INITIAL-BOUNDARY VALUE PROBLEM FOR 2D MICROPOLAR EQUATIONS WITHOUT ANGULAR VISCOSITY

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Abstract. This paper concerns the initial-boundary value problem to 2D micropolar equations without angular viscosity in a smooth bounded domain. It is shown that such a system admits a unique and global weak solution. The main idea of this paper is to fully exploit the structure of this system and establish high order estimates via introducing an auxiliary field which is at the energy level of one order lower than micro-rotation.

1. Introduction and main results

This paper is devoted to the initial-boundary value problem to the two-dimensional (2D) micropolar equations without angular viscosity. The micropolar equations were introduced in 1965 by C.A. Eringen to model micropolar fluids (see, e.g., [6]). Micropolar fluids are fluids with microstructure. Certain anisotropic fluids, e.g. liquid crystals which are made up of dumbbell molecules, are of this type. The standard 3D incompressible micropolar equations are given by

\[
\begin{align*}
\mathbf{u}_t - (\nu + \kappa)\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi &= 2\kappa \nabla \times \mathbf{w}, \\
\mathbf{w}_t - \gamma \Delta \mathbf{w} + 4\kappa \mathbf{w} - \mu \nabla \nabla \cdot \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{w} &= 2\kappa \nabla \times \mathbf{u}, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( \mathbf{u} = \mathbf{u}(x,t) \) denotes the fluid velocity, \( \pi(x,t) \) the scalar pressure, \( \mathbf{w}(x,t) \) the micro-rotation field, and the parameter \( \nu \) represents the Newtonian kinematic viscosity, \( \kappa \) the micro-rotation viscosity, \( \gamma \) and \( \mu \) the angular viscosities.

Roughly speaking, they belong to a class of non-Newtonian fluids with nonsymmetric stress tensor (called polar fluids) and include, as a special case, the classical fluids modeled by the Navier-Stokes equations. In fact, when micro-rotation effects are neglected, namely \( w = 0 \), (1.1) reduces to the incompressible Navier-Stokes equations. The micropolar equations are significant generalizations of the Navier-Stokes equations and cover many more phenomena such as fluids consisting of particles suspended in a viscous medium. The micropolar equations have been extensively applied and studied by many engineers and physicists.

Key words and phrases. Initial-boundary value problem, 2D micropolar equations, partial viscosity.

2010 Mathematics Subject Classification. 35Q35, 76D03.
Because of their physical applications and mathematical significance, the well-posedness problem on the micropolar equations have attracted considerable attention recently from the community of mathematical fluids \[1, 2, 4, 13\]. Lukaszewicz in his monograph \[15\] studied the well-posedness problem on the 3D stationary model as well as the time-dependent micropolar equations. In spite of previous progress on the 3D case, just like the 3D Navier-Stokes equations, the problem of global regularity or finite time singularity for strong solutions of the 3D micropolar fluid is still widely open. Therefore, more attention is focused on the 2D micropolar equations, which are a special case of the 3D micropolar equations. In the special case when \[u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0), \pi = \pi(x_1, x_2, t), w = (0, 0, w(x_1, x_2, t)),\]

the 3D micropolar equations reduce to the 2D micropolar equations,

\[
\begin{align*}
\mathbf{u}_t - (\nu + \kappa)\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi &= -2\kappa \nabla^\perp \mathbf{w}, \\
\mathbf{w}_t - \gamma \Delta \mathbf{w} + 4\kappa \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{w} &= 2\kappa \nabla^\perp \cdot \mathbf{u}, \\
\nabla \cdot \mathbf{u} &= 0.
\end{align*}
\]

(1.2)

Here \(\mathbf{u} = (u_1, u_2)\) is a 2D vector with the corresponding scalar vorticity \(\Phi\) given by

\[\Phi \equiv \nabla^\perp \cdot \mathbf{u} = \partial_1 u_2 - \partial_2 u_1,\]

while \(\omega\) represents a scalar function with

\[\nabla^\perp \mathbf{w} = (-\partial_2 \mathbf{w}, \partial_1 \mathbf{w}).\]

In \[3\], Dong and Chen obtained the global existence and uniqueness, and sharp algebraic time decay rates for the 2D micropolar equations \[12\]. Despite all this, the global regularity problem for the inviscid equation is currently out of reach. Therefore, more recent efforts are focused on the 2D micropolar equation with partial viscosity, which naturally bridge the inviscid micropolar equation and the micropolar equation with full viscosity. In \[3\], Dong and Zhang examined \(1.2\) with the micro-rotation viscosity \(\gamma = 0\) and established the global regularity. Another partial viscosity case, \(1.2\) with \(\nu = 0, \gamma > 0, \kappa > 0\) and \(\kappa \neq \gamma\), was examined by Xue, who was able to obtain the global well-posedness in the frame work of Besov spaces \[20\]. Recently, Dong, Li and Wu took on the case when \(1.2\) involves only the angular viscosity \[4\], in which they proved the global (in time) regularity.

Most of the results we mentioned above are for the whole space \(\mathbb{R}^2\) or \(\mathbb{R}^3\). In many real-world applications, the flows are often restricted to bounded domains with suitable constraints imposed on the boundaries and these applications naturally lead to the studies of the initial-boundary value problems. In addition, solutions of the initial-boundary value problems may exhibit much richer phenomena than those of the whole space counterparts. Up to now, the case when \(\nu > 0, \kappa > 0\) and \(\gamma > 0\) has been
extensively analyzed by [18] for 2D case with periodic boundary conditions and [21] for 3D case with small initial data respectively.

However, the progress on initial-boundary value problem for (1.2) with partial viscosity is quite limited. For the case with only the angular viscosity, it has been solved by Jiu, Liu, Wu and Yu in [12]. While, the initial-boundary value problem for the opposite case, namely
\[
\begin{align*}
\begin{cases}
    u_t - (\nu + \kappa) \Delta u + u \cdot \nabla u + \nabla \pi &= -2\kappa \nabla^\perp w, \\
    w_t + 4\kappa w + u \cdot \nabla w &= 2\kappa \nabla^\perp \cdot u, \\
    \nabla \cdot u &= 0,
\end{cases}
\end{align*}
\] (1.3)

is still open. In this paper, we investigate the initial-boundary value problem of system (1.3) with the natural boundary condition
\[
\left. u \right|_{\partial \Omega} = 0 \quad (1.4)
\]
and the initial condition
\[
(u, w)(x, 0) = (u_0, w_0)(x), \quad \text{in } \Omega, \quad (1.5)
\]
where \( \Omega \subset \mathbb{R}^2 \) represents a bounded domain with smooth boundary. Besides, we also impose the following compatibility conditions
\[
\begin{align*}
\begin{cases}
    u_0|_{\partial \Omega} = 0, \ \nabla \cdot u_0 = 0, \\
    -(\nu + \kappa) \Delta u_0 + u_0 \cdot \nabla u_0 + \nabla \pi_0 &= -2\kappa \nabla^\perp w_0, \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\] (1.6)
where \( \pi_0 \) is determined by the Neumann boundary problem
\[
\begin{align*}
\begin{cases}
    \Delta \pi_0 &= -\nabla \cdot [u_0 \cdot \nabla u_0], \\
    \nabla \pi_0 \cdot n &= [(-\nu + \kappa) \Delta u_0 - 2\kappa \nabla^\perp w_0 - u_0 \cdot \nabla u_0] \cdot n, \quad \text{on } \partial \Omega.
\end{cases}
\end{align*}
\] (1.7)

