2-Distance List \((\Delta + 3)\)-Coloring of Sparse Graphs

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Abstract

A 2-distance list \(k\)-coloring of a graph is a proper coloring of the vertices where each vertex has a list of at least \(k\) available colors and vertices at distance at most 2 cannot share the same color. We prove the existence of a 2-distance list \((\Delta + 3)\)-coloring for graphs with maximum average degree less than \(\frac{8}{3}\) and maximum degree \(\Delta \geq 4\) as well as graphs with maximum average degree less than \(\frac{14}{5}\) and maximum degree \(\Delta \geq 6\).

Keywords 2-Distance coloring · Square coloring · Sparse graphs · Discharging method

1 Introduction

A \(k\)-coloring of the vertices of a graph \(G = (V, E)\) is a map \(\phi : V \to \{1, 2, \ldots, k\}\). A \(k\)-coloring \(\phi\) is a proper coloring, if and only if, for all edge \(xy \in E\), \(\phi(x) \neq \phi(y)\). In other words, no two adjacent vertices share the same color. The chromatic number of \(G\), denoted by \(\chi(G)\), is the smallest integer \(k\) such that \(G\) has a proper \(k\)-coloring. A generalization of \(k\)-coloring is \(k\)-list-coloring. A graph \(G\) is \(L\)-list colorable if for a given list assignment \(L = \{L(v) : v \in V(G)\}\) there is a proper coloring \(\phi\) of \(G\) such that for all \(v \in V(G)\), \(\phi(v) \in L(v)\). If \(G\) is \(L\)-list colorable for every list assignment \(L\) with \(|L(v)| \geq k\) for all \(v \in V(G)\), then \(G\) is said to be \(k\)-choosable or \(k\)-list-colorable. The list chromatic number of a graph \(G\) is the smallest integer \(k\) such that \(G\) is \(k\)-choosable.

In 1969, Kramer and Kramer introduced the notion of 2-distance coloring \([5, 6]\). This notion generalizes the “proper” constraint (that does not allow two adjacent vertices to have the same color) in the following way: a 2-distance \(k\)-coloring is such that no pairs of vertices at distance at most 2 have the same color (similarly to proper \(k\)-list-coloring, one can also define 2-distance \(k\)-list-coloring). The 2-distance
The chromatic number of a graph, denoted by $\chi^2(G)$, is the smallest integer $k$ so that $G$ has a 2-distance $k$-coloring. We denote $\chi^2_l(G)$ the 2-distance list chromatic number of $G$.

For all $v \in V$, we denote $d_G(v)$ the degree of $v$ in $G$ and by $\Delta(G) = \max_{v \in V} d_G(v)$ the maximum degree of a graph $G$. For brevity, when it is clear from the context, we will use $\Delta$ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$). One can observe that, for any graph $G$, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex $v$ with degree $\Delta$, and $v$ itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta^2 + 1$. Moreover, this bound is tight for some graphs (see [7] for some examples).

By nature, 2-distance colorings and the 2-distance chromatic number of a graph depend a lot on the number of vertices in the neighborhood of every vertex. More precisely, the “sparser” a graph is, the lower its 2-distance chromatic number will be. One way to quantify the sparsity of a graph is through its maximum average degree. The average degree $\text{ad}(G)$ of a graph $G = (V, E)$ is defined by $\text{ad}(G) = \frac{2|E|}{|V|}$. The maximum average degree $\text{mad}(G)$ is the maximum, over all subgraphs $H$ of $G$, of $\text{ad}(H)$. Another way to measure the sparsity is through the girth, i.e. the length of a shortest cycle. We denote by $g(G)$ the girth of $G$. Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of planar graphs.

**Proposition 1** (Folklore) For every planar graph $G$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$.

A graph is planar if one can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When $G$ is a planar graph, Wegner conjectured in 1977 that $\chi^2(G)$ becomes linear in $\Delta(G)$:

**Conjecture 1** (Wegner [11]) Let $G$ be a planar graph with maximum degree $\Delta$. Then,

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

The conjecture was proven for some cases [3, 4, 10, 11] and some subfamilies of planar graphs [8].

