Reynolds number of transition and large-scale properties of strong turbulence.

Victor Yakhot
Department of Mechanical Engineering, Boston University, Boston, Massachusetts 02215, USA.
(Dated: September 16, 2014)

A turbulent flow is characterized by velocity fluctuations excited in an extremely broad interval of wave numbers \( k > \Lambda_f \) where \( \Lambda_f \) is a relatively small set of the wave-vectors where energy is pumped into fluid by external forces. Iterative averaging over small-scale velocity fluctuations from the interval \( \Lambda_f < k \leq \Lambda_0 \), where \( \eta = 2\pi/\Lambda_0 \) is the dissipation scale, leads to an infinite number of “relevant” scale-dependent coupling constants (Reynolds numbers) \( Re(k) = O(1) \). It is shown that in the i.r. limit \( k \to \Lambda_f \), the Reynolds numbers \( Re(k) \to Re_{tr} \), where \( Re_{tr} \) is the recently numerically and experimentally discovered universal Reynolds number of “smooth” transition from Gaussian to anomalous statistics of spatial velocity derivatives. The calculated relation \( Re(\Lambda_f) = Re_{tr} \), “selects” the lowest - order non-linearity as the only relevant one. This means that in the infra-red limit \( k \to \Lambda_f \) all high-order nonlinearities generated by the scale-elimination sum up to zero.

PACS numbers 47.27

Introduction. “The turbulence problem” can be formulated as follows: consider the Navier-Stokes (NS) equations driven by the large-scale force \( F(\Lambda_f) \) where \( \Lambda_f \) denotes a relatively small set of wave-vectors \( |k| \approx 2\pi/L \). We fix both force \( F = O(1) \) and the integral scale \( L = 2\pi/\Lambda_f = O(1) \) independent upon Reynolds number, and by decreasing kinematic viscosity \( \nu \), vary the Reynolds number \( Re = uL/\nu \), where \( u(r,t) \) is a solution to the NS equations of motion. As long as \( \nu > \nu_{tr} (Re < Re_{tr}) \) the flow is laminar, i.e. \( u = u(k) \) with \( k \approx \Lambda_f \). At \( Re = Re_{tr} (\nu = \nu_{tr}) \) the solution \( u = u_0(\Lambda_f) \) becomes unstable and at \( Re > Re_{tr} \), the velocity field can be written as: \( u(k) = u_0(\Lambda_f) + v(k,t) \) where \( k > \Lambda_f \). Formation of the small-scale time-dependent velocity components \( v(k,t) \) is the main manifestation of transition to turbulence. We would like to stress a relatively trivial, but extremely important for what follows, statement: at \( Re \leq Re_{tr} \), the laminar flow pattern \( u_0(\Lambda_f) \) is a solution to the Navier-Stokes equations characterized by a single coupling constant which is a properly chosen Reynolds number.

When the Reynolds number \( Re \gg Re_{tr} \), the flow is characterized by velocity fluctuations \( v(k) \) excited in a broad interval of scales \( \Lambda_f \leq k \leq \Lambda_0 \), and in the limit \( Re \to \infty \) the ratio \( \Lambda_f/\Lambda_0 \to 0 \). Below we set the u.v. cut-off \( \Lambda_0 \) equal to Kolmogorov’s dissipation scale \( \Lambda_0 \approx \Lambda_f e^{\frac{3}{2}} \). In this case, as will be shown below, the equation for turbulent fluctuations is similar to the Navier-Stokes equation with the broad-band “force” \( f_j = -\nu_i \partial_i u_{0,j} \) in the right side. Following K.G. Wilson [1] we can average the governing NS equations over velocity fluctuations from a thin “slice” in the wave-vector space \( \Lambda(r) = \Lambda_0 e^{-r} \leq k \leq \Lambda_0 \) with \( r \to 0 \). In this case this exact procedure leads to equations for the remaining long wave-length modes \( v(k) \) from the interval \( k < \Lambda_0 e^{-r} \). The main problem is that the derived equation, in addition to corrections to viscosity (\( \nu(r) \to \nu + \Delta \nu(r) \) ) and driving force \( \Delta f \), includes an infinite number of coupling constants (see below) which, unlike in the theory of critical phenomena, are relevant when \( r \gg 1 \). Since \( \Delta \nu(r) > 0 \), the \( r \)-dependent Reynolds number \( Re(\Lambda(r)) < Re(\Lambda_0) \). Iterating this procedure one can derive equations for the modes with \( k \ll \Lambda_0 \) belonging to the so called inertial range where molecular (“bare”) viscosity \( \nu \) is irrelevant and all coupling constants can depend on the “dressed” Reynolds number \( Re(r) = \frac{2\nu_{rms}(r)}{\nu(r)\Lambda(r)} \). By dimensional reasoning \( \nu(r) \approx \nu_{rms}(r)/\Lambda(r) \) and we see that in the inertial range \( Re(r) = O(1) \). This means that the resulting equations include infinite number of non-linearities generated by the scale-eliminating procedure. In the inertial range all these terms are relevant and are responsible for anomalous scaling and non-analyticity of velocity increments.

