Abstract. We introduce the gonosomal algebra. Gonosomal algebra extend the evolution algebra of the bisexual population (EABP) defined by Ladra and Rozikov. We show that gonosomal algebras can represent algebraically a wide variety of sex determination systems observed in bisexual populations. We illustrate this by about twenty genetic examples, most of these examples cannot be represented by an EABP. We give seven algebraic constructions of gonosomal algebras, each is illustrated by genetic examples. We show that unlike the EABP gonosomal algebras are not dibaric. We approach the existence of dibaric function and idempotent in gonosomal algebras.

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Key words. Baric algebra, dibaric algebra, commutative duplication, non commutative duplication, bisexual population, gonosomal gene, sex determining systems.

1. INTRODUCTION

To construct an algebraic model of sex-linked inheritance (i.e. controlled by sex), it is necessary to solve two problems: the asymmetry of the transmission of genes and the wide variety of sex-determination systems.

In populations with separate sexes, bisexual and diploid, the sexual differentiation is controlled in the majority of cases by a pair of chromosomes: the sex chromosomes also called gonosomes (or heterochromosomes, or heterosomes). The gonosomes often have different shapes (dimorphism) which induces an asymmetry in the transmission of sex-linked traits.

In the animal kingdom, sex is determined by different systems, we distinguish five systems: $XY$, $WZ$, $X0$, $Z0$ and $WXY$, which can be added their multiple variants.

In the $XY$ sex-determination system observed in most mammals, homogametic $XX$ individuals are females and heterogametic $XY$ are males. The $XY$-system has multiple variants as $X_1X_2Y$ and $XY_1Y_2$. In the $X_1X_2Y$-system, females are $X_1X_2X_3X_4$ and males $X_1X_2Y$ while in the system $XY_1Y_2$ females are $XX$ and males $XY_1Y_2$. Systems with more gonosomes as $X_1X_2X_3X_4X_5Y$ or $X_1X_2X_3X_4X_5Y_1Y_2Y_3Y_4Y_5$ have also been observed.

The $WZ$-system is mainly found in birds, females are heterogametic $WZ$ while males are homogametic $ZZ$. This system also accepts multiple variants such that $W_1W_2Z$, $WZ_1Z_2$, $W_1W_2Z_1Z_2$.

The $X0$-system is mainly observed in Hymenoptera (bees, wasps, ...). In this system sex is controlled by a single $X$ chromosome, females have two $X$ chromosomes, their genotype is $XX$, while males have one, they are $X0$ (zero indicates the absence of the second $X$ chromosome). It has been observed in several species of spiders the following systems $X_1X_20$, $X_1X_2X_30$, $X_1X_2X_3X_4$. As for $Z0$ sex determination system mainly observed among Lepidoptera, females are $Z0$ and males $ZZ$.

Finally, the $WXY$-system observed in several species of tropical fish is more complex: an individual having a gonosome $Y$ is male unless it is coupled with a $W$ chromosome, so
in this system individuals with genotypes \( XY \) or \( YY \) are males and those of genotypes \( XX \), \( WX \) and \( WY \) are female.

However we shall see later in this work through numerous examples that there are many other sex determination systems.

A gene is said sex-linked or gonosomal if it is located on the sex chromosomes. Because to the dimorphism of gonosomes there are two kinds of gonosomal gene. Indeed, in the \( XY \) and \( WZ \) systems two parts on gonosomes are observed: one part homologous (or pseudo-autosomal) where genes are common to both gonosome types and a differential part where a locus located on a gonosome has no counterpart on the other. There are therefore two types of gonosomal genes: a gene is pseudo-autosomal (or partially sex-linked) if the locus is on the homologous parts; it is gonosomal (or completely sex-linked) when the locus is on the differential part of the sex chromosomes (cf. \([5]\)).

In this work, after recalling the algebraic models of sex-linked inheritance, we give a genetic example that can not be represented by these algebras, this leads us to extend these algebras and define the gonosomal algebras. Then we give seven algebraic constructions of gonosomal algebras and illustrating them with examples we see that these algebras can represent algebraically a wide variety of genetic phenomena related to sex as: i) temperature-dependent sex determination; ii) sequential hermaphrodi

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\[ WZ \]

\[ WXY \]

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\[ Y-linked \]

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The second definition is due to Gonshor [5], it does not depend on a given base, it is built from the gametic state by the sex duplication method. Starting from a baric algebra \((A, \omega)\), we define on the space \(A \oplus A \oplus A\) the structure algebra:

\[
(x \otimes y \otimes z) (x' \otimes y' \otimes z') = \frac{1}{2} (xy \otimes z' + x' y' \otimes z) \oplus \frac{1}{2} (\omega (z') xy + \omega (z) x' y').
\]

The resulting algebra is called the sex-linked duplicate.

In [15], Wörz-Busekros showed that the Etherington and Gonshor definitions are equivalent.

Recently Ladra and Rozikov [11] introduced a more general definition. Starting from the canonical basis \((e_1, \ldots, e_{n+v})\) of \(\mathbb{R}^{n+v}\), \(e_i^{(f)} = e_i, \ i = 1, \ldots, n\) and \(e_i^{(m)} = e_{n+i}, \ i = 1, \ldots, v\), they provide \(\mathbb{R}^{n+v}\) with the structure algebra:

\[
e_i^{(f)} e_p^{(m)} e_j^{(f)} = \frac{1}{2} \left( \sum_{k=1}^{n} p_{ip,k} e_k^{(f)} + \sum_{l=1}^{v} p_{ip,l} e_l^{(m)} \right),
\]

\[
e_i^{(f)} e_j^{(f)} = 0, \quad 1 \leq i, j \leq n,
\]

\[
e_p^{(f)} e_q^{(m)} = 0, \quad 1 \leq p, q \leq v,
\]

\[
\sum_{k=1}^{n} p_{ip,k} = \sum_{l=1}^{v} p_{ip,l} = 1, \quad (1 \leq i \leq n, 1 \leq p \leq v).
\]

This algebra is named evolution algebra of the bisexual population by their authors.

3. GONOSOMAL ALGEBRA, DEFINITION AND EXAMPLES

In this section we give a definition of a sex-linked algebra that extends the one given in [11]. This extension of the definition of [11] finds its source in the following example.

3.1. Introductory example.

**Example 1. Heredity of hemophilia.**

Hemophilia is a genetic disorder linked to the X chromosome, it is due to mutations in two genes located at the end of the long arm of gonosome X. This is a lethal recessive genetic disease that is lethal in the homozygous state, it follows that if \(X^h\) denotes the \(X\) chromosome carrying hemophilia, there are only two female genotypes: \(XX\) and \(XX^h\) (genotype \(X^hX^h\) is lethal) and two male genotypes: \(XY\) and \(X^hY\). The results of the four kinds of crosses are:

\[
XX \times XY \Rightarrow \frac{1}{2}XX, \frac{1}{2}XY; \quad XX \times X^hY \Rightarrow \frac{1}{2}XX^h, \frac{1}{2}XY;
\]

\[
XX^h \times XY \Rightarrow \frac{1}{2}XX, \frac{1}{2}XX^h, \frac{1}{2}XY, \frac{1}{4}X^hY; \quad XX^h \times X^hY \Rightarrow \frac{1}{2}XX^h, \frac{1}{2}XY, \frac{1}{4}X^hY.
\]

Algebraically we represent this by the following commutative algebra defined on the basis \((e_1, e_2, \tilde{e}_1, \tilde{e}_2)\) of \(\mathbb{R}\)-vector space:

\[
e_1 \tilde{e}_1 = \frac{1}{3} e_1 + \frac{1}{3} \tilde{e}_1, \quad e_1 \tilde{e}_2 = \frac{1}{3} e_2 + \frac{1}{3} \tilde{e}_1,
\]

\[
e_2 \tilde{e}_1 = \frac{1}{3} e_1 + \frac{1}{3} e_2 + \frac{1}{3} \tilde{e}_1 + \frac{1}{3} \tilde{e}_2, \quad e_2 \tilde{e}_2 = \frac{1}{3} e_2 + \frac{1}{3} \tilde{e}_1 + \frac{1}{3} \tilde{e}_2,
\]

\[
e_i e_j = 0 \quad \tilde{e}_i \tilde{e}_j = 0, \quad (i, j = 1, 2).
\]

It is clear that this algebra is not an evolution algebra of the bisexual population as defined in [11].
3.2. Definition of a gonosomal algebra.

The above example leads us to put the following definition.

**Definition 2.** A $K$-algebra $A$ is gonosomal if there is a basis $\mathcal{B} = \{e_i; 1 \leq i \leq n\} \cup \{\tilde{e}_p; 1 \leq p \leq m\}$ of $A$ verifying for all $1 \leq i, j \leq n$ and $1 \leq p, q \leq m$:

$$e_i\tilde{e}_p = \tilde{e}_p e_i = \sum_{k=1}^{n} \gamma_{pk} e_k + \sum_{r=1}^{m} \tilde{\gamma}_{pr} \tilde{e}_r,$$

$$e_i e_j = 0,$$

$$\tilde{e}_p \tilde{e}_q = 0$$

$$\sum_{k=1}^{n} \gamma_{pk} + \sum_{r=1}^{m} \tilde{\gamma}_{pr} = 1.$$

The basis $\mathcal{B}$ is called gonosomal basis of $A$.

In this definition the vectors of $(e_i)_{1 \leq i \leq n}$ (resp. $(\tilde{e}_p)_{1 \leq p \leq m}$) are interpreted as genetic types observed in females (resp. in males), the structure constant $\gamma_{pk}$ (resp. $\tilde{\gamma}_{pr}$) represents the female (resp. male) proportion of type $e_k$ (resp. $\tilde{e}_r$) in the progeny of a female type $e_i$ with a male type $\tilde{e}_p$.

**Remark 3.** If in the definition we have $\sum_{k=1}^{n} \gamma_{pk} = \sum_{r=1}^{m} \tilde{\gamma}_{pr}$ for all $1 \leq i \leq n$ and $1 \leq p \leq m$, then noting $P_{ip}^{(f)} = 2\gamma_{ip} k$ and $P_{ip}^{(m)} = 2\tilde{\gamma}_{ip}$ we have $e_i\tilde{e}_p = \sum_{k=1}^{n} P_{ip}^{(f)} e_k + \sum_{r=1}^{m} P_{ip}^{(m)} \tilde{e}_r$ with $\sum_{k=1}^{n} P_{ip}^{(f)} e_k = \sum_{r=1}^{m} P_{ip}^{(m)} \tilde{e}_r = 1$, we find the definition of the evolution algebra of the bisexual population given in [11].

**Remark 4.** A gonosomal algebra is commutative by definition. It is not in general associative, so for the algebra $A$ of example we have: $e_1(\tilde{e}_1 e_2) - (e_1 \tilde{e}_2) e_1 = \frac{1}{4}(e_1 + \tilde{e}_1) \neq 0$.

The gonosomal algebras form a new class of non-associative algebra, indeed the algebra $A$ given in example is not Lie because $e_1(\tilde{e}_1 e_2) + e_1(e_2 e_1) + e_2(e_1 \tilde{e}_1) \neq 0$ and taking in $A$, $x = \frac{1}{4} e_1 + \frac{1}{4} \tilde{e}_1$ and $y = e_1$ we obtain $x^2(yx) - (x^2 y)x = \frac{1}{16} (e_1 + e_1)$ therefore $A$ is not either Jordan, nor power associative because $x^2 x^2 - x^4 = -\frac{1}{16} (e_1 + e_1)$, nor alternative because we have $x^2 y - x (yx) = \frac{1}{32} (e_1 + e_1)$.

3.3. Examples of gonosomal algebras.

We will show that the gonosomal algebras allows to represent algebraically a wide variety of genetic phenomena linked to sex. In these examples, unless otherwise specified, the term genetic type represents as well alleles, genotypes or collection of genes whose loci are on gonosomes. We start with some unusual examples.

**Example 5. Temperature-dependent sex determination.**

In reptiles (snakes, crocodiles, turtles, lizards) we find two types of sex determination, either a genotypic determination controlled according to the species by $XY$- or $ZW$-system, either a determination depending on the incubation temperature of eggs (or TSD: temperature-dependent sex determination). TSD is observed in all species of crocodilians and most turtles. TSD is controlled by three temperature ranges, the eggs subject to feminizing temperatures (TF) (resp. masculinising (TM)) give rise to 100% or a majority of females (resp. males) and those subject to transition temperatures (TRT) provide 50% females and 50% males.

Algebraically, we consider the space $A$ with basis $(e_i, \tilde{e}_i)_{1 \leq i \leq n}$ where $e_i$ (resp. $\tilde{e}_i$) are female (resp. male) genetic types present in a population and subject to TSD. We note $\tau_1$, $\tau_2$ and $\tau_3$ the probability that eggs are incubated at TF, TM and TRT respectively, thus we have $\tau_1 + \tau_2 + \tau_3 = 1$. For each $r = 1, 2$ ($r = 1$ for TF, $r = 2$ for TM) we note $\mu_r$ respectively the proportions of females and males arising from eggs
placed in the environment \( r \), thus we have \( \mu_r + \bar{\mu}_r = 1 \) with \( \mu_1 > \bar{\mu}_1 \geq 0 \) and \( 0 \leq \mu_2 < \bar{\mu}_2 \).

Finally, if for all \( 1 \leq i, p \leq n \) we denote \( \theta_{ipk} \) the egg proportion of \( e_k \) type in the laying of a female \( e_i \) crossed with a male \( \bar{e}_p \), thus \( \sum_{k=1}^{n} \theta_{ipk} = 1 \). Then the space \( A \) equipped with the following algebra structure:

\[
e_i e_j = \bar{e}_p \bar{e}_q = 0,
\]

\[
e_i \bar{e}_p = \left( \mu_1 \tau_1 + \mu_2 \tau_2 + \frac{1}{2} \tau_3 \right) \sum_{k=1}^{n} \theta_{ipk} e_k + \left( \bar{\mu}_1 \tau_1 + \bar{\mu}_2 \tau_2 + \frac{1}{2} \tau_3 \right) \sum_{k=1}^{n} \theta_{ipk} \bar{e}_k
\]

is a gonalosomal algebra and the product \( e_i \bar{e}_p \) gives the genetic distribution of progeny of a female \( e_i \) with a male \( \bar{e}_p \).

**Example 6. Sequential hermaphroditism.**

It is observed in many species of fish that the sex of an individual changes during his life. We distinguish two cases of sequential hermaphroditism: protogyny and prootandry. In protogyny (9% of fish families) individuals are first female then become male, the opposite occurs in the protoandry (1% of fish families) when a male change sex to female. This sex change occurs as a result of the disappearance of the dominant male or female.

This situation can be represented by a gonalosomal algebra. For this we consider in a protogynous hermaphroditic population, a gene whose types are noted \( e_1, \ldots, e_n \) when they are observed in females and \( \bar{e}_1, \ldots, \bar{e}_n \) in males. The progeny of a female type \( e_i \) with a male \( \bar{e}_j \) consists of individuals having a female phenotype. Therefore if \( \lambda_{ijk} \) is the frequency of type \( e_k \) in the progeny of the cross between \( e_i \) with \( \bar{e}_j \) before sexual inversion, then the distribution of types in offspring is \( \sum_{k=1}^{n} \lambda_{ijk} e_k \).

Androgenesis is a rare process observed in fish, molluscs and insects. Androgenesis is the production of an offspring containing exclusively the nuclear genome of the fathering male via the maternal eggs. During fertilization spermatozoon penetrates the ovum which causes the expulsion of the female genome. In some species the eggs give haploid individuals, in others chromosomes are immediately duplicated after fertilization to give diploid individuals and in other species, there is a polysemry that is to say, the ovum is simultaneously fertilized by several spermatozoa, in this case we obtain diploid or polyploid individuals.

