Abstract. We prove that a countable group with an effective minimal non-
elementary convergence group action is a Powers group. More strongly we
prove that it is a strongly Powers group and thus its non-trivial subnormal
subgroups are $C^*$-simple.

Keywords: convergence group actions; Powers groups; reduced group $C^*$-
algebras; relatively hyperbolic groups.

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1. Introduction

$C^*$-simplicity for countable groups have been studied very much since Powers
proved that non-abelian free groups are $C^*$-simple ([14]), where a countable group
is $C^*$-simple if the reduced $C^*$-algebra of it is simple. For example, it is known that
torsion-free hyperbolic groups which are not cyclic are $C^*$-simple ([9]). Also some
criteria for $C^*$-simplicity for groups are established (refer to [3], [10] and [11]). In
particular Powers groups are $C^*$-simple ([10 Theorem 13]). See [10, Definition 9]
about the definition of Powers groups.

The following is our main theorem:

Theorem 1.1. Let $G$ be a countable group. If $G$ has an effective minimal non-
elementary convergence group action, then $G$ is a Powers group.

On the above the case where $G$ is torsion-free is known ([10 Corollary 12 (iv)]).
See [11 Corollary 12] and [11] for other examples of Powers groups.

We give two corollaries. First one is the following:

Corollary 1.2. Let $G$ be a countable group with a minimal non-elementary con-
vergence group action. Then the following are equivalent:

(i) the action is effective;
(ii) $G$ is a Powers group;
(iii) the reduced $C^*$-algebra of $G$ is simple;
(iv) the reduced $C^*$-algebra of $G$ has a unique normalized trace;
(v) $G$ has infinite conjugacy classes;
(vi) $G$ does not have non-trivial amenable normal subgroups;
(vii) $G$ does not have non-trivial finite normal subgroups;
(viii) all minimal non-elementary convergence group actions of $G$ are effective.

See Section 2 about convergence group actions. Corollary 1.2 implies [1 Corollary
2], which deals with the case of relatively hyperbolic groups in the sense of Osin.
We remark that if a countable group $G$ is a properly relatively hyperbolic group in
the sense of Osin and is not virtually cyclic, then $G$ has a minimal non-elementary convergence group action (see Section 2).

Recall that each group has a unique maximal amenable normal subgroup, which is called its amenable radical. The second corollary is the following:

**Corollary 1.3.** Let $G$ be a countable group with a minimal non-elementary convergence group action. Then the amenable radical $R_a(G)$ is finite and equal to the kernel of the action. Also $G/R_a(G)$ is a strongly Powers group.

Here a group is called a strongly Powers group when its non-trivial subnormal subgroups are Powers groups (see [11, 1. Introduction]).

It follows from Corollaries 1.2 and 1.3 that a countable group with an effective minimal non-elementary convergence group action is a strongly Powers group and thus its non-trivial subnormal subgroups are $C^*$-simple (see Corollary 3.4).

2. Properties of convergence group actions

The study of convergence groups was initiated in [7]. In this section we recall some definitions and properties related to convergence group actions (refer to [15], [6] and [4]).

Let $G$ be a countable group, $X$ be a compact metrizable space and $\rho : G \to \text{Homeo}(X)$ be a homomorphism. The pair $(\rho, X)$ is a convergence group action if $X$ has at least three points, and for any infinite sequence $\{g_i\}$ of mutually different elements of $G$, there exist a subsequence $\{g_{i_j}\}$ of $\{g_i\}$ and two points $r, a \in X$ such that $\rho(g_{i_j})|_{X \setminus \{r\}}$ converges to $a$ uniformly on compact subsets of $X \setminus \{r\}$ and also $\rho(g_{i_j}^{-1})|_{X \setminus \{a\}}$ converges to $r$ uniformly on compact subsets of $X \setminus \{a\}$. The sequence $\{g_{i_j}\}$ is called a convergence sequence of $(\rho, X)$ and also the points $r$ and $a$ are called the repelling point of $\{g_{i_j}\}$ and the attracting point of $\{g_{i_j}\}$, respectively. When we consider a convergence group action $(\rho, X)$, for $l \in G$, we call $\rho(l)$ a loxodromic element if $\rho(l)$ is of infinite order and has exactly two fixed points. The sequence $\{l^i\}_{i \in \mathbb{N}}$ is a convergence sequence of $(\rho, X)$ with the repelling point $r$ and the attracting point $a$, which are distinct and fixed by $\rho(l)$. Hence we call $r$ the repelling fixed point of $\rho(l)$ and $a$ the attracting fixed point of $\rho(l)$.

