Dirac quantization of the massless Thirring model:
energy-momentum tensor anomaly

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Abstract

The Dirac method of quantizing Hamiltonian systems with constraints is applied to the massless Thirring model. We solve the quantum Hamiltonian equation for the energy-momentum tensor and obtain a violation of the classical conservation law. A previously noticed problem with the equal-time anticommutators can be fixed using this Hamiltonian method.

Classification codes: PACS 02.10.-v,03.65-w
Keywords: quantum anomaly, Thirring model

1. Introduction

Since the late 1960’s an extensive literature has evolved on the massless Thirring model. We would like to fill a gap, however.

The Thirring model is a typical system with constraints and so we quantize it here according to Dirac’s special Hamiltonian formalism. Unlike most authors, however, we don’t use solutions of the normal-ordered Lagrange equations [1-4] in our treatment, since such solutions are not part of the Dirac-Hamiltonian formalism [5]. We only consider operators, commutators, and normal ordering in initial time-like elements. Only Hamiltonian language is used and the quantum Hamiltonian equations of motion are solved. We don’t mix elements of the Lagrange and Hamiltonian formalisms.

This method was successfully tested in the quantum integrable sine-Gordon, Zhilber-Shabat, nonlinear Schrodinger, Korteweg-de Vries and modified Korteweg-de Vries systems [6,7]. Quantum commutative integrals of motion for these models were constructed as solutions of the quantum Hamiltonian equations. This

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is the unique known nonperturbative way of constructing quantum integrals of motion, at the present time [6]. In addition, using this method in the massless Thirring model helps to reveal a new infinite dimensional symmetry in the sine-Gordon theory, explaining the power law behavior of its correlation functions [7].

Furthermore using this Hamiltonian approach in the massless Thirring model we avoid a problem with equal-time anticommutators. In fact we exchange this problem in its hidden form (in the equal-time anticommutator) for an obvious anomaly. Indeed the difference between the standard equal-time anticommutator from the canonical one is of order $g^2$ for $(g \ll 1)$. We fix this problem in the anticommutator, but get instead a correction in the same order of $g$ in another place: in the degree of non-analyticity of the operator $T_{++}$. That is the main result of this article. It is important because this anomaly breaks the well known conformal symmetry of the massless Thirring model.

In Sec. 2 we consider the naive solution of this model in a modern, adapted version [3] and explain the unsolved problem with this approach. In Sec. 3 we use the classical Hamiltonian formalism for systems with constraints on the massless Thirring model. We solve the simple constraints and calculate their Dirac brackets. More complicated constraints have complicated Dirac brackets and so we work with them in a special way: very useful operators $P_{\pm}$ are introduced. In Sec. 4 we establish an important property of initial functions, and so we use these functions in all operators of our theory. In Sec. 5 we calculate a quantum anomaly of the components of the energy-momentum tensor, a quantum correction to the classical conservation law. Sec. 6 is devoted to concluding remarks and a brief discussion of the connection of this work with the Thirring/sine-Gordon equivalence.

### 2. Problem with equal-time anticommutator

Let us recall aspects of the massless Thirring model in the modern conformal field theory approach. We have the vertex operator [3]

$$V_{m,n}(z, \bar{z}) =: \exp 2i \left( \frac{\beta_+ m + \beta_- n}{2} \epsilon(z) + \frac{\beta_+ m - \beta_- n}{2} \bar{\epsilon}(\bar{z}) \right)$$

so that $\epsilon(z, \bar{z})$ satisfies equation $\partial_z \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0$, where $\epsilon(z, \bar{z}) = \epsilon(z) + \bar{\epsilon}(\bar{z})$. Components of the energy-momentum tensor are given by $T(z) \sim: (\epsilon(z))^2 :$ and $\bar{T}(\bar{z}) \sim: (\bar{\epsilon}(\bar{z}))^2 :$. In the standard approach these are holomorphic and antiholomorphic quantities. Conformal dimensions have the form

$$(\Delta, \bar{\Delta}) = (\beta_+ m + \beta_- n)^2, (\beta_+ m - \beta_- n)^2), \quad \beta_+ \beta_- = \frac{1}{2}, \quad (2.2)$$
β+ = \left(\frac{1 + g}{2(1 - g)}\right)^{\frac{1}{2}}, \quad \beta_- = \left(\frac{1 - g}{2(1 + g)}\right)^{\frac{1}{2}}. \quad (2.3)

