SKEW BRACES OF SIZE \( p^2q \)

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Abstract. In this paper we enumerate the skew braces of size \( p^2q \) for \( p, q \) odd primes by the classification of regular subgroups of the holomorph of the groups of size \( p^2q \). In particular, we provide explicit formulas for the skew braces of abelian type.

1. Introduction

The study of the solutions to the set-theoretical Yang–Baxter equation (YBE) has started with \cite{12} as a discrete version of the braid equation. We say that for a given set \( X \) and a function \( r : X \times X \to X \times X \), the pair \((X, r)\) is a set-theoretical solution to the Yang–Baxter equation if

\[
(id_X \times r)(r \times id_X)(id_X \times r) = (r \times id_X)(id_X \times r)(r \times id_X)
\]

holds. A particular family of solutions is the family of non-degenerate solutions, i.e. solutions obtained as

\[
r : X \times X \to X \times X, \quad (x, y) \mapsto (\sigma_x(y), \tau_y(x)),
\]

where \( \sigma_x, \tau_x \) are permutations of \( X \) for every \( x \in X \). A particular case is given by non-degenerate involutive solutions, that is \( r^2 = id_{X \times X} \). Non-degenerate solutions have been studied by several different authors \cite{13, 14, 16, 22}. Rump introduced the notion of brace, a binary algebraic structure providing examples of non-degenerate involutive solutions to YBE. These algebraic structures have been generalized later to skew (left) braces by Guarnieri and Vendramin in \cite{15}, which provide non-involutive non-degenerate solutions.

A skew (left) brace is a triple \((A, +, \circ)\) where \((A, +)\) and \((A, \circ)\) are groups (not necessarily abelian) such that

\[
a \circ (b + c) = a \circ b - a + a \circ c
\]

holds for every \( a, b, c \in A \). Braces are skew braces for which the additive group is abelian and a skew brace \((A, +, \circ)\) is said to be a bi-skew brace if also \((A, \circ, +)\) is a skew brace \cite{10}.

The classification of skew braces is strictly related to the problem of finding non-degenerate solutions to \cite{1}. Indeed, given an involutive non-degenerate solution as in \cite{2}, the group generated by \( \{\sigma_x : x \in X\} \) has a canonical structure of brace and in \cite{3} it has been shown how to recover all the solutions with a given associated brace. Later, this approach was extended to skew braces in \cite{4}, therefore, in a sense, the study of the solution of \cite{1} can be reduced to the classification of skew braces, since there is a way to construct all the solutions with an associated skew brace structure.

Recent progress on the classification problem for (skew) braces are, for instance: the classification of braces with cyclic additive group \cite{19, 20}, skew braces of size \( pq \) for \( p, q \) different primes \cite{1}, braces of size \( p^2q \) for \( p, q \) primes with \( q > p + 1 \) \cite{11}, braces of order \( p^2, p^3 \) where \( p \) is a prime \cite{3}, skew braces of order \( p^3 \) \cite{17} and skew braces of squarefree size \cite{2}.

In this paper we enumerate the skew braces of size \( p^2q \) where \( p, q \) are odd primes. We also provide an explicit formula for braces.

Key words and phrases. Yang-Baxter equation, set-theoretic solution, skew brace, Hopf–Galois extensions.
Skew braces are also related to Hopf-Galois extensions as explained in [21], since they are both in connection with regular subgroups of the holomorph of a group. The results of this paper have already been partially proved in [7] (the authors claim that the missing cases are the subject of a second paper in preparation), through the connection between skew braces and Hopf-Galois extensions.

We employ the same techniques we used in [1], which are based on the algorithm for the construction of skew braces with a given additive group developed in [15]. Indeed, the algorithm allows to obtain all the skew braces with additive group $A$ from regular subgroups of its holomorph $\text{Hol}(A) = A \rtimes \text{Aut}(A)$. The isomorphism classes of skew braces are parametrized by the orbits of such subgroups under the action by conjugation of the automorphism group of $A$ in $\text{Hol}(A)$ [15, Section 4].

In particular, we enumerate the skew braces according to their additive group, and the structure of the paper is displayed in the following table. The classification of the groups of size $p^2q$ can be found in [6] and the description of their automorphism groups in [8].

| $p = 1 \pmod{q}$         | Groups                        | Sections |
|--------------------------|-------------------------------|----------|
| $\mathbb{Z}_{p^2q}$      |                               | 3.1      |
| $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ |                               | 3.2      |
| $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_q$ | 3.3      |
| $G_k$                     |                               |          |

| $p = -1 \pmod{q}$        | $\mathbb{Z}_{p^2q}$          | 4.1      |
| $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | 4.2      |
| $G_F$                    |                               | 4.3      |

| $q = 1 \pmod{p}$, $q \neq 1 \pmod{p}$ | $\mathbb{Z}_{p^2q}$ | 5.1      |
|                                       | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | 5.2      |
|                                       | $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$ | 5.3      |
|                                       | $\mathbb{Z}_q \rtimes \mathbb{Z}_p^2$ | 5.4      |

| $q = 1 \pmod{p^2}$          | $\mathbb{Z}_{p^2q}$          | 6.1      |
|                            | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | 6.2      |
|                            | $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p^2)$ | 6.3      |
|                            | $\mathbb{Z}_q \rtimes \mathbb{Z}_p^2$ | 6.4      |
|                            | $\mathbb{Z}_q \times \mathbb{Z}_p^2$ | 6.5      |

| $p, q$ algebraically independent | $\mathbb{Z}_{p^2q}$ and $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | 7        |

The enumeration results are collected in suitable tables in each section.

2. Preliminaries

A triple $(A, +, \circ)$ is said to be a skew (left) brace if both $(A, +)$ and $(A, \circ)$ are groups and

$$a \circ (b + c) = a \circ b + a \circ c$$

holds for every $a, b, c \in A$. Following the standard terminology for Hopf–Galois extensions, if $\chi$ is a group theoretical property, we say that a skew brace $(A, +, \circ)$ is of $\chi$-type if $(A, +)$ has the property $\chi$.

A bi-skew brace is a skew brace $(A, +, \circ)$ such that $(A, \circ, +)$ is also a skew brace (see [10]). Equivalently

$$x + (y \circ z) = (x + y) \circ x' \circ (x + z)$$

holds for every $x, y, z \in A$, where $x'$ denotes the inverse of $x$ in $(A, \circ)$.

Given a skew brace $(A, +, \circ)$, the mapping

$$\lambda : (A, \circ) \to \text{Aut}(A, +), \quad \lambda_a(b) = -a + a \circ b,$$
Lemma 2.1. Let \((A,+,\circ)\) be a skew brace. Then \(\ker \lambda\) is a subgroup of \((A,+)\).

Proof. Let \(a, b \in \ker \lambda\). Then
\[
\lambda_{a+b} = \lambda_{a} + \lambda_{b} = \lambda_{a} + \lambda_{b} = \lambda_{a} \lambda_{b} = 1, \quad \lambda_{-a} = \lambda_{\lambda_{a}(a')} = \lambda_{a}^{-1} = 1.
\]
Therefore \(\ker \lambda\) is a subgroup of \((A,+)\).

Let \((A,+,\circ)\) be a skew brace. A subgroup \(I\) of \((A,+)\) is a left ideal of \(B\) if \(\lambda_{a}(I) \leq I\) for every \(a \in A\). Every left ideal is a subgroup of \((I,\circ)\). If \(I\) is a normal subgroup of both \((A,+)\) and \((A,\circ)\) then \(I\) is an ideal.

Lemma 2.2. Let \((A,+,\circ)\) be a finite skew brace of size \(p^2q\). Then

(i) if \(p = \pm 1 \pmod{q}\) then the unique \(p\)-Sylow subgroup of \((A,+)\) is an ideal. In particular, the \(p\)-Sylow subgroup of \((A,+)\) and \((A,\circ)\) are isomorphic.

(ii) if \(q = 1 \pmod{p}\) then the unique \(q\)-Sylow subgroup of \((A,+)\) is an ideal.

Proof. The groups of size \(p^2q\) with \(p = \pm 1 \pmod{q}\) has a unique \(p\)-Sylow subgroup, therefore it is characteristic. If we denote by \(P\) the unique \(p\)-Sylow subgroup of \((A,+)\), we have that for every \(x \in A\), \(\lambda_{x}(P) = P\). Since it has order \(p^2\), \(P\) is the unique \(p\)-Sylow subgroup of \((A,\circ)\). Then it is an ideal. The same argument applies for (ii).

2.1. Skew braces and regular subgroups.

From [15], we know that given a group \((A,+)\) (not necessarily abelian) we have a bijective correspondence between isomorphism classes of skew braces \((A,+,\circ)\) and orbits of regular subgroups of \(\text{Hol}(A,+)\) under conjugation by \(\text{Aut}(A,+)\) identified with the subgroup \(\{1\} \times \text{Aut}(A) \leq \text{Hol}(A,+)\). If \(G\) is a regular subgroup of \(\text{Hol}(A,+)\), then it is easy to verify that \(\pi_{1}|_{G} : G \rightarrow A\) is a bijective map.

Theorem 2.3. [15] Theorem 4.2, Proposition 4.3] Let \((A,+)\) be a group. If \(\circ\) is an operation such that \((A,+,\circ)\) is a skew brace, then \(\{(a,\lambda_{a}) : a \in A\}\) is a regular subgroup of \(\text{Hol}(A,+)\). Conversely, if \(G\) is a regular subgroup of \(\text{Hol}(A,+)\), then \(A\) is a skew brace with
\[
a \circ b = a + f(b)
\]
where \((\pi_{1}|_{G})^{-1}(a) = (a,f) \in G\) and \((A,\circ) \cong G\).

Moreover, isomorphism classes of skew braces over \(A\) are in bijective correspondence with the orbits of regular subgroups of \(\text{Hol}(A)\) under the action of \(\text{Aut}(A)\) by conjugation.

The classification of groups of size \(p^2q\), where \(p,q\) are odd primes, is given in [6] and the description of their automorphism in [8]. So, we can construct all skew braces with additive group isomorphic to \(A\) by finding representatives of the orbits of regular subgroups under the action by conjugation of \(\text{Aut}(A)\) on \(\text{Hol}(A)\) using Theorem 2.3.

In some cases we will provide an explicit formula for the \(\circ\) operation. The connection between a regular group and the multiplicative group structure of the associated skew braces is the following.

Remark 2.4. Let \((A,+)\) be a group and \(G\) a regular subgroup of \(\text{Hol}(A)\). According to Theorem 2.3 \((A,+,\circ)\) where
\[
a \circ b = a + \pi_{2}((\pi_{1}|_{G})^{-1}(a))(b)
\]
for every \(a, b \in A\) is a skew brace. In other words, \(\lambda_{a} = \pi_{2}(\pi_{1}^{-1}|_{G}(a))\) and so \(|\ker \lambda| = \frac{|G|}{|\pi_{2}(G)|}\).

The following lemma will be used to compute the formulas of the multiplicative operation of skew braces through all the paper.
Lemma 2.5. Let \( A \) be a skew brace such that \( A = \ker \lambda + \mathrm{Fix}(A) \). Then
\[
(a + b) \circ (c + d) = a + b + \lambda_b(c) + d
\]
for every \( a, c \in \ker \lambda \) and every \( b, d \in \mathrm{Fix}(A) \). In particular, \( \lambda_{nb} = \lambda_b^n \) for every \( n \in \mathbb{N} \).

Proof. Let \( a, c \in \ker \lambda \) and \( b, d \in \mathrm{Fix}(A) \). Then
\[
(a + b) \circ (c + d) = a + b + \lambda_{ob}(c + d) = a + b + \lambda_b(c + d) = a + b + \lambda_b(c) + d.
\]
The last statement follows since \( b \circ b \circ \ldots \circ b = nb \) for every \( b \in \mathrm{Fix}(A) \) and every \( n \in \mathbb{N} \). \[\square\]

2.2. Regular subgroups. Recall that a group of permutations \( G \) acting on a set \( X \) is said to be regular if, for any \( x, y \in X \), there exist a unique \( g \in G \) such that \( g(x) = y \). The holomorph of a group \( A \) can be embedded in the group of permutations on \( A \). In fact, an element \((a, f) \in \text{Hol}(A)\) acts on \( A \) by \((a, f) \cdot x = a + f(x)\) for all \( x \in A \) where the operation on \( A \) is written additively. So, a subgroup \( G \) of \( \text{Hol}(A) \) is said to be regular if the image of \( G \) in the group of permutations on \( A \) is regular as above. It is easy to see that a subgroup \( G \leq \text{Hol}(A) \) is regular if and only if \( G \cdot 1_A = A \) and \(|G| = |A|\), where \( 1_A \) is the identity element of \( A \).

In this section we describe the general method to search for regular subgroups of the holomorph of a given finite group \( A \), inspired by [18 Section 2.2] and we fix some notation and terminology. Let
\[
\pi_1 : \text{Hol}(A) \longrightarrow A, \quad (a, f) \mapsto a \quad \text{and} \quad \pi_2 : \text{Hol}(A) \longrightarrow \text{Aut}(A), \quad (a, f) \mapsto f
\]
the canonical surjections. The map \( \pi_1 \) controls the regularity of a subgroup \( G \leq \text{Hol}(A) \) with \(|G| = |A|\). Indeed, it is easy to check that the orbit of \( 1 \in A \) with respect to the action of \( G \) is given by \( \pi_1(G) \). Therefore, for \( G \leq \text{Hol}(A) \) with \(|G| = |A|\), \( G \) is regular if and only if \( \pi_1(G) = A \). In particular, if \( G \) is regular, then \( G \cap 1 \times \text{Aut}(A) = 1 \).

The map \( \pi_2 \) is a group homomorphism and we are going to search for regular subgroups of \( \text{Hol}(A) \) according to the size of their image under \( \pi_2 \). The size of the image of \( \pi_2 \) divides both \(|(A, +)|\) and \(|\text{Aut}(A, +)|\), so it divides their greatest common divisor. If the image under \( \pi_2 \) is trivial, the associated skew brace is the unique trivial skew brace over \( A \).

We are going to apply the following two-step strategy to compute a set of representative of conjugacy classes of regular subgroups of \( \text{Hol}(A) \).

Let assume that the image under \( \pi_2 \) has size \( k \). The first step is to provide a list of non-conjugate regular subgroups with size of image under \( \pi_2 \) equal to \( k \). In order to show that two groups are not conjugate we are using some properties which are invariants under conjugation. The first one is the conjugacy class of the image under \( \pi_2 \) in \( \text{Aut}(A) \). Once we fix this first invariant, we can also fix the kernel of \( \pi_2 \) up to the action of the normalizer of the image of \( \pi_2 \). Therefore, for the first step we need to:

- find the conjugacy classes of subgroups of size \( k \) of \( \text{Aut}(A, +) \). In case \( \text{Aut}(A) \) is abelian, we have to compute all its the subgroups of the desired size.
- According to Lemma 2.1, the kernel of \( \pi_2 \) is a subgroup of \( A \). So we need to find a set of representatives of the orbits of subgroups of size \( \frac{|A|}{k} \) of \( A \), under the action of the normalizer of the image of \( \pi_2 \) on \( A \).

In order to show that two groups with the same image under \( \pi_2 \) and the same kernel are not conjugate, sometimes we need to introduce some ad-hoc invariant, which depends on the particular structure of \( A \) and its automorphisms.

The second step is to show that any regular subgroup is conjugate to one in the list computed in step one. In particular, we need to describe the subgroups of \( \text{Hol}(A) \) according to the possible values of the invariants mentioned above.
Let \( \{ \alpha_i : 1 \leq i \leq n \} \) be a set of generators of \( \pi_2(G) \) and \( \{ k_j : 1 \leq j \leq m \} \) be a set of generators of the kernel of \( \pi_2|_G \) then the regular subgroup \( G \) can be presented as follows:

\[
G = \langle k_1, \ldots, k_m, u_1\alpha_1, \ldots, u_n\alpha_n \rangle,
\]

for some \( u_i \in A \). In particular, \( u_i \neq 1 \), since \( G \) is regular (indeed, \( u_i\alpha_i \cdot 1 = u_i \neq 1 \)). We will call this presentation the standard presentation of \( G \). Note that we can choose any representative of the coset of \( u_i \) with respect to the kernel, without changing the group \( G \) (this is equivalent to multiply on the left the elements \( u_i\alpha_i \) by elements of the kernel). Moreover, any generator can be multiplied by any element in \( G \), without changing the group. We will use these operations to make computations easier without further explanation.

The group \( G \) has to satisfy the following necessary conditions, that will provide constrains over the choice of the elements \( u_i \):

(K) The kernel of \( \pi_2|_G \) is normal in \( G \).
(R) The generators \( \{ u_i\alpha_i : 1 \leq i \leq n \} \) satisfy the same relations as \( \{ \alpha_i : 1 \leq i \leq n \} \) modulo \( \ker \pi_2|_G \) (e.g. if \( \alpha_i^p = 1 \) then \( (u_i\alpha_i)^n \in \ker \pi_2|_G \)).

Given one of such groups, we can conjugate by the normalizer of \( \pi_2(G) \) in \( \text{Aut}(A) \) that stabilizes the kernel of \( \pi_2|_G \) in order to show that \( G \) is conjugate to one of the chosen representatives.

3. Skew braces of size \( p^2q \) with \( p = 1 \pmod{q} \)

In this section we assume that \( p \) and \( q \) are odd primes such that \( p = 1 \pmod{q} \), unless we explicitly state otherwise. We denote by \( \mathcal{B} \) the subset of \( \mathbb{Z}_q \) that contains \( 0, 1, -1 \) and one out of \( k \) and \( -1 \) for \( k \neq 0, 1, -1 \). In particular, \( |\mathcal{B}| = \frac{q-1}{2} \). We denote by \( g \) a fixed element of order \( q \) in \( \mathbb{Z}_p^\times \) and by \( t \) a fixed element of order \( q \) in \( \mathbb{Z}_p^\times \).

The following lemma collects all the isomorphism classes of groups of size \( p^2q \).

**Lemma 3.1.** [3] Proposition 21.17] The groups of size \( p^2q \) are the following:

(i) \( \mathbb{Z}_{p^2} \times \mathbb{Z}_q = \langle \sigma, \tau \mid \sigma^p = \tau^q = 1, \sigma \tau = \tau \sigma \rangle. \)

(ii) \( \mathbb{Z}_p \times \mathbb{Z}_q = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^q = \epsilon^q = 1, [\sigma, \tau] = [\sigma, \epsilon] = [\tau, \epsilon] = 1 \rangle. \)

(iii) \( \mathbb{Z}_{p^2} \rtimes_\mathbb{Z}_q = \langle \sigma, \tau \mid \sigma^p = \tau^q = 1, \tau \sigma \tau^{-1} = \sigma^t \rangle. \)

(iv) \( \mathcal{G}_k = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^q = \epsilon^q = 1, \epsilon \sigma \epsilon^{-1} = \sigma^q, \epsilon \tau \epsilon^{-1} = \tau^q \rangle \cong \mathbb{Z}_p^2 \times_{\mathcal{D}_k} \mathbb{Z}_q \), where \( \mathcal{D}_k \) is the diagonal matrix with eigenvalues \( g, g^k \) for \( k \in \mathcal{B} \).

In particular, there are \( \frac{q+1}{2} \) isomorphism classes of such groups.

**Remark.** The groups in item (iv) of Lemma 3.1 are parametrized by the set \( \mathcal{B} \) since \( \mathcal{G}_k \) and \( \mathcal{G}_{k-1} \) are isomorphic. We will take into account such isomorphism for the enumeration of skew braces according to the isomorphism class of their additive and multiplicative groups.

The following table collects the enumeration of skew braces according to their additive and multiplicative groups.
3.1. **Skew braces of cyclic type.** In this section we will denote by $A$ the cyclic group $\mathbb{Z}_{p^2q}$. The automorphism group of $A$ is isomorphic to $\mathbb{Z}_{p^2}^\times \times \mathbb{Z}_q^\times$. We denote by $\varphi_{i,j}$ the automorphism given by
\[
\sigma \mapsto \sigma^i \quad \text{and} \quad \tau \mapsto \tau^j
\]
where $i \in \mathbb{Z}_{p^2}^\times$ and $j \in \mathbb{Z}_q^\times$.

The size of $\text{Aut}(A)$ is $p(p-1)(q-1)$, so if $G$ is a regular subgroup of $\text{Hol}(A)$, then $|\pi_2(G)|$ belongs to $\{q,p,pq\}$. First, we see that there are not regular subgroup with $|\pi_2(G)| = pq$.

**Proposition 3.2.** Let $G$ be a regular subgroup of $\text{Hol}(A)$. Then $|\pi_2(G)| \neq pq$.

**Proof.** The unique subgroup of $\text{Aut}(A)$ of size $pq$ is generated by $\alpha = \varphi_{g(p+1),1}$ and the unique subgroup of order $p$ of $A$ is $\langle \sigma^p \rangle$. Assume that $G$ is a subgroup of size $p^2q$ of $\text{Hol}(A)$ with $\pi_2(G) = \langle \alpha \rangle$. Then, the standard presentation of $G$ is
\[
G = \langle \sigma^p, \sigma^a \tau^b \alpha \rangle
\]
for some $a,b$. An element of $G$ is
\[
\sigma^a \tau^b \alpha = (\sigma^p)^m (\sigma^a \tau^b \alpha)^m = \sigma^{mn+a \frac{m(p+1)m-1}{2}} \tau^b m \alpha^m.
\]
If $b = 0$ then $\pi_1(G) \subseteq \langle \sigma \rangle$, so $G$ is not regular. If $m = 0 \pmod{q}$ then $q^m(p+1)^m-1 = 0 \pmod{p}$. So, the elements $\sigma^i$ with $i \neq 0 \pmod{p}$ are not in $\pi_1(G)$, therefore $G$ is not regular. \hfill \Box

The following proposition is more general than the case $p = 1 \pmod{q}$ and it will be used in the sequel. The only condition needed is $q \neq 1 \pmod{p}$, which is indeed the case of this section.

**Proposition 3.3.** Let $p,q > 2$ be primes with $q \neq 1 \pmod{p}$. The unique skew brace of cyclic type with $|\ker \lambda| = pq$ is $(B,+,\circ)$ where
\[
\binom{n}{m} \circ \binom{s}{r} = \binom{n + s}{m + r}, \quad \binom{n}{m} \circ \binom{s}{r} = \binom{n + s + pns}{m + r}
\]
for every $0 \leq n,s \leq p^2-1$ and $0 \leq m,r \leq q-1$. In particular, $(B,\circ) \cong \mathbb{Z}_{p^2q}$.

**Proof.** The unique subgroup of size $p$ of $\text{Aut}(A)$ is the subgroup generated by $\varphi_{p+1,1}$ and $A$ has a unique subgroup of size $pq$, namely $\langle \sigma^p, \tau \rangle$. Assume that $G$ is a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p$. Then $G$ has the following standard presentation:
\[
G = \langle \sigma^p, \tau, \sigma^a \tau^b \varphi_{p+1,1} \rangle = \langle \sigma^p, \tau, \sigma^a \varphi_{p+1,1} \rangle,
\]
where $1 \leq a \leq p-1$ and in particular $G$ is abelian. According to [9, Corollary 4.3], since both the additive and the multiplicative group of the skew brace associated to $G$ are abelian, then such

| $+ \circ$ | $\mathbb{Z}_{p^2q}$ | $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | $G_k$, $k \neq 0,\pm1,2$ | $G_0$ | $G_{-1}$ | $G_1$ | $G_2$
|-------|--------|----------------|----------------|-----------------|--------|--------|--------|--------|
| $\mathbb{Z}_{p^2q}$ | 2 | 1 | - | - | - | - | - | - |
| $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ | 4 | 2(q-1) | - | - | - | - | - | - |
| $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | - | - | 2 | 1 | 1 | 1 | 1 | 2 |
| $G_k$ | - | - | 4 | 8(q+1) | 8(q+1) | 4(q+1) | 4(q-1) | 8(q+1) |
| $G_0$ | - | - | 2 | 4q | 4q | 2q | 2(q-1) | 4q |
| $G_{-1}$ | - | - | 3 | 4q + p + 2 | 4q + p + 2 | 3q + p - 1 | 2(q-1) | 4q + p |
| $G_1$ | - | - | 5 | 3(q + 2) | 4(q+1) | 2(q+1) | 3q - 1 | 6q |

**Table 1.** Total number of skew braces of size $p^2q$ according to the additive and multiplicative isomorphism class of groups where $p = 1 \pmod{q}$. Note that for $q = 3$ we have that $\mathfrak{A} = \{0,-1,-1\}$, $2 = -1$ and so the table has to be read accordingly.
skew brace splits as a direct product of the skew brace of size $p^2$ and a trivial skew brace of size $q$. According to the classification of skew braces of size $p^2$ given in [3] Proposition 2.4, there is just one non-trivial such brace with cyclic additive group and it leads to formula (4). The group $(B, \circ)$ is cyclic, since according to Lemma 2.2 its $p$-Sylow subgroup is cyclic.

**Proposition 3.4.** The unique skew brace of cyclic type with $|\ker \lambda| = p^2$ is $(A, +, \circ)$ where

$$\left( \begin{array}{c} n \\ m \end{array} \right) + \left( \begin{array}{c} s \\ r \end{array} \right) = \left( \begin{array}{c} n + s \\ m + r \end{array} \right), \quad \left( \begin{array}{c} n \\ m \end{array} \right) \circ \left( \begin{array}{c} s \\ r \end{array} \right) = \left( \begin{array}{c} n + t^m s \\ m + r \end{array} \right),$$

for every $0 \leq n, s \leq p^2 - 1$ and $0 \leq m, r \leq q - 1$. In particular, $(A, \circ) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and $(A, +, \circ)$ is a bi-skew brace.

**Proof.** The unique subgroup of order $q$ of Aut($A$) is the subgroup generated by $\varphi_{t,1}$. The unique subgroup of order $p^2$ of $A$ is generated by $\sigma$. Let $G$ be a regular subgroup with $|\pi_2(G)| = q$, then $G$ has the following standard presentation:

$$G_b = \langle \sigma, \tau b \varphi_{t,1} \rangle,$$

with $1 \leq b \leq q - 1$. The group $G_b$ is conjugate to the subgroup

$$H = \langle \sigma, \tau \varphi_{t,1} \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$$

by $\varphi_{t,1}^{-1}$. It is easy to check that $H$ is regular. Therefore there exist a unique skew brace with the desired properties up to isomorphism.

Also, according to [1 Corollary 1.2], we have that (5) defines a bi-skew brace.

In the following table we summarize the results of this subsection. The columns refer to the isomorphism class of the multiplicative group.

| $|\ker \lambda|$ | $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$ | $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$ |
|------------------|------------------------|------------------------|
| $pq$ | 1 | - |
| $p^2$ | - | 1 |
| $p^2q$ | 1 | - |

Table 2. Number of skew braces of cyclic type of size $p^2q$ for $p = 1 \pmod{q}$.