Our goal here is to establish the global existence and uniqueness of weak solutions to (1.3)-(1.5) by given the least regularity assumptions on the initial data, and obtain the following result.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Assume \((u_0, w_0)\) satisfies
\[
u_0 \in H^2(\Omega), \quad w_0 \in W^{1,4}(\Omega)
\]
and the compatibility conditions (1.6) and (1.7). Then (1.3)-(1.5) has a unique global smooth solution \((u, w)\) satisfying
\[
\begin{align*}
    u \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; W^{2,4}(\Omega)), \quad w \in L^\infty(0, T; W^{1,4}(\Omega))
\end{align*}
\] (1.8)
for any \( T > 0 \).
We remark that the initial-boundary value problem on (1.3) is not trivial and quite different from the Cauchy problem. The difficulty is due to the dynamic micro-rotational term $\nabla_\perp w$ in the velocity equation, which prevents us to obtain any high order estimates except the basic energy estimate. For the Cauchy problem, there is no boundary conditions and therefore the equation of vorticity $\Phi$

$$\Phi_t - (\nu + \kappa)\Delta \Phi + \mathbf{u} \cdot \nabla \Phi + 2\kappa \Delta w = 0$$

is available. To overcome this difficulty, the authors in [5] observe that the sum of the vorticity and micro-rotation angular velocity

$$Z = \Phi - \frac{2\kappa}{\nu + \kappa} w$$

satisfies the transport-diffusion equation

$$\partial_t Z - (\nu + \kappa)\Delta Z + \mathbf{u} \cdot \nabla Z = \left( \frac{8\kappa^2}{\nu + \kappa} - \frac{8\kappa^3}{(\nu + \kappa)^2} \right) w - \frac{4\kappa^2}{\nu + \kappa} Z,$$

which helps them to obtain the global bound $\|\Phi(t)\|_{L^\infty(\mathbb{R}^2)}$ via the global bound of $\|Z(t)\|_{L^\infty(\mathbb{R}^2)}$, and therefore establish the desired high order estimates.

However, for the initial-boundary value problem, this method does not work. This is due to the presence of no-slip boundary condition for $\mathbf{u}$, and hence the transport-diffusion equation satisfied by $\Phi$ and $Z$ would not work any more. To overcome the difficulty caused by the term $\nabla_\perp w$, our strategy is to utilize an auxiliary field $\mathbf{v}$ which is at the energy level of one order lower than $\mathbf{w}$ and with appropriate boundary condition. Keep this in mind, we then introduce the vector field $\mathbf{v} = -\frac{2\kappa}{\nu + \kappa} A^{-1} \nabla_\perp w$ be the unique solution of the stationary Stokes system with source term $-\frac{2\kappa}{\nu + \kappa} A^{-1} \nabla_\perp w$

$$\begin{cases}
-\Delta \mathbf{v} + \nabla \pi = -\frac{2\kappa}{\nu + \kappa} \nabla_\perp w & \text{in } \Omega, \\
\nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\
\mathbf{v} = 0 & \text{on } \partial \Omega.
\end{cases}$$

Thanks to (1.9), it is clear to deduce the field $\mathbf{v}$ also solves, after taking the operator $A^{-1} \nabla_\perp$ on (1.3)$^2$, that

$$\partial_t \mathbf{v} + 4\kappa \mathbf{v} - 2\kappa A^{-1} \nabla_\perp (\nabla_\perp \cdot \mathbf{u}) + A^{-1} \nabla_\perp (\mathbf{u} \cdot \nabla w) = 0. \quad (1.10)$$

On this basis, we further discover that the new field $\mathbf{g} = \mathbf{u} - (\nu + \kappa) \mathbf{v}$ satisfies the system

$$\begin{cases}
\partial_t \mathbf{g} - (\nu + \kappa)\Delta \mathbf{g} + \nabla p = Q & \text{in } \Omega, \\
\nabla \cdot \mathbf{g} = 0 & \text{in } \Omega, \\
\mathbf{g} = 0 & \text{on } \partial \Omega, \quad (1.11)
\end{cases}$$

according to (1.3), (1.9) and (1.10), where $Q = -\mathbf{u} \cdot \nabla \mathbf{u} - A^{-1} \nabla_\perp (\mathbf{u} \cdot \nabla \mathbf{w}) + 2\kappa A^{-1} \nabla_\perp (\nabla_\perp \cdot \mathbf{u}) - 4\kappa \mathbf{v}$. The obvious advantage of doing so lies in that it provides the cornerstone of establishing $H^1$-norm estimates of velocity $\mathbf{u}$. As a result, after noticing that $\mathbf{v}$ is at the
energy level of one order lower than \( w \) and some careful \textit{a priori} estimates for \( g \), we can successfully establish the desired high order estimates, which guarantees the global existence and uniqueness of weak solutions to the system \((1.3)-(1.5)\).

The rest of this paper is divided into four sections. The second section serves as a preparation and presents a list of facts and tools for bounded domains such as embedding inequalities and logarithmic type interpolation inequalities. Section 3 establishes the \textit{a priori} estimates, which is necessary in the proof of Theorem 1.1. Section 4 completes the proof of Theorem 1.1.

2. Preliminaries

This section serves as a preparation. We list a few basic tools for bounded domains to be used in the subsequent sections. In particular, we provide the Gagliardo-Nirenberg type inequalities, the logarithmic type interpolation inequalities and regularization estimates for elliptic equations and Stokes system in bounded domains. These estimates will also be handy for future studies on PDEs in bounded domains.

We start with the well-known Gagliardo-Nirenberg inequality for bounded domains (see, e.g., [17]).

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. Let \( 1 \leq p, q, r \leq \infty \) be real numbers and \( j \leq m \) be non-negative integers. If a real number \( \alpha \) satisfies

\[
\frac{1}{p} - \frac{j}{n} = \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \alpha) \frac{1}{q}, \quad \frac{j}{m} \leq \alpha \leq 1,
\]

then

\[
\|D^j f\|_{L^p(\Omega)} \leq C_1 \|D^m f\|_{L^r(\Omega)} \|f\|_{L^q(\Omega)}^{1-\alpha} + C_2 \|f\|_{L^s(\Omega)},
\]

where \( s > 0 \), and the constants \( C_1 \) and \( C_2 \) depend upon \( \Omega \) and the indices \( p, q, r, m, j, s \) only.

Especially, the following special cases will be used.

