A control problem for a speculative investor in a target zone model

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Abstract
We consider a stochastic control problem for a trader who wishes to maximize the expected local time through generating price impact. The local time can be regarded as a proxy for the inventory of a central bank whose aim is to maintain a target zone.

1 Model setup and main results
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions and supporting a standard Brownian motion $W$. We suppose that in the absence of a large-investor intervention the exchange rate $\tilde{S}$ between two currencies is given by a Bachelier model,

$$\tilde{S}_t = S_0 + \sigma W_t, \quad t \geq 0,$$

where $S_0$ and $\sigma$ are two positive constants. This unaffected exchange rate process will be impacted by the price impact components of two types of “big players”. One of these “big players” models the central bank of one of the two currencies, whereas the second one stands for a strategic speculative investor or for the accumulated price impact of a group of such investors. It is the goal of the central bank to maintain a (one-sided) target zone in which the actual exchange rate must stay above a specified level $c$. Such target zones are frequently observed on financial markets. The central bank keeps up the target zone through the permanent price impact of trades that are executed as soon as the exchange rate threatens to fall below the level $c$, thereby creating an ever increasing inventory. This accumulation of inventory is often problematic for the central bank and frequently leads to the abandonment of the target zone regime. A notorious example is the “breaking of the Bank of England” by the investor George Soros on September 16, 1992.

The strategy of the strategic investor over the time horizon $[0, T]$ is described through the trading speed $(\xi_t)_{t \geq 0}$ in a linear Almgren–Chriss-type model, so that $X_t = \int_0^t \xi_s ds$ is

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Figure 1: Plot of the GBP/DEM exchange rate from January 1, 1992 to December 31, 1992. On September 16, 1992, the British government announced on exit from the European Exchange Rate Mechanism (ERM), which was followed by a rapid drop of the exchange rate.

Figure 2: Plot of the EUR/CZK exchange rate from August 1, 2016 to August 1, 2017. On April 6, 2017, the Czech National Bank removed the EUR/CZK floor, this was followed by a rapid drop of the exchange rate.
the accumulated inventory after trading over the time interval $[0, t]$. Here, we use a Markovian control that may depend on the current time $t$ and the current exchange rate $S_t$. It is important to note here that $S$ denotes the actual exchange rate after the central bank intervention and not the unaffected exchange rate process $\tilde{S}$, which will typically not be observable to any market participant. Thus, we assume that $\xi_t$ is of the form

$$\xi_t = v(t, S_t), \quad t \in [0, T], \quad (1.1)$$

where $v(t, x)$ is a continuous function on the domain $D := [0, T] \times [c, \infty) \setminus \{(0, c)\}$ satisfying the following two properties:

- For every compact subset $K$ of $D$ there exists $L_K \geq 0$ such that $|v(t, x) - v(t, y)| \leq L_K|x - y|$ for all $(t, x), (t, y) \in K$.

- There exists a constant $C \geq 0$ such that $|v(t, x)| \leq C(1 + |x|)$ for all $(t, x) \in D$.

By $V$ we denote the class of all such functions $v$.

In the linear Almgren–Chriss model, the permanent price impact generated at time $t$ by the strategy $\xi$ in (1.1) is of the form $\gamma \int_0^t \xi_s \, ds = \gamma \int_0^t v(r, S_r) \, dr$, where $\gamma > 0$ is the permanent impact parameter $[1]$. Thus, the actual exchange rate process is of the form

$$S_t^v = \tilde{S}_t + \gamma \int_0^t v(r, S_r) \, dr + R_t, \quad (1.2)$$

where $R_t$ is the permanent price impact generated by the response strategy of the central bank. This strategy must be nondecreasing and such that the stochastic integral equation (1.2) admits a solution $S$ satisfying $S_t \geq c$ for all $t \in [0, T]$ $P$-a.s. Moreover, the response $R$ must be adapted to the natural filtration of $S$. As for the strategic investor’s strategy, we could insist that $R_t$ is absolutely continuous in $t$, but since central banks typically face less restrictions on transaction costs than regular investors, we will only assume that $t \mapsto R_t(\omega)$ is of bounded variation for $P$-a.e. $\omega \in \Omega$. Let us denote by $\mathcal{R}(v)$ the class of all processes $R$ satisfying the preceding conditions for a given strategy $v \in V$.

