Site Percolation and Phase Transitions in Two Dimensions

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Abstract:
The properties of the pure-site clusters of spin models, i.e. the clusters which are obtained by joining nearest-neighbour spins of the same sign, are here investigated. In the Ising model in two dimensions it is known that such clusters undergo a percolation transition exactly at the critical point. We show that this result is valid for a wide class of bidimensional systems undergoing a continuous magnetization transition. We provide numerical evidence for discrete as well as for continuous spin models, including SU($N$) lattice gauge theories. The critical percolation exponents do not coincide with the ones of the thermal transition, but they are the same for models belonging to the same universality class.

The idea to explain the mechanism of a phase transition in terms of the interplay of geometrical structures which can be identified in the system is quite old: the growth of such structures close to the transition represents the increase of the range of the correlation between different parts of the system, so that the formation of a spanning structure corresponds to a divergent correlation length and to the transition to a new state of global order for the system.

Here we exclusively refer to lattice models and the geometrical structures of interest are usually clusters of neighbouring particles (spins). Percolation theory is the ideal framework to deal with clusters. Near the critical point of the percolation transition there is power-law behaviour for the percolation variables and one can define an analogous set of critical exponents as for standard thermal transitions. The main percolation variables are:

- the percolation strength $P$, i.e. the probability that a site chosen at random belongs to a percolating cluster;
- the average cluster size $S$,

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s},$$ (1)

where $n_s$ is the number of clusters with $s$ sites and the sums exclude eventual percolating clusters.

One can in principle define the clusters arbitrarily, but, in order to reproduce the critical behavior of the model, the following conditions must be satisfied:

- the percolation point must coincide with the thermal critical point;
- the connectedness length (average cluster radius) diverges as the thermal correlation length (same exponent);
- the percolation strength $P$ near the threshold varies like the order parameter $m$ of the model (same exponent);
- the average cluster size $S$ diverges as the physical susceptibility $\chi$ (same exponent).

The first system to be investigated was of course the Ising model. If one takes an Ising configuration one can immediately isolate the "classical" magnetic domains, i.e. the clusters formed by binding to each other all nearest-neighbouring sites carrying spins of the same sign. It turns out that these clusters have indeed interesting properties in two dimensions, since their percolation temperature coincides with the critical temperature of the magnetization transition; the percolation exponents, however, do not coincide with the Ising exponents.
In this paper we show that the result is quite general in two dimensions, being valid for several models undergoing a continuous magnetization transition. In particular we shall see that the presence of antiferromagnetic interactions does not affect the result, at least to the extent of the cases studied here. Moreover the critical percolation exponents do not randomly vary from one model to the other but are uniquely fixed by the universality class the model belongs to.

Our results are based on Monte Carlo simulations on square lattices of various systems near criticality. The models we studied can be divided in two groups: models with $Z(2)$ global symmetry and a magnetization transition with Ising exponents and models with $Z(3)$ global symmetry and a magnetization transition with exponents belonging to the 2-dimensional 3-state Potts model universality class. The systems belonging to the first group are:

1. the Ising model, $\mathcal{H} = -J \sum_{ij} s_i s_j$ ($J > 0$, $s_i = \pm 1$);
2. a model with nearest-neighbour (NN) ferromagnetic coupling and a weaker next-to-nearest (NTN) antiferromagnetic coupling: $\mathcal{H} = -J_1 \sum_{NN} s_i s_j - J_2 \sum_{NTN} s_i s_j$ ($J_1 > 0$, $J_2 < 0$, $|J_2/J_1| = 1/10$, $s_i = \pm 1$);
3. the continuous Ising model, $\mathcal{H} = -J \sum_{ij} S_i S_j$ ($J > 0$, $-1 \leq S_i \leq +1$);
4. SU(2) pure gauge theory in 2+1 dimensions.

