EELLS-SAMPSON TYPE THEOREMS FOR SUBELLIPTIC HARMONIC MAPS FROM SUB-RIEMANNIAN MANIFOLDS*

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Abstract. In this paper, we consider critical maps of a horizontal energy functional for maps from a sub-Riemannian manifold to a Riemannian manifold. These critical maps are referred to as subelliptic harmonic maps. In terms of the subelliptic harmonic map heat flow, we investigate the existence problem for subelliptic harmonic maps. Under the assumption that the target Riemannian manifold has non-positive sectional curvature, we prove some Eells-Sampson type existence results for this flow when the source manifold is either a step-2 sub-Riemannian manifold or a step-\(r\) sub-Riemannian manifold whose sub-Riemannian structure comes from a tense Riemannian foliation. Finally, some Hartman type results are also established for the flow.

Introduction

Sub-Riemannian geometry is a natural generalization of Riemannian geometry, whose birth dates back to Carathéodory’s 1909 seminal paper on the foundations of Carnot thermodynamics. Geometric analysis on sub-Riemannian manifolds has been received much attention during the past decades (cf. [BBS1,2]). By a sub-Riemannian manifold we mean a triple \((M, H, g_H)\), where \(M\) is a connected smooth manifold, \(H\) is a subbundle of \(TM\), and \(g_H\) is a smooth fiberwise metric on \(H\). The subbundle \(H\) is usually assumed to have the bracket generating property for \(TM\). More precisely, one may introduce a generating order for the sub-Riemannian manifold, that is, \(M\) is called a step-\(r\) sub-Riemannian manifold if sections of \(H\) together with their Lie brackets up to order \(r\) span \(T_x M\) at each point \(x\) (see §1 for the detailed definition). This is a remarkable property, which makes both the geometry and analysis on sub-Riemannian manifolds more interesting and rich.

The present paper is devoted to the study of a natural counterpart of harmonic maps in the realm of sub-Riemannian geometry. A smooth map \(f : (M, H, g_H) \to (N, h)\) from a sub-Riemannian manifold with a smooth measure \(d\mu\) to a Riemannian manifold is called a subelliptic harmonic map if it is a critical map of the following energy functional

\[
E_H(f) = \frac{1}{2} \int_M |df_H|^2 d\mu,
\]

where \(df_H\) is the restriction of \(df\) to \(H\). To make the above geometric variational problem manageable, we will restrict our attention in this paper to a relative simple case that the source

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sub-Riemannian manifold is endowed with a Riemannian extension \(g\) of \(g_H\), and \(d\mu = dv_g\) (the Riemannian volume measure). We will find that the Euler-Lagrange-equations of the functional (0.1) is a nonlinear subelliptic system of partial differential equations (see §4 for its concrete expression)

\[
\tau_H(f) = 0,
\]

which justifies the terminology for the critical map of \(E_H\). The principal part in (0.2) is actually the sub-Laplacian \(\Delta_H\), which is a hypoelliptic operator.

Recall that Jost-Xu [JX] first introduced subelliptic harmonic maps associated with a Hörmander system of vector fields on a domain of \(R^n\) into Riemannian manifolds, and obtained an existence and regularity theorem for these subelliptic maps under Dirichlet condition and the same convexity condition of [HKW] on the images. A related uniqueness result for subelliptic harmonic maps in the sense of [JX] was given later by [Zh1]. As a global formulation of Jost-Xu’s subelliptic harmonic maps, E. Barletta at al. introduced subelliptic harmonic maps from strictly pseudoconvex CR manifolds into Riemannian manifolds, which were referred to as pseudoharmonic maps in [BDU]; see also [DP] and [Zh2] for some discussions on subelliptic harmonic maps from almost contact Riemannian manifolds and sub-Riemannian manifolds respectively. On the other hand, Wang [Wa] established some regularity results for subelliptic harmonic maps from Carnot groups, see also [HS], [ZF] for some regularity results of subelliptic \(p\)-harmonic maps.

In the theory of harmonic maps, the Eells-Sampson theorem is a fundamental theorem which has many essential applications in Riemannian and Kählerian geometry (cf. [JY], [Tol]). It therefore seems natural and important to generalize this theorem to the case of subelliptic harmonic maps from sub-Riemannian manifolds. Note that step-1 sub-Riemannian manifolds are just Riemannian manifolds. The simplest non-trivial sub-Riemannian manifolds are step-2 sub-Riemannian manifolds, which includes strictly pseudoconvex CR manifolds, contact metric manifolds, quaternionic contact manifolds, or more general Heisenberg manifolds, etc. (cf. [CC]). In [ChC], S. Chang and T. Chang gave an Eells-Sampson type result for pseudoharmonic maps from compact strictly pseudoconvex CR manifolds to compact Riemannian manifolds with nonpositive curvature under an additional analytic condition \(\Delta_H, \xi = 0\), where \(\Delta_H\) and \(\xi\) are respectively the sub-Laplacian and Reeb vector field of the source CR manifolds. Later, Y. Ren and G. Yang [RY] obtained a general Eells-Sampson type result for pseduoharmonic maps without Chang-Chang’s condition. The main purpose in this paper is to establish Eells-Sampson type theorems for subelliptic harmonic maps from more general sub-Riemannian manifolds. Therefore we will investigate the following subelliptic harmonic map heat flow

\[
\begin{cases}
\frac{\partial f}{\partial t} = \tau_H(f) \\
f|_{t=0} = \varphi
\end{cases}
\]

for any given map \(\varphi : (M, H, g_H; g) \rightarrow (N, h)\). Our main results include the short-time, long-time and homotopy existence theorems for (0.3).

The paper is organized as follows. In §1 and §2, we collect some basic notions and results about sub-Riemannian manifolds and hypoelliptic PDEs from the literature. In §3, we first give the structure equations of the generalized Bott connection \(\nabla_B\) on a sub-Riemannian manifold \((M, H, g_H; g)\); and then introduce the second fundamental form of a
map \( f : (M, H, g_H; g) \rightarrow (N, h) \) with respect to the generalized Bott connection on the source manifold and the Levi-Civita connection on the target manifold. Using the moving frame method, we are able to deduce some commutation relations for the derivatives of the second fundamental form and thus some Bochner type formulas for the map. In §4, we first give the Euler-Lagrange-equations (0.2) in terms of the second fundamental form of a map. Next, by means of the Nash embedding of the target manifolds, we derive the explicit formulations for both (0.2) and (0.3). §5 is devoted to existence problems. Using the heat kernel associated with \( \triangle_H - \partial_t \) and the Duhamel’s principle, we may establish a short time existence of (0.2) for any initial map from a compact sub-Riemannian manifold to a compact Riemannian manifold. When \( N \) has nonpositive curvature, we have the following long-time existence.

**Theorem A.** Let \( (M, H, g_H; g) \) be a compact sub-Riemannian manifold and let \( (N, h) \) be a compact Riemannian manifold with nonpositive sectional curvature. Then for any smooth map \( \varphi : M \rightarrow N \), the subelliptic harmonic map heat flow (0.3) admits a global smooth solution \( f : M \times [0, \infty) \rightarrow N \).

Under the nonpositive curvature condition on \( N \), the above theorem shows that the flow (0.3) does not blow up at any finite time. Furthermore, in order to establish Eells-Sampson type results for (0.3), one needs to have a uniform upper bound for the energy density \( e(f(\cdot, t)) \) of the solution \( f(\cdot, t) \) for (0.3). We are able to give these uniform upper bounds in the following two cases: the source manifolds are either step-2 sub-Riemannian manifolds or step-\( r \) sub-Riemannian manifolds whose sub-Riemannian structures come from some Riemannian foliations. For both these cases, we have the Eells-Sampson type results, which assert that there exists a sequence \( t_i \rightarrow \infty \) such that \( f(x, t_i) \rightarrow f_\infty(x) \) uniformly, as \( t_i \rightarrow \infty \), to a \( C^\infty \) subelliptic harmonic map \( f_\infty : M \rightarrow N \). In §6, we establish Hartman type results for the subelliptic harmonic map heat flow. Combining the Eells-Sampson and Hartman type results, we have the following result for the first case.

**Theorem B.** Let \( (M, H, g_H; g) \) be a compact step-2 sub-Riemannian manifold and let \( N \) be a compact Riemannian manifold with non-positive sectional curvature. Then the subelliptic harmonic map heat flow (0.3) exists for all \( t \in [0, \infty) \) and converges uniformly to a subelliptic harmonic map \( f_\infty \) as \( t \rightarrow \infty \). In particular, any map \( \varphi \in C^\infty(M, N) \) is homotopic to a \( C^\infty \) subelliptic harmonic map.

Riemannian foliations provide an important source of sub-Riemannian manifolds. For a Riemannian foliation \( (M, g; F) \) with a bundle-like metric \( g \), let \( H = (T_F)^\perp \) (the horizontal subbundle of the foliation \( F \) with respect to \( g \)) and \( g_H \) be the restriction of \( g \) to \( H \). Then we have a sub-Riemannian manifold \( (M, H, g_H; g) \) corresponding to \( (M, g; F) \). The Riemannian foliation \( (M, g; F) \) will be said to be tense if the mean vector field of \( F \) is parallel with respect to the Bott connection. This is the second case in which we establish an Eells-Sampson type result. Consequently we have

**Theorem C.** Let \( (M, H, g_H; g) \) be a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation with the property that \( H \) is bracket generating for \( TM \). Let \( N \) be a compact Riemannian manifold with non-positive sectional curvature. Then the subelliptic harmonic map heat flow (0.3) exists for all \( t \in [0, \infty) \) and converges uniformly to a subelliptic harmonic map \( f_\infty \) as \( t \rightarrow \infty \). In particular, any map \( \varphi \in C^\infty(M, N) \) is homotopic to a \( C^\infty \) subelliptic harmonic map.
Hopefully these existence results will be useful for studying either step-2 sub-Riemannian manifolds, such as contact and quaternionic contact manifolds, or tense Riemannian foliations with bracket generating horizontal subbundles. Besides their possible geometric applications, we believe that it is reasonable to investigate first the formulation for subelliptic harmonic maps considered in this paper before studying more general formulations, such as taking arbitrary smooth measures on the source sub-Riemannian manifolds.

1. Sub-Riemannian geometry

Let $M$ be a connected $(m + d)$-dimensional manifold of class $C^\infty$ and let $H$ be a rank $m$ subbundle of the tangent bundle $TM$. We say that $H$ satisfies the bracket generating condition if vector fields which are sections of $H$ together with all their brackets span $T_xM$ at each point $x$. More precisely, for any $x \in M$ and any open neighborhood $U$ of $x$, we let $\Gamma(U, H)$ denote the space of smooth sections of $H$ on $U$, and define $\{\Gamma^j(U, H)\}_{j \geq 1}$ inductively by $\Gamma^{j+1}(U, H) = \Gamma^j(U, H) + [\Gamma^1(U, H), \Gamma^j(U, H)]$ for each positive integer $j$, where $\Gamma^1(U, H) = \Gamma(U, H)$. Here $[,]$ denotes the Lie bracket of vector fields. By evaluating $\Gamma^j(U, H)$ at $x$, we have a subspace $H_x^{(j)}$ of the tangent space $T_xM$, that is,

$$H_x^{(j)} = \{X(x) : X \in \Gamma^j(U, H)\}. \tag{1.1}$$

According to [St], [Mon], $H$ is said to be $r$-step bracket generating for $TM$ if $H_x^{(r)} = T_xM$ for each $x \in M$.

A sub-Riemannian manifold is a triple $(M, H, g_H)$, where $g_H$ is a fiberwise metric on the subbundle $H$. When $H$ is 1-step bracket generating, that is, $H = TM$, the sub-Riemannian manifold is just a Riemannian manifold. Henceforth we will always assume that $H$ satisfies the $r$-step bracket generating condition for some $r \geq 2$. For a sub-Riemannian manifold, the subbundle $H$ is also referred to as a horizontal distribution. We say that a Lipschitz curve $\gamma : [0, \delta] \to M$ is horizontal if $\gamma'(t) \in H_{\gamma(t)}$ a.e. in $[0, \delta]$. The sub-Riemannian metric $g_H$ induces a natural structure of metric space, where the distance is the so-called Carnot–Carathéodory distance

$$d_{CC}(x_0, x_1) = \inf \left\{ \int_0^\delta \sqrt{g_H(\gamma'(t), \gamma'(t))} \, dt \mid \gamma : [0, \delta] \to M \text{ is a horizontal curve,} \right. \\
\gamma(0) = x_0, \gamma(\delta) = x_1 \right\}. \tag{1.2}$$

By the theorem of Chow-Rashevsky ([Ch], [Ra]), there always exist such curves joining $x_0$ and $x_1$, so the distance is finite and continuous, and induces on $M$ the original topology. It turns out that the distance $d_{CC}$ plays an essential role in geometric analysis on sub-Riemannian manifolds. According to this distance, we have a corresponding family of balls on $M$ given by

$$B_{CC}(x, \delta) = \{y \in M \mid d_{CC}(x, y) < \delta\}. \tag{1.3}$$

These balls not only determine the metric topological properties of $(M, d_{CC})$, but also reflect the non-isotropic feature of the sub-Riemannian structure (cf. [NSW]).

One difficulty in sub-Riemannian geometry is the absence of a canonical measure such as the Riemannian volume measure. Whenever $M^{m+d}$ is endowed with a Riemannian metric $g$,
we can compute the volume of the $d_{CC}$-balls. One of the main results in [NSW] is an estimate of the volume of these balls. To describe this result, we choose a local frame field \( \{e_1, \ldots, e_m\} \) of \( H \) on a connected open subset \( U \subset M \). Let

\[
E^{(1)} = \{e_1, \ldots, e_m\}, \\
E^{(2)} = \{[e_1, e_2], \ldots, [e_1, e_m], \ldots, [e_{m-1}, e_m]\},
\]

so that the components of \( E^{(l)} \) are the commutators of length \( l \). Clearly \( E^{(1)}, \ldots, E^{(l)} \) span \( H^{(l)} \) at each point of \( U \) (\( 1 \leq l \leq r \)). Consequently, by the assumption for \( H \), we see that \( E^{(1)}, \ldots, E^{(r)} \) span \( TM \) at each point of \( U \). Let \( Y_1, \ldots, Y_q \) be some enumeration of the components of \( E^{(1)}, \ldots, E^{(r)} \). A degree is assigned to each \( Y_i \), namely the corresponding length of the commutator. For each \( (m + d) \)-tuple of integers \( I = (i_1, \ldots, i_{m+d}) \) with \( 1 \leq i_j \leq q \), following [NSW], one defines

\[
d(I) = \sum_{A=1}^{m+d} \deg(Y_{i_A}) \quad \text{and} \quad a_I(x) = |Y_{i_1}(x) \wedge \cdots \wedge Y_{i_{m+d}}(x)|, \quad x \in U.
\]

The Nagel-Stein-Wainger polynomial is defined by

\[
\Lambda(x, r) = \sum_I a_I(x) r^{d(I)}, \quad r > 0,
\]

where the sum is over all \( (m + d) \)-tuples.