The central bank has two main goals. First, the target zone must be maintained by guaranteeing that $S_t^v \geq c$ for all $t$. Second, the inventory accumulated by keeping up the target zone must be controlled. We assume that this inventory at time $t$ is given proportional to the local time

$$L_t^v(S^v) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[c, c+\varepsilon]}(S^v_r) \, d\langle S^v, S^v \rangle_r = \lim_{\varepsilon \downarrow 0} \frac{\sigma^2}{\varepsilon} \int_0^t 1_{[c, c+\varepsilon]}(S^v_r) \, dr, \quad P$-a.s.,

of the semimartingale $S^v$ at $c$.

Next we consider the optimal strategy of a speculative trader who tries to maximize the central bank’s inventory with the goal of pushing it to its risk limits and so to force it to abandon the target zone. The accumulation of excessive risk is indeed one of the most common reasons why a central bank would abandon a target zone. Thus, the speculative investor aims to maximize the expected central bank inventory at a given future time $T$. That is, the goal is
to maximize $E[L_T^c(S^v)]$. According to the linear Almgren–Chriss model, the investor’s trading strategy $\xi_t = v(t, S^v_t)$ creates transaction costs, sometimes also called “slippage”, proportional to

$$\int_0^T \xi_t^2 dt = \int_0^T (v(t, S^v_t))^2 dt.$$ 

These costs arise, e.g., from short-term price impact effects and from the need to increase the proportion of market vs. limit orders in a strategy with high trading speed. We therefore assume that the goal of the investor is to

$$\text{maximize } E \left[ L_T^c(S^v) - \kappa \int_0^T v(t, S^v_t)^2 dt \right] \text{ over } v \in \mathcal{V}. \quad (1.3)$$

Our main result provides a closed-form solution to the preceding problem of stochastic optimal control and thus establishes a Stackelberg equilibrium in our stochastic differential game between trader and central bank. As a result, we have singled out the worst-case scenario a central bank may be facing when keeping up a (one-sided) target zone.

**Theorem 1.1.** Suppose that $S_0 > c$ and let $\beta = \gamma^2/(2\kappa \sigma^2)$ and

$$U(t, z) = \frac{1}{\beta} \log \left( E \left[ \exp \left( \beta \sigma L^c_t(z-c)/\sigma(W) \right) \right] \right), \quad z \geq c, \quad (1.4)$$

where $L^c_t(W)$ is the local time of the Brownian motion $W$ at level $x \in \mathbb{R}$. Then we have

$$U(t, S_0) = \sup_{v \in \mathcal{V}} E \left[ L^c_T(S^v) - \kappa \int_0^T v(r, S^v_r)^2 dr \right].$$

Moreover, $U(t, z)$ belongs to $C^{1,2}((0, T] \times [c, \infty))$, and there exists a unique strategy $v^* \in \mathcal{V}$ for which the supremum is attained. It is given by

$$v^*(t, z) = \frac{\gamma}{2\kappa} \frac{\partial_z U(T - t, z)}. \quad (1.5)$$

**Remark 1.2.** From Formula 1.3.3 on p. 161 in [5] we get a closed-form expression for $U$,

$$U(t, z) = \frac{1}{\beta} \log \left( \text{erf} \left( \frac{z - c}{\sigma \sqrt{2t}} \right) + e^{-\beta(z-c)+\beta^2 t/2} \left[ 1 - \text{erf} \left( \frac{z - c - \beta \sqrt{t}}{\sigma \sqrt{2}} \right) \right] \right), \quad (1.6)$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ is the Gaussian error function. See Figure 3 for a plot of $U(t, z)$ and $v^*(t, z).$
Figure 3: The value function $U(t, z)$ (left) and the optimal strategy $v^*(t, z)$ (right) for $\sigma = \gamma = \kappa = 1$ and $c = 0$.

2 An approximate control problem

In this section, we consider a regularized version of the control problem (1.3). This control problem provides the basis for the informed guess of the value function and optimal strategy in Theorem 1.1. It is also interesting in its own right. To this end, we define

$$G_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-(x-c)^2/(2\varepsilon)},$$

and the “regularized local time”

$$L_{t}^{c,\varepsilon} (S^v) := \int_{0}^{t} G_\varepsilon (S^v_r) \, dr.$$

Then we consider the following regularized version of the control problem (1.3):

$$\max_{v \in V} E\left[ L_{t}^{c,\varepsilon} (S^v) - \kappa \int_{0}^{t} v(u)^2 \, du \right] \quad \text{over } v \in V. \quad (2.1)$$

