On the universal $R$-matrix of $U_q\widehat{sl}_2$ at roots of unity

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Abstract

We show that the action of universal $R$-matrix of affine $U_q\widehat{sl}_2$ quantum algebra, when $q$ is a root of unity, can be renormalized by some scalar factor to give a well defined nonsingular expression, satisfying Yang-Baxter equation. It reduced to intertwining operators of all representations, corresponding to Chiral Potts, if the parameters of these representations lie on well known algebraic curve.

We also show that affine $U_q\widehat{sl}_2$ for $q$ is a root of unity form the autoquasitriangular Hopf algebra in the sense of Reshetikhin.

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1 Introduction

The intertwining operators of quantum groups ([1, 2, 3, 4]) lead to solutions of Yang-Baxter equation, which play the crucial role in two dimensional field theory and integrable statistical systems ([4, 5]). It is well known that the most of them can be obtained from the universal $R$-matrix ([1]) for a given quantum group: the solutions of spectral parameter dependent Yang-Baxter equation can be obtained from the universal $R$-matrix of affine quantum groups ([6]) and the solutions of non-spectral parameter dependant Yang-Baxter equations can be obtained from the universal $R$-matrix of finite quantum groups.

The situation is not the same for the case, when the parameter $q$ of quantum group is a root of unity.

In this case the center of quantum group is larger and new type of representations appear, which have no a classical analog ([5, 7, 8, 9]). It was shown in [10, 11] that the cyclic representations lead to solutions of Yang-Baxter equation with a spectral parameter, lying on some algebraic curve. These solutions correspond to Chiral Potts Model ([12, 13, 14]) and its generalizations (for quantum groups $U_{q}sl_{n}$).

The formal expression of the universal $R$-matrix fails in this case: it have a singularities at $q$ is a root of unity. Recently in [15] Reshetikhin introduced the notion of autoquasitriangular Hopf algebra to avoid these singularities. He treated the $U_{q}sl_{2}$ case.

The main goal of this paper is to show that after suitable renormalization by scalar factor the universal $R$-matrix produces $R$-matrices for concrete representations.

In section 2, we consider the universal $R$-matrix on Verma modules of $U_{q}sl_{2}$ for $q$ is a root of unity. We prove that it is well defined and make a connection with the $R$-matrix of autoquasitriangular Hopf algebra, founded by Reshetikhin.

In section 3 we consider the algebra $U_{q}\hat{sl}_{2}$ at roots of unity. We found the central elements of its Poincaré-Birkoff-Witt (PBW) basis, generalizing the results of [1] for affine case. It appears that new type of central elements appear for some imaginary roots, which have no analog for finite quantum groups. After this we prove the autoquasitriangularity of $U_{q}\hat{sl}_{2}$, generalizing the results of [13] for affine case. Then we consider the action of affine universal $R$-matrix on $U_{q}\hat{sl}_{2}$- and $U_{q}sl_{2}$-Verma modules. On $U_{q}\hat{sl}_{2}$-Verma modules
it is well defined. For $U_q sl_2$-Verma modules (evaluation representation) we renormalize its expression by scalar factor to exclude the singularities. The rest part lead to solutions of infinite dimensional spectral parameter dependent Yang-Baxter equation. We showed that under the certain condition this $R$-matrix can be restricted to semicyclic representations, giving the Boltzmann weights of Chiral Potts model, corresponding to such type representations, which was considered in [16, 17, 18, 19]. The condition, mentioned above, is on the parameters of representations: they must lie on well known algebraic curve. It is integrability condition of Chiral Potts model.

In the last section we made same type suggestion for the cyclic representations.

2 The $U_q sl_2$ case

2.1 The Universal $R$-matrix on Verma Modules at root of unity

The quantum group $U_q sl_2$ is a $[q, q^{-1}]$-algebra, generated by the elements $E$, $F$, $K$ with the following relations between them

$$[K, K^{-1}] = 0 \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \quad (1)$$

$$KEK^{-1} = q^2E \quad KFK^{-1} = q^{-2}F,$$

On $U_q sl_2$ there is a Hopf algebra structure with comultiplication $\Delta : U_q sl_2 \rightarrow U_q sl_2 \otimes U_q sl_2$ defined by

$$\Delta(K) = K \otimes K \quad \Delta(E) = E \otimes 1 + K^{-1} \otimes E \quad \Delta(F) = F \otimes K + 1 \otimes F$$

We denote $K = q^H$, $q = e^h$, as usually, and consider the $[[h]]$-algebra $U_h sl_2$ with the same defining relations. $U_h sl_2$ is a quasitriangular Hopf algebra, i.e. it possess the universal $R$-matrix $R \in U_h sl_2 \otimes U_h sl_2$ connecting the comultiplication $\Delta$ with the opposite comultiplication $\Delta' = \sigma \circ \Delta$, where $\sigma(x \otimes y) := y \otimes x$:

$$\Delta'(a) = R \Delta(a) R^{-1}, \quad \forall a \in U_q sl_2 \quad (2)$$
It satisfies the quasitriangularity relations

$$(\Delta \otimes 1)R = R_{13}R_{23} \quad (1 \otimes \Delta)R = R_{13}R_{12} \quad (3)$$

and Yang-Baxter equation (1)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (4)$$

Here we used the usual notation: if $R = \sum_i a_i \otimes b_i$, $a_i, b_i \in U_q sl_2$, then

$$R_{12} = \sum_i a_i \otimes b_i \otimes 1 \quad R_{13} = \sum_i a_i \otimes 1 \otimes b_i \quad R_{23} = \sum_i 1 \otimes a_i \otimes b_i$$

The explicit expression of $R$ in terms of formal power series is

$$R = \exp_{q^{-2}}((q - q^{-1})(E \otimes F))q^{\frac{1}{2}H \otimes H} \quad (5)$$

where the $q$-exponent is defined by $\exp_q(z) = \sum_{n \geq 0} \frac{z^n}{(z)_q^n}$, $(z)_q := \frac{1 - q^n}{1 - q}$.