**Theorem 1.1.** (cf. [NSW]) Let \( \{e_i\}_{i=1}^m \) be a local frame field of \( H \) on an open subset \( U \) of \( (M, g) \). Then, for every open subset \( V \) of \( U \) such that \( \overline{V} \subset U \) is compact, there exist constants \( 0 < C, R_0 < 1 \), such that for any \( x \in V \), and \( 0 < r \leq R_0 \), one has

\[
CA(x, r) \leq \text{vol}_g(B_{CC}(x, r)) \leq C^{-1} \Lambda(x, r).
\]

To describe the local growth order of \( \text{vol}_g(B_{CC}(x, r)) \), we let

\[
Q(x) = \inf\{d(I) \mid a_I(x) \neq 0\}, \quad Q = \sup\{d(I) \mid |a_I(x)| \neq 0, \ x \in V\}.
\]

According to [Ga], the numbers \( Q(x) \) and \( Q \) are respectively called the pointwise homogeneous dimension of \( \{e_i\}_{i=1}^m \) at \( x \) and the local homogeneous dimension of \( \{e_i\}_{i=1}^m \) on \( U \). By the definitions of \( Q(x) \) and \( Q \), one gets from (1.6) that

\[
t^Q \Lambda(x, r) \leq \Lambda(x, tr) \leq t^Q(x) \Lambda(x, r), \quad 0 < t \leq 1.
\]

**Corollary 1.2.** (cf. also [Ga], [DGN]) For any \( x \in V \), \( 0 < t \leq 1, \ 0 < r \leq R_0 \), we have

\[
C_1 t^Q \leq \frac{\text{vol}_g(B_{CC}(x, tr))}{\text{vol}_g(B_{CC}(x, r))} \leq C_1^{-1} t^Q(x)
\]

where \( C_1 = C^2 \). Besides, there exists a positive constant \( C_2 \) such that

\[
\text{vol}_g(B_{CC}(x, r)) \geq C_2 r^Q, \quad x \in V.
\]
Proof. Clearly (1.9) follows immediately from Theorem 1.1 and (1.8). Next Theorem 1.1 also yields
\[
\text{vol}_g(B_{CC}(x,r)) \geq C r^Q \sum_I a_I(x).
\]
Since \(\sum_I a_I(x) > 0\) on the compact set \(\overline{V}\), there exists a positive number \(\tilde{C}\) such that \(\sum_I a_I(x) \geq \tilde{C}\) for any \(x \in V\). Therefore \(\text{vol}_g(B_{CC}(x,r)) \geq C_2 r^Q\) with \(C_2 = C \tilde{C}\). \(\square\)

For our purpose, we will consider compatible Riemannian metrics on a sub-Riemannian manifold \((M,H,g_H)\). A Riemannian metric \(g\) on \(M\) is called a Riemannian extension of \(g_H\) if \(g|_H = g_H\). It is a known fact that such extensions always exist (cf. [St]). Actually we may choose any Riemannian metric \(\tilde{g}\) on \(M\) and let \(V\) be the orthogonal complement of \(H\) with respect to \(\tilde{g}\). Set \(g_V = \tilde{g}|_V\). Then we have a Riemannian extension of \(g_H\)
\[
(1.11) \quad g = g_H + g_V
\]
by requiring \(g(u,v) = 0\) for any \(u \in H\) and \(v \in V\). Clearly such a Riemannian extension for \(g_H\) is not unique. From now on, we always fix a Riemannian extension \(g\) on the sub-Riemannian manifold \((M,H,g_H)\), and consider the quadruple \((M,H,g_H;g)\). According to \(g\), the tangent bundle \(TM\) has the following orthogonal decomposition:
\[
(1.12) \quad TM = H \oplus V.
\]
The distribution \(V\) will be referred to as the vertical distribution or bundle on \((M,H,g_H;g)\).

It would be convenient to introduce a suitable linear connection compatible to the sub-Riemannian structure on \((M,H,g_H;g)\) in some sense. The generalized Bott connection is one of such connections given by
\[
(1.13) \quad \nabla^B_X Y = \begin{cases} 
\pi_H(\nabla^R_X Y), & X,Y \in \Gamma(H) \\
\pi_H([X,Y]), & X \in \Gamma(V), Y \in \Gamma(H) \\
\pi_V([X,Y]), & X \in \Gamma(H), Y \in \Gamma(V) \\
\pi_V(\nabla^R_X Y), & X,Y \in \Gamma(V) 
\end{cases}
\]
where \(\nabla^R\) denotes the Riemannian connection of \(g\). Clearly \(\nabla^B\) preserves the decomposition (1.12), and it also satisfies
\[
(1.14) \quad \nabla^B_X g_H = 0 \quad \text{and} \quad \nabla^B_Y g_V = 0
\]
for any \(X \in H\) and \(Y \in V\). However, \(\nabla^B\) does not preserve the Riemannian metric \(g\) in general. The readers are referred to [BF], [Ba2] for some discussions about this connection on Riemannian foliations with totally geodesic leaves.

Example 1.1. Let \(G\) be a simply connected Lie group whose Lie algebra \(\mathfrak{g}\) admits a direct sum decomposition of vector spaces:
\[
\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \quad (r \geq 1)
\]
such that \([V_1,V_j] = V_{j+1} \text{ for } 1 \leq j \leq r - 1\) and \([V_1,V_r] = \{0\}\). Then \(G\) is referred to as a Carnot group. We may define a distribution \(H\) on \(G\) by \(H_g = dL_g(V_1) \subset T_g G, \forall g \in G\). Let \(g_H\) be a left-invariant metric on \(H\). Clearly \((G,H,g_H)\) is a step-\(r\) sub-Riemannian manifold. It is known that Carnot groups play an important role in sub-Riemannian geometry and related geometric analysis.
Example 1.2. Let $(M^{2n+1}, \theta)$ be a (strict) contact manifold, that is, $\theta$ is a global 1-form satisfying
\[ \theta \wedge (d\theta)^n \neq 0 \]
everywhere on $M$. Then the contact subbundle $H := \ker \theta$ is a 2-step bracket generating subbundle of rank $2n$. The Reeb vector field associated with $\theta$ is a unique vector field $\xi$ on $M$ satisfying
\[ \theta(\xi) = 1 \quad \text{and} \quad d\theta(\xi, \cdot) = 0. \]
An almost complex structure $J$ in $H$ is said to be compatible with $d\theta$ if
\[ d\theta(J\cdot, J\cdot) = d\theta(\cdot, \cdot) \quad \text{and} \quad d\theta(J\cdot, \cdot) > 0. \]
Then the contact subbundle $H$ and the Levi form $L_\theta = d\theta(J\cdot, \cdot)$ define a sub-Riemannian structure on $M$. We extend $J$ to an endomorphism of $TM$ by setting $J\xi = 0$. The Webster metric defined by
\[ g_\theta = L_\theta + \theta \otimes \theta \]
is a Riemannian extension of $L_\theta$. We call $(M, \theta, \xi, J, g_\theta)$ a contact metric manifold. A contact metric manifold $(M, \theta, \xi, J, g_\theta)$ for which $J$ is integrable is referred to as a strictly pseudoconvex CR manifold.

Example 1.3. (cf. [Biq1,2]) A quaternionic contact manifold $M$ is a $(4n + 3)$-dimensional manifold with a rank $4n$ distribution $H$ locally given as the kernel of 1-form $\eta = (\eta^1, \eta^2, \eta^3)$ with values in $R^3$. In addition, $H$ is equipped with a Riemannian metric $g_H$ and three local almost complex structures $I_i$ $(i = 1, 2, 3)$ satisfying the identities of the imaginary unit quaternions. These structures also satisfy the following compatible conditions: $g_H(I_i\cdot, I_i\cdot) = g_H(\cdot, \cdot)$ and $d\eta_i = g(I_i\cdot, \cdot)$. When the dimension of $M$ is at least eleven, Biquard [Biq1] also described the supplementary distribution $V$ by the so-called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$. These Reeb vector fields are determined by
\[ \eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \cdot d\eta_s)|_H = 0, \quad (\xi_s \cdot d\eta_k)|_H = -(\xi_k \cdot d\eta_s)|_H. \]
Consequently $(H, g_H)$ defines a 2-step bracket generating sub-Riemannian structure on $M$. Using the triple of Reeb vector fields, we may extend $g_H$ to a Riemannian metric $g$ on $M$ by requiring $\text{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_s, \xi_k) = \delta_{sk}$.

Example 1.4. (cf. [GW], [Mo]) A foliation on a manifold is the collection of integral manifolds of an integrable distribution on the manifold. Let $\mathcal{F}$ be a foliation on a Riemannian manifold $(M, g)$. Set $V = T\mathcal{F}$, $H = V^\perp$ (w.r.t. $g$) and $g_H = g|_H$. Then $(H, g_H)$ defines a sub-Riemannian structure on $M$. The foliation is called a Riemannian foliation if $\nabla^g_\xi g_H = 0$ for any $\xi \in V$. In this case, following [Re], $g$ is referred to as a bundle-like metric. Note that we are only interested in a Riemannian foliation $\mathcal{F}$ whose horizontal distribution $H$ is bracket generating for $TM$ in this paper.

For a sub-Riemannian manifold $(M, H, g_H; g)$, we may define a global vector field by
\[ \zeta = \pi_H(\sum_{\alpha} \nabla^R_{e_\alpha} e_\alpha) \]
which will be called the mean curvature vector field of the vertical distribution $V$. When $V$ is the tangent bundle of a foliation $\mathcal{F}$ on $M$ as in Example 1.4, $\zeta$ is just the usual mean curvature vector field along each leaf in $(M, g, \mathcal{F})$. It is easy to verify by (1.13) that if $\mathcal{F}$ is a Riemannian foliation with totally geodesic fibers, then $\nabla^\mathcal{B}$ is a metric connection for $g$ (cf. [BF], [Ba2]).
2. Analysis for hypoelliptic operators

In [Hö], Hörmander considered the following type of differential operator:

\begin{equation}
\mathcal{D} = \sum_{i=1}^{m} X_i^2 + Y
\end{equation}

where \(X_1, \ldots, X_m, Y\) are smooth vector fields on a manifold \(\widetilde{M}\) with the property that their commutators up to certain order span the tangent space at each point. He proved that \(\mathcal{D}\) is hypoelliptic in the sense that if \(u\) is a distribution defined on any open set \(\Omega \subset \widetilde{M}\), such that \(\mathcal{D}u \in C^\infty(\Omega)\), then \(u \in C^\infty(\Omega)\). Due to this celebrated result, hypoelliptic operators have since been the subject of intense study (cf. [RS], [Br]). In following, we will discuss two important hypoelliptic operators arising in sub-Riemannian geometric analysis, namely, the sub-Laplacian and its heat operator.

Let \((M^{m+d}, H, g_H; g)\) be a sub-Riemannian manifold with the rank \(m\) subbundle \(H\) satisfying the \(r\)-step bracket generating condition. A smooth vector field \(X\) on \(M\) is said to be horizontal if \(X_p \in H_p\) for each \(p \in M\). For a smooth function \(u\), its horizontal gradient is the unique horizontal vector field \(\nabla^H u\) satisfying \(g_H(\nabla^H u)_q, X) = du(X)\) for any \(X \in H_q\), \(q \in M\). We choose a local orthonormal frame field \(\{e_A\}_{A=1}^{m+d}\) on an open domain \(\Omega\) of \((M, g)\) such that \(\text{span}\{e_i\}_{i=1}^{m} = H\), and thus \(\text{span}\{e_\alpha\}_{\alpha=m+1}^{m+d} = V\). Such a frame field is referred to as an adapted frame field for \((M, H, g_H; g)\). Consequently

\begin{equation}
\nabla^H u = \sum_{i=1}^{m} (e_i u) e_i.
\end{equation}

Due to the Hörmander’s condition, we see that \(f\) is constant if and only if \(\nabla^H u = 0\).

By definition, in terms of the Riemannian connection \(\nabla^R\), the divergence of a vector field \(X\) on \(M\) is given by

\begin{equation}
\text{div}_g X = \sum_{A=1}^{m+d} \{e_A \langle X, e_A \rangle - \langle X, \nabla^R_{e_A} e_A \rangle\}.
\end{equation}

Then the sub-Laplacian of a function \(u\) on \((M, H, g_H; g)\) is defined as

\begin{equation}
\triangle_H u = \text{div}_g(\nabla^H u).
\end{equation}

Using the divergence theorem, we see that \(\triangle_H\) is a symmetric operator, that is,

\begin{equation}
\int_M v(\triangle_H u) dv_g = \int_M u(\triangle_H v) dv_g = -\int_M |\nabla^H u|^2 dv_g
\end{equation}

for any \(u, v \in C_0^\infty(M)\). Using (2.2), (2.3), and (1.15), we may rewrite (2.4) as

\begin{equation}
\begin{aligned}
\triangle_H u &= \sum_{i=1}^{m} \{e_i \langle \nabla^H u, e_i \rangle - \langle \nabla^H u, \nabla^g_{e_i} e_i \rangle \} - \langle \nabla^H u, \zeta \rangle \\
&= \sum_{i=1}^{m} e_i^2(u) - (\sum_{i=1}^{m} \nabla^g_{e_i} e_i + \zeta) u.
\end{aligned}
\end{equation}
This shows that $\triangle_H$ is an operator of Hörmander type, and thus it is hypoelliptic on $M$. Clearly the operator $\triangle_H - \frac{\partial}{\partial t}$ is also an operator given locally in the form of (2.1) with $X_1,\ldots,X_m,Y$ satisfying the Hörmander’s condition on $M \times R$. Therefore the heat operator corresponding to $\triangle_H$ is hypoelliptic too.

In 1976, L. Rothschild and E.M. Stein [RS] established a more precise regularity theory for hypoelliptic operators. Define

$$S_k^p(\triangle_H,\Omega) = \{ u \in L^p(\Omega) | e_{i_1} \cdots e_{i_s}(u) \in L^p(\Omega), \ 1 \leq i_1,\ldots,i_s \leq m, \ 0 \leq s \leq k \}$$

and

$$S_k^p(\triangle_H - \frac{\partial}{\partial t},\Omega \times (0,T)) = \{ u \in L^p(\Omega \times (0,T)) | \partial_t^i e_{i_1} \cdots e_{i_s}(u) \in L^p(\Omega \times (0,T)), \ 1 \leq i_1,\ldots,i_s \leq m, \ 2l + s \leq k \}$$

(2.7)

for any non-negative integer $k$. By the theory of Rothschild and Stein, we have

**Theorem 2.1.** Let $\mathcal{D} = \triangle_H$ (resp. $\triangle_H - \frac{\partial}{\partial t}$) and $\tilde{M} = \Omega$ (resp. $\Omega \times (0,T)$). Suppose $f \in L^p_{loc}(\tilde{M})$, and

$$\mathcal{D}f = g \quad \text{on} \quad \tilde{M}.$$ 

If $g \in S^p_k(\mathcal{D},\tilde{M})$, then $\chi f \in S^p_{k+2}(\mathcal{D},\tilde{M})$ for any $\chi \in C_0^\infty(\tilde{M})$, $1 < p < \infty$, $k = 0, 1, 2, \ldots$. In particular, the following inequality holds

$$\| \chi f \|_{S^p_{k+2}(\mathcal{D},\tilde{M})} \leq C_\chi \left( \| g \|_{S^p_k(\mathcal{D},\tilde{M})} + \| f \|_{L^p(\tilde{M})} \right)$$

where $C_\chi$ is a constant independent of $f$ and $g$.

**Remark 2.1.** Let $L^p_k(\tilde{M})$, $1 < p < \infty$, be the classical Sobolev space. From [RS], we know that $S^p_k(\triangle_H,\tilde{M}) \subset L^p_k(\tilde{M})$ for any $k \geq 0$, while $S^p_k(\triangle_H - \frac{\partial}{\partial t},\tilde{M}) \subset L^p_{k/r}(\tilde{M})$ if $k$ is even or a multiple of $r$. For any positive integer $l$, $\alpha \in (0,1)$ and $1 < p < \infty$, if $k$ is large enough, then $S^p_k(\mathcal{D},\tilde{M}) \subset C^{l,\alpha}(\tilde{M})$ (the Hölder space) for $\mathcal{D} = \triangle_H$ or $\triangle_H - \frac{\partial}{\partial t}$.

Now we give some results about the heat kernel on compact sub-Riemannian manifolds, which will be needed in §5. Let $K(x,y,t)$ be the heat kernel for $\triangle_H$ on a compact sub-Riemannian manifold $(M,H,g_H;g)$, that is,

$$\triangle_H - \frac{\partial}{\partial t}K(x,y,t) = 0$$

$$\lim_{t \to 0} K(x,y,t) = \delta_x(y).$$

(2.9)

The readers may refer to [Ba1,3], [Bi] and [St] for the existence of $K(x,y,t)$. We list some basic properties of $K(x,y,t)$ as follows:

1. $K(x,y,t) \in C^\infty(M \times M \times R^+)$;
2. $K(x,y,t) = K(y,x,t)$ for $x,y \in M$ and $t > 0$;
3. $K(x,y,t) > 0$ for $x,y \in M$ and $t > 0$;
4. $\int_M K(x,y,t) dv_g(y) = 1$ for any $x \in M$;
5. $K(x,y,t+s) = \int_M K(x,z,t)K(y,z,s)dv_g(z)$ (semi-group property).

The following result is a special case of a somewhat more general theorem proved in [Sá].
**Theorem 2.2.** (cf. [Sá]) Let $K(x,y,t)$ be the heat kernel of $\triangle_H$ on $(M,H,g_H;g)$. Set $w(x;\delta) = \text{vol}(B_{CC}(x;\delta))$. Then

\begin{equation}
|\nabla^H_x K(x,y,t)| \leq A_P t^{-\frac{1}{2}} w(x;t^{1/2})^{-1} \left(1 + \frac{d_{CC}(x,y)^2}{t}\right)^{-P}
\end{equation}

and

\begin{equation}
K(x,y,t) \leq B_P w(x;t^{1/2})^{-1} \left(1 + \frac{d_{CC}(x,y)^2}{t}\right)^{-P}
\end{equation}

for $0 < t < 1$, all nonnegative integer $P$, and some positive constants $A_P$ and $B_P$ depending on $P$, where $\nabla^H_x$ denotes the horizontal gradient of $K$ with respect to $x$.

**Lemma 2.3.** For any $\beta \in (0,1/2)$, there exists a $C_\beta > 0$ such that

\[
\int_0^t \int_M |\nabla^H_x K(x,y,s)| dv_g(y) ds \leq C_\beta t^{\beta}
\]

for $0 < t < R_0$ for some positive constant $R_0$.

