NO EIGENVALUE IN FINITE QUANTUM ELECTRODYNAMICS

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Abstract

We re-examine Quantum Electrodynamics (QED) with massless electron as a finite quantum field theory as advocated by Gell-Mann-Low, Baker-Johnson, Adler, Jackiw and others. We analyze the Dyson-Schwinger equation satisfied by the massless electron in finite QED and conclude that the theory admits no non-trivial eigenvalue for the fine structure constant.

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1 Introduction

Subsequent to the work of Gell-Mann and Low [1] in their classic 1954 paper, the possibility that spin-$\frac{1}{2}$ Quantum Electrodynamics (QED) may be a finite quantum field theory has been investigated by Johnson, Baker and Wiley [2] and by Adler [3]. The basic premise of such a finite field theory is that the renormalization constants of QED, to wit, the electron bare mass $m_0$, the electron wave function renormalization constant $Z_2$ and the photon wavefunction renormalization constant $Z_3$ tend to finite limits as the cut-off $\Lambda$ used in the calculations of the theory tends to infinity. It has been shown by Johnson et al that the renormalization constant $Z_2$, a gauge variant quantity, can be rendered finite by an appropriate choice of gauge, while the gauge invariant quantity $m_0$ obeys the scaling law

$$m_0(\Lambda) = \text{const.} \left( \frac{\Lambda^2}{m^2} \right)^{-\epsilon}, \quad \epsilon = \frac{3}{2} \left( \frac{\alpha_0}{2\pi} \right) + \frac{3}{8} \left( \frac{\alpha_0}{2\pi} \right)^2 + \cdots$$

(1)

where $\alpha_0$ is the bare fine-structure constant in QED, realized as an eigenvalue of the Gell-Mann-Low equation, $\psi(x) = 0$ at $x_0 = \alpha_0$. The bare mass $m_0$ thus vanishes as $\Lambda \rightarrow \infty$, provided $\epsilon > 0$. This implies that the physical mass $m$ arises solely from its interaction. With $m_0 = 0$, as $\Lambda \rightarrow \infty$, the axial-vector current $J_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$ is formally conserved, except for the Bell-Jackiw-Adler [4] anomaly: $\partial^\mu J_5^\mu = 2im_0J_5^5 = 0$. It was shown by Baker and Johnson [5] in 1971 that although the bare mass tends to zero in the limit of infinite cut-off, the matrix elements of $J_5^\mu$ diverge in just such a fashion as to render the matrix elements of $\partial^\mu J_5^\mu$ nonvanishing and finite. In this case therefore the chiral symmetry is explicitly broken in spite of the vanishing bare mass. Hence there is no Nambu-Goldstone boson and such a zero mass state is not observed in nature either.

To demonstrate that one can construct a finite theory of QED that will allow non-trivial eigenvalues of the Gell-Mann-Low equation has remained somewhat of an outstanding problem. In our earlier work on the subject [6], we had concluded that such a self-consistent finite theory does not exist, reinforcing the 1979 conjecture of Baker and Johnson [7, 8]. On the other hand there have been other investigations which conjecture that such a theory is feasible, especially the existence of a strong coupling phase of QED [9, 10, 11]. We thought it worthwhile to re-examine the question of non-trivial solutions of the Gell-Mann-Low function in the finite QED theory, and embark on another careful investigation of the standard finite QED on the basis of the Dyson-Schwinger equation.

In this investigation we shall begin with a solution to the Ward-Takahashi (WT) identity satisfied by the vector vertex function in QED and develop the Dyson-Schwinger equation satisfied by the
electron propagator by employing the gauge technique. This will be presented in Sec.2. We shall
circumvent the problems introduced by the Bell-Jackiw-Adler anomaly by not dealing with the axial-
vector current. We shall be working in the Landau gauge throughout. The solution to the WT
identity for the vector vertex function in QED is arbitrary up to the existence of an undetermined
transverse part, as is well-known. We shall not neglect such a transverse part and we shall see
that it plays a crucial role in our investigation. In Sec.3, we shall present a complete and detailed
analysis of the Dyson-Schwinger equation for the electron inverse propagator which will lead to
an examination of the eigenvalue problem in QED. The last section, Sec.4 will be devoted to the
conclusion that there is no non-trivial solution to the eigenvalue in finite QED followed by some
important remarks.

