SOME RESULTS ON $\star$-RICCI FLOW

Dipankar Debnath and Nirabhra Basu

Abstract. In this paper we have introduced the notion of $\star$-Ricci flow and shown that $\star$-Ricci soliton which was introduced by Kaimakamis and Panagiotidou in 2014 is a self similar soliton of the $\star$-Ricci flow. We have also found the deformation of geometric curvature tensors under $\star$-Ricci flow. In the last two sections of the paper, we have found the $\mathcal{F}$-functional and $\omega$-functional for $\star$-Ricci flow respectively.

Keywords: $\star$-Ricci flow, Conformal Ricci flow, $\mathcal{F}$-functionals, $\omega$-functionals.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(M, g)$ by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0,$$

where $\mathcal{L}$ denotes the Lie derivative operator, $\lambda$ is a constant and $S$ is the Ricci tensor of the metric $g$. Tachibana [3] first introduced $\star$-Ricci tensor on almost Hermitian manifolds and Hamada [1] apply this to almost contact manifolds by defining

$$S^\star(X, Y) = \frac{1}{2} \text{trace}(Z \rightarrow R(X, \phi Y)\phi Z),$$

for any $X, Y \in TM$. In 2014, Kaimakamis and Panagiotidou [2] introduced the concept of $\star$-Ricci solitons within the background of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor $S$ in (1.1) with the $\star$-Ricci tensor $S^\star$. More precisely, a $\star$-Ricci soliton on $(M, g)$ is defined by

$$\mathcal{L}_V g + 2S^\star + 2\lambda g = 0.$$
have obtained deformation of geometric curvature tensor under \(\ast\)-Ricci flow. We have also provided the rate of change of \(F\)-functionals and \(\omega\)-entropy functional with respect to time under this flow.

We have defined \(\ast\)-Ricci flow as follows

\[
\frac{\partial g}{\partial t} = -2S^\ast(X, Y).
\]

(1.3)

In this paper we have shown that just like Ricci soliton; \(\ast\)-Ricci soliton is a self-similar soliton of the \(\ast\)-Ricci flow. We have also found the deformation of geometric curvature tensors under \(\ast\)-Ricci flow.

**Proposition 1.1.** Defining \(\bar{g}(t) = \sigma(t) \phi^*_t(g) + \sigma(t) \psi^*_t \Big( \frac{\partial g}{\partial t} \Big) + \sigma(t) \psi^*_t(\mathcal{L}X g),\) we have

\[
\frac{\partial \bar{g}}{\partial t} = \bar{\sigma}(t) \psi^*_t(g) + \sigma(t) + \psi^*_t \Big( \frac{\partial g}{\partial t} \Big) + \sigma(t) \psi^*_t(\mathcal{L}X g).
\]

(1.4)

**Proof:** This follows from the definition of Lie derivative. If we have a metric \(g\), a vector field \(Y\) and \(\lambda \in \mathbb{R}\) such that

\[-2 \text{Ric}^*(g_0) = \mathcal{L}Y g_0 - 2\lambda g_0\]

after setting \(g(t) = g_0\) and \(\sigma(t) = 1 - 2\lambda t\) and then integrating the \(t\)-dependent vector field \(X(t) = \frac{1}{\sigma(t)}Y\). To give a family of deffeomorphism \(\psi_t\) with \(\psi_0\) the identity then \(\bar{g}\) defined previously is a Ricci flow with

\[\bar{g} = g_0 \frac{\partial \bar{g}}{\partial t} = \sigma'(t) \phi^*_t(g_0) + \sigma(t) \phi^*_t(\mathcal{L}X g_0)\]

\[= \phi^*_t(-2\lambda g_0 + \mathcal{L}Y g_0) = \phi^*_t(-2 \text{Ric}^*(g_0)) = -2 \text{Ric}^*(\bar{g}).\]

**Proposition 1.2.** Under \(\ast\)-Ricci flow

\[g(\frac{\partial}{\partial y} \nabla_X Y, Z) = -2(\nabla_X S^\ast)(Y, Z) + 2S^\ast(Y, \nabla_X Z) + 2S^\ast(\nabla_X Y, Z).\]

**Proof.** Let us consider

\[\frac{\partial}{\partial t} \nabla_X Y = \pi(X, Y).\]

