Smoothness of Algebraic Supervarieties and Supergroups

R. Fioresi

Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato, 5.
40126 Bologna. Italy.
e-mail: fioresi@dm.unibo.it

Abstract

In this paper we discuss the notion of smoothness in complex algebraic supergeometry and we prove that all affine complex algebraic supergroups are smooth. We then prove the stabilizer theorem in the algebraic context, providing some useful applications.

1 Introduction

The category of differentiable supermanifolds was introduced and discussed in several works among which [2, 3, 10, 11, 12] from different point of views, especially in connection with the important physical applications, which stem from string theory and ultimately are related with the problem of the classification of elementary particles.

In this paper we are interested in algebraic supergeometry and its relation with its differential counterpart. In his foundational work [12] on supermanifolds, Manin defined the notion of superscheme and discussed some important examples.

Along the same lines we want to understand the concept of smoothness in complex algebraic supergeometry. Given the algebraic nature of the problems in the theory of supermanifolds, we believe that a deep analysis of the superalgebraic category can shed light also on the differential one. Moreover it is the correct category to work with, when one wants to discuss quantum deformations of the geometric objects.
In ordinary algebraic geometry smoothness is a local notion, strongly linked to the dimension of the local ring of the variety at the point. Unfortunately, due to the presence of the odd nilpotents, it is not easy to generalize the idea of dimension of a ring to the super context. To overcome this problem, we define smoothness as a property of the completion of the local superring of the supervariety at a given point; namely we say that a point is smooth if the local super ring is isomorphic to a power series super ring. We are then able to show that any supervariety admits a unique supermanifold structure in a neighbourhood of a smooth point, as in the classical case, through the application of the implicit function theorem, after reduction to local complete intersection.

Using Cartier’s Theorem adapted to supergeometry we can then prove that all algebraic supergroups are smooth, in other words, all affine algebraic supergroups are also Lie supergroups. We apply this result to the case of the stabilizer supergroup functor of an action of an affine supergroup on an affine supervariety. After showing that the stabilizer is representable, that is it is a supergroup, we show that classical supergroups are smooth (for a list of classical supergroups see for example [5] pg 70). This fact is generally known, it is treated for example in a different context by Gruson in [7] and by Varadarajan in [13] pg 289. We however provide an independent proof using algebraic techniques, which we believe can be of help also in other differentiable supermanifold questions and can also give other examples of algebraic Lie supergroups.

This paper is organized as follows.

In Section 2, we review some basic facts of algebraic and differential supergeometry, among which the definition of supermanifolds, supervarieties and their functor of points.

In Section 3 we give the definition of smooth point of a supervariety. We then prove the super version of the classical result which states that a smooth point of a complex algebraic variety admits a supermanifold structure in a suitable neighbourhood.

In Section 4 we prove that all (closed) points of complex algebraic groups are smooth.

In Section 5 we prove the Stabilizer Theorem, which states that the stabilizer functor for the action of an affine algebraic supergroup on an affine
supervariety is representable by a supergroup hence it is a smooth variety i.e. a supermanifold.

As an application, in Section 6, we show that the classical supergroup functors as described in [5] pg 70 are representable, i.e. they are algebraic supergroups, and consequently, they are Lie supergroups.

Acknowledgements. We wish to thank Prof. V. S. Varadarajan, Dr. L. Caston, Prof. D. Gieseker and Prof. M. Duflo for helpful comments.

2 Basic definitions of Supergeometry

In this section we want to recall some basic definitions and facts in supergeometry. For more details see [13, 4, 5, 12].

Let \( k \) be the ground field.

A superalgebra \( A \) is a \( \mathbb{Z}_2 \)-graded algebra, \( A = A_0 \oplus A_1 \), \( p(x) \) denotes the parity of an homogeneous element \( x \). \( A \) is said to be commutative if \( xy = (-1)^{p(x)p(y)}yx \). \( I^{\odd} \) denotes the ideal generated by the odd nilpotents.

Definition 2.1. A superspace \( S = (|S|, \mathcal{O}_S) \) is a topological space \( |S| \) endowed with a sheaf of superalgebras \( \mathcal{O}_S \) such that the stalk \( \mathcal{O}_{S,x} \) is a local superalgebra for all \( x \in |S| \). A morphism \( \phi : S \to T \) of superspaces is given by \( \phi = (|\phi|, \phi^*) \), where \( \phi : |S| \to |T| \) is a map of topological spaces and \( \phi^* : \mathcal{O}_T \to \phi^*\mathcal{O}_S \) is a sheaf morphism such that \( \phi^*(m_{|\phi|(x)}) = m_x \) where \( m_{|\phi|(x)} \) and \( m_x \) are the maximal ideals in the stalks \( \mathcal{O}_{T,|\phi|(x)} \) and \( \mathcal{O}_{S,x} \) respectively.

The most important examples of superspaces are given by supermanifolds and superschemes.

Definition 2.2. Let’s consider the superspace \( \mathbb{C}^{p|q} = (\mathbb{C}^p, \mathcal{H}_{\mathbb{C}^{p|q}}) \), where \( \mathcal{H}_{\mathbb{C}^{p|q}}|_U = \mathcal{H}_{\mathbb{C}^p}|_U \otimes \mathbb{C}[\xi_1 \ldots \xi_q], \quad U \) open in \( \mathbb{C}^p \)

where \( \mathbb{C}[\xi_1 \ldots \xi_q] \) is the exterior algebra generated by \( \xi_1 \ldots \xi_q \) and \( \mathcal{H}_{\mathbb{C}^p} \) denotes the sheaf of holomorphic functions on \( \mathbb{C}^p \).

