Solutions with single radial interface of the generalized Cahn–Hilliard flow

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Abstract
We consider the generalized parabolic Cahn–Hilliard equation
\[ u_t = -\Delta \left[ \Delta u - W'(u) \right] + W''(u) \left[ \Delta u - W'(u) \right], \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^n, \]
where \( n = 2 \) or \( n \geq 4 \), \( W(\cdot) \) is the typical double-well potential function and \( \mathbb{R} \) is given by
\[ \mathbb{R} = \begin{cases} (0, \infty), & \text{if } n = 2, \\ (-\infty, 0), & \text{if } n \geq 4. \end{cases} \]
We construct a radial solution \( u(t, x) \) possessing an interface. At main order this solution consists of a traveling copy of the steady state \( \omega(|x|) \), which satisfies \( \omega''(y) - W'(\omega(y)) = 0 \). Its interface is resemble at main order copy of the sphere of the following form
\[ |x| = \sqrt{-2(n-3)(n-1)^2t}, \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^n, \]
which is a solution to the Willmore flow in Differential Geometry. When \( n = 1 \) or \( 3 \), the result consists of trivial interface solutions.

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1 Introduction

We are interested in the construction of solutions of the following generalized parabolic Cahn–Hilliard equation

$$u_t = -\Delta [\Delta u - W'(u)] + W''(u)[\Delta u - W'(u)], \quad \forall (t, x) \in \tilde{\mathbb{R}} \times \mathbb{R}^n. \quad (1.1)$$

where $n = 2$ or $n \geq 4$, $\tilde{\mathbb{R}}$ is given by

$$\tilde{\mathbb{R}} = \begin{cases} (0, \infty), & \text{if } n = 2, \\ (-\infty, 0), & \text{if } n \geq 4, \end{cases}$$

and the functions $W'(s)$ and $W''(s)$ denote the derivatives of first and second orders of $W$ respectively. The potential $W(s)$ is a smooth function, which satisfies the following assumptions

$$\begin{cases} W(s) > W(-1) = W(1) \quad \text{in } (-1, 1), \\ W(s) = W(-s), \quad \text{for all } s \in \mathbb{R}, \\ W'(-1) = W'(1) = 0, \\ W''(-1) = W''(1) > 0. \end{cases} \quad (1.2)$$

The potential $W(u)$ has two non-degenerate local minimum points $u = +1$ and $u = -1$, which are stable equilibria of (1.1). In particular, the function $W(s) = \frac{1}{4}(1 - s^2)^2$ obviously satisfies the above conditions in (1.2). Up to a scaling, the function $W(s) = \cos(s)$ also satisfies (1.2).

1.1 Backgrounds

Generally speaking, the Cahn–Hilliard equation means the following equation

$$\begin{cases} u_t = -\Delta [\Delta u - W'(u)], \quad \forall (t, x) \in (-\infty, +\infty) \times \Omega, \\ u(0, x) = u_0(x), \quad \forall x \in \Omega, \end{cases} \quad (1.3)$$
where $\Omega$ denotes a smooth bounded domain in $\mathbb{R}^n$ or the whole space $\mathbb{R}^n(n \geq 1)$, which describes phase separation processes of binary alloys in [7]. Various kinds of problems of this equation have been extensively studied in recent 30 years. By applying a priori estimate and continuity argument, the existence and asymptotic behaviors of global smooth solutions of Cauchy problem (1.3) have been proved in [35] when the initial value $u_0$ is close to stable equilibria $\bar{u}$ ($W(\bar{u}) = 0$) in the $L^\infty \cap L^1(\mathbb{R}^n)$ space. Using Fourier transform and estimates on the kernel of a linear parabolic operator, a uniform $L^\infty$ bound estimate for solutions of perturbed Cauchy problem (1.3) with additional nonlinear terms, was established by Caffarelli and Muler in [6]. In a bounded domain, the global well-posedness and long-time behavior of solutions to problem (1.3) with several types of dynamic boundary conditions were studied in [8, 9, 29, 36, 40]. See also [10, 34] for some recent advances of the Cahn–Hilliard equation with logarithmic potentials and related phase field models.

Based on the works of De Giorgi in [17, 18], numerous authors [3, 42, 43] studied the diffuse approximation of the Willmore functional:

$$W(S, \Omega) = \frac{1}{2} \int_{\partial S \cap \Omega} |H_{\partial S}(x)|^2 \, d\mu^{n-1},$$

(1.4)

where $\Omega$ is a given open set in $\mathbb{R}^n$, the set $S \subset \mathbb{R}^n$ with smooth boundary $\partial S \subset \Omega$, and $H_{\partial S}(x)$ is the mean curvature of surface $\partial S$ at point $x \in \partial S$. The approximating functional is defined by

$$W_\varepsilon(u) = \begin{cases} \frac{1}{2} \int_{\Omega} \left( \varepsilon \Delta u - \frac{W'(u)}{\varepsilon} \right)^2 \, dx, & \text{if } u \in L^1(\Omega) \cap W^{2,2}(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases}$$

(1.5)

where the function $W$ satisfies (1.2). The essential and more challenging work is to rigourously prove that the approximating functional $W_\varepsilon(u)$ $\Gamma$-converges to the Willmore functional $W(S, \Omega)$ as $\varepsilon$ goes to 0. Bellettini and Paolini in [4] proved the $\Gamma$-lim sup inequality for smooth Willmore hypersurfaces. However, the $\Gamma$-lim inf inequality is more hard to prove. Up to now, it has been proved in $\mathbb{R}^n$ with $n = 2, 3$ in [42] or $n = 2$ in [39]. This problem is still open in $\mathbb{R}^n$ with $n \geq 4$. The relation of the critical points of (1.4) and (1.5) was exemplified by Rizzi [41] and also Malchiodi et al. [38]. On the other hand, for the parabolic Cahn–Hilliard equation (1.3), the gamma convergence results have obtained by Le [33] under suitable conditions. And see [2] for the case of (1.3) on the one-dimensional torus.

The $L^2$ gradient flow of the approximating energy $W_\varepsilon(u)$ is equivalent to the evolution equation:

$$\begin{aligned} \partial_t u_\varepsilon &= - \Delta \left[ \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right] \\ &\quad + \frac{1}{\varepsilon^2} W''(u_\varepsilon) \left[ \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right] \quad \text{in } (-\infty, +\infty) \times \Omega, \end{aligned}$$

(1.6)

which was introduced in [24] to describe the deformation of a vesicle membrane under the elastic bending energy, with prescribed bulk volume and surface area. The well-posedness of the phase field model (1.6) with fixed $\varepsilon$ has been proved in [11] providing a volume constraint for the average of $u$, or in [12] with both volume and area constraints. By applying formal method of matched asymptotic expansions, Loreti and March [37] (or Wang [44]) showed that if $\Gamma(t) \subset \mathbb{R}^n$ with $n = 2$ or $3$, is a family of compact closed smooth interfaces and evolves by Willmore flow, it can be approximated by nodal set of the solution $u_\varepsilon$ to the phase
field (1.6) when ε goes to 0. The Willmore flow equation is given by

$$V(t) = \Delta \Sigma(t) H - \frac{1}{2} H^3 + H \|A\|^2,$$

(1.7)

which is the $L^2$ gradient flow for (1.4) with $\partial S(t) = \Sigma(t)$, where $V(t)$ denotes the outer normal velocity at $x \in \Sigma(t)$, $\Delta \Sigma(t)$ is the Laplace–Beltrami operator on surface $\Sigma(t)$, $H$ and $A$ are the mean curvature and the second fundamental form of $\Sigma(t)$ respectively, and $\|A\|^2$ is the sum of squared coefficients of $A$. Fei and Liu [25] proved that for given a solution $\Gamma_0(t)$ of (1.7) in $\mathbb{R}^n$ ($n = 2, 3$), there exists a solution $u_\varepsilon$ of equation (1.6) with Neumann boundary condition such that its level set convergence to $\Gamma_0(t)$ as the parameter $\varepsilon$ goes to zero. Moreover, a variety of problems for the Willmore flow (1.7) has been investigated by Kuwert and Schätzle in several papers (for example see [30–32]). However, the study of connections between equation (1.6) and the Willmore flow (1.7) is a challenging work when $n \geq 4$. The present paper is one of the first attempts in this direction.

### 1.2 Main results

In this paper, we want to find solutions of (1.1) whose values lie at all times in $[-1, 1]$, and approach either $+1$ or $-1$ in the most of the space $\mathbb{R}^n$. This type of solution corresponds to a continuous realization of a material, in which the two states ($u = -1$ and $u = +1$) coexist. The main difficult point of the study of this type solution of (1.1), is to derive qualitative information on the interface region (the walls separating the two phases). It is easy to find that $u(t, x)$ is a solution of (1.1) if and only if $u_\varepsilon(t, x) := u(\varepsilon^{-t}t, \varepsilon^{-1}x)$ satisfies equation (1.6). Basing on the results in [37] or [44] with $n = 2$ and 3, we know that the nodal set of $u_\varepsilon(t, x)$ approximates to the solution of Willmore flow (1.7). Let us consider the sphere $\Gamma_n(t)$ evolving by the Willmore flow in (1.7). Then we have that the radius $\gamma_n(t)$ of $\Gamma_n(t)$ satisfies the equation

$$\gamma_n'(t) = -\frac{1}{2} \left( \frac{n-1}{\gamma_n(t)} \right)^3 + \frac{(n-1)^2}{(\gamma_n(t))^3},$$

(1.8)

which has a solution

$$\gamma_n(t) := \sqrt[3]{-2(n-3)(n-1)^2 t},$$

(1.9)

where $t \leq 0$ when $n \geq 3$ and $t \geq 0$ when $n = 2$. Due to the self-similarity, there holds that the sphere $|x| = \gamma_n(t)$ is also the interface (nodal set) for $u(t, x)$, which is a solution of (1.1).

Our aim is to construct solutions to Eq. (1.1) with one transition layer closes to the sphere $|x| = \gamma_n(t)$. In which, $t \leq 0$ when $n \geq 4$, these solutions are called ancient solutions. And $t \geq 0$ for $n = 2$, they are long time solutions. We shall mention that the ancient radially symmetric solutions for the parabolic Allen–Cahn equation

$$u_t = \Delta u + u - u^3 \quad \text{in } (-\infty, 0] \times \mathbb{R}^n,$$

(1.10)

have been obtained by del Pino and Gkikas in [19] for $n = 1$ and [20] with $n \geq 2$.

Let us introduce a layer function by considering the following semilinear elliptic equation

$$\omega''(y) - W'(\omega(y)) = 0, \quad \omega'(y) > 0, \quad y \in \mathbb{R}, \quad \omega(0) = 0 \quad \text{and} \quad \lim_{y \to \pm \infty} \omega(y) = \pm 1,$$

(1.11)
which has a unique smooth solution $\omega(y)$ obtained in [1], where $W$ satisfies the conditions in (1.2). In particular if we choose $W'(s) = s^3 - s$, then (1.11) is the elliptic Allen–Cahn equation, and its solution is written as

$$
\omega(y) = \tanh \left( \frac{y}{\sqrt{2}} \right).
$$

For general function $W$ satisfying (1.2), the solution $\omega$ of (1.11) has no explicit expression. In [1], the inverse function of $\omega$ is given by

$$
\lambda(s) := \int_0^s \frac{1}{\sqrt{2[W(\tau) - W(1)]}} \, d\tau, \quad s \in (-1, 1).
$$

(1.12)

Thanks to (1.2), the function $\lambda(s)$ is well-defined. Taking $\omega$ as a basic layer (the name layer is motivated by the fact that $\omega$ approaches the limits 1 and $-1$ at $\pm \infty$), we will construct a solution of (1.1) with a ‘transition layer’ which closes to the sphere $|x| = r_n(t)$ in (1.9).

More precisely, we want to find a solution of equation (1.1), which has the following asymptotical behavior

$$
u(t, x) \approx \omega(|x| - \rho(t)),
$$

(1.13)

where $\omega$ is given by (1.11). The function $\rho(t)$ satisfies

$$
\rho(t) = r_n(t) + h(t), \quad \text{with} \quad h(t) = O\left(\frac{1}{\log |t|}\right),
$$

(1.14)

where $h(t)$ is a $C^1$ function with respect to $t$ and the function $r_n(t)$ is defined by (1.9). In fact, $\rho(t)$ can be chosen by solving the following ODE

$$
\rho'(t) + \frac{(n - 3)(n - 1)^2}{2\rho^3(t)} = Q(\rho(t), \rho'(t)), \quad \text{with} \quad Q(\rho(t), \rho'(t)) = O\left(\frac{1}{|t| \left[ \log |t| \right]^{2(p-1)}}\right),
$$

for all $t \leq 0$ when $n \geq 4$ or $t \geq 0$ when $n = 2$, where $p \in (n, n + 1)$, see Sect. 5.

Our main results can be stated as follows.

**Theorem 1.1** When $n = 2$, there exists a radial solution $u(t, x)$ of equation (1.1) with $t \geq 0$, which has the form $u(t, x) = \omega(|x| - \hat{\rho}(t)) + \varphi(t, |x|)$, with

$$
\hat{\rho}(t) = \sqrt[2]{2t} + \tilde{h}(t).
$$

(1.15)

Moreover, $\tilde{h}(t)$ is a $C^1$ function with the decay of order $O(1/\log |t|)$ as $t \to +\infty$ and $\lim_{t \to +\infty} \varphi(t, |x|) = 0$ uniformly in $x \in \mathbb{R}^2$.

**Theorem 1.2** When $n \geq 4$, there exists an ancient solution $u(t, x)$ of equation with $t \leq 0$, which has the form $u(t, x) = \omega(|x| - \rho(t)) + \varphi(t, |x|)$ with

$$
\rho(t) = \sqrt[2]{-2(n - 3)(n - 1)^2} + h(t).
$$

(1.16)

Moreover, $h(t)$ is a $C^1$ function with the decay of order $O(1/\log |t|)$ as $t \to -\infty$ and $\lim_{t \to -\infty} \phi(t, |x|) = 0$ uniformly in $x \in \mathbb{R}^n$. □
Remark 1.3 In this paper, we mainly prove that the results of Theorem 1.2 hold. For Theorem 1.1, its proof is similar and we just now notice that \( t > 0 \) in this case \( n = 2 \). When \( n = 1 \) and 3, it is easy to check that problem (1.1) has a radial solution of the form

\[
u(t, |x|) = \omega(|x| - c),
\]

with any constant \( c \in \mathbb{R} \), where \( \omega(y) \) is the solution of (1.11). It is an interesting problem to construct solutions with multiple interfaces. The reader can refer to [19–23] and the references therein. \( \Box \)

Remark 1.4 We prove Theorem 1.2 by using Lyapunov–Schmidt reduction method. This method has been widely applied to solving the existence problem of solutions to various kinds of equations [13–16, 20]. Comparing to the previous works, we are facing three difficulties during the process of dealing with problem (1.1) as the following:

1. Firstly, the main term \( \omega(|x| - \rho(t)) \) in (1.14) is not a good approximate solution since it does not satisfy the boundary conditions in (2.3) and does not provide enough decay of the error. To modify it, we introduce a cut-off function around the origin and a correction function, see (2.14) in Sect. 2. This main idea comes from [41].

2. Secondly, equation (1.1) is a parabolic equation of fourth order, which does not satisfy the Maximum Principle since the heat kernel of the biharmonic parabolic operator is sign-changing, see [26]. To overcome it, we employ the blow-up technique together with the representation of parabolic kernel in Sect. 3.

3. Lastly, the correlative heat kernel of the linearization operator of the equation (1.1) is more intricate, which leads to hard task to get suitable a priori estimates of linearized problem of equation (1.1). To solve it, we modify the estimate of the error term \( E(t, r) \) in (2.18) with a polynomial decay in Sect. 2, and perform more delicate calculations in Sect. 3. \( \Box \)

The paper is organized as follows.

◮ In the first part of Sect. 2, we deduce some estimates of decay for \( \omega \) in (1.11) and its derivatives. After that, an approximate solution, say \( z(t, |x|) \) with a parameter \( \rho(t) \) in (2.14) and (2.7), will be defined. By the perturbation of \( z(t, |x|) + \phi(t, |x|) \), the setting-up of a projected form of (1.1) will be derived, see (2.16)–(2.17) with the Lagrange multiplier \( c(t) \). With the introduction of a suitable norm, the estimates of the error will be provided in the last part of Sect. 2.

◮ Section 3 is devoting to the collection of some results of linear parabolic equations with a biharmonic operator and then obtain the solvability of a linear projected problem in (3.61).

◮ In Section 4, we solve the nonlinear problem (2.16)–(2.17) by applying an argument of the fixed-point theorem.

◮ In Section 5, in order to obtain a radial solution to (1.1) we choose a suitable parameter \( h(t) \) (in other words, adjusting the parameter \( \rho \) given by (2.7)) such that \( c(t) \) equals to zero, in problem (2.16)–(2.17).

2 The setting-up: ansatz, the nonlinear projected problem for perturbation term

2.1 Some estimates of the basic layer

Before proving the main theorems, we first derive the decay estimates of the basic layer \( \omega \), which is the solution to equation (1.11).
Lemma 2.1  Let $\omega$ be the solution of problem (1.11), then we have

\[
\begin{align*}
\lim_{y \to +\infty} \frac{\omega(y) - 1 + \beta e^{-\sqrt{W''(1)y}}}{\beta^2 e^{-2\sqrt{W''(1)y}}} &= \frac{W^{(3)}(1)}{6W''(1)}, \\
\lim_{y \to -\infty} \frac{\omega(y) + 1 - \beta e^{\sqrt{W''(-1)y}}}{\beta^2 e^{2\sqrt{W''(-1)y}}} &= -\frac{W^{(3)}(-1)}{6W''(-1)}, \\
\lim_{y \to +\infty} \frac{\omega(y)}{e^{\sqrt{W''(1)y}}} &= \beta \sqrt{W''(1)} \quad \text{and} \quad \lim_{y \to -\infty} \frac{\omega(y)}{e^{\sqrt{W''(-1)y}}} = \beta \sqrt{W''(-1)},
\end{align*}
\] (2.1)

where $W''(s)$ and $W^{(3)}(s)$ denote the derivatives of second and third orders of the function $W(s)$ respectively, and

\[
\beta := \exp \left\{ \sqrt{W''(1)} \int_0^1 \left[ \frac{1}{\sqrt{2(W(s) - W(1))}} - \frac{1}{\sqrt{W''(1)(1 - s)}} \right] ds \right\}. \tag{2.2}
\]

Proof  Since the function $W$ satisfies the conditions in (1.2), especially $W''(1) = W'(-1) > 0$, the above limits in (2.1) are well-defined and $\beta$ is finite. Recall that the inverse function of $\omega$ is the function $\lambda(s)$ given by (1.12). Using L’Hospital’s rule, Taylor’s formula, the function $W \in C^{3,1}_{\text{loc}}(\mathbb{R})$ satisfies (1.2) and (1.12), we find that

\[
\lim_{s \to 1} - \frac{1}{1 - s} \left\{ \lambda(s) + \frac{\ln(1 - s)}{\sqrt{W''(1)}} - \int_0^1 \frac{1}{\sqrt{2(W(s) - W(1))}} - \frac{1}{\sqrt{W''(1)(1 - s)}} ds \right\} = -\frac{W^{(3)}(1)}{6(W''(1))^{3/2}},
\]

and

\[
\lim_{s \to -1} \frac{1}{1 + s} \left\{ \lambda(s) - \frac{\ln(1 + s)}{\sqrt{W''(-1)}} - \int_{-1}^0 \frac{1}{\sqrt{2(W(s) - W(1))}} - \frac{1}{\sqrt{W''(1)(1 + s)}} ds \right\} = \frac{W^{(3)}(-1)}{6(W''(-1))^{3/2}}.
\]

According to the fact that if $x = \lambda(s)$, then $s = \omega(y)$, using Taylor’s formula again and the evenness of $W$ in $(-1, 1)$, we easily deduce that the first two limits in (2.12) hold. So we have

\[
\lim_{y \to +\infty} \frac{1 - \omega(y)}{e^{-\sqrt{W''(y)}}} = \beta \quad \text{and} \quad \lim_{y \to -\infty} \frac{1 + \omega(y)}{e^{\sqrt{W''(-y)}}} = \beta.
\]

Using L’Hospital’s rule again, the condition (1.2) and the equation (1.11), we have

\[
\lim_{y \to +\infty} \frac{\omega'(y)}{e^{-\sqrt{W''(y)}}} = \lim_{y \to +\infty} \left\{ \frac{\omega(y) - 1}{-\sqrt{W''(1)e^{-\sqrt{W''(y)}}}} \times \frac{W'(-\omega) - W'(1)}{\omega(-\omega) - 1} \right\} = \beta \sqrt{W''(1)},
\]

and

\[
\lim_{y \to -\infty} \frac{\omega'(y)}{e^{\sqrt{W''(-y)}}} = \lim_{y \to -\infty} \left\{ \frac{\omega(y) + 1}{\sqrt{W''(-1)e^{\sqrt{W''(-y)}}}} \times \frac{W'(\omega) - W'(-1)}{\omega(\omega) + 1} \right\} = \beta \sqrt{W''(-1)}.
\]  

Next we consider the kernel of a fourth order linear operator. The main result is stated as the following.
Lemma 2.2 Let \( W(s) \) be a smooth function satisfying the conditions in (1.2). Then we have that any smooth solution of the homogeneous problem
\[
\left[ \partial_{xx} - W'(\omega(x)) \right] \varphi = 0, \quad |\varphi| \leq 1, \quad |\varphi_{xx}| \leq 1 \quad \text{in } \mathbb{R},
\]
has the form
\[
\varphi(x) = \mu \omega'(x),
\]
with some constant \( \mu \in \mathbb{R} \).

