An equivalent system of Einstein Equations

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Abstract. We provide the full set of equations governing the evolution of self-gravitating systems by using a tetrad formalism in General Relativity and the orthogonal splitting of the Riemann tensor. We apply this formalism to spherical case and found that: the only static solution with homogeneous energy density, under the physical reasonable considerations, is the Schwarzschild solution; the conditions under which a non-dissipative fluid is static and finally that shear-free and isotropic pressure conditions are equivalent, for non-dissipative fluids with homogeneous energy density.

1. Introduction
Timelike congruence is one of the main subjects of General Relativity because matter particles follow timelike trajectories. Therefore, the study of geometrical and kinematic aspects of timelike curves is fundamental in the analysis of the evolution of self-gravitating fluids and there it is common the use of a framework based on the well known 1+3 formalism [1, 2, 3, 4, 5, 6, 7]. In this formalism, any tensor quantity can be split in its components along of a tangent vector to a timelike congruence \( V \), and the components totally orthogonal to it.

From the orthogonal splitting of the first covariant derivative of \( V \) into irreducible parts, we obtain the kinematical variables, the expansion rate \( \Theta \), the shear rate \( \sigma_{\alpha\beta} \) and the rotation rate \( \omega_{\alpha\beta} \). Next, from the Ricci identities we find the evolution for these variables, the constraint equations. Finally from the Bianchi identities we obtain the evolution and constraint equations for the electric and magnetic part of the Weyl tensor.

In general we have a system of six evolution and six constraint equations of first order; considered as a set of evolution equations for the 1 + 3 covariant variables. This system is solved for a given the equation of state for the material. However, they do not form a complete set of equations guaranteeing the existence of a corresponding metric and connections. Therefore we need to use a tetrad description.

In this work we shall use a tetrad formalism to obtain two set equations of first-order differential equations:

1) The first one is obtained from Ricci identities, which can be used to define the physical variables and the scalars of Weyl tensor.

2) The second one is obtained from Bianchi identities and can be interpreted as evolution equations for the physical variables and the scalars of Weyl tensor.

Later, we use this two set equations to analyze the spherical case and find some new results.
The paper is organized as follows: In the next section we present the strategy and general formalism and write all relevant equations. Section 3 is devoted to the analysis of spherical case and we end with a short final remark for future works.

2. The strategy and general formalism
The strategy we shall follow is to compile two independent sets of equations, expressed in terms of scalar functions, which contain the same information than the Einstein system.

Let us also choose an orthogonal unitary tetrad:

\[ e_α^{(0)} = V_α, e_α^{(1)} = K_α, e_α^{(2)} = L_α \text{ and } e_α^{(3)} = S_α. \]  

As usual, \( \eta_{αβ} = g_αβ e_α^{(a)} e_β^{(b)} \), with \( a = 0, 1, 2, 3 \), i.e. latin indices label different vectors of the tetrad. Thus, the tetrad satisfies the standard relations:

\[
V_α V^α = -K_α K^α = -L_α L^α = -S_α S^α = -1 \tag{2}
\]

\[ V_α K^α = V_α L^α = V_α S^α = K_α L^α = K_α S^α = S_α L^α = 0 \]

With the above tetrad (1) we shall also define the corresponding directional derivatives operators

\[ f^* = V^α \partial_α f; \quad f^† = K^α \partial_α f \quad \text{and} \quad f^* = L^α \partial_α f. \tag{3} \]

The first set can be considered purely geometrical and emerges from the projection of the Riemann tensor along the tetrad [8], i.e.

\[
2V_α;[βγ] = R_δαβγ V^δ, \quad 2K_α;[βγ] = R_δαβγ K^δ, \tag{4}
\]

\[ 2L_α;[βγ] = R_δαβγ L^δ \quad \text{and} \quad 2S_α;[βγ] = R_δαβγ S^δ; \]

where \( e_α^{(a)} \) are the second covariant derivatives of each tetrad (7) vector indicated with \( a = 0, 1, 2, 3 \).

The second set emerges from the Bianchi identities i.e.

\[
R_αβγδ;μ = R_αβγδ;μ + R_αβδμγ;δ + R_αβδμγ;γ = 0. \tag{5}
\]

3. Spherical Case
In this section we shall present the relevant equations, for the spherically symmetric, locally anisotropic, disipative, collapsing matter configuration, written in terms of the kinematical variables—the four acceleration \( a_α \), the expansion scalar \( Θ \), and the shear tensor \( \sigma - \) and the structure scalars, which are some scalars functions related to the splitting of the Riemann Tensor.