We can consider the obvious equations

\partial_z V_{m,n}(z, \bar{z}) =: i(\beta_+ m - \beta_- n) \partial_{\bar{z}} \bar{\epsilon}(\bar{z}) V_{m,n} :,
\partial_{\bar{z}} V_{m,n}(z, \bar{z}) =: i(\beta_+ n + \beta_- m) \partial_z \epsilon(z) V_{m,n} :. \quad (2.4)

After identifying certain vertex operators with fields of theory

\psi_1 = V_{\frac{1}{4}, -\frac{1}{4}}, \quad \psi_2 = V_{\frac{1}{4}, \frac{1}{4}}, \quad \psi_1^+ = V_{\frac{1}{2}, \frac{1}{2}}, \quad \psi_2^+ = V_{\frac{1}{2}, -\frac{1}{2}}, \quad (2.5)

we have equations for massless Thirring model

\partial_z \psi_1 = g : J\psi_1 :, \quad \partial_{\bar{z}} \psi_2 = g : J\psi_1 :. \quad (2.6)

These solutions for \psi_1, \psi_2 satisfy normal ordered Lagrange equations.

But for a quantum solution we also must demand the correct equal-time anticommutator

[\psi_1(x, t), \psi_1^+(y, t)]_+ = i\delta(x - y). \quad (2.7)

This anticommutator is a fundamental property of the quantum theory. In the considered case above the operator algebra of the solutions has the form

\psi_1^+(z, \bar{z}) \psi_1(0, 0) = z^{-\frac{1}{2 - 2g}} \bar{z}^{-\frac{g^2}{4 - 2g}}. \quad (2.8)

In fact it is possible to prove the equal-time property (which follows from the above) only for the \( g = 0 \) case. And so this solution has a problem, as indeed we will show the quantum solution does.

Let us consider one example of an explanation in the literature [4]. The author considers the solution

\psi(x) = \exp[-ib\gamma^5 \phi(-)] \exp[i a \phi(-)(x)] \exp[i a \phi(+)(x)] \exp[-ib\gamma^5 \phi(+) (x)] u \quad (2.9)

of the Thirring model, where \( u \) is a “two component c-number quantity”. For the equal-time anticommutator

\langle 0 [\psi_r(x), \psi_r^+(y)]_+ | 0 \rangle = u_r(x) u_r^+(y) \delta(x - y)(x - y)^{h-1} \quad (2.10)

is obtained. Here \( h = \frac{a^2 + b^2}{2\pi} \geq 1 \), \( a \) and \( b \) are constants in the theory. By requiring

\quad u_r(x) u_r^+(y) \sim (x - y)^{-(h-1)}, \quad (2.11)

the author gets the ordinary \( \delta \) function result for the equal-time anticommutator. But we believe that it is not possible to construct this “c-number quantity” in a space of functions. Indeed we have from (2.11)

\log u_r(x) + \log u_r^+(y) = (1 - h) \log(x - y), \quad (2.12)
(for noncommutative operators $\hat{a}, \hat{b}$; we have $\log(\hat{a}\hat{b}) \neq \log\hat{a} + \log\hat{b}$). If $\partial_x\partial_y$ operator acts on the left side, (2.12) we get 0; on the right side we get $\frac{(1-h)}{(x-y)^2} \neq 0$.

In the Hamiltonian approach, the canonical equal-time relation is postulated at the beginning of the calculation and is preserved in time evolution, and so this problem disappears.

If we consider the massive Thirring model we get a similar problem with the equal-time relation for fermions when $\sigma \neq 0$ [8]

$$[\psi(x, t), \psi^+(y, t)]_+ = (x - y)^\sigma(x - y) \neq \delta(x - y). \quad (2.13)$$

This problem can also be fixed by the Hamiltonian method of quantization. We will consider the massive case in a separate work.

3. Classical Hamiltonian formalism

We use light cone coordinates with the notation

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \quad (3.1)$$

and represent the $\gamma^\pm$-matrices by

$$\gamma^+ = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma^- = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.2)$$

If we think of $x^0$ as the imaginary time $i\tau$, we work in Euclidean space, and $x^+ \sim z, x^- \sim \bar{z}$ are its complex coordinates.

