GLOBAL CONSERVATION LAW FOR AN EVEN ORDER ELLIPTIC SYSTEM WITH ANTIMETRIC POTENTIAL

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Abstract. In this note, we refine the local conservation law obtained by Lamm-Rivière for fourth order systems and de Longueville-Gastel for general even order systems to a global conservation law.

Keywords: Global conservation law, Uhlenbeck’s gauge transform, Even order elliptic system, Antisymmetric potential, weak compactness

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1. Introduction

Interesting geometric partial differential equations are usually of critical or even supercritical nonlinearity in nature. For instance, consider the harmonic mappings equation

\[ -\Delta u = A(u)(\nabla u, \nabla u), \quad u \in W^{1,2}(B^m, N), \]

where \( N \) is a closed Riemannian manifold embedded isometrically in some \( \mathbb{R}^n \) and \( A \) denotes the second fundamental form \( N \) in \( \mathbb{R}^n \). Two fundamental analysis issues regarding (1.1) are the regularity of weak solutions and the weak sequential compactness of solutions with uniformly bounded energy. However, the \( L^1 \) integrability of the right hand side of (1.1) makes it on the borderline of the usual elliptic regularity theory. In supercritical dimensions, that is when \( m \geq 3 \), weakly harmonic mappings even could be everywhere discontinuous; see the surprising examples constructed in [16]. Another equation with similar regularity difficulties is the so called prescribed mean curvature equation

\[ -\Delta u = -2H(u)u_x \wedge u_y, \quad u \in W^{1,2}(B^2, \mathbb{R}^3), \]

where \( H \in L^\infty(\mathbb{R}^3) \).

On the other hand, the lack of maximum principle for elliptic systems makes the regularity issues rather difficult problems. To solve both analysis issues mentioned above, the approach of conservation law turns out to be a very precious tool that is remained to be effective. Suppose for simplicity \( N = \mathbb{S}^n \) is the sphere. Shatah [24] proved that \( u \) is a solution of (1.1) if and only if it satisfies the following conservation law

\[ \text{div}(u^i\nabla u^j - u^j\nabla u^i) = 0 \quad \text{for all} \ i, j \in \{1, \cdots, n\}. \]
Combining (1.3) with the Rellich-Kondrachov compactness theorem leads rather directly to the affirmative answer of the weak compactness question. In the conformal dimension $m = 2$, the answer to the regularity issue was settled by Hélein using again (1.3): by Poincaré’s lemma, there exists $B \in W^{1,2}$ such that $\nabla \perp B_{ij} = u_i \nabla u_j - u_j \nabla u_i$, where $\nabla \perp = (-\partial_y, \partial_x)$. Thus (1.1) is equivalent to

$$-\Delta u = \nabla \perp B \cdot \nabla u. \tag{1.4}$$

Now the curl-grad structure in the right-hand side of (1.4) implies better regularity than being simply in $L^1$, which was a key observation discovered by Wente [29] when studying constant mean curvature equations, see also [3] using the language of Hardy spaces. Using Wente’s Lemma, one easily concludes that $u \in C^0(B^n, S^n)$. For general target spaces other than $\mathbb{S}^n$, Hélein [12] also solved in dimension two the regularity issue via his famous moving frame method; based on the idea of conservation law but permits to avoid a direct conservation law. However, it seems that the approach of Hélein does not apply to general conformally invariant second order elliptic variational problems with quadratic growth; in particular, does not apply to the prescribed mean curvature equation (1.2) under the weakest assumption that $H \in L^\infty(\mathbb{R}^3)$.

A direct conservation law was not found until the significant work [17] of Rivière, where the author found that not only the harmonic mappings equation (1.1), but also the prescribed mean curvature equation (1.2) and the Euler-Lagrange equation of the general conformally invariant second order elliptic variational problems with quadratic growth in dimension two. More precisely, he introduced the second order linear elliptic system

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in } B^2, \tag{1.5}$$

where $u \in W^{1,2}(B^2, \mathbb{R}^n)$ and $\Omega = (\Omega_{ij}) \in L^2(B^2, \mathfrak{so}_n \otimes \Lambda^1 \mathbb{R}^2)$. One can verify (see [17]) that (1.5) includes the Euler-Lagrange equations of critical points of all second order conformally invariant variational functionals which act on mappings $u \in W^{1,2}(B^2, N)$ from $B^2 \subset \mathbb{R}^2$ into a closed Riemannian manifold $N \subset \mathbb{R}^n$. In particular, (1.5) includes the equations of weakly harmonic mapping equation (1.1) and the prescribed mean curvature equation (1.2). Since $\Omega \in L^2$, system (1.5) is critical in the sense that $\Omega \cdot \nabla u \in L^1(B^2)$, which allows for discontinuous weak solutions in general. Due to antisymmetry of $\Omega$, the application of (an adapted version of) Uhlenbeck’s gauge theory [26] allows Rivière to write

$$\Omega = P^{-1}dP + P^{-1}d^*\xi P,$$

where $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, \mathfrak{so}_n \otimes \Lambda^2 \mathbb{R}^2)$. Then Rivière succeeded in finding functions $A \in L^\infty \cap W^{1,2}(B^2, GL(n))$ and $B \in W^{1,2}(B^2, M_n)$ such that

$$\nabla A - A\Omega = \nabla \perp B. \tag{1.6}$$

Once such $A$ and $B$ were found, it is straightforward to check that system (1.5) can be written equivalently as the direct conservation law

$$\text{div} (A \nabla u + B \nabla \perp u) = 0,$$

from which everywhere continuity of weak solutions of system (1.5) can be derived. As applications, this recovered the famous regularity result of Hélein [12], and confirmed affirmatively two long-standing regularity conjectures by Hildebrandt and Heinz on conformally
invariant geometrical problems and the prescribed bounded mean curvature equations respectively, see [17] for details.