**Proof.** Since $M$ is compact, there are two finite open coverings $\{V_a\}$ and $\{U_a\}$ ($a = 1,\ldots,l$) of $M$ such that $\overline{V}_a \subset U_a$, $\overline{V}_a$ is compact, and Corollary 1.2 holds for each pair $(V_a,U_a)$. In particular, there exist positive constants $C_a$, $D_a$ and $R_a$ such that for any $x \in V_a$, and $0 < r < R_a$, one has

\begin{equation}
\frac{\text{vol}_g(B_{CC}(x,tr))}{\text{vol}_g(B_{CC}(x,r))} \geq C_a t^{Q_a}
\end{equation}

for $0 < t \leq 1$ and

\begin{equation}
\text{vol}_g(B_{CC}(x,r)) \geq D_a r^{Q_a}
\end{equation}

where $Q_a$ is the local homogeneous dimension on $U_a$. For any given $\beta \in (0,\frac{1}{2})$, we let $x \in V_a$ and $\gamma_a = \frac{2\beta+Q_a-1}{2Q_a}$. Note that $0 < \gamma_a < \frac{1}{2}$, and thus $s^{(\frac{1}{2}-\gamma_a)} < 1$ for any $0 < s < 1$. For any $0 < s < R_a^2$, we obtain from (2.12) that

\[
\text{Vol}_g(B_{CC}(x,s^{\frac{1}{2}})) = \text{Vol}_g(B_{CC}(x,s^{\frac{1}{2}}-\gamma_a s^{\gamma_a})) \\
\geq C_a s^{(\frac{1}{2}-\gamma_a)Q_a} \text{Vol}_g(B_{CC}(x,s^{\gamma_a}))
\]

that is,

\begin{equation}
\frac{\text{vol}_g(B_{CC}(x,s^{\gamma_a}))}{\text{Vol}_g(B_{CC}(x,s^{\frac{1}{2}}))} \leq C_a^{-1} s^{(\gamma_a-\frac{1}{2})Q_a}.
\end{equation}
Then we complete the proof of this lemma. □

the subelliptic heat equation satisfying

\[ \text{Let Lemma 2.4.} \]

and a mean value type inequality for subsolutions of the subelliptic heat equation.

type. In following lemma, we provide both a maximum principle (whose proof is routine),

\[ \text{in } M \]

where \( \tilde{\alpha} \) is a uniform positive constant. Set \( C_\beta = \max_{1 \leq a \leq l} \{ \tilde{C}_a \} \) and \( R_0 = \min_{1 \leq a \leq l} \{ R_a \} \).

Then we complete the proof of this lemma. □

In [Bo], Bony showed that the maximum principle holds for an operator of Hörmander type.

In following lemma, we provide both a maximum principle (whose proof is routine),

a mean value type inequality for subsolutions of the subelliptic heat equation.

**Lemma 2.4.** Let \( M \) be a compact sub-Riemannian manifold. Suppose \( \phi \) is a subsolution of the subelliptic heat equation satisfying

\[ \left( \Delta_H - \frac{\partial}{\partial t} \right) \phi \geq 0 \]

on \( M \times [0,T) \) with initial condition \( \phi(x,0) = \phi_0(x) \) for any \( x \in M \). Then

\[ \sup_{M} \phi(x,t) \leq \sup_{M} \phi_0(x) . \]

Furthermore, if \( \phi(x,t) \) is nonnegative, then there exist a constant \( B \) and an integer \( Q \) such that

\[ \sup_{x \in M} \phi(x,t) \leq B t^{-\frac{Q}{2}} \int_{M} \phi_0(y) dv_g(y) \]

for \( 0 < t \leq \min \{ R_0^2, T \} \), where \( R_0 \) is as in Lemma 2.3.

**Proof.** First we assume that \( \phi \) is a subsolution of the subelliptic heat equation. Set \( c = \sup_{M} \phi_0(x) \). For any fixed \( \varepsilon > 0 \), one may introduce a function \( \phi_\varepsilon = \phi - \varepsilon (1 + t) \). Clearly \( \phi_\varepsilon < c \) at \( t = 0 \). We claim that \( \phi_\varepsilon < c \) for all \( t > 0 \). In order to prove this, let us suppose the result is false. This means that there exists \( \varepsilon > 0 \) such that \( \phi_\varepsilon \geq c \) somewhere in \( M \times [0,T) \). Since \( M \) is compact, there exists a point \( (x_0, t_0) \in M \times [0,T) \) such that \( \phi_\varepsilon(x_0, t_0) = c \) and \( \phi_\varepsilon(x, t) \leq c \) for all \( x \in M \) and \( t \in [0, t_0) \). It follows that \( (\frac{\partial \phi_\varepsilon}{\partial t})(x_0, t_0) \leq 0 \) and \( (\Delta_H \phi_\varepsilon)(x_0, t_0) \leq 0 \), so that

\[ 0 > (\Delta_H \phi_\varepsilon)(x_0, t_0) - \varepsilon \geq (\frac{\partial \phi_\varepsilon}{\partial t})(x_0, t_0) \geq 0 \]
which is a contradiction. Hence $\phi < c$ on $M \times [0, T)$ for any $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\phi \leq c$ on $M \times [0, T)$. This proves the maximum principle.

Next we assume that $\phi$ is a nonnegative subsolution of the subelliptic heat equation. Set

$$\tilde{\phi}(x, t) = \int_M K(x, y, t)\phi_0(y)dv_g(y). \quad (2.15)$$

Then $\tilde{\phi}$ solves the subelliptic heat equation

$$\left(\triangle_H - \frac{\partial}{\partial t}\right)\tilde{\phi} = 0$$

with initial data $\tilde{\phi}(x, 0) = \phi_0(x)$ for any $x \in M$. By (2.15), we get

$$\sup_{x \in M} \tilde{\phi}(x, t) \leq \sup_{x, y \in M} K(x, y, t) \int_M \phi_0(y)dv_g(y). \quad (2.16)$$

The semi-group property of $K(x, y, t)$ yields

$$K(x, y, t) = \int_M K(x, z, \frac{t}{2})K(y, z, \frac{t}{2})dv_g(z) \leq \left(\int_M K^2(x, z, \frac{t}{2})dv_g(z)\right)^{\frac{1}{2}} \left(\int_M K^2(y, z, \frac{t}{2})dv_g(z)\right)^{\frac{1}{2}} = K^{\frac{1}{2}}(x, x, t)K^{\frac{1}{2}}(y, y, t). \quad (2.17)$$

According to Theorem 2.2, we have

$$K(x, x, t) \leq \tilde{B} \cdot vol_g(B_{CC}(x, \sqrt{t}))^{-1} \quad (2.18)$$

for some constant $\tilde{B}$. Now we cover $M$ by two finite open coverings \(\{V_a\}_{a=1}^l\) and \(\{U_a\}_{a=1}^l\) as in the proof of Lemma 2.3. Let $Q_a$ be the local homogeneous dimension on $U_a$. Set $Q = \max_{1 \leq a \leq l}\{Q_a\}$. Then we know from (2.13) that

$$vol_g(B_{CC}(x, r)) \geq Dr^Q \quad (2.19)$$

for $0 < r \leq R_0 = \min\{R_a\}$, where $D = \min_{1 \leq a \leq l}\{D_a\}$ and $Q = \max_{1 \leq a \leq l}\{Q_a\}$. In terms of (2.16), (2.17), (2.18) and (2.19), we conclude that

$$\sup_{x \in M} \tilde{\phi}(x, t) \leq Bt^{-\frac{Q}{2}} \int_M \phi_0(y)dv_g(y)$$

for $0 < t \leq R_0^2$. Since $\phi$ is a subsolution, the maximum principle implies that $\phi \leq \tilde{\phi}$ for $0 < t < \min\{R_0^2, T\}$. Hence we complete the proof of this lemma. }
3. Second fundamental forms and their covariant derivatives

We will use the moving frame method to perform local computations on maps from sub-Riemannian manifolds. For a sub-Riemannian manifold \((M, H, g_H; g)\), let us first give the structure equations for the generalized Bott connection \(\nabla^B\) defined by (1.13). Let \(\{e_A\}_{A=1}^{m+p}\) be an adapted frame field in \(M\), and let \(\{\omega^A\}_{A=1}^{m+p}\) be its dual frame field. From now on, we shall make use of the following convention on the ranges of indices in \(M\):

\[
1 \leq A, B, C, ..., \leq m + p; \quad 1 \leq i, j, k, ..., \leq m;
\]

\[
m + 1 \leq \alpha, \beta, \gamma, ..., \leq m + p,
\]

and we shall agree that repeated indices are summed over the respective ranges. The connection 1-forms \(\{\omega^B_A\}\) of \(\nabla^B\) with respect to \(\{e_A\}_{A=1}^{m+p}\) are given by

\[
(3.1) \quad \nabla^B_X e_A = \omega^B_A(X)e_B
\]

for any \(X \in TM\). Since \(\nabla^B\) preserves the decomposition (1.12), we have

\[
(3.2) \quad \nabla^B_X e_i = \omega^j_i(X)e_j, \quad \nabla^B_X e_\alpha = \omega^\beta_\alpha(X)e_\beta
\]

and thus

\[
(3.3) \quad \omega^\alpha_i = 0, \quad \omega^j_\alpha = 0.
\]

Let \(T(\cdot, \cdot)\) and \(R(\cdot, \cdot)\) be the torsion and curvature of \(\nabla^B\) given respectively by

\[
(3.4) \quad T(X, Y) = \nabla^B_X Y - \nabla^B_Y X - [X, Y],
\]

\[
R(X, Y)Z = \nabla^B_X \nabla^B_Y Z - \nabla^B_Y \nabla^B_X Z - \nabla^B_{[X,Y]} Z
\]

where \(X, Y, Z \in \Gamma(TM)\). Write

\[
(3.5) \quad T(X, Y) = T^A(X, Y)e_A, \quad R(X, Y)e_A = \Omega^B_A(X, Y)e_B.
\]

Note that (3.2) implies

\[
(3.6) \quad \Omega^\alpha_i = \Omega^j_\alpha = 0.
\]

As a linear connection, the structure equations of \(\nabla^B\) are (cf. [KN])

\[
(3.7) \quad d\omega^A = -\omega^A_B \wedge \omega^B + T^A, \quad d\omega^A_B = -\omega^A_C \wedge \omega^C_B + \Omega^A_B.
\]
Lemma 3.1. For any $X,Y \in \Gamma(TM)$, we have

$$T(X,Y) = -\pi_V([\pi_H(X), \pi_H(Y)]) - \pi_H([\pi_V(X), \pi_V(Y)]).$$

Proof. If $X,Y \in \Gamma(H)$, we verify by means of (1.13) that

$$\pi_H(T(X,Y)) = [\pi_H(X), \pi_H(Y)] = 0$$

and

$$\pi_V(T(X,Y)) = -\pi_V([X,Y]).$$

Similarly, if $X,Y \in \Gamma(V)$, then

$$\pi_V(T(X,Y)) = 0 \quad \text{and} \quad \pi_H(T(X,Y)) = -\pi_H([X,Y]).$$

Finally, if $X \in \Gamma(V), Y \in \Gamma(H)$, then (1.13) implies directly that $T(X,Y) = 0$. Combining these cases, we prove this lemma. □

Using the dual frame field and Lemma 3.1, one may express the torsion as

$$(3.8) \quad T(\cdot, \cdot) = \frac{1}{2}(T_{ij}^I \omega^i \wedge \omega^j) \otimes e_I + \frac{1}{2}(T_{ij}^I \omega^\alpha \wedge \omega^\beta) \otimes e_i$$

We also write

$$(3.9) \quad \Omega^A_B = \frac{1}{2} R^A_{BCD} \omega^C \wedge \omega^D, \quad R^A_{BCD} = -R^A_{BDC}.$$
In terms of the frame fields in $M$ and $N$, the differential $df$ may be expressed as

$$df = f^I_A \omega^A \otimes \tilde{e}_I.$$ 

Consequently

(3.13) $$f^* \tilde{\omega}^I = f^I_A \omega^A = f^I_i \omega^i + f^I_\alpha \omega^\alpha.$$ 

By taking the exterior derivative of (3.13) and making use of the structure equations in $M$ and $N$, we get

(3.14) $$Df_A \wedge \omega^A + \frac{1}{2} f^I_C T^C_{AB} \omega^A \wedge \omega^B = 0$$

where

(3.15) $$Df^I_A = df^I_A - f^I_C \omega^C + f^K_A \tilde{\omega}^I_K = f^I_{AB} \omega^B.$$

Clearly the second fundamental form $\beta$ can be expressed as

(3.16) $$\beta = f^I_{AB} \omega^A \otimes \omega^B \otimes \tilde{e}_I.$$ 

From (3.14), (3.15) and Lemma 3.1, it follows that

(3.17)

\[
\begin{align*}
&f^I_{ij} - f^I_{ji} = f^I_n T^a_{ij} \\
&f^I_{\alpha\beta} - f^I_{\beta\alpha} = f^I_k T^k_{\alpha\beta} \\
&f^I_{i\alpha} - f^I_{\alpha i} = 0.
\end{align*}
\]

By taking the exterior derivative of (3.15), we deduce that

(3.18) $$Df^I_{AB} \wedge \omega^B = - f^I_D \Omega^D_A + f^K_A \tilde{\Omega}^I_K - f^I_{AD} T^D_A$$

\[
= - \frac{1}{2} f^I_D R^D_{ABC} \omega^B \wedge \omega^C + \frac{1}{2} f^I_A \tilde{R}^I_{KJL} f^I_B f^I_C \omega^B \wedge \omega^C \\
- \frac{1}{2} f^I_{AD} T^D_{BC} \omega^B \wedge \omega^C,
\]

where

(3.19) $$Df^I_{AB} = df^I_{AB} - f^I_{CB} \omega^C_A - f^I_{AC} \omega^C_B + f^K_A \tilde{\omega}^I_K.$$ 

By putting

(3.20) $$Df^I_{AB} = f^I_{ABC} \omega^C,$$

we get from (3.18) the commutation relation

(3.21) $$f^I_{ABC} - f^I_{ACB} = f^I_D R^D_{ABC} + f^I_{AD} T^D_{BC} - f^K_A \tilde{R}^I_{KJL} f^I_B f^I_C.$$
For the map $f$, besides the differential $df$, one may also introduce two partial differentials $df_H = df \mid_{H} \in \Gamma(H^* \otimes f^{-1}TN)$ and $df_V = df \mid_{V} \in \Gamma(V^* \otimes f^{-1}TN)$. By the definition of Hilbert–Schmidt norm for a linear map, we have

\begin{equation}
|df_H|^2 = (f_i^I)^2, \quad |df_V|^2 = (f_{\alpha}^I)^2, \quad |df|^2 = (f_{\alpha}^I)^2.
\end{equation}

Set

\[ e_H(f) = \frac{1}{2} |df_H|^2, \quad e_V(f) = \frac{1}{2} |df_V|^2, \quad e(f) = \frac{1}{2} |df|^2. \]

