Classification of involutions on finitary incidence algebras of non-connected posets

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Abstract

Let FI(X, K) be the finitary incidence algebra of a non-connected partially ordered set X over a field K of characteristic different from 2. For the case where every multiplicative automorphism of FI(X, K) is inner, we present necessary and sufficient conditions for two involutions on FI(X, K) to be equivalent.

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1 Introduction

Algebras with involution play an important role in the general theory of algebras (cf. [8][11]). In [2] was proved that the finitary incidence algebra of a partially ordered set X over a field K, FI(X, K), has an involution if and only if X has an involution. Involutions on FI(X, K) were first studied by Scharlau [13] and thirty years later by Spiegel [14][15]. Their results about the classification of involutions on FI(X, K) ([13] Theorem 2.1(b)] and [15] Theorem 1) are incorrect as indicated in [5].

Let K be a field of characteristic different from 2. In the particular case where X is a chain of cardinality n, FI(X, K) ∼ UTn(K), the algebra of upper triangular matrices over K, and the classification of involutions on UTn(K) was given in [6]. When X has an element that is comparable to all of its elements, Brusamarelo et al. obtained the classification of involutions on FI(X, K) for the case where X is finite [1], and later, locally finite [2]. In [5], they generalized this classification for the case where X is connected (not necessarily locally finite) and every multiplicative automorphism of FI(X, K) is inner. This allowed us to

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obtain the classification of involutions on the idealization of the incidence space $I(X, K)$ over the finitary incidence algebra $FI(X, K)$ adding the hypothesis that every derivation from $FI(X, K)$ to $I(X, K)$ is inner. In this paper, we consider a field $K$ of characteristic different from 2 and a non-connected partially ordered set $X$ such that every multiplicative automorphism of $FI(X, K)$ is inner, and we give the classification of involutions on $FI(X, K)$. For this, we use some results and ideas from [13].

Our work is organized as follows. In Section 2 we recall some definitions and results about partially ordered sets and finitary incidence algebras, and we also present new results about automorphisms of finitary incidence algebras. In Section 3 we consider a partially ordered set $X$ written as the disjoint union of its connected components, and we give some properties of restrictions of automorphisms, anti-automorphisms and involutions of $X$ and $FI(X, K)$. In Section 4 we consider a field $K$ of characteristic different from 2 and a non-connected partially ordered set $X$ such that every multiplicative automorphism of $FI(X, K)$ is inner and we give necessary and sufficient conditions for two involutions on $FI(X, K)$ to be equivalent via inner automorphism (Theorems 4.3 and 4.4). So, we use this classification to obtain the general classification of involutions on $FI(X, K)$ (Theorems 4.7 and 4.11).

2 Preliminaries

2.1 Posets

Let $(X, \leq)$ be a partially ordered set (poset, for short). Two elements $x, y \in X$ are comparable if either $x \leq y$ or $y \leq x$. If an element $x_0 \in X$ is comparable to every element of $X$, then it is called an all-comparable element. Any subset $Y$ of $X$ is also a poset with the relation $\leq$ restricted to the elements of $Y$, and in this case $Y$ is said to be a subposet of $X$. A subposet $Y$ of $X$ is a chain if any two elements of $Y$ are comparable. On the other hand, $Y$ is an antichain if any two distinct elements of $Y$ are not comparable.

Given $x, y \in X$, the interval from $x$ to $y$ is the set $[x, y] = \{z \in X : x \leq z \leq y\}$. If all intervals of $X$ are finite, then $X$ is said to be locally finite.

The elements $x, y \in X$ are connected if for some positive integer $n$, there exists $x = x_0, x_1, \ldots, x_n = y$ in $X$ with $x_i$ and $x_{i+1}$ comparable for $i = 0, 1, \ldots, n-1$. It is easy to see that the connectedness of elements of $X$ is an equivalence relation whose equivalence classes are called connected components of $X$. Then $X$ can be written as the disjoint union of its connected components. When $X$ has only one connected component it is said to be connected.

Let $X$ and $Y$ be posets. An isomorphism (resp. anti-isomorphism) from $X$ to $Y$ is a bijective map $\varphi : X \to Y$ that satisfies the following property for any $x, y \in X$:

$$x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y) \quad (\text{resp. } \varphi(y) \leq \varphi(x)).$$

When $X = Y$, $\varphi$ is also called an automorphism (resp. anti-automorphism). An anti-automorphism $\varphi : X \to X$ of order 2 is an involution on $X$. 

2
Let $X$ be a poset with an involution $\lambda$. By [3 Theorem 4.7], there is a triple of disjoint subsets $(X_1, X_2, X_3)$ of $X$ with $X = X_1 \cup X_2 \cup X_3$ satisfying:

(i) $X_3 = \{x \in X : \lambda(x) = x\}$;
(ii) if $x \in X_1$ ($X_2$), then $\lambda(x) \in X_2$ ($X_1$);
(iii) if $x \in X_1$ ($X_2$) and $y \leq x$ ($x \leq y$), then $y \in X_1$ ($X_2$).

In this case, $(X_1, X_2, X_3)$ is called a $\lambda$-decomposition of $X$.

### 2.2 Finitary incidence algebras

Let $K$ be a field. Throughout the paper $K$-algebras are associative with unity. The center of a $K$-algebra $A$ is denoted by $Z(A)$ and the set of invertible elements of $A$ is denoted by $U(A)$. For each $a \in U(A)$, $\Psi_a$ denotes the inner automorphism defined by $a$, i.e., $\Psi_a : A \to A$ is such that $\Psi_a(x) = axa^{-1}$ for all $x \in A$. The group of automorphisms of $A$ is denoted by $\text{Aut}(A)$ and the subgroup of inner automorphisms is denoted by $\text{IAut}(A)$.

*Involutions* on a $K$-algebra $A$ are ($K$-linear) anti-automorphisms of order 2. Two involutions $\rho_1$ and $\rho_2$ are *equivalent* if there exists $\phi \in \text{Aut}(A)$ such that $\phi \circ \rho_1 = \rho_2 \circ \phi$.

Let $X$ be a poset and let $K$ be a field. The *incidence space* $I(X, K)$ of $X$ over $K$ is the $K$-space of functions $f : X \times X \to K$ such that $f(x, y) = 0$ if $x \nleq y$. Let $FI(X, K)$ be the subspace of functions $f \in I(X, K)$ such that for any $x \leq y$ in $X$, there is only a finite number of subintervals $[u, v] \subseteq [x, y]$ such that $u \neq v$ and $f(u, v) \neq 0$. Then $FI(X, K)$ is a $K$-algebra with the product (convolution)

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y),$$

for any $f, g \in FI(X, K)$, called the *finitary incidence algebra* of $X$ over $K$. Furthermore, $I(X, K)$ is an $FI(X, K)$-bimodule [10]. If $X$ is locally finite, then $FI(X, K) = I(X, K)$ is the *incidence algebra* of $X$ over $K$ [12]. The unity of $FI(X, K)$ is the function $\delta$ defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. An element $f \in FI(X, K)$ is invertible if and only if $f(x, x) \neq 0$ for all $x \in X$, by [10] Theorem 2. A function $f \in FI(X, K)$ is *diagonal* if $f(x, y) = 0$ whenever $x \neq y$. A diagonal function $f$ is *constant on a set $Y$* if $f(x, x) = f(y, y)$ for all $x, y \in Y$.

