Critical interface: twisting spin glasses at $T_c$

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Abstract We consider identical copies of spin glasses in finite dimension coupled at the boundaries. This allows to identify the spin glass analogous of twisted boundary conditions in ferromagnetic system and leads to the definition of an interface free-energy that is positively defined and that should scale with a positive power of the system size in the spin glass phase. In this note we study the behavior of the interface at the spin glass critical temperature $T_c$ within mean field theory. We show that the leading scaling of the interface free-energy does not depend on replica symmetry breaking, and can be obtained by simple scaling arguments using a cubic theory for critical spin glasses.
1 Introduction

Sensitivity to boundary conditions is a fundamental tool to study the nature of the Gibbs states of extended physical systems. If ergodicity is broken and Gibbs states are not unique, different boundary conditions can select pure phases or induce interfaces in the system. For example, in Ising-like ferromagnets in the ferromagnetic phase, the choice of homogeneous “plus” (respectively “minus”) boundary conditions where all the spins outside a large region are fixed to point in the up (resp. down) direction is enough to select the pure state with positive (resp. negative) magnetization. Twisted boundary conditions, where along a specific direction + and − conditions are chosen at the opposite boundaries (while neutral boundary condition, e.g. periodic ones, are chosen in the other directions) induce an interface with positive tension. On the contrary, in the paramagnetic phase the system is insensitive to boundaries and the surface tension is equal to zero. Using boundary conditions one can therefore investigate the stability of low temperature phases and the lower critical dimension below which the ferromagnetic phase is unstable. Around the critical temperature, the behavior of the interface tension is described by finite size scaling, which implies that right at the critical temperature, the interface tension vanishes, and the free-energy cost to impose twisted boundary conditions remains finite in the thermodynamic limit.

In trying to extend this kind of considerations to study the stability of spin glass phases, one is confronted with the fact pure phases are in this case glassy random states strongly correlated with the quenched disorder. They cannot therefore be selected using boundary conditions uncorrelated with the quenched disorder. To project on some pure phases boundary conditions should be chosen that depend self-consistently on the quenched disorder. One way to do that, is to consider two “clones” of a spin glass with identical disorder coupled at the boundaries. In ref. [10] it was considered a situation where different values of the overlap (a measure of correlation between configurations) was imposed on the boundaries along one direction. In that paper it was investigate the stability of low temperature phase replica symmetry broken (RSB) phases below the critical temperature, against spatial fluctuations of the overlap that restore replica symmetry, and it was found a value of the lower critical dimension below which the fluctuations destroy RSB equal to $D_{LCD} = 5/2$.

In the present note we we consider systems exactly at the critical temperature, where the techniques developed in [10] do not apply.

In the case of ferromagnetic systems the behavior of the interface free-energy at the critical temperature is a well understood issue in critical fluctuation theory [11]. One has that below dimension 4 the interfacial free-energy
scales with the system size as $\sigma \sim L^{-(d-1)}$ while it scales as $\sigma \sim L^{-3}$ above dimension 4.

We then compare two kinds of boundary conditions. We consider two identical copies of cylindrical spin glasses (i.e. we impose p.b.c. in the transverse directions) and we couple them on the boundaries. In a first kind of b.c. the systems are chosen to be identical on both boundaries. In a second kind of b.c., the two systems are still imposed to be identical on one of the boundaries, but they are opposite in the other boundary.

We have already remarked that in ref. [10] it was considered a situation where different values of the overlap were fixed at the boundary. In this situation the interface energy is a function of the temperature, of the geometrical parameters and of the values of the overlap. In ref. [11] it was showed that it may be interesting to consider the extreme case where the overlap is 1 at one boundary and -1 (or 1) at the other boundary. These extreme choice simplifies the analysis and it can be implemented in numerical simulations in a very simple way.

The plan of the paper is the following: In section II we discuss in detail the Ising ordered case. This is well understood and it will guide us in the analysis of the more complex spin glass case. Section III which constitute the core of this paper is devoted to the analysis of the Edwards-Anderson model at $T_c$. Finally we draw our conclusions.

