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RESULTANTS AND THE BORCHERDS $\Phi$-FUNCTION

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Abstract. The Borcherds $\Phi$-function is the automorphic form on the moduli space of Enriques surfaces characterizing the discriminant locus. In this paper, we give an algebro-geometric construction of the Borcherds $\Phi$-function.

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References.

1. Introduction. In [2], Borcherds discovered a beautiful automorphic form on the moduli space of Enriques surfaces, which is closely analogous to the Dedekind $\eta$-function. In this paper, this automorphic form is called the Borcherds $\Phi$-function. Up to a constant, the Borcherds $\Phi$-function is characterized as the automorphic form of weight 4 on the period domain for Enriques surfaces with respect to the automorphism group of the Enriques lattice, whose zero divisor is exactly the discriminant locus.

Besides the original constructions [2, 3], several distinct understandings of the Borcherds $\Phi$-function are known [14, 23, 27]. However, even though its direct connection with the moduli space of Enriques surfaces, no explicit algebro-geometric construction of the Borcherds $\Phi$-function has been known until now. The purpose of the present paper is to give such a construction modeled after the following classical result for the Dedekind $\eta$-function: For an elliptic curve $y^2 = 4x^3 - g_2x - g_3$ equipped with a symplectic basis $\{\alpha, \beta\}$ of its first integral homology group, the value of the Dedekind $\eta$-function evaluated at its period...
\[ \tau_{\alpha, \beta} = (\int_{\beta} dx/y) / (\int_{\alpha} dx/y), \Im \tau_{\alpha, \beta} > 0 \]

is given by the formula

\[ (1.1) \eta(\tau_{\alpha, \beta})^{24} = (g_2^3 - 27g_3^2) \left( \frac{1}{2\pi} \int_{\alpha} \frac{dx}{y} \right)^{12}. \]

Since the period of an Enriques surface, i.e., the period of its universal covering \( K3 \) surface, does depend on the choice of a marking, the value of the Borcherds \( \Phi \)-function makes sense only for marked Enriques surfaces. However, the Petersson norm of the Borcherds \( \Phi \)-function is independent of the choice of a marking and hence is well defined even for non-marked Enriques surfaces. For an Enriques surface \( Y \), the Petersson norm of the Borcherds \( \Phi \)-function evaluated at its period point is denoted by \( \| \Phi(Y) \| \). If \( Y \) has rational double points (RDP’s for short), then \( \| \Phi(Y) \| \) is defined as the value \( \| \Phi(\tilde{Y}) \| \), where \( \tilde{Y} \to Y \) is the minimal resolution.

Firstly, we give an algebrao-geometric expression of \( \| \Phi(Y) \| \). For this, we need explicit equations defining Enriques surfaces. Let \( f_1, g_1, h_1 \in \mathbb{C}[x_1, x_2, x_3] \) and \( f_2, g_2, h_2 \in \mathbb{C}[x_4, x_5, x_6] \) be quadratic forms and define \( f, g, h \in \mathbb{C}[x_1, ..., x_6] \) as

\[ f = f_1 + f_2, \quad g = g_1 + g_2, \quad h = h_1 + h_2. \]

We set

\[ X_{(f, g, h)} = \{ [x] \in \mathbb{P}^5; \ f(x) = g(x) = h(x) = 0 \}, \]

which is preserved by the involution on \( \mathbb{P}^5 \)

\[ \iota(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, x_3, -x_4, -x_5, -x_6). \]

If \( (f, g, h) \) is not special, then \( X_{(f, g, h)} \) is a \( K3 \) surface on which acts \( \iota \) freely. Hence

\[ Y_{(f, g, h)} = X_{(f, g, h)}/\iota \]

is an Enriques surface. In fact, every Enriques surface is expressed as the minimal resolution of some \( Y_{(f, g, h)} \) by Verra [25] and Cossec [5].

We define a canonical differential \( \omega \) on \( X_{(f, g, h)} \) as the residue of \( (f, g, h) \) (cf. [10]). Namely, let \( F, G, H \in \mathbb{C}[x_1, ..., x_5] \) be the inhomogeneous equations of \( f, g, h \) obtained by setting \( x_6 = 1 \), respectively. The meromorphic canonical form on \( \mathbb{C}^5 \)

\[ \frac{d x_1 \wedge \cdots \wedge d x_5}{F(x)G(x)H(x)} \]

extends to a meromorphic canonical form on \( \mathbb{P}^5 \) with logarithmic poles along the divisor \( \text{div}(fgh) \subset \mathbb{P}^5 \). We set

\[ \omega = \Upsilon|_{X_{(f, g, h)}}, \]
where $\Upsilon$ is a locally defined holomorphic 2-form on $\mathbf{P}^5$ such that
\[
\frac{dF}{F} \wedge \frac{dG}{G} \wedge \frac{dH}{H} \wedge \Upsilon = \frac{dx_1 \wedge \cdots \wedge dx_5}{FGH}.
\]
Then $\omega = \Upsilon|_{X_{(f,g,h)}}$ is independent of the choice of $\Upsilon$ as above.

Our first theorem is stated as follows (cf. Theorem 3.1).

**Theorem 1.1.** If $Y_{(f,g,h)}$ is an Enriques surface with possibly RDP’s, then
\[
\|\Phi(Y_{(f,g,h)})\|^2 = |R(f_1, g_1, h_1)R(f_2, g_2, h_2)| \left(\frac{2}{\pi^4} \int_{X_{(f,g,h)}} \omega \wedge \omega\right)^4,
\]
where $R(f_i, g_i, h_i)$ is the resultant of $f_i, g_i, h_i$.

We remark that a weaker version of (1.2) was obtained by Maillot-Rössler [17]. As a consequence of Theorem 1.1, we get an extension of (1.1) to the Borcherds $\Phi$-function (cf. Theorem 4.1). Write $\langle \cdot , \cdot \rangle$ for the cup-product on $H^2(X_{(f,g,h)}, \mathbb{Z})$.

**Theorem 1.2.** Assume that $Y_{(f,g,h)}$ is smooth. Let $v \in H^2(X_{(f,g,h)}, \mathbb{Z})$ be an anti-$\ell$-invariant primitive isotropic vector of level $\ell \in \{1, 2\}$ and let $v^\vee \in H^2(X_{(f,g,h)}, \mathbb{Z})$ be its Poincaré dual. Let $v' \in H^2(X_{(f,g,h)}, \mathbb{Z})$ be another anti-$\ell$-invariant primitive isotropic vector of level $\ell$ with $\langle v, v' \rangle = \ell$. Then, by choosing a suitable marking $\alpha_{v,v'}$ of $X_{(f,g,h)}$, the vector
\[
z_{v,v'} = \frac{\omega - \langle \omega, v' / \ell \rangle v - \langle \omega, v \rangle (v' / \ell)}{\langle \omega, v \rangle} \in (\mathbb{Z}v + \mathbb{Z}v')^\perp \otimes \mathbb{C}
\]
is the period of $Y_{(f,g,h)}$ via $\alpha_{v,v'}$ (cf. Section 4) and the following equality holds
\[
\Phi_\ell(\alpha_{v,v'}(z_{v,v'}))^2 = R(f_1, g_1, h_1)R(f_2, g_2, h_2) \left(\frac{2}{\pi^2} \int_{v'} \omega\right)^8.
\]
Here $\Phi_\ell(z)$ is the Borcherds $\Phi$-function with respect to the level $\ell$ cusp (cf. Section 2.2).

We point out a similarity of (1.3) to the Thomae type formula of Matsumoto-Terasoma [19] for certain $K3$ surfaces. As another application of Theorem 1.1, we obtain a new relation between the Borcherds $\Phi$-function and theta series as follows.

Let $M_{3,6}^0(\mathbb{C})$ be the set of complex $3 \times 6$-matrices without vanishing $3 \times 3$-minors. Considering the loci of the moduli space of Enriques surfaces parametrized by $M_{3,6}^0(\mathbb{C})$, we get an explicit relation between the Borcherds $\Phi$-function and certain theta functions. For $N = (n_{ij}) = (n_1, \ldots, n_6) \in M_{3,6}^0(\mathbb{C})$, we define
\[
X_N = \{ [x] \in \mathbf{P}^5 ; n_1x_1^2 + n_2x_2^2 + n_3x_3^2 + n_4x_4^2 + n_5x_5^2 + n_6x_6^2 = 0 \}.
\]
Then the involution $\iota_{(pqr)}(x_p, x_q, x_r, x_s, x_t, x_u) = (x_p, x_q, x_r, -x_s, -x_t, -x_u)$ acts on $X_N$. Here $(pqr)$ denotes the partition \{p, q, r\} $\sqcup$ \{s, t, u\} = \{1, \ldots, 6\}. For all $N \in M_3^o(C)$ and for all partitions $(pqr)$, $X_N/\iota_{(pqr)}$ is an Enriques surface.

There is another $K3$ surface $Z_N$ associated to $N \in M_3^o(C)$

$$Z_N = \left\{ (w_1 : w_2 : w_3, y) \in O_{p^2}(3); \ y^2 = \prod_{k=1}^{6} (n_{1k} w_1 + n_{2k} w_2 + n_{3k} w_3) \right\},$$

which is identified with its minimal resolution $\tilde{Z}_N$. After a suitable choice of a system of transcendental cycles of $\tilde{Z}_N$, the period of $\tilde{Z}_N$, denoted by $\Omega_N$, lies in

$$D = \{ T \in M_2(C); \ (T - {\bar{T}})/2\sqrt{-1} > 0 \} \subset M_2(C),$$

where $M_2(C)$ is the set of complex $2 \times 2$ matrices (cf. [26, 19] and Section 5.3).

For $\Omega \in D$ and $(pqr)$, the theta function $\Theta_{(pqr)}(\Omega)$ is defined as a certain Fourier series on $D$ (cf. Section 5.3), whose Petersson norm is denoted by $\|\Theta_{(pqr)}(\Omega)\|$.

**Theorem 1.3.** For all $N \in M^o(3, 6)$, the following equality holds

$$\left\| \Phi \left( X_N/\iota_{(pqr)} \right) \right\| = \left\| \Theta_{(pqr)}(\Omega_N) \right\|^4.$$

By Freitag-Salvati Manni [7], the Borcherds $\Phi$-function is expressed as an additive Borcherds lift. Even more, they proved recently that the Borcherds $\Phi$-function itself can be understood as a theta series [8]. After the works of Freitag-Salvati Manni, it is not surprising that the restriction of the Borcherds $\Phi$-function to a subdomain isomorphic to $D$ can be expressed as a theta series.

As a by-product of Theorem 1.3, we shall show that $\theta_{a,b}(T)^8$ is expressed as an infinite product of Borcherds type, where $\theta_{a,b}(T)$ is an arbitrary even theta constant of genus 2 (cf. Corollary 6.4). We remark that the product of all even theta constants of genus 2 is expressed as a Borcherds product by Gritsenko-Nikulin [12].

To prove Theorem 1.1, we compare the $\partial \bar{\partial} \log(\cdot)$ of the both sides of (1.2) as currents on the Grassmann variety parametrizing 3-dimensional subspaces of the $\iota$-invariant quadratic forms in the variables $x_1, \ldots, x_6$. Outside the locus of vanishing resultants, the comparison of the curvature is easy. We give a criterion that the period map for a one-parameter degenerating family of Enriques surfaces intersects the discriminant locus transversally, and we use it to determine the singularity of $\|\Phi\|$ near the locus of vanishing resultants. In this way, we prove Theorem 1.1, up to a constant. To determine the constant, we evaluate the both sides of (1.2) for those Enriques surfaces studied by Mukai [20] and Ohashi [22].

This paper is organized as follows. In Section 2, we recall Enriques surfaces and Borcherds $\Phi$-function. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3. In Section 6, we study
infinite product expansion of theta constants of genus 2. In Section 7, we study the Borcherds $\Phi$-function for those Enriques surfaces studied by Mukai and Ohashi.

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2. Enriques surfaces and the Borcherds $\Phi$-function. A free $\mathbb{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. For a lattice $L = (\mathbb{Z}^r, \langle \cdot, \cdot \rangle)$ and $k \in \mathbb{Q}^\times$, we define $L(k) := (\mathbb{Z}^r, k\langle \cdot, \cdot \rangle)$. The group of isometries of $L$ is denoted by $O(L)$. The set of roots of $L$ is defined by $\Delta_L := \{ l \in L; \langle l, l \rangle = -2 \}$.

Let $U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and let $E_8$ be the even unimodular negative-definite lattice of rank 8. The $K3$-lattice $L_{K3}$ and the Enriques lattice $\Lambda$ are defined as

$$L_{K3} := U \oplus U \oplus U \oplus E_8 \oplus E_8,$$

$$\Lambda := U(2) \oplus U \oplus E_8(2).$$

We fix a primitive embedding $\Lambda \subset L_{K3}$. Then $\Lambda_{-L_{K3}} \cong U(2) \oplus E_8(2)$.

2.1. $K3$ and Enriques surfaces. A $K3$ surface $X$ is a smooth compact complex surface with $H^1(X, \mathcal{O}_X) = 0$ and the trivial canonical line bundle $K_X \cong \mathcal{O}_X$. By [1, p. 241], $H^2(X, \mathbb{Z})$ endowed with the cup-product is isometric to $L_{K3}$.

An Enriques surface $Y$ is a compact connected complex surface with $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$, $K_Y \not\cong \mathcal{O}_X$ and $K_Y^{\otimes 2} \cong \mathcal{O}_X$. Let $X$ be the universal covering of $Y$ and let $\theta: X \to X$ be the non-trivial covering transformation of $X$ over $Y$. Then $X$ is a $K3$ surface and $\theta$ is an anti-symplectic holomorphic involution without fixed points. By, e.g., [1], there is an isometry of lattices $\alpha: H^2(X, \mathbb{Z}) \cong L_{K3}$ such that

$$\alpha(H^2(X, \mathbb{Z})_+) = \Lambda_{-L_{K3}}, \quad \alpha(H^2(X, \mathbb{Z})_-) = \Lambda,$$

where $H^2(X, \mathbb{Z})_\pm := \{ l \in H^2(X, \mathbb{Z}); \theta^*(l) = \pm l \}$. The pair $(Y, \alpha)$ is called a marked Enriques surface if the isometry $\alpha$ satisfies (2.1).

We set

$$\Omega_\Lambda := \{ [\eta] \in \mathbb{P}(\Lambda \otimes \mathbb{C}); \langle \eta, \eta \rangle = 0, \langle \eta, \bar{\eta} \rangle > 0 \}.$$

Let $\Omega_\Lambda^+$ be one of the connected components of $\Omega_\Lambda$, which we give in Section 2.2. Then $\Omega_\Lambda^+$ is a bounded symmetric domain of type IV of dimension 10. Set

$$O^+(\Lambda) := \{ g \in O(\Lambda); g(\Omega_\Lambda^+) = \Omega_\Lambda^+ \}.$$

The projective action of $O^+(\Lambda)$ on $\Omega_\Lambda^+$ is proper and discontinuous. We define

$$\mathcal{M} := \Omega_\Lambda^+/O^+(\Lambda).$$
The period of a marked Enriques surface \((Y, \alpha)\) is defined as
\[
\varpi(Y, \alpha) := [\alpha(H^0(X, \Omega_X^2))] \in \Omega_+^\Lambda
\]
and the period of an Enriques surface \(Y\) is defined as the \(O^+(\Lambda)\)-orbit of \(\varpi(Y, \alpha)\):
\[
\varpi(Y) := [\varpi(Y, \alpha)] \in \mathcal{M}.
\]
For \(d \in \Delta_\Lambda\), set \(H_d := \{\eta \in \Omega_+^\Lambda; \langle \eta, d \rangle = 0\}\). The \(O^+(\Lambda)\)-invariant reduced divisor
\[
\mathcal{D} := \sum_{d \in \Delta_\Lambda/\pm 1} H_d \subset \Omega_+^\Lambda
\]
is called the discriminant locus. We set \(\overline{\mathcal{D}} := \mathcal{D}/O^+(\Lambda)\). Then \(\varpi(Y) \notin \overline{\mathcal{D}}\). By [1],
\[
\mathcal{M}^o := (\Omega_+^\Lambda \setminus \mathcal{D})/O^+(\Lambda) = \mathcal{M} \setminus \overline{\mathcal{D}}
\]
is the coarse moduli space of Enriques surfaces via the period map.

2.2. The Borcherds \(\Phi\)-function and its basic properties.

2.2.1. The Borcherds \(\Phi\)-function. For a subset \(S \subset \mathbf{P}(\Lambda \otimes \mathbb{C})\), the cone over \(S\) is denoted by \(C(S) := \{\eta \in (\Lambda \otimes \mathbb{C}) \setminus \{0\}; [\eta] \in S\}\). Up to a constant, the Borcherds \(\Phi\)-function is defined as the holomorphic function \(\Phi(Z)\) on \(C(\Omega_+^\Lambda)\) with the following properties (C1), (C2), and (C3):

(C1) \(\Phi(\lambda Z) = \lambda^{-4} \Phi(Z)\) for all \(\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}\).

(C2) \(\Phi(g(Z)) = \chi(g) \Phi(Z)\) for all \(g \in O^+(\Lambda)\), where \(\chi \in \text{Hom}(O^+(\Lambda), \mathbb{C}^*)\).

(C3) The zero divisor of \(\Phi\) is the cone \(C(\mathcal{D})\).

In [2], \(\Phi(Z)\) (precisely speaking \(\Phi_2(w)\) below) was constructed. In [3, Example 13.7], \(\Phi(Z)\) (precisely speaking \(\Phi_1(z), \Phi_2(w)\) below) was constructed as the Borcherds lift of certain vector valued modular form for \(M_{p_2}(Z)\). Since the lattice used in [3, Example 13.7] is distinct from \(\Lambda\), we also refer to [28, Example 8.9], where \(\Lambda\) is used in the construction of \(\Phi\) as the Borcherds lift.

The Petersson norm of \(\Phi(Z)\) is the \(C^\infty\) function on \(C(\Omega_+^\Lambda)\) defined by
\[
\|\Phi(Z)\|^2 := 2^{-4} \langle Z, \overline{Z} \rangle^4_\Lambda \Phi(Z)^2.
\]

**Lemma 2.1.** The character of \(\Phi^2\) is trivial.

**Proof.** Since \(O^+(\Lambda^\vee(2)) \cong O^+(\Lambda^\vee(2)) \cong O^+(\Lambda)\), the result follows from the fact that \(O^+(\Lambda^\vee(2))\) is generated by reflections [11, proof of Prop. 5.6].

Since \(\chi^2\) is trivial by Lemma 2.1, \(\|\Phi(Z)\|^2\) is invariant under the actions of the groups \(O^+(\Lambda)\) and \(\mathbb{C}^*\). We regard \(\|\Phi\|^2\) as a \(C^\infty\) function on \(\mathcal{M}\) in what follows.
For an Enriques surface $Y$, we define

$$\|\Phi(Y)\| := \|\Phi(\overline{Y})\|.$$ 

Let $\kappa$ be the Kähler form of the Bergmann metric on $M$ and let $\delta_D$ be the Dirac $\delta$-current associated to the divisor $D$. By the Poincaré-Lelong formula, we get

$$-dd^c \log \|\Phi\|^2 = 4\kappa - \frac{1}{2} \delta_D$$

as currents on $M$. Here the coefficient $1/2$ enters in the formula, because the projection $\Omega^+_A \to M$ is doubly ramified along $D$.

### 2.2.2. The tube domain realization of $\Omega_A$ and the automorphic factor.

We define the positive cone of a Lorentzian lattice $L$ as $C_L = \{ x \in L \otimes \mathbb{R} : x^2 > 0 \}$. Then $C_L$ consists of two connected components $C^+_L$ and $C^-_L$ such that $C^-_L = -C^+_L$.

For $\ell = \{ 1, 2 \}$, set

$$M_\ell := \mathbb{U}(2/\ell) \oplus \mathbb{E}_8(2).$$

Let $\{e_\ell, f_\ell\}$ be a basis of $\mathbb{U}(\ell)$ such that $\langle e_\ell, e_\ell \rangle = \langle f_\ell, f_\ell \rangle = 0$ and $\langle e_\ell, f_\ell \rangle = \ell$. Regarding $\mathbb{U}$ and $\mathbb{U}(2)$ as direct summands of $\Lambda$, we get $e_\ell^+ / \mathbb{Z}e_\ell \cong M_\ell$.

We define the isomorphism $\iota_\ell : M_\ell \otimes \mathbb{R} + iC_{M_\ell} \ni u \to [\iota_\ell(u)] \in \Omega_A$ by

$$\iota_\ell(u) := -(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u.$$ 

Let $C^+_{M_1}$ be the component of $C_{M_1}$ whose closure contains $e_2, f_2$. Then $\Omega^+_A$ is defined as the component corresponding to $M_2 \otimes \mathbb{R} + iC^+_{M_1}$, and $C^+_{M_2}$ is defined as the component such that $M_2 \otimes \mathbb{R} + iC^+_{M_2} \cong \Omega^+_A$ via (2.3). Replacing $\{e_1, f_1\}$ by $\{-e_1, -f_1\}$ if necessary, we may and will assume that $e_1, f_1 \in C^+_{M_2}$.

Through the isomorphism (2.3), $O^+(\Lambda)$ acts on $M_\ell \otimes \mathbb{R} + iC^+_{M_\ell}$. For $u \in M_\ell \otimes \mathbb{R} + iC^+_{M_\ell}$ and $g \in O^+(\Lambda)$, we define $g \cdot u \in M_\ell \otimes \mathbb{R} + iC^+_{M_\ell}$ by the formula

$$\iota_\ell(g \cdot u) = -\frac{(g \cdot u)^2}{2}e_\ell + \frac{f_\ell}{\ell} + (-1)^{2/\ell}g \cdot u := \frac{g(\iota_\ell(u))}{\langle g(\iota_\ell(u)), e_\ell \rangle_\Lambda}.$$ 

The automorphic factor for the $O^+(\Lambda)$-action on $M_\ell \otimes \mathbb{R} + iC^+_{M_\ell}$ is defined by

$$\iota_\ell(g, u) := \langle g(\iota_\ell(u)), e_\ell \rangle_\Lambda = \langle g(-(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u), e_\ell \rangle_\Lambda.$$ 

Then $j_\ell(gg', u) = j_\ell(g, g' \cdot u) j_\ell(g', u)$ for all $g, g' \in O^+(\Lambda)$. Since

$$\langle 2(\mathfrak{F}u, \mathfrak{F}u)_{M_\ell} \rangle,$$

we get

$$\langle -(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u, -(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u \rangle_\Lambda$$

$$= \langle \iota_\ell(u), \iota_\ell(u) \rangle_\Lambda.$$
we get the automorphic property of $\langle \mathfrak{S} u, \mathfrak{S} u \rangle_{\mathbb{M}_\ell}$ by (2.4), (2.5)
\[ \langle \mathfrak{S} (g \cdot u), \mathfrak{S} (g \cdot u) \rangle_{\mathbb{M}_\ell} = |j_\ell(g, u)|^{-2} \langle \mathfrak{S} u, \mathfrak{S} u \rangle_{\mathbb{M}_\ell} \quad (g \in O^+(\Lambda)). \]

2.2.3. The Borcherds $\Phi$-function with respect to the level $\ell$ cusp. We define the Borcherds $\Phi$-function with respect to the level $\ell$ cusp as the pullback of $\Phi(Z)$ to $\mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}^+_{\mathbb{M}_\ell}$ via the embedding $\iota_\ell$, i.e.,
\[ \Phi_\ell(u) := \Phi(\iota_\ell(u)) = \Phi\left(-\left(u^2/2\right)e_\ell + \left(f_\ell/\ell\right) + (-1)^{2/\ell}u\right). \]

By (C1), $\Phi(Z)$ can be recovered from $\Phi_\ell(u)$. In this sense, $\Phi$ and $\Phi_\ell$ are equivalent.

By (2.4), (2.5), (C1), (C2), $\Phi_\ell(u)$ satisfies the following functional equation
\[ \Phi_\ell(g \cdot u) = \chi(g) j_\ell(g, u)^4 \Phi_\ell(u) \]
on $\mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}^+_{\mathbb{M}_\ell}$ for all $g \in O^+(\Lambda)$.

By (2.6), the Petersson norm of $\Phi$ is expressed as follows on $\mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}^+_{\mathbb{M}_\ell}$
\[ \|\Phi_\ell(u)\| := \|\Phi(\iota_\ell(u))\|^2 = \langle \mathfrak{S} u, \mathfrak{S} u \rangle_{\mathbb{M}_\ell}^4 |\Phi_\ell(u)|^2. \]

The relation between $\Phi_1$ and $\Phi_2$ is as follows. For $z \in \mathbb{M}_1 \otimes \mathbb{R} + i \mathcal{C}^+_{\mathbb{M}_1}$, the point $w \in \mathbb{M}_2 \otimes \mathbb{R} + i \mathcal{C}^+_{\mathbb{M}_2}$ corresponding to $z$ is given by $\iota_1(z)/\langle z, e_2 \rangle_{\mathbb{M}_1} = \iota_2(w)$.

