Re-interpretations of an experiment on the back-action in a weak measurement

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Interpretations of an experiment on the back-action in a weak measurement in [M. Inuma et al., New J. Phys. 13 (2011), 033041] are revisited. We show two different but essentially equivalent interpretations for this experiment along the original scenario of weak measurements proposed by Aharonov, Albert, and Vaidman. To do this, we introduce the notion of extended weak values which is associated not only with the states of the system but also the state of the measuring device. We also evaluate fluctuations in this experiment and found that an optimal measurement strength exists for a fixed polarization angle prepared as an initial state, at which fluctuations in measurement results vanish. The consistency of this evaluation with the experimental results is discussed.

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I. INTRODUCTION

Weak measurements are topical subjects in modern quantum mechanics. Since the proposal of the weak measurement by Aharonov, Albert, and Vaidman (AAV)\textsuperscript{1} in 1988, many theoretical and experimental researches have been carried out. The idea of weak measurement has been used to resolve fundamental issues in quantum mechanics such as Hardy’s paradox\textsuperscript{2}, violation of the Leggett-Garg inequality\textsuperscript{3,4}, and the error-disturbance relation\textsuperscript{5} derived by Ozawa\textsuperscript{6}. On the other hand, many experiments\textsuperscript{7-10} show that the weak measurement is also very useful for high-precision measurements. Weak measurements are topical subjects in modern quantum mechanics. Since the proposal of the weak measurement by Aharonov, Albert, and Vaidman (AAV)\textsuperscript{1} in 1988, many theoretical and experimental researches have been carried out. The idea of weak measurement has been used to resolve fundamental issues in quantum mechanics such as Hardy’s paradox\textsuperscript{2}, violation of the Leggett-Garg inequality\textsuperscript{3,4}, and the error-disturbance relation\textsuperscript{5} derived by Ozawa\textsuperscript{6}. On the other hand, many experiments\textsuperscript{7-10} show that the weak measurement is also very useful for high-precision measurements.

The original proposal by AAV is based on von-Neumann’s measurement theory\textsuperscript{11} in which the whole system involves a quantum system to be measured and a measuring device to measure it. According to the AAV scenario, weak measurements have four steps: the first one is the preparation of the initial state of the system (pre-selection) as well as the measuring device; the second one is to induce an interaction weakly between the system and the measuring device; the third one is the measurement of the state of the measuring device; and the final one is the measurement of the final state of the system (post-selection) by the projection measurement; and the final one is the measurement of the state of the measuring device. Through these four steps, the experimentally obtained value in the linear-order of the weak interaction between the system and the measuring device is not an eigenvalue of the operator $\hat{O}$ of the system but the weak value of the operator $\hat{O}$:

$$
\langle \hat{O}_w \rangle = \frac{\langle \psi_2 | \hat{O} | \psi_1 \rangle}{\langle \psi_2 | \psi_1 \rangle},
$$

(1.1)

where $|\psi_{1,2}\rangle$ are the pre- and post-selected states of the system, respectively.

Along this AAV scenario of weak measurement, one of the authors revealed some properties of weak measurements taking into account of non-linear effects of the von-Neumann interaction\textsuperscript{12}. In Ref.\textsuperscript{12}, the arguments are restricted to an operator $\hat{O}$ which satisfies the property $\hat{O}^2 = 1$ and the all-order evaluations with respect to the coupling constant in the von-Neumann interaction $H = g\hat{O}\delta(t - t_0)$ have been carried out. As the results, it is shown that the final results of weak measurements are given by

$$
\frac{\langle q \rangle'}{g} = \frac{\text{Re} \langle \hat{O}_w \rangle}{1 + \frac{1}{2} \left(1 - \langle \hat{O}_w \rangle^2\right) (e^{-s} - 1)},
$$

(1.2)

$$
g(p)' = \frac{s e^{-s} \text{Im} \langle \hat{O}_w \rangle}{1 + \frac{1}{2} \left(1 - \langle \hat{O}_w \rangle^2\right) (e^{-s} - 1)}.
$$

(1.3)

Here, $q$ and $p$ are the canonical variables for the measuring device ($\{q,p\} = i$, $\hbar = 1$), the initial state of the measuring device was assumed to be the zero mean-value Gaussian, and $s$ is a parameter of the measurement strength defined by $s := 2g^2(p^2)$ with the initial momentum variance $\langle p^2 \rangle$ of the measuring device, $\langle q \rangle'$ and $\langle p \rangle'$ are the expectation value of the pointer variable and its conjugate momentum after the post-selection.

On the other hand, the experimental group of one of the authors\textsuperscript{13} realized a weak measurement of a photon polarization close to ideal and showed that the back-action effects in the weak measurement have an important role in the regime where the weak value becomes large. We refer this reference as ISTKH\textsuperscript{13} in this paper. In ISTKH, it is claimed that the weak value which obtained by the experiment is different from Eq. (1.1).