The models belonging to the second group are:

1. the 3-state Potts model, $\mathcal{H} = -J \sum_{ij} \delta(s_i, s_j)$ ($J > 0$, $s_i = 1, 2, 3$);
2. a model obtained by adding to 1) a weaker next-to-nearest (NTN) antiferromagnetic coupling: $\mathcal{H} = -J_1 \sum_{NN} \delta(s_i, s_j) - J_2 \sum_{NTN} \delta(s_i, s_j)$ ($J_1 > 0$, $J_2 < 0$, $|J_2/J_1| = 1/10$, $s_i = 1, 2, 3$);

In each case we produced the thermal equilibrium configurations by using standard Monte Carlo algorithms, like Metropolis or heat bath; whenever we could (Ex. models 1 and 3 of the first group, model 1 of the second) we adopted cluster algorithms in order to reduce the autocorrelation time. At each iteration we calculated the energy and the magnetization; the latter is necessary to study the thermal transition. For the identification of the pure-site clusters in each configuration we made use of the algorithm devised by Hoshen and Kopelman [5], with free boundary conditions. Finally we determined the percolation strength $P$, the average cluster size $S$ and the size $S_M$ of the largest cluster of each configuration, from which we can calculate the fractal dimension $D$ of the percolating cluster. We say that a cluster percolates if it spans the lattice from top to bottom. The finite size scaling laws at the critical temperature $T_p$ for the percolation variables read

\begin{align*}
P(T_p) &\propto L^{-\beta_p/\nu_p} \\
S(T_p) &\propto L^{\gamma_p/\nu_p} \\
S_M(T_p) &\propto L^D,
\end{align*}

where $L$ is the lattice side and $\beta_p$, $\gamma_p$, $\nu_p$ are critical exponents of the percolation transition. Moreover, to study the percolation transition it is helpful to define also the percolation cumulant. It is the probability of having percolation at a given temperature and lattice size, i.e., the quantity obtained by dividing the number of "percolating" samples by the total number of analyzed configurations. This variable has three remarkable properties:

- if one plots it as a function of $T$, all curves corresponding to different lattice sizes cross at the same temperature $T_p$, which marks the threshold of the percolation transition;
- the percolation cumulants for different values of the lattice size $L$ coincide, if considered as functions of $t_p L^{1/\nu_p}$ ($t_p = (T - T_p)/T_p$, $\nu_p$ is the exponent of the percolation correlation length);
- the value of the percolation cumulant at $T_p$ is a universal quantity, i.e. it labels a well defined set of critical indices.

\*For the models 2 of both groups a cluster dynamics exists, but it does not bring great advantages because of the huge size of the clusters to be flipped at the critical point.
These features allow a rather precise determination of the critical point and a fairly good estimate of the critical exponent $\nu_p$.

![Graph](image)

**FIG. 1.** Percolation cumulant of pure-site clusters in the 2D Ising model as a function of $\beta = J/kT$ for different lattice sizes. The curves cross remarkably at the same point, in agreement with the critical point of the magnetization transition (vertical dotted line in the plot).

We start to expose our results from the models with an Ising-like transition. We included the Ising model itself in order to reproduce the results of [3,4] and to precisely determine the percolation exponents. We simulated the model on several lattices, from $600^2$ to $2000^2$. The number of measurements we have taken ranged from a minimum of 20000 for the large $2000^2$ lattice to over 100000 for the others. From the crossing of the cumulants (Fig. 1), we obtained for the percolation temperature $J/kT_p = 0.44069(1)$, in excellent agreement with the critical temperature $J/kT_c = \log(1 + \sqrt{2})/2 = 0.44068679\ldots$. For all models we investigated it is possible to perform a precise interpolation of the raw data by applying the density of state method [6] both to the thermal and to the percolation variables. The curves in Fig. 1 are examples of such interpolations; we did not put the errors to show more clearly the crossing point and the fact that the expected behaviour is already very well represented by the mean values, which is due to the high statistics we could reach in this simple case.