A complex supermanifold of dimension \( p|q \) is a superspace \( M = (|M|, \mathcal{H}_M) \) which is locally isomorphic to \( \mathbb{C}^{p|q} \), i.e. for all \( x \in |M| \) there exist open sets \( V_x \subset |M|, U \subset \mathbb{C}^p \) such that:

\[
\mathcal{O}_M|_{V_x} \cong \mathcal{H}_{\mathbb{C}^{p|q}}|_U
\]
**Definition 2.3.** A *superscheme* $S$ is a superspace $(|S|, \mathcal{O}_S)$ such that $(|S|, \mathcal{O}_{S,0})$ is a quasi-coherent sheaf of $\mathcal{O}_{S,1}$-modules. A *morphism* of supermanifolds or of superschemes is a morphism of the corresponding superspaces.

Superschemes can be characterized by a local model as we shall presently see.

**Definition 2.4.** $\text{Spec} A$.

Let $A$ be a superalgebra and let $\mathcal{O}_{A_0}$ be the structural sheaf of the ordinary scheme $\text{Spec}(A_0) = (\text{Spec}A_0, \mathcal{O}_{A_0})$ ($\text{Spec}A_0$ denotes the prime spectrum of the commutative ring $A_0$). The stalk of the sheaf at the prime $p \in \text{Spec}(A_0)$ is the localization of $A_0$ at $p$. As for any superalgebra, $A$ is a module over $A_0$. We have indeed a sheaf $\mathcal{O}_A$ of $\mathcal{O}_{A_0}$-modules over $\text{Spec}A_0$ with stalk $A_p$, the localization of the $A_0$-module $A$ over the prime $p \in \text{Spec}(A_0)$:

$$A_p = \left\{ \frac{f}{g} \mid f \in A, g \in A_0 - p \right\}.$$ 

$A_p$ contains a unique two-sided maximal ideal generated by the maximal ideal in the local ring $(A_p)_0$ and the generators of $(A_p)_1$ as $A_0$-module.

$\mathcal{O}_A$ is a sheaf of superalgebras and $(\text{Spec}A_0, \mathcal{O}_A)$ is a superscheme that we denote with $\text{Spec}A$.

The next proposition shows that $\text{Spec}A$ is the local model for superschemes.

**Proposition 2.5.** A superspace $S$ is a superscheme if and only if it is locally isomorphic to $\text{Spec}A$ for some superalgebra $A$, i. e. for all $x \in |S|$, there exists $U_x \subset |S|$ open such that $(U_x, \mathcal{O}_S|_{U_x}) \cong \text{Spec}A$. (Clearly $A$ depends on $U_x$).

*Proof.* See [4] §3. \qed

**Definition 2.6.** We say that a superscheme $X$ is *affine* if it is isomorphic to $\text{Spec}A$ for some algebra $A$ and we call $k[X] =_{def} A$ the *coordinate ring* of the affine superscheme $X$. If $k[X]/I^{\text{odd}}$ is the coordinate ring of an ordinary affine algebraic variety (called the *reduced variety* associated to $X$) and $(|X|, \mathcal{O}_{X,0})$ is a coherent sheaf of $\mathcal{O}_{X,1}$-modules, we say that $X$ is an *affine algebraic variety*.

**Remark 2.7.** There is an equivalence of categories between superalgebras and affine superschemes. This equivalence is treated in detail in [4] §3.
We now want to introduce the concept of functor of points associated to an affine supervariety.

**Definition 2.8.** Let $X$ be a supervariety. Its *functor of points* is given by:

$$ h_X : \text{salg} \rightarrow \text{sets}, \quad h_X(A) = \text{Hom}(\text{Spec} A, X) $$

where $\text{salg}$ is the category of commutative superalgebras. If $X$ is an affine supervariety $h_X(A) = \text{Hom}(k[X], A)$. If $h_X$ is group valued we say that $X$ is an *affine supergroup*. This is equivalent to the fact that $k[X]$ is a Hopf superalgebra. This is also the same as giving a multiplication $m : X \times X \rightarrow X$ and an inverse $i : X \rightarrow X$ satisfying the usual commutative diagrams.

More in general, we say that $G : \text{salg} \rightarrow \text{sets}$ is a *supergroup functor* if it is group valued. Clearly, a representable supergroup functor is an affine supergroup.

### 3 Smoothness of complex algebraic supervarieties

Let $k = \mathbb{C}$.

Let $X = (|X|, \mathcal{O}_X)$ be a supervariety and let $P \in |X|$ be a *closed point* i.e. $P$ corresponds to a maximal ideal. Let $m_P$ be the maximal ideal in $\mathcal{O}_{X,P}$.

**Definition 3.1.** We say that $P$ is *smooth* if

$$ \widehat{\mathcal{O}_{X,P}} \cong C[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]], \quad \widehat{\mathcal{O}_{X,P}} = \lim_{\leftarrow} \mathcal{O}_{X,P}/m_P^n $$

where $x_i$'s and $\xi_j$'s are respectively even and odd variables. In this case we say that the dimension of the supervariety $X$ at $P$ is $r|s$. Notice that the dimension is well defined, that is if $C[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]] \cong C[[x_1 \ldots x_m, \xi_1 \ldots \xi_n]]$ then $r = m$, $n = s$.

Smoothness of a point of a supervariety cannot be checked at the classical level as the next examples show.
Example 3.2. 1. Consider the supervariety $X$ with coordinate ring $\mathbb{C}[X] = \mathbb{C}[x, y, \xi, \eta]/(\xi \eta)$. Its reduced variety is the affine plane, where all the closed points are smooth in the classical sense. It is immediate to check that this supervariety has no smooth points according to Definition 3.1.

2. Consider the supervariety with coordinate ring $\mathbb{C}[x, y, \xi, \eta]/(\xi x + \eta y)$. Again its reduced variety is the affine plane. One can check that all (closed) points are smooth except the origin.

Since the notion of smoothness is local we can assume that $X$ is an affine supervariety, with coordinate ring $\mathbb{C}[X] = \mathbb{C}[x_1 \ldots x_m, \xi_1 \ldots \xi_n]/I$, where $I = (f_1 \ldots f_p, \phi_1 \ldots \phi_q)$. In this case $\mathcal{O}_{X, P}$ is the localization of $\mathbb{C}[X]$ at the point $P$ (see Definition 2.4).