For example, in bivalves of genus *Corbiculla*, androgenesis is observed in four species which are simultaneous hermaphroditic, i.e. male and female reproductive organs are mature at the same time. In these species ova are fertilized by unreduced spermatozoa, i.e. whose genetic composition is identical to that of the somatic cells of the male transmitter. This mode of reproduction allows the coexistence and interbreeding of individuals di-, tri- and tetraploid within the same species of *Corbiculla*.

Algebraically, let be \( e_1, \ldots, e_n \) and \( \bar{e}_1, \ldots, \bar{e}_n \) respectively the ova and spermatozoa genetic types carried by *Corbiculla*. Let \( \theta \) the proportion of eggs among the gametes of an individual of type \( i \), we define on the space \( A \) with basis \( \{ e_i, \bar{e}_j \}_{1 \leq i, p \leq n} \) the products \( e_i \bar{e}_j = \theta_i e_j + (1 - \theta_j) \bar{e}_j \), other products are zero, then the algebra \( A \) is gonalosomal.

**Example 8. Sex determination by thelytokous parthenogenesis or by gynogenesis.**

In some species a female gamete develops an embryo without fertilization by a male gamete, this form of asexual reproduction is called parthenogenesis. There are several forms of parthenogenesis, one is the thelytoky. In thelytokous parthenogenesis a diploid female gives birth only to diploid females and therefore the population consists solely of females.
Another phenomenon related to the thelytokous parthenogenesis is gynogenesis (or pseudogamy or merospermy) which is observed in fish, amphibians and insects. Gynogenesis requires the fertilization of the ovum by a sperm cell of a close species what activates its development in zygote, the male genome degenerates or is eliminated, thus it is not expressed in the offspring which consists only of females.

To represent algebraically these two situations, we consider the gonosomal \( \mathbb{R} \)-algebra defined on a basis \( \{ e_i \}_{1 \leq i \leq n} \cup \{ e \} \) by \( e_i e = \sum_{k=1}^{n} \mu_{ik} e_k \) where \( e_1, \ldots, e_n \) symbolize female genotypes and \( \mu_{ik} \) the mutation rate from type \( e_i \) to type \( e_k \).

**Example 9. Bacterial conjugation.**

Bacterial conjugation is a transfer of genetic material between bacterial cells, a donor (called male) to a recipient (called female), that is why bacterial conjugation is often regarded as the bacterial equivalent of sexual reproduction. It is controlled by a conjuga-

tive factor carried by a plasmid. Plasmids are double-stranded, circular DNA molecules, present in the cytoplasm of bacteria that replicate autonomously. Plasmids are not essential to a normal activity of bacteria but they carry genes that provide a selective advantage to the holder (antibiotic resistance, increased pathogenicity, bacteriocins synthesis inhibiting the growth of other bacteria, acquisition of new metabolic properties). In addition to these genes, some plasmids carry a conjugative factor composed of several genes that control their transfer to another bacterial cell. The most studied conjugative factor is the factor \( F \), it contains genes that encode the synthesis of pili allowing a bacterium \( F^+ \) to dock with a bacterium \( F^- \), surface exclusion genes that prevent two bacteria \( F^- \) to moor and genes that allow the synthesis and transfer of one copy of the plasmid into a bacterium \( F^- \). At the end of the conjugation, the factor \( F \) persists in the donor bacterium which stays \( F^+ \) and a copy of the plasmid carrying this factor is acquired by the recipient bacterium which becomes \( F^- \).

Algebraically, let \( a_1, \ldots, a_n \) be the different types of bacterial chromosomes and \( b_1, \ldots, b_m \) the plasmid types carrying a conjugative factor observed in a colony of bacteria. If we put \( e_i = a_i \) and \( e_{p,q} = a_p \otimes b_q \) then the space \( A \) with the basis \( \{ e_i \} \cup \{ e_{p,q} \} \) and the commu-
tative product \( e_i e_j = 0, e_{p,q} e_{r,s} = 0, e_{p,q} e_{r,s} = \frac{1}{\tau} e_{p,q} + \frac{1}{\tau} e_{r,s} \) is a gonosomal algebra which models the bacterial conjugation. If \( 0 \leq \tau \leq 1 \) denote conjugation rate, starting from two populations of bacteria \( x = \sum a_i e_i + \sum_{p,q} a_{p,q} e_{p,q} \) and \( y = \sum b_i e_i + \sum_{p,q} b_{p,q} e_{p,q} \) where \( \sum a_i + \sum_{p,q} a_{p,q} = 1 \) and \( \sum b_i + \sum_{p,q} b_{p,q} = 1 \), then the product obtained after conjuga-
tion of \( x \) and \( y \) is given by the product \( x \ast y = \tau xy + \frac{1}{\tau^2} (x + y) \).

**4. Gonosomal algebras constructions**

The definition of gonosomal algebra is very general, it allows to represent a wide variety of sex-linked inheritance situations, but the counterpart is that we can not give an intrinsic definition similar to that of Gonshor for zygotic algebra of sex-linked inheritance.

In the following we give seven methods to construct gonosomal algebras. These con-
structions are obtained:

- by reduction of gonosomal algebra;
- from the duplicate of a baric algebra;
- from two baric algebras;
- from two algebra laws;
- from three linear forms and three linear maps;
- from a baric algebra and two linear maps;
- from a baric algebra and a gonosomal algebra.

We can see each construction as a particular gonosomal algebra class that can be studied for its own interest. Each construction is illustrated by genetic examples.
In the following constructions we often use the the notion of baric algebra. A $K$-algebra $A$ is baric if it admits a non trivial algebra morphism $\omega : A \to K$, called weight morphism of $A$, we note $(A, \omega)$ to indicate that $A$ is weighted by $\omega$ and for $x \in A$, the scalar $\omega(x)$ is called the weight of $x$. We also use the following result (cf. [10], lemma 1.10) : a finite-dimensional $K$-algebra $A$ is baric if and only if $A$ has a basis $(e_1, \ldots, e_n)$ such that $e_ie_j = \sum_{k=1}^n \gamma_{jk} e_k$ with $\gamma_{jk} \in K$ and $\sum_{k=1}^n \gamma_{jk} = 1$ $(i, j = 1, \ldots, n)$.

4.1. Construction by reduction of gonosomal algebra.

Starting from a gonosomal algebra we can construct others by reducing the gonosomal basis. In the result below, for any integer $k \geq 1$ we note $[1,k] = \{1, \ldots, k\}$.

**Proposition 10.** Let $A$ be a gonosomal $K$-algebra, $(e_i)_{i \in [1,n]} \cup (\tilde{e}_p)_{p \in [1,m]}$ a gonosomal basis of $A$ with $e_i e_p = \sum_{k=1}^n \gamma_{pk} e_k + \sum_{r \in J} \tilde{y}_{pr} \tilde{e}_r$. If there is $I \subseteq [1,n]$ and $J \subseteq [1,m]$ such that for all $i \in [1,n] \setminus I$ and $p \in [1,m] \setminus J$ we have $\sigma_{ip} = 1 - \sum_{k \in I} \gamma_{pk} + \sum_{r \in J} \tilde{y}_{pr} \neq 0$, then the subspace spanned by $(e_i)_{i \in [1,n] \setminus I} \cup (\tilde{e}_p)_{p \in [1,m] \setminus J}$ with multiplication

$$e_i * e_p = \sigma_{ip}^{-1} \left( \sum_{k \in I} \gamma_{pk} e_k + \sum_{r \in J} \tilde{y}_{pr} \tilde{e}_r \right)$$

and $e_i * e_j = \tilde{e}_p * \tilde{e}_q = 0$ for all $(i, j \in [1,n] \setminus I; p, q \in [1,m] \setminus J)$, is a gonosomal algebra.

**Proof.** This follows immediately from $\sum_{k \in I} \gamma_{pk} = \sum_{r \in J} \tilde{y}_{pr} = \sigma_{ip}$. \hfill \Box

This result is very useful to show that the transmission of a recessive lethal character can be represented by a gonosomal algebra.

**Example 11.** Recessive lethal gonosomal allele in XY-system.

A gene is lethal if one of its alleles causes the death of organisms that carry it. We study in XY-system, a gonosomal gene having two alleles $a$ and $b$. We are going to consider two cases.

**Case 1.** Allele $b$ is recessive lethal to females.

In this case there is no genotype female $bb$. Therefore we have only four crosses:

- $aa \times aY \Rightarrow \frac{1}{4}aa, \frac{1}{2}aY; \quad aa \times bY \Rightarrow \frac{1}{4}aa, \frac{1}{2}bY$;
- $ab \times aY \Rightarrow \frac{1}{4}ab, \frac{1}{2}aY, \frac{1}{4}bY; \quad ab \times bY \Rightarrow \frac{1}{4}ab, \frac{1}{2}aY, \frac{1}{4}bY$.

**Case 2.** Allele $b$ is lethal to males.

In this case there is no genotype male $bY$. There are therefore only three crosses:

- $aa \times aY \Rightarrow \frac{1}{4}aa, \frac{1}{2}aY; \quad ab \times aY \Rightarrow \frac{1}{4}aa, \frac{1}{2}ab, \frac{1}{4}aY; \quad bb \times aY \Rightarrow ab$.

Algebraically, we consider the basic algebra $A$ with basis $(e_1, e_2, e_3, \tilde{e}_1, \tilde{e}_2)$ defined for $i, j = 1, 2, i \neq j$ by

$$e_i e_j = \tilde{e}_p e_i = \frac{1}{2}e_i + \frac{1}{2}e_j; \quad e_i \tilde{e}_p = \tilde{e}_p e_i = \frac{1}{2}e_i + \frac{1}{2}e_j; \quad e_3 \tilde{e}_1 = \tilde{e}_2 e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_3 + \frac{1}{2}e_1 + \frac{1}{2}e_2,$$

other products are zero. It is clear that $A$ is a gonosomal algebra. Using one hand, the correspondences $e_1 \leftrightarrow aa$, $e_2 \leftrightarrow bb$, $e_3 \leftrightarrow ab$, $\tilde{e}_1 \leftrightarrow aY$, $\tilde{e}_2 \leftrightarrow bY$ and taking on the other hand in the proposition [10] $I = \{2\}$, $J = \emptyset$ we recognize the case 1, with $I = \emptyset$, $J = \{2\}$ we obtain the case 2.

4.2. Construction from the duplicate of a baric algebra.

We recall that if $(A, \omega)$ is a commutative $K$-algebra, the non commutative duplicate of $A$ is the space $A \otimes A$ and the commutative duplicate of $A$ is the quotient space of $A \otimes A$ by the ideal spanned by $\{x \otimes y - y \otimes x ; x, y \in A\}$. They are both noted $D(A)$ and equipped with the algebra law: $(x \otimes y)(x' \otimes y') = (xy) \otimes (x'y')$. The surjective morphism $\mu : D(A) \to A^2$,
Proof. From $A_1,A_2 \neq \{0\}$ and $A_1,A_2 \subseteq \ker(\omega_D)$ we deduce there are $e_i \in A_1$ and $\tilde{e}_i \in A_2$ such that $\omega_D(e_i) = \omega_D(\tilde{e}_i) = 1$. Let us complete $\{e_1, \ldots, e_n\}$ of $A_1$, replacing in $\tilde{B}$ each element $e_i$ such that $\omega_D(e_i) \neq 0$ by $\omega_D(e_i)^{-1} e_i$ and each term $e_i$ such that $\omega_D(e_i) = 0$ by $e_i + e_i$, we can suppose that we have $\omega_D(e_i) = 1$ for all $1 \leq i \leq n$. Analogously we complete $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$ of $A_2$ verifying $\omega_D(\tilde{e}_j) = 1$ for all $1 \leq j \leq m$. By $A_1 \cap A_2 = \{0\}$ it follows that $B \cup \tilde{B}$ is a basis of $A_1 \oplus A_2$. Let us show that $A_1 \oplus A_2$ equipped with the product given in the statement is gonosomal for this basis. From the multiplication definition it occurs immediately that $e_i \tilde{e}_j = \tilde{e}_j e_i = 0$. Then for all $e_i \in B$ and $\tilde{e}_j \in \tilde{B}$ we have $e_i \tilde{e}_j = \mu(e_i) \otimes \mu(\tilde{e}_j)$, but it follows from $\mu(e_i) \otimes \mu(\tilde{e}_j) = \sum_{k=1}^{n} \alpha_{ijk} e_k$ and $\omega_D(\mu(e_i) \otimes \mu(\tilde{e}_j)) = \omega(\mu(e_i) \mu(\tilde{e}_j)) = \omega_D(e_i) \omega_D(\tilde{e}_j) = 1$, we obtain $\sum_{k=1}^{n} \alpha_{ijk} \otimes \tilde{e}_j$. 

Remark 13. If in the proposition $A$ is a $K$-algebra with basis $(a_i)_{1 \leq i \leq n}$ and $(a_i \otimes Y)_{1 \leq i \leq n}$. Then the Etherington morphism $\mu$ gives the gametogenesis results for females $\mu(a_i \otimes a_j) = \sum_{k=1}^{n} \gamma_{ik} a_k$ and for males $\mu(a_i \otimes Y) = \sum_{k=1}^{n} \gamma_{ik} a_k + \frac{1}{2} Y$, with $\sum_{i=1}^{n} \gamma_{ik} = \frac{1}{2}$. With this, the algebra law of $A_1 \oplus A_2$ is

$$(a_i \otimes a_j)(a_k \otimes Y) = \sum_{r,s=1}^{n} \gamma_{rs} \gamma_{sk} a_r \otimes a_s + \frac{1}{2} \sum_{r=1}^{n} \gamma_{jr} a_r \otimes Y = \sum_{0 \leq r \leq r \leq n} (\gamma_{jr} \gamma_{sk} + \gamma_{jr} \gamma_{sr}) a_r \otimes a_s + \frac{1}{2} \sum_{1 \leq r \leq n} \gamma_{jr} a_r \otimes Y,$$

we find the multiplication (2.1).

Sex determination in the great majority of species is controlled by two genotypes (e.g. XX or XY). However there are cases where sex is encoded by more than two genotypes, we are going to give several examples and see that these cases also obey the definition.

Example 14. Heredity in the WXY-system.

We consider a population whose sex is determined by the WXY-system, in this system male genotypes are XY or YY and female WX, WY or XX. This situation is more complex than other systems because here a gene can be completely gonosomal for a pair of sex chromosomes and partially gonosomal (i.e. pseudo-autosomal) for another pair. To take this into account in an algebraic model of this system we introduce the following formalism.

Gonosomes $W$, $X$, $Y$ are denoted respectively by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$. Considering a gonosomal gene, its alleles are noted $a_1, \ldots, a_N$ and $a_0$ is used to indicate that the gene is not present on a gonosome.

For each $1 \leq s \leq 3$ we put $I_s = \{1, \ldots, N\}$ if gonosome $\Gamma_s$ carries the gene, otherwise we put $I_s = \{0\}$. For each $i \in I_s$, the notation $a_{is}^{(r)}$ represents a gamete containing gonosome $\Gamma_i$ and allele $a_i$, then for all $i \in I_s$, $j \in I_s$ such that $r \leq s$, the genotype of an individual of sex $\Gamma_i \Gamma_j$ with alleles $a_i$ on $\Gamma_i$ and $a_j$ on $\Gamma_j$ is denoted by $a_{is}^{(r)} a_{js}^{(s)}$. Next we note $a_{rk}^{(rs)}$ (resp. $b_{jk}^{(rs)}$) the probability that an individual of genotype $a_{is}^{(r)} a_{js}^{(s)}$ produces a gamete carrying allele $a_k^{(r)}$ (resp. $a_k^{(s)}$), thus we have $\sum_{s \in I_s} a_{rk}^{(rs)} = \sum_{s \in I_s} b_{jk}^{(rs)} = 1$ and we agree
that $a_{ij}^{(r)} = 0$ if $I_r \neq \{0\}$ and $e_{ij}^{(r,s)} = a_{ij}^{(r,s)} = 0$ for all $k \neq 0$ (similar conventions are used to $\beta_{ij}^{(r,s)}$, in other words if the gene locus is present on a gonosome it can not disappear and if absent it can not appear. It follows from this that the progeny of two individuals, one of genotype $a_i^{(s)} \cdot a_j^{(s)}$ and the other of genotype $a_p^{(u)} \cdot a_q^{(v)}$ is:

$$\frac{1}{4}a_{ij}^{(r,s)} \beta_{pql}^{(u,v)} a_k^{(s)} a_l^{(s)} + \frac{1}{4}a_{ij}^{(r,s)} \beta_{pql}^{(u,v)} a_k^{(s)} a_l^{(s)}$$

We can associate with this situation a gonosomal algebra by setting in proposition 12 the vector space $A$ with basis $\{a_i^{(r)}; 1 \leq r \leq 3, i \in I_r\}$ and commutative multiplication $a_i^{(r)} a_j^{(s)} = \frac{1}{4} \sum_{k \in I_k} a_{ij}^{(r,s)} a_k^{(s)} + \frac{1}{4} \sum_{l \in I_l} \beta_{ijkl}^{(r,s)} a_k^{(s)}$, the algebra $A$ is weighted by $\omega (a_i^{(r)}) = 1$.

