Generalized Einstein gravities and generalized AdS symmetries

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Abstract

We consider the curvatures 2-form asociated with AdSL₄-valued one-form gauge connection, and then we construct a four-dimensional action that generalize the Einstein-Hilbert gravity. It is shown t hat the Maxwell extension of Einstein gravity can be obtained from AdSL₄-gravity making use of the Inönü-Wigner contraction method. In the same way, by gauging the AdSL₅-spacetime algebra, the Einstein-Hilbert gravity is exten ded including the vector fields $k_{µ}^{ab}$ and $h_{µ}^{a}$ which are associated with non-Abelian tensors and non Abelian vectors charges in the AdSL₅ algebra.

The $B₅$ extension of Einstein gravity can be obtained from AdSL₅-gravity using of the above mentioned contraction procedure. Some aspects of a gravity based on the algebra AdSL₆ are considered in an appendix.

1 Introducción

Some time ago was extended the standard geometric framework of Einstein gravity by gauging the Maxwell algebra [1], [2] which led to a generalized cosmological term that includes a contribution from the six four-vector fields $k_{µ}^{ab}$ which introduce a new set of curvatures denoted by $F_{µν}^{ab}$ (in addition to the well-known torsion and Lorentz curvature) that allow to build generalizations of the Einstein action.

The Maxwell algebra [3], [4] can also be obtained from (A)dS algebra using the Lie algebra expansion procedure developed in Refs. [5], [6], [7], [8], [9]. This procedure allows obtaining two families of algebras, which are known as generalized Poincaré algebras (also called $Bₙ$ algebra, where Maxwell’s algebra corresponds to the particular case known as $B₄$ algebra ) and generalized AdS algebras (also called AdSLₙ algebras) [11], [12], [13].

An Expansion is, in general, an algebra dimension-changing process, i.e., is a way to obtain new algebras of increasingly higher dimensions from a given
one. A physical motivation for increasing the dimension of Lie algebras is that increasing the number of generators of an algebra is a non-trivial way of enlarging spacetime symmetries. Examples of this can be found in [1], [2], [14], [15].

The generalized Poincaré algebra \( \mathfrak{B}_n \) [10], [11], [12], [13] can be obtained from anti-de-Sitter algebra and the semigroup \( S_{E}^{2n-1} = \{ \lambda_0, \cdots, \lambda_{2n} \} \) whose multiplication law is given by \( \lambda_a \lambda_b = \lambda_{a+b} \) when \( \alpha + \beta \leq 2n \) and \( \lambda_a \lambda_b = 0 \) when \( \alpha + \beta > 2n \), where \( \lambda_{2n} \) corresponds to the zero element of the semigroup. The generators of \( \mathfrak{B}_n \) denoted by \( \{ P_a, J_{ab}, Z_{ab}^{(i)}, Z_{a}^{(i)} \} \) satisfy the following commutation relations

\[
\begin{align*}
[P_a, P_b] &= Z_{ab}^{(1)}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \\
[J_{ab}, Z_{c}^{(i)}] &= \eta_{bc} Z_{a}^{(i)} - \eta_{ac} Z_{b}^{(i)}, \\
[Z_{ab}^{(i)}, P_c] &= \eta_{bc} Z_{a}^{(i)} - \eta_{ac} Z_{b}^{(i)}, \\
[Z_{a}^{(i)}, Z_{c}^{(j)}] &= \eta_{bc} Z_{c}^{(i+j)} - \eta_{ac} Z_{b}^{(i+j)}, \\
[J_{ab}, Z_{cd}^{(i)}] &= \eta_{bc} Z_{cd}^{(i)} + \eta_{ad} Z_{bc}^{(i)} - \eta_{ac} Z_{bd}^{(i)} - \eta_{bd} Z_{ac}^{(i)}, \\
[Z_{i}^{(i)}, Z_{jd}^{(j)}] &= \eta_{bc} Z_{i}^{(i+j)} + \eta_{ad} Z_{bc}^{(i+j)} - \eta_{ac} Z_{bd}^{(i+j)} - \eta_{bd} Z_{ac}^{(i+j)}, \\
[P_a, Z_{b}^{(i)}] &= Z_{ab}^{(i+1)}, \\
[Z_{i}^{(i)}, Z_{b}^{(j)}] &= Z_{ab}^{(i+j+1)}.
\end{align*}
\]

(1)

where, \( J_{ab} \) and \( \tilde{P}_a \) are the generators of the anti-de-Sitter algebra and \( J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}, Z_{ab}^{(i)} = \lambda_{2i} \otimes \tilde{J}_{ab}, P_a = \lambda_1 \otimes \tilde{P}_a \) and \( Z_{a}^{(i)} = \lambda_{2i+1} \otimes \tilde{P}_a \), with \( i, j = 0, 1, \cdots, n - 1 \).

From the commutation relations (1) we see that the set \( \mathfrak{B}_n^I = \{ P_a, Z_{ab}^{(i)}, Z_{a}^{(i)} \} \) satisfies the conditions \( \mathfrak{B}_n^I \subset \mathfrak{B}_n^I, \{ \text{so}(3, 1), \mathfrak{B}_n^I \} \subset \mathfrak{B}_n^I \), i.e., \( \mathfrak{B}_n^I \) is an ideal of the generalized Poincaré Algebra \( \mathfrak{B}_n \) generated by \( \{ P_a, Z_{ab}^{(i)}, Z_{a}^{(i)} \} \). This means that the generalized Poincaré Algebra \( \mathfrak{B}_n \) is the semidirect sum of the Lorentz algebra \( \text{so}(3, 1) \) and the ideal \( \mathfrak{B}_n^I \), that is, \( \mathfrak{B}_n = \text{so}(3, 1) \oplus \mathfrak{B}_n^I \).

From (1) it is also possible to see that for \( i = 0, n = 1 \) we have the Poincaré algebra that in this nomenclature we will call \( \mathfrak{B}_i \) (also called \( \mathfrak{B}_3 \)); for \( i = 1, n = 2 \) we have Maxwell’s algebra which in this nomenclature we will call \( \mathfrak{B}_2 \) (also called \( \mathfrak{B}_4 \)); for \( i = 2, n = 3 \) we have the algebra, \( \mathfrak{B}_3 \) (also called \( \mathfrak{B}_5 \)); for \( i = 3, n = 4 \) we have the algebra \( \mathfrak{B}_4 \) (also called \( \mathfrak{B}_6 \)), etc. These algebras and their relation to AdSL\(_n\) algebras are shown in Appendix A.

On the other hand the AdSL\(_n\) algebras [11], [12], [13] can be obtained from the anti-de-Sitter algebra and the semigroup \( \lambda_0 \{ \lambda_0 \}^{N} = \{ \lambda_0 \}^{N} \) whose multiplication law is given by \( \lambda_a \lambda_b = \lambda_{a+b} \) when \( \alpha + \beta \leq 2N \) and \( \lambda_a \lambda_b = \lambda_{a+b-2[N+1/2]} \) when \( \alpha + \beta > 2N \), where \( [x] \) is the integer part of \( x \). Note that for \( N \) odd the semigroup
$S_{M}^{(N)}$ coincides with the cyclic group of $(N + 1)$ elements $\mathbb{Z}_{N+1}$.

It might be of interest to mention that for the AdSL$_n$ algebras with $n = 3, 4, 5, 6$ it is found that in the cases $n = 3$ and 5 they cannot be expressed as a semidirect sum of the Lorentz algebra and an ideal, while for the cases $n = 4$ and 6 it is straightforward to see that they can be written as the semidirect sum of the Lorentz algebra and an ideal (see Appendix A).

The consequences of considering a space-time with Maxwell symmetries in the description of the gravitational field have been studied, in the context of Chern-Simons gravities, in Refs. [11], [16], [19], and in the context of four-dimensional gravity, in Refs. [1], [2].