2 Dyson-Schwinger equation at the QED fixed point

To investigate the Dyson-Schwinger equation of QED at the Gell-Mann-Low fixed point, we begin
with the renormalized electron propagator whose general invariant form

\[ S^{-1}(p) = \frac{1}{p^2} A(p^2) + \Sigma(p^2), \]

follows from Lorentz invariance and parity invariance (PT invariance will suffice [12]). In the finite
QED theory, we set \( \Sigma(p^2) \equiv 0 \), for any \( p \) and chiral symmetry remains an exact symmetry. The
proper renormalized vertex function in QED satisfies the Ward-Takahashi identity

\[ (p - p')^\mu \Gamma_\mu(p, p') = S^{-1}(p) - S^{-1}(p'). \]

We would like to refer the reader to our earlier work [6] where we have reviewed the well-known
consequences of the Gell-Mann-Low eigenvalue equation \( \psi(x) = 0 \), which may or may not have a
non-trivial zero at \( x_0 = \alpha_0 \) at the position of the bare fine structure constant of QED, \( \alpha_0 = Z^{-1} \alpha \).
The important premise of the finite theory of QED is that the position of the zero, \( x = x_0 \) can
be determined by working with QED with zero physical mass [2]. This is predicated upon the
application of Weinberg’s theorem [13], which ensures that terms vanishing asymptotically in each
order of perturbation theory in the massive case do not sum to dominate over the asymptotic parts.

It can be easily shown that at the Gell-Mann-Low fixed point with \( m = 0 \) in finite QED, the
full, exact, renormalized electron propagator has the simple scaling form [14]

\[ S^{-1}(p) = \frac{1}{p^2} \phi A(p^2) = \frac{1}{p^2} \phi \left( \frac{p^2}{\mu^2} \right)^\gamma \]
where \( \gamma(\alpha) \) is the anomalous dimension of the electron in the massless theory given in the Landau gauge by

\[
\gamma = \mu \frac{\partial}{\partial \mu} \ln Z_2 = O(\alpha^2) + \cdots.
\]  

(5)

and \( \mu \) is the subtraction point. This can be established as follows. If we begin with the Callan-Symanzik renormalization group equation \([15]\), specialize to the Landau gauge and set \( m = 0 \) (massless electron) and \( \beta(\alpha) = 0 \) at the fixed point, then we have the equation satisfied by the two-point function which is essentially the same as the inverse electron propagator

\[
\left( \mu \frac{\partial}{\partial \mu} + 2\gamma \right) \Gamma^{(2)}(p, \alpha, 0, \mu, 0) = 0.
\]  

(6)

The solution for the two-point function can be expressed as

\[
A(p^2) = \left( \frac{p^2}{\mu^2} \right)^\gamma.
\]  

(7)

This is customarily expressed in terms of the Euclidean momenta, as a function of \((-p^2)\), which is how we shall employ it later in this investigation.

This can be confirmed by examining the trace anomaly in QED \([16]\). At a fixed point, \( \beta(\alpha) = 0 \), when we set the physical electron mass equal to zero, we have for the divergence of the scale current:

\[
\partial^\mu D_\mu = \frac{\beta(\alpha)}{2\alpha} F^\mu_\nu F^\nu_\mu + [1 + \gamma_\theta(\alpha)] m \bar{\psi} \psi = 0
\]  

(8)

and hence scale invariance becomes exact. We further assume that scale invariance is not spontaneously broken, \( Q_D|0 > = 0 \), where

\[
Q_D = \int d^3x D_0(x, t),
\]  

(9)

\((i.e., no dilatons are present in QED). Therefore Eq.(4) follows.