We start with the Lagrangian for the massless Thirring model in light cone coordinates

$$\mathcal{L} = \frac{i\sqrt{2}}{2} (\psi_1^+ \partial_+ \psi_2 + \psi_1^+ \partial_- \psi_1 - \partial_- \psi_1^+ \psi_1 - \partial_+ \psi_2^+ \psi_2) - 2g \psi_1^+ \psi_1 \psi_2^+ \psi_2. \quad (3.3)$$

The canonical conjugates of the fields are the following:

$$\pi_{\psi_1} = \frac{\partial L}{\partial (\partial_- \psi_1)} = -\frac{i\sqrt{2}}{2} \psi_1^+, \quad f_1^1 = \frac{i\sqrt{2}}{2} \psi_1^+ + \pi_{\psi_1},$$

$$\pi_{\psi_1^+} = \frac{\partial L}{\partial (\partial_- \psi_1^+)} = -\frac{i\sqrt{2}}{2} \psi_1, \quad f_2^1 = \frac{i\sqrt{2}}{2} \psi_1 + \pi_{\psi_1^+}, \quad (3.4)$$

$$\pi_{\psi_2} = \frac{\partial L}{\partial (\partial_- \psi_2)} = 0, \quad f_3^1 = \pi_{\psi_2},$$
\[ \pi_{\psi_2^+} = \frac{\partial L}{\partial (\dot{\psi}_2^+)} = 0, \quad f_4^1 = \pi_{\psi_2^+}. \]

We must have the canonical Poisson brackets (more exactly Poisson-Berezin brackets for our anti-commutative variables) for the fields and their conjugate fields:

\[
\{\psi_1(x); \pi_{\psi_1}(y)\} = \delta(x - y), \quad \{\psi_1(x); \pi_{\psi_1}(y)\} = \delta(x - y), \\
\{\psi_2(x); \pi_{\psi_2}(y)\} = \delta(x - y), \quad \{\psi_2^+(x); \pi_{\psi_2^+}(y)\} = \delta(x - y). \tag{3.5}
\]

The expressions \( f_i^1 = 0, (i = 1, \ldots, 4) \) are the primary constraints. We have the first step Hamiltonian density,

\[
\mathcal{H}_1 = -\frac{i\sqrt{2}}{2} (\psi_2^+ \partial_+ \psi_2 - \partial_+ \psi_2^+ \psi_2) + 2g\psi_1^+\psi_1^\dagger \psi_2 + \lambda_1^1 \left( \frac{i\sqrt{2}}{2} \psi_1^+ + \pi_{\psi_1} \right) + \lambda_2^1 \left( \frac{i\sqrt{2}}{2} \psi_1 + \pi_{\psi_1^+} \right) + \lambda_3^1 \pi_{\psi_2} + \lambda_4^1 \pi_{\psi_2^+}, \tag{3.6}
\]

constructed in the usual way for systems with constraints [5]. We must demand the conservation of constraints in time (a dot indicates a derivative with respect to “time” \( t \)). From \( \dot{f}_1^1 = \dot{f}_2^1 = 0 \), we can obtain \( \lambda_1^1, \lambda_2^1 \), and from \( \dot{f}_3^1 = \dot{f}_4^1 = 0 \), we find new (secondary) constraints: \( f_3^2, f_4^2 \)

\[
\dot{f}_1^1 = \{H_1; f_1^1\} = 0; \quad \lambda_1^1 = \frac{2g}{i\sqrt{2}} \psi_1^+ \psi_2^+ \psi_2, \\
\dot{f}_2^1 = \{H_1; f_2^1\} = 0; \quad \lambda_2^1 = \frac{2g}{i\sqrt{2}} \psi_1^+ \psi_2^+ \psi_2, \\
\dot{f}_3^1 = \{H_1; f_3^1\} = 0; \quad f_3^2 = i\sqrt{2} \partial_+ \psi_2 - 2g\psi_1^+ \psi_1 \psi_2 = 0, \\
\dot{f}_4^1 = \{H_1; f_4^1\} = 0; \quad f_4^2 = i\sqrt{2} \partial_+ \psi_2^+ + 2g\psi_1^+ \psi_1 \psi_2^+ = 0. \tag{3.7}
\]

Here we have introduced \( H_i = \int h_i dz \). The constraints \( f_3^2, f_4^2 \) are part of the Lagrange equations but in the Hamiltonian sense, they are only constraints of second class. We must introduce the second step Hamiltonian density \( \mathcal{H}_2 \):

\[
\mathcal{H}_2 = -\frac{i\sqrt{2}}{2} (\psi_2^+ \partial_+ \psi_2 - \partial_+ \psi_2^+ \psi_2) + 2g\psi_1^+\psi_1^\dagger \psi_2 + 2g\psi_1^+ \psi_2^+ \psi_2 \left( \frac{i\sqrt{2}}{2} \psi_1^+ + \pi_{\psi_1} \right) \\
-2g\psi_1^+ \psi_2^+ \psi_2 \left( \frac{i\sqrt{2}}{2} \psi_1 + \pi_{\psi_1^+} \right) + \lambda_3^1 \pi_{\psi_2} + \lambda_4^1 \pi_{\psi_2^+} + \lambda_3^2 \left( i\sqrt{2} \partial_+ \psi_2 - 2g\psi_1^+ \psi_1 \psi_2 \right) + \lambda_4^2 \left( i\sqrt{2} \partial_+ \psi_2^+ + 2g\psi_1^+ \psi_1 \psi_2^+ \right). \tag{3.8}
\]