Motivated by problems from conformal geometry, to search higher order conformally invariant mappings in higher dimensions that play a similar role as that of harmonic mapping in dimension two, it is natural to consider the \( m \)-polyharmonic energy functionals

\[
E_m(u) := \frac{1}{2} \int_{B^{2m}} |D^m u|^2 dx, \quad u \in W^{m,2}(B^{2m}, N),
\]

or

\[
I_m(u) := \frac{1}{2} \int_{B^{2m}} |\nabla^{m-1} Du|^2 dx, \quad u \in W^{m,2}(B^{2m}, N),
\]

where \( \nabla \) is the Levi-Civita connection on \( N \) and \( m \geq 2 \). Critical points of \( E_m \) (\( I_m \) resp.) are called extrinsically (intrinsically resp.) \( m \)-polyharmonic mappings. When \( m = 2 \), critical points of \( E_2 \) (\( I_2 \) resp.) are called extrinsically (intrinsically resp.) biharmonic mappings.

Historically, Chang, Wang and Yang [2] established a regularity theory for biharmonic mappings into the sphere and later Wang [27, 28] extended this regularity theory to biharmonic mappings into closed Riemannian manifolds. Inspired by Rivièrè’s direct conservation law approach [17] for the second order elliptic system (1.5), Lamm and Rivièrè proposed in [14] the fourth order elliptic system

(1.7) \[ \Delta^2 u = \Delta(V \cdot \nabla u) + \text{div}(w \nabla u) + W \cdot \nabla u \quad \text{in} \ B^4, \]

where

\[ V \in W^{1,2}(B^4, M_n \otimes \Lambda^1 \mathbb{R}^4), w \in L^2(B^4, M_n) \]

and \( W \) is of the form \( W = \nabla \omega + F \) with

\[ \omega \in L^2(B^4, s_0) \]

and \( F \in L^4(B^4, M_n \otimes \Lambda^1 \mathbb{R}^4) \). They verified that system (1.7) indeed includes both extrinsic and intrinsic biharmonic mappings from \( B^4 \) into closed Riemannian manifolds as special cases. Similar to the second order case [17], if we were able to find \( A \in W^{2,2} \cap L^{\infty}(B^4, M_n) \) and \( B \in W^{1,4/3}(B^4, M_n \otimes \Lambda^2 \mathbb{R}^4) \) satisfying

(1.8) \[ d \Delta A + \Delta AV - \nabla Aw + AW = d^* B \quad \text{in} \ B^4, \]

then it is immediate to check that \( u \) is a solution of (1.7) if and only if it satisfies the direct conservation law

(1.9) \[ \text{div} \{ \nabla (A \Delta u) - 2 \nabla A \Delta u + \Delta A \nabla u - Aw \nabla u + \nabla A (V \cdot \nabla u) - A \nabla (V \cdot \nabla u) - B \cdot \nabla u \} = 0 \]

in \( B^4 \). Due to some technical difficulties, Lamm and Rivièrè [14] only succeeded in finding \( A \in W^{2,2} \cap L^{\infty}(B^{1/2}_{1/2}, M_n) \) and \( B \in W^{1,4/3}(B^{1/2}_{1/2}, M_n \otimes \Lambda^2 \mathbb{R}^4) \) such that (1.8) holds in the smaller ball \( B^{1/2}_{1/2} \subset B^4 \). Consequently, their conservation law is “local” in the sense that it only holds in a strictly smaller region \( B^{1/2}_{1/2} \) rather than on the whole domain \( B^4 \), where system (1.7) is defined.

The study on general \( m \)-polyharmonic mappings have also attracted great attention in the last decades. For instance, Gastel and Scheven [5] obtained a regularity theory for both extrinsic and intrinsic polyharmonic mappings in critical dimensions via the moving frame method of Helein [12]. For more progress in this respect, see e.g. [11, 15] and the
references therein. Based on the works of Rivière [17] for $m = 1$ and Lamm-Rivière [14] for $m = 2$, it is natural to find a unified treatment for general $m$-polyharmonic mappings via the direct conservation law approach.

This was achieved very recently by de Longueville and Gastel in their interesting work [4]. To includes (both extrinsic and intrinsic) $m$-polyharmonic mappings, they introduced the even order linear elliptic system

$$\Delta^m u = \sum_{l=0}^{m-1} \Delta^l (V_l, du) + \sum_{l=0}^{m-2} \Delta^l \delta (w_l du) \quad \text{in } B^{2m}.$$  

System (1.10) reduces to (1.7) when $m = 2$, and to (1.5) when $m = 1$. The coefficient functions are assumed to satisfy

$$w_k \in W^{2k+2-m,2} (B^{2m}, \mathbb{R}^{n \times n}) \quad \text{for } k \in \{0, \ldots, m-2\}$$
$$V_k \in W^{2k+1-m,2} (B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^1 \mathbb{R}^{2m}) \quad \text{for } k \in \{0, \ldots, m-1\}.$$  

Moreover, the first order potential $V_0$ has the decomposition $V_0 = d\eta + F$ with

$$\eta \in W^{-2-m,2} (B^{2m}, \mathfrak{so}(n)), \quad F \in W^{-2-m,2-1} (B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^1 \mathbb{R}^{2m}).$$

To formulate the conservation law of de Longueville and Gastel [4], we set

$$\theta_D := \sum_{k=0}^{m-2} \|w_k\|_{W^{2k+2-m,2}(D)} + \sum_{k=1}^{m-1} \|V_k\|_{W^{2k+1-m,2}(D)}$$
$$+ \|\eta\|_{W^{-2-m,2}(D)} + \|F\|_{W^{-2-m,2-1}(D)}$$

for $D \subset \mathbb{R}^{2m}$. Then, under the smallness assumption

$$\theta_{B^{2m}} < \epsilon_m,$$

they were able to find $A \in W^{m,2} \cap L^\infty (B^{2m}_{1/2}, \text{Gl}(n))$ and $B \in W^{-2-m,2} (B^{2m}_{1/2}, \mathbb{R}^{n \times n} \otimes \Lambda^2 \mathbb{R}^{2m})$ such that

$$\Delta^{m-1} dA + \sum_{k=0}^{m-1} (\Delta^k A) V_k - \sum_{k=0}^{m-2} (\Delta^k A) w_k = \delta B \quad \text{in } B^{2m}_{1/2}.$$  