We now consider the Dyson-Schwinger equation satisfied by the inverse of the full, exact, renormalized electron propagator in \( m = 0 \) QED:

\[
S^{-1}(p) = Z_2 \not{p} - i Z_2 e^2 (2\pi)^{-4} \int d^4k \gamma^\mu D^\nu_{\mu\nu} \not{k} S(k) \Gamma^\nu(p, k).
\]  

(10)
It is well-known that in a theory of $m = 0$, spin-$\frac{1}{2}$ QED, the full, exact, renormalized photon propagator is exactly given by the free photon propagator, as established by Eguchi [17] and thus we have the form for the photon propagator in the Landau gauge:

$$D_{\mu \nu}(q) = \left( \frac{g_{\mu \nu}}{q^2} - g_{\mu \nu} \right) \frac{1}{q^2} \tag{11}$$

where $q_\mu = p_\mu - k_\mu$. The solution to the Ward-Takahashi identity satisfied by the renormalized, proper vector vertex function in QED can be determined in the standard manner by the gauge technique [18] to yield

$$\Gamma^\nu = \Gamma^\nu_L + \Gamma^\nu_T, \tag{12}$$

where the longitudinal part of the vertex function admits the general, kinematical singularity-free solution [19, 20] given by

$$\Gamma^\nu_L(p,k) = \frac{1}{2} \left( A + \tilde{A} \right) \gamma^\nu + \frac{\Sigma - \tilde{\Sigma}}{k^2 - p^2} \left( \gamma^\nu \phi + \bar{k} \gamma^\nu \right) - \frac{1}{2} \frac{\left( A - \tilde{A} \right)}{k^2 - p^2} \left[ 2 \bar{\phi} \gamma^\nu k + (p^2 + k^2) \gamma^\nu \right], \tag{13}$$

and the transverse piece obeys the condition

$$(p - p')_\mu \Gamma^\mu_{T}(p, p') = 0 \tag{14}$$

and consequently undetermined by the Ward-Takahashi identity. Here we have employed the notation $\tilde{A} = A(k^2)$ etc. The transverse piece may be expressed by a kinematical-singularity free decomposition in terms of invariant functions but we find it is not necessary to do so at this point. However it must be stressed that we shall retain it throughout our calculation. We shall see that, undetermined and arbitrary as it is, the transverse piece of the vertex has a significant bearing on our conclusions. In massless finite QED, when the chiral symmetry is exact, the above solution for the vertex function reduces to

$$\Gamma^\nu(p, k) = \frac{1}{2} \left( A + \tilde{A} \right) \gamma^\nu - \frac{1}{2} \frac{\left( A - \tilde{A} \right)}{k^2 - p^2} \left[ 2 \bar{\phi} \gamma^\nu k + (p^2 + k^2) \gamma^\nu \right] + \Gamma^\nu_{T}(p, k). \tag{15}$$

The Dyson-Schwinger equation, Eq.(10) reduces to

$$S^{-1}(p) = \phi A(p^2) = Z_2 \phi - i Z_2 e^2 (2\pi)^{-4} \int d^4 k \left( \frac{g_{\mu \nu}}{q^2} - \gamma_{\mu \nu} \right) \frac{1}{q^2 k^2 A} \left[ \Gamma^\nu_L(p,k) + \Gamma^\nu_{T}(p,k) \right]. \tag{16}$$
We may now multiply the above equation by $\rho$, divide by $4p^2$ and evaluate the trace over the Dirac matrices. After a tedious computation, we arrive at the following equation:

$$A(p^2) - Z_2 = iZ_2 e^2 (2\pi)^{-4} \int d^4k \frac{1}{p^2 k^2 A(k^2)} \left\{ \frac{1}{2q^4} (A + \tilde{A})[2p^2 k^2 - (p \cdot k)(p^2 + k^2)] + \frac{1}{q^4 k^2 - p^2} (p \cdot k)(p^2 - k^2)^2 \right. \\
- (A + \tilde{A}) \frac{(p \cdot k)}{q^2} - \frac{(A - \tilde{A})}{q^2 (k^2 - p^2)} \left[ 4(p \cdot k)^2 - (p \cdot k)(p^2 + k^2) \right] + \Gamma_1^T(p^2, k^2, p \cdot k) \right\}, \quad (17)$$

where the last term, $\Gamma_1^T$ arises from the trace calculation of the term containing the transverse vertex piece.