**Proposition 2.1.** Let $X$ be a poset and $K$ a field. Then $Z(FI(X, K))$ is the set of all diagonal functions which are constant on each connected component of $X$. Consequently, $X$ is connected if and only if $FI(X, K)$ is a central algebra.

**Proof.** The proof is exactly the same as in [16 Theorem 1.3.13].

Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components. For a nonempty subset $L$ of $J$, we set $X_L = \bigcup_{j \in L} X_j$ and $f_L = f|_{X_L \times X_L}$ for each $f \in I(X, K)$. Note that $f_L \in I(X_L, K)$, moreover, $f_L \in FI(X_L, K)$
whenever \( f \in FI(X, K) \). When \( L = \{ l \} \) we denote \( f_L = f_l \). The following result is well-known and easy to prove.

**Proposition 2.2.** Let \( X = \bigcup_{j \in J} X_j \) be the decomposition of the poset \( X \) into its connected components. The map

\[
\phi : I(X, K) \to \prod_{j \in J} I(X_j, K)
\]

\[
f \mapsto (f_j)_{j \in J}
\]

is an isomorphism of \( K \)-vector spaces and

\[
\phi|_{FI(X,K)} : FI(X, K) \to \prod_{j \in J} FI(X_j, K)
\]

is an isomorphism of \( K \)-algebras.

When convenient, we will identify \( f \in I(X, K) \) with \( (f_j)_{j \in J} \in \prod_{j \in J} I(X_j, K) \).

The next corollary follows directly from the proposition above.

**Corollary 2.3.** Let \( X = \bigcup_{j \in J} X_j \) be the decomposition of the poset \( X \) into its connected components. Let \( L \) be a nonempty subset of \( J \) and \( L^c = J - L \). Then, for any \( f, g \in I(X, K) \),

(i) \( (f + g)_L = f_L + g_L \);
(ii) \( (af)_L = af_L \) for all \( a \in K \);
(iii) \( (fg)_L = f_L g_L \) if \( f, g \in FI(X, K) \);
(iv) \( f \in U(FI(X, K)) \iff f_L \in U(FI(X_L, K)) \) and \( f_{L^c} \in U(FI(X_{L^c}, K)) \). In this case, \( (f^{-1})_L = (f_L)^{-1} \);
(v) \( f \in Z(FI(X, K)) \iff f_L \in Z(FI(X_L, K)) \) and \( f_{L^c} \in Z(FI(X_{L^c}, K)) \).

**Corollary 2.4.** Let \( X = \bigcup_{j \in J} X_j \) be the decomposition of the poset \( X \) into its connected components. Let \( L \) be a nonempty subset of \( J \) and \( u, v \in U(FI(X, K)) \). Then

(i) \( [\Psi_u(f)]_L = \Psi_{u_L}(f_L) \) for all \( f \in FI(X, K) \);
(ii) \( \Psi_u = \Psi_v \iff \Psi_{u_j} = \Psi_{v_j} \; \forall j \in J \iff \Psi_{u_L} = \Psi_{v_L} \) and \( \Psi_{u_{L^c}} = \Psi_{v_{L^c}} \).

**Proof.** It follows from the previous corollary and [7, Proposition 2.2].

An element \( \sigma \in I(X, K) \) such that \( \sigma(x, y) \neq 0 \) for all \( x \leq y \), and \( \sigma(x, y)\sigma(y, z) = \sigma(x, z) \) whenever \( x \leq y \leq z \) is called multiplicative. For each map \( h : X \to K^\ast \), we define a fractional element \( \tau_h \in I(X, K) \) by \( \tau_h(x, y) = \frac{h(x)}{h(y)} \) for all \( x \leq y \) in \( X \). Its easy to check that any fractional element is multiplicative.
Lemma 2.5. Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components and let $\sigma \in I(X,K)$. Then $\sigma$ is fractional (multiplicative) if and only if $\sigma_j \in I(X_j,K)$ is fractional (multiplicative) for all $j \in J$.

Proof. Clearly, $\sigma$ is multiplicative if and only if $\sigma_j$ is multiplicative for all $j \in J$, because if $x \leq y \leq z$, then $x, y$ and $z$ are in a same connected component of $X$.

If $\sigma = \tau_h$ for some $h : X \to K^*$, then $\sigma_j = \tau_{h_j}$ for all $j \in J$, where $h_j = h|_{X_j}$. Conversely, if for each $j \in J$ there is $g_j : X_j \to K^*$ such that $\sigma_j = \tau_{g_j}$, then $\sigma = \tau_g$ where $g : X \to K^*$ is given by $g(x) = g_j(x)$ if $x \in X_j$.

For each multiplicative element $\sigma \in I(X,K)$ consider $M_\sigma : FI(X,K) \to FI(X,K)$ defined by $M_\sigma(f)(x,y) = \sigma(x,y)f(x,y)$, for all $f \in FI(X,K)$ and $x, y \in X$. Then $M_\sigma$ is an automorphism which is said to be a multiplicative automorphism. We denote the group of all multiplicative automorphisms of $FI(X,K)$ by $\text{MAut}(FI(X,K))$. The subset $\text{Frac}(FI(X,K)) = \{M_\sigma : \sigma$ is fractional$\}$ is a subgroup of $\text{MAut}(FI(X,K))$ and its elements are called fractional automorphisms.

As in the case where $X$ is locally finite, we have the following relationship between the inner, multiplicative and fractional automorphisms of $FI(X,K)$.

Proposition 2.6. For any field $K$ and any poset $X$,

$$\text{IAut}(FI(X,K)) \cap \text{MAut}(FI(X,K)) = \text{Frac}(FI(X,K)).$$

Proof. It is analogous to the proof of [116, Proposition 7.3.3].

Proposition 2.7. Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components. The multiplicative automorphism $M_\sigma$ of $FI(X,K)$ is inner if and only if the multiplicative automorphism $M_{\sigma_j}$ of $FI(X_j,K)$ is inner for each $j \in J$.

Proof. It follows directly from Lemma 2.5 and Proposition 2.6.

Corollary 2.8. Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components. Then $\text{MAut}(FI(X,K)) \subseteq \text{IAut}(FI(X,K))$ if and only if $\text{MAut}(FI(X_j,K)) \subseteq \text{IAut}(FI(X_j,K))$ for all $j \in J$.