Demanding the conservation of the new constraints in time yields

\[
\dot{f}_3^3 = \{H_2; f_3^3\} = 0; \quad \lambda_3^1, \\
\dot{f}_4^3 = \{H_2; f_4^3\} = 0; \quad \lambda_4^1. 
\]
\[ f_1^2 = \{H_2; f_1^2\} = 0; \quad \lambda_1^1, \quad (3.9) \]
\[ f_1^1 = \{H_2; f_1^1\} = 0; \quad \lambda_2^1 = 0, \]
\[ f_1^3 = \{H_2; f_1^3\} = 0; \quad \lambda_3^1 = 0. \]

From \( f_1^2 = f_1^3 = 0 \) we can determine \( \lambda_3^1, \lambda_4^1 \), but their forms are not important for this work. Similarly, \( f_1^1 = f_1^3 = 0 \) determine \( \lambda_3^1, \lambda_2^1 \). Thus all the constants in the \( \mathcal{H}_2 \) Hamiltonian density (3.8) are determined and we have no new constraints.

It is very useful to resolve the constraints \( f_1^1 = f_1^3 = 0 \) and so we must calculate Dirac brackets. Using the notation \( f_1 = f_1^1; f_2 = f_2^1, \alpha, \beta = 1, 2 \) the expressions for the Dirac brackets are [5]

\[ \{\psi_1, \psi_1^+\}_{\text{Dirac}} = \{\psi_1, \psi_1^+\} - \sum_{\alpha, \beta} \{\psi_1, f_{\alpha}\} \{f_{\alpha}, f_{\beta}\}^{-1} \{f_{\alpha}, \psi_1^+\}, \quad (3.10) \]
\[ \{\psi_1(x), \psi_1^+(y)\}_{\text{Dirac}} = \frac{i}{\sqrt{2}} \delta(x - y), \]
\[ \{\psi_1(x), \psi_1(y)\}_{\text{Dirac}} = \{\psi_1(x)^+, \psi_1^+(y)\}_{\text{Dirac}} = 0. \]

In our case, the matrix of the constraints \( \{f_\alpha f_\beta\} \) is not degenerate, and so has an inverse. We used \( \delta^{-1}(x - y) = \delta(x - y) \). In our theory the physical anticommutative variables are \( \psi_1, \psi_1^+ \) only, and the anti-commutative variables \( \psi_2, \psi_2^+ \) are dependent. We don’t resolve the \( f_1^1, f_1^3, f_3^1, f_4^1 \) constraints, and so we will use the ordinary Poisson (3.5) brackets between the fields \( \psi_2, \psi_2^+ \) and \( \pi_{\psi_2}, \pi_{\psi_2^+} \), but then impose the constraints.

After resolving the constraints \( f_1^1 = f_1^3 = 0 \), the Hamiltonian density \( \mathcal{H}_2 \) has the form:

\[ \mathcal{H}_2 = -\frac{i\sqrt{2}}{2} \left( \psi_2^+ \partial_+ \psi_2 - \partial_+ \psi_2^+ \psi_2 \right) + 2g\psi_1^+ \psi_1 \psi_2^+ \psi_2 + \lambda_3^1 \pi_{\psi_2} + \lambda_4^1 \pi_{\psi_2^+}. \quad (3.11) \]

In our research, we are only interested in functionals \( I[\psi_1, \psi_1^+] \). The important part of \( \mathcal{H}_2 \) is therefore \( \mathcal{H}_2' = 2g\psi_1^+ \psi_1 \psi_2^+ \psi_2 \), since \( \{\mathcal{H}_2, I[\psi_1, \psi_1^+]\} \subset \{\mathcal{H}_2, I[\psi_1, \psi_1]\} \).