We observe that this is a non-linear integral equation satisfied by the function $A(p^2)$ and it is the exact consequence of the Dyson-Schwinger equation for the electron propagator in finite QED at the Gell-Mann-Low fixed point since we have not introduced any approximations and we have not discarded the transverse piece.

To proceed further, we need to evaluate the various angular integrals appearing in the above equation. For this purpose, let us first transform to Euclidean momenta by the following transformations:

$$d^4k \rightarrow i d^4k = i \int k^3dkd\Omega; \quad p^2 \rightarrow -p^2; \quad k^2 \rightarrow -k^2; \quad p \cdot k \rightarrow -p \cdot k. \quad (18)$$

Identifying the various angular integrals which occur as $I_1, I_2, \cdots$ which are defined in Appendix A, we obtain

$$A(-p^2) - Z_2 = -Z_2 e^2 (2\pi)^{-4} \int d^4k \frac{1}{p^2 k^2 A(-k^2)} \left\{ \frac{1}{2}(A + \tilde{A}) \left[ 2p^2 k^2 I_4 - (p^2 + k^2)I_5 \right] - (p^2 - k^2)(A - \tilde{A})I_5 \right. \\
- (A + \tilde{A})I_2 - \frac{(A - \tilde{A})}{(p^2 - k^2)} \left[ 4I_3 - (p^2 + k^2)I_2 \right] + \Gamma_1^T(-p^2, -k^2, -p \cdot k) \right\}. \quad (19)$$

It is understood that all momenta are Euclidean, defined by $p_E = \sqrt{-p^2}$, $k_E = \sqrt{-k^2}$ in what follows. Henceforward we shall drop the subscript $E$ for Euclidean momenta in order to avoid
Performing the angular integrations and making use of the results in Appendix A [21], we arrive at the following result

\[
A(p^2) - Z_2 = \frac{Z_0 e^2}{16\pi^2} \int_0^\infty k \frac{dk}{p^2 A(k^2)} \left\{ \frac{A(p^2) - A(k^2)}{(p^2 - k^2)} \left[ \frac{2\sigma^2(p^2 - k^2)^2}{p_2^2(1 - \sigma^2)} - p_2^2(1 + \sigma^2) \right] + \Gamma^2_T(p^2, k^2) \right\},
\] (20)

where \(\Gamma^2_T(p^2, k^2)\) arises from the angular integral of the transverse vertex part, and \(\sigma = p_</p_\) and \(p_< = \min\{p, k\}, p_> = \max\{p, k\}\) (see Appendix A). This equation contains the essential ingredients of finite QED, with no approximations nor additional assumptions. Now in order to ascertain whether the Dyson-Schwinger equations of finite theory of QED, as we have developed thus far, admit of a solution to the Gell-Mann-Low eigenvalue equation, we proceed as follows. We may check the self-consistency of the theory constructed in this manner, of massless QED at the fixed point by making the replacement

\[
A(p^2) = \left(\frac{p^2}{\mu^2}\right)^\gamma
\] (21)
in accordance with Eq.(11) where \(\gamma = \gamma(\alpha)\) is the QED anomalous dimension in the \(m = 0\) theory. After some algebra, we thus obtain

\[
\left(\frac{p^2}{\mu^2}\right)^\gamma - Z_2 = \frac{Z_0 e^2}{16\pi^2} \int_0^p k \frac{dk}{p^2 A(k^2)} \left\{ \left[ (p^2/k^2)^\gamma - 1 \right] \frac{(p^2k^2 - 3k^4)}{p^4(p^2 - k^2)} + \Gamma^3_T(p^2, k^2) \right\} + \frac{Z_0 e^2}{16\pi^2} \int_p^\infty k \frac{dk}{p^2 A(k^2)} \left\{ \left[ (p^2/k^2)^\gamma - 1 \right] \frac{(p^2k^2 - 3p^4)}{p^4k^2(p^2 - k^2)} + \Gamma^4_T(p^2, k^2) \right\},
\] (22)

