Birkhoff-James orthogonality and its local symmetry in some sequence spaces

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Abstract
We study Birkhoff-James orthogonality and its local symmetry in some sequence spaces namely \(\ell_p\), for \(1 \leq p \leq \infty\), \(p \neq 2\), \(c\), \(c_0\) and \(c_{00}\). Using the characterization of the local symmetry of Birkhoff-James orthogonality, we characterize isometries of each of these spaces onto itself and obtain the Banach-Lamperti theorem for onto operators on the sequence spaces.

Keywords
Birkhoff-James orthogonality · Smooth points · Left-symmetric points · Right-symmetric points · Onto isometries · Ultrafilters

Mathematics Subject Classification
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1 Introduction
In recent times, symmetry of Birkhoff-James orthogonality has been a topic of considerable interest [1, 8, 9, 12–14, 19]. It is now well known that the said symmetry plays an important role in the study of the geometry of Banach spaces. The present article aims to explore Birkhoff-James orthogonality and its local symmetry in some well studied sequence spaces. As an outcome of our exploration, we acquire the Banach-Lamperti Theorem [15] for onto
operators on some classical sequence spaces by characterizing the onto isometries of the same. We would like to mention that recently such a study has been carried out in the context of $\ell_p^n$ spaces for $1 \leq p \leq \infty$, $p \neq 2$ in [3]. It should also be noted that Birkhoff-James orthogonality is closely related to the norm derivatives and their various properties, which find applications in understanding the geometry of Banach spaces. Recently, such a study has been carried out in [22], where the notion of local smoothness induced by the norm derivatives has been completely characterized. We refer the readers to some of the related works [4, 5, 25] for more information in this context.

Let us now establish the relevant notations and terminologies to be used throughout the article. Denote the scalar field $\mathbb{R}$ or $\mathbb{C}$ by $K$ and recall the sign function $\text{sgn} : K \to K$, given by

$$\text{sgn}(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Consider a normed linear space $X$ over $K$ and denote its continuous dual by $X^*$. Let $J(x)$ denote the collection of all support functionals of a non-zero $x \in X$, i.e.,

$$J(x) := \{ f \in X^* : \| f \| = 1, \ |f(x)| = \| x \| \}.\quad (1.1)$$

A non-zero element $x \in X$ is said to be smooth if $J(x)$ is singleton.

Given $x, y \in X$, $x$ is said to be Birkhoff-James orthogonal to $y$ [2], denoted by $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\|, \quad \text{for all } \lambda \in K.$$

James proved in [11] that $x \perp_B y$ if and only if $x = 0$ or there exists $f \in J(x)$ such that $f(y) = 0$. In the same article he also proved that a non-zero $x \in X$ is smooth if and only if Birkhoff-James orthogonality is right additive at $x$, i.e.,

$$x \perp_B y, \ x \perp_B z \Rightarrow x \perp_B (y + z), \quad \text{for every } y, z \in X.$$

Birkhoff-James orthogonality is not symmetric in general, i.e., $x \perp_B y$ does not necessarily imply that $y \perp_B x$. In fact, James proved in [10] that Birkhoff-James orthogonality is symmetric in a normed linear space of dimension higher than 2 if and only if the space is an inner product space. However, the importance of studying the local symmetry of Birkhoff-James orthogonality in describing the geometry of normed linear spaces has been illustrated in [3, Theorem 2.11], [18, Corollary 2.3.4.]. Let us recall the following definition in this context from [17], which is of paramount importance in our present study.

**Definition 1** An element $x$ of a normed linear space $X$ is said to be left-symmetric (resp. right-symmetric) if

$$x \perp_B y \Rightarrow y \perp_B x \quad (\text{resp. } y \perp_B x \Rightarrow x \perp_B y),$$

for every $y \in X$.

The left-symmetric and the right-symmetric points of $\ell_p^n$ spaces where $1 \leq p \leq \infty$, $p \neq 2$, were characterized in [3]. Here we take a step forward towards generalizing these results in the following sequence spaces: $\ell_p$, for $1 \leq p \leq \infty$ and $p \neq 2$, $c$, $c_0$ and $c_{00}$. Characterizations of the smooth points, the left-symmetric points and the right-symmetric points of a given Banach space are of paramount importance in understanding the geometry of the Banach space. We refer the readers to [1, 8, 9, 12–14, 19–21, 23, 24] for some prominent work in this direction.
The local symmetry of Birkhoff-James orthogonality in a Banach space also plays an important role in determining the isometric isomorphisms on the space. Let us observe that Corollary 2.3.4. of [18] in this regard can be stated in the following generalized form:

**Corollary 1.1** Let $X$ and $Y$ be two normed linear spaces and let $T : X \rightarrow Y$ be an onto linear isometry. Then $x \in X$ is left-symmetric (resp. right-symmetric) if and only if $T(x) \in Y$ is left-symmetric (resp. right-symmetric).

This result is used for proving the Banach-Lamperti Theorem for onto operators on the sequence spaces, i.e., for the case where the measure space is $\mathbb{N}$ equipped with the counting measure by finding the onto isometries of $\ell_p$, for $1 \leq p \leq \infty$ and $p \neq 2$. We also do the same for the spaces $c$, $c_0$ and $c_{00}$ as a direct consequence of the results characterizing the local symmetry of Birkhoff-James orthogonality in these spaces. It can be noted that Lamperti’s idea in [15] uses the concept of convexity, concavity and Radon-Nikodym derivatives along with the properties of the integral involved in the definition of the $L_p$ norm and therefore cannot be generalized in case of $p = \infty$. Our approach using the local symmetry of Birkhoff-James orthogonality however, has no such restrictions and hence is applied for the $p = \infty$ case as well.

In the first section we completely characterize Birkhoff-James orthogonality in $\ell_\infty$ over $\mathbb{K}$ and then characterize the left-symmetric and the right-symmetric points of the space. As a corollary of our results, we obtain characterizations of Birkhoff-James orthogonality and the left-symmetric and the right-symmetric points in $c$, $c_0$ and $c_{00}$. Using Corollary 1.1, we find the isometries of each of these spaces onto itself.

In the second and third sections we obtain the same characterizations in $\ell_1$ and $\ell_p$ spaces for $1 < p < \infty$ and $p \neq 2$ respectively. Observe that the $p = 2$ case is trivial since $\ell_2$ is a Hilbert space. We also find all the isometries of these spaces onto themselves using Corollary 1.1.

Since we are proving the Banach-Lamperti Theorem for onto operators on the sequence spaces by this isometry characterization, we define signed permutation operators on $\mathbb{K}^\mathbb{N}$, the vector space of all sequences in $\mathbb{K}$.

**Definition 2** A map $T : \mathbb{K}^\mathbb{N} \rightarrow \mathbb{K}^\mathbb{N}$ is said to be a signed permutation operator if there exists a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$T(x) = (c_n x_{\sigma(n)}) , \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^\mathbb{N},$$

where $|c_n| = 1$, for every $n \in \mathbb{N}$.

### 2 Geometry of $\ell_\infty$

In this section, we characterize Birkhoff-James orthogonality between two elements of $\ell_\infty$ and then obtain characterizations of the smooth points, the left-symmetric points and the right-symmetric points of the space. To serve our purpose, we review some basic facts about the convergence of a $\mathbb{K}$-valued sequence under an ultrafilter on $\mathbb{N}$. A detailed treatment on ultrafilters can be found in [6, 7].

