Two analytical formulae of the temperature inside a body by using partial lateral and initial data

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Abstract
This paper considers the problem of computing the value of a solution of the heat equation at a given point inside a bounded domain after the initial time. It is assumed that the initial value of the solution inside the domain (possibly in a part of the domain) is known; the boundary value and the normal derivative on a part of the boundary of the domain over a finite time interval are known. Two analytical formulae for the problem are given. Both formulae make use of a special fundamental solution having a large parameter of the backward heat equation.

1. Introduction
In this paper we consider the following inverse problem. Assume that we have a known heat conductive body and know the initial temperature inside the body. We are in the situation that we cannot access the whole boundary, however, want to know the time evolution of the temperature at a given point inside the body (possibly close to an inaccessible part of the boundary) after the initial time. How can one know this from the temperature and heat flux on an accessible part of the boundary of the body for a finite observation time?

This is a typical inverse and ill-posed problem and appears in many areas of engineering and medicine. The aim of this paper is to develop an analytical approach to the problem. We consider the simplest, but important, case: the heat conductive body has a known isotropic and homogeneous heat conductivity; there is no heat source or sink inside the body. In this case, using Fourier's law, after a scaling, one may assume that the temperature at a given point and time inside the body satisfies the heat equation. Then the problem is formulated as follows.

Let \( \Omega \subset \mathbb{R}^n (n = 1, 2, 3) \) be a bounded domain with a smooth boundary and \( 0 < T < \infty \). Let \( u = u(x, t) \) satisfy

\[
\partial_t u = \Delta u \quad \text{in} \quad \Omega \times ]0, T[.
\]  

(1.1)
Given non-empty open subsets $\Gamma \subset \partial \Omega \times ]0, T[\,$ and $U \subset \Omega$ (typically $U = \Omega$) find a formula for calculating $u(x, t)$ in $\Omega \times ]0, T[\,$ from the data $u(x, 0)$ for $x \in U$ and $(u(x, t), \partial u/\partial \nu(x, t))$ for $(x, t) \in \Gamma$. Here $\nu$ denotes the unit outward normal vector field to $\partial \Omega$.

We do not assume that $u(x, t)$ for $(x, t) \in (\partial \Omega \times ]0, T[\) \setminus \Gamma$ and $u(x, 0)$ for $x \in \Omega \setminus U$ are known. These are considered as unknown input or output that one cannot control. In this paper we give two solutions to the problem under a restriction on the location of $U$ and $\Gamma$ relative to the point $(x, t)$ in the spacetime where we want to know the value of $u$. The consequence of the results stated in sections 4 and 5 is the following.

Let $(x_0, t_0) \in \Omega \times ]0, T[\,$ and $c > 0$. If $(\Omega \setminus U) \times \{0\}$, $\Omega \times \{T\}$ and $(\partial \Omega \times ]0, T[\) \setminus \Gamma$ are contained in the half space $(x, t) \cdot \omega(c) \leq (x_0, t_0) \cdot \omega(c) - \delta$ in $\mathbb{R}^{n+1}$ with $\delta > 0$ and $\omega(c) = (1/\sqrt{1+c^2})(\cos, -1)^T \in S^n$, then there are two explicit formulae to calculate $u(x_0, t_0)$ from the data. The restriction on $\Omega \times \{T\}$ is not serious and note that the restriction on $U$ and $\Gamma$ are equivalent to the conditions $\{x \in \Omega \mid (x, 0) \cdot \omega(c) < (x_0, t_0) \cdot \omega(c) - \delta\} \subset U$ and $\{(x, t) \in \partial \Omega \times ]0, T[ \mid (x, t) \cdot \omega(c) < (x_0, t_0) \cdot \omega(c) - \delta\} \subset \Gamma$, respectively. See figure 1 for the configuration of $U$, $\Gamma$ and the half space $(x, t) \cdot \omega(c) \leq (x_0, t_0) \cdot \omega(c) - \delta$.

For the detail of the formulae see (4.5) of theorem 4.1 and (5.18) of corollary 5.1 in sections 4 and 5, respectively. Here we explain our approach together with the construction of the paper.

1.1. First approach

First we observe that the complex exponential function $e^{z_1(t-2)}$, $z \in \mathbb{C}^n$ satisfies the backward heat equation $(\partial_t + \Delta)w = 0$ in $\mathbb{R}^{n+1}$.

Given $c > 0$ and $\omega, \omega^\perp \in S^{n-1}$ with $\omega \cdot \omega^\perp = 0$ if $n = 2, 3$ and $\omega \in \{-1, 1\}$ if $n = 1$ define a special $z$ depending on a parameter $\tau$ with $c^2 \tau > 1$

$$z = \begin{cases} c \tau \left(\omega + i\sqrt{1 - \frac{1}{c^2 \tau}} \omega^\perp\right), & \text{if } n = 2, 3 \\ c \tau \left(1 + i\sqrt{1 - \frac{1}{c^2 \tau}}\right) \omega, & \text{if } n = 1. \end{cases}$$

(1.2)
One can write \( \text{Re} \{ x \cdot z - t(z \cdot z) \} = \tau \sqrt{1 + c^2(x, t)} \cdot \omega(c) \) since \( \text{Re} z \cdot z = \tau \). This yields that if \((x, t)^T \cdot \omega(c) > 0\), then \(|e^{x \cdot z - t(z \cdot z)}| \longrightarrow \infty \) as \( \tau \longrightarrow \infty \); if \((x, t)^T \cdot \omega(c) < 0\), then \(|e^{x \cdot z - t(z \cdot z)}| \longrightarrow 0 \) as \( \tau \longrightarrow \infty \). This means that the asymptotic behaviour of \( e^{x \cdot z - t(z \cdot z)} \) divides the whole spacetime into two parts whose common boundary is the hyper plane \((x, t)^T \cdot \omega(c) = 0\).

Second we choose \( D \subset \mathbb{R}^{n+1} \) with \( \overline{D} \subset \Omega \times [0, T] \) and \((x_0, t_0) \in \partial D\) in such a way that for all \( \rho \) smooth on \( \overline{D} \) and for constants \( \mu \) and \( C \) independent of \( \rho \)

\[
\lim_{\tau \to \infty} \tau^\mu e^{x_0 \cdot z + b(t)} \int_D e^{x \cdot z - t(z \cdot z)} \rho(x, t) \, dx \, dt = C \rho(x_0, t_0). \tag{1.3}
\]

The examples of the choice of \( D \) are given in subsections in section 4.

Third we construct a special solution of the backward heat equation with a special inhomogeneous term

\[
\partial_t u + \Delta v + e^{x \cdot z - t(z \cdot z)} \chi_D(x, t) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}. \tag{1.4}
\]

The point of the property of \( v \) is: the growth order of a suitable norm of \( e^{-x \cdot z + t(z \cdot z)} v \) as \( \tau \longrightarrow \infty \) is at most algebraic. This yields that if \((x, t)^T \cdot \omega(c) < (x_0, t_0)^T \cdot \omega(x)\), then the function \( e^{x_0 \cdot z + b(t)} \cdot v(x, t) \) is exponentially decaying as \( \tau \longrightarrow \infty \). For the construction of \( v \) we basically follow a Fourier transform method for a reduced equation to (1.4) which has been used in [16] since it is constructive. It starts with introducing a special fundamental solution denoted by \( G_z \) for the reduced equation. This together with necessary estimates are described in section 3.

Finally, integrating equation (1.1) multiplied by \( v \) over \( \Omega \times [0, T] \) and applying an integration by parts formula described in section 2, we know that the quantity

\[
\tau^\mu e^{x_0 \cdot z + b(t)} \int_D e^{x \cdot z - t(z \cdot z)} u(x, t) \, dx \, dt
\]

which is coming from the inhomogeneous term in (1.4) multiplied by \( \tau^\mu e^{x_0 \cdot z + b(t)} \), is divided into two parts: the first part consists of the data on \( \Gamma \) and \( U \times [0] \); the second unknown data on \((\Omega \setminus U) \times [0], (\partial \Omega \times [0, T]) \setminus \Gamma \) and \( \Omega \times [T] \) relative to the hyper plane \((x, t)^T \cdot \omega(c) = (x_0, t_0)^T \cdot \omega(c)\) we know that the second part tends to 0 as \( \tau \longrightarrow \infty \). Combining this with (1.3) for \( \rho = u \) on \( \overline{D} \), one gets a formula to calculate \( u(x_0, t_0) \) from the data on \( \Gamma \) and \( U \times [0] \) only. See section 4 for the precise description of the formula. This approach can be considered as an application of the enclosure method [8] to the heat equation (1.1).

1.2. Second approach

In the final section we give an integral representation formula of \( G_z \) that is important for calculating \( v \). Moreover, as a byproduct we present a Carleman-type formula for the heat equation, which instead of \( v \), makes use of \( e^{x \cdot z - t(z \cdot z)} G_z(x, t) \) directly which is a special fundamental solution of the backward heat equation. This approach is classical and the formula can be considered as an extension to the heat equation of Yarmukhamedov’s formula [18] which has been established for the Laplace equation.