The implications of the existence of space-times with symmetries given both, by generalized Poincaré algebras and by generalized AdS algebras, were studied, in the case of odd dimensions, in references [17], [18], [10], and in the even dimensions, in the context of WZW terms, in Refs. [20], [21], [22], [23].

An interesting open problem is to obtain an understanding of the gravitational field from the direct construction of 4-dimensional invariant actions both under the generalized Poincaré algebras and the AdSL$_n$ algebras.

The geometry of space-times based on generalized Poincare algebras and generalized AdS-Lorentz algebras involve new gauge fields and therefore new tensor curvatures that allow to construct new gravity actions which lead to modifications of the standard gravity.

This paper is organized as follows: In Section 2 we consider the construction of the curvatures 2-form associated with AdSL$_4$-valued one-form gauge connection, and a four-dimensional gravity based in the AdSL$_4$ algebra is constructed using the procedure described in references [1], [2]. In Section 3 we construct a four-dimensional action, using an AdSL$_5$-valued one-form gauge connection following the same procedure of Section 2. Actions based on the generalized Poincare algebras ($\mathcal{B}_4$ and $\mathcal{B}_5$) are obtained in Section 4 by making use of the well-known contraction methods. Finally Concluding Remarks are presented in Section 5. Two Appendices are included, where are considered details about the algebras $\mathcal{B}_n$ and AdSL$_m$ as well as some aspects of the construction of an action gravity based in an AdSL$_6$-valued one-form gauge connection.

2 AdSL$_4$ extension of Einstein-Hilbert-Cartan gravity

In this section we will study the extension of Einstein’s theory of gravitation considering AdSL$_4$ symmetries.

2.1 Gauging the AdSL$_4$-algebra

In order to write down the two-form curvatures we start from the AdSL$_4$-valued one-form gauge connection
\[
A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{T} e^a P_a + \frac{1}{2} k^{ab} Z_{ab}
\]  
(2)

where \(a, b = 0, 1, 2, 3\) are tangent space indices raised and lowered with the Minkowski metric \(\eta_{ab}\), and where

\[
e^a = e^a_\mu dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu dx^\mu, \quad k^{ab} = k^{ab}_\mu dx^\mu,
\]

are the \(e^a_\mu\) vierbein, the \(\omega^{ab}_\mu\) spin connection and the \(k^{ab}_\mu\) new non-abelian gauge fields. The generators \(P_a, J_{ab}, Z_{ab}\) satisfy the AdS\(_4\)-algebra commutation relations (64).

The corresponding two-form curvature can be obtained from

\[
F = dA + \frac{1}{2} [A, A].
\]
(3)

In fact, from (2) and (3) we find

\[
F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{T} T^a P_a + \frac{1}{2} F^{ab} Z_{ab},
\]
(4)

with

\[
\begin{align*}
T^a &= T^a + k^a_c e^c, \\
R^{ab} &= d\omega^{ab} + \omega^a_c \omega^{cb}, \\
F^{ab} &= Dk^{ab} + k^{[a}_c k^{b]}_c + \Lambda e^a e^b,
\end{align*}
\]
(5)

where \(T^a\) and \(R^{ab}\) are the standard torsion and curvature, \(F^{ab}\) is the curvature of the non-abelian gauge fields \(k^{ab}\), \(Dk^{ab} = dk^{ab} + \omega^{[a}_c k^{b]}_c\) is the covariant derivative with respect to \(\omega^{ab}\), and \(\Lambda = 1/l^2\).

The term \(k^{[a}_c k^{b]}_c\) comes from the commutator \([Z_{ab}, Z_{cd}]\), which is zero by Maxwell’s algebra. \(F^{ab}\) corresponds to the curvature 2-form associated with gauge fields \(k^{ab}\) for the AdS\(_4\) algebra. The term \(k^{a}_c k^{b}\) comes from the commutator \([P_a, Z_{cd}]\), which for the Maxwell algebra is zero. We can therefore recover the structure equations of Maxwell’s algebra by setting these commutators to zero.

The corresponding Bianchi identities are obtained in the usual form. The covariant derivative of the generic curvature \(F\) is given by

\[
DF = dF + [A, F].
\]
(6)

From (2), (3) and (6) we find

\[
\begin{align*}
DT^a + k^a_b T^b &= (R^a_b + F^a_b) e^b, \\
DR^{ab} &= 0, \\
DF^{ab} + k^{[a}_c F^{c][b]} + \Lambda e^{[a} T^{b]} + R^{[a}_c k^{a]_b} &= 0,
\end{align*}
\]
(7) (8) (9)
where $DT^a = dT^a + \omega^a_b T^b$.

Taking into account that the gauge potentials transform as $\delta \varepsilon A = D\varepsilon$, where $\varepsilon$ is a $\text{AdS}_4$ algebra valued parameter given by

$$\varepsilon(x) = \frac{1}{2} \pi^{ab} J_{ab} + \frac{1}{4} \pi^a P_a + \frac{1}{2} \varepsilon^{ab} Z_{ab},$$

(10)

it is direct to find

$$\delta \varepsilon e^a = D\varepsilon e^a + \pi^a_d e^d + k^a c \varepsilon e^c + \xi^a c e^c,$$

(11)

$$\delta \varepsilon \omega^{ab} = D\pi^{ab},$$

(12)

$$\delta \varepsilon k^{ab} = D\varepsilon^{ab} + k^{[a c [b]} + k^{a c e]} + \Lambda e^{[a \rho b]}.$$ (13)

On the other hand, from $\delta \varepsilon F = [F, \varepsilon]$ we find that the curvatures transform as

$$\delta R^{ab} = R^{ab} - \frac{1}{2} \varepsilon^{abc} \pi^{c[b]},$$

(14)

$$\delta T^a = \pi^a c T^c + R^a c \varepsilon^c + \xi^a c T^c,$$

(15)

$$\delta F^{ab} = F^{ab} + R^{ab} - \frac{1}{2} \varepsilon^{abc} \pi^{c[b]} + \Lambda T^{[a \rho b]}.$$ (16)

It might be of interest to note that the terms $F^a c \varepsilon^c$ have their origin in the commutators $[Z_{ab}, P_c]$ and $[P_a, Z_{cd}]$ respectively, which are null in the case of Maxwell’s algebra. Similarly the terms $\pi^a c T^c$, $R^a c \varepsilon^c$, $F^{a c e}_{[b]}$, $F^{a c e}_{[b]}$, $R^{a c e}_{[b]}$, $\frac{1}{2} \pi T^{[a \rho b]}$, have their origin in the commutators $[P_a, J_{cd}]$, $[J_{ab}, P_c]$, $[Z_{ab}, Z_{cd}]$, $[Z_{ab}, J_{cd}]$, $[J_{ab}, Z_{cd}]$, $[P_a, P_b]$, which is null for Maxwell’s algebra, respectively.

