Affine Strings

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Abstract

A new model of bosonic strings is considered. An action of the model is the sum of the standard string action and a term describing an interaction of a metric with a linear (affine) connection. The Lagrangian of this interaction is an arbitrary analytic function \( f(R) \) of the scalar curvature. This is a classically integrable model. The space of classical solutions of the theory consists from sectors with constant curvature. In each sector the equations of motion reduce to the standard string equations and to an additional constant curvature equation for the linear connection. A bifurcation in the space of all Lagrangians takes place. Quantization of the model is briefly discussed. In a quasiclassical approximation one gets the standard string model with a fluctuating cosmological constant. The Lagrangian \( f(R) \), like Morse function, governs transitions between manifolds with different topologies.
1 Introduction

There are several motivations to look for a new string model. One is an old idea that gauge fields might be related to strings. In particular, the Nielsen-Olesen string-like solution of the Higgs model [1], 't Hooft large N expansion of gauge theory [2], the Wilson strong coupling expansion for lattice gauge theory [3] and loop space formulation of gauge theory, see for example [4], give indications to such relation. Still it is an open problem what could be an explicit relation between gauge theory and string theory and even more - the standard string theory is not appropriate for describing gauge fields, because of longitude oscillators outside the critical dimension, for a recent discussion about this point see [3], [4]. String theory also could be applied to solid state physics and to the theory of defects in solid bodies [4]. Considerations of modified strings could help also in understanding unique properties of the standard string and in an investigation of the space of all two dimensional field theories, see [5].

A string model that could correspond to gauge theory will be probably more complicated than the standard one and/or will have additional fields on the worldsheet. To be useful a modified string model should be tractable and be natural from a geometrical viewpoint. It is not so easy to propose such a theory. It seems one can try different modifications of the standard string model hoping to extend our understanding of two dimensional field theories and finally to find a string model which could be appropriate for describing gauge fields and also to other problems noted above. What could be additional fields on the worldsheet? From a geometrical viewpoint the most natural fields are only a metric and a (linear) connection.

In this paper a modification of the standard string theory involving a linear connection on the worldsheet is considered. Recall the action for bosonic string [4]

\[ S = \frac{1}{2} \int_M \partial_\mu X \partial_\nu X g^{\mu\nu} \sqrt{g} d^2 \xi \]  

(1)

Here \( M \) is a two dimensional manifold with local coordinates \( \xi = (\xi^\mu) \) and with a metric \( g_{\mu\nu}(\xi) \), \( \mu, \nu = 1, 2 \), \( X = (X^k(\xi)) \), \( k = 1, ..., D \) are coordinates of the string.

The Euler-Lagrange equations for the action (1) are

\[ \Box_g X = 0 \]  

(2)

\[ \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\alpha X \partial_\beta X g^{\alpha\beta} = 0 \]  

(3)

where \( \Box_g \) is the Laplace-Beltrami operator.

A modification of the string action (1) which we consider in this paper consists in adding to (1) a term with a Lagrangian \( f(R) \), where \( R \) is the scalar curvature depending on a linear (affine) connection \( \Gamma_\nu^{\mu\tau} \) and a metric \( g_{\mu\nu} \), \( f \) is an analytic function of real variable. So we have the metric \( g_{\mu\nu} \) and the connection \( \Gamma_\nu^{\mu\tau} \) as additional fields on the string worldsheet. We modify the internal geometry of string introducing the affine connection as a new geometrical field on the worldsheet. We are not going to discuss here a modification of external geometry of string like in models of rigid strings.

The model of 2d gravity with the Lagrangian \( f(R) \) has been considered recently in [4], this Lagrangian in an arbitrary dimension \( d \geq 3 \) has been considered in [4]. It was shown that for almost arbitrary analytic Lagrangian \( f(R) \) one always gets Einstein...
equations (if \( d \geq 3 \)) or the equation of constant curvature (if \( d = 2 \)). We will see in this paper that this universality, i.e. almost independence of equations of motion on \( f(R) \) takes place also for string theory. Actually the traceless of the energy-momentum tensor of the matter is the crucial property for the universality at any dimension.

