ORTHOSYMPELECTIC SATAKE EQUIVALENCE, II

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To Mikhail Kapranov on his 60th birthday

Abstract. This is a companion paper of [BFGT, BFT]. We prove an equivalence relating representations of a degenerate orthosymplectic supergroup with the category of twisted $\text{Sp}(2n, \mathbb{C}[t])-\text{equivariant } D$-modules on the so called mirabolic affine Grassmannian of $\text{Sp}(2n)$. We also discuss (conjectural) extension of this equivalence to the case of quantum supergroups and to some exceptional supergroups.

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arXiv:2207.03115v2 [math.RT] 20 Sep 2022
1. **Introduction**

1.1. **Orthosymplectic Satake equivalence.** This paper is a sequel to [BFT] where a conjectural version of the Satake equivalence for orthosymplectic groups $\text{SO}(2k|2n)$ was studied. The main goal of the present paper is to extend this study to the case of $\text{SO}(2k+1|2n)$. The ‘central’ case is $k = n$, and this is the only case where we prove an equivalence of categories proposed by D. Gaiotto.

Namely, we construct a geometric realization of the category $\text{Rep}(\text{SO}(2n+1|2n))$ of finite-dimensional representations of the degenerate supergroup $\text{SO}(2n+1|2n)$. Recall that the even part of $\text{SO}(2n+1|2n)$ is $\text{SO}(2n+1) \times \text{Sp}(2n)$, and its Lie superalgebra $\mathfrak{osp}(2n+1|2n)$ has trivial supercommutator of two odd elements, while the supercommutator of an even element with another element is the same as in $\mathfrak{osp}(2n+1|2n)$.

This geometric realization makes use of the ‘mirabolic’ affine Grassmannian $V_F \times \text{Gr}_G$ of the symplectic group $G = \text{Sp}(V)$, where $V$ is a $2n$-dimensional symplectic complex vector space. The symplectic form on $V$ induces a (complex-valued) symplectic form on the Tate vector space $V_F$, and it gives rise to the Weyl algebra $W$ of $V_F$. The symplectic mirabolic category is the category of $G_O$-equivariant objects in the tensor product of $D$-modules on $\text{Gr}_G$ and $W$-modules. The category $W$-mod is equipped with a twisted action of $D$-mod$_{-1/2}(\text{Gr}_G)$ at the level $-1/2$. Hence there are two equivalent incarnations of the symplectic mirabolic category. First, $D W$-mod$_{G_O}$ are the $G_O$-equivariant objects of $D$-mod$_{1/2}(\text{Gr}_G) \otimes W$-mod with respect to the untwisted strong action of $G_F$. Second, $D W'$-mod$_{G_O}$ are the $G_O$-equivariant objects of $D$-mod$(\text{Gr}_G) \otimes W$-mod with respect to the $-1/2$-twisted strong action of $G_F$.

Our construction of the equivalence $\text{Rep}(\text{SO}(2n+1|2n)) \rightarrow D W$-mod$_{G_O}$ follows the pattern of [BFT]. Namely, we first construct an equivalence from $D^\text{perf}(\text{Sym}^*(\Pi(V_0 \otimes V)\{[-1]\}))$ to a derived version of $D W$-mod$_{G_O}$. Here $V_0$ is a $2n+1$-dimensional complex vector spaces equipped with a nondegenerate symmetric bilinear form. Then we precompose this equivalence with the Koszul equivalence $D^\text{b} \text{Rep}(\text{SO}(2n+1|2n)) \rightarrow D^\text{perf}_\text{perf}(\text{Sym}^*(\Pi(V_0 \otimes V)\{[-1]\}))$. Finally, we check that the composed equivalence respects the natural $t$-structures.

Compared to [BFT] we have to overcome two new difficulties. First, $W$-mod is equivalent to the category of $D$-modules on $V_F/V_O$, but the set of $G_O$-orbits in $(V_F/V_O) \times \text{Gr}_G$ is uncountable, and we have to classify the discrete family of
relevant orbits which do carry $G_\mathcal{O}$-equivariant $D$-modules. It turns out that this family is naturally numbered by the dominant weights of $\text{SO}(V_0) \times \text{Sp}(V)$. Second, the computation of equivariant Ext’s in [BFT] was reduced to the calculations with equivariant cohomology, but in the present case, due to the twisting, the equivariant (De Rham) cohomology simply vanish, and we have to resort to the endoscopic arguments of [DLYZ].

Otherwise, the arguments are quite parallel to those of [BFT], and they are just briefly indicated after introducing an appropriate setup.

1.2. Further generalizations: general orthosymplectic case, quantum supergroups and exceptional supergroups. The above results only involve the category of representations of the degenerate supergroup $\text{SOSp} (2n + 1 | 2n)$. Following D. Gaiotto, we formulate in §3.1 similar conjectures relating $\text{Rep}(\text{SOSp}(2k + 1 | 2n))$ and $\text{Rep}(\text{SOSp}(2n + 1 | 2k))$ (for $k \leq n$) with certain equivariant objects of $D\text{-mod}(G_{\mathcal{O}}) \otimes W(F^{2k})\text{-mod}$ (the Weyl algebra of a symplectic Tate space $F^{2k}$). The equivariance is taken with respect to the semidirect product $\text{Sp}(2k, \mathcal{O}) \ltimes U_k(F)$ for a certain unipotent subgroup $U_k \subset \text{Sp}(2n)$ related to a ‘hook’ nilpotent $e \in \text{sp}(2n)$ of Jordan type $(2n − 2k, 1^{2k})$.

We also formulate a quantum version of these conjectures for representations of quantum supergroups $\text{Rep}_q(\text{SOSp}(2k + 1 | 2n))$ and $\text{Rep}_q(\text{SOSp}(2n + 1 | 2k))$. In §3.2 we reformulate the Gaiotto conjectures [BFGT, §2.6] for representations of quantum supergroups $\text{Rep}_q(\text{GL}(K | N))$ in a similar vein. Namely, we replace a Whittaker-type equivariance condition with respect to a certain unipotent group by a plain equivariance condition (no character) with respect to a larger unipotent group, at the expense of adding a factor of $W(F^{2k})\text{-mod}$, cf. [TY, §5.3].

Finally, in §§3.3,3.4 we formulate similar conjectures for representations of the exceptional quantum supergroups $\text{Rep}_q(F(4))$ and $\text{Rep}_q(G(3))$.

We refer the reader to [BFGT, §2] where the meaning of very similar statements (for the supergroup $\text{GL}(M | N)$) is explained from the point of view of local geometric Langlands correspondence.

1.3. Acknowledgments. We are grateful to G. Dhillon, A. Elashvili, D. Gaiotto, A. Hanany, M. Jibladze, D. Leites, I. Motorin, H. Nakajima, V. Ostrik, S. Raskin, V. Serganova, D. Timashev and R. Yang for very useful discussions.

A.B. was partially supported by NSERC. M.F. was partially funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project ‘5-100’.

2. A COHERENT REALIZATION OF $D\text{-mod}^{\text{Sp}(2n, \mathcal{O}), \text{lc}}$

2.1. Symplectic mirabolic category.
2.1.1. Weyl algebra. We fix a $2n$-dimensional complex vector space $V$ with symplectic form $\langle , \rangle$. Let $G = \text{Sp}(2n) = \text{Sp}(V)$, and $\mathfrak{g} = \mathfrak{sp}(2n) = \mathfrak{sp}(V)$. The symplectic form on $V$ extends to the same named $\mathbb{C}$-valued symplectic form on $V = V_F: \langle f, g \rangle = \text{Res} \langle f, g \rangle_F dt$. We denote by $\mathcal{W}$ the completion of the Weyl algebra of $(V, \langle , \rangle)$ with respect to the left ideals generated by the compact subspaces of $V$.

We consider the dg-category $\mathcal{W}$-mod of discrete $\mathcal{W}$-modules. More concretely, we identify $\mathcal{W}$ with the ring of differential operators on a Lagrangian discrete lattice $L \subset V$, e.g. $L = t^{-1}V_{\mathbb{C}[t^{-1}]}$. Then $\mathcal{W}$-mod is the inverse limit of $\text{D-mod}(U)$ over finite dimensional subspaces $U \subset L$ with respect to the functors $i_U^! U \hookrightarrow U'$. Equivalently, $\mathcal{W}$-mod is the colimit of $\text{D-mod}(U)$ with respect to the functors $i_U^! U \hookrightarrow U'$.

There is a twisted action $\text{D-mod}_{-1/2}(G_F) \circlearrowright \mathcal{W}$-mod $[R, \S 10]$. Here $-1/2$ stands for the $-1/2$-multiple of the level of $G_F$ corresponding to the trace form of $V$ on $\mathfrak{g}$.

2.1.2. Satake equivalence. We consider the dg-category $\text{D-mod}^{G_{\mathcal{O}, \text{lc}}}_{1/2}(\text{Gr}_{G}) \subset \text{D-mod}^{G_{\mathcal{O}}}_{1/2}(\text{Gr}_{G})$ of locally compact $G_{\mathcal{O}}$-equivariant $D$-modules on the affine Grassmannian $\text{Gr}_{G}$ twisted by the square root of the very ample determinant line bundle $D$. Here ‘locally compact’ means compact when regarded as a plain $D$-module on $\text{Gr}_{G}$. The twisted Satake equivalence of [DLYZ] is a monoidal equivalence

$$\beta^g: D^G_{\text{perf}}(\text{Sym}^\bullet(\mathfrak{g}[-2])) \sim \text{D-mod}^{G_{\mathcal{O}, \text{lc}}}_{1/2}(\text{Gr}_{G})$$

from the dg-category of $G$-equivariant perfect dg-modules over the dg-algebra $\text{Sym}^\bullet(\mathfrak{g}[-2])$ equipped with trivial differential. It extends to the same named monoidal equivalence of the Ind-completions

$$\beta^g: D^G(\text{Sym}^\bullet(\mathfrak{g}[-2])) \sim \text{D-mod}^{G_{\mathcal{O}, \text{ren}}}_{1/2}(\text{Gr}_{G}).$$

Here $\text{D-mod}^{G_{\mathcal{O}, \text{ren}}}_{1/2}(\text{Gr}_{G})$ stands for the renormalized category [AG, §12.2.3]. It also extends to the same named monoidal equivalence

$$(2.1.1) \quad \beta^g: D^G_{\text{nilp}}(\text{Sym}^\bullet(\mathfrak{g}[-2])) \sim \text{D-mod}^{G_{\mathcal{O}}}_{1/2}(\text{Gr}_{G}),$$

similarly to [AG, Corollary 12.5.5]. Here $D^G_{\text{nilp}}(\text{Sym}^\bullet(\mathfrak{g}[-2]))$ stands for the category of $G$-equivariant dg-modules over $\text{Sym}^\bullet(\mathfrak{g}[-2])$ with nilpotent support.

We will also use the untwisted Satake equivalence of [BF]

$$\beta^{g'}: D^{G'}_{\text{perf}}(\text{Sym}^\bullet(\mathfrak{g}'[-2])) \sim \text{D-mod}^{G_{\mathcal{O}, \text{lc}}}(\text{Gr}_{G}),$$

along with its Ind-completion

$$\beta^{g'}: D^{G'}(\text{Sym}^\bullet(\mathfrak{g}'[-2])) \sim \text{D-mod}^{G_{\mathcal{O}, \text{ren}}}(\text{Gr}_{G}).$$
Here \( G^\vee = \text{SO}(2n + 1) = \text{SO}(V_0) \) is the Langlands dual Lie group, while \( \mathfrak{g}^\vee = \mathfrak{so}(2n + 1) = \mathfrak{so}(V_0) \) is the Langlands dual Lie algebra, and the \( 2n + 1 \)-dimensional vector space \( V_0 \) is equipped with a nondegenerate symmetric bilinear form \((\cdot, \cdot)\).

It also extends to the same named monoidal equivalence \([\text{AG}, \text{Corollary 12.5.5}]\)

\[
(2.1.2) \quad \mathcal{B}^\vee : D_{\text{nilp}}^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee[-2])) \cong D\text{-mod}^{G_0}(G) \tag{2.1.2}
\]

2.1.3. *Symplectic mirabolic category.* We denote by \( \mathcal{D}W\text{-mod} \) the tensor product \( D\text{-mod}_{1/2}(\text{Gr}_G) \otimes W\text{-mod} \). We consider the category \( \mathcal{D}W\text{-mod}^{G_0,lc} \) of locally compact \( G_0 \)-equivariant objects and its Ind-completion \( \mathcal{D}W\text{-mod}^{G_0,\text{ren}} \).

The *right convolution action* of \( D\text{-mod}_{1/2}^{G_0,lc}(\text{Gr}_G) \) on \( \mathcal{D}W\text{-mod}^{G_0,lc} \) gives rise to the action of \( D\text{-mod}_{1/2}^{G_0,lc}(\text{Gr}_G) \) on \( \mathcal{D}W\text{-mod}^{G_0,lc} \). Furthermore, recall the twisted action \( D\text{-mod}_{-1/2}(G_F) \) on \( \mathcal{D}W\text{-mod}^{G_0,lc} \). The action of \( G_F \) on \( \text{Gr}_G \) gives rise to the action \( D\text{-mod}_{1/2}(G_F) \) on \( D\text{-mod}_{1/2}(G) \). These two actions combine to give the diagonal *non-twisted action* \( D\text{-mod}(G_F) \otimes \mathcal{D}W\text{-mod} \). Finally, this latter action gives rise to the left action \( D\text{-mod}^{G_0,lc}(\text{Gr}_G) \otimes \mathcal{D}W\text{-mod}^{G_0,lc} \).

All in all, we obtain an action

\[
D\text{-mod}^{G_0,lc}(\text{Gr}_G) \otimes D\text{-mod}_{1/2}^{G_0,lc}(\text{Gr}_G) \otimes \mathcal{D}W\text{-mod}^{G_0,lc}.
\]

We will also need another realization \( \mathcal{D}'W\text{-mod}^{G_0,lc} \cong \mathcal{D}W\text{-mod}^{G_0,lc} \): we have the natural equivalences

\[
(2.1.3) \quad \mathcal{D}W\text{-mod}^{G_0} = (D\text{-mod}_{1/2}(\text{Gr}_G) \otimes W\text{-mod})^{G_0} \\
\cong (D\text{-mod}(\text{Gr}_G) \otimes D\text{-mod}_{1/2}(\text{Gr}_G) \otimes W\text{-mod})^{G_F} \\
\cong (D\text{-mod}(\text{Gr}_G) \otimes W\text{-mod})^{G_0} =: \mathcal{D}'W\text{-mod}^{G_0}.
\]

Here the action of \( G_F \) on \( D\text{-mod}(\text{Gr}_G) \otimes W\text{-mod} \) is *twisted*: \( D\text{-mod}_{-1/2}(G_F) \) on \( D\text{-mod}(\text{Gr}_G) \otimes W\text{-mod} \). So \( D\text{-mod}(\text{Gr}_G) \otimes W\text{-mod} \) denotes the category of \( G_0 \)-equivariant objects with respect to this twisted action of \( G_0 \subset G_F \).

It is equipped with the right action of \( D\text{-mod}^{G_0}(\text{Gr}_G) \) and the left action of \( D\text{-mod}_{-1/2}^{G_0}(\text{Gr}_G) \).

The equivalence \( \mathcal{D}W\text{-mod}^{G_0} \cong \mathcal{D}'W\text{-mod}^{G_0} \) is compatible with the actions of \( D\text{-mod}^{G_0}(\text{Gr}_G) \) and \( D\text{-mod}_{-1/2}^{G_0}(\text{Gr}_G) \) in the following way. First of all, the inversion \( g \mapsto g^{-1} : G_F \rightarrow G_F \), gives rise to a monoidal anti-involution of \( D\text{-mod}^{G_0}(\text{Gr}_G) \). The left action of \( D\text{-mod}^{G_0}(\text{Gr}_G) \) on \( \mathcal{D}W\text{-mod}^{G_0} \) goes to the right action of \( D\text{-mod}^{G_0}(\text{Gr}_G) \) on \( \mathcal{D}'W\text{-mod}^{G_0} \) composed with the above anti-involution. Let \( \mathcal{C} \) denote the Chevalley involution of \( G \) (the canonical outer automorphism of \( G \) interchanging conjugacy classes of \( g \) and \( g^{-1} \)). We keep the same notation for the induced involution of \( D\text{-mod}^{G_0}(\text{Gr}_G) \). Then the above anti-involution coincides with \( \mathcal{C} \) (due to the commutativity constraint on \( D\text{-mod}^{G_0}(\text{Gr}_G) \), there is no difference between involutions and
anti-involutions). Finally, notice that $C$ is trivial since the Dynkin graph of $G$ has no automorphisms. Second, in a similar vein, the right action of $\mathcal{D}$-mod$_{1/2}^G(\text{Gr}_G)$ on $\mathcal{D}$-mod$_{1/2}^G(\text{Gr}_G)$ goes to the left action of $\mathcal{D}$-mod$_{1/2}^{G_0}(\text{Gr}_G)$ on $\mathcal{D}$-mod$_{1/2}^{G_0}(\text{Gr}_G)$ composed with the twisting by the inverse determinant line bundle $\mathcal{D}^{-1}$ on $\text{Gr}_G$.