1.3. Sideways heat equation

There are extensive mathematical studies in the case when the one dimensional heat equation is appropriate. This case is also referred to as the sideways heat equation. Carasso [2] considered
the heat equation in the half line

$$\partial_t u = u_{xx} \quad \text{in } [0, \infty[ \times ]0, \infty[$$

with the initial condition $u(x, 0) = 0$ for $0 \leq x < \infty$. He gave a Tikhonov regularization procedure that yields an approximation to $u(x, t)$ for $0 < t < \infty$ at an arbitrary fixed $0 < x < 1$ from noisy $u(1, t)$ for $0 \leq t < \infty$. Note that by solving an initial boundary value problem for the heat equation in $1 < x < \infty$, one gets the heat flux $u_x(1, t)$ from $u(1, t)$ and thus his problem is reduced to a one-dimensional version of our problem. Motivated by Carasso’s work Levine [14] considered a radially symmetric solution of the heat equation in higher dimensions and established a Tikhonov regularization procedure. Their methods are based on an explicit integral representation for the solution of the heat equation and not time local in the sense: to determine the value of the solution of the heat equation at a fixed point in the body one needs the data for all the time ($T = \infty$). In contrast to their methods our method does not make use of any representation of the solution of the heat equation and needs data only on an appropriate finite time interval depending on the location of the point and time where and when we want to know the value of the solution and covers fully multi-dimensional cases. For possible applications and numerical studies of the sideways heat equation, see [1, 4] and references therein.

Some further remarks are in order.

• The formulae obtained in this paper contain a large parameter $\tau$. If the data is noisy, then the formulae break down as $\tau \to \infty$ and thus one has to choose $\tau$ carefully depending on the upper bound of the noise and error. This means that parameter $\tau$ plays a role of the regularization parameter. It is clear that one can give a theoretical way of choosing such $\tau$ by using the idea in [10, 12]. Therein this way for the enclosure method applied to an inverse boundary value problem and the Cauchy problem for elliptic equations is given. Anyway this point should be considered in a numerical study of the formulae which belongs to our future plan.

• Our method can be considered as a non-iterative method or direct method and for iterative methods for general heat operators see [13] and references therein.

A brief outline of this paper is as follows. Two formulae are given in sections 4 and 5. The first formula is based on a weighted $L^2$-estimate of a special fundamental solution for the backward heat equation; second on the pointwise decaying estimate in a hyper space: these estimates are found in sections 3 and subsection 5.3, respectively. To describe the formulae rigorously and cover more general cases we require the weak formulation of the direct problem for the heat equation and related integration by parts formula. These are found in section 2.

2. A weak formulation of the direct problem and integration by parts formula

In this section we describe what we mean by a solution of (1.1) together with an integration by parts formula. We follow the formulation of the direct problem described in [3].

Given $\rho \in L^\infty(\partial \Omega)$, $f_0 \in L^2(0, T; (H^1(\Omega))')$ and $h_0 \in L^2(0, T; H^{-1/2}(\partial \Omega))$ we say that $u \in W(0, T; H^1(\Omega), (H^1(\Omega))')$ satisfy

$$\partial_t u - \Delta u = f_0 \quad \text{in } \Omega \times ]0, T[,$$

$$\nabla u \cdot v + \rho u = h_0 \quad \text{on } \partial \Omega \times ]0, T[$$

(2.1)
in the weak sense if \( u \) satisfies

\[
\langle u'(t), \varphi \rangle + \int_{\Omega} \nabla u(x, t) \cdot \nabla \varphi(x) \, dx + \int_{\partial \Omega} u(t)|_{\partial \Omega} \cdot \varphi|_{\partial \Omega} \rho \, dS = \langle f_0(t), \varphi \rangle + \langle h_0(t), \varphi|_{\partial \Omega} \rangle \quad \text{in (0, T),}
\]

(2.2)
in the sense of distribution on (0, T) for all \( \varphi \in H^1(\Omega) \).

Note that by theorem 1 on p 473 in [3] we see that every \( u \in W(0, T; H^1(\Omega), (H^1(\Omega))') \) is almost everywhere equal to a continuous function of [0, T] in \( L^2(\Omega) \). Further, we have

\[
W(0, T; H^1(\Omega), (H^1(\Omega))') \hookrightarrow C^0([0, T]; L^2(\Omega)),
\]

the space \( C^0([0, T]; L^2(\Omega)) \) being equipped with the norm of uniform convergence. Thus one can consider \( u(0) \) and \( u(T) \) as elements of \( L^2(\Omega) \). Then by theorems 1 and 2 on p 512 and 513 in [3] we see that given \( u_0 \in L^2(\Omega) \) there exists a unique \( u \) such that \( u \) satisfies (2.1) in the weak sense and satisfies the initial condition \( u(0) = u_0 \).

Given \( \rho \in L^\infty(\partial \Omega) \), \( f_1 \in L^2(0, T; (H^1(\Omega))') \) and \( h_1 \in L^2(0, T; H^{-1/2}(\partial \Omega)) \) we say that a \( v \in W(0, T; H^1(\Omega), (H^1(\Omega))') \) satisfy

\[
\begin{align*}
\partial_t v + \Delta v &= f_1 \quad \text{in } \Omega \times ]0, T[, \\
\nabla v \cdot v + \rho v &= h_1 \quad \text{on } \partial \Omega \times ]0, T[ \tag{2.3}
\end{align*}
\]
in the weak sense if \( v \) satisfies

\[
\langle v'(t), \varphi \rangle - \int_{\Omega} \nabla v(x, t) \cdot \nabla \varphi(x) \, dx - \int_{\partial \Omega} v(t)|_{\partial \Omega} \cdot \varphi|_{\partial \Omega} \rho \, dS = \langle f_1(t), \varphi \rangle - \langle h_1(t), \varphi|_{\partial \Omega} \rangle \quad \text{in (0, T),}
\]

in the sense of distribution on (0, T) for all \( \varphi \in H^1(\Omega) \).

**Proposition 2.1.** Let \( u \in W(0, T; H^1(\Omega), (H^1(\Omega))') \) satisfy (2.1) in a weak sense; let \( v \in W(0, T; H^1(\Omega), (H^1(\Omega))') \) satisfy (2.3) in a weak sense. Then the formula

\[
\int_0^T (\langle h_1(t), u(t)|_{\partial \Omega} \rangle - \langle h_0(t), v(t)|_{\partial \Omega} \rangle) \, dt
\]

\[
= \int_\Omega u(x, 0)v(x, 0) \, dx - \int_\Omega u(x, T)v(x, T) \, dx
\]

\[
+ \int_0^T (\langle f_1(t), u(t) \rangle + \langle f_0(t), v(t) \rangle) \, dt, \tag{2.4}
\]

is valid.

**Proof.** Substituting \( \varphi = v(t) \) into (2.2), we have

\[
\langle u'(t), v(t) \rangle + \int_{\Omega} \nabla u(x, t) \cdot \nabla v(x, t) \, dx + \int_{\partial \Omega} u(t)|_{\partial \Omega} \cdot v(t)|_{\partial \Omega} \rho \, dS = \langle f_0(t), v(t) \rangle + \langle h_0(t), v(t)|_{\partial \Omega} \rangle \quad \text{a.e. in (0, T).}
\]

(2.5)

Similarly we have also

\[
\langle v'(t), u(t) \rangle - \int_{\partial \Omega} \nabla v(x, t) \cdot \nabla u(x, t) \, dx - \int_{\partial \Omega} v(t)|_{\partial \Omega} \cdot u(t)|_{\partial \Omega} \rho \, dS = \langle f_1(t), u(t) \rangle - \langle h_1(t), u(t)|_{\partial \Omega} \rangle \quad \text{a.e. in (0, T).}
\]

(2.6)
Taking the sum of (2.5) and (2.6), we have

\[
\langle u'(t), v(t) \rangle + \langle v'(t), u(t) \rangle = (f_0(t), v(t)) + \langle h_0(t), v(t) \rangle_{\partial \Omega} + (f_1(t), u(t)) - \langle h_1(t), u(t) \rangle_{\partial \Omega}.
\]

Integrating both sides of this equation over the interval \((0, T)\) and using the formula (theorem 2 on p 477 in [3])

\[
\int_0^T \langle u'(t), v(t) \rangle \, dt + \int_0^T \langle v'(t), u(t) \rangle \, dt = \int_\Omega u(x, T)v(x, T) \, dx - \int_\Omega u(x, 0)v(x, 0) \, dx,
\]

we obtain (2.4). □

In particular, let \(u \in W(0, T; H^1(\Omega), (H^1(\Omega))')\) satisfy

\[
\partial_t u - \Delta u = 0 \quad \text{in} \ \Omega \times [0, T],
\]

\[
\nabla u \cdot \nu + \rho u = h_0 \quad \text{on} \ \partial \Omega \times [0, T],
\]

in the weak sense and \(v \in W(0, T; H^1(\Omega), (H^1(\Omega))')\) satisfy

\[
\partial_t v + \Delta v = f_1 \quad \text{in} \ \Omega \times [0, T],
\]

\[
\nabla v \cdot \nu + \rho v = h_1 \quad \text{on} \ \partial \Omega \times [0, T],
\]

in the weak sense. Then (2.4) gives the formula

\[
\int_0^T \langle (h_1(t), u(t)) \rangle_{\partial \Omega} - \langle (h_0(t), v(t)) \rangle_{\partial \Omega} \, dt = \int_0^T (f_1(t), u(t)) \, dt + \int_\Omega u(x, 0)v(x, 0) \, dx - \int_\Omega u(x, T)v(x, T) \, dx.
\]

3. A solution of the equation \(\partial_t v + \Delta v + f = 0\)

In this section given \(f = f(x, t)\) we construct a special solution of the equation

\[
-\partial_t v - \Delta v = f \quad \text{in} \ \mathbb{R}^{n+1}.
\]

(3.1)

Note that \(-\partial_t - \Delta\) is the formal adjoint for \(-\partial_t + \Delta\).

