Gradation of Fuzzy Preconcept Lattices

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Abstract: Noticing certain limitations of concept lattices in the fuzzy context, especially in view of their practical applications, in this paper, we propose a more general approach based on what we call graded fuzzy preconcept lattices. We believe that this approach is more adequate for dealing with fuzzy information than the one based on fuzzy concept lattices. We consider two possible gradation methods of fuzzy preconcept lattice—an inner one, called D-gradation and an outer one, called M-gradation, study their properties, and illustrate by a series of examples, in particular, of practical nature.

Keywords: fuzzy context; fuzzy preconcept; fuzzy preconcept lattice; fuzzy concept; fuzzy concept lattice; graded fuzzy preconcept lattice

1. Introduction

Formal concept analysis, or just concept analysis for short, was developed mainly in eighties of the previous century by R. Wille and B. Ganter. The principles and fundamental results of concept analysis were exposed in detail in Reference [1] and further expanded in Reference [2]. The concept analysis starts with the notion of a (formal) context, i.e., a triple \((X, Y, R)\), where \(X\) and \(Y\) are sets, and \(R \subseteq X \times Y\) is a relation between the elements of these sets. The elements of \(X\) are interpreted as some abstract objects, the elements of \(Y\) are interpreted as some abstract properties or attributes, and the entry \((x, y) \in R\) means that an object \(x\) has attribute \(y\). The idea of the concept analysis is to reveal all pairs \((A, B)\) of sets \(A \subseteq X\) and \(B \subseteq Y\) (called concepts) such that every object \(x \in A\) has all properties \(y \in B\) and every property \(y \in B\) holds for all objects \(x \in A\).

The set of all such pairs in a given context \((X, Y, R)\) endowed with a certain partial order makes a lattice, called a concept lattice, the principal object of research in concept analysis.

In the second half of nineties, and especially in the first decade of the 21st century, different fuzzy counterparts of the formal concept were introduced and studied. In the fuzzy case, a context is a tuple \((X, Y, L, R)\), where \(X\) and \(Y\) are non-empty sets, \(L\) is a lattice, and \(R : X \times Y \rightarrow L\) is an \(L\)-fuzzy relation. Fuzzy concepts in this fuzzy context are pairs \((A, B)\), where \(A\) and \(B\) are \(L\)-fuzzy subsets of the sets \(X\) and \(Y\), respectively, which are interrelated in a way, regarding the relation in the crisp case (see Definition 4). The most important work in the first decade of the 21st century in the field of fuzzy concept analysis was carried out by R. Bělohlávek; see, e.g., Reference [3–10], etc. In particular, Reference [3,10] are probably the first works where fuzzy concept lattices appear.

Concept analysis and concept lattices, crisp, as well as fuzzy, aroused great interest both among theorists in mathematics and among practicing researchers. The theoretical interest in concept lattices can be explained, in particular, by the fact that they form interesting non-trivial internal connections with other mathematical structures. As an example, we mention here that every complete lattice can be obtained as a concept lattice for some formal context [2].

Approximately at the same time when the mathematical notion of a concept was introduced and concept analysis started to develop, the notion of a rough set appeared, and the
theory of rough sets was initiated in the works by Z. Pawlak; see Reference [11], etc. Ten years later, a fuzzy counterpart of a rough set was introduced [12], and the corresponding theory was started in the works by various authors; see, e.g., Reference [13,14]. Although, outwardly, the two approaches—(fuzzy) concept analysis and (fuzzy) rough sets—are essentially different, these theories are deeply interrelated, in particular, on the theory-categorical level; see Reference [15] for the crisp case and Reference [8] for the fuzzy counterpart. Specifically, it was established in these works that fuzzy concept lattices have “stronger impressive power” than lattices based on fuzzy rough sets, while, in the crisp case, both structures are, in a certain sense, equivalent.

Another, at the first glance, unexpected fact is the existence of interesting relations between fuzzy concept analysis and structures of fuzzy mathematical morphology; see, e.g., Reference [7].

Since its inception, crisp concept analysis has found important applications in the study of “real-world” problems. Starting with illustrative examples of application of crisp lattices given in Reference [1], there appeared many serious works in which concept lattices were used in the research of medical-related problems [16–18], etc.; problems related to biology [19,20], etc.; social type problems [21], etc.; and in other applied sciences. On the other hand, we found only a few works, where fuzzy concept analysis is used in the research of any practical-type problems. Moreover, all examples known to us are based on “small” (say 5 element subsets of the unit interval [0, 1]) lattices and actually can be reformulated and solved with the tools of the so-called multi-level crisp concept lattice [1,2]; see, for instance, the examples given in Reference [6]. However, the need to use lattices with infinite or finite but with a huge number of elements appears when we have to deal with such objects or properties like location, temperature, wind direction and its strength, color, soil acidification, etc.

In our opinion, the problem to use fuzzy concept lattices in the case when the scale value $L$ is continuous (like $L = [0, 1]$) or lattice having many, possibly incomparable elements, is that the request in the concept analysis of the precise correspondence between the fuzzy set $A$ of objects and the fuzzy set $B$ of attributes in “real-world” situations is (almost) impracticable. In this case, one sooner has to deal with the weaker request asking that the correspondence between $A$ and $B$ must hold up to a certain degree. In order to provide a theoretical basis for the research of the problem in this situation, we first replace the notion of a fuzzy concept by a much weaker notion of a fuzzy preconcept, and then propose technique, allowing to evaluate “how far a fuzzy preconcept is from the nearest fuzzy concept”. We evaluate this “nearness” by operators of gradation on the lattices of fuzzy preconcepts for a given fuzzy context $(X, Y, L, R)$. It was the main goal of this paper to propose two approaches for gradation of fuzzy preconcepts and to define the corresponding lattices of graded fuzzy preconcepts. Later, we initiate the study of these lattices and illustrate them by a series of examples, both of a theoretical and a practical nature. We believe that the approach based on graded fuzzy preconcept lattices will be more appropriate when dealing with fuzzy information than the traditional one which is based on fuzzy concept lattices.

The paper is organized as follows. In the second, preliminary, section, we briefly recall the notions related to lattices, residuated lattices or quantales, fuzzy sets, and fuzzy relations, and these concepts constitute the language for our research. Further in this section, we remind the reader of the concept of a fuzzy inclusion of one fuzzy set into another; just the measure of such inclusion lies in the base of the grades of fuzzy preconcepts—the main concepts further introduced in this paper.

In the third section, we define fuzzy preconcepts, introduce partial order relation $\preceq$ on the family of all fuzzy preconcepts of a given fuzzy context $(X, Y, L, R)$, and show that the resulting structure $(P(X, Y, L, R), \preceq)$ is a lattice. We describe some properties of such fuzzy preconcept lattices. In the fourth section, we consider operators $R^\uparrow$ and $R^\downarrow$ on fuzzy preconcept lattices; these operators play fundamental role in our work, and, in particular, they are used in order to distinguish “real” fuzzy concepts from arbitrary fuzzy preconcepts. This is done in Section 5, where the family of all fuzzy concepts
A lattice is called complete, if, for every non-empty set, its infimum or the greatest lower bound (also called meet) is defined. A partially ordered set is called a lattice, if any two its elements have the upper and the lower bound. Let \( L \) be a lattice. The infimum or the greatest lower bound of \( L \) is called the meet or the greatest lower bound. For every \( a \in L \), the element \( \bot \) is called the bottom element of \( L \). For every \( a \in L \), the element \( \top \) is called the top element of \( L \). A complete lattice is called infinitely distributive or join-distributive if \( L \) is its top element and every \( \{b_i | i \in I\} \subseteq L \). A complete lattice is called join-distributive if it is infinitely and co-infinitely distributive. It is known that every completely distributive lattice is infinitely bi-distributive. Every MV-algebra, which is join-distributive, is also meet-distributive.

In the sixth section, the definition of a grade of a fuzzy preconcept is based on the evaluation of mutual “contentment” of the fuzzy set of objects and the fuzzy set of attributes. We call this approach “inner”, the corresponding evaluation of a fuzzy preconcept from “being a real fuzzy concept” and study the corresponding graded fuzzy preconcept lattices. In the seventh section, we consider an alternative approach to the evaluation of the measure of conceptuality of a fuzzy preconcept from its conceptual kernel—the notions introduced in this section. The value obtained in this way is called the measure of conceptuality of a fuzzy preconcept from its conceptual hull and the measure of distinction of a fuzzy preconcept from its conceptual kernel. The following, sixth and seventh, sections are the central ones in the work. Here, we sketch one example of practical nature related to modeling of income in public transportation services.

In the last, conclusion, section, we briefly summarize main results obtained and survey some directions for the future work.

2. Preliminaries

2.1. Lattices

We recall here some well known concepts from the theory of lattices that will be used in the paper; see, e.g., Reference [22–25], for details.

Given a set \( L \), a binary relation \( \leq \) on \( L \) is called a partial order if it is (1) reflexive, i.e., \( a \leq a \) for all \( a \in L \); (2) anti-symmetric, i.e., \( a \leq b \) and \( b \leq a \) implies that \( a = b \) for all \( a,b \in L \); and (3) transitive, i.e., \( a \leq b \) and \( b \leq c \) implies \( a \leq c \) for all \( a,b,c \in L \).

