Holographic dark energy in Rastall theory

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Bearing holographic dark energy hypothesis in mind, the ability of vacuum energy in describing the current accelerated universe is studied in the framework of Rastall theory. Here, in addition to the ordinary approach in which it is expected that this energy plays the role of dark energy, we also address a new approach where the sum of this energy and Rastall term is responsible for the current accelerated universe. We also investigate the cosmological outcomes of using Tsallis entropy in quantifying the energy of fields in vacuum for both mentioned approaches. The implications of considering an interaction between the various segments of cosmic fluid have been addressed in each studied cases.

I. INTRODUCTION

Motivated by the long-range nature of gravity, the use of generalized entropies to study Cosmos has recently been proposed [1, 2]. In this regard, it has been shown that the Tsallis entropy can make a bridge between Verlinde and Padmanabhan hypothesis, and additionally, be combined with thermodynamic laws to modify Friedmann equations in a way that the outcomes can explain the accelerated universe without considering a mysterious dark energy source [3]. In this regard, relying on the holographic dark energy hypothesis [4–10], and using Tsallis entropy, a new holographic dark energy, namely Tsallis holographic dark energy model (THDE), is also introduced in Ref. [11].

In the Einstein framework, where the horizon entropy meets the Bekenstein entropy, while there is not any mutual interaction between the Cosmos sectors, both HDE models, obtained by taking into account the radius of apparent horizon as the IR cutoff, and pressureless source are scaled with the same function of Hubble parameter [2]. This difficulty may be solved by using generalized entropy based on HDE models such as THDE [11]. There are observational and theoretical works admitting the breakdown of the conservation law (the backbone of general relativity) in curved spacetime [4, 12, 14], a point which has firstly been noted by P. Rastall [15]. Although, Rastall gravity produces acceptable and suitable predictions and explanations for various gravitational and cosmological phenomena [14, 16, 18], it is shown that a dark energy-like source is still needed to describe the current accelerated universe in this framework [20].

In fact, using the holographic hypothesis, one can get an estimation for the vacuum energy helping us in studying its cosmological consequences [3]. In the next section, we are going to study the cosmological implications of vacuum energy in the Rastall framework by employing the holographic hypothesis, and using horizon entropy obtained by applying the thermodynamic laws to horizon in Rastall theory [16, 17]. In Sec. (III), applying Tsallis entropy to horizon, we shall investigate the ability of modelling dark energy by using THDE in the Rastall framework. In each cases, the effects of considering a mutual interaction between dark energy candidate and other segments of cosmos are also studied. Moreover, we also propose a new candidate for dark energy i.e., the sum of vacuum energy and Rastall term. The ability of considering this hypothesis in describing the current Cosmos has also been studied in Secs. (II) and (III). The last section includes the summary of work, and the units have been set so that \( c = \hbar = k_B = 1 \).

II. HDE IN RASTALL THEORY

Since the WMAP data indicates a flat universe [21], here, we only focus on flat FRW universe with line element

\[
\begin{aligned}
ds^2 &= -dt^2 + a(t)^2 [dr^2 + r^2(d\theta^2 + \sin(\theta)^2d\phi^2)],
\end{aligned}
\]

in which \( a(t) \) is the scale factor. The apparent horizon of this spacetime, as a proper casual boundary, is also located at \( \frac{r_A}{r} = \frac{1}{H} \), (2)

and hence, \( A = \frac{4\pi r_A^2}{H^2} \) is the horizon area. The Rastall field equations are written as [15]

\[
G_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = \kappa T_{\mu\nu},
\]

leading to

\[
(12\kappa \lambda - 3)H^2 + 6\kappa \lambda \dot{H} = -\kappa \rho,
\]

and

\[
(12\kappa \lambda - 3)H^2 + (6\kappa \lambda - 2)\dot{H} = \kappa p,
\]
as the corresponding Friedmann equations. In the above equations, \( \lambda \) and \( \kappa \) denote the Rastall parameter and the Rastall gravitational coupling, respectively. Considering the Newtonian limit, one can obtain \([13, 16]\)

\[
\kappa = \frac{4\eta - 1}{6\eta - 1} \kappa_G, \quad \lambda = \frac{\eta(6\eta - 1)}{(4\eta - 1)\kappa_G},
\]

in which \( \kappa_G = 8\pi G \) is the Einstein gravitational coupling and \( \eta = \kappa \lambda \). Continuity equation governing the cosmic fluid with energy density \( \rho \) and pressure \( p \) is \([4, 17]\)

\[
\frac{3\eta - 1}{4\eta - 1}\dot{\rho} + \left(\frac{3\eta}{4\eta - 1}\right)\dot{p} + 3H(\rho + p) = 0.
\]