Wegner’s conjecture motivated extensive researches on 2-distance chromatic number of sparse graphs, either of planar graphs with high girth or of graphs with upper bounded maximum average degree which are directly linked due to Proposition 1. For a survey of the work done on 2-distance coloring of planar graphs with high girth, see [7]. See [2, 9] for more results about 2-distance list coloring.

In this article, we prove the following theorems:
Theorem 1 If $G$ is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \geq 4$, then $\chi^2(G) \leq \Delta(G) + 3$.

Theorem 2 If $G$ is a graph with $\text{mad}(G) < \frac{14}{3}$ and $\Delta(G) \geq 6$, then $\chi^2(G) \leq \Delta(G) + 3$.

Theorem 1 improves upon a previous result for graphs with $\Delta(G) = 5$ [1].

Due to Proposition 1, we get the following corollaries for planar graphs:

Corollary 1 If $G$ is a graph with $g(G) \geq 8$ and $\Delta(G) \geq 4$, then $\chi^2(G) \leq \Delta(G) + 3$.

Corollary 2 If $G$ is a graph with $g(G) \geq 7$ and $\Delta(G) \geq 6$, then $\chi^2(G) \leq \Delta(G) + 3$.

Notations and drawing conventions For $v \in V(G)$, the 2-distance neighborhood of $v$, denoted $N^2_G(v)$, is the set of 2-distance neighbors of $v$, which are vertices at distance at most two from $v$, not including $v$. We also denote $d^2_G(v) = |N^2_G(v)|$. We will drop the subscript and the argument when it is clear from the context. Also for conciseness, from now on, when we say “to color” a vertex, it means to color such vertex differently from all of its colored neighbors at distance at most two. Similarly, any considered coloring will be a 2-distance list-coloring. We will also say that a vertex $u$ “sees” another vertex $v$ if $u$ and $v$ are at distance at most 2 from each other.

As a drawing convention, black vertices will have all of their neighbors represented while white vertices may have a higher degree than what is shown.

Some more notations:

- A $d$-vertex ($d^+$-vertex, $d^-$-vertex) is a vertex of degree $d$ (at least $d$, at most $d$).
- A $k$-path ($k^+$-path, $k^-$-path) is a path of length $k + 1$ (at least $k + 1$, at most $k + 1$) where the $k$ internal vertices are 2-vertices and the endvertices are $3^+$-vertices.
- A $(k_1, k_2, \ldots, k_d)$-vertex is a $d$-vertex incident to $d$ different paths, where the $i^{th}$ path is a $k_i$-path for all $1 \leq i \leq d$.

In both proofs, we will consider $G_1$ (resp. $G_2$) a counter-example to Theorem 1 (resp. Theorem 2) with the fewest number of vertices. The purpose of the proofs is to prove that $G_1$ (resp. $G_2$) cannot exist. We will always start by studying their structural properties, then apply a discharging procedure.

2 Proof of Theorem 1

2.1 Structural Properties of $G_1$

Lemma 1 Graph $G_1$ is connected.

Otherwise a component of $G_1$ would be a smaller counterexample.

Lemma 2 The minimum degree of $G_1$ is at least 2.

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Proof By Lemma 1, the minimum degree is at least 1 or \( G_1 \) would be a single isolated vertex which is \( (\Delta(G_1) + 3) \)-colorable. If \( G_1 \) contains a degree 1 vertex \( v \) (Figure 1a), then we can simply remove such vertex and 2-distance color the resulting graph, which is possible by minimality of \( G_1 \). Then, we add \( v \) back and color \( v \) (at most \( \Delta(G_1) \) constraints and \( \Delta(G_1) + 3 \) colors).

□

Lemma 3 Graph \( G_1 \) has no \( 2^+ \)-paths.

Proof Suppose that \( G_1 \) contains a \( 2^+ \)-path \( uvwx \) (Fig. 1b). We color \( G_1 - \{v, w\} \) by minimality of \( G_1 \). Observe that \( v \) and \( w \) each sees at most \( \Delta + 1 \) colors so they have at least two available colors left each. Thus, we can easily extend the coloring to \( v \) and \( w \).

□

Lemma 4 Graph \( G_1 \) has no \((1,1,1)\)-vertices.