Given \( a,b \in L \), the element \( c = a \lor b \in L \) is called suprema and the least upper bound of \( a \) and \( b \) if \( a \leq c \), \( b \leq c \) and \( c \leq d \in L \) whenever \( a \leq d \) and \( b \leq d \) for every \( d \in L \). The infimum or the greatest lower bound \( a \land b \) is defined in the dual way. A partially ordered set is called a lattice, if any two its elements have the upper and the lower bound.

Let \( M \subseteq L \). Element \( c = \bigvee M \in L \) is called the supremum or the least upper bound of \( M \) if (1) \( k \leq c \) for every \( k \in M \) and (2) if \( d \in L \) and \( k \leq d \) for every \( k \in M \), then \( c \leq d \). The infimum or the greatest lower bound \( d = \bigwedge M \in L \) of \( M \subseteq L \) is defined in a dual way. A lattice is called complete, if, for every non-empty \( M \subseteq L \), there exists the least upper and the greatest lower bounds. In particular, the least upper bound of \( L \) is its top element \( \top \), and the greatest lower bound of \( L \) is its bottom element \( \bot \).

A complete lattice \( L \) is called infinitely distributive or join-distributive if \( a \land (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \land b_i) \) for every \( a \in L \) and every \( \{b_i | i \in I\} \subseteq L \). A complete lattice \( L \) is called infinitely co-distributive or meet-distributive, if \( a \lor (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \lor b_i) \) for every \( a \in L \) and every \( \{b_i | i \in I\} \subseteq L \). A complete lattice is called infinitely bi-distributive or join-meet-distributive if it is infinitely and co-infinitely distributive. It is known that every completely distributive lattice is infinitely bi-distributive. Every MV-algebra, which is join-distributive, is also meet-distributive.
2.2. Quantales and Residuated Lattices

The notion of a quantale first appears in Rosenthal’s paper [26]. Here, we recall the information related to quantales that is needed in the work.

Let \( \mathbb{L} \) be a complete lattice and \( \ast : \mathbb{L} \times \mathbb{L} \to \mathbb{L} \) be a binary associative monotone operation. Then, the tuple \( (\mathbb{L}, \leq, \land, \lor, \ast) \) is called a quantale if \( \ast \) distributes over arbitrary joins:

\[
\forall a \in \mathbb{L}, \{ b_i | i \in I \} \subseteq \mathbb{L} \implies \bigvee_{i \in I} (a \ast b_i) = \bigvee_{i \in I} (a \ast \bigvee_{i \in I} b_i).
\]

Operation \( \ast \) will be referred to as the product in \( \mathbb{L} \). A quantale is called integral if the top element \( \top \) acts as the unit, i.e., \( \top \ast a = a \ast \top = a \) for every \( a \in \mathbb{L} \); in this case, we write \( \top = 1_{\mathbb{L}} \) and \( \bot = 0_{\mathbb{L}} \). A quantale is called commutative, if the product is commutative. In what follows, in saying quantale, we mean a commutative integral quantale.

A typical example of a quantale is the unit interval endowed with a lower semi-continuous \( t \)-norm; see, e.g., Reference [27].

In a quantale, a further binary operation \( \mapsto : \mathbb{L} \times \mathbb{L} \to \mathbb{L} \), the residuum, can be introduced as associated with operation \( \ast \) of the quantale \( (\mathbb{L}, \leq, \land, \lor, \ast) \) via the Galois connection, i.e.,

\[
a \ast b \leq c \iff a \leq b \mapsto c \text{ for all } a, b, c \in \mathbb{L}.
\]

Explicitly, the residuum can be defined by

\[
a \mapsto b = \bigvee \{ l \in \mathbb{L} | l \ast a \leq b \}.
\]

A quantale \( (\mathbb{L}, \leq, \land, \lor, \ast, \mapsto) \) provided with the derived operation \( \mapsto \), i.e., the tuple \( (\mathbb{L}, \leq, \land, \lor, \ast, \mapsto) \), is known also as a (complete) residuated lattice [24].

In the following proposition, we collect well-known properties of the residuum which will be used in the main text:

**Proposition 1.** See, e.g., Reference [28,29].

1. \( (a \mapsto b) \mapsto c = \bigwedge_{i \in I} (a_i \mapsto b) \) for all \( \{ a_i | i \in I \} \subseteq \mathbb{L} \) for all \( b \in \mathbb{L} \);
2. \( a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i) \) for all \( a \in \mathbb{L} \) for all \( \{ b_i | i \in I \} \subseteq \mathbb{L} \);
3. \( 1_{\mathbb{L}} \mapsto a = a \) for all \( a \in \mathbb{L} \);
4. \( a \mapsto b = 1_{\mathbb{L}} \) whenever \( a \leq b \);
5. \( a \ast (a \mapsto b) \leq b \) for all \( a, b \in \mathbb{L} \);
6. \( (a \mapsto b) \ast (b \mapsto c) \leq a \mapsto c \) for all \( a, b, c \in \mathbb{L} \);
7. \( a \mapsto b \leq (a \ast c \mapsto b \ast c) \) for all \( a, b, c \in \mathbb{L} \);
8. \( a \ast b \leq a \land b \) for any \( a, b \in \mathbb{L} \);
9. \( (a \ast b) \mapsto c = a \mapsto (b \mapsto c) \) for any \( a, b, c \in \mathbb{L} \).

2.3. Fuzzy Sets and Fuzzy Relations

The concept of a fuzzy set was introduced by L.A. Zadeh [30] and then extended to a more general concept of an \( L \)-fuzzy set by J.A. Goguen [31], where \( L \) is a complete lattice, in particular, a quantale. We assume that the reader is well acquainted with terminology related to (\( L \))-fuzzy sets. Just to clarify the details, we reproduce here some notions.

Given a set \( X \) its \( L \)-fuzzy subset is a mapping \( A : X \to L \). The lattice and the quantale structure of \( L \) is extended point-wise to the \( L \)-exponent of \( X \); that is to the set \( L^X \) of all \( L \)-fuzzy subsets of \( X \). Specifically, the union and intersection of a family of \( L \)-fuzzy sets \( \{ A_i | i \in I \} \subseteq \mathbb{L} \) are defined by their join \( \bigvee_{i \in I} A_i \) and meet \( \bigwedge_{i \in I} A_i \), respectively. An \( L \)-fuzzy relation between two sets \( X \) and \( Y \) is an \( L \)-fuzzy subset of the product \( X \times Y \), i.e., a mapping \( R : X \times Y \to \mathbb{L} \); see, e.g., Reference [32,33].

**Remark 1.** In this work, we will deal with several different lattices and quantales. The symbol \( \mathbb{L} \) is used as the common notation for any lattice (quantale) which appears in our work. On the other hand, notation \( L \) is used when we speak about \( L \)-fuzzy subsets of sets and about \( L \)-fuzzy relations.
as they are defined above. Besides, since $L$, as the lattice of values for $L$-fuzzy subsets and $L$-fuzzy relations, is in the paper an arbitrary but a fixed lattice, we shall omit the prefix $L$ and speak just of fuzzy sets and fuzzy relations.

2.4. Measure of Inclusion of $L$-Fuzzy Sets

The gradation of a preconcept lattice presented below is based on the fuzzy inclusion between fuzzy sets. We present here a brief introduction into this field.

In order to fuzzify the inclusion relation $A \subseteq B$, “a fuzzy set $A$ is a subset of a fuzzy set $B$”, and we have to interpret it as a certain fuzzy inclusion $\rightarrow$ based on “if-then” rule, i.e., on some implication $\Rightarrow$ on the lattice $L$. In the result, we come to the formula $A \rightarrow B = \inf_{x \in X} (A(x) \Rightarrow B(x))$. As far as we know, for the first time this approach was applied in Reference [34], where it was based on the Kleene-Dienes implication $\Rightarrow$. Later, the fuzzified relation of inclusion between fuzzy sets was studied and used by many authors; see, e.g., Reference [35–39]. In most of the papers, the implication $\Rightarrow$ was defined by means of residuum $\rightarrow$ of the underlying quantale $(L, \wedge, \vee, *)$. This implication behaves in this situation “much better” than the Kleene-Dienes and some other implication on $(L, \wedge, \vee, *)$, although some authors use also other implications for the measuring of the inclusion of one fuzzy set into another. In our paper, we stick to the residuum-based measure of inclusion specified in the following definition:

**Definition 1.** By setting $A \rightarrow B = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$ for all $A, B \in L^X$, we obtain a mapping $\rightarrow: L^X \times L^X \rightarrow L$. Equivalently, relation $\rightarrow$ can be defined by $A \rightarrow B = \inf (A \Rightarrow B)$, where the infimum of the $L$-fuzzy set $A \rightarrow B \in L^X$ is taken in the lattice $L$. We call $A \rightarrow B$ by the ($L$-valued) measure of inclusion of the $L$-fuzzy set $A$ into the $L$-fuzzy set $B$. In the text, we use also notation $A \leftarrow B$ in the place of $B \rightarrow A$. Besides, we denote $A \cong B =_{def} (A \rightarrow B) \wedge (B \rightarrow A)$, and view it as the degree of equality of $L$-fuzzy sets $A$ and $B$.