Then we take the following subspaces of $D(A)$:

$$A_1 = \text{span} \left\{ a_i^{(1)} \otimes a_j^{(2)} + a_i^{(2)} \otimes a_j^{(1)} \right\}_{i,j \in I_i, j \neq i} \cup \left\{ a_i^{(2)} \otimes a_j^{(3)} : i \neq j \right\}_{i,j \in I_i}$$

$$A_2 = \text{span} \left\{ a_i^{(2)} \otimes a_j^{(3)} + a_j^{(3)} \otimes a_i^{(2)} : i,j \in I_i \right\}$$

**Example 15. Heredity in the WZ-system with male feminization.**

By 1940, the French biologist Albert Vandel noted that some female woodlice of the species *Armadillidium vulgare* generate 80% to 100% of females instead of the expected 50% and that this seems hereditary. The explanation of this phenomenon was discovered in 1973: these females with almost exclusively female descent are in reality males infected by a bacterium which transforms them into females. This bacterium was identified in 1992, it belongs to the genus *Wolbachia*, it is an endosymbiotic bacterial cell widespread in insects which mainly transmitted from mother to offspring and changes the reproduction of its guests either by making sterile the matings, or by killing embryos, or still by feminizing males.

Biologically, the sex of woodlice follows the WZ-system. When a male ZZ is infected by Wolbachia, what is denoted ZZ + w, it becomes female and can cross with a male ZZ, and as transmission of Wolbachia does not happen 100% there is a majority of ZZ + w females and a minority of ZZ males. Finally, in this population three kinds of females is observed: WZ, WZ + w, ZZ + w. For the crosses, if we note $\eta$ ($0.5 < \eta < 1$) the transmission rate of Wolbachia in the offspring, we have:

$$\begin{align*}
WZ \times ZZ & \rightarrow \frac{1}{2} WZ, \frac{1}{2} ZZ; \\
(WZ + w) \times ZZ & \rightarrow \frac{1}{2} (WZ + w), \frac{1}{2} (ZZ + w), \frac{1}{2} (1 - \eta) WZ, \frac{1}{2} (1 - \eta) ZZ; \\
(ZZ + w) \times ZZ & \rightarrow \frac{1}{2} (ZZ + w), (1 - \eta) ZZ.
\end{align*}$$

Algebraically, we take a vector space $A$ with basis $(e_1, \ldots, e_4)$ equipped with the multiplication:

$$\begin{align*}
e_1^2 &= e_i, \quad (i = 1, \ldots, 4) \\
e_1 e_2 &= (1 - \eta) e_1 + e_2, \\
e_1 e_3 &= \frac{1}{2} e_1 + \frac{1}{2} e_3, \\
e_1 e_4 &= e_2 e_3 = e_2 e_4 = \frac{1}{2} (e_1 + e_3) + \frac{1}{2} (e_2 + e_4), \\
e_3 e_4 &= \frac{1}{2} e_3 + \frac{1}{2} e_4.
\end{align*}$$

The algebra $A$ is weighted by $\omega (e_i) = 1$. We take in the proposition 12 the subspaces $A_1 = \mathbb{R} \langle e_1 \otimes e_i \rangle$ and $A_2 = \mathbb{R} \langle e_1 \otimes e_2, e_1 \otimes e_3, e_1 \otimes e_4 \rangle$, if in the products obtained from the proposition 12 we make the correspondences $e_1 \leftrightarrow Z, e_2 \leftrightarrow Z + w, e_3 \leftrightarrow W$ and $e_4 \leftrightarrow W + w$, we retrieve the results of crosses given in the biological model.

**Example 16. Heredity in the XY-system with fertile XY females.**

In rodents, cases where the XY system is atypical have been found. For example, in *Myopus schisticolor* (wood lemming) and *Mus minutoides* (african pygmy mouse), we
described three female genotypes: \( XX, XX^* \) and \( X^*Y \). In these genotypes, notation \( X^* \) refers to a chromosome carrying a gene having two actions: it inactivates the action of gonosome \( Y \) and it causes the elimination of gonosome \( Y \) during gametogenesis, so \( X^*Y \) females give only ova of \( X^* \) type. Therefore the results of crosses are:

\[
\begin{align*}
XX \times XY &\Rightarrow \frac{1}{2}XX, \frac{1}{2}XY; \\
XX^* \times XY &\Rightarrow \frac{1}{2}XX, \frac{1}{2}XX^*, \frac{1}{2}X^*Y, \frac{1}{2}XY; \\
X^*Y \times XY &\Rightarrow \frac{1}{2}XX^*, \frac{1}{2}X^*Y.
\end{align*}
\]

To interpret this in algebraic terms, we define the space \( A \) with basis \((e_1, e_2, e_3)\) and the multiplication:

\[
e_i^2 = e_i, \quad e_ie_j = \frac{1}{2}e_1 + \frac{1}{2}e_i, \quad e_2e_3 = e_2, \quad (i = 1, \ldots, 3).
\]

Next we take in proposition 12 the subspaces \( A_1 = \mathbb{R}(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_3) \) and \( A_2 = \mathbb{R}(e_1 \otimes e_3) \), then the product defined in proposition 12 and the relations \( e_1 \leftrightarrow X, e_2 \leftrightarrow X^*, e_3 \leftrightarrow Y \), allow to find the results of crosses.

In \textit{Dicrostonyx torquatus} (Arctic lemming) three female genotypes are also described: \( XX, XX^* \) and \( X^*Y \), but unlike the previous case, the females \( X^*Y \) give normal male \( XY \).

In this case the results of crosses are:

\[
\begin{align*}
XX \times XY &\Rightarrow \frac{1}{2}XX, \frac{1}{2}XY; \\
XX^* \times XY &\Rightarrow \frac{1}{2}XX, \frac{1}{2}XX^*, \frac{1}{2}X^*Y, \frac{1}{2}XY; \\
X^*Y \times XY &\Rightarrow \frac{1}{2}XX^*, \frac{1}{2}X^*Y, \frac{1}{2}XY.
\end{align*}
\]

And in this case the algebraic model is defined from the algebra \( A = \mathbb{R}(e_1, e_2, e_3) \) where

\[
e_i^2 = e_i, \quad e_1e_i = \frac{1}{2}e_1 + \frac{1}{2}e_i, \quad e_2e_3 = e_2, \quad (i = 1, \ldots, 3).
\]

Then we take in proposition 12 the subspaces \( A_1 = \mathbb{R}(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_3) \) and \( A_2 = \mathbb{R}(e_1 \otimes e_3) \), next we apply proposition 10 to obtain a gonosomal algebra structure on \( A_1 \oplus \mathbb{R}(e_1 \otimes e_3) \) from which relations \( e_1 \leftrightarrow X, e_2 \leftrightarrow X^*, e_3 \leftrightarrow Y \) allow to find the results of crosses.

### 4.3. Construction from two baric algebras.

**Proposition 17.** Let \((A, \omega)\) and \((\tilde{A}, \tilde{\omega})\) be (not necessarily commutative) finite-dimensional baric \( K \)-algebras, the Etherington morphism \( \mu : A \otimes A \rightarrow \tilde{A}^2, a \otimes a' \mapsto aa' \) and \( \varphi : A \otimes \tilde{A} \rightarrow A, \varphi : A \otimes \tilde{A} \rightarrow A \) two linear maps such that \( \omega \circ \varphi + \tilde{\omega} \circ \varphi = \omega \otimes \tilde{\omega} \), then the \( K \)-space \( A \otimes A \) \( \otimes \tilde{A} \) equipped with the algebra structure:

\[
(x, y) (x', y') = (\mu (x) \otimes \varphi (y') + \mu (x') \otimes \varphi (y), \mu (x) \otimes \tilde{\varphi} (y') + \mu (x') \otimes \tilde{\varphi} (y))
\]

is a gonosomal algebra.

**Proof.** For all \( x \in A \otimes \tilde{A} \) and \( y' \in A \otimes \tilde{A} \) identifying \((x,0)\) to \( x \) and \((0,y')\) to \( y' \), the multiplications in the statement becomes:

\[
xy' = \mu (x) \otimes \varphi (y') + \mu (x') \otimes \tilde{\varphi} (y').
\]

So for all \( x,x' \in A \otimes \tilde{A} \) and \( y,y' \in A \otimes \tilde{A} \) we have \( y'x = xy' = 0 \) and \( yy' = 0 \). Algebras \( A \) and \( \tilde{A} \) being finite-dimensional weighted there is a basis \((a_i)_{1 \leq i \leq n}\) of \( A \) and a basis \((\tilde{a}_p)_{1 \leq p \leq m}\) of \( \tilde{A} \) such that \( \omega (a_i) = 1 \) and \( \tilde{\omega} (\tilde{a}_p) = 1 \). Let \( \tau \) be a bijection from \([1, \ldots, n]^2\) to \([1, \ldots, n^2]\) ordering the basis \((a_i \otimes a_j)_{1 \leq i \leq n}\) of \( A \otimes A \), we put \( e_{1i,j} = (a_i \otimes a_j, 0) \), \((1 \leq i, j \leq n)\). Let \( \sigma \) be a bijection from \([1, \ldots, n] \times [1, \ldots, m]\) to \([1, \ldots, nm]\) ordering the basis \((a_i \otimes \tilde{a}_p)_{i,p}\) of \( A \otimes \tilde{A} \), we put \( e_{\sigma(i,p)} = (0, a_i \otimes \tilde{a}_p) \). Next for all \( 1 \leq i, j \leq n \) and \( 1 \leq p \leq m \) let \( \varphi (e_{\sigma(i,p)}) = \)
\[ \sum_{k=1}^{n} \alpha_k \sigma(i,p) \lambda_k = \sum_{k=1}^{n} \beta_k \sigma(i,p) \lambda_k \text{ and } \mu(a_i \otimes a_j) = a_i a_j = \sum_{k=1}^{n} \lambda_{jk} \alpha_k. \]

With this identity \((*)\) is written:

\[ e_{\tau(i,j)} \sum_{(p,q)} = (a_i \otimes a_j - 0) (0, a_p \otimes \bar{a}_p) \]

\[ = \sum_{k=1}^{n} \lambda_{jk} \alpha_k \sigma(i,p) e_{\tau(k,r)} \oplus \sum_{s=1}^{m} \lambda_{sk} \beta_k \sigma(p,q) \tilde{e}_{\sigma(k,s)}. \quad (**) \]

So noting:

\[ \gamma_{\tau(i,j), \sigma(p,q), \tau(k,r)} = \lambda_{jk} \alpha_k \sigma(i,p) \] and \( \tilde{\gamma}_{\tau(i,j), \sigma(p,q), \sigma(k,s)} = \lambda_{jk} \beta_k \sigma(p,q), \)

the identity \((**)\) becomes:

\[ e_{\tau(i,j)} \sum_{(p,q)} = \sum_{\tau(k,r)}^{n^2} \gamma_{\tau(i,j), \sigma(p,q), \tau(k,r)} e_{\tau(k,r)} + \sum_{\sigma(k,s)}^{nm} \tilde{\gamma}_{\tau(i,j), \sigma(p,q), \sigma(k,s)} \tilde{e}_{\sigma(k,s)}. \]

Now, we have:

\[ \sum_{k=1}^{n} \alpha_k \sigma(i,p) + \sum_{j=1}^{m} \beta_k \sigma(i,p) = \omega(\varphi(a_i \otimes \bar{a}_p)) + \tilde{\omega}(\tilde{\varphi}(a_i \otimes \bar{a}_p)) = \omega \otimes \tilde{\omega}(a_i \otimes \bar{a}_p) = \omega(a_i) \tilde{\omega}(\bar{a}_p) = 1 \]

and \( \sum_{k=1}^{n} \lambda_{jk} = \omega(a_i a_j) = 1, \) we deduce that

\[ \sum_{\tau(k,r)} \gamma_{\tau(i,j), \sigma(p,q), \tau(k,r)} + \sum_{\sigma(k,s)} \tilde{\gamma}_{\tau(i,j), \sigma(p,q), \sigma(k,s)} = 1, \quad (1 \leq i, j \leq n, 1 \leq p, q \leq m), \]

what establishes that \( A \otimes A \times A \otimes \bar{A} \) is a gonosomal algebra. \( \square \)

**Remark 18.** In the above proposition we take \((A, \omega)\) a baric algebra, \( \bar{A} \) the \( K \)-algebra spanned by an element \( Y \) verifying \( Y^2 = Y \) weighted by \( \bar{\omega}(Y) = 1 \), the maps \( \varphi : A \otimes \bar{A} \to A, \varphi(x \otimes Y) = \frac{1}{2} x \) and \( \tilde{\varphi} : A \otimes \bar{A} \to \bar{A}, \tilde{\varphi}(x \otimes Y) = \frac{1}{2} \omega(x) Y \). Then we have:

\( (x \otimes y \otimes z \otimes Y) \left( x' \otimes y' \otimes z' \otimes Y \right) = \frac{1}{2} \left( xy \otimes z' + x' y \otimes z \right) + \frac{1}{2} \left( \omega(z') xy \otimes Y + \psi(z) x' y' \otimes Y \right) \)

and after identification of \( A \otimes \bar{A} \) with \( A \) we find the law (2.2).

**Example 19.** X-linked sex ratio distortion.

In some species the X chromosome carries alleles referred as distorters which disrupt in male the production of gametes carrying Y chromosome. In this case the males offspring consists only of females. However in some species suppressors of distortion are observed on the Y chromosome, in this case depending on the suppressor type the percentage of females in the male offspring is between 50% (Y totally suppressor) and 100% (Y no suppressor).

If we denote \( X^d \) the X chromosome carrying an distorter allele and \( Y^s \) the Y chromosome carrying a suppressor gene, there are three female genotypes \( XX, XX^d, X^dX^d \) and four male genotypes \( XY, XY^s, X^dY, X^dY^s \) and gametogenesis of these genotypes are \( XY \rightarrow 4 X + 4 Y, XY^s \rightarrow 4 X + 4 Y, X^dY \rightarrow X^d + 4 X^dY, X^dY^s \rightarrow \frac{4}{2} X^d + \frac{4}{2} Y^s \) where \( 0 \leq \theta \leq 1 \) is the suppression rate of distortion. For example, the cross \( XX \times X^dY \) gives only \( XX^d \) females and \( XX \times X^dY^s \rightarrow \frac{2}{2} X^d + \frac{4}{2} XY^s \).