Now we want to derive the Bochner formulas of $\Delta_H e_H(f)$, $\Delta_H e_V(f)$ and $\Delta_H e(f)$. For a function $u : M \to R$, one gets easily from (2.6) and (3.12) that

\begin{equation}
\Delta_H u = \beta(u)(e_k, e_k) - \zeta(u) = u_{kk} - \zeta^k u_k
\end{equation}

where $\zeta = \zeta^k e_k$. Using (3.22), we compute

\begin{equation}
e_H(f)k = f_i^I f_{ik}^I
\end{equation}

and

\begin{equation}
(e_H(f))_{kk} = (f_{ik}^I)^2 + f_i^I f_{ikk}^I.
\end{equation}

Consequently, in terms of (3.17) and (3.21), we derive that

\begin{equation}
f_{ikk}^I = [f_{ik}^I + f_{\alpha k}^I T_{ik}^\alpha]k = f_{ikk}^I + f_{\alpha k}^I T_{ik}^\alpha + f_{\alpha}^I T_{ik,k}
\end{equation}

where $\tau_{ik}^I = f_{ik}^I - \zeta^k f_{ik}^I$ (see Proposition 4.1 below for its geometric meaning). Then it follows from (3.23), (3.25), (3.26) and (3.17) that

\begin{equation}
\Delta_H e_H(f) = [e_H(f)]_{kk} - \zeta^k [e_H(f)]_k
\end{equation}

\begin{equation}
= (f_{ik}^I)^2 + f_i^I f_{ikk}^I - \zeta^k f_i^I f_{ik}^I
\end{equation}

\begin{equation}
= (f_{ik}^I)^2 + f_i^I \tau_{ik}^I, + f_i^I \zeta_i^k f_k + \zeta^k f_i^I (f_{ik}^I - f_{ik}^I)
\end{equation}

\begin{equation}
+ f_i^I f_j^I R_{ik}^D + f_i^I f_{ij}^D T_{ik}^D - f_i^I f_{ik}^K \tilde{R}_{KJL} f_{ij}^I j_k^L + f_i^I f_{\alpha k}^I T_{ik}^\alpha + f_i^I f_{\alpha}^I T_{ik,k}
\end{equation}

\begin{equation}
= (f_{ik}^I)^2 + f_i^I \tau_{ik}^I + f_i^I \zeta_i^k f_k + \zeta^k f_i^I f_{\alpha k}^I T_{ik}^\alpha + f_i^I f_{ij}^D R_{ik}^D
\end{equation}

\begin{equation}
+ f_i^I f_{ik}^I T_{ik}^\alpha - f_i^I f_{ik}^K \tilde{R}_{KJL} f_{ij}^I j_k^L + f_i^I f_{\alpha k}^I T_{ik}^\alpha + f_i^I f_{\alpha}^I T_{ik,k}
\end{equation}

\begin{equation}
= (f_{ik}^I)^2 + f_i^I \tau_{ik}^I + f_i^I \zeta_i^k f_k + \zeta^k f_i^I f_{\alpha k}^I T_{ik}^\alpha + f_i^I f_{ij}^D R_{ik}^D
\end{equation}

\begin{equation}
+ 2 f_i^I f_{jk}^I T_{ik}^\alpha - f_i^I f_{ik}^K \tilde{R}_{KJL} f_{ij}^I j_k^L + f_i^I f_{\alpha k}^I T_{ik}^\alpha + f_i^I f_{\alpha}^I T_{ik,k}.
\end{equation}
Similarly, using (3.17) and (3.21), we have

\[
\begin{align*}
(e^\nu(f))_{kk} &= (f^l_{\alpha k} f^l_{\alpha k})_k = (f^l_{\alpha k})^2 + f^l_{\alpha l} f^l_{\alpha k} \\
&= (f^l_{\alpha k})^2 + f^l_{\alpha l} (f^l_{k \alpha} + f^l_{\alpha D} R^D_{k \alpha} + f^l_{\alpha D} T^D_{k \alpha} - f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k}) \\
&= (f^l_{\alpha k})^2 + f^l_{\alpha} \tau^l_{H, \alpha} + f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} \zeta^l_{\alpha} f^l_{\alpha} + f^l_{\alpha} f^l_{\alpha} D^D_{k \alpha} \\
&+ f^l_{\alpha} f^l_{k} T^D_{k \alpha} - f^l_{\alpha} f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k}.
\end{align*}
\]

(3.28)

It follows that

\[
\Delta_H [e^\nu(f)] = (e^\nu(f))_{kk} - \zeta^l f^l_{\alpha} f^l_{\alpha} = (f^l_{\alpha k})^2 + f^l_{\alpha} \tau^l_{H, \alpha} + f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} f^l_{\alpha} D^D_{k \alpha} - f^l_{\alpha} f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k}.
\]

(3.29)

From (3.27), (3.29), we conclude that

\[
\begin{align*}
\Delta_H e(f) &= (f^l_{ik})^2 + (f^l_{\alpha k})^2 + f^l_{\alpha} \tau^l_{H, \alpha} + f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} f^l_{\alpha} D^D_{k \alpha} \\
&+ f^l_{\alpha} f^l_{k} T^D_{k \alpha} + f^l_{\alpha} \tau^l_{H, \alpha} + f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} f^l_{\alpha} D^D_{k \alpha} - f^l_{\alpha} f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k}
\end{align*}
\]

(3.30)

Lemma 3.2. Let \((M, H, g^H; g)\) be a compact sub-Riemannian manifold and let \((N, h)\) be a Riemannian manifold with non-positive sectional curvature. Let \(f : M \to N\) be a smooth map. Set \(\tau^l_{H} = f^l_{kk} - \zeta^l f^l_{k} \). Then one has

\[
\Delta_H e(f) - f^l_{\alpha} \tau^l_{H, \alpha} - f^l_{\alpha} \tau^l_{H, \alpha} \geq -C_\varepsilon e_H(f) - \varepsilon e_V(f) + (f^l_{ik})^2 + \frac{1}{2} (f^l_{\alpha k})^2
\]

(3.31)

for any given \(\varepsilon > 0\), where \(C_\varepsilon\) is a positive number depending only on \(\varepsilon\) and

\[
\sup_{M, i, j, k, \alpha} \{ |\zeta^l_{k, \alpha}|, |\zeta^l_{k, i}|, |\zeta^l_{\alpha, \alpha}|, |T^alpha_{ij}|, |T^alpha_{ij,k}|, |R^j_{kik}|, |R^j_{kak}| \}.
\]

In particular, we have

\[
\Delta_H e(f) - f^l_{\alpha} \tau^l_{H, \alpha} - f^l_{\alpha} \tau^l_{H, \alpha} \geq -C_\varepsilon e(f).
\]

(3.32)

Proof. For any \(\varepsilon > 0\), we deduce, by Schwarz inequality, that

\[
\begin{align*}
f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} f^l_{k} R^j_{kik} &\geq -C_1 e_H(f), \\
\zeta^l_{k, \alpha} f^l_{\alpha} T^alpha_{ki} + f^l_{\alpha} f^l_{k} T^alpha_{ij,k} + f^l_{\alpha} \zeta^l_{k, \alpha} f^l_{k} + f^l_{\alpha} f^l_{k} R^j_{kak} &\geq -C_2 (\varepsilon) e_H(f) - \varepsilon e_V(f), \\
2 f^l_{\alpha} f^l_{k} T^alpha_{ik} &\geq -C_3 e_H(f) - \frac{1}{2} (f^l_{\alpha k})^2,
\end{align*}
\]

(3.33)

for some positive constants \(C_1, C_2(\varepsilon)\) and \(C_3\). Since \((N, h)\) has non-positive sectional curvature, we have

\[
f^l_{\alpha} f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k} + f^l_{\alpha} f^l_{k} \tilde{R}^l_{KJL} f^l_{\alpha} f^l_{k} \leq 0.
\]

(3.34)
From (3.30), (3.33), (3.34), we obtain (3.31) and thus (3.32) too. □

We will also need similar commutation relations as (3.17) and (3.21) for maps from the product manifold \( M \times (0, \delta) \). Here the product manifold \( M \times (0, \delta) \) is endowed with the direct sum connection of \( \nabla^M \) on \( M \) and the trivial connection on \( (0, \delta) \). Now let \( f : M \times (0, \delta) \to N \) be a smooth map. Write

\[
(3.35) \quad f^* \tilde{\omega} = f^A_A \omega^A + f^I_t dt.
\]

Taking the exterior derivative of (3.35), one has

\[
(3.36) \quad Df^A_A \wedge \omega^A + Df^I_t \wedge dt + \frac{1}{2} f^C_C T^C_{AB} \omega^A \wedge \omega^B = 0
\]

where

\[
(3.37) \quad Df^I_A = df^I_A - f^I_B \omega^B_A + f^K_A \omega^I_K = f^I_{AB} \omega^B + f^I_A dt
\]

\[
Df^I_t = df^I_t + f^K_t \omega^I_K = f^I_{At} \omega^A + f^I_t dt.
\]

Consequently \( \{ f^I_{AB} \} \) satisfy (3.17) and

\[
(3.38) \quad f^I_{At} = f^I_{tA}.
\]

Similarly taking derivative of the first equation in (3.37) gives

\[
(3.39) \quad Df^I_{AB} \wedge \omega^B + Df^I_{At} \wedge dt = -f^I_D \Omega^D_A + f^K_A \Omega^I_K - f^K_{AD} T^D
\]

where

\[
(3.40) \quad Df^I_{AB} = df^I_{AB} - f^I_{CB} \omega^C_A - f^I_{AC} \omega^C_B + f^K_{AB} \omega^I_K = f^I_{ABC} \omega^C + f^I_{AB} dt
\]

\[
Df^I_{At} = df^I_{At} - f^I_{Bt} \omega^B_A + f^K_{At} \omega^I_K = f^I_{AtC} \omega^C + f^I_{At} dt.
\]

Clearly \( \{ f^I_{ABC} \} \) satisfy (3.21) and

\[
(3.41) \quad f^I_{ABt} - f^I_{ABt} = -f^K_A \tilde{R}^I_K J^J_J^I J^f_I.
\]

4. Subelliptic harmonic maps and their heat flows

For a map \( f : (M^{m+p}, H, g_H; g) \to (N^n, h) \), besides the usual energy \( E(f) \), we have the following two partial energies:

\[
(4.1) \quad E_H(f) = \int_M e_H(f) dv_g = \frac{1}{2} \int_M \langle df(e_i), df(e_i) \rangle dv_g
\]

and

\[
(4.2) \quad E_V(f) = \int_M e_V(f) dv_g = \frac{1}{2} \int_M \langle df(e_\alpha), df(e_\alpha) \rangle dv_g
\]

where the integrands in the second equality of (4.1) (resp. (4.2)) are summed over the range of the index \( i \) (resp. \( \alpha \)). The partial energies \( E_H(f) \) and \( E_V(f) \) are called horizontal and vertical energies respectively. Clearly

\[
E(f) = E_H(f) + E_V(f).
\]
Definition 4.1. A map \( f : (M, H, g_H; g) \rightarrow (N, h) \) is referred to as a subelliptic harmonic map if it is a critical point of the energy \( E_H(f) \).

Proposition 4.1. Let \( \{ f_t \}_{|t| < \varepsilon} \) be a family of maps from \( (M, H, g_H; g) \) to \( (N, h) \) with \( f_0 = f \) and \( \frac{\partial f_t}{\partial t} \big|_{t=0} = \nu \in \Gamma (f^{-1}TN) \). Suppose the variation vector field \( \nu \) has compact support. Then

\[
\frac{dE_H(f_t)}{dt} \big|_{t=0} = - \int_M \langle \nu, \tau_H(f) \rangle dv_g
\]

where \( \tau_H(f) = \beta(e_i, e_i) - df(\zeta) \) is called the subelliptic tension field of \( f \).

Proof. We shall denote by \( F : M \times (-\varepsilon, \varepsilon) \rightarrow N \) the map defined by \( F(x, t) = f_t(x) \). Let \( \tilde{\nabla}^F \) be the pull-back connection of \( \nabla \) by \( F \). Since \( \nabla \) is torsion-free, we have

\[
\frac{dF}{\partial t} = \frac{dF}{\partial t}(e_i) \frac{\partial}{\partial t} = \frac{\partial}{\partial t}(e_i) \frac{dF}{e_i}(e_i)
\]

for any \( X \in TM \) (cf. [EL], page 14). Applying (4.1) to \( f_t \) and using (4.4), we derive that

\[
\frac{d}{dt} E_H(f_t) \big|_{t=0} = \int_M \langle \tilde{\nabla}_X^F dF(e_i), dF(e_i) \rangle dv_g \big|_{t=0}
\]

\[
= \int_M \langle \tilde{\nabla}_i^F \nu, df(e_i) \rangle dv_g
\]

(4.5)

\[
= \int_M \langle e_i \nu, df(e_i) \rangle - \langle \nu, \tilde{\nabla}^F_i df(e_i) \rangle dv_g
\]

\[
= \int_M \langle e_i \nu, df(e_i) \rangle - \langle \nu, df(\nabla_{e_i}^e e_i) \rangle dv_g
\]

\[
- \int_M \langle \nu, \beta(e_i, e_i) \rangle dv_g,
\]

where the terms with the index \( i \) are summed over \( 1 \leq i \leq m \). Set \( \theta(X) = \langle \nu, df \circ \pi_H(X) \rangle \) for any \( X \in TM \). The codifferential of \( \theta \) is given by

\[
\delta \theta = - (\nabla_{e_A}^R \theta)(e_A)
\]

(4.6)

\[
= - \langle e_A(\Theta(e_A)) - \theta(\nabla_{e_A}^R e_A) \rangle
\]

\[
= - \langle e_i \theta(e_i) - \theta(\pi_H(\nabla_{e_i}^R e_i)) + \theta(\pi_H(\nabla_{e_i}^R e_i)) \rangle
\]

\[
= - \langle e_i \theta(e_i) - \theta(\nabla_{e_i}^R e_i) \rangle + \theta(\zeta)
\]

where \( \zeta = \pi_H(\nabla_{e_A}^R e_A) \) (a sum w.r.t. \( \alpha \)). It follows from (4.6) and the divergence theorem that

\[
\int_M \langle e_i \nu, df(e_i) \rangle - \langle \nu, df(\nabla_{e_i}^R e_i) \rangle dv_g = \int_M \langle \nu, df(\zeta) \rangle dv_g.
\]

(4.7)

By (4.5) and (4.7), we obtain

\[
\frac{d}{dt} E_H(f_t) \big|_{t=0} = \int_M \langle \nu, df(\zeta) \rangle dv_g - \int_M \langle \nu, \beta(e_i, e_i) \rangle dv_g
\]

\[
= - \int_M \langle \nu, \tau_H(f) \rangle dv_g.
\]
Corollary 4.2. A map \( f : (M, H, g_H; g) \rightarrow (N, h) \) is a subelliptic harmonic map if and only if it satisfies the Euler-Lagrange equation

\[
\tau_H(f) = 0.
\]

Remark 4.1. If \( f : (M, H, g_H; g) \rightarrow R \) is a smooth function, we find from (3.23) that \( \tau_H(f) = \Delta_H f \). Therefore \( f \) is a subelliptic harmonic function if and only if \( \Delta_H f = 0 \).

We will introduce a subelliptic heat flow for maps from a sub-Riemannian manifold \((M, H, g_H; g)\) to a Riemannian manifold \((N, h)\) in order to find subelliptic harmonic maps between these manifolds. Henceforth we assume that both \( M \) and \( N \) are compact. As in the theory of harmonic maps, our strategy to solve (4.8) is to deform a given smooth map \( \varphi : M \rightarrow N \) along the gradient flow of the energy \( E_H \). This is equivalent to solving the following subelliptic harmonic map heat flow:

\[
\begin{cases}
\frac{\partial f}{\partial t} = \tau_H(f) \\
|f|_{t=0} = \varphi
\end{cases}
\]

where \( \tau_H(f(\cdot, t)) \) is the subelliptic tension field of \( f(\cdot, t) : (M, H, g_H; g) \rightarrow (N, h) \).

Now we want to give the explicit formulations for both (4.8) and (4.9), which are convenient for proving the existence theory. In view of the Nash embedding theorem, one can always assume that \( \mathcal{J} : (N, h) \rightarrow (R^K, g_E) \) is an isometric embedding in some Euclidean space, where \( g_E \) denotes the standard Euclidean metric. Let \( \nabla \) and \( D \) denote the Riemannian connections of \((N, h)\) and \((R^K, g_E)\) respectively. The second fundamental form of \( \mathcal{J} \) with respect to \((\nabla, D)\) is

\[
\beta(\mathcal{J}; \nabla, D)(Z, W) = D_W d\mathcal{J}(Z) - d\mathcal{J}(\nabla_W Z)
\]

where \( Z, W \) are any vector fields on \( N \). Recall that for a map \( f : (M, \nabla^B) \rightarrow (N, \nabla) \), we have defined its second fundamental form \( \beta(f; \nabla^B, \nabla) \) by (3.12). Applying the composition formula for second fundamental forms (see Proposition 2.20 on page 16 of [EL]) to the maps \( f : (M, \nabla^B) \rightarrow (N, \nabla) \) and \( \mathcal{J} : (N, \nabla) \rightarrow (R^K, D) \), we have

\[
\beta(\mathcal{J} \circ f; \nabla^B, D)(\cdot, \cdot) = d\mathcal{J}(\beta(f; \nabla^B, \nabla)(\cdot, \cdot)) + \beta(\mathcal{J}; \nabla, D)(df(\cdot), df(\cdot)).
\]

For simplicity, we shall identify \( N \) with \( \mathcal{J}(N) \), and write \( \mathcal{J} \circ f \) as \( u \), which is a map from \( M \) to \( R^K \). Set

\[
\tau_H(u; \nabla^B, D) = \sum_i \beta(u; \nabla^B, D)(e_i, e_i) - du(\zeta).
\]

It follows from (4.11), (4.12) that

\[
\tau_H(u; \nabla^B, D) - tr_g \beta(\mathcal{J}; \nabla, D)(df_H, df_H) = d\mathcal{J}(\tau_H(f)) -
\]

By compactness of \( N \), there exists a tubular neighborhood \( B(N) \) of \( N \) in \( R^K \) which can be realized as a submersion \( \Pi : B(N) \rightarrow N \) over \( N \). Actually the projection map \( \Pi \) is simply given by mapping any point in \( B(N) \) to its closest point in \( N \). Clearly its differential
\(d\Pi : T_yR^K \to T_yR^K\) when evaluated at a point \(y \in N\) is given by the identity map when restricted to the tangent space \(TN\) of \(N\) and maps all the normal vectors to \(N\) to the zero vector. Since \(\Pi \circ J = J : N \rightarrow R^K\) and \(\beta(J; \tilde{\nabla}, D)\) is normal to \(N\), we have

\[
\beta(J; \tilde{\nabla}, D)(\cdot, \cdot) = d\Pi(\beta(J; \tilde{\nabla}, D)(\cdot, \cdot)) + \beta(\Pi; D, D)(dJ, dJ)
\]

and thus

\[(4.14) \quad \beta(J; \tilde{\nabla}, D)(\cdot, \cdot) = \beta(\Pi; D, D)(dJ, dJ).
\]

