Quantitative statistical stability, speed of convergence to equilibrium and partially hyperbolic skew products.

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Abstract

We consider a general relation between fixed point stability of suitably perturbed transfer operators and convergence to equilibrium (a notion which is strictly related to decay of correlations). We apply this relation to deterministic perturbations of a class of (piecewise) partially hyperbolic skew products whose behavior on the preserved fibration is dominated by the expansion of the base map. In particular we apply the results to power law mixing toral extensions. It turns out that in this case, the dependence of the physical measure on small deterministic perturbations, in a suitable anisotropic metric is at least Hölder continuous, with an exponent which is explicitly estimated depending on the arithmetical properties of the system. We show explicit examples of toral extensions having actually Hölder stability and non differentiable dependence of the physical measure on perturbations.

1 Introduction

The concept of statistical stability of a dynamical system deals with the stability of the statistical properties of its trajectories when the system is perturbed or changed in some way. Since the statistical properties of systems and their behavior are important in many fields of mathematics and in applied science, the study of this kind of stability has important applications in many fields. Many important statistical properties of dynamics are encoded in suitable probability measures which are invariant for the action of the dynamics; because of this the mathematical approach to statistical stability is often related to the stability of such invariant measures under perturbations of the system. In this context it is important to get quantitative estimates, such as the differentiability of the statistical properties under small perturbations (called Linear Response) or other

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quantitative statements, such as the Lipschitz or Hölder dependence. These questions are well understood in the uniformly hyperbolic case, where the system’s derivative has uniformly expanding or contracting directions. In this case quantitative estimates are available, proving the Lipschitz and even differentiable dependence of the relevant invariant measures under perturbations of the system (see e.g. [3], [7] or [28] and related references, for recent surveys where also some result beyond the uniformly hyperbolic case are discussed). For systems having not a uniformly hyperbolic behavior, and in presence of discontinuities, the situation is more complicated and much less is known. Qualitative results and some quantitative ones (providing precise information on the modulus of continuity) are known under different assumptions or in families of cases, and there is not yet a general understanding of the statistical stability in those cases (see. e.g. [1], [2], [5], [9], [10], [11], [16], [18], [19], [25], [32], [33]).

We approach this question from a general point of view, using a functional analytic perturbation lemma (see Theorem 5) which relates the convergence to equilibrium speed of the system to the stability of its invariant measures belonging to suitable spaces. In our case we consider a space of signed measures equipped with a suitable anisotropic norm adapted to the system, in which the relevant invariant measures are proved to exist. We show how, with some technical work, the approach can be applied to slowly mixing partially hyperbolic skew products. The functional analytic perturbation lemma we use is quite flexible and was applied in [19] to the study of the statistical stability of maps with indifferent fixed points. A similar functional analytic construction was also applied to piecewise hyperbolic skew product maps and Lorenz like two dimensional maps in [20].

**Main results.** The paper has the following structure and main contents:

a) A general quantitative relation between speed of convergence to equilibrium of the system and its statistical stability (Section 2).

b) The application of this relation to a general class of skew products allowing discontinuities and a sort of partial hyperbolicity, getting quantitative estimates for their statistical stability in function of their convergence to equilibrium. Here a main ingredient is the construction of suitable spaces of regular measures adapted to these systems. (Sections 3, 4, 5).

c) The application of this construction to the stability under deterministic perturbations of a class of piecewise constant toral extensions having slow convergence to equilibrium, getting Hölder stability for these examples (Section 6).

d) Finally, we show examples of mixing piecewise constant toral extensions where a perturbation of the map of size $\delta$ results in a change of physical invariant measure of the size of order $\delta^\beta$, where $\beta \leq 1$ depends on the Diophantine properties of the map (Section 6.3).

Let us explain in more details the content of the items listed above:
a) This relation is a fixed point stability statement we apply to the system’s transfer operators, giving nontrivial information for systems having different kinds of speed of convergence to equilibrium, implying for instance, that in a system with power law convergence to equilibrium speed, under quite general additional assumptions, the physical measure is Hölder stable (see Theorem 5 and Remark 7).

b) We apply the relation to show a general quantitative stability statement for perturbations of a class of skew products. We consider skew products in which the base dynamics is expanding and dominates the behavior of the dynamics on the fibers. For this purpose we introduce suitable spaces of signed measures adapted to such systems. We consider spaces of signed measures having absolutely continuous projection on the base space $[0,1]$ (corresponding to the strongly expanding direction) and equip them with suitable anisotropic norms: the weak norm $||| \cdot |||_{1}$ and the strong norm $||| \cdot |||_{p-BV}$, which are defined by disintegrating along the central foliation (preserved by the skew product) and considering the regularity of the disintegration. These spaces have properties that make them work quite like $L^1$ and $p$-Bounded Variation real functions spaces in the classical theory for the statistical properties of one dimensional dynamics. In Section 4 we prove a kind of Lasota Yorke inequality in this framework. This will be used together with a kind of Helly selection principle proved in Section 3 to estimate the regularity of the invariant measures (like it is done in the classical construction for one dimensional, piecewise expanding maps). We summarize this in the following result (see Proposition 22 for a precise and more general statement).

**Theorem 1** Consider a family of skew product maps $F_{\delta} : [0,1] \times M \to [0,1] \times M$, where $M$ is a compact manifold with boundary, $F_{\delta} = (T_{\delta}, G_{\delta})$ is such that the following hold uniformly on $\delta$

- **A1** $T_{\delta} : [0,1] \to [0,1]$ is piecewise expanding with $C^2$ onto branches;
- **A2** the behavior of $G_{\delta}$ on the fibers is dominated by the expansion of $T_{\delta}$;
- **A3** $G_{\delta}$ satisfies a sort of $BV$ regularity: there is $A > 0$, such that

$$\sup_{r \leq A} \frac{1}{r} \int_{y \in M, x_1, x_2 \in B(x,r)} |G(x_1, y) - G(x_2, y)| dx < \infty$$

(see Sk1, ..., Sk3 at beginning of Section 3 for precise statements of these assumptions). Then the maps $F_{\delta}$ have invariant probability measures $f_{\delta}$ having an absolutely continuous projection on the base space $[0,1]$ and uniformly bounded $||| \cdot |||_{p-BV}$ norms (they are uniformly regular in the strong space).

In Section 5 we consider a class of perturbations of our skew products such that the related transfer operators are near in some sense when applied to (regular) measures and state a first general statement on the statistical stability of such skew products (see Proposition 25).
c) The statement is then applied to slowly mixing piecewise constant toral extensions: systems of the kind \((X, F)\) where \(X = [0, 1] \times \mathcal{T}^d\), \(\mathcal{T}^d\) is the \(d\) dimensional torus and \(F : X \to X\) is defined by

\[
F(\omega, t) = (T\omega, t + \tau(\omega))
\]

where \(T : [0, 1] \to [0, 1]\) is expanding and \(\tau : [0, 1] \to \mathcal{T}^d\) is a piecewise constant function. The qualitative ergodic theory of this kind of systems was studied in several papers (see e.g. [13], [14]). Quantitative results appeared more recently ([17], [15], [29]), proving from different points of view that the speed of correlation decay is generically fast (exponential), but in some cases where \(\tau\) is piecewise constant, this decay follows a power law whose exponent depend on the diophantine properties of \(\tau\) (see [21] or Section 6.1).

We apply our general result to deterministic perturbations of these maps, showing that the physical measure of those systems varies at least Hölder continuously in our anisotropic “\(L^1\) like” distance. We state informally an example of such an application (see Proposition 32 for a more general statement and the required definitions).

**Theorem 2** Consider a family of skew product maps \(F_\delta : [0, 1] \times \mathcal{T}^d \to [0, 1] \times \mathcal{T}^d\) satisfying A1,...,A3, as in Theorem 1. Let us assume \(F_0\) is a piecewise constant toral extension as in (1), with

\[
T_0(x) = 2x \mod(1)
\]

and

\[
G_0(x, t) = (Tx, t + \theta \varphi(x))
\]

where \(\theta = (\theta_1, ..., \theta_d) \in \mathcal{T}^d\) has linear Diophantine type\(^1\) \(\gamma_1(\theta)\) and \(\varphi = 1_{[0, \frac{1}{2}]}\) is the characteristic function of \([0, \frac{1}{2}]\). Suppose \(F_\delta\) is a small perturbation of \(F_0\) in the following sense

\begin{itemize}
  \item[D1] for each \(\delta\), \(T_\delta = T_0 \circ \sigma\) for some diffeomorphism \(\sigma\) near to the identity \(\|\sigma - I\|_\infty \leq \delta\), \(\|\frac{\partial}{\partial x} - 1\|_\infty \leq \delta\);
  \item[D2] for each \(\delta\) and \(x \in [0, 1], y \in \mathcal{T}^d\) : \(|G_0(x, y) - G_\delta(x, y)| \leq \delta\).
\end{itemize}

Then and for each \(\gamma > \gamma_1(\theta)\) there is \(K \geq 0\) such that,

\[
\|f_\delta - f_0\|_{1^*} \leq K\delta^{\frac{1}{\gamma_1(\theta)}}.
\]

We remark that the perturbations allowed are quite general. In particular they allow discontinuities, and the invariant measure to become singular with respect to the Lebesgue measure after perturbation. We also remark that for a class of smooth toral extensions with fast decay of correlations, a differentiable dependence statement was proved in [18].

\(^1\)See Definition 26 for a recall about this Diophantine type for vectors of real numbers.

\(^2\)For such map the Lebesgue measure \(f_0\) is invariant for the system.
d) We finally show examples of piecewise constant, mixing toral extensions where the physical measure of the system actually varies in a Hölder way (and hence not in a differentiable way) with an exponent depending on the arithmetical properties of the system. We state informally the main result about this, see Propositions 33 and 34 for precise statements.

**Theorem 3** Let us consider a piecewise constant toral extension map $F_0 : [0,1] \times T^1$, as in (2), (3), where $\theta$ is a well approximable Diophantine irrational with $\gamma_I(\theta) > 2$. For every $\gamma < \gamma_I(\theta)$ there is a sequence of reals $\delta_j \geq 0$, $\delta_j \to 0$ and a sequence of maps $\tilde{F}_\delta_j(x, y) = (\tilde{T}_0(x), \tilde{G}_\delta_j(x, y))$ satisfying A1,...,A3, D1,D2 such that

$$||\mu_0 - \mu_j||_1 \geq \frac{1}{9} \delta_j$$

holds for every $j$ and every $\mu_j$, invariant borel probability measure of $\tilde{F}_\delta_j$ with absolutely continuous projection on $[0,1]$.

This shows that in some sense, the general statistical stability result is sharp. We remark that recently, in [33] examples of $C^r$ families of mostly contracting diffeomorphisms with strictly Hölder behavior have been given (see also [16] for previous results on Hölder stability of these kinds of partially hyperbolic maps).

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2 Quantitative fixed point stability and convergence to equilibrium.

Let us consider a dynamical system $(X, T)$ where $X$ is a metric space and the space $SM(X)$ of signed Borel measures on $X$. The dynamics $T$ naturally induces a function $L : SM(X) \to SM(X)$ which is linear and is called transfer operator. If $\nu \in SM(X)$ then $L[\nu] \in SM(X)$ is defined by

$$L[\nu](B) = \nu(T^{-1}(B))$$

for every measurable set $B$. If $X$ is a manifold, the measure is absolutely continuous ($d\nu = f \, dm$, where $m$ represents the Lebesgue measure) and $T$ is nonsingular, the operator induces another operator $\tilde{L} : L^1(m) \to L^1(m)$ acting on measure densities ($\tilde{L}f = \frac{dL(f \, m)}{dm}$). By a small abuse of notation we will still indicate by $L$ this operator.