As the next proposition shows, the measure of inclusion $\rightarrow: L^X \times L^X \rightarrow L$ has properties in a certain sense resembling the properties of the residuum:

**Proposition 2.** (see, e.g., Reference [40,41].) Mapping $\rightarrow: L^X \times L^X \rightarrow L$ satisfies the following properties:

1. $(\bigvee_i A_i) \rightarrow B = \bigwedge_i (A_i \rightarrow B)$ for all $\{A_i \mid i \in I\} \subseteq L^X$ and for all $B \in L^X$;
2. $A \rightarrow (\bigwedge_i B_i) = \bigwedge_i (A \rightarrow B_i)$ for all $A \in L^X$, and for all $\{B_i \mid i \in I\} \subseteq L^X$;
3. $A \rightarrow B = 1_L$ whenever $A \leq B$;
4. $1_X \rightarrow A = \bigwedge_x A(x)$ for all $A \in L^X$, where $1_X: X \rightarrow L$ is a constant function with the value $1_L \in L$;
5. $(A \rightarrow B) \leq (A \ast C) \rightarrow (B \ast C)$ for all $A, B, C \in L^X$;
6. $(A \rightarrow B) \ast (B \rightarrow C) \leq (A \rightarrow C)$ for all $A, B, C \in L^X$;
7. $(\bigwedge_i A_i) \rightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \rightarrow B_i)$ for all $\{A_i \mid i \in I\}, \{B_i \mid i \in I\} \subseteq L^X$;
8. $(\bigvee_i A_i) \rightarrow (\bigvee_i B_i) \geq \bigvee_i (A_i \rightarrow B_i)$ for all $\{A_i \mid i \in I\}, \{B_i \mid i \in I\} \subseteq L^X$.

3. Preconcepts and Preconcept Lattices

Let $L$ be a complete lattice (in particular, a quantale) with the top and the bottom elements 1 and 0, respectively. Further, let $X, Y$ be sets and $R: X \times Y \rightarrow L$ be a fuzzy relation.

Following terminology accepted in the theory of (fuzzy) concept lattices, as in, e.g., Reference [1,3–5], we refer to the tuple $(X, Y, L, R)$ as a fuzzy context.

**Definition 2.** Given a fuzzy context $(X, Y, L, R)$, a pair $P = (A, B) \in L^X \times L^Y$ is called a fuzzy preconcept (The notion of a fuzzy preconcept is not related to the notion of a preconcept as it is defined in Reference [2] (p. 59).).

On the set $L^X \times L^Y$ of all fuzzy preconcepts, we introduce a partial order $\preceq$ as follows. Given $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$, we set $P_1 \preceq P_2$ if and only if $A_1 \leq A_2$ and $B_1 \geq B_2$. 
Let \((\mathcal{P}, \preceq)\) be the set \(L^X \times L^Y\) endowed with this partial order. Further, given a family of fuzzy concepts \(\{P_i = (A_i, B_i) : i \in I\} \subseteq L^X \times L^Y\), we define its join (supremum) by \(\forall i \in I P_i = (\bigwedge_{i \in I} A_i, \bigvee_{i \in I} B_i)\) and its meet (infimum) as \(\forall i \in I P_i = (\bigvee_{i \in I} A_i, \bigwedge_{i \in I} B_i)\).

**Theorem 1.** \(\mathcal{P}\) is a complete lattice. Besides, if \(L\) is an infinitely bi-distributive lattice, then \((\mathcal{P}, \preceq, \wedge, \vee)\) is also a infinitely bi-distributive lattice.

**Proof.** Let \(\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathcal{P}\). Then, \(\bigwedge_{i \in I} P_i = (\bigwedge_{i \in I} A_i, \bigvee_{i \in I} B_i)\) and \(\bigvee_{i \in I} P_i = (\bigvee_{i \in I} A_i, \bigwedge_{i \in I} B_i)\) is a complete lattice. So, \((\mathcal{P}, \preceq, \wedge, \vee)\) is a complete lattice. Its top and bottom elements are \(\top_p = (1_A, 0_Y)\) and \(\bot_p = (0_X, 1_Y)\), where \(a_X\) and \(a_Y\) are, respectively, constant fuzzy subsets of \(X\) and \(Y\) with values \(a \in L\).

Further, assume that \(L\) is infinitely bi-distributive. To show that \((\mathcal{P}, \preceq, \wedge, \vee)\) is infinitely distributive, let \(\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathcal{P}\) and \(P = (A, B) \in \mathcal{P}\).

\[
\forall i \in I P_i \quad \forall P = (A, B) \quad \bigwedge_{i \in I} P_i \quad \bigvee_{i \in I} P_i
\]

In the same manner, we prove that \((\mathcal{P}, \preceq, \wedge, \vee)\) is an infinitely co-distributive lattice:

\[
\bigwedge_{i \in I} P_i \quad \bigvee_{i \in I} P_i
\]

\[\square\]

In the sequel, we write just \(\mathcal{P}\) or \((\mathcal{P}, \preceq)\) instead of \((\mathcal{P}, \preceq, \wedge, \vee)\), if no misunderstanding is possible, or \((\mathcal{P}, X, Y, L, R)\), in the case when we need to specify the fuzzy context we are working in.

**4. Operators \(R^\uparrow\) and \(R^\downarrow\) on \(L\)-Powersets**

Let \(X\) and \(Y\) be sets and let \(R : X \times Y \rightarrow L\) be a fuzzy relation, where \(L\) is a fixed quantale. Given a fuzzy context \((X, Y, L, R)\), we define operators \(R^\uparrow : L^X \rightarrow L^Y\) and \(R^\downarrow : L^Y \rightarrow L^X\) as follows:

**Definition 3.** (see, e.g., Reference [4]). Given \(A \in L^X\), we define \(A^\uparrow \in L^Y\) by setting

\[ A^\uparrow(y) = \bigwedge_{x \in X} (A(x) \rightarrow R(x, y)) \quad \forall y \in Y.\]

Given \(B \in L^Y\), we define \(B^\downarrow \in L^X\) by setting

\[ B^\downarrow(x) = \bigwedge_{y \in Y} (B(y) \rightarrow R(x, y)) \quad \forall x \in X.\]

By changing \(A\) over \(L^X\), we get an operator \(R^\uparrow : L^X \rightarrow L^Y\), and, by changing \(B\) over \(L^Y\), we get an operator \(R^\downarrow : L^Y \rightarrow L^X\).

**Remark 2.** In the crisp case, i.e., when \(A \subseteq X, B \subseteq Y\) and \(R : X \times Y \rightarrow \{0,1\}\), this definition is obviously equivalent to the original definition of operators \(A \rightarrow A'\) and \(B \rightarrow B'\) in Reference [1].

From the properties of the residuum, one can easily justify the following.

**Proposition 3.** Operators \(R^\uparrow : L^X \rightarrow L^Y\) and \(R^\downarrow : L^Y \rightarrow L^X\) are non-increasing:

\[ A_1 \leq A_2 \Rightarrow A_1^\uparrow \geq A_2^\uparrow; \quad B_1 \leq B_2 \Rightarrow B_1^\downarrow \geq B_2^\downarrow.\]
In the sequel, we write $A^{\uparrow \downarrow}$ instead of $(A^\uparrow)^\downarrow$ and $B^{\downarrow \uparrow}$ instead of $(B^\downarrow)^\uparrow$. We also write $R^{\uparrow \downarrow}$ for the composition $R^\uparrow \circ R^\downarrow : L^X \to L^X$, and $R^{\downarrow \uparrow}$ for the composition $R^\downarrow \circ R^\uparrow : L^Y \to L^Y$.

**Proposition 4.** (cf., e.g., Reference [1] in crisp case, Reference [5].) $A^{\downarrow \uparrow} \geq A$ for every $A \in L^X$ and $B^{\downarrow \uparrow} \geq B$ for every $B \in L^Y$.

**Proof.** Given $A \in L^X$ and $x \in X$, we have $A^{\downarrow \uparrow}(x) = \bigwedge_{y \in Y} (A^\uparrow(y) \to R(x, y)) = \bigwedge_{y \in Y} (A^\uparrow(y) \to R(x, y))$.

In a similar way, the second inequality can be established. □

**Proposition 5.** (cf., e.g., Reference [1] in crisp case, Reference [5].) $A^\uparrow = A^{\downarrow \uparrow \uparrow}$ for every $A \in L^X$ and $B^\uparrow = B^{\downarrow \uparrow \uparrow}$ for every $B \in L^Y$.

**Proof.** $A \leq A^{\downarrow \uparrow}$ by Proposition 4; hence, $A^\uparrow \geq A^{\downarrow \uparrow \uparrow}$. On the other hand, applying Proposition 3, we have $A^\uparrow \leq A^{\downarrow \uparrow \uparrow}$.

