Exact solution for core-collapsed isothermal star clusters

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Star clusters in isothermal spheres are studied from a thermodynamic point of view. New density profiles are presented, that describe the collapsed phase at low temperatures. At the transition a set of binaries is formed that carries 13% of the gravitational energy, while also a huge latent heat is generated. The total energy of the binaries is fixed by thermodynamics. In the canonical ensemble all specific heats are positive, while previously discussed negative specific heat solutions are metastable. In a microcanonical ensemble the latter remain partly dominant.

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Globular star clusters typically have $10^5$-$10^6$ stars of solar mass. In our Galaxy about 150 of them have been observed, while their total number is estimated to be several hundreds. Their most intriguing property is core collaps: Some 20% of the globulars in our Galaxy have undergone a “violent relaxation” towards a collapsed state. The core collaps is stopped by the formation of strongly bound binary stars (“hard binaries”), and possibly also more exotic objects like blue stragglers or black holes, that provide energy to the remaining core.

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The temperature and energy scales are set by

$$T_G = \frac{G M m}{R} \quad U_G = \frac{G M^2}{R}$$  \hspace{1cm} (1)

Let us introduce the reduced temperature

$$\tau = \frac{2T}{T_G} = \frac{2RT}{GMm}$$  \hspace{1cm} (2)

The lowest temperature for which solutions are known is $T_c = 0.396741 T_G$, corresponding to $\tau_c = 0.79346$. We shall address the open question how the system behaves at lower $T$. It should be described by the Poisson equation for the gravitational potential $\phi(r) = -Gm \int d^3r' n(r')/|r-r'|$. For isothermal spheres one has $n(r) = A e^{-\beta m \phi(r)}$, leading to

$$\nabla^2 \phi(r) = 4\pi G m n(r) = 4\pi G m A e^{-\beta m \phi(r)}$$  \hspace{1cm} (3)

where $n$ is the number density and $\beta = 1/T$. The regular solutions are spherically symmetric and can be mapped on the Emden equation

$$y'' + \frac{2}{x} y' = e^{-y}$$  \hspace{1cm} (4)

with $y(0) = y'(0) = 0$. They are well understood, see [5]. Chandrasekhar [4] and later Lynden-Bell and Wood [5] mention the singular solution

$$n(r) = \frac{N}{4\pi R r^2}; \quad \beta m \phi(r) = 2 \ln \frac{r}{R} - 2.$$  \hspace{1cm} (5)

Though these authors do not pay much attention to it, it will become the cornerstone of our analysis. The solution (5) has $\tau = 1$, and thus only exists at $T = T_G/2$. To describe the physics at low $T$, we have asked ourselves: Can eq. (3) be deformed continuously for $\tau \neq 1$?
yielding some algebra shows that eq. (8) simplifies to
\[ \tau \]
where
\[ h(\tau) = 1 \]. From (3) it follows that
\[ h(\tau) = 1 \], while \( \mathcal{N}_h(\tau) \to \nu \ln 2 + \pi^2 / (24\nu) \)
for large \( \nu \). From the second equality it is easily seen that
\[ \nu^2 \mathcal{N}_h''(\nu) = \mathcal{N}_h''(1/\nu) \], which implies the symmetry
\[ \mathcal{N}_h(1) = \frac{1}{\tau} \mathcal{N}_h(\tau) \] (15)

It is important to know when the solution is stable. Let us consider angular variations of the form \( n(r) \to n(r, \theta)[1 + f(\tau)] \), where \( y = \text{atanh}(\cos \theta) / \tau \), and satisfying \( \int d^3 r n f = 0 \). For small \( f \) the free energy is invariant to first order, while to second order it reads
\[ \delta F = - \frac{Gm^2}{2} \int d^3 r d^3 r' \frac{n f n' f'}{|r - r'|} + \frac{T}{2} \int d^3 r n f^2 \]
\[ = \int d^3 r d^3 r' n' f' \left( -2\pi Gm^2 n f - \frac{T}{2} \nabla^2 f \right) \] (16)
with \( n' = (\nu', \theta') \) and \( f' = f(y') \). For our profile \( (\ref{eq:profile}) \) the term between brackets leads to the Schrödinger equation
\[ - f''(y) - \frac{2\pi}{\cosh^2 y} f(y) = E f(y) \] (17)

For \( \tau > 1 \) there is a bound state, \( f = A \sinh y / \cosh^\lambda y \) with \( \lambda = (-1 + \sqrt{1 + 8\tau}) / 2 \). Since \( E < 0 \) such fluctuations would lower the free energy, thus making the solution unstable. This has a simple physical interpretation: if \( \tau > 1 \) the density diverges on the north-south axis; this unnatural situation is now seen to be unstable. If no other instabilities occur, the solution \( (\ref{eq:profile}), (\ref{eq:profile1}) \) is thus stable for \( 0 < \tau < 1 \).