2.1.4. Theta-sheaf. Let $\text{Gr}_G$ be the Kashiwara (infinite type) scheme version of the affine Grassmannian of $G$: it is the moduli space of $G$-bundles on $\mathbb{P}^1$ equipped with a trivialization in the formal neighbourhood of $\infty \in \mathbb{P}^1$. Recall the Radon Transform $\text{RT}_1: \mathcal{D}$-mod$_{-1/2}^G(\text{Gr}_G) \xrightarrow{\sim} \mathcal{D}$-mod$_{-1/2}^{G_0}(\text{Gr}_G)$!

We will use the inverse equivalence $\text{RT}_1^{-1}$. We will also use the $G_0$-equivariant $-1/2$-twisted $D$-module $\Theta$ on $\text{Gr}_G$, see [BDFRT, §2.2]. It was introduced in [La, §2]. The perverse sheaf corresponding to $\Theta$ under the Riemann-Hilbert correspondence was introduced in [Ly] and studied in [LL] (over the base field $\mathbb{F}_q$). We will use the following key relation between the theta-sheaf and the twisted Satake equivalence. First we introduce the notation

$$R := \text{RT}_1^{-1}(\Theta)$$

We choose a pair of opposite maximal unipotent subgroups $U_G, U_G^{-} \subset G$, their regular characters $\psi, \psi^{-}$, and denote by $\kappa^0: D^G(\text{Sym}^\bullet(\mathfrak{g}[-2])) \to D(\mathbb{C}[\Xi_\mathfrak{g}])$ the functor of Kostant-Whittaker reduction with respect to $(U_G^-, \psi^-)$ (see e.g. [BF, §2]). Here $\Xi_\mathfrak{g}$ with grading disregarded is the tangent bundle $T\Sigma_\mathfrak{g}$ of the Kostant slice $\Sigma_\mathfrak{g} \subset \mathfrak{g}^*$. Let us write $\kappa$ for the Ad-invariant bilinear form on $\mathfrak{g}$, i.e. level, corresponding to the central charge of $-1/2$. Explicitly, if we write $\kappa_b$ for the basic level giving the short coroots of $\mathfrak{g}$ squared length two, and $\kappa_c$ for the critical level, then $\kappa$ is defined by

$$\kappa = -1/2 \cdot \kappa_b - \kappa_c.$$ 

If we consider the Langlands dual Lie algebra $\mathfrak{g}' \simeq \mathfrak{so}_{2n+1}$, the form $\kappa$ gives rise to identifications $\Sigma_\mathfrak{g} \cong \Sigma_\mathfrak{g}'$ and $\Xi_\mathfrak{g} \cong \Xi_\mathfrak{g}'$. Also, we have a canonical isomorphism $H^\bullet_{G_0}(\text{Gr}_G) \cong \mathbb{C}[\Xi_{\mathfrak{g}'}] \cong \mathbb{C}[\Xi_\mathfrak{g}]$. This is a theorem of V. Ginzburg [Gi] (for a published account see e.g. [BF, Theorem 1]).

Now given $\mathcal{F} \in \mathcal{D}$-mod$_{1/2}^G(\text{Gr}_G)$ we consider the tensor product $\mathcal{F} \otimes \mathcal{R}$. Since the twistings of the factors cancel out, the tensor product is an untwisted $G_0$-equivariant $D$-module, and we can consider its equivariant De Rham cohomology.

The aforementioned key property is a canonical isomorphism [DLYZ]

$$H^\bullet_{G_0, \text{DR}}(\text{Gr}_G, \mathfrak{g}^\mathfrak{g}(M) \otimes \mathcal{R}) \cong \kappa^0 M$$

of $H^\bullet_{G_0}(\text{Gr}_G) \cong \mathbb{C}[\Xi_\mathfrak{g}]$-modules for any $M \in D^G(\text{Sym}^\bullet(\mathfrak{g}[-2]))$. 
Similarly, we have the Kostant-Whittaker reduction functor (see [BF, §2])

$$\kappa^{\vee}: D^{G^{\vee}}(\text{Sym}^\bullet(\mathfrak{g}^{\vee}[-2])) \to D(\mathbb{C}[\Xi^{\vee}_{\mathfrak{g}}]) \cong D(\mathbb{C}[\Xi_{\mathfrak{g}}]).$$

By [BF, Theorem 4], we have a canonical isomorphism

$$(2.1.6) \quad H^{\bullet}_{G_{dR}}(\text{Gr}_G; \beta^{\vee}(M)) \cong \kappa^{\vee} M$$

of $H^{\bullet}_{G_{dR}}(\text{Gr}_G) \cong \mathbb{C}[\Xi_{\mathfrak{g}}]$-modules for any $M \in D^{G^{\vee}}(\text{Sym}^\bullet(\mathfrak{g}^{\vee}[-2]))$.

2.2. Constructible realization of $\mathcal{D}W$-mod$^{G_{dR,lc}}$. Let $\text{Heis}$ stand for the Heisenberg central extension of $V_F$ with $\mathbb{G}_a$ (canonically split after restriction to $V_O$). Let $\chi$ be the character $D$-module on $G_{O} \ltimes V_{O} \times \mathbb{G}_a$ equal to the pullback of the exponential $D$-module on $\mathbb{G}_a$ with respect to the projection $G_{O} \times V_{O} \times \mathbb{G}_a \to \mathbb{G}_a$. We consider the category $D$-mod$_{1/2}(\text{Gr}_G \times \text{Heis}) := D$-mod$_{1/2}(\text{Gr}_G) \otimes D$-mod(Heis). It is equipped with the strong diagonal action of $G_{O} \ltimes V_{O} \times \mathbb{G}_a$ (the action on $D$-mod$_{1/2}(\text{Gr}_G)$ factors through the quotient $G_{O}$). We consider the category $D$-mod$^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis})$ of locally compact $\chi$-equivariant objects.

**Lemma 2.2.1.** There is an equivalence of categories

$$D$$-mod$_{1/2}^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis}) \cong \mathcal{D}W$$-mod$^{G_{dR,lc}}$.

**Proof.** Let $\text{IC}_0 \in D$-mod$_{1/2}(\text{Gr}_G)^{G_{dR,lc}, \chi}$ denote the irreducible twisted $D$-module supported at the base point of $\text{Gr}_G$. Then the tensor product $\text{IC}_0 \otimes \mathbb{C}[V_{O}] \in \mathcal{D}W$-mod is strongly $(G_{O} \ltimes V_{O} \times \mathbb{G}_a, \chi)$-equivariant, and so gives rise to a functor from $D$-mod$_{1/2}^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis})$ to $\mathcal{D}W$-mod$^{G_{dR,lc}}$ that is the desired equivalence. \qed

**Remark 2.2.2.** A similar category of $\ell$-adic sheaves over the base field $\mathbb{F}_q$ was studied in [LL].

We consider the following $\mathbb{G}_m$-action on $\text{Heis} \simeq V_F \oplus \mathbb{G}_a: c \cdot (v, a) = (cv, c^2a)$. Then [Ga2, §1.6] defines the Kirillov model $\mathcal{K}ir$ (a category equivalent to $D$-mod$_{1/2}^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis})$) as follows. First we consider the full subcategory $\mathcal{C} \subset D$-mod$_{1/2}^{G_{O} \times V_{O}, \chi,lc}(\text{Gr}_G \times \text{Heis})$ formed by the objects killed by the averaging functor $A^{\mathbb{G}_a}$ (averaging without the exponential character $\chi$). Then we define $\mathcal{K}ir := \mathcal{C}^{\mathbb{G}_m}$. According to loc.cit., there is a canonical equivalence $\mathcal{K}ir \cong D$-mod$_{1/2}^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis})$.

Finally, applying the Riemann-Hilbert equivalence, we obtain the constructible version $\mathcal{K}ir^{\text{constr}} \cong \mathcal{K}ir \cong D$-mod$_{1/2}^{G_{O} \times V_{O} \times \mathbb{G}_a, \chi,lc}(\text{Gr}_G \times \text{Heis}) \cong \mathcal{D}W$-mod$^{G_{dR,lc}}$.

**Remark 2.2.3.** Another incarnation $\mathcal{D}W$-mod$^{G_{dR,lc}}$ of the mirabolic category $\mathcal{D}W$-mod$^{G_{dR,lc}}$ (see §2.1.3) has a similar constructible realization. We consider the category $D$-mod$(\text{Gr}_G \times \text{Heis}) = D$-mod$(\text{Gr}_G) \otimes D$-mod(Heis).
It is equipped with the strong diagonal action of $G_\mathbb{O} \times V_\mathbb{O} \times \mathbb{G}_a$ (the action on $D\text{-mod}(Gr_G)$ factors through the quotient $G_\mathbb{O}$). We consider the category $D\text{-mod}^{G_\mathbb{O} \times V_\mathbb{O} \times \mathbb{G}_a, \chi}$ of locally compact $\chi$-equivariant objects. We have the corresponding Kirillov category $\mathcal{K}ir' \cong D\text{-mod}^{G_\mathbb{O} \times V_\mathbb{O} \times \mathbb{G}_a, \chi}(Gr_G \times \text{Heis}) \cong D'W\text{-mod}^{G_\mathbb{O}, \chi}.$

2.3. Fusion. Let $X$ be a smooth curve. For any integer $k > 0$, and a collection $x = (x_i)_{i=1}^k$ of $S$-points of $X$, we denote by $\mathcal{D}_x$ the formal neighborhood of the union of graphs $|x| := \bigcup_{i=1}^k \Gamma_{x_i} \subset S \times X$, and we denote by $\mathcal{D}_x^2 := \mathcal{D}_x \setminus |x|$ the punctured formal neighborhood.

The mirabolic version of the Beilinson-Drinfeld Grassmannian is the ind-scheme $Gr_{G,BD,k}^{\text{mir}}$ over $X^k$ parametrizing the following collections of data:

$$(x_i)_{i=1}^k, \mathcal{E}, \phi: \mathcal{E}_{\text{triv}}|_{\mathcal{D}_x^2} \sim \mathcal{E}|_{\mathcal{D}_x^2}, v \in \Gamma(\mathcal{D}_x^2, \mathcal{E}), a \in \mathbb{G}_a$$

where $\mathcal{E}$ is a rank $2n$ vector bundle on $\mathcal{D}_x$ equipped with a symplectic form. In case $X = \mathbb{A}^1$, over the complement to the diagonals we have a canonical isomorphism

$$(\mathbb{A}^k \setminus \Delta) \times \mathbb{A}^k Gr_{G,BD,k}^{\text{mir}} \cong (\mathbb{A}^k \setminus \Delta) \times (Gr_G \times V_F)^k \times \mathbb{G}_a.$$ 

We denote the projection $(\mathbb{A}^k \setminus \Delta) \times (Gr_G \times V_F)^k \times \mathbb{G}_a \to (Gr_G \times V_F)^k \times \mathbb{G}_a$ by $pr_2$. Given $\mathcal{F}_1, \mathcal{F}_2 \in D\text{-mod}^{G_\mathbb{O} \times V_\mathbb{O} \times \mathbb{G}_a, \chi}(Gr_G \times \text{Heis})$, we take $k = 2$ and define the fusion

$$\mathcal{F}_1 \ast \mathcal{F}_2 := pr_2^*\psi_{x-y} pr_2^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)[1],$$

where $x, y$ are coordinates on $\mathbb{A}^2$ (so that $x - y = 0$ is the equation of the diagonal $\Delta \subset \mathbb{A}^2$), and $\psi_{x-y}$ is the nearby cycles functor for the pullback of the function $x - y$ to $Gr_{G,BD,2}^{\text{mir}}$. Note that the leftmost occurrence of $pr_2$ in the above definition projects $\mathbb{A}^1 \times Gr_G \times V_F \times \mathbb{G}_a = \mathbb{A}^1 \times Gr_G \times \text{Heis}$ to $Gr_G \times \text{Heis}$, while the rightmost occurrence of $pr_2$ projects $(\mathbb{A}^2 \setminus \Delta) \times (Gr_G \times V_F)^2 \times \mathbb{G}_a$ to $(Gr_G \times \text{Heis})^2$ (and the latter copy of $pr_2$ coincides with the diagonal embedding $\mathbb{G}_a \hookrightarrow \mathbb{G}_a \times \mathbb{G}_a$ of the $\mathbb{G}_a$-factor of $\text{Heis}$).

2.4. Orthogonal and symplectic Lie algebras. We adapt [BFT, §2.1] to our present setup. The tensor product $V_0 \otimes V$ is equipped with a nondegenerate skew-symmetric bilinear form $\langle , \rangle \otimes \langle , \rangle$. It is preserved by the action of the group $\text{SO}(V_0) \times \text{Sp}(V)$.

Our nondegenerate bilinear forms on $V_0, V$ define identifications $V_0 \cong V_0^*, V \cong V^*$. In particular, $V_0 \otimes V$ is identified with $V_0^* \otimes V = \text{Hom}(V_0, V)$. Given $A \in \text{Hom}(V_0, V)$ we have the adjoint operator $A^t \in \text{Hom}(V,V_0)$. We have the moment maps

$$q_0: V_0 \otimes V \to \mathfrak{so}(V_0)^*, A \mapsto A^t A, \text{ and } q: V_0 \otimes V \to \mathfrak{sp}(V)^*, A \mapsto AA^t,$$
where we make use of the identification $\mathfrak{so}(V_0) \cong (\mathfrak{so}(V_0))^*$ (resp. $\mathfrak{sp}(V) \cong (\mathfrak{sp}(V))^*$) via the trace form (resp. negative trace form) of the defining representation. Note also that the complete moment map ($\mathfrak{q}_0, \mathfrak{q}$) coincides with the “square” (half-self-supercalculator) map on the odd part $g_1$ of the orthosymplectic Lie superalgebra $\mathfrak{g}$. We define the odd nilpotent cone $N_\mathfrak{g} \subset V_0 \otimes V$ as the reduced subscheme cut out by the condition of nilpotency of $A^iA$ (equivalently, by the condition of nilpotency of $\mathfrak{AA}^i$).

We choose Cartan subalgebras $\mathfrak{t}_0 \subset \mathfrak{so}(V_0)$ and $\mathfrak{t} \subset \mathfrak{sp}(V)$. We choose a basis $\varepsilon_1, \ldots, \varepsilon_n$ in $\mathfrak{t}_0^*$ such that the Weyl group $W_0 = W(\mathfrak{so}(V_0), \mathfrak{t}_0)$ acts by permutations of basis elements and by the sign changes of basis elements, and the roots of $\mathfrak{so}(V_0)$ are given by $\{ \pm \varepsilon_i, \varepsilon_j, \varepsilon_i \neq j; \pm \varepsilon_i \}$. We set $\Sigma_{q^\vee} = \mathfrak{t}_0^*/W_0$. We also choose a basis $\delta_1, \ldots, \delta_n$ in $\mathfrak{t}^*$ such that the Weyl group $W = W(\mathfrak{sp}(V), \mathfrak{t})$ acts by permutations of basis elements and by the sign changes of basis elements, and the roots of $\mathfrak{sp}(V)$ are given by $\{ \pm \delta_i, \delta_j, \varepsilon_i \neq j; \pm 2\delta_i \}$. We set $\Sigma_q = \mathfrak{t}^*/W$.

We identify $\mathfrak{t}_0^* \cong \mathfrak{t}^*$, $\varepsilon_i \mapsto \delta_i$, and this identification gives rise to an isomorphism $\Pi : \Sigma_{q^\vee} \to \Sigma_q$.

Recall (see e.g. [BF, §§2.1.2, 6]) that $\Sigma_{q^\vee}$ is embedded as a Kostant slice into the open set of regular elements $(\mathfrak{so}(V_0))^*_{\text{reg}} \subset \mathfrak{so}(V_0)^*$, and $\Sigma_q$ is embedded into $(\mathfrak{sp}(V))^*_{\text{reg}}$. Furthermore, these slices $\Sigma_{q^\vee}, \Sigma_q$ carry the universal centralizer sheaves of abelian Lie algebras $\mathfrak{z}_{q^\vee}, \mathfrak{z}_q$. Given an $\text{SO}(V_0)$-module $M$ (resp. an $\text{Sp}(V)$-module $M'$), we have the corresponding graded $\Gamma(\Sigma_{q^\vee}, \mathfrak{z}_{q^\vee})$-module $\mathbf{k}^\mathfrak{z}_{q^\vee}(M \otimes \text{Sym}^\bullet(\mathfrak{g}[-2]))$ (resp. the $\Gamma(\Sigma_q, \mathfrak{z}_q)$-module $\mathbf{k}^\mathfrak{z}_q(M' \otimes \text{Sym}^\bullet(\mathfrak{g}[-2]))$) (the Kostant functor of loc.cit., cf. notation of §2.1.4). Since the universal enveloping algebra $\text{U}(\mathfrak{z}_{q^\vee})$ (resp. $\text{U}(\mathfrak{z}_q)$) is identified in loc.cit. with the sheaf of functions on the tangent bundle $T \Sigma_{q^\vee} \cong \Xi_{q^\vee}$ (resp. $T \Sigma_q \cong \Xi_q$), we will use the same notation $\mathbf{k}^\mathfrak{z}_{q^\vee}(M \otimes \text{Sym}^\bullet(\mathfrak{g}[-2]))$, $\mathbf{k}^\mathfrak{z}_q(M' \otimes \text{Sym}^\bullet(\mathfrak{g}[-2]))$ for the corresponding coherent sheaves on $\Xi_{q^\vee}, \Xi_q$. Finally, according to the previous paragraph, we have the isomorphism $d\Pi : \Xi_{q^\vee} = T\Sigma_{q^\vee} \to T\Sigma_q = \Xi_q$.

