Quantum vacua:  
Momentum space topology of fermion zero modes

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Abstract. Quantum vacua are characterized by the topological structure of their fermion zero modes. The vacua are distributed into universality classes protected by topology in momentum space. The vacua whose manifold of fermion zero modes has co-dimension 3 are of special interest because in the low-energy corner the fermionic excitations become the Weyl relativistic chiral fermions, while the dynamical bosonic collective modes of the fermionic vacuum interact with the chiral fermions as the effective gravity and gauge fields. The relativistic invariance, the chirality of fermions, the gauge and gravity fields, the relativistic spin, etc., are the emergent low-energy properties of the quantum vacuum with such fermion zero modes. The vacuum of the Standard Model and the vacuum of superfluid $^3$He-A belong to this universality class and thus they are described by similar effective theories. This allows us to use this quantum liquid for the theoretical and experimental simulations of many problems related to the quantum vacuum, such as the chiral anomaly and the cosmological constant problems.

1 Universality classes of fermionic vacua

1.1 Fermion zero modes of quantum vacuum

All known elementary particles and gauge bosons are either massless or have the mass which is extremely small compared to the Planck energy scale. This suggests that all the particles and fields originate from the fermionic or bosonic zero modes of the quantum vacuum. If so, our goal must be to describe and classify the possible zero modes of quantum vacua and their interactions. This does not require the knowledge of the exact ‘microscopic’ (trans-Planckian) structure of the quantum vacuum and can be done on the phenomenological grounds.

The classification of quantum vacua in terms of their zero modes is simplified if we assume that only fermions are fundamental, while the bosons are composite representing the collective modes of the fermionic quantum vacuum. As distinct from the symmetry classification of the vacuum states, it is the topology of the propagator of the fermionic field which distinguishes between different quantum vacua. The fermionic vacua and their fermion zero
modes are thus distributed into universality classes protected by momentum space topology. If the space dimension is $D=3$ and if we consider only translationally invariant vacua, there are only two universality classes of fermion zero modes which are protected by topology. The topologically stable zeros of the fermionic energy spectrum form in the 3D momentum space the manifolds of either co-dimension 1 or co-dimension 3. The universality class corresponding to co-dimension 1 contains the vacua whose fermion zero modes form the 2D surface in the 3D momentum space – the Fermi surface. The class of co-dimension 3 contains the vacua whose fermion zero modes are concentrated near the distinguished topologically stable points in the 3D momentum space – the Fermi points.

The elementary particles of our quantum vacuum originate from the massless chiral Weyl fermions of the Standard Model, left and right quarks and lepton. The Weyl fermions are typical representatives of the fermion zero modes of co-dimension 3. The Hamiltonian for the $a$-th chiral fermionic species is

$$\mathcal{H}_a = c C_a \sigma^i p_i ,$$

where $C_a = \pm 1$ is the chirality of the fermion. This Hamiltonian vanishes at the point $p = 0$ in 3D momentum space.

1.2 Momentum space topological invariant for Fermi point

To characterize this singular point in the 3D momentum space (or any other manifold of zeros of co-dimension 3) let us introduce a somewhat more general $2 \times 2$ Hamiltonian which does not obey the Lorentz invariance:

$$\mathcal{H} = \sigma^i M_i(p) .$$

Here $M$ is an arbitrary function of the momentum $p$. The vector field $M(p)$ can have hedgehogs – the points where $|M(p)| = 0$ and thus the fermionic energy $E^2(p) = |M(p)|^2 = 0$. Such a point is topologically stable if the following integral over the 2-surface $\sigma_2$ around this point is non-zero:

$$N_3 = \frac{1}{8\pi} \epsilon_{ijk} \int_{\sigma_2} d^2k \frac{1}{|M(p)|^3} M \cdot \left( \frac{\partial M}{\partial p_i} \times \frac{\partial M}{\partial p_j} \right) .$$

In the case of chiral fermions in Eq. (1), the field $M(p) = cp$ represents the hedgehog in momentum space, and the topological charge of this hedgehog coincides with the chirality of fermions, $N_3 = C_a = \pm 1$. Investigation of the topological properties of the Fermi points was started in 1981 for fermions on a lattice [1], where these points were referred to as generic degeneracy points, and in 1982 in superfluid $^3$He-A [2] where they were called boojums on Fermi surface. According to Abrikosov and Beneslavskii [3] and Nielsen and Ninomiya [4] such points can also appear in semiconductors as the crossing (diabolic) points of two energy bands. In condensed matter examples the spin is effective: the Pauli matrices $\sigma^i$ act in the space of two crossing bands.
2 Emergent relativistic quantum field theory