For the proof of this lemma, see Lemma 3.6 which is a more general result. \( \square \)

2.2 The setting-up of the problem

We will prove that Theorem 1.2 holds. Thus, we always assume that \( n \geq 4 \) and \( t < 0 \) in the rest of the present paper.

Let \( \tilde{u}(t, |x|) \) be a solution of (1.1), by a translation \( u(t, |x|) = \tilde{u}(t - T, |x|) \) with some abuse of notation, then we have that \( u(t, r) \) satisfies the following problem
\[
\begin{aligned}
& u_t = -u_{rrrr} - \frac{2(n-1)}{r} u_{rr} + \left( 2W''(u) - \frac{(n-1)(n-3)}{r^2} \right) u_{rr} + \left( \frac{2(n-1)W''(u)}{r} + \frac{(n-1)(n-3)}{r^3} \right) u_r \\
& + W'''(u)u_r^2 - W'(u)W''(u), \quad \forall (t, r) \in (-\infty, -T] \times (0, +\infty),
\end{aligned}
\]
(2.3)
where \( r = |x| \) and \( T \) is a large positive number whose value can be adjusted at different steps. For convenience, we denote the right hand side of the first equation in (2.3) by \( F(u) \), that is
\[
F(u) := -u_{rrrr} - \frac{2(n-1)}{r} u_{rr} + \left( 2W''(u) - \frac{(n-1)(n-3)}{r^2} \right) u_{rr} + \left( \frac{2(n-1)W''(u)}{r} + \frac{(n-1)(n-3)}{r^3} \right) u_r + W'''(u)u_r^2 - W'(u)W''(u).
\]
(2.4)

Our purpose is to find a solution of (2.3) with the property
\[
u(t, r) \approx \omega(r - \rho(t)),
\]
where \( \omega(\gamma) \) is the solution of problem (1.11).

Firstly, we notice that \( \omega(r - \rho(t)) \) does not satisfy the boundary conditions in (2.3). A smooth cut-off function \( \chi(r) \) can be defined in the form
\[
\chi(r) = 0, \quad \text{for } r \leq \frac{\delta_0}{2} \quad \text{and} \quad \chi(r) = 1, \quad \text{for } r \geq \delta_0,
\]
(2.5)
for some small fixed positive number \( \delta_0 \). We define the first approximate solution of (2.3) as the following
\[
\tilde{\omega}(t, r) = \omega(r - \rho(t)) \chi(r) + \chi(r) - 1.
\]
(2.6)
Here we assume that the function \( \rho(t) \) has the form
\[
\rho(t) = \gamma_n(t) + h(t),
\]
(2.7)
where the function \( h(t) = O((\log |t|)^{-1}) \) as \( t \) goes to \( -\infty \) and the function \( \gamma_n(t) \) is defined in (1.9), i.e.
\[
\gamma_n(t) := \sqrt{-2(n-1)^2(n-3)t}, \quad t < 0,
\]
(2.8)
which is a radial solution of Willmore flow equation (1.7) with \( n \geq 4 \). More precisely, we assume that \( h(t) \) satisfies the following constraint

\[
\sup_{t \leq 1} |h(t)| + \sup_{t \leq 1} \left\{ \frac{|t|}{\log |t|} |h'(t)| \right\} \leq 1.
\] (2.9)

Secondly, by using (1.11), we have that

\[
-\partial_r \tilde{\omega}(t, r) + F(\tilde{\omega}(t, r)) = \rho'(t)\omega'(r - \rho(t)) - \frac{(n - 3)(n - 1)}{r^2} \omega''(r - \rho(t))
\] (2.10)

\[
+ \frac{(n - 1)(n - 3)}{r^3} \omega'(r - \rho(t)), \quad \text{for } r > \delta_0.
\]

where the operator \( F(u) \) is defined by (2.4). By (2.8), we find that the second term in the right hand side of equality (2.10), that is

\[
- \frac{(n - 1)(n - 3)}{r^2} \omega''(r - \rho(t)),
\]

has a slow decay of order \( O \left( |t|^{-\frac{1}{2}} \right) \) as \( t \) goes to negative infinity. However, it is not enough to solve equation (2.10) since this term is much bigger than other terms in (2.10). To cancel it and improve the approximate solution, inspired by [41], we define a correction function

\[
\tilde{\omega}(y) := -\omega'(y) \int_0^y \left[ \omega'(\hat{y}) \right]^{-2} \int_{-\infty}^{\hat{y}} \frac{s}{2} \omega'(s) \, ds \, d\hat{y}.
\] (2.11)

Then there hold that

\[
L^*(\tilde{\omega}) := \left[ - \partial_{yy} + W''(\omega(y)) \right] \tilde{\omega}(y) = \frac{1}{2} y \omega'(y), \quad (L^*)^2[\tilde{\omega}(y)] = -\omega''(y), \quad \forall \, y \in \mathbb{R},
\] (2.12)

and \( \tilde{\omega}(y) \) is an odd function with exponential decay such that

\[
\int_{\mathbb{R}} \omega'(y) \tilde{\omega}(y) \, dy = 0 \quad \text{and} \quad |\tilde{\omega}(y)| \leq Ce^{-\frac{3}{4} \sqrt{W''(y)}}, \quad \text{for } y \in \mathbb{R}.
\] (2.13)

At last, we define an approximate solution of problem (2.3) as the following

\[
z(t, r) := \omega(r - \rho(t))\chi(r) + \chi(r) - 1 + \frac{(n - 1)(n - 3)}{r^2} \tilde{\omega}(r - \rho(t))\chi(r),
\] (2.14)

where the cut-off function \( \chi(r) \) and the function \( \rho(t) \) are given by (2.5) and (2.7) respectively.

We will look for a solution of equation (2.3) of the form

\[
u(t, r) = z(t, r) + \phi(t, r),
\] (2.15)

where \( \phi \) is a small perturbation term. This can be done by using the Lyapunov–Schmidt reduction method in two steps.

1. The first step (see Sects. 3, 4) is solving the following projected version of problem (2.3) in terms of \( \phi(t, r) \):

\[
\phi_t = L[\phi] + E(t, r) + N(\phi) - c(t)\partial_r \tilde{\omega}(t, r) \quad \text{in } (-\infty, -T) \times (0, \infty),
\] (2.16)

and

\[
\int_0^\infty \phi(t, r)\omega'(r - \rho(t)) r^{n-1} \, dr = 0, \quad \text{for all } t < -T,
\] (2.17)
where the function \( \tilde{\omega}(t, r) \) is defined by (2.6), the error term \( E(t, r) \) and nonlinear term \( N(\phi) \) are defined respectively by

\[
E(t, r) := F\left(z(t, r)\right) - \frac{\partial z(t, r)}{\partial t},
\]

and

\[
N(\phi) := F\left(z(t, r) + \phi(t, r)\right) - F\left(z(t, r)\right) - F'(z(t, r))\phi.
\]

In the above, \( F(u) \) is defined by (2.4) and the linear operator \( L[\phi] := F'(z(t, r))\phi \) is defined as follows

\[
F'(z(t, r))\phi := -\phi_{rrr} - \frac{2(n-1)}{r} \phi_{rr} + \left[ 2W''(z(t, r)) - \frac{(n-1)(n-3)}{r^2} \right] \phi_r \\
- \left( W''(z(t, r)) \right)^2 \phi + \left[ \frac{2(n-1)W''(z(t, r))}{r} - \frac{(3-n)(n-1)}{r^3} \right] \phi_r + 2W'''(z(t, r)) \phi_r z_r \\
+ 2W'''(z(t, r)) z_{rr} \phi + 2\frac{n-1}{r} W''(z(t, r)) z_r \phi.
\]

(2) The second step is to consider the following relation

\[
c(t) \int_0^\infty \partial_r \tilde{\omega}(t, r) \omega'(r - \rho(t)) r^{n-1} \, dr \\
= \int_0^\infty \left[ \omega''(r - \rho(t)) + \frac{n-1}{r} \omega'(r - \rho(t)) - W''(z(t, r)) \omega'(r - \rho(t)) \right] \\
\times \left( -\phi_{rr} - \frac{n-1}{r} \phi_r + W''(z(t, r)) \phi \right) r^{n-1} \, dr \\
+ \int_0^\infty \left[ \partial_r z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] \phi \omega'(r - \rho(t)) r^{n-1} \, dr \\
+ \int_0^\infty \left( \partial_t \phi(t, r) \partial_t \left[ \omega'(r - \rho(t)) \right] \right) r^{n-1} \, dr \\
+ \int_0^\infty \left( E(t, r) + N(\phi) \right) \omega'(r - \rho(t)) r^{n-1} \, dr,
\]

for all \( t < -T \). Later on, in Sect. 5, we will choose \( h(t) \) such that \( c(t) = 0 \). This means that the function \( u \) in (2.15) will exactly solve (2.3).

### 2.3 Estimates of the error terms

We will establish some estimates for the error term \( E(t, r) \) in (2.18). By Taylor’s formula, the definitions in (2.4) and (2.14), we have that

\[
E(t, r) = F\left(z(t, r)\right) - \frac{\partial z(t, r)}{\partial t} \\
= F\left(\tilde{\omega}(t, r) + \tilde{z}(t, r)\right) - \frac{\partial [\tilde{\omega}(t, r) + \tilde{z}(t, r)]}{\partial t} \\
= F\left(\tilde{\omega}(t, r)\right) + F'\left(\tilde{\omega}(t, r)\right)[\tilde{z}(t, r)] + F''\left(\tilde{\omega}(t, r) + \theta\tilde{z}(t, r)\right)[\tilde{z}(t, r), \tilde{z}(t, r)]
\]
where $\theta \in (0, 1)$. In the above, the operators $F(u)$ and $F'(u)[v]$ are given by (2.4) and (2.20) respectively, and the function $\hat{\omega}(t, r)$ is given by (2.6) and $\tilde{z}(t, r)$ is defined by

$$\tilde{z}(t, r) := \frac{(n - 1)(n - 3)}{r^2} \tilde{\omega}(r - \rho(t)) \chi(r),$$

with $\rho(t)$ and $\chi(r)$ given by (2.7) and (2.5) respectively. The operator $F''(u)[v_1, v_2]$ is defined as the following

$$F''(u)[v_1, v_2] := \Delta \left[ W''(u) v_1 v_2 \right] - \left\{ W''(u)W''(u) + W^{(4)}(u) - \Delta u + W'(u) \right\} v_1 v_2$$

$$+ W''(u) \left\{ \Delta v_1 - W''(u)v_1 \right\} v_2 + \left\{ \Delta v_2 - W''(u)v_2 \right\} v_1.$$  

The main result is given by the following lemma.

**Lemma 2.3** Let $\alpha := \sqrt{|W''(1)|}$, $p \in (n, n + 1]$ and $T > 1$, we set

$$\Phi(t, r) := \begin{cases} \log |t| |r|^{-\frac{1}{2}} \left( 1 + |r - \gamma_n(t) - \frac{1}{4\alpha} \log |t| \right)^{-p}, & \text{if } r \in [\delta_0, +\infty), \\ \log |t| |r|^{-\frac{1}{2}} \overline{K} \left\{ \frac{\delta_0}{2} \leq r < \delta_0 \right\}, & \text{if } r \in (0, \delta_0), \end{cases}$$

where the function $\gamma_n(t)$ is defined by (2.8) and $\delta_0$ is a small positive number given in (2.5). Here $\overline{K}_A$ is the characteristic function of the set $A$. Then there exists constant $C > 0$ which depends only on $\delta_0$, $\alpha$, and $n$, such that

$$|E(t, r)| \leq C \frac{\Phi(t, r)}{\log |t|},$$

for all $(t, r) \in (-\infty, -T] \times (0, +\infty)$.

**Proof** We first estimate the term $E_1(t, r)$ in (2.24). By the definitions of $F(u)$ and $\hat{\omega}(t, r)$ in (2.4) and (2.6), Lemma 2.1, we have that

$$E_1(t, r) = \left[ \rho'(t) + \frac{(n - 1)(n - 3)}{2r^3} \right] \omega'(r - \rho(t)) \overline{K}_{[r \geq \delta_0]} + O \left( \frac{1}{|\gamma_n(t)|^2} \right) \overline{K} \left\{ \frac{\delta_0}{2} \leq r < \delta_0 \right\},$$

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where $\chi_A$ is the characteristic function of the set $A$. Furthermore, using the assumption of $\rho(t)$ in (2.7)–(2.9), the definition of $\gamma_n(t)$ in (2.8) and Lemma 2.1, we have that

$$
|E_1(t, r)| \leq C \left[ \frac{1}{|t|^2} \chi_{|r| \leq \delta_0} + \frac{1}{r^3} \chi_{|r| > \delta_0} \right] e^{-n \rho(t)} + \frac{C}{|t|^2} \chi_{\left\{ \frac{3\delta_0}{2} < r < \delta_0 \right\}}
$$

$$
\leq C \left[ \frac{1}{|t|^2} \chi_{|r| \leq \delta_0} + \frac{1}{r^3} \chi_{|r| > \delta_0} \right] e^{-n \rho(t)} + \frac{C}{|t|^2} \chi_{\left\{ \frac{3\delta_0}{2} < r < \delta_0 \right\}}
$$

$$
\leq C \frac{\Phi(t, r)}{\log |t|}, \quad \text{for all} \ (t, r) \in (-\infty, -T] \times (0, +\infty), \quad (2.27)
$$

where $C$ is a positive constant only depending on $\delta_0$, $\alpha$ and $n$. Here we used the fact that there exists a positive constant $C > 0$ only depending on $\delta_0$, $\alpha$ and $n$ such that

$$
\frac{1}{r^3} e^{-\frac{3\delta_0}{2}|r-\rho(t)|} \chi_{\left\{ \frac{3\delta_0}{2} \leq r \leq \delta_0 \right\}} \leq C e^{-\frac{n}{4} \rho(t)} \quad \text{and} \quad e^{-\frac{3\delta_0}{2}|x|} \leq \frac{C}{(1 + |x|)^\beta}, \quad (2.28)
$$

for all $x \in \mathbb{R}$ and $p \in (n, n + 1]$, where $C > 0$ does not depend on $x$ and $t$.

Next we consider the term $E_2(t, r)$ in (2.25). Using the definitions of linear operators $F'(u)[v]$ and $F''(u)[v, v]$ in (2.20) and (2.23), Lemma 2.1 and estimate in (2.13), we derive that

$$
E_2(t, r) = \left\{ F'(\omega(r-\rho(t)))\bar{z}(t, r) - \frac{(n-1)(n-3)}{r^2} \omega''(r-\rho(t)) + \left( F''(\omega(r-\rho(t))) + \omega''(r-\rho(t)) \right)\bar{z}(t, r), \bar{z}(t, r) + \rho'(t)\omega'(r-\rho(t)) \right\} \chi_{|r| \leq \delta_0}
$$

$$
= \left\{ \left[ -\left( \delta_r - W''(\omega(r-\rho(t))) \right)^2 \omega(r-\rho(t)) - \omega''(r-\rho(t)) \right] \frac{(n-1)(n-3)}{r^2}
$$

$$
+ O \left( \frac{1}{|\gamma_n(t)|^2} \right) \chi_{\left\{ \frac{3\delta_0}{2} < r < \delta_0 \right\}}
$$

$$
= O \left( \frac{1}{r^3} + \frac{\rho'(t)}{r^2} \right) e^{-\frac{3\delta_0}{2}|r-\rho(t)|} \chi_{|r| \leq \delta_0} + O \left( \frac{1}{|\gamma_n(t)|^2} \right) \chi_{\left\{ \frac{3\delta_0}{2} < r < \delta_0 \right\}}
$$

where we used the equalities in (2.12) and the fact that the function $\tilde{\omega}(t, r)$ in (2.11) and its derivatives are all exponentially decaying. By the same argument in (2.27) and the equality in (2.28), we can get that

$$
|E_2(t, r)| \leq C \frac{\Phi(t, r)}{\log |t|}, \quad \text{for all} \ (t, r) \in (-\infty, -T] \times (0, +\infty),
$$

where $C$ is a positive constant only depending on $\alpha$ and $n$.

Eventually, combining the above estimates of the terms $E_1(t, r)$ and $E_2(t, r)$, we can obtain the desired results. □

**Remark 2.4** According to the above proof of Lemma 2.3, it is easy to find that the error term $E(t, r)$ has an exponentially decaying in space variable

$$
|E(t, r)| \leq C \left| \frac{1}{|t|^2} e^{\frac{3\delta_0}{2}|r-\rho(t)| - \frac{1}{3\alpha} \log |t|} \right|, \quad (2.29)
$$
for all $r \geq \delta_0$, where $C > 0$ does only depend on $n$ and $\alpha$. Our goal is to solve nonlinear problem (2.16)–(2.17). Hence, according to (2.29), we may consider the following linear parabolic problem of fourth order

$$- \partial_t \varphi + F'(z(t, r))\varphi = f(t, r), \quad (t, r) \in (-\infty, -T) \times (0, +\infty), \quad (2.30)$$

with the function $f(t, r)$ satisfying

$$\sup_{(t, r) \in (-\infty, -T) \times (0, +\infty)} \left| f(t, r) \right| \leq C e^{\|r - \gamma_n(t) - \frac{1}{2\delta} \log|t|\|} < +\infty,$$

where the linear operator $F'(z(t, r))$ is defined in (2.20). However, the heat kernel of linear parabolic operator $-\partial_t + F'(z(t, r))$ would not have the exponential decay in (2.29). As far as we known, it however has the polynomial decay. Hence, we provide an estimate of the error term $E(t, r)$ with the polynomial decay in the above lemma.

$$\square$$

3 The linear problem

In this section, firstly, we will obtain the solvability of a class of semilinear biharmonic parabolic equations by applying some properties of biharmonic heat kernel and arguments of fixed-point theorem, which is given in Proposition 3.4. Secondly, we will prove that the linear projected problem (3.61) is solvable by using Proposition 3.4 and a priori estimate in Lemma 3.5. The main result is given by Proposition 3.9.

3.1 A few results of linear parabolic equations with a biharmonic operator

We first collect some known results for homogeneous biharmonic parabolic equation, from [26–28]. Its solution can be represented by the convolution of a biharmonic heat kernel and initial function.

The biharmonic heat kernel is given by

$$p_n(t, x) := \bar{\alpha}_n t^{-n/4} f_n\left(\frac{|x|}{t^{1/4}}\right), \quad \forall (t, x) \in \mathbb{R}^{n+1}_+ := (0, +\infty) \times \mathbb{R}^n. \quad (3.1)$$

Here the function $f_n$ is given by

$$f_n(s) := s^{1-n} \int_0^{+\infty} e^{-\eta^4} (s\eta)^n J_{\frac{\nu-2}{2}}(s\eta) d\eta, \quad \text{for } s > 0,$$

where $J_{\nu}$ denotes the $\nu$-th Bessel function of the first kind. $\bar{\alpha}_n$ denotes a suitable positive normalization number which depends on $n$ and satisfies

$$\bar{\alpha}_n t^{-n/4} \int_{\mathbb{R}^n} f_n\left(\frac{|y|}{t^{1/4}}\right) dy = 1, \quad \text{for all } t > 0.$$

There exist two constants $K_n$ and $\mu_n$ depending on $n$ such that

$$|f_n(s)| \leq K_n \exp(-\mu_n s^{4/3}), \quad \text{for all } s \geq 0. \quad (3.2)$$

For the derivative of $f_n$, the following formula holds

$$f'_n(s) = -sf_{n+2}(s). \quad (3.3)$$
Proposition 3.1 We consider the following Cauchy problem for the biharmonic heat equation:

$$
\begin{cases}
  u_t + (-\Delta)^2 u = 0 & \text{in } \mathbb{R}^{n+1}_+ := (0, +\infty) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x) & \text{in } \mathbb{R}^n,
\end{cases}
$$

(3.4)

where $n \geq 1$ and $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then (3.4) admits a unique global in time solution explicitly given by

$$
u(t, x) = \int_{\mathbb{R}^n} u_0(y) p_n(t, x - y) dy, \quad \forall (t, x) \in \mathbb{R}^{n+1}_+,$$

where the function $p_n$ is given in (3.1).