Proof Suppose by contradiction that there exists a \((1, 1, 1)\)-vertex \( u \) with three \( 2 \)-neighbors \( u_1, u_2, \) and \( u_3 \) (Fig. 1c). We color \( G_1 - \{u, u_1, u_2, u_3\} \) by minimality of \( G_1 \), then we extend this coloring to the remaining vertices by coloring \( u_1, u_2, u_3, \) and \( u \) in this order. Observe that this possible since we have \( \Delta + 3 \geq 7 \) colors as \( \Delta \geq 4 \).

□

Lemma 5 Graph \( G_1 \) has no \( 3 \)-vertices with a \( 2 \)-neighbor and a \((1,1,0)\)-neighbor.

Proof Suppose by contradiction that there exists a \( 3 \)-vertex \( u \) with a \( 2 \)-neighbor \( v \) and a \((1, 1, 0)\)-neighbor \( w \). Let the \( 2 \)-neighbors of \( w \) be \( w_1 \) and \( w_2 \).

First, observe that if two adjacent \( 3 \)-vertices share a common \( 2 \)-neighbor, for example, if \( u \) is also adjacent to \( w_1 \) (Fig. 2a), then we color \( G_1 - \{u, w, w_1\} \) by minimality of \( G_1 \) and finish by coloring \( u, w, \) and \( w_1 \) in this order. This is possible since we have \( \Delta + 3 \) colors and \( \Delta \geq 4 \). Hence, all named vertices are distinct.

Now, we color \( G_1 - \{u, v, w, w_1, w_2\} \) by minimality (Fig. 2b). Let \( L(x) \) be the list of available colors left for a vertex \( x \in \{u, v, w, w_1, w_2\} \). Since we have \( \Delta + 3 \) colors and \( \Delta \geq 4 \), \( |L(v)| \geq 2 \), \( |L(u)| \geq 2 \), \( |L(w)| \geq 4 \), \( |L(w_1)| \geq 3 \), and \( |L(w_2)| \geq 3 \). We remove the extra colors so that \( |L(x)| \) reaches the lower bound for each \( x \in \{u, v, w, w_1, w_2\} \). Consider the two following cases.

- If \( L(u) \neq L(v) \), then we color \( u \) with \( c \in L(u) \setminus L(v) \). We finish by coloring \( w_1, w_2, w, \) and \( v \) in this order.
If \( L(u) = L(v) \), we color \( w_1 \) with \( c \in L(w_1) \setminus L(u) \) (which is possible since \( |L(w_1)| = 3 \) and \( |L(u)| = 2 \)). Then, we color \( w \) with \( d \in L(w) \setminus (L(u) \cup \{c\}) \) (which is possible as \( |L(w)| = 4 \)). Finally, we finish by coloring \( w_2, u, \) and \( v \) in this order.

We thus obtain a valid coloring of \( G_1 \), which is a contradiction. \( \square \)

**Lemma 6** Graph \( G_1 \) has no 3-vertices with two \((1,1,0)\)-neighbors and another 3-neighbor.

**Proof** Suppose by contradiction that there exists a 3-vertex \( u \) with two \((1,1,0)\)-neighbors \( v \) and \( w \) and another 3-neighbor \( t \). Let \( v_1 \) and \( v_2 \) (resp. \( w_1 \) and \( w_2 \)) be \( v \)'s (resp. \( w \)'s) 2-neighbors.

If \( v \) and \( w \) share a common 2-neighbor, say \( v_1 = w_1 \) (Fig. 3a), then we color \( G_1 - \{u, v, w, v_1, v_2, w_2\} \) by minimality of \( G_1 \) and finish by coloring \( u, v, w, v_2, w_2, \) and \( v_1 \) in this order. This is possible since we have \( \Delta + 3 \) colors and \( \Delta \geq 4 \). Note that this coloring also work when \( v_2 = w_2 \). Hence, all named vertices are distinct.