#### 2.1 Ultrafilters on $\mathbb{N}$ and convergence of sequences under them

We begin by recalling a few definitions.
Definition 3 (Filters and Ultrafilters) A non-empty subset \( F \) of the power set of a non-empty set \( X \) is said to be a filter on \( X \) if

(i) \( \emptyset \notin F \).

(ii) \( A, B \in F \Rightarrow A \cap B \in F \).

(iii) \( A \in F \) and \( A \subseteq B \Rightarrow B \in F \).

A filter \( U \) on \( X \) is said to be an ultrafilter on \( X \) if any filter on \( X \) containing \( U \) must be \( U \).

Note that a filter \( U \) is an ultrafilter if and only if for every \( A \subseteq X \), \( A \in U \) or \( X \setminus A \in U \) if and only if for every \( n \in \mathbb{N} \) and \( A_1, A_2, \ldots, A_n \subseteq X \), \( X = \bigcup_{i=1}^{n} A_i \) implies \( A_i \in U \), for some \( 1 \leq i \leq n \). An ultrafilter \( U \) on \( X \) is called a principal ultrafilter if there exists \( x \in X \) such that \( \{x\} \in U \). An ultrafilter which is not a principal ultrafilter is called a free ultrafilter.

We also recall the definition of a filter base.

Definition 4 A non-empty subset \( B \) of the power set of a non-empty set \( X \) is said to be a filter base if

(i) \( \emptyset \notin B \).

(ii) If \( A, B \in B \), there exists \( C \in B \) such that \( C \subseteq A \cap B \).

Note that every filter base \( B \) is contained in a unique minimal filter given by \( \{A \subseteq X : B \subseteq A, \text{ for some } B \in B\} \). Since every filter is contained in some ultrafilter by Zorn’s lemma, every filter base is also contained in some ultrafilter.

We now focus on the case \( X = \mathbb{N} \). Recall the definition of convergence along a filter:

Definition 5 (Convergence along a filter) Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{K} \) and let \( F \) be a filter on \( \mathbb{N} \). Then we say \( x_n \) converges to some point \( x_0 \in \mathbb{K} \), denoted by \( \lim_{F} x_n \), under \( F \) if for every \( \epsilon > 0 \),

\[ \{ n \in \mathbb{N} : |x_n - x_0| < \epsilon \} \in F. \]

Let us state a few well-known results pertaining to the convergence of a sequence under a filter without proof.

Theorem 2.1 Let \( F \) is a filter on \( \mathbb{N} \) and \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) be two sequences in \( \mathbb{K} \). Then the following hold true:

(i) \( \lim_{F} x_n \), if exists, is unique.

(ii) If \( x = \lim_{F} x_n \), and \( f : \mathbb{K} \to \mathbb{K} \) is continuous, then \( f(x) = \lim_{F} f(x_n) \).

(iii) If \( x = \lim_{F} x_n \) and \( y = \lim_{F} y_n \), then \( x + \lambda y \in \lim_{F} (x_n + \lambda y_n) \), for any \( \lambda \in \mathbb{K} \). Also, \( xy = \lim_{F} x_n y_n \) and \( \frac{x}{y} = \lim_{F} \frac{x_n}{y_n} \) if \( y_n \neq 0 \neq y \), for every \( n \in \mathbb{N} \).

(iv) If \( F \) is an ultrafilter and \( (x_n)_{n \in \mathbb{N}} \) is bounded, then \( \lim_{F} x_n \) exists.

It is trivial to see that if \( U \) is the principal ultrafilter containing \( \{N\} \), for some \( N \in \mathbb{N} \), \( \lim_{U} x_n = x_N \). We establish the following result pertaining to the limit of a bounded sequence under any free ultrafilter.

Proposition 2.2 Let \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) be two bounded sequences in \( \mathbb{K} \). Then the following hold:

(i) \( x_0 = \lim_{U} x_n \), for some free ultrafilter \( U \) on \( \mathbb{N} \) if and only if \( x_0 \) is a subsequential limit of \( x_n \).

(ii) \( x_0 = \lim_{U} x_n \) and \( y_0 = \lim_{U} y_n \), for some free ultrafilter \( U \) on \( \mathbb{N} \) if and only if there exists an increasing sequence \( (n_k)_{k \in \mathbb{N}} \) on \( \mathbb{N} \) such that \( x_{n_k} \to x_0 \) and \( y_{n_k} \to y_0 \) as \( k \to \infty \).
Proof. Recall that an ultrafilter is free if and only if it contains no finite subset of \( \mathbb{N} \).

(i) We first prove the necessity. Suppose \( x_0 = \lim_U x_n \), for some free ultrafilter \( U \) on \( \mathbb{N} \). If \( x_0 \) is not a subsequential limit of \( x_n \), there exists \( \delta > 0 \) such that \( \{ n \in \mathbb{N} : |x_n - x_0| < \delta \} \) is finite which is a contradiction since \( U \) is a free ultrafilter.

Now, assume that \( (n_k)_{k \in \mathbb{N}} \) is an increasing sequence in \( \mathbb{N} \) such that \( x_{n_k} \to x_0 \), for some \( x_0 \in \mathbb{K} \) and consider the following set:

\[ \{ A \subset \mathbb{N} : \{ n_k : k \in \mathbb{N} \} \setminus A \text{ is finite} \}. \tag{2.1} \]

Observe that the set defined by (2.1) is a filter base and therefore is contained in some ultrafilter \( U \) on \( \mathbb{N} \). Clearly, no finite subset of \( \mathbb{N} \) is an element of \( U \) and hence \( U \) is a free ultrafilter. Since, \( \lim_U x_n = x_0 \), the sufficiency is established.

(ii) The sufficiency can be proved like (i). To prove the necessity, let there be no sequence \( (n_k)_{k \in \mathbb{N}} \) in \( \mathbb{N} \) such that \( x_{n_k} \to x_0 \) and \( y_{n_k} \to y_0 \). Then there exist \( \delta_1, \delta_2 > 0 \) such that

\[ \{ n \in \mathbb{N} : |x_n - x_0| < \delta_1 \} \cap \{ n \in \mathbb{N} : |y_n - y_0| < \delta_2 \} \text{ is finite}, \]

a contradiction since \( U \) is a free ultrafilter and therefore contains no finite subset of \( \mathbb{N} \). This proves the necessity.