where \(\Gamma^3_T\) and \(\Gamma^4_T\) represent the contributions arising from the transverse piece vertex function. With a change in variables, \(s = p^2, k^2 = sx\), this can be rewritten in the form

\[
\left(\frac{p^2}{\mu^2}\right)^\gamma - Z_2 = \frac{Z_0 e^2}{32\pi^2} \int_0^1 dx \left\{ \frac{x(1 - 3x)(x^{-\gamma} - 1)}{(1 - x)} + \Gamma^5_T(s, x) \right\} + \frac{Z_0 e^2}{32\pi^2} \int_1^\infty dx \left\{ \frac{x(x - 3)(x^{-\gamma} - 1)}{(1 - x)} + \Gamma^6_T(s, x) \right\},
\] (23)

where \(\Gamma^5_T\) and \(\Gamma^6_T\) are contributions arising from the transverse vertex piece. For general values of \(\gamma\), we can evaluate the integrals [22] and we obtain the result
in terms of the hypergeometric functions, where $\Gamma^7_T(s)$ arises from the integral of the contribution from the transverse vertex piece.

If we evaluate Eq.\((24)\) at $s = \mu^2$, we obtain

$$Z_2 = 1 - \frac{Z_2 e^2}{32\pi^2} \left\{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) - 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) + F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma^7_T(\mu^2) \right\},$$

which can be rewritten as

$$Z_2^{-1} = 1 + \frac{e^2}{32\pi^2} \left\{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) - 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) + F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma^7_T(\mu^2) \right\},$$

This has been obtained in the Landau gauge to all orders in $\alpha$. We are now ready to analyze the results contained in Eqs.\((24, 26)\), a major consequence of the Dyson-Schwinger equations of finite QED with massless electron at a fixed point and draw conclusions.
3 Conclusion and Summary

Let us examine Eq. (24) and determine what are the allowed values of $\gamma$, the anomalous dimension in massless QED. From Eqs. (24) and (25), we obtain

$$\left(\frac{s}{\mu^2}\right)^\gamma = 1 - \frac{Z_2 e^2}{32\pi^2} \left\{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\
- 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\
+ F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_7^T(\mu^2) \right\}$$

which simplifies to

$$\left(\frac{s}{\mu^2}\right)^\gamma = 1 - \frac{Z_2 e^2}{32\pi^2} \left\{ \Gamma_7^T(\mu^2) - \Gamma_7^T(s) \right\}. \quad (27)$$

We note that either $\gamma \neq 0$ or $\gamma = 0$. Let us first consider the case $\gamma \neq 0$. Since $\gamma$ is a finite function of $\alpha$, the left hand side of Eq. (28) is finite. On the other hand, the right hand side is unity since, according to Eq. (26), $Z_2^{-1}$ diverges, which means that $Z_2^{-1}$ is infinite for arbitrary non-zero values of $\gamma$. Eq. (28) cannot be satisfied unless $\gamma = 0$. We are therefore forced to conclude [24] that $\gamma = 0$ and thus $\gamma = 0$ is the only solution. Incidentally, one can show explicitly that the divergent terms of the longitudinal part in Eq. (26) do not cancel except when $\gamma = 0$. (see Appendix B for details). With the choice $\gamma = 0$, it then follows from Eq. (26) that $Z_2^{-1}$ is finite and

$$Z_2^{-1} = Z_2^{-1}(\mu^2) = 1 + \frac{e^2}{32\pi^2} \Gamma_7^T(\mu^2) \quad (29)$$

and hence we determine that $\Gamma_7^T(\mu^2)$ is finite.