2 Pure system

We consider an Ising-like system in a cylindrical geometry. The d-dimensional cylinder has a length $L$ and a cross-section area $A \sim L^{d-1}$. The boundary conditions are periodic in the directions transverse to the axis of the cylinder, but on the surfaces at the end of the longitudinal direction the spins may be all up or all down. Below $T_c$ there is an interfacial tension $\sigma$ defined as the difference of free energy per unit area

$$\sigma = \frac{F_{\uparrow\downarrow} - F_{\uparrow\uparrow}}{A}. \quad (1)$$

It is a function of the temperature $t \sim (T - T_c)$ , and of the aspect ratio $L/A^{1/d-1}$. This tension vanishes near $T_c$ as

$$\sigma \sim (-t)\mu. \quad (2)$$

A scaling law due to Widom [1] relates $\mu$ to the correlation length exponent $\nu$

$$\mu = \nu(d - 1) \quad (3)$$
and it may be derived in the standard renormalization group framework \([2]\) (for \(d \leq 4\)).

Assume that we want to know the behavior of this tension for finite \(L\) at \(T_c\). We use finite size scaling
\[
\sigma(t, L) = (-t)^\mu f(L/\xi)
\]
and, since \(\sigma\) is non singular at \(T_c\) for \(L\) finite, it implies that
\[
f(x) \sim x^{-\mu/\nu}.
\]
Therefore at \(T_c\) the interfacial tension vanishes with \(1/L\) as
\[
\sigma(0, L) \sim L^{-\mu/\nu} = L^{-(d-1)}.
\]
In other words at \(T_c\)
\[
F_{↓↑} - F_{↑↓} \sim L^0.
\]
In appendix A we give the details of the verification of this behavior \((7)\) in a simple Landau theory (i.e. tree level of a \(\phi^4\) theory) : i.e. for a free energy at \(T_c\)
\[
F = A \int_0^L dz \left( \frac{1}{2} \left( \frac{d\varphi}{dz} \right)^2 + \frac{g}{4} \varphi^4 \right)
\]
(we have assumed translation invariance in the \((d-1)\) transverse directions). Plus-plus b.c. are defined be the fact that at both boundaries the order parameter is fixed to the value \(m\), plus-minus conditions correspond to impose the value \(m\) on the left boundary and \(-m\) on the right one. One finds that independently of the values of \(m\), the surface tension behaves as
\[
\frac{F_{↓↑} - F_{↑↓}}{A} = \frac{C}{gL^3}
\]
with a positive constant \(C\) given by
\[
C = \frac{5}{192\pi^2} \Gamma^8 \left( \frac{1}{4} \right) - \frac{1}{12\pi^{3/2}} \Gamma^6 \left( \frac{1}{4} \right) \simeq 44.8.
\]
Clearly this mean field theory should only be compared with \((7)\) in four dimensions, the result \((9)\) is indeed in agreement since \(A \sim L^3\).

### 3 Spin glass

Twisting fixed boundary conditions uncorrelated with the disordered couplings for a spin glass does not produce any interface since it can be gauged away in a change of the random couplings. It may increase or decrease the free energy, depending on the random instance..

In this paper, we would like to consider two copies of a similar cylindrical sample of spin glass with identical couplings \(J_{ij}\) and we assume that we fix two successive boundary conditions as follows:
• In the up-up situation the overlap between the spins of the two copies in the left ($x = 0$) plane and in the $x = L$ plane are equal to a positive value $l_0$.

• In the up-down situation the overlap is still equal $l_0$ in the left plane, but it is $-l_0$ in the right plane.

In ref. [11] Contucci et al. considered boundary conditions of this kind in the extreme case $l_0 = 1$. Notice that in this case, the construction leads to effective interactions within the spins in the planes $x = 0$ and $x = L$ are doubled. Choosing the interactions in these planes with the same statistics as the others would lead to stronger effective interactions between the spins in these planes then all the other interactions. In order to avoid such strong difference, in [11], the strength of the interaction among spins on the boundary planes was reduced by half.