Consequently, they obtained the local conservation law: $u$ solves (1.10) in $B^{2m}_{1/2}$, if and only if it is a distributional solution of

$$0 = \delta \left[ \sum_{l=0}^{m-1} (\Delta^l A) \Delta^{m-l-1} du - \sum_{l=0}^{m-2} (d\Delta^l A) \Delta^{m-l-1} u - \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} (\Delta^l A) \Delta^{k-l-1} d \langle V_k, du \rangle + \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} (d\Delta^l A) \Delta^{k-l-1} \langle V_k, du \rangle$$
$$- \sum_{k=0}^{m-2} \sum_{l=0}^{m-k-2} (\Delta^l A) d\Delta^{k-l-1} \delta (w_k du) + \sum_{k=0}^{m-2} \sum_{l=0}^{m-k-2} (d\Delta^l A) \Delta^{k-l-1} \delta (w_k du)$$
$$- \langle B, du \rangle \right],$$
in $B_{1/2}^{2m}$, where $d\Delta^{-1}\delta$ denotes the identity map. We mention that a similar local conservation law but with a slightly different form of $A$ is deduced by Hörter and Lamm [13].

Comparing with the "global" conservation law (1.6) of Rivièrë for the second order problem (1.5), it is natural to formulate an open problem as follows.

Problem A. Can we establish a global conservation law for (1.10)? That is, can we find $A \in W^{m,2} \cap L^{\infty}(B^{2m}, GL(n))$ and $B \in W^{2-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^2 \mathbb{R}^{2m})$ in the whole domain $B^{2m}$ such that (1.15) and thus (1.16) holds in $B^{2m}$.

The aim of this note is to give an affirmative answer to the above Problem. Our main result reads as follows.

**Theorem 1.1** (Global conservation law). There exist constants $\epsilon_m, C_m > 0$ such that under the smallness assumption (1.14), there exist $A \in W^{m,2} \cap L^{\infty}(B^{2m}, GL(n))$ and $B \in W^{2-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^2 \mathbb{R}^{2m})$ satisfying (1.15) in $B^{2m}$. Moreover,

$$\|A\|_{W^{m,2}(B^{2m})} + \|\text{dist}(A, SO(m))\|_{L^{\infty}(B^{2m})} + \|B\|_{W^{2-m,2}(B^{2m})} \leq C_m \theta B^{2m}.$$

Consequently, $u$ solves (1.10) if and only if it satisfies the conservation law (1.16) on $B^{2m}$.

We remark that it is able to find, by combining the proof of de Longueville and Gas-...
Theorem 1.3. Let \( f_i \) be a sequence in \( H^{-m} := (W^{m,2}(B^{2m}, \mathbb{R}^n))^* \) that converges to 0 in \( H^{-m} \) and \( u_i \) be a bounded sequence in \( W^{m,2}(B^{2m}, \mathbb{R}^n) \) solving
\[
\Delta^m u_i = \sum_{l=0}^{m-1} \Delta^l \langle V_{i,l}, du_i \rangle + \sum_{l=0}^{m-2} \Delta^l \delta (w_{l,i} du_i) + f_i \quad \text{in } B^{2m},
\]
where
\[
w_{k,i} \in W^{2k+2-m,2} \left( B^{2m}, \mathbb{R}^{n \times n} \right) \quad \text{for } k \in \{0, \ldots, m-2\}
\]
\[
V_{k,i} \in W^{2k+1-m,2} \left( B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m} \right) \quad \text{for } k \in \{0, \ldots, m-1\}.
\]
Moreover, the first order potential \( V_{0,i} \) has the decomposition \( V_{0,i} = du_i + F_i \) with
\[
\eta_i \in W^{2-m,2} \left( B^{2m}, so(n) \right) , \quad F_i \in W^{2-m,2/m+1} \left( B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m} \right).
\]
Assume that all of coefficients \( V_{l,i}, w_{l,i}, \eta_i, F_i \) converge weakly to \( V_l, w_l, \eta, F \) in the respective defining spaces. Then there exists a subsequence of \( u_i \) which converges weakly in \( W^{m,2} \) to a solution \( u \) of (1.10).

Our notations are standard. By \( A \lesssim B \) we mean there exists a universal constant \( C > 0 \) such that \( A \leq CB \).

2. Proofs of the main results

2.1. Proof of Corollary 1.2. In this section, we shall give a complete proof of Corollary 1.2. Since there is only one technical idea different from that of [14], our proof is almost verbatim to [14, Proof of Theorem 1.5]. In the key point we will point out the difference between our proof and theirs. We encourage the readers to have a closer look at the proof there for a comparison.

Proof of Corollary 1.2. Write
\[
\theta = \|V\|_{W^{1,2}(B^4)} + \|w\|_{L^2(B^4)} + \|\omega\|_{L^2(B^4)} + \|F\|_{L^{4/3,1}(B^4)}.
\]
To ease our notation, we shall often omit the defining domain in the various norms. For instance, the notation $W^{2,2}$ without a precise defining domain always means on the entire defining domain of the referred objects.