Similarly, the second equality can be proved. □

**Example 1.** Let a fuzzy context $(X, Y, L, R)$ be given and let $A \subseteq X$ (Here, and in the sequel, we do not distinguish between a crisp set $A \subseteq X$ and its characteristic function $\chi_A : X \to \{0, 1\}$). Then, for every $y \in Y$, $A^\uparrow(y) = \bigwedge_{x \in X} A(x) \to R(x, y) = \bigwedge_{x \in A} R(x, y)$. In the same way, we prove that, if $B \subseteq Y$, then $B^\downarrow(x) = \bigwedge_{y \in B} R(x, y)$.

Hence, even in the case when $A \subseteq X$, $B \subseteq Y$, the pair $(A, B)$ can be a concept (either crisp or fuzzy) only in the case when $R$ is also crisp, i.e., $R : X \times Y \to \{0, 1\}$. This already shows the limitation for the use of concept lattices in the case of a fuzzy context and gives an additional evidence in favor of the graded approach to fuzzy preconcept lattices.

Continuing the previous example, we calculate $A^{\uparrow \downarrow}$ and $B^{\downarrow \uparrow}$ in case of crisp sets $A$ and $B$:

$A^{\uparrow \downarrow}(x) = \bigwedge_{y \in Y} \left( \bigwedge_{x' \in A} (R(x', y) \to R(x, y)) \right)$,

$B^{\downarrow \uparrow}(y) = \bigwedge_{x \in X} \left( \bigwedge_{y' \in B} (R(x, y') \to R(x, y)) \right)$.$\square$

**Proposition 6.** (cf., e.g., Reference [1] for the crisp case, Reference [5].) Given a family $\{A_i \mid i \in I\} \subseteq L^X$, we have $(\bigvee_{i \in I} A_i)^\uparrow = \bigwedge_{i \in I} A_i^\downarrow$. Given a family $\{B_i \mid i \in I\} \subseteq L^Y$, we have $(\bigvee_{i \in I} B_i)^\uparrow = \bigwedge_{i \in I} B_i^\downarrow$.

**Proof.** Take any $y \in Y$. Then, $(\bigvee_{i \in I} A_i)^\uparrow(y) = \bigwedge_{x \in X} (\bigvee_{i \in I} (A_i(x) \to R(x, y)))$.

The second equality can be proved in a similar way. □

5. Concepts and Concept Lattices

Let, as before, $L$ be a quantale with the top and bottom elements 1 and 0, respectively, and let $(X, Y, L, R)$ be a fuzzy context. Referring to the definition of a (fuzzy) concept given in Reference [1,5], we reformulate it as follows:

**Definition 4.** A fuzzy preconcept $(A, B)$ is called a (formal) fuzzy concept if $A^\uparrow = B$ and $B^\downarrow = A$. 

Let $C = \mathcal{C}(X, Y, L, R)$ be the subset of $\mathcal{P} = \mathcal{P}(X, Y, L, R)$ consisting of fuzzy concepts $(A, B)$ and let $\preceq$ be the partial order on $C$ induced by the partial order $\preceq$ on the lattice $(\mathcal{P}, \preceq)$. Then, $(C, \preceq)$ is a partially ordered subset of the lattice $(\mathcal{P}, \preceq)$. However, generally $(C, \preceq)$ is not a sublattice of the lattice $(\mathcal{P}, \preceq, \wedge, \vee)$, and we need to define joins and meets in $(C, \preceq)$ differently from $\wedge$ and $\vee$ in order to view $(C, \preceq)$ as a lattice. To do this, first, we show the following simple lemma:
Lemma 1. Let \((A_1, B_1), (A_2, B_2)\) be fuzzy concepts. If \(A_1 \leq A_2\), then \(B_1 \geq B_2\) and if \(B_1 \geq B_2\) then \(A_1 \leq A_2\).

Proof. If \(A_1 \leq A_2\), then, from Proposition 3, it follows that \(A_1^\uparrow \geq A_2^\uparrow\), hence, \(B_1 \geq B_2\). The proof of the second statement is similar. \(\square\)

Corollary 1. Let \((A_1, B_1), (A_2, B_2) \in \mathbb{C}\). Then, \((A_1, B_1) \preceq (A_2, B_2)\) if and only if \(A_1 \leq A_2\) if and only if \(B_1 \geq B_2\).

Proposition 7. If \(A \in L^X\), then \((A^\uparrow, A^\downarrow)\) is the smallest concept containing \(A\) as the fuzzy set of objects. If \(B \in L^Y\), then \((B^\downarrow, B^\uparrow)\) is the smallest concept containing \(B\) as the fuzzy set of attributes.

Proof. From Proposition 5, it follows that \((A^\uparrow, A^\downarrow)\) is a fuzzy context. Further, from Proposition 4, we know that \(A \leq A^\uparrow\). Assume that there is a fuzzy concept \((A_o, A_o^\uparrow)\) such that \(A \leq A_o \leq A^\uparrow\). Then, \(A^\uparrow \geq A_o^\uparrow \geq A^\uparrow\), hence, \(A_o^\uparrow = A^\uparrow\). Therefore, \(A_o = A^\uparrow\) and \((A_o, A_o^\uparrow) = (A^\uparrow, A^\downarrow)\). In a similar way, we can prove that \((B^\downarrow, B^\uparrow)\) is the smallest context containing \(B\) as the fuzzy set of attributes. \(\square\)

Remark 3. Some topology-related comments

Given a fuzzy set \(A \in L^X\) let \(c_X(A) = A^\uparrow\). Note first that

1. \(A \subseteq c_X(A)\) (by Proposition 4), i.e., operator \(c_X : L^X \to L^X\) is extensional,
2. \(A_1 \leq A_2 \implies c_X(A_1) \leq c_X(A_2)\), i.e., operator \(c_X : L^X \to L^X\) is isotone,
3. \(c_X(c_X(A)) = (c_X(A)^\uparrow)^\uparrow = (A^\uparrow)^\uparrow = A^\uparrow = c_X(A)\), i.e., operator \(c_X : L^X \to L^X\) is idempotent.

Following the accepted terminology, as in, e.g., Reference [22,42], this means that \(c_X : L^X \to L^X\) is a closure operator. We call \(c_X(A)\) by the closure of the fuzzy set \(A\) in the fuzzy context \((X, Y, L, R)\) and say that \(A\) is closed in the fuzzy context \((X, Y, L, R)\), if \(c_X(A) = A\). Let \(S_X\) be the family of all closed fuzzy subsets of \(L^X\) in the fuzzy context \((X, Y, L, R)\).

We show that \(S_X\) is closed under arbitrary joins. Indeed, \(c_X(A_1 \cup A_2) \leq c_X(A_1) = A_1\) for every \(i \in I\); hence, \(c_X(A_i) \leq \bigwedge_{i \in I} A_i\). On the other hand, since, obviously, \(c_X(A_i) \geq \bigwedge_{i \in I} A_i\), we get the equality \(c_X(A_i) = \bigwedge_{i \in I} A_i\).

Following the standard terminology accepted in general topology, we call a family \(S \subseteq L^X\) for some set \(X\) a fuzzy supra co-topology, if it is closed under arbitrary meets (intersections), and contains \(0_X\) and \(1_X\) (Thus, the distinction of a fuzzy supra co-topology from a fuzzy co-topology is that the axiom of finite meets is not requested). Thus, in our case, the family \(S_X\) is already a fuzzy supra co-topology up to the question whether \(0_X\) is closed. Therefore, in order to conclude that the family \(S = \{A^\downarrow \mid A \in L^X\}\) is a supra co-topology, we have to find out whether \(0_X^\downarrow = 0_X\). We calculate \(0_X^\downarrow\) as follows: \(0_X^\downarrow(x) = \bigwedge_{x \in X}(0_X(x) \Rightarrow R(x, y)) = 1_Y\) and further \(0_X^\downarrow(x) = 1_Y = \bigwedge_{y \in Y}(1_Y \Rightarrow R(x, y)) = \bigwedge_{y \in Y} R(x, y)\). So, to get the desired \(0_X^\downarrow = 0_X\), we have to request that, for every \(x \in X\), it holds \(\bigwedge_{y \in Y} R(x, y) = 0\). This, obviously, is not true, in general, but holds in some important cases. In particular, it is fulfilled if, for every object \(x \in X\), there exists some property \(y \in Y\) not satisfied by \(x\); such a situation seems to be quite natural in all “practical” cases.

In a similar way, we can consider the closure operator \(c_Y : L^Y \to L^Y\) in the fuzzy context \((X, Y, L, R)\) defined by \(c_Y(B) = B^\downarrow\) and define the system \(S_Y \subseteq L^Y\) of \(c_Y\) closed fuzzy sets that constitutes an (almost) fuzzy supra co-topology on \(L^Y\) in the fuzzy context \((X, Y, L, R)\). The difference here from the above case is that \(0_Y^\downarrow(y) = \bigwedge_{x \in X} R(x, y)\), and it is equal to \(0_Y\), in particular, in the case when, for every property \(y \in Y\), one can find an object \(x \in X\), which does not have this property.

Theorem 2. Let \((X, Y, L, R)\) be a fuzzy context and let \(\leq\) be the partial order induced from the lattice \(\mathbb{P}(X, Y, L, R, \preceq)\). Then, \(C(X, Y, L, R, \preceq)\) is a complete lattice.
We know already that \( \mathbb{C}(X, Y, L, R, \preceq) \) is a partially ordered set. So, the proof will follow directly from the next proposition.