To show that this situation is depicted by a gonosomal algebra we use the correspondences \( X \leftrightarrow e_1, X^d \leftrightarrow e_2, Y \leftrightarrow \bar{e}_1 \) and \( Y^s \leftrightarrow \bar{e}_2 \) and we apply proposition (17) with \( A = \mathbb{R} (e_1, e_2), \bar{A} = \mathbb{R} (\bar{e}_1, \bar{e}_2) \) equipped with the algebraic structures: \( e_i e_j = \frac{1}{2} e_i + \frac{1}{2} e_j \) and \( \bar{e}_i \bar{e}_j = \frac{1}{2} \bar{e}_i + \frac{1}{2} \bar{e}_j \), then \( A \) and \( \bar{A} \) are weighted by \( \omega(e_i) = 1 \) and \( \tilde{\omega}(\bar{e}_i) = 1 \). We define
\( \varphi : A \otimes \tilde{A} \rightarrow A \) and \( \tilde{\varphi} : A \otimes \tilde{A} \rightarrow \tilde{A} \) by
\[
\varphi(e_i \otimes \tilde{e}_j) = \begin{cases} 
\frac{e_1}{2} & \text{if } i = 1 \\
\frac{e_2}{2} & \text{if } (i, j) = (2, 1) \\
\frac{2 - \theta}{2} e_2 & \text{if } (i, j) = (2, 2)
\end{cases},
\tilde{\varphi}(e_i \otimes \tilde{e}_j) = \begin{cases} 
\frac{\tilde{e}_1}{2} & \text{if } i = 1 \\
0 & \text{if } (i, j) = (2, 1) \\
\frac{\tilde{e}_2}{2} & \text{if } (i, j) = (2, 2).
\end{cases}
\]

Example 20. Kleptogenesis.
Kleptogenesis is observed in fishes (Poeciliopsis), frogs (Pelophylax) and insects (Bacillus). These bisexual species are present in various forms that can hybridise, that is to say cross-fertilizations is observed between two related species. During oogenesis hybrid females produce eggs that contain only the genome of one of his parents transmitted without recombination, the genome of the other parental species being totally evacuated. This mode of reproduction is thus characterized by a clonal transmission of the genome, whereas the other genome is acquired by sexual reproduction.

The first observation of this phenomenon was made by L. Berger in three species of European green frogs (genus Pelophylax): P. lessonae (LL) small, P. ridibundus (RR) large and P. esculentus (LR) whose size is intermediate. If frogs (LL) and (RR) are crossed, hybrids (LR) are obtained, but if we cross a female (LR) with a male (LL) only (LR) are obtained while the Mendel law provides 50% (LL) and 50% (LR). This result is explained by the systematic elimination during the (LR) frogs gametogenesis of the entire genome inherited from parents (LL).

Algebraically, we apply the proposition [7] by taking \( A = \tilde{A} \) an algebra with basis \( (a_1, a_2) \) and multiplication \( a a_j = \frac{1}{2} (a_i + a_j) \) weighted by \( \varphi(a_i) = 1 \). Let \( 0 < \theta < 1 \) be the female proportion in the population, we define \( \varphi, \tilde{\varphi} : A \otimes \tilde{A} \rightarrow A \) by
\[
\varphi(a_i \otimes a_j) = \begin{cases} 
(1 - \theta) a_i & \text{if } i = j \\
(1 - \theta) a_1 & \text{if } (i, j) = (1, 2), (2, 1),
\end{cases}
\tilde{\varphi}(a_i \otimes a_j) = \begin{cases} 
\theta a_i & \text{if } i = j \\
\theta a_1 & \text{if } (i, j) = (1, 2), (2, 1).
\end{cases}
\]

Noting \( e_{ij} = (0, a_i \otimes a_j) \) female genotypes and \( \tilde{e}_{ij} = (a_i \otimes a_j, 0) \) male genotypes, the product defined in the proposition [7] coupled to the relations \( a_1 \leftrightarrow R, a_2 \leftrightarrow L \) give the results of crosses observed in species Pelophylax.

The following three examples show that this construction is also useful to give algebraic models of genetic processes influenced by sex.

Example 21. Recombination between two pseudo-autosomal genes.
This case was investigated for zygotic algebra in [2]. We consider two pseudo-autosomal genes \( a \) and \( b \) in the XY system which allelic forms are \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_m \); it is also assumed that the locus of the gene \( a \) is closer to the centromere and does not occur recombination between the locus and the centromere, it follows that the alleles of \( a \) does not change chromosome during recombination.

We note \((a_i b_j, a_k b_l)\) the genotype of an individual where the haplotype \(a_i b_j\) is transmitted by the mother and \(a_k b_l\) by the father. Let \( \theta \) and \( \tilde{\theta} \) respectively the recombination rates between these two loci in females and males. During meiosis a female \((a_i b_j, a_k b_l)\) produces ova in the following proportions: \(\{\frac{\theta}{2} a_i b_j, \frac{\theta}{2} a_k b_l, \frac{\theta}{2} a_i b_l, \frac{\theta}{2} a_k b_j\}\). In a male \((a_i b_q, a_k b_t)\), the distribution of spermatozoa is \(\{\frac{1 - \theta}{2} a_i b_q, \frac{1 - \theta}{2} a_k b_t, \frac{\theta}{2} a_i b_t, \frac{\theta}{2} a_k b_q\}\) where, taking into consideration the notation, genotypes \(a_i b_q\) and \(a_k b_t\) are carried by a gonosome \( X \) and \(a_k b_t\), \(a_i b_q\) are on a gonosome \( Y \). It results for example, that the frequency distribution of the eight genotypes of girls born from the cross between a female \((a_i b_j, a_k b_l)\) and a male \((a_p b_q, a_r b_s)\) is:
we denote by 

\[
\frac{(1-\theta)\theta}{4} (a_ib_j, a_rb_j), \quad \frac{(1-\theta)\theta}{4} (a_ib_j, a_rb_j), \quad \frac{(1-\theta)\theta}{4} (a_kb_l, a_pb_q), \quad \frac{(1-\theta)\theta}{4} (a_kb_l, a_pb_q), \\
\frac{\theta(1-\theta)}{4} (a_kb_l, a_pb_q), \quad \frac{\theta(1-\theta)}{4} (a_kb_l, a_pb_q),
\]

With proposition [17] we show that this situation is modeled by a gonosomal algebra. Indeed, let \(A, \tilde{A}\) be the vector spaces spanned by \((a_i \otimes b_j)\) respectively equipped with algebraic structures:

\[
(a_i \otimes b_j)(a_k \otimes b_l) = \frac{1}{4} \theta (a_i \otimes b_j + a_k \otimes b_l) + \frac{\theta}{4} (a_i \otimes b_l + a_k \otimes b_j), \\
(a_i \otimes b_j)(a_k \otimes b_l) = \frac{1}{4} \theta (a_i \otimes b_j + a_k \otimes b_l) + \frac{\theta}{4} (a_i \otimes b_l + a_k \otimes b_j).
\]

Then \(A\) is weighted by \(\omega(a_i \otimes a_j) = 1\) and \(\tilde{A}\) by \(\tilde{\omega}(a_i \otimes a_j) = 1\). We define the maps

\[
\varphi : A \otimes \tilde{A} \to A \text{ by } \varphi((a_i \otimes b_j) \otimes (a_k \otimes b_l)) = \frac{(1-\theta)}{2} a_i \otimes b_j + \frac{\theta}{2} a_k \otimes b_l \text{ and } \tilde{\varphi} : \tilde{A} \otimes A \to \tilde{A} \text{ by } \\
\tilde{\varphi}((a_i \otimes b_j) \otimes (a_k \otimes b_l)) = \frac{(1-\theta)}{2} a_i \otimes b_j + \frac{\theta}{2} a_k \otimes b_l, \text{ we have } \omega \circ \varphi + \tilde{\omega} \circ \tilde{\varphi} = \omega \circ \tilde{\omega}.
\]

**Example 22. Genetic mutation with sex-dependent mutation rate in a multiple XY-system.**

The inheritance of a gonosomal gene was studied for the XY system in the absence of mutation in [4, 7, 9] and with mutation in [5]. Nevertheless it is observed in all sex determination systems, in the following it is described in the multiple XY-system.

To interpret this model in gonosomal algebra terms, we apply the proposition 17. For \(1 \leq k \leq n\), we put \(A_k = \text{span}\{a_{i}; i \in I_k\}\) and \(A = \bigotimes_{k=1}^{n} A_k\). When \(m \geq 1\), for \(1 \leq r \leq m\) we note \(A_r = \text{span}\{a_{i}; i \in J_r\}\) and \(\tilde{A} = \bigotimes_{k=1}^{n} \tilde{A}_k\); when \(m = 0\) we put \(\tilde{A} = \text{span}\{a_0\}.

\[
(a_i \otimes b_j)(a_k \otimes b_l) = \frac{1}{4} \theta (a_i \otimes b_j + a_k \otimes b_l) + \frac{\theta}{4} (a_i \otimes b_l + a_k \otimes b_j), \\
(a_i \otimes b_j)(a_k \otimes b_l) = \frac{1}{4} \theta (a_i \otimes b_j + a_k \otimes b_l) + \frac{\theta}{4} (a_i \otimes b_l + a_k \otimes b_j).
\]
First we provide every space $A_k$ with the algebraic structure $a_i a_j = \frac{1}{2} \sum_{c \in I_k} (\mu_{c,i} + \mu_{c,j}) a_r$ and the weight function $\omega_k (a_i) = 1$ for all $i, j \in I_k$, and every space $A_k$ with the multiplication $a_i a_j = \frac{1}{2} \sum_{c \in I_k} (\mu_{c,i} + \mu_{c,j}) a_r$ and the weight function $\omega_k (a_i) = 1$. Next we equip the spaces $A$ and $A$ with the algebraic structure $(\otimes_{c \in I_k} (x_{c,y_k})) = \otimes_{c \in I_k} (x_{c,y_k})$, the weight functions $\omega = \otimes_{c \in I_k} \omega_k$ for $A$ and $\tilde{\omega} = \otimes_{c \in I_k} \tilde{\omega}_k$ for $\tilde{A}$. Finally we define the maps $\varphi : A \otimes \tilde{A} \to A$ and $\tilde{\varphi} : A \otimes \tilde{A} \to \tilde{A}$ by $\varphi (\otimes_{c \in I_k} a_{c,i} \otimes \otimes_{c \in I_k} a_{c,j}) = \otimes_{c \in I_k} (\sum_{c \in I_k} \mu_{c,i} a_{c,j})$ and $\tilde{\varphi} (\otimes_{c \in I_k} a_{c,i} \otimes \otimes_{c \in I_k} a_{c,j}) = \otimes_{c \in I_k} (\sum_{c \in I_k} \tilde{\mu}_{c,i} a_{c,j}).$

**Example 23.** Transposable elements with sex-dependent transposition rate.

Since the Barbara McClintock’s pioneering works, we know that a more or less important part of the genome (45% in humans) consists of repeated sequences that can move along chromosomes. These sequences called transposable elements or transposons (also known as mobile genetic elements or jumping genes). The transposon displacement is called a transposition. According to the transposon there are two transposition modes: conservative or replicative, some transposons use both. Transposition is conservative when the transposon moves from one site to another without being replicated. It is replicative when one or more copies of the transposon is transferred to another site, in this case there is an increase of the number of copies of the transposon inside the genome.

We consider a bisexual population composed of diploid individuals with chromosomal number $2n \geq 4$. Chromosomes are numbered from 1 to $n$, where $n$ is the number reserved for the $X$ and $Y$ sex chromosomes. We study the copies number of a transposon into the genome of individuals. The notation $c_{i,j}$ is the chromosome number $i$ carrying $j$ copies of the transposon and we assume that each chromosome can carry at most $N$ copies.

We note $\tau_{c,j}$ (resp. $\bar{\tau}_{c,j}$) the insertion (in this case $k > 0$) or deletion (in this case $k < 0$) probability in a female (resp. male) of $k$ copies on a chromosome carrying $j$ copies. Thus we have $0 \leq k + j \leq N$ and we put $\tau_{c,j} = \bar{\tau}_{c,j} = 0$ as soon as $k + j < 0$ or $k + j > N$, all this allows to take $-N \leq k \leq N$ in the definitions of $\tau_{c,j}$ and $\bar{\tau}_{c,j}$.

Let $c_{1,i_1} \ldots c_{n,i_n}$ the chromosomes transmitted by a parent, we note $c_{1,i_1} \otimes \ldots \otimes c_{n,i_n}$ the genome of an individual is represented by $c_{1,i_1} \otimes c_{j_1} \ldots \otimes c_{j_n}$, where $c_{i_1} \ldots c_{i_n}$ (resp. $c_{j_1} \ldots c_{j_n}$) are chromosomes inherited from the mother (resp. the father). Let $c_{1,i_1} \otimes \ldots \otimes c_{j_1} \ldots c_{j_n}$ the genome of an individual, after transposition, if the individual is a female then the egg distribution is

$$\frac{1}{2^n} \sum_{p=1}^{n} \left( \sum_{k \in \mathbb{Z}} \tau_{k,j_p} c_{p,j_p+k} + \sum_{l \in \mathbb{Z}} \tau_{l,j_p} c_{p,j_p+l} \right),$$

if the individual is a male, the distribution of spermatozoa carrying $X$ gonosome is

$$\frac{1}{2^n} \sum_{p=1}^{n-1} \left( \sum_{k \in \mathbb{Z}} \tau_{k,j_p} c_{p,j_p+k} + \tau_{k,j_p} c_{p,j_p+k} \right) \otimes \sum_{k \in \mathbb{Z}} \tau_{k,i_n} c_{n,i_n+k}$$

and that of the sperm cells carrying the $Y$ gonosome is

$$\frac{1}{2^{n-1}} \sum_{p=1}^{n-1} \left( \sum_{k \in \mathbb{Z}} \tau_{k,j_p} c_{p,j_p+k} + \tau_{k,j_p} c_{p,j_p+k} \right) \otimes \sum_{k \in \mathbb{Z}} \tau_{k,i_n} c_{n,i_n+k}.$$

The result after crossing a female $c_{1,i_1} \otimes c_{j_1} \ldots c_{j_n}$ with a male $c_{1,i_1} \otimes c_{j_1} \ldots c_{j_n}$, the distribution of girls is given by

$$\frac{1}{2^n} \sum_{p=1}^{n} \left( \sum_{k \in \mathbb{Z}} \tau_{k,j_p} c_{p,j_p+k} + \tau_{k,j_p} c_{p,j_p+k} \right) \otimes \sum_{k \in \mathbb{Z}} \tau_{k,i_n} c_{n,i_n+k},$$

and that of boys is given by

$$\frac{1}{2^{n-1}} \sum_{p=1}^{n-1} \left( \sum_{k \in \mathbb{Z}} \tau_{k,j_p} c_{p,j_p+k} + \tau_{k,j_p} c_{p,j_p+k} \right) \otimes \sum_{k \in \mathbb{Z}} \tau_{k,i_n} c_{n,i_n+k}.$$. 

