Spacetime as a deformable solid

M. O. Tahim, R. R. Landim, and C. A. S. Almeida

1Departamento de Física - Universidade Federal do Ceará
C.P. 6030, 60455-760 Fortaleza-Ceará-Brazil
2Departamento de Ciências da Natureza, Faculdade de Ciências, Educação e Letras do Sertão Central (FECLESC), Universidade Estadual do Ceará, 63900-000 Quixadá-Ceará-Brazil

In this letter we discuss the possibility of treating the spacetime by itself as a kind of deformable body for which we can define an fundamental lattice, just like atoms in crystal lattices. We show three signs pointing in that direction. We simulate the spacetime manifold by a very specific congruence of curves and use the Landau-Raychadhuri equation to study the behavior of such a congruence. The lattice appears because we are forced to associate to each curve of the congruence a sort of fundamental "particle". The world-lines of these particles should be identified with the congruence fulfilling the spacetime manifold. The conclusion is that when describing the deformations of the spacetime the Einstein equations emerge and the spacetime metric should be treated as a secondary (not fundamental) object of the theory.

PACS numbers: 04.50.+h, 04.20.Cv, 04.20.Gz
Keywords: spacetime model, Landau-Raychadhuri equation, Einstein gravity

I. INTRODUCTION

The idea pointing out some resemblances between the spacetime and the dynamics of deformable bodies is not new. There are authors that have discussed these observations some decades ago. The idea generally discussed is related with the real meaning of the spacetime: can it be quantized in the sense we generally work in these modern days (the canonical approach and the others methods)? If the spacetime is some sort of fluid, how can we identify its fundamental constituents (a real crystal, from the macroscopic viewpoint, looks like a continuum, but from the microscopic viewpoint, is made of small parts, i.e., atoms and molecules)? If these ideas are reliable then perhaps it does not make sense to quantize the spacetime using the standard procedures. In this way the spacetime as we observe and describe would be a secondary entity, originating from an collectivity of more fundamental objects. The first realization of these ideas has appeared with the Sakharov’s work [1] related with what is known as Emergent Gravity. Following this line, the Einstein’s gravitational theory would appear after a direct quantization of matter. The dynamics of the gravitational field is generated as a secondary effect associated to radiative corrections at one loop. Several lines of research have started through the years derived from Sakharov’s idea (an example is the so called Stochastic Gravity [2]). On the other hand, in the last years a Ted Jacobson’s paper have shown how to understand the Einstein’s equations from a thermodynamical viewpoint [3]: the Einstein’s equations are equations of state for the spacetime. That conclusion strongly suggests that the spacetime may really be compared to a special kind of deformable body. It is just in these lines that we propose in this work to analyze a total of three signs pointing in that direction, i.e, that the spacetime is a kind of deformable body. The first sign is related to the deformations of the spacetime. The second one shows an "elastic" origin for the Einstein-Hilbert action, and the third one shows a relation between the Newton’s law of gravitation and the Hooke’s law of elasticity. In order to discuss these signs, we adopt the spacetime as modeled by a congruence of very specific curves: at all points of the spacetime manifold, (it is defined by such a curve). The characteristics of these curves can be studied by the use of the Landau-Raychadhuri equation [4,5,6] revealing if the spacetime, as modeled in this way, is or is not a curved manifold. It is important to note that is very common in physics to study a continuum system by discretizing it. In canonically quantizing a field theory, for example, we treat the field as a collection of interacting harmonic oscillators and postulate some commutation rules in order to construct the physical spectrum. How to do this for spacetime itself? The usual procedure is based on the metric tensor field $g_{\mu\nu}$. We again just try the same idea of discretization of this field. The result everybody know: Gravity is nonrenormalizable. Nevertheless, note that $g_{\mu\nu}$ does not represent spacetime by