The presence of the metric and the connection on the world sheet looks rather natural from a geometrical point of view. We will investigate the corresponding classical equations of motion and show that in fact equations (2) and (3) are preserved in the modified theory. The whole effect of adding the Lagrangian \( f(R) \) to the action (1) reduces to an interaction of the metric \( g \) with the connection \( \Gamma \) which has a form of constant curvature equation. The space of classical solutions of the model consists from sectors of constant curvature \( R = c_i, i = 1, 2, \ldots \). Constants \( c_i \) are roots of the equation

\[
f'(R)R - f(R) = 0
\]

If \( f'(c_i) = 0 \) one has a sector with the metric interacting to the connection. If \( f'(c_i) \neq 0 \) one has a sector with the metric interacting to a vector field which is a solution of an equation for the connection. This means that a bifurcation in the space of solutions takes place.

What is a deep reason for the universality discussed is unclear to me at the moment. There are important works which could help to clarify different aspects of universality. Universality of quantum gauge theory was suggested by Bennet and Nielsen in their works on random dynamics and gauge glasses [12]. Recently Fairlie, Govaerts and Morosov [13] have found a universal field equation which is invariant under field redefinition and can be derived from infinity of inequivalent lagrangians. We see that Einstein equations also enjoy this property [14]. Universality in theory of phase transitions is well known, for an appropriate consideration in field theory see [14].

In this paper we consider only classical equations of motion. Quantization of the model we hope to consider in another paper. Here we discuss quantization of the model very briefly and find that in a semiclassical approximation the model is reduced to the standard bosonic string model with a fluctuating cosmological constant. We will see that the Lagrangian \( f(R) \) selects an appropriate topology of the manifold \( M \) and for a good understanding of quantization of the model one needs a better understanding of topology change in quantum gravity, for a recent discussion see [13, 16]. In fact the Lagrangian \( f(R) \) plays the role of the Morse function [17], its critical points are related with change of topology.

Recall that there are other modifications of the action (1). In this paper we consider the connection without torsion. But one can add to (1) a term describing an interaction of the metric with a connection with torsion. Adding of this term leads to a more drastic modification of the standard string theory, in particular, equation (3) is modified. The corresponding model of non-critical strings was considered in [15, 19]. This model is purely geometrical one as well as discussed in the present paper. Another modification of the standard string model which involves an additional scalar field [20] was considered in [21] and has been further generalized in [22]. For a recent consideration of \( R^2 \)-gravity see [23].
2 The Action and the Equations of Motion

We consider the following action

\[ S(X, g, \Gamma) = \frac{1}{2} \int_M \partial_\mu X \partial_\nu X g^{\mu\nu} \sqrt{g} d^2 \xi + \int_M f(R) \sqrt{g} \ d^2 \xi \] (4)

where \( M \) is a two–dimensional manifold endowed with a metric \( g_{\mu\nu} \) and a symmetric linear connection \( \Gamma^\sigma_{\mu\nu}, \mu, \nu, \sigma = 1, 2 \). One defines the Riemann curvature tensor by

\[ R^\lambda_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\sigma} - \Gamma^\lambda_{\alpha\sigma} \Gamma^\alpha_{\mu\nu} \] (5)

The Ricci tensor is defined by

\[ R_{\mu\sigma}(\Gamma) = R_{\mu\nu\rho\sigma}(\Gamma) \] (6)

and the scalar curvature

\[ R = R(\Gamma, g) = R_{\mu\nu\rho\sigma}(\Gamma) g^{\mu\nu} \] (7)

The function \( f \) in (4) is a function of one real variable, which we assume to be analytic on the real line.

The Euler–Lagrange equations for the action (2.1) with respect to \( X, g \) and \( \Gamma \) can be written in the following form

\[ \Box g X = 0, \] (8)

\[ \frac{1}{2} (\partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\alpha X \partial_\beta X g^{\alpha\beta}) + f'(R) R_{(\mu\nu)}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = 0 \] (9)

\[ \nabla_\alpha (f'(R) \sqrt{g} g^{\mu\nu}) = 0, \] (10)

where \( \nabla_\alpha \) is the covariant derivative with respect to \( \Gamma \) and \( R_{(\mu\nu)}(\Gamma) \) denotes the symmetrical part of \( R_{\mu\nu}(\Gamma) \).

Let us consider equations (9) and (10) by using the same method as in [11]. Multiplying equation (9) by \( g^{\mu\nu} \) one obtains

\[ f'(R) R - f(R) = 0 \] (11)

If equation (11) is identically satisfied we have

\[ f(R) = cR, \] (12)

where \( c \) is an arbitrary constant.