Next we will use the above results to study the following inhomogeneous problem:

$$
\begin{cases}
  u_t + (-\Delta)^2 u = f(t, x) & \text{in } (t_0, t_1) \times \mathbb{R}^n, \\
  u(t_0, x) = u_0(x) & \text{in } \mathbb{R}^n,
\end{cases}
$$

(3.5)

where $f(t, x) \in L^\infty((t_0, t_1) \times \mathbb{R}^n)$ and $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Definition 3.2 Assume that $t_1 \in (t_0, +\infty)$. We say that $u \in L^\infty((t_0, t_1) \times \mathbb{R}^n)$ is a mild solution of (3.5) if

$$u(t, x) = \Gamma_{t-t_0}[u_0](x) + \int_{t_0}^t \Gamma_{t-\tau} [f(\tau, \cdot)](x) d\tau, \quad \text{for } (t, x) \in (t_0, t_1) \times \mathbb{R}^n. \quad (3.6)$$

Here $\Gamma_t$ is a linear operator defined by

$$
\Gamma_t[u](x) := (p_n(t, \cdot) * u)(x) = \int_{\mathbb{R}^n} p_n(t, x - y) u(y) dy,
$$

where the biharmonic heat kernel $p_n(t, x)$ is given by (3.1). In addition, when $t_1 = +\infty$, a mild solution also can be defined by (3.6) provided $u(t, x) \in L^\infty_{\text{loc}}(t_0, +\infty) \times L^\infty(\mathbb{R}^n)$.

Notice that it is not hard to verify that $u$ is a mild solution of (3.5) if and only if $u$ solves it in the pointwise sense. We give some regularity estimates for mild solutions.

Proposition 3.3 Let $u$ be a mild solution of problem (3.5). Assume that $t_1 < +\infty$. Then the following estimates are valid:

- If $u_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, then for every $t_* \in (t_0, t_1)$, $t_* > 0$, $\theta \in (0, 4)$ and $\theta \in (0, 1)$, it holds that

$$
\sup_{t \in (t_*, t_1)} \|u(t, \cdot)\|_{C^\theta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|u(\cdot, x)\|_{C^\theta(t_*, t_1)} \leq C_1 \left( \|f\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|u_0\|_{L^\infty(\mathbb{R}^n)} \right),
$$

(3.7)

for some positive constant $C_1$ depending on $n$, $\theta$, $t_1 - t_0$, $t_1 - t_*$, $\theta$ and $t_*$.

- If $u_0 \in C^\vartheta(\mathbb{R}^n)$ for some $\vartheta \in (0, 4)$ and $t_0 > 0$, then it holds

$$
\sup_{t \in (t_0, t_1)} \|u(t, \cdot)\|_{C^\vartheta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|u(\cdot, x)\|_{C^\vartheta(t_0, t_1)} \leq C_2 \left( \|f\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|u_0\|_{C^\vartheta(\mathbb{R}^n)} \right),
$$

(3.8)

for some positive constant $C_2$ depending on $t_1 - t_0$, $n$ and $\vartheta$. 

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\textbf{Proof} According to the formula (3.3) and some direct computations, we can derive some estimates for the derivatives of biharmonic heat kernel \( p_n(t, x) \) as follows:

- \(|\partial_t p_n(t, x)| \leq C_n \left( t^{-1} |p_n(t, x)| + t^{-1} |x|^2 |p_{n+2}(t, x)| \right) ;
- |\nabla_x p_n(t, x)| \leq C_n |x| |p_{n+2}(t, x)| ;
- |\nabla^2_x p_n(t, x)| \leq C_n \left( |p_{n+2}(t, x)| + |x|^2 |p_{n+4}(t, x)| \right) ;
- |\nabla^3_x p_n(t, x)| \leq C_n \left( |p_{n+4}(t, x)| + |x|^3 |p_{n+6}(t, x)| \right) ;
- |\nabla^4_x p_n(t, x)| \leq C_n \left( |p_{n+4}(t, x)| + |x|^2 |p_{n+6}(t, x)| + |x|^4 |p_{n+8}(t, x)| \right) ,

where \( C_n > 0 \) only depends on \( n \).

We decompose the mild solution in (3.6) as follows

\[ u(t, x) = U_1(t, x) + U_2(t, x), \]

where

\[ U_1(t, x) = \Gamma_{t-t_0}[u_0](x), \quad U_2(t, x) = \int_{t_0}^{t} \Gamma_{t-t}[f(\tau, \cdot)](x) d\tau. \]

The analysis will begin with the first term \( U_1(t, x) \). For any \( t > t_0 \) and \( x \in \mathbb{R}^n \), using the above estimates and (3.2), we derive that

\[ |\partial_t U_1(t, x)| = \left| \int_{\mathbb{R}^n} \partial_t p_n(t-t_0, x-y)u_0(y)dy \right| \leq \frac{C_n}{t-t_0} \| u_0 \|_{L^\infty(\mathbb{R}^n)}. \]

Moreover, using (3.2), we have that

\[ |U_1(t, x)| \leq \int_{\mathbb{R}^n} |p_n(t-t_0, x-y)u_0(y)| dy \leq C_n \| u_0 \|_{L^\infty(\mathbb{R}^n)}. \]

For any \( \vartheta \in (0, 4) \) and \( t, t_1 \in (t_0, t_1) \), using the fact \( p_n(t, x) = t^{-n/4} p_n(\frac{x}{t^{1/4}}, 1) \), it can be checked that

\[ \left| \frac{U_1(t, x) - U_1(t_1, x)}{|t-t_1|^{\vartheta / 4}} \right| \leq \left| \int_{\mathbb{R}^n} u_0((t-t_0)^{1/4} y) - u_0((t_1-t_0)^{1/4} y) p_n(1, y) dy \right| \leq \frac{C_n}{|t-t_1|^{\vartheta / 4}} \| u_0 \|_{C^\vartheta(\mathbb{R}^n)}. \]  

(3.9)

Here we have used the inequality \( a^{\vartheta / 4} - b^{\vartheta / 4} \leq (a-b)^{\vartheta / 4} \) for any \( a \geq b \geq 0 \) and \( \vartheta \in (0, 4) \) in the second inequality, \( C_n > 0 \) only depends on \( n \).

Moreover, for the derivatives of \( U_1 \) with respect to \( x \), by (3.2), we have that

\[ |\nabla_x U_1(t, x)| = \left| \int_{\mathbb{R}^n} \nabla_x p_n(t-t_0, x-y)u_0(y)dy \right| \leq C_n \| u_0 \|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |y| |p_{n+2}(y, t-t_0)| dy \leq \frac{C_n}{(t-t_0)^{1/4}} \| u_0 \|_{L^\infty(\mathbb{R}^n)}, \]

where \( C_n \) does only depend on \( n \). Similarly, we can get that

\[ |\nabla^2_x U_1(t, x)| \leq \frac{C_n}{(t-t_0)^{1/2}} \| u_0 \|_{L^\infty(\mathbb{R}^n)} \quad \text{and} \quad |\nabla^3_x U_1(t, x)| \leq \frac{C_n}{(t-t_0)^{3/4}} \| u_0 \|_{L^\infty(\mathbb{R}^n)}. \]
and

$$|\nabla^4_x U_1(t, x)| \leq \frac{C_n}{(t - t_0)} \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$ 

Thus for any \(x, y \in \mathbb{R}^n\) and \(\theta \in (0, 1)\), we have

$$[U_1]_\theta(t) := \frac{|U_1(t, x) - U_1(t, y)|}{|x - y|^\theta} \leq \begin{cases} 2\|U_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)}, & \text{if } |x - y| \geq 1, \\ \|\nabla_x U_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)}, & \text{if } |x - y| \leq 1. \end{cases}$$

Hence, for any \(t_0 > t_0\), we have

$$\sup_{t \in (t_*, t_1)} [U_1]_\theta(t) \leq \frac{C_n}{(t_0 - t_0)^{1/4}} \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$ 

By similar arguments, we can obtain the following estimates

$$\sup_{t \in (t_*, t_1)} [\nabla U_1]_\theta(t) \leq \frac{C_n}{(t_0 - t_0)^{1/2}} \|u_0\|_{L^\infty(\mathbb{R}^n)}, \quad \sup_{t \in (t_*, t_1)} [\nabla^2 U_1]_\theta(t) \leq \frac{C_n}{(t_0 - t_0)^{3/4}} \|u_0\|_{L^\infty(\mathbb{R}^n)},$$

and

$$\sup_{t \in (t_*, t_1)} [\nabla^3 U_1]_\theta(t) \leq \frac{C_n}{(t_0 - t_0)^{1/4}} \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$ 

Combining the above estimates, we have that for any \(\vartheta \in (0, 4), t_* > t_0\) and \(\theta \in (0, 1)\),

$$\sup_{t \in (t_*, t_1)} \|U_1(t, \cdot)\|_{C^\vartheta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|U_1(\cdot, x)\|_{C^\vartheta(t_*, t_1)} \leq C_n \left[1 + \vartheta(t_* - t_0)\right] \|u_0\|_{L^\infty(\mathbb{R}^n)} \quad (3.10)$$

where \(\vartheta(t_* - t_0) = \frac{1}{(t_0 - t_0)} + \frac{1}{(t_0 - t_0)^{1/4}} + \frac{1}{(t_0 - t_0)^{1}}\).

Using the formula

$$U_1(t, x) = \int_{\mathbb{R}^n} p_n(t - t_0, y) u_0(x - y) dy$$

and (3.9), for \(\vartheta \in (0, 4)\), we have

$$\sup_{t \in (t_0, t_1)} \|U_1(t, \cdot)\|_{C^\vartheta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|U_1(\cdot, x)\|_{C^\vartheta(t_0, t_1)} \leq C_n \|u_0\|_{C^\vartheta(\mathbb{R}^n)}, \quad (3.11)$$

where \(C_n\) is a positive constant which depends on \(n\) and \(\vartheta\).

Next we will estimate \(U_2(t, x)\). First we find that

$$|U_2(t, x)| \leq C_n \int_{t_0}^t \int_{\mathbb{R}^n} |p_n(t - \tau, x - y) f(\tau, y)| dy d\tau$$

$$\leq C_n (t_1 - t_0) \|f\|_{L^\infty((t_1, t) \times \mathbb{R}^n)}.$$ 

For any \(t, t^1 \in (t_0, t_1)\), by the previous estimate of \(\partial_t p_n(t, x)\), we have

$$|p_n(t, x) - p_n(t^1, x)| \leq C_n \left|\log t - \log t^1\right|$$

$$\left(\|p_n((1 - \theta_1) t + \theta_1 t^1, x)\| + |x|^2 \|p_{n+2}((1 - \theta_1) t + \theta_1 t^1, x)\|\right),$$

for some \(\theta_1 \in (0, 1)\). Using the above inequality and (3.2), for \(\theta \in (0, 1)\), we have

$$\frac{|U_2(t, x) - U_2(t^1, x)|}{|t - t^1|^\theta} \leq \int_{t_0}^{t^1} \int_{\mathbb{R}^n} \left|p_n(t - \tau, x - y) - p_n(t^1 - \tau, x - y)\right| f(y, \tau) dy d\tau \quad |t - t^1|^\theta$$
parabolic equations. Let $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and we consider the initial value problem

$$\begin{aligned}
& u_t + (-\Delta)^2 u = G[\Delta u, \nabla u, u, t, x] \quad \text{in } (t_0, t_1) \times \mathbb{R}^n, \\
& u(t_0, x) = u_0(x) 
\end{aligned}$$

and for any $t \in (t_0, t_1,)$, we have

$$\begin{aligned}
& |\nabla_x U_2(t, x)| \leq C_n(t - t_0)^{1/4} \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)} , \\
& |\nabla_x^2 U_2(t, x)| \leq C_n(t - t_0)^{1/4} \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)},
\end{aligned}$$

and for any $\theta \in (0, 1),$

$$\begin{aligned}
& |\nabla_x^3 U_2(t, x_1) - \nabla_x^3 U_2(t, x_2)| \\
& \leq n \frac{\| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)}}{|x_1 - x_2|^\theta}
\end{aligned}$$

where $C_n$ is a positive constant which only depends on $n$. Second, we study the derivatives of $U_2(t, x)$ with respect to $x$. By some direct computations and (3.2), we have

$$\begin{aligned}
& \left| \left| \nabla^2_x U_2(t, x) \right| \right|_{L^\infty((t_0, t) \times \mathbb{R}^n)} \\
& \leq C_n(t - t_0)^{1/4} \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)}
\end{aligned}$$

By the same arguments as above, we can derive that

$$\begin{aligned}
& |\nabla_x^3 U_2(t, x)| \\
& \leq C_n(t - t_0)^{1/4} \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)},
\end{aligned}$$

and for any $\theta \in (0, 1),$

$$\begin{aligned}
& |\nabla_x^3 U_2(t, x_1) - \nabla_x^3 U_2(t, x_2)| \\
& \leq C_n(t - t_0)^{(1-\theta)/4} \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)}
\end{aligned}$$

Combing the above estimates of $U_2$, for any $\vartheta \in (0, 4)$ and $\theta \in (0, 1)$, we have that

$$\begin{aligned}
& \sup_{t \in (t_0, t_1)} \| U_2(t, \cdot) \|_{C^\vartheta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \| U_2(\cdot, x) \|_{C^\theta((t_0, t_1)} \\
& \leq C_n \left[ u(t_1 - t_0) + (t_1 - t_0)^{1-\theta} \right] \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)}. \quad (3.12)
\end{aligned}$$

On the other hand, for $\vartheta \in (0, 4)$, there holds

$$\begin{aligned}
& \sup_{t \in (t_0, t_1)} \| U_2(t, \cdot) \|_{C^\vartheta(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \| U_2(\cdot, x) \|_{C^\theta((t_0, t_1)} \leq C_n u(t_1 - t_0) \| f \|_{L^\infty((t_0, t) \times \mathbb{R}^n)}, \quad (3.13)
\end{aligned}$$

where $u(t - t_0) = (t - t_0) + (t - t_0)^{1-\vartheta}$ and $C_n$ is a positive constant which depends on $n$. The results in Proposition 3.3 follow from (3.10), (3.11), (3.12) and (3.13). \hfill \square

We will apply the above proposition to study the solvability of a class of semilinear parabolic equations. Let $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and we consider the initial value problem

$$\begin{aligned}
& u_t + (-\Delta)^2 u = G[\Delta u, \nabla u, u, t, x] \quad \text{in } (t_0, t_1) \times \mathbb{R}^n, \\
& u(t_0, x) = u_0(x) 
\end{aligned}$$

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where $G[p, q, s, t, x] : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function satisfying

1. For every $M > 0$ and $T > t_0$, there exists $C_{T, M} > 0$ such that
   \[
   \left| G[p, q, s, t, x] \right| \leq C_{T, M},
   \]
   for all $x \in \mathbb{R}^n$, $t \in [t_0, T]$ and $p, q, s \in [-M, M]$.

2. There is a constant $\sigma > 0$ in such a way that
   \[
   \left| G[p_1, q_1, s_1, t, x] - G[p_2, q_2, s_2, t, x] \right| \leq \sigma \left( |p_1 - p_2| + |q_1 - q_2| + |s_1 - s_2| \right),
   \]
   for all $x \in \mathbb{R}^n$, $t \geq t_0$, $p_1, p_2 \in \mathbb{R}$, $q_1, q_2 \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$.

In particular, we can take

\[
G[\Delta u, \nabla u, u, t, x] = a(t, x)\Delta u + \sum_{i=1}^{n} b^i(t, x) \nabla x_i u + c(t, x) u + g(t, x),
\]

where those functions $a(t, x)$, $b^1(t, x)$, $\ldots$, $b^n(t, x)$, $c(t, x)$ are all belonging to $L^\infty([t_0, +\infty), C^1(\mathbb{R}^n))$ and $g \in L^\infty([t_0, +\infty) \times \mathbb{R}^n)$.

Assume that $u(t, x)$ satisfies that

\[
\Delta u, \ |\nabla u|, \ u \in L^\infty((t_0, t_1) \times \mathbb{R}^n),
\]

then by (3.6), $u$ is a mild solution of (3.14) if and only if

\[
u(t, x) = \Gamma_{t-t_0}[u_0](x) + \int_{t_0}^{t} \Gamma_{t-\tau} \left[ G[\Delta u, \nabla u, u, \tau, \cdot] \right](\tau, \cdot) d\tau,
\]

for $(t, x) \in (t_0, t_1) \times \mathbb{R}^n$. (3.17)

Notice that $G[\Delta u, \nabla u, u, t, x] \in L^\infty((t_0, t_1) \times \mathbb{R}^n)$ by the assumptions on $u$ and $G$. Thus (3.17) is well defined by Proposition 3.3.

Now we devote our efforts to the study of the solvability of problem (3.14). For convenience, we define a map $N_{G, u_0}$ from $L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$ to itself as follows

\[
N_{G, u_0}[u](t, x) := \Gamma_{t-t_0}[u_0](x) + \int_{t_0}^{t} \Gamma_{t-\tau} \left[ G[\Delta u, \nabla u, u, \tau, \cdot] \right](\tau, \cdot) d\tau, \quad \forall (t, x) \in \mathbb{R}^n \times (t_0, t_1),
\]

for $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $u \in L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$. The main result is the following:

**Proposition 3.4** Given any $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, there exists a unique mild solution $u$ to problem (3.14) with $t_1 = +\infty$. Moreover, if $u_0 \in C^2(\mathbb{R}^n)$, then the solution $u$ has the property: for $s > t_0$, there exists $C$ independent of $s$ such that

\[
\sup_{t \in (t_0, s)} \|u(t, \cdot)\|_{C^2(\mathbb{R}^n)} \leq C \left( \left( s - t_0 \right) + \left( s - t_0 \right)^{1/2} \right) \|G(0, 0, 0, x, t)\|_{L^\infty((t_0, s) \times \mathbb{R}^n)} + \|u_0\|_{C^2(\mathbb{R}^n)}.
\]

**Proof** Notice that this proposition is equivalent to that problem (3.14) has a unique solution $u$ for any $t_1 \in (t_0, +\infty)$. According to the definition in (3.6), $u$ is a mild solution of (3.14) if and only if $u$ is a fixed point of the map $N_{G, u_0}$. Thus we will study the existence and uniqueness of fixed point for the map $N_{G, u_0}$.
Using Proposition 3.3 and the assumptions on $G$, we have that $\mathcal{N}_{G,u_0}$ is an operator from the space $L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$ to itself. We claim that the map $\mathcal{N}_{G,u_0}$ is a contraction when $T_1 := t_1 - t_0$ is small enough.

Indeed, by Proposition 3.1, the assumptions on $G$ in (3.15) and some direct computations, we have that for any $u, w \in L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$,

\[
|\mathcal{N}_{G,u_0}[u](t, x) - \mathcal{N}_{G,u_0}[w](t, x)| \leq \int_{t_0}^{t} \int_{\mathbb{R}^n} |p_n(t - \tau, x - y)| \cdot \left| G[\Delta u, \nabla u, u](\tau, y) - G[\Delta w, \nabla w, w](\tau, y) \right| dy \ d\tau
\]

\[
\leq \sigma T_1 C_n \left( \|\Delta u - \Delta w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|\nabla u - \nabla w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|u - w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} \right),
\]

for a.e. $x \in \mathbb{R}^n$, $t \in (t_0, t_1)$, where $C_n$ is a positive number which only depends on $n$. Hence, we can choose $T_1 \leq \frac{1}{2\sigma C_n}$ such that

\[
\|\mathcal{N}_{G,u_0}[u](t, x) - \mathcal{N}_{G,u_0}[w](t, x)\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} \leq \frac{1}{2} \left( \|\Delta u - \Delta w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|\nabla u - \nabla w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} + \|u - w\|_{L^\infty((t_0, t_1) \times \mathbb{R}^n)} \right).
\]

Thus, the map $\mathcal{N}_{G,u_0}$ is a contraction.

Next we consider the following iterative sequence

\[
u_m(t, x) = (\mathcal{N}_{G,u_0})^m[0](t, x),
\]

by Banach fixed point theorem, then there exists $u \in L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$ such that $\nu_m(t, x)$ converges to $u(t, x)$ in the space $L^\infty((t_0, t_1); C^2(\mathbb{R}^n))$ and $\mathcal{N}_{G,u_0}(u) = u$, which is the unique solution of problem (3.14) with $t_1 = t_0 + \frac{1}{2\sigma C_n}$.