Let $(\rho, X)$ be a convergence group action of a countable group $G$. Since $X$ has at least three points, $\text{Ker}\rho$ has no convergence sequences and thus $\text{Ker}\rho$ is finite. The set of all repelling points and attracting points is equal to the limit set $\Lambda(\rho)$ ([15, Lemma 2M]). The cardinality of $\Lambda(\rho)$ is 0, 1, 2 or $\infty$ ([15, Theorem 2S, Theorem 2T]). We remark that $\#\Lambda(\rho) = 0$ if $G$ is finite by definition. Also it is well-known that $G$ is virtually infinite cyclic if $\#\Lambda(\rho) = 2$ (see [15, Lemma 2Q, Lemma 2N, Theorem 2I] and also [6, Example 1.3]). We call $(\rho, X)$ a non-elementary convergence group action if $\#\Lambda(\rho) = \infty$. We note that if a countable group $G$ has a non-elementary convergence group action, then the induced action on the limit set is a minimal non-elementary convergence group action.

We briefly recall the notion of relatively hyperbolic group, which was introduced in [8]. Let $G$ be a countable group with a finite family of infinite subgroups $\mathbb{H}$. Assume that $G$ is not virtually cyclic for simplicity. We call $(G, \mathbb{H})$ a properly relatively hyperbolic group if there exists a geometrically finite convergence group action $(\rho, X)$ such that $\mathbb{H}$ is a set of representatives of conjugacy classes of maximal parabolic subgroups (refer to [5, Definition 1], [16, Theorem 0.1] and [12, Definition 3.1] for details). We remark that geometrically finite convergence group actions
are minimal and non-elementary. Also we note that several definitions of relative hyperbolicity for a pair \((G, \mathbb{H})\) of a countable group and a family of subgroups are mutually equivalent if \(\mathbb{H}\) is finite and each \(H \in \mathbb{H}\) is infinite (refer to [12]).

**Remark 2.1.** If we consider a relatively hyperbolic group \((G, \mathbb{H})\) in the sense of Osin ([13, Definition 2.35]), then \(\mathbb{H}\) can be infinite and each \(H \in \mathbb{H}\) can be finite. However even if \(G\) is hyperbolic relative to an infinite family \(\mathbb{H}\) in the sense of Osin, then \(G\) can be realized as a free product of two infinite subgroups \(A\) and \(B\) ([12, Theorem 2.44]), and thus \(G\) is hyperbolic relative to \(\{A, B\}\), which is a finite family of infinite subgroups.

**Remark 2.2.** We do not know whether there exists a countable group \(G\) such that it is not hyperbolic relative to any family of proper subgroups in the sense of Osin, but it has a non-elementary convergence group action.

We need two facts in order to prove Corollary 1.3. First one is claimed in [6, Section 1].

**Proposition 2.3.** Let \(G\) be a countable group with a minimal non-elementary convergence group action \((\rho, X)\). Then \(\text{Ker}\rho\) is the maximal finite normal subgroup of \(G\).

We give its proof for readers in Appendix A. Second one is [6, Proposition 3.1]. We give a proof for readers here.

**Proposition 2.4.** Let \(G\) be a countable group with a convergence group action \((\rho, X)\). Let \(N\) be an infinite normal subgroup of \(G\). We consider the restricted convergence group action \((\rho|_N, X)\). Then we have \(\Lambda(\rho|_N) = \Lambda(\rho)\). In particular if \((\rho, X)\) is non-elementary, then \((\rho|_N, X)\) is also non-elementary.

