LAGRANGIAN SKELETONS, COLLARS AND DUALITY

E. BALLICO, E. GASPARIM, F. RUBILAR, B. SUZUKI

ABSTRACT. We present a geometric realization of the duality between skeleta in $T^*\mathbb{P}^n$ and collars of local surfaces. Such duality is predicted by combining two auxiliary types of duality: on one side, symplectic duality between $T^*\mathbb{P}^n$ and a crepant resolution of the $A_n$ singularity; on the other side, toric duality between two types of isolated quotient singularities. We give a correspondence between Lagrangian submanifolds of a cotangent bundle and vector bundles on a collar, and describe those birational transformations within the skeleton which are dual to deformations of vector bundles.

CONTENTS

1. Skeleton to collar duality 1
2. Lagrangian skeleton of $T^*\mathbb{P}^n$ 2
3. Potentials on the cotangent bundle 5
4. Birational maps within the skeleton 6
5. Duality for multiplicity $n$ singularities 7
6. Vector bundles on local surfaces 8
7. Deformations 11
Acknowledgements 12
References 12

1. SKELETON TO COLLAR DUALITY

The simplest example of symplectic duality is the one between the cotangent bundle of projective space $T^*\mathbb{P}^{n-1}$ and the crepant resolution $\overline{Y}_n$ of the $A_{n-1}$ singularity obtained as a quotient $Y_n = \mathbb{C}^2/Z_n$ [BLPW, BF]. There exists also a duality between $\overline{Y}_n$ and the surface $Z_n = \text{Tot} O_{\mathbb{P}^1}(-n)$, in the sense that they are both minimal resolutions of quotient singularities, but their respective singularities have dual toric fans. In fact, the singular surface $\mathcal{X}_n$ obtained from $Z_n$ by contracting the zero section is also a quotient of $\mathbb{C}^2$ by the cyclic group of $n$ elements, but the singularity of $\mathcal{X}_n$ is of type $\frac{1}{n}(1, 1)$ whereas the singularity of $Y_n$ is of type $\frac{1}{n}(1, n - 1)$. Motivated by these two dualities we discuss some features of the resulting duality between $T^*\mathbb{P}^{n-1}$ and $Z_n$. On one side, we consider $T^*\mathbb{P}^{n-1}$ together with a complex potential, thus forming a Landau–Ginzburg model, and we study the Lagrangian skeleton of the corresponding Hamiltonian flow; on the other side, we describe the behaviour of vector bundles on the surfaces $Z_n$ considered as algebraic varieties.

In both cases we will focus our attention on building blocks used for those types of gluing procedures which may be viewed as surgery operations. We will see that vector bundles on what we call the collar of $Z_n$ (see Sec. 6) behave similarly to components of the Lagrangian skeleton of $T^*\mathbb{P}^{n-1}$.

Denoting by bir a birational transformation applied to a compactified Lagrangian and def a deformation of the complex structure of a vector bundle (without describing a categorial equivalence) we
give a geometric description of a 1-1 correspondence between objects and some essential morphisms. Such a duality is described by the diagram in the following theorem.

**Theorem 1.1.** The following diagram commutes:

\[
\begin{array}{ccc}
L_j \subset T^*\mathbb{P}^{n-1} & \xrightarrow{\text{dual}} & \sigma_{Z_0}^n(j) \oplus \sigma_{Z_0}^n(-j) \\
\xrightarrow{\text{birk}} & & \uparrow \text{def} \\
L_{j+1} \subset T^*\mathbb{P}^{n-1} & \xleftarrow{\text{dual}} & \sigma_{Z_0}^n(j+1) \oplus \sigma_{Z_0}^n(-j-1).
\end{array}
\]

(1.2)

Duality between Lagrangians and vector bundles.

The surfaces $Z_n$ have rich moduli spaces of vector bundles, but it is mainly the restriction of a vector bundle to the collar of $Z_n^0$ (see 6.4) that plays a role in this duality. The cotangent bundle is taken with the canonical symplectic structure and Lagrangian skeleta are described in Sec. 2. Vector bundles on the local surfaces $Z_n$ are building blocks for vector bundles on compact surfaces. In fact, a new gluing procedure called grafting introduced in [GS] explores the local contribution of these building blocks to the top Chern class. This grafting procedure was successful in explaining the physics mechanism underlying the phenomenon of instanton decay around a complex line with negative self-intersection, showing that instantons may decay by inflicting curvature to the complex surface that holds them [GS, Sec. 7]. For a line with self-intersection $-n$, grafting is done via cutting and gluing over a collar $Z_n^0$. The set of isomorphisms classes of rank 2 vector vector bundles over such a collar $Z_n^0$ presents a behaviour similar to that of the Lagrangian skeleton of the cotangent bundle $T^*\mathbb{P}^{n-1}$. Therefore our construction here offers a geometric interpretation of this particular instance of duality by exploring building blocks of surgery operations on both sides. When considered in families, one Lagrangian in the skeleton is taken to the next via a birational transformation (Sec. 4) whereas a bundle on the collar is taken to another via deformation of the complex structures (Sec. 7). In this sense we may say that when considering objects of this duality, birational transformations on Lagrangian skeleta occur as dual to deformations of vector bundles.

### 2. Lagrangian Skeleton of $T^*\mathbb{P}^n$

In this section we will calculate skeleta of certain Landau–Ginzburg models. By a Landau–Ginzburg model we mean a complex manifold together with a complex valued function.

Let $(M, \omega)$ be a symplectic manifold together with a potential $h$. We assume that $h$ is a Morse function. In the case when $h$ is a real valued function, the stable manifold of a critical point $p$ consists of all the points in $M$ that are taken to $p$ by the gradient flow of $h$. However, when $h$ is a complex valued function, even though the stable manifold of a point $p$ is still formed by points that flow to $p$, the natural choice is to use the Hamiltonian flow of $h$ (which can be thought of as the symplectic gradient). Furthermore, in the cases considered here, the Hamiltonian flow is given by a torus action (as described in Sec. 3) and the critical points of $h$ are the fixed points of such action.

Let $L$ be the union of the stable manifolds of the Hamiltonian flow of $h$ with respect to the Kähler metric. Then $L$ is the isotropic skeleton of $(M, \omega)$. When $L$ is of middle dimension, it is called the **Lagrangian skeleton** of $(M, \omega)$. In the case of exact symplectic manifolds, the Lagrangian skeleton of $M$ is the complement of the locus escaping to infinity under the natural Liouville flow, see [Ru, STW].

