Do bosons obey Bose-Einstein distribution: two iterated limits of Gentile distribution

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Abstract: It is a common impression that by only setting the maximum occupation number to infinity, which is the demand of the indistinguishability of bosons, one can achieve the statistical distribution that bosons obey — the Bose-Einstein distribution. In this letter, however, we show that only with an infinite maximum occupation number one cannot uniquely achieve the Bose-Einstein distribution, since in the derivation of the Bose-Einstein distribution, the problem of iterated limit is encountered. For achieving the Bose-Einstein distribution, one needs to take both the maximum occupation number and the total number of particles to infinities, and, then, the problem of the order of taking limits arises. Different orders of the limit operations will lead to different statistical distributions. For achieving the Bose-Einstein distribution, besides setting the maximum occupation number, we also need to state the order of the limit operations.

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1 Introduction

The indistinguishability of identical particles demands that for Bose-Einstein statistics, a quantum state can be occupied by any number of particles, and for Fermi-Dirac statistics, a quantum state can be occupied by only one particle. Starting from this point, as a common impression, in statistical mechanics, one can obtain statistical distributions by only setting the various values of the maximum occupation number $n$: by $n \to \infty$, one achieves the Bose-Einstein distribution, and by $n = 1$, one achieves the Fermi-Dirac distribution. In this letter, however, we show that, contrary to this common impression, only from a maximum occupation number, one cannot uniquely construct the statistical distribution in the Bose-Einstein case, though such a treatment works well in the Fermi-Dirac case. This difference between Bose-Einstein and Fermi-Dirac cases comes from the fact that the maximum occupation number in the Bose-Einstein case is $\infty$, but in the Fermi-Dirac case is a finite number.

In section 2, we point out that there exists not only one possibility to take limits in the derivation of the Bose-Einstein distribution. In sections 3 and 4, we discuss the various orders of taking limits and show which one will lead to the Bose-Einstein distribution. The conclusion is summarized in Sec. 5.
2 The order of taking limits

The reason why the Bose-Einstein distribution cannot be uniquely achieved by only setting \( n \to \infty \) is that there are two infinities involved in the derivation of the Bose-Einstein distribution — the maximum occupation number and the total number of particles. The infinite maximum occupation number is the demand of the indistinguishability of bosons; the infinite total number of particles, i.e., the thermodynamic limit, guarantees the extensivity of a thermodynamic system. As a result, during the derivation, we need to perform two limits, \( \lim_{n \to \infty} \) and \( \lim_{\langle N \rangle \to \infty} \), where \( \langle N \rangle \) denotes the mean total number of particles in the grand canonical ensemble. This means that in the Bose-Einstein case, we have to first make a choice of the order of the limit operations.

For simultaneously taking the maximum occupation number \( n \) and the total number of particles \( \langle N \rangle \) into account, we start with Gentile statistics \([1, 2]\) in which the maximum occupation number \( n \) can take on any value. Different values of \( n \) correspond to different kinds of statistics, and the Bose-Einstein case is a special case of Gentile statistics with \( n \to \infty \) \([3]\). The Gentile distribution reads

\[
f_G = \frac{1}{z^{-1}e^{\varepsilon/\left(kT\right)} - 1} - \frac{n + 1}{\left[z^{-1}e^{\varepsilon/\left(kT\right)}\right]^{n+1} - 1},
\]

where \( T \) is the temperature, \( \varepsilon \) is the energy, and \( z \) is the fugacity. In the grand canonical ensemble, the fugacity \( z \) is determined by \([3]\)

\[
\langle N \rangle = \frac{V}{\lambda^3} h_{3/2} \left( z \right) + \frac{1}{z^{-1} - 1} - \frac{n + 1}{z^{-\left(n+1\right)} - 1},
\]

where \( \lambda = h/\sqrt{2\pi mkT} \) is the mean thermal wavelength,

\[
h_{3/2} \left( z \right) = g_{3/2} \left( z \right) - \frac{1}{\sqrt{n+1}} g_{3/2} \left( z^{n+1} \right),
\]