Note, that to be precise, $U_q sl_2$ is not a quasitriangular Hopf algebra, because the term $q^{H \otimes H}$ in (5) do not belong to $U_q sl_2 \otimes U_q sl_2$, but it is an autoquasitriangular Hopf algebra ([15]). The latter is a Hopf algebra $A$, where the condition (2) is generalised by

$$\Delta' = \hat{R}(\Delta),$$

where $\hat{R}$ is an automorphism of $A \otimes A$ (not inner, in general). So, although (5) is ill defined on $U_q sl_2$, but the action

$$\hat{R}(a) = RaR^{-1}, \quad (6)$$

where $a \in U_q sl_2 \otimes U_q sl_2$ is till well defined.

For two representations of $U_q sl_2$ $V_1$ and $V_2$ one can consider two $U_q sl_2$-actions on $V_1 \otimes V_2$ by means of both comultiplications $\Delta$ and $\Delta'$. If $R$ is defined on $V_1 \otimes V_2$, then both $\Delta$- and $\Delta'$-actions are equivalent via intertwining operator $R_{V_1 \otimes V_2} = R|_{V_1 \otimes V_2}$. For general $q$ the restriction of (3) on tensor product of two irreducible representations (in general, of any highest weight representations) is well defined. And all solutions of Yang-Baxter equation (4), having $U_q sl_2$- symmetry in sense of (2) can be obtained from the universal $R$-matrix (5) in such way.
The situation is different for \( q \) being a root of unity. In this case the singularities appear in the formal expression of \( R \).

Recall that for \( q = \exp\left(\frac{2\pi i}{N'}\right) \) the elements \( F^N, K^N, K_{-1}^N \), where \( N = N' \) for odd \( N' \) and \( N = \frac{N'}{2} \) for even \( N' \), belong to the center of \( U_q\mathfrak{sl}_2 \). In irreducible representations they are multiples of identity. Recall that every \( N \)-dimensional irreducible representation is characterized by the values \( x, y, z \) of these central elements (and also by the value of \( q \)-deformed Casimir \( c = \frac{K_q + K_{q^{-1}}}{q - q^{-1}} + EF \), which for the fixed \( x, y, z \) can have in general \( N \) discrete values ([9])).

Although the expression of \( R \)-matrix ([5]) of \( U_q\mathfrak{sl}_2 \) has singularities for \( q^{N'} \rightarrow 1 \) in all terms \( \frac{1}{(n)_{q^{-2}}!} E^n \otimes F^n \) for \( n \geq N \), its restriction on tensor product of Verma modules \( M_{\lambda_1} \otimes M_{\lambda_2} \) is well defined.

Recall that \( M_{\lambda} \) is formed by the basic vectors \( v^\lambda_m, m = 0, 1, \ldots \), satisfying

\[
Ev^\lambda_0 = 0 \quad Fv^\lambda_m = v^\lambda_{m+1} \quad Hv^\lambda_0 = \lambda v^\lambda_0, \quad \lambda \in \mathbb{C}
\]

To consider the action of \( R \) on \( M_{\lambda_1} \otimes M_{\lambda_2} \) we use the formula, which can be obtained from the defining relations (1) ([9]):

\[
\left[E^n, F^s\right] = \sum_{j=1}^{\min(n,s)} \begin{bmatrix} n \cr j \end{bmatrix} \begin{bmatrix} s \cr j \end{bmatrix} [j]! F^{s-j} \left( \prod_{r=1}^{j} [H + j - n - s + r] \right) E^{n-j},
\]

where

\[
\begin{bmatrix} a \cr b \end{bmatrix} = \frac{[a]!}{[b]! [a-b]!}
\]

and \([n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\).

So, for \( n > s \) \( \frac{E^n}{(n)_{q^{-2}}!} v^\lambda_s = 0 \) and for \( n \leq s \):

\[
\frac{E^n}{(n)_{q^{-2}}!} v^\lambda_s = q^{\frac{n(n-1)}{2}} \begin{bmatrix} s \cr n \end{bmatrix} \prod_{r=1}^{n} [\lambda - s + r] v^\lambda_{s-n}
\]

The \( q \)-binomial \( \begin{bmatrix} s \cr n \end{bmatrix} \) has a non-infinity limit for \( q^{N'} \rightarrow 1 \). So,

\[
R(v^\lambda_s \otimes v^\lambda_{s'}) = \sum_{n=0}^{s} q^{\frac{(\lambda_1 - 2s)(\lambda_2 - 2s')}{2}} q^{\frac{n(n-1)}{2}} (q - q^{-1})^n \begin{bmatrix} s \cr n \end{bmatrix} \times \prod_{r=1}^{n} [\lambda_1 - s + r] v^\lambda_{s-n} \otimes v^\lambda_{s'+n}
\]
is well defined for \( q \) is a root of unity.

## 2.2 The connection with Reshetikhins \( R \)-matrix of autoquasitriangular Hopf algebra

This \( R \)-matrix can be presented in another form by using resent results of Reshetikhin ([13]). He used an asymptotic formula for \( q \)-exponent in the limit \( q^{N'} \to 1 \) to bring out multiplicatively singularities from \( \exp_q(\langle q^{-1} - 1 \rangle E \otimes F) \).

The expression of universal \( R \)-matrix in this limit then acquires the form:

\[
R = \exp \left( \frac{1}{2N^2 \hbar} \text{Li}_2(E^N \otimes F^N) \right) (1 - E^N \otimes F^N)^{-\frac{1}{2}} \times \prod_{m=0}^{N-1} (1 - \varepsilon^m E \otimes F)^{-\frac{N}{N'} q^{\frac{1}{2}} H \otimes H} \cdot O(\hbar)
\]

Here \( q = \exp(\hbar) \varepsilon, \varepsilon = \exp\left(\frac{2\pi i}{N'}\right) \) and \( \text{Li}_2(x) = \int_0^x \frac{\ln(1-y)}{y} dy \) is a dilogarithmetic function.