Let $u^1(t, x)$ denote the mild solution of (3.14), which can be obtained by the above arguments. We consider the following initial problem:

\[
\begin{cases}
\nu_t + (-\Delta)^2 u = G[\Delta u, \nabla u, u, x, t] & \text{in } (t_0 + \frac{1}{2\sigma C_n}, t_0 + \frac{1}{\sigma C_n}) \times \mathbb{R}^n, \\
u(t_0 + \frac{1}{\sigma C_n}, x) = u^1(t_0 + \frac{1}{2\sigma C_n}, x) & \text{in } \mathbb{R}^n.
\end{cases}
\]

Repeating the previous arguments, we can get that the above equation has a unique solution $u^2(t, x)$. Thus we can extend the time interval of existence to $(t_0, t_0 + \frac{l}{2\sigma C_n})$ with any positive integer $l \geq 1$ step by step. In this process, we used the fact that the initial function is belonging to $C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ on each step. Indeed, the solutions are well-defined bounded continuous functions of $t$ and $x$, and are continuously differentiable in $x$, due to the estimate (3.8) with $\vartheta = 1$ in Proposition 3.3. The estimate (3.18) follows from Proposition 3.3. \hfill \Box

### 3.2 A priori estimate of a linear problem involving the operator $L$

Recall the functions $\rho$, $\gamma_n$ and $h$ in (2.7)-(2.9). By defining

\[
\|\psi\|_{C_\phi((t_1, t_2) \times (0, \infty))} := \|\frac{\psi}{\Phi}\|_{L^\infty((t_1, t_2) \times (0, \infty))},
\]

we choose a set consisting of continuous functions as the following

$C_\phi((t_1, t_2) \times (0, \infty))$
where \( t_1 < t_2 < 0, \delta_0 > 0 \) is a small fixed number given in (2.5), and \( \Phi(t, r) \) is given by (2.26).

Let us consider the following Cauchy problem:

\[
\begin{align*}
\psi_t &= L[\psi] + g(t, r) & \text{in} \ (s, -T) \times (0, \infty), \\
\psi(s, r) &= 0, & \forall r \in (0, +\infty), \\
\partial_r^j \psi(t, 0) &= 0, & \forall t \in (s, -T) \text{ and } j = 1, 3,
\end{align*}
\]

(3.20)

where \( g \in C_{\Phi} ((-\infty, -T) \times (0, \infty)) \), \( T > 0 \) and \( s + 1 < -T \). Here are some facts.

(1) Problem (3.20) has a unique solution \( \psi^s(t, r) \).

Indeed, by the definition of the operator \( L[\psi] \) in (2.20), we can rewrite \( L[\psi] \) as follows

\[
L[\psi] = -\Delta^2 \psi + 2W''(z(t, r)) \Delta \psi + 2W'''(z(t, r)) \Delta z(t, r) \psi + W^{(4)}(z(t, r)) |\nabla z(t, r)|^2 \psi
+ 2W'''(z(t, r)) \nabla z(t, r) \cdot \nabla \psi - \left[ W''(z(t, r)) \right]^2 - W'(z(t, r)) W'''(z(t, r)) \right] \psi,
\]

where we recall that approximate solution

\[
z(t, r) = \left( \omega(r - \rho(t)) + \frac{(n - 1)(n - 3)}{r^2} \omega(r - \rho(t)) \right) \chi(r) + \chi(r) - 1,
\]

and the functions \( \chi(r) \) and \( \rho(t) \) are given by (2.5) and (2.7) respectively. According to Lemma 2.1 and the assumptions on \( W(s) \) in (1.2), those coefficients in front of \( \Delta \psi, \nabla \psi \) and \( \psi \) are all smooth and bounded for \( x \in \mathbb{R}^n \) and \( t < -2 \). Thus we know that problem (3.20) is uniquely solvable by applying Proposition 3.4, and denote this solution by \( \psi^s(t, r) \).

(2) We have that \( \psi^s(t, r) = 0 \) when \( r \in (0, \frac{\delta_0}{2}) \).

In fact, we consider

\[
\psi^s(t, r) := \begin{cases} 
\psi^s(t, r), & \text{if } r \in (0, \frac{\delta_0}{2}], \\
0, & \text{if } r \in (\frac{\delta_0}{2}, +\infty),
\end{cases}
\]

which satisfies that

\[
\begin{align*}
\psi_t &= -(\Delta)^2 \psi + 2W''(1) \Delta \psi - \left[ W''(1) \right]^2 \psi & \text{in} \ (s, -T) \times (0, +\infty), \\
\psi(s, r) &= 0, & \forall r \in (0, +\infty), \\
\partial_r^j \psi(t, 0) &= 0, & \forall t \in (s, -T) \text{ and } j = 1, 3,
\end{align*}
\]

where \( n \geq 4 \) and we have used the fact that \( z(t, r) = -1 \) when \( r \in (0, \frac{\delta_0}{2}) \). According to the Proposition 3.4, the above equation has a unique mild solution \( \psi(t, r) = 0 \). Hence \( \psi^s(t, r) = 0 \), that is \( \psi^s(t, r) = 0 \) when \( r \in (0, \frac{\delta_0}{2}) \).

We next establish a priori estimate for the solutions of problem (3.20).

**Lemma 3.5** Let \( g \in C_{\Phi} ((s, -T) \times (0, \infty)) \) and \( \psi^s \) be a solution of problem (3.20) which satisfies the orthogonality condition

\[
\int_0^\infty \psi^s(t, r) \omega' (r - \rho(t)) r^{n-1} \, dr = 0, \quad s < t < -T.
\]

(3.21)
Then there exists a uniform constant $T_0 > T$ such that for any $t \in (s, -T_0]$, the following estimate is valid
\[
\sum_{l=0}^{3} \| \partial_t^l \psi \|_{C_{\Phi}((s,t) \times (0,\infty))} \leq C \| g \|_{C_{\Phi}((s,t) \times (0,\infty))},
\]
(3.22)
where $C > 0$ is a uniform positive constant independent of $t$ and $s$.

The proofs of this lemma consist of tedious analysis, which will be given in the sequel.

### 3.2.1 Proof of Lemma 3.5

Here are the details.

**Proof** We prove the above lemma by contradiction. Assume that there exist two sequences $\{s_i\}$ and $\{t_i\}$ such that $s_i + 1 < t_i < 0$ and $s_i \to -\infty$, $t_i \to -\infty$ when the sub-index $i$ goes to infinity. We assume that: for each given $i$, there exists $g^i \in C_{\Phi}((s_i, t_i) \times (0, \infty))$, such that
\[
\sum_{l=0}^{3} \| \partial_t^l \psi^i \|_{C_{\Phi}((s_i, t_i) \times (0, \infty))} = 1,
\]
(3.23)
and
\[
\| g^i \|_{C_{\Phi}((s_i, t_i) \times (0, \infty))} \to 0, \quad \text{as } i \to +\infty,
\]
(3.24)
where $\psi^i$ is the solution of problem (3.20)–(3.21) with $g = g^i$, $s = s_i$ and $-T = t_i$.

**Assertion:** For any $R > 0$, we have that
\[
\lim_{i \to \infty} \sum_{l=0}^{3} \left\| \frac{\partial_t^l \psi^i}{\Phi} \right\|_{L^\infty(A^{(s_i, t_i)}_{\Phi})} = 0,
\]
(3.25)
where $\Phi(t, r)$ is given by (2.26) and $A^{(s_i, t_i)}_{\Phi}$ is defined as
\[
A^{(s_i, t_i)}_{\Phi} := \left\{ (t, r) \in (s_i, t_i) \times (0, \infty) : |r - \gamma_n(t)| < R + 1 \right\},
\]
and the function $\gamma_n(t)$ is defined by (2.8).

The proof of (3.25) will be provided in Sect. 3.2.2. We first accept the validity of (3.25). According to (3.23), we have that there exists $l \in \{0, 1, 2, 3\}$ such that
\[
\| \partial_t^l \psi^i \|_{C_{\Phi}((s_i, t_i) \times (0, \infty))} \geq \frac{1}{4}
\]
Without loss of generality, we assume that $\| \psi^i \|_{C_{\Phi}((s_i, t_i) \times (0, \infty))} \geq \frac{1}{4}$. For the other situations, the following arguments can be adapted in a simple way.

Recall the definition of $C_{\Phi}((s_i, t_i) \times (0, \infty))$ in (3.19), then we derive that
\[
\| \psi^i \|_{C_{\Phi}((s_i, t_i) \times (0, \infty))} = \sup_{(t, r) \in (s_i, t_i) \times (0, \infty)} \frac{|\psi^i(t, r)|}{\Phi(t, r)} \geq \frac{1}{4}.
\]
Thus, there exists $(\bar{t}_i, \bar{r}_i) \in (s_i, t_i) \times (0, \infty)$ such that
\[
\frac{|\psi^i(\bar{t}_i, \bar{r}_i)|}{\Phi(\bar{t}_i, \bar{r}_i)} \geq \frac{1}{4}.
\]
Furthermore, by (3.25), we have
\[
\lim_{i \to +\infty} \left| \bar{r}_i - \gamma_n(\bar{t}_i) \right| = +\infty. \tag{3.26}
\]

Let us define
\[
\phi^i(v, y) := \frac{\psi^i(v + \bar{t}_i, y + \bar{y}_i + \gamma_n(v + \bar{t}_i))}{\Phi(\bar{t}_i, \bar{y}_i + \gamma_n(\bar{t}_i))}, \quad \text{where } \bar{y}_i = \bar{r}_i - \gamma_n(\bar{t}_i),
\]
where the functions \(\gamma_n(t)\) and \(\Phi(t, r)\) are given by (2.8) and (2.26) respectively. Thus \(\phi^i(v, y)\) satisfies the following problem
\[
\begin{align*}
\partial_v \phi^i &= -\phi^i_{yyyyy} - \frac{2(n-1)}{y + \bar{y}_i + \gamma_n(v + \bar{t}_i)} \phi^i_{yyyy} + \frac{2 W''(\bar{z}(v, y)) - (n-3)(n-1)}{(y + \bar{y}_i + \gamma_n(v + \bar{t}_i))^2} \phi^i_{yy} \\
&\quad - \left( W''(\bar{z}(v, y)) \right)^2 \phi^i + \left[ \frac{2(n-1)W''(\bar{z}(v, y))}{y + \bar{y}_i + \gamma_n(v + \bar{t}_i)} - \frac{(3-n)(n-1)}{(y + \bar{y}_i + \gamma_n(v + \bar{t}_i))^3} \right] \phi^i_y \\
&\quad + 2 W''(\bar{z}(v, y)) \bar{z}_y \phi^i_y - W'''(\bar{z}(v, y)) W'(\bar{z}(v, y)) \phi + W''(\bar{z}(v, y)) \bar{z}_y^2 \phi^i \\
&\quad + \frac{\phi^i(v + \bar{t}_i, y + \bar{y}_i + \gamma_n(v + \bar{t}_i))}{\Phi(\bar{t}_i, \bar{y}_i + \gamma_n(\bar{t}_i))} \quad \text{in } B_i^{(\bar{s}_i, \bar{t}_i)}, \tag{3.27}
\end{align*}
\]
with the conditions
\[
\left| \phi^i(0, 0) \right| \geq \frac{1}{5}, \quad \phi^i(s_i - \bar{t}_i, y) = 0, \quad \text{for } y \in (\bar{y}_i - \gamma_n(v + \bar{t}_i), +\infty), \tag{3.28}
\]
where
\[
\bar{z}(v, y) := z(v + \bar{t}_i, y + \bar{y}_i + \gamma_n(v + \bar{t}_i))
\]
and the set \(B_i^{(s_i, t_i)}\) is defined by
\[
B_i^{(s_i, t_i)} := (s_i - \bar{t}_i, 0] \times (\bar{y}_i - \gamma_n(v + \bar{t}_i), +\infty).
\]

Since \(\lim_{i \to +\infty} |\bar{y}_i| = +\infty\) due to (3.25), we have that \(\lim_{i \to +\infty} \bar{y}_i = +\infty\) or \(-\infty\) up to a subsequence. Moreover, notice that there exist two cases:

**Case 1** \(\lim_{i \to -\infty} s_i - \bar{t}_i = -\infty.\)

**Case 2** \(\lim_{i \to -\infty} s_i - \bar{t}_i = -\zeta\) for some positive constant \(\zeta > 0.\)

Next we will prove that Lemma 3.5 holds, and the process will be divided into the following two steps.

**Step one** We first consider **Case 1**, and then have that \(\phi^i \to \phi\) locally uniformly. Moreover, \(|\phi(0, 0)| > \frac{1}{5}\) and \(\phi\) satisfies
\[
\phi_v = -\phi_{yyyy} + 2 W''(1) \phi_{yy} - \left[ W''(1) \right]^2 \phi \quad \text{in } (-\infty, 0] \times \mathbb{R}. \tag{3.29}
\]

By the assumption on \(\psi^i\) in (3.23), we have that for all \((v, y) \in B_i^{(s_i, t_i)},\)
\[
\left| \phi^i(v, y) \right| = \left| \frac{\psi^i(v + \bar{t}_i, y + \bar{y}_i + \gamma_n(v + \bar{t}_i))}{\Phi(\bar{t}_i, \bar{y}_i + \gamma_n(\bar{t}_i))} \right| \leq \left| \frac{\Phi(v + \bar{t}_i, y + \bar{y}_i + \gamma_n(v + \bar{t}_i))}{\Phi(\bar{t}_i, \bar{y}_i + \gamma_n(\bar{t}_i))} \right|, \tag{3.30}
\]
where the sets $B_i^j$ with $j = 0, 1$, are defined by

$$B_i^0 := (s_i - t_i, 0) \times \left( - \tilde{y}_i - \gamma_n(v + \tilde{t}_i), \delta_0 - \gamma_n(v + \tilde{t}_i) - \tilde{y}_i \right),$$

$$B_i^1 := (s_i - t_i, 0) \times (\delta_0 - \gamma_n(v + \tilde{t}_i) - \tilde{y}_i, +\infty).$$

(3.31)

Next we estimate the right hand side of the above inequality (3.30). There exist the following different situations:

1. When $j = 1$ and $t_i = \tilde{y}_i + \gamma_n(\tilde{t}_i) > \delta_0$, by the definitions of $\gamma_n(t)$ and $\Phi(t, r)$ in (2.8) and (2.26), we have that

$$\left| \frac{\Phi(v + \tilde{t}_i, y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))}{\Phi(\tilde{t}_i, \tilde{y}_i + \gamma_n(\tilde{t}_i))} \right| \leq C.$$

(3.32)

2. When $j = 1$ and $\frac{s_i}{2} < \tilde{t}_i \leq \delta_0$, by the same argument as above, we have that

$$\left| \frac{\Phi(v + \tilde{t}_i, y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))}{\Phi(\tilde{t}_i, \tilde{y}_i + \gamma_n(\tilde{t}_i))} \right| \leq C(1 + |y|)^p.$$

(3.33)

3. When $j = 0$, by the definitions of $\gamma_n(t)$ and $\Phi(t, r)$ in (2.8) and (2.26), we have that

$$\left| \frac{\Phi(v + \tilde{t}_i, y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))}{\Phi(\tilde{t}_i, \tilde{y}_i + \gamma_n(\tilde{t}_i))} \right| \leq C$$

where we have used the fact that $|\tilde{y}_i - \alpha| \leq |y|$ since $(v, y) \in B_i^0$ given by (3.31).

Combining above estimates, we derive that

$$|\phi^j(v, y)| \leq C(|y| + 1)^p,$$

(3.34)

for all $(v, y) \in B_i^j$ with $j = 0, 1$, where $p \in (n, n + 1)$ and $C$ is a positive constant only depending on $\alpha$ and $n$. Similarly, we have that

$$|\phi_{y}^j(v, y)| \leq C(|y| + 1)^p, \quad |\phi_{y}^j(v, y)| \leq C(|y| + 1)^p \quad \text{and} \quad |\phi_{yy}^j(v, y)| \leq C(|y| + 1)^p,$$

(3.35)

for all $(v, y) \in B_i^j$ and $C$ is a positive constant which only depends on $\alpha$ and $n$. Notice that

$$\bigcup_{j=0}^{1} B_i^j = (s_i - t_i, 0) \times \left( - \tilde{y}_i - \gamma_n(v + \tilde{t}_i), +\infty \right),$$

due to (3.31). Owing to (3.33) and (3.34), we have that

$$\phi^j \to \phi \quad \text{in } L^2_{loc}([\xi, 0); H^2_{loc}(\mathbb{R})].$$
where $\tilde{\zeta} = \zeta$ or $\infty$. Moreover, we note that there holds that

$$\lim_{i \to +\infty} -\gamma_n (v + \tilde{t}_i) - \tilde{y}_i = M,$$

up to a subsequence, where $M = -\infty$ or $M \in (-\infty, 0]$. For the second case, that is $M \in (-\infty, 0]$, we can get that $\phi$ satisfies that

$$\phi_v = -\phi_{yyyy} + 2W''(1)\phi_{yy} - \left[ W''(1) \right]^2 \phi \quad \text{in} \ (-\infty, 0] \times [0, +\infty).$$

Then, we can consider the function

$$\tilde{\phi}(v, x) := \phi(v, x) \quad \text{when} \ x \geq 0, \ \text{and} \ \tilde{\phi}(v, x) := \phi(v, -x) \quad \text{when} \ x < 0,$$

which is the even extension of $\phi$ with respect to $x$. It is easy to check that $\tilde{\phi}$ satisfies equation (3.29).

For Case 2: $\lim_{i \to \infty} s_i - \tilde{t}_i = -\zeta$ for some constant $\zeta > 0$, we have $\phi^i \to \phi$ locally uniformly, and $\phi$ satisfies

$$\begin{cases} 
\phi_v = -\phi_{yyyy} + 2W''(1)\phi_{yy} - \left[ W''(1) \right]^2 \phi & \text{in} \ (-\zeta, 0] \times \mathbb{R}, \\
\phi(-\zeta, y) = 0 & \text{for all} \ y \in \mathbb{R}.
\end{cases}$$

By Proposition 3.4, the above equation has a unique solution $\phi \equiv 0$. However, we have that

$$|\phi(0, 0)| = \lim_{i \to \infty} |\phi^i(0, 0)| \geq \frac{1}{5},$$

owing to (3.27), which leads to a contradiction. Hence the Case 2 cannot happen.

**Step two** We claim that

$$\phi(v, y) \equiv 0, \quad (3.35)$$

for all $(v, y) \in (-\infty, 0] \times \mathbb{R}$. This result contradicts with $|\phi(0, 0)| > \frac{1}{5}$, thus we can derive that Lemma 3.5 holds.

In the rest part of this proof, the job is to show that the above conclusion in (3.35) holds. Recall that $\alpha = \sqrt{W''(1)} > 0$, we then consider the following parabolic equation

$$u_v = -u_{yyyy} + 2\alpha^2 u_{yy}, \quad \forall (v, y) \in (0, +\infty) \times \mathbb{R}.$$ 

Using the Fourier transformation for $y$, the above equation has a formula solution

$$Q(v, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left\{ -\nu \left( |\xi|^4 + 2\alpha^2 |\xi|^2 \right) \right\} e^{i\xi y} d\xi; \quad (3.36)$$

Hence, for any $T > 0$ and $f \in L^\infty((-T, 0) \times \mathbb{R})$, the following initial value problem

$$u_v = -u_{yyyy} + 2\alpha^2 u_{yy} - \alpha^4 u + f(v, y), \quad (v, y) \in (-T, 0) \times \mathbb{R}, \quad u(-T, y) = 0, \quad \forall y \in \mathbb{R},$$

has a solution of the form

$$u(v, y) = \frac{1}{2\pi} \int_0^{v-T_1} \int_{\mathbb{R}} e^{-\alpha^4 \tau} Q(\tau, x) f(v - \tau, y - x) dx d\tau, \quad \forall (v, y) \in (-T, 0) \times \mathbb{R}. \quad (3.37)$$

According to formula (3.37) and Eq. (3.27), we have that

$$\phi'(v, y) = \int_0^{v-s_i+\tilde{t}_i} \int_{\mathbb{R}} e^{-\alpha^4 \tau} Q(\tau, x) f_i(\phi'(v - \tau, y - x), v - \tau, y - x) dx d\tau,$$
for all $(v, y) \in (s_i - \tilde{t}_i, 0) \times \mathbb{R}$, where the function $f_i(\phi^i, v, y)$ is defined as follows

$$f_i(\phi^i, v, y) := -\frac{2(n-1)}{y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i)} \phi_{yy}^i - \frac{(n-3)(n-1)}{(y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))^2} \phi_{yy}^i$$

$$+ \left[ \frac{2(n-1)W''(\bar{z}(v, y))}{y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i)} - \frac{(3-n)(n-1)}{(y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))^3} \right] \phi_y^i$$

$$+ 2W''(\bar{z}(v, y))\bar{z}_y \phi_y^i - W''(\bar{z}(v, y))W'(\bar{z}(v, y))\phi + W^{(4)}(\bar{z}(v, y)) |\bar{z}_y|^2 \phi^i$$

$$+ 2W''(\bar{z}(v, y))\bar{z}_{yy} \phi_y^i + 2 \frac{n-1}{y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i)} W''(\bar{z}(v, y))\bar{z}_y \phi_y^i + \phi_y^i \partial_y \gamma_n(v + \tilde{t}_i)$$

$$+ \frac{g^i(v + \tilde{t}_i, y + \tilde{y}_i + \gamma_n(\tilde{t}_i))}{\Phi(\tilde{t}_i, \tilde{y}_i + \gamma_n(\tilde{t}_i))}$$

$$+ 2\left[ W''(\bar{z}(v, y)) - W''(1) \right] \phi_y^i - \left( [W''(\bar{z}(v, y))]^2 - [W''(1)]^2 \right) \phi^i,$$

where $\bar{z}(v, y) = z(v + \tilde{t}_i, y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i))$, the function $z(v, r)$ is given by (2.14).