**Step 1.** Find a suitable Gauge transform

By [14, Equation (2.3)], there exists $\Omega \in W^{1,2}(B^4, \mathfrak{so}_m \otimes \wedge^1 \mathbb{R}^4)$ such that

$$
\begin{align*}
\Omega = P^{-1} dP + P^{-1} d^* \xi P \\
\|\Omega\|_{W^{1,2}} \leq c \|\omega\|_{L^2},
\end{align*}
$$

Then an application of [14, Theorem A.5] gives that, for $\epsilon_m$ sufficiently small with $\|\omega\|_{L^2} \leq \epsilon_m$, there exist $U \in W^{2,2}(B^4, \mathfrak{so}_m)$ and $P = e^U \in W^{2,2}(B^4, SO_m)$ and $\xi \in W^{2,2}(B^4, \mathfrak{so}_m \otimes \wedge^2 \mathbb{R}^4)$ and $c_m > 0$ such that

$$
\begin{align*}
\Omega &= P^{-1} dP + P^{-1} d^* \xi P \\
ed (i_{\partial B^4}^* \xi) &= 0 \\
\|P\|_{W^{2,2}} + \|\xi\|_{W^{2,2}} \leq c_m \|\Omega\|_{W^{1,2}} \leq C_m \epsilon_m.
\end{align*}
$$

**Step 2.** Rewrite $\omega$ and $W$

Direct computation gives

$$
\omega = d^* \Omega = d^* \left( P^{-1} dP + P^{-1} d^* \xi P \right).
$$

Thus

$$
W = d\omega + F = -P^{-1} d\Delta P + K_1,
$$

where

$$
K_1 = -dP^{-1} \Delta P - d\langle dP^{-1}, dP \rangle - d\langle P^{-1} d^* \xi, dP \rangle - d\langle dP^{-1}, d^* \xi P \rangle + F.
$$

Similar to [14, Equation (2.9)], using the improved Sobolev embedding

$$
W^{1,2}(B^4) \subset L^{4,2}(B^4),
$$

one can easily verify that $K_1 \in L^{4/3,1}(B^4)$ with

$$
\|K_1\|_{L^{4/3,1}(B^4)} \leq c \|\omega\|_{L^2(B^4)} + c \|F\|_{L^{4/3,1}(B^4)} \leq c \|\theta\|.
$$

**Step 3.** Reduce to an equivalent problem

Suppose now $A, B$ solves (1.8). Let $\tilde{A} = AP^{-1}$. Then, using (2.2), $\tilde{A}$ and $B$ solves the equivalent equation

$$
\begin{align*}
&d \Delta \tilde{A} + \Delta \tilde{A} K_2 + \nabla^2 \tilde{A} K_3 + d \tilde{A} K_4 + \tilde{A} K_5 = d^* BP^{-1}
\end{align*}
$$

in $B^4$, where

$$
\begin{align*}
&\|K_2\|_{W^{1,2}(B^4)} + \|K_3\|_{W^{1,2}(B^4)} + \|K_4\|_{L^{2}(B^4)} + \|K_5\|_{L^{4/3,1}(B^4)} < c \|\theta\|.
\end{align*}
$$

Note that by (2.3),

$$
\begin{align*}
d^* B &= \left( d \Delta \tilde{A} + \Delta \tilde{A} K_2 + \nabla^2 \tilde{A} K_3 + d \tilde{A} K_4 + \tilde{A} K_5 \right) P.
\end{align*}
$$

---

1Note that $d^* \alpha = -\text{div} (\alpha)$ for any 1-form $\alpha$. Hence $d^* (f \alpha) = fd^* \alpha - df \cdot \alpha$ for any function $f$ and 1-form $\alpha$. 

At this moment, different from the cut-off argument of [14, last paragraph on page 251], we shall use an extension argument as follows. Firstly, we extend \( K_i \) (2 \( \leq i \leq 5 \)) into \( \mathbb{R}^4 \) with compact support in \( B_2^4 \) in a norm-bounded way, for which we denoted by \( \tilde{K}_i \), such that
\[
\| \tilde{K}_2 \|_{W^{1,2}(B_2^4)} + \| \tilde{K}_3 \|_{W^{1,2}(B_2^4)} + \| \tilde{K}_4 \|_{L^2(B_2^4)} + \| \tilde{K}_5 \|_{L^{4/3,1}(B_2^4)} < c\theta.
\]
We next let \( \tilde{U} \) be a norm-bounded extension of \( U \) into \( B_2^4 \) with compact set so that \( \tilde{U} \in W^{2,2}(B_2^4, so_m) \). Then \( \tilde{P} = e^{\tilde{U}} \in W^{2,2}(B_2^4, SO_m) \) with norm bounded by the corresponding norm of \( P \). In order to find a solution \((\tilde{A}, \tilde{B})\) for (2.3)-(2.5), we solve the corresponding problem in the extended region \( B_2^4 \) as follows:

\[
d\Delta \tilde{A} + \Delta \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{A} \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 = d^* B\tilde{P}^{-1} \quad \text{in } B_2^4.
\]

It is clear that each solution of (2.7) automatically solves (2.3) in \( B^4 \).

To solve (2.7), we turn to look for \((\tilde{A}, \tilde{B})\) such that
\[
\begin{cases}
\tilde{A} = \tilde{A} + id & \text{in } B_2^4, \\
\tilde{B} = 0 & \text{in } B_2^4.
\end{cases}
\]

Then \((\tilde{A}, \tilde{B})\) necessarily solves the equivalent system
\[
\begin{cases}
\Delta^2 \tilde{A} = d^* \left( \Delta \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{A} \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 - d^* B\tilde{P}^{-1} \right) & \text{in } B_2^4, \\
\Delta \tilde{B} = d \left[ \left( \nabla^2 \tilde{A} + \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 \right) \tilde{P} \right] & \text{in } B_2^4, \\
\tilde{B} = 0 & \text{in } B_2^4.
\end{cases}
\]

To solve this system, we impose the following boundary value assumptions
\[
\begin{cases}
\tilde{A} = \frac{\partial \tilde{A}}{\partial r} = 0 & \text{on } \partial B_2^4, \\
\int_{B_2^4} \Delta \tilde{A} = 0, \\
\int_{\partial B_2^4} (*B) = 0.
\end{cases}
\]