Proof. Suppose $\text{MAut}(FI(X,K)) \subseteq \text{IAut}(FI(X,K))$. Let $i \in J$ and $\sigma \in I(X_i,K)$ be a multiplicative element. By Lemma 2.5, $\tau \in I(X,K)$ defined by

$$\tau(x,y) = \begin{cases} \sigma(x,y) & \text{if } x \leq y \in X_i \\ 1 & \text{if } x \leq y \in X_j, j \neq i \\ 0 & \text{if } x \not\leq y \end{cases}$$

is multiplicative. Thus, $M_\sigma \in \text{IAut}(FI(X,K))$, therefore $M_{\tau_i} \in \text{IAut}(FI(X_i,K))$, by Proposition 2.7. Since $M_\sigma = M_{\tau_i}$, then $M_\sigma \in \text{IAut}(FI(X_i,K))$.

The converse follows directly from Proposition 2.7.
Corollary 2.9. Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components. If $X_j$ has an all-comparable element for each $j \in J$, then $\text{MAut}(FI(X, K)) \subseteq \text{IAut}(FI(X, K))$.

Proof. It follows directly from Corollary 2.8 and [3, Proposition 2.4].

Let $X$ and $Y$ be posets and let $\varphi : X \to Y$ be an isomorphism (resp. anti-isomorphism). Then $\varphi$ induces an isomorphism (resp. anti-isomorphism) $\hat{\varphi}$ (resp. $\rho_{\varphi}$) from $FI(X, K)$ to $FI(Y, K)$ given by

$$\hat{\varphi}(f)(x, y) = f(\varphi^{-1}(x), \varphi^{-1}(y)) \quad \text{and} \quad \rho_{\varphi}(f)(x, y) = f(\varphi^{-1}(y), \varphi^{-1}(x)),$$

for all $f \in FI(X, K)$ and $x, y \in Y$. In particular, when $X = Y$, $\hat{\varphi}$ is an automorphism (resp. anti-automorphism) of $FI(X, K)$. Moreover, if $\varphi$ is an involution, so is $\rho_{\varphi}$.

The theorem below follows from [9, Lemma 3] for automorphisms and it is [3, Theorem 3.5] for anti-automorphisms and involutions.

Theorem 2.10. Let $X$ be a poset and let $K$ be a field. If $\Phi$ is an automorphism (anti-automorphism, involution) of $FI(X, K)$, then $\Phi = \Psi \circ \hat{\varphi}$ (resp. $\Phi = \Psi \circ \rho_{\varphi}$), where $\Psi \in \text{IAut}(FI(X, K))$, $M \in \text{MAut}(FI(X, K))$ and $\phi$ is the automorphism (anti-automorphism, involution) induced by an automorphism (anti-automorphism, involution) of $X$.

Remark 2.11. The automorphism (anti-automorphism, involution) $\phi$ in the theorem above is equal to $\hat{\varphi}$ (resp. $\rho_{\varphi}$), where $\varphi$ is the automorphism (anti-automorphism, involution) of $X$ induced by $\Phi$. (See proofs of [9, Lemma 3] and [3, Theorem 3.5]).

Let $X = \bigcup_{j \in J} X_j$ be the decomposition of the poset $X$ into its connected components. If $X_j$ has an all-comparable element for each $j \in J$, then $\text{MAut}(FI(X, K)) \subseteq \text{IAut}(FI(X, K))$, by Corollary 2.8. In this case, the decomposition of $\Phi$ given by Theorem 2.10 takes the simplest form $\Phi = \Psi \circ \phi$. However, the existence of an all-comparable element in $X_j$ for each $j \in J$ is not necessary to ensure that every multiplicative automorphism of $FI(X, K)$ is inner. For example, if $X_j$ is a finite tree for all $j \in J$, then $\text{MAut}(FI(X, K)) \subseteq \text{IAut}(FI(X, K))$, by [4, Corollary 6] and Corollary 2.8. Necessary and sufficient conditions for a multiplicative automorphism of $FI(X, K)$ to be inner were given in [4] for the case when $X$ is finite and connected.

3 Restrictions of automorphisms, anti-automorphisms and involutions

From now on we consider a non-connected poset $X$ such that $\text{MAut}(FI(X, K)) \subseteq \text{IAut}(FI(X, K))$ and we write $X = \bigcup_{j \in J} X_j$ its decomposition into connected components. In this case, if $\Phi$ is an automorphism (anti-automorphism, involution) of $FI(X, K)$ that induces the automorphism (anti-automorphism, involution) $\varphi$ of $X$, then there is $u \in U(FI(X, K))$ such that $\Phi = \Psi_u \circ \hat{\varphi}$ (resp. $\Phi = \Psi_u \circ \rho_{\varphi}$), by Theorem 2.10 and Remark 2.11.
Remark 3.1. Let $\varphi : X \to X$ be an automorphism (anti-automorphism, involution).

(i) For each $i \in J$ there exists $j \in J$ such that $\varphi(X_i) = X_j$.
(ii) If $L$ is a nonempty subset of $J$ such that $\varphi(X_L) = X_L$, then $\varphi|_{X_L} : X_L \to X_L$ is an automorphism (anti-automorphism, involution) which will be denoted by $\varphi_L$, and $(\varphi_L)^{-1} = (\varphi^{-1})_L$. When $L = \{l\}$ we just write $\varphi_L = \varphi_l$.

Definition 3.2. Let $\alpha$ and $\lambda$ be an automorphism and an anti-automorphism of $X$, respectively, $L$ a nonempty subset of $J$ such that $\alpha(X_L) = X_L$ and $\lambda(X_L) = X_L$, and $u \in U(FI(X, K))$. For $\Phi = \Psi_u \circ \hat{\alpha}$ and $\rho = \Psi_u \circ \rho_\lambda$ we define

$$
\Phi_L := \Psi_{u_L} \circ \hat{\alpha}_L \quad \text{and} \quad \rho_L := \Psi_{u_L} \circ \rho_{\lambda_L}.
$$

When $L = \{l\}$ we write $\Phi_L = \Phi_l$ and $\rho_L = \rho_l$.

Proposition 3.3. Let $\alpha$ and $\lambda$ be an automorphism and an anti-automorphism of $X$, respectively, and let $L$ be a nonempty subset of $J$ such that $\alpha(X_L) = X_L$ and $\lambda(X_L) = X_L$. Let $\Phi$ be an automorphism of $FI(X, K)$ that induces $\alpha$ on $X$ and $\rho$ an anti-automorphism of $FI(X, K)$ that induces $\lambda$ on $X$. Then

(i) $[\rho(f)]_L = \rho_L(f_L)$ for all $f \in FI(X, K)$;
(ii) $[\Phi(f)]_L = \Phi_L(f_L)$ for all $f \in FI(X, K)$;
(iii) $[\Phi \circ \rho]_L = \Phi_L \circ \rho_L$ and $[\rho \circ \Phi]_L = \rho_L \circ \Phi_L$.