Using (3.19), (3.24), (3.33), (3.34), the same arguments in proof of Lemma 3.6, we have that

$$|f_i(\phi^i(v, y), v, y)| \leq C(1 + |y|)^p \left[ \exp \left\{ -\alpha |y + \tilde{y}_i| \right\} + \|g^i\|_{C_\Phi((s_i, \tilde{t}_i) \times (0, +\infty))} \right]$$

$$+ \frac{1}{[-(v + \tilde{t}_i)]^{3/4}} + \sum_{l=1}^{3} \left( y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i) \right)^{-l} \mathbb{I}_{\{y > \frac{\alpha \gamma_n(v + \tilde{t}_i) - \tilde{y}_i}{\alpha} \}}$$

for all $(v, y) \in (s_i - \tilde{t}_i, 0] \times (-\tilde{y}_i - \gamma_n(v + \tilde{t}_i), +\infty)$, where $C$ depends on $\alpha$ and $n$, and $\mathbb{I}_A$ denotes the characteristic function of the set $A$.

Moreover a straightforward shift of contour gives that there exists a positive constant $C$ such that

$$|Q(v, y)| \leq C v^{-\frac{3}{4}} \exp \left\{ -\frac{|y|}{v^{1/4}} \right\},$$

or see (3.2) in [5]. Thus, using (3.38) and the above inequality, we have that

$$|\phi^i(v, x)| \leq C \int_0^{v - s_i + \tilde{t}_i} \int_{\frac{s_i - \tilde{y}_i - \gamma_n(v + \tilde{t}_i)}{2}}^{+\infty} \tau^{-\frac{3}{4}} \exp \left\{ -\frac{|x - y|}{\tau^{1/4}} - \alpha^4 \tau \right\} \exp \left\{ -\alpha |y + \tilde{y}_i| \right\}$$

$$+ \frac{1}{[-(v + \tilde{t}_i - \tau)]^{3/4}} + \|g^i\|_{C_\Phi((s_i, \tilde{t}_i) \times \mathbb{R})} + \sum_{l=1}^{3} \left( y + \tilde{y}_i + \gamma_n(v + \tilde{t}_i) \right)^{-l}$$

$$\left[1 + |y| \right]^p \text{d}y \text{d}\tau.$$

Thus, by Lebesgue’s dominated convergence theorem, the facts

$$|\tilde{y}_i| \to \infty, \quad \tilde{t}_i \to -\infty \quad \text{and} \quad \|g^i\|_{C_\Phi((s_i, \tilde{t}_i) \times (0, +\infty))} \to 0,$$

when $i$ goes to infinity in (3.24), we can obtain (3.35). □
3.2.2 Proof of Assertion

We assume that (3.25) is not valid. Then there exists a sequence \( \{i_m \in \mathbb{N} \} \) with \( \lim_{m \to \infty} i_m = \infty \) and \( R > 0 \) such that there exists \( l \in \{0, 1, 2, 3\} \) satisfying
\[
\left\| A^{(i_m, t_m)} \right\|_{L^\infty} > 0.
\]
Without loss of the generality, we assume that
\[
\left\| \frac{\psi^{i_m}}{\Phi} \right\|_{L^\infty} > 0.
\]
For the other situations, the following arguments are similar.

Let
\[
(\hat{t}_m, r_m) \in A^{(s_m, t_m)} = \{(t, r) \in (s_m, t_m) \times \mathbb{R} : |r - \gamma_n(t)| < R + 1\}
\]
such that there exist \( \delta > 0 \) and
\[
\left| \frac{\psi^{i_m}(\hat{t}_m, r_m)}{\Phi(\hat{t}_m, r_m)} \right| > \delta > 0. \tag{3.40}
\]
Let us introduce the following change of variables
\[
v = t - \hat{t}_m, \quad y = r - y_m - \rho(t + \hat{t}_m) \quad \text{and} \quad y_m = r_m - \gamma_n(\hat{t}_m),
\]
and set
\[
\hat{\psi}^{i_m}(v, y) := \frac{\psi^{i_m}(v + \hat{t}_m, y + y_m + \rho(v + \hat{t}_m))}{\Phi(\hat{t}_m, y_m + \rho(\hat{t}_m))}, \tag{3.41}
\]
where we recall that \( \rho(t) = \gamma_n(t) + h(t) \) defined in (2.7). Hence, there exists \( \gamma_0 \) such that
\[
\lim_{m \to \infty} y_m = \gamma_0 \quad \text{and} \quad |\gamma_0| < R + 2.
\]
According to (3.20), we have that \( \hat{\psi}^{i_m} \) satisfies the following problem
\[
\hat{\psi}^{i_m}_v = -\hat{\psi}^{i_m}_{yyy} - \frac{2(n - 1)}{y + y_m + \rho(v + \hat{t}_m)} \hat{\psi}^{i_m}_{yy} + \left[ \frac{2(n - 1)}{y + y_m + \rho(v + \hat{t}_m)} \right] \hat{\psi}^{i_m}_{y}
\]
\[
- \left( W''(z(v, y)) \right)^2 \hat{\psi}^{i_m} + \left[ \frac{2(n - 1)W''(z(v, y))}{y + y_m + \rho(v + \hat{t}_m)} \right] \hat{\psi}^{i_m}_{y} - \left( \frac{(3 - n)(n - 1)}{y + y_m + \rho(v + \hat{t}_m)} \right)^2 \hat{\psi}^{i_m}_{yy}
\]
\[
+ \frac{2}{y + y_m + \rho(v + \hat{t}_m)} W''(z(v, y)) \hat{\psi}^{i_m}_y + \left[ \frac{n - 1}{y + y_m + \rho(v + \hat{t}_m)} \right] \hat{\psi}^{i_m}_y - \frac{2W''(z(v, y))}{y + y_m + \rho(v + \hat{t}_m)} \hat{\psi}^{i_m}_y
\]
\[
+ \frac{g^{i_m}(v + \hat{t}_m, y + y_m + \rho(v + \hat{t}_m))}{\Phi(\hat{t}_m, y_m + \rho(\hat{t}_m))} \right) \partial_y \rho(v + \hat{t}_m) \hat{\psi}^{i_m}_{y} \quad \text{in} \, \Gamma^{(i_m, t_m)} . \tag{3.42}
\]
with the conditions
\[
\left| \hat{\psi}^{i_m}(0, 0) \right| \geq \delta, \quad \hat{\psi}^{i_m}(s_m - \hat{t}_m, y) = 0, \quad y \in (-y_m - \rho(v + \hat{t}_m), +\infty), \tag{3.43}
\]
where
\[
\hat{\zeta}(v, y) = z(v + \hat{t}_{im}, y + y_{im} + \rho(v + \hat{t}_{im})),
\]
and
\[
\Gamma^{s_{im} - \hat{t}_{im}} := \left\{ (v, y) \in (s_{im} - \hat{t}_{im}, 0] \times (-y_{im} - \rho(v + \hat{t}_{im}), +\infty) \right\}.
\]
As the same as previous part, there holds
\[
\lim_{m \to \infty} (s_{im} - \hat{t}_{im}) = \hat{\zeta},
\]
where \(\hat{\zeta}\) equal to \(-\infty\) or a negative number.

A contradiction will be derived by arguments in the following five steps.

**Step 1** We first consider the case: \(\lim_{m \to \infty} (s_{im} - \hat{t}_{im}) = -\infty\). We have that \(\hat{\psi}_{im} \to \hat{\psi}\) locally uniformly, \(|\hat{\psi}(0, 0)| > \delta > 0\), which \(\hat{\psi}\) satisfies the following equation
\[
\hat{\psi}_{v} = -\hat{\psi}_{yyyy} + 2W''(\omega(y + y_0))\hat{\psi}_{yy} + 2W'''(\omega(y + y_0))\omega'(y + y_0)\hat{\psi}_y - \left[ W''(\omega(y + y_0)) \right]^2 \hat{\psi} + \left[ W'''(\omega(y + y_0))(\omega'(y + y_0))^2 + W''(\omega(y + y_0))\omega''(y + y_0) \right] \hat{\psi} \text{ in } (-\infty, 0] \times \mathbb{R}.
\]
According to (2.26), (3.23) and (3.41), by the definition of \(\rho\) in (2.7), similar as (3.33), we have
\[
|\hat{\psi}_{im}(v, y)| \leq \frac{\psi_{im}(v + \hat{t}_{im}, y + y_{im} + \rho(v + \hat{t}_{im}))}{\Phi(\hat{t}_{im}, y_{im} + \rho(\hat{t}_{im}))} \leq \frac{\Phi(v + \hat{t}_{im}, y + y_{im} + \rho(v + \hat{t}_{im}))}{\Phi(\hat{t}_{im}, y_{im} + \rho(\hat{t}_{im}))} \leq C(\alpha, n, R, \|h\|_{L^\infty})(1 + |y|)^p, \quad \forall (v, y) \in B_{\hat{t}_{im}, j}, \tag{3.45}
\]
for all \(j = 0, 1\), where \(\rho(t) = \gamma_n(t) + h(t)\) in (2.7) and the sets \(B_{\hat{t}_{im}, j}\) are defined as the following
\[
B_{\hat{t}_{im}, 0} := \left\{ (v, y) \in (s_{im} - \hat{t}_{im}, 0] \times \mathbb{R} : 0 < y + y_{im} + \rho(v + \hat{t}_{im}) \leq \delta_0 \right\},
\]
\[
B_{\hat{t}_{im}, 1} := \left\{ (v, y) \in (s_{im} - \hat{t}_{im}, 0] \times \mathbb{R} : \delta_0 \leq y + y_{im} + \rho(v + \hat{t}_{im}) < +\infty \right\}.
\]
We notice that
\[
\Gamma^{s_{im} - \hat{t}_{im}} = \bigcup_{l=0}^1 B_{\hat{t}_{im}, j} = (s_{im} - \hat{t}_{im}, 0] \times (-y_{im} - \rho(v + \hat{t}_{im}), +\infty). \tag{3.46}
\]
Similarly, we have that
\[
\sum_{l=1}^3 \left| \partial_y^l \hat{\psi}_{im}(v, y) \right| \leq C(1 + |y|)^p, \tag{3.47}
\]
for all \((v, y) \in \Gamma^{s_{im} - \hat{t}_{im}}\), where \(p \in (n, n + 1)\) and \(C\) depends on \(\alpha, R, n\) and \(\|h\|_{L^\infty}\). Since \(\gamma_n(v + \hat{t}_{im}) \to +\infty\) as \(i\) goes to infinity, by (3.47), (3.42)–(3.43), and \(y_{im} \to y_0\), we can get the limiting equation in (3.44).
If $\xi \in (-\infty, 0)$, we have $\hat{\psi}(\xi, y) = 0$. According to Proposition 3.4, we have that equation (3.44) has a unique solution $\hat{\psi}(v, y) \equiv 0$, which contradicts with $|\hat{\psi}(0, 0)| > 0$. Hence, it only happens that $\xi = -\infty$.

**Step 2** We will prove the following orthogonality condition for $\hat{\psi}$

$$
\int_{\mathbb{R}} \hat{\psi}(v, y) \omega'(y + y_0) dy = 0, \quad \text{for all } v \in (-\infty, 0].
$$

(3.48)

In fact, according to (3.21) and (3.41), we get that

$$
0 = \left[ y_{im} + \rho(v + \hat{i}_{im}) \right]^{1-n} \int_{0}^{\infty} \psi^{im}(v + \hat{i}_{im}, r) \omega'(r - \rho(v + \hat{i}_{im})) r^{n-1} dr
$$

$$
= \int_{-y_{im} - \rho(v + \hat{i}_{im})}^{\infty} \hat{\psi}^{im}(v, y) \omega'(y + y_{im}) [ y + y_{im} + \rho(v + \hat{i}_{im}) ]^{n-1} dy,
$$

where $\rho(t) = \gamma_1(t) + h(t)$ with $\|h(t)\|_{L_\infty} < 1$ and the function $\omega$ is given by (1.11). And using (3.45), (3.46) and Lemma 2.1, we also get that

$$
|\hat{\psi}^{im}(v, y) \omega'(y + y_{im})| \leq C(R, n, \alpha, \|h\|_{L_\infty})(1 + |y|)^\alpha \exp \{ -\alpha |y + y_{im}| \}.
$$

Since $y_{im} \to y_0$ and $\hat{i}_{im} \to -\infty$, as $m$ goes to infinity, using Lebesgue’s dominated convergence theorem, we can obtain that (3.48) holds.

**Step 3** In this step, we will prove the following decay of $\hat{\psi}(v, y)$: there exists a constant $C$ of the form

$$
C = C(\alpha, n, R, \|h\|_{L_\infty}) > 0
$$

such that

$$
|\hat{\psi}(v, y)| \leq \frac{C}{(1 + |y|)^\tau}, \quad \forall (v, y) \in (-\infty, 0] \times \mathbb{R}.
$$

(3.49)

In fact, for any $(v, y) \in B_{i_{im}, j}$, by the definition of $\rho(t)$ in (2.7), in view of the proof of (3.45), we have

$$
\left| g^{im}(v + \hat{i}_{im}, y + y_{im} + \rho(v + \hat{i}_{im})) \right| \leq C \|g^{im}\|_{C_{\Phi}(\Gamma_{s_{im}, t_{im}} \times (0, +\infty))} (1 + |y|)^\alpha p,
$$

(3.50)

for all $(v, y) \in \Gamma_{s_{im}, t_{im}}$ given by (3.46), where $C$ depends on $\alpha, R, n$ and $\|h\|_{L_\infty}$.

By formula (3.37), the solution of equation (3.42) has the form

$$
\hat{\psi}^{im}(v, y) = \int_{0}^{v - s_{im} + \hat{i}_{im}} \int_{\mathbb{R}} e^{-\alpha^4 \tau} Q(\tau, x) \hat{f}_{i_{im}}(\hat{\psi}^{im}(v - \tau, y - x), v - \tau, y - x) dx d\tau,
$$

(3.51)

for all $(v, y) \in (s_{im} - \hat{i}_{im}, 0) \times \mathbb{R}$, where $\alpha = \sqrt{W''(1)}$ and the function $\hat{f}_{i_{im}}$ is given by

$$
\hat{f}_{i_{im}} = 2 \left[ W''(\hat{\xi}(v, y)) - W''(1) \right] \hat{\psi}^{im}_y + 2 W'''(\hat{\xi}(v, y)) \partial_{\hat{\xi}} \hat{\xi}(v, y) \hat{\psi}^{im}_y
$$

$$
- \left[ W''(\hat{\xi}(v, y))^2 - [W''(1)]^2 \right] \hat{\psi}^{im}_y
$$

$$
+ \left[ \partial_{yy} W''(\hat{\xi}(v, y)) + W'''(\hat{\xi}(v, y)) \partial_{\hat{\xi}} \hat{\xi}(v, y) - W'(\hat{\xi}(v, y)) \right] \hat{\psi}^{im}_y
$$

$$
\hat{\psi}^{im}(v, y) = 0, \quad \text{for all } v \in (-\infty, 0].
$$

(3.48)
\[
\begin{align*}
-\frac{2(n-1)}{y + y_0 + \rho(v + \hat{t}_i)} \hat{\psi}_{yy}^{im} - \frac{(n-3)(n-1)}{(y + y_0 + \rho(v + \hat{t}_i))^2} \hat{\psi}_{yy}^{im} \\
+ \left[ \frac{2(n-1)W''(\hat{z}(v, y))}{y + y_0 + \rho(v + \hat{t}_i)} - \frac{(3-n)(n-1)}{(y + y_0 + \rho(v + \hat{t}_i))^2} \right] \hat{\psi}_{yy}^{im} \\
+ 2 \frac{n-1}{y + y_0 + \rho(v + \hat{t}_i)} W''(\hat{z}(v, y)) \hat{z}_y \hat{\psi}_{yy}^{im} + \frac{g^{im}(v + \hat{t}_i, y + y_0 + \rho(v + \hat{t}_i))}{\Phi(t, y_0 + \rho(\hat{t}_i))} \\
- \partial_v \rho(v + \hat{t}_i) \hat{\psi}_{yy}^{im}
\end{align*}
\]

and \( \hat{z}(v, y) = z(v + \hat{t}_i, y + y_0 + \rho(v + \hat{t}_i)) \).

According to the Lemma 2.1, (3.19), (3.50), (3.45) and (3.47), we have that

\[
|f^{im}_m| \leq C(1 + |y|)^p \left\{ \exp \left\{ -\alpha \left| y + y_0 \right| \right\} + \| g^{im} \|_{C_\Phi((s_0, t_m) \times (0, +\infty))} \right\} \\
+ \frac{1}{\left( (v + \hat{t}_m) \right)^{3/4}} + \sum_{l=1}^{3} \left( y + y_0 + \rho(v + \hat{t}_i) \right)^{-l} \left\{ H \left| y - \frac{t_0}{\rho(v + \hat{t}_i)} - y_0 \right|, \right\}
\]

for all \((v, y) \in (y_0 - \hat{t}_m, 0) \times (-\rho(v + \hat{t}_i) - y_0, +\infty)\), where \( C \) depends on \( \alpha, R, n \) and \( \| h \|_{L^\infty} \), and \( \Phi_A \) is the characteristic function of the set \( A \).

Thus by (3.51), similar arguments as (3.39) and \( |y_0| \leq R + 2 \), let \( m \) goes to infinity, and then we can get the desired result since \( \| g^{im} \|_{C_\Phi((s_0, \hat{t}_m) \times (0, +\infty))} \rightarrow 0 \) as index \( m \) goes to infinity in (3.24) and the following fact that

\[
\int_0^{v + \hat{t}_m} \int_\mathbb{R} e^{-\alpha^4 \tau} |Q(\tau, y - x)| (1 + |x|)^p \exp \left\{ -\alpha \left| x + y_0 \right| \right\} dx d\tau \\
\leq C \int_0^{v + \hat{t}_m} \int_\mathbb{R} \exp \left\{ -\alpha^4 \tau - \frac{1}{4} |x| \right\} \exp \left\{ -\frac{\alpha}{2} \left| y - \tau^{1/4} x \right| \right\} dx d\tau \\
\leq C \int_0^{v + \hat{t}_m} \int_\mathbb{R} \exp \left\{ -\alpha^4 \tau - \frac{1}{4} |x| \right\} \frac{1}{1 + \left| y - \tau^{1/4} x \right|} \, dx d\tau \\
\leq C \int_0^{v + \hat{t}_m} \int_\mathbb{R} \exp \left\{ -\alpha^4 \tau - \frac{1}{4} |x| \right\} \left( \frac{1 + \left| \tau^{1/4} x \right|}{1 + |y|} \right)^2 \, dx d\tau \\
\leq \frac{C}{(1 + |y|)^2},
\]

where \( \hat{t}_m := \hat{t}_m - s_0 \), and \( C > 0 \) only depends on \( \alpha \).

**Step 4** To proceed further, we need the following well-known result. However, we can’t find the proof of this result in references. Here we give a proof in details.

**Lemma 3.6** Considering the Hilbert space

\[
H = \left\{ \phi(y) \in H^2(\mathbb{R}) : \int_\mathbb{R} \phi(y) \omega'(y) dy = 0 \right\},
\]

then the following inequality is valid

\[
\int_\mathbb{R} |\phi''(y) - W''(\omega)\phi(y)|^2 dy \geq c \int_\mathbb{R} |\phi(y)|^2 dy, \quad \forall \phi \in H, \quad (3.52)
\]

where \( c > 0 \) is a uniform constant.
Proof For any $\phi \in H$, since $\omega' > 0$, we can set $\phi(y) = \zeta_1(y)\omega'(y)$, then
\[
\int_\mathbb{R} \left| \phi''(y) - W''(\omega)\phi(y) \right|^2 dy \\
= \int_\mathbb{R} \left| \omega'''(y)\zeta_1(y) + 2\zeta_1'(y)\omega''(y) + \zeta_1''(y)\omega'(y) - W''(\omega)\zeta_1(y)\omega'(y) \right|^2 dy \\
= \int_\mathbb{R} \left| 2\omega''(y)\zeta_1'(y) + \omega'(y)\zeta_1''(y) \right|^2 dy \geq 0,
\]
where we used $\omega''' = W''(\omega)\omega'$ and integration by parts in the last equality.

