We report on the use of Feynman-Hellmann techniques to calculate the off-forward Compton amplitude (OFCA) in lattice QCD. At leading-twist, the Euclidean OFCA is parameterised by the Mellin moments of generalised parton distributions (GPDs). Hence we extract GPD moments for two values of the soft momentum transfer, \( t = -1.10, -2.20 \ \text{GeV}^2 \) and zero-skewness kinematics at an unphysical pion mass of \( m_\pi = 470 \ \text{MeV} \). This includes the first determination of the \( n = 4 \) moments.
1. Introduction

Generalised parton distributions (GPDs) [1–3] are observables that contain a staggering amount of hadronic information, including the spatial distribution [4] and spin structure [2] of constituent quarks and gluons, and the pressure distributions within hadrons [5]. However, experimental probes of GPDs are fraught with difficulties. In particular, global fits require assumptions about the functional form of GPDs that are beyond our current understanding [6]. For this reason, there has been strong interest in lattice QCD studies of GPDs. Historically, lattice studies have been limited to their lowest Mellin moments; the highest calculated so far are the $n = 3$ moments [7–11]. More recently, there has been a great deal of interest in calculating parton distributions from equal-time, non-local correlators in lattice QCD [12, 13], including calculations of quasi-GPDs [14–16].

Here, we report on a lattice QCD calculation of the off-forward Compton amplitude (OFCA),

$$T^{\mu\nu} \equiv i \int d^4 z e^{i (q+q') \cdot z} \langle P' | T \{ j^\mu (z/2) j^\nu (-z/2) \} | P \rangle,$$

(1.1)

which describes the process of nucleon-photon scattering: $\gamma^*(q)N(P) \rightarrow \gamma^*(q')N(P')$, with $q_\mu \neq q'_\mu$ (see Figure 1). At high energies ($|q^2|$ and/or $|q'^2| \gg \Lambda_{\text{QCD}}^2$), this amplitude is dominated by a convolution of GPDs [2]. Therefore, we can use a lattice calculation of this amplitude to determine GPD-related quantities.

The method we use to calculate the OFCA is an extension of Feynman-Hellmann methods, which have previously been used to calculate the forward Compton amplitude [17–21], and off-forward elastic form factors [22]. This involves computing nucleon propagators in the presence of weakly-coupled background fields. By isolating the contribution that is quadratic in this coupling, we can calculate four-point functions. As such, Feynman-Hellmann methods provide a realistic alternative to the direct computation of four-point functions.

The numerical results presented here are at the SU(3) flavour symmetric point and a larger-than-physical pion mass [23]. In terms of kinematics, we are at the zero-skewness point, which is not accessible to experiment but is the limit in which GPDs encode spatial distributions of quarks [4]. Moreover, we calculate two values of the soft momentum transfer, $t = -1.10, -2.20 \text{ GeV}^2$, with a hard momentum transfer of $\hat{Q}^2 \approx 6 - 7 \text{ GeV}^2$.

For this preliminary calculation, we consider this hard scale sufficiently large to assume that the extracted amplitude is dominated by its GPD contributions. Therefore, we also present Mellin moment fits, which we interpret as GPD moments. This includes the first determination of the $n = 4$ moments. A more detailed discussion of the work presented here can be found in Ref. [24].
2. Feynman-Hellmann Methods

In this section we will give a brief derivation of the Feynman-Hellmann relation that allows us to access the OFCA. We start with the perturbed quark propagators that we calculate:

\[
\mathcal{S}_\Lambda = \left[ \begin{array}{c}
\mathbf{M} & -\lambda_1 \mathcal{F}_1(\vec{q}_1) - \lambda_2 \mathcal{F}_2(\vec{q}_2) \\
\mathbf{F} & \mathbf{M}^{-1}
\end{array} \right]^{-1} = \mathbf{M}^{-1} + \sum_i \lambda_i \mathbf{M}^{-1} \mathcal{F}_i(\vec{q}_i) \mathbf{M}^{-1} + \sum_{i,j} \lambda_i \lambda_j \mathbf{M}^{-1} \mathcal{F}_i(\vec{q}_i) \mathbf{M}^{-1} \mathcal{F}_j(\vec{q}_j) \mathbf{M}^{-1} + \ldots
\]

(2.1)

Here, our couplings, \(\lambda_{1,2}\), are small, and \(\vec{q}_1 \neq \vec{q}_2\) are our inserted momenta. We choose our perturbing matrices to be \([\mathcal{F}_i(\vec{q}_j)]_{n,m} = \delta_{x_n x_m} 2 \cos(\vec{q}_j \cdot \vec{x}_n)\).