Proof. Suppose $\Phi = \Psi_u \circ \hat{\alpha}$ and $\rho = \Psi_u \circ \rho_\lambda$, for some $v, u \in U(FI(X, K))$.

(i) Let $f \in FI(X, K)$. Firstly, if $\lambda(X_L) = X_L$, then for all $x \leq y$ in $X_L$ we have

$$
[r_\lambda(f)]_L(x, y) = \rho_\lambda(f)(x, y) = f(\lambda^{-1}(y), \lambda^{-1}(x)) = f_L(\lambda^{-1}(y), \lambda^{-1}(x)) = f_L((\lambda^{-1})_L(y), (\lambda^{-1})_L(x)) = f_L((\lambda_L)^{-1}(y), (\lambda_L)^{-1}(x)) = \rho_{\lambda_L}(f_L)(x, y).
$$

Thus, $[\rho_\lambda(f)]_L = \rho_{\lambda_L}(f_L)$. Therefore,

$$
[r_\lambda(f)]_L = [\Psi_u(r_\lambda(f))]_L = [\Psi_{u_L}(r_\lambda(f))]_L = \Psi_{u_L}(\rho_{\lambda_L}(f_L)) = \rho_L(f_L),
$$

by Corollary 2.4 (i) and Definition 3.2.

(ii) It is similar to (i).

(iii) Note that $\Phi \circ \rho$ induces the anti-automorphism $\alpha \circ \lambda$ of $X$, because

$$
\Phi \circ \rho = \Psi_v \circ \hat{\alpha} \circ \Psi_u \circ \rho_\lambda = \Psi_v \circ \Psi_{\hat{\alpha}(u)} \circ \hat{\alpha} \circ \rho_\lambda = \Psi_{v \hat{\alpha}(u)} \circ \rho_{\alpha \circ \lambda}.
$$

Moreover, $(\alpha \circ \lambda)(X_L) = X_L$. Given $g \in FI(X_L, K)$, consider $f \in FI(X, K)$ such that $f_L = g$. By (i) and (ii) we have

$$(\Phi_L \circ \rho_L)(g) = \Phi_L(\rho_L(f_L)) = [\Phi(\rho(f))]|_L = [\Phi \circ \rho]|_L(f_L) = [\Phi \circ \rho]|_L(g).$$

Therefore, $[\Phi \circ \rho]|_L = \Phi_L \circ \rho_L$. Analogously, $[\rho \circ \Phi]|_L = \rho_L \circ \Phi_L$. 

\[7\]
Proposition 3.4. Let $\lambda$ be an involution on $X$, $L$ a nonempty subset of $J$ such that $\lambda(X_L) = X_L$, and $u \in U(\text{FI}(X, K))$. Let $\rho = \Psi_u \circ \rho_X$. Then $\rho$ is an involution on $\text{FI}(X, K)$ if and only if $\rho_L = \Psi_{u_L} \circ \rho_{\lambda_L}$ is an involution on $\text{FI}(X_L, K)$ and $\rho_{L^c} = \Psi_{u_{L^c}} \circ \rho_{\lambda_{L^c}}$ is an involution on $\text{FI}(X_{L^c}, K)$.

Proof. We have

\[ \rho \text{ is an involution } \iff \Psi_u = \Psi_{\rho_X(u)} \text{ by } \text{[7] Proposition 2.2} \]

\[ \iff \Psi_{u_L} = \Psi_{\rho_X(u)_L} \text{ and } \Psi_{u_{L^c}} = \Psi_{\rho_X(u)_{L^c}} \text{ by Corollary } \text{[2, Theorem 2.2]}(\text{ii}) \]

\[ \iff \Psi_{u_L} = \Psi_{\rho_{\lambda_L}(u_L)} \text{ and } \Psi_{u_{L^c}} = \Psi_{\rho_{\lambda_{L^c}}(u_{L^c})} \text{ by Proposition } \text{[3, Theorem 3.3]}(\text{i}) \]

\[ \iff \rho_L \text{ and } \rho_{L^c} \text{ are involutions by } \text{[7] Proposition 2.2}. \]

\[ \square \]

Proposition 3.5. Let $\lambda$ be an involution on $X$ such that $\lambda(X_j) = X_j$ for all $j \in J$, and $u \in U(\text{FI}(X, K))$. Let $\rho = \Psi_u \circ \rho_X$. Then $\rho$ is an involution on $\text{FI}(X, K)$ if and only if $\rho_j = \Psi_{u_j} \circ \rho_{\lambda_j}$ is an involution on $\text{FI}(X_j, K)$ for all $j \in J$.

Proof. It is analogous to the proof of Proposition 3.4. \[ \square \]

4 Classification of involutions

We recall that the involutions $\rho_1$ and $\rho_2$ on $\text{FI}(X, K)$ are equivalent if there exists $\phi \in \text{Aut}(\text{FI}(X, K))$ such that $\phi \circ \rho_1 = \rho_2 \circ \phi$. In this section we consider a field $K$ of characteristic different from 2 and a non-connected poset $X$ such that $\text{MAut}(\text{FI}(X, K)) \subseteq \text{IAut}(\text{FI}(X, K))$ and we give necessary and sufficient conditions for two involutions on $\text{FI}(X, K)$ to be equivalent. For this, we use some results and ideas from [1] and [3].

4.1 Classification of involutions via inner automorphisms

We start by considering the classification via inner automorphisms. Let $\rho_1$ and $\rho_2$ be involutions on $\text{FI}(X, K)$. If there exists $\Psi \in \text{IAut}(\text{FI}(X, K))$ such that $\Psi \circ \rho_1 = \rho_2 \circ \Psi$, then $\rho_1$ and $\rho_2$ induce the same involution on $X$, by [3, Theorem 5.1]. Thus, two involutions on $\text{FI}(X, K)$ that induce different involutions on $X$ are not equivalent via inner automorphisms. For that, in this subsection we fix an involution $\lambda$ on $X$ and give necessary and sufficient conditions for two involutions on $\text{FI}(X, K)$ that induce $\lambda$ on $X$ to be equivalent via inner automorphisms.

We denote by $\text{Inv}_\lambda(\text{FI}(X, K))$ the set of all involutions on $\text{FI}(X, K)$ that induce $\lambda$ on $X$.