According to the picture described above, the integral scale \( L = 2\pi/\Lambda_f \) is the largest scale of turbulence, i.e. \( \Lambda(r) \geq \Lambda_f \). It will be shown below that as \( \Lambda(r) \to \Lambda_f \) the effective viscosity \( \nu(\Lambda(r)) \to \nu_{tr} \) and \( Re(\Lambda(r)) \to Re_{tr} \). Since at this Reynolds number the marginally stable flow, described by the Navier-Stokes equations with \( \nu = \nu_{tr} \) is laminar, we may conclude that all high-order terms, additional to the NS equations, must disappear and only the lowest order quadratic non-linearity, characteristic of the NS equations survives. This way transition to turbulence ”selects” the only relevant non-linearity. Assuming validity of Landau’s theory of transition we will be able conclude that in the limit \( \Lambda(\Lambda) \to \Lambda_f \), high-order non-linearities generated by the scale-eliminating procedure are \( O(\sqrt{Re(\Lambda(r)) - Re_{tr}}) \to 0 \).

Transition to turbulence. There exist a huge literature on this topic which, together with the theory of dynamical systems, evolved into a separate field of research. Typically, one searches for instabilities in laminar flow \( u_0 \) manifested by exponential growth of perturbations \( u(k,t) \). We will loosely identify laminar flow as a pattern \( u_0 \) formed by a small set of excited modes supported in the range of wave-numbers \( k \approx \Lambda_f \). All modes with \( k > \Lambda_f \) are strongly overdamped, i.e. \( u(k) = 0 \) for both \( k < \Lambda_f \) and \( k > \Lambda_f \).
Landau’s theory. Here we mention just one work which is relevant for considerations presented below [2]. Assuming that in the vicinity of a transition point imaginary part of complex frequency is much smaller than the real one, Landau considered the linearized Navier-Stokes equations for incompressible fluid. Denoting the velocity field at a transition point \( \mathbf{u}_0 \) and introducing an infinitesimal perturbation \( \mathbf{u}_1 \) he wrote \( \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \) with \( \mathbf{u}_1 = A(t) f(r) \). Based on general qualitative considerations, Landau proposed:

\[
\frac{d|A|^2}{dt} = 2\gamma |A|^2 - \alpha |A|^4
\]

where in the vicinity of transition point \( \gamma = \epsilon (Re - Re_{tr}) \) and \( \alpha > 0 \). In principle, \( |A|^2 \) must be considered as time-averaged. Landau noted, however, that \( \mathbf{u}_1(k) \) is a slow mode and, since the averaging is taken over relatively short time -intervals, the averaging sign in the above equations is not necessary. At small times the solution exponentially grows and then reaches the maximum \( A_{max} \propto \sqrt{Re - Re_{tr}} \). When \( \gamma = Re - Re_{tr} < 0 \), any initial perturbation decays. In this theory, the magnitude of transitional Reynolds number is a free parameter and since the large-scale field \( \mathbf{u}_0 \) strongly depends on geometry, external forces and stresses, the transition Reynolds number \( Re_{tr} \) is not expected to be a universal constant.

Landau assumed that further increase of the Reynolds number leads to instability of first unstable mode generating next two excited modes with the wave-vectors \( k_2 > k_1 \) etc. In modern lingo, this process can be perceived as an onset of the energy cascade toward small- scale excitations with \( k > \lambda_f \). This leads to formation of “inertial range” and strongly intermittent small-scale dissipation rate \( \mathcal{E} \).

Transition to turbulence: a new angle. A new way of looking at phenomenon of transition to turbulence was introduced in numerical simulations of a flow at a relatively low Reynolds number \( R_{\lambda} = \sqrt{\frac{5}{2\pi} u_{rms}^2} \geq 4.0 \) [3]. In this approach transition to turbulence is identified with the first appearance of non-gaussian anomalous fluctuations of velocity derivatives including those of the dissipation rate \( \mathcal{E} \). As will be shown below, in a sense, it is a transition between two different states : Gaussian ( structureless ) and anomalous (structured ), resembling those observed in experiments on Benard convection. On a first glance the transition is smooth meaning that no “jumps” in velocity field were detected. However, the precise nature of this transition is yet to be investigated. All we can state at this point is that the transformation happens in a narrow range of the Reynolds number variation. The homogeneous and isotropic turbulence (HIT) was generated in a periodic box by a force in the right-side of the Navier-Stokes equation \( \mathbf{F}(\mathbf{k}, t) = \mathcal{P} \sum \frac{u(k, t)}{|u(k, t)|^2} \delta_{\mathbf{k}, \mathbf{k}'} \), where summation is carried over \( \mathbf{k}_f = (1, 1, 2); (1, 2, 2) \). It is easy to see that the model with this forcing generates flows with constant energy flux \( \mathcal{P} = \mathcal{E} = \nu \frac{\partial u}{\partial x}^2 = const \) and the variation of the Reynolds number is achieved by variation of viscosity.