Definition 3.3. As in the classical setting we define the jacobian of $f_1 \ldots f_p, \phi_1 \ldots \phi_q \in \mathbb{C}[x_1 \ldots x_m, \xi_1 \ldots \xi_n]$ at a point $P$ as:

$$\text{Jac}(f, \phi) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(P) & \ldots & \frac{\partial f_1}{\partial x_m}(P) & \frac{\partial f_1}{\partial \xi_1}(P) & \ldots & \frac{\partial f_1}{\partial \xi_n}(P) \\
\vdots & & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial \phi_1}{\partial x_1}(P) & \ldots & \frac{\partial \phi_1}{\partial x_m}(P) & \frac{\partial \phi_1}{\partial \xi_1}(P) & \ldots & \frac{\partial \phi_1}{\partial \xi_n}(P)
\end{pmatrix}$$

(for the definition of $\frac{\partial f}{\partial x}$ see for example [13]). The rank of the jacobian is given by $a|b$ where $a$ and $b$ are the ranks of the $m \times p$, $n \times q$ diagonal blocks.

Lemma 3.4. Let the notation be as above. Let $P \in |X|$ be a closed point i.e. a maximal ideal $m_P$ in $\mathbb{C}[X]$. Then

$$\text{rk}(\text{Jac}(f, \phi))(P) = m|n - \dim(m_P/m_P^2).$$

Proof. The proof is the same as in ordinary case, (see for example [8] pg 32), let’s sketch it. We have a natural identification:

$$F : M_P/M_P^2 \cong \mathbb{C}^{m|n}
\quad f \quad \mapsto \quad df_P = \text{def} \left( \frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial \xi_n}(P) \right)$$

where $M_P$ denotes the maximal ideal corresponding to the point $P$ in $\mathbb{C}[x_1 \ldots x_m, \xi_1 \ldots \xi_n]$. Viewing the rows of $\text{Jac}(f, \phi)$ as vectors in $\mathbb{C}^{m|n}$ the above identification tells us immediately that

$$\text{rk}(\text{Jac}(f, \phi))(P) = \dim(I + M_P^2)/M_P^2.$$
where $I = (f_1 \ldots f_p, \phi_1 \ldots \phi_q)$. Since localizations commute with quotients we have that:

$$m_P/m_P^2 \cong (M_P/I)/(M_P^2 + I)/I = M_P/(M_P^2 + I).$$

Hence we have:

$$\text{rk}(\text{Jac}(f, \phi))(P) = \dim(I + M_P^2)/M_P = \dim M_P/M_P^2 - \dim M_P/(M_P^2 + I).$$

**Proposition 3.5.** If $P$ is a smooth point of an affine supervariety $X$ with dimension $r|s$ in $P$ then:

1. $m_P/m_P^2$ has dimension $r|s$.
2. $Gr(\mathcal{O}_{X,P}) = \mathbb{C}[x_1 \ldots x_r, \xi_1 \ldots \xi_s]$ where $Gr(\mathcal{O}_{X,P}) = \bigoplus_i m_P^i/m_P^{i+1}$.
3. $\text{rk}(\text{Jac}(f, \phi))(P) = m|n - r|s$.

**Proof.** Parts (1) and (2) are immediate by Lemma A.5 in the Appendix, (3) is a consequence of Lemma 3.4.

**Remark 3.6.** 1. The proof of this result resembles the one for the commutative setting. One difference that may generate confusion is the following. When we are localizing $\mathbb{C}[X]$ to obtain $\mathcal{O}_{X,P}$ we are using a maximal ideal of the even part $\mathbb{C}[X]_0$ that is $(x_1 - a_1, \ldots, x_m - a_m, \xi_i \xi_j, \forall i > j), a_i \in \mathbb{C}$. On the other hand, when we are completing the local superalgebra $\mathcal{O}_{X,P}$ we are taking the inverse limit of the system $\mathcal{O}_{X,P}/m_P^n$, where $m_P$ is the maximal ideal of this superalgebra, hence it is a graded object and it will necessarily contain all the odd generators.

2. If $P$ is smooth, $m_P/m_P^2$ is generated by $r|s$ elements, hence by the super Nakayama’s Lemma A.6, we have that $m_P$ is generated by $r|s$ elements.

**Observation 3.7.** The affine supervariety $X$ is embedded in $\mathbb{C}^{m|n}$ via the chosen explicit presentation of its coordinate ring $\mathbb{C}[X]$. Hence we can give to the set of closed points of $X$ a complex topology inherited from this embedding. However this topology is independent from the embedding; this is a classical fact, still valid in this setting since it is a topological question. We want to show that the closed points of the supervariety $X$ equipped with this complex topology, admit a unique supermanifold structure in a suitable complex neighbourhood $U$ of the smooth point $P$. In other words we want to show that:

$$\mathcal{H}_{\mathbb{C}^{m|n}}|_U = (\mathcal{H}_{\mathbb{C}^{m|n}/\mathcal{K}})|_U \cong \mathcal{H}_{\mathbb{C}^{m|n}}|_U \otimes \mathbb{C}[\xi_1 \ldots \xi_s] \quad (\ast)$$

7
where $\mathcal{K}$ is the ideal sheaf whose global sections are generated in $\mathcal{H}_{C^{m|n}}$ by the ideal $I$ of the supervariety $X$. The whole question in the super setting is to show the existence of a local splitting ($\ast$). To settle this problem our strategy is to use the implicit functions theorem, which is still valid in this setting. Let’s recall the statement from [11] pg 52.