3.1. The tetrad, the source and kinematical variables
To illustrate the above procedure we shall restrict to a spherically symmetric line element given by

\[
ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 (dθ^2 + \sin^2(θ) dφ^2), \tag{6}\]

where the coordinates are: \( x^0 = t, x^1 = r, x^2 = θ \), and \( x^3 = φ \); with \( A(t, r), B(t, r) \) and \( R(t, r) \) functions of their arguments.

For this case the tetrad can written as:

\[
V_α = (-A, 0, 0, 0) \quad K_α = (0, B, 0, 0) \]

\[ L_α = (0, 0, R, 0) \quad \text{and} \quad S_α = (0, 0, 0, R \sin θ). \tag{7} \]
The covariant derivatives of the orthonormal tetrad are:

\[ V_{\alpha;\beta} = -a_1 K_{\alpha} V_{\beta} + \sigma_1 K_{\alpha} K_{\beta} + \sigma_2 (L_{\alpha} L_{\beta} + S_{\alpha} S_{\beta}), \]
\[ K_{\alpha;\beta} = -a_1 V_{\alpha} V_{\beta} + \sigma_1 V_{\alpha} K_{\beta} + J_1 (L_{\alpha} L_{\beta} + S_{\alpha} S_{\beta}), \]
\[ L_{\alpha;\beta} = \sigma_2 V_{\alpha} L_{\beta} - J_1 K_{\alpha} L_{\beta} + J_2 S_{\alpha} S_{\beta} \quad \text{and} \]
\[ S_{\alpha;\beta} = \sigma_2 V_{\alpha} S_{\beta} - J_1 K_{\alpha} S_{\beta} - J_2 L_{\alpha} S_{\beta}; \]

where \( J_1, J_2, \sigma_1, \sigma_2 \) and \( a_1 \) are expressed in terms of the metric functions and their derivatives as:

\[ J_1 = \frac{1}{B} \frac{R'}{R}, \quad J_2 = \frac{1}{R} \cot \theta, \]
\[ \sigma_1 = \frac{1}{A} \frac{B'}{B}, \quad \sigma_2 = \frac{1}{A} \frac{R'}{R} \]

and \( a_1 = \frac{1}{B} \frac{A'}{A} \);

with primes and dots representing radial and time derivatives, respectively.

As we mentioned before we shall assume our source as a bounded, spherically symmetric, locally anisotropic, dissipative, collapsing matter configuration, described by a general energy momentum tensor, written in the “canonical” form, as

\[ T_{\alpha\beta} = (\rho + P) V_{\alpha} V_{\beta} + Pg_{\alpha\beta} + \Pi_{\alpha\beta} + F_{\alpha} V_{\beta} + F_{\beta} V_{\alpha}. \]

It is immediate to see that the physical variables can defined –at the Eckart frame where fluid elements are at rest– as:

\[ \rho = T_{\alpha\beta} V^\alpha V^\beta, \quad F_{\alpha} = -\rho V_{\alpha} - T_{\alpha\beta} V^\beta, \]
\[ P = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}, \quad \Pi_{\alpha\beta} = h^\alpha_{\mu} h^\beta_{\nu} (T_{\mu\nu} - Ph_{\mu\nu}); \]

with \( h_{\mu\nu} = g_{\mu\nu} + V_{\mu} V_{\nu} \).

Observe that from the condition \( F^\mu V_{\mu} = 0 \), and the symmetry of the problem, Einstein equations imply \( T_{03} = 0 \), thus

\[ F_{\mu} = FK_{\mu} \iff F_{\mu} = \left( \frac{F}{B}, 0, 0 \right). \]

Clearly \( \rho \) is the energy density (the eigenvalue of \( T_{\alpha\beta} \) for eigenvector \( V^\alpha \)), \( F_{\alpha} \) represents the energy flux four vector; \( P \) corresponds to the isotropic pressure, and \( \Pi_{\alpha\beta} \) is the anisotropic tensor, which can be expressed as

\[ \Pi_{\alpha\beta} = \Pi_1 \left( K_{\alpha} K_{\beta} - \frac{h_{\alpha\beta}}{3} \right), \]

with

\[ \Pi_1 = \left( 2K^\alpha K^\beta + L^\alpha L^\beta \right) T_{\alpha\beta}. \]
Finally, we shall express the kinematical variables—the four acceleration, the expansion scalar and the shear tensor—for a self-gravitating fluid as:

\[ a_\alpha = V^\beta V_{\alpha,\beta} = aK_\alpha = \left(0, \frac{A'}{A}, 0, 0\right), \]  

\[ \Theta = V^\alpha_{,\alpha} = \frac{1}{A} \left(\frac{2\dot{B}}{B} + \frac{\dot{R}}{R}\right), \]  

\[ \sigma = \frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right), \]  

### 3.2. The orthogonal splitting of Riemann tensor and structure scalars

In this section we shall introduce a set of scalar functions, the structure scalars, obtained from the orthogonal splitting of the Riemann tensor (see [9, 10, 11]) which has proven to be very useful to express Einstein Equations.

