NON-LINEAR EVOLUTION EQUATIONS DRIVEN BY ROUGH PATHS

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Abstract. We prove existence and uniqueness results for (mild) solutions to some non-linear parabolic evolution equations with a rough forcing term. Our method of proof relies on a careful exploitation of the interplay between the spatial and time regularity of the solution by capitalising some of Kato’s ideas in semigroup theory. Classical Young integration theory is then shown to provide a means of interpreting the equation. As an application we consider the three dimensional Navier-Stokes system with a stochastic forcing term arising from a fractional Brownian motion with $h > 1/2$.

1. Introduction

In this paper we study the initial-value problem of the following non-linear evolution equation of parabolic type

\begin{equation}
\frac{d}{dt} u + Au + \tilde{Q}(u) = F(u)\dot{w}
\end{equation}

in a separable Hilbert space $X$ with an initial value $u_0 \in X$, where $-A$ is the infinitesimal generator of an analytic semigroup $\{P_t\}_{t \geq 0}$, $\tilde{Q} : D(\tilde{Q}) \subset X \to X$ is a non-linear operator, and $F : X \to L_2(Z, X)$ satisfies a Lipschitz type condition which we specify later. The driving path for the perturbation $w = (w_t)_{t \geq 0}$ is an $\alpha$-Hölder continuous path in a separable Hilbert space $Z$, where $\alpha \in (\frac{1}{2}, 1]$. The class of paths includes sample paths of fractional Brownian Motion with Hurst parameter $h > \frac{1}{2}$.

Our motivation for studying equations of this form is the three dimensional Navier-Stokes system

\begin{align*}
\frac{\partial}{\partial t} u + u \cdot \nabla u &= \Delta u - \nabla p + F(u)\dot{w} \\
\nabla \cdot u &= 0
\end{align*}

in a bounded domain $\Omega$ with compact, smooth boundary $\Gamma$. The force term $F(u)\dot{w}$ is modeled by a fractional Brownian Motion in the Hilbert space $Z$, and $u_t$ describes the velocity of the fluid flow under the influence of a stochastic force $F(u)\dot{w}$ and subject to the no-slip condition. By projecting to the $L^2$-space of solenoidal vector fields on $\Omega$ via the Leray-Hopf projection $P_\infty$, the above equation can be written as the following evolution equation

\begin{equation}
\frac{d}{dt} u + Au + P_\infty(u, \nabla u) = P_\infty F(u)\dot{w}
\end{equation}

where $A = -P_\infty \circ \Delta$ is the Stokes operator, which is a self adjoint operator.

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If we let \( \{P_t\}_{t \geq 0} \) be the semigroup generated by \(-A, h(s) = P_t u(s)\) and we consider the formal calculation
\[
F'(s) = (AP_{t-s})u(s) + P_{t-s}u'(s) = (AP_{t-s})u(s) - (P_{t-s}A)u(s) - P_{t-s}\tilde{Q}(u(s)) + P_{t-s}(F(u(s))\dot{w}_s) = -P_{t-s}\tilde{Q}(u(s)) + P_{t-s}F(u(s))\dot{w}_s,
\]
then integrating from 0 to \( t \) one obtains
\[(1.3) \quad u(t) = P_t u_0 - \int_0^t P_{t-s}\tilde{Q}(u(s))ds + \int_0^t P_{t-s}F(u(s))\dot{w}_sds.\]
The first integral on the right-hand side is considered as the Bochner integral, the second one needs to be interpreted as some sort of stochastic integration. Hence the previous has to be written as
\[
u(t) = P_t u_0 - \int_0^t P_{t-s}\tilde{Q}(u(s))ds + \int_0^t P_{t-s}F(u(s))dw_s
\]
so that if \( u \) satisfies the above, we call it a mild solution to (1.1). For Navier-Stokes equations in the classical case that \( u \) is differentiable Kato made the following observation (see [4]) which proves significant. If \( t-s > 0 \) and \( x \in X \), then \( P_{t-s}x \) belongs to the domain \( D(A^\tau) \) for any real \( \tau \). On the other hand, if \( \tau > 0 \), \( A^{-\tau} \) is bounded, the integral involving the non-linear terms of (1.3) can be rewritten to enhance the regularity. For example
\[
\int_0^t P_{t-s}\tilde{Q}(u(s))ds = \int_0^t A^\tau P_{t-s}A^{-\tau}\tilde{Q}(u(s))ds.
\]
As long as \( A^{-\tau}\tilde{Q}(u(s)) \) is bounded on \([0,t]\), then, since \( \|A^\tau P_{t-s}\| \leq C(t-s)^{-\tau} \) which is still integrable on \([0,t]\) for any \( \tau < 1 \). Therefore, we may consider the non-linear operator \( Q = A^{-\tau}\tilde{Q} \) instead of \( \tilde{Q} \), where \( Q(x) = A^{-\tau}\tilde{Q}(x) \) for \( x \in D(Q) \), but in general the domain of definition \( D(Q) \) can be extended to be a little bit larger than \( D(\tilde{Q}) \) due to the fact that \( A^{-\tau} \) is a bounded linear operator. Essentially the same idea applies to the stochastic case, although the difficulty is in the stochastic term.

Now, according to Sobolevski [135], we have the estimate
\[(1.4) \quad \|A^{-\tau}P_{\infty}(w,\nabla u)\| \leq C\|\sqrt{Au}\|\|\sqrt{Aw}\|
\]
where \( \|\cdot\| \) is the \( L^2 \)-norm and \( A \) is the Stokes operator (with Dirichlet condition) in a bounded domain with smooth boundary \( \Gamma \). Hence as we are motivated by the stochastic version of this situation, we find fixed points in a certain space of the map
\[
Lu(t) = P_t u_0 - \int_0^t A^\tau P_{t-s}Q(u(s))ds + \int_0^t P_{t-s}F(u(s))dw_s
\]
where \( w \) is a fractional Brownian Motion.

