Suppression of large-scale perturbations
by stiff solid

Vladimír Balek* and Matej Škovrań†

Department of Theoretical Physics, Comenius University, Bratislava, Slovakia

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Abstract

Evolution of large-scale scalar perturbations in the presence of stiff solid (solid with pressure
to energy density ratio \( > 1/3 \)) is studied. If the solid dominated the dynamics of the universe long
enough, the perturbations could end up suppressed by as much as several orders of magnitude. To
avoid too steep large-angle power spectrum of CMB, radiation must have prevailed over the solid
long enough before recombination.

1 Introduction

In standard cosmology, large-scale perturbations stay unchanged throughout the Friedmann ex-
pansion that started after inflation, except for the last period before recombination when the
Newtonian potential was suppressed, due to the transition from radiation to matter, by the factor
9/10 (see, for example, [1]). The potential is not affected even by phase transitions and annihilations
taking place in the hot universe, as long as the matter filling the universe can be regarded as
ideal fluid. Among alternative scenarios considered in the literature there are some that relax that
assumption, introducing a solid component of the universe formed in the early stage of Friedmann
expansion [2, 3, 4, 5, 6, 7, 8]. The solid is supposed to have negative pressure to energy density
ratio \( w \); in particular, it can consist of cosmic strings (\( w = -1/3 \)) or domain walls (\( w = -2/3 \)).
Such matter starts to influence the dynamics of the universe at late times only and has no effect
on the evolution of perturbations during the hot universe period.

* e-mail address: balek@fmph.uniba.sk
† e-mail address: skovrań@fmph.uniba.sk
To obtain large-scale perturbations whose magnitude at recombination differs from their magnitude at the end of inflation, we need a solid with $w \geq 1/3$. A scenario with radiation-like solid ($w = 1/3$) was considered in [9], where it was shown that the solid produces an additional term in the gravitational potentials that can be large at the beginning but decays afterwards. If one introduces stiff solid ($w > 1/3$) instead, the character of the expansion of the universe changes for a limited period and a question arises whether this cannot cause a shift in the nondecaying part of the potentials, in analogy to what we observe in a universe filled with ideal fluid as it passes from one expansion regime to another due to a jump in $w$. If so, the incorporation of the solid into the theory, with the value of its shear modulus left free, would enlarge the interval of admissible values of the primordial potential, extending in such a way the parameter space of inflationary scenarios.

A possible realization of stiff solid would be a system of equally charged particles with anisotropic short-range interaction. By using Yukawa potential, one obtains stiff fluid [10, 11]; however, if the potential is squeezed in some direction and the particles are arranged into a lattice, the system acquires nonzero transversal as well as longitudinal shear modulus with respect to that direction.

In order that a solid, radiation-like or stiff, has an effect on large-scale perturbations, the solidification has to be anisotropic, producing a solid with flat internal geometry and nonzero shear stress. Such solidification might possibly take place in case the Friedmann expansion was preceded by solid inflation, driven by a solid with $w < 0$ rather than by a scalar field [12, 13, 14, 15].

In the paper we study how a stiff solid formed during Friedmann expansion would influence the evolution of large-scale perturbations. In section 2 we derive solution for such perturbations in a one-component universe and establish matching conditions in a universe whose matter content has changed abruptly; in section 3 we determine the behavior of perturbations after the solid has been formed and find both nondecaying and decaying part of Newtonian potential after radiation prevailed again; and in section 4 we discuss the results. Signature of the metric tensor is $(+−−−)$ and a system of units is used in which $c = 16\pi G = 1$.

## 2 Perturbations in the presence of solid

### 2.1 Evolution equations

Consider a flat FRWL universe filled with an elastic medium, fluid or solid, with energy density $\rho$ and pressure $p$, and denote the conformal time by $\eta$ and the scale parameter by $a$. Expansion of the universe is described by the equations

$$a' = \left(\frac{1}{6} \rho a^4 \right)^{1/2}, \quad \rho' = -3H\rho + p,$$

where the prime denotes differentiation with respect to $\eta$, $H = a'/a$ and $\rho + p$. 

2
In a perturbed universe, spacetime metric and stress-energy tensor acquire small space-dependent corrections $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$. We will use the proper-time gauge in which $\delta g_{00} = 0$ (the cosmological time $t = \int a d\eta$ coincides with the proper time of local observers). The metric in this gauge is

$$ds^2 = a^2[d\eta^2 + 2B_i d\eta dx^i - (\delta_{ij} - 2\psi \delta_{ij} - 2E_{ij}) dx^i dx^j],$$

(2)

where the effective equality indicates that only the scalar part of the quantity in question is given.

Suppose the matter filling the universe has Euclidean internal geometry and contains no entropy perturbations. The perturbation to $T_{\mu\nu}$ is then given solely by the perturbation to $g_{\mu\nu}$ and the shift vector of matter $\xi$. We will use the remaining gauge freedom to impose the condition $\xi = 0$, so that our gauge will be also comoving. In this gauge, the perturbation of mass density $\delta \rho = \delta T_{00}$, the energy flux density $S^i = -T_{i0}$ and the perturbation of stress tensor $\delta \tau_{ij} = \delta T_{ij}$ are

$$\delta \rho = \rho_+(3\psi + \mathcal{E}), \quad S^i = -\rho_+ B_i, \quad \delta \tau_{ij} = -K(3\psi + \mathcal{E})\delta_{ij} - 2\mu E^T_{ij}.$$  

(3)

where $K$ is the compressional modulus, $\mu$ is the shear modulus and the index ‘T’ denotes the traceless part of the matrix. (Our $K$ is 2 times greater and our $\mu$ is 4 times greater than $K$ and $\mu$ in [2]. We have defined them so in order to be consistent with the standard definitions in Newtonian elasticity.)

