Mean curvature flow for pinched submanifolds
in rank one symmetric spaces

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Abstract

G. Pipoli and C. Sinestrari considered the mean curvature flow starting from a closed submanifold in the complex projective space. They proved that if the submanifold is of small codimension and satisfies a suitable pinching condition for the second fundamental form, then the flow has two possible behaviors: either the submanifold collapses to a round point in finite time, or it converges smoothly to a totally geodesic submanifold in infinite time. In this paper, we prove the similar results for the mean curvature flow starting from pinched closed submanifolds in (general) rank one symmetric spaces of compact type. Also, we prove that closed submanifolds in (general) rank one symmetric spaces of non-compact type collapse to a round point along the mean curvature flow under certain strict pinching condition for the norm of the second fundamental form.

1 Introduction

Let \( f : M \hookrightarrow \overline{M} \) be a smooth immersion of a closed connected manifold \( M \) into a Riemannian manifold \( \overline{M} \). Denote by \( h \) and \( H \) the second fundamental form and the mean curvature vector of the immersion, respectively. The mean curvature flow starting from \( f \) is the one-parameter family \( \{ F_t \}_{t \in [0,T_{\text{max}})} \) of immersions of \( M \) into \( \overline{M} \) satisfying

\[
\begin{aligned}
& \frac{\partial F_t}{\partial t} = H_t \quad (0 \leq t < T_{\text{max}}) \\
& F_0 = f,
\end{aligned}
\]

where \( H_t \) is the mean curvature vector of \( F_t \). Define a map \( F : M \times [0,T_{\text{max}}) \rightarrow \overline{M} \) by \( F(p,t) = F_t(p) \) ((\( p, t \)) \( \in M \times [0,T_{\text{max}}) \)). Denote by \( M_t \) the image \( F_t(M) \). In the case where \( F_t \)'s are embeddings, we call \( \{ M_t \}_{t \in [0,T_{\text{max}}]} \) the mean curvature flow starting from \( M_0 \). It is well-known that a smooth solution of this flow equation (1.1) exists uniquely in short time because \( M \) is closed. Furthermore, if \( T_{\text{max}} \) is finite, the curvature of \( M_t \) necessarily becomes unbounded as \( t \rightarrow T_{\text{max}} \), that is, the flow blows up in finite time.

In 1984, the study of the mean curvature flow treated as the evolution of immersions was originated by Huisken (\cite{H1}). He (\cite{H1}) proved that any closed convex hypersurface in Euclidean space collapses to a round point in finite time along the mean curvature flow. In 1986, he (\cite{H2}) proved that the same fact holds for the mean curvature flow starting from closed hypersurfaces in general Riemannian manifolds (of bounded curvature) satisfying a certain kind of convexity condition, where this convexity condition coincides with the usual convexity condition in the case where the ambient space is a Euclidean space. In 2010, Andrews and Baker (\cite{AB}) considered the mean curvature flow of an arbitrary codimension in a Euclidean space (and a sphere) and proved the collapse to a round point of the mean curvature flow starting from a closed submanifold of an arbitrary codimension in the Euclidean space (and a sphere) satisfying a suitable pinching condition for the norm of the second fundamental form. In 2011, Liu, Xu, Ye and Zhao (\cite{LXYZ}) proved the collapse of the flow to a round
point in finite time under the similar pinching condition in the case where the ambient space is the hyperbolic space. Also, in 2012, Liu, Xu and Zhao ([LXZ]) proved the collapse of the flow to a round point in finite time under the similar pinching condition in the case where the ambient space is a general Riemannian manifold. Recently, Pipoli and Sinestrari ([PS]) have proved that, if the maximal time of the flow is finite, then the flow collapses to a round point under the similar pinching condition in the case where the ambient space is the complex projective space, where we note that their pinching condition is weaker than the condition in [LXZ]. By contrast, very few authors have considered the cases where the mean curvature flow converges to a stationary limit. It is known that this kind of results are obtained only in special cases. For the curve shortening flow, Grayson ([G]) showed that an embedded curve in a Riemannian surface either collapses to a round point or converges smoothly to geodesic along the mean curvature flow. For the mean curvature flow starting from the higher dimensional submanifold in a Riemannian manifold, other kinds of singularities can occur and an analogous statement can be expected only under suitable restrictions. Higher dimensional analogues of the results of [G] were obtained for submanifolds in the sphere by Huisken ([H3]) (codimension one-case) and by Baker ([Ba]) (higher codimension-case). Their results can be stated together as follows.

**Fact 1.** ([H3] [Ba]) Let $M$ be an $m(\geq 2)$-dimensional closed submanifold in the $(m + k)$-dimensional unit sphere $S^{m+k}(1)$ and $\{M_t\}_{t \in [0,T_{\text{max}})}$ be the mean curvature flow starting from $M$. Assume that $M$ satisfies

\[
\begin{align*}
||h||^2 &< \frac{1}{m-1}||H||^2 + 2 & (\text{when } m \geq 4 \text{ or } "m = 3 \text{ and } k = 1") \\
||h||^2 &< \frac{3}{4}||H||^2 + \frac{4}{3} & (\text{when } m = 2 \text{ and } k = 1) \\
||h||^2 &< \frac{4}{3m}||H||^2 + \frac{2(m-1)}{3} & (\text{when } m = 2, 3 \text{ and } k > 1),
\end{align*}
\]

where $h$ and $H$ denote the second fundamental form and the mean curvature vector of $M$, respectively. Then this condition is preserved along this flow and one of the following statements holds:

(i) $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}},$

(ii) $T_{\text{max}} = \infty$ and $M_t$ converges to a totally geodesic submanifold as $t \to \infty$. This limit submanifold is isometric to $S^m(1)$.

![Figure 1: The possible two behaviors of the mean curvature flow](image)

Pipoli and Sinestrari ([PS]) have recently proved the similar fact in the $n$-dimensional complex projective space $\mathbb{C}P^n(4)$ of constant holomorphic sectional curvature 4.
Fact 2. (PS) Let $M$ be a real $m$-dimensional closed submanifold in $\mathbb{C}P^n(4)$ and $\{M_t\}_{t \in [0,T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Set $k := 2n - m$. Assume that “$m \geq 5$ and $k = 1$” or “$m \geq 11$ and $2 \leq k < \frac{m - 3}{4}$” hold and that $M$ satisfies

\[
\|h\|^2 < \begin{cases} \frac{1}{m-1}\|H\|^2 + 2 & \text{(when } m \geq 5 \text{ and } k = 1) \\ \frac{1}{m-1}\|H\|^2 + \frac{m - 4k - 3}{m} & \text{(when } m \geq 11 \text{ and } 2 \leq k < \frac{m - 3}{4}) \end{cases}
\]

where $h$ and $H$ denote the second fundamental form and the mean curvature vector of $M$, respectively. Then this condition is preserved along this flow and one of the following statements holds:

(i) $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}}$,

(ii) $T_{\text{max}} = \infty$ and $M_t$ converges to a totally geodesic submanifold as $t \to \infty$. The case (ii) can occur only in the case where $m$ is even, and the limit submanifold $M_{\infty}$ is isometric to $\mathbb{C}P^{n-1}(4)$.

Denote by $\mathbb{H}P^n(4)$ the quaternionic projective space of constant quaternionic sectional curvature.

Fact 3. (PS) Let $M$ be a closed real hypersurface in $\mathbb{H}P^n(4)$ ($n \geq 3$) and $\{M_t\}_{t \in [0,T_{\text{max}}]}$ be the mean curvature flow starting from pinched closed real hypersurfaces in $\mathbb{H}P^n(4)$ (see Theorem 7.1 in [PS]).

\[
\text{(1.4) } \|h\|^2 < \frac{1}{4n-2} \|H\|^2 + 2,
\]

where $h$ and $H$ denote the second fundamental form and the mean curvature vector of $M$, respectively. Then this condition is preserved along this flow, $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}}$.

On the other hand, K. Liu, Y. Xu and E. Zhao ([LXZ]) proved the following fact in an $n$-dimensional complete Riemannian manifold $\overline{M}$ satisfying the following bounded curvature condition:

\[
\text{(1.5) } -K_1 \leq K_{\overline{M}} \leq K_2, \quad \|\nabla \overline{R}\| \leq L \text{ and } \text{inj}(\overline{M}) > 0,
\]

where $K_{\overline{M}}$ denotes the sectional curvature of $\overline{M}$ (which is a function over the Grassmann bundle of $\overline{M}$ consisting of 2-planes), $\nabla$ and $\overline{R}$ denote the Levi-Civita connection and the curvature tensor of $\overline{M}$, respectively, $\text{inj}(\overline{M})$ denotes the injective radius of $\overline{M}$, $K_1, K_2, L$ are some non-negative constants and $\nu_{\overline{M}}$ is some positive constant.

Fact 4. (LXZ) Let $M$ be an $m(\geq 2)$-dimensional closed submanifold in an $n$-dimensional Riemannian manifold $\overline{M}$ satisfying the above bounded curvature condition (1.5) and $\{M_t\}_{t \in [0,T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Then, for some positive constant $b_0$ described explicitly in terms of $n, m, K_1, K_2$ and $L$, the following statement holds:

If $M_0$ satisfies

\[
\text{(1.6) } \|h\|^2 \leq \begin{cases} \frac{4}{3m} \|H\|^2 - b_0 & \text{(when } m = 2, 3) \\ \frac{1}{m-1} \|H\|^2 - b_0 & \text{(when } m \geq 4) \end{cases}
\]

where $h$ and $H$ are the second fundamental form and the mean curvature vector of $M$, respectively, then this condition is preserved along this flow, $T_{\text{max}} < \infty$ and $M_t$’s collapses to a round point as $t \to T_{\text{max}}$.
Denote by $\mathbb{CP}^n(4c), \mathbb{HP}^n(4c)$ and $\mathbb{OP}^2(4c)$ the complex projective space of constant holomorphic sectional curvature $4c$, the quaternionic projective space of constant quaternionic sectional curvature $4c$ and the Cayley plane of constant octonian sectional curvature $4c$, respectively. For $\mathbb{CH}^n(4c)$, $\mathbb{HH}^n(-4c)$ and $\mathbb{OH}^2(-4c)$ the complex hyperbolic space of constant holomorphic sectional curvature $-4c$, the quaternionic hyperbolic space of constant quaternionic sectional curvature $-4c$ and the Cayley hyperbolic plane of constant octonian sectional curvature $-4c$. Throughout this paper, $\mathbb{F}$ denotes the complex number field $\mathbb{C}$, the quaternionic algebra $\mathbb{H}$ or the Cayley algebra $\mathbb{O}$, $\mathbb{FP}^n(c)$ denotes one of rank one symmetric spaces of compact type:

$$\mathbb{CP}^n(4c), \mathbb{HP}^n(4c) \text{ or } \mathbb{OP}^2(4c)$$
and $\mathbb{FH}^n(c)$ denotes one of rank one symmetric spaces of non-compact type:

$$\mathbb{CH}^n(-4c), \mathbb{HH}^n(-4c) \text{ or } \mathbb{OH}^2(-4c).$$

Also, throughout this paper, $\overline{M}$ denotes $\mathbb{FP}^n(c)$ or $\mathbb{FH}^n(c)$. Set

$$d := \begin{cases} 2 & \text{(when } \overline{M} = \mathbb{CP}^n(4c), \mathbb{CH}^n(-4c)) \\
4 & \text{(when } \overline{M} = \mathbb{HP}^n(4c) \text{ and } k \geq 2) \\
8 & \text{(when } \overline{M} = \mathbb{OP}^2(4c) \text{ and } k = 1) \\
2c(m - 4(d - 1)k - 3)c & \text{(when } \overline{M} = \mathbb{CH}^n(-4c), \mathbb{HH}^n(-4c) \text{ and } k \geq 2) \\
-8c/m & \text{(when } \overline{M} = \mathbb{CP}^n(4c) \text{ and } k = 1) \\
-(8m + 4(d - 1)k + 3)c/m & \text{(when } \overline{M} = \mathbb{OP}^2(4c) \text{ and } k = 1). 
\end{cases}$$

Let $M$ be an $m$-dimensional closed submanifold in $\overline{M}$ and set $k := dn - m$. Set

$$b := \begin{cases} 2c(m - 4(d - 1)k - 3)c & \text{(when } \overline{M} = \mathbb{CP}^n(4c) \text{ and } k = 1) \\
-8c/m & \text{(when } \overline{M} = \mathbb{CP}^n(4c), \mathbb{HP}^n(4c) \text{ and } k \geq 2) \\
-(8m + 4(d - 1)k + 3)c/m & \text{(when } \overline{M} = \mathbb{CH}^n(-4c), \mathbb{HH}^n(-4c) \text{ and } k \geq 2). 
\end{cases}$$

We consider the following condition:

$$(*) \quad ||h||^2 < \frac{1}{m - 1}||H||^2 + b,$$

where $h$ and $H$ denote the second fundamental form and the mean curvature vector of $M$, respectively. In general, the inequality $||h||^2 \geq \frac{||H||^2}{m}$ holds. Hence this condition $(*)$ implies a pinching condition for $||h||^2$. In this paper, we prove the following facts for the mean curvature flows starting from closed submanifolds in rank one symmetric spaces $\overline{M}$ satisfying the above pinching condition $(*)$.

**Theorem 1.1.** Let $M$ be a closed real hypersurface in $\mathbb{FP}^n(4c)$ ($n \geq 3$) and $\{M_t\}_{t \in [0, T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Assume that $M$ satisfies the above pinching condition $(*)$ (for $b = 2c$). Then the condition $(*)$ is preserved along this flow, $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}}$.

**Theorem 1.2.** Let $M$ be an $m$-dimensional closed submanifold of codimension greater than one in $\mathbb{CP}^n(4c)$ or $\mathbb{HP}^n(4c)$, and $\{M_t\}_{t \in [0, T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Assume that $m \geq \max\{\frac{2d}{k}, \frac{3d}{2} + 5\}$, $M$ satisfies the pinching condition $(*)$. Then the condition $(*)$ is preserved along this flow and one of the following statements holds:

(i) $T_{\text{max}} < \infty$, and $M_t$ collapses to a round point as $t \to T_{\text{max}}$.

(ii) $T_{\text{max}} = \infty$, and $M_t$ converges to a totally geodesic submanifold as $t \to T_{\text{max}}$.

**Theorem 1.3.** Let $M$ be a closed real hypersurface in $\mathbb{FH}^n(-4c)$ ($n \geq 3$) and $\{M_t\}_{t \in [0, T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Assume that $M$ satisfies the pinching condition $(*)$. Then the condition $(*)$ is preserved along this flow, $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}}$.
Theorem 1.4. Let $M$ be an $m$-dimensional closed submanifold of codimension greater than one in $C\mathbb{H}^n(-4c)$ or $\mathbb{H}H^m(-4c)$, and $\{M_t\}_{t\in[0,T_{\text{max}}]}$ be the mean curvature flow starting from $M$. Assume that $m \geq \max\{\frac{3d}{2}, \frac{3d}{2} + 5\}$, $M$ satisfies the pinching condition ($\ast$). Then the condition ($\ast$) is preserved along this flow, $T_{\text{max}} < \infty$ and $M_t$ collapses to a round point as $t \to T_{\text{max}}$.

Remark 1.1. (i) The positive constant $b_0$ in the statement of the result in [LXZ] is given in the proof of Theorem 3.2 of [LXZ] (see Page 10 of [LXZ]). By comparing the definitions of $b_0$ and $b$, we have $-b_0 < b$. Hence Theorems 1.1 – 1.4 improve the above result (Fact 4) in [LXZ].

(ii) In their result (Fact 2), Pipoli and Sinistrari ([PS]) imposed the small codimension condition $k < \frac{m-3}{4}$ (in the case of $k \geq 2$) to claim that the term $\frac{m-4k-3}{m}$ in their pinching condition (1.3) is positive. In our results (Theorems 1.2 and 1.4), we need not to impose such a small codimension condition because we do not claim that the term $b$ in our pinching condition ($\ast$) is positive. On the other hand, we need to impose the lower bound condition $m \geq \max\{\frac{3d}{2}, \frac{3d}{2} + 5\}$ for the dimension of the submanifold to prove the preservability of the condition ($\ast$) along the mean curvature flow. In fact, since we use an orthonormal frame of type (II) (as in Lemma 3.1) to prove the preservability of the condition ($\ast$), we need to impose $m \geq \frac{3d}{2}+5$. Also, according to the proof of Proposition 3.8, we need to impose $m \geq \frac{3d}{2}+5$.

(iii) The conditions ($\ast$) implies that

$$||H||^2 > \begin{cases} 8(dn-1)(dn-2)c & \text{when } M = \mathbb{F}H^n(-4c) \text{ and } k = 1 \\ (m-1)(8m+4(d-1)k+3)c & \text{when } M = \mathbb{C}H^n(-4c), \mathbb{H}H^n(-4c) \text{ and } k \geq 2 \end{cases}$$

Thus the conditions ($\ast$) in Theorems 1.3 and 1.4 imply that $||H||$ is rather big (see Figure 2).

(iv) In our method of the proof, we cannot derive the result similar to Theorems 1.2 and 1.4 in the case of $M = \partial P^2(4c)$ or $\partial H^2(-4c)$. For, in these cases, $m$ must be larger than or equal to $\frac{3d}{2}+5 = \frac{48}{2}+5 = 17$ in order that the inequality (3.19) in Section 3 holds. However, this is impossible because $\dim \partial P^2(4c) = \dim \partial H^2(-4c) = 16$. Also, is negative the constant $\alpha = \frac{(11-2d)m-19}{9m(m+2)}$ in (4.1) of Section 4 in these cases. Hence the evolution inequality (4.2) for $f_\alpha$ in Section 4 does not hold.

![Figure 2](image)

$\overline{M} = \mathbb{F}H^n(-4c)$

$\overline{M} = \mathbb{F}H^n(-4c)$

$M$ is almost umbilic and $|H|$ is rather big.

$M_t$ collapses to $\{p_0\}$ as $t \to T_{\text{max}}(< \infty)$.

$M$ is almost umbilic but $|H_p|$ is rather small.

How is the behaviour of $M_t$ as $t \to T_{\text{max}}$?

This paper is organized as follows. In Section 2, we recall some basic notions and facts. In Section 3, we prove the preservability of the above pinching condition ($\ast$) along the mean curvature flow. In Section 4, we study the behavior of the norm of the traceless part of the second fundamental form, which will be used to measure the improvement of the pinching as $t \to T_{\text{max}}$. In Section 5, we prove the finiteness of the maximal time of the mean curvature flow starting from closed submanifolds in
rank one symmetric spaces of non-compact type. In Section 6, we prove Theorems 1.1, 1.2 (the case of $T_{\text{max}} < \infty$), Theorems 1.3 and 1.4. In Section 7, we prove Theorem 1.2 (the case of $T_{\text{max}} = \infty$).

The second author together with the first author studied the mean curvature flow in the quaternionic projective space and the third author together with the first author studied the mean curvature flow in the complex hyperbolic space.

2 Basic notions and facts

Set

\[ \tau := \begin{cases} 
1 & \text{(when } \overline{M} = \mathbb{P}^n(4c)\text{)} \\
-1 & \text{(when } \overline{M} = \mathbb{P}^n(-4c)\text{)}.
\]  

Denote by $\overline{\mathcal{g}}$ and $\overline{R}$ the metric and the curvature tensor of $\overline{M}$, respectively. First we recall that $\overline{R}$ is given by

\[ \overline{R}(X,Y,Z,W) = \tau c(\overline{\mathcal{g}}(Y,Z)\overline{\mathcal{g}}(X,W) - \overline{\mathcal{g}}(X,Z)\overline{\mathcal{g}}(Y,W)) + \sum_{\xi=1}^{d-1} (\overline{\mathcal{g}}(Y,J_\xi Z)\overline{\mathcal{g}}(X,J_\xi W) - \overline{\mathcal{g}}(X,J_\xi Z)\overline{\mathcal{g}}(Y,J_\xi W) - 2\overline{\mathcal{g}}(X,J_\xi Y)\overline{\mathcal{g}}(Z,J_\xi W)) \]  

for all tangent vector fields $X, Y, Z, W$ of $\overline{M}$, where $(J_1, \ldots, J_{d-1})$ is the complex structure, a canonical local frame field of the quaternionic structure or the Octonian structure of $\overline{M}$. Hence the sectional curvature $\overline{\mathcal{K}}(X,Y)$ of the tangent plane spanned by orthonormal tangent system $X, Y$ of $\overline{M}$ is given by

\[ \overline{\mathcal{K}}(X,Y) = \overline{\mathcal{K}}(X,Y,Y,X) = \tau c \left( 1 + 3 \sum_{\xi=1}^{d-1} \overline{\mathcal{g}}(X,J_\xi Y)^2 \right), \]

that is, $c \leq \tau \overline{\mathcal{K}} \leq 4c$. Furthermore, $\overline{M}$ is a symmetric space (hence $\nabla \overline{R} = 0$) and an Einstein manifold with Einstein constant $\tau c (dn + 3d - 4)$, which is denoted by $\tau$.

