Adiabatic Cooper-Pair Pumping in a Linear Array of Cooper Pair Boxes

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We present a study of adiabatic Cooper pair pumping in one dimensional array of Cooper pair boxes. We do a detailed theoretical analysis of an experimentally realizable stabilized charge pumping scheme in a linear array of Cooper pair boxes. Our system is subjected to synchronized flux and voltage fields and travel along a loop which encloses their critical ground state of the system in the flux-voltage plane. The locking potential in the sine-Gordon model slides and changes its minimum which yields the Cooper pair pumping. Our analytical methods are the Berry phase analysis and Abelian bosonization studies.

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1. Introduction:

The adiabatic pumping physics gets more attraction after the pioneering work of Thouless [1, 2]. Quantum adiabatic pumping physics is related with many systems like open quantum dots [3, 4, 5], superconducting quantum wires [6, 7], Josephson junctions [8, 9], the Luttinger quantum wire [10], interacting quantum wire [11] and also to the quantum spin pump [12].

An adiabatic parametric quantum pumping is a device that generates a dc current by a cyclic variation of system parameters, the variation being slow enough that the system remains close to the ground state throughout the pumping cycle. It is well known to us that when a quantum mechanical system evolves then it acquires a time dependent dynamical phase and time independent geometrical phase [13]. Geometrical phase dependent on the geometry of the path turned in the parameter space. For the closed path encircling the critical ground state differ by the phase factor and the cyclic parametric variation introduce non-vanishing transport in the system. Quantum pumps are to transform an ac signal of frequency \( f \) into dc current given by the relation \( I = fQ \), where \( Q = ne \) for electron pumping and \( Q = 2ne \) for Cooper pair pumping. Quantum pumping with perfect accuracy could be utilized to establish a standard of current. Here we have considered the perfect pumping condition. The error in the pumping procedure arises due to the current reversal and the spontaneous charge excitations. In Ref. (14), the authors have discussed the sources of errors (the non-adiabatic correction leave the system in an unknown superposition of the charge state, instead of definite charge state, after the full cycles) and their minimization. Here we would like to explain the theoretical detailed of that experimental proposals stabilized charge pumping [14] in a linear array of Cooper pair boxes. Before we start our full swing quantum field theoretical calculations. We would like to present a derivation to illustrate the Cooper pair pumping in one dimensional array of Cooper pair boxes.

2. Model Hamiltonians and Continuum Field Theoretical Study:

Our Hamiltonian \( H = H_c + H_J \) consists of two parts one is the Josephson coupling and the other is the Coulomb charging energy. Where \( H_J = - \sum_{k=1}^{N} E_{J,k} \cos \phi_k \) and \( H_c = \sum_{k=1}^{N} Q_k^2 / 2 \epsilon_k \). \( Q_k \) is the charge occurs at the kth junction, this charge is measured w.r.t the gate charge. So the effective charge of the kth junction can be varied by varying the gate voltage of the system connected to the junctions. The charging Hamiltonian is diagonal in the basis formed by the charge eigen state \( |\bar{n} > = |n_1, n_2, ..., n_{N-1} > \), \( n_k \) is the number of Cooper pair in each island. We would like to explain the transport of Cooper pair of this one dimensional tunnel junction through the analysis of Berry phase [13]. We will see that the Cooper pair current in the system consists of two parts one is the conventional supercurrent and the other is adiabatic Cooper pair pumping. Here we follow the seminal paper of Berry [13] for our theoretical analysis. In our system, we are varying the Josephson junction couplings through the applied flux and the charge on the dot by applying the gate voltage. So the adiabatic varying parameters are, \( \bar{R} = (E_J, Q_k) \). The state \( |\psi(t) > \) of the systems evolves according to Schrodinger’s equation

\[
H(R(t))|\psi(t) > = i\hbar |\psi(t) > .
\]

At any instant, the natural basis consists of the eigenstates \( |n(R) > = |n_1 R, n_2 R, ..., n_{N-1} R > \). \( |\psi(t_0) > = |n(t_0) > \).