Several existing papers study evolution equations driven by a fractional Brownian Motion (fBM): we summarize some of the main results here. In [4] the authors prove path-wise existence and uniqueness of mild solutions to equations of the form (1.1) where the perturbation operator \( \tilde{Q} : X \to X \) is taken to be continuous with linear growth. Additionally they assume the stochastic force term \( F \) satisfies \( F : X \to L(X) \) and the composition \( P_tF(\cdot) \) has linear growth and is Lipschitz with a constant proportional to \( t\gamma_1, t^{-\gamma_2} \) respectively for \( \gamma_1, \gamma_2 \in (0,1) \). The solutions are obtained from
fractional calculus methods for fBM with Hurst parameter \( h > \frac{1}{2} \) and are elements of the space of \( X \)-valued paths such that the norm
\[
\|u\| := \sup_{t \in [0,T]} \|u(t)\|_X + \sup_{t \in [0,T]} \int_0^t \|u(t) - u(s)\|_X \frac{ds}{(t-s)^{1+\alpha}}
\]
\( \alpha \in (1-h, \frac{1}{2}) \), is finite. Both of the cases \( h > \frac{1}{2} \) and \( h < \frac{1}{2} \) are treated in [16] where necessary and sufficient conditions are obtained for existence of solution \( s \) to stochastically forced linear evolution equations with linear noise, i.e. of the form
\[
\frac{d}{dt} u + Au = F \dot{w}
\]
where \( F \) does not depend on \( u \). Here the authors do not solve path-wise but instead use Skorohod type integration to produce a solution which is a square integrable (in the stochastic sense) \( X \)-valued process and then show space regularity of the solution as a continuous map into the domain of \( -A \).

More recently, in [5], the equation (1.5) with nonlinear \( F \) is studied (i.e. \( F : X \to L_2(Z, X) \) with some Lipschitz type conditions) and path-wise mild solutions are obtained using Young integration.

In this paper we treat a non-linear evolution equation forced by non-linear noise. The ideas used are straightforward while the difficulty lies in the delicate interplay between the space and time regularity of the paths involved.

The paper is organized as follows. In the second section, we set out the conditions on the operators involved and define the space in which we seek a mild solution. In section 3, we make some initial estimates which follow from our conditions and are used in the fixed point proof; in the fourth and fifth sections we discuss the Bochner and Young integral terms of our fixed point map. Subsequently we prove the existence of a mild solution and finally discuss the application to the Navier-Stokes equations.

2. Preliminaries

We would like to set up the technical conditions on various terms appearing in the previous evolution equation. The model example is randomly enforced Navier-Stokes equation in two or three dimensions, in which \( A = -P_\infty \circ \Delta \) (or \( A = -P_\infty \circ \Delta + I \) where \( I \) is the identity operator) is the Stokes operator on a bounded domain \( \Omega \) together with Dirichlet boundary conditions, and \( \tilde{Q}(u) = P_\infty (u, \nabla u) \). Therefore our assumptions will be motivated with the aim of applying the abstract setting to this case.

With this in mind, we assume \( A \) is a positive-definite, self-adjoint operator on \( X \), with positive spectral gap, so that the spectrum of \( A \) lies in the half line \([\lambda_0, \infty)\). Its domain is denoted by \( D(A) \).

For every real \( \epsilon \), \( A^\epsilon \) is again a self-adjoint operator, and \( A^\epsilon \) is bounded if \( \epsilon < 0 \). The domain \( D(A^\epsilon) \) is decreasing, so that \( D(A) \subset D(A^\epsilon) \) for \( \epsilon \in [0,1] \) and \( D(A^\epsilon) \) is a Hilbert space under the norm \( ||A^\epsilon x|| \).

Let \( P_t = e^{-tA} \) be the \( C_0 \)-semigroup of contractions on \( X \) generated by \( -A \). We will frequently use the following facts: for any \( t > 0 \) and \( x \in X \), \( P_t x \in D(A^\epsilon) \) for any \( \epsilon \leq 1 \) and \( ||A^\epsilon P_t|| \leq C \) for some constant \( C \) depending on \( \epsilon \).

For every \( t > 0 \) and \( x \in D(A^\epsilon) \) for some \( \epsilon \in (0,1] \), we have
\[
x - P_t x = \int_0^t A P_s x ds
= \int_0^t A^{1-\epsilon} P_s A^\epsilon x ds
\]
so that
\[ \| x - P tx \| \leq \| A' x \| \int_0^t \| A^{1-\epsilon} P_s \| ds \]
\[ \leq C(\epsilon) \| A' x \| \int_0^t s^{-(1-\epsilon)} ds \]
\[ = \frac{C(\epsilon)}{\epsilon} t^\epsilon \| A' x \| \]
(2.1)

an estimate which will be needed in order to derive useful a priori estimates. Now we are in a position to formulate the technical conditions on the non-linear term \( Q \) as follows.

**Condition 1.** Let \( \tau, \delta \in [0,1) \), \( \delta \leq \alpha \in (\frac{1}{2},1) \) be three parameters such that \( \delta + \tau < 1 \), \( \alpha + \tau < 1 \), \( \alpha + \delta > 1 \), and let \( K_j : [0,\infty) \to [0,\infty) \) \( (j = 0,1,2) \) be three increasing functions.

1. The domain of definition of \( Q, D(Q) = D(A^\delta) \).
2. \( Q \) is weakly Gateaux differentiable on \( D(A^\delta) \): if \( x \in D(A^\delta) \), then there is a linear operator \( DQ(x) : D(A^\delta) \to X \), such that \( \epsilon \to \langle Q(x + \epsilon \xi), z \rangle \) is differentiable for any \( \xi \in D(A^\delta) \), \( z \in X^* \),
\[ \langle DQ(x)\xi, z \rangle = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \langle H(x + \epsilon \xi), z \rangle \]
and \( \| DQ(x)\xi \| \leq K_1(\| A^\delta x \|) \| A^\delta \xi \| \). That is, \( DQ(x) \) is a bounded linear operator in \( D(A^\delta) \).

**Example 1.** If \( A = -P_\infty \Delta \) is the Stokes operator with Dirichlet boundary condition on a bounded domain with smooth boundary \( \Gamma \), and \( Q(u) = A^{-\frac{1}{4}}P_\infty(u, \nabla u) \), on the Hilbert space \( \mathcal{K}_2(\Omega) \), then we can choose \( \delta = \frac{1}{2}, \tau = \frac{1}{4} \).

Now let us set out the conditions on the non-linear operator \( F : X \to L_2(Z,X) \).

**Condition 2.** We assume the following for \( \epsilon = \max \{ \alpha + \delta, 2\alpha \} \).

1. For every \( x \in X, \xi \in Z \) we have \( F(x)\xi \in D(A^\epsilon) \) and \( A^{\epsilon} F \) is globally Lipschitz in the sense that
\[ \| A^{\epsilon} [F(x) - F(y)] \|_{L_2(Z,X)} \leq C \| x - y \|_X \forall x, y \in X. \]

2. The composition \( A^{\epsilon+\tau} F \) is globally relatively Lipschitz in the sense that
\[ \| A^{\epsilon+\tau} [F(x) - F(y)] \|_{L_2(Z,X)} \leq C \| A^\delta [x - y] \|_X \forall x, y \in D(A^\delta) \].