Let $M$ be an $m$-dimensional closed submanifold in $\overline{M}$. Denote by $g$, $\nabla$ and $R$ the induced metric, the Levi-Civita connection and the curvature tensor of $M$, respectively. The tangent and normal space of $M$ at a point $p$ are denoted by $T_pM$ and $N_pM$, respectively. Set $k := n - m$. Unless otherwise mentioned, Latin letters $i, j, l, \ldots$ run from 1 to $m$, Greek letters $\alpha, \beta, \gamma, \ldots$ run from $m + 1$ to $n$. Unless necessary, we abbreviate $S_p$ as $S$ for a tensor field $S$ on $M$. Let $(e_1, \ldots, e_n)$ be an orthonormal frame of $\overline{M}$ at a point of $M$, such that the first $m$ vectors are tangent to $M$ and the other ones are normal. With respect to this orthonormal frame, the second fundamental form $h$ can be written as

\[ h_{ij} = \sum_{\alpha} h^\alpha_{ij} e_\alpha \quad (h = \sum_{\alpha} h^\alpha \otimes e_\alpha) \]

for some symmetric $(0,2)$-tensor fields $h^\alpha$. The mean curvature vector field $H$ of $M$ is written as

\[ H = \sum_{\alpha} \text{trace}_g h^\alpha e_\alpha = \sum_{\alpha} \sum_{i,j} g^{ij} h^\alpha_{ij} e_\alpha. \]

Set $H^\alpha := \text{trace}_g h^\alpha = \sum_{r,s} g^{rs} h^\alpha_{rs}$. Denote by $\hat{h}$ the traceless part $h - \frac{1}{m} H \otimes g$ of the second fundamental form. Clearly we have $||\hat{h}||^2 = ||h||^2 - \frac{1}{m} ||H||^2$. In the case where $M$ is a hypersurface, the mean curvature vector field $H$ is a multiple of the unit normal vector field $\nu$ and $H = -(\lambda_1 + \cdots + \lambda_m) \nu$.
holds, where \(\lambda_1 \leq \cdots \leq \lambda_m\) are the principal curvatures of \(M\). In addition, we have \(||h||^2 = \lambda_1^2 + \cdots + \lambda_m^2\) and

\[
(2.4) \quad ||h||^2 = ||h||^2 - \frac{1}{m}||H||^2 = \frac{1}{m} \sum_{i<j}(\lambda_i - \lambda_j)^2.
\]

Thus the smallness of \(||\hat{h}||^2\) implies that the principal curvatures are close to one another.

Let \(\{M_t = F_t(M)\}_{t \in [0,T_{\text{max}}]}\) be the mean curvature flow an \(m\)-dimensional closed submanifold \(M\) in \(\tilde{M}\). Denote by \(g_t, \nabla^t, R_t, h_t, H_t, d\mu_t\) the induced metric, the Levi-Civita connection, the curvature tensor, the second fundamental form, the mean curvature vector and the volume element of \(M_t\), respectively. The evolution equations of the various geometric quantities along the mean curvature flow in a general Riemannian manifold were computed in [AB] and [Ba]. In our case, they take a simpler form because the ambient space \(\tilde{M}\) is a locally symmetric space. In our case, the evolution equations of \(||H_t||^2, ||h_t||^2\) and \(d\mu_t\) are as follows.

**Lemma 2.1.** The quantities \(||H_t||^2, ||h_t||^2\) and \(d\mu_t\) satisfy the following evolution equations:

\[
(2.5) \quad \frac{\partial}{\partial t}||H_t||^2 = \Delta ||H_t||^2 - 2||\nabla^t H_t||^2 + 2 \sum_{\alpha,\beta} \left( \sum_{i,j} H^\alpha_{ij} h^\beta_{ij} \right)^2 2 \sum_{l,\alpha,\beta} R^\alpha_{l,\alpha\beta l} H^\alpha H^\beta,
\]

\[
(2.6) \quad \frac{\partial}{\partial t}||h_t||^2 = \Delta ||h_t||^2 - 2||\nabla^t h_t||^2 + 2 \sum_{\alpha,\beta} \left( \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \right)^2 2 \sum_{\alpha,\beta} \left( \sum_{p} h^\alpha_{ip} h^\beta_{jp} - h^\beta_{ip} h^\alpha_{jp} \right)^2
\]

\[
+ 4 \sum_{i,j,p,q} R_{ipqi} \left( \sum_{\alpha} h^\alpha_{ip} h^\alpha_{iq} \right) - 4 \sum_{j,l,p} R_{ljpl} \left( \sum_{\alpha} h^\alpha_{pl} h^\alpha_{lj} \right)
\]

\[
+ 2 \sum_{l,\alpha,\beta} R^\alpha_{l,\alpha\beta l} \left( \sum_{i,j} h^\alpha_{ij} h^\alpha_{ij} \right) - 8 \sum_{j,p,\alpha,\beta} R^\alpha_{jp\beta\alpha} \left( \sum_{i} h^\alpha_{ij} h^\beta_{ij} \right),
\]

\[
(2.7) \quad \frac{\partial}{\partial t}d\mu_t = -||H_t||^2 d\mu_t.
\]

In the case where \(M\) is a hypersurface, these equations have the following simpler forms.

**Lemma 2.2.** Assume that \(M\) is a hypersurface. Then we have

\[
(2.8) \quad \frac{\partial}{\partial t}||H_t||^2 = \Delta ||H_t||^2 - 2||\nabla^t H_t||^2 + 2||H_t||^2 (||h_t||^2 + \overline{\text{Ric}}(\nu_t, \nu_t))
\]

\[
(2.9) \quad \frac{\partial}{\partial t}||h_t||^2 = \Delta ||h_t||^2 - 2||\nabla^t h_t||^2 + 2||h_t||^2 (||h_t||^2 + \overline{\text{Ric}}(\nu_t, \nu_t)) - 4 \sum_{i,j,p,l} (h_{ij} h^p_{pl} - h^p_{ij} h^p_{pl})
\]

where \(\overline{\text{Ric}}\) is the Ricci tensor of \(\tilde{M}\).
3 The preservability of pinching condition

In this section, we prove that the pinching conditions in Theorem 1.1–1.4 are preserved along the mean curvature flow under the settings of Theorem 1.1–1.4, respectively. Let \( M \) be an \( m \)-dimensional closed submanifold in \( \overline{M} \). Set \( k := dn - m \). Denote by \( h \) and \( H \) the second fundamental form and the mean curvature vector of \( M \), respectively.

To obtain the desired estimates, it is important to perform the computations by using a special orthonormal frame with suitable properties. Let \( p \) be a point of \( M \) with \( ||H_p|| \neq 0 \). A first kind of orthonormal frame is an orthonormal frame of \( T_p \overline{M} \) satisfying

\[
e_{m+1} = \frac{H_p}{||H_p||}.
\]

Then we can choose \( e_{m+2}, \ldots, e_{dn} \) such that \((e_{m+1}, \ldots, e_{dn})\) is an orthonormal frame of \( N_pM \) and choose any orthonormal frame \((e_1, \ldots, e_m)\) of \( T_pM \). An orthonormal frame obtained in this way will be said to be of type (I). For the components of the second fundamental form \( h \) and its traceless part \( h \) with respect to an orthonormal frame \((e_1, \ldots, e_m)\) of type (I), the following relations hold:

\[
\begin{cases}
\text{trace}_g h^{m+1} = ||H||, \\
\text{trace}_g h^\alpha = 0, & \alpha \geq m + 2
\end{cases}
\]

and

\[
\begin{cases}
\hat{h}^{m+1} = h^{m+1} - \frac{||H||}{m} q, \\
\hat{h}^\alpha = h^\alpha, & \alpha \geq m + 2.
\end{cases}
\]

With respect to an orthonormal frame \((e_1, \ldots, e_m)\) of type (I), we adopt the following notation:

\[
||h_1||^2 := ||h^{m+1}||^2, \quad ||h_1||^2 := ||\hat{h}^{m+1}||^2, \quad ||h_-||^2 := ||\hat{h}_-||^2 := \sum_{\alpha=m+2}^{n} ||\hat{h}^\alpha||^2.
\]

A second kind of orthonormal frame is one more linked with the geometry of \( \overline{M} \), which is useful to compute explicitly the components of the curvature tensor of \( \overline{M} \). The properties required for this kind of orthonormal frame are described in the following lemma.

**Lemma 3.1.** If \( k \leq m \), then for each \( p \in M \) and each \( \xi \in \{1, \ldots, d-1\} \), there exists an orthonormal frame \((e_1^\xi, \ldots, e_m^\xi)\) of \( T_pM \) and an orthonormal frame \((e_{m+1}^\xi, \ldots, e_{dn}^\xi)\) of \( N_pM \) satisfying the following conditions:

(i) For every \( r \leq \left\lfloor \frac{k}{2} \right\rfloor \), we have

\[
\begin{cases}
J_\xi e_{m+2r-1}^\xi = \tau_{r}^\xi e_{2r-1}^\xi + \nu_{r}^\xi e_{m+2r}^\xi, \\
J_\xi e_{m+2r}^\xi = \tau_{r}^\xi e_{2r}^\xi - \nu_{r}^\xi e_{m+2r-1}^\xi,
\end{cases}
\]

where \( \xi \in \{1, \ldots, d-1\} \), \( \tau_{r}^\xi, \nu_{r}^\xi \in [0, 1] \), \((\tau_{r}^\xi)^2 + (\nu_{r}^\xi)^2 = 1 \) and \( \lfloor \cdot \rfloor \) denotes the Gauss’s symbol of \( \cdot \);

(ii) If \( k \) is odd, then \( J_\xi e_{m+k}^\xi = e_k^\xi \);

(iii) The remaining vectors satisfy

\[
e_{k+1}^\xi, \ J_\xi e_{k+1}^\xi = e_{k+2}^\xi, \ J_\xi e_{k+3}^\xi = e_{k+4}^\xi, \ldots, J_\xi e_{m-1}^\xi = e_m^\xi.
\]

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See the proof of Lemma 3.1 in [PS] about the proof of this lemma. An orthonormal frame satisfying the properties of this lemma will be said to be of type (II). Since \( J_\xi^2 = -id \), from (3.3) it follows easily that such an orthonormal frame also satisfies

\[
\begin{align*}
J_\xi e_2^{2r-1} &= -\nu^r e_2, \\
J_\xi e_2^r &= \nu^r e_2^{2r-1} - \tau^r e_m^{2r},
\end{align*}
\tag{3.4}
\]

If \( k \) is odd, it is convenient to define \( \tau^r_1 = 1, \nu^r = 0 \) for \( r = \frac{k+1}{2} \). In this way, the first equations in (3.3) and in (3.4) hold also for this value of \( r \).

In general, the requirements for orthonormal frames of type (I) and (II) are incompatible. In case of \( k \geq 2 \), we introduce the following notations in [AB]

\[
\begin{align*}
R_1 &:= \sum_{\alpha, \beta} \left( \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta} \right)^2 + \sum_{i,j,\alpha, \beta} \left( \sum_p h_{ip}^{\alpha} h_{jp}^{\beta} - h_{ip}^{\beta} h_{jp}^{\alpha} \right)^2, \\
R_2 &:= \sum_{i,j} \left( \sum_{\alpha} H_{ij}^{\alpha} \right)^2.
\end{align*}
\tag{3.5}
\]

If we use an orthonormal frame of type (I), it is easy to check that

\[
R_2 = \begin{cases} 
||h_1||^2||H||^2 = ||h_1||^2||H||^2 + \frac{1}{m}||H||^4 \quad \text{(when } H \neq 0) \\
0 \quad \text{ (when } H = 0) \end{cases}
\]

The following result, which was proved in [AB] §3 and in [Ba] §5.2, is useful in the estimation of the reaction term occurring in the evolution equations of Lemma 2.1. In the proof, is used only the algebraic properties of \( R_1 \) and \( R_2 \) (the properties of the flow is not used).

**Lemma 3.2.** At a point where \( H \neq 0 \) we have

\[
2R_1 - 2aR_2 \leq 2||h_1||^4 - 2 \left( a - \frac{2}{m} \right) ||h_1||^2||H||^2 - \frac{2}{m} \left( a - \frac{1}{m} \right) ||H||^4 + 8||h_1||^2||h_-||^2 + 3||h_-||^4
\]

for any \( a \in \mathbb{R} \). In addition, if \( a > \frac{1}{m} \) and if \( ||h||^2 = a||H||^2 + b \) holds for some \( b \in \mathbb{R} \), we have

\[
2R_1 - 2aR_2 \leq \left( 6 - \frac{2}{ma} \right) ||h||^2||h_-||^2 - 3||h_-||^4
+ \frac{2ma}{ma - 1} ||h_1||^2 + \frac{4b}{ma - 1} ||h_-||^2 - \frac{2b^2}{ma - 1}.
\]

Now we shall derive a sharp estimate on the gradient terms appearing in the evolution equations for \( ||h||^2 \) and \( ||H||^2 \), which will be used many times in the rest of this paper. Observe that the results are independent of the property of the flow. Our starting point is the following inequality, which was originally proved by Huisken (see Lemma 2.2 of [H2]) in the case of hypersurfaces, and later extended to general codimension by Liu, Xu and Zhao (see Lemma 3.2 of [LXZ]).

**Lemma 3.3.** Let \( M \) be an \( m \)-dimensional submanifold in \( \mathbb{M} \). Then

\[
||\nabla h||^2 \geq \left( \frac{3}{m + 2} - \eta \right) ||\nabla H||^2 - \frac{2}{m + 2} \left( \frac{2}{(m + 2)\eta} - \frac{m}{m - 1} \right) ||\omega||^2
\tag{3.6}
\]
holds for any $\eta > 0$. Here $\omega = \sum_{i,j} \mathbf{R}_{i,j} e_i \otimes \omega_j$, where $\omega_j$ is the dual frame to $e_i$. In particular, if $\overline{M}$ is $\mathbb{H}^m$ (in more general, Einstein) and if $M$ is a hypersurface, then $\omega = 0$ and as $\eta \to 0$ in Lemma 3.4, we find

$$||\nabla h||^2 \geq \frac{3}{m+2}||\nabla H||^2.$$  

(3.7)

For submanifolds of higher codimension, $\omega$ is in general nonzero. For any tangent vector field $X$ on $M$, we write $J_\xi X = P_\xi X + F_\xi X$, where $P_\xi X$ and $F_\xi X$ are the tangent and normal components of $J_\xi X$, respectively. Similarly, for any normal vector field $V$, we write $J_\xi V = t_\xi V + f_\xi V$, where $t_\xi V$ and $f_\xi V$ are tangent and normal components of $J_\xi V$, respectively. Let $P$ and $Q$ be elements of $\Gamma(T^*M \otimes TM)$, $\Gamma(T^*M \otimes T^\perp M)$, $\Gamma((T^\perp M)^* \otimes TM)$ and $\Gamma((T^\perp M)^* \otimes T^\perp M)$, where $TM$ (resp. $T^\perp M$) denotes the tangent (resp. normal) bundle of $M$, $(\bullet)^*$ denotes the dual bundle of $(\bullet)$ and $\Gamma(\bullet)$ denotes the space of all sections of the vector bundle $(\bullet)$. Define $\langle P, Q \rangle$ by

$$\langle P, Q \rangle := \begin{cases} 
\sum g(\mathbf{P}e_i, Qe_i) & (P, Q \in \Gamma(T^*M \otimes TM)) \\
\sum (\mathbf{P}e_i, Qe_i) & (P, Q \in \Gamma(T^*M \otimes T^\perp M)) \\
\sum g(\mathbf{P}e_\alpha, Qe_\alpha) & (P, Q \in \Gamma((T^\perp M)^* \otimes TM)) \\
\sum (\mathbf{P}e_\alpha, Qe_\alpha) & (P, Q \in \Gamma((T^\perp M)^* \otimes T^\perp M)),
\end{cases}$$

where $(e_i)$ is an orthonormal tangent frame of $M$ (with respect to $g$) and $(e_\alpha)$ is an orthonormal normal frame of $M$ (with respect to $\overline{g}$). Set $||P|| := \sqrt{\langle P, P \rangle}$.

Now we shall derive a relation among $||P_\xi||$, $||F_\xi||$ and $||P_\xi F_\xi||$.

**Lemma 3.4.** For $||P_\xi||$, $||F_\xi||$ and $||P_\xi F_\xi||$, the following relation holds:

$$||P_\xi||^2 \cdot ||F_\xi||^2 \geq m||P_\xi F_\xi||^2.$$  

(3.8)

**Proof.** We discuss the cases where $k$ is even and where $k$ is odd separately. First we consider the case of $k = 2k'$ (even). By using the relations (3.3) and (3.4), we can derive

$$||P_\xi||^2 = (m - k) + 2 \sum_{r \leq k'} (\nu^2)^2 = m - 2 \sum_{r \leq k'} (\tau^2)^2, \quad ||t_\xi||^2 = 2 \sum_{r \leq k'} (\tau^2)^2.$$ 

Therefore, by using $(\tau^2)^2 + (\nu^2)^2 = 1$ and $k \leq m$, we find

$$||P_\xi||^2 ||t_\xi||^2 = 2m \sum_{r \leq k'} (\tau^2)^2 - 4 \sum_{r \leq k'} (\tau^2)^2 (\nu^2)^2 \geq 2m \sum_{r \leq k'} (\tau^2)^2 - 2 \sum_{r \leq k'} (\tau^2)^4 + (\tau^2)^4 = 2m \sum_{r \leq k'} (\tau^2)^4 - 2k \sum_{r \leq k'} (\tau^2)^4 \geq 2m \sum_{r \leq k'} (\tau^2)^4 - (\tau^2)^4 = 2m \sum_{r \leq k'} (\tau^2)^4.$$ 

Similarly, in the case of $k = 2k' + 1$ (odd), we can derive

$$||P_\xi||^2 = (m - k) + 2 \sum_{r \leq k'} (\nu^2)^2 = (m - 1) - 2 \sum_{r \leq k'} (\tau^2)^2, \quad ||t_\xi||^2 = 1 + 2 \sum_{r \leq k'} (\tau^2)^2.$$ 

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and hence
\[
\|P_ξ\|^2 \|t_ξ\|^2 \geq (m - 1) + 2(m - 2) \sum_{r \leq k'} (τ_ξ^r)^2 - 2(k - 1) \sum_{r \leq k'} (τ_ξ^r)^4
\]
\[
\geq (m - 1) + 2(m - 2) \sum_{r \leq k'} (τ_ξ^r)^2 (ν_ξ^r)^2.
\]

For any \(r\), we have \((τ_ξ^r)^2 (ν_ξ^r)^2 \leq \frac{1}{4}\) by \((τ_ξ^r)^2 + (ν_ξ^r)^2 = 1\). Therefore, by using \(m - 1 \geq k - 1 = 2k'\), we can derive
\[
\|P_ξ\|^2 \|t_ξ\|^2 \geq 2k' + 2(m - 2) \sum_{r \leq k'} (τ_ξ^r ν_ξ^r)^2
\]
\[
\geq 8 \sum_{r \leq k'} (τ_ξ^r ν_ξ^r)^2 + 2(m - 2) \sum_{r \leq k'} (τ_ξ^r ν_ξ^r)^2
\]
\[
= 2(m + 2) \sum_{r \leq k'} (τ_ξ^r ν_ξ^r)^2.
\]

On the other hand, we have
\[
\|F_ξ P_ξ\|^2 = 2 \sum_{r \in \{\frac{1}{2}\}} (τ_ξ^r ν_ξ^r)^2
\]

in both cases where \(k\) is even and odd. Hence we obtain
\[
\|P_ξ\|^2 \cdot \|F_ξ\|^2 \geq m \|F_ξ P_ξ\|^2
\]
in both cases where \(k\) is even and odd.

\[\Box\]

**Lemma 3.5.** Let \(M\) be an \(m\)-dimensional submanifold in \(\overline{M}\). If \(k \leq m\), then, at any point of \(M\), we have
\[
\|\nabla h\|^2 \geq \frac{2(m + 1)}{9(d - 1)^2} \|ω\|^2.
\]

**Proof.** We first compute \(\|ω\|^2\) by using an orthonormal frame of type (II). Define \((0,4)\)-tensor field \(ρ(ξ) (ξ = 0,1,2,3)\) on \(\overline{M}\) by
\[
ρ(0)(X,Y,Z,W) := τc \{\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W)\}
\]
and
\[
ρ(ξ)(X,Y,Z,W) := τc \{\overline{g}(Y,J_ξ Z)\overline{g}(X,J_ξ W) - \overline{g}(X,J_ξ Z)\overline{g}(Y,J_ξ W)
\] - \(2\overline{g}(X,J_ξ Y)\overline{g}(Z,J_ξ W)\}) \quad (ξ = 1,2,3),
\]
for \(X,Y,Z,W \in T \overline{M}\). By using (2.2) and \((ρ(0))_{α,j,j,i} = 0\), we have
\[
\|ω\|^2 = \sum_{α,j,i} \left(\sum_{ξ=1}^{d-1} (ρ(ξ))_{α,j,i}^2\right)^2 \leq (d - 1) \sum_{ξ=1}^{d-1} \sum_{α,j,i} (ρ(ξ))_{α,j,i}^2.
\]

On the other hand, by using (3.4), we have
\[
\sum_{α,j,i} (ρ(ξ))_{α,j,i}^2 = 18c^2 \sum_{r \leq \{\frac{1}{2}\}} (τ_ξ^r ν_ξ^r)^2 = 9c^2 \|F_ξ P_ξ\|^2.
\]

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By using the Codazzi equation, we have

\[(3.11)\quad \|\omega\|^2 \leq 9(d - 1)c^2 \sum_{\xi=1}^{d-1} \|F_\xi P_\xi\|^2,\]

where we use also (3.9). Define a \((0, 3)\)-tensor field \(T\) on \(M\) by

\[T(X, Y, Z) := (\nabla_X h)(Y, Z) + \frac{c}{d-1} \sum_{\xi=1}^{d-1} \left\{ g(P_\xi X, Y) F_\xi Z + g(P_\xi X, Z) F_\xi Y \right\} \quad (X, Y, Z \in TM).\]

Then we have

\[\|T\|^2 = \|\nabla h\|^2 + \frac{4c}{d-1} \sum_{\xi=1}^{d-1} \sum_{i,j} \mathcal{G}(\nabla_{e_i} h)(P_\xi e_i, e_j, F_\xi e_j) \]

\[+ \frac{2c^2}{d-1} \sum_{\xi=1}^{d-1} (\|P_\xi\|^2 \cdot \| F_\xi \|^2 + \|F_\xi P_\xi\|^2).\]