We aim here at characterizing the behavior of the difference $\Delta F = \frac{F_{\uparrow \downarrow} - F_{\uparrow \uparrow}}{A}$ at $T_c$ as a function of $L$. Again we shall limit ourselves to mean field theory. We submit to test the Widom scaling law, verifying that above the upper critical dimension $D_{UCD} = 6$ the interface tension scales as $\sigma \sim L^{-5}$.

We would like to study this problem within a Landau-like small order parameter expansion of the free-energy close to $T_c$. Unfortunately, this can not be done directly if the value $l_0$ of the order parameter at the boundary is chosen to be large. In particular, we could not directly consider the conditions chosen in [11]. However, the small order parameter expansion can be saved even in this case, arguing that, analogously to the ferromagnetic case, the result for $\Delta F$ should not depend on the value $|l_0|$ imposed at the boundaries. We will see that this is true within the Landau model. We have also confirmed this in a spherical spin-glass model [5], which has a Landau expansion identical to the one of the standard Ising spin glass, and for which one can write equations valid even if for large values of the order parameter.

We start from a short range standard Edwards-Anderson model. A After tracing out over the spins we can rewrite it in the long wave length limit in terms of a $2n \times 2n$ matrix $Q$ (for the bulk we need only a $n \times n$ matrix $Q$) when $a$ and $b$ are both smaller than $n$, $Q_{ab}$ refers to the overlap between the spins in the first copy; when the two indexes are larger than $n$ we are dealing with the overlaps $Q_{a,b}$ in the second copy and for one index lower than $n$ and one larger, $Q_{aa}$ is a cross-overlap. The critical mean field free-energy is here

$$F = \int d^d x \left[ \frac{1}{2} \text{tr} (\nabla Q)^2 + \frac{w}{3} \text{tr} Q^3 + \frac{u}{4} \sum_{ab} Q_{ab}^4 \right]$$  \hspace{1cm} (11)

Assuming again translation invariance along the transverse directions $Q(\vec{x}) = Q(z)$, we assume that at $T_c$ the replica-symmetry is unbroken. Therefore we
assume
\[ Q_{ab} = Q_{\alpha\beta} = q(z) \]  
and
\[ Q_{aa} = k(z)(1 - \delta_{a,a}) + l(z)\delta_{a,a}. \]

Let us note that the free energy (11) is invariant under \( S_n \times S_n \), i.e. independent permutations of the \( n \) replicas of system one, and of the \( n \) replicas of system two. However the last term in (13) breaks this symmetry down to \( S_n \). Indeed one may imagine the boundary conditions as follows: one adds a coupling \( J_{\sigma\tau} \) between the spins of the two systems located in the plane \( z = L \). The boundary conditions \( \sigma = \tau \) would correspond to \( J \) going to \(+\infty\), whereas \( \sigma = -\tau \) is enforced by the limit \( J \rightarrow -\infty \). After replicating the coupling becomes \( J \sum_{\sigma,\tau} a_{\sigma,\tau}^a \) and thus the boundary conditions lead to a \( \delta_{a,a} \) term in (13) \[6\]

Then one has
\[
\text{tr} \left( \frac{dQ}{dz} \right)^2 = 2n(n-1)q'^2 + 2n(n-1)k'^2 + 2nl'^2 \]
(14)

\[
\text{tr} \ Q^3 = 2n(n-1)(n-2)q^3 + 6n(n-1)(n-2)qk^2 + 12n(n-1)ql \]
(15)

\[
\sum Q^4_{ab} = n(n-1)q^4 + \frac{1}{2}n(n-1)k^4 + nl^4 \]
(16)

and thus, in the zero replica limit, the free energy reads
\[
\frac{F}{n} = A \int_0^L dz \left[ -q'^2 - k'^2 + l'^2 + \frac{w}{3}(4q^3 + 12qk^2 - 12ql) + \frac{u}{4}(-q^4 - \frac{1}{2}k^4 + l^4) \right] \]
(17)

The quartic term may be dropped at \( T_c \) since the functions \( q(z), k(z), l(z) \) remain small for large \( L \). The equations of motion are thus simply
\[
2q'' + 4w(q^2 + k^2 - kl) = 0 \]
\[
2k'' + 4w(2kq - ql) = 0 \]
\[
2l'' + 4wqk = 0 \]
(18)

The plus-plus, and plus-minus boundary conditions are different in the two cases for the functions \( l(z) \) and \( k(z) \). In the plus-plus case, if one imposes the same boundary conditions on \( q \) and \( k \), \( q(z) = k(z) \) is solution and we are left with two equations. There is one constant of motion, namely \( 4q^2 - l^2 + \frac{w}{3}(8q^3 - 3ql) \), but the last integration has to be done numerically.

