A conjectured duality between supergravity and $N = \infty$ gauge theories gives predictions for the glueball masses as eigenvalues for a supergravity wave equations in a black hole geometry, and describes a physics, most close to a high-temperature expansion of a lattice QCD. We present an analytical solution for eigenvalues and eigenfunctions, with eigenvalues given by zeroes of a certain well-computable function $r(p)$, the zeroes of which signifies that two solutions with desired behaviour at two singular points become linearly dependent. Our computation shows corrections to the WKB formula $m^2 = 6n(n + 1)$ for eigenvalues corresponding to glueball masses in 3 dimensional QCD, and gives the first states with masses $m^2 = 11.58766; 34.52698; 68.974962; 114.91044; 172.33171; 241.23607; 321.62549, \ldots$. In $QCD_4$, our computation gives squares of masses $37.169908; 81.354363; 138.473573; 208.859215; 292.583628; 389.671368; 500.132850; 623.97315 \ldots$ for $O^{++}$. In both cases, we have a powerful method which allows to compute eigenvalues with an arbitrary precision, if so needed, which may provide quantative tests for the duality conjecture. Our results matches well with the numerical computation of \cite{5} withing precision reported there in both $QCD_3$ and $QCD_4$ cases. As an additional curiosity, we report that for eigenvalues of about 7000, the power series, although convergent, has coefficients of orders $10^{34}$; and also the final answer gets small, of order $10^{-6}$ in $QCD_4$. Tricks were used to get reliably the function $r(p)$ above 7000. In principle we can go to infinitely high eigenavals at an expence of computer sufferings; although eventually such computation will slow down to make it impractical, also as corrections may be expected for higher states, since fine cancellations of very big terms is what produces higher eigenvalues.
1 Glueball masses in $QCD_3$

The conjectures of Maldacena and Witten \[3\], \[1\] gives, in particular, a prediction for glueball masses in $N = \infty$ QCD in 3 dimensions from supergravity, namely, from certain solutions of classical equations of motion in a black hole metric for a massless scalar dilaton field $\Phi$. The dilaton shows up here as it couples to the $O^{++}$ glueball operator; other glueballs, which couples to other fields, say a two-form in supergravity theory, are described, for example, in \[3\]. The classical equations of motion for dilaton are:

$$ \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \Phi) $$

in a black hole metric

$$ \frac{ds^2}{l_s^2 \sqrt{4\pi g_s N}} = \left( \rho^2 - \frac{1}{\rho^2} \right)^{-1} d\rho^2 + \left( \rho^2 - \frac{1}{\rho^2} \right) d\tau^2 + \rho^2 \sum_{i=1}^{3} dx_i^2 + d\Omega_5^2 $$

Here $x_1, x_2, x_3$ is where our $QCD_3$ lives, and it roughly should be imagined as a 4-ball $x_1, x_2, x_3, \rho$ with $x_1, x_2, x_3$ on $S^3$ boundary; $\tau$ is a circle, on which we pose antiperiodic boundary conditions for fermions and periodic for bosons, thus breaking supersymmetry; $d\Omega_5$ is a sphere $S^5$ with a standard metric. We need to take solutions of the form

$$ \Phi = f(\rho)e^{ikx} $$

with the appropriate physical conditions for $\Phi$ at $\rho = \infty$, (normalizability), and at $\rho = 1$ (single-valuedness for complex $\rho$) \[4\]. By introducing a variable $X = \rho^2$, (and calling it just $x$ below) we get certain well-posed Sturm-Liouville problem:

$$ \frac{d^2 f}{dx^2} + \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \right) \frac{df}{dx} - \frac{p}{x(x^2-1)} f = 0 \quad (1) $$

together with the boundary data:

- $f$ decays as $x \to \infty$, and:
- $f$ is regular at $x = 1$. Here $4p = k^2$ is the mass of glueball states, $4p = -k^2$.

The above equation has regular singularities, and its solution can be written as a series, with radius of convergence determined by distance between two singularities. For the above equation, the radii of convergencies of expansions at 1 and infinity overlap, therefore, a reliable computation is possible.