By using the Codazzi equation, we have

\[(3.12)\quad \tau \sum_{\xi=1}^{d-1} \sum_{i,j} \mathcal{G}(\nabla_{e_i} h)(P_\xi e_i, e_j, F_\xi e_j) = \sum_{\xi=1}^{d-1} \sum_{i,j} \mathcal{G}(\nabla_{e_i} h)(e_j, P_\xi e_i, F_\xi e_j)
\]

\[= \tau \sum_{\xi=1}^{d-1} \sum_{i,j} \mathcal{G}(\nabla_{e_j} h)(P_\xi e_j, e_i, F_\xi e_j) + c \sum_{\xi=1}^{d-1} (P_\xi^2, t_\xi F_\xi \]

\[- c \sum_{\xi=1}^{d-1} \|P_\xi\|^2 \cdot \| F_\xi \|^2 - 2c \sum_{\xi=1}^{d-1} \|F_\xi P_\xi\|^2\]

\[\leq -c \sum_{\xi=1}^{d-1} (\|P_\xi\|^2 \cdot \| F_\xi \|^2 + \|F_\xi P_\xi\|^2),\]

where we use the fact that \((\nabla_{e_j} h)(P_\xi e_j, e_i)\) vanishes because \(\nabla_{e_j} h\) is symmetric and \(P_\xi\) is skew-symmetric. From (3.8), (3.11) and (3.12), we obtain

\[\|T\|^2 \leq \|\nabla h\|^2 - \frac{2c^2}{d-1} \sum_{\xi=1}^{d-1} (\|P_\xi\|^2 \cdot \| F_\xi \|^2 + \|F_\xi P_\xi\|^2) \leq \|\nabla h\|^2 - \frac{2(m + 1)c^2}{d-1} \sum_{\xi=1}^{d-1} \|F_\xi P_\xi\|^2\]

and hence

\[(3.13)\quad \|\nabla h\|^2 \geq \frac{2(m + 1)c^2}{d-1} \sum_{\xi=1}^{d-1} \|F_\xi P_\xi\|^2.\]

From (3.10) and (3.13), we obtain the desired inequality.

\[\square\]

**Lemma 3.6.** Let \(M\) be an \(m\)-dimensional submanifold in \(\overline{M}\). If

\[m \geq \begin{cases} 8 & \text{when } \overline{M} = \mathbb{C}P^n(4c), \mathbb{C}H^n(-4c) \\ 11 & \text{when } \overline{M} = \mathbb{H}P^n(4c), \mathbb{H}H^n(-4c) \\ 1 & \text{when } \overline{M} = \varnothing P^2(4c), \varnothing H^2(-4c), \end{cases}\]

then we have

\[\|\nabla h\|^2 \geq \frac{2(10 - d)}{9(m + 2)}\|\nabla H\|^2.\]
Proof. If the codimension is one, then the result follows directly from \( \text{(3.14)} \). In the case of higher codimension, it follows from Lemmas 3.3 and 3.5 that

\[
3\|\nabla h\|^2 = 2\|\nabla h\|^2 + \|\nabla h\|^2 \\
\geq 2 \left( \frac{3}{m+2} - \eta \right) \|\nabla H\|^2 + \left( \frac{2(m+1)}{9(d-1)^2} - \frac{4}{m+2} \left( \frac{2}{(m+2)\eta} - \frac{m}{m-1} \right) \right) \|\omega\|^2.
\]

Now we choose \( \eta = \frac{d-1}{3(m+2)} \) to obtain

\[
3\|\nabla h\|^2 \geq 2 \left( \frac{10-d}{3(m+2)} \right) \|\nabla H\|^2 + \frac{2}{9(d-1)^2} \left( m+1 - \frac{(d-1)(18(7-d)m-108)}{(m-1)(m+2)} \right) \|\omega\|^2.
\]

Then the coefficient of \( \|\omega\|^2 \) in the right-hand side of this inequality is positive when \( m \) is as in the statement of this lemma. Hence we can derive the desired inequality. \( \square \)

For the real number \( b \) as in Introduction and a sufficiently small positive number \( \varepsilon \), define a real number \( b_\varepsilon \) by

\[
b_\varepsilon := \begin{cases} 
(1-\varepsilon)b & \text{(when } M = \mathbb{P}^n(4c) \text{)} \\
(1+\varepsilon)b & \text{(when } M = \mathbb{P}^n(-4c) \text{)}.
\end{cases}
\]

For simplicity, set \( a_\varepsilon := \frac{1}{m-1+\varepsilon} \). We consider the following pinching condition:

\[
(\ast_\varepsilon) \quad \|h\|^2 \leq \frac{1}{m-1+\varepsilon} \|H\|^2 + b_\varepsilon.
\]

Now we shall prove the preservability of the pinching condition in Theorems 1.1 and 1.3.

**Proposition 3.7.** Let \( M \) be a closed hypersurface in \( \overline{M} \) as in Theorem 1.1 or 1.3. Then the pinching condition \((\ast)\) is preserved along the mean curvature flow.

**Proof.** Since \( M \) satisfies the condition \((\ast)\) and it is closed, it satisfies the condition \((\ast_\varepsilon)\) for a sufficiently small positive number \( \varepsilon \). Define \( Q_\varepsilon \) by \( Q_\varepsilon := \|h\|^2 - a_\varepsilon \|H\|^2 - b_\varepsilon \). From Lemma 2.2, we obtain

\[
\frac{\partial}{\partial t} Q_\varepsilon = \Delta Q_\varepsilon - 2\|\nabla h\|^2 - a_\varepsilon \|\nabla H\|^2 + 2\|h\|^2 - a_\varepsilon \|H\|^2(\|h\|^2 + \tau)
\]

\[
\quad - 4 \sum_{i,j,p,l} (h_{ij}h_{pl}^{\rho}R_{pil} - h_{ij}h_{pl}^{\rho}R_{pil})
\]

\[
= \Delta Q_\varepsilon - 2\|\nabla h\|^2 - a_\varepsilon \|\nabla H\|^2 + 2\|h\|^2 - a_\varepsilon \|H\|^2(\|h\|^2 + \tau) + 2b_\varepsilon (\|h\|^2 + \tau)
\]

\[
\quad - 4 \sum_{i,j,p,l} (h_{ij}h_{pl}^{\rho}R_{pil} - h_{ij}h_{pl}^{\rho}R_{pil}),
\]

where \( \tau \) denotes the Einstein constant \( \tau cd(n+1) \). Also, it follows from (3.7) that

\[
\|\nabla h\|^2 - a_\varepsilon \|\nabla H\|^2 \geq \left( \frac{3}{m+2} - \frac{1}{m-1+\varepsilon} \right) \|\nabla H\|^2 \geq 0
\]

because of \( m \geq 3 \). Thus the gradient term in the evolution equation (3.14) is non-positive. Next we shall investigate the reaction term of (3.14). Fix an orthonormal tangent frame \( (e_1, \ldots, e_m) \) of \( M_t \) consisting of eigenvectors of the shape operator \( A_t \) of \( M_t \). Let \( \lambda_i \) be the eigenvalue corresponding to \( e_i \). First we consider the case of Theorem 1.1. From \( c \leq \overline{K}_{ij} \leq 4c \), we can derive

\[
-4 \sum_{i,j,p,l} (h_{ij}h_{pl}^{\rho}R_{pil} - h_{ij}h_{pl}^{\rho}R_{pil}) = -4 \sum_{i<p} R_{ip}(\lambda_i^\alpha - \lambda_p^\alpha)^2
\]

\[
\leq -4mc \left( \|h\|^2 - \frac{1}{m} \|H\|^2 \right) \leq -4mc\|h\|^2.
\]
Since \( \frac{2c}{a_\varepsilon} \geq \pi \) by \( n \geq 3 \geq 1 + \frac{4}{d} \), we obtain
\[
2b_\varepsilon(||h||^2 + \bar{r}) - 4 \sum_{i,j,p,l} (h_{ij} h_{j}^{p} R_{pi}^{l} - h_{ij} h_{j}^{p} R_{pl}^{i}) \leq -\frac{4c}{a_\varepsilon} Q_\varepsilon.
\]
From (3.14), (3.15) and this inequality, we can derive
\[
\frac{\partial}{\partial t} Q_\varepsilon \leq \Delta Q_\varepsilon + 2Q_\varepsilon \left(||h||^2 + \bar{r} - \frac{4c}{a_\varepsilon}\right).
\]
Therefore, by the maximum principle, the condition \((*)_{\varepsilon}\) is preserved along the mean curvature flow.

Next we consider the case of Theorem 1.3. From \(-4c \leq K_{ij} \leq -c\), we can derive
\[
-4 \sum_{i,j,p,l} (h_{ij} h_{j}^{p} R_{pi}^{l} - h_{ij} h_{j}^{p} R_{pl}^{i}) = -4 \sum_{i<p} K_{ip} (\lambda_i^\alpha - \lambda_p^\alpha)^2
\]
\[
\leq 16mc \left(||h||^2 - \frac{1}{m} ||H||^2\right) \leq 16mc ||h||^2.
\]
Since \( \frac{8c}{a_\varepsilon} \geq -\pi \) by \( n \geq 2 > \frac{d+16}{16} \), we obtain
\[
2b_\varepsilon(||h||^2 + \bar{r}) - 4 \sum_{i,j,p,l} (h_{ij} h_{j}^{p} R_{pi}^{l} - h_{ij} h_{j}^{p} R_{pl}^{i}) \leq \frac{16c}{a_\varepsilon} Q_\varepsilon.
\]
From (3.14), (3.15) and this inequality, we can derive
\[
\frac{\partial}{\partial t} Q_\varepsilon \leq \Delta Q_\varepsilon + 2Q_\varepsilon \left(||h||^2 + \bar{r} + \frac{8c}{a_\varepsilon}\right).
\]
Therefore, by the maximum principle, the condition \((*)_{\varepsilon}\) is preserved along the mean curvature flow.

\[\square\]

Now we shall prove the preservability of the pinching condition of Theorem 1.2.

**Proposition 3.8.** Let \( M \) be an \( m \)-dimensional closed submanifold in \( \overline{M} \) as in Theorems 1.2. Then the pinching condition \((*)\) is preserved along the mean curvature flow.

**Proof.** Since \( M \) satisfies the condition \((*)\) and it is closed, it satisfies the condition \((*)_{\varepsilon}\) for a sufficiently small positive number \( \varepsilon \). Define \( Q_\varepsilon \) by \( Q_\varepsilon := ||h||^2 - a_\varepsilon ||H||^2 - b_\varepsilon \). From Lemma 2.1, we can derive
\[
\frac{\partial}{\partial t} Q_\varepsilon = \Delta Q_\varepsilon - 2(||\nabla h||^2 - a_\varepsilon ||\nabla H||^2) + 2R_1 - 2a_\varepsilon R_2 + P_{ae}.
\]
Here \( P_{ae} := P_I + P_{II,a_\varepsilon} + P_{III} \), where
\[
P_I := 4 \sum_{i,j,p,q} R_{ipql} \left( \sum_{\alpha} h_{pq}^{\alpha} h_{ij}^{\alpha} \right) - 4 \sum_{j,l,p} R_{ljp} \left( \sum_{i,\alpha} h_{pi}^{\alpha} h_{ij}^{\alpha} \right),
\]
\[
P_{II,a_\varepsilon} := 2 \sum_{l,\alpha,\beta} R_{l\alpha\beta l} \left( \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta} \right) - 2a_\varepsilon \sum_{l,\alpha,\beta} R_{l\alpha\beta l} H^\alpha H^\beta,
\]
\[
P_{III} := -8 \sum_{j,p,\alpha,\beta} R_{j\alpha\beta p} \left( \sum_{i} h_{ip}^{\alpha} h_{ij}^{\beta} \right).
\]
Thus the gradient terms in the evolution equation (3.19) are non-positive.

Assume that there exists $t_0 \in [0, T_{\text{max}})$ and $p_0 \in M_{t_0}$ with $((Q_x)_{t_0})_{p_0} = 0$, where we take $t_0$ as small as possible. We shall investigate the reaction term of (3.18) at $(p_0, t_0)$. Take any orthonormal normal frame $(e_{m+1}, \cdots, e_{dn})$ of $M_{t_0}$ at $p_0$ and, for arbitrarily fixed $\alpha \in \{m+1, \cdots, dn\}$, take an orthonormal tangent frame $(e_1, \cdots, e_m)$ of $T_{p_0}M_{t_0}$ consisting of eigenvectors of the shape operator $(A_{t_0})_{e_\alpha}$, which is not necessarily that of type (I) or (II). Let $\lambda_i$ be the eigenvalue of $A_{e_\alpha}$ corresponding to $e_i$. In similar to (3.16), we have

\[
4 \sum_{i, j, p, q} R_{ipjq} h^\alpha_{pj} h^\alpha_{ij} - 4 \sum_{j, l, p} R_{ijlp} \left( \sum_i h^\alpha_i h^\alpha_{ip} \right) = -4 \sum_{i, p} R_{ip} \left( \lambda^\alpha_i - \lambda^\alpha_p \right)^2 \leq -4mc||\dot{h}||^2.
\]

Hence we can derive

\[
P_I \leq -4mc||\dot{h}||^2
\]

at $(p_0, t_0)$. Next we shall estimate the terms $P_{II, a_e}$ and $P_{III}$ at $(p_0, t_0)$. We shall use an orthonormal frame of type (II) to estimate these terms at $(p_0, t_0)$. Take an orthonormal frame $(e_1^\xi, \cdots, e_{d-1}^\xi)$ $(\xi = 1, \cdots, d-1)$ of type (II) at $p_0 \in M_{t_0}$. Set $\mathbf{K}_{s, \alpha} := \mathbf{K}(e_s^\xi, e_\alpha^\xi)$. From (2.3) and (3.3), we have

\[
\mathbf{K}_{s, \alpha} = c \left( 1 + 3\tilde{c}(e_s^\xi, J_\xi e_\alpha^\xi) \right)^2 = c \left( 1 + 3\delta_{s, \alpha} - m \right) \leq c(1 + 3\delta_{s, \alpha} - m).
\]

On the other hand, it follows from $((Q_x)_{t_0})_{p_0} = 0$ that $||h||^2 = a_e||H||^2 + b_e$, that is, $(a_e - 1/m)||H||^2 = ||h||^2 - b_e$ holds at $(p_0, t_0)$. Hence, by noticing $a_e \geq 1/m$, we can derive

\[
P_{II, a_e} = 2 \sum_{s, \alpha} \mathbf{K}_{s, \alpha} (||h^\alpha||^2 - a_e||H^\alpha||^2) = 2 \sum_{s, \alpha} \mathbf{K}_{s, \alpha} \left( ||\dot{h}^\alpha||^2 - (a_e - 1/m)||H^\alpha||^2 \right) \leq 2 \sum_{s, \alpha} \mathbf{K}_{s, \alpha} (||\dot{h}^\alpha||^2) \leq 2c \sum_{s, \alpha} (1 + 3\delta_{s, \alpha} - m)||\dot{h}^\alpha||^2 \leq 2(m + 3)c||\dot{h}||^2.
\]

We use the notations $\rho(\xi)$ $(\xi = 0, 1, \cdots, d-1)$ in the proof of Lemma 3.5. The part $P_{III}$ is written as

\[
P_{III} := -8 \sum_{\xi=1}^{d-1} \sum_{j, p, \alpha, \beta} (\rho(\xi))_{jp\alpha} \left( \sum_i h^\alpha_{ip} h^\beta_{ij} \right),
\]

where we note that $(\rho(0))_{jp\alpha} = 0$. We shall estimate

\[
(P_{III})(\xi) := -8 \sum_{j, p, \alpha, \beta} (\rho(\xi))_{jp\alpha} \left( \sum_i h^\alpha_{ip} h^\beta_{ij} \right) \quad (\xi = 1, \cdots, d-1).
\]
By the same calculation as the estimate of the part (III) in the proof of Proposition 3.6 in [PS], we can derive

\[(P_{III}(\xi)) \leq 8kc||\ddot{h}||^2 \quad (\xi = 1, \cdots, d - 1).\]

Hence we obtain

\[(P_{III}) \leq 8(d - 1)kc||\ddot{h}||^2.\]

By (3.20), (3.21) and (3.23), we obtain

\[(P_{a\varepsilon}) \leq -2(m - 4(d - 1)k - 3)c||\ddot{h}||^2.\]

Set \(R := 2R_1 - 2a\varepsilon R_2 + P_{a\varepsilon}.\) First we consider the case of \((H_{t_0})_{p_0} \neq 0.\) We use an orthonormal frame of type (I) at \(p_0 \in M_{t_0}.\) Then it follows from Lemma 3.2 that

\[
R = 2R_1 - 2a\varepsilon R_2 + P_{a\varepsilon}
\leq \left( 6 - \frac{2}{ma\varepsilon - 1} \right) ||\ddot{h}||^2||\ddot{h}_-||^2 - 3||\ddot{h}_-||^4
+ \left( \frac{2ma\varepsilon b\varepsilon}{ma\varepsilon - 1} - 2(m - 4(d - 1)k - 3)c \right) ||\ddot{h}_1||^2
+ \left( \frac{4b\varepsilon}{ma\varepsilon - 1} - 2(m - 4(d - 1)k - 3)c \right) ||\ddot{h}_-||^2 - \frac{2b\varepsilon^2}{ma\varepsilon - 1}.
\]

By the assumption \(m \geq \frac{3d}{2} + 5\) in Theorem 1.2, the coefficients of \(||\ddot{h}||^2||\ddot{h}_-||^2\) is negative. It is easy to show that the coefficient of \(||\ddot{h}_1||^2\) vanishes. Also, it follows from \(((Q_{\varepsilon})_t)_{p_0} = 0\) that \(||\ddot{h}||^2 \geq b\varepsilon\) holds at \((p_0, t_0)\). These facts imply

\[
R \leq -3||\ddot{h}_-||^4 + \left( 6 - \frac{2}{ma\varepsilon - 1} \right) b\varepsilon + \frac{4b\varepsilon}{ma\varepsilon - 1} - 2(m - 4(d - 1)k - 3)c \right) ||\ddot{h}_-||^2 - \frac{2b\varepsilon^2}{ma\varepsilon - 1}
\leq -3||\ddot{h}_-||^4 + 4b\varepsilon||\ddot{h}_-||^2 + 2b\varepsilon(b\varepsilon - (m - 4(d - 1)k - 3)c).
\]

Furthermore, from \(4b\varepsilon||\ddot{h}||^2 \leq 3||\ddot{h}||^4 + \frac{4}{3}b\varepsilon^2\), we can derive

\[
R \leq 2b\varepsilon \left( \frac{5}{3}b\varepsilon - (m - 4(d - 1)k - 3)c \right).
\]

The right-hand side of this inequality is negative by the assumption \(m \geq \frac{3d}{2} + 5\) in Theorem 1.2. Hence we obtain \(R < 0\) at \((p_0, t_0)\). Next we consider the case of \((H_{t_0})_{p_0} = 0.\) Then we have \(||\ddot{h}||^2 = ||\ddot{h}||^2 = b\varepsilon\) and \(R_2 = 0.\) Furthermore, by using Theorem 1 in [LL], we find \(2R_1 \leq 3||\ddot{h}||^4 = 3b\varepsilon^2.\) These together with (3.24) imply

\[
R \leq 3b\varepsilon^2 - 2(m - 4(d - 1)k - 3)b\varepsilon.
\]

The right-hand side of this inequality is negative by the assumption \(m \geq \frac{3d}{2} + 5\) in Theorem 1.2. Hence we obtain \(R < 0\) at \((p_0, t_0)\). Therefore, since \(R < 0\) at \((p_0, t_0)\) in both cases, it is shown that the condition \((\ast_\varepsilon)\) is preserved along the mean curvature flow by using the maximum principle. Hence the statement of this proposition follows from the arbitrariness of \(\varepsilon.\)

Now we shall prove the preservability of the pinching condition of Theorem 1.4.
Proposition 3.9. Let $M$ be an $m$-dimensional closed submanifold in $\overline{M}$ as in Theorems 1.4. Then the pinching condition $(\ast)$ is preserved along the mean curvature flow.

Proof. Since $M$ satisfies the condition $(\ast)$ and it is closed, it satisfies the condition $(\ast_\epsilon)$ for a sufficiently small positive number $\epsilon$. Define $Q_\epsilon$ by $Q_\epsilon := \|h\|^2 - a_\epsilon\|H\|^2 - b_\epsilon$. From Lemma 2.1, we can derive the evolution equation (3.18) for $Q_\epsilon$. By Lemma 3.6 and the assumption for $m$ in Theorem 1.4, we obtain the inequality (3.19). Thus the gradient terms in the evolution equation (3.18) are non-positive.

Assume that there exists $t_0 \in [0, T_{\text{max}})$ and $p_0 \in M_{t_0}$ with $((Q_\epsilon(t_0))_{p_0} = 0$, where we take $t_0$ as small as possible. We shall investigate the reaction term of (3.24) at $(p_0, t_0)$. Take any orthonormal normal frame $(e_{m+1}, \cdots, e_{dn})$ of $M_{t_0}$ at $p_0$ and, for arbitrarily fixed $\alpha \in \{m+1, \cdots, dn\}$, take an orthonormal tangent frame $(e_1, \cdots, e_m)$ of $T_{p_0}M_{t_0}$ consisting of eigenvectors of the shape operator $(A_{t_0})_{e_\alpha}$, which is not necessarily that of type (I) or (II). Let $\lambda_i$ $(1 \leq i \leq m)$ be the eigenvalue corresponding to $\tilde{e}_i$, that is, In similar to (3.17), we have

$$
4 \sum_{i,j,p,q} R_{ipjq} h^\alpha_{pq} h^\alpha_{ij} - 4 \sum_{j,l,p} R_{ijlp} \left( \sum_i h^\alpha_{pi} h^\alpha_{ij} \right)
$$

$$
= -4 \sum_{i,p} R_{ipip} \left( (\lambda^\alpha_i)^2 - \lambda^\alpha_i \lambda^\alpha_p \right)
$$

$$
= -2 \sum_{i<p} K_{ip} (\lambda^\alpha_i - \lambda^\alpha_p)^2 \leq 16mc||\dot{h}^\alpha||^2.
$$

Hence we can derive

$$
P_I \leq 16mc||\dot{h}||^2
$$

at $(p_0, t_0)$.

