GENERALISED VERMA MODULES FOR
THE ORTHOSYMPELECTIC LIE SUPERALGEBRA \(osp_{k|2}\)

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Abstract. The composition factors and their multiplicities are determined for generalised Verma modules over the orthosymplectic Lie superalgebra \(osp_{k|2}\). The results enable us to obtain explicit formulae for the formal characters and dimensions of the finite-dimensional irreducible modules. Applying these results, we also compute the first and second cohomology groups of the Lie superalgebra with coefficients in finite-dimensional Kac modules and irreducible modules.

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1. Introduction

Lie superalgebras continue to attract much interest in both mathematics and physics. These algebras originated from quantum field theory in the 1970s, and provide the mathematical foundation for the physical notion of supersymmetry. [The Large Hadron Collider built by the European Organization for Nuclear Research is currently in operation to verify supersymmetry experimentally.] It was already clear from the foundational works of Kac [13, 14, 15] that the representation theory of finite-dimensional simple Lie superalgebras was very different from that of the ordinary simple Lie algebras. In particular, finite-dimensional representations of such a Lie superalgebra over \(\mathbb{C}\) are not semi-simple in general, in sharp contrast to those of the latter. This renders the development of the representation theory of Lie superalgebras a more interesting but harder task.

2000 Mathematics Subject Classification. Primary 17B37, 20G42, 17B10.
The finite-dimensional irreducible representations of some Lie superalgebras \[28, 29\] and classes of representations of the general linear superalgebra \[9, 11, 12, 18\] were understood long ago, and very precise conjectures were also formulated on the representation theory of the general linear superalgebra \[30\]. However, it was within the last fifteen years that major advances were achieved. Serganova in \[22, 23\] further developed the geometric approach \[17, 18\] to the representation theory of Lie superalgebras to derive algorithms for computing characters of finite-dimensional irreducible representations. Brundan \[3\] discovered a remarkable connection between the general linear superalgebra and the quantum group of \(\mathfrak{gl}_\infty\). He reformulated the generalised Kazhdan-Lusztig theory for the Lie superalgebra in terms of canonical bases of the quantum group, proving the algorithm for determining the composition numbers of the Kac modules conjectured in \[31\]. The approach of \[3\] was further developed in \[24, 27, 7, 5\]. In particular, a closed character formula was obtained for the finite-dimensional irreducible representations of the general linear superalgebra in \[27\], and the generalised Kazhdan-Lusztig polynomials of \(\mathfrak{gl}(m|N)\) were shown to be equal to those of \(\mathfrak{gl}(m + N)\) (when \(N \to \infty\)) in \[7\] (and earlier in \[8\] for polynomial representations). This suggested an equivalence \[7\] Conjecture 6.10] between the appropriate parabolic category \(\mathcal{O}\) of \(\mathfrak{gl}(m|\infty)\) and that of \(\mathfrak{gl}(m + \infty)\), which was recently established in \[7, 4\] independently using very different methods.

Very recently, Gruson and Serganova \[12\] improved upon earlier work of Serganova \[23\] to give combinatorial algorithms for computing characters of finite-dimensional irreducible representations of \(\mathfrak{osp}_{k|2n}\). An interesting fact is that references \[22, 23\] worked entirely within the category of finite-dimensional \(\mathfrak{osp}_{m|2n}\)-modules, where certain virtual modules built from cohomology groups of locally free equivariant sheaves over super Grassmannians played the role of Kac modules in the case of type \(A\). In this paper we further study the representation theory of the orthosymplectic Lie superalgebras.

In this paper we study the structure of the generalised Verma modules for the orthosymplectic Lie superalgebras. The goal here is somewhat different from that of references \[22, 23\], as generalised Verma modules do not play a role within the framework of these papers. We shall focus on \(\mathfrak{osp}_{k|2n}\) for \(n = 1\) and \(k \geq 3\). We choose the distinguished Borel subalgebra for \(\mathfrak{osp}_{k/2}\) and consider the maximal parabolic subalgebra obtained by removing the unique odd simple root. The generalised Verma modules studied here are those induced from finite-dimensional irreducible modules over the parabolic subalgebra.

