Polylogarithms from the Bound-State S-matrix
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Abstract—Higher-point functions of gauge invariant composite operators in $\mathcal{N} = 4$ super Yang–Mills theory can be computed via triangulation. The elementary tile in this process is the hexagon introduced for the evaluation of structure constants. A gluing procedure welding the tiles back together is needed to return to the original object. We re-analyse previous work on five-point functions of half-BPS operators. At one loop this involves dressing a pentagonal matter skeleton graph by virtual exchanges. There are two types of contributing processes: the gluing of two adjacent tiles by one virtual magnon, and the gluing of three adjacent tiles by two virtual magnons. The latter process is of the utmost interest, because it is the first instance in which virtual particles scatter on a hexagon. While we keep the restricted kinematics used in the original article on the problem, we employ a different “mirror rotation”, thus rendering a large part of the four-variable problem accessible to analytic methods. For the resulting multiple series of hypergeometric type two summation techniques are developed: integration in the modulus, and substitution of the integral representation of $\Gamma_p F_q$ functions. All solvable contributions individually yield hyperlogarithms of weight two.

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1. INTRODUCTION

The AdS/CFT conjecture in the original form connects $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions to IIB string theory on $\text{AdS}_5 \times \text{S}_5$ [1]. In particular, the spectrum of anomalous dimensions in field theory should correspond to the energy levels of the string. The BMN construction [2] gave the first class of composite operators in field theory and the dual string states. Subsequently, the computation of the planar part of the one-loop anomalous dimensions of the BMN operators has been interpreted as the diagonalisation of the Heisenberg XXX chain [3], thereby linking the AdS/CFT correspondence to an integrable system. A deformation of the corresponding Bethe equations can incorporate the effects of planar higher-loop corrections in the field theory. This is achieved by using the Zhukowski variables $x(u)$ defined by

\begin{equation}
 x + \frac{g^2}{2x} = u \tag{1}
\end{equation}

instead of the Bethe rapidity $u$ of the original spin chain. Since the Bethe ansatz involves the quantities $u^{\pm} = u \pm \frac{i}{2}$ one further introduces $x^{\pm}(u) = x(u^{\pm})$. In the definition (1), $g^2 = \frac{g_{\text{YM}}^2 N}{(8\pi^2)}$ is the 't Hooft coupling involving $N$, the rank of the gauge group $SU(N)$. This deformed integrable system has been extended to the full set of single-trace gauge-invariant composite operators of the theory. When supplied with the “dressing phase” it apparently correctly captures the planar anomalous dimensions to any desired order in the coupling constant [4], where “finite size effects” do not have to be taken into account.

Viewing the dual string as a two dimensional field theory, an approach to finite size corrections was devised using the thermodynamic Bethe ansatz (TBA) [5]. As in the original work, a double Wick rotation exchanges space and time. This “mirror transformation” is (here the scaling is adapted to the string side)

\begin{equation}
 \gamma: x^+ \rightarrow \frac{1}{x^-} \tag{2}
\end{equation}

while $x^-$ stays inert. More generally, we may define [6]

\begin{equation}
 2\gamma: x^{\pm} \rightarrow \frac{1}{x^{\mp}}, \tag{3}
\end{equation}

\begin{equation}
 3\gamma: x^+ \rightarrow x^+, \quad x^- \rightarrow \frac{1}{x^-}, \tag{4}
\end{equation}

\begin{equation}
 4\gamma: x^{\pm} \rightarrow x^{\mp} + x^{\pm}. \tag{5}
\end{equation}

The TBA for the AdS/CFT correspondence is very complex; for once in the mirror theory one has to consider the scattering of all bound states of excitations of the original chain. Bound states come in two forms named the symmetric and the antisymmetric repre-
The motivation for our study is evaluated in [11] by matching a truncated residue matrix. This is depicted in Fig. 2. The process has been simplest case probing the bound state scattering particles (or “virtual magnons”) on two edges is the hopefully manageable sum. The OPE, this yields a complicated but — as we shall see hexagons that captures the virtual corrections. Unlike the strands of propagators. The greatest achievement to together by the exchange of virtual particles between the second hexagon with the operator \( e^{z_p} = e^{i p \cdot x} \) \( (\rho \text{ denotes the momentum of the bound state particles) have to be introduced. We would eventually like to answer whether the choice of braiding adopted there is the only possible one.

2. ELEMENTS OF THE CALCULATION

Let us first consider gluing the left and the central hexagon in Fig. 2 by the exchange of a single bound state particle as described in [10]. By conformal transformations we can always move the points into a plane as \( x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = \{0, -\zeta(1/z), \bar{\zeta}(1/\bar{z}), 0\} \). The first hexagon now connects the points \( 0, 1, \infty \) as in the defining three-point problem studied in [6]; the configuration is “canonical”. The second hexagon shares the edge between \( 0, \infty \) over which we glue, but its third point is parametrised by the variables \( z, \bar{z} \).