The results of Ref.[3] can be briefly summarized as follows: 1. Extremely well-resolved simulations of the low-Reynolds number flows at \( R_{\lambda} \geq 9 – 10 \) revealed a clear scaling range \( M_\alpha = \frac{\langle \frac{\partial u}{\partial x} \rangle^\alpha}{\langle \frac{\partial u}{\partial x} \rangle^2} \propto Re^\rho_\alpha \) with anomalous scaling exponents \( \rho_\alpha \) consistent with the inertial range exponents typically observed only in very high Reynolds number flows \( Re \gg Re_{tr} \). Identical scaling exponents \( \rho_\alpha \) were later obtained in isotropic turbulence generated by a different forcing [4], channel and pipe flows [5] and, more recently, in Benard convection [6] indicating possibility of a broad universality. 2. For \( R_{\lambda} < 9 – 10 \) all flows were subgaussian indicating a dynamical system consisting of a small number of modes with the small-scale fluctuations strongly overdamped. This flow can be called “quasilaminar” or coherent. 3. At a transition point \( R_{\lambda, tr} \approx 9 – 10 \) the fluctuating velocity derivatives obey gaussian statistics and at \( R_{\lambda} > 9 – 10 \) a strongly anomalous scaling of the moments, typical of high-Reynolds number turbulence, is clearly seen. 4. It has also been noticed that transition was smooth, i.e. velocity field at \( \mathbf{u}(R_{\lambda, tr}) - \mathbf{u}(R_{\lambda, tr}^+) \rightarrow 0 \).

**FIG. 1.** (Color online) Normalized moments of the dissipation rate \( \frac{\partial u}{\partial x}^\alpha \) in homogeneous and isotropic turbulence. Refs.[3]-[4]. DSY stand for Donzis, Sreenivasan and Yeung.
On Fig.1 the moments of the dissipation \( \frac{\partial E}{\partial t} \) rate computed in [3] are combined with the data obtained by Donzis et. al [4] in HIT generated by a completely different large-scale forcing. We can see that the scaling exponents, found in the range of very low Reynolds number in [3] hold in a much wider range of the Reynolds number variation. This means that in the range \( R_\lambda \geq 10 \) turbulence can be considered as fully developed.

On Fig.3 the moments \( M_\nu \) of velocity derivatives are shown in the vicinity of a transition point. One can see that at \( R_\lambda = 9 - 10 \) a relatively sharp transformation from a sub-Gaussian at \( R_\lambda < 9 - 10 \) to anomalous scaling of the dissipation rate moments occurs independently on the driving force. This surprising result will be used below as a constraint on development of turbulence models.

**The model.**

Based on these results, we consider a flow generated by the Navier-Stokes equations with a force \( \mathbf{F}(\Lambda f) \). Keeping the force \( \mathbf{F} = \text{const} \) and the length-scale \( L = 2\pi/\Lambda f = \text{const} \), let us vary viscosity in the interval \( 0 \leq \nu \leq \infty \). In the range \( \nu > \nu_tr \) or \( Re < Re_tr \) the flow is laminar in a sense that it is described by a relatively small number of modes with \( u(k) \) with \( k \approx \Lambda f \). At the transition point \( Re_{\lambda tr} \approx 9 - 10 \) the transitional pattern \( \mathbf{u}_0(\Lambda f) \) is formed, so that:

\[
L(\mathbf{u}_0, \nu_tr) = \frac{D\mathbf{u}_0}{Dt} + \nabla p - \nu_tr \nabla^2 \mathbf{u}_0 - \mathbf{F}(\Lambda f) = 0
\]

Here \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \). When \( \nu \ll \nu_tr \) the Navier-Stokes equations read:

\[
\frac{D\mathbf{u}}{Dt} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}(\Lambda f) = 0
\]

If we write \( \mathbf{u} = \mathbf{u}_0 + \mathbf{v} \), the equation for the “turbulent” component \( (k > \Lambda f) \) of velocity field:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f}
\]

with

\[
f = f_1 + f_2 + f_3 = -\mathbf{u}_0 \cdot \nabla \mathbf{v} - \nabla \mathbf{u}_0 + (\nu - \nu_tr) \nabla^2 \mathbf{u}_0(\Lambda f)
\]

The first term \( f_1 \) in this expression describes kinematic transfer of small “eddies” by the large ones. The second, \( f_2 \), is responsible for turbulence production due to interaction of small-scale fluctuations with the large-quasicoherent flow \( \mathbf{u}_0 \). This effect is well-known in the turbulence modeling literature.

Thus, the total energy production rate is:

\[
\mathcal{P} = \frac{f_2}{\nu} = -\nu Tr \partial^2 \mathbf{u}_{0,i} \partial x_j \approx \nu Tr (u^2_{0,i})^2 \approx \Lambda f^2 (\nu)^2
\]

The balance can be written for each scale \( l = 2\pi/\Lambda(r) \) but with “turbulent viscosity” \( \nu(l) \propto \lambda^{3/4} l^{1/4} \) and introducing the projection operator \( \mathcal{P}_{lmn} = \frac{k_m}{k} P_{ln}(k) + k_n P_{lm}(k) \) with \( P_{ij} = (\delta_{ij} - \frac{k_i k_j}{k^2}) \) we have with \( k = (k, \omega) \):

\[
f_{2,i} = -\frac{i}{2} \mathcal{P}_{lmn}(k) \int \nu(\hat{\omega}) u_{0,n}(\hat{k} - \hat{q}) d\hat{q}
\]

so that \( \mathcal{F}_2 = 0 \) and

\[
\mathcal{P}(k) \propto k^2 \int \frac{q^{-12}}{\Omega^2 + q^2} (u^4_{0,i})^2 \delta(\hat{k} - \hat{q}) \delta(\omega - \Omega) dq d\omega \propto k^{-3}
\]

The \( \nu \)-fluctuations are driven by a pumping with algebraic spectrum!