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and that of the boys is
\[
\frac{1}{2^{2n-1}} \bigotimes_{p=1}^{n} \left( \sum_{k=-N}^{N} (\tau_{k,ip}c_{ip,jp+k} + \bar{\tau}_{k,ip}c_{ip,jp+k}) \right) \otimes \\
\bigotimes_{p=1}^{n-1} \left( \sum_{k=-N}^{N} (\tau_{k,ip}c_{ip,jp+k} + \bar{\tau}_{k,ip}c_{ip,jp+k}) \right) \otimes \sum_{k=-N}^{N} \tau_{k,jn}c_{jn+k}.
\]

Algebraically, let $A$ be the $\mathbb{R}$-vector space with basis $(c_{i_1,\ldots,i_n})_{0 \leq i_1,\ldots,i_n \leq N}$ equipped with the algebra structure:
\[
c_{i_1,\ldots,i_n}c_{j_1,\ldots,j_n} = \frac{1}{2^n} \bigotimes_{p=1}^{n} \left( \sum_{k=-N}^{N} (\tau_{k,ip}c_{ip,jp+k} + \bar{\tau}_{k,ip}c_{ip,jp+k}) \right),
\]
and $\tilde{A}$ the $\mathbb{R}$-vector space with basis $(\tilde{c}_{i_1,\ldots,i_n})_{0 \leq i_1,\ldots,i_n \leq N}$ equipped with the algebra structure:
\[
\tilde{c}_{i_1,\ldots,i_n}\tilde{c}_{j_1,\ldots,j_n} = \frac{1}{2^n} \bigotimes_{p=1}^{n} \left( \sum_{k=-N}^{N} (\tau_{k,ip}\tilde{c}_{ip,jp+k} + \bar{\tau}_{k,ip}\tilde{c}_{ip,jp+k}) \right).
\]

Algebras $A$ and $\tilde{A}$ are weighted by $\omega(c_{i_1,\ldots,i_n}) = \tilde{\omega}(\tilde{c}_{i_1,\ldots,i_n}) = 1$. By defining the maps $\varphi : A \otimes A \to A$,
\[
\varphi(c_{i_1,\ldots,i_n} \otimes \tilde{c}_{j_1,\ldots,j_n}) = \bigotimes_{p=1}^{n} \left( \sum_{k=-N}^{N} (\tau_{k,ip}c_{ip,jp+k} + \bar{\tau}_{k,ip}c_{ip,jp+k}) \right) \otimes \sum_{k=-N}^{N} \tau_{k,jn}c_{jn+k}
\]
and $\tilde{\varphi} : A \otimes \tilde{A} \to \tilde{A}$ by
\[
\tilde{\varphi}(c_{i_1,\ldots,i_n} \otimes \tilde{c}_{j_1,\ldots,j_n}) = \bigotimes_{p=1}^{n} \left( \sum_{k=-N}^{N} (\tau_{k,ip}c_{ip,jp+k} + \bar{\tau}_{k,ip}c_{ip,jp+k}) \right) \otimes \sum_{k=-N}^{N} \tau_{k,jn}\tilde{c}_{jn+k},
\]
we obtain an algebra structure that describes the situation exposed above, it is gonosomal by proposition $[17]$. 

4.4. Three constructions from two algebra laws.

**Proposition 24.** Let $A$ be a finite-dimensional $K$-vector space equipped with two algebra laws $\circ$ and $\bullet$ to which $A$ is not necessarily commutative and such that $(A, \circ)$ and $(A, \bullet)$ have the same weight function $\omega$. Then for all $\theta \in K$ and for all linear maps $\varphi, \varphi' : A \to A$ verifying $\omega \circ \varphi = \omega \circ \varphi' = \omega$, the $K$-vector space $A \times A$ with the multiplication
\[
(x, y) (x', y') = \begin{cases} 
\theta \varphi (x \circ y' + x' \circ y), & (1 - \theta) \varphi' (x \bullet y' + x' \bullet y), 
\end{cases}
\]
is a gonosomal algebra.

**Proof.** It is clear that the multiplicative law defined above is commutative. According to the hypotheses, the algebra $(A, \circ)$ admits a basis $(a_1, \ldots, a_n)$ and $(A, \bullet)$ admits a basis $(a'_1, \ldots, a'_n)$ such that $\omega(a_i) = \omega(a'_i) = 1$. If we put $a_i \circ a'_j = \sum_{k=1}^{n} \gamma_{ij}a_k$ and $a_i \bullet a'_j = \sum_{k=1}^{n} \gamma_{ij}a_k'$, we have $\sum_{k=1}^{n} \gamma_{jk} = \sum_{k=1}^{n} \gamma_{jk}' = 1$ for all $1 \leq i, j \leq n$. Let $\varphi (a_k) = \sum_{p=1}^{n} \alpha_{kp}a_p$ and $\varphi' (a'_k) = \sum_{p=1}^{n} \alpha'_{kp}a'_p$, the assumption $\omega \circ \varphi = \omega \circ \varphi' = \omega$ implies $\sum_{p=1}^{n} \alpha_{kp} = \sum_{p=1}^{n} \alpha'_{kp}' = 1$ for all $1 \leq k \leq n$. For each $1 \leq i \leq n$, we put $e_i = (a_i, 0)$ and $\tilde{e}_i = (0, a'_i)$, it follows immediately from the definition of the algebra law defined on $A \times A$ that $e_i e_j = \tilde{e}_i \tilde{e}_j = (0, 0)$ and
\[
e_i \tilde{e}_j = \left( \theta \sum_{k=1}^{n} \gamma_{jk} \varphi (a_k), (1 - \theta) \sum_{k=1}^{n} \gamma_{jk}' \varphi' (a_k) \right)
= \sum_{p=1}^{n} \sum_{k=1}^{n} \theta \gamma_{jk} \alpha_{kp} e_k + \sum_{p=1}^{n} \sum_{k=1}^{n} (1 - \theta) \gamma_{jk}' \alpha'_{kp} \tilde{e}_k,
\]

with $\sum_{p=1}^{n} \sum_{k=1}^{n} \theta \eta_{ijk} a_{pk} + \sum_{p=1}^{n} \sum_{k=1}^{n} (1 - \theta) \gamma_{ij} a'_{pk} = 1$. □

**Example 25.** Heredity of an autosomal gene with sex-dependent segregation and mutation.

We consider the alleles $a_1, \ldots, a_n$ an autosomal gene in a diploid bisexual population. Genotypes $(a_i, a_j)$ are ordered, i.e. in $(a_i, a_j)$ allele $a_i$ is transmitted by the mother and $a_j$ is received from the father. It is assumed that the segregation coefficients and the mutation rates during meiosis are sex-dependent, we note $\eta_{ijk}$ (resp. $\eta'_{ijk}$) the segregation coefficient of allele $a_i$ in a female (resp. male) $(a_i, a_j)$ and $\mu_{pk}$ (resp. $\mu'_{pk}$) the mutation rate of allele $a_i$ to $a_j$ in females (resp. males). Finally it is assumed that each generation the female rate in the population remains constant equal to $\theta$. Then after crossing a female $(a_i, a_j)$ with a male $(a_p, a_q)$, the proportion of females (resp. males) $(a_k, a_i)$ in the offspring is $\theta \sum_{q=1}^{n} \mu'_{pk} \eta'_{ijk} \eta'_{jpk}$ (resp. $(1 - \theta) \sum_{q=1}^{n} \mu_{pk} \eta_{ijk} \eta_{jpk}$).

Algebraically, we define on the vector space $V$ with basis $(a_1, \ldots, a_n)$ the algebra laws:

$$a_i \circ a_j = \sum_{k=1}^{n} \eta_{ijk} a_k \text{ and } a_i \bullet a_j = \sum_{k=1}^{n} \eta'_{ijk} a_k$$

where $\sum_{k=1}^{n} \eta_{ijk} = \sum_{k=1}^{n} \eta'_{ijk} = 1$, algebras $(V, \circ)$ and $(V, \bullet)$ are weighted by $a(a_i) = 1$. Then by applying the proposition[24] with $A = V \otimes V$ the non commutative duplicate of $V$ equipped with laws $(x \otimes y) \circ (x' \otimes y') = (x \circ y) \otimes (x' \otimes y')$ and $(x \otimes y) \bullet (x' \otimes y') = (x \bullet y) \otimes (x' \bullet y')$, the linear maps $\varphi, \varphi' : A \rightarrow A$ defined by $\varphi(a_i \otimes a_j) = \sum_{q=1}^{n} \mu_{pq} a_q \otimes a_i$, $\varphi'(a_i \otimes a_j) = \sum_{p=1}^{n} \mu'_{pq} a_i \otimes a_q$, we obtain the frequency distributions of genetic types after crossing a female $e_{ij} = (a_i \otimes a_j, 0)$ with a male $\tilde{e}_{ij} = (0, a_i \otimes a_j)$.

**Example 26.** Reproduction in ciliates.

Ciliates (Ciliophora) are unicellular organisms. Ciliates can reproduce in two ways: by fission (asexual reproduction) or by conjugation (called sexual reproduction). Ciliates have two nuclei: a large nucleus (macronucleus) polyploid which is involved in the control of metabolism and a smaller nucleus (micronucleus) diploid which is involved in reproduction.

Conjugation is a cross-fertilization process, it begins when two ciliates come together side by side and form a cytoplasmic bridge between them, this cytoplasmic bridge ensures the exchange of genetic material. The macronuclei degenerate while each micronucleus undergoes two meiosis resulting in four haploid nuclei which three are destroyed, the remaining nucleus divides to produce two haploid gametic nuclei: a resident and a mobile. The mobile nuclei are exchanged between the two conjugants, then the conjugants separate, the gametic nuclei fuse in a zygotic nucleus which divides several times, among the nuclei obtained one becomes micronucleus and the others form a macronucleus. Finally, after conjugation we have two genetically identical individuals.

Given a diploid ciliate species with $2n$ chromosomes in the micronucleus. For each $1 \leq k \leq n$, we note $a^k_1, \ldots, a^k_{n_k}$ the genetic types that can be found on the chromosome $k$ and $A_k$ the vector space with basis $(a^k_1, \ldots, a^k_{n_k})$ equipped with the algebra structure $a^k_i a^k_j = \frac{1}{2} a^k_i + \frac{1}{2} a^k_j$. We define on the space $\otimes_{k=1}^{n} A_k$ the algebra structure:

$$\left( \otimes_{k=1}^{n} a^k_{i_k} \right) \ast \left( \otimes_{k=1}^{n} a^k_{j_k} \right) = \otimes_{k=1}^{n} \left( a^k_i a^k_j \right)$$

which gives the distribution of gametic nuclei produced by ciliates whose micronucleus genotype is $(a^i_1, \ldots, a^n_{n_i}, a^j_1, \ldots, a^n_{n_j})$. Then by applying the proposition[24] with $A = \left( \otimes_{k=1}^{n} A_k \right) \ast \left( \otimes_{k=1}^{n} A_k \right)$, taking as laws $\circ$ and $\bullet$ the law defined by $(x \otimes y) (x' \otimes y') = (x \circ y) \otimes (x' \circ y')$. If there is no mutation, we take the maps $\varphi = \psi = 1d$ and $\theta = \frac{1}{2}$, we obtain the genotypes distribution after conjugation of two ciliates.

**Proposition 27.** Let $A$ be a $K$-vector space equipped with two algebra laws $\circ$ and $\bullet$ to which $A$ is not necessarily commutative or baric. The $K$-space $A \otimes A$ is equipped with the multiplication:

$$(a \otimes b) \ast (c \otimes d) = (a \circ b) \otimes (c \bullet d).$$
If there is a subalgebra $G$ of $A \otimes A$ such that $G$ is finite-dimensional, weighted by a map $\omega$, then for all $\theta \in K$ and for all linear maps $\varphi, \psi : G \to G$ such that $\omega \circ \varphi = \omega \circ \psi = \omega$, the $K$-space $G \times G$ equipped with the law:

$$(x, y) (x', y') = \left(\theta \varphi (x \ast y' + x' \ast y), (1 - \theta) \psi (x \ast y' + x' \ast y)\right)$$

is a gonalosomal algebra.

**Proof.** The algebra $G$ admits a basis $(a_1, \ldots, a_n)$ such that $\omega (a_i) = 1$ for all $1 \leq i \leq n$. It follows that if $a_i \ast a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k$ then $\sum_{k=1}^{n} \gamma_{ijk} = 1$ for $1 \leq i, j \leq n$. Next if $\varphi (a_i) = \sum_{k=1}^{n} \alpha_k a_k$ and $\psi (a_i) = \sum_{k=1}^{n} \beta_k a_k$, from $\omega \circ \varphi = \omega \circ \psi = \omega$ it comes $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$ for all $1 \leq i \leq n$. It is clear that the law defined on $G \times G$ is commutative. If for all $1 \leq i \leq n$ we put $e_i = (a_i, 0)$ and $e_i = (0, a_i)$, then we have $e_i e_j = 0$, $e_i \varphi q e_q = 0$ and

$$e_i \varphi p = \theta \sum_{k=1}^{n} \gamma_{ipk} \alpha_k e_r + (1 - \theta) \sum_{k=1}^{n} \gamma_{ijk} \beta_k e_r,$$

with $\theta \sum_{k=1}^{n} \gamma_{ipk} \alpha_k + (1 - \theta) \sum_{k=1}^{n} \gamma_{ijk} \beta_k = 1$. □

**Example 28. Genomic imprinting (or parental imprinting)**

In many diploid placental mammals, we observe a functional asymmetry for some autosomal genes according to their paternal or maternal origin, it results in the offspring by the expression of only one allele on both. This phenomenon is called genomic (or parental) imprinting. The genomic imprinting is submitted to the cycle: deletion, installation, maintenance. In every generation in each individual at the time of gamete formation, all parental imprints are erased in the germ cells and when these cells become mature gametes the genes subject to imprinting are inactivated or not according to the sex of the individual, next the imprints are transmitted by fertilization to the next generation where they are transmitted through cell divisions throughout the life.

For example, the mutation *brachyury* ($T$) of the mouse is known since 1927, it is a dominant mutation that results in a shortening of the tail. One of its alleles, called hairpin tail (denoted $T^{hp}$) has a strange inheritance: the cross $(\varphi, +/+) \times (\sigma, T^{hp}/+)$, sign + denoting the normal allele, gives a descent made up 50% of short-tailed mice and 50% of normal tail, while the symmetric cross $(\varphi, T^{hp}/+) \times (\sigma, +/+)$ gives only normal tail mice. This observation which is contradictory to the Mendel laws, is explained by maternal imprint which inactivates allele $T^{hp}$ and thereby silencing it, which means that the embryos $T^{hp}/+$ die in utero. It follows that we cannot observe homozygote $T^{hp}/T^{hp}$, so the cross $(T^{hp}/+) \times (T^{hp}/+)$ produces only $T^{hp}/+$ and $+/+$ descendants.

Algebraically, we establish coding $e_1 \leftrightarrow +$, $e_2 \leftrightarrow T^{hp}$, we define on the space $A$ with basis $(e_1, e_2)$ the laws $\circ$ and $\bullet$:

- $e_1 \circ e_1 = e_1$, $e_2 \circ e_2 = 0$, $e_1 \circ e_2 = e_2 \circ e_1 = e_1$.
- $e_1 \bullet e_1 = e_1$, $e_2 \bullet e_2 = 0$, $e_1 \bullet e_2 = e_2 \bullet e_1 = \frac{1}{2} e_1 + e_2$.

If we note $e_{ij} = e_i \otimes e_j$, by applying the proposition\(^{27}\) we have

$$e_{11} e_{ij} = e_{21} e_{ij} = e_{21} e_{ij} = \begin{cases} e_{11} & \text{si } (i, j) = (1, 1) \\ \frac{1}{2} e_{11} + \frac{1}{2} e_{12} & \text{si } (i, j) = (1, 2), (2, 1) \\ 0 & \text{si } (i, j) = (2, 2) \end{cases}$$

and taking for $G$ the space with basis $(e_{11}, e_{12})$ weighted by $\omega (e_{11}) = \omega (e_{12}) = 1$, the maps $\varphi = \psi = Id$ and $\theta$ the female proportion by generation, we obtain an algebraic model for the transmission of *brachyury* mutation.

**Proposition 29.** Let $A$ be a $K$-vector space, $A_1, A_2 \subseteq A$, $A_1 \cap A_2 = \{0\}$ two finite-dimensional subspaces with respective bases $(a_{1i})_{1 \leq i \leq n_1}$ and $(a_{2i})_{1 \leq i \leq n_2}$. If it exists on $A_1 \oplus A_2$ two
algebra laws \( \circ \) and \( \bullet \) verifying \( A_1 \circ A_1 \cup A_1 \circ A_2 \subset A_1, A_1 \bullet A_2 \cup A_2 \bullet A_2 \subset A_2 \) and if the linear map \( \eta : A_1 + A_2 \to K \) defined by \( \eta (a_{1,j}) = \eta (a_{2,j}) = 1 \), \( \eta (a_{1,j} \bullet a_{2,j}) = 1 \), then for all \( \theta \in K \) and for all linear maps \( \phi_1 : A_1 \to A_1 \) and \( \phi_2 : A_2 \to A_2 \) such that \( \eta \circ \phi_1 = \eta \circ \phi_2 = \eta \), the space \( A_1 \times A_2 \) with multiplication

\[
(x_1, x_2)(y_1, y_2) = (\theta \phi_1 (x_1 \circ y_1 + x_1 \circ y_2), (1 - \theta) \phi_2 (x_1 \circ y_2 + y_1 \circ x_2))
\]
is a gonosomal algebra.

