ON THE SPECTRUM OF MAGNETOHYDRODYNAMIC TURBULENCE

Stanislav Boldyrev
Department of Astronomy and Astrophysics, University of Chicago, 5640 South Ellis Avenue, Chicago, IL 60637; boldyrev@uchicago.edu

Received 2005 March 9; accepted 2005 May 5; published 2005 May 19

ABSTRACT

We propose a phenomenological model for incompressible magnetohydrodynamic turbulence. We argue that nonlinear-wave interaction weakens as the energy cascade proceeds to small scales; however, the anisotropy of fluctuations along the large-scale magnetic field increases, which makes turbulence strong at all scales. To explain the weakening of the interaction, we propose that small-scale fluctuations of the velocity and magnetic fields become increasingly dynamically aligned as their scale decreases, so that turbulent “eddies” become locally anisotropic in the plane perpendicular to the large-scale magnetic field. In the limit of weak anisotropy, that is, weak large-scale magnetic field, our model reproduces the Goldreich-Sridhar spectrum, while the limit of strong anisotropy, that is, strong large-scale magnetic field, corresponds to the Iroshnikov-Kraichnan scaling of the spectrum. This is in good agreement with recent numerical results.

Subject headings: MHD — turbulence

1. INTRODUCTION

Magnetohydrodynamic (MHD) turbulence is a state of a randomly stirred conducting fluid in the limit of very small fluid viscosity and resistivity; it plays an essential role in a variety of astrophysical systems, from planets and stars to interstellar and intergalactic media (see, e.g., Biskamp 2003). Despite more than 35 years of analytical, numerical, and observational investigations, the spectrum of MHD turbulence remains a subject of controversy. The standard results and recent developments in the theory of MHD turbulence can be found in many excellent texts (e.g., Iroshnikov 1963; Kraichnan 1965; Goldreich & Sridhar 1995, 1997; Ng & Bhattacharjee 1996; Cho & Vishniac 2000; Biskamp & Müller 2000; Maron & Goldreich 2001; Cho et al. 2002; Galtier et al. 2000, 2002; Müller et al. 2003; Biskamp 2003; Kulsrud 2005). In the present Letter, we first analyze the approaches of Iroshnikov (1963) and Kraichnan (1965) and Goldreich & Sridhar (1995) and point out the discrepancies of these theories with numerical results. Then we propose a new model for the turbulent MHD cascade, which is free of these discrepancies and which is in good agreement with recent high-resolution numerical findings of Maron & Goldreich (2001), Müller et al. (2003), and Padoan et al. (2004). Our results are summarized in § 3.

The Iroshnikov-Kraichnan energy cascade. —The MHD equations describing evolution of the velocity field, \( \mathbf{v}(r, t) \), and fluctuations of the magnetic field, \( \mathbf{b}(r, t) \), have an especially simple form when expressed in the Elsässer variables, \( z = \mathbf{v} - \mathbf{b} \) and \( w = \mathbf{v} + \mathbf{b} \):

\[
\begin{align*}
\partial_t z + (V_r \cdot \nabla) z + (w \cdot \nabla) z &= -\nabla P, \\
\partial_t w - (V_r \cdot \nabla) w + (z \cdot \nabla) w &= -\nabla P,
\end{align*}
\]

where the pressure \( P \) is determined from the incompressibility condition, \( \nabla \cdot z = 0 \) or \( \nabla \cdot w = 0 \). In equations (1) and (2), \( V_r = B_0/(4\pi \rho)^{1/2} \) is the Alfvén velocity, \( \rho \) is the fluid density, \( B_0 \) is the large-scale magnetic field, and we have neglected small viscosity and resistivity. Note that for \( w = 0 \), any function \( z = f(r - V_r t) \) is a solution of the system (eqs. [1]–[2]); analogously, for \( z = 0 \) the solution is \( w = g(r + V_r t) \), where the function \( g \) is arbitrary.