An invariant measure is a fixed point for the transfer operator. Let us now see a quantitative stability statement for these fixed points under suitable perturbations of the operator. Let us consider a certain system having a transfer operator $L_0$ for which we know the speed of convergence to equilibrium (see [4] below). Consider a "nearby" system $L_1$ having suitable properties: suppose
there are two normed vector spaces of measures with sign \( B_s \subseteq B_w \subseteq SM(X) \) (the strong and weak space) with norms \( \| \cdot \|_w \leq \| \cdot \|_s \) and suppose the operators \( L_0 \) and \( L_1 \) preserve the spaces: \( L_i(B_s) \subseteq B_s \) and \( L_i(B_w) \subseteq B_w \) with \( i \in \{0,1\} \).

Let us consider
\[
V_s := \{ f \in B_s, f(X) = 0 \}
\]
the space of zero average measures in \( B_s \). The speed of convergence to equilibrium of a system will be measured by the speed of contraction to 0 of this space by the iterations of the transfer operator.

**Definition 4** Let \( \phi(n) \) be a real sequence converging to zero. We say that the system has convergence to equilibrium with respect to norms \( \| \cdot \|_w \), \( \| \cdot \|_s \) and speed \( \phi \) if \( \forall g \in V_s \)
\[
\| L_0^n(g) \|_w \leq \phi(n) \| g \|_s.
\] (4)

Suppose \( f_0, f_1 \in B_s \) are fixed probability measures of \( L_0 \) and \( L_1 \). The following statement relates the distance between \( f_0 \) and \( f_1 \) with the distance between \( L_0 \) and \( L_1 \) and the speed of convergence to equilibrium of \( L_0 \). The proof is elementary, we include it for completeness. Similar quantitative stability statements are used in [22], [20] and [19] to support rigorous computation of invariant measures, get quantitative estimates for the statistical stability of Lorenz like maps and intermittent systems.

**Theorem 5** Suppose we have estimates on the following aspects of the operators \( L_0 \) and \( L_1 \):

1. (speed of convergence to equilibrium) there is \( \phi \in C^0(\mathbb{R}) \), \( \phi(t) \) decreasing to 0 as \( t \to \infty \) such that \( L_0 \) has convergence to equilibrium with respect to norms \( \| \cdot \|_w \), \( \| \cdot \|_s \) and speed \( \phi \).

2. (control on the norms of the invariant measures) There is \( \tilde{M} \geq 0 \) such that
\[
\max(\| f_1 \|_s, \| f_0 \|_s) \leq \tilde{M};
\]

3. (iterates of the transfer operator are bounded for the weak norm) there is \( \tilde{C} \geq 0 \) such that for each \( n \),
\[
\| L_0^n \|_{B_w \rightarrow B_w} \leq \tilde{C}.
\]

4. (control on the size of perturbation in the strong-weak norm) Denote
\[
\epsilon := \sup_{\| f \|_s \leq 1} \| (L_1 - L_0)f \|_w
\]
consider the decreasing function \( \psi \) defined as \( \psi(z) = \frac{\phi(z)}{z} \), then we have the following explicit estimate
\[
\| f_1 - f_0 \|_w \leq 2\tilde{M}\tilde{C} \epsilon^{-1}(\frac{\epsilon\tilde{C}}{2}) + 1).
\]
Proof. The proof is a direct computation from the assumptions

\[ ||f_1 - f_0||_w \leq ||L^N_1 f_1 - L^N_0 f_0||_w \]
\[ \leq ||L^N_1 f_1 - L^N f_1||_w + ||L^N f_1 - L^N_0 f_0||_w \]
\[ \leq ||L^N_0 (f_1 - f_0)||_w + ||L^N_1 f_1 - L^N_0 f_1||_w. \]

Since \( f_1 - f_0 \in V_s \), \( ||f_1 - f_0||_s \leq 2\tilde{M} \),

\[ ||f_1 - f_0||_w \leq 2\tilde{M}\phi(N) + ||L^N_1 f_1 - L^N_0 f_1||_w \]
but

\[ (L^N_1 - L^N_0) = \sum_{k=1}^{N} L^N_0(-k)(L_0 - L_1)L_1^{k-1} \]
hence

\[ -(L^N_1 - L^N_0)f_1 = \sum_{k=1}^{N} L^N_0(-k)(L_0 - L_1)L_1^{k-1}f_1 \]
\[ = \sum_{k=1}^{N} L^N_0(-k)(L_0 - L_1)f_1 \]
then

\[ ||f_1 - f_0||_w \leq 2\tilde{M}\phi(N) + \epsilon\tilde{M}NC. \]

Now consider the function \( \psi \) defined as \( \psi(x) = \frac{\phi(x)}{x} \), choose \( N \) such that
\[ \psi^{-1}(\frac{\epsilon\tilde{C}}{2}) \leq N \leq \psi^{-1}(\frac{\epsilon\tilde{C}}{2}) + 1, \]
in this way \( \frac{\phi(N)}{N} \leq \frac{\epsilon\tilde{C}}{2} \leq \frac{\phi(N-1)}{N-1} \) and

\[ ||f_1 - f_0||_w \leq 2\tilde{M}\tilde{C}\epsilon(\psi^{-1}(\frac{\epsilon\tilde{C}}{2}) + 1). \]

Theorem 5 implies that a system having convergence to equilibrium is statistically stable (in the weak norm).

**Corollary 6** Let \( L_\delta, \delta \in [0, \tilde{\delta}) \) be a family of transfer operators under the assumptions of Theorem 4 including \( \lim_{n \to \infty} \phi(n) = 0. \) Let \( f_\delta \) be a fixed probability measure of \( L_\delta. \) Suppose there is \( C \geq 0 \) such that for every \( \delta \)

\[ \sup_{||f||_s \leq 1} ||(L_\delta - L_0)f||_w \leq C\delta. \]

Then \( f_0 \) is the unique fixed probability measure in \( B_s \) and it holds

\[ \lim_{\delta \to 0} ||f_\delta - f_0||_w = 0. \]
Proof. The uniqueness of \( f_0 \) is trivial from the definition of convergence to equilibrium. For the stability, suppose there was a sequence \( \delta_n \to 0 \) and \( l \geq 0 \) such that \( ||f_{\delta_n} - f_0||_w \geq l \ \forall n \), then
\[
2\tilde{M}\tilde{C}C\delta_n(\psi^{-1}(\frac{\delta_nC\tilde{C}}{2}) + 1) \geq l
\]
\[
\psi^{-1}(\frac{\delta_nC\tilde{C}}{2}) \geq \frac{l}{\delta_n2MCC} - 1
\]
\[
\frac{\delta_nC\tilde{C}}{2} \leq \frac{\phi(\frac{l}{\delta_n2MCC} - 1)}{\delta_n2MCC} - 1
\]
\[
\frac{l}{\delta_n2MCC} - 1) \frac{\delta_nC\tilde{C}}{2} \leq \phi(\frac{l}{\delta_n2MCC} - 1)
\]
\[
\frac{l}{4M} - \delta_nC\tilde{C} \leq \phi(\frac{l}{\delta_n2MCC} - 1)
\]
which is impossible to hold as \( \delta_n \to 0 \). \( \blacksquare \)

Remark 7 In Theorem 6, if \( \phi(x) = Cx^{-\alpha} \) then \( \psi(x) = Cx^{-\alpha-1} \), \( \epsilon(\psi^{-1}(\epsilon) + 1) \sim \epsilon^{1+\frac{1}{\alpha}} \) and we have the estimate for the modulus of continuity
\[
||f_1 - f_0||_w \leq K_1\epsilon^{1+\frac{1}{\alpha}}
\]
where the constant \( K_1 \) depends on \( \tilde{M}, \tilde{C}, C \) and not on the distance between the operators measured by \( \epsilon \).

3 Spaces we consider

Our approach is based on the study of the transfer operator restricted to a suitable space of measures with sign. We introduce a space of regular measures where we can find the invariant measure of our systems, and the ones of suitable perturbations of it. We hence consider some measure spaces adapted to skew products. The approach is taken from [20] (see also [3]) where it was used for Lorenz-like two dimensional maps. Let us consider a map \( F : X \to X \) where \( X = [0,1] \times M \), and \( M \) is a compact manifold with boundary, such that
\[
F(x,y) = (T(x), G(x,y)). \tag{5}
\]
Suppose \( F \) satisfies the following conditions:

Sk1 Suppose \( T \) is \( \frac{1}{\lambda} \)-expanding\(^3\) and it has \( C^{1+\xi} \) branches\(^4\) which are onto. The branches will be denoted by \( T_i, i \in [1,..,q] \).

\(^3\)We suppose that \( \inf_{x \in [0,1]} T'(x) \geq \frac{1}{\lambda} \) for some \( \lambda < 1 \).

\(^4\)More precisely we suppose that there are \( \xi, C_\lambda \geq 0 \) such that
\[
\frac{1}{|T_i \circ T_i^{-1}(\gamma_2)|} - \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \leq C_\lambda d(\gamma_1, \gamma_2)^{\xi}, \forall \gamma_1, \gamma_2 \in [0,1].
\]
Consider the $F$-invariant foliation $\mathcal{F}^s := \{\{x\} \times M\}_{x \in [0,1]}$. We suppose that the behavior on $\mathcal{F}^s$ is dominated by $\lambda$: there is $\alpha \in \mathbb{R}$ with $\lambda^\alpha < 1$, such that for all $x \in [0,1]$ holds

$$|G(x,y_1) - G(x,y_2)| \leq \alpha |y_1 - y_2| \text{ for all } y_1, y_2 \in M. \quad (6)$$

For each $p \leq \xi$ there is $A > 0$, such that

$$\hat{H} := \sup_{r \leq A} \frac{1}{r^p} \int \sup_{y \in M, x_1, x_2 \in B(x,r)} |G(x_1, y) - G(x_2, y)| dx < \infty$$

Remark 8 We remark that $Sk3$ allows discontinuities in $G$, provided a kind of bounded variation regularity is respected. $Sk2$ allows a dominated expansion or contraction in the fibers direction. Furthermore, by $Sk1$ the transfer operator of the map $T$ satisfies a Lasota Yorke inequality of the kind

$$||L^n_T(\mu)||_{BV} \leq A_T \lambda^n ||\mu||_{BV} + B_T ||\mu||_1 \quad (7)$$

where $||\mu||_{BV}$ is the generalized bounded variation norm (see [26]); for some constant $A_T$ and $B_T$ depending on the map.

Definition 9 We say that a family of maps $F_\delta = (T_\delta(x), G_\delta(x,y))$ satisfies $Sk1, ..., Sk3$ uniformly, if each $T_\delta$ is piecewise expanding, with onto $C^{1+\xi}$ branches, admitting a uniform expansion rate $\frac{1}{\lambda}$, a uniform $\alpha$, a uniform Hölder constant $C$, a uniform second coefficient of the Lasota Yorke inequality $B_T$ and furthermore the family $G_\delta$ satisfies $Sk3$ with a uniform bound on the constant $\hat{H}$.