**Proof.** We remark that \(\Lambda(\rho|_N)\) is not empty since \(N\) is infinite. Also obviously we have \(\Lambda(\rho|_N) \subseteq \Lambda(\rho)\). Hence we have \(\Lambda(\rho|_N) = \Lambda(\rho)\) if \(\#\Lambda(\rho) = 1\).

If \(\#\Lambda(\rho) = 2\), then \(G\) is virtually infinite cyclic. Then \(N\) is also virtually infinite cyclic. When we take an element \(n \in N\) of infinite order, \(\rho(n)\) is loxodromic and fixes \(\Lambda(\rho)\) ([15, Lemma 2Q, Theorem 2G]). Thus we have \(\Lambda(\rho|_N) \supset \Lambda(\rho)\).

Suppose that \((\rho, X)\) is non-elementary. Moreover we can assume that it is minimal without loss of generality. We take a convergence sequence \(\{n_i\}_{i \in \mathbb{N}}\) of \((\rho|_N, X)\) with the attracting point \(a \in X\) and the repelling point \(r \in X\). Then for any \(g \in G\), the infinite sequence \(\{gn_i g^{-1}\}_{i \in \mathbb{N}}\) is a convergence sequence of \((\rho, X)\) with the attracting point \(\rho(g)a\in X\) and the repelling point \(\rho(g)r \in X\). The infinite sequence \(\{gn_i g^{-1}\}_{i \in \mathbb{N}}\) can be regarded as a convergence sequence of \((\rho|_N, X)\) since \(N\) is normal. In particular we have \(\rho(G)a \subseteq \Lambda(\rho|_N)\). Moreover the closure \(\overline{\rho(G)a}\) of \(\rho(G)a\) in \(X\) is contained in \(\Lambda(\rho|_N)\) since \(\Lambda(\rho|_N)\) is closed. Since \(\Lambda(\rho) = \rho(G)a\) ([15, Theorem 2S]), we have \(\Lambda(\rho|_N) \supset \Lambda(\rho)\). \(\Box\)

### 3. Proofs of results

In this section we prove Theorem 1.1 and Corollaries 1.2 and 1.3.

We show the following in order to prove Theorem 1.1.

**Proposition 3.1.** Let \(G\) be a countable group with a minimal non-elementary convergence group action \((\rho, X)\). Then \((\rho, X)\) is effective if and only if there exists a point \(x \in X\) such that \(\rho(g)x \neq x\) for any element \(g \in G \setminus \{1\}\).
This claims that effectiveness of a minimal non-elementary convergence group action can be detected by some single point.

The following is a key lemma for Proof of Proposition 3.1.

Lemma 3.2. Let $G$ be a countable group with a minimal non-elementary convergence group action $(\rho, X)$. Then for any element $g \in G$ such that $\rho(g)$ is not the identity map, the fixed point set $\text{Fix}(\rho(g)) \subset X$ has an empty interior.

Proof. We take an element $g \in G$ such that $\rho(g)$ is not the identity map. We put $U := X \setminus \text{Fix}(\rho(g))$, which is clearly open and not empty. We assume that the interior $V$ of $\text{Fix}(\rho(g))$ is not empty. Since $U$ and $V$ are mutually disjoint open sets which are not empty, we have an element $l \in G$ such that $\rho(l)$ is loxodromic, a fixed point $r$ of $\rho(l)$ is in $U$ and the other fixed point $a$ of $\rho(l)$ is in $V$ ([14, Theorem 2R]). Then $\rho(glg^{-1})$ is loxodromic and satisfies $\text{Fix}(\rho(glg^{-1})) = \{\rho(g)r, \rho(g)a\}$. Since $r \in U$ and $a \in V$, we have $\rho(g)r \neq r$ and $\rho(g)a = a$. This contradicts the fact that the fixed point sets of two loxodromic elements either coincide or have an empty intersection ([15, Theorem 2G]). □