We choose Borel subalgebras $\mathfrak{t}_0 \subset \mathfrak{b}_0 \subset \mathfrak{so}(V_0)$ corresponding to the choice of positive roots $R_0^+ = \{ \varepsilon_i \pm \varepsilon_j, \varepsilon_i \neq j; \varepsilon_i \}$ and $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{sp}(V)$ corresponding to the choice of positive roots $R^+ = \{ \delta_i \pm \delta_j, \varepsilon_i \neq j; 2\delta_i \}$. We set $\rho_0 = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha$ and $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. We denote by $\Lambda_0$ (resp. $\Lambda_1$) the weight lattice of $\text{SO}(V_0)$ (resp. of $\text{Sp}(V)$). We denote by $\Lambda_0^+ \subset \Lambda_0$ (resp. $\Lambda_1^+ \subset \Lambda_1$) the monoids of dominant weights. For $\lambda \in \Lambda_0^+$ (resp. $\lambda \in \Lambda_1^+$) we denote by $V_\lambda$ the irreducible representation of $\text{SO}(V_0)$ (resp. of $\text{Sp}(V)$) with highest weight $\lambda$.

2.5. The main theorem. We keep the notation of [BFT, §2.2] with a single exception: we replace $V_1$ of loc.cit. with $V$. In particular, $\mathbb{G}$ stands for the subgroup $\text{SOSp}(V_0|V)$ with the even part $\mathbb{G}_0 = \text{SO}(V_0) \times \text{Sp}(V)$, and $\mathfrak{g}$ stands for its
Lie superalgebra $\mathfrak{osp}(V_0|V)$. Also, $\mathfrak{G}^*$ stands for the dg-algebra\(^1\) $\text{Sym}^*(\mathfrak{g}_1[-1])$ with trivial differential, and $\Lambda$ stands for the exterior algebra $\Lambda(V_0 \otimes V)$. Finally, $D^b\text{Rep}(\mathcal{G})$ stands for the dg-category of finite dimensional complexes of $\mathcal{G}$-modules.

Our goal is the following

**Theorem 2.5.1.** (a) There exists an equivalence of triangulated categories $\Phi: D^G_{\text{perf}}(\mathfrak{G}^*) \sim \mathbb{D}^W\text{-mod}^{\text{Go},lc}$ commuting with the convolution action of the monoidal spherical Hecke category

$$D^G_{\text{perf}}(\text{Sym}^*(\mathfrak{g}^*[-2])) \otimes D^G_{\text{perf}}(\text{Sym}^*(\mathfrak{g}[-2])) \cong \mathbb{D}^\text{mod}^{\text{Go},lc}(\text{Gr}_G) \otimes \mathbb{D}^\text{mod}^{\text{Go},lc}(\text{Gr}_G).$$

(b) The composed equivalence

$$\Phi \circ \kappa: D^G_{\text{id}}(\Lambda) \sim \mathbb{D}^W\text{-mod}^{\text{Go},lc}$$

is exact with respect to the tautological $t$-structure on $D^G_{\text{id}}(\Lambda)$ and the tautological $t$-structure on $\mathbb{D}^W\text{-mod}^{\text{Go},lc}$.

(c) This equivalence is monoidal with respect to the tensor structure on $D^G_{\text{id}}(\Lambda)$ and the fusion $\star$ on $\mathbb{D}^W\text{-mod}^{\text{Go},lc}$.

(d) The equivalence of (b) extends to a monoidal equivalence from $SD^G_{\text{id}}(\Lambda) = D^b\text{Rep}(\mathcal{G})$ to $S\mathbb{D}^W\text{-mod}^{\text{Go},lc} := \mathbb{D}^W\text{-mod}^{\text{Go},lc} \otimes_{\text{Vect}} \text{SVect}$.

(e) The category $\mathbb{D}^W\text{-mod}^{\text{Go},lc}$ is equivalent to the dg-category of bounded complexes of the abelian category $\mathbb{D}^W\text{-mod}^{\text{Go},lc,\vee}$ (the heart of the tautological $t$-structure).

The proof will be given in §2.14.

2.6. **Irreducibles in $\mathbb{D}^W\text{-mod}^{\text{Go},lc,\vee}$**. We identify $\text{Gr}_G$ with the moduli space of parahoric subalgebras in $\mathfrak{g}_F$ conjugate to $\mathfrak{g}_0$, and we identify the cotangent bundle $T^*\text{Gr}_G$ with the space of pairs $(p, x)$ where $p$ is a parahoric subalgebra, and $x$ lies in the nilpotent radical of $p$. We identify $\mathfrak{g}_0^*$ with $\mathfrak{g}_F/t\mathfrak{g}_0$. The moment map $\mu: V_F \times T^*\text{Gr}_G \to \mathfrak{g}_0$ of the $G_0$-action takes $(v, p, x)$ to $S^2v + x \pmod{t\mathfrak{g}_0}$. Here we identify $\mathfrak{g}_F$ with $\text{Sym}^2V_F$. Thus the zero level $\mu^{-1}(0)$ of the moment map is the space of triples $(v, p, x)$ such that $S^2v + x \in t\mathfrak{g}_0$.

**Lemma 2.6.1.** The zero level $\mu^{-1}(0) \subset V_F \times T^*\text{Gr}_G$ of the moment map is Lagrangian.

**Proof.** We consider the auxiliary mirabolic affine Grassmannian $V_F \times \text{Gr}_{G_\text{GL}(2n)}$ and its cotangent bundle $T^*V_F \times T^*\text{Gr}_{G_\text{GL}(2n)} \cong V_F \times V_F \times T^*\text{Gr}_{G_\text{GL}(2n)}$. It is equipped with the moment map of the $\text{GL}(2n, \mathcal{O})$-action $\mu_{\text{GL}(2n)}: T^*V_F \times T^*\text{Gr}_{G_\text{GL}(2n)} \to$

---

\(^1\)We view $\mathfrak{g}_1$ as an odd vector space, so that $\text{Sym}^*(\mathfrak{g}_1[-1])$ (with grading disregarded) is really a symmetric (infinite-dimensional) algebra, not an exterior algebra.
\( \mathfrak{gl}(2n, O)^* \). The zero level \( \mu^{-1}_{GL(2n)}(0) \) is Lagrangian since the set of \( GL(2n, O) \)-orbits in the mirabolic affine Grassmannian \( V_F \times \text{Gr}_{GL(2n)} \) is discrete.

We have an involution \( \sigma \circ \text{GL}(2n) \) whose fixed point set is \( \text{Sp}(2n) \subset \text{GL}(2n) \). It induces an involution of \( \text{Gr}_{GL(2n)} \) and of \( T^*\text{Gr}_{GL(2n)} \), and we extend it to the same named involution \( \sigma \circ (T^*V_F \times T^*\text{Gr}_{GL(2n)} \cong V_F \times V_F \times T^*\text{Gr}_{GL(2n)}) \) permuting the two factors \( V_F \). The moment map \( \mu_{GL(2n)} \) is \( \sigma \)-equivariant, and the \( \sigma \)-fixed point set of \( \mu_{\text{GL}(2n)} : T^*V_F \times T^*\text{Gr}_{GL(2n)} \to \mathfrak{gl}(2n, O)^* \) is nothing but \( \mu : V_F \times T^*\text{Gr}_G \to \mathfrak{g}_O \).

Now since the zero level \( \mu^{-1}_{GL(2n)}(0) \) is Lagrangian (and hence isotropic), the zero level \( \mu^{-1}(0) \) is isotropic as well. Finally, \( \mu^{-1}(0) \) is coisotropic as the zero level of any moment map. □

Recall that the irreducible components of \( \mu^{-1}_{GL(2n)}(0) \) are numbered by bisignatures \( (\lambda, \nu) = (\lambda_1 \geq \ldots \geq \lambda_n, \nu_1 \geq \ldots \geq \nu_2n) \): a component is the closure of the conormal bundle to the corresponding \( GL(2n, O) \)-orbit in \( V_F \times \text{Gr}_{GL(2n)} \) [FGT, Proposition 8]. The component numbered by \( (\lambda, \nu) \) will be denoted \( \Lambda^{(\lambda, \nu)}_{GL(2n)} \), and its open subset (the conormal bundle) will be denoted \( \hat{\Lambda}^{(\lambda, \nu)}_{GL(2n)} \). For a signature \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \) we denote by \( \lambda^* \) the signature \( (-\lambda_2 \geq \ldots \geq -\lambda_1) \). Recall the involution \( \sigma \) introduced in the proof of Lemma 2.6.1. The next lemma follows immediately from definitions:

**Lemma 2.6.2.** The involution \( \sigma \) takes \( \Lambda^{(\lambda, \nu)}_{GL(2n)} \) to \( \Lambda^{(\lambda^*, \nu^*)}_{GL(2n)} \). □

The set of self-dual signatures \( \lambda = \lambda^* \) is in bijection with the set of length \( n \) partitions \( \theta = (\theta_1 \geq \ldots \geq \theta_n \geq 0) \); namely, \( \theta \mapsto \lambda := (\theta_1 \geq \ldots \geq \theta_n \geq -\theta_n \geq \ldots \geq -\theta_1) \). Given partitions \( \theta = (\theta_1 \geq \ldots \geq \theta_n \geq 0) \) and \( \zeta = (\zeta_1 \geq \ldots \geq \zeta_n \geq 0) \) we consider the corresponding self-dual bisignature

\[
(\lambda, \nu) := (\theta_1 \geq \ldots \geq \theta_n \geq -\theta_n \geq \ldots \geq -\theta_1, \ z_1 \geq \ldots \geq \zeta_n \geq -\zeta_n \geq \ldots \geq -\zeta_1)
\]

and the corresponding \( GL(2n, O) \)-orbit \( \mathcal{O}_{(\lambda, \nu)} \) in \( V_F \times \text{Gr}_{GL(2n)} \).

**Lemma 2.6.3.** The fixed point set of \( \sigma \) in the conormal bundle \( \hat{\Lambda}^{(\lambda, \nu)}_{GL(2n)} \) to \( \mathcal{O}_{(\lambda, \nu)} \) is irreducible.

**Proof.** The orbit \( \mathcal{O}_{(\lambda, \nu)} \) projects onto the \( GL(2n, O) \)-orbit \( \text{Gr}^{\lambda, \nu}_{GL(2n)} \) in the affine Grassmannian \( \text{Gr}_{GL(2n)} \). Any fiber of this projection is open in an appropriate lattice in \( V_F \). Hence any fiber of the projection \( \Pi \) of \( \hat{\Lambda}^{(\lambda, \nu)}_{GL(2n)} \) to \( \text{Gr}^{\lambda \vee \nu}_{GL(2n)} \) is open in an appropriate vector space. Now the fixed point set of \( \sigma \) in \( \text{Gr}^{\lambda \vee \nu}_{GL(2n)} \) is the corresponding \( \text{Sp}(2n, O) \)-orbit \( \text{Gr}^{\theta^* \vee \zeta}_{G} \) in \( \text{Gr}_{G} \); in particular, it is irreducible. The fixed point set of \( \sigma \) in the fiber of \( \Pi \) over a point of \( \text{Gr}^{\theta^* \vee \zeta}_{G} \) is irreducible as well. The irreducibility of the fixed point set (\( \hat{\Lambda}^{(\lambda, \nu)}_{GL(2n)} \)) follows. □
We define an irreducible component $\Lambda^{(\theta, \zeta)} \subset \mu^{-1}(0)$ as the closure of the fixed point set $(\hat{\lambda}_{\text{GL}(2n)})^\sigma$ of Lemma 2.6.3.

**Proposition 2.6.4.** Any irreducible component of $\mu^{-1}(0)$ is of the form $\Lambda^{(\theta, \zeta)}$ for a pair of length $n$ partitions $(\theta, \zeta)$.

We start the proof with the following elementary linear algebraic lemma.

**Lemma 2.6.5.** Let $\Lambda \subset U$ be a Lagrangian subspace of a symplectic vector space $U$. Let $\sigma \subset U$ be an involution respecting the symplectic form, but not necessarily preserving $\Lambda$. Suppose the invariant subspace $\Lambda^\sigma$ is Lagrangian in the symplectic vector space $U^\sigma$. Then $\sigma(\Lambda) = \Lambda$. \hfill $\square$

To prove the proposition, we fix a point $g$ in an $\text{Sp}(2n, \mathbf{O})$-orbit $\text{Gr}^\theta_{\mathbf{O}} \subset \text{Gr}^\lambda_{\text{GL}(2n)}$, and consider the fiber $W$ of $T^*V_F \times T^*\text{Gr}_{\text{GL}(2n)}$ over $g$. It contains the conormal $W'$ to the $\text{GL}(2n, \mathbf{O})$-orbit $\text{Gr}^\lambda_{\text{GL}(2n)}$, and the quotient $W/W'$ is symplectic. The involution $\sigma$ of the proof of Lemma 2.6.1 acts on $W/W'$. Now we apply Lemma 2.6.5 to $U := W/W'$ and to a Lagrangian subspace $\Lambda^\sigma \subset U^\sigma$ arising from the fiber over $g$ of an irreducible component of $\mu^{-1}(0)$. Then the lemma guarantees that $\Lambda \subset U$ arises from the fiber over $g$ of a $\sigma$-invariant irreducible component of $\mu^{-1}(0)$.

The proposition is proved. \hfill $\square$

We can view irreducibles in $\mathcal{D}W_{\text{mod}}^{G_{\mathbf{O}, \text{lc}, \wp}}$ as irreducible $G_{\mathbf{O}}$-equivariant $D$-modules on $(V_F/V_{\mathbf{O}}) \times \text{Gr}_{\mathbf{G}}$. Proposition 2.6.4 implies that among the uncountably many $\text{Sp}(2n, \mathbf{O})$-orbits in $(V_F/V_{\mathbf{O}}) \times \text{Gr}_{\mathbf{G}}$, only those orbits whose conormal bundle closures are of type $\Lambda^{(\theta, \zeta)}$ (so called relevant orbits) carry irreducibles in $\mathcal{D}W_{\text{mod}}^{G_{\mathbf{O}, \text{lc}, \wp}}$. For example, let $E_0 \in \mathcal{D}W_{\text{mod}}^{G_{\mathbf{O}, \text{lc}, \wp}}$ stand for the irreducible module $IC_{\mathbf{O}} \otimes \mathbb{C}[V_{\mathbf{O}}]$. Then the conormal bundle to its support is the irreducible component $\Lambda^{(0,0)}$. More generally, we view a length $n$ partition $\theta$ (resp. $\zeta$) as a dominant weight of $\text{Sp}(V)$ (resp. of $\text{SO}(V)$), and consider the corresponding irreducible $D$-modules $IC_{\theta} \in \text{D-mod}^{G_{\mathbf{O}, \text{lc}, \wp}}(\text{Gr}_{\mathbf{G}})$ (resp. $IC_{\zeta} \in \text{D-mod}^{G_{\mathbf{O}, \text{lc}, \wp}}(\text{Gr}_{\mathbf{G}})$) corresponding to $V_{\theta} \in \text{Rep}(\text{Sp}(V))$ (resp. to $V_{\zeta} \in \text{Rep}(\text{SO}(V��)$) under the (twisted) Satake equivalence of §2.1.2. Then the support of $IC_{\theta} \ast E_0 \ast IC_{\zeta}$ is the closure of the relevant $\text{Sp}(2n, \mathbf{O})$-orbit $\mathcal{O}_{(\theta, \zeta)} \subset (V_F/V_{\mathbf{O}}) \times \text{Gr}_{\mathbf{G}}$ whose conormal bundle closure is the component $\Lambda^{(\theta, \zeta)}$.

**Proposition 2.6.6.** For any length $n$ partitions $\theta, \zeta$, there is a unique irreducible module $IC_{(\theta, \zeta)}$ in $\mathcal{D}W_{\text{mod}}^{G_{\mathbf{O}, \text{lc}, \wp}}$ whose support is the closure of $\mathcal{O}_{(\theta, \zeta)}$.