3. Additionally, we assume that \( A^{\epsilon} F : X \to L_2(Z,X) \) is weakly Gateaux differentiable, i.e. \( \tau \to \langle (A^{\epsilon} F(x + \tau \xi), \xi) \rangle \) is differentiable and there exists a linear operator \( DA^{\epsilon} F(x) : X \to L_2(Z,X) \) such that \( \frac{d}{d\tau} \langle (A^{\epsilon} F(x + \tau \xi), \xi) \rangle \bigg|_{\tau = 0} = \langle (DA^{\epsilon} F(x) y, \xi) \rangle \) for all \( \xi \in L_2(Z,X)' \), and satisfies the bounds
\[ \sup_x \| DA^{\epsilon} F(x) \|_{L(X,L_2(Z,X))} < \infty \]
and
\[ \| DA^{\epsilon} F(x) - DA^{\epsilon} F(y) \|_{L(X,L_2(Z,X))} \leq C \| x - y \|. \]

Throughout this paper, the non-negative constants \( C_j \) may be different from place to place, and \( C_j \) may depend on only the parameters \( \tau, \delta, \alpha \) and the operator \( A \), but are independent of \( u, w \), and \( T \).
3. Initial Estimates

The main goal of this section is to derive technical estimates which are used to show the existence and uniqueness of solutions to the evolution equation \(\text{(1.1)}\). Let us define \(\mathbb{H}_T\) to be the collection of all \(\alpha\)-Hölder continuous paths \(u = (u_t)\) in \(X\) such that for each \(t \in [0, T]\), \(u_t \in D(A^\beta)\) and \(t \rightarrow A^\beta u_t\), \((s, t) \rightarrow \frac{u_t - u_s}{|t-s|^\alpha}\) are bounded. If \(u \in \mathbb{H}_T\) we define the norm

\[
||u||_{\mathbb{H}_T} = \sup_{t \in [0, T]} ||A^\beta u_t|| + \sup_{s, t \in [0, T], s \neq t} \frac{||u_t - u_s||}{|t-s|^\alpha}.
\]

By definition, if \(||u||_{\mathbb{H}_T} \leq \beta\) then

\[
||A^\beta u_t|| \leq \beta, \quad ||u_t - u_s|| \leq \beta |t-s|^\alpha
\]

for any \(s, t \in [0, T]\). And since \(||x|| \leq C_\delta ||A^\delta x||\) we also have \(||u_t|| \leq C_\delta \beta\). It is also obvious that \((\mathbb{H}_T, || \cdot ||_{\mathbb{H}_T})\) is a Banach space.

**Lemma 1.** \(Q\) is locally Lipschitz continuous with respect to \(A^\delta:\)

\[
||Q(x) - Q(y)|| \leq K_1(||A^\delta x|| + ||A^\delta y||)||A^\delta(x - y)||.
\]

**Proof.** Let \(\theta(s) = sx + (1-s)y\). Then \(\theta \in C^1([0, 1], X), \theta'(s) \in D(A^\delta)\) and \(||A^\delta \theta(s)|| \leq ||A^\delta x|| + ||A^\delta y||\). For every \(\xi \in X^*\)

\[
\langle Q(x) - Q(y), \xi \rangle = \int_0^1 \frac{d}{ds} \langle Q(\theta(s)), \xi \rangle = \int_0^1 \langle DQ(\theta(s))\theta'(s), \xi \rangle
\]

which yields that

\[
||\langle Q(x) - Q(y), \xi \rangle|| \leq \int_0^1 ||DQ(\theta(s))\theta'(s)|| ||\xi|| ds
\]

so \(3.2\) follows. \(\square\)

We turn to the operator \(F\).

**Lemma 2.** If \(F\) satisfies Condition \(2\) then we have for every \(x, y \in X\) and \(\xi \in Z\)

\[
||A^\tau [F(x) - F(y)] \xi||_X \leq C ||x - y||_X ||\xi||_Z
\]

and for all \(x \in D(A^\delta), \xi \in Z\)

\[
||A^{\delta + \tau} F(x) \xi||_X \leq C_1 ||A^\delta x|| ||\xi||_Z + C_1 ||\xi||_Z.
\]

**Proof.** Since the operator norm is bounded above by the Hilbert-Schmidt norm it follows immediately from \(2\) that

\[
||A^\tau [F(x) - F(y)] \xi||_X \leq C ||A^\tau [F(x) - F(y)]||_{L^2(Z, X)} ||\xi||_Z.
\]
Similarly, using (2.3) we have for any $y \in X$

$$\|A^{4+\varepsilon}F(x)\xi\|_X \leq \|A^{4+\varepsilon}[F(x) - F(y)]\xi\|_X + \|A^{4+\varepsilon}F(y)\xi\|_X \leq C\|A^4x\|_X \|\xi\| + \|A^{4+\varepsilon}F(y)\|_{L_2(Z,X)} \|\xi\| \leq C_1\|A^4x\|_X \|\xi\| + C_2.$$

□

**Lemma 3.** Suppose that $F : X \rightarrow L_2(Z,X)$ satisfies Condition (2.3). If $u,v \in H_T$ we have for $0 \leq s < t \leq T$

$$||[A^\varepsilon F(u) - A^\varepsilon F(u_s)] - [A^\varepsilon F(v) - A^\varepsilon F(v_s)]||_{L_2(Z,X)} \leq C \left( \sup_{x \in X} \|DA^\varepsilon F(x)\| + \frac{3}{2} \|v\|_{H_T} \right) \|u - v\|_{H_T} (t - s)^\alpha,$$

which implies, in particular, that $A^\varepsilon P_{t-}[F(u) - A^\varepsilon F(v)] : [0,t] \rightarrow L_2(Z,X)$ is $\alpha$–Hölder continuous with

$$||A^\varepsilon P_{t-}[F(u) - A^\varepsilon F(v)]||_{\alpha-Hölder}[0,T] \leq \left( \sup_{x \in X} \|DA^\varepsilon F(x)\| + \frac{3}{2} \|v\|_{H_T} \right) \|u - v\|_{H_T}.$$

**Remark 1.** Here as elsewhere we denote the $\alpha$–Hölder norm of a path $u : [s,t] \rightarrow W$ in a Banach space $(W,\|\cdot\|_W)$ by

$$\|u\|_{\alpha-Hölder}[s,t] = \sup_{t_1 < t_2} \frac{\|u(t_2) - u(t_1)\|_W}{(t_2 - t_1)^\alpha}.$$