When $\gamma = 0$, we observe from Eq. (28), that

$$\frac{Z_2 e^2}{32\pi^2} \left\{ \Gamma_7^T(\mu^2) - \Gamma_7^T(s) \right\} = 0, \quad (30)$$

where $Z_2$ is given by Eq. (29) and hence $\Gamma_7^T(s)$ is finite. The only way to obtain a non-trivial eigenvalue, $e^2 \neq 0$, is if $\Gamma_7^T(s) = \Gamma_7^T(\mu^2)$. If the latter were valid, then this transverse vertex piece is
a constant, independent of s which amounts to an artificial and contrived constraint on the arbitrary and unknown transverse vertex part. We therefore conclude that the fine structure constant has only the trivial eigenvalue.

Returning to our main result, we see that $\gamma = 0$ is based on the non-pertubative method stemming from the Dyson-Schwinger equations, and is thus valid to all orders in $\alpha$ in the Landau gauge. It therefore follows that for weak coupling ($\alpha << 1$), $\alpha$ must vanish identically since $\gamma$ in the Landau gauge is of order $\alpha^2$. Next we may raise the question of what happens in strong coupling QED. Now we appeal to the gauge independence of the result. Since $\beta(\alpha)$ is gauge independent (since $Z_3$ is gauge independent) but $\gamma(\alpha)$ is gauge dependent (since $Z_2$ is gauge-dependent), the vanishing of $\beta(\alpha)$ is valid in all gauges, but if $\gamma = 0$ in one gauge, it need not be so in other gauges. It then follows that the only gauge independent solution which is a simultaneous zero of both $\beta(\alpha) = 0$ and $\gamma(\alpha) = 0$ is $\alpha = 0$ which is the trivial solution. We cannot invoke the Federbush-Johnson theorem [24] which only applies to the gauge independent sector [25] in QED. Since the gauge technique employed in this investigation is non-perturbative, we reiterate that our conclusion is valid for arbitrarily strong coupling [9, 10, 11].

We may recall that Adler et al [8] and Baker-Johnson [7] had argued from the triangle anomaly [26, 27] that the Gell-Mann-Low function cannot have a zero at all. It is interesting that our present investigation establishes the absence of a non-trivial zero of the Gell-Mann-Low function by an entirely different approach [28], in which the existence of the non-vanishing transverse vertex part plays a crucial role.

We have carried out our investigation in the Landau gauge. The theory of QED is gauge invariant in content and we believe that our conclusions should prevail in any gauge. The form of the solution $A(p^2) = (p^2/\mu^2)^\gamma$ remains valid in all gauges in the minimal subtraction scheme [29] at $\beta(\alpha) = 0$. It would be interesting and worthwhile, however, to demonstrate the manifest gauge invariance of the theory. This complicated extension is under investigation and will be reported in a future communication.
Appendix A: Angular integrals

The computation of the right hand side of Eq. (19) requires the evaluation of several angular integrals. A few but not all of these integrals are contained in [21]. We employ an expansion in terms of the Gegenbauer functions,

$$\frac{1}{|p-k|^2} = \sum_{n=0}^{\infty} \frac{1}{p_>^n} \sigma^n C_n^1(x); \quad x = \cos \theta; \quad \sigma = \frac{p_<}{p_>},$$

where $p_> = \max\{p,k\}, p_< = \min\{p,k\}$. The evaluation of the angular integrals is facilitated by employing the standard properties of the Gegenbauer functions [30], including in particular, the following:

$$\int_{-1}^{1} dx \sqrt{1-x^2} C_n^1(x) = \frac{\pi}{2} \delta_{n0},$$

$$\int_{-1}^{1} dx \sqrt{1-x^2} C_n^1(x) C_m^1(x) = \frac{\pi}{2} \delta_{mn},$$

$$xC_n^1(x) = \frac{1}{2} \left[ C_{n+1}^1 + C_{n-1}^1(x) \right]. \quad (31)$$

The integrals identified below can thus be evaluated and the results are as follows.