Let us now see what is the physical meaning of the new solution. When coming from high \( T \) the system is, of course, in the regular solution. It can be described in terms of the Milne variables \((u, v)\), with \( v = \beta T G = 2 / \tau \), that satisfy \( (\ref{eq:profile}), (\ref{eq:profile1}) \)
\[ \frac{v}{u} \frac{du}{dv} = - \frac{u + v - 3}{u - 1} \] (18)
with initial condition \( u(0) = 3 \). In the \( u-v \) plane the solution spirals counter clockwise around the point \((1,2)\), corresponding to infinite density contrast \( d_0 \).

The entropy is generally defined as
\[ S = \int d^3 r n(r) \left( \frac{5}{2} - \ln n(r) \Lambda^3 \right) \] (19)
where \( \Lambda(T) \sim T^{-1/2} \) is the thermal wavelength. We shall subtract the constant \( S_0 = N \ln [\sqrt{2\pi R^3 / N \Lambda^3(T_G)}] \) from \( S \) in order to condense our notation.

The high temperature phase then has the properties...
\[
\frac{F_{\text{regular}}}{U_G} = \frac{1}{v} \left( 2 - u - v + \ln u + \frac{3}{2} \ln \frac{v}{2} \right) \quad (20)
\]
\[
\frac{U_{\text{regular}}}{U_G} = 2u - 3 \quad (21)
\]
\[
\frac{S_{\text{regular}}}{N} = 2u + v - \frac{7}{2} - \ln u - \frac{3}{2} \ln \frac{v}{2} \quad (22)
\]
\[
\frac{C_{\text{regular}}}{N} = \frac{u(u + v - 3)}{u - 1} + u - \frac{3}{2} \quad (23)
\]

We can now identify the binodal temperature \(T_B = \frac{1}{3} T_G\) and the spinodal temperature \(T_S = T_c\). In the interval \((T_s, T_B)\) (i.e. \(\tau_c < \tau < 1\)) there are two stable phases: the regular one and the new core state. To lower the free energy, the system may collapse at \(T_B\). The free and internal energy, the entropy and the specific heat are then

\[
\frac{F_{\text{core}}}{U_G} = -\frac{1}{2} - 5\tau \ln \tau - \frac{1 - \tau}{2} N_h(\tau) \quad (24)
\]
\[
\frac{U_{\text{core}}}{U_G} = -\frac{1}{2} + 5\tau \ln \frac{1}{\tau} - \frac{1 + \tau}{2} N_h(\tau) \quad (25)
\]
\[
\frac{S_{\text{core}}}{N} = \frac{5}{2} \left( 1 + \ln \tau \right) - 2N_h(\tau) \quad (26)
\]
\[
\frac{C_{\text{core}}}{N} = \frac{5}{2} - N_h(\tau) - (1 + \tau)N_h'(\tau) \quad (27)
\]

When we lower \(T\) the system can also stay in the undercooled regular state, and undergo a core collapse at some transition point \(T_i = \frac{1}{3} T_G\) between \(T_S\) and \(T_B\). If it did not yet do so, the system must collapse at \(T_S\), simply because there is no other option.

At each of these transitions points the free energy is discontinuous, a truly uncomfortable feature. The solution to this paradox is to assume that also binary stars are formed with free energy bridging this gap. The collapse can thus be seen as a phase separation of the regular phase into a core and binaries. It is well known that only a few hard binaries can be involved. Indeed, a finite central mass fraction would lead to a behavior \(\beta \phi \sim -R/r\), which is incompatible with the Poisson equation (4). We therefore assume that the entropy of the binaries can be neglected, even though they must have a large energy. Using \(U_{\text{binaries}} = F_{\text{binaries}}\) we obtain from the difference between (20) and (24)

\[
\frac{U_{\text{binaries}}(T_i^-)}{U_G} = \frac{1 - \tau_i}{2} N_h(\tau_i) - \frac{1}{2} + (2 - u_i + \ln \tau_i u_i) \frac{\tau_i}{2} \quad (28)
\]

where \(u_i\) is the \(u\)-value corresponding to \(v = 2/\tau_i\). The above expression ranges from \(-0.0765\) at \(T_B\) to \(-0.10193\) at \(T_S\), and is roughly equal to 13 % of the pre-collapses gravitational energy.

As usual at first order phase transitions, there is also a latent heat

\[
\frac{\Delta U(T_i)}{U_G} = 1 + (u_i - 3 + N_h(\tau_i) - \frac{1}{2} \ln u_i \tau_i) \tau_i \quad (29)
\]

At \(T_B\) this equals 0.4062, being no less than 60.6 % of the precollaps gravitational energy, while at \(T_S\) it equals 0.22043, still 27.8 %. These striking effects are induced by the strong binding nature of the core, expressed by its \(1/r^2\) density profile (6). It leads to a low value of the core energy (25). This is the manifestation of Lynden-Bell’s violent relaxation (6), and it is violent indeed!

