The advantage of Lévy strategies in intermittent search processes

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Search strategies based on random walk processes with long-tailed jump length distributions (Lévy walks) on the one hand and intermittent behavior switching between local search and ballistic relocation phases on the other, have been previously shown to be beneficial in stochastic target finding problems. We here study a combination of both mechanisms: an intermittent process with Lévy distributed relocations. We demonstrate how Lévy distributed relocations reduce oversampling and thus further optimize the intermittent search strategy in the critical situation of rare targets.

Random search processes occur in many areas. The simplest example is that of passive particles immersed in a thermal bath subjecting them to Brownian motion until encounter, for instance, in chemical reactions \[4\]. This Brownian search dynamics may be accelerated in various ways: (i) By a drift toward the reaction center, for instance, in the time-dependent Onsager problem for diffusion in an attractive Coulomb potential, or in chemotaxis of biological cells \[2\; 3\]. (ii) By combining more than one search mechanism due to available interfaces as known from gene regulation \[4\]: to find their target sequence on the DNA molecule more efficiently, proteins switch between 3D bulk diffusion, and 1D sliding along the DNA \[3\]. (iii) By performing a Lévy walk, i.e., a random walk whose jump lengths are drawn from a long-tailed distribution \[2\; 3\]: \(|x|^{-1-\alpha}\; (0 < \alpha < 2)\; \alpha\]. The resulting trajectory has diverging variance \((\langle x^2 \rangle) = \infty\), unless a velocity is introduced, and fractal dimension \(d_f = \alpha\), covering space less densely to reduce oversampling, an advantage over Brownian search \[2\; 3\; 6\; 10\; 11\]. (iv) By intermittent strategies during which local (Brownian) search switches with ballistic relocations \[12\].

We here demonstrate for a searcher without orientational memory how intermittent and Lévy search strategies can be combined to produce a synergistic strategy, that for rare targets is more efficient than previously introduced intermittent search models. Similarly to Refs. \[12\; 13\; 14\; 15\] we focus on the 1D case, that is relevant, for instance, for animals searching for food at ecological interfaces (forest edges, coastlines etc.).

Generalizing the search model from Ref. \[14\], we consider two phases: In phase 1 the searcher looks for the target performing diffusive motion with diffusion constant \(D\). There is a probability per time \(\tau_1^{-1}\) that the searcher leaves this search phase and switches to phase 2, the relocation phase, where it moves ballistically with velocity \(v\) in a random direction. The time spent relocating is drawn from the waiting time distribution \(\psi(t)\), that previously was taken to be exponential (leading to a Markovian process \[12\; 14\]), but we relax this assumption here to show the advantage of Lévy strategies. The purpose of the relocation phase is to move as quickly as possible away from the area that has just been searched, and thus the searcher is not scanning for the target in this phase. To compare with previous results we take a closed cell approach: the search is performed on an interval of length \(L\) with periodic boundary conditions, corresponding to regularly spaced targets with density \(1/L\). The model can be formulated as an equation for the probability density \(P(x,t)\) for the position \(x\) of the searcher in the search phase:

\[
\frac{\partial P}{\partial t} = \frac{1}{\tau_1} \int_{-L/2}^{L/2} dx' \int_0^t dt' W(x-x', t-t') P(x', t') - \frac{1}{\tau_1} P(x,t) + D \frac{\partial^2 P}{\partial x^2} - p_{\text{fa}}(t) \delta(x).
\]

The role of the last term on the right hand side is to remove the particle when it arrives at the target placed at \(x = 0\). The density \(p_{\text{fa}}(t)\) thus represents the first arrival time at the target, which is determined implicitly by the absorbing boundary condition \(P(x = 0, t) = 0\). The kernel \(W(x,t)\) representing relocations is given by

\[
W(x,t) = \frac{\psi(t)}{2} \sum_{n=-\infty}^{\infty} \delta(|x + nL| - vt).
\]

The \(\delta\)-coupling enforces that the distance traveled in time \(t\) is \(vt\), and the sum over \(n\) renders \(W(x,t)\) \(L\)-periodic in \(x\). \(\psi(t)\) is related to the spatial distribution of the relocations \(\lambda(x)\) by \(\psi(t) = 2v \lambda(vt)\). \(\lambda(x)\) is assumed to be symmetric around \(x = 0\) (no orientational memory).