Next we shall estimate the terms $P_{II,\alpha}$ and $P_{III}$ at $(p_0, t_0)$. We shall use an orthonormal frame of type (II) to estimating these terms at $(p_0, t_0)$. Take an orthonormal frame $(e_1^\xi, \cdots, e_{dn}^\xi)$ $(\xi = 1, \cdots, d-1)$ of type (II) at $p_0 \in M_{t_0}$. Set $\mathcal{K}_{s\alpha} := \mathcal{K}(e_s^\xi, e_\alpha^\xi)$. From (2.3) and (3.3), we have

$$
\mathcal{K}_{s\alpha}^\xi = -c \left( 1 + 3\overline{\mathcal{F}}(e_s^\xi, J_{s\alpha} e_\alpha^\xi) \right)^2 = -c \left( 1 + 3\overline{\mathcal{F}}s_{\alpha} - m \tau_{s+1}^\xi \right)^2 \leq -c.
$$

On the other hand, it follows from $((Q_\epsilon(t_0))_{p_0} = 0$ that $||h\|^2 = a_\epsilon ||H\|^2 + b_\epsilon$, that is, $(a_\epsilon - \frac{1}{m})||H\|^2 = ||\dot{h}||^2 - b_\epsilon$ holds at $(p_0, t_0)$. Therefore, by noticing $a_\epsilon \geq \frac{1}{m}$, we can derive

$$
P_{II,\alpha} = 2 \sum_{s,\alpha} \mathcal{K}_{s\alpha}^\xi (||h^\alpha||^2 - a_\epsilon ||H^\alpha||^2)
$$

$$
= 2 \sum_{s,\alpha} \mathcal{K}_{s\alpha}^\xi (||\dot{h}^\alpha||^2 - (a_\epsilon - \frac{1}{m}) ||H^\alpha||^2)
$$

$$
\leq -2c \sum_{s,\alpha} ||h^\alpha||^2 + 2c \sum_{s,\alpha} (1 + 3\overline{\mathcal{F}}s_{\alpha} - m \tau_{s+1}^\xi) \left( a_\epsilon - \frac{1}{m} \right) ||H^\alpha||^2
$$

$$
= -2mc||\dot{h}||^2 + 2(m + 3)c \left( a_\epsilon - \frac{1}{m} \right) ||H||^2
$$

$$
= -2(m + 3)c b_\epsilon + 6c||\dot{h}||^2.
$$

As in (3.22), we can derive

$$
P_{III} \leq 8(d-1)kc||\dot{h}||^2.
$$
By (3.25), (3.26) and (3.27), we obtain
\[(3.28) \quad P_{a_e} \leq 2(8m + 4(d - 1)k + 3)c||\dot{h}||^2 - 2(m + 3)b_e c.\]

Set \( R := 2R_1 - 2a_eR_2 + P_{a_e} \). Note that \( ||(H(t_0))_{p_0}|| > 0 \) because \( b_e < 0 \). We use an orthonormal frame of type (I) at \( p_0 \in M_0 \). It follows from Lemma 3.2 that
\[
R \leq \left(6 - \frac{2}{ma_e - 1}\right)||\dot{h}||^2||\dot{h}\| - 3||\dot{f}\| + \left(\frac{2ma_e b_e}{ma_e - 1} + 8m + 4(d - 1)k + 3c\right)||\dot{h}\|^2 + \left(\frac{4b_e}{ma_e - 1} + 2(8m + 4(d - 1)k + 3c)\right)||\dot{h}\|^2 - \frac{2\epsilon^2}{ma_e - 1} - 2(m + 3)b_e c.\]

By the assumption \( m \geq \frac{3d}{2} + 5 \) in Theorem 1.4, the coefficient of \( ||\dot{h}||^2||\dot{h}\|^2 \) is negative. It is easy to show that the coefficient of \( ||\dot{h}||^2 \) vanishes and that the coefficient of \( ||\dot{h}\|^2 \) is negative. Hence we have
\[
R \leq -\frac{2\epsilon^2}{ma_e - 1} - 2(m + 3)b_e < 0
\]
by the assumption \( m \geq \frac{3d}{2} + 5 \) in Theorem 1.4. Therefore, since \( R < 0 \) at \((p_0, t_0)\), it is shown that the condition \((*)\) is preserved along the mean curvature flow by using the maximum principle. Hence the statement of this proposition follows from the arbitrariness of \( \varepsilon \).

\( \square \)

4 Evolution of the traceless second fundamental form

Let \( M \) be a closed submanifold in \( \overline{M} \) as in Theorems 1.1–1.4 and \( \{M_t\}_{t \in [0, T_{\text{max}}]} \) be the mean curvature flow starting from \( M \). Following to the discussion in [Ha, HH, PS], we shall analyze the traceless part of the second fundamental form and show that it becomes small in a suitable sense if the extrinsic curvature becomes unbounded. Since the initial manifold \( M \) satisfies the condition \((*)\), it satisfies the condition \((*_{\varepsilon})\) for some \( \varepsilon \in [0, 1) \). Hence it follows from Propositions 3.7–3.9 that this condition is preserved along the mean curvature flow. So, as in [H3, Ha, PS], set
\[
W := \alpha ||H||^2 + \beta \quad \text{and} \quad f_\sigma := \frac{||\dot{h}||^2}{W^{1-\sigma}},
\]
where \( \sigma \) is a suitably small non-negative constant, \( \beta := b \) and
\[
\alpha := \begin{cases} 
\frac{2c}{(m + 1 + \varepsilon)(1 + 2c(1 - \varepsilon))} & \text{(when } \overline{M} = \mathbb{H}^n(4c) \text{ and } k = 1) \\
\frac{9m(m + 2)}{11 - 2d - 8c} & \text{(when } \overline{M} = \mathbb{C}P^n(4c) \text{, } \mathbb{H}P^n(4c) \text{ and } k \geq 2) \\
\frac{(m + 1 + \varepsilon)(1 + 2c)(1 - \varepsilon)}{(11 - 2d)m - 19} & \text{(when } \overline{M} = \mathbb{F}H^n(-4c) \text{ and } k = 1) \\
\frac{9m(m + 2)}{(11 - 2d)m - 19} & \text{(when } \overline{M} = \mathbb{C}H^n(-4c) \text{, } \mathbb{H}H^n(-4c) \text{ and } k \geq 2),}
\end{cases}
\]

By using Lemmas 2.1, 2.2, 3.2, 3.6, Propositions 3.7–3.9, we can derive the following result by the same discussion as the proof of Proposition 4.1 in [PS].

**Proposition 4.1.** For any \( \sigma \in [0, \frac{1}{4}] \), the following inequality
\[
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2\alpha(1 - \sigma)}{W} (\nabla f_\sigma, \nabla ||H||^2) - 2C_1W^{\sigma - 1}||\nabla H||^2 + 2\sigma ||h||^2 f_\sigma + 2C_2\sigma + 2C_3W^{\sigma - 1}
\]
holds for some constants \( C_1 > 0, C_2 \) and \( C_3 \) depending only on \( m \) and \( M \).
Proof. By straightforward calculations, we can derive

\begin{equation}
\frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma = W^{\sigma - 1} \left( \frac{\partial}{\partial t} ||h||^2 - \Delta ||h||^2 \right) - \alpha (1 - \sigma) f_\sigma \left( \frac{\partial}{\partial t} ||H||^2 - \Delta ||H||^2 \right) + \frac{2 \alpha (1 - \sigma)}{W} (\nabla f_\sigma, \nabla ||H||^2) - \alpha^2 (1 - \sigma) \frac{f_\sigma}{W^2} ||\nabla||^2.
\end{equation}

First we consider the case $M = \mathbb{F} P^n(4c)$ or $\mathbb{F} H^n(-4c)$ and $k = 1$ (i.e., Theorems 1.1 and 1.3-case). By using the evolution equations in Lemma 2.2 and neglecting the $||\nabla||^2$ term, we have

\begin{equation}
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2 \alpha (1 - \sigma)}{W} (\nabla f_\sigma, \nabla ||H||^2) - 2W^{\sigma - 1} ||\nabla h||^2
\end{equation}

\begin{align*}
&+ 2W^{\sigma - 1} \left( \frac{1}{m} + f_0 (1 - \sigma) \alpha \right) ||\nabla H||^2 + 2\beta \frac{1 - \sigma}{W} f_\sigma (||h||^2 + \bar{r}) \\
&+ 2\sigma f_\sigma (||h||^2 + \bar{r}) - 4W^{\sigma - 1} (h_{ij} h_j^p R_{lpi} - h^i j h^b R_{pili}).
\end{align*}

Our choice of $\alpha$ and $\beta$ gives $0 \leq f_0 < 1$. Hence, from the inequality (3.7) in Lemma 3.3, we have

\begin{equation}
- ||\nabla h||^2 + \left( \frac{1}{m} + f_0 (1 - \sigma) \alpha \right) ||\nabla H||^2
\end{equation}

\begin{align*}
&\leq - ||\nabla h||^2 + \left( \frac{1}{m} + \alpha \right) ||\nabla H||^2 \leq - C_1 ||\nabla H||^2,
\end{align*}

where $C_1 := \frac{3}{m+2} - \frac{1}{m} - \alpha$. We have $C_1 > 0$ by our choice of $\alpha$ and $m(= dn - 1)$. Denote by $R$ the reaction term in (4.4), that is,

\[ R := 2\beta \frac{(1 - \sigma)}{W} f_\sigma (||h||^2 + \bar{r}) + 2\sigma f_\sigma (||h||^2 + \bar{r}) - 4W^{\sigma - 1} (h_{ij} h_j^p R_{lpi} - h^i j h^b R_{pili}). \]

By using inequalities (3.16) and (3.17), we can derive

\[ R \leq \begin{cases} 2f_\sigma \left( \beta (1 - \sigma) \frac{(||h||^2 + \bar{r})}{W} + \sigma (||h||^2 + \bar{r}) - 2mc \right) & \text{when } M = \mathbb{F} P^n(4c) \\
2f_\sigma \left( \beta (1 - \sigma) \frac{(||h||^2 + \bar{r})}{W} + \sigma (||h||^2 + \bar{r}) + 8mc \right) & \text{when } M = \mathbb{F} H^n(-4c) \end{cases} \]

Easily we have

\[ ||h||^2 + \bar{r} \leq \begin{cases} \frac{1}{m - 1 + \epsilon} ||H||^2 + 2c (1 - \epsilon) + \bar{r} = \frac{\bar{r} + 2c (1 - \epsilon)}{\beta} W & \text{when } \mathbb{F} P^n(4c) \\
\frac{1}{m - 1 + \epsilon} ||H||^2 - 8c (1 + \epsilon) + \bar{r} = \frac{\bar{r} - 8c (1 + \epsilon)}{\beta} W & \text{when } \mathbb{F} H^n(-4c) \end{cases} \]

Hence we can derive

\[ R = \begin{cases} 2f_\sigma (-m + 3d - 1 - 2\epsilon - 2\sigma (1 - \epsilon)) c + 2\sigma f_\sigma ||h||^2 & \text{when } M = \mathbb{F} P^n(4c) \\
2f_\sigma (7m - 3d - 5 - 8\epsilon - 8\sigma (1 + \epsilon)) c + 2\sigma f_\sigma ||h||^2 & \text{when } M = \mathbb{F} H^n(-4c) \end{cases} \]

and hence

\[ R \leq \begin{cases} 2f_\sigma (-m + 3d - 1) c + 2\sigma f_\sigma ||h||^2 & \text{when } M = \mathbb{F} P^n(4c) \\
2f_\sigma (7m - 3d - 5) c + 2\sigma f_\sigma ||h||^2 & \text{when } M = \mathbb{F} H^n(-4c) \end{cases} \]

This together with (4.4) and (4.5) implies the statement of this proposition with $C_3 = 0$ and

\[ C_2 = \begin{cases} (-m + 3d - 1) c & \text{when } M = \mathbb{F} P^n(4c) \\
(7m - 3d - 5) c & \text{when } M = \mathbb{F} H^n(-4c). \end{cases} \]
Next we consider the case of $\mathcal{M} = \mathbb{C}P^n(4c)$ or $\mathbb{H}P^n(4c)$ and $k \geq 2$ (i.e., Theorem 1.2-case). By straightforward calculations, we can derive (4.3). By using Lemma 2.1 and the properties of the curvature tensor $\mathcal{R}$, we can derive
\[
\frac{\partial}{\partial t}||H||^2 = \Delta||H||^2 - 2||\nabla H||^2 + 2R_2 + 2\sum_{s,\alpha}K_{s\alpha}||H^\alpha||^2 
\geq \Delta||H||^2 - 2||\nabla H||^2 + 2R_2 + 2mc||H||^2 
\]
and
\[
\frac{\partial}{\partial t}||h||^2 = \Delta||h||^2 - 2(||\nabla h||^2 - \frac{1}{m}||\nabla H||^2) + 2\left(R_1 - \frac{1}{m}R_2\right) + P_M, 
\]
These together with (3.8) (which holds also for $a_* = \frac{1}{m}$) implies
\[
\frac{\partial}{\partial t}f_\sigma = W^{\sigma-1}\frac{\partial}{\partial t}||h||^2 - \alpha(1 - \sigma)f_\sigma \frac{\partial}{\partial t}||H||^2 
\leq W^{\sigma-1}\left\{\Delta||h||^2 - 2(||\nabla h||^2 - \frac{1}{m}||\nabla H||^2) 
+ 2\left(R_1 - \frac{1}{m}R_2\right) - 2(m - 4(d - 1)k - 3)c||h||^2\right\} 
- \alpha(1 - \sigma)f_\sigma (\Delta||H||^2 - 2||\nabla H||^2 + 2R_2 + 2mc||H||^2).
\]
On the other hand, we have
\[
W^{\sigma-1}\Delta||h||^2 - \alpha(1 - \sigma)f_\sigma\frac{\partial}{\partial t}||H||^2 \leq \Delta f_\sigma + \frac{2\alpha(1 - \sigma)}{W}\langle \nabla f_\sigma, \nabla ||H||^2 \rangle. 
\]
From these inequalities, we can estimate the evolution of $f_\sigma$ as follows:
\[
\frac{\partial}{\partial t}f_\sigma \leq \Delta f_\sigma + \frac{2\alpha(1 - \sigma)}{W}\langle \nabla f_\sigma, \nabla ||H||^2 \rangle - 2W^{\sigma-1}||\nabla h||^2 
+ 2W^{\sigma-1}\left(\frac{1}{m} + \alpha(1 - \sigma)f_0\right)||\nabla H||^2 + 2W^{\sigma-1}\left(R_1 - \frac{1}{m}R_2\right) 
- 2\alpha(1 - \sigma)f_\sigma \frac{\partial}{\partial t}||H||^2 
- 2(m - 4(d - 1)k - 3)cW^{\sigma-1}||h||^2.
\]
Now we shall estimate the gradient terms in the right-hand side of this evolution inequality. By using Lemma 3.6 and $0 \leq f_0 < 1$, we can derive
\[
-||\nabla h||^2 + \left(\frac{1}{m} + f_0(1 - \sigma)\alpha\right)||\nabla H||^2 
\leq \left(-\frac{2(10 - d)}{9(m + 2)} + \frac{1}{m} + \alpha\right)||\nabla H||^2 
\leq -C_1||\nabla H||^2, 
\]
where $C_1 := -\frac{2(10 - d)}{9(m + 2)} + \frac{1}{m} + \alpha = \frac{11(1 - 2d)m - 18}{9m(m + 2)} - \frac{(11 - 2d)m - 19}{9m(m + 2)} = \frac{1}{9m(m + 2)}(> 0)$. Next we shall analyze the reaction term of (4.8). We can write them as
\[
R := 2W^{\sigma-2}R' + 2\alpha f_\sigma \frac{\partial}{\partial t}||H||^2.
\]
Easily we can show

\[ R' = \left( R_1 - \frac{1}{m}R_2 \right) W - \alpha \| \hat{h} \|^2 R_2 - m\alpha (1 - \sigma)c \| \hat{h} \|^2 \leq (m - 4(d - 1)k - 3)c \| \hat{h} \|^2 W. \]

(4.12) \[ \alpha \sigma f_\sigma \frac{W}{2} R_2 \leq \sigma f_\sigma \| \hat{h} \|^2. \]

We take \( \sigma \) as \( 0 < \sigma < \frac{1}{4} \). Then, by using Lemma 3.2, \( \| \hat{h} \|^2 = \| \hat{h}_1 \|^2 + \| \hat{h}_- \|^2 \), \( R_2 = \| \hat{h}_1 \|^2 \| H \|^2 + \frac{1}{m} \| H \|^4 \) and the pinching condition (*) (which holds for all time by Proposition 3.8), we can derive

\[ R' \leq -\alpha \frac{m - 4}{m(m - 1)} \| \hat{h}_- \|^2 \| H \|^4 + \left( \beta m + 1 \right) \| \hat{h}_1 \|^2 \| H \|^2 \]
\[ + \left( \frac{2\beta}{m(m - 1)} - \alpha (m(2 - \sigma) - 4(d - 1)k - 3c) \right) \| \hat{h}_- \|^2 \| H \|^2 \]
\[ + \beta (2\beta - (m - 4(d - 1)k - 3c)) \| \hat{h}_1 \|^2 + \| \hat{h}_- \|^2. \]

Furthermore, by using \( m \geq \frac{3d}{4} + 5 \), \( \beta \leq \frac{m - 4(4d - 1k - 3c)}{m} \leq (1), 0 < \sigma < \frac{1}{4} \) and \( k \geq 2 \), we can derive

\[ R' \leq -(m - 2)\beta^2 \| \hat{h} \|^2 + \left\{ \frac{2\beta}{m^2 + 5} - \frac{3m\alpha c}{4} \right\} \| \hat{h} \|^2 \| H \|^2 \]
\[ \leq -(m - 2)\beta^2 \| \hat{h} \|^2 + \left\{ \frac{2(m - 8d + 5)}{(m + 5)(m - 8d + 5)} - \frac{3m\alpha c}{4} \right\} \| \hat{h} \|^2 \| H \|^2 \]
\[ \leq -(\frac{3m\alpha c}{4} - \frac{2(m - 8d + 5)}{(m + 5)m}) c \| H \|^2 + (m - 2)\beta^2 \| \hat{h} \|^2. \]

Easily we can show \( \frac{3m\alpha c}{4} - \frac{2(m - 8d + 5)}{(m + 5)m} > 0 \). Hence we can derive that

\[ R' \leq C_2 \| \hat{h} \|^2 W \]

holds for some negative constant \( C_2 \) depending only on \( m \) and \( d \). This together with (4.12) implies that

\[ R \leq 2\sigma f_\sigma \| h \|^2 + 2C_2 f_\sigma. \]

(4.13) \[ R \leq 2\sigma f_\sigma \| h \|^2 + 2C_2 f_\sigma. \]

From (4.10), (4.11) and (4.13), we can derive the desired inequality.

Next we consider the case of \( \overline{M} = \mathbb{C}H^n(-4c) \) or \( \mathbb{H}H^n(-4c) \) and \( k \geq 2 \). (i.e., Theorem 1.4-case).