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The “experimental weak value” \(\langle \hat{O}_w \rangle_{\text{exp}}\) in ISTKH is given by

\[
\langle \hat{O}_w \rangle_{\text{exp}} = \frac{\langle \hat{O}_w \rangle}{1 + \eta \left( \langle \hat{O}_w \rangle^2 - 1 \right)},
\]

where \(\eta\) is a parameter describing the back-action effect. In ISTKH, the expression of Eq. (1.3) is derived from the formalism of the positive operator valued measure (POVM) and the combination of unitary transformations representing optical components, but this derivation does not directly follow the original scenario proposed by AAV. In particular, the ISTKH experiment explicitly utilizes a maximally entangled state of the system and the measuring device, which is produced before the weak interaction between them. In spite of the above difference, the results Eq. (1.2) (or Eq. (1.3)) and Eq. (1.4) look similar, but their identification is still not confirmed.

In this paper, we discuss the consistency between the ISTKH experiment and results in Ref. [12]. The essential difference between the analysis in Ref. [12] and the ISTKH experiment is in the treatment of the entangled state. In the AAV scenario, the weakly entangled state is created through the weak von-Neumann interaction, while the ISTKH experiment produces the maximally entangled state of the system and the measuring device. To overcome this difference in the re-interpretation of the ISTKH, we choose the state just after the entanglement creation as a pre-selected state. This choice gives rise to the other difficulty that we cannot distinguish the pre-selected state of only the system, because the entangled state involves both initial states of the measuring device and the system. To treat this state as the pre-selected state, we introduce the notion of \textit{extended weak values}

\[
\langle \hat{O} \otimes \hat{P}_w \rangle_{\text{pre}} := \frac{\langle \psi_{f(i)} | \hat{O} \otimes \hat{P} | \psi_{\text{comp}} \rangle}{\langle \psi_{f(i)} | \psi_{\text{comp}} \rangle}
\]

instead of the original definition of the weak value (1.1), where \(|\psi_{\text{comp}}\rangle\) is the pre-selected state of a composite system of the system and the measuring device, \(\hat{P}\) is an operator for the measuring device, and \(|\psi_{f(i)}\rangle\) is a product state of the post-selected state of the system and the final state of the measuring device. [See Eqs. (2.2), (3.9)–(3.13), (3.24), and (3.25).] This extended definition of weak values is a natural extension of the original definition (1.1) and very useful when we equivalently treat the system and the measurement device. In our case, \(|\psi_{\text{comp}}\rangle\) is chosen as a maximally entangled state. Through these newly defined weak values, it becomes possible to re-interpret the experiments in ISTKH along the original AAV scenario and to show that the results Eq. (1.2) (or Eq. (1.3)) and Eq. (1.4) are consistent. In the same manner, we also evaluated the fluctuations of the results in ISTKH and consequently found that an optimal measurement strength exists for a fixed weak value, on which the fluctuations in the final measurement vanish. We also discuss the physical meaning of this optimal condition and made a confirmation of the consistency with the experiment.

Organization of this paper is as follows: In section II, we briefly review the experimental setup in ISTKH. In section III, the ISTKH experiment is re-examined along the original scenario of weak measurements proposed by AAV through the introduction of an extended definition of weak values. We show two different interpretations of this experiment: the first one is a weak measurement with a real weak value; the second one is a weak measurement with an imaginary weak value. These two interpretations are essentially equivalent to each other. In section IV, we discuss the behavior of the fluctuations in the final measurement in ISTKH. Final section (section V) is devoted to summary and discussions which include discussions on the consistency with experimental results.

Throughout this paper, we use the natural unit \(\hbar = 1\).

II. EXPERIMENTAL SETUP IN ISTKH

Here, we briefly summarize the experimental setup in ISTKH. The details of the setup is given in ISTKH, but we restrict the explanation to only significant points in the experiment. In ISTKH, the observable in the system is the polarization of photons and the pointer in the measuring device is the which-path information. As shown in Figure 1, the incident photons from the laser go through the Glan-Thompson prism (GTP) and the half-wave plate (HWP) before the polarized beam splitter (PBS). The GTP and HWP can transmit the photon with the linear polarization and control the polarization angle of the incident
photon polarization into the horizontal state \(|H\rangle\) and the vertical state \(|V\rangle\). Thus, the PBS produces the entangled state of the photon polarization and the path as

\[
|\psi_{\text{ent}}\rangle = C_V|V\rangle \otimes |a1\rangle + C_H|H\rangle \otimes |a2\rangle. \tag{2.2}
\]

After the PBS, the photons go through the other HWP on each path \(a1\) and \(a2\) inside of the interferometer. In the path \(a1\), where the polarization state just before the HWP is \(|V\rangle\), the polarization is rotated by \(-2\theta\) to the clockwise direction by the HWP. On the other hand, in the path \(a2\), where the polarization state just before the HWP is \(|H\rangle\), the polarization is rotated by \(2\theta\) to the counter-clockwise direction by the HWP. These rotations of the polarizations are regarded as the weak interaction in the weak measurement.