For the two independent conformal cross-ratios on which the process will depend we find

\[
x_{13}x_{23} = z\bar{z}, \quad x_{12}x_{24} = (1-z)(1-\bar{z}).
\]

In [10] it is suggested to obtain the non-standard coordinates \( z, \bar{z} \) from the usual situation \( 0, 1, \infty \) acting on the second hexagon with the operator

\[
W(z, \bar{z}) = e^{-D \log(\zeta) \frac{L}{\xi}} L = \frac{1}{2} \left( L_1^2 - L_2^2 - L_1^2 + L_2^2 \right),
\]

where \( D \) is the dilatation generator. The idea is now to trade the representation of \( D \) on the coordinates for its representation by spin chain data

\[
\frac{1}{2} (D - J) = E = i \rho + i u + \ldots
\]

where \( \rho \) refers to the momentum in mirror kinematics, and the dots indicate terms of higher order in the coupling. The operator \( W(z, \bar{z}) \) will now be applied to the
bound state inserted on the second hexagon. Since the generators employed are diagonal this introduces a multiplicative weight factor. The hexagons themselves can finally be evaluated as in the three-point problem.

According to [6] we have to choose sl(2) sector bound states. This “antisymmetric representation” at level (or length) $a$ has the parts

$$\left(\psi^1\right)^{a-k-1}\left(\psi^2\right)^k \phi, \quad \left(\psi^1\right)^{a-k}\left(\psi^2\right)^2 \phi^2. \quad (7)$$

In the last formula, $\phi^2$ are a doublet of bosonic scalar constituents, while $\psi^a$ are two-component spinors. Customarily, for the first class of states one separately considers $i = 1, 2$. Now,

$$W(z, \bar{z})(\psi^1)^{a-k}(\psi^2)^k = (z\bar{z})^{-ia}\left(\frac{z}{\bar{z}}\right)^{a-k}\left(\psi^1\right)^{a-k}(\psi^2)^k. \quad (8)$$

because $L$ (the Cartan generator of the Lorentz transformation) attributes weight $1, -1$ to $\psi^1, \psi^2$, respectively. We focus on re-summing the infinite series arising from the Minkowski-space part of the problem. For now, we turn a blind eye on the internal space operations like the $J$-charge. At this stage, we can also send $\psi^{a-k-1} \rightarrow \psi^{a-k}$ etc. since these are constant shifts. On the other hand, the summation ranges for the $k$-counter in (7) must be respected to obtain sensible results.

In the five-point process in Fig. 2 the central tile is glued to two neighboring hexagons. Full fledged five-point kinematics cannot be parametrised using only the coordinates of the 1,2 plane, so that the Cartan generators used above are not enough to recover it. Clearly, the fifth cross ratio is lost. Nonetheless, in this article we follow [11] in using restricted kinematics. As done there, we postulate that introducing one weight factor for either gluing, so $W(z_1, \bar{z}_1)W(z_2, \bar{z}_2)$ correctly captures all features of the five-point process in reduced kinematics.

On the left and the right hexagon there is only one bound state and thus no scattering. Yet, the contraction rule [6] for the outer hexagons enforces the scattering on the middle tile to be diagonal. Further, let us choose $3\gamma;\gamma'$ kinematics on the middle hexagon in which case the scalar factor $h$ from [6] becomes

$$h(u^{3\gamma}, v^{\gamma}) = \Sigma(u^{\gamma}, v^{\gamma}) \quad (9)$$

with the “improved” BES dressing phase [4, 8] in mirror/mirror kinematics.

$$\Sigma^{ab} = \frac{1 + \frac{a + iu}{2}}{1 + \frac{b + iv}{2}} \Gamma \frac{1 + \frac{a + iv}{2}}{1 + \frac{b + iu}{2}} \times \frac{1 + \frac{a + b - i(u - v)}{2}}{1 + \frac{a + b + i(u - v)}{2}} + O(g). \quad (10)$$

A comprehensive discussion of the bound state $S$-matrix is given in [7], although in the “symmetric representation” in which the role of bosons and fermions is exchanged. Hence in (7) one would literally swap $\phi \leftrightarrow \psi$. By way of example, we consider the scattering of two states of the first type in (7), both with $\alpha = 1$ or both with $\alpha = 2$. The relevant scattering matrix is called $\mathcal{K}_{\gamma}^{ik}(a, u, b, v)$ in [7]. Here we associate $a, k, u, x^\pm(u)$ with the first state and $b, l, v, y^\pm(v)$ with the second. At bound state length one (fundamental particles) one has $k = l = 0$ so that the complicated part of the $S$-matrix (see (11) below) reduces to one. This describes the scattering of two equal fundamental fermions and in agreement with the nomenclature of [12] the remaining overall factor is called $D$. The entire $S$-matrix can be changed by an overall factor, and indeed this $D$ is equal to the $A$-element in [12].