**Proof.** Let $\theta^u_\tau = \tau u_t + (1 - \tau) u_s$, $\theta^v_\tau = \tau v_t + (1 - \tau) v_s$ and $\xi \in L_2(Z,X)'$ then we have

$$\langle \langle [A^\varepsilon F(u) - A^\varepsilon F(u_s)] - [A^\varepsilon F(v) - A^\varepsilon F(v_s)], \xi \rangle \rangle = \int_0^1 \frac{d}{d\tau} \langle \langle A^\varepsilon F(\theta^u_\tau), \xi \rangle \rangle d\tau - \int_0^1 \frac{d}{d\tau} \langle \langle A^\varepsilon F(\theta^v_\tau), \xi \rangle \rangle d\tau$$

$$= \int_0^1 \left\langle \left\langle DA^\varepsilon F(\theta^u_\tau) \left( \frac{d}{d\tau} \theta^u_\tau - \theta^v_\tau \right), \xi \right\rangle \right\rangle d\tau$$

$$+ \int_0^1 \left\langle \left\langle DA^\varepsilon F(\theta^u_\tau) - DA^\varepsilon F(\theta^v_\tau), \frac{d}{d\tau} \theta^v_\tau, \xi \right\rangle \right\rangle d\tau.$$
This implies
\[
\|\langle [A^\varepsilon F(u_t) - A^\varepsilon F(u_s)] - [A^\varepsilon F(u_t) - A^\varepsilon F(v_s)], \xi \rangle \|
\leq \int_0^1 \left\| DA^\varepsilon F(\theta^\varepsilon_r) \left( \frac{d}{dr} [\theta^\varepsilon_r - \theta^\varepsilon_s] \right) \right\| \|\xi\| \, dr
\]
\[
+ \int_0^1 \left\| [DA^\varepsilon F(\theta^\varepsilon_r) - DA^\varepsilon F(\theta^\varepsilon_s)] \left( \frac{d}{dr} \theta^\varepsilon_r \right) \right\| \|\xi\| \, dr
\]
\leq \int_0^1 \sup_{x \in X} \|DA^\varepsilon F(x)\| \|u_t - u_s\| - [v_t - v_s]\| \|\xi\| \, dr
\]
\[
+ \int_0^1 \|\theta^\varepsilon_s - \theta^\varepsilon_r\| \|v_t - v_s\| \|\xi\| \, dr
\]
\[
= \int_0^1 \sup_{x \in X} \|DA^\varepsilon F(x)\| \|u_t - u_s\| - [v_t - v_s]\| \|\xi\| \, dr
\]
\[
+ \int_0^1 \|\theta^\varepsilon_s - \theta^\varepsilon_r\| \|v_t - v_s\| \|\xi\| \, dr
\]
\leq \sup_{x \in X} \|DA^\varepsilon F(x)\| \|u_t - u_s\| - [v_t - v_s]\| \|\xi\| \, dr
\]
\[
+ \int_0^1 \|\theta^\varepsilon_s - \theta^\varepsilon_r\| \|v_t - v_s\| \|\xi\| \, dr
\]
\leq \left( \sup_{x \in X} \|DA^\varepsilon F(x)\| + \frac{3}{2} \|v_t - v_s\| \|\xi\| \, dr \right) \|u_t - u_s\| \|\xi\| \|\xi\| \]
and the result follows. \qed

4. The Bochner Integral

Proposition 1. Let \( u \in \mathbb{H}_T \) such that \( \sup_{t \in (0,T]} \| A^\varepsilon u_t \| < \infty \). Let \( \varepsilon \in [0,1) \). Then
(1) For any \( t \in (0,T] \), \( s \rightarrow A^\varepsilon P_{t-s} Q(u(s)) \) is continuous on \((0,t)\).
(2) For every \( s < t \in (0,T] \), \( \int_s^t A^\varepsilon P_{t-r} Q(u(r)) \, dr \) exists and

\[
\int_s^t A^\varepsilon P_{t-r} Q(u(r)) \, dr \leq CK_2 \left( \sup_{t \in (0,T]} \| A^\varepsilon u_t \| \right) (t-s)^{1-\varepsilon}.
\]

Proof. Let \( t \in (0,T] \) and consider \( f(s) = A^\varepsilon P_{t-s} Q(u(s)) \). Then for \( s_1 \in (0,t) \), \( s_1 > s_2 \), one has
\[
\|f(s_1) - f(s_2)\| \leq \|A^\varepsilon P_{t-s_1} (Q(u(s_1) - Q(u(s_2)))\|
\leq C(t-s_1)^{-\varepsilon} \|Q(u(s_1) - Q(u(s_2))\|
\leq C(t-s_1)^{-\varepsilon} (I - P_{s_1-s_2}) \|Q(u(s_2))\|
\leq C(t-s_1)^{-\varepsilon} K_1 \left( \sup_{t \in (0,T]} \| A^\varepsilon u_t \| \right) \|A^\varepsilon (u(s_1) - u(s_2))\|
\leq C(t-s_1)^{-\varepsilon} K_1 \left( \sup_{t \in (0,T]} \| A^\varepsilon u_t \| \right) \|A^\varepsilon (u(s_1) - u(s_2))\|
\leq C(t-s_1)^{-\varepsilon} K_1 \left( \sup_{t \in (0,T]} \| A^\varepsilon u_t \| \right) \|A^\varepsilon (u(s_1) - u(s_2))\|
\]
Since \( A^\delta u \in C((0, T], X) \) so is \( u \) and letting \( s_1 \to s_2 \in (0, t) \) the continuity of the semigroup and of \( u \) give \( \lim_{s_1 \to s_2} ||f(s_1) - f(s_2)|| = 0 \), i.e. \( f \) is continuous on \((0, t)\). Moreover,

\[
||f(s)|| \leq C(t - s)^{-\varepsilon}||Qu(s)|| \\
\leq C(t - s)^{-\varepsilon}K_2(||A^\delta u(s)||) \\
\leq CK_2 \left( \sup_{t \in (0, T]} ||A^\delta u_t|| \right) (t - s)^{-\varepsilon},
\]

from which (4.1) follows immediately.