### 2.2 Four-dimensional $\text{AdS}_4$-action gravity

Following the procedure developed in Refs. [1], [2] we postulate the following lagrangians

$$\mathcal{L}_1 = \varepsilon_{abcd} R^{ab} F^{cd}, \quad \mathcal{L}_2 = \frac{1}{2} \varepsilon_{abcd} F^{ab} F^{cd},$$

(17)

from where we find that a gravitational action can be constructed from the lagrangian

$$-\frac{1}{2\kappa \Lambda} \mathcal{L}_1 = \mathcal{L}_{E-\Lambda} - \frac{1}{2\kappa \Lambda} \varepsilon_{abcd} R^{ab} k^{[a c [b]} + \text{boundary terms},$$

(18)

where

$$\mathcal{L}_{E-\Lambda} = -\frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} e^c e^d,$$

is the Einstein-Hilbert term, and the lagrangian

$$\frac{\lambda}{2\kappa \Lambda^2} \mathcal{L}_2 = \mathcal{L}_{\text{cosm}} + \frac{\lambda}{4\kappa \Lambda^2} \varepsilon_{abcd} D k^{ab} D k^{cd} + \frac{\lambda}{2\kappa \Lambda^2} \varepsilon_{abcd} D k^{ab} k^{[c} e^{d]} + \frac{\mu}{2\kappa} \varepsilon_{abcd} D k^{ab} e^c e^d + \frac{\lambda}{4\kappa \Lambda^2} \varepsilon_{abcd} k^{[a} f^{j} e^{k] j} k^{e} e^{d]} + \frac{\mu}{2\kappa} \varepsilon_{abcd} k^{[a e} k^{b]} e^c e^d,$$

(19)
where $\Lambda \equiv 1/l^2$, $\lambda$ is the cosmological constant, $\mu = \lambda/\Lambda$ and

$$\mathcal{L}_{\text{cosm}} = \frac{\lambda}{4\kappa} \varepsilon_{abcd} e^a e^b e^c e^d,$$

is the standard cosmological term. This means $\mathcal{L}_2$ can be identified, following \[1\], as an extension of the cosmological term for the AdS$\mathcal{L}_4$ algebra. So that a Lagrangian for the AdS$\mathcal{L}_4$ gravity is given by

$$\mathcal{L}_{\text{AdS} \mathcal{L}_4} = \frac{\mu}{2\kappa} \varepsilon_{abcd} Dk^{ab} e^c e^d$$

which, after some algebra, takes the form

$$\mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_{\text{cosm}} + \frac{\mu}{2\kappa} \varepsilon_{abcd} Dk^{ab} e^c e^d$$

where

$$F_{ab} = \frac{1}{\kappa} \left[ Dk^{ab} + \Lambda e^a e^b + k^{[a} k^{b]} e^c - R^{ab} e^c \right],$$

i.e., leads to the field equation

$$\varepsilon_{abcd} \left( \mu F^{ab} e^c - R^{ab} e^c \right) = 0,$$

where $F^{ab}$ is given by the last equation of \[5\].

### 2.3 Field Equations for AdS$\mathcal{L}_4$-extended Einstein gravity

The variation of the action \[22\] with respect to the vielbein $e^a$ leads to

$$\delta_{e^a} \mathcal{L} = \frac{1}{\kappa} \varepsilon_{abcd} \left( \mu \left[ Dk^{ab} + \Lambda e^a e^b + k^{[a} k^{b]} e^c \right] e^c - R^{ab} e^c \right) \delta e^d = 0,$$

i.e., leads to the field equation

$$\varepsilon_{abcd} \left( \mu F^{ab} e^c - R^{ab} e^c \right) = 0,$$

where $F^{ab}$ is given by the last equation of \[5\].

The variation of the action \[22\] with respect to the spin connection $\omega^{ab}$ leads to

$$\delta_{\omega^{ab}} \mathcal{L} = \delta \omega^{ab} \left\{ -\frac{1}{\kappa} \varepsilon_{abcd} \left[ Dk^{ed} e^d - \frac{\mu}{\Lambda} k^e \left( Dk^{cd} + k^{[c} k^{d]} e^e \right) \right] \right\} = 0.$$
so that, the field equation corresponding to the variation with respect to $\omega^{ab}$ is

$$\frac{1}{\kappa} \varepsilon_{abcd} \left( T^c e^d - \frac{\mu}{\Lambda} k^c e F^{ed} \right) = 0 \quad (25)$$

The variation of the action (22) with respect to the vielbein $k^{ab}$ leads to

$$\delta k^{ab} \mathcal{L} = -\frac{\mu}{\kappa} \varepsilon_{abcd} \delta k^{ab} D e^d - \frac{\mu}{2\kappa\Lambda} \varepsilon_{abcd} D k^{ab} \delta k^{cd}$$

$$- \frac{2\mu}{\kappa\Lambda} \varepsilon_{abcd} R^{ab} k^e \delta k^{ed} + \frac{2\mu}{\kappa\Lambda} \varepsilon_{abcd} k^{a[b} k^{c]} k^e \delta k^{ed}$$

$$+ \frac{2\mu}{\kappa} \varepsilon_{abcd} k^a e \delta k^{eb} + \frac{2\mu}{\kappa} \varepsilon_{abcd} D k^{ab} k^c \delta k^{ed}$$

$$+ \text{boundary terms} = 0, \quad (26)$$

and therefore the corresponding field equation is given by

$$\frac{1}{\kappa} \varepsilon_{abcd} \left( T^c e^d - \frac{3}{2} \frac{\mu}{\Lambda} k^c e \mathcal{R}^{de} - \frac{2\mu}{\Lambda} k^c e F^{ed} \right) = 0, \quad (27)$$

where $T^c, \mathcal{R}^{ce}$ and $F^{ed}$ are given by (5).

3 AdS$\mathcal{L}_5$-generalizacien of the Einstein–Hilbert–Cartan gravity

Now we will study the generalizacion of Einstein gravity considering AdS$\mathcal{L}_5$ symmetries.

3.1 Geometric aspect of the gauging of the AdS$\mathcal{L}_5$-algebra

In this case, the AdS$\mathcal{L}_5$-valued one-form gauge connection is given by

$$A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a,$$

where now the generators $P_a, J_{ab}, Z_{ab}, Z_a$ satisfy the AdS$\mathcal{L}_5$-algebra commutation relations (65).

Following the same procedure of the previous section we find that the corresponding curvature 2-form is given by

$$F = \frac{1}{2} \mathcal{R}^{ab} J_{ab} + \frac{1}{l} \mathcal{T}^a P_a + \frac{1}{2} \mathcal{F}^{ab} Z_{ab} + \frac{1}{l} H^a Z_a$$

with

$$\mathcal{R}^{ab} = R^{ab} + k^{a[c} k^{e]} + 2\Lambda e^a h^b \quad (28)$$

$$\mathcal{T}^a = T^a + k^a h^b = D e^a + k^a h^b \quad (29)$$

$$\mathcal{F}^{ab} = D k^{ab} + \Lambda e^a h^b + \Lambda e^b e^b \quad (30)$$

$$H^a = D h^a + k^a h^b. \quad (31)$$
where $T^a$ and $R^{ab}$ are the standard torsion and curvature, $\hat{F}^{ab}$ is the curvature of the non-abelian gauge fields $k^{ab}$ with $Dk^{ab} = dk^{ab} + \omega^c [k^{ab}]$ and $H^a$ is the curvature associated to the generator $Z_a$.

The covariant derivative of the curvatures lead to the following Bianchi identities

\begin{align}
DR^{ab} + k^{[a} F^{c][b]} + \Lambda h^{[a} \tilde{T}^{b]} + \Lambda e^{[a} H^{b]} = 0, \\
D\tilde{T}^a + R^a_{\ b} e^b + \tilde{F}^a_{\ b} h^b + k^a_{\ b} H^b = 0, \\
D\tilde{F}^{ab} + R^{[a}_{\ c} k^{c][b]} + \Lambda e^{[a} \tilde{T}^{b]} + \Lambda h^{[a} H^{b]} = 0, \\
DH^a + R^a_{\ b} h^b + k^a_{\ b} \tilde{T}^b + \tilde{F}^a_{\ b} e^b = 0,
\end{align}

where $D\tilde{T}^a = d\tilde{T}^a + \omega^a \tilde{T}^b$.