The proper-time gauge is not defined uniquely since one can shift the cosmological time by an arbitrary function $\delta t(x)$. Under such shift, $E$ stays unaltered and $B$ and $\psi$ transform as

$$B \rightarrow B + \delta \eta, \quad \psi \rightarrow \psi - \mathcal{H}\delta \eta,$$

where $\delta \eta = a^{-1}\delta t$. This suggests that we represent $B$ and $\psi$ as

$$B = B + \chi, \quad \psi = -\mathcal{H}\chi,$$  

(4)

where $B$ stays unaltered by the time shift and $\chi$ transforms as $\chi \rightarrow \chi + \delta \eta$.

We will restrict ourselves to perturbations of the form of plane waves with the wave vector $k$, $B$ and $\mathcal{E} \propto e^{ik \cdot x}$. The action of the Laplacian then reduces to the multiplication by $-k^2$; in particular, the definition of $\mathcal{E}$ becomes $\mathcal{E} = -k^2 E$. For simplicity, we will suppress the factor $e^{ik \cdot x}$ in $B$ and $\mathcal{E}$, as well as in other functions describing the perturbation. They will be regarded as functions of $\eta$ only.

Evolution of scalar perturbations is governed by two differential equations of first order for the functions $B$ and $\mathcal{E}$, coming from equations $T_{i\mu \eta \nu} = 0$ and $2G_{00} = T_{00}$. The equations are [16]

$$B' = (3c^2_{S0} + \alpha - 1)\mathcal{H}B + c^2_{S\parallel} \mathcal{E}, \quad \mathcal{E}' = -(k^2 + 3\alpha \mathcal{H}^2)B - \alpha \mathcal{H}\mathcal{E},$$

(5)

where $\alpha = \rho_+/(2\mathcal{H})^2 = (3/2)\rho_+ / \rho$, $c_{S0}$ is the “fluid” sound speed (sound speed of the solid with suppressed contribution of shear modulus), $c^2_{S0} = K/\rho_+$, and $c_{S\parallel}$ is the longitudinal sound speed,
\[ c_{s}^{2} = c_{s0}^{2} + (4/3)\mu/\rho. \] The only place where the shear modulus enters equations (5) is the term \[ c_{s}^{2} \] in the equation for \( B \).

Consider a one-component universe filled with a solid that has both \( p \) and \( \mu \) proportional to \( \rho \). The quantity \( K \) is then proportional to \( \rho \), too, since \( K = \rho_+ c_{s0}^{2} \) and \( c_{s0}^{2} = dp/d\rho \). Mechanical properties of such solid are given completely by two dimensionless constants \( w = p/\rho \) and \( \tilde{n} = \mu/\rho \). To simplify formulas, we will often use the constant \( \beta = \mu/\rho_+ = \tilde{n}/w_+, \) where \( w_+ = 1+w \), instead of \( \tilde{n} \).

For constant \( w \) and \( \tilde{n} \), the quantities appearing in the equations for \( B \) and \( E \) are all constant, except for the Hubble parameter that is proportional to \( \eta^{-1} \). Explicitly,

\[
\alpha = \frac{3}{2} w_+, \quad c_{s0}^{2} = w, \quad c_{s}^{2} = w + \frac{4}{3} \beta \equiv \tilde{w}, \quad H = 2w\eta^{-1},
\]

where \( u = 1/(1 + 3w) \). With these expressions, equations for \( B \) and \( E \) simplify to

\[
B' = u(1 + 9w)\eta^{-1}B + \tilde{w}E, \quad E' = -(k^2 + 18u^2 w_+ \eta^{-2})B - 3uw_+ \eta^{-1}E,
\]

and after excluding \( E \), we arrive at an equation of second order for \( B \),

\[
B'' + 2v\eta^{-1}B' + [q^2 - (2v - b)\eta^{-2}]B = 0,
\]

where \( q = \sqrt{u}k, \quad v = u(1 - 3w) \) and \( b = 24u^2\tilde{n} \). The equation is solved by Bessel functions of the argument \( q\eta \), multiplied by a certain power of \( \eta \). We are interested only in large-scale perturbations, that is, perturbations stretched far beyond the sound horizon. Such perturbations have \( q\eta \ll 1 \), hence we can skip the term \( q^2 \) in the square brackets in (7) to obtain

\[
B \approx \eta(c_J\eta^{-m} + c_Y\eta^{-M}),
\]

where the parameters \( m \) and \( M \) are defined in terms of the parameters \( v = v + 1/2 = (3/2)u(1-w) \) and \( n = \sqrt{v^2 - b} \) as \( m = v - n \) and \( M = v + n \). The constants are denoted \( c_J \) and \( c_Y \) to remind us that the two terms in (8) come from the Bessel functions \( J \) and \( Y \).

The function \( B \) is non-oscillating for \( b < v^2 \) and oscillating for \( b > v^2 \). Solutions of the second kind are well defined if the solid was not present in the universe from the beginning, but was formed at a finite time. Here we will restrict ourselves to the solutions of the first kind, which means that we will consider only values of the dimensionless shear stress \( \tilde{n} \leq (3/32)(1-w)^2 \).