Hence
\[ w = -\frac{1}{\langle z, e_2 \rangle_{\mathbb{M}_1}} \left\{ -\frac{z^2}{2} e_1 + f_1 + \left( z - \langle z, f_2 \rangle_{\mathbb{M}_1} e_2 \right) f_2 - \langle z, e_2 \rangle_{\mathbb{M}_1} \right\}. \]

By the inequality $\langle \mathfrak{S} w, e_1 \rangle_{\mathbb{M}_2} = \mathfrak{S}(-1/\langle z, e_2 \rangle_{\mathbb{M}_1}) > 0$, we get $\mathfrak{S} w \in \mathcal{C}^+_{\mathbb{M}_2}$. Since $\iota_1(z)/\langle z, e_2 \rangle_{\mathbb{M}_1} = \iota_2(w)$, we deduce from (C1) that
\[ \Phi_2(w) = \langle z, e_2 \rangle_{\mathbb{M}_1}^4 \Phi_1(z). \]

2.2.4. The Borcherds $\Phi$-function with respect to the level 1 cusp. By [3, Example 13.7], [28, (8.5), Example 8.9], $\Phi_1(z)$ is expressed as the following infinite product
\[ \Phi_1(z) = \prod_{\lambda \in \mathbb{M}_1 \cap \mathcal{C}^+_{\mathbb{M}_1} \setminus \{0\}} \left( \frac{1 - e^{\pi i (\lambda, z)_{\mathbb{M}_1}}}{1 + e^{\pi i (\lambda, z)_{\mathbb{M}_1}}} \right) \ell(\lambda^2/2) \]
when $|\Im z|^2 \gg 0$. Here the series $\{c(n)\} \subset \mathbb{Z}$ is defined by the generating function:
\[ \sum_{n \in \mathbb{Z}} c(n) e^{2\pi i n \tau} = \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8}, \quad \eta(\tau) := e^{2\pi i \tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}). \]

By this explicit expression, $\Phi(Z)$ is defined without an ambiguity of constant now.
2.2.5. The Borcherds $Φ$-function with respect to the level 2 cusp. Let $W \subset C_{M_2}^+$ be a Weyl chamber with $e_1 \in W$ and set $Π^+ := \{ λ ∈ M_2; (λ, W)_{M_2} > 0, λ^2 ≥ -2 \}$.

**Theorem 2.2.** If $(3w)^2 ≫ 0$, $Φ_2(w)$ is expressed as the following infinite product:

$$Φ_2(w) = 2^8 e^{2πi(e_1, w)_{M_2}} \prod_{λ ∈ Π^+} \left(1 - e^{2πi(λ, w)_{M_2}}\right)^{(-1)\langle λ_1, f_1 \rangle_{M_2} e(λ^2/2)} \right).$$

**Proof.** By [2], [3, Th. 13.3(5)], [28, (8.5), Example 8.9], there is a constant $C$ with $|C| = 2^8$ such that

$$Φ_2(w) = C e^{2πi(e_1, w)_{M_2}} \prod_{λ ∈ Π^+} \left(1 - e^{2πi(λ, w)_{M_2}}\right)^{(-1)\langle λ_1, f_1 \rangle_{M_2} e(λ^2/2)} \right).$$

For $z ∈ M_1 ⊗ R + iC_{M_1}$, write $z = (z, f_2)_{M_1} e_2/2 + (z, e_2)_{M_1} f_2/2 + z_{E_8(2)}$ with $z_{E_8(2)} ∈ E_8(2) ⊗ C$ and $τ := (e_2, z)_{M_1}$. Consider the limit $S(f_2, z)_{M_1} → +∞$ with $(λ, e_2)_{M_1} ≠ 0$, we get $exp(2πi(λ, z)_{M_1}) → 0$ for all $λ ∈ C_{M_1}^+ \setminus R ≥ 0$ as $S(f_2, z)_{M_1} → +∞$. We set $Φ_F(τ) := lim_{S(f_2, z)_{M_1} → +∞} Φ_1(z)$. Then we get by (2.10) (2.13)

$$Φ_F(τ) = \prod_{n ∈ Z_+} \left(1 - e^{πi(ne_2, z)_{M_1}}\right) c\left(\frac{n^2}{τ}\right) = \prod_{n > 0} \left(1 - e^{πinτ}\right) = \frac{η(τ/2)^16}{η(τ)^8}.$$  

We set $w_{E_8(z)} := w - (f_1, w)_{M_2} e_1 - (e_1, w)_{M_2} f_1 ∈ E_8(2) ⊗ C$. Since $z^2 = (z, e_2)_{M_1} (z, f_2)_{M_1} + z_{E_8(2)}^2$, we get $S(f_1, w)_{M_2} → +∞$ and $(e_1, w)_{M_2} = -1/τ, w_{E_8(2)} = -z_{E_8(2)}/τ$ by (2.8). We set $σ := (e_1, w)_{M_2}$ and $Φ_F(σ) := lim_{S(w, f_1)_{M_2} → +∞} Φ_2(w)$. In the same way as above, we deduce from (2.12) that

$$Φ_F(σ) = C e^{2πiσ} \prod_{n ∈ Z_+} \left(1 - e^{2πi(ne_1, w)_{M_2}}\right)^{(-1)n^2 c\left(\frac{n^2}{2}\right)} \right).$$

Since $σ = -1/τ$, we get the following by substituting (2.13) and (2.14) into (2.9)

$$C η(-2/τ)^2/η(-1/τ)^2 = Φ_F(-1/τ) = τ^4 Φ_F(τ) = τ^4 \cdot η(τ/2)^4 / η(τ)^8.$$  

Since $η(-1/τ)^2 = τ^4 η(τ)^8$, we get $C = 2^8$ by (2.15).
2.3. Degenerations of Enriques surfaces and the Borcherds $\Phi$-function.

**Theorem 2.3.** Let $\Delta \subset \mathbb{C}$ be the unit disc. Let $Z_0$ be a K3 surface with at most nodes as its singularities and let $f : (Z, Z_0) \to (\Delta, 0)$ be a flat deformation of $Z_0$. Let $\nu : Z \to \Delta$ be a holomorphic involution preserving the fibers of $f$. For $t \in \Delta$, set $Z_t := f^{-1}(t)$, $\nu_t := \nu|_{Z_t}$ and $S_t := Z_t/\nu_t$. Assume the following:

(i) $Z$ is smooth.

(ii) The fixed-point-set of the $\nu$-action on $Z$ consists of one node of $Z_0$, say $O$.

(iii) The map $f : Z \to \Delta$ is projective.

Then $S_t$ is an Enriques surface for $t \not= 0$ and the following holds

$$\log \|\Phi(S_t)\|^2 = \frac{1}{2} \log |t|^2 + O(1) \quad (t \to 0).$$

**Proof.**

**Step 1.** Let $\tilde{\Delta}$ be another disc and set $\tilde{\Delta}^* := \tilde{\Delta} \setminus \{0\}$. Let $Z \times_{\Delta} \tilde{\Delta}$ be the family over $\tilde{\Delta}$ induced from $f : Z \to \Delta$ by the map $\tilde{\Delta} \ni t \to t^2 \in \Delta$. A resolution $\pi : \tilde{Z} \to Z \times_{\Delta} \tilde{\Delta}$ of the singularities of $Z \times_{\Delta} \tilde{\Delta}$ is called a simultaneous resolution of $f : Z \to \Delta$ if the following property is satisfied: Set $\tilde{\pi} := \text{pr}_1 \circ \pi : \tilde{Z} \to Z$ and $\tilde{f} := \text{pr}_2 \circ \pi : \tilde{Z} \to \tilde{\Delta}$. For $t \in \tilde{\Delta}$, we set $\tilde{Z}_t := \tilde{f}^{-1}(t)$ and $\tilde{\pi}_t := \tilde{\pi}|_{\tilde{Z}_t} : \tilde{Z}_t \to Z_{t^2}$. Then $\tilde{\pi}_t$ is an isomorphism for $t \in \tilde{\Delta}^*$ and is the minimal resolution for $t = 0$.

There exists a simultaneous resolution $\tilde{f} : \tilde{Z} \to \tilde{\Delta}$ of $f : Z \to \Delta$. Then $\tilde{f}$ is a smooth (possibly non-projective) morphism and $Z_t$ is a smooth K3 surface for $t \not= 0$. By (ii), $S_t$ is an Enriques surface for $t \not= 0$.

**Step 2.** We recall the simultaneous resolution of $f : Z \to \Delta$. By (ii), we can write $\text{Sing}(Z_0) = \{O, p_1, q_1, \ldots, p_m, q_m\}$ with $\nu(p_i) = q_i$. By (iii) and an argument using morsification, there is a system of coordinates $(O, (z_1, z_2, z_3))$ near $O$ such that

$$f(z) = z_1^2 + z_2^2 + z_3^2, \quad \nu(z) = -z.$$  

Similarly, there is a system of coordinates $(U_\alpha, (u_1, u_2, u_3))$ (resp. $(V_\alpha, (v_1, v_2, v_3))$) around $p_\alpha$ (resp. $q_\alpha$) such that

$$f(u) = u_1^2 + u_2^2 + u_3^2, \quad f(v) = v_1^2 + v_2^2 + v_3^2, \quad \nu(U_\alpha) = V_\alpha, \quad \nu^* v_\alpha^i = u_\alpha^i.$$ 

Define $O' := O \times_{\Delta} \tilde{\Delta} = \{(z, t) \in O \times \tilde{\Delta} : z_1^2 + z_2^2 + z_3^2 - t^2 = 0\}$ and

$$\tilde{O} := \{((z, t), (\zeta_0 : \zeta_1)) \in O' \times \mathbb{P}^1 : (\zeta_0 : \zeta_1) = (z_1 + \sqrt{-1}z_2 : z_3 + t)\}.$$ 

We define $\tilde{U}_\alpha$ and $\tilde{V}_\alpha$ in the same manner. By [4, Section 2.7], $\tilde{O}$ is a complex manifold and is the closure of the graph of the rational map

$$O \times_{\Delta} \tilde{\Delta} \ni (z, t) \dashrightarrow (z_1 + \sqrt{-1}z_2 : z_3 + t) = (t - z_3 : z_1 - \sqrt{-1}z_2) \in \mathbb{P}^1.$$
Moreover, the obvious projection \( \pi := \text{pr}_1: \tilde{O} \to O \times \Delta \tilde{\Delta} \) gives a resolution of the singularity of \( O \times \Delta \tilde{\Delta} \) with \( \pi^{-1}(o) \cong \mathbb{P}^1 \). Similarly, \( \pi: \tilde{U}_\alpha \to U_\alpha \times \Delta \tilde{\Delta} \) and \( \pi: \tilde{V}_\alpha \to V_\alpha \times \Delta \tilde{\Delta} \) are resolutions with \( \pi^{-1}(p_\alpha) \cong \mathbb{P}^1 \) and \( \pi^{-1}(q_\alpha) \cong \mathbb{P}^1 \). Then \( \tilde{Z} \) is obtained from \( Z \times \Delta \tilde{\Delta} \) by replacing \( O \times \Delta \tilde{\Delta}, U_\alpha \times \Delta \tilde{\Delta}, V_\alpha \times \Delta \tilde{\Delta} \) by \( \tilde{O}, \tilde{U}_\alpha, \tilde{V}_\alpha \), respectively. We set

\[
E_0 := \tilde{\pi}_0^{-1}(o), \quad F_\alpha := \tilde{\pi}_0^{-1}(p_\alpha), \quad F'_\alpha := \tilde{\pi}_0^{-1}(q_\alpha).
\]

Then \( E_0, F_1, F'_1, \ldots, F_m, F'_m \) are mutually disjoint \((-2)\)-curves of \( \tilde{Z}_0 \).

Step 3. Let \( \tau \) be the meromorphic involution on \( \tilde{Z} \) induced by the involution \( t \times \text{id}_\Delta \) on \( Z \times \Delta \tilde{\Delta} \). Then \( \tau \) is a holomorphic involution on \( \tilde{Z} \setminus E_0 \) exchanging \( F_\alpha \) and \( F'_\alpha \). For \( t \in \hat{\Delta}^* \), set \( \tilde{t}_\alpha := \tilde{\tau}|_{\tilde{Z}_\alpha} \) and \( \tilde{S}_t := \tilde{Z}_t/\tilde{t}_\alpha \). Since the family \( \tilde{f}: \tilde{Z} \to \tilde{\Delta} \) is differentiably trivial, it admits a marking \( \mu \), i.e., a trivialization of the local system \( \mu: \mathbb{R}^2 \tilde{f}_* \mathbb{Z} \cong \mathbb{L}_{K_3} \) such that the condition (2.1) is satisfied for all \( t \in \hat{\Delta}^* \). Let

\[
\omega: \tilde{\Delta} \ni t \to \omega(\tilde{S}_t, \mu) \in \Omega_{\tilde{\Delta}}^+
\]

be the period map for the marked family \( (\tilde{f}: (\tilde{Z}, \tilde{\tau}) \to \tilde{\Delta}, \mu) \). Let \( \Pi: \Omega_{\tilde{\Delta}}^+ \to \mathcal{M} \) be the projection. Since \( \tilde{Z}_t = Z_{t_2}, \tilde{t}_t = t_{t_2} \) and hence \( \tilde{S}_t = S_{t_2} \), we have

\[
(2.18) \quad \Pi \circ \omega(t) = \omega(S_{t_2}).
\]

Step 4. Let \( \Sigma := \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\} \) be the unit sphere of \( \mathbb{R}^3 \). Define the embedding \( i: \Sigma \times \tilde{\Delta} \hookrightarrow \tilde{O} \subset \tilde{Z} \) by

\[
(2.19) \quad i(\{(x_1, x_2, x_3), t\}) := \{(t x_1, t x_2, t x_3, t), (x_1 + \sqrt{-1} x_2 : x_3 + 1)\}.
\]

We define the embeddings \( j_\alpha: \Sigma \times \Delta \hookrightarrow \tilde{U}_\alpha \) and \( j'_\alpha: \Sigma \times \Delta \hookrightarrow \tilde{V}_\alpha \) in the same way. For \( t \in \tilde{\Delta} \), define submanifolds \( E_t, F_{\alpha, t}, F'_{\alpha, t} \subset \tilde{Z}_t \) diffeomorphic to \( \Sigma \) as

\[
E_t := i(\{(\Sigma \times \{t\})\}), \quad F_{\alpha, t} := j_\alpha(\{(\Sigma \times \{t\})\}), \quad F'_{\alpha, t} := j'_\alpha(\{(\Sigma \times \{t\})\}).
\]

Then the 2-spheres \( E_t, F_{\alpha, t}, F'_{\alpha, t} \) are transcendental cycles for \( t \neq 0 \), while \( E_0, F_{\alpha, 0} = F'_\alpha, F_{\alpha, 0} = F'_{\alpha, 0} \subset \tilde{Z}_0 \) are \((-2)\)-curves contracted by \( \tilde{f}_0 \). By (2.17) and the definitions of \( F_{\alpha, t}, F'_{\alpha, t} \), we get for all \( t \in \tilde{\Delta} \)

\[
(2.20) \quad \tilde{\tau}_t(F_{\alpha, t}) = F'_{\alpha, t}, \quad \tilde{\tau}_t(F'_{\alpha, t}) = F_{\alpha, t}.
\]

Let \( c_1(E_t) \in H^2(\tilde{Z}_t, \mathbb{Z}) \) be the Poincaré dual of \( E_t \) and define

\[
\delta := \mu(c_1(E_0)) = \mu(c_1(E_t)) \in \Delta_{\mathbb{L}_{K_3}}.
\]
Similarly, we define roots $d_\alpha, d'_\alpha \in \Delta_{\mathbb{L}K_3}$ as

$$d_\alpha := \mu(c_1(F_{\alpha,0})) = \mu(c_1(F_{\alpha,t})), \quad d'_\alpha := \mu(c_1(F'_{\alpha,0})) = \mu(c_1(F'_{\alpha,t})).$$

Since $\tilde{\eta}_t$ preserves $E_t$ and reverses its orientation for $t \neq 0$ by (2.16), (2.19), we get $d, \delta \in \Delta_A$. Since $d_\alpha$ and $d'_\alpha$ are not eigenvectors of the involution $\mu \circ \tilde{\eta}_t^* \circ \mu^{-1} \in O(\mathbb{L}K_3)$ by (2.20), we get $d_\alpha, d'_\alpha \not\in \Lambda$.

**Step 5.** Since $E_0$ is an algebraic cycle of $\tilde{Z}_0$ and $\delta \in \Delta_A$, we get $\omega(0) \in H_\delta$. Set $H_\delta^0 := H_\delta \setminus \bigcup_{d \in \Delta(A) \setminus \{\pm \delta\}} H_d$. Let us see that $\omega(0) \in H_\delta^0$. By (iii), there is an $\nu$-invariant relatively ample line bundle $L$ on $Z$. Set $\tilde{L} := \tilde{\pi}^*L$ and $\tilde{L}_t := \tilde{L}|_{\tilde{Z}_t}$ for $t \in \tilde{A}$. We get $l := \mu(c_1(\tilde{L}_0)) = \mu(c_1(\tilde{L}_t)) \in \Delta^\perp$ by the $\tilde{\eta}_t$-invariance of $\tilde{L}_t$.

Let $d \in \Delta_A$ be such that $\omega(0) \in H_\delta$. By [1, Chap. VIII, Prop. 3.7(i)], either $d$ or $-d$ is effective. For simplicity, assume that $d$ is effective. There is an effective divisor $\Gamma$ on $\tilde{Z}_0$ such that $\mu(c_1(\Gamma)) = d \in \Delta_A$. If $\tilde{\pi}_0(\Gamma)$ contains a curve, then we get $0 < \deg(L_0|_{\tilde{\pi}_0(\Gamma)}) = \deg(\tilde{L}_0|_{\Gamma}) = \langle \mu(c_1(\tilde{L}_0)), \mu(c_1(\Gamma)) \rangle = \langle l, d \rangle = 0$, a contradiction. Hence $\dim \tilde{\pi}_0(\Gamma) = 0$ and $\text{Supp}(\Gamma)$ is contained in the exceptional divisor of $\tilde{\pi}_0$.

Write $\Gamma = \nu E_0 + \sum_{\alpha} \nu_\alpha F_\alpha + \sum_{\alpha} \nu'_\alpha F'_\alpha$, where $\nu, \nu_\alpha, \nu'_\alpha \in \mathbb{Z}$. Since $\mu(c_1(\Gamma)) = d \in \Delta_A$ and since $E_0, F_1, F'_1, \ldots, F_m, F'_m$ are mutually disjoint $(-2)$-curves, we get $\nu^2 + \sum_{\alpha} \nu_\alpha^2 + \sum_{\alpha} (\nu'_\alpha)^2 = 1$. Hence $\Gamma = E_0, F_\alpha, F'_\alpha$. Since $d_\alpha, d'_\alpha \not\in \Lambda, \delta \in \Delta$ by Step 4 and since $d \in \Delta$ by assumption, we get $\Gamma \neq F_\alpha, F'_\alpha$ and $\Gamma = E_0$. This proves that, if $d \in \Delta_A$ and $\omega(0) \in H_\delta$, then $d = \pm \delta$. Namely, $\omega(0) \in H_\delta^0$.

**Step 6.** Let $K_Z$ be the canonical bundle of $Z$. Since $K_Z$ is a trivial line bundle on $Z$, there is a nowhere vanishing holomorphic 3-form $\xi$ on $Z$. For $t \in \Delta$, we set

$$\eta_t := \text{Res}_{Z_t} \left[ \xi / (f(z) - t) \right] \in H^0(Z_t, K_{Z_t}) \setminus \{0\}.$$

Then $\tilde{\eta}_t := \eta_t^*$ is regarded as a non-zero canonical form on $\tilde{Z}_t$ for $t \neq 0$. By (2.16), we can express $\eta_t = e^{\psi(z)}dz_2 \wedge dz_3/z_1$ on a neighborhood of $0$, where $\psi(z)$ is a holomorphic function near $0$. By this expression of $\eta_t$ and (2.19), we get

$$\langle \alpha(\tilde{\eta}_t), \delta \rangle = \int_{E_t} \tilde{\eta}_t = \int_{t\Sigma} e^{\psi(z)} \frac{dz_2 \wedge dz_3}{z_1} = t \left\{ e^{\psi(0)} \int_{\Sigma} \frac{dz_2 \wedge dz_3}{z_1} \right\} + O(t^2),$$

where $t\Sigma := \{(tx_1, tx_2, tx_3) \in \mathbb{C}^3; (x_1, x_2, x_3) \in \Sigma\}$. This proves that $\omega(t)$ intersects $H_\delta^0$ transversally at $\omega(0)$. Since $\Phi(z)$ vanishes of order one on $H_\delta^0$,

$$\log \| \Phi(\omega(S_t, \alpha)) \|^2 = \log \| \Phi(\omega(t)) \|^2 = \log |t|^2 + O(1) \quad (t \to 0).$$

This, together with (2.18), yields the result. \qed
3. An algebraic expression of $\|\Phi\|$.

3.1. The $(2,2,2)$-model of an Enriques surface. Let $\text{Sym}(3, \mathbb{C})$ be the set of complex $3 \times 3$-symmetric matrices. For $S = (s_{ij}) \in \text{Sym}(3, \mathbb{C})$, we set

$$Q(x; S) := \sum_{i,j=1}^{3} s_{ij}x_i x_j \in \mathbb{C}[x_1, x_2, x_3].$$

For $A = (A_1, A_2, A_3) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^3$, the resultant of $Q(x; A_1), Q(x; A_2), Q(x; A_3)$

$$R(A) = R(A_1, A_2, A_3)$$

is defined as the unique irreducible polynomial in the entries of $A_1, A_2, A_3$ satisfying the following conditions (cf. [6, Chap. 3, Th. (2.3)]):

(i) The system of equations $Q(x; A_1) = Q(x; A_2) = Q(x; A_3) = 0$ has a non-trivial solution if and only if $R(A) = 0$.

(ii) If $A_1 = \text{diag}(1,0,0), A_2 = \text{diag}(0,1,0), A_3 = \text{diag}(0,0,1)$, then $R(A) = 1$. By, e.g., [6, Chap. 3, p. 89, Eq. (2.8)], $R(A)$ is a homogeneous polynomial of degree 4 in the entries of $A_i$, $1 \leq i \leq 3$, so that $\deg R(A) = 12$. The polynomial $R(A)$ is too large to write here. See [15, p. 215, Table 1] for an explicit formula for $R(A)$.

For vectors $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ in $\text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^3$, we define

$$X_{(A,B)} := \{ (x_1 : x_2 : x_3 : y_1 : y_2 : y_3) \in \mathbb{P}^5; Q(x; A_i) + Q(y; B_i) = 0 (i = 1, 2, 3) \}.$$

If $X_{(A,B)}$ is a smooth complex surface, then $X_{(A,B)}$ is a $K3$ surface by the adjunction formula. We say that $(A, B)$ is admissible if $R(A)R(B) \neq 0$ and $X_{(A,B)}$ is a reduced complex surface with at most RDP’s. When $(A, B)$ is admissible, $X_{(A,B)}$ is a $K3$ surface with at most RDP, i.e., the minimal resolution of $X_{(A,B)}$ is a $K3$ surface.

By definition, $X_{(A,B)}$ is preserved by the involution on $\mathbb{P}^5$

$$(3.1) \quad \iota: (x_1 : x_2 : x_3 : y_1 : y_2 : y_3) \mapsto (x_1 : x_2 : x_3 : -y_1 : -y_2 : -y_3).$$

When $\iota$ preserves a subset $V \subset \mathbb{P}^5$, we define $V^{\iota} := \{ p \in V; \iota(p) = p \}$. We have $\mathbb{P}^5^{\iota} = P_1 \amalg P_2$, where $P_1 = \{ x_1 = x_2 = x_3 = 0 \}$ and $P_2 = \{ y_1 = y_2 = y_3 = 0 \}$ are projective planes. Since the three conics $Q(x; A_i) = 0 (i = 1, 2, 3)$ have no points in common if and only if $R(A) \neq 0$, $X_{(A,B)}^{\iota} = 0$ if and only if $R(A)R(B) \neq 0$. Hence

$$Y_{(A,B)} := X_{(A,B)}/\iota.$$
is an Enriques surface with at most RDP’s (i.e., the minimal resolution of $Y_{(A,B)}$ is an Enriques surface) for admissible $(A,B)$. Let $L_{(A,B)}$ be the ample line bundle of degree 4 on $Y_{(A,B)}$ induced from the $\iota$-invariant ample line bundle $O_{5}(1)$ on $X_{(A,B)}$. When $(A,B)$ is admissible, the polarized variety $(Y_{(A,B)}, L_{(A,B)})$ or equivalently $(X_{(A,B)}, \iota, O_{5}(1))$ is called a $(2,2,2)$-model in this paper. For simplicity, we often omit $L_{(A,B)}$, $O_{5}(1)$. Since every Enriques surface is birational to some $Y_{(A,B)}$ with admissible $(A,B)$ by [25, 5], every point of $M^{\omega}$ admits a $(2,2,2)$-model.