The gauge potentials transform as

\[ \delta_\varepsilon A = D\varepsilon = d\varepsilon + [A, \varepsilon], \]

where $\varepsilon$ is a AdS$_5$ algebra valued parameter given by

\[ \varepsilon = \frac{1}{2} \pi^{ab} J_{ab} + \frac{1}{l} \rho^a P_a + \frac{1}{2} e^{ab} Z_{ab} + \frac{1}{l} \sigma^a Z_a. \]

Using the commutation relation of the AdS$_5$-algebra we find

\begin{align}
\delta \omega^{ab} &= d\pi^{ab} + \omega^{[a} \pi^{c][b]} + k^{[a} e^{c][b]} + \Lambda e^{[a} \sigma^{b]} + \Lambda h^{[a} \rho^{b]}, \\
\delta e^a &= d\rho^a + \pi^a e^c + \omega^a \rho^c + \xi^a e^c + k^a \sigma^c, \\
\delta k^{ab} &= d\kappa^{ab} + k^{[a} \pi^{c][b]} + \omega^{[a} e^{c][b]} + \Lambda e^{[a} \rho^{b]} + \Lambda h^{[a} \sigma^{b]}, \\
\delta h^a &= d\sigma^a + \pi^a h^c + k^a \rho^c + \xi^a e^c + \omega^a \sigma^c.
\end{align}

On the other hand, from

\[ \delta_\varepsilon F = [F, \varepsilon] \]

we find that the curvatures transform as

\begin{align}
\delta R^{ab} &= R^{[a}_{\ c} \pi^{c][b]} + \tilde{F}^{[a}_{\ c} \xi^{c][b]} + \Lambda H^{[a} \rho^{b]} + \Lambda \tilde{T}^{[a} \sigma^{b]}, \\
\delta \tilde{T}^a &= \pi^a \tilde{T}^c + R^a_{\ b} \rho^c + \xi^a H^c + \tilde{F}^a_{\ c} \sigma^c, \\
\delta \tilde{F}^{ab} &= \tilde{F}^{[a}_{\ c} \pi^{c][b]} + R^{[a}_{\ c} e^{c][b]} + \Lambda \tilde{T}^{[a} \rho^{b]} + \Lambda H^{[a} \sigma^{b]}, \\
\delta H^a &= \pi^a H^c + \tilde{F}^a_{\ c} \rho^c + \xi^a \tilde{T}^c + R^a_{\ c} \sigma^c.
\end{align}
3.2 Four-dimensional AdS$_5$-Einstein-Hilbert action

The same procedure of the previous section leads to an action for the AdS$_5$-gravity constructed from the following 4-forms invariant

\[
\mathcal{L}_1 = \varepsilon_{abcd} R^{ab} \wedge \tilde{F}^{cd}, \quad \mathcal{L}_2 = \frac{1}{2} \varepsilon_{abcd} \tilde{F}^{ab} \wedge \tilde{F}^{cd}.
\] (46)

\[
-\frac{1}{2\kappa\Lambda} \mathcal{L}_1 = -\frac{1}{2\kappa\Lambda} \varepsilon_{abcd} R^{ab} \wedge \tilde{F}^{cd}
= \mathcal{L}_{E-H} - \frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} h^c h^d + \frac{\mu}{2\kappa\lambda} \varepsilon_{abcd} k^a e^b D k^{cd}
- \frac{1}{2\kappa} \varepsilon_{abcd} k^a e^b e^c e^d
- \frac{1}{2\kappa} \varepsilon_{abcd} k^a e^b h^c h^d - \frac{1}{\kappa} \varepsilon_{abcd} e^b D k^{cd}
- \frac{\lambda}{\kappa} \varepsilon_{abcd} e^b e^c e^d - \frac{\lambda}{\kappa} \varepsilon_{abcd} e^b h^c h^d,
\] (47)

\[
\frac{\lambda}{2\kappa\Lambda^2} \mathcal{L}_2 = \frac{\lambda}{4\kappa\Lambda^2} \varepsilon_{abcd} \tilde{F}^{ab} \wedge \tilde{F}^{cd}
= \mathcal{L}_{\text{cosm}} + \frac{\lambda}{4\kappa\Lambda} \varepsilon_{abcd} D k^{ab} D k^{cd} + \frac{\lambda}{2\kappa\Lambda} \varepsilon_{abcd} D k^{ab} h^c h^d
+ \frac{\lambda}{4\kappa\Lambda} \varepsilon_{abcd} D k^{ab} e^c e^d + \frac{\lambda}{2\kappa\Lambda} \varepsilon_{abcd} e^b e^c e^d + \frac{\lambda}{4\kappa\Lambda} \varepsilon_{abcd} e^b h^c h^d.
\] (48)

Defining $\mu = \lambda/\Lambda$ we propose

\[
\mathcal{L}_{\text{AdS}_5} = \mathcal{L}_{E-H} + \mathcal{L}_{\text{cosm}} - \frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} h^c h^d - \frac{\mu}{2\kappa\lambda} \varepsilon_{abcd} k^a e^b D k^{cd}
- \frac{1}{2\kappa} \varepsilon_{abcd} k^a e^b e^c e^d
- \frac{1}{2\kappa} \varepsilon_{abcd} k^a e^b h^c h^d - \frac{1}{\kappa} \varepsilon_{abcd} e^b D k^{cd}
- \frac{\lambda}{\kappa} \varepsilon_{abcd} e^b e^c e^d
- \frac{\lambda}{\kappa} \varepsilon_{abcd} e^b h^c h^d + \frac{\mu^2}{4\kappa\Lambda} \varepsilon_{abcd} D k^{ab} D k^{cd}
+ \frac{\lambda}{2\kappa\Lambda} \varepsilon_{abcd} D k^{ab} e^c e^d
+ \frac{\lambda}{2\kappa\Lambda} \varepsilon_{abcd} D k^{ab} h^c h^d
+ \frac{\lambda}{2\kappa} \varepsilon_{abcd} e^b h^c h^d + \frac{\lambda}{4\kappa} \varepsilon_{abcd} e^b e^c e^d,
\] (49)

where,

\[
\mathcal{L}_{E-H} = -\frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} e^c e^d,
\] (50)

\[
\mathcal{L}_{\text{cosm}} = \frac{\lambda}{4\kappa} \varepsilon_{abcd} e^a e^b e^c e^d.
\] (51)
3.3 Field Equations for AdS\(_{5}\)-extended Einstein gravity

The variation of \( \mathcal{L}_{\text{AdS}\_5} \) with respect to \( e^a \) leads to the following equations of motion

\[
\delta_e \mathcal{L}_{\text{AdS}\_5} = [\mathcal{L}_{\text{AdS}\_5}]_{e^a} \delta e^d = \frac{1}{\kappa} \varepsilon_{abcd} \left\{ - (R^{ab} + k^a e k^{eb} + 2\Lambda e^a h^b) e^c - (Dk^{ab} + \Lambda e^a e^b + \Lambda h^a h^b) h^c \\
+ \mu \left( Dk^{ab} + \Lambda e^a e^b + \Lambda h^a h^b \right) e^c \right\} \delta e^d,
\]

so that

\[
[\mathcal{L}_{\text{AdS}\_5}]_{e^a} = \frac{1}{\kappa} \varepsilon_{abcd} \left\{ - R^{ab} e^c - \tilde{F}^{ab} e^c + \mu \tilde{F}^{ab} e^c \right\} = 0
\]

The variation of \( \mathcal{L}_{\text{AdS}\_5} \) with respect to \( h^a \) leads to the following equations of motion

\[
\delta_h \mathcal{L}_{\text{AdS}\_5} = [\mathcal{L}_{\text{AdS}\_5}]_{h^a} \delta h^d = \frac{1}{\kappa} \varepsilon_{abcd} \left\{ - (R^{ab} + k^a e k^{eb} + 2\Lambda e^a h^b) h^c - (Dk^{ab} + \Lambda e^a e^b + \Lambda h^a h^b) e^c \\
+ \mu \left( Dk^{ab} + \Lambda e^a e^b + \Lambda h^a h^b \right) h^c \right\} \delta h^d,
\]

i.e.,

\[
[\mathcal{L}_{\text{AdS}\_5}]_{h^a} = \frac{1}{\kappa} \varepsilon_{abcd} \left\{ - R^{ab} e^c - \tilde{F}^{ab} e^c + \mu \tilde{F}^{ab} e^c \right\}.
\]