Taking a mixed, second-order derivative gives

\[
\frac{\partial^2}{\partial \lambda_1 \partial \Lambda_2} \mathcal{S}_\Lambda \bigg|_{\lambda=0} = \mathbf{M}^{-1} \mathcal{F}_1(\vec{q}_1) \mathbf{M}^{-1} \mathcal{F}_2(\vec{q}_2) \mathbf{M}^{-1} + (1 \leftrightarrow 2),
\]

(2.2)

which is a four-point function with momentum transfer.

We can insert these quark propagators either as up or down quarks into a nucleon propagator:

\[
\mathcal{G}_A^d \simeq \langle S^d u^a S^d_A \rangle, \quad \mathcal{G}_A^u \simeq \langle S^u u^a S^u_A \rangle,
\]

where we have suppressed the spin and flavour structure of the nucleon propagators.

Then, as in Eq. (2.2), we can take a mixed, second-order derivative to get

\[
\frac{\partial^2}{\partial \lambda_1 \partial \Lambda_2} \mathcal{G}_0(\tau, \vec{p}' \prime) \bigg|_{\lambda=0} \simeq \frac{\tau}{2 E_N(\vec{p}' \prime)} \frac{\sum_{s',s} \text{tr} \left[ \Gamma u(\vec{p}' \prime, s') T_{\mu \nu}(\vec{q}_1, \vec{q}_2) \bar{u}(\vec{p}' \prime, s) \right]}{\sum_s \text{tr} \left[ \Gamma u(\vec{p}' \prime, s) \bar{u}(\vec{p}' \prime, s) \right]},
\]

(2.3)

where

\[
T_{\mu \nu}(\vec{q}_1, \vec{q}_2) = \sum_{z} e^{\frac{i}{2} (\vec{q}_1 + \vec{q}_2) \cdot \vec{z}} \langle N(\vec{p}' \prime)| T\{J_{\mu}(z) J_{\nu}(0)\}| N(\vec{p}' - \vec{q}_1 + \vec{q}_2) \rangle,
\]

a discretisation of the OFCA, Eq. (1.1). Note that a more complete derivation of Eq. (2.3) is presented in Ref. [24].
We approximate the mixed, second-order derivative with the ratio

\[ R_{\lambda} \triangleq \frac{\mathcal{G}(\lambda, \lambda) + \mathcal{G}(-\lambda, -\lambda) - \mathcal{G}(\lambda, -\lambda) - \mathcal{G}(-\lambda, \lambda)}{\mathcal{G}(0, 0)}, \]

and use a linear fit in Euclidean time, \( \tau \), to extract the OFCA (Fig. 2).

After fitting in Euclidean time, we can fit \( R_{\lambda} \) across multiple \( \lambda \) to a quadratic function,

\[ g_{\lambda} = b\lambda^2, \]

as shown in Fig. 2. The results are well-described by a quadratic, which confirms that
we are extracting the \( \lambda_1\lambda_2 \) contribution that is proportional to the OFCA.

3. Parameterisation of the Compton amplitude

In the previous section, we outlined a method to calculate the OFCA in lattice QCD. Now, we
briefly discuss how to parameterise the OFCA in terms of GPD moments.

To begin, we define four linearly independent Lorentz scalars that our OFCA is a function of:

\[ \tilde{\omega} = \frac{2(P + P') \cdot (q + q')}{(q + q')^2}, \quad \xi = \frac{q^2 - q'^2}{(P + P') \cdot (q + q')}, \quad t = (P' - P)^2, \quad \tilde{Q}^2 = -\frac{1}{4}(q + q')^2. \] (3.1)

It is well-known that, for large \( \tilde{Q}^2 \), the off-forward Compton amplitude is dominated by
contributions from GPDs [2]:

\[ T_{\mu\nu}(\tilde{\omega}, \xi, t, \tilde{Q}^2) \approx g_{\mu\nu} \tilde{\omega}^2 \int dx \frac{xG(x, \xi, t)}{1 - x^2\tilde{\omega}^2 - i\epsilon} + \cdots + O(1/\tilde{Q}^2), \]

where \( G \) is a GPD. Or in the Euclidean region, \(|\tilde{\omega}| < 1\),

\[ T_{\mu\nu}(\tilde{\omega}, \xi, t, \tilde{Q}^2) \approx g_{\mu\nu} \sum_n \tilde{\omega}^n \int dx x^{n-1} G(x, \xi, t) + \cdots + O(1/\tilde{Q}^2). \]

A complete leading-twist operator product expansion (OPE) of the OFCA with leading-order
Wilson coefficients has been calculated, and is presented in Ref. [24]. Here, we will present the
final result of that work, which is relevant to interpreting the lattice results.