Theorem 4.1. Let $\rho, \eta \in \text{Inv}_\lambda(\text{FI}(X, K))$ and let $L$ be a nonempty subset of $J$ such that $\lambda(X_L) = X_L$. Consider the involutions $\rho_L$ and $\eta_L$ on $\text{FI}(X_L, K)$, and $\rho_{L^c}$ and $\eta_{L^c}$ on $\text{FI}(X_{L^c}, K)$. Then $\rho$ and $\eta$ are equivalent via inner automorphism, if and only if $\rho_L$ and $\eta_L$ are equivalent via inner automorphism, and $\rho_{L^c}$ and $\eta_{L^c}$ are equivalent via inner automorphism.
Proof. Note that if \( \lambda(X_L) = X_L \), then \( \lambda(X_{L^c}) = X_{L^c} \). Therefore, if \( \Psi_u \circ \rho = \eta \circ \Psi_u \) for some \( u \in U(FI(X, K)) \), then \( \Psi_{u_L} \circ \rho_L = \eta_L \circ \Psi_{u_L} \) and \( \Psi_{u_{L^c}} \circ \rho_{L^c} = \eta_{L^c} \circ \Psi_{u_{L^c}} \), by Proposition 3.3(iii).

Conversely, let \( w_1 \in U(FI(X_L, K)) \) and \( w_2 \in U(FI(X_{L^c}, K)) \) such that \( \Psi_{w_1} \circ \rho_L = \eta_L \circ \Psi_{w_1} \) and \( \Psi_{w_2} \circ \rho_{L^c} \). Let \( w \in FI(X, K) \) such that \( w_L = w_1 \) and \( w_{L^c} = w_2 \). Then \( w \in U(FI(X, K)) \). By Corollary 2.3(i) and Proposition 3.3(i), we have

\[
[(\Psi_w \circ \rho)(f)]_L = \Psi_{w_L}(\rho_L(f)) = \eta_L(\Psi_{w_1}(f)) = \eta_L(\Psi_{w_{L^c}}(f)) = (\Psi_w \circ \rho)_L(f)
\]

for all \( f \in FI(X, K) \). Analogously, \([[(\Psi_w \circ \rho)(f)]_{L^c} = [(\eta \circ \Psi_w)(f)]_{L^c} \) for all \( f \in FI(X, K) \). Therefore, \( (\Psi_w \circ \rho)(f) = (\eta \circ \Psi_w)(f) \) for all \( f \in FI(X, K) \), that is, \( \Psi_w \circ \rho = \eta \circ \Psi_w \).

Consider the following relation defined on the set \( J \) (such that \( X = \bigcup_{j \in J} X_j \)):

\[
i \leq j \Leftrightarrow X_i = X_j.
\]

Then \( (J, \leq) \) is a poset (antichain). Note that the involution \( \lambda : X \to X \) induces an involution \( \overline{\lambda} : J \to J \) such that, given \( j \in J \), one has \( \lambda(X_j) = X_{\overline{\lambda}(j)} \), by Remark 3.1(i). Let \((J_1, J_2, J_3)\) be a \( \overline{\lambda} \)-decomposition of \( J \). Then

\[
J_3 = \{j \in J : \overline{\lambda}(j) = j\} = \{j \in J : \lambda(X_j) = X_j\}.
\]

Lemma 4.2. If \( \rho = \Psi_u \circ \rho_\lambda \) is an involution on \( FI(X, K) \), then there is a central unit \( v \) of \( FI(X, K) \) such that \( \rho_\lambda(u) = vu \). Moreover, for each \( j \in J \), there is \( k_j \in K^* \) such that \( v(x, x) = k_j \) for all \( x \in X_j \), and \( k_j k_{\overline{\lambda}(j)} = 1 \).

Proof. By (i) and (ii) of \([7, \text{Proposition 2.2}]\), there is a central unit \( v \in FI(X, K) \) such that \( \rho_\lambda(u) = vu \). So

\[
u = \rho_\lambda(u) = \rho_\lambda(vu) = \rho_\lambda(u)\rho_\lambda(v) = vu\rho_\lambda(v) = uv\rho_\lambda(v)
\]

and, consequently, \( v\rho_\lambda(v) = 1 \). Since \( v \in Z(FI(X, K)) \), then \( v \) is diagonal and for each \( j \in J \) there is \( k_j \in K^* \) such that \( v(x, x) = k_j \) for all \( x \in X_j \), by Proposition 2.3. Thus, if \( x \in X_j \), then \( \lambda(x) \in \lambda(X_j) = X_{\overline{\lambda}(j)} \) and so

\[
1 = \delta(x, x) = (v\rho_\lambda(v))(x, x) = v(x, x)v(\lambda(x), \lambda(x)) = k_j k_{\overline{\lambda}(j)}.
\]

In order to classify the involutions on \( FI(X, K) \) that induce \( \lambda \) on \( X \), via inner automorphisms, we consider the cases \( J_3 = \emptyset \) and \( J_3 \neq \emptyset \). We start by considering \( J_3 = \emptyset \) or, equivalently, \( \lambda(X_j) \neq X_j \) for all \( j \in J \).

Theorem 4.3. Let \( \lambda \) be an involution on \( X \) such that \( J_3 = \emptyset \). Every \( \rho \in \text{Inv}_\lambda(FI(X, K)) \) is equivalent to \( \rho_\lambda \) via inner automorphism.
Proof. Let \( u \in U(\text{FI}(X, K)) \) such that \( \rho = \Psi_u \circ \rho_\lambda \). By Lemma 4.2, there is a central unit \( v \) of \( \text{FI}(X, K) \) such that \( \rho_\lambda(u) = vu \) and \( k_j k_{\lambda(j)} = 1 \), where \( v(x, x) = k_j \) for all \( x \in X_j \), for each \( j \in J \). Define \( w \in \text{FI}(X, K) \) by

\[
w(x, y) = \begin{cases} 
k_j & \text{if } x = y \in X_j \text{ and } j \in J_1 \\
1 & \text{if } x = y \in X_j \text{ and } j \in J_2 \\
0 & \text{if } x \neq y
\end{cases}
\]

Then \( w \) is a central unit of \( \text{FI}(X, K) \) and

\[
\rho_\lambda(w)(x, y) = w(\lambda(y), \lambda(x)) = \begin{cases} 
k_j & \text{if } \lambda(x) = \lambda(y) \in X_j \text{ and } j \in J_1 \\
1 & \text{if } \lambda(x) = \lambda(y) \in X_j \text{ and } j \in J_2 \\
0 & \text{if } x \neq y
\end{cases}
\]

\[
= \begin{cases} 
k_j & \text{if } x = y \in X_{\lambda(j)} \text{ and } j \in J_1 \\
1 & \text{if } x = y \in X_{\lambda(j)} \text{ and } j \in J_2 \\
0 & \text{if } x \neq y
\end{cases}
\]

Since \( \rho_\lambda(w) \) and \( v \) are diagonal, so is \( \rho_\lambda(w)v \), and by (3) and (4), we have

\[
(\rho_\lambda(w)v)(x, x) = \rho_\lambda(w)(x, x)v(x, x)
\]

\[
= \begin{cases} 
k_j k_{\lambda(j)} & \text{if } x \in X_{\lambda(j)} \text{ and } j \in J_1 \\
k_{\lambda(j)} & \text{if } x \in X_{\lambda(j)} \text{ and } j \in J_2 \\
1 & \text{if } x \in X_{\lambda(j)} \text{ and } j \in J_1 \\
k_j & \text{if } x \in X_{\lambda(j)} \text{ and } j \in J_2 \\
k_j & \text{if } x \in X_j \text{ and } j \in J_1 \\
k_j & \text{if } x \in X_j \text{ and } j \in J_2 \\
= w(x, x)
\end{cases}
\]

for all \( x \in X \). Therefore, \( \rho_\lambda(w)v = w \).