An approximate expression for \( E \) is obtained by inserting the approximate expression for \( B \) into the first equation in (6). In this way we find

\[
E \approx \dot{c}_J\eta^{-m} + \dot{c}_Y\eta^{-M},
\]

where \( \dot{c}_J \) and \( \dot{c}_Y \) are defined in terms of \( c_J \) and \( c_Y \) as \( \dot{c}_J = -(1/\tilde{n})(3/2 - n)c_J \) and \( \dot{c}_Y = -(1/\tilde{n})(3/2 + n)c_Y \).
2.2 Potentials $\Phi$ and $\Psi$

Scalar perturbations we are interested in are most easily interpreted in the Newtonian gauge, in which the metric is

$$ds^2 = a^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)d\mathbf{x}^2].$$

Let us express the potentials $\Phi$ and $\Psi$ in terms of the functions $B$ and $E$. If we perform explicitly the coordinate transformation from the proper-time to Newtonian gauge, we find (see equation (7.19) in [1])

$$\Psi = \mathcal{H}(B - E').$$

For $\Phi$ we could proceed analogically, but it is simpler to use Einstein equations. If we write the scalar part of the stress tensor as a sum of pure trace and traceless part, $\tau^{ij} = \tau^{(1)}\delta_{ij} + \tau^{(2)}T,ij$, from equations $2G_{ij} = T_{ij}$ we obtain that the difference of $\Phi$ and $\Psi$ is given by the latter quantity (see equation (7.40) in [1]),

$$\Delta \Phi \equiv \Phi - \Psi = \frac{1}{2}\tau^{(2)}a^2.$$ 

By inserting here from the third equation (3) we find

$$\Delta \Phi = -\mu a^2 E.$$

We can see that in a universe filled with an ideal fluid ($\mu = 0$) the potentials $\Phi$ and $\Psi$ coincide.

After inserting into the expression for $\Psi$ from the second equation in (5) and into the expression for $\Delta \Phi$ from the first equation in (1), we arrive at

$$\Psi = -k^{-2}\Omega \mathcal{H}^2(3\mathcal{H}B + \mathcal{E}), \quad \Delta \Phi = 6\bar{\mu}k^{-2}\mathcal{H}^2\mathcal{E}.$$ 

For the one-component universe introduced before, expressions for $\Psi$ and $\Delta \Phi$ become

$$\Psi = -6u^2 w_+(k\eta)^{-2}(6u\eta^{-1}B + \mathcal{E}), \quad \Delta \Phi = 24u^2 \bar{\mu}(k\eta)^{-2}\mathcal{E}.$$ 

With $B$ and $\mathcal{E}$ given in (8) and (9), both $\Phi$ and $\Psi$ are linear combinations of $\eta^{-2-m}$ and $\eta^{-2-M}$. For an ideal fluid $m = 0$ and $M = 2\nu$, so that we expect the function $\Phi$ to be linear combination of $\eta^{-2}$ and $\eta^{-2\nu}$, where $\nu_+ = 1 + \nu$. This is, however, not true because the coefficient in front of $\eta^{-2}$ turns out to be zero. Thus, if we want to establish how $\Phi$ looks like for an ideal fluid, or how $\Phi$ and $\Psi$ look like for a solid with small $\bar{\mu}$, we must add the next-to-leading term to the $J$-part of both expressions (8) and (9). The term is suppressed by the factor $(q\eta)^2$, therefore the $J$-part of $\Phi$ for an ideal fluid is constant and the $J$-part of $\Phi$ and $\Psi$ for a solid with small $\bar{\mu}$ acquires a term proportional to $\eta^{-m}$. For a universe filled with an ideal fluid we have

$$B \doteq \eta(c_J + c_Y \eta^{-2\nu}), \quad \mathcal{E} \doteq \dot{c}_J + \dot{c}_Y \eta^{-2\nu},$$

(15)
where \( \hat{c}_J \) and \( \hat{c}_Y \) are defined in terms of \( c_J \) and \( c_Y \) as \( \hat{c}_J = -6uc_J \) and \( \hat{c}_Y = -3u(w_+/\nu)c_Y \). After computing the additional terms in \( B \) and \( E \) and inserting the resulting expressions into equations (14), we arrive at

\[
\Phi = C_J + C_Y \eta^{-2\nu},
\]

(16)

where \( C_J \) and \( C_Y \) are defined in terms of \( c_J \) and \( c_Y \) as \( C_J = 3u^2(w_++\nu+c_J) \) and \( C_Y = 12u^2w_+\nuq^{-2}c_Y \).

### 2.3 Transitions with jump in \( w \) and \( \tilde{\mu} \)

Suppose the functions \( w_\eta \) and \( \tilde{\mu}_\eta \) change at the given moment \( \eta_{tr} \) ("transition time") from \((w_I, \tilde{\mu}_I)\) to \((w_{II}, \tilde{\mu}_{II}) = (w_I + \Delta w, \tilde{\mu}_I + \Delta \tilde{\mu})\). (We have attached the index \( \eta \) to the symbols \( w \) and \( \tilde{\mu} \) in order to distinguish the functions denoted by them from the values these functions assume in a particular era.) Rewrite the first equation in (5) as

\[
B' = c_{2S}^2(3HB + \mathcal{E}) + \left(\frac{3}{2}w_\eta + 1\right)HB + \frac{4}{3}\beta_\eta \mathcal{E},
\]

(17)

where

\[
c_{2S}^2 = \frac{dp}{d\rho} = w_\eta + \rho \frac{dw_\eta}{d\rho}.
\]

(18)

Because of the jump in \( w_\eta \), there appears \( \delta \)-function in \( c_{2S}^2 \), and to account for it we must assume that \( B \) has a jump, too. However, on the right hand side of equation (17) we then obtain an expression of the form "\( \theta \)-function \times \( \delta \)-function"; and if we rewrite \( B' \) as

\[
B' = \frac{dB}{d\rho} \rho' = -3H\rho w_\eta + \frac{dB}{d\rho},
\]

on the left hand side there appears another such expression. To give meaning to the equation we must suppose that \( w_\eta \) changes from \( w_I \) to \( w_{II} \) within an interval of the length \( \Delta \rho \ll \rho_{tr} \), and send \( \Delta \rho \) to zero in the end. If we retain just the leading terms in equation (17) in the interval with variable \( w \), we obtain

\[
w_\eta + \frac{dB}{d\rho} = -\left( B + \frac{E_{tr}}{3H_{tr}} \right) \frac{dw_\eta}{d\rho},
\]

(19)

where we have used the fact that, as seen from the second equation in (5), the function \( \mathcal{E} \) is continuous at \( \eta = \eta_{tr} \). The solution is

\[
B + \frac{E_{tr}}{3H_{tr}} = \frac{C}{w_\eta+}.
\]

Denote the jump of the function at the moment \( \eta_s \) by square brackets. To determine \( [B] \), we express \( B_I \) and \( B_{II} \) in terms of \( w_{I+} \) and \( w_{II+} \), compute the difference \( B_{II} - B_I \) and use the expression for \( B_I \) to exclude \( C \). In this way we find

\[
[B] = -\frac{\Delta w}{w_{II+}} \left( B_I + \frac{E_{tr}}{3H_{tr}} \right),
\]

(20)
Note that the same formula is obtained if we assume that the functions with jump are equal to the mean of their limits from the left and from the right at the point where the jump occurs.