Given a complex vector \(z \in \mathbb{C}^n\) set

\[
v(x, t) = e^{iz \cdot z(t)} w(x, t), \quad f(x, t) = e^{iz \cdot z(t)} g(x, t).
\]

Then (3.1) becomes

\[
-\partial_t w - 2z \cdot \nabla w - \Delta w = g.
\]

(3.2)

Taking the Fourier transform of both sides, we obtain

\[
(-i\eta - 2iz \cdot \xi + |\xi|^2) \hat{w}(\xi, \eta) = \hat{g}(\xi, \eta).
\]

This motivates us to study the meaning of \(1/P_z(\xi, \eta)\) where

\[
P_z(\xi, \eta) = -i\eta - 2iz \cdot \xi + |\xi|^2.
\]

Let

\[
z = a + ib, \quad a, b \in \mathbb{R}^n.
\]
Since
\[ P_z(\xi, \eta) = |\xi + b|^2 - |b|^2 - i(\eta + 2\alpha \cdot \xi), \]
we have: \( P_z(\xi, \eta) = 0 \) if and only if \( |\xi + b| = |b| \) and \( \eta + 2\alpha \cdot \xi = 0 \). Therefore the set \( \{(\xi, \eta) \in \mathbb{R}^{n+1} | P_z(\xi, \eta) = 0 \} \) is a compact set and forms a submanifold of \( \mathbb{R}^{n+1} \) with codimension 2 provided \( b \neq 0 \). This yields that for \( z \in \mathbb{C}^n \) with \( \text{Im} z \neq 0 \) the function \( \frac{1}{P_z(\xi, \eta)} \) defines a tempered distribution on \( \mathbb{R}^{n+1} \).

**Definition 3.1.** Define \( G_z(x, t) \) as the inverse Fourier transform of \( \frac{1}{P_z(\xi, \eta)} \)
\[ G_z(x, t) = \frac{1}{(2\pi)^{n+1}} \int e^{i(x \cdot \xi + it \eta)} \frac{d\xi}{P_z(\xi, \eta)} \quad \text{Im} z \neq 0 \] (3.3)
In this section we study the convolution operator \( g \mapsto G_z \ast g \) acting on the set of all rapidly decreasing functions on \( \mathbb{R}^{n+1} \).

### 3.1. Purely imaginary \( z \)

It is easy to see that from (3.3) we obtain the translation formula: for all \( z \in \mathbb{C}^n \) with \( \text{Im} z \neq 0 \)
\[ G_z(x, t) = G_{i\tau\omega}(x - 2t \text{Re } z, t). \] (3.4)
Thus we starts with considering the case when \( \text{Re } z = 0 \). More precisely, given \( \omega \in S^{n-1} \) and \( \tau > 0 \) set
\[ z = i\tau \omega. \]

We set \( F_{\tau}(x, t) = G_z(x, t) \) for this \( z \), that is
\[ F_{\tau}(x, t) = \frac{1}{(2\pi)^{n+1}} \int e^{i(x \cdot \xi + it \eta)} \frac{d\xi}{-i\eta + 2\tau \omega \cdot \xi + |\xi|^2}. \]

Let us study the property of \( F_{\tau}(x, t) \). For the purpose we employ the argument done in [7]. The points are: a relationship between the operators \( F_{\tau} \ast \cdot \) and \( F_1 \ast \cdot \); an estimation of a scaling effect on weighted \( L^2 \)-norms; an weighted \( L^2 \)-estimate for the operator \( F_1 \ast \cdot \).

#### 3.1.1. Scaling laws

Here we summarize some simple facts needed for the next subsection.

It is easy to see that \( F_{\tau}(x, t) \) satisfies the scaling law
\[ \forall \lambda > 0 F_{\lambda \tau}(x, t) = \lambda^n F_{\tau}(\lambda x, \lambda^{2} t). \] (3.5)

Given a distribution \( g(x, t) \) define
\[ g_{\lambda}(x, t) = g(\lambda x, \lambda^{2} t), \quad \lambda > 0. \]

It follows from (3.5) that
\[ F_{\tau} \ast g = \frac{1}{\tau^{2}} F_{1} \ast (g_{1/\tau}), \] (3.6)
and also
\[ D_x^\alpha D_t^\beta F_{\tau} \ast g = \tau^{-2+|\alpha|+2\beta} \{ D_x^\alpha D_t^\beta F_{1} \ast (g_{1/\tau}) \}. \] (3.7)

Given \( s \in \mathbb{R} \) we denote by \( L^2_s(\mathbb{R}^{n+1}) \) the set of all tempered distributions \( g = g(x, t) \) that satisfies \( (1 + |x|^2 + t^2)^{s/2} g \in L^2(\mathbb{R}^{n+1}) \) and set
\[ \| g \|_s = \left( \int |g(x, t)|^2 (1 + |x|^2 + t^2)^{s} \, dx \, dt \right)^{1/2}, \quad s \in \mathbb{R}. \]

Note that the set of all rapidly decreasing functions on \( \mathbb{R}^{n+1} \) is dense in \( L^2_s(\mathbb{R}^{n+1}) \).
Given $R > 0$ let $\tau \geq R$. Set

$$C(R) = \min\{R^4, R^2, 1\} (>0).$$

Since we have

$$1 + \frac{|x|^2}{\tau^2} + \frac{t^2}{\tau^4} \geq \frac{C(R)}{\tau^4}(1 + |x|^2 + t^2)$$

and

$$1 + \tau^2|x|^2 + \tau^4t^2 \leq \frac{\tau^4}{C(R)}(1 + |x|^2 + t^2),$$

it follows that

$$\|g_\tau\|_s \leq \frac{C(R)^{s/2}}{\tau^{s/2}} \|g\|_s, \quad s < 0 \quad (3.8)$$

and

$$\|g_{\tau^{-1}}\|_{s'} \leq \frac{\tau^{2s'+(n+2)/2}}{C(R)^{s'/2}} \|g\|_{s'}, \quad s' > 0. \quad (3.9)$$

### 3.1.2. Weighted $L^2$-estimates.

**Lemma 3.1.** Let $-1 < \delta < 0$. Given a rapidly decreasing function $g$ on $\mathbb{R}^{n+1}$ the tempered distribution $F_1 * g$ belongs to $L^2_\delta(\mathbb{R}^{n+1})$ and there exists a positive constant $C_\delta$ independent of $g$ and $\omega$ such that

$$\|D^\alpha_x D^\beta_t F_1 * g\|_\delta \leq C_\delta \|g\|_{1+\delta}, \quad |\alpha| + 2\beta \leq 2. \quad (3.10)$$

**Proof.** For $z = i\omega$, we have $P_z(\xi, \eta) = (\xi + \omega)^2 - 1 - i\eta$. Let $|\xi| \geq 8$. We have

$$|P_z(\xi, \eta)|^2 = (|\xi|^4 + 2\xi \cdot \omega)^2 + \eta^2$$

$$\geq \begin{cases} |\xi|^4 + \eta^2 - 4|\xi|^3 = |\xi|^4 \left(1 - \frac{4}{|\xi|^3}\right) + \eta^2 \\ \geq \frac{1}{2}|\xi|^4 + \eta^2 \geq \frac{1}{2}(|\xi|^4 + |\eta|^2) \geq \frac{1}{4}(|\xi|^2 + |\eta|)^2. \end{cases}$$

Next let $|\xi| \leq 8$ and $|(\xi, \eta)| \geq 8\sqrt{1 + 8^2}$. We have

$$|\eta|^2 \geq 8^2(1 + 8^2) - |\xi|^2 \geq 8^4 \geq |\xi|^4.$$ 

This yields $|\eta| \geq (|\xi|^2 + |\eta|)/2$ and thus one gets

$$|P_z(\xi, \eta)|^2 \geq \eta^2 \geq \frac{1}{2}(|\xi|^2 + |\eta|)^2.$$ 

Therefore it holds that, for all $(\xi, \eta) \in \mathbb{R}^{n+1}$ with $|(\xi, \eta)| \geq 8\sqrt{1 + 8^2}$

$$|P_z(\xi, \eta)| \geq \frac{1}{2}(|\xi|^2 + |\eta|).$$

Using this inequality, a local representation of $1/P_z(\xi, \eta)$ in each neighbourhood of some zero points of $P_z(\xi, \eta)$ and lemma 3.1 in [16], we have the desired conclusion. □
A combination of (3.6), (3.7), (3.8), (3.9) and (3.10) yields

**Proposition 3.1.** Let \(-1 < \delta < 0\) and \(R > 0\). For all rapidly decreasing functions \(g\) on \(\mathbb{R}^{n+1}\) and \(\tau \geq R\) we have

\[
\|D^n_\tau D^\beta G_{\tau\alpha} \ast g\|_\delta \leq \frac{C \tau^{\alpha+2\beta}}{C(R)^{1/2}} \|g\|_{1+\delta}, \quad |\alpha| + 2\beta \leq 2. \tag{3.11}
\]

**Proof.** First consider the case when \(|\alpha| = \beta = 0\). We have

\[
\|F_\tau \ast g\|_\delta = \frac{1}{\tau^2} \|\{F_1 \ast (g_{\tau^{-1}})\}_\tau\|_\delta
\]

\[
\leq \frac{C(R)^{3/2}}{\tau^{2n+2+(n+2)/2}} \|G_1 \ast (g_{\tau^{-1}})\|_\delta
\]

\[
\leq \frac{C(R)^{3/2} C_f}{\tau^{2n+2+(n+2)/2}} \|g_{\tau^{-1}}\|_{1+\delta}
\]

\[
\leq \frac{C(R)^{3/2} C_f}{\tau^{2n+2+(n+2)/2}} \frac{C(R)^{1+\delta}}{C(R)(1+\delta)^2} \|g\|_{1+\delta}.
\]