By straightforward calculations, we can derive (4.3). By using Lemma 2.1 and the properties of the curvature tensor \( R \), we can derive

\[ \frac{\partial}{\partial t} \| H \|^2 \geq \Delta \| H \|^2 - 2\| \nabla H \|^2 + 2R_2 - 8mc \| H \|^2 \]

and (4.7). Since (3.28) holds for any \( \varepsilon \in [0, 1] \), as \( \varepsilon = 0 \), we have

\[ P_\frac{1}{m} \leq 2(8m + 4(d - 1)k + 3)\| \hat{h} \|^2 + \frac{2(m + 3)(8m + 4(d - 1)k + 3)c^2}{m}. \]
Furthermore, by using $\sigma > 0$, we can derive
\[
\frac{\partial}{\partial t} f_\sigma \leq W^{\sigma - 1} \left( \Delta \| h \|^2 - 2 \left( \| \nabla h \|^2 - \frac{1}{m} \| \nabla H \|^2 \right) \right) + 2W^{\sigma - 1} \left\{ \left( R_1 - \frac{1}{m} R_2 \right) + (8m + 4(d - 1)k + 3)c \| h \|^2 \right. \\
+ \left( \frac{m + 3(8m + 4(d - 1)k + 3)c^2}{m} \right) \right\} - \alpha(1 - \sigma) \frac{f_\sigma}{W} \left( \Delta \| H \|^2 - 2 \| \nabla H \|^2 + 2R_2 - 8mc\| H \|^2 \right).
\]

(4.16)

Furthermore, by using (4.9), we can estimate the evolution of $f_\sigma$ as follows:
\[
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2\alpha(1 - \sigma)}{W} \langle \nabla f_\sigma, \nabla \| H \|^2 \rangle - 2W^{\sigma - 1} \| \nabla h \|^2 \\
+ 2W^{\sigma - 1} \left( \frac{1}{m} + \alpha(1 - \sigma)f_0 \right) \| \nabla H \|^2 + 2W^{\sigma - 1} \left( R_1 - \frac{1}{m} R_2 \right) \\
- 2\alpha(1 - \sigma) \frac{f_\sigma}{W} R_2 + 8m\alpha(1 - \sigma)c \frac{f_\sigma}{W} \| H \|^2 \\
+ \left( 2(8m + 4(d - 1)k + 3)\| h \|^2 + \frac{(m + 3)(8m + 4(d - 1)k + 3)c^2}{m} \right) cW^{\sigma - 1}.
\]

(4.17)

By using Lemma 3.6 and $0 \leq f_0 < 1$, we can derive the estimate (4.11) of the gradient term in the right-hand side of this evolution inequality. We shall analyze the reaction term of (4.17). We can write them as
\[
R := 2W^{\sigma - 2} R' + 2\alpha \sigma \frac{f_\sigma}{W} R_2,
\]
where
\[
R' = \left( R_1 - \frac{1}{m} R_2 \right) W - \alpha \| h \|^2 R_2 + 4m\alpha(1 - \sigma)c \| h \|^2 \| H \|^2 \\
+ (8m + 4(d - 1)k + 3)\| h \|^2 cW + \frac{(m + 3)(8m + 4(d - 1)k + 3)c^2}{m} W.
\]

We take $\sigma$ as $0 \leq \sigma < \frac{4}{3}$. Then, by using Lemma 3.2, $\| h \|^2 = \| h_1 \|^2 + \| h_2 \|^2$, $R_2 = \| h_1 \|^2 \| H \|^2 + \frac{1}{m} \| H \|^4$ and the pinching condition ($\star$) (which holds for all time by Proposition 3.9), we can derive
\[
R' \leq -\frac{4\alpha}{m(m - 1)} \| h \|_2^2 \| H \|_4^4 \\
+ \left( \frac{\beta(m + 1)}{m(m - 1)} + \alpha(4m(3 - \sigma) + 4(d - 1)k + 3)c \right) \| h_1 \|^2 \| H \|^2 \\
+ \left( \frac{2\beta}{m(m - 1)} + \alpha(4m(3 - \sigma)c + 4(d - 1)kc + 3\beta) \right) \| h_2 \|^2 \| H \|^2 \\
+ \beta(2\beta + (8m + 4(d - 1)k + 3)c)\| h \|^2 \\
+ \frac{(m + 3)(8m + 4(d - 1)k + 3)c^2}{m} W.
\]

Furthermore, by using $\sigma > 0$, $\alpha > 0$ and $\beta < 0$, we can derive
\[
R' \leq \left( \frac{\alpha - \frac{2}{m^2(m - 1)}}{8m + 4(d - 1)k + 3 + 4am} \right) c\| h \|^2 \cdot \| H \|^2 \\
+ \beta(2\beta + (8m + 4(d - 1)k + 3)c)\| h \|^2 + \frac{(m + 3)(8m + 4(d - 1)k + 3)c^2}{m} W.
\]

(4.18)
Since $\alpha - \frac{2}{m^2(m-1)} > 0$ by $m \geq \frac{3d}{2} + 5$, we see that the coefficient of $||\hat{h}||^2 \cdot ||H||^2$ in the right-hand side of (4.18) is positive. Also, we see that the coefficient of $||\hat{h}||^2$ in the right-hand side of (4.18) is negative. Hence we can derive that

$$R' \leq C_2||\hat{h}||^2W + C_3W$$

for some positive constants $C_2$ and $C_3$ depending only on $m$ and $d$. This together with (4.12) (which holds also in this case) implies that

(4.19) \quad $R \leq 2\sigma f_\sigma ||h||^2 + 2C_2f_\sigma + 2C_3W^{\sigma-1}$.

From (4.11), (4.17) and (4.19), we can derive the desired inequality.

By using Lemmas 2.1, 2.2, 3.3 and 3.6, we can derive the following evolution inequalities by the same calculation as the proof of Lemma 4.2 in [PS].

Lemma 4.2. In the case of $\overline{M} = \mathbb{F}P^n(4c)$ (i.e., Theorems 1.1 and 1.2-cases), we have

(i) $\frac{\partial}{\partial t}||\hat{h}||^2 \leq \Delta||\hat{h}||^2 - 2C_4||\nabla h||^2 + 4||h||^2||\hat{h}||^2$ for some $C_4 > 0$ only depending on $m$,

(ii) $\frac{\partial}{\partial t}||H||^4 \geq \Delta||H||^4 - 12||H||^2||\nabla H||^2 + \frac{4}{m}||H||^6$.

Lemma 4.3. In the case of $\overline{M} = \mathbb{F}H^n(-4c)$ (i.e., Theorems 1.3 and 1.4-cases), we have

(i) $\frac{\partial}{\partial t}||\hat{h}||^2 \leq \Delta||\hat{h}||^2 - 2C_4||\nabla h||^2 + 4||h||^2||\hat{h}||^2 + 2(7m + 4(d - 1)k)c||\hat{h}||^2$

for some $C_4 > 0$ only depending on $m$,

(ii) $\frac{\partial}{\partial t}||H||^4 \geq \Delta||H||^4 - 12||H||^2||\nabla H||^2 + \frac{4}{m}||H||^6 - 16mc||H||^4$.

Proof of Lemmas 4.2 and 4.3. First we consider the case of $k = 1$ (i.e., Theorems 1.1 and 1.3-cases). From Lemma 2.2, (3.7) (in Lemma 3.3), (3.16) and (3.17), we obtain

$$\frac{\partial}{\partial t}||\hat{h}||^2 - \Delta||\hat{h}||^2$$

$$= -2(||\nabla h||^2 - \frac{1}{m}||\nabla H||^2) + 2||\hat{h}||^2(||\hat{h}||^2 + \bar{r}) - 4 \sum_{i,j,p,l} (h_{ij}h_{pl}T_{iplj} - h_{ij}h_{pl}T_{plij})$$

$$\leq \begin{cases} -\frac{4(m-1)}{3m}||\nabla h||^2 + 2||\hat{h}||^2(||\hat{h}||^2 + \bar{r}) - 4mc||\hat{h}||^2 \quad \text{(when } \overline{M} = \mathbb{F}P^n(4c)) \\ -\frac{4(m-1)}{3m}||\nabla h||^2 + 2||\hat{h}||^2(||\hat{h}||^2 + \bar{r}) + 16mc||\hat{h}||^2 \quad \text{(when } \overline{M} = \mathbb{F}H^n(-4c)) \end{cases}$$

and hence

$$\frac{\partial}{\partial t}||\hat{h}||^2 - \Delta||\hat{h}||^2$$

$$\leq \begin{cases} -\frac{4(m-1)}{3m}||\nabla h||^2 + 2||\hat{h}||^2||\hat{h}||^2 - (4m - 6d + 6)c||\hat{h}||^2 \quad \text{(when } \overline{M} = \mathbb{F}P^n(4c)) \\ -\frac{4(m-1)}{3m}||\nabla h||^2 + 2||\hat{h}||^2||\hat{h}||^2 + 2(7m - 3d + 3)c||\hat{h}||^2 \quad \text{(when } \overline{M} = \mathbb{F}H^n(-4c)) \end{cases}$$

Therefore we obtain the evolution inequalities in (i) of Lemmas 4.2 and 4.3. Also, from Lemma 2.2, we obtain

$$\frac{\partial}{\partial t}||H||^4 = 2||H||^2(\Delta||H||^2 - 2||\nabla H||^2 + 2||H||^2(||H||^2 + \bar{r}))$$

$$= \Delta||H||^4 - 2||\nabla ||H||^2||^2 - 4||H||^2||\nabla H||^2 + 4||H||^4(||H||^2 + \bar{r})$$

$$\geq \Delta||H||^4 - 12||H||^2||\nabla H||^2 + 4||H||^4(||H||^2 + \bar{r})$$

$$\geq \Delta||H||^4 - 12||H||^2||\nabla H||^2 + \frac{4}{m}||H||^6 + 4\bar{r}||H||^4.$$
Since \( \tau = c(dn + 3d - 4) - c(m + 3d - 3) \), we have
\[
4\tau > \begin{cases} 
0 & \text{(when } M = F^P_n(4c) \text{)} \\
-16mc & \text{(when } M = F^H_n(-4c) \text{)}. 
\end{cases}
\]
Therefore we obtain the evolution inequalities in (ii) of Lemmas 4.2 and 4.3.

Next we consider the case of \( k \geq 2 \) (i.e., Theorems 1.2 and 1.4-cases). From Lemma 2.1, we have
\[
\frac{\partial}{\partial t} ||\dot{h}||^2 \leq \Delta ||\dot{h}||^2 - 2 \left( ||\nabla h||^2 - \frac{1}{m} ||\nabla H||^2 \right) + 2 \left( R_1 - \frac{1}{m} R_2 \right) + P_m.
\]
From Lemma 3.6, we have
\[
||\nabla h||^2 - \frac{1}{m} ||\nabla H||^2 \geq \frac{(11 - 2d)m - 18}{9m(m + 2)} ||\nabla h||^2.
\]
Furthermore, from Lemma 3.2, we obtain
\[
R_1 - \frac{1}{m} R_2 \leq ||\dot{h}_1||^4 + 4||\dot{h}_1||^2 ||\dot{h}_2||^2 + \frac{3}{2} ||\dot{h}_2||^4 + \frac{1}{m} ||\dot{h}_1||^2 ||H||^2 \\
\leq 2 \left( ||\dot{h}_1||^2 + ||\dot{h}_2||^2 \right)^2 + \frac{2}{m} ||H||^2 \left( ||\dot{h}_1||^2 + ||\dot{h}_2||^2 \right) = 2 ||\dot{h}||^2 ||h||^2.
\]
By simple calculations, we have
\[
P_{Hi} \frac{1}{m} \leq \begin{cases} 
2(m + 3)c||\dot{h}||^2 & \text{(when } M = F^P_n(4c) \text{)} \\
-2mc||\dot{h}||^2 & \text{(when } M = F^H_n(-4c) \text{)}. 
\end{cases}
\]
This together with (3.20), (3.23), (3.25), and (3.27) implies that
\[
P_{Hi} \frac{1}{m} \leq \begin{cases} 
-2(m - 4(d - 1)k - 3)c||\dot{h}||^2 < 0 & \text{(when } M = F^P_n(4c) \text{)} \\
2(7m + 4(d - 1)k)c||\dot{h}||^2 & \text{(when } M = F^H_n(-4c) \text{)}. 
\end{cases}
\]
Therefore we obtain the evolution inequalities (with \( C_4 = \frac{(11 - 2d)m - 18}{9m(m + 2)} \)) in (i) of Lemmas 4.2 and 4.3. Next we shall derive the evolution inequality in (ii) of Lemmas 4.2 and 4.3. From Lemma 2.1, we have
\[
\frac{\partial}{\partial t} ||H||^4 = \Delta ||H||^4 - 2||\nabla ||H||^2||^2 - 4||H||^2||\nabla ||H||^2 \\
+ 2||H||^2 \left( 2R_2 + 2 \sum_{s,\alpha} K_{s\alpha} ||H^\alpha||^2 \right).
\]
Also, we have
\[
2R_2 = 2||H||^2 \left( ||\dot{h}_1||^2 + \frac{1}{m} ||H||^2 \right) \geq \frac{2}{m} ||H||^4, \quad ||\nabla ||H||^2||^2 = 4||H||^2||\nabla H||^2
\]
and
\[
\sum_{s,\alpha} K_{s\alpha} ||H^\alpha||^2 \geq \begin{cases} 
0 & \text{(when } M = F^P_n(4c) \text{)} \\
-4mc||H||^2 & \text{(when } M = F^H_n(-4c) \text{)}. 
\end{cases}
\]
From these relations, we obtain the evolution inequalities in (ii) of Lemmas 4.2 and 4.3. \(\square\)

Finally, we give the evolution inequality for \( ||\nabla H||^2 \). By the same discussion as the proof of Corollary 5.10 in [Ba], we can derive the following result.

**Lemma 4.4.** There exists a constant \( C_5 \) depending only on \( M \) such that
\[
\frac{\partial}{\partial t} ||\nabla H||^2 \leq \Delta ||\nabla H||^2 + C_5(||H||^2 + 1)||\nabla h||^2.
\]
5 Finiteness of maximal time

In this section, we shall show the finiteness of the maximal time of any mean curvature flow starting from a closed submanifold in $\mathbb{F}H^n(-4c)$. Denote by $S_p(a)$ the geodesic sphere of radius $a$ centered at $p$ in $\mathbb{F}H^n(-4c)$, and by $h^{p,a}$ and $H^{p,a}$ the second fundamental form and the mean curvature vector of $S_p(a)$. Let $M$ be an $m$-dimensional closed submanifold in $\mathbb{F}H^n(-4c)$ and $\{M_t = F_t(M)\}_{t \in [0,T_{\text{max}})}$ be the mean curvature flow starting from $M$.

Lemma 5.1. $T_{\text{max}} < \infty$ holds.

Proof. Take a geodesic sphere $S_{p_0}(a)$ surrounding $f(M)$. Denote by $\varphi : CH^n(-4) \to \mathbb{R}$ the (Riemannian) distance function from $p_0$ and set $r_t := \varphi \circ F_t$. Then we can show

\[ \nabla \nabla \varphi \Big|_{F_t(p)} = \|H^{p_0,r_t(p)}\| \quad (5.1) \]

and

\[ (\Delta r_t)_p = (\nabla \nabla \varphi)_{F_t(p)} + \|h^{p_0,r_t(p)}((\nu_t)_p)_{TS},((\nu_t)_p)_{TS}\| + dT((H_t)_p) \quad (5.2) \]

for any $p \in M$, where $\Delta$ denotes the Laplace operator of the Levi-Civita connection $\nabla$ of $\mathbb{F}H^n(-4c)$ and $((\nu_t)_p)_{TS}$ denotes the $T_{F_t(p)}S_{p_0}(r_t(p))$-component of the unit normal vector $(\nu_t)_p$ of $M_t$ at $F_t(p)$. Set

\[ D_{1,p} := \text{Span} \left\{ J_\xi \left( \frac{\partial}{\partial \varphi} \right)_{F_t(p)} \mid \xi = 1, \ldots, d-1 \right\} ^\perp \]

and

\[ D_{2,p} := \text{Span} \left\{ J_\xi \left( \frac{\partial}{\partial \varphi} \right)_{F_t(p)} \mid \xi = 1, \ldots, d-1 \right\} , \]

where $(\bullet)^\perp$ denotes the orthogonal complement of $(\bullet)$ in $T_pS_{p_0}(r_t(p))$. Then we can show

\[ \|h^{p_0,a}(X,X)\| = \left\{ \begin{array}{ll} \|X\|^2 \tan a & (X \in D_{1,p}) \\ \frac{2\|X\|^2}{\tan 2a} & (X \in D_{2,p}) \end{array} \right. \quad (5.3) \]

and hence

\[ \|H^{p_0,a}\| = \frac{(n-1)d}{\tan a} + \frac{2(d-1)}{\tan 2a} \quad (5.4) \]

From (5.1) – (5.4), we obtain

\[ (\Delta r_t)_p = \frac{2(d-1)}{\tan 2r_t(p)} + \frac{(n-1)d}{\tan r_t(p)} + dT((H_t)_p) + \frac{2}{\tan 2r_t(p)}||(\nu_t)_{TS}(2)||^2 + \frac{1}{\tan r_t(p)}||(\nu_t)_{TS}(1)||^2, \quad (5.5) \]

where $((\nu_t)_{TS}(i))$ $(i = 1, 2)$ denotes the $D_{i,p}$-component of $((\nu_t)_p)_{TS}$. On the other hand, we have

\[ \frac{\partial r_t}{\partial t} = dT((H_t)_p). \]

Hence we obtain

\[ \left( \frac{\partial}{\partial t} \right)_p (\Delta r_t)_p = (\Delta r_t)_p - \frac{2(d-1)}{\tan 2r_t(p)} - \frac{(n-1)d}{\tan r_t(p)} + \frac{2}{\tan 2r_t(p)}||(\nu_t)_{TS}(2)||^2 - \frac{1}{\tan r_t(p)}||(\nu_t)_{TS}(1)||^2 \]

\[ \leq (\Delta r_t)_p - ((n+1)d - 2). \quad (5.6) \]

Therefore, by the maximum principle, we can derive $r_t \leq \max r_0 - ((n+1)d - 2)t$ for all time $t$. This implies that $T_{\text{max}} \leq \frac{\max r_0}{(n+1)d - 2} (< \infty)$.

\[ \Box \]
6 Proof of Theorems 1.3 and 1.4

In this section, we shall prove Theorems 1.3 and 1.4. Also, we prove that if \( T_{\text{max}} < \infty \), then the mean curvature flows in Theorems 1.1 and 1.2 collapse to a round point as \( t \to T_{\text{max}} \). Throughout this section, let \( M \) be as in Theorems 1.1 – 1.4 and we assume that \( T_{\text{max}} < \infty \) in the case where \( M \) is as in Theorems 1.1 and 1.2. Since \( M \) satisfies the condition \((*)\) and it is compact, it satisfies the condition \((*_{\varepsilon})\) for a sufficiently small positive number \( \varepsilon \). By Propositions 3.7 – 3.9, the condition \((*_{\varepsilon})\) is preserved along the mean curvature flow. As in the previous section, set \( W = \alpha \|H\|^2 + \beta \), where \( \beta = b \) and \( \alpha \) and \( \beta \) is as in \((\ref{1.1})\).

**Theorem 6.1.** Let \( M \) be as above. Then there exist positive constants \( C_0 \) and \( \sigma_0 \) depending only on the initial submanifold \( M \) such that, for all \( t \in [0, T_{\text{max}}) \), the following inequality holds:

\[
\int_{M_t} \int_M \int_{M_t} f_\sigma W d\mu \leq C_0(\|H\|^2 + 1)^{1-\sigma_0}.
\]

Since there exists the positive term \( 2\sigma\|h\|^2 f_\sigma \) among the reaction term of the evolution inequality (4.2) in Proposition 4.1, we cannot use the maximum principle to show the uniform boundedness of \( \{(f_\sigma)_{t} \}_{t \in [0, T_{\text{max}}]} \). So, as in Huisken [H3], Baker [B] and Pipoli-Sinestrari [PS], we shall estimate the \( L^p \)-norm of \( f_\sigma \) from above by exploiting the good negative term of \( \|\nabla H\|^2 \). By using this \( L^p \)-estimate, the Sobolev’s inequality for submanifolds and the Stampacchia’s iteration lemma, we shall derive the uniform boundedness of \( \{(f_\sigma)_{t} \}_{t \in [0, T_{\text{max}}]} \).

For a function \( \overline{p} \) over \( M \times [0, T) \), we denote \( \int_M \overline{p}(\cdot, t) d\mu_t \) by \( \int_M \overline{p} d\mu \) for the simplicity. By the same discussion as the proof of Proposition 5.4 in [PS], we shall derive the following Poincaré-type inequality for \( f_\sigma \).

**Proposition 6.2.** There exists a positive constant \( C_6 \) depending only on \( m, k \) and the initial submanifold \( M \) such that, for any \( p \geq 2, 0 < \sigma < \frac{1}{2} \) and \( \eta > 0 \), we have

\[
\frac{\varepsilon \rho}{2} \int_{M_t} f_\sigma^p W d\mu \leq (\eta(p + 1) + 5) \int_{M_t} W^{\sigma - 1} f_\sigma^{p - 1} \|\nabla H\|^2 d\mu + \frac{p + 1}{\eta} \int_{M_t} f_\sigma^{p - 2} \|\nabla f_\sigma\|^2 d\mu + 2^{p + 1} m b \int_{M_t} f_\sigma^p d\mu + \frac{1}{p} C_6^p.
\]

First we show the following fact in the same method as the proof of Proposition 5.2 in [PS].

**Lemma 6.3.** In the case of \( k = 1 \) (i.e., Theorems 1.1 and 1.3-cases), there exists a positive constant \( C_7 \) depending only on \( m \) such that the intrinsic sectional curvature \( K: G_2(M_t) \to \mathbb{R} \) of \( M_t \) satisfies

\[
K > \varepsilon C_7 W,
\]

where \( G_2(M_t) \) denotes the Grassmann bundle of \( M_t \) consisting of the 2-planes.

**Proof.** Let \( (e_1, \ldots, e_m) \) be an orthonormal tangent frame consisting of eigenvectors of the shape operator \( A_t \) of \( M_t \). Let \( A_t e_i = \lambda_i e_i \ (i = 1, \ldots, n) \). For any \( i \neq j \), the Gauss equation gives

\[
K_{ij} = K_{ij} + \lambda_i \lambda_j.
\]

Like in [H3], we can use the following algebraic property: for any \( i \neq j \)

\[
\|h\|^2 - \frac{1}{m - 1} \|H\|^2 = -2\lambda_i \lambda_j + \left( \lambda_i + \lambda_j - \frac{\|H\|^2}{m - 1} \right) + \sum_{i \neq j} \left( \lambda_i - \frac{\|H\|^2}{m - 1} \right) 
\]

\[
\geq -2\lambda_i \lambda_j.
\]

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In the case of $\mathcal{M} = \mathbb{F}P^n(4c)$, we have

$$2K_{ij} \geq 2c - ||h||^2 + \frac{1}{m-1}||H||^2$$

$$\geq \left( \frac{1}{m-1} - \alpha \varepsilon \right) ||H||^2 + 2c - b \varepsilon$$

$$= \frac{\varepsilon}{(m-1)(m-1+\varepsilon)} ||H||^2 + 2c\varepsilon.$$ 

In the case of $\mathcal{M} = \mathbb{F}H^n(-4c)$, we have

$$2K_{ij} \geq -8c - ||h||^2 + \frac{1}{m-1}||H||^2$$

$$\geq \left( \frac{1}{m-1} - \alpha \varepsilon \right) ||H||^2 - 8c - b \varepsilon$$

$$= \frac{\varepsilon}{(m-1)(m-1+\varepsilon)} ||H||^2 + 8c\varepsilon.$$ 

Thus, in both cases, we see that

$$K_{ij} > \varepsilon C^7 W$$

for a suitable positive constant $C^7$ depending only on $m$. 

By using (23) in [AB], we obtain

$$\Delta ||\hat{h}||^2 \geq 2||\nabla \hat{h}||^2 + 2\langle \hat{h}, \nabla \nabla H \rangle + 2Z - C||h||^2,$$

where $C$ is a suitable positive constant depending only on $m, k$ and $Z$ is given by

$$Z = \sum_{i,j,p,\alpha,\beta} H^\alpha h_{ip}^\alpha h_{pj}^\beta h_{ij}^\beta - \sum_{\alpha,\beta} \left( \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 - \sum_{i,j,\alpha,\beta} \left( \sum_{p} (h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{ip}^\beta) \right)^2.$$ 

Since the condition $(\varepsilon \varepsilon)$ is preserved along the mean curvature flow, we can derive

$$(6.1) \quad \Delta ||\hat{h}||^2 \geq 2||\nabla \hat{h}||^2 + 2\langle \hat{h}, \nabla \nabla H \rangle + 2Z - \gamma W,$$

where $\gamma$ is a suitable positive constant depending only on $m$ and $k$.