Let \( u_1 = wu \in U(\text{FI}(X, K)) \). Then \( \rho_\lambda(u_1) = \rho_\lambda(w)\rho_\lambda(u) = \rho_\lambda(w)vu = wu = u_1 \). Moreover, \( \Psi_u = \Psi_{u_2} \), by [7 Proposition 2.2]. Thus, \( \rho = \Psi_{u_1} \circ \rho_\lambda \) with \( \rho_\lambda(u_1) = u_1 \). Since \( \lambda(X_j) \neq X_j \) for all \( j \in J \), then \( \lambda(x) \neq x \) for all \( x \in X \), by Remark 3.1 (i). So, by [3 Lemma 5.3], there exists \( v_1 \in U(\text{FI}(X, K)) \) such that \( u_1 = v_1 \rho_\lambda(v_1) \). Therefore, \( \rho = \Psi_{v_1}\rho_\lambda(v_1) \circ \rho_\lambda \), thus \( \rho \) and \( \rho_\lambda \) are equivalent via inner automorphism, by [7 Proposition 2.2].

Now, we suppose \( J_3 \neq \emptyset \). By (2),

\[
\lambda(X_{J_3}) = \lambda \left( \bigcup_{j \in J_3} X_j \right) = \bigcup_{j \in J_3} \lambda(X_j) = \bigcup_{j \in J_3} X_j = X_{J_3}.
\]

Theorem 4.4. Let \( \lambda \) be an involution on \( X \) such that \( J_3 \neq \emptyset \). Let \( \rho, \eta \in \text{Inv}_\lambda(\text{FI}(X, K)) \). The following statements are equivalent:
(i) \( \rho \) and \( \eta \) are equivalent via inner automorphism.

(ii) \( \rho_{J_3} \) and \( \eta_{J_3} \) are equivalent via inner automorphism.

(iii) \( \rho_j \) and \( \eta_j \) are equivalent via inner automorphism, for all \( j \in J_3 \).

**Proof.** (i) \( \Rightarrow \) (iii) By [2], \( \lambda(X_j) = X_j \) for all \( j \in J_3 \). Thus, if \( \rho \) and \( \eta \) are equivalent via inner automorphism, then \( \rho_j \) and \( \eta_j \) are equivalent via inner automorphism for all \( j \in J_3 \), by Theorem 4.4

(iii) \( \Rightarrow \) (ii) Let \( w_j \in FI(X_j, K) \) such that \( \Psi_{w_j} \circ \rho_j = \eta_j \circ \Psi_{w_j} \) for all \( j \in J_3 \). Let \( w \in U(FI(X_{J_3}, K)) \) such that \( w = \prod_{j \in J_3} FI(X_j, K) \). As in (1), we can show that \( \Psi_w \circ \rho_{J_3} = \eta_{J_3} \circ \Psi_w \) and, therefore, \( \rho_{J_3} \) and \( \eta_{J_3} \) are equivalent via inner automorphism.

(ii) \( \Rightarrow \) (i) Suppose \( \rho_{J_3} \) and \( \eta_{J_3} \) are equivalent via inner automorphism. Since \( \lambda(X_{J_3}) = X_{J_3} \), then \( \lambda(X_{(J_3)^c}) = X_{(J_3)^c} \). Thus \( \lambda(X_j) \neq X_j \) for all \( j \in (J_3)^c \). By Theorem 4.4, the involutions \( \rho_{(J_3)^c} \) and \( \eta_{(J_3)^c} \) on \( FI(X_{(J_3)^c}, K) \) are equivalent via inner automorphism. Therefore, \( \rho \) and \( \eta \) are equivalent via inner automorphism, by Theorem 4.4.

Denote by \( \approx \) the equivalence of elements of \( \text{Inv}_\lambda(FI(X, K)) \) via inner automorphisms, that is, for \( \rho, \eta \in \text{Inv}_\lambda(FI(X, K)) \),

\[
\rho \approx \eta \iff \Psi \circ \rho = \eta \circ \Psi \quad \text{for some } \Psi \in \text{IAut}(FI(X, K)).
\]

As usual, we denote the quotient set by \( \text{Inv}_\lambda(FI(X, K)) / \approx \) and the equivalence class of \( \rho \in \text{Inv}_\lambda(FI(X, K)) \) by \( \bar{\rho} \).

**Lemma 4.5.** The following map \( \Omega \) is a bijection:

\[
\Omega : \frac{\text{Inv}_\lambda(FI(X, K)) \approx}{\bar{\rho}} \rightarrow \prod_{j \in J_3} \frac{\text{Inv}_{\lambda_j}(FI(X_j, K)) \approx}{(\bar{\rho}_j)}.
\]

**Proof.** By Theorem 4.4, \( \Omega \) is well-defined and injective. For each \( j \in J_3 \), let \( \phi_j \in \text{Inv}_{\lambda_j}(FI(X_j, K)) \). Since \( \text{MAut}(FI(X_j, K)) \subseteq \text{IAut}(FI(X_j, K)) \), by Corollary 2.4, there is \( u_j \in U(FI(X_j, K)) \) such that \( \phi_j = \Psi_{u_j} \circ \rho_{\lambda_j} \), by Theorem 2.10.

Let \( v \in U(FI(X, K)) \) such that \( v_j = u_j \) if \( j \in J_3 \), and \( v_j = \delta_j \) if \( j \notin J_3 \). Consider the anti-automorphism \( \Phi = \Psi_v \circ \rho \) of \( FI(X, K) \). By Definition 3.2, for each \( j \in J_3 \), \( \Phi_j = \Psi_{v_j} \circ \rho_{\lambda_j} = \Psi_{u_j} \circ \rho_{\lambda_j} = \phi_j \). Thus, since \( \lambda(X_j) = X_j \) for all \( j \in J_3 \), then \( \Phi_{J_3} \) is an involution on \( FI(X_{J_3}, K) \), by Proposition 3.6. On the other hand, \( \Phi_{(J_3)^c} = \Psi_{v_{(J_3)^c}} \circ \rho_{\lambda_{(J_3)^c}} = \rho_{\lambda_{(J_3)^c}} \) which is an involution on \( FI(X_{(J_3)^c}, K) \). By Proposition 3.6, \( \Phi \) is an involution on \( FI(X, K) \). Therefore, \( \Phi \in \text{Inv}_\lambda(FI(X, K)) \) and \( \Omega(\Phi) = (\bar{\phi}_j)_{j \in J_3} = (\bar{\phi}_j)_{j \in J_3} \).
Let \((P_1, P_2, P_3)\) be a \(\lambda\)-decomposition of \(X\). For each \(j \in J_3\), we set \(P_n^j = P_n \cap X_j\) for \(n = 1, 2, 3\), and we have

\[
X_j = X \cap X_j = (P_1 \cup P_2 \cup P_3) \cap X_j = (P_1 \cap X_j) \cup (P_2 \cap X_j) \cup (P_3 \cap X_j) = P_1^j \cup P_2^j \cup P_3^j.
\]

Moreover,

- \(P_3^j = \{x \in X_j : \lambda_j(x) = x\}\);
- if \(x \in P_1^j (P_2^j)\), then \(\lambda_j(x) \in P_1^j (P_2^j)\);
- if \(x \in P_1^j (P_2^j)\) and \(y \leq x \leq y\), then \(y \in P_1^j (P_2^j)\).