**Proof.** This multiplication defined on \( A_1 \times A_2 \) is commutative. If we put \( a_{1,j} \circ a_{2,j} = \sum_{k=1}^{n_1} \gamma_{jk} a_{1,k} \) and \( a_{1,j} \bullet a_{2,j} = \sum_{k=1}^{n_2} \gamma'_{jk} a_{2,k} \), from \( \eta (a_{1,j} \circ a_{2,j}) = 1, \eta (a_{1,j} \bullet a_{2,j}) = 1 \) it comes \( \sum_{k=1}^{n_1} \gamma_{jk} = \sum_{k=1}^{n_2} \gamma'_{jk} = 1 \). If \( (\alpha_{ij})_{1 \leq i, j \leq n_1} \) and \( (\alpha'_{ij})_{1 \leq i, j \leq n_2} \) are respectively matrices of the maps \( \phi_1 \) and \( \phi_2 \) in bases \( (a_{1,j})_{1 \leq i \leq n_1} \) and \( (a_{2,j})_{1 \leq i \leq n_2} \), from \( \eta \circ \phi_1 = \eta \circ \phi_2 = \eta \) it comes \( \sum_{j=1}^{n_1} \alpha_{ij} = 1 \) and \( \sum_{j=1}^{n_2} \alpha'_{ij} = 1 \). Then if we put \( e_i = (a_{1,j}, 0) \) and \( \tilde{e}_p = (0, a_{2,p}) \) we have \( e_i \tilde{e}_p = 0, \tilde{e}_p e_q = 0 \) and

\[
e_i \tilde{e}_p = \left( \theta \sum_{k=1}^{n_1} \gamma_{jk} \phi_1 (a_{1,k}), (1 - \theta) \sum_{k=1}^{n_2} \gamma'_{jk} \phi_2 (a_{2,k}) \right)
\]

\[
\theta \sum_{k=1}^{n_1} \gamma_{jk} \alpha_{jk} e_i + (1 - \theta) \sum_{k=1}^{n_2} \gamma'_{jk} \alpha'_{jk} \tilde{e}_p
\]

with \( \theta \sum_{k=1}^{n_1} \gamma_{jk} \alpha_{jk} + (1 - \theta) \sum_{k=1}^{n_2} \gamma'_{jk} \alpha'_{jk} = 1 \).

**Example 30.** X-inactivation (or lyonization).

In most placental or marsupial mammals, one of the X chromosome in the female genome is inactive: genes carried by this chromosome are not expressed throughout the lifetime. The X-inactivation occurs upon implantation of the egg, it is random in placental mammals and in marsupials it is always the X chromosome inherited from the father who is inactivated.

We use the proposition\(^{29}\) to give an algebraic model of the X-inactivation. We consider the alleles \( a_1, \ldots, a_n \) of a gonosomal gene, we note \( a_1, \ldots, a_n \) (resp. \( a'_1, \ldots, a'_n \)) when these alleles are active (resp. silencer). Let \( V \) be the vector space with basis \( (a_i, a'_i)_{1 \leq i \leq n} \) and \( A = V + V \otimes V \). The space \( A_1 \) spanned by \( \{ a_i \otimes a_j, a_i \otimes a'_j; 1 \leq i, j \leq n \} \) represents ordered female genotypes, the space \( A_2 = V \) gives male genotypes. If the X chromosome inactivation rate of maternal origin is noted by \( \tau \), we define on \( A_1 \oplus A_2 \) the laws \( \circ \) and \( \bullet \):

\[
a_i \circ a_j = a_i \otimes a_j \otimes a_k = \frac{\tau}{2} \left( a_i \otimes a_k + a_j \otimes a_k \right) + \frac{1 - \tau}{2} \left( a_i \otimes a'_k + a_j \otimes a'_k \right)
\]

\[
a_i \bullet a_j = \frac{1}{2} (a_i + a_j).
\]

If \( \theta \) means the rate of females in the population, and if we take \( \phi_i = id_{A_1}, i = 1, 2 \), then the multiplication defined in the proposition\(^{29}\) gives the genotype distribution in the absence of mutation of a cross between a female genotype \( (a_i \otimes a_j, 0) \) or \((a_i \otimes a'_j, 0)\) with a male genotype \((0, a_k)\). In the presence of mutation, we define \( \phi_1 (a_i \otimes a_j) = \sum_{k=1}^{n} \mu_k a_i \otimes a_k_{k} = \sum_{j=1}^{n} \tilde{\mu}_k a_j \) and \( \phi_2 (a_i) = \sum_{j=1}^{n} \tilde{\mu}_k a_k \) where \( \mu_k \) (resp. \( \tilde{\mu}_k \)) is the mutation rate of the allele \( a_i \) to the allele \( a_k \) in females (resp. males).

**Example 31.** Sex determination by elimination of sex chromosomes.

In Sciaridae sex is determined by the gonosome \( X \): females are \( XX \) and males \( X0 \) with the peculiarity that \( Sciaridae \) males are obtained by elimination of all chromosomes coming from the father. We note respectively \( A^m \) and \( A^p \) the set of autosomes coming from the mother and father, \( X^m \) and \( X^p \) a gonosome \( X \) transmitted by the mother and
father. In Sciaridae after fertilization zygotes are of genotype $A^m A^f X^m X^f Y^f$, during the formation of the somatic lineage, one gonosome $X^f$ is eliminated from the cells of the somatic female lineages while both $X^f$ are eliminated in the somatic male lineages. In male germ line during spermatogenesis cells lose chromosomes $A^p$ and $X^p$, the $X^m$ gonosome is replicated, finally we obtain a $A^m X^m X^m$ type spermatozoo. In females, oogenesis proceeds normally and it leads to $AX$ type ovules.

We must add that according to the composition of female progeny two types of Sciaridae are distinguished: digenic or monogenic. In digenic Sciaridae the progeny of a female consists of males and females with a sex-ratio different from 1. In monogenic Sciaridae, descendants are all of the same sex, the females are called androgenic when they produce only males and gynogenic when they produce only females. There are also Sciaridae which are monogenic and digenic.

Sex determination in digenic Sciaridae can be represented algebraically as follows. Let $V, G, O$ spaces with respective bases $(a_1, \ldots, a_n)$, $(g_1, \ldots, g_m)$ and $(o)$. We consider the space $A = V \otimes V \otimes G \otimes (G + O)$. An element $a_i \otimes a_j \otimes g_k \otimes g$ of the basis of $A$, or $(a_i a_j g_k g)$ in an abbreviated form, is the genotype of an adult, where $a_i$ (resp. $a_j$) represents autosomes coming from the mother (resp. father), $g_k$ a gonosome $X$ transmitted by the mother and $g$ indicate the sex of the individual with $g \in \{ g_1, \ldots, g_m \}$ if female and in this case $g$ was transmitted by the father, or $g = o$ if male. We apply the proposition 29 by taking $A_1 = V \otimes V \otimes G \otimes G, A_2 = V \otimes V \otimes G \otimes O$ equipped with multiplications:

$$
(a_i a_j g_k g) \circ (a_p a_q g r g') = \begin{cases}
\frac{1}{4} (a_i + a_j) \otimes a_p \otimes (g_k + g) \otimes g, & \text{if } (g, g') \in G \times O,
0, & \text{otherwise},
\end{cases}

(a_i a_j g_k g) \bullet (a_p a_q g r g') = \begin{cases}
\frac{1}{4} (a_i + a_j) \otimes a_p \otimes (g_k + g) \otimes o, & \text{if } (g, g') \in G \times O,
0, & \text{otherwise},
\end{cases}
$$

with $\eta (a_i a_j g_k g l) = \eta (a_i a_j g_k o) = 1$, $\varphi = \psi = Id$ and $\theta = \frac{1}{1 + \sigma}$ where $\sigma$ is the sex-ratio of the population.

The Sciaridae monogeny depends on a particular gonosome $X$ noted $X^1$: $X^X X^X$ females are gynogenic while $XX$ females are androgenic. Algebraically, let $V, G^o, O$ spaces with respective bases $(a_1, \ldots, a_n)$, $(g_1, \ldots, g_m)$ and $(o)$, where $a_k$ represents an autosomal type, $g_k$ a gonosome determining gynogeny, $g_k$ a X gonosome type and $o$ is associated with the male sex. Applying the proposition 29 with the spaces $A = V \otimes V \otimes (G^o + G) \otimes (G + O), A_1 = V \otimes V \otimes (G^o + G) \otimes G, A_2 = V \otimes V \otimes G \otimes O$, we define the laws $\circ$ and $\bullet$ by:

$$
(a_i a_j g_k g) \circ (a_p a_q g r g') = \frac{1}{4} (a_i + a_j) \otimes a_p \otimes (g_k^* + g) \otimes g, \text{ if } (g, g') \in G \times O,

(a_i a_j g_k g) \circ (a_p a_q g r g') = 0 \text{ if } g' \in G + O,

(a_i a_j g_k g) \bullet (a_p a_q g r g') = 0 \text{ if } g' \in G + O,

(a_i a_j g_k g) \circ (a_p a_q g r g') = \frac{1}{4} (a_i + a_j) \otimes a_p \otimes (g_k + g) \otimes o, \text{ if } (g, g') \in G \times O,
$$

with $\eta (a_i a_j g_k g l) = \eta (a_i a_j g_k o) = 1$, $\varphi = \psi = Id$ and $0 \leq \theta \leq 1$ corresponds to the proportion of gynogeny ($\theta = 1$ in the case of gynogenic Sciaridae, $\theta = 0$ for monogenic).

4.5. Construction from three linear forms and three linear maps.

This construction applies to the case of ordered genotypes where take into account of maternal and paternal origin of genes.

**Proposition 32.** Let $A, A', \widetilde{A}$ be finite-dimensional $K$-vector spaces; $\omega : A \rightarrow K, \omega' : A' \rightarrow K, \omega : A \rightarrow K$ nonzero linear forms and $\varphi : A \otimes A' \rightarrow A, \varphi' : A \otimes A \rightarrow A', \tilde{\varphi} : A \otimes \tilde{A} \rightarrow \tilde{A}$
three linear maps such that \( \omega \circ \varphi = \omega \otimes \omega' \) and \( \omega' \circ \varphi' + \tilde{\omega} \circ \tilde{\varphi} = \omega \otimes \tilde{\omega} \), then the K-space \( A \otimes A' \otimes A \otimes A \) equipped with the algebra structure:

\[
(x \otimes y) \cdot (x' \otimes y') = [\varphi(x) \otimes \varphi'(y') + \varphi'(x') \otimes \varphi(y)] \otimes [\varphi(x) \otimes \tilde{\varphi}(y') + \varphi(x') \otimes \bar{\varphi}(y)]
\]

is a gonosomal algebra.

**Proof.** Note that for all \( x \in A \otimes A' \) and \( y \in A \otimes \tilde{A} \), identifying \( x \boxplus 0 \) to \( x \) and \( 0 \boxplus y' \) to \( y' \), the multiplication \( A \otimes A' \otimes A \otimes \tilde{A} \) given in the statement becomes:

\[
x'y' = \varphi(x) \otimes \varphi'(y') \otimes \varphi(x') \otimes \bar{\varphi}(y'), \quad (*)
\]

and furthermore we have \( y'y = xy' \), \( xx' = yy' = 0 \) for all \( x, x' \in A \otimes A' \) and \( y, y' \in A \otimes \tilde{A} \).

Linear forms \( \omega, \omega', \tilde{\omega} \) being nonzero and spaces \( A, A', \tilde{A} \) being finite-dimensional, they admit bases \( (a_i)_{1 \leq i \leq n}, (a'_j)_{1 \leq j \leq m}, (\tilde{a}_k)_{1 \leq k \leq \mu} \) such that \( \omega(a_i) = 1, \omega(a'_j) = 1 \) and \( \tilde{\omega}(\tilde{a}_k) = 1 \) (cf. proof of the proposition [12]). Let \( \sigma : \llbracket 1, n \rrbracket \times \llbracket 1, m \rrbracket \to \llbracket 1, \mu \rrbracket \) bijections, we define the maps \( \varphi(a_i \otimes a'_j) = \sum_{i=1}^{n} \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s, \)

\[
\varphi'(a_i \otimes \tilde{a}_k) = \sum_{i=1}^{n} \lambda_{\tilde{\varphi}(i)} a_r \otimes \tilde{a}_s, \quad \tilde{\omega}(a_i \otimes \tilde{a}_k) = \sum_{s=1}^{n} \lambda_{\tilde{\omega}(s)} a_r \otimes \tilde{a}_s,
\]

and with this, according to \( (*) \) we have:

\[
(a_i \otimes a'_j)(a_i \otimes \tilde{a}_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{\tilde{\omega}(i)} a_r \otimes \tilde{a}_s = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{\tilde{\omega}(i)} a_r \otimes \tilde{a}_s, \quad (**)
\]

but we have

\[
\sum_{i=1}^{n} \lambda_{\varphi(i)} = \omega(\varphi(a_i \otimes a'_j)) = \omega \otimes \omega'(a_i \otimes a'_j) = \omega(a_i) \omega'(a'_j) = 1,
\]

and also

\[
\sum_{s=1}^{n} \lambda_{\tilde{\omega}(s)} = \tilde{\omega}(\varphi(a_i \otimes \tilde{a}_k)) = \omega \otimes \omega'(a_i \otimes \tilde{a}_k) = \omega(a_i) \omega'(a_i \otimes \tilde{a}_k) = 1,
\]

hence

\[
\sum_{r=1}^{n} \sum_{s=1}^{m} \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s + \sum_{r=1}^{n} \sum_{s=1}^{m} \lambda_{\tilde{\omega}(i)} a_r \otimes \tilde{a}_s = \sum_{r=1}^{n} \sum_{s=1}^{m} \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s + \sum_{r=1}^{n} \sum_{s=1}^{m} \lambda_{\tilde{\omega}(i)} a_r \otimes \tilde{a}_s = 1.
\]

Finally, by putting \( e_{\sigma(i,j)} = a_i \otimes a'_j, \tilde{e}_{\sigma(i,k)} = a_i \otimes \tilde{a}_k, \gamma_{\sigma(i,j), \sigma(i,k)} = \lambda_{\varphi(i,j)} a_r \otimes \tilde{a}_s, \gamma_{\sigma(i,j), \sigma(i,k)} = \lambda_{\tilde{\omega}(i)} a_r \otimes \tilde{a}_s \) and \( \tilde{\gamma}(i,j), \gamma(i,k), \tilde{e}(i,k) \), the relation \( (**) \) is written in the form:

\[
e_{\sigma(i,j)} \tilde{e}_{\sigma(i,k)} = \sum_{\sigma(i,r)} \gamma_{\sigma(i,j), \sigma(i,k)} e_{\sigma(i,r)} + \sum_{\sigma(i,k)} \tilde{\gamma}_{\sigma(i,j), \sigma(i,k)} \tilde{e}_{\sigma(i,k)}
\]

with \( \sum_{\sigma(i,r)} \gamma_{\sigma(i,j), \sigma(i,k)} e_{\sigma(i,r)} + \sum_{\sigma(i,k)} \tilde{\gamma}_{\sigma(i,j), \sigma(i,k)} \tilde{e}_{\sigma(i,k)} = 1 \), which establishes that the algebra \( A \otimes A' \otimes A \otimes \tilde{A} \) is gonosomal.

\[ \square \]

We are going to apply this result to to an exceptional mode of reproduction in the living world.

**Example 33. Reproduction of a triploid (Bufo baturae).**

In nature triploid individuals appear from a cross between a tetraploid (4n) and a diploid (2n) as the result of the union between a diploid (2n) and a haploid (n) gamete which gives a triploid zygote (3n). Triploids are sterile, this is due to the mechanism of gamete formation. Indeed, during meiosis chromosomes are associated by homologous pairs before being divided equally for each group in different gametes and this independently for each group. As a result, in triploids for each homologous pair, a gamete receives two chromosomes and an other only one. Since the distribution of chromosomes is independent of a homologous pair to another, the probability of having a haploid or diploid gamete in a 3n triploid is \( (1/2)^{3n-1} \), for example for \( 3n = 33 \) this probability is \( 1/1024 \). In theory there exists a possibility that two haploid or diploid gametes unite to give an euploid zygote, but the
probability of this event is very low. The triploid sterility is used in agriculture to produce seedless fruits (banana, watermelon, grapes . . .).