By using Lemma 6.3 and noticing $1 \leq \tau K \leq 4$, we can derive the following fact in the same method as the proof of Lemma 5.3 in [PS].

**Lemma 6.4.** (i) In the case of $\mathcal{M} = \mathbb{F}P^n(4c)$, there exists a positive constant $\rho$ depending only on $m$ and $k$ satisfying

$$Z + 2mb||\hat{h}||^2 \geq \rho \varepsilon ||\hat{h}||^2 W.$$ 

(ii) In the case of $\mathbb{F}H^n(-4c)$, there exists a constant $\rho$ depending only on $m$ and $k$ satisfying

$$Z - \frac{mb}{2}||\hat{h}||^2 \geq \rho \varepsilon ||\hat{h}||^2 W.$$
Proof. First we consider the case of $k = 1$. Take an orthonormal frame such that diagonalizes the shape operator. By using the Gauss equation, Lemma 6.3 and $1 \leq \tau K \leq 4$, we have

$$Z = \left( \sum_i \lambda_i \right) \left( \sum_i \lambda_i^3 \right) - \left( \sum_i \lambda_i^2 \right)^2 \leq \sum_i \lambda_i \lambda_j (\lambda_i - \lambda_j)^2$$

$$= \sum_{i<j} K_{ij} (\lambda_i - \lambda_j)^2 - \sum_{i<j} K_{ij} (\lambda_i - \lambda_j)^2 \geq \text{\varepsilon}_m W \| \hat{h} \|^2 - \frac{\text{\varepsilon}_m W}{8} \| \hat{h} \|^2 \quad \text{(when } \mathbb{F} P^m (4c))$$

$$\geq \left\{ \begin{array}{ll}
\varepsilon c m W \| \hat{h} \|^2 - 2m |\hat{b}| |\hat{h}|^2 \\
\varepsilon c m W \| \hat{h} \|^2 - \frac{mb}{\text{\varepsilon}} |\hat{b}| |\hat{h}|^2 \\
\end{array} \right. \quad \text{(when } \mathbb{F} H^n (-4c))$$

Thus the statements (i) and (ii) of this lemma follows.

Next we consider the case of $k \geq 2$. Take any $(p, t) \in M \times [0, T_{\text{max}})$. We need to distinguish into the cases where $H \neq 0$ and $H = 0$ at $(p, t)$. First we consider the case where $H \neq 0$ at $(p, t)$. In this case, by using the estimate in page 384 in [AB], we have

$$Z \geq -\frac{m}{2} |\hat{h}_1|^4 - \frac{3}{2} |\hat{h}_-|^4 - \frac{m + 2}{2} |\hat{h}_1|^2 |\hat{h}_-|^2 + \frac{1}{2(m - 1)} \left( |\hat{h}_1|^2 + |\hat{h}_-|^2 \right) |H|^2.$$

Since $(*)$ is preserved along the mean curvature flow, we have

$$|H|^2 > \frac{m(m - 1 + \varepsilon)}{1 - \varepsilon} \left( |\hat{h}_1|^2 + |\hat{h}_-|^2 - b \varepsilon \right)$$

$$\Rightarrow m(m - 1 + \varepsilon) \left( |\hat{h}_1|^2 + |\hat{h}_-|^2 - b \right).$$

Therefore we obtain

$$Z \geq \frac{m}{2} |\hat{h}_1|^4 - \frac{3}{2} |\hat{h}_-|^4 - \frac{m + 2}{2} |\hat{h}_1|^2 |\hat{h}_-|^2$$

$$+ \frac{m}{2(1 - \varepsilon)} \left( |\hat{h}_1|^2 + |\hat{h}_-|^2 \right) \left( |\hat{h}_1|^2 + |\hat{h}_-|^2 - b \right)$$

$$\geq \frac{\varepsilon m}{2(1 - \varepsilon)} |\hat{h}_1|^4 + \frac{m - 3 + 3\varepsilon}{2(1 - \varepsilon)} |\hat{h}_-|^4$$

$$+ \frac{m - 2 + \varepsilon (m + 2)}{2(1 - \varepsilon)} |\hat{b}| |\hat{h}|^2 - \frac{m}{2(1 - \varepsilon)} b |\hat{h}|^2.$$

From this estimate, it follows that there exists a positive constant $\mu_1$ depending only on $m$ satisfying

$$Z + \frac{mb \varepsilon}{2(1 - \varepsilon)} |\hat{h}|^2 \geq Z + \frac{m}{2(1 - \varepsilon)} b |\hat{h}|^2 \geq \varepsilon \mu_1 |\hat{h}|^4.$$

On the other hand, by using the definition of $Z$ and estimating various terms by Peter-Paul’s inequality, we can derive

$$Z \geq \mu_2 |\hat{h}|^2 |H|^2 - \mu_3 |\hat{h}|^4$$

for some positive constants $\mu_2$ and $\mu_3$ depending on $m$. Hence we obtain

$$Z + \frac{mb \varepsilon}{2(1 - \varepsilon)} |\hat{h}|^2 \geq \hat{C} \left( \mu_2 |\hat{h}|^2 |H|^2 - \mu_3 |\hat{h}|^4 + \frac{mb \varepsilon}{2(1 - \varepsilon)} |\hat{h}|^2 \right) + (1 - \hat{C}) \varepsilon \mu_1 |\hat{h}|^4.$$
for any \( \tilde{C} \in [0,1] \). Choose \( \frac{\varepsilon\mu_1}{\varepsilon\mu_1 + \mu_3} \) as \( \tilde{C} \). Then, we have

\[
Z + \frac{mb\bar{c}}{2}||\dot{h}||^2 \geq \frac{\varepsilon\mu_1}{\varepsilon\mu_1 + \mu_3} \left( \mu_2||H||^2 + \frac{mb\bar{c}}{2} \right) ||\dot{h}||^2.
\]

From this inequality, we can derive the statements (i) and (ii) of this lemma. Next we consider the case where \( H = 0 \) at \((p,t)\). Then, since \((*)_\varepsilon\) holds in all time, this case cannot happen in the case of \( M = \mathbb{H}^n(-4c) \). Hence we may assume that \( M = \mathbb{H}^n(4c) \). Then we have \( ||h||^2 = ||\dot{h}||^2 \leq b \) and \( W = \beta = b \) because \((*)_\varepsilon\) holds in all time. Hence by using Theorem 1.1 in [LL], we can derive

\[
Z \geq -\frac{3}{2}||h||^4 \geq -\frac{3}{2}b||\dot{h}||^2.
\]

Therefore we obtain

\[
Z + 2mb||\dot{h}||^2 \geq \left( 2m - \frac{3}{2} \right) ||\dot{h}||^2W.
\]

Thus we can derive the statements (i) and (ii) of this lemma. \( \square \)

**Proof of Proposition 6.2.** By using (6.1), we have

\[
\Delta f_\sigma \geq 2W^{\sigma - 1}||\nabla \dot{h}||^2 + 2W^{\sigma - 1}\dot{h}, \nabla \nabla H) + 2W^{\sigma - 1}Z - \gamma W^{\sigma} - \alpha(1 - \sigma)\frac{f_\sigma}{W} \Delta ||H||^2
\]

\[
- \frac{2\alpha(1 - \sigma)}{W} \langle \nabla f_\sigma, \nabla ||H||^2 \rangle + \alpha^2(1 - \sigma)\frac{f_\sigma}{W^2} ||\nabla ||H||^2||^2.
\]

Since the term \( 2W^{\sigma - 1}||\nabla \dot{h}||^2 \) and \( \alpha^2(1 - \sigma)\frac{f_\sigma}{W^2} ||\nabla ||H||^2||^2 \) are positive, we can omit them. By using Lemma 6.4, we have

\[
\Delta f_\sigma \geq 2W^{\sigma - 1}\langle \dot{h}, \nabla \nabla H \rangle - \alpha(1 - \sigma)\frac{f_\sigma}{W} \Delta ||H||^2 - \frac{2\alpha(1 - \sigma)}{W} \langle \nabla f_\sigma, \nabla ||H||^2 \rangle
\]

\[
- \tau \mathcal{F}^{\sigma + 1}mbf_\sigma + 2\rho \varepsilon f_\sigma W - \gamma W^{\sigma}.
\]

By multiplying \( f_\sigma^{p - 1} \) to this inequality and integrating on \( M_t \) with respect to \( d\mu_t \), we can derive

\[
2\rho \varepsilon \int_{M_t} f_\sigma^p W d\mu_t \leq (\eta(p + 1) + 5) \int_{M_t} W^{\sigma - 1}f_\sigma^{p - 1}||\nabla H||^2 d\mu_t + \frac{p + 1}{\eta} \int_{M_t} f_\sigma^{p - 2}||\nabla f_\sigma||^2 d\mu_t
\]

\[
+ \tau \mathcal{F}^{\sigma + 1}mb \int_{M_t} f_\sigma^p d\mu_t + \gamma \int_{M_t} f_\sigma^{p - 1}W^{\sigma} d\mu_t.
\]

By the Young’s inequality, we obtain the following estimate with respect to the last term of the right-hand side of this inequality:

\[
\gamma f_\sigma^{p - 1}W^{\sigma} \leq \gamma W \left( \frac{r^p}{p} W^{(\sigma - 1)p} + \frac{p - 1}{p} r^{- \frac{\rho}{\varepsilon p}} f_\sigma^{p} \right),
\]

where \( r := \left( \frac{(p - 1)\gamma}{\varepsilon \rho p} \right)^{\frac{1}{p - 1}} \). Note that \( r \leq \frac{\gamma}{\varepsilon \rho} \). From \( 0 < \sigma < \frac{1}{4} \) and \( p \geq 2 \), we have \((\sigma - 1)p + 1 < 0\) and hence \( W^{(\sigma - 1)p + 1} \leq \beta^{(\sigma - 1)p + 1} \). Therefore we have

\[
\frac{1}{p} \gamma r^p \int_{M_t} W^{(\sigma - 1)p + 1} d\mu_t \leq \frac{1}{p} \gamma r^p \beta^{(\sigma - 1)p + 1} \text{vol}(M_t)
\]

\[
\leq \gamma r^p \beta^{(\sigma - 1)p + 1} \text{vol}(M_0).
\]
Set $C_6 := \left(\gamma r^p \beta^{(\alpha-1)p+1} \text{vol}(M_0)\right)^{1/p}$. Then we obtain the desired inequality.

Set

$$C_8 := \begin{cases} 1 & (dn)^2 - 3dn + 2 \quad \text{(when } \mathcal{M} = \mathbb{R}P^n(4c)) \\ \frac{(dn)^2 - 4dn + 4 - \varepsilon}{m(m-1)} & (\text{when } \mathcal{M} = \mathbb{R}H^n(-4c), \; k = 1) \\ \frac{m^2 - 2m + 1 - \varepsilon}{m^2 - 2m + 1} & (\text{when } \mathcal{M} = \mathbb{R}H^n(-4c), \; k \geq 2). \end{cases}$$

From Propositions 4.1 and 6.2, we can derive the following result for the estimate of the $L^p$-norm of $f_\sigma$ by the same discussion as the proof of Proposition 5.5 in [PS].

**Proposition 6.5.** There exists a constant $C_9(= C_9(p))$ depending only on $\sigma, p, m, k, \epsilon, \rho, \beta, \text{Vol}(M_0)$ and $T_{\text{max}}$ such that, if $p \geq \frac{8C_8}{C_1} + 1$ and $\sigma < \frac{\varepsilon \sqrt{C_1 p}}{2^m \sqrt{p}}$, then the inequality

$$\left(\int_{M_t} f_\sigma^p \, d\mu\right)^{1/p} \leq C_9 \text{ holds for all } t \in [0, T_{\text{max}}].$$

**Proof.** By multiplying $pf_\sigma^{p-1}$ to the inequality (4.2) in Proposition 4.1 and integrating on $M$ with respect to $d\mu$, we obtain

$$\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu + pf_\sigma^{p-1} \int_{M_t} f_\sigma^{p-1} \, d\mu + 2C_3p \int_{M_t} \|\nabla f_\sigma\|^2 W^{\sigma-1} f_\sigma^{p-1} \, d\mu$$

$$\leq 4p \int_{M_t} ||H||W^{-1}\|\nabla H\||\nabla f_\sigma||f_\sigma^{p-1} \, d\mu + 2\int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu.$$

Also, we have

$$\alpha ||H|| \leq \left\{ \begin{array}{ll} \sqrt{W} & (\text{when } \mathcal{M} = \mathbb{R}P^n(4c)) \\ \sqrt{W - \beta} & (\text{when } \mathcal{M} = \mathbb{R}H^n(-4c)). \end{array} \right.$$
which holds by our assumption for \( m \). From (6.2), (6.3) and these inequalities, we obtain
\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} \| \nabla f_\sigma \|^2 \, d\mu + C_1 p \int_{M_t} \| \nabla H \|^2 W^{\sigma-1} f_\sigma^{p-1} \, d\mu 
\]
(6.4)
\[
\leq 2\sigma p \int_{M_t} \| h \|^2 f_\sigma^p \, d\mu + 2C_2 p \int_{M_t} f_\sigma^p \, d\mu + 2C_3 p \int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu 
\]
\[
\leq 4\sigma p m \int_{M_t} W f_\sigma^p \, d\mu + 2C_2 p \int_{M_t} f_\sigma^p \, d\mu + 2C_3 p \int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu ,
\]
whch together with Proposition 6.2 derives
\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} \| \nabla f_\sigma \|^2 \, d\mu + C_1 p \int_{M_t} \| \nabla H \|^2 W^{\sigma-1} f_\sigma^{p-1} \, d\mu 
\]
(6.5)
\[
\leq \frac{8\sigma p m}{\varepsilon \rho} (\eta(p+1) + 5) \int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \| \nabla H \|^2 \, d\mu 
\]
\[
+ \frac{p-1}{\eta} \int_{M_t} f_\sigma^{p-2} \| \nabla f_\sigma \|^2 \, d\mu + \frac{C_6^p}{\rho} \int_{M_t} f_\sigma^p \, d\mu + 2C_2 p \int_{M_t} f_\sigma^p \, d\mu + 2C_3 p \int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu .
\]

Let \( \eta = \frac{\sqrt{\nu}}{4p^2} \). Then, by using the assumptions for \( p \) and \( \sigma \), we have
\[
\frac{8\sigma p m}{\varepsilon \rho} (\eta(p+1) + 5) \leq C_1 p \quad \text{and} \quad \frac{8\sigma p(p+1)m}{\varepsilon \rho \eta} \leq \frac{p(p-1)}{2} .
\]

From (6.5) and these inequalities, we obtain
\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu \leq \left( \frac{8\sigma^{2-1} \sigma p m^2}{\varepsilon \rho} + 2C_2 p \right) \int_{M_t} f_\sigma^p \, d\mu 
\]
(6.6)
\[
+ \frac{8\sigma m}{\varepsilon \rho} \cdot C_6^p + 2C_3 p \int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu .
\]

By the Young’s inequality, we have
\[
W^{\sigma-1} \cdot f_\sigma^{p-1} \leq \frac{W^{(\sigma-1)p}}{p} + \frac{p-1}{p} \cdot f_\sigma^p .
\]

From \( 0 < \sigma < \frac{1}{4} \), we have \( (\sigma-1)p < 0 \) and hence \( W^{(\sigma-1)p} \leq \beta^{(\sigma-1)p} \). Hence we obtain
\[
\int_{M_t} W^{\sigma-1} f_\sigma^{p-1} \, d\mu \leq \frac{1}{p} \beta^{(\sigma-1)p} \cdot \operatorname{Vol}(M_0) + \frac{p-1}{p} \int_{M_t} f_\sigma^p \, d\mu .
\]
(6.7)

From (6.6) and (6.7), we can derive
\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu \leq \tilde{C}_1 \int_{M_t} f_\sigma^p \, d\mu + \tilde{C}_2
\]
for some positive constants \( \tilde{C}_1 \) and \( \tilde{C}_2 \) depending only on \( p, \sigma, m, k, \varepsilon, \rho, \beta \) and \( \operatorname{Vol}(M_0) \). Therefore, since \( T_{\text{max}} \) is finite, we obtain the assertion for a constant \( C_9 \) depending only on \( p, \sigma, m, k, \varepsilon, \rho, \beta, \operatorname{Vol}(M_0) \) and \( T_{\text{max}} \). □

From this proposition, we can derive the following result.
Corollary 6.6. Assume that $T_{\max} < \infty$. Then the following statements (i) and (ii) hold:

(i) Let $r$ be any positive number. For any $p > \frac{8C_0}{C_1} + \frac{1}{p} + 1$ and any $\sigma < \frac{\sqrt{C_0r+1}}{2mp\sqrt{r}} + \frac{1}{p}$, we have

$$
\left( \int_{M_t} W^{(\sigma-1)r} f^{(p-1)r}_\sigma \, d\mu \right)^{\frac{1}{p(p-1)r}} \leq C_9 (\sigma - \frac{1}{p} - 1) (p - 1)r \quad (t \in [0, T_{\max})).
$$

(ii) For any $p > \frac{8C_0}{C_1} + 1$ and any $\sigma < \frac{\sqrt{C_0r+1}}{2mp\sqrt{r}} - \frac{q}{p}$, we have

$$
\left( \int_{M_t} ||h||^{2q} f^{p}_\sigma \, d\mu \right)^{\frac{1}{p}} \leq (2m)^{\frac{q}{p}} C_9 (\sigma + \frac{q}{p}, p) \quad (t \in [0, T_{\max})).
$$

Proof. First we shall show the statement (i). Easily we have $W^{(\sigma-1)r} f^{(p-1)r}_\sigma = f^{(p-1)r}_{\sigma-\frac{1}{p-1}}$. Hence it follows from Proposition 6.5 that the desired inequality holds for any $p$ and $\sigma$ as in the statement (i). Next we shall show the statement (ii). From $||h||^2 \leq 2mW$ and Proposition 6.5, we obtain

$$
\left( \int_{M_t} ||h||^{2q} f^{p}_\sigma \, d\mu \right)^{\frac{1}{p}} \leq (2m)^{\frac{q}{p}} \left( \int_{M_t} W^{q} f^{p}_\sigma \, d\mu \right)^{\frac{1}{p}} \leq (2m)^{\frac{q}{p}} C_9 (\sigma + \frac{q}{p}, p).
$$

Here we recall the Stampacchia’s iteration lemma.

Lemma 6.7. Let $\phi : [s_0, \infty) \to \mathbb{R}$ be a non-negative and non-increasing function satisfying

$$
\phi(s_2) \leq \frac{C}{(s_2 - s_1)^p} ||\phi(s_1)||^\gamma
$$

for any $s_1, s_2$ with $s_0 < s_1 < s_2$, where $C, p$ are positive constant and $\gamma$ is a constant with $\gamma > 1$. Then $\phi(s_0 + d_0) = 0$ holds, where

$$
d_0 = \left( C||\phi(s_0)||^{\gamma - 1} 2p^{p/(\gamma - 1)} \right)^{1/p}.
$$

Also, we recall the Sobolev inequality for submanifolds.

Theorem 6.8. ([HS]) Let $M$ be an $m$-dimensional submanifold in a Riemannian manifold $(\overline{M}, \overline{g})$, where $M$ may have the boundary. Denote by $H$ the mean curvature vector field of $M$, $K$ the maximal sectional curvature of $\overline{M}$, $R(M)$ the injective radius of $\overline{M}$ restricted to $M$ and $\omega_m$ the volume of the unit ball in the Euclidean space $\mathbb{R}^m$. Let $b$ be a positive real number or a purely imaginary one satisfying $b^2 \geq K$ and $\psi$ a non-negative $C^1$ function on $M$ vanishing on $\partial M$. Then the following inequality holds:

\begin{equation}
(6.8) \quad \left( \int_M \psi^{\frac{m-1}{m}} \, d\mu \right)^{\frac{m-1}{m}} \leq \tilde{C}(m) \int_M (||\nabla \psi|| + \psi ||H||) \, d\mu
\end{equation}

if

\begin{equation}
(6.9) \quad b^2 (1 - \alpha)^{-\frac{m}{2}} (\omega_m^{-1} \cdot \text{vol}(\text{supp} \, \psi))^{\frac{2}{m}} \leq 1 \quad \text{and} \quad 2\rho_0 \leq R(M),
\end{equation}

where

$$
\rho_0 := \begin{cases} 
\frac{b^{-1} \sin^{-1} b \cdot (1 - \alpha)^{-\frac{1}{2}} \cdot (\omega_m^{-1} \cdot \text{vol}(\text{supp} \, \psi))^{\frac{1}{m}}} {(1 - \alpha)^{-\frac{1}{2}} (\omega_m^{-1} \cdot \text{vol}(\text{supp} \, \psi))^{\frac{1}{m}}} & \text{(for $b$ is real)}, \\
\frac{b^{-1} \sin^{-1} b \cdot (1 - \alpha)^{-\frac{1}{2}} \cdot (\omega_m^{-1} \cdot \text{vol}(\text{supp} \, \psi))^{\frac{1}{m}}} {(1 - \alpha)^{-\frac{1}{2}} (\omega_m^{-1} \cdot \text{vol}(\text{supp} \, \psi))^{\frac{1}{m}}} & \text{(for $b$ is purely imaginary)}.
\end{cases}
$$

Here $\alpha$ is a free parameter with $0 < \alpha < 1$, and

$$
\tilde{C}(m) = \tilde{C}(m, \alpha) := \frac{\pi}{2} \cdot 2^{m-2} \alpha^{-1} (1 - \alpha)^{-\frac{1}{2}} \cdot \frac{m}{m - 1} \omega_m^{\frac{1}{m}}.
$$
Now we shall prove Theorem 6.1.