To justify the expression for $[\mathcal{B}]$, let us compute the jump in $\Psi$. It holds

$$[\Psi] = -\frac{3}{2}k^{-2}H_{tr}^2(3H_{tr}[w_\eta+\mathcal{B}]+\Delta w\mathcal{E}_{tr}),$$

and if we write $[w_\eta+\mathcal{B}] = w_{1+}[\mathcal{B}]+\Delta w\mathcal{B}_{1}$ and insert for $[\mathcal{B}]$, we find that $[\Psi]$ vanishes. This must be so because for $\Psi$ we have (see equation (7.40) in [1])

$$\Psi'' + \mathcal{H}(2\Psi' + \Phi') + (2H' + H^2)\Psi = -\frac{1}{4}\delta\tau^{(1)},$$

where the bar indicates that the quantity $\delta\tau^{(1)}$ is computed in Newtonian gauge. A jump in $\Psi$ would produce a derivative of $\delta$-function in the first term, but no such expression with opposite sign appears in the other terms.

The jump in $B'$ can be found from equation (17) by computing the jump of the right hand side, with no need for the limiting procedure we have used when determining the jump in $B$. The result is

$$[B'] = 4\frac{\Delta w}{w_{1+}}\mathcal{H}_{tr}\mathcal{B}_{tr} + \left(\frac{5-3w_{1+}}{6w_{1+}}\Delta w + \frac{4}{3}\Delta \beta\right)\mathcal{E}_{tr}. \quad (21)$$

### 3 Scenario with stiff solid

#### 3.1 Expansion of the universe

Suppose at some moment $\eta_s$ the hot universe underwent a phase transition during which a part of radiation ($w = 1/3$) instantaneously turned into a stiff solid ($w > 1/3$). In a one-component universe with given parameter $w$, the density of matter falls down the faster the greater the value of $w$. As a result, if the solid acquired a substantial part of the energy of radiation at the moment it was formed, it dominated the evolution of the universe for a limited period until radiation took over again. Let us determine the function $a(\eta)$ for such universe.

Denote the part of the total energy that remained stored in radiation after the moment $\eta_s$ by $\epsilon$. In the period with pure radiation ($\eta < \eta_s$) the mass density was $\rho = \rho_s(a_s/a)^4$, so that from the first equation in (1) we obtain

$$a = C\eta, \quad C = \left(\frac{1}{6}\rho_s a_s^4\right)^{1/2}. \quad (22)$$

In the period with a mix of radiation and solid ($\eta > \eta_s$) the mass density is

$$\rho = \epsilon\rho_s(a_s/a)^4 + (1-\epsilon)\rho_s(a_s/a)^{3w_+} = \rho_s(a_s/a)^4[\epsilon + (1-\epsilon)(a_s/a)\Delta],$$

where $\Delta = 3w_+ - 4$. As a result, the first equation in (1) transforms into

$$a' = C[\epsilon + (1-\epsilon)(a_s/a)\Delta]^{1/2}. \quad (23)$$
For \( w > 1/3 \) the parameter \( \Delta \) is positive, therefore the second term eventually becomes less than the first term even if \( \epsilon \ll 1 \).

Suppose radiation retained less than one half of the total energy at the moment of radiation-to-solid transition (\( \epsilon < 1/2 \)). The subsequent expansion of the universe can be divided into two eras, solid-dominated and radiation-dominated, separated by the time \( \eta_{\text{rad}} \) at which the mass densities of the solid and radiation were the same. The value of \( \eta_{\text{rad}} \) is given by

\[
a_{\text{rad}} = a_s (\epsilon^{-1} - 1)^{1/\Delta}.
\]  

(24)

Suppose now that the post-transitional share of energy stored in radiation was small (\( \epsilon \ll 1 \)). The universe then expands by a large factor between the times \( \eta_{\text{s}} \) and \( \eta_{\text{rad}} \),

\[
a_{\text{rad}} = a_s \epsilon^{-1/\Delta} \gg a_s,
\]

and can be described in a good approximation as if it was filled first with pure solid and then with pure radiation. Thus, equation (23) can be replaced by

\[
a' = \begin{cases} 
C(a_s/a)^{\Delta/2} & \text{for } \eta < \eta_{\text{rad}}, \\
\sqrt{C} & \text{for } \eta > \eta_{\text{rad}}
\end{cases}
\]

(25)

The solution is

\[
a = \begin{cases} 
[(\Delta/2 + 1)a_s^{\Delta/2}C\tilde{\eta}]^{1/\Delta+1} & \text{for } \eta < \eta_{\text{rad}}, \\
\sqrt{C}\tilde{\eta} & \text{for } \eta > \eta_{\text{rad}}
\end{cases}
\]

(26)

where \( \tilde{\eta} \) and \( \tilde{\eta} \) are shifted time variables, \( \tilde{\eta} = \eta - \eta_{\text{s}} \) and \( \tilde{\eta} = \tilde{\eta} - \eta_{\text{ss}} \). From the approximate expression for \( a_{\text{rad}} \) we obtain

\[
\tilde{\eta}_{\text{rad}} = \frac{1}{\Delta/2 + 1} \epsilon^{-\Delta/2+1} \eta_{\text{s}},
\]

(27)

and by matching the solutions at \( \eta_{\text{s}} \) and \( \eta_{\text{rad}} \) we find

\[
\eta_{\text{s}} = \frac{\Delta/2}{\Delta/2 + 1}\eta_{\text{s}}, \quad \eta_{\text{ss}} = -\frac{\Delta}{2} \eta_{\text{rad}},
\]

(28)

Note that equation (23) solves analytically for \( w = 2/3 \) and \( w = 1 \), when \( \Delta = 1 \) and \( \Delta = 2 \). We do not give these solutions here since will not need them in what follows.