An important remark can now be made. The constraints \( f_3^2, f_4^2 \) must be “hamiltonized”. We must introduce fields \( P_\pm \) as integrals over certain densities:

\[ P_+ = \frac{2g}{i\sqrt{2}} \int \psi_1^+ \psi_1 \pi_{\psi_2} \pi_{\psi_2^+} dz, \quad P_- = -\frac{2g}{i\sqrt{2}} \int \psi_1^+ \psi_1 \pi_{\psi_2} \pi_{\psi_2^+} dz. \quad (3.12) \]

Here \( \pi_{\psi_2} \) and \( \pi_{\psi_2^+} \) are our constraints \( f_3^1 \) and \( f_4^1 \). The operators \( P_\pm \) do not vanish, however, because their action is defined so that the constraints are imposed only after calculating the Poisson brackets.
After quantization, the operators $\hat{P}_\pm$ help to remove singularities in the theory. We calculate

$$\partial_+ \psi_2 = \{ P_+, \psi_2 \} = \frac{2g}{i\sqrt{2}} \{ \int \psi_1^+ \psi_2 \pi_{\psi_2} dz, \psi_2 \} = \frac{2g}{i\sqrt{2}} \psi_1^+ \psi_1 \psi_2$$

(3.13)

using $\{ \psi_2(x), \pi_{\psi_2(y)} \} = \delta(x-y)$, and obtain the constraint $f_3^2$. Another important example of the action of these operators is

$$\partial_+ (\psi_2) \psi_1 = \{ P_+, \psi_2 \psi_1 \} = \frac{2g}{i\sqrt{2}} \{ \int \psi_1^+ \psi_2 \pi_{\psi_2} dz, \psi_2 \psi_1 \} =$$

$$= \frac{2g}{i\sqrt{2}} \psi_1^+ \psi_1 \psi_2 \psi_1 + g \psi_1 \psi_2 \pi_{\psi_2} \psi_2 = \frac{2g}{i\sqrt{2}} \psi_1^+ \psi_1 \psi_2 \psi_1,$$

(3.14)

where we used

$$\{ \psi_1(x), \psi_1^+(y) \}_{\text{Dirac}} = \frac{i}{\sqrt{2}} \delta(x-y), \quad \{ \psi_1(x), \psi_1(y) \}_{\text{Dirac}} = 0,$$

$$\{ \psi_2(x), \pi_{\psi_2}(y) \} = \delta(x-y), \quad \pi_{\psi_2}(x) = 0.$$

(3.15)

While $\partial_+$ acts on $\psi_2$ only, $P_+$ acts on all fields $\psi_2$ and $\psi_1$, and imposing the constraints removes the extra parts in the classical case. We will consider the quantum analog of this example, and will find a nontrivial action of this operator. Expression (3.14) is equal to zero because $\psi_1^2 = 0$. We must remember here that constraints $\pi_{\psi_2} = \pi_{\psi_2}^\dagger = 0$ must be imposed after calculating Poisson brackets (or commutators (anti-commutators) in the quantum case).

4. Quantization

Recall the solution of the quantum Hamilton equation

$$\hat{\psi}(z, \bar{z}) = \exp \left( -i \bar{z} \hat{H} \right) \hat{\psi}^0(z, \bar{z}) \exp \left( i \bar{z} \hat{H} \right).$$

(4.1)

The superscript 0 indicates an initial quantum field, and $\hat{\psi}(z, \bar{z})$ denotes an operator solution. Let us consider the operator Hamilton equation of motion:

$$i \partial_- \hat{\psi}(z, \bar{z}) = [ \hat{H}, \hat{\psi}(z, \bar{z}) ]$$

(4.2)

Inserting the solution gives

$$i \partial_- \hat{\psi}(z, \bar{z}) = [ \hat{H}, \hat{\psi}(z, \bar{z}) ] + i \exp \left( -i \bar{z} \hat{H} \right) \partial_- \hat{\psi}^0(z, \bar{z}) \exp \left( i \bar{z} \hat{H} \right)$$

(4.3)

For consistency then, we must demand that the initial operator obey

$$\partial_- \hat{\psi}^0(z, \bar{z}) = 0.$$

(4.4)
This is a very useful property (that of analytical functions) for quantization.

It is a simple exercise to check that the solution above satisfies fermionic properties too. For example, \( \hat{\psi}^2(z, \bar{z}) = 0 \) (this follows from the fermionic properties of the initial quantum fields). It is easy to check property in Hamiltonian approach \( \{ \hat{\psi}(z, \bar{z}), \hat{\psi}^+(z', \bar{z}) \} = i\delta(z - z') \) (we have postulated for initial operators \( \{ \hat{\psi}^0(z, \bar{z}), \hat{\psi}^{0+}(z', \bar{z}) \} = i\delta(z - z') \)). And so we don’t have the problems like in naive approach.

