CASIMIR OPERATORS FOR LIE SUPERALGEBRAS

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Abstract. Casimir operators — the generators of the center of the enveloping algebra — are described for simple or close to them “classical” finite dimensional Lie superalgebras with nondegenerate symmetric even bilinear form in: Sergeev A., The invariant polynomials on simple Lie superalgebras. Represent. Theory 3 (1999), 250–280; math-RT/9810111 and for the “queer” series in: Sergeev A., The centre of enveloping algebra for Lie superalgebra $Q(n, C)$. Lett. Math. Phys. 7, no. 3, 1983, 177–179.

Here we consider the remaining cases, and state conjectures proved for small values of parameter.

Under deformation (quantization) the Poisson Lie superalgebra $po(0|2n)$ on purely odd superspace turns into $gl(2^{n-1}|2^{n-1})$ and, conjecturally, the lowest terms of the Taylor series expansion with respect to the deformation parameter (Planck’s constant) of the Casimir operators for $gl(2^{n-1}|2^{n-1})$ are the Casimir operators for $po(0|2n)$.

Similarly, quantization sends $po(0|2n−1)$ into $q(2^{n−1})$ and the above procedure makes Casimir operators for $q(2^{n−1})$ into same for $po(0|2n−1)$.

Casimir operators for the Lie superalgebra $vect(0|m)$ of vector fields on purely odd superspace are only constants for $m > 2$. Conjecturally, same is true for the Lie superalgebra $svect(0|m)$ of divergence free vector fields, and its deform, for $m > 3$.

Invariant polynomials on $po(0|2n−1)$ are also described. They do not correspond to Casimir operators.

This is a version of the paper published in: Ivanov E. et. al. (eds.) Supersymmetries and Quantum Symmetries (SQS’99, 27–31 July, 1999), Dubna, JINR, 2000, 409–411. Meanwhile we have understood how to prove our conjecture for $vect(0|m)$ and precisely same idea was used by Noam Shomron [Sh], who additionally described, to an extent, indecomposable $vect(0|m)$-modules. The only difference with the Dubna version is this and similar remarks.

As always, speaking about invariant polynomials on Lie superalgebras (or their duals), one should bear in mind a more natural from super point of view approach due to Shander [Sha]: description of nonpolynomial invariant functions. Actually, even in non-super cases people did consider non-polynomial elements, say, localizations of the enveloping algebras, but there it does not lead to totally new invariants.

0. Recapitulations. Recall that by the Casimir elements for the Lie algebra or superalgebra $g$ we mean the generators of the center $Z(g)$ of $U(g)$. Their applications in description of physical phenomena are well-known and numerous.

If $g$ possesses an invariant nondegenerate supersymmetric bilinear form $B$, then $g \simeq g^*$. In this case, the description of the Casimir elements for $g$ is equivalent to the description of the algebra $I(g)$ of invariant polynomials on $g$ because $Z(g)$ is isomorphic to $I(g)$ (as vector spaces). To describe $I(g)$ is, however, easier than $Z(g)$. For the most complete description of $I(g)$ for simple finite dimensional Lie superalgebras and/or their “relatives”, i.e., close to them “classical” algebras, such as central extensions, etc., see [St].
This description of $I(\mathfrak{g})$ does not cover $\mathfrak{q}(n)$, and $\mathfrak{po}(0|m)$, and their relatives: it is a much more difficult problem to be solved by other means and ideas than those of [S1].

Thus, Casimir operators are described at the moment for

- simple Lie superalgebras with Cartan matrix and their “relatives” [S1]; (the first results are due to Berezin [3] and Kac [K1, K2]);
- $\mathfrak{q}(n)$ (but not for relatives), see [3];
- $\mathfrak{pe}(n)$ (no Casimir operators) and $\mathfrak{spec}(n)$, see [usc] (implicitly) and [2] (for $n = 3$); now we can refer to an interesting paper of Maria Gorelik [G], where the $\mathfrak{spec}(n)$-invariants are completely and explicitly described (though the author herself does not like the result and did not even put on arXive, to say nothing about publishing, we find it a very nice one; it is based on her previous papers [G2]);
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- $\mathfrak{vect}(0|m)$ (for $m > 2$), see Shomron’s paper [Sh].