**Theorem 3.8.** Let $M$ be a complex supermanifold, $P \in |U|$, where $U \subset M$, is isomorphic to an open in $C^{r|s}$. Let $K$ be the ideal in $\mathcal{H}_M(U)$ generated by $g_1 \ldots g_p, \gamma_1 \ldots \gamma_q$ vanishing at $P$ and with linearly independent differentials at $P$. Then there exists a unique subsupermanifold:

$$N = (|N|, \mathcal{H}_N), \quad \mathcal{H}_N = \mathcal{H}_M|U/K$$

where $\mathcal{K}$ is the sheaf of ideals with global sections $K$ and $|N|$ is the topological space whose existence is granted by the classical result.

**Remark 3.9.** The key for the proof of this result is the fact that any set of functions $g_1 \ldots g_p, \gamma_1 \ldots \gamma_q$ with linearly independent differentials at $P$ can be completed to obtain a set of local coordinates in a neighbourhood of $P$. More details on this can be found in [13] pg 148.

This theorem allows us, in a special case, to obtain immediately the result we are after.

**Corollary 3.10.** Let $P \in |X|$ be a smooth point, and let $X$ have dimension $r|s$ at $P$. Let’s assume that the ideal $I$ of the supervariety $X$ is given by $I = (f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s})$ (in this case we say that $X$ is a complete intersection). Then in a neighbourhood of $P$, $X$ admits a complex supermanifold structure (in the sense of Observation 3.7).

**Proof.** This is a direct application of the Theorem 3.8. The supervariety $X$ is defined in $C^{m|n}$ by the polynomials $f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s}$ with $\text{rk}(\text{Jac}(f_i, \phi_j)(P)) = m|n-r|s$. Consider the ideal $K$ generated in $\mathcal{H}(C^{m|n})$ by the $f_i$’s and $\phi_j$’s. Then there exists a unique subsupermanifold $N$ of $C^{m|n}$ such that $\mathcal{H}_N = (\mathcal{H}_{C^{m|n}}/\mathcal{K})|U$ for a suitable neighbourhood $U$ of $P$, $\mathcal{K}$ is the ideal sheaf whose global sections are $K$. \hfill $\square$

---

1We say that $f \in \mathcal{H}_M(U)$ vanishes at $P$ if it is zero under the morphism:

$$\mathcal{H}_M(U) \rightarrow \mathcal{H}_{M,P} \rightarrow \mathcal{H}_{M,P}/m_{h,P} \cong \mathbb{C}$$

$m_{h,P}$ being the maximal ideal in $\mathcal{H}_{M,P}$. 

8
In general the ideal $I$ of the supervariety $X$ is given by $(f_1 \ldots f_p, \phi_1 \ldots \phi_q)$ where $p \mid q >\mid n - r \mid s$. We want to show that, as it happens for the classical setting, $X$ is locally a complete intersection, so that we can conclude our discussion with the same reasoning as in Corollary 3.10. Let $P \in X$ be a smooth point and assume $f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s}$ are such that:

$$\text{rk}(\text{Jac}(f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s}))(P) = m \mid n - r \mid s.$$ 

Let $X'$ be the variety corresponding to the ring:

$$C[X'] = C[x_1 \ldots x_m, \xi_1 \ldots \xi_n] / (f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s})$$

and let $O_{X',P}$ denote its local ring at the closed point $P$. We are going to show the following:

1. $P$ is a smooth point of $X'$. Moreover $X'$ has the same dimension of $X$ i.e. $O_{X',P} = C[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]$. This implies that $X'$ is a complete intersection.

2. $X$ and $X'$ are locally isomorphic, in other words $O_{X,P} \cong O_{X',P}$. Since this result is true for all the points in a neighbourhood of $P$, we have that $O_X(U) \cong O_{X'}(U)$. Hence we can apply the result 3.8 to $X'$ to conclude that $X$ admits a supermanifold structure near $P$.

**Lemma 3.11.** Let the notation be as above. We have the following commutative diagram:

$$
\begin{array}{ccc}
O_{X',P} & \rightarrow & O_{X,P} \\
\downarrow & & \downarrow \\
\widehat{O_{X',P}} & \rightarrow & \widehat{O_{X,P}}
\end{array}
$$

where the horizontal arrows are surjections, while the vertical ones injections.

**Proof.** Observe that since we have a surjection $C[X'] \rightarrow C[X]$ we also have a surjective morphism (this is a property of localizations):

$$O_{X',P} \rightarrow O_{X,P}$$

mapping the maximal ideal onto the maximal ideal. This will give raise to a surjective system:

$$O_{X',P}/m_P^n \rightarrow O_{X,P}/m_P^n.$$
where \( m_P \) and \( m'_P \) denote the maximal ideals in \( \mathcal{O}_{X,P} \) and \( \mathcal{O}_{X',P} \). Hence \( \widehat{\mathcal{O}}_{X',P} \rightarrow \widehat{\mathcal{O}}_{X,P} \) is a surjective map. The vertical arrows are injections since \( \cap m_P^i = \cap m_P'^i = (0) \). This happens since this is true in the ordinary case and since the odd variables disappear for large \( i \)'s.

**Remark 3.12.** By Lemma 3.4 we get immediately that \( \dim(m_P/m_P^2) = \dim(m'_P/m_P'^2) \) and since the point \( P \) is smooth

\[
\dim(m_P/m_P^2) = \dim(m'_P/m_P'^2) = r|s.
\]

Hence by the super Nakayama’s Lemma A.6 we have that both \( m_P \) and \( m'_P \) are generated by \( r|s \) elements.

**Lemma 3.13.** Let the notation be as above.

\[
\hat{\mathcal{O}}_{X',P} \cong \mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]/I
\]

for a suitable ideal \( I \).