If the Cosmos is filled by a pressureless source with energy density \( \rho_m \) and another fluid with energy density \( \rho_\Lambda \) and pressure \( p_\Lambda \), then the Friedmann equations take the form

\[
(12\eta - 3)H^2 + 6n \dot{H} = \frac{4\eta - 1}{1 - 6\eta} \kappa_G(\rho_\Lambda + \rho_m),
\]

\[
(12\eta - 3)H^2 + (6\eta - 2)\dot{H} = \frac{4\eta - 1}{6\eta - 1} \kappa_G p_\Lambda,
\]

leading to

\[
\dot{H} = \frac{4\eta - 1}{2(1 - 6\eta)} \kappa_G(\rho_\Lambda + \rho_m + p_\Lambda).
\]

Whenever there is not any mutual interaction between cosmic fluid. Eq. \( \mathbf{8} \) is decomposed to

\[
\left(\frac{3\eta - 1}{4\eta - 1}\right)\dot{\rho}_m + 3H\rho_m = 0 \Rightarrow \rho_m = \rho_0 a^{-\frac{3(1-4\eta)}{4\eta - 1}},
\]

where \( \rho_0 \) is the integration constant, and

\[
\left(\frac{3\eta - 1}{4\eta - 1}\right)\dot{\rho}_\Lambda + \left(\frac{3\eta}{4\eta - 1}\right)\dot{p}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = 0.
\]

Since, in the Rastall framework, we have \( S = (\frac{6\eta - 1}{4\eta - 1})\frac{A}{G} \) for the horizon entropy \([16, 17]\), by following the original HDE hypothesis \([3]\), one easily reaches

\[
\rho_\Lambda = \frac{3B}{8\pi G} \left(\frac{6\eta - 1}{4\eta - 1}\right)H^2,
\]

where \( B \) is a numerical constant as usual. Inserting this result in the first line of Eq. \( \mathbf{8} \), we can check that \( \rho_m \approx H^2 \) whenever \( \eta = 0 \) or even whenever \( \eta \neq 0 \) and \( w_\Lambda = \frac{\rho_\Lambda}{\rho} = \text{constant} \). Therefore, the same as the Einstein framework (\( \eta = 0 \) \([3]\)), dark energy and \( \rho_m \) will be scaled by the same function of \( H \) in the Rastall framework if \( \eta \neq 0 \) and \( w_\Lambda = \text{constant} \). It means that we should have \( w_\Lambda \neq \text{constant} \) to avoid this difficulty in the Rastall framework.

**A. Common approach**

From Eq. \( \mathbf{5} \), by defining critical density \( \rho_c \) as \( \rho_c = \frac{3H^2}{8\pi G} \), we get

\[
1 = \frac{4\eta - 1}{6\eta - 1} (\Omega_m + \Omega_\Lambda) + \Omega_\eta + 4\eta,
\]

in which

\[
\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_\eta = \frac{6\eta \dot{H}}{8\pi G},
\]

\[
\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = B \frac{6\eta - 1}{4\eta - 1}.
\]

Now, using Eqs. \( \mathbf{8}, \mathbf{14} \) and \( \mathbf{12} \), one can obtain the EoS and deceleration parameters as

\[
\omega_\Lambda = \frac{1}{B} \left( \frac{\Omega_\eta(3\eta - 1)}{3\eta} + 4\eta - 1 \right),
\]

and

\[
q = -1 - \frac{\dot{H}}{H^2} = -1 - \frac{\Omega_\eta}{2\eta},
\]

respectively, which are, in fact, the same for both non-interacting and interacting cases. At the classical level, the stability of an energy source with energy density \( \rho \) and pressure \( p \) is also determined by the sign of the sound speed square \( (\upsilon_s^2) \) evaluated as

\[
\upsilon_s^2 = \frac{dP_\Lambda}{d\rho_\Lambda} = \frac{\dot{\rho}_\Lambda}{\dot{\rho}_\Lambda} = w_\Lambda + \frac{\dot{\omega}_\Lambda}{\dot{\rho}_\Lambda}.
\]

In the following, we study the HDE model in the Rastall theory for both non-interacting and interacting cases .