### 3.2. Skew braces of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$-type.

In this section $A$ will denote the group $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ with presentation:

$$\langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, [\sigma, \tau] = [\sigma, \epsilon] = [\tau, \epsilon] = 1 \rangle.$$

The automorphism group of $A$ is isomorphic to $GL_2(p) \times \mathbb{Z}_q^\times$ and it has size $p(p-1)^2(p+1)(q-1)$. For a fixed basis of $\mathbb{Z}_p^2$ we can think of the automorphisms of $A$ as $M\alpha$ where $M$ is an invertible matrix in such basis and $\alpha \in \mathbb{Z}_q^\times$.

If $G$ is a regular subgroup of Hol($A$) then $|\pi_2(G)|$ divides $p^2q$ and $|\text{Aut}(A)|$. So $|\pi_2(G)|$ divides $pq$ since $p = 1 \pmod{q}$.

**Remark 3.5.** Then the automorphism of $A$ of order $p, q$ or $pq$ acts trivially on it $q$-Sylow subgroup, since $p$ and $q$ does not divide $q - 1$. Hence, if $G$ is a regular group of size $p^2q$ then the action of $\pi_2(G)$ on the $q$-Sylow subgroup of $A$ is trivial.

**Remark 3.6.** It is a well-know fact that, up to conjugation the subgroups of order $q$ of $GL_2(p)$ are generated by one of the matrices

$$D_s = \left[ \begin{array}{cc} g & 0 \\ 0 & g^s \end{array} \right], \quad \bar{D} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & g \end{array} \right],$$
for \( s \in \mathcal{B} \).

The subgroups of order \( p \) of \( GL_2(p) \) are the \( p \)-Sylow subgroups, and then they are all conjugate to the group generated by

\[
C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

A set of representatives of conjugacy classes of subgroups of order \( pq \) of \( GL_2(p) \) is given by

\[
H_s = \langle C, D_s \rangle \text{ and } \bar{H} = \langle C, \bar{D} \rangle, \text{ where } 0 \leq s \leq q - 1.
\]

Indeed, up to conjugation, we can assume that the \( p \)-Sylow subgroup of a group \( H \) of size \( pq \) is generated by \( C \). Then the generator of order \( q \) is an upper triangular matrix, since it is in the normalizer of \( C \) and then we can assume that \( H \) is generated by \( C \) and by diagonal matrix of order \( q \) which can be taken to be \( D_s \) for \( 0 \leq s \leq q - 1 \) or \( \bar{D} \). Such matrices are not conjugate by the elements of the normalizer of \( C \) and so the correspondent groups are not conjugate.

In the following proposition we assume once more that \( q \neq 1 \) (mod \( p \)) as we did in Proposition 3.5. This is the case of this Section.

**Proposition 3.7.** Let \( p, q \) be primes such that \( q \neq 1 \) (mod \( p \)). The unique skew brace of \( A \)-type with \( |\ker \lambda| = pq \) is \((B, +, \circ)\) where

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix},
\]

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + x_2 y_2 \\ x_2 + y_2 + x_3 y_3 \\ x_3 + y_3 \end{pmatrix}
\]

for every \( 0 \leq x_1, x_2, y_1, y_2 \leq p - 1, 0 \leq x_3, y_3 \leq q - 1 \). In particular, \((B, \circ) \cong A\).

**Proof.** Let \( G \subseteq \text{Hol}(A) \) be a regular subgroup and let \( |\pi_2(G)| = p \). According to Remarks 3.5 and 3.6, we can assume that \( \pi_2(G) \) is generated by \( C \), up to conjugation. Therefore the group \( G \) has the following standard presentation:

\[
G = \langle v, \epsilon, \sigma^a \tau^b C \rangle
\]

for some \( a, b \). The condition (K) implies that \( v \in \langle \sigma \rangle \) and then \( G \) is abelian. Therefore both the additive and the multiplicative group of the skew braces obtained by \( G \) are abelian, then according to [3, Corollary 4.3] such skew braces split as a direct product of a skew brace of size \( p^2 \) and a trivial skew brace of size \( q \). According to the classification of skew braces of size \( p^2 \) given in [3, Proposition 2.4], there is just one non-trivial such brace and it leads to formula (6). According to Lemma 2.2, the group \((B, \circ)\) is isomorphic to \( A \) since its \( p \)-Sylow subgroup is elementary abelian.

**Lemma 3.8.** The skew braces of \( A \)-type with \( |\ker \lambda| = p^2 \) are \( A_s = (A, +, \circ) \) for \( s \in \mathcal{B} \) where

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix},
\]

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^{x_2} y_1 \\ x_2 + (g^a)^{x_3} y_2 \\ x_3 + y_3 \end{pmatrix}
\]

for every \( 0 \leq x_1, x_2, y_1, y_2 \leq p - 1, 0 \leq x_3, y_3 \leq q - 1 \). In particular \((A_s, \circ) \cong G_s\), and \( A_s \) is a bi-skew brace.

**Proof.** The groups

\[
H_s = \langle \sigma, \tau, \epsilon D_s \rangle \cong G_s,
\]

for \( s \in \mathcal{B} \) are not pairwise conjugated since their image under \( \pi_2 \) are not and \( |\pi_2(H_s)| = q \). Assume that \( h = \sigma^n \tau^m (\epsilon D_s)^l \in H \) is in the stabilizer of 1, i.e., \( \sigma^n \tau^m (\epsilon D_s)^l \cdot 1 = \sigma^n \tau^m (\epsilon D_s)^l \cdot 1 = \sigma^n \tau^m \epsilon^l = 1 \).

Then \( l = n = m = 0 \) and so \( h = 1 \). Since \( |H| = p^2 q \) then \( H \) is regular.

We show that every regular subgroup \( G \) with \( |\pi_2(G)| = q \) is conjugate to one of them. The kernel \( \ker \pi_2|_G \) is the \( p \)-Sylow subgroup of \( A \) and the image of \( \pi_2|_G \) is generated by an automorphism of
order $q$. Then, according to Remarks 3.5 and 3.6 we can assume that the automorphism is given by $D_s$ for $s \in \mathcal{B}$. Therefore
\[ G = \langle \sigma, \tau, e^a D_s \rangle \]
for some $a \neq 0$ since $G$ is regular. Then $G$ is conjugate to $H_s$ by the automorphism of $\mathbb{Z}_q$ mapping $e$ to $e^{-a}$.

Let $A_s$ be the skew brace associated to $H_s$. Then $e \in \text{Fix}(A_s)$ and so, $A_s = \ker \lambda + \text{Fix}(A_s)$. Therefore we can apply Lemma 2.5 and so $\lambda_{a+1} \tau \in \mathbb{C} \epsilon \lambda_{a+2} \in \mathbb{C} \epsilon \lambda_{a+3} \in \mathbb{C} \epsilon \lambda_{a+4}$ for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1, 0 \leq x_3, y_3 \leq q-1$. Then (7) follows and the skew braces $A_s$ are bi-skew braces according to [1, Proposition 1.1], since $(A, +) = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ and $(A, \circ) = \mathbb{Z}_p^2 \times \mathbb{Z}_q$.

\[ \square \]

**Lemma 3.9.** The unique skew brace of $A$-type with $|\ker \lambda| = p$ is $(B, +, \circ)$ where
\[ (x_1, y_1) + (y_2, y_2) = (x_1 + y_1, x_2 + y_2), \quad (x_1, y_1) \circ (y_2, y_2) = (x_1 + g^{x_2} y_1 + g^{2-1} x_2 y_2, x_2 + g^{2-1} x_2 y_2) \]
for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1, 0 \leq x_3, y_3 \leq q-1$. In particular, $(B, \circ) \cong G_2$.

**Proof.** The image under $\pi_2$ of the group
\[ H = \langle \sigma, \tau C, \epsilon D_{2-1} \rangle \cong G_{2-1} \cong G_2 \]
has size $pq$. Assume that $h = \sigma^n(\tau C)^m(\epsilon D_{2-1})^l = \sigma^n \epsilon^{m(l-1)} \tau^m \epsilon^l D_{2-1}$ is in the stabilizer of 1, i.e. $\sigma^{n+m(l-1)} \tau^m \epsilon^l = 1$. Therefore $n = m = l = 1$, so $h = 1$ and then, since $|H| = p^2 q$ we have that $H$ is regular. Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. According to Remarks 3.5 and 3.6 we can assume that $\pi_2(G) = \langle C, D_s \rangle$ for some $s$. Then the group $G$ has the following standard presentation
\[ G = \langle v, u^a C, \ we^b D_s \rangle, \]
where $v, u, w \in \langle \sigma, \tau \rangle$. Checking condition (R) for $(u^a C)^p \in \ker \pi_2 | G$ we have $a = 0$ and by condition (K) we have $v \in \langle \sigma \rangle$. Therefore
\[ G = \langle \sigma, \tau, \epsilon^a C, \tau^b e^b D_s \rangle, \]
where $a, b \neq 0$ since $G$ is regular. Conjugating by the automorphism $a^{-1} I$ of $\mathbb{Z}_p^2$ and the automorphism $b^{-1}$ of $\mathbb{Z}_q$ we can assume that $a = b = 1$. Therefore
\[ G = \langle \sigma, \tau C, \tau^b e D_s \rangle = \langle \sigma, \tau C, \epsilon C^{-c} D_s \rangle, \]
where the second equality follows by multiplying the last generator by a suitable power of the second one. Since
\[ \epsilon C^{-c} D_s(\tau) D_s C D_s^{-1} C^{-c} \epsilon^{-1} = \tau \epsilon^c C \epsilon^{-1} = (\tau C)^{g^{1-s}} = \tau g^{1-s} C g^{1-s} \quad (\text{mod } \langle \sigma \rangle) \]
then $s = 2^{-1}$. Then $G$ is conjugate to $H$ by $C^n$ where $n = \frac{c}{1-g^{2-1}}$.

Let $B$ be the skew brace associated to the subgroup $H$. Then $e \in \text{Fix}(B)$ and $\sigma \in \ker \lambda$, and so
\[ e \circ e \circ \ldots \circ e = e^{x_3}, \quad \tau \circ \tau \circ \ldots \circ \tau = \sigma^{x_2(x_2-1)/2} \tau^{x_2} = \sigma^{x_2(x_2-1)/2} \circ \tau^{x_2} \]
Hence $\lambda_{a+1} \tau \in \mathbb{C} \epsilon \lambda_{a+2} \in \mathbb{C} \epsilon \lambda_{a+3} \in \mathbb{C} \epsilon \lambda_{a+4}$ for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1, 0 \leq x_3, y_3 \leq q-1$. Then (9) follows.

\[ \square \]

**Remark 3.10.** In order to show that a subgroup of the holomorph is regular we will always check that the stabilizer of the identity is trivial as we did in Corollary 3.8 and Lemma 3.9 and in the sequel we will omit the explicit computations.
The following table summarizes the results of this subsection.

| | \(|\ker \lambda|\) | \(\mathbb{Z}_p^2 \times \mathbb{Z}_q\) | \(G_0\) | \(G_{-1}\) | \(G_{+1}\) | \(G_2\) | \(G_k\), \(k \neq 0, \pm 1, 2^{-1}\) |
|---|---|---|---|---|---|---|---|
| \(p\) | - | - | - | - | 1 | - | |
| \(pq\) | 1 | - | - | - | - | - | |
| \(p^2\) | - | 1 | 1 | 1 | 1 | 1 | |
| \(p^2q\) | 1 | - | - | - | - | 1 | |

Table 3. Number of skew braces of \(\mathbb{Z}_p^2 \times \mathbb{Z}_q\)-type of size \(p^2q\) for \(p = 1 \pmod{q}\).

Note that, for \(q = 3\) we have that \(2 = -1 \pmod{3}\).

3.3. **Skew brace of \(\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q\)-type.** In this section we denote by \(A\) the group \(\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q\) where \(t\) is a fixed element of order \(q\) in \(\mathbb{Z}_{p^2}^\times\).

According to the description of the automorphism group of groups of order \(p^2q\) given in \([8]\) Theorem 3.4], we have that

\[ \phi : \text{Hol}(\mathbb{Z}_{p^2}) = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}^\times \rightarrow \text{Aut}(A), \quad (i, j) \mapsto \varphi_{i,j} = \begin{cases} \tau \mapsto \sigma^j \tau, \\ \sigma \mapsto \sigma^j \end{cases} \]

is an isomorphism of groups. In particular \(|\text{Aut}(A)| = p^3(p - 1)|, and if \(G\) is a regular subgroup of \(\text{Hol}(A)\) then \(|\pi_2(G)|\) divides \(p^2q|\).

The following lemma provides an invariant for subgroups of \(\text{Hol}(A)\) up to conjugation.

**Lemma 3.11.** The group

\[ H = \langle \sigma, \varphi_{1,1} \rangle \]

is normal in \(\text{Hol}(A)\) and \(\text{Hol}(A)/H \cong \mathbb{Z}_q \times \mathbb{Z}_{p^2}^\times\). In particular, \(h \tau^a f h^{-1} = \tau^a f \pmod{H}\) for every \(f, h \in \text{Aut}(A)\).

If \(G\) is a regular subgroup of \(\text{Hol}(A)\) then \(|\pi_2(G)|\) divides \(p^2q\) and \(p^3(p - 1)|\), so divides \(p^2q\). Let us show that such size can not be equal to \(p\).

**Lemma 3.12.** Let \(G\) be a regular subgroup of \(\text{Hol}(A)\). Then \(|\pi_2(G)| \neq p\).

**Proof.** Assume that \(G\) is a subgroup of size \(p^2q\) of \(\text{Hol}(A)\) and \(|\pi_2(G)| = p\). Up to conjugation, we can choose the generators of \(\ker \pi_2|_G\) to be \(\tau\) and \(\sigma^p\). Hence the standard presentation is

\[ G = \langle \tau, \sigma^p, \sigma^a \rangle, \]

where \(\alpha\) is an automorphism of order \(p\) and so \(\alpha(\tau) = \sigma^n \tau\) for some \(n\). By condition (K) we have that

\[ \sigma^a \alpha \tau \alpha^{-1} = \sigma^a \tau = \sigma^{(1-t)a+n} \tau \in \langle \tau, \sigma^p \rangle. \]

Then \((1-t)a\) has to be a multiple of \(p\) and since \(1-t\) is invertible modulo \(p^2\) then also \(a\) is a multiple of \(p\). Hence \(\pi_1(G) \lesssim \langle \sigma^p, \tau \rangle\) and then \(G\) is not regular. \(\square\)

**Lemma 3.13.** The skew braces of \(A\)-type with \(|\ker \lambda| = p^2\) are \(A_s = (A, +, \circ)\) for \(1 \leq s \leq q - 1\) where

\[ (x_1 \quad x_2) + (y_1 \quad y_2) = (x_1 + t x_2 y_1 \quad x_2 + y_2), \quad (x_1 \quad x_2) \circ (y_1 \quad y_2) = (x_1 + t(s+1)x_2 y_1 \quad x_2 + y_2) \]

for every \(0 \leq x_1, x_2 \leq p^2 - 1, 0 \leq y_1, y_2 \leq q - 1\). In particular, \(A_s\) is a bi-skew brace and

\[ (A_s, \circ) \cong \begin{cases} \mathbb{Z}_{p^2q}, & \text{if } s = -1, \\ A, & \text{otherwise}. \end{cases} \]
Proof. According to Remark 3.10 arguing as in Lemma 3.9 it is easy to show that the groups
\[ G_s = \langle \sigma, \tau^s \varphi_{0,t} \rangle \leq \text{Hol}(A) \]
are regular. If \( G_s \) and \( G_{s'} \) are conjugate by an element \( h \in N_{\text{Aut}(A)}(\varphi_{0,t}) \). Then \( h \tau^s \varphi_{0,t} h^{-1} \in G_{s'} \).

According to Lemma 3.11 then \( h \tau^s \varphi_{0,t} h^{-1} = \tau^s \varphi_{0,t} \) (mod \( \langle \sigma, \varphi_{1,1} \rangle \)) and since \( h \) normalizes \( \varphi_{0,t} \), they are equal (mod \( \langle \sigma \rangle \)). Therefore \( \pi_2(\tau^s \varphi_{0,t}) = \varphi_{0,t} \), then \( \tau^s \varphi_{0,t} = \tau^s \varphi_{0,t} \) (mod \( \sigma \)) and so \( s = s' \).

Let \( G \) be a regular subgroup of \( \text{Hol}(A) \) with \( |\pi_2(G)| = q \). Up to conjugation, the unique subgroup of order \( q \) in \( \text{Aut}(A) \) is generated by \( \varphi_{0,q} \) and the unique subgroup of order \( p^2 \) of \( A \) is normal \( p \)-Sylow \( \langle \sigma \rangle \). Then \( G \) is conjugate to \( G_s \) for some \( s \neq 0 \).

For the skew brace \( A_s \) associated to \( G_s \) we have that \( A_s = k \lambda + \text{Fix}(A_s) \). So according to Lemma 2.5 the multiplicative group structure of \( A_s \) is given by \( \langle \sigma, \varphi_{1,1} \rangle \). According to Corollary 1.2, \( A_s \) is a bi-skew brace, since \( (A_s, +) = \mathbb{Z}_{p^2} \times \mathbb{Z}_q \) and \( (A_s, \circ) = \mathbb{Z}_{p^2} \times_{t+1} \mathbb{Z}_q \). \( \square \)

Lemma 3.14. A set of representatives of regular subgroups \( G \) of \( \text{Hol}(A) \) with \( |\pi_2(G)| = p^2 \) is
\[ G_s = \langle \tau, (\sigma^{1/s} \varphi_{1,(p+1)^s}) \rangle \cong \mathbb{Z}_{p^2} q \]
for \( s = 0, 1 \).

Proof. The groups \( G_s \) are regular and they are not conjugate since their images under \( \pi_2 \) are not.

If \( G \) is a regular subgroup of \( \text{Hol}(A) \) with \( |\pi_2(G)| = p^2 \), then \( |\ker \pi_2| = q \) and so \( \ker \pi_2 \langle G \rangle = \langle a^a \tau \rangle \) for some \( a \). So \( G \) has a normal \( q \)-Sylow subgroup and then \( G \) is abelian. According to Lemma 2.2 the \( p \)-Sylow subgroup of the skew brace associated to \( G \) is cyclic and then so it is \( \pi_2(G) \). The unique cyclic subgroups of order \( p^2 \) of \( \text{Aut}(A) \) up to conjugation are \( \langle \varphi_{1,(p+1)^s} \rangle \) for \( s = 0, 1 \). Up to conjugation by \( \varphi_{1,1} \), we can assume that \( \ker \pi_2 \langle G \rangle \) is generated by \( \tau \). Therefore
\[ G = \langle \tau, (\sigma^{1/s} \varphi_{1,(p+1)^s}) \rangle \]
for some \( b \neq 0 \). From the abelianess of \( G \) we have that \( b = \frac{1}{\tau-1} \). \( \square \)

Lemma 3.15. There exists a unique conjugacy class of regular subgroups \( G \) of \( \text{Hol}(A) \) with \( |\pi_2(G)| = pq \). A representative is
\[ H = \langle \sigma^{p+1} \tau^{-1} \varphi_{0,(p+1)} \rangle \cong \mathbb{Z}_{p^2} q \].

Proof. The group \( H \) is regular. Let \( G \) be a regular subgroup of \( \text{Hol}(A) \) with \( |\pi_2(G)| = pq \). Then \( \ker \pi_2 \langle G \rangle = \langle \sigma^p \rangle \). Up to conjugation, the subgroups of size \( pq \) of \( \text{Aut}(A) \) are \( \langle \varphi_{p,1}, \varphi_{0,t} \rangle \) and \( \langle \varphi_{0,t(p+1)} \rangle = \langle \varphi_{0,t}, \varphi_{0,p+1} \rangle \).

In the first case, we have
\[ G = \langle \sigma^p, \sigma^a \tau^b \varphi_{p,1}, \sigma^c \tau^d \varphi_{0,t} \rangle. \]

From the condition (R) for \( (\sigma^a \tau^b \varphi_{p,1})^p \in \ker \pi_2 \langle G \rangle \) it follows that \( b = 0 \). According to Lemma 2.2 the \( p \)-Sylow subgroup of \( G \) is cyclic and the action by conjugation of \( h = \sigma^e \tau^d \varphi_{0,t} \) on it is given by:
\[ h \sigma^p h^{-1} = \sigma^{pe} h \sigma^a \tau^b \varphi_{p,1} h^{-1} = (\sigma^a \varphi_{p,1})^t. \]

Then \( e^{d+1} = t \), so \( d = 0 \) and \( \pi_1 \langle G \rangle \subseteq \langle \sigma \rangle \). Therefore \( G \) is not regular, contradiction.

Let \( \pi_2(G) = \langle \varphi_{0,t}, \varphi_{0,p+1} \rangle \). Then,
\[ G = \langle \sigma^p, \sigma^a \tau^b \varphi_{0,p+1}, \sigma^c \tau^d \varphi_{0,t} \rangle. \]

From condition (K) we have that \( b = 0, d = -1 \) (mod \( q \)), and \( c = 0 \) (mod \( p \)) and so \( a \neq 0 \) (mod \( p \)) since \( G \) is regular. Conjugating by \( \varphi_{0,a^{-1}} \), we can assume that \( a = 1 \). So
\[ G = \langle \sigma^p, \sigma \varphi_{0,p+1}, \tau^{-1} \varphi_{0,t} \rangle. \]

Therefore \( H \leq G \) and since \( |H| = |G| \) equality holds. \( \square \)
Lemma 3.16. A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is

$$G_d = \langle \sigma \tau^d \varphi_{1,1}, \tau^d \varphi_{0,t} \rangle \cong A,$$

where $1 \leq d \leq q - 1$.

**Proof.** It is easy to check that the groups $G_d$ for $1 \leq d \leq q - 1$ have the desired properties. The same argument used in Lemma 3.11 shows that they are not pairwise conjugate.

Let $G$ be a regular subgroup with $|\pi_2(G)| = p^2q$. According to Lemma 2.2, the $p$-Sylow subgroup of the multiplicative group of the skew brace associated to $G$ is cyclic, and then so it is the $p$-Sylow subgroup of $\pi_2(G)$. Up to conjugation we can assume that $\pi_2(G)$ is $\langle \varphi_{1,1}, \varphi_{0,t} \rangle \cong A$, i.e.

$$G = \langle \sigma \tau^h \varphi_{1,1}, \sigma \tau^d \varphi_{0,t} \rangle.$$

From condition (K) we have that the standard presentation of $G$ is

$$\langle \sigma \tau^d \varphi_{0,t} \rangle$$

where $d \neq 0$ since $G$ is regular. If $d = -1$ then from the condition $(\sigma \tau^d \varphi_{0,t})^q = 1$ it follows that $c = 0$. Otherwise, $G$ is conjugate to $G_d$ by $\varphi_{n,1}$ where $n = \frac{c(t-1)}{1-\tau^{d+1}}$. \hfill $\square$

We summarize the contents of this subsection in the following table.

| $|\ker \lambda|$ | $\mathbb{Z}_{p^2q}$ | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
|------------------|-----------------|------------------|
| 1                | -               | $q - 1$          |
| $p$              | 1               | -                |
| $q$              | 2               | -                |
| $p^2$            | 1               | $q - 2$          |
| $p^2q$           | -               | 1                |

Table 4. Number of skew braces of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$-type of size $p^2q$ for $p = 1 \pmod{q}$.

3.4. Skew brace of $G_k$-type for $k \neq 0, \pm 1$. In this section we assume that $k \neq 0, 1, -1$. Recall that a presentation of the group $G_k$ is the following

$$(11) \quad G_k = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, \epsilon \sigma \epsilon^{-1} = \sigma^q, \epsilon \tau \epsilon^{-1} = \tau^q \rangle.$$

An automorphism of $G_k$ is determined by its action on the generators, i.e. by its restriction to $\langle \sigma, \tau \rangle$ given by a matrix and by the image on $\epsilon$. According to [8] Subsections 4.1, 4.2], the mapping

$$\phi : \mathbb{Z}_p^2 \times_{\rho} (\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times) \rightarrow \text{Aut}(G_k),$$

$$[(n, m), (a, b)] \mapsto h = \left\{ \begin{array}{cc} h|_{(\sigma, \tau)} = & \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \\ \epsilon \rightarrow \sigma^n \tau^m \epsilon, \end{array} \right. \vspace{1cm}$$

where $\rho(a, b)(n, m) = (an, bm)$ is a group isomorphism. In particular, $|\text{Aut}(G_k)| = p^2(p - 1)^2$ and the unique $p$-Sylow subgroup of $\text{Aut}(G_k)$ is generated by $\alpha_1 = [(1, 0), (1, 1)]$ and $\alpha_2 = [(0, 1), (1, 1)]$.

The following lemma is analogous to Lemma 3.11.

**Lemma 3.17.** The group

$$H = \langle \sigma, \tau, \alpha_1, \alpha_2 \rangle$$

is normal in $\text{Hol}(G_k)$ and $\text{Hol}(G_k)/H \cong \mathbb{Z}_q \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. In particular, $he^a fh^{-1} = e^af$ (mod $H$) for every $f, h \in \text{Aut}(G_k)$. \vspace{1cm}
Let $G$ be a regular subgroup of $\text{Hol}(G_k)$. Then $|\pi_2(G)|$ divides $p^2q$ and $|\text{Aut}(G_k)|$ and we need to discuss all possible cases $(p, p^2, q, pq, p^2q)$. 

**Proposition 3.18.** The skew braces of $G_k$-type with $|\ker\lambda| = p^2$ are $A_{a,b} = (A, +, \circ)$ for $0 \leq a, b \leq q - 1$ : $(a, b) \neq (0, 0)$, where 
\[
\begin{align*}
(A_{a,b}, \circ) &\cong \begin{cases} 
\mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = -1, b = -k, \\
G_0, & \text{if } a = -1, b \neq -k, \\
G_{a+k}, & \text{otherwise.}
\end{cases}
\end{align*}
\]

Proof. The groups $G_{a,b} = \langle \sigma, \tau, \epsilon \gamma_1^{a}, \epsilon \gamma_2^{b} \rangle$ where $\gamma_1 = [(0, 0), (g, 1)]$ and $\gamma_2 = [(0, 0), (1, g)]$ are regular. If $G_{a,b}$ and $G_{c,d}$ are conjugate by $h \in \text{Aut}(G_k)$, then $h(\gamma_1^{a}, \gamma_2^{b})h^{-1} = (\gamma_1^{c}, \gamma_2^{d})$ and $h\epsilon \gamma_1^{a} \gamma_2^{b}h^{-1} = \epsilon \gamma_1^{c} \gamma_2^{d}$. Since $h\epsilon \gamma_1^{a} \gamma_2^{b}h^{-1} = \epsilon \gamma_1^{a} \gamma_2^{b}$ (mod $\langle \sigma, \tau \rangle$), then $\pi_2(\epsilon \gamma_1^{a} \gamma_2^{b}) = \pi_2(\epsilon \gamma_1^{c} \gamma_2^{d})$ and so $(a, b) = (c, d)$.