In the plus-minus case, one has to deal with three different functions.

We impose the boundary conditions in the following way: we consider a sample of size \( (1 + 2r)L \) with \(-rL \leq x \leq (1+r)L\) and fix the value of \( l(x) \) to
preassigned values on the regions $-rL < x < 0$, that we call left boundary, and $L < x < (1 + r)L$, that we call right boundary. In the case of plus-plus, the value $l_0 > 0$ is imposed both right and left boundaries, in the plus-minus case the value $l_0 > 0$ is chosen on the left boundary and $-l_0$ is chosen on the right boundary. The value of $q(x)$ and $k(x)$ on the boundaries are not fixed by the constraint and should correspond to extremization of the free-energy. Since $r > 0$ and $L$ is large, $q(0)$ and $k(0)$ and $q(L)$ and $k(L)$ should take the same values that that they would take if a uniform overlap profile $l(x) = \pm l_0$ for all $x$ was chosen, namely $q(0) = k(0) = q(L) = k(L) = l_0/2$ for plus-plus conditions and $q(0) = k(0) = q(L) = l_0/2$, $k(L) = -l_0/2$ for plus-minus conditions.

We expect that, as verified explicitly in the case of the ferromagnet, the interface free-energy does not depend on the imposed value $l_0$. In that case, a scaling argument based on the fact that the dominant interaction terms in the free-energy are the cubic ones (namely the transformation $x \rightarrow xL$ $q \rightarrow w^{-1}L^{-2}q$ and analogously for $k$ an $l$), suggests that the interface tension, in absence of accidental cancellations should behave as $AL^{-5}$ for large $L$, i.e. for $A \sim L^{D-1}$, $\Delta F \sim L^{D-6}$. Indeed, as it is easy to see by the above rescaling, we can write $F_{++} = \frac{A}{wL^2}g_{++}(l_0wL^2)$, $F_{+-} = \frac{A}{wL^2}g_{-}(l_0wL^2)$. This is of course true only if no accidental cancellations appear. If this is the case, we are indeed authorized to neglect the quartic term. Since this is the term responsible for RSB, we can neglect RSB effects to the leading order.

In order to verify that $g_{+-} - g_{++}$ remains finite for large $L$, we have integrated numerically the rescaled equations (18), for various values of the parameter $v = l_0wL^2$. This is done by simple a relaxation method, where initial profiles are iterated until one observes convergence to a solution of a discretized version of the equations (18). In fig. (2) we plot the function $f(v) = g_{+-}(v) - g_{++}(v)$, together with a power law fit of the kind $f(v) = a + \frac{b}{v^c}$. While $f(v)$ continue to have an appreciable dependence on $v$ (and thus on $L$) for very large values of $v$, the fit indicates that $f_\infty = \lim_{v \to \infty} f(v)$ is different from zero. We have in addition explicitly verified that the inclusion of the quartic term in the free-energy gives a subleading contribution to $\Delta F$.