To describe the Lagrangian skeleton of $T^*\mathbb{P}^n$, we will use the Hamiltonian torus action. We start out with $\mathbb{P}^n$ described by homogeneous coordinates $[x_0, x_1, \ldots, x_n]$, covered by the usual open charts $U_j = \{x_j \neq 0\}$. We then write trivializations of the cotangent bundle $T^*\mathbb{P}^n$ taking products $V_i = U_i \times \mathbb{C}^n$.
and over the $V_0$ chart we write coordinates as $V_0 = \{(x_0, x_1, \ldots, x_n), (y_1, \ldots, y_n)\}$. In this chart, we write the Hamiltonian action of the torus $\mathbb{T} := \mathbb{C} \setminus \{0\}$ on $T^*\mathbb{P}^n$ as

$$\mathbb{T} \cdot V_0 = \{(1, t^{-1}x_1, \ldots, t^{-n}x_n), (ty_1, \ldots, t^ny_n)\}. \quad (2.1)$$

Note that the same action can be written as

$$\mathbb{T} \cdot V_0 = \{(t^n, t^{-n-1}x_1, \ldots, x_n), (ty_1, \ldots, t^ny_n)\}.$$

We will now describe the Lagrangian skeleton corresponding to this Hamiltonian action. We start by showing an example, i.e. the case of $T^*\mathbb{P}^3$ and then we present the general procedure.

**Example: skeleton of $T^*\mathbb{P}^3$.** We take $\mathbb{P}^3$ with homogeneous coordinates $[x_0, x_1, x_2, x_3]$, and cover it by open sets $U_i = \{x_i \neq 0\}$ and charts $\varphi_i : U_i \to \mathbb{C}^3$ given by $\varphi_i([x_0, x_1, x_2, x_3]) = \left(\frac{x_i}{x_0}, \ldots, \frac{x_3}{x_0}\right)$. The transition matrices for the cotangent bundle $T_{ij} : \varphi_i(U_i \cap U_j) \to \text{Aut}(\mathbb{C}^3)$ are

$$T_{01} = \begin{pmatrix} -x_1^2 & -x_1x_2 & -x_1x_3 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}, \quad T_{02} = \begin{pmatrix} -x_1x_2 & -x_2^2 & -x_2x_3 \\ x_2 & 0 & 0 \\ 0 & x_2 & 0 \end{pmatrix}, \quad T_{03} = \begin{pmatrix} -x_1x_3 & -x_2x_3 & -x_3^2 \\ x_3 & 0 & 0 \\ 0 & x_3 & 0 \end{pmatrix}.$$  

Consequently, we can write down a cover for the cotangent bundle as $V_i = U_i \times \mathbb{C}^3$, and in coordinates

$$V_0 = \{(x_0, x_1, x_2, x_3), (y_1, y_2, y_3)\},$$

$$V_1 = \{(x_1^{-1}, 1, x_1^{-1}x_2, x_1^{-1}x_3), (-x_1^2y_1 - x_1x_2y_2 - x_1x_3y_3, x_1y_2, x_1y_3)\},$$

$$V_2 = \{(x_2^{-1}, x_2^{-1}x_1, 1, x_2^{-1}x_3), (-x_1x_2y_1 - x_2^2y_2 - x_2x_3y_3, x_2y_1, x_2y_3)\},$$

$$V_3 = \{(x_3^{-1}, x_3^{-1}x_1, x_3^{-1}x_2, 1), (-x_1x_3y_1 - x_2x_3y_2 - x_3^2y_3, x_3y_1, x_3y_2)\}.$$

Now we take the Hamiltonian action of the torus $\mathbb{T}$ on $T^*\mathbb{P}^3$ given by

$$\mathbb{T} \cdot V_0 = \{(1, t^{-1}x_1, t^{-2}x_2, t^{-3}x_3), (ty_1, t^2y_2, t^3y_3)\} = \{(t^3, t^2x_1, tx_2, x_3), (ty_1, t^2y_2, t^3y_3)\},$$

and compatibility on the intersections implies that

$$\mathbb{T} \cdot V_1 = \{(tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, t^{-2}x_1^{-1}x_3), (-t^{-1}(x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3), tx_1y_2, t^2x_1y_3)\},$$

$$\mathbb{T} \cdot V_2 = \{(t^2x_2^{-1}, tx_2^{-1}x_1, 1, t^{-1}x_2^{-1}x_3), (-t^{-2}(x_1x_2y_1 + x_2^2y_2 + x_2x_3y_3), t^{-1}x_2y_1, tx_2y_3)\},$$

$$\mathbb{T} \cdot V_3 = \{(t^3x_3^{-1}, t^2x_3^{-1}x_1, tx_3^{-1}x_2, 1), (-t^{-3}(x_1x_3y_1 + x_2x_3y_2 + x_3^2y_3), t^{-2}x_3y_1, t^{-1}x_3y_2)\}.$$

Using these, we calculate the Lagrangians.

**Stable manifold of $e_0$ - on $V_0$** we find the points satisfying

$$\lim_{t \to 0} \{(1, t^{-1}x_1, t^{-2}x_2, t^{-3}x_3), (ty_1, t^2y_2, t^3y_3)\} = \{(0, 0, 0)\}.$$

This requires $x_1 = x_2 = x_3 = 0$ and we obtain the fibre over the point $[1, 0, 0, 0]$, that is,

$$L_0 = \mathbb{T}^+_{[1,0,0,0]} \mathbb{P}^3 \sim \mathbb{E}^3.$$

**Stable manifold of $e_1$ - on $V_1$** we look for the points satisfying

$$\lim_{t \to 0} \{(tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, t^{-2}x_1^{-1}x_3), (-t^{-1}(x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3), tx_1y_2, t^2x_1y_3)\} = \{(0, 1, 0, 0)\}.$$

This requires $x_1^{-1}x_2 = x_1^{-1}x_3 = 0 = x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3$, but since $x_1 \neq 0$ in this chart, we get $x_2 = x_3 = 0 = y_1$. So, we are left with points having coordinates $[1, x_1, 0, 0]$, $(0, y_2, y_3)$ on $V_0$ which on
\[ V_1 \text{ become } \{x_1^{-1}, 1, 0, 0\}, (0, x_1 y_2, x_1 y_3) \]. We obtain (after taking the closure, that is, by adding the point \([1, 0, 0, 0], (0, 0, 0)\)) the set of points \([1, x_1, 0, 0], (0, y_2, y_3) \mapsto [x_1^{-1}, 1, 0, 0], (0, x_1 y_2, x_1 y_3)\) so that
\[
L_1 = \Theta_{p_1}(-1) \neq \Theta_{p_1}(-1).
\]