Let $G$ be a regular subgroup of $\text{Hol}(A)$ such that $|\pi_2(G)| = q$. Up to conjugation, we can assume that $\pi_2(G)$ is generated by $\gamma_1^{a} \gamma_2^{b}$ where $(a, b) \neq (0, 0)$. Therefore 
\[G = \langle \sigma, \tau, \epsilon \gamma_1^{a} \gamma_2^{b} \rangle\]
where $n \neq 0$ because $G$ is regular. Therefore $G = G_{a,n-1, b,n-1}$.

Let $A_{a,b}$ be the skew brace associated to the regular subgroup $G_{a,b}$. Note that $\epsilon \in \text{Fix}(A_{a,b})$ and $\sigma, \tau \in \ker\lambda$. Therefore we can apply Lemma 2.23 and (12) follows. According to [11, Proposition 1.1], $A_{a,b}$ is a bi-skew braces since $(A_{a,b}, +) = \mathbb{Z}_p^2 \times \mathbb{Z}_q$, $(A_{a,b}, \circ) = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ and the images of $\rho$ and $\eta$ commute.

**Lemma 3.19.** A set of representative regular subgroups $G$ of $\text{Hol}(G_k)$ with $|\pi_2(G)| = p$ is 
\[
H_1 = \langle \epsilon, \tau, \sigma ^{-1} \alpha_1 \rangle \cong G_0, \quad K_1 = \langle \epsilon, \tau, \sigma ^{-1} \alpha_1 \alpha_2 \rangle \cong G_0, 
\]
\[
H_2 = \langle \epsilon, \sigma, \tau ^{-1} \alpha_2 \rangle \cong G_0, \quad K_2 = \langle \epsilon, \sigma, \tau ^{-1} \alpha_1 \alpha_2 \rangle \cong G_0.
\]

Proof. Up to conjugation, the subgroups of order $p$ of $\text{Aut}(G_k)$ are $\langle \alpha_1 \rangle, \langle \alpha_2 \rangle, \langle \alpha_1 \alpha_2 \rangle$. The groups in the list of the statement have the desired properties and they are not conjugate since either their images or their kernels with respect to $\pi_2$ are not.

Let $G$ be a regular subgroup of $\text{Hol}(G_k)$ with $|\pi_2(G)| = p$. The kernel of $\pi_2$ has size $pq$ and therefore it has an element of order $q$ of the form $uw$ where $u \in \langle \sigma, \tau \rangle$. The subgroup $L = \langle \alpha_1, \alpha_2 \rangle$ normalizes $\pi_2(G)$, so we can conjugate $G$ by a suitable element of $L$ and assume that $u = 0$. Then the kernel has the form $\langle \epsilon, v \rangle$ for $v \in \langle \sigma, \tau \rangle$. The $p$-Sylow subgroup of $\ker\pi_2$ is normal and therefore $v = \sigma$ or $\tau$. Therefore the group $G$ has the following form 
\[G = \langle \epsilon, v, w\theta \rangle\]
where $\theta \in \langle \alpha_1, \alpha_2, \alpha_1 \alpha_2 \rangle$ and, either $v = \sigma$ and $w \in \langle \tau \rangle$ or $v = \tau$ and $w \in \langle \sigma \rangle$. By condition (K) and the fact that $w \neq 1$ because $G$ is regular, it follows that $G$ is either $H_i$ or $K_i$ for $i = 1, 2$. 

**Lemma 3.20.** There exists a unique conjugacy class of regular subgroups $G$ of $\text{Hol}(G_k)$ with $|\pi_2(G)| = p^2$. A representative is 
\[
H = \langle \epsilon, \sigma ^{-1} \alpha_1, \tau ^{-1} \alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q.\]
Lemma 3.21. A set of representatives of conjugacy classes of subgroups of \( Aut(G_k) \) of order \( pq \) and order \( p^2q \) are

| Size     | \( G \)                                               | Parameters                                      | Class                                      |
|----------|-------------------------------------------------------|------------------------------------------------|--------------------------------------------|
| \( pq \) | \( H_{1,s} = \langle \alpha_1, \beta_s \rangle \)    | \( 0 \leq s \leq q - 1 \)                      | \( \mathbb{Z}_p \rtimes \mathbb{Z}_q \)   |
| \( H_{2,s} = \langle \alpha_2, \beta_s \rangle \) | \( 0 \leq s \leq q - 1 \)                      | \( \mathbb{Z}_{pq}, \text{ if } s = 0, \mathbb{Z}_p \rtimes \mathbb{Z}_q, \text{ otherwise.} \) |%
| \( K_i = \langle \alpha_i, \beta \rangle \)   | \( i = 1, 2 \)                                  | \( \mathbb{Z}_{pq}, \text{ if } i = 1, \mathbb{Z}_p \rtimes \mathbb{Z}_q, \text{ if } i = 2. \) |
| \( W = \langle \alpha_1\alpha_2, \beta_1 \rangle \) | -                                                   | \( \mathbb{Z}_p \rtimes \mathbb{Z}_q \)   |
| \( p^2q \) | \( T_s = \langle \alpha_1, \alpha_2, \beta_s \rangle \) | \( 0 \leq s \leq q - 1 \)                      | \( G_s \)                                  |
| \( U = \langle \alpha_1, \alpha_2, \beta \rangle \) | -                                                   | \( G_0 \)                                  |

where \( \beta_s = [(0, 0), (g, g^s)] \) and \( \tilde{\beta} = [(0, 0), (1, g)] \).

Proof. Let \( G \) be a subgroup of size \( pq \) of \( Aut(G_k) \). Then \( G \) is generated by an element of order \( p \) and an element of order \( q \). The element of order \( p \) belongs to the subgroup generated by \( \alpha_1, \alpha_2 \) and up to conjugation there are three elements of order \( p \), namely \( \alpha_1, \alpha_2, \alpha_1\alpha_2 \). The elements of order \( q \) are of the form \( \alpha_1^n\alpha_2^m\beta_s \) for \( s \neq 0 \), \( \alpha_1^n\beta_0 \) or \( \alpha_2^n \tilde{\beta} \).

If the \( p \)-Sylow subgroup of \( G \) is generated by \( \alpha_i \), \( i = 1, 2 \), using that

\[
(a, b), (x, y)\alpha_1^n\alpha_2^m\beta_s[(a, b), (x, y)]^{-1} = \alpha_1^{xn+(1-g)a}\alpha_2^{ym+(1-g^s)b}\beta_s,
\]
\[
(a, b), (x, y)\alpha_1^n\beta_0[(a, b), (x, y)]^{-1} = \alpha_1^{xn+(1-g)a}\beta_0,
\]
\[
(a, b), (x, y)\alpha_2^m\tilde{\beta}[(a, b), (x, y)]^{-1} = \alpha_2^{ym+(1-g)b}\tilde{\beta},
\]

we have that \( G \) is conjugate to \( H_{i,s} \) or to \( K_i \). If the \( p \)-Sylow subgroup of \( G \) is generated by \( \alpha_1\alpha_2 \) then necessarily the element of order \( q \) has form \( \alpha_1^n\alpha_2^m\beta_1 \), since the \( p \)-Sylow subgroup has to be normal. In such case, then \( G = \langle \alpha_1\alpha_2, \alpha_1^n\alpha_2^m\beta_1 \rangle = \langle \alpha_1\alpha_2, \alpha_1^n\beta_1 \rangle \) and according to (13), \( G \) is conjugate to \( W \) by a suitable power of \( \alpha_1 \).

If \( G \) is a subgroup of order \( p^2q \), then \( G \) is generated by \( \alpha_1, \alpha_2 \) and an element of order \( q \), which can be chosen to be \( \beta_s \) for \( 0 \leq s \leq q - 1 \) or \( \tilde{\beta} \). Such groups are not conjugate, since their restriction to \( \langle \sigma, \tau \rangle \) are not.

In the following we are employing the same notation as in Lemma 3.21.

Lemma 3.22. A set of representatives of conjugacy classes of subgroups \( G \) of \( Hol(G_k) \) with \( |\pi_2(G)| = \) \( pq \) is...
Subgroups

| Subgroup | Parameters | Isomorphism class | # |
|----------|------------|------------------|---|
| $H_{1,s}$ | $\bar{H}_{1,s,d} = \langle \tau, \sigma^{s-d} \alpha_1, e^d \beta_s \rangle$ | $0 \leq s \leq q-1$, $1 \leq d \leq q-1$ | $G_{dk+s}$ | $q(q-1)$ |
| $H_{1,s}$ | $\bar{H}_{1,s} = \langle \sigma, \tau \alpha_1, e^{-k} \beta_s \rangle$ | $2 \leq s \leq q$ | $G_{k+1}$ | $q-1$ |
| $\bar{H}_{1,s}$ | $\langle \sigma, \tau \alpha_1, e^{-k} \beta_s \rangle$ | $2 \leq s \leq q$ | $G_{k+1}$ | $q-1$ |
| $H_{2,s}$ | $\bar{H}_{2,s,d} = \langle \sigma, \tau \alpha_1, e^d \beta_s \rangle$ | $0 \leq s \leq q-1$, $1 \leq d \leq q-1$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $s = 0, d = -1$, $G_0$, if $s \neq 0, d = -1$, $G_{(k+1)-k}$, otherwise. | $q(q-1)$ |
| $H_{2,s}$ | $\langle \sigma, \tau \alpha_2, e^s \beta_s \rangle$ | $2 \leq s \leq q$ | $G_0$, if $s = 0$, $G_{(k+1)-k}$, otherwise. | $q-1$ |
| $\bar{H}_{2,s}$ | $\langle \sigma, \tau \alpha_2, e^s \beta_s \rangle$ | $2 \leq s \leq q$ | $G_0$, if $s = 0$, $G_{(k+1)-k}$, otherwise. | $q-1$ |
| $K_1$ | $\bar{K}_{1,d} = \langle \tau, \sigma^{s-d} \alpha_1, e^d \beta_s \rangle$ | $1 \leq d \leq q-1$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ if $d = -k^{-1}$, $G_0$, otherwise. | $q-1$ |
| $K_1$ | $\langle \sigma, \tau \alpha_1, e^{-k+d} \beta \rangle$ | - | $G_0$ | $1$ |
| $\bar{K}_{1}$ | $\langle \sigma, \tau \alpha_1, e^{k-d} \beta \rangle$ | - | $G_0$ | $1$ |
| $K_2$ | $\bar{K}_{2,d} = \langle \sigma, \tau \alpha_2, e^d \beta \rangle$ | $1 \leq d \leq q-1$ | $G_d$ | $q-1$ |
| $K_2$ | $\langle \sigma, \tau \alpha_2, e^d \beta \rangle$ | - | $G_{k+1}$ | $1$ |
| $\bar{K}_{2}$ | $\langle \sigma, \tau \alpha_2, e^d \beta \rangle$ | - | $G_{k+1}$ | $1$ |
| $W$ | $\bar{W}_d = \langle \sigma, \tau \alpha_2, e^d \beta \rangle$ | $1 \leq d \leq q-1$ | $G_{d+1}$ | $q-1$ |
| $\bar{W}_d$ | $\langle \sigma, \tau \alpha_2, e^d \beta \rangle$ | $1 \leq d \leq q-1$ | $G_{d+1}$ | $q-1$ |

Proof. All the subgroups in the table have the desired properties. The subgroups with the same image under $\pi_2$ and belonging to different rows of the tables are not conjugate, since their kernels with respect to $\pi_2$ are not. Indeed, a set of representative of the orbits of the elements of $\langle \sigma, \tau \rangle$ under the action of $\text{Aut}(\mathcal{G}_k)$ is $\{\sigma, \tau, \sigma \tau\}$.

We focus on the case when the image under $\pi_2$ is $H_{1,s}$, for the other cases we can apply the same ideas, so we omit the computation.

If $\bar{H}_{1,s,c}$ and $\bar{H}_{1,s,d}$ are conjugate by $h \in \text{Aut}(\mathcal{A})$ then $he^c \beta_s h^{-1} \in \bar{H}_{1,s,d}$. According to Lemma 3.17, $he^c \beta_s h^{-1} \equiv e^c \beta_s \pmod{H}$ where $H = \langle \sigma, \tau \alpha_1, \alpha_2 \rangle$. Then, since $\pi_2(he^c \beta_s h^{-1}) = \beta_s \pmod{\alpha_1}$ then $e^c \beta_s = e^d \beta_s \pmod{H}$ and so $c = d$.

Let $G$ be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$ with $\pi_2(G) = H_{1,s}$. The kernel of $\pi_2$ is a subgroup of order $p$ of $\mathcal{G}_k$ and we can choose it up to the action of the normalizer of $H_{1,s}$ on the subgroups of $\langle \sigma, \tau \rangle$. Since $\{(0,0), (a,b) : 1 \leq a, b \leq p-1\} \leq N_{\text{Aut}(\mathcal{G}_k)}(H_{1,s})$, we can assume that the kernel is generated by $\sigma$, by $\tau$ or by $\sigma \tau$.

(i) Assume that the kernel of $\pi_2$ is generated by $\sigma$. Then $\tau^a e^b \alpha_1 \in G$ for some $a, b$. From the (R) condition $(\tau^a e^b \alpha_1)^p \in \ker \pi_2$ we have that $b = 0$ and since $G$ is regular then $a \neq 0$. Hence, up to conjugating by $[(0,0), (1, a^{-1})]$ we can assume that $a = 1$, therefore $G$ has the form

$$G = \langle \sigma, \tau \alpha_1, \tau e^d \beta_s \rangle = \langle \sigma, \tau \alpha_1, \alpha_1^{-1} e^d \beta_s \rangle.$$

From the (R) conditions $(\alpha_1^{-1} e^d \beta_s) \tau \alpha_1 (\alpha_1^{-1} e^d \beta_s)^{-1} = (\tau \alpha_1)^q \pmod{\ker \pi_2}$ it follows that $d = \frac{1-s}{k}$. The group $G$ is regular, then $d \neq 0$ and so $s \neq 1$. If $dk + s = 0$ then $c = 0$ since $(\tau e^d \beta_s)^q \in \ker \pi_2$, otherwise, if $c \neq 0$ we have that $G$ is conjugate to $\bar{H}_{1,s}$ by a suitable power of $\alpha_1$. 

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(ii) Assume that the kernel of $\pi_2$ is generated by $\tau$. Then

$$G = \langle \tau, \sigma^a e^b \alpha_1, \sigma^c e^d \beta_s \rangle.$$  

From the (R) conditions we have that $b = 0$, $a = \frac{1}{q-1}$ and $d \neq 0$ since $G$ is regular. If $d + 1 = 0$ then $c = 0$ since $(\sigma^c e^d \beta_s)^q \in \ker \pi_2$, otherwise, we have that $G$ is conjugate to $\tilde{H}_{1,s,d}$ by a suitable power of $\alpha_1$.

(iii) Assume that the kernel of $\pi_2$ is generated by $\sigma \tau$. Then

$$G = \langle \sigma \tau, \sigma^a e^b \alpha_1, \sigma^c e^d \beta_s \rangle.$$  

The (R) conditions imply that $b = 0$, $a = \frac{1}{q-1}$ and $d \neq 0$ since $G$ is regular. From the (K) condition it follows that $d = \frac{s-1}{k}$ and so $s \neq 1$. If $d + 1 = 0$ then $c = 0$ since $(\sigma^c e^d \beta_s)^q \in \ker \pi_2$, otherwise, we have that $G$ is conjugate to $\mathcal{T}_{1,s}$ by a suitable power of $\alpha_1$. □

Lemma 3.23. A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(G_k)$ with $|\pi_2(G)| = p^2 q$ is

| $\pi_2(G)$ | Subgroups | Parameters | Isomorphism class | $\#$ |
| --- | --- | --- | --- | --- |
| $T_s$ | $\tilde{T}_{c,s} = \langle \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, e^c \beta_s \rangle$ | $0 \leq s \leq q-1$ | $G_s$ | $q(q-1)$ |
|  |  | $1 \leq c \leq q-1$ |  |  |
|  | $\tilde{T}_s = \langle \tau \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, e^s \beta_s \rangle$ | $2 \leq s \leq q$ | $G_s$ | $q-1$ |
|  |  |  |  |  |
| $U$ | $\tilde{U}_c = \langle \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, e^c \beta \rangle$ | $1 \leq c \leq q-1$ | $G_0$ | $q-1$ |
|  |  |  |  |  |
|  | $\tilde{U} = \langle \tau \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, e^{-k-1} \beta \rangle$ |  | $G_0$ | 1 |
|  |  |  |  |  |
|  | $\tilde{U} = \langle \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, e^c \beta \rangle$ |  | $G_0$ | 1 |

Proof. First, we show that the subgroups in the statement are not pairwise conjugated and then we will see that every regular subgroup of $\text{Hol}(G_k)$ is conjugated to one of these.

If two regular subgroups of $\text{Hol}(G_k)$ are conjugate, so the corresponding $p$-Sylow are conjugated too. Therefore the groups of the families $\{T_{c,s} : 0 \leq s, c \leq q-1, c \neq 0\}$, $\{\tilde{T}_s : 2 \leq s \leq q\}$ and $\{U_s : 2 \leq s \leq q\}$ are not conjugate since their $p$-Sylow subgroups are not. The same argument used in Lemma 3.22 shows that if $\tilde{T}_{c,s}$ and $\tilde{T}_{d,s}$ are conjugate then $c = d$.

Let $G$ be a regular subgroup of $\text{Hol}(G_k)$ with $\pi_2(G) = T_s$. Then

$$G = \langle u e^a \alpha_1, u e^b \alpha_2, u e^c \beta_s \rangle,$$

for $u, v, w \in \langle \sigma, \tau \rangle$ and $0 \leq a, b, c \leq q-1$. From the (R) conditions we have that $c = b = 0$, so $c \neq 0$ since $G$ is regular and $u = \sigma^{\frac{1}{s-1}} \tau^n$ and $v = \sigma^m \tau^{s-1}$ where either $n = 0$ or $c = k^{-1}(1-s)$ and either $m = 0$ or $c = s-1$.

If $n = m = 0$, then we have that

$$G = \langle \sigma^{\frac{1}{s-1}} \alpha_1, \tau^{s-1} \alpha_2, \sigma^a \tau^b e^c \beta_s \rangle$$

for some $a, b$.

Let $(n, m) \neq (0, 0)$. Note that $c = k^{-1}(1-s) = s-1$ implies $s = 1$ and then $c = 0$, contradiction. So either $c = s-1$ and $m = 0$ or $c = k^{-1}(1-s)$ and $n = 0$. Consequently, in both cases $s \neq 1$.

Conjugating $G$ respectively by the automorphism $h = [(0, 0), (n^{-1}, 1)]$ or $g = [(0, 0), (1, m^{-1})]$ we
have that $G$ has one of the following form

$$G = \langle \tau \sigma \alpha_1, \tau g^k \alpha_2, \sigma a, \tau b \epsilon^{1-s} \beta_s \rangle,$$

$$G = \langle \sigma \alpha_1, \sigma g \alpha_2, \sigma a, \tau b \epsilon^{1-s} \beta_s \rangle,$$

for some $a, b$. In both cases, since $(\omega c^s \beta_s) = 1$ we have that if $c+1 = 0$ then $a = 0$ and if $ck + s = 0$ then $b = 0$. Hence $G$ is conjugate to $\tilde{T}_c, \tilde{T}_s$ or $\tilde{T}_n$ respectively, by $h = [(r, t), (1, 1)]$ where $(r, t)$ is a solution of the following system of linear equations:

$$\begin{cases}
    r(g^{c+1} - 1) = a(1 - g) \\
    t(1 - g^{k+s}) = b(1 - g^k)
\end{cases}.$$

We can use the same ideas when $\pi_2(G) = U$. In such case, $G$ is conjugate to $\tilde{U}_c, \tilde{U}$ or $\tilde{U}$.

We summarize the content of this section in the following table:

| $|\ker \lambda|$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | $G_k$ | $G_s, s \neq 0, \pm 1, k$ | $G_0$ | $G_{-1}$ | $G_1$ |
|---|---|---|---|---|---|---|
| $1$ | $-1$ | $2(q+1)$ | $2(q+1)$ | $q+1$ | $q-1$ |
| $p$ | $2$ | $4(q+2)$ | $4(q+2)$ | $4(q+1)$ | $2(q+2)$ | $2(q-1)$ |
| $q$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $pq$ | $2$ | $2q - 3$ | $2(q-1)$ | $2(q-1)$ | $q-1$ | $q-1$ |
| $p^2q$ | $3$ | $1$ | $1$ | $1$ | $1$ | $1$ |

Table 5. Number of skew braces of $G_k$-type of size $p^2q$ for $p = 1 \pmod{q}$.

3.5. Skew braces of $G_0$-type. Let us consider the group $G_0$ with presentation

$$G_0 = \langle \sigma, \tau, \epsilon, \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = [\tau, \epsilon] = 1, \epsilon \sigma \epsilon^{-1} = \sigma^q \rangle.$$

According to the description of the automorphisms of $A$ provided in \[8\] Theorem 3.1, 3.4, the map

$$\phi : \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \text{Aut}(G_0), \quad (n, (a, b)) \mapsto h = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \epsilon \mapsto \sigma^n \epsilon,$$

where $\rho(a, b)(n) = an$, is a group isomorphism. In particular, $|\text{Aut}(G_0)| = p(p-1)^2$ and its $p$-Sylow is generated by $\alpha = [1, (1, 1)]$.

If $G$ be a regular subgroup of $\text{Hol}(G_0)$, then $|\pi_2(G)|$ divides $p^2q$ and $|\text{Aut}(G_0)|$, so it divides $pq$.

Proposition 3.24. The skew braces of $G_0$-type with $|\ker \lambda| = p^2$ are $A_{a,b} = (A, +, \circ)$ for $0 \leq a, b \leq q - 1 : (a, b) \neq (0, 0)$, where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^{x_2}y_1 \\ x_2 + (g^{x_3})^{-1}y_2 \\ x_3 + y_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^{(a+1)x_3}y_1 \\
                        x_2 + g^{b x_3}y_2 \\
                        x_3 + y_3 \end{pmatrix}$$

for every $0 \leq x_1, x_2, y_1, y_2 \leq p - 1, 0 \leq x_3, y_3 \leq q - 1$. In particular, there are $q^2 - 1$ such skew braces, they are bi-skew braces and

$$(A_{a,b}, \circ) \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = -1, b = 0, \\
G_0, & \text{if } a = -1, b \neq 0, \\
G_{\frac{a}{a+1}}, & \text{otherwise}. \end{cases}$$
Proof. We can use the very same argument of the proof of Proposition 3.18 to compute the conjugacy classes of regular subgroups of $\text{Hol}(G_0)$ with size of the image under $\pi_2$ equal to $q$. It turns out that the representatives are

$$G_{a,b} = \langle \sigma, \tau, \epsilon^a \beta \rangle$$

where $\gamma_1 = [0, (g, 1)], \gamma_2 = [0, (1, g)], (a, b) \neq (0, 0)$. The formula follows using the same argument as in Proposition 3.18. \qed

Lemma 3.25. A set of representative regular subgroups $G$ of $\text{Hol}(G_0)$ with $|\pi_2(G)| = p$ is

$$H = \langle \epsilon, \tau, \sigma \epsilon \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q, \quad K = \langle \epsilon, \sigma, \tau \rangle \cong G_0.$$ 

Proof. The subgroup $H$ and $K$ have the desired properties and they are not conjugate, since their kernel are not in the same orbit with respect to the action of $\text{Aut}(G_0)$.

The unique subgroup of order $p$ of $\text{Aut}(G_0)$ is $\langle \alpha \rangle$ and it is normal. The kernel has the form $\langle \sigma^b \epsilon, u \rangle$ for some $b$ and some $v \in \langle \sigma, \tau \rangle$. Up to conjugation we can assume that $b = 0$ and that $v = \sigma, \tau$ or $\sigma \tau$. If $u = \sigma \tau$, then the $p$-Sylow of the kernel of $G$ is not normal.

If $u = \tau$, then

$$G = \langle \epsilon, \tau, \sigma \epsilon \rangle.$$ 

From the condition (R) it follows that $a = \frac{1}{g-1}$, i.e. $G = H$. If $u = \sigma$, then

$$G = \langle \epsilon, \sigma, \tau \rangle,$$

for some $a \neq 0$. So $G$ is conjugate to $K$ by $[(0, 0), (1, a^{-1})]$ \qed

Arguing similarly to Lemma 3.21 we have that a set of representative of conjugacy classes of subgroups of order $pq$ of $\text{Aut}(G_0)$ is

$$T_s = \langle \alpha, \beta_s \rangle, \quad U = \langle \alpha, \bar{\beta} \rangle$$

where $\beta_s = [0, (g, g_s)]$ and $0 \leq s \leq q - 1$ and $\bar{\beta} = [0, (1, g)]$.

Lemma 3.26. A set of representatives of regular subgroups $G$ of $\text{Hol}(G_0)$ with $|\pi_2(G)| = pq$ is

| $\pi_2(G)$ | Subgroups | Isomorphism class | Parameters | # |
|-------------|-----------|------------------|------------|---|
| $T_s$       | $\tilde{T}_{s,c} = \langle \tau, \sigma^c \beta_s \rangle$ | $G_0$ | $0 \leq s \leq q - 1, 1 \leq c \leq q - 1$ | $q(q-1)$ |
|             | $\tilde{T} = \langle \sigma \tau, \sigma^c \beta_s \rangle$ | $G_0$ | $2 \leq s \leq q$ | $q - 1$ |
| $T_1$       | $\tilde{T}_{1,c} = \langle \sigma \tau, \epsilon \beta_1 \rangle$ | $G_{c+1}$ | $1 \leq c \leq q - 1$ | $q - 1$ |
|             | $\tilde{U}_c = \langle \sigma \tau, \epsilon \beta \rangle$ | $G_0$ | $1 \leq c \leq q - 1$ | $q - 1$ |
| $U$         | $\tilde{U} = \langle \sigma \tau, \epsilon \beta \rangle$ | $G_0$ | - | $1$ |

In particular, there are $q^2 + 2q - 2$ such groups.

Proof. The groups in the table have the desired properties and they are not conjugate (using the same argument used in the previous section).