Recall that although the elements

\[
\frac{E^N}{(N)_{q^{-2}}}, \frac{F^N}{(N)_{q^{-2}}} \quad \text{and} \quad H
\]

don’t belong to \( U_q sl_2 \) for \( q^{N'} \to 1 \), but their adjoint actions

\[
\text{ad}(x)a = [x, a], \quad \text{Ad}(\exp(x))a = \exp(x)a \exp(-x) = \exp(\text{ad}(x))a
\]
on \( U_q sl_2 \) are well defined in this limit and give rise to some derivations ([29]). Let’s denote them by \( e, f \) and \( \hbar \) correspondingly.

The element \( \frac{1}{2hN^2} \text{Li}_2(E^N \otimes F^N) \) in the exponent of (8) in the adjoint representation also acts on \( U_q sl_2 \otimes U_q sl_2 \) as a derivation in the limit \( \hbar \to 0 \).

It can be expressed by means of the derivations \( e \) and \( f \) as follows:

\[
\lim_{\hbar \to 0} \text{ad} \left( \frac{1}{2\hbar N^2} \text{Li}_2(E^N \otimes F^N) \right) = c_{N'} \ln(1 - E^N \otimes F^N) \times (e \otimes F^N + F^N \otimes f),
\]
where

\[ c_{N'} = \begin{cases} 
- (1 - \varepsilon^{-2})^{-N} & \text{for odd } N \quad (N' = N) \\
- (1 - \varepsilon^{-2})^{-N} & \text{for even } N \quad (N' = 2N)
\end{cases} \tag{8} \]

Note, that

\[ \text{Ad}(\varepsilon^{\frac{1}{2}H \otimes H}) = \varepsilon^{\frac{1}{2}(h \otimes H + H \otimes h)} = K^{1 \otimes \frac{1}{2}h} \otimes K^{rac{1}{2}h \otimes 1} \]

is well defined in the adjoint representation.

So, one can write down the automorphism \( \hat{R} \) \( \text{(3)} \) in the limit \( \hbar \to 0 \), obtained in \cite{15}, in the following form \( \text{(4)} \)

\[ \hat{R} = \prod_{m=0}^{N-1} \text{Ad}\left( (1 - \varepsilon^{m}E \otimes F)^{-\frac{1}{2}} \right) \]
\[ \times \exp\left( (1 - \varepsilon^{-2})^{-N} \frac{\ln(1 - E^{N} \otimes F^{N})}{E^{N} \otimes F^{N}} (e \otimes F^{N} + E^{N} \otimes f) \right) \]
\[ \times K^{1 \otimes \frac{1}{2}h} \otimes K^{rac{1}{2}h \otimes 1} \tag{9} \]

Let us now consider the restriction of \( \text{(3)} \) on the quotient algebra obtained from \( U_{q}sl_{2} \) by factorisation on the ideal, generated by \( E^{N} \), i.e. impose \( E^{N} = 0 \). Although this ideal is not stable with respect to derivations \( e, f, h \), it is easy to see that it is stable with respect to \( \hat{R} \).

Moreover, the left \( U_{q}sl_{2} \otimes U_{q}sl_{2} \)-module

\[ I_{\lambda_{1}, \lambda_{2}} = (U_{q}sl_{2} \otimes I_{\lambda_{2}}) \bigoplus (I_{\lambda_{1}} \otimes U_{q}sl_{2}) \]

is also stable with respect to \( \hat{R} \). Here we denoted by \( I_{\lambda} \) the left \( U_{q}sl_{2} \)-module, generated by \( E \) and \( (K - \varepsilon^{h}) \). This fact allows to restrict \( \text{(3)} \) on Verma modules, because we have the left \( U_{q}sl_{2} \)-module equivalence

\[ (U_{q}sl_{2} \otimes U_{q}sl_{2})/I_{\lambda_{1}, \lambda_{2}} \cong M_{\lambda_{1}} \otimes M_{\lambda_{2}} \]

So, one can derive from \( \text{(3)} \) the restriction of \( R \) on this factormodule is given by the multiplication on \( \text{(5)} \)

\^\footnote{For quantum groups one can introduce 4 equivalent comultiplications: \( \Delta_{q}, \Delta'_{q}, \Delta_{q^{-1}}, \Delta'_{q^{-1}} \). In \( \text{(3)} \) the comultiplication \( \Delta'_{q^{-1}} \) had been used as a basic one. So, \( R \)-matrix, used there, is \( R_{q^{-1}} \) in our notations and differs from \( \Delta_{q} \)-case used here by permutation of \( q \)-exponent and \( \frac{1}{2}H \otimes H \).}

\^\footnote{Note that both \( \hbar \) and \( e \) are well defined on \( M_{\lambda} \) in contrast to \( f \)}
This expression is another form of expression of universal $R$-matrix (5) on Verma modules and coincide with (7).

3 The case of affine $U_q\hat{sl}_2$

3.1 The PBW basis and the universal $R$-matrix

The affine quantum universal enveloping algebra $U_q\hat{sl}_2$ is a $[q, q^{-1}]$-Hopf algebra, generated by elements $E_i := E_{\alpha_i}, F_i = F_{\alpha_i}, K_i = q^{H_i}, i = 0, 1$ and $q^d$ with defining relations (11):

$$
[q^{H_i}, q^{H_j}] = 0 \quad q^d q^{H_i} = q^{H_i} q^d \quad [E_i, F_j] = \delta_{ij} [H_i]_q
$$

$$
q^{H_i} E_j q^{-H_i} = q^{a_{ij}} E_j \quad q^{H_i} F_j q^{-H_i} = q^{-a_{ij}} \quad q^d E_1 q^{-d} = q E_1
$$

$$
q^d F_1 q^{-d} = q^{-1} E_1 \quad q^d E_0 q^{-d} = E_0 \quad q^d F_0 q^{-d} = F_0
$$

$$
(ad_q E_i)^{1-a_{ij}} E_j = 0 \quad (ad_q F_i)^{1-a_{ij}} F_j = 0
$$

and comultiplication

$$
\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i} \quad \Delta(q^d) = q^d \otimes q^d
$$

$$
\Delta(E_i) = E_i \otimes 1 + q^{-H_i} \otimes E_i \quad \Delta(F_i) = F_i \otimes q^{H_i} + 1 \otimes F_i
$$