The experimental data of Refs.[3]-[6] point to independence of small-scale features of turbulence on the nature of production mechanism. On the Fig.2 the energy spectrum in a flow past circular cylinder of diameter \( D \) is shown for the large - scale Reynolds number \( Re = UD/\nu \approx 10^6 - 10^7 \). The onset of Kolmogorov's inertial range can be clearly seen at the wave number \( k \approx \Lambda_f = 2\pi/L \) separating inertial and non-universal, geometry-dependent energy - containing range of scales. A somewhat striking feature of the plot is a very narrow intermediate range which points to the smallness of sub-leading contributions to the inertial range scaling of the energy spectrum. Therefore, for \( \nu < \nu_tr \) we choose the well-known and well-studied model (1): where the random force, mimicking small-scale fluctuations is defined by the correlation function:

\[
f_i(k, \omega)f_j(k', \omega') = 2 D_0 (2\pi)^{d+1} k^{-d} P_{ij}(k) \delta(k + k') \delta(\omega + \omega');
\]

(2)
Based on the above argument, the force (2) is extrapolated onto interval \( \lambda_1 < k \leq \Lambda \equiv \Lambda_0 \), so that \( \int f_j = 0 \) and, by construction, \( f_i(k \leq \Lambda_f, t) = 0 \).

The renormalization or coarse graining. The renormalization group for fluid flows has been developed in Refs.[7]-[8] and was generalized to enable computations of various dimensionless amplitudes in the low order in the \( \epsilon \)-expansion in Refs.[9]-[12]. Introducing velocity and length-and-time scales \( U = \sqrt{D_0/\nu_0 \Lambda_0^2} \), \( X = 1/\Lambda_0 \) and \( T = t \nu_0 \Lambda_0^2 \), respectively, the equation (1) can be written as (for simplicity we do not change notations for dimensionless variables):

\[
\frac{\partial \mathbf{v}}{\partial t} + \hat{\lambda}_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\hat{\lambda}_0 \nabla p + \nabla^2 \mathbf{v} + \frac{\mathbf{f}}{\sqrt{D_0 \nu_0 \Lambda^2}}
\]

where the single dimensionless coupling constant (“bare” Reynolds number) is: \( \hat{\lambda}_0^2 = \frac{D_0}{\nu_0 \Lambda_0^2} \).

**Projecting Navier-Stokes equation onto domain** \( k \leq \Lambda_0 e^{-r} \) **where** \( r \to 0 \). Technical details of all calculations presented below are best described in [10]. Formally introducing modes \( \mathbf{v}^< (k, t) \) and \( \mathbf{v}^> (k, t) \) with \( k \) from the intervals \( k \leq \Lambda_0 e^{-r} \) and \( \Lambda_0^{-r} \leq k \leq \Lambda_0 \), respectively, and averaging over small-scale fluctuations \( \mathbf{v}^> \), leads to equation for the large-scale modes:

\[
\frac{\partial \mathbf{v}^<}{\partial t} + \mathbf{v}^< \cdot \nabla \mathbf{v}^< = -\nabla p^< + \frac{\partial \sigma^{(2)}_{ij}}{\partial x_j} + \nu \nabla^2 \mathbf{v}^< + f_i + \Delta f_i \tag{3}
\]

where the second-order correction to the Reynolds stress \( \sigma_{ij} = -\mathbf{v}^< \mathbf{v}^<_j \) is:

\[
\sigma^{(2)}_{ij} = \hat{\lambda}_1^4 (r) \Delta \nu(r) S_{ij} - \hat{\lambda}_1^4 (r) \nu(r) \frac{D}{Dt} \left[ \tau(r) S_{ij} \right] - \hat{\lambda}_1^4 (r) \nu(r) \tau(r) \left[ \beta_1 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \beta_2 \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \beta_3 \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \right] + O(\hat{\lambda}_1^6) + \cdots \tag{4}
\]

where \( \hat{\lambda}_1 = O(\hat{\lambda}_0 (e^{-r} - 1)) \) is a coupling constant generated by the scale-elimination and the time-constant \( 1/\tau(r) = \nu(r) \Lambda^2 (r) \). In this limit the coefficients \( \beta_i \) can be explicitly calculated [13] The “dressed” viscosity is denoted as \( \nu(r) = \nu + \Delta \nu(r) \) with correction to “viscosity” written in the wave-number space as:

\[
\Delta \nu = A_d D_0 \nu_0^2 \left[ \frac{e^{\epsilon r} - 1}{\epsilon \Lambda_0^2} + O \left( \frac{k^2}{\Lambda_0^{\epsilon + 2}} \frac{e^{(\epsilon + 2) r} - 1}{e + 2} \right) + O(\hat{\lambda}_1^6) \right] \tag{5}
\]
\[ e = 4 + y - d \text{ and } A_d = \frac{S_d}{2\pi^2}; \quad \dot{A}_d = \frac{1}{2} \frac{d^2 - d}{d + 2}. \]

On the interval \( k < \Lambda_0 e^{-r} \) the equations (3)-(5) are equivalent to the original equations of motion defined on the interval \( k \leq \Lambda_0 \).