From the above we can understand the surprise of biologists discovering that the toad Bufo baturae, who lives in the desert mountains of northern Pakistan, is triploid (3n = 33) and reproduces. The explanation of this particularism was found in male and female gametogenesis. The B. baturae genome contains two sets of chromosomes carrying a nuclear organizer (nucleolus-organizing region or NOR) and one set without NOR, noted NOR" in the following. In males, the cells at the origin of the sperm undergo at first elimination of the set of NOR" chromosomes, so we obtain diploid spermatocytes which, after a normal meiosis with possibilities of recombination, give haploid sperm of NOR" type. In females, there are two kinds of ovogenesis. In one, as in males, there is at first elimination of recombination rates between chromosome sets.

In the other, there is at first duplication of the set of NOR" chromosomes, followed by a meiosis without recombinations. In the other, there is at first elimination of the set NOR" followed by a normal meiosis with recombination between NOR" chromosomes. In both cases it leads to diploid eggs of NOR" / NOR" type whose fusion with the sperm gives triploid zygotes containing two sets of NOR" and one of NOR".

We can apply the proposition 34 to this situation. Note \((a_i)_{1\leq i \leq n}\) (resp. \((\tilde{a}_j)_{1\leq j \leq m}\)) the sets of chromosomes NOR" containing the chromosome X (resp. Y) and \((a_i')_{1\leq i \leq p}\) the sets NOR". We consider the spaces A with basis \((a_i \otimes a_k')_{i,k}A'\) with basis \((a_i)\), and A with basis \((\tilde{a}_j)\), we give the linear forms \(\omega\) \((a_i \otimes a_k') = 1\), \(\omega'(a_i) = 1\) and \(\omega(\tilde{a}_j) = 1\). In this model an element \(a_i \otimes a_k' \otimes a_j\) (resp. \(a_i \otimes a_k' \otimes \tilde{a}_j\)) in the basis of the space \(A \otimes A'\) (resp. \(A \otimes \tilde{A}\)) represents a female (resp male) karyotype in which sets \(a_i\) and \(a_k'\) were transmitted by the mother and sets \(a_j\) and \(\tilde{a}_j\) come from father.

Let \(\theta\) the rate of oocytes losing a set NOR" and \(\rho_{ijq}\) the recombination rate between the sets \(a_i\) and \(a_j\) resulting in the set \(a_q\), thus we have \(\sum_{q=1}^{n} \rho_{ijq} = 1\), then the map \(\varphi : A \otimes A' \to A, \varphi \left( a_i \otimes a_k' \otimes a_j \right) = \frac{\theta}{\sum} \left( a_i a_k' + a_i a_k' \right) + (1 - \theta) \sum_{q=1}^{n} \rho_{ijq} a_q \otimes a_k'\) gives the distribution of egg types produced by a female karyotype \(a_i \otimes a_k' \otimes a_j\). If we note \(\tilde{\rho}_{ijq}\) and \(\tilde{\rho}_{ijq}\) the recombination rates between chromosome sets \(a_i\) and \(a_j\) in spermatocytes of karyotype \(a_i \otimes a_k' \otimes \tilde{a}_j\) that give sperm of \(a_i'\) and \(\tilde{a}_q\), type, we have \(\sum_{q=1}^{n} \left( \tilde{\rho}_{ijq} + \tilde{\rho}_{ijq} \right) = 1\). Then \(\varphi' : A \otimes \tilde{A} \to A', \tilde{\varphi} : \tilde{A} \otimes \tilde{A} \to \tilde{A}\) where \(\varphi' \left( a_i \otimes a_k' \otimes \tilde{a}_j \right) = \sum_{q=1}^{n} \rho_{ijq} a_q \otimes a_k'\) and \(\tilde{\varphi} \left( a_i \otimes a_k' \otimes \tilde{a}_j \right) = \sum_{q=1}^{n} \tilde{\rho}_{ijq} a_q \otimes a_k'\), respectively give the distributions of sperm carrying the X and Y gonosome produced by a male of karyotype \(a_i \otimes a_k' \otimes \tilde{a}_j\).

4.6. Construction from a baric algebra and two linear maps.

**Proposition 34.** Let \((A, \omega)\) be a baric (not necessarily commutative) K-algebra and \(\varphi, \tilde{\varphi} : A \to A\) two linear maps such that \(\omega \circ (\varphi + \tilde{\varphi}) = \omega\). Then the K-vector space \(A \times A\) equipped with multiplication:

\[
(x,y) (x',y') = (\varphi(x y' + x' y), \tilde{\varphi}(x y' + x' y))
\]

is a gonoosomal algebra.

**Proof.** The algebra A being weighted there is a basis \((a_i)_{1\leq i \leq n}\) in A such that \(a_i a_j = \sum_{k=1}^{n} \delta_{ij} a_k\) with \(\sum_{k=1}^{n} a_k = 1\) thus \(\omega(a_i) = 1\). We put \(e_1 = (a_i,0)\) and \(e_1 = (0,a_i)\), it is clear that \(e_i e_j = e_i e_j = 0\) for all \(1 \leq i, j \leq n\). Next with \(\varphi(a_k) = \sum_{p=1}^{n} a_{pk} a_k\) and \(\tilde{\varphi}(a_k) = \sum_{p=1}^{n} \tilde{a}_{pk} a_k\), the assumption \(\omega \circ (\varphi + \tilde{\varphi}) = \omega\) is translated by \(\sum_{p=1}^{n} (a_{pk} + \tilde{a}_{pk}) = 1\). Then
we have:

\[ e_i \tilde{e}_j = (\varphi(a_i a_j), \tilde{\varphi}(a_i a_j)) \]

\[ = \left( \sum_{k=1}^{n} \gamma_{jk} \varphi(a_k), \sum_{k=1}^{n} \gamma_{jk} \tilde{\varphi}(a_k) \right) \]

\[ = \sum_{k,p=1}^{n} \gamma_{jk} \alpha_{pk} (a_k, 0) + \sum_{k,p=1}^{n} \gamma_{jk} \alpha_{pk} (0, a_k) \]

with \( \sum_{k,p=1}^{n} \gamma_{jk} (\alpha_{pk} + \tilde{\alpha}_{pk}) = 1 \), thus the algebra \( A \times A \) so defined is gonosomal. \( \square \)

As illustrated in the following example, this construction is very useful when sex determination is polygenetic that is to say that sex determination factors are distributed on several chromosomes.

**Example 35. Sex determination in Musca domestica.**

In *Musca domestica* (2n = 12), sex determination follows the XY-system. On the Y chromosome is a diallelic locus: \( M, + \). The allele \( M \) determine male sex, allele + is neutral. The female sex is determined by an allele \( F \) present on the autosome IV, allele \( F \) is recessive compared to \( M \). Therefore male genotypes are \((M+, FF)\) or \((MM, FF)\) and female \((++, FF)\). However in some fly populations locus \( M, + \) is also located on one of the autosomes or even on the \( X \) chromosome, in these populations there is a mutation of the factor \( F \) denoted by \( F^D \) which is dominant compared to \( M \), in these cases the sex is determined by 18 genotypes including 10 for the sex female (see table below).

| Autosomes | Gonosomes |
|-----------|-----------|
| IV        | I-V       |
| FF        | ++        |
| FF        | M+, MM    |
| FFD       | ++, M+, MM|

Algebraically, given three spaces \( A, B \) and \( C \) with respective bases \( (a_1, a_2) \), \( (b_1, b_2) \) and \( (c_1, c_2) \) and with the algebra law: \( xy = \frac{1}{4}x + \frac{1}{4}y \), the space \( A \otimes B \otimes C \) is equipped with the algebraic structure \( (a_i \otimes b_j \otimes c_k)(a_p \otimes b_q \otimes c_r) = (a_i a_p) \otimes (b_j b_q) \otimes (c_k c_r) \) and the weight function \( \omega(a_i \otimes b_j \otimes c_k) = 1 \). For \( i, j, k \in \{1, 2\} \) we note \( e_{i,j,k} = a_i \otimes b_j \otimes c_k \) and we put:

\[ \varphi = \begin{cases} 
\varepsilon(1,1,1) \otimes \varepsilon(1,1,1) \otimes \varepsilon(2,1,1) \otimes \varepsilon(1,1,1) \otimes \varepsilon(2,1,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(2,2,1), \\
\varepsilon(1,1,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(2,1,1) \otimes \varepsilon(2,2,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(2,2,1), \\
\varepsilon(1,1,2) \otimes \varepsilon(2,2,1) \otimes \varepsilon(2,2,1) \otimes \varepsilon(2,2,1) 
\end{cases} \]

\[ \sigma = \begin{cases} 
\varepsilon(1,1,1) \otimes \varepsilon(1,2,2) \otimes \varepsilon(1,2,1) \otimes \varepsilon(1,1,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(1,1,1) \otimes \varepsilon(1,2,1) \\
\varepsilon(1,1,1) \otimes \varepsilon(1,2,2) \otimes \varepsilon(1,2,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(1,2,1) \otimes \varepsilon(1,2,1) 
\end{cases} \]

then applying the proposition [34] with \( \varphi, \tilde{\varphi} \) defined on \( A \otimes B \otimes C \) by \( \varphi_Q = id, \varphi_{\sigma} = 0 \) and \( \tilde{\varphi}_{\sigma} = 0, \tilde{\varphi}_{\sigma} = id \), the algebra \( A \otimes B \otimes C \) is gonosomal. Using the coding \( a_1 \leftrightarrow F, b_1 \leftrightarrow +, b_2 \leftrightarrow M, c_1 \leftrightarrow X, c_2 \leftrightarrow Y \), we obtain the frequency distribution of crosses.

**Example 36. Cytoplasmic heredity.**

Cytoplasmic heredity is the inheritance of genes that are not carried by chromosomes in the nucleus cells, that is why it is called as extrachromosomal or extranuclear heredity. These genes are located on the genome of mitochondria (or chloroplasts in plants). Mitochondria are organelles present in the cytoplasm, they play a key role in the production and energy storage in eukaryotic cells. Mitochondria have their own DNA, they divide continuously regardless of the division of the cell that contains them, and they are distributed randomly between the daughter cells during cell division. The mitochondrial genome mutates, this translates into a cell by a genetically heterogeneous mitochondria population and
during oogenesis by a heterogeneous oocyte population: some oocytes contain only one type of mitochondria (homoplasy) others contain several types (heteroplasy). At the time of the sexual reproduction, only the head of a sperm cell enters into the oocyte, it is possible that some mitochondria of the flagellum are transferred at the same time, however the number of mitochondria of paternal origin being very low compared to those present in the oocyte, it is considered that the mode of transmission of the mitochondrial genome is maternal.

Classify oocytes of a diploid sexual species in categories \(c_1, \ldots, c_n\) according to their mitochondrial populations. During fertilization, the fusion of a \(c_i\) type oocyte with a sperm gives a \(c_k\) type egg with a frequency \(v_{ik}\) (thus \(\sum_{k=1}^{n} v_{ik} = 1\)), this egg becomes female with a frequency \(\sigma_i\) or male in a proportion \(1 - \sigma_i\).

Algebraically, we apply the proposition \(\text{[34]}\) on the algebra \(A\) with basis \((c_1, \ldots, c_n)\) defined by \(c_ic_j = c_l\) for \(1 \leq i, j \leq n\), weighted by \(\omega(c_i) = 1\) and the maps \(\sigma_i(c_l) = \sigma_i \sum_{k=1}^{n} v_{ik} c_k\) and \(\phi(c_i) = (1 - \sigma_i) \sum_{k=1}^{n} v_{ik} c_k\).

Example 37. Sex determination by deuterotokous parthenogenesis.

This type of parthenogenesis is known as cyclical because it alternates a parthenogenetic phase and a sexual phase. In this case a diploid female gives birth to diploid males or females.

In some species of aphids (Aphidoidea), the homogametous sex female 2A-2X are divided into two types: gynoparous females only lay eggs that develop into females and androparous females only lay eggs that develop into males. In gyniparous females a mitosis replaces meiosis and produces 2A-2X genotype eggs that develop into females carrying A-X type oocytes. In androparous females meiosis is abnormal: one of the X chromosome is lost, the resulting spermatocytes 2A-X give males with A-X type sperm (those of type A-0 are eliminated). The crosses give only 2A-2X females.

Algebraically, we consider the algebra \(V\) with basis \((a_1, \ldots, a_n)\), multiplication \(a_i a_j = \frac{1}{2} (a_i + a_j)\) and weight function \(\omega(a_i) = 1\). Let \(A = D(V)\) be the (commutative or non commutative) duplicate of \(V\), we apply the proposition \(\text{[34]}\) with \(\phi(x) = \theta x, \phi(x) = (1 - \theta) x\). Let \(e_{ij} = (a_i \otimes a_j, 0)\) and \(\tilde{e}_{ij} = (0, a_i \otimes a_j)\) we obtain:

\[
e_{ij} \tilde{e}_{pq} \frac{\theta}{\theta} \left((e_{i} + e_{j}) \otimes (e_{p} + e_{q}), 0\right) + \frac{1}{1 - \theta} \left(0, (e_{i} + e_{j}) \otimes (e_{p} + e_{q})\right).
\]

4.7. Construction from a baric algebra and a gonosomal algebra.

**Proposition 38.** Let \((A, \alpha)\) be a finite-dimensional (not necessarily commutative) baric \(K\)-algebra and \(G\) a gonosomal \(K\)-algebra. The space \(A \otimes G\) equipped with the multiplication

\[
(x \otimes y)(x' \otimes y') = \frac{1}{2} (xx' + x'y') \otimes yy',
\]

is a gonosomal algebra.

**Proof.** The algebra \(A\) admits a basis \((a_1, \ldots, a_n)\) such that \(a_i a_q = \sum_{r=1}^{N} \lambda_{pqr} a_r\) with \(\sum_{r=1}^{N} \lambda_{pqr} = 1\), and \(G\) has a gonosomal basis \(\{e_i; 1 \leq i \leq n\} \cup \{f_j; 1 \leq j \leq m\}\). We put \(a_{p,i} = a_p \otimes e_i\) and \(\tilde{a}_{p,j} = a_p \otimes f_j\) for \(1 \leq p \leq N, 1 \leq i \leq n\) and \(1 \leq j \leq m\). We have \(a_{p,i} a_{q,j} = 0, \tilde{a}_{p,i}, \tilde{a}_{q,j} = 0\) and

\[
a_{p,i} \tilde{a}_{q,j} = \frac{1}{2} \left(\sum_{r,k} \left(\lambda_{pqr} + \lambda_{qpr}\right) \gamma_{rk}\right) \tilde{a}_{r,k} + \frac{1}{2} \left(\sum_{k,r} \left(\lambda_{pqr} + \lambda_{qpr}\right) \beta_{rk}\right) \tilde{a}_{r,k}
\]

with

\[
\frac{1}{2} \left(\sum_{r,k} \left(\lambda_{pqr} + \lambda_{qpr}\right) \gamma_{rk}\right) + \frac{1}{2} \left(\sum_{k,r} \left(\lambda_{pqr} + \lambda_{qpr}\right) \beta_{rk}\right) = 1.
\]

This proves that \(A \otimes G\) is a gonosomal algebra. \(\square\)
Example 39. Crosses with distribution of autosomal types according to sexes.

In a bisexual population we consider autosomal genetic types of autosomal \(a_1, \ldots, a_n\) and we assume that over the generations the proportion of females in the population has a constant constant value \(0 \leq \sigma \leq 1\). This frequency \(\sigma\) is connected to the population sex-ratio \(\rho\) (proportion of males to females per generation), indeed we have \(\rho = \frac{1-\sigma}{\sigma}\).

