GLOBAL EXISTENCE AND EXPONENTIAL DECAY OF STRONG SOLUTIONS TO THE CAUCHY PROBLEM OF 3D DENSITY-DEPENDENT NAVIER-STOKES EQUATIONS WITH VACUUM

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Abstract. This paper deals with the 3D incompressible Navier-Stokes equations with density-dependent viscosity in the whole space. The global well-posedness and exponential decay of strong solutions is established in the vacuum cases, provided the assumption that the bound of density is suitably small, which extends the results of [Nonlinear Anal. Real World Appl., 46:58–81, 2019] to the global one. However, it’s entirely different from the recent work [arXiv:1709.05608v1, 2017] and [J. Math. Fluid Mech., 15:747–758, 2013], there is not any smallness condition on the velocity.

1. Introduction. The motion of a three-dimensional (3D) density-dependent incompressible fluid is governed by the following Navier-Stokes equations:

\[\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \pi &= \text{div}(\mu(\rho) \nabla u), \\
\text{div} u &= 0,
\end{align*}\]

Here, \(t \geq 0\) is time, \(x \in \mathbb{R}^3\) is the spatial coordinates, and the unknown functions \(\rho = \rho(x,t), \quad u = (u^1, u^2, u^3)(x,t), \quad \pi = \pi(x,t)\) denote the density, velocity, and pressure of the fluid, respectively. The viscosity \(\mu(\rho)\) satisfies the following hypothesis:

\[\mu(\xi) \in C^1[0, \infty), \quad 0 < \underline{\mu} \leq \mu(\xi) \leq \overline{\mu} < \infty \quad \text{for} \quad \forall \xi \in [0, \infty).\]

We consider the Cauchy problem of (1.1) with \((\rho, u)\) vanishing at infinity and the initial conditions:

\[\begin{align*}
\rho(x,0) &= \rho_0(x), \\
\rho u(x,0) &= m_0(x), \\
x &\in \mathbb{R}^3,
\end{align*}\]

for given initial data \(\rho_0\) and \(m_0\).

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In the past decades, there has been lots of literatures on the mathematical study of nonhomogeneous incompressible flow, which was initiated by the Russian school (cf. [1, 2, 12, 14]) when the viscosity coefficient $\mu$ is a positive constant and the density is strictly away from vacuum (i.e., the density $\rho$ is strictly positive). In particular, Kazhikov [1, 2, 12] studied the global weak solutions and local strong solutions. Later, Ladyzenskaja-Solonnikov [14] constructed the unique solvability to the nonhomogeneous Navier-Stokes equations in a bounded domain of $\mathbb{R}^N (N = 2, 3)$. It is also well known that the local strong solution is indeed a global one in either two dimensions or three dimensions with small data.

When the initial density allows for vacuum (i.e. the initial density may vanish in some open sets), the issue of the existence of solutions becomes much more complicated due to the possible degeneracy near vacuum. The global weak solutions of (1.1) were constructed by Simon [19]. To overcome the difficulties caused by the presence of vacuum, Choe-Kim [4] proposed the following compatibility condition:

$$-\mu \Delta u_0 + \nabla \pi_0 = \sqrt{\rho_0} g,$$

for some $(\nabla \pi_0, g) \in L^2$, (1.4)

and studied the local existence of strong solutions of (1.1) with large data in dimension two and three. Very recently, Li [15] observed that condition (1.4) can be removed, but the density must still be regular and only local-in-time solutions are produced. Huang-Wang [9] and Lv-Shi-Zhong [17] obtained the 2D global strong solutions with large initial data for the initial boundary value problem and the Cauchy one, respectively. For the three-dimensional case, Kim [13] obtained a Serrin’s type blow-up criterion of strong solutions in weak Lebesgue space, and showed that if $\|\nabla u_0\|_{L^2}$ is sufficiently small, then (1.1) has a unique global strong solution. This result was recently improved by Craig-Huang-Wang [5] by requiring that the norm of $\|u_0\|_{\dot{H}^{1/2}}$ is small enough, where $\dot{H}^{1/2}$ denotes the homogeneous fractional Sobolev space with $1/2$-order.

For the viscosity coefficient $\mu(\rho)$ depends on $\rho$, Lions [16] derived the global existence of weak solutions. Later, Desjardins [6] proved the global weak solution with more regularity for the two-dimensional case provided that $\mu(\rho_0)$ is a small perturbation of a positive constant in $L^\infty$-norm. Cho-Kim [3] established the local strong solution by using some compatibility condition. Huang-Wang [10] obtained global strong solution to the 2D Navier-Stokes equations provided the $\|\nabla \mu(\rho_0)\|_{L^q} (q > 2)$ is small enough. This result was later generalized to the 3D case by Zhang [21] and Huang-Wang [11] when $\|\nabla u_0\|_{L^2}$ is small, Yu-Zhang [20] when the bound of density is suitably small, or when the total mass is small with large oscillations. More recently, He-Li-Lv [8] proved the global existence of strong solutions provided $\|u_0\|_{\dot{H}^{1/2}} (\beta \in (1/2, 1])$ is small over $\mathbb{R}^3$.

Before stating the main results, we first explain the notations and conventions used throughout this paper. Set

$$\int f \, dx := \int_{\mathbb{R}^3} f \, dx.$$

Moreover, for $1 \leq r \leq \infty$, $k \geq 1$, and $\beta > 0$, the standard Sobolev spaces are defined as follows:
Theorem 1.1. For some constant $q > 3$, assume that the initial data $(\rho_0, m_0)$ satisfy

$$0 \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in L^1 \cap L^2 \cap H^1, \quad \nabla \mu(\rho_0) \in L^q, \quad u_0 \in D^{1}_{0,\sigma}, \quad m_0 = \rho_0 u_0. \quad (1.5)$$

