Are there any problem for large values of the action like there were for small ones?

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In this paper we show that a method of quantization proposed few years ago (Ann. of Physics (314) 2005, 24) is equivalent to studying the system for values of the action close to zero. In this paper we also study the behaviour when the action gets very very large which could be the regime where dark energy and dark matter are invoked.

BRIEF REVIEW

In the 30’s Koopman and von Neumann [1] proposed an Hilbert space and operatorial approach to classical mechanics (CM). In the 80’s and 90’s this approach was turned into a path integral. All this work is summerized in ref. [2] to which we will refer the reader for more details. In the same paper we showed how quantization can be achieved in this framework.

 Skipping all the details, the generating functional for this path integral is given by [2]:

\[ Z = \int \mathcal{D}\Phi \exp \left[ i \int dt \, d\theta \, d\bar{\theta} \, i L(\Phi) \right], \]  

(1)

where \( L \) is the usual Lagrangian of the system, \( \theta \) and \( \bar{\theta} \) are two Grassmannian partners of time \( t \), the \( \Phi \) are superfield extensions of the phase space \((q,p)\) of the system. These extensions are defined by

\[ \Phi^a(t,\theta,\bar{\theta}) = \varphi^a(t) + \theta c^a(t) + \bar{\theta} \omega^{ab} \bar{c}_b(t) + i \theta \omega^{ab} \lambda_b(t), \]  

(2)

where \( \varphi^a \) are the phase space variables of the system, \( \omega^{ab} \) the symplectic matrix of the Hamiltonian equations of motion, and \( c^a, \bar{c}_b, \lambda_a \) are auxiliary variables whose geometrical nature has been clarified in several old papers quoted in ref. [2].

The quantization of the system is achieved by multiplying \( L(\Phi^a) \) by \( \frac{i}{\hbar} \). In fact, as proved in ref. [2], this procedure brings the classical generating functional \[1\] to the following one:

\[ Z = \mathcal{N} \int \mathcal{D}\varphi \exp \left[ \frac{i}{\hbar} \int dt \, L(\varphi) \right], \]  

(3)

which is the generating functional of the associated quantum system (QM) modulo a normalization constant \( \mathcal{N} \). In ref. [2] we studied the geometrical meaning of this procedure. We proved that it is equivalent to “geometric quantization” [2].

In this paper we shall now study the physical meaning of the procedure above.

SMALL VALUES OF THE ACTION

Let us first do a dimensional analysis. The argument of the exponential in \[1\] is just a phase. So \( d\theta \, d\bar{\theta} \) has the dimension of the inverse of the action \( \int dt \, L(\Phi) \). Next let us notice that the factor \( \theta \bar{\theta} \) by which we multiply the \( L(\Phi) \), in order to get quantum mechanics (modulo \( \hbar \)), is equivalent to \( \delta(\bar{\theta}) \delta(\theta) \), which has the dimension of the inverse of \( d\theta \, d\bar{\theta} \) because \( \int d\theta \, d\bar{\theta} \delta(\bar{\theta}) \delta(\theta) = 1 \). As a consequence \( \theta \bar{\theta} \) has the dimension of an action. Multiplying by this factor, as it is equivalent to \( \delta(\bar{\theta}) \delta(\theta) \), means choosing values of the action close to zero. This is exactly what QM does: it gives the dynamics for small values of the action. This is the reason why multiplying by \( \theta \bar{\theta} \) we get QM. Somehow this procedure projects out \[1\] the contribution for small values of the action.

LARGE VALUES OF THE ACTION

In this section we will try to derive which is the theory for large values of the action. Why do we do this? The reason is that those systems for which we have to invoke dark matter and dark energy are systems with very large values of the action. Think of stars rotating very fast around the center of their galaxy, or of clusters of galaxies possessing huge masses and as a consequence a huge action or think of the universe when it started accelerating in its expansion 5 billions years ago.

We showed in the previous section that \( \theta \bar{\theta} \) has the dimension of an action. Multiplying the action by \( \theta \bar{\theta} \) is the same as multiplying by \( \delta(\theta) \delta(\bar{\theta}) \). The product of these Dirac deltas is equivalent to \( \delta(\theta \bar{\theta}) \). In fact \( \delta(\theta \bar{\theta}) = \delta(\theta) \delta(\bar{\theta}) = \delta(\theta) \bar{\theta} = \theta \bar{\theta} \). Now if we search for large values of the action, we should multiply the action in \[1\] by \( \delta(\frac{1}{\theta \bar{\theta}}) \). We better pay attention in doing the inverse of a Grassmannian number like \( \theta \bar{\theta} \). As explained in ref. [1] first we should add a complex number \( \epsilon \) to \( \theta \bar{\theta} \rightarrow \epsilon + \theta \bar{\theta} \). The inverse of \( \epsilon + \theta \bar{\theta} \) is \( \frac{1}{\hbar} \left( 1 - \frac{\theta \bar{\theta}}{\epsilon} \right) \). So in the action \[1\] we have to insert

\[ \delta \left[ \frac{1}{\hbar} \left( 1 - \frac{\theta \bar{\theta}}{\epsilon} \right) \right]. \]  

(4)

The Dirac delta \[1\] has its zero in \( \theta \bar{\theta} = \epsilon \) and with \( \epsilon \rightarrow 0 \),
this implies $\overline{\theta} = 0$. So for large values of the action we have the same distribution as for small values like in QM. This is very surprising but at least it tells us that the distribution is not the one of classical mechanics. In fact where dark matter and dark energy are invoked is because CM fails.

One last point: In the quantization procedure we had to divide the $\theta\bar{\theta}$ by $\hbar$ in order to get QM. We wondered by which factor we should divide in the case of large actions. It has to have the dimension of the action but be very large. One such factor could be $B = M c^2 T$ where $M$ is the mass of the universe and $T$ the age of the universe. (This factor was suggested to me by Stefano Baroni.)

We conclude this work with a question that we have not been able to answer yet and which is the title of the paper: ”are there problems for large values of the action like there were for small ones”. This is a natural question to ask as the two distributions are so similar.

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