Assume that $G$ is a regular subgroup of $\text{Hol}(G_0)$ and that $\pi_2(G) = T_s$. Then

$$G = \langle v, u^\epsilon \alpha, w^\epsilon \beta \rangle$$ 

for $u, v, w \in \langle \sigma, \tau \rangle$ and $0 \leq a, b \leq q - 1$. From the (R) condition $(ue^a \alpha)p \in \langle v \rangle$ it follows that $a = 0$ and therefore $b \neq 0$ since $G$ is regular. Since

$$\beta_s(v) \epsilon^b = D^b_0 \beta_s(v),$$
and since \( D_0^b \beta_s \) is a diagonal matrix with entries \( g^{b+1}, g^s \), the condition (K) implies that either \( b = s - 1 \) or \( v = \sigma, \tau \).

Let assume that \( v = \sigma \). Then
\[
G = \langle \sigma, \tau^{a_0} \alpha, \tau^{c_0} \beta \rangle,
\]
where \( a_0 \neq 0 \). From the (K) conditions it follows that \( s = 1 \) and \( G \) is conjugate to \( \overline{T}_{1,b} \) by \( \alpha^n[0, (1, a^{-1})] \) for a suitable \( n \).

If \( v = \tau \) a similar argument show that \( G \) is conjugate to \( \overline{T}_{b,s} \).

Assume that \( b = s - 1 \) or \( v = \sigma, \tau \).

Let assume that \( v = \sigma \). Then
\[
G = \langle \sigma, \sigma^{1/2} \alpha, \omega \beta \rangle,
\]
where \( \alpha \neq 0 \). From the (K) conditions it follows that \( s = 1 \) and \( G \) is conjugate to \( T_{1,b} \).

A similar argument shows that if \( \tau_2(G) = U \) then \( G \) is conjugate to \( \widetilde{U}_c \) or to \( \hat{U} \).

**Remark 3.27.** The skew brace \( B \) is a direct product of if and only if there exists \( I, J \) ideals of \( B \) such that \( I + J = B \) and \( I \cap J = 0 \).

The following are the skew braces of \( G_0 \)-type which are direct products of the trivial skew brace of size \( p \) and a skew brace of size \( pq \).

(i) The skew braces \( B \cong A_{a,0} \) for \( a \neq -1 \) as defined in Corollary 3.24 are direct product of the trivial skew brace of size \( p \) and a skew brace of size \( pq \) with \( |\ker \lambda| = p \) (\cite{1}, Theorem 3.9).

(ii) The skew brace associated to the group \( H \) of Lemma 3.25 is the direct product of the trivial skew brace of size \( p \) and the unique skew brace of size \( pq \) with \( |\ker \lambda| = q \) (\cite{1}, Theorem 3.6).

(iii) The skew brace associated to \( \overline{T}_{0,c} \) for \( c \neq 0 \) as defined in Lemma 3.26 is the direct product of the trivial skew brace of size \( p \) and a skew brace of size \( pq \) with \( |\ker \lambda| = 1 \) (\cite{1}, Theorem 3.12).

We summarize the contents of this subsection in the following table:

| \(|\ker \lambda|\) | \(\mathbb{Z}_p^2 \times \mathbb{Z}_q\) | \(G_k\) | \(G_0\) | \(G_{-1}\) | \(G_1\) |
|---|---|---|---|---|---|
| \(p\) | - | \(2(q+1)\) | \(2q+1\) | \(q+1\) | \(q-1\) |
| \(pq\) | 1 | - | 1 | - | - |
| \(p^2\) | 1 | \(2(q-1)\) | \(2q-3\) | \(q-1\) | \(q-1\) |
| \(p^2 q\) | - | - | 1 | - | - |

**Table 6.** Number of skew braces of \( G_0 \)-type of size \( p^2 q \) for \( p = 1 \) (mod \( q \)).

### 3.6. Skew braces of \( G_{-1} \)-type

Let us consider the group \( G_{-1} \) with presentation
\[
G_{-1} = \langle \sigma, \tau, \epsilon, \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = 1, \epsilon \sigma \epsilon^{-1} = \sigma^q, \epsilon \tau \epsilon^{-1} = \tau^q \rangle.
\]

Let
\[
T = (\mathbb{Z}_p^2 \times_p (\mathbb{Z}_p^2 \times \mathbb{Z}_p^2)) \times_p \mathbb{Z}_2
\]
where \( \rho(a, b)(n, m) = (an, bm) \), \( \rho'(-1) [(n, m), (a, b)] = [(g^{-1}m, -gn), (b, a)] \) for every \( 0 \leq n, m \leq p - 1, 0 \leq m \leq q - 1 \). According to [4 Subsections 4.1, 4.3], the mapping

\[
\phi : T \longrightarrow \text{Aut}(G_{-1}) \quad [(n, m), (a, b), \pm 1] \mapsto h_\pm = \begin{cases} h_\pm |(\sigma, \tau) = H_\pm(a, b), \\ e \mapsto \sigma^m \tau^m e^{-1} \end{cases},
\]

where

\[
H_+(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad H_-(a, b) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},
\]

is an isomorphism.

Since the \( p \)-Sylow subgroup of \( G_{-1} \) is characteristic, there exist a group homomorphism

\[
\nu : \text{Aut}(G_{-1}) \longrightarrow \text{Aut}(G_{-1}/\langle \sigma, \tau \rangle), \quad h_\pm \mapsto \pm 1.
\]

The kernel of \( \nu \) is denoted by \( \text{Aut}(G_{-1})^+ \) and it contains the \( p \)-Sylow of \( \text{Aut}(G_{-1}) \), generated by \( \alpha_1 = [(1, 0), (1, 1)] \) and \( \alpha_2 = [(0, 1), (1, 1)] \), and the elements of odd order of \( \text{Aut}(G_{-1}) \). In particular, \( \text{Aut}(G_{-1}) \) is generated by \( \text{Aut}(G_{-1})^+ \) and \( \psi = [(0, 0), (1, 1), -1] \), defined by

\[
(15) \quad \psi = \begin{cases} \sigma \mapsto \tau, & \tau \mapsto \sigma, & \epsilon \mapsto \epsilon^{-1} \end{cases}.
\]

**Proposition 3.28.** The skew braces of \( G_{-1} \)-type with \( |\ker \lambda| = p^2 \) are \( A_{a,b} = (A, +, \circ) \) for \( (a, b) \neq (0, 0) \), where

\[
(16) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \circ g^{x_3 y_1} \\ x_2 + (g^k)^{x_3 y_2} \\ x_3 + y_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^{(a+1)x_3 y_1} \\ x_2 + g^{(b-1)x_3 y_2} \\ x_3 + y_3 \end{pmatrix}
\]

for every \( 0 \leq x_1, x_2, y_1, y_2 \leq p - 1, 0 \leq x_3, y_3 \leq q - 1 \). In particular, they are bi-skew braces and

\[
(A_{a,b}, \circ) \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = -1, b = 1, \\ G_0, & \text{if } a = -1, b \neq 1, \\ G_{a,b}, & \text{otherwise.} \end{cases}
\]

Moreover, \( A_{a,b} \cong A_{c,d} \) if and only if \( (c, d) = (-b, -a) \) and so there are \( \frac{(q-1)(q+2)}{2} \) such skew braces.

**Proof.** Let \( G \) be a regular subgroup of \( \text{Hol}(G_{-1}) \) such that \( |\pi_2(G)| = q \) and then \( \pi_2(G) \leq \text{Aut}(G_{-1})^+ \). Arguing as in Proposition 3.18 we can show that every such group is conjugate to a subgroup

\[
G_{a,b} = \langle \sigma, \tau, \epsilon \rangle_{\gamma_1, \gamma_2}.
\]

This subgroup is conjugate to \( G_{-b,-a} \) by \( \psi \) as defined in (15). Hence the groups of the form \( G_{a,-a} \) are normalized by \( \psi \) and the other orbits have length 2. Therefore there are

\[
q - 1 + \frac{q(q - 1)}{2} = \frac{(q - 1)(q + 2)}{2}
\]

orbits under the action of \( \psi \). The formula follows by the same argument of Proposition 3.18 \( \square \)

**Lemma 3.29.** A set of representative regular subgroups \( G \) of \( \text{Hol}(G_{-1}) \) with \( |\pi_2(G)| = p \) is

\[
H_1 = \langle \epsilon, \tau, \sigma^{\frac{1}{a+1}} \alpha_1 \rangle, \quad K_1 = \langle \epsilon, \tau, \sigma^{\frac{1}{a+1}} \alpha_1 \alpha_2 \rangle.
\]

**Proof.** Arguing as in Lemma 3.19 we can show that every regular subgroup of \( \text{Hol}(G_{-1}) \) with \( |\pi_3(G)| = p \) is conjugate to \( H_i, K_i \) for \( i = 1, 2 \). It is easy to see that \( H_1 \) is conjugate to \( H_2 \) by the automorphism \( \psi \) and that \( K_1 \) is conjugate to \( K_2 \) by the automorphism \( [(0, 0), (g^2, 1), -1] \). \( \square \)
Lemma 3.30. There exists a unique conjugacy class of regular subgroups $G$ of $\text{Hol}(G_{-1})$ with $|\pi_2(G)| = p^2$. A representative is

$$H = \langle \epsilon, \sigma^{s^{-1}} \alpha_1, \tau^{\frac{1}{q}} \alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q,$$

Proof. The group $H$ has the desired properties. Let $G$ be a regular subgroup of $\text{Hol}(G_{-1})$ such that $|\pi_2(G)| = p^2$. Then the kernel is a normal $q$-Sylow subgroup of $G$ and so $G$ is abelian. The image $\pi_2(G)$ coincide with the normal $p$-Sylow of $\text{Aut}(G_{-1})$, generated by $\alpha_1, \alpha_2$. Up to conjugation by a suitable element of $\pi_2(G)$, we can choose the generator of the kernel to be $\epsilon$. Therefore

$$G = \langle \epsilon, \sigma^b \alpha_1, \sigma^d \alpha_2 \rangle.$$

The group is abelian, therefore $b = c = 0$, $a = \frac{1}{q-1}$ and $t = \frac{1}{q-1}$. □

Lemma 3.31. A set of representatives of conjugacy classes of subgroups of $\text{Aut}(G_{-1})$ of order $pq$ and $p^2q$ is

| Size | $G$ | Parameters | Class |
|------|-----|------------|-------|
| $pq$ | $H_{1,s} = \langle \alpha_1, \beta_s \rangle$ | $0 \leq s \leq q - 1$ | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
|      | $H_{2,0} = \langle \alpha_2, \beta_0 \rangle$ | - | $\mathbb{Z}_p$ |
|      | $W = \langle \alpha_1 \alpha_2, \beta_1 \rangle$ | - | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
| $p^2q$ | $T_s = \langle \alpha_1, \alpha_2, \beta_s \rangle \cong G_s$ | $s \in \mathbb{B}$ | $G_s$ |

where $\beta_s = [(0,0), (g,g^s), 1]$.

Proof. The subgroups of size $pq$ of $\text{Aut}(G_{-1})^+$ up to conjugation are collected in Lemma 3.21. We need to compute the orbits of such groups under the action by conjugation of $\psi$. It is easy to see that $\psi(\alpha_1, \beta_s)\psi = (\alpha_2, \beta_{s-1})$ for $s \neq 0$, $\psi(\alpha_2, \beta_0)\psi = (\alpha_1, \beta)$, $\psi T_s \psi = T_{s-1}$, $\psi U \psi = T_0$ and $\psi W \psi$ is conjugate to $W$ by an element of $\text{Aut}(G_{-1})^+$. □

Lemma 3.32. A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(G_{-1})$ with $|\pi_2(G)| = pq$ is the following:

| $\pi_2(G)$ | Subgroups | Parameters | Isomorphism class | $\#$ |
|------------|------------|------------|------------------|-----|
| $H_{1,s}$  | $\bar{H}_{1,s,d} = \langle \tau, \sigma^{s^{-1}} \alpha_1, e^d \beta_s \rangle$ | $0 \leq s \leq q - 1$, $1 \leq d \leq q - 1$ | $G_{s-d}$ | $q(q-1)$ |
| $H_s$      | $\bar{H}_s = \langle \sigma, \tau \alpha_1, e^{s^{-1}} \beta_s \rangle$ | $2 \leq s \leq q$ | $G_s$ | $q - 1$ |
| $\bar{H}_{s+1}$ | $\bar{H}_s = \langle \sigma, \tau \alpha_1, e^{s^{-1}} \beta_s \rangle$ | $2 \leq s \leq q$ | $G_{s+1}$ | $q - 1$ |
| $H_{2,0}$  | $\bar{H}_{2,0,d} = \langle \sigma, \tau^{s^{-1}} \alpha_2, e^d \beta_0 \rangle$ | $1 \leq d \leq q - 1$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $d = -1$ | $G_0$, otherwise | $q - 1$ |
| $H_{2,0}$  | $\bar{H}_{2,0} = \langle \sigma, \alpha_2, e^{-1} \beta_0 \rangle$ | - | $G_0$ | 1 |
| $\bar{H}_{2,0}$ | $\bar{H}_{2,0} = \langle \sigma, \tau^{s^{-1}} \alpha_2, e^{-\frac{1}{2}} \beta_0 \rangle$ | - | $G_0$ | 1 |
| $W$        | $\bar{W}_d = \langle \sigma, \tau^{s^{-1}} \alpha_1 \alpha_2, e^d \beta_1 \rangle$ | $1 \leq d \leq q - 1$ | $G_{d+1}$ | $q - 1$ |

Proof. In order to compute the regular subgroups of $\text{Hol}(G_{-1})^+$ up to conjugation we need to compute the conjugacy classes of subgroup with a fixed image under $\pi_2$ with respect to the action of the normalizer of the image. If $\pi_2(G) \neq W$ we can conclude as in Lemma 3.22, since its normalizer is contained in $\text{Aut}(G_{-1})^+ \cong \text{Aut}(G_k)$ for $k \neq 0, \pm 1$. Otherwise, it is easy to see that the groups $\bar{W}_d$ and $\bar{W}_{-d}$ as in the table of Lemma 3.22 are conjugate by $[(0,0), (g^2, 1), -1]$. □

Lemma 3.33. A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(G_{-1})$ with $|\pi_2(G)| = p^2q$ is
\[\pi_2(G)\] | \text{Subgroups} | \text{Class} | \text{Parameters} | \# \\
\hline
\(T_s\) & \(\bar{\mathcal{T}}_{s,d} = \langle \sigma \overline{g}^{-1} \alpha_1, \tau \overline{g}^{-1} \alpha_2, \epsilon^d \beta_s \rangle\) & \(G_s\) & \(s \in \mathbb{N} \setminus \{1\}, 1 \leq d \leq q - 1\) & \(\frac{q^2 - 1}{2}\) \\
\hline
\(\bar{\mathcal{T}}_{s,c} = \langle \tau \sigma \overline{g}^{-1} \alpha_1, \epsilon^c \overline{g}^{-1} \alpha_2, \epsilon^1 \beta_s \rangle\) & \(G_s\) & \(s \in \mathbb{N} \setminus \{1, -1\}, 0 \leq c \leq p - 1, c \neq -\frac{q}{(q - 1)^2}\) & \(\frac{(p - 1)(q - 1)}{2}\) \\
\hline
\(\bar{\mathcal{T}}_s = \langle \sigma \overline{g}^{-1} \alpha_1, \sigma \epsilon^s \overline{g}^{-1} \alpha_2, \epsilon^1 \beta_s \rangle\) & \(G_s\) & \(s \in \mathbb{N} \setminus \{1, -1\}\) & \(\frac{q - 1}{2}\) \\
\hline
\(T_1\) & \(\bar{\mathcal{T}}_{1,d} = \langle \sigma \overline{g}^{-1} \alpha_1, \epsilon^d \beta_1 \rangle\) & \(G_1\) & \(1 \leq d < -d \leq q - 1\) & \(\frac{q - 1}{2}\) \\
\hline
\(T_{1,c} = \langle \tau \sigma \overline{g}^{-1} \alpha_1, \epsilon^{-2} \beta_{-1} \rangle\) & \(\mathcal{G}_{-1}\) & \(0 \leq c \leq p - 1, c \neq -\frac{q}{(q - 1)^2}\) & \(p - 1\) \\

Proof. A regular subgroup of \(\text{Hol}(G_{-1})\) with \(\pi_2(G) = T_s\) has the form
\[G = \langle \sigma \overline{g}^{-1} \alpha_1, \sigma \epsilon^d \alpha_2, \sigma^x \epsilon^z \beta_s \rangle.
\]
From the (R) conditions we have that \(a' = c' = 0, d = \frac{q}{1 - g}\) and either \(b = 0\) or \(z = s - 1\) and either \(c = 0\) or \(z = s - 1\).

If \(c = d = 0\), then \(G\) has the following form:
\[G = \langle \sigma \overline{g}^{-1} \alpha_1, \sigma \epsilon^d \alpha_2, \sigma^x \epsilon^z \beta_s \rangle.
\]
The last generator has order \(q\), so if \(z = -1\) then then \(x = 0\) and if \(z = s\) then \(y = 0\). We have that \(G\) is conjugate to \(\bar{\mathcal{T}}_{s,d}\) by a suitable element of \(\langle \alpha_1, \alpha_2 \rangle\). If \(s = 1\) then \(\bar{\mathcal{T}}_{1,d}\) is conjugate to \(\bar{\mathcal{T}}_{1,-d}\) by \(\psi\).

Assume that \((b, c) \neq (0, 0)\) and so \(z = s - 1\). In particular, \(s \neq 1\) since \(z \neq 0\). Since
\[\pi_2((\sigma \overline{g}^{-1} \alpha_1)^n) = \pi_2((\sigma \overline{g}^{-1} \tau \overline{g}^{-1} \alpha_2)^n) = \alpha_1^n, \quad \pi_2(\sigma \overline{g}^{-1} \tau \overline{g}^{-1} \alpha_2) = \alpha_2,
\]
for every \(0 \leq n \leq p - 1\) and since \(\pi_2\) is an isomorphism then \(\sigma \overline{g}^{-1} \tau \overline{g}^{-1} \alpha_2 \neq (\sigma \overline{g}^{-1} \tau \overline{g}^{-1} \alpha_2)^n\) and so \(c \neq -\frac{q}{(q - 1)^2}\).

It is easy to see that, up to conjugation we can assume that either \(b = 0\) and \(c = 1\) or that \(b = 1\) and \(c \neq 0\). In the first case \(G\) is conjugate to \(\bar{\mathcal{T}}_s\) and in the second case it is conjugate to \(\bar{\mathcal{T}}_{s,c}\) by a suitable element of \(\langle \alpha_1, \alpha_2 \rangle\). If \(s = -1\) we have that \(\bar{\mathcal{T}}_{-1,0}\) and \(\bar{\mathcal{T}}_{-1}\) are conjugate by \([(0, 0), (-g, -g), -1]\).

\[\begin{array}{c|c|c|c|c|c|c}
|k\ker \lambda| & \mathbb{Z}_p^2 \times \mathbb{Z}_q & \mathcal{G}_k & \mathcal{G}_0 & \mathcal{G}_{-1} & \mathcal{G}_1 \\
\hline
1 & - & p + q - 1 & p + q - 1 & p + q - 2 & \frac{q^2 - 1}{2} \\
\hline
p & 1 & 2(q + 2) & 2(q + 1) & q + 2 & q - 1 \\
\hline
q & 1 & - & - & - & - \\
\hline
pq & - & - & 2 & - & - \\
\hline
p^2 & 1 & q - 1 & q - 1 & q - 2 & \frac{q^2 - 1}{2} \\
\hline
p^2q & - & - & - & 1 & - \\
\end{array}
\]

Table 7. Number of skew braces of \(G_{-1}\)-type of size \(p^2q\) for \(p = 1\) (mod \(q\)).

3.7. Skew braces of \(G_1\)-type. A presentation of the group \(G_1\) is the following
\[G_1 = \langle \sigma, \tau, \epsilon \mid \sigma^p = \epsilon^q = [\sigma, \tau] = 1, \epsilon \sigma \epsilon^{-1} = \sigma^q, \epsilon \tau \epsilon^{-1} = \tau^q \rangle.
\]

According to [S], Subsections 4.1, 4.2, the mapping
\[\phi : \mathbb{Z}_p^2 \times GL_2(p) \rightarrow \text{Aut}(G_1), \quad [(n, m), H] \mapsto h = \{h|_{(\alpha, \beta)} = H, \quad \epsilon \mapsto \epsilon \sigma^n \tau^m\}
\]
is a group isomorphism. In particular, \(|\text{Aut}(G_1)| = p^3(p - 1)^2(p + 1)\) and a \(p\)-Sylow of \(\text{Aut}(G_1)\) is generated by \(\alpha_1 = [(1, 0), Id], \alpha_2 = [(0, 1), Id], \gamma = [(0, 0), C]\), where \(C\) is defined as in Remark 3.6.
Proposition 3.34. The skew braces of size $G_1$-type with $|\ker \lambda| = p^2$ are $A_{a,b} = (A, +, \circ)$ for $0 \leq a, b \leq q - 1$, with $(a, b) \neq (0, 0)$, where

\[
(17) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^3x_1 y_1 \\ x_2 + (g^k)^4x_2 y_2 \\ x_3 + y_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g(a+1)x_1 y_1 \\ x_2 + g(b+1)x_2 y_2 \\ x_3 + y_3 \end{pmatrix}
\]

for every $0 \leq x_1, x_2, y_1, y_2 \leq p - 1$, $0 \leq x_3, y_3 \leq q - 1$. In particular, they are bi-skew braces and

\[
(A_{a,b}, \circ) \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = b = -1, \\ \mathcal{G}_0, & \text{if } a = -1, b \neq -1, \\ \mathcal{G}_{b+1}, & \text{otherwise.} \end{cases}
\]

Moreover, $A_{a,b} \cong A_{c,d}$ if and only if $(c, d) = (b, a)$ and so there are $\frac{(q-1)(q+2)}{2}$ such skew braces.

Proof. Let $G$ be a regular subgroup of Hol($A$) such that $|\pi_2(G)| = q$. Arguing as in Proposition 3.18 we can show that every such group is conjugate to a subgroup

\[
G_{a,b} = \langle \sigma, \tau, \gamma_1^a \gamma_2^b \rangle.
\]

Hence $G_{a,b}$ and $G_{c,d}$ are conjugate if and only if

\[
h(\sigma)h_1^a \gamma_2^bh^{-1} = eh_1^a \gamma_2^bh^{-1} \equiv e \gamma_1^a \gamma_2^d \text{ (mod } \ker \pi_2|_{G_{c,d}}).
\]

i.e. $\gamma_1^a \gamma_2^b$ and $\gamma_1^c \gamma_2^d$ are conjugate, i.e. $(a, b) = (c, d)$ or $(a, b) = (d, c)$. In particular there are $\frac{(q-1)(q+2)}{2}$ such classes. \qed

Lemma 3.35. A set of representatives of conjugacy classes of regular subgroups $G$ of Hol($G_1$) with $|\pi_2(G)| = p$ is

\[
H = \langle \epsilon, \tau, \sigma \gamma^{-1} \rangle \cong \mathcal{G}_0, \quad K = \langle \epsilon, \sigma, \tau \gamma^{-1} \rangle \cong \mathcal{G}_0.
\]

Proof. Up to conjugation the subgroups of order $p$ of Aut($G_1$) are $\langle \alpha_1 \rangle$, $\langle \gamma \rangle$ and $\langle \alpha_2 \gamma \rangle$. So $H$ and $K$ are non-conjugate regular subgroups of Hol($G_1$) and $|\pi_2(H)| = |\pi_2(K)| = p$.

Let $G$ be a regular subgroup of Hol($G_1$). If $\pi_2(G) = \langle \alpha_1 \rangle$, arguing as in Lemma 3.19 we can show that $G$ is conjugate to $H$.

Assume that $\pi_2(G) = \langle \gamma \rangle$. Then

\[
G = \langle u \epsilon, v, w \gamma \rangle,
\]

for some $u, v, w \in \langle \sigma, \tau \rangle$. According to (K), then $v \in \sigma$ and so we can assume that $v = \sigma$, $u = \tau^b$ and $w = \tau^c$ for some $b, c$ with $c \neq 0$. Then

\[
\tau^c \gamma^b \epsilon \tau^{-c} = a^b \tau^{b+1-g^c} \epsilon \in \ker \pi_2
\]

which implies $c = 0$, contradiction.

Let $\pi_2(G) = \langle \alpha_2 \gamma \rangle$. Arguing as in the previous case we can assume that

\[
G = \langle \tau^b \epsilon, \sigma, \tau^c \alpha_2 \gamma \rangle.
\]

Since $\ker \pi_2$ is normal then $c = \frac{1}{g-1}$. Hence $G$ is conjugate to $K$ by $\alpha_2^{-b} \gamma^{-b}$. \qed

Lemma 3.36. A set of representatives of conjugacy classes of regular subgroups $G$ of Hol($G_1$) with $|\pi_2(G)| = p^2$ is

\[
H = \langle \epsilon, \sigma \gamma^{-1} \alpha_1, \tau \gamma^{-1} \alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q, \quad K = \langle \epsilon, \sigma \gamma^{-1} \alpha_1, \tau \gamma^{-1} \alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q.
\]
Proof. The subgroups of size $p^2$ are $\langle \alpha_1, \alpha_2 \rangle$, $\langle \alpha_1, \gamma \rangle$ and $\langle \alpha_1, \alpha_2 \gamma \rangle$. Hence the groups $H$ and $K$ have the desired properties and they are not conjugate.

Let $G$ be a regular subgroup of $\text{Hol}(G_1)$ such that $|\pi_2(G)| = p^2$. The kernel of $\pi_2$ is a normal $q$-Sylow of $G$ and so $G$ is abelian.

Let $\pi_2(G) = \langle \alpha_1, \alpha_2 \rangle$. Then, up to conjugation by the normalizer of $\pi_2(G)$ the kernel is $\langle \epsilon \rangle$. Hence

$$G = \langle \epsilon, u\alpha_1, v\alpha_2 \rangle$$

for some $u, v \in \langle \sigma, \tau \rangle$. From abelianess of $G$ it follows that $u = \sigma^{b\theta}$ and $v = \tau^{b\theta}$, i.e. $G = H$.

Let $\pi_2(G) = \langle \alpha_1, \gamma \rangle$. Then, up to conjugation by the normalizer of $\pi_2(G)$ we have

$$G = \langle \tau^b\epsilon, v\alpha_1, w\gamma \rangle,$$

for some $b$ and some $v, w \in \langle \sigma, \tau \rangle$. From the abelianness of $G$ we have that $v = \sigma^{b\theta - 1}$ and $w = \sigma^{b\theta - 1}$ and then $b \neq 0$. On the other hand, since $\tau^b\epsilon$ has order $q$ then $\langle \tau \rangle$ is not contained in $\pi_1(G)$, contradiction.