Then for

$$\|\nabla \mu(\rho_0)\|_{L^q} := M_1, \quad \|\nabla u_0\|_{L^2}^2 := M_2,$$

there exists some small positive constant $\epsilon_0$ depending only on $q, \mu, \pi, \|\rho_0\|_{L^1 \cap L^2}$, $M_1$, and $M_2$ such that if

$$\bar{\rho} \leq \epsilon_0, \quad (1.6)$$

the Cauchy problem (1.1)-(1.3) admits a unique strong solution $(\rho, u, \pi)$ satisfying that for any $0 < \tau < T < \infty$ and $p \in [2, p_0)$ with $p_0 := \min\{6, q\}$,

$$0 \leq \rho \in C([0, T]; L^1 \cap H^1), \quad \nabla \mu(\rho) \in C([0, T]; L^q), \quad \nabla u \in L^\infty(0, T; L^2) \cap L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau, T]; H^1 \cap W^{1,p}),$$

$$\pi \in L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau, T]; H^1 \cap W^{1,p}), \quad (1.7)$$

Moreover, it holds that

$$\sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2} \leq 2\|\nabla \rho_0\|_{L^2}, \quad \sup_{0 \leq t < \infty} \|\nabla \mu(\rho)\|_{L^q} \leq 2\|\nabla \mu(\rho_0)\|_{L^q},$$

and that there exists some positive constant $\sigma$ depending only on $\|\rho_0\|_{L^1 \cap L^2}$ and $\mu$ such that for all $t \geq 1$,

$$\|\nabla u_l(t, \cdot)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^1 \cap W^{1,p_0}}^2 + \|\pi(\cdot, t)\|_{H^1 \cap W^{1,p_0}}^2 \leq C e^{-\sigma t}, \quad (1.9)$$

where $C$ depends only on $q$, $\|\rho_0\|_{L^1 \cap L^2}$, $\mu$, $\pi$, $M_1$, $M_2$, and $\|\nabla \rho_0\|_{L^2}$.

Remark 1.1. It should be noted here that our Theorem 1.1 holds for arbitrarily large initial velocity with a smallness assumption only on the bound of density, which is in sharp contrast to Craig-Huang-Wang [5] and He-Li-Lv [8] where they need the smallness assumptions on $\|u_0\|_{H^\frac{1}{2}}$ and $\|u_0\|_{H^\beta} (\beta \in (1/2, 1))$, respectively.
2. Preliminaries. The following local existence theorem which has been proved in [18].

**Lemma 2.1.** Assume that \((\rho_0, u_0)\) satisfies \((1.5)\). Then there exists a small time \(T_0 > 0\) and a unique strong solution \((\rho, u, \pi)\) to the problem \((1.1)-(1.3)\) in \(\mathbb{R}^3 \times (0, T_0)\) satisfies \((1.7)\).

To derive the estimates of the derivatives of the solutions, we need the following regularity on the Stokes equations, whose proof can be found in [8].

**Lemma 2.2.** For positive constants \(\mu, \overline{\mu},\) and \(q \in (3, \infty),\) in addition to \((1.2)\), assume that \(\mu(\rho)\) satisfies

\[
\nabla \mu(\rho) \in L^q. \tag{2.1}
\]

Then, if \(F \in L^2 \cap L^r\) with \(r \in \left[\frac{2q}{q+2}, q\right],\) there exists some positive constant \(C\) depending only on \(\mu, \overline{\mu}, r,\) and \(q\) such that the unique weak solution \((u, \pi) \in D^1_{0,\sigma} \times L^2\) to the following Cauchy problem

\[
\begin{aligned}
&-\text{div}(\mu(\rho)\nabla u) + \nabla \pi = F, \quad x \in \mathbb{R}^3, \\
&\text{div} u = 0, \quad x \in \mathbb{R}^3, \\
&u(x) \to 0, \quad |x| \to \infty,
\end{aligned} \tag{2.2}
\]

satisfies

\[
\|\nabla u\|_{L^2} + \|\pi\|_{L^2} \leq C\|F\|_{L^{\frac{q}{6}}}, \tag{2.3}
\]

\[
\|\nabla^2 u\|_{L^{r'}} + \|\nabla \pi\|_{L^{r'}} \leq C\|F\|_{L^r} + C\|\nabla \mu(\rho)\|_{L^{\frac{2q+6}{q-3}}} \|F\|_{L^{\frac{q}{6}}}. \tag{2.4}
\]

Moreover, if \(F = \text{div} g\) with \(g \in L^2 \cap L^\tilde{r}\) for some \(\tilde{r} \in \left(\frac{6q}{q+8}, q\right),\) there exists a positive constant \(C\) depending only on \(\mu, \overline{\mu}, q,\) and \(\tilde{r}\) such that the unique weak solution \((u, \pi) \in D^1_{0,\sigma} \times L^2\) to \((2.2)\) satisfies

\[
\|\nabla u\|_{L^{2} \cap L^{\tilde{r}}} + \|\pi\|_{L^{2} \cap L^{\tilde{r}}} \leq C\|g\|_{L^{2} \cap L^{\tilde{r}}} + C\|\nabla \mu(\rho)\|_{L^{\frac{q(\tilde{r}-2)}{\tilde{r}(\tilde{r}-3)}}} \|g\|_{L^2}. \tag{2.5}
\]

3. A priori estimates. In this section, we will establish some necessary a priori bounds of local strong solutions \((\rho, u, \pi)\) to the Cauchy problem \((1.1)-(1.3)\) whose existence is guaranteed by Lemma 2.1. Thus, let \(T > 0\) be a fixed time and \((\rho, u, \pi)\) be the smooth solution to \((1.1)-(1.3)\) on \(\mathbb{R}^3 \times (0, T]\) with smooth initial data \((\rho_0, u_0)\) satisfies \((1.5).\) For simplicity, we shall use the letters \(C\) and \(C_i (i = 1, 2, \ldots)\) to denote the generic constants which may be dependent on \(q, \mu, \overline{\mu},\) and \(\|\rho_0\|_{L^{1} \cap L^{\frac{3}{2}}},\) but independent of \(T.\)

We aim to get the following key a priori estimates on \((\rho, u, \pi)\).