One of our main results is determining the composition factors of the generalised Verma modules for \(\mathfrak{osp}_{k/2}\) and also obtaining their multiplicities (see Theorem \[3.16\] for the precise statement of the result). Using this result we construct explicit formulae for the characters and dimensions of the finite-dimensional irreducible modules in Theorem \[3.17\]. The formulae are sums of a finite number of terms, each of which resembles Weyl’s character or dimension formula for semi-simple Lie algebras. We point out that a character formula (in the form of an infinite sum) for the finite dimensional irreducible \(\mathfrak{osp}_{k|2}\)-modules was obtained very recently by Luo \[16\] using a different method.

The maximal finite-dimensional quotients of the generalised Verma modules with dominant highest weights are the Kac modules \[13, 14, 15\] for the orthosymplectic Lie superalgebras. Kac modules naturally arise as the zeroth cohomology groups of some locally free equivariant sheaves over the super Grassmannian corresponding to the parabolic subalgebra. However, if the highest weight is atypical, the Kac module can
be irreducible in some special cases but is not irreducible in general. Therefore it is also interesting to understand the structure of Kac modules. We determine the characters of the Kac modules in Proposition 3.18.

By using results obtained on the structure of the generalised Verma modules and Kac modules, we compute explicitly the first and second cohomology groups of the orthosymplectic Lie superalgebra with coefficients in finite-dimensional irreducible modules and Kac modules in Theorem 4.3. This part of the paper is a continuation of [21, 26] on the study of Lie superalgebra cohomology.

Closely related to our work is the very recent paper by Cheng, Lam and Wang [6], which proved an equivalence between the parabolic $\mathcal{O}$ categories of $\mathfrak{osp}_{m|\infty}$ and $\mathfrak{so}_{m|\infty}$ in analogy to the type-$A$ case. Equipped with this equivalence of categories, one in principle can study the representation theory of $\mathfrak{osp}_{m|2n}$ for finite $n$ by first understanding representations of $\mathfrak{so}_{m|\infty}$, then transcribing the results to the corresponding representations of $\mathfrak{osp}_{m|\infty}$, and finally deducing precise results for $\mathfrak{osp}_{m|2n}$-representations by truncating the $\mathfrak{osp}_{m|\infty}$-representations to finite $n$. Such a method worked reasonably well in the case of the polynomial representations of $\mathfrak{gl}(m|n)$ [8], and it is expected to work in the case of $\mathfrak{osp}_{m|2n}$ as well.

2. Preliminaries

This section contains some background material which will be needed later. It also serves to fix notation. We work over the field of complex numbers throughout the paper.

2.1. The Lie superalgebras $\mathfrak{osp}_{k|2}$. We denote by $\mathfrak{g}$ the orthosymplectic Lie superalgebra $\mathfrak{osp}_{k|2}$ for $k > 2$. The algebra $\mathfrak{osp}_{2|2}$ is of type I [13], whose finite-dimensional irreducible modules have long been understood [29]. Choose the distinguished Borel subalgebra [13, 20] for $\mathfrak{g}$. Then the corresponding Dynkin diagram is given by

$$
\begin{align*}
\mathfrak{g} = D(m, 1) & : \delta_{-\varepsilon_1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m-2} \varepsilon_{m-1} \varepsilon_{m-1} \varepsilon_m, & \text{if } k = 2m, \\
\mathfrak{g} = B(m, 1) & : \delta_{-\varepsilon_1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m-1} \varepsilon_m, & \text{if } k = 2m + 1,
\end{align*}
$$

where the grey node is associated with the unique odd simple root. We shall denote by $\Pi$ the set of simple roots. We also denote by $\Delta_0^+$ and $\Delta_1^+$ the sets of positive even roots and positive odd roots respectively, which are given by

$$
\begin{align*}
\Delta_0^+ &= \{2\delta, \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq m\}, & \Delta_1^+ &= \{\delta \pm \varepsilon_i | 1 \leq i \leq m\} & \text{if } k = 2m, \\
\Delta_0^+ &= \{2\delta, \varepsilon_i, \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq m\}, & \Delta_1^+ &= \{\delta, \delta \pm \varepsilon_i | 1 \leq i \leq m\} & \text{if } k = 2m + 1.
\end{align*}
$$

We define the order

$$0 < \delta - \varepsilon_1 < \varepsilon_1 - \varepsilon_2 < \cdots < \varepsilon_{m-1} - \varepsilon_m \text{ and } 0 < \delta, \varepsilon_i,$$

which gives a total order on the set of roots.