Let $\pi_2(G) = \langle \alpha_1, \alpha_2 \gamma \rangle$ and as before we can assume that

$$G = \langle \tau^b\epsilon, v\alpha_1, w\alpha_2 \gamma \rangle.$$ 

Since $G$ is abelian then $v = \sigma^{b\theta}$ and $w = \sigma^{b\theta - 1}$. Therefore $G$ is conjugate to $K$ by $\alpha_{2}^{-b}\gamma^{-b}$. □

The following lemma collects the conjugacy classes of subgroups of size $pq$ and $p^2q$ of $\text{Aut}(G_1)$.

**Lemma 3.37.** A set of representatives of conjugacy classes of subgroups of size $pq$ and $p^2q$ of $\text{Aut}(G_1)$ is

| Size | $G$                                      | Parameters | Class            |
|------|------------------------------------------|------------|------------------|
| $pq$ | $H_s = \langle \alpha_1, \beta_s \rangle$ | $0 \leq s \leq q - 1$ | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
|      | $K_s = \langle \gamma, \beta_s \rangle$   | $0 \leq s \leq q - 1$ | $\mathbb{Z}_p \times q$, if $s = 1$, $\mathbb{Z}_p \times \mathbb{Z}_q$, otherwise |
|      | $U = \langle \alpha_2, \beta_0 \rangle$    | -           | $\mathbb{Z}_p$    |
|      | $W = \langle \alpha_1, \beta_0 \rangle$    | -           | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
|      | $M = \langle \alpha_1, \alpha_2 \gamma, \beta_2^{-1} \rangle$ | -           | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
|      | $V = \langle \gamma, \beta \rangle$       | -           | $\mathbb{Z}_p \times \mathbb{Z}_q$ |
| $p^2q$ | $I_s = \langle \alpha_1, \alpha_2, \beta_s \rangle$ | $s \in \mathcal{B}$ | $\mathcal{G}_S$ |
|      | $R_s = \langle \alpha_1, \gamma, \beta_s \rangle$ | $0 \leq s \leq q - 1$ | $\mathcal{G}_{1-s}$ |
|      | $N = \langle \alpha_1, \gamma, \beta \rangle$   | -           | $\mathcal{G}_0$   |
|      | $L = \langle \alpha_1, \alpha_2 \gamma, \beta_2^{-1} \rangle$ | -           | $\mathcal{G}_2$   |

where $\beta_s = [(0, 0), D_s]$ and $\tilde{\beta} = [(0, 0), \tilde{D}]$ and $D_s$ and $\tilde{D}$ as defined in Remark 3.6.

Proof. The groups in the statement are not conjugate since their restriction to $\langle \sigma, \tau \rangle$ are not. Let $H$ be a subgroup of order $pq$ of $\text{Aut}(G_1)$. If $|H|_{\langle \sigma, \tau \rangle} = q$ then, according to Remark 3.6 we have that, $H|_{\langle \sigma, \tau \rangle}$ is generated by $D_s$ or by $\tilde{D}$ for $0 \leq s \leq q - 1$. Hence

$$H = \langle \alpha_1^n \alpha_2^m, \alpha_1^r \alpha_2^t \theta \rangle,$$

where $\theta$ is either $\beta_s = [(0, 0), D_s]$ or $\tilde{\beta} = [(0, 0), \tilde{D}]$. If $\theta = \beta_0$ then $t = 0$ and if $\theta = \tilde{\beta}$ then $r = 0$. Conjugating by a suitable element of $\langle \alpha_1, \alpha_2 \rangle$ we can assume that $r = t = 0$, and then $H$ has the following form

$$H = \langle \alpha_1^n \alpha_2^m, \theta \rangle.$$
From the normality of the $p$-Sylow of $H$ it follows that either $n = 0$ or $m = 0$. Moreover, the pairs $H = \langle \alpha_1, \beta_s \rangle$ and $H = \langle \alpha_2, \beta_{s-1} \rangle$, and $H = \langle \alpha_1, \beta_0 \rangle$ and $H = \langle \alpha_2, \beta \rangle$ are conjugate by the automorphism $\psi$ swapping $\sigma$ and $\tau$. Hence, up to conjugation the list of such groups is

$$H_s = \langle \alpha_1, \beta_s \rangle \quad U = \langle \alpha_2, \beta_0 \rangle$$

for $0 \leq s \leq q - 1$.

If $|H|_{(\sigma, \tau)} = pq$ then, according to Remark 3.35, up to conjugation have that $H|_{(\sigma, \tau)} = \langle C, B_s \rangle$ or $H|_{(\sigma, \tau)} = \langle C, B \rangle$. Therefore

$$H = \langle \alpha_1^n \alpha_2^m \gamma, \alpha_1^r \alpha_2^t \theta \rangle.$$

As in the previous case, up to conjugation we can assume that $r = t = 0$.

The $p$-Sylow of $H$ has to be normal and so it follows that $n, m$ are solutions of the following system of linear equations:

$$\begin{cases} 2n(1-s - g) + mg(1-s - 1) = 0 \\ (g^s - 1) m = 0. \end{cases}$$

Hence if $\theta = \beta_s$ for $s \neq 0, 2^{-1}$ then $n = m = 0$. If $\theta = \beta_0$ then either $n = m = 0$ or, up to conjugation by $\langle (0,0), n^{-1}Id \rangle$, $m = 0$ and $n = 1$. If $\theta = \beta_{2^{-1}}$ then $n = 2n$ and then, either $n = m = 0$ or up to conjugation by $\langle (0,0), n^{-1}Id \rangle$, $n = 1, m = 2$.

Let $H$ be a subgroup of $\text{Aut}(G_1)$ of size $p^2 q$ and assume that $|H|_{(\sigma, \tau)} = q$. Then, up to conjugation,

$$H = \langle \alpha_1, \alpha_2, \beta_s \rangle,$$

for $s \in \mathbb{G}$.

Assume that $|H|_{(\sigma, \tau)} = pq$. Then, up to conjugation,

$$H = \langle \alpha_1^n \alpha_2^m \alpha_1^r \alpha_2^t \gamma, \alpha_1^u \alpha_2^w \theta \rangle,$$

where $\theta$ is either $\beta_s$ or $\beta$. From the abelianness of the $p$-Sylow it follows that $m = 0$ and therefore

$$H = \langle \alpha_1, \alpha_2^r \gamma, \alpha_2^t \theta \rangle.$$

If $\theta = \beta_0$ then $w = 0$, otherwise, up to conjugation by some power of $\alpha_2$ we can assume that $w = 0$. From the normality of the $p$-Sylow it follows that $t(1-g) = 0$ if $\theta = \beta_s$ or $(1-g)t = 0$ if $\theta = \beta$. Hence if $\theta \neq \beta_{2^{-1}}$ then $t = 0$, otherwise up to conjugation we can assume that $t \in \{0, 1\}$. 

**Lemma 3.38.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(G_1)$ with $|\pi_2(G)| = pq$ is

| $\pi_2(G)$ | Subgroups | Parameters | Class | # |
|------------|-----------|------------|-------|---|
| $H_s$      | $H_s = \langle \sigma, \tau \alpha_1, \epsilon^{1-s} \beta_s \rangle$ | $2 \leq s \leq q$ | $\mathcal{G}_{2-s}$ | $q - 1$ |
| $H_{s,c}$  | $H_{s,c} = \langle \tau, \sigma^{u-1} \alpha_1, \epsilon^c \beta_s \rangle$ | $0 \leq s \leq q - 1$, $1 \leq c \leq q - 1$ | $\mathcal{G}_{c+s}$ | $q(q - 1)$ |
| $U$        | $U_c = \langle \sigma, \tau^{u-1} \alpha_2, \epsilon^c \beta_0 \rangle$ | $1 \leq c \leq q - 1$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_p$, if $c = -1$, $\mathcal{G}_0$, otherwise. | $q - 1$ |
| $K_s$      | $K_s = \langle \sigma, \tau \alpha_1 \gamma, \epsilon^{1-2s} \beta_s \rangle$ | $0 \leq s \leq q - 1$, $s \neq 2^{-1}$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_p$, if $s = 1$, $\mathcal{G}_2$, otherwise. | $q - 1$ |
| $W$        | $W = \langle \sigma, \tau \alpha_1 \gamma, \epsilon \beta_0 \rangle$ | $-$ | $\mathcal{G}_2$ | $1$ |
| $M$        | $M_c = \langle \sigma, \tau^{u-1} \alpha_1 \alpha_2^c \gamma, \epsilon^c \beta_2^{-1} \rangle$ | $1 \leq c \leq q - 1$ | $\mathcal{G}_{2(c+1)}$ | $q - 1$ |
| $V$        | $V = \langle \sigma, \tau \gamma, \epsilon^{-2} \beta \rangle$ | $-$ | $\mathcal{G}_2$ | $1$ |
Proof. Assume that $\pi_2(G) = H_s$. Up to conjugation by elements of the normalizer of $H_s$ we can assume that the kernel is be generated by $\sigma$, $\tau$ or $\sigma \tau$. In the first case, $G$ has the form

$$G = \langle \sigma, \tau^a \alpha_1, \tau^b \epsilon \beta_s \rangle.$$ 

The (R) conditions imply that $a' = 0$, $c = 1 - s$ and so $a \neq 0$ and $s \neq 1$ since $G$ is regular. The group $G$ is conjugate to $\tilde{H}_s$ by

$$h = \alpha_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix}.$$ 

In the second case, $G$ has the form

$$G = \langle \tau, \sigma^a \alpha_1, \sigma^b \epsilon \beta_s \rangle,$$

and the (R) conditions imply $a' = 0$ and $a = \frac{1}{g-1}$, and $c \neq 0$ since $G$ is regular. If $c = -1$, then since $(\sigma^b \epsilon \beta_s)^q = 1 \pmod{\ker \pi_2}$, then $b = 0$. Otherwise, $G$ is conjugate to $\tilde{H}_{s,c}$ by $h = \alpha_1^n$ for $n = -b g^{-1} \frac{q-1}{g^2 - 1}$. Assume that the kernel is generated by $\sigma \tau$. Then

$$G = \langle \sigma \tau, \sigma^a \alpha_1, \sigma^b \epsilon \beta_s \rangle.$$

The (K) condition implies that $s = 1$. Then $\gamma$ normalizes $\pi_2(G)$ and so $G$ is conjugate by $\gamma^{-1}$ to a group with kernel generated by $\tau$, and then we are back to the previous case.

If $\pi_2(G) = U$, we again consider three cases. If the kernel is generated by $\sigma$, the standard presentation of $G$ is

$$G = \langle \sigma, \tau^a \alpha_2, \tau^b \epsilon \beta_0 \rangle.$$ 

From the (R) conditions we have that $a' = 0$, $a = \frac{1}{g-1}$ and so $c \neq 0$ since $G$ is regular. Then $G$ is conjugate to $\tilde{U}_c$ by $\alpha_2^n$ where $n = -b g^{-1} \frac{q-1}{g^2 - 1}$.

If the kernel is generated by $\tau$ we have that

$$G = \langle \tau, \sigma^a \alpha_2, \sigma^b \epsilon \beta_0 \rangle.$$ 

The conditions (R) implies that $c = 0$ and therefore $G$ is not regular, contradiction.

If the kernel is generated by $\sigma \tau$ then the condition (K) is not satisfied.

Let $\pi_2(G) = K_s$. Then $G$ has the form

$$G = \langle v, u \gamma, we \epsilon \beta_s \rangle,$$

where $v, u, w \in \langle \sigma, \tau \rangle$. From (K) we have that $v \in \langle \sigma \rangle$ and so

$$G = \langle \sigma, \tau^a \gamma, \tau^b \epsilon \beta_s \rangle$$

where $c \neq 0$. The conditions (R) imply that $c = 1 - 2s$ and then $s \neq 2^{-1}$. Up to conjugation by $h = [(0, 0), a^{-1}Id]$ we can assume that $a = 1$. If $s = 1$ then $b = 0$ since $(\tau^b \epsilon \beta_s)^q \in \ker \pi_2$. Otherwise, $G$ is conjugate to $\tilde{K}_s$ by $\gamma^n$ where $n = \frac{b}{1 - g^{-1}}$.

Let $\pi_2(G) = W$. As in the previous case, the kernel is to be generated by $\sigma$. So $G$ has the form

$$G = \langle \sigma, \tau^a \alpha_1 \gamma, \tau^b \epsilon \beta_0 \rangle.$$ 

From the (R) conditions we have $c = 1$ and up to conjugation we can assume $a = 1$. Then $G$ is conjugate to $\tilde{W}$ by $\alpha_1^m$ for $n = \frac{b}{1 - g^{-1}}$ and $m = \frac{b}{1 - g^{-1}}$.

Let $\pi_2(G) = M$. If the kernel is generated by $\sigma$, then $G$ has the form

$$G = \langle \sigma, \tau^a \alpha_1 \alpha_2 \gamma, \tau^b \epsilon \beta_2 \gamma \rangle.$$
From the (R) conditions we have \( a = \frac{2}{g - 1} \). If \( c + \frac{1}{2} = 0 \), then \( b = 0 \) by since the \( q \)-th power of the third generator lies on the kernel. Otherwise, \( G \) is conjugate to \( \tilde{M} \) by \( \alpha_1^n \alpha_2^m \gamma^m \) where
\[
n = \frac{1 - g^\frac{1}{2}}{1 + g^\frac{1}{2}} (1 - 2g^\frac{1}{2})m^2 \quad \text{and} \quad m = -\frac{b-g-1}{g^\frac{1}{2}+1}.
\]
If the kernel is generated by \( \tau \) or \( \sigma \tau \), the condition (K) is violated.

Assume that \( \pi_2(G) = V \). Condition (K) implies that the kernel of \( \pi_2 \) is generated by \( \sigma \) and so
\[
G = \langle \sigma, \sigma^a \gamma, \tau^b e^\epsilon \beta \rangle.
\]
From the (R) conditions we have \( c = -2 \). Then \( G \) is conjugate to \( \tilde{V} \) by \( \gamma^a [(0, 0), \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}] \) where
\[
n = \frac{b}{g^{a(g-1)}}.
\]

**Lemma 3.39.** A set of representatives of conjugacy classes of regular subgroups \( G \) of \( \text{Hol}(G) \) with \( |\pi_2(G)| = p^2 q \) is

| \( \pi_2(G) \) | Subgroups | Parameters | Class | # |
|---|---|---|---|---|
| \( T_s \) | \( \tilde{T}_{s,d} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, e^d \beta_s \rangle \) | \( s \in \mathbb{B} \setminus \{-1\} \) \( 1 \leq d \leq q - 1 \) | \( G_s \) | \( (q-1)(q+1) \) |
| \( \tilde{T}_{-1,d} \) | \( \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, e^d \beta_1 \rangle \) | \( 1 \leq d < -d \leq q - 1 \) | \( G_{-1} \) | \( \frac{q-1}{2} \) |
| \( \tilde{T}_s \) | \( \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, e^{-\frac{1}{2}} \beta_s \rangle \) | \( s \in \mathbb{B} \setminus \{1\} \) | \( G_s \) | \( \frac{q+1}{2} \) |
| \( \tilde{T}_s \) | \( \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, e^{-\frac{1}{2}} \beta_s \rangle \) | \( s \in \mathbb{B} \setminus \{1, -1\} \) | \( G_s \) | \( \frac{q-1}{2} \) |
| \( R_s \) | \( \tilde{R}_s = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{2}} \alpha, e^{-\frac{1}{2}} \beta \rangle \) | \( 0 \leq s \leq q - 1 \), \( s \neq 2^{-1} \) | \( G_{1-s} \) | \( q - 1 \) |
| \( N \) | \( \tilde{N} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{2}} \alpha, e^{-\frac{1}{2}} \beta \rangle \) | - | \( G_0 \) | 1 |
| \( L \) | \( \tilde{L}_d = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, e^d \beta_2 \rangle \) | \( 1 \leq d \leq q - 1 \) | \( G_2 \) | \( q - 1 \) |

**Proof.** Assume that \( \pi_2(G) = H_s \). Then \( G \) has the form
\[
G = \langle \sigma^n \tau^b e^c \alpha_1, \sigma^{-d} \tau e^d \alpha_2, \sigma^e \tau e^u \beta_s \rangle.
\]
The (R) conditions imply that \( a' = b' = 0 \), \( a = d = \frac{1}{g-1} \) and \( b(g^u - 1) = c(g^{1-u} - 1) = 0 \), and so \( u \neq 0 \) since \( G \) is regular. The last generator has order \( q \), so if \( u = -1 \) then \( e = 0 \) and if \( u = -s \) then \( f = 0 \).

Assume that \( b = c = 0 \). Then \( G \) is conjugate to \( \tilde{T}_{s,g} \) by \( h = \alpha_1^n \alpha_2^m \) where
\[
n = \begin{cases} 0, & \text{if } u = -1, \\ \frac{e(1-g)}{g^{1-u}}, & \text{otherwise}, \end{cases} \quad m = \begin{cases} 0, & \text{if } u = -s, \\ \frac{f(1-g)}{g^{1-u}}, & \text{otherwise}. \end{cases}
\]
If \( s = -1 \) then \( \tilde{T}_{-1,d} \) is conjugate to \( \tilde{T}_{-1,-d} \) by \( \psi = \left[ (0, 0), \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \).

Assume that both \( b \) and \( c \) are not zero. Then \( s + u - 1 = u - s + 1 = 0 \), which implies \( s = 1 \) and then \( u = 0 \), contradiction.

Let assume that \( b = 0 \) and \( c \neq 0 \). Then \( u = s - 1 \) and so \( s \neq 1 \). Up to conjugation by \( h = [(0, 0), c^{-1} Id] \) we can assume that \( c = 1 \). The third generator has order \( q \), so if \( s = 0 \) then \( e = 0 \) and if \( s = 2^{-1} \) then \( f = 0 \). Then \( G \) is conjugate to \( \tilde{T}_s \) by \( h = \alpha_1^n \alpha_2^m \) where
\[
m = \begin{cases} 0, & \text{if } s = 2^{-1}, \\ \frac{f(1-g)}{g^{1-u}}, & \text{otherwise}, \end{cases} \quad n = \begin{cases} 0, & \text{if } s = 0, \\ (1-g)g^{1-u}(1-m), & \text{otherwise}. \end{cases}
\]
Similarly, if \( c = 0 \) and \( b \neq 0 \) we have that \( G \) is conjugate to \( \overline{T}_s \). In particular, \( \overline{T}_{-1} \) and \( \overline{T}_{-2} \) are conjugate by \( \psi \). If \( s \neq -1 \) and \( \overline{T}_s \) and \( \overline{T}_{-s} \) are conjugate, then the induced map between the quotients with respect to the \( p \)-Sylow subgroups maps \( \epsilon^{1-s} \beta_s \) to itself and then \( \epsilon^{1-s} \beta_s = (\epsilon^{s-1} \beta_s)^n \) for some \( n \). Then \( n = 1 \) and \( s = 1 \), contradiction.

Assume that \( \pi_2(G) = R_s \). Then \( G \) has the form

\[
G = (\sigma^a \tau^b \epsilon^c \alpha_1, \sigma^c \tau^d \epsilon^b' \gamma, \sigma^e \tau^f \epsilon^u \beta_s).
\]

From the (R) conditions we have \( a' = b' = b = 0 \) and then \( u, d \neq 0 \) since \( G \) is regular.

The group \( G \) is conjugate by \( \gamma^{-\frac{a}{2}} \) to a group of the form

\[
G = (\sigma^a \alpha_1, \tau^d \gamma, \sigma^e \tau^f \epsilon^u \beta_s).
\]

From the (R) conditions, it follows that \( a = \frac{1}{q-1}, d(g^{u+2s-1} - 1) = 0 \) and \( f = \frac{d(1-q^{1-s})}{2} \). Therefore \( u = 1 - 2s \) and then \( s \neq 2^{-1} \). Up to conjugation by \( h = [0, 0, d^{-1}Id] \) we can assume that \( d = 1 \). Hence \( G \) is conjugate to \( \overline{R}_s \) by a suitable power of \( \alpha_1 \).

Assume that \( \pi_2(G) = N \). Using the same ideas we used in the previous cases we can assume that \( G \) has the following form

\[
G = (\sigma^a \alpha_1, \tau^d \gamma, \sigma^e \tau^f \epsilon^u \beta_s).
\]

and then \( G \) is conjugate to \( \overline{N} \) by a suitable power of \( \alpha_1 \).

Assume that \( \pi_2(G) = L \). Using the same ideas we used in the previous we can assume that \( G \) has the following form

\[
G = (\sigma^a \alpha_1, \tau^d \gamma, \sigma^e \tau^f \epsilon^u \beta_{2-1}).
\]

Up to conjugate by \( h = \alpha_2^{-\zeta} \gamma^{-\frac{\theta}{4}} \) we can assume that \( c = 0 \) From the normality of the \( p \)-Sylow it follows that \( d = \frac{1}{q-1} \) and \( f = 0 \). So the group \( G \) has the form

\[
G = (\sigma^a \alpha_1, \tau^d \gamma, \sigma^e \epsilon^u \beta_{2-1}).
\]

Then \( G \) is conjugate to \( \overline{L}_u \) by a suitable power of \( \alpha_1 \).

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{ker } \lambda & \mathbb{Z}_p^2 \times \mathbb{Z}_q & \mathcal{G}_k, k \neq 0, \pm 1, 2 & \mathcal{G}_2 & \mathcal{G}_0 & \mathcal{G}_{-1} & \mathcal{G}_1 \\
\hline
1 & - & 5 & 2q + 1 & q + 3 & \left\lceil \frac{q+3}{2} \right\rceil & q \\
p & 2 & 2(q + 1) & 3q & 2q & q + 1 & q \\
q & 2 & - & - & - & - & - \\
pq & - & - & 2 & - & - & - \\
p^2 & 1 & q - 1 & q - 1 & \left\lceil \frac{q-1}{2} \right\rceil & q - 2 & 1 \\
p^2q & - & - & - & - & - & 1 \\
\hline
\end{array}
\]

**Table 8.** Number of skew braces of \( \mathcal{G}_1 \)-type of size \( p^2q \) for \( p = 1 \) (mod \( q \)) and \( q > 3 \). For \( q = 3 \) then \( \mathfrak{S} = \{0, 1, -1\} \) and \( 2 = -1 \) and so the table has to be read accordingly.

4. Skew braces of size \( p^2q \) with \( p = -1 \) (mod \( q \))

In this section we assume that \( p \) and \( q \) are odd primes such that \( p = -1 \) (mod \( q \)) and that \( H = x^2 + \xi x + 1 \) is an irreducible polynomial over \( \mathbb{Z}_p \) such that its companion matrix

\[
F = \begin{bmatrix}
0 & -1 \\
1 & -\xi
\end{bmatrix}
\]
has order $q$. In particular,

\[(19) \quad H(F^n) = 0 \text{ if and only if } n = \pm 1.\]

The groups of size $p^2q$ are the following:

**Lemma 4.1.** [6, Proposition 21.17] The groups of size $p^2q$ are the following:

1. $\mathbb{Z}_{p^2q}$.
2. $\mathbb{Z}_p \times \mathbb{Z}_q$.
3. $\mathcal{G}_F = \langle \sigma, \tau, \epsilon | \sigma^p = \tau^q = \epsilon^q = 1, \epsilon \sigma \epsilon^{-1} = \tau, \epsilon \tau \epsilon^{-1} = \sigma^{-1} \tau^{-\xi} \rangle \cong \mathbb{Z}_p^2 \rtimes \mathbb{Z}_q$.

The enumeration of the skew braces of size $p^2q$ and $p = -1 \pmod q$ according to their additive and multiplicative group is the following:

| $\mathbb{Z}_{p^2q}$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | $\mathcal{G}_F$ |
|---------------------|---------------------------------|----------------|
| $+\setminus \circ$  | 2                               | 1              |
| $\mathbb{Z}_{p^2q}$ | -                               | $p + 2q - 4$   |
| $\mathbb{Z}_p \times \mathbb{Z}_q$ | - | 2 |
| $\mathcal{G}_F$ | - | 2 |

**Table 9.** Number of skew braces of size $p^2q$ with $p = -1 \pmod q$.

### 4.1 Skew braces of cyclic type.

In this section we denote by $A$ the cyclic group $\mathbb{Z}_{p^2q}$. In this case, the image of regular subgroups of $\text{Hol}(A)$ under $\pi_2$ is $p$. Since in this case $q \neq 1 \pmod p$, we can apply Proposition 3.3 and accordingly the enumeration of skew braces of $A$-type is collected in the following table.

| $\ker \lambda$ | $\mathbb{Z}_{p^2q}$ |
|-----------------|---------------------|
| $pq$            | 1                   |
| $p^2q$          | 1                   |

**Table 10.** Number of skew braces of $\mathbb{Z}_{p^2q}$-type of size $p^2q$ for $p = -1 \pmod q$.

### 4.2 Skew braces of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$-type.

Let $A = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ and let $G$ be a regular subgroup of $\text{Hol}(A)$. Then the size of $\pi_2(G)$ divides $p^2q$ and $|\text{Aut}(A)|$, so divides $pq$, since $p = -1 \pmod q$.

**Proposition 4.2.** Let $G$ be a regular subgroup of $\text{Hol}(A)$. Then $|\pi_2(G)| \neq pq$.

**Proof.** Let $G$ be such group then $\ker \pi_2$ is a normal subgroup of size $p$ of $G$. Therefore $G$ is not isomorphic to $\mathcal{G}_F$, since it has no normal subgroup of size $p$. On the other hand $G$ is not abelian since $\text{Aut}(A)$ has no abelian subgroup of order $pq$. \hfill $\Box$

**Lemma 4.3.** The unique skew brace of $A$-type with $|\ker \lambda| = p^2$ is $(B, +, \circ)$ where

\[(20) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & -\xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},\]

for every $0 \leq x_1, x_2, y_1, y_2 \leq p - 1$, $0 \leq x_3, y_3 \leq q - 1$.

**Proof.** Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = q$. The automorphisms of order $q$ of $A$ act trivially on the $q$-Sylow of $A$. Up to conjugation, we can assume that $\pi_2(G)$ is generate by $F$. Therefore,

\[G = \langle \sigma, \tau, \sigma^n \tau^m \epsilon^F \rangle = \langle \sigma, \epsilon^a F \rangle,\]
where \( a \neq 0 \) since \( G \) is regular. We can conjugate \( G \) by the automorphism taking \( e^a \) to \( e \) and so we can assume that \( a = 1 \). It is straightforward to verify that such group is regular. Let \( B \) the skew brace associated to \( G \). Then \( \sigma, \tau \in \ker \lambda \) and \( \epsilon \in \Fix(B) \) and then \( B = \ker \lambda + \Fix(B) \). So, formula \( (20) \) follows by Lemma 2.5.