Since we have (3.7), a similar argument yields (3.11) for the case when \(|\alpha| \neq 0\) or \(\beta \neq 0\). □

### 3.2. General \(z\) and construction of \(v\)

Now given a real vector \(c\) set

\[g_c(x, t) = g(x - tc, t).\]

From (3.4) we have, for \(z = a + ib\)

\[G_z \ast g = \{G_{ib} \ast (g_{-2a})\}_{2a} \tag{3.12}\]

and also

\[D^n_\alpha (G_z \ast g) = (D^n_\alpha G_{ib} \ast g_{-2a})_{2a}, \]

\[D_j (G_z \ast g) = -\sum_{j=1}^n \left( \frac{\partial}{\partial x_j} G_{ib} \ast g_{-2a} \right)_{2a} a_j + (D_j G_{ib} \ast g_{-2a})_{2a}. \tag{3.13}\]

**Remark 3.1.** Equation (3.12) corresponds to the simple fact: a function \(w(x, t)\) satisfies equation (3.2) if and only if the function \(\tilde{w}(x, t) = w(x + 2ta, t)\) satisfies the equation

\[-\partial_t \tilde{w} - 2ib \cdot \nabla \tilde{w} - \Delta \tilde{w} = g(x + 2ta, t).\]

Now we give an estimate for \(G_z \ast g\).

**Theorem 3.1.** Let \(-1 < \delta < 0\) and \(R > 0\). Let \(z = a + ib\). For all rapidly decreasing functions \(g\) on \(\mathbb{R}^{n+1}\) and \(b \neq 0\) with \(|b| \geq R\) we have

\[
\|D^n_\tau G_z \ast g\|_\delta \leq C(R, \delta) \left( \sqrt{1 + |a|^2 + |b|} \right) |b|^{n} \|g\|_{1+1}, \quad |\alpha| \leq 2
\]

\[
\|D_j G_z \ast g\|_\delta \leq C(R, \delta) \left( \sqrt{1 + |a|^2 + |b|} \right) (2|a||b| + |b|^2) \|g\|_{1+1}.
\]

**Proof.** It suffices to consider only the case when \(a \neq 0\). Set \(f = G_{ib} \ast (g_{-2a})\). From (3.12) we have

\[
\|G_z \ast g\|_\delta^2 = \|f \|_\delta^2
\]

\[
= \int |f(x - 2ta, t)|^2 (1 + |x|^2 + t^2)^d \, dx \, dt
\]

\[
= \int |f(y, t)|^2 (1 + |y + 2ta|^2 + t^2)^d \, dy \, dt.
\]

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One can write

\[
|y + 2ia|^2 + t^2 = A \begin{pmatrix} y \\ t \end{pmatrix} \cdot \begin{pmatrix} y \\ t \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} I_n & 2a \\ 2a^T & 1 + 4|a|^2 \end{pmatrix}.
\]

It is easy to see that the eigenvalues \( \lambda \) of \( A \) coincide with the roots of the equation

\[
\lambda^2 - 2(1 + 2|a|^2)\lambda + 1 = 0.
\]

Solving this equation, we have

\[
\lambda = (1 + 2|a|^2) \pm 2|a|\sqrt{1 + |a|^2}.
\]

Since the minimum eigenvalue has the form

\[
(1 + 2|a|^2) - 2|a|\sqrt{1 + |a|^2} = \frac{1}{(\sqrt{1 + |a|^2} + |a|)^2},
\]

one has

\[
\frac{|y|^2 + t^2}{(\sqrt{1 + |a|^2} + |a|)^2} \leq A \begin{pmatrix} y \\ t \end{pmatrix} \cdot \begin{pmatrix} y \\ t \end{pmatrix} \leq (\sqrt{1 + |a|^2} + |a|)^2(|y|^2 + t^2).
\]

(3.14)

Since \(-1 < \delta < 0\), from (3.14) we have

\[
\|f_{2a}\|_1^2 \leq (\sqrt{1 + |a|^2} + |a|)^{-2\delta} \|f\|_2^2
\]

and

\[
\|g_{-2a}\|_{p+1}^2 \leq (\sqrt{1 + |a|^2} + |a|)^{2(1+\delta)} \|g\|_{p+1}^2.
\]

These together with (3.11) yield

\[
\|G \ast g\|_2^2 \leq (\sqrt{1 + |a|^2} + |a|)^{-2\delta} \|f\|_2^2
\]

\[
\leq C(R, \delta)(\sqrt{1 + |a|^2} + |a|)^{-2\delta} \|g_{-2a}\|_{p+1}^2
\]

\[
\leq C(R, \delta)(\sqrt{1 + |a|^2} + |a|)^{-2\delta + 2(1+\delta)} \|g\|_{p+1}^2.
\]

Other cases also can be proved as above since we have (3.13).

Therefore the map \( g \mapsto G \ast g \in \mathcal{L}^2_2(\mathbb{R}^{n+1}) \) can be uniquely extended as a bounded linear operator of \( L^2_{p+1}(\mathbb{R}^{n+1}) \) into \( L^2_2(\mathbb{R}^{n+1}) \). We denote it by the same symbol. Then we see that, given \( \z \) with \( \operatorname{Im} \z \neq 0 \) and \( g \in L^2_{p+1}(\mathbb{R}^{n+1}) \) the function

\[
v(x, t) = e^{\z \cdot x - t|z|^2}(G \ast g)(x, t)
\]

satisfies the backward heat equation (3.1) with the source term \( f(x, t) = e^{\z \cdot x - t|z|^2}g(x, t) \) in the sense of distribution. Note that \( e^{-z \cdot \partial_t - t|z|^2}v(x, t) \in \mathcal{L}^2_2(\mathbb{R}^{n+1}) \) and this \( v \) is unique. This is a consequence of theorem 7.1.27 of [6] and the facts that the set of all zero points of \( P(\xi, \eta) \) has codimension 2 in \( \mathbb{R}^{n+1} \) and \(-1 < \delta < 0\). See also corollary 3.4 in [16] for this type of argument.

**Remark 3.2.** Hsieh [5] developed the scattering theory associated with the operator \( \partial_t - \Delta \) in \( \mathbb{R}^{2+1} \). For the purpose he studied the operator \( L^I(\mathbb{R}^{2+1}) \) \( \ni f \mapsto (1/Q_\z(\xi, \eta))\hat{q} \ast f \in \mathcal{L}^2(\mathbb{R}^{2+1}) \) where \( \hat{q} \) the Fourier transform of a function \( q \) on \( \mathbb{R}^{2+1} \), \( \z \in \mathcal{C}^2 \) and \( Q_\z(\xi, \eta) \) the symbol of the operator \( e^{-z \cdot \partial_t - t|z|^2}(\partial_t - \Delta)e^{-z \cdot \partial_t - t|z|^2} \). Note that \( e^{-z \cdot \partial_t - t|z|^2} \) satisfies the heat equation not the backward heat equation. In the paper there is no result related with theorem 3.1.
4. A computation formula of \( u \) in \( R^{n+1} \) with \( n = 1, 2, 3 \)

In this section we assume that \( h_0 \) in (2.7) satisfies \( h_0 \in L^2(0, T; H^{-1/2}(\partial \Omega)) \) not just \( h_0 \in L^2(0, T; \Omega) \).

Let \( u \in W(0, T; H^1(\Omega), (H^1(\Omega)') \text{ satisfy (2.7) in the weak sense. Let } D \text{ a bounded open subset of } R^{n+1} \text{ with } \bar{D} \subset \Omega \times [0, T]. \) We denote by \( \chi_D \) the characteristic function of \( D \).

In our method the concept introduced in the following plays an important role.

**Definition 4.1.** We say that \( D \) is visible at \((x_0, t_0) \in R^{n+1} \) as \( \tau \to \infty \) from the complex direction \( z \) given by (1.2) if \((x_0, t_0) \in \partial D \) and there exist \( \mu > 0 \) and constant \( C \neq 0 \) such that for all \( \rho \in C^\infty(\bar{D}) \)

\[
\lim_{\tau \to \infty} e^{i\tau x_0 - \tau t_0 + |z|} \tau^\mu \int_D e^{i\tau x - \tau t + |z|} \rho(x, t) \, dx \, dt = C \rho(x_0, t_0).
\]

(4.1)

The constant \( C \) is unique if it exists.