Now, we color \( G_1 - \{u, v, w, w_1, w_2\} \) by minimality (Fig. 3b). Let \( L(x) \) be the list of available colors left for a vertex \( x \in \{u, v, w, w_1, w_2\} \). Since we have \( \Delta + 3 \) colors and \( \Delta \geq 4 \), \( |L(u)| \geq 2 \) (as \( d(t) = 3 \)), \( |L(v)| \geq 2 \), \( |L(w)| \geq 4 \), \( |L(w_1)| \geq 3 \), and \( |L(w_2)| \geq 3 \). Note that we obtain the same lower bounds on the lists of colors as
in Lemma 5. Thus, the exact same proof holds and we have a valid coloring of $G_1$, which is a contradiction.

2.2 Discharging Rules

Since $\text{mad}(G_1) < \frac{\delta}{3}$, we have

$$\sum_{u \in V(G)} (3d(u) - 8) < 0$$

(1)

We assign to each vertex $u$ the charge $\mu(u) = 3d(u) - 8$. To prove the non-existence of $G_1$, we will apply discharging rules that preserve the sum of charge to obtain a non-negative total charge, which will contradict Eq. 1.

R0 Every $3^+$-vertex gives 1 to each of its 2-neighbors.
R1 Every $4^+$-vertex gives 1 to each of its 3-neighbors.
R2 Every $(0, 0, 0)$-vertex gives 1 to each of its $(1, 1, 0)$-neighbors.

2.3 Verifying that Charges on Each Vertex are Non-negative

Let $\mu^*$ be the assigned charges after the discharging procedure. In what follows, we will prove that:

$$\forall u \in V(G_1), \mu^*(u) \geq 0.$$

Let $u \in V(G_1)$.

Case 1: If $d(u) = 2$, then $u$ receives charge 1 from each endvertex of the 1-path it lies on by R0 (as there are no 2$^+$-paths by Lemma 3); Thus we get $\mu^*(u) = \mu(u) + 2 \cdot 1 = 3 \cdot 2 - 8 + 2 = 0$.

Case 2: If $d(u) = 3$, then $\mu(u) = 3 \cdot 3 - 8 = 1$. Since there are no 2$^+$-paths due to Lemma 3 and no $(1, 1, 1)$-vertices due to Lemma 4, we have the following cases.

- If $u$ is a $(1, 1, 0)$-vertex, then $u$ only gives 1 to its 2-neighbor by R0. Hence,

  $$\mu^*(u) \geq 1 - 2 \cdot 1 + 1 = 0.$$

- If $u$ is a $(1, 0, 0)$-vertex, then $u$ only gives 1 to its 2-neighbor by R0. Hence,

  $$\mu^*(u) \geq 1 - 1 = 0.$$

- If $u$ is a $(0, 0, 0)$-vertex, then $u$ only gives charge to $(1, 1, 0)$-vertices by R2. Let $t$, $v$, and $w$ be $u$’s $3^+$-neighbors.

  If $u$ is adjacent to a $4^+$-neighbor then it receives 1 by R1 and at worst, it gives 1 to each of the two other neighbors by R2. As a result,
\[ \mu^*(u) \geq 1 + 1 - 2 \cdot 1 = 0. \]

If \( u \) is adjacent to three 3-vertices, then at most one of them can be a \((1, 1, 0)\)-vertex due to Lemma 6. So, \( u \) only gives at most 1 to a \((1, 1, 0)\)-neighbor by \( R2 \). Consequently,

\[ \mu^*(u) \geq 1 - 1 = 0. \]

**Case 3:** If \( 4 \leq d(u) \leq \Delta \), then, at worst, \( u \) gives 1 to each of its neighbors by \( R0 \) and \( R1 \). As a result,

\[ \mu^*(u) \geq 3d(u) - 8 - d(u) \geq 2 \cdot 4 - 8 = 0. \]

To conclude, we started with a charge assignment with a negative total sum, but after the discharging procedure, which preserved this sum, we end up with a non-negative one, which is a contradiction. In other words, there exists no counter-examples to Theorem 1.