Let \(\{y^a\}_{1 \leq a \leq K}\) be the natural Euclidean coordinate system of \(R^K\). Set \(u^a = y^a \circ u, \Pi^a = y^a \circ \Pi\). From (4.12), Remark 4.1 and (4.14), we have

\[(4.15) \quad \tau_H(u; \nabla^a, D) = \Delta_H u^a \frac{\partial}{\partial y^a},
\]

and

\[(4.16) \quad tr_g \beta(J; \tilde{\nabla}, D)(df_H, df_H) = tr_g \beta(\Pi; D, D)(du_H, du_H) = \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \frac{\partial}{\partial y^a}.
\]

where \(\Pi^a_{bc} = \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c}\). Consequently (4.13), (4.15) and (4.16) imply that

\[(4.17) \quad dJ(\tau_H(f)) = (\Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle) \frac{\partial}{\partial y^a}.
\]

Thus \(f\) is a subelliptic harmonic map if and only if \(u = (u^a) : M \rightarrow R^K\) satisfies

\[(4.18) \quad \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle = 0, \quad 1 \leq a, b, c \leq K.
\]

Inspired by the above explicit formulation for \(\tau_H(f)\), we will establish the fact that in order to solve (4.9), it suffices to solve the following system

\[(4.19) \quad \left\{ \begin{array}{l}
\frac{\partial u^a}{\partial t} = \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle, \\
u^a|_{t=0} = \varphi^a
\end{array} \right.
\]

where \(\varphi^a = y^a \circ \varphi\). Let us define a map \(\rho : B(N) \rightarrow R^K\) by

\[\rho(y) = y - \Pi(y), \quad y \in B(N).
\]

Clearly, \(\rho(y)\) is normal to \(N\) and \(\rho(y) = 0\) if and only if \(y \in N\).
Lemma 4.3. Let $u(x,t) = (u^a(x,t)) ((x,t) \in M \times [0,\delta])$ be a solution of (4.19) with initial condition $\varphi = (\varphi^a) : M \to \mathbb{R}^K$. Then the quantity

$$\int_M \rho(u(x,t)) dv_g$$

is a nonincreasing function of $t$. In particular, if $\varphi(M) \subset N$, then $u(x,t) \in N$ for all $(x,t) \in M \times [0,\delta)$.

Proof. Since $\rho(y) = y - \Pi(y)$, we have

(4.20) \[ \rho^a_b = \delta^a_b - \Pi^a_b \]

and

(4.21) \[ \rho^a_{bc} = -\Pi^a_{bc} \]

where $\rho^a_b = \frac{\partial \rho^a}{\partial y^b}$ and $\rho^a_{bc} = \frac{\partial^2 \rho^a}{\partial y^b \partial y^c}$. By applying the composition law ([EL]) to the maps $u_t : (M, \nabla^M) \to (B(N), D)$ and $\rho : (B(N), D) \to (\mathbb{R}^K, D)$, we have

(4.22) \[ \triangle_H \rho(u) = d\rho(\triangle_H u) + tr_g \beta(\rho; D, D)(du_H, du_H). \]

It follows from (4.20), (4.21), (4.22) and (4.19) that

(4.23) \[
\begin{align*}
(\triangle_H \rho(u))^a &= \rho^a_b \triangle_H u^b + \rho^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \\
&= \triangle_H u^a - \Pi^a_b \triangle_H u^b - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \\
&= \frac{\partial u^a}{\partial t} - \Pi^a_b \triangle_H u^b \\
&= \rho^a_b \frac{\partial u^b}{\partial t} + \Pi^a_b (\frac{\partial u^b}{\partial t} - \triangle_H u^b). 
\end{align*}
\]

Since $d\Pi(\frac{\partial u}{\partial t} - \triangle_H u)$ is tangent to $N$ and $\rho(u)$ is normal to $N$, we find from (4.23) that

(4.24) \[ \rho^a(u)(\triangle_H \rho(u))^a = \rho^a(u) \rho^a_b(u) \frac{\partial u^b}{\partial t}. \]

Using (4.24), (2.5), we deduce that

$$\frac{\partial}{\partial t} \int_M (\rho^a(u))^2 dv_g = 2 \int_M \rho^a \rho^a_b(u) \frac{\partial u^b}{\partial t} dv_g$$

$$= 2 \int_M \rho^a(u)(\triangle_H \rho(u))^a dv_g$$

$$= -2 \int_M |\nabla_H \rho(u)|^2 dv_g$$

$$\leq 0$$

which proves this lemma. \qed

In terms of (4.17) and Lemma 4.3, we conclude that
Theorem 4.4. Let \( \varphi : M \to N \subset R^K \) be a smooth map given by \( \varphi = (\varphi^1, ..., \varphi^K) \) in the Euclidean coordinates. If \( u : M \times [0, \delta) \to N \) is a solution of the following system
\[
\frac{\partial u^a}{\partial t} = \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle, \quad 1 \leq a \leq K,
\]
with initial condition \( u^a(x, 0) = (\varphi^a(x)) \) for all \( x \in M \), then \( u \) solves the subelliptic heat flow
\[
\frac{\partial u}{\partial t} = \tau_H(u)
\]
with initial condition \( u(x, 0) = \varphi(x) \).

A general version of the second variation formula for \( E_H \) is useful for our purpose. Although its derivation is routine, we now derive this formula for the convenience of the readers.

Proposition 4.5. Let \( F : (M, H, g_H; g) \times (-\varepsilon, \varepsilon) \to N \) be a family of maps with \( F(\cdot, 0) = f \) and \( \frac{\partial F}{\partial t} \mid_{t=0} = \nu \in \Gamma(f^{-1}TN) \). Then
\[
\left. \frac{d^2 E_H(F(\cdot, t))}{dt^2} \right|_{t=0} = - \int_M \langle \xi, \tau_H(f) \rangle dv_g + \int_M \{ \langle \tilde{\nabla}_{e_i} \nu, \tilde{\nabla}_{e_i} \nu \rangle - \tilde{\nabla}(df(e_i), \nu, df(e_i), \nu) \} dv_g
\]
where \( \xi = \tilde{\nabla}_{\frac{\partial}{\partial t}} df(\frac{\partial}{\partial t}) \mid_{t=0} \in \Gamma(f^{-1}TN) \).

Proof. At each \( t \), we compute
\[
\frac{\partial e_H(F(\cdot, t))}{\partial t} = \langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df(e_i), df(e_i) \rangle = \langle \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}), df(e_i) \rangle
\]
and
\[
\frac{\partial^2 e_H}{\partial t^2} = \langle \tilde{\nabla}_{e_i} \tilde{\nabla}_{\frac{\partial}{\partial t}} df(\frac{\partial}{\partial t}), df(e_i) \rangle + \langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df(\frac{\partial}{\partial t}), \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}) \rangle
\]
where \( \{e_i\} \) is a local orthonormal frame field for \( (H, g_H) \). Note that
\[
\tilde{\nabla}_{e_i} \tilde{\nabla}_{\frac{\partial}{\partial t}} df(\frac{\partial}{\partial t}) = \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}) + \tilde{\nabla}(df(e_i), \nu, df(e_i), \nu)
\]
(4.27)
\[
= \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}) + \tilde{\nabla}(df(X), df(\frac{\partial}{\partial t})).
\]
From (4.26) and (4.27), we obtain
\[
\frac{\partial^2 e_H}{\partial t^2} = \langle \tilde{\nabla}_{e_i} \tilde{\nabla}_{\frac{\partial}{\partial t}} df(\frac{\partial}{\partial t}), df(e_i) \rangle + \langle \tilde{\nabla}(df(\frac{\partial}{\partial t}), df(e_i)) df(\frac{\partial}{\partial t}), df(e_i) \rangle
\]
\[
+ \langle \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}), \tilde{\nabla}_{e_i} df(\frac{\partial}{\partial t}) \rangle
\]
and thus

\[
\frac{d^2 E_H(F)}{dt^2}\bigg|_{t=0} = \int_M \langle \tilde{\nabla}_{e_i} \tilde{\nabla}_{\frac{\partial}{\partial t}} F, dF(e_i) \rangle dv_g |_{t=0} \\
+ \int_M \{ \langle \tilde{\nabla}_{e_i} F, \tilde{\nabla}_{e_i} dF(\frac{\partial}{\partial t}) \rangle + \langle \tilde{R}(dF(\frac{\partial}{\partial t}), dF(e_i))dF(\frac{\partial}{\partial t}), dF(e_i) \rangle \} dv_g |_{t=0} \\
= \int_M \langle \tilde{\nabla}_{e_i} \xi, df(e_i) \rangle dv_g + \int_M \{ \langle \tilde{\nabla}_{e_i} \nu, \tilde{\nabla}_{e_i} \nu \rangle - \tilde{R}(df(e_i), \nu, df(e_i), \nu) \} dv_g
\]

(4.28)

where \( \xi = \tilde{\nabla}_{\frac{\partial}{\partial t}} F |_{t=0} \in \Gamma(f^{-1}TN) \). By (4.5) and (4.7), we have

\[
\int_M \langle \tilde{\nabla}_{e_i} \xi, df(e_i) \rangle dv_g = - \int_M \langle \xi, \tau_H(f) \rangle dv_g.
\]

In terms of (4.28) and (4.29), we complete the proof of this proposition. \( \square \)

**Corollary 4.6.** Suppose \( f : M \times [0, \delta) \to N \) is a solution of the subelliptic harmonic map heat flow \( \frac{\partial f}{\partial t} = \tau_H(f(\cdot, t)) \) for \( t \in [0, \delta) \). Then

\[
\frac{d^2 E_H(f(\cdot, t))}{dt^2} = 2 \int_M \{ \langle \tilde{\nabla}_{e_i} \tau_H(f), \tilde{\nabla}_{e_i} \tau_H(f) \rangle - \tilde{R}(df(e_i), \tau_H(f), df(e_i), \tau_H(f)) \} dv_g.
\]

**Proof.** Applying Proposition 4.5 to \( \{ f(\cdot, t) \} \) at each \( t \in [0, \delta) \), we get

\[
\frac{d^2 E_H(f(\cdot, t))}{dt^2} = - \int_M \langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \tau_H(f), \tau_H(f) \rangle dv_g \\
+ \int_M \{ \langle \tilde{\nabla}_{e_i} \tau_H(f), \tilde{\nabla}_{e_i} \tau_H(f) \rangle - \tilde{R}(df(e_i), \tau_H(f), df(e_i), \tau_H(f)) \} dv_g.
\]

Note that Proposition 4.1 gives

\[
\frac{dE_H(f(\cdot, t))}{dt} = \int_M \langle \frac{\partial f}{\partial t}, \tau_H(f) \rangle dv_g = - \int_M |\tau_H(f)|^2 dv_g.
\]

Consequently

\[
\frac{d^2 E_H(f(\cdot, t))}{dt^2} = -2 \int_M \langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \tau_H(f), \tau_H(f) \rangle dv_g.
\]

(4.31)

This corollary follows immediately from (4.30) and (4.31). \( \square \)
5. Existence of Subelliptic Harmonic Maps

5.1 Short-time Existence

For bounded functions \( f : M \times [0, \delta) \to R \) and \( \psi : M \to R \), let us consider the subelliptic heat flow

\[
\begin{align*}
\begin{cases}
(\Delta H - \frac{\partial}{\partial t}) w &= f(x,t), \\
 w|_{t=0} &= \psi.
\end{cases}
\end{align*}
\]

(5.1)

By Duhamel’s principle, we know that one solution of (5.1) is given by

\[
w(x,t) = \int_M K(x,y,t)\psi(y)dv_g - \int_0^t \int_M K(x,y,t-s)f(y,s)dv_g(y)ds.
\]

(5.2)

First we establish the following short-time existence theorem.

**Theorem 5.1.** Let \((M^{m+d}, H, g_H, g)\) be a compact sub-Riemannian manifold, and \((N^n, h) \subset R^K\) be a compact submanifold with the induced Euclidean metric. For any smooth map \( \varphi : M \to N \), there exists \( \delta_0 > 0 \) such that the subelliptic harmonic map heat flow with initial condition

\[
\begin{align*}
\begin{cases}
(\Delta H - \frac{\partial}{\partial t}) u^a(x,t) &= \Pi_{bc}^a (\nabla^H u^b, \nabla^H u^c) \\
u^a(x,0) &= \varphi^a(x), \quad 1 \leq a, b, c \leq K.
\end{cases}
\end{align*}
\]

admits a smooth solution on \( M \times [0, \delta_0) \), where \( \delta_0 \) is a constant depending only on \( \sup_M e(\varphi) \) and geometric quantities of both \( M \) and \( N \).

**Proof.** Writing \( u = (u^a(x,t))_{1 \leq a \leq K} \), the subelliptic harmonic map heat flow may be expressed as

\[
\begin{align*}
\begin{cases}
(\Delta H - \frac{\partial}{\partial t}) u &= F(x,t) \\
u(x,0) &= \varphi(x)
\end{cases}
\end{align*}
\]

(5.3)

where \( F(x,t) = (\Pi_{bc}^a (\nabla^H u^b, \nabla^H u^c)) \) depends on the unknown solution \( u \) itself. In terms of (5.2), we can define a sequence of approximate solutions for (5.3) inductively as follows:

\[
u_0(x,t) = \int_M K(x,y,t)\varphi(y)dv_g(y)
\]

(5.4)

\[
u_k(x,t) = u_0(x,t) - \int_0^t \int_M K(x,y,t-s)f_{k-1}(y,s)dv_g(y)ds
\]

where

\[
f_{k-1}(y,s) = (\Pi_{bc}^a (\nabla^H u_{k-1}^b, \nabla^H u_{k-1}^c))(y,s), \quad k = 1, 2, 3, \cdots.
\]

(5.5)

Clearly \( u_0 \) and \( u_k : M \to R^K \) satisfy respectively

\[
\begin{align*}
\begin{cases}
(\Delta H - \frac{\partial}{\partial t}) u_0 &= 0, \\
u_0(x,0) &= \varphi(x)
\end{cases}
\end{align*}
\]

(5.6)
and

\begin{equation}
(5.7) \quad \left\{ \begin{array}{l}
(\Delta_H - \frac{\partial}{\partial t}) u_k = F_{k-1}(x,t) \\
u_k(x,0) = \varphi(x), \quad k = 1, 2, \ldots.
\end{array} \right.
\end{equation}

We set

\begin{equation}
(5.8) \quad \Lambda = \sup_{B(N),a,b,c,d} \left\{ |\Pi^a_{bc}|, \left| \frac{\partial \Pi^a_{bc}}{\partial y^d} \right| \right\}
\end{equation}

where \((y^1, \ldots, y^K)\) are coordinates of \(R^K\), and \(B(N)\) is the tubular neighborhood of \(N\) on which \(\Pi\) is defined. Let us also introduce

\begin{equation}
(5.9) \quad p_{k-1}(t) = \sup_{M \times [0,t]} \sqrt{e_H(u_{k-1})}, \quad k = 1, 2, \ldots.
\end{equation}

which is obviously non-decreasing in \(t\). From (5.5) and (5.9), we have

\begin{equation}
(5.10) \quad \sup_{M \times [0,t]} |F_{k-1}(x,s)| \leq \Lambda p_{k-1}^2(t).
\end{equation}

Note that

\begin{equation}
(5.11) \quad |u_0| \leq \|\varphi\|_{C^0} = \sup_{x \in M} \sqrt{\sum_{a=1}^{K} (\varphi^a(x))^2},
\end{equation}

since \(\int_M K(x,y,t)dy = 1\). Here and afterwards, \(\| \cdot \|_{C^0}\) denotes the \(C^0\)-norm of functions or tensor fields on \(M\). From (5.4), (5.10) and (5.11), we derive that

\begin{equation}
(5.12) \quad |u_k - u_0| \leq \Lambda tp_{k-1}^2
\end{equation}

and

\begin{equation}
(5.13) \quad |u_k| \leq \Lambda tp_{k-1}^2(t) + \|\varphi\|_{C^0}.
\end{equation}

Note that \(\tau_H(u_0) = \Delta_H u_0\) for the map \(u_0 : M \to R^K\). In view of (3.32), we have

\begin{equation}
\left(\Delta_H - \frac{\partial}{\partial t}\right) e(u_0) \geq -Ce(u_0),
\end{equation}

or equivalently,

\begin{equation}
(5.14) \quad \left(\Delta_H - \frac{\partial}{\partial t}\right) (e^{-Ct}e(u_0)) \geq 0.
\end{equation}