Iroshnikov and Kraichnan used this fact to propose that the interacting Alfvén-wave packets (or “eddies”) are those propagating in opposite directions along the large-scale magnetic field lines (Iroshnikov 1963; Kraichnan 1965). From that, they deduced that the energy transfer time from wave packets of size \( \lambda \) to smaller ones is increased compared with the simple dimensional estimate \( \tau(\lambda) \propto \lambda/\delta b_\lambda \). (We denote by \( \delta v_r \) and \( \delta b \) the velocity and magnetic field fluctuations in the “eddy.”) In the Alfvén wave, \( \delta v_r \sim \delta b_\lambda \). Indeed, consider two wave packets of size \( \lambda \) propagating in opposite directions along a magnetic field line with the Alfvén velocities. Assuming that the “eddies” are decorrelated at the field-parallel scale \( \lambda \), one can estimate from equations (1) and (2) that during one collision the distortion of the “eddy” is \( \Delta \delta v_r \sim (\delta v_r/\lambda)(\lambda/V_r) \). The distortions add up randomly; therefore, the “eddy” will be distorted relatively strongly after \( N \sim (\delta v_r/\Delta \delta v_r)^2 \sim (V_r/\delta v_r)^2 \) collisions. The energy transfer time is, therefore,

\[
\tau_{IK}(\lambda) \sim N \lambda/V_r \sim \lambda/\delta v_r (V_r/\delta v_r). \tag{3}
\]

It is important that in the Iroshnikov-Kraichnan interpretation, an “eddy” has to experience many uncorrelated interactions with oppositely moving “eddies” before its energy is transferred to a smaller scale. Moreover, turbulence becomes progressively weaker as the energy cascade proceeds toward smaller scales. The requirement of constant energy flux over scales \( \delta v_r/\tau_{IK}(\lambda) = \text{const} \) immediately leads to the scaling of fluid fluctuations \( \delta v_r \propto \lambda^{3/2} \), which results in the energy spectrum (the Iroshnikov-Kraichnan spectrum),

\[
E_{IK}(k) = |\delta v_r|^2 k^2 \propto k^{-3/2}. \tag{4}
\]

Iroshnikov and Kraichnan did not consider anisotropy of the spectrum, so \( E_{IK}(k) \) was assumed to be three-dimensional and isotropic.

Anisotropy: The Goldreich-Sridhar energy cascade. —Over the years, isotropy of the MHD spectrum in a strong external magnetic field seemed to contradict analytical and numerical findings (see, e.g., Biskamp 2003). A theory of anisotropic MHD turbulence was proposed by Goldreich & Sridhar (1995). They argued that “eddies” are strongly anisotropic; they are
elongated along the large-scale magnetic field lines. As a consequence, the time of relatively strong “eddy” distortion (the energy transfer time) is on the order of a crossing time required for two oppositely moving “eddies” to pass each other.

Suppose that the “eddy” has a transverse (to the large-scale field) size λ. Then, its field-parallel size l can be found from the so-called critical-balance condition, proposed by Goldreich & Sridhar (1995). This condition has two explanations that are equivalent in the Goldreich-Sridhar picture. First, the critical balance can be understood as a formal balance of the second and third terms in equations (1) and (2): \( V_\lambda l \sim \delta \beta v_\perp l. \) Second, it follows from the geometric distortion of magnetic field lines in the turbulent “eddy.” Indeed, the “eddy” displaces the lines in their perpendicular direction by a distance \( \xi \sim \delta \beta v_\perp l/V_\perp, \) and this displacement equals the perpendicular “eddy” size λ.

The critical-balance condition is the same for all scales, so, contrary to the Iroshnikov-Kraichnan picture, the turbulence strength does not change with the scale. The energy transfer time predicted in the Goldreich-Sridhar theory is

\[
\tau_{\text{GS}}(\lambda) \sim l/V_\perp \sim \lambda/\delta v_\perp. \tag{5}
\]

Assuming that the energy cascade is independent of the scale, \( \delta v_\perp/\tau_{\text{GS}}(\lambda) = \text{const}, \) one obtains the scaling of fluid fluctuations, \( \delta v_\perp \propto \lambda^{-1}. \) The corresponding energy spectrum is

\[
E_{\text{GS}}(k_\perp) = \|\delta v_\perp\|^2 k_\perp \propto k_\perp^{-5/3}. \tag{6}
\]

This spectrum coincides with the Kolmogorov spectrum of incompressible nonmagnetized fluid turbulence, as it should, since the energy transfer time coincides with the “eddy” turnover time, \( \tau(\lambda) \sim \lambda/\delta v_\perp, \) in both approaches. The anisotropy of fluctuations is described by the condition that follows from the critical balance, \( l \propto \lambda^{2/3}. \) One can therefore write that the fluctuations scale with the field-parallel size of the “eddy” as \( \delta v_\perp \propto l^{1/3}. \)