We repeated the steps of [7] to re-derive the $S$-matrix in the antisymmetric representation. Flipping the statistics means exchanging Poincaré and conformal supersymmetry, and also Lorentz and internal symmetry generators. Sticking to the same algebra conventions one obtains a sign flip on the rapidity parameters, so in particular $x^\pm \leftrightarrow x^\mp$. The $\mathcal{K}$-element at bound state length one now describes the scattering of two equal bosons. We observe that what was called $D$ before now becomes $A^{-1}$. Hence for the antisymmetric representation the construction yields $S^{-1}$ without any rescaling.

Next, by observation—at least in $3\gamma, 1\gamma$ kinematics and at leading order in $g$—the diagonal elements of our $S^{-1}$ in the antisymmetric representation are related to those of [7] by flipping the sign of the rapidities, which has the interpretation of a complex conjugation or of taking a second inverse. In fact, this statement is true up to some global factors $(-1)^F$ which we reject as unphysical; they will be undone by the contraction prescription on the hexagons. Hence we can use the $S$-matrix of [7] for our purposes, without any changes!

Below we reproduce the expression for $\mathcal{K}$ from [7], because it is the only one for which the original work gives a completely explicit writing. The other matrix elements are written as a sum over $x$’s with slightly shifted counters with certain matrices built from $x^\pm, y^\pm, a, b, k, l, n$ as coefficients, so they are similar albeit more complicated.

\footnote{It follows from here that the bound state length 1 part of the $S$-matrix of [7] is in fact the inverse of that derived in [12].}
\(\mathcal{X}^{k,l}_n = D \prod_{j=1}^{n} (a - j) \prod_{j=1}^{k+l-n} (b - j) \prod_{j=1}^{k} (a - j) \prod_{j=1}^{n} (b - j) \left(-\delta + \frac{a + b}{2} - j\right) \ast \sum_{m=0}^{k} \binom{k}{m} \left[l(1) \binom{l(1)}{n-m} \prod_{j=1-m}^{m} c_j \prod_{j=1}^{k-m} d_j \prod_{j=1}^{n-m} \bar{d}_{k-l-m-j+2}\right] \)

where \(\delta = u - v\) and

\[c_j = -i \delta + \frac{a - b}{2} - j + 1, \quad d_j = \frac{a + 1 - j}{2}, \quad \bar{d}_j = \frac{b + 1 - j}{2}.

Further,

\[D = \frac{x^- - y^+}{x^+ - y^-} \sqrt{\frac{x^+}{x^-} y^+} \sqrt{\frac{u^- - v^+}{u^+ - v^-} \sqrt{u^+ u^-} \sqrt{v^+ v^-} + O(g^2)}.\]

At fixed bound state lengths \(a, b\), let us denote the tensor product of two states of the first type listed in (7) as \([k, l]\). Their scattering takes the form

\[\otimes |k, l\rangle = \sum_{n=0}^{k+l} \mathcal{X}^{k,l}_n |n, k + l - n\rangle,\]

where \(n = k\) is diagonal, so we adopt the convention in which the state with bound state length \(a\) is written on the left before and after scattering.

Any virtual particle used in the gluing procedure of [6] is endowed with a “mirror measure” which comprises a factor \((g^2)^{l+1}\). Here \(l\) is the “width” of the edge crossed by the virtual particle, i.e. the number of propagators forming that edge. At \(O(g^2)\) the order estimate of [13] only allows gluing over edges of width zero. For the pentagon frame in Figure 2 these are the edges 13 or 14 or both. In the latter case one obtains \(g^4\) from the measure factors, while the boundstate \(S\)-matrix on the central hexagon contains terms of order \(l/g^2\).