For \( u \in \mathbb{H}_T \) we define

(4.2) \[ Lu(t) = P_t u_0 - \int_0^t A^\tau P_{t-s} Q(u(s)) ds + \int_0^t P_{t-s} F(u(s)) dw_s \]

where \( w = (w_t) \) is an \( \alpha \)-Hölder continuous path in \( Z \), where \( \alpha \in (\frac{1}{2}, 1] \) such that \( \alpha + \tau < 1 \), so that \( 2\alpha > 1 \). With these constraints, and recalling that \( \delta \leq \alpha \) we see that from the previous proposition that

\[
\sup_{t \in [0, T]} ||A^\delta T^t || \leq CK_2(\beta) T^{1-\delta - \tau}
\]

and

\[
|| \int_0^{t_2} A^\tau P_{t_2-s} Q(u(s)) ds - \int_0^{t_1} A^\tau P_{t_1-s} Q(u(s)) ds || \\
= \left| \left| \int_0^{t_1} (P_{t_2-t_1} - I) A^\tau P_{t_1-s} Q(u(s)) ds + \int_{t_1}^{t_2} A^\tau P_{t_2-s} Q(u(s)) ds \right| \right| \\
\leq C_1 (t_2 - t_1)^\alpha K_2 \left( \sup_{t \in [0, T]} A^\delta u_t \right) t_1^{1-(\alpha + \tau)} + \sup_{t \in [0, T]} A^\delta u_t \right) (t_2 - t_1)^{1-\tau} \\
\leq CT^{1-(\alpha + \tau)} K_2(\beta) (t_2 - t_1)^\alpha,
\]

so that the Bochner integral appearing in (4.2) has the right properties to be in \( \mathbb{H}_T \). We now turn our attention to the the properties of the Young integral featuring in (4.2).

5. The Young Integral

We will use the Young integral to make a rigorous path-wise definition of the stochastic integral above provided the process has sample paths of the correct regularity. The example to keep in mind is fractional Brownian motion. The following is due to L.C. Young from the 1930s when he made the definition following investigations into the convergence of Fourier series (see [17]). It is an extension of Stieltjes integration to the case of paths of finite \( p \) variation for \( p > 1 \).

Definition 1. A continuous path \( f : [0, T] \to X \) where \( X \) is a Banach space is said to have finite \( p \) variation on \([0, T]\) if

\[
||f||_{p, [0, T]} := \left[ \sup_D \sum_k \left| \left| f_{t_k} - f_{t_{k-1}} \right| \right|^p_X \right]^{1/p} < \infty
\]

where by \( \sup_D \) we understand supremum over all partitions of \([0, T]\).
In analogy with the Riemann sums for Stieltjes integrals of paths of finite variation, we have the following general definition of the integral in the case of real valued functions \( f \) and \( w \) defined on the interval \([0, T] \).

**Remark 2.** It is obvious that if a path \( f \) is Hölder continuous with exponent \( h \in (0, 1) \) on the interval \([s, t] \) then \( f \) has finite \( \frac{1}{h} \) variation and

\[
\|f\|_{\frac{1}{h}, [s, t]} \equiv \left( \sup_{D([s, t])} \sum_{k} \|f_{t_k} - f_{t_{k-1}}\|^h \right)^{\frac{1}{h}} \leq \|f\|_{H^h, [s, t]} (t-s)^h
\]

We will use this fact several times in later proofs.

**Definition 2.** The Stieltjes integral

\[
\int_0^T f(t) \, dw_t
\]

is said to exist in the Riemann sense with value \( I \) provided that for all \( \delta > 0 \) there exists a \( \varepsilon_\delta > 0 \) such that if the partition \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) satisfies \( |t_i - t_{i-1}| < \delta \) for all \( i \), then

\[
\left| \sum_{k=0}^{N-1} f(t_k) (w_{t_{k+1}} - w_{t_k}) - I \right| < \varepsilon_\delta
\]

with \( \varepsilon_\delta \to 0 \) as \( \delta \to 0 \).

**Theorem 1.** If the real valued functions \( f \) and \( w \) have finite \( p \) and \( q \) variation respectively such that \( p, q > 0 \) and \( \frac{1}{p} + \frac{1}{q} > 1 \), then the Stieltjes integral

\[
\int_0^T f(t) \, dw_t
\]

exists in the Riemann sense.

In fact the definition of the integral and Young’s theorem can be extended to the infinite dimensional case when \( w : [0, T] \to Z \) and \( f : [0, T] \to L(\mathbb{Z}, X) \) are bounded paths with finite \( p \) and \( q \) variation respectively. In this case, the integral \( \int_0^T f(s) \, dw_s : [0, T] \to X \) is a bounded path of finite \( q \) variation and we have the estimate

\[
\left\| \int_0^T (f(s) - f(0)) \, dw_s \right\|_{p, [0, T]} \leq C \|f\|_{p, [0, T]} \|w\|_{p, [0, T]}
\]

known as Young’s inequality. The details can be found in [1]. Using this, we can in a path-wise way define the integral \( \int_0^T A^q P_t \, F(u(s)) \, dw_s \) for processes \( w \) with sufficiently regular sample paths.

**Lemma 4.** Let \( u \in \mathbb{H}_T, w : [0, T] \to Z \) be \( \alpha \)-Hölder continuous, \( \epsilon \in [0, \alpha] \) and \( \delta + \alpha > 1 \). If \( F : X \to L_2(Z, X) \) satisfies conditions (2.2) and (2.3) then

\[
\left\| \sum_{r \in (0, T]} \|A^q P_t \, F(u_r)\|_{L_2(Z, X)} \right\|_{L_2(\mathbb{Z}, X)} \leq C \left( \sup_{r \in (0, T]} \|A^\epsilon u_r\| + T^{\alpha - \delta} + 1 \right) |s - r|^\delta
\]

for all \( r, s \in [0, t], t \in [0, T] \). In particular, \( A^q P_t \, F(u.) : [0, t] \to L_2(Z, X) \) is \( \delta \)-Hölder continuous and hence the Young integral

\[
\int_0^t A^q P_t \, F(u_s) \, dw_s
\]
exists and satisfies
\begin{equation}
\left| \int_0^t A^r P_{t-s} F(u_s) \, dw_s - A^r P_t F(u_0) (w_t - w_0) \right| \leq C \left( \sup_{r \in (0, T)} \| A^\delta u_r \| + T^{\alpha-\delta} + 1 \right).
\end{equation}

Proof. For $0 \leq r < s \leq t$ let us observe by using (2.2) and (3.4)
\begin{align*}
\| (A^r P_{t-s} F(u_s) - A^r P_{t-r} F(u_r)) \|_{L^2(Z,X)} &\leq C \left( \| (I - P_{s-r}) A^r F(u_r) \|_{L^2(Z,X)} + \| A^r [F(u_s) - F(u_r)] \|_{L^2(Z,X)} \right) \\
&\leq \frac{C(\delta)}{\delta} (s-r)^\delta \| A^{r+\delta} F(u_s) \|_{L^2(Z,X)} + C \| u_s - u_r \|
\end{align*}
From which it follows that
\begin{equation*}
\| (A^r P_{t-s} F(u_s) - A^r P_{t-r} F(u_r)) \| \leq C \left( \sup_{r \in (0, T)} \| A^\delta u_r \| + T^{\alpha-\delta} + 1 \right) (s-r)^\delta.
\end{equation*}
Estimate (5.1) follows at once. From (1) this is sufficient to guarantee the existence of the Young integral and (5.2) follows from Young’s inequality.