For an admissible $(A,B)$, let $K_{X_{(A,B)}}$ be the dualizing sheaf on $X_{(A,B)}$. We define $\omega_{(A,B)} \in H^{0}(X_{(A,B)}, K_{X_{(A,B)}})$ as the residue of $Q(x; A_{i}) + Q(y; B_{i})$ $(i = 1, 2, 3)$. Let $q_{i}$ $(i = 1, 2, 3)$ be the inhomogeneous equation of $Q(x; A_{i}) + Q(y; B_{i})$ obtained by setting $y_{3} = 1$. Then $dx_{1} \wedge \cdots \wedge dy_{2}/q_{1}q_{2}q_{3}$ is a canonical form on $C^{5} \setminus \text{Sing} \ X_{(A,B)}$ with logarithmic poles along $\text{div} \ q_{1} \cup \text{div} \ q_{2} \cup \text{div} \ q_{3}$, which extends to canonical form on $P^{5} \setminus \text{Sing} \ X_{(A,B)}$ with logarithmic poles along $\bigcup_{i=1}^{3} \text{div}(Q(x; A_{i}) + Q(y; B_{i}))$. Then there exists a locally defined 2-form $\Upsilon$ on $P^{5} \setminus \text{Sing} \ X_{(A,B)}$ such that

\begin{equation}
\label{eq:3.2}
dx_{1} \wedge \cdots \wedge dy_{2}/q_{1}q_{2}q_{3} = (dq_{1}/q_{1}) \wedge (dq_{2}/q_{2}) \wedge (dq_{3}/q_{3}) \wedge \Upsilon.
\end{equation}

Let $j : X_{(A,B)} \setminus \text{Sing} \ X_{(A,B)} \to X_{(A,B)}$ be the inclusion. We define

\begin{equation}
\label{eq:3.3}
\omega_{(A,B)} = j_{*}(\Upsilon|_{X_{(A,B)} \setminus \text{Sing} \ X_{(A,B)}}),
\end{equation}

which is independent of $\Upsilon$. In this section, we prove the following theorem.

**Theorem 3.1.** For every admissible $(A,B) \in \text{Sym}(3, C) \otimes C^{6}$,

\begin{equation}
\label{eq:3.4}
\|\Phi(Y_{(A,B)})\|^{2} = |R(A)R(B)| \cdot \left(\frac{2}{\pi^{4}}\int_{X_{(A,B)}} \omega_{(A,B)} \wedge \omega_{(A,B)}\right)^{4}.
\end{equation}

**3.2. A Grassmannian and a family of $K3$ surfaces with involution.** For a finite dimensional complex vector space $U$ with dual space $U^{\vee}$, the Grassmann variety of $k$-dimensional subspaces of $U$ is defined set-theoretically as

$$\text{Gr}(k, U) := \text{GL}(C^{k}) \setminus \{f \in \text{Hom}(U^{\vee}, C^{k}) ; \text{rk}(f) = k\}.$$ 

Let $V$ be the complex vector space of quadratic forms in the variables $x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}$ invariant under the involution (3.1)

\[ V := Cx_{1}^{2} + Cx_{2}^{2} + Cx_{3}^{2} + Cx_{1}x_{2} + Cx_{1}x_{3} + Cx_{2}x_{3} + Cy_{1}^{2} + Cy_{2}^{2} + Cy_{3}^{2} + Cy_{1}y_{2} + Cy_{1}y_{3} + Cy_{2}y_{3}. \]

Let $\Psi : P^{5} \to P^{11}$ be the morphism defined by

$$\Psi(x : y) := (x_{1}^{2} : x_{2}^{2} : x_{3}^{2} : x_{1}x_{2} : x_{1}x_{3} : x_{2}x_{3} : y_{1}^{2} : y_{2}^{2} : y_{3}^{2} : y_{1}y_{2} : y_{1}y_{3} : y_{2}y_{3}).$$
Then $Ψ$ induces the embedding $\mathbb{P}^5/\iota \hookrightarrow \mathbb{P}^{11}$. Hence $\text{Sing}(Ψ(\mathbb{P}^5)) = Ψ((\mathbb{P}^5)^*)$.

Let $S$ be the Grassmann variety of 3-dimensional linear subspaces of $V$:

$$S := \text{Gr}(3, V).$$

We identify $S$ with the Grassmann variety of 9-dimensional subspaces of $V^\vee = \mathbb{C}^{12}$. Let $L \subset \mathbb{C}^{12} \times \text{Gr}(9, \mathbb{C}^{12})$ be the tautological vector bundle of rank 9 over $\text{Gr}(9, \mathbb{C}^{12})$ and let

$$\pi : \mathbb{P}(L) \longrightarrow \text{Gr}(9, \mathbb{C}^{12})$$

be the projective-space bundle associated to $L$. We regard $\mathbb{P}(L)$ as a complex submanifold of codimension 3 of $\mathbb{P}^{11} \times \text{Gr}(9, \mathbb{C}^{12})$. Then $\pi = \text{pr}_2|_{\mathbb{P}(L)}$. By the canonical identification between $S$ and $\text{Gr}(9, \mathbb{C}^{12})$, the fiber $L_s := p^{-1}(s) \subset \mathbb{P}^{11}$ is the linear subspace of codimension 3 corresponding to $s \in S$.

In $\mathbb{P}^{11} \times S$, we have two subvarieties $Ψ(\mathbb{P}^5) \times S$ and $\mathbb{P}(L)$. We define $\mathcal{Y} \subset \mathbb{P}^{11} \times S$ and $\mathcal{X} \subset \mathbb{P}^5 \times S$ by

$$\mathcal{Y} := (Ψ(\mathbb{P}^5) \times S) \cap \mathbb{P}(L)$$

$$\mathcal{X} := (Ψ \times \text{id}_S)^{-1}(\mathcal{Y}),$$

which are equipped with the projections $\pi : \mathcal{Y} \to S$ and $\pi : \mathcal{X} \to S$. For $s \in S$, set

$$Y_s := \pi^{-1}(s) = Ψ(\mathbb{P}^5) \cap \mathbb{P}(L_s), \quad X_s := \pi^{-1}(s) = Ψ^{-1}(Y_s).$$

If $\{F_s, G_s, H_s\} \subset V$ is a basis of the 3-dimensional subspace of $V$ corresponding to $s \in S$, then we have the expressions

$$X_s = \{(x, y) \in \mathbb{P}^5; F_s(x, y) = G_s(x, y) = H_s(x, y) = 0\}, \quad Y_s = X_s/\iota.$$

Here $X_s$ is regarded as the scheme defined by the ideal sheaf generated by $F_s, G_s, H_s$. It is clear that for every $s \in S$, there exists $(A, B) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^6$ such that $X_s = X_{(A, B)}$. Conversely, $X_{(A, B)} = X_s$ for some $s \in S$ if and only if the three vectors $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^2$ are linearly independent. When $(A, B) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^6$ satisfies this condition, we set

$$s(A, B) := \text{span}\{Q(x; A_i) + Q(y; B_i)\}_{i=1,2,3} \in S.$$

Hence $X_{(A, B)} = X_{s(A, B)}$ in this case.

We define

$$S^* := \{s \in S; \pi: \mathcal{X} \longrightarrow S \text{ is flat at any } x \in X_s\},$$

$$S^o := \{s \in S^*; \text{Sing}(X_s) = 0\}.$$

Since the non-flat locus and the critical locus of $\pi$ are Zariski closed subsets of $\mathcal{X}$, $S^*$ and $S^o$ are Zariski open subsets of $S$. Then $\pi: \mathcal{X}|_{S^o} \to S^*$ is a flat family of
projective surfaces, which is smooth over $S^0$. We define $D^* := S^* \setminus S^0$ and

\[ D' := \{ s \in D^*; \dim \text{Sing} (X_s) = 0 \}, \]
\[ D'' := \{ s \in D^*; \text{emb} (X_s, x) = 3 \ (\forall x \in \text{Sing} (X_s)) \}, \]
\[ D^o := \{ s \in D^*; \text{Sing} (X_s) \text{ consists of nodes} \}, \]
\[ E := S \setminus (S^o \cup D^o) = (S \setminus S^*) \cup (D^* \setminus D^o), \]

where $\text{emb} (X_s, x)$ denotes the embedding dimension of the germ $(X_s, x)$. Then $D'$ is a Zariski open subset of $D^*$. By Jacobi’s criterion, $D''$ is a Zariski open subset of $D^*$. Hence $D' \cap D''$ is a Zariski open subset of $D^*$.

**Lemma 3.2.** The following hold:

1. $D^o$ is a Zariski open subset of $D^*$.
2. $E$ is a Zariski closed subset of $S$ with $\dim E \leq \dim S - 2$.

**Proof.** (1) Let $s \in D' \cap D''$. Then $\text{Sing} (X_s)$ consists of isolated hypersurface singularities. Let $(Z, 0) \in \text{Sing} (X_s)$. Let $\text{Def} (Z, 0)$ be its miniversal deformation space and let $f: (\mathcal{Z}, (Z, 0)) \to (\text{Def} (Z, 0), 0)$ be the miniversal deformation of $(Z, 0)$. There is a map $\mu_Z: (S, s) \to (\text{Def} (Z, 0), 0)$ such that $\pi: (\mathcal{X}, (Z, 0)) \to (S, s)$ is induced from $f: (\mathcal{Z}, (Z, 0)) \to (\text{Def} (Z, 0), 0)$ by $\mu_Z$. Let $(D_Z, 0)$ be the discriminant locus of $f$. By [16, Props. 4.10 and 6.11, Cor. 4.11], there is a proper Zariski closed subset $(F_Z, 0) \subset (D_Z, 0)$ such that $\text{Sing} (f^{-1} (t))$ consists of a unique node for $t \in D_Z \setminus F_Z$. Then $D^o = (D' \cap D'') \setminus \bigcup_{(Z, 0) \in \text{Sing} (X_s)} \mu^{-1}_Z (F_Z)$ on a small neighborhood $s$ in $S$. By this expression, $D^o$ is a Zariski open subset of $D^*$.

(2) Let $s \in S^o \cup D^o$ be an arbitrary point. Then $X_s$ has at most nodes as its singular points. Since any deformation of a node is either a smoothing or a trivial deformation, $s$ is an interior point of $S^o \cup D^o$. Hence $E$ is a closed subset of $S$ with respect to the Euclidean topology. We get $E = \overline{E} = (S \setminus S^*) \cup (\overline{D^*} \setminus D^o)$, where $\overline{A}$ denotes the closure of a subset $A \subset S$ with respect to the Euclidean topology on $S$. Since $\overline{D^*} \setminus D^o$ is a Zariski closed subset of $S$ by (1), so is $E$.

Assume that there is a component $E' \subset E$ with $\dim E' = \dim S - 1$. Regard $E'$ as a reduced effective divisor of $S$. Since $S$ is a Grassmann variety and hence $\text{Pic} (S) = \mathbb{Z}$, $E'$ must be an ample divisor of $S$. Then, for any irreducible subvariety $T \subset S$ with $\dim T > 0$, $T \cap E'$ is either a divisor of $T$ or $T$ itself. In particular, we always have $\dim [T \cap E] \geq \dim T - 1$ for any proper subvariety $T \subset S$ with $\dim T > 0$. However, by Lemma 3.3 below, there exists a subGrassmannian $T \subset S$ of dimension 9 such that $\dim [T \cap E] = \dim [T \setminus (S^o \cup D^o)] \leq \dim T - 2$. This contradicts the inequality $\dim [T \cap E] \geq \dim T - 1$. This proves that $\dim E \leq \dim S - 2$. \qed

**3.3. A subfamily parametrized by a subGrassmannian.** In this subsection, we use the coordinates $(x_1, \ldots, x_6)$ instead of $(x_1, x_2, x_3, y_1, y_2, y_3)$. Write $M_{p, q} (C)$ for the set of complex $p \times q$-matrices. For $N = (n_1, \ldots, n_6) \in M_{3, 6} (C)$,
\[ n_i \in \mathbb{C}^3, \text{ we set } \Delta_{ijk}(N) := \det(n_i, n_j, n_k) \text{ and we define} \]
\[ M^o_{3, 6}(\mathbb{C}) := \{ N \in M(3, 6; \mathbb{C}); \Delta_{ijk}(N) \neq 0 \ (\forall 1 \leq i < j < k \leq 6) \}. \]

Write \( T \) for the Grassmannian variety of 3-dimensional subspaces of \( \mathbb{C}^6 \):
\[ T = \text{Gr}(3, \mathbb{C}^6) = \text{GL}(\mathbb{C}^3) \setminus \{ N \in M(3, 6; \mathbb{C}); \text{rk}(N) = 3 \}. \]

For \( N \in M(3, 6; \mathbb{C}) \) with \( \text{rk}(N) = 3 \), the corresponding point of \( T \) is denoted by \([N]\). For \( 1 \leq i < j < k \leq 6 \), we set
\[ T_{ijk} := \text{GL}(\mathbb{C}^3) \setminus \{ N \in M(3, 6; \mathbb{C}); \text{rk}(N) = 3, \Delta_{ijk}(N) = 0 \}, \]
\[ T^o := \text{GL}(\mathbb{C}^3) \setminus M^o_{3, 6}(\mathbb{C}) = T \setminus \bigcup_{i < j < k} T_{ijk}. \]

Then \( T_{ijk} \) is a divisor of \( T \) and \( T^o \) is a non-empty Zariski open subset of \( T \). We set
\[ T^o_{ijk} := T_{ijk} \setminus \bigcup_{l < m < n, \{l, m, n\} \neq \{i, j, k\}} T_{lmn}, \]
\[ D^o_T := \bigcup_{i < j < k} T^o_{ijk}. \]

Then \( T^o_{ijk} \) is a non-empty Zariski open subset of \( T_{ijk} \) and hence \( T \setminus (T^o \cup D^o_T) = \bigcup_{\{i, j, k\} \neq \{l, m, n\}} T_{ijk} \cap T_{lmn} \) is a Zariski closed subset of \( T \).

We regard \( T \) as a subGrassmannian of \( S \) by the embedding
\[ i: T \ni [N] = [(n_{ij})] \longmapsto \left[ \mathbb{C} \sum_{j=1}^{6} n_{1j}x_j^2 + \mathbb{C} \sum_{j=1}^{6} n_{2j}x_j^2 + \mathbb{C} \sum_{j=1}^{6} n_{3j}x_j^2 \right] \in S. \]

Then we get for \( N = (n_1, \ldots, n_6) \) with \( \text{rk}(N) = 3 \)
\[ X_N = X_{[N]} := X_{i(N)} = \{ [x] \in \mathbb{P}^5; \sum_{k=1}^{6} x_k^2n_k = 0 \}. \]

**Lemma 3.3.** The following hold.

1. For \([N] \in T^o\), \( X_{[N]} \) is smooth. In particular, \( T^o \subset S^o \).
2. For \([N] \in D^o_T\), \( \text{Sing}(X_{[N]}) \) consists of nodes. In particular, \( D^o_T \subset D^o \).
3. The Zariski closed subset \( T \setminus (T^o \cup D^o_T) \) has codimension at least 2 in \( T \).

**Proof:** Let \( N = (n_{ij}) = (n_1, \ldots, n_6) \in M(3, 6; \mathbb{C}) \) and assume that \([N] \in T^o \cup D^o_T\). Then there exists \( i < j < k \) such that \( \Delta_{lmn}(N) \neq 0 \) for any \( \{l, m, n\} \neq \{i, j, k\} \). In particular, any two column vectors \( n_l, n_m \) of \( N \) are linearly independent for \( l \neq m \).

Let \( x = (x_1 : \cdots : x_6) \in X_{[N]} \). Then \( \sum_i n_i x_i^2 = 0 \). If 4 of the coordinates of \( x \) vanish, then there are \( l < m \) such that \( n_l x_l^2 + n_m x_m^2 = 0 \). Since \( n_l, n_m \) are linearly
independent, we get \( x_i = x_m = 0 \) and hence all coordinates of \( x \) vanish. This contradicts \( x \in \mathbb{P}^5 \). Thus the condition \( x = (x_i) \in X[N] \), \( [N] \in T^o \cup D_T^o \) implies the existence of \( 1 \leq l < m < n \leq 6 \) with \( x_l x_m x_n \neq 0 \).

Since the Jacobian matrix of the defining equations of \( X[N] \) at \( x \) is given by the \( 3 \times 6 \)-matrix \( 2N \cdot \text{diag}(x_1, \ldots, x_6) \), since any two column vectors of \( N \) are linearly independent, and since 3 of \( x_1, \ldots, x_6 \) are non-zero, there is a pair \( 1 \leq p < q \leq 3 \) such that \( df_p \wedge df_q \neq 0 \) at \( x \). For simplicity, assume that \( df_1 \wedge df_2 \neq 0 \) at \( x \). Then \( f_1, f_2 \) are part of local coordinates of \( \mathbb{P}^5 \) around \( x \). Choose a non-vanishing \( 2 \times 2 \)-minor det \((n_1, n_{1j}) - (n_1 n_{2j} - n_{1j} n_{2i}) \neq 0 \). Then we get

\[
(n_1 n_{2j} - n_{1j} n_{2i}) \cdot f_3 \equiv \sum_{k \neq i, j, 1 \leq k \leq 6} \Delta_{ijk}(N) x_k \mod f_1, f_2.
\]

Hence, on a neighborhood of \((x, [N]) \in \mathcal{X}|_{T^o \cup D_T^o} \cap \{x_i x_j \neq 0\} \), the total space \( \mathcal{X}|_{T^o \cup D_T^o} \) is locally isomorphic to the hypersurface of \( \mathbb{P}^3 \times (T^o \cup D_T^o) \) defined by

\[
(3.4) \quad \Delta_{k_{ij}}(N) z_1^2 + \Delta_{k_{2ij}}(N) z_2^2 + \Delta_{k_{3ij}}(N) z_3^2 + \Delta_{k_{4ij}}(N) z_4^2 = 0,
\]

where \( \{i, j\} \cup \{k_1, k_2, k_3, k_4\} = \{1, \ldots, 6\} \). Since \( (\Delta_{ijk}(N))_{i < j < k} \) are the Plücker coordinates of \( T \) and since at most one of \( \Delta_{ijk}(N) \) can vanish on \( T^o \cup D_T^o \), we deduce from (3.4) the assertions (1), (2). Since \( T_{ijk} \) is a Schubert cycle and hence is an irreducible ample divisor on \( T \), we get \( \dim[T_{ijk} \cap T_{lmn}] = \dim T - 2 \) when \( T_{ijk} \neq T_{lmn} \). This proves (3). \( \Box \)

3.4. A natural pluricanonical differential associated to a \((2, 2, 2)\)-model.

3.4.1. Equivalence of \((2, 2, 2)\)-models. We set

\[
\text{GL} \left( \mathbb{C}^6 \right)^t := \{ \gamma \in \text{GL} \left( \mathbb{C}^6 \right) ; \gamma t = t \gamma \} = \text{GL} \left( \mathbb{C}^3 \right) \times \text{GL} \left( \mathbb{C}^3 \right).
\]

For \( A = (A_1, A_2, A_3) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^3 \) and \( P = (p_{ij}) \in \text{GL}(\mathbb{C}^3) \), we define

\[
A \cdot P := A \cdot (I_{\text{Sym}(3, \mathbb{C})} \otimes P) = \left( \sum_{i=1}^{3} A_i p_{i1}, \sum_{i=1}^{3} A_i p_{i2}, \sum_{i=1}^{3} A_i p_{i3} \right),
\]

\[
A^P := (t^P A_1, t^P A_2, t^P A_3 P).
\]

By definition, \( s(A, B) = s(A', B') \) in \( S \) if and only if \( (A', B') = (A \cdot P, B \cdot P) \) for some \( P \in \text{GL}(\mathbb{C}^3) \). Since every isomorphism \( (X_{(A, B)}, t, O_{\mathbb{P}^5(1)}) \cong (X_{(A', B')}, t, O_{\mathbb{P}^5(1)}) \) is induced by a projective transform of \( \mathbb{P}^5 \) associated to an element of \( \text{GL}(\mathbb{C}^6)^t \), we see that \( (Y_{(A, B)}, \mathcal{L}_{(A, B)}) \cong (Y_{(A', B')}, \mathcal{L}_{(A', B')}) \) if and only if \( Y_{(A', B')} = Y_{(A, B') \cdot P} \) for some \( (P, P') \in \text{GL}(\mathbb{C}^6)^t \).
3.4.2. A pluricanonical differential on $X_s$ and its invariance property.

Define

$$D^+ := \{ s \in D^*; \ \text{Sing}(X_s) \ \text{consists of RDP's} \}. $$

If $s \in D^+$, then $X_s$ is a $K3$ surface with RDP’s. Since the singular fiber of any flat deformation of a RDP contains at most RDP’s, $S^0 \cup D^+$ is a Zariski open subset of $S$.

For an admissible $(A, B) \in \text{Sym}(3, C) \otimes C^6$, we define

$$\Xi_{X(A, B)} := R(A)R(B)\omega^{\otimes 8}_{(A, B)} \in H^0 \left( X_{(A, B)}; K_{X(A, B)}^{\otimes 8} \right).$$

Then $\Xi_{X(A, B)}$ is an $ι$-invariant pluricanonical form of weight 8 on $X(A, B)$ and is identified with a pluricanonical form of weight 8 on $Y_{(A, B)}$.

**Lemma 3.4.** If $s(A, B) = s(A', B') \in S^0 \cup D^+$, then

$$\Xi_{X(A, B)} = \Xi_{X(A', B')}. $$

**Proof.** We can write $(A', B') = (A \cdot P, B \cdot P)$, $P \in \text{GL}(C^3)$. By (3.3), (3.2), we get

$$\omega_{(A \cdot P, B \cdot P)} = (\det P)^{-1}\omega_{(A, B)}. $$

By the explicit formula [15, p. 215, Table 1] for $R(A)$, we get

$$R(A \cdot P) = (\det P)^4 R(A). $$

The result follows from these two equalities. □

After Lemma 3.4, it makes sense to define for $s = s(A, B) \in S^0 \cup D^+$

$$\Xi_s := \Xi_{X(A, B)}. $$

Then we get a section $\Xi \in H^0(S^0 \cup D^+, \pi_* K_{\mathcal{X}/S}^{\otimes 8})$ such that $\Xi(s) := \Xi_s$. Here $K_{\mathcal{X}/S}$ denotes the relative dualizing sheaf of the family $\pi: \mathcal{X}|_{S^0 \cup D^+} \to S^0 \cup D^+$.

The group $\text{GL}(C^6)^+ = \text{GL}(C^3) \times \text{GL}(C^3)$ acts on $\mathcal{X}$ and $S$ by

$$(P, P') \cdot \left( \begin{pmatrix} x \\ y \end{pmatrix}, s \right) := \left( \begin{pmatrix} P_x \\ P_y \\ P' \end{pmatrix}, (P, P') \cdot s \right),

(P, P') \cdot s(A, B) := s(A^P, B^{P'}).$$

Then the projection $\pi: \mathcal{X} \to S$ is $\text{GL}(C^6)^+$-equivariant, and $\pi_* K_{\mathcal{X}/S}^{\otimes 8}$ is equipped with the structure of a $\text{GL}(C^6)^+$-equivariant line bundle on $S^0 \cup D^+$.

**Lemma 3.5.** The section $\Xi \in H^0(S^0 \cup D^+, \pi_* K_{\mathcal{X}/S}^{\otimes 8})$ is $\text{GL}(C^6)^+$-invariant.
Proof. Let \( g = (P, P') \in \text{GL}(\mathbb{C}^6)^t \) and \( s = s(A, B) \in S^o \cup D^+ \). The projective transform \( T_g(x) := (P_{x})_{g} \) of \( \mathbb{P}^5 \) induces an isomorphism \( T_g : X_{gs} \to X_s \). Since \( X_s \) and \( X_{gs} \) are K3 surfaces with possibly RDP’s, there exists \( \chi(g) \in \mathbb{C}^* \) with \( T_g^* \omega_{(A, B)} = \chi(g) \omega_{(A, B)} \). Since \( T_{gg'} = T_g T_{g'} \), we see that \( \chi : \text{GL}(\mathbb{C}^6)^t \to \mathbb{C}^* \) is a character. Letting \( g \) be a scalar matrix, we get \( \chi(g) = \det(g) \). Hence

\[
(T_{(P, P')})^* \omega_{(A, B)} = \det(P) \det(P') \omega_{(A, B)}.
\]

It is classical that

\[
R(A^P) = \det(P)^8 R(A).
\]

By these two equalities, we get \( T_{g}^* \Xi_{g,s} = \Xi_s \). This proves the lemma. \( \square \)

We define the \( \text{GL}(\mathbb{C}^6)^t \)-invariant divisor \( \mathcal{R} \) on \( S^o \cup D^o \) by

\[
\mathcal{R} := \text{div}(\Xi) = \{ s = s(A, B) \in S^o \cup D^o; R(A)R(B) = 0 \}.
\]

By the definition of resultants, \( R(A)R(B) = 0 \) if and only if \( X^t_{(A, B)} \neq 0 \). Hence \( \Xi_s \neq 0 \) if and only if \( \ell|_{X_s} \) is free from fixed points, i.e., \( Y_s = X_s/\ell \) is an Enriques surface (with possibly RDP’s). By this interpretation, we get

\[
\mathcal{R} = \{ s \in S^o \cup D^o; X^t_s \neq 0 \}.
\]

We define

\[
\mathcal{R}^o := \{ s \in \mathcal{R}; \#X^t_s = 1 \}.
\]

Lemma 3.6. The following hold:

1. \( \mathcal{R} \subset D^o \).
2. \( \mathcal{R}^o \) is a dense Zariski open subset of \( \mathcal{R} \).