\[
|\psi(t) > = e^{-iE_n \delta t} |n(t_0) > + \sum_{l \neq n} \frac{\hbar}{i} \frac{F(E_l, E_n)}{E_{l0} - E_{n0}} \times \nabla_{Rn} |n(t_0) > . \delta l |l(t_0) >
\]

\[
F(E_l, E_n) = (e^{-iE_l(t_0)} - e^{-iE_n(t_0)})
\]

\[
|\psi(t) > = |n(t) > + |\delta n(t) >
\]

or \( \sum_{l \neq n} \frac{\hbar}{i} \frac{F(E_l, E_n)}{E_{l0} - E_{n0}} \times \nabla_{Rn} |n(t_0) > . \delta l |l(t_0) >\)
The total charge transport due to Cooper pair transport is nothing but the collection of tunable Josephson coupling and the electrostatic potential of superconducting island in a Cooper pair box. The adiabatic quantum Cooper pair pumping procedure, is nothing but the transport of Cooper pair from one end of the system to the other end. The basic of Cooper pair tunneling in an array of Cooper pair boxes can be understood from the analysis of Cooper pair transport in a single Cooper pair box with two terminal Squid. One of the terminal (say left) to transport a Cooper pair into the box and another to transport the same pair to the reservoir. This process generate the current. The electrostatic energy of the system can be expressed as

$$E = \sum_i E_{\xi(i)}(n_i - n_{g(i)})^2 + \sum_i E_{m(i,i+1)}(n_i - n_{g(i)})(n_{i+1} - n_{g(i+1)}) \tag{8}$$

Here $E_{\xi(i)}$ and $E_{m(i,i+1)}$ are respectively the charging energy and the electrostatic couplings between two islands. We are interested in the charge degeneracy point, i.e., when the gate charge is close to $1/2$, the lowest energy states are characterized by either zero or one Cooper pair on each island. With this assumption they have reduced the Hilbert space and map the system to a finite anisotropic Heisenberg spin-$1/2$ chain in an external magnetic field. In the spin language Cooper pair pumping is nothing but the transport of spin (Jordan-Wigner fermions) from one end of chain to the other end. They have defined the Hamiltonian.

$$H = -\frac{1}{2} B_x^1 \sigma_x^1 - \frac{1}{2} B_x^N \sigma_x^N - \sum_i \frac{1}{2} B_z^i \sigma_z^i + \sum_{i=1}^{N-1} [\Delta_{i,i+1} \sigma_z^i \sigma_z^{i+1} - J_{i,i+1}(\sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1})] \tag{9}$$

$\Delta_{i,i+1}$ is the constant electrostatic coupling amplitude. The tunable parameters of the system: $B_x^i$, the Josephson coupling of the leftmost and rightmost Squids. $B_z^i$ is the electrostatic potential of the island and $J_{i,i+1}$ is the Josephson coupling between the neighboring island. We are interested in the charge degeneracy point at this point the most favorable state of the system is the antiferromagnetic configuration (|010101,...> and |101010,...>). They start with one of the antiferromagnetic states and transfer the charge of every island to the right by two sites to achieve pumping. They have implemented it by applying identical pulse sequence to every second island. So we can write $B^1 = J_{1,2} = J_{4,5} = J_{e,o}$ and $B^N = J_{1,2} = J_{4,5} = J_{e,o}$ and $J_{e,o}$ and $J_{o,e}$ are respectively the even-odd and odd-even Josephson couplings. They have also considered a difference between the charging energies between the odd and even sites. In our theoretical analysis, we consider the Josephson couplings for even and odd sites as respectively $E_{J_1} = E_{J_2}(1 - \delta_1(t))$.
and \( E_{k} = E_{k}(1 + \delta_{1}(t)) \). The charging energies of even and odd sites are respectively \( B_{1} = B_{0}(1 - \delta_{2}(t)) \) and \( B_{2} = B_{0}(1 + \delta_{2}(t)) \). One can write the Hamiltonian in terms of spin operators:

\[
H = -\sum_{n} E_{kn}(1 - (-1)^{n}\delta_{1}(t))(S_{z}^{+}S_{z}^{-} + S_{z}^{-}S_{z}^{+}) + \sum_{n} \Delta S_{z}^{+}S_{z}^{-} + \frac{1}{2} \sum_{n} B_{0}(1 - (-1)^{n}\delta_{2}(t))S_{y}^{+}S_{y}^{-}
\]