**Corollary 2.1.** Suppose \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary, then

1. \( \|f\|_{L^4(\Omega)} \leq C (\|f\|_{L^2(\Omega)}^{\frac{3}{2}} \|\nabla f\|_{L^2(\Omega)}^{\frac{1}{2}} + \|f\|_{L^2(\Omega)}), \forall f \in H^1(\Omega); \)
2. \( \|\nabla f\|_{L^4(\Omega)} \leq C (\|f\|_{L^2(\Omega)}^{\frac{3}{2}} \|\nabla^2 f\|_{L^2(\Omega)}^{\frac{1}{2}} + \|f\|_{L^2(\Omega)}), \forall f \in H^2(\Omega); \)
3. \( \|f\|_{L^\infty(\Omega)} \leq C (\|f\|_{L^2(D)}^{\frac{3}{2}} \|\nabla^2 f\|_{L^2(D)}^{\frac{1}{2}} + \|f\|_{L^2(D)}), \forall f \in H^2(\Omega); \)
4. \( \|f\|_{L^\infty(\Omega)} \leq C (\|f\|_{L^2(D)}^{\frac{3}{2}} \|\nabla^3 f\|_{L^2(D)}^{\frac{1}{2}} + \|f\|_{L^2(D)}), \forall f \in H^3(\Omega). \)
The next lemmas state the regularization estimates for elliptic equations and Stokes system defined on bounded domains (see, e.g., [8, 9, 11, 13, 19]).

Lemma 2.2. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Consider the elliptic boundary value problem

\[
\begin{align*}
-\Delta f &= g \quad \text{in } \Omega, \\
  f &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(2.1)

If, for \( p \in (1, \infty) \) and an integer \( m \geq -1 \), \( g \in W^{m,p}(\Omega) \), then (2.1) has a unique solution \( f \) satisfying

\[
\|f\|_{W^{m+2,p}(\Omega)} \leq C\|g\|_{W^{m,p}(\Omega)},
\]

where \( C \) depending only on \( \Omega, m \) and \( p \).

Lemma 2.3. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Consider the stationary Stokes system

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega, \\
  \nabla \cdot u &= 0 \quad \text{in } \Omega, \\
  u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(2.2)

If, for \( q \in (1, \infty) \), \( f \in L^q(\Omega) \), then there exists a unique solution \( u \in W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \) of (2.2) satisfying

\[
\|u\|_{W^{2,q}(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)},
\]

(2.3)

If \( f = \nabla \cdot F \) with \( F \in L^q(\Omega) \), then

\[
\|u\|_{W^{1,q}(\Omega)} \leq C\|F\|_{L^q(\Omega)}.
\]

(2.4)

Besides, if \( f = \nabla \cdot F \) with \( F_{ij} = \partial_k H^k_{ij} \) and \( H^k_{ij} \in W^{1,q}_0(\Omega) \) for \( i, j, k = 1, \ldots, N \), then

\[
\|u\|_{L^q(\Omega)} \leq C\|H\|_{L^q(\Omega)}.
\]

(2.5)

Here, all the above constants \( C \) depend only on \( \Omega \) and \( q \).

Lemma 2.4. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary and \( f = \nabla \cdot F \) be the same as in system (2.2), then for \( F \in W^{1,q}(\Omega) \) with \( q \in (2, \infty) \), the solution \( u \) of system (2.2) satisfies

\[
\|\nabla u\|_{L^\infty(\Omega)} \leq C(1 + \|F\|_{L^\infty(\Omega)})\ln(e + \|\nabla F\|_{L^q(\Omega)}),
\]

(2.6)

where \( C \) depending only on \( \Omega \).
Lemma 2.5. Let $1 < p, q < \infty$, and suppose that $f \in L^p(0, T; L^q(\Omega))$, $u_0 \in W^{2,p}(\Omega)$. If $(u, p)$ is the solution of the Stokes system

$$
\begin{align*}
&\partial_t u - \Delta u + \nabla p = f \quad \text{in } \Omega, \\
&\nabla \cdot u = 0 \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial\Omega, \\
&u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{align*}
$$

(2.7)

then there holds that

$$
\|\partial_t u, \nabla^2 u, \nabla p\|_{L^p(0,T;L^q(\Omega))} \leq C(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{W^{2,p}(\Omega)}).
$$

(2.8)

3. A priori estimates

This section is devoted to establishing the a priori estimates of (1.3)-(1.5), which is an important step in the proof of Theorem 1.1. To be more precise, we first provide the definition of weak solutions of (1.3)-(1.5) and then state the main result of this section as a proposition.

Definition 3.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, w_0) \in H^1(\Omega)$. A pair of measurable functions $(u, w)$ is called a weak solution of (1.3)-(1.5) if

1. $u \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \ w \in C(0, T; L^2(\Omega));$

2. $\int_\Omega u_0 \cdot \varphi_0 dx + \int_0^T \int_\Omega \left[ u \cdot \varphi_t - (\nu + \kappa) \nabla u \cdot \nabla \varphi + u \cdot \nabla \varphi \cdot u + 2\kappa \nabla \perp w \cdot \varphi \right] dx dt = 0,$

$$
\int_\Omega w_0 \cdot \psi_0 dx + \int_0^T \int_\Omega \left[ w \psi_t + 4\kappa w \psi + u \cdot \nabla \psi \cdot w - 2\kappa \nabla \perp \cdot u \psi \right] dx dt;
$$

holds for any $(\varphi, \psi) \in C^\infty([0, T] \times \Omega)$ with $\nabla \cdot \varphi = \varphi|_{\partial\Omega} = \varphi(x, T) = 0$ and $\psi(x, T) = 0$.

The main result of this section is stated in the following proposition.

Proposition 3.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $(u, w)$ be the smooth solution of (1.3)-(1.5). Assume $u_0 \in H^2(\Omega)$ and $w_0 \in W^{1,4}(\Omega)$, then there holds that

$$
\|u\|_{L^\infty(0,T;H^1_0(\Omega))} + \|u\|_{L^2(0,T;W^{2,4}(\Omega))} + \|w\|_{L^\infty(0,T;W^{1,4}(\Omega))} \leq C,
$$

(3.1)

where $C$ depends only on $T$, $\|u_0\|_{H^2(\Omega)}$ and $\|w_0\|_{W^{1,4}(\Omega)}$.

The proof of this proposition relies on the following basic energy estimates.
Proposition 3.2. Suppose $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $(u, w)$ be the smooth solution of (1.3) - (1.5). If, in addition, $u_0 \in L^2(\Omega)$ and $w_0 \in L^2(\Omega)$, then it holds that
\[
\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T; H^1_0(\Omega))} + \|w\|_{L^\infty(0,T;L^2(\Omega))} \leq C,
\]
where $C$ depends only on $T$, $\|u_0\|_{L^2(\Omega)}$ and $\|w_0\|_{L^2(\Omega)}$.