4 Conclusions

In this paper we have considered the interface free-energy to impose different boundary conditions in a spin glass. We have concentrated in the behavior at the critical point and found that the behavior of the interface follows the usual pattern of critical phenomena. The free-energy difference above 6 dimensions is of order $L^{D-6}$ as naive scaling suggests. This implies that RSB effects can be neglected to the leading order at the critical temperature. This is very different from what found at low temperature [10], where the scaling
Figure 1: The order parameters in the two solutions. Upper panel: the functions $l(x)/l_0$ and $q(x)/l_0 = k(x)/l_0$ as a function of $x/L$ for $v = 1000$ and $r = 1/2$. Lower Panel: the functions $l(x)/l_0$ and $q(x)/l_0$ and $k(x)/l_0$ for the same values of the parameters.

of the interface was found to depend critically on zero modes associated to RSB.
Figure 2: The rescaled free-energy difference, \( f = (F_{+-} - F_{++})w^2L^{6-D} \) as a function of \( v = l_wL^2 \), together with a power law fit of the form \( f(v) = a + \frac{b}{v^c} \). Chi-square fitting for \( v > 10^4 \) gives the parameters \( a = 480 \pm 20 \), \( b = 1806 \pm 15 \) and \( c = 0.19 \pm 0.01 \).

Appendix A : Pure systems
The free energy is
\[
F = A \int_{-L/2}^{+L/2} dz \left[ \frac{1}{2} \varphi'^2 + \frac{u}{4} \varphi^4 \right]
\] (19)

1. Up-up boundary conditions
The magnetization \( m \) in the planes \( z = \pm L/2 \) is given and we solve the equation of motion
\[ -\varphi'' + u\varphi^3 = 0 \] (20)
with the b.c.
\[ \varphi(L/2) = \varphi(+L/2) = m \] (21)
The mechanical analog is a negative energy bounce off the potential \(-\frac{u}{4}\varphi^4\) starting at \( \varphi = m \) bouncing at \( \varphi_0 \) and returning to \( \varphi = m \). It is given by
\[
\int_m^{\varphi(z)} \frac{d\psi}{\sqrt{\psi^4 - \varphi_0^4}} = -\sqrt{\frac{u}{2}}(L/2 + z)
\] (22)
Then \( \varphi_0 \) is determined by
\[
\int_m^{\varphi_0} \frac{d\psi}{\sqrt{\psi^4 - \varphi_0^4}} = -\sqrt{\frac{u}{8}}L
\] (23)
For $L$ large the order parameter $\varphi_0$ at the centre of the sample is small, much smaller than the given finite value $m$ on the boundaries, and it is thus given asymptotically by

$$L\varphi_0 = \sqrt{\frac{8}{u}} \int_1^\infty \frac{dt}{\sqrt{t^4 - 1}} + O(1/L)$$  \hspace{1cm} (24)

The free energy is then given by

$$\frac{F_{\uparrow \downarrow}}{A} = \frac{u}{2} \int_{-L/2}^{+L/2} dz (\varphi^4 - \varphi_0^4) + \frac{u}{4} L \varphi_0^4$$

$$= -\sqrt{2} u \int_m^{\varphi_0} d\psi \sqrt{\psi^4 - \varphi_0^4} + \frac{u}{4} L \varphi_0^4$$

$$= \frac{1}{3} \sqrt{2} u m^3 - \sqrt{2} u \varphi_0^3 + \frac{u}{4} L \varphi_0^4 + \sqrt{2} u \varphi_0^3 \int_1^\infty \frac{dt}{\sqrt{t^4 - 1 + t^2}}.$$  \hspace{1cm} (25)

The corrections to the first leading term are of order $1/L^3$. We finally note that

$$\int_1^\infty \frac{dt}{\sqrt{t^4 - 1}} = \frac{\Gamma^2(1/4)}{\sqrt{32\pi}}$$  \hspace{1cm} (26)

and

$$\int_1^\infty \frac{dt}{\sqrt{t^4 - 1 + t^2}} = \frac{\Gamma^2(1/4)}{\sqrt{72\pi}} - \frac{1}{3}.$$  \hspace{1cm} (27)

2. Up-down boundary conditions

We are still dealing with the mechanical analog of a motion in the inverted potential $-u\varphi^4/4$ but, now with a positive energy solution going from $\varphi(L/2) = m$ to $\varphi(-L/2) = -m$. The solution is given by

$$\int_m^\varphi \frac{d\psi}{\sqrt{\psi^4 + \varphi_1^4}} = -\sqrt{\frac{u}{2}} (L/2 - z)$$  \hspace{1cm} (28)

and $\varphi_1$ is determined by

$$\int_{-m}^m \frac{d\psi}{\sqrt{\psi^4 + \varphi_1^4}} = \sqrt{\frac{u}{2}} L$$  \hspace{1cm} (29)