**Proof.** By the Theorem A.2 in the Appendix, we have that there exist a unique map:

\[
\mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]] \rightarrow \widehat{\mathcal{O}}_{X',P}
\]

sending \( x_i \)'s and \( \xi_j \)'s into \( r|s \) generators of the maximal ideal \( m'_P \). So the map is surjective and we obtain our result. \( \square \)

**Proposition 3.14.** Let the notation be as above.

\[
\widehat{\mathcal{O}}_{X',P} \cong \widehat{\mathcal{O}}_{X,P} = \mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]].
\]

**Proof.** By the Lemma 3.11 we have that:

\[
\widehat{\mathcal{O}}_{X',P}/J \cong \widehat{\mathcal{O}}_{X,P} = \mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]].
\]

By the Theorem A.4 in the Appendix, we get the result. \( \square \)

We have proven the local isomorphism in the completions, now we turn our attention to the local rings.

**Lemma 3.15.** Let the notation be as above.

\[
\mathcal{O}_{X',P} \cong \mathcal{O}_{X,P}.
\]
Proof. By Proposition 3.14 and Lemma 3.11.

This concludes the proof of the following:

**Theorem 3.16.** Let $X$ be a complex algebraic supervariety, $P$ a smooth point of $X$. Then, there exist a neighbourhood of $P$ where we can give to $X$ a unique structure of a complex supermanifold.

**Proof.** Assume without loss of generality that $X$ is affine and has dimension $r|s$ at $P$. Let

$$\mathbb{C}[X] = \mathbb{C}[x_1 \ldots x_m, \xi_1 \ldots \xi_n]/(f_1 \ldots f_p, \phi_1 \ldots \phi_q)$$

be the coordinate ring of $X$. Let $X'$ be the algebraic supervariety defined by the coordinate ring:

$$\mathbb{C}[X'] = \mathbb{C}[x_1 \ldots x_m, \xi_1 \ldots \xi_n]/(f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s})$$

where $\text{rk}(\text{Jac}(f_1 \ldots f_{m-r}, \phi_1 \ldots \phi_{n-s})) = m|n - r|s$. Then by Corollary 3.10 the result holds for $X'$ and by Lemma 3.15 $X$ and $X'$ are locally isomorphic.

The next lemma will be crucial in the discussion of smoothness of algebraic supergroups in Section 4.

**Lemma 3.17.** Let the notation be as above. Let $m_P$ be generated by $r|s$ elements. If

$$\text{Gr}(\mathcal{O}_{X,P}) = \mathbb{C}[x_1 \ldots x_r, \xi_1 \ldots \xi_s]$$

then $P$ is smooth.

**Proof.** By the Theorem [A.2] in the Appendix, we have that there exist a surjective map:

$$\mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]] \longrightarrow \widehat{\mathcal{O}_{X,P}}$$

sending $x_1 \ldots x_r, \xi_1 \ldots \xi_s$ into the generators of the maximal ideal of $\widehat{\mathcal{O}_{X,P}}$. Hence we have that $\widehat{\mathcal{O}_{X,P}} = \mathbb{C}[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]/J$ for some ideal $J$. Since $\text{Gr}(\widehat{\mathcal{O}_{X,P}}) = \text{Gr}(\widehat{\mathcal{O}_{X,P}})$ by Lemma [A.5] the result follows by Lemma [A.3] in the Appendix.
4 Smoothness of Supergroups

In this section we want to show that affine algebraic supergroups are smooth, that is all closed points are smooth. In other words we show that the set of closed points of an affine supergroup has a supermanifold structure in the sense of Observation 3.7, hence it is a Lie supergroup. We will do this by using an argument appearing in the classical Cartier’s theorem which states that Hopf algebras over a field of characteristic zero are reduced.

It is enough to prove that the identity is a smooth point, since, because of the multiplication law, all closed points have the same local structure.

Let $G$ be an affine algebraic supergroup, $\mathcal{C}[G]$ its Hopf superalgebra with comultiplication $\Delta$, counit $\epsilon$ and antipode $S$. Let $m_1 = k$ $\epsilon e$ be the maximal ideal of the identity element and let:

$$m_1/m_1^2 = \text{span}_\mathbb{C}\{t_1 \ldots t_{r+s}\}$$

where $t_1 \ldots t_r$ are even, $t_{r+1} \ldots t_{r+s}$ are odd.

By an abuse of notation let $t_1 \ldots t_{r+s}$, denote also the image of these elements modulo $m_1^N$.

**Lemma 4.1.** The monomials $t_1^{n_1} \ldots t_{r+s}^{n_{r+s}}, \sum_{i=1}^{r+s} n_i = N$ form a basis for the superspace $m_1^N/m_1^{N+1}$. (Clearly $n_i = 0, 1$ if $i$ is the index of an odd element, $i = r + 1 \ldots s$).

**Proof.** The proof is the same as in the classical case, we include it here for completeness (for more details see [14] pg. 86). Let $t_1^* \ldots t_{r+s}^*$ be the dual basis of $t_1 \ldots t_{r+s}$. Define the map:

$$d_l : \mathcal{C}[G] = \mathbb{C} + m_1 \rightarrow m_1/m_1^2 \rightarrow \mathbb{C}$$

as $d_l = t_l^* \cdot p$, $l = 1 \ldots r + s$, where $p : \mathcal{C}[G] \rightarrow m_1/m_1^2$ is the natural projection.