Proof. We start with the global $L^2$-bound. Taking the inner product of (1.3) with $(u, w)$ yields
\[
\frac{1}{2} \frac{d}{dt}(\|u\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) + (\nu + \kappa)\|\nabla u\|_{L^2(\Omega)}^2 + 4\kappa\|w\|_{L^2(D)}^2 = -2\kappa \int_D \nabla \cdot u \cdot w \, dx + 2\kappa \int_\Omega \nabla \cdot w \, dx.
\]
Noticing that $\nabla \cdot u = \partial_1 u_2 - \partial_2 u_1$ and $\nabla \cdot w = (\partial_2 w, \partial_1 w)$, we have
\[
-\nabla \cdot u = u_1 \partial_2 w - u_2 \partial_1 w = \partial_2(u_1 w) - \partial_1(u_2 w) + \nabla \cdot w.
\]
Integrating by parts and applying the boundary condition for $u$, we have
\[
-2\kappa \int_\Omega \nabla \cdot u \cdot w \, dx + 2\kappa \int_\Omega \nabla \cdot w \, dx = 4\kappa \int_\Omega \nabla \cdot w \, dx - 2\kappa \int_{\partial\Omega} u \cdot n^\perp w \, ds
\]
\[
\leq \frac{(\nu + \kappa)}{2} \|\nabla u\|_{L^2(\Omega)}^2 + C\|w\|_{L^2(\Omega)}^2,
\]
where $n^\perp = (-n_2, n_1)$. It then follows, after integration in time, that
\[
\|u\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 + (\nu + \kappa) \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 \, dt + 8\kappa \int_0^T \|w\|_{L^2(\Omega)}^2 \, dt
\]
\[
\leq e^{CT}(\|u_0\|_{L^2(\Omega)}^2 + \|w_0\|_{L^2(\Omega)}^2) \equiv A(T, \|u_0\|, \|w_0\|_{L^2}),
\]
where $C = C(\gamma, \kappa)$. This completes the proof of Proposition 3.2.

Our next goal is to show the global bound for $\|u\|_{H^1(\Omega)}$. As stated in the introduction, $v$ is at the energy level of one order lower than $w$. Then for system (1.9), by setting
\[
F = 2\kappa \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix},
\]
we can then invoke Lemma 2.3 to build up the estimates that
\[
\|v\|_{W^{1,q}(\Omega)} \leq C\|w\|_{L^q(\Omega)}
\]
for any $q \in (1, \infty)$, which also yields, after applying Lemma 3.2
\[
\|v\|_{L^\infty(0,T;H^1(\Omega))} \leq C\|w\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|u_0\|_{L^2(\Omega)}^2 + \|w_0\|_{L^2(\Omega)}^2).
\] (3.5)

Therefore, to establish the $H^1(\Omega)$ estimates of velocity, it suffices to do the $H^1$-norm estimates of $g = u - (\nu + \kappa)v$ as follows.

**Lemma 3.1.** Under the assumptions of Proposition 3.2, we further assume $u_0 \in H^1(\Omega)$ and $w_0 \in L^4(\Omega)$, then we obtain
\[
\|\nabla g\|_{L^\infty(0,T;L^2(\Omega))} + \|\Delta g\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{L^\infty(0,T;L^4(\Omega))} + \|w\|_{L^2(0,T;L^4(\Omega))} \leq C,
\]
where $C$ depends only on $T$, $\|u_0\|_{H^1(\Omega)}$ and $\|w_0\|_{L^4(\Omega)}$.

**Proof.** Taking inner product of $(1.11)\ ^1$ with $-\Delta g$, and applying the boundary condition $g|_{\partial\Omega} = 0$, the Cauchy-Schwarz inequality, we have
\[
\frac{1}{2} \frac{d}{dt}\|\nabla g\|_{L^2(\Omega)}^2 + (\nu + \kappa)\|\Delta g\|_{L^2(\Omega)}^2 = -\int_\Omega Q \cdot \Delta g \, dx
\]
\[
\leq \|Q\|_{L^2(\Omega)} \|\Delta g\|_{L^2(\Omega)}
\]
\[
\leq \frac{(\nu + \kappa)}{8} \|\Delta g\|_{L^2(\Omega)}^2 + C\|Q\|_{L^2(\Omega)}^2,
\]
(3.6)

with
\[
\|Q\|_{L^2(\Omega)}^2 \leq C\|u \cdot \nabla u\|_{L^2(\Omega)}^2 + C\|A^{-1}\nabla^\perp(u \cdot \nabla w)\|_{L^2(\Omega)}^2
\]
\[
+ C\|A^{-1}\nabla^\perp(\nabla^\perp \cdot u)\|_{L^2(\Omega)}^2 + C\|v\|_{L^2(\Omega)}^2
\]
\[
= \sum_{i=1}^4 I_i.
\] (3.7)

Next, we will estimate the four terms one by one. By applying Hölder inequality, Corollary 2.1 (3.4), Lemma 2.2 and Young inequality, it follows that
\[
I_1 \leq C\|u \cdot \nabla g\|_{L^2(\Omega)}^2 + C\|u \cdot \nabla v\|_{L^2(\Omega)}^2
\]
\[
\leq C\|u\|_{L^2(\Omega)}^2\|\nabla g\|_{L^2(\Omega)}^2 + C\|u \cdot \nabla v\|_{L^2(\Omega)}^2
\]
\[
\leq C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)}\|\Delta g\|_{L^2(\Omega)}
\]
\[
+ C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}^2
\]
\[
\leq C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)}\|\Delta g\|_{L^2(\Omega)}
\]
\[
+ C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}^2
\]
\[
\leq \frac{(\nu + \kappa)}{8} \|\Delta g\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\|\nabla u\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}^2 \|\nabla g\|_{L^2(\Omega)}^2
\]
\[
+ C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}^2.
\] (3.8)
Regarding the left terms, from the incompressible condition $\nabla \cdot u = 0$ and the boundary condition $u|_{\partial \Omega} = 0$, we can infer that $u \cdot \nabla w = \nabla \cdot (uw)$ and $uw|_{\partial \Omega} = 0$. Therefore, by using H"older inequality, Corollary 2.1 and Lemma 2.3, we obtain

$$I_2 + I_3 + I_4$$

$$\leq C||A^{-1}\nabla \cdot (uw)||_{L^2(\Omega)}^2 + C||A^{-1}\nabla (\nabla \cdot u)||_{L^2(\Omega)}^2 + C||v||_{L^2(\Omega)}^2$$

$$\leq C||uw||_{L^2(\Omega)}^2 + C||u||_{L^2(\Omega)}^2 + C||w||_{L^2(\Omega)}^2$$

$$\leq C||u||_{L^2(\Omega)}^2 ||\nabla u||_{L^2(\Omega)} ||w||_{L^2(\Omega)}^2 + C(||u||_{L^2(\Omega)}^2 + ||w||_{L^2(\Omega)}^2).$$