**Iterating scale-elimination procedure.** The relations (3)-(5) are exact as long as the eliminated “slice” in the wave-vector space is very thin. As will be shown below, in the limit \( \Lambda(r) \ll \Lambda_0 \) the high-order contributions to the Reynolds stress are not small. The problem is that due to proliferation of tensorial indexes, these terms, while can be qualitatively analyzed using Wyld’s diagrammatic expansion [14], are very hard to calculate.

Eliminating the modes from the interval \( \Lambda_0 e^{-r} \leq k \leq \Lambda_0 \) the equations can be formally written:

\[
\frac{\partial v^<}{\partial t} + v^< \cdot \nabla f_i^< = -\nabla p + (\nu + \Delta \nu) \nabla^2 v^< + \mathbf{F} + \Delta \mathbf{f}_i + \text{HOT}
\]

where, by Galileo invariance, high-order \((n > 1)\) contributions generated by the scale-elimination can be formally written:

\[
\text{HOT} = \sum_{n=2}^{\infty} \hat{\lambda}_1^n(r)^n-1(\partial_r v^< + v^< \cdot \nabla^n) v^< + O(\hat{\lambda}_2(r)^4 \nabla^2 \frac{1}{\Lambda^2(r)} e^{(\epsilon + 2) r} - 1) + \cdots \tag{7}
\]

with \( \tau(r) \approx 1/(\nu(r) \Lambda^2(r)) \) and \( \hat{\lambda}_1 = \hat{\lambda}_0(e^{\epsilon r} - 1) \). In addition, the expressions (6)-(7) include various products of time- and space-derivatives responsible, for example, for the rapid distortion effects (RDE). The high-order non-linearities generated by the procedure are small if the eliminated shell is very thin but, as will be shown below, they exponentially grow with increase of \( r \). It is clear that the procedure generates an infinite number of coupling constants which are the factors in front of non-linear terms.

**Recursion relations.** As a result of elimination of the first shell \( \Lambda_0 e^{-r} \leq k \leq \Lambda_0 \), the original uncorrected “bare” viscosity \( \nu_0 \) disappears and instead the equations include only “dressed” viscosity \( \nu(r) \). Then, starting with the equations (4)-(7) defined on the interval \( k < \Lambda_0 e^{-r} \), we can eliminate the modes from the next shell of wave-numbers \( \Lambda_0 e^{-(r+\delta r)} \leq k \leq \Lambda_0 e^{-r} \) and derive equations of motion with another set of corrected transport coefficients. The procedure can be iterated resulting in the cut-off-dependent viscosity, induced force etc. Therefore, with \( \delta r \to 0 \) the parameters in the coarse-grained “Navier-Stokes equations” for the “resolved” velocity field \( \mathbf{u}^< \), defined at the scales \( l \geq 2\pi/\Lambda(r) \), satisfy the differential relations:

\[
\frac{\nu(r + \delta r) - \nu(r)}{\delta r} = A_d \frac{D_0}{\nu(r)^2} \frac{1}{\Lambda^2(r)} \left[ \sum_{n=0}^{\infty} \alpha_n \hat{\lambda}^n(r) + O(\frac{k^2}{\Lambda(r)^2}) \right] \tag{8}
\]

and

\[
HOT = \sum_{n=2}^{\infty} \hat{\lambda}_1^n(r)^n-1(\partial_r u^< + u^< \cdot \nabla^n) u^< + O(\hat{\lambda}_2(r)^4 \nabla^2 \frac{1}{\Lambda^2(r)} + \cdots) \tag{9}
\]

with \( \tau(r) \approx 1/(\nu(r) \Lambda^2(r)) \). To asses the role of different contributions to (8)-(9), first we assume \( \hat{\lambda}(r) \ll 1 \) and \( \hat{\lambda}_1(r) \ll 1 \) and analyze the lowest-order terms only.

**Low-order truncation of the expansion (8)-(9).** This leads to differential recursion equations: recalling that \( \Lambda(r) = \Lambda_0 e^{-r} \), one obtains:

\[
\frac{\nu(r + \delta r) - \nu(r)}{\delta r} = d\nu(r) = A_d \nu(r) \hat{\lambda}^2(r)
\]

where

\[
\frac{d\hat{\lambda}^2}{dr} = \epsilon \hat{\lambda}^2 - 3A_d \hat{\lambda}^4
\]

and:

\[
\nu(r) = \nu_0 \left[ 1 + \frac{3A_d D_0 S_d}{\epsilon \nu_0^2 (2\pi)^d} \left( \frac{1}{\Lambda^2(r)} - \frac{1}{\Lambda_0^2} \right) \right]^\frac{1}{2}
\]

\[
\hat{\lambda} = \hat{\lambda}_0 e^{\frac{\epsilon r}{2}} \left[ 1 + \frac{3A_d D_0 S_d}{\epsilon \nu_0^2 (2\pi)^d} \left( \frac{1}{\Lambda^2(r)} - \frac{1}{\Lambda_0^2} \right) \right]^{-\frac{1}{2}}
\]

The solution for the “induced” coupling constant \( \hat{\lambda}_1 \) is:

\[
\hat{\lambda}_1(r) = \frac{\sqrt{\epsilon e^{\frac{\epsilon r}{2}}}}{\sqrt{\frac{\epsilon}{\Lambda_0^2} + 3A_d (e^{\frac{\epsilon r}{2}} - 1)}} \tag{10}
\]