We construct now some function spaces which are suitable for the systems we consider. The idea is to consider spaces of measures with sign, with suitable norms constructed by disintegrating measures along the central foliation. In this way a measure on $X$ will be seen as a collection (a path) of measures on the leaves. In the central direction (on the leaves) we will consider a norm which is the dual of the Lipschitz norm. In the expanding direction we will consider the $L^1$ norm and a suitable variation norm. These ideas will be implemented in the next paragraphs.

Let $(X,d)$ be a compact metric space, $g : X \to \mathbb{R}$ be a Lipschitz function and let $Lip(g)$ be its best Lipschitz constant, i.e.

$$Lip(g) = \sup_{x, y \in X} \left\{ \frac{|g(x) - g(y)|}{d(x, y)} \right\}.$$ 

Definition 10 Given two signed Borel measures $\mu$ and $\nu$ on $X$, we define a Wasserstein-Kantorovich Like distance between $\mu$ and $\nu$ by

$$W_1(\mu, \nu) = \sup_{Lip(g) \leq 1, ||g||_\infty \leq 1} \left| \int g d\mu - \int g d\nu \right|. \quad (8)$$
Let us denote
\[ \|\mu\|_{W_1} := W_1(0, \mu). \]
As a matter of fact, \( \|\cdot\|_{W_1} \) defines a norm on the vector space of signed measures defined on a compact metric space.

Let \( SB(\Sigma) \) be the space of Borel signed measures on \( \Sigma \). Given \( \mu \in SB(\Sigma) \) denote by \( \mu^+ \) and \( \mu^- \) the positive and the negative parts of it (\( \mu = \mu^+ - \mu^- \)).

Denote by \( AB \) the set of signed measures \( \mu \in SB(\Sigma) \) such that its associated marginal signed measures, \( \mu^\pm = \pi_x^\pm \mu^\pm \) are absolutely continuous with respect to the Lebesgue measure \( m \), on \([0,1]\) i.e.
\[ AB = \{ \mu \in SB(\Sigma) : \pi_x^+ \mu^+ << m \text{ and } \pi_x^- \mu^- << m \} \]
where \( \pi_x : X \to [0,1] \) is the projection defined by \( \pi_x(x,y) = x \) and \( \pi_x^\pm \) is the associated pushforward map.

Let us consider a finite positive measure \( \mu \in AB \) on the space \( X \) foliated by the preserved leaves \( F^c = \{ \gamma_l \}_{l \in [0,1]} \) such that \( \gamma_l = \pi_x^{-1}(l) \). We will also call \( F^c \) as the central foliation.

Let us denote \( \mu_x = \pi_x^* \mu \) and let \( \phi_\mu \) be its density (\( \mu_x = \phi_\mu \cdot m \)). The Rokhlin disintegration theorem describes a disintegration of \( \mu \) by a family \( \{ \mu_\gamma \}_{\gamma} \) of probability measures on the central leaves in a way that the following holds.

**Remark 11** The disintegration of a measure \( \mu \) is the \( \mu_x \)-unique measurable family \( \{ \{ \mu_\gamma \}, \phi_\mu \} \) such that, for every measurable set \( E \subset X \) it holds
\[ \mu(E) = \int_{[0,1]} \mu_\gamma(E \cap \gamma) d\mu_x(\gamma). \]  

**Definition 12** Let \( \pi_{x,y} : \gamma \to M \) be the restriction \( \pi_y|_\gamma \), where \( \pi_y : X \to M \) is the projection defined by \( \pi_y(x,y) = y \) and \( \gamma \in F^c \). Given a positive measure \( \mu \in AB \) and its disintegration along the stable leaves \( F^c \), \( \{ \{ \mu_\gamma \}, \phi_\mu \} \), we define the **restriction of \( \mu \) on \( \gamma \)** as the positive measure \( \mu|_\gamma \) on \( M \) (not on the leaf \( \gamma \)) defined as
\[ \mu|_\gamma = \pi_{x,y}^* (\phi_\mu(\gamma) \mu_\gamma). \]

**Definition 13** For a given signed measure \( \mu \in AB \) and its decomposition \( \mu = \mu^+ - \mu^- \), define the **restriction of \( \mu \) on \( \gamma \)** by
\[ \mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma. \]

**Definition 14** Let \( L^1 \subseteq AB \) be defined as
\[ L^1 = \left\{ \mu \in AB : \int_{[0,1]} \|\mu_\gamma\|_{W_1} \, dm(\gamma) < \infty \right\}. \]

\[ {^{\text{In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with } \gamma.}} \]
and define norm on it, \( \| \cdot \|_{-1} : \mathcal{L}^1 \rightarrow \mathbb{R} \), by

\[
\| \mu \|_{-1} = \int_{[0,1]} \| \mu |\gamma \|_{W_1} \, dm(\gamma).
\]  

(12)

The notation we use for this norm is similar to the usual \( L^1 \) norm. Indeed this is formally the case if we associate to \( \mu \), by disintegration, a path \( G_{\mu} : [0,1] \rightarrow (SB(M), \| \cdot \|_{W_1}) \) defined by \( G_{\mu}(\gamma) = \mu |\gamma \). In this case, this will be the \( L^1 \) norm of the path.

3.1 The transfer operator associated to \( F \) and basic properties of \( \mathcal{L}^1 \)

Let us now consider the transfer operator \( L_F \) associated with \( F \). Being a push forward map, the same function can be also denoted by \( F^* \) we will use this notation sometime. There is a nice characterization of the transfer operator in our case, which makes it work quite like a one dimensional operator. For the proof see [20].

Proposition 15 (Perron-Frobenius like formula) Let us consider a skew product map \( F \) satisfying \( Sk_1 \) and \( Sk_2 \). For a given leaf \( \gamma \in \mathcal{F}^s \), define the map \( F_{\gamma} : M \rightarrow M \) by

\[
F_{\gamma} = \pi_y \circ F |\gamma \circ \pi_{-1}^{-1} \gamma.
\]

For all \( \mu \in \mathcal{L}^1 \) and for almost all \( \gamma \in [0,1] \) it holds

\[
(L_F \mu) |\gamma = \sum_{i=1}^q \frac{F_{T_i^{-1}(\gamma)}^* \mu |T_i^{-1}(\gamma)}{|T_i^f \circ T_i^{-1}(\gamma)|}.
\]  

(13)

We recall some results showing that the transfer operator associated to a Lipschitz function is also Lipschitz with the same constant, for the "1" distance, and moreover, that the transfer operator of a map satisfying \( Sk_1, ..., Sk_3 \) is also Lipschitz with the same constant for the \( \| \cdot \|_{1} \) norm. In particular, if \( \alpha \leq 1 \) the transfer operator is a weak contraction, like it happen for the \( L^1 \) norm on the one dimensional case (for the proof and more details see [20]).

Lemma 16 If \( G : Y \rightarrow Y \), where \( Y \) is a metric space is \( \alpha \)-Lipschitz, for every Borel measure with sign \( \mu \) it holds

\[
\| L_G \mu \|_{W_1} \leq \alpha \| \mu \|_{W_1}.
\]

If \( \mu \in \mathcal{L}^1 \) and \( F : [0,1] \times M \rightarrow [0,1] \times M \) satisfies \( Sk_1, ..., Sk_3 \) then

\[
\| L_F \mu \|_{-1} \leq \alpha \| \mu \|_{-1}.
\]  

(14)

11
3.2 The strong norm

We consider a norm which is stronger than the $L^1$ norm. The idea is to consider a disintegrated measure as a path of measures on the preserved foliation and define a kind of bounded variation regularity for this path, in a way similar to what was done in [26] for real functions.

For this strong space we will prove a regularization inequality, similar to the Lasota Yorke ones. We will use this inequality to prove the regularity of the invariant measure of the family of skew products we consider.

Let us consider $\mu \in L^1$. Let us define

$$osc(\mu, x, r) = \operatorname{esssup}_{\gamma_2, \gamma_1 \in B(x, r)} (W_1(\mu|_{\gamma_1}, \mu|_{\gamma_2}))$$

and

$$var_p(\mu, r) = \int_{[0,1]} r^{-p} osc(\mu, x, r) \, dx.$$ 

Now let us choose $A > 0$ and consider $var_p(\mu) := \sup_{r \leq A} var_p(\mu, r)$. Finally we define $p - BV$ norm as:

$$||\mu||_{p-BV} = ||\mu||_{1-p} + var_p(\mu).$$

Let us consider $1 \geq p \geq 0$ and the following space of measures

$$p - BV = \{ \mu \in L^1, ||\mu||_{p-BV} < \infty \}.$$

This will play the role of the strong space in the present case.

**Remark 17** If $\mu \in p - BV$, then it follows that

$$\operatorname{esssup}_{\gamma} ||\mu|_{\gamma}||_{W_1} \leq A^{p-1} ||\mu||_{p-BV}.$$ 

See [4], Lemma 2 for a proof in the case of real functions which also works in our case.

We now prove a sort of Helly selection principle for sequences of positive measures with bounded variation. This principle will be used, together with a regularization inequality, proved in next section, to get information on the variation of invariant measures. First we need a preliminary lemma:

**Lemma 18** If $\mu_n$ is a sequence of positive measures such that for each $n$, $||\mu_n||_{1-p} \leq C$, $var_p(\mu_n) \leq M$, and $\mu_n|_{\gamma} \rightharpoonup \mu|_{\gamma}$ for a.e. $\gamma$, in the Wasserstein distance, then

$$||\mu||_{1-p} \leq C, var_p(\mu) \leq M.$$
Proof. Let us consider the \( |||\cdot|||^1\) norm. Since by Remark \[\text{17}\] it holds \( |||\mu_n|||^1_W \leq AP^{-1}(C + M) \forall \gamma\), by the dominated convergence theorem, \( |||\mu|||^1_W \leq C\). Let us now consider the oscillation. We have that \( \lim_{n \to \infty} osc(\mu_n, x, r) \geq osc(\mu, x, r) \) for all \( x, r \). Indeed, consider a small \( \epsilon \geq 0 \). Because of the definition of \( osc() \), we have that for all \( l \geq osc(\mu, x, r) - \epsilon \), there are positive measure sets \( A_1 \) and \( A_2 \) such that \( W_1(\mu|_{\gamma_1}, \mu|_{\gamma_2}) \geq l \) for all \( \gamma_1 \in A_1, \gamma_2 \in A_2 \). Since \( \mu_n|_{\gamma} \to \mu|_{\gamma} \) for a.e. \( \gamma \), if \( n \) is big enough there must be sets \( A_1^n \) and \( A_2^n \) of positive measure, such that \( W_1(\mu_n|_{\gamma_1}, \mu_n|_{\gamma_2}) \geq l \) for all \( \gamma_1 \in A_1^n, \gamma_2 \in A_2^n \), by this \( osc(\mu_n, x, r) \geq l \), then for all \( x, r \), \( \lim inf osc(\mu_n, x, r) \geq osc(\mu, x, r) \). By Fatou’s Lemma, \( \lim inf \varphi(\mu_n, r) = \lim inf \int r^{-p} osc(\mu_n, x, r) \, dx \geq \varphi(\mu, r) \). From which the statement follows directly. \[\Box\]

Theorem 19 (Helly-selection-like theorem) Let \( \mu_n \) be a sequence of probability measures on \( X \) such that \( |||\mu_n|||^p_{BV} \leq M \) for some \( M \geq 0 \). Then there is \( \mu \) with \( |||\mu|||^p_{BV} \leq M \) and subsequence \( \mu_{n_k} \) such that

\[
|||\mu_{n_k} - \mu|||^1_W \to 0.
\]

Proof. Let us discretize in the vertical direction. Let us consider a continuous projection of the space of probability measures on \( M \) on a finite dimensional space \( \pi_y : \gamma \to U_\delta \) (\( U_\delta \) is finite dimensional). Suppose \( \pi_y, \delta \) is such that \( |||\pi_y(\nu) - \nu|||^1_W \leq C\delta, \forall \nu \in PM(M) \) (such projection can be constructed discretizing the space by a partition of unity made of Lipschitz functions with support on sets whose diameter is smaller than \( \delta \), see [23] for example). Let us consider the natural extension of this projection to the whole \( L^1(X) \) space \( \pi_\delta : L^1(X) \to L^1(X) \), defined by \( \pi_\delta(\mu) = \pi_y(\mu|_\gamma) \).