It will be convenient to make the dependence of controlled reflecting diffusion $S^v$ on its initial value $z := S_0 \geq c$ explicit by writing $S^{v,z}$. With this notation, we define the value function of the problem (2.1) by

$$V^{\varepsilon}(t, z) := \sup_{v \in V} E\left[ L_{t}^{c,\varepsilon} (S^{v,z}) - \kappa \int_{0}^{t} v(u)^2 \, du \right]. \quad (2.2)$$

The generator of $S^{v,z}$ is formally given by

$$G = \gamma v(t, z) \partial_z + \frac{\sigma^2}{2} \partial_{zz}, \quad (2.3)$$
with Neumann boundary condition at $c$. Hence, standard heuristic arguments suggest that the function $V_\epsilon$ should solve the following Hamilton–Jacobi–Bellman equation,

$$\partial_t U = \frac{\sigma^2}{2} \partial_{zz} U + G_\epsilon + \sup_{v \in \mathbb{R}} (\gamma v \partial_z U - \kappa v^2), \quad \text{in } (0, T) \times (c, \infty),$$

with initial condition

$$U(0, z) = 0 \quad \text{for all } z \geq c,$$

and Neumann boundary condition

$$\partial_z U(t, c) = 0 \quad \text{for all } 0 \leq t \leq T.$$  

The maximum over $v \in \mathbb{R}$ on the right-hand side of (2.4) is attained in

$$v^* = \frac{\gamma \partial_z U}{2\kappa},$$

and so (2.4) becomes

$$\partial_t U = \frac{\sigma^2}{2} \partial_{zz} U + G_\epsilon + \frac{\gamma^2}{4\kappa} (\partial_z U)^2, \quad \text{in } (0, T) \times (c, \infty).$$

Let $h(t, z)$ be such that $U(t, z) = \frac{2a\sigma^2}{\gamma^2} \log h(t, z)$. Then, $h$ must solve

$$\partial_t h = \frac{\sigma^2}{2} \partial_{zz} h + \frac{\gamma^2}{2\kappa\sigma^2} h G_\epsilon, \quad \text{in } (0, T) \times (c, \infty),$$

with initial condition $h(0, z) = 1$ and boundary condition $\partial_z h(t, c+) = 0$.

**Proposition 2.1.** Let $\beta = \gamma^2/(2\kappa\sigma^2)$. Then the function

$$U_\epsilon^\beta(t, z) = \frac{1}{\beta} \log \mathbb{E}\left[ e^{\beta \int_0^t G_\epsilon(z + \sigma W_r) \, dr} \right],$$

belongs to $C^{1,2}([0, \infty) \times \mathbb{R})$ and its restriction to $[0, T] \times [c, \infty)$ is a classical solution to the initial value problem (2.4)–(2.6).

Equation (2.7) suggests that the optimal strategy $v^*_\epsilon$ for the problem (2.1) is given by

$$v_\epsilon^*(t, x) = \frac{\gamma \partial_x U_\epsilon^\beta(T - t, x)}{2\kappa}.$$

To make this statement more precise, note first that

$$\partial_z U_\epsilon^\beta(t, z) = \frac{E\left[ \int_0^t G_\epsilon'(z + \sigma W_r) \, dr \cdot e^{\beta \int_0^t G_\epsilon(z + \sigma W_r) \, dr} \right]}{E\left[ e^{\beta \int_0^t G_\epsilon(z + \sigma W_r) \, dr} \right]}.$$

This function is clearly satisfies a uniform Lipschitz condition in $z$. 

6
Theorem 2.2. The function $U^\varepsilon$ is equal to the value function $V^\varepsilon$ in (2.2) and the strategy $v^\varepsilon_t$ is the $P_z$-a.s. unique optimal strategy in $\mathcal{V}$.

Now we show that the approximate value functions approximate our original value function (1.4), which can also be represented as

$$U(t, z) = \frac{1}{\beta} \log \left( E \left[ \exp \left( \beta L^{c-z}_t(\sigma W) \right) \right] \right). \quad (2.12)$$

As in Proposition 2.1, we let $\beta = \gamma^2/(2\kappa\sigma^2)$.

Proposition 2.3. We have $U^\varepsilon(t, z) \to U(t, z)$ uniformly in $(t, z) \in [0, T] \times \mathbb{R}$ as $\varepsilon \downarrow 0$.