Finally, we add up the estimates from (3.6) to (3.9), it yields that

$$\frac{1}{2} \frac{d}{dt} ||\nabla g||_{L^2(\Omega)}^2 + \frac{3(\nu + \kappa)}{4} ||\Delta g||_{L^2(\Omega)}^2 \leq C(\frac{1}{2} ||u||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}^2 ||\nabla u||_{L^2(\Omega)}^2) ||\nabla g||_{L^2(\Omega)}^2$$

$$+ C(||u||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)} ||w||_{L^2(\Omega)}^2).$$

Clearly, (3.10) is not a closed estimate still because the bound of $||w||_{L^4(\Omega)}$ is unknown. However, we discover that the estimate of $||w||_{L^4(\Omega)}$ can be bounded in turn by $||\nabla g||_{L^2(\Omega)}$ and $||\Delta g||_{L^2(\Omega)}$. This motivates us to search for the closed estimates of $||\nabla g||_{L^2(0,T;L^2(\Omega))}$ and $||w||_{L^2(0,T;L^4(\Omega))}$. To start with, by multiplying (1.3) with $|w|^3 w$ and integrating on $\Omega$, we have

$$\frac{1}{4} \frac{d}{dt} ||w||_{L^4(\Omega)}^4 + 4\kappa ||w||_{L^4(\Omega)}^4 = 2\kappa \int_{\Omega} \nabla \cdot u |w|^3 w \, dx$$

$$\leq C||\nabla u||_{L^4(\Omega)} ||w||_{L^4(\Omega)}^3$$

$$\leq C(||\nabla g||_{L^4(\Omega)} + ||\nabla v||_{L^4(\Omega)}) ||w||_{L^4(\Omega)}^3$$

$$\leq C(||\nabla g||_{L^2(\Omega)} + ||\nabla g||_{L^2(\Omega)}^2 ||\Delta g||_{L^2(\Omega)}^2 + ||w||_{L^4(\Omega)} ||w||_{L^4(\Omega)}^3).$$

which further implies, after dividing $||w||_{L^4(\Omega)}^2$ on both sides, that

$$\frac{1}{2} \frac{d}{dt} ||w||_{L^4(\Omega)}^2 + 4\kappa ||w||_{L^4(\Omega)}^2$$

$$\leq C(||\nabla g||_{L^2(\Omega)} + ||\nabla g||_{L^2(\Omega)}^2 ||\Delta g||_{L^2(\Omega)}^2 + ||w||_{L^4(\Omega)} ||w||_{L^4(\Omega)}$$

$$\leq \frac{3(\nu + \kappa)}{4} ||\Delta g||_{L^2(\Omega)}^2 + C(||\nabla g||_{L^2(\Omega)}^2 + ||w||_{L^4(\Omega)}^2).$$

Subsequently, by summing up (3.10)-(3.12) and some basic calculations, we finally obtain

$$\frac{1}{2} \frac{d}{dt} (||\nabla g||_{L^2(\Omega)}^2 + ||w||_{L^4(\Omega)}^2) + \frac{(\nu + \kappa)}{2} ||\Delta g||_{L^2(\Omega)}^2 + 4\kappa ||w||_{L^4(\Omega)}^2$$

$$\leq C(1 + ||u||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)} ||\nabla g||_{L^2(\Omega)} + ||w||_{L^4(\Omega)}^2).$$
This together with Gronwall’s inequality and (3.3) then yield the following bound
\[ \| \nabla g \|^2_{L^2(\Omega)} + \| w \|^2_{L^4(\Omega)} + (\nu + \kappa) \int_0^T \| \Delta g \|^2_{L^2(\Omega)} dt + 8\kappa \int_0^T \| \nabla w \|^2_{L^4(\Omega)} dt \leq C_1 e^{C_2 T} (\| \nabla u_0, w_0 \|^2_{L^2(\Omega)} + \| w_0 \|^2_{L^4(\Omega)}) \equiv A_2(T), \]
where \( C_1 = C_1(\nu, \kappa), C_2 = C_2(\nu, \kappa, \| u_0, w_0 \|_{L^2(\Omega)}) \). This completes the proof of Lemma 3.1.

Although we have derived the estimate of \( \| u \|_{L^\infty(0,T; H^1(\Omega))} \), to prove the global existence of solutions, we still need the estimate of \( \| u \|_{L^2(0,T; H^2(\Omega))} \). Therefore, even with the help of estimate \( \| g \|_{L^2(0,T; H^2(\Omega))} \), we still need the global bound of \( \| v \|_{L^2(0,T; H^2(\Omega))} \).

Namely, we should prove that \( \| w \|_{L^2(0,T; H^1(\Omega))} \) is globally bound according to Lemma 2.3. To achieve this, we firstly establish the bound of \( \| w \|_{L^\infty(0,T; L^2(\Omega))} \).

**Proposition 3.3.** In addition to the conditions in Lemma 3.1, if we further assume \( w_0 \in L^p(\Omega) \) for any \( 2 \leq p \leq \infty \), then the micro-rotation \( w \) obeys the global bound

\[ \| w \|_{L^\infty(0,T; L^2(\Omega))} \leq C, \]

where \( C \) depends only on \( T \), \( \| u_0 \|_{H^1(\Omega)} \) and \( \| w_0 \|_{L^2(\Omega)} \).

**Proof.** We start with the equation of \( w \), namely (1.3)\(^2\). For any \( 2 \leq q < \infty \), multiplying (1.3)\(^2\) with \( |w|^{q-2}w \) and integrating on \( \Omega \), we obtain

\[ \frac{1}{q} \frac{d}{dt} \| w \|^q_{L^q(\Omega)} + 4\kappa \| w \|^q_{L^q(\Omega)} \leq 2\kappa \| \nabla u \|_{L^q(\Omega)} \| w \|_{L^q(\Omega)}^{q-1}, \]

i.e.,

\[ \frac{d}{dt} \| w \|_{L^q(\Omega)} + 4\kappa \| w \|_{L^q(\Omega)} \leq 2\kappa \| \nabla u \|_{L^q(\Omega)}. \]

Then, by employing the definition of \( g \), (3.4) and Sobolev embedding inequalities, we further have

\[ \frac{d}{dt} \| w \|_{L^q(\Omega)} + 4\kappa \| w \|_{L^q(\Omega)} \leq C \| \nabla g \|_{L^q(\Omega)} + C \| \nabla v \|_{L^q(\Omega)} \leq C \| g \|_{W^{1,q}(\Omega)} + C \| w \|_{L^q(\Omega)} \leq C \| g \|_{H^2(\Omega)} + C \| w \|_{L^q(\Omega)} \leq C \| u \|_{L^2(\Omega)} + C \| w \|_{L^2(\Omega)} + C \| \Delta g \|_{L^2(\Omega)} + C \| w \|_{L^q(\Omega)} \]

which, according to Gronwall’s inequality, Proposition 3.2 and Lemma 3.1 implies

\[ \| w \|_{L^q(\Omega)} + 4\kappa \int_0^T \| w \|_{L^q(\Omega)} dt \leq e^{CT} \left[ \| w_0 \|_{L^q(\Omega)} + \int_0^T (\| u \|_{L^2(\Omega)} + \| w \|_{L^2(\Omega)} + \| \Delta g \|_{L^2(\Omega)}) dt \right]. \]
This completes the proof of Proposition 3.3.