**Stable Manifold of \(e_2\) on \(V_2\)** We look for the points satisfying
\[
\lim_{t \to 0} [t^2 x_2^{-1}, t x_2^{-1} x_1, 1, t^{-1} x_2^{-1} x_3], (-t^{-2} x_3 y_1 + x_2^2 y_2 + x_2 x_3 y_3), t^{-1} x_2 y_1, t x_2 y_3) = [0, 0, 1, 0], (0, 0, 0).
\]
This requires \(x_2^{-1} x_3 = 0 = x_1 x_2 y_1 + x_2^3 y_2 + x_2 x_3 y_3 = x_2 y_1\) but since \(x_2 \neq 0\) in this chart, we get \(x_3 = 0\) and \(x_1 x_2 y_1 + x_2^3 y_2 = 0 = x_2 y_1\) and since on this chart \(x_2 \neq 0\) it follows that \(y_1 = y_2 = 0\).

We obtain (after taking the closure) the set of points \([1, x_1, x_2, 0\}, (0, 0, y_3) \mapsto [x_2^{-1}, x_2^{-1} x_1, 1, 0], (0, x_2 y_3)\), so
\[
L_2 = \Theta_{p_2}(-1).
\]

**Stable Manifold of \(e_3\) on \(V_3\)** We find the points satisfying
\[
\lim_{t \to 0} [t^3 x_3^{-1}, t^2 x_3^{-1} x_1, t x_3^{-1} x_2, 1\), (-t^{-3} x_3 y_1 + x_2 x_3 y_2 + x_2^2 y_3), t^{-2} x_3 y_1, t^{-1} x_3 y_2) = [0, 0, 0, 1], (0, 0).
\]
This requires \(x_1 x_3 y_1 + x_2 x_3 y_2 + x_2^2 y_3 = x_3 y_1 = x_3 y_2 = 0\) and since \(x_3 \neq 0\) in this chart, we get that \(y_1 = y_2 = y_3 = 0\). We obtain the set of points \([x_0, x_1, x_2, x_3\}, (0, 0, 0)\], so
\[
L_3 = \mathbb{P}^3.
\]

The generalization of this procedure to higher dimensions now becomes evident, giving:

**General case: the skeleton of \(T^*\mathbb{P}^n\).** We take \(\mathbb{P}^n\) with homogeneous coordinates \([x_0, x_1, x_2, \ldots, x_n]\), and cover it by standard open sets \(U_i = \{j \neq 0\}\) and charts \(\varphi_i : U_i \to \mathbb{C}^n\) given by \(\varphi_i([x_0, x_1, x_2, \ldots, x_n]) = (\tilde{x}_i, \ldots, \tilde{x}_i, \tilde{x}_i, \ldots)\). The transition matrices for the cotangent bundle \(T_{ij} : \varphi_i(U_i \cap U_j) \to \mathrm{Aut}(\mathbb{C}^n)\) are

\[
T_{01} = \begin{pmatrix}
-x_1^2 & -x_1 x_2 & \cdots & -x_1 x_n \\
0 & x_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_1
\end{pmatrix}, \\
T_{0n} = \begin{pmatrix}
-x_1 x_n & -x_2 x_n & \cdots & -x_2 x_n & -x_2 x_n \\
x_n & 0 & \cdots & 0 & 0 \\
0 & x_n & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x_n & 0
\end{pmatrix}.
\]
\[
T_{0j} = \begin{pmatrix}
-x_j x_1 & -x_j x_2 & \cdots & -x_j x_n \\
x_j & 0 & \cdots & 0 & 0 \\
0 & x_j & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & x_j
\end{pmatrix}.
\]

Consequently, we can write down a cover for the cotangent bundle as \(V_i = U_i \times \mathbb{C}^n\), and in coordinates
\[
V_0 = \{[x_0, \ldots, x_n], (y_1, \ldots, y_n)\}, \\
V_1 = \{[x_1^{-1}, 1, x_1^{-1} x_2, \ldots, x_1^{-1} x_{n-1}, x_1^{-1} x_n], (-x_1^2 y_1 - x_1 x_2 y_2 - x_1 x_3 y_3, x_1 y_2, \ldots, x_1 y_n)\}, \\
\vdots \]
\[
V_j = \{[x_j^{-1}, x_j^{-1} x_1, \ldots, 1, \ldots, x_j^{-1} x_n], (-x_j x_1 y_1 - \ldots - x_j x_n y_n, x_j y_2, \ldots, x_j y_n)\},
\]

\[ V_n = \{ [x_1, x_2, \ldots, x_n, 1], (-x_n x_1 y_1 - \ldots, -x_n^2 y_0, x_0 y_2, \ldots, x_n y_n) \}. \]

Now we take the Hamiltonian action of the torus \( T \) on \( T^* \mathbb{P}^n \) given by
\[ T \cdot V_0 = \{ [1, t^{-1} x_1, t^{-2} x_2, \ldots, t^{-n} x_n], (t y_1, t^2 y_2, \ldots, t^n y_n) \} = \{ [t^n, t^{n-1} x_1, t^{n-2} x_2, \ldots, x_n], (t y_1, t^2 y_2, \ldots, t^n y_n) \}, \]
and compatibility on the intersections implies that
\[ T \cdot V_1 = \{ [t x_1^{-1}, 1, t^{-1} x_1 x_2, \ldots, t^{-(n-1)} x_1 x_n], (-t^{-1} (x_1^2 y_1 + \ldots + x_1 x_n y_n), t x_1 y_2, \ldots, t^{-(n-1)} x_1 y_n) \}, \]
\[ \vdots \]
\[ T \cdot V_n = \{ [t^n x_n^{-1}, t^{n-1} x_n^{-1} x_1, \ldots, t x_n^{-1} x_n^{-1} x_n, 1], (-t^{-n} (x_n x_n y_1 + \ldots + x_n^2 y_n), t^{-(n-1)} x_n y_1, \ldots, t^{-1} x_n y_n) \}. \]

Using these, we calculate the Lagrangians.