Proof of Proposition 3.1. The ‘if’ part is trivial. We prove the ‘only if’ part. Suppose that $(\rho, X)$ is effective. Then for each $g \in G \setminus \{1\}$, $\rho(g)$ is not the identity map. Hence the interior $\text{Int}(\text{Fix}(\rho(g)))$ is empty by Lemma 3.2. Then $\text{Int} \left( \bigcup_{g \in G \setminus \{1\}} \text{Fix}(\rho(g)) \right)$ is also empty since $X$ is a Baire space. Thus $\bigcup_{g \in G \setminus \{1\}} \text{Fix}(\rho(g))$ must be a proper subset of $X$ because $X$ is not empty. □

Remark 3.3. Since Lemma 3.2 claims that an effective minimal non-elementary convergence group action is ‘slender’ (see the paragraph just before [11 Corollary 10]), the argument of the proof of Proposition 3.1 is parallel to an observation in the proof of [11 Corollary 10].

Proof of Theorem 1.1. Suppose that $G$ has an effective minimal non-elementary convergence group action $(\rho, X)$. There exist two elements $l_1, l_2 \in G$ such that $\rho(l_1)$ and $\rho(l_2)$ are loxodromic and have no common fixed points ([15, Theorem 2T]). Since $\{l_1\}_{i \in \mathbb{N}}$ and $\{l_2\}_{i \in \mathbb{N}}$ are convergence sequences, then $\rho(l_1)$ and $\rho(l_2)$ are hyperbolic in the sense of [10, Definition 10]. Therefore $(\rho, X)$ is strongly hyperbolic in the sense of [10, Definition 10]. Also for any finite subset $F \subset G \setminus \{1\}$, there exists $x \in X$ such that $\rho(f)x \neq x$ for all $f \in F$ by Proposition 3.1. Therefore $(\rho, X)$ is strongly faithful in the sense of [10, Definition 10]. Hence $G$ is a Powers group by [10 Proposition 11 and the following remark]. □

Proof of Corollary 1.2. Clearly (viii) implies (i). Also Theorem 1.1 claims that (i) implies (ii).

For general countable groups, the following relations hold among properties (ii), (iii), (iv), (v), (vi), (vii) and (viii): It is well-known that (ii) implies both (iii) and (iv) (see [10, Theorem 13 and Remark (i) on it]); each of (iii) and (iv) implies both (v) and (vi) (see [3, Proposition 3]); each of (v) and (vi) obviously implies (vii); (vii) implies (viii) by definition of convergence group actions (see Section 2). □

We have the following:

Corollary 3.4. Let $G$ be a countable group with an effective minimal non-elementary convergence group action $(\rho, X)$. Then for every non-trivial subnormal subgroup $N$ of $G$, the restricted action $(\rho|_N, X)$ is an effective minimal non-elementary convergence group action. In particular $G$ is a strongly Powers group.
Proof: We take a non-trivial subnormal subgroup $N$ of $G$. There exists a finite chain of subgroups $N = N_k < N_{k-1} < \cdots < N_0 = G$ such that $N_j$ is normal in $N_{j-1}$ for any $j = 1, \ldots, k$. We suppose that $G$ has an effective minimal nonelementary convergence group action $(\rho, X)$. Then $N_1$ is infinite by Proposition 2.3 and thus the restricted action $(\rho|_{N_1}, X)$ is an effective minimal nonelementary convergence group action by Proposition 2.4. By induction on $j$, the restricted action $(\rho|_N, X)$ is an effective minimal non-elementary convergence group action. Hence $N$ is a Powers group by Theorem 1.1.

Proof of Corollary 1.3. Suppose that $G$ has a minimal non-elementary convergence group action $(\rho, X)$. $K\rho$ is the maximal finite normal subgroup of $G$ by Proposition 2.3. Now we prove that the amenable radical $R_a(G)$ is finite. We assume that $G$ has an infinite amenable normal subgroup $N$. Since $N$ does not contain any nonabelian free subgroups, the restricted convergence group action $\rho|_N$ is elementary ([15, Theorem 2U]). This contradicts Proposition 2.4.