We now move on to the next lemma asserting the global bound for $\|g\|_{L^2(0,T;W^{2,q}(\Omega))}$.

**Lemma 3.2.** Under the assumptions of Proposition 3.3, if in addition, $u_0 \in H^2(\Omega)$ and $w_0 \in H^1(\Omega)$, then the inequality

$$
\|g\|_{L^2(0,T;W^{2,q}(\Omega))} \leq C
$$

holds for any $2 < q < \infty$, where $C$ depends only on $T$, $\|u_0\|_{H^2(\Omega)}$ and $\|w_0\|_{H^1(\Omega)}$.

**Proof.** Initially, by applying Lemma 2.5 to (1.11) and Lemma 2.3, it is clear that

\[
\|w\|_{L^\infty(\Omega)} + 4 \kappa \int_0^T \|w\|_{L^\infty(\Omega)} dt 
\leq C(T).
\]

Noting that $C$ is independent of $p$, we then derive, by letting $p \to \infty$,

\[
\|w\|_{L^\infty(\Omega)} + 4 \kappa \int_0^T \|w\|_{L^\infty(\Omega)} dt 
\leq e^{CT} \left[ \|w_0\|_{L^\infty(\Omega)} + \int_0^T \left( \|u\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} + \|\Delta g\|_{L^2(\Omega)} \right) dt \right]
\leq C(T).
\]

For the first term, by employing Hölder inequality, Sobolev embedding inequalities, and Proposition 3.3, we have

\[
I_1 \leq \|u\|_{L^\infty(0,T;L^{2n}(\Omega))} \|\nabla u\|_{L^2(0,T;L^{2n}(\Omega))}
\leq C \|u\|_{L^\infty(0,T;H^1(\Omega))} \|\nabla g\|_{L^2(0,T;L^{2n}(\Omega))} + \|\nabla v\|_{L^2(0,T;L^{2n}(\Omega))}
\leq C \left( \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \right) \left( \|g\|_{L^2(0,T;H^2(\Omega))} + \|w\|_{L^2(0,T;L^2(\Omega))} \right)
\leq C \left( \|\nabla g\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla v\|_{L^\infty(0,T;L^2(\Omega))} \right) \left( \|g\|_{L^2(0,T;L^2(\Omega))} + \|\Delta g\|_{L^2(0,T;L^2(\Omega))} \right)
\leq C \left( \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \right) \left( \|g\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;L^2(\Omega))} \right)
\leq C(T)
\]
As for the remaining terms, by using the equality $\mathbf{u} \cdot \nabla \mathbf{w} = \nabla \cdot (\mathbf{uw})$ and same tools as estimating (3.17), it follows that

$$I_2 + I_3 + I_4 \leq \|A^{-1} \nabla \perp (\mathbf{uw})\|_{L^2(0,T;L^q(\Omega)))} + \|A^{-1} \nabla \perp (\nabla \perp \cdot \mathbf{u})\|_{L^2(0,T;L^q(\Omega)))} + \|\mathbf{v}\|_{L^2(0,T;L^q(\Omega)))}$$

$$\leq C\|\mathbf{uw}\|_{L^2(0,T;L^q(\Omega)))} + C\|\mathbf{u}\|_{L^2(0,T;L^q(\Omega)))} + C\|\mathbf{w}\|_{L^2(0,T;L^q(\Omega)))}$$

$$\leq C\|\mathbf{u}\|_{L^2(0,T;L^q(\Omega)))} + C\|\mathbf{w}\|_{L^2(0,T;L^q(\Omega)))} + C\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega)))} + C\|\mathbf{w}\|_{L^2(0,T;H^1(\Omega)))}$$

$$\leq C\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega)))} + C\|\mathbf{w}\|_{L^2(0,T;H^1(\Omega)))} + C\|w\|_{L^2(0,T;L^q(\Omega)))}$$

$$\leq C(T). \quad (3.18)$$

Thus, through summing up the estimates from (3.15) to (3.18) and applying Proposition 3.2 again, we finally prove that $\|g\|_{L^2(0,T;W^{2,q}(\Omega)))} \leq C(T)$. \hfill \Box

Finally, to guarantee the global existence and uniqueness of weak solutions both, we further need the global bound for $\|\nabla \mathbf{w}\|_{L^q(0,T;L^q(\Omega)))}$. And now, we get to work on it.

**Proposition 3.4.** In addition to the conditions in Lemma 3.2, we further assume $\nabla \mathbf{w}_0 \in L^q(\Omega)$ for any $2 < q < \infty$, we then derive the global bound

$$\|\nabla \mathbf{w}\|_{L^q(0,T;L^q(\Omega)))} \leq C,$$

where $C$ depends only on $T$, $\|\mathbf{u}_0\|_{H^2(\Omega)}$, $\|\mathbf{w}_0\|_{H^1(\Omega)}$ and $\nabla \mathbf{w}_0 \in L^q(\Omega)$.

**Proof.** Taking the first-order partial $\partial_i$ of (3.3) yields,

$$\partial_i \mathbf{w}_i + 4\kappa \partial_i \mathbf{w} + \mathbf{u} \cdot \nabla \partial_i \mathbf{w} + \partial_i \mathbf{u} \cdot \nabla \mathbf{w} = 2\kappa \partial_i \nabla \perp \cdot \mathbf{u}. \quad (3.19)$$

Then, for any $2 < q < \infty$, multiplying (3.19) with $|\partial_i \mathbf{w}|^{q-2} \partial_i \mathbf{w}$, summing over $i$ and integrating on $\Omega$, we obtain

$$\frac{1}{q} \frac{d}{dt} \|\nabla \mathbf{w}\|_{L^q(\Omega))}^q + 4\kappa \|\nabla \mathbf{w}\|_{L^q(\Omega))}^q \leq \|\nabla \mathbf{u}\|_{L^q(\Omega)} \|\nabla \mathbf{w}\|_{L^q(\Omega))}^q + 2\kappa \|\nabla \mathbf{u}\|_{L^q(\Omega)} \|\nabla \mathbf{w}\|_{L^q(\Omega))}^{q-1},$$

i.e.,

$$\frac{d}{dt} \|\nabla \mathbf{w}\|_{L^q(\Omega))} + 4\kappa \|\nabla \mathbf{w}\|_{L^q(\Omega))} \leq \|\nabla \mathbf{u}\|_{L^q(\Omega)} \|\nabla \mathbf{w}\|_{L^q(\Omega))} + 2\kappa \|\nabla \mathbf{u}\|_{L^q(\Omega)}.$$