Here we use the $q$-deformed adjoint action $(ad_q x) y := \sum_i x_i y s(x^i)$, where $\Delta(x) = \sum_i x_i \otimes x^i$ and $s : U_q\hat{sl}_2 \rightarrow U_q\hat{sl}_2$ is antipode of $U_q\hat{sl}_2$, defined by

$$
s(E_i) = -K_i E_i \quad s(F_i) = -F_i K_i^{-1} \quad s(K_i) = K_i^{-1}
$$

Also we denoted by $a_{ij}$ the Cartan matrix of affine $\hat{sl}(2)$ Lie algebra

$$
a_{ij} = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
$$
Let’s denote by \( c \) the central element \( c = H_1 + H_2. \)

Define on \( U_q \hat{sl}_2 \) an antiinvolution \( \iota \) by

\[
\iota(K_i) = K_i^{-1}, \quad \iota(E_i) = F_i, \quad \iota(F_i) = E_i, \quad \iota(q) = q^{-1}
\]

As above, denote by \( U_h \hat{sl}_2 \) the \([h]\)-algebra with the same relations but
the elements \( H_i \) instead of \( q H_i \).

The PBW basis of \( U_h \hat{sl}_2 \) is formed by elements \( H_i, d, E_{\alpha_i+n\delta}, F_{\alpha_i+n\delta}, E'_{n\delta}, F'_{n\delta} \), which are inductively defined by the relations

\[
E_{\alpha_0+n\delta} = (-1)^n (\text{ad}_{E'_\delta})^n E_0 \quad E_{\alpha_1+n\delta} = (\text{ad}_{E'_\delta})^n E_1 \\
E'_{n\delta} = [2]^{-1}(E_{\alpha_0+(n-1)\delta}E_1 - q^{-2}E_1E_{\alpha_0+(n-1)\delta}) \quad (12)
\]

The expression of the universal \( R \)-matrix of \( U_h \hat{sl}_2 \) is simpler if one redefine
\( E'_{n\delta} \) and \( F'_{n\delta} \) by means of Schur polynomials \((6)\):

\[
E'_{n\delta} = \sum_{0<k_1<\ldots<k_m \atop k_1p_1+\ldots+k_mp_m=n} (q^2-q^{-2})^{\sum p_i^{-1}} (E_{k_1\delta})^{p_1} \ldots (E_{k_m\delta})^{p_m} \\
F'_{n\delta} = \iota(E'_{n\delta})
\]

In order to rewrite all the relations between \((12)\) in compact form it is suitable to change slightly the basis as follows:

\[
E_{\alpha_0+n\delta} = (-1)^n q^{-2n}x_{n+1}^{-1}k^{-1} \quad E_{\alpha_1+n\delta} = (-1)^n q^{-(c+2)n}x_n^+ \\
E'_{n\delta} = (-1)^n q^{-\frac{c}{2}n-2n}q^{-2}\psi_nk^{-1} \quad E_{n\delta} = (-1)^n q^{-(\frac{c}{2}+2)n}a_n \quad (13)
\]

Then the elements \( x_n^{\pm}, (n \in \mathbb{Z}), a_k, (k \in \mathbb{Z}, k \neq 0), \psi_m, \varphi_m, (m \geq 1) \)
and \( \psi_0 = \varphi_0^{-1} = k \) satisfy the following relations

\[
[a_m, a_n] = \delta_{m,-n} \frac{[2m][mc]}{m} \quad [a_m, k] = 0 \\
x_m^{\pm}k^{-1} = q^{\pm2}x_m^\pm \\
[a_m, x_n^{\pm}] = \pm \frac{[2m][mc]}{m} q^{2\frac{c}{2}n}x_m^\pm
\]
\[ \begin{aligned}
&x_{m+1}^+ x_n^+ - q^\pm 2 x_n^+ x_{m+1}^+ = q^\pm 2 x_{m+1}^+ x_n^+ - x_{n+1}^+ x_m^+ \\
&[x_m^+, x_n^+] = \frac{1}{q-q^{-1}} (q^{2(m-n)} \psi_{m+n} - q^{-2(m-n)} \varphi_{m+n}) \\
&\sum_{m=0}^{\infty} \psi_m z^{-m} = k \exp ((q - q^{-1}) \sum_{m=1}^{\infty} a_m z^{-m}) \\
&\sum_{m=0}^{\infty} \varphi_m z^m = k^{-1} \exp (- (q - q^{-1}) \sum_{m=1}^{\infty} a_m z^m)
\end{aligned} \] (14)

These relations had been introduced by Drinfeld in [20] and define another realization of affine algebra \( U_q \widehat{sl}_2 \). The antiinvolution \( \iota \) in this notations is

\[ \begin{aligned}
\iota(x_n^+) = x_{-n}^+ & \quad \iota(\psi_n) = \varphi_{-n} & \quad \iota(a_n) = a_{-n} & \quad \iota(q) = q^{-1}
\end{aligned} \]

We choose the normal ordering of positive root system \( \Delta_+ \) of \( U_q \widehat{sl}_2 \) as follows:

\[ \alpha_0, \alpha_0 + \delta, \ldots, \alpha_0 + n\delta, \ldots, \delta, 2\delta, \ldots, n\delta, \ldots, \alpha_1 + n\delta, \ldots, \alpha_1 + \delta, \ldots, \alpha_1 \] (15)

Then the universal \( R \)-matrix has the form [6]:

\[ \begin{aligned}
R &= \left( \prod_{n \geq 0} \exp q^{-2} ((q - q^{-1})(E_{a_0+n\delta} \otimes F_{a_0+n\delta})) \right) \\
&\times \exp \left( \sum_{n>0} \frac{n E_{n\delta} \otimes F_{n\delta}}{q^{2n} - q^{-2n}} \right) \times \left( \prod_{n \geq 0} \exp q^{-2} ((q - q^{-1})(E_{a_1+n\delta} \otimes F_{a_1+n\delta})) \right) \right) q^{2H_0 \otimes H_0 + c \otimes d + d \otimes c},
\end{aligned} \] (16)

where the product is given according to the normal order (15).