One can express spin chain systems to a spinless fermion systems through the application of Jordan-Wigner transformation. In Jordan-Wigner transformation the relation between the spin and the electron creation and annihilation operators are

\[
S_{z}^{+} = \psi_{n}^{\dagger}\psi_{n} - \frac{1}{2}, \quad S_{z}^{-} = \psi_{n}^{\dagger}\exp[i\pi\sum_{j=-\infty}^{n-1} n_{j}], \quad S_{y}^{+} = \psi_{n}^{\dagger}\exp[-i\pi\sum_{j=-\infty}^{n-1} n_{j}]
\]

where \( n_{j} = \psi_{j}^{\dagger}\psi_{j} \) is the fermion number at site \( j \).

\[
H = -\frac{E_{10}}{2} \sum_{n} (1 - (-1)^{n}\delta_{1}(t))(\psi_{n+1}^{\dagger}\psi_{n} + \psi_{n}^{\dagger}\psi_{n+1}) + \Delta \sum_{n} (\psi_{n}^{\dagger}\psi_{n} - 1/2)(\psi_{n+1}^{\dagger}\psi_{n+1} - 1/2) - \frac{B_{0}}{2} \sum_{n} (1 - (-1)^{n}\delta_{2}(t))(\psi_{n}^{\dagger}\psi_{n} - 1/2).
\]

Our approach is completely analytical. Before we proceed further for continuum field theoretical study of these model Hamiltonians, we would like to explain the basic aspects of quantum Cooper pair pumping in terms of spin pumping physics of our model Hamiltonians: An adiabatic sliding motion of one dimensional potential, in gapped Fermi surface (insulating state), pumps an integer numbers of particle per cycle. In our case the transport of Jordan-Wigner fermions (spinless fermions) is nothing but the transport of spin from one end of the chain to the other end because the number operator of spinless fermions is related with the \( z \)-component of spin density \[18\]. We shall see that non-zero \( \delta(t) \) introduces the gap at around the Fermi point and the system is in the insulating state (Peierls insulator). In this phase spinless fermions form the bonding orbital between the neighboring sites, which yields a valance band in the momentum space. It is well known that the physical behavior of the system is identical at these two Fermi points. One can analyses this double degeneracy point, from the seminal paper of Berry \[13\]. It appears as source and sink vector fields defined in the generalized crystal momentum space. \( B_{n}(K) = \nabla_{K} \times A_{n}(K), \) and \( A_{n}(K) = \frac{1}{2\pi} < n(K)|\nabla_{K}|n(K)> \), where \( K = (k, \delta(t)) \). Here \( B_{n} \) and \( A_{n} \) are the fictitious magnetic field (flux) and vector potential of the \( n \)th Bloch band respectively. The degenerate points behave as a magnetic monopole in the generalized momentum space \[13\], whose magnetic unit can be shown to be 1, analytically \[12,13\]

\[
\int_{S_{1}} dS \cdot B_{\pm} = \pm 1
\]

positive and negative signs of the above equations are respectively for the conduction and valance band meet at the degeneracy points. \( S_{1} \) represent an arbitrary closed surface which enclose the degeneracy point. In the adiabatic process the parameter \( \delta(t) \) is changed along a loop (\( \Gamma \)) enclosing the origin (minima of the system). We define the expression for spin current \( I \) from the analysis of Berry phase. Then according to the original idea of quantum adiabiatic particle transport \[1,2,12,17\], the total number of spinless fermions \( (I) \) which are transported from one side of this system to the other is equal to the total flux of the valance band, which penetrates the 2D closed sphere \( (S_{2}) \) spanned by the \( \Gamma \) and Brillouin zone \[12\].

\[
I = \int_{S_{2}} dS \cdot B_{+1} = 1
\]

We have already understood that quantized spinless fermion transport is equivalent to the spin transport \[19\]. We will interpret this equation more physically after Eq. \[18\]. This quantization is topologically protected against the other perturbation as long as the gap along the loop remains finite \[12,17\]. Studies of spin pumping explain the stabilization of quantized spin pumping against \( z \)-component of exchange interactions.