**Proposition 3.1.** There exist some small positive constant \(\epsilon_0\) depending only on \(q, \mu, \overline{\mu}, \|\rho_0\|_{L^{1} \cap L^{\frac{3}{2}}}, M_1,\) and \(M_2\) such that if \((\rho, u, \pi)\) is a smooth solution of \((1.1)-(1.3)\) on \(\mathbb{R}^3 \times (0, T]\) satisfying

\[
\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_{L^q} \leq 4M_1, \quad \sup_{0 \leq t \leq T} e^{\sigma t} \|\nabla u\|_{L^2}^2 \leq 4\overline{\mu} M_2, \tag{3.1}
\]
the following estimates hold

\[ \sup_{0 \leq t \leq T} \| \nabla \mu(\rho) \|_{L^2} \leq 2M_1, \quad \sup_{0 \leq t \leq T} e^{\sigma t} \| \nabla u \|_{L^2}^2 \leq \frac{T}{\mu} M_2, \]  

(3.2)

provided

\[ \bar{\rho} \leq \epsilon_0, \]  

(3.3)

where

\[ \epsilon_0 := \min \left\{ \frac{1}{(2C_1)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(4C_2 M_1)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(4C_3)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(4C_4)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_5)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_6)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_7)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_8)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_9)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_10)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_11)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_12)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_13)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_14)^{\frac{2}{3}} M_2^\frac{4}{3}}, \frac{1}{(5C_15)^{\frac{2}{3}} M_2^\frac{4}{3}}, \right\}. \]

Before proving Proposition 3.1, we establish some necessary a priori estimates, see Lemmas 3.1–3.4.

We begin with the nonnegativity and boundedness of density and basic energy estimates for \((\rho, u, \pi)\).

**Lemma 3.1.** Let \((\rho, u, \pi)\) be a smooth solution to (1.1)–(1.3) satisfy (3.1). Then for

\[ \sigma := \frac{3\mu}{4\|\rho_0\|_{L^2}^{\frac{1}{3}}}, \]  

(3.4)

it holds that

\[ 0 \leq \rho(x, t) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) = \bar{\rho}, \quad \| \rho \|_{L^\frac{2}{3} \cap L^1} = \| \rho_0 \|_{L^\frac{2}{3} \cap L^1}, \]  

(3.5)

\[ \sup_{0 \leq t \leq T} \| \sqrt{\rho} u \|_{L^2}^2 + 2\mu \int_0^T \| \nabla u \|_{L^2}^2 dt \leq \frac{4}{3} \| \rho_0 \|_{L^\frac{2}{3} \cap L^1}^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M_2, \]  

(3.6)

\[ \sup_{0 \leq t \leq T} e^{\sigma t} \| \sqrt{\rho} u \|_{L^2}^2 + \mu \int_0^T e^{\sigma t} \| \nabla u \|_{L^2}^2 dt \leq \frac{4}{3} \| \rho_0 \|_{L^\frac{2}{3} \cap L^1}^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M_2. \]  

(3.7)

**Proof.** Note that (3.5) follows from the transport equation (1.1) and making use of (1.1) \((\text{see Lions [16, Theorem 2.1]}))\]. To prove (3.6), multiplying (1.1) by \(u\) and integrating by parts lead to

\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} u \|_{L^2}^2 + \| \mu(\rho) \nabla u \|_{L^2}^2 = 0, \]  

(3.8)

which, integrated over \([0, T]\), by Sobolev inequality [7, II.3.11] and interpolation inequality, and (3.5), give

\[ \sup_{0 \leq t \leq T} \| \sqrt{\rho} u \|_{L^2}^2 + 2\mu \int_0^T \| \nabla u \|_{L^2}^2 dt \leq \| \sqrt{\rho_0} u_0 \|_{L^2}^2 \leq \frac{4}{3} \| \rho_0 \|_{L^\frac{2}{3} \cap L^1}^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} \| \nabla u_0 \|_{L^2}^2. \]  

(3.9)
Moreover, according to result of He-Li-Lv [8, (3.26)], yields that
\[ \frac{d}{dt} \| \sqrt{\rho} u \|_{L^2}^2 + \sigma \| \sqrt{\rho} u \|_{L^2}^2 + \mu \| \nabla u \|_{L^2}^2 \leq 0, \]
with \( \sigma \) is defined as in (3.4), and by integrating over \((0, T)\), gives
\[ \sup_{0 \leq t \leq T} e^{\sigma t} \| \sqrt{\rho} u \|_{L^2}^2 + \mu \int_0^T e^{\sigma t} \| \nabla u \|_{L^2}^2 dt \leq \| \sqrt{\rho_0} u_0 \|_{L^2}^2 \leq \frac{4}{3} \| \rho_0 \|_{L^2}^{\frac{5}{2}} \| \rho \|_{L^2}^{\frac{1}{2}} \| \nabla u_0 \|_{L^2}^2. \]  
(3.10)

The proof of Lemma 3.1 is completed. \( \square \)

**Lemma 3.2.** Let \((\rho, u, \pi)\) be a smooth solution to (1.1)–(1.3) satisfying (3.1). Then, it holds that
\[ \mu \sup_{0 \leq t \leq \zeta(T)} \| \nabla u \|_{L^2}^2 + \int_0^{\zeta(T)} \| \sqrt{\rho} u_t \|_{L^2}^2 dt \leq \overline{\rho} M_2, \]  
(3.11)
\[ \mu \sup_{\zeta(T) \leq t \leq T} e^{\sigma t} \| \nabla u \|_{L^2}^2 + \int_{\zeta(T)}^T e^{\sigma t} \| \sqrt{\rho} u_t \|_{L^2}^2 dt \leq \underline{\rho} e^{\sigma \zeta(T)} M_2 + \frac{4 \overline{\rho}}{3} \| \rho_0 \|_{L^2}^{\frac{5}{2}} \| \rho \|_{L^2}^{\frac{1}{2}} M_2, \]  
(3.12)
provided \( \overline{\rho} \leq \epsilon_2 := \min \left\{ \frac{1}{(2C_1)^{\frac{2}{q}}} \frac{1}{M_2^{\frac{2}{q}}}, \frac{1}{(4C_2)^{\frac{2}{q}}} \frac{1}{M_2^{\frac{2}{q}}}, \frac{1}{(4C_2 M_1)^{\frac{2}{q}}} \frac{1}{M_2^{\frac{2}{q}}} \right\}. \) Here, \( \zeta(t) \) is defined by
\[ \zeta(t) := \min \{1, t\}. \]