### 3 Proof of Theorem 2

#### 3.1 Structural Properties of \( G_2 \)

Observe that in Lemmas 1–4, we only use the fact that \( G_1 \) is a minimal counter-example to Theorem 1 and that we have \( \Delta(G_1) + 3 \) colors, which is at least 7. Consequently, since \( G_2 \) is a minimal counter-example to Theorem 2 and we have \( \Delta(G_2) + 3 \geq 9 \) colors as \( \Delta(G_2) \geq 6 \), the same arguments allow us to obtain Lemmas 7–10:

**Lemma 7**  *Graph \( G_2 \) is connected.*

**Lemma 8**  *The minimum degree of \( G_2 \) is at least 2.*

**Lemma 9**  *Graph \( G_2 \) has no \( 2^+ \)-paths.*

**Lemma 10**  *Graph \( G_2 \) has no \((1,1,1)\)-vertices.*

**Lemma 11**  *A \((1,1,0)\)-vertex can only share a common 2-neighbor with a \( \Delta \)-vertex.*

![Reducible configurations from Lemmas 11 to 13](image-url)
Proof Suppose by contradiction that there exists a \((1, 1, 0)\)-vertex \(u\), with two 2-neighbors \(v\) and \(w\), and let \(x\) be the other endvertex of the 1-path \(uvw\) with \(d(x) \leq \Delta - 1\) (Fig. 4a). We color \(G_2 - \{u, v, w\}\) by minimality of \(G_2\). Then, it suffices to finish coloring \(u, v,\) and \(w\) in this order as we have \(\Delta + 3\) colors.

Lemma 12 A \((1,1,0)\)-vertex has a \((\Delta - 1)^+\)-neighbor.

Proof Suppose by contradiction that there exists a \((1, 1, 0)\)-vertex \(u\) with two 2-neighbors \(v\) and \(w\) and a \((\Delta - 2)^-\)-neighbor \(t\) (Fig. 4b). We color \(G_2 - \{u, v, w\}\) by minimality of \(G_2\). Then, it suffices to finish coloring \(v, w,\) and \(u\) in this order as we have \(\Delta + 3\) colors.

Lemma 13 A \((1,0,0)\)-vertex cannot have two 3-neighbors.

Proof Suppose by contradiction that there exists a \((1, 0, 0)\)-vertex \(u\) with a 2-neighbor \(v\) and two 3-neighbors (Fig. 4c). We color \(G_2 - \{v\}\) by minimality of \(G_2\). We uncolor \(u\) then it suffices to finish coloring \(v, u\) in this order as we have \(\Delta + 3\) colors and \(\Delta \geq 6\).

Lemma 14 A \((1,0,0)\)-vertex with a 3-neighbor and a 4-neighbor can only share a common 2-neighbor with a \(\Delta\)-vertex.

Proof Suppose by contradiction that there exists a \((1, 0, 0)\)-vertex \(u\), with a 2-neighbor \(v\), a 3-neighbor, and a 4-neighbor, and let \(x\) be the other endvertex of the 1-path \(uvx\) with \(d(x) \leq \Delta - 1\) (Fig. 5a). We color \(G_2 - \{v\}\) by minimality of \(G_2\). We uncolor \(u\) then it suffices to finish coloring \(u, v\) in this order as we have \(\Delta + 3\) colors and \(\Delta \geq 6\).

Lemma 15 A \((1,1,1,1)\)-vertex can only share a common 2-neighbor with a \(\Delta\)-vertex.

Proof Suppose by contradiction that there exists a \((1, 1, 1, 1)\)-vertex \(u\), with four 2-neighbors \(v_1, v_2, v_3,\) and \(v_4\), and let \(x\) be the other endvertex of the 1-path \(uv_4x\) with \(d(x) \leq \Delta - 1\) (Fig. 5b). We color \(G_2 - \{u, v_1, v_2, v_3, v_4\}\) by minimality of \(G_2\). Then, it suffices to finish coloring \(v_1, v_2, v_3, v_4,\) and \(u\) in this order as we have \(\Delta + 3\) colors and \(\Delta \geq 6\).

