THE NATURE OF THE SOFT EXCITATION AT THE CRITICAL END POINT OF QCD *

A. JAKOVÁC
HAS † and Budapest Univ. of Technology and Economics, Research Group “Theory of Condensed Matter”, H-1521 Budapest, Hungary

A. PATKÓS
Department of Atomic Physics, Eötvös University, H-1117 Budapest

ZS. SZÉP
Research Group for Statistical Physics, HAS, H-1117 Budapest
E-mail: szepzs@antonius.elte.hu

P. SZÉPFALUSY
Dept. of Physics of Complex Systems, Eötvös University, H-1117 Budapest
Research Inst. for Solid State Physics and Optics, HAS, H-1525 Budapest

Using a large flavor number expansion and a gap equation for the pion mass the chiral quark-meson model is solved at the lowest order in the fermion contributions. In the chiral limit the tricritical point (TCP) is determined analytically. The softening of the sigma particle is verified at this point and is further investigated for a physical pion mass in the neighbourhood of the critical endpoint (CEP).

1. Introduction

Recently the location of the CEP in the $\mu - T$ phase-diagram of QCD was obtained for $N_f = 2 + 1$ dynamical staggered quarks with physical masses $^1$. One might investigate this problem using an effective low energy model also, with some nonperturbative technique. Its low energy nature might prevent us from giving the accurate location of the endpoint and the shape of the

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†Hungarian Academy of Sciences
phase boundary, but one can still obtain insights into the physics near the CEP. In this contribution we focus on the nature of the soft mode.

2. The method for solving the model

In the presence of the vacuum expectation value $\Phi$ the requirement of a finite constituent quark mass $m_q = g\Phi$ as $N \to \infty$ defines the Lagrangian

$$L[\sigma, \pi^a, \psi] = -\left[\frac{\lambda}{24} \Phi^4 + \frac{1}{2} m^2 \Phi^2\right] N - \left[\frac{\lambda}{6} \Phi^3 + m^2 \Phi - h\right] \sigma \sqrt{N}$$

$$+ \frac{1}{2} \left[(\partial \sigma)^2 + (\partial \pi)^2\right] - \frac{1}{2} m_{\sigma 0}^2 \sigma^2 - \frac{1}{2} m_{\pi 0}^2 \pi^2 - \frac{\lambda}{6 \sqrt{N}} \Phi \sigma \rho^2 - \frac{\lambda}{24 N} \rho^4$$

$$+ \bar{\psi} \left[i \partial^\mu \gamma_\mu - m_q - \frac{g}{\sqrt{N}} \left(\sigma + i \sqrt{2 N_f} \gamma^\alpha T^a \pi^a\right)\right] \psi,$$  \(1\)

where $N = N_f^2$, $m_{\sigma 0}^2 = m^2 + \frac{\lambda}{2} \Phi^2$, $m_{\pi 0}^2 = m^2 + \frac{\lambda}{2} \Phi^2$, $\rho^2 = \sigma^2 + \pi^2$ and $h$ is the external field. The fermions, for which the chemical potential is introduced by the shift $\partial_t \to \partial_t - i \mu$, start contributing in the large $N$ expansion at level $O(1/\sqrt{N})$, which is between LO and NLO of the mesonic sector. To calculate their corrections to the LO, which has $O(N)$ symmetry, we disregarded the second independent quartic coupling of the model, present for $N_f > 2$.

Due to the proliferation of graphs a resummation is needed. The best thing to do would be to use a selfconsistent pion and fermion propagator. This makes renormalisation difficult, and a consistent method for it was proposed only recently \(^2\). Our method consists in taking into account the $N_c$ coloured quarks to $O(1/\sqrt{N})$ perturbatively at one loop order and using for the propagator of the pions a simplified form, parameterised by a mass $M$ determined from the gap-equation:

$$M^2 = -iG^{-1}_a(p = 0) = m^2 + \frac{\lambda}{6} \Phi^2 + \frac{\lambda}{6} T_B(M) - 4 \frac{g^2 N_c}{\sqrt{N}} T_F(m_q).$$  \(2\)

The pion and fermion tadpoles ($T_{B,F}$) determine the equation of state (EoS)

$$V^{\text{eff}}_\text{eff} = \Phi H(\Phi^2) - h = \Phi \left[m^2 + \frac{\lambda}{6} (\Phi^2 + T_B(M)) - 4 \frac{g^2 N_c}{\sqrt{N}} T_F(m_q)\right] - h = 0.$$  \(3\)

The renormalisation method was presented in \(^3\). The relation $M^2 = H(\Phi^2)$ shows that in the symmetry broken regime defined by $H(\Phi^2) = h/\Phi$ and in the chiral limit ($h = 0$) the physical pion mass $M$ is always zero, i.e. Goldstone’s theorem is fulfilled in the minimum of the effective potential. In what follows the number of flavors is set to two: $N_f = 2$. 