The two equations in (28) can be rewritten to formulas for the ratios of shifted and unshifted times,

\[
\frac{\tilde{\eta}_{\text{s}}}{\eta_{\text{s}}} = \frac{1}{\Delta/2 + 1} = \frac{u}{u_0}, \quad \frac{\tilde{\eta}_{\text{rad}}}{\eta_{\text{rad}}} = \frac{\Delta}{2} + 1 = \frac{u_0}{u},
\]

where \( u_0 \) is the value of \( u \) in the radiation-dominated era, \( u_0 = 1/2 \). These equations stay valid also after we replace radiation by an ideal fluid with an arbitrary pressure to energy density ratio \( w_0 \). To demonstrate that, let us derive them from the condition of continuity of the Hubble parameter. If the universe is filled in the given period with matter with the given value of \( w \), its scale parameter depends on a suitably shifted time \( \tilde{\eta} \) as \( a \propto \tilde{\eta}^{2w} \). Thus, its Hubble parameter is \( \mathcal{H} = 2u\tilde{\eta}^{-1} \) and the requirement that \( \mathcal{H} \) is continuous at the moment when \( w \) changes from \( w_I \) to \( w_{II} \) is equivalent to \( \tilde{\eta}_{II}/\tilde{\eta}_I = u_{II}/u_I \).
3.2 Behavior of the function \( \mathcal{B} \)

We are interested in large-scale perturbations in a universe in which the parameters \( w \) and \( \tilde{\mu} \) assume values \((w_0, 0)\) before \( \eta_s \), \((w, \tilde{\mu})\) between \( \eta_s \) and \( \eta_{rad} \), and \((w_0, 0)\) after \( \eta_{rad} \). (Most of the time we will leave \( w_0 \) free, only at the end we will put \( w_0 = 1/3 \).) Denote the functions describing the perturbation before \( \eta_s \) and after \( \eta_{rad} \) by the indices 0 and 1 respectively, and keep the functions referring to the interval between \( \eta_s \) and \( \eta_{rad} \) without index. If only the nondecaying part of the perturbation (the part with constant \( \Phi \)) survives at the moment \( \eta_s \), the functions \( \mathcal{B}_0 \) and \( \mathcal{E}_0 \) can be replaced by their \( J \)-parts,

\[
\mathcal{B}_0 = c_{J0} \eta, \quad \mathcal{E}_0 = \dot{c}_{J0} = -6u_0 c_{J0}.
\]  

For the functions \( \mathcal{B} \) and \( \mathcal{E} \) we have expressions (8) and (9) with \( \eta \) replaced by \( \tilde{\eta} \) and for the function \( \mathcal{B}_1 \) we have the first equation (15) with \( c_J \) and \( c_Y \) replaced by \( c_{J1} \) and \( c_{Y1} \), \( \nu \) replaced by \( \nu_0 \) and \( \eta \) replaced by \( \tilde{\eta} \). All we need to obtain the complete description of the perturbation is to match the expressions for \( \mathcal{B}_0 \), \( \mathcal{B} \) and \( \mathcal{B}_1 \) with the help of the expressions for \( \mathcal{E}_0 \) and \( \mathcal{E} \) at the moments \( \eta_s \) and \( \eta_{rad} \).

At the moment \( \eta_s \), the jumps in \( w_\eta \) and \( \tilde{\mu}_\eta \) are \( \Delta w_\eta = w - w_0 = \Delta w \) and \( \Delta \tilde{\mu}_\eta = \tilde{\mu} \). By using these values and the identity \( \mathcal{E}_0 = -3\mathcal{H}_s \mathcal{B}_0 \), we find

\[
[B]_s = 0, \quad [B']_s = -\left( \frac{1}{2} \Delta w - \frac{4}{3} \beta \right) \mathcal{E}_0,
\]

de not \( x_0 = c_{J0} \). Equations for the unknowns \( \tilde{x} = c_J \tilde{\eta}_s^{-m} \) and \( \tilde{y} = c_Y \tilde{\eta}_s^{-M} \) are

\[
\tilde{x} + \tilde{y} = \frac{u_0}{u} x_0, \quad (1 - m) \tilde{x} + (1 - M) \tilde{y} = \left[ 1 + 8u_0 \left( \frac{3}{8} \Delta w - \beta \right) \right] x_0,
\]

and their solution is

\[
\tilde{x} = \frac{u_0}{u} \frac{1}{2n} (M - 8u\beta) x_0, \quad \tilde{y} = -\frac{u_0}{u} \frac{1}{2n} (m - 8u\beta) x_0.
\]

At the moment \( \eta_{rad} \), the jumps in \( w_\eta \) and \( \tilde{\mu}_\eta \) are \( \Delta w_{rad} = -\Delta w \) and \( \Delta \beta_{rad} = -\beta \). By inserting these values into the expressions for \( [B] \) and \( [B'] \) we obtain

\[
[B]_{rad} = \frac{\Delta w}{w_0^+} (\mathcal{B}_{rad} + \frac{\mathcal{E}_{rad}}{3\mathcal{H}_{rad}}), \quad [B']_{rad} = -4 \frac{\Delta w}{w_0^+} \mathcal{H}_{rad} \mathcal{B}_{rad} - \left( \frac{5 - 3w_0}{6w_0^+} \Delta w + \frac{4}{3} \beta \right) \mathcal{E}_{rad}.
\]