The variation of \( \mathcal{L}_{\text{AdS}\_5} \) with respect to \( \omega^{ab} \) leads to the following equations of motion

\[
\delta_\omega \mathcal{L}_{\text{AdS}\_5} = \delta^{ab} [\mathcal{L}_{\text{AdS}\_5}]_{\omega^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} \delta \omega^{ab} \left\{ - T^{cd} - Dh^c h^d - \frac{\mu^2}{\lambda} k^c e \tilde{F}^{ed} - \frac{\mu}{\lambda} k^c e \left( k^f k^{fd} + 2\Lambda e^e h^d \right) \right\}.
\]

From (28) we know that \( R^{ed} - R^{ed} = k^c e k^{ed} + 2\Lambda e^e h^d \), so

\[
\varepsilon_{abcd} k^c e (R^{ed} - R^{ed}) = \varepsilon_{abcd} k^c e \left[ k^f k^{fd} + 2\Lambda e^e h^d \right],
\]

and therefore

\[
[\mathcal{L}_{\text{AdS}\_5}]_{\omega^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} \delta \omega^{ab} \left\{ - T^{cd} - Dh^c h^d - \frac{\mu^2}{\lambda} k^c e \tilde{F}^{ed} + \frac{\mu}{\lambda} k^c e R^{ed} - \frac{\mu^2}{\lambda} k^c e \tilde{F}^{ed} \right\} = 0
\]

The variation of \( \mathcal{L}_{\text{AdS}\_5} \) with respect to \( k^{ab} \) leads to field equation
\[ \delta_k L_{\text{AdS}} = \delta k^{ab} \left[ L_{\text{AdS}} \right]_{\delta k^{ab}} \]
\[ = \frac{1}{\kappa} \varepsilon_{abcd} \delta k^{ab} \left( \frac{2\mu}{\lambda} k^c e Dk^{ed} + k^c e^e e^d + k^c e h^e h^d - Dh^e e^d - De^e h^d + \mu De^e e^d + \mu Dh^e h^d - \frac{\mu^2}{2\lambda} R^e e k^{ed} \right), \] (58)

so that,
\[ \left[ L_{\text{AdS}} \right]_{\delta k^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} \left( \frac{\mu}{\lambda} k^c e \tilde{F}^{ed} + \frac{\mu}{\lambda} k^c e Dk^{ed} - \frac{\mu^2}{2\lambda} R^e e k^{ed} \right) + Dh^c (\mu h^d - e^d) + De^c (\mu e^d - h^d) = 0 \] (59)

4 \( \mathcal{B}_4 \) and \( \mathcal{B}_5 \) actions from AdS\( \mathcal{L}_4 \) and AdS\( \mathcal{L}_5 \) gravities

In this Section we obtain the well-known Maxwell gravity from the AdS\( \mathcal{L}_4 \) action and a four-dimensional gravity action from the AdS\( \mathcal{L}_5 \) gravity action together with their corresponding field equations.

4.1 Maxwell gravity from AdS\( \mathcal{L}_4 \)-action

From (22) we see that the Lagrangian AdS\( \mathcal{L}_4 \) differs from the Maxwell Lagrangian, of Refs. [1], [2], in the following four terms
\[ + \frac{\lambda}{\kappa \Lambda^2} \varepsilon_{abcd} Dk^{ab} k^c e k^d - \frac{\mu}{\kappa} \varepsilon_{abcd} k_f e k^d - \frac{\lambda}{\kappa \Lambda^2} \varepsilon_{abcd} e^{k^a k^b} e^{k^c} e^{k^d} + \frac{\mu}{\kappa} \varepsilon_{abcd} k^a k^b k^c e^d, \] (60)
coming from the commutators \([Z_{ab}, P_c]\) and \([Z_{ab}, Z_{cd}]\), which in the case of Maxwell’s algebra commute. The natural question is how to obtain the Lagrangian corresponding to the Maxwell algebra from the Lagrangian for the AdS\( \mathcal{L}_4 \) algebra? Just as it is possible to obtain the Maxwell algebra from the AdS\( \mathcal{L}_4 \) algebra by means of an Inonu-Wigner contraction in the Weimar-Woods sense, it is also possible to find the Lagrangian for the Maxwell algebra. Indeed, carrying out the rescaling of the generators \( P_a \rightarrow \xi P_a \), \( Z_{ab} \rightarrow \xi^2 Z_{ab} \) and of the fields \( e^a \rightarrow \xi^{-1} e^a \), \( k^{ab} \rightarrow \xi^{-2} k^{ab} \) in the Lagrangian (21) for the AdS\( \mathcal{L}_4 \) algebra, we obtain
\[ \mathcal{L} = \mathcal{L}_E - H + \mathcal{L}_{\text{cosm}} + \frac{\mu}{2\kappa} \varepsilon_{abcd} Dk^{ab} e^{c d} + \frac{\mu^2}{4\kappa \lambda} \varepsilon_{abcd} D\omega k^{ab} D\omega k^{cd}, \] (61)
result that coincides with equation (29) of the reference [1]. It is straightforward to see that the application of the Inonu-Wigner contraction procedure to the AdSL₄-field equations leads to the equations (31), (34) and (37) of the reference [1].

4.2 ℬ₅ gravity from AdSL₅-gravity

Considering that the generalized Poincare algebra ℬ₅ can be obtained from the AdSL₅ algebra, by means of an Inonu-Wigner contraction in the Weimar-Woods sense, the natural question is how to obtain the corresponding Lagrangian to the ℬ₅ algebra from the Lagrangian for the AdSL₅ algebra? Following the same procedure of the previous section we carry out the rescaling of the generators $P_a \rightarrow \xi P_a$, $Z_{ab} \rightarrow \xi^2 Z_{ab}$, $Z_a \rightarrow \xi^3 Z_a$ and of the fields $e^a \rightarrow \xi^{-1} e^a$, $k^{ab} \rightarrow \xi^{-2} k^{ab}$, $h^a \rightarrow \xi^{-3} h^a$ in the Lagrangian (49) for the AdSL₅ algebra, we obtain the four-dimensional Lagrangian for ℬ₅ algebra

$$L^{(4D)}_{ℬ₅} = L_{E-H} + L_{cosm} + \frac{\lambda}{2\kappa \Lambda} \varepsilon_{abcd} Dk^{ab} e^c e^d + \frac{\mu^2}{4\kappa \lambda} \varepsilon_{abcd} Dk^{ab} Dk^{cd}$$

$$- \frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} h^c h^d - \frac{1}{\kappa} \varepsilon_{abcd} Dk^{ab} h^c e^d - \frac{\Lambda}{\kappa} \varepsilon_{abcd} h^a e^b e^c e^d$$

$$- \frac{\Lambda}{\kappa} \varepsilon_{abcd} e^a h^b h^c h^d + \frac{\lambda}{2\kappa \Lambda} \varepsilon_{abcd} Dk^{ab} h^c h^d + \frac{\lambda}{2\kappa} \varepsilon_{abcd} e^a h^b e^c h^d$$

$$+ \frac{\lambda}{4\kappa} \varepsilon_{abcd} h^a h^b h^c h^d,$$

(62)

which contains the Lagrangian corresponding to the ℬ₅ algebra.

Applying the Inonu-Wigner contraction procedure to the AdSL₅-field equations we obtain

$$\frac{1}{\kappa} \varepsilon_{abcd} \left\{ -R^{ab} e^c - 2\Lambda e^a h^b e^c - \tilde{F}^{ab} h^c + \mu \tilde{F}^{ab} e^c \right\} = 0$$

$$\frac{1}{\kappa} \varepsilon_{abcd} \left\{ -R^{ab} h^c - 2\Lambda e^a h^b h^c - \tilde{F}^{ab} e^c + \mu \tilde{F}^{ab} h^c \right\} = 0$$

$$\frac{1}{\kappa} \varepsilon_{abcd} \left\{ -T^c e^d - D_c h^e h^d - \frac{\mu^2}{\lambda} k^c e^d \tilde{F}^{ed} - 2\Lambda^2 k^c e^e h^d \right\} = 0$$

$$\frac{1}{\kappa} \varepsilon_{abcd} \left( D h^c (\mu h^d - e^d) + D e^c (\mu e^d - h^d) - \frac{\mu^2}{2\lambda} R^c e^d \right) = 0,$$

(63)

which correspond to the field equations for ℬ₅-extended Einstein gravity.

