Center Smooth Sets and Center Smooth Numbers of Graphs

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Abstract. For any proper set S of V in a graph G, the S-eccentricity, (in short eS(x)) of a vertex x in G is maxd(x,y). The S-center of G is \( C_s(G) = \{ v \in V \mid e_s(v) \leq e_s(x) \forall x \in S \} \) and \( S_2 \)-eccentricity, \( e_{S2}(v) \) (in short \( e_{S2}[v] \)) of a vertex v in S is maxd(v,u). \( S_2 \)-center of G is \( C_{S2}(G) = \{ v \in V \mid e_{S2}(v) \leq e_{S2}(x) \forall x \in V \} \). Then G is called a center-smooth graph if \( C_s(G) = C_{S2}(G) \) and the set S is defined to be a center-smooth set. We identify the center smooth sets of certain classes of graphs namely, \( K_{p,q} \), wheel graphs and lollipop graph and enumerate them for many of these graph classes. We also introduce the concept of center smooth number, which is defined as the number of distinct center smooth set of a graph G, and determine the center smooth number of some graph classes.

1. Introduction [2]

We consider only finite simple undirected connected graphs. For the graph G, V(G) denotes its vertex set and E(G) denotes its edge set. As usual, p=|V| and q=|E| denote the number of vertices and edges of a graph G, respectively. A pair \( u, v \) of vertices of G, the distance d(u,v) between u and v is the length of a shortest u-v path in G. The degree of a vertex u, denoted by deg(u) is the number of vertices adjacent to u. the eccentricity \( e(u) \)of a vertex u is \( \max\{d(u,v) \mid v \in V(G)\} \) A vertex v is an eccentric vertex of u if \( e(u) = d(u,v) \). The diameter of the graph G, \( diam(G) = \max \{ e(v) \mid v \in V(G) \} \) and the radius, \( r(G) = \min \{ e(v) \mid v \in V(G) \} \). The center of G, \( C(G) = \{ v \in V(G) \mid e(v) = r(G) \} \). G is said to be a self-centered graph if the radius and diameter of G are equal.

Definition 1.1.[3]. A vertex v is said to be an antipodal vertex of u if \( d(u,v) = diam(G) \).

Definition 1.2.[6] A vertex u of a graph G is called a universal vertex if u is adjacent to all other vertices of G.

Example 1.

Figure 1. universal vertex is v4
In figure 1, a vertex $v_3$ is a universal vertex.  

**Definition 1.3.**[6]. A graph $G$ is a universal graph if every vertex in $G$ is universal vertex.

The complete graph $K_n$ is universal graph.  

**Definition 1.4.**[4]. A graph $G$ is a unique eccentric vertex graph (in short, a UEV), if every vertex of $G$ has a unique eccentric vertex. The unique eccentric vertex of a vertex $u$ is denoted by $u_e$.  

**Note:** A unique eccentric vertex graph $G$ become a self centered graph if and only if each vertex of $G$ is an eccentric vertex of some other vertex in $G$ has been proved in[4]. A UEV graph need not be self-centered and a self-centered graph need not be a UEV graph. For example, all even paths are UEV graphs, but are not self-centered and the complete graph and complete bipartite graph are self-centered, but are not UEV graphs.  

**Definition 1.5.**[6]. An $S \subseteq V$ is a dominating set in $G$ if every vertex in $V \setminus S$ is adjacent to a vertex in $S$. If $S$ is a dominating set then $V$-$S$ need not be a dominating set.  

**Definition 1.6.**[5]. The $S$-eccentricity $e_S(v)$ of a vertex $v$ in $G$ is $\max_{x \in V}(d(v,x))$. The $S$-center of $G$ is $C_S(G) = \{v \in V \mid e_S(v) \leq e_S(x) \forall x \in V\}$.  

**Example 3.** In figure 2, $S=\{u_2, u_4, u_5\}$ and $V-S=\{u_1, u_3, u_6, u_7\}$. Then the $S$-center $C_S(G) = \{u_4\}$.  