Let \(\gamma_{jk}\) the frequency of type \(a_k\) in the progeny of two individuals of types \(a_i\) and \(a_j\). Let \(A\) be the algebra with basis \((a_1, \ldots, a_n)\), the commutative law \(a_i a_j = \sum_{k=1}^{n} \gamma_{jk} a_k\), \(\sum_{k=1}^{n} \gamma_{jk} = 1\) and \(G\) the gonosomal algebra with basis \((f, m)\) defined by \(f^2 = m^2 = 0\) and \(fm = mf = (1-\sigma)m\). If we apply the proposition \(38\) to \(A \otimes G\) we find \((a_i \otimes f)(a_j \otimes m) = \sigma \sum_{k=1}^{n} \gamma_{jk} a_k \otimes m\), therefore after the cross of types \(a_i\) and \(a_j\), the frequency of type \(a_k\) in the female (resp. male) offspring is \(\sigma \gamma_{jk}\) (resp. \((1-\sigma) \gamma_{jk}\)).

Example 40. Transmission law of a couple of autosomal and gonosomal genes.

We consider in a bisexual population the autosomal types \(a_1, \ldots, a_N\); the female \(e_1, \ldots, e_n\) and male \(\tilde{e}_1, \ldots, \tilde{e}_m\) gonosomal types. Let \(A\) be algebra with basis \((a_1)_{1 \leq i \leq N}\) with commutative law \(a_i a_j = \sum_{k=1}^{N} \lambda_{ijk} a_k\), \(\sum_{k=1}^{N} \lambda_{ijk} = 1\) and \(G\) the gonosomal algebra with basis \((e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_m)\) defined by \(e_i \tilde{e}_j = \sum_{k=1}^{n} \gamma_{ijk} e_i + \sum_{k=1}^{m} \tilde{\gamma}_{ijk} \tilde{e}_k\). If we apply the proposition \(38\) to \(A \otimes G\) we find:

\[
(a_i \otimes e_p)(a_j \otimes \tilde{e}_q) = \sum_{k=1}^{N} \left( \sum_{r=1}^{n} \lambda_{ijk} \gamma_{pqr} a_k \otimes e_r + \sum_{r=1}^{m} \tilde{\lambda}_{ijk} \tilde{\gamma}_{pqr} a_k \otimes \tilde{e}_r \right),
\]

in other words, after crossing a \((a_i, e_p)\) female type with a \((a_j, \tilde{e}_q)\) male the frequency of type \((a_k, e_r)\) in females is \(\tilde{\lambda}_{ijk} \tilde{\gamma}_{pqr}\) and type \((a_k, \tilde{e}_r)\) in males is \(\lambda_{ijk} \gamma_{pqr}\).

5. ON BARICITY AND DIBARICITY OF GONOSOMAL ALGEBRAS

The notion of baric algebra (see definition in section 4) was introduced by Etherington \(3\), it plays a fundamental role in the study of genetic algebras. But as zygotic algebras for sex-linked inheritance admit bases whose elements are nilpotent these means that they are not baric. In order to remedy this, Holgate \(9\) introduced the notion of dibaric algebra.

Definition 41. A \(K\)-algebra \(A\) is dibaric if there is a surjective morphism \(\chi : A \to \mathcal{S}\) where \(\mathcal{S}\) is the \(K\)-algebra with basis \((f, m)\) such that \(f^2 = m^2 = 0\) and \(fm = mf = \frac{1}{2}(f + m)\).

Remark 42. A \(K\)-algebra with basis \(\{a, b\}\) such that \(a^2 = b^2 = 0\), \(ab = ba = \theta a + (1-\theta)b\) where \(\theta \in K, \theta \neq 0, 1\), is isomorphic to the dibaric algebra \(\mathcal{S}\) with basis \((f, m)\) by \(a \mapsto 2(1-\theta)f\) and \(b \mapsto 2bm\).

If \(A\) is a dibaric algebra then \(A^2\) is baric. There are other results concerning dibaric algebras in \(11\) and \(13\).

Remark 43. In \(11\) the authors showed that the evolution algebra of the bisexual population are dibaric. The example \(11\) shows that gonosomal algebras are not in general dibaric. Indeed, let us assume that the algebra \(A\) defined in example \(11\) is dibaric, let \(\chi : A \to \mathcal{S}\) a dibaric function such that \(\chi(e_i) = \tilde{\lambda}_i f + \mu_i m\) and \(\chi(\tilde{e}_i) = \lambda_i f + \mu_i m\) for all \(i = 1, 2\).

From \(\chi(e_i)\) \(\chi(e_j) = \chi(e_i e_j) = 0\) and \(\chi(\tilde{e}_p)\) \(\chi(\tilde{e}_q) = \chi(\tilde{e}_p \tilde{e}_q) = 0\) it follows 6 identities:

\[
\begin{align*}
\lambda_i \mu_j + \mu_i \lambda_j &= 0, & 1 \leq i \leq j \leq 2, \\
\lambda_p \mu_q + \mu_p \lambda_q &= 0, & 1 \leq p \leq q \leq 2.
\end{align*}
\]

From \(\eta(e_i \tilde{e}_p) = \eta(e_i) \eta(\tilde{e}_p)\) it follows 8 identities:
Proposition 44. Let $A$ be a gonosomal $K$-algebra with a gonosomal basis $(e_i)_{1 \leq i \leq n} \cup (\tilde{e}_p)_{1 \leq p \leq m}$ such that $e_i \tilde{e}_j = \sum_{k=1}^{n} \gamma_{ik} e_k + \sum_{p=1}^{m} \tilde{y}_{ijp} \tilde{e}_p$, if it exists $\sigma \in K$, $\sigma \neq 0$, 1 verifying $\sum_{k=1}^{n} \gamma_{pk} = \sigma$ for all $1 \leq i \leq n$ and $1 \leq p \leq m$ then $A$ is dibaric.

Proof. Indeed, the algebra $A$ is dwieighted by $\chi(e_i) = \frac{1}{2(1-\sigma)} f$ and $\chi(\tilde{e}_p) = \frac{1}{2(1-\sigma)} m$. \hfill \Box

Proposition 45. Let $A$ be a gonosomal algebra with gonosomal basis $(e_i)_{1 \leq i \leq n} \cup (\tilde{e}_p)_{1 \leq p \leq m}$ and multiplication $e_i \tilde{e}_j = \sum_{k=1}^{n} \gamma_{ik} e_k + \sum_{p=1}^{m} \tilde{y}_{ijp} \tilde{e}_p$. The algebra $A$ is dibaric if and only if the system of $2nm$ quadratic equations in unknowns $x_1, \ldots, x_n$, $y_1, \ldots, y_m$:

\[
\begin{align*}
&x_iy_j - 2\sum_{p=1}^{m} \gamma_{ijp} x_p = 0, \\
&x_iy_j - 2\sum_{p=1}^{m} \tilde{y}_{ijp} y_p = 0, \quad (1 \leq i \leq n, 1 \leq j \leq m)
\end{align*}
\]

admits a non-zero solution.

Proof. Suppose that $A$ is dwieighted by $\chi$ with $\chi(e_i) = \alpha_i f + \beta_i m$ and $\chi(\tilde{e}_i) = \tilde{\alpha}_i f + \tilde{\beta}_i m$. From $\chi(e_i \tilde{e}_j) = \chi(e_i) \chi(\tilde{e}_j) = 0$ and $\chi(\tilde{e}_p \tilde{e}_q) = \chi(\tilde{e}_p) \chi(\tilde{e}_q) = 0$ it comes:

\[
\begin{align*}
(5.3) \quad &\alpha_i \beta_j + \beta_i \alpha_j = 0, \quad (1 \leq i, j \leq n) \\
(5.4) \quad &\tilde{\alpha}_p \tilde{\beta}_q + \tilde{\beta}_p \tilde{\alpha}_q = 0, \quad (1 \leq p, q \leq m)
\end{align*}
\]

From $\chi(e_i \tilde{e}_j) = \chi(e_i) \chi(\tilde{e}_j)$ it results:

\[
\begin{align*}
(5.5) \quad &\sum_{p=1}^{n} \gamma_{ijp} \alpha_p + \sum_{p=1}^{m} \tilde{y}_{ijp} \tilde{\alpha}_p = \frac{1}{2} \left( \alpha_i \tilde{\beta}_j + \beta_i \tilde{\alpha}_j \right), \\
&\sum_{p=1}^{n} \gamma_{ijp} \beta_p + \sum_{p=1}^{m} \tilde{y}_{ijp} \tilde{\beta}_p = \frac{1}{2} \left( \alpha_i \tilde{\beta}_j + \beta_i \tilde{\alpha}_j \right), \quad (1 \leq i \leq n, 1 \leq j \leq m).
\end{align*}
\]

There is $1 \leq i_0 \leq n$ or $1 \leq p_0 \leq m$ such that $\alpha_{i_0} \neq 0$ or $\tilde{\alpha}_{p_0} \neq 0$ because otherwise it would $\chi(A) = K(m)$ and $\chi$ would not surjective. Suppose for example that $\alpha_{i_0} \neq 0$ for an integer $1 \leq i_0 \leq n$, then from the equations $(5.5)$ it comes $\alpha_{i_0} \beta_j = 0$ from where $\beta_i = 0$, it follows from $(5.5)$ that for all $j \neq i_0$ we have $\alpha_{i_0} \beta_j = 0$ thus $\beta_j = 0$ for all $1 \leq j \leq m$, we deduce that there is $1 \leq j_0 \leq m$ such that $\beta_{j_0} \neq 0$ otherwise we would have $\chi(A) = K(f)$ and $\chi$ would not surjective and by the same way as above this leads to $\tilde{\alpha}_j = 0$ for all $1 \leq j \leq m$. 


It follows that the system (5.5) becomes:

\[
\begin{aligned}
&\sum_{p=1}^{n} \gamma_{jp} \alpha_{p} = \frac{1}{2} \alpha_{i} \tilde{\beta}_{j}, \\
&\sum_{p=1}^{m} \gamma_{jp} \tilde{\beta}_{p} = \frac{1}{2} \alpha_{j}, \quad (1 \leq i \leq n, 1 \leq j \leq m)
\end{aligned}
\]

Conversely, if the system (5.6) admits a non-zero solution, then we verify without difficulty the map \( \chi \) defined on \( A \) by \( \chi(e_i) = \alpha_i f \) and \( \chi(\tilde{e}_i) = \tilde{\beta} m \) is a dibaric function. \( \square \)

6. IDEMPOTENTS IN GONOSOMAL ALGEBRAS

In order to study the idempotents of a gonosomal algebra we introduce the following definition:

**Definition 46.** Let \( A \) be a gonosomal \( K \)-algebra with gonosomal basis \( \{e_i\}_{1 \leq i \leq n} \cup \{\tilde{e}_j\}_{1 \leq j \leq m} \). For \( x = \sum_{i=1}^{n} \alpha_i e_i + \sum_{j=1}^{m} \alpha_j \tilde{e}_j \) in \( A \), the scalar \( \mu(x) = \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \alpha_j \) is called mass of \( x \).

**Proposition 47.** For a gonosomal \( K \)-algebra admits an idempotent of mass \( \mu \neq 0 \) it is necessary that the field \( K \) contains the roots of \( X^2 - (\mu - 1)^2 + 1 \).

**Proof.** Let \( A \) be a gonosomal \( K \)-algebra with gonosomal basis \( \{e_i\}_{1 \leq i \leq n} \cup \{\tilde{e}_j\}_{1 \leq j \leq m} \). Let us suppose that there exists an idempotent \( e = \sum_{i=1}^{n} \alpha_i e_i + \sum_{j=1}^{m} \alpha_j \tilde{e}_j \) of mass \( \mu(e) = \mu \). We have \( e^2 = e \) if and only if we have \( 2 \sum_{i,j} \alpha_i \alpha_j e_i \tilde{e}_j = e \), which results in the system of quadratic equations with \( n + m \) unknowns \( \alpha_1, \ldots, \alpha_m \) and \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m \):

\[
\begin{aligned}
2 \sum_{i,j} \gamma_{ip} \alpha_j &= \alpha_p, \quad 1 \leq p \leq n, \\
2 \sum_{i,j} \tilde{\gamma}_{ip} \tilde{\alpha}_j &= \tilde{\alpha}_q, \quad 1 \leq q \leq m.
\end{aligned}
\]

If we make the sum of all equations of this system we obtain:

\[
2 \left( \sum_{i=1}^{n} \alpha_i \right) \left( \sum_{j=1}^{m} \tilde{\alpha}_j \right) = \sum_{p=1}^{n} \alpha_p + \sum_{q=1}^{m} \tilde{\alpha}_q = \mu,
\]

in other words \( \sum_{p=1}^{n} \alpha_p \) and \( \sum_{q=1}^{m} \tilde{\alpha}_q \) are roots of \( 2X^2 - 2\mu X + \mu \) whose discriminant is \( \mu^2 - 2\mu = (\mu - 1)^2 - 1. \) \( \square \)

In the particular case of a population with a dominant male where a single male involved in reproduction, we can specify the set of idempotents. This case was studied in [10] and [12] for evolution algebras of the bisexual population, for gonosomal algebras we have:

**Proposition 48.** Let \( A \) be a gonosomal algebra with gonosomal basis \( \{e_i\}_{1 \leq i \leq n} \cup \{\tilde{e}_j\} \) where \( e_i \tilde{e}_j = \sum_{p=1}^{n} \gamma_{ip} e_p + \tilde{\gamma}_e \tilde{e} \). Noting \( \Gamma = (\gamma_{ip})_{1 \leq i \leq p \leq n} \in M_n(K) \) and \( \tilde{\Gamma} = (\tilde{\gamma}_e)_{1 \leq e \leq n} \in M_{1,n}(K) \), the set of nonzero idempotents of \( A \) is \( \{(\alpha, \tilde{\alpha}) \in K^n \times K^m : \alpha \tilde{\alpha} \neq 0 \} \), where \( \tilde{\alpha} \neq 0 \) is such that \( (2\tilde{\alpha})^{-1} \in \text{Spec}(\Gamma) \), \( \alpha \in K^n \) verify \((\Gamma - (2\tilde{\alpha})^{-1} I_n) \alpha^T = 0 \) and \( \tilde{\Gamma} \alpha^T = \frac{1}{2} \).

**Proof.** We have \( n \geq 2 \) and \( m = 1 \), let \( x = \sum_{i=1}^{n} \alpha_i e_i + \tilde{\alpha} \tilde{e} \) be an idempotent of \( A \), with this the system (6.1) is written:

\[
\begin{aligned}
2\tilde{\alpha} \sum_{i=1}^{n} \gamma_{ip} \alpha_i &= \alpha_p, \quad 1 \leq p \leq n, \\
2\tilde{\alpha} \sum_{i=1}^{m} \tilde{\gamma}_i \alpha_i &= \tilde{\alpha}.
\end{aligned}
\]

If \( \tilde{\alpha} = 0 \) we have \( \alpha_1 = \ldots = \alpha_n = 0 \).

If \( \tilde{\alpha} \neq 0 \), putting \( \Gamma = (\gamma_{ip})_{1 \leq i \leq p \leq n} \in M_n(K), \tilde{\Gamma} = (\tilde{\gamma}_e)_{1 \leq e \leq n} \in M_{1,n}(K) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), the above linear system becomes:

\[
(\Gamma - (2\tilde{\alpha})^{-1} I_n) \alpha^T = 0, \quad \tilde{\Gamma} \alpha^T = \frac{1}{2} \quad (\ast).
\]
We have \( \det \left( \Gamma - (2\tilde{\alpha})^{-1}I_n \right) = 0 \), otherwise the system (*) has no solution, therefore \((2\tilde{\alpha})^{-1} \in \text{Spec} (\Gamma)\) and \(\alpha^T \in \ker \left( \Gamma - (2\tilde{\alpha})^{-1}I_n \right)\).

\[\square\]

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