The search efficiency is quantified by the mean search time \(\langle t \rangle = \int_0^\infty dt \; p_{\text{fa}}(t)\). To obtain \(\langle t \rangle\) we Fourier expand \(P(n,t) = \int_{-L/2}^{L/2} dx \; e^{ikn \cdot x} P(x,t)\) \((n\) is an integer with corresponding wavenumber \(k_n = 2\pi n/L\), and Laplace transform \(P^\prime(n,u) = \int_0^\infty dt \; e^{-\nu t} P(n,t)\), to find

\[
u P(n,u) - \delta_{n,0} = \frac{1}{\tau_1} W(n,u) P(n,u) - \frac{1}{\tau_1} P(n,u) - D k_n^2 P(n,u) - p_{\text{fa}}(u).
\]

The initial distribution is uniform, \(P(x,t = 0) = 1/L\), since the searcher initially has no information on the position of the target. Isolating \(P(n,u)\), summing over \(n\)
(note that \( \sum_{n=1}^{\infty} P(n, u) = P(x = 0, u) = 0 \)), and solving for \( p_{\text{fa}}(u) \) we find
\[
p_{\text{fa}}(u) = \left\{ \sum_{n=1}^{\infty} \frac{u + \frac{1}{2} - \psi(u)}{u + Dk_n^2 + \frac{1}{2} - W(n, u)} \right\}^{-1}.
\]

In Laplace space the mean search time \( \langle t \rangle \) can be found by expanding \( p_{\text{fa}} \) at small \( u \) since \( p_{\text{fa}}(u) \sim 1 - \langle t \rangle u + \ldots \). Be \( \tau_2 \) the average time spent in one relocation event we have \( \psi(u) \sim 1 - \tau_2 u + \ldots \), and thus arrive at
\[
\langle t \rangle = \sum_{n=1}^{\infty} \frac{2(\tau_1 + \tau_2)}{D\tau_1 k_n^2 + 1 - \lambda(k_n)}.\]

Here \( \lambda(k_n) = W(n, u = 0) = \int_{-\infty}^{\infty} dx \, e^{ik_n x} \lambda(x) \) is the Fourier transform of the relocation length distribution at the discrete wavenumbers \( k_n = 2\pi n/L \). We now use Eq. \((5)\) to determine the search efficiency of (i) Lévy and (ii) exponentially distributed relocations:

(i) For Lévy distributed relocations we use the symmetric Lévy stable law with characteristic function \((10)\)
\[
\lambda(k) = e^{-\sigma |k|^{\alpha}}, \quad \sigma = \frac{\pi \nu \tau_2}{2(1-1/\alpha)}.
\]

The index \( \alpha \) is restricted to \( 1 < \alpha < 2 \) so that the mean relocation time \( \tau_2 \) is finite. Fig. \(1\) depicts trajectories for cases of exponential and Lévy relocations, distinguishing the Lévy case with its occasional long relocations.

We introduce three approximations valid for large \( L \):

(a) Assume that \( \nu \tau_2 \gg D\tau_1 \), i.e., that the mean relocation distance is much longer than the average distance scanned in a typical search phase. We will see that this is self-consistent with the obtained optimal values of \( \tau_1 \) and \( \tau_2 \) that have the same \( L \)-scaling for large \( L \). This assumption means that \( D\tau_1 k_n^2 \) and \( \lambda(k_n) \) are to a good approximation non-zero at different \( n \), and we expand
\[
\frac{1}{D\tau_1 k_n^2 + 1 - \lambda(k_n)} \sim \frac{1}{D\tau_1 k_n^2 + 1} + \frac{1}{1 - \lambda(k_n)} - 1.
\]

(b) Assuming that the search range \( \sqrt{D\tau_1} \) is much smaller than \( L \), we replace the sum over the first term on the right hand side of Eq. \((7)\) by an integral, yielding
\[
\sum_{n=1}^{\infty} \frac{1}{D\tau_1 k_n^2 + 1} \sim \int_0^{\infty} \frac{1}{D\tau_1 k_n^2 + 1} \, dn = \frac{L}{4\sqrt{D\tau_1}}.
\]

(c) Approximate the last two terms of Eq. \((7)\): as the contribution from the singularity at small \( n \) dominates the sum (note that \( k_n |_{n=1} \to 0 \) in the limit of large \( L \)),
\[
\sum_{n=1}^{\infty} \left( \frac{1}{1 - \lambda(k_n)} - 1 \right) \sim \left( \frac{L}{2\pi \sigma} \right)^{\alpha} \zeta(\alpha).
\]

Here \( \zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha} \) is the Riemann \( \zeta \) function.