For \( \epsilon = 4 \), corresponding to Kolmogorov’s energy spectrum, in the limit \( cr \gg 1 \) the coupling constants tend to the fixed point

\[
\hat{\lambda}, \rightarrow \left( \frac{\epsilon}{3A_d} \right)^{\frac{1}{2}} \approx 1.29 \sqrt{\epsilon} \approx 2.58. \tag{11}
\]

It is also clear that \( \hat{\lambda}_1 \approx \hat{\lambda}_* \). This result means that for \( k > \Lambda_1 \), the above truncation of the expansion is, in general, incorrect and high-order non-linearities generated
by procedure are not small. Now we consider a special case of the transport approximation \( \Lambda(r) \to \Lambda_f = 2\pi/L \).

Parameters. All low-order calculations leading to dimensionless amplitudes described below, are best described in great detail in Ref.[10]. Eliminating all modes from the interval \( k \geq \Lambda_0 \) and setting \( \epsilon \to 4 \) gives:

\[
\nu(k) = \left( \frac{3}{8} A_d 2 D_0 \right) \frac{3}{2} k^{-\frac{5}{2}} \approx 0.42 \left( \frac{2 D_0 S_d}{(2\pi)^d} \right) k^{-\frac{5}{2}}
\]

and from the linearized equation at the fixed point we derive Kolmogorov’s spectrum valid in the range \( k \geq \Lambda_f \):

\[
E(k) = \frac{1}{2} \left( \frac{3}{8} A_d \right) \frac{2}{(2\pi)^d} \int_0^\infty Tr V_{ij}(k\omega) d\omega = \\
1.186(2 D_0 S_d/(2\pi)^d) \frac{3}{2} k^{-\frac{5}{2}}
\]

where \( (2\pi)^{d+1} V_{ij}(k,\omega) \). In the so called EDQNM approximation, which is exact at the Gaussian fixed point (see below), the force amplitude \( D_0 \) can be related to the mean dissipation rate [10], [11];

\[
2 D_0 S_d/(2\pi)^d \approx 1.59 \varepsilon; \quad E(k) = C_K \varepsilon \frac{3}{2} k^{-\frac{5}{2}}; \quad C_K = 1.61
\]

(13)

Let us identify the infrared cut-off \( \Lambda_f = \Lambda(r) \approx 2\pi/L \) with the wave-number corresponding to the top of the inertial range. In the large Re-limit \( \Lambda_0/\Lambda_f \gg 1 \), the total energy of the inertial range turbulent fluctuations is evaluated readily:

\[
\mathcal{K} = \int_{\Lambda_f}^\infty E(k) dk = \frac{3}{2} C_K \left( \frac{\varepsilon}{\Lambda_f} \right)^2 = \\
3 \left( \frac{2}{2} \right) 1.61 \left( \frac{3}{8} A_d 1.59 \right) \frac{3}{2} \frac{\varepsilon}{\nu(\Lambda_f) \Lambda_f^2} \approx 1.19 \frac{\varepsilon}{\nu(\Lambda_f) \Lambda_f^2}
\]

(14)

and, setting \( k = \Lambda_f \) gives the expression for effective viscosity in equation for the large-scale dynamics in the interval of scales \( k < \Lambda_f \):

\[
\nu_T \equiv \nu(\Lambda_f) = 0.084 \frac{K^2}{\varepsilon}; \quad 10.0 \times \nu(\Lambda_f)^2 \Lambda_f^2 = \mathcal{K}
\]

(15)

Fixed-point Reynolds number and irrelevant variables. The expression (15) gives effective viscosity accounting for all turbulent fluctuations from the interval \( 1/\Lambda_0 \leq r < L = 1/\Lambda_f \) acting on the almost-coherent-large scale flow on the scales \( r \approx L = 1/\Lambda_f \).

Using (13) -(15) we can calculate the effective \( R_{\lambda,f} = 2K\sqrt{5/(3E\nu(\Lambda_f))} = \sqrt{20/(3 \times 0.084)} = 9.0 \). The same parameter can be expressed in terms of the fixed-point coupling constant:

\[
\hat{\nu}^* = \sqrt{\frac{D_0 S_d/(2\pi)^d}{\nu^2(\Lambda_f)}} = \sqrt{\frac{0.8 \varepsilon}{\nu^2(\Lambda_f)}} = \sqrt{\frac{0.8 \times 400 \varepsilon \nu_T}{u_{rms}^2}}
\]

where \( R_{\lambda,f} \approx 9.0 \) very close to Reynolds number of transition \( R_{\lambda} \approx 9.0 \), obtained from direct numerical simulations of Refs.[3]-[4]. This result agrees with observation that in the flows past various bluff bodies, the Reynolds number based on the measured “turbulent viscosity” and large-scale velocity field is \( R_{\lambda,T} = O(10) \), independent on the “bare” (classic) Reynolds number calculated with molecular viscosity.