3 Proofs

3.1 Proofs of the results from Section 2

Proof of Proposition 2.1. Let

$$\tilde{h}(t, z) := E \left[ e^{\beta \int_0^t G_\varepsilon(z + \sigma W_r) \, dr} \right], \quad t \geq 0, \ z \in \mathbb{R}.$$ 

Since $G_\varepsilon$ is bounded and smooth, we may apply Theorem 3.6 in Chapter 4 of [7] to conclude that $\tilde{h}$ belongs to $C^{1,2}([0, \infty) \times \mathbb{R})$ and satisfies

$$\partial_t \tilde{h} = \frac{\sigma^2}{2} \partial_{zz} \tilde{h} + \frac{\gamma^2}{2\kappa\sigma^2} \tilde{h} G_\varepsilon$$

in $[0, \infty) \times \mathbb{R}$ with initial condition $\tilde{h}(0, z) = 1$. Since $G_\varepsilon$ is symmetric around $c$, the same is true for $\tilde{h}$, and it follows that $\partial_z \tilde{h}(t, c) = 0$. Hence, the restriction of $\tilde{h}$ to $[0, T] \times [c, \infty)$ satisfies (2.9) together with the initial and boundary conditions $\tilde{h}(0, z) = 1$ and $\partial_z \tilde{h}(t, c+) = 0$. Retracing the steps that led to (2.9) now completes the proof of the assertion.

Proof of Theorem 2.2. Itô’s formula yields that for all $v \in \mathcal{V}$ $P_z$-a.s.,

$$U^\varepsilon(t, z) = U^\varepsilon(t, z) + \sigma \int_0^t \partial_2 U^\varepsilon(t - r, S^v_r) \, dW_r + \int_0^t \partial_2 U^\varepsilon(t - r, S^v_r) \, dL^\varepsilon_r(S^v_r)$$

$$+ \int_0^t \left( - \partial_1 U^\varepsilon(t - r, S^v_r) + \gamma v(r, S^v_r) \partial_2 U^\varepsilon(t - r, S^v_r) + \frac{\sigma^2}{2} \partial_{zz} U^\varepsilon(t - r, S^v_r) \right) \, dr$$

$$\leq U^\varepsilon(t, z) + \sigma \int_0^t \partial_2 U^\varepsilon(t - r, S^v_r) \, dW_r - \int_0^t \left( G_\varepsilon(S^v_r) - \kappa v(r, S^v_r)^2 \right) \, dr,$$

where we have used the HJB equation (2.4) as well as (2.6) together with the fact that $dL^\varepsilon_r(S^v_r)$ is supported on $\{r \mid S^v_r = c\}$. Since $G_\varepsilon$ and $G^\prime_\varepsilon$ are bounded, it follows from (2.11) that $\partial_2 U^\varepsilon$
is bounded, and so \( \int_0^t \partial_z U^z(t - r, S_r^u) dW_r \) is a true martingale. Using the initial condition \( U^z(0, \cdot) = 0 \) and taking expectations in (3.1) gives

\[
U^z(t, z) \geq E \left[ \int_0^t \left( G_x(S_r^u - z) - \kappa v(r)^2 \right) dr \right].
\]  

(3.2)

Taking the supremum over \( v \in V \) yields \( U^z(t, z) \geq V^z(t, z) \) for all \( t \) and \( z \).

Next, by (2.7), we will have equality in (3.1), and hence in (3.2), if and only if

\[
v(r, S_r^v) = \frac{\gamma}{2R} \partial_z U^z(t - r, S_r^v)
\]

for a.e. \( r \in [0, t] \), which gives \( v = v^*_z \). Recall that \( \partial_z U^z(t, z) \) is bounded and continuously differentiable in both variables, hence \( v^*_z \in V \).

\[ \square \]

The following lemma will be needed for the proof of Proposition 2.3.

**Lemma 3.1.** (a) Let \( p \geq 1 \). For every \( 0 \leq t \leq T \), there exists \( C > 0 \) such

\[
\sup_{x \in \mathbb{R}} E \left[ \left( \int_0^t G_x(z + \sigma W_u) du - L_t^x(z + \sigma W_u) \right)^p \right] \leq C \varepsilon^{p/4}, \quad \text{for all } \varepsilon \in (0, 1),
\]

(b) For every \( \lambda > 0 \) we have

\[
\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}} E \left[ e^{\lambda \int_0^t G_x(z + \sigma W_u) du} \right] \leq E \left[ \exp \left( \lambda \sup_{x \in \mathbb{R}} L_T^x(\sigma W) \right) \right] < \infty.
\]

**Proof.** The proof uses ideas from Lemma 2.2 in [3]. For simplicity, in this proof we will write \( L_t^x \) for \( L_t^x(\sigma W) \).