Proof. (1) Since \( X^t_s \subset \text{Sing}(X_s) \) for any \( s \in S \), we get (1).

(2) It is clear that \( \mathcal{R}^o \) is a Zariski open subset of \( \mathcal{R} \). We see the density of \( \mathcal{R}^o \) in \( \mathcal{R} \). Let \( s \in \mathcal{R} \) be an arbitrary point. It suffices to see that there is a curve \( \gamma : (\Delta, 0) \to (\mathcal{R}, s) \) such that \( \gamma(\Delta \setminus \{0\}) \subset \mathcal{R}^o \). Write \( s = s(A, B) \). Since \( X^t_{(A, B)} \neq 0 \), either the system of equations \( Q(x; A_1) = Q(x; A_2) = Q(x; A_3) = 0 \) or \( Q(y; B_1) = Q(y; B_2) = Q(y; B_3) = 0 \) has a non-trivial solution. Assume that the former has a non-trivial solution \( c \in \mathbb{P}^2 \). Since generic three conics of \( \mathbb{P}^2 \) have no points in common, we can find a vector-valued holomorphic function \( \Delta \ni t \to (A(t), B(t)) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^6 \) satisfying the following conditions:

(i) \( A_i(0) = A_i \) and \( B_i(0) = B_i \) for \( i = 1, 2, 3 \).
(ii) For \( t \neq 0 \), \( \{ x \in \mathbb{P}^2; Q(x; A_1(t)) = Q(x; A_2(t)) = Q(x; A_3(t)) = 0 \} = \{ c \} \).
(iii) For \( t \neq 0 \), \( \{ x \in \mathbb{P}^2; Q(x; B_1(t)) = Q(x; B_2(t)) = Q(x; B_3(t)) = 0 \} = \emptyset \).
Set $\gamma(t) := s(A(t), B(t)) \in S$. Since $S^\circ \cup D^o = S \setminus E$ is a Zariski open subset of $S$ by Lemma 3.2(2), we may assume $\gamma(\Delta) \subset S^\circ \cup D^o$. We get $\gamma(0) = 0$ by (i) and $\gamma(\Delta \setminus \{0\}) \subset \mathcal{R}^o$ by (ii), (iii). This proves (2). \hfill \Box

3.4.3. A canonical Hermitian metric on the space of pluricanonical forms. Let $X$ be a $K3$ surface with possibly RDP’s. For every $\nu \in \mathbb{Z}_{>0}$, $H^0(X, K_X^\otimes \nu)$ is equipped with the Hermitian structure $\| \cdot \|_{L^2/\nu}$, which depends only on the complex structure on $X$, such that

$$\| \xi \|_{L^2/\nu} := \left( \int_X |\xi \wedge \overline{\xi}|^{1/\nu} \right)^{\nu/2}, \quad \xi \in H^0(X, K_X^\otimes \nu).$$

When $s = s(A, B) \in S^\circ \cup D^+$, we get

$$\| \Xi_s \|_{L^1/4} = \| \Xi_{X(A, B)} \|_{L^1/4} = \left| R(A) R(B) \right| \left| \int_{X_{(A, B)}} \omega(A, B) \wedge \overline{\omega(A, B)} \right|^4.$$

The Hermitian structure on the the invertible sheaf $\pi_* K_X^\otimes \nu |_{S^\circ \cup D^+}$ induced from this fiberwise Hermitian structure is also denoted by $\| \cdot \|_{L^2/\nu}$.

By Lemma 3.5, the norm $\| \Xi \|_{L^1/4}$ is a $\text{GL}(\mathbb{C})^4$-invariant, nowhere-vanishing $C^\infty$ function on $S^\circ$. By the existence of simultaneous resolution of the family $\pi: \mathcal{X}|_{S^\circ \cup D^+} \to S^\circ \cup D^+$, $\| \Xi \|_{L^1/4}$ extends to a $C^\infty$ function on $S^\circ \cup D^+$.

3.5. A comparison of $\| \Phi \|$ and $\| \Xi \|_{L^1/4}$ over $S$. Recall that $\mathcal{Y} = \mathcal{X}/\iota$. Let

$$\overline{\omega}: S^\circ \to \mathcal{M}^o$$

be the period map for the family of Enriques surfaces $\pi: \mathcal{Y}|_{S^\circ} \to S^\circ$. Then $\overline{\omega}$ is $\text{GL}(\mathbb{C})^4$-equivariant and $\overline{\omega}$ is dominant. By the existence of simultaneous resolution of the family $\pi: \mathcal{X}|_{S^\circ \cup D^+} \to S^\circ \cup D^+$, $\overline{\omega}$ extends to a holomorphic map from $S^\circ \cup D^+$ to $\mathcal{M}$. This extension is again denoted by $\overline{\omega}$. Then, for $s \in D^+$, we have

$$\| \Phi(Y_s) \| := \| \Phi(\overline{\omega}(s)) \| = \| \Phi(\overline{Y}_s) \|,
$$

where $\overline{Y}_s \rightarrow Y$ is the minimal resolution. Since $s \in \mathcal{R}$ if and only if $Y_s$ is not an Enriques surface, i.e., $\overline{\omega}(s) \in \overline{\mathcal{D}}$, we get on $S^\circ \cup D^o$

$$\text{Supp}(\overline{\omega}^o \text{div}(\Phi)) = \text{Supp}(\overline{\omega}^o \overline{\mathcal{D}}) = \text{Supp}(\mathcal{R}).$$

Theorem 3.7. The following equality of $\text{GL}(\mathbb{C})^4$-invariant functions on $S^\circ \cup D^+$ holds

$$\overline{\omega}^o \| \Phi \|^2 = (2\pi^{-4})^4 \| \Xi \|_{L^1/4}.$$
Proof.

Step 1. Recall that $\kappa$ is the Kähler form of the Bergman metric on $\mathcal{M}$. Let $\delta_{\mathcal{R}}$ be the Dirac $\delta$-current on $S^o \cup D^o$ associated to the divisor $\mathcal{R}$ on $S^o \cup D^o$. Since $\log \|\omega(A,B)\|_{L^2}^2$ is a local potential function of $\omega^\wedge \kappa$, we get the following equation of currents on $S^o \cup D^o$ by the Poincaré-Lelong formula

$$-dd^c \log \|\Xi\|_{L^{1/4}} = 4\omega^\wedge \kappa - \frac{1}{2} \delta_{\mathcal{R}}. \tag{3.6}$$

Step 2. Let $\sigma \in \mathcal{R}^0 \smallsetminus \text{Sing}(\mathcal{R}^0)$ be an arbitrary point. Let $C \subset S$ be a compact Riemann surface intersecting $\mathcal{R}^0$ transversally at $\sigma$. Let $(\Delta, t)$ be a coordinate neighborhood of $C$ centered at $\sigma$ such that $\Delta \smallsetminus \{0\} \subset C \cap S^o$. For $t \in \Delta$, let $s(t) \in S$ be the corresponding point in $S$. Write $s(t) = s(A(t), B(t))$, where the map $\Delta \ni t \mapsto (A(t), B(t)) \in \text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6$ is holomorphic. Since $s(0) = \sigma \in \mathcal{R}^0$ and $s(\Delta \smallsetminus \{0\}) \subset S^o$, the flat family $\pi : \mathcal{X} \rightarrow \Delta$ has the following properties:

(i) $X_{s(t)}$ is smooth for $t \in \Delta \smallsetminus \{0\}$ and $\text{Sing}(X_{\sigma})$ consists of nodes.

(ii) $\mathcal{X}^t|_\Delta = X^t_0 \subset \text{Sing}(X_{\sigma})$ and $\#X^t_0 = 1$.

We see that $\mathcal{X}^t|_C$ is smooth on a neighborhood of $X_{\sigma}$ if the curve germ $(C, 0) \subset (S, \sigma)$ is generic. It suffices to prove the smoothness of $\mathcal{X}$ at $\text{Sing}(X_{\sigma})$.

Let $p = (x_0, y_0) \in \text{Sing}(X_{\sigma})$. Set $f_i(x, y; t) := Q(x; A_i(t)) + Q(y; B_i(t))$ for $i = 1, 2, 3$. For $(\widetilde{A}, \widetilde{B}) \in \text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6$, we define

$$J_{p, (\widetilde{A}, \widetilde{B})} = (\partial_{x_i} f_i(x_0, y_0; 0), \partial_{y_i} f_i(x_0, y_0; 0), Q(x_0; \widetilde{A}_i) + Q(y_0; \widetilde{B}_i))_{i,j,k=1,2,3}.$$ 

Since $X_{s(t)}$ is defined by the system of equations $f_i(x, y; t) = Q(x; A_i(t)) + Q(y; B_i(t))$ ($i = 1, 2, 3$) and hence the corresponding Jacobian matrix at $p = (x_0, y_0)$ is given by the $3 \times 7$-matrix $J_{p, (\partial_t A(0), \partial_t B(0))}$, $\mathcal{X}^t|_C$ is smooth at $p = (x_0, y_0)$ if $\text{rank} J_{p, (\partial_t A(0), \partial_t B(0))} = 3$.

Since $p = (x_0, y_0) \in X_{\sigma}$ is a node by (i) and hence $\text{emb}(X_{\sigma}, p) = 3$, we get

$$\text{rank} (\partial_{x_i} f_i(x_0, y_0; 0), \partial_{y_i} f_i(x_0, y_0; 0)) = 2 \tag{3.7}$$

by Jacobi’s criterion. Since $(x_0, y_0) \neq (0, 0)$ and hence the linear map

$$\text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6 \ni (\widetilde{A}, \widetilde{B}) \mapsto (Q(x_0; \widetilde{A}_i) + Q(y_0; \widetilde{B}_i))_{i=1,2,3} \in \mathcal{C}^3$$

is surjective, we deduce from (3.7) that $\text{rank} J_{p, (\widetilde{A}, \widetilde{B})} = 3$ for generic $(\widetilde{A}, \widetilde{B}) \in \text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6$. This implies that the subset of $\text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6$ defined by

$$\mathcal{Q}_p := \{(A, B) \in \text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6; \text{rank} J_{p, (A, B)} < 3\}$$

is a proper Zariski closed subset. Hence $\mathcal{X}$ is smooth at every point of $\text{Sing}(X_{\sigma})$ if $(\partial_t A(0), \partial_t B(0)) \in [\text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6] \setminus \bigcup_{p \in \text{Sing}(X_{\sigma})} \mathcal{Q}_p$. This proves that $\mathcal{X}^t|_C$ is smooth on a neighborhood of $X_{\sigma}$ for a generic curve germ $(C, 0) \subset (S, \sigma)$. 


Step 3. Let \((C, 0) \subset (S, o)\) be a generic curve germ such that \(\mathcal{X}|_C\) is smooth on a neighborhood of \(X_o\). By Step 2 (i), (ii), we may apply Theorem 2.3 to the deformation germ of K3 surfaces with involution \(f: (\mathcal{X}|_C, \iota) \to C\). Then we get

\[
\log \|\Phi\|^2|_C(t) = \frac{1}{2} \log |t|^2 + O(1) \quad (t \to 0).
\]

Since \(\mathcal{R}^o\) is a dense Zariski open subset of \(\mathcal{R}\) by Lemma 3.6(2), we get the following equation of currents on \(S^o \cup D^o\) by (2.2), (3.5), (3.8)

\[
-dd^c \log \left(\omega^*\|\Phi\|^2\right) = 4\omega^* \kappa - \frac{1}{2} \delta_{\mathcal{R}^o}.
\]

Comparing (3.6) and (3.9), we get the following equation of currents on \(S^o \cup D^o\)

\[
-dd^c \log \left(\omega^*\|\Phi\|^2/\|\Xi\|_{L/4}\right) = 0,
\]

which implies that \(\partial \log (\omega^*\|\Phi\|^2/\|\Xi\|_{L/4})\) is a holomorphic 1-form on \(S^o \cup D^o\). Since \(\dim [S \setminus (S^o \cup D^o)] = \dim E \leq \dim S - 2\) by Lemma 3.2(2), it follows from the Hartogs extension principle that \(\partial \log \omega^*\|\Phi\|^2/\|\Xi\|_{L/4}\) extends to a holomorphic 1-form on the Grassmann variety \(S^o\). Since \(H^0(S, \Omega^1_S) = 0\) by the rationality of \(S\), we get \(\partial \log (\omega^*\|\Phi\|^2/\|\Xi\|_{L/4}) = 0\) on \(S^o\). Since \(\omega^*\|\Phi\|^2/\|\Xi\|_{L/4}\) is real-valued, we get \(d \log (\omega^*\|\Phi\|^2/\|\Xi\|_{L/4}) = 0\) on \(S^o\). This proves the existence of a constant \(c \in \mathbb{R}\) such that the following equality of functions on \(S^o\) holds

\[
\omega^*\|\Phi\|^2 = c (2\pi^{-4})^4 \|\Xi\|_{L/4}.
\]

Restricting (3.10) to a certain subset of \(S\), we get \(c = 1\) by Theorem 7.5 below. 

Let \(\lambda\) be the Hodge bundle on \(\mathcal{M}\). By Lemma 2.1, \(\Phi^2\) is a holomorphic section of \(\lambda^{\otimes 8}\). Regarding \(\Phi^2 \in H^0(\mathcal{M}, \lambda^{\otimes 8})\), we get \(\text{div}(\Phi^2) = \overline{D}\). To emphasize, denote by \(\| \cdot \|_{\text{Pet}}\) the Petersson norm on \(\lambda\). Theorem 3.7 implies the following.

**Corollary 3.8.** There exists a \(\text{GL}(\mathbb{C}^6)^t\)-equivariant holomorphic isometry

\[
f: (\omega^* (\lambda^{\otimes 8}), \omega^* \| \cdot \|_{\text{Pet}}^{\otimes 8}) \cong \left(\pi_*(K^{\otimes 8}_{\mathcal{X}/S}), \| \cdot \|_{L/4}\right)
\]

of holomorphic Hermitian line bundles on \(S^o \cup D^+\) such that

\[
f(\omega^* \Phi^2) = (2\pi^{-4})^4 \Xi.
\]

**Proof.** By Theorem 3.7, \((\omega^* \Phi^2)^{-1} \otimes \Xi\) is a nowhere vanishing \(\text{GL}(\mathbb{C}^6)^t\)-invariant holomorphic section of \(\omega^* (\lambda^{\otimes 8})^{-1} \otimes \pi_*(K^{\otimes 8}_{\mathcal{X}/S})\) on \(S^o \cup D^+\). Defining \(f(\xi) := (\omega^* \Phi^2)^{-1} \otimes \Xi \otimes \xi\), we see that \(\omega^* (\lambda^{\otimes 8}) \cong \pi_*(K^{\otimes 8}_{\mathcal{X}/S})\) on \(S^o \cup D^+\) via \(f\). The desired equality follows from Theorem 3.7. 

\(\square\)
Proof of Theorem 3.1. If \((A, B) \in \text{Sym}(3, C) \otimes C^6\) is admissible, then \(s(A, B) \in S^0 \cup D^+\) and \(X_{(A,B)} = X_{s(A,B)}\). Theorem 3.1 follows from Theorem 3.7. \(\square\)

4. An algebraic expression of the Borcherds \(\Phi\)-function. In this section, we fix the following notation. Let \((A, B) \in \text{Sym}(3, C) \otimes C^6\) be admissible. Hence \(s(A, B) \in S^0 \cup D^+\). Let \(v \in H^2(X_{(A,B)}, Z)\) be a primitive isotropic vector of level \(\ell\). Here the level of \(v\) is the positive generator of \(\langle v, A \rangle \subset Z\). Then \(\ell = 1\) or \(\ell = 2\) by [24, Prop. 4.5]. Let \(v' \in H^2(X_{(A,B)}, Z)\) be another primitive isotropic vector of level \(\ell\) such that \(\langle v, v' \rangle = \ell\) and set

\[
\mathcal{L}_{v, v'} := (Zv + Zv')^\perp_{H^2(X_{(A,B)}, Z)}.
\]

We get the orthogonal decomposition \(H^2(X_{(A,B)}, Z) = (Zv + Zv') \oplus \mathcal{L}_{v, v'}\), from which \(\mathcal{L}_{v, v'}\) is 2-elementary with \(\mathcal{L}_{v, v'} \cong \mathbb{M}_\ell = \mathbb{U}(2/\ell) \oplus \mathbb{E}_8(2)\). We define

\[
(-1)^{2/\ell} z_{A,B,v,v'} := \frac{\omega(A,B) - \langle \omega(A,B), v'/\ell \rangle v - \langle \omega(A,B), v \rangle (v'/\ell)}{\langle \omega(A,B), v \rangle}.
\]

Then \(z_{A,B,v,v'} \in \mathcal{L}_{v, v'} \otimes \mathbb{R} + i\mathcal{C}_{\mathcal{L}_{v, v'}}\), and the following equality holds:

\[
\omega(A,B)/\langle \omega(A,B), v \rangle = -\frac{(z_{A,B,v,v'})^2}{2} + \frac{v'}{\ell} + (-1)^{2/\ell} z_{A,B,v,v'}.
\]

Let \(\alpha_{v,v'} : \mathcal{L}_{v, v'} \cong \mathbb{M}_\ell\) be an isometry of lattices, called a marking of \(\mathcal{L}_{v, v'}\), such that \(\alpha_{v,v'}(z_{A,B,v,v'}) \in \mathbb{M}_\ell \otimes \mathbb{R} + i\mathcal{C}_{\mathbb{M}_\ell}\). (If \(\alpha_{v,v'}(z_{A,B,v,v'}) \in \mathbb{M}_\ell \otimes \mathbb{R} - i\mathcal{C}_{\mathbb{M}_\ell}\), then we replace \(\alpha_{v,v'}\) by \(-\alpha_{v,v'}\).)

Recall that \(\{\varepsilon_\ell, f_\ell\}\) is a basis of \(\mathbb{U}(\ell)\) with \(\varepsilon_\ell^2 = f_\ell^2 = 0\) and \(\langle \varepsilon_\ell, f_\ell \rangle = \ell\). We extend \(\alpha_{v,v'}\) to an isometry \(\tilde{\alpha}_{v,v'} : H^2(X_{(A,B)}, Z) \cong A\) by setting

\[
\tilde{\alpha}_{v,v'}(mv + nv' + x) := m\varepsilon_\ell + n\varepsilon_\ell + \alpha_{v,v'}(x),
\]

where \(m, n \in \mathbb{Z}\), \(x \in \mathcal{L}_{v, v'}\). Hence \(\tilde{\alpha}_{v,v'}(v) = \varepsilon_\ell\) and \(\tilde{\alpha}_{v,v'}(v') = f_\ell\). Since

\[
\tilde{\alpha}_{v,v'}(\omega(A,B))/\langle \omega(A,B), v \rangle = -\frac{(\alpha_{v,v'}(z))^2}{2}\varepsilon_\ell + \frac{f_\ell}{\ell} + (-1)^{2/\ell}\alpha_{v,v'}(z) = i_\ell(\alpha_{v,v'}(z)),
\]

\(\alpha_{v,v'}(z_{A,B,v,v'})\) is the period of the marked Enriques surface \((Y_{(A,B), \tilde{\alpha}_{v,v'}})\) under the isomorphism (2.3). In this section, we prove the following theorem.

**Theorem 4.1.** Let \(v' \in H_2(X_{(A,B)}, Z)\) be the Poincaré dual of \(v\). Then the following equality holds:

\[
\Phi_\ell(\alpha_{v,v'}(z_{A,B,v,v'}))^2 = R(A)R(B) \left( \frac{2}{\pi^2} \int_{v'} \omega(A,B) \right)^8.
\]

For the proof, we need some intermediary results.
LEMMA 4.2. The following equality holds:

\[ \Phi_{\ell}(\alpha_{v,v'}(z_{A,B},v'))^2 = \left| R(A)R(B) \right| \cdot \left| \frac{2}{\pi^2} \int_{\mathcal{V}} \omega(A,B) \right|^8. \]  

In particular, there exists a unique angle \( \theta_{A,B,v,v',\alpha_{v,v'}} \in \mathbb{R}/2\pi\mathbb{Z} \) such that

\[ \Phi_{\ell}(\alpha_{v,v'}(z_{A,B},v'))^2 = e^{i\theta_{A,B,v,v',\alpha_{v,v'}}} R(A)R(B) \cdot \left( \frac{2}{\pi^2} \int_{\mathcal{V}} \omega(A,B) \right)^8. \]

**Proof.** For simplicity, write \( z := \alpha_{v,v'}(z_{A,B},v') \in \mathbb{M}_\ell \otimes \mathbb{R} + i\mathcal{C}^+_{\mathbb{M}_\ell} \). By the definition of the Petersson norm (cf. Section 2.2), we get

\[ \| \Phi(Y_{A,B}) \| = \left\langle \mathcal{S} z, \mathcal{S} z \right\rangle_{\mathbb{M}_\ell}^2 |\Phi_{\ell}(z)|. \]

By (2.6) and the equality \( \left\langle \omega(A,B), v \right\rangle = \int_{\mathcal{V}} \omega(A,B) \), we get

\[ \int_{X(A,B)} \omega(A,B) \wedge \overline{\omega(A,B)} = \left| \left\langle \omega(A,B), v \right\rangle \right|^2 \left( \int_{X(A,B)} \frac{\omega(A,B)}{\left\langle \omega(A,B), v \right\rangle} \wedge \frac{\overline{\omega(A,B)}}{\left\langle \omega(A,B), v \right\rangle} \right) \]

\[ = \left| \left\langle \omega(A,B), v \right\rangle \right|^2 \left\langle \ell_{\ell}(z), \overline{\ell_{\ell}(z)} \right\rangle_{\Lambda} \]

\[ = 2\left\langle \mathcal{S} z, \mathcal{S} z \right\rangle_{\mathbb{M}_\ell} \left| \int_{\mathcal{V}} \omega(A,B) \right|^2. \]

Substituting (4.3) and (4.4) into the formula in Theorem 3.1, we get (4.1). \( \square \)

**LEMMA 4.3.** Let \( v'' \in H^2(X_{A,B},\mathbb{Z}) \) be another primitive isotropic vector of level \( \ell \) such that \( \left\langle v, v'' \right\rangle = \ell \) and let \( \alpha_{v,v''} : \Sigma_{v,v''} \cong \mathbb{M}_\ell \) be an isometry of lattices such that \( \alpha_{v,v''}(z_{A,B},v') \in \mathbb{M}_\ell \otimes \mathbb{R} + i\mathcal{C}^+_{\mathbb{M}_\ell} \). Then

\[ \Phi_{\ell}(\alpha_{v,v''}(z_{A,B},v'))^2 = \Phi_{\ell}(\alpha_{v,v''}(z_{A,B},v'))^2. \]

**Proof.** For simplicity, write \( z_{v,v'} \) (resp. \( z_{v,v''} \)) for \( z_{A,B,v,v'} \) (resp. \( z_{A,B,v,v''} \)). Set \( g_{v,v''} := \tilde{\alpha}_{v,v} \circ \tilde{\alpha}_{v,v''}^{-1} \in O^+(\Lambda) \). Since

\[ \tilde{\alpha}_{v,v'}(\omega(A,B)) = g_{v,v''}(\tilde{\alpha}_{v,v''}(\omega(A,B))), \]

\[ \tilde{\alpha}_{v,v'}(\omega(A,B))/\left\langle \omega(A,B), v \right\rangle = \ell_{\ell}(\alpha_{v,v'}(z_{v,v})), \]

\[ \tilde{\alpha}_{v,v'}(\omega(A,B))/\left\langle \omega(A,B), v \right\rangle = \ell_{\ell}(\alpha_{v,v''}(z_{v,v''})) \]

and hence \( g_{v,v''}(\ell_{\ell}(\alpha_{v,v''}(z_{v,v''}))) = \ell_{\ell}(\alpha_{v,v''}(z_{v,v''})) \), it follows from the definition (2.4) of the \( O^+(\Lambda) \)-action on \( \mathbb{M}_\ell \otimes \mathbb{R} + i\mathcal{C}^+_{\mathbb{M}_\ell} \) that

\[ \alpha_{v,v'}(z_{v,v'}) = g_{v,v''} \cdot \alpha_{v,v''}(z_{v,v''}). \]
Since $g_{\nu',\nu'}(e_\ell) = e_\ell$, $g_{\nu',\nu'}$ induces an isometry $\overline{g}_{\nu',\nu'} \in O(e_\ell^+ / Z \epsilon_\ell)$ of the lattice $e_\ell^+ / Z \epsilon_\ell \cong \mathbb{M}_\ell$. Since $f_\ell / e_\ell \in \mathbb{M}_\ell'$, there exists $\lambda \in \mathbb{M}_\ell'$ such that

$$g_{\nu',\nu'} \left( m \epsilon_\ell + n \frac{f_\ell}{e_\ell} + x \right) = \left( m - n \frac{(\lambda)^2}{2} - \langle \lambda, \overline{g}_{\nu',\nu'}(x) \rangle \right) \epsilon_\ell + n \frac{f_\ell}{e_\ell} + \overline{g}_{\nu',\nu'}(x) + n \lambda$$

for all $m, n \in \mathbb{Z}$ and $x \in \mathbb{M}_\ell$. By this expression and (2.5), we get $j_\ell(g_{\nu',\nu'}, z) = 1$ for all $z \in \mathbb{M}_\ell \otimes \mathbb{R} + i Q^+$. By (2.7) and Lemma 2.1, we get

$$\Phi_\ell(g_{\nu',\nu'}, z)^2 = \Phi_\ell(z)^2.$$  

Since $\Phi_\ell(\alpha_{\nu',\nu'}(z_{\nu',\nu'})) = \Phi_\ell(g_{\nu',\nu'} \cdot \alpha_{\nu',\nu'}(z_{\nu',\nu'}))^2$, we get the desired equality

$$\Phi_\ell(\alpha_{\nu',\nu'}(z_{\nu',\nu'}))^2 = \Phi_\ell(\alpha_{\nu',\nu'}(z_{\nu',\nu'}))^2. \quad \square$$

After Lemma 4.3 and (4.2), $\theta_{\nu,v',\alpha_{\nu,v}} \in \mathbb{R} / 2\pi \mathbb{Z}$ depends only on $(A, B, v)$. Set

$$c(A, B, v) := e^{2 \pi i \theta_{\nu,v',\alpha_{\nu,v}}} = \Phi_\ell(\alpha_{\nu',\nu'}(z_{A,B,v,v}))/R(A)R(B) (2\pi^{-2} \int_{v'} \omega_{A,B})^8.$$

**Lemma 4.4.** Let $w \in H^2(X_{(A,B)}, \mathbb{Z})_-$ be an arbitrary primitive isotropic vector of level $\ell$. Then

$$c(A, B, v) = c(A, B, w).$$

In particular, $c(A, B, v)$ depends only on $(A, B)$ and $\ell$.