By rescaling the percolation cumulant curves as described above, we obtain Fig. 2 if we choose for the exponent $\nu_p$ the value of the thermal 2D Ising exponent $\nu_{Is} = 1$. This time we have also drawn the errors, and we see that all data relative to different lattice sizes fall onto one and the same curve, which shows that $\nu_p = \nu_{Is} = 1$, as suggested in [7].

![Graph](image)

**FIG. 2.** Rescaling of the percolation cumulant curves of Fig. 1 by setting $\nu_p = \nu_{Is} = 1$. The curves fall on top of each other.
The percolation exponents and the threshold value of the cumulant are listed in Table I (Model 1), where we also put the critical percolation indices of the other three models of this group. The values of $\gamma_p$, $\beta_p$ and of the fractal dimension $D$ are in good accord with the theoretical predictions of [8] (first line of Table I).

For the model with competitive interactions we plot the percolation cumulant for two lattice sizes, $100^2$ and $200^2$ (Fig. 3). Our statistics for this system is of 20000 measurements for each temperature and lattice size, we could not increase it because the algorithms one can use are not very efficient in this case. However, the result is clear: we see that the percolation temperature agrees with the magnetization one (dotted vertical line in the plot). More precisely we find $J_1/kT_p = 0.51422(12)$, to be compared with $J_1/kT_c = 0.51418(10)$. The values of the critical percolation exponents agree with the ones of the 2D Ising model as we can see in Table I (Model 2).

The magnetization transition of the continuous Ising model was studied in [9]. The value of the critical temperature is $J/kT_c = 1.09312(1)$. For our simulations we made use of the same algorithm described in [9], which consists in a combination of heat bath steps and cluster flippings; this algorithm is quite efficient and we could push our statistics up to 100000 measurements for each lattice size. We remark that in this case we have to do with a continuous spin variable, so that we join nearest-neighbouring like-signed spins, although their absolute values are, in general, different. Fig. 4 shows the percolation cumulant for the four lattice sizes we took: $100^2$, $200^2$, $300^2$, $400^2$. Due to our high statistics, we did not draw the errors in our plot, and we see that the curves of the mean values cross remarkably at the same point, which coincides with the thermal critical point (dotted vertical line in the figure). Our estimate of the percolation temperature is $J/kT_p = 1.09311(3)$.

FIG. 3. Percolation cumulant of pure-site clusters for the $Z(2)$ model with nearest-neighbour ferromagnetic and next-to-nearest-neighbour antiferromagnetic interactions.

FIG. 4. Percolation cumulant of pure-site clusters for the continuous Ising model.
We notice once again that the critical indices are, within errors, the same we found for the previous two models (Model 3 in Table I).

Finally we considered the more involved case of SU(2) lattice gauge theory, in the pure gluonic sector. It is known that in this case there is a confinement-deconfinement transition with Ising exponents. The order parameter is the lattice average $L$ of the Polyakov loop, which is a continuous variable like the spins of the continuous Ising model we examined above. We analyzed the 2+1 dimensional case, which corresponds to two space dimensions. The number of lattice spacings in the imaginary time direction is $N_{\tau} = 2$. We took four lattices: $64^2$, $96^2$, $128^2$ and $200^2$. The statistics ranged from 20000 to 50000 measurements. Our value for the critical coupling is $\beta_c = 4/\gamma_c^2 = 3.4504(11)$, which is by an order of magnitude more precise than the value reported in [10]. The original SU(2) matrix configurations on the links of the 2+1 dimensional lattice are projected onto Polyakov loop configurations on the sites of a 2-dimensional lattice. For the percolation analysis we investigated the Polyakov loop configurations. In order to build the clusters we proceeded as for the continuous Ising model, by joining nearest-neighbouring sites carrying like-signed values of the Polyakov loop variable. In Fig. 5 we plotted the percolation cumulant curves as a function of the coupling $\beta$ for two lattice sizes, $96^2$ and $200^2$. Also here the coincidence of the percolation with the critical point seems to be clear. The percolation coupling we determined is $\beta_p = 4/\gamma_p^2 = 3.4501(14)$. The critical percolation indices are again the same we found so far (Model 4 in Table I).