Let us consider the sequence \( \pi_\delta(\mu_n) \). We have \( |||\pi_\delta(\mu_n)|||^p_{BV} \leq K_\delta M \) where \( K_\delta \) is the modulus of continuity of \( \pi_\delta \). Indeed

\[
\esssup_{\gamma_2, \gamma_1 \in B(x, r)} (W_1(\pi_\delta(\mu)|_{\gamma_1}, \pi_\delta(\mu|_{\gamma_2})) \leq K_\delta \esssup_{\gamma_2, \gamma_1 \in B(x, r)} (W_1(\mu|_{\gamma_1}, \mu|_{\gamma_2})).
\]

Since after projecting we now are in a space of functions with values in a finite dimensional space, to the sequence \( \pi_\delta(\mu_n) \) we can apply the classical Helly selection theorem and get that there is a limit measure \( \mu_\delta \) and a sub sequence \( \mu_{n_k} \) such that \( \pi_\delta(\mu_{n_k}) \to \mu_\delta \) in \( L^1 \) and \( \pi_\delta(\mu_{n_k})|_{\gamma} \to \mu_\delta|_{\gamma} \) almost everywhere. Let us consider a sequence \( \delta_i \to 0 \) and select inductively at every step from the previous selected subsequence \( \mu_i \) such that \( \pi_\delta_{i-1}(\mu_i) \to \pi_\delta_{i-1} \) a further subsequence \( \mu_{i_k} \), such that \( \pi_\delta_{i_k} (\mu_{i_k}) \to \mu_\delta \) in \( L^1 \) and almost everywhere. Since \( \forall \gamma \) and \( m \leq i \),

\[
|||\pi_\delta_{i_k}(\mu_{i_k}|_{\gamma}) - \pi_\delta_{i}(\mu_{i_k}|_{\gamma})|||^1_W \leq C\delta_{m,i,j} \leq \gamma \text{ such that } \delta_{m,i,j} \geq \delta_i
\]

and then \( \forall \gamma \)

\[
|||\mu_{n_k}|_{\gamma} - \mu_{\delta_k}|_{\gamma}|||^1_W \leq C\delta_{m,i,j} \]

uniformly in \( \gamma \). Since \( \mu_{n_k} \) are positive measures, this shows that there is a \( \mu \) such that \( \mu_{\delta_k} \to \mu \) in \( L^1 \) and \( \mu_{\delta_k}|_{\gamma} \to \mu|_{\gamma} \) almost everywhere. This shows that
Proof. By the Perron Frobenius like formula (Lemma 15) we have to estimate \(||\mu||_{p-BV} \leq M. \]

4 A regularization inequality

In this section we prove an inequality, showing that iterates of a bounded variation positive measure are of uniform bounded variation. This will play the role of a Lasota Yorke inequality. A consequence will be a bound on the variation of invariant measures in \(L^1\). This will be used when applying Theorem 3 to provide the estimate needed at Item 2 of. The following regularization inequality can be proved.

**Proposition 20** Let \(F\) be a skew product map satisfying assumptions \(Sk1, ..., Sk3\) and let us suppose that \(\mu\) is a positive measure. Let \(p \leq \xi\) (the Hölder exponent as in \(Sk1\)). It holds

\[
\var_p(LF\mu) \leq \lambda^p \alpha \var_p(\mu) + (\hat{H})||\mu_x||\infty + 3q\alpha C_h A^{p-\xi}||\mu_x||\infty).
\]

We recall that here \(\mu_x\) is the marginal of the disintegration of \(\mu\) (see Equation 15) and \(||\mu_x||\infty\) is the supremum norm for its density.

**Proof.** By the Perron Frobenius like formula (Lemma 15)

\[
(LF\mu)|_\gamma = \sum_{i=1}^{q} \frac{F_{T_i^{-1}(\gamma)}^* \mu_{T_i^{-1}(\gamma)}}{|T_i \circ T_i^{-1}(\gamma)|} \text{ for almost all } \gamma \in [0,1] \quad (15)
\]

we have to estimate

\[
I := \sup_{r \leq A} \frac{1}{r^p} \int \sup_{\gamma_2, \gamma_1 \in B(x,r)} ||(LF\mu)|_{\gamma_1} - (LF\mu)|_{\gamma_2}||_{W_1} \ dm(x)
\]

\[
= \sup_{r \leq A} \frac{1}{r^p} \int \sup_{\gamma_2, \gamma_1 \in B(x,r)} \left| \sum_{i=1}^{q} \left( \frac{F_{T_i^{-1}(\gamma)}^* \mu_{T_i^{-1}(\gamma)}}{|T_i \circ T_i^{-1}(\gamma)|} - \frac{F_{T_i^{-1}(\gamma)}^* \mu_{T_i^{-1}(\gamma)}}{|T_i' \circ T_i^{-1}(\gamma)|} \right) \right| \ dm(x).
\]

To compact notations let us set in next equations

\[
F^*(a,b) := F_{T_i^{-1}(a)}^* \mu_{T_i^{-1}(b)}, \ g_i(a) := T_i' \circ T_i^{-1}(a)
\]

By the triangular inequality

\[
I \leq \sup_{r \leq A} \sum_{i=1}^{q} \frac{1}{r^p} \int \sup_{\gamma_2, \gamma_1 \in B(x,r)} \left| \frac{F^*(\gamma_1, \gamma_1)}{|g_i(\gamma_1)|} - \frac{F^*(\gamma_2, \gamma_2)}{|g_i(\gamma_2)|} \right| \ dm(x)
\]

\[
+ \sup_{r \leq A} \sum_{i=1}^{q} \frac{1}{r^p} \int \sup_{\gamma_2, \gamma_1 \in B(x,r)} \left| \int \frac{F^*(\gamma_2, \gamma_2)}{|g_i(\gamma_1)|} - \frac{1}{|g_i(\gamma_2)|} \right| \ dm(x).
\]
Recalling that \( \frac{1}{|g_1(\gamma_1)|} - \frac{1}{|g_1(\gamma_2)|} \leq C_h d(\gamma_1, \gamma_2)^\xi \), then
\[
I \leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \sup_{\gamma_2, \gamma_1 \in B(x, r)} \| F^*(\gamma_1, \gamma_1) - F^*(\gamma_2, \gamma_2) \|_{W_1} dm
\]
\[
+ \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int C_h r^\xi \sup_{\gamma_2} \| F^*(\gamma_2, \gamma_2) \|_{W_1} dm.
\]
And
\[
I \leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \sup_{\gamma_2, \gamma_1 \in B(x, r)} \| F^*(\gamma_1, \gamma_1) - F^*(\gamma_2, \gamma_2) \|_{W_1} dm
\]
\[
+ \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int C_h r^\xi \sup_{\gamma_2} \| F^*(\gamma_2, \gamma_2) \|_{W_1} dm.
\]
Hence
\[
I \leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \sup_{\gamma_2, \gamma_1 \in B(x, r)} \| F^*(\gamma_1, \gamma_1) - F^*(\gamma_2, \gamma_2) \|_{W_1} dm
\]
\[
+ \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int C_h r^\xi \sup_{\gamma_2} \| F^*(\gamma_2, \gamma_2) \|_{W_1} dm
\]
\[
+ 3 \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int C_h r^\xi \sup_{\gamma_2} \| F^*(\gamma_2, \gamma_2) \|_{W_1} dm
\]
\[
= I_a + I_b + I_c.
\]
Now
\[
I_a \leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \sup_{\gamma_2, \gamma_1 \in B(x, r)} \| F^*_{T_i^{-1}\gamma_1}\left(\mu_{|T_i^{-1}\gamma_1}\right) - \mu_{|T_i^{-1}\gamma_2}\|_{W_1}
\]
We recall that by Lemma \[16\] \( \| F^\mu \|_{W_1} \leq c \| \mu \|_{W_1} \) then
\[
I_a \leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \sup_{\gamma_2, \gamma_1 \in B(x, r)} \alpha \| \mu_{|T_i^{-1}\gamma_1}\|_{W_1}
\]
\[
\leq \sup_{r \leq A} \sum_{i=1}^q \frac{1}{r^p} \int_{I_i, y_1, y_2 \in B(x, \lambda r)} \alpha \| \mu_{y_1} - \mu_{y_2}\|_{W_1} dx.
\]
\[
\leq \lambda^p \sup_{k \leq \Lambda A} \sum_{i=1}^q \frac{1}{r^p} \int_{I_i, y_1, y_2 \in B(x, \lambda k)} \alpha \| \mu_{y_1} - \mu_{y_2}\|_{W_1} dx
\]
and 
\[ \text{var}_p(L_F \mu) \leq \lambda \alpha \text{var}_p(\mu) + (I_b + I_c). \]

By Sk3
\[
I_b = \sup_{r \leq A} \sum_i \frac{1}{r^p} \int \frac{1}{|g_i(x)|} \text{esssup}_{y_1, y_2 \in B(x, r)} \| F^*(\gamma_1, \gamma_2) - F^*(\gamma_2, \gamma_2) \| w_1 \, dm.
\]
\[
\leq \hat{H} \| \mu_x \|_\infty.
\]

Now, let us remark that since we are working with positive measures
\[
\text{esssup}_{\gamma_2} \| F^*(\gamma_2, \gamma_2) \| w_1 \leq \alpha \| \mu_x \|_\infty,
\]
then
\[
I_c = 3 \sup_{r \leq A} \sum_q \frac{1}{r^p} \int C h r^\xi \text{esssup}_{\gamma_2} \| F^*(\gamma, \gamma_2) \| w_1 \, dm
\]
\[
\leq 3 \sup_{r \leq A} \sum_q \frac{1}{r^p} \int C h r^\xi \alpha \| \mu_x \|_\infty \, dm \quad (16)
\]
\[
\leq 3 q C h A^{\xi - p} \alpha \| \mu_x \|_\infty. \quad (17)
\]

Summarizing
\[
\text{var}_p(L_F \mu) \leq \lambda \alpha \text{var}_p(\mu) + \left( \hat{H} \| \mu_x \|_\infty + 3 q \alpha C h A^{\xi - p} \| \mu_x \|_\infty \right). \quad (18)
\]

\[\blacksquare\]