and \( g_{3/2} \left( z \right) \) is the Bose-Einstein integral. In principle, from eq. \([2]\), one can solve the expression of \( z \) as

\[
z = w \left( T, n, \langle N \rangle \right),
\]

and then write the distribution function as

\[
f_G \left( T, n, \langle N \rangle \right) = \frac{1}{w^{-1} e^{\varepsilon/\left(kT\right)} - 1} - \frac{n + 1}{w^{-1} e^{\varepsilon/\left(kT\right)} \left[w^{-1} e^{\varepsilon/\left(kT\right)}\right]^{n+1} - 1}.
\]

There are two infinities, \( n \) and \( \langle N \rangle \), in the distribution function \( f_G \left( T, n, \langle N \rangle \right) \). For recovering the Bose-Einstein distribution, we need to take both \( n \) and \( \langle N \rangle \) tend to \( \infty \) in eq. \([4]\). However, there are three possible ways to take the limits: two iterated limits

\[
\lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} f_G \left( T, n, \langle N \rangle \right)
\]
and
\[ \lim_{n \to \infty} \lim_{\langle N \rangle \to \infty} f_G(T, n, \langle N \rangle), \] (6)
and the double limit
\[ \lim_{n \to \infty} f_G(T, n, \langle N \rangle). \] (7)

The first thing is thus to choose the way how to take the limit from the above three. Nevertheless, the indistinguishability of bosons only requires an infinite maximum occupation number, but does not say anything about the order of the limits. Generally speaking, different orders of the two limit operations will lead to different results \[4, 5\]. At first sight, one may think that a natural choice is the double limit (7). However, in the following, we will show that the double limit (7) does not exist, since the iterated limits (5) and (6) exist but are not equal to each other. It is the iterated limit (5) that leads to the Bose-Einstein distribution. That is to say, for obtaining the Bose-Einstein distribution, besides setting \( n \to \infty \), the demand of indistinguishability, we also need an additional condition that the order of the iterated limit should be \( \lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} \).

3 \ \lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} f_G: the Bose-Einstein distribution

We first discuss the case of the iterated limit \( \lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} f_G \). In this case, the limit \( \lim_{n \to \infty} \) is first performed.

In principle, for obtaining the statistical distribution under the limit of \( n \to \infty \), we need to calculate the explicit expression for the fugacity \( z \), substitute it into eq. (1) to achieve eq. (4), and then perform the limit \( \lim_{n \to \infty} \) in eq. (4). However, it is difficult to obtain the expression of \( z \), so we will do this in an indirect way.

We first determine the maximum value of the fugacity \( z \) at \( n \to \infty \). Since \( \frac{1}{z-1} - \frac{n+1}{z(n+1)-1} \) is a monotonically increasing function of \( z \), from eq. (2), the maximum value of \( z \) appears at \( T = 0 \), or, \( \lambda \to \infty \), i.e.,

\[ \langle N \rangle = \lim_{n \to \infty} \left[ \frac{1}{z_{\text{max}}-1} - \frac{n+1}{z_{\text{max}}(n+1)-1} \right]. \] (8)

The solution of eq. (8) is
\[ z_{\text{max}} = \frac{\langle N \rangle}{\langle N \rangle + 1}. \] (9)

Based on this result, we can take the limit \( \lim_{n \to \infty} \) on eq. (2):

\[ \langle N \rangle = \frac{V}{\lambda^2} g_{3/2}(z) + \frac{1}{z-1-1}, \] (10)
or
\[ 1 = \frac{1}{\rho \lambda^2} g_{3/2}(z) + \frac{1}{\langle N \rangle z-1-1}. \] (11)
where \( \rho = \langle N \rangle / V \) is the particle number density.