2.2 Birkhoff-James orthogonality and smoothness of a point in \( \ell_\infty \)

We begin this sub-section by recalling a few known results:

**Theorem 2.3** The space \( \ell_\infty \) is isometrically isomorphic to \( C(\beta \mathbb{N}) \), the Banach space of all \( \mathbb{K} \)-valued continuous functions on \( \beta \mathbb{N} \) equipped with the supremum norm, where \( \beta \mathbb{N} \) denotes the Stone-Čech compactification of \( \mathbb{N} \). Recalling the homeomorphism between \( \beta \mathbb{N} \) and the space of all ultrafilters on \( \mathbb{N} \) equipped with the Stone topology, one can explicitly write down such an isometric isomorphism \( T : \ell_\infty \to C(\beta \mathbb{N}) \), given by

\[ T ((x_n)_{n \in \mathbb{N}}) (U) = \lim_U x_n, \quad U \text{ an ultrafilter on } \mathbb{N}. \tag{2.2} \]

Also, since \( \beta \mathbb{N} \) is compact Hausdorff, by an application of the Riesz representation Theorem in measure theory, we have the following result:

**Theorem 2.4** The dual space of \( C(\beta \mathbb{N}) \) is isometrically isomorphic to the space of all regular \( \mathbb{K} \)-valued Borel measures on \( \beta \mathbb{N} \) equipped with the total variation norm and the functional corresponding to a regular \( \mathbb{K} \)-valued Borel measure \( \mu \) acting on \( C(\beta \mathbb{N}) \) is given by

\[ \mu : f \mapsto \int_{\beta \mathbb{N}} f d\mu, \quad f \in C(\beta \mathbb{N}). \]

We note that by Theorem 2.3 and Theorem 2.4, \( \ell_\infty^* \) is isometrically isomorphic to the space of all regular \( \mathbb{K} \)-valued Borel measures on \( \beta \mathbb{N} \) equipped with the total variation norm. We begin by characterizing the support functionals of a non-zero \( f \in C(\beta \mathbb{N}) \) and introduce the following definition in this regard.

**Definition 6** For a given \( f \in C(\beta \mathbb{N}) \), we define \( M_f \) to be the collection of all the points in \( \beta \mathbb{N} \) where \( f \) attains its norm, i.e.,

\[ M_f := \{ U \in \beta \mathbb{N} : |f(U)| = \| f \| \}. \]

Using the above, we now characterize \( J(f) \), (see (1.1) for definition) for a non-zero \( f \in C(\beta \mathbb{N}) \).
Theorem 2.5 Let $f \in C(\mathbb{B} \mathbb{N})$ be non-zero. Then $\mu \in J(f)$, for some regular $\mathbb{K}$-valued Borel measure $\mu$ if and only if
\[
|\mu|(\mathbb{B} \mathbb{N} \setminus M_f) = 0, \quad |\mu|(M_f) = 1 \text{ and } d\mu(\mathcal{U}) = \text{sgn}(f(\mathcal{U}))d|\mu|(\mathcal{U}),
\]
for almost every $\mathcal{U} \in M_f$ with respect to the measure $\mu$, where $|\mu|$ denotes the total variation of $\mu$.

Proof The sufficiency follows by elementary computations. Now, if $\mu \in J(f)$, then $d\mu(\mathcal{U}) = e^{i\theta(\mathcal{U})}d|\mu|(\mathcal{U})$, for some measurable function $\theta : \mathbb{B} \mathbb{N} \to \mathbb{R}$. Note that
\[
\|f\| = \left\| \int_{\beta \mathbb{N}} f(\mathcal{U})e^{i\theta(\mathcal{U})}d|\mu|(\mathcal{U}) \right\| \leq \int_{\beta \mathbb{N}} |f(\mathcal{U})|d|\mu|(\mathcal{U}) \leq \|f\|.
\]
Hence equality must hold in both the inequalities involved. Equality in the second inequality implies that $|f(\mathcal{U})| = \|f\|$, for almost every $\mathcal{U} \in \mathbb{B} \mathbb{N}$, giving $|\mu|(\mathbb{B} \mathbb{N} \setminus M_f) = 0$ (and hence, $|\mu|(M_f) = 1$) and equality in the first inequality gives that for almost every $\mathcal{U} \in M_f$, with respect to $\mu$,
\[
f(\mathcal{U})e^{i\theta(\mathcal{U})} = \|f\| \quad \Rightarrow \quad e^{i\theta(\mathcal{U})} = \text{sgn}(f(\mathcal{U})).
\]

We now come to our characterization of Birkhoff-James orthogonality in $\ell_\infty$.

Theorem 2.6 Let $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$ be two elements of $\ell_\infty$. Then the following are equivalent:

(i) $x \perp_B y$.

(ii) $0 \in \text{conv}\{\lim_{\mathcal{U}} \overline{x_n}y_n : \lim_{\mathcal{U}}|x_n| = \|x\|\}$.

(iii) $0$ lies in the convex hull of $\{\overline{x_n}y_n : |x_n| = \|x\|\}$ and $\{\lim_{k \to \infty} x_{n_k}y_{n_k} : n_k \text{ is an increasing sequence in } \mathbb{N}, \lim_{k \to \infty} |x_{n_k}| = \|x\|, y_{n_k} \text{ converge as } k \to \infty\}$

Proof The result holds trivially if $x = 0$. Hence, we assume that $x \neq 0$.

(i) $\Leftrightarrow$ (ii)

If $0 \in \text{conv}\{\lim_{\mathcal{U}} \overline{x_n}y_n : \lim_{\mathcal{U}}|x_n| = \|x\|\}$, then there exist ultrafilters $\mathcal{U}_i$ on $\mathbb{N}$ and $\lambda_i \in [0, 1]$, for $1 \leq i \leq m$ such that $\sum_{i=1}^m \lambda_i = 1$ and
\[
\sum_{i=1}^m \lambda_i \lim_{\mathcal{U}_i} \overline{x_n}y_n = 0, \quad \lim_{\mathcal{U}_i}|x_n| = \|x\|, \quad \text{for every } 1 \leq i \leq m.
\]

Consider the functional $\Psi : \ell_\infty \to \mathbb{K}$, given by
\[
\Psi((z_n)_{n \in \mathbb{N}}) = \sum_{i=1}^n \lambda_i \lim_{\mathcal{U}_i} \text{sgn}(x_n)z_n, \quad (z_n)_{n \in \mathbb{N}} \in \ell_\infty.
\]

Then clearly $\Psi$ has norm 1 and $\Psi(x) = \|x\|$. Hence $\Psi$ is a support functional of $x$ that annihilates $y$ and establishes the sufficiency.

Now, let us assume that $x \perp_B y$. Then by Theorem 2.3, we have the two maps $f, g : \beta \mathbb{N} \to \mathbb{K}$, given by
\[
f(\mathcal{U}) := \lim_{\mathcal{U}} x_n, \quad g(\mathcal{U}) := \lim_{\mathcal{U}} y_n, \quad \mathcal{U} \in \beta \mathbb{N},
\]
satisfying $f \perp_B g$ in $C(\beta \mathbb{N})$. Observe that $M_f = \{\mathcal{U} \in \beta \mathbb{N} : \lim_{\mathcal{U}}|x_n| = \|x\|\}$. Now, by Theorem 2.5, there must exist a regular positive Borel measure $\nu$ on $M_f$ with $\nu(M_f) = 1$ such that
\[
\int_{M_f} \text{sgn}(f(\mathcal{U}))g(\mathcal{U})d\nu(\mathcal{U}) = 0 \quad \Leftrightarrow \quad \int_{M_f} f(\mathcal{U})g(\mathcal{U})d\nu(\mathcal{U}) = 0.
\]
Let \( \Lambda \) be the collection of all regular positive Borel measures \( \mu \) on \( M_f \) with \( \mu(M_f) = 1 \). Consider the map \( \Phi : \Lambda \to \mathbb{K} \), given by

\[
\Phi(\mu) := \int_{M_f} f(U)g(U)\,d\mu(U), \quad \mu \in \Lambda.
\]

Then clearly \( \Phi(\Lambda) \) must be a convex subset of \( \mathbb{K} \) since \( \Lambda \) is convex. Observe that by Theorem 2.5, \( \Lambda \) is the collection of all support functionals of \( |f| \in C(\beta \mathbb{N}) \) and hence is weak* compact by the Banach-Alaoglu Theorem. Since the map \( \Phi \) is continuous under the weak* topology, \( \Phi(\Lambda) \) will be compact. Hence by the Krein-Milman Theorem, \( \Phi(\Lambda) \) must be the closed convex hull of its extreme points.