(a) Let \( p, t \) and \( \varepsilon \) as in the hypothesis. Taking \( c = 0 \) in Exercise 1.33 in Chapter VI.1 of [13], we obtain the existence of a constant \( C_0 \) depending only on \( p \) and \( T \) such that

\[
E \left[ \left( \sup_{t \leq T} |L_t^x - L_t^y|^p \right) \right] \leq C_0 |x - y|^{p/2}, \quad \text{for all } x, y \in \mathbb{R}.
\]  

(3.3)

From the occupation time formula we have P-a.s.

\[
\int_0^t G_x(z + \sigma W_u) du = \int_{\mathbb{R}} G_x(z + x) L_t^x dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} L_t^{\varepsilon z + \sqrt{\varepsilon} x} dx.
\]  

(3.4)

Using (3.3) and Jensen’s inequality we therefore have

\[
E \left[ \sup_{t \leq T} \left( \int_0^t G_x(z + \sigma W_u) du - L_t^{\varepsilon z} \right)^p \right] \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} E \left[ \sup_{t \leq T} |L_t^{\varepsilon z + \sqrt{\varepsilon} y} - L_t^{\varepsilon z}|^p \right] dy
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} \left( C_0 \varepsilon^{p/2} y^p \right)^{1/2} dy
\]

\[
= C \varepsilon^{p/4}.
\]

(b) From (3.4) we get that for any \( \varepsilon \in (0, 1) \),

\[
E \left[ e^{\lambda \int_0^t G_x(z + \sigma W_r) dr} \right] = E \left[ e^{\lambda \sup_{x \in \mathbb{R}} \int_0^t e^{-y^2/2} L_t^{\varepsilon z + \sqrt{\varepsilon} y} dy} \right] \leq E \left[ \exp \left( \lambda \sup_{x \in \mathbb{R}} L_T^x(\sigma W) \right) \right],
\]

and the right-hand side is finite according to Lemma 1 in [6].

\[ \square \]

8
**Proof of Proposition 2.3.** We write again \( L_t^{x,z} \) for \( L_t^{x,z}(\sigma W) \). Using the Lipschitz continuity of the function \( \log x \) for \( x \geq 1 \), we get

\[
|U^\varepsilon(t, z) - U(t, z)| \leq \frac{1}{\beta} E \left[ \sup_{s \leq T} \left| e^{\beta L_t^{x,z}} - e^{\beta \int_0^T G_\varepsilon(z + \sigma W_r) \, dr} \right| \right] \\
\leq \frac{1}{\beta} E \left[ e^{\beta (L_T^{x,z} - \int_0^T G_\varepsilon(z + \sigma W_r) \, dr)} \sup_{s \leq T} \left| L_s^{x,z} - \int_0^s G_\varepsilon(z + \sigma W_r) \, dr \right| \right].
\]

The Cauchy–Schwarz inequality thus yields

\[
|U^\varepsilon(t, z) - U(t, z)| \\
\leq \frac{1}{\beta} \left( E \left[ e^{2\beta (L_T^{x,z} + \int_0^T G_\varepsilon(z + \sigma W_r) \, dr)} \right] \right)^{1/2} \left( E \left[ \sup_{s \leq T} \left( L_s^{x,z} - \int_0^s G_\varepsilon(z + \sigma W_r) \, dr \right)^2 \right] \right)^{1/2}. \tag{3.5}
\]

Lemma 3.1(b) shows that

\[
\lim_{\varepsilon \to 0} \sup_{z \in \mathbb{R}} E \left[ e^{2\beta (L_T^{x,z} + \int_0^T G_\varepsilon(s_r + z) \, dr)} \right] \leq \lim_{\varepsilon \to 0} \sup_{z \in \mathbb{R}} \left( E \left[ e^{4\beta L_T^{x,z}} \right] \right)^{1/2} \left( E \left[ \sup_{s \leq T} \left( L_s^{x,z} - \int_0^s G_\varepsilon(z + \sigma W_r) \, dr \right)^2 \right] \right)^{1/2} < \infty.
\]

Using this bound along with Lemma 3.1(a) in (3.5) thus gives that \( U^\varepsilon(t, z) \to U(t, z) \), uniformly in \( (t, z) \in [0, T] \times \mathbb{R} \), as \( \varepsilon \downarrow 0 \). \qed