$$I_1 = \int \frac{d\Omega}{|p-k|^2} = 2\pi^2 \frac{1}{p_>^2},$$

$$I_2 = \int \frac{p \cdot kd\Omega}{|p-k|^2} = \pi^2 \sigma^2,$$

$$I_3 = \int \frac{(p \cdot k)^2 d\Omega}{|p-k|^2} = \frac{1}{2} \pi^2 p_<^2 (1 + \sigma^2),$$

$$I_4 = \int \frac{d\Omega}{|p-k|^4} = \frac{2\pi^2}{p_>^4} \frac{1}{1 - \sigma^2},$$

$$I_5 = \int \frac{p \cdot kd\Omega}{|p-k|^4} = \frac{2\pi^2 \sigma^2}{p_>^2 (1 - \sigma^2)}.$$

The following notations have been used here: $p_> = \max\{p,k\}, p_< = \min\{p,k\}, \sigma = p_</p_>$. It is understood that $p, k$ stand for the Euclidean lengths $p_E = \sqrt{-p^2}, k_E = \sqrt{-k^2}$ respectively.
Appendix B: Analysis of the hypergeometric functions

Here we shall analyze the behavior of the hypergeometric functions appearing in Eq. (26). For this purpose we focus our attention on the coefficient of \((e^2/32\pi^2)\) which is manifestly divergent on the right hand side of the equation. This may be expressed in terms of the gamma functions as follows:

\[
\frac{3}{\Gamma(4)} \frac{\Gamma(\epsilon)}{\Gamma(1)} - \frac{\Gamma(3)}{\Gamma(1)} \frac{\Gamma(\epsilon)}{\Gamma(1)} + \frac{\Gamma(3 - \gamma)}{\Gamma(1)} \frac{\Gamma(\epsilon)}{\Gamma(1)} - \frac{3}{\Gamma(4 - \gamma)} \frac{\Gamma(\epsilon)}{\Gamma(1)} \\
+ \frac{3}{\Gamma(1)} \frac{\Gamma(-1 + \epsilon)}{\Gamma(-1 + \epsilon)} - \frac{\Gamma(-1 + \epsilon)}{\Gamma(-2 + \epsilon)} \frac{\Gamma(\epsilon)}{\Gamma(1)} + \frac{\Gamma(\gamma - 1)}{\Gamma(\gamma - 1)} \frac{\Gamma(\epsilon)}{\Gamma(1)} - \frac{3}{\Gamma(4 - \gamma)} \frac{\Gamma(\epsilon)}{\Gamma(1)} + \Gamma_T(\mu^2)
\]

where \(\lim \epsilon \to 0\) is understood. Each of the divergent terms can be dealt with by using the identity derived by Ryder [31] for the divergent gamma function

\[\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\epsilon} + \psi(n + 1) + O(\epsilon) \right],\]

where

\[\psi(n + 1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma_E,\]

is the logarithmic derivative of the gamma function [30] and

\[\gamma_E = \lim_{n \to \infty} \left( \sum_{m=1}^{n} \frac{1}{m} - \ln n \right) = 0.5772156649 \cdots\]

is the Euler-Mascheroni constant. The conclusion stated in the beginning of Section 3 immediately follows, namely: \(Z_{2}^{-1}\) is infinite when \(\gamma \neq 0\); \(Z_{2}^{-1}\) is finite only if \(\gamma = 0\). Note that \(\Gamma_T(\mu^2)\) is finite when \(\gamma = 0\).
References and Footnotes

1. M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954); N.N.Bogoliubov and D.Shirkov, *Introduction to the theory of quantized fields*, Interscience, New York, 1959; K. Wilson, Phys. Rev. D3, 1818 (1971).