Next, by employing Lemma 2.4 for (1.9), it clearly holds

$$\|\nabla \mathbf{w}\|_{L^q(\Omega))} \leq C(1 + \|\nabla \mathbf{w}\|_{L^q(\Omega))}) \ln(e + \|\nabla \mathbf{w}\|_{L^q(\Omega))}) \quad (3.20)$$

for any $q \in (2, \infty)$. Subsequently, by recalling the definition of $\mathbf{g}$, applying Lemma 2.3 and Sobolev embedding inequalities, we further deduce that

$$\frac{d}{dt} \|\nabla \mathbf{w}\|_{L^q(\Omega))} + 4\kappa \|\nabla \mathbf{w}\|_{L^q(\Omega))} \leq \|\nabla \mathbf{u}\|_{L^q(\Omega)} \|\nabla \mathbf{w}\|_{L^q(\Omega))} + 2\kappa \|\nabla \mathbf{u}\|_{L^q(\Omega)} \|\nabla \mathbf{w}\|_{L^q(\Omega))} + (\nu + \kappa) \|\nabla \mathbf{w}\|_{L^q(\Omega))} \|\nabla \mathbf{w}\|_{L^q(\Omega))}.$$
where $\varphi$ of $\|w\|_{L^q(\Omega)}$ and Lemma 3.2, it is clear that $\varphi(t) = (1 + \|w\|_{L^q(\Omega)})(1 + \|g\|_{W^{2,q}(\Omega)})$. According to Proposition 3.3 and Lemma 3.2, it is clear that $\varphi(t) \in L^1(0, T)$. This, together with Gronwall’s inequality yield that

$$
\|\nabla w\|_{L^q(\Omega)} + 4\kappa \int_0^T \|\nabla w\|_{L^q(\Omega)} dt \leq C(T).
$$

This completes the proof of Proposition 3.4. □

**Proof of Proposition 3.1** According to the assumptions on the initial data, Proposition 3.2 and Proposition 3.4, it is clear that $\|w\|_{L^\infty(0,T;W^{1,4}(\Omega))} \leq C$. Then, by definition of $g$ and Lemma 2.3, we have

$$
\|u\|_{L^\infty(0,T;W^{1,4}(\Omega))} + \|u\|_{L^2(0,T;W^{2,4}(\Omega))} \\
\leq \|g\|_{L^\infty(0,T;\Omega)} + \|g\|_{L^2(0,T;W^{2,4}(\Omega))} + (\nu + \kappa) \left[ \|v\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^2(0,T;W^{2,4}(\Omega))} \right] \\
\leq \|g\|_{L^\infty(0,T;\Omega)} + \|g\|_{L^2(0,T;W^{2,4}(\Omega))} + (\nu + \kappa) \left[ \|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;W^{2,4}(\Omega))} \right].
$$

The terms $\|w\|_{L^\infty(0,T;L^2(\Omega))}$, $\|g\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|g\|_{L^2(0,T;W^{2,4}(\Omega))}$ is globally bounded due to Proposition 3.2, Lemma 3.1 and Lemma 3.2 respectively. To bound the term $\|w\|_{L^2(0,T;W^{1,4}(\Omega))}$, it suffices to apply Proposition 3.2, Proposition 3.4 with $q = 4$ and Hölder inequality. This completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.1

The goal of this section is to complete the proof of Theorem 1.1. To do so, we first establish the global existence of weak solutions by Schauder’s fixed point theorem. Then the *a priori* estimates obtained in the previous section for $u$ and $w$ allow us to prove the uniqueness of weak solutions.

**Existence:** The proof is a consequence of Schauder’s fixed point theorem. We shall only provide the sketches.

To define the functional setting, we fix $T > 0$ and $R_0$ to be specified later. For notational convenience, we write

$$
X \equiv C(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))
$$

with $\|g\|_X \equiv \|g\|_{C(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;H^1_0(\Omega))}^2$, and define

$$
B = \{ g \in X \mid \|g\|_X \leq R_0 \}.
$$
Clearly, \( B \subset X \) is closed and convex.

We fix \( \epsilon \in (0, 1) \) and define a continuous map on \( B \). For any \( \mathbf{v} \in B \), we regularize it and the initial data \((u_0, w_0)\) via the standard mollifying process,

\[
\mathbf{v}^\epsilon = \rho^\epsilon * \mathbf{v}, \quad u_0^\epsilon = \rho^\epsilon * u_0, \quad w_0^\epsilon = \rho^\epsilon * w_0,
\]

where \( \rho^\epsilon \) is the standard mollifier. Initially, the transport equation with smooth external forcing \( 2\kappa \nabla^\perp \cdot \mathbf{v}^\epsilon \) and smooth initial data \( w_0^\epsilon \)

\[
\begin{aligned}
&\left\{
\begin{array}{l}
  w_t + \mathbf{v}^\epsilon \cdot \nabla w + 4\kappa w = 2\kappa \nabla^\perp \cdot \mathbf{v}^\epsilon, \\
  w(x, 0) = w_0^\epsilon(x),
\end{array}
\right.
\end{aligned}
\]

has a unique solution \( w^\epsilon \). We then solve the nonhomogeneous (linearized) Navier-Stokes equation with smooth initial data \( u_0^\epsilon \)

\[
\begin{aligned}
&\left\{
\begin{array}{l}
  \mathbf{u}_t + \mathbf{v}^\epsilon \cdot \nabla \mathbf{u} - (\nu + \kappa)\Delta \mathbf{u} + \nabla \pi = -2\kappa \nabla^\perp w^\epsilon, \\
  \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\partial \Omega} = 0, \\
  \mathbf{u}(x, 0) = \mathbf{u}_0^\epsilon(x),
\end{array}
\right.
\end{aligned}
\]

and denote the solution by \( \mathbf{u}^\epsilon \). This process allows us to define the map

\[
F^\epsilon(\mathbf{v}) = \mathbf{u}^\epsilon.
\]

We then apply Schauder’s fixed point theorem to construct a sequence of approximate solutions to \((1.3)-(1.5)\). It suffices to show that, for any fixed \( \epsilon \in (0, 1) \), \( F^\epsilon : B \to B \) is continuous and compact. More precisely, we need to show

(a) \( \| \mathbf{u}^\epsilon \|_B \leq R_0 \);

(b) \( \| \mathbf{u}^\epsilon \|_{C(0,T;H^1_0(\Omega))} + \| \mathbf{u}^\epsilon \|_{L^2(0,T;H^2(\Omega))} \leq C \);

(c) \( \| F^\epsilon(\mathbf{v}_1) - F^\epsilon(\mathbf{v}_2) \|_B \leq C \| \mathbf{v}_1 - \mathbf{v}_2 \|_B \) for \( F \) independent of \( \epsilon \) and any \( g_1, g_2 \in B \).