**Proof.** We have to check that the stabilizer in $\text{Sp}(2n, \mathbf{O})$ of a point in $\mathcal{O}_{(\theta, \zeta)}$ is connected. The argument is similar to the proof of [BFT, Lemma 2.3.3]. Namely, a representative of the orbit $\mathcal{O}_{(\theta, \zeta)}$ is given by

$$v_{\theta, \zeta} := t^{-\theta_1}e_1 + \ldots + t^{-\theta_n}e_n \in V_F/V_{\mathbf{O}};$$
\[ L_{\theta, \zeta} := \text{O}t^{-\theta_1 - \zeta_1}e_1 \oplus \ldots \oplus \text{O}t^{-\theta_n - \zeta_n}e_n \oplus \text{O}t^{\theta_n + \zeta_n} \oplus \ldots \oplus \text{O}t^{\theta_1 + \zeta_1}e_{2n} \in \text{Gr}_G. \]

Here we chose a basis \( e_1, \ldots, e_{2n} \) of \( V \) such that \( \langle e_i, e_{2n+1-j} \rangle = \delta_{ij} \) for \( 1 \leq i, j \leq n \).

The stabilizer \( \text{Stab}_{\text{Sp}(2n, \mathbb{O})}(L_{\theta, \zeta}) \) is the semidirect product of its connected unipotent radical and the reductive part \( \text{Stab}^{\text{red}}_{\text{Sp}(2n, \mathbb{O})}(L_{\theta, \zeta}) \) isomorphic to a Levi subgroup \( L \subset G \).

More precisely, we write the partition \( \theta + \zeta \) in the form \( (0^{m_0}1^{m_1} \ldots) \) (where almost all multiplicities \( m_i \) are zero). Then \( L \simeq \text{Sp}(2m_0) \times \prod_{i>0} \text{GL}(m_i) \). Now we have to find the stabilizer of \( v_{\theta, \zeta} \) in \( \text{Stab}_{\text{Sp}(2n, \mathbb{O})}(L_{\theta, \zeta}) \). The stabilizer of \( v_{\theta, \zeta} \) in the unipotent radical of \( \text{Stab}^{\text{red}}_{\text{Sp}(2n, \mathbb{O})}(L_{\theta, \zeta}) \) is connected, and it remains to check that the stabilizer of \( v_{\theta, \zeta} \) in \( \text{Stab}^{\text{red}}_{\text{Sp}(2n, \mathbb{O})}(L_{\theta, \zeta}) \) is connected. The latter stabilizer is the product of stabilizers of certain vectors (summands of \( v_{\theta, \zeta} \), like e.g. \( e_1 + e_2 + \ldots + e_{m_{\text{max}}} \)) in the GL-factors of \( L \simeq \text{Sp}(2m_0) \times \prod_{i>0} \text{GL}(m_i) \).

Finally, the stabilizer of a vector in \( \text{GL}(m_i) \) is a connected mirabolic subgroup. \( \square \)

**Remark 2.6.7.** In fact, we will see that \( \text{IC}_{\zeta} \ast E_0 \ast \text{IC}_{\theta} \simeq \text{IC}_{(\theta, \zeta)} \) in Corollary 2.14.2 below.

### 2.7. De-equivariantized Ext algebra

Recall the notation of the end of §2.4. For dominant weights \( \lambda_0 \in \Lambda_0^+ \) and \( \lambda_1 \in \Lambda_1^+ \), we denote by \( \text{IC}_{\lambda_0} \in \text{D-mod}^{\mathbb{G}_O,lc,\vee}(\text{Gr}_G) \) (resp. by \( \text{IC}_{\lambda_1} \in \text{D-mod}^{\mathbb{G}_O,lc,\vee}_{1/2}(\text{Gr}_G) \)) the irreducible \( D \)-modules corresponding to \( V_{\lambda_0} \in \text{Rep}(\text{SO}(V_0)) \) (resp. to \( V_{\lambda_1} \in \text{Rep}(\text{Sp}(V)) \)) under the (twisted) Satake equivalence of §2.1.2.

Also, recall that \( E_0 \in \mathcal{D}W\text{-mod}^{\mathbb{G}_O,lc} \) stands for the irreducible module \( \text{IC}_0 \otimes \mathbb{C}[V_0] \).

We restrict the action

\[
\text{D-mod}^{\mathbb{G}_O,lc}(\text{Gr}_G) \otimes \text{D-mod}^{\mathbb{G}_O,lc}_{1/2}(\text{Gr}_G) \otimes \mathcal{D}W\text{-mod}^{\mathbb{G}_O,lc}
\]

of §2.1.3 to the action of

\[
\text{D-mod}^{\mathbb{G}_O,lc,\vee}(\text{Gr}_G) \otimes \text{D-mod}^{\mathbb{G}_O,lc,\vee}_{1/2}(\text{Gr}_G) \cong \text{Rep}(\text{SO}(V_0)) \otimes \text{Rep}(\text{Sp}(V)).
\]

Let \( \mathcal{D}W\text{-mod}^{\mathbb{G}_O,\text{deeq}} \) denote the corresponding de-equivariantized category, see [Gal]. We have

\[
(2.7.1) \quad \text{RHom}_{\mathcal{D}W\text{-mod}^{\mathbb{G}_O,\text{deeq}}}(\mathcal{F}, \mathcal{G}) = \bigoplus_{\lambda_0 \in \Lambda_0^+, \lambda_1 \in \Lambda_1^+} \text{RHom}_{\mathcal{D}W\text{-mod}^{\mathbb{G}_O,lc}}(\mathcal{F}, \text{IC}_{\lambda_0} \ast \mathcal{G} \ast \text{IC}_{\lambda_1}) \otimes V_{\lambda_1}^* \otimes V_{\lambda_0}^*.
\]

**Lemma 2.7.1.** The dg-algebra \( \text{RHom}_{\mathcal{D}W\text{-mod}^{\mathbb{G}_O,\text{deeq}}}(E_0, E_0) \) is formal, i.e. it is quasiisomorphic to the graded algebra \( \text{Ext}^*_{\mathcal{D}W\text{-mod}^{\mathbb{G}_O,\text{deeq}}}(E_0, E_0) \) with trivial differential.
Proof. The argument is the same as in the proof of [BFGT, Lemma 3.9.1]. The only difference is in the process of changing the setting to the base field $\mathbb{F}_q$. To this end we apply the equivalence $\mathcal{K}ir_{\text{constr}} \cong \mathcal{D}W\text{-mod}^{G_0,lc}$ of the end of §2.2. In $\mathcal{K}ir_{\text{constr}}$ we can change the setting to the base field $\mathbb{F}_q$, and again apply the equivalence $\mathcal{K}ir_{\text{constr},\mathbb{F}_q} \cong D^\text{constr}_{\text{const}} \cong D^G_{\text{const}} \simeq D_{\text{const}}(\mathcal{G}_G \times \text{Heis})_{\mathbb{F}_q}$. Here $\widetilde{\mathcal{G}}_G$ stands for the $\mu_2$-gerbe of square roots of the determinant line bundle on $\mathcal{G}_G$ (and we consider only the genuine sheaves: such that $-1 \in \mu_2$ acts by $-1$), and $\psi$ stands for the Artin-Schreier character sheaf on $\mathbb{G}_a$. Now the sheaves $IC_{\lambda_0} \ast E_0 \ast IC_{\lambda}$ can be equipped with a pure Weil structure, and the rest of the argument is identical to the one in the proof of [BFGT, Lemma 3.9.1].

We denote the dg-algebra $\text{Ext}^\bullet_{\mathcal{D}W\text{-mod}^{G_0,deeq}}(E_0, E_0)$ (with trivial differential) by $\mathfrak{e}^\bullet$. Since it is an Ext-algebra in the de-equivariantized category between objects induced from the original category, it is automatically equipped with an action of $\text{SO}(V_0) \times \text{Sp}(V) = G_0$ (notation of §2.4), and we can consider the corresponding triangulated category $D_{\text{perf}}^{G_0}(\mathfrak{e}^\bullet).

Lemma 2.7.2. There is a canonical equivalence $D_{\text{perf}}^{G_0}(\mathfrak{e}^\bullet) \sim \mathcal{D}W\text{-mod}^{G_0,lc}$.

Proof. Same as the one of [BFGT, Lemma 3.9.2].

2.8. Equivariant De Rham cohomology and universal objects. We will work with the category $\mathcal{D}W\text{-mod}^{G_0,lc} \cong \mathcal{D}W\text{-mod}^{G_0,lc}$ in the realization of Remark 2.2.3, i.e. in the equivalent incarnation $\text{D-mod}^{G_0 \times V_0 \times \mathbb{G}_a,\chi,lc}(\mathcal{G}_G \times \text{Heis})$. Let $\iota: \mathcal{G}_G \times V_0 \times \mathbb{G}_a \hookrightarrow \mathcal{G}_G \times \text{Heis}$ denote the closed embedding.

Let $\mathcal{M}_{\chi}$ stand for a unique $(G_0 \times V_0 \times \mathbb{G}_a,\chi)$-equivariant $D$-module on Heis that is a free rank one $\mathcal{O}_{\text{Heis}}$-module. For a $D$-module $\mathcal{F} \in \text{D-mod}^{G_0 \times V_0 \times \mathbb{G}_a,\chi,lc}(\mathcal{G}_G \times \text{Heis})$, its restriction $\iota^\dagger \mathcal{F}$ decomposes as $\iota^\dagger \mathcal{F} = \mathcal{F}_0 \boxtimes \mathcal{M}_{\chi}$ for some $\mathcal{F}_0 \in \text{D-mod}^{G_0,lc}(\mathcal{G}_G)$. We will denote $\iota^0 \mathcal{F} := \mathcal{F}_0$. We will keep the same notation $\iota^0$ for the functor $\mathcal{D}W\text{-mod}^{G_0,lc} \rightarrow \text{D-mod}^{G_0,lc}(\mathcal{G}_G)$ obtained by composing $\iota^0$ with the equivalence $\mathcal{D}W\text{-mod}^{G_0,lc} \cong \mathcal{D}W\text{-mod}^{G_0,lc} \cong \text{D-mod}^{G_0 \times V_0 \times \mathbb{G}_a,\chi,lc}(\mathcal{G}_G \times \text{Heis})$ of Remark 2.2.3. We set

\begin{equation}
H_{G_0,\text{DR}}^\bullet(\mathcal{G}_G,\iota^0-): \mathcal{D}W\text{-mod}^{G_0,lc} \cong \mathcal{D}W\text{-mod}^{G_0,lc} \cong \text{D-mod}^{G_0 \times V_0 \times \mathbb{G}_a,\chi,lc}(\mathcal{G}_G \times \text{Heis}) \rightarrow \mathbb{C}[\Xi_g]\text{-mod}.
\end{equation}

We introduce the following object

\begin{equation}
\mathcal{U} := \wp_{\mathcal{G}_G} \boxtimes \mathbb{C}[V_0] \in \mathcal{D}W\text{-mod}^{G_0}
\end{equation}

of the ‘big’ (not renormalized) category $\mathcal{D}W\text{-mod}^{G_0}$. Under the equivalence $\mathcal{D}W\text{-mod}^{G_0} \cong \mathcal{D}W\text{-mod}^{G_0}$ of §2.1.3, the object $\mathcal{U}$ goes to the ‘quantization of
the universal lattice' 

\[ (2.8.3) \quad \mathcal{U} \in \mathcal{D}W\text{-mod}^{G_0}. \]

For a future use we formulate a relation between \( \mathcal{U} \) and the left convolution action of \( \text{D-mod}^{G_0,lc}(\text{Gr}_G) \) on \( \mathcal{D}W\text{-mod}^{G_0,lc} \).

**Lemma 2.8.1.** (a) For \( \mathcal{F}_0 \in \text{D-mod}^{G_0,lc}(\text{Gr}_G) \), we have \( \mathcal{F}_0 \ast E_0 = \mathcal{F}_0 \otimes \mathcal{U} \in \mathcal{D}W\text{-mod}^{G_0,lc} \).

(b) For \( \mathcal{F}_1 \in \text{D-mod}_{1/2}^{G_0,lc}(\text{Gr}_G) \), we have \( E_0 \ast \mathcal{F}_1 = \mathcal{F}_1 \boxtimes C[V_0] \in \mathcal{D}W\text{-mod}^{G_0,lc} \).

**Proof.** (b) is clear. To prove (a), we invoke the ‘symmetric’ realization \( \mathcal{D}W\text{-mod}^{G_0} \cong (\text{D-mod}(\text{Gr}_G) \otimes \text{D-mod}_{1/2}(\text{Gr}_G) \otimes \mathcal{W}\text{-mod})^G \) of (2.1.3). Under this equivalence, \( E_0 \in \mathcal{D}W\text{-mod}^{G_0} \) goes to \( \Delta_\ast \mathcal{U} \in (\text{D-mod}(\text{Gr}_G) \otimes \text{D-mod}_{1/2}(\text{Gr}_G) \otimes \mathcal{W}\text{-mod})^G \), where \( \Delta: \text{Gr}_G \hookrightarrow \text{Gr}_G \times \text{Gr}_G \) stands for the diagonal embedding. Also, \( \mathcal{F}_0 \ast E_0 \) goes to \( \text{pr}_1^! \mathcal{F}_0 \otimes \Delta_\ast \mathcal{U} \). \( \square \)

**Remark 2.8.2.** Let \( \mathcal{U}_{\text{Heis}} \) denote the image of \( \mathcal{U} \) under the equivalence \( \mathcal{D}W\text{-mod}^{G_0} \cong \text{D-mod}_{1/2}^{G_0} \times \mathcal{W}\text{-mod}(\text{Gr}_G \times \text{Heis}) \), cf. Lemma 2.2.1. Let also \( \omega_{\text{Heis}} \) denote the dualizing sheaf on Heis. Then for \( \mathcal{F}_0 \in \text{D-mod}^{G_0,lc}(\text{Gr}_G) \), the convolution action \( \mathcal{F}_0 \ast E_0 \in \mathcal{D}W\text{-mod}^{G_0,lc} \) goes under the equivalence \( \mathcal{D}W\text{-mod}^{G_0,lc} \cong \text{D-mod}_{1/2}^{G_0} \times \mathcal{W}\text{-mod}(\text{Gr}_G \times \text{Heis}) \) of Lemma 2.2.1 to \( (\mathcal{F}_0 \boxtimes \omega_{\text{Heis}}) \otimes \mathcal{U}_{\text{Heis}} \in \text{D-mod}_{1/2}^{G_0} \times \mathcal{W}\text{-mod}(\text{Gr}_G \times \text{Heis}) \).

We have a natural morphism of dg-algebras \( R\Gamma_{\text{Gr}_G,\text{DR}}(\text{Gr}_G, \mathcal{O}_{\text{Gr}_G}) \to R\text{Hom}_{\text{D-mod}^{G_0}(\text{Gr}_G)}(\omega_{\text{Gr}_G}, \omega_{\text{Gr}_G}) \). Since the dg-algebra \( R\Gamma_{\text{Gr}_G,\text{DR}}(\text{Gr}_G, \mathcal{O}_{\text{Gr}_G}) \) is formal and quasi-isomorphic to \( \mathbb{C}[\Xi] \), we obtain the morphisms of dg-algebras 

\[ (2.8.4) \quad \mathbb{C}[\Xi] \to R\text{Hom}_{\text{D-mod}^{G_0}(\text{Gr}_G)}(\omega_{\text{Gr}_G}, \omega_{\text{Gr}_G}) \quad \text{and} \quad \mathbb{C}[\Xi] \to R\text{Hom}_{\mathcal{D}W\text{-mod}^{G_0}}(\mathcal{U}', \mathcal{U}'). \]

Also, let \( \nabla \in (\mathcal{W} \hat{\otimes} \mathcal{W})\text{-mod} \) be the diagonal module (the image of the regular bimodule \( W \in (\mathcal{W} \hat{\otimes} \mathcal{W}^{\text{opp}})\text{-mod} \) under the isomorphism \( W^{\text{opp}} \cong W \) that multiplies all the generators in \( V_F \) by \( \sqrt{-1} \). Given \( \mathcal{F} \in \mathcal{D}'\mathcal{W}\text{-mod}^{G_0,lc} \), \( \mathcal{G} \in \mathcal{D}'\mathcal{W}\text{-mod}^{G_0} \), we define a pairing with values in \( \mathbb{C}[\Xi] \text{-mod} \): 

\[ \langle \mathcal{F}, \mathcal{G} \rangle = \text{Hom}_{\mathcal{W} \hat{\otimes} \mathcal{W}}(\nabla, H^{\bullet}_{G_0,\text{DR}}(\text{Gr}_G, \Delta^!(\mathcal{F} \boxtimes \mathcal{G}))). \]

Here \( \Delta: \text{Gr}_G \hookrightarrow \text{Gr}_G \times \text{Gr}_G \) is the diagonal embedding, so that \( H^{\bullet}_{G_0,\text{DR}}(\text{Gr}_G, \Delta^!(\mathcal{F} \boxtimes \mathcal{G})) \) carries a structure of a module over \( H^{\bullet}_{G_0}(\text{Gr}_G) \otimes \mathcal{W} \hat{\otimes} \mathcal{W} \).