We claim that the only extreme points of the set \( \Phi(\Lambda) \) are of the form \( \bar{f}(U)g(U) \), for some \( U \in M_f \). Let \( \Phi(\mu) \) be an extreme point of \( \Phi(\Lambda) \) where \( \mu \) is not a Dirac delta measure at any point on \( M_f \). Clearly, if \( \bar{f}g \) is constant at every point of the support of \( \mu \), then \( \Phi(\mu) = \bar{f}(U)g(U) \), for any \( U \) in the support of \( \mu \). Therefore, let us assume that \( U \) \( \forall \) are two points in the support of \( \mu \) with \( \bar{f}(U)g(U) \neq \bar{f}(V)g(V) \). Fix \( 0 < \epsilon < \frac{1}{2}\| \bar{f}(U)g(U) - \bar{f}(V)g(V) \| \) and set:

\[
G_\epsilon := \left\{ V \in M_f : \| \bar{f}(U)g(U) - \bar{f}(V)g(V) \| < \epsilon \right\}.
\]

Then \( G_\epsilon \) is an open subset of \( M_f \) containing \( U \) and \( M_f \setminus G_\epsilon \) contains a neighbourhood of \( V \) in \( M_f \). Hence \( \mu(G_\epsilon) \), \( \mu(M_f \setminus G_\epsilon) > 0 \). Now since \( \mu \) can be written as a convex combination of \( \frac{1}{\mu(G_\epsilon)}\mu|_{G_\epsilon} \) and \( \frac{1}{\mu(M_f \setminus G_\epsilon)}\mu|_{M_f \setminus G_\epsilon} \),

\[
\Phi(\mu) = \frac{1}{\mu(G_\epsilon)} \int_{G_\epsilon} \bar{f}(W)g(W)\,d\mu(W), \quad \text{for every} \ 0 < \epsilon < \frac{1}{2}\| \bar{f}(U)g(U) - \bar{f}(V)g(V) \|,
\]

as \( \Phi(\mu) \) is an extreme point of \( \Phi(\Lambda) \). Hence,

\[
\left| \Phi(\mu) - \bar{f}(U)g(U) \right| = \frac{1}{\mu(G_\epsilon)} \left| \int_{G_\epsilon} \left( \bar{f}(W)g(W) - \bar{f}(U)g(U) \right)\,d\mu(W) \right| 
\]

\[
\leq \frac{1}{\mu(G_\epsilon)} \int_{G_\epsilon} \| \bar{f}(W)g(W) - \bar{f}(U)g(U) \|\,d\mu(W) \leq \epsilon,
\]

giving \( \Phi(\mu) = \bar{f}(U)g(U) \) since \( 0 < \epsilon < \frac{1}{2}\| \bar{f}(U)g(U) - \bar{f}(V)g(V) \| \) is arbitrary.

Therefore,

\[
0 \in \overline{\text{conv}} \left\{ \lim_{U} x_n y_n : \lim_{U} |x_n| = \|x\| \right\}, \quad \text{as} \ \Phi(\nu) = 0.
\]

Clearly, since \( M_f \) is compact, \( \{ \lim_{U} x_n y_n : \lim_{U} |x_n| = \|x\| \} \) is also compact. We now prove that the convex hull of a compact subset of \( \mathbb{K} \) is closed. Indeed, using Caratheodory’s theorem, for any element \( x \) in \( \text{conv}(K) \), where \( K \subset \mathbb{K} \) is compact, there exist \( x_1, x_2, x_3 \in K \) and \( \lambda_1, \lambda_2, \lambda_3 \in [0, 1] \) such that

\[
\sum_{i=1}^{3} \lambda_i x_i = x \quad \text{and} \quad \sum_{i=1}^{3} \lambda_i = 1.
\]

Hence if \( x_0 \in \text{conv}(K) \), there exist sequences \( x_i^{(n)} \in K \) and \( \lambda_i^{(n)} \in [0, 1] \), for \( 1 \leq i \leq 3 \) and \( n \in \mathbb{N} \) such that

\[
\lim_{n \to \infty} \sum_{i=1}^{3} \lambda_i^{(n)} x_i^{(n)} = x_0, \quad \text{and} \quad \sum_{i=1}^{3} \lambda_i^{(n)} = 1, \quad \text{for every} \ n \in \mathbb{N}.
\]
Since $K$ and $[0, 1]$ are compact, considering a convergent subsequence of all the six sequences and passing on to the limits, we have $x_0 \in \text{conv}(K)$. Combining all the results, we therefore obtain

$$0 \in \text{conv} \left\{ \lim_{\mathcal{U}} \frac{x_n}{y_n} : \lim_{\mathcal{U}} |x_n| = \|x\| \right\}.$$ 

(ii) $\leftrightarrow$ (iii) This is immediate from Proposition 2.2 and the fact that if $\mathcal{U}$ is a principal ultrafilter containing $\{N\}$, for some $N \in \mathbb{N}$, then $\lim_{\mathcal{U}} x_n = x_N$, for every sequence $(x_n)_{n \in \mathbb{N}}$ in $\ell_\infty$.

We record the characterization of Birkhoff-James orthogonality in $\ell_\infty$, for the real case as a corollary below.

**Corollary 2.7** Let $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$ be two elements of $\ell_\infty$ over $\mathbb{R}$. Then $x \perp_B y$ if and only if any of the following holds true:

(i) There is an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $|x_{n_k}| \to \|x\|$ and $x_{n_k}y_{n_k} \to 0$ as $k \to \infty$.

(ii) There are increasing sequences of natural numbers $(n_k)_{k \in \mathbb{N}}$ and $(m_k)_{k \in \mathbb{N}}$ such that $|x_{n_k}| \to \|x\|$ and $|x_{m_k}| \to \|x\|$ as $k \to \infty$ with $x_{n_k}y_{n_k} \geq 0 \geq x_{m_k}y_{m_k}$, for every $k \in \mathbb{N}$.

(iii) There is an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $|x_{n_k}| \to \|x\|$ as $k \to \infty$ and a natural number $N$ such that $|x_N| = \|x\|$ and $x_Ny_N$ and $x_{n_k}y_{n_k}$ are of different signs for every $k \in \mathbb{N}$.

(iv) There are $N, M \in \mathbb{N}$ such that $|x_N| = \|x\| = |x_M|$ and $x_Ny_N \geq 0 \geq x_My_M$.

We conclude this section by characterizing the smooth points of $\ell_\infty$.