Each path \(a1\) and \(a2\) is overlapped at the beam splitter (BS) with the 50:50 ratio of the transmittance to the reflectance and the photon’s path-state is transformed into the states \(|b1\rangle\) and \(|b2\rangle\) as

\[
|b1\rangle = \frac{1}{\sqrt{2}} (|a1\rangle + |a2\rangle), \quad |b2\rangle = \frac{1}{\sqrt{2}} (|a1\rangle - |a2\rangle). \tag{2.3}
\]

The transformation (2.3) of the Hilbert space \(|{a1, a2}\rangle\) to the Hilbert space \(|{b1, b2}\rangle\) is an unitary transformation which satisfies

\[
|a1\rangle\langle a1| + |a2\rangle\langle a2| = |b1\rangle\langle b1| + |b2\rangle\langle b2| = 1. \tag{2.4}
\]

On the path \(b2\) from the output port of BS, a HWP is installed to disentangle the system and the measuring device. After the BS, the photons go through a polarizer on each path \(b1\) and \(b2\) for the post-selection of the weak measurement. In the ISTKH experiment, the final polarization state is selected to the horizontal state \(|H\rangle\) by these polarizers.

The photons in each path \(b1\) and \(b2\) are detected by the single photon detector on each path. The number of photons are counted and their results give the conditional probabilities associated with the which-path information \(|b1\rangle\) and \(|b2\rangle\). Here, we denote these probability distributions \(P(b1)\) and \(P(b2)\), respectively.

Finally, the difference \(P(b1) - P(b2)\) is calculated, because the weak value is obtained by dividing an average of the pointer variable with the obtained conditional probabilities by the measurement strength. Therefore, the difference of the conditional probabilities \(P(b1) - P(b2)\) can be considered as the final result in this experiment.

According to ISTKH, the expression of \(P(b1) - P(b2)\) is given by

\[
P(b1) - P(b2) = \frac{\sin(4\theta) \text{Re}(\tilde{\sigma}_x)}{1 + \frac{1}{2} (1 - |\langle \tilde{\sigma}_x \rangle|)^2 (\cos(4\theta) - 1)}, \tag{2.5}
\]

where \(\langle \tilde{\sigma}_x \rangle = C_V/C_H\). Eq. (2.5) is the precise expression of Eq. (2.4), which was derived from the POVM formalism and the combination of unitary transformations. One of the purposes of this paper is re-derivation of Eq. (2.5) along the original scenario by AAV [1].

III. INTERPRETATIONS OF THE EXPERIMENT IN ISTKH THROUGH AAV SCENARIO

In the original AAV argument [1], the weak measurement consists of four processes, pre-selection, weak interaction, post-selection, and the final measurement of the measuring device. However, these serial processes do not explicitly include the entanglement creation. Therefore, the correspondence between a sequence of the above processes and an alternative sequence in the ISTKH experiment remains unclear.

One of ways for excluding the entanglement creation from the whole process of the weak measurement is to use the state Eq. (2.2) as the pre-selected state instead of Eq. (2.1). This requires a mathematically equivalent treatment for the device’s state as the system’s one. In this paper, we resolved this problem by the introduction of two extended weak values associated also with the final state of the measuring device, which are defined in later Eqs. (3.10) and (3.11) in section III A or Eqs. (3.24) and (3.25) in section III B.

On the process of the post-selection, the final state \(|H\rangle\) is selected by the polarizers just before the photon detection in the ISTKH experiment. It is therefore straightforward to treat the photon polarization with the basis of \(|{H, V}\rangle\) as the “system”. For the selection of \(|H\rangle\) as the final state, the effect of the HWP installed just after the BS does not consequently affect the final result \(P(b1) - P(b2)\), because this HWP just gives the \(\pi\) phase difference between the state \(|H\rangle\) and the state \(|V\rangle\). This final result can be obtained by the final measurement of the “measuring device”, that is the projection measurement to the basis \(|{b1, b2}\rangle\) representing which-path information. This is easily understood by rewriting the probability distribution \(P(b1) - P(b2)\) in the form

\[
P(b1) - P(b2) := \langle b1|\rho_f|b1\rangle - \langle b2|\rho_f|b2\rangle = \text{Tr}[[|b1\rangle\langle b1| - |b2\rangle\langle b2|] \rho_f], \tag{3.1}
\]

where \(\rho_f\) is the conditional density matrix after the post-selection. The above equation shows that \(P(b1) - P(b2)\) is the expectation value of the which-path operator \(|b1\rangle\langle b1| - |b2\rangle\langle b2|\) and the final measurement cor-
responds to the projection measurement of the Hilbert space \{\ket{b1}, \ket{b2}\).

The remaining correspondence is on the process of the weak interaction between the system and the measuring device. The identification of the interaction Hamiltonian in the ISTKH experiment is necessary to understand the full process along the AAV’s scenario. We found that two effective interaction Hamiltonians, which provide the explanation of the ISTKH experiment in a different but essentially equivalent way. One is the interpretation by a real extended weak value for the operator \(\hat{A}\) and another is by an imaginary extended weak value for \(\hat{A}\).