Following [9], we can express the splitting of the Riemann tensor as

\[ R_{\alpha\beta\mu\nu} = 2V_\mu V_{[\alpha}Y_{\beta]\nu} + 2h_{\alpha}[\nu X_{\mu]\beta] + 2V_\nu V_{[\beta}Y_{\alpha]\mu} + h_{\beta\nu}(X_\alpha h_{\mu\beta} - X_{\alpha\mu}) + h_{\beta\mu}(X_\nu h_{\alpha\nu} - X_0 h_{\alpha\nu}) + 2V_\mu Z_{\nu\mu}^\gamma \varepsilon_{\alpha\beta\gamma} + 2V_{\nu}Z_{\alpha}^{\gamma} \varepsilon_{\mu\nu\gamma} \]  

and its corresponding Ricci contraction as

\[ R_{\alpha\mu} = Y_0 V_{\alpha} V_{\mu} - X_{\alpha\mu} - X_0 h_{\alpha\mu} + Z_{\alpha}^{\nu\beta} \varepsilon_{\mu\nu\beta} V_{\alpha} + V_\mu Z_{\nu\beta}^\gamma \varepsilon_{\alpha\nu\beta}; \]  

where the quantities \( Y_{\alpha\beta}, X_{\alpha\beta} \) and \( Z_{\alpha\beta} \) can be expressed as

\[ Y_{\alpha\beta} = \frac{1}{3} Y_0 h_{\alpha\beta} + Y_1 (K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta}) , \]  

\[ X_{\alpha\beta} = \frac{1}{3} X_0 h_{\alpha\beta} + X_1 (K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta}) \]  

and

\[ Z_{\alpha\beta} = Z(L_\alpha S_\beta - L_\beta S_\alpha) \]

with,

\[ Y_0 = 4\pi(\rho + 3P), \quad Y_1 = \mathcal{E}_1 - 4\pi \Pi_1, \]  

\[ X_0 = 8\pi\rho, \quad X_1 = - (\mathcal{E}_1 + 4\pi \Pi_1), \]  

\[ Z = 4\pi \mathcal{F}, \]

and the electric part of the Weyl tensor is written as

\[ E_{\alpha\beta} = C_{\alpha\nu\beta\delta} V^\nu V^\delta = \mathcal{E}_1 (K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta}). \]

### 3.3. Projections of Riemann Tensor

From the above system (4), by using the covariant derivative of equations (8) and the projections of the orthogonal splitting of the Riemann tensor— we can obtain first set of
3.4. Equations from Bianchi identities

The second set of equations for the spherical case, emerges from the independent Bianchi identities (5), and can be written as

\[
\begin{align*}
\sigma_1^\bullet - a_1^\dagger - a_1^2 + \sigma_1^2 &= -\frac{1}{3}(Y_0 + 2Y_1), \\
\sigma_2^\bullet + \sigma_2^2 - a_1J_1 &= \frac{1}{3}(Y_1 - Y_0), \\
\sigma_2^\dagger + J_1(\sigma_2 - \sigma_1) &= -Z, \\
J_1^\bullet + J_1\sigma_2 - a_1\sigma_2 &= -Z, \\
J_1^\dagger + J_2\sigma_2 &= \frac{1}{3}(X_1 - X_0), \\
J_2^\bullet + J_2\sigma_2 &= 0, \\
J_2^\dagger + J_1J_2 &= 0 \quad \text{and} \\
J_1^2 - \frac{1}{R^2} - \sigma_2^2 &= -\frac{1}{3}(X_0 + 2X_1).
\end{align*}
\]

3.5. Spherical Cases of Study

3.5.1. The static case

In the line element (6) we can assume, without any loss of generality, \( R = r \) and integrate (28) to obtain

\[
A = C_1 e^{\int B^2 r(Y_0 - Y_1) dr}
\]

and, from (30) it follows at once that

\[
B^2 = \frac{1}{1 - \frac{r^2}{3}(X_0 + 2X_1)}
\]

or, by using (29)

\[
B^2 = \frac{1}{1 + C_2 r^2 + 2r^2 \int \frac{\sigma_1^\bullet}{r} dr},
\]

where \( C_1 \) and \( C_2 \) are constants of integration.

As we can see from (39) and (41), this metric describes any static anisotropic sphere (see reference [12]) and can be expressed in terms of structure scalars \( X_1 \) and \( Y_0 - Y_1 \).