**Non-interacting Case**

Taking the time derivative of the first Friedmann Eq. \( \mathbf{8} \), and combining the result with Eqs. \( \mathbf{10}, \mathbf{11}, \mathbf{12} \) and \( \mathbf{14} \), one finds

\[
\frac{\dot{H}}{H^3} = \frac{4\eta - 1}{2\eta(1 - 6\eta)} \left[ \frac{3\Omega_m(6\eta - 1)}{3\eta - 1} \right] \left[ (\Omega_\Lambda + 6\eta - 1) \left( \frac{\Omega_m}{\eta} + \frac{3(4\eta - 1)}{3\eta - 1} \right) \right],
\]

which can be used to write

\[
\Omega'_\eta = \frac{\dot{\Omega}_\eta}{H} = \frac{4\eta - 1}{1 - 6\eta} \left[ \frac{3\Omega_m(6\eta - 1)}{3\eta - 1} \right] \left[ (\Omega_\Lambda + 6\eta - 1) \left( \frac{\Omega_m}{\eta} + \frac{3(4\eta - 1)}{3\eta - 1} \right) \right] - \frac{\Omega^2_\eta}{\eta},
\]
where prime denotes derivative with respect to $x = \ln a$, and we used $\dot{\Omega}_\eta = H\Omega'_\eta$. We have plotted the evolution of $\Omega_\eta$ versus redshift parameter $z$ in Fig. 1 for some values of $\eta$ and $B$. The behavior of $\omega_\Lambda$ and $q$ are also plotted in Fig. 2. As it is obvious, Universe has a transition from a deceleration phase to the current accelerated phase at the redshift $z \approx 0.6$ in agreement with the recent observation data [23], and the EoS parameter can not cross the phantom line. For studying the classical stability of the dark energy model ($\rho_\Lambda$), by taking the time derivative of Eq. (15) and employing Eqs. (12) and (19), we reach

$$v_s^2 = \frac{4\eta - 1}{B} + \frac{4\eta - 1}{3B\eta(1 - 6\eta)\Omega_\eta} \left(3\eta(6\eta - 1)\Omega_\eta + (\Omega_\Lambda + 6\eta - 1)(3\eta(4\eta - 1) + (3\eta - 1)\Omega_\eta)\right),$$

plotted in Fig. 3 versus redshift parameter. We see that the square of sound speed is always negative during the history evolution of Universe meaning that the non-interacting HDE in Rastall theory is classically unstable, a behavior obtained in the framework of standard cosmology in which HDE plays the role of dark energy [10].

An interacting Case

In the presence of mutual interaction between DM and HDE, the conservation equations are given as [22–25]

$$\left(\frac{3\eta - 1}{4\eta - 1}\right)\dot{\rho}_m + 3H\rho_m = Q,$$

$$\left(\frac{3\eta - 1}{4\eta - 1}\right)\dot{\rho}_\Lambda + \left(\frac{3\eta}{4\eta - 1}\right)\dot{p}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = -Q,$$

where $Q$ denotes the interaction term and we assume that it has the form $Q = H(\alpha \rho_m + \beta p_D)$, in which $\alpha$ and $\beta$ are
unknown coupling constants [22]. Using Eqs. (12), (21), and the time derivative of Eq. (8), we reach

\[
\frac{\ddot{H}}{H^3} = \frac{4\eta - 1}{2(1 - 6\eta)} \left[ \frac{\Omega(\alpha - 3)(6\eta - 1)}{1 - 3\eta} \right] + (\Omega + 6\eta - 1) \left( \frac{\Omega(\alpha - 3)(4\eta - 1)}{1 - 3\eta} \right) + (4\eta - 1)\beta \Omega \left( \frac{3\eta - 1}{3\eta - 1} \right), \tag{22}
\]

used in order to write

\[
\Omega' = \frac{4\eta - 1}{(1 - 6\eta)} \left[ \frac{\Omega(\alpha - 3)(6\eta - 1)}{1 - 3\eta} \right] + (\Omega + 6\eta - 1) \left( \frac{\Omega(\alpha - 3)(4\eta - 1)}{1 - 3\eta} \right) + (4\eta - 1)\beta \Omega \left( \frac{3\eta - 1}{3\eta - 1} \right) - \frac{\Omega^2}{\eta}. \tag{23}
\]