2.1 Emergence of chiral fermions, gauge field and gravity

The Fermi points with the elementary topological charge, $N_3 = \pm 1$, have a remarkable property. Near such a point $p_i^{(a)}$ the vector field $M(p)$ and the Hamiltonian have the following general form:

$$M_n(p) \approx e_i n(p_i - p_i^{(a)}) \quad , \quad H_a = e_i \sigma^a (p_i - p_i^{(a)}) . \quad (4)$$

After the shift of the momentum and the diagonalization of the $3 \times 3$ matrix $e_i^j$, one again obtains $H_a = C_a \epsilon^i p_i$, where the chirality $C_a$ is determined by the sign of the determinant of the matrix $e_i^j$. This means that fermions near such a Fermi point are always the Weyl fermions – the relativistic chiral fermions – even if the original system is not relativistic. The quantities $e_i^j$ (or the effective metric $g_{ik} = \sum_n e_i^n e_k^n$) and $p_i^{(a)}$ emerging at low energy are dynamical variables characterizing the bosonic collective modes of the fermionic vacuum. They interact with chiral fermions as the effective gravity and gauge field, respectively.

Thus the relativistic invariance, the chirality of fermions, the gauge and gravity fields, and also the relativistic spin are the emergent low-energy properties of the quantum vacuum with fermion zero mode of co-dimension 3, if its topological charge is elementary, $N_3 = +1$ or $N_3 = -1$.

2.2 Role of discrete symmetries: emergence of parity and non-Abelian local symmetry

What happens if the topological charge of the Fermi point is not elementary? For instance, the positions of two Fermi points with the charges $N_3 = -1$ and $N_3 = +1$ can coincide, so that the total topological charge of the singular point is trivial, $N_3 = 0$. It appears that all the above properties of the emerging relativistic quantum field theory will survive if there is some discrete symmetry between these two fermionic species which protects zeros of the Hamiltonian. In the low-energy ‘relativistic’ limit this discrete symmetry manifests itself as the space parity $P$ which transforms the left-handed fermion to the right-handed one. If this discrete symmetry is violated or spontaneously broken, the zeros in momentum space are protected neither by topology nor by symmetry; zeros disappear which means that in the relativistic low-energy limit the fermion zero modes acquire the Dirac mass. Such situation occurs in the Standard Model and in the so-called planar phase of superfluid $^3$He. In the planar phase, the corresponding space parity $P$ of the low-energy fermions is effective: it evolves from some approximate internal symmetry of the ‘high-energy’ atomic physics of the quantum liquid.

Another important role of discrete symmetries shows up if we consider the Fermi point with the multiple charge, say $N_3 = +2$. If there is a proper discrete
symmetry between the fermions, the singular point becomes the combination of two elementary Fermi points describing two right-handed fermionic species coupled by this symmetry. In this case the effective relativistic invariance is not distorted, but in addition one finds that the double degenerate Fermi point generates the effective non-Abelian gauge field. Since the positions $p^{(1)}$ and $p^{(2)}$ of constituent Fermi points with $N_3 = +1$ can oscillate separately, the $4 \times 4$ Hamiltonian for two fermionic species becomes

$$\mathcal{H} = \epsilon^i_n \sigma^n (p_i - A_i - \tau_b A^b_i) .$$

(5)

Here $\tau_b$ is the Pauli matrix in the isotopic space of two species of fermion zero modes, and the new collective mode $A^b_i$ plays the role of the Yang-Mills field. Such situation occurs in the Standard Model and in the superfluid $^3$He-A. The higher discrete symmetry groups of the Fermi points can generate the higher-order effective local symmetry in the fermionic sector. For example the Fermi point with $N_3 = +4$ and with the discrete symmetry $Z_4$ or $Z_2 \times Z_2$ will lead to the local $SU(4)$ symmetry in the low-energy corner.

According to Eq. (5), in the vicinity of Fermi points the fermionic sector acquires the local $U(1)$ and non-Abelian symmetry groups, and also the general covariance. This, however, does not imply that such symmetries will automatically emerge in the bosonic sector, i.e. in the effective action for the collective fields $A_i$, $A^b_i$ and $\epsilon^i_n$. But this is possible at least in principle, since the corresponding action for the bosonic fields come from the integration over the fermionic fields in the same manner as in Sakharov’s induced gravity [5]. The problem is that because of the ultraviolet divergences the bosonic action is generated not only by fermion zero modes but also by the high-energy degrees of freedom of the fermionic vacuum which are not necessarily Lorentz invariant (see the detailed discussion in Ref. [6]).