These two interpretations give the essentially same results. Thus, both of these interpretations reproduce the result of the ISTKH experiment along the original proposal of AAV [1].

A. Interpretation with a real extended weak value for \(\hat{A}\)

The polarization in each path a1 and a2 is rotated in opposite direction by the HWPs inside the interferometer. As the effective Hamiltonian describing the interaction between the polarization and the which-path operator, we consider the following form:

\[
\hat{H}_{HW} = \Omega \left( \sin(2\theta) \frac{\hat{\sigma}_x}{2} \otimes \hat{A} + \cos(2\theta) \frac{\hat{\sigma}_z}{2} \otimes \hat{1} \right),
\]

where \(\theta\) is an angle of the fast axis of HWP from the horizontal axis and \(\Omega \Delta t\) represents a phase difference between the fast component and the slow component after passing through the HWP. In the case of HWP, the phase difference is \(\Omega \Delta t = \pi\) with \(\Delta t = l/c\), where \(l\) is the length of the HWP. The operator \(\hat{A}\) is defined by

\[
\hat{A} := -|a1\rangle \langle a1| + |a2\rangle \langle a2|,
\]

which has an eigenvalue of \(\pm 1\) for each path eigenstate of \(|a1\rangle\) and \(|a2\rangle\). Therefore, the which-path operator determines a sign of the first term including \(\hat{\sigma}_x\) in Eq. (3.2).

From the interaction Hamiltonian Eq. (3.2), the evolution operator \(\hat{U}_1\) through this interaction is given by

\[
\hat{U}_1 = \exp \left(-i\hat{H}_{HW} \Delta t \right)
= \exp \left[-i \frac{\pi}{2} \left( \sin(2\theta) \hat{\sigma}_x \otimes \hat{A} + \cos(2\theta) \hat{\sigma}_z \otimes \hat{1} \right) \right].
\]

(3.4)

Here, we note that

\[
\left( \sin(2\theta) \hat{\sigma}_x \otimes \hat{A} + \cos(2\theta) \hat{\sigma}_z \otimes \hat{1} \right)^2 = \hat{1} \otimes \hat{1}.
\]

(3.5)

From this property, the unitary operator \(\hat{U}_1\) can be written as

\[
\hat{U}_1 = -i \left( \sin(2\theta) \hat{\sigma}_x \otimes \hat{A} + \cos(2\theta) \hat{\sigma}_z \otimes \hat{1} \right).
\]

(3.6)

This unitary operator transforms the polarization vector into in the line-symmetric position with respect to the fast axis of the HWP, which angle is \(-\theta\) from the vertical axis in path a1 and \(+\theta\) from the horizontal axis in path a2. As the result, these transformations correspond to the effective rotation of the polarization in a1 with the angle of \(-2\theta\) from the vertical axis and of the polarization in a2 with the angle of \(+2\theta\) from the horizontal axis. If the which-path operator \(\hat{A}\) is absent, the unitary operator \(\hat{U}_1\) becomes a well-known form for HWP [17]. By the unitary transformation \(\hat{U}_1\), the total density matrix is evolved from the initial density matrix \(\rho_{\text{init}} = |\psi_{\text{ent}}\rangle \langle \psi_{\text{ent}}|\) to \(\hat{U}_1 \rho_{\text{init}} \hat{U}_1^\dagger\), where the initial state \(|\psi_{\text{ent}}\rangle\) is given by Eq. (2.2).

Following this unitary evolution, the 50:50 BS transforms the basis of which-path information \(|a1\rangle, |a2\rangle\) into the other basis \(|b1\rangle, |b2\rangle\). Since this transformation can be written as the multiplication of the identity operator \(\hat{I}\), we can consider this BS transformation after the post-selection of the system.

After selecting to the final state \(|H\rangle\) by the polarizers downstream of the BS, the conditional density matrix can be expressed as

\[
\rho_f = \left|\psi_f\right| \langle \psi_f | \left( |\psi_f\rangle \langle \psi_f | \right),
\]

(3.7)

\[
|\psi_f\rangle = -i |\psi_{\text{ent}}\rangle \langle \psi_{\text{ent}}| \left( \sin(2\theta) \hat{\sigma}_x \otimes \hat{A} + \cos(2\theta) \hat{\sigma}_z \otimes \hat{1} \right) |\psi_{\text{ent}}\rangle,
\]

(3.8)

where the identity \(\hat{1}\) in Eq. (3.8) is given by (2.4).