Therefore, \((P_1^j, P_2^j, P_3^j)\) is a \(\lambda_j\)-decomposition of \(X_j\) for each \(j \in J_3\). Consider the subset \(J_3^* = \{j \in J_3 : P_3^j = \emptyset\}\) of \(J_3\). By [3, Theorem 5.4], for each \(j \in J_3^*\) there are only two equivalence classes of involutions on \(FI(X_j, K)\) that induce \(\lambda_j\), via inner automorphism. On the other hand, if \(j \in J_3 \setminus J_3^*\), then \(P_3^j \neq \emptyset\). In this case, by [3, Theorem 5.5], the number of equivalence classes of involutions on \(FI(X_j, K)\) that induce \(\lambda_j\), via inner automorphism, is equal to \(|S_K|^{\|P_j^3\| - 1}\), where \(S_K = K^*/(K^*)^2\). Thus, by Lemma 4.6,

\[
\left| \frac{\text{Inv}_\lambda(FI(X, K))}{\approx} \right| = \prod_{j \in J_3^*} \left| \frac{\text{Inv}_{\lambda_j}(FI(X_j, K))}{\approx} \right| = 2^{|J_3|} \prod_{j \in J_3 \setminus J_3^*} |S_K|^{|P_j^3| - 1}.
\]

Therefore we have the following result.

**Theorem 4.6.** The number of equivalence classes of involutions on \(FI(X, K)\) that induce \(\lambda\), via inner automorphism, is equal to \(2^{|J_3|} \prod_{j \in J_3 \setminus J_3^*} |S_K|^{|P_j^3| - 1}\).

### 4.2 General classification of involutions

The general classification of involutions on \(FI(X, K)\) follows directly from the classification via inner automorphism, by the following theorem, whose proof is the same as in [3, Theorem 5.6].

**Theorem 4.7.** The involutions \(\rho_1\) and \(\rho_2\) on \(FI(X, K)\) are equivalent if and only if there exists an automorphism \(\alpha\) of \(X\) such that \(\rho_1 = \Phi \circ \rho_2 \circ \Phi^{-1}\) are equivalent via inner automorphism.

**Proof.** The involutions \(\rho_1\) and \(\rho_2\) on \(FI(X, K)\) are equivalent if and only if there exists an automorphism \(\Phi\) of \(FI(X, K)\) such that \(\rho_1 = \Phi \circ \rho_2 \circ \Phi^{-1}\). By Theorem 2.10 \(\Phi = \Psi \circ \hat{\alpha}\) where \(\Psi \in \text{IAut}(FI(X, K))\) and \(\alpha\) is an automorphism of \(X\). Therefore, \(\rho_1\) and \(\rho_2\) are equivalent if and only if \(\rho_1 = \Psi \circ \hat{\alpha} \circ \rho_2 \circ \hat{\alpha}^{-1} \circ \Psi^{-1}\). \(\square\)
As in [1, p.1953], we consider the equivalence relation $\sim$ on the set of all involutions on $X$ as follows:

$$\lambda \sim \mu \iff \alpha \circ \lambda = \mu \circ \alpha$$

for some automorphism $\alpha$ of $X$.

**Remark 4.8.** Let $\lambda$ and $\alpha$ be an involution and an automorphism of $X$, respectively, and $\mu \in U(FI(X,K))$. If $\rho = \Psi_\mu \circ \rho_\lambda$ is an involution on $FI(X,K)$, then $\hat{\alpha} \circ \rho \circ \hat{\alpha}^{-1} = \Psi_{\hat{\alpha}(\mu)} \circ \rho_{\alpha \circ \lambda \circ \alpha^{-1}}$. In particular, $\alpha \circ \lambda \circ \alpha^{-1}$ is the involution on $X$ induced by the involution $\hat{\alpha} \circ \rho \circ \hat{\alpha}^{-1}$.

**Corollary 4.9.** (i) If $\rho_1$ and $\rho_2$ are equivalent involutions on $FI(X,K)$, then $\lambda_{\rho_1} \sim \lambda_{\rho_2}$.

(ii) If $\lambda_1 \sim \lambda_2$ and $\rho$ is an involution on $FI(X,K)$ that induces $\lambda_1$ on $X$, then $\rho$ is equivalent to some involution $\eta$ that induces $\lambda_2$.

**Proof.** The proof is the same as in [1, Corollary 27], just replacing Theorem 25, Corollary 6, and Proposition 26 of [1] by Theorem 4.7, [3, Theorem 5.1], and Remark 4.8 respectively.

We have seen that if $\lambda$ is an involution on $X$ such that $\lambda(X_j) \neq X_j$ for all $j \in J$, then every involution on $FI(X,K)$ that induces $\lambda$ is equivalent to $\rho_\lambda$ (via inner automorphism) (Theorem 4.3). On the other hand, if $\lambda(X_i) = X_i$ for some $i \in J$, then the equivalence via inner automorphism of two involutions on $FI(X,K)$ that induce $\lambda$ is given by the restriction to the set $J_3 = \{ j \in J : \lambda(X_i) = X_j \}$ (Theorem 4.4). Finally, we will see that the general classification, in the latter case, is also given by the restriction to $J_3$ (Theorem 4.11).

**Lemma 4.10.** Let $\lambda$ be an involution on $X$ such that $J_3 \neq \emptyset$ and $\alpha$ an automorphism of $X$ such that $\alpha \circ \lambda = \lambda \circ \alpha$. Then $\alpha(X_{J_3}) = X_{J_3}$.

**Proof.** Let $j \in J_3$. By Remark 3.1 there exists $i \in J$ such that $\alpha(X_j) = X_i$, and by (2), $\lambda(X_j) = X_j$. So

$$X_i = \alpha(X_j) = \alpha(\lambda(X_j)) = \lambda(\alpha(X_j)) = \lambda(X_i)$$

and then $i \in J_3$. Thus, $\alpha(X_j) = X_i \subseteq X_{J_3}$. Therefore, $\alpha(X_{J_3}) \subseteq X_{J_3}$. On the other hand, since $\alpha^{-1}$ is an automorphism such that $\alpha^{-1} \circ \lambda = \lambda \circ \alpha^{-1}$, then $\alpha^{-1}(X_{J_3}) \subseteq X_{J_3}$.