2.3 Chiral anomaly

There are some properties of the quantum vacuum which do not depend on whether the bosonic sector of the effective theory has all the symmetries of the fermionic sector or not. An example is provided by the chiral anomaly, which is completely determined by the spectral flow through the Fermi point and thus by its momentum space topology. The fermionic charge $B$ produced per unit time per unit volume from the vacuum in the presence of the effective magnetic and electric fields is given by the following generalization of the equation derived by Adler [7] and Bell and Jackiw [8] for axial anomaly:

$$\dot{B} = \frac{1}{16\pi^2} F_{\mu\nu} F^{*\mu\nu} \sum_a N_{3a} B_a q_a^2 .$$

(6)

Here $q_a$ is the charge of the $a$-th fermion with respect to the effective gauge field $F^{*\mu\nu}$; $F^{*\mu\nu} = (1/2) \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}$ is the dual field strength; and $N_{3a}$ is the topological charge of the Fermi point. This relativistic, gauge invariant and
covariant equation is applicable to all systems with Fermi points including $^3$He-A where the bosonic sector certainly does not obey these symmetries. In $^3$He-A the Adler–Bell–Jackiw equation has been verified in experiments with continuous vortex-skyrmions [9].

3 Vacuum energy

The fermion zero modes of quantum vacuum can say nothing on the microscopic properties of the quantum vacuum, such as the value of the vacuum energy and thus of the cosmological constant $\rho_{\Lambda}$. The naive estimation of the vacuum energy density, as the zero-point energy $(1/2)E(p)$ of bosonic modes plus the negative energy of fermion modes in the Dirac sea, gives

$$\rho_{\Lambda}\sqrt{g} = \frac{1}{V} \left( \nu_{\text{bosons}} \sum_p \frac{1}{2} cp - \nu_{\text{fermions}} \sum_p cp \right), \tag{7}$$

where $V$ is the volume of the system; $\nu_{\text{bosons}}$ and $\nu_{\text{fermions}}$ is the number of bosonic and fermionic zero modes. Because of the ultraviolet divergence this estimate is in a huge disagreement with observations which is known as the cosmological constant problem.

To calculate the real energy of the vacuum state one must know the ultraviolet (microscopic) structure of the given vacuum. We have no information on our quantum vacuum, but we can consider as a guide the well known quantum vacua of the same universality class. The result is rather unexpected and so general that it can be applicable to any vacuum. It appears that the contribution of zero modes in Eq. (7) is completely cancelled by the contribution from all other microscopic degrees of freedom without any fine tuning. Applying this to our vacuum one may conclude that the equilibrium quantum vacuum does not gravitate.

This cancellation is the result of the local stability of the vacuum state, and it is exact if the vacuum is not perturbed. The vacuum responds to perturbations, and the energy density of the perturbed vacuum is on the order of the energy density of perturbations. Thus the cosmological constant is not a constant at all but is the dynamical quantity which is either continuously or in a stepwise manner adjusted to perturbations. Since all the perturbations of the vacuum in the present Universe (gravitating matter, expansion, curvature, non-zero temperature, inhomogeneity, etc.) have energy scales much lower than the Planck scale, our vacuum is extremely close to the local equilibrium, and thus the dark energy today is extremely small.

What happens if a phase transition occurs in which the symmetry of the vacuum is broken, as is supposed to happen in the early Universe when, say, the electroweak symmetry was broken? In the effective theory, such a transition must be accompanied by a change of the vacuum energy, which means that the vacuum must have a huge energy either above or below the phase transition.
However, the exact microscopic theory suggests the phase transition does not disturb the zero value of the vacuum energy. After the vacuum relaxes to a new equilibrium state, its energy density will be zero again. The energy change is completely compensated by the change of the internal (microscopic) quantities characterizing the microscopic structure of the vacuum. In the quantum vacuum of quantum liquids this is the chemical potential and density of the atoms of the liquid [6].

4 Other systems

We considered here the effective theory of fermion zero modes emerging in the low-energy corner of the quantum vacuum belonging to the universality class of the co-dimension 3. This class is of special interest because the corresponding effective theory is the quantum field theory of chiral fermions interacting with gauge fields and gravity. However, in the condensed matter physics the more popular is the universality class of the co-dimension 1. It contains the Fermi liquids with Fermi surfaces. The effective theory of such vacua has been constructed by Landau [10] and is known now as the Landau theory of Fermi liquids. A similar effective theory emerges for fermion zero modes in the core of quantized vortices; in the mixed state of superconductors; and in relativistic theories in the presence of strong fields, for instance, in the vicinity and beyond the black-hole horizon. In multi-dimensional systems the higher universality classes are also topologically protected, such as the class of co-dimension 5; in addition the fermion zero modes of the co-dimension 3 appear on branes, and all the properties of the relativistic quantum field theory emerge for the brane matter at low energy. In the systems with even space dimension the fully gapped vacua also have non-trivial momentum space topology, which leads to quantization of physical parameters, and other exotic phenomena. Discussion of all these universality classes of quantum vacua can be found in Ref. [6].

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