Introduce the constants

\[
\tilde{X} = c_J \tilde{\eta}_{rad}^{-m} = p^{-m} \tilde{x}, \quad \tilde{Y} = c_Y \tilde{\eta}_{rad}^{-M} = p^{-M} \tilde{y},
\]

where \( p \) is the ratio of final and initial moments of the period during which the solid affects the dynamics of the universe, \( p = \tilde{\eta}_{rad}/\eta_s \). Equations for the unknowns \( \tilde{\tilde{x}} = c_{J1} \) and \( \tilde{\tilde{y}} = c_{Y1} \tilde{\eta}_{rad}^{-2\nu_0} \) are

\[
\tilde{\tilde{x}} + \tilde{\tilde{y}} = \frac{u}{u_0} (K_J \tilde{X} + K_Y \tilde{Y}), \quad \tilde{\tilde{x}} + (1 - 2\nu_0) \tilde{\tilde{y}} = L_J \tilde{X} + L_Y \tilde{Y},
\]

\[
\tilde{\tilde{x}} + \tilde{\tilde{y}} = \frac{u}{u_0} (K_J \tilde{X} + K_Y \tilde{Y}), \quad \tilde{\tilde{x}} + (1 - 2\nu_0) \tilde{\tilde{y}} = L_J \tilde{X} + L_Y \tilde{Y},
\]

\[
\tilde{\tilde{x}} + \tilde{\tilde{y}} = \frac{u}{u_0} (K_J \tilde{X} + K_Y \tilde{Y}), \quad \tilde{\tilde{x}} + (1 - 2\nu_0) \tilde{\tilde{y}} = L_J \tilde{X} + L_Y \tilde{Y},
\]
where the coefficients on the right hand side are defined as

\[ K_J = \frac{1}{w_0} \left[ w_+ - \frac{\Delta w}{6uw} (m + 6uw) \right], \quad K_Y = \text{ditto with } m \to M, \]

and

\[ L_J = 1 - m - \frac{8u\Delta w}{w_0} + \frac{m + 6uw}{w_0} \left( \frac{5 - 3w_0}{6w_0} \Delta w + \frac{4}{3} \beta \right), \quad L_Y = \text{ditto with } m \to M, \]

The solution is

\[ \begin{align*}
\tilde{x} &= \frac{1}{2\nu_0} (M_J \tilde{X} + M_Y \tilde{Y}), \\
\tilde{y} &= -\frac{1}{2\nu_0} (N_J \tilde{X} + N_Y \tilde{Y})
\end{align*} \tag{34} \]

with the constants \( M_\alpha \) and \( N_\alpha, \alpha = J, Y \), defined in terms of the constants \( L_\alpha \) and \( K_\alpha \) as

\[ M_\alpha = L_\alpha - (1 - 2\nu_0) \frac{u}{u_0} K_\alpha, \quad N_\alpha = L_\alpha - \frac{u}{u_0} K_\alpha. \]

### 3.3 Behavior of potentials

Knowing how the function \( \mathcal{B} \) looks like, we can establish the time dependence of the Newtonian potential \( \Phi \) and the potential describing the curvature of 3-space \( \Psi \). Before the time \( \eta_s \), both potentials are the same, \( \Phi_0 \) as well as \( \Psi_0 = C_{J0} \sim x_0 \). Between the times \( \eta_s \) and \( \eta_{rad} \), the potentials are given by the two equations in (14) with \( \eta \) replaced by \( \tilde{\eta} \). With \( \mathcal{B} \) and \( \mathcal{E} \) inserted from equations (8) and (9), both \( \Phi \) and \( \Psi \) become sums of terms proportional to \( \tilde{\eta}^{-2-m} \) and \( \tilde{\eta}^{-2-M} \). We have already mentioned that for \( \tilde{\mu} = 0 \) the coefficient in the first term in \( \Phi = \Psi \) is zero, and one easily verifies that for \( w > 1/3 \) and \( \tilde{\mu} \) close to zero the first coefficient in both \( \Phi \) and \( \Psi \) is proportional to \( \tilde{\mu} \). (After a simple algebra we find that it is proportional to \( m(1 - 4\beta) - 8\beta \) and \( m - 8u\beta \) for \( \Phi \) and \( \Psi \) respectively, with \( m \) reducing to \( b/(2\nu) = 8uw_+\beta/(1-w) \) in the limit \( \beta \ll 1 \).) The coefficients contain the constants \( c_J \) and \( c_Y \) and if we use \( c_J \propto \tilde{x} \) and \( c_Y \propto \tilde{y} \) with \( \tilde{x} \) and \( \tilde{y} \) given in equation (31), we find that the second coefficient is proportional to \( \tilde{\mu} \), too. (In the expression for \( \tilde{y} \) we encounter the factor \( m - 8u\beta \) again.) Both coefficients contain also the factor \( x_0 \sim \Phi_0 \), therefore for \( \eta \) close to \( \eta_s \) we have \( \Phi \) as well as \( \Psi \sim \tilde{\mu}(k\tilde{\eta})^{-2} \Phi_0 \). As \( \eta \) grows, the first correction to the term proportional to \( \tilde{\eta}^{-2-m} \), which is of order \( \Phi_0 \), may take over while the perturbation still remains stretched over the horizon. However, in order that our approximation is valid, this term must be negligible in the first period after the moment \( \eta_s \). (Note that this does not hold for the potential \( \Psi \) just after \( \eta_s \): it equals \( \Phi_0 \) at \( \eta_s \), hence it is dominated by the correction term for a short period afterwards.) As a result, \( \tilde{\mu} \) must be not too close to zero, \( \tilde{\mu} \gg (k\tilde{\eta}_s)^2 \).