**Remark 21** By Equation [7] it holds that for each \( n \)
\[
\| L^n \mu_x \|_\infty \leq C_{\mu} := A^{p-1}(A_T \lambda \| \mu_x \|_{BV} + B_T \| \mu_x \|_1 + 1).
\]

Iterating (18) we obtain
\[
\text{var}_p(L^n_F \mu) \leq (\lambda \alpha)^n \text{var}_p(L^{n-1}_F \mu) + \left( \hat{H} + 3 q \alpha C h A^{\xi - p} \right) C_{\mu} \quad (19)
\]
\[
\leq \ldots \leq (\lambda \alpha)^n \text{var}_p(\mu) + \frac{\hat{H} + 3 q \alpha C h A^{\xi - p}}{1 - \lambda \alpha} C_{\mu}
\]

By the Helly-like selection principle (Theorem [19]) we then have

**Proposition 22** In a system as above there is at least an invariant positive measure in \( p - BV \). For every such invariant measure \( \mu \)
\[
\text{var}_p(\mu) \leq \frac{B_T(\hat{H} + 3 q \alpha C h A^{\xi - p})}{1 - \lambda \alpha} \| \mu \|_{-1'}.
\]
Proof. We consider the sequence of positive measures \( \mu_n = \frac{1}{n} \sum L^n \nu \). By Equation 19, this sequence has uniformly bounded variation. Applying Theorem 19, we deduce the existence of an invariant measure \( \mu \) in \( p - BV \). By the Lasota Yorke inequality relative to the map \( T \), we have that

\[
A^{p-1}B_T ||\mu||_{-1} \geq A^{p-1}B_T ||\mu_x||_1 \geq A^{p-1}||\mu_x||_{BV} \geq ||\mu_x||_\infty.
\]

This gives that

\[
\text{var}_p(\mu) = \text{var}_p(LF \mu) \leq \lambda^p \alpha \text{var}_p(\mu) + B_T ||\mu||_{-1} A^{p-1}(\hat{H} + 3a \beta C A^{\xi-p})
\]

from which we get the statement. ■

5 Distance between the operators and a general statement for skew products

Here we consider a suitable class of perturbations of a map satisfying \( Sk_1, ..., Sk_3 \) such that the associated transfer operators are near in the strong-weak topology, providing one of the estimates needed to apply Theorem 5 (item 4). In this section and in the following we set \( p = 1 \). Now we define a topology on the space of piecewise expanding maps to have a notion of "allowed perturbations" for these maps.

Definition 23 Let \( T_1 \) and \( T_2 \) be to piecewise expanding maps. Denote by

\[ \text{Int}_n = \{ A \in [0,1]^2, s.t. A = I_1 \cup \ldots \cup I_n, \text{where } I_i \text{ are intervals} \} \]

the set of subsets of \([0,1]\) which is the union of at most \( n \) intervals. Set

\[ C(n,T_1,T_2) = \{ \epsilon : \exists A_1 \in \text{Int}_n \text{ and } \exists \sigma : I \to I \text{ a diffeomorphism s.t.} \]

\[ m(A_1) \geq 1 - \epsilon, T_1|_{A_1} = T_2 \circ \sigma|_{A_1} \]

\[ \text{and } \forall x \in A_1, |\sigma(x) - x| \leq \epsilon, |\frac{1}{\sigma'(x)} - 1| \leq \epsilon \} \]

and define a kind of distance from \( T_1 \) to \( T_2 \) as:

\[ d_{S,n}(T_1,T_2) = \inf \{ \epsilon | \epsilon \in C(n,T_1,T_2) \}. \]

It holds that one dimensional piecewise expanding maps which are near in the sense of \( d_{S,n} \) also have transfer operators which are near as operators from \( BV \) to \( L^1 \). If we denote by \( d_S \) the classical notion of Skorokhod distance (see 12 e.g.), it is obvious that \( \forall n d_{S,n} \geq d_S \). By 12, Lemma 11.2.1, it follows that \( \forall n \) there is \( C_{Sk} \geq 0 \) such that for each pair of piecewise expanding maps \( T_1, T_2 \)

\[ ||L_{T_1} - L_{T_2}||_{BV \to L^1} \leq C_{Sk} d_{n,S}(T_1,T_2). \]

Let us see a statement of this kind for our skew products.

Proposition 24 Let \( F_\delta = (T_\delta, G_\delta), 0 \leq \delta \leq D \) be a family of maps satisfying \( Sk_1, ..., Sk_3 \) uniformly with \( \xi = 1 \) and:

\[ ||L_{T_\delta} - L_{T_\varepsilon}||_{BV \to L^1} \]

\[ \leq C_{Sk} d_{n,S}(T_\delta, T_\varepsilon). \]

17
1. There is \( n \in \mathbb{N} \) such that for each \( \delta \leq D \), \( d_{n,S}(T_0,T_\delta) \leq \delta \). (thus for each \( \delta \) there is a set \( A_1 \in \text{Int}_n \) as in the definition of \( C(n,T_1,T_2) \))

2. For each \( \delta \leq D \) there is a set \( A_2 \in \text{Int}_n \) such that \( m(A_2) \geq 1 - \delta \) and for all \( x \in A_2, y \in M : |G_0(x,y) - G_\delta(x,y)| \leq \delta \).

Let us denote by \( F_\delta^* \) the transfer operators of \( F_\delta \) and by \( f_\delta \) a family of probability measures with uniformly bounded variation

\[
\text{var}_1(f_\delta) \leq M_2.
\]

Then, there is a constants \( C_1 \) such that for \( \delta \) small enough

\[
\|(F_0^* - F_\delta^*)f_\delta\|_{\gamma} \leq C_1 \delta (M_2 + 1).
\]

**Proof.** Let us set \( A = A_1 \cap A_2 \). Note that \( m(A^c) \leq 2\delta \). Let us estimate

\[
\|(F_0^* - F_\delta^*)f_\delta\|_1 = \int_I \|(F_0^*f_\delta - F_\delta^*f_\delta)\|_\gamma \ dm(\gamma) \quad (22)
\]

\[
= \int_I \|F_0^*(1_Af_\delta)\|_\gamma - F_\delta^*(1_Af_\delta)\|_\gamma \ dm(\gamma)
\]

\[
+ \int_I |F_0^*(1_Af_\delta)\|_\gamma - F_\delta^*(1_Af_\delta)\|_\gamma \ dm(\gamma). \quad (23)
\]

By the assumptions, for a.e. \( \gamma, \|f_\delta\|_\gamma \leq (M_2 + 1) \) and \( \|1_A f_\delta\|_1 \leq (M_2 + 1) \). Since \( F^* \) is \( \alpha \)-Lipschitz for the \( \mathcal{L}^1 \) norm then

\[
\int_I \|F_0^*(1_Af_\delta)\|_\gamma - F_\delta^*(1_Af_\delta)\|_\gamma \ dm(\gamma) \leq 2\alpha (M_2 + 1) \delta.
\]

Let us now estimate the first summand of \((22)\). Let us set \( \mu = 1_A f_\delta \) and let us estimate

\[
\|(F_0^* - F_\delta^*)\mu\|_1 = \int_I \|(F_0^*\mu - F_\delta^*\mu)\|_\gamma \ dm(\gamma).
\]

Let us denote by \( T_{0,i}, \) with \( 0 \leq i \leq q \) the branches of \( T_0 \) defined in the sets \( P_i \), partition of \( I \), and set \( T_{\delta,i} = T_\delta|_{P_i \cap A} \) these functions will play the role of the branches for \( T_\delta \). Since in \( A, T_0 = T_\delta \circ \sigma_\delta \) (where \( \sigma_\delta \) is the diffeomorphism in the definition of the Skorokhod distance), then \( T_{\delta,i} \) are invertible. Then for \( \mu = \text{a.e.} \gamma \in I \)

\[
(F_0^*\mu - F_\delta^*\mu)\|_\gamma = \sum_{i=1}^q \frac{F_{0,T_{\delta,i}^{-1}(\gamma)}^{*}\mu|_{T_{\delta,i}^{-1}(\gamma)} X_{T_0(P_i \cap A)}}{|T_{\delta,i}^{-1}(\gamma)|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^{*}\mu|_{T_{\delta,i}^{-1}(\gamma)} X_{T_\delta(P_i \cap A)}}{|T_{\delta,i}^{-1}(\gamma)|}.
\]

Let us now consider \( T_0(P_i \cap A) \) and \( T_\delta(P_i \cap A) \), and remark that \( T_0(P_i \cap A) = \sigma_\delta(T_\delta(P_i \cap A)) \) and \( \sigma_\delta \) is a diffeomorphism near to the identity. Let us denote
\( B_t = T_0(P_t \cap A) \cap T_\delta (P_t \cap A), \ C_t = T_0(P_t \cap A) \triangle T_\delta (P_t \cap A). \)

\[
\| (F^*_0 - F^*_\delta) \mu \|_1 = \int_I \| (F^*_0 \mu - F^*_\delta \mu) \|_W \, dm(\gamma)
\]

\[
\leq \int_I \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm

+ \int_I \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{C_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{C_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm.
\]

And since there is \( K_1 \) such that \( m(C_i) \leq K_1 \delta \), then

\[
\int_I \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{C_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{C_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm \leq qK_1(M_2 + 1)\delta.
\]

Now we have to consider the first summand of (24). We have

\[
\int_I \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm

\leq \int_I \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm

+ \int_I \left\| \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm

= \int_I I(\gamma) \, dm(\gamma) + \int_I II(\gamma) \, dm(\gamma).
\]

The two summands will be treated separately.

\[
I(\gamma) = \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W
\]

\[
\leq \left\| \sum_{i=1}^q \frac{F^*_{0, T_{0,i}^{-1}(\gamma)} \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W

+ \left\| \sum_{i=1}^q \frac{F^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W

= I_a(\gamma) + I_b(\gamma).
\]
Then by assumption (2),

$$\int I_a(\gamma) dm = \left\| \sum_{i=1}^{q} \frac{F_{0,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} - \sum_{i=1}^{q} \frac{F_{\delta,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} \right\|_W dm(\gamma)$$

$$\leq \int \left\| \sum_{i=1}^{q} \frac{F_{0,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} - \frac{F_{\delta,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} \right\|_W dm dm(\beta)$$

$$\leq \sum_{i=1}^{q} \int_{T_0^{-1}(B_i)} \left\| F_{0,\beta}^*\mu_{\beta} - F_{\delta,T_0^{-1}(\gamma)}^*\mu_{\beta} \right\|_W dm(\beta)$$

Remark that $T_0^{-1}(B_i) \subseteq P_i \cap A$ and $T_0^{-1}(T_0^{-1}(B_i)) \subseteq P_i \cap A$. Since $|T_0^{-1}(\beta) - T_0^{-1}(\beta)| \leq \delta$ and $T_0^{-1}$ is a contraction, then $|T_0^{-1} \circ T_0^{-1}(\beta) - \beta| \leq \delta$. Then

$$\left\| F_{0,\beta}^*\mu_{\beta} - F_{\delta,T_0^{-1}(\gamma)}^*\mu_{\beta} \right\|_W \leq \left\| F_{0,\beta}^*\mu_{\beta} - F_{\delta,\beta}^*\mu_{\beta} \right\|_W + \left\| F_{\delta,\beta}^*\mu_{\beta} - F_{\delta,T_0^{-1}(\gamma)}^*\mu_{\beta} \right\|_W.$$