**Theorem 2.8** Let $x = (x_n)_{n \in \mathbb{N}}$ be a non-zero element in $\ell_\infty$. Then $x$ is a smooth point if and only if there is no subsequence of $(|x_n|)_{n \in \mathbb{N}}$ that converges to $\|x\|$ and there exists a unique $N \in \mathbb{N}$ such that $|x_N| = \|x\|$.

**Proof** We first prove the sufficiency. Suppose, there is no subsequence of $(|x_n|)_{n \in \mathbb{N}}$ that converges to $\|x\|$ and there exists a unique $N \in \mathbb{N}$ such that $|x_N| = \|x\|$. Then it follows from Theorem 2.6 that $x \perp_B y$, for some $y = (y_n)_{n \in \mathbb{N}} \in \ell_\infty$, if and only if $y_N = 0$. Consequently, $x \perp_B y$ and $x \perp_B z$ implies $x \perp_B (y + z)$, proving $x$ is smooth.

Again, if $x \in \ell_\infty$ is smooth, then the function $f : \beta\mathbb{N} \to \mathbb{K}$, given by

$$f(\mathcal{U}) := \lim_{\mathcal{U}} x_n, \quad \mathcal{U} \in \beta\mathbb{N},$$

is smooth in $C(\beta\mathbb{N})$ (Theorem 2.3). Therefore, if $\mathcal{U}, \mathcal{V} \in M_f$, then $\Psi, \Phi : C(\beta\mathbb{N}) \to \mathbb{K}$, given by

$$\Psi(g) := \text{sgn}(f(\mathcal{U}))g(\mathcal{U}), \quad \Phi(g) := \text{sgn}(f(\mathcal{V}))g(\mathcal{V}), \quad g \in C(\beta\mathbb{N}),$$

are two distinct support functionals of $f$ because there exists a continuous map $F : \beta\mathbb{N} \to [0, 1]$ such that $F(\mathcal{U}) = 0$ and $F(\mathcal{V}) = 1$, since $\beta\mathbb{N}$ is compact Hausdorff. Hence $M_f$ must be singleton.

Now, if $\mathcal{U} \in M_f$ is a free ultrafilter, then by Theorem 2.2, there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that $|x_{n_k}| \to \|x\|$ as $k \to \infty$. Consider the following two collections:

$$\{ A \subset \mathbb{N} : \{n_{2k} : k \in \mathbb{N}\} \text{\ is\ finite}\} \quad \text{and} \quad \{ A \subset \mathbb{N} : \{n_{2k-1} : k \in \mathbb{N}\} \text{\ is\ finite}\}.$$
Clearly, both of these sets are filter bases on \( \mathbb{N} \) and are therefore contained in free ultrafilters. Also note that since the first collection contains \( \{ n_{2k} : k \in \mathbb{N} \} \) and the second collection contains \( \mathbb{N} \setminus \{ n_{2k} : k \in \mathbb{N} \} \), the two ultrafilters must be distinct. However, the sequence \((x_n)\) has the same limit under both of the ultrafilters. Hence if \( M_f \) is singleton, it can contain only a principle ultrafilter and thus the necessity follows. \( \square \)

### 2.3 Local symmetry of Birkhoff-James orthogonality in \( \ell_\infty \)

In this sub-section we characterize the left-symmetric and the right-symmetric points of the space \( \ell_\infty \).

**Theorem 2.9** The only non-zero left-symmetric points of \( \ell_\infty \) are scalar multiples of \( e_n \) for \( n \in \mathbb{N} \), where \( e_n \) denotes the sequence having the \( n \)-th term 1 and the rest of the terms 0.

**Proof** Since \( \lim_{\mathcal{U}} e_n = 0 \), for every ultrafilter other than the principal ultrafilter containing \( \{ n \} \), by Theorem 2.6, \( e_n \perp_B x \), for some \( x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty \) if and only if \( x_n = 0 \). Hence for \( \lambda \in \mathbb{K} \),

\[
\| x + \lambda e_n \| = \max \{ |\lambda|, \| x \| \} \geq \| x \|
\]

and this proves the necessity.

Now, let \( x = (x_n)_{n \in \mathbb{N}} \) be a left-symmetric point of \( \ell_\infty \). If \( \lim_{\mathcal{U}} \| x_n \| = \| x \| \), for some free ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \), then set \( y = (y_n)_{n \in \mathbb{N}} \in \ell_\infty \), given by \( y_n := \frac{1}{2} \text{sgn}(x_n) \). Then clearly, \( \lim_{\mathcal{U}} y_n x_n = 0 \), since \( \mathcal{U} \) is a free ultrafilter. Hence, \( x \perp_B y \). However, by Theorem 2.6, for some \( z = (z_n)_{n \in \mathbb{N}} \in \ell_\infty \), \( y \perp_B z \) if and only if \( z_N = 0 \) where \( N = \min \{ n \in \mathbb{N} : x_n \neq 0 \} \).

Thus \( y \perp_B x \), implying that \( \lim_{\mathcal{U}} \| x_n \| \neq \| x \| \), for any free ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \).

Hence, there exists \( N \in \mathbb{N} \) such that \( \| x_N \| = \| x \| \). Now, suppose that \( x_M \neq 0 \), for some \( M \neq N \in \mathbb{N} \). Then clearly, setting \( y = e_M \), yields that \( x \perp_B y \) and \( y \perp_B x \), by Theorem 2.6, which establishes the necessity. \( \square \)

Now, from Corollary 1.1, if \( T : \ell_\infty \to \ell_\infty \) is an onto linear isometry, then for every \( n \in \mathbb{N} \), \( T(e_n) = c e_m \), for some \( m \in \mathbb{N} \) and some unimodular constant \( c \). Also, since \( T \) is onto, \( T \) is invertible and \( T^{-1} \) is also an onto isometry and hence \( T \) must be a signed permutation operator. Note that this extends the result of Lamperti [15] on \( L_p \) spaces for \( 1 \leq p < \infty \) to the \( p = \infty \) case for onto operators with the measure space as \( \mathbb{N} \) under counting measure. We record this result as a corollary.

**Corollary 2.10** Let \( T : \ell_\infty \to \ell_\infty \) be an onto linear isometry. Then \( T \) must be a signed permutation operator.

We conclude this sub-section by characterizing the right-symmetric points of \( \ell_\infty \).

**Theorem 2.11** \( x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty \) is a right-symmetric point if and only if \( |x_n| = \| x \| \) for every \( n \in \mathbb{N} \).

**Proof** Note that if \( |x_n| = \| x \| \), for every \( n \in \mathbb{N} \), then \( \lim_{\mathcal{U}} |x_n| = \| x \| \), for every ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) and hence by Theorem 2.6, \( y \perp_B x \), for \( y = (y_n)_{n \in \mathbb{N}} \in \ell_\infty \) if and only if

\[
0 \in \text{conv} \left\{ \lim_{\mathcal{U}} y_n x_n : \lim_{\mathcal{U}} |y_n| = |y| \right\}
\]

\[
\Rightarrow 0 \in \text{conv} \left\{ \lim_{\mathcal{U}} x_n y_n : \lim_{\mathcal{U}} |x_n| = \| x \| \right\}
\]
\[ \Rightarrow x \perp_B y. \]

Hence the sufficiency is clear and we therefore focus on proving the necessity.