There remain the following cases:

- the divergence free sublagebras: the “standard” one, $\mathfrak{vect}(0|m)$, that preserves the volume element $\text{vol}$ and its deformation $\mathfrak{vect}(0|m)$ preserving $(1 + \theta_1 \cdot \theta_m)\text{vol}$ for which, conjecturally, there are no Casimir operators (for $m > 3$);
- $\mathfrak{sq}(n)$ and $\mathfrak{psq}(n)$;
- $\mathfrak{po}(0|m)$ considered below (but the general proof is still lacking), $\mathfrak{spo}(0|m)$ as well as $\mathfrak{h}(0|m)$ and $\mathfrak{s}\mathfrak{h}(0|m)$.

**Example 1.** For $\mathfrak{g} = \mathfrak{gl}(n)$ the Casimir operators corresponding to the first $n$ invariant polynomials $\text{tr}(X^k)$, where $X \in \mathfrak{g}$, i.e., for $k = 1, \ldots, n$, generate the center of $U(\mathfrak{g})$. The form $B$ is defined by the trace.

**Example 2.** The Lie superalgebra $\mathfrak{gl}(m|n)$ is a super analog of $\mathfrak{gl}(n)$; the form $B$ is defined by the supertrace. It is known that the center of its universal enveloping algebra is NOT finitely generated. For the generators we may take $\text{str}(X^k)$ for all positive integer values of $k$. (A comment: suppose we somehow proved that apart from traces there are no invariant polynomials on $\mathfrak{gl}(n)$. To prove that only finitely many of them generate the algebra, consider the power series expansion of $\det(X - \lambda 1_n)$ with respect to $\lambda$. The coefficients of $\lambda$ are polynomials in $\text{tr}(X^k)$; the characteristic polynomial yields a recursive formula for $\text{tr}(X^m)$.

Contrarywise, the superdeterminant, the Berezinian, is a rational function in $\text{str}(X^k)$, to express the Berezinian in terms of supertraces we need all of them.)

On the other hand, any rational function on $\mathfrak{gl}(m|n)$ can be expressed as a rational function of the first $n + m$ supertraces, cf. [S1] with [3] and [K1, K2]).

**Example 3.** The Lie superalgebra $\mathfrak{po}(0|2n)$, the Poisson superalgebra, is one more super analog of $\mathfrak{gl}(n)$. Indeed, there exists a well-known deformation (quantization) with parameter $\hbar$ such that at $\hbar = 0$ we have $\mathfrak{po}(0|2n)$ whereas at $\hbar \neq 0$ the Lie superalgebras from the parametric family are isomorphic to $\mathfrak{gl}(2^{n-1}|2^{n-1})$.

Recall, that the superspace of $\mathfrak{po}(0|2n)$ is isomorphic to the Grassmann superalgebra on $2n$ generators $\xi, \eta$ and the bracket is given by the formula

$$\{f, g\} = (-1)^{p(f)} \sum \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right).$$

The form $B$ on $\mathfrak{po}(0|m)$ for any $m$ is given by the integral $B(f, g) = \int fg \text{vol}$, where $\text{vol}$ is the volume element, and, clearly, the following are invariant polynomials on $\mathfrak{po}(0|m)$:

$$r_k = \int f^k \text{vol}(\xi, \eta), \text{ where } k = 1, 2, \ldots \quad (*)$$
Observe, however, that, unlike \( \mathfrak{gl}(m|n) \) case, these polynomials \( r_k \) do not generate the whole algebra of invariant polynomials for \( n \geq 2 \). A counterexample of least degree is given for \( n = 2 \) by the Casimir operator whose radial part is equal to \( x_1^2x_2^2(x_1^2 - x_2^2) \).