Now we can write

\[g(\frac{\partial}{\partial t} \nabla_X Y, Z) = g(\pi(X, Y), Z).\]

(1.5)

Again

\[g(\frac{\partial}{\partial t} \nabla_X Y, Z) = \frac{\partial}{\partial t} g(\nabla_X Y, Z) - \frac{\partial g}{\partial t}(\nabla_X Y, Z).\]

(1.6)

\[g(\pi(X, Y), Z) = \frac{\partial}{\partial t} g(\nabla_X Y, Z) + 2S^\ast(\nabla_X Y, Z).\]
We have
\[ Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \]  
\[ (1.7) \]

From (1.5) we have
\[ g(\pi(X, Y), Z) = \partial_t [Xg(Y, Z) - g(Y, \nabla_X Z)] + 2S^*(\nabla_X Y, Z) \]
\[ g(\pi(X, Y), Z) = X\frac{\partial g}{\partial t}(Y, Z) - (\frac{\partial g}{\partial t})(Y, \nabla_X Z) + 2S^*(\nabla_X Y, Z) \]
or
\[ g(\pi(X, Y), Z) = -2(\nabla_X S^*)(Y, Z) + 2S^*(Y, \nabla_X Z) + 2S^*(\nabla_X Y, Z) \]
i.e.
\[ (1.8) \]
\[ g(\partial g \nabla_X Y, Z) = -2(\nabla_X S^*)(Y, Z) + 2S^*(Y, \nabla_X Z) + 2S^*(\nabla_X Y, Z). \]

2. The $\mathcal{F}$-functional for the $*$-Ricci flow

Let $M$ be a fixed closed manifold, $g$ a Riemannian metric and $f$ a function defined on $M$ to the set of real numbers $\mathbb{R}$.

Then the $\mathcal{F}$-functional on pair $(g, f)$ is defined as
\[ \mathcal{F}(g, f) = \int (-1 + |\nabla f|^2)e^{-f}dV. \]
\[ (2.1) \]

Now, we will establish how the $\mathcal{F}$-functional changes according to time under $*$-Ricci flow.

**Theorem 2.1.** In $*$-Ricci flow the rate of change of $\mathcal{F}$-functional with respect of time is given by
\[ \frac{d}{dt} \mathcal{F}(g, f) = \int [-2\text{Ric}^*(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(|\nabla f|^2) 
+ (-1 + |\nabla f|^2)(-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV \]
where
\[ \mathcal{F}(g, f) = \int (-1 + |\nabla f|^2)e^{-f}dV. \]

**Proof.** We may calculate
\[ \frac{\partial}{\partial t} |\nabla f|^2 = \frac{\partial}{\partial t} g(\nabla f, \nabla f) = \frac{\partial g}{\partial t}(\nabla f, \nabla f) + 2g(\frac{\partial f}{\partial t}, \nabla f). \]
\[ (2.2) \]
So using proposition 2.3.12 of [13] we can write

\[
\frac{d}{dt} \mathcal{F}(g, f) = \int \left[ \frac{\partial g}{\partial t}(\nabla f, \nabla f) + 2g(\nabla \frac{\partial f}{\partial t}, \nabla f) \right] e^{-f} dV \\
+ \int (-1 + |\nabla f|^2)[-\frac{\partial f}{\partial t} + \frac{1}{2} \text{tr} \frac{\partial g}{\partial t}] e^{-f} dV.
\]  

(2.3)

Using integration by parts of equation (2.2), we get

\[
\int 2g(\nabla \frac{\partial f}{\partial t}, \nabla f)e^{-f} dV = -2 \int \frac{\partial f}{\partial t} (\Delta f - |\nabla f|^2) e^{-f} dV.
\]  

(2.4)

Now putting (2.4) in (2.3), we get

\[
\frac{d}{dt} \mathcal{F}(g, f) = \int \left[ \frac{\partial g}{\partial t}(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2) \\
+ (-1 + |\nabla f|^2)[-\frac{\partial f}{\partial t} + \frac{1}{2} \text{tr} \frac{\partial g}{\partial t}] \right] e^{-f} dV.
\]  

(2.5)

Using (1.3) in (2.5), we get the following result for conformal Ricci flow, as

\[
\frac{d}{dt} \mathcal{F}(g, f) = \int [-2\text{Ric}^*(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2) \\
+ (-1 + |\nabla f|^2)[-\frac{\partial f}{\partial t} + \frac{1}{2} \text{tr} \frac{\partial g}{\partial t}] \right] e^{-f} dV.
\]  

(2.6)

Hence the proof.