Collecting (a) to (c), Eq. \((5)\) is approximated by
\[
\langle t \rangle \sim 2(\tau_1 + \tau_2) \left[ \frac{L}{4\sqrt{D\tau_1}} + \left( \frac{L}{2\pi \sigma} \right)^{\alpha} \zeta(\alpha) \right].
\]

For honest comparison between Lévy and exponential strategies, we determine the respective optimal \( \tau_1 \) and \( \tau_2 \). Solving \( \partial \langle t \rangle / \partial \tau_1 = 0 \) and \( \partial \langle t \rangle / \partial \tau_2 = 0 \) simultaneously, we obtain from Eq. \((10)\) that at large \( L \)
\[
\tau_1 \sim (b/a^\alpha)^{1/(\alpha-1/2)}, \quad \tau_2 \sim (b/\sqrt{a})^{1/(\alpha-1/2)},
\]
where (using \( \Omega \equiv \sqrt{1 + 4(\alpha - 1)\alpha} \))
\[
a = (1 + \Omega)/(2[\alpha - 1]),
\]
\[
b = 2\sqrt{D(\alpha - \Omega - 3)} \zeta(\alpha) L^{\alpha-\frac{1}{2}} \left[ \frac{\Gamma(1 - \alpha)}{\pi^2 \nu} \right]^{\alpha}
\]
such that the optimal \( \tau_i \) scale with \( L \) like \( L^{(\alpha-1)/(\alpha-1/2)} \).

According to Eq. \((10)\), \( \langle t \rangle \) will then scale like \( L^{(3\alpha-2)/(2\alpha-1)} \), implying that for large \( L \) the more efficient search will occur for \( \alpha \) close to \( 1 \). However, the
prefactor to the $L$-scaling diverges as $α → 1$, so the optimal choice of $α$ will be somewhat larger than 1 for any finite $L$, as demonstrated in Fig. 2.

(ii) For exponentially distributed relocation with

$$ψ(t) = τ_2^{-1} e^{-t/τ_2},$$ (13)

approximations (a) to (c) also apply, with $σ = ντ_2$. The corresponding results for $(t)$ and optimal $τ_i$ obtain by replacing $Γ(1-1/α)$ by $π/2$ and taking $α = 2$:

$$(t) \sim \frac{τ_1 + τ_2}{12} \left[\frac{6L}{\sqrt{Dτ_1}} + \left(\frac{L}{ντ_2}\right)^2\right],$$ (14)

$$τ_1 \sim \frac{1}{2} \left(\frac{D}{18v^4}\right)^{1/3} L^{2/3}, \quad τ_2 \sim 2τ_1.$$ (15)

These expressions agree with those of Ref. 14, 17.

The search time $(t)$ with $L$ for exponential strategies scales like $L^{1/3}$ for optimal $τ_1$ and $τ_2$. This proves that Lévy strategies with $1 < α < 2$ are increasingly more efficient than the exponential strategies for decreasing target density. In Fig. 3 we show $(t)$ as function of relocation time $τ_2$.

To understand better the $α$-dependence of the Lévy strategy we study the first arrival density $p_{fa}(t)$ for large $L$, where again $L >> ντ_2 >> \sqrt{Dτ_1}$. We consider times much longer than one relocation-search cycle such that $ψ(u) \sim 1 - τ_2u + \ldots$, and rewrite Eq. (4) as

$$p_{fa}(u) \sim \frac{1}{u} \frac{τ_1}{τ_1 + τ_2} \frac{1}{W_0(u)} \frac{1}{L},$$ (16)

where we have introduced the term

$$W_0(u) = \frac{1}{L} \sum_{n=-∞}^{∞} \frac{1}{u + Dk_n^2 + [1 - W(n, u)]/τ_1}. $$ (17)

The last expression can be simplified following similar approximations as for $(t)$ before. The separation of length scales leading to approximation (a) allows us to write

$$W_0(u) \sim \frac{τ_1}{L} \sum_{n=-∞}^{∞} \left[\frac{1}{Dτ_1 k_n^2 + 1} + \frac{1}{τ_1 u + 1 - W(n, u)}\right].$$ (18)

For the last two terms in Eq. (18), the contribution at small $n$ again dominates the sum (approximation (c)), and we expand $W(n, u)$ at small $k_n$ and $u$, finding $W(n, u) \sim 1 - σ^α |k_n|^α - τ_2 u$. Collecting the results, we see that

$$W_0(u) \sim \frac{τ_1}{L} \sum_{n=-∞}^{∞} \left[\frac{1}{Dτ_1 k_n^2 + 1} + \frac{1}{(τ_1 + τ_2)u + σ^α |k_n|^α}\right].$$ (19)