Consequently the Maximum principle (see Lemma 2.4) implies that

\[ e^{-Ct}e(u_0) \leq e(\varphi) \]
and thus
\[(5.15)\quad p_0(t) \leq \sqrt{e^{Ct}e(\varphi)}.\]

Using Lemma 2.3, (5.4) and (5.10), we may deduce
\[
|\nabla^H u_k(x,t)| \leq \left|\nabla^H u_0\right| + \int_0^t \int_M \left|\nabla^H K(x,y,t-s)\right| |F_{k-1}(y,s)| dv_g(y)
\leq \left|\nabla^H u_0\right| + C_1 \Lambda p_{k-1}^2(t) t^\beta
\]
hence implying
\[(5.16)\quad p_k(t) \leq C_1 \Lambda p_{k-1}^2(t) t^\beta + p_0(t).\]

For any \(0 < \varepsilon < 1\), by choosing \(\delta\) sufficiently small, (5.15) yields that
\[
C_1 \Lambda \delta^\beta p_0(\delta) \leq C_1 \Lambda \delta^\beta \sqrt{e^{C\delta}e(\varphi)} \leq \frac{\varepsilon}{4}.
\]

By an inductive argument, we get
\[(5.17)\quad C_1 \Lambda \delta^\beta p_k(\delta) \leq \frac{\varepsilon}{2}\]
since (5.16) gives
\[
C_1 \Lambda \delta^\beta p_k(\delta) \leq \left(C_1 \Lambda \delta^\beta p_{k-1}(\delta)\right)^2 + C_1 \Lambda \delta^\beta p_0(\delta)
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}
= \frac{\varepsilon}{2}.
\]

Consequently
\[(5.18)\quad p_k(\delta) \leq C_2 \varepsilon \delta^{-\beta}.\]

We define the following space of functions,
\[C^1_H(M, R^K) = \{f : M \rightarrow R^K \mid f \in C^0, \quad \nabla^H f \in C^0\}\]
which is endowed with the norm
\[
\|f\|_{C^1_H} = \|f\|_{C^0} + \|\nabla^H f\|_{C^0}.
\]

It is known that \((C^1_H(M, R^K), \|\cdot\|_{C^1_H})\) is a Banach space. From (5.13) and (5.18), one has
\[
\|u_k\|_{C^1_H(M, R^K)} \leq C_3(C_2, \varepsilon, \delta).
\]
In terms of (5.12) on \( M \times [0, \delta] \) and using (5.17), we deduce that

\[
|u_k(x, t) - u_0(x, t)| \leq \Lambda \delta^2 p_{k-1}^2(\delta)
\]

(5.19)

\[
\leq \frac{\varepsilon^2 \delta^{1-2\beta}}{4C_1 \Lambda}.
\]

The validity of the inequality (5.17) depends on choosing a sufficiently small \( \delta \). Note also that \( 1 - 2\beta > 0 \). From (5.19), we find that all maps \( u_k \) \( (k = 1, 2,...) \) will map \( M \) into \( B(N) \) by choosing both \( \varepsilon \) and \( \delta \) sufficiently small since \( |u_0(x, t) - \varphi(x)| \) can be chosen to be sufficiently small for small \( t \) by continuity of \( u_0 \).

Now we want to show that \( \{u_k(x, t)\} \) form a Cauchy sequence in \( C^1_H(M, R^K) \) for sufficiently small \( t \). Let us define

\[
X_k(t) = \sup_{M \times [0, t]} \{ |u_k(x, s) - u_{k-1}(x, s)| + |\nabla^H_x u_k(x, s) - \nabla^H_x u_{k-1}(x, s)| \}
\]

(5.20)

which is a non-decreasing function of \( t \). Note that

\[
F_k(x, t) - F_{k-1}(x, t) = (\Pi_{bc}^b(u_k)(\nabla^H_{x} u^b_{k}, \nabla^H_{x} u^c_{k}) - \Pi_{bc}^b(u_{k-1})(\nabla^H_{x} u^b_{k-1}, \nabla^H_{x} u^c_{k-1})
\]

\[
= (\Pi_{bc}^b(u_k) - \Pi_{bc}^b(u_{k-1}))(\nabla^H_{x} u^b_{k}, \nabla^H_{x} u^c_{k})
\]

\[
+ (\Pi_{bc}^b(u_k) - \Pi_{bc}^b(u_{k-1}))(\nabla^H_{x} u^b_{k} - \nabla^H_{x} u^b_{k-1}, \nabla^H_{x} u^c_{k})
\]

\[
+ (\Pi_{bc}^b(u_k) - \Pi_{bc}^b(u_{k-1}))(\nabla^H_{x} u^b_{k-1}, \nabla^H_{x} u^c_{k} - \nabla^H_{x} u^c_{k-1})
\]

(5.21)

Using (5.18) and the estimate

\[
|\Pi_{bc}^b(u_k) - \Pi_{bc}^b(u_{k-1})| \leq \Lambda |u_k - u_{k-1}|,
\]

we may derive from (5.21) that

\[
\sup_{M \times [0, t]} |F_k(x, t) - F_{k-1}(x, t)| \leq C_4 X_k(t)(p_k^2(t) + p_{k-1}(t) + p_k(t) + p_{k-1}(t))
\]

(5.22)

\[
\leq C_5 X_k(t)
\]

for any \( t \leq \delta \). Consequently we get the following two estimates

\[
|u_k - u_{k-1}| \leq \int_0^t \int_M K(x, y, t-s)|F_{k-1}(y, s) - F_{k-2}(y, s)| \, dv_g(s) \, ds
\]

\[
\leq C_5 t X_{k-1}(t)
\]

and

\[
|\nabla^H_x u_k - \nabla^H_x u_{k-1}|
\]

\[
\leq \int_0^t \int_M |\nabla^H_x K(x, y, t-s)| \cdot |F_{k-1}(y, s) - F_{k-2}(y, s)| \, dv_g(s) \, ds
\]

\[
\leq C_6 t^\beta X_{k-1}(t)
\]

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which imply

\[(5.23)\quad X_k(t) \leq C_7 t^\beta X_{k-1}(t)\]

for \(k \geq 2\). For \(k = 1\), using \(t < 1\), we have from (5.4) and (5.15) that

\[(5.24)\quad |u_1(x,t) - u_0(x,t)| \leq \int_0^t \int_M K(x,y,t - s) |F_0(y,s)| dv_g(y) ds \leq t \Lambda p_0^2(t) \leq t \Lambda e^C e(\varphi)\]

and

\[(5.25)\quad |\nabla^H_x u_1(x,t) - \nabla^H_x u_0(x,t)| \leq \int_0^t \int_M |\nabla^H_x K(x,y,t - s)| |F_0(y,s)| dv_g(y) ds \leq C_1 t^\beta \Lambda p_0^2(t) \leq C_1 t^\beta \Lambda e^C e(\varphi).\]

It follows that

\[(5.26)\quad X_1(t) \leq C_8(C_7 t^\beta) e(\varphi).\]

By iterating (5.23) and using (5.26), we get

\[(5.27)\quad X_k(t) \leq C_8(C_7 t^\beta)^k e(\varphi).\]

We may choose a sufficiently small positive number \(\delta_0\) such that \(\delta_0 \leq \delta\) and \(C_7 \delta_0^\beta < 1\). Hence (5.27) implies that for any \(i < j\)

\[
\sup_{[0,\delta_0]} \| u_i(\cdot,t) - u_j(\cdot,t) \|_{C^1_H(M)} \leq \sum_{k=i+1}^j X_k(\delta_0) \leq C_9 \sum_{k=i+1}^j (C_7 \delta_0^\beta)^k
\]

which tends to 0 as \(i, j \to \infty\). Hence there exists \(u \in C^0(M \times [0,\delta_0], B(N))\) with \(u(\cdot,t) \in C^1_H(M, B(N))\) for each \(t \in [0,\delta_0]\), such that \(u_k \to u\) and \(\nabla^H u_k \to \nabla^H u\) uniformly on \(M \times [0,\delta_0]\). Consequently

\[F_k(x,t) \to F(x,t) = (\Pi^a_{bc}(u) (\nabla^H u^b, \nabla^H u^c))\]

and thus (5.4) implies that \(u\) is given by

\[u(x,t) = \int_M K(x,y,t) \varphi(y) dv_g(y) - \int_0^t \int_M K(x,y,t - s) F(y,s) dv_g(y) ds.\]

Clearly \(u\) is a weak solution of the subelliptic harmonic map heat flow. In terms of Theorem 2.1 and Remark 2.1, by a bootstrapping argument, we find that \(u \in C^\infty(M \times (0,\delta_0), N)\) satisfies (4.19). \(\square\)

Next we give the following uniqueness theorem.
Theorem 5.2. Let $u$ and $v$ be solutions on $M \times [0, \delta)$ to the subelliptic harmonic map heat flow with the same initial condition: $u(x, 0) = v(x, 0) = \varphi(x)$. Then $u$ and $v$ are identical.

Proof. Set $\Psi = \sum_{a=1}^{K} (u^a - v^a)^2$. A direct computation gives

$$
\left( \Delta_H - \frac{\partial}{\partial t}\right) \Psi = 2 \sum_a (u^a - v^a) \left( \Delta_H - \frac{\partial}{\partial t}\right) (u^a - v^a) + 2 \sum_a | \nabla^H (u^a - v^a) |^2
$$

(5.28)

$$
= 2 \sum_A (u^a - v^a)(F^a(u) - F^a(v)) + 2 \sum_a | \nabla^H (u^a - v^a) |^2 .
$$

For any $0 < \delta_1 < \delta$, we set

$$
p_{\delta_1} = \sup_{M \times [0, \delta_1]} \sqrt{e_H(u)}, \quad q_{\delta_1} = \sup_{M \times [0, \delta_1]} \sqrt{e_H(v)}.
$$

Writing $F^a(u) - F^a(v)$ in a similar way as (5.21), one may get

$$
| F^a(u) - F^a(v) | \leq C(\delta_1, \Lambda, p_{\delta_1}, q_{\delta_1}) \left( \Psi^{\frac{1}{2}} + \sum_b | \nabla^H (u^b - v^b) | \right)
$$

(5.29)

on any $[0, \delta_1]$ with $\delta_1 < \delta$, where $C(\delta_1, \Lambda, p_{\delta_1}, q_{\delta_1})$ is a constant depending on $\delta_1, \Lambda, p_{\delta_1}$ and $q_{\delta_1}$. It follows immediately from (5.28) and (5.29) that

$$
\left( \Delta_H - \frac{\partial}{\partial t}\right) \Psi \geq -\tilde{C}(\delta_1, \Lambda, p_{\delta_1}, q_{\delta_1}) \Psi
$$

on $[0, \delta_1]$ for some positive constant $\tilde{C}(\delta_1, \Lambda, p_{\delta_1}, q_{\delta_1})$. This implies

$$
\left( \Delta_H - \frac{\partial}{\partial t}\right) (e^{-\tilde{C}t} \Psi) \geq 0
$$

on $[0, \delta_1]$ and thus the maximum principle asserts that $\Psi = 0$ on $[0, \delta_1]$. Since $\delta_1$ is arbitrary, we conclude that $\Psi = 0$ on $[0, \delta)$. □

5.2 Long-time Existence

We first give a criteria for the long-time existence of the subelliptic harmonic map heat flow.

Lemma 5.3. Suppose $u = J \circ f$ is a solution of the subelliptic harmonic heat flow on $M \times [0, \delta_{\max})$, where $\delta_{\max}$ is the maximal existence time for the solution $u$. If $\delta_{\max} < \infty$, then

$$
\liminf_{t \to \delta_{\max}} \{ \sup_M e(u(\cdot, t)) \} = +\infty.
$$

In other words, if

$$
\liminf_{t \to \delta - 0} \{ \sup_M e(u(\cdot, t)) \} < +\infty
$$

on any $M \times [0, \delta)$, where the solution $u$ exists, then $\delta_{\max} = \infty$ (long-time existence).
Proof. Suppose \( u \) is a solution of the subelliptic harmonic map heat flow on \( M \times [0, \delta_{\text{max}}) \) with \( \delta_{\text{max}} < \infty \). We want to prove that

\[
\liminf_{t \to \delta_{\text{max}}} \{ \sup_M e(u(\cdot, t)) \} = +\infty.
\]

Otherwise, there is a sequence \( t_k \to \delta_{\text{max}} \) such that \( \sup_M e(u(\cdot, t_k)) \leq C_0 \) for some positive number \( C_0 \). By Theorem 5.1, there exists a positive number \( \delta(C_0, M, N) \) depending only on \( C_0 \) and the geometric quantities of \( M \) and \( N \) such that the subelliptic harmonic heat flow admits a solution with \( u_{t_k} \) as its initial condition on \([t_k, t_k + \delta(C_0, M, N)]\). Taking a sufficiently large \( k \), the uniqueness in Theorem 5.2 enables us to obtain a solution on \( M \times [0, \delta_{\text{max}} + \delta'] \) for some positive number \( \delta' \). This contradicts to the assumption that \( \delta_{\text{max}} \) is the maximal existence time. \( \Box \)

From now on, we assume that \((N, h)\) has non-positive sectional curvature. Let \( f : M \to N \) be a solution of the subelliptic harmonic map heat flow on \([0, \delta)\). By (3.32) we get

\[
(\triangle_H - \frac{\partial}{\partial t})e(f) \geq -Ce(f)
\]

for some constant, that is,

(5.30) \[
(\triangle_H - \frac{\partial}{\partial t})(e^{-Ct}e(f)) \geq 0.
\]

Lemma 5.4. Let \( f : M \to N \) be a solution of the subelliptic harmonic map heat flow on \([0, \delta)\). Suppose \((N, h)\) has non-positive sectional curvature. Set \( \alpha = \min\{R_0, \sqrt{\delta}\} \), where \( R_0 \) is given by Lemma 2.4. Then

\[
e(f(\cdot, t)) \leq C(\varepsilon_0)E(f(\cdot, t - \varepsilon_0))
\]

for \( t \in [\varepsilon_0, \delta) \), where \( \varepsilon_0 \) is a fixed number in \((0, \alpha^2/2)\).

Proof. Using (5.30) and applying the mean value inequality in Lemma 2.4 to \( e^{-C(s+t)}e((x, s + t)) \) for \( t \in (0, \alpha^2) \) and \( s + t < \delta \), we obtain

\[
e^{-C(s+t)}e(f(x, s + t)) \leq Bt^{-\frac{\alpha^2}{2}} \int_M e^{-Cs}e(f(y, s))dv(y),
\]

which implies

(5.31) \[
e(f(x, s + t)) \leq Bt^{-\frac{\alpha^2}{2}}e^{Ct} \int_M e(f(y, s))dv(y).
\]

By choosing a fixed \( t = \varepsilon_0 \in (0, \alpha^2/2) \), we get the estimate

\[
e(f(x, s + \varepsilon_0)) \leq C(\varepsilon_0)E(f(\cdot, s))
\]

where \( C(\varepsilon_0) \) is a constant depending on \( \varepsilon_0 \). \( \Box \)

In view of Lemmas 5.3 and 5.4, one needs to estimate \( E(f) \) for a solution \( f \) of the subelliptic harmonic map heat flow in order to obtain a long-time existence result. Note that Proposition 4.1 implies

\[
\frac{d}{dt} E_H(f(\cdot, t)) = - \int_M |\tau_H(f(\cdot, t))|^2 dv_y \leq 0.
\]

Consequently \( E_H(f(\cdot, t)) \leq E_H(\varphi) \), where \( \varphi \) is the initial map of \( f \). Therefore it is enough to estimate \( E_V(f(\cdot, t)) \) for the long-time existence.
Theorem 5.5. Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold and let \((N, h)\) be a compact Riemannian manifold with nonpositive sectional curvature. Then for any map \(\varphi : M \to N\), the subelliptic harmonic map heat flow (4.9) admits a global smooth solution \(f : M \times [0, \infty) \to N\).

Proof. By Schwarz inequality and the curvature assumption on \(N\), we get immediately from (3.29) that

\[
(\triangle_H - \frac{\partial}{\partial t})e_V(f) \geq -C_1e_H(f) - C_2e_V(f)
\]

for some positive constants \(C_1\) and \(C_2\). Integrating (5.33) gives

\[
\frac{d}{dt}E_V(f) \leq CE_H(f) + C_2E_V(f) \\
\leq CE_H(\varphi) + C_2E_V(f) \\
= C_1 + C_2E_V(f)
\]

which implies

\[
\int_0^t \frac{dE_V(f)}{C_1 + C_2E_V(f)} \leq t.
\]

It follows that

\[
\ln (C_1 + C_2E_V(f)) - \ln (C_1 + C_2E_V(\varphi)) \leq C_2t
\]

that is,

\[
E_V(f) \leq \frac{1}{C_2} \left\{ e^{C_2t} (C_1 + C_2E_V(\varphi)) - C_1 \right\}
\]

Hence we find that the solution \(f(\cdot, t)\) does not blow up at any finite time. \(\square\)

5.3 Eells-Sampson type results

We will establish Eells-Sampson type results in following two cases: the source manifolds are either step-2 sub-Riemannian manifolds or step-\(r\) sub-Riemannian manifolds whose sub-Riemannian structures come from some Riemannian foliations.