**Proposition 8** (cf. Reference [1] for the crisp case, Reference [5]). Let \( \{ C = \{ C_i = (A_i, B_i) \} \subseteq \mathbb{C}(X, Y, L, R, \preceq) \) be a family of fuzzy concepts.

Then

1. \( \bigwedge_{i \in I} C_i = \left( \bigwedge_{i \in I} A_i, \left( \bigvee_{i \in I} B_i \right) \right) \) is its infimum in the partially ordered set \((\mathbb{C}, \preceq)\).

2. \( \bigvee_{i \in I} C_i = \left( \left( \bigvee_{i \in I} A_i \right)^\uparrow, \bigwedge_{i \in I} B_i \right) \) is its supremum in the partially ordered set \((\mathbb{C}, \preceq)\).

**Proof.** 1. We have to prove only that \( \bigwedge_{i \in I} C_i \) is a fuzzy concept; its minimality will be clear from its definition since \( \left( \bigwedge_{i \in I} A_i, \bigvee_{i \in I} B_i \right) \) is the meet of \( C \) in \( \mathbb{P}(X, Y, L, R) \). Indeed

\[
\left( \bigvee_{i \in I} B_i \right)^\uparrow = \left( \bigwedge_{i \in I} A_i \right)^\uparrow = \left( \bigwedge_{i \in I} A_i \right)^\uparrow = \left( \bigwedge_{i \in I} A_i \right)^\uparrow \;
\]

2. We have to prove only that \( \bigvee_{i \in I} C_i \) is a fuzzy concept; its maximality will be clear from its definition since \( \left( \bigvee_{i \in I} A_i, \bigwedge_{i \in I} B_i \right) \) is the join of \( C \) in \( \mathbb{P}(X, Y, L, R) \). Indeed

\[
\left( \bigvee_{i \in I} A_i \right)^\uparrow = \left( \bigwedge_{i \in I} B_i \right)^\uparrow = \left( \bigwedge_{i \in I} B_i \right)^\uparrow = \left( \bigwedge_{i \in I} B_i \right)^\uparrow \;
\]

\[\square\]

Taking into account that, in a fuzzy concept \( (A_i, B_i) \), it holds \( A_i^\uparrow = B_i \) and \( B_i^\downarrow = A_i \), we get the following corollary from the previous Proposition (8):

**Corollary 2.** Let \( \{ C_i = (A_i, B_i) \mid i \in I \} \subseteq \mathbb{C} \) be a family of fuzzy concepts. Then

1. \( \bigwedge_{i \in I} C_i = \left( \bigwedge_{i \in I} A_i, \left( \bigvee_{i \in I} B_i \right) \right) \) is its infimum in the lattice \((\mathbb{C}, \preceq)\).

2. \( \bigvee_{i \in I} C_i = \left( \left( \bigvee_{i \in I} A_i \right)^\downarrow, \bigwedge_{i \in I} B_i \right) \) is its supremum in the lattice \((\mathbb{C}, \preceq)\).

6. Conceptuality Degree of a Fuzzy Preconcept and \( \mathbb{D} \)-Graded Preconcept Lattices

6.1. Degrees of Object and Attribute Based Contentments and the Degree of Conceptuality of a Fuzzy Preconcept

Let \((X, Y, L, R)\) be a fuzzy context, \( \mathbb{P}(X, Y, L, R) \) be the corresponding fuzzy preconcept lattice and \((A, B) \in \mathbb{P}(X, Y, L, R)\).

**Definition 5.** The degree of contentment of the fuzzy set \( A \) of objects by the fuzzy set \( B \) of attributes or the degree object-based contentment of the fuzzy preconcept \((A, B)\) for short is defined by \( \mathbb{D}^\uparrow (A, B) =_{\text{def}} A^\uparrow \cong B \).

**Definition 6.** The degree of contentment of a fuzzy set \( B \) of attributes by the fuzzy set \( A \) of objects or the attribute-based contentment of the fuzzy preconcept \((A, B)\) is defined by \( \mathbb{D}^\downarrow (A, B) =_{\text{def}} A \cong B^\downarrow \).

**Definition 7.** The degree of conceptuality of a fuzzy preconcept \((A, B)\) in the fuzzy preconcept lattice \( \mathbb{P} \) is defined by \( \mathbb{D}(A, B) = \mathbb{D}^\uparrow (A, B) \wedge \mathbb{D}^\downarrow (A, B) \).

Changing pairs \((A, B) \in \mathbb{P}(X, Y, L, R)\), we obtain mappings \( \mathbb{D}^\uparrow : \mathbb{P} \rightarrow L \), \( \mathbb{D}^\downarrow : \mathbb{P} \rightarrow L \) and \( \mathbb{D} : \mathbb{P} \rightarrow L \).
Definition 8. The pair \((\mathcal{P}, \mathcal{D})\) is called the graded preconcept lattice of the fuzzy context \((X, Y, L, R)\).

We illustrate the evaluation of conceptuality degree in the fuzzy context \((X, Y, L, R)\) in some simple situations. To simplify calculations, we distinguish the following special conditions. The first one concerns the properties of the product \(\ast\) of the quantale, while the next has to do with the fuzzy relation \(R\) or with the preconcept \((A, B)\) itself.

\((\diamond_\ast)\) Operation \(\ast\) has no zero divisors, i.e.,

\[a \ast b = 0 \Rightarrow a = 0 \text{ or } b = 0\]

for any \(a, b \in L\).

The next conditions are applicable only for calculating gradation degrees of crisp pairs of preconcepts \((A, B) \in \mathcal{P}(X, Y, L, R)\). In this case, we denote \(A^c = X \setminus A\) and \(B^c = Y \setminus B\), i.e., the complements of the sets \(A\) and \(B\), respectively.

\((\diamond_A)\) \(\forall y \in B, \lambda_{x \in A} R(x, y) = 0\). In particular, this relation holds if \(B = Y\).

\((\diamond_B)\) \(\forall y \in A, \lambda_{x \in B} R(x, y) = 0\). In particular, this relation holds if \(A = X\).

\((\diamond_R)\) Thus, both conditions \((\diamond_A)\) and \((\diamond_B)\) are satisfied.

\((\diamond_{AB})\) \(A = X\) and \(B = Y\).

Note that, obviously \((\diamond_{AB}) \Rightarrow (\diamond_R)\).

Example 2. Let \(A \subseteq X, B \subseteq Y\), let \((L, \leq, \wedge, \vee, \ast)\) be an arbitrary quantale, \(\mapsto: L \times L \rightarrow L\) its residuum, and \(R: X \times Y \rightarrow L\) a fuzzy relation. Then

\[A^\dagger \mapsto B = \lambda_{y \in Y} \left(\lambda_{x \in X} (A(x) \mapsto R(x, y)) \mapsto B(y)\right) = \lambda_{y \in B} \left(\lambda_{x \in A} (A(x) \mapsto R(x, y)) \mapsto 0\right) = \lambda_{y \in B} \left(\lambda_{x \in A} (R(x, y) \mapsto 0)\right);
\]

\[B \mapsto A^\dagger = \lambda_{y \in Y} (B(y) \mapsto R(y, x)) = \lambda_{y \in Y} (\lambda_{x \in X} (A(x) \mapsto R(x, y))) = \lambda_{y \in Y} (\lambda_{x \in A} R(x, y)) = \lambda_{y \in B} \lambda_{x \in A} R(x, y);
\]

\[D^\dagger (A, B) = \left(\lambda_{y \in B} \left(\lambda_{x \in X} (R(x, y) \mapsto 0)\right)\right) \land \left(\lambda_{x \in A} (R(x, y))\right).
\]

Thus, in case of crisp object and attribute sets \(A, B\) and under assumption that either \((\diamond_\ast)\) or condition \((\diamond_A)\) holds, then \(B \mapsto A^\dagger = 1\), and, hence, \(D^\dagger (A, B) = \lambda_{x \in A, y \in B} R(x, y)\).

In a similar way, we prove that

\[A \mapsto B^\dagger = \lambda_{x \in A} (\lambda_{y \in B} R(x, y)) = A^c \mapsto A = \lambda_{x \in A^c} \left(\lambda_{y \in B} (R(x, y) \mapsto 0)\right),\]

\[D^\dagger (A, B) = \left(\lambda_{x \in A} (\lambda_{y \in B} R(x, y) \mapsto 0)\right) \land \left(\lambda_{x \in A} (\lambda_{y \in B} R(x, y))\right)
\]

and

\[D^\dagger (A, B) = \lambda_{x \in A} (\lambda_{y \in B} R(x, y))\]

in the case when either \((\diamond_\ast)\) or \((\diamond_B)\) holds.