From eqs. (24) and (27) it is clear that \(C_{\text{core}} \neq T dS_{\text{core}}/dT\), which looks quite disturbing. However, we should not forget that the binary content is also a thermal part of the system. Assuming that equilibrium thermodynamics applies to the full system, we infer

\[
C_{\text{binaries}} - T \frac{dS_{\text{binaries}}}{dT} = T \frac{dS_{\text{core}}}{dT} - C_{\text{core}} \quad (30)
\]

Neglecting \(S_{\text{binaries}}\) and using eqs. (24), (27), this yields

\[
\frac{C_{\text{binaries}}}{N} = N_h(\tau) + (1 - \tau)N_h'(\tau) \quad (31)
\]

It is positive, implying that upon lowering \(T\) more and more energy is stored in binaries. Together with (25) this shows that thermodynamics relates the energy of the binaries to the properties of the core. Both contributions to the specific heat are positive, and so is their sum.
\[
\frac{C_{\text{collaps}}}{N} = \frac{5}{2} - 2\tau N_h'(\tau) \tag{32}
\]

At \(T_B\) one has \(C_{\text{core}} = N/2\), \(C_{\text{binaries}} = N\), while at \(T_S\): \(C_{\text{core}} = 0.79504N\), \(C_{\text{binaries}} = 0.99432N\). Finally, at \(T = 0\) we find the Dulong-Petit type behaviors \(C_{\text{core}} = (5/2 - \ln 2)N\), \(C_{\text{binaries}} = \ln 2N\).

The total internal energy of the collapsed phase reads

\[
\frac{U_{\text{collaps}}}{U_G} = \frac{\tau_i}{2}(2 - u_t + \ln \tau_i u_t) - 1 + \frac{5\tau}{4} - \tau N_h(\tau) - \int_{\tau}^{\infty} d\nu N_h(\nu) \tag{33}
\]

The final effect of metastability, leading to a \(T_t < T_B\), is to make the energy content of the binaries smaller. In other words, knowledge of their energy allows the determination of the collapse temperature \(T_t\).

In Figure 1 we plot the total free energies of the problem. Our approach solves the often discussed paradox of negative heat capacities of certain regular solutions. Indeed, as shown in Fig. 1, for \(T\) in a region above \(T_B\) there are two regular solutions. The one with larger free energy is also the one with the largest density contrast \((d_0 > 32.2)\), and a negative heat capacity. It is metastable and cannot be reached by adiabatic cooling.

In Figure 2 we present the specific heat as function of temperature for the regular solution, as well as for the binaries, the core and their sum.

FIG. 2. Specific heat \(C\) as function of temperature \(T\). All contributions are positive. Upon cooling the system may collapse at or below \(T_B\). The largest chance occurs at \(T_S\), where the regular solution has diverging energy fluctuations.

Let us finally discuss the microcanonical situation. We have to consider the entropy curves \(S(U)\) on lowering \(U\). For the regular solution it follows by eliminating \(T\) between (32) and (31). The entropy of the stable collapsed phase has a maximum \(S_{\text{collaps}}(\tau = 1) = \frac{1}{2}\). The regular phase has at that entropy already a negative heat capacity, but that is not forbidden in the microcanonical ensemble. For the energy of the collapsed phase we must adjust onset value for the binaries, leading to the binodal value \(U_B = -0.2679U_G\). The spinodal point is the point of instability of the regular solution, \(U_S = -0.3342U_G\). The microcanonical ensemble thus exploits the regular solutions more. But, as soon as the collapsed solutions are stable, they are also in the microcanonical approach the dominant ones.

In conclusion, we have presented new stable density profiles that describe the core collapsed phase of isothermal clusters. Using standard thermodynamics the energy content stored in binaries is determined. In the canonical ensemble all specific heats are positive. Standard notions such as binodal and spinodal temperatures, phase separation, undercooling and latent heat find their natural place, just as in condensed matter physics.

Our results probably pertain to realistic clusters, whose core is to a good approximation isothermal. Also there we expect that thermodynamics fixes the energy of the binaries. For clusters with north-south axis perpendicular to our line of sight, the projected density of the profile (6) is proportional to \(e^{-h(\cos\theta)\text{acos}(r/R)/r}\). Such a \(1/r\) law is known from observations [12] [3].

FIG. 3. Entropy \(S\) as function of energy \(U\). Coming from high \(U\), the system enters the region with negative heat capacity at \(U_c = -0.1965U_G\). At \(U_B\) it can go to the collapsed state. If it fails to find it, it can ultimately collapse at \(U_S\), where the regular solution becomes unstable.

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