By way of example, including the two measure and the two weight factors as well as the dressing phase, the scattering process involving the \(\mathcal{X}\)-matrix yields the sum-integral

\[I(\mathcal{X}) = \sum_{a, b, l = 0}^{\infty} \sum_{k, l = 0}^{a-1, b-1} \int \frac{dudvabg^4}{4\pi^2 \left(u^2 + \frac{a^2}{4}\right) \left(v^2 + \frac{b^2}{4}\right)} W(W') \mathcal{X}^{a,b, k,l}.\]

This is a priori an \(O(g^4)\) contribution, so it should drop from the one-loop result. However, we expect the scattering of the scalar constituents of the bound states to introduce braiding factors like

\[e^{i\delta} e^{-\frac{\delta^2}{4}} \left(\sum_{k, l = 0}^{\infty} \sum_{a, b, l = 0}^{\infty} \int \frac{dudvabg^4}{4\pi^2 \left(u^2 + \frac{a^2}{4}\right) \left(v^2 + \frac{b^2}{4}\right)} W(W') \mathcal{X}^{a,b, k,l}\right)\]

where we have scaled back to the field theory convention of (1) to meet the weak-coupling expansion. Importantly, this factor does not only adjust the leading power in the coupling constant to \(g^2\), but it also removes the square-root branchcuts that would render inefficient the residue theorem as a means of evaluating the integrals over the rapidities \(u, v\). With the present work we hope to eventually shed some light on whether the “averaging prescription” [11] for such additional braiding factors is the only valid choice.

Despite of the appearance, the \(\mathcal{X}\)-matrix has singularities in \(\delta\) only in the lower half-plane. Poles in \(\mathcal{X}\) can therefore be avoided simply by closing the integration contour over the upper half-plane for \(u\) and the lower half-plane for \(v\). Doing so, the poles \(u^-, v^+\) from the measure can contribute. Likewise, in the numerator of the phase, \(\Gamma \left[1 + \frac{a + iu}{2}\right]\) and \(\Gamma \left[1 + \frac{b - iv}{2}\right]\) develop singularities. Note however, that we cannot localise both rapidities by poles from the phase:

\[u = i \left(m + \frac{a}{2}\right), \quad v = -i \left(n + \frac{b}{2}\right) \Rightarrow \Gamma \left[1 + \frac{a + b}{2} + i(u - v)\right] = \Gamma \left[1 - m - n\right]\]

for \(m, n \in \mathbb{N}\) so that this denominator \(\Gamma\)-function creates a zero in these cases. Thus at least one pole, perhaps a higher one, must come from the measure.

Then, e.g. with \(u = \frac{a}{2}\),
Table 1. Simple diagonal $S$-matrix elements and their dressing by $u^\pm, v^\pm$

| $\mathcal{Y}_i, \mathcal{X}_j$ | $f$ | $P$ | $\prod u^\pm v^\pm$ |
|---|---|---|---|
| $\mathcal{Y}_{11}$ | $-i \frac{a-k}{a(a-k+b-l)}$ | $\frac{1}{P_2}$ | $\frac{1}{(u^-)^2 u^+ v^+}$ |
| $\mathcal{Y}_{22}$ | $i \frac{b-l}{b(a-k+b-l)}$ | $P_1$ | $\frac{1}{u^+ u^- v^+ (v^+)^2}$ |
| $\mathcal{X}_{11}$ | $-\frac{1}{a^2 b^2 (k+l)(a-k+b-l)}$ | $1$ | $\frac{1}{u^+ u^- v^+ (v^+)^2}$ |
| $\mathcal{X}_{22}$ | $\frac{l(b-l)}{b^2 (k+l)(a-k+b-l)}$ | $1$ | $\frac{1}{(u^-)^2 v^+ v^+}$ |
| $\mathcal{X}_{33}$ | $-\frac{k(a-k)}{a^2 (k+l)(a-k+b-l)}$ | $1$ | $\frac{1}{(u^-)^2 u^+ v^+}$ |
| $\mathcal{X}_{55}$ | $\frac{l(a-k)}{ab(k+l)(a-k+b-l)}$ | $\frac{1}{(P_1 P_2)}$ | $\frac{1}{(u^-)^2 u^+ v^+}$ |
| $\mathcal{X}_{55}^+$ | $\frac{l(a-k)}{ab(k+l)(a-k+b-l)}$ | $P_1 P_2$ | $\frac{1}{u^+ u^- (u^+)^2}$ |
| $\mathcal{X}_{66}$ | $\frac{k(b-l)}{ab(k+l)(a-k+b-l)}$ | $\frac{1}{(P_1 P_2)}$ | $\frac{1}{(u^-)^2 u^+ v^+}$ |
| $\mathcal{X}_{66}^+$ | $\frac{k(b-l)}{ab(k+l)(a-k+b-l)}$ | $P_1 P_2$ | $\frac{1}{u^+ v^+ (v^+)^2}$ |

$$\Sigma^{ab} = \frac{\Gamma[1+b/2-iv] \Gamma[1+a+b/2+iv]}{\Gamma[1+a] \Gamma[1+b/2+iv] \Gamma[1+b/2-iv]}$$

and therefore the term in the phase that could create a pole at $v_n = -i(\frac{b}{2} + n)$ actually drops. We therefore consider the measure poles only, i.e. $u = i \frac{a}{2}, v = -i \frac{b}{2}$.