### 3.2 \( U_q \widehat{sl}_2 \) at roots of unity

For \( q \) being a root of unity \( \left( q = \varepsilon, \varepsilon = e^{\frac{2\pi i}{N}} \right) \) the center of \( U_q \widehat{sl}_2 \) is enlarged by the \( N \)-th power of the root vectors, as for finite quantum groups:

\[ \begin{aligned}
[E_{\gamma}^N, x] = 0 & \quad [F_{\gamma}^N, x] = 0 & \quad [K_{\gamma}^N, x] = 0,
\end{aligned} \] (17)

where \( \gamma \in \Delta_+ := \{ \alpha_i + n\delta, m\delta | n \geq 0, m > 0 \} \) and \( x \in U_{\varepsilon} \widehat{sl}_2 \).
These conditions for the simple roots $\gamma = \alpha_i$ can be proven by using the defining relations of Cartan-Weyl basis \([11]\) as for finite quantum algebras it had been done in \([9]\). Indeed, using
\[
\Delta(E_i^N) = K_1^{-N} \otimes E_i^N + E_i^N \otimes 1,
\]
recalling that $q$-deformed adjoint action $\text{ad}_q$ is a $U_q \hat{sl}_2$-representation:
\[
\text{ad}_q(ab)c = \text{ad}_q(a)\text{ad}_q(b)c, \forall a, b, c \in U_q \hat{sl}_2
\]
and using Serre relations in \([12]\), we obtain for $i \neq j, N \geq 3$:
\[
[E_i^N, E_j] = \text{ad}_q(E_i^N)E_j = (\text{ad}_q(E_i))^N E_j = 0
\]
Other commutations in \([17]\) for $\gamma = \alpha_i$ can be verified easily.

To carry out \([17]\) for other roots one can try to use the isomorphism, induced by the $q$-deformed Weyl group. In affine case it had considered in \([21]\). But it is easier to use the symmetries of Drinfeld realization of $U_q \hat{sl}_2$ directly. It is easy to see from \([14]\) that the operation $\omega \pm$ on $U_q \hat{sl}_2$ defined by
\[
\begin{align*}
\omega \pm (x_m^\pm) &= x_{m \pm 1}^\pm & \omega \pm (a_m) &= a_m & \omega \pm (q) &= q \\
\omega \pm (\psi_n) &= q^c \psi_n & \omega \pm (\varphi_n) &= q^{-c} \varphi_n & \omega \pm (c) &= c
\end{align*}
\]
is an algebra automorphism. As the roots can be obtained by applying $\omega \pm$ from the simple ones, we finished the proof.

In addition to this, the elements $E_{kN\delta}, F_{kN\delta}$ are central for $k \in \mathbb{N}_+$. This can be seen from \([14]\) and \([13]\). These central elements have no analog for finite algebras.

The adjoint action of
\[
\frac{E_i^N}{(N)_q^{-2}}, \frac{E_j^N}{(N)_q^{-2}}, \gamma \in \Delta_+ \quad \text{and} \quad \frac{kNE_{kN\delta}}{q^{2kN} - q^{-2kN}}, \frac{kNF_{kN\delta}}{q^{2kN} - q^{-2kN}}
\]
lead in the limit $\hbar \to 0$ to derivations of $U_{\varepsilon \hat{sl}_2}$, which we denote by $e_\gamma$, $f_\gamma$, $\hat{e}_k$, $\hat{f}_k$ correspondingly. The action of automorphism $\omega$ on these derivations inherits from its action on corresponding root vectors.

### 3.3 The Universal $R$-matrix at roots of 1

Now let’s consider the expression of universal $R$-matrix \([18]\) in the limit $\hbar \to 0$. The singularities, which appear in all $q$-exponents, are the same type as in
the expression of universal $R$-matrix of $U_{h}sl_{2}$. New type singularities appear due to the factor $\frac{kN}{q^{kN_{1}} - q^{-kN_{1}}}$ in the exponent before all term $E_{kN_{\delta}} \otimes F_{kN_{\delta}}$ for any natural $k$.

But as in $U_{\varepsilon}sl_{2}$ case, the adjoint action $\hat{R}$ of $R$ on $U_{\varepsilon}\hat{sl}_{2} \otimes U_{\varepsilon}\hat{sl}_{2}$ is well defined.

Indeed, the adjoint action of every $q$-exponent term 

$$R_{\gamma} = \exp_{q^{-2}}((q - q^{-1})(E_{\gamma} \otimes F_{\gamma})),$$

where $\gamma = \alpha_{i} + n\delta$ in (16) can be treated as it has been done in $U_{\varepsilon}sl_{2}$ case:

$$\lim_{\bar{h} \to 0} \text{Ad}(R_{\gamma}) = \prod_{m=0}^{N-1} \text{Ad}((1 - \varepsilon^{m}E_{\gamma} \otimes F_{\gamma})^{-\frac{2}{N_{1}}}) \times \exp \left( c_{N_{1}} \ln(1 - E_{\gamma} \otimes F_{\gamma})^{-\frac{2}{N_{1}}} \right),$$

where $c_{N_{1}}$ is defined by (8).