Quantization turns the integral on functions that generate \( \mathfrak{po}(0|m) \) into the supertrace on supermatrices from \( \mathfrak{gl}(2n-1|2n-1) \) for \( m = 2n \) and into queertrace on the general queer superalgebra \( \mathfrak{q}(2n-1) \) for \( m = 2n - 1 \). This and other arguments, see [31], indicates that \( \mathfrak{q}(n) \) is one more super analog of \( \mathfrak{gl}(n) \). Recall that \( \text{qtr} \left( \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right) = \text{tr}B \), see [31].

**Remark.** For any Lie superalgebra \( \mathfrak{g} \) and its finite dimensional representation \( \rho \), the polynomials

\[
P_{k,\rho} = \text{str}\rho(X)^k
\]

are invariant ones. For \( \mathfrak{gl}(m|n) \) any degree \( k \) homogeneous invariant polynomial can be represented as

\[
\sum_{\rho} c_{\rho}P_{k,\rho},
\]

where the sum runs over a finite number of finite dimensional representations, see [31]. For Lie superalgebras \( \mathfrak{po}(0|2n) \) the situation is different: when \( n \geq 2 \) and \( k \geq 6 \) the polynomial \((*)\) can not be obtained as any finite sum of the form \((***)\), so we have strict inclusions \( I(\mathfrak{g}) \supset I_{(*)}(\mathfrak{g}) \supset I_{(***)}(\mathfrak{g}) \), where \( I_{(*)} \) and \( I_{(***)} \) are algebras of polynomials of the form \((*)\) and \((***)\), respectively.

To describe the complete set of \( \mathfrak{po}(0|2n) \)-invariants, recall that the Grassmann superalgebra \( \mathbb{C}[\xi, \eta] \) can be deformed into the Clifford superalgebra \( \mathcal{Cliff}_h(\hat{\xi}, \hat{\eta}) \), where

\[
\hat{\xi}_i\hat{\eta}_j + \hat{\eta}_j\hat{\xi}_i = \delta_{ij}h.
\]

For any subset \( I = \{i_1 < \cdots < i_l\} \subset \{1, 2, \ldots, n\} \) set \( \xi_I = \xi_{i_1} \cdots \xi_{i_l} \) and \( \eta_I = \eta_{i_1} \cdots \eta_{i_l} \); notations \( \hat{\xi}_I \) and \( \hat{\eta}_I \) are similar. Define the linear map \( Q : \mathbb{C}[\xi, \eta] \rightarrow \mathcal{Cliff}_h(\hat{\xi}, \hat{\eta}) \) by setting

\[
Q(\xi_I\eta_J) = \hat{\xi}_I\hat{\eta}_J;
\]

before applying \( Q \) to a monomial one has to reduce it to a normal form, say, to \( \xi\eta \)-form.

**4. Lemma.** \( [Q(f), Q(g)] = hQ(\{f, g\}) + \mathcal{O}(h^2) \), where in the left hand side stands the supercommutator in the Clifford superalgebra, and \( \{\cdot, \cdot\} \) in the right hand side is the Poisson bracket.

The Lie superalgebra associated with the associative superalgebra \( \mathcal{Cliff}_h(\hat{\xi}, \hat{\eta}) \) is \( \mathfrak{gl}(2n-1|2n-1) \); for it, the invariant polynomials are described above and in [31]. The Lie superalgebra associated with the Clifford superalgebra generated by 2\( n - 1 \) elements is \( \mathfrak{q}(2n-1) \) (see [1]); for it, the Casimir elements are described in [32].

**5. Lemma.** Let \( F \) be an invariant polynomial on \( \mathcal{Cliff}_h \) considered as Lie superalgebra, i.e., on \( \mathfrak{gl}(2n-1|2n-1) \) or on its subalgebra \( \mathfrak{q}(2n-1) \). Let us represent it in the form \( F = \sum_{k \geq k_0} F_k h^k \).

If \( P \in I(\mathfrak{po}(0|2n)) \) is such that \( Q(P) = F_{k_0} \), then \( P \) is an invariant polynomial on \( \mathfrak{po}(0|m) \).