3. \(\omega\)-entropy functional for the \(*\)-Ricci flow

Let \(M\) be a closed manifold, \(g\) a Riemannian metric on \(M\) and \(f\) a smooth function defined from \(M\) to the set of real numbers \(\mathbb{R}\). We define \(\omega\)-entropy functional as

\[
\omega(g, f, \tau) = \int [\tau \text{R}^* + |\nabla f|^2] + f - n] udV
\]

where \(\tau > 0\) is a scale parameter and \(u\) is defined as \(u(t) = e^{-f(t)} \int_M udV = 1\).

We would also like to define heat operator acting on the function \(f: M \times [0, \tau] \to \mathbb{R}\) by \(\diamond := \frac{\partial}{\partial t} - \Delta\) and also, \(\diamond^* := -\frac{\partial}{\partial t} - \Delta + R^*\), conjugate to \(\diamond\).

We choose \(u\), such that \(\diamond^* u = 0\).

Now we prove the following theorem.

**Theorem 3.1:** If \(g, f, \tau\) evolve according to

\[
\frac{\partial g}{\partial t} = -2\text{Ric}^*
\]  

(3.2)
Some Results on $\ast-$Ricci Flow

(3.3) \[\frac{\partial \tau}{\partial t} = -1\]

(3.4) \[\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R^* + \frac{n}{2\tau}\]

and the function $v$ is defined as $v = [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]u$, the rate of change of $\omega$-entropy functional for conformal Ricci flow is

\[\frac{d\omega}{dt} = -\int_M \bigstar v,\]

where

\[\bigstar v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - u\tau[4 < \text{Ric}^*, \text{Hess} f > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|\text{Hess} f|^2].\]

**Proof:** We find that

\[\bigstar v = \bigstar(u\frac{v}{u}) = \frac{v}{u}\bigstar u + u\bigstar(u\frac{v}{u}).\]

We have defined previously that $\bigstar u = 0$,

so

\[\bigstar v = u\bigstar(u\frac{v}{u})\]

\[\bigstar v = u\bigstar\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n].\]

We shall use the conjugate of heat operator, as defined earlier as $\bigstar = -\left(\frac{\partial}{\partial t} + \Delta - R^*\right)$.

Therefore

\[\bigstar v = -u\left(\frac{\partial}{\partial t} + \Delta - R^*\right)[\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]\]

\[\Rightarrow u^{-1}\bigstar v = -\left(\frac{\partial}{\partial t} + \Delta\right)[\tau(2\Delta f - |\nabla f|^2 + R^*)]

\[\qquad \quad - (\frac{\partial}{\partial t} + \Delta)f - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n].\]

Using equation (3.3), we have

\[u^{-1}\bigstar v = (2\Delta f - |\nabla f|^2 + R^*) - \tau(\frac{\partial}{\partial t} + \Delta)(2\Delta f - |\nabla f|^2 + R^*)\]

\[\quad - \frac{\partial f}{\partial t} - \Delta f - \frac{v}{u}.\]
Now
\[
\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2 \frac{\partial}{\partial t}(\Delta f) - \frac{\partial}{\partial t}|\nabla f|^2.
\]

Using proposition (2.5.6) of [13], we have
\[
\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta \frac{\partial f}{\partial t} + 4 <\text{Ric}^*, \text{Hess} f > - \frac{\partial g}{\partial t}(\nabla f, \nabla f) - 2g(\frac{\partial}{\partial t}\nabla f, \nabla f).
\]

Now using the *–Ricci flow equation (1.3), we have
\[
\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta \frac{\partial f}{\partial t} + 4 <\text{Ric}^*, \text{Hess} f > + 2\text{Ric}^*(\nabla f, \nabla f) - 2g(\frac{\partial}{\partial t}\nabla f, \nabla f).
\]

Using (3.4) in (3.6), we get
\[
\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta(-\Delta f + |\nabla f|^2 - R^* + \frac{n}{2\tau}) + 4 <\text{Ric}^*, \text{Hess} f > + 2\text{Ric}^*(\nabla f, \nabla f)
\]