**Definition 1.7.** The $S_1$-eccentricity, $e_{S_1}(v)$ of a vertex $v$ in $S$ is $\max_{x \in V}(d(v,x))$. The $S_1$-center of $G$ is $C_{S_1}(G) = \{v \in V \mid e_{S_1}(v) \leq e_{S_1}(x) \forall x \in V\}$. In figure 2, $S=\{u_1, u_2, u_3\}$ and $V-S=\{u_4, u_5, u_6, u_7\}$. Then the $S_1$-center $C_{S_1}(G) = \{u_2\}$.  

**Definition 1.8.** Let $S$ be a proper set of $G$. $G$ is called a center-smooth graph if $C_S(G) = C_{S_1}(G)$ and $S$ is said to be a center-smooth set.  

**Definition 1.9.** The lollipop graph $LQP_{p,d}$ is a graph obtained from the complete graph $K_{p-d}$ and the path $P_d$ by joining one end-vertex of $P_d$ to each vertex of $K_{p-d}$.  

**2. Results on center smooth sets**

**Theorem 2.1.** Let $K_{a,p}$ be a complete bipartite graph with bipartition...
(X, Y) where |X| = q > 1 and |Y| = p > 1. Then $K_{q, p}$ is center-smooth graph if S are
(i). $\{x\}, x \in X$
(ii). $\{y\}, y \in Y$
(iii). $\{x, y\}, x \in X, y \in Y$
Proof: (i) Let S=$\{x\}, x \in X$. for each vertex w, which are adjacent to x, then $e_S(w)=1=e_{x1}(w)$ and for each $z \neq w$, $e_S(z)=2=e_{x1}(z)$. Thus, we have $C_s(K_{q, p}) = \{w\} = C_{x1}(K_{q, p})$.
(ii) Let S=$\{y\}, y \in Y$. Then the result is similar to (i).
(iii) Let S=$\{x, y\}, x \in X$ and $y \in Y$. Then for all $v \in V$ $e_S(v)=e_{x1}(v)$ and therefore, $C_s(K_{q, p}) = C_{x1}(K_{q, p})$.

**Proposition 2.1.** A $v \in G$ is a universal vertex iff $\{v\}$ is a dominating set in G.
Proof: Let $v \in G$ be an universal vertex. Then by definition, v is adjacent to all vertices of G. By definition, $\{v\}$ is a dominating set.

**Proposition 2.2.** Let $G = K_2 - e(xy)$ be a graph. Then G is center-smooth graph.
Proof: Let S be a proper set of G. Then for every $u \neq x, y \in V$, $e_S(u)=1=e_S(u)$. Since x and y are antipodal vertices of G, then $e(x)=e(y)=2$. Thus $C_s(G) = \{v\} = C_s(G)$.

**Theorem 2.2.** The universal graph G is center-smooth graph.
Proof: Since each vertex v of G is an universal vertex. From the proposition (2.1), every vertex v is dominate N(v). Therefore, $e_{x1}(v) = V = e_S(v)$ and hence G is center-smooth graph. Now we shall identify the center smooth of wheel graphs. The wheel graph $W_p$ is $K_5$ and their result have already been identified. Therefore, we prove the case for $p \geq 5$.

**Theorem 2.3.** Let $W_p, p \geq 5$ be wheel graph on the vertex set
$\{v_1, v_2, \ldots, v_p\}$ where $v_p$ is the universal vertex. Then
$C_s(W_p) = C_s(W_p)$ if S are
(i). $\{v_i\}, 1 \leq i \leq p$
(ii). $\{v_i, v_{i+1}\}, v_i, v_{i+1} \in E(C_{p-1})$
(iii). $\{v_i, v_j\}, 1 \leq i \leq p-1$
(iv). $\{v_i, v_j, v_p\}, v_i, v_j \in V(C_{p-2})$
(v). $\{v_i, v_{i+1}, v_p\}, v_i, v_{i+1} \in E(C_{p-1})$