**Theorem 4.1.** Let \((x_0, t_0) \in \Omega \times [0, T] \) be an arbitrary fixed point. Assume that \( T > 0, \omega, \Gamma \) and \( U \) satisfy the following conditions:

\[
\sup_{x \in \Omega} \left\{ \frac{x}{\tau} \cdot \omega(c) < \frac{x_0}{0} \cdot \omega(c); \right. \tag{4.2}
\]

\[
\sup_{x \in \Omega, U} \left\{ \frac{x}{\tau} \cdot \omega(c) < \frac{x_0}{0} \cdot \omega(c); \right. \tag{4.3}
\]

\[
\sup_{x(x, t) \in (\Omega \times [0, T])} \left\{ \frac{x}{\tau} \cdot \omega(c) < \frac{x_0}{0} \cdot \omega(c). \right. \tag{4.4}
\]

Assume that \( D \) with \( \bar{D} \subset \Omega \times [0, T] \) is visible at \((x_0, t_0) \) as \( \tau \to \infty \) from the complex direction \( z \) given by (1.2). Let \( v \) be given by (3.15) with \( g = \chi_D \). Then we have

\[
u(x_0, t_0) = \frac{1}{C} \lim_{\tau \to \infty} \tau^\mu e^{-i\tau x_0 t_0 + |z|} I(\tau), \tag{4.5}\]

where

\[
I(\tau) = \int_{\Omega} \left\{ \left( \frac{\partial v}{\partial t} (x, t) + \rho(x) v(x, t) \right) u(x, t) - h_0(x, t) v(x, t) \right\} ds \, dt \, - \int_U v(x, 0) u(x, 0) \, dx.
\]

**Proof.** Choose a sequence \( g_j \in C^\infty_0(\mathbb{R}^{n+1}) \) in such a way that \( g_j \to \chi_D \) in \( L^2_{\delta}(\mathbb{R}^{n+1}) \) as \( j \to \infty \). Define

\[
v_j(x, t) = e^{i\tau x_0 t_0 + |z|} (G_z \ast g_j)(x, t).
\]

(4.6)

A combination of theorem 3.1 and the Sobolev imbedding theorem in \( R^{n+1} \) we see that \( v_j \in C^\infty(\mathbb{R}^{n+1}) \) and \( \{v_j\}_{\Omega \times [0, T]} \) is Cauchy in \( H^{2,1}(\Omega \times [0, T]) \equiv L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \text{ see pages 6–7 in } [15] \). Since \( v_j|_{\Omega \times [0, T]} \to v|_{\Omega \times [0, T]} \) in \( L^2(\Omega \times [0, T]) \), we conclude that \( v \in H^{2,1}(\Omega \times [0, T]) \) and this \( v \) satisfies

\[
\left\| D^{\alpha}_x D^{\beta}_t (e^{i\tau x_0 t_0 + |z|} v(x, t)) \right\|_{L^2(\Omega \times [0, T])} = O(|z|^{-\delta}), \quad |\alpha| + 2\beta \leq 2. \tag{4.7}
\]

By the trace theorem (theorem 2.1 on p 9 in [15]) we have \( v_j|_{\Omega \times [0, T]} \to v|_{\Omega \times [0, T]} \) in \( H^{3/2,3/2}(\partial \Omega \times [0, T]) \equiv L^2(0, T; H^{3/2}(\partial \Omega)) \cap H^{3/4}(0, T; L^2(\partial \Omega)); \partial \nu v_j|_{\partial \Omega \times [0, T]} \to \partial \nu v|_{\partial \Omega \times [0, T]} \) in \( H^{1/2,1/2}(\partial \Omega \times [0, T]) \equiv L^2(0, T; H^{1/2}(\partial \Omega)) \cap H^{1/4}(0, T; L^2(\partial \Omega)); v_j(x, 0) \to v(x, 0) \text{ and } v_j(x, T) \to v(x, T) \in H^1(\Omega). \)
Note that \( v_j \in W(0, T; H^1(\Omega), (H^1(\Omega)^\prime))^\prime \) and satisfies (2.8) in the weak sense with \( f_1 = -e^{z-z(t, z)}g_j(x, t) \) and \( h_1 = \partial v_j/\partial v + \rho v_j \) on \( \partial \Omega \times [0, T] \).

Thus (2.9) yields

\[
\int_0^T \int_{\partial \Omega} \left\{ \frac{\partial v_j}{\partial \nu} + \rho v_j \right\} u(t)|_{\partial \Omega} - h_0(t)v_j(t)|_{\partial \Omega} \, dS \, dt = - \int_0^T \int_{\Omega} e^{z-z(t, z)}g_j(x)u(x, t) \, dx \, dt \\
+ \int_{\Omega} u(x, 0)v_j(x, 0) \, dx - \int_{\Omega} u(x, T)v_j(x, T) \, dx.
\]

Taking the limit \( j \to \infty \), we obtain

\[
\int_{\partial \Omega \times [0, T]} \left\{ \frac{\partial v}{\partial \nu}(x, t) + \rho(x)v(x, t) \right\} u(x, t) - h_0(x, t)v(x, t) \, dS \, dt \\
+ \int_{\Omega} u(x, T)v(x, T) \, dx - \int_{\Omega} u(x, 0)v(x, 0) \, dx \\
= - \int_D e^{z-z(t, z)}u(x, t) \, dx \, dt.
\]

(4.8)

Note that we made use of the fact that: every \( \phi \in L^2(0, T; L^2(\partial \Omega)) \) can be identified with \( \phi(x, t) \equiv \phi(t)(\tau) \in L^2(\partial \Omega \times [0, T]) \).

Divide \( \partial \Omega \times [0, T] = \Gamma \cup \{ (\partial \Omega \times [0, T]) \setminus \Gamma \} \) and \( \Omega = U \cup (\Omega \setminus U) \). From (4.8) we have

\[
I(\tau) = - \int_D e^{z-z(t, z)}u(x, t) \, dx \, dt + R
\]

(4.9)

where

\[
R = - \int_{\Omega} v(x, T)u(x, T) \, dx + \int_{\Omega \setminus U} v(x, 0)u(x, 0) \, dx \\
- \int_{\partial \Omega \times [0, T] \setminus \Gamma} \left\{ \frac{\partial v}{\partial \nu}(x, t) + \rho(x)v(x, t) \right\} u(x, t) - h_0(x, t)v(x, t) \, dS \, dt.
\]

(4.10)

Assumptions (4.2), (4.3), (4.4) together with the trace theorem, (4.7) and (4.10) ensure that \( |\tau^\mu e^{-\gamma z-z(t, z)}R| \) is decaying as \( \tau \to \infty \). Since the heat operator \( \partial_t - \Delta \) is hypoelliptic, we know \( u \in C^\infty(D) \). This together with (4.9) and (4.1) yields (4.5).

Formula (4.5) can be considered as an application of an idea in [8, 9] that was originally developed for the Cauchy problem for the stationary Schrödinger equation.

So the problem is reduced to: how to choose \( D \) that is visible at \((x_0, t_0)\) as \( \tau \to \infty \) from the complex direction \( z \) given by (1.2). In the following subsections we consider this problem.

### 4.1. The case \( n = 1 \)

Let \( \delta > 0 \). We denote by \( D(x_0, t_0, \omega(c), \delta) \) the inside of the triangle with vertices \( P = (x_0, t_0), P_0 = (x_0 - \delta/(\sqrt{1 + c^2}) \sqrt{1 + c^2} \omega, t_0) \) and \( P_1 = (x_0, t_0 + \delta \sqrt{1 + c^2}) \) in the spacetime \( \mathbb{R}^{1+1} \).

The two points \( P_0 \) and \( P_1 \) are located on the line \((x, t)^T \cdot \omega(c) = (x_0, t_0)^T \cdot \omega(c) - \delta \).

In [11] we have already known the following. For the proof see that of theorem 2.3 in [11].
Proposition 4.1. Let $D = D(x_0, t_0, \omega(c), \delta)$. If $\rho \in C^2(\mathbb{D})$, then
\[
\lim_{\tau \to -\infty} 2(c\tau)^2 e^{-(c\tau)^2 (\tau/c + \delta)} \int_D e^{z(t' z' \tau)} \rho(x, t) \, dx \, dt = -\frac{i||P_1 - P_0||\rho(P)}{|P_1 - P|(|\sqrt{c^2 + 1} + i(c/\delta)|P_0 - P|)}.
\]

Therefore this $D$ is visible at $(x_0, t_0)$ from complex direction $z$. The constants $\mu$ and $C$ in (4.1) are given by $\mu = 3$ and
\[
C = \frac{1 + i}{4e^3}.
\] (4.11)

4.2. The cases when $n = 2, 3$

The cases when $n = 2, 3$ start with describing the following which is easily derived by the proof of theorem 2.2 and lemma 4.1 in [11].

Proposition 4.2. Let $n \geq 2$. Let $D \subset \mathbb{R}^{n+1}$ be a finite cone with a vertex at $P = (x_0, t_0)$ and a bottom face $Q \neq \emptyset$ that is a bounded open subset of $n$-dimensional hyper plane $(x, t)^T \cdot \omega(c) = (x_0, t_0)^T \cdot \omega(c) - \delta$. If $\rho \in C^0_{\theta} (\mathbb{D})$ with $0 < \theta \leq 1$, then
\[
\lim_{\tau \to -\infty} \frac{2}{n!} (c\tau)^n e^{-(c\tau)^2 (\tau + \delta)} \int_D e^{z(t' z' \tau)} \rho(x, t) \, dx \, dt = K_D \rho(P),
\]
where
\[
K_D = 2\delta \int_Q \frac{dS(y)}{(\sqrt{c^2 + 1} - i(y - P) \cdot (\omega(c)))^{n+1}}.
\]

Therefore if $K_D \neq 0$, then constants $\mu$ and $C$ in (4.1) are given by $\mu = n + 1$ and
\[
C = \frac{n! K_D}{2c^{n+1}}.
\] (4.12)

However, it is not easy to show that $K_D \neq 0$ for $D$ with general $Q$. In the following we specify $Q$ and show that $K_D \neq 0$.