3. Chiral case \((h = 0)\): analytical determination of TCP

Since in this case \(M = 0\), one obtains the phase boundary by expanding the fermion tadpole in powers of \(\Phi\). The 2nd order line is given by

\[
m_T^2 := m^2 + \frac{g^2 \mu^2}{4\pi^2} N_c + \left( \frac{\lambda}{12} + \frac{g^2 N_c}{12} \right) T^2 = 0,
\]

while the location of TCP comes from combining (4) with the condition

\[
\frac{\lambda}{6} := \frac{\lambda}{6} + \frac{g^2 N_c}{4\pi^2} \left[ \frac{\partial}{\partial n} \left( \text{Li}_n(-e^{-\Phi}) + \text{Li}_n(-e^{-\Phi}) \right) \right]_{n=0} - \ln \frac{c_1 T}{M_B} = 0.
\]

\(n=0\) \(\ln c_1/2 = 1 - \gamma_E + \eta\), and \(\eta = \ln M_B/M_F\) gives a relation between \(M_B\) and \(M_F\) the renormalisation scales of the bosonic and fermionic tadpoles. The parameters are fixed at \(T = \mu = 0\). The choice \(m_\Phi = m_N/3 \approx 312.67\) MeV and \(\Phi_0 = f_\pi/2\) gives \(g = 6.72\). \(\lambda = 400\) is fixed by requiring the agreement between the location of the complex \(\sigma\)-pole on the 2nd Riemann sheet and the experimentally favoured mass and width of the \(\sigma\) particle. Solving (4) and (5) the 2nd order phase transition, the first spinodal line and the location of TCP are found (l.h.s. of Fig. 1). On the r.h.s. we see

Figure 1. L.h.s.: The \(T - \mu\) phase diagram for \(M_B = 886\) MeV and \(\eta = 0\). R.h.s. the dependence of \(T_c(\mu = 0)\) (upper curves) and of \(T_{TCP}\) (lower curves) on \(M_B\).

the variation of TCP and of the critical temperature with \(\eta\) and \(M_B\). The pole structure of the \(\sigma\) propagator restricts the allowed range of \(M_B\) to the right from the arrow, seen on the horizontal axis, where no low scale imaginary poles appear besides the well known large scale tachyon, whose scale increases with decreasing values of \(\lambda\). Although in this region one can easily obtain \(T_c(\mu = 0) \in (150, 170)\) MeV, the TCP stays robustly below 70 MeV. Previous effective model studies give similar low values compared
with the CEP temperature obtained in \[^1\], which could mean that the phase transition is driven by higher excited hadronic states rather than by the light d.o.f. This is supported by a resonance gas model calculation \[^6\] used to reproduce lattice results on quark number susceptibilities and pressure.

An approximate effective potential is obtained by integrating the EoS away from equilibrium and subsequently Taylor expanding it around \(\Phi = 0\). This value becomes a minimum as \(T \to T_c\) and so \(M \to 0\). With this, one can show the vanishing of the effective coupling in the large \(N\) limit \[^7\] at the critical point. Here we just quote the result of \[^8\]

\[
V_{\text{approx}}(\Phi) = \frac{m_f^2}{4U^2W} \left[ \frac{m_T^2}{2} \Phi^2 + \frac{\lambda}{12} \Phi^4 + \left( \frac{W\lambda^2}{216m_T^2} + \frac{\kappa}{3} \right) \Phi^6 \right],
\]

where \(\kappa = \frac{g^2N_c}{(4\pi T)^2\sqrt{N}} \frac{\partial}{\partial n} \left[ \ln_n(-e^{\frac{\mu}{T}}) + \ln_n(-e^{-\frac{\mu}{T}}) \right] \bigg|_{n=-2} \), \(W = 1 - \frac{\lambda^2}{48\pi^2} \times \ln \frac{c_T^2}{4\pi} \) and \(\ln \frac{c_T^2}{4\pi} = \frac{1}{2} - \gamma\). The shape of \(V_{\text{approx}}(\Phi)\) depends on the values of the parameters. In the temperature range of a 2nd order phase transition \(W < 0\). For increasing values of \(\mu/T\), \(\kappa\) changes sign at 1.91. This sign change restricts the location of the TCP to the region \(\mu > 2T\), where \(\kappa > 0\).