Each $d_l$ gives rise to a derivation $D_l : \mathcal{C}[G] \rightarrow \mathcal{C}[G]$ in the following way:

$$D_l(a) = \text{def} \sum a^{(1)} d_l(a^{(2)}), \quad \text{where} \quad \Delta(a) = \sum a^{(1)} \otimes a^{(2)}.$$  

Observe that

$$\epsilon(D_l(a)) = \sum \epsilon(a^{(1)}) d_l(a^{(2)}) = d_l \sum \epsilon(a^{(1)}) a^{(2)} = d_l(a).$$
Hence we have that $D_i(t_j) \equiv \delta_{ij}$ and modulo $m_1$ (since $ker \epsilon = m_1$). Let $P(T_1 \ldots T_{r+s})$ be a homogeneous polynomial of degree $n$ over $C$.

$$D_i(P)(t_1 \ldots t_{r+s}) = \sum D_i(t_j) \frac{\partial P}{\partial T_j}(t_1 \ldots t_{r+s})$$

Since $\frac{\partial P}{\partial T_j}(t_1 \ldots t_{r+s}) \in m_1^{n-1}$ we have $D_i(P) \equiv \frac{\partial P}{\partial T_j}$ modulo $m_1^n$. Now, since if $x \equiv y$ modulo $m_1^n$ it implies $D_i(x) \equiv D_i(y)$ modulo $m_1^{n-1}$, we have that:

$$D_{r+s}^{n_{r+s}} \ldots D_1^{n_1} t_1^{n_1} \ldots t_{r+s}^{n_{r+s}} = n_1! \ldots n_{r+s}! \mod m_1$$

while on all other monomials the composition of $D$’s will give zero. Hence given a relation $P$ in $m_1^{n+1}$ applying the correct sequence of $D$’s one can single out the coefficient of any monomial. 

\[\square\]

**Corollary 4.2.** The identity $1 \in |G|$ is a smooth point.

**Proof.** By the Lemma 4.1 and Lemma A.5 we have that the graded associated ring to $O_{G,1}$ is

$$Gr(O_{1,G}) = C[t_1 \ldots t_r, \theta_1 \ldots \theta_s].$$

This implies by Lemma 3.17 that the identity point is smooth. \[\square\]

**Corollary 4.3.** If $G$ is an affine supergroup, then $G$ is smooth, that is all its closed points are smooth.

**Proof.** Let $h_G$ denote the functor of points of $G$ and $\mu : h_G \times h_G \rightarrow h_G$ the natural transformation corresponding to the group law. Let $g \in |G|$ be a closed point. $g$ can be identified with an element of $h_G(C) \subset h_G(A)$. Hence we can define a natural transformation:

$$l_g : h_G \rightarrow h_G, \quad l_g,A(x) = m_A(g, x), \forall x \in h_G(A)$$

This natural transformation corresponds to an isomorphism of $G$ into itself, hence $O_{G,1} \cong O_{G,g}$, so $g$ is smooth. (For more details on the correspondence between natural transformations between functor of points and morphisms of the supervarieties see [4] Chapter 3). \[\square\]
5 The Stabilizer Theorem

Notation: In this section we use the same letter $X$ to denote both a supervariety $X$ and its functor of points $h_X$.

Let $G$ be an affine algebraic supergroup acting on an affine supervariety $X$, in other words we have a morphism

$$\rho: G \times X \longrightarrow X, \quad (g, x) \mapsto g \cdot x, \quad \forall g \in G(A), x \in X(A)$$

satisfying the usual properties, viewed in the category of supervarieties. Let $u$ be a topological point of $X$, that is $u \in |X|$ or equivalently $u \in X(C) = \text{Hom}(C[X], C)$. Let $m_u$ be the maximal ideal corresponding to $u$. Notice that $u$ can be viewed naturally as an $A$-point $u_A$ for all superalgebras $A$ since $C \subset A$. So we have a morphism:

$$\tau: G \longrightarrow X, \quad g \mapsto g \cdot u_A$$

or equivalently:

$$\tilde{\tau}: C[X] \longrightarrow C[G].$$

**Definition 5.1.** We define the stabilizer supergroup functor of the point $u \in |X|$ with respect to the action $\rho$, the group valued functor $\text{Stab}_u: (\text{salg}) \longrightarrow (\text{sets})$ defined by:

$$\text{Stab}_u(A) = \{g \in G(A) \mid \tau_A(g) = g \cdot u_A = u_A\}$$

where $\tau_A: G(A) \longrightarrow X(A)$, or equivalently:

$$\text{Stab}_u(A) = \{g \in G(A) = \text{Hom}(C[G], A) \mid g \cdot \tilde{\tau} = u_A\}$$

We want to prove that this functor is representable by an affine supergroup.

**Theorem 5.2.** Let $G$ be an affine supergroup acting on an affine supervariety $X$ and let $u$ be a topological point of $X$. Then $\text{Stab}_u$ is an affine supergroup.

**Proof.** The stabilizer can be described in an equivalent way as:

$$\text{Stab}_u(A) = \{g \in G(A) \mid (g \cdot \tau)|_{m_u} = 0\}$$

where $m_u \subset C[X]$ is the ideal of $u$. Let $I$ be the ideal in $C[G]$ generated by $\tilde{\tau}(x)$ for all $x \in m_u$. One can immediately check that $g \in G(A) = \text{Hom}(C[G], A)$ is in $\text{Stab}_u(A)$ if and only if $g$ factors via $C[G]/I$, that is $g: C[G] \longrightarrow C[G]/I \longrightarrow A$. So we have that $\text{Stab}_u(A) = \text{Hom}(C[G]/I, A)$. 

We want to describe some important applications of this result.
6 The classical series of Lie supergroups

In [9], Kac proved a classification theorem for simple Lie superalgebras. The description of the supergroup functors, corresponding to the classical super series of Lie superalgebras introduced by Kac, appeared in [5] pg 70; however no representation statement was proved there.

In this section we want to describe the supergroup functors corresponding to the classical super series and to show they are representable i.e. they are algebraic supergroups, hence Lie supergroups by the results of Section 4. For the series $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$ this result was proved in [13] pg 289 with differential techniques.

One should also prove that the Lie superalgebras of these Lie supergroups coincide with the classical series mentioned above; however this goes beyond the scope of this paper and we leave it as an exercise to the reader.