6. THE NON-LINEAR MAPPING $L$

We now prove that the non-linear mapping $L$ is a (non-linear) bounded operator. More precisely:

**Theorem 2.** If $u \in \mathbb{H}_T$ and $u_0 \in D(A^\alpha)$, then $Lu \in \mathbb{H}_T$ and the following estimates hold: if $\| u \|_{\mathbb{H}_T} \leq \beta$
\begin{align*}
\| Lu \|_{\mathbb{H}_T} \leq C_1 (1 + T^\alpha) \| A^\alpha u_0 \| + C_2 \left( T^{1-\delta-r} + T^{1-r-\alpha} \right) K_2(\beta) \\
+ C_3 [\beta + 1] \left( T^\alpha + T^\delta + T^{\alpha+\delta} + T^{2\alpha} \right) + C_4 \left( 1 + T^\alpha \right) \| A^\alpha F(u_0) \|_{op}.
\end{align*}

Proof. There are three terms appearing in the definition of $L$, namely $P_t u_0$, the ordinary integral
\begin{equation*}
J_t = \int_0^t A^r P_{t-s} Q(u_s) \, ds
\end{equation*}
and the Young integral
\begin{equation*}
U_t = \int_0^t P_{t-s} F(u_s) \, dw_s.
\end{equation*}
Let us estimate their $\mathbb{H}_T$ norms one by one. Firstly,
\begin{equation*}
\| A^\delta P_t u_0 \| \leq \| A^\delta u_0 \| \leq \beta
\end{equation*}
and
\begin{align*}
\| P_t u_0 - P_s u_0 \| &= \| (P_{t-s} - I) P_s u_0 \| \\
&\leq C_\alpha (t-s)^\alpha \| A^\alpha P_s u_0 \| \\
&\leq C_\alpha \| A^\alpha u_0 \| (t-s)^\alpha.
\end{align*}
Secondly we consider the ordinary integral $J_t$. It is elementary that

$$||A^δ J_t|| \leq C_1 \int_0^t (t-s)^{-(\delta+\tau)}||Q(u_s)||ds$$

$$\leq C_1 \int_0^t (t-s)^{-(\delta+\tau)}K_2(||A^δ u_s||)ds$$

$$\leq C_2K_2(\beta)s^{1-\delta-\tau}.$$

To estimate the Hölder norm we use the following elementary formula

$$J_t - J_s = \int_s^t A^\tau P_{t-r}Q(u_r)dr + (P_{t-s} - I) \int_s^t A^\tau P_{s-r}Q(u_r)dr.$$

While it is easy to see that

$$\left|\int_s^t A^\tau P_{t-r}Q(u_r)dr\right| \leq \frac{C_4K_2(\beta)}{1-\tau}(t-s)^{1-\tau}$$

$$\leq \frac{C_4K_2(\beta)}{1-\tau}(t-s)^{\alpha}(t-s)^{1-\tau-\alpha}$$

$$\leq \frac{C_4K_2(\beta)}{1-\tau}T^{1-\tau-\alpha}(t-s)^{\alpha}.$$

Using the bound

$$||x - P_t x|| \leq \frac{C(\delta)}{\delta}t^\delta ||A^\delta x||$$

we deduce that

$$\left|\int_0^s A^\tau P_{t-s}Q(u_r)dr\right| \leq \frac{C(\alpha)}{\alpha}A^\alpha \int_0^s A^\tau P_{t-r}Q(u_r)dr$$

$$\leq \frac{C(\alpha)}{\alpha(1-\tau-\alpha)}K_2(\beta)s^{1-\tau-\alpha}(t-s)^{\alpha}$$

$$\leq \frac{C(\alpha)}{\alpha(1-\tau-\alpha)}K_2(\beta)T^{1-\tau-\alpha}(t-s)^{\alpha}.$$

Finally we handle the Young integral. Let $[s, t] \subset [0, T]$ then

$$U_t - U_s = \int_0^t P_{t-r}F(u_r)dr_t - \int_0^s P_{s-r}F(u_r)dr_r$$

$$= \int_s^t P_{t-r}F(u_r)dr_r + (P_{t-s} - I) \int_s^t P_{s-r}F(u_r)dr_r,$$

so that

$$||U_t - U_s|| \leq \left|\int_s^t P_{t-r}F(u_r)dr_r\right| + \left|\int_0^s (P_{t-s} - I) \int_0^s P_{s-r}F(u_r)dr_r\right|$$

$$\leq \left|\int_s^t P_{t-r}F(u_r)dr_r\right| + \frac{C(\alpha)}{\alpha} (t-s)^{\alpha} \left|\int_0^s A^\alpha P_{s-r}F(u_r)dr_r\right|$$

To handle the two integrals, let us consider

$$R_{s,t}^\delta = A^\delta \int_s^t P_{t-r}F(u_r)dr_r.$$
for \( \varepsilon \in [0, \alpha] \). From Young’s inequality we have that

\[
\|R_{s,t}^\varepsilon\| = \left\| \int_s^t A^\varepsilon P_{t-r} F(u_r) dr \right\| \\
\leq C \|A^\varepsilon P_{t-s} F(u_s)\|_{\delta-HöL} \|u_s\|_{\alpha-HöL} (t-s)^{\delta+\alpha} + \|A^\varepsilon P_{t-s} F(u_s) (w_t - w_s)\| \\
\leq C (\beta + 1 + T^{\alpha-\delta}) (t-s)^{\delta+\alpha} + \|A^\varepsilon P_{t-s} (F(u_s) - F(u_0)) (w_t - w_s)\| \\
+ \|A^\varepsilon P_{t-s} F(u_0) (w_t - w_s)\| 
\]

Hence, using condition 2 and lemma 2 we can deduce

\[
(6.2) \quad \|R_{s,t}^\varepsilon\| \leq C (\beta + 1 + T^{\alpha-\delta}) (t-s)^{\delta+\alpha} + C \beta s^\alpha (t-s)^\alpha + C \|A^\alpha F(u_0)\|_{op} (t-s)^\alpha. 
\]

An application of this with \( \varepsilon = \delta \) yields

\[
\|A^\delta U_t\| \leq \left\| \int_0^t A^\delta P_{r-s} F(u_r) dr \right\| \\
\leq C (\beta + 1 + T^{\alpha-\delta}) T^{\delta+\alpha} + C \beta T^{2\alpha} + C \|A^\alpha F(u_0)\|_{op} T^\alpha. 
\]