According to Proposition 3.7 there exists a unique non trivial skew brace of size \( p^2q \) with the kernel of \( \lambda \) of size \( pq \). Hence we have the following enumeration:

\[
\begin{array}{c|cc}
\text{ker } \lambda & \mathbb{Z}_p^2 \times \mathbb{Z}_q & \mathcal{G}_F \\
p^2 & - & 1 \\
pq & 1 & - \\
p^2q & 1 & - \\
\end{array}
\]

Table 11. Number of skew braces of \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \)-type of size \( p^2q \) for \( p = -1 \pmod q \).

4.3. Skew braces of \( \mathcal{G}_F \)-type. Recall that \( F \) is the companion matrix of the irreducible polynomial \( H = x^2 + \xi x + 1 \) and it has order \( q \). A presentation of the group \( \mathcal{G}_F \) is the following:

\( (21) \)

\[
\mathcal{G}_F = \langle \sigma, \tau, \epsilon, \sigma^p = \tau^p = \epsilon^q = 1, \epsilon \sigma \epsilon^{-1} = \tau, \epsilon \tau \epsilon^{-1} = \sigma^{-1}\tau^{-\xi} \rangle.
\]

An automorphism of \( \mathcal{G}_F \) is determined by its image on the generators, i.e. by its restriction to \( \langle \sigma, \tau \rangle \) given by a matrix and by the image on \( \epsilon \). According to \cite{3} subsections 4.1, 4.4], the map

\[
\phi : \mathbb{Z}_p^2 \rtimes \rho \left( \text{N}_{\text{GL}_2(p)}(F) \right) \longrightarrow \text{Aut}(\mathcal{G}_F), \quad [(n, m), M] \mapsto h_{\pm} = \begin{cases} h_{\langle \sigma, \tau \rangle} = M \\ \epsilon \mapsto \sigma^n \tau^m \epsilon^{\pm 1} \end{cases},
\]

where \( \phi([(n, m), M]) = h_{\pm} \) if \( M \in C_{\text{GL}_2(p)}(F) \) and \( h_{\text{otherwise}} \) otherwise, is a group isomorphism. The form of \( M_{\pm} = h_{\pm} \langle \sigma, \tau \rangle \) is the following:

\( (22) \)

\[
M_+ = \begin{bmatrix} x & -y \\ y & x - \xi y \end{bmatrix}, \quad M_- = \begin{bmatrix} x & y - \xi y \\ y & -x \end{bmatrix},
\]

where \( x, y \in \mathbb{Z}_p \) and \( x^2 + y^2 - \xi xy \neq 0 \). The \( p \)-Sylow subgroup of \( \mathcal{G}_F \) is characteristic, so the map

\[
\nu : \text{Aut}(\mathcal{G}_F) \longrightarrow \text{Aut}(\mathcal{G}_F/\langle \sigma, \tau \rangle), \quad h_{\pm} \mapsto \pm 1,
\]

is a group homomorphism. The kernel of \( \nu \) is \( \text{Aut}(\mathcal{G}_F)^+ = \mathbb{Z}_p^2 \rtimes C_{\text{GL}_2(p)}(F) \) and it contains the \( p \)-Sylow of \( \text{Aut}(\mathcal{G}_F) \), generated by \( \alpha_1 = [(1, 0), Id] \) and \( \alpha_2 = [(0, 1), Id] \), and the elements of odd order of \( \text{Aut}(\mathcal{G}_F) \). The centralizer of \( F \) is cyclic and it acts transitively on the non zero elements of \( \langle \sigma, \tau \rangle \).

**Proposition 4.4.** Let \( G \) be a regular subgroup of \( \text{Hol}(\mathcal{G}_F) \). Then \( |\pi_2(G)| \neq p, pq \).

**Proof.** Let \( G \) be a regular subgroup of \( \text{Hol}(\mathcal{G}_F) \).

If \( |\pi_2(G)| = p \), according to Lemma 2.1, the kernel of \( \pi_2 \) is a subgroup of order \( pq \) of \( \mathcal{G}_F \). But \( \mathcal{G}_F \) has no such subgroups, contradiction.

If \( |\pi_2(G)| = pq \), then ker \( \pi_2 \) is a normal subgroup of \( G \) of size \( p \). Therefore \( G \) is abelian, since \( \mathcal{G}_F \) has no such normal subgroups. On the other hand, \( \text{Aut}(\mathcal{G}_F) \) has no abelian subgroup of size \( pq \), contradiction. \( \square \)

**Lemma 4.5.** There is just one conjugacy class of regular subgroup \( G \) of \( \text{Hol}(\mathcal{G}_F) \) with \( |\pi_2(G)| = p^2 \). A representative is

\[
H = \langle \epsilon, u\alpha_1, w\alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q,
\]

where \( v = (1 - F)^{-1}(\sigma) \) and \( w = (1 - F)^{-1}(\tau) \).
Proof. The group $H$ has the desired property. Assume that $G$ is a regular subgroup of $\text{Hol}(G_F)$ with $|\pi_2(G)| = p^2$. Then the image of $\pi_2$ is the normal $p$-Sylow of $\text{Aut}(G_F)^+$ generated by $\alpha_1$ and $\alpha_2$. The kernel of $\pi_2$ is a normal subgroup of size $q$ of $G$, then $G$ is abelian. Up to conjugation, we can assume that the kernel is generated by $\epsilon$ and so we have $G = \langle \epsilon, uf, wg \rangle$. From the fact that $G$ is abelian, it follows that $v = (1 - F)^{-1}(\sigma)$ and $w = (1 - F)^{-1}(\tau)$. $\square$

Lemma 4.6. A set of representative regular subgroups $G$ of $\text{Hol}(G_F)$ with $|\pi_2(G)| = q$ is

$$G_d = \langle \sigma, \tau, e^d f \rangle \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } d = -1, \\ G_F, & \text{otherwise}, \end{cases}$$

where $f = [(0, 0), F]$ and $1 \leq d \leq q - 1$.

Proof. The groups $G_d$ have size $p^2q$ and they are regular. Assume that $G_d$ and $G_e$ are conjugate, then

$$g(\epsilon)^e g f g^{-1} = \epsilon^{+e} f^{\pm 1} = e^{sd} f^s \pmod{\langle \sigma, \tau \rangle}.$$ 

Therefore $s = \pm 1$ and $c = d$.

Let $G$ be a regular subgroup of $\text{Hol}(G_F)$ such that $|\pi_2(G)| = q$. Up to conjugation, we can assume that $\pi_2(G)$ is generated by $f$. The kernel of $\pi_2$ is the $p$-Sylow of $G_F$ and then we can assume that

$$G = \langle \sigma, \tau, e^d f \rangle$$

where $d \neq 0$. $\square$

Lemma 4.7. A set of representatives of regular subgroups $G$ of $\text{Hol}(G_F)$ such that $|\pi_2(G)| = p^2q$ is

| Groups       | Parameters | Class | #   |
|--------------|------------|-------|-----|
| $G_e = \langle u e \alpha_1, f(u)e \alpha_2, e^cf \rangle$ | $1 \leq c \leq q - 1$, $c \neq -2$ | $G_F$ | $q - 2$ |
| $H_a = \langle f(v_\sigma^{-1})\sigma^{-1}v_\alpha \alpha_2, e^2f \rangle$ | $1 \leq a \leq p - 1$ | $G_F$ | $p - 1$ |

where $f = [(0, 0), F]$, $u_e = H(F^{c+1})^{-1}(F - 1)^{-1}(1 - F^c)(1 - F^{c+2})(\sigma)$ and $v_\alpha = (x, y)$ is a fixed element such that $\Psi(v_\alpha) = x^2 + y^2 - x + y - \xi xy = a$.

Proof. The subgroups in the statement have the desired properties. The unique subgroup of order $p^2q$ of $\text{Aut}(A)$ is generated by $\alpha_1, \alpha_2$ and $f = [(0, 0), F]$ and it is isomorphic to $G_F$. Therefore, the standard presentation of the regular subgroup $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is

$$G = \langle u e^c \alpha_1, v e^b \alpha_2, w e^c f \rangle$$

where $u, v, w \in \langle \sigma, \tau \rangle$. From the (R) conditions it follows that $a = b = 0$ and then $c \neq 0$ since $G$ is regular, and that

$$(23) \quad \begin{cases} v = F^{c+1}(u) + (F - 1)^{-1}(1 - F^c)F(\sigma) \\ h(F^{c+1})(u) = (F - 1)^{-1}(1 - F^c)(1 - F^{c+2})(\sigma). \end{cases}$$

According to (19), if $c \neq -2$ then $h(F^{c+1}) \neq 0$ and therefore it is an invertible automorphism of $\mathbb{Z}_p^2$ (ker $h(F^{c+1})$ is an $F$-invariant subspace, and therefore it is either 0 or $\mathbb{Z}_p^2$). If $c \neq -2$ then (23) is equivalent to

$$(24) \quad \begin{cases} v = F(u) \\ u = h(F^{c+1})^{-1}(F - 1)^{-1}(1 - F^c)(1 - F^{c+2})(\sigma), \end{cases}$$

and so

$$G = \langle u \alpha_1, f(u)\alpha_2, w e^c f \rangle$$

where $c \neq -2$. If $c + 1 = 0$, the condition $(w e^c f)^q = 1$ implies that $w = 0$ and so $G = G_{-1}$. Otherwise, $tGt^{-1} = G_e$ where $t = \alpha_1^n \alpha_2^m$ for some $n, m$. Indeed, this is equivalent to have $w = 0$. 

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\[ nx + mF(x), \text{ where } x = (1 - F)u - (1 - F)^{-1}(F^c - 1)F(\sigma). \] Since \( x \neq 0 \) for every \( c \neq -1, -2 \) then \( \{x, F(x)\} \) is a basis of \( \mathbb{Z}_p^2 \) and therefore \( w \in \{x, F(x)\} \).

Let \( c = -2 \). In such case, the second condition in (23) is trivial according to (19) and the first one is equivalent to

\[ u = F(v) - (F + 1)(\sigma). \]

Since \( G \) is regular, then \( u \) and \( v \) are linearly independent. If \( v = (x, y) \) then \( u = (-y - 1, x - ky - 1) \) and therefore \( u \) and \( v \) are linearly independent if and only if \( \Psi(x, y) \neq 0 \). Since the number of solution of the equation \( \Psi(x, y) = 0 \) is \( p + 1 \), there are \( p^2 - p - 1 \) choices for \( v \).

Let \( P_v = \langle u\alpha_1, v\alpha_2 \rangle \). The group \( P_v \) where \( \bar{v} = (\frac{1}{\tau^2}, -\frac{1}{\tau^2}) \) is normalized by \( \text{Aut}(A) \) and for the other admissible values of \( v \) the index of the normalizer of \( P_v \) is \( p + 1 \). Hence, there are \( 1 + \frac{p^2 - p - 2}{p+1} = p - 1 \) different conjugacy classes of such subgroups.

In particular, \( P_v \) and \( P_{v'} \) are conjugate if and only if \( M_\pm(P_v) = P_{v'} \) for some \( M_\pm \) as in (22). This condition turns out to be equivalent to have \( \Psi(v) = \Psi(v') \). So to choose a set of representatives of \( P_v \) it is enough to find a set of admissible elements with different values under \( \Psi \).

We show that \( G \) is conjugate to some \( H_a \) by \( h \in \langle \alpha_1, \alpha_2 \rangle \). Indeed, this is equivalent to have that

\[ w \in U = \langle u - v - (F - 1)^{-1}(F^{-2} - 1)(\sigma), u + (\xi + 1)v + (F - 1)^{-1}(F^{-2} - 1)(1 + \xi F)(\sigma) \rangle. \]

If \( v = (x, y) \) and \( \Psi(v) \neq 0 \), the generators of \( U \) are linearly independent and so \( G \) is conjugate to \( H_a \) for \( a = \Psi(v) \).

We summarize the contents of this subsection in the following table:

| \( |\ker \lambda| \) | \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) | \( G_F \) |
|-----------------|-----------------|-----------------|
| 1               | -               | \( p + q - 3 \) |
| \( q \)         | 1               | -               |
| \( p^2 \)       | 1               | \( q - 2 \)     |
| \( p^2q \)      | -               | 1               |

**Table 12.** Number of skew braces of \( G_F \)-type of size \( p^2q \) for \( p = -1 \pmod q \).

5. Skew braces of size \( p^2q \) with \( q = 1 \pmod p \)

In this section we assume that \( p \) and \( q \) are odd primes such that \( q = 1 \pmod p \) and \( q \neq 1 \pmod {p^2} \), unless otherwise indicated. We also fix an element \( r \) of order \( p \) in \( \mathbb{Z}_q^\times \). The relevant groups for this section are the following.

**Lemma 5.1.** [6, Proposition 21.17] The groups of size \( pq \) are the following:

- (i) \( \mathbb{Z}_{p^2q} = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \ \tau \sigma = \sigma \tau \rangle. \)
- (ii) \( \mathbb{Z}_p^2 \times \mathbb{Z}_q = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^q = \epsilon^q = 1, \ \sigma \tau = [\sigma, \epsilon] = [\tau, \epsilon] = 1 \rangle. \)
- (iii) \( \mathbb{Z}_p \times \langle \mathbb{Z}_q \times \mathbb{Z}_p \rangle = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^q = \epsilon^q = 1, \ \sigma \tau = [\sigma, \epsilon] = [\tau, \sigma] = 1, \ \sigma \epsilon \sigma^{-1} = \epsilon^{-1} \rangle. \)
- (iv) \( \mathbb{Z}_q \times \langle \mathbb{Z}_p \times \mathbb{Z}_q \rangle = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \ \sigma \tau \sigma^{-1} = \tau^r \rangle. \)

We summarize in the following table the total number of skew braces according to the additive and multiplicative isomorphism class of groups.
Table 13. Number of skew braces of size $p^2q$ with $q = 1 \pmod{p}$ and $q \neq 1 \pmod{p^2}$.

|   | $\mathbb{Z}_{p^2q}$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$ | $\mathbb{Z}_q \rtimes \mathbb{Z}_p^2$ |
|---|-----------------|------------------|------------------|------------------|
| $\oplus \circ$ | 2 | - | - | $p$ |
| $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | - | 2 | 4 | - |
| $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$ | - | 4 | $6p - 4$ | - |
| $\mathbb{Z}_q \rtimes \mathbb{Z}_p^2$ | $2p$ | - | - | $2p(p - 1)$ |

5.1. **Skew braces of cyclic type.** In this section we denote by $A$ the cyclic group $\mathbb{Z}_{p^2q}$. For the automorphism group we employ the same notation as in Subsection 3.1.

If $G$ is a regular subgroup of $\text{Hol}(A)$ then $|\pi_2(G)| \in \{p, p^2\}$. First we show that there are not regular subgroups in the case $p^2$.

**Proposition 5.2.** Let $G$ be a regular subgroup of $\text{Hol}(A)$. Then $|\pi_2(G)| \neq p^2$.

**Proof.** Let $G$ be such group. The image by $\pi_2$ of $G$ is the unique $p$-Sylow subgroup $\langle \varphi_{p+1,1}, \varphi_{1,r} \rangle$ of $\text{Aut}(A)$. The unique subgroup of $A$ of size $q$ is $\langle \tau \rangle$, therefore $G$ has the following standard presentation:

$$\langle \tau, \sigma^a \varphi_{p+1,1}, \sigma^b \varphi_{1,r} \rangle.$$  

The group $\pi_2(G)$ is elementary abelian and so, by the (R) conditions we have $a, b = 0 \pmod{p}$. Hence, $\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle$ and so $G$ is not regular, contradiction. $$\square$$

The group $\text{Aut}(A)$ has a unique $p$-Sylow subgroup, which is generated by $\varphi_{p+1,1}$ and $\varphi_{1,r}$, which is elementary abelian. So the subgroups of order $p$ in $\text{Aut}(A)$ are

$$\langle \varphi_{p+1,1} \rangle \quad \text{and} \quad \langle \varphi_{jp+1,r} \rangle$$

where $0 \leq j \leq p - 1$.

**Proposition 5.3.** The skew braces of cyclic type of size $p^2q$ with $|\ker \lambda| = pq$ are $A_{j,s} = (A, +, \circ)$ for $(j, s) \in \{(1, 0) \cup \{(j, 1) : 0 \leq j \leq p - 1\}$ where

$$\binom{n}{m} + \binom{s}{t} = \binom{n + s}{m + t}, \quad \binom{n}{m} \circ \binom{s}{t} = \binom{n + (njp + 1)}{m + r^sn t}$$

for every $0 \leq n, s \leq p^2 - 1$ and $0 \leq m, t \leq q - 1$. In particular,

$$\langle A_{(j,s)}, \circ \rangle \cong \begin{cases} \mathbb{Z}_{p^2q} & \text{if } (j, s) = (1, 0), \\ \mathbb{Z}_q \rtimes \mathbb{Z}_p^2 & \text{otherwise.} \end{cases}$$

**Proof.** The groups

$$H = \langle \sigma^p, \tau, \sigma \varphi_{p+1,1} \rangle \cong \mathbb{Z}_{p^2q}, \quad G_j = \langle \sigma^p, \tau, \sigma \varphi_{jp+1,r} \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p^2,$$

for $0 \leq j \leq p - 1$ are regular subgroups and they are not conjugate, since their image under $\pi_2$ are not.

Let $G$ be a regular subgroup with $|\pi_2(G)| = p$. According to (26), the subgroups of size $p$ of $\text{Aut}(\mathbb{Z}_{p^2q})$ are $\langle \varphi_{p+1,1} \rangle$ and $\langle \varphi_{jp+1,r} \rangle$ with $0 \leq j \leq p - 1$. In the first case we can argue as in Proposition 5.3 and the associate skew brace is $A_{1,0}$.

Assume that $\pi_2(G) = \langle \varphi_{jp+1,r} \rangle$ for some $0 \leq j \leq p - 1$. Then the kernel of $\pi_2$ is the unique subgroup of order $pq$, namely $\langle \sigma^p, \tau \rangle$ and so $G$ has the following standard presentation:

$$G = \langle \sigma^p, \tau, \sigma^a \varphi_{jp+1,r} \rangle,$$
where \( a \neq 0 \) since \( G \) is regular. Then, \( G \) is conjugate to \( G_j \) by \( \varphi_{a^{-1},1} \). Let \( B_j \) be the skew brace associate to \( G_j \). Then, since \( \tau, \sigma^p \in \ker \pi_2 \) and \( \sigma \circ \sigma \circ \ldots \circ \sigma = \sigma^{n(n-1)p} \) we have that
\[
\lambda_{\sigma^n \tau^m} = \lambda_{\tau^m \sigma^n} = \lambda_{\sigma^n} = \lambda_{\tau^m} = \varphi_{j,1}\tau
\]
and so the formula in the statement follows.

Proof. Let \( \pi_0 \) for every \( a \in G \), \( \sigma \) associate to \( G \) and so the formula in the statement follows.

We summarizes the results of this subsection in the following table:

| \( \ker \lambda \) | \( \mathbb{Z}_q \times \mathbb{Z}_p \) | \( \mathbb{Z}_q \times \mathbb{Z}_p \) |
|------------------|------------------|------------------|
| \( pq \)         | \( 1 \)          | \( p \)          |
| \( p^2 q \)      | \( 1 \)          | \( - \)          |

Table 14. Number of skew braces of \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \)-type of size \( p^2 q \) for \( q = 1 \mod p \) and \( p \neq 1 \mod p^2 \).

5.2. Skew braces of \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \)-type. In this section we denote by \( A \) the group \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) and by \( B \) the matrix defined in Remark 3.6. We can think of the elements of the automorphism group of \( A \) as \( xM \) where \( x \) is an element in \( \mathbb{Z}_q^* \) and \( M \) is an invertible matrix, as we did in Subsection 3.2.

Proposition 5.4. The skew braces of \( A \)-type with \( |\ker \lambda| = pq \) are \((A_{i,j},+\circ)\) where \( 0 \leq i, j \leq 1 \) and \((i,j) \neq (0,0)\), where
\[
\begin{pmatrix} n \\ m \\ l \end{pmatrix} + \begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} n+s \\ m+t \\ l+u \end{pmatrix}, \quad \begin{pmatrix} n \\ m \\ l \end{pmatrix} \circ \begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} n+s \\ m+jm \\ l+r^m u \end{pmatrix},
\]
for every \( 0 \leq n, m, s, t, u \leq p-1 \) and \( 0 \leq l, u \leq q-1 \). In particular,
\[(A_{i,j},\circ) \cong \begin{cases} \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q), & \text{otherwise.} \\
A, & \text{if } (i,j) = (0,1),
\end{cases}
\]

Proof. Let \( G \leq \text{Hol}(A) \) be a regular subgroup with \( |\pi_2(G)| = p \). Then, up to conjugation the image \( \pi_2(G) \) is generated by \( \alpha = r^2C^3 \), for \( i, j \in \{0,1\} \) and \((i,j) \neq (0,0)\). The kernel of \( \pi_2|_G \) has order \( pq \), so \( \ker \pi_2|_G = \langle \epsilon, v \rangle \) for some \( v \in \langle \sigma, \tau \rangle \). Therefore we have the following standard presentation:
\[G = \langle \epsilon, v, u \rangle,\]
for some \( u \in \langle \sigma, \tau \rangle \). If \((i,j) = (0,1)\) we can argue as in Proposition 3.7 and then the formula for this case follows.

Assume that \((i,j) = (1,0)\). Then, up to conjugation by an element of \( GL_2(p) \) we can assume that \( v = \sigma \) and \( u = \tau \).

Assume that \((i,j) = (1,1)\). By condition (K) we have \( v \in \langle \sigma \rangle \) and then
\[G = \langle \sigma, \epsilon, \tau^a C \rangle,\]
for \( a \neq 0 \). We can assume \( a = 1 \), otherwise we conjugate by \( a^{-1}\text{Id} \).

Let \( A_{i,j} \) be the skew brace associated to \( G \). In both cases, we have that \( \lambda_{\alpha_{n,m}\epsilon} = \lambda_{\tau^m} = \lambda_{\tau}^m \), since \( \sigma, \epsilon \in \ker \pi_2 \) and \( \tau_{\sigma} = \tau \) (mod \( |\sigma| \)). Then the claim follows.

Proposition 5.5. Let \( w \) a fixed quadratic non residue modulo \( p \). The skew braces of \( A \)-type with \( |\ker \lambda| = q \) are \((B_{s,+\circ})\) for \( s \in \{1, w\} \), where
\[
\begin{pmatrix} n \\ m \\ l \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} n+x \\ m+y \\ l+z \end{pmatrix}, \quad \begin{pmatrix} n \\ m \\ l \end{pmatrix} \circ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} n+x+y^2 \\ m+y \\ l+wr^m \end{pmatrix},
\]
and so the formula in the statement follows.
for every $0 \leq n, m, x, y \leq p - 1$ and $0 \leq l, z \leq q - 1$. In particular, $(B_s, \circ) \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times_r \mathbb{Z}_p)$.

**Proof.** The groups

$$G_s = \langle \epsilon, \tau^s C, \sigma \rangle \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times_r \mathbb{Z}_p)$$

for $s \in \{1, w\}$ are regular subgroup of Hol$(A)$ and $|\pi_s(G_s)| = p^2$. They are not conjugate, since if $hG_1h^{-1} = G_w$ for some $h \in \text{Aut}(A)(C)$, say $h = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ then

$$\sigma^y \tau^z C_x^z = (\tau^w C)^{\frac{z}{w}} \pmod{\langle \epsilon \rangle} \quad \text{and} \quad \sigma^y r = \sigma r \pmod{\langle \epsilon \rangle},$$

which implies $x = 1$ and then $w = z^2$, contradiction.

Let $G \leq \text{Hol}(A)$ be a regular subgroup and let $|\pi_s(G)| = p^2$. Then, up to conjugation we have that $\pi_s(G) = \langle C, r \rangle$. Hence the group has the standard presentation

$$G = \langle \epsilon, uC, vr \rangle,$$

for some $u, v \in \langle \sigma, \tau \rangle$. The (R) conditions imply that $v \in \langle \sigma \rangle$, and so

$$G_u = \langle \epsilon, uC, \sigma^m r \rangle$$

where $m \neq 0$ and $u = \sigma^s \tau^t$ for $t \neq 0$ since $G$ is regular. Conjugation by $m^{-1}Id$ allows us to assume $m = 1$. We can conjugate by $C^\frac{1}{w}$ to assume that $u = \tau^t$. If $t = x^2$ then $G_{x^2}$ is conjugate to $G_1$ by $h$, otherwise if $w = tz^2$ then $G_{x^2}$ is conjugate to $G_w$ by $g$ where

$$h = \begin{bmatrix} 1 & 2^{-1} x(x - 1) \\ 0 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 & 2^{-1} (1 - z) \\ 0 & z \end{bmatrix}.$$

Let $B_s$ be the skew brace associated to $G_s$. Then, since $\epsilon \in \ker \lambda$ and $\lambda_s(\sigma) = \sigma$ we have that $\sigma \circ \cdots \circ \sigma = \sigma^n$ and $\epsilon \circ \cdots \circ \epsilon = \epsilon^t$. On the other hand, $(\sigma r)^{-\frac{1}{w}} (\tau^s C)^{\frac{s}{w}} = \tau r C^\frac{s}{w} C^\frac{1}{w} \in G_s$ and moreover

$$(\sigma r)^{-\frac{m}{w}} (\tau r C^\frac{s}{w}) = \tau^m r C^\frac{s}{w} C^\frac{1}{w} \in G_s,$$

so $\lambda_{r^m} = r^m C^\frac{m}{w} C^\frac{1}{w}$. So we have

$$\lambda_{r^m} = \lambda_{r^m} \lambda_{r^m} = \lambda_{r^m} \lambda_{r^m} = \lambda_{r^m} \lambda_{r^m} = \lambda_{r^m} \lambda_{r^m} = r^{n + \frac{m - 1}{2w}} C^\frac{m}{w},$$

where we used that $\lambda_{r^m}(\tau) = \tau$. Hence, the formula follows. \hfill $\square$

We summarize the contents of this subsection in the following table:

| $|\ker \lambda|$ | $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ | $\mathbb{Z}_p \times (\mathbb{Z}_q \times_r \mathbb{Z}_p)$ |
|------------------|---------------------------------|---------------------------------|
| $q$              | $-$                            | $2$                            |
| $pq$             | $1$                            | $2$                            |
| $p^2q$           | $1$                            | $-$                            |

**Table 15.** Number of skew braces of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$-type of size $p^2q$ for $q = 1 \pmod{p}$ and $q \neq 1 \pmod{p^2}$. 