We complete our investigation by studying the models of the second group. Now we have three spin states and the clusters are simply obtained by binding nearest-neighbouring sites in the same spin state. The fact that the pure-site clusters percolate at $T_c$ for the 3-state Potts model in two dimensions was found, though not rigorously, in [11]. The percolation temperature that we determined is $J/kT_p = 1.00511(9)$, in agreement with the thermal threshold $J/kT_c = \log(1 + \sqrt{3}) = 1.00505...$. The exponents of the percolation transition are listed in Table I (Model 1). We see that they are different from the thermal exponents ($\beta/\nu = 2/15$, $\gamma/\nu = 26/15$, $\nu = 5/6$), except $\nu_p$, which, like in the Ising case, coincides with the thermal value $5/6$. We remark that the values of the critical indices are in very good accord with the predictions of [12] (first line of Table I). The percolation exponents are also different from the ones we have found for the models in the 2D Ising universality class.

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We complete our investigation by studying the models of the second group. Now we have three spin states and the clusters are simply obtained by binding nearest-neighbouring sites in the same spin state. The fact that the pure-site clusters percolate at $T_c$ for the 3-state Potts model in two dimensions was found, though not rigorously, in [11]. The percolation temperature that we determined is $J/kT_p = 1.00511(9)$, in agreement with the thermal threshold $J/kT_c = \log(1 + \sqrt{3}) = 1.00505...$. The exponents of the percolation transition are listed in Table I (Model 1). We see that they are different from the thermal exponents ($\beta/\nu = 2/15$, $\gamma/\nu = 26/15$, $\nu = 5/6$), except $\nu_p$, which, like in the Ising case, coincides with the thermal value $5/6$. We remark that the values of the critical indices are in very good accord with the predictions of [12] (first line of Table I). The percolation exponents are also different from the ones we have found for the models in the 2D Ising universality class.

The analysis of the model with competitive interactions leads to the same conclusions: the percolation point is $J_1/kT_p = 1.1670(7)$, in accord with the magnetization temperature $J_1/kT_c = 1.1665(9)$. In Fig. 6 we plotted both the Binder and the percolation cumulant, so that we can have a visual comparison of the two thresholds. The critical percolation indices agree with the ones we have found for the 3-state Potts model, as we can see in Table I (Model 2).
In conclusion, we have found that the percolation point of the pure-site clusters of various models coincides with the critical point of the thermal magnetization transition. We examined theories with discrete and continuous spins, involving interactions beyond the fundamental nearest-neighbour one, and we included competitive interactions by means of antiferromagnetic couplings. In particular, we stress that the effective theory of the Polyakov loop for SU(2) gauge theory (and in general for SU(N)) includes many couplings, not only nearest-neighbour spin-spin interactions, and that such couplings are ferromagnetic but also antiferromagnetic (see [13]). Therefore, the fact that our result is valid even for SU(2) suggests that it is a rather general feature of bidimensional models with a continuous magnetization transition. It would be interesting to check whether this property of the pure-site clusters always holds in two dimensions. We remark as well that, even if we analyzed models with magnetization transitions, the result is probably also true for systems undergoing a phase transition towards antiferromagnetic ground states, since in this case the definition of the clusters can be trivially extended by joining nearest-neighbouring spins of opposite signs $\dagger$.

Moreover, we found that the critical percolation indices are the same for systems in the same universality class, although they do not coincide with the thermal indices. This shows that, at criticality, the pure-site clusters of all models in the same universality class are, virtually, indistinguishable from each other as far as their size distributions are concerned (although they could differ in their topological properties). So, in many respects, the behaviour of the pure-site clusters of a model is as universal as other features of the model, e.g., its critical exponents. That indicates that there is a close relationship between such clusters and the critical behaviour of the system. It is indeed possible to show that the pure-site clusters of a model "contain" the clusters which reproduce its critical behaviour in the sense explained at the beginning of this paper $\ddagger$.