Proof: (i) For each vertex $v \neq \{v_i\}, 1 \leq i \leq p-1$, $e_{x1}(v) = 1 = e_S(v)$ and hence $C_s(W_p) = \{v\} = C_s(W_p)$. Further, for all $S \subseteq V$ be such that $S \neq \{v_p\}$, for each vertex $v_j, 1 \leq j \leq p-1$ which are adjacent to $v_i$, then $e_{x1}(v_j) = 1 = e_S(v_j)$ and for each $v_j \neq v_p$, $e_{x1}(v_j) = 2 = e_S(v_j)$. Thus, clearly $C_s(W_p) = \{v_j\} = C_s(W_p)$.
(ii) Let S=$\{v_1, v_2, \ldots, v_i, v_{i+1}\}$. Then
and for all $v \in V(G)$. That is, $v$ is a universal vertex. Then obviously, $\mathcal{C}_v(V_p) = \{v_p\} = \mathcal{C}_v(W_p)$

(iii) For $S = \{v_i, v_j\}, 1 \leq i \leq p-1$, we have three cases:

Case 1. If $v_i$ is adjacent to $v_j$, then $v_i, v_j \in V(C_{p-1})$. Thus the proof is similar to (ii)

Case 2. If $v_i$ is adjacent to $v_j$, then $v_i, v_j \in V(C_{p-1})$, and there exist a vertex $v_k$ in $V(C_{p-1})$ such that $d(v_i, v_k) = d(v_j, v_k) = 1$ and for all other vertices $v \neq v_k$ $d(v, v) = 2$. But $d(v_i, v) = 1$ because $v_j \in V(C_{p-1})$. Thus, clearly $e_{v_i}(v_k) = 1 = e_{v_i}(v_k) = e_{v_i}(v_j) = e_{v_i}(v_j)$ and hence $\mathcal{C}_{v_i}(W_p) = \{v_j, v_k\} = \mathcal{C}_{v_i}(W_p)$.

Case 3. Suppose $v_i$ is not adjacent to $v_j$, then $v_i, v_j \in V(G)$. Let $v \in V(C_{p-1})$ be adjacent to both the vertices $v_i$ and $v_j$. Obviously, $e_{v_i}(v) = 1 = e_{v_i}(v)$. Since $v_i$ is adjacent to all other vertices in $W_p$, $e_{v_i}(v) = 1 = e_{v_i}(v)$. Hence, $\mathcal{C}_{v_i}(W_p) = \{v, v_i\} = \mathcal{C}_{v_i}(W_p)$.

(iv) Let $S = \{v_i, v_j, v_p\}, v_i, v_j \in V(C_{p-1})$. Then the arguments are similar to (iii).

(v) For $S = \{v_i, v_i+1, v_p\}, v_i, v_i+1 \in E(G)$, no one is adjacent to each vertex in $S$. But, since, $v_p$ is adjacent to all other vertices in $W_p$, then $e_{v_i}(v_p) = 1 = e_{v_i}(v_p)$. Therefore, $\mathcal{C}_{v_i}(W_p) = \{v_p\} = \mathcal{C}_{v_i}(W_p)$.

**Theorem 2.4.** Let $G$ be a cubic graph. Then $\mathcal{C}_{v_i}(G) = \mathcal{C}_v(G)$ if $S$ are

(i) $\{x\}$, $x \in V$

(ii) $\{x, y\}$, $xy \in E(G)$

(iii) $\{x, y, z\}$, $xy$ and $xz$ $z \in E(G)$

(iv) $\{x, y, z, r\}$, $xy$, $xz$, $zr$ and $yr \in E(G)$

Proof: (i) Let $S = \{x\}$, $x \in V$. For each vertex $v(\neq x \in V(G))$ which is adjacent to $x$, $e_{v_i}(v) = 1 = e_{v_i}(v)$ and $e_{v_i}(u) = e_{v_i}(u) = 1$. Hence $\mathcal{C}_{v_i}(G) = \{v\} = \mathcal{C}_v(G)$.