5 Concluding Remarks

In this article we have considered local gauge theories based on the AdSL₄ and AdSL₅ algebras with vierbein, spin connection and six additional geometric $k^a_{\mu}$.
non-Abelian gauge fields in the first case and with ten additional geometric
\((k^{ab}_{\mu}, h^{a}_{\mu})\) non-Abelian gauge fields, in the second case.

The geometry of space-times based on the above mentioned algebras involve
new curvature tensors that allow to construct new gravity actions which lead to
modifications of the Einstein gravity.

We have constructed the curvatures 2-form asociated with \(AdS_{L4}\) and \(AdS_{L5}\)
valued one-form gauge connections, which allow us to construct four-dimensional
gravities that generalize the Einstein Hilbert gravity. From these gravitational
actions we find that the Maxwell extension as well as the \(2_{5}\) extension of
Einstein gravity can be obtained using the Inönü-Wigner contraction method.

The field equations (24) and (25) allow us to express the spin connection as
a function of the vierbein \(e^{a}\) and the new non-Abelian gauge fields \(k^{ab}_{\mu}\). This
would allow us to obtain a second order formulation for the \(AdS_{L4}\)-gravity, with
independent fields \(e^{a}\) and \(k^{ab}_{\mu}\). Similarly, from the equations (53), (55) and (57)
it might be possible to express the spin connection as a function of the fields \(e^{a}\),
\(k^{ab}_{\mu}\) and \(h^{a}_{\mu}\) and obtain a second order formulation for the \(AdS_{L5}\)-gravity, with
independent fields \(e^{a}\), \(k^{ab}_{\mu}\) and \(h^{a}_{\mu}\).

It might be of interest to note that some years ago, another generalization of
the Einstein-Hilbert-Cartan action, invariant under \(AdS_{L4}\), i.e., invariant under
the local non-Abelian gauge symmetries associated with the \(Z_{ab}\) generators, was
proposed in Ref. [24]. The Lagrangian (33) of this reference coincides with the
Lagrangian (22) only in the Einstein-Hilbert and the cosmological terms, but it
does not contain terms involving the field \(k^{ab}_{\mu}\) \((h^{a}_{\mu}\) in Ref. [24]), i.e. it contains
no terms that could contribute to a cosmological term. The reason for this,
according to [24], would be that the starting point for the construction of action
(33) was the so called BF theory, which is a geometric theory.

In this context, it might also be of interest to note that the Randall-Sundrum
compactification procedure could allow obtaining the action (22) from the five-
dimensional Chern-Simons gravity invariant under \(AdS_{L4}\). This procedure could
also be used to obtain the Lagrangian (29) of reference [1] from the Maxwell
Chern-Simons gravity action [25].

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6 Appendix A: Generalized AdS and Poincare
algebras

6.1 \(AdS_{L4}\), \(AdS_{L5}\) and \(AdS_{L6}\)-algebra

\(AdS_{L4}\)-algebra: The \(S\)-expansion of the Lie AdS algebra using as semigroup
\(S^{(2)}_{M}\) = \{\(\lambda_{0}, \lambda_{1}, \lambda_{2}\}\} endowed with the multiplication rule \(\lambda_{\alpha}\lambda_{\beta} = \lambda_{\alpha+\beta}\) if \(\alpha + \beta \leq 2\); \(\lambda_{\alpha}\lambda_{\beta} = \lambda_{\alpha+\beta-2}\) if \(\alpha + \beta > 2\), lead us to the so called \(AdS_{L4}\)-algebra,
whose generators satisfy the following commutation relations

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \quad [P_a, P_b] = Z_{ab}, \\
[Z_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b.
\end{align*}
\] (64)

This algebra was also reobtained in Ref. [13] from Maxwell algebra through a procedure known as deformation.

**AdS\(\mathcal{L}_5\)-algebra:** the same procedure, but now using the semigroup \(S^{(3)} = \mathbb{Z}_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}\) endowed with the multiplication rule \(\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}\) if \(\alpha + \beta \leq 3; \lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta-4}\) if \(\alpha + \beta > 3\), lead to the AdS\(\mathcal{L}_5\)-algebra, whose generators satisfy the following commutation relations

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \quad [P_a, P_b] = Z_{ab}, \\
[Z_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \quad [Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b.
\end{align*}
\] (65)

**AdS\(\mathcal{L}_6\)-algebra:** This algebra is obtained using the semigroup \(S^{(4)} = \mathbb{Z}_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) endowed with the multiplication rule \(\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}\) if \(\alpha + \beta \leq 4; \lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta-4}\) if \(\alpha + \beta > 4\). Their generators satisfy the following commutation relations

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, Z_{cd}^{(1)}] &= \eta_{bc}Z_{ad}^{(1)} + \eta_{ad}Z_{bc}^{(1)} - \eta_{ac}Z_{bd}^{(1)} - \eta_{bd}Z_{ac}^{(1)}, \\
[J_{ab}, Z_{cd}^{(2)}] &= \eta_{bc}Z_{ad}^{(2)} + \eta_{ad}Z_{bc}^{(2)} - \eta_{ac}Z_{bd}^{(2)} - \eta_{bd}Z_{ac}^{(2)}, \\
[Z_{ab}^{(1)}, Z_{cd}^{(1)}] &= \eta_{bc}Z_{ad}^{(1)} + \eta_{ad}Z_{bc}^{(1)} - \eta_{ac}Z_{bd}^{(1)} - \eta_{bd}Z_{ac}^{(1)}, \\
[Z_{ab}^{(2)}, Z_{cd}^{(2)}] &= \eta_{bc}Z_{ad}^{(2)} + \eta_{ad}Z_{bc}^{(2)} - \eta_{ac}Z_{bd}^{(2)} - \eta_{bd}Z_{ac}^{(2)}.
\end{align*}
\] (66)
7 \( \mathcal{B}_4, \mathcal{B}_5 \) and \( \mathcal{B}_6 \) algebras

\( \mathcal{B}_4 \)-algebra: This algebra normally called Maxwell algebra can be obtained from \( \text{AdS}_4 \) by means of Inönü–Wigner contraction. In fact, rescaling \( P_a \rightarrow \lambda P_a, Z_{ab} \rightarrow \lambda^2 Z_{ab} \) in (64) and then taking the limit \( \lambda \rightarrow \infty \) we obtain the Maxwell algebra.

\( \mathcal{B}_5 \)-algebra: This algebra can be obtained from \( \text{AdS}_5 \) algebra rescaling \( P_a \rightarrow \lambda P_a, Z_{ab} \rightarrow \lambda^2 Z_{ab}, Z_a \rightarrow \lambda^3 Z_a \) in Eq. (65) and then taking the limit \( \lambda \rightarrow \infty \).

\( \mathcal{B}_5 \)-algebra: This algebra can also obtained from \( \text{AdS}_6 \) by means of rescaling \( P_a \rightarrow \lambda P_a, Z_a \rightarrow \lambda^3 Z_a, Z_{ac}^{(1)} \rightarrow \lambda^2 Z_{ac}^{(1)}, Z_{ac}^{(2)} \rightarrow \lambda^4 Z_{ac}^{(2)} \) in (66) and then taking the limit \( \lambda \rightarrow \infty \).

7.1 Semidirect sum structure of the \( \mathcal{B}_4, \mathcal{B}_5 \) and \( \mathcal{B}_6 \) algebras

Poincaré algebra: the generators of the Poincaré algebra are the generator of the translation group \( T^4 \) and the generator of the Lorentz rotation group \( \text{so}(3,1): (P_a, J_{ab}) \) which satisfy the following commutation relations

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \\
[P_a, P_b] &= 0.
\end{align*}
\]

(67)

From (67) we see that \( T^4 \) is an ideal of the Poincaré algebra since \( [T^4, T^4] \subset T^4 \) and \( [\text{so}(3,1), T^4] \subset T^4 \), which means that \( \text{iso}(3,1) = \text{so}(3,1) \oplus T^4 \).