To treat the post-selected state of the system and the projected states of the measuring device equivalently, we introduce following two final states

\[
|\psi_{f(1)}\rangle := |H\rangle \otimes |b1\rangle, \quad |\psi_{f(2)}\rangle := |H\rangle \otimes |b2\rangle.
\]

(3.9)

Further, we also introduce two extended weak values which are defined by

\[
\left( \hat{\sigma}_x \otimes \hat{A} \right)_{w(1)} := \frac{\langle \psi_{f(1)} | \hat{\sigma}_x \otimes \hat{A} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle} = -\frac{C_V}{C_H},
\]

(3.10)

\[
\left( \hat{\sigma}_x \otimes \hat{A} \right)_{w(2)} := \frac{\langle \psi_{f(2)} | \hat{\sigma}_x \otimes \hat{A} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(2)} | \psi_{\text{ent}} \rangle} = \frac{C_V}{C_H}.
\]

(3.11)

The above weak values have real values. In addition, the following relations are satisfied,

\[
\frac{\langle \psi_{f(1)} | \hat{\sigma}_x \otimes \hat{1} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle} = \frac{\langle \psi_{f(2)} | \hat{\sigma}_x \otimes \hat{1} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(2)} | \psi_{\text{ent}} \rangle} = 1,
\]

(3.12)

\[
\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle = \langle \psi_{f(2)} | \psi_{\text{ent}} \rangle = \frac{1}{\sqrt{2}} C_H.
\]

(3.13)

By using Eq. (3.10)–(3.13), the normalization factor \(\langle \psi_f | \psi_f \rangle\) of the final density matrix Eq. (3.7) is expressed
Using Eq. (3.7)–(3.17), we obtain the expression of the HWP in the path a2 is also described by the active extended weak values with the conventional weak values. The appearance of the which-path operator $\hat{A}$ transforms the conditional density matrix from the initial state $\rho_{init} = |\psi_{ent}\rangle\langle\psi_{ent}|$ to $U_2 \rho_{init} U_2^\dagger$, where the pre-selected state $|\psi_{ent}\rangle$ is given by Eq. (2.22). In the same way as the AAV scenario, the conditional density matrix $\rho_f$ after the post-selection can be written as

\[
P(b_1) - P(b_2) = \frac{\text{Re} \left< \hat{\sigma}_x \otimes \hat{A} \right>_{w(1)} \sin(4\theta)}{1 + \frac{i}{4} \left(1 - \left< \hat{\sigma}_x \otimes \hat{A} \right>_{w(1)}^2 \right) (\cos(4\theta) - 1)},
\]

which is essentially equivalent to Eq. (2.15), replacing the extended weak values with the conventional weak values. The appearance of the which-path operator $\hat{A}$ in the definition of the extended weak value is a natural consequence of our extension of their definitions (3.10) and (3.11). The result of Eq. (3.18) shows that the ISTKH experiment can be regarded as the measurement of the real extended weak value for the operator $\hat{\sigma}_x$ as commented in Ref. [12].

### B. Interpretation with an imaginary weak value for $\hat{\sigma}_y$

The effective rotation of the photon polarization by the HWP in the path a2 is also described by the active transformation of $\{ C_H, C_V \}$ as

\[
\begin{pmatrix} C_H \\ C_V \end{pmatrix} \rightarrow \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} C_H \\ C_V \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2\theta \\ -2\theta & 0 \end{pmatrix} + O(\theta^2) \begin{pmatrix} C_H \\ C_V \end{pmatrix}.
\]

Since the unitary operator generated by an operator $\hat{G}$ is expressed as $U(\theta) = e^{-i\alpha\hat{G}} = 1 - i\alpha \hat{G} + O(\alpha^2)$, the rotation expressed as Eq. (3.19) is generated by the operator $\hat{G} = \hat{\sigma}_y = -i[H] \langle V \rangle + i[V] \langle H \rangle$ with the parameter $\alpha = 2\theta$. In the path a1, the polarization can be also rotated as Eq. (3.19) with the opposite direction ($\theta \to -\theta$). Therefore, the unitary evolution by the HWP on each path a1 and a2 is written as

\[
\hat{U}_2 := \exp \left[-i(2\theta)\hat{\sigma}_y \otimes \hat{A} \right].
\]

The effective interaction Hamiltonian can be expressed as

\[
\hat{H}_2 = (2\theta)\hat{\sigma}_y \otimes \hat{A}(t-t_0).
\]

Since the operator $\hat{\sigma}_y \otimes \hat{A}$ satisfies the property $\left( \hat{\sigma}_y \otimes \hat{A} \right)^2 = 1 \otimes 1$, the unitary operator Eq. (3.20) is also represented by

\[
\hat{U}_2 = \cos(2\theta) \hat{1} \otimes \hat{1} - i \sin(2\theta) \hat{\sigma}_y \otimes \hat{A}.
\]

This unitary operator describes the rotation of photon polarizations on each path a1 and a2. Therefore, instead of the operator $\hat{U}_1$ given by Eq. (3.6), we can use the operator $\hat{U}_2$ given by Eq. (3.22). This unitary evolution transforms the conditional density matrix from the initial density matrix $\rho_{init} = |\psi_{ent}\rangle\langle\psi_{ent}|$ to $\hat{U}_2 \rho_{init} \hat{U}_2^\dagger$, where the pre-selected state $|\psi_{ent}\rangle$ is given by Eq. (2.22).
Thus, the expression of $P$ extended weak final states given by Eq. (3.9) and two

\[ \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} := \frac{\langle \psi_{f(1)} | \sigma_y \otimes \hat{A} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle} = i \frac{C_V}{C_H} \] (3.24)

\[ \langle \sigma_y \otimes \hat{A} \rangle_{w(2)} := \frac{\langle \psi_{f(2)} | \sigma_y \otimes \hat{A} | \psi_{\text{ent}} \rangle}{\langle \psi_{f(2)} | \psi_{\text{ent}} \rangle} = -i \frac{C_V}{C_H} \] (3.25)