Example 3. Now, let \(\mathcal{P}(X, Y, L, R)\) be a fuzzy context, where \(L = [0, 1]\), \(a \in (0, 1)\), \(X_a \subseteq X\), \(B \subseteq Y\), and let a fuzzy set \(A: X \rightarrow [0, 1]\) be defined by

\[A(x) = \begin{cases} a & \text{if } x \in X_a, \\ 0 & \text{if } x \notin X_a. \end{cases}
\]

Then,

\[B \mapsto A^\dagger = \lambda_{y \in Y} (B(y) \mapsto A^\dagger (y)) = \lambda_{y \in B} A^\dagger (y) = \lambda_{y \in B, x \in X_a} (a \mapsto R(x, y));\]

\[A^\dagger \mapsto B = \lambda_{y \in B} \left(\lambda_{x \in X_a} (a \mapsto R(x, y)) \mapsto 0; \text{ hence,}\right)
\]

\[D^\dagger (A, B) = \left(\lambda_{y \in B, x \in X_a} (a \mapsto R(x, y))\right) \land \left(\lambda_{y \in X} (a \mapsto R(x, y))\right) \mapsto 0\]

and

\[D^\dagger (A, B) = \lambda_{y \in B, x \in X_a} (a \mapsto R(x, y))\]

if condition \((\diamond_\ast)\) or condition \((\diamond_A)\) is satisfied.

In a similar way, in order to calculate \(D^\dagger (A, B)\), we have \(A \mapsto B^\dagger = \lambda_{x \in X} (A(x) \mapsto B^\dagger (x)) = \lambda_{x \in X_a} (a \mapsto \lambda_{y \in B} R(x, y));\)

\[B^\dagger \mapsto A = \left(\lambda_{x \in X_a} (\lambda_{y \in B} (R(x, y) \mapsto a))\right) \land \left(\lambda_{x \in X} (\lambda_{y \in B} (R(x, y) \mapsto 0))\right);\]
Axioms 2021, 10, 41

\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X} \left( \bigwedge_{y \in Y} (R(x, y) \Rightarrow 0) \right) \right) \wedge \\
\left( \bigwedge_{x \in X} \left( \bigwedge_{y \in Y} (R(x, y) \Rightarrow a) \wedge (a \Rightarrow \bigwedge_{y \in Y} R(x, y)) \right) \right).
\]

Under assumption of \((\ast_1)\) or \((\ast_2)\), the formula can be simplified and we get
\[
\mathcal{D}(A, B) = \bigwedge_{x \in X} \left( \bigwedge_{y \in Y} (R(x, y) \Rightarrow a) \wedge (a \Rightarrow \bigwedge_{y \in Y} R(x, y)) \right).
\]

**Example 4.** Now, let \(A \subseteq X, Y_b \subseteq Y, L = [0, 1], b \in (0, 1)\) and \(B : Y \to L = [0, 1]\) be defined by
\[
B(y) = \begin{cases} 
    b & \text{if } y \in Y_b \\
    0 & \text{if } y \notin Y_b
\end{cases}.
\]

Then, under assumption of \((\ast_1)\) or \((\ast_2)\), calculating similar as in the previous example, we get:
\[
\mathcal{D}(A, B) = \left( \bigwedge_{y \in Y_b} (b \Rightarrow \bigwedge_{x \in A} R(x, y)) \right) \wedge \left( \bigwedge_{y \in Y_b} (\bigwedge_{x \in A} (R(x, y) \Rightarrow b)) \right).
\]

**Example 5.** We demonstrate the previously obtained formulas for calculating \(\mathcal{D}(A, B)\) in case of the three basic \(t\)-norms \(*\) on \([0, 1]\): \(*_\wedge = \wedge\) - the minimum \(t\)-norm, \(*_L\) - the Łukasiewicz \(t\)-norm, and \(*_P\) - the product \(t\)-norm; see, e.g., Reference [27].

(1) Łukasiewicz \(t\)-norm has zero divisors. Therefore, to simplify situation, we will consider the case when \(X_a = X, B = Y, i.e., \) in the case when assumption \((\ast_{AB})\) is satisfied. Then, from the above formulas, we have
\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X, y \in Y} (1 - a + R(x, y)) \right) \wedge 1;
\]
\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X} (1 - \bigwedge_{y \in Y} R(x, y) + a) \right) \wedge 1;
\]
\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X} (\bigwedge_{y \in Y} (R(x, y) \Rightarrow 1) \wedge (a \Rightarrow \bigwedge_{y \in Y} R(x, y))) \right).
\]

(2) The product \(t\)-norm has no zero-divisors, i.e., it satisfies assumption \((\ast_1)\). Hence, under this assumption, we can apply formulas obtained in Example 3 and have
\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X} (a \Rightarrow \bigwedge_{y \in Y} R(x, y)) \right) \wedge \left( \bigwedge_{x \in X} \left( \bigwedge_{y \in Y} (R(x, y) \Rightarrow a) \right) \right).
\]

To describe \(\mathcal{D}(A, B)\) for the product \(t\)-norm in this situation, we denote
\[
X_1 = \{ x \in X \mid a < \bigwedge_{y \in Y} R(x, y) \}, X_2 = \{ x \in X \mid a \geq \bigwedge_{y \in Y} R(x, y) \}.
\]

Then
\[
\mathcal{D}(A, B) = \left( \bigwedge_{x \in X_1} a \right) \wedge \left( \bigwedge_{x \in X_2} \frac{\bigwedge_{y \in Y} R(x, y)}{a} \right).
\]

(3) The minimum \(t\)-norm has no zero-divisors, i.e., it satisfies assumption \((\ast_1)\). Therefore, using formulas obtained in Example 3 and notations from the previous paragraph, we have
\[
\mathcal{D}(A, B) = \begin{cases} 
    \bigwedge_{x \in X_2, y \in Y} R(x, y) & \text{if } X_2 \neq \emptyset \\
    a & \text{otherwise}
\end{cases}.
\]

**Example 6.** Here, we will sketch calculation of \(\mathcal{D}(A, B)\) and \(\mathcal{D}(A, B)\) in the case when \(A\) and \(B\) are 3-valued fuzzy sets. For simplicity of exposition, we assume that condition \((\ast_1)\) is satisfied. Besides, we restrict to calculation only the basic expressions, i.e., \(A^\dagger \hookrightarrow B, B \hookrightarrow A^\dagger, A \hookrightarrow B^\dagger,\) and \(B^\dagger \hookrightarrow A\).

Let \(X, Y\) be sets and \(R : X \times Y \to [0, 1]\) be a fuzzy relation. Let \(0 < a < 1\) and let \(X = X_0 \cup X_a \cup X_1\), where the sets \(X_0, X_a, X_1\) are disjoint. Further, let \(0 < b < 1\) and let \(Y = Y_0 \cup Y_b \cup Y_1\), where the sets \(Y_0, Y_b, Y_1\) are disjoint. We define fuzzy sets \(A : X \to [0, 1]\) and
Thus, the degree of attribute-based contentment

\[ D \] of the separate fuzzy preconcepts.

6.2. D-Graded Preconcept Lattices

Recall that, given a fuzzy context \((X, Y, L, R)\), the fuzzy preconcept lattice \(\mathcal{P}(X, Y, L, R, \preceq)\) endowed with operators of \(D\)-gradation, that the tuple \((\mathcal{P}(X, Y, L, R), \preceq, D^\downarrow, D^\uparrow)\) is called a \(D\)-graded fuzzy preconcept lattice. In the next two theorems, we prove the basic properties of \(D\)-graded fuzzy preconcept lattices.

**Theorem 3.** Let \(\mathcal{P} = (\mathcal{P}, \preceq, \wedge, \vee)\) be a fuzzy preconcept lattice. Given a family of fuzzy preconcepts \(\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathcal{P}\), it holds that:

\[
D^\downarrow(\bigvee_{i \in I} P_i) \geq \bigwedge_{i \in I} D^\uparrow(P_i).
\]

Thus, the degree of object-based contentment \(D^\downarrow\) of the union of fuzzy preconcepts is not less than the infimum (in \(L\)) of the degrees of object-based contentments \(D^\uparrow\) of the separate fuzzy preconcepts.

**Proof.** The proof follows from the following series of (in)equalities that are justified by applying Proposition 6 and Proposition 2:

\[
D^\downarrow(\bigvee_{i \in I} (A_i, B_i)) = \left(\bigvee_{i \in I} A_i\right)^\uparrow \cong \left(\bigwedge_{i \in I} B_i\right) = \left(\bigwedge_{i \in I} A_i^\uparrow\right) \cong \left(\bigwedge_{i \in I} B_i\right) = \\
\left(\bigwedge_{i \in I} A_i^\uparrow\right) \wedge \left(\bigwedge_{i \in I} A_i^\downarrow\right) \preceq \left(\bigwedge_{i \in I} B_i\right) \geq \\
\left(\bigwedge_{i \in I} A_i^\uparrow\right) \leftarrow B_i \wedge \left(\bigwedge_{i \in I} A_i^\downarrow\right) \leftarrow B_i = \bigwedge_{i \in I} D^\downarrow(A_i, B_i).
\]

\[ \square \]

**Theorem 4.** Given a family of fuzzy preconcepts \(\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathcal{P}\) it holds

\[
D^\downarrow(\bigwedge_{i \in I} P_i) \geq \bigvee_{i \in I} D^\uparrow(P_i).
\]