Maxwell algebra: the generators of Maxwell algebra \( (P_a, J_{ab}, Z_{ab}) \) satisfy the commutation relations

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}, \\
[Z_{ab}, Z_{cd}] &= 0, \\
[Z_{ab}, P_c] &= 0.
\end{align*}
\]

(68)

from where we see that the subset of generators \( \mathcal{M}' = (P_a, Z_{ab}) \) is an ideal of Maxwell’s algebra since \( [\mathcal{M}', \mathcal{M}'] \subset \mathcal{M}', [\text{so}(3,1), \mathcal{M}'] \subset \mathcal{M}' \). This means that the Maxwell algebra \( \mathcal{M} \) is the semidirect sum of the Lorentz algebra \( \text{so}(3,1) \) and the ideal \( \mathcal{M}' \), that is \( \mathcal{M} = \text{so}(3,1) \oplus \mathcal{M}' \).

Generalized Poincaré algebra \( \mathcal{B}_6 \): the generators of this algebra \( (P_a, J_{ab}, Z_{ab}, Z_a) \) satisfy the commutation relations
satisfy the commutation relations

\[ [P_a, P_b] = \Lambda Z_{ab}, \]
\[ [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \]
\[ [J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \]
\[ [J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}, \]
\[ [J_{ab}, Z_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \]
\[ [Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \]
\[ [Z_{ab}, Z_c] = 0, \]
\[ [P_a, Z_b] = 0, \]
\[ [Z_{ab}, Z_c] = 0, \]
\[ [Z_{ab}, Z_c] = 0, \]

(69)

from where we see that the subset of generators \( g_5^f = (P_a, Z_{ab}, Z_a) \) is an ideal of \( g_5 \) algebra since \( [g_5^f, g_5] \subset g_5^f, [so(3, 1), g_5^f] \subset g_5^f \). This means that the Maxwell algebra \( g_5 \) is the semidirect sum of the Lorentz algebra \( so(3, 1) \) and the ideal \( g_5^f \), that is \( g_5 = so(3, 1) \oplus g_5^f \).

**Generalized Poincaré algebra** \( g_6 \): the generators of this algebra \( (P_a, J_{ab}, Z_{cd}^{(1)}, Z_{ab}^{(2)}, Z_a) \) satisfy the commutation relations

\[
[ J_{ab}, J_{cd} ] = \eta_{bc} J_{ad} + \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \\
[ J_{ab}, Z_{cd}^{(1)} ] = \eta_{bc} Z_{ad}^{(1)} + \eta_{ac} Z_{bd}^{(1)} - \eta_{bd} Z_{ac}^{(1)}, \\
[ J_{ab}, Z_{cd}^{(2)} ] = \eta_{bc} Z_{ad}^{(2)} + \eta_{ac} Z_{bd}^{(2)} - \eta_{bd} Z_{ac}^{(2)}, \\
[ J_{ab}, P_c ] = \eta_{bc} P_a - \eta_{ac} P_b, \\
[ J_{ab}, Z_c ] = \eta_{bc} Z_a - \eta_{ac} Z_b, \\
[ Z_{ab}^{(1)}, Z_{cd}^{(1)} ] = \eta_{bc} Z_{ad}^{(2)} + \eta_{ac} Z_{bd}^{(2)} - \eta_{bd} Z_{ac}^{(2)}, \\
[ Z_{ab}^{(1)}, P_c ] = \eta_{bc} Z_a - \eta_{ac} Z_b, \\
[ Z_{ab}^{(2)}, Z_{ce}^{(1)} ] = 0, \\
[ Z_{ab}^{(2)}, Z_{ce}^{(2)} ] = 0, \\
[ Z_{ab}^{(2)}, P_c ] = 0 \\
[ Z_{ab}^{(2)}, Z_c ] = 0, \\
[ Z_{ab}^{(2)}, Z_c ] = 0, \\

(70)

from where we see that the subset of generators \( g_6^f = (P_a, Z_{cd}^{(1)}, Z_{ab}^{(2)}, Z_a) \) is an ideal of \( g_6 \) algebra since \( [g_6^f, g_6^f] \subset g_6^f, [so(3, 1), g_6^f] \subset g_6^f \). This means that the Maxwell algebra \( g_6 \) is the semidirect sum of the Lorentz algebra \( so(3, 1) \) and the ideal \( g_6^f \), that is \( g_6 = so(3, 1) \oplus g_6^f \).

8 Appendix B: About \( AdS_L^6 \)-gravity

In this case, the \( AdS_L^6 \)-valued one-form gauge connection is given by
\[ A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} \epsilon^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a + \frac{1}{2} q^{ab} \dot{Z}_{ab} \]

where now the generators \( P_a, J_{ab}, Z_{ab}, Z_a, \dot{Z}_{ab} \) satisfy the \( AdS_L_6 \)-algebra commutation relations \([60]\). Following the same procedure of the previous sections we find that the associated two-form curvature are given by

\[ F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2} \bar{F}^{ab} Z_{ab} + \frac{1}{l} \hat{H}^a Z_a + \frac{1}{2} Q^{ab} \dot{Z}_{ab}, \]

with

\[
\begin{align*}
\bar{T}^a &= T^a + q^a e^b + k^a h^b, \\
R^{ab} &= d\omega^{ab} + \omega^a \omega^b, \\
\bar{F}^{ab} &= Dk^{ab} + q^a [k^{[b]} + \Lambda e^a e^b + \Lambda h^a h^b], \\
Q^{ab} &= Dq^{ab} + k^a e^b + q^a q^b + \Lambda e^b h^b, \\
\hat{H}^a &= Dh^a + k^a e^c + q^a h^c, \end{align*}
\]

where \( T^a \) and \( R^{ab} \) are the standard torsion and curvature, \( \bar{F}^{ab} \) is the curvature associated to the generator \( Z_{ab} \) and \( Q^{ab}, \hat{H}^a \) are the curvature associated to the generators \( \dot{Z}_{ab} \) and \( Z_a \) respectively.

The covariant derivative of the curvatures lead to the following Bianchi identities

\[ DF = \frac{1}{2} DR^{ab} J_{ab} + \frac{1}{l} D\bar{T}^a P_a + \frac{1}{2} D\bar{F}^{ab} Z_{ab} + \frac{1}{l} D\hat{H}^a Z_a + \frac{1}{2} DQ^{ab} \dot{Z}_{ab}, \]

with

\[
\begin{align*}
dR^{ab} + \omega^a R^{cb} - \omega^b R^{ca} &= DR^{ab} = 0, \\
D\bar{T}^a + R^a c^b + q^a \bar{T}^b + Q^a e^b + k^a H^c &= 0, \\
D\bar{F}^{ab} + k^a [e^b + k^a \bar{F}^{[b]}] + q^a [\bar{F}^{c]} + k^a [q^{[b]} + 2 \frac{1}{l^2} e^a \bar{T}^b + 2 \frac{1}{l^2} h^a \bar{H}^b &= 0, \\
D\hat{H}^a + R^a h^b + k^a \bar{T}^b + \bar{F}^a e^b + q^a \bar{H}^b + Q^a h^b &= 0, \\
DQ^{ab} + q^a [e^b + k^a \bar{F}^{[b]}] + q^a [q^{[b]} + 2 \frac{1}{l^2} e^a \bar{T}^b + 2 \frac{1}{l^2} h^a \bar{H}^b &= 0. \end{align*}
\]