The evolution of \(\Omega\) against redshift \(z\) has been plotted in Fig. 4 for the initial condition \(\Omega(\eta = 0) = 0.73, B = 1.2\). We have also plotted the evolution of the EoS and deceleration parameters against redshift \(z\) for some values of \(\alpha\) and \(\beta\). From this figure, one can clearly see that the transition redshift \(z_t\) lies within the interval \(0.5 < z < 0.8\).

By taking the time derivative of Eq. (15) and combining the result with Eqs. (12) and (23), we get

\[
v^2_s = \frac{4\eta - 1}{B} + \frac{4\eta - 1}{3B(1 - 6\eta)} \left[ (3 - \alpha)(6\eta - 1)\Omega \right] + (\Omega + 6\eta - 1) \left( (3 - \alpha)(4\eta - 1) + \frac{(3\eta - 1)\Omega}{\eta} \right) + (4\eta - 1)\beta \Omega \left( \frac{3\eta - 1}{3\eta - 1} \right). \tag{24}
\]

The evolution of \(v^2_s\) versus redshift parameter is plotted in Fig. 6, showing that the interacting HDE model in Rastall theory, unlike the non-interacting case, is classically stable.

**B. A new approach**

Here, we assume that DE is a combination of the Rastall term and vacuum energy. In this case, Eqs. (8) can be rewritten as

\[
3H^2 = \kappa(\rho_m + \rho_D),
\]

\[
H^2 + \frac{2}{3} \dot{H} = -\frac{\kappa}{3} p_D, \tag{25}
\]

where

\[
\rho_D = \rho_\Lambda + \frac{6\eta(6\eta - 1)}{\kappa G(4\eta - 1)} (2H^2 + \dot{H}),
\]

\[
p_D = p_\Lambda - \frac{6\eta(6\eta - 1)}{\kappa G(4\eta - 1)} (2H^2 + \dot{H}). \tag{26}
\]
It is easy to see that the standard cosmology is restored at the appropriate limit $\eta \to 0$. Indeed, we wrote the Friedmann equations in a form easily comparable with the standard Friedmann equations. In this case, by defining critical density as

$$\rho_c = \frac{3H^2(6\eta - 1)}{\kappa G(4\eta - 1)},$$

and using Eq. (25), one can find

$$\Omega_m + \Omega_D = 1,$$  \hspace{1cm} (27)

where

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{\kappa \rho_m}{3H^2}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = B,$$

$$\Omega_D = \Omega_\Lambda + 4\eta + 2\eta \frac{\dot{H}}{H^2}. \hspace{1cm} (28)$$

The use of Eq. (25) leads also to

$$\omega_D = -\frac{1}{\Omega_D} \left( 1 + \frac{2\dot{H}}{3H^2} \right). \hspace{1cm} (29)$$

Taking the time derivative of Eq. (28), we obtain

$$\dot{\Omega}_D = 2\eta \left( \frac{\ddot{H}}{H^2} - 2 \frac{\dot{H}^2}{H^3} \right). \hspace{1cm} (30)$$

For the limiting case $\eta \to 0$, the equation of motion of HDE in standard cosmology ($\Omega_D = \Omega_\Lambda = \text{const}$), as a desired result, is restored.

Substituting Eq. (28) into (29), one can obtain the EoS as

$$\omega_D = -\frac{1}{3\eta \Omega_D} \left( \Omega_D - \Omega_\Lambda - \eta \right), \hspace{1cm} (31)$$

Using Eq. (28), one also finds

$$q = -1 - \frac{\dot{H}}{H^2} = -\frac{1}{2\eta} (\Omega_D - \Omega_\Lambda - 2\eta). \hspace{1cm} (32)$$

Hence, it is obvious that at $\eta \to 0$ ($\Omega_D = \Omega_\Lambda$) limit, the EoS and deceleration parameters for the HDE in standard cosmology are recovered. It is also useful to mention that Eq. (31) and (32) are the same for both non-interacting and interacting cases.