4.2.1. The case when $n = 2$. Let $\delta > 0$. Choose arbitrary two points $x_1, x_2$ on the line $x \cdot \omega = x_0 \cdot \omega - (\delta/c)\sqrt{1 + c^2}$ in such a way that the orientation of the two vectors $\omega, x_1 - x_2$ coincides with that of the standard basis $e_1, e_2$. We denote by $D(x_0, x_1, x_2, \omega(c), \delta)$ the inside of the tetrahedron in $\mathbb{R}^{n+1}$ with the vertices $(x_0, t_0), (x_1, t_0), (x_2, t_0)$ and $(x_0, t_0 + \delta \sqrt{1 + c^2})$. We see that the three points $(x_1, t_0), (x_2, t_0)$ and $(x_0, t_0 + \delta \sqrt{1 + c^2})$ are on the plane $(x, t)^T \cdot \omega(c) = (x_0, t_0)^T \cdot \omega(c) - \delta$. Therefore $D(x_0, x_1, x_2, \omega(c), \delta)$ coincides with the finite cone with a vertex at $(x_0, t_0)$ and a bottom face $Q$ that is the triangle in $\mathbb{R}^{n+1}$ with the vertices $(x_1, t_0), (x_2, t_0)$ and $(x_0, t_0 + \delta \sqrt{1 + c^2})$.

From (4.2) in [11] we obtain the formula
\[
K_D = K_D \vartheta \cdot (0, -1)^T
\]
\[
= c^3 \sum_{j=1}^{3} \frac{[(\nu_j \times \nu_j - 1) \times (\nu_j \times \nu_j - 1)]}{[(\nu_j \times \nu_j - 1) \cdot \vartheta][[(\nu_j \times \nu_j - 1) \cdot \vartheta]}} \nu_j \cdot (0, -1)^T,
\] (4.13)
where $D = D(x_0, x_1, x_2, \omega(c), \delta), \nu_1, \nu_2, \nu_3 = \nu_0$ are the unit outward normal vector to the faces of $D$ that are triangles $\Delta_1$ with the vertices $(x_0, t_0), (x_1, t_0), (x_0, t_0 + \delta \sqrt{1 + c^2}), \Delta_2$ with the vertices $(x_0, t_0), (x_2, t_0), (x_0, t_0 + \delta \sqrt{1 + c^2}), \Delta_3$ with the vertices $(x_0, t_0), (x_1, t_0), (x_2, t_0); \vartheta = (c(\omega + i\omega^0), -1)^T$. 


Since corollary 4.1 in [11] ensures this $K_D \neq 0$, we conclude that $D$ is visible at $(x_0, t_0)$ as $\tau \to \infty$ from the complex direction $z$. Note that $\nu_3 = (0, -1)^T$ and $\nu_1 \cdot \nu_3 = \nu_2 \cdot \nu_3 = 0$. Therefore from (4.12) and (4.13) we have the simpler expression

$$C = \frac{[(\nu_1 \times \nu_2) \times (\nu_1 \times \nu_3)]}{[(\nu_1 \times \nu_2) \cdot \partial][(\nu_1 \times \nu_3) \cdot \partial]}.$$  \hspace{1cm} (4.14)

Now choose $x_1, x_2, \delta$ in such a way that $D(x_0, x_1, x_2, \omega(c), \delta) \subset \Omega \times [0, T]$. Then we obtain formula (4.5) for $D = D(x_0, x_1, x_2, \omega(c), \delta), \mu = 3$ and $C$ given by (4.14).

4.2.2. The case when $n = 3$. Let $\delta > 0$. Choose an arbitrary three points $x_1, x_2$ and $x_3$ on the plane $x \cdot \omega = x_0 \cdot \omega = (\delta/c)\sqrt{1 + c^2}$ in such a way that the orientation of the three vectors $\omega, x_1 - x_2, x_3 - x_2$ coincides with that of the standard basis $e_1, e_2, e_3$. The four points $x_0, x_1, x_2$ and $x_3$ form a tetrahedron $\Delta$ in $\mathbb{R}^3$. We denote by $\nu$ the unit outward normal vector field to $\partial\Delta$. $\partial\Delta$ consists of four triangles: $T_1$ with the vertices $x_0, x_1$ and $x_2$; $T_2$ with the vertices $x_0, x_3$ and $x_2$; $T_3$ with the vertices $x_0, x_1$ and $x_3$; $T_4$ with the vertices $x_1, x_2$ and $x_3$. Since $\nu$ takes a constant vector on each $T_j$, we denote the vector by $\nu_j$. In particular, we have $\nu_4 = -\omega$.

We denote by $D(x_0, x_1, x_2, x_3, \omega(c), \delta)$ the inside of the finite cone in $\mathbb{R}^{3+1}$ with a vertex $(x_0, t_0 + \delta\sqrt{1 + c^2})$ and the bottom that is the inside of the tetrahedron in the space $\tau = t_0$ with vertices $(x_0, t_0), (x_1, t_0), (x_2, t_0)$ and $(x_3, t_0)$. Then the boundary of $D(x_0, x_1, x_2, x_3, \omega(c), \delta)$ consists of five tetrahedrons: $\Delta_1$ with the vertices $(x_0, t_0), (x_1, t_0), (x_2, t_0)$ and $(x_3, t_0 + \delta\sqrt{1 + c^2})$; $\Delta_2$ with the vertices $(x_0, t_0), (x_2, t_0), (x_3, t_0)$ and $(x_0, t_0 + \delta\sqrt{1 + c^2})$; $\Delta_3$ with the vertices $(x_0, t_0), (x_3, t_0), (x_1, t_0)$ and $(x_0, t_0 + \delta\sqrt{1 + c^2})$; $\Delta_4$ with the vertices $(x_0, t_0), (x_2, t_0), (x_1, t_0)$ and $(x_0, t_0 + \delta\sqrt{1 + c^2})$; $Q$ with vertices $(x_1, t_0), (x_2, t_0), (x_3, t_0)$ and $(x_0, t_0 + \delta\sqrt{1 + c^2})$.

We see that $D = D(x_0, x_1, x_2, x_3, \omega(c), \delta)$ coincides with the finite cone with a vertex at $(x_0, t_0)$ and the bottom $Q$. Let $\alpha$ be an arbitrary constant vector in $C^{3+1}$. Since

$$\nabla_{(x,t)} \cdot (e^{z - t'(x,t)}) \alpha = (z, -\tau)^T \cdot \alpha e^{z - t'(x,t)},$$

we have

$$(z, -\tau)^T \cdot \alpha \int_D e^{z - t'(x,t)} \, dx \, dt = \int_D \nabla_{(x,t)} \cdot (e^{z - t'(x,t)}) \alpha \, dx \, dt$$

$$= \sum_{j=1}^3 \alpha \cdot \nu_j, 0) \int_{\Delta_j} e^{z - t'(x,t)} \, dS(x, t) + \alpha \cdot (0, -1)^T \int_{\Delta_4} e^{z - t'(x,t)} \, dS(x, t)$$

$$- \alpha \cdot \omega(c) \int_Q e^{z - t'(x,t)} \, dS(x, t),$$

where we made use of the fact that the unit outward normal vector to $\partial D$ takes $(\nu_j, 0)^T$ on $\Delta_j$ for each $j = 1, 2, 3$; $-e_4 = (0, -1)^T$ on $\Delta_4$; $-\omega(c)$ on $Q$. Since $\alpha$ is arbitrary, one obtains

$$\left(\begin{array}{c} z \\ -\tau \end{array} \right) \int_D e^{z - t'(x,t)} \, dx \, dt$$

$$= \sum_{j=1}^3 \left(\begin{array}{c} \nu_j \\ 0 \end{array} \right) \int_{\Delta_j} e^{z - t'(x,t)} \, dS(x, t) + \left(\begin{array}{c} 0 \\ -1 \end{array} \right) \int_{\Delta_4} e^{z - t'(x,t)} \, dS(x, t)$$

$$- \omega(c) \int_Q e^{z - t'(x,t)} \, dS(x, t).$$ \hspace{1cm} (4.15)
Since $Q$ is included in the hyper plane $(x, t)^T \cdot \omega(c) = (x_0, t_0)^T \cdot \omega(c) - \delta$, we have
\[
e^{-x_0 \cdot z+b_0(z-c)} \omega(c) \int_{Q} e^{x}z^{-t}dS(x, t) = O(e^{-\delta \sqrt{\tau z}}).
\] (4.16)

We compute the integral
\[
I_j(\tau) = \int_{\Delta_j} e^{x}z^{-t}dS(x, t), \quad j = 1, 2, 3, 4.
\]
On $\Delta_j$ for each $j$ one can write
\[
\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_0 \\ t_0 \end{pmatrix} + \alpha_a + \beta b_j + \gamma c_j,
\]
where $(\alpha, \beta, \gamma) \in \Delta_0 = \{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma \leq 1, \alpha, \beta, \gamma \geq 0 \}$ and $a, b_j, c_j$ are suitable linearly independent vectors in $\mathbb{R}^{31}$ and satisfy the condition
\[
a_j \cdot \omega(c) < 0, \quad b_j \cdot \omega(c) < 0, \quad c_j \cdot \omega(c) < 0.
\] (4.17)

Writing $A_j = (a, b_j, c_j)$ which is a $4 \times 3$-matrix and $\tau \Delta_0 = \{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma \leq \tau, \alpha, \beta, \gamma \geq 0 \}$, we have
\[
e^{-x_0 \cdot z+b_0(z-c)} I_j(\tau) = \sqrt{\det A_j A_j} \int_{\Delta_0} e^{(a, b_j, c_j)} \cdot (z, -t)^T d\alpha d\beta d\gamma
\]
\[
= \frac{1}{\tau} \int_{\tau \Delta_0} e^{(a, b_j, c_j)} \cdot (z, -t)^T d\alpha d\beta d\gamma.
\]
Thus together with (4.17) yields
\[
\lim_{\tau \to \infty} \tau^3 e^{-x_0 \cdot z+b_0(z-c)} I_j(\tau) = \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma e^{(a, b_j, c_j)} \cdot \theta
\]
\[
= \frac{(-1)^3}{(a_j \cdot \theta)(b_j \cdot \theta)(c_j \cdot \theta)}
\] (4.18)

where $\theta = (c(\omega + i\omega^T), -1)^T$.