**Theorem 4.11.** Let $\lambda$ be an involution on $X$ such that $J_3 \neq \emptyset$. Let $\rho, \eta \in \text{Inv}_\lambda(FI(X,K))$. Then $\rho$ and $\eta$ are equivalent if and only if the involutions $\rho_{J_3}$ and $\eta_{J_3}$ on $FI(X_{J_3},K)$ are equivalent.

**Proof.** Suppose $\rho$ and $\eta$ are equivalent. By Theorem 4.7, there exists an automorphism $\alpha$ of $X$ such that $\rho$ and $\hat{\alpha} \circ \eta \circ \hat{\alpha}^{-1}$ are equivalent via inner automorphism. Thus, by [3, Theorem 5.1] and Remark 1.8 $\lambda = \alpha \circ \lambda \circ \alpha^{-1}$, and then $\alpha(X_{J_3}) = X_{J_3}$ by Lemma 4.10. It follows from Theorem 4.3, Proposition 3.3 (iii) and (5) that $\rho_{J_3}$ and $\hat{\alpha}_{J_3} \circ \eta_{J_3} \circ \hat{\alpha}_{J_3}^{-1}$ are equivalent via inner automorphism. Therefore, $\rho_{J_3}$ and $\eta_{J_3}$ are equivalent by Theorem 4.7.
Conversely, suppose \( \rho_J \) and \( \eta_J \) are equivalent. By Theorem 4.7, there exists an automorphism \( \beta \) of \( X_J \) such that \( \rho_J \) and \( \beta \circ \eta_J \circ \beta^{-1} \) are equivalent via inner automorphism. By [3, Theorem 5.1], Definition 3.2, and Remark 4.8,

\[
\lambda_J = \beta \circ \lambda_J \circ \beta^{-1}. \tag{6}
\]

Consider the automorphism \( \alpha : X \to X \) defined by

\[
\alpha(x) = \begin{cases} 
\beta(x) & \text{if } x \in X_J \\
x & \text{if } x \not\in X_J
\end{cases}. \tag{7}
\]

Then \( \alpha(X_J) = \beta(X_J) = X_J \) and \( \alpha_J = \beta \). Thus, by Proposition 3.3 (iii),

\[
(\widehat{\alpha} \circ \eta \circ \widehat{\alpha^{-1}})_J = \widehat{\alpha_J} \circ \eta_J \circ \widehat{\alpha^{-1}}_J = \widehat{\beta} \circ \eta_J \circ \widehat{\beta^{-1}}_J.
\]

It follows that \( \rho_J \) and \( (\widehat{\alpha} \circ \eta \circ \widehat{\alpha^{-1}})_J \) are equivalent via inner automorphism. Moreover, by (6) and (7), \( \lambda = \alpha \circ \lambda \circ \alpha^{-1} \), therefore \( \rho \) and \( \widehat{\alpha} \circ \eta \circ \widehat{\alpha^{-1}} \) induce the same involution on \( X \), by Remark 4.8. Thus, by Theorem 4.4, \( \rho \) and \( \widehat{\alpha} \circ \eta \circ \widehat{\alpha^{-1}} \) are equivalent via inner automorphism, whence \( \rho \) and \( \eta \) are equivalent, by Theorem 4.7.

\[\square\]

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References

[1] Brusamarello, R., Fornaroli, E. Z., Santulo Jr., E. A., Classification of involutions on incidence algebras, *Comm. Algebra* **39** (2011) 1941–1955. DOI: https://doi.org/10.1080/00927872.2010.480958

[2] Brusamarello, R., Fornaroli, E. Z., Santulo Jr., E. A., Anti-automorphisms and involutions on (finitary) incidence algebras, *Linear Multilinear Algebra* **60** (2012) 181–188. DOI: https://doi.org/10.1080/03081087.2011.576393

[3] Brusamarello, R., Fornaroli, E. Z., Santulo Jr., E. A., Classification of involutions on finitary incidence algebras, *Internat. J. Algebra Comput.* **24**(8) (2014), 1085–1098. DOI: https://doi.org/10.1142/S0218196714500477

[4] Brusamarello, R., Fornaroli, E. Z., Santulo Jr., E. A., Multiplicative automorphisms of incidence algebras, *Comm. Algebra* **43**(2) (2015), 726–736. DOI: https://doi.org/10.1080/00927872.2013.847951

[5] Brusamarello, R., Lewis, D. W., Automorphisms and involutions on incidence algebras, *Linear Multilinear Algebra* **59**(11) (2011), 1247–1267. DOI: https://doi.org/10.1080/03081087.2010.496113
[6] Di Vincenzo, O. M., Koshlukov, P., La Scala, R., Involutions for upper triangular matrix algebras, *Adv. in Appl. Math.* **37**(4) (2006), 541–568. DOI: https://doi.org/10.1016/j.aam.2005.07.004

[7] Fornaroli, E. Z., Pezzott, R. E. M., Anti-isomorphisms and involutions on the idealization of the incidence space over the finitary incidence algebra, *Linear Algebra Appl.* **637** (2022), 82–109. DOI: https://doi.org/10.1016/j.laa.2021.12.005

[8] Jacobson, N., *Finite-Dimensional Division Algebras over Fields*, Springer-Verlag, Berlin, 1996.

[9] Khripchenko, N. S., Automorphisms of finitary incidence rings, *Algebra Discrete Math.* **9**(2) (2010), 78–97.

[10] Khripchenko, N. S., Novikov, B. V., Finitary incidence algebras, *Comm. Algebra* **37**(5) (2009), 1670–1676. DOI: https://doi.org/10.1080/00927870802210019

[11] Knus, M.-A., Merkurjev, A., Rost, M., Tignol, J.-P., *The Book of Involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998.

[12] Rota, G.-C., On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2**(4) (1964), 340–368. DOI: https://doi.org/10.1007/BF00531932

[13] Scharlau, W., Automorphisms and involutions of incidence algebras, *Lectures Notes in Mathematics* **488** (1975), 340–350. DOI: https://doi.org/10.1007/BFb0081233

[14] Spiegel, E., Involutions in incidence algebras, *Linear Algebra Appl.* **405** (2005), 155–162. DOI: https://doi.org/10.1016/j.laa.2005.03.003

[15] Spiegel, E., Upper-triangular embeddings of incidence algebras with involution, *Comm. Algebra* **36**(5) (2008), 1675–1681. DOI: https://doi.org/10.1080/00927870801937224

[16] Spiegel, E., O’Donnell, C. J., *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 206, Marcel Dekker, Inc., New York, 1997.