**Stable manifold of \( e_0 \) on \( V_0 \)** we find the points satisfying
\[ \lim_{t \to 0} [1, t^{-1} x_1, \ldots, t^{-n} x_n], (t y_1, t^2 y_2, \ldots, t^n y_n) = [1, 0, \ldots, 0], (0, \ldots, 0) \]
this requires \( x_1 = x_2 = \cdots = x_n = 0 \) \( \)and we obtain the fibre over the point \([1, 0, \ldots, 0]\), that is,
\[ L_0 = T^*_{[1,0,\ldots,0]} \mathbb{P}^n \sim \mathbb{C}^n. \]

**Stable manifold of \( e_1 \) on \( V_1 \)** we find the points satisfying
\[ \lim_{t \to 0} [t x_1^{-1}, 1, t^{-1} x_1 x_2, \ldots, t^{-(n-1)} x_1 x_n], (-t^{-1} (x_1^2 y_1 + \ldots + x_1 x_n y_n), t x_1 y_2, \ldots, t^{-(n-1)} x_1 y_n) = [0, 1, 0 \ldots, 0], (0, \ldots, 0). \]
This requires \( x_1^{-1} x_2 = \cdots = x_1^{-1} x_n = 0 = x_1^2 y_1 + \cdots + x_1 x_n y_n \), but since \( x_1 \neq 0 \) in this chart, we get \( x_2 = \cdots = x_n = 0 = y_1 \). So, we are left with points having coordinates \([1, x_1, 0, \ldots, 0], (0, y_2, \ldots, y_n) \) on \( V_0 \) which on \( V_1 \) become \([x_1^{-1}, 1, 0 \ldots, 0], (0, x_1 y_2, \ldots, x_1 y_n) \). We obtain (after taking the closure, that is adding the point \([1, 0, \ldots, 0], (0, \ldots, 0)\)\( \))
\[ L_1 = \{ [1, x_1, 0, \ldots, 0], (0, y_2, \ldots, y_n) \} \sim C^1 \mathbb{P}^1. \]

**Other stable manifolds**
Using similar computations, we have that the Lagrangian \( L_j \) corresponding to the fixed point \( e_j \) is
\[ L_j = \begin{cases} \mathbb{C}^n & \text{if } j = 0, \\ a^{n-j} \mathbb{P}^{j-1} & \text{if } 0 < j < n, \\ \mathbb{P}^n & \text{if } j = n. \end{cases} \]

(2.2)

**3. Potentials on the cotangent bundle**

In this section we consider the question: what choices of potential \( h \) for a Landau–Ginzburg model \((T^* \mathbb{P}^n, h)\) is compatible with the Hamiltonian action considered in the previous sections, and hence gives rise to the same skeleton? We obtain the following result.

**Proposition 3.1.** Consider \((T^* \mathbb{P}^n, h_c)\) with coordinates \([1, x_1, \ldots, x_n], (y_1, \ldots, y_n)\). Each potential
\[ h_c([1, x_1, \ldots, x_n], (y_1, \ldots, y_n)) = \sum_{i=1}^n -2i x_i y_i + c, \]
has a corresponding Hamiltonian flow that coincides with the flow obtained by the torus action given in (2.1), that is
\[ T \cdot V_0 = \{(1, t^{-1}x_1, \ldots, t^{-n}x_n), (ty_1, \ldots, t^n y_n)\}. \]

To prove this, first consider the vector field on \( T^*\mathbb{P}^n \) corresponding to the Hamiltonian action given in coordinates by
\[ T \cdot V_0 = \{(1, t^{-1}x_1, \ldots, t^{-n}x_n), (ty_1, \ldots, t^n y_n)\}. \]

On the image of the \( V_0 \) chart, the right hand side becomes
\[ \alpha(t) = (t^{-1}x_1, \ldots, t^{-n}x_n, ty_1, \ldots, t^n y_n) \]
so that the derivative gives
\[ \alpha'(t) = (-t^{-2}x_1, \ldots, -nt^{-n-1}x_n, y_1, \ldots, nt^{n-1}y_n) \]
and evaluating at 1 we get
\[ \alpha'(1) = (-x_1, \ldots, -nx_n, y_1, \ldots, ny_n). \]

From the action of this 1-parameter subgroup, we have obtained the flow \( \alpha'(1) \). Now we wish to calculate a potential \( h \) corresponding to the vector field \( X = \alpha'(1) \).

Let \( \omega \) be the canonical symplectic form on \( T^*\mathbb{P}^n \), then \( h \) must satisfy, for all vector fields \( Z \in \mathfrak{X}(M) \)
\[ dh(Z) = \omega(X, Z). \]

In coordinates this gives
\[ \left( \frac{\partial h}{\partial x} \right) (a) = \sum_{i=1}^n dx_i \wedge dy_i \left( (-x_1, \ldots, -nx_n, y_1, \ldots, ny_n), (a_1, \ldots, a_n, b_1, \ldots, b_n) \right) = -2 \sum i x_i b_i + i y_i a_i. \]

where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \). Comparing the terms multiplying \( a_k \) and \( b_k \) on each side of the equation, for \( i = 1, \ldots, n \) we obtain the differential equations
\[ \frac{\partial h}{\partial x_i} = -2i y_i, \quad \frac{\partial h}{\partial y_i} = -2i x_i. \]

For \( c \in \mathbb{C} \), the solutions are:
\[ h_c = -2x_1 y_1 - \ldots - 2nx_n y_n + c = \sum_{i=1}^n -2i x_i y_i + c. \]

We thus conclude that any Landau–Ginzburg model of the form \( (T^*\mathbb{P}^n, h_c) \) will give rise to the same skeleta described above. This concludes the description of our Landau–Ginzburg models and their skeleta on \( T^*\mathbb{P}^n \) and in the next section we discuss birational maps within each skeleton.

4. Birational maps within the skeleton

In this section we present the birational transformations between components of the skeleton that justify the vertical downarrow appearing on the left hand side of diagram (1.2). As we saw in (2.2), the component \( L_j \) of the skeleton of \( T^*\mathbb{P}^n \) has the form
\[ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \]
with \( n - j \) factors. Projectivizing we obtain \( \mathbb{P}^{1} \times \mathbb{P}^{n-j-1} \). Thus, the component \( L_{j+1} \) has the form
\[ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \]
with \( n - j - 1 \) factors. Projectivizing we obtain \( \mathbb{P}^{1+1} \times \mathbb{P}^{n-j-2} \). The projectivizations are birationally equivalent, as we describe next, and up to tensoring by \( \mathcal{O}(+1) \), we may choose a birational map taking \( L_j \) to \( L_{j+1} \).