Then, two further applications of (6.2) with \( \varepsilon = 0 \) and \( \varepsilon = \alpha \) show

\[
(6.3) \quad \left\| \int_s^t P_{t-r} F(u_r) dr \right\| \leq C (\beta + 1 + T^{\alpha-\delta}) (t-s)^{\delta+\alpha} + C \beta s^\alpha (t-s)^\alpha + C \|A^\alpha F(u_0)\|_{op} (t-s)^\alpha \\
(6.4) \quad \left\| \int_0^s A^\alpha P_{s-r} F(u_r) dr \right\| \leq C (\beta + 1 + T^{\alpha-\delta}) T^{\delta+\alpha} + C \beta T^{2\alpha} + C \|A^\alpha F(u_0)\|_{op} T^\alpha. 
\]

Assembling (6.1) and (6.3) provides

\[
\|U_t - U_s\| \leq C (\beta + 1 + T^{\alpha-\delta}) (T^\delta + T^{\alpha+\delta}) + C \beta T^\alpha (1 + T^\alpha) + C \|A^\alpha F(u_0)\|_{op} (1 + T^\alpha). 
\]

The result then follows by collecting together the appropriate terms. \( \square \)

Next we prove that \( L \) is locally Lipschitz.

**Theorem 3.** Let \( u, v \in \mathbb{H}_T^\beta \) such that \( \|u\|_{\mathbb{H}_T^\beta} \leq \beta, \|v\|_{\mathbb{H}_T^\beta} \leq \beta, u_0 = v_0 \in D(A^\alpha) \), then

\[
\|L u - L v\|_{\mathbb{H}_T^\beta} \leq C \left\{ (T^\delta + T^{\alpha-\delta}) K_1 (2\beta) + (T^\alpha + T^{2\alpha}) \left( \sup_{x \in X} |DA^\alpha F(x)| + \frac{3}{2} \beta + 1 \right) \right\} \|u - v\|_{\mathbb{H}_T^\beta}. 
\]

**Proof.** Let \( u, v \in \mathbb{H}_T^\beta \) such that \( \|u\|_{\mathbb{H}_T^\beta} \leq \beta \) and \( \|v\|_{\mathbb{H}_T^\beta} \leq \beta \). Let \( \theta_t = u_t - v_t \) and consider

\[
D_t = \int_0^t A^\alpha P_{t-r} (Q(u_r) - Q(v_r)) dr. 
\]

If \( t > s \) then

\[
D_t - D_s = \int_s^t A^\alpha P_{t-r} [Q(u_r) - Q(v_r)] dr \\
+ (P_{t-s} - I) \int_s^t A^\alpha P_{t-r} [Q(u_r) - Q(v_r)] dr. 
\]
It follows that
\[
||D_t - D_s|| \leq C \int_s^t (t-r)^{-\tau} ||Q(u_r) - Q(v_r)|| \, dr \\
+ C |t-s|^\alpha \int_0^s (s-r)^{-\tau-\alpha} ||Q(u_r) - Q(v_r)|| \, dr
\]
together with the estimate
(6.5)
\[
||Q(u_r) - Q(v_r)|| \leq K_1(2\beta)||A^\delta \theta_r||
\]
we obtain
\[
||D_t - D_s|| \leq CK_1(2\beta) \int_s^t (t-r)^{-\tau} ||A^\delta \theta_r|| \, dr \\
+ CK_1(2\beta) |t-s|^\alpha \int_0^s (s-r)^{-\tau-\alpha} ||A^\delta \theta_r|| \, dr \\
\leq \frac{C}{1-\tau} K_1(2\beta) T^{1-\tau-\alpha} |t-s|^\alpha ||\theta||_{H^1_T} \\
+ \frac{C}{1-\tau-\alpha} K_1(2\beta) T^{1-\tau-\alpha} ||\theta||_{H^1_T} |t-s|^\alpha.
\]
Using the same argument we can obtain for all \( t \in [0, T] \)
\[
||A^\delta D_t|| \leq \frac{T^{1-\delta-\tau}}{1-\delta-\tau} K_1(2\beta) ||\theta||_{H^1_T}.
\]
We now define
\[
\Delta_t \equiv \int_0^t P_{t-r} [F(u_r) - F(v_r)] \, dw_r
\]
and consider
\[
\Delta_t - \Delta_s = \int_s^t P_{t-r} [F(u_r) - F(v_r)] \, dw_r \\
+ (P_{t-s} - I) \int_0^s P_{s-r} [F(u_r) - F(v_r)] \, dw_r,
\]
which satisfies
\[
||\Delta_t - \Delta_s|| \leq \left| \left| \int_s^t P_{t-r} [F(u_r) - F(v_r)] \, dw_r \right| \right| \\
+ C (t-s)^\alpha \left| \left| \int_0^s A^\alpha P_{s-r} [F(u_r) - F(v_r)] \, dw_r \right| \right|.
\]
As before, we consider for \( \epsilon \in [0, \alpha] \)
\[
R_{s,t}^\epsilon \equiv \int_s^t A^\epsilon P_{t-r} [F(u_r) - F(v_r)] \, dw_r,
\]
and note that Young’s inequality gives
\begin{equation}
\|R_{s,t}\| \leq C \|A^\alpha P_{t-s} [F(u_s) - F(v_s)]\|_{\alpha-H^1} \|u_s|_{\alpha-H^1} (t-s)^{2\alpha}
+ \|A^\alpha P_{t-s} (F(u_s) - F(v_s)) \|_{\alpha-H^1} \|u_s - v_s\|\)
\leq C \|A^\alpha P_{t-s} [F(u_s) - F(v_s)]\|_{\alpha-H^1} \|u_s|_{\alpha-H^1} (t-s)^{2\alpha}
+ C \| (u_s - u_0) - (v_s - v_0) \| (t-s)^\alpha.
\end{equation}
(6.6)

From lemma we have
\begin{equation}
\|A^\alpha [F(u_s) - F(v_s)] - A^\alpha [F(u_t) - F(v_t)]\|_{L^2(Z,X)}
\leq \left( \sup_x ||DA^\alpha F(x)|| + \frac{3}{2} ||v||_{H^T} \right) ||\theta||_{H^T} (t-s)\alpha.
\end{equation}
(6.7)