By assumption (2),

$$\left\| F_{0,\beta}^*\mu_{\beta} - F_{\delta,\beta}^*\mu_{\beta} \right\|_W \leq \delta(M_2 + 1).$$

By assumption Sk3,

$$\left\| F_{\delta,\beta}^*\mu_{\beta} - F_{\delta,T_0^{-1}(\gamma)}^*\mu_{\beta} \right\|_W \leq \sup_{y \in M, x_1, x_2 \in B(\beta, \delta)} |G(x_1, y) - G(x_2, y)| (M_2 + 1).$$

Thus,

$$I_a(\gamma) \leq \delta^p (\bar{H} + 1)(M_2 + 1) + \delta(M_2 + 1).$$

To estimate $I_b(\gamma)$ we have:

$$I_b(\gamma) = \left\| \sum_{i=1}^{q} \frac{F_{\delta,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} - \frac{F_{\delta,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)}^i T_0^{-1}(\gamma) \chi_{B_i}}{|T_0^{-1}(\gamma)|} \right\|_W$$

$$\leq \sum_{i=1}^{q} \left\| \frac{\chi_{B_i}(\gamma)}{|T_0^{-1}(\gamma)|} - \frac{\chi_{B_i}(\gamma)}{|T_0^{-1}(\gamma)|} \right\|_W \left\| F_{\delta,T_0^{-1}(\gamma)}^*\mu_{T_0^{-1}(\gamma)} \right\|_W$$

and

$$\int I_b dm \leq [(P_{T_0} - P_{T_2})(1)] \alpha(M_2 + 1) + qK_1 \delta.$$
by Equation \(21\) then

\[
\int_{A_1} I_\delta(\gamma) \, dm(\gamma) \leq [C_{Sk} \alpha(M_2 + 1) + qK_1] \delta.
\]

Now, let us estimate the integral of the second summand

\[
II(\gamma) = \left\| \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}}^*(\gamma) \mu |_{T_{\delta,i}^{-1}(\gamma) \cap B_i}}{|T_{\delta,i}^\prime(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}}^*(\gamma) \mu |_{T_{\delta,i}^{-1}(\gamma) \cap B_i}}{|T_{\delta,i}^\prime(T_{\delta,i}^{-1}(\gamma))|} \right\|_W.
\]

Then, setting \(g_i(a) := T_{\delta,i}^\prime \circ T_{\delta,i}^{-1}(a)\) to compact notations

\[
\int_I II(\gamma) \, dm(\gamma) \leq \sum_{i=1}^q \int_{B_i} \frac{1}{g_i(\gamma)} \left\| F_{\delta,T_{\delta,i}^{-1}}^*(\gamma) \left( \mu |_{T_{\delta,i}^{-1}(\gamma)} - \mu |_{T_{\delta,i}^{-1}(\gamma)} \right) \right\|_W \, dm(\gamma)
\]

\[
\leq \sum_{i=1}^q \int_{B_i} \frac{\alpha}{g_i(\gamma)} \left\| \mu |_{T_{\delta,i}^{-1}(\gamma)} - \mu |_{T_{\delta,i}^{-1}(\gamma)} \right\|_W \, dm(\gamma)
\]

Let us consider the change of variable \(\gamma = T_{\delta,i}(\beta)\), then

\[
\int_I II(\gamma) \, dm(\gamma) \leq \alpha \sum_{i=1}^q \int_{T_{\delta,i}^{-1}(B_i)} \left\| \mu |_{T_{\delta,i}^{-1}(\beta)} - \mu |_{\beta} \right\|_W \, dm(\beta).
\]

Since \(|T_{\delta,i}(\beta) - T_{\delta,i}(\beta)| \leq \delta\) and \(T_{\delta,i}^{-1}\) is a contraction, then \(|T_{\delta,i}^{-1} \circ T_{\delta,i}(\beta) - \beta| \leq \delta\) then

\[
\int_I II(\gamma) \, dm(\gamma) \leq \alpha \int_{B(\delta,\beta)} \sup_{x,y \in B(\beta,\delta)} (||x| - |y||_W) \, dm(\beta) \leq \alpha \int osc(\mu, \beta, \delta) \, d\beta
\]

and then

\[
\int_I II(\gamma) \, dm(\gamma) \leq \alpha 2\delta(M_2 + 1).
\]

Summing all, the statement is proved. \(\blacksquare\)

The last statement, together with the results of the previous sections allows to prove the following quantitative statement for skew product maps.

**Proposition 25** Consider a family of skew product maps \(F_\delta = (T_\delta, G_\delta), 0 \leq \delta \leq D\) satisfying Sk1, ..., Sk3 uniformly, with \(\xi = 1\), and let \(f_\delta \in L^1\) invariant probability measures of \(F_\delta\), suppose:

1. There is \(\phi \in C^0(\mathbb{R}), \phi(t)\) decreasing to 0 as \(t \to \infty\) such that \(L_{F_\delta}\) has convergence to equilibrium with respect to norms \(||\cdot||_{1-BV}, ||\cdot||_{1^*}\) and speed \(\phi\);
2. there is \(\tilde{C} \geq 0\) such that for each \(n\)

\[
\|L_{F_\delta}^n\|_{L^1 \to L^1} \leq \tilde{C};
\]
3. there is \( n \in \mathbb{N} \) such that for each \( \delta \leq D \), \( d_{n,S}(T_0, T_\delta) \leq \delta \);

4. for each \( \delta \leq D \) there is a set \( A_2 \in \text{Int}_n \) such that \( m(A_2) \geq 1 - \delta \) and for all \( x \in A_2, y \in M : |G_0(x, y) - G_\delta(x, y)| \leq \delta \).

Let \( B = \frac{B_T(\hat{H} + 3q \alpha C_h)}{1 - \lambda^p \alpha} + 1 \). Consider the function \( \psi \) defined as \( \psi(x) = \frac{2(x)}{x} \), then

\[
\|f_\delta - f_0\|_{-1} \leq 2C_B^2 \|C_1 \delta^{-1}(\frac{C_B C_1 \delta}{2}) + 1\|.
\]

where \( C_1 \) is the constant in the statement of Proposition 24.

**Proof.** The proof is a direct application of the estimates given in the previous section into Theorem 5. The quantity \( \tilde{M} \) appearing at Item 2 of Theorem 5 is estimated by Proposition 22:

\[
\tilde{M} \leq \frac{B_T(\hat{H} + 3q \alpha C_h)}{1 - \lambda^p \alpha}.
\]

By Proposition 24 the distance between the operators appearing at Item 4 of Theorem 5 is bounded by \( \epsilon \leq C_1 \delta (M_2 + 1) \) Where \( M_2 \) bounds the strong norm of \( f_\delta \).

We remark that the quantitative stability is proved here in the \( \| \|_{-1} \) topology. This topology is strong enough to control the behavior of observables which are discontinuous along the preserved central foliation, see [10] for other results on quantitative stability of the statistical properties of discontinuous observables and related applications.

In the following section we show a class of nontrivial partially hyperbolic skew products having power law convergence to equilibrium and will apply this statement to these examples.

6 Application to slowly mixing toral extensions

To give an example of application of Proposition 25 to a class of nontrivial system, we consider a class of "partially hyperbolic" skew products with some discontinuities, having slow (power law) decay of correlations and convergence to equilibrium.

We will consider a class of skew products \( F = (T, G) \) (piecewise constant toral extensions) defined as follows:

Tel l \( \in \mathbb{N} \). We assume that \( T \) is the piecewise expanding map on \([0, 1]\) defined as

\[
T(x) = lx \mod(1);
\]
The system is extended by a skew product to a system \((X, F)\) where \(X = [0, 1] \times T^d\), where \(T^d\) is the \(d\) dimensional torus and \(F : X \to X\) is defined by

\[
F(x, t) = (Tx, t + \varphi(x))
\]

where \(\varphi = \varphi \chi_I\) is the characteristic function of a set \(I \subset [0, 1]\) which is an union of the sets \(P_i\) where the branches of \(T\) are defined. In this system the second coordinate is translated by \(\varphi\) if the first coordinate belongs to \(I\).

We remark that on the system \((X, F)\) the Lebesgue measure is invariant. We will suppose that \(\varphi\) is of finite Diophantine type. Let us recall the definition of Diophantine type for the linear approximation. The definition tests the possibility of approximating \(0\) by an integer linear combination of its components.

The notation \(\| \cdot \|\) will indicate the distance to the nearest integer vector (or number) in \(\mathbb{R}\), and \(|k| = \sup_{0 \leq i \leq d} |k_i|\) indicates the supremum norm.

**Definition 26** The Diophantine type of \(\varphi = (\theta_1, \ldots, \theta_d)\) for the linear approximation is

\[
\gamma_l(\varphi) = \inf \{ \gamma , s.t. \exists c_0 > 0 \ s.t. \| k \cdot \varphi \| \geq c_0 |k|^{-\gamma} \ \forall 0 \neq k \in \mathbb{Z}^d \}.
\]

**6.1 The decay of correlations**

In [31], it was observed that piecewise constant toral extensions cannot have exponential decay of correlations (in [30] by the way it is shown that for some piecewise constant \(SU_2(\mathbb{C})\) extensions there can be exponential decay of correlations). Quantitative estimates for the speed of decay of correlations by the arithmetical properties of the angles, have been given in [21].

In this section we recall those results and see that the systems defined above have at least polynomial decay of correlations, while for some choice of the angles the speed of decay is proved to be actually polynomial.

**Definition 27 (Decay of correlations)** Let \(\phi, \psi : X \to \mathbb{R}\) be observables on \(X\) belonging to the Banach spaces \(B, B'\), let \(\nu\) be an invariant measure for \(T\). Let \(\Phi : \mathbb{N} \to \mathbb{R}\) such that \(\Phi(n) \to 0\). A system \((X, T, \nu)\) is said to have decay of correlations with speed \(\Phi\) with respect to observables in \(B\) and \(B'\) if

\[
\left| \int \phi \circ T^n \psi d\nu - \int \phi d\nu \int \psi d\nu \right| \leq ||\phi||_B ||\psi||_{B'} \Phi(n)
\]

where \(|| \cdot ||_B, || \cdot ||_{B'}\) are the norms in \(B\) and \(B'\).

The decay of correlations depends on the class of observables considered. On the skew products satisfying conditions \(Te1\) and \(Te2\) as above, it is possible to establish an explicit upper bound for the rate of decay of correlations which depend on the linear type of the translation angle (see [21], Lemma11).
Proposition 28  In the piecewise constant toral extensions described above, for Lipschitz observables the rate of decay is

\[ \Phi(n) = O(n^{-\frac{1}{2\gamma}}) \]

for any \( \gamma > \gamma_l(\theta) \).

For \( C^p, C^q \) observables, the rate of decay is

\[ \Phi(n) = O(n^{-\frac{1}{2}\max(p,q,p+q-d)}) \]

for any \( \gamma > \gamma_l(\theta) \).