Now, if \( x \neq 0 \) is right-symmetric and \( |x_N| < \|x\| \) for some \( N \in \mathbb{N} \), set \( y = (y_n)_{n \in \mathbb{N}} \in \ell_\infty \) given by

\[
y_n := \begin{cases} 
\text{sgn}(x_n), & n \neq N, \\
-\text{sgn}(x_N), & n = N \text{ and if } x_N \neq 0, \\
1, & n = N \text{ and if } x_N = 0.
\end{cases}
\]

Then clearly, by Theorem 2.6, \( y \perp_B x \). However,

\[
x_n y_n := \begin{cases} 
|x_n|, & n \neq N, \\
-|x_N|, & n = N.
\end{cases}
\]

Hence, if \( U \) is an ultrafilter not containing \( \{N\} \),

\[
\lim_{U} x_n y_n = \lim_{U} |x_n|.
\]

(2.3)

Therefore, since \( |x_N| < \|x\| \), the limit of \( |x_n| \) under the principal ultrafilter containing \( \{N\} \) is not \( \|x\| \) and hence by (2.3) and Theorem 2.6, \( x \not\perp_B y \) establishing the necessity.

### 2.4 Geometry of \( c, c_0 \) and \( c_{00} \)

Recall that \( c, c_0 \) and \( c_{00} \) are the collections of all convergent sequences, convergent to zero sequences and eventually zero sequences respectively. Let us denote the limit of a sequence \( x = (x_n)_{n \in \mathbb{N}} \in c \) by \( \lim x \). Since all the three spaces are subspaces of \( \ell_\infty \), two elements in any of these spaces are Birkhoff-James orthogonal if and only if they are Birkhoff-James orthogonal in \( \ell_\infty \). Keeping this fundamental principle in mind we observe that if \( x = (x_n)_{n \in \mathbb{N}} \in c, \lim_{U} x_n = \lim x \) and if \( x \in c_0, \lim_{U} x_n = 0 \), for every free ultrafilter \( U \) on \( \mathbb{N} \). Thus we obtain the following result from Theorem 2.6.

**Theorem 2.12** Suppose that \( x = (x_n)_{n \in \mathbb{N}} \) and \( y = (y_n)_{n \in \mathbb{N}} \) are two sequences in \( K \). Then

(i) If \( x, y \in c \), then \( x \perp_B y \) if and only if

\[
0 \in \text{conv}\{||x|| : |x_n| = \|x\| \} \cup \{\lim x \}, \quad \text{if } \lim |x| = \|x\|,
\]

and \( 0 \in \text{conv}\{||y|| : |y_n| = \|y\| \} \cup \{\lim y \}, \quad \text{if } \lim |y| \neq \|y\|. \)

In particular, for \( K = \mathbb{R} \), \( x \perp_B y \) if and only if any of the following is true:

1. There exists \( N \in \mathbb{N} \) such that \( |x_N| = \|x\| \) and \( y_N = 0 \) or \( \lim |x| = \|x\| \) and \( \lim y = 0 \).
2. There exists \( N \in \mathbb{N} \) such that \( |x_N| = \lim |x| = \|x\| \) and \( x_N y_N \) and \( \lim xy \) are of different signs.
3. There exist \( N, M \in \mathbb{N} \) such that \( |x_N| = |x_M| = \|x\| \) and \( x_N y_N < 0 \neq x_M y_M \).

(ii) If \( x, y \in c_0 \) or \( c_{00} \), \( x \perp_B y \) if and only if

\[
0 \in \text{conv}\{||x|| : |x_n| = \|x\| \}.
\]

In particular, for \( K = \mathbb{R} \), \( x \perp_B y \) if and only if one of the following holds:

1. There exists \( N \in \mathbb{N} \) such that \( |x_N| = \|x\| \) and \( y_N = 0 \).
2. There exist \( N, M \in \mathbb{N} \) such that \( |x_N| = |x_M| = \|x\| \) and \( x_N y_N < 0 \neq x_M y_M \).
The characterization of the smooth points of these spaces requires a little bit more work.

**Theorem 2.13**  
(i) \( x = (x_n)_{n \in \mathbb{N}} \in c \) is smooth if and only if \( |x_n| \leq \|x\| \), for every \( n \in \mathbb{N} \), or there exists a unique \( N \in \mathbb{N} \), such that \( |x_N| = \|x\| \) and \( \lim|x| \neq \|x\| \).

(ii) \( x = (x_n)_{n \in \mathbb{N}} \in c_0 \) or \( c_{00} \) is smooth if and only if there exists a unique \( N \in \mathbb{N} \) such that \( |x_N| = \|x\| \).

**Proof**  
(i) If there are distinct \( N, M \in \mathbb{N} \) such that \( |x_N| = |x_M| = \|x\| \), then the two functionals \( \Psi, \Phi : c \to \mathbb{K} \) given by  

\[
\Psi(y) := \text{sgn}(x_N)y_N, \quad \Phi(y) := \text{sgn}(x_M)y_M, \quad y = (y_n)_{n \in \mathbb{N}} \in c,
\]

are distinct support functionals of \( x \). Again if \( |x_N| = \lim|x| = \|x\| \), for some \( N \in \mathbb{N} \), then \( \Psi', \Phi' : c \to \mathbb{K} \) given by  

\[
\Psi'(y) := \text{sgn}(x_N)y_N, \quad \Phi'(y) := \lim_{n \to \infty} \text{sgn}(x_n)y_n, \quad y = (y_n)_{n \in \mathbb{N}} \in c,
\]

are two distinct support functionals of \( x \), establishing the necessity.

Now, by Theorem 2.12, if there exists a unique \( N \in \mathbb{N} \) with \( |x_N| = \|x\| \) and \( \lim|x| < \|x\| \), then \( x \perp_B y \), for some \( y = (y_n)_{n \in \mathbb{N}} \in c \) if and only if \( y_N = 0 \) and clearly, Birkhoff-James orthogonality is right additive at \( x \). Again if \( |x_n| < \|x\| \), for every \( n \in \mathbb{N} \), then \( \lim|x| = \|x\| \) and again by Theorem 2.12, \( x \perp_B y \), for some \( y \in c \) if and only if \( \lim y = 0 \) proving the right additivity of Birkhoff-James orthogonality at \( x \) and the sufficiency is established.

(ii) The result for \( c_0 \) and \( c_{00} \) follows from part (i) of this theorem and the observation that for \( x \in c_0 \) or \( c_{00} \), \( \lim|x| = 0 \neq \|x\| \) unless \( x = 0 \).

Using these two results and going through the proofs of Theorem 2.9, Theorem 2.11 and Corollary 2.10, we obtain the following result.

**Theorem 2.14**  
1. The left-symmetric points of each of these spaces are \( \lambda e_n \) for \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{K} \).
2. The right-symmetric points of \( c \) are the sequences \( x = (x_n)_{n \in \mathbb{N}} \) such that \( |x_n| = \|x\| \), for every \( n \in \mathbb{N} \) and the spaces \( c_0 \) and \( c_{00} \) have no non-zero right-symmetric point.
3. The linear isometries of each of these spaces onto itself are the signed permutations operators.