*Electronic address: mktahim@fisica.ufc.br
†Electronic address: renan@fisica.ufc.br
‡Electronic address: carlos@fisica.ufc.br
itself. It is a field that depends on the coordinates of the spacetime. An example may help. Consider a solid with a well-defined lattice. We know that there are some excitations of this solid called "phonons". These objects can be studied by a scalar field theory in the continuum limit. In this case, what is the "spacetime"? We can choose a single atom of the lattice as reference frame and in this way we get time and distance information. The scalar field as a collective excitation of the lattice will depend on distance and time related to that reference frame. Then we claim that spacetime is the own lattice and from the microscopic viewpoint it is discrete because the lattice is made of "atoms". The conclusion is that modeling the spacetime manifold by a congruence of curves requires us to associate to each such a curve a fundamental "particle". The world-lines of such particles will compose the congruence fulfilling the spacetime manifold. Let us continue with the example of the solid and its lattice. Because the lattice is made of several atoms we know how to compute distances, i.e., we have a metric defined at every point in the continuum limit. Now which field is more fundamental, the metric one or the scalar field describing the lattice excitations? We claim that the more fundamental field is the scalar field because it is related directly with the atoms composing the lattice. These atoms are true reference frames in the sense that they are just matter reference frames: the physical notion of distance comes after the notion of matter, then the metric is a secondary object. In what follows we will consider the spacetime as a kind of solid for which we can define such a lattice. In field theory we already do this. We regard gauge fields, the metric tensor field and spinor fields as fundamental fields defined at every point of spacetime. The approach we will follow here is to consider any field except the metric tensor field as fundamental and this field will give us the notion of "spacetime" in the sense discussed above. The consequence of this idea is that if we want to study the deformations of such a spacetime we arrive at the Einstein’s equations as emergent equations. The organization of this work is the following: in the first section we make a review of the Landau-Raychaudhuri equation and discuss deformations of such a spacetime we will be using the Landau-Raychaudhuri equation for timelike or spacelike curves, depending on what sign we will be discussing. So, we have respectively

\[ \frac{d\Theta}{d\tau} = -\frac{1}{3} \Theta^2 - (\sigma^{ab})^2 + (\omega^{ab})^2 - R_{cd} \xi^c \xi^d \]  

and

\[ \frac{d\Theta}{d\lambda} = -\frac{1}{2} \Theta^2 - (\bar{\sigma}^{ab})^2 + (\bar{\omega}^{ab})^2 - R_{cd} \kappa^c \kappa^d, \]

where \(\tau, \lambda\) are the parameters used to describe the curves of the congruences. \(\xi^c\) and \(\kappa^c\) are the tangent vectors to the curves (generators of the congruence) and they are, in that order, a timelike and a lightlike vector. \(\Theta\) is a scalar and describes expansion of volume, \(\sigma^{ab}/\bar{\sigma}^{ab}\) measures the distortion of volume and \(\omega^{ab}/\bar{\omega}^{ab}\) measures the rotation of the curves. In the case of this work, we will be interested in small distortions of volume in such a way that the quadratic terms in the Landau-Raychaudhuri equation may be disregarded (they are like second order corrections). In these conditions, the Landau-Raychaudhuri equation can be easily integrated giving the scalar of expansion as a function of the Ricci tensor:

\[ \Theta = -\tau R_{cd} \xi^c \xi^d \equiv -\lambda R_{cd} \kappa^c \kappa^d. \]

The objects that appear in the Landau-Raychaudhuri equation can be obtained from a kinematical decomposition of general tensors. A general tensor can be decomposed into a symmetric plus an antisymmetric part:

\[ B_{ab} = B_{(ab)} + B_{[ab]}, \]

The antisymmetric part is associated with the measure of rotation of the congruence. The symmetric part can yet be decomposed into a trace (the scalar of expansion) and a symmetric traceless piece which is associated with the measure of the distortion of volume. The full symmetric part will be identified with the tensor of deformation of spacetime in analogy with the case of mechanics of deformable bodies. In the sections that follows the spacetime will be simulated by a congruence of timelike or lightlike curves, depending on the sign we will be discussing.
III. FIRST SIGN: THE DEFORMATIONS OF THE SPACETIME

Consider a small region of the spacetime containing the point $P$. This region defines the volume element $dV$. The question here is how to study the deformations of this volume caused by some external agent? One way to do this is by the use of the Lie derivative. In fact, if we get a volume $V$ and we propagate it using a congruence of integral curves we will obtain the modified volume $V'$. The difference between these two configurations gives us a way to measure the total deformation. With this in mind we postulate the following action for points of the spacetime:

$$S = k \int dV.$$  \hspace{1cm} (5)