**Step 4.** Solve the equivalent system (2.8) with boundary value (2.9).

We will use the fixed point argument as in [14] to find a solution \((\tilde{A}, \tilde{B})\) for problem (2.8)-(2.9). To this end, we introduce the Banach space \( \mathbb{H} = (\mathbb{H}, \| \cdot \|_{\mathbb{H}}) \) as follows:
\[
\mathbb{H} = \left\{ (u, v) \in W^{2,2} \cap L^\infty(B_2^4, M_m) \times W^{1,4/3}(B_2^4, M_m \otimes \Lambda^2 \mathbb{R}^4) : (u, v) \text{ satisfies (2.9)} \right\}
\]

with norm given by
\[
\|(u, v)\|_{\mathbb{H}} \equiv \|u\|_{W^{2,2}(B_2^4)} + \|u\|_{L^\infty(B_2^4)} + \|v\|_{W^{1,4/3}(B_2^4)}.
\]

Then, for any \((u, v)\) \( \in \mathbb{H} \), there exists a unique solution \( \bar{u} \in W^{2,2}(B_2^4) \) satisfying
\[
\begin{cases}
\Delta^2 \bar{u} = d^* \left( \Delta u \tilde{K}_2 + \nabla^2 u \tilde{K}_3 + du \tilde{K}_4 + u \tilde{K}_5 - d^* v \tilde{P}^{-1} + \tilde{K}_5 \right) & \text{in } B_2^4, \\
\bar{u} = \frac{\partial \bar{u}}{\partial r} = 0 & \text{on } \partial B_2^4, \\
\int_{B_2^4} \Delta \bar{u} = 0.
\end{cases}
\]

Note that \( u \in W^{2,2} \) and \( v \in W^{1,4/3} \) implies that
\[
f \equiv \Delta u \tilde{K}_2 + \nabla^2 u \tilde{K}_3 + du \tilde{K}_4 + u \tilde{K}_5 + \tilde{K}_5 \in L^{4/3,1}(B_2^4)
\]
with
\[ \| f \|_{L^{4/3,1}} \leq c\theta (\| u \|_{W^{2,2}} + \| u \|_{L^\infty} + 1) . \]
and
\[ d^*(d^* v\tilde{P}^{-1}) = -(d^* v, d\tilde{P}^{-1}) = \pm (d * v \wedge d\tilde{P}^{-1}). \]
Hence, applying \[14, \text{Lemma A.3}\] (with \( w = \tilde{P}^{-1}, \ p = 4/3, \ q = 4 \)), we deduce that \( \tilde{u} \in W^{3, (4/3,1)}(B^4_2) \) with
\[ \| \tilde{u} \|_{L^\infty} + \| \tilde{u} \|_{W^{2,2}} + \| \Delta \tilde{u} \|_{L^{4,1}} + \| d\Delta \tilde{u} \|_{L^{4,1}} \leq c\theta (\| u \|_{W^{2,2}} + \| u \|_{L^\infty} + 1 + \| v \|_{W^{1,4}}). \]
Remind that all the norms in the above and below are taken over \( B^4_2 \). Then \[14, \text{Lemma A.1}\] implies that the equations

\[
\begin{align*}
\Delta \tilde{v} &= d \left[ (d \Delta \tilde{u} + \Delta \tilde{u} \tilde{K}_2 + \nabla^2 \tilde{u} \tilde{K}_3 + d \tilde{u} \tilde{K}_4 + \tilde{u} \tilde{K}_5 + \tilde{K}_5) \tilde{P} \right] \quad \text{in } B^4_2, \\
d\tilde{v} &= 0 \quad \text{in } B^4_2, \\
i_{\partial B^4_2}(\tilde{v}) &= 0
\end{align*}
\]
has a unique solution \( \tilde{v} \in W^{1,4/3}(B^4_2, M_m \otimes \wedge^2 \mathbb{R}^4) \) satisfying
\[ \| d\tilde{v} \|_{L^{4/3,1}} \leq c \left( \| \tilde{u} \|_{W^{3,4/3,1}} + \theta \right) \leq c\theta \left( \| u \|_{W^{2,2}} + \| u \|_{L^\infty} + \| v \|_{W^{1,4}} + 1 \right). \]
Hence, for any \((u, v) \in \mathbb{H}\), there exists a unique \((\tilde{u}, \tilde{v}) \in \mathbb{H}\) satisfying

\[
\begin{align*}
\Delta \tilde{u} &= d \left[ (d \Delta \tilde{u} + \Delta \tilde{u} \tilde{K}_2 + \nabla^2 \tilde{u} \tilde{K}_3 + d \tilde{u} \tilde{K}_4 + \tilde{u} \tilde{K}_5 + \tilde{K}_5 + \tilde{K}_5) \tilde{P} \right] \quad \text{in } B^4_2, \\
d\tilde{u} &= 0 \quad \text{in } B^4_2, \\
i_{\partial B^4_2}(\tilde{u}) &= 0
\end{align*}
\]
with boundary value given by (2.9). Moreover, there exists \( c > 0 \), such that
\[ (2.10) \quad \| (\tilde{u}, \tilde{v}) \|_\mathbb{H} \leq c\theta (\| (u, v) \|_\mathbb{H} + 1). \]