3. RE-SUMMATION OF THE RESIDUA

At leading order in the coupling in $3\gamma_1 \gamma_k$ kinematics, the diagonal $S$-matrix elements factor out a product of $u^\pm, v^\pm$. Importantly for our residue strategy, also a $(u^+ u^- - v^+ v^-)$ denominator in the so-called $\mathcal{X}$-elements cancels which would otherwise lead to an expansion in terms of fractional powers of the cross-ratios. Writing the rapidities in terms of $\delta = u - v, s = u + v$, we can completely factor out the dependence on $s$.

Further, localising both rapidities at the poles of the measure, so $u = i \frac{a}{2}, v = -i \frac{b}{2} \Rightarrow \delta = i \frac{a+b}{2}$, the remaining $\delta$-dependent part of most of the diagonal matrix elements reduces to a simple ratio of $\Gamma$-functions of the counters. In all these cases one finds the same kernel

$$K = \begin{pmatrix} k+l & (a-k)+(b-l) \\ l & a+k \end{pmatrix} / \begin{pmatrix} a+b \\ a \end{pmatrix},$$

though with simple index shifts. We can express these functions as a product $f(a,k,b,l)K$. Conveniently, the dressing phase (10)

$$\Sigma^{ab} \big|_{u=i \frac{a}{2}, v=-i \frac{b}{2}} = \left( \frac{a+b}{a} \right) + \ldots$$

even compensates the denominator of (18). For the diagonal matrix elements behaving in this way (their definition is given in [7]), Table 1 lists the factor $f$, the dressing by $P_i = e^{\frac{i}{3} \gamma_i}$ necessary to obtain a rational integrand of order $O(g^2)$, and the total $s$-dependent part arising from this dressing, the measure, and the matrix elements including $D$.

There are two possible choices marked as $\pm$ for $\mathcal{X}_{55}, \mathcal{X}_{66}$. These are related by a factor $P_1 P_2 = (u^+ u^-)/(v^+ v^-) + O(g^2)$. Further multiplication with
this combination would neither change the order in \( g \) nor re-introduce cuts. Yet, the listed possibilities are singled out by yielding maximally one double pole in \( u^- \) or \( v^- \) (not both). This is required by field theory at one loop, which does not allow \( \log^n \) singularities in OPE limits.

When the derivative acts on \((z_i \bar{z}_i)^{-ix}\) or \((z_i \bar{z}_i)^{-iy}\), the large logarithm it creates is a constant factor w.r.t. the quadruple sum over \(a, k, b, l\), which must have polylogarithm weight one. In the cases in Table 1, and also for the diagonal \( \mathcal{X} \)-elements, this sum can be computed in closed form as we shall explain below. Our techniques also apply to the analytic evaluation of those terms in which the derivative acts on \(1/u^+ \) or \(1/v^+ \). Linear combinations of weight two hyperlogarithms are found.

On the other hand, when the derivative acts on the \( \delta \)-dependent part of the matrix elements it destroys the factorisation properties yielding the kernel \( K \). Yet, we could fit all of the resulting series—and also those for the remaining matrix elements—on an ansatz built from the functions found in those terms amenable to explicit integration.

### 3.1. Integrating \( \mathcal{Y}_{11} \) Element

The \( \mathcal{Y}_{11} \) is relevant to the scattering of \( \phi^i (\psi^i)^{a-1} \) (\( \psi^i \))^\( b-l \) over \( (\psi^i)^{a-1} (\psi^i)^{b-l} \). We put aside the factor \(\sqrt{\bar{z}_i/z_i}\) that arises from the action of the \( L \) generator as well as a similar factor \(\sqrt{\bar{a}_i/a_i}\) from the internal space rotation. We use the notation \(\{z_i, \bar{z}_i, z_2, \bar{z}_2\} \rightarrow \{z_i, \bar{z}_i, 1/a_i, 1/y_2\}\) for ease of reading.