The volume described above is just the Riemannian volume form, invariant under general coordinate transformations. The constant $k$ has the necessary dimension in order to give the correct dimension for the action, which is "energy×time". This action is quite different from the usual actions in field theory because there is no "a priori" Lagrangian density. The usual actions carry information about energy due to some fields distributed along some regions of spacetime and it seems that it does not happen with our proposed action. This is not a problem if we remember the idea discussed in the introduction above: the volume actually comprises part of the lattice associated with some field. This means that the volume of this spacetime carries energy associated with the lattice. Now, we minimize that action in the following manner: we take its Lie variation and requires that it should be stationary, just like the usual procedure. Then, the equation of motion for the points of that specified region is

$$\delta_{\text{Lie}} S = k \int \Theta dV = 0,$$  \hspace{1cm} (6)

where $\Theta$ is the scalar of expansion associated to the volume $dV$. Using the Landau-Raychaudhuri equation with the conditions cited in the first section, i.e., in a situation where $\sigma^{ab} = \sigma^{ab} = 0$, then we have $\Theta = -\lambda R_{ab} \kappa^a \kappa^b$. Substituting this result back in the equation of motion above we obtain

$$-k \int \lambda R_{ab} \kappa^a \kappa^b dV = 0.$$  \hspace{1cm} (7)

We can establish the equality above for all null vector $\kappa^a$ (we assume here the congruence is null-like which means the fundamental particles are massless), i.e.,

$$R_{ab} = f(g) g_{ab},$$  \hspace{1cm} (8)

where $f(g)$ is a function that just depends on the metric of the spacetime. This function can be easily found by requiring the disappearance of the covariant divergence of the last equation (it is like a type of "conservation of deformation") which leads to the result

$$R_{ab} - \frac{1}{2} R g_{ab} \pm \Lambda g_{ab} = 0,$$  \hspace{1cm} (9)

which is just the side corresponding to the geometry of the spacetime in the Einstein’s equations. Conclusion: when we deform the volume of the spacetime, the Einstein’s equations give us a way to understand such deformations.

IV. SECOND SIGN: THE ELASTIC ORIGIN OF THE EINSTEIN-HILBERT ACTION

Consider now the following functional which we will identify as an action:

$$S = k \int d^4x \sqrt{g} \Theta.$$  \hspace{1cm} (10)

We can see clearly that this functional obeys the requirement of being invariant under general coordinate transformations because the volume element is the Riemannian one and the quantity $\Theta$ is a scalar, in this case, the scalar of volume expansion. The meaning of that functional is the following: it has the same mathematical form as the measure of the linear deformation of a "fluid". Now, the scalar of expansion is given by $\Theta = -\lambda R_{ab} \xi^a \xi^b$ in the conditions already cited (for a congruence of timelike curves, i.e., if the fundamental particles of the lattice are massive).
The spacetime metric can be decomposed as $g_{ab} = h_{ab} + \xi_a \xi_b$. Then, the scalar of expansion can be rewritten as $\Theta \sim -\tau R_{ab} g^{ab} \equiv -\tau R$. Substituting this result in the functional defined above we arrive at

$$S \sim -k \int d^4 \tau \sqrt{g} R,$$

which is just the Einstein-Hilbert action multiplied by the factor $\tau$ (parameter of the curves). Nevertheless, in the standard minimization procedure we make a variation of the action with the fields to get the equations of motion. The parameter $\tau$ does not have a functional variation and, therefore, the equations of motion remain unchanged. The conclusion of this section is that when we minimize the Einstein-Hilbert action we are indeed looking for deformations of a “fluid” that minimize the functional of deformation described above.

**V. THIRD SIGN: THE NEWTON’S LAW AND THE HOOKE’S LAW**

Regarding the conclusions of the sections above it is natural to ask if the spacetime obeys some mathematical relation similar to the equations describing phenomena related with material bodies. The answer is positive as we will see. Consider that the spacetime is a special kind of deformable body. We will postulate that for linear deformations (for small dislocations of the “constituents” of that body) we can write a Hooke’s law that links the deformation tensor $\varepsilon^{ab}$ to the tensions $\tau^{ab}$ applied on the body in discussion (the spacetime). The tensors $\varepsilon^{ab}$ and $\tau^{ab}$ are symmetric, in analogy with the definitions of these objects in mechanics of the deformable bodies. Then, the Hooke’s law is:

$$\tau^{ab} = -k \varepsilon^{ab}.$$  \hfill (12)

The deformation tensor $\varepsilon^{ab}$ is defined, in this case, as the symmetrical part in the kinematical decomposition of the tensor $R^{ab} = \nabla^a \kappa^b$ in the construction of the Landau-Raychaudhuri equation. In this way, its trace will obey the following rule:

$$Tr \varepsilon^{ab} = \varepsilon^a_a \equiv \Theta = -\lambda R_{ab} \kappa^a \kappa^b.$$  \hfill (13)