For an arbitrary $p$, the fundamental system of solutions of the equation (1) is the following:

For $x$ with $1 < x < \infty$, it is given by a linear combination of two convergent series
\[ y_1^{(\infty)}(x) = 1/(x^2) + \sum_{n=1}^{\infty} a[n]x^{-n} \]
\[ y_2^{(\infty)}(x) = \frac{\nu^2}{2} \text{Log}(x)y_1^{(\infty)}(x) + \sum_{n=1}^{\infty} b[n]x^{-n} \]

where \( a[n], b[n] \) are given by a recursion formula
\[
a[<0] = 0; a[0] = 1; a[n+1] = \frac{1}{(n+2)^2 - 1} (a[n-1]((n+1)^2) + pa[n]) ;
\]
\[
b[<0] = 0; b[0] = 0; \]
\[
b[n+1] = \frac{1}{(n)^2 - 1} \left( \frac{\nu^2}{2} (2n^2[n-1] - (2n-2)a[n-3]) + b[n-1](n-1)^2 + pb[n] \right) \]

For \( x \) with \( 0 < x < 2 \), the fundamental system of solutions is given by a linear combination of two convergent series
\[
y_1^{(1)}(x) = 1 + \sum_{n=1}^{\infty} a[n](x-1)^n \]
\[
y_2^{(1)}(x) = \text{Log}(x-1)y_1^{(1)}(x) + \sum_{n=1}^{\infty} b[n](x-1)^n \]

where \( a[n], b[n] \) are given by a recursion formula
\[
a[<0] = 0; a[0] = 1; a[n+1] = -\frac{1}{2(n+1)^2} \left( (3n(n+11) - p)a[n] + a[n-1](n^2 - 1) \right) ;
\]
\[
b[<0] = 0; b[0] = 0; \]
\[
b[n+1] = -\frac{1}{2(n+1)^2} \left( 4(n+1)a[n+1] + 3(1+2n)a[n] + 2na[n-1] + (3n(n+1) - p)b[n] + b[n-1](n^2 - 1) \right) \]

The solution of the boundary problem (4) can be described as follows: For certain \( p \) there exist a solution which is proportional to \( y_1^{(\infty)}(x) \) for \( x > 1 \) and to \( y_1^{(1)}(x) \) as \( |x-1| < 1 \). The condition for such a solution to exist is that the Wronskian of two solutions \( y_1^{(\infty)}(x) \) and \( y_1^{(1)}(x) \), which depend from \( p \) as a parameter,
\[
\text{Wronskain}(p,x) = \begin{pmatrix}
  y_1^{(1)}(x) & y_1^{(\infty)}(x) \\
  \frac{d}{dx} y_1^{(1)}(x) & \frac{d}{dx} y_1^{(\infty)}(x)
\end{pmatrix}
\]

must be zero. For \( x \) such that \( 1 < x < 2 \) both the series defining \( y_1^{(\infty)}(x) \) and the series defining \( y_1^{(1)}(x) \) are convergent, therefore, the Wronskian can be computed. It
is easy to see that the Wronskian depend from $x$ as
\[
Wronskain(p, x) = \frac{r(p)}{x(x - 1)(x + 1)},
\]
and therefore, the function $r(p)$
\[
r(p) = x(x - 1)(x + 1)Wronskain(p, x)
\]
can be computed at any point $x$ with $1 < x < 2$, which allows to determine the function $r(p)$, with any desired accuracy, as the series (2), (3) are convergent, and also the solutions $y_1$ are analytic functions of $x$.

The spectrum of (??) correspond to zeroes of the function $r(p)$. By standart oscillation theorems, there are no zeroes of $r(p)$ for $p \geq 0$, and there is an infinite discrete set of zeroes for $p < 0$. We do not know an analytic formula for the roots of $r(p)$, however, we can find their numerical values, for example by plotting function $r(p)$ accurately and looking where the zeroes are, as one would do for a transcendental equation of the sort $\tan(x) = x$; we can also find roots numerically with an arbitrary precision, for example by invoking Newton method of finding a root. Such computations shows that the first several roots are located at $-11.58766; -34.52698; -68.974962; -114.91044; -172.33171; -241.236607; -321.626549, \ldots$.

We also believe that numbers $4p = -6n(n + 1)$, $n = 1, 2..., \ldots$, which WKB method give, are not the roots, although they are close to those roots listed.

### 1.1 Orthogonality

For the discrete set of values $p_n$, $n = 1, 2, \ldots$ such that $r(p) = 0$ we have normalizable at $1 \leq x \leq \infty$ wave functions $\{F_n(x)\}$. Those functions are orthogonal, which here says
\[
\int_1^{+\infty} x(1 - x^2)F_n(x)F_m(x)dx = 0, \quad n \neq m \tag{4}
\]

### 1.2 Checking that -12 is not a root

Since a WKB computation, which may or may not have corrections in this case, shows the first eigenstate at $4p = -12$, we were interested to check does our function $r(p)$ has a root exactly at $4p = -12$. It is easy to check whether or not it does, as our series are convergent (and pretty fast, faster then the worst of of $\sum(-1)^nn(x - 1)^n$ and $\sum(-1)^nn(1/x)^n$ do), for the $p$ we are interested in). Therefore, we can compute the function $r(p)$, using finite number of terms in the series, and estimate the error. Thus,
using our method, we can answer the question whether or not the WKB formula gets corrections. We did such computation, and our result is that

\[-0.022482 < r(p)|_{4p=-12} < -0.022481\]

(with the sign of \(r(p)\) here consistent with our expectation to get the first root close by with \(4p > -12\), as \(r(p)\) is positive and of order 1 for small negative \(p\)). Thus we believe there is no root at exactly \(4p = -12\), and there is a correction to the WKB formula, with the exact root at about \(-11.59\).