In \( \gamma_1 \) kinematics the \( \mathcal{Y}_{11} \) element is of leading order \(1/g\), further \(1/P_2\) is also \(O(g^{-1})\) and the measure contributes a factor \( g^4 \). Next, we pick residua at \( u^- \) (double pole) and \( v^- \), whereby \(1/(u^- v^-) \rightarrow 1/(ab)\), which cancels against the numerator of the measure. Integrating the derivative onto \((z_i \bar{z}_i)^{-ix}\) one obtains a global factor \(-i \log(z_i \bar{z}_i)\). From \(f \hat{K} \Sigma\) as listed in Table 1 we find the quadruple sum

\[
\frac{S_{\log}(\mathcal{Y}_{11})}{\log(z_i \bar{z}_i)} = \sum_{a,b,k,l} z_i^{a-k} b_1^{k_1} b_2^{b-l} a_2^l
\]

\[
\times \frac{\prod[a - k - b - l]}{a! \prod[a - k - b - l]} \prod[l + k + l]
\]

where \(a, b, k, l = 0 \ldots a - 1, b\). To obtain independent sums we would like to shift \(a \rightarrow a + k, b \rightarrow b + l\), but the explicit factor \(1/a\) is a hinderance. We can eliminate it by differentiation in the absolute value of \(z_i\): Defining

\[
r^2 = z_i \bar{z}_i, \quad p^2 = \frac{z_i}{\bar{z}_i} \Rightarrow \frac{\partial}{\partial r} z_i a_k^{-k} b_k^{-l} = a_i^a b_k^l.
\]

The inverse operation is \(\int dr/r\), which is, of course, only defined up to a function of the phase \(z_i/\bar{z}_i\) and the other two variables \(y_1, a_2\). Differentiating and shifting the sums we obtain

\[
-r \frac{\partial}{\partial r} \frac{S_{\log}(\mathcal{Y}_{11})}{\log(z_i \bar{z}_i)} = \sum_{a,b,k,l} \left( \sum_{a=0}^{\infty} \left( \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{b=0}^{\infty} \right) \right)
\]

\[
\times \frac{z_i^a y_2^b}{(a + b - 1)} \left( \frac{k + l}{k a - l} \right)
\]

The sums are of geometric type and can easily be taken in closed form:

\[
-r \frac{\partial}{\partial r} \frac{S_{\log}(\mathcal{Y}_{11})}{\log(z_i \bar{z}_i)} = \frac{z_i (a_2 + y_2 - a_2 y_2 - b_1 y_2 - a_2 z_i)}{(1 - a_2)(1 - a_2 - b_1)(1 - z_i)(1 - y_2 - z_i)}.
\]

Expressing the r.h.s. in terms of modulus and phase of \(z_i\) we can apply the integral operator \(\int dr/r\) which, by definition, adds a letter to a hyperlogarithm. The result is

\[
S_{\log}(\mathcal{Y}_{11}) \left( \frac{z_i}{\log[1 - b_l] - \log[1 - z_i]} \right) + \left( \frac{z_i}{b_l - z_i} \right)
\]

\[
+ \left( \frac{z_i}{b_l - z_i} \right) \left( \frac{z_i}{b_l - y_2 + a_2 z_i} \right) + c(z_i/b_l, y_2, a_2),
\]

where \(c\) is a fairly complicated expression constant in \(r\). Re-expanding the result of the integration and comparing to the original series we see that \(c \rightarrow 0\) is the right result, so the entire constant part has to be subtracted. This happened in all instances in which we applied the differentiation/integration trick, presumably for the reason that all the denominators in the set of functions do depend on the modulus \(r\) in question. It would be important to better understand this point.

We can thus eliminate \(a\) from the denominator of (20) without loss of information. The root of the procedure is then a rational function (23) and we add polylogarithm levels by the integration in the modulus \(r\). For instance, the contribution from the derivative
falling onto \(1/u^+\) as is the r.h.s. of (20) but with a second factor of \(1/(ia)\). To find it, we can apply the operation \(\int \, dr/r\) a second time, on (24):

\[
S_{\text{max}}(\mathcal{Y}_{11}) = -z_i (L_{12}[b] - L_{12}[z_i])
\]

\[
+ \frac{z_i}{(b - z_i - b_1 y_2 + a z_i)}
\]

(25)

At this point we cannot present an algorithm for the evaluation of the remaining contribution, say, \(S_{\text{max}}(\mathcal{Y}_{11})\) in which the scattering matrix (including the phase) is differentiated.

The situation is strictly analogous for all other entries in Table 1 barring for \(\mathcal{F}_{11}\) which is special in that there is no double pole. The integration algorithm described here can therefore catch the entire contribution:

\[
S(\mathcal{F}_{11}) = -y_2 L_{12}[a_z] - a_z L_{12}[y_2]
\]

\[
- z_i L_{12}[b] - b_i L_{12}[z_i]
\]

\[
+ (1 - b_i) y_2 L_{12}[a_z] - a_z (1 - z_i) L_{12}[y_2]
\]

(26)