The fact that the pure-site clusters may percolate at the critical point also for systems different from the Ising model was suggested in [15]; however, this prediction concerned just models with ferromagnetic interactions and is a conjecture. This work extends the result to models with competitive interactions to which the theorems and arguments used in [15] cannot be applied. We guess that the property we have illustrated here is due to the fact that the pure-site clusters of the Ising model, for instance, are quite compact objects. This is confirmed by the fact that, in two dimensions, even if one randomly breaks the bonds between nearest-neighbour spins of the same sign with a probability $p_B = \exp(-2J/kT)$, the corresponding site-bond clusters keep percolating at the critical temperature $T_c$. That means that the infinite pure-site cluster at $T_c$ remains an infinite cluster even if we randomly break many of the bonds between the spins belonging to it, which considerably reduces its size. The presence of antiferromagnetic couplings also acts like a bond-breaking factor, which tends to reduce the size of the original pure-site clusters because of the fact that they favour the presence of anti-aligned spins in the system. Nevertheless, if the anti-alignement interactions are weaker that the main nearest-neighbour ferromagnetic coupling, it is likely that they do not succeed in breaking the original spanning structure into finite clusters. That could also explain the fact that the

$\dagger$ The couplings can involve spins which are far from each other, a higher number of spins (plaquette interactions, six-spins couplings, etc.), and also self-interactions.

$\ddagger$ In the Ising model the equivalence of the two cases is evident.
critical percolation exponents are the same as in the original model, since we can imagine that the antiferromagnetic interactions do not perturb enough the renormalization group trajectories in the (percolation) coupling space, so that they would keep on converging to the same fixed point, exactly as it happens for the thermal transition. That naturally leads to the question: how small should the antiferromagnetic couplings be compared to the fundamental ferromagnetic one, so that the pure-site clusters still percolate at \( T_c \)? For the systems we have studied the ratio was of 1:10; we have evidence that the result holds also for a ratio 3:10, we did not present it here because the statistics is quite low. It would be particularly interesting to check whether the property holds as long as the competition between the different interactions still allows a phase transition with spin-ordering to take place.

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| \( \beta_p/\nu_p \) | \( \gamma_p/\nu_p \) | \( \nu_p \) | Fractal Dimension \( D \) | Cumulant at \( T_p \) |
|------------------|-----------------|-------|----------------|------------|
| Predictions      | 5/96=0.052083.. | 91/48=1.89583.. | 1     | 187/96=1.947916.. |          |
| Model 1          | 0.052(2)        | 1.901(11)     | 1.004(9) | 1.947(2)     | 0.9832(4)  |
| Model 2          | 0.051(4)        | 1.908(16)     | 1.02(4)  | 1.947(3)     | 0.9821(22) |
| Model 3          | 0.053(3)        | 1.902(14)     | 0.99(3)  | 1.946(4)     | 0.9837(18) |
| Model 4          | 0.051(4)        | 1.907(18)     | 0.993(35)| 1.946(4)     | 0.9811(32) |

**TABLE I.** Critical percolation indices for the models in the 2-dimensional \( Z(2) \) universality class.

| \( \beta_p/\nu_p \) | \( \gamma_p/\nu_p \) | \( \nu_p \) | Fractal Dimension \( D \) | Cumulant at \( T_p \) |
|------------------|-----------------|-------|----------------|------------|
| Predictions      | 7/80=0.0875     | 73/40=1.825 | 5/6=0.8333.. | 153/80=1.9125 |        |
| Model 1          | 0.092(11)       | 1.832(18)  | 0.82(2)     | 1.910(4)    | 0.932(2)  |
| Model 2          | 0.085(14)       | 1.842(21)  | 0.84(3)     | 1.914(5)    | 0.929(5)  |

**TABLE II.** Critical percolation indices for the models in the 2-dimensional \( Z(3) \) universality class.
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