For large enough \( \tilde{\mu}, \Phi \) and \( \Psi \) can become much greater in absolute value not only than \( \Phi_0 \), but also than 1. The theory then seems to collapse, but it does not because, as can be checked by direct computation, \( kB, \psi \) and \( \mathcal{E} \) remain much less than 1. (A detailed discussion for \( \Delta w = 0 \) can be found in [17].) Thus, the proper-time comoving gauge which we have implemented instead of more common, and intuitively more appealing, Newtonian gauge, is not only convenient.
computationally, but also preferable on principal grounds. Without it we would not know that the perturbations stay small and the linearized theory stays applicable after a solid with above-critical parameter $\tilde{\mu}$ was formed, causing the potentials $\Phi$ and $\Psi$ to rise beyond control.

We are interested in the potential $\Phi$ after the moment $\eta_{\text{rad}}$, when both potentials coincide again. Denote the nondecaying part of $\Phi$ in that period as $\Phi_{1,\text{nd}}$. It holds $\Phi_{1,\text{nd}} = C_1$, and by using the relation between $C_j$ and $c_j$ we obtain

$$\Phi_{1,\text{nd}} = 3u_0^2 \frac{w_0 + \tilde{z}}{\nu_0 + \tilde{x}}.$$  \hspace{1cm} (35)

Here we must insert for $\tilde{x}$ from equation (34), with $\tilde{X}$ and $\tilde{Y}$ given in equation (32), $\tilde{x}$ and $\tilde{y}$ given in equation (31) and $x_0$ given by

$$\Phi_0 = 3u_0^2 \frac{w_0 + \nu_0}{\nu_0 + x_0}.$$  \hspace{1cm} (36)

The resulting expression for $\Phi_{1,\text{nd}}$ is

$$\Phi_{1,\text{nd}} = \frac{1}{2} \frac{u_0}{\nu_0} \frac{1}{2n} (\tilde{M}p^{-m} - \tilde{M}p^{-M})\Phi_0,$$  \hspace{1cm} (37)

with the coefficients $\tilde{M}_J$ and $\tilde{M}_Y$ defined as

$$\tilde{M}_J = M_J(M - 8u\beta), \quad \tilde{M}_Y = M_Y(m - 8u\beta).$$

After some algebra the coefficients reduce to

$$\tilde{M}_J = 2\nu_0 \frac{u}{\nu_0} M - b, \quad \tilde{M}_Y = \text{ditto with } M \to m.$$  \hspace{1cm} (38)

Let us now determine how fast the function $\Phi$ approaches its limit value. Denote $\tilde{z} = q_0 \tilde{\eta}$, where $q_0 = \sqrt{w_0 k}$. The decaying part of $\Phi$ in the period under consideration is

$$\Delta \Phi_1 = -2\nu_0 \frac{u}{\nu_0} \frac{1}{2n} (\tilde{N}_Jp^{-m} - \tilde{N}_Yp^{-M})\tilde{z}_{\text{rad}}^{-2} \Phi_0,$$  \hspace{1cm} (39)

where $\tilde{z}$ is rescaled time normalized to 1 at the moment $\eta_{\text{rad}}$, $\zeta = \tilde{\eta}/\tilde{\eta}_{\text{rad}}$, and the coefficients $\tilde{N}_J$ and $\tilde{N}_Y$ are defined in terms of $N_J$ and $N_Y$ in the same way as the coefficients $\tilde{M}_J$ and $\tilde{M}_Y$ in terms of $M_J$ and $M_Y$. After rewriting the former coefficients similarly as we did with the latter ones, we obtain

$$\tilde{N}_J = \tilde{N}_Y = -\frac{u_0}{w_0 + 2b}. $$  \hspace{1cm} (40)

From these equations and equations (36) and (37) we find that the ratio of the decaying and nondecaying part of $\Phi$ at the moment of solid-to-radiation transition is

$$\left. \frac{\Delta \Phi_1}{\Phi_{1,\text{nd}}}_{\text{rad}} \right| = R \tilde{z}_{\text{rad}}^{-2}, \quad R = 4\nu_0 \frac{u_0}{\nu_0} \frac{u_0}{\nu_0 + 2n u [n \coth(n \log p) + \nu] - u_0 b}.$$  \hspace{1cm} (41)

The ratio is greater than one for $\beta \gtrsim \tilde{z}_{\text{rad}}^2$. The function $\Phi$ is then dominated by the decaying term at the moment $\eta_{\text{rad}}$, the nondecaying term taking over later, at the moment $\eta_{\text{nd}}$ given by

$$\tilde{z}_{\text{nd}} = R^{\frac{u_0}{2n u + 2n u [n \coth(n \log p) + \nu] - u_0 b}}.$$  \hspace{1cm} (42)
The exponent at $\tilde{z}_{rad}$ is positive for any $w_0 < 1$ (it equals $1/3$ for $w_0 = 1/3$) and the constant $R$ is of order 1 or less. Thus, if the perturbation was stretched over the horizon at the moment the fluid originally filling the universe started to be dominating again ($\tilde{z}_{rad} \ll 1$), it will stay so at the moment the nondecaying term prevails over the decaying one ($\tilde{z}_{nd} \ll 1$).