From (13) and (14) it follows that the operations

$$\hat{e}_{k} = \lim_{\bar{h} \to 0} \text{ad} \left( \frac{kN_{1}E_{kN_{\delta}}}{q^{kN_{1}} - q^{-kN_{1}}} \right), \hat{f}_{k} = \lim_{\bar{h} \to 0} \text{ad} \left( \frac{kNF_{kN_{\delta}}}{q^{kN_{1}} - q^{-kN_{1}}} \right)$$

also are the derivations on $U_{\varepsilon}\hat{sl}_{2}$, as it was mentioned above. So,

$$\hat{R}_{kN_{\delta}} = \lim_{\bar{h} \to 0} \text{Ad}(R_{kN_{\delta}}) = \lim_{\bar{h} \to 0} \text{Ad} \left( \exp \left( \frac{kN_{1}E_{kN_{\delta}}}{q^{kN_{1}} - q^{-kN_{1}}}F_{kN_{\delta}} \otimes E_{kN_{\delta}} \right) \right)$$

$$= \exp(\hat{e}_{k} \otimes F_{kN_{\delta}} + E_{kN_{\delta}} \otimes \hat{f}_{k}),$$

gives rise to an outer automorphism of $U_{\varepsilon}\hat{sl}_{2}$.

Finally, the right term in (14) has the following adjoint action

$$\hat{K} = \text{Ad}(\varepsilon^{\frac{1}{2}H_{0} \otimes H_{0} + c \otimes d + d \otimes c}) = K_{0}^{1} \mathbb{H}_{0}^{\otimes 1} \varepsilon^{c \otimes d + d \otimes c},$$

Here $h_{0} = \text{ad}(H_{0})$ is a derivation on $U_{q}\hat{sl}_{2}$.

So, we proved, that the quantum algebra $U_{\varepsilon}\hat{sl}_{2}$ is autoquasitriangular Hopf algebra with the automorphism

$$\hat{R} = \left( \prod_{\gamma \in \Delta^{+}} \hat{R}_{\gamma} \right) \hat{K},$$

where the product over positive roots is ordered according to the normal order (13).
3.4 The universal $R$-matrix on Verma modules

Consider now Verma module $M_{\hat{\lambda}}$ over $U_{\hat{\epsilon}\hat{sl}_2}$ with highest weight $\hat{\lambda}$. It is generated by vectors

$$v_{k_1...k_n}^\hat{\lambda} = F_{\gamma_n}^{k_n} ... F_{\gamma_1}^{k_1} v_0^\hat{\lambda} \quad k_1, ..., k_n = 0, 1, ... \quad \gamma \in \Delta_+ \quad \gamma_1 < ... < \gamma_n,$$

where $v_0^\hat{\lambda}$ is a highest weight vector:

$$E_{\gamma} v_0^\hat{\lambda} = 0 \quad H v_0^\hat{\lambda} = \hat{\lambda}(H) v_0^\hat{\lambda}$$

As for $U_q sl_2$-case all terms $R_\gamma$ and $K$ in the product of universal $R$-matrix $(16)$ are well defined in the limit $\hbar \to 0$. Indeed, there is a well defined action of derivations $e_i, \hat{e}_i$ on $M_{\hat{\lambda}}$ by

$$e_i g v_0^\hat{\lambda} := e_i(g) v_0^\hat{\lambda} \quad \hat{e}_i g v_0^\hat{\lambda} := \hat{e}_i(g) v_0^\hat{\lambda} \quad \forall g \in U_{\hat{\epsilon}\hat{sl}_2}$$

Moreover, in the action of $(16)$, on any vector $x \in M_{\hat{\lambda}_1} \otimes M_{\hat{\lambda}_1}$ the term $R_\gamma$ with sufficiently large $\gamma$ give rise to identity and only finite number of $R_\gamma$ survive. In the decomposition of each such $R_\gamma$ also survive only finitely many terms. So, the action of $R$ on $x \in M_{\hat{\lambda}_1} \otimes M_{\hat{\lambda}_1}$ is well defined.

To define the action of the Universal $R$-matrix $(16)$ on $U_{q\hat{sl}_2}$-Verma modules, the spectral parameter dependent homomorphism $\rho_x: U_{q\hat{sl}_2} \to U_q sl_2$ must be introduced [3]:

$$\rho_x(E_{\alpha_0}) = E \quad \rho_x(F_{\alpha_0}) = F \quad \rho_x(H_0) = H$$

$$\rho_x(E_{\alpha_1}) = xF \quad \rho_x(F_{\alpha_1}) = x^{-1}E \quad \rho_x(H_1) = -H$$

Note, that in this representation the central charge $c$ is zero. Under the action of $\rho_x$ the root vectors acquire the form (22):

$$E_{\alpha_0+n\delta} = (-1)^n x^n q^{-nh} E \quad F_{\alpha_0+n\delta} = (-1)^n x^n F q^{nh}$$

$$E_{\alpha_1+n\delta} = (-1)^n x^{n+1} F q^{-nh} \quad F_{\alpha_1+n\delta} = (-1)^n x^{n+1} q^{nh} E$$

$$E'_{n\delta} = \left(\frac{(-1)^n}{[2]_q}\right) x^n q^{(n-1)h}(EF - q^{-2}FE)$$

$$F'_{n\delta} = \left(\frac{(-1)^{n-1}}{[2]_q}\right) x^{-n} q^{(n-1)h}(FE - q^{-2}EF)$$

(20)
Substituting this in the expression of affine universal $R$-matrix following \cite{22}, one can obtain the spectral parameter $R$-matrix:

\[
R \left( \frac{x}{y} \right) = (\rho_x \otimes \rho_y) R = R^+ \left( \frac{x}{y} \right) R^0 \left( \frac{x}{y} \right) R^- \left( \frac{x}{y} \right) K, \tag{21}
\]

where

\[
R^+ (z) = \prod_{n \geq 0} \exp_{q^{-2}} \left( (q - q^{-1}) z^n (q^{-nH} E \otimes F q^{nH}) \right)
\]

\[
R^0 (z) = \exp \left( \sum_{n > 0} \frac{n}{q^{2n} - q^{-2n}} z^n E_{n\delta} \otimes F_{n\delta} \right) \tag{22}
\]

\[
R^- (z) = \prod_{n \geq 0} \exp_{q^{-2}} \left( (q - q^{-1}) z^{n+1} (F q^{-nH} \otimes q^{nH} E) \right)
\]

\[
K = q^{\frac{1}{2} H \otimes H}
\]

Now we consider \cite{22} on Verma modules $M_\lambda$ of $U_q sl_2$ and its behavior at roots of unity.