Again $\varphi_1$ is of order $1/L$ and given asymptotically by

$$L\varphi_1 = \sqrt{\frac{2}{u}} \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{t^4 + 1}}$$  \hspace{1cm} (30)

Then the free energy

$$\frac{F_{\uparrow \downarrow}}{A} = -\frac{u}{4} L \varphi_1^4 + \sqrt{2} u \varphi_1^3 \int_0^{+m/\varphi_1} dt \sqrt{t^4 + 1}$$  \hspace{1cm} (31)
The last asymptotic estimate
\[
\int_0^{+m/\varphi_1} dt \sqrt{t^4 + 1} = \frac{1}{3} \left( \frac{m}{\varphi_1} \right)^3 + \int_0^{\infty} \frac{dt}{\sqrt{t^4 + 1 + t^2}}
\] (32)
and the values of the integrals
\[
\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{t^4 + 1}} = \frac{\Gamma^2(1/4)}{\sqrt{16\pi}}
\] (33)
\[
\int_0^{\infty} \frac{dt}{\sqrt{t^4 + 1 + t^2}} = \frac{\Gamma^2(1/4)}{\sqrt{36\pi}}
\] (34)
completes the calculation.

We may now compute \( F_{1,\downarrow} - F_{1,\uparrow} \); the constant \( m^3 \) term cancels and we are left with a difference proportional to \( A/L^3 \) as expected
\[
\frac{F_{1,\downarrow} - F_{1,\uparrow}}{A} = \frac{1}{uL^3} \left[ \frac{5}{3 \cdot 2^{10/2} \pi^2} \Gamma^8(1/4) - \frac{1}{3 \cdot 4^3 \pi \sqrt{2\pi}} \Gamma^6(1/4) \right]
\] (35)

Appendix B: A spherical Spin Glass Model

As we observed in the main text, the boundary conditions defined in ref. [11], imply high overlaps on the boundaries, and cannot be analyzed in the context of Landau small order parameter expansions of the free-energy. We define here a model with the following properties:

- The replica free-energy of the model admits a simple closed form in terms of the overlap matrix \( Q_{a,b}(x) \), and it can be explicitly analytically continued to \( n \to 0 \) when the Parisi ansatz is assumed.
- The Landau expansion close to \( T_c \) coincides up to the fourth order term with the one of the SK model.

Consider a system where on each site, labelled by its longitudinal and transverse coordinates \( x \) and \( y \) respectively, there are \( K \) spins \( \sigma_{r,x,y} \), \( r = 1, \ldots, K \) subjected to the spherical constraint \( \sum_r (S_{r,x,y})^2 = K \). We write the Hamiltonian of the model as a sum of terms that couple spins on the same plane and terms that couple spins in adjacent planes.

\[
H = \sum_{x=0}^{L} H_{x}^{\text{ort}} + \sum_{x=0}^{L-1} H_{x}^{\text{par}}
\] (36)
where the Hamiltonians \( H_{x}^{\text{ort}} \) and \( H_{x}^{\text{par}} \) are gaussian random variables with variances