The Goldreich-Sridhar picture, however, does not fully agree with numerical simulations. As was recently discovered by Müller et al. (2003), the anisotropic spectrum depends on the strength of the external magnetic field. Denote \( \gamma = B_0^2/(\rho \delta v_\perp^2), \) where \( \delta v_\perp \) is the velocity field at the outer scale of turbulence, L. It was found that the field-perpendicular scaling of fluctuations changes from the Goldreich-Sridhar form to the Iroshnikov-Kraichnan form as the field increased from \( \gamma \ll 1 \) to \( \gamma \gg 1. \) A similar result for \( \gamma = 1 \) was obtained earlier by Maron & Goldreich (2001). These intriguing numerical findings motivated our interest in the problem. In the next section, we propose a phenomenological model of MHD turbulence, which agrees well with available numerical results, for any strength of the external field. In the limiting case of a weak external field, our model reproduces the anisotropic spectrum of Goldreich & Sridhar. In the other limiting case of a very strong external field, anisotropy of the spectrum is stronger; however, the field-perpendicular spectrum formally coincides with the spectrum predicted in the Iroshnikov-Kraichnan model.

2. A MODEL FOR MHD TURBULENCE

To begin with, we make a certain assumption about reduction of the nonlinear interaction, which is not present in either the Iroshnikov-Kraichnan or Goldreich-Sridhar picture. We postpone the justification of this assumption until the end of this section, when we obtain the corresponding solution for the turbulent spectra and compare it with numerical simulations.

**Reduction of nonlinear interaction.**—Let us assume that the nonlinear interaction of the counterpropagating fluctuations is reduced by a factor \( (\delta v_\perp/V_\perp)^\alpha, \) where \( \alpha \) is some undetermined exponent, \( 0 \leq \alpha \leq 1. \) In other words, we assume that the nonlinearity in equations (1) and (2) is “depleted,” so that the interaction terms are of order

\[
(w \cdot \nabla)z \sim (z \cdot \nabla)w \sim (\delta v_\perp l/\lambda)(\delta v_\perp V_\perp). \tag{7}
\]

Thus, the fluid fluctuations at the transverse distance \( \lambda \) become decorrelated on the timescale

\[
\tau_\alpha(\lambda) \sim (\lambda/\delta v_\perp)(V_\perp/\delta v_\perp)^\alpha. \tag{8}
\]

Their decorrelation length along the magnetic field line can be found from the causality principle. For \( \delta v_\perp \ll V_\perp, \) the perturbation cannot propagate along the field line faster than \( V_\perp; \) therefore, the correlation length along the field line is on the order of \( l \sim V_\perp \tau_\alpha(\lambda). \) This condition is analogous to the critical-balance condition of the previous section in that it satisfies the same balance between the linear and nonlinear interaction terms in equations (1) and (2). However, its geometric meaning is different and will be discussed below.

We see that the interaction strength (eq. [7]) decreases for smaller scales; however, the field-parallel “eddy” size \( l \) is adjusted in such a way that the energy transfer to the smaller scales always takes one “eddy” crossing time. Note that contrary to the Iroshnikov-Kraichnan formalism, and similar to the Goldreich-Sridhar approach, in our picture turbulence is strong and fluctuations are highly anisotropic. Now, we require that the energy cascade be constant over scales \( \delta v_\perp/\tau_\alpha(\lambda) = \text{const}. \) We derive \( \delta v_\perp \propto \lambda^{1/(3 + \alpha)} \) and the anisotropy condition reads \( l \propto \lambda^{3/(3 + \alpha)} \). The corresponding energy spectrum is given by

\[
E(k_\perp) = \|\delta v_\perp\|^2 k_\perp \propto k_\perp^{5/(3 + \alpha)}. \tag{9}
\]

One can formally define the corresponding longitudinal spectrum of fluctuations, from the condition \( E(k_\parallel dk_\parallel = E(k)dk_\parallel \) with \( k_\parallel \propto k_\perp^{\alpha/(3 + \alpha)} \). The answer is \( E(k_\parallel) \propto k_\parallel^{-5/2}; \) note that it is independent of \( \alpha. \) Therefore, the scaling of fluid fluctuations with respect to the field-parallel distance \( l \) is always \( \delta v_\parallel \propto l^{1/(2 + \alpha)} \).