Now let us compute
\[
\Delta(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta^2 f - \Delta|\nabla f|^2.
\]

Using (3.7) and (3.8) in (3.5) we obtain after a brief calculation
\[
w^{-1}\ast v = (2\Delta f - |\nabla f|^2 + R^*) - \tau[-2\Delta^2 f + 2\Delta|\nabla f|^2 + 4 <\text{Ric}^*, \text{Hess} f >
\]
\[
+2\text{Ric}^*(\nabla f, \nabla f) - 2g(\frac{\partial f}{\partial t}\nabla f, \nabla f) + 2\Delta^2 f - \Delta|\nabla f|^2 - \frac{\partial f}{\partial t} - \Delta f - \frac{\Delta f}{\tau} - \frac{\Delta f}{\tau} - \frac{\Delta f}{\tau}
\]
\[
= \Delta f - |\nabla f|^2 + R^* - \tau|\Delta|\nabla f|^2 + 4 <\text{Ric}^*, \text{Hess} f > + 2\text{Ric}^*(\nabla f, \nabla f)
\]
\[
- 2g(\frac{\partial f}{\partial t}\nabla f, \nabla f) - \frac{\partial f}{\partial t} - \frac{\Delta f}{\tau} - \frac{\Delta f}{\tau}
\]
\[
= \Delta f - |\nabla f|^2 + R^* - \tau|\Delta|\nabla f|^2 + 4 <\text{Ric}^*, \text{Hess} f > + 2\text{Ric}^*(\nabla f, \nabla f)
\]
\[
- 2g(\frac{\partial f}{\partial t}\nabla f, \nabla f) + \Delta f - |\nabla f|^2 + R^* - \frac{\Delta f}{\tau} - \frac{\Delta f}{\tau}
\]
\[
= 2(\Delta f - |\nabla f|^2 + R^*) - \frac{\Delta f}{\tau} - \frac{\Delta f}{\tau} - \tau|\Delta|\nabla f|^2 + 4 <\text{Ric}^*, \text{Hess} f >
\]
\[
+ 2\text{Ric}^*(\nabla f, \nabla f) - 2g(\frac{\partial f}{\partial t}\nabla f, \nabla f)
\]
\( u^{-1} \diamond v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n] - \tau|\Delta|\nabla f|^2 + 4 < Ric^*, Hess f > + 2Ric^*(\nabla f, \nabla f) - 2g(\frac{\partial f}{\partial t}, \nabla f) \). \]

Using (3.4), we get

\[
\begin{align*}
u^{-1} \diamond v &= 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^*] \\
&\quad + R^* + \Delta|\nabla f|^2 + 4 < Ric^*, Hess f > + 2Ric^*(\nabla f, \nabla f) \\
&\quad - 2g(\nabla(-\Delta f + |\nabla f|^2 + \frac{n}{2\tau} - R^*), \nabla f). \tag{3.10}
\end{align*}
\]

We can rewrite (3.10) in the following way

\[
\begin{align*}
u^{-1} \diamond v &= 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^*] \\
&\quad + 4 < Ric^*, Hess f > - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) \\
&\quad + \tau[-\Delta|\nabla f|^2 - 2Ric^*(\nabla f, \nabla f) + 2g(\nabla(\Delta f), \nabla f)]. \tag{3.11}
\end{align*}
\]

and using Bochner formula in (3.11) and simplifying it, we get

\[
\begin{align*}
\Rightarrow \quad u^{-1} \diamond v &= 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^*] \\
&\quad + 4 < Ric^*, Hess f > - 2g(\nabla|\nabla f|^2, \nabla f) \\
&\quad + 4g(\nabla(\Delta f), \nabla f) - 2\tau|Hess f|^2. \\
\end{align*}
\]

i.e.

\[
\begin{align*}
u^{-1} \diamond v &= 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - \frac{v}{u} - \tau[4 < Ric^*, Hess f > \\
&\quad - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hess f|^2. \tag{3.12}
\end{align*}
\]
So finally we have

\[ \hat{\Delta}^*v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - ur[4 < Ric^*, Hess f > - 2g(\nabla |\nabla f|^2, \nabla f) + 4g(\nabla (\Delta f), \nabla f) + 2|Hess f|^2]. \]  

(3.13)

Now using remark (8.2.7) of [13], we get

\[ \frac{d\omega}{dt} = -\int_M \hat{\Delta}^*v. \]

So the evolution of \( \omega \) with respect to time can be found by this integration.