In the broken symmetry phase \(m_T^2 < 0\) and it vanishes at \(T_c\) signalling that the coefficient of the quartic term in \(\Phi\) also vanishes. Around \(T_c\) from (4) one gets \(m_T^2 \sim T - T_c\), which through \(m_{\text{eff}} = \left| \frac{m_T^2}{2UW} \right|\) gives the correct LO critical exponent \(\nu = 1\) for the O(\(N\)) model at large \(N\). The usual Landau type analysis applies to the square bracket in (6). The scaling exponent of the order parameter on the 2nd order line is given by the first two terms. The minimum condition gives \(\Phi^2 = -\frac{3m_T^2}{\lambda} \Rightarrow \Phi \sim (T - T_c)^{\beta}\), \(\beta = \frac{1}{2}\). At TCP one can set \(\lambda = 0\) in (6) and keep the sixth order term. The minimum condition gives \(\Phi^4 = -\frac{m_T^2}{2\lambda} \Rightarrow \Phi \sim (T - T_c)^{\beta}\), \(\beta = \frac{1}{4}\). This mean field estimates for \(\beta\) were checked numerically using the exact EoS (3).

4. Case of the physical pion mass: the soft mode at CEP

In view of the gap-equation (2) and EoS (3) the form of the \(\sigma\) propagator can be inferred from the consistency condition \(-G^{-1}_\sigma(p = 0) = \frac{d^2V_{\text{eff}}(\phi)}{d\phi^2}\):

\[
G^{-1}_\sigma(p) = p^2 - \frac{h}{\phi} - 2\Phi^2 \left[ \frac{\lambda}{\sqrt{N}} \frac{\partial}{\partial m_q} I_F(p, m_q) \right] \frac{1}{1 - \frac{4\sqrt{N} I_B(p, M)}{1}}.
\]

where \(I_{B,F}\) are the pion and fermion one-loop “fish” integrals. Requiring \(M = 140\) MeV at \(T = \mu = 0\) one has from (2) \(h/f_\sigma^2 = 1.13\). The location of CEP is determined numerically by simultaneously solving (2) and (3):
($\mu_B, T)_{\text{CEP}} = (987, 12.34) \text{ MeV. } T_{\text{CEP}} \text{ is unrealistically low, nevertheless it is of interest to study which mode becomes soft at this point. Two minima meet at CEP which flattens the effective potential. This is shown also by the diverging peak of the susceptibility } d\Phi/dh. \text{ This static information has to show up consistently when investigating } G_\sigma(p) \text{ in the static limit i.e. first taking } p_0 \to 0, \text{ then } |p| \to 0. \text{ The question is in which } (p_0, |p|) \text{ region one has to look for the responsible excitations. The behaviour of the spectral function } \rho(p_0, |p|) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \text{Im } G_\sigma(p_0 + i\varepsilon, |p|) \text{ for fixed } |p| \text{ gives a hint in this respect: a peak is found in the } p_0 < |p| \text{ region. The location of its maximum moves toward the origin with decreasing values of } |p| \text{ and at CEP eventually the peak diverges at } p_0 = 0 \text{ as } |p| \to 0. \text{ The nature of the analytical object on the second Riemann sheet which produces this peak will be explored further. In the NJL model a pole on the negative imaginary axis, approaching the origin as } |p| \to 0, \text{ is responsible for the peak observed in the spectral function}^9.\]

5. Conclusions

Contrary to the chiral case, where the zero temperature $\sigma$ pole continuously moves to the origin of the 2nd Riemann sheet and becomes soft at TCP, for a physical pion mass the $\sigma$ pole stays massive at CEP. A space-like ($0 \leq p_0 < |p|$) excitation is responsible for the flattening of the effective potential at this point. Before a more detailed investigation of the analytic structure at CEP we have to improve our method of solving the model by dynamically generating the fermion mass through a Dyson-Schwinger equation for the fermion propagator, but also one has to extend our method to the physically more appealing $SU(3)_L \times SU(3)_R$ case.

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