Remark 29 We remark that the rate is actually polynomial in some cases. In [21], Section 5 (using a result of [24]) it is proved that if the Diophantine type is large, then the mixing rate of the systems we consider is actually slow, with a power law speed which depends on the Diophantine type. In a system satisfying (26), let the exponent of power law decay be defined by

\[ p = \lim \inf_{n \to \infty} \frac{-\log \Phi(n)}{\log n} \]

Let us consider the skew product of the doubling map and a circle rotation endowed with the Lebesgue (invariant) measure. For this example the exponent \( p \) satisfies

\[ \frac{1}{2\gamma(\theta)} \leq p \leq \frac{6}{\max(2, \gamma(\theta)) - 2} \]

6.2 Convergence to equilibrium

We will use the decay of correlations of the toral extensions to get a convergence to equilibrium result with respect to the strong and weak norm of our anisotropic spaces. We have from Proposition 28 that for Lipschitz observables the rate of decay is \( O(n^{-\frac{1}{2\gamma}}) \) for any \( \gamma > \gamma_l(\alpha) \) and for any Lipschitz observables with \( \int f = 0 \):

\[ |\int g \circ F^n f \, dm| \leq C||f||_{lip}||g||_{lip} n^{-\frac{1}{2\gamma}}. \]

From this we will prove that

\[ ||L^n \mu||_{-1} \leq C4n^{-\frac{1}{2\gamma}} ||\mu||_{1-BV}. \]

For this purpose our strategy is to approximate a \( 1 - BV \) measure \( \mu \) which is meant to be iterated with a Lipschitz density and use the decay of correlations with Lipschitz observables to estimate its convergence to equilibrium. We remark that a statement of this kind extend greatly the kinds of measures which are meant to be iterated, as the space of \( 1 - BV \) measures contains measures with singular behavior in the neutral direction.

The first step in the strategy is approximating the disintegration of \( \mu \) with a kind of "piecewise constant one" in next Lemma.
Lemma 30 Let us consider a uniform grid of size $\epsilon = \frac{1}{m}$, $m \in \mathbb{N}$, on the interval $[0, 1]$. Given a measure $\mu$ with $||\mu||_{1-BV} < \infty$. There is a measure $\mu_\epsilon$ such that $\mu_\epsilon$ is piecewise constant on the $\epsilon$-grid ($\mu_\epsilon|_x$ is constant on each element of the grid as $x$ varies) and

$$\text{var}_1(\mu_\epsilon) \leq 2\text{var}_1(\mu), \quad ||\mu_\epsilon||_1 \leq ||\mu||_1$$

furthermore suppose

$$||\mu - \mu_\epsilon||_1 \leq 2\epsilon\text{var}_p(\mu).$$

Proof. Let us consider $\mu_\epsilon$ defined by averaging in the following way: let $x \in [0, 1]$ and $I_i$ be the element of the $\epsilon$-grid containing $x$. Then for a measurable set $A \subseteq M \mu|_x(A)$ is defined as

$$\mu_\epsilon|_x(A) = \int_{I_i} \mu_\epsilon|_x(A) d\gamma.$$

We remark that $\mu|_x - \mu_\epsilon|_x \leq \text{osc}(\epsilon, x_i(x), \mu)$ where $x_i(x)$ is the grid center closest to $x$, and $\text{osc}(\epsilon, x_i(x), \mu) \leq \text{osc}(2\epsilon, x, \mu)$ then

$$\int ||\mu|_x - \mu_\epsilon|_x|| W \leq 2\epsilon \int \frac{\text{osc}(2\epsilon, x, \mu)}{2\epsilon} \leq 2\epsilon \sup_{2\epsilon \leq A} \frac{\text{osc}(2\epsilon, x, \mu)}{2\epsilon} \leq 2\epsilon\text{var}_p(\mu)$$

The other inequalities are analogous. ■

Proposition 31 The convergence to equilibrium of a system satisfying Te1, Te2 can be estimated as

$$||L^n \nu||_1 \leq C_4 n^{-\frac{1}{m}} ||\nu||_{1-BV}.$$

Proof. Consider a $1 - BV$ measure $\nu$, without loss of generality we can suppose $||\nu||_{1-BV} = 1$. Let us approximate $\nu$ it with a Lipschitz measure. First let us approximate it with a piecewise constant measure $\nu_\epsilon$ as before. We have

$$||\nu - \nu_\epsilon||_1 \leq 2\epsilon\text{var}_1(\nu) \leq 2\epsilon.$$

Let $\nu_i$ be such that $\nu_i = \nu_\epsilon|_{x_i}$ with $x_i$ center of the $\epsilon$ grid as before, and $f_i$ be the convolution $\gamma * \nu_i$ where $\gamma$ is a $e_2^{-1}$ Lipschitz mollifier supported in $[-\epsilon_2, \epsilon_2]^d$. $f_i$ is a $e_2^{-1}$ Lipschitz function. Let

$$f(x, y) = \begin{cases} f_i(y) & \text{if } |x - x_i| \leq (1 - \epsilon_2)\epsilon \\ \phi_i(x)f_i(y) + (1 - \phi_i(x))f_{i+1}(y) & \text{if } x_i + (1 - \epsilon_2)\epsilon \leq x \leq x_{i+1} - (1 - \epsilon_2)\epsilon \end{cases}$$
where \( \phi_i \) is a linear function s.t. \( \phi_i(x_i+(1-\epsilon_2)e) = 0 \) and \( \phi_i(x_i+1-(1-\epsilon_2)e) = 1 \). We remark that \( f \) is \( \sqrt{2e^{-1}} \) Lipschitz, \( \int f \, dm = 0 \) and \( ||\nu - f m||_{1''} \leq 3\epsilon_2 \). Hence

\[
||\nu - f m||_{1''} \leq 2\text{var}_1(\nu) + 3\epsilon_2. \tag{27}
\]

Since the convolution with a Lipschitz kernel is a weak contraction in the Wasserstein norm, applying Lemma \( \ref{30} \) we get \( \text{var}_1(f m) \leq 2\text{var}_1(\nu) \) and \( ||f||_{1''} \leq ||\nu||_{1''} \). Now we apply Proposition \( \ref{28} \) in an efficient way. The proposition concerns the behavior of the correlation of two observables. We will consider \( f \) as one of them, and the other will be constructed in a suitable way to get the desired statement.

Let \( f \) be the Lipschitz density found above. Let \( \mu = L^n f m \). Let \( \mu_\epsilon \) its piece-wise constant approximation defined as in Lemma \( \ref{30} \) and \( \mu_i = \mu_\epsilon |_{x_i} \). Consider 1-Lipschitz functions \( l_i : T^d \to \mathbb{R} \) such that \( ||l_i\mu_i| - ||\mu_i||_{W} \leq \xi \), consider functions \( h_i : [0, 1] \to \mathbb{R} \) such that \( h_i = 1 \) on the central third of the \( i \) interval of the \( \epsilon \)-grid and zero elsewhere, and \( l_{lip}(h_i) = 3\epsilon^{-1} \). Consider \( g_i : X \to \mathbb{R} \) defined by \( g_i(x,y) = l_i(y)h_i(x) \) and \( g = \sum_i g_i \). By what is said above

\[
||\mu_\epsilon||_1 \leq 3(\xi + \int g \mu_\epsilon)
\]

and by Lemma \( \ref{30} \) \( ||L^n f m - \mu_\epsilon||_1 \leq 2\text{var}_1(\mu) \). Then

\[
||L^n(f m)||_1 \leq ||\mu_\epsilon||_1 + 2\text{var}_1(\mu) \\
\leq 3(\xi + \int g \mu_\epsilon) + 2\text{var}_1(\mu).
\]

Now consider \( \int g L^n f m \). Since \( g \) is 1-Lipschitz in the \( y \) direction, we have that

\[
|\int g L^n f - \int g \mu_\epsilon| \leq ||L^n f m - \mu_\epsilon||_1 \leq 2\text{var}_1(\mu)
\]

and

\[
||L^n f||_1 \leq 3(\xi + \int g dL^n f \mu_0 + 2\text{var}_1(\mu)) + 2\text{var}_1(\mu).
\]

Now, since \( \int f dm = 0 \), by Proposition \( \ref{28} \)

\[
|\int g dL^n(f m)| \leq C||f||_{lip}||g||_{lip} n^{-\frac{1}{\alpha}}
\]

then

\[
||L^n f||_1 \leq 3(\xi + C||f||_{lip}3\epsilon^{-1}n^{-\frac{1}{\alpha}} + 2\text{var}_1(L^n f m)) + 2\text{var}_1(L^n(f m)).
\]

We recall that the Lebesgue measure is invariant for the system. Then if \( K \) is a constant density such that \( f + K \geq 0 \) it holds \( L^n(f m + K m) = L^n(f m) + K m \). It holds \( \text{var}_1(L^n f) = \text{var}_1(L^n(f + K)) \), since it is a positive measure, to \( (f + K)m \) we can apply the regularization inequality. Setting \( B = \frac{2\gamma_l(\lambda^n + \beta + 3\epsilon C_\lambda)}{1-\lambda^n} \) we get

\[
\text{var}_1(L^n(f m)) \leq \lambda^n \alpha^n \text{var}_1(f) + B(||f||_{1''} + K) \tag{28}
\]
since $f$ is $\sqrt{2}e^{-1}e^{-1}$-Lipschitz and $||f||_{1-BV} \leq 1$, $||f||_\infty \leq \sqrt{2}e^{-1}e^{-1} + 1$, then $\text{var}(L^n(fm)) \leq \lambda^n \alpha^n \text{var}(f) + B(1 + \sqrt{2}e^{-1}e^{-1} + 1)$ and

$$||L^n(fm)||_1 \leq 3\xi + 3C||f||_{lip}3e^{-1}n^{-\frac{1}{n}} + 8\text{var}_1(L^n f) \leq 3\xi + 3C||f||_{lip}3e^{-1}n^{-\frac{1}{n}} + 8[\lambda^n \alpha^n \text{var}_1(f) + B(1 + \sqrt{2}e^{-1}e^{-1} + 1)] \leq 3\xi + 3C\sqrt{2}e^{-1}e^{-1}3e^{-1}n^{-\frac{1}{n}} + 16[\lambda^n \alpha^n \text{var}_1(f) + B(1 + \sqrt{2}e^{-1}e^{-1} + 1)]$$

Taking $\xi = n^{-\frac{1}{n}}, \epsilon_2 = n^{-\frac{1}{n}}$, $\epsilon = n^{-\frac{1}{n}}$ recalling that $\text{var}_1(f) \leq \text{var}_1(\nu) \leq 1$

$$||L^n(fm)||_1 \leq 3n^{-\frac{1}{n}} + 9C\sqrt{2}n^{-\frac{1}{n}} + 16n^{-\frac{1}{n}}(\alpha \lambda^n + 2B) + \sqrt{2}Bn^{-\frac{1}{n}} \leq C_3n^{-\frac{1}{n}}.$$

Finally, by Equation 27

$$||L^n \nu||_1 \leq ||L^n(\nu - fm)||_{1-1} + ||L^n(fm)||_{1-1} \leq 2\epsilon||\nu||_{1-BV} + ||L^n(fm)||_{1-1} + 3\epsilon_2 \leq C_4n^{-\frac{1}{n}}.$$

Once we have an estimate for the speed of convergence to equilibrium, by Proposition 29 and Remark 6, the following holds directly:

**Proposition 32** Consider a family of skew product maps $F_\delta = (T_\delta, G_\delta)$, $0 \leq \delta \leq D$ satisfying uniformly Sk1, ..., Sk3 and let $f_\delta \in L^1$ its invariant probability measures, suppose

1. $F_0$ is a piecewise constant toral extension as defined in Section 2 with linear Diophantine type $\gamma_1(\theta)$;
2. There is $n \in \mathbb{N}$ such that for each $\delta \leq D$, $d_{n, S}(T_0, T_\delta) \leq \delta$;
3. for each $\delta \leq D$ there is a set $A_2 \in \text{Int}_{\mu}$ such that $m(A_2) \geq 1 - \delta$ and for all $x \in A_2$, $y \in T^d : |G_0(x,y) - G_\delta(x,y)| \leq \delta$.