Hence we have
\[
\int_\mathbb{R} \left| \phi''(y) - W''(\omega)\phi(y) \right|^2 dy = 0 \text{ if and only if } 2\omega''(y)\zeta_1'(y) + \omega'(y)\zeta_1''(y) = 0.
\]
All solutions of the last equation have the following form
\[
\zeta_1(x) = c_1 \int_0^x \frac{1}{(\omega'(y))^2} dy + c_2, \quad c_1, c_2 \in \mathbb{R}.
\]
Since
\[
\phi(x) = \zeta_1(x)\omega'(x) \in H^2(\mathbb{R}) \quad \text{and} \quad \int_\mathbb{R} \phi(y)\omega'(y)dy = 0,
\]
we have that $c_1 = 0$ and $c_2 = 0$.

Now we assume that there exists a sequence $\{\phi_m\} \in H$ such that
\[
\int_\mathbb{R} |\phi_m|^2 dy = 1 \quad \text{and} \quad \int_\mathbb{R} |\phi'' - W''(\omega)\phi_m|^2 dy \leq \frac{1}{m}. \quad (3.53)
\]
Thus, $\phi_m \rightharpoonup \phi$ in $H(\mathbb{R})$ and $\phi_m \rightarrow \phi$ in $L^2(K)$ for any compact subset $K \subset \mathbb{R}$, which implies that
\[
0 = \int_\mathbb{R} \phi_m(y)\omega'(y)dy \rightarrow \int_\mathbb{R} \phi\omega' dy = 0,
\]
for some $\phi \in H(\mathbb{R})$. By Fatou’s Lemma, we have
\[
\int_\mathbb{R} |\phi'' - W''(\omega)\phi|^2 dy = 0.
\]
Hence $\phi = 0$.

On the other hand, we first have
\[
|W''(1)|^2 \int_\mathbb{R} |\phi_m|^2 dy \leq \int_\mathbb{R} |\phi'' - W''(1)\phi_m|^2 dy \\
\leq 2 \left\{ \int_\mathbb{R} |\phi'' - W''(\omega)\phi_m|^2 dy + \int_\mathbb{R} \left| [W''(1) - W''(\omega)]\phi_m \right|^2 dy \right\}.
\]
Thus, by (3.53), we get that
\[
|W''(1)|^2 \leq \int_\mathbb{R} \left| [W''(1) - W''(\omega)]\phi \right|^2 dy.
\]
According to the assumptions on $W$ in (1.2), $W''(1) \neq 0$. The above last inequality implies that $\phi \neq 0$, which is a contradiction. \qed
Step 5 Multiplying (3.44) by \( \hat{\psi}(y) \) and integrating with respect to \( y \), and using (3.52), we have
\[
0 = \frac{1}{2} \int_{\mathbb{R}} (\hat{\psi}^2)_y dy + \int_{\mathbb{R}} |\hat{\psi}_{yy} - W''(\omega(y + y_0))\hat{\psi}(v, y)|^2 dy
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}} (\hat{\psi}^2)_y dy + c \int_{\mathbb{R}} |\hat{\psi}(v, y)|^2 dy,
\]
where the constant \( c > 0 \). Then, according to Gronwall’s inequality, we get that, for the term
\[
a(v) := \int_{\mathbb{R}} |\hat{\psi}(v, y)|^2 dy,
\]
there holds
\[
a(v) \geq a(0)e^{-2cv}, \quad \forall v < 0,
\]
This is a contradiction since (3.49).
The proof of (3.25) is completed.

3.3 The projected linear problem involving the operator \( L \)

We consider the following projection problem:
\[
\begin{align*}
\psi_t &= L[\psi] + g(t, r) - c(t)\partial_r \hat{\omega}(t, r) \quad \text{in} \ (s, -T] \times (0, \infty), \\
\psi(s, r) &= 0, \quad \forall r \in (0, +\infty), \\
\partial_r^j \psi(t, 0) &= 0, \quad \forall t \in (s, -T], \ j = 1, 3,
\end{align*}
\tag{3.54}
\]
where the linear operator \( L[\psi] = F'(z(t, r))[\psi] \) is defined by (2.20), the function \( g \in C_\Phi((s, -T) \times (0, +\infty)) \) and \( c(t) \) satisfies the following relation
\[
c(t) \int_0^\infty \partial_r \hat{\omega}(t, r)\omega'([r - \rho(t)])r^{n-1} dr
\]
\[
= \int_0^\infty \left[ \omega''([r - \rho(t)]) + \frac{n-1}{r} \omega''([r - \rho(t)]) - W''(z(t, r))\omega'([r - \rho(t)]) \right] dr
\]
\[
\times \left[ -\psi_{rr} - \frac{n-1}{r} \psi_r + W''(z(t, r))\psi \right] r^{n-1} dr
\]
\[
+ \int_0^\infty \left[ \partial_{rr} z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] \psi \omega'([r - \rho(t)])r^{n-1} dr
\]
\[
+ \int_0^\infty g(t, r)\omega'([r - \rho(t)])r^{n-1} dr + \int_0^\infty g(t, r)\omega'([r - \rho(t)])r^{n-1} dr, \tag{3.55}
\]
for all \( t \in (s, -T] \), where the functions \( \hat{\omega}(t, r) \) and \( z(t, r) \) are defined by (2.6) and (2.14) respectively.

If \( \psi \) is a solution of (3.54) and \( c(t) \) satisfies (3.55), integration by parts will imply that \( \psi \) satisfies the following orthogonality condition
\[
\int_0^\infty \psi(t, r)\omega'([r - \rho(t)])r^{n-1} dr = 0, \quad \forall t \in (s, -T). \tag{3.56}
\]
For (3.55), we have the following result:
Lemma 3.7 Let $T > 0$ big enough and $\psi, \psi_r, \psi_{rr}$ and $g \in C_{\Phi}(s, -T) \times (0, +\infty)$. Then there exists $c(t)$ such that (3.55) holds. Furthermore the following estimates are valid

$$|c(t)| \leq \frac{C_0}{\left[\log |t|\right]^{p-1}} |t|^{1/2} \left\{ \frac{1}{|t|^{1/4}} \left[ \sum_{l=0}^{2} \left\| \partial_t^l \psi \right\|_{C_{\Phi}((s, -T) \times (0, \infty))} \right] + \|g\|_{C_{\Phi}((s, -T) \times (0, \infty))} \right\},$$

(3.57)

and

$$\left| \frac{c(t)\partial_t \hat{\omega}(t, r)}{\Phi(t, r)} \right| \leq \frac{C_0}{|t|^{1/4}} \sum_{l=0}^{2} \left\| \partial_t^l \psi \right\|_{C_{\Phi}((s, -T) \times (0, \infty))} + C_0 \|g\|_{C_{\Phi}((s, -T) \times (0, \infty))},$$

(3.58)

for any $t \in [s, -T]$, where the function $\hat{\omega}(t, r)$ is given by (2.6), $p \in (n, n + 1]$ and $C_0$ is a positive constant which does not depend on $s, t, T, \psi$ and $g$.

**Proof** We first consider the left hand side of system (3.55). By Lemma 2.1 and the definition of $\hat{\omega}(t, r)$ in (2.6), we have that

$$\int_0^\infty \partial_t \hat{\omega}(t, r) \omega'(r - \rho(t)) r^{n-1} dr = \int_0^\infty \omega'(r) \omega'(r + \rho(t)) r^{n-1} dr + O \left( [\gamma_n(t)]^{n-2} \right),$$

where the functions $\rho(t)$ and $\gamma_n(t)$ are given by (2.7) and (2.8) respectively.

For the first and second terms in the right hand side of (3.55), we can obtain that

$$I(t) := \int_0^\infty \left[ -\omega''(r - \rho_i(t)) - \frac{n-1}{r} \omega''(r - \rho_i(t)) + W''(z(t, r)) \omega'(r - \rho(t)) \right] \times \left( \psi_{rr} + \frac{n-1}{r} \psi_r - W''(z(t, r)) \psi \right) r^{n-1} dr$$

$$+ \int_0^\infty \left[ \partial_{rr} z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] \psi \omega'(r - \rho(t)) r^{n-1} dr$$

$$\leq C \sum_{l=0}^{2} \left\| \partial_t^l \psi \right\|_{C_{\Phi}((s, -T) \times (0, \infty))} \int_0^\infty \left[ \left| \omega''(r - \rho(t)) - W''(z(t, r)) \omega'(r - \rho(t)) \right| \right] \Phi(t, r) r^{n-1} dr$$

$$+ \int_0^\infty \Phi(t, r) \sum_{l=1}^{3} \left| \omega^{(l)}(r - \rho(t)) \right| r^{n-2} dr,$$

deue to the facts that the functions $\psi, \psi_r, \psi_{rr}$ belong to $C_{\Phi}(s, -T) \times (0, +\infty)$. By similar arguments in the proof of Lemma 2.3 and the definition of $\Phi(t, r)$ in (2.26), we have that

$$I(t) \leq C \sum_{l=0}^{2} \left\| \partial_t^l \psi \right\|_{C_{\Phi}((s, -T) \times (0, \infty))} \left\{ \int_0^\infty \Phi(t, r) \sum_{l=1}^{3} \left| \omega^{(l)}(r - \rho(t)) \right| r^{n-2} dr \right.$$
Lemma 3.8 Let \( g \in C\Phi((s, -T) \times (0, \infty)) \), then there exists a uniform constant \( T_0 > 0 \) and a unique solution \( \psi^s \) of problem (3.54). Moreover, \( \psi^s \) satisfies the orthogonality condition in (3.56) with \( t \in (s, -T_0) \), and

\[
\sum_{l=0}^{3} \left\| \partial^l_t \psi^s \right\|_{C\Phi((s, t) \times (0, \infty))} \leq C \left\| g \right\|_{C\Phi((s, t) \times (0, \infty))},
\]

(3.59)

where \( t \in (s, -T_0) \) and \( C \) is a uniform constant which does not depend on \( t, s \).

**Proof** We will use a fixed-point argument to prove this lemma. Let

\[
X^s := \left\{ \psi : \sum_{l=0}^{3} \left\| \partial^l_t \psi \right\|_{C\Phi((s, s+1) \times (0, \infty))} < +\infty \right\},
\]

and \( T^s(g) \) be the solution of (3.20) with \( -T = s+1 \). We consider the operator \( A^s : X^s \to X^s \) defined by

\[
A^s(\psi) := T^s(g - C(\psi)), \quad C(\psi) = c(t) \partial_t \hat{\omega}(t, r),
\]

where \( \hat{\omega}(t, r) \) is given by (2.6) and the function \( c(t) \) satisfies (3.55). Hence \( c(t) \) depends on \( \psi \), and \( \psi \) satisfies the orthogonality condition in (3.56). By Lemma 3.5, we have that

\[
\sum_{l=0}^{3} \left\| \partial^l_t A^s(\psi) \right\|_{C\Phi((s, s+1) \times (0, \infty))} \leq C_1 \left\| g - C(\psi) \right\|_{C\Phi((s, s+1) \times (0, \infty))},
\]

(3.60)
for a uniform constant $C_1 > 0$.

Set
\[ c := (C_0 + C_1)\|g\|_{C_0((s,s+1)\times(0,\infty))} \]
and
\[ X^s_c := \{ \psi : \|\psi\|_{C_0((s,s+1)\times(0,\infty))} < 2c \}, \]
where $C_0$ and $C_1$ are given by Lemma 3.7 and (3.60) respectively. It is easy to derive the following two facts:

(1) By (3.60) and Lemma 3.7, we have
\[
\sum_{l=0}^{3} \|\partial_r^l A^s(\psi)\|_{C_0((s,s+1)\times(0,\infty))} \leq C_0 \left( \|C(\psi)\|_{C_0((s,s+1)\times(0,\infty))} + \|h\|_{C_0((s,s+1)\times(0,\infty))} \right) 
\leq \frac{CC_0}{\sqrt{|s+1|}} \left( \sum_{l=0}^{2} \|\partial_r^l \psi\|_{C_0((s,s+1)\times(0,\infty))} \right) + c,
\]
where $C$ is given by Lemma 3.7.

(2) For any $\psi_1, \psi_2 \in X^s_c$, by (3.60) and Lemma 3.7 again, we deduce that
\[
\sum_{l=0}^{3} \|\partial_r^l [A^s(\psi_1) - A^s(\psi_2)]\|_{C_0((s,s+1)\times(0,\infty))} \leq C_0 \|C(\psi_1) - C(\psi_2)\|_{C_0((s,s+1)\times(0,\infty))} 
\leq C_0 \|C(\psi_1) - C(\psi_2)\|_{C_0((s,s+1)\times(0,\infty))} 
\leq \frac{CC_0}{\sqrt{|s+1|}} \|\psi_1 - \psi_2\|_{C_0((s,s+1)\times(0,\infty))},
\]
where $C_0$ is given by Lemma 3.7.

Hence, taking $s$ large enough, we have that the operator $A^s$ is a contraction map from $X^s_c$ to itself. According to the Banach fixed point theorem, we know that there exists a unique $\psi^s \in X^s_c$ such that $A^s(\psi^s) = \psi^s$, which is a solution to problem (3.54) with $-T = s + 1$.

Next we will extend the solution $\psi^s(t, r)$ in $(s, s + 1) \times (0, +\infty)$ to $(s, -T_0) \times (0, +\infty)$, which will satisfy the orthogonality condition in (3.56) and the a priori estimate in Lemma 3.5. We choose $T_0$ large enough such that $\frac{CC_0}{\sqrt{|s|}} < 1$, where $C$ is given by Lemma 3.7 and $C_0$ is given by (3.60). Thus the above fixed-point argument can be repeated when $s + 1 \leq -T_0$. Hence passing finite steps of fixed-point arguments, the solution $\psi^s(t, r)$ can be extended to $(s, -T_0]$. Moreover the solution $\psi^s$ satisfies (3.59) and the orthogonality condition.

### 3.4 The solvability of a linear projected problem

This section is devoted to building the solvability of the following linear parabolic projected problem:

\[
\begin{align*}
\psi_t &= L[\psi] + g(t, r) - c(t)\partial_r \tilde{\omega}(t, r) \quad \text{in } (-\infty, -T) \times (0, \infty), \\
\int_{\mathbb{R}} \psi(t, r) (r - \rho(t)) r^{n-1} dr &= 0, \quad \text{for all } t \in (-\infty, -T],
\end{align*}
\]

for a bounded function $g$, and $T > 0$ fixed sufficiently large. In the above, the linear operator $L$ is given by (2.20), the functions $\rho(t)$ and $\tilde{\omega}(t, r)$ are defined by (2.7) and (2.6) respectively.
The function $c(t)$ solves the following relation:

$$
c(t) \int_0^\infty \partial_r \tilde{\omega}(t, r) \omega'(r - \rho(t)) r^{n-1} dr
= \int_0^\infty \left[ \omega''(r - \rho(t)) + \frac{n-1}{r} \omega'(r - \rho(t)) - W''(z(t, r)) \omega'(r - \rho(t)) \right] \times \left[ -\psi_{rr} - \frac{n-1}{r} \psi_r + W'(z(t, r)) \right] r^{n-1} dr
+ \int_0^\infty \left[ \partial_r z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] \psi'(r - \rho(t)) r^{n-1} dr
+ \int_0^\infty \psi(t, r) \partial_r [\omega'(r - \rho(t))] r^{n-1} dr + \int_0^\infty g(t, r) \omega'(r - \rho(t)) r^{n-1} dr,
$$

(3.62)

for all $t < -T$, where $z(t, r)$ is given by (2.14). Indeed, this can be solved uniquely since if $T$ is taken sufficiently large, the coefficient $\int_0^\infty \partial_r \tilde{\omega}(t, r) \omega'(r - \rho(t)) r^{n-1} dr$ is strictly positive.

The main result in this section is as follows:

**Proposition 3.9** There exist positive constants $T_1$ and $C$ such that for any $t \leq -T_1$ and $g \in C_\varphi((-\infty, t) \times (0, \infty))$, problem (3.61)–(3.62) has a solution $\psi = A(g)$, which defines a linear operator of $g$ and satisfies the following estimate

$$
\sum_{l=0}^3 \| \partial_r^l \psi \|_{C_\varphi((-\infty, t) \times (0, \infty))} \leq C \| g \|_{C_\varphi((-\infty, t) \times (0, \infty))}, \quad \text{for all } t \leq -T_1,
$$

(3.63)

where we denote the $l$-th order derivative of $\psi(t, r)$ with respect of $r$ by $\partial_r^l \psi$.

**Proof** We choose a sequence $s_j \to -\infty$. Let $\psi^{s_j}$ be the solution to problem (3.54) with $s = s_j$, according to Lemma 3.8. By (3.59), we can find that the sequence $\{\psi^{s_j}\}$ converges to $\psi$ (up to subsequence) locally uniformly in $(-\infty, -T_1) \times (0, \infty)$. Using (3.22) and Proposition 3.3, we have that $\psi$ is a solution of (3.61) and satisfies (3.63). The proof is completed. \qed

### 4 Solving the nonlinear projected problem

In this section, we mainly solve the nonlinear problem (2.16)–(2.17) by using the fixed-point arguments. According to the result in Proposition 3.9, $\phi$ is a solution of (2.16)–(2.17) if only if $\phi \in C_\Phi((-\infty, -T_1) \times (0, \infty))$ is a fixed point of the following operator

$$
T(\phi) := A(E(t, r) + N(\phi) - c(t) \partial_r \tilde{\omega}(t, r)),
$$

(4.1)

where $T_1$ and $A$ are given by Proposition 3.9.

Let $T \geq T_1$. We define a set

$$
\Lambda_T := \left\{ h \in C^1(-\infty, -T) : \| h \|_{\Lambda_T} < 1 \right\},
$$

(4.2)

with the norm

$$
\| h \|_{\Lambda_T} := \sup_{t \leq -T} | h(t) | + \sup_{t \leq -T} \left( \frac{| t |}{\log | t |} | h'(t) | \right),
$$

(4.3)
and also a close domain

\[ X_T := \left\{ \phi : \phi \in C_{\Phi}((\infty, -T) \times (0, \infty)) \text{ and } \sum_{l=0}^{2} \| \partial_l^l \phi \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} \leq \frac{2\hat{C}}{\log T} \right\} \tag{4.4} \]

where the space \( C_{\Phi}((-\infty, -T) \times (0, \infty)) \) is defined by (3.19) and \( \hat{C} \) is a fixed constant.

The main result is given by the following proposition

**Proposition 4.1** There exists \( T_2 \geq T_1 \) such that for any given function \( h(t) \in \Lambda_{T_2} \), there is a solution \( \phi(t, r, h) \) to equation \( \phi = T(\phi) \) with respect to \( \rho(t) = \gamma_n(t) + h(t) \). The solution \( \phi(t, r, h) \) also satisfies problem (2.16)–(2.17). Furthermore, the following estimate holds

\[ \sum_{l=0}^{2} \| \partial_l^l \phi(t, r, h_1) - \partial_l^l \phi(t, r, h_2) \|_{C_{\Phi}((-\infty, -T_1) \times (0, \infty))} \leq C \frac{1}{\log T_2} \| h_1 - h_2 \|_{\Lambda_{T_2}} \tag{4.5} \]

where \( T_1 \) and \( T \) are given by Proposition 3.9 and (4.1) respectively, \( C \) is a positive constant which does not depend on \( h_1 \), \( h_2 \) and \( T_2 \).

To prove the above proposition, we first prepare two lemmas. We note that the error term \( E(t, r) \) and the nonlinear term \( N(\phi) \) in (2.18)–(2.19) are all dependent of \( h \), due to the setting \( \rho(t) = \gamma_n(t) + h(t) \) in (2.7). So we denote \( E(t, r) \) and \( N(\phi) \) by \( E(t, r, h) \) and \( N(\phi, h) \) respectively.