Note that one can represent the terms $R^\pm, R^0$ of universal $R$-matrix in a more suitable way by performing infinite sum and infinite product in \cite{22}. So, we have \cite{23}:

\[
R^+ (z) = 1 + (E \otimes F) \frac{(q - q^{-1})}{1 - zq^{-2} K^{-1} \otimes K} + \frac{(E \otimes F)^2}{(2)_{q^{-2}}!} \frac{(q - q^{-1})^2}{(1 - zq^{-2} K^{-1} \otimes K)(1 - zq^{-4} K^{-1} \otimes K)} + \ldots \tag{23}
\]

\[
R^- (z) = 1 + \frac{z(q - q^{-1})}{1 - zq^{-2} K^{-1} \otimes K} F \otimes E + \frac{z^2(q - q^{-1})^2}{(2)_{q^{-2}}!} \frac{(F \otimes E)^2}{(1 - zq^{-2} K^{-1} \otimes K)(1 - zq^{-4} K^{-1} \otimes K)} + \ldots \tag{24}
\]
and

\[ R^0(z) = f(z) \bar{R}^0(z) \]  \hspace{1cm} (25)

where

\[
f(z) = \exp \sum_{n \geq 1} \left( (q - q^{-1}) \frac{[\lambda_1 n]_q [\lambda_2 n]_q}{[2n]_q} \right) \frac{z^n}{n}
\]

\[
= \frac{(z q^{\lambda_1 - \lambda_2 - 2}; q^{-4})_{\infty} (z q^{\lambda_1 - \lambda_2 - 2}; q^{-4})_{\infty}}{(z q^{\lambda_1 + \lambda_2 - 2}; q^{-4})_{\infty} (z q^{\lambda_1 - \lambda_2 - 2}; q^{-4})_{\infty}},
\]  \hspace{1cm} (26)

\[
(z; q)_{\infty} = \prod_{i=0}^{\infty} (1 - z q^k)
\]

\[
\bar{R}^0(z) = \exp \sum_{n \geq 1} \left( \frac{q^n + q^{-n}}{(q^n - q^{-n})} \left( q^{-\lambda_1 n} - K^{-n} \right) \otimes \left( K^n - q^{\lambda_2 n} \right) \right) \frac{z^n}{n}
\]

\[
\times \exp \sum_{n \geq 1} \left( (q^{ln} - K^{-n}) \otimes q^{-n} \frac{[\lambda_2 n]_q}{[n]_q} + q^n \frac{[\lambda_1 n]_q}{[n]_q} \otimes (K^n - q^{\lambda_2 n}) \right) \frac{z^n}{n}
\]  \hspace{1cm} (27)

By performing the infinite sum in (27) one can easily show that the term \( \bar{R}^0(z) \) acting on \( v_1^{\lambda_1} \otimes v_2^{\lambda_2} \) gives rise to the following expression, which is well defined in the limit \( q^N \to 1 \):

\[
\bar{R}^0(z) v_1^{\lambda_1} \otimes v_2^{\lambda_2} = \prod_{i=j-i+1}^{j} \left( 1 - q^{-2l} q^{\lambda_2 - \lambda_1} z \right)
\]

\[
\prod_{i=j-i+1}^{j} \left( 1 - q^{2l} q^{\lambda_2 - \lambda_1} z \right)
\]

\[
\times \left( \frac{\prod_{i=0}^{j-1} (1 - q^{-2l} q^{\lambda_2 + \lambda_1} z) v_1^{\lambda_1} \otimes v_2^{\lambda_2}}{\prod_{i=0}^{j-1} (1 - q^{2l} q^{\lambda_2 - \lambda_1} z) v_1^{\lambda_1} \otimes v_2^{\lambda_2}} \right)
\]  \hspace{1cm} (28)

The scalar factor \( f(z) \) \((26)\) is singular for \( q^N = 1 \). It can be omitted from the expression of \( R \)-matrix. So, the regular expression of \( R \)-matrix for \( q^N = 1 \) on \( M_{\lambda_1} \otimes M_{\lambda_2} \) has the form

\[
R_{\lambda_1, \lambda_2}(z) = R^+(z) \bar{R}^0(z) R^-(z)
\]  \hspace{1cm} (29)

Note that it satisfies \( R_{\lambda_1, \lambda_2}(z) v_0^{\lambda_1} \otimes v_0^{\lambda_2} = v_0^{\lambda_1} \otimes v_0^{\lambda_2} \). This renormalized expression of \( R \)-matrix doesn’t satisfy the quasitriangularity condition \((3)\). The
The intertwining property (2) and spectral parameter dependent Yang-Baxter equation

$$R_{\lambda_1,\lambda_2}(\frac{x_1}{x_2})R_{\lambda_1,\lambda_3}(\frac{x_1}{x_3})R_{\lambda_2,\lambda_3}(\frac{x_2}{x_3}) = R_{\lambda_2,\lambda_3}(\frac{x_2}{x_3})R_{\lambda_1,\lambda_3}(\frac{x_1}{x_3})R_{\lambda_1,\lambda_2}(\frac{x_1}{x_2})$$

are satisfied.