**Acknowledgement.** The authors are thankful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.

**REFERENCES**

1. T. Hamada: *Real hypersurfaces of complex space forms in terms of Ricci \(-\star\)-tensor*, Tokyo J. Math., 2002, 25, 473-483.
2. G. Kaimakamis and K. Panagiotidou: *\(-\star\)-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys., 2014, 86, 408-413.
3. S. Tachibana: *On almost-analytic vectors in almost \(\star\)-Kählerian manifolds*, Tohoku Math. J., 1959, 11, 247-265.
4. X. Dai, Y. Zhao and U. C. De: *\(-\star\)-Ricci soliton on \((\kappa; \mu)\)'-almost Kenmotsu manifolds*, Open Math. 17 (2019), 74-882.
5. V. Venkatesha, H. A. Kumara and D. M. Naik: *Almost \(-\star\)-Ricci soliton on para Kenmotsu manifolds*, Arab. J. Math. (2019). https://doi.org/10.1007/s40065-019-00269-7.
6. A. K. Huchchappa, D. M. Naik and V. Venkatesha: *Certain results on contact metric generalized \((\kappa; \mu)\)-space forms*, Commun. Korean Math. Soc. 34 (4) (2019), 1315-1328.
7. G. Perelman: *The entropy formula for the Ricci flow and its geometric applications*, arXiv.org/abs/math/0211159, (2002) 1-39.
8. N. Basu and D. Debnath: *Characteristic of conformal Ricci-soliton and conformal gradient Ricci soliton in LP-Sasakian manifold*, accepted in PJM(Palestine Journal of Mathematics).
9. Venkatesha, D. M. Naik and H. A. Kumara: *\(-\star\)-Ricci solitons and gradient almost \(-\star\)-Ricci solitons on Kenmotsu manifolds*, arXiv.org/abs/arXiv:1901.05222v [Math.DG] 16 Jan 2019.
10. G. Perelman: *Ricci flow with surgery on three manifolds*, arXiv.org/abs/math/0303109, (2002), 1-22.
11. R. S. Hamilton: *Three Manifold with positive Ricci curvature*, J.Differential Geom.17(2), (1982), 255-306.
12. B. CHOW, P. LU, L. NI: *Hamilton's Ricci Flow*, American Mathematical Society Science Press, 2006.

13. P. Topping: *Lecture on The Ricci Flow*, Cambridge University Press; 2006.

14. A. E. Fischer: *An introduction to conformal Ricci flow*, Class. Quantum Grav.21(2004), S171 - S218.

15. K. Mandal and S. Makhal: *-Ricci solitons on three dimensional normal almost contact metric manifolds*, Lobachevskii Journal of Mathematics, 40, 189-194, 2019.

16. A. Ghosh and D. S. Patra: *-Ricci Soliton within the frame-work of Sasakian and (κ, μ)-contact manifold*, International Journal of Geometric Methods in Modern Physics, Vol. 15, No. 07, 1850120 (2018).

17. K. De and C. Dey: *-Ricci solitons on (ε) - para Sasakian manifolds*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics. 2019, Vol. 12 Issue 2, p265-274.

18. P. Majhi, U. C. De and Y. J. Suh: *-Ricci solitons on Sasakian 3-manifolds*, Publicationes mathematicae, 2018, 93(1-2):241-252.

19. D. Dey and P. Majhi: *N(k)-contact metric as *-conformal Ricci soliton*, arXiv.org.math arXiv:2005.02194.

20. D. G. Prakasha and P. Veeresha: *Para-Sasakian manifolds and *-Ricci solitons*, Afrika Matematika volume 30, pages. 989–998(2019).

Dipankar Debnath
Department of Mathematics
Bamanpukur High School(H.S)
Bamanpukur, Sree Mayapur
Nabadwip, Nadia
West Bengal, Pin-741313
India
dipankardebnath123@gmail.com

Nirabhra Basu
Department of Mathematics
Bhawanipur Education Society College
Kolkata-700020
West Bengal
India
nirabhra.basu@thebges.edu.in