**Proof.** First, by (1.1)_1, we obtain the following equations
\[ \mu(\rho)_t + u \cdot \nabla \mu(\rho) = 0. \]  
(3.13)
Next, multiplying (1.1)_2 by \( u_t \) and integrating the resulting equality by parts. It follows from Gagliardo-Nirenberg inequality and Young’s inequality, (3.5), and (3.1) that
\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} & \int \mu(\rho) \| \nabla u \|_{L^2}^2 dx + \int \rho |u_t|^2 dx \\
& = - \int \rho u \cdot \nabla u \cdot u_t dx - \frac{1}{2} \int u \cdot \nabla \mu(\rho) |\nabla u|^2 dx \\
& \leq C \overline{\rho} \frac{1}{2} \| \sqrt{\rho} u_t \|_{L^2} \| u \|_{L^6} \| \nabla u \|_{L^3} + C \| \nabla \mu(\rho) \|_{L^3} \| u \|_{L^6} \| \nabla u \|_{L^2}^{\frac{1}{2}} + C \overline{\rho} \frac{1}{2} \| \sqrt{\rho} u_t \|_{L^2} \| \nabla u \|_{L^6} \| \nabla u \|_{H^1} \\
& \leq \frac{1}{4} \| \sqrt{\rho} u_t \|_{L^2}^2 + C \overline{\rho} \| \nabla u \|_{L^6} \| \nabla u \|_{H^1} + C \| \nabla \mu(\rho) \|_{L^3} \| u \|_{L^6} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1} \\
& \leq \frac{1}{4} \| \sqrt{\rho} u_t \|_{L^2}^2 + C(\overline{\rho} M_2 + M_1^{\frac{1}{2}} M_1^{\frac{1}{2}}) \| \nabla u \|_{H^1}^2. 
\end{align*} \]  
(3.14)
According to Lemma 2.2 with \( F = \rho u_t + \rho u \cdot \nabla u \) and by (3.5) and (3.1), we derive
\[ \| \nabla u \|_{H^1} + \| \pi \|_{H^1} \leq C(\| \rho u_t \|_{L^2} + \| \rho u \cdot \nabla u \|_{L^2}) + C(\| \rho u_t \|_{L^2} + \| \rho u \cdot \nabla u \|_{L^2}) \]
\[ \leq C(\overline{\rho} \frac{1}{2} + \frac{1}{2} \| \sqrt{\rho} u_t \|_{L^2} \frac{1}{2} + \| \sqrt{\rho} u_t \|_{L^2} \| \nabla u \|_{L^6} \| \nabla u \|_{L^2}) \]
\[ \leq C(\overline{\rho} \frac{1}{2} + \| \rho_0 \|_{L^2} \frac{1}{2} + \frac{1}{2} \| \sqrt{\rho} u_t \|_{L^2} \| \nabla u \|_{L^6} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1}) \]
\[ \leq C(\bar{\rho}^\frac{\alpha}{2} + \|\rho_0\|_{L^\frac{2}{1}}^{\frac{3}{2}} \bar{\rho}^\frac{\alpha}{2}) \|\sqrt{\rho}u_t\|_{L^2} + CM_2^{\frac{3}{2}} (\bar{\rho} + \|\rho_0\|_{L^{\frac{2}{1}}}) \|\nabla u\|_{H^1}, \]
\[ \leq C_1 \bar{\rho}^\frac{\alpha}{2} \|\sqrt{\rho}u_t\|_{L^2} + C_1 M_2^{\frac{3}{2}} \bar{\rho}^\frac{\alpha}{2} \|\nabla u\|_{H^1}, \]

which directly yields that
\[ \|\nabla u\|_{H^1} \leq C_1 \bar{\rho}^\frac{\alpha}{2} \|\sqrt{\rho}u_t\|_{L^2}, \]  
provided \( \bar{\rho} \leq \epsilon_1 := \min \left\{ 1, \frac{1}{(2C_3)^\frac{\alpha}{2} M_2^{\frac{3}{2}}} \right\}. \)

Substituting (3.15) into (3.14), it is easy to deduce that
\[ \frac{d}{dt} \int \mu(\rho)|\nabla u|^2 dx + \frac{3}{2} \int \rho|u_t|^2 dx \leq C_2 (\bar{\rho}^\frac{\alpha}{2} M_2 + \bar{\rho}^\frac{\alpha}{2} M_2^{\frac{3}{2}} M_1) \|\sqrt{\rho}u_t\|_{L^2}^2, \]  
which, integrating over \([0, \zeta(T)]\), gives
\[ \mu \sup_{0 \leq t \leq \zeta(T)} \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^{\zeta(T)} \|\sqrt{\rho}u_t\|_{L^2}^2 dt \leq \overline{\mu} M_2, \]
provided \( \bar{\rho} \leq \epsilon_2 := \min \left\{ \epsilon_1, \frac{1}{(4C_3)^\frac{\alpha}{2} M_2^{\frac{3}{2}}}, \frac{1}{(4C_3 M_2)^\frac{\alpha}{2}} M_2^{\frac{3}{2}} \right\}. \)

Next, multiplying (3.16) by \( e^{\sigma t} \) and integrating the result inequality over \([\zeta(T), T] \)
lead to
\[ \mu \sup_{\zeta(T) \leq t \leq T} e^{\sigma t} \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_{\zeta(T)}^T e^{\sigma t} \|\sqrt{\rho}u_t\|_{L^2}^2 dt \]
\[ \leq \overline{\mu} e^{\sigma \zeta(T)} M_2 + \overline{\mu} \int_{\zeta(T)}^T e^{\sigma t} \|\nabla u\|_{L^2}^2 dt \leq \overline{\mu} e^{\sigma \zeta(T)} M_2 + \frac{4}{3} \frac{\overline{\mu}}{\mu} \|\rho_0\|_{L^1}^{\frac{3}{2}} \bar{\rho}^\frac{\alpha}{2} M_2. \]