Note the type of relation between the deformation tensor $\varepsilon^{ab}$ and the Ricci tensor $R^{ab}$: it is the trace of the deformation tensor that is linked to the Ricci tensor. This, in a sense, denotes that the deformation tensor is an object “bigger than” the Ricci tensor. Taking the trace in the expression for the Hooke’s law we obtain

$$\tau^a_a = k \lambda R_{ab} \kappa^a \kappa^b,$$  \hfill (14)

where we used the relation for the trace of the deformation tensor. We see now that the side of the deformations in the Hooke’s law is related with the geometry of the spacetime. The side of the tensions must be, therefore, related to the material content that produces the tensions on the spacetime. This is a reasonable assumption in the sense that if we want to equate in the same way the two sides of the Hooke’s law we must require that the trace of the tension tensor satisfies $\tau^a_a = \lambda T_{ab} \kappa^a \kappa^b$. This is not difficult to accept if we remember the relation between the deformation tensor and the Ricci tensor discussed above. Another way to see this result is by the substitution of $\kappa^a \kappa^b$ by the metric of the spacetime together the quantities that define projections in this spacetime. This will result in a trace as we want. The parameter $\lambda$ enters in the expression by dimensional reasons. Because of this we see that the tensor of tension is, in the same way as the deformation tensor, an object “bigger than” $T_{ab}$. Equating in this way deformations and tensions we arrive at

$$(R_{ab} - k^{-1} T_{ab}) \kappa^a \kappa^b = 0,$$  \hfill (15)

that is valid for all null vector $\kappa^a$, i. e.,

$$R_{ab} - k^{-1} T_{ab} = f(g) g_{ab}.$$  \hfill (16)

The function $f(g)$ again depends only on the metric $g_{ab}$ of the spacetime and can be determined by requiring the disappearance of the covariant divergence of the geometric part. But this only happens if we require in addition the validity of $\nabla^a T_{ab} = 0$, i. e., the quantity $T_{ab}$ must satisfies an equation of conservation. Note that we have assumed a null-like congruence in discussing this signal. Concluding, we obtain the equation

$$R_{ab} - \frac{1}{2} R g_{ab} \pm \Lambda g_{ab} = k^{-1} T_{ab},$$  \hfill (17)

which is just the Einstein’s equation if we identify $T_{ab}$ with the energy-momentum tensor. The identification is correct because $T_{ab}$ is a symmetric object due to the symmetry of $R_{ab}$ and it obeys a conservation law. If we take seriously these analogies we are forced to declare that the Hooke’s law for the spacetime is, in this viewpoint, more fundamental than the Einstein’s equation. In other words, the Newton’s law of gravitation comes from the Hooke’s law describing the deformations of the spacetime.
VI. CONCLUSIONS AND PERSPECTIVES

In this work we discussed the relations between the idea of spacetime curvature and deformations of solids. Interesting signs can be constructed in analogy with the physics of material bodies. The first sign shows that if we deform the spacetime volume using Lie propagation through integral curves we obtain the geometrical part of the Einstein’s equation. In the second sign, we proof a relation between the Einstein-Hilbert action and the volume deformation of a kind of "solid". In the third sign, we postulate the validity of a Hooke’s law for the spacetime and show that the Newton’ law of gravitation appears through Einstein’s equation. In all of these signs, the Landau-Raychaudhuri equation plays important role in the description of the spacetime as fulfilled by a congruence of curves. Nevertheless, we make use of a time-like congruence in just one sign while in the others we use null-like congruences. This means that it is important to decide if the fundamental particles of the lattice are massive or non-massive in order to establish the characteristics of the proposed lattice. Until now, there is no fundamental reason to choose one or other sort of congruence. We regard all of these signs as important steps in order to compose the idea of this paper. This question will be better addressed in a forthcoming paper.

The authors would like to thank Fundação Cearense de apoio ao Desenvolvimento Científico e Tecnológico (FUN-CAP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support.

[1] A. D. Sakharov, ”Vacuum quantum fluctuations in curved space and the theory of gravitation”, Soviet Physics Doklady, 12 (1968) 1040.
[2] B.L. Hu, E. Verdaguer, ”Stochastic gravity: Theory and applications”, Living Rev. Relativity, 7 (2004) 3.
[3] T. Jacobson, ”Thermodynamics of spacetime: The Einstein equation of state”, Phys. Rev. Lett. 75 (1995) 1260.
[4] L. Landau and E. M. Lifshitz, Classical theory of fields, Pergamon Press, Oxford, UK, 1975.
[5] A. Raychaudhuri, Phys. Rev. 98 (1955) 1123.
[6] R. M. Wald, General Relativity, Chicago, USA: Univ. Press (1984)