Since $K\rho$ is equal to $R_a(G)$, the quotient $G/R_a(G)$ has the induced effective minimal non-elementary convergence group action $(\tilde{\rho}, X)$. Hence Corollary 3.4 can be applied to $G/R_a(G)$.

Appendix A. Proof of Proposition 2.3

In this appendix we prove Proposition 2.3. In fact we prove Proposition A.1 (compare with [2, Lemma 3.3] for the case of relatively hyperbolic groups).

Let $G$ be a countable group with a convergence group action $(\rho, X)$ and $l$ be an element of $G$ such that $\rho(l)$ is loxodromic. We put $E^+_\rho(l) := \text{Stab}_\rho(r) \cap \text{Stab}_\rho(a)$ and $E_\rho(l) := \text{Stab}_\rho(\{r, a\})$, where $\text{Stab}_\rho(r)$, $\text{Stab}_\rho(a)$ and $\text{Stab}_\rho(\{r, a\})$ are the stabilizer of subsets $\{r\}$, $\{a\}$ and $\{r, a\}$ in $G$, respectively. Also we define $E^+_\rho(G)$ (resp. $E_\rho(G)$) by the intersection of all the sets in the family $\{E^+_\rho(l) | l \in G, \rho(l) \text{ is loxodromic}\}$ (resp. $\{E_\rho(l) | l \in G, \rho(l) \text{ is loxodromic}\}$).

Proposition A.1. Let $G$ be a countable group with a non-elementary convergence group action. If $(\rho, X)$ is a minimal non-elementary convergence group action, then $K\rho$ is the maximal finite normal subgroup of $G$ and equal to both $E^+_\rho(G)$ and $E_\rho(G)$.

In order to prove the above, we need some lemmas.

Lemma A.2. Let $G$ be a countable group with a non-elementary convergence group action $(\rho, X)$. Then $E^+_\rho(G)$ is a finite normal subgroup of $G$.

Proof. For any element $l \in G$ such that $\rho(l)$ is loxodromic and for any element $h \in G$, $\rho(hlh^{-1})$ is loxodromic. Clearly we have $E^+_\rho(hlh^{-1}) = hE^+_\rho(l)h^{-1}$.

We can take two elements $l_1, l_2 \in G$ such that $\rho(l_1)$ and $\rho(l_2)$ are loxodromic and have no common fixed points ([15, Theorem 2R]). Then $E^+_\rho(l_1) \cap E^+_\rho(l_2)$ is finite because $E^+_\rho(l_1)$ and $E^+_\rho(l_2)$ have no common elements of infinite order ([15, Theorem 2G]) and they are virtually infinite cyclic by [15, Theorem 2L].

We can prove the following in the same way as the proof of Lemma A.2.

Lemma A.3. Let $G$ be a countable group with a non-elementary convergence group action $(\rho, X)$. Then $E_\rho(G)$ is a finite normal subgroup of $G$. 
Lemma A.4. Let $G$ be a countable group with a non-elementary convergence group action $(\rho, X)$, $l \in G$ be an element such that $\rho(l)$ is loxodromic and $g$ be an element of $G$. If there exists a positive integer $n$ such that $gl^n g^{-1} = l^n$, then $g$ is an element of $E^+_\rho(l)$.

Proof. We take the repelling fixed point $r$ of $\rho(l)$ and the attracting fixed point $a$ of $\rho(l)$. Then $\rho(gl^n g^{-1})$ is a loxodromic element with the repelling fixed point $\rho(g)r$ and the attracting fixed point $\rho(g)a$. $gl^n g^{-1} = l^n$ implies that $\rho(g)r = r$ and $\rho(g)a = a$. \hfill \Box

Lemma A.5. Let $G$ be a countable group with a non-elementary convergence group action $(\rho, X)$. Then any finite normal subgroup $M$ of $G$ is contained in $E^+_\rho(G)$.