We will use the initial fields in all operators (\( \hat{H}, \hat{P}_{\pm} \)) and will drop the “0” and \( \bar{z} \) in the notation \( \hat{\psi}^0(z, \bar{z}) \). The Poisson brackets above (for the classical initial fields) can be quantized in the usual way, and we obtain the standard singular parts of operator product expansions:

\[
\hat{\psi}_1(z) \hat{\psi}_1^+(z') = \frac{1}{(z - z')}, \quad \hat{\psi}_2(z) \hat{\pi}_{\psi_2}(z') = 1 \quad \pi_{\psi_2}(z) \hat{\psi}_2^+(z') = \frac{1}{(z - z')}.
\]  

(4.5)

Here we used the standard expansions of the analytical fields:

\[
\psi_1(\xi) = \sum_n \psi_1^n (\xi - \alpha)^{-\frac{n}{2}}, \quad \psi_1^+(\xi) = \sum_n \psi_1^+ n (\xi - \alpha)^{-\frac{n}{2}},
\]

\[
\psi_2(\xi) = \sum_n \psi_2^n (\xi - \alpha)^{-\frac{n}{2}}, \quad \psi_2^+(\xi) = \sum_n \psi_2^+ n (\xi - \alpha)^{-\frac{n}{2}},
\]

(4.6),

\[
\pi_{\psi_2}(\xi) = \sum_n \pi_{\psi_2}^n (\xi - \alpha)^{-\frac{n}{2}}, \quad \pi_{\psi_2}^+(\xi) = \sum_n \pi_{\psi_2}^+ n (\xi - \alpha)^{-\frac{n}{2}},
\]

where \( \alpha \) is the center of the expansion. We have also introduced some redetermine all of the fields multiply by unimportant constants.

Poisson brackets for the complicated operators \( \hat{H}, \hat{P}_{\pm} \) must also be quantized. The integration contours for both parts of the commutator \( \{ \hat{Q}, \hat{A}(\xi') \} = \hat{Q}\hat{A}(\xi') - \hat{A}(\xi')\hat{Q}, \quad \hat{Q} = \int \hat{q}(\xi) d\xi \) need to be determined. Recall also that we are interested in only a certain part of the Hamiltonian, \( \hat{H}'_2 \). Our quantum operators are

\[
\hat{H}'_2 = 2g \int \hat{\psi}_1^+ \hat{\psi}_1 \hat{\psi}_2^+ \hat{\psi}_2 : d\xi, \quad \hat{P}_+ = \frac{2g}{i\sqrt{2}} \int \hat{\psi}_1^+ \hat{\psi}_1 \hat{\pi}_{\psi_2} \hat{\psi}_2 : dx, \quad \hat{P}_- = -\frac{2g}{i\sqrt{2}} \int \hat{\psi}_1^+ \hat{\psi}_1 \hat{\pi}_{\psi_2}^+ : d\xi.
\]

(4.8)
Fig. 1 Contour of integration for commutator.

where :: denotes normal ordering at the initial time. Let us consider the contour integration in our commutators. In the first part of the commutator, we choose the contour closing above the point $\xi'$ (see Fig.1).

In the second part, we choose the contour closing below the same point $\xi'$ (see Fig.2).

Letting the radius $R$ of the semicircles go to $\infty$, we have

$$
\int_{C_R\to\infty} \hat{H}_2'(\xi) \hat{A}(\xi') d\xi = 0, \quad \int_{C_R\to\infty} \hat{p}_\pm(\xi) \hat{A}(\xi') d\xi = 0,
$$

(4.9)

using the asymptotic behavior of the fundamental fields $\hat{\psi}_1, \hat{\psi}_1^+ \sim \frac{1}{R}$. So for our operators $\hat{H}_2', \hat{P}_\pm$, (using analytic property (4.4)):

$$
[\hat{Q}, \hat{A}(\xi')] = \oint_{\xi'} \hat{q}(\xi) \hat{A}(\xi') d\xi,
$$

(4.10)

where the notation indicates closed-contour integration around the point $\xi = \xi'$.