The probability difference $P(b1) - P(b2)$ can be evaluated with Eq. (3.15). From Eq. (3.13), and Eq. (3.23–3.25), we get

\[ |b1\psi_f\rangle^2 = |\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle|^2 \left\{ \frac{\sin^2(2\theta) + \cos^2(2\theta) - \text{Im} \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \sin(4\theta)}{1 + \frac{1}{2} \left( 1 - \left| \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \right|^2 \right)} \right\}, \] (3.27)

\[ |b2\psi_f\rangle^2 = |\langle \psi_{f(1)} | \psi_{\text{ent}} \rangle|^2 \left\{ \frac{\sin^2(2\theta) + \cos^2(2\theta) + \text{Im} \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \sin(4\theta)}{1 + \frac{1}{2} \left( 1 - \left| \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \right|^2 \right)} \right\}. \] (3.28)

Thus, the expression of $P(b1) - P(b2)$ can be obtained as

\[ P(b1) - P(b2) = \frac{-\text{Im} \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \sin(4\theta)}{1 + \frac{1}{2} \left( 1 - \left| \langle \sigma_y \otimes \hat{A} \rangle_{w(1)} \right|^2 \right)(\cos(4\theta) - 1)}, \] (3.29)

which coincides with Eq. (2.5) by the replacement of $\text{Re} \langle \sigma_x \rangle_{w} \to \text{Im} \langle \sigma_y \otimes \hat{A} \rangle_{w(1)}$. The appearance of the which-path operator $\hat{A}$ is a natural consequence of our extension of weak values. It is obvious that the result Eq. (3.29) corresponds to Eq. (1.3) which was derived in [12]. Eq. (3.18) and Eq. (3.29) show that the ISTKH experiment can be interpreted not only as a weak measurement of a real weak value but also that of an imaginary weak value.

**IV. FLUCTUATIONS**

In the previous section, we showed that the two expectation values $P(b1) - P(b2)$ derived from the two different interpretations are the essentially same. In this section, we discuss the fluctuations in $P(b1) - P(b2)$. Since $P(b1) - P(b2)$ is the expectation value of the which-path operator $|b1\rangle \langle b1| - |b2\rangle \langle b2|$, the corresponding variance is given by

\[ \text{Var} (|b1\rangle \langle b1| - |b2\rangle \langle b2|) := \text{Tr} \left[ (|b1\rangle \langle b1| - |b2\rangle \langle b2|)^2 \rho_f \right] - \left( \langle |b1\rangle \langle b1| - |b2\rangle \langle b2| \right)^2. \] (4.1)

Using the property $(|b1\rangle \langle b1| - |b2\rangle \langle b2|)^2 = \hat{1}$, Eq. (4.1) provides the same variance for the same value $P(b1) - P(b2)$. Therefore, we show the fluctuations in $P(b1) - P(b2)$ obtained only from the interpretation in section IIIA.

Using Eq. (3.18), we can easily obtain

\[ \text{Var} (|b1\rangle \langle b1| - |b2\rangle \langle b2|) = \left( \frac{1 - \zeta^2}{1 + \zeta^2} \right)^2, \] (4.2)

where we used $\text{Im} \langle \sigma_x \otimes \hat{A} \rangle_{w(1)} = 0$ and defined $\zeta$ by

\[ \zeta := \tan(2\theta) \text{Re} \langle \sigma_x \otimes \hat{A} \rangle_{w(1)}. \] (4.3)
Thus, the fluctuation $\Delta(P(b1) - P(b2)) := \sqrt{\text{Var}(|b1\rangle\langle b1| - |b2\rangle\langle b2|)}$ is given by

$$\Delta(P(b1) - P(b2)) = \left| \frac{1 - \zeta^2}{1 + \zeta^2} \right|. \quad (4.4)$$

It is obvious that the fluctuations expressed by Eq. (4.4) vanishes when the condition

$$\zeta = \pm 1 \quad (4.5)$$

is satisfies. We will discuss the physical meaning of these optimal conditions later.