**Proof of Theorem 6.1.** Define a function \( f_{\sigma,l} : M \times [0,T_{\max}) \to \mathbb{R} \) by \( f_{\sigma,l}(x,t) := \max\{f_{\sigma}(x,t) - l, 0\} \), where \( l \) is any positive number with \( l \geq l_0 := \max_{x \in M} f_{\sigma}(x,0) \). Set \( A_l(t) := \{x \in M \mid f_{\sigma}(x,t) \geq l\} \). For a function \( v \) over \( M \times [0,T_{\max}) \), we denote \( \int_{A_l(t)} v(\cdot,t) d\mu \) by \( \int_{A_l(t)} v d\mu \) for the simplicity. In similar to (6.4), we can derive the following evolution equation:

\[
\frac{\partial}{\partial t} \int_{A_l(t)} f_{\sigma,l}^p d\mu \leq - \frac{p(p-1)}{2} \int_{A_l(t)} f_{\sigma,l}^{p-2} ||\nabla f_{\sigma,l}\|^2 d\mu - C_1 p \int_{A_l(t)} W^{\sigma-1} f_{\sigma,l}^{p-1} ||\nabla H||^2 d\mu \\
+ 2Cp \int_{A_l(t)} ||h||^2 f_{\sigma,l}^p d\mu - 2C_2 p \int_{A_l(t)} f_{\sigma,l}^p d\mu \\
+ 2C_3 p \int_{A_l(t)} W^{\sigma-1} f_{\sigma,l}^{p-1} d\mu.
\]

(6.10)

For \( p \geq 2 \), we have the following estimate:

\[
\frac{p(p-1)}{2} f_{\sigma,l}^{p-2} ||\nabla f_{\sigma,l}||^2 \geq ||\nabla \tilde{f}_{\sigma,l}||^2
\]

on \( A_l(t) \). Set \( \tilde{v}_l := \tilde{f}_{\sigma,l}^\sigma \). By using this estimate and discarding some terms in the right-hand side of (6.10), we have

\[
\frac{\partial}{\partial t} \int_{A_l(t)} v_l^2 d\mu + \int_{A_l(t)} ||\nabla v_l||^2 d\mu \\
\leq 2Cp \int_{A_l(t)} ||h||^2 v_l^2 d\mu + 2C_2 p \int_{A_l(t)} v_l^2 d\mu + 2C_3 p \int_{A_l(t)} W^{\sigma-1} v_l^{2(p-1)/p} d\mu.
\]

By integrating both sides of this inequality from 0 to any \( t_0 \in [0,T_{\max}) \), we have

\[
\int_{A_{t_0}(t)} v_l^2 d\mu + \int_0^{t_0} \left( \int_{A_l(t)} ||\nabla v_l||^2 d\mu \right) dt \\
\leq 2Cp \int_0^{t_0} \left( \int_{A_l(t)} ||h||^2 v_l^2 d\mu \right) dt + 2C_2 p \int_0^{t_0} \left( \int_{A_l(t)} v_l^2 d\mu \right) dt \\
+ 2C_3 p \int_0^{t_0} \left( \int_{A_l(t)} W^{\sigma-1} v_l^{2(p-1)/p} d\mu \right) dt,
\]

where we use \( l \geq l_0 \). By the arbitrariness of \( t_0 \), we have

\[
\sup_{t \in [0,T_{\max})} \int_{A_l(t)} v_l^2 d\mu + \int_{0}^{T_{\max}} \left( \int_{A_l(t)} ||\nabla v_l||^2 d\mu \right) dt \\
\leq 4Cp \int_0^{T_{\max}} \left( \int_{A_l(t)} ||h||^2 v_l^2 d\mu \right) dt + 4C_2 p \int_0^{T_{\max}} \left( \int_{A_l(t)} v_l^2 d\mu \right) dt \\
+ 4C_3 p \int_0^{T_{\max}} \left( \int_{A_l(t)} W^{\sigma-1} v_l^{2(p-1)/p} d\mu \right) dt.
\]

(6.11)
By applying the Sobolev inequality (6.8) to \( v \) and using the Hölder inequality, we can derive
\[
\left( \int_{M_t} v_{l_1}^{2q} d\mu \right)^{\frac{1}{2q}} \leq \tilde{C}(m) \text{Vol}(M_0) \left( \int_{M_t} ||\nabla v||^2 d\mu \right)^{\frac{1}{2}} + \tilde{C}(m) \left( \int_{M_t} ||H||^m d\mu \right)^{\frac{1}{m}} \left( \int_{M_t} v_{l_1}^{2q} d\mu \right)^{\frac{1}{2q}},
\]
where \( q := \frac{m}{2(m-1)} \). We want to take advantage of the good gradient term in the left-hand side of (6.11). By squaring both sides of this inequality and using \((a + b)^2 \leq 2(a^2 + b^2)\), we obtain
\[
\left( \int_{M_t} v_{l_1}^{2q} d\mu \right)^{\frac{1}{q}} \leq C_{10} \int_M ||\nabla v||^2 d\mu + C_{11} \left( \int_{M_t} ||H||^m d\mu \right)^{\frac{2}{m}} \left( \int_{M_t} v_{l_1}^{2q} d\mu \right)^{\frac{1}{q}},
\]
where \( C_{10} = 2\tilde{C}(m)^2 \text{Vol}(M_0)^2 \) and \( C_{11} = 2\tilde{C}(m)^2 \). Since \( f_\sigma(\cdot, t) \geq l \) on \( A_t(l) \), it follows from Corollary 6.6 that
\[
\left( \int_{A_t(l)} ||H||^m d\mu \right)^{\frac{2}{m}} \leq m \left( \int_{A_t(l)} ||h||^m \frac{f_\sigma^p}{\nu^p} d\mu \right)^{\frac{2}{m}} = m \cdot l^{-\frac{2m}{p}} \left( \int_{A_t(l)} ||h||^m f_\sigma^p d\mu \right)^{\frac{2}{m}} \leq 2m^2 \left( C_0(\sigma + \frac{m}{2p}, p) l \right)^{\frac{2m}{p}}.
\]
Fix \( l_1 > l_0 > 0 \), where we take \( l_1 \) as a sufficiently large number satisfying \( 2m^2 C_{11} (C_0(\sigma + \frac{m}{2p}, p) / l_1)^{\frac{2m}{p}} < 1 \). In the sequel, let \( l \geq l_1 \). Then by absorbing the second term in the right-hand side of (6.12) into the left-hand side, we obtain
\[
\tilde{C} \left( \int_{M_t} v_{l_1}^{2q} d\mu \right)^{\frac{1}{q}} \leq \int_{M_t} ||\nabla v||^2 d\mu,
\]
where \( \tilde{C} := \frac{1 - 2m^2 C_{11} (C_0(\sigma + \frac{m}{2p}, p) / l_1)^{\frac{2m}{p}}}{C_{10}} \). From (6.11) and (6.14), we obtain
\[
\sup_{t \in [0, T_{\text{max}}]} \int_{A_t(l)} v_{l_1}^{2q} d\mu + \tilde{C} \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_{l_1}^{2q} d\mu \right)^{\frac{1}{q}} dt \leq 4\sigma p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} ||h||^2 v_{l_1}^2 d\mu \right) dt + 4C_2 p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_{l_1}^2 d\mu \right) dt + 4C_3 p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} W^\sigma v_{l_1}^{2(p-1)/p} d\mu \right) dt.
\]
We need to estimate the second term of the left-hand side. According to the interpolation inequality for the \( L^p \) spaces, we have
\[
||| \cdot |||_{L^{q_0}} \leq ||| \cdot |||_{L^1}^{1-\theta} ||| \cdot |||_{L^q}^{\theta},
\]
where \( q_0 := 2 - \frac{1}{q} \) and \( \theta := \frac{q}{2q - 1} \). By this interpolating inequality, we obtain
\[
\left( \int_{A_t(l)} v_{l_1}^{2q_0} d\mu \right)^{\frac{1}{q_0}} \leq \left( \int_{A_t(l)} v_{l_1}^2 d\mu \right)^{\frac{q_0 - 1}{q_0}} \left\{ \left( \int_{A_t(l)} v_{l_1}^{2q} d\mu \right)^{\frac{1}{q}} \right\}^{\frac{1}{q_0}},
\]
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that is,
\[ \int_{A_t(l)} v_l^{2q_0} \, d\mu \leq \left( \int_{A_t(l)} v_l^2 \, d\mu \right)^{q_0-1} \left( \int_{A_t(l)} v_l^{2q} \, d\mu \right)^{\frac{1}{q}}. \]

By using this inequality and the Young inequality, we can derive

\[ \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^{2q_0} \, d\mu \right) \, dt \right)^{\frac{1}{q_0}} \]

\[ \leq \left( \int_0^{T_{\text{max}}} \left( \left( \int_{A_t(l)} v_l^2 \, d\mu \right)^{q_0-1} \left( \int_{A_t(l)} v_l^{2q} \, d\mu \right)^{\frac{1}{q}} \right) \, dt \right)^{\frac{1}{q_0}} \]

(6.16)

\[ \leq q_0 - 1 \sup_{t \in [0,T_{\text{max}}]} \left( \int_{A_t(l)} v_l^2 \, d\mu \right) + \frac{1}{q_0} \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^{2q} \, d\mu \right)^{\frac{1}{q}} \, dt \]

\[ \leq q_0 - 1 \left\{ \sup_{t \in [0,T_{\text{max}}]} \left( \int_{A_t(l)} v_l^2 \, d\mu \right) + \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^{2q} \, d\mu \right)^{\frac{1}{q}} \, dt \right\}. \]

We may assume that \( \hat{C} < 1 \) by taking \( l_1 \) as a larger positive number if necessary. From (6.15), (6.16) and \( \hat{C} < 1 \), we have

\[ \frac{q_0 \hat{C}}{q_0 - 1} \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^{2q_0} \, d\mu \right) \, dt \right)^{\frac{1}{q_0}} \]

(6.17)

\[ \leq 4\sigma p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} ||h||^2 v_l^2 \, d\mu \right) \, dt + 4C_2 p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^2 \, d\mu \right) \, dt \]

\[ + 4C_3 p \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} W^{\sigma-1} v_l^{2(p-1)} \, d\mu \right) \, dt. \]

Set \( \||A(l)|| := \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} \, d\mu \right) \, dt \). By the Hölder inequality, we have

(6.18)

\[ \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^{2q_0} \, d\mu \right) \, dt \right)^{\frac{1}{q_0}} \geq \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_l^2 \, d\mu \right) \, dt \right) \cdot \||A(l)||^{-\frac{q_0}{q_0-1}}. \]
By using (6.17), (6.18) and the H"older inequality again, we obtain
\[
\int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_t^2 \, d\mu \right) dt \\
\leq \frac{q_0 - 1}{q_0 C} \|A(l)\|^2 \left( q_0 - 1 \right)^{-\frac{1}{r}} \times \left\{ 4\sigma_p \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} \|h\|^2 v_t^2 r \, d\mu \right) dt \right)^{\frac{1}{r}} \\
+ 4C_2p \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} v_t^2 \, d\mu \right) dt \right)^{\frac{1}{r}} \\
+ 4C_3p \left( \int_0^{T_{\text{max}}} \left( \int_{A_t(l)} W^{(\sigma-1)r} v_t^{2\frac{(p-1)r}{p}} \, d\mu \right) dt \right)^{\frac{1}{r}} \right\},
\]
(6.19)
where \( r \) is a sufficiently large positive number so that \( \gamma := 2 - \frac{1}{q_0} - \frac{1}{r} > 1 \). According to Proposition 6.5 and Corollary 6.6, the second factor \( \{ \cdots \} \) of the right-hand side of this inequality can be bounded by a positive constant. Take any positive constants \( s_1, s_2 \) with \( s_2 > s_1 \geq l_1 \). Clearly we have
\[
\int_0^{T_{\text{max}}} \left( \int_{A_t(s_1)} v_s^2 \, d\mu \right) dt \geq \int_0^{T_{\text{max}}} \left( \int_{A_t(s_1)} (f_{\sigma,s_1} - f_{\sigma,s_2})^p \, d\mu \right) dt \\
\geq (s_2 - s_1)^p \|A(s_2)\|.
\]
This together with (6.19) derives
\[
(s_2 - s_1)^p \|A(s_2)\| \leq \overline{C} \|A(s_1)\|^\gamma,
\]
which holds for all \( s_2 > s_1 \geq l_1 \), where \( \overline{C} \) is a positive constant which is independent of the choices of \( s_1 \) and \( s_2 \). It follows from Lemma 6.7 that \( \|A(l_1 + d_0)\| = 0 \), where \( d_0 = (\overline{C}^{\frac{\gamma}{\gamma-1}} \|A(l_1)\|^{-1})^{1/p} \). This implies that
\[
\sup_{t \in [0,T_{\text{max}}]} \max_M f_{\sigma}(\cdot,t) \leq l_1 + d_0.
\]
This together with \( f_{\sigma} \geq \max_M \frac{1}{\max(\alpha,\|H\|+1)^{\frac{1}{p}}} \cdot \frac{\|h\|^2}{\|H\|+1} \) implies that
\[
\sup_{t \in [0,T_{\text{max}}]} \max_M \frac{\|h\|^2}{\|H\|+1} \leq (l_1 + d_0) \cdot \max\{\alpha, \|\beta\|\}^{1-\sigma}.
\]
Thus the statement of Theorem 6.1 follows.

Next we shall derive a gradient estimate for the mean curvature. This estimate is required to compare with the mean curvature oneself. First we prepare some technical inequalities. By the discussion similar to the proof of Lemma 5.6 in [PS], we can derive the following technical inequality by using Lemmas 2.1, 4.2, 4.3 and Theorem 6.1.

**Lemma 6.9.** The family \( \{||H_t||^2, ||\dot{h}_t||^2\}_{t \in [0,T_{\text{max}}]} \) satisfies
\[
\frac{\partial}{\partial t} (||H||^2||\dot{h}||^2) \leq \Delta (||H||^2||\dot{h}||^2) - C_4 ||H||^2 ||\nabla h||^2 + C_{12} ||\nabla h||^2 \\
+ 2||H||^2||\dot{h}||^2 \{6||h||^2 - (\sigma \cdot 1 - 1)(7m + 4(d - 1)k)c\}
\]
for some positive constant \( C_{12} \).
Figure 3: Proof of uniform boundedness by Stampacchia’s iteration lemma

**Proof.** By Lemmas 2.1, 4.2 and 4.3,

\[
\frac{\partial}{\partial t} \left( ||H||^2 ||\dot{h}||^2 \right) = \left( \frac{\partial}{\partial t} ||H||^2 \right) \cdot ||\dot{h}||^2 + ||H||^2 \cdot \frac{\partial}{\partial t} ||\dot{h}||^2 \\
\leq (\Delta ||H||^2 - 2 ||\nabla H||^2 + 2 ||H||^2 (||h||^2 + \bar{r})) ||\dot{h}||^2 \\
+ ||H||^2 \{\Delta ||\dot{h}||^2 - 2C_4 ||\nabla h||^2 + 4||h||^2 ||\dot{h}||^2 \\
- (\tau - 1)(7m + 4(d - 1)k)c||\dot{h}||^2 \} \\
= \Delta (||H||^2 ||\dot{h}||^2) - 2 \langle \nabla ||H||^2, \nabla ||\dot{h}||^2 \rangle - 2C_4 ||H||^2 ||\nabla h||^2 - 2 ||\dot{h}||^2 ||\nabla H||^2 \\
+ ||\dot{h}||^2 ||H||^2 \{6||h||^2 - (\tau - 1)(7m + 4(d - 1)k)c \}. \tag{6.20}
\]

Furthermore, by using Lemma 3.6, we have

\[-2 \langle \nabla ||H||^2, \nabla ||\dot{h}||^2 \rangle = -8||H|| \cdot ||\dot{h}|| \langle \nabla ||H||, \nabla ||\dot{h}|| \rangle \\
\leq 8||H|| \cdot ||\dot{h}|| \cdot ||\nabla H|| \cdot ||\nabla \dot{h}|| \leq 24 \left( \frac{m + 2}{2(10 - 2d)} \right)^{\frac{1}{2}} ||H|| \cdot ||\dot{h}|| \cdot ||\nabla h||^2.
\]

By using Theorem 6.1 and Young inequality, we can show that there exists a positive constant $C_{12}$ satisfying

\[24 \left( \frac{m + 2}{2(10 - 2d)} \right)^{\frac{1}{2}} ||H|| ||\nabla h||^2 ||\dot{h}|| \leq 24 \left( \frac{m + 2}{2(10 - 2d)} \right)^{\frac{1}{2}} ||H|| \cdot ||\nabla h||^2 \sqrt{C_0(||H||^2 + 1)}^{1 - \frac{\tau}{2}} \\
\leq C_4 ||H||^2 ||\nabla h||^2 + C_{12} ||\nabla h||^2.
\]

These relations together with (6.20) implies the desired inequality. \qed

Define a function $g$ by

\[g := ||H||^2 ||\dot{h}||^2 + \frac{1}{2} \left( \frac{C_{12}}{C_4} + 1 \right) ||\dot{h}||^2. \tag{6.21}\]
By using Lemma 4.2, 4.3, 6.9 and \( ||H||^2 \leq m||h||^2 \), we obtain

\[
(6.22) \quad \frac{\partial}{\partial t} g \leq \Delta g - C_4(||H||^2 + 1)||\nabla h||^2 + 2||h||^2 \cdot ||\hat{h}||^2(6m||h||^2 + C_{13}) + C_{14}||\hat{h}||^2,
\]

where \( C_{13} := \frac{C_4}{2}(1 + m(\bar{r} \cdot 1 - 1)(7m + 4(d-1)k)c \) and \( C_{14} := \frac{1}{2}(\frac{C_4}{2} + 1)(\bar{r} \cdot 1 - 1)(7m + 4(d-1)k)c \).

**Proposition 6.10.** For any sufficiently small positive number \( \eta \), there exists a constant \( C_\eta > 0 \) depending only on \( \eta \) such that the inequality

\[
||\nabla H||^2 \leq \eta||H||^4 + C_\eta
\]

holds for all \( t \in [0, T_{\text{max}}] \).

**Proof.** Set \( f := ||\nabla H||^2 + \frac{1}{C_4}(C_5 + 1)g - \eta||H||^4 \), where \( \eta \) is a sufficiently small positive number. From Lemmas 4.2 - 4.4 and (6.22), we can derive

\[
(6.23) \quad \frac{\partial}{\partial t} f \leq - \Delta f - (||H||^2 + 1)||\nabla h||^2 + \frac{2}{C_4}(C_5 + 1)||\hat{h}||^2||h||^2(6m||h||^2 + C_{13})
\]

\[
+ \frac{C_{14}}{C_4}(C_5 + 1)||\hat{h}||^2 - \eta \left( \frac{4}{m}||H||^6 + 12||H||^2||\nabla H||^2 + 8(\bar{r} \cdot 1 - 1)m \eta||H||^4 \right).
\]

Since \( ||\nabla h||^2 \geq \frac{2(10 - d)}{9(m + 2)}||\nabla H||^2 \) by Lemma 3.6, we have

\[
-(||H||^2 + 1)||\nabla h||^2 + 12\eta||H||^2||\nabla H||^2 \leq \left( -||H||^2 - 1 + \frac{108(m + 2)}{2(10 - 2d)} \eta||H||^2 \right)||\nabla h||^2.
\]

Hence we have

\[
-(||H||^2 + 1)||\nabla h||^2 + 12\eta||H||^2||\nabla H||^2 < 0
\]

for a sufficiently small positive number \( \eta \). Denote by \( R \) the reaction terms in (6.23), that is,

\[
R := \frac{2}{C_4}(C_5 + 1)||\hat{h}||^2||h||^2(6m||h||^2 + C_{13}) + \frac{C_{14}}{C_4}(C_5 + 1)||\hat{h}||^2
\]

\[
- \eta \left( \frac{4}{m}||H||^6 + 8(\bar{r} \cdot 1 - 1)m \eta||H||^4 \right).
\]

By using the pinching condition \( (\ast \varepsilon) \), we have

\[
R \leq \frac{2}{C_4}(C_5 + 1)||\hat{h}||^2(a_\varepsilon||H||^2 + b_\varepsilon)(6ma_\varepsilon||H||^2 + C_{13})
\]

\[
+ \frac{C_{14}}{C_4}(C_5 + 1)||\hat{h}||^2 - \frac{4\eta}{m}||H||^6 - 8(\bar{r} \cdot 1 - 1)m \eta||H||^4.
\]

Hence, from Theorem 6.1 and the Young inequality, we obtain

\[
R \leq \frac{2}{C_4}(C_5 + 1)C_0(||H||^2 + 1)^{1-\sigma}(a_\varepsilon||H||^2 + b_\varepsilon)(6ma_\varepsilon||H||^2 + C_{13})
\]

\[
+ \frac{C_{14}}{C_4}(C_5 + 1)C_0(||H||^2 + 1)^{1-\sigma} - \frac{4\eta}{m}||H||^6 - 8(\bar{r} \cdot 1 - 1)m \eta||H||^4,
\]

where \( \mu \) is any positive constant. Thus we have

\[
(6.24) \quad R \leq \left( -\frac{4\eta}{m} + \tilde{C}_6(\eta, \mu) \right)||H||^6 + \tilde{C}_4(\eta, \mu)||H||^4 + \tilde{C}_2(\eta, \mu)||H||^2 + \tilde{C}_0(\eta, \mu),
\]

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where \( \hat{C}_i(\eta, \mu) \) \((i = 0, 2, 4, 6)\) are constants depending only on \( \eta \) and \( \mu \). Since \( \hat{C}_6(\eta, \mu) \to 0 \) as \( \mu \to 0 \), we can find such positive number \( \mu_0 \) as \( \hat{C}_6(\eta, \mu_0) < \frac{4}{m} \). Set \( \hat{C}_6(\eta) := \hat{C}_6(\eta, \mu_0) \). Then the coefficient \((-\frac{4}{m} + \hat{C}_6(\eta))\) of the term of the highest degree in the right-hand side (which is regarded as a polynomial with variable \( ||H|| \)) of (6.24) is negative if we take \( \eta > 0 \) sufficiently small. Hence, if \( ||H|| \) is sufficiently large, then we have \( R < 0 \). Therefore, we can find a positive constant \( C_{15}(\eta) \) depending only on \( \eta \) such that \( R < C_{15}(\eta) \) always holds even if \( ||H|| \) take any value. Hence we have

\[
\frac{\partial}{\partial t} f \leq \Delta f + C_{15}(\eta).
\]

This together with \( T_{\max} < \infty \) implies that there exists a constant \( C_\eta \) depending only on \( \eta \) such that \( f \leq C_\eta \). Then, from the definition of \( f \), we obtain

\[
||\nabla H||^2 \leq ||\nabla H||^2 + \frac{1}{C_4}(C_5 + 1)g \leq \eta||H||^4 + C_\eta.
\]

Next we recall the well-known Myers theorem.