Thus, the proof of Lemma 3.2 is finished. \( \square \)

**Lemma 3.3.** Let \((\rho, u, \pi)\) be a smooth solution to (1.1)–(1.3) satisfying (3.1). Then there exists a generic positive constant \( C \) depending only on \( q, \mu, \overline{\mu}, \) and \( \|\rho_0\|_{L^{1} \cap L^\frac{2}{1}} \), such that
\[ \sup_{0 \leq t \leq \zeta(T)} t \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^{\zeta(T)} \|\nabla u_t\|_{L^2}^2 dt \leq 2\overline{\mu} M_2 + CM_1^{\frac{3\alpha}{2}} M_2^{\frac{3}{2}} \bar{\rho}^\frac{\alpha}{2}, \]
\[ \sup_{\zeta(T) \leq t \leq T} e^{\sigma t} \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_{\zeta(T)}^T e^{\sigma t} \|\nabla u_t\|_{L^2}^2 dt \]
\[ \leq 2\overline{\mu} e^{\sigma \zeta(T)} M_2 + C(M_1^{\frac{3\alpha}{2}} M_2^{\frac{3}{2}} + M_2) \bar{\rho}^\frac{\alpha}{2}, \]
\[ \sup_{\zeta(T) \leq t \leq T} e^{\sigma t} (\|\nabla u\|_{H^1}^2 + \|\pi\|_{H^1}^2) \leq CM_2 \bar{\rho}^\frac{\alpha}{2} + CM_1^{\frac{3\alpha}{2}} M_2^{\frac{3}{2}} \bar{\rho}. \]

provided \( \bar{\rho} \leq \epsilon_4 := \min \left\{ \epsilon_2, \frac{1}{(4C_3)^\frac{\alpha}{2} M_2^{\frac{3}{2}}}, \frac{1}{(4C_3 M_2)^{\frac{3}{2}}} \frac{1}{(4C_4)^{\frac{3}{2}}} M_2^{\frac{3}{2}}, \frac{1}{C_4 M_2} \right\}. \)

**Proof.** First, differentiating (1.1)\(_2\) with respect to \( t \) yields that
\[ \rho u_{tt} + p u \cdot \nabla u_t - \text{div}(\mu(\rho) \nabla u_t) + \nabla \pi_t \]
\[ = -\rho_t u_t - (\rho u)_t \cdot \nabla u + \text{div}(\mu(\rho) \nabla u). \]
Multiplying the above equality by \(u_t\), we obtain after using integration by parts and (1.1) that

\[
\frac{1}{2} \frac{d}{dt} \int \rho|u_t|^2 dx + \int \mu(\rho)|\nabla u|^2 dx
\]

\[= \int (u \cdot \nabla \mu(\rho)) \nabla u \cdot \nabla u_t dx - 2 \int \rho u \cdot \nabla u_t \cdot u_t dx
\]

\[- \int \rho u_t \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t) dx
\]

\[= : \sum_{i=1}^4 I_i. \tag{3.22}
\]

Now, we will use the Gagliardo-Nirenberg inequality and (3.1) to estimate each term on the right hand of (3.22) as follows:

\[I_1 \leq C \|\nabla \mu(\rho)\|_{L^4} \|u\|_{L^4} \|\nabla u\|\| \nabla u\|_{L^{4}} \|\nabla u\|_{L^{2}} \]

\[\leq C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^4} \|\nabla^2 u\|_{L^2}
\]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 + CM \frac{\mu_0}{2}
\]  

\[\|\nabla u\|_{L^2} \tag{3.23}
\]

\[I_2 \leq C \|\nabla u_t\|_{L^2}^2 \|u\|_{L^2} \|\nabla u_t\|_{L^2}
\]

\[\leq C \|\nabla u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \]

\[\|\nabla u\|_{L^2} \tag{3.24}
\]

\[I_3 \leq C \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^2}
\]

\[\leq C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} \]

\[\|\nabla u\|_{L^2} \tag{3.25}
\]

\[I_4 \leq C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} + C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \]

\[\leq C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2} \]

\[\leq \delta \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \]

\[\|\nabla u\|_{L^2} \tag{3.26}
\]

Substituting (3.23)–(3.26) into (3.22), using (3.15) and choosing \(\delta\) suitably small, we have

\[
\frac{d}{dt} \int \rho|u_t|^2 dx + \mu \int |\nabla u_t|^2 dx
\]

\[\leq C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \]

\[\leq CM \rho \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C\rho \|\nabla \mu(\rho)\|_{L^4} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \]