The birational maps \( \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{n+m} \) we need here are well known, but we recall one construction for completeness. We take homogeneous coordinates \( y_0, \ldots, y_n \) on \( \mathbb{P}^n \) and \( z_0, \ldots, z_m \) on \( \mathbb{P}^m \). Set \( r := (n+1)(m+1) - 1 \) and take homogeneous coordinates \( u_{ij}, 0 \leq i \leq n, 0 \leq j \leq m \), of \( \mathbb{P}^r \). Let \( \nu : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^r \).
\( \mathbb{P}^r \) be the Segre embedding of \( \mathbb{P}^n \times \mathbb{P}^m \) into \( \mathbb{P}^r \) given by the equations \( u_{ij} = y_i z_j \). The birational map (not a morphism, since there is no birational morphism between these two varieties) is induced by a linear projection \( \ell_M : \mathbb{P}^r \setminus M \to \mathbb{P}^{n+m} \), where \( M \) is an \( (r - n - m - 1) \)-dimensional linear subspace whose equations are coordinates \( u_{ij} = 0 \) for some \( i, j \) and the \( n + m + 1 \) homogeneous coordinates of \( \mathbb{P}^{n+m+1} \) are the ones used to describe \( M \). Recall that linear projections in suitable coordinates are just rational maps which forget some of the coordinates.

We start by considering the simplest example, that is, \( n = m = 1 \) and hence \( r = 3 \). Therefore \( M \) is a point, say \( ([0 : 1], [0 : 1]) \). Take for \( \mathbb{P}^3 \) homogeneous coordinates

\[
x_0 = y_0 z_0, \quad x_1 = y_1 z_0, \quad x_2 = y_0 z_1, \quad x_3 = y_1 z_1
\]

with \( M = [0 : 0 : : 1] \) and use \( x_0, x_1, x_2 \) for coordinates of \( \mathbb{P}^2 \).

The next step is to consider \( n = 2, m = 1 \) and hence \( r = 5 \). Take \( \mathbb{P}^5 \) with homogeneous coordinates

\[
x_0 = y_0 z_0, \quad x_1 = y_0 z_1, \quad x_2 = y_1 z_0, \quad x_3 = y_1 z_1, \quad x_4 = y_2 z_0, \quad x_5 = y_2 z_1.
\]

Then \( M \) is a line contained in the first ruling of \( \mathbb{P}^2 \times \mathbb{P}^1 \) so it has the form \( L \times \langle p \rangle \) where \( L \subset \mathbb{P}^2 \) is a line, and \( p \in \mathbb{P}^1 \) is a point. If we take \( L = [y_0 : 0] \) and \( p = [0 : 1] \) we get the equations \( x_0 = x_1 = x_2 = x_4 = 0 \) and the coordinates of \( \mathbb{P}^3 \) should be \( x_0, x_1, x_2, x_4 \). We blow-up \( L \times \langle p \rangle \subset \mathbb{P}^2 \times \mathbb{P}^1 \) and then we contract the strict transform of \( \mathbb{P}^2 \times \langle p \rangle \) and \( L \times \mathbb{P}^1 \). So, the birational map is clear.

Now, to take one Lagrangian to the next one, we argue in generality. Suppose we have 2 quasi-projective varieties \( X, X' \), with Zariski open subsets \( U \subseteq X, V \subseteq X' \), \( U \neq \emptyset \), such that there exists an isomorphism

\[
s : U \to V.
\]

If for a fixed quasi-projective variety \( Y \), we need two proper birational morphisms

\[
u_1 : Y \to X \quad \text{and} \quad u_2 : Y \to X'
\]

compatible with \( s \), then we have a single choice: take first the graph

\[
W := \{(x, s(x)) \in U \cup V,
\]

then take the closure \( T \) of \( W \) in \( X \times X' \). Then \( T \) has the two morphisms

\[
v_1 : T \to X \quad \text{and} \quad v_2 : T \to X'
\]

and any other \( (Y, u_1, u_2) \) must be obtained by composing \( (T, v_1, v_2) \) with a morphism \( f : Y \to T \), in such a way that we obtain

\[
u_1 := f \circ v_1 \quad \text{and} \quad u_2 := f \circ v_2.
\]

The argument in this section shows that we have a birational transformation taking \( L_j \) to \( L_{j+1} \), thus justifying the vertical downarrow \( \text{bir} \) appearing in Thm. 1.1 we now proceed to discuss the other side of the duality in focus here, namely singularities and vector bundles on their resolutions.

5. Duality for multiplicity \( n \) singularities

We describe a duality between vector bundles on 2 distinct minimal resolutions of toric singularities of multiplicity \( n \), which are both quotients of \( \mathbb{C}^2 \) by the cyclic group of \( n \) elements \( \mathbb{Z}/n\mathbb{Z} \), and whose toric cones are dual, they are:

\[
\mathcal{X}_n := \frac{1}{n} (1, 1) \quad \text{and} \quad \mathcal{X}_n' := \frac{1}{n} (1, n-1).
\]

These singularities are obtained by the following actions:

\[
\begin{pmatrix}
\rho & 0 \\
0 & \rho
\end{pmatrix}
\]

for \( \mathcal{X}_n \) and

\[
\begin{pmatrix}
\rho & 0 \\
0 & \rho^{-1}
\end{pmatrix}
\]

for \( \mathcal{X}_n' \),

where \( \rho \) is a primitive \( n \)-th root of unity, that is, \( \rho = e^{2 \pi i / n} \). In general the singularity \( \frac{1}{n} (1, a) \) is obtained from the action \((x, y) \mapsto (\rho x, \rho^a y)\).
A resolution of singularities $\tilde{X} \to X$ is called **minimal** if $\tilde{X} \to X$ with $X'$ smooth imply $\tilde{X} \cong X'$. Let $Z_n$ and $\overline{Y}_n$ denote the minimal toric resolutions of $\mathcal{X}_n$ and $\mathcal{X}_n^\vee$, respectively, depicted in Fig. 1 and 2. We observe that, in particular we have $Z_2 \cong \overline{Y}_2$, but $Z_n \not\cong \overline{Y}_n$ for $n \neq 2$.

The surface $Z_n = \text{Tot} \mathcal{O}_p(-n)$ contains a single rational curve with self-intersection $-n$, whereas $\mathcal{X}_n^\vee$ contains an isolated $A_{n-1}$-singularity and $\overline{Y}_n$ contains a chain of $n-1$ curves $E_i \cong \mathbb{P}^1$ for $1 \leq i \leq n-1$ whose intersection matrix $(E_i \cdot E_j)$ coincides with the negative of the Cartan matrix of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of type $A_{n-1}$.

Note that the Dynkin diagram $\cdots \to 1 \to -1 \to \cdots$ of type $A_{n-1}$ is precisely the graph dual to the system of curves $E_i$ in the resolution of $\overline{Y}_n$.