From (4.15), (4.16), (4.18) we obtain
\[
\lim_{\tau \to \infty} \tau^3 e^{-x_0 \cdot z+b_0(z-c)} \int_D e^{x}z^{-t}dS(x, t)
\]
\[
= -\sum_{j=1}^3 \sqrt{\det A_j A_j} (a_j \cdot \theta)(b_j \cdot \theta)(c_j \cdot \theta) (\nu_j)
\]
\[
= -\frac{\sqrt{\det A_4 A_4}}{(a_4 \cdot \theta)(b_4 \cdot \theta)(c_4 \cdot \theta)} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]
This yields the formula
\[
\lim_{\tau \to \infty} \tau^4 e^{-x_0 \cdot z+b_0(z-c)} \int_D e^{x}z^{-t}dS(x, t) = -\frac{\sqrt{\det A_4 A_4}}{(a_4 \cdot \theta)(b_4 \cdot \theta)(c_4 \cdot \theta)}.
\]
By choosing $\rho \equiv 1$ in proposition 4.2, one concludes
\[
K_0 = -\frac{2}{3!} \tau^4 \frac{\sqrt{\det A_4 A_4}}{(a_4 \cdot \theta)(b_4 \cdot \theta)(c_4 \cdot \theta)}.
\]
Therefore $D$ is visible at $(x_0, t_0)$ as $\tau \to \infty$ from the complex direction $z$ and (4.1) is valid for $\mu = 4$ and $C$ given by the formula
\[
C = -\frac{\sqrt{\det A_4 A_4}}{(a_4 \cdot \theta)(b_4 \cdot \theta)(c_4 \cdot \theta)}.
\]
Therefore (4.5) is valid for this $C$ and $\mu = 4$ provided $\delta$ is chosen in such a way that $D(x_0, x_1, x_2, x_3, \omega(c), \delta) \subset \Omega \times ]0, T].$
5. An integral representation of $G_z$ and a byproduct

It is quite important for us to compute $v$ given by (3.15) with $g = \chi_D$. In this section we give an integral representation of the distribution $G_z(x, t)$ together with

$$K_z(x, t) = e^{i x \cdot z - t|z|^2} G_z(x, t), \quad z = a + ib, \quad b \neq 0$$

(5.1)

which is a solution of the equation $\partial_t v + \Delta v + \delta(x, t) = 0$ in $\mathbb{R}^{n+1}$. Using the representation of $K_z$, we show that $K_z$ is in the hyper space $(x, t)^T \cdot \omega(c) < 0$ with $z$ given by (1.2) is exponentially decaying as $t \to \infty$. As a byproduct of this fact we see that $K_z$ yields a Carleman-type formula for the heat equation.

5.1. Representation of $G_z$

Let $H(t)$ denote the Heaviside function.

**Proposition 5.1.** It holds that

$$G_z(x, t) = e^{-i(x - 2\alpha b) \cdot |b|^2 t}$$

$$\times \left\{ -\left(\frac{|b|}{2\pi}\right)^n \int_{|\xi| < 1} e^{i(|b(x - 2\alpha b) \cdot |b|^2 t)|\xi|} d\xi + H(-t) \left(\frac{1}{2\sqrt{\pi|t|}}\right)^n e^{\frac{i\pi|b|^2}{2|t|}} \right\}.$$  \hspace{1cm} (5.2)

**Proof.** First we give a representation for $F_1(x, t) = G_{i\omega}(x, t)$ where $\omega = b/|b|$. The starting point is the following formulae for the Fourier transform. Let $\Re c \neq 0$. It is easy to see that if $\Re c > 0$, then

$$\frac{1}{-i\eta - c} = -\int H(t) e^{-ct} e^{-\eta t} dt;$$

if $\Re c < 0$, then

$$\frac{1}{-i\eta - c} = \int H(-t) e^{-ct} e^{\eta t} dt.$$

Thus $F_1(x, t)$ becomes

$$F_1(x, t) = \frac{1}{(2\pi)^{n+1}} \int e^{i(x \cdot \xi + i\eta t)} d\xi d\eta$$

$$= \frac{1}{(2\pi)^n} \int_{|\xi| < 1} e^{i\xi \cdot x} \left\{ -H(t) e^{i(|\xi + \omega|)^2 - 1} \right\} d\xi + \int_{|\xi| > 1} e^{i\xi \cdot x} \left\{ H(-t) e^{i(|\xi + \omega|)^2 - 1} \right\} d\xi$$

$$= -\frac{H(t)}{(2\pi)^n} \int_{|\xi| < 1} e^{i\xi \cdot x} e^{i(|\xi + \omega|)^2 - 1} d\xi + \frac{H(-t)}{(2\pi)^n} \int_{|\xi| > 1} e^{i\xi \cdot x} e^{i(|\xi + \omega|)^2 - 1} d\xi.$$  \hspace{1cm} (5.3)

This together with a change of variables yields

$$F_1(x, t) = -\frac{H(t) e^{-ix \omega}}{(2\pi)^n} \int_{|\xi| < 1} e^{ix \cdot \xi} e^{i|\xi|^2 - 1} d\xi$$

$$+ \frac{H(-t) e^{-ix \omega}}{(2\pi)^n} \int_{|\xi| > 1} e^{ix \cdot \xi} e^{i|\xi|^2 - 1} d\xi$$

and thus we obtain

$$e^{ix \cdot \omega t} F_1(x, t) = -\frac{H(t)}{(2\pi)^n} \int_{|\xi| < 1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi + \frac{H(-t)}{(2\pi)^n} \int_{|\xi| > 1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi.$$  \hspace{1cm} (5.4)

Since

$$\int e^{ix \cdot \xi} e^{-|\xi|^2 a} d\xi = \left(\frac{\pi}{a}\right)^n e^{-|x|^2/\pi}, \quad a > 0.$$
we have, for all \( t < 0 \)
\[
\frac{1}{(2\pi)^n} \int_{|\xi|>1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi = \frac{1}{(2\pi)^n} \left\{ \left(\frac{\pi}{|t|}\right)^n e^{i\frac{|x|^2}{2|t|}} - \int_{|\xi|<1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi \right\}
\]
\[
= \left(\frac{1}{2\sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}} - \frac{1}{(2\pi)^n} \int_{|\xi|<1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi.
\]
This together with (5.3) yields
\[
e^{ix \cdot \omega t} F_1(x, t)
\]
\[
= -\frac{1}{(2\pi)^n} \left[ (H(t) + H(-t)) \int_{|\xi|<1} e^{ix \cdot \xi} e^{i|\xi|^2 t} d\xi + H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}} \right] + H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}}
\]
\[
= -\frac{1}{(2\pi)^n} \left(\frac{|b|}{2\pi}\right)^n \int_{|\xi|<1} e^{ib \cdot \xi} e^{i|b|^2 |\xi|^2 t} d\xi + H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}}
\]
\[ (5.5) \]
It follows from (3.5) that
\[
G_{ib}(x, t) = |b|^n F_1(|b|x, |b|^2 t).
\]
A combination of this and (5.5) gives
\[
G_{ib}(x, t) = e^{-ix \cdot b - |b|^2 t} \left\{ \left(\frac{|b|}{2\pi}\right)^n \int_{|\xi|<1} e^{ib \cdot \xi} e^{i|b|^2 |\xi|^2 t} d\xi + H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}} \right\}
\]
From this we immediately obtain (5.2) since we have (3.4).

\[ \square \]

5.2. Representation of \( K_z \)

From (5.1), (5.2) and the equation
\[
x \cdot a - t|a|^2 + \frac{1}{4t} |x - 2\imath a|^2 = \frac{|x|^2}{4t},
\]
\[ (5.6) \]
it follows that
\[
K_z(x, t) = H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}} + w_z(x, t),
\]
\[ (5.7) \]
where
\[
w_z(x, t) = -e^{x \cdot a - |a|^2} \left(\frac{|b|}{2\pi}\right)^n \int_{|\xi|<1} e^{ib \cdot (x - 2\imath a) \cdot \xi} e^{i|b|^2 |\xi|^2 t} d\xi.
\]
\[ (5.8) \]
Remarks are in order.

- The \( w_z \) does not depend on the direction of the vector \( b \).
- Since the distribution
\[
H(-t) \left(\frac{1}{2 \sqrt{\pi |t|}}\right)^n e^{i\frac{|x|^2}{4|t|}}
\]
is the fundamental solution of the equation \( \partial_t v + \Delta v = 0 \), the \( w_z(x, t) \) is an entire solution of the backward heat equation. Moreover \( w_z \) is a smooth function on the whole space. Note also that \( w_z \) has an impressive representation:
\[
w_z(x, t) = -\left(\frac{|b|}{2\pi}\right)^n \int_{|\xi|<1} e^{i(z(\xi) - t\imath(\xi))} d\xi,
\]
where \( z(\xi) = a + i|b|\xi \).
From (5.8) and a change of variables we know that \( w_z(x, t) = w_z(x, t) \). Thus \( w_z \) is real valued and hence we have

\[
w_z(x, t) = e^{x - t|a|^2} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} \cos(|b|(x - 2\tau a) \cdot \xi) e^{ib(\xi^2 - 1) |\xi|^2} d\xi.
\]

Write

\[
|b|^2 |\xi|^2 t + i|b|(x - 2\tau a) \cdot \xi = t \left(|b|\xi + i \frac{1}{2t} (x - 2\tau a)\right)^2 + \frac{1}{4t} |x - 2\tau a|^2.
\]

Combining this with (5.6), we can rewrite (5.8) as

\[
w_z(x, t) = -e^{x - t|a|^2} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} e^{i(|\xi|^2 + (x - 2\tau a)/(2t))^2} d\xi.
\]

We study more the expression (5.7).