Now set
\[ \mathbb{X} = \{ (u, v) \in \mathbb{H} : \| (u, v) \|_\mathbb{H} \leq 1 \} \]
and define the mapping \( T : \mathbb{H} \to \mathbb{H} \) by
\[ T(u, v) = (\tilde{u}, \tilde{v}). \]
The apriori estimate (2.10) implies that we can choose \( \epsilon_m \ll 1 \) such that \( c\theta \leq 1/2 \), and so \( T(\mathbb{X}) \subset \mathbb{X} \). Similar argument implies that \( T \) is a contraction operator on \( \mathbb{X} \), and
\[ \| T(u_1, v_1) - T(u_2, v_2) \|_\mathbb{H} \leq \frac{1}{2} \| (u_1, v_1) - (u_2, v_2) \|_\mathbb{H}. \]
Therefore \( T \) is a contraction mapping on \( \mathbb{X} \). The standard fixed point theorem gives a unique \((\tilde{A}, B) \in \mathbb{X}\) with
\[ T(\tilde{A}, B) = (\tilde{A}, B). \]
This solves (2.8)-(2.9). Moreover, by (2.10), we get \( \tilde{A} \in W^{3,4/3,1}(B^4_2) \) and \( B \in W^{1,4/3,1}(B^4_2) \), and
\[ \| (\tilde{A}, B) \|_\mathbb{H} \leq c\theta \leq \epsilon_m. \]

**Step 5.** Solve the orginal system
Let \((\tilde{A}, B)\) be the solution founded in Step 4 and set \(\tilde{A} = \tilde{A} + id\).

Then \(\tilde{A} \in W^{3,4/3,1}(B_2^4), B \in W^{1,4/3,1}(B_2^4)\) and they satisfy
\[
d^* \left( d\Delta \tilde{A} + \Delta \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{A} \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 - d^* B \tilde{P}^{-1} \right) = 0 \quad \text{in } B_2^4.
\]

Thus the nonlinear Hodge decomposition (see \cite[Theorem 2.4.14]{21}) implies that
\[
d\Delta \tilde{A} + \Delta \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{A} \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 - d^* B \tilde{P}^{-1} = d^* C + h,
\]
where \(C \in W^{1,4/3,1}(B_2^4, M_m \otimes \wedge^2 \mathbb{R}^4)\) satisfies \(i_{\partial B_2^4}^*(\ast C) = 0\) and \(h\) is a harmonic 2-form in \(B_2^4\). As \(\tilde{K}_i \equiv 0 \ (2 \leq i \leq 5)\) in a \(\delta\)-neighborhood of \(\partial B_2^4\), we have
\[
h + d^* C = d\Delta \tilde{A} - d^* B \quad \text{for } 2 - \delta < |x| \leq 2.
\]

Since \(i_{\partial B_2^4}^* \circ d = d \circ i_{\partial B_2^4}^*\) and \(i_{\partial B_2^4}^*(\ast C) = 0\), we deduce
\[
i_{\partial B_2^4}^*(\ast h) = (d\Delta \tilde{A} \cdot \nu)^{vol_{\partial B_2^4}} + d i_{\partial B_2^4}^* B = 0.
\]

This implies that \(h \equiv 0\), since \(B_2^4\) has trivial homotopy groups, from which we conclude that
\[
d\Delta \tilde{A} + \Delta \tilde{A} \tilde{K}_2 + \nabla^2 \tilde{A} \tilde{K}_3 + d\tilde{A} \tilde{K}_4 + \tilde{A} \tilde{K}_5 - d^* B \tilde{P}^{-1} = d^* C
\]
with \(i_{\partial B_2^4}^*(\ast C) = 0\). By the same argument as that of \cite[14], we infer that \(C \equiv 0\) by choosing \(\epsilon_m\) sufficiently small. The proof is complete. \(\square\)

2.2. Proof of Theorem 1.1. In this section, we give a sketch of the proof of Theorem 1.1, since the proof is very similar to \cite[Proof of Theorem 4.1(i)]{4}.

**Step 1.** Find a suitable Gauge transform.

Given \(\eta \in W^{2-m,2}(B_2^m, so(n))\), we extended it in a norm-bounded way to \(B_2^{2m}\) with compact support so that its extension \(\hat{\eta} \in W^{2-m,2}(B_2^{2m}, so(n))\). Reasoning as in \cite[Second paragraph, page 11]{4}, we can find \(\Omega \in W^{m-1,2}(B_2^{2m}, so(n) \wedge \mathbb{R}^{2m})\) such that

\[
\begin{cases}
\Delta^{m-2} d^* \Omega = -\hat{\eta} \\
\|\Omega\|_{W^{m-1,2}(B_2^{2m})} \leq c\|\hat{\eta}\|_{W^{2-m,2}(B_2^{2m})}.
\end{cases}
\]

When the smallness assumption (1.14) is satisfied for some sufficiently small \(\epsilon_m\), we may apply \cite[Theorem 2.4]{4} to find \(P \in W^{m,2}(B_2^m, SO(n))\) and \(\xi \in W^{m,2}(B_2^m, so(n) \otimes \mathbb{R}^{2m})\) such that
\[
\Omega = P^{-1} dP + P^{-1} d^* \xi P \quad \text{on } B_2^m.
\]

Moreover, \(\|dP\|_{W^{m-1,2}(B_2^m)} + \|d^* \xi\|_{W^{m-1,2}(B_2^m)} \leq c\|\Omega\|_{W^{m-1,2}(B_2^{2m})}\).

**Step 2.** Rewrite \(V_0\).

By definition, we have
\[
V_0 = d\eta + F = -d\Delta^{m-2} d^* (P^{-1} dP + P^{-1} d^* \xi P) + F \quad \text{on } B_2^m.
\]

Similar to \cite[Third paragraph, page 11]{4}, we could further write it as
\[
V_0 = -P^{-1} d\Delta^{m-1} P + K,
\]
with \(K \in W^{2-m,2}_{m+1,1}(B_2^{2m})\).