The time $\eta_{rad}$ must not be too close to the time of recombination $\eta_{rec}$, if the spectrum of large-angle CMB anisotropies is not to be tilted too much. If we denote the wave number of $\tilde{z}_{rad}$ the nondecaying term prevails over the decaying one ($n$ must be multiplied by 0.01 of the scalar spectrum. If we allow for a tilt of the primordial spectrum, too, the expression for the longest wavelength that can be observed in CMB have $k = 0.01 k(0)$. For $w_0 = 1/3$, Newtonian potential after the moment $\eta_{rad}$ is $\Phi_1 = (1 + R_{rad}^{-2}\tilde{z}^3)\Phi_{1,nd} = (1 + R_{rad}\tilde{z}^{-3})\Phi_{1,nd}$, and if we take into account that the value of $\tilde{z}_{rad}$ is approximately 1, we find

$$r = \frac{\Phi_1(k(0))}{\Phi_1(0.01 k(0))} = \frac{1 + R_{rad}^{\tilde{z}}(0)}{1 + 10^4 R_{rad}^{\tilde{z}}(0)} = 1 - 10^4 R_{rad}^{\tilde{z}}(0).$$

The observational value of $r$ is 0.01$^{ns_{s-1}}$, where $ns$ is the scalar spectral index, a characteristic of perturbations whose deviation from 1 (about $-0.04$ according to observations) describes the tilt of the scalar spectrum. If we allow for a tilt of the primordial spectrum, too, the expression for $r$ must be multiplied by 0.01$^{ns_{0-1}}$. Denote $p_* = 1/\tilde{z}_{rad} = \tilde{z}_{rec}/\tilde{z}_{rad} = a_{rec}/a_{rad} = T_{rad}/T_{rec}$ and require that $ns$ differs from $ns_0$ at most by some $\Delta ns \ll 1$. To ensure that, $p_*$ must satisfy

$$p_* > 2 \times 10^3 R \Delta ns^{-1}.$$  (42)

For numerical calculations we need the value of $p$. It is a ratio of times, but can be rewritten in terms of a ratio of scale parameters or temperatures, $P = a_{rad}/a_* = T_*/T_{rad}$, as

$$p = P^{\frac{1}{\Delta \nu}}.$$  (43)

The value of $p$, or equivalently, $P$, determines the interval of admissible $w$’s. To obtain it, note that for $w_0 = 1/3$ equation (24) yields $P = (\epsilon^{-1} - 1)^{1/\Delta} \pm \epsilon^{-1/\Delta}$, or

$$P \pm \epsilon^{1/\Delta}.$$  (44)

(This is consistent with equation (27), which can be rewritten as $p = \epsilon^{-\Delta/2 + 1} = \epsilon^{-\frac{1}{6 \Delta \nu}}$. Thus, the jump in the parameter $w$ for the given ratio $P$ must satisfy

$$\Delta w \geq \frac{\log 1/\epsilon}{3 \log P} \geq \frac{1}{3 \log P}.$$  (45)

The dependence of the quantities $\phi = \Phi_{1,nd}/\Phi_0$ and $R$ on the parameter $\beta$ is depicted in fig. 1. The values of $w_0$ and $w$ are 1/3 and 2/3 on both panels and the solid and dotted lines correspond to $P = 10^3$ and $P = 10^{13}$ respectively. The lines are terminated at $\beta = 1/160$, which is the maximum value of $\beta$ admitting non-oscillating solutions in a solid with $w = 2/3$. 

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For completeness, we have included also values $\beta < 0$ into the graphs. The transversal sound speed squared is negative for such $\beta$, so that the vector perturbations start to grow exponentially once they have appeared. As a result, the theory is acceptable only if such perturbations are produced neither during inflation (which is the case in simplest models) nor in the subsequent phase transitions.

The parameter $P$ assumes the smaller value if, for example, the solid dominated the dynamics of the universe between the electroweak and confinement scale, and the greater value, if the solid was formed as soon as at the GUT scale and dominated the dynamics of the universe up to the electroweak scale. Unless the parameter $w$ of the solid is close to that of radiation, the fraction of energy which remains stored in radiation after the solid has been formed must be quite small in the former case and very small in the latter case. For $w = 2/3$ this fraction equals $1/P$, so that for the greater $P$ the mechanism of the radiation-to-solid transition must transfer to the solid all but one part in 10 trillions of the energy of radiation.

The quantity $\phi$ is the factor by which the value of the potential $\Phi$ changes due to the presence of stiff solid in the early universe. From the left panel of fig. 1 we can see that $\Phi$ is shifted upwards for $\beta < 0$ and downwards for $\beta > 0$, and the enhancement factor decreases monotonically with $\beta$, the steeper the larger the value of $P$. For maximum $\beta$ the function $\Phi$ is suppressed by the factor 0.41 if $P = 10^3$ and by the factor 0.004 if $P = 10^{13}$. The quantity $R$ determines, together with the parameter $\Delta n_S$, the minimal duration of the period between the moment when radiation took over again and recombination. According to the right panel of fig. 1, the temperature at the beginning of this period had to be at least $8 \times 10^3 \Delta n_S^{-1} T_{re} \simeq 0.2 (\Delta n_S/0.01)^{-1}$ MeV for maximum $\beta$ and $P = 10^3$. 

Fig. 1: Final value of Newtonian potential in a universe with stiff solid (left) and normalized ratio of decaying to nondecaying part of the potential at solid-to-radiation transition (right), plotted as functions of dimensionless shear modulus.
4 Conclusion

We have studied a scenario with stiff solid appearing in the hot universe and dominating the evolution of the universe during a limited period before recombination. In comparison with the scenario containing radiation-like solid [9], a new effect is that the nondecaying part of Newtonian potential becomes suppressed. This might raise hope that the tensor-to-scalar ratio is enhanced, which would surely be interesting from the observational point of view. However, a straightforward calculation shows that the tensor perturbations are suppressed by exactly the same factor as the scalar ones. The shift in Newtonian potential towards less values means that the rms of primordial potential was in fact greater than supposed. As a result, there appears an additional freedom in the choice of the parameters of inflaton potential; for example, one can use potentials with smaller inclination of the plateau than in the case without solid when implementing slow-roll inflation.

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