(ii) For $S = \{x, y\}$, $xy \in E(G)$ such that $e_{v_i}(v) = 3 = e_{v_i}(v)$ and for $v(\neq u)$ is non-adjacent vertex of both $x$ and $y$, $e_{v_i}(u) = 2 = e_{v_i}(u)$. Hence, $\mathcal{C}_{v_i}(G) = \{u\} = \mathcal{C}_v(G)$.

(iii) Let $S = \{x, y\}$ be such that $x, y, z \in S$. Then there exists a vertex $v(\neq u)$ such that $d(v, S) = 3$. That is, $v$ is an eccentric vertex of $S$ and the $d(u, S) = 2$. Clearly, $\mathcal{C}_{v_i}(G) = \{u\} = \mathcal{C}_v(G)$.

(iv) For $S = \{x, y, z\}$ be such that $xy$, $xz$, $zr$ and $yr \in E(G)$. Then for every $v \in V$, $e_{v_i}(v) = 3 = e_{v_i}(v)$ and therefore $\mathcal{C}_{v_i}(G) = V = \mathcal{C}_v(G)$.

**Theorem 2.5.** Let $G$ be a lollipop graph with vertex set $\{v_1, v_2, \ldots, v_p\}$. Then $\mathcal{C}_{v_i}(G) = \mathcal{C}_v(G)$ if $S$ are
(i) \( \{ v_i \}, 1 \leq i \leq p \)

(ii) \( \{ v_i, v_j \}, v_i v_j \in E(G) \)

(iii) \( \{ v_i, v_j, v_k \}, v_i, v_j, v_k \in V(G) \)

Proof (i) Let \( S \subset V \) be such that \( v \in S \). For each vertex \( v \neq u \), \( e_{S_1}(u) = 1 = e_{S}(u) \) and \( e_{S_1}(v) = e_{S}(v) \). Hence \( e_{S_1}(G) = \{ v \} = e_{S}(G) \).

(ii) Let \( S = \{ v_i, v_j \}, v_i v_j \in E(G) \). If \( v_i \) and \( v_j \) are the eccentric vertices of \( G \), then there exist a vertex \( v \neq v_i, v_j \). \( e_{S_1}(v) = e_{S}(v) = 2 = e_{S_1}(v_i) = e_{S}(v_i) \). So, \( e_{S_1}(G) = \{ v \} = e_{S}(G) \).

Suppose \( v_i \) is an eccentric vertex and \( v_j \) is a central vertex of \( G \), then there exist a vertex \( v \neq v_i, v_j \). \( e_{S_1}(v) = e_{S}(v) = 2 = e_{S_1}(v_i) = e_{S}(v_i) \). Thus \( e_{S_1}(G) = \{ v_i, v_j \} = e_{S}(G) \).

(iii) Let \( S = \{ v_i, v_j, v_k \}, v_i, v_j, v_k \in V(G) \). If \( v_i, v_j, v_k \in V(G) \) and \( v_k \in V(G) \), then \( \{ v_k \} = e_{S_1}(G) \) and \( e_{S_1}(G) = V \). So \( v_k \in V(G) \) and therefore \( e_{S_1}(G) = \{ v_k \} = e_{S}(G) \).

Theorem 2.6. For any graph \( G \) having \( n-1 \) end vertices, \( e_{S_1}(G) = e_{S}(G) \) iff \( S \) does not contains \( n-1 \) end vertices.

Proof: Suppose that \( G \) has \( n-1 \) end vertices. If not it has \( e_{S_1}(G) \neq e_{S}(G) \). Then we have to prove \( S \) contains only \( n-1 \) end vertices. Clearly, all the end vertices are eccentric vertices. Thus, all the end vertices i.e., \( n-1 \) vertices are adjacent to exactly one vertex \( v \).

Then \( v \) is a universal vertex or a central vertex. If \( S = \{ v \} \), then \( e_{S_1}(G) = e_{S}(G) \). It implies a contradiction. Suppose \( S = \{ v_i, v_j \} \), \( v_i v_j \in E(G) \). Then \( e_{S_1}(G) = e_{S}(G) \). It also implies a contradiction.