2. K. Johnson and M. Baker, Phys. Rev. D8 1110 (1973) and references cited therein.

3. S.L.Adler, Phys. Rev. D5, 3021 (1972); J. Bernstein, Nucl.Phys.B95, 461 (1975).

4. J.S.Bell and R.Jackiw, Nuovo Cimento A60, 47 (1969); S.L.Adler, Phys. Rev. 177, 2426 (1969); R.Jackiw and K.Johnson, Phys.Rev.182, 1459 (1969); S.L.Adler and W.Bardeen, Phys. Rev. 182, 1517 (1969); C.R.Hagen, Phys. Rev. 177, 2622 (1969); B. Zumino, *Proceedings of the Topical conference on Weak Interactions*, CERN, Geneva 1969), p.361.

5. M.Baker and K.Johnson, Phys. Rev. D3, 2516 (1971).

6. R.Acharya and P. Narayana Swamy, Nuovo Cimento A103, 1131 (1990).

7. M. Baker and K.Johnson, Physica A 96, 120 (1979). See also N. Krasnikov, Phys.Lett. B225, 284 (1989).

8. S.L.Adler, C.Callan, R.Jackiw and D.Gross, Phys.Rev. D6, 2982 (1972).

9. V.Miransky, Nuovo Cimento A90, 149 (1985).

10. C.Leung, S.Love and W.Bardeen, Nucl.Phys.B273, 649 (1986).

11. J.Kogut, E.Dagotto and A.Kocic, Phys.Rev.Lett. 62, 1001 (1989); K.-I. Kondo, International Journal of Modern Physics A 11, 77 (1996) and references therein.

12. J.Bernstein, *Elementary Particles and their Currents*, W.H.Freeman and Co., 1968.

13. S. Weinberg, Phys.Rev. 118, 838 (1960).

14. See e.g., S.L.Adler and W.Bardeen, Phys.Rev. D4 3045 (1971).

15. C. Callan, *Summer School of Theoretical Physics, Les Houche 1971* editors C.Dewitt and C.Itzykson. Gordon Breach publishers (1973) New York; see also S. Weinberg, Phys.Rev. D8, 3497 (1973)
16. S.Adler, J.C.Collins and A.Duncan, Phys. Rev. D15, 1712 (1977).
17. T.Eguchi, Phys.Rev.D17, 611 (1978).
18. A.Salam, Phys.Rev.130, 1287 (1963); R.Delburgo and P. West, Phys.Lett.B72, 3413; ibid J.Phys.A 10, 1049 (1977). See also D.W.Atkinson and H.A.Slim, Nuovo Cimento A50, 555 (1979).
19. J.Ball and F.Zachariasen, Phys.Lett.B 106, 133 (1981); J.Ball and T.Chiu, Phys.Rev. D 22, 2542 (1980)
20. It is more expedient to express the solution in terms of the functions A and Σ: see e.g., R. Acharya and P. Narayana Swamy, Nuovo Cimento A98, 773 (1987).
21. Some of these results are contained in R.Arnowitt and S.Deser, Phys.Rev 138B , 712 (1965)
22. I.S.Gradshteyn and I.M.Ryzhik, Table of Integrals, Series and Products, Academic Press, New York (1980).
23. This circumstance in finite QED is not to be confused with the fact that $Z_2$ is not finite in the Landau gauge beyond the second order in perturbation theory.
24. P.G. Federbush and K. Johnson, Phys.Rev.120, 1926 (1960).
25. F. Strocchi, Phys.Rev.D6, 1193 (1972): see in particular footnote 12.
26. N. Christ, Phys.Rev. D4, 946 (1973).
27. See also K.Huang, in Asymptotic Realms of Physics, edited by A.Guth et al, M.I.T.Press, Cambridge (1983).
28. We thank Professor R. Jackiw for a kind communication pointing out to us the significance of the triangle anomaly in establishing the absence of a non-trivial solution to the Gell-Mann-Low function, in connection with our earlier work, ref. 6.
29. J. Zinn-Justin, Quantum Field theory and critical phenomena, second edition, Clarendon Press, (1993) Oxford.
30. H.Bateman, Higher Transcendental Functions, Volume I, McGraw-Hill Book Company, New York (1953).
31. L. Ryder, Quantum Field Theory, Cambridge University Press, Cambridge (1985).