Then we have a natural isomorphism of \( \mathbb{C}[\Xi] \text{-modules} \)

\[ (2.8.5) \quad \forall \mathcal{E} \in \mathcal{D}'\mathcal{W}\text{-mod}^{G_0,lc}, \quad H^{\bullet}_{G_0,\text{DR}}(\mathcal{E}) \cong \langle \mathcal{E}, \mathcal{U}' \rangle. \]
2.9. Idempotents. According to (2.8.4), $\omega_{Gr_\\mathbb{G}} \in \text{D-mod}^{G_\\mathbb{G}}(Gr_\\mathbb{G})$ carries a structure of a module over a formal dg-algebra $C[\Sigma_\\mathbb{G}] = H^*_G(Gr_\\mathbb{G})$. The equivariant cohomology $C[\Sigma_\\mathbb{G}] = H^*_G(Gr_\\mathbb{G})$ also carries a structure of a formal $C[\Sigma_\\mathbb{G}]$-module. We set

$$
(2.9.1) \quad \tilde{\omega}_{Gr_\\mathbb{G}} := R\text{Hom}_{C[\Sigma_\\mathbb{G}]}(C[\Sigma_\\mathbb{G}], \omega_{Gr_\\mathbb{G}}) \in \text{D-mod}^{G_\\mathbb{G}}(Gr_\\mathbb{G}),
$$

where $R\text{Hom}_{C[\Sigma_\\mathbb{G}]}$ stands for the right adjoint to $\otimes_{C[\Sigma_\\mathbb{G}]}$.

Recall that $\Theta$ and hence $R$ (notation of (2.1.4)) is a direct sum of two sheaves: $R = R_0 \oplus R_1 = RT^{-1}(\Theta_0) \oplus RT^{-1}(\Theta_1)$; and $R_0$ has zero costalks at all the odd $G_\\mathbb{G}$-orbits in $Gr_\\mathbb{G}$, while $R_1$ has zero costalks at all the even $G_\\mathbb{G}$-orbits in $Gr_\\mathbb{G}$. Similarly to (2.8.4), we have a morphism of dg-algebras

$$
\omega_{Gr_\\mathbb{G}} := R\text{Hom}_{C[\Sigma_\\mathbb{G}]}(C[\Sigma_\\mathbb{G}], \omega_{Gr_\\mathbb{G}}) \in \text{D-mod}^{G_\\mathbb{G}}(Gr_\\mathbb{G}),
$$

and we set

$$
(2.9.2) \quad \tilde{\mathcal{R}} = \tilde{R}_0 \oplus \tilde{R}_1 := R\text{Hom}_{C[\Sigma_\\mathbb{G}]}(C[\Sigma_\\mathbb{G}], \mathcal{R}) \in \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G}).
$$

Finally, we set (cf. (2.8.2))

$$
(2.9.3) \quad \tilde{\mathcal{U}}' := R\text{Hom}_{C[\Sigma_\\mathbb{G}]}(C[\Sigma_\\mathbb{G}], \mathcal{U}') = \tilde{\omega}_{Gr_\\mathbb{G}} \boxtimes C[V_0] \in \text{D'}\mathcal{W}\text{-mod}^{G_\\mathbb{G}}.
$$

Lemma 2.9.1. We have natural isomorphisms

(a) $\tilde{\omega}_{Gr_\\mathbb{G}} \ast \tilde{\omega}_{Gr_\\mathbb{G}} \simeq \tilde{\omega}_{Gr_\\mathbb{G}}$.

(b) $\tilde{\mathcal{R}} \ast \tilde{\mathcal{R}} \simeq \tilde{\mathcal{R}}$.

Proof. (a) Under the equivalence $\beta^\circ: D^G_{\text{nilp}}(\text{Sym}^*(\mathfrak{g}'[-2])) \tilde{\rightarrow} \text{D-mod}^{G_\\mathbb{G}}(Gr_\\mathbb{G})$ of (2.1.2), $\tilde{\omega}_{Gr_\\mathbb{G}}$ corresponds to $j^\text{reg} \ast \tilde{\omega}_{\text{reg}}^\circ \mathcal{O}_{\mathfrak{g}'[2]}$. Here $j^\text{reg}: \mathcal{O}_{\text{reg}} \hookrightarrow N_{\mathfrak{g}'}$ stands for the open embedding of the formal neighbourhood of the regular nilpotent orbit into the formal neighbourhood of the nilpotent cone in $\mathfrak{g}'[2]$. Furthermore, $i^\text{reg}$ stands for the closed embedding of $\mathcal{O}_{\text{reg}}$ into $\mathfrak{g}'[2]$. Clearly, $j^\text{reg} \ast i^\text{reg} \mathcal{O}_{\mathfrak{g}'[2]}$ is an idempotent.

(b) Similarly, under the composed equivalence

$$
D^G_{\text{nilp}}(\text{Sym}^*(\mathfrak{g}[-2])) \tilde{\rightarrow} \text{D-mod}^{G_\\mathbb{G}}_{1/2}(Gr_\\mathbb{G}) \tilde{\rightarrow} \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G}),
$$

$\tilde{\mathcal{R}}$ corresponds to $j^\text{reg} \ast \mathcal{O}_{\mathfrak{g}[2]}$. Here the first equivalence is $\beta^\circ$ of (2.1.1), and the second one is the twisting by the inverse determinant line bundle $D^{-1}$. \hfill \Box

Remark 2.9.2. Under the equivalence $D^G_{\text{nilp}}(\text{Sym}^*(\mathfrak{g}[-2])) \tilde{\rightarrow} \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G})$ of the proof of Lemma 2.9.1(b), $\tilde{R}_1 \in \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G})$ corresponds to $j^\text{reg}(\mathcal{L} \otimes i^\text{reg} \mathcal{O}_{\mathfrak{g}[2]})$. Here $\mathcal{L}$ is the $G$-equivariant line bundle on $\mathcal{O}_{\text{reg}}$ with the nontrivial action of the center $\{\pm 1\} \subset G$.

The category $\text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G})$ splits into direct sum of two subcategories: $\text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G}) = \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G})_0 \oplus \text{D-mod}^{G_\\mathbb{G}}_{-1/2}(Gr_\\mathbb{G})_1$. The objects of the
first (resp. second) subcategory have zero costalks at the odd (resp. even) $G_0$-orbits in $\text{Gr}_G$. The category $\text{D-mod}_{G_0}^0(\text{Gr}_G)_0$ is monoidal. According to [DLYZ], it is monoidally equivalent to $\text{D-mod}_{\text{Spin}(2n+1)}^0(\text{Gr}_{\text{Spin}(2n+1)})$, and $\mathcal{R}_0$ goes to the dualizing sheaf (`endoscopy').

**Lemma 2.9.3.** The internal Hom objects are computed as follows:

(a) $\text{Hom}_{\text{D-mod}^0(\text{Gr}_G)}(\tilde{\omega}_{\text{Gr}_G}, \tilde{\omega}_{\text{Gr}_G}) = \tilde{\omega}_{\text{Gr}_G}$.

(b) $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}^{0}(\mathcal{R}_0, \mathcal{R}_0) = \mathcal{R}_0$.

**Proof.** (a) follows from $\text{Hom}_{\text{D-mod}^0(\text{Gr}_G)}(\omega_{\text{Gr}_G}, \omega_{\text{Gr}_G}) = \omega_{\text{Gr}_G} \otimes H^*(\text{Gr}_G)$.

(b) follows from (a) and endoscopy. □

Since the monoidal categories $\text{D-mod}^0(\text{Gr}_G)$ and $\text{D-mod}_{G_0}^{0}(\text{Gr}_G)_0$ act on $\mathcal{D}'\mathcal{W}$-$\text{mod}^0_G$, we can consider the corresponding internal Hom objects $\text{Hom}_{\text{D-mod}^0(\text{Gr}_G)}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \in \text{D-mod}^0(\text{Gr}_G)$ and $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}^{0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \in \text{D-mod}_{G_0}^0(\text{Gr}_G)_0$, and $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \in \text{D-mod}_{G_0}^0(\text{Gr}_G)_0$.

**Lemma 2.9.4.** We have canonical isomorphisms

(a) $\text{Hom}_{\text{D-mod}^0(\text{Gr}_G)}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \cong \tilde{\omega}_{\text{Gr}_G}$.

(b) $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}^{0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \cong \mathcal{R}_0$.

(c) $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \cong \mathcal{R}_0$.

**Proof.** (a) follows from Lemma 2.9.3(a) since $\tilde{\mathcal{U}}' = \tilde{\omega}_{\text{Gr}_G} \boxtimes \mathbb{C}[\mathcal{V}_0]$ and $\text{D-mod}^0_{G_0}(\text{Gr}_G)$ acts on $\mathcal{D}'\mathcal{W}$-$\text{mod}^0_G$ via its action on $\text{D-mod}^0_{G_0}(\text{Gr}_G)$ (on itself).

(b) We have $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\mathbb{C}[\mathcal{V}_0], \mathbb{C}[\mathcal{V}_0]) = \mathcal{R}$ according to [BDFRT, Proposition 3.2.1]. Hence

$$\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}') \cong \tilde{\omega}_{\text{Gr}_G} \otimes \text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\mathbb{C}[\mathcal{V}_0], \mathbb{C}[\mathcal{V}_0]) = \tilde{\omega}_{\text{Gr}_G} \otimes \mathcal{R} = \mathcal{R}.$$  

(c) follows from (b) since $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}^{0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}')$ is obtained from $\text{Hom}_{\text{D-mod}_{G_0}^0(\text{Gr}_G)_0}(\tilde{\mathcal{U}}', \tilde{\mathcal{U}}')$ by applying the projection functor $\text{D-mod}^0_{G_0}^{0}(\text{Gr}_G) \to \text{D-mod}^0_{G_0}(\text{Gr}_G)_0$ (adjoint to the embedding $\text{D-mod}^0_{G_0}(\text{Gr}_G)_0 \to \text{D-mod}^0_{G_0}(\text{Gr}_G)$). □

We define the full monoidal subcategory $\mathcal{C}_{\tilde{\omega}} \subset \text{D-mod}_{G_0}(\text{Gr}_G)$ (resp. $\mathcal{C}_\mathcal{R} \subset \text{D-mod}_{G_0}^0(\text{Gr}_G)_0$) generated by $\tilde{\omega}_{\text{Gr}_G}$ (resp. $\mathcal{R}_0$) with respect to finitely many...
operations of taking direct sums and cones. Similarly, we define the full subcategory $\mathcal{E}_U \subset D^W\text{-mod}^{G_0}$ generated by $\bar{U}'$ with respect to finitely many operations of taking direct sums and cones. Then both $\mathcal{E}_\omega$ and $\mathcal{E}_R$ act on $\mathcal{E}_U$. The following corollary immediately follows from Lemmas 2.9.1 and 2.9.4.

**Corollary 2.9.5.** The actions of $\mathcal{E}_\omega$ and $\mathcal{E}_R$ on $\bar{U}'$ give rise to the equivalences $\mathcal{E}_\omega \xrightarrow{\sim} \mathcal{E}_U$ and $\mathcal{E}_R \xrightarrow{\sim} \mathcal{E}_U$. □

### 2.10. Hecke eigenproperty

Recall the classical Satake monoidal equivalence [MV]

$$H^\vee := H_{DR}^\bullet(G_0, -) : D\text{-mod}^{G_0, \vee}(G) \to \text{Rep}(G^\vee).$$

Also, recall the twisted Satake monoidal equivalence [Ly]

$$H := H_{DR}^\bullet(G_0, - \otimes \mathcal{R}) : D\text{-mod}^{G_0, \vee}_{1/2}(G) \to \text{Rep}(G).$$

Twisting by the determinant line bundle $D$ defines a monoidal equivalence $D\text{-mod}^{G_0, \vee}_{1/2}(G) \xrightarrow{\sim} D\text{-mod}^{G_0, \vee}_{1/2}(G)$. We will keep the same notation for the composed twisted Satake monoidal equivalence

$$H := H_{DR}^\bullet(G_0, - \otimes D \otimes \mathcal{R}) : D\text{-mod}^{G_0, \vee}_{1/2}(G) \to \text{Rep}(G).$$

For $\mathcal{F}_0 \in D\text{-mod}^{G_0, \vee}(G)$ and $\mathcal{F}_1 \in D\text{-mod}^{G_0, \vee}_{1/2}(G)$, we have isomorphisms of $\mathbb{C}[\Xi]$-modules (notation of §2.1.4)

$$H_{G_0, DR}^\bullet(G_0, \mathcal{F}_0) = \kappa^\vee(H^\vee(\mathcal{F}_0) \otimes \text{Sym}^\bullet(g^\vee[-2])), \quad \text{and}$$

$$H_{G_0, DR}^\bullet(G_0, \mathcal{F}_1 \otimes \mathcal{R}) = \kappa(\mathcal{H}(\mathcal{F}_1) \otimes \text{Sym}^\bullet(g[-2])).$$

**Lemma 2.10.1.** For $\mathcal{F}_0 \in D\text{-mod}^{G_0, \vee}(G)$, and $\mathcal{F}_1, \mathcal{F}_0 \in D\text{-mod}^{G_0, \vee}_{1/2}(G)$, and $\mathcal{F}_{1,1} \in D\text{-mod}^{G_0, \vee}_{1/2}(G)$, we have natural isomorphisms of $\mathbb{C}[\Xi]$-modules

(a) $\mathcal{F}_0 \ast \tilde{\omega}_{G_0} \cong H_{G_0, DR}^\bullet(G_0; \mathcal{F}_0) \otimes_{\mathbb{C}[\Xi]} \tilde{\omega}_{G_0}$.

(b) $\mathcal{F}_{1,0} \ast \tilde{\mathcal{R}}_0 \cong H_{G_0, DR}^\bullet(G_0; \mathcal{F}_{1,0} \otimes D \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Xi]} \tilde{\mathcal{R}}_0$, and

$$\mathcal{F}_{1,0} \ast \tilde{\mathcal{R}}_0 \cong H_{G_0, DR}^\bullet(G_0; \mathcal{F}_{1,0} \otimes D \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Xi]} \tilde{\mathcal{R}}_1.$$

(c) $\mathcal{F}_{1,1} \ast \tilde{\mathcal{R}}_0 \cong H_{G_0, DR}^\bullet(G_0; \mathcal{F}_{1,1} \otimes D \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Xi]} \tilde{\mathcal{R}}_0$, and

$$\mathcal{F}_{1,1} \ast \tilde{\mathcal{R}}_0 \cong H_{G_0, DR}^\bullet(G_0; \mathcal{F}_{1,1} \otimes D \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Xi]} \tilde{\mathcal{R}}_1.$$

These isomorphisms are compatible with the monoidal structures of the (twisted) Satake equivalences.

**Proof.** Follows from the fact used in the proof of Lemma 2.9.1 that under the (twisted) derived Satake equivalence, $\tilde{\omega}_{G_0}$ (resp. $\tilde{\mathcal{R}}_0$) corresponds to $j_{\text{reg}}^\text{reg} \circ \mathcal{O}_{\text{reg} g^\vee[2]}$ (resp. $j_{\text{reg}}^\text{reg} \circ \mathcal{O}_{\text{reg} g[2]}$). □
Recall the convolution action of §2.1.3:
\[ \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G) \otimes \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \otimes \text{D'}W\text{-mod}^{G_0}. \]

**Proposition 2.10.2.** For \( \mathcal{F}_0 \in \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \) and \( \mathcal{F}_1 \in \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G) \) we have a natural isomorphism of \( \mathbb{C}[\Xi_g] \)-modules

\[ \mathcal{F}_1 \ast \mathcal{U}' \ast \mathcal{F}_0 \cong H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_1 \otimes \mathcal{D} \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_0) \otimes_{\mathbb{C}[\Sigma_g]} \mathcal{U}'. \]

These isomorphisms are compatible with the monoidal structures of the (twisted) Satake equivalences.

**Proof.** From Corollary 2.9.5 and Lemma 2.10.1 we deduce

\[ \mathcal{F}_1 \ast \mathcal{U}' \ast \mathcal{F}_0 \cong H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_1 \otimes \mathcal{D} \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_0) \otimes_{\mathbb{C}[\Sigma_g]} \mathcal{U}'. \]

But \( \omega_{\text{Gr}} = \omega_{\text{Gr}} \ast \tilde{\omega}_{\text{Gr}} \), hence \( \mathcal{U}' \cong \omega_{\text{Gr}} \ast \tilde{\mathcal{U}}' \), and the desired result follows. \( \square \)

**Corollary 2.10.3.** (a) For any \( \mathcal{E} \in \text{D'}W\text{-mod}^{G_0}_{\text{lc}}, \mathcal{F}_0 \in \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \) and \( \mathcal{F}_1 \in \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G) \), we have a natural isomorphism of \( \mathbb{C}[\Xi_g] \)-modules

\[ H_{G_0,\text{DR}}^*(\mathcal{F}_1 \ast \mathcal{E} \ast \mathcal{F}_0) \]

\[ \cong H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_1 \otimes \mathcal{D} \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{F}_0) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\mathcal{E}). \]

These isomorphisms are compatible with the monoidal structures of the (twisted) Satake equivalences.