Let us consider the action of the quantum operators $\hat{P}_\pm$ (4.8). In our case we must calculate expressions like $\partial_+ (\hat{\psi}_2) \hat{\psi}_1^+$. The naive way (without introducing $\hat{P}_\pm$ operators) is would be

$$
\lim_{z \to z'} \partial_+ (\hat{\psi}_2)(z) \hat{\psi}_1^+(z') = \frac{2g}{i\sqrt{2}} \lim_{z \to z'} (\hat{\psi}_1^+ \hat{\psi}_1 \hat{\psi}_2)(z) \hat{\psi}_1^+(z').
$$

(4.11)

We find a singularity for $z \to z'$, arising from the product $\hat{\psi}_1(z) \hat{\psi}_1^+(z') \sim \frac{1}{z-z'}$, so that

$$
\lim_{z \to z'} (\hat{\psi}_1^+ \hat{\psi}_1 \hat{\psi}_2)(z) \hat{\psi}_1^+(z') = \infty.
$$

(4.12)
If we use the action of the quantum operators \( \hat{P}_\pm \), we instead obtain
\[
\partial_+(\hat{\psi}_2)\hat{\psi}_1^+ = i\hbar \partial_+(\hat{\psi}_1^+\hat{\psi}_2). \tag{4.13}
\]
The singularity has been removed, but there is a quantum correction to the classical result (\( \hbar \) is Planck’s constant). This is the idea behind the introduction of these operators \( \hat{P}_\pm \). Using the quantum Hamiltonian \( \hat{H} \), when we calculate \( [\hat{H}, \hat{I}] \) we find singularities. The analytic property of initial fields (4.4), however, with the choice of the contour of the integration described above help to remove those singularities. Incidentally, this way (in \( [\hat{H}, \hat{I}] \)) of removing the singularity gives the correct commutative integrals of motion (elements of Hamiltonian formalism too) for quantum sine-Gordon theory [7].

In this article we will solve the quantum Hamilton equation
\[
i\partial_- \hat{A} = [\hat{H}, \hat{A}], \tag{4.14}
\]
for the “++” component of the energy-momentum tensor. That is, we put \( \hat{A} = \hat{T}_{++} = \hat{T} \), with
\[
\hat{T} = : \hat{\psi}_1 \partial_+ \hat{\psi}_1^+ : + : \hat{\psi}_1^+ \partial_+ \hat{\psi}_1 :, \tag{4.15}
\]
where \( \hat{\psi}_1, \hat{\psi}_1^+ \) and \( \hat{T} \) are initial functions. We must calculate \( [\hat{H}, \hat{T}] \).

5. Calculation of anomaly

Let us consider
\[
\hat{i}_2 = : \hat{\psi}_1 \partial_+ \hat{\psi}_1^+ :. \tag{5.1}
\]
After a simple calculation we obtain
\[
[\hat{H}, \hat{i}_2] = \tag{5.2}
\]
\[
= 2g_i(-) : \hat{\psi}_1 \hat{\psi}_2^+ \hat{\psi}_2 \partial_+ \hat{\psi}_1 : + 2g_i(-) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+ \hat{\psi}_2) \hat{\psi}_1^+ : + g_i \hbar \partial_2^2 (\hat{\psi}_2^+ \hat{\psi}_2)(-).
\]
Similarly, for
\[
\hat{i}_2' = - : \hat{\psi}_1^+ \partial_+ \hat{\psi}_1 :, \tag{5.3}
\]
we find
\[
[\hat{H}, \hat{i}_2'] = \tag{5.4}
\]
\[
= 2g_i(-) : \hat{\psi}_1 \hat{\psi}_2^+ \hat{\psi}_2 \partial_+ \hat{\psi}_1 : + 2g_i(-) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+ \hat{\psi}_2) \hat{\psi}_1^+ : + g_i \hbar \partial_2^2 (\hat{\psi}_2^+ \hat{\psi}_2)(-).
\]
The result is
\[
[\hat{H}, \hat{T}] = 4g_i : \hat{\psi}_1 \partial_+ (\hat{\psi}_2^+ \hat{\psi}_2) \hat{\psi}_1^+ :. \tag{5.5}
\]
Now we can calculate \( \hat{\psi}_1 \hat{\psi}_1^+ \partial_+ (\hat{\psi}_2 \hat{\psi}_2^+) \) using our operators \( \hat{P}_\pm \). We find
\[
(5.6)
\]
\[
\begin{align*}
: \partial_+ \hat{\psi}_2^+ \hat{\psi}_2^+ \hat{\psi}_1^+ : &= (-i\hbar) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+) \hat{\psi}_2^+ \hat{\psi}_1^+ : + (-i\hbar) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+) \hat{\psi}_2^+ \hat{\psi}_1^+ :
+ (-i\hbar^2) \partial_+^2 \hat{\psi}_2^+ \hat{\psi}_2^+, \\
: \hat{\psi}_2^+ \partial_+ \hat{\psi}_2^+ \hat{\psi}_1^+ : &= (-i\hbar) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+) \hat{\psi}_2^+ \hat{\psi}_1^+ : + (-i\hbar) : \partial_+ (\hat{\psi}_1^+ \hat{\psi}_2^+) \hat{\psi}_2^+ \hat{\psi}_1^+ :
+ (-i\hbar^2) \partial_+^2 \hat{\psi}_2^+ \hat{\psi}_2^+.
\end{align*}
\]