We verify (a) first. A simple \( L^2 \)-estimate on \((4.1)\) leads to

\[
\begin{align*}
\| w^\epsilon \|^2_{L^2(\Omega)} + 4\kappa \int_0^T \| w^\epsilon \|^2_{L^2(\Omega)} dt &\leq \| w_0^\epsilon \|^2_{L^2(\Omega)} + 4\kappa \int_0^T \| \nabla w^\epsilon \|^2_{L^2(\Omega)} dt \\
&\leq \| w_0 \|^2_{L^2(\Omega)} + 4\kappa \int_0^T \| \nabla w \|^2_{L^2(\Omega)} dt \\
&\leq \| w_0 \|^2_{L^2(\Omega)} + 4\kappa R_0.
\end{align*}
\]

Then by taking inner product of \((4.2)\) with \( \mathbf{u}^\epsilon \) and simple calculations, we have

\[
\| \mathbf{u}^\epsilon \|^2_{L^2(\Omega)} + (\nu + \kappa) \int_0^T \| \nabla \mathbf{u}^\epsilon \|^2_{L^2(\Omega)} dt \leq \| \mathbf{u}_0 \|^2_{L^2(\Omega)} + \frac{4\kappa^2}{\nu + \kappa} \int_0^T \| w^\epsilon \|^2_{L^2(\Omega)} dt.
\]

In order for \( F^\epsilon \) to map \( B \) to \( B \), it suffices for the right-hand side to be bounded by \( R_0 \). Invoking the bounds for \( \| w^\epsilon \|_{L^2} \), we obtain a condition for \( T \) and \( R_0 \),

\[
\| \mathbf{u}_0 \|^2_{L^2(\Omega)} + CT (\| w_0 \|^2_{L^2(\Omega)} + R_0) \leq R_0,
\]

(4.3)
where the constant $C$ depends only on the parameters $\kappa$ and $\gamma$. It is not difficult to see that, if $CT < 1$ and $R_0 > \|u_0\|_{L^2(\Omega)} + \|w_0\|_{L^2(\Omega)}$, (4.3) would hold. Similarly, we can verify (c) under the condition that $T$ is sufficiently small. Besides, (b) can be verified by the similar way as estimating (3.1). Schauder’s fixed point theorem then allows us to conclude that the existence of a solution on a finite time interval $T$. These uniform estimates would allow us to pass the limit to obtain a weak solution $(u, w)$.

We remark that the local solution obtained by Schauder’s fixed point theorem can be easily extended into a global solution via Picard type extension theorem due to the global bounds obtained in (3.1). This allows us to obtain the desired global weak solutions. □

Now, we are in the position to prove the uniqueness of weak solutions through a usual way. To be more precise, we will consider the difference between two solutions and then establish the energy estimates for the resulting system of the difference at the level of basic energy.

**Uniqueness:** Assume $(u, w, \pi)$ and $(\tilde{u}, \tilde{w}, \tilde{\pi})$ are two solutions of (1.3)-(1.5) with the regularity specified in (1.8). Consider their difference

$$U = u - \tilde{u}, \quad W = w - \tilde{w}, \quad \Pi = \pi - \tilde{\pi},$$

which solves the following initial-boundary value problem

$$\begin{cases}
U_t - (\nu + \kappa)\Delta U + u \cdot \nabla U + U \cdot \nabla \tilde{u} + \nabla \Pi = -2\kappa \nabla^\perp W, \\
W_t + u \cdot \nabla W + U \cdot \nabla \tilde{w} + 4\kappa W = 2\kappa \nabla^\perp \cdot U, \\
\nabla \cdot U = 0, \quad U|_{\partial \Omega} = 0, \\
(U, W)(x, 0) = 0.
\end{cases} \tag{4.4}$$

Dotting the first two equations of (4.4) with $(U, W)$ yields

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|U\|_{L^2(\Omega)}^2 + \|W\|_{L^2(\Omega)}^2) + (\nu + \kappa)\|\nabla U\|_{L^2(\Omega)}^2 + 4\kappa \|W\|_{L^2(\Omega)}^2 & = -\int_\Omega U \cdot \nabla \tilde{u} \cdot U dx - \int_\Omega U \cdot \nabla \tilde{w} \cdot W dx - 2\kappa \int_\Omega \nabla^\perp W \cdot U dx \\
& + 2\kappa \int_\Omega \nabla^\perp \cdot UW dx.
\end{align*} \tag{4.5}$$

By the divergence theorem and the boundary condition $U|_{\partial \Omega} = 0,$

$$\begin{align*}
-2\kappa \int_\Omega \nabla^\perp W \cdot U dx + 2\kappa \int_\Omega \nabla^\perp \cdot UW dx & = 4\kappa \int_\Omega \nabla^\perp \cdot UW dx \\
& \leq \frac{(\nu + \kappa)}{4} \|\nabla U\|_{L^2(\Omega)}^2 + C \|W\|_{L^2(\Omega)}^2.
\end{align*}$$
To bound the first and second term on the right side of (4.5), we integrate by parts and invoke the boundary condition \(U\vert_{\partial\Omega} = 0\) to obtain

\[
-\int_{\Omega} U \cdot \nabla \tilde{u} \cdot U dx - \int_{\Omega} U \cdot \nabla \tilde{w} \cdot W dx 
\leq \|\nabla \tilde{u}\|_{L^2(\Omega)} \|U\|_{L^4(\Omega)}^2 + \|\nabla \tilde{w}\|_{L^4(\Omega)} \|U\|_{L^4(\Omega)} \|W\|_{L^2(\Omega)}
\leq C\|\nabla \tilde{u}\|_{L^2(\Omega)} \|U\|_{L^2(\Omega)} \|\nabla \tilde{U}\|_{L^2(\Omega)} + \|\nabla \tilde{w}\|_{L^4(\Omega)} \|U\|_{L^6(\Omega)}^2 \|\nabla \tilde{U}\|_{L^2(\Omega)} \|W\|_{L^2(\Omega)}
\leq \frac{\nu + \kappa}{4} \|\nabla \tilde{U}\|_{L^2(\Omega)}^2 + C (1 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{w}\|_{L^4(\Omega)}^2) (\|U\|_{L^2(\Omega)}^2 + \|W\|_{L^2(\Omega)}^2).
\]

Inserting the estimates above in (4.5) yields

\[
\frac{d}{dt} (\|U\|_{L^2(\Omega)}^2 + \|W\|_{L^2(\Omega)}^2) 
\leq C (1 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{w}\|_{L^4(\Omega)}^2) (\|U\|_{L^2(\Omega)}^2 + \|W\|_{L^2(\Omega)}^2)
\]

By Gronwall’s inequality, we obtain

\[
\|U(t)\|_{L^2(\Omega)}^2 + \|W(t)\|_{L^2(\Omega)}^2 
\leq e^{C\int_0^t (1 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{w}\|_{L^4(\Omega)}^2) \, dr} (\|U_0\|_{L^2(\Omega)}^2 + \|W_0\|_{L^2(\Omega)}^2)
\]

for any \(t \in (0, T)\). According to Proposition 3.2 Proposition 3.4 and noting that \(U_0 = W_0 = 0\), we obtain the desired uniqueness \(U = W \equiv 0\). This finishes the proof of Theorem 1.1. \(\square\)

**Acknowledgments**

J. Liu is supported by the Connotation Development Funds of Beijing University of Technology. S. Wang is supported by National Natural Sciences Foundation of China (No. 11371042, No. 11531010).

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