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5.3. **Skew braces of** $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$-**type.** In this section we denote by $A$ the group $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$. A presentation of such group is

$$A = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, \ [\epsilon, \tau] = [\tau, \sigma] = 1, \sigma \epsilon \sigma^{-1} = \epsilon^r \rangle.$$ 

According to [8, Subsection 4.6] the mapping

$$
\phi : (\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times) \times (\mathbb{Z}_q \rtimes \mathbb{Z}_q^\times) \rightarrow \text{Aut}(A), \quad [(l, i), (s, j)] \mapsto \varphi_{l, i, s, j} = \begin{cases} 
\epsilon \mapsto \epsilon^3 \\
\tau \mapsto \tau^i \\
\sigma \mapsto \tau^j \epsilon^8 \sigma
\end{cases},
$$

is an isomorphism of groups. Since $\text{Aut}(A)$ is a direct product we can write $\varphi_{l, i, s, j} = \alpha_{l, i} \beta_{s, j}$, where

$$
\alpha_{l, i} = \begin{cases} 
\epsilon \mapsto \epsilon \\
\tau \mapsto \tau^i \\
\sigma \mapsto \tau^j \epsilon^8 \sigma
\end{cases} \quad \text{and} \quad \beta_{s, j} = \begin{cases} 
\epsilon \mapsto \epsilon^3 \\
\tau \mapsto \tau \\
\sigma \mapsto \epsilon^8 \sigma.
\end{cases}
$$

In particular, $|\text{Aut}(A)| = pq(p-1)(q-1)$. Hence $|\pi_2(G)|$ divides $p^2q$ for each regular subgroup $G$.

The conjugacy classes of subgroups of $\text{Aut}(A)$ are given by:

| Size | Generators | Class |
|------|------------|-------|
| $p$  | $\alpha_{1, 1}$, $\beta_{0, r}$, $\alpha_{1, 1} \beta_{0, r}$ | $\mathbb{Z}_p$ |
| $q$  | $\beta_{1, 1}$ | $\mathbb{Z}_q$ |
| $pq$ | $\alpha_{1, 1}$, $\beta_{1, 1}$, $\beta_{0, r}$, $\alpha_{1, 1} \beta_{0, r}$, $\beta_{0, r}$, $\alpha_{1, 1} \beta_{0, r}$, $\beta_{1, 1}$ | $\mathbb{Z}_{pq}$, $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ |
| $p^2$ | $\alpha_{1, 1}$, $\beta_{0, r}$ | $\mathbb{Z}_p^2$ |
| $p^2q$ | $\alpha_{1, 1}$, $\beta_{1, 1}$, $\beta_{0, r}$ | $A$ |

**Table 16.** Conjugacy classes of subgroups of $\text{Aut}(A)$.

**Proposition 5.6.** Let $(B, +, \circ)$ a skew brace of $A$-type. The following are equivalent:

(i) $|\ker \lambda_B| = p^2$.

(ii) $B \cong B_1 \times B_2$, where $B_1$ is the trivial skew brace of size $p$ and $B_2$ is the unique skew brace of size $pq$ with $|\ker \lambda_{B_2}| = p$.

In particular, $(B, \circ) \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$.

**Proof.** (i) $\Rightarrow$ (ii) The unique subgroup of size $q$ of $\text{Aut}(A)$ is generated by $\beta_{1, 1}$ and the kernel is a $p$-Sylow subgroup of $A$ which can be taken as $\langle \sigma, \tau \rangle$ up to conjugation. Therefore $\tau \in \ker \lambda_B \cap \text{Fix}(B)$ and so it easy to see that $\tau \in Z(B, \circ)$ and then $I = \langle \tau \rangle = \langle \tau \rangle_\circ$ is an ideal of $B$. Moreover, $J = \langle \epsilon, \sigma \rangle_\circ$ is a left ideal and since $\tau \in Z(B, \circ)$ then $J$ is an ideal of $B$. Therefore $B = I + J$ and so it is a direct product of the trivial skew brace of size $p$ and a skew brace $B_2$ of size $pq$ with $|\ker \lambda_{B_2}| = p$. According to [1, Theorem 3.6], there exists a unique such brace and $(B_2, \circ) \cong \mathbb{Z}_{pq}$.

(ii) $\Rightarrow$ (i) It follows since $|\ker \lambda_{B_1 \times B_2}| = |\ker \lambda_{B_1}| |\ker \lambda_{B_2}| = p^2$. \qed

**Lemma 5.7.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p$ is

$$H = \langle \epsilon, \tau, \sigma \alpha_{1, 1} \rangle \cong A, \quad L = \langle \epsilon, \sigma, \tau \beta_{0, r} \rangle \cong A, \quad G_{c, \epsilon, \tau, \sigma} = \langle \epsilon, \tau, \sigma \epsilon \tau \rangle \cong \begin{cases} 
\mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } c = -1, \\
A, & \text{otherwise,}
\end{cases}$$

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for \(1 \leq c \leq p - 1\) and \(\theta \in \{\beta_{0,r}, \alpha_{1,1}\beta_{0,r}\}\).

**Proof.** The groups \(H, L\) and \(G_{c,\theta}\) are regular. If \(G_{c,\beta_{0,r}}\) and \(G_{d,\beta_{0,r}}\) are conjugate by some \(h \in \text{Aut}(A)\), then \(h\) normalizes \(\beta_{0,r}\) and so it centralizes it. So \(h(\sigma)^{c}\beta_{0,r} = \sigma^{c}\beta_{0,r} \pmod{\langle \epsilon, \tau \rangle}\). Then \(\sigma^{c}\beta_{0,r} = \sigma^{d}\beta_{0,r} \pmod{\langle \epsilon, \tau \rangle}\) which implies that \(c = d\). The same argument applies if \(\theta = \alpha_{1,1}\beta_{0,r}\).

Let \(G\) be a regular subgroup of \(\text{Hol}(A)\) such that \(|\pi_{2}(G)| = p\). According to Table 16 we need to discuss three cases.

Assume that \(\pi_{2}(G) = \langle \alpha_{1,1} \rangle\). Then the kernel has order \(pq\) and so, up to conjugation by a power of \(\alpha_{1,1}\) we can assume that

\[ G = \langle \epsilon, \sigma^{n}\tau^{m}, \sigma^{a}\tau^{b}\alpha_{1,1} \rangle. \]

By condition (K) we have \(n = 0\). So \(G = \langle \epsilon, \tau, \sigma^{a}\alpha_{1,1} \rangle\) and \(G\) is conjugate to \(H\) by \(\alpha_{0,a^{-1}}\).

Assume that \(\pi_{2}(G) = \langle \beta_{0,r} \rangle\). Then \(G\) has the following standard presentation:

\[ G = \langle \epsilon, \sigma^{n}\tau^{m}, \sigma^{a}\tau^{b}\beta_{0,r} \rangle. \]

If \(n \neq 0\), we conjugate \(G\) by \(\alpha_{-\frac{m}{n},1}\) and then by \(\alpha_{0,b^{-1}}\) and we get \(L\). If \(n = 0\) we have

\[ G = \langle \epsilon, \tau, \sigma^{b}\beta_{0,r} \rangle = G_{b,\beta_{0,r}}. \]

Assume that \(\pi_{2}(G) = \langle \alpha_{1,1}\beta_{0,r} \rangle\). Then

\[ G = \langle \epsilon, \sigma^{n}\tau^{m}, \sigma^{a}\tau^{b}\alpha_{1,1}\beta_{0,r} \rangle. \]

The condition (K) implies that \(n = 0\) and so \(G = \langle \epsilon, \tau, \sigma^{a}\alpha_{1,1}\beta_{0,r} \rangle = G_{a,\alpha_{1,1}\beta_{0,r}}\).

**Lemma 5.8.** A set of representatives of conjugacy classes of regular subgroups \(G\) of \(\text{Hol}(A)\) with \(|\pi_{2}(G)| = p^{2}\) is

\[ G_{a} = \langle \epsilon, \sigma^{a}\alpha_{1,1}, \tau\beta_{0,r} \rangle, \]

for \(1 \leq a \leq p^{2} - 1\).

**Proof.** The groups \(G_{a}\) are regular. If \(G_{a}\) is conjugate to \(G_{b}\) by \(h = \alpha_{l,i}\beta_{s,j}\) then \(h\sigma^{a}\alpha_{1,1}h^{-1} = \tau^{l_{a}}\alpha^{a}\alpha_{1,1} \in G_{b}\). Then \(l = 0\) and so \(\sigma^{a}\alpha_{1,1} = (\sigma^{b}\alpha_{1,1})^{i} \pmod{\langle \epsilon \rangle}\). Therefore \(i = 1\) and \(a = b\).

The kernel of \(\pi_{2}|_{G}\) is the subgroup generated by \(\epsilon\). Hence, up to conjugation we can assume that

\[ G = \langle \epsilon, \sigma^{a}\tau^{b}\alpha_{1,1}, \sigma^{c}\tau^{d}\beta_{0,r} \rangle. \]

From the (R) conditions we have \(c = 0\) and then \(a, d \neq 0\) since \(G\) is regular. Up to conjugation by \(\alpha_{0,d^{-1}}\) we can assume that \(d = 1\). Then \(G\) is conjugate to \(G_{a}\) by \(\alpha_{1,1}^{-1}\).

**Lemma 5.9.** A set of representatives of conjugacy classes of regular subgroups \(G\) of \(\text{Hol}(A)\) with \(|\pi_{2}(G)| = pq\) is

\[ H_{1} = \langle \tau, \sigma\alpha_{1,1}, \epsilon^{1}\beta_{1,1} \rangle \cong \mathbb{Z}_{p}^{2} \times \mathbb{Z}_{q}, \quad H_{2} = \langle \sigma, \tau\beta_{0,r}, \epsilon^{1}\beta_{1,1} \rangle \cong A, \]

\[ G_{a,\theta} = \langle \tau, \sigma^{\theta}, \epsilon^{1}\beta_{1,1} \rangle \cong A, \]

for \(1 \leq a \leq p - 1\) and \(\theta \in \{\beta_{0,r}, \alpha_{1,1}\beta_{0,r}\}\).

**Proof.** The same argument as in Lemma 5.7 shows that the groups in the statement are regular and not conjugate.

Let \(G\) be a regular subgroup of \(\text{Hol}(A)\) with \(|\pi_{2}(G)| = pq\). We have to check three cases and for all of them we can assume that \(\ker \pi_{2}|_{G}\) is generated by \(\tau\) or \(\sigma\), up to the action of the normalizer of \(\pi_{2}(G)\).

Let \(\pi_{2}(G) = \langle \alpha_{1,1}, \beta_{1,1} \rangle\). If \(\ker \pi_{2}|_{G} = \langle \sigma \rangle\) then the kernel is not normal in \(G\). Otherwise,

\[ G = \langle \tau, \epsilon^{a}\beta^{b}\alpha_{1,1}, \epsilon^{d}\beta_{1,1} \rangle. \]
The condition (R) implies that $d = 0$ and $c = \frac{1}{r-1}$ and therefore $b \neq 0$ since $G$ is regular. So $G$ is conjugate to $H_1$ by $\alpha_0 b^\beta_{1,1}$ where $n = -a \frac{r-1}{p-1}$.

Let $\pi_2(G) = \langle \beta_{0,r}, \beta_{1,1} \rangle$. If $\ker\pi_2|_G$ is generated by $\tau$ then

$$G = \langle \tau, \epsilon^a \sigma^b \beta_{0,r}, \epsilon^c \sigma^d \beta_{1,1} \rangle.$$ 

From the (R) conditions we have that $d = 0$ and $c = \frac{1}{r-1}$. Moreover $b \neq 0$ since $G$ is regular. Then $G$ is conjugate to $G_a$ by $\beta_{1,1}^{-a}$.

If $\ker\pi_2|_G = \langle \sigma \rangle$, then

$$G = \langle \sigma, \epsilon^a r^b \beta_{0,r}, \epsilon^c r^d \beta_{1,1} \rangle.$$ 

According to the (K) conditions we have $c = \frac{1}{r-1}$ and $a = 0$ and by (R) we have $d = 0$ and $b = 1$.

Let $\pi_2(G) = \langle \alpha_{1,1} \beta_{0,r}, \beta_{1,1} \rangle$. If $\ker\pi_2|_G = \langle \tau \rangle$, then

$$G = \langle \tau, \epsilon^a r^b \alpha_{1,1} \beta_{0,r}, \epsilon^c r^d \beta_{1,1} \rangle.$$ 

The (R) conditions imply that $d = 0$, $c = \frac{1}{r-1}$ and if $b = -1$ then $a = 0$. Otherwise, $G$ is conjugate to $G_{a,\alpha_{1,1} \beta_{0,r}}$ by $\beta_{1,1}^n$ where $n = \frac{a(r-1)}{r-1}$.

**Lemma 5.10.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is

$$G_a = \langle \sigma \alpha_{1,1}, \tau^a \beta_{0,r}, \epsilon^c r^d \beta_{1,1} \rangle \cong A,$$

for $1 \leq a \leq p - 1$.

**Proof.** The groups in the statement are regular. Using the same argument of Lemma 5.7 we can show that the groups $G_a$ are not conjugate.

Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$. The unique subgroup of order $p^2q$ up to conjugation is $\langle \alpha_{1,1}, \beta_{0,r}, \beta_{1,1} \rangle \cong A$. Hence a standard presentation of $G$ is the following:

$$G = \langle u^{e^a \alpha_{1,1}}, v^{b \beta_{0,r}}, w^{c \beta_{1,1}} \rangle$$

for some $u, v, w \in \langle \tau, \sigma \rangle$. From (R) we have that $w = 1$. Denote $u = \sigma^x \tau^y$ and $v = \sigma^z \tau^t$ for some $0 \leq x, y, z, t \leq p - 1$. Since $\alpha_{1,1}$ is central in $\pi_2(G)$ and $\pi_2|_G$ is a bijection, we have $z = 0$ (and then $x \neq 0$ by regularity), $a = b \frac{1-x}{r(1-r)}$ and $c = \frac{1}{r-1}$. Then

$$G = \langle \sigma^A e^{b \frac{1-x}{r(1-r)} \alpha_{1,1}}, \tau^t e^{b \beta_{0,r}}, \epsilon^c r^d \beta_{1,1} \rangle,$$

where $t \neq 0$ since $G$ regular. Hence $G$ is conjugate to $G_{xt}$ by $h = \left[\left(\frac{x-1}{x}, \frac{1}{x} x, x, (-b, 1)\right]\right]$.

The following remark is analogous to Remark 3.27.

**Remark 5.11.** The skew braces of $A$-type which decompose as direct products are the following:

(i) the skew brace associated to the group $G_{c,\beta_{0,r}}$ for $1 \leq c \leq p - 1$ as defined in Lemma 5.7 is the direct product of the trivial skew brace of size $p$ and a skew brace of size $pq$ with $|\ker\lambda_{B_2}| = q$, see [1] Theorem 3.9.

(ii) The skew brace associated to $G_{a,\beta_{0,r}}$ for $1 \leq a \leq p - 1$ as defined in Lemma 5.9 is the direct product of the trivial skew brace of size $p$ and a skew brace of size $pq$ with $|\ker\lambda_{B_2}| = 1$, see [1] Theorem 3.12.

We summarize the contents of this subsection in the following table.
\[
\begin{array}{c|ccc}
| \ker \lambda | & \mathbb{Z}_p^2 \times \mathbb{Z}_q & \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p) \\
\hline
1 & - & p-1 \\
p & 1 & 2p-1 \\
q & - & p-1 \\
pq & 2 & 2(p-1) \\
p^2 & 1 & - \\
p^2q & - & 1 \\
\end{array}
\]

Table 17. Number of skew braces of \(\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)\)-type of size \(p^2 q\) for \(q = 1 \pmod{p}\) and \(q \neq 1 \pmod{p^2}\).

5.4. **Skew braces of \(\mathbb{Z}_q \rtimes \mathbb{Z}_p^2\)-type.** In this section we denote by \(A\) the group \(\mathbb{Z}_q \rtimes \mathbb{Z}_p^2\). Such group has the following presentation:

\[
A = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \sigma \tau \sigma^{-1} = \tau^r \rangle.
\]

According to [8, Subsection 4.5], the map

\[
\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_q^\times) \longrightarrow \text{Aut}(A), \quad (k, j) \mapsto \varphi_{ij}^k = \begin{cases} \tau \mapsto \tau^i \\ \sigma \mapsto \tau^j \sigma \tau^{-1} \end{cases},
\]

is an isomorphism. In particular, \(|\text{Aut}(A)| = pq(q - 1)\) and so if \(G\) is a regular subgroup of \(\text{Hol}(A)\) then \(|\pi_2(G)|\) divides \(p^2q\).

The conjugacy classes of subgroups of \(\text{Aut}(A)\) are given by:

| Size | Groups | Parameters | Class |
|------|--------|------------|-------|
| \(p\) | \(\langle \varphi_{1,0}^k \rangle\) | \(0 \leq k \leq p-1\) | \(\mathbb{Z}_p\) |
| \(q\) | \(\langle \varphi_{1,0}^{q} \rangle\) | - | \(\mathbb{Z}_q\) |
| \(p^2\) | \(\langle \varphi_{1,1}^q, \varphi_{1,0}^q \rangle\) | - | \(\mathbb{Z}_p^2\) |
| \(pq\) | \(\langle \varphi_{1,1}^q, \varphi_{1,0}^q \rangle\) | \(0 \leq k \leq p-1\) | \(\mathbb{Z}_q \rtimes \mathbb{Z}_p\) |
| \(p^2q\) | \(\langle \varphi_{1,0}^q, \varphi_{1,0}^q, \varphi_{1,1}^q \rangle\) | - | \(\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)\) |

Table 18. Conjugacy classes of subgroups of \(\text{Aut}(A)\).

**Proposition 5.12.** Let \(G\) be a regular subgroup of \(\text{Hol}(A)\). Then \(|\pi_2(G)| \notin \{p^2, p^2q\}\).

**Proof.** Let \(G\) be a subgroup of \(\text{Hol}(A)\) of size \(p^2q\). If \(|\pi_2(G)| = p^2\), then according to Table 18 and the fact that the kernel of \(\pi_2\) is the normal \(q\)-Sylow subgroup of \(A\), we have the following standard presentation for \(G\):

\[
G = \langle \tau, \sigma^a \sigma^b \varphi_{1,0}^1, \sigma^c \sigma^d \varphi_{r,0}^1 \rangle = \langle \tau, \sigma^a \varphi_{1,0}^1, \sigma^c \varphi_{r,0}^0 \rangle
\]

where, from condition (R) we have \(a = c = 0 \pmod{p}\). Therefore \(\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle\) and so \(G\) is not regular.

Assume that \(|\pi_2(G)| = p^2q\). According to Table 18, \(G\) has the standard presentation:

\[
G = \langle \sigma^a \varphi_{1,0}^1, \sigma^c \varphi_{1,1}^1, \sigma^e \varphi_{r,0}^0 \rangle,
\]

where, according to (R), \(a = e = 0 \pmod{p}\) and \(b = c = f = 0\). Hence, \(\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle\) and so \(G\) is not regular. \(\square\)
Lemma 5.13. There is a unique conjugacy class of regular subgroups \(G\) of \(\text{Hol}(A)\) with \(|\pi_2(G)| = q\). A representative is
\[
G = \langle \sigma, \tau^{-1}\varphi_{1,1}^0 \rangle \cong \mathbb{Z}_p^2 q.
\]

Proof. Let \(G\) be such group. The subgroups of size \(p^2\) of \(\mathbb{Z}_q \rtimes \mathbb{Z}_p^2\) are all conjugated to \(\langle \sigma \rangle\). Then, according to Table \(\text{I}8\) we have that \(G\) has the standard presentation:
\[
G = \langle \sigma, \tau^a \varphi_{1,1}^0 \rangle = \langle \sigma, \varphi_{1,1}^0 \rangle.
\]
By condition (K), we have that \(a = \frac{1}{r-1}\).

Lemma 5.14. A set of representatives of conjugacy classes of regular subgroups \(G\) of \(\text{Hol}(A)\) with \(|\pi_2(G)| = pq\) is
\[
G_a = \langle \sigma^p, \sigma^a \varphi_{1,0}^1, \tau^{-1} \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_p^2 q, \quad H_{a,k} = \langle \sigma^p, \sigma^a \varphi_{0,0}^1, \tau^{-1} \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p^2,
\]
where \(1 \leq a \leq p - 1\) and \(0 \leq k \leq p - 1\).

Proof. The groups \(G_a\) and \(H_{a,k}\) are regular. Let \(G_a\) and \(G_b\) be conjugate by \(h \in \text{Aut}(A)\) and let \(K = \langle \tau, \varphi_{1,1}^0 \rangle \leq \text{Hol}(A)\). Since \(h(\sigma^a)h\varphi_{1,0}^1 h^{-1} = \sigma^a \varphi_{1,0}^1\) (mod \(K\)) and \(h\sigma^a \varphi_{1,0}^1 h^{-1} \in G_b\) then \(\sigma^a \varphi_{1,0}^1 = \sigma^b \varphi_{1,0}^1\) (mod \(K\)) and so \(a = b\). Similarly, if \(H_{a,k}\) is conjugate to \(H_{b,k}\) then \(a = b\).
Let \(G\) be a regular subgroup of \(\text{Hol}(A)\) such that \(|\pi_2(G)| = pq\). According to Table \(\text{I}8\) the image of \(\pi_2\) is normal. Up to conjugation, the unique subgroup of size \(p\) of \(A\) is \(\langle \sigma^p \rangle\) and so \(\ker \pi_2 = \langle \sigma^p \rangle\). So we have the following cases.

If \(\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{1,1}^0 \rangle\). Then
\[
G = \langle \sigma^p, \sigma^a \tau^b \varphi_{1,0}^1, \sigma^c \tau^d \varphi_{1,1}^0 \rangle.
\]
From condition (K) we have \(d = \frac{1}{r-1}\) and \(c = 0\) and so \(a \neq 0\) since \(G\) is regular. Thus,
\[
G = \langle \sigma^p, \sigma^a \tau^b \varphi_{1,0}^1, \tau^{-1} \varphi_{1,1}^0 \rangle,
\]
and \(G\) is conjugate to \(G_a\) by \(h = \varphi_{1,n}^0\) where \(n = bg_a (r-1) \tau^{-1}\).

If \(\pi_2(G) = \langle \varphi_{r,0}^1, \varphi_{1,1}^0 \rangle\) then, according to condition (K) the standard presentation of \(G\) is
\[
G = \langle \sigma^p, \sigma^a \tau^b \varphi_{r,0}^1, \tau^{-1} \varphi_{1,1}^0 \rangle
\]
where \(a \neq 0\) since \(G\) is regular. If \(a = p - 1\), from \(\langle \sigma^{p-1} \tau^b \alpha_k \rangle \in \langle \sigma^p \rangle\) it follows that \(b = 0\). Otherwise, \(G\) is conjugate to \(H_{a,k}\) by \(h = \varphi_{1,n}^0\) where \(n = -br^a (r-1) \tau^{-1}\).

Lemma 5.15. A set of representatives of conjugacy classes of regular subgroups \(G\) of \(\text{Hol}(A)\) with \(|\pi_2(G)| = p\) is
\[
G_a = \langle \tau, \sigma^p, \sigma^a \varphi_{1,0}^1 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p^2, \quad H_{a,k} = \langle \tau, \sigma^p, \sigma^a \varphi_{0,0}^1 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p^2 \text{ if } a = p - 1,
\]
where \(1 \leq a \leq p - 1\) and \(0 \leq k \leq p - 1\).

Proof. The groups \(G_a\) and \(H_{a,k}\) are regular and the same argument of Lemma \(\text{5.14}\) shows that they are not conjugate. The only subgroup of size \(pq\) in \(A\) is \(\langle \tau, \sigma^p \rangle\). If \(G\) is a regular subgroup of \(\text{Hol}(A)\), we have two cases to consider according to Table \(\text{I}8\) if \(\pi_2(G) = \langle \varphi_{1,0}^1 \rangle\), then \(G\) has the following standard presentation:
\[
G = \langle \tau, \sigma^p, \sigma^a \tau^b \varphi_{1,0}^1 \rangle = \langle \tau, \sigma^p, \sigma^a \varphi_{1,0}^1 \rangle,
\]
where \(1 \leq a \leq p - 1\), i.e. \(G = G_a\).
If \( \pi_2(G) = \langle \varphi_{r,0}^k \rangle \) for \( 0 \leq k \leq p - 1 \), a standard presentation for \( G \) is

\[
G = \langle \tau, \sigma^p, \sigma^a \varphi_{r,0}^k \rangle
\]

with \( 1 \leq a \leq p - 1 \), i.e. \( G = H_{a,k} \).

We summarize the contents of this subsection in the following table:

| \( |\ker \lambda| \) | \( \mathbb{Z}_{p^2} \times \mathbb{Z}_q \) | \( \mathbb{Z}_q \times_{r} \mathbb{Z}_{p^2} \) |
|---|---|---|
| \( p \) | \( p - 1 \) | \( p(p-1) \) |
| \( pq \) | \( p \) | \( p^2 - p - 1 \) |
| \( p^2 \) | \( 1 \) | - |
| \( p^2q \) | - | 1 |

**Table 19.** Number of skew braces of \( \mathbb{Z}_q \times_r \mathbb{Z}_{p^2} \)-type of size \( p^2q \) for \( q = 1 \pmod{p} \) and \( q \neq 1 \pmod{p^2} \).

6. Skew braces of size \( p^2q \) with \( q = 1 \pmod{p^2} \)

In this section we assume that \( q = 1 \pmod{p^2} \) and we will denote by \( h \) a fixed element of order \( p^2 \) in \( \mathbb{Z}_q^\times \). Accordingly, we have the following groups of size \( p^2q \).

**Lemma 6.1.** [6] Proposition 21.17 The groups of size \( p^2q \) are the following:

(i) \( \mathbb{Z}_{p^2q} = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \tau \sigma = \sigma \tau \rangle \).

(ii) \( \mathbb{Z}_p \times \mathbb{Z}_q = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^q = \epsilon^q = 1, [\sigma, \tau] = [\sigma, \epsilon] = [\tau, \epsilon] = 1 \rangle \).

(iii) \( \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p) = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, [\epsilon, \tau] = [\sigma, \epsilon] = 1, \epsilon \tau \epsilon^{-1} = \sigma h^p \rangle \).

(iv) \( \mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2} = \langle \sigma, \tau \mid \tau^q = \sigma^{p^2} = 1, \sigma \tau \sigma^{-1} = \tau h^p \rangle \).

(v) \( \mathbb{Z}_q \rtimes_{h} \mathbb{Z}_{p^2} = \langle \sigma, \tau \mid \tau^q = \sigma^{p^2} = 1, \sigma \tau \sigma^{-1} = \tau h \rangle \).