(i) The case when \( t > 0 \). Since

\[
x \cdot a - t|a|^2 + t|b|^2 = \tau \sqrt{1 + e^2} \left( \frac{x}{t} \right) \cdot \omega(c),
\]

it follows from (5.7) and (5.8) that

\[
K_z(x, t) = -e^{\tau \sqrt{1 + e^2} (x,t) \cdot \omega(c)} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} e^{ib(\xi - 2\tau a) \cdot \xi} e^{i1 - |\xi|^2 |b|^2} d\xi.
\]

(ii) The case when \( t < 0 \). It follows from (5.4) that

\[
\int_{|\xi|<1} e^{ib(\xi - 2\tau a) \cdot \xi} e^{ib|\xi|^2} d\xi
\]

\[
= \frac{1}{|b|^n} \left( \frac{\pi}{2} \right)^n e^{i(x - 2\tau a)^2/(4\tau)} - \int_{|\xi|>1} e^{ib(\xi - 2\tau a) \cdot \xi} e^{ib|\xi|^2} d\xi.
\]

This together with (5.7) and (5.8) yields

\[
K_z(x, t) = e^{\tau \sqrt{1 + e^2} (x,t) \cdot \omega(c)} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} e^{ib(\xi - 2\tau a) \cdot \xi} e^{ib|\xi|^2} d\xi.
\]

Remark 5.1. Using the well-known formula

\[
\int_{|\xi|=\eta} e^{ib\xi} d\xi = (2\pi)^n \frac{n/2^{n/2}}{|\eta|^{n/2}} J_{(n-2)/2}(|\eta| r), \quad \forall \eta \in \mathbb{R}^n,
\]

one can rewrite (5.8), (5.10) and (5.11) as one-dimensional integrals.

5.3. Exponential decaying of \( K_z \) in the hyper space \( (x,t) \cdot \omega(c) < 0 \) and a Carleman-type formula for the heat equation

In this subsection first we show that \( K_z(x, t) \) is exponentially decaying as \( \tau \to \infty \) if \( (x,t) \cdot \omega(c) < 0 \) and \( z \) is given by (1.2).

Proposition 5.2. Given \( \delta > 0 \) we have, as \( \tau \to \infty \)

\[
\sup_{(x,t) \cdot \omega(c) < -\delta} |K_z(x, t)| = O(e^{-\sqrt{\tau \delta^2}}). 
\]

Proof. Let \( (x,t) \) satisfy \( (x,t) \cdot \omega(c) < -\delta \).
Remark 5.2. All the derivatives of $K_z(x, t)$ are at most algebraically growing as $\tau \to \infty$.\(\square\)

(ii) The case when $t < 0$. We divide this case into two subcases: (a) $|b|^2(-t)$ is large; (b) $|b|^2(-t)$ is not large and can be arbitrary small.

First consider (a). Given $R > 0$ let $t$ satisfy $|b|^2 t < -R$. From (5.4) and (5.11) we obtain

$$|K_z(x, t)| \leq C_n e^{-\tau \sqrt{1+c^2}} |b|^n.$$ \hfill (5.12)

Next consider (b). We employ expressions (5.7) and (5.8). Let $t$ satisfy $-R \leq |b|^2 t < 0$. Using (5.9), we can rewrite (5.8) as

$$w_z(x, t) = -e^{i \sqrt{1+c^2} (x, t)\cdot \omega(c)} e^{-|b|^2 t} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} e^{ib|x-x_0|\xi} e^{ib|x|^2} d\xi.$$ \hfill (5.14)

Since $t < 0$ and $|b|^2 t \leq R$, it follows from (5.14) that

$$|w_z(x, t)| \leq C_n e^{-\tau \sqrt{1+c^2}} |b|^n e^R.$$ \hfill (5.15)

Since $cx \cdot \omega < cx \cdot \omega - t < -\sqrt{1+c^2}$, we have $|x \cdot \omega| > (\delta/c) \sqrt{1+c^2}$ and thus $|x| > (\delta/c) \sqrt{1+c^2}$. Using this together with $|b|^2 |t| \leq R$, we obtain

$$|t|^{-n} e^{i \sqrt{1+c^2} (x, t)\cdot \omega(c)} e^{-|b|^2 t} \left( \frac{|b|}{2\pi} \right)^n \int_{|\xi|<1} e^{ib|x-x_0|\xi} e^{ib|x|^2} d\xi \leq \frac{|b|^{2n}}{R^n} e^{-|b|^2 (\delta/c) \sqrt{1+c^2})/4R}.$$ \hfill (5.16)

This together with (5.7) and (5.15) yields that

$$|K_z(x, t)| \leq C_n \left( e^{-\tau \sqrt{1+c^2}} |b|^n e^R + \frac{|b|^{2n}}{R^n} e^{-|b|^2 (\delta/c) \sqrt{1+c^2})/4R} \right).$$ \hfill (5.17)

A combination of (5.13) and (5.16) gives

$$|K_z(x, t)| \leq C_n \left\{ e^{-\tau \sqrt{1+c^2}} |b|^n e^R \left( \frac{1}{R^n} + 1 \right) + \frac{|b|^{2n}}{R^n} e^{-|b|^2 (\delta/c) \sqrt{1+c^2})/4R} \right\}. \hfill (5.18)$$

Now proposition 5.2 is a direct consequence of (5.12) and (5.17). \(\square\)

Remark 5.2. All the derivatives of $K_z(x, t)$ also have a similar property: for each $\alpha \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}^n_+$

$$e^{i \sqrt{1+c^2} \xi} \sup_{(x, t)\cdot \omega(c) < \delta} \left| \partial_\alpha^\beta K_z(x, t) \right|$$

is at most algebraically growing as $\tau \to \infty$.

As a corollary of proposition 5.1 and remark 5.2 we obtain a Carleman-type formula.

Corollary 5.1. Let $(x_0, t_0) \in \Omega \times [0, T]$ be an arbitrary fixed point. Assume that $T > 0$, $\omega$, $\Gamma$ and $U$ satisfy (4.2), (4.3) and (4.4). Let $v(x, t) = K_z(x - x_0, t - t_0)$ for $z$ given by (1.2) and $u$ be a solution of (2.7). Then we have

$$u(x_0, t_0) = -\lim_{\tau \to \infty} J(\tau),$$ \hfill (5.19)

where

$$J(\tau) = \int_{\Gamma} \left\{ \left( \frac{\partial v}{\partial v}(x, t) + \rho(x)v(x, t) \right) u(x, t) - h_0(x, t) v(x, t) \right\} dS \ d\tau - \int_{U} v(x, 0) u(x, 0) dx.$$ \hfill (5.20)
Remark 5.3. Yarmukhamedov [18] considered the Cauchy problem for the Laplace equation in a three-dimensional bounded domain $D$ that is bounded by the plane $x_3 = 0$ and by smooth surfaces lying in the half-space $x_3 > 0$. He gave a formula for calculating the value of the solution at a given point inside the domain from the Cauchy data on the portion in $x_3 > 0$ of $\partial D$. For the purpose he made use of a special fundamental solution for the Laplace operator which has been introduced by himself in [17] and is parameterized by an entire function $E(w)$, $w \in \mathbb{C}$ satisfying $E(0) = 1$, $E(w) = E(\overline{w})$ and for each $R > 0$ and $m = 0, 1, 2 \sup |E^{(m)}(w)| |\text{Re } w| < R < \infty$. His fundamental solution $\Phi_E$ takes the form for $x = (x_1, x_2, x_3)$ with $x' = (x_1, x_2) \neq 0$:
\[
\Phi_E(x) = -\frac{1}{2\pi^2} \int_0^\infty \text{Im} \left( \frac{E(x_3 + i\sqrt{|x'|^2 + u^2})}{x_3 + i\sqrt{|x'|^2 + u^2}} \right) \frac{du}{|x'|^2 + u^2}.
\]
He chose the special $E(w) = e^{|w|}$ with $\tau > 0$. Then $\Phi_E(x)$ has the representation
\[
\Phi_E(x) = \frac{e^{\tau |x'|}}{2\pi^2} \int_0^\infty \left( x_3 \frac{\sin(\tau \sqrt{|x'|^2 + u^2})}{\sqrt{|x'|^2 + u^2}} - \cos(\tau \sqrt{|x'|^2 + u^2}) \right) \frac{du}{|x'|^2 + u^2}.
\]
From this one sees that the Cauchy data of $\Phi_E(y - x)$ on the portion in $y_3 = 0$ of $\partial D$ for an arbitrary fixed $x \in D$ decays exponentially as $\tau \to \infty$. This fact corresponds to proposition 5.2 and is an evidence that formula (5.18) can be considered as an extension to the heat equation of his formula.

It would be interesting to find a hidden ‘parameter’ for $K_z$ like $E$ for $\Phi_E$. That is: can one find a family of special fundamental solutions for the backward heat operator $\partial_t + \triangle$ that contains $K_z$ as a special member? In other words, can one find another fundamental solution for the backward heat operator that is decaying in one side of a hyper surface not plane?

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