**Step 3.** Reduce to an equivalent problem.
Suppose $A, B$ solves (1.15). Let $\tilde{A} = AP^{-1}$. Then $(\tilde{A}, B)$ solves the equivalent equation (see [4, Equation (11)])

\begin{equation}
(2.12) \quad d\Delta^{m-1}\tilde{A} + \sum_{j=0}^{2m-2} \langle D^j\tilde{A}, K_j \rangle + K_0 = d^*BP^{-1},
\end{equation}

with all the coefficient functions $K_0, \cdots, K_{m-2}$ bounded by $c\theta$ in the respective norm spaces $K_j \in W^{j+1-m,2}(B^{2m})$ for $j \in \{1, \cdots, m-2\}$.

Different from the cut-off argument used in [4, Last paragraph, page 11], we again use an extension argument similar to the proof of Corollary 1.2. Namely, we extend all the coefficient functions $K_j, j = 0, \cdots, m-2$ into $\mathbb{R}^{2m}$ with compact support in $B^{2m}_{2}$ in a norm-bounded way, for which we denoted by $\tilde{K}_j$. Then we use a similar extension $\tilde{P}$ of $P$ such that $\tilde{P} \in W^{m,2}(B^{2m}_2, SO(n))$.

In order to find a solution $(\tilde{A}, B)$ for (2.12), we solve the corresponding problem in the extended region $B^{2m}_{2}$ as follows:

\begin{equation}
(2.13) \quad d\Delta^{m-1}\tilde{A} + \sum_{j=0}^{2m-2} \langle D^j\tilde{A}, \tilde{K}_j \rangle + \tilde{K}_0 = d^*BP^{-1} \quad \text{in } B^{2m}_{2}.
\end{equation}

It is clear that each solution of (2.13) automatically solves (2.12) in $B^{2m}_{2}$.

From now on, the remaining steps are identical to that used in [4]. One uses a fix point argument to find solutions $(\tilde{A}, B)$ that solves (2.13); for details, see [4, Proof of Lemma 4.2]. The proof is thus complete.

2.3. **Proof of Theorem 1.3.** In this section, we shall give a proof of Theorem 1.3, following closely the approach of Rivière [17] for the second order system (1.5).

**Step 1.** Cover the ball $B^{2m}_{2}$ with small “good balls” $B^{2m}_j \subset B^{2m}_{2}$.

We may cover the ball $B^{2m}_{2}$ by balls $B^{2m}_j \subset B^{2m}_{2}$ in such a way that in every $B^{2m}_j$ we may assume that

$$\theta_{B^{2m}_j} < \epsilon_m.$$ 

This collection of balls will cover the whole $B^{2m}_{2}$ outside a finite set of points.

**Step 2.** Derive the convergence on each good ball $B_j := B^{2m}_j$.

Since $\theta_{B^{2m}_j} < \epsilon_m$, we may apply Theorem 1.1 to find $A_i = A_{i,j} \in W^{m,2}(B^{2m}_{2}, \text{GL}(n))$ and $B_i = B_{i,j} \in W^{2-m,2}(B^{m}_{j}, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^{2m})$ such that

\begin{equation}
(2.14) \quad \Delta^{m-1}dA_i + \sum_{k=0}^{m-1} (\Delta^k A_i)V_{k,i} - \sum_{k=0}^{m-2} (\Delta^k dA_i)w_{k,i} = \delta B_i \quad \text{in } B_j.
\end{equation}
Moreover, $u_i$ is a distributional solution of
\[ -A_i f_i = \delta \sum_{l=0}^{m-1} \left( \Delta^l A_i \right) \Delta^{m-l-1} du_i - \sum_{l=0}^{m-2} \left( d\Delta^l A_i \right) \Delta^{m-l-1} u_i - \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( \Delta^l A_i \right) \Delta^{k-l-1} d \langle V_{k,i}, du_i \rangle + \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( d\Delta^l A_i \right) \Delta^{k-l-1} \langle V_{k,i}, du_i \rangle \\
- \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( \Delta^l A_i \right) d\Delta^{k-l-1} \delta (w_{k,i} du_i) + \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( d\Delta^l A_i \right) \Delta^{k-l-1} \delta (w_{k,i} du_i) - \langle B_i, du_i \rangle. \]

Up to a subsequence if necessary, we may assume that $A_i$ converges weakly to $A \in W^{m,2}$ and $B_i$ converges weakly in $W^{3-m,2m}$.

It is easy to check that
\[ \Delta^{m-1} dA_i \to \Delta^{m-1} dA \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \langle \Delta^k A_i \rangle V_{k,i} \to \langle \Delta^k A \rangle V_k \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \langle \Delta^k dA_i \rangle w_{k,i} \to \langle \Delta^k dA \rangle w_k \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \delta(B_{k,i}) \to \delta(B_i) \quad \text{in} \quad \mathcal{D}'(B_j). \]

Similarly, we have
\[ \sum_{l=0}^{m-1} \left( \Delta^l A_i \right) \Delta^{m-l-1} du_i \to \sum_{l=0}^{m-1} \left( \Delta^l A \right) \Delta^{m-l-1} du \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \sum_{l=0}^{m-2} \left( d\Delta^l A_i \right) \Delta^{m-l-1} u_i \to \sum_{l=0}^{m-2} \left( d\Delta^l A \right) \Delta^{m-l-1} u \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( \Delta^l A_i \right) \Delta^{k-l-1} d \langle V_{k,i}, du_i \rangle \to \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( \Delta^l A \right) \Delta^{k-l-1} d \langle V_k, du \rangle \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( d\Delta^l A_i \right) \Delta^{k-l-1} \langle V_{k,i}, du_i \rangle \to \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( d\Delta^l A \right) \Delta^{k-l-1} \langle V_k, du \rangle \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( \Delta^l A_i \right) d\Delta^{k-l-1} \delta (w_{k,i} du_i) \to \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( \Delta^l A \right) d\Delta^{k-l-1} \delta (w_k du) \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( d\Delta^l A_i \right) \Delta^{k-l-1} \delta (w_{k,i} du_i) \to \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( d\Delta^l A \right) \Delta^{k-l-1} \delta (w_k du) \quad \text{in} \quad \mathcal{D}'(B_j), \]
\[ \langle B_i, du_i \rangle \to \langle B, du \rangle \quad \text{in} \quad \mathcal{D}'(B_j) \]
and
\[ A_i f_i \to 0 \quad \text{in} \quad \mathcal{D}'(B_j). \]
Combining (2.14) - (2.18) all together, we infer from Theorem 1.1 that \( u \) is a solution to (1.10) in \( B_j \).