**Non-interacting Case**

Combining Eqs. (10), (12), (27) and (28) with the time derivative of the first Friedmann equation (25), we arrive

$$\frac{\ddot{H}}{H^3} = \frac{3(4\eta - 1)(\Omega_D - 1)}{2\eta(1 - 3\eta)} - \frac{1}{2\eta^2} (\Omega_\Lambda + 4\eta - 1)(\Omega_D - \Omega_\Lambda - 4\eta). \hspace{1cm} (33)$$

Inserting Eqs. (28) and (33) into (30), we can also obtain the evolution of dimensionless DE density parameter as

$$\Omega_D' = (\Omega_D - 1) \left( \frac{3(4\eta - 1)}{1 - 3\eta} - \frac{\Omega_D - \Omega_\Lambda - 4\eta}{\eta} \right). \hspace{1cm} (34)$$

The evolution of $\Omega_D$ against redshift $z$ has been plotted in Fig. 7 for the initial condition $\Omega_D(z = 0) = 0.73$ and some values of $\eta$, addressing us that $\Omega_D \to 0$ and $\Omega_D \to 1$ at the early time and late time, respectively. Moreover, the behavior of $\omega_D(z)$ and $q(z)$ are shown numerically in Fig. 8 for some values of the parameter $\eta$ which implies that Universe has experienced a transition at the redshift $z \approx$...
(35)

\[ v_s^2 = \frac{\rho_D}{3} \left( \Omega_D - \Omega_A - \eta \right) - \frac{(\Omega_A + \eta)(1 - \Omega_D)}{3} \left( \frac{\Omega_D - \Omega_A - 4\eta}{\eta} \right) - \frac{1 - 3\eta}{\eta} \times \left( \frac{3(4\eta - 1)(\Omega_D - 1) + (1 - 3\eta)(\Omega_D - \Omega_A - 4\eta)}{(3\eta(4\eta - 1))(\Omega_D - 1) + (1 - 3\eta)(\Omega_D - \Omega_A - 4\eta)} \right). \]

plotted in Fig. 9 for non-interacting case. It is therefore seen that the stability of this case depends on the value of Rastall parameter so that there exists a critical value for this parameter that lies within the interval \( 0.25 < \eta < 0.26 \) and separates the stable and unstable regimes. Therefore we have classical stability (un-stability) for \( \eta > \eta_c \) (\( \eta < \eta_c \)).

### An interacting Case

Taking the time derivative of Eq. (25) along with using Eqs. (12), (21), (27) and (28), we reach at

\[
\frac{\dot{H}}{H^2} = \frac{4\eta - 1}{2\eta(1 - 3\eta)} ((\alpha - 3)(1 - \Omega_D) + \beta \Omega_A) - \frac{1}{2\eta^2} (\Omega_A + 4\eta - 1)(\Omega_D - \Omega_A - 4\eta). \tag{36}
\]

Inserting Eq. (36) into (30), one gets

\[
\Omega_D' = \frac{4\eta - 1}{1 - 3\eta} ((\alpha - 3)(1 - \Omega_D) + \beta \Omega_A) - \frac{1}{\eta} (\Omega_D - 1)(\Omega_D - \Omega_A - 4\eta), \tag{37}
\]

plotted as a function of \( z \) in Fig. 10 for the initial condition \( \Omega_D(z = 0) = 0.73 \). In this manner, while we have \( \Omega_D \to 1 \) at the late time \( (z \to -1) \), \( 0.2 < \Omega_D < 0.3 \) at the early time \( (z \to \infty) \), in agreement with Eq. (25), where the dark energy DE density parameter is a combination of the Rastall term and vacuum energy.

Figure. 11 includes the evolution of \( \omega_A(z) \) and \( q(z) \) for some values of \( \alpha \) and \( \beta \) parameters, and indicates that there is a transition from the deceleration phase to the accelerated phase in the interval \( 0.5 < z_t < 0.9 \). It is also seen that, in this case, the interacting HDE model behaves like the phantom DE at the late time.