As \( \Lambda(r) \to L = 2\pi/\Lambda_f \), the effective viscosity \( \nu_T(r) \to \nu_r \), and \( Re(r) \to Re_{tr} \), which is the most important and surprising outcomes of the theory. If, as was found numerically, transition to turbulence is “smooth”, the velocity field \( \mathbf{u} \) must come out from equations of motion obtained by the scale - elimination and Navier-Stokes equations for quasi-laminar flow at a transition point: therefore

\[
\hat{L}(\mathbf{u}_0, \nu_{\Lambda_f}) - HOT = \hat{L}(\mathbf{u}_0, \nu_{tr}) \equiv 0
\]

(16)

If, in addition, the transition is universal, i.e. is independent on initial conditions we conclude that as \( \Lambda(r) \to \Lambda_f \), and, according to the above derivation \( \nu(\Lambda_f) \to \nu_{tr} \), the nonlinearities generated by the scale elimination procedure

\[
HOT \to 0
\]

are irrelevant. The role of the induced noise will be discussed in detail below.

All we can definitely say is: if indeed transition is continuous in the limit \( \Lambda(r) \to \Lambda_f \), the nonlinearities \( HOT(\Lambda(r)) \to 0 \). To understand the way it tends to zero, let us consider the linearized equation of motion in the vicinity of the fixed point where \( \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \):

\[
\frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u} \cdot \nabla \mathbf{u}_1 = - \nabla p_1 + \nu \nabla^2 \mathbf{u}_1 + HOT
\]

(17)

If \( \mathbf{u}_1 \propto \mathbf{A} e^{i\omega t} \), then according to purely phenomenological Landau’s theory of transition: \( \mathbf{u}_1 \propto A_{max} \propto \sqrt{Re - Re_{tr}} \) and

\[
HOT \approx \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 \approx \mathbf{u}_0^2 \Lambda_f \sqrt{Re - Re_{tr}}
\]
Large-scale dynamics. Now we would like to discuss the large-scale flow in the interval $k \approx \Lambda_f$, where the bare force $f(k) = 0$ and therefore the equation of motion is:

$$\frac{\partial \psi^<}{\partial t} + \psi^< \cdot \nabla \psi^< = -\nabla p + \nu(\Lambda_f) \nabla^2 \psi^< + F + \psi(18)$$

with induced noise evaluated in Ref.[10];

$$\psi_i(k) \omega_j(k', \omega') = (2\pi)^d+1 D_L k^2 P_{ij}(k) \delta(k + k') \delta(\omega + \omega') \quad (19)$$

where

$$D_L = D_0 \frac{d^2 - 2}{20d(d + 2)} \frac{\Lambda^*\Lambda^*}{\Lambda_f^2} = D_0 \frac{0.155}{\Lambda_f^2} \quad (20)$$

The induced force $\psi$ is the result of small-scale turbulent fluctuations on the large-scale dynamics which is often called “backscattering”. In the most important limit $k \rightarrow \Lambda_f$ the ratio $D_0/D_L \approx 1/0.155 \approx 7.0$, which, while being numerically not too large, is responsible for both blurring of the large-scale transitional patterns and for the observed gaussian statistics of the large-scale velocity fluctuations in the high-Reynolds number flows.

Summary and conclusions. The theory of critical phenomena is one of the most spectacular applications of renormalization group to statistical mechanics. In this approach the procedure is not applied to microscopic Hamiltonians, like those for ferromagnetic solids or liquid helium, but to macroscopic Hamiltonians (free energies) reflecting basic large - scale symmetries of a system. The magnitude of critical temperature $T_c$, depending on the details of intermolecular microscopic interactions remains undetermined and the size of the system does not appear in the theory where all correlation functions are expressed in terms of $\tau = \frac{|T - T_c|}{T_c}$. In this respect, hydrodynamics are different, for large-and small-scale dynamics responsible for transition at $Re = Re_{tr}$ and behavior of turbulent fluctuations at $Re >> Re_{tr}$, are contained in the Navier-Stokes equations. Therefore, it is not surprising that the recently discovered universal $Re_{tr} \approx 9. - 10$ can serve as a dynamic constraint on the theory.

1. One of most difficult, not yet understood properties of turbulence, is anomalous scaling of velocity increments on the small scales $\eta \ll l \ll L = 2\pi/\Lambda_f$. It was shown above that in this range of scales the flow is described by an infinite number of $O(1)$ coupling constants. At this point we do not know how to deal with them.

2. The selection of relevant variables is possible in the limit $l \rightarrow L$. Keeping only the lowest - order contributions, the calculated fixed-point Reynolds number $Re_f \approx Re_{\Lambda, tr}$, where $Re_{\Lambda, tr} = 9. - 10$ is the numerically computed Reynolds number of transition to turbulence. Since the numerically discovered transition is “smooth”, i.e. $u_0 = u_{fp}$ and $(\nabla_i u_{0,j} = \nabla_i u_{fp,j})$, at this point all additional to the NS equations high-order nonlinearities are irrelevant.

3. Comparison with Landau’s theory of transition to turbulence shows that in the vicinity of transition point, the neglected nonlinear terms are $O(\sqrt{Re - Re_{tr}}) \rightarrow 0$.