From Eq. (2.3), we can see that the quantity of interest is

\[ \mathcal{R}(\tilde{\omega}, t, \tilde{Q}^2) \equiv \frac{\sum_{s,s'} \text{tr}[\Gamma u(P', s') T_{33} \bar{u}(P, s)]}{\sum_s \text{tr}[\Gamma u(P', s') \bar{u}(P', s)]}. \] (3.2)

First, we note a few extra conditions we apply to our numerical results

1. We use zero-skewness kinematics (\( \xi = 0 \)). From Eq. (3.1), this is equivalent to \( \tilde{q}_1^2 = \tilde{q}_2^2 \).
2. We use the spin-parity projector \( \Gamma = \frac{1}{2}(\mathbb{1} + \gamma_4) \).
3. We subtract off the \( \tilde{\omega} = 0 \) contribution: \( \mathcal{R}(\tilde{\omega}, t, \tilde{Q}^2) = \mathcal{R}(\tilde{\omega}, t, \tilde{Q}^2) - \mathcal{R}(\tilde{\omega} = 0, t, \tilde{Q}^2) \).

The final parameterisation we fit to is then

\[ \mathcal{R}'(\tilde{\omega}, t, \tilde{Q}^2) = 2K_{33} \sum_{n=2,4,6} \tilde{\omega}^n M^q_n(t), \]

(3.3)
Table 1: Details of the gauge ensemble used in this work.

| \(N_f\) | \(\kappa_1\) | \(\kappa_s\) | \(L^3 \times T\) | \(a\)  | \(m_\pi\) | \(m_\pi L\) | \(Z_V\) | \(N_{\text{cfg}}\) |
|-------|------------|------------|---------------|------|--------|---------|-------|-------------|
| 2 + 1 | 0.1209     | 0.1209     | 32^3 \times 64 | 0.074(2) | 0.467(12) | \(\sim 5.6\) | 0.8611(84) | 1763        |

where \(K_{33}\) is a kinematic factor we can divide out, and we define

\[
M_n^q(x, t) \equiv \int_{-1}^{1} dx x^{n-1} \left[ H^q(x, t) + \frac{t}{8m_N^2} E^q(x, t) \right],
\]

the moments of a linear combination of the unpolarised GPDs at zero-skewness. Calculating the independent contributions of \(H\) and \(E\) GPD moments is a goal of future work.

4. Results: Moment Fits and Compton Amplitude

Details of the gauge ensembles used in this work are given in Tab. 1. We calculate two sets of perturbed correlators, with two pairs of inserted momenta, \(\vec{q}_{1,2}\):

| Set | \(\vec{q}_1, \vec{q}_2\) | \(t [\text{GeV}^2]\) | \(\vec{Q}^2 [\text{GeV}^2]\) | \(N_{\text{meas}}\) |
|-----|----------------|----------------|----------------|---------|
| #1  | (1, 5, 1), (−1, 5, 1) | −1.10 | 7.13 | 996 |
| #2  | (4, 2, 2), (2, 4, 2) | −2.20 | 6.03 | 996 |

We then vary the \(\vec{\omega}\) variable by varying our sink momentum \(\vec{p}'\). Since \(|\vec{\omega}| < 1\), we can truncate the expansion in \(\vec{\omega}\), Eq. (3.3), at some power \(J\) for \(\vec{\omega}^{2J}\). Then, using Markov chain Monte Carlo methods [25, 26], we can fit the moments defined in Eq. (3.4). We assume monotonically decreasing moments:

\[
M_2(t) \geq M_4(t) \geq \ldots \geq M_{2J}(t).
\]

However, future work will aim to incorporate model-independent constraints on GPDs [27–29] and on the Compton amplitude [30] to derive better prior conditions.

In our case, we fit the first four moments, \(n = 2, 4, 6, 8\), and report the first two. Plots of \(\overline{\mathcal{R}}\)
and the extracted moments are given in Fig. 3, where we observe that the values of $\mathcal{R}$ from our lattice simulation are well described by the parameterisation with the moments. Moreover, the $M_2$ moments are consistent with those calculated using three-point methods [11]. The $M_4$ moments have never been calculated before, and hence this study is a first look at the $t$ behaviour of these moments. We observe a decrease with $-t$ for both moments, as expected. At present, however, the results are too noisy and exploratory to draw strong distinctions between the $t$ dependence of the two moments.

5. Conclusion and Outlook

In these proceedings, we have reported on the determination of the off-forward Compton amplitude in lattice QCD. This calculation employed an extension of Feynman-Hellmann methods that have previously been applied to numerous matrix elements, including the forward Compton amplitude.

Although this calculation is highly exploratory, the initial results are very promising. Future work will be aimed at:

1. Controlling the systematic errors, including higher-twist corrections and the anomalous asymptotic behaviour of the subtraction function [21].

2. Separating out the moments of the helicity-conserving and -flipping GPDs, $H$ and $E$, respectively (see Eq. (3.4)).

3. Calculating a greater kinematic spread of $t$ and $Q^2$ values.

This will allow us to fit many more GPD moments, and report their higher-twist contributions more accurately. Moreover, it would allow us to constrain GPD models, and apply other methods to access GPDs directly from the Euclidean OFCA [31].

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