**Proof.** Let $w' \in H^2(X_{(A,B)}, \mathbb{Z})_-$ be a primitive isotropic vectors of level $\ell$ with $\langle w, w' \rangle = \ell$. For simplicity, write $z_{v,v'}$ (resp. $z_{w,w'}$) for $z_{A,B,v,v}$ (resp. $z_{A,B,w,w}$). Let $\alpha_{w,w'}: \mathbb{L}_{w,w} \cong \mathbb{M}_\ell$ be a marking and set $g_{v,v';w,w'} := \overline{\alpha}_{v,v} \circ \overline{\alpha}^{-1}_{w,w'} \in O^+(\Lambda)$. Since

$$\overline{\alpha}_{v,v'}(\omega_{A,B}) = g_{v,v';w,w'}(\overline{\alpha}_{w,w'}(\omega_{A,B}));$$

$$\overline{\alpha}_{v,v'}(\omega_{A,B})/\langle \omega_{A,B}, v \rangle = \ell_\ell (\alpha_{v,v'}(z_{v,v'}));$$

$$\overline{\alpha}_{w,w'}(\omega_{A,B})/\langle \omega_{A,B}, w \rangle = \ell_\ell (\alpha_{w,w'}(z_{w,w'}))$$

and hence $g_{v,v';w,w'}(\ell_\ell (\alpha_{w,w'}(z_{w,w'}))) = \ell_\ell (\alpha_{v,v'}(z_{v,v'}))$, it follows from the definition (2.4) of the $O^+(\Lambda)$-action on $\mathbb{M}_\ell \otimes \mathbb{R} + i C^+_\mathbb{M}_\ell$ that $z_{v,v'} = g_{v,v';w,w'} \cdot z_{w,w'}$. By (2.5), the automorphic factor for the Borcherds $\Phi$-function $\Phi_\ell$ is expressed as

$$j_\ell(g, [\eta]) = \langle g(\eta), e_\ell \rangle / \langle \eta, e_\ell \rangle, \quad [\eta] \in \Omega^+_A, \quad g \in O^+(\Lambda)$$
under the identification $\Omega^+_A \cong \mathbb{M}_\ell \otimes \mathbb{R} + i\mathcal{C}^+_\ell$ given by (2.3). We get

$$\mathfrak{j}_\ell \left( g_{v,v',w,w'} \left[ (\bar{\alpha}_{w,w'}(\omega_{(A,B)})) \right] \right) = \frac{\langle g_{v,v',w,w'}(\bar{\alpha}_{w,w'}(\omega_{(A,B)})), \epsilon_\ell \rangle}{\langle \bar{\alpha}_{w,w'}(\omega_{(A,B)}), \epsilon_\ell \rangle} = \frac{\langle \bar{\alpha}_{v,v'}(\omega_{(A,B)}), \epsilon_\ell \rangle}{\langle \bar{\alpha}_{w,w'}(\omega_{(A,B)}), \epsilon_\ell \rangle} = \frac{\langle \omega_{(A,B)}, \nu \rangle}{\langle \omega_{(A,B)}, \nu \rangle}.$$  

By Lemma 2.1 and the automorphic property of $\Phi^2_\ell$, we get

$$\Phi_\ell \left( \alpha_{v,v'}(z_{v,v'}) \right)^2 = \Phi_\ell \left( g_{v,v',w,w'} \cdot z_{w,w} \right)^2 = \mathfrak{j}_\ell \left( g_{v,v',w,w'} \left[ (\bar{\alpha}_{w,w'}(\omega_{(A,B)})) \right] \right) \Phi_\ell \left( \alpha_{w,w'}(z_{w,w}) \right)^2 = \left( \frac{\langle \omega_{(A,B)}, \nu \rangle}{\langle \omega_{(A,B)}, \nu \rangle} \right)^8 \mathfrak{c}(A,B,\nu) R(A) R(B) \left( \frac{2}{\pi^2} \int_{\omega_{(A,B)}} \right)^8 = \mathfrak{c}(A,B,\nu) R(A) R(B) \left( \frac{2}{\pi^2} \int_{\omega_{(A,B)}} \right)^8.$$  

Comparing this with the definition of $\mathfrak{c}(A,B,\nu)$, we get $\mathfrak{c}(A,B,\nu) = \mathfrak{c}(A,B,\nu)$. □

After Lemma 4.4 and the relations in the proof of Lemma 3.4, one can define a function $\mathfrak{c}_\ell(\cdot)$ on $S^0$ by

$$\mathfrak{c}_\ell \left( s(A,B) \right) := \mathfrak{c}(A,B,\nu).$$

**Lemma 4.5.** The function $\mathfrak{c}_\ell(\cdot)$ on $S^0$ is constant. In particular, there exists a constant $\mathfrak{c}_\ell$ with $|\mathfrak{c}_\ell| = C$ such that

$$\Phi_\ell \left( \alpha_{v,v'}(z_{A,B,v,v'}) \right)^2 = \mathfrak{c}_\ell R(A) R(B) \left( \frac{2}{\pi^2} \int_{\nu} \omega_{(A,B)} \right)^8$$

for all $(A,B)$ with $s(A,B) \in S^0$ and primitive isotropic vector $\nu \in H^2(X_{(A,B)}, \mathbf{Z})_{\text{red}}$ of level $\ell$. Here $C$ is the constant in (3.10).

**Proof:** Since $S^0$ is connected, it suffices to show that $\mathfrak{c}_\ell(\cdot)$ is a locally constant function on $S^0$. Let $s_0 \in S^0$. There is a small neighborhood $\mathcal{U}$ of $s_0$ in $S^0$ such that the family of K3 surfaces $\pi : \mathcal{X}|_\mathcal{U} \to \mathcal{U}$ is topologically trivial. By the topological triviality, we can choose $v, v'$ and $\alpha_{v,v'}$ to be constant under the identification of the fiber $X_s = \pi^{-1}(s)$ with $X_{s_0}$. Choosing $\mathcal{U}$ sufficiently small, we may assume that there exists a holomorphic section $\sigma : \mathcal{U} \ni u \to (A(u), B(u)) \in \text{Sym}(3, \mathcal{C}) \otimes \mathcal{C}^6$ such that $s(A(u), B(u)) = u$ for all $u \in \mathcal{U}$. Then both $\Phi_\ell \left( \alpha_{v,v'}(z_{A(u),B(u),v,v'}) \right)^2$ and $R(A(u)) R(B(u)) \left( \frac{2\pi^2}{\nu} \int_{\nu} \omega_{(A(u),B(u))} \right)^8$ are holomorphic functions on $\mathcal{U}$. As a result, $\mathfrak{c}_\ell(u) := \mathfrak{c}(A(u), B(u), \nu)$ is a holomorphic function on $\mathcal{U}$. Since
Proposition 4.7 for $\omega$ is a certain integer. Then there exists a simultaneous resolution map such that $s$ is a primitive isotropic vector of level $\ell$ with compact support in $\widetilde{X}_{(C,D)} \setminus p^{-1}(\text{Sing} X_{(C,D)})$. Let $w \in H^2(\widetilde{X}_{(C,D)}, \mathbb{Z})_-$ be another primitive isotropic vector of level $\ell$ with $\langle w, w' \rangle = \ell$. Then the following equality holds

$$
(4.5) \quad \Phi_\ell(\alpha_{w,w'}(z_{C,D,w}w'))^2 = c_\ell R(C)R(D) \left( \frac{2}{\pi^2} \int_{w'} p^*\omega_{(C,D)} \right)^8.
$$

Proof: Let $\gamma \colon \Delta \ni t \to (A(t), B(t)) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^6$ be a holomorphic map such that $s(A(t), B(t)) \in S^\circ$ for $t \neq 0$ and $(A(0), B(0)) = (C,D)$. Let $\pi \colon \mathcal{X} \times_S \Delta \to \Delta$ be the family of $K3$ surfaces induced from $\pi \colon \mathcal{X} \to S$ by $\gamma$. Then $\pi^{-1}(t) = X(A(t), B(t))$ for $t \in \Delta$. Let $\omega$ be the holomorphic section of the relative dualizing sheaf $K_{\mathcal{X} \times_S \Delta/\Delta}$ such that $\omega(t) = \omega(A(t), B(t))$ for $t \in \Delta$. There exists a simultaneous resolution $\pi' \colon \mathcal{X}' \to \Delta$ of the family $\pi \colon \mathcal{X} \times_S \Delta \to \Delta$ such that $(\pi')^{-1}(t) = X(A(t'), B(t'))$ for $t \neq 0$ and $(\pi')^{-1}(0) = \widetilde{X}_{(C,D)}$, where $\nu$ is a certain integer. Then $\omega$ induces a nowhere vanishing holomorphic section $\omega'$ of the relative dualizing sheaf of the family $\pi' \colon \mathcal{X}' \to \Delta$ such that $\omega'(t) = \omega(A(t'), B(t'))$ for $t \neq 0$ and $\omega'(0) = p^*\omega_{(C,D)}$. Since the family $\pi \colon \mathcal{X}' \to \Delta$ is topologically trivial, $w$ and $w'$ can be regarded as 2-cocycles of $(\pi')^{-1}(t)$ for all $t \in \Delta$. Hence the functions $f(t) := \Phi_\ell(\alpha_{w,w'}(z_{A(t),B(t),w}w'))^2$ and $g(t) := R(A(t))R(B(t)) \left( \frac{2}{\pi^2} \int_{w'} \omega_{A(t),B(t)} \right)^8$ extends to holomorphic functions on $t$ such that $f(0) = \Phi_\ell(\alpha_{w,w'}(z_{C,D,w}w'))^2$ and $g(0) = R(C)R(D) \left( \frac{2}{\pi^2} \int_{w'} p^*\omega_{(C,D)} \right)^8$. Since $f(t)/g(t) = \epsilon_{\ell}t$ for $t \neq 0$ by Lemma 4.5, we get $f(0)/g(0) = \epsilon_{\ell}$. This proves the result.

**Proposition 4.7.** There exists a non-zero constant $\epsilon_{\ell}$ with $|\epsilon_{\ell}| = \mathcal{C}$ such that

$$
\Phi_\ell(\alpha_{\nu,\nu'}(z_{A,B,\nu,\nu'}))^2 = \epsilon_{\ell} R(A)R(B) \left( \frac{2}{\pi^2} \int_{\nu'} p^*\omega_{(A,B)} \right)^8
$$

for all admissible $(A,B) \in \text{Sym}(3, \mathbb{C}) \otimes \mathbb{C}^6$ and primitive isotropic vector $\nu \in H^2(\widetilde{X}_{(A,B)}, \mathbb{Z})_-$ of level $\ell$ with compact support in $\widetilde{X}_{(A,B)} \setminus p^{-1}(\text{Sing} X_{(A,B)})$.

**Proof.** The result follows from Lemmas 4.5 and 4.6.

**Proof of Theorem 4.1.** In Theorem 7.5 below, we shall prove that $\epsilon_{\ell} = 1$ in Proposition 4.7 for $\ell = 1,2$. The result follows from Proposition 4.7.
5. Borcherds $\Phi$-function and theta function. In this section, we give a new relation between the Borcherds $\Phi$-function and certain theta functions closely related to the configuration space $X^o(3, 6)$.

5.1. The configuration space $X^o(3, 6)$. Identify $\lambda \in (\mathbb{C}^*)^6$ with $\text{diag}(\lambda) \in \text{GL}(\mathbb{C}^6)$ and consider the $\text{GL}(\mathbb{C}^3) \times (\mathbb{C}^*)^6$-action on $M_{3, 6}(\mathbb{C})$ defined by $(g, \lambda) \cdot N := gN\lambda^2$. We define

$$X^o(3, 6) := \text{GL}(\mathbb{C}^3) \setminus M_{3, 6}^o(\mathbb{C})/(\mathbb{C}^*)^6.$$ 

By [26, p. 149], $X^o(3, 6)$ is a Zariski open subset of $\mathbb{C}^4$. The image of $N \in M^o(3, 6)$ in $X^o(3, 6)$ is denoted by $[N]$. We consider the following group actions on $X^o(3, 6)$:

The symmetric group $S_6$ acts on $M(3, 6)$ by $N^{\sigma} := (n_{\sigma(1)}, \ldots, n_{\sigma(6)})$ for $N = (n_1, \ldots, n_6) \in M(3, 6)$ and $\sigma \in S_6$. This $S_6$-action descends to an $S_6$-action on $X^o(3, 6)$. Following [26, Chap. VII, Sect. 3], we define for $N = (N_1, N_2) \in M_{3, 6}^o(\mathbb{C})$

$$N^\vee := (tN_1^{-1}, tN_2^{-1}) \in M_{3, 6}^o(\mathbb{C}).$$

The involution $\vee: M^o(3, 6) \ni N \to N^\vee \in M^o(3, 6)$ on $M^o(3, 6)$ descends to an involution $\vee: X^o(3, 6) \to X^o(3, 6)$. By [26, Chap. 7 Prop.3.3], the actions of $S_6$ and $\vee$ on $X^o(3, 6)$ commute. We refer to [26, Chap. 7–9] for more details about $X^o(3, 6)$.

5.2. Ten families of Enriques surfaces parametrized by $X^o(3, 6)$. In this subsection, we use the coordinates $(x_1, \ldots, x_6)$ instead of $(x_1, x_2, x_3, y_1, y_2, y_3)$. For $N = (n_{ij}) \in M_{3, 6}^o(\mathbb{C})$, we set $f_i(x; N) = \sum_{j=1}^{6} n_{ij}x_j^2$. Recall that

$$X_N = \{(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \in \mathbb{P}^5; f_1(x; N) = f_2(x; N) = f_3(x; N) = 0\}.$$ 

If $N = N'$ in $X^o(3, 6)$, then $X_N \cong X_{N'}$.

In this section, the holomorphic 2-form on $X_N$ defined as the residue of $f_1(x; N)$, $f_2(x; N)$, $f_3(x; N)$ is denoted by $\omega_N \in H^0(X_N, K_{X_N})$. For a later use, let us give an explicit formula for $\omega_N$. Setting $x_3 = 1$, we consider the affine coordinates $(x_1, x_2, x_4, x_5, x_6)$ of $\mathbb{P}^5$. Set

$$N = \begin{pmatrix}
n_{11} & n_{12} & n_{13} & 1 & 0 & 0 \\
n_{21} & n_{22} & n_{23} & 0 & 1 & 0 \\
n_{31} & n_{32} & n_{33} & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{cases} 
\ f_1 = n_{11}x_1^2 + n_{12}x_2^2 + n_{13} + x_4^2 \\
\ f_2 = n_{21}x_1^2 + n_{22}x_2^2 + n_{23} + x_5^2 \\
\ f_3 = n_{31}x_1^2 + n_{32}x_2^2 + n_{33} + x_6^2.
\end{cases}$$
Since \( df_1 \wedge df_2 \wedge df_3 = 2^3 x_4 x_5 x_6 \, dx_4 \wedge dx_5 \wedge dx_6 + \cdots \), we deduce from the relation
\[
\omega = 2^{-3} \frac{dx_1 \wedge dx_2}{x_4 x_5 x_6} = (2i)^{-3} \frac{dx_1 \wedge dx_2}{\sqrt{\prod_{i=1}^{3}(n_i x_i^2 + n_i x_i^2 + n_i^3)}} \mod dx_4, dx_5, dx_6.
\]

By the definition \( \omega_N = \gamma|_{X_N} \), we get the expression on \( X_N \setminus \{x_4 x_5 x_6 = 0\} \)
\[
(5.1) \quad \omega_N = \frac{dx_1 \wedge dx_2}{2^3 x_4 x_5 x_6} = (2i)^{-3} \frac{dx_1 \wedge dx_2}{\sqrt{\prod_{i=1}^{3}(n_i x_i^2 + n_i x_i^2 + n_i^3)}}.
\]

For \( J = \{j_1, j_2, j_3\} \subset \{1, \ldots, 6\} \) with \( j_1 < j_2 < j_3 \), let \( \langle J \rangle \) denote the partition
\( \{1, \ldots, 6\} = J \amalg J^c \). Write \( J^c = \{j_4, j_5, j_6\}, j_4 < j_5 < j_6 \). Then \( \langle J \rangle \) is also written as \( \{j_1, j_2, j_3\} \). Hence \( \langle J \rangle = \{j_1, j_2, j_3\} \). For a partition \( \langle J \rangle = \{j_1, j_2, j_3\} \) and \( N \in M(3, 6) \), we set \( \Delta_{\langle J \rangle}(N) := \Delta_{j_1 j_2 j_3}(N) \Delta_{j_4 j_5 j_6}(N) \). We define an involution \( \iota_{\langle J \rangle} \) on \( \mathbb{P}^5 \) by
\[
\iota_{\langle J \rangle}(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}, x_{j_5}, x_{j_6}) := (x_{j_1}, x_{j_2}, x_{j_3}, -x_{j_4}, -x_{j_5}, -x_{j_6}).
\]

Then \( \iota_{\langle J \rangle} \) acts on \( X_N \). For \( N \in M^o_{3, 6}(\mathbb{C}) \), we define an Enriques surface \( Y_{N,\langle J \rangle} \) by
\[
Y_{N,\langle J \rangle} := X_N / \iota_{\langle J \rangle}.
\]

Then \( Y_{N,\langle J \rangle} \cong Y_{N',\langle J \rangle} \) if \( [N] = [N'] \). We define a map \( \overline{\omega}_{\langle J \rangle} : X^o(3, 6) \to \mathcal{M}^o \) by
\[
\overline{\omega}_{\langle J \rangle}([N]) := \overline{\omega}(Y_{N,\langle J \rangle}).
\]

For all \( [N] \in X^o(3, 6) \), we have
\[
(\overline{\omega}_{\langle J \rangle}(\Phi)([N])) = \|\Phi(Y_{N,\langle J \rangle})\|.
\]

**Theorem 5.1.** For all \( N \in M^o_{3, 6}(\mathbb{C}) \) and for all partitions \( \langle J \rangle \),
\[
\|\Phi(Y_{N,\langle J \rangle})\| = |\Delta_{\langle J \rangle}(N)|^2 \left( \frac{2}{\pi^4} \left| \int_{X_N} \omega_N \wedge \omega_N \right| \right)^2.
\]

**Proof.** The symmetric group \( S_6 \) acts on \( \mathbb{P}^5 \) by \( \sigma(x_i) = x_{\sigma(i)}, \, (1 \leq i \leq 6) \). There is \( \sigma \in S_6 \) with \( \sigma^{-1} \sigma = \iota_{\langle J \rangle} \). Since \( \sigma(X_N) = X_{N^\sigma} \) and \( \sigma^* \omega_{N^\sigma} = \omega_N \), we get
\[
|\Delta_{\langle 123 \rangle}(N^\sigma)|^2 \left( \frac{2}{\pi^4} \left| \int_{X_{N^\sigma}} \omega_{N^\sigma} \wedge \omega_{N^\sigma} \right| \right)^2 = |\Delta_{\langle J \rangle}(N)|^2 \left( \frac{2}{\pi^4} \left| \int_{X_N} \omega_N \wedge \omega_N \right| \right)^2.
\]

Since \( R(Q) = \det(A)^4 \) for \( Q = (Q_1, Q_2, Q_3), \, Q_i(x) = \sum_{j=1}^{3} a_{ij} x_j^2 \) and since
\( Y_{N,\langle J \rangle} = Y_{N^\sigma} \), the left-hand side is equal to \( \|\Phi(Y_{N^\sigma})\| \) by Theorem 3.1. \( \Box \)
5.3. Borcherds $\Phi$-function and theta function. We consider another families of $K3$ surfaces over $X^0(3,6)$ (cf. [26, 19]). For $N = (n_{ij}) \in M_{3,6}^o(\mathbb{C})$, define

$$Z_N := \{((z_1 : z_2 : z_3), w) \in \mathcal{O}_{\mathbb{P}^2}(3); \ w^2 = \prod_{i=1}^6 (n_{1i}z_1 + n_{2i}z_2 + n_{3i}z_3)\},$$

where $w$ denotes the coordinate of the fibers of $\mathcal{O}_{\mathbb{P}^2}(3)$. When $N \in M_{3,6}^o(\mathbb{C})$, $Z_N$ is the singular $K3$ surface with 15 nodes defined as the double covering of $\mathbb{P}^2$, whose branch divisor is the union of 6 lines $n_{1i}z_1 + n_{2i}z_2 + n_{3i}z_3 = 0$ ($i = 1, \ldots, 6$). Let $\pi : W_N \to Z_N$ be the minimal resolution of $Z_N$. Then $W_N$ is a smooth $K3$ surface. By construction, $W_N \cong W_{N'}$ if $[N] = [N']$ in $X^0(3,6)$.

We define a holomorphic section of the dualizing sheaf of $Z_N$ by

$$\eta_N := \frac{z_1 dz_2 \wedge dz_3 - z_2 dz_1 \wedge dz_3 + z_3 dz_1 \wedge dz_2}{w}.$$  

On the open subset $Z_N \cap \{z_3 \neq 0\}$, we get the expression

$$\eta_N = \frac{dy_1 \wedge dy_2}{\sqrt{\prod_{i=1}^6 (n_{1i}y_1 + n_{2i}y_2 + n_{3i})}} \big|_{Z_N \cap \{z_3 \neq 0\}},$$

where $(y_1, y_2)$ is the inhomogeneous coordinates of $\mathbb{P}^2$.