4. The infra-red divergencies appearing in the each term of the expansion do not disappear but are summed up into equations of motion for the large-scale features of the flow. To stress how accurate the derived transport approximation is, we would like to reproduce our old result on decay of isotropic and homogeneous turbulence [10]. Since in this flow $S_{ij} = 0$, the equations governing the decay are very simple:

$$\frac{\partial K}{\partial t} = -\mathcal{E}; \quad \frac{\partial \mathcal{E}}{\partial t} = -C_{\varepsilon,2} \mathcal{E}^2 K$$

with $C_{\varepsilon,2} = 1.68$ calculated at the integral scale in the lowest order of renormalized perturbation expansion [10]. The present paper justifies the approximation and procedure leading to this and all other constants calculated in [10] by an ad hoc neglecting of HOT. The above equations give:

$$\frac{K}{K_0} = \left( \frac{t}{t_0} \right)^{-\gamma_c,2} = \left( \frac{t}{t_0} \right)^{-\gamma_c} \approx \left( \frac{t}{t_0} \right)^{-1.47}$$

This result has a long and difficult history. First, it was shown by Kolmogorov that $\gamma = 10/7 \approx 1.43$, very close to the one shown above. Somewhat later, Kolmogorov’s construction has been interpreted by Landau as a consequence of conservation of the angular momentum [2]. This theory has been criticized by Batchelor et. al. [15] and the early experiments, yielding $\gamma \approx 1.0 - 1.3$ (see for example Ref.[16]), seemed to support Batchelor’s conclusions. A huge number of experimental, theoretical and later numerical papers dealt with this subject [16]. As a consequence, the constant $C_{\varepsilon,2}$ in the $K - \mathcal{E}$ model, widely used for engineering simulations, was taken as $C_{\varepsilon,2} \approx 1.92$ corresponding $\gamma \approx 1.1$. This led to the over-dissipated turbulent velocity field computed with this model. It took many years to realize that Kolmogorov’s theory was developed for a finite patch of turbulence in an infinite fluid and the exponent $\gamma$ was very sensitive to the finite size effects, geometry etc. This long-standing confusion has recently been resolved by a remarkable (4096^3) numerical simulation by Ishida et. al. [17], who showed that when the initially prepared flow satisfied constraints of Kolmogorov’s theory, the
exponent of kinetic energy decay was indeed $\gamma \approx 10/7$.

5. To conclude we would like to mention that if, in general, a turbulent flow is generated by an instability of a large-scale quasi-coherent flow pattern (dynamical system) $u_0$, then the equations of motion governing anomalous velocity fluctuations are given by (1) with $f_2 = \mathbf{v} \cdot \nabla u_0$. This may explain a broad universality of small-scale features of strong turbulence discovered in Refs.[3]-[6].

I am grateful to A. M. Polyakov, N. Goldenfeld, V. Lebedev, I. Kolokolov, Y. Sinai, U. Frisch, E. Titi, H. Chen, I. Staroselsky and J. Wanderer for their interest in this work and numerous suggestions. Many ideas leading to this paper emerged from numerical investigations of transition conducted jointly with J. Schumacher, D. Donzis and K. R. Sreenivasan.

References.

1. K.G. Wilson, Rev. Mod. Phys., 12, 75 (1974).
2. L.D. Landau & E.M. Lifshits, “Fluid Mechanics”, Pergamon, New York, 1982.
3. J. Schumacher, K.R. Sreenivasan & V. Yakhot; New J. of Phys. 9, 89 (2007).
4. D.A. Donzis, P.K. Yeung and K.R. Sreenivasan, “Dissipation and enstrophy in homogeneous turbulence: resolution effects and scaling in direct numerical simulations”, Phys. Fluids 20, 045108 (2008).
5. P.E. Hamlington, D. Krasnov, T. Boeck and J. Schumacher, “Local dissipation scales and energy dissipation - moments in channel flow”, J. Fluid. Mech. 701, 419-429 (2012).
6. J. Schumacher, J. D. Scheel, D. Krasnov, D. A. Donzis, V. Yakhot and K. R. Sreenivasan, Proc. Natl. Acad. Sci. USA 111, 10961-10965 (2014).
7. D. Forster, D. Nelson & M.J. Stephen, Phys. Rev. A 16, 732 (1977).
8. C. DeDominisis & P.C. Martin, Phys. Rev. A 19, 419 (1979).
9. V. Yakhot & S.A. Orszag, Phys. Rev. Lett. 57, 1722 (1986).
10. V. Yakhot & L. Smith, J. Sci. Comp. 7, 35 (1992).
11. V. Yakhot, S.A. Orszag, T. Gatski, S. Thangam & C. Speciale, Phys. Fluids A 4, 1510 (1992).
12. W.P. Dannevik, V. Yakhot & S.A. Orszag, Phys. Fluids 30, 2021 (1987).
13. R. Rubinstein & M. Barton, Phys. Fluids, A 12, 1472 (1990); H. Chen, S.A. Orszag, I. Staroselsky & S. Succi, J. Fluid Mech. 519, 301 (2004).
14. H.W. Wyld, Annals of Physics 14, 143-165 (1961).
15. G.K. Batchelor and I. Proudman, Trans. R. So. Lond. A 248, 369-405 (1956).
16. G. Compte-Bellot and S. Corrsin, “The use of a contraction to improve the isotropy of grid-generated turbulence”, J. Fluid Mech 25, 657-682 (1966); A.S. Monin and A.M. Yaglom, “Statistical Fluid Mechanics”, The MIT Press, v2, Cambridge, MA., 1975.
17. T. Ishida, P.A. Davidson and Y. Kaneda, J. Fluid Mech. 564, 455-475 (2006).