Proof. When we consider a finite normal subgroup $M$ of $G$, we have $[G : C_G(M)] < \infty$, where $C_G(M)$ is the centralizer of $M$ in $G$. Hence for any element $l \in G$, there exists a positive integer $n$ such that $l^n \in C_G(M)$, that is, $ml^m m^{-1} = l^n$ for any $m \in M$. Thus we have $m \in E^+_\rho(l)$ by Lemma A.4. \hfill \Box

Proof of Proposition A.4. $E^+_\rho(G)$ is the maximal finite normal subgroup by Lemma A.2 and Lemma A.5.

Since $E^+_\rho(G)$ is a finite normal subgroup of $G$ by Lemma A.3, we have $E^+_\rho(G) \subset E^+_\rho(G)$. Also we have $E^+_\rho(G) \supset E^+_\rho(G)$ by definition.

We take an element $l \in G$ such that $\rho(l)$ is loxodromic and the repelling fixed point $r \in X$ of $\rho(l)$. Then for any $g \in E^+_\rho(G)$, $\rho(g)$ fixes every point of the orbit $\rho(G)r$. Since $(\rho, X)$ is non-elementary and minimal, $\rho(g)$ fixes every point of $X$ by [15, Theorem 2S]. Thus we have $\ker \rho \supset E^+_\rho(G)$. Also we have $\ker \rho \subset E^+_\rho(G)$ by definition. \hfill \Box

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References

[1] G. Arzhantseva; A. Minasyan, Relatively hyperbolic groups are $C^*$-simple. J. Funct. Anal. 243 (2007), no. 1, 345–351.
[2] G. Arzhantseva; A. Minasyan; D. Osin, The SQ-universality and residual properties of relatively hyperbolic groups. J. Algebra 315 (2007), no. 1, 165–177.
[3] M. Bachir Bekka; Pierre de la Harpe, Groups with simple reduced $C^*$-algebras. Expo. Math. 18 (2000), no. 3, 215–230.
[4] B. H. Bowditch, Convergence groups and configuration spaces. Geometric group theory down under (Canberra, 1996), 23–54, de Gruyter, Berlin, 1999.
[5] B. H. Bowditch, Relatively hyperbolic groups. Preprint 1997, University of Southampton. [http://www.maths.soton.ac.uk/pure/preprints.php]
[6] Eric M. Freden, Properties of convergence groups and spaces. Conform. Geom. Dyn. 1 (1997), 13–23 (electronic).
[7] F. W. Gehring; G. J. Martin, Discrete quasiconformal groups. I. Proc. London Math. Soc. (3) 55 (1987), no. 2, 331–358.
[8] M. Gromov, Hyperbolic groups, Essays in group theory (S. Gersten, ed.), 75–263, MSRI Publications 8, Springer-Verlag, 1987.
[9] Pierre de la Harpe, Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint. (French) [Hyperbolic groups, operator algebras and Jolissaint’s theorem] C. R. Acad. Sci. Paris Ser. I Math. 307 (1988), no. 14, 771–774.
[10] Pierre de la Harpe, On simplicity of reduced $C^*$-algebras of groups. Bull. Lond. Math. Soc. 39 (2007), no. 1, 1–26.
[11] Pierre de la Harpe; Jean-Philippe Préaux, $C^*$-simple groups: amalgamated free products, HNN extensions, and fundamental groups of 3-manifolds, arXiv:0909.3528v2
[12] G Christopher Hruska, Relative hyperbolicity and relative quasiconvexity for countable groups. Algebr. Geom. Topol. 10 (2010), no. 3, 1807–1856.
[13] Denis V. Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. Mem. Amer. Math. Soc. 179 (2006), no. 843
[14] Robert T. Powers, Simplicity of the $C^*$-algebra associated with the free group on two generators. Duke Math. J. 42 (1975), 151–156.
[15] Pekka Tukia, Convergence groups and Gromov’s metric hyperbolic spaces. New Zealand J. Math. 23 (1994), no. 2, 157–187.
[16] Asli Yaman, A topological characterization of relatively hyperbolic groups. (English summary) J. Reine Angew. Math. 566 (2004), 41–89.

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