**Comparison with numerical simulations.**—Obviously, our result with \( \alpha = 0 \) corresponds to the Goldreich-Sridhar scaling, while \( \alpha = 1 \) produces the Iroshnikov-Kraichnan scaling. Simulations by Müller et al. (2003) for a range of large-scale external magnetic fields have shown that the scaling of the second-order structure function with respect to the field-perpendicular scale \( \lambda \) changes from the Goldreich-Sridhar value in the case of a weak external field, \( \gamma \ll 1, \) to the Iroshnikov-Kraichnan value in the case of a strong field, \( \gamma \gg 1. \) At the same time, the scaling of the second-order structure function with respect to the field-parallel distance \( l \) does not change much and stays close to \( \langle (\delta v_\parallel)^2 \rangle \propto l^{3/2}. \) This result is consistent with our prediction, \( \langle (\delta v_\parallel)^2 \rangle \propto l \). The same scalings for the case of a strong external magnetic field were earlier observed in simulations by Maron & Goldreich (2001). In these simulations, it was also found that the scaling of the energy-cascade time was \( \tau_\alpha \propto l^{\alpha/(3 + \alpha)}. \) This contradicts the Goldreich-Sridhar picture (see the discussion in § 6.1.3 of Maron & Goldreich 2001) but coincides with our formula \( \tau_\alpha(\lambda) \propto \lambda^{1/(3 + \alpha)} \) for \( \alpha = 1. \)

It is important to note that when the perpendicular Iroshnikov-Kraichnan scaling is formally reproduced in our model, the turbulent fluctuations are strong and anisotropic, with \( l \propto \lambda^{1/2}. \)
Moreover, the turbulence possessing the Iroshnikov-Kraichnan spectrum is more anisotropic than the turbulence in the Goldreich-Sridhar model, which is not surprising, since, as just discussed, the Iroshnikov-Kraichnan spectrum corresponds to a much stronger external magnetic field.

Scale-dependent dynamic alignment.—We next explain a possible physical origin for the proposed reduction of the non-linear interaction (eq. [7]). A hint toward the explanation can be obtained from geometric considerations. The displacement of magnetic field lines in the direction perpendicular to the large-scale magnetic field, produced by the wave packet with \( l \propto \lambda^{1/(3+\alpha)} \) and \( \delta B_0 \propto \lambda^{\alpha/(3+\alpha)} \), is given by \( \xi \propto l \delta B_0 / \lambda \propto \lambda^{\alpha/(3+\alpha)} \). We thus obtain that the transverse displacement of magnetic field lines in the shear-wave packet is on the order of \( \xi \) at distances \( \lambda \ll \xi \). Therefore, this packet should be highly anisotropic in the plane perpendicular to the large-scale magnetic field. In this plane, it is elongated in the direction of the field.

In our model, this means that \( \nu_l \) is isotropic in the plane \( \nu_l \). In this plane, it is elongated in the direction of the field. In the other limit, \( \nu_l \) approaches the so-called Alfvenic state, where either \( \nu_l \) or \( \nu_r \) tends to zero large-scale magnetic field. In this limit, we can obtain the following approximate evolution equations: \( \partial \delta v_{ll} \approx \delta v_{ll} \theta / \lambda \) and \( \partial \delta v_{lr} \approx \delta v_{lr} \theta / \lambda \). The alignment exponent \( \alpha \) would be determined by numeric coefficients in these equations, which cannot be obtained from the dimensional analysis.

As an important analogy, we note that decaying MHD turbulence approaches the so-called Alfvénic state, where either \( \nu_l \) or \( \nu_r \) is zero depending on the initial conditions (see, e.g., Grappin et al. 1982, 1983). Based on our analysis, we propose that driven turbulence behaves in a similar manner, although \( \nu_l \) does not become exactly equal to \( \pm \nu_r \). Rather, magnetic fluctuations \( \delta B_0 \) tend to align their direction, but not their magnitude, with that of the velocity fluctuations, \( \pm \delta v \). As a result, the fluctuations \( \delta w_{ll} \) and \( \delta z_{lr} \) are of the same order, and the directions of \( \delta w_{ll} \) and \( \pm \delta z_{lr} \) are aligned within the angle \( \theta_{ll} \sim \lambda / \xi \propto \lambda^{\alpha/(3+\alpha)} \). The degree of the alignment increases progressively as the scale decreases. Such scale-dependent dynamic alignment (and the associated depletion of nonlinearity) can in principle be checked numerically, although the numerical analysis may be complicated by the rather slow dependence of the alignment angle \( \theta_{ll} \) on the scale. There is, however, a numerical indication that MHD turbulence indeed has a tendency to create correlated regions of polarized fluctuations (Maron & Goldreich 2001).

On the nonuniversality of the turbulent spectrum.—Our analysis points to an interesting possibility that MHD turbulence is nonuniversal, in that it depends on the large-scale magnetic field. We may further speculate that, in principle, other large-scale conditions may affect the scaling properties of turbulence. For example, the dynamic alignment may be sensitive to the level of cross-helicty fluctuations; an analogous result is known for the case of decaying turbulence (see, e.g., Grappin et al. 1982, 1983). An alternative possibility may be that the spectrum is universal and has the Iroshnikov-Kraichnan scaling, but, for \( \gamma \ll 1 \), the dynamic alignment with \( \alpha = 1 \) is established rather slowly as the scale decreases, and the resolution of numerical simulations is not large enough to reach the universal regime.