**Step 3.** Remove the singular set on \( B^{2m} \).

By **Step 2**, we know the distribution

\[
T := \Delta^m u - \sum_{l=0}^{m-1} \Delta^l \langle V_l, du \rangle - \sum_{l=0}^{m-2} \Delta^l \delta (w_l du)
\]

vanishes on each good ball \( B_j \) and hence \( T \) only supports at finitely many points in \( B^{2m} \). On the other hand, we know that \( T \in H^{-m} + L^1 \) and so it has to be identically zero on \( B^{2m} \). This completes the proof.

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**References**

[1] R.C. Adams and J.F. Fournier, *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] S.-Y. A. Chang, L. Wang and P.C. Yang, *A regularity theory of biharmonic maps*. Commun. Pure Appl. Math. 52(9) (1999), 1113-1137.

[3] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*. J. Math. Pures Appl. (9) 72 (1993), 247-286.

[4] F.L. de Longueville and A. Gastel, *Conservation laws for even order systems of polyharmonic map type*. Calc. Var. Partial Differential Equations 60, 138 (2021).

[5] A. Gastel and C. Scheven, *Regularity of polyharmonic maps in the critical dimension*. Comm. Anal. Geom. 17 (2009), no. 2, 185-226.

[6] C.-Y. Guo, Changyou Wang and C.-L. Xiang, *\( L^p \)-regularity for fourth order elliptic systems with antisymmetric potentials in higher dimensions*. Preprint at arXiv:2111.07227v2, 2021.

[7] C.-Y. Guo and C.-L. Xiang, *Regularity of solutions for a fourth order linear system via conservation law*. J. Lond. Math. Soc. (2) 101 (2020), no. 3, 907-922.

[8] C.-Y. Guo and C.-L. Xiang, *Regularity of weak solutions to higher order elliptic systems in critical dimensions*. Trans. Amer. Math. Soc. 374 (2021), no. 5, 3579-3602.

[9] C.-Y. Guo, C.-L. Xiang and G.-F. Zheng, *The Lamm-Rivière system I: \( L^p \) regularity theory*. Calc. Var. Partial Differential Equations 60, 213 (2021).

[10] C.-Y. Guo, C.-L. Xiang and G.-F. Zheng, *\( L^p \) regularity theory for even order elliptic systems with antisymmetric first order potentials*. ArXiv-preprint https://arxiv.org/abs/2106.10818, 2021.

[11] P. Goldstein, P. Strzelecki and A. Zatorska-Goldstein, *On polyharmonic maps into spheres in the critical dimension*. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 1387-1405.

[12] F. Hélein, *Harmonic maps, conservation laws and moving frames*. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[13] J. Hörter and T. Lamm, *Conservation laws for even order elliptic systems in the critical dimensions - a new approach*. Calc. Var. Partial Differential Equations 60, 125 (2021).

[14] T. Lamm and T. Rivière, *Conservation laws for fourth order systems in four dimensions*. Comm. Partial Differential Equations 33 (2008), 245-262.

[15] T. Lamm and C. Wang, *Boundary regularity for polyharmonic maps in the critical dimension*. Adv. Calc. Var. 2 (2009), 1-16.

[16] T. Rivière, *Everywhere discontinuous harmonic maps into spheres*. Acta Math. 175 (1995), no. 2, 197-226.

[17] T. Rivière, *Conservation laws for conformally invariant variational problems*. Invent. Math. 168 (2007), 1-22.
[18] T. Rivière, *Analysis aspects of Willmore surfaces*. Invent. Math. **174**:1 (2008), 1-45.

[19] T. Rivière, *The role of integrability by compensation in conformal geometric analysis*. Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging, 93-127, Sémin. Congr., 22, Soc. Math. France, Paris, 2011.

[20] T. Rivière, *Conformally invariant variational problems*. Lecture notes at ETH Zurich, available at https://people.math.ethz.ch/ riviere/lecture-notes, 2012.

[21] G. Schwarz, *Hodge decomposition—a method for solving boundary value problems*. Lecture Notes in Mathematics, 1607. Springer-Verlag, Berlin, 1995.

[22] T. Rivière and M. Struwe, *Partial regularity for harmonic maps and related problems*. Comm. Pure Appl. Math. **61** (2008), 451-463.

[23] B. Sharp and P. Topping, *Decay estimates for Rivière’s equation, with applications to regularity and compactness*. Trans. Amer. Math. Soc. **365** (2013), no. 5, 2317-2339.

[24] J. Shatah, *Weak solutions and development of singularities of the SU(2) σ-model*. Commun. Pure Appl. Math. **41** (1988), 459-469.

[25] M. Struwe, *Partial regularity for biharmonic maps, revisited*. Calc. Var. Partial Differential Equations **33** (2008), 249-262.

[26] K. Uhlenbeck, *Connections with Lp bounds on curvature*. Comm. Math. Phys. **83** (1982), 31-42.

[27] C.Y. Wang, *Biharmonic maps from R4 into a Riemannian manifold*. Math. Z. **247** (2004), 65-87.

[28] C.Y. Wang, *Stationary biharmonic maps from Rm into a Riemannian manifold*. Comm. Pure Appl. Math. **57** (2004), 419-444.

[29] H.C. Wente, *An existence theorem for surfaces of constant mean curvature*. J. Math. Anal. Appl. **26** (1969), 318-344.

[30] W.P. Ziemer, *Weakly differentiable functions*. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

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