**Lemma 4.2** Let \( h_1, h_2 \in \Lambda_T \) and \( \phi_1, \phi_2 \in X_T \), then there exists a constant \( C \) depending on \( \hat{C} \) such that

\[ \| N(\phi_1, h_1) - N(\phi_2, h_2) \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} \leq \frac{C}{\log T} \| h_1 - h_2 \|_{\Lambda_T} \tag{4.6} \]

and

\[ \| E(t, r, h_1) - E(t, r, h_2) \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} \leq \frac{C}{\log T} \| h_1 - h_2 \|_{\Lambda_T} \tag{4.7} \]

**Proof** By the definition of \( N(\phi, h) \) in (2.19) and the smoothness of \( W \), we have that

\[ |N(\phi_1, h) - N(\phi_2, h)| \]

\[ = \left| \int_{0}^{1} \left\{ F'(z(t, r) + y\phi_1 + (1 - y)\phi_2)[\phi_1 - \phi_2] - F'(z(t, r))[\phi_1 - \phi_2] \right\} dy \right| \]

\[ = \left| \int_{0}^{1} \left\{ F''(z(t, r) + \theta[y\phi_1 + (1 - y)\phi_2])[y\phi_1 + (1 - y)\phi_2] \right\} dy \right| \]

\[ \leq C \left( \sum_{l=0}^{2} \| \partial_l^l (\phi_1 - \phi_2) \|_{C_{\Phi}((-\infty, T_1) \times (0, \infty))} \right) \left( \sum_{l=0}^{2} \sum_{j=1}^{2} \| \partial_l^j (\phi_1) \|_{C_{\Phi}((-\infty, T_1) \times (0, \infty))} \right) \Phi(t, r), \tag{4.8} \]

and

\[ |N(\phi, h_1) - N(\phi, h_2)| \]

\[ \leq \frac{C}{\log T} \| h_1 - h_2 \|_{\Lambda_T}. \]
\[\int_0^1 \int_0^1 F''(yz_1(t, r) + (1 - y)z_2(t, r) + s)[z_1(t, r) - z_2(t, r), \phi] dy ds \leq C\|h_1 - h_2\|_{\Lambda_T} \|\Phi\|_{C\Phi((-\infty, T_1) \times (0, \infty))} \Phi(t, r), \]

(4.9)

where \(z_i(t, r)\) is given by (2.14) with \(\rho(t) = \gamma_t(t) + h_i(t)\), for \(i = 1, 2\). In the above, \(C\) is a positive constant independent of \(h, h_1, h_2\) and \(\phi, \phi_1, \phi_2\), and the linear operator \(F''(u)[v_1, v_2]\) is defined by (2.23). Hence combining (4.8) and (4.9), we can obtain that (4.6) holds.

On the other hand, by (2.24) and (2.25), similar arguments in Lemma 2.3 will imply that

\[|E_1(t, r, h_1) - E_1(t, r, h_2)| \leq C \left\{ \left| \rho_1'(t)\omega'(r - \rho_1(t)) - \rho_2'(t)\omega'(r - \rho_2(t)) \right| + \frac{(n - 1)^2(n - 3)}{r^3} \left| \omega'(r - \rho_1(t)) - \omega'(r - \rho_2(t)) \right| \right\} \left| 1 \right|_{r \geq \delta_0}
\]

\[+ \frac{C}{|\gamma_n(t)|^2} \left\{ |h_1(t) - h_2(t)| + |h_1'(t) - h_2'(t)| \right\} \left| 1 \right|_{2 < r < \delta_0}
\]

\[\leq C\|h_1 - h_2\|_{\Lambda_T} \Phi(t, r) \log T,
\]

and

\[|E_2(t, r, h_1) - E_2(t, r, h_2)|
\]

\[\leq C \left\{ \left| \rho_1'(t)\omega'(r - \rho_1(t)) - \rho_2'(t)\omega'(r - \rho_2(t)) \right| + \frac{1}{r^3} \sum_{l=0}^{3} \left| \partial_{r}^l \omega(r - \rho_1(t)) - \partial_{r}^l \omega(r - \rho_2(t)) \right| \right\} \left| 1 \right|_{r \geq \delta_0}
\]

\[+ \frac{1}{r^3} \sum_{l=0}^{3} \left| \partial_{r}^l \omega(r - \rho_1(t)) - \partial_{r}^l \omega(r - \rho_2(t)) \right| \left| 1 \right|_{r \geq \delta_0}
\]

\[\leq C\|h_1 - h_2\|_{\Lambda_T} \Phi(t, r) \log T,
\]

where \(\Phi(t, r)\) is defined by (2.26) and \(C > 0\) is an uniform constant independent of \(h_1, h_2\) and \(t\). Using the fact that \(E(t, r) = E_1(t, r) + E_2(t, r)\) and combining the above estimates, we can get (4.7).

\[\square
\]

**Lemma 4.3** Let \(h_1, h_2 \in \Lambda_T, \phi_1, \phi_2 \in X_T, c(\phi, h, t)\) satisfy equation (3.55) with respect to \(\phi\) and \(\rho = \gamma_n + h\). Then

\[|c(\phi_1, h_1, t) - c(\phi_2, h_2, t)| \leq \frac{C}{T^{3/4}[\log T]^{p-1}}\|\phi_1 - \phi_2\|_{C\Phi((-\infty, -T) \times (0, \infty))}
\]

\[+ \frac{C}{T^{1/2}[\log T]^{p-1}}\|h_1 - h_2\|_{\Lambda_T},
\]

(4.10)

where \(C\) depends on \(\hat{C}\) in (4.4), and \(p \in (n, n + 1]\).

**Proof** Proving this lemma just need to do some similar calculations in Lemmas 3.7 and 4.2, we omit it here.

\[\square
\]

**Proof of Proposition 4.1** We consider the operator \(T\) defined by (4.1) from the domain \(X_T\) in (4.4) to itself. We will prove \(T\) is a contraction mapping. Thus by fixed-point theorem, the operator \(T\) has a unique fixed point \(\phi\), i.e. \(T(\phi) = \phi\).
For any $\phi_1, \phi_2 \in X_T$, according to Lemmas 2.3, 4.2 and Proposition 3.9, we find that

$$\sum_{l=0}^{2} \| \partial^l T(0) \|_{C \Phi((\infty,-T) \times (0,\infty))} \leq \frac{\widehat{C}}{\log T}$$

and

$$\sum_{l=0}^{2} \| \partial^l [T(\phi_1) - T(\phi_2)] \|_{C \Phi((\infty,-T) \times (0,\infty))} \leq \frac{\widehat{C}}{\log T} \| \phi_1 - \phi_2 \|_{C \Phi((\infty,-T) \times (0,\infty))}.$$

Hence, $T$ is a contraction mapping in $X_T$ for any $T$ large enough. Hence, according to Banach fixed-point theorem, there exists a unique $\phi \in X_T$ such that $T(\phi) = \phi$.

Next we will prove the estimate (4.5). Choosing $h_1, h_2 \in \Lambda_T$, according to the above proof, we know that there exist $\phi_i = \phi(t, r, h_i), i = 1, 2$, which are solutions to problem (2.16)–(2.17) with $\rho = \gamma_n + h_1$ respectively.

We note that $\phi_1 - \phi_2$ does not satisfy the orthogonality condition (2.17). Let us consider a function $\tilde{\phi} = \phi_1 - \phi_2$, where

$$\tilde{\phi}_2 = \phi_2 - \tilde{c}(t) \partial_r \tilde{\phi}_1(t, r),$$

where $\tilde{\omega}_1(t, r) := \omega(r - \rho_i(t)) \chi(y_n(t)r)$ with $\rho_i(t) = \gamma_n(t) + h_i(t)$, the cut-off function $\chi$ is defined by (2.5), and $\tilde{c}(t)$ is defined by the following relation

$$\tilde{c}(t) = \int_0^{\infty} \partial_t \tilde{\omega}_1(t, r) \omega'(r - \rho_1(t)) r^{n-1} dr = \int_0^{\infty} \phi_2(t, r) \omega'(r - \rho_1(t)) r^{n-1} dr,$$

for all $t \leq -T$. According to the proof of Lemma 3.7, the coefficient of $\tilde{c}(t)$ in the left hand side of above equality is strictly positive, hence the function $\tilde{c}(t)$ is well-defined.

Thus $\tilde{\phi}$ satisfies the following problem

$$\begin{align*}
\tilde{\phi}_t &= F'(z_1(t, r)) [\tilde{\phi}] + [E(t, r, h_1) - E(t, r, h_2)] + \left[N(\phi_1, h_1) - N(\phi_2, h_2)\right] \\
&+ c_2(t) \left[ \partial_r \tilde{\omega}_1(t, r) - \partial_r \tilde{\omega}_1(t, r) \right] + R(h_1, h_2) - \left[ c_1(t) - c_2(t) \right] \partial_t \tilde{\omega}_1(t, r) \\
&\text{in } (-\infty, -T) \times (0, \infty), \\
\int_0^{\infty} \tilde{\phi}(t, r) \omega'(r - \rho_1(t)) r^{n-1} dr &= 0, \quad \text{for all } t < -T,
\end{align*}$$

where $\tilde{\omega}_1(t, r) := \omega(r - \rho_i(t)) \chi(y_n(t)r)$ with $i = 1, 2$, and the term $R(h_1, h_2)$ is defined by

$$R(h_1, h_2) = \tilde{c}(t) \partial_r \tilde{\omega}_1(t, r) + \tilde{c}(t) \partial_t \left[ \partial_r \tilde{\omega}_1(t, r) \right] + \tilde{c}(t) F'(z_1(t, r)) \left[ \partial_t \tilde{\omega}_1(t, r) \right] + F'(z_2(t, r)) [\phi_2] - F'(z_1(t, r)) [\phi_2].$$

where $z_i(t, r)$ is defined by (2.14) with $\rho_i(t) = \gamma_n(t) + h_i(t)$ for $i = 1, 2$.

Recall that the linear operator $L[\phi] = F'(z(t, r)) [\phi]$ is written as

$$L[\phi] := -\phi_{rrrr} - \frac{2(n-1)}{r} \phi_{rr} + \left[ 2W''(z(t, r)) - \frac{(n-3)(n-1)}{r^2} \right] \phi_{rr} - \left( W''(z(t, r)) \right)^2 \phi$$

$$+ \left[ \frac{2(n-1)W''(z(t, r))}{r} - \frac{(3-n)(n-1)}{r^3} \right] \phi_r + 2W'''(z(t, r)) \phi_r z_r + W^{(4)}(z(t, r)) |z_r|^2 \phi$$

$$- W'''(z(t, r)) W'(z(t, r)) \phi + 2W'''(z(t, r)) z_{rr} \phi + \frac{n-1}{r} W''(z(t, r)) z_r \phi,$$

where the approximate solution $z(t, r)$ is given by (2.14).
where we have used the following fact.

By the definition of $F$, we have

$$
\sum_{l=0}^{2} \| \partial^l \tilde{\phi} \|_{C_\phi((-\infty,-T) \times (0,\infty))} \leq C \frac{1}{\log T} \left\{ \| h_1 - h_2 \|_{L_T} + \| \phi_1 - \phi_2 \|_{C_\phi((-\infty,-T) \times (0,\infty))} \right\} + C \sup_{t \leq -T} \frac{|t|^{1/2}}{(\log |t|)^{1-p}} \left( |\tilde{c}(t)| + |\tilde{c}'(t)| \right). \tag{4.11}
$$

By the orthogonality condition (2.17), there holds that

$$
\int_{0}^{\infty} \phi_2(t,r) \partial_r \tilde{\omega}_1(t,r) r^{-n} dr = \int_{0}^{\infty} \phi_2(t,r) \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] r^{-n} dr
$$

where we have used the following fact

$$
\frac{\partial_r \tilde{\omega}(t,r)}{\Phi(t,r)} \leq C |t|^{1/2} \left[ \log |t| \right]^{p-1}, \quad \text{for all } r > 0,
$$

with $\tilde{\omega}(t,r)$ is defined by (2.6).

We consider

$$
\left| \frac{d}{dt} \int_{0}^{\infty} \phi_2(t,r) \partial_r \tilde{\omega}_1(t,r) r^{-n} dr \right| = \left| \frac{d}{dt} \int_{0}^{\infty} \phi_2(t,r) \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] r^{-n} dr \right|.
$$

By the definition of $F'(z(t,r)) [\phi]$ and integration by parts, we derive that

$$
\int_{0}^{\infty} (\phi_2)_r \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] r^{-n} dr
$$

$$
= \int_{0}^{\infty} \left[ (\phi_2)_r - F'(z_2(t,r)) [\phi] \right] \left( \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right) r^{-n} dr
$$

$$
+ \int_{0}^{\infty} \left( -\phi_{rr} - \frac{n-1}{r} \phi_r + W''(z_2(t,r)) \right) \partial_{rr} \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] r^{-n} dr
$$

$$
+ (n-1) \int_{0}^{\infty} \left( -\phi_{rr} - \frac{n-1}{r} \phi_r + W''(z_2(t,r)) \right) \partial_{rr} \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] r^{-n-2} dr
$$

$$
+ \int_{0}^{\infty} W''(z_2(t,r)) \left[ \phi_{rr} + \frac{n-1}{r} \phi_r - W''(z_2(t,r)) \phi \right] \left( \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right) r^{-n} dr
$$

$$
+ \int_{0}^{\infty} \left( \partial_{rr} z_2 + \frac{n-1}{r} \partial_r z_2 - W'(z_2(t,r)) W''(z_2(t,r)) \phi \left( \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right) \right) r^{-n} dr.
$$

By the previous fixed-point arguments and the above equality (4.14), we have

$$
\left| \int_{0}^{\infty} r^{n-1} (\phi_2)_r \left[ \partial_r \tilde{\omega}_2(t,r) - \partial_r \tilde{\omega}_1(t,r) \right] dr \right| \leq C \frac{\| h_1 - h_2 \|_{L_T}}{|t|^{1/2} \left[ \log |t| \right]^{p-1}} |\gamma_n(t)|^{n-1}.
$$

By the above estimates (4.12), (4.13), (4.15) and the definition of $\tilde{c}(t)$, we obtain that

$$
|\tilde{c}(t)| + |\tilde{c}'(t)| \leq C \frac{\| h_1 - h_2 \|_{L_T}}{|t|^{1/2} \left[ \log |t| \right]^{p-1}}, \quad \forall t < -T.
$$
Hence
\begin{align*}
\sum_{l=0}^{2} \| \partial^l \phi \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} & \leq \frac{C}{\log T} \left[ \| \phi_1 - \phi_2 \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} + \| h_1 - h_2 \|_{L^1_T} \right],
\end{align*}
where $C$ is a uniform positive constant independent of $T$.

Eventually, we have that
\begin{align*}
\sum_{l=0}^{2} \| \partial^l [\phi_1 - \phi_2] \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} & \leq \sum_{l=0}^{2} \| \partial^l \phi \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} + C \sup_{t \leq -T} \left( \frac{|t|^{1/2}}{\log |t|} \right)^{p-1} |\tilde{c}(t)| \\
& \leq C \frac{1}{\log T} \left[ \| \phi_1 - \phi_2 \|_{C_{\Phi}((-\infty, -T) \times (0, \infty))} + \| h_1 - h_2 \|_{L^1_T} \right],
\end{align*}
where we choose $T$ large enough. Thus we can obtain the estimate (4.5). \qed

5 The reduction procedure: choosing the parameter $h$

As we stated in Sect. 2.2, the left job is to choose suitable $h(t)$, i.e. $\rho(t) = \gamma_n(t) + h(t)$, such that the function $c(t)$ vanishes in Eq. (2.16).

5.1 Deriving the reduced equation involving $h$

According to (2.21), the relation $c(t) = 0$ is equivalent to
\begin{align*}
0 &= \int_{0}^{\infty} \left[ \omega''(r - \rho(t)) + \frac{n-1}{r} \omega''(r - \rho(t)) - W''(z(t, r))\omega'(r - \rho(t)) \right] \\
& \times \left( -\phi_{rr} - \frac{n-1}{r} \phi_r + W''(z(t, r)) \phi \right) r^{n-1} dr \\
& + \int_{0}^{\infty} \left[ \partial_r z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] \phi \omega'(r - \rho(t)) r^{n-1} dr \\
& + \int_{0}^{\infty} \phi(t, r) \partial_t \left[ \omega'(r - \rho(t)) \right] r^{n-1} dr + \int_{0}^{\infty} \left( E(t, r) + N(\phi) \right) \omega'(r - \rho(t)) r^{n-1} dr,
\end{align*}
(5.1)
where the error term $E(t, r)$ and the nonlinear term $N(\phi)$ are defined by (2.18)–(2.19), $z(t, r)$ is given by (2.14) and $\phi, \phi_r, \phi_{rr} \in C_{\Phi}((-\infty, -T) \times (0, +\infty))$ defined by (3.19).

The tedious computations of all terms in (5.1) will be given in the following parts.

5.1.1 We first estimate the projection of the error term $E(t, r)$. According to (2.24) and (2.25), we have that
\begin{align*}
\int_{0}^{\infty} E(t, r) \omega'(r - \rho(t)) r^{n-1} dr &= \int_{0}^{\delta_0} E(t, r) \omega'(r - \rho(t)) r^{n-1} dr \\
&+ \sum_{i=1}^{5} \int_{\delta_0}^{\infty} \tilde{E}_i(t, r) \omega'(r - \rho(t)) r^{n-1} dr, \quad (5.2)
\end{align*}
where $\gamma_n(t)$ is given by (2.8) and $\delta_0$ is a fixed small positive number in (2.5), and the functions $\tilde{E}_1(t, r), \ldots, \tilde{E}_5(t, r)$ are defined as follows

$$\tilde{E}_1(t, r) := \omega'(r - \rho(t)) \left( \rho'(t) + \frac{(n - 3)(n - 1)^2}{2r^3} \right),$$

$$\tilde{E}_2(t, r) := -\partial_r \left( \partial_r \omega(t, r) - W'(\tilde{\omega}(t, r)) + W''(\tilde{\omega}(t, r)) \right),$$

$$\tilde{E}_3(t, r) := \frac{2(n - 1)}{r} \left[ W''(\tilde{\omega}(t, r)) - W''(\omega(r - \rho(t))) \right] \omega'(r - \rho(t)),$$

$$\tilde{E}_4(t, r) := F'\left(\tilde{\omega}(t, r), \tilde{\zeta}(t, r)\right) - \frac{(n - 1)(n - 3)}{r^2} \left[ \frac{(n - 3)}{2r} \omega'(r - \rho(t)) + \omega''(r - \rho(t)) \right],$$

$$\tilde{E}_5(t, r) := F''(\tilde{\omega}(t, r), \partial_t \tilde{\zeta}(t, r)) \left[ \tilde{\zeta}(t, r) - \frac{\partial \tilde{\zeta}(t, r)}{\partial t} \right],$$

where the functions $\tilde{\omega}(t, r)$ and $\tilde{\zeta}(t, r)$ are defined by (2.6) and (2.22) respectively.

For the first term in the right hand side of (5.2), by Lemma 2.3, we have that

$$\left| \int_0^{\delta_0} E(t, r) \partial_r \left[ \omega(r - \rho(t)) \right] r^{n-1} dr \right| \leq C \int_0^{\delta_0} \Phi(t, r) \omega'(r - \rho(t)) r^{n-1} dr$$

$$\leq C \left[ \rho(t) \right]^{n-1} \log |t| \left\{ - \frac{\gamma_n(t)}{2} \right\},$$

(5.4)

where $\gamma_n(t)$ is defined by (2.8) and $C > 0$ only depends on $n$ and $\alpha = \sqrt{W''(1)}$.

Next we will estimate other terms in the right hand side of (5.2). Using the definition of $\tilde{\omega}(t, r)$ in (2.6) and equation in (1.11), we have that

$$\tilde{E}_2(t, r) = \tilde{E}_3(t, r) = 0, \quad \text{for all } r > \delta_0.$$

For $\tilde{E}_1(t, r)$ in (5.3), by Lemma 2.1, we have that

$$\int_{\delta_0}^{\infty} \tilde{E}_1(t, r) \omega'(r - \rho(t)) r^{n-1} dr$$

$$= \int_{\delta_0}^{\infty} \omega'(r - \rho(t)) \left( \rho'(t) + \frac{(n - 3)(n - 1)^2}{2r^3} \right) \omega'(r - \rho(t)) r^{n-1} dr$$

$$= \left( \rho'(t) + \frac{(n - 3)(n - 1)^2}{2r^3} \right) \left[ \rho(t) \right]^{n-1} \int_{\mathbb{R}} \left[ \omega'(x) \right]^2 dx + O \left( \frac{\left[ \rho(t) \right]^{n-1} \log |t|}{|t|^{5/4}} \right),$$

(5.5)

where we have used the fact that

$$\int_{\mathbb{R}} \left[ \omega'(y) \right]^2 dy = 0,$$

in the last equality.