3. CONCLUSIONS

To conclude, we summarize our main results:

1. Our consideration is motivated by the recent numerical observations (Maron & Goldreich 2001; Müller et al. 2003) that incompressible MHD turbulence is not completely described by either the Iroshnikov-Kraichnan or the Goldreich-Sridhar model. The scaling of velocity fluctuations was found in these papers to depend on the strength of the large-scale external magnetic field: the Iroshnikov-Kraichnan scaling appeared in the limit of strong external magnetic field, while the Goldreich-Sridhar scaling appeared in the limit of weak field.

2. To explain these numerical findings, we propose that turbulent fluctuations become increasingly dynamically aligned as the energy cascade proceeds to smaller scales. Velocity fluctuations \( \delta v \) tend to align their direction with that of magnetic field fluctuations, \( \pm \delta B_0 \), and the smaller the scale \( \lambda \), the stronger the alignment. The dynamic alignment leads to reduction
of the nonlinear-wave interaction (so-called depletion of non-linearity).

3. As a result of point 2, fluctuating “eddies” are three-dimensionally anisotropic. The “eddies,” whose smallest scale is \( \lambda \), have the scale \( \xi \propto \lambda^{3+\alpha} \) in the direction of the shear (the direction of the magnetic field line distortion), and the scale \( \ell \propto \lambda^{3+\alpha} \) in the direction of the large-scale magnetic field, as is sketched in Figure 1. The scaling and anisotropy of fluctuations are described by a single parameter \( \alpha \), which depends on the strength of the external magnetic field, and which is determined in our work from comparison with numerical simulations by Maron & Goldreich (2001) and Müller et al. (2003).

4. The energy distribution is given by

\[
E(k) \propto k^{-(5+\alpha)\xi^{3+\alpha}}.
\]

This coincides with the Goldreich-Sridhar spectrum in the limit of weak anisotropy (\( \alpha = 0 \)), and with the Iroshnikov-Kraichnan spectrum in the limit of strong anisotropy (\( \alpha = 1 \)).

As the external magnetic field increases from \( \gamma \ll 1 \) to \( \gamma \gg 1 \), the anisotropy of turbulent fluctuations increases, and the parameter \( \alpha \) changes from \( \alpha = 0 \) to \( \alpha = 1 \). According to point 3, the corresponding scalings of the fluctuations with respect to their field-perpendicular and field-parallel dimensions are

\[
\delta v \propto \lambda^{1+\alpha}, \quad \delta v \propto \xi^{\frac{1}{2}}, \quad \text{and} \quad \delta v \propto \ell^{\frac{1}{2}}.
\]

5. The smallest-scale “eddies” in our turbulent cascade (\( \lambda \rightarrow 0 \) for \( \alpha \neq 0 \)) have a sheetlike morphology, in agreement with micro–current-sheet dissipative structures, numerically observed in MHD turbulence (Biskamp & Müller 2000). In the case of zero external magnetic field, \( \alpha = 0 \), the dissipative structures are filaments, which also agrees with numerical simulations (Padoan et al. 2004).

6. Previous attempts to explain the numerically observed Iroshnikov-Kraichnan scaling (and the associated cascade-time increase, eq. [3]), in anisotropic sub-Alfvénic MHD turbulence (\( \gamma \gg 1 \)) essentially invoked intermittency effects (see, e.g., § 6.6.4 of Maron & Goldreich 2001; Müller et al. 2003). The intermittency effects are essential for explaining higher order statistics of MHD turbulence, that is, scaling of higher order structure functions of \( \omega \) and \( \zeta \). However, they usually provide only small corrections to the scaling of the second-order structure functions, and of the energy spectra. Such effects are not addressed in the present Letter; our derivation is based on the idea of scale-dependent dynamic alignment, which does not require intermittency.

I am grateful to Fausto Cattaneo and Samuel Vainshtein for many discussions of MHD turbulence, to Jason Maron, the referee of this Letter, for important comments on the results of Maron & Goldreich (2001), to Wolf-Christian Müller for discussion of the results of Müller et al. (2003), and to Dmitri Uzdensky for helpful comments on the physics and the style of the paper. This work was supported by the NSF Center for Magnetic Self-Organization in Laboratory and Astrophysical Plasmas at the University of Chicago. The hospitality of the Aspen Center for Physics, where part of this work was done, is gratefully acknowledged.