We therefore have
\[
: -\hat{\psi}_1^+ \partial_+ (\hat{\psi}_2^+ \hat{\psi}_2^+) : = -i\hbar (\partial_+^2 \hat{\psi}_2^+ + \partial_+^2 \hat{\psi}_2^+) \quad (5.7)
\]
yielding
\[
[\hat{H}, \hat{T}] = -2\mu^2 (\partial_+^2 \hat{\psi}_2^+ + \partial_+^2 \hat{\psi}_2^+) , \quad \mu = i\hbar. \quad (5.8)
\]

If \([\hat{H}, \hat{T}]\) is not zero, then we have a quantum anomaly, \(\partial_- \hat{T} \neq 0\). This is easily established:
\[
(1 - \frac{\mu^2}{2}) (\partial_+^2 \hat{\psi}_2^+ + \partial_+^2 \hat{\psi}_2^+) = 2ig : \partial_+ \hat{\psi}_1^+ \hat{\psi}_1^+ \hat{\psi}_2^+ : + : \hat{\psi}_1^+ \partial_+ \hat{\psi}_1^+ \hat{\psi}_2^+ : \neq 0. \quad (5.9)
\]

The “++” component of the energy-momentum tensor (4.15) is therefore not conserved in time. However, if we have \(h = 0\) (classical limit) or \(g = 0\) (free massless fermions), we do have
\[
[\hat{H}, \hat{T}] = 0, \quad (5.10)
\]
and so \(\hat{T}\) is conserve for a quantum free massless fermion theory.

In the usual quantum case only one mode (momentum) is conserved, and indeed we can make a simple transformation to obtain
\[
[\hat{H}, \hat{T}] = 2\mu^2 (\partial_+ \hat{\psi}_2^+ \hat{\psi}_2^+ + \partial_+ \hat{\psi}_2^+ \hat{\psi}_2^+). \quad (5.11)
\]
If we introduce notation for the momentum operator \(\hat{I}_2 = \int \hat{T} dz\), we have \([\hat{H}, \hat{I}_2] = 0\). If we want to calculate \(\hat{T}_{\text{++}}\) component of the energy-momentum tensor we must consider “+” variable like the time, from the beginning of the calculation.

\section*{6. Conclusion}

An important conclusion can now be drawn. We can bosonize the initial functions using
\[
\hat{\psi}_1(z) = : \exp \left( \hat{\phi}(z) \right) : , \quad \hat{\psi}_1^+(z) = : \exp \left( -\hat{\phi}(z) \right) :, \quad (6.1)
\]
where \(\hat{\phi}\) is the initial bosonic function (operator). After a simple transformation we find
\[
\hat{T} = : \hat{\psi}_1 \partial_+ \hat{\psi}_1^+ : + : \hat{\psi}_1^+ \partial_+ \hat{\psi}_1 : = - : (\partial_+ \hat{\phi})^2 : . \quad (6.2)
\]
We see the equivalence between the “++” component of the energy-momentum tensor for bosonic and Thirring fermionic theories at the initial time. A similar calculation can be found in [9]. In the quantum massless Thirring model we have \( \partial_+ \tilde{T}_{\text{Thirring}} \neq 0 \), however. So, notwithstanding the equivalence of the initial operators \( \tilde{T} \), the equivalence between the massless Thirring model and the massless free bosonic field is lost, if the Hamiltonian formalism is used throughout. Indeed let us consider the free massless bosonic theory. It has the Hamiltonian \( \hat{H}_{\text{boson}}^{\text{free}} = 0 \) in light cone coordinates, and so \( \tilde{T}_{\text{boson}}^{\text{free}} \) (6.2) is conserved in time, \( \partial_+ \tilde{T}_{\text{boson}}^{\text{free}} = 0 \).

We see then, the violation of the zeroth-order approximation (in the sense of [10]) of the sine-Gordon /Thirring model equivalence [8,10]. Of course, the solution [1-4] can still sometimes be useful, because the violation is very weak, \( \sim g^2 h^2 \), for \( g \ll 1 \).

**Acknowledgements**

This work was supported by a NATO Science Fellowship, the University of Lethbridge, and by NSERC of Canada. We thank Mark Walton and Jian-Ge Zhou for discussions.

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