**Theorem 6.11.** (M) Let \((M, g)\) be an \( m \)-dimensional complete connected Riemannian manifold. If its Ricci curvature \( \text{Ric} \) satisfies

\[
\text{Ric} \geq (m - 1)\kappa g
\]

for some positive constant \( \kappa \), then the diameter of \((M, g)\) is smaller than or equal to \( \frac{\pi}{\sqrt{\kappa}} \).

By using Theorem 6.1 and Proposition 6.10, we shall prove that, if time is sufficiently close to \( T_{\max} \), then the sectional curvature \( K_t : G_2(M_t) \to \mathbb{R} \) of \( M_t \) is positive.

**Proposition 6.12.** For any \( \mu \in (0, \min\{\frac{1}{2m(m-1)}, \frac{1}{\delta}\}) \) and any a positive constant \( \tilde{b} \), there exists a constant \( \theta(\mu, \tilde{b}) \in [0, T_{\max}) \) satisfying the following two conditions:

(I) for all \( t \in [\theta(\mu, \tilde{b}), T_{\max}) \), \( K_t > \mu W_t \) holds;

(II) for all \( t \in [\theta(\mu, \tilde{b}), T_{\max}) \), \( ||h_t||^2 \leq \frac{1}{m-\delta}||H_t||^2 - \tilde{b} \) holds.

**Proof.** Fix an orthonormal basis of type (I) with the additional condition that \( A_{m+1}(e_i) = \lambda_i e_i \) \((i = 1, \cdots, m)\), where \( A = A_t \) denotes the shape operator of \( M_t \) and \( \lambda_1 \leq \cdots \leq \lambda_m \). According to the Gauss equation, we have

\[
(6.25) \quad K_{ij} = \overline{K}_{ij} + \sum_{a=m+1}^{dn} (h_{ij}^a h_{ij}^a - (h_{ij}^0)^2),
\]

where \( K_{ij} \) denotes the sectional curvature \( K_t(e_i, e_j) \) of \( M_t \) for the plane spanned by the orthonormal system \((e_i, e_j)\), and \( \overline{K}_{ij} \) is the sectional curvature of \( \overline{M} \) for the same plane, which is regarded as an element of the Grassmann bundle \( G_2(\overline{M}) \) of \( \overline{M} \) consisting of the 2-planes.

First we consider the case of \( \overline{M} = F P^n(4c) \). From (6.25) and \( \overline{K} \geq 1 \), we have

\[
K_{ij} \geq 1 + \lambda_i \lambda_j + \sum_{a=m+2}^{dn} (h_{ij}^a h_{ij}^a - (h_{ij}^0)^2)
\]
\[
\geq 1 + \frac{1}{2} \left( \frac{1}{m-1}||H||^2 - ||h_1||^2 \right) - ||h_-||^2
\]
\[
\geq 1 + \frac{1}{2(m-1)}||H||^2 - \frac{1}{2} \left( ||h_1||^2 + \frac{1}{m}||H||^2 \right) - ||h_-||^2
\]
\[
\geq 1 + \frac{1}{2m(m-1)}||H||^2 - ||h||^2.
\]
Furthermore, it follows from Theorem 6.1 that
\begin{equation}
K_{ij} \geq 1 + \frac{1}{2m(m-1)} \|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma}.
\end{equation}

Also, it follows from Theorem 6.1 that
\begin{equation}
\|h\|^2 - \frac{1}{m-1} \|H\|^2 + \hat{b} \leq \|\partial h\|^2 - \frac{1}{2m(m-1)} \|H\|^2 + \hat{b}
\end{equation}
\begin{equation}
\leq C_0(\|H\|^2 + 1)^{1-\sigma} - \frac{1}{m(m-1)} \|H\|^2 + \hat{b}.
\end{equation}

On the other hand, it is shown that there exists a positive constant \(C_\ast(\mu, \hat{b})\) depending only on \(\mu\) and \(\hat{b}\) such that, if \(\|H\| \geq C_\ast(\mu, \hat{b})\), then
\begin{equation}
1 + \frac{1}{2m(m-1)} \|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma} - \mu W > 0
\end{equation}
and
\begin{equation}
C_0(\|H\| + 1)^{1-\sigma} - \frac{1}{m(m-1)} \|H\|^2 + \hat{b} < 0.
\end{equation}

because is positive (resp. negative) the coefficient \(\frac{1}{2m(m-1)} - \mu \alpha\) (resp. \(-\frac{1}{m(m-1)}\)) of the term of the highest degree (with respect to \(\|H\|\)) of the right-hand side of (6.28) (resp. (6.29)). Hence, if \(\|H\| \geq C_\ast(\mu, \hat{b})\), then we have \(K > \mu W\) and \(\|h\|^2 < \frac{1}{m-1} \|H\|^2 - \hat{b}\). According to Proposition 6.10, there exists a constant \(C_\eta\) with \(\|\nabla H\|^2 \leq \eta \|H\|^4 + C_\eta\). Set \(\|H\|_{\max}(t) := \max_{M} \|H(t)\|\). Since \(T_{\max} < \infty\), we have \(\lim_{t \to T_{\max}} \|H\|_{\max}(t) = \infty\). Hence there exists a positive constant \(\theta(\mu, \hat{b})\) such that, for all \(t \in [\theta(\mu, \hat{b}), T_{\max}]\), \(\|H\|_{\max}(t) \geq \max \left\{ \left( \frac{C_\eta}{\eta} \right)^{\frac{1}{2}}, 2C_\ast(\mu, \hat{b}) \right\}\) holds. By using Proposition 6.10, we can show that \(\|\nabla^4 H(t)\|_{\max} \leq \|\nabla^4 H_{\max}(t)\| \leq \sqrt{27} \|H\|_{\max}(t)^2\) holds on \(M_t\) for all \(t \in [\theta(\mu, \hat{b}), T_{\max}]\). Fix \(t_0 \in [\theta(\eta), T_{\max}]\) and let \(x_0\) be a point of \(M_{t_0}\) attaining the maximum \(\|H\|_{\max}(t_0)\). Then, along any geodesic \(\gamma\) in \(M_{t_0}\) starting from \(x_0\), we have
\begin{equation}
\|H_{\gamma(s)}(t_0)\| \geq \|H\|_{\max}(t_0) - \sqrt{2\eta} \|H\|_{\max}(t_0)^2 s \geq \frac{1}{2} \|H\|_{\max}(t_0)
\end{equation}
for all \(s \in [0, (2\sqrt{2\eta} \max(t_0))^{-1}]\). For the simplicity, set \(r_{t_0} : = (2\sqrt{2\eta} \max(t_0))^{-1}\). Then we have \(\|H_{t_0}\| > \frac{1}{2} \|H\|_{\max}(t_0) \geq C_\ast(\mu, \hat{b})\) holds on the geodesic ball \(B_{x_0}(r_{t_0})\) of radius \(r_{t_0}\) centered at \(x_0\) in \(M_{t_0}\). Therefore, \(K_{t_0} > \mu W_{t_0}\) and \(\|h_{t_0}\|^2 < \frac{1}{m-1} \|H(t_0)\|^2 - \hat{b}\) hold on \(B_{x_0}(r_{t_0})\). Furthermore, it follows that
\begin{equation}
K_{t_0} > \mu W_{t_0} > \mu \alpha \|H_{t_0}\|^2 \geq \frac{\mu \alpha}{4} \|H\|_{\max}(t_0)^2
\end{equation}
holds on \(B_{x_0}(r_{t_0})\). Hence we see that
\begin{equation}
\text{Ric}_{t_0} \geq (m - 1) \frac{\mu \alpha}{4} \|H\|_{\max}(t_0)^2 g_{t_0}
\end{equation}
holds on \(B_{x_0}(r_{t_0})\). Hence, by using Myers theorem, we can derive that the diameter of \(B_{x_0}(r_{t_0})\) is smaller than or equal to \(\sqrt{\frac{2\mu \alpha \|H\|_{\max}(t_0)^2}{\text{Ric}_{t_0}}}\). Here we note that, even if \(B_{x_0}(r_{t_0})\) is possible to be not complete, we can apply Myers theorem to \(B_{x_0}(r_{t_0})\) according to its proof. By taking \(\eta\) as a sufficiently small positive number, we may assume \(\sqrt{\frac{2\mu \alpha \|H\|_{\max}(t_0)^2}{\text{Ric}_{t_0}} \leq r_{t_0}\). This implies that \(M_{t_0} = B_{x_0}(r_{t_0})\). Thus
\[ K_{t_0} > \mu W_{t_0} \text{ and } ||h_{t_0}||^2 < \frac{1}{m-1}||H_{t_0}||^2 - \hat{b} \text{ hold on } M_{t_0}. \] Therefore the statement of this proposition follows from the arbitrariness of \( t_0 \).

Next we consider the case of \( \overline{M} = \mathbb{F}H^n(-4c) \). From (6.25), \( \overline{K} \geq -4 \) and Theorem 6.1, we can derive

\[ K_{ij} \geq -4 + \frac{1}{2m(m-1)}||H||^2 - C_0(||H||^2 + 1)^{1-\sigma}. \]

and (6.27). On the other hand, it is shown that there exists a positive constant \( C_*(\mu, \hat{b}) \) depending only on \( \mu \) and \( \hat{b} \) such that, if \( ||H|| \geq C_*(\mu, \hat{b}) \), then

\[ -4 + \frac{1}{2m(m-1)}||H||^2 - C_0(||H||^2 + 1)^{1-\sigma} - \mu W > 0 \]

and (6.29) hold. Hence, if \( ||H|| \geq C_*(\mu, \hat{b}) \), then we have \( K > \mu W \) and \( ||h||^2 < \frac{1}{m-1}||H_t||^2 - \hat{b} \). Hence we can derive the statement of this proposition by using Myers theorem as in the above proof of the case of \( \overline{M} = \mathbb{F}P^n(4c) \).

\[ \text{Figure 4: Estimate on the whole of the submanifold by Myers theorem} \]
Next we shall recall the main result of [LXZ].

**Theorem 6.13.** For any Riemannian manifold with bounded curvature (for example, Riemannian homogeneous spaces), there exists a positive constant $b_0$ such that, if an $m$-dimensional submanifold in the Riemannian manifold satisfies

\begin{equation}
||h||^2 < \frac{1}{m-1}||H||^2 - b_0,
\end{equation}

then the submanifold collapses to a round point in finite time along the mean curvature flow.

By using these results, we can derive the statements of Theorems 1.3, 1.4 and the finite maximal time-case of Theorems 1.1 and 1.2.

**Proof of Theorems 1.3, 1.4 and $T_{\text{max}} < \infty$-case of Theorems 1.1 and 1.2** The pinching conditions (*) in Theorems 1.1–1.4 are weaker than (6.33), but it follows from Proposition 6.12 that (6.33) holds for all $t$ sufficiently close to $T_{\text{max}}$. Therefore the statements of Theorems 1.3, 1.4 and the finite maximal time-case of Theorems 1.1 and 1.2 follow from Theorem 6.13.

## 7 Proof of Theorems 1.1 and 1.2

In this section, we shall complete the proof of Theorems 1.1 and 1.2. Throughout this section, we assume that $T_{\text{max}} = \infty$. In this case, the discussion is simpler than the case of finite maximal time.

**Proposition 7.1.** There exist positive constants $C_0$ and $\delta_0$ depending only on the initial manifold $M_0$ such that

\begin{equation}
||h_t||^2 \leq C_0(||H_t||^2 + 1)e^{-\delta_0 t}
\end{equation}

holds for any time $t \in [0, \infty)$.

**Proof.** According to Proposition 4.1 with $\sigma = 0$, we have

\begin{equation}
\frac{\partial f_0}{\partial t} \leq \Delta f_0 + \frac{2\alpha}{W} (\nabla f_0, \nabla ||H||^2) - 2C_1 \frac{||\nabla H||^2}{W} + 2C_2f_0 + 2C_3.
\end{equation}

Since $M = \mathbb{F}P^n(4c)$, we have $C_2 < 0$ and $C_3 = 0$. Also, we have $C_1 > 0$. Hence we have

\begin{equation}
\frac{\partial f_0}{\partial t} \leq \Delta f_0 + \frac{2\alpha}{W} (\nabla f_0, \nabla ||H||^2) + 2C_2f_0.
\end{equation}

From this evolution inequality, we can derive $f_0(\cdot, t) \leq \tilde{C}_e^{2C_2t}$ ($0 \leq t < \infty$) for some $\tilde{C}$ depending only on $M_0$. Since $C_2 < 0$, the statement of this proposition follows.

From this estimate, we can prove that the intrinsic sectional curvature $K_\ell$ of the evolving submanifold $M_\ell$ is positive for sufficiently large time as in the case of finite maximal time.

**Proposition 7.2.** There exist positive constants $\mu$ and $\theta$ such that, for any time $t \in [0, \infty)$, $K_\ell > \mu W_t(> 0)$ holds.

**Proof.** As stated in the proof of Proposition 6.12, we have $K_{ij} \geq 1 + \frac{1}{2m(m-1)}||H||^2 - ||h||^2$. Furthermore, according to Proposition 7.1, we have

\begin{equation}
K_{ij} \geq 1 + \frac{1}{2m(m-1)}||H_t||^2 - C_0(||H_t||^2 + 1)e^{-\delta_0 t}.
\end{equation}

From this inequality, we can derive the statement of this proposition by the discussion similar to the proof of Proposition 6.12.
According to Lemma 6.9, we have

\begin{equation}
(7.1) \quad \frac{\partial}{\partial t} \|H\|^2 \|h\|^2 \leq \Delta (\|H\|^2 \|h\|^2) - C_4 \|H\|^2 \|\nabla h\|^2 + C_1 \|H\|^2 \|h\|^2 \|h\|^2.
\end{equation}

Now we consider the function $g$ defined in (6.21). By using Lemma 4.2 and (7.1), we can repeat the computations of the previous sections to conclude that the inequality (6.22) holds also in this case. We shall give a gradient estimate for the mean curvature.

**Proposition 7.3.** For any sufficiently small positive constant $\eta$, there exists a positive constant $C_\eta$ depending only on $\eta$ such that the inequality

\[ \|\nabla H_t\|^2 \leq (\eta \|H_t\|^4 + C_\eta) e^{-\delta t/2}, \]

holds for all $t \in [0, \infty)$.

**Proof.** Define $f$ by

\[ f = e^{\frac{\delta t}{2}} \left( \|\nabla H_t\|^2 + \frac{1}{C_4}(C_5 + \delta_0 m)g \right) - \eta \|H_t\|^4, \]

where $\eta$ is a sufficiently small positive number. Then, by the same discussion as the proof of Proposition 6.10, it follows from Lemma 3.6, 4.2, 4.4, Proposition 7.1 and (6.22) that $\frac{\partial}{\partial t} f \leq \Delta f + C_{16}(\eta) e^{-\frac{\delta t}{2}}$ holds for some positive constant $C_{16}(\eta)$ depending only on $\eta$. From this evolution inequality, we can derive that there exists a constant $C_\eta$ depending only on $\eta$ such that $f \leq C_\eta$ holds for all time. From the definition of $f$, we obtain the desired inequality.

Next we shall show the uniform boundedness of the mean curvature.

**Lemma 7.4.** If $T_{\text{max}} = \infty$, then $\{\|H_t\|^2\}_{t \in [0, \infty)}$ is uniform bounded.

**Proof.** Let $b_0$ be the positive constant in Theorem 6.13. From Proposition 7.1, we have

\[ \|H_t\|^2 - \frac{1}{m-1} \|H_t\|^2 + b_0 = \|H_t\|^2 - \frac{1}{m(m-1)} \|H_t\|^2 + b_0 \]

\[ \leq C_0(\|H_t\|^2 + 1) e^{-\delta t} - \frac{1}{m(m-1)} \|H_t\|^2 + b_0. \]

Notice that the right-hand side is negative if $t$ and $\|H_t\|^2$ are sufficiently large. Suppose that $\{\|H_t\|^2\}_{t \in [0, \infty)}$ is not uniform bounded. Then there exists a sequence $\{t_i\}_{i=1}^\infty$ satisfying $\lim_{i \to \infty} t_i = \infty$ and $\lim_{i \to \infty} \|H\|_{\text{max}}(t_i) = \infty$. By using Propositions 7.1, 7.3 and Myers theorem as in the proof of Proposition 6.12, we can show that there exists $i_0$ such that $\|H_{t_{i_0}}\|^2 - \frac{1}{m-1} \|H_{t_{i_0}}\|^2 + b_0 < 0$ holds on the whole of $M_{t_{i_0}}$. According to Theorem 6.13, the mean curvature flow starting from $M_{t_{i_0}}$ collapses to a round point in finite time. That is, so is also the mean curvature flow starting from $M_0$. This contradicts $T_{\text{max}} = \infty$. Therefore $\{\|H_t\|^2\}_{t \in [0, \infty)}$ is uniform bounded.

**Proof of Theorems 1.1 and 1.2.** Let $M$ be as in Theorem 1.1 or 1.2. In the previous section, we have already proved that, if $T_{\text{max}} < \infty$, then the mean curvature flow starting from $M$ collapses to a round point as $t \to T_{\text{max}}$. Hence we suffice to consider the case of $T_{\text{max}} = \infty$. In this case, since $\{\|H_t\|^2\}_{t \in [0, \infty)}$ is uniform bounded by Lemma 7.4, it follows from Propositions 7.1 and 7.3 that there exists a positive constant $C$ satisfying

\[ \|\nabla H\|^2 \leq Ce^{-\delta t} \text{ and } \|\nabla H\|^2 \leq Ce^{-\frac{\delta t}{2}}. \]
As in the proof of Proposition 6.12, it follows from Proposition 7.2 that there exists a positive constant $C$ such that $\text{Ric}_t \geq \frac{C}{t} g_t$ holds for all $t \in [0, \infty)$. Hence, by using Myers theorem, we can derive $\sup_{t \in [0, \infty)} d_t < \infty$, where $d_t$ is the diameter of $M_t$. Set $d^* := \sup_{t \in [0, \infty)} d_t$. By using this fact and integrating the above second estimate along geodesics, we obtain

\begin{equation}
\|H\|_{\max}(t) - \|H\|_{\min}(t) \leq d^* \sqrt{C} e^{-\frac{\delta_0}{4t}}. \tag{7.2}
\end{equation}

Suppose that $\|H\|_{\min}(t_1) \neq 0$ for some time $t_1$. Then, from (2.5) and (3.5), we obtain

$$
\frac{\partial \|H\|^2}{\partial t} \geq \Delta \|H\|^2 - 2\|\nabla H\|^2 + \frac{1}{m} \|H\|^4.
$$

From this evolution inequality, we can show that $\|H\|^2$ blows up in finite time by a standard comparison argument. This contradicts $T_{\max} = \infty$. Hence we can derive that $\|H\|_{\min}(t) = 0$ for all time $t$. Therefore, from (7.2), we obtain

$$
\|H\|_{\max}(t) \leq d^* \sqrt{C} e^{-\frac{\delta_0}{4t}} \quad (0 \leq t < \infty).
$$

This derives

$$
\|h_t\|^2 = \|\dot{h}_t\|^2 + \frac{1}{m} \|H_t\|^2 \leq \hat{C} e^{-\frac{\delta_0}{4t}} \quad (0 \leq t < \infty)
$$

for some positive constant $\hat{C}$. Furthermore, since the induced metrics $g_t$ on $M_t$ satisfies the evolution equation $\frac{\partial g}{\partial t} = -2\|H\| g$, we have

$$
\int_0^\infty \left| \frac{\partial \|H\|}{\partial t} \right| dt \leq 2 \int_0^\infty \|H\| \|h\| dt \leq 2\sqrt{m} \int_0^\infty \|h\|^2 dt \leq 2\sqrt{m} \hat{C} \int_0^\infty e^{-\frac{\delta_0}{4t}} dt = \frac{4\sqrt{m} \hat{C}}{\delta_0}.
$$

So we can apply a result by Hamilton [Ha, Lemma 14.2] to show that $g_t$ converges uniformly to a continuous metric $g_\infty$ as $t \to \infty$. By using the interpolation inequalities as in [H1, §10], we can show that the exponential decay for $\|h\|^2$ gives the exponential decay for the norms $\|\nabla^k h\|$ of $k$-th covariant derivatives of $h$ for any $k$. From this fact, we can derive that the flow $M_t$ converges to a $(C^\infty)$ totally geodesic submanifold $M_\infty$ in the $C^\infty$-topology as $t \to \infty$. However, if $M$ is as in Theorem 1.1 (hence $M$ is a hypersurface), then this case cannot happen because there exists no totally geodesic hypersurface in $\mathbb{F} P^n(4c)$. This completes the proof of Theorems 1.1 and 1.2. \hfill \Box

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