(b) Recall an equivalent realization

\[ \text{D'}W\text{-mod}^{G_0}_{\text{lc}} \cong \text{D}W\text{-mod}^{G_0}_{\text{lc}} \otimes \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \otimes \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G). \]

Then equivalently, for any \( \mathcal{A} \in \text{D}W\text{-mod}^{G_0}_{\text{lc}}, \mathcal{G}_0 \in \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \) and \( \mathcal{G}_1 \in \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G) \), we have a natural isomorphism of \( \mathbb{C}[\Xi_g] \)-modules

\[ H_{G_0,\text{DR}}^*(\mathcal{G}_0 \ast \mathcal{A} \ast \mathcal{G}_1) \]

\[ \cong H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{G}_1 \otimes \mathcal{R}) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\text{Gr}_G, \mathcal{G}_0) \otimes_{\mathbb{C}[\Sigma_g]} H_{G_0,\text{DR}}^*(\mathcal{A}). \]

**Proof.** Compare Proposition 2.10.2 and (2.8.5). \( \square \)

In particular, taking \( \mathcal{A} = E_0 \in \text{D}W\text{-mod}^{G_0}_{\text{lc}}, \mathcal{G}_1 = IC_{\lambda_1} \in \text{D-mod}^{G_0}_{-1/2}(\text{Gr}_G), \mathcal{G}_0 = IC_{\lambda_0} \in \text{D-mod}^{G_0}_{\text{lc}}(\text{Gr}_G) \), we obtain

**Corollary 2.10.4.** For \( \lambda_0 \in \Lambda_0^+, \lambda_1 \in \Lambda_1^+ \), we have a canonical isomorphism of \( \mathbb{C}[\Xi_g] \)-modules

\[ \kappa^\theta(V_{\lambda_1} \otimes \text{Sym}^*(g[-2])) \otimes_{\mathbb{C}[\Sigma_g]} \kappa^{\theta'}(V_{\lambda_0} \otimes \text{Sym}^*(g'[-2])) \iso H_{G_0,\text{DR}}^*(\text{IC}_{\lambda_0} \ast E_0 \ast \text{IC}_{\lambda_1}). \]
2.11. Injectivity.

**Lemma 2.11.1.** For $\lambda_0 \in \Lambda_0^+$, $\lambda_1 \in \Lambda_1^+$, the natural morphism

$$
\text{Ext}_{\text{D-mod}^{\mathcal{O}, \kappa}}(E_0, IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1})
\rightarrow \text{Hom}_{C[S]}(H_{G_0, \text{DR}}^\bullet(Gr_G, E_0), H_{G_0, \text{DR}}^\bullet(Gr_G, IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1}))
$$

is injective.

**Proof.** Note that the line $Gr^0_G \times \{0\} \times \mathbb{G}_a \subset Gr_G \times \text{Heis}$ is a connected component of the fixed point set $(Gr_G \times \text{Heis})^{G_m}$ of the loop rotation. Let us denote by $\mathcal{M}_{\lambda^0, \lambda^1}$ the D-mod$^{G_0 \times \mathbb{G}_a \times \delta}$-counterpart (see Remark 2.2.3) of the sheaf $IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1}$. Then $\mathcal{M}_{\lambda^0, \lambda^1}$ is semisimple and equivariant with respect to the loop rotation. Furthermore, we have a canonical isomorphism

$$(2.11.1) \quad \text{Ext}_{\text{D-mod}^{\mathcal{O}, \kappa}}(E_0, IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1}) \simeq \Phi_0 \mathcal{M}_{\lambda^0, \lambda^1},$$

where $\Phi_0$ stands for the $G_0$-equivariant De Rham cohomology of the hyperbolic restriction to the line $Gr^0_G \times \{0\} \times \mathbb{G}_a \subset Gr_G \times \text{Heis}$.

It suffices to check that the natural morphism

$$
\text{Ext}_{\text{D-mod}^{\mathcal{O}, \kappa}}(E_0, IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1})
\rightarrow \text{Hom}_{C[S]}(H_{G_0, \text{DR}}^\bullet(Gr_G, E_0), H_{G_0, \text{DR}}^\bullet(Gr_G, IC_{\lambda_0} \ast E_0 \ast IC_{\lambda_1}))
$$

(in the RHS we take Hom over the $G_0$-equivariant cohomology of the point) is injective. Making use of Lemma 2.8.1, we rewrite the LHS as

$$
\text{Ext}_{\text{D-mod}^{\mathcal{O}, \kappa}}(IC_{\lambda_0} \ast E_0, E_0 \ast IC_{\lambda_1}) \simeq H_{G_0, \text{DR}}^\bullet(Gr_G, IC_{\lambda_0} \otimes IC_{\lambda_1} \otimes \mathcal{R}).
$$

Due to the purity reasons (2.11.1), all the $C[S]$-modules in question are free, and it suffices to establish the injectivity of the similar morphism for *non*-equivariant Ext’s and cohomology:

$$
H_{\text{DR}}^\bullet(Gr_G, IC_{\lambda_0} \otimes IC_{\lambda_1} \otimes \mathcal{R}) \rightarrow H_{\text{DR}}^\bullet(Gr_G \times Gr_G, IC_{\lambda_0} \boxtimes (IC_{\lambda_1} \otimes \mathcal{R})).
$$

Recall that $\Theta$ and hence $\mathcal{R}$ is a direct sum of two sheaves: $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 = RT^{-1}_1(\Theta_\gamma) \oplus RT^{-1}_1(\Theta_\delta)$; and $\mathcal{R}_0$ has zero costalks at all the odd $G_0$-orbits in $Gr_G$, while $\mathcal{R}_1$ has zero costalks at all the *even* $G_0$-orbits in $Gr_G$. All the costalks of $\mathcal{R}_0$ (resp. $\mathcal{R}_1$) live in the even (resp. odd) cohomological degrees. Finally, all the $G_0$-orbits in $Gr_G$ are even-dimensional. By the parity vanishing reasons, the Cousin spectral sequence of the filtration of $Gr_G$ by $G_0$-orbits (resp. of the filtration of $Gr_G \times Gr_G$ by $G_0 \times G_0$-orbits) degenerates. Hence it suffices to check the desired injectivity for the similar morphism for $Gr_G$ (resp. $Gr_G \times Gr_G$) replaced by a $G_0$-orbit $Gr^\lambda_G$ (resp. $Gr^\lambda_G \times Gr^\lambda_G$). The latter injectivity is clear. $\square$
2.12. **Some Invariant Theory.** The following proposition is proved similarly to [BFT, Proposition 2.8.3].

**Proposition 2.12.1.** Given an $\text{SO}(V_0)$-module $M$ and an $\text{Sp}(V)$-module $M'$, the Kostant functors of $\S2.4$ induce an isomorphism

$$\text{Hom}_{\text{SO}(V_0) \times \text{Sp}(V) \times \mathbb{C}[V_0 \otimes V]}(\mathbb{C}[V_0 \otimes V], M \otimes \mathbb{C}[V_0 \otimes V] \otimes M')$$

$$\xrightarrow{\sim} \text{Hom}_{\mathbb{C}[\Sigma_\varpi]}\left(\mathbb{C}[\Sigma_\varpi], \kappa^0(M \otimes \text{Sym}^\bullet(\mathfrak{g}[-2])) \otimes_{\mathbb{C}[\Sigma_\varpi]} \kappa^0(M' \otimes \text{Sym}^\bullet(\mathfrak{g}[-2]))\right).$$

Here we identify $\Sigma_\varpi$ and $\Sigma_{\varpi'}$ as in $\S2.4$, and we view $\Sigma_\varpi$ as the zero section of the tangent bundle $\Sigma_\varpi = T\Sigma_\varpi$.

\[\square\]

2.13. **The generators of $\mathfrak{e}^\bullet$.** Let $\omega \in \Lambda_\varpi^+ \ (\text{resp. } \varpi \in \Lambda_\varpi^+) \ be \ the \ first \ fundamental \ weight \ such \ that \ V_\omega \ (\text{resp. } V_\varpi) \ is \ the \ tautological \ representation \ of \ V_0 \ of \ \text{SO}(V_0) \ (\text{resp. } \text{Sp}(V)). \ We \ will \ describe \ the \ objects \ IC_\omega \ast E_0 \ and \ E_0 \ast IC_\varpi \ of \ \mathcal{D} \mathcal{W}\text{-mod}^{\mathcal{G}_0}$.

Along the Grassmannian $\text{Gr}_G$ they are both supported on the minimal Schubert variety $\overline{\text{Gr}}_G^\omega$. If we identify the Weyl algebra $\mathcal{W}$ with differential operators on $V_{\mathbb{F}}/V_0$, then along this factor, both $IC_\omega \ast E_0$ and $E_0 \ast IC_\varpi$ are supported on $(t^{-1}V_{\mathbb{O}})/V_{\mathbb{O}}$.

More precisely, we consider a symplectic vector space $(t^{-1}V_{\mathbb{O}})/V_0$ along with its projection $pr$ onto $(t^{-1}V_{\mathbb{O}})/V_0$. Then the minimal Schubert variety $\overline{\text{Gr}}_G^\omega$ possesses a resolution of singularities $\overline{\text{Gr}}_G^\omega \xrightarrow{\pi} \overline{\text{Gr}}_G^\omega$ formed by all the collections of lines $\ell \subset (t^{-1}V_{\mathbb{O}})/V_0$ and Lagrangian subspaces $L \subset (t^{-1}V_{\mathbb{O}})/tV_0$ such that $L \subset pr^{-1}(\ell)$. The projection $p: \overline{\text{Gr}}_G^\omega \to \mathbb{P}((t^{-1}V_{\mathbb{O}})/V_0), (\ell, L) \mapsto \ell$, identifies $\overline{\text{Gr}}_G^\omega$ with a $\mathbb{P}^1$-bundle over $\mathbb{P}((t^{-1}V_{\mathbb{O}})/V_0)$, namely the projectivization of a rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(2)$. The exceptional divisor $E \subset \overline{\text{Gr}}_G^\omega$ (the preimage of the singular base point $0 \in \overline{\text{Gr}}_G^\omega$) is the infinite section of the above $\mathbb{P}^1$-bundle. The pullback of the determinant line bundle $\pi^*\mathcal{D} \cong (p^*\mathcal{O}(2))(E)$.

We denote by $X \subset \overline{\text{Gr}}_G^\omega \times ((t^{-1}V_{\mathbb{O}})/V_0)$ the ‘universal line’ whose fiber over $(\ell, L) \in \overline{\text{Gr}}_G^\omega$ is $L \subset (t^{-1}V_{\mathbb{O}})/V_0$. The open subset $X^o \subset X$ is the preimage of the $\mathcal{G}_\otimes$-orbit $\overline{\text{Gr}}_G^\omega \subset \overline{\text{Gr}}_G^\omega$. The orbit $\overline{\text{Gr}}_G^\omega$ is identified with the total space of the line bundle $\mathcal{O}(2)$ over $\mathbb{P}((t^{-1}V_{\mathbb{O}})/V_0)$, and $X^o$ is identified with the pullback of $\mathcal{O}(-1)$ to this total space. Hence we have a regular function $f: X^o \to \mathcal{G}_\otimes$ whose restriction to any fiber of the former (resp. latter) line bundle is linear (resp. quadratic). This function $f$ has a simple pole along the preimage in $X$ of the exceptional divisor $E \subset \overline{\text{Gr}}_G^\omega$. We consider the $D$-module $\exp(f)$ on $X^o$ with irregular singularity at the preimage of $E$. Also, we have the smooth irreducible $\sqrt{D}$-twisted $D$-module on $\overline{\text{Gr}}_G^\omega$ with 1-dimensional fibers to be denoted $\sqrt{D}$. We will keep the same notation for its pullback to $X^o$. We define an irreducible $\sqrt{D}$-twisted $D$-module $\mathcal{F}_0^\omega$ on $X^o$ as $\sqrt{D} \otimes \exp(f)$. Finally, we view $\mathcal{F}_0^\omega$ as a $\sqrt{D}$-twisted
D-module on $\text{Gr}_G^\omega \times ((t^{-1}V_O)/V_O)$ and define $\mathcal{F}_0$ as the minimal extension of $\mathcal{F}_0$ to $\text{Gr}_G^\omega \times ((t^{-1}V_O)/V_O)$. Then $\text{IC}_\omega * E_0$ is nothing but $\mathcal{F}_0$.

Furthermore, we have the smooth irreducible $\sqrt{D}$-twisted $D$-module $\mathcal{F}_1^\omega$ with 1-dimensional fibers on $\text{Gr}_G^\omega \times \{0\} \subset \text{Gr}_G^\omega \times ((t^{-1}V_O)/V_O)$ (it was denoted $\sqrt{D}$ in the previous paragraph). It extends cleanly to a $\sqrt{D}$-twisted $D$-module on $\text{Gr}_G^\omega \times \{0\} \subset X$, and we define $\mathcal{F}_1$ as the direct image of this $D$-module under the projection $X \to \text{Gr}_G^\omega \times ((t^{-1}V_O)/V_O)$. Then $E_0 * \text{IC}_\omega$ is nothing but $\mathcal{F}_1$. In particular, the minimal extension $\mathcal{F}_1$ of $\mathcal{F}_1^\omega$ is clean.

Since $\text{Gr}_G^\omega \times \{0\}$ is a smooth divisor in $X^\circ$, we have the canonical elements $h \in \text{Ext}^1(\mathcal{F}_1, \mathcal{F}_0^\omega)$, $h^* \in \text{Ext}^1(\mathcal{F}_0, \mathcal{F}_0^\omega)$. Since $E_0 * \text{IC}_\omega$ is the clean extension of $\mathcal{F}_1^\omega$, the elements $h, h^*$ give rise to the same named elements

$$h \in \text{Ext}^1_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0 * \text{IC}_\omega, \text{IC}_\omega * E_0), \quad h^* \in \text{Ext}^1_{\text{DW-mod}^{\text{Gr}_G^\omega}}(\text{IC}_\omega * E_0, E_0 * \text{IC}_\omega).$$

The composition $h^* \circ h \in \text{Ext}^2_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0 * \text{IC}_\omega, E_0 * \text{IC}_\omega)$ is the multiplication by the first Chern class of the normal bundle $\mathcal{N}_{\text{Gr}_G^\omega \times \{0\}/X^\circ}$ equal to $c_1(D)$.

Since $\text{Ext}^1_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0 * \text{IC}_\omega, \text{IC}_\omega * E_0) = \text{Ext}^1_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0, \text{IC}_\omega * E_0 * \text{IC}_\omega)$, we obtain the subspace $h \otimes V_0^* \otimes V^* \cong h \otimes V_0 \otimes V \subset \mathcal{C}^1 := \text{Ext}^1_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0, E_0)$ (notation of §2.7). We will denote this subspace simply by $V_0 \otimes V$.

### 2.14. Proof of Theorem 2.5.1.

Due to the results of §2.13, we obtain a homomorphism from the free tensor algebra

$$\phi^* : T(\Pi(V_0 \otimes V)[-1]) \to \mathcal{C} := \text{Ext}^*_{\text{DW-mod}^{\text{Gr}_G^\omega}}(E_0, E_0).$$

**Lemma 2.14.1.** The homomorphism $\phi^*$ factors through the projection

$$T(\Pi(V_0 \otimes V)[-1]) \to \text{Sym}(\Pi(V_0 \otimes V)[-1]) = \mathcal{G}^*$$

(notation of §2.5) and induces an isomorphism $\mathcal{G}^* \xrightarrow{\sim} \mathcal{C}^*$. 

**Proof.** The same as the proof of [BFT, Lemma 2.6.2], granted Proposition 2.12.1 along with Lemma 2.11.1 and Corollary 2.10.4. \qed

Now Theorem 2.5.1 is proved the same way as [BFT, Theorem 2.2.1] is proved in [BFT, §§2.7.2.10]. Moreover, the following corollary is proved the same way as [BFT, Corollary 2.6.3].