2 Glueball spectrum in \(QCD_4\)

The conjectures of Maldacena and Witten [3], [1] gives, in particular, a prediction for glueball masses in \(N = \infty\) QCD in 4 dimensions from supergravity, namely, from certain solutions of classical equations of motion in a black hole metric for a massless scalar dilaton field \(\Phi\). The dilaton shows up here as it couples to the \(O^{++}\) glueball operator; other glueballs, which couples to other fields, say a two-form in supergravity theory, are described, for example, in [5]. The classical equations of motion for dilaton are:

\[
\partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \Phi)
\]

where the metric is a black hole

\[
\frac{ds^2}{l_s^2 g_5^2 N/4\pi} = \frac{d\rho^2}{4\rho^2 \left(1 - \frac{1}{\rho^8}\right)} + \rho^2 \left(1 - \frac{1}{\rho^8}\right) dx_2^2 + \rho^4 \sum_{i=1}^{4} dx_i^2 + \rho^4 d\Omega_4^2
\]

Here \(x_1, x_2, x_3, x_4\) is where our \(QCD_4\) lives, and, oversimplifying a bit, it can be imagined as a 5-ball \(x_1, x_2, x_3, x_4, \rho\) with \(x_1, x_2, x_3, x_4\) on \(S^4\) boundary; \(\tau\) is a circle and \(d\Omega_4\) is a sphere \(S^4\) with a standard metric. We need to take solutions of the form

\[\Phi = F(\rho) e^{ikx}\]

with the appropriate physical conditions for \(\Phi\) at \(\rho = \infty\), (normalizability), and at \(\rho = 1\) (single-valuedness for complex \(\rho\)) [1]. By introducing a variable \(X = \rho^2\), (and calling it just \(x\) below) we get certain well-posed Sturm-Liouville problem:

\[(x^7 - x) \frac{d^2 F}{dx^2} + \left(10x^6 - 4\right) \frac{dF}{dx} - px^3 F = 0, \quad (5)\]

(which is the equation of motion of a massless scalar field, dilaton, which couples to the relevant glueball operator)
together with the boundary data:

\[ F(x) \text{ decays as } x \to \infty, \text{ and:} \]

\[ F(x) \text{ is regular at } x = 1 \]

Similarly to the equation (1), the singularities at 1 and infinity for this equation are regular, and the fundamental system of solutions of the equation (2) can be written as power series with the appropriate radius of convergence, namely, for \( 1 < x < \infty \) the fundamental system of solutions is is

\[
y^{(\infty)}_1(x) = \frac{1}{x^9} + \sum_{n=1}^{\infty} a[n]x^{-n-9}
\]

\[
y^{(\infty)}_2(x) = \frac{\gamma^2}{2} \log(x)y^{(\infty)}_1(x) + \sum_{n=1}^{\infty} b[n]x^{-n}, \tag{6}
\]

where \( a[n] \) are given by a recursion formula

\[
a[< 0] = 0; a[0] = 1;
\]

\[
a[2n] = \frac{1}{(2n)(2n + 9)} (pa[2n - 2] + (2n)(2n + 3)a[2n - 6]); n = 1, 2, \ldots
\]

The series are convergent for \( 1 < x < \infty \).

For \( x \) with \( |x - 1| \leq \frac{\sqrt{3} - 1}{\sqrt{2}} \), (with \( \frac{\sqrt{3} - 1}{\sqrt{2}} \) being the distance between 1 and the next nearest sixth roots of one, \( e^{\pm \frac{\pi i}{6}} \), on a complex plane of \( x \), as can be discovered by looking at the coefficient in front of the second derivative in our equation) the fundamental system of solutions is given by a linear combination of two convergent series

\[
y^{(1)}_1(x) = 1 + \sum_{n=1}^{\infty} a[n](x - 1)^n
\]

\[
y^{(1)}_2(x) = \log(x - 1)y^{(1)}_1(x) + \sum_{n=1}^{\infty} b[n](x - 1)^n, \tag{7}
\]