In the last formula, the pure function \(L_{\mathcal{F}_{11}}\) is:

\[
L_{\mathcal{F}_{11}} = \log[1 - a_z]^2 - \log[1 - b_i]^2 - \log[1 - a_z] \log[1 - y_2] + \log[1 - b_i] \log[1 - z_i]
\]

\[
+ \log[1 - a_z - b_i] (-\log[1 - a_z] + \log[1 - b_i] + \log[1 - y_2] - \log[1 - z_i])
\]

\[
+ L_{12}[a_z] - L_{12}[y_2] - L_{12}[b_i] + L_{12}[z_i] - L_{12} \left[ \frac{a_z}{1 - b_i} \right] + L_{12} \left[ \frac{y_2}{1 - z_i} \right] + L_{12} \left[ \frac{b_i}{1 - a_z} \right] - L_{12} \left[ \frac{z_i}{1 - y_2} \right]
\]

\[
+ L_{12} \left[ -\frac{a_z - y_2}{1 - a_z} \right] - L_{12} \left[ -\frac{b_i - z_i}{1 - b_i} \right] - L_{12} \left[ -\frac{a_z - y_2 + b_i y_2 - a z_i}{(1 - a_z - b_i)(1 - z_i)} \right] + L_{12} \left[ -\frac{b_i - z_i - b_i y_2 + a z_i}{(1 - a_z - b_i)(1 - y_2)} \right]
\]

(27)

This result can be recovered by our second technique. Suffice it here to give a brief sketch of the method on the example of a part of this calculation: first, since the \(\mathcal{F}_{11}\) element has the summation range \(\sum_{z=1}^\infty \sum_{k=0}^a\) and similarly for \(b,l\) we have to split the sums as in (22) at both ends when shifting the counters as in the \(\mathcal{Y}_{11}\) computation. This introduces a number of special cases, which we do not discuss here, although they cannot be omitted, of course. Second, expanding the numerator factor \((a l - b k)^2\) from Table 1 we obviously find three terms. Let us focus on the third of these:

\[
\tilde{S}_3 = \sum_{a,b,k,l=1}^\infty a b^k c^k z^a (a + k)^2 \frac{k^2 \Gamma[a + b] \Gamma[k + l]}{(a + k)^2 \Gamma[a + b] \Gamma[k + l]}
\]

(28)

Due to the denominator factor \((a + k)^2\) (before shifting the counter \(a,b\)), holomorphic and the antiholomorphic part do not decouple anymore so that multiple hypergeometric series arise. On the other hand, the \(b\) and \(l\) sums still yield simple geometric series. We find

\[
\tilde{S}_3 = \sum_{a,k=1}^\infty b^k (1 - (1 - a_z)^{-k}) z_i^a (1 - (1 - y_2)^{-a})
\]

\[
\times \left( -\frac{1}{a(a + k)} + \frac{1}{a(a + k)} \right)
\]

(29)

with the two types of sums

\[
\Sigma_1 = \sum_{a,k=1}^\infty \frac{u^a v^k}{(a + k)^2} = \sum_{a=1}^\infty \frac{u^a}{(a + 1)^2} F_3[1,l + a,1 + a],[2 + a,2 + a],v] = \int_0^1 ds dt \sum_{a=1}^\infty u^a (1 - s)^s v^a (1 - stv)^{(1+a)}
\]

\[
= \int_0^1 ds dt \frac{(1 - s)uv}{(1 - stv)(1 - tu + stv - stv)} = v L_{12}(u - u L_{12}(v))
\]

(30)
The final result for \( \bar{S}_1 \) from substituting the parts of (29) is a little unwieldy so that we refrain from presenting it here.

For \( S(\bar{X}) \) we can obtain all three pieces \( S_{\log}, S_{\text{mes}}, S_{\text{mat}} \) in closed form by similar methods. In particular,

\[
S_{\log}(\bar{X}) = \frac{y_2 z_1 (\log[1 - a_2] + \log[1 - b_1] - \log[1 - a_2 - b_1] - \log[1 - y_2] - \log[1 - z_1] + \log[1 - y_2 - z_1])}{\log[z_1 h_2 a_2 z_2]},
\]

and \( S_{\text{mes}}(\bar{X}) \) contains two pure functions (whose difference is closely related to \( L_{y_1 z_1} \)) over the denominators

\[
\{a_2 h_1 - a_2 h_1 y_2 - a_2 h_1 z_1 - a_2 y_2 z_1 + a_2 y_2 z_1 + h_1 y_2 z_1, \}
\]

respectively. Note that hypergeometric functions have also been found to be useful in re-summing integrability results for amplitudes [15].