5.3.1 Step-2 sub-Riemannian manifolds

Recall that \(T(\cdot, \cdot)\) denotes the torsion of the Bott connection \(\nabla^B\) on \((M, H, g_H, g)\). Let \(\pi : S(V) \to M\) be the unit sphere bundle of the vertical bundle \(V\), that is, \(S(V) = \{ v \in V : \|v\|_g = 1 \}\). For any \(v \in S(V)\), the \(v\)-component of \(T(\cdot, \cdot)\) is given by \(T^v(\cdot, \cdot) = \langle T(\cdot, \cdot), v \rangle\). Then we have a smooth function \(\eta(v) = \frac{1}{2}\|T^v\|^2_g : S(V) \to R\). Using Lemma 3.1 and an adapted frame field \(\{e_A\}_{A=1, \ldots, m+d}\) for \((M^{m+d}, H, g_H; g)\), we obtain

\[
\eta(v) = \sum_{1 \leq i < j \leq m} (T_{ij}^\alpha)^2 \langle e_\alpha, v \rangle^2 \\
\]

(5.34)

\[
= \sum_{1 \leq i < j \leq m} \langle [e_i, e_j], v \rangle^2.
\]

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Lemma 5.6. \( H \) is 2-step bracket generating if and only if \( \eta(v) > 0 \) for each \( v \in S(V) \).

Proof. For any \( v \in S(V) \) with \( \pi(v) = x \), we let \( X, Y \) be any local sections of \( H \) around \( x \). Writing \( X = X^i e_i \) and \( Y = Y^j e_j \), we get

\[
[X, Y]_x \equiv X^i(x)Y^j(x)[e_i, e_j]_x \mod \{H_x\} \\
\equiv X^i(x)Y^j(x)[e_i, e_j, e_\alpha]_x e_\alpha(x) \mod \{H_x\}
\]

Hence \( H \) is 2-step bracket generating for \( TM \) if and only if

\[
\text{span}_{1 \leq i, j \leq m} \{[e_i, e_j]_x \} \equiv V_x \mod \{H_x\}
\]

at each point \( x \in M \). By (5.34), this is equivalent to \( \eta(v) > 0 \). \( \square \)

Lemma 5.7. Let \((M, H, g_H; g)\) be a compact step-2 sub-Riemannian manifold and set \( \eta_{\min} = \min_{v \in S(V)} \eta(v) \). Let \( N \) be a compact Riemannian manifold with non-positive sectional curvature. Suppose \( f : M \times [0, \delta) \to N \) is a solution of the subelliptic harmonic map heat flow. Then, for any given \( t_0 \in (0, \delta) \), we have

\[
E_V(f(\cdot, t)) \leq E_V(f(\cdot, t_0)) + \frac{4}{\eta_{\min}} \left( \int_M \left| \tau_H(f(\cdot, t_0)) \right|^2 + CE_H(f(\cdot, t_0)) \right).
\]

for any \( t \in (t_0, \delta) \).

Proof. The compactness of \( M \) implies that \( S(V) \) is compact, so there exists a point \( v \in S(V) \) such that \( \eta_{\min} = \eta(v) \). Since \( H \) is 2-step bracket generating, we know from Lemma 5.6 that \( \eta_{\min} > 0 \). Let \( \varepsilon \) be a fixed positive number with \( \varepsilon \leq \frac{\eta_{\min}}{4} \). From (3.31), (3.17) and (5.34), one has

\[
(\Delta_H - \frac{\partial}{\partial t})e(f) \geq -C_\varepsilon e_H(f) - \varepsilon e_V(f) + (f_{ik})^2 + \frac{1}{2}(f_{\alpha k})^2 \\
\geq -C_\varepsilon e_H(f) - \varepsilon e_V(f) + \sum_l \sum_{i<j} ((f_{ij}^l)^2 + (f_{ji}^l)^2) \\
= -C_\varepsilon e_H(f) - \varepsilon e_V(f) + \frac{1}{2} \sum_l \sum_{i<j} ((f_{ij}^l + f_{ji}^l)^2 + (f_{ij}^l - f_{ji}^l)^2) \\
\geq -C_\varepsilon e_H(f) - \varepsilon e_V(f) + \frac{1}{2} \sum_l \sum_{i<j} (f_{ij}^l)^2(T_{ij}^\alpha)^2 \\
= -C_\varepsilon e_H(f) - \varepsilon e_V(f) + \frac{1}{2} \sum_l \sum_{\alpha} \sum_{i<j} (f_{ij}^l)^2 \eta(e_\alpha) \\
\geq -C_\varepsilon e_H(f) - \varepsilon e_V(f) + \frac{1}{2} \eta_{\min} e_V(f).
\]

Integrating (5.35) over \( M \) yields

\[
\frac{d}{dt} E(f) \leq C_\varepsilon E_H(f) + \varepsilon E_V(f) - \frac{\eta_{\min}}{2} E_V(f) \\
\leq C_\varepsilon E_H(f) - \frac{\eta_{\min}}{4} E_V(f).
\]
Consequently
\[(5.36)\quad \frac{d}{dt} E_H(f) + \frac{d}{dt} E_V(f) + \frac{\zeta_{\min}}{4} E_V(f) \leq C \varepsilon E_H(f(\cdot, t_0)).\]

By Corollary 4.6, we have
\[\frac{d^2}{dt^2} E_H(f) \geq 0,\]
which implies that
\[(5.37)\quad \frac{d}{dt} E_H(f(\cdot, t)) \geq \frac{d}{dt} E_H(f(\cdot, t_0)) = -\int_M |\tau_H(f(\cdot, t_0))|^2.\]

Set \(A = \int_M |\tau_H(f(\cdot, t_0))|^2 + C \varepsilon E_H(f(\cdot, t_0)).\) From (5.36) and (5.37), it follows that
\[\frac{d}{dt} E_V(f(\cdot, t)) + \frac{\eta_{\min}}{4} E_V(f(\cdot, t)) \leq A\]
that is,
\[(5.38)\quad \frac{d}{dt} \left( e^{\frac{\eta_{\min}}{4} t} E_V(f(\cdot, t)) \right) \leq A e^{\frac{\eta_{\min}}{4} t}.\]

By integrating (5.38) over \([t_0, t],\) we find
\[e^{\frac{\eta_{\min}}{4} t} E_V(f(\cdot, t)) - e^{\frac{\eta_{\min}}{4} t_0} E_V(f(\cdot, t_0)) \leq \frac{4A}{\eta_{\min}} \left( e^{\frac{\eta_{\min}}{4} t} - e^{\frac{\eta_{\min}}{4} t_0} \right).\]

Hence
\[E_V(f(\cdot, t)) \leq e^{\frac{\eta_{\min}}{4}(t_0-t)} E_V(f(\cdot, t_0)) + \frac{4A}{\eta_{\min}} (1 - e^{\frac{\eta_{\min}}{4}(t_0-t)})\]
\[\leq E_V(f(\cdot, t_0)) + \frac{4A}{\eta_{\min}}.\]

\[\square\]

**Theorem 5.8.** Let \((M, H, g_H; g)\) be a compact step-2 sub-Riemannian manifold and let \(N\) be a compact Riemannian manifold with non-positive sectional curvature. Then, for any smooth map \(\varphi : M \to N,\) there exists a \(C^\infty\) solution \(f(x, t)\) of the subelliptic harmonic map heat flow (4.9) on \(M \times [0, \infty).\) Moreover, there exists a sequence \(t_i \to \infty\) such that \(f(x, t_i) \to f_\infty(x)\) uniformly, as \(t_i \to \infty,\) to a \(C^\infty\) subelliptic harmonic map \(f_\infty : M \to N.\)

**Proof.** Let \(\mathcal{J} : N \hookrightarrow R^K\) be an isometric embedding. Theorem 4.4 tells us that solving (4.9) is equivalent to solving (4.19). In view of Theorem 5.1 and Lemmas 5.3, 5.4, 5.7, we conclude that (4.19) admits a global \(C^\infty\) solution \(u = \mathcal{J} \circ f : M \times [0, \infty) \to N \subset R^K\) with \(f\) solving (4.9).

Now we investigate the convergence of \(u\) as \(t \to \infty.\) First, one observes that the compactness of \(N\) and the uniform boundedness of \(e(u_t)\) implies that the 1-parameter family of maps \(u(\cdot, t)\) form a uniformly bounded and equicontinuous family of maps. Therefore, by Arzela-Ascoli Theorem, there exists a sequence \(t_i \to \infty\) such that
\[(5.39)\quad u(\cdot, t_i) \to u_\infty(\cdot)\]
to a Lipschitz map $u_\infty : M \to N \subset R^K$.

Let us now deduce the equation which $|f_t|^2 = |df(\frac{\partial}{\partial t})|^2$ satisfies. By a direct computation, using the commutation formulas (3.38) and (3.41), we have

\[
\left(\triangle_H - \frac{\partial}{\partial t}\right) (f^I_t)^2 = 2 (f^I_{tk})^2 + 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

\[
= 2 (f^I_{tk})^2 + 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

\[
= 2 (f^I_{tk})^2 + 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

\[
= 2 (f^I_{tk})^2 + 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

\[
= 2 (f^I_{tk})^2 + 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

\[
= 2 (f^I_{tk})^2 - 2 f^I_t j^I_{tk} - 2 \zeta^I f^I_t j^I_{tt} - 2 f^I_t f^I_{tt}
\]

In terms of the curvature condition of $N$, (5.40) yields

\[
\left(\triangle_H - \frac{\partial}{\partial t}\right) |f_t|^2 \geq 0.
\]

By integrating (5.32) on any $[0, \delta]$, we get

\[
\int_0^\delta \int_M |f_s|^2 dv_g ds = E_H(\varphi) - E_H(\delta)
\]

which implies that

\[
\int_0^\infty \int_M |f_s|^2 dv_g ds < \infty.
\]

Therefore there exists a sequence $s_n \to \infty$ such that $\int_M |f_{s_n}|^2 dv_g \to 0$. From Corollary 4.6, we see that

\[
\frac{d^2 E_H(f_t)}{dt^2} = - \frac{d}{dt} \left\{ \int_M |f_t|^2 dv_g \right\} \geq 0.
\]

Consequently $\int_M |f_t|^2 dv_g$ is decreasing in $t$. Hence we find that

\[
\int_M |f_t|^2 dv_g \to 0
\]

as $t \to \infty$. Clearly the function $\phi(x, t) = |f_{s+t}|^2$ also satisfies (5.41) for any given $s > 0$. Applying Lemma 2.4 to the function $\phi(x, t)$ for $0 < t < R^2_0$, we obtain

\[
|f_{s+t}|^2 \leq Bt^{\frac{Q}{2}} \int_M |f_s|^2 dv_g.
\]

Then, for $t = \frac{R^2_0}{2}$, (5.43) gives that

\[
|f_{s + \frac{R^2_0}{2}}|^2 \leq \frac{2^Q B}{R^Q_0} \int_M |f_s|^2 dv_g
\]

(5.44)
for any $s > 0$. From (5.42) and (5.44), it follows that

\[(5.45) \sup_{x \in M} |u_t|^2 (x,t) \to 0\]

as $t \to \infty$. Clearly (5.39) and (5.45) imply that $u_\infty$ is a weak solution of (4.18). By Theorem 2.1, we can now conclude that $u_\infty$ is smooth, that is, $f_\infty$ is a smooth subelliptic harmonic map from $M$ to $N$. \(\square\)

**Remark 5.1.** It would be interesting to note that the existence for Theorem 5.8 is independent of the choice of the extension $g$ for $g_H$.

### 5.3.2 Riemannian foliations with basic mean curvature vector

Let $(M,H,g_H;g)$ be a sub-Riemannian manifold corresponding to a Riemannian foliation $\mathfrak{F}$ on $(M,g)$ as in Example 1.4. A foliation being Riemannian means that it is locally a Riemannian submersion. In order to describe the local geometry of $(M,g;\mathfrak{F})$, we may assume temporarily that the foliation is given by a Riemannian submersion $\pi: (M,g) \to (Z,g_Z)$. Then a vector field $X$ on $M$ is said to be projectable if it is $\pi$-related to a vector field $\tilde{X}$ on $B$, that is, $\tilde{X} \circ \pi = \pi_* (X)$.

**Lemma 5.9.** Let $(M,H,g_H;g)$ be a sub-Riemannian manifold corresponding to a Riemannian submersion $\pi: (M,g) \to (Z,g_Z)$. Let $X$ be a horizontal vector field on $(M,H,g_H;g)$. Then $X$ is projectable if and only if $\nabla^B_\xi X = 0$ for any $\xi \in V$.

**Proof.** Let $\Gamma(V)$ denote the space of vertical vector fields. From [Mo], [GW], we know that a vector field $X$ on $M$ is projectable if and only if $[\xi,X] \in \Gamma(V)$ for any $\xi \in \Gamma(V)$, that is $\pi_H([\xi,X]) = 0$. According to (1.13), the lemma follows. \(\square\)

In what follows, given a Riemannian submersion $\pi: (M,g) \to (Z,g_Z)$, a vector field $X$ on $M$ is said to be basic if it is both horizontal and projectable.

**Lemma 5.10.** (cf. Lemma 1.4.1 in [GW]) Let $(M,H,g_H;g)$ be as in Lemma 5.9. If $X,Y \in \Gamma(M)$ are basic, then so is $\nabla^B_X Y$.

Now we consider the general case that $(M,g;\mathfrak{F})$ is a Riemannian foliation. One says that $(M,g;\mathfrak{F})$ is tense if its mean curvature vector field $\zeta$ is parallel with respect to $\nabla^B$ along the leaves, that is, $\nabla^B_\xi \zeta = 0$ for any $\xi \in V$. In view of Lemma 5.9, we know that this condition means that $\zeta$ is (locally) basic.

**Lemma 5.11.** Let $(M^{m+d},H,g_H;g)$ be a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation $(M,g;\mathfrak{F})$. Let $N$ be a compact Riemannian manifold with non-positive sectional curvature. If $f: M \times [0,\delta] \to N$ is a solution of the subelliptic harmonic map heat flow, then $E_V(f_t)$ is decreasing. In particular, $E_V(f_t) \leq E(\varphi)$.

**Proof.** We first show that the curvature tensor of $\nabla^B$ satisfies

\[(5.46) \quad R^A_{jakl} = 0\]

with respect to an adapted frame $\{e_A\}_{A=1}^{m+d}$. For any point $p \in M$, there exists a neighborhood $U$ of $p$ such that the restriction of $\mathfrak{F}$ to $U$ corresponds to a Riemannian submersion $\pi: (U,g) \to Z$.
(Z, gZ), since Φ is Riemannian. Clearly we may choose an adapted frame field \{e_A\}_{A=1}^{m+d} such that \{e_1, ..., e_m\} are basic with respect to π, that is, e_j ∈ Γ(U, H) and ∇^u_\xi e_j = 0 for any \xi ∈ V (1 ≤ j ≤ m) due to Lemma 5.9. In view of Lemmas 5.9 and 5.10, we also have ∇^u_\xi e_j = ∇^u_\xi e_k = 0 and ∇^\xi [e_\alpha, e_k] = ∇^\xi [e_\alpha, e_k] = 0, where \[e_\alpha, e_k\]^V denotes the vertical component of \[e_\alpha, e_k\]. Consequently

\[
R^A_{jak} = \langle R(e_\alpha, e_k)e_j, e_A \rangle
\]

(5.47)

In particular, one has \(R^i_{kak} = 0\). Using the assumptions that \((M, g; \mathcal{F})\) is tense and \(N\) has non-positive curvature, we conclude from (3.29), (5.47) that

\[
(\triangle_H - \frac{\partial}{\partial t})e_V(f_i) = (f^I_{iak})^2 + f^I_{ak}f^k_I + f^I_{ak} f^I_{j} R^j_{kak} - f^I_{ak} f^I_{k} R^I_{k, j, L} f^j_{a} f^L_{k} \geq 0.
\]

Integrating (5.48) then gives this lemma. □

Remark 5.2. In [Dom], Dominguez showed that every Riemannian foliation \(\mathcal{F}\) on a compact manifold \(M\) admits a bundle-like metric \(g\) for which the mean curvature vector field \(\zeta\) is basic. Hence tense Riemannian foliations exist in abundance.