We focus on times short enough such that the $L$-periodicity of the problem does not yet play a role, so that Laplace space $u \gg (σ^α |k_n|^α|_{n=1})/(τ_1 + τ_2)$. In this approximation we replace the sum $L^{-1} \sum_{n=-∞}^{∞}$ by the integral $\int_{-∞}^{∞} dk_n/(2π)$, obtaining

$$W_0(u) \sim \frac{1}{2\sqrt{Dτ_1}} \frac{τ_1}{L(τ_1 + τ_2)}$$ (20)

For shorter times (corresponding to larger $u$) we discard the subdominant second term in Eq. (20). Laplace inversion of Eq. (10) then produces

$$p_{fa}(t) \sim \frac{2\sqrt{Dτ_1}}{L(τ_1 + τ_2)}.$$ (21)

At later times (smaller $u$) the second term in Eq. (20) dominates, and the plateau (21) turns into

$$p_{fa}(t) \sim \frac{α}{2} \left[\sin\left(\frac{πt}{α}\right)\right]^2 \frac{ντ_2}{L(τ_1 + τ_2)^{1/α} t^{1-1/α}}.$$ (22)

The crossover between these two regimes occurs when the values of expressions (22) and (21) become equal, i.e., at

$$t \sim (τ_1 + τ_2) \left\{\frac{α[sin(π/α)]^2 ντ_2}{4\sqrt{Dτ_1}}\right\}^{α/(α-1)}.$$ (23)

Note that in Eq. (21), $2\sqrt{Dτ_1}$ is the average length scanned in a search event. Division by $L$ yields the probability to find the target during this phase, and $1/(τ_1 + τ_2)$ is the rate at which the search phase itself occurs. A crucial part in this interpretation is that the probability of searching in a previously scanned area is negligible. This assumption will break down at some point because of the searcher’s lack of orientational memory. The searcher will then begin to revisit explored regions with a reduced probability of finding the target as a result. This causes the crossover to the power-law behavior (22). Fig. 4 shows the turnover from plateau to inverse...
power-law of the first arrival. At even longer times, finite size effects cause a turnover to an exponential decay.

From Eq. (22) the advantage of having $\alpha$ close to unity at large $L$ becomes evident: the presence of rare but long relocation events reduces the risk of rescanning already visited areas which will be important for large $L$. However, the downside to choosing an $\alpha$-value too close to 1 is that an increased amount of very long relocations implies an increased amount of very short ones too, as the average distance is fixed by $v\tau_2$. This means that the crossover to the less favorable situation described by Eq. (22) happens earlier, so that larger $\alpha$ becomes more efficient for shorter search times relevant at smaller $L$.

From a more general perspective, intermittent strategies are beneficial when purely diffusive search would slow down over time due to the increasing number of returns to previously scanned areas (oversampling). Choosing an exponential strategy for the relocation events, however, only partially solves this problem, as this strategy is still governed by the central limit theorem (CLT). Thus, the problem of oversampling merely becomes postponed to later times. Conversely, Lévy-intermittent strategies are not bound to the CLT, rendering them advantageous in the search for rare targets. Although less pronounced, the problem of oversampling still occurs in two dimensional search studied in Ref. [19]. Thus, Lévy strategies are expected to be advantageous in this case, as well.

We have shown that for intermittent search strategies Lévy distributed relocations are advantageous over exponential distributions when targets are sparse, because rare long relocations reduce the eventually occurring problem of oversampling. Thus we advocate that intermittent strategies should not be thought of as alternatives to Lévy strategies, as suggested in Ref. [19]. In contrast, the combination of intermittent search and Lévy relocation strategies turns out to be beneficial.

Our analysis relies on the assumption that each relocation is pointed toward a random direction. This will be a good model for “non-intelligent” search, similar to bacterial movement in absence of chemical or temperature gradients during which tumbling motion changes with directed motion $\mathbf{R}$. Intelligent creatures will improve the target search by partial or complete memory, avoiding previously visited locations. It will be interesting to study in more detail models with search memory.

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[17] Note the typo in the expression for $\langle t \rangle$ in Eq. (5) of Ref. [14]: coth(1/2$\alpha$L)] should be coth($\alpha$L/2).
[18] Corresponding to the fact that the central part close to $x = 0$ of Lévy stable laws with a smaller index $\alpha$ is more pronounced than for larger $\alpha$ or a Gaussian.
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