### 3.2 The Non-Factoring Matrix Elements and Educated Guesses

The total set of denominators is thus

\[
\{a - y, b - z, by - az, a - y + by - az, b - by - z + az, \}
\]

\[
ab - aby - abz - yz + ayz + byz, \quad ab - aby - yz + ayz, ab - abz - yz + byz, \}
\]

where we have dropped the 1, 2 subscripts, because the sums have all been done whence there can be no confusion between \( a_2, b_1 \) and the bound state lengths \( a, b \) which have disappeared from the problem. The logarithmic functions that we have observed at transcendentality weight one are

\[
\{\log[1 - a], \log[1 - b], \log[1 - a - b], \log[1 - y], \log[1 - z], \log[1 - y - z]\}. \tag{35}
\]

Importantly, all the functions that we have found by integration have the following features:

- They are sums \( \sum_i f_i/d_i \) with \( d_i \) in the set (34).
- Every numerator \( f_i \) is a sum of rational factors times pure functions, where the rational factors are terms of the denominator \( d_i \).
- At weight one the pure functions are linear combinations of the logarithms in (35).

We can now write an ansatz using the most general linear combination of denominator terms times logarithms in the numerator of every \( d_i \). This is constrained by two requirements:

- If \( l_i \) is the linear combination acting as a numerator for \( d_i \), the Taylor expansion\(^2\) of \( l_i/d_i \) must not have any denominator.
- Allowing all relevant denominator terms as rational factors in the numerator of \( l_i \) obviously allows for a complete cancellation of \( d_i \), which makes the eight parts linearly dependent. To avoid this we do not allow the respective last denominator term in any numerator but \( l_i \).

Indeed, the transcendentality one part of \( S(\bar{Y}_{13}), S(\bar{Y}_{44}), S(\bar{F}_{44}) - \) on which we cannot run our integration scheme as even the undifferentiated matrix elements do not factor into \( \Gamma \)-functions—fits such an ansatz! To compute \( S_{\text{mes}}(\bar{Y}_{13}) \) and \( S_{\text{mes}}(\bar{Y}_{44}) \) from these formulae one straightforwardly proceeds by integration; for \( \bar{F}_{44} \) there is no such contribution due to the simple structure of the \( u^\pm v^\pm \) factor in Table 2. On the other hand, \( S_{\text{mat}(\ldots)} \) remains inaccessible once again.

However, the explicit weight two results so far obtained—in particular \( S(\bar{F}_{11}) \) as stated in (27), \( S_{\text{mes}}(\bar{X}) \) and \( S_{\text{mat}(\ldots)} \)—offer a range of hyperlogarithms that we can express by a set of basis functions

\[
\sum_i \frac{u^a v^b}{a(a + k)} \propto \sum_i \frac{u^a v^b}{a} = \sum_i \frac{u^a v^b}{a} \mathcal{F}_1[1, 1 + a, 2 + a, v] = \int_0^1 ds \sum_i \frac{u^a (1 - s)^a (1 - sv)^{(l + a)}}{a} \tag{31}
\]
with the help of the symbol [14]. The letters in their symbols are
\[
\{1 - a, 1 - b, 1 - a - b, 1 - y, 1 - z, 1 - y - z\}
\] (36)
and
\[
\{a, b, y, z, a - y, b - z, a - y + by - az, b - by - z + az, ab - aby - abz - yz + ayz + byz\}.
\] (37)
Since we obtained the functions from a Taylor expansion for small symbols obey a first entry condition: only the letters in the set (36) occur in the first entry. A basis of functions is given by
- products of the logarithms in (35), yielding 21 functions,
- the eight dilogarithms in the third line of (27),
- the four dilogarithms in the fourth line of (27) and a fifth variant:
\[
\text{Li}_2\left[\frac{ab - aby - abz - yz + ayz + byz}{(1 - a)(1 - b)(1 - y - z)}\right].
\] (38)
In fact, we can separately fit every contribution onto an ansatz of the same type as for the weight one problem, though employing the 34 weight two functions listed above.

4. CONCLUSIONS

In the evaluation of \(n\)-point functions in \(\mathcal{N} = 4\) super Yang–Mills theory by hexagon tessellations, the first complicated process is the gluing of three adjacent tiles by two single mirror magnons. On the central tile this necessitates the evaluation of diagonal scattering of two so-called \(sl(2)\) bound states.

We find a beautiful and efficient integration scheme for this two-magnon problem, although we cannot yet ascertain that the outcome is the physical result. To answer this question must be one aim of future work.

Remarkably, the problem yields a multilinear alphabet of letters in the symbol of the relevant generalised polylogarithms, suggesting that the two-magnon problem can be integrated in closed form also beyond the leading order in the coupling constant\(^3\). Integration in the modulus—one of the two methods here presented—induces moves on the rational factors and the arguments of hyperlogarithms which recall the transmutations in cluster algebras [16].

Last, another direction of future research will be to simplify the bound state scattering matrix in the various kinematical regimes in order to be able to address higher processes, too.

\(^3\) We thank O. Schnetz for a discussion on this point.