Let $H_2(W_N, Z)_-$ be the anti-invariant part of $H_2(W_N, Z)$ with respect to the involution induced by the one $(z, w) \mapsto (z, -w)$ on $Z_N$. In [19, Sect. 3.1], Matsumoto-Terasoma constructed a basis $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_{12}, \gamma'_{13}, \gamma'_{14}, \gamma'_{23}, \gamma'_{24}, \gamma'_{34}\}$ of $H^2(W_N, Z)_-$. (See also Sect. 6.2 below.) Let $2H$ be the Gram matrix of the intersection form on $H^2(W_N, Z)_-$ with respect to $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_{12}, \gamma'_{13}, \gamma'_{14}, \gamma'_{23}, \gamma'_{24}, \gamma'_{34}\}$. By [19, Prop. 3], we have

$$\left(\int_{W_N} \gamma'_{ij} \wedge \gamma'_{kl}\right)_{1 \leq i < j \leq 4, 1 \leq k < l \leq 4} = 2H, \quad H := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We set $\eta_{ij}(N) := \langle \gamma'_{ij}, \eta_N \rangle / 2$ for $1 \leq i < j \leq 4$ and

$$\eta(N) := (\eta_{12}(N), \eta_{13}(N), \eta_{14}(N), \eta_{23}(N), \eta_{24}(N), \eta_{34}(N)) \in \mathbb{C}^6.$$

Since $-\eta_N = \eta_{34}(N)\gamma'_{12} + \eta_{13}(N)\gamma'_{13} + \eta_{23}(N)\gamma'_{14} + \eta_{42}(N)\gamma'_{23} + \eta_{24}(N)\gamma'_{24} + \eta_{12}(N)\gamma'_{34}$, it follows from the Riemann-Hodge bilinear relations that

$$\eta(N)H^t\eta(N) = 0, \quad \int_{W_N} \eta_N \wedge \overline{\eta}_N = \eta(N) \cdot 2H \cdot \overline{\eta}(N).$$
Following [19, Eq. (15)], we define the normalized period of \( W_N \) by

\[
\Omega_N := \frac{1}{\eta_{34}(N)} \left( -\frac{\eta_{14}(N)}{\eta_{13}(N)} - \frac{\eta_{13}(N) - i\eta_{24}(N)}{1 + i} \right) \in \mathbb{D},
\]

where \( \mathbb{D} := \{ \tau \in M_{2,2}(\mathbb{C}); \ (\tau - i\bar{\tau})/2\sqrt{-1} \text{ is positive-definite} \} \) is a symmetric bounded domain of type \( I_2.2 \). Notice that \( \mathbb{D} \) is isomorphic to a symmetric bounded domain of type \( IV \) of dimension 4 (cf. [19, Sect. 4.1]).

Let \( U_{22}^M (1 + i) \) be the discrete group acting on \( \mathbb{D} \) defined in [19, Prop. 6]. For \( \Omega \in \mathbb{D} \), write \( [\Omega] \in \mathbb{D}/U_{22}^M (1 + i) \) for the \( U_{22}^M (1 + i) \)-orbit of \( \Omega \). By [19, Prop. 6], the period map for the family of \( K3 \) surfaces \( \bigcup_{N \in M^{s}(3,6)} W_N \to M^{s}(3, 6) \) induces a holomorphic map

\[
\mathcal{P}: X^o(3, 6) \ni [N] \to [\Omega_N] \in \mathbb{D}/U_{22}^M (1 + i).
\]

Set \( e(x) := \exp(2\pi i x) \). Following [19, p. 137], we set

\[
E_v := \left\{ (a, b) \in (\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i])^4; \ a^t \bar{b} \equiv 0 \ (\bmod \ (1 + i)) \right\}.
\]

For \( (a, b) \in E_v \), the theta function \( \Theta_{[a, b]}(\Omega) \in \mathcal{O}(\mathbb{D}) \) is defined as (cf. [19, Sect. 4.4])

\[
\Theta_{[a, b]}(\Omega) := \sum_{n \in \mathbb{Z}[i]^2} e \left[ \left( n + \frac{a}{1 + i} \right) \frac{\Omega t}{2} \left( n + \frac{a}{1 + i} \right) + \Re \left( n + \frac{a}{1 + i} \right) \right] \left( \frac{b}{1 + i} \right) \right].
\]

By Matsumoto-Terasoma [19, p. 138], there is a one-to-one correspondence between the set of partitions \( \{ \langle J \rangle \} \) and \( E_v = \{ (a) = (a_1a_2) \} \) as follows:

| \langle J \rangle | \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} | \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} | \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} | \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} | \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \begin{pmatrix} 135 & 146 \\ 246 & 235 \end{pmatrix} | \begin{pmatrix} 136 & 125 \\ 245 & 256 \end{pmatrix} | \begin{pmatrix} 125 & 134 \\ 256 & 236 \end{pmatrix} | \begin{pmatrix} 145 & 156 \\ 236 & 234 \end{pmatrix} | \begin{pmatrix} 124 & 156 \\ 356 & 345 \end{pmatrix} | \begin{pmatrix} 126 & 156 \\ 345 & 356 \end{pmatrix} | \begin{pmatrix} 123 & 156 \\ 456 & 345 \end{pmatrix} |

When a partition \( \langle J \rangle \) corresponds to \( (a, b) \in E_v \) by the above rule, we define

\[
\Theta_{\langle J \rangle}(\Omega) := \Theta_{[a, b]}(\Omega).
\]

The Petersson norm of \( \Theta_{[a, b]}(\Omega) \) is the function on \( \mathbb{D} \) defined as

\[
\| \Theta_{[a, b]}(\Omega) \|^2 := \det \left( \frac{\Omega - i\bar{\Omega}}{2i} \right) |\Theta_{[a, b]}(\Omega)|^2.
\]

Since \( \| \Theta_{[a, b]} \| \) is an \( U_{22}^M (1 + i) \)-invariant function on \( \mathbb{D} \) by [19, Prop. 7], we regard \( \| \Theta_{[a, b]} \| \) as a function on \( \mathbb{D}/U_{22}^M (1 + i) \). Then \( \mathcal{P}^* \| \Theta_{[a, b]} \| \) is a function on \( X^o(3, 6) \).
THEOREM 5.2. For all \( N \in M_{3,6}^o(\mathbf{C}) \) and for all partitions \( \langle J \rangle \),

\[
\| \Phi(Y_{N,\langle J \rangle}) \| = \| \Theta(J)(\Omega_N) \|^4.
\]

Namely, one has the equality of functions \( \Phi(J) \| = \Theta^* \| \Theta(J) \| ^4 \) on \( X^o(3,6) \).

Proof.

Step 1. Firstly, we set \( \langle J \rangle = (\frac{123}{456}) \) and prove the assertion in this case. For \( K = (k_{ij}) \in M_{3,3}(\mathbf{C}) \) with \( (K,I) \in M_{3,6}^o(\mathbf{C}) \), set

\[
V_K := \{(y_1, y_2, y_4, y_5, y_6) \in \mathbf{C}^5; \quad k_{11}y_1 + k_{12}y_2 + k_{13} + y_1 + y_2 = 0 \quad (i = 1, 2, 3)\}.
\]

Then \( V_K \) is an affine subspace of \( \mathbf{C}^5 \), whose closure in \( \mathbf{P}^5 \) is denoted by \( \overline{V}_K \). Set

\[
\xi_K = 2^{-5} \frac{dy_1 \wedge dy_2}{\sqrt{y_1y_2 \prod_{i=1,2,3}(k_{1i}y_1 + k_{12}y_2 + k_{13})}} \bigg|_{V_K}.
\]

Let \( \varphi : \mathbf{P}^5 \to \mathbf{P}^5 \) be the map defined as \( \varphi(x_1 : \ldots : x_6) = (x_1^2 : \ldots : x_6^2) \). Since \( \overline{V}_K = \varphi(X_{(K,I)}) \) and \( \omega_{(K,I)} = i^{-3} \varphi^* \xi_K \), we get by Theorem 5.1 and (5.1)

\[
\begin{align*}
\| \Phi(Y_{(K,I)}) \|^2 &= |\Delta_{123}(K,I)|^4 \cdot \left( \frac{2}{\pi^4} \left| \int_{X_{(K,I)}} \varphi^*(\xi_K \wedge \overline{\xi}_K) \right| \right)^4 \\
&= |\det(K)|^4 \cdot \left( \frac{2}{\pi^4} \left| \int_{V_K} \xi_K \wedge \overline{\xi}_K \right| \right)^4 \\
&= \left( \frac{2^{-9} \deg(\varphi)|\det(K)|}{\pi^4} \int_{\overline{V}_K} \frac{(\sqrt{-1})^2 dy_1 \wedge dy_1 \wedge dy_2 \wedge dy_2}{|y_1y_2 \prod_{i=1}^{3}(k_{1i}y_1 + k_{12}y_2 + k_{13})|} \right)^4 \\
&= \left( \frac{2^{-4} |\det(K)|}{\pi^3} \int_{\mathbf{P}^2} \eta_{(K,I)} \wedge \overline{\eta}_{(K,I)} \right)^4.
\end{align*}
\]

Here \( \eta_{(K,I)} \wedge \overline{\eta}_{(K,I)} \) is regarded as a volume form on \( \mathbf{P}^2 \) and the last line follows from (5.2) and the formula \( \deg(\varphi) = 2^5 \).

For \( g \in \text{GL}(\mathbf{C}^3) \), let \( f_g \in \text{PGL}(\mathbf{C}^3) \) be the projective transform \( f_g[z] = [t^g z] \). Regarding \( \eta_N \wedge \overline{\eta}_N \) as a volume form on \( \mathbf{P}^2 \) for \( N \in M_{3,6}^o(\mathbf{C}) \), we get

\[
(5.5) \quad f_g^*(\eta_N \wedge \overline{\eta}_N) = |\det(g)|^{-2} \eta_N \wedge \overline{\eta}_N.
\]
Since \(X_{gN} = X_N\) and thus \(\|\Phi(Y_{gN})\| = \|\Phi(Y_N)\|\) for all \(g \in \text{GL}(\mathbb{C}^3), N \in M_{3,6}^0(\mathbb{C})\), we deduce from (5.4), (5.5) that for any \(N = (N_1, N_2) = N_2(N_1^{-1}, I) \in M_{3,6}^0(\mathbb{C})\)

\[
\|\Phi(Y_N)\|^2 = \left(\frac{2^{-4} \left| \det(N_2^{-1}N_1) \right|}{\pi^4} \int_{\mathbb{P}^2} \eta(t_{N_1^tN_2^{-1}}, I) \wedge \overline{\eta(t_{N_1^tN_2^{-1}}, I))} \right)^4
\]

\[
= \left(\frac{2^{-4} \left| \det(t_{N_1^{-1}}) \det(t_{N_2^{-1}}) \right|}{\pi^4} \int_{\mathbb{P}^2} \eta(t_{N_2^{-1}, t_{N_2^{-1}}}) \wedge \overline{\eta(t_{N_2^{-1}, t_{N_2^{-1}}})} \right)^4
\]

\[
= \left(\frac{2^{-4} |\Delta(J)(N^\vee)|}{\pi^4} \cdot \frac{1}{2} \int_{Z_{N^\vee}} \eta_{N^\vee} \wedge \overline{\eta_{N^\vee}} \right)^4.
\]

Here \(\eta_{N^\vee}\) is regarded as a canonical form on \(Z_{N^\vee}\) in the last line. By (5.6), we get

\[
\|\Phi(Y_N)\|^2 = \left\{\frac{2^{-3} \left| \Delta(J)(N^\vee) \eta_{34}(N^\vee) \right|}{4\pi^4} \int_{Z_{N^\vee}} \frac{\eta_{N^\vee}}{\eta_{34}(N^\vee)} \wedge \frac{\overline{\eta_{N^\vee}}}{\eta_{34}(N^\vee)} \right\}^4
\]

\[
= \left\{\frac{2^{-3} |\Theta(J)(\Omega_{N^\vee})|}{\pi^4} \cdot 8 \det \left(\frac{\Omega_{N^\vee} - \overline{\Omega_{N^\vee}}}{2i} \right) \right\}^4
\]

\[
= \|\Theta(J)(\Omega_{N^\vee})\|^8.
\]

Here the second equality follows from [19, Eq.(4)] and the following identity

\[
\det \left(\frac{\Omega_{N^\vee} - \overline{\Omega_{N^\vee}}}{2i} \right) = -\left\{3 \frac{\eta_{14}}{\eta_{34}} \cdot 3 \frac{\eta_{23}}{\eta_{34}} - \frac{1}{2} \left\{3 \left(\frac{\eta_{13}}{\eta_{34}}\right)^2 + \left(3 \frac{\eta_{24}}{\eta_{34}}\right)^2 \right\} \right\}
\]

\[
= \frac{1}{2} \left\{3 \left(\frac{\eta}{\eta_{34}}\right) \cdot H \cdot t \left\{3 \left(\frac{\eta}{\eta_{34}}\right) \right\} \right\}
\]

\[
= \frac{1}{8} \left\{8 \left(\frac{\eta}{\eta_{34}}\right) \cdot 2H \cdot t \left(\frac{\eta}{\eta_{34}}\right) \right\}
\]

\[
= \frac{1}{8} \int_{Z_{N^\vee}} \frac{\eta_{N^\vee}}{\eta_{34}} \wedge \frac{\overline{\eta_{N^\vee}}}{\eta_{34}},
\]

where we wrote \(\eta_{ij} = \eta_{ij}(N^\vee)\) and \(\eta = \eta(N^\vee)\) and used (5.3) to get the last equality. Since \([\Omega_{N^\vee}] = [^t\Omega_{N}]\) by the equality \(\psi(ax) = T \cdot \psi(x)\) in [26, p. 260] and since \(\Theta_{[a,b]}(\Omega)^2 = \Theta_{[a,b]}(^t\Omega)^2\) by [18, Lemma 3.1.3], we deduce from (5.7) that

\[
\|\Phi(Y_N)\| = \|\Theta(J)(\Omega_{N^\vee})\|^4 = \|\Theta(J)(^t\Omega_N)\|^4 = \|\Theta(J)(\Omega_N)\|^4.
\]
Step 2. Let \( \langle J \rangle \) be an arbitrary partition and let \( \sigma \in S_6 \) be a permutation such that \( \langle J \rangle = (\sigma(1) \sigma(2) \sigma(3)) (\sigma(4) \sigma(5) \sigma(6)) \). By [19, Th. 2] and (5.8), we get for all \( N \in M^o(3,6) \),

\[
\left\| \Theta_{\langle J \rangle} (\Omega_N)^2 \right\|^2 = \left\| \frac{1}{4\pi^2} \Delta_{\langle J \rangle}(N) \eta_{34}(N)^2 \left( \int_{Z_N} \eta_N^* \right)^2 \right\|^2 = \left\| \frac{1}{4\pi^2} \Delta_{\langle J \rangle}(N) \eta_{34}(N)^2 \left( \frac{1}{8} \int_{Z_N} \eta_N^* \right)^2 \right\|^2 = \left\| \Theta_{\langle J \rangle} (\Omega_N)^2 \right\|^2.
\]

(5.10)

Since \( Y_{N,\langle J \rangle} = Y_{N^\sigma} \), the result follows from (5.10) and (5.9) applied to \( N^{\sigma} \). \( \square \)

5.4. The case where \( X_N \) is a Jacobian Kummer surface. Let \( \lambda := (\lambda_k) \in \mathbb{C}^6 \) be such that \( \lambda_k \neq \lambda_l \) for all \( k \neq l \). Then

\[ C_\lambda := \{(x, y) \in \mathbb{C}^2; y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5)(x - \lambda_6)\} \]

is a curve of genus 2 with the ordered set of branch points \( \lambda = (\lambda_k) \). We identify \( C_\lambda \) with the corresponding projective curve. Then \( H^0(C_\lambda, \Omega^1_{C_\lambda}) \) is equipped with the basis \( \{\omega_1 = dx/y, \omega_2 = xdx/y\} \). Let \( \{A_1, A_2, B_1, B_2\} \) be the canonical basis of \( H_1(C_\lambda, \mathbb{Z}) \) as in [19, Sect. 3.1]. Since \( \{\lambda_k\} \) is an ordered set, \( C_\lambda \) is equipped with a level 2-structure. The period of \( C_\lambda \) is defined by

\[
[T_\lambda] := \left[ \left( \int_{B_1} \omega_1 \int_{B_2} \omega_1 \right)^{-1} \left( \int_{A_1} \omega_1 \int_{A_2} \omega_1 \right)^{-1} \left( \int_{B_1} \omega_2 \int_{B_2} \omega_2 \right)^{-1} \left( \int_{A_1} \omega_2 \int_{A_2} \omega_2 \right)^{-1} \right] \in \mathfrak{S}_2/\Gamma(2),
\]

where \( \mathfrak{S}_2 \) is the Siegel upper half-space of degree 2 and \( \Gamma(2) \subset Sp_4(\mathbb{Z}) \) is the principal congruence subgroup of level 2.

Let \( K(C_\lambda) \) be the Kummer surface associated to the Jacobian variety of \( C_\lambda \). There are two models of \( K(C_\lambda) \). For each \( \lambda_k \), the point \( (1 : \lambda_i : \lambda_i^2) \) lies on the conic \( x_0x_2 - x_1^2 = 0 \) with tangent line \( \ell_i = \{\lambda_i^2 x_0 - 2\lambda_i x_1 + x_2 = 0\} \). We set

\[
N_\lambda := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{bmatrix}, \quad N'_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} N_\lambda.
\]

Then \( K(C_\lambda) \) is isomorphic to the minimal resolution of the double covering of \( \mathbb{P}^2 \) with branch divisor \( \ell_1 \cup \cdots \cup \ell_6 \). Hence

\[ K(C_\lambda) \cong W_{N_\lambda}' \cong W_{N_\lambda}. \]
On the other hand, it follows from [9, pp. 769–770, p. 789] that

$$K(C_{\lambda}) \cong X_{N_{\lambda}}.$$  

By the second expression of $K(C_{\lambda})$, we have ten free involutions $\{\iota_{(J)}\}$ on $K(C_{\lambda})$.

Recall that, for $a, b \in \{0, \frac{1}{2}\}^2$, the Riemann theta constant $\theta_{a,b}(T)$ is defined as

$$\theta_{a,b}(T) := \sum_{n \in \mathbb{Z}^2} e^{\left((n+a)(T/2))^t(n+a) + (n+a)b\right)}, \quad T \in \mathbb{G}_2,$$

whose Petersson norm is defined as $\|\theta_{a,b}(T)\|^2 := (\det T)^{\frac{1}{2}}|\theta_{a,b}(T)|^2$ and whose parity is defined as $4^t\alpha b \bmod 2$. By [18, Lemma 2.1.1(vi) and p. 399 1.2-4], we get

$$\Theta_{[a,b]}(T) = \theta_{\mathbb{R}(\frac{a}{1+\tau}),\mathbb{R}(\frac{b}{1+\tau})}(T^2) = \theta_{\mathbb{Z}(\frac{a}{1+\tau}),\mathbb{Z}(\frac{b}{1+\tau})}(T^2).$$

**Theorem 5.3.** Let $\langle J \rangle$ be a partition corresponding to $(a, b) \in Ev$. Then

$$\|\Phi(K(C_{\lambda})/\iota_{(J)})\| = \|\theta_{\mathbb{R}(\frac{a}{1+\tau}),\mathbb{R}(\frac{b}{1+\tau})}(T_{\lambda})\|^8 = \|\theta_{\mathbb{Z}(\frac{a}{1+\tau}),\mathbb{Z}(\frac{b}{1+\tau})}(T_{\lambda})\|^8.$$

**Proof.** Since $[\Omega_{N_{\lambda}}] = [T_{\lambda}]$ by [26, Chap. IX, Remark 10.2], the result follows from Theorem 5.2 and (5.11). □

6. An infinite product expansion of theta constants of genus two.

6.1. Periods of principally polarized Abelian surfaces. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$. For $T = (t_1, t_2) \in \mathbb{G}_2$ with $t_1 = \begin{pmatrix} T_{11} \\ T_{12} \end{pmatrix}, t_2 = \begin{pmatrix} T_{21} \\ T_{22} \end{pmatrix} \in \mathbb{C}^2$, we define

$$A_T := \mathbb{C}^2/Zt_1 + Zt_2 + Ze_1 + Ze_2.$$  

Then $A_T$ is an Abelian surface with period matrix $(T, I_2)$. Let $\theta(z, T) \in O(\mathbb{C}^2 \times \mathbb{G}_2)$ be the Riemann theta function and let $C_T$ be the theta divisor of $A_T$:

$$C_T := \{[z] \in A_T; \theta(z, T) = 0\}.$$  

Let $\mathbb{G}_{2}^a$ be the complement of the $Sp_4(\mathbb{Z})$-orbit of the diagonal locus in $\mathbb{G}_2$. Let $T \in \mathbb{G}_{2}^a$. Then $C_T$ is a curve of genus 2 with Jacobian variety $A_T$. Let $i_T: C_T \hookrightarrow A_T$ be the inclusion, which induces an isomorphism $H^1(A_T, \mathbb{Z}) \cong H^1(C_T, \mathbb{Z})$. Let $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ be the canonical basis of $H_1(C_T, \mathbb{Z})$ (cf. [9, pp. 227–228]) such that $(i_T)_*\alpha_i$, $(i_T)_*\beta_i$ correspond to the cycles $\tilde{e}_i, \tilde{t}_i \in H_1(A_T, \mathbb{Z})$. Then

$$\int_{(i_T)_*\alpha_i} dz_j = T_{ij}, \quad \int_{(i_T)_*\beta_i} dz_j = \delta_{ij}.$$  

Let $\alpha_1^\vee, \alpha_2^\vee, \beta_1^\vee, \beta_2^\vee \in H^1(C_T, \mathbb{Z})$ be the Poincaré duals of $\alpha_1, \alpha_2, \beta_1, \beta_2$, respectively. Then $\{\alpha_1^\vee, \alpha_2^\vee, \beta_1^\vee, \beta_2^\vee\}$ is a symplectic basis of $H^1(C_T, \mathbb{Z})$. 
Let \( \{a_1^\gamma, a_2^\gamma, b_1^\gamma, b_2^\gamma\} \subset H^1(A_T, \mathbb{Z}) \) be the basis such that \( i^*_T a_1^\gamma = \alpha_1^\gamma, i^*_T b_1^\gamma = \beta_1^\gamma \). By [9, p. 310, Lemma], \( [C_T] = a_1^\gamma \wedge b_1^\gamma + a_2^\gamma \wedge b_2^\gamma \in H^2(A_T, \mathbb{Z}) \) is the Poincaré dual of \( C_T \subset A_T \). Set

\[
A := \langle [C_T] \rangle = \{ \alpha \in H^2(A_T, \mathbb{Z}); \langle [C_T], \alpha \rangle = 0 \} \subset H^2(A_T, \mathbb{Z}).
\]

Equipped with the basis \( \{a_1^\gamma \wedge a_2^\gamma, b_1^\gamma \wedge b_2^\gamma, a_1^\gamma \wedge b_1^\gamma, a_2^\gamma \wedge b_2^\gamma, a_1^\gamma \wedge b_1^\gamma - a_2^\gamma \wedge b_2^\gamma\} \), \( A \) is regarded as the lattice \( \mathbb{U}(-1) \oplus \mathbb{U}(-1) \oplus (-2) \).

The domain of type IV associated to the lattice \( A \) is defined as

\[
\Omega_A := \{ [x] \in \mathbb{P}(A \otimes \mathbb{C}); \langle x, x \rangle_A = 0, \langle x, \gamma \rangle_A > 0 \}.
\]

By the Riemann-Hodge bilinear relations, \( [dz_1 \wedge dz_2] \in \Omega_A^+ \), where \( \Omega_A^+ \) is the component of \( \Omega_A \) containing \( [dz_1 \wedge dz_2] \). We define an isomorphism of complex manifolds \( \varpi_A : \mathcal{G}_2 \to \Omega_A^+ \) by \( \varpi_A(T) := [dz_1 \wedge dz_2] \). Since \( [dz_j] = a_1^\gamma - b_1^\gamma T_{ij} - b_2^\gamma T_{2j} \), we have

\[
\varpi_A(T) = a_1^\gamma + \det T b_1^\gamma \wedge b_2^\gamma + T_{11} a_2^\gamma \wedge b_1^\gamma - T_{22} a_1^\gamma \wedge b_2^\gamma - T_{12} (a_1^\gamma \wedge b_1^\gamma - a_2^\gamma \wedge b_2^\gamma).
\]

### 6.2. 2-cycles on Jacobian Kummer surfaces

Let \( A_T[2] \) be the points of order 2 of \( A_T \). Let \( p : \tilde{A}_T \to A_T \) be the blowing-up of \( A_T[2] \). Let \( \tilde{-1} \) be the involution on \( \tilde{A}_T \) induced by \( -1(z) = -z \) on \( A_T \). Then \( K_T := \tilde{A}_T/\tilde{-1} \) is a Kummer surface associated to \( A_T \). Since \( A_T \) is the Jacobian variety of \( C_T \), we have \( K_T = K(C_T) \). Let \( q : \tilde{A}_T \to K_T \) be the projection. By [1, VIII, Prop. 5.1], the injective homomorphism

\[
(6.1) \quad \phi := q \circ p^* : H^2(A_T, \mathbb{Z}) \to H^2(K_T, \mathbb{Z})
\]

satisfies the equality \( \langle \phi(l), \phi(m) \rangle = 2 \langle l, m \rangle \) for all \( l, m \in H^2(K_T, \mathbb{Z}) \). If \( T \in \mathcal{G}_2^0 \) is generic enough, then \( \phi(A) \subset H^2(K_T, \mathbb{Z}) \) is the transcendental lattice of \( K_T \). Set

\[
\Gamma_{12} := \phi(a_1^\gamma \wedge a_2^\gamma), \quad \Gamma_{13} := \phi(a_1^\gamma \wedge b_1^\gamma), \quad \Gamma_{14} := \phi(a_1^\gamma \wedge b_2^\gamma),
\]

\[
\Gamma_{23} := \phi(a_2^\gamma \wedge b_1^\gamma), \quad \Gamma_{24} := \phi(a_2^\gamma \wedge b_2^\gamma), \quad \Gamma_{34} := \phi(b_1^\gamma \wedge b_2^\gamma).
\]

In what follows, \( \phi(A) \) is equipped with the basis \( \{\Gamma_{12}^\gamma, \Gamma_{13}^\gamma, \Gamma_{14}^\gamma, \Gamma_{23}^\gamma, \Gamma_{13}^\gamma - \Gamma_{24}^\gamma\} \). Hence \( \phi(A) = \mathbb{A}(2) \). Let us see the relation between the cycles \( \gamma_{ij} \) and \( \Gamma_{ij}^\gamma \).