### 3 Geometry of \( \ell_1 \)

In this section, we characterize Birkhoff-James orthogonality and its local symmetry in \( \ell_1 \). We begin with a known result.

**Theorem 3.1**  
The dual space of \( \ell_1 \) is isometrically isomorphic to \( \ell_\infty \) with the functional \( \Psi_a \) corresponding to an element \( a = (a_n)_{n \in \mathbb{N}} \in \ell_\infty \), given by  

\[
\Psi_a(x) = \sum_{n=1}^{\infty} a_n x_n, \quad x = (x_n)_{n \in \mathbb{N}} \in \ell_1.
\]

In order to avoid confusion, we denote the norm of an element in \( \ell_1 \) by \( \| \cdot \|_1 \) and the norm of an element in its dual, identified with the \( \ell_\infty \) space by \( \| \cdot \|_\infty \). We come to the following preliminary lemma before giving a complete characterization of Birkhoff-James orthogonality in \( \ell_1 \).
Lemma 3.2 Let $x = (x_n)_{n \in \mathbb{N}} \in \ell_1$ be non-zero. Then $a = (a_n)_{n \in \mathbb{N}} \in \ell_\infty$ is a support functional of $x$ if and only if $a_n = \text{sgn}(x_n)$ if $x_n \neq 0$ and $|a_n| \leq 1$ if $x_n = 0$.

Proof The sufficiency can be established by elementary calculations. For the necessity, note that

$$
\|x\|_1 = \sum_{n=1}^{\infty} a_n x_n = \sum_{x_n \neq 0} a_n x_n \leq \sum_{a_n \neq 0} |a_n x_n| \leq \|a\|_\infty \sum_{x_n \neq 0} |x_n| = \|x\|_1.
$$

Hence, equality holds in both the inequalities, giving $|a_n| = \|a\|_\infty = 1$ and $a_n x_n = |a_n x_n|$, i.e., $a_n = \text{sgn}(x_n)$, for every $n \in \mathbb{N}$ with $x_n \neq 0$. Also, since $\|a\|_\infty = 1$, $|a_n| \leq 1$, for $n \in \mathbb{N}$ with $x_n = 0$.

We now come to the characterization of Birkhoff-James orthogonality in $\ell_1$.

Theorem 3.3 Let $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}} \in \ell_1$. Then $x \perp_B y$ if and only if

$$
\left| \sum_{n=1}^{\infty} \text{sgn}(x_n) y_n \right| \leq \sum_{x_n = 0} |y_n|,
$$

(3.1)

where sum over an empty set is defined to be 0.

Proof If $x \perp_B y$, then there exists $a = (a_n)_{n \in \mathbb{N}} \in \ell_\infty$ with $\|a\|_\infty = 1$ such that $\Psi_a \in J(x)$ and $\Psi_a(y) = 0$. This yields

$$
\left| \sum_{x_n \neq 0} \text{sgn}(x_n) y_n \right| \leq \sum_{x_n = 0} |a_n y_n| \leq \sum_{x_n = 0} |y_n|,
$$

proving the necessity.

Again, if (3.1) holds, set $k \in \mathbb{K}$, $|k| \leq 1$ such that

$$
\sum_{n=1}^{\infty} \text{sgn}(x_n) y_n = k \left( \sum_{x_n = 0} |y_n| \right).
$$

Consider $a = (a_n)_{n \in \mathbb{N}}$ given by

$$
a_n := \begin{cases} 
\text{sgn}(x_n), & a_n \neq 0, \\
-k \text{sgn}(y_n), & a_n = 0.
\end{cases}
$$

Then clearly $a \in \ell_\infty$ and by Lemma 3.2, $\Psi_a \in J(x)$. Since $\Psi_a(y) = 0$, the sufficiency follows.

As a corollary to the above result, we mention the case when $\mathbb{K} = \mathbb{R}$.

Corollary 3.4 Suppose that $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}} \in \ell_1$ are two members of $\ell_1$ over $\mathbb{R}$. Let $\mathcal{N}_1 := \{n \in \mathbb{N} : x_n y_n > 0\}$, $\mathcal{N}_2 := \{n \in \mathbb{N} : x_n y_n < 0\}$ and $\mathcal{N}_0 := \{n \in \mathbb{N} : x_n = 0\}$. Then $x \perp_B y$ if and only if

$$
\left| \sum_{n \in \mathcal{N}_1} |y_n| - \sum_{n \in \mathcal{N}_2} |y_n| \right| \leq \sum_{n \in \mathcal{N}_0} |y_n|.
$$

We now come to the characterization of the smooth points of $\ell_1$. 

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Theorem 3.5 \textit{The smooth points of }$\ell_1$\textit{ are the sequences having no zero term.}

\textbf{Proof} Suppose $x = (x_n)_{n \in \mathbb{N}} \in \ell_1$ has no zero term. Then by Theorem 3.3, if $x \perp_B y$ and $x \perp_B z$ where $y = (y_n)_{n \in \mathbb{N}}, z = (z_n)_{n \in \mathbb{N}} \in \ell_1$, then

$$\sum_{n=1}^{\infty} \text{sgn}(x_n) y_n = \sum_{n=1}^{\infty} \text{sgn}(x_n) z_n = 0.$$ 

Hence

$$\sum_{n=1}^{\infty} \text{sgn}(x_n)(y_n + z_n) = 0 \Rightarrow x \perp_B (y + z),$$

proving $x$ is smooth.

If $x = (x_n)_{n \in \mathbb{N}} \in \ell_1$ has a zero term $x_N$, for some $N \in \mathbb{N}$, by Lemma 3.2, we obtain that $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}} \in \ell_\infty$, given by

$$a_n := \begin{cases} \text{sgn}(x_n), & x_n \neq 0, \\ 0, & x_n = 0, \end{cases} \quad \text{and} \quad b_n := \begin{cases} \text{sgn}(x_n), & x_n \neq 0, \\ 0, & x_n = 0 \text{ and } n \neq N, \\ 1, & n = N. \end{cases}$$

are two distinct support functionals of $x$. Hence $x$ cannot be smooth.

We finally come to the characterization of local symmetry of Birkhoff-James orthogonality in $\ell_1$.

Theorem 3.6 \textit{No non-zero point of }$\ell_1$\textit{ is left-symmetric.}

\textbf{Proof} Suppose that $x = (x_n)_{n \in \mathbb{N}} \in \ell_1$ is non-zero. If there exists $N \in \mathbb{N}$ such that $x_N = 0$, then setting $a = (a_n)_{n \in \mathbb{N}}$, given by $y_n := \text{sgn}(x_n) \frac{1}{\|x\|_1}$, we have $\|y\|_1 < \infty$. Thus considering $z := y + 2\|y\|_1 e_N$. Theorem 3.3 implies $x \perp_B z$ and $z \not\perp_B x$. If $x_n \neq 0$, for every $n \in \mathbb{N}$, then there exists $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{M} |x_n| \neq \sum_{n=M+1}^{\infty} |x_n|.$$

Set $y = (y_n)_{n \in \mathbb{N}}$, given by,

$$y_n := \begin{cases} \text{sgn}(x_n), & 1 \leq n \leq M, \\ -\text{sgn}(x_n) \frac{M}{\|x\|_1}, & n \geq M + 1. \end{cases}$$

Then $y \in \ell_1$. Again, by Theorem 3.3, $x \perp_B y$ and $y \not\perp_B x$ and hence $x$ is not a left-symmetric point.