In all other cases, for a given analytic function \( f(R) \) equation (11) can have no more than a countable set of solutions

\[ R = c_i, \] (13)

where \( c_i \) are constants, \( i = 1, 2, ... \).

For a given \( c_i \) (13) one can have either \( f'(c_i) = 0 \) or \( f'(c_i) \neq 0 \). First consider the case \( f'(c_i) = 0 \). Then from (11) one gets \( f(c_i) = 0 \) and equation (9) will take the form (3) and equation (10) satisfies identically. Therefore the full system of equations for \( X, g \) and \( \Gamma \) in this case is

\[ \Box g X = 0 \] (14)
\[ \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\alpha X \partial_\beta X g^{\alpha\beta} = 0 \quad (15) \]

\[ R(\Gamma, g) = c_i. \quad (16) \]

Equations (14) and (15) are standard equation of motion and constraints for the bosonic string. The metric \( g \) interacts with the connection \( \Gamma \) only by means of equation (16). The whole dependence on the form of the Lagrangian \( f \) presents only in the constant \( c_i \) which is a root of equation (11). There are no other restrictions on \( g \) and \( \Gamma \).

Now consider the second possibility when \( f'(c_i) \neq 0 \). Then equation (10) takes the form

\[ \nabla_\mu (\sqrt{g} g^{\alpha\beta}) = 0 \quad (17) \]

Equation (17) was investigated in [10] and we present here the result. Write equation (17) in the form

\[ \nabla_\mu g_{\alpha\beta} = - \frac{1}{2} g^{\sigma\tau} \partial_\mu g_{\sigma\tau} g_{\alpha\beta} \quad (18) \]

Denoting

\[ B_\mu = \Gamma^{\sigma}_{\mu\sigma} - \frac{1}{2} g^{\sigma\tau} \partial_\mu g_{\sigma\tau} \quad (19) \]

one can rewrite equation (18) as

\[ (\nabla_\mu + B_\mu) g_{\alpha\beta} = 0 \quad (20) \]

Clearly, the system of equations (19) and (20) for \( \Gamma, g, B \) is equivalent to equation (17) or (18) for \( \Gamma \) and \( g \).

Let us then consider the system (19), (20). The general solution of equation (20) for the connection \( \Gamma \) is

\[ \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} ((\partial_\mu + B_\mu) g_{\nu\alpha} + (\partial_\nu + B_\nu) g_{\mu\alpha} - (\partial_\alpha + B_\alpha) g_{\mu\nu}) \quad (21) \]

Equation (19) follows from (21). Let us stress that we consider the fields \( \Gamma, g \) and \( B \) in equations (19) and (20) as a priori arbitrary fields being the subject to these equations. Therefore we obtain that the general solution of equation (17) has the form (21) where \( g \) and \( B \) are arbitrary fields.

Let us remark that equation (17) is invariant under conformal (Weyl) transformation

\[ g_{\alpha\beta} \to e^\lambda g_{\alpha\beta}, \quad \Gamma \to \Gamma \quad (22) \]

and the solution (21) is invariant under transformations

\[ g_{\alpha\beta} \to e^\lambda g_{\alpha\beta}, \quad B_\mu \to B_\mu - \partial_\mu \lambda. \quad (23) \]

We need an expression for the Ricci tensor. Using the definition (6) of the Ricci tensor for the connection (21) one gets

\[ R_{\alpha\beta} = R_{(\alpha\beta)} + F_{\alpha\beta} \quad (24) \]

where

\[ R_{(\alpha\beta)} = \frac{1}{2} (R(g) - D_\sigma B^\sigma) g_{\alpha\beta} \quad (25) \]
\[ F_{\alpha\beta} = \frac{1}{2} (\partial_{\alpha} B_{\beta} - \partial_{\beta} B_{\alpha}) \]  
\[ R = R_{\alpha\beta} g^{\alpha\beta} = R(g) - D_{\sigma} B^{\sigma} = c_i. \]

Therefore

\[ R_{(\alpha\beta)} = \frac{1}{2} c_i g_{\alpha\beta}. \]

Now by using (11), (13) and (28) we have

\[ f'(R) R_{(\mu\nu)} (\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = f'(c_i) \frac{1}{2} c_i g_{\mu\nu} - \frac{1}{2} f(c_i) g_{\mu\nu} = 0 \]

and equation (8) reduces to (3).