Suppose \( S = \{ v_i, v_k \}, v_i, v_k \) are nonadjacent vertices then \( e_{S_1}(G) = e_{S}(G) \). Again it implies a contradiction. Assume \( S = \{ v_i, v_j, ..., v_{n-1} \} \) where \( v_i, v_j, ..., v_{n-1} \) are \( n-1 \) end vertices. Then all the eccentric vertices have eccentricity 2. But the distance between eccentric vertex to central vertex is 1. Therefore, \( \{ v \} = e_{S_1}(G) \) and \( e_{S_1}(G) = V \). Hence, \( e_{S_1}(G) \neq e_{S}(G) \) when \( S \) contains only \( n-1 \) end vertices. The converse is obvious.

Theorem 2.7. If \( G \) is both self-centered and a UEV graph, then \( G \) is center-smooth graph.

Proof: By the result in [4], for every \( x \in V \) there exists a \( y \in V \) such that \( x = y \). Since \( G \) is self-centered, we have \( r(G) = r = \text{diam}(G) \). That is, \( e(v) = r, \forall v \in V \). Let \( S \subseteq V \) and \( x \in V \setminus S \). Then \( e_S(x) = r \) and since \( y = x \in V \setminus S \), \( e_{S-\{x\}}(x) = r \). Since \( G \) is self-centered, then for every \( x \in V \), \( e_S(x) = r = e_{S-\{x\}}(x) \). Hence \( e_{S-\{x\}}(G) = e_S(G) \) which shows that \( G \) is center-smooth.

Remark 1. The converse of the theorem (2.7) is false. Here we shall take two cases:
Case 1: For a graph $G$ in figure 3, $G$ is center-smooth and UEV graph but it is not self-centered.

![Figure 3. UEV-graph](image)

Case 2: For a graph $G$ in figure 4, $G$ is center-smooth and self-centered graph but it is not UEV graph.

![Figure 4. Self-centered graph](image)

3. Enumerating center smooth set

In this section we enumerate the center smooth sets of various classes of graphs. We first give the following definition. The number of distinct center smooth sets of a graph $G$ is defined as the center smooth number of $G$ and is denoted by $\text{csn}(G)$. The following results gives center smooth number of some familiar classes of graphs. The proofs of the lemma 1 and lemma 2 follows from the theorem (2.2) and proposition (2.2) respectively, so we leave the proofs.

Lemma 3.1. The center smooth number $\text{csn}(G)$ when $G=K_p+q+2k$ where $k$ is the number of intersect lines in $G$.

Lemma 3.2. For the graph $G-e$ where $e \in E$, $\text{csn}(G-e)=p+q+2k$ where $k$ is the number of intersect lines($I_i$) in $G$.

Lemma 3.3. For the graph $K_p-e$ where $e \in E$,

$$\text{csn}(K_p-e)=\begin{cases} p+q & \text{if } p=3 \\ p+q+2k & \text{if } p>3 \text{ and } e \notin I_1 \\ p+q+2k+2 & \text{if } p>3 \text{ and } e \in I_1 \end{cases}$$

where $k$ is the number of intersect lines($I_i$) in $K_p-e$.

Proof: Let $C_n$, $n \geq 3$ be a graph, since $C_n$ is self-centered graph, each vertex is adjacent to exactly two vertices of $C_n$. Let $S$ be a proper set of $C_n$. By the definition of $\text{csn}$, If $n=3$, then $C_3$ has 3 center smooth sets. If $n=4$, then $C_4$ has 4 center smooth sets. And so if $n=5$, then $C_5$ has 5 center smooth sets. These arguments go on that way. Also $S$ contains $n-1$ vertices which is true only if $n=3$ and 4. Finally, we get center smooth sets of $C_n$. Hence the result.

Conclusion

In this paper, $S_1$-eccentricity of a vertex is defined and the properties of a center-smooth set are obtained. Also the center-smooth graph have been introduced and their properties are studied. The center
smooth sets of various classes of graphs were enumerated. In future, these concepts can be studied with respect to edges to develop more results.

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