\[
\langle H_{x}^{\text{ort}}[\sigma] H_{x}^{\text{ort}}[\tau] \rangle = \sum_{y} f_x \left( \frac{1}{2^{d-1}} \sum_{z \in V_y} q_{y,x,z} \right)
\]
\[
\langle H_{x}^{\text{par}}[\sigma] H_{x}^{\text{par}}[\tau] \rangle = \sum_{y} f (q_{y,x+1})
\] (37)
where we denoted by $S_y^x$ the value of the spin on site $y$ of the $x$-th plane, $q_{y,x} = \frac{1}{K} \sum_{r=1}^{K} \sigma_{y,x}^r \tau_{y,x}^r$, the overlaps between spin configurations on different sites. The different functions of the overlap are choosen to be $f(q) = \frac{1}{2}(q^2 + yq^4)$, $f_x(q) = f(q)$ for $x \neq 0$, and $f(0) = \frac{1}{2}f(q)$ for $x = 0$. Notice that with this choice it is possible to express Hamiltonian (36) in terms of two body and four body gaussian couplings between the spins. We can now introduce two copies of the system with ++ and +- boundary conditions. The ++ conditions consist in considering two copies $\sigma_{x,y}^r$ and $\tau_{x,y}^r$ constrained to be identical for $x = 0, L$: $\sigma_{0,y}^r = \tau_{0,y}^r$ and $\sigma_{L,y}^r = \tau_{L,y}^r$. The +- conditions consist in considering two copies $\sigma_{x,y}^r$ and $\tau_{x,y}^r$ constrained to be identical for $x = 0$ but opposed for $x = L$: $\sigma_{0,y}^r = \tau_{0,y}^r$ and $\sigma_{L,y}^r = -\tau_{L,y}^r$. As for the reduced model, the replica treatment of the problem involves the introduction of two local $n \times n$ overlap matrices, which under the replica symmetric ansatz and the assumption of independence of the overlap profiles of the transverse spatial coordinate, can be parametrized in terms of the functions $q(x)$, $l(x)$ and $k(x) x = 0, ..., L$ of the main text. The resulting $F_{++}$ free-energies and $F_{+-}$ as a function of these parameters can be decomposed as

$$F_{++}[q, l, k] = F_{++}^{bulk}[q, l, k] + F_{++}^{boundary} + F_{++}^{boundary}$$

(38)

$$F_{+-}[q, l, k] = F_{+-}^{bulk}[q, l, k] + F_{+-}^{boundary} + F_{+-}^{boundary}$$

(39)

where

$$-\beta F_{++}^{bulk} = \frac{1}{eta} \sum_{x=1}^{L-1} [f(1) + f(l(x)) - f(q(x)) - f(k(x))]$$

$$+\frac{1}{2} \sum_{x=1}^{L-1} \frac{q(x) + k(x)}{1 - l(x) - q(x) + k(x)}$$

$$+\frac{1}{2} \sum_{x=1}^{L-1} \log [1 + l(x) - q(x) - k(x)]$$

$$+\log [1 - l(x) - q(x) + k(x)] + \frac{q(x) - k(x)}{1 - l(x) - q(x) + k(x)}$$

(40)

$$-\beta F_{++}^{boundary} = \frac{1}{\beta} \log [1 - q(0)]$$

$$+\frac{1}{2} \log [1 - q(0)] + \frac{q(L)}{1 - q(0)} + \log [1 - q(L)]$$

$$+\log [1 - q(L)]$$

(41)
$$-\beta F_{boundary}^{+-} =$$
$$+\beta^2 (f(1) + f((1 + l(L - 1))/2) - f((q(L - 1) + q(L))/2) - f((q(L) - k(L - 1))/2)$$

(43)

Given the complexity of the expression, we have manipulated them through the Mathematica software in order to obtain the equations of motion, and integrated the resulting equations numerically by the relaxation method. The integration gives result fully compatible with the analysis of the reduced model, confirming the independence on the detailed boundary conditions imposed. As a proxy of the free-energy difference $F_{+-} - F_{++}$, in figure 3 we plot the difference in free-energy density in the center of the box multiplied by $L^6$, which according to the argument given in the main text should tend to a constant for large $L$. As in the case of the reduced model, fitting curves of the kind $f(L) = a + b/L^c$ confirm this behavior.

Figure 3: The difference of free-energy density in the center of the box for the two kinds of boundary conditions as a function of the system size $L$. Notice that even and odd values of $L$ give rise to different curves. The curves are plotted together with a power law fit of the form $f(L) = a + b/L^c$. Chi-square fitting for $L > 20$ gives the parameters $a = 0.15 \pm 0.04$, $b = 3.1 \pm 0.3$ and $c = 0.44 \pm 0.04$ for $L$ even and $a = 1.3 \pm 0.1$, $b = 46.9 \pm 1.7$ and $c = 0.58 \pm 0.04$ for $L$ odd.

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