\[\|\nabla u\|_{L^2} \tag{3.27}
\]
Next, multiplying (3.27) by \( t \), we get
\[
\frac{d}{dt} \left[ t \int \rho |u_t|^2 \, dx \right] + \mu t \int |\nabla u_t|^2 \, dx \\
\leq CM_2 \bar{\rho} \frac{\dot{\bar{\rho}}}{\sqrt{T}} \left[ (t) \| \sqrt{\rho u_t} \|_{L^2}^2 \right] + \| \sqrt{\rho u_t} \|_{L^2}^2 + CM_1 \frac{3\bar{\rho}}{T^2} t \| \nabla u \|_{L^2}^2 \\\n+ C \bar{\rho} \| \nabla u \|_{L^2} \| \sqrt{\rho u_t} \|_{L^2}^2,
\]  
which, integrated over \([0, \zeta(T)]\), from (3.6) and (3.1), gives
\[
\sup_{0 \leq t \leq \zeta(T)} (t) \| \sqrt{\rho u_t} \|_{L^2}^2 + \frac{\mu}{\sqrt{T}} \int_0^{\zeta(T)} t \| \nabla u_t \|_{L^2}^2 \, dt \\
\leq CM_2 \bar{\rho} \frac{\dot{\bar{\rho}}}{\sqrt{T}} \sup_{0 \leq t \leq \zeta(T)} (t) \| \sqrt{\rho u_t} \|_{L^2}^2 \int_0^{\zeta(T)} \| \nabla u \|_{L^2}^2 \, dt \\\n+ C \bar{\rho} \sup_{0 \leq t \leq \zeta(T)} (t) \| \sqrt{\rho u_t} \|_{L^2}^2 \left( \int_0^{\zeta(T)} \| \sqrt{\rho u_t} \|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^{\zeta(T)} \| \nabla u \|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \\\n+ \int_0^{\zeta(T)} \| \sqrt{\rho u_t} \|_{L^2}^2 \, dt + CM_1 \frac{3\bar{\rho}}{T^2} \sup_{0 \leq t \leq \zeta(T)} \| \nabla u \|_{L^2}^2 \int_0^{\zeta(T)} \| \nabla u \|_{L^2}^2 \, dt \\\n\leq C_3 (M_2^2 \bar{\rho} + M_2 \bar{\rho}^2) \sup_{0 \leq t \leq \zeta(T)} (t) \| \sqrt{\rho u_t} \|_{L^2}^2 \\\n+ \bar{\mu} M_2 + CM_1 \frac{3\bar{\rho}}{T^2} M_2^2 \bar{\rho}^\frac{3}{2},
\]
which directly yields that
\[
\sup_{0 \leq t \leq \zeta(T)} (t) \| \sqrt{\rho u_t} \|_{L^2}^2 + \mu \int_0^{\zeta(T)} t \| \nabla u_t \|_{L^2}^2 \, dt \leq 2\bar{\mu} M_2 + CM_1 \frac{3\bar{\rho}}{T^2} M_2^2 \bar{\rho}^\frac{3}{2},
\]  
provided \( \bar{\rho} \leq \epsilon_3 := \min \left\{ \epsilon_2, \frac{1}{(4C_3)^\frac{1}{2} M_2}, \frac{1}{(4C_3 M_2)^\frac{1}{2}} \right\} \).

Similarly, multiplying (3.27) by \( e^{\sigma t} \) and integrating the result inequality over \([\zeta(T), T]\), we derive that
\[
\sup_{\zeta(T) \leq t \leq T} e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2 \leq CM_2 \bar{\rho} \frac{\dot{\bar{\rho}}}{\sqrt{T}} \sup_{\zeta(T) \leq t \leq T} (e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2) \int_\zeta(T)^T \| \nabla u \|_{L^2}^2 \, dt \\\n+ C \bar{\rho} \sup_{\zeta(T) \leq t \leq T} (e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2) \sup_{\zeta(T) \leq t \leq T} (e^{\sigma t} \| \nabla u \|_{L^2}^2) \left( \int_\zeta(T)^T e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \left( \int_\zeta(T)^T e^{-2\sigma t} \, dt \right)^{\frac{1}{2}} \\\n+ \int_\zeta(T)^T e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2 \, dt + CM_1 \frac{3\bar{\rho}}{T^2} \sup_{\zeta(T) \leq t \leq T} e^{\sigma t} \| \nabla u \|_{L^2}^2 \, dt \\\n\leq C_4 (M_2^2 \bar{\rho} + M_2 \bar{\rho}^2 + M_2 \bar{\rho}) \sup_{\zeta(T) \leq t \leq T} (e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2) \\\n+ \bar{\mu} e^{\sigma \zeta(T)} M_2 + C(M_1 \frac{3\bar{\rho}}{T^2} M_2 + M_2) \bar{\rho}^\frac{3}{2}.
\]
\[
\leq C_4(M_2^2 \bar{\rho}^3 + M_2 \bar{\mu}) \sup_{\zeta(T) \leq T} (e^{\sigma_T} ||\sqrt{\rho} u_t||^2_{L^2}) + 2\mu e^{\sigma_T} M_2 \\
+ C(M_1^3 M_2^2 + M_2)\bar{\rho}^3,
\]
that is
\[
\sup_{\zeta(T) \leq T} e^{\sigma_T} ||\sqrt{\rho} u_t||^2_{L^2} + \mu \int_{\zeta(T)}^{T} e^{\sigma_t} ||\nabla u_t||^2_{L^2} dt \\
\leq 2\mu e^{\sigma_T} M_2 + C(M_1^3 M_2^2 + M_2)\bar{\rho}^3,
\]
provided \(\bar{\rho} \leq \epsilon_4 := \min \left\{ \epsilon_3, \frac{1}{(4C_4)^\frac{1}{2} M_2^2}, \frac{1}{4C_4 M_2} \right\} \).

Finally, it follows from (3.15), (3.29) and (3.30) that (3.21) holds. Therefore, the proof of Lemma 3.3 is completed.

\textbf{Lemma 3.4.} Let \((\rho, u, \pi)\) be a smooth solution to (1.1)–(1.3) satisfying (3.1). Then there exists a generic positive constant \(C\) depending only on \(q, \mu, \bar{\rho}\), and \(||\rho_0||_{L^1 \cap L^2}\), such that
\[
\int_{0}^{T} ||\nabla u||_{L^\infty} dt \leq CM_2^\frac{1}{2} \bar{\rho}^\frac{1}{6} + CM_1^{\frac{3q}{q-3}} M_2 \bar{\rho}^\frac{1}{6} + CM_2 \bar{\rho} \\
+ C M_1^{\frac{3q}{q-3}} M_2^2 \bar{\rho}^\frac{1}{6} + CM_2^{\frac{3q}{q-3}} \bar{\rho}^{\frac{1}{6} - \frac{1}{3}}.
\]