### III. THDE IN RASTALL GRAVITY

Following (11), THDE density in Rastall theory is obtained as

\[
\rho_T = \frac{3B}{8\pi G} \frac{6\eta - 1}{4\eta - 1} H^{4 - 2\delta},
\]

where \( \delta \) and \( B \) are unknown parameters.
FIG. 10: $\Omega_D$ versus $z$ for interacting HDE, considering the initial condition $\Omega_{D0} = 0.73$, $B = 0.68$ and some values of $\alpha$ and $\beta$.

FIG. 11: The evolution of the $\omega_D$ and $q$ parameters with respect to the $z$ for interacting HDE in Rastall theory.

FIG. 12: $v_s^2(z)$ for the interacting HDE for some values of $\alpha$ and $\beta$.

FIG. 13: The $\Omega_T$ and $\Omega_D$ against $z$ for the non-interacting THDE model for the common (solid line) and new (dash line) approaches in Rastall theory.

A. The common approach

Using definition (14), we can write THDE dimensionless density parameter and its evolution as

$$\Omega_T = B \left( \frac{6\eta - 1}{4\eta - 1} \right) H^{2-2\delta},$$

and

$$\Omega_T' = \frac{(1 - \delta)\Omega_T\Omega_q}{\eta},$$
respectively. One can also use Eqs. (8), (21), (38) and (40) in order to obtain EoS and deceleration parameters as

$$\omega_T = \frac{(4\eta - 1)\Omega_T}{B^2(6\eta - 1)}\left(\Omega_D - \Omega_T - \eta\right),$$

(41)

and

$$q = \frac{-1}{2\eta}(\Omega_D - \Omega_T - 2\eta),$$

(42)

respectively, which are, indeed, the same for both non-interacting and interacting cases.

**B. Non-interacting case**

Taking the time derivative of the first Friedmann equation, (8), and putting Eqs. (10), (14), (38) and (39) in the result, we reach at

$$\Omega''_\eta = \frac{3(4\eta - 1)\Omega_\eta}{1 - 3\eta} - \frac{\Omega^2_\eta}{\eta} + \frac{4\eta - 1}{\eta(1 - 6\eta)}\left[\frac{3\eta(4\eta - 1)}{3\eta - 1}(\Omega_T + 6\eta - 1) + (2 - \delta)\Omega_T + 6\eta - 1\right].$$

(43)

The behavior of the dimensionless density parameter, EoS and deceleration parameters for interacting THDE in Rastall theory are plotted in Figs. 15 and 16. Figure 15 shows that $\Omega_T \to 0$ at the early time and $\Omega_T \to 1$ at future ($z \to -1$), as a desired result. It is also obvious that Universe has a transition from the deceleration

**C. An interacting case**

For interacting case, by using Eqs. (8), (14), (21), (38) and (40), we can find

$$\Omega''_\eta = \frac{4\eta - 1}{(1 - 6\eta)(3\eta - 1)}\left[(\alpha - 3)(1 - 6\eta)\Omega_\eta + (4\eta - 1)\left(\alpha - 3)(1 - \Omega_T + 6\eta) + \beta\Omega_T\right) + \frac{(3\eta - 1)\Omega_\eta}{\eta}(2 - \delta)\Omega_T + 6\eta - 1\right] - \frac{\Omega^2_\eta}{\eta}.$$  

(44)

FIG. 14: The $\omega$ and $q$ against $z$ for the non-interacting THDE model for the common (solid line) and new (dash line) approaches in Rastall theory.

FIG. 15: $\Omega_T$ against $z$ for the interacting THDE model in Rastall theory. We have taken $\Omega_T(z = 0) = 0.73$ as the initial condition.
phase to the current accelerated phase at the redshift $z \approx 0.6$, and unlike the non-interacting case, the interacting THDE model can produce acceptable behavior at future, and its EoS parameter can cross the phantom line.

Combining the time derivative of Eq. (41) with Eqs. (38), (40) (44) and (17), one can also obtain $v_s^2$ for interacting THDE in the Rastall theory. Since this expression is too long, we shall not present it here, and only a plot of it in Fig. 17 is presented, showing that the THDE model in Rastall theory is classically stable.