The surfaces $\mathcal{X}_n$ and $\mathcal{X}_n^\vee$ are toric varieties having fans formed by a single cone, calling $\sigma_{\mathcal{X}_n}$ and $\sigma_{\mathcal{X}_n}^\vee$ their respective fans, we have that $\sigma_{\mathcal{X}_n}$ is dual to $\sigma_{\mathcal{X}_n}^\vee$. In particular, for the case of $n = 2$ we also have that $\sigma_{\mathcal{X}_n} \cong \sigma_{\overline{Y}_2}$ is self-dual.

We now describe the coordinate rings of the singularities. We have $\mathcal{X}_n = \text{Spec } A$, where
\[
A = H^0(Z_n, \mathcal{O}) \cong \mathbb{C}[x_0, \ldots, x_n]/(x_i x_{j+1} - x_{i+1} x_j)_{0 \leq i < j < n}.
\]

Given that $\mathcal{X}_n \cong \mathbb{C}^2/\Gamma$, where $\Gamma$ is the group generated by $\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$ for $\rho$ a primitive $n$-th root of unity, we have $\Gamma \cong \mathbb{Z}/n\mathbb{Z}$, with $j \in \mathbb{Z}/n\mathbb{Z}$ corresponding to $\begin{pmatrix} \rho^j & 0 \\ 0 & \rho^j \end{pmatrix}$.

Functions on the quotient $\mathbb{C}^2/\Gamma$ are given by those functions on $\mathbb{C}^2$ which are invariant under $\Gamma$.

The algebra of functions on $\mathbb{C}^2$ is $\mathbb{C}[a, b]$ and $\Gamma$ acts by multiplication by $\rho$ on both $a$ and $b$. We thus have that $a^i b^j = (\rho a)^i (\rho b)^j = \rho^{i+j} a^i b^j$ if and only if $\rho^{i+j} = 1$, i.e. if and only if $i + j$ is a multiple of $n$.

One sees that $\mathbb{C}[a, b]^\Gamma$ (functions on $\mathbb{C}^2$ invariant under $\Gamma$) are generated by $a^n, a^{n-1} b, \ldots, a b^{n-1}, b^n$.

Now one can check that the invariants are
\[
\mathbb{C}[a, b]^\Gamma = \mathbb{C}[a^n, a^{n-1} b, \ldots, a b^{n-1}, b^n] \cong A
\]

with the resolution mapping $a^i b^{n-i} \mapsto x_i$ for $0 \leq i \leq n$.

so that $\mathbb{C}^2/\Gamma \cong \mathcal{X}_n$. This map looks quite similar to the Veronese embedding. In fact, $\mathcal{X}_n$ is the so-called affine cone over the Veronese curve (or rational normal curve) of degree $n$, i.e. $\mathcal{X}_n \cong \mathbb{C}^2/\Gamma$ is the affine cone over the image of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^n$ given by $[a : b] \mapsto [a^n : a^{n-1} b : \cdots : a b^{n-1} : b^n]$.

The duality between $\mathcal{X}_n$ and $\mathcal{X}_n^\vee$ is made clear by their toric fans. Just observe that each fan consists of a single cone, and the vectors forming the fan of $\mathcal{X}_n$ are perpendicular to those of the fan of $\mathcal{X}_n^\vee$ as depicted in Fig. 1 and 2.

6. Vector bundles on local surfaces

We now describe vector bundles on $Z_n$, the resolution of the isolated singularity $\mathcal{X}_n$. The surface $Z_n$ is the local model of the neighborhood of a rational line $\ell$ with self-intersection $-n$ in a complex
surface $X$. Thus, vector bundles on $Z_n$ model vector bundles around such a line $\ell$ in $X$. The case $n = 1$ occurs when blowing-up a smooth point, and was explored in [G].

Recently, a new complex surgery operation on vector bundles over $Z_n$, named grafting, was introduced in the context of mathematical physics (see [GS]). It provided an original explanation for the phenomenon of instanton decay in terms of curvature of the underlying space. Here we explore the geometric features of this grafting procedure. When considered from the point of view of grafting, bundles on $Z_n$ occur as building blocks of vector bundles on surfaces, in a sense somewhat analogue (and dual) to the use of the Lagrangian skeleton for building a symplectic manifold.

Let $E$ be a vector bundle on a compact complex surface $X$ which contains a $-n$ line. Let $F = E|_N$ be the restriction of $E$ to an open neighborhood $N$ of $\ell$ in the analytic topology. Grafting is obtained by replacing $F$ by another vector bundle $F'$, which is then glued to $E|_{X \setminus N}$. Note that after grafting the top Chern class of $E$ will in general change, but not the first one. Therefore, this surgery procedure is not obtained by an elementary transformation. The gluing itself is done over $N \setminus \ell$ which is identified with the complement of the zero section in $Z_n$ called the collar defined below; such a gluing is possible because vector bundles on $Z_n$ are completely determined by their restriction to a finite formal neighborhood of $\ell$, see [BGK]. We now describe explicit local data on $Z_n$ used to classify vector bundles on them. These vector bundles restricted to the collars will give rise to the dual objects to the components of the skeleton described above.

For each integer $n$, we have the surface $Z_n = \text{Tot}(\mathcal{O}_p\langle -n \rangle)$. The complex manifold structure can be described by gluing the open sets

$$ U = \mathbb{C}[z, u] \quad \text{and} \quad V = \mathbb{C}[\xi, v] $$

by the relation

$$ (\xi, v) = (z^{-1}, z^n u) \quad (6.1) $$

whenever $z$ and $\xi$ are not equal to 0. We call (6.1) the canonical coordinates for $Z_n$.

Using canonical coordinates, the contraction $Z_n := \text{Tot}(\mathcal{O}_p\langle -n \rangle) \to \mathcal{X}_n$ sends $z^i u \mapsto x_i$, where $x_i$ are the coordinates of $\mathcal{X}_n$ as described in (5.1).

Let $E$ be a rank $r$ holomorphic vector bundle on $Z_n$. The restriction of $E$ to the zero section $\ell \simeq \mathbb{P}^1$ is a rank $r$ bundle on $\mathbb{P}^1$, which by Grothendieck’s lemma splits as a direct sum of line bundles. Thus, $E|_{\ell} \simeq \mathcal{O}_{\mathbb{P}^1}(j_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(j_r)$. Following [B], we call $(j_1, \ldots, j_r)$ the splitting type of $E$. When $E$ is a rank 2 bundle with first Chern class 0, then the splitting type is $(j, -j)$ for some $j \geq 0$ and we say for short that $E$ has splitting type $j$.