Let \( T \in \mathcal{G}^0 \). Define the map \( \sigma : C_T \times C_T \to A_T \) by \( \sigma(x, y) := i_T(x) + i_T(y) \) and let \( \Pi : A_T \to A_T/\pm 1 \) be the projection. We define \( \text{sym} := \Pi \circ \sigma : C_T \times C_T \to A_T/\pm 1 \) (cf. [19, p. 128]). Set \( \gamma_1 := \alpha_1, \gamma_2 := \alpha_2, \gamma_3 := \beta_1, \gamma_4 := \beta_2 \). Let \( \gamma_{ij} \in H^2(K_T, \mathbb{Z}) \) be the Poincaré dual of the image of \( \text{sym}(\gamma_i \times \gamma_j) \) with respect to the resolution \( r : K_T \to A_T/\pm 1 \). Namely, if \( E_1 \Pi \ldots \Pi E_16 = r^{-1}(A_T[2]) \subset K_T \) are the exceptional curves of \( r : K_T \to A_T/\pm 1 \), then \( \gamma_{ij} \) is the
unique element of \( H^2_c(K_T \setminus \Pi_{k=1}^{16} E_k, Z) = H^2(K_T, Z) \cap c_1(E_1)^{-1} \cap \cdots \cap c_1(E_{16})^{-1} \) satisfying

\[
(6.2) \quad \langle r^* y, \mathbf{\bar{\gamma}}_{ij} \rangle = \int_{\text{sym}_*(\gamma_i \times \gamma_j)} y, \quad \forall y \in H^2_c(\{A_T \setminus A_T[2]\}/ \pm 1, R).
\]

Recall that if \( C_T \cong C_\lambda \) with some \( \lambda \in C^6 \), then \( K_T \cong K(C_\lambda) \cong W_{N_\lambda} \). Set \( \mathbb{T} := H^2(W_{N_\lambda}, Z)_- \) and regard \( \mathbb{T} \) as a primitive sublattice of \( H^2(K_T, Z) \) via the isomorphism \( H^2(K_T, Z) \cong H^2(W_{N_\lambda}, Z) \), where \( H^2(W_{N_\lambda}, Z)_- \) was introduced in Sect. 5.3. By [19, Sect. 3.1] \( \gamma_{ij}^\prime \) is defined as the orthogonal projection of \( \mathbf{\bar{\gamma}}_{ij} \) to \( \mathbb{T} \).

**Lemma 6.1.** One has \( \Gamma_{ij}^\prime - \gamma_{ij}^\prime \in \phi(A)^{-1} \).

**Proof:** If \( T \in \mathcal{S}_2 \) is generic, \( \phi(A) \) is the smallest sublattice of \( H^2(K_T, Z) \) satisfying \( \phi(A) \otimes C \subset H^0(K_T, \Omega^2_{K_T}) \). Since \( H^0(W_{N_\lambda}, \Omega^2_{W_{N_\lambda}}) \subset \mathbb{T} \otimes C \), this implies \( \phi(A) \subset \mathbb{T} \).

Set \( \mathbf{c}_1^\gamma := a_1^\gamma, \mathbf{c}_2^\gamma := a_2^\gamma, \mathbf{c}_3^\gamma := b_1^\gamma, \mathbf{c}_4^\gamma := b_2^\gamma \) and \( \gamma_1^\gamma := \alpha_1^\gamma, \gamma_2^\gamma := \alpha_2^\gamma, \gamma_3^\gamma := \beta_1^\gamma, \gamma_4^\gamma := \beta_2^\gamma \). For every \( x \in H^2(A_T, Z) \cap [C_T]^{-1} \), we get

\[
\langle \phi(x), \Gamma_{ij}^\gamma \rangle = 2 \langle x, \mathbf{c}_1^\gamma \wedge \mathbf{c}_2^\gamma \rangle_{H^2(A_T, Z)} = \frac{2}{\pm 1} \int_{C_T \times C_T} \sigma^* x \wedge \sigma^* \mathbf{c}_1^\gamma \wedge \sigma^* \mathbf{c}_2^\gamma
\]

\[
= \int_{C_T \times C_T} \sigma^* x \wedge (pr_1^\gamma \gamma_1^\gamma \wedge pr_2^\gamma \gamma_2^\gamma - pr_1^\gamma \gamma_2^\gamma \wedge pr_2^\gamma \gamma_1^\gamma) = 2 \int_{\gamma_i \times \gamma_j} \sigma^* x.
\]

By the Mayer-Vietoris exact sequence, we get \( H^2(A_T, Z) = H^2_c(A_T \setminus A_T[2], Z) \). Let \( x \in A = H^2_c(A_T \setminus A_T[2], Z) \cap [C_T]^{-1} \). Since \( x = (-1)^* x \), we get

\[
(6.4) \quad \int_{\gamma_i \times \gamma_j} \sigma^* x = \int_{\text{sym}_*(\gamma_i \times \gamma_j)} x \int_{\gamma_i \times \gamma_j} \frac{x + (-1)^* x}{2} = \frac{1}{2} \int_{\text{sym}_*(\gamma_i \times \gamma_j)} \Pi_s x.
\]

Since \( r^* \Pi_s (x) = q_s \mathbf{L} (x) \) for all \( x \in H^2(A_T \setminus A_T[2], Z) \), we get by (6.2), (6.3), (6.4)

\[
(6.5) \quad \langle \phi(x), \Gamma_{ij}^\gamma \rangle = \int_{\text{sym}_*(\gamma_i \times \gamma_j)} \Pi_s x = \langle r^* \Pi_s x, \mathbf{\bar{\gamma}}_{ij} \rangle = \langle \phi(x), \mathbf{\bar{\gamma}}_{ij} \rangle.
\]

Since \( \gamma_{ij}^\prime - \mathbf{\bar{\gamma}}_{ij} \in \mathbb{T}^{\perp} \subset \phi(A)^{-1} \) by the definition of \( \gamma_{ij}^\prime \) and \( \Gamma_{ij}^\gamma - \mathbf{\bar{\gamma}}_{ij} \in \phi(A)^{-1} \) by (6.5), we get the result. \( \square \)

### 6.3. Switches and periods of Jacobian Kummer surfaces.

Let \( \pi: K \to \mathcal{S}_2 \) be the universal family of Kummer surfaces with \( \pi^{-1}(T) = K_T = K(C_T) \) for all \( T \in \mathcal{S}_2 \). By fixing an order of odd characteristics \( (a, b) \), \( a, b \in \{0, 1\}^2 \), the branch points of the canonical map \( \Phi_{|K_{C_T}|} : C_T \to \mathbb{P}^1 \) are also ordered, because they are given by the 6 points \( \{(\frac{\partial}{\partial z}(a + T b), T) : \frac{\partial}{\partial z}(a + T b), T)\} \langle a, b \rangle: \text{odd} \). Hence \( K(C_T) \cong \ldots \)
the equation \( \omega \) which extends to an embedding 
\( (6.6) \)

\[
\pi: K|_{\mathcal{S}_2^0} \to \mathcal{S}_2^0
\]

is equipped with the 10 fixed-point-free involutions \( \{ \iota_{(J)} \} \).

Let \( T_0 \in \mathcal{S}_2^0 \). Since \( \phi(\mathbb{A}) \) is the smallest sublattice of \( H^2(K_{T_0}, \mathbb{Z}) \) whose complexification contains \( \phi(dz_1 \wedge dz_2) \) if \( T_0 \in \mathcal{S}_2^0 \) is generic, we get the inclusion 
\( \phi(\mathbb{A}) \subset H^2(K_{T_0}, \mathbb{Z})_\ell = \{ l \in H^2(K_{T_0}, \mathbb{Z}); \ i_{(J)}^* l = l \} \). Let \( \alpha_{(J),T_0}: H^2(K_{T_0}, \mathbb{Z}) \cong \mathbb{L}_{K3} \) be a marking satisfying (2.1) with respect to \( \iota_{(J)} \). Then \( \alpha_{(J),T_0}(\phi(\mathbb{A})) = \alpha_{(J),T_0}(\phi(\mathbb{A})) \) for all \( T \in \mathcal{S}_2 \).

The period map for the marked family of Enriques surfaces 
\( (6.6) \)

\[
\alpha_{(J)}(\phi(\omega_A(T))) = \left[ \alpha_{(J)}(\Gamma_{12}) + \det T \alpha_{(J)}(\Gamma_{34}) + T_{11} \alpha_{(J)}(\Gamma_{23}) - T_{22} \alpha_{(J)}(\Gamma_{14}) - T_{12} \alpha_{(J)}(\Gamma_{13} - \Gamma_{24}) \right],
\]

which extends to an embedding \( \omega_{(J)}: \mathcal{S}_2 \to \Omega^*_A \) defined by

\[
\omega_{(J)} := \alpha_{(J)} \circ \phi \circ \omega_A.
\]

6.4. Infinite product expansion of theta constants of genus 2. Let \( \ell \in \{1, 2\} \) be the level of the primitive isotropic vector \( \alpha_{(J)}(\Gamma_{34}') \) in \( \Lambda \). Choosing \( \alpha_{(J)} \) suitably, we may and will assume by [24, Prop. 4.5] that \( \alpha_{(J)}(\Gamma_{34}') = \varepsilon_\ell \). Let \( \Gamma_{34} \in H_2(K_T, \mathbb{Z}) \) be the Poincaré dual of \( \Gamma_{34}' \). We define \( z_{(J)}(T) \in \mathbb{M}_\ell \otimes \mathbb{R} + i\mathbb{C}\mathbb{M}_\ell \) by the equation

\[
(6.7) \quad \omega_{(J)}(T) = \left\{ -\left(z_{(J)}(T)^2/2\right)\varepsilon_\ell + \left(\frac{f_\ell}{\ell}\right) + z_{(J)}(T) \right\}.
\]

**Lemma 6.2.** There exist constants \( A, B, C, D \in \mathbb{M}_\ell \) depending on \( \langle J \rangle \) such that

\[
z_{(J)}(T) = (A + BT_{11} + CT_{12} + DT_{22})/2, \quad \forall T \in \mathcal{S}_2.
\]

**Proof.** By (6.6) and (6.7), there exists \( \lambda \in \mathbb{C}^* \) with

\[
\alpha_{(J)}(\Gamma_{12}') + \det T \varepsilon_\ell + T_{11} \alpha_{(J)}(\Gamma_{23}') - T_{22} \alpha_{(J)}(\Gamma_{14}') - T_{12} \alpha_{(J)}(\Gamma_{13}' - \Gamma_{24}')
\]

\[
= \lambda \left\{ -\left(z_{(J)}(T)^2/2\right)\varepsilon_\ell + \left(\frac{f_\ell}{\ell}\right) + z_{(J)}(T) \right\}.
\]

Comparing the inner products with \( \varepsilon_\ell \), we get \( \lambda = 2 \). Set

\[
f' := \alpha_{(J)}(\Gamma_{12}'), \quad a := \alpha_{(J)}(\Gamma_{23}'), \quad b := \alpha_{(J)}(\Gamma_{14}'), \quad c := \alpha_{(J)}(\Gamma_{13}' - \Gamma_{24}').
\]

Since \( \langle f_\ell/\ell, a \rangle, \langle f_\ell/\ell, b \rangle, \langle f_\ell/\ell, c \rangle, \langle f_\ell/\ell, f' \rangle \in \mathbb{Z} \), we get the desired expression

\[
z_{(J)}(T) = \frac{f'}{2} - \langle f_\ell/\ell, f' \rangle + T_{11}a - T_{22}b - T_{12}c \frac{\varepsilon_\ell}{2} - \frac{f_\ell}{\ell} + \frac{1}{2}(T_{11}a - T_{22}b - T_{12}c)
\]

\[
= (A + BT_{11} + CT_{12} + DT_{22})/2
\]

with \( A = f' - (2/\ell)f_\ell, B = -\langle f_\ell/\ell, a \rangle \varepsilon_\ell + a, C = \langle f_\ell/\ell, c \rangle \varepsilon_\ell - c, D = \langle f_\ell/\ell, b \rangle \varepsilon_\ell - b. \) \( \square \)
**Theorem 6.3.** Let \((a, b) \in E_v\) correspond to \(\langle J \rangle\). Then for all \(T \in \mathfrak{S}_2\),
\[
\left( \varpi_{\langle J \rangle} \Phi_\ell \right) (T) = \Phi_\ell (z_{\langle J \rangle}(T)) = \pm \theta_{\mathfrak{R}(\frac{a}{1+i})} \mathfrak{R}(\frac{b}{1+i}) (T)^8 = \pm \theta_{3(\frac{a}{1+i})3(\frac{b}{1+i})} (T)^8.
\]

**Proof.** For short, write \(z(T)\) for \(z_{\langle J \rangle}(T)\) and \(\alpha\) for \(\alpha_{\langle J \rangle}\). Let \(\lambda \in \mathbb{C}^6\) be such that \(K_T = K(C_\lambda)\). Since \(K_T \cong W_{N_\lambda}, \eta_{N_\lambda}/\int_{\Gamma_{34}} \eta_{N_\lambda}\) is a canonical form on \(K_T\) whose integration over \(\Gamma_{34}\) is normalized to 1. Hence we get the equality
\[
\alpha (\eta_{N_\lambda}/\Gamma_{34}, \eta_{N_\lambda}) = -(z(T)^2/2) \xi_\ell + (f_\ell/\ell) + z(T),
\]
from which we get

\[
2(\mathfrak{S} z(T), \mathfrak{S} z(T)) = \left\langle -(z(T)^2/2) \xi_\ell + (f_\ell/\ell) + z(T), -(z(T)^2/2) \xi_\ell + (f_\ell/\ell) + z(T) \right\rangle
= \langle \alpha(\eta_{N_\lambda}), \alpha(\overline{\eta}_{N_\lambda}) \rangle \left| \int_{\Gamma_{34}} \eta_{N_\lambda} \right|^{-2} = \left( \int_{Z_{N_\lambda}} \eta_{N_\lambda} \wedge \overline{\eta}_{N_\lambda} \right) \left| \int_{\Gamma_{34}} \eta_{N_\lambda} \right|^{-2}.
\]

Since \(\eta_{34}(N_\lambda) = \langle \gamma_{34}', \eta_{N_\lambda} \rangle/2 = \langle \Gamma_{34}', \eta_{N_\lambda} \rangle/2 = (\int_{\Gamma_{34}} \eta_{N_\lambda})/2\) by Lemma 6.1, we get
\[
\det \mathfrak{S} T = \frac{1}{8} \int_{Z_{N_\lambda}} \left( \frac{\eta_{N_\lambda}}{\eta_{34}(N_\lambda)} \right) \wedge \left( \frac{\overline{\eta}_{N_\lambda}}{\eta_{34}(N_\lambda)} \right) = (\mathfrak{S} z(T), \mathfrak{S} z(T))
\]
by (5.8), (6.8). We deduce from Theorem 5.2 and (6.9) that
\[
\| \Phi(Y_{N_\lambda, \langle J \rangle}) \| = \| \Theta_{\langle J \rangle}(T) \|^4 = (\mathfrak{S} z(T), \mathfrak{S} z(T))^2 |\Theta_{\langle J \rangle}(T)|^4.
\]

Comparing (6.10) with the definition \(\| \Phi(Y_{N_\lambda, \langle J \rangle}) \| = (\mathfrak{S} z(T), \mathfrak{S} z(T))^2 |\Phi_\ell(z(T))|\),
we get the equality of functions on \(\mathfrak{S}_2\):
\[
|\Phi_\ell(z(T))| = |\Theta_{\langle J \rangle}(T)|^4.
\]

Hence there exists by (5.11) a constant \(C \in \mathbb{C}\) with \(|C| = 1\) such that for all \(T \in \mathfrak{S}_2\)
\[
\Phi_\ell(z(T))^2 = C \Theta_{\langle J \rangle}(T)^8 = C \theta_{\mathfrak{R}(\frac{a}{1+i})} \mathfrak{R}(\frac{b}{1+i}) (T)^{16} = C \theta_{3(\frac{a}{1+i})3(\frac{b}{1+i})} (T)^{16}.
\]

Set \(q_{mn} := \exp(\pi i T_{mn})\) and \((a, b) := (\mathfrak{R}(\frac{a}{1+i}), \mathfrak{R}(\frac{b}{1+i}))\). By (2.10), (2.11) and Lemma 6.2, we get \(\Phi_\ell(z(T)) \in \mathbb{Z}\{q_{11}, q_{12}, q_{22}\}\). By definition, we get \(\theta_{a,b}(T)^8\) is a Siegel modular form for the principal congruence subgroup of level 2 by [21, p. 190, Eq. (5.2)], we get \(\theta_{a,b}(T + 2B)^8 = \theta_{a,b}(T)^8\) for any integral symmetric \(2 \times 2\)-matrix \(B\). Hence \(\theta_{a,b}(T)^8 \in \mathbb{Z}\{q_{11}, q_{12}, q_{22}\}\) for any even \((a, b)\). Since \(\Phi_\ell(z(T)) \in \mathbb{Z}\{q_{11}, q_{12}, q_{22}\}\) and \(\theta_{a,b}(T)^8 \in \mathbb{Z}\{q_{11}, q_{12}, q_{22}\}\), we get \(C \in \mathbb{Q}\) by (6.11). Since \(|C| = 1\), this implies \(C = \pm 1\). By (6.11) again, \(\Phi_\ell(z(T))^2 = \pm \theta_{a,b}(T)^{16}\). If \(\Phi_\ell(z(T))^2 = -\theta_{a,b}(T)^{16}\),
then \( \Phi_{\ell}(z(T)) = \pm \sqrt{-1} \theta_{a,b}(T)^8 \in \sqrt{-1} \mathbb{Z}\{q_{11}, q_{12}, q_{22}\} \). Since \( \Phi_{\ell}(z(T)) \in \mathbb{Z}\{q_{11}, q_{12}, q_{22}\} \), we get a contradiction \( \Phi_{\ell}(z(T)) \equiv 0 \). This proves \( C = 1 \). □

**Corollary 6.4.** For all even \((a, b) \in \{0, \frac{1}{2}\}^4\), \( \theta_{a,b}(T)^8 \) admits an infinite product expansion of Borcherds type.

Let \( \Delta_5(T) := \prod\limits_{(a, b) \text{ even}} \theta_{a,b}(T) \) be the product of all even theta constants. By Corollary 6.4, \( \Delta_5(T)^8 \) is expressed as an infinite product of Borcherds type. In fact, Gritsenko-Nikulin [12] proved that \( \Delta_5(T) \) is a Borcherds product.

7. **Kummer surfaces of product type and involutions of even type.** In this section, we prove \( C = c_1 = c_2 = 1 \) in (3.10) and Proposition 4.7 by studying \( \Phi \) for those Enriques surfaces which are the quotient of a Kummer surface of product type by involutions of even type. Let \( \mathfrak{H} \) be the complex upper-half plane.

7.1. **Elliptic functions.** For \( \tau \in \mathfrak{H} \), set \( E_{\tau} := \mathbb{C}/2\mathbb{Z} + 2\tau \mathbb{Z} \). Let \( \wp(u; 1, \tau) \) be the Weierstrass \( \wp \)-function on the \( u \)-plane associated to the lattice \( \mathbb{Z} + \tau \mathbb{Z} \). The map \( E_{\tau} = \mathbb{C}/2\mathbb{Z} + 2\tau \mathbb{Z} \ni [u] \rightarrow (\wp(\frac{u}{2}; 1, \tau) : \wp'(\frac{u}{2}; 1, \tau) : 1) \in \mathbb{P}^2 \) is an isomorphism from \( E_{\tau} \) to the cubic curve defined by the affine equation \( w^2 = 4(z - e_1)(z - e_2)(z - e_3) \), where \( e_1 = \wp(\frac{1}{2}; 1, \tau) \), \( e_2 = \wp(1 + \frac{1}{2}; 1, \tau) \), \( e_3 = \wp(\frac{3}{2}; 1, \tau) \).

By [13, p. 213 Eq. (3)], Jacobi’s elliptic functions and the \( \wp \)-function are related as follows:

\[
\begin{align*}
\text{sn}(\sqrt{e_1 - e_3} u) &= \frac{\sqrt{(e_1 - e_3)/(\wp(u; 1, \tau) - e_3))}}{\wp(u; 1, \tau) - e_3),} \\
\text{cn}(\sqrt{e_1 - e_3} u) &= \frac{\sqrt{(\wp(u; 1, \tau) - e_1)/(\wp(u; 1, \tau) - e_3))}}{\wp(u; 1, \tau) - e_3),} \\
\text{dn}(\sqrt{e_1 - e_3} u) &= \frac{\sqrt{(\wp(u; 1, \tau) - e_2)/(\wp(u; 1, \tau) - e_3))}}{\wp(u; 1, \tau) - e_3).}
\end{align*}
\]

By [13, p. 215, Tabelle II], \( \text{sn}(\sqrt{e_1 - e_3} u) \) is an odd function with period lattice \( \Gamma_1 := 2\mathbb{Z} + \tau \mathbb{Z} \) and \( \text{cn}(\sqrt{e_1 - e_3} u) \) (resp. \( \text{dn}(\sqrt{e_1 - e_3} u) \)) is an even function with period lattice \( \Gamma_2 := 2\mathbb{Z} + (1 + \tau) \mathbb{Z} \) (resp. \( \Gamma_3 := \mathbb{Z} + 2\tau \mathbb{Z} \)). We regard \( \text{sn}(\sqrt{e_1 - e_3} u), \text{cn}(\sqrt{e_1 - e_3} u), \text{dn}(\sqrt{e_1 - e_3} u) \) as periodic meromorphic functions on \( \mathbb{C} \) with period lattice \( \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = 2\mathbb{Z} + 2\tau \mathbb{Z} \). Namely, they are meromorphic functions on \( E_{\tau} \).