### D. The second approach

In this case, using \( \rho_c = \frac{3H^2(6\eta-1)}{\kappa G(4\eta-1)} \) in rewriting the dimensionless density parameters of THDE as

\[
\Omega_T = BH^{2-2\delta},
\]

where $B$ is an known parameter as usual \[11\], and using Eq. (28) leading to

\[
\Omega_D = \Omega_T + 4\eta + 2\eta \frac{\dot{H}}{H^2},
\]

and combining it with Eq. (45), one reaches at

\[
\Omega_T' = \frac{(1-\delta)(\Omega_D - \Omega_T - 4\eta)\Omega_T}{\eta}.
\]

Calculations for the EoS parameter and $q$ also yield

\[
\omega_D = \frac{-1}{3\eta \Omega_D} \left( \Omega_D - \Omega_T - \eta \right),
\]

and

\[
q = \frac{-1}{2\eta} \left( \Omega_D - \Omega_T - 2\eta \right),
\]

respectively. It is finally useful to mention here that Eqs. (47) (48) and (49) are valid for both interacting and non-interacting cases.

### Non-interacting Case

Taking the derivative with respect to time from Eq. (25) and using Eqs. (10), (38), (46) and (27), we reach at

\[
\frac{\ddot{H}}{H^3} = \frac{3(4\eta - 1)(\Omega_D - 1)}{2\eta(1 - 3\eta)} - \frac{1}{2\eta^2}((2 - \delta)\Omega_T + 4\eta - 1)(\Omega_D - \Omega_T - 4\eta).
\]
Now by taking the time derivative of Eq. (46) and using Eqs. (47) and (33), one finds

\[ \Omega_D' = (\Omega_D - 1) \left( \frac{3(4\eta - 1)}{1 - 3\eta} - \frac{\Omega_D - \Omega_T - 4\eta}{\eta} \right). \]  

(51)

We have also plotted the behavior of \( \Omega_D(z) \), \( \omega_D(z) \) and \( q(z) \) in Figs. 13 and 14 which show that, even in the second approach, the non-interacting THDE model cannot still produce suitable behavior at future (\( z < 0 \)).

An interacting Case

The evolution of \( \Omega_D \), by combining Eqs. (21), (38), (46) and (27) with the time derivative of the first Friedmann equation (25), is obtained as

\[ \Omega_D' = \frac{4\eta - 1}{1 - 3\eta} ((\alpha - 3)(1 - \Omega_D) + \beta \Omega_A) - \frac{1}{\eta}((\Omega_D - 1)(\Omega_D - \Omega_A - 4\eta)). \]  

(52)

\( \omega_D \) and \( \Omega_T \) as the functions of redshift have been plotted in Fig. 18 indicating that \( \Omega_D \to 1 \) and \( \Omega_T \to 1 \) in future and we have \( \Omega_T \to 0 \) and \( 0.1 \leq \Omega_D < 0.3 \) in the past. Indeed, while the Rastall term have more contribution in \( \Omega_D \) in the past, the portion of vacuum energy (\( \Omega_T \)) is increased by decreasing \( z \) and will get dominant at future. In Fig. 18, the behavior of \( \omega_D(z) \) and \( q(z) \) are also plotted. Taking the time derivative of relation (48) and considering Eqs. (17) and (39), one can obtain \( v_s^2 \) for interacting THDE model in the second approach. Since this expression is too long, we have only plotted it in Fig. 20 rather than presenting it.
We found out that, the same as the original HDE in standard cosmology [9], the density dimensionless parameter of HDE in Rastall framework is constant in common approach. The behavior of deceleration and EoS parameters is suitable in both interacting and non-interacting regimes. In the second approach, while the sum of HDE and Rastall parameter plays the role of dark energy, unlike the common approach, the density dimensionless parameter is not constant during the cosmic evolution. In this manner and in the presence of interaction, all cosmological parameters show acceptable behavior with the same values of the coupling constants including $\eta$, $\alpha$ and $\beta$, an outcome which cannot be obtained in the absence of the assumed mutual interaction between vacuum energy and dark matter.

Our study implies the fact that although the density dimensionless parameter of THDE ($\Omega_T$) is not constant during the cosmic evolution, the cosmic parameters show proper behavior by themselves only whenever the mutual interaction is present between the vacuum energy and dark matter. The same conclusion is also valid in the second approach. It is finally useful to mention that only the non-interacting HDE in common approach is always classically unstable.

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