\textbf{Proof.} First, it follows from the Lemma 2.2 and Gagliardo-Nirenberg inequality, (3.15) that for \(r \in (3, \min(q, 6))\),
\[
||\nabla^2 u||_{L^r} + ||\nabla \pi||_{L^r} \leq C||\rho u_t + \rho u \cdot \nabla u||_{L^r} + C||\rho u_t + \rho u \cdot \nabla u||_{L^2} \\
\leq C\rho^{\frac{5r-6}{6-r}} ||\sqrt{\rho} u_t||_{L^2} ||\nabla u||_{L^\frac{6r}{6-r}} + C\bar{\rho} ||u||_{L^\infty} ||\nabla u||_{L^\frac{6r}{6-r}} \\
+ C||\rho||_{L^2} \left( ||\sqrt{\rho} u_t||_{L^2} + \bar{\rho}^\frac{1}{6} ||u||_{L^6} ||\nabla u||_{L^2} \right) \\
\leq C\rho^{\frac{5r-6}{6-r}} ||\sqrt{\rho} u_t||_{L^2} ||\nabla u||_{L^\frac{6r}{6-r}} + C\bar{\rho}^{\frac{5r-6}{6-r}} ||\nabla u||_{L^\frac{6r}{6-r}} \\
+ C||\rho_0||_{L^\frac{1}{2}} \bar{\rho}^\frac{1}{6} \left( ||\sqrt{\rho_0} u_t||_{L^2} + M_2^\frac{1}{2} \bar{\rho}^\frac{1}{6} ||\nabla u||_{H^1} \right) + \frac{1}{2} ||\nabla^2 u||_{L^r} \\
\leq C(\bar{\rho}^\frac{1}{6} + M_2^\frac{1}{2} \bar{\rho}) ||\sqrt{\rho} u_t||_{L^2} + \frac{1}{2} ||\nabla^2 u||_{L^r} + C\rho^{\frac{5r-6}{6-r}} ||u||_{L^\frac{6r}{6-r}} \\
+ C\rho^{\frac{5r-6}{6-r}} ||\sqrt{\rho} u_t||_{L^2} ||\nabla u||_{L^\frac{6r}{6-r}},
\]
which yields that
\[
||\nabla^2 u||_{L^r} + ||\nabla \pi||_{L^r} \leq C(\bar{\rho}^\frac{1}{6} + M_2^\frac{1}{2} \bar{\rho}) ||\sqrt{\rho} u_t||_{L^2} + C\rho^{\frac{5r-6}{6-r}} ||u||_{L^\frac{6r}{6-r}} \\
+ C\rho^{\frac{5r-6}{6-r}} ||\sqrt{\rho} u_t||_{L^2} ||\nabla u||_{L^\frac{3q-6}{3q}}.
\]
Then, one derives from the Gagliardo-Nirenberg inequality and (3.33) that
\[
||\nabla u||_{L^\infty} \leq C||\nabla^2 u||_{L^\frac{6r}{6-r}} ||\nabla u||_{L^\frac{3q-6}{3q}} \leq C||\nabla u||_{L^2} + C||\nabla^2 u||_{L^r} \\
\leq C(\bar{\rho}^\frac{1}{6} + M_2^\frac{1}{2} \bar{\rho}) ||\sqrt{\rho} u_t||_{L^2} + C\rho^{\frac{5r-6}{6-r}} ||u||_{L^\frac{6r}{6-r}} + C||\nabla u||_{L^2} \\
+ C\rho^{\frac{5r-6}{6-r}} ||\sqrt{\rho} u_t||_{L^2} ||\nabla u||_{L^\frac{3q-6}{3q}}.
\]
On the one hand, using (3.11), (3.19), and (3.6) that for \( t \in [0, \zeta(T)] \),
\[
\int_0^{\zeta(T)} \| \nabla u \|_{L^\infty} dt 
\leq C \rho^{5-\varepsilon} \sup_{0 \leq t \leq \zeta(T)} \left( t \| \sqrt{\rho u_t} \|_{L^2}^2 \right)^{6/5} \left( \int_0^{\zeta(T)} t \| \nabla u_t \|_{L^2}^2 dt \right)^{3/5} \left( \int_0^{\zeta(T)} t^{-2r} dt \right)^{2/5} 
+ C (\rho^{1/2} + M_2^{1/2} \tilde{\rho}) \sup_{0 \leq t \leq \zeta(T)} \left( t \| \sqrt{\rho u_t} \|_{L^2}^2 \right)^{1/2} \left( \int_0^{\zeta(T)} t^{-\frac{1}{2}} dt + C \left( \int_0^{\zeta(T)} \| \nabla u \|_{L^2}^2 dt \right)^{1/2} 
+ C \rho^{5-\varepsilon} \left( \sup_{0 \leq t \leq \zeta(T)} \| \nabla u \|_{L^2}^2 \right)^{2r-3} \left( \int_0^{\zeta(T)} \| \nabla u \|_{L^2}^2 dt \right) 
\leq CM_2^{1/2} \rho^{5-\varepsilon} + CM_1^{3/4-\sigma_1} M_2 \rho^{17r-18} + CM_2^{1/2} \tilde{\rho}^\varepsilon + CM_1^{3/4-\sigma_1} M_2 \tilde{\rho}^\varepsilon 
+ CM_2 \tilde{\rho} + CM_1^{3/4-\sigma_1} M_2^{3/2} \tilde{\rho}^\varepsilon + CM_2^{3r-3} \rho^{16r-18} 
\leq CM_2^{1/2} \rho^{1/2} + CM_1^{3/4-\sigma_1} M_2 \rho^{1/2} + CM_2 \tilde{\rho} + CM_1^{3/4-\sigma_1} M_2^{3/2} \tilde{\rho}^\varepsilon + CM_2^{3r-3} \rho^{16r-18}. \tag{3.35}
\]