We then estimate the term $\tilde{E}_4(t, r)$ given in (5.3). By the definition of $\tilde{\zeta}(t, r)$ in (2.22) and the same arguments in proof of Lemma 2.3, (2.20), for all $r > \delta_0$, we have that

$$\tilde{E}_4(t, r) = \frac{(n - 1)(n - 3)}{r^3} \left\{ 4 \partial_{rr} \tilde{\omega}(r - \rho(t)) - 4W''(\omega(r - \rho(t))) \partial_r \tilde{\omega}(r - \rho(t)) \right\}$$

$$- \frac{(n - 3)}{2} \omega'(r - \rho(t)).$$
\[-2(n-3)\left[ \partial_{rrr} \tilde{\omega}(r-\rho(t)) - W''(\omega(r-\rho(t))) \partial_r \tilde{\omega}(r-\rho(t)) \omega'(r-\rho(t)) \right] \right. \\
- W'''(\omega(r-\rho(t))) \tilde{\omega}(r-\rho(t)) \right\} \\
+ \frac{1}{r^4} \left\{ \left[ 12(n-4)-(n-1)(n-3) \right] \partial_r \tilde{\omega}(r-s\rho(t)) - 4(n-4)W''(\omega(r-\rho(t))) \tilde{\omega}(r-\rho(t)) \right\} \\
+ \rho'(t) \frac{(n-1)(n-3)}{r^2} \tilde{\omega}'(r-\rho(t)) + O \left( \frac{1}{r^5} \sum_{j=1}^{k} e^{-\frac{2}{r^2} \left| r-\rho(t) \right|} \right),

where we have used the properties of \( \tilde{\omega}(x) \) in (2.11) and the exponential decay of its derivatives.

Thus, by same arguments in above estimate of the projection of \( \tilde{E}_2(t, r) \) and the equalities in (2.12), (2.11) and (1.11), integrating by parts, we have that

\[
\int_{\delta_0}^{\infty} \tilde{E}_4(t, r) \omega'(r-\rho(t)) r^{n-1} dr \\
= 4(n-3)(n-1) \int_{\frac{n}{2}-\rho(t)}^{\infty} \omega'(y) \left[ \tilde{\omega}'''(y) - W''(\omega(y)) \tilde{\omega}'(y) \right] \left[ y + \rho(t) \right]^{n-4} dy \\
+ \frac{(n-3)^2(n-1)}{2} \int_{\delta_0-\rho(t)}^{\infty} \omega'(y) \left[ 2y \omega''(y) + \omega'(y) \right] \left[ y + \rho(t) \right]^{n-4} dy \\
+ \int_{\delta_0-\rho(t)}^{\infty} \omega'(y) \left[ y + \rho(t) \right]^{n-6} dy \\
+ \int_{\delta_0-\rho(t)}^{\infty} \left[ (12(n-4)-4(n-1)(n-3)) \tilde{\omega}''(y) - 4(n-4)W''(\omega(y)) \tilde{\omega}(y) \right] \omega'(y) \left[ y + \rho(t) \right]^{n-4} dy \\
+ \frac{\rho'(t)}{\rho(t)^2} \int_{\delta_0-\rho(t)}^{\infty} \tilde{\omega}'(y) \omega'(y) dy \\
= O \left( \frac{[\rho(t)]^{n-1}}{|t|^{5/4}} \right),
\]

where we have used (2.10), (2.12), (2.13) and the fact that

\[
\int_{\mathbb{R}} \omega'(y) \left[ 2y \omega''(y) + \omega'(y) \right] dy = 0.
\]

For the term \( \tilde{E}_5(t, r) \) in (5.3), by the definitions in (2.22) and (2.23), the properties of \( W \) in (1.2), and the equalities in (2.12) together with the oddness of \( \tilde{\omega} \) in (2.11), we get that

\[
\int_{\delta_0}^{\infty} \tilde{E}_5(t, r) \omega'(r-\rho(t)) r^{n-1} dr \\
= \int_{\delta_0}^{\infty} \left\{ \partial_r \left[ W'''(\omega(r-\rho(t))) (\omega(r-\rho(t)))^2 \right] \\
- W'''(\omega(r-\rho(t))) W''(\omega(r-\rho(t))) (\omega(r-\rho(t)))^2 \\
+ 2 \partial_{rr} \tilde{\omega}(r-\rho(t)) - W''(\omega(r-\rho(t))) \tilde{\omega}(r-\rho(t)) \right\}
\]
\[ W'''(\omega(r - \rho(t)))\omega'(r - \rho(t)) r^{n-5} dr \]
\[ + O \left( \int_{\delta_0 - \rho(t)}^{\infty} \omega'(y)[y + \rho(t)]^{n-6} \exp \left\{ -\frac{\alpha |y|}{2} \right\} dy \right) \]
\[ = O \left( \frac{[\rho(t)]^{n-1}}{|t|^{5/4}} \right). \]

5.1.2. We next estimate other terms in the right hand side of (5.1). By \( \phi \in C_{\Phi}((-\infty, -T) \times (0, +\infty)) \) and the definitions of \( \Phi(t, r) \) and \( \rho(t) \) in (2.26) and (2.7) respectively, we have that
\[ \int_0^{\infty} |N(\phi)| \omega'(r - \rho(t)) r^{n-1} dr \]
\[ \leq C \int_0^{\infty} (\Phi(t, r))^2 \omega'(r - \rho(t)) r^{n-1} dr \]
\[ \leq C \frac{\log |t|}{|t|} \int_{\gamma(t)-\rho(t)}^{\infty} \left( 1 + |x + \rho(t) - \frac{\alpha}{3} \log |t| \right)^{-2p} |x + \rho(t)|^{n-1} \omega'(x) dx \]
\[ + C \frac{\log |t|}{|t|} \left[ \rho(t) \right]^{n-1} \exp \left\{ -\frac{\gamma_n(t)}{2} \right\}. \]

Thus, by Lemma 2.1 and direct computations, we have that
\[ \int_0^{\infty} |N(\phi)| \omega'(r - \rho(t)) r^{n-1} dr \leq C \frac{[\rho(t)]^{n-1}}{|t| \left[ \log |t| \right]^{2(p-1)}}, \tag{5.6} \]
where \( C \) only depends on \( \alpha, n \) and \( \|h\|_{L^\infty} \).

We estimate the first and second term in (5.1). According to Taylor expansion and (1.11), the same argument as the proof of Lemma 3.6, we have that
\[ \left| \int_0^{\infty} \left[ -\omega'''(r - \rho(t)) - \frac{n-1}{r} \omega''(r - \rho(t)) + W''(z(t, r)) \omega'(r - \rho(t)) \right] \right| \]
\[ \times \left\{ \phi_{rr} + \frac{n-1}{r} \phi_r - W''(z(t, r)) \phi \right\} r^{n-1} dr \]
\[ + \int_0^{\infty} \left[ \partial_{rr} z(t, r) + \frac{n-1}{r} \partial_r z(t, r) - W'(z(t, r)) \right] W'''(z(t, r)) \phi \omega'(r - \rho(t)) r^{n-1} dr \]
\[ \leq C \frac{[\rho(t)]^{n-1}}{|t| \left[ \log |t| \right]^{p-2}}. \]

By Lemma 2.1 and the definition of \( \rho \) in (2.7), similar arguments as in (5.6), we have
\[ \left| \rho'(t) \int_0^{\infty} \phi(t, r) \omega''(r - \rho(t)) r^{n-1} dr \right| \leq \frac{C}{|t|^{3/4}} \int_0^{\infty} \Phi(t, r) \omega''(r - \rho(t)) r^{n-1} dr \]
\[ \leq C \frac{[\rho(t)]^{n-1}}{|t|^{5/4}}. \]
Combining the above estimates, we can obtain that (5.1) is equivalent to the following ODE:

$$\rho'(t) + \frac{(n-3)(n-1)^2}{2\rho^3(t)} = Q(\rho(t), \rho'(t)), \quad (5.7)$$

for all $t < -T$, where we recall that $\rho(t) = \gamma_n(t) + h(t)$, and $\gamma_n(t)$ is given by (2.8). The function $h(t)$ belongs to the set $\Lambda_T$ with $T > T_2$, where $T_2$ is given by Proposition 4.1 and we also recall the following two definitions in (4.2)–(4.3)

$$\Lambda_T = \left\{ h(t) : h \in C^1(-\infty, -T] \text{ and } \|h\|_{\Lambda_T} < 1 \right\}.$$

where

$$\|h\|_{\Lambda_T} = \sup_{t \leq -T} |h(t)| + \sup_{t \leq -T} \left[ \frac{|t|}{\log |t|} \right] |h'(t)|. \quad (5.8)$$

According to the above arguments, Proposition 4.1 and Lemma 2.3, we have that

**Proposition 5.1** Let $p \in (n, n + 1]$ and $P(h(t), h'(t)) := Q(\rho(t), \rho'(t))$ in (5.7). Then for all $h, h_1, h_2 \in \Lambda_T$, there hold that

$$|P(h, h')| \leq \frac{C(n, \alpha)}{|t| \left[ \frac{1}{\log |t|} \right]^{2(p-1)}},$$

and

$$|P(h_1, (h_1)') - P(h_2, (h_2)')| \leq \frac{C(n, \alpha)}{|t| \left[ \frac{1}{\log |t|} \right]^{2(p-1)} \|h_1 - h_2\|_{\Lambda_T}},$$

where we recall that $\alpha = \sqrt{W''(1)} > 0$ and $C(n, \alpha)$ is a uniform constant depending on $n$ and $\alpha$.

### 5.2 Solving the reduced equation involving $h$

In the rest of this section we will study the ODE in (5.7). We look for solutions of (5.7) of the form $\rho(t) = \gamma_n(t) + h(t)$, then $h(t)$ satisfies that

$$h'(t) + \gamma_n'(t) + \frac{(n-3)(n-1)^2}{2[\gamma_n(t) + h(t)]^3} = P(h(t), h'(t)) \quad \text{in } (-\infty, -\tilde{T}_0),$$

where $\tilde{T}_0 > T_2$, where $T_2$ is given by Proposition 4.1. By the definition of $\gamma_n(t)$ in (2.8), the above equation is equivalent to

$$h'(t) + \frac{3h(t)}{4t} = \tilde{P}(h(t), h'(t)) \quad \text{in } (-\infty, -\tilde{T}_0), \quad (5.9)$$

where

$$\tilde{P}(h(t), h'(t)) := P(h(t), h'(t)) + \frac{(n-3)(n-1)^2}{2} \times \left[ \frac{1}{(\gamma_n(t))^3} - \frac{1}{(\gamma_n(t) + h(t))^3} - \frac{3h(t)}{(\gamma_n(t))^4} \right]. \quad (5.10)$$
We will solve equation (5.9) by applying the fixed-point theorem in a suitable space with \( h(-\hat{T}_0) = 0 \). It is easily to check that if \( h(t) \) is a solution of (5.9) with initial data 0, then it has the form
\[
h(t) = -\frac{1}{(-t)^\frac{3}{4}} \int_t^{-\hat{T}_0} (-s)^\frac{3}{4} \tilde{P}(h(s), h'(s)) \, ds,
\]
with \( t \leq -\hat{T}_0 \).

Let us define two operators as following
\[
P(h(t)) := -\frac{1}{(-t)^\frac{3}{4}} \int_t^{-\hat{T}_0} (-s)^\frac{3}{4} \tilde{P}(h(s), h'(s)) \, ds \quad \text{and} \quad P'(h(t)) := \partial_t P(h(t)).
\]

Then using Proposition 5.1 and (5.10), we have that
\[
|P(0)| \leq \frac{\tilde{C}(n, \alpha)}{\log \hat{T}_0} \quad \text{and} \quad \frac{|t|}{\log |t|} |P'(0)| \leq \frac{\tilde{C}(n, \alpha)}{\log \hat{T}_0},
\]
with \( \hat{T}_0 > e^2 \), where \( \tilde{C}(n, \alpha) \) is a positive constant depending on \( n \) and \( \alpha \). We consider the domain
\[
Y := \left\{ h(t) \in C^4(-\infty, -\hat{T}_0] \, \| h(t) \|_{\Lambda \hat{T}_0} \leq \frac{2\tilde{C}(n, \alpha)}{\log \hat{T}_0} \right\},
\]
where the norm \( \| \cdot \|_{\Lambda T} \) is given by (5.8) and \( \tilde{C}(n, \alpha) \) is a positive constant in (5.12).

According to Proposition 5.1 and (5.10), we get that for all \( h_1, h_2 \in Y \),
\[
|P(h_1) - P(h_2)| \leq \frac{C(n, \alpha)}{\log \hat{T}_0} \| h_1(t) - h_2(t) \|_{\Lambda \hat{T}_0}
\]
and
\[
|P'(h_1) - P'(h_2)| \leq \frac{C(n, \alpha)}{\log \hat{T}_0} \| h_1(t) - h_2(t) \|_{\Lambda \hat{T}_0},
\]
where \( C(n, \alpha) > 0 \) only depends on \( n \) and \( \alpha \). Thus by Banach fixed point theorem, there exists \( h(t) \in Y \) such that \( P(h(t)) = h(t) \), if we choose \( \hat{T}_0 \) big enough. Thus we obtain that the ODE in (5.7) is solvable. Furthermore, by the formula in (5.11), there holds that
\[
|h(t)| \leq \frac{C}{\log |t|}, \quad \text{as } t \to -\infty.
\]

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**Declaration**

**Conflict of interest** The authors declared that they have no conflict of interest to this work.

**References**

1. Ambrosio, L., Cabré, X.: Entire solutions of semilinear elliptic equations in \( \mathbb{R}^3 \) and a conjecture of De Giorgi. J. Am. Math. Soc. 13(4), 725–739 (2000)
2. Bellettini, G., Bertini, L., Mariani, M., Novaga, M.: Convergence of the one-dimensional Cahn–Hilliard equation. SIAM J. Math. Anal. 44(5), 3458–3480 (2012)
3. Bellettini, G., Mugnai, L.: On the approximation of the elastica functional in radial symmetry. Calc. Var. Partial Differ. Equi. 24(1), 1–20 (2005)
4. Bellettini, G., Paolini, M.: Approssimazione variazionale di funzioni con curvatura, Seminario di analisi matematica. Univ, Bologna (1993)
5. Bricmont, J., Kupiainen, A., Taskinen, J.: Stability of Cahn–Hilliard fronts. Commun. Pure Appl. Math. 52(7), 839–871 (1999)
6. Caffarelli, L., Muler, N.E.: An $L^\infty$ bound for solutions of the Cahn–Hilliard equation. Arch. Ration. Mech. Anal. 133(2), 129–144 (1995)
7. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system, I. Interfacial free energy. J. Chem. Phys. 28(2), 258–267 (1958)
8. Cavaterra, C., Grasselli, M., Wu, H.: Non-isothermal viscous Cahn–Hilliard equation with inertial term and dynamic boundary conditions. Commun. Pure Appl. Anal. 13(5), 1855–1890 (2014)
9. Cherfils, L., Gatti, S., Miranville, A.: A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls. J. Math. Sci. 189, 604–636 (2013)
10. Cherfils, L., Miranville, A., Zelik, S.: The Cahn–Hilliard equation with logarithmic potentials. Milan J. Math. 79, 561–596 (2011)
11. Colli, P., Laurencot, P.: A phase-field approximation of the Willmore flow with volume constraint. Interfaces Free Bound. 13(3), 341–351 (2011)
12. Colli, P., Laurencot, P.: A phase-field approximation of the Willmore flow with volume and area constraints. SIAM J. Math. Anal. 44(6), 3734–3754 (2012)
13. Cortázar, C., del Pino, M., Musso, M.: Green’s function and infinite-time bubbling in the critical nonlinear heat equation. J. Eur. Math. Soc. 22(1), 283–344 (2020)
14. Cozzi, M., Dávila, J., del Pino, M.: Long-time asymptotics for evolutionary crystal dislocation models. Adv. Math. 371, 107242 (2020)
15. Daskalopoulos, P., del Pino, M., Sesum, N.: Type II ancient compact solutions to the Yamabe flow. J. Reine Angew. Math. 738, 1–71 (2018)
16. Davila, J., Del Pino, M., Musso, M., Wei, J.: Gluing methods for vortex dynamics in Euler flows. Arch. Ration. Mech. Anal. 235(3), 1467–1530 (2020)
17. De Giorgi, E.: Convergence problems for functionals and operators. In: Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis, pp. 131–188. Bologna: Pitagora (1979)
18. De Giorgi, E.: Some remarks on $\Gamma$-convergence and least square methods. In: Composite Media and Homogenization Theory, Progr. Nonlinear Differential Equations Appl., vol. 5, pp. 135–142. Birkhäuser Boston, Boston, MA (1991)
19. del Pino, M., Gkikas, K.: Ancient multiple-layer solutions to the Allen–Cahn equation. Proc. R. Soc. Edinb. Sect. A 148(6), 1165–1199 (2018)
20. del Pino, M., Gkikas, K.: Ancient shrinking spherical interfaces in the Allen–Cahn flow. In: Ann. Inst. H. Poincaré Anal. NonLinéaire, vol. 35, No. 1, pp. 187–215 (2018)
21. del Pino, M., Kowalczyk, M., Wei, J.: The Toda system and clustering interfaces in the Allen–Cahn equation. Arch. Ration. Mech. Anal. 190(1), 141–187 (2008)
22. del Pino, M., Kowalczyk, M., Wei, J.: Traveling waves with multiple and nonconvex fronts for a bistable semilinear parabolic equation. Commun. Pure Appl. Math. 66(4), 481–547 (2013)
23. del Pino, M., Kowalczyk, M., Wei, J., Yang, J.: Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature. J. Comput. Phys. 198(2), 450–468 (2004)
24. Fei, M., Liu, Y.: Phase-field approximation of the Willmore flow. Arch. Ration. Mech. Anal. 241(3), 1655–1706 (2021)
25. Ferrero, A., Gazzola, F., Grunau, H.-C.: Decay and eventual local positivity for biharmonic parabolic equations. Discrete Contin. Dyn. Syst. 21(4), 1129–1157 (2008)
26. Galaktionov, V.A.: On regularity of a boundary point for higher-order parabolic equations: towards Petrovskii-type criterion by blow-up approach. NoDEA Nonlinear Differ. Equ. Appl. 16(5), 597–655 (2009)
27. Gazzola, F., Grunau, H.-C.: Eventual local positivity for a biharmonic heat equation in $R^n$. Discrete Contin. Dyn. Syst. Ser. S 1(1), 83–87 (2008)
28. Goldstein, G., Miranville, A., Schimperna, G.: A Cahn–Hilliard model in a domain with non-permeable walls. Physica D 240(8), 754–766 (2011)
29. Kuwert, E., Schätzle, R.: The Willmore flow with small initial energy. J. Differ. Geom. 57(3), 409–441 (2001)
31. Kuwert, E., Schätzle, R.: Gradient flow for the Willmore functional. Commun. Anal. Geom. 10(2), 307–339 (2002)
32. Kuwert, E., Schätzle, R.: Reiner removability of point singularities of Willmore surfaces. Ann. Math. 160(1), 315–357 (2004)
33. Le, Nam Q.: A gamma-convergence approach to the Cahn–Hilliard equation. Calc. Var. Partial Differ. Equ. 32(4), 499–522 (2008)
34. Li, D.: A regularization-free approach to the Cahn–Hilliard equation with logarithmic potentials. Discrete Contin. Dyn. Syst. 42(5), 2453–2460 (2022)
35. Liu, S., Wang, F., Zhao, H.: Global existence and asymptotics of solutions of the Cahn–Hilliard equation. J. Differ. Equ. 238(2), 426–469 (2007)
36. Liu, C., Wu, H.: An energetic variational approach for the Cahn–Hilliard equation with dynamic boundary condition: model derivation and mathematical analysis. Arch. Ration. Mech. Anal. 233(1), 167–247 (2019)
37. Loreti, P., March, R.: Propagation of fronts in a nonlinear fourth order equation. Eur. J. Appl. Math. 11(2), 203–213 (2000)
38. Malchiodi, A., Mandel, R., Rizzi, M.: Periodic solutions to a Cahn–Hilliard–Willmore equation in the plane. Arch. Ration. Mech. Anal. 228(3), 821–866 (2018)
39. Nagase, Y., Tonegawa, Y.: A singular perturbation problem with integral curvature bound. Hiroshima Math. J. 37(3), 455–489 (2007)
40. Racke, R., Zheng, S.: The Cahn–Hilliard equation with dynamical boundary conditions. Adv. Differ. Equ. 8(1), 83–110 (2003)
41. Rizzi, M.: Clifford Tori and the singularly perturbed Cahn-Hilliard equation. J. Differential Equations 262(10), 5306–5362 (2017)
42. Röger, M., Schätzle, R.: On a modified conjecture of De Giorgi. Math. Z. 254(4), 675–714 (2006)
43. Tonegawa, Y.: Phase field model with a variable chemical potential. Proc. R. Soc. Edinb. Sect. A 132(4), 993–1019 (2002)
44. Wang, X.: Asymptotic analysis of phase field formulations of bending elasticity models. SIAM J. Math. Anal. 39(5), 1367–1401 (2008)

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