We recast the spinless fermions operators in terms of field operators by this relation

\[
\psi(x) = [e^{ik_{F}x}\psi_{R}(x) + e^{-ik_{F}x}\psi_{L}(x)]
\]

where \( \psi_{R}(x) \) and \( \psi_{L}(x) \) describe the second-quantized fields of right- and left-moving fermions respectively. We want to express the fermionic fields in terms of bosonic field by this relation

\[
\psi_{r}(x) = \frac{U_{r}}{\sqrt{2\pi}} e^{-i(\rho(x) - \theta(x))}
\]

\( r \) is denoting the chirality of the fermionic fields, right (1) or left movers (-1). The operators \( U_{r} \) are operators that commute with the bosonic field. \( U_{r} \) of different species commute and \( U_{r} \) of the same species anticommute. \( \phi \) field corresponds to the quantum fluctuations (bosonic) of spin and \( \theta \) is the dual field of \( \phi \). They are related by this relation \( \phi_{R} = \theta - \phi \) and \( \phi_{L} = \theta + \phi \).

Using the standard machinery of continuum field theory \[18\], we finally get the bosonized Hamiltonians as

\[
H_{0} = v_{0} \int_{0}^{L} \frac{dx}{2\pi} \left\{ \pi^{2} : \Pi^{2} : + : [\partial_{x}\phi(x)]^{2} : \right\} + \frac{2\Delta}{\pi^{2}} \int dx : [\partial_{x}\phi_{L}(x)]^{2} : + : [\partial_{x}\phi_{R}(x)]^{2} : + \frac{4\Delta}{\pi^{2}} \int dx (\partial_{x}\phi_{L}(x))(\partial_{x}\phi_{R}(x))
\]
Hamiltonian is the gapless Tomonaga-Luttinger liquid part of the Hamiltonian with $v_0 = \sin k_F$.

After continuum field-theory the Hamiltonian become

$$H = H_0 + \frac{E_0 \delta_1(t)}{2\pi^2 \alpha^2} \int dx : \cos[2\sqrt{K}\phi(x)] :$$

$$+ \frac{B_0 \delta_1(t)}{2\pi \alpha} \int dx : \cos[2\sqrt{K}\phi(x)] :$$

$$+ \frac{\Delta}{2\pi^2 \alpha^2} \int dx : \cos[4\sqrt{K}\phi(x)] : - \frac{B_0}{2} \int dx \partial_x \phi(x)$$

The second term of the Hamiltonian for NN exchange interaction has originated from the XY interaction. This dimerization is the spontaneous dimerization, i.e., the infinitesimal variation of Josephson coupling in lattice sites, is sufficient to produce a gap around the Fermi points. The third term of the Hamiltonian arises due to the site dependent on-site charging energies modulated by the gate voltage. It yields the staggered phase of the system. The effect of applied gate voltage on the Cooper pair box appears as an effective magnetic field and also as a staggered magnetic in the spin representation of the model Hamiltonian. The system is in the mixed phase when both interactions (2nd and 3rd terms of the Hamiltonian) are in equal magnitude otherwise the system is in any one of the states of the mixed phase depending on the strength of the couplings. The last term can be absorbed in the Hamiltonian through the proper shifting of the wave function. So when $1/2 < K < 1$ only these (2nd and 3rd terms of the Hamiltonian) time dependent dimerizing field (third term of Eq. 17) is relevant and lock the phase operator at $\phi = 0 + \frac{n\pi}{\sqrt{K}}$. Now the locking potential slides adiabatically (here the cyclic magnetic flux and gate voltage fields that produces the dimerization). Speed of the sliding potential is low enough such that system stays in the same valley, i.e., there is no scope to jump onto the other valley. The system will acquire $2\pi$ phase during one complete cycle of dimerizing field. This is the basic mechanism of Cooper pair (spin pumping) of our system. The quantized Cooper pair transport of this scenario can be generalized up to the value of $\Delta$ for which $K$ is greater than $1/2$. In this limit, z-component of exchange interaction has no effect on the Cooper pair pumping physics of our system.

In summary, we have presented the theoretical explanation of adiabatic Cooper pair pumping of experimentally reliable stabilized charge pumping scheme of an array of Cooper pair boxes. The charge state of our system is definite.

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