There are many rank 2 vector bundles on $Z_n$. For each fixed splitting type, they can be obtained as a quotient of $\text{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$. Considering isomorphism classes of vector bundles modulo holomorphic equivalence, moduli spaces were obtained as follows.

**Proposition 6.2.** [BGK, Thm. 4.11] *The moduli space of irreducible SU(2) instantons on $Z_n$ with charge (and splitting type) $j$ is a quasi-projective variety of dimension $2j - n - 2$.***

An equivalent formulation in terms of vector bundles is:
Corollary 6.3. The moduli space of (stable) rank 2 bundles on $Z_n$ with vanishing first Chern class and local second Chern class $j$ is a quasi-projective variety of dimension $2j - n - 2$.

Even though vector bundles on $Z_n$ are many, their restrictions to the collars have very simple behaviour, as we now shall demonstrate.

We denote by $\ell$ the $\mathbb{P}^1$ contained in $Z_n$ corresponding to the zero section of the corresponding vector bundles, and we set

$$Z_n^0 := Z_n \setminus \ell.$$  \hfill (6.4)

We call $Z_n^0$ the collar of $\ell$ in $Z_n$. Using the canonical coordinates for $Z_n$ we obtain canonical coordinates for the collar by setting

$$Z_n^0 = U^o \cup V^o,$$

with the complex manifold structure obtained by gluing the open sets

$$U^o = \mathbb{C} \times \mathbb{C} - (0) \cong \mathbb{C}[z, u, u^{-1}] \quad \text{and} \quad V^o = \mathbb{C} \times \mathbb{C} - (0) \cong \mathbb{C}[\xi, v, v^{-1}]$$

by the relation

$$(\xi, v) = (z^{-1}, z^n u).$$

Lemma 6.5. The homotopy type of $Z_n^0$ is that of an $S^1$-bundle over $S^2$, and $\pi_1(Z_n^0) = \mathbb{Z}/n\mathbb{Z}$.

Proof. Let $D = \{z, |z| \leq 1\}$ be the unit disc in $\mathbb{C}$, denoted $D^+$ when oriented positively, and $D^-$ when oriented negatively. The homotopy type of $Z_n^0$ is then that of

$$U^o \sim U^+ = D^+ \times S^1 = [z, u = e^{i\theta}] \quad \text{and} \quad V^o \sim U^- = D^- \times S^1 = [\xi, v = e^{i\phi}],$$

with identification in $U^+ \cap V^-$ given by

$$(\xi = e^{i\alpha}, \nu = e^{i\phi}) = (z^{-1} = e^{-i\alpha}, \nu = z^n u = e^{i(\theta + na)}).$$

The result of the identification is an $S^1$-bundle over $S^2 = D^+ \cup D^-$, with the $S^1$ fibers identified via the gluing map $z^n$ which has degree $n \in \pi_1(SO(2)) = \mathbb{Z}$ since $SO(2) \simeq S^1$.

Let $\iota : Z_n^0 \to Z_n$ denote the inclusion, and set

$$L_n(j) := \iota^* \Theta Z_n(j).$$

Proposition 6.6. For each $n$, the group of all isomorphism classes of line bundles $\{L_n(j), j \in \mathbb{Z}\}$ is cyclic of order $n$, hence $\mathbb{Z}/n\mathbb{Z}$.

Proof. Note that $\text{Pic}(Z_n) = Z$. Each line bundle over $Z_n$ with first Chern class $j$ is isomorphic to $\Theta Z_n(j)$ and therefore can be represented by a transition matrix $(z^{-j})$. Since in canonical coordinates we have that $u^{-1} \neq 0$ and $v \neq 0$ on the collar $Z_n^0$, we may change coordinates as follows

$$(z^{-j}) \simeq (v)(z^{-j})(u^{-1}) = (z^n u \cdot z^{-j} \cdot u^{-1}) = (z^{-j+n}),$$

i.e., over $Z_n^0$, the bundles $L_n(j)$ and $L_n(j-n)$ (defined by $(z^{-j})$ and $(z^{-j+n})$ respectively) are isomorphic. Moreover, if $j_1 \equiv j_2 \mod n$, then $L_n(j_1)$ and $L_n(j_2)$ are isomorphic. The proof that the cases $1, 2, ..., n-1$ are not pairwise isomorphic is included in the proof of Prop. 6.7. \hfill $\Box$

The following is a slightly rephrased version of [GKM, Prop. 4.1].

Proposition 6.7. Let $E_1$ and $E_2$ be rank 2 bundles over $Z_n$ with vanishing first Chern classes and splitting types $j_1$ and $j_2$, respectively. There exists an isomorphism $E_1|Z_2 \simeq E_2|Z_2$ if and only if $j_1 \equiv j_2 \mod n$. In particular, $E_1$ is trivial over $Z_n$ if and only if $j_1 \equiv 0 \mod n$.

Proof. We first claim that the bundle $\Theta \ell(-n)$ is trivial on $Z_n^0$. In fact, if $u = 0$ is the equation of $\ell$, then $s(z, u) = u$ determines a section of $\Theta \ell(-n)$ that does not vanish on $Z_n^0$.

If a bundle $E$ over $Z_n$ has splitting type $j$, then by definition, $E|\ell \cong \Theta \ell(-j) \oplus \Theta \ell(j)$. So there is a surjection $\rho : E|\ell \rightarrow \Theta \ell(j)$, and a corresponding elementary transformation, resulting in a vector bundle $E' = \text{Elm}_{\ell(j)}(E)$ which splits over $\ell$ as $\Theta \ell(-n) \oplus \Theta \ell(j + n)$, see [BGK, Sec. 3]. Therefore we can use the surjection $\rho : E|\ell \rightarrow \Theta \ell(j + n)$ to perform a second elementary transformation, and we
obtain the bundle $E'' = \text{Elm}_{\Theta_t(j+n)}(E')$, which splits over $t$ as $\Theta_t(-j) \oplus \Theta_t(j+2n)$ and has first Chern class $2n$. Tensoring by $\Theta_t(-n)$ we get back to a $sl_2(\mathbb{C})$-bundle with splitting type $j+n$. Hence, the transformation
\[
\Phi(E) = \Theta(-n) \circ \text{Elm}_{\Theta_t(j+n)} \circ \text{Elm}_{\Theta_t(j)}(E)
\]
increases the splitting type by $n$ while keeping the isomorphism type of $E$ over $Z_n^0$. So we need only to analyze bundles with splitting type $j < n$.