1. $A(n)$ series. Define $\text{GL}_{m|n}(A)$ as the set of all invertible morphisms $g : A^{m|n} \to A^{m|n}$. This is equivalent to ask that the Berezinian or superdeterminant

\[
\text{Ber}(g) = \text{Ber} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \det(p - qs^{-1}r) \det(s^{-1})
\]

is invertible in $A$ (where $p$ and $s$ are $m \times m$, $n \times n$ matrices of even elements in $A$, while $q$ and $r$ are $m \times n$, $n \times m$ matrices of odd elements in $A$). A necessary and sufficient condition for $g \in \text{GL}_{m|n}(A)$ to be invertible is that $p$ and $s$ are invertible. The group valued functor

\[
\text{GL}_{m|n} : \text{(salg)} \longrightarrow \text{(sets)}
\]

\[
A \longrightarrow \text{GL}_{m|n}(A).
\]

is an affine supergroup called the general linear supergroup and it is represented by the algebra

\[
\mathbb{C}[\text{GL}_{m|n}] := \mathbb{C}[x_{ij}, y_{\alpha \beta}, \xi_{i\beta}, \gamma_{\alpha j}, z, w]/(w \det(x) - 1, z \det(y) - 1),
\]

$i, j = 1, \ldots m, \quad \alpha, \beta = 1, \ldots n.$

Consider the morphism:

\[
\rho : \text{GL}_{m|n} \times \mathbb{C}^{0|1} \longrightarrow \mathbb{C}^{0|1} \quad (g, c) \longrightarrow \text{Ber}(g)c.
\]

\[\text{For the definition of Lie superalgebra of an algebraic supergroup see } [4] \text{ Ch. 5}\]
The stabilizer of the point $1 \in \mathbb{C}^{0|1}$ coincides with all the matrices in $\text{GL}_{m|n}(A)$ with Berezinian equal to 1, that is $\text{SL}_{m|n}(A)$ the special linear supergroup. By the Theorem 5.2 we have immediately that $\text{SL}_{m|n}$ is representable and by the result of Section 4 we have that it is a complex supermanifold. Moreover one can check that $A(m, n) = \text{Lie}(\text{SL}_{m|n})$.

2. $B(m, n)$, $C(n)$, $D(m, n)$ series. Consider the morphism:

$$\rho : \text{GL}_{m|2n} \times \mathcal{B} \longrightarrow \mathcal{B} \quad (g, \psi(\cdot, \cdot)) \longrightarrow \psi(g\cdot, g\cdot),$$

where $\mathcal{B}$ is the supervector space of all the symmetric bilinear forms on $\mathbb{C}^{m|2n}$. The stabilizer of the point $\Phi$ the standard bilinear form on $\mathbb{C}^{m|2n}$ is the supergroup functor $\text{Osp}_{m|2n}$. Again this is an algebraic supergroup by Theorem 5.2 and it is also a complex supermanifold. One can check that $B(m, n) = \text{Lie}(\text{Osp}_{2m+1|2n})$, $C(n) = \text{Lie}(\text{Osp}_{2|2n-2})$ and $D(m, n) = \text{Lie}(\text{Osp}_{2m|2n})$.

3. $P(n)$ series. Define the algebraic supergroup $\pi \text{Sp}_{m|n}$ as we did for $\text{Osp}_{m|n}$, by taking antisymmetric bilinear forms instead of symmetric ones. Consider the action:

$$\pi \text{Sp}_{m|n} \times \mathbb{C}^{1|0} \longrightarrow \mathbb{C}^{1|0} \quad (g, c) \mapsto \text{Ber}(g)c.$$ 

By Theorem 5.2 we have that $\text{Stab}_1$ is an affine algebraic supergroup, hence it is a Lie supergroup. It is corresponding to the $P(n)$ series.

3. $Q(n)$ series. Let $D = \mathbb{C}[\eta]/(\eta^2 + 1)$. This is a non commutative superalgebra. Define the supergroup functor $GL_n(D) : (\text{salg}) \longrightarrow (\text{sets})$, with $GL_n(D)(A)$ the group of automorphisms of the left supermodule $A \otimes D$. In [5] is proven the existence of a morphism called the odd determinant

$$\text{odet} : GL_n(D) \longrightarrow \mathbb{C}^{0|1}.$$ 

Reasoning as before define:

$$GL_n(D) \times \mathbb{C}^{0|1} \longrightarrow \mathbb{C}^{0|1}, \quad g, c \longrightarrow \text{odet}(g)c.$$ 

Then $G = \text{Stab}_1$ is an affine algebraic supergroup and for $n \geq 2$ we define $Qg(n)$ as the quotient of $G$ and the diagonal subgroup $GL_{1|0}$. This is an algebraic and Lie supergroup and its Lie superalgebra is $Q(n)$. 

16
A Appendix: Commutative Superalgebra

In this Appendix we collect some facts about commutative superalgebra very similar to the equivalent facts in commutative algebra.

Let $k$ be the ground field.

All superalgebras are assumed to be commutative. Let’s denote (as before) with latin letter the even elements and with greek letters the odd elements of a superalgebra.

**Theorem A.1.** Let $A$ be a finitely generated superalgebra. Then there exists a unique superalgebra morphism $\phi : k[x_1 \ldots x_m, \xi_1 \ldots \xi_n] \to A$ (where $k[x_1 \ldots x_m, \xi_1 \ldots \xi_n]$ denotes the polynomial superalgebra with even indeterminates $x_i$’s and odd indeterminates $\xi_j$’s) sending the $x_i$’s and the $\xi_j$’s to chosen elements in $A$ of the correct parity.

This comes from the universality of the construction of the polynomial superalgebra as it is done for example in [5] pg 49.

**Theorem A.2.** Let $A$ be a finitely generated superalgebra and let $\hat{A} = \lim\leftarrow A/n_i$ be its completion with respect an ideal $n_i$. Let $\hat{n}$ be the ideal in $\hat{A}$ corresponding to $n$. Then there exist a unique superalgebra morphism $\phi : k[[x_1 \ldots x_m, \xi_1 \ldots \xi_n]] \to \hat{A}$ sending the $x_i$’s and the $\xi_j$’s to chosen elements in $\hat{n}$ of the correct parity.