In the following, we discuss the behaviors of $P(b1) - P(b2)$ and its fluctuation $\Delta(P(b1) - P(b2))$. The difference $P(b1) - P(b2)$ given by Eq. (4.18) can be rewritten as the form

$$P(b1) - P(b2) = \frac{2\zeta}{1 + \zeta^2}, \quad (4.6)$$

where we used $\text{Im} \left\langle \hat{\sigma}_x \otimes \hat{A} \right\rangle_{w(1)} = 0$. When $|\zeta| \gg 1$, $P(b1) - P(b2) \sim 2/\zeta$ and $\Delta(P(b1) - P(b2)) \sim 1$ are given from Eq. (4.13) and Eq. (4.14). Therefore, in the regime $|\zeta| \gg 1$ the fluctuation dominates and clearly makes it difficult to measure the signal $P(b1) - P(b2)$. On the other hands, at $\zeta = 0$ and $|\zeta| = \infty$, $P(b1) - P(b2) = 0$ is satisfied. This leads to the presence of extremal points in $P(b1) - P(b2)$. Differentiating $P(b1) - P(b2)$ with respect to $\zeta$, we can easily confirm that these extremal points coincide with the optimal conditions Eq. (4.15), i.e., $P(b1) - P(b2) = 1$ at $\zeta = 1$ and $P(b1) - P(b2) = -1$ at $\zeta = -1$.

Since $0 \leq P(b1), P(b2) \leq 1$ is satisfied, each relation of $P(b1) - P(b2) = 1$ and $P(b1) - P(b2) = -1$ results in $(P(b1), P(b2)) = (1, 0)$ and $(P(b1), P(b2)) = (0, 1)$, respectively. This indicates that the state $|\psi_f\rangle/\sqrt{|\langle \psi_f |\psi_f \rangle|}$ of the total system just after the post-selection should be an eigenstate of the which-path operator $|b1\rangle\langle b1| - |b2\rangle\langle b2|$ in the optimal condition Eq. (4.5). We can easily see it by rewriting the state $|\psi_f\rangle$ as follows:

$$|\psi_f\rangle = -i \cos(2\theta) |\psi_f^{(1)}\rangle |\psi_{\text{ent}}\rangle \{(1 + \zeta) |b1\rangle + (1 - \zeta) |b2\rangle\}. \quad (4.7)$$

This expression clearly shows $\rho_f = |b1\rangle\langle b1|$ at $\zeta = 1$ and $\rho_f = |b2\rangle\langle b2|$ at $\zeta = -1$, respectively. Thus, the measurement of the which-path observable in the optimal conditions Eq. (4.5) results in the measurement of the eigenstates of the operator $|b1\rangle\langle b1| - |b2\rangle\langle b2|$ and the fluctuations in $P(b1) - P(b2)$ vanish.

V. SUMMARY AND DISCUSSIONS

In summary, we showed re-interpretations of the ISTKH experiment on the back-action in a weak measurement through the AAV original scenario which was discussed in Ref. [12]. Although the ISTKH experiment is different from the usual argument due to the explicit entanglement creation between the system and the measuring device, the introduction of the extended weak value makes us understand the ISTKH experiment along the original scenario proposed by AAV, i.e., pre-selection, weak interaction, post-selection, and the final measurement of the measuring device. The extended weak value is generally applicable to other experiments requiring the mathematically equivalent treatment for the device and the system. As the result, we achieved two re-interpretations to the ISTKH experiment, which correspond to two types of weak measurements: one is for the real extended weak value and another is for the imaginary extended weak value, but these re-interpretations give the essentially same results. These show that these two interpretations are essentially equivalent and the ISTKH experiment is consistent with the original AAV argument and the results in Ref. [12].

The equivalence of two re-interpretation is very interesting, because the usual argument on weak measurement [13] shows that the real and imaginary parts of weak value emerges as the shift in the average of the pointer variable and its conjugate momentum, respectively. Unlike this argument, the ISTKH experiment is the first example of the experiment which can be regarded as not only the measurement of a real weak value, but also that of an imaginary weak values at the same time. This interesting feature is obviously provided by the properties of Pauli operators. Therefore, the similar weak measurements of qubit systems have a possibility of including the same feature. However, many experiments of qubit systems do not need a knowledge of the explicit form of the effective Hamiltonian in the aspect of practical implementations, so that there have been no such arguments yet.

In addition, we also showed that the fluctuations in the expectation value for both interpretations result in the completely same formula. We found the optimal condition in the measurement strength $\theta$, where the fluctuations vanish for a fixed weak value. This vanishing fluctuations do not certainly provide the infinity of the signal to noise ratio in the optimal condition because of the presence of other noise sources (for example, shot noise, imperfection in the setup, and so on). However, this feature of the fluctuation in the optimal condition is potentially useful for the improvement of the signal to noise ratio. In the reference [18], they discussed the precision of a phase measurement by the weak measurement in the shot-noise limited interferometer and found that the photo-detector dominantly produces the photon shot-noise which determines the signal to noise ratio. In the ISTKH experiment, the state just after the post-selection may be identified to the eigenstates of the which-path operator in the final measurement, so that the fluctuation becomes very small. This is a new theoretical result on fluctuations and predicts that there are in principle rooms of the improvement of the fluctuations in the
The behavior of these fluctuations is consistent with the current experiment [4]. In the similar condition to Ref. [4], the above calculation gives the optimal condition of $\theta = 11.25^\circ$ for the vanishing fluctuation. From Ref. [4], the experimental data at $\theta = 11.0^\circ$ shows $P(b_1) - P(b_2) = 0.857$ and $\Delta(P(b_1) - P(b_2)) = 0.00537$, which corresponds to the minimum fluctuation in all experimental data obtained by changing $\theta$ discretely. In comparison with $\theta = 2.2^\circ$, for example, they obtained $P(b_1) - P(b_2) = 0.311$ and $\Delta(P(b_1) - P(b_2)) = 0.0131$. The error at $\theta = 2.2^\circ$ is 2.44 times as large as the error at $\theta = 11.0^\circ$. This factor corresponds to 6 times lower statistics at $\theta = 11.0^\circ$, but the statistics at any $\theta$ is almost same and the difference is absolutely less than the factor two. Taking into account the presence of noise sources, such as imperfection of visibility, shot noise, etc., the minimum in fluctuation at $\theta = 11.0^\circ$ is consistent with the above calculation. However, no systematic experiments have been carried out yet for the test of these theoretical predictions on fluctuations in detail.