**Corollary 2.14.2.** For any $\lambda_0 \in \Lambda_0^+$, $\lambda_1 \in \Lambda_1^+$, the convolution $\text{IC}_{\lambda_0} * E_0 * \text{IC}_{\lambda_1}$ lies in the heart $\text{DW-mod}^{\text{Gr}_G^\omega, \text{IC}^\omega}$ and coincides with the irreducible object $\text{IC}_{(\lambda_1, \lambda_0)}$ (notation of §2.6).
3. Classical simple Lie superalgebras with invariant even pairing

3.1. Orthosymplectic Gaiotto conjectures. Given a positive integer $m \leq n$ we set $k = n - m$ and consider a nilpotent element $e \in \mathfrak{sp}(2n)$ of Jordan type $(2m, 1^{2k})$. We fix a maximal reductive subgroup $\text{Sp}(2k)$ in the centralizer of $e$ (i.e. an orthogonal decomposition $V = \mathbb{C}^{2n} = \mathbb{C}^{2m} \oplus \mathbb{C}^{2k}$ compatible with $e$). We choose an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{sp}(2n)$. In the theory of finite $W$-algebras, the Slodowy slice $S_e$ is obtained as the Hamiltonian reduction of $\mathfrak{sp}(2n)$ with respect to a certain unipotent subgroup $U(e) \subset \text{Sp}(2n)$ with a character. The Lie algebra $\text{Lie} U(e) = \mathfrak{sp}(2n)_{-1} \simeq \mathbb{C}^{2k}$. In case $k = n - 1$, we have $\mathfrak{sp}(2n)_{-1} \simeq \mathfrak{h} \mathfrak{e} \mathfrak{i} \mathfrak{s} (\mathfrak{sp}(2n)_{-1})$. We denote by $U_k$ the unipotent subgroup of $\text{Sp}(2n)$ with the Lie algebra $\mathfrak{sp}(2n)_{-1}$.

As in §2.1.1, we consider the completed Weyl algebra $\mathcal{W}(\mathfrak{sp}(2n)_{-1} \otimes \mathcal{F}) = \mathcal{W}(\mathcal{F}^{2k})$. There is a twisted action $\text{D-mod}_{1/2}(\text{Sp}(2k, \mathcal{F})) \otimes \mathcal{W}(\mathcal{F}^{2k})$-mod. Let $\kappa_\text{b}$ stand for the basic level, i.e. the bilinear form $\text{Tr}(X \cdot Y)$ on $\mathfrak{sp}(2n)$. It corresponds to the determinant line bundle on $\text{Gr}_G$ (the ample generator of the Picard group). Given $c \in \mathbb{C}^\times$ we consider the categories $\mathcal{D} \mathcal{W}(\mathcal{F}^{2k})$-mod$_{1/2+c^{-1}} : = \text{D-mod}_{1/2+c^{-1}}(\text{Gr}_G) \otimes \mathcal{W}(\mathcal{F}^{2k})$-mod (resp. $\mathcal{D} \mathcal{W}(\mathcal{F}^{2k})$-mod$_{-c^{-1}} : = \text{D-mod}_{-c^{-1}}(\text{Gr}_G) \otimes \mathcal{W}(\mathcal{F}^{2k})$-mod), and the categories $\mathcal{D} \mathcal{W}(\mathcal{F}^{2k})$-mod$_{1/2+c^{-1}}^{\text{Sp}(2k, \mathcal{O}) \times U_k(\mathcal{F}), \text{lc}}$ (resp. $\mathcal{D} \mathcal{W}(\mathcal{F}^{2k})$-mod$_{-c^{-1}}^{\text{Sp}(2k, \mathcal{O}) \times U_k(\mathcal{F}), \text{lc}}$) of locally compact $\text{Sp}(2k, \mathcal{O}) \ltimes U_k(\mathcal{F})$-equivariant objects.

On the dual side, we consider the quantum groups

$U_q(\mathfrak{osp}(2k + 1|2n))$ and $U_q(\mathfrak{osp}(2n + 1|2k))$, $q = \exp(\pi \sqrt{-1}/c)$.

More precisely, when $k = 0$, we consider the modified covering quantum supergroup $U_q(\mathfrak{osp}(1|2n))$ of [CFLW]. We denote by $\text{Rep}_q(\text{SOSp}(2k + 1|2n))$ (resp. $\text{Rep}_q(\text{SOSp}(2n + 1|2k))$) the dg-category of finite dimensional complexes of $U_q(\mathfrak{osp}(2k + 1|2n))$-modules (resp. $U_q(\mathfrak{osp}(2n + 1|2k))$-modules).

Conjecture 3.1.1. (a) For $c \notin \mathbb{Q}^\times$, the categories

$\mathcal{D} \mathcal{W}(\mathcal{F}^{2k})$-mod$_{1/2+c^{-1}}^{\text{Sp}(2k, \mathcal{O}) \times U_k(\mathcal{F}), \text{lc}}$ and $\text{Rep}_q(\text{SOSp}(2k + 1|2n))$

are equivalent as braided tensor categories, and this equivalence is compatible with the tautological $t$-structures.
(b) For \( c \not\in \mathbb{Q}^* \), the categories
\[
\mathcal{D}W(F^{2k})\text{-mod}_{c^{-1}}^{\text{Sp}(2k,O)\times U_k(F),lc} \text{ and } \text{Rep}_q(\text{SOSp}(2n + 1|2k))
\]
are equivalent as braided tensor categories, and this equivalence is compatible with the tautological t-structures.

Remark 3.1.2. Consider the extremal case \( m = n \) (that is, \( k = 0 \)). We set \( q' = \exp(\pi \sqrt{-1/c} - \pi \sqrt{-1/2}) \). Then the equivalence of Conjecture 3.1.1(b) is nothing but the Fundamental Local Equivalence \( \text{Rep}_q(\text{SO}(2n + 1)) \rightarrow \text{D-mod}_{c^{-1}}^{U(F),lc}(\text{Gr}_G) \) for the category of twisted Whittaker \( D \)-modules on \( \text{Gr}_G \). On the other hand, Conjecture 3.1.1(a) proposes an equivalence \( \text{Rep}_q(\text{SOSp}(1|2n)) \rightarrow \text{D-mod}_{c^{-1}}^{U(F),lc}(\text{Gr}_G) \). Combining the two equivalences we obtain a purely algebraic equivalence \( \text{Rep}_q(\text{SO}(2n + 1)) \approx \text{Rep}_q(\text{SOSp}(1|2n)) \) that is nothing but the equivalence induced by the twistor isomorphism \( \Psi \) between the integral forms of the modified quantum (super) groups of [CFLW, Theorem 4.3].

Remark 3.1.3. More generally, for arbitrary \( k \), combining the equivalences of Conjecture 3.1.1(a,b), we arrive at an equivalence \( \text{Rep}_q(\text{SOSp}(2n + 1|2k)) \approx \text{Rep}_q(\text{SOSp}(2k + 1|2n)) \). We are unaware of its algebraic proof.

Remark 3.1.4. Let \( S_e \subset \text{sp}(2n) \) denote the Slodowy slice through a nilpotent element \( e \) of “hook” Jordan type \((2n - 2k, 1^{2k})\). Then the symplectic variety \( \text{Sp}(2n) \times S_e \) is obtained by the Hamiltonian reduction \((T^* \text{Sp}(2n) \times \mathbb{C}^{2k})//U_k\). It is a hyperspherical variety of the group \( \text{Sp}(2n) \times \text{Sp}(2k) \) (i.e. the algebra of invariant functions \( \mathbb{C}[\text{Sp}(2n) \times S_e]^{\text{Sp}(2n)\times\text{Sp}(2k)} \) is Poisson commutative).

The twisted \( S \)-dual of \( \text{Sp}(2n) \times S_e \) is \( \mathbb{C}^{2n} \otimes \mathbb{C}^{2k+1} \) (twisting along \( \text{Sp}(2n) \)) is \( \mathbb{C}^{2n} \otimes \mathbb{C}^{2k} \otimes \text{SO}(2k + 1) \) (the bifundamental representation of the even part of \( \text{SOSp}(2k + 1|2n) \) on \( \text{osp}(2k + 1|2n)_1 \)). If the twisting is taken along \( \text{Sp}(2k) \), then the twisted \( S \)-dual is \( \mathbb{C}^{2n+1} \otimes \mathbb{C}^{2k} \otimes \text{SO}(2n + 1) \times \text{Sp}(2k) \) (the bifundamental representation of the even part of \( \text{SOSp}(2n + 1|2k) \) on \( \text{osp}(2n + 1|2k)_1 \)).

3.2. Gaiotto conjectures for \( \text{GL}(K|N) \) revisited. Let \( 0 < K < N - 1 \) be some positive integers. We will reformulate [BFGT, Conjecture 2.6.1] similarly to §3.1. Namely, we consider the unipotent subgroup \( U_{M,N}^r \subset \text{GL}_N \) of [BFGT, Remark 2.4.1] for \( M = K - 1 \). The \( 1_{M+1} = 1_K \) block in the middle of the matrix
\[
\begin{pmatrix}
U_r & * & * \\
0 & 1_{M+1} & * \\
0 & 0 & U_s
\end{pmatrix}
\]
of \textit{loc.cit.} has a row of \( K \) matrix elements right above it; a column of \( K \) matrix elements immediately to the right of it, and a matrix element at the intersection of these row and column; \( 2K + 1 \) matrix elements altogether. Annihilating all the other above-diagonal matrix elements of \( U_{K-1,N}^r \) we obtain a surjective homomorphism \( \phi' : U_{K-1,N}^r \rightarrow \text{Heis}(\mathbb{C}^{2K}) \) onto the Heisenberg group.
of a 2\(K\)-dimensional symplectic vector space. This symplectic vector space is equipped with a natural polarization \(\mathbb{C}^{2K} \cong \mathbb{C}^K \oplus \mathbb{C}^{K^*}\) and an action of \(\text{GL}_K\), and \(\phi\) is \(\text{GL}_K\)-equivariant.

Furthermore, the character \(\chi^{(r,s)}_{M,N}\) of [BFGT, Remark 2.4.1] is the sum of certain matrix elements of \(U'_{M,N}\). Let \(\chi^{(r,s)}_{K-1,N}\) denote the partial sum excluding the summands belonging to \(2K + 1\) matrix elements described in the previous paragraph. We view \(\chi^{(r,s)}_{K-1,N}\) as a homomorphism to the center of the Heis-berg group: \(U'_{K-1,N} \to \mathbb{C} \hookrightarrow \text{Heis}(\mathbb{C}^{2K})\). We define a surjective homomorphism \(\phi: U'_{K-1,N} \twoheadrightarrow \text{Heis}(\mathbb{C}^{2K})\) as \(\phi := \phi + \chi^{(r,s)}_{K-1,N}\). It is still \(\text{GL}_K\)-equivariant.

We consider the completed Weyl algebra \(\mathcal{W}(\mathbb{F}^{2K})\). Given \(c \in \mathbb{C}^\times\) we consider the category \(\mathcal{D}W(\mathbb{F}^{2K})\)-mod\(_{c^{-1}}\) := \(\text{D-mod}_{c^{-1}}(\text{Gr}_{\text{GL}_N}) \otimes \mathcal{W}(\mathbb{F}^{2K})\)-mod and the category \(\mathcal{D}W(\mathbb{F}^{2K})\)-mod\(_{c^{-1}}^{\text{GL}(2K,O) \ltimes U'_{K-1,N}(\mathbb{F}),\text{lc}}\) of locally compact \(\text{GL}(2K,O) \ltimes U'_{K-1,N}(\mathbb{F})\)-equivariant objects (here \(U'_{K-1,N}(\mathbb{F})\) is understood to act on \(\mathcal{W}(\mathbb{F}^{2K})\) via \(\phi\)).

**Conjecture 3.2.1.** For \(c \not\in \mathbb{Q}^\times\), \(q = \exp(\pi \sqrt{-1}/c)\), the categories

\[
\mathcal{D}W(\mathbb{F}^{2K})\)-mod\(_{c^{-1}}^{\text{GL}(2K,O) \ltimes U'_{K-1,N}(\mathbb{F}),\text{lc}}\) and \(\text{Rep}_q(\text{GL}(K|N))\)

are equivalent as braided tensor categories, and this equivalence is compatible with the tautological \(t\)-structures.

**Remark 3.2.2.** Let \(S_e \subset \mathfrak{gl}_N\) denote the Slodowy slice through a nilpotent element \(e\) of “hook” Jordan type \((N - K, 1^K)\). Then the symplectic variety \(\text{GL}_N \times S_e\) is obtained by the Hamiltonian reduction \((T^*\text{GL}_N \times \mathbb{C}^{2K})//U'_{K-1,N}\). Alternatively, it can be obtained by the Hamiltonian reduction \((T^*\text{GL}_N)//(U'_{K,N},\chi^{(r,s)}_{M,N})\). It is a hyperspherical variety of the group \(\text{GL}_N \times \text{GL}_K\) (i.e. the algebra of invariant functions \(\mathbb{C}[\text{GL}_N \times S_e]^\text{GL}_N\times\text{GL}_K\) is Poisson commutative).

The \(S\)-dual of \(\text{GL}_N \times S_e \otimes \text{GL}_N \times \text{GL}_K\) is \(T^*\text{Hom}(\mathbb{C}^N, \mathbb{C}^{K}) \otimes \text{GL}_N \times \text{GL}_K\) (the representation of the even part of \(\text{GL}(N|K)\) on \(\mathfrak{gl}(N|K)_1\)).

### 3.3. Exceptional Lie superalgebra \(\mathfrak{f}(4)\)

We consider a nilpotent element \(e \in \mathfrak{sp}(6)\) of Jordan type \((3, 3)\), so that \(e\) lies in a 14-dimensional nilpotent orbit. We fix a maximal reductive subgroup \(\text{SL}(2)\) in the centralizer of \(e\). We choose an \(\mathfrak{sl}_2\)-triple \((e, h, f)\) in \(\mathfrak{sp}(6)\). The adjoint action of \(h\) on \(\mathfrak{sp}(6)\) equips it with a grading, and \(\mathfrak{sp}(6)_{-1} \simeq \mathbb{C}^2\) carries a canonical symplectic form. The 8-dimensional nilpotent Lie algebra \(u = \mathfrak{sp}(6)_{\leq -1}\) projects onto \(\text{Heis}(\mathfrak{sp}(6)_{-1})\). We denote by \(U \subset \text{Sp}(6)\) the unipotent subgroup with Lie algebra \(u\). It is normalized by \(\text{SL}(2) \subset Z_{\mathfrak{sp}(6)}(e)\).

As in §2.1.1, we consider the completed Weyl algebra \(\mathcal{W}(\mathbb{F}^2)\). There is a twisted action \(\text{D-mod}_{-1/2}(\text{SL}(2, \mathbb{F})) \circ \mathcal{W}(\mathbb{F}^2)\)-mod. Given \(c \in \mathbb{C}^\times\) we consider the category \(\mathcal{D}W(\mathbb{F}^2)\)-mod\(_{c^{-1}}\) := \(\text{D-mod}_{c^{-1}}(\text{Gr}_{\text{PSp}(6)}) \otimes \mathcal{W}(\mathbb{F}^2)\)-mod, and the category...
\[ \mathcal{D}W(F^2)\text{-mod}^{SL(2,O)\ltimes U(F),lc}_{c=1} \] of locally compact \( SL(2,O) \ltimes U(F) \)-equivariant objects.

On the dual side, we consider the quantum group \( U_q(\mathfrak{f}(4)) \), \( q = \exp(\pi \sqrt{-1}/c) \), and we denote by \( \text{Rep}_q(F(4)) \) the dg-category of finite dimensional complexes of \( U_q(\mathfrak{f}(4)) \)-modules.

**Conjecture 3.3.1.** For \( c \not\in \mathbb{Q}^\times \), the categories
\[ \mathcal{D}W(F^2)\text{-mod}^{SL(2,O)\ltimes U(F),lc}_{c=1} \text{ and } \text{Rep}_q(F(4)) \]
are equivalent as braided tensor categories, and this equivalence is compatible with the tautological \( t \)-structures.

**Remark 3.3.2.** Let \( S_e \subset \mathfrak{sp}(6) \) denote the Slodowy slice through \( e \). Then the symplectic variety \( \text{PSp}(6) \times S_e \) is obtained by the Hamiltonian reduction \( (T^* \text{PSp}(6) \times \mathbb{C}^2)/U \). It is a hyperspherical variety of the group \( \text{PSp}(6) \times SL(2) \) (i.e. the algebra of invariant functions \( \mathbb{C}[\text{PSp}(6) \times S_e]^{\text{PSp}(6)\times SL(2)} \) is Poisson commutative). The twisted \( S \)-dual of \( \text{PSp}(6) \times S_e \) is \( \mathbb{C}^8 \otimes \mathbb{C}^2 \otimes \text{Spin}(7) \times SL(2) \) (the bispinor representation of the even part of \( F(4) \) on \( \mathfrak{f}(4) \)).

**Remark 3.3.3 (A. Elashvili).** There is an order 4 outer automorphism \( \sigma \) of the simply connected group \( E_6 \) such that its fixed points coincide with the even part of \( F(4) \), and the \( \sqrt{-1} \)-eigenspace of \( \sigma \) on \( \mathfrak{e}_6 \) coincides with \( \mathfrak{f}(4)_1 \).