If $j = 0$, the bundle is globally trivial on $Z_n$. If $j \neq 0$, then $E|_{Z_n^0}$ induces a non-zero element on the fundamental group $\pi_1(Z_n^0) = \mathbb{Z}/n\mathbb{Z}$.

By Lem. 6.5 the collar $Z_n^0$ has the homotopy type of an $S^1$-bundle over $S^2$ and $\pi_1(Z_n^0) = \mathbb{Z}/n\mathbb{Z}$. Therefore $H_1(Z_n^0, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and by Poincaré duality $H^2(Z_n^0, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. The exponential sheaf sequence
\[
0 \to \mathcal{O} \to \Theta \to \Theta^* \to 0
\]
induces the first Chern class map
\[
H^1(Z_n^0, \Theta^*) \to H^2(Z_n^0, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z},
\]
and
\[
L_n(j) \equiv j \mod n.
\]

In this section we have described vector bundles on $Z_n$, their moduli, and behavior on collars. We will see next that each splitting type is connected to the lower ones by deformations.

### 7. Deformations

In this section we justify the vertical upwards arrow appearing in diagram (1.2). We start with a vector bundle $E$ with splitting type $(j, -j)$ on $Z_n$, so that $E$ may be written as an extension
\[
0 \to \Theta(-j) \to E \to \Theta(j) \to 0. \tag{7.1}
\]
Alternatively, we may also choose to write $E$ as an extension of $\Theta(j+s)$ by $\Theta(-j-s)$ for any $s > 0$. To see this, just observe that there exist inclusions
\[
H^1(\Theta(-2j)) = \text{Ext}^1(\Theta(j), \Theta(-j)) \leftarrow \text{Ext}^1(\Theta(j+s), \Theta(-j-s)) = H^1(\Theta(-2j-2s)).
\]

Let $p$ be the extension class corresponding to representing the bundle $E$ by the exact sequence (7.1). Next, fixing an injection $t$, consider the family $t \cdot t(p)$ of extensions of $\Theta(j+1)$ by $\Theta(-j-1)$. For such a family, when $t = 0$ we obtain $\Theta(j+1) \oplus \Theta(-j-1)$ but when $t = 1$ we obtain $E$.

Now, using induction on $j$, we conclude that every bundle on $Z_n$ occurs as a deformation of another bundle with splitting type as high as desired. In particular, such behavior of lowering the splitting type via deformations is also observed over the collars, justifying the vertical uparrow $\text{bir}$ appearing in Thm. 1.1. We now combine this vertical uparrow with the vertical downarrow $\text{bir}$ described in Sec. 4. There is a 1-1 correspondence between elements of the skeleton and splitting types on the collar. Given that this correspondence is obtained via a combination of 2 dualities, we call it a duality transformation. We denote it by a horizontal double arrow:
\[
L_j \leftrightarrow \Theta_{Z_n}(j) \oplus \Theta_{Z_n}(-j).
\]

Collecting horizontal and vertical arrows together, we obtain the commutative diagram claimed in Thm. 1.1.

In conclusion, we have given an explicit geometric description of a duality between Lagrangians in the skeletons of cotangent bundles and vector bundles on collars. The symplectic side of the duality studies the components of the Lagrangian skeleton of cotangent bundles over $n$-dimensional projective spaces. The complex algebraic side considers only 2-dimensional complex varieties. These 2 are rather different types of objects. So, a priori this duality was not at all evident, but was abstractly predicted by a combination of 2 other types of duality.
In future work, we intend to pursue a generalization of this type of duality to the realm of Calabi–Yau threefolds, investigating what symplectic manifolds and Lagrangians are dual to vector bundles on local Calabi–Yau varieties and what operations occur as dual to deformations of vector bundles, see [GR, GKRS]. The latter promises to be a challenging question, given the existence of infinite dimensional families of deformations in the case of 3-dimensional varieties, see [BGS].

Acknowledgements

E. Ballico is a member of MIUR and GNSAGA of INdAM (Italy). E. Gasparim was partially supported by the Vicerrectoría de Investigación y Desarrollo Tecnológico de la Universidad Católica del Norte (Chile) and by the Department of Mathematics of the University of Trento. B. Suzuki was supported by the ANID-FAPESP cooperation 2019/13204-0. F. Rubilar acknowledges support from Beca Doctorado Nacional – Folio 21170589. Gasparim, Rubilar and Suzuki acknowledge support of MathAmSud GS&MS 21-MATH-06 - 2021/2022. We are grateful to I. Cheltsov for inviting us to contribute to this volume.

References

[B] E. Ballico, Rank 2 vector bundles in a neighborhood of an exceptional curve of a smooth surface, Rocky Mountain J. Math. 29 n.4 (1999) 1185–1193.
[BGK] E. Ballico, E. Gasparim, T. Köppe, Vector bundles near negative curves: moduli and local Euler characteristic, Comm. Algebra 37 (2009) 2688–2713.
[BGS] E. Ballico, E. Gasparim, B. Suzuki, Infinite dimensional families of Calabi-Yau threefolds and moduli of vector bundles, J. Pure Appl. Algebra 225 n.4 (2021) 106554.
[BF] M. Brion, B. Fu, Symplectic Resolutions for Conical Symplectic Varieties, Int. Math. Res. Not. IMRN 2015 n.12 (2014) 4335–4343.
[BLPW] T. Braden, A. Licata, N. Proudfoot, B. Webster, Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality, arXiv:1407.0964.
[G] E. Gasparim, The Atiyah–Jones conjecture for rational surfaces, Advances Math. 218 (2008) 1027–1050.
[GKM] E. Gasparim, T. Köppe, P. Majumdar, Local Holomorphic Euler Characteristic and Instanton Decay, Pure Appl. Math. Q. 4 n.2 (2008) 363–382.
[GKRS] E. Gasparim, T. Köppe, F. Rubilar, B. Suzuki, Deformations of noncompact Calabi–Yau threefolds, Rev. Colombiana de Matemáticas 52 n.1 (2018) 41–57.
[GR] E. Gasparim, R. Rubilar, Deformations of Noncompact Calabi–Yau Manifolds, Families and Diamonds, Contemp. Math. 766 (2021) 117–132.
[GS] E. Gasparim, B. Suzuki Curvature grafted by instantons, Indian J. Phys. 95 n.8 (2021) 1631–1638.
[Ru] W.-D. Ruan, The Fukaya category of symplectic neighborhood of a non-Hausdorff manifold, arXiv:0602119.
[STW] V. Shende, D. Treumann, H. Williams, On the combinatorics of exact Lagrangian surfaces, arXiv:1603.07449.