Thus, the degree of attribute-based contentment \(D^\downarrow\) of the meet of fuzzy preconcepts in the lattice \((\mathcal{P}, \preceq, \wedge, \vee)\) is not less than the infimum of degrees of attribute-based contentment \(D^\uparrow\) of the separate fuzzy preconcepts.
The proof follows from the next series of (in)equalities that are justified by applying Proposition 6 and Proposition 2:

\[ D^\downarrow(\bar{\bigwedge}_{i \in I}(A_i, B_i)) = \left( \bigwedge_{i \in I} A_i \right) \cong \left( \bigvee_{i \in I} B_i \right) = \left( \bigwedge_{i \in I} A_i \right) \cong \left( \bigvee_{i \in I} B_i \right) = \left( \left( \bigwedge_{i \in I} A_i \right) \hookrightarrow \left( \bigvee_{i \in I} B_i \right) \right) \land \left( \left( \bigwedge_{i \in I} A_i \right) \hookleftarrow \left( \bigvee_{i \in I} B_i \right) \right) \geq \left( \bigwedge_{i \in I} A_i \hookrightarrow \bar{B}_i \right) \land \left( \bigwedge_{i \in I} A_i \hookleftarrow \bar{B}_i \right) = \bigwedge_{i \in I} D^\downarrow(A_i, B_i). \]

\[ \square \]

Using terminology accepted in Topology, the previous two theorems can be reformulated in the following united way:

**Theorem 5.** The mapping \( D^\downarrow : \mathbb{P} \to L \) is an \( L \)-valued point-free fuzzy Alexandrov supra topology on the lattice \((\mathbb{P}, \preceq, \bar{\vee}, \bar{\wedge})\), the mapping \( D^\uparrow : \mathbb{P} \to L \) is an \( L \)-valued point-free fuzzy Alexandrov supra co-topology on the lattice \((\mathbb{P}, \preceq, \vee, \wedge)\) and the pair of mappings \((D^\downarrow, D^\uparrow)\) is an \( L \)-valued point-free fuzzy Alexandrov supra-di-topology on the lattice \((\mathbb{P}, \preceq, \vee, \wedge)\).

### 7. Measure of Conceptuality of a Fuzzy Preconcept and \( \mathcal{A} \)-Graded Preconcept Lattices

In the previous section, we estimated the “deviation” of a fuzzy preconcept \((A, B)\) from its being a “real” concept by analyzing the “mutual” contentment of the given fuzzy sets \(A\) and \(B\) in the fuzzy context \((X, Y, L, R)\). We did not take care of the location of the pair \((A, B)\) in respect to the fuzzy conceptual lattice \(C(X, Y, \preceq, \bar{\vee}, \bar{\wedge})\) that, in a certain sense, “surrounds” this pair. Therefore, we referred to that approach as an inner one.

On the other hand, in this section, we consider the “closest” fuzzy concepts to a given fuzzy preconcept \((A, B)\) and estimate their distinction. In this sense, the approach proposed here looks like an outer one. In order to realize this idea, we introduce the concepts of a fuzzy conceptual kernel and a fuzzy conceptual hull of a fuzzy preconcept \((A, B)\).

#### 7.1. Conceptional Hull and Conceptional Kernel of a Fuzzy Preconcept

Let \((X, Y, L, R)\) be a fixed fuzzy context, \((\mathbb{P}(X, Y, L, R), \preceq)\) be the corresponding fuzzy preconcept lattice and let \((C(X, Y, L, R), \preceq)\) be its partial ordered subset of fuzzy concepts. Further, let \((A, B) \in \mathbb{P}\) be a preconcept, i.e., just a pair of fuzzy sets \(A \in L^X, B \in L^Y\). A natural question arises: how far is this preconcept \((A, B)\) from a “real” concept? To state this question more precisely, we are interested to find the largest (in the sense of preorder \(\preceq\) on \(\mathbb{P}\)) fuzzy concept which is smaller or equal than \((A, B)\) and to find the smallest fuzzy concept that is larger or equal than \((A, B)\).

**Definition 9.** A fuzzy concept \(K(A, B) =_{def} (A^0, B^0) \in C\) is called the conceptual kernel of a fuzzy preconcept \((A, B)\) if
1. \((A^0, B^0) \preceq (A, B)\) and
2. for every \((C, D) \in C\) such that \((C, D) \preceq (A, B)\) it holds \((A^0, B^0) \succeq (C, D)\).

**Definition 10.** A fuzzy concept \(H(A, B) =_{def} (A^\uparrow, B^\uparrow) \in C\) is called the conceptual hull of a fuzzy preconcept \((A, B)\) if
1. \((A, B) \succeq (A^\uparrow, B^\uparrow)\) and
2. for every \((C, D) \in C\) such that \((C, D) \preceq (A, B)\) it holds \((A^\uparrow, B^\uparrow) \preceq (C, D)\).

The answer to the question of the existence of conceptual hulls and kernels for fuzzy preconcepts is given in the next theorem.

**Theorem 6.** Let a fuzzy preconcept \((A, B)\) be given. If there exists a fuzzy concept \((C, D) \preceq (A, B)\), then there exists also the kernel \(K(A, B)\). If there exists a fuzzy concept \((C, D) \succeq (A, B)\), then there exists also the hull \(H(A, B)\).
Proof. To prove the first statement, let $C = \{(C_i, D_i) \mid i \in I\} \subseteq \mathcal{C}$ be the family of all fuzzy subsets such that $(C_i, D_i) \preceq (A, B)$ and assume that this family is not empty. Now, take $\gamma_{i \in I}(C_i, D_i)$. According to Theorem 2, $\gamma_{i \in I}(C_i, D_i) \in \mathcal{C}$, and besides from the construction, it is clear that $\gamma_{i \in I}(C_i, D_i) \preceq (A, B)$. Hence, $\gamma_{i \in I}(C_i, D_i) = K(A, B)$.

To prove the second statement, let $C = \{(C_i, D_i) \mid i \in I\} \subseteq \mathcal{C}$ be the family of all fuzzy subsets such that $(C_i, D_i) \succeq (A, B)$ and assume that this family is not empty. Now, take $\lambda_{i \in I}(C_i, D_i)$. According to Theorem 2, $\lambda_{i \in I}(C_i, D_i) \in \mathcal{C}$, and besides, obviously, $\lambda_{i \in I}(C_i, D_i) \preceq (A, B)$. From the construction, it is clear that $\lambda_{i \in I}(C_i, D_i) = H(A, B)$.

As different from the problem of existence, the problem of finding the conceptional kernel and hull for a fuzzy preconcept seems to be quite difficult. However, we have some special cases when the kernel and the hull for a fuzzy preconcept $(A, B)$ can be easily found. Namely, let a fuzzy preconcept $(A, B)$ be given. Reasoning about the fuzzy conceptual hull of a fuzzy preconcept $(A, B)$, we have to minimally enlarge (in the sense of the order $\preceq$) the pair $(A, B)$ in order to get a fuzzy concept. This leads to the idea to take $A \lor B^\perp$ as the set of objects, thus minimally expanding $A$ $(\preceq)$ in order to satisfy all attributes from $B$ and to take $A^\perp \land B$ as the set of attributes minimally reducing $B$ $(\preceq)$ in order to keep in accordance with all objects from $A$. Now, if we are lucky and $(A \lor B^\perp, A^\perp \land B)$ is a fuzzy concept, then it is obviously just the hull $H(A, B)$ of the fuzzy preconcept $(A, B)$.

Reasoning in a dual way, the pair $(A \land B^\perp, A^\perp \lor B)$ can pretend to be the fuzzy conceptual kernel of a fuzzy preconcept $(A, B)$.

We analyze this idea in two cases. First, take $(0_X, 1_Y)$, i.e., the minimal element in $(\mathcal{P}, \preceq)$. Then, $1^\perp_Y(x) = \bigwedge_{y \in Y} R(x, y)$, $0^\perp_Y(y) = 1_Y(y)$; hence, in this situation,

\[
\left((A \lor B^\perp), (A^\perp, \land B)\right) = \left(\bigwedge_{y \in Y} R(\cdot, y), 1_Y\right).
\]

Directly checking, we get that

\[
\left(\bigwedge_{y \in Y} R(\cdot, y)\right)^\perp = 1_Y(\cdot) \text{ and } 1^\perp_Y(\cdot) = \left(\bigwedge_{y \in Y} R(\cdot, y)\right)
\]

hence, $H(1_X, 0_Y) = \left(\bigwedge_{y \in Y} R(\cdot, y), 1_Y\right)$ is the fuzzy conceptional hull of the minimal fuzzy preconcept $(0_X, 1_Y) \in \mathcal{P}$. Obviously, the conceptional kernel of the minimal preconcept $(0_X, 1_Y)$ does not exist unless $(0_X, 1_Y)$ is a fuzzy concept itself.