By [13, p. 204], [21, p. 69], the theta constants are defined as

\[
\begin{align*}
\theta_0(\tau) &= \theta_{01}(\tau) := \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})(1 - e^{-\pi i (2n-1)\tau})^2, \\
\theta_2(\tau) &= \theta_{10}(\tau) := 2e^{\pi i \tau/4}\prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})(1 + e^{2\pi i n\tau})^2, \\
\theta_3(\tau) &= \theta_{00}(\tau) := \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})(1 + e^{-\pi i (2n-1)\tau})^2.
\end{align*}
\]

Then \( \sqrt{e_2 - e_3} = \pi \theta_2(\tau)^2 \) and \( \sqrt{e_1 - e_3} = \pi \theta_3(\tau)^2 \) by [13, p. 202, Eq. (6)].
7.2. A $(2, 2, 2)$-model of a Kummer surface of product type. For $\tau_1, \tau_2 \in \mathcal{H}$ and $z_1, z_2 \in \mathbb{C}$, we define functions $x_1, x_2, y_0, y_1, y_2$ on $E_{\tau_1} \times E_{\tau_2}$ by

$$
x_1 := \text{cn} \left( \sqrt{e_1^3 - e_3^3} z_1 \right) = \frac{\varphi(z_1; 1, \tau_1) - e_3^1}{\varphi(z_1; 1, \tau_1) - e_3^3},
$$

$$
x_2 := \text{dn} \left( \sqrt{e_1^3 - e_3^3} z_1 \right) = \frac{\varphi(z_1; 1, \tau_1) - e_2^1}{\varphi(z_1; 1, \tau_1) - e_3^3},
$$

$$
y_0 := \frac{\text{sn} \left( \sqrt{e_1^3 - e_3^3} z_1 \right)}{\text{sn} \left( \sqrt{e_1^3 - e_3^3} z_2 \right)} = \frac{e_1^1 - e_3^1}{e_1^3 - e_3^3} \sqrt{\varphi(z_2; 1, \tau_2) - e_3^2},
$$

$$
y_1 := \frac{\text{sn} \left( \sqrt{e_1^3 - e_3^3} z_1 \right)}{\text{cn} \left( \sqrt{e_1^3 - e_3^3} z_2 \right)} = \frac{e_1^1 - e_3^1}{e_1^3 - e_3^3} \sqrt{\varphi(z_2; 1, \tau_2) - e_3^2},
$$

$$
y_2 := \frac{\text{sn} \left( \sqrt{e_1^3 - e_3^3} z_1 \right)}{\text{dn} \left( \sqrt{e_1^3 - e_3^3} z_2 \right)} = \frac{e_1^1 - e_3^1}{e_1^3 - e_3^3} \sqrt{\varphi(z_2; 1, \tau_2) - e_3^2},
$$

where $e_i^1 := \varphi \left( \frac{1}{2}; 1, \tau_i \right)$, $e_i^2 := \varphi \left( \frac{1+\tau_i}{2}; 1, \tau_i \right)$, $e_i^3 := \varphi \left( \frac{\tau_i}{2}; 1, \tau_i \right)$. Consider the map

$$
f : E_{\tau_1} \times E_{\tau_2} \ni ([z_1], [z_2]) \mapsto (1 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathbb{P}^5.
$$

Then $f(-[z_1], -[z_2]) = f([z_1], [z_2])$ and $f$ is well defined if and only if $\varphi(z_1; 1, \tau_1) \neq \infty$ or $\varphi(z_2; 1, \tau_2) \neq \infty$. Since $\varphi(z_1; 1, \tau_1) = \varphi(z_2; 1, \tau_2) = \infty$ if and only if $([z_1], [z_2])$ is a point of order 2 of $E_{\tau_1} \times E_{\tau_2}$ and since the indeterminacy of $f$ is resolved by blowing-up the points of order 2, $f$ is a birational morphism from $K(E_{\tau_1} \times E_{\tau_2})$ to

$$
X_{(\lambda_1, \lambda_2)} := \begin{cases} (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathbb{P}^5; & \lambda_2 x_0^2 - \lambda_2 x_0^2 - y_0^2 + y_2^2 = 0, \\
1 - \lambda_1) x_0^2 + \lambda_1 x_1^2 - x_2^2 = 0, & x_0^2 - x_1^2 - y_0^2 + y_1^2 = 0, \end{cases}
$$

where $\lambda_i = (e_i^2 - e_3^1)/(e_i^1 - e_3^3) = \theta_2(\tau_i)^4/\theta_3(\tau_i)^4$ is the cross ratio of the branch points of $\varphi_{E_{\tau_1}}$. Then $K(\tau_1, \tau_2)$ is isomorphic to the minimal resolution of $X_{(\lambda_1, \lambda_2)}$. In the notation of Sect. 5.2, we deduce from (7.3) that $X_{(\lambda_1, \lambda_2)} = X_{M(\lambda_1, \lambda_2)}$, where

$$
M(\lambda_1, \lambda_2) := \begin{pmatrix} \lambda_1 - 1 & -\lambda_1 & 1 & 0 & 0 & 0 \\
\lambda_2 & -\lambda_2 & 0 & -1 & 0 & 1 \\
1 & -1 & 0 & -1 & 1 & 0 \end{pmatrix} \in M_{3,6}(\mathbb{C}).
$$
Since $\Delta_{123}(M(\lambda_1, \lambda_2)) = \Delta_{456}(M(\lambda_1, \lambda_2)) = 0$, we get $M(\lambda_1, \lambda_2) \not\in M_{3, 6}^a(\mathbb{C})$. Indeed $X(\lambda_1, \lambda_2)$ has nodes. For $\langle J \rangle \neq (\frac{123}{456})$, the minimal resolution of

$$Y_{(\lambda_1, \lambda_2), \langle J \rangle} := X(\lambda_1, \lambda_2)/\iota_{\langle J \rangle}$$

is an Enriques surface and the value $\|\Phi(Y_{(\lambda_1, \lambda_2), \langle J \rangle})\|$ is well defined.

The involutions $\iota_{\langle J \rangle}$ are involutions of even type on $K(E_{\tau_1} \times E_{\tau_2})$ in the sense of [20]. Let $\epsilon$ be the involution on $K(E_{\tau_1} \times E_{\tau_2})$ induced by the involution $-1_{E_{\tau_1}} \times \text{id}_{E_{\tau_2}}$ on $E_{\tau_1} \times E_{\tau_2}$. For $a = (a_1, a_2) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{12} \setminus \{0\}$ and $b = (b_1, b_2) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{12} \setminus \{0\}$, let $\sigma(a, b)$ be the involution on $K(E_{\tau_1} \times E_{\tau_2})$ induced by the translation $(z_1, z_2) \mapsto (z_1 + a_1 + a_2 \tau_1, z_2 + b_1 + b_2 \tau_2)$ on $E_{\tau_1} \times E_{\tau_2}$. Using the transformation rules for $\text{sn}(\sqrt{e_1 - e_3 u})$, $\text{cn}(\sqrt{e_1 - e_3 u})$, $\text{dn}(\sqrt{e_1 - e_3 u})$ under the translations $u \mapsto u + 1$, $u \mapsto u + \tau$, $u \mapsto u + 1 + \tau$ (cf. [13, p. 215 Tabelle I]), we have

$$\epsilon \circ \sigma(a, b) = \iota_{\langle J \rangle},$$

where the correspondence between $(a \ b) = (\frac{a_1 a_2}{b_1 b_2})$ and $\langle J \rangle \neq (\frac{123}{456})$ is given as follows:

| $\langle J \rangle$ | $10$ | $10$ | $10$ | $10$ | $01$ | $01$ | $11$ | $10$ | $11$ | $10$ |
|----------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $(a \ b)$            | $135$ | $134$ | $136$ | $146$ | $156$ | $145$ | $125$ | $124$ | $126$ | $134$ |

Here we used the notation $(\frac{a_1 a_2}{b_1 b_2}) = \epsilon((\frac{a_1 + a_2 \tau_1}{b_1 + b_2 \tau_2}))/2 \in E_{\tau_1} \times E_{\tau_2}$.

### 7.3. Periods of Kummer surfaces of product type

For $(\tau_1, \tau_2) \in \mathfrak{S} \times \mathfrak{S}$, set

$$\mathbb{K} := \phi(\text{pr}_1^*H^1(E_{\tau_1}, \mathbb{Z}) \wedge \text{pr}_2^*H^1(E_{\tau_2}, \mathbb{Z})) \subset H^2(E_{\tau_1} \times E_{\tau_2}, \mathbb{Z}),$$

where $\text{pr}_i : E_{\tau_1} \times E_{\tau_2} \to E_{\tau_i}$ is the projection. If $(\tau_1, \tau_2) \in \mathfrak{S} \times \mathfrak{S}$ is generic enough, then $\mathbb{K}$ is the transcendental lattice of $K(E_{\tau_1} \times E_{\tau_2})$.

Let $\{\alpha_i, \beta_i\}$ be the canonical basis of $H_1(E_{\tau_i}, \mathbb{Z})$ such that

$$\int_{\alpha_i} dz_i = 2\tau_i, \quad \int_{\beta_i} dz_i = 2,$$

where $z_i$ is the coordinate of $E_{\tau_i}$. We define the cocycles $\alpha_i^\vee, \beta_i^\vee, a_i^\vee, b_i^\vee, \Gamma_i^\vee$ in the same way as in Sections 6.1 and 6.2. Then $\mathbb{K}$ is equipped with the basis $\{\Gamma_{i1}^\vee, \Gamma_{i2}^\vee, \Gamma_{i3}^\vee, \Gamma_{i4}^\vee\}$ and is regarded as $\mathbb{U}(-2) \oplus \mathbb{U}(-2)$. Since $[dz_i] = 2(\alpha_i^\vee - \tau_i \beta_i^\vee)$ by (7.5), we get

$$\phi(C[dz_1 \wedge dz_2]) = C(\Gamma_{i1}^\vee + \tau_1 \tau_2 \Gamma_{i3}^\vee + \tau_1 \Gamma_{i2}^\vee - \tau_2 \Gamma_{i4}^\vee).$$
7.4. The restriction of $\Phi$ to $\mathcal{H} \times \mathcal{H}$. Let $\pi: \mathcal{X} \to \mathcal{H} \times \mathcal{H}$ be the family of Kummer surfaces such that $\pi^{-1}(\tau_1, \tau_2) = X_{(\lambda_1, \lambda_2)}$, where $X_{(\lambda_1, \lambda_2)}$ is the minimal resolution of $X_{(\lambda_1, \lambda_2)}$. For a partition $\langle J \rangle$, the fiberwise involution $\iota_{\langle J \rangle}$ on $X_{(\lambda_1, \lambda_2)}$ induces an involution on $\mathcal{X}$, which is again denoted by $\iota_{\langle J \rangle}$. Since $\mathcal{H} \times \mathcal{H}$ is contractible, the family $\pi: \mathcal{X} \to \mathcal{H} \times \mathcal{H}$ is topologically trivial and hence there exists a marking $\alpha_{\langle J \rangle}: R^2 \pi_1, Z \cong \mathbb{L}_K^3$ such that the condition (2.1) is satisfied fiberwise for $\iota_{\langle J \rangle}$. Choosing $\alpha_{\langle J \rangle}$ suitably, we may also assume by [24, Prop. 4.5] that $\alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3}) = e^J = 0$ for all $\lambda \in \Lambda$, where the equality holds if and only if $\lambda \in R_{\geq 0}$. We fix such a marking $\alpha_{\langle J \rangle}$ of $\pi: \mathcal{X} \to \mathcal{H} \times \mathcal{H}$ and set

$$\Omega_{\alpha_{\langle J \rangle}(\mathcal{K})} := \{[\eta] \in \Omega_{\alpha_{\langle J \rangle}(\mathcal{K})}^+ ; \eta \in \alpha_{\langle J \rangle}(\mathcal{K}) \otimes \mathbb{C}\}.$$ 

Let $\varpi_{\langle J \rangle}: \mathcal{H} \times \mathcal{H} \to \Omega_{\alpha_{\langle J \rangle}(\mathcal{K})}^+$ be the period map for the marked family of Enriques surfaces $(\pi: \mathcal{X}/\iota_{\langle J \rangle} \to \mathcal{H} \times \mathcal{H}, \alpha_{\langle J \rangle})$. By (7.6), we get

$$\varpi_{\langle J \rangle}(\tau_1, \tau_2) = \left[\alpha_{\langle J \rangle}(\square_{\mathcal{H}^3}) + \tau_1 \tau_2 \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3}) + \tau_1 \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3}) - \tau_2 \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3})\right].$$

Set $\mathcal{F}^\prime := \alpha_{\langle J \rangle}(\square_{\mathcal{H}^3})$, $\mathcal{F}_\ell := \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3})$, $a := \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3})$, $b := \alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3})$ as in the proof of Lemma 6.2. Define $z_{\langle J \rangle} \in \mathbb{M}_\ell \otimes \mathbb{R} + iC_{\mathbb{M}_\ell}^+$ by

$$z_{\langle J \rangle}(\tau_1, \tau_2) := z_{\langle J \rangle}(\mathrm{diag}(\tau_1, \tau_2)) = \mathcal{F}^\prime + \langle \mathcal{F}^\prime/\ell, \mathcal{F}_\ell + \tau_1 a - \tau_2 b \rangle \frac{\mathcal{F}_\ell}{2} - \frac{\mathcal{F}^\prime}{\ell} + \frac{1}{2}(\tau_1 a - \tau_2 b).$$

Then $z_{\langle J \rangle}(\tau_1, \tau_2) = (A + B\tau_1 + D\tau_2)/2$ by Lemma 6.2, where the constants $A, B, D \in \mathbb{M}_\ell$ were given in the proof of Lemma 6.2. Since $z_{\langle J \rangle}(\tau_1, \tau_2)$ satisfies the relation (6.7), it follows from the definition of $\Phi_{\ell}(z)$ that

$$(\varpi_{\langle J \rangle}^\prime \Phi_{\ell})(\tau_1, \tau_2) = \Phi_{\ell}(z_{\langle J \rangle}(\tau_1, \tau_2)).$$

**Lemma 7.1.** Let $\ell \in \{1, 2\}$ be the level of $\alpha_{\langle J \rangle}(\Gamma_{\mathcal{H}^3})$ in $\Lambda$. Then

$$\lim_{(\tau_1, \tau_2) \to (\pm \infty, \pm \infty)} (\varpi_{\langle J \rangle}^\prime \Phi_{\ell})(\tau_1, \tau_2) = 2 - \ell.$$ 

**Proof.** Since $\mathbb{S} z_{\langle J \rangle} / \mathbb{S} \tau_1 \in C_{\mathbb{M}_\ell}^+$, we get $B = \lim_{\tau_1 \to \pm \infty} 2\mathbb{S} z_{\langle J \rangle} / \mathbb{S} \tau_1 \in C_{\mathbb{M}_\ell}^+$. Hence $\langle \lambda, B \rangle \geq 0$ for all $\lambda \in C_{\mathbb{M}_\ell}^+$, where the equality holds if and only if $\lambda \in R_{\geq 0}$. Similarly, we get $\langle \lambda, D \rangle \geq 0$ for all $\lambda \in C_{\mathbb{M}_\ell}^+$, where the equality holds if and only if $\lambda \in R_{\geq 0}$. Since the isotropic vectors $B, D$ of the Lorentzian lattice $\mathbb{M}_\ell$ are not
parallel and thus \([\mathcal{C}_{M_{\ell}}^+ \setminus \mathbf{R}_{\geq 0} B] \cup [\mathcal{C}_{M_{\ell}}^+ \setminus \mathbf{R}_{\geq 0} D] = \mathcal{C}_{M_{\ell}}^+ \setminus \{0\}\), we get

\[
\lim_{(\tau_1, \tau_2) \to (+i\infty, +i\infty)} \left| e^{\pi i \langle \lambda, z_{\langle J \rangle} \rangle (\tau_1, \tau_2)} \right| = \lim_{(\tau_1, \tau_2) \to (+i\infty, +i\infty)} e^{-\frac{\pi}{2} ((B, \lambda) \Im \tau_1 + (D, \lambda) \Im \tau_2)} = 0
\]  

(7.7)

for all \(\lambda \in \mathcal{C}_{M_{\ell}}^+ \setminus \{0\}\). Since (2.10), (2.11) converge absolutely for \(\Im z \in \mathcal{M}_{\ell} \gg 0\), we get the result by substituting (7.7) into the explicit expressions (2.10), (2.11).

Recall that the holomorphic 2-form \(\omega_{M(\lambda_1, \lambda_2)}\) on (the regular part of) \(X_{(\lambda_1, \lambda_2)}\) was defined in Sect. 5.2. For simplicity, write \(\omega_{(\lambda_1, \lambda_2)}\) for \(\omega_{M(\lambda_1, \lambda_2)}\).

**Lemma 7.2.** The following equality holds

\[
f^* \omega_{(\lambda_1, \lambda_2)} = 2^{-3} \pi^2 \theta_3(\tau_1)^2 \theta_3(\tau_2)^2 \, dz_1 \wedge dz_2.
\]

(7.8)

**Proof:** In the coordinates of \(\mathbf{P}^5\) given by \((x_1, y_0, x_0, x_2, y_2, y_1)\), \(M(\lambda_1, \lambda_2)\) is of the form \((K, I_3)\), \(K \in M_3(\mathbf{C})\) and the uniformization (7.2) satisfies \(x_0 = 1\) as required in Sect. 5.2. Since \(f^* dx_1 = -\sqrt{e_1^1 - e_1^3} \, \text{sn}(z_1 \sqrt{e_1^1 - e_1^3}) \, \text{dn}(z_1 \sqrt{e_1^1 - e_1^3}) \, dz_1\) and

\[
f^* dy_0 = -\sqrt{e_2^1 - e_2^3} \frac{\text{sn}(z_1 \sqrt{e_1^1 - e_1^3})}{\text{sn}(z_2 \sqrt{e_2^1 - e_2^3})} \times \text{cn}(z_2 \sqrt{e_2^1 - e_2^3}) \, \text{dn}(z_2 \sqrt{e_2^1 - e_2^3}) \, dz_2 \mod dz_1
\]

by the definitions of \(x_1, x_2, y_0, y_1, y_2\), we deduce from (5.1) the desired equality:

\[
f^* \omega_{(\lambda_1, \lambda_2)} = f^* \left( \frac{dx_1 \wedge dy_0}{2^3 x_2 y_1 y_2} \right) = \frac{1}{2^3} \prod_{i=1,2} \sqrt{e_i^1 - e_i^3} \, dz_1 \wedge dz_2
\]

\[
= \frac{\pi^2}{2^3} \theta_3(\tau_1)^2 \theta_3(\tau_2)^2 \, dz_1 \wedge dz_2,
\]

where the last equality follows from [13, p. 202 Eq.(6)].
Lemma 7.3. Set $\Delta_{\langle J \rangle} := \Delta_{\langle J \rangle}(M(\lambda_1, \lambda_2))$. Then

$$\Delta^2_{\langle J \rangle} = \begin{cases} 
(\lambda_2 - 1)^2 \\
\lambda_2^2 \\
1 \\
\lambda_1^2(\lambda_2 - 1)^2 \\
\lambda_1^2 \lambda_2^2 \\
\lambda_1^2 \\
(\lambda_1 - 1)^2 \\
(\lambda_1 - 1)^2 \lambda_2^2 \\
(\lambda_1 - 1)^2(\lambda_2 - 1)^2 
\end{cases} = \begin{cases} 
\frac{\theta_0(\tau_2)^8}{\theta_3(\tau_2)^8} \\
\frac{\theta_2(\tau_2)^8}{\theta_3(\tau_2)^8} \\
1 \\
\frac{\theta_2(\tau_1)^8 \theta_0(\tau_2)^8}{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8} \\
\frac{\theta_2(\tau_1)^8 \theta_2(\tau_2)^8}{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8} \\
\frac{\theta_0(\tau_1)^8}{\theta_3(\tau_1)^8} \\
\frac{\theta_0(\tau_1)^8 \theta_2(\tau_2)^8}{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8} \\
\frac{\theta_0(\tau_1)^8 \theta_0(\tau_2)^8}{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8} \\
\frac{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8}{\theta_3(\tau_1)^8 \theta_3(\tau_2)^8} 
\end{cases} \text{ if } \langle J \rangle = \begin{cases} 
124 \\
356 \\
125 \\
346 \\
126 \\
345 \\
134 \\
256 
\end{cases}
$$

Proof. By (7.4), the first equality is elementary. The second equality follows from the first one because $\lambda_i = \theta_2(\tau_i)^4/\theta_3(\tau_i)^4$ and $1 - \lambda_i = -\theta_0(\tau_i)^4/\theta_3(\tau_i)^4$. □

Lemma 7.4. Let $g: K_{(\tau_1, \tau_2)} \rightarrow X(\lambda_1, \lambda_2)$ be the minimal resolution. If $\langle J \rangle \neq \binom{123}{456}$, then

$$\Delta^2_{\langle J \rangle} \cdot \left( \int_{\Gamma_{34}} \frac{2}{\pi^2} g^* \omega(\lambda_1, \lambda_2) \right)^4 = \theta_0(\tau_1)^8 \theta_0(\tau_2)^8,$$

where the correspondence between $(\epsilon, \delta)$ and $\langle J \rangle$ is given as follows:

| $(\epsilon, \delta)$ | $(2, 2)$ | $(2, 0)$ | $(2, 3)$ | $(0, 2)$ | $(0, 0)$ | $(0, 3)$ | $(3, 2)$ | $(3, 0)$ | $(3, 3)$ |
|---------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\langle J \rangle$ | $\binom{135}{246}$ | $\binom{134}{256}$ | $\binom{136}{245}$ | $\binom{146}{235}$ | $\binom{156}{234}$ | $\binom{145}{236}$ | $\binom{125}{346}$ | $\binom{124}{356}$ | $\binom{124}{345}$ |

Proof. The cycles $\beta_i \in H_1(E_{\tau_i}, \mathbb{Z})$ is represented by the circle $(R + \epsilon \sqrt{-1})/2\mathbb{Z}$ with $0 < \epsilon \ll 1$, which does not pass through any points of order 2 of $E_{\tau_i}$. Regard $f^* \omega(\lambda_1, \lambda_2)$ as the 2-form on $A(\tau_1, \tau_2)$ by the formula (7.8). Since $\phi(f^* \omega(\lambda_1, \lambda_2)) = \ldots$
2g^*ω(λ_1,λ_2) and since b^*_1 ∧ b^*_2 is the Poincaré dual of \( β_1 \times β_2 \subset A_1 \), we get

\[
\int_{Γ_{34}} g^*ω(λ_1,λ_2) = \frac{1}{2} \left< φ(f^*ω(λ_1,λ_2)), Γ_{34}^* \right> = \left< f^*ω(λ_1,λ_2), b^*_1 ∧ b^*_2 \right>
\]

\[
= \int_{β_1 × β_2} f^*ω(λ_1,λ_2)
\]

\[
= \int_{β_1 × β_2} \frac{π^2}{2θ_3(τ_1)^2θ_3(τ_2)^2} dτ_1 ∧ dτ_2
\]

\[
= \frac{π^2}{2θ_3(τ_1)^2θ_3(τ_2)^2}
\]

by Lemma 7.2 and (7.5). This, together with Lemma 7.3, implies the result. □

**Theorem 7.5.** One has \( c_1 = c_2 = C = 1 \) in (3.10) and Proposition 4.7.

**Proof.** Since \( C = |c| \), it suffices to prove \( c_1 = c_2 = 1 \). Let \( ⟨J⟩ \) be a partition such that \( α_{⟨J⟩}(Γ_{34}) = c_ℓ \). Then we get

\[
(7.9) \quad \sqrt{c_ℓ} = \frac{ω^*_{⟨J⟩}Φ_ℓ}{Δ^2_{⟨J⟩} \cdot \left( ∫_{Γ_{34}} 2π^2ω(λ_1,λ_2) \right)^4} = \frac{(ω^*_{⟨J⟩}Φ_ℓ)(τ_1,τ_2)}{θ_e(τ_1)^8θ_δ(τ_2)^8},
\]

where the first equality follows from Proposition 4.7 and the second from Lemma 7.4. By (7.1), we get \( \lim_{(τ_1,τ_2)→(+i∞,+i∞)} θ_e(τ_1)^8θ_δ(τ_2)^8 = 0 \) or 1 and both cases occur. Since \( c_ℓ \) is a non-zero constant on \( \mathfrak{f} | \mathfrak{f} \), by Proposition 4.7, it follows from (7.9) and Lemma 7.1 that for any \( ℓ \in \{1,2\} \), there is a choice \( ⟨J⟩ \) such that \( α_{⟨J⟩}(Γ_{34}) \) has level \( ℓ \) in \( Λ \). Hence \( α_{⟨J⟩}(Γ_{34}) = c_ℓ \).

Let \( ℓ = 1 \). By (7.9) and Lemma 7.1, we get \( \lim_{(τ_1,τ_2)→(+i∞,+i∞)} \sqrt{c_1} = ±1 \), since \( c_1 \) is a non-zero constant by Proposition 4.7. This proves that \( c_1 = 1 \).

Assume \( ℓ = 2 \). Set \( q_m := \exp(πiτ_m) \). Since \( Φ_2(τ) \) has integral Fourier coefficients by (2.11) and thus \( ⟨ω^*_{⟨J⟩}Φ_ℓ⟩(τ_1,τ_2) \in \mathbb{Z} \{q_1,q_2\} \), we get \( \sqrt{c_2} \in q_1^{-2}q_2^{-2} \mathbb{Z} \{q_1,q_2\} \) by (7.1), (7.9). Hence \( \sqrt{c_2} \in \mathbb{Z} \). Since \( |c_2| = |c_1| \) and since \( c_1 = 1 \), we get \( c_2 = 1 \). □

**Corollary 7.6.** If \( α_{⟨J⟩}(Γ_{34}) \) is a primitive isotropic vector of \( Λ \) of level \( ℓ \), then

\[
Φ_ℓ(ω_{⟨J⟩}(τ_1,τ_2)) = ±θ_e(τ_1)^8θ_δ(τ_2)^8
\]

under the correspondence between \( ⟨J⟩ \) and \( (ε,δ) \) in Lemma 7.4. In particular;

\[
\|Φ(Y_{⟨J⟩}(λ_1,λ_2))\| = \|θ_e(τ_1)θ_δ(τ_2)\|^8.
\]

**Proof.** Since \( c_1 = c_2 = 1 \) by Theorem 7.5, the result follows from (7.9). □
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