In other words, the lowest (with respect to \( h \)-degree) components of invariant polynomials in the operators, i.e., on \( \mathfrak{gl}(2n-1|2n-1) \) or \( \mathfrak{q}(2n-1) \), are invariant polynomials in their symbols — the elements from \( \mathfrak{po}(0|m) \) for \( m = 2n \) or \( 2n - 1 \), respectively. In particular, it is easy to see that

\[
\text{str}(Q(f)^k) \quad \text{(or \( \text{qtr}(Q(f)^k) \)) = \frac{1}{k} \left( \int f^k \text{vol} \right) h^n + \mathcal{O}(h^n).}
\]

\((***)\)
6. Conjecture. ([S2]) All the invariant polynomials on $\mathfrak{po}(0|m)$ for any $m$ (odd as well as even) may be obtained in the way indicated in Lemma 5.

This 15 years old conjecture is now proven for $m$ even and $\leq 6$. To prove it in full generality, we need new ideas as compared with those we know and successfully used in [S1]: here the degree of complexity of computations performed à la [S1] grows steeply with $m$.

References

[B] Berezin, F. Representations of the supergroup $U(p,q)$. Funkcional. Anal. i Prilozhen. 10 (1976), n. 3, 70–71 (in Russian); Berezin, F. Laplace–Cazimir operators on Lie supergroups. The general theory. Preprints ITEPh 77, Moscow, ITEPh, 1977; Berezin F. Analysis with anticommuting variables, Kluwer, 1987

[BL] Bernstein J. N., Leites D. A., The superalgebra $Q(n)$, the odd trace and the odd determinant. C. R. Acad. Bulgare Sci. 35 (1982), no. 3, 285–286

[G] Gorelik M., The center of a simple $P$-type Lie superalgebra. (preprint, 23 Sep 2001)

[G2] Gorelik M., On the Ghost Centre of Lie Superalgebras, math.RT/9910114; id., Annihilation Theorem and Separation Theorem for basic classical Lie superalgebras, math.RA/0008143

[K1] Kac V. G., Characters of typical representations of classical Lie superalgebras. Commun. Alg. v. 5, 1977, 889–897

[K2] Kac V. G., Representations of classical Lie superalgebras. In: Bleuler K. et al. (eds.) Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), Lecture Notes in Math., 676, Springer, Berlin, 1978, 597–626.

[L] Leites, D. A. Clifford algebras as superalgebras, and quantization. (Russian. English summary) Teoret. Mat. Fiz. 58 (1984), no. 2, 229–232. (English translation: Theoret. and Math. Phys. 58 (1984), no. 2, 150–152.)

[Sch] Scheunert M., Invariant supersymmetric multilinear forms and the Casimir elements of $P$-type Lie superalgebras. J. Math. Phys. 28, no. 5, 1987, 1180–1191; id., Casimir elements of the general linear and the $P$-type Lie superalgebras. Proceedings of the XV International Conference on Differential Geometric Methods in Theoretical Physics (Clausthal, 1986), 275–288, World Sci. Publishing, Teaneck, NJ, 1987

[S1] Sergeev A.. The invariant polynomials on simple Lie superalgebras. Represent. Theory 3 (1999), 250–280; math-RT/9810111

[S2] Sergeev A., Laplace operators and Lie superalgebra representations. Ph.D. thesis, Moscow University, 1985. (an expanded English translation in: Leites D. (ed.) Seminar on Supermanifolds, Reports of Stockholm University, n. 32/1988–15, 44–95

[S3] Sergeev A., The centre of enveloping algebra for Lie superalgebra $Q(n, C)$. Lett. Math. Phys. 7, no. 3, 1983, 177–179

[Sha] Shander V., Invariant functions on supermatrices, math.RT/9810112 (a detailed version of Shander, V. N. Orbits and invariants of the supergroup $GQ_n$. (Russian) Funktsional. Anal. i Prilozhen. 26 (1992), no. 1,69–71 translation in Funct. Anal. Appl. 26 (1992), no. 1, 55–56)

[Sh] Shomron N., Blocks of Lie Superalgebras of Type $W(n)$, math.RT/0009103

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