We now come to the characterization of the right-symmetric points of $\ell_1$ and thereby find the collection of onto isometries of the space.

Theorem 3.7 \textit{The only right-symmetric points of }$\ell_1$\textit{ are scalar multiples of }$e_n$, \textit{for }$n \in \mathbb{N}$.

\textbf{Proof} Observe that for $y = (y_n)_{n \in \mathbb{N}} \in \ell_1$, by Theorem 3.3, $y \perp_B \lambda e_n$, for some $\lambda \neq 0$ if and only if $y_n = 0$ and hence $\lambda e_n \perp_B y$. Again if $x = (x_n)_{n \in \mathbb{N}} \in \ell_1$ has $x_n, x_m \neq 0$, for $n \neq m$, we consider $r \in \mathbb{N}$ such that

$$0 < |x_r| \leq \sum_{k \neq r} |x_k|.$$ 

Hence by Theorem 3.3, $x \not\perp_B e_r$ but $e_r \perp_B x$. 

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We use this result to characterize the onto isometries of \( \ell_1 \) with the help of Corollary 1.1 and thereby establish the Banach-Lamperti theorem for onto operators on \( \ell_1 \).

**Corollary 3.8** Let \( T : \ell_1 \to \ell_1 \) be an onto linear isometry. Then \( T \) must be a signed permutation operator.

### 4 Geometry of \( \ell_p \), for \( 1 < p < \infty, p \neq 2 \)

In this final section, we characterize Birkhoff-James orthogonality and its local symmetry in \( \ell_p \), for \( p \in (1, \infty) \setminus \{2\} \). We begin with a well-known result.

**Theorem 4.1** The dual space of \( \ell_p \) is isometrically isomorphic to \( \ell_q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) with the functional \( \Psi_a \) corresponding to \( a = (a_n)_{n \in \mathbb{N}} \in \ell_q \), given by

\[
\Psi_a(x) = \sum_{n=1}^{\infty} a_n x_n, \quad x = (x_n)_{n \in \mathbb{N}} \in \ell_p.
\]

Hence, as was done in the case of \( \ell_1 \), we continue to denote the norm of an element of \( \ell_p \) by \( \| \cdot \|_p \) and the norm of an element in the dual of \( \ell_p \), identified with \( \ell_q \), by \( \| \cdot \|_q \). To begin with, we note down a corollary of this theorem pertaining to the characterization of the support functional of an element of \( \ell_p \).

**Corollary 4.2** Let \( a = (a_n)_{n \in \mathbb{N}} \in \ell_q \) be a support functional of \( x = (x_n)_{n \in \mathbb{N}} \in \ell_p \setminus \{0\} \). Then

\[
a_n = \frac{1}{\|x\|_p^{p-1}} \text{sgn}(x_n)|x_n|^{p-1}, \quad n \in \mathbb{N}.
\]

The proof of this result involves elementary computations and the equality criteria of Holder’s inequality. Observe also that the corollary establishes the uniqueness of the support functional of any non-zero element in the space, thereby proving its smoothness.

We now come to our characterization of Birkhoff-James orthogonality in \( \ell_p \) which follows as a direct consequence of Corollary 4.2 and James’ characterization of Birkhoff-James orthogonality.

**Theorem 4.3** Let \( x = (x_n)_{n \in \mathbb{N}} \) and \( y = (y_n)_{n \in \mathbb{N}} \in \ell_p \). Then \( x \perp_B y \) if and only if

\[
\sum_{n=1}^{\infty} \text{sgn}(x_n)|x_n|^{p-1} y_n = 0.
\]

We now characterize the local symmetry of Birkhoff-James orthogonality in \( \ell_p \).

**Theorem 4.4** \( x = (x_n)_{n \in \mathbb{N}} \in \ell_p \) is a left-symmetric point if and only if \( x \) is a non-zero right-symmetric point if and only if \( x = \lambda e_N + \lambda_1 e_M \), for some \( N, M \in \mathbb{N} \) and \( \lambda, \lambda_1, \lambda_2 \in \mathbb{K} \), \( |\lambda_1| = |\lambda_2| \).

**Proof** We first characterize the left-symmetric points. Let \( x \neq 0 \). The sufficiency can be verified by elementary computations. Now, suppose \( x_N, x_M \neq 0 \) and \( |x_N| \neq |x_M| \), for some \( N, M \in \mathbb{N} \). Set \( y = \text{sgn}(x_N)|x_M|^{p-1} e_N - \text{sgn}(x_M)|x_N|^{p-1} e_M \) and note that by Theorem 4.4, \( x \perp_B y \). However,

\[
\sum_{n=1}^{\infty} \text{sgn}(y_n)|y_n|^{p-1} x_n = |x_N x_M| \left( |x_N|^{p^2-2p} - |x_M|^{p^2-2p} \right) \neq 0.
\]
since $p \neq 2$, proving $y \not\perp_B x$. Again if $x_N, x_M, x_K \neq 0$, we may assume that $|x_N| = |x_M| = |x_K|$, as otherwise, by the previous argument, $x$ cannot be left-symmetric. But then, setting $y = sgn(x_N)e_N - \frac{1}{2}(sgn(x_M)e_M - sgn(x_K)e_K)$ clearly yields $x \perp_B y$ and $y \not\perp_B x$ by Theorem 4.4.

Now, Proposition 2.1 of [16] states that in a smooth, strictly convex space, a point is left-symmetric if and only if it is right-symmetric. Hence the proof for the right-symmetric case follows from the left-symmetric case.

As a consequence of this result, we characterize all the onto isometries $T : \ell_p \to \ell_p$ and prove the Banach-Lamperti Theorem for onto operators on $\ell_p$.

**Theorem 4.5** Let $T : \ell_p \to \ell_p$ be an onto linear isometry. Then $T$ must be a signed permutation operator.

**Proof** Observe that $T$ and $T^{-1}$ are both onto isometries and hence $T(x)$ and $T^{-1}(x)$ are both left-symmetric points of $\ell_p$, for any $x \in \ell_p$ left-symmetric, by Corollary 1.1. Hence it is sufficient to show that for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $T(e_n) = \lambda e_m$, for some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Suppose by contradiction, $T(e_n) \neq \lambda e_m$, for any $m \in \mathbb{N}$ and $|\lambda| = 1$. Then $T(e_n) = \frac{1}{2\pi}(\lambda_1 e_1 + \lambda_2 e_j)$, for $i \neq j \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, $|\lambda_1| = |\lambda_2| = 1$. Now, $T^{-1}\left(\frac{1}{2\pi}(\lambda_1 e_1 - \lambda_2 e_j)\right)$ is a left-symmetric point other than any unimodular multiple of $e_n$ and hence must be $\lambda e_m$ for some $m \neq n$ and $|\lambda| = 1$ or $\frac{1}{2\pi}(v_1 e_k + v_2 e_l)$, for some $k \neq l \in \mathbb{N}$ and $|v_1| = |v_2| = 1$. In either case, we clearly obtain that $T^{-1}(e_i)$ is not a left-symmetric point, which establishes the desired contradiction.

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