The gauge potentials transform as

\[ \delta_\epsilon A = D\epsilon = d\epsilon + [A, \epsilon], \]

where \( \epsilon \) is a \( AdS_L_6 \) algebra valued parameter given by

\[ \epsilon = \frac{1}{2} \pi^{ab} J_{ab} + \frac{1}{l} \rho^a P_a + \frac{1}{2} \xi^{ab} Z_{ab} + \frac{1}{l} \sigma^a Z_a + \frac{1}{2} \lambda^{ab} \dot{Z}_{ab}, \]

Using the commutation relation of the \( AdS_L_6 \)-algebra we find
\[ \delta \omega^{ab} = d\pi^{ab} + \omega^{[a c} \pi^{b]} \], 
\[ \delta e^a = d\rho^a + \pi^a \rho^c + \omega^a \rho^c + \xi^a \rho^c + \chi^a \rho^c, \] 
\[ \delta k^{ab} = d\xi^{ab} + k^{[a c} \xi^{b]} + \omega^{[a c} \xi^{b]} + q^{[a c} \xi^{b]} + k^{a c} \chi^{b]} + \Lambda e^{a \rho^b} + \Lambda h^{a \sigma^b}, \] 
\[ \delta h^a = d\sigma^a + \pi^a \sigma^c + k^a \sigma^c + \omega^a \sigma^c + q^a \sigma^c + \chi^a \sigma^c + \chi^a h^c, \] 
\[ \delta q^{ab} = d\chi^{ab} + q^{[a c} \chi^{b]} + k^{[a c} \chi^{b]} + \omega^{[a c} \chi^{b]} + q^{[a c} \chi^{b]} + \Lambda h^{a \rho^b} + \Lambda e^{a \sigma^b}. \] 

On the other hand, from \( \delta \varepsilon F = [F, \varepsilon] \) we find that the curvatures transform as

\[ \delta R^{ab} = R_{\epsilon}^{[a c} \pi^{b]} \], 
\[ \delta \tilde{T}^a = \pi^a \tilde{T}^c + R^a \rho^c + Q^a \rho^c + \xi^a \tilde{H}^c + \tilde{F}^a \sigma^c + \chi^a \tilde{T}^c, \] 
\[ \delta \tilde{F}^{ab} = \tilde{F}_{\epsilon}^{[a c} \pi^{b]} + R^a \xi^{c]b} + q^{[a c} \xi^{b]} + \tilde{F}^{a \lambda} \chi^{b]} + \Lambda \tilde{T}^{a \rho^b} + \Lambda \tilde{H}^{a \sigma^b}, \] 
\[ \delta \tilde{H}^a = \pi^a \tilde{H}^c + \tilde{F}^a \rho^c + \epsilon^a \tilde{T}^c + R^a \sigma^c + Q^a \sigma^c + \chi^a \tilde{H}^c, \] 
\[ \delta \tilde{Q}^{ab} = Q_{\epsilon}^{[a c} \pi^{b]} + \tilde{F}_{\epsilon}^{[a c} \pi^{b]} + R_{\epsilon}^{[a c} \pi^{b]} + Q_{\epsilon}^{[a c} \pi^{b]} + \Lambda \tilde{H}^{a \rho^b} + \Lambda \tilde{T}^{a \sigma^b}. \]

### 8.1 Field Equations for AdSL6-gravity

The same procedure of the previous section leads to an action for the AdSL\(_6\)-gravity constructed from the following 4-forms invariant

\[ \mathcal{L}_1 = \frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} F^{cd} - \frac{1}{2\kappa} \varepsilon_{abcd} h^{c} h^{d} - \frac{1}{2\kappa} \varepsilon_{abcd} q^{[c} k^{e]} \] 

\[ \mathcal{L}_2 = \frac{\lambda}{4\kappa^{2}} \varepsilon_{abcd} \tilde{F}^{ab} \tilde{F}^{cd} \]

\[ = \mathcal{L}_{\cosm} + \frac{\lambda}{4\kappa^{2}} \varepsilon_{abcd} \left( Dk^{ab} Dk^{cd} + 2Dk^{ab} q^{[c} k^{e]} \right) + 2\Lambda Dk^{ab} e^{c} e^{d} + 2\Lambda Dk^{ab} h^{c} h^{d} + 2\Lambda q^{[a} k^{b]} e^{c} e^{d} + 2\Lambda q^{[a} k^{b]} h^{c} h^{d} + 2\Lambda^{2} e^{c} h^{d} + q^{[a} k^{b]} q^{c} k^{d} + \Lambda^{2} h^{c} h^{d} \] 

Defining \( \mu = \lambda/\Lambda \) we propose

\[ \mathcal{L}_{AdSL_6} = -\frac{\mu}{2\kappa} \mathcal{L}_1 + \frac{\mu^{2}}{2\kappa^{2}} \mathcal{L}_2, \]
\[ \mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_{\text{cosm}} - \frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} h^c h^d - \frac{1}{2\kappa \Lambda} \varepsilon_{abcd} R^{ab} q^{[c} e^{d]} \]

\[ + \frac{\mu^2}{4\kappa \lambda} \varepsilon_{abcd} D k^{ab} D k^{cd} + \frac{\mu^2}{2\kappa \lambda} \varepsilon_{abcd} D k^{ab} R^{cd} + \frac{\mu}{2\kappa} \varepsilon_{abcd} R^{ab} h^c h^d \]

\[ + \frac{\mu}{2\kappa} \varepsilon_{abcd} q^{[a} e^{[b]} e^{c]} e^{d]} + \frac{\mu}{2\kappa} \varepsilon_{abcd} q^{[a} e^{[b]} e^{c]} e^{d]} h^c h^d \]

\[ + \frac{\lambda}{2\kappa} \varepsilon_{abcd} e^a e^b e^c e^d + \frac{\lambda}{\kappa} q^{[a} e^{[b]} e^{[c]} e^{d]} + \frac{\lambda}{4\kappa} \varepsilon_{abcd} h^a h^b h^c h^d, \]

\( (94) \)

The variation of \( \mathcal{L}_{\text{AdS}} \) with respect to \( e^a, h^a, \omega^{ab}, k^{ab}, \) and \( q^{ab} \) leads to the following equations of motion

\[ [\mathcal{L}]_{e^a} = \frac{1}{\kappa} \varepsilon_{abcd} \left( \mu \left( D k^{ab} + q^{[a} e^{[b]} \right) + \Lambda h^a h^b + \Lambda e^a e^b \right) e^c - R^{ab} e^c = 0 \] (95)

\[ = \frac{1}{\kappa} \varepsilon_{abcd} \left( \mu \bar{F}^{ab} e^c - R^{ab} e^c \right) = 0 \] (96)

\[ [\mathcal{L}]_{h^a} = \frac{1}{\kappa} \varepsilon_{abcd} \left( -R^{ab} h^c + \mu \left( D k^{ab} + q^{[a} e^{[b]} \right) + \Lambda e^a e^b + \Lambda h^a h^b \right) h^c = 0 \] (97)

\[ = \frac{1}{\kappa} \varepsilon_{abcd} \left( -R^{ab} h^c + \mu \bar{F}^{ab} h^c \right) = 0 \] (98)

\[ [\mathcal{L}]_{\omega^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} \left\{ -\frac{\mu^2}{\lambda} k^a e^b \bar{F}^{eb} + \frac{1}{\Lambda} D \left( q^{[a} e^{[b]} \right) - T^{ab} e^b - Dh^a h^b \right\} = 0 \]

\[ [\mathcal{L}]_{k^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} \left( \mu T^{ab} e^b + \mu D h^a h^b + \frac{\mu^2}{2\lambda} R^{ab} e^b + \frac{\mu^2}{\lambda} q^{[a} e^{[b]} \bar{F}^{eb} + \frac{\mu^2}{\lambda} D \left( q^{[a} e^{[b]} \right) \right) = 0 \]

\[ [\mathcal{L}]_{q^{ab}} = \frac{1}{\kappa} \varepsilon_{abcd} k^a e \left\{ R^{eb} - \frac{\mu^2}{\lambda} \bar{F}^{eb} \right\} = 0. \]

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