Then for each $\gamma > \gamma_1(\theta)$ there is $K_1$ such that for $\delta$ small enough

$$||f_\delta - f_0||_{1-1} \leq K_1 \delta^{\frac{1}{2}}.$$

### 6.3 An example having Hölder behavior

In this section we show a simple example of perturbation of toral extensions satisfying assumptions Te1 and Te2 for which the statistical behavior is actually, only Hölder stable. This shows how that Propositions 29 and 32 give a general estimate, which is quite sharp in the case of piecewise constant toral extensions.
Proposition 33 Consider a well approximable Diophantine irrational \( \theta \) with \( \gamma_l(\theta) > 2 \). Let us consider the map \( F_0 : [0,1] \times T^1 \) defined as a skew product \( F_0(T_0(x), G_0(x,y)) \) where

\[
T_0(x) = 2x \mod(1)
\]

and

\[
G_0(x,y) = y + \theta \varphi(x)
\]

where \( \varphi = \chi_{[\frac{1}{2},1]} \). Consider \( \gamma' < \gamma_l(\theta) \); there is is a sequence of reals \( \delta_j \geq 0 \), \( \delta_j \to 0 \) and a sequence of perturbed of maps \( F_{\delta,j}(x,y) = (T_{\delta,j}(x), G_{\delta,j}(x,y)) \) satisfying \( S1, ..., S3 \), with \( T_{\delta,j}(x) = T_0(x) \) and \( ||G_{\delta,j}(x,y) - G_0(x,y)||_\infty \leq 2\delta_j \) such that

\[
||\mu_0 - \mu_j||^{1'} \geq \frac{1}{9} \delta_j^{\frac{1}{\gamma' - 1}}
\]

holds for every \( j \) and every \( \mu_j \), invariant measure of \( F_{\delta,j}(x,y) \) in \( L^1 \).

Proof. We remark that since there is convergence to equilibrium for \( F_0 \), the Lebesgue measure \( \mu_0 \) on \([0,1] \times T^1 \) is the unique invariant measure in \( L^1 \) for \( F_0 \). Consider \( F_{\delta} = (T_0(x), y + (\delta + \theta) \varphi(x)) \). For a sequence of values of \( \delta \) converging to 0 it holds that \((\delta + \theta)\) is rational. For this sequence the map \( y \to y + (\delta + \theta) \) (: \( T^1 \to T^1 \)) is such that, 0 has a periodic orbit. Let \( y_1 = 0, ..., y_k \) be this orbit. For these parameters, consider the product measure \( \mu_n = \frac{1}{k} \sum_{i \leq k} \mu \otimes \delta_{y_i} \), where \( m \) is the Lebesgue measure on \([0,1] \) and \( \delta_{y_i} \) is the delta measure placed on \( y_i \). The measure \( \mu_n \) is invariant for \( F_{\delta}(x,y) \), and is in \( L^1 \). It is easy to see that \( ||\mu_0 - \mu_n||^{1'} \geq \frac{1}{9} \frac{1}{k} \). Now the Diophantine type of \( \theta \) will give an estimate for the relation between \( \delta \) and \( k \). Indeed let \( \gamma' < \gamma(\theta) \), by the Diophantine type of \( \theta \) we know that there are infinitely many \( k_j \) and integers \( p_j \) such that \( |k_j \theta - p_j| \leq \frac{1}{k_j^{\gamma'}} \) then \( |\theta - \frac{p_j}{k_j}| \leq \frac{1}{k_j^{\gamma' - 1}} \). Let us now consider \( \delta_j = -\theta + \frac{p_j}{k_j} \), it holds \( |\delta_j| \leq \frac{1}{k_j^{\gamma' - 1}} \) and the angle \((\delta_j + \theta)\) generates a periodic orbit of period \( k_j \). This happens by perturbing the second coordinate of the map by a quantity which is less than \( \frac{1}{k_j^{\gamma' - 1}} \). Summarizing, for the map \( F_{\delta,j} \) we have that there is no perturbation on the first coordinate, for the second coordinate, \( ||G_0 - G_{\delta,j}||_\infty \leq \delta_j \) and denoting as \( \mu_j \) the invariant measure on the periodic orbit defined before it holds

\[
||\mu_0 - \mu_j||^{1'} \geq \frac{1}{9} \delta_j^{\frac{1}{\gamma' - 1}}.
\]

This example can further be improved by perturbing the map \( F_{\delta,j} \) to a new map \( \hat{F}_{\delta,j} \) in a way that \( \mu_j \) (a measure supported on the attractor of \( \hat{F}_{\delta,j} \)) and \( \mu_j + \frac{1}{2k_j} \) (supported on the repeller of \( \hat{F}_{\delta,j} \)) are the only invariant measures in \( L^1 \) for \( \hat{F}_{\delta,j} \) and \( \mu_j \) is the unique physical measure for the system. This can be

\[\text{Defined as } ||\mu_j + \frac{1}{2k_j}||(A) = \mu_j(A - \frac{1}{2k_j}) \text{ for each measurable set } A \text{ in } T^1 \text{. Where } A - \frac{1}{2k_j} \text{ is the translation of the set } A \text{ by } -\frac{1}{2k_j}.\]
done by making a small further $C^\infty$ perturbation on $G$. Let us denote again by $y_1, ..., y_{k_j}$ the periodic orbit of 0 as before. Let us consider a $C^\infty$ function $g : [0, 1] \to [0, 1]$ such that:

- $g$ is negative on each interval $[y_i, y_i + \frac{1}{2k_j}]$ and positive on each interval $[y_i + \frac{1}{2k_j}, y_{i+1}]$ (so that $g(y_i + \frac{1}{2k_j}) = 0$);
- $g'$ is positive in each interval $[y_i + \frac{1}{3k_j}, y_{i+1} - \frac{1}{3k_j}]$ and negative in $[y_{i+1} - \frac{1}{3k_j}, y_i + \frac{1}{3k_j}]$.

Considering $D_\delta : T^1 \to T^1$ defined by $D_\delta(x) = x + \delta g(x) \mod(1)$ it holds that the iteration of this map send all the space but the set $\{ y_i \text{ s.t. } i \leq k_j \}$ (which is a repeller) to the set $\{ y_i \text{ s.t. } i \leq k_j \}$ (the attractor). Then define $\tilde{F}_j$ as:

$$\tilde{F}_j(x, y) = (T_j(x), D_j(y + (\delta_j + \theta) \varphi(x))).$$

The claim directly follows from the remark that for the map $(\tilde{F}_j)^k$ the sets $\Gamma_1 := [0, 1] \times \{ y_i \text{ s.t. } i \leq k_j \}$ and $\Gamma_2 := [0, 1] \times \{ y_i + \frac{1}{2k_j} \text{ s.t. } i \leq k_j \}$ are invariant and the set $\Gamma_1$ attracts the whole $[0, 1] \times T^1 - \Gamma_2$. ■

The construction done in the previous proof can be extended to show Hölder behavior for the average of a given regular observable. We show an explicit example of such an observable for a skew product with a particular angle $\theta$.

**Proposition 34** Consider a map $F_0$ as above with the rotation angle $\theta = \sum_1^{\infty} 2^{-2i}$. With

$$T_0(x) = 2x \mod(1)$$

and

$$G_0(x, y) = y + \theta \varphi(x)$$

as in Proposition 33. Let $\tilde{F}_j$ be its perturbations as described in the proof of the proposition and $\mu_j$ their invariant measures in $L^1$. There is an observable $\psi : [0, 1] \times T^1 \to \mathbb{R}$ with derivative in $L^2$ and $C \geq 0$ such that

$$| \int \psi d\mu_0 - \int \psi d\mu_j | \geq C \sqrt{\delta_j}.$$

**Proof.** We recall that that $\sum_{n+1}^{\infty} 2^{-2i} \leq 2^{-2^{(n+1)+1}}$. By this $\|2^{2^n} \theta\| \leq 2^{-2^{(n+1)+1}}$ and the Diophantine type of $\theta$ is greater than 4. Following the construction above, we have that with a perturbation of size less than $2^{-2^{(n+1)+1}}$ the angles $\theta_j = \sum_{i=1}^{n} 2^{-2i}$ generate on the second coordinate of the skew product orbits of period $2^{2j}$. Now let us construct a suitable observable which can ”see” the change of the invariant measure under this perturbation. Let us consider

$$\psi(x, y) = \sum_{i=1}^{\infty} \frac{1}{(2^{2i})^2} \cos(2^{2i}2\pi y)$$  (29)
and \( \psi_k(x,y) = \sum_{i=1}^{k} \frac{1}{2^{2i}} \cos(2^{2i} 2\pi y) \). Since for the observable \( \psi \), the \( i \)-th Fourier coefficient decreases like \( i^{-2} \), then \( \psi \) has a derivative in \( L^2 \). Let \( x_1 = 0, \ldots, x_{2^{2i}} \) be the periodic orbit of 0 for \( y \to y + \theta_j \), and \( \mu_j = \frac{1}{2^{2i}} \sum x_i \), the physical measure supported on it. Since \( 2^{2i} \) divides \( 2^{2(j+1)} \), then \( \sum_{i=1}^{2^{2i}} \psi_k(x_i) = 0 \) for every \( k < j \), thus \( \int \psi_{j-1} d\mu_j = 0 \). Then

\[
v_j := \int \psi \, d\mu_j \geq \frac{1}{(2^{2i})^2} - \sum_{j=1}^{\infty} \frac{1}{(2^{2i})^2} \geq 2^{-2^{2i+1}} - 2^{-2^{2(j+1)+1}}.
\]

And for \( j \) big enough

\[
2^{-2^{2i+1}} - 2^{-2^{2(j+1)+1}} \geq \frac{1}{2} (2^{-2^{2i}})^2.
\]

Summarizing, with a perturbation of size \( \delta_j = \sum_{j=1}^{\infty} 2^{-2^{2i}} = 2 \sum_{j=1}^{2^{2j}} 2^{-2^{2j}} = 2^{-2^{2j+1}} = 2(2^{-2^{2j}})^2 \) we get a change of average for the observable \( \psi \) from \( \int \psi dm = 0 \) to \( v_n \geq \frac{1}{2} (2^{-2^{2j}})^2 \). Hence we have that there is a \( C \geq 0 \) such that with a perturbation of size \( \delta_j \) we get a change of average for the observable \( \psi \) of size bigger than \( C \delta_j \).

\textbf{Remark 35} Using \( \frac{1}{(2^{2i})^2} \) instead of \( \frac{1}{(2^{2i+1})^2} \) in (29) we can obtain a smoother observable. Using rotation angles with bigger and bigger Diophantine type it is possible to obtain a dependence of the physical measure to perturbations with worse and worse Hölder exponent. Using angles with infinite Diophantine type it is possible to have a behavior whose modulus of continuity is worse than the Hölder one.

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