It is easy to check that the Schwarzschid interior solution corresponds to the case \( X_1 = Y_1 = 0 \). To illustrate the simplicity of the method, let us consider \( X_0 = \text{const} \) and feeds this condition back into equation (38) to obtain

\[
X_1 = \frac{C}{r^3}
\]
Now, the regularity condition at \( r = 0 \) via (40), implies \( C = 0 \), then

\[-\mathcal{E}_1 - 4\pi \Pi_1 = 0 \Rightarrow \mathcal{E}_1 = \Pi_1 = 0. \tag{43}\]

Given that \( \mathcal{E}_1(t,0) = 0 \), by the conditions of regularity at the origen, and \( \mathcal{E}_1(t,R_\Sigma) = \frac{3M}{R_\Sigma} > 0 \) imposed by the boundary conditions, we demand that \( \mathcal{E}_1(t,r) > 0 \). On the other hand \( \Pi_1 \) must be positive to guarantee the decreasing of radial pressure, via T.O.V. equation (See the reference [13])

Thus, we can see that necessarily the only static solution with homogeneous energy density, under the above considerations, is the Schwarzschild solution.

3.5.2. The non-static case

(i) The Shear-free solutions \( \sigma_1 = \sigma_2 \)

Under this condition, i.e., \( \sigma = 0 \), the equation (17) turns out to be

\[ \frac{\dot{B}}{B} = \frac{\dot{R}}{R} \tag{44} \]

Thus we have two options:

Option 1 If

\[ R = bB \tag{45} \]

where \( b \) is an arbitrary function of \( r \) which can be taken as 1 without loss of generality, then we get

\[ R = rB, \tag{46} \]

but it does not satisfy the regularity condition at \( r = 0 \), nor the boundary condition.

Option 2

\[ B = T\dot{B} \text{ and } R = T\dot{R}, \tag{47} \]

where \( T \) is an arbitrary function of \( t \) and \( \dot{B}, \dot{R} \) are arbitrary functions of \( r \). Feeding back (47) into the equation (29), with \( Z = 0 \), we get

\[ \sigma = \sigma(t) \Rightarrow A = \frac{T}{\sigma T}, \tag{48} \]

which implies that \( a_1 = 0 \). Therefore, we conclude that non-dissipative spherically symmetric shear-free fluids are necessarily geodesic.

(ii) The Case \( \sigma_1 = 0 \text{ and } Z = 0 \)

If one assumes that \( \sigma_1 = 0 \) and that the fluid is non-dissipative then, the equation (29) can be integrated to obtain

\[ \sigma_2 = \frac{\tilde{\sigma}(t)}{R}, \tag{49} \]

where \( \tilde{\sigma}(t) \) is an arbitrary function of integration. Next, by demanding regularity of \( \sigma_2 \) at \( r = 0 \), a fact obvious from (34), we have to put \( \tilde{\sigma} = 0 \), which implies that \( \sigma_2 = 0 \), therefore we are lead to the static case.

(iii) The Case \( \sigma_2 = 0 \text{ and } Z = 0 \)

If one assume that \( \sigma_2 = 0 \) and that the fluid is non-dissipative then, from (29) it follows at once that

\[ J_1\sigma_1 = 0 \tag{50} \]

but, \( J_1 \) can not vanish, thus we get \( \sigma_1 = 0 \) i.e., static case.

From results obtained in ii) and iii) it follows that in the absence of dissipation (\( Z = 0 \)), using the regular center condition, if one of the functions \( B \) or \( R \) does not depend on \( t \), the metric (6) is static.
The Case $X_0 = X_0(t)$ and $Z = 0$ In this case, we obtain from the equation (38) that $X_1 = 0$ and the equations (35)-(37) become:

$$
a_1(Y_1 - X_0 - Y_0) + 3J_1Y_1 = (Y_0 - Y_1)^† \tag{51}
$$

$$
X^*_0 + (X_0 + Y_0 - Y_1)\sigma_1 + (X_0 + Y_0 + 2Y_1)\sigma_2 = 0 \tag{52}
$$

$$
X^*_0 + 2(X_0 + Y_0 - Y_1)\sigma_2 = 0 \tag{53}
$$

Notice that if $X_0 = \text{const.}$, we get the static case, that we have analyzed before. Combining the equations (52) and (53) we find that

$$
(Y_0 + X_0 - Y_1)(\sigma_1 - \sigma_2) + 3Y_1\sigma_2 = 0 \tag{54}
$$

From the equation (54), it follows that

$$
\sigma_1 - \sigma_2 = 0 \iff Y_1 = 0. \tag{55}
$$

Again this leads us to the static case. In other words, shear-free and isotropic pressure conditions are equivalent, for non-dissipative fluids with homogeneous energy density.

4. Final Remark

We have shown, through very simple examples in the spherical case, how easily it is possible to obtain information from this Einstein-equivalent system and this encourage us to explore more general cases far from the spherically symmetric toy model we have developed in this work. The axisymmetric cases and also more general matter configurations are undergoing works that will be completed shortly.

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