Therefore finally in the case \( f'(c_i) \neq 0 \) we obtain the following system of equations for fields \( X, g \) and \( B \):

\[ \Box_g X = 0, \]

\[ \partial_{\mu} X \partial_{\nu} X - \frac{1}{2} g_{\mu\nu} \partial_{\alpha} X \partial_{\beta} X g^{\alpha\beta} = 0 \]

\[ R(g) - D_{\sigma} B^{\sigma} = c_i. \]

The connection \( \Gamma \) is expressed in terms of \( g \) and \( B \) by means of formula (21).

We find that locally the space of solutions of equations (8)–(10) for any analytical function \( f \) decomposes into sectors \( H_n \) describing by equations (14)–(16) and sectors \( H_\rho \) describing by equations (30)–(32). Every sector corresponds to a connected manifold with a given constant \( c_i \) because the curvature is a continuous function. To take into account all sectors one should consider the original manifold \( M \) in (4) as a disjoint union of connected manifolds perhaps with different topologies. Therefore a natural setting for consideration of the action (4) is a collection of connected manifolds. At classical level there is no interaction between these manifolds. But after quantization transitions between sectors with different topologies could appear. One can compare the Lagrangian \( f(R) \) with Morse function [17] describing transitions between manifolds with different topologies.

The case of quadratic function \( f(R) \) was considered in detail in [11]. Another simple case admitting an explicit solution is \( f(R) = aR^n + bR + c. \)

### 3 Conclusion

We have no intention to consider in this paper the quantization of the model. Here we make only some preliminary remarks about quantum properties of the model. The partition function for the action (4) is

\[ Z = \int \exp \{ i S(X, g, \Gamma) \} D X D g D \Gamma \]

An effective action for string variables \( X \) after integrating out the metric and the connection is defined as
\[ \exp\{iS_{\text{eff}}(X)\} = \int \exp\{iS(X, g, \Gamma)\} \, Dg D\Gamma \]  

(34)

In a naive quasiclassical approximation one has a sum over all critical points of the action, i.e. over all classical solutions of eqs. (16) and (32). The space of classical solutions of these equations consists from sectors $H_n$ and $H_{\rho}$ which are infinite dimensional functional spaces. Therefore in the quasiclassical approximation still one has functional integrals because there is sum over the space of all solutions:

\[ e^{iS_{\text{eff}}(X)} = \sum_n \int e^{iS_n} \delta(R(g, \Gamma) - c_n) \, Dg D\Gamma + \]

\[ \sum_\rho \int e^{iS_\rho} \delta(R(g) - D_\sigma B^\sigma - c_\rho) \, Dg DB, \]  

(35)

where

\[ S_{\text{cl}}^i = \frac{1}{2} \int \partial_\mu X \partial_\nu X g^{\mu \nu} \sqrt{g} \, d^2 \xi + f(c_i) \int \sqrt{g} \, d^2 \xi \]

The term \( f(c_i) \) is a cosmological constant. So one can interpret the sums as a string theory with the fluctuating cosmological constant. Recall that \( f(c_n) = 0 \) in the sector $H_n$. Note that this effect is different from the worm-holes approach [24, 25] where it comes from a summation over nontrivial topologies.

In string theory one considers the sum over different genus to get a unitary S-matrix in the target space. There are no obvious reasons why one should have topology change for the Lagrangian (1) from the point of view of 2-dim gravity. However if one has the Lagrangian \( f(R) \) then even from a point of view of 2-dim gravity one should take into account all possible critical points of the action, i.e. \( R = c_i, i = 1, 2, \ldots \) like in the Goldstone-Higgs model one should take into account all critical points of potential energy \( V(\phi) \). Therefore the original manifold \( M \) should be a disconnected sum of connected manifolds with constant curvatures \( R = c_i \) and consequently of different topology (genus). The same arguments one can apply to the Lagrangian \( \phi f(R) \) [23] containing an additional scalar field \( \phi \) and to any other Lagrangian leading to the equations \( R = c_i \).

To develope a nonperturbative quantum approach to this model one needs a lattice formulation of gravity with a connection. Such a formulation was suggested in [26] and would be very interesting to use it for this model and also for the string with torsion [18, 19].

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