As the second case, we take $(A, B) = (1_X, 0_Y)$, i.e., the maximal element in the preconcept lattice $(\mathcal{P}, \preceq)$, and we are looking for its fuzzy conceptional kernel $K(1_X, 0_Y)$. Now, we get $A = 1_X$, $B^\perp(x) = 1_X(x)$, $B = 0_Y$, $A^\perp(y) = \bigwedge_{x \in X} R(x, y)$; hence, in this situation,

\[
\left((A \land B^\perp), (A^\perp \lor B)\right) = \left(1_X, \bigwedge_{x \in X} R(x, \cdot)\right).
\]

Directly checking, we get that

\[
\left(\bigwedge_{x \in X} R(x, \cdot)\right)^\perp = 1_Y(\cdot) \text{ and } 1_Y(\cdot) = \left(\bigwedge_{x \in X} R(x, \cdot)\right);
\]

hence,

\[
K(1_X, 0_Y) = \left(1_X, \bigwedge_{x \in X} R(x, \cdot)\right)
\]

is the conceptional kernel of the maximal fuzzy preconcept $(1_X, 0_Y) \in \mathcal{P}$. Obviously, the conceptional hull of the maximal preconcept $(1_X, 0_Y)$ does not exist unless $(1_X, 0_Y)$ is a fuzzy concept itself.

Now, we can make further clarification of Theorem 2:
Theorem 7. Let \((X, Y, L, R)\) be a fuzzy context and let \(\preceq\) be the partial order on \(C\) induced from the lattice \(P(X, Y, L, R, \preceq)\). Then, \((C(X, Y, L, R, \lambda, \gamma, \preceq)\) is a complete lattice. Its top and bottom elements are, respectively, \(\top_C = (1_X(\cdot), \Lambda_{x \in X} R(x, \cdot))\) and \(\bot_C = (\Lambda_{y \in Y} R(\cdot, y), 1_Y(\cdot))\).

For the future needs, we denote \(P^C = \{(A, B) \in P \mid \bot_C \leq (A, B) \leq \top_C\}\). The meaning of the lattice \((P^C, \preceq)\) is that it is just the family of all fuzzy preconcepts that have both conceptional kernels and conceptional hulls. We call \(P^C\) the conceptional interior of the preconcept lattice \(P = \text{def} (P(X, Y, L, R), \preceq)\).

7.2. Measure of Conceptuality of a Fuzzy Preconcept and \(M\)-Graded Preconcept Lattices

In this subsection, we introduce measures of lower and upper conceptual approximations of a fuzzy preconcept \((A, B)\), which are defined as a certain measure of distinctions between \((A, B)\) and its fuzzy conceptual kernel \(K(A, B)\) and hull \(H(A, B)\), respectively.

Let \((X, Y, L, R)\) be a fuzzy context, \(P(X, Y, L, R)\) the corresponding fuzzy preconcept lattice and \(P(X, Y, L, R)\) be its conceptional interior. We start with the following definition.

Definition 11. Let \((C, D), (E, F) \in (P(X, Y, L, R), \preceq)\) and \((C, D) \preceq (E, F)\). We define the measure of inclusion of \((E, F)\) in \((C, D)\) by

\[(E, F) \subseteq (C, D) = (E \hookrightarrow C) \land (F \hookleftarrow D)\]

and the measure of covering \((E, F)\) by \((C, D)\) as

\[(C, D) \supseteq (E, F) = (C \hookleftarrow E) \land (D \hookrightarrow F)\].

Definition 12. Given a preconcept \((A, B)\) in a fuzzy preconcept lattice \((P^C, \preceq)\), we define its lower measure of conceptuality by \(O(A, B) = (A, B) \subseteq K(A, B)\) and its upper measure of conceptuality by \(U(A, B) = (A, B) \supseteq H(A, B)\). Finally, the measure of conceptuality of \((A, B)\) is defined by \(M(A, B) = O(A, B) \land U(A, B)\).

Thus, the lower measure of conceptuality of a fuzzy preconcept \((A, B)\) is defined as the measure of its inclusion in its kernel \(K(A, B)\) and the upper measure of conceptuality is defined as the measure describing how its conceptional hull \(H(A, B)\) is covered by \((A, B)\).

Let a fuzzy context \((X, Y, L, R)\) be given, and let \(P^C = \text{def} (P^C(X, Y, L, R), \preceq)\) be the conceptional interior of the corresponding fuzzy preconcept lattice \(P = (P^C(X, Y, L, R), \preceq)\).

Definition 13. An \(M\)-graded fuzzy preconcept lattice of the fuzzy context \((X, Y, L, R)\) is the triple \((P^C, O, U)\).

The detailed study of \(M\)-graded fuzzy preconcept lattices, in particular, their topological and categorical properties and their relations with \(D\)-graded fuzzy preconcept lattices will be the subject of another paper (at present in preparation).

8. Appendix

The graded fuzzy preconcept lattices may be used in practical cases when multiple properties are assigned to multiple objects with each object having several properties and vice versa. In real life, such construction is rather limited since each object usually has only one exact property or value measured by appropriate tools or based on, e.g., observational data. Therefore, we propose to analyze the forecasting models using different assumptions or expert opinions and allowing to assign the values to selected objects and related properties. A rather simple and obvious example can be introduced via estimating consumption levels and prices for particular goods or services.

We consider the model from transportation industry by estimating the number of passengers \(A\) on, e.g., particular railway route and level of fares \(B\) for the forecasting of
annual income from the service. Below, Table 1 represents five estimates of number of passengers and prices for a single ticket.

Table 1. Estimated number of passengers $A$ and prices $B$ of a single ticket.

| A     | B  |
|-------|----|
| 40,000| 30 |
| 60,000| 35 |
| 80,000| 40 |
| 90,000| 45 |
| 100,000| 50 |

We construct the function of $R(x, y)$ described in the Example 2 by choosing Łukasiewicz $t$-norm expressed as follows:

$$R(x, y) = \max \left( \frac{x}{x_{\text{max}}} + \frac{y}{y_{\text{max}}} - 1; 0 \right),$$

where $x \in A$, $y \in B$, $x_{\text{max}}$ is the largest element of set $A$, and $y_{\text{max}}$ is the largest element of set $B$. Therefore, we obtain the following set of values: $\{0, 0.2, 0.4, 0.5, 0.6, 0.1, 0.3, 0.5, 0.6, 0.7, 0.2, 0.4, 0.6, 0.7, 0.8, 0.3, 0.5, 0.7, 0.8, 0.9, 0.4, 0.6, 0.8, 0.9, 1\}$.

The degree of conceptuality calculated in the Example 2 is

$$D(A, B) = \bigwedge_{x \in A, y \in B} R(x, y).$$

Therefore, we select the smallest of the above values of function $R(x, y)$ and obtain that the conceptuality degree equals to 0.

This example can be further extended to scenario when certain percentage of passengers is exempted from paying for the tickets. In this case, we use Example 3 and apply the value $a = 0.8$ which means that 80 percent of all passengers are paying for their tickets in full. We calculate the degree of conceptuality in case of Łukasiewicz $t$-norm:

$$D(A, B) = D^a(A, B) = \bigwedge_{x \in A, y \in B} (1 - |a - R(x, y)|).$$

We obtain the following set of values $\{0.2, 0.4, 0.6, 0.7, 0.8, 0.3, 0.5, 0.7, 0.8, 0.9, 0.4, 0.6, 0.8, 0.9, 1, 0.5, 0.7, 0.9, 1, 0.9, 0.6, 0.8, 1, 0.9, 0.8\}$ and the minimum value 0.2 which corresponds to the degree of conceptuality.

9. Conclusions

Noticing the limitation of the concept lattices in case of a fuzzy context in view of the possible applications, especially for “real world” problems, we introduce here a very general notion of a preconcept, and on the other hand restrict it by assigning to a preconcept a certain “degree of its conceptuality”. In the result, we come to what we call a graded fuzzy preconcept lattice. We develop two approaches of gradation. The first one is based on the evaluation of a certain mutual contentment of the fuzzy set of potential objects and the fuzzy set of potential attributes; we call it an inner approach, and the graded fuzzy preconcept lattice obtained in this way $D$-graded. The second approach is based on the evaluation of the “proximity” of a given fuzzy preconcept to the “closest” fuzzy concepts, namely the conceptual hull and conceptual kernel of the given fuzzy preconcept. We view this approach as an outer one and call the fuzzy preconcept lattice graded in this way $M$-graded. We study basic properties of $D$-graded and $M$-graded fuzzy preconcept lattices and illustrate $D$-gradation of fuzzy preconcepts with a series of theoretical examples and with one example of practical nature related to modeling of income in public transportation services.

Concerning the future plans for our work, we consider theoretical, as well as practical, issues. As one of the main tasks we view further investigation of the $M$-graded fuzzy preconcept lattice, the properties of which are less disclosed in this paper than the properties
of $D$-graded fuzzy preconcept lattices. In particular, topological and algebraic properties of $M$-graded fuzzy preconcept lattices should be studied, in particular, on dependence of the underlying lattice $L$. More examples of evaluation in $M$-graded fuzzy preconcept should be found. Quite important, in our opinion, is the research of categorical properties of both versions of the graded fuzzy preconcept lattices, in particular, to describe the products and coproducts, subobjects, initial and final objects, etc., in these categories.

Without doubt, we will continue the work on finding other examples of applications of graded fuzzy preconcept lattices in practically important problems. Specifically, meanwhile, we are working on scenario for assessing the risk of pandemic impact and interrelations between such factors as numbers of infected versus hospitalized people and availability of the medical staff. These factors can be analyzed using fuzzy preconcept lattices in order to estimate the risk level of the pandemic spread. This application will be the subject of the consequent paper.

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