Using Lemma 5.11 and a similar argument for Theorem 5.8, we obtain

**Theorem 5.12.** Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation with the property that \(H\) is bracket generating for \(TM\). Let \(N\) be a compact Riemannian manifold with non-positive sectional curvature. Then, for any smooth map \(\varphi : M → N\), there exists a \(C^\infty\) solution \(f(x, t)\) of the subelliptic harmonic map heat flow (4.9) on \(M \times [0, \infty)\). Moreover, there exists a sequence \(t_i → \infty\) such that \(f(x, t_i) → f_\infty(x)\) uniformly, as \(t_i → \infty\), to a \(C^\infty\) subelliptic harmonic map \(f_\infty : M → N\).

Before ending this section, we would like to mention that Z.R. Zhou [Zh2] announced an Eells-Sampson type result for subelliptic harmonic maps from a sub-Riemannian manifold with vanishing \(\Gamma\)-tensor. Here the \(\Gamma\)-tensor was introduced by Strichartz in [St]. However, \(\Gamma ≡ 0\) if and only if the horizontal distribution \(H\) is integrable.

### 6. Hartman type Results

First, we show the smoothness of a family of solutions to the subelliptic harmonic map heat flow with a family of smooth maps as its initial value. Our proof is similar to that in [Ha] for the harmonic map heat flow and that in [RY] for the pseudo-harmonic map heat flow, but with suitable modifications.

**Lemma 6.1.** Let \(\varphi(x, \lambda) : M × [0, a] → N \subset R^K\) be a smooth map and, for each fixed \(\lambda ∈ [0, a]\), let \(u(x, t, \lambda)\) be a solution of the subelliptic harmonic map heat flow on \(M × [0, \delta]\) such that \(u(x, 0, \lambda) = \varphi(x, \lambda)\). Then \(u : M × (0, \delta) × (0, a) → N\) is smooth.
Proof. Suppose \( u(x, t, \lambda) \) satisfies
\[
\begin{align*}
\triangle_H u - \frac{\partial u}{\partial t} &= F(x, t, \lambda) \\
u(x, 0, \lambda) &= \varphi(x, \lambda)
\end{align*}
\]
for \((x, t, \lambda) \in M \times (0, \delta) \times (0, a)\), where \( F(x, t, \lambda) = (\Pi_{bc}^\delta (\nabla^H u^b, \nabla^H u^c)) \). First, we assert that for any integer \( l \geq 1 \), \( u(x, t, \lambda), \partial^l u/\partial \lambda^j \) and \( \nabla^H \partial^j u/\partial \lambda^j \) \((j = 1, 2, \cdots, l)\) are continuous on \( M \times [0, \delta) \times [0, a] \). This can be proved by a re-examination (and differentiations with respect to \( \lambda \)) of the successive approximations used in the proof of the short time existence theorem (Theorem 5.1). In terms of Theorem 2.1, we see that for any fixed \( \lambda, u(\cdot, \cdot, \lambda) \in C^\infty(M \times (0, \delta), N) \) and all partial derivatives of \( u \) with respect to \((x, t)\) are bounded on any compact subsets of \( M \times (0, \delta) \times (0, a) \). Besides, by an inductive argument on \( l \) and the uniqueness theorem for the subelliptic harmonic heat flow (Theorem 5.2), we see that \( u(x, t, \lambda) \) is smooth in \( \lambda \) for each \((x, t) \in M \times (0, \delta)\), and all partial derivatives of \( u \) with respect to \( \lambda \) are bounded on any compact subsets of \( M \times (0, \delta) \times (0, a) \) too. Therefore we may use the ‘joint smoothness lemma’ in [RS] (Lemma 6.2 on page 266 in [RS]) to conclude that \( u : M \times (0, \delta) \times (0, a) \to N \) is smooth. \( \square \)

Next, we have the following lemma.

**Lemma 6.2.** Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold and \( N \) be a compact Riemannian manifold with non-positive sectional curvature. Let \( \varphi(x, \lambda) : M \times [0, a] \to N \) be a family of smooth maps and for fixed \( \lambda \), let \( f(x, \lambda, t) \) be the solution of the subelliptic harmonic map heat on \([0, \delta]\) such that \( f(x, 0, \lambda) = \varphi(x, \lambda) \). Then for each \( \lambda \in [0, a] \),
\[
\sup_{M \times \{t\} \times \{\lambda\}} | df(\frac{\partial}{\partial \lambda}) |^2
\]
is non-increasing in \( t \).

**Proof.** For the map \( f : M \times [0, \delta] \times [0, a] \to N \), we define the following function
\[
Q(x, t, \lambda) = \langle df(\frac{\partial}{\partial \lambda}), df(\frac{\partial}{\partial \lambda}) \rangle
\]
(6.2)
\[
= f^l \frac{\partial}{\partial \lambda}
\]
where \( df(\frac{\partial}{\partial \lambda}) = f^l \frac{\partial}{\partial \lambda} \). In terms of (3.23), (3.38) and (3.41), we deduce from (6.2) that
\[
\left( \triangle_H - \frac{\partial}{\partial t} \right) Q = 2 f^l \frac{\partial}{\partial \lambda} f^l_k + 2 f^l \frac{\partial}{\partial \lambda} f^l_m - 2 \zeta^k f^l_j f^l_k - 2 f^l \frac{\partial}{\partial \lambda}
\]
\[
= 2 \left( f^l \frac{\partial}{\partial \lambda} \right)^2 + 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 \zeta^k f^l_j f^l_k - 2 f^l \frac{\partial}{\partial \lambda}
\]
\[
= 2 \left( f^l \frac{\partial}{\partial \lambda} \right)^2 + 2 f^l \frac{\partial}{\partial \lambda} f^l_m - 2 \zeta^k f^l_j f^l_k - 2 \zeta^k f^l_j f^l_k - 2 \zeta^k f^l_j f^l_k - 2 f^l \frac{\partial}{\partial \lambda}
\]
\[
= 2 \left( f^l \frac{\partial}{\partial \lambda} \right)^2 + 2 f^l \left( f^l_k - \zeta^k f^l_k \right) - 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 \zeta^k f^l_j f^l_k - 2 f^l \frac{\partial}{\partial \lambda}
\]
\[
= 2 \left( f^l \frac{\partial}{\partial \lambda} \right)^2 + 2 f^l \left( f^l_k - \zeta^k f^l_k \right) - 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 f^l \frac{\partial}{\partial \lambda} f^l_k
\]
\[
= 2 \left( f^l \frac{\partial}{\partial \lambda} \right)^2 - 2 f^l \frac{\partial}{\partial \lambda} f^l_k - 2 f^l \frac{\partial}{\partial \lambda} f^l_k
\]
\[
\geq 0.
\]
\begin{align*}
\sup_{x \in M} Q(x, t, \lambda) & \leq \sup_{x \in M} Q(x, \tau, \lambda) \\
\text{for every fixed } \lambda & \in [0, a]. \text{ Hence the desired quantity is non-increasing.} \quad \square
\end{align*}

Suppose \( f_0 \) and \( f_1 \) are any two maps from \( M \) to \( N \). In terms of the Riemannian distance \( d_N \) of \( N \), we have the following distance between these two maps
\begin{equation}
(6.4) \quad d_N^\infty(f_0, f_1) = \sup_{x \in M} d_N(f_0(x), f_1(x)).
\end{equation}

Next, when \( f_0 \) and \( f_1 \) are homotopic, we may introduce the homotopy distance between them as follows: If \( F: M \times [0, 1] \to N \) is a smooth homotopy from \( f_0 \) to \( f_1 \), so that \( F(x, 0) = f_0(x) \) and \( F(x, 1) = f_1(x) \), then the length of \( F \) is defined by
\begin{equation}
(6.5) \quad L(F) = \sup_{x \in M} \int_0^1 \left| df(F(\partial/\partial \lambda)) \right|_{(x, \lambda)} d\lambda.
\end{equation}

One defines the homotopy distance \( \tilde{d}(f_0, f_1) \) to be the infimum of the lengths over all homotopies from \( f_0 \) and \( f_1 \). When \( N \) has non-positively sectional curvature, the homotopy distance can be attained by a smooth homotopy \( G \) between \( f_0 \) and \( f_1 \) in which \( \lambda \mapsto G(x, \lambda) \) is a geodesic for each \( x \in M \), and in this case \( L(G) = \sup_{x \in M} | dG(\partial/\partial \lambda) | \) for each \( \lambda \in [0, 1] \) (cf. \cite{Jo2}, \cite{SY}). It is easy to see that
\begin{equation}
(6.6) \quad d_N^\infty(f_0, f_1) \leq \tilde{d}(f_0, f_1),
\end{equation}

and if \( d_N^\infty(f_0, f_1) < inj(N) \) (the injective radius of \( N \)), then \( d_N^\infty(f_0, f_1) = \tilde{d}(f_0, f_1) \). Note that in order to define \( d_N^\infty(f_0, f_1) \) or \( \tilde{d}(f_0, f_1) \), we only need a Riemannian metric on \( N \), while \( M \) can be any compact smooth manifold without any metric.

**Proposition 6.3.** Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold and let \( N \) be a Riemannian manifold with non-positive sectional curvature. Suppose \( f_0(x, t) \) and \( f_1(x, t) \) are solutions of the subelliptic harmonic map heat flow on \([0, \delta]\) with homotopic initial data. Then \( t \mapsto \tilde{d}(f_0(\cdot, t), f_1(\cdot, t)) \) is non-increasing.

**Proof.** For any fixed \( t_0 \in [0, \delta] \), let \( F \) be the minimizing homotopy from \( f_0(\cdot, t_0) \) to \( f_1(\cdot, t_0) \), that is, \( L(F) = \tilde{d}(f_0(\cdot, t_0), f_1(\cdot, t_0)) \). By Theorem 5.1, we have a solution \( f(x, t, \lambda) \) of the subelliptic harmonic map heat flow on \([t_0, t_0 + \delta_0]\) for some \( \delta_0 > 0 \) such that \( f(x, t_0, \lambda) = F(x, \lambda) \) for any \( t \in [t_0, t_0 + \delta_0] \), it is clear that \( f(x, t, \lambda) \) is a homotopy between \( f_0(x, t) \) and \( f_1(x, t) \). For any \( t \in [t_0, t_0 + \delta_0] \), using Lemma 6.2, we derive that
\begin{align}
\tilde{d}(f_0(\cdot, t), f_1(\cdot, t)) & \leq L(f(\cdot, t, \cdot)) = \sup_{x \in M} \int_0^1 \left| df(\frac{\partial}{\partial \lambda}) \right|_{(x, t, \lambda)} d\lambda \\
& \leq \int_0^1 \sup_{x \in M} \left| df(\frac{\partial}{\partial \lambda}) \right|_{(x, t, \lambda)} d\lambda \\
& \leq \int_0^1 \sup_{x \in M} \left| df(\frac{\partial}{\partial \lambda}) \right|_{(x, t_0, \lambda)} d\lambda \\
& = \tilde{d}(f_0(\cdot, t_0), f_1(\cdot, t_0)).
\end{align}

This completes the proof of Proposition 6.3. \quad \square
**Theorem 6.4.** Let \((M, H, g_H; g)\) be either as in Theorem 5.8 or Theorem 5.12. Suppose \((N, h)\) is a compact Riemannian manifold with non-positive sectional curvature. Then the subelliptic harmonic map heat flow (4.9) exists for all \(t \in [0, \infty)\) and converges uniformly to a subelliptic harmonic map \(f_\infty\) as \(t \to \infty\). In particular, any map \(\varphi \in C^\infty(M, N)\) is homotopic to a subelliptic harmonic map.

**Proof.** According to either Theorems 5.8 or 5.12, we know that the subelliptic harmonic map heat flow (4.9) admits a global solution \(f : M \times [0, \infty) \to N\), and there exists a sequence \(\{t_k\}\) such that \(f(x, t_k)\) converges uniformly to a subelliptic harmonic map \(f_\infty(x)\) as \(t_k \to \infty\).

The uniform convergence implies that \(d_N^t(f(\cdot, t_k), f_\infty(\cdot)) < \text{inj}(N)\) for sufficiently large \(k\), and thus there is a unique minimizing geodesic from \(f(x, t_k)\) to \(f_\infty(x)\), which depends smoothly on \(x\). These geodesics define a homotopy from \(f(\cdot, t_k)\) to \(f_\infty(\cdot)\). This means that the maps \(f(\cdot, t_k)\) with large \(k\) (and hence all, since \(f(\cdot, t)\) is continuous in \(t\)) are homotopic to \(f_\infty\). In view of Proposition 6.3, we have

\[
\tilde{d}(f(\cdot, t_k + t), f_\infty(\cdot)) \leq \tilde{d}(f(\cdot, t_k), f_\infty(\cdot)) = d_N^\infty(f(\cdot, t_k), f_\infty(\cdot))
\]

for all \(t \geq 0\). Hence we conclude that the selection of the subsequence is not necessary and that \(f(\cdot, t)\) uniformly converges to \(f_\infty\) as \(t \to \infty\). \(\square\)

In previous existence results, the initial map \(\varphi : M \to N\) is assumed to be smooth. Similar to the case of the harmonic map heat flow, we may take a continuous map as the initial value for the subelliptic harmonic map heat flow.

**Corollary 6.5.** Let \(M\) and \(N\) be as in Theorem 6.4. Then any continuous map \(\varphi : M \to N\) is homotopic to a subelliptic harmonic map \(f\).

**Proof.** One just need to smooth out the map \(\varphi\) to a smooth map \(\tilde{\varphi}\) such that \(\tilde{\varphi}\) is homotopic to \(\varphi\) (cf. [Jo1], page 103-104). By applying Theorem 6.4 to \(\tilde{\varphi}\), we get this corollary immediately. \(\square\)

**Remark 6.1.** Alternatively, one may check the proof for local existence (Theorem 5.1), since after any positive time \(t\), the approximate solutions become automatically smooth. The remaining arguments are as in Theorems 5.1, 5.8 and 5.12.

**Corollary 6.6.** Let \(M\) and \(N\) be as in Theorem 6.4. Let \(\varphi : M \to N\) be a continuous map. Then the space of subelliptic harmonic maps homotopic to \(\varphi\) is connected, and subelliptic harmonic maps in \([\varphi]\) are all minimizers of \(E_H(\cdot)\) having the same horizontal energy.

**Proof.** First, let us choose a minimizing sequence \(\varphi_k\) \((k = 1, 2, \ldots)\) in \([\varphi]\) for \(E_H(\cdot)\). Then we get subelliptic harmonic maps \(f_k\) \((k = 1, 2, \ldots)\) by the preceding corollary. It follows from Lemmas 5.4, 5.7, 5.11 and (5.32) that \(e(f_k)\) \((k = 1, 2, \ldots)\) are uniformly bounded. Hence there exists a sequence of \(\{f_k\}\) converges uniformly to a Lipschitz map \(f_{\text{min}}\). Clearly \(J \circ f\) is a weak solution of (4.18) with

\[
E_H(f_{\text{min}}) = \lim_{t \to \infty} E_H(f_k)
\]

and thus \(f_{\text{min}}\) is subelliptic by Theorem 2.1.

Now let \(f\) be any subelliptic harmonic map in \([\varphi]\). Then there is a homotopy \(F : M \times [0, 1] \to N\) between \(f\) and \(f_{\text{min}}\). It is known that \(F\) determines a smooth geodesic homotopy \(G : M \times [0, 1] \to N\) between these two maps. In [Zhou2], Zhou used the second variation
formula to show that each map in a geodesic homotopy between two subelliptic harmonic maps has the same horizontal energy. Consequently $E_H(G(\cdot, t)) = E_H(f_{\min})$. Therefore we may conclude that each map $G(\cdot, t)$ is a minimizing subelliptic harmonic map for $E_H(\cdot)$, and the space of subelliptic harmonic maps in $[\varphi]$ is connected. □

From Examples 1.2, 1.3 and Theorems 5.8, 6.4, we immediately get

**Corollary 6.7.** Let $(M, H, g_H)$ be either a compact contact manifold or a compact quaternionic contact manifold with a compatible metric $g$ and let $N$ be a compact Riemannian manifold with non-positive sectional curvature. Then, for any continuous $\varphi : M \to N$, there exists a $C^\infty$ subelliptic harmonic map $f_\infty : M \to N$ homotopic to $\varphi$, which is a minimizer of $E_H$ in $[\varphi]$.

**Remark 6.2.** If $M$ is in particular a strictly pseudoconvex CR manifold, the pseudoharmonic maps considered in [ChC] and [RY], are subelliptic harmonic maps defined with respect to the Webster metrics, while these metrics are only special Riemannian extensions of the sub-Riemannian metrics determined by the Levi forms. Hence, even in the CR case, the above Corollary 6.7 generalizes their results to the case that $g$ may be arbitrary Riemannian extensions of the sub-Riemannian metrics (see also Remark 5.1). This may provide some convenience for considering further geometric analysis problems for subelliptic harmonic maps on these manifolds.

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