Combining (6.6) and (6.7) we can deduce
\begin{equation}
\frac{||\Delta_t - \Delta_s||}{(t-s)\alpha} \leq C \left( T^\alpha + T^{2\alpha} \right) \left( \sup_x ||DA^\alpha\Delta_t|| + \|v\|_{H^T} \right) \||\theta||_{H^T},
\end{equation}
and hence
\begin{equation}
\|Lu - Lv\|_{H^T} \leq \sup_{t \in [0,T]} \|A^\delta D_t\| + \sup_{t \in [0,T]} \|A^\delta \Delta_t\|
+ \sup_{0 < s < t \leq T} \frac{||D_t - D_s||}{(t-s)^\alpha} + \sup_{0 < s < t \leq T} \frac{||\Delta_t - \Delta_s||}{(t-s)^\alpha}
\leq C \left( T^{1-\delta - \tau} + T^{1-\alpha - \tau} \right) K_1 (2\beta) \||\theta||_{H^T}
+ C \left( T^\alpha + T^{2\alpha} \right) \left( \sup_x ||DA^\alpha F(x)|| + \frac{3}{2} ||\theta||_{H^T}
\end{equation}

\textbf{Theorem 4.} Let $u_0 \in D(A^\alpha)$ then for some $T^* > 0$ depending only on $||A^\alpha u_0||$ and $||A^\alpha F(u_0)||_{op}$ there is a unique $u \in H^T$ such that
\begin{equation}
u_t = P_t u_0 - \int_0^t A^\alpha P_{t-s} Q(u_s)ds + \int_0^t P_{t-s} F(u_s)dw_s.
\end{equation}

\textbf{Proof.} By choosing $\beta = \beta \left( ||A^\alpha u_0||, ||A^\alpha F(u_0)||_{op} \right)$ sufficiently large Theorem 2 allows us to ensure that
\begin{equation}L (H^T \cap \{ u : ||u||_{H^T} \leq \beta \}) \subseteq H^T \cap \{ u : ||u||_{H^T} \leq \beta \} \text{ for all } T \in [0,T_1] \text{ and some } T_1 > 0.
\end{equation}
From Theorem 3 we see that $L$ is a contraction on $H^T \cap \{ u : ||u||_{H^T} \leq \beta \}$ for all $T \in [0,T_2]$ some $T_2 > 0$. By taking $T^* = T_1 \land T_2$ the result follows by a standard contraction-mapping fixed point argument. \hfill \Box

7. Randomly forced Navier-Stokes equations

In particular this theorem applies to Navier-Stokes equation driven by fractional Brownian motion with $h > \frac{1}{2}$. In this case, the equation we want to solve is given together with the Dirichlet boundary conditions and no-slip condition by
\begin{equation}\frac{\partial}{\partial t} u + u \cdot \nabla u = \Delta u - \nabla p + F(u)\hat{w}, \nabla . u = 0, \ u|_\Gamma = 0
\end{equation}
in some bounded domain \( \Omega \) with compact, smooth boundary \( \Gamma \). As is well known (see for instance [8]) the orthogonal complement of the set \( K^\infty(\Omega) = \{ u \in C^\infty(\Omega) : \nabla u = 0 \} \) in \( L^2(\Omega) \) is given by 
\[
G(\Omega) = \{ u : u = \nabla p, \ p, \nabla p \in L^2(\Omega) \},
\]
and if we denote the closure of \( K^\infty(\Omega) \) in \( L^2(\Omega) \) by \( K(\Omega) \) then we have the decomposition \( L^2(\Omega) = K(\Omega) \oplus G(\Omega) \). Letting \( P_\infty : L^2(\Omega) \rightarrow K(\Omega) \) be the orthogonal projection onto \( K(\Omega) \) the above equation can be written as
\[
\frac{\partial}{\partial t} u + Au + P_\infty(u, \nabla u) = P_\infty F(u) \dot{w},
\]
where \( A = -P_\infty \circ \Delta \) is the Stokes operator. More precisely we can define the usual Stokes operator (i.e. with Dirichlet boundary conditions) in the following manner.

**Definition 3.** The Stokes operator acting in \( K(\Omega) \) is given by the self-adjoint linear operator 
\[-P_\infty \circ \Delta \text{ with domain } D(-P_\infty \circ \Delta) = \{ u : u \in W^{2,2}(\Omega), u|_{\partial \Omega} = 0 \} \]
where \( \Delta \) is the usual trace Laplacian on functions (or vector fields) where the derivative is taken in the generalized sense.

The following result is well known and we refer the reader to [8] for the details.

**Theorem 5.** The Stokes operator with Dirichlet boundary conditions is self-adjoint and its inverse is a compact operator in \( K(\Omega) \).

Hence, for the Dirichlet boundary condition case, the Stokes operator satisfies the conditions required by our fixed point theorem. It is also clear that if \( w \) is a fractional Brownian motion with Hurst parameter \( h > \frac{1}{2} \) then it satisfies the conditions required by the fixed point theorem. It remains to verify the condition on \( Q := P_\infty ((u \cdot \nabla) u) \).

**Lemma 5.** The operator \( Q(u) = A^\dagger P_\infty ((u \cdot \nabla) u) \) is weakly Gateaux differentiable on \( D(A^\dagger) \) where \( A \) is either the Stokes operator associated with either the Dirichlet or Navier and kinematic boundary conditions. Its derivative is given by
\[
DQ(u) \xi = A^\dagger P_\infty ((u \cdot \nabla) \xi) + A^\dagger P_\infty ((\xi \cdot \nabla) u)
\]

**Proof.** Let \( z \in K(\Omega)^* = K(\Omega) \), then it is clear that
\[
\frac{d}{de} \left\langle A^\dagger P_\infty \left( (|u| + \varepsilon |\xi|) \cdot \nabla \right) (u + \varepsilon \xi) \right| z \right\rangle = \frac{d}{de} \left\langle A^\dagger P_\infty \left( (u \cdot \nabla) u \right) , z \right\rangle + \varepsilon \left\langle A^\dagger P_\infty \left( (u \cdot \nabla) \xi \right) , z \right\rangle + \varepsilon \left\langle A^\dagger P_\infty \left( (\xi \cdot \nabla) u \right) , z \right\rangle + \varepsilon^2 \left\langle A^\dagger P_\infty \left( (\xi \cdot \nabla) \xi \right) , z \right\rangle
\]
which implies
\[
\frac{d}{de} \left\langle A^\dagger P_\infty \left( (|u| + \varepsilon |\xi|) \cdot \nabla \right) (u + \varepsilon \xi) \right| z \right\rangle |_{\varepsilon=0} = \left\langle A^\dagger P_\infty \left( (u \cdot \nabla) \xi \right) + A^\dagger P_\infty \left( (\xi \cdot \nabla) u \right) , z \right\rangle.
\]

Then the Sobolev inequality [1.3] implies the relative boundedness conditions on \( Q \) hold in the case of Dirichlet boundary conditions and so we can apply our fixed point theorem to this case.

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