### 3.2 Proof of Theorem 1.1

**Proposition 3.2.** The function \( U \) satisfies the following partial differential equation,

\[
\partial_t U(t, z) = \frac{\sigma^2}{2} \partial_{zz} U(t, z) + \frac{\gamma^2}{4\kappa} (\partial_z U(t, z))^2, \quad \text{in } (0, T) \times [c, \infty), \tag{3.6}
\]

with boundary condition

\[
\partial_z U(t, c) = -1, \quad t \in (0, T]. \tag{3.7}
\]

**Proof.** Let

\[
\psi(t, x) := \text{erf} \left( \frac{x}{\sigma \sqrt{2t}} \right) + e^{-\beta x + \beta^2 \sigma^2 t/2} \left[ 1 - \text{erf} \left( \frac{x - \beta \sigma^2 t}{\sigma \sqrt{2t}} \right) \right].
\]

For \( t > 0 \), we have

\[
\partial_t \psi(t, x) = \frac{\beta \sigma}{\sqrt{2\pi t}} e^{-x^2/(2\sigma^2)} + \frac{\beta \sigma}{2} e^{(x - \beta^2 \sigma^2 t)/(2\sigma^2)} \left[ 1 - \text{erf} \left( \frac{x - \beta \sigma^2 t}{\sigma \sqrt{2t}} \right) \right],
\]

\[
\partial_x \psi(t, x) = -\beta e^{-\beta x + \beta^2 \sigma^2 t/2} \left[ 1 - \text{erf} \left( \frac{x - \beta \sigma^2 t}{\sigma \sqrt{2t}} \right) \right],
\]

\[
\partial_{xx} \psi(t, x) = \frac{2}{\sigma^2} \partial_t \psi(t, x).
\]

9
From (1.6) we have \( U(t, z) = \frac{1}{\beta} \log \psi(t, z - c) \). It follows that

\[
\partial_t U(t, z) = \frac{\partial_t \psi(t, z - c)}{\beta \psi(t, z - c)}, \quad \partial_z U(t, z) = \frac{\partial_z \psi(t, z - c)}{\beta \psi(t, z - c)}. \tag{3.8}
\]

In particular,

\[
\partial_z U(t, c) = \frac{\partial_z \psi(t, 0)}{\beta \psi(t, 0)} = -1.
\]

Next, the second \( z \)-derivative of \( U \) corresponds to

\[
\partial_{zz} U(t, z) = \frac{\partial_{xx} \psi(t, z - c)}{\beta \psi(t, z - c)} - \beta \left( \frac{\partial_z \psi(t, z - c)}{\beta \psi(t, z - c)} \right)^2.
\]

Plugging everything together yields the assertion. \( \square \)

**Proof of Theorem 1.1**  
Recall that we write \( S^{v,z} \) for the reflecting diffusion starting from \( z = S_0 > c \) with given \( v \in \mathcal{V} \). Itô's formula gives

\[
U(0, S^{v,z}_t) = U(t, z) + \sigma \int_0^t \partial_z U(t - r, S^{v,z}_r) \, dW_r + \int_0^t \partial_z U(t - r, c) \, dL^c_r(S^{v,z})
\]

\[
= U(t, z) + \sigma \int_0^t \partial_z U(t - r, S^{v,z}_r) \, dW_r - \int_0^t \gamma v(r, S^{v,z}_r) \, dr
\]

\[
\leq U(t, z) + \sigma \int_0^t \partial_z U(t - r, S^{v,z}_r) \, dW_r - \int_0^t \kappa v(r, S^{v,z}_r) \, dr,
\]

where we have used (3.6) and (3.7) in the second step. It follows from (3.8) that \( \partial_z U \) is bounded, and so \( \sigma \int_0^t \partial_z U(t - r, S^{v,z}_r) \, dW_r \) is a true \( P_z \)-martingale. Using the initial condition \( U(0, \cdot) = 0 \) and taking expectations gives

\[
U(t, z) \geq E \left[ L^c_t(S^{v,z}) - \kappa \int_0^t v(r, S^{v,z}_r)^2 \, dr \right]. \tag{3.9}
\]

Taking the supremum over \( v \in \mathcal{V} \) shows the inequality “\( \geq \)” in Theorem 1.1.

Note that we will have an equality in (3.9) if and only if

\[
v(r, x) = \frac{\gamma}{2\kappa} \partial_z U(t - r, x) \quad \text{for a.e. } r \in [0, t].
\]

Finally, the formulas derived in the proof of Proposition 3.2 easily yield that \( v \in \mathcal{V} \). \( \square \)

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