Let us consider now the possibility to restrict (29) on finite dimensional semicyclic modules. Recall that the semicyclic module $V_{\alpha,\lambda}$ is obtained by factorisation of $M_\lambda$ on $I_{\alpha,\lambda} = (F^N - \alpha M_\lambda$ for some $\alpha \in \mathbb{C}$:

$$V_{\alpha,\lambda} = M_\lambda / I_{\alpha,\lambda}$$

The $R$-matrix (29) is well defined on $V_{\alpha_1,\lambda_1} \otimes V_{\alpha_2,\lambda_2}$ if it preserves this factorization, i.e.

$$R_{\lambda_1,\lambda_2}(z)(M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \subset (M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \bigoplus (I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2})$$

and

$$R_{\lambda_1,\lambda_2}(z)(I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2}) \subset (M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \bigoplus (I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2})$$

The conditions above follow from

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) (\lambda_2^N \cdot F^N \otimes 1 + 1 \otimes F^N)$$

$$= (\lambda_1^N \cdot 1 \otimes F^N + F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)$$

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) (x^N \cdot F^N \otimes 1 + y^N \lambda_1^N \cdot 1 \otimes F^N)$$

$$= (y^N \cdot 1 \otimes F^N + x^N \lambda_2^N \cdot F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)$$

Here we used the intertwining property (2) for

$$\Delta(E_i^N) = E_i^N \otimes 1 + K_i^{-N} \otimes E_1 \quad \Delta(F_i^N) = F_i^N \otimes K_i^N + 1 \otimes F_i^N$$

So, one can express the operators

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) (F^N \otimes 1) \quad \text{and} \quad R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) (1 \otimes F^N)$$
as a linear combination of the operators

\[(F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) \quad \text{and} \quad (1 \otimes F^N)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)\]

(if \(\frac{x}{y}^N \neq \lambda_1^N \lambda_2^N\)). In the same way,

\[R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)((F^N - \lambda_1) \otimes 1) \quad \text{and} \quad R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)(1 \otimes (F^N - \lambda_2))\]

are a linear combinations of terms

\[((F^N - \lambda_1) \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) \quad \text{and} \quad (1 \otimes (F^N - \lambda_2))R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)\]

with the same coefficients if parameters \(x, y, \lambda_1, \lambda_2, \alpha_1, \alpha_2\) lie on the algebraic curve

\[
\frac{\alpha_1}{1 - \lambda_1^N} = \frac{\alpha_2}{1 - \lambda_2^N} \quad z^N = \left(\frac{x}{y}\right)^N = 1 \quad (33)
\]

In this case the factorisation conditions (31), (32) are fulfilled and \(R\)-matrix (29) can be reduced to \(R\)-matrix \(R_{V_{\alpha_1,\lambda_1} \otimes V_{\alpha_2,\lambda_2}}\) of semicyclic representations of \(U_\varepsilon \hat{sl}_2\), considered in [16, 18, 19]. The condition (33) on parameters of representations appears naturally as a consistency of factorisation \(V_{\alpha,\lambda} = M_{\lambda}/I_{\alpha,\lambda}\) with the intertwining property (2) of \(R\)-matrix.

Note that the formulae (23), (24), (27), (29) can be applied directly to semicyclic modules, using the constraint \(F^N = \alpha \cdot \text{id}\) on \(V_{\alpha,\lambda}\).

### 4 Discussions

Let’s consider now the possibility of restriction of the automorphism (19) in evaluation representation (20) to cyclic modules. Recall that their intertwining operators are Boltsman weight of Chiral Potts model (10). The cyclic modules are representations of quotient algebra \(Q_\xi = Q_{\beta,\alpha,\lambda}, \xi = (\beta, \alpha, \lambda)\), which is obtained from \(U_\varepsilon sl_2\) by factorisation on ideal \(I_{\beta,\alpha,\lambda}\), generated by \((F^N - \alpha), (E^N - \beta), (K^N - \lambda N)\), \((\beta, \alpha, \lambda \in C)\) (10):

\[Q_{\beta,\alpha,\lambda} = U_\varepsilon sl_2/I_{\beta,\alpha,\lambda}\]
The necessary condition for restriction of $\hat{R}(z)$ to $Q_\xi$ is the constraint on parameters of representation to lie on the algebraic curve, defined by

$$\frac{\alpha_1}{1-\lambda_1} = \frac{\alpha_2}{1-\lambda_2} = \left(\frac{z}{y}\right)^N = 1$$

$$\frac{\beta_1}{1-\lambda_1} = \frac{\beta_2}{1-\lambda_2} = \left(\frac{z}{y}\right)^N$$

(34)

We expect that this condition is sufficient also and automorphism $\hat{R}$ can be restricted on some automorphism (outer, in general) of quotient algebra $Q_\xi_1 \otimes Q_\xi_2$, which we denote by $\hat{R}^{Q_\xi_1} \otimes Q_\xi_2$.

Consider now its action on tensor product of cyclic modules $V_\xi_1 \otimes V_\xi_2$. $\hat{R}^{Q_\xi_1} \otimes Q_\xi_2$ reduced here to matrix algebra automorphism. Recall that every automorphism of matrix algebra is inner. So,

$$\hat{R}^{Q_\xi_1} \otimes Q_\xi_2 = \text{Ad}(R_{\xi_1,\xi_2})$$

with some matrix $R_{\xi_1,\xi_2}$. This $R$-matrix is nothing but to the Boltsmann weights of Chiral Potts model.

For quotients $Q_{0,0,\lambda}$, corresponding to semicyclic irreps, this suggestion is true.

Note that in case of $q^4 = 1$ there is a Hopf algebra homomorphism between different quotients, as it was observed in [24]. This fact was used there to construct $R$-matrices of quotient algebras for $q^4 = 1$ from the $R$-matrix of $Q_{0,0,\lambda_1} \otimes Q_{0,0,\lambda_1}$, which corresponds to nilpotent irreps.

Another question is to extend these results in case of other quantum algebras.

Then we had finished this work, we saw the papers [25] and [26], where the center of quantum Kac-Moody algebras was studied also. As it was observed there the automorphisms $\omega_\pm$ (18) corresponds to translations of quantum Weyl group.

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