On the other hand, using (3.12), (3.20), and (3.7), we obtain that for \( t \in [\zeta(T), T] \),
\[
\int_{\zeta(T)}^{T} \| \nabla u \|_{L^\infty} dt 
\leq C \rho^{5-\varepsilon} \int_{\zeta(T)}^{T} \left( \| \sqrt{\rho u_t} \|_{L^2} + \| \nabla u_t \|_{L^2} \right) \, dt + C \left( \int_{\zeta(T)}^{T} e^{-\sigma t} \, dt \right)^{1/2} 
\left( \int_{\zeta(T)}^{T} e^{\sigma t} \| \nabla u \|_{L^2}^2 \, dt \right)^{1/2} + CM_2^{1/2} \rho^{1/2} + \frac{M_2^{1/2} \tilde{\rho}}{\zeta(T) \leq T} \left( e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2 \right)^{1/2} \int_{\zeta(T)}^{T} e^{-\frac{\sigma t}{2}} \, dt 
+ C \rho^{5-\varepsilon} \left( \sup_{\zeta(T) \leq t \leq T} \left( e^{\sigma t} \| \nabla u \|_{L^2}^2 \right) \right)^{2r-3} \int_{\zeta(T)}^{T} e^{-\frac{2r-3}{2} \sigma t} \, dt 
\leq CM_1^{3/4-\sigma_1} M_2 \rho^{1/2} + CM_2 \tilde{\rho} + CM_1^{3/4-\sigma_1} M_2^{3/2} \tilde{\rho}^\varepsilon + CM_2^{3r-3} \rho^{16r-18} 
+ CM_2^{1/2} \rho^{1/2} + CM_2 \tilde{\rho} + C \rho^{5-\varepsilon} \left( \int_{\zeta(T)}^{T} e^{-\sigma t} \, dt \right)^{1/2} \left( \int_{\zeta(T)}^{T} e^{\sigma t} \| \nabla u_t \|_{L^2}^2 \, dt \right)^{1/2} 
+ C \rho^{5-\varepsilon} \left( \sup_{\zeta(T) \leq t \leq T} \left( e^{\sigma t} \| \sqrt{\rho u_t} \|_{L^2}^2 \right) \right)^{1/2} \int_{\zeta(T)}^{T} e^{-\frac{\sigma t}{2}} \, dt 
\leq CM_1^{3/4} \rho^{1/2} + CM_1^{3/4-\sigma_1} M_2 \rho^{1/2} + CM_2 \tilde{\rho} + CM_1^{3/4-\sigma_1} M_2^{3/2} \tilde{\rho}^\varepsilon + CM_2^{3r-3} \rho^{16r-18}. \tag{3.36}
\]

Combining this with (3.35) gives (3.31) and finishes the proof of Lemma 3.4. \( \Box \)

With Lemmas 3.1–3.4 at hand, we are in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** First, it follows from (1.1) that
\[
\frac{d}{dt} \| \nabla \mu(\rho) \|_{L^q} \leq q \| \nabla u \|_{L^\infty} \| \nabla \mu(\rho) \|_{L^r}, \tag{3.37}
\]
which together with Gronwall’s inequality and (3.31) yields
\[
\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_{L^2} \leq \|\nabla \mu(\rho_0)\|_{L^\infty} \exp \left\{ q \int_0^T \|\nabla u\|_{L^\infty} dt \right\} \\
\leq \|\nabla \mu(\rho_0)\|_{L^2} \exp \left\{ C_5 \left( M_2^2 \tilde{\rho}^\frac{3}{2} + M_1 \frac{3q}{2q-1} M_2^{\frac{3}{2}} \right) + M_2 \tilde{\rho} + M_1 \frac{3q}{2q-1} M_2^{\frac{3}{2}} \tilde{\rho} \right\} \\
\leq 2M_1, \tag{3.38}
\]
provided \( \tilde{\rho} \leq \epsilon_5 := \min \left\{ \epsilon_4, \frac{(\ln 2)^6}{(5C_5)^6 M_2^3}, \frac{(\ln 2)^{\frac{3}{2}}}{(5C_5)^{\frac{3}{2}} M_1^{\frac{3q}{2q-1}} M_2^3}, \frac{\ln 2}{5C_5 M_2^3} \right\} .

Next, we deduce from (3.16) and (3.6) that
\[
\sup_{0 \leq t \leq T} e^{\sigma t} \|\nabla u\|_{L^2}^2 \leq \frac{\mu}{\mu} M_2 + \frac{\mu}{\mu} \int_0^T e^{\sigma t} \|\nabla u\|_{L^2}^2 dt \leq \frac{\mu}{\mu} M_2 + \frac{4}{3q} \frac{\mu}{\mu} \|\rho_0\|_L^2 \tilde{\rho}^\frac{3}{2} M_2 \\
\leq \frac{\mu}{\mu} M_2, \tag{3.39}
\]
provided \( \tilde{\rho} \leq \epsilon_6 := \min \left\{ \epsilon_5, \left( \frac{3q}{4\|\rho_0\|_L^2} \right)^\frac{1}{3} \right\} .

Choosing \( \epsilon_0 := \min \left\{ \epsilon_6, \frac{\epsilon}{2} \right\} \), we directly obtain (3.2) from (3.39), and (3.38). The proof of Proposition 3.1 is finished. \( \square \)

The following Lemma 3.5 is necessary for further estimates on the higher-order derivatives of the strong solution \((\rho, u, \pi)\), which can be found in [8].

**Lemma 3.5.** Let \((\rho, u, \pi)\) be a smooth solution to (1.1)–(1.3) satisfying (3.1). Then there exists a generic positive constant \( C \) depending only on \( q, \mu, \bar{\mu}, \|\rho_0\|_{L^1 \cap L^\infty} \), \( M_1 \), and \( M_2 \) such that for \( \rho_0 := \min\{6, q\} \) and \( q_0 := \frac{4q}{q-3} \),
\[
\sup_{0 \leq t \leq T} e^{\sigma t} \left( \zeta \|\nabla u\|^2_{H^1} + \zeta \|\pi\|^2_{H^1} \right) + \int_0^T e^{\sigma t} \|\nabla u\|^2_{L^2} dt \\
+ \int_0^T e^{\sigma t} \left( \|\nabla u\|^2_{H^1} + \|\pi\|^2_{H^1} + \zeta \|\nabla u\|^2_{W^{1, p_0}} + \zeta \|\pi\|^2_{W^{1, p_0}} \right) \leq C, \tag{3.40}
\]
\[
\sup_{0 \leq t \leq T} e^{\sigma_0 t} \left( \zeta \|\nabla u\|^2_{W^{1, p_0}} + \zeta \|\pi\|^2_{W^{1, p_0}} + \|\nabla u\|^2_{L^2} \right) \\
+ \int_0^T e^{\sigma_0 t} \left( \|\rho u_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|\pi_t\|^2_{L^2 \cap L^{p_0}} \right) dt \leq C. \tag{3.41}
\]

**Proof of Theorem 1.1.** With the a priori estimates obtained in Lemmas 3.1–3.5 at hand, Theorem 1.1 can be established similarly to that in [8]. Here, we omit it for simplicity. \( \square \)

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