Lemma 16 A \((1,1,1,0)\)-vertex with a 3-neighbor can only share a common 2-neighbor with a \((\Delta - 1)^+\)-vertex.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Reducible configurations from Figs. 14 to 16}
\end{figure}
Proof Suppose by contradiction that there exists a (1, 1, 1, 0)-vertex \( u \), with three 2-neighbors \( v_1, v_2, \) and \( v_3 \) and a 3-neighbor, and let \( x \) be the other endvertex of the 1-path \( uv_3x \) with \( d(x) \leq \Delta - 2 \) (Fig. 5c). We color \( G_2 - \{ u, v_1, v_2, v_3 \} \) by minimality of \( G_2 \). Then, it suffices to finish coloring \( v_1, v_2, u, \) and \( v_3 \) in this order as we have \( \Delta + 3 \) colors and \( \Delta \geq 6 \).

3.2 Discharging rules

Since \( \text{mad}(G_2) < \frac{14}{5} \), we have

\[
\sum_{u \in V(G)} (5d(u) - 14) < 0
\]

(2)

We assign to each vertex \( u \) the charge \( \mu(u) = 5d(u) - 14 \). To prove the non-existence of \( G_2 \), we will apply the following discharging rules that preserve the sum of charges to obtain a non-negative total charge, which will contradict Eq. 2.

**R0** Every \( 3^+ \)-vertex gives 2 to each of its 2-neighbors.

**R1**

(i) Every 4-vertex gives \( \frac{1}{2} \) to each of its 3-neighbors.

(ii) Every 5\(^+\)-vertex gives 2 to each of its 3-neighbors.

**R2** Let \( uvw \) be a 1-path:

(i) If \( d(u) = 5 \) and \( d(w) \leq 4 \), then \( u \) gives \( \frac{1}{2} \) to \( w \).

(ii) If \( d(u) \geq 6 \) and \( d(w) \leq 4 \), then \( u \) gives \( \frac{2}{3} \) to \( w \).

3.3 Verifying that Charges on Each Vertex are Non-negative

Let \( \mu^* \) be the assigned charges after the discharging procedure. In what follows, we will prove that:

\[
\forall u \in V(G_2), \mu^*(u) \geq 0.
\]

Let \( u \in V(G_2) \).

**Case 1**: If \( d(u) = 2 \), then \( u \) receives charge 2 from each endvertex of the path it lies on by **R0** (as there are no \( 2^+ \)-paths by Lemma 9); Thus we get \( \mu^*(u) = \mu(u) + 2 \cdot 2 = 5 \cdot 2 - 14 + 2 \cdot 2 = 0 \).

**Case 2**: If \( d(u) = 3 \), then recall \( \mu(u) = 5 \cdot 3 - 14 = 1 \). Since there are no \( 2^+ \)-paths due to Lemma 9 and no (1, 1, 1)-vertices due to Lemma 10, we have the following cases.
If \( u \) is a \((1, 1, 0)\)-vertex, then \( u \) gives 2 to each of its two 2-neighbors. At the same time, the other endvertices of \( u \)'s incident 1-paths must be \( \Delta \)-vertices due to Lemma 11. So, \( u \) receives \( \frac{2}{3} \) from each of these \( \Delta \)-endvertices by \( R_2(\text{ii}) \) as \( \Delta \geq 6 \). Moreover, \( u \)'s \( 3^+ \)-neighbor must be a \((\Delta - 1)^+\)-vertex due to Lemma 12. As a result, \( u \) also receives 2 from its \((\Delta - 1)^+\)-neighbor as \( \Delta \geq 6 \). To sum up,

\[
\mu^+(u) \geq 1 - 2 \cdot 2 + 2 \cdot \frac{2}{3} + 2 = \frac{1}{3}.
\]

If \( u \) is a \((1, 0, 0)\)-vertex, then \( u \) only gives 2 to its 2-neighbor by \( R_0 \). We distinguish the two following cases.