Now, we can take the limit \( \lim_{\langle N \rangle \to \infty} \) and obtain the range of value of the fugacity
\[
0 < z \leq 1.
\] (12)

This is just the Bose-Einstein case \([6]\). The corresponding statistical distribution is the Bose-Einstein distribution:
\[
f = \frac{1}{z^{-1}e^{\frac{\varepsilon}{(kT)}} - 1}.
\] (13)

The above result shows that if the order of the limits is chosen as \( \lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} f_G \), one achieves the Bose-Einstein distribution.

4 \quad \lim_{n \to \infty} \lim_{\langle N \rangle \to \infty} f_G: beyond the Bose-Einstein distribution

An alternative choice for the iterated limit is \( \lim_{n \to \infty} \lim_{\langle N \rangle \to \infty} \), i.e., the limit \( \lim_{\langle N \rangle \to \infty} \) is first performed.

Rewrite eq. (2) as
\[
1 = \frac{1}{\rho \lambda^3} h_{3/2} (z) + \frac{1}{\langle N \rangle} \left[ \frac{1}{z^{-1} - 1} - \frac{n + 1}{z^{-(n+1)} - 1} \right] .
\] (14)

Here \( \frac{1}{z^{-1} - 1} - \frac{n + 1}{z^{-(n+1)} - 1} \) is the number of the particles occupying the ground state, whose maximum value is the maximum occupation number \( n \). When taking the limit \( \lim_{\langle N \rangle \to \infty} \), the ground-state term vanishes. Consequently, taking the limit \( \lim_{\langle N \rangle \to \infty} \) on eq. (14) gives
\[
\rho \lambda^3 = h_{3/2} (z).
\] (15)

\( h_{3/2} (z) \) is a monotonically increasing function of \( z \) and \( 0 < h_{3/2} (z) < \infty \) \([3]\), so the range of the value of \( z \) is \( 0 < z < \infty \).

Next, take the limit \( \lim_{n \to \infty} \) on eq. (15).

After taking the limit \( \lim_{n \to \infty} \), eq. (14) becomes eq. (15). The left-hand side of eq. (15) can take on any value from 0 to \( \infty \), so the fugacity \( z \) can take on the value that is greater than 1 due to the fact that \( h_{3/2} (z) \) is a monotonically increasing function of \( z \) and when \( z = 1 \), \( \lim_{n \to \infty} h_{3/2} (z) = \zeta (3/2) \), where \( \zeta (3/2) \) is the zeta function.

The distribution corresponding to \( z > 1 \) must not be the Bose-Einstein distribution because in the Bose-Einstein case, \( 0 < z \leq 1 \). Concretely, taking \( \lim_{\langle N \rangle \to \infty} \) in the distribution function gives
\[
f_G = \begin{cases} \frac{1}{z^{-1}e^{\frac{\varepsilon}{(kT)}} - 1}, & \varepsilon > \mu (T), \\ \infty, & \varepsilon < \mu (T), \end{cases}
\] (16)

where \( \mu \) is the chemical potential. Notice that before taking \( \lim_{n \to \infty} \), the limit \( \lim_{\langle N \rangle \to \infty} \) has been already taken. Eq. (16) is a statistical distribution with an infinite maximum occupation number, i.e., \( n \to \infty \), but is not the Bose-Einstein distribution.
The above results show that both the iterated limits $\lim_{\langle N \rangle \to \infty} \lim_{n \to \infty} f_G$ and $\lim_{n \to \infty} \lim_{\langle N \rangle \to \infty} f_G$ exist, but are not equal to each other. As a result, the double limit $\lim_{n \to \infty} f_G$ does not exist.

5 Conclusions

In conclusion, in the derivation of the Bose-Einstein distribution, two infinities are encountered. Thus, we have to choose the way how to take the limits. Different orders of the limit operations give different statistical distributions. In the common derivation of Bose-Einstein distribution, the order of the iterated limit has been unstatedly chosen as $\lim_{\langle N \rangle \to \infty} \lim_{n \to \infty}$. That is to say, to achieve the Bose-Einstein distribution, besides setting the maximum occupation number to infinity, one also needs to choose the order of taking limits.

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