Finally, we give a comment on the other possibility of interpretations of the ISTKH experiment. In this paper, we choose the maximal entangled state as the pre-selected state and the new problem arisen from the in-distinguishability between the system and the measuring device was overcome by introducing the extended weak values. Apart from such way, it may be also possible to assign the initial polarization state (2.1) to the pre-selected state in the same way as the ISTKH setup. In this case, although it is not necessary to introduce the extended weak values, we have to include the entanglement creation into the four processes of the weak measurements. Since the creation of the entangled state is also represented by the unitary operator, the effective Hamiltonian corresponding to the unitary transformation included the creation process should be expressed as the specific mathematical form. If we find this form, it will become possible to make the other re-interpretation along the original AAV argument with treating the initial polarization state (2.1) as the pre-selected state. Although we leave this possibility of the interpretation as future works, we note that many other interpretations for this experiment will also be possible.

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[1] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60 (1988), 1351.
[2] Y. Aharonov, A. Botero, S. Pospescu, B. Reznik, and J. Tolkaksen, Phys. Lett. A 301 (2002), 130; J. S. Lundeen and A. M. Steinberg, Phys. Rev. Lett. 102 (2009), 020404; K. Yokota, T. Yamamoto, M. Koashi, and N. Imoto, New J. Phys. 11 (2009), 033011.
[3] N. S. Williams and A. N. Jordan, Phys. Rev. Lett. 100 (2008), 026804.
[4] Y. Suzuki, M. Inouma, and H. F. Hofmann, New J. Phys. 14 (2012), 103922.
[5] A. P. Lund and H. M. Wiseman, New J. Phys. 12 (2010), 093011; L. A. Rozema, A. Darabi, D. H. Mahler, A. Hayat, Y. Soudager, and A. M. Steinberg, Phys. Rev. Lett. 109 (2012), 100404.
[6] M. Ozawa, Phys. Rev. A 67 (2003), 042105.
[7] N. W. M. Ritchie, J. G. Story, and R. G. Hulet, Phys. Rev. Lett. 66 (1991), 1107.
[8] G. J. Pryde, J. L. O’Brien, A. G. White, T. C. Ralph, H. M. Wiseman, Phys. Rev. Lett. 94 (2005), 220405.
[9] O. Hosten and P. Kwiat, Science 319 (2008), 787; K. J. Resch, Science 319 (2008), 733.
[10] P. B. Dixon, D. J. Starling, A. N. Jordan, and J. C. Howell, Phys. Rev. Lett. 102 (2009), 173601; J. C. Howell, D. J. Starling, P. B. Dixon, P. K. Vudyasetu, and A. N. Jordan, Phys. Rev. A 81 (2010), 033813; D. J. Starling, P. B. Dixon, A. N. Jordan, and J. C. Howell, Phys. Rev. A 80 (2009), 041803(R).
[11] von Neumann J., *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, NJ, 1955).
[12] K. Nakamura, A. Nishizawa, and M.-K. Fujimoto, Phys. Rev. A 85 (2012), 012113.
[13] M. Inouma, Y. Suzuki, G. Taguchi, Y. Kadoya, and H. F. Hofmann, New J. Phys. 13 (2011), 033041.
[14] Although Stokes parameters are used to explain the photon polarizations in ISTKH, throughout this paper, we describe the polarization states by the basis $\{|H\rangle, |V\rangle\}$ and we also introduce the operators $\hat{\sigma}_x = |H\rangle\langle V| + |V\rangle\langle H|$, $\hat{\sigma}_y = -i|H\rangle\langle V| + i|V\rangle\langle H|$, and $\hat{\sigma}_z = |H\rangle\langle H| - |V\rangle\langle V|$. The difference of the overall sign in $P(b_1) - P(b_2)$ does not matter, because the overall sign is experimentally controllable in the actual ISTKH experiment.
[15] A. Nishizawa, K. Nakamura, and M.-K. Fujimoto, Phys. Rev. A 85 (2012), 062108.
[16] See, for example, A. Yariv *Optical Electronics in Modern Communications 5th edition* (Oxford University Press, New York, NY, 1997).
[17] M. Jozsa, Phys. Rev. A 76 (2007) 044103.