**Proof.** This is the same as Theorem 7.16 in [6] pg 200. Let’s briefly recall it. By Theorem A.1 we have that there is a unique map $k[x_1 \ldots x_m, \xi_1 \ldots \xi_n] \to \hat{A}/\hat{n}^i$ sending the $x_i$’s and the $\xi_j$’s to chosen elements in $n$. Clearly this maps factors in the following way:

$$k[x_1 \ldots x_m, \xi_1 \ldots \xi_n] \to k[x_1 \ldots x_m, \xi_1 \ldots \xi_n]/(x_1 \ldots x_m, \xi_1 \ldots \xi_n)^i \to \hat{A}/\hat{n}^i.$$

One can check that

$$\frac{k[x_1 \ldots x_m, \xi_1 \ldots \xi_n]}{(x_1 \ldots x_m, \xi_1 \ldots \xi_n)^i} \cong \frac{k[[x_1 \ldots x_m, \xi_1 \ldots \xi_n]]}{(x_1 \ldots x_m, \xi_1 \ldots \xi_n)^i},$$

hence by the universal property of the inverse limit we have obtained the required map and the uniqueness.

If $A$ is a local superring with maximal ideal $m$, let $Gr(A) = \oplus m^i/m^{i+1}$. 

17
Lemma A.3. If
\[ \text{Gr}(k[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]/I) \cong \text{Gr}(k[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]) \]
then \( I = (0) \).

Proof. Let \( m \) be the maximal ideal in \( k[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]] \). There exist \( i \) such that \( I \subset m^i \) but \( I \not\subset m^{i+1} \) otherwise we are done since \( I \subset \cap m^i = (0) \). Then
\[ \frac{m^i/I}{m^{i+1} + I/I} = \frac{m^i/(m^{i+1} + I)}{m^i/m^{i+1}} \]
which gives a contradiction. \( \square \)

Theorem A.4. If
\[ k[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]]/I \cong k[[x_1 \ldots x_r, \xi_1 \ldots \xi_s]] \]
then \( I = (0) \).

Proof. This is a consequence of Lemma [A.3] \( \square \)

Lemma A.5. Let \( A \) be a commutative superalgebra and \( m \) a maximal ideal. Let \( A_m \) be the localization of \( A \) into the even part \( m_0 \) of the maximal ideal \( m \) and \( \hat{A}_m \) the completion of \( A_m \) with respect to the maximal ideal \( \hat{m} \) in \( A_m \). Then:
\[ m^i/m^{i+1} \cong \hat{m}^i/\hat{m}^{i+1} \]

Proof. This is the same as in the commutative case, because localization and completion commute with quotients. \( \square \)

Theorem A.6. Super Nakayama’s Lemma.

Let \( A \) be a local commutative super ring with maximal (homogeneous) ideal \( m \). Let \( E \) be a finitely generated module for the ungraded ring \( A \).

(i) If \( mE = E \), then \( E = 0 \); more generally, if \( H \) is a submodule of \( E \) such that \( E = mE + H \), then \( E = H \).

(ii) Let \((v_i)_{1 \leq i \leq p}\) be a basis for the \( k \)-vector space \( E/mE \) where \( k = A/m \). Let \( e_i \in E \) be above \( v_i \). Then the \( e_i \) generate \( E \). If \( E \) is a supermodule for the super ring \( A \), and \( v_i \) are homogeneous elements of the super vector space \( E/mE \), we can choose the \( e_i \) to be homogeneous too (and hence of the same
parity as the $v_i$).

(iii) Suppose $E$ is projective, i.e. there is a $A$-module $F$ such that $E \oplus F = A^N$ where $A^N$ is the free module for the ungraded ring $A$ of rank $N$. Then $E$ (and hence $F$) is free, and the $e_i$ above form a basis for $E$.

Proof. See [4] Appendix.

References

[1] C. Bartocci, U. Bruzzo, D. Hernandez-Ruiperez. The Geometry of Supermanifolds. MAIA 71, Kluwer Academic Publishers, 1991.

[2] M. Batchelor. Graded manifolds and vector bundles: a functorial correspondence. J. Math. Phys. 26 (1985), no. 7, 1578–1582. Graded manifolds and supermanifolds. Mathematical aspects of superspace (Hamburg, 1983), 91–133, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 132, Reidel, Dordrecht, 1984.

[3] F. A. Berezin, Introduction to superanalysis. Edited by A. A. Kirillov. D. Reidel Publishing Company, Dordrecht (Holland) (1987). With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leîtes.

[4] L. Caston and R. Fioresi, Mathematical Foundation of Supersymmetry. To appear.

[5] P. Deligne and J. Morgan, Notes on supersymmetry (following J. Bernstein), in “Quantum fields and strings. A course for mathematicians”, Vol 1, AMS, 1999.

[6] D. Eisenbud, Commutative Algebra with a view toward Algebraic Geometry Springer Graduate Text in Mathematics, 150, 1994.

[7] C. Gruson, Description de certains super groupes classiques. Ann. Inst. Fourier (Grenoble) 44, no. 1, 39-63, 1994.

[8] R. Hartshorne, Algebraic geometry. Graduate Text In Mathematics. Springer-Verlag, New York, 1977.
[9] V. G. Kac, *Lie superalgebras* Adv. in Math. **26** (1977) 8-26.

[10] B. Kostant, *Graded manifolds, Graded Lie theory and prequantization.* Springer Lecture Notes in Math. **570**, 1977.

[11] D. A. Leites, *Introduction to the theory of supermanifolds.* Russian Math. Survey. **35**:1, 1-64, 1980.

[12] Y. Manin, *Gauge field theory and complex geometry.* Springer Verlag, 1988.

[13] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction.* Courant Lecture Notes,1. AMS, 2004.

[14] W. Waterhouse, *Introduction to affine group schemes.* Graduate Text In Mathematics. Springer-Verlag, New York, 1979.