### 3.4. Exceptional Lie superalgebra \( \mathfrak{g}(3) \)

We consider a nilpotent element \( e \in \mathfrak{g}_2 \) corresponding to a short root vector, so that \( e \) lies in the 8-dimensional nilpotent orbit. We fix a maximal reductive subgroup \( SL(2) \) in the centralizer of \( e \). We choose an \( \mathfrak{sl}_2 \)-triple \((e, h, f)\) in \( \mathfrak{g}_2 \). The adjoint action of \( h \) on \( \mathfrak{g}_2 \) equips it with a grading, and \((\mathfrak{g}_2)_{-1} \simeq \mathbb{C}^2 \) carries a canonical symplectic form. The 5-dimensional nilpotent Lie algebra \( \mathfrak{u} = (\mathfrak{g}_2)_{\leq -1} \) projects onto \( \mathfrak{hei}(\mathfrak{g}_2)_{-1} \). We denote by \( U \subset G_2 \) the unipotent subgroup with Lie algebra \( \mathfrak{u} \). It is normalized by \( SL(2) \subset Z_{G_2}(e) \). Note that \( SL(2) \ltimes U \) is the derived group of a parabolic subgroup of \( G_2 \) (the stabilizer of the highest weight line in the 7-dimensional representation of \( G_2 \)).

As in \( \S 2.1.1 \), we consider the completed Weyl algebra \( W(F^2) \). There is a twisted action \( \text{D-mod}_{1/2}(SL(2,F)) \otimes W(F^2)\text{-mod} \). Given \( c \in \mathbb{C}^\times \) we consider the category \( \mathcal{D}W(F^2)\text{-mod}_{c=1} := \text{D-mod}_{c=1}(\text{Gr}_{G_2}) \otimes W(F^2)\text{-mod} \), and the category \( \mathcal{D}W(F^2)\text{-mod}^{SL(2,O)\ltimes U(F),lc}_{c=1} \) of locally compact \( SL(2,O) \ltimes U(F) \)-equivariant objects.

On the dual side, we consider the quantum group \( U_q(\mathfrak{g}(3)) \), \( q = \exp(\pi \sqrt{-1}/c) \), and we denote by \( \text{Rep}_q(G(3)) \) the dg-category of finite dimensional complexes of \( U_q(\mathfrak{g}(3)) \)-modules.
Conjecture 3.4.1. For $c \not\in \mathbb{Q}^\times$, the categories
\[
\mathcal{D}W(F^2)\text{-mod}_{e^{-i}(2,O) \times U(F),c}^{\mathrm{SL}(2,O)} \text{ and } \text{Rep}_q(G(3))
\]
are equivalent as braided tensor categories, and this equivalence is compatible with the tautological $t$-structures.

Remark 3.4.2. Let $S_e \subset g_2$ denote the Slodowy slice through $e$. Then the symplectic variety $G_2 \times S_e$ is obtained by the Hamiltonian reduction $(T^* G_2 \times \mathbb{C}^2) / U$. It is a hyperspherical variety of the group $G_2 \times \mathbb{C}^2$ (i.e. the algebra of invariant functions $\mathbb{C}[G_2 \times S_e]_{G_2 \times \mathbb{C}^2}$ is Poisson commutative). The twisted $S$-dual of $G_2 \times S_e \supset G_2 \times \mathbb{C}^2$ is $\mathbb{C}^7 \otimes \mathbb{C}^2 \otimes G_2 \times \mathbb{C}^2$ (the bifundamental representation of the even part of $G(3)$ on $g(3)_1$).

3.5. Kostka polynomials.

3.5.1. Case of $\mathfrak{osp}(2n + 1|2n)$. Following [GL, §4], we consider the mixed Borel subalgebra $b \subset g = \mathfrak{osp}(V_0|V)$ with the set of positive simple roots

\[
\{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \ldots, \varepsilon_{n-1} - \delta_{n-1}, \delta_{n-1} - \varepsilon_n, \varepsilon_n - \delta_n, \delta_n\}. \]

All the above simple roots are odd, and all but the last one are isotropic. The odd part $n_1$ of the nilpotent radical $n$ of $b$ has Cartan eigenvalues

\[
R_1^+ = \{\varepsilon_i + \delta_j\}_{1 \leq i, j \leq n} \cup \{\varepsilon_i - \delta_j\}_{1 \leq i, j \leq n} \cup \{\delta_i - \varepsilon_j\}_{1 \leq i, j \leq n} \cup \{\delta_i\}_{1 \leq i \leq n}. \]

Following [BFT, Definition 3.3.1], for $\alpha \in t^* \oplus t^*_0$ (notation of §2.4) we consider a polynomial $L_{\alpha}(q) := \sum p_d q^d$ where $p_d$ is the number of unordered partitions of $\alpha$ into a sum of $d$ elements of $R_1^+$. We say that $\Lambda^+_1 \times \Lambda_0^+ \ni (\lambda_1, \lambda_0) \geq (\mu_1, \mu_0)$ if $(\lambda_1, \lambda_0) - (\mu_1, \mu_0) \in \mathbb{N}(R_1^+)$. We set

\[
K_{(\lambda_1, \lambda_0), (\mu_1, \mu_0)}(q) := \sum_{w \in W, \ w_0 \in W_0} (-1)^{w_0} (-1)^w L_{(w(\lambda_1 + \rho) - \rho - \mu_1, w_0(\lambda_0 + \rho_0) - \rho_0 - \mu_0)}(q),
\]

notation of §2.4.

Recall the notation of §2.6, especially the paragraph before Proposition 2.6.6. Given $(\lambda_1, \lambda_0) \in \Lambda^+_1 \times \Lambda_0^+$, we will view the irreducible object $(\text{IC}_{\lambda_1} \boxtimes \text{IC}_{\lambda_0} \ast E_0 \simeq \text{IC}_{(\lambda_1, \lambda_0)} \subset \mathcal{D}W_{\text{mod}}^{G_O, i} \subset \mathcal{D}W_{\text{mod}}^{G_O, i}_{G_O} \subset (V_F / V_O) \times \text{Gr}_G$. Given $(\mu_1, \mu_0) \in \Lambda^+_1 \times \Lambda_0^+$, we are interested in the stalks of $\text{IC}_{(\lambda_1, \lambda_0)}$ at the relevant $G_O$-orbit $O_{(\mu_1, \mu_0)} \subset (V_F / V_O) \times \text{Gr}_G$. The proof of the following theorem is entirely similar to the one of [BFT, Theorem 3.3.5].

Theorem 3.5.1. (a) A relevant $G_O$-orbit $O_{(\mu_1, \mu_0)} \subset (V_F / V_O) \times \text{Gr}_G$ lies in the closure of a relevant $G_O$-orbit $O_{(\lambda_1, \lambda_0)}$ iff $(\lambda_1, \lambda_0) \geq (\mu_1, \mu_0)$.

(b) We have

\[
q^{-\dim O_{(\mu_1, \mu_0)}} K_{(\lambda_1, \lambda_0), (\mu_1, \mu_0)}(q^{-1}) = \sum_i \dim(\text{IC}_{(\lambda_1, \lambda_0)})^{-i} \text{IC}_{(\mu_1, \mu_0)} q^{-i},
\]
3.5.2. Case of $f(4)$. In the setup of §3.3, let us consider the limit case $c^{-1} \to 0$, and the category $\mathcal{D}'W(\mathbb{F}^2)\text{-mod} := D\text{-mod}(\text{Gr}_{\mathbb{P}^6}) \otimes W(\mathbb{F}^2)\text{-mod}$ \footnote{There is a typo in [M, Exercise 4.7.11]: the definitions of simple roots $\alpha_1, \alpha_3$ of Spin(7) should be swapped: $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_3 = \varepsilon_3$.} (untwisted $D$-modules), along with the category $\mathcal{D}'W(\mathbb{F}^2)\text{-mod}^{\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F}), \text{lc}}$ of locally compact $\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F})$-equivariant objects. Similarly to Theorem 2.5.1(d), we expect that there is a monoidal equivalence from the category of representations of the degenerate supergroup $D^\text{b}\text{Rep}(\mathbb{F}(4))$ to the category $\mathcal{D}'W(\mathbb{F}^2)\text{-mod}^{\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F}), \text{lc}} := \mathcal{D}'W(\mathbb{F}^2)\text{-mod}^{\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F}), \text{lc}} \otimes_{\text{Vect}} \text{SVect}$.

Similarly to §2.6, we can view irreducible objects of $\mathcal{D}'W(\mathbb{F}^2)\text{-mod}^{\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F}), \text{lc}}$ as irreducible $\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F})$-equivariant $D$-modules on $(\mathbb{F}^2/\mathcal{O}) \times \text{Gr}_{\mathbb{P}^6}$ supported on certain relevant $\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F})$-orbits. We expect that as in Theorem 3.5.1(a), the adjacency relation of relevant orbits is governed by a choice of a certain Borel subalgebra in $f(4)$. Namely, in notation of [FSS, §2.18] (cf. also [M, Exercise 4.7.12]) the Borel subalgebra $B_4$ in question has positive simple roots $\left\{ \frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) \right\}$.

Let $B' \subset \text{Spin}(7)$, $B'' \subset \text{SL}(2)$ be the Borel subgroups whose Lie algebras are contained in the Borel subalgebras $B_4 \subset f(4)$. Let $\mathfrak{n}_1$ be the odd part of the nilpotent radical of $B_4$. Let $\mathfrak{n}_1' \subset \mathfrak{n}_1$ be the 6-dimensional subspace spanned by the root vectors of weights

$$\frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$

$$\frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

Then the natural action $B' \times B'' \odot \mathfrak{n}_1'$ extends to the action $P' \times B'' \odot \mathfrak{n}_1'$ where $P' \supset B'$ is the subminimal parabolic subgroup corresponding to the (long) middle simple root $\varepsilon_2 - \varepsilon_3$ of Spin(7). In other words, $\text{Spin}(7)/P'$ is the isotropic flag space $\mathcal{F}(1, 3, 7)$ of the 7-dimensional space equipped with the symmetric bilinear form preserved by Spin(7). Let $\widetilde{N}_1$ be the vector bundle over $(\text{Spin}(7)/P') \times (\text{SL}(2)/B'') = \mathcal{F}(1, 3, 7) \times \mathbb{P}^1$ associated to the representation $P' \times B'' \odot \mathfrak{n}_1'$. We have a natural morphism $\widetilde{N}_1 \to N_1$ to the odd nilpotent cone of $f(4)$. This is a particular case of Hesselink’s desingularization [H] (in particular, both $\widetilde{N}_1$ and $N_1$ have dimension 15, cf. [E, row 4 of Table 6]). We are grateful to A. Elashvili and M. Jibladze for this observation.

3.5.3. Case of $\mathfrak{g}(3)$. In the setup of §3.4, let us consider the limit case $c^{-1} \to 0$, and the category $\mathcal{D}'W(\mathbb{F}^2)\text{-mod} := D\text{-mod}(\text{Gr}_{\mathbb{C}^2}) \otimes W(\mathbb{F}^2)\text{-mod}$ (untwisted $D$-modules), along with the category $\mathcal{D}'W(\mathbb{F}^2)\text{-mod}^{\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F}), \text{lc}}$ of locally compact $\text{SL}(2, \mathcal{O}) \ltimes U(\mathbb{F})$-equivariant objects. Similarly to Theorem 2.5.1(d),
we expect that there is a monoidal equivalence from the category of representations of the degenerate supergroup $D^b\text{Rep}(G(3))$ to the category $S^DW(F^2)-\text{mod}^{SL(2,O)\times U(F),lc} := D^W(F^2)-\text{mod}^{SL(2,O)\times U(F),lc} \otimes_{\text{Vect}} \text{SVect}$.

Similarly to §2.6, we can view irreducible objects of $D^\prime W(F^2)-\text{mod}^{SL(2,O)\times U(F),lc,\otimes} \cong D^\prime W(F^2)-\text{mod}^{SL(2,O)\times U(F),lc} \otimes_{\text{Vect}} \text{SVect}$ as irreducible $SL(2,O)\times U(F)$-equivariant $D^\prime$-modules on $(F^2/O^2)\times Gr_{G_2}$ supported on certain relevant $SL(2,O)\times U(F)$-orbits. We expect that as in Theorem 3.5.1, the adjacency relation of relevant orbits is governed by a choice of a certain Borel subalgebra in $g(3)$. Namely, in notation of [FSS, §2.19] (cf. also [M, Exercise 4.7.10]) the Borel subalgebra $B_3$ in question has positive simple roots $\{\delta, -\delta + \varepsilon_1, \varepsilon_2 - \varepsilon_1\}$.

Let $B' \subset G_2$, $B'' \subset SL(2)$ be the Borel subgroups whose Lie algebras are contained in the Borel subalgebra $B_3 \subset g(3)$. Let $n_1$ be the odd part of the nilpotent radical of $B_3$. Then the natural action $B' \times B'' \circ n_1$ extends to the action $P' \times B'' \circ n_1$ where $P' \supset B'$ is the parabolic subgroup whose Lie algebra contains the negative simple short root space. Let $\tilde{N}_1$ be the vector bundle over $(G_2/P') \times (SL(2)/B'') = (G_2/P') \times P^1$ associated to the representation $P' \times B'' \circ n_1$. We have a natural morphism $\tilde{N}_1 \rightarrow N_1$ to the odd nilpotent cone of $g(3)$. This is a particular case of Hesselink’s desingularization [H] (in particular, both $\tilde{N}_1$ and $N_1$ have dimension 13, cf. [E, row 8 of Table 6]). We are grateful to A. Elashvili and M. Jibladze for this observation.

References

[AG] D. Arinkin, D. Gaitsgory, Singular support of coherent sheaves and the geometric Langlands conjecture, Selecta Math. (N.S.) 21 (2015), no. 1, 1–199.

[BDFRT] A. Braverman, G. Dhillon, M. Finkelberg, S. Raskin, R. Travkin, Coulomb branches of noncotangent type (with appendices by Gurbir Dhillon and Theo Johnson-Freyd), arXiv:2201.09475.

[BF] R. Bezrukavnikov, M. Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction, Moscow Math. Journal 8 (2008), no. 1, 39–72.

[BFGT] A. Braverman, M. Finkelberg, V. Ginzburg, R. Travkin, Mirabolic Satake equivalence and supergroups, Compos. Math. 157 (2021), no. 8, 1724–1765.

[BFT] A. Braverman, M. Finkelberg, R. Travkin, Orthosymplectic Satake equivalence, arXiv:1912.01930.

[CFLW] S. Clark, Z. Fan, Y. Li, W. Wang, Quantum supergroups III. Twistors, Comm. Math. Phys. 332 (2014), no. 1, 415–436.

[DLYZ] G. Dhillon, Y.-W. Li, Z. Yun, X. Zhu, Endoscopy for affine Hecke algebras, in preparation.

[E] A. G. Elashvili, Stationary subalgebras of points of the common state for irreducible linear Lie groups, Funct. Anal. Appl. 6 (1972), no. 2, 139–148.

[FGT] M. Finkelberg, V. Ginzburg, R. Travkin, Mirabolic affine Grassmannian and character sheaves, Selecta Math. (N.S.) 14 (2009), no. 3-4, 607–628.

[FSS] L. Frappat, A. Sciarrino, P. Sorba, Dictionary on Lie algebras and superalgebras, Academic Press Inc., San Diego, CA (2000).
[Ga1] D. Gaitsgory, *Sheaves of categories and the notion of 1-affineness*, Contemp. Math. **643** (2015), 127–225.

[Ga2] D. Gaitsgory, *The local and global versions of the Whittaker category*, Pure Appl. Math. Q. **16** (2020), no. 3, 775–904.

[Gi] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, arXiv:alg-geom/9511007.

[GL] C. Gruson, S. Leidwanger, * Cônes nilpotentes des super algèbres de Lie orthosymplectiques*, Ann. Math. Blaise Pascal **17** (2010), no. 2, 303–326.

[H] W. H. Hesselink, *Desingularizations of Varieties of Nullforms*, Invent. Math. **55** (1979), 141–163.

[La] V. Lafforgue, *Correspondance Theta pour les D-modules*, unpublished manuscript.

[Ly] S. Lysenko, *Moduli of metaplectic bundles on curves and theta-sheaves*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 3, 415–466.

[LL] V. Lafforgue, S. Lysenko, *Geometric Weil representation: local field case*, Compos. Math. **145** (2009), no. 1, 56–88.

[MV] I. Mirković, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143; *Erratum*: Ann. of Math. (2) **188** (2018), no. 3, 1017–1018.

[M] I. Musson, *Lie superalgebras and enveloping algebras*, Graduate Studies in Mathematics **131**, AMS, Providence, RI (2012), xx+488pp.

[R] S. Raskin, *Homological methods in semi-infinite contexts*, https://web.ma.utexas.edu/users/sraskin/topalg.pdf (2019).

[TY] R. Travkin, R. Yang, *Untwisted Gaiotto equivalence*, arXiv:2201.10462.