- If \( u \) has a \( 5^+ \)-neighbor, then \( u \) receives 2 by \( R_1(\text{ii}) \). Thus,

\[
\mu^+(u) \geq 1 - 2 + 2 = 1.
\]

- If \( u \) only has \( 4^- \)-neighbors, then \( u \) must have at least one 4-neighbor due to Lemma 13. Now, if \( u \) has two 4-neighbors, then it receives \( \frac{1}{2} \) twice by \( R_1(\text{i}) \). Hence,

\[
\mu^+(u) \geq 1 - 2 + 2 \cdot \frac{1}{2} = 0.
\]

If \( u \) has exactly one 4-neighbor and the other one is a 3-neighbor, then the other endvertex of the 1-path incident to \( u \) must be a \( \Delta \)-vertex due to Lemma 14. As a result, \( u \) also receives \( \frac{1}{2} \) from its 4-neighbor by \( R_1(\text{i}) \) and \( \frac{2}{3} \) from the \( \Delta \)-endvertex by \( R_2(\text{ii}) \) as \( \Delta \geq 6 \). To sum up,

\[
\mu^+(u) \geq 1 - 2 + \frac{1}{2} + \frac{2}{3} = \frac{1}{6}.
\]

- If \( u \) is a \((0, 0, 0)\)-vertex, then \( u \) does not gives any charge away. Thus,

\[
\mu^+(u) = \mu(u) = 1.
\]

**Case 3:** If \( d(u) = 4 \), then recall \( \mu(u) = 5 \cdot 4 - 14 = 6 \) and observe that \( u \) only gives charge 2 or \( \frac{1}{2} \) away respectively by \( R_0 \) or \( R_1(\text{i}) \). We have the following cases.

- If \( u \) is a \((1, 1, 1, 1)\)-vertex, then \( u \) gives 2 to each of its four 2-neighbors by \( R_0 \). At the same time, the other endvertices of the 1-paths incident to \( u \) are all \( \Delta \)-vertices due to Lemma 15. As a result, \( u \) also receives \( \frac{2}{3} \) from each of the four \( \Delta \)-endvertices by \( R_2(\text{ii}) \). To sum up,

\[
\mu^+(u) \geq 6 - 4 \cdot 2 + 4 \cdot \frac{2}{3} = \frac{2}{3}.
\]
If \( u \) is a \((1, 1, 1, 0)\)-vertex, then \( u \) gives 2 to each of its three 2-neighbors by \( R0 \). Let \( v \) be the \( 3^+ \)-neighbor. If \( v \) is a \( 4^+ \)-vertex, then \( u \) does not give anything to \( v \). Thus,

\[
\mu^*(u) \geq 6 - 3 \cdot 2 = 0.
\]

If \( v \) is a \( 3 \)-vertex, then \( u \) gives \( \frac{1}{2} \) to \( v \) by \( R1(i) \). Due to Lemma 16, the other endvertices of the 1-paths incident to \( v \) must be \((A - 1)^+\)-vertices. As a result, \( v \) receives at least \( \frac{1}{2} \) from each of the three \((A - 1)^+\)-endvertices as \( A \geq 6 \) by \( R2 \). To sum up,

\[
\mu^*(u) \geq 6 - 3 \cdot 2 - \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}.
\]

If \( u \) is a \((1^-, 1^-, 0, 0)\)-vertex, then at worst \( u \) gives 2 twice by \( R0 \) and \( \frac{1}{2} \) twice by \( R1(i) \). Thus,

\[
\mu^*(u) \geq 6 - 2 \cdot 2 - 2 \cdot \frac{1}{2} = 1.
\]

Case 4: If \( d(u) = 5 \), then, at worst, \( u \) gives \( 2 + \frac{1}{2} \) away along each incident edge by \( R0 \) (or \( R1(ii) \)) and \( R2(i) \). As a result,

\[
\mu^*(u) \geq 5d(u) - 14 - \left( 2 + \frac{1}{5} \right) d(u) = \left( 2 + \frac{4}{5} \right) \cdot 5 - 14 = 0.
\]

Case 5: If \( 6 \leq d(u) \leq A \), then, at worst, \( u \) gives \( 2 + \frac{2}{3} \) away along each incident edge by \( R0 \) (or \( R1(ii) \)) and \( R2(ii) \). As a result,

\[
\mu^*(u) \geq 5d(u) - 14 - \left( 2 + \frac{2}{3} \right) d(u) \geq \left( 2 + \frac{1}{3} \right) \cdot 6 - 14 = 0.
\]

To conclude, we started with a charge assignment with a negative total sum, but after the discharging procedure, which preserved this sum, we end up with a non-negative one, which is a contradiction. In other words, there exists no counter-examples to Theorem 2.

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