where \( a[n] \) are given by a recursion formula

\[
a[< 0] = 0; a[0] = 1;
\]

\[
a[n + 1] = -\frac{1}{6(n + 1)^2} ( (n)(21(n - 1) + 60) - p)a[n] + \\
( (n - 1)(35(n - 2) + 150) - 3p)a[n - 1] + ( (n - 2)(35(n - 3) + 200) - 3p)a[n - 2] + \\
+ ( (n - 3)(21(n - 4) + 150) - p)a[n - 3] + ((n - 4)(7(n - 5) + 60))a[n - 4] + \\
+ ((n - 5)(n - 6 + 10))a[n - 5])
\]
Our boundary conditions require that the solution must be proportional \( y_1^{(\infty)}(x) \) for \( x > 1 \) and to \( y_1^{(1)}(x) \) as \( |x - 1| < 1 \), and therefore, the Wronskian of two solutions \( y_1^{(\infty)}(x) \) and \( y_1^{(1)}(x) \),

\[
Wronskain(p, x) = \begin{pmatrix}
y_1^{(1)}(x) & y_1^{(\infty)}(x) \\
\frac{dy_1^{(1)}}{dx} & \frac{dy_1^{(\infty)}}{dx}
\end{pmatrix}
\]

which depend from \( p \) as a parameter, must be zero. For \( x \) such that \( 1 < x < \frac{\sqrt{3} - 1}{\sqrt{2}} \) the series defining \( y_1^{(\infty)}(x) \) and the series defining \( y_1^{(1)}(x) \) are both convergent, (for any \( p \)), and therefore the Wronskian can be effectively computed for such \( x \) using our series. The Wronskian depend from \( x \) as follows:

\[
Wronskain(p, x) = \frac{r(p)}{x^4(x^6 - 1)}
\]

where

\[
r(p) = x^4(x^6 - 1)Wronskain(p, x)
\]

does not depend from \( x \), and our eigenvalues are zeroes of the function \( r(p) \).

Since we can compute the Wronskian at any point \( 1 < x < \frac{\sqrt{3} - 1}{\sqrt{2}} \), using the series, the function \( r(p) \) is also well- determined, and we can examine where that function have zeroes, for example by plotting it and looking where it have zeroes, or, for a more precise computation, invoking for example Newton method to find a root numerically.

### 2.1 First few states

The computation gives the first few zeroes at -37.169908; -81.354363; -138.473573; -208.859215; -292.583628; -389.671368 -500.1328 -623.97315. . . . There is an infinite discrete set of roots, all roots are negative. It is not difficult to get to roots for values of \( p \) up to about -4600. Thereafter, computation is becoming increasingly difficult, as the coefficients in the series grow very large, before being killed by powers of \( x - 1 \) or \( 1/x \). Also, the final product, our function \( r(p) \), becomes pretty small, with coefficients of orders \( 10^{34} \) and results of orders \( 10^{-6} \), and we used tricks to go ahead.

### 2.2 Orthogonality

For the discrete set of values \( p_n \), \( n = 1, 2 \ldots \) such that \( r(p) = 0 \) we have normalizable at \( 1 \leq x \leq \infty \) wave functions \( \{F_n(x)\} \). Those functions are orthogonal, which here
\[\int_1^{+\infty} x^4(1-x^6)F_n(x)F_m(x)dx = 0, \quad n \neq m \quad (8)\]

3 Some pictures: \(r(p)\) for \(QCD_4\)

The graph of function \(r(p)\) up to about \(p = -7600\) for \(QCD_4\) follows. The roots of this function correspond to glueball eigenstate masses. It is not difficult to get to roots for values of \(p\) up to about \(-4600\). Thereafter, computation is becoming increasingly difficult, as for example for \(p\) around 7000 there are terms of order \(10^{34}\) in the series, and although due to powers of \(x - 1\) or \(1/x\) present the series is convergent, the first terms are very big, and in computing \(r(p)\) they combine into an expression of order \(10^{-6}\); as brut-force computation failed here, tricks were used to make it run for such (and higher) eigenvalues. The eigenfunctions look pretty weird here, and it shouldn’t take much for them and eigenvalues to be corrected by about anything; thus we suspect our hunt for higher eigenvalues is for sport, mostly. Surprisingly, function \(r(p)\) keeps to be pretty nice there. (Assistance of I.V.G. in putting up the pictures and fighting with latex is gratefully acknowledged.)
\[ r(p) \times 5000 \]

- \( p \) is multiplied by 5000.
\[ r(p) \times (10^7) \]
$r(p) \times 10^7$
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