We summarize in the following table the total number of skew braces according to the additive and multiplicative isomorphism class of groups that we will obtain in this section.

| \( + \times \) | \( \mathbb{Z}_{p^2q} \) | \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) | \( \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p) \) | \( \mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2} \) | \( \mathbb{Z}_q \rtimes_{h} \mathbb{Z}_{p^2} \) |
|---|---|---|---|---|---|
| \( \mathbb{Z}_{p^2q} \times \mathbb{Z}_q \) | \( 2p \) | \( p \) | \( p \) |
| \( \mathbb{Z}_p \times \mathbb{Z}_q \) | \( 2 \) | \( 4 \) | \( 2p - p - 1 \) |
| \( \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p) \) | \( -4 \) | \( 6p - 4 \) | \( -4 \) |
| \( \mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2} \) | \( 2p \) | - | \( 2p(p-1) \) |
| \( \mathbb{Z}_q \rtimes_{h} \mathbb{Z}_{p^2} \) | \( -2 \) | - | \( 2(p-1) \) |

**Table 20.** Number of skew braces of size \( p^2q \) with \( q = 1 \pmod{p^2} \).

6.1. Skew braces of cyclic type. In this section we denote by \( A \) the cyclic group of size \( p^2q \). The size of the image of \( \pi_2 \) of regular subgroups of \( \text{Aut}(A) \) divides \( p^2 \). The automorphism group of \( A \) is as in Subsection 3.1.

**Lemma 6.2.** The subgroups of size \( p^2 \) of \( \text{Aut}(A) \) are

\[
H_j = \langle \varphi_{jp+1,h} \rangle, \quad T = \langle \varphi_{p+1,1}, \varphi_{1,h^p} \rangle, \quad 0 \leq j \leq p - 1.
\]
Proof. Every subgroup of size $p^2$ in $\text{Aut}(A)$ embeds into the $p$-Sylow subgroup of $\text{Aut}(A)$, generated by $\varphi_{p+1,1}$ and $\varphi_{1,\lambda}$ which is isomorphic to $Z_p \times Z_{p^2}$. According to [23, Theorem 3.3], the group $Z_p \times Z_{p^2}$ has $p+1$ subgroups of size $p^2$. The subgroups $\langle \varphi_{p+1,1}, \varphi_{1,\lambda} \rangle$ and $\langle \varphi_{jp+1,\lambda} \rangle$ for $0 \leq j \leq p-1$ are $p+1$ distinct subgroups of size $p^2$. □

Lemma 6.3. The mapping

$$f_j : Z_{p^2} \rightarrow Z_{p^2}, \quad m \mapsto \frac{m(m-1)}{2}jp + m$$

is a bijection for every $0 \leq j \leq p-1$.

Proof. Clearly, we have $f(m) = m \pmod{p}$ and we can prove inductively that $f_j(m + kp) = f_j(m) + kp$. Since every element in $Z_{p^2}$ is of the form $m + kp$ for suitable $m, k$ then $f_j$ is surjective. □

Proposition 6.4. The skew braces of cyclic type with $|\ker\lambda| = q$ are $(B_j, +, \circ)$ for $0 \leq j \leq p - 1$ where

$$(n \pmod{m}) + (x \pmod{y}) = (n + x \pmod{m + y}), \quad (m \circ (x \pmod{y})) = \left(\frac{n + (f_j^{-1}(n)pj + 1)x}{m + hj^{-1}(n)y}\right),$$

for every $0 \leq n, x \leq p - 1$ and $0 \leq m, y \leq q - 1$. In particular, $(B_j, \circ) \cong Z_q \rtimes h Z_{p^2}$.

Proof. The groups $G_j = \langle \tau, \sigma \varphi_{jp+1,\lambda} \rangle \cong Z_q \rtimes h Z_{p^2}$, for $0 \leq j \leq p - 1$ are regular and they are not conjugate since their image under $\pi_2$ are not. According to Lemma 6.2, we have the following cases:

(i) $\pi_2(G) = T$: arguing as in Proposition 5.2, we can show that there are no regular subgroups with this projection.

(ii) $\pi_2(G) = H_j$: in this case a standard presentation of $G$ is

$G = \langle \tau, \sigma a \varphi_{jp+1,\lambda} \rangle = \langle \tau, \sigma a \varphi_{jp+1,\lambda} \rangle$,

where $a \neq 0$ since $G$ is regular. Then $G$ is conjugate to $G_j$ by $\varphi_{a^{-1},1}$.

Let $(B_j, +, \circ)$ be the skew brace associated to $G_j$, then

$$\sigma \circ \ldots \sigma = \sigma f_j(n),$$

where $f_j$ is defined as in Lemma 6.3. Therefore $\lambda_{\sigma a^{n}\tau^{m}} = \lambda_{\tau^{m} \circ \sigma^{n}} = \lambda_{f_j^{-1}(n)}$ and so the formula follows. □

For the case $|\pi_2(G)| = p$, we can argue as in Proposition 5.3 and then we have $p + 1$ regular subgroups. In particular we have $p + 1$ skew braces of $A$-type with $|\ker\lambda| = pq$.

We summarize the contents of this subsection in the following table:

| $|\ker\lambda|$ | $Z_{p^2q}$ | $Z_q \rtimes h Z_{p^2}$ | $Z_q \rtimes h^2 Z_{p^2}$ |
|-----------------|-------------|-----------------|-----------------|
| $q$             | -           | $p$             | -               |
| $pq$            | 1           | -               | $p$             |
| $p^2q$          | 1           | -               | -               |

Table 21. Number of skew braces of $Z_{p^2q}$-type of size $p^2q$ for $q = 1 \pmod{p^2}$. 42
6.2. Skew braces of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$-type. In this section we denote by $A$ the group $\mathbb{Z}_p^2 \times \mathbb{Z}_q$. The size of the kernel of $\lambda$ of non trivial skew braces of $A$-type is either $q$ or $pq$. The conjugacy classes of subgroups of size $p$ of $\text{Aut}(A)$ are the same as in the case $q = 1 \pmod{p}$ and therefore, if $|\ker\lambda| = pq$ we can argue as in Proposition 5.4 indeed it applies whenever $q = 1 \pmod{p}$.

On the other hand, up to conjugation, the subgroups of order $p^2$ of $\text{Aut}(A)$ are $(C, h^p)$ and $\langle C^l h \rangle$ where $l = 0, 1$. $C$ is as in Remark 3.6 and $h$ is identified with the automorphism of $\mathbb{Z}_q$ given by $x \mapsto x^h$ for all $x \in \mathbb{Z}_q$. If $G$ is a regular subgroup with $\pi_2(G) = \langle C^l h \rangle$ then

$$G = \langle \epsilon, v C^l h \rangle,$$

for some $v = \sigma^a \tau^b$. The group $G$ is not regular, since $\pi_1(G) = \{\epsilon^n \sigma^{ma+b} \tau^{bm} : 0 \leq n \leq q-1, 0 \leq m \leq p-1\} \neq G$. Otherwise, we can argue as in Proposition 5.5 and therefore it provides a description of skew braces of $A$-type with $|\ker\lambda| = q$.

Hence, the enumeration of skew braces of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$-type of size $p^2q$ for $q = 1 \pmod{p^2}$ is as in Table 15.

6.3. Skew braces of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$-type. In this section we denote by $A$ the group $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$. The regular subgroups of $\text{Hol}(A)$ with size of the image under $\pi_2$ equal to $q, p, pq$ are the same as the one described in Proposition 5.6, Lemma 5.7 and Lemma 5.9, respectively, since the subgroups of the automorphism group of $A$ of order $p, q$ and $pq$ coincide with the ones in the case $q = 1 \pmod{p}$.

**Lemma 6.5.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is

$$G_a = \langle \epsilon, \sigma^a \alpha_{1,1}, \tau \beta_{0, h^p} \rangle \cong A,$$

for $1 \leq a \leq p-1$.

**Proof.** If the image of $\pi_2(G)$ is not cyclic we can argue as in Lemma 5.8 to get $G_a$ as in the statement. We show that if $\pi_2(G)$ is generated by $\tau \in \{\beta_{0, h^p}, \alpha_{1,1} \} \beta_{0, h^p}$ then $G$ is not regular. Indeed if

$$G = \langle \epsilon, \sigma^a \tau^b \theta \rangle$$

and then $\pi_1(G) = \{\epsilon^n \sigma^{am} \tau^{bm} : 0 \leq n \leq q-1, 0 \leq m \leq p-1\} \neq G$. Therefore, $G$ is not regular. \hfill $\Box$

**Lemma 6.6.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is

$$G_a = \langle \sigma \alpha_{1,1}, \tau^a \beta_{0, h^p}, \epsilon^{-1} \beta_{1,1} \rangle \cong A,$$

for $1 \leq a \leq p-1$.

**Proof.** If the $p$-Sylow of the image of $\pi_2(G)$ is not cyclic we can conclude as in Lemma 5.10. We show that if $\pi_2(G) = \langle \beta_{1,1}, \theta \rangle$, for $\theta \in \{\beta_{0, h^p}, \alpha_{1,1} \beta_{0, h^p}\}$ then $G$ is not regular. Indeed if

$$G = \langle \epsilon^a \beta_{1,1}, \epsilon^b v \theta \rangle$$

for some $u, v \in \langle \sigma, \tau \rangle$, from the condition (R) we have $u = 1$, i.e. $G = \langle \epsilon^a \beta_{1,1}, \epsilon^b v \theta \rangle$. So we have that

$$\pi_1(G) = \{\epsilon^{an+f(v,b,m)}v^m : 0 \leq n \leq q-1, 0 \leq m \leq p^2-1\} \subseteq \langle \epsilon, v \rangle$$

and so $G$ is not regular. \hfill $\Box$

Therefore, according to Lemma 6.5 and Lemma 6.6 the enumeration of the skew braces of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$-type of size $p^2q$ for $q = 1 \pmod{p^2}$ according to size of the kernel of $\lambda$ is as in Table 17.
6.4. Skew braces of $\mathbb{Z}_q \rtimes_{h,p} \mathbb{Z}_{p^2}$-type. In this section we denote by $A$ the group $\mathbb{Z}_q \rtimes_{h,p} \mathbb{Z}_{p^2}$. The automorphism of $A$ can be described as in Section 5.4 and we employ the same notation. In particular, the subgroups of order $p$, $q$ and $pq$ of $\text{Aut}(A)$ coincide with the previous case. Therefore, if $G$ is a regular subgroup with $|\pi_2(G)| \in \{q, pq, p\}$ we can apply respectively, Lemma 5.13, 5.14 and 5.15.

A $p$-Sylow subgroup of $\text{Aut}(A)$ is given by $\langle \varphi_{1,0}^1, \varphi_{h,0}^0 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p^2$. So, according to [23, Theorem 3.3], it has $p + 1$ subgroups of order $p^2$, namely

$$(27) \quad \langle \varphi_{h,0}^k \rangle \quad \text{and} \quad \langle \varphi_{1,0}^1, \varphi_{h,p,0}^0 \rangle,$$

for $0 \leq k \leq p - 1$.

On the other hand, since $\langle \varphi_{1,1}^0 \rangle$ is the unique $q$-Sylow subgroup of $\text{Aut}(A)$, we have that the subgroups of size $p^2q$ in $\text{Aut}(A)$ are the following $p + 1$ subgroups:

$$(28) \quad \langle \varphi_{1,0}^1, \varphi_{h,p,0}^0, \varphi_{1,1}^0 \rangle \quad \text{and} \quad \langle \varphi_{h,0}^k, \varphi_{1,1}^0 \rangle,$$

for $0 \leq k \leq p - 1$.

**Lemma 6.7.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is

$G_{b,k} = \langle \tau, \sigma^b \varphi_{h,0}^k \rangle \cong \mathbb{Z}_q \times_h \mathbb{Z}_{p^2},$

for $0 \leq k \leq p - 1$ and $1 \leq b \leq p - 1$.

**Proof.** The groups in the statement are regular and arguing as in Lemma 5.14 we can show that they are not conjugate. Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$. The unique subgroup of order $q$ of $A$ is $\langle \tau \rangle$. According to (27) we have two cases to consider.

(i) If $\pi_2(G) = \langle \varphi_{h,0}^k \rangle$, then

$G = \langle \tau, \sigma^b \varphi_{h,0}^k \rangle = \langle \tau, \sigma^b \varphi_{h,0}^k \rangle,$

where $b \not\equiv 0 \pmod{p}$, i.e. $G = G_{b,k}$.

(ii) If $\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{h,p,0}^0 \rangle$, then we can apply Proposition 5.12 and so there are no such regular subgroups.

$\square$

**Lemma 6.8.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is

$G_{b,k} = \langle \sigma^b \varphi_{h,0}^k, \tau \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_q \times_h \mathbb{Z}_{p^2},$

for $1 \leq b \leq p - 1$ and $0 \leq k \leq p - 1$.

**Proof.** The groups in the statement are regular and not conjugate (we can employ the same argument of 5.14 again). Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$. According to (28) we have two cases to consider. In the first one we have no regular subgroups by the same argument of Proposition 5.12. In the second case, $G$ has the standard presentation:

$G = \langle \tau^a \sigma^b \varphi_{h,0}^k, \tau \varphi_{1,1}^0 \rangle.$

From the (R) conditions we have that

$G = \langle \tau^a \sigma^b \varphi_{h,0}^k, \tau \varphi_{1,1}^0 \rangle,$

with $b \not\equiv 0 \pmod{p}$. Let $1 \leq c \leq p - 1$ such that $b = c \pmod{p}$. Then $G$ is conjugate to $G_{c,k}$ by a suitable power of $\varphi_{1,0}^1$. $\square$
We summarize the contents of this subsection in the following table:

| ker λ | Z₂ | Zₚ × h | Z₂ | Zₚ × h Z₂ |
|-------|----|-------|----|-----------|
| 1     | -  | -     | p(p - 1) | - |
| p     | p - 1 | p(p - 1) | - |
| q     | -  | -     | p(p - 1) | - |
| p²    | 1  | -     | - |
| pq    | p  | p² - p - 1 | - |
| p²q   | -  | 1     | - |

Table 22. Number of skew braces of $\mathbb{Z}_q \rtimes h \mathbb{Z}_{p^2}$-type of size $p²q$ for $q = 1 \pmod{p²}$.

6.5. Skew braces of $\mathbb{Z}_q \rtimes h \mathbb{Z}_{p^2}$-type. In this section, we denote by $A$ the group $\mathbb{Z}_q \rtimes h \mathbb{Z}_{p^2}$. A presentation of such group is

$$G = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \sigma \tau \sigma^{-1} = \tau^h \rangle.$$ 

According to [8, Theorem 3.4], the map

$$\phi : \mathbb{Z}_q \rtimes \mathbb{Z}_q \rightarrow \text{Aut}(A), \quad (i, j) \mapsto \varphi_{i,j} = \begin{cases} \tau \mapsto \tau^j, \\ \sigma \mapsto \tau^i \sigma \end{cases},$$

is a group isomorphism. In particular $|\text{Aut}(G)| = q(q - 1)$. Since $q = 1 \pmod{p²}$, then $|\pi_2(G)|$ divides $p²q$.

A set of representatives of the conjugacy classes of subgroups of $\text{Aut}(A)$ is the following:

| Size | Group | Class |
|------|-------|-------|
| p    | $\langle \varphi_{0,h} \rangle$ | $\mathbb{Z}_p$ |
| q    | $\langle \varphi_{1,1} \rangle$ | $\mathbb{Z}_q$ |
| pq   | $\langle \varphi_{0,h}, \varphi_{1,1} \rangle$ | $\mathbb{Z}_q \rtimes h \mathbb{Z}_p$ |
| p²   | $\langle \varphi_{0,h} \rangle$ | $\mathbb{Z}_{p^2}$ |
| p²q  | $\langle \varphi_{1,1}, \varphi_{0,h} \rangle$ | $A$ |

Table 23. Conjugacy classes of groups of $\text{Aut}(A)$.

Lemma 6.9. A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p$ is

$$G_c = \langle \tau, \sigma^c, \sigma^c \varphi_{0,h} \rangle \cong A,$$

where $1 \leq c \leq p - 1$.

Proof. The subgroup $G_c$ is regular. If $G_c$ and $G_d$ are conjugate, then

$$(\varphi_{i,j})(\sigma^c \varphi_{0,h})(\varphi_{i,j})^{-1} = (\tau^j \sigma)^c \varphi_{j(1-h)p}, h \in G_d$$

for some $i, j$. Then $\varphi_{j(1-h)p}, h \in \pi_2(G_d)$ and so $j = 0$. Therefore $\sigma^c \varphi_{0,h} \in G_d$ and so $\sigma^c \varphi_{0,h} = \sigma^d \varphi_{0,h} \pmod{\ker \pi_2|G_c}$, i.e. $d = c$.

The unique subgroup of $A$ of order $pq$ is $\langle \tau, \sigma^p \rangle$. Hence if $G$ is a regular subgroup of $\text{Hol}(A)$, we can assume that $G = G_c$ for some $c \neq 0$. □

Lemma 6.10. There is a unique conjugacy class of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = q$. A representative is

$$G = \langle \sigma, \tau \pi_{1,1}, \varphi_{1,1} \rangle \cong \mathbb{Z}_{p^2q}.$$
Proof. Since all $p$-Sylow subgroups of $A$ are conjugated to each other and have order $p^2$, we can suppose without loss of generality, that a regular subgroup of $\text{Hol}(A)$ has the standard presentation

$$G = \langle \sigma, \tau^a \sigma^b \varphi_{1,1} \rangle = \langle \sigma, \tau^a \varphi_{1,1} \rangle,$$

for some $a' \neq 0$. The (K) condition is fulfilled if and only if $a = \frac{1}{p-1}$.

**Lemma 6.11.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ such that $|\pi_2(G)| = pq$ is

$$G_c = \langle \sigma^c, \phi_{0,h}, \tau^b \varphi_{1,1} \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{pq^2},$$

for $1 \leq c \leq p - 1$.

**Proof.** The subgroups $G_c$ for $1 \leq c \leq p - 1$ are regular and it is easy to check that they are not pairwise conjugated.

Let $G$ be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. According to the Table 23, we can assume that $\pi_2(G) = \langle \varphi_{0,h}, \varphi_{1,1} \rangle$. The subgroups of order $p$ of $A$ are $\langle \tau^a \sigma^p \rangle$ for some $a$. Since $\varphi_{1,1}$ is the normalizer of $\pi_2(G)$, up to conjugation by $\varphi_{1,1}$ we can assume that $a = 0$. So, $G$ has the standard presentation

$$G = \langle \sigma^p, \tau^b \sigma^c \varphi_{0,h}, \tau^d \sigma^e \varphi_{1,1} \rangle$$

for some $0 \leq b, d \leq p - 1$ and $0 \leq c, e \leq q - 1$. From the (R) conditions it follows that $e = 0$ (mod $p$), $d = \frac{1}{p-1}$ and $b = 0$. Thus $G = G_c$.

**Lemma 6.12.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ such that $|\pi_2(G)| = p^2$ is

$$G_b = \langle \tau, \sigma^b \varphi_{0,h} \rangle \cong \begin{cases} \mathbb{Z}_{pq^2}, & \text{if } b = -1 \pmod{p^2}, \\ \mathbb{Z}_q \rtimes \mathbb{Z}_{pq^2}, & \text{if } b = -1 \pmod{p} \text{ and } b \neq -1 \pmod{p^2}, \\ \mathbb{Z}_q \rtimes \mathbb{Z}_{pq^2}, & \text{otherwise,} \end{cases}$$

where $b \neq 0 \pmod{p}$.

**Proof.** Since $A$ has a unique subgroup of order $q$ and according to the Table 23, we have that every regular subgroup $G$ has the standard presentation:

$$G_b = \langle \tau, \sigma^b \varphi_{0,h} \rangle,$$

for $b \neq 0 \pmod{p}$. It is straightforward to check that they are all not pairwise conjugated. Checking condition (K) we have $\sigma^b \varphi_{0,h} \tau \varphi_{0,h}^a \sigma^{-b} = \tau^{b+1}$, so if $b = 0 \pmod{p^2}$ then $G_b$ is cyclic. If $b + 1 = 0 \pmod{p}$ but $b \neq -1 \pmod{p^2}$, then $G_b$ is isomorphic to $\mathbb{Z}_q \rtimes \mathbb{Z}_{pq^2}$. Otherwise, $G_b \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{pq^2}$.

**Lemma 6.13.** A set of representatives of conjugacy classes of regular subgroups $G$ of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2 q$ is

$$G_d = \langle \tau^{\frac{1}{n-1}} \varphi_{1,1}, \sigma^d \varphi_{0,h} \rangle \cong A$$

for $d \neq 0 \pmod{p}$.

**Proof.** Let $G$ be a regular subgroup with $|\pi_2(G)| = p^2 q$. According to the Table 23, we can assume that a standard presentation is:

$$G = \langle \tau^a \sigma^b \varphi_{1,1}, \tau^c \sigma^d \varphi_{1,h} \rangle.$$

According to the (R) conditions, then $b = 0$, $a = \frac{1}{n-1}$ and $d \neq 0 \pmod{p}$ because $G$ is regular. Hence,

$$G = \langle \tau^{\frac{1}{n-1}} \varphi_{1,1}, \tau^c \sigma^d \varphi_{0,h} \rangle,$$

for $d \neq 0 \pmod{p}$. If $d = -1$, then $c = 0$ since $\tau^c \sigma^d \varphi_{0,h}$ has order $p$, otherwise $G$ is conjugate to $G_{c,d}$ by $\varphi_{1,-c \frac{1}{n-1} - d+1-1}$.

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We summarize the contents of this subsection in the following table:

| ker λ | \( \mathbb{Z}_{pq} \) | \( \mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2} \) | \( \mathbb{Z}_q \rtimes_{h} \mathbb{Z}_{p^2} \) |
|-------|----------------|-----------------|----------------|
| 1     | -              | -               | \( p(p-1) \)   |
| \( p \) | \( p-1 \)     | -               | -              |
| \( q \) | 1             | \( p-1 \)     | \( p(p-2) \)   |
| \( p^2 \) | 1             | -               | -              |
| \( pq \) | -             | -               | \( p-1 \)     |
| \( p^2q \) | -             | -               | 1              |

Table 24. Number of skew braces of \( \mathbb{Z}_q \rtimes_{h} \mathbb{Z}_{p^2} \)-type of size \( p^2q \) for \( q = 1 \text{ (mod } p^2) \).

7. \( p,q \) ALGEBRAICALLY INDEPENDENT

Let \( p,q \) be primes. If none of the following congruences holds

\[
p = 1 \pmod{q}, \quad p = -1 \pmod{q}, \quad q = 1 \pmod{p}, \quad q = -1 \pmod{p},
\]

then \( p \) and \( q \) are said to be algebraically independent. In such case, the only groups of size \( p^2q \) are the abelian ones, \( \mathbb{Z}_{pq} \) and \( \mathbb{Z}_p \times \mathbb{Z}_q \) and the size of the kernel of \( \lambda \) of any skew brace is \( pq \).

According to Proposition 3.3 and to Proposition we have: 3.7

| \( +\circ \) | \( \mathbb{Z}_{pq} \) | \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) |
|-----------|----------------|-----------------|
| \( \mathbb{Z}_{pq} \) | 2             | -               |
| \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) | -             | 2               |

Table 25. Number of skew braces of size \( p^2q \) with \( p \) and \( q \) algebraically independent.

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REFERENCES

[1] E. Acri and M. Bonatto. Skew braces of size \( pq \). arXiv e-prints, page arXiv:1908.03228, Aug 2019.
[2] A. A. Alabdali and N. P. Byott. Skew Braces of Squarefree Order. arXiv e-prints, page arXiv:1910.07814, Oct 2019.
[3] D. Bachiller. Classification of braces of order \( p^3 \). J. Pure Appl. Algebra, 219(8):3568–3603, 2015.
[4] D. Bachiller. Solutions of the Yang-Baxter equation associated to skew left braces, with applications to racks. J. Knot Theory Ramifications, 27(8):1850055, 36, 2018.
[5] D. Bachiller, F. Cedó, and E. Jespers. Solutions of the Yang-Baxter equation associated with a left brace. J. Algebra, 463:80–102, 2016.
[6] S. R. Blackburn, P. M. Neumann, and G. Venkataraman. Enumeration of finite groups, volume 173 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2007.
[7] E. Campedel, A. Caranti, and I. Del Corso. Hopf-Galois structures on extensions of degree \( p^2q \) and skew braces of order \( p^2q \): the cyclic Sylow \( p \)-subgroup case. arXiv e-prints, page arXiv:1912.00901, Dec 2019.
[8] E. Campedel, A. Caranti, and I. Del Corso. The automorphism groups of groups of order \( p^2q \). arXiv e-prints, page arXiv:1911.11567, Nov 2019.
[9] F. Cedó, A. Smoktunowicz, and L. Vendramin. Skew left braces of nilpotent type. Proc. Lond. Math. Soc. (3), 118(6):1367–1392, 2019.
[10] L. N. Childs. Bi-skew braces and Hopf Galois structures. arXiv e-prints, page arXiv:1904.08814, Apr 2019.
[11] C. Dietzel. Braces of order \( p^2q \). arXiv e-prints, page arXiv:1801.06911, Jan 2018.
[12] V. G. Drinfel’d. On some unsolved problems in quantum group theory. In Quantum groups (Leningrad, 1990), volume 1510 of Lecture Notes in Math., pages 1–8. Springer, Berlin, 1992.
[13] P. Etingof, T. Schedler, and A. Soloviev. Set-theoretical solutions to the quantum Yang-Baxter equation. Duke Math. J., 100(2):169–209, 1999.
[14] T. Gateva-Ivanova and M. Van den Bergh. Semigroups of $I$-type. *J. Algebra*, 206(1):97–112, 1998.

[15] L. Guarnieri and L. Vendramin. Skew braces and the Yang–Baxter equation. *Math. Comp.*, 86(307):2519–2534, 2017.

[16] J.-H. Lu, M. Yan, and Y.-C. Zhu. On the set-theoretical Yang-Baxter equation. *Duke Math. J.*, 104(1):1–18, 2000.

[17] K. Nejabati Zenouz. *On Hopf-Galois Structures and Skew Braces of Order $p^3$*. PhD thesis, The University of Exeter, https://ore.exeter.ac.uk/repository/handle/10871/32248, 2018.

[18] K. Nejabati Zenouz. Skew braces and Hopf-Galois structures of Heisenberg type. *J. Algebra*, 524:187–225, 2019.

[19] W. Rump. Classification of cyclic braces. *J. Pure Appl. Algebra*, 209(3):671–685, 2007.

[20] W. Rump. Classification of cyclic braces, ii. *Transactions of the American Mathematical Society*, page 1, 03 2018.

[21] A. Smoktunowicz and L. Vendramin. On skew braces (with an appendix by N. Byott and L. Vendramin). *J. Comb. Algebra*, 2(1):47–86, 2018.

[22] A. Soloviev. Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation. *Math. Res. Lett.*, 7(5-6):577–596, 2000.

[23] M. Tărnăuceanu. An arithmetic method of counting the subgroups of a finite abelian group. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 53 (101)(4):373–386, 2010.

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