**Notation 2.1.** For $\alpha \in \Pi$, we fix $e_\alpha, f_\alpha$ to be respectively the positive, negative root vectors. For any $\beta \in \Delta^+ \setminus \Pi$, where $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, we uniquely define the positive, negative root vectors $e_\beta, f_\beta$ as follows: Let $\alpha \in \Pi$ be the unique minimal simple root such that $\beta - \alpha \in \Delta^+$, then $e_\beta = [e_\alpha, e_{\beta - \alpha}], f_\beta = [f_{\beta - \alpha}, f_\alpha]$. We also write $e_{-\beta} = f_\beta$ and define $h_\beta = [e_\beta, f_\beta]$ for $\beta \in \Delta^+$.
Then $h_\alpha$, $\alpha \in \Pi$ forms a basis of the Cartan subalgebra $\mathfrak{h}$. The bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ is defined by

$$(\delta, \delta) = -1, \quad (\delta, \varepsilon_i) = 0, \quad (\varepsilon_i, \varepsilon_j) = \delta_{i,j} \quad \text{for} \quad 1 \leq i, j \leq m.$$ 

It is clear that $\mathfrak{g} = \bigoplus_{i=2}^{\infty} \mathfrak{g}_i$ is a $\mathbb{Z}_2$-consistent $\mathbb{Z}$-graded Lie superalgebra such that $\mathfrak{g}_0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$. Here $\mathfrak{g}_{\pm 2} = \mathbb{C}e_{\pm 2\delta}$ and $\mathfrak{g}_0 = \mathfrak{so}(m)$ with the set of positive roots $\Delta^+_0 = \Delta^+_0 \setminus \{2\delta\}$.

A weight $\lambda = \lambda_0 + \sum_{i=1}^m \lambda_i \varepsilon_i \in \mathfrak{h}^*$ is written as

$$\lambda = (\lambda_0 \mid \lambda_1, ..., \lambda_m).$$

We shall call $\lambda_i$ the $i$-th coordinate of $\lambda$. Denote by $\rho$ the half signed sum of positive roots, and by $\rho_1$ the half sum of positive odd roots. Then

$$\rho = (1 - m \mid m - 1, m - 2, ..., 0), \quad \rho_1 = (m \mid 0, ..., 0), \quad \text{if} \quad k = 2m,$$

$$\rho = (\frac{1}{2} - m \mid m - \frac{1}{2}, m - \frac{3}{2}, ..., \frac{1}{2}), \quad \rho_1 = (m + \frac{1}{2} \mid 0, ..., 0), \quad \text{if} \quad k = 2m + 1.$$  

(2.1)

For convenience, we always denote $s = 1$ if $k = 2m$ and $s = \frac{1}{2}$ if $k = 2m + 1$. Then $\rho = (s - m \mid m - s, ..., 1 - s)$ and $\rho_1 = (m + 1 - s \mid 0, ..., 0)$. Sometimes, it will be convenient to use the so-called $\rho$-translated notation of a weight $\lambda$, which will be always denoted by the same symbol with a tilde:

$$\tilde{\lambda} = \lambda + \rho = (\tilde{\lambda}_0 \mid \tilde{\lambda}_1, ..., \tilde{\lambda}_m).$$  

(2.2)

Then $\tilde{\lambda}_0 = \lambda_0 + s - m$ and $\tilde{\lambda}_i = \lambda_i + m + 1 - s - i$ for $i > 0$.

Denote by $W$ and $W_0$ the Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively. Then $W = W_0 \times \mathbb{Z}_2$, where the nontrivial element $\sigma \in \mathbb{Z}_2$ changes the sign of the 0-th coordinate when acting on a weight. $W_0$ is the Weyl group of $\mathfrak{so}(m)$, which acts on a weight $\lambda \in \mathfrak{h}^*$ by permuting the coordinates $\lambda_1, ..., \lambda_m$ and also changing their signs (the number of sign changes must be in $2s\mathbb{Z}_+$. Define the dot action of $W$ on $\mathfrak{h}^*$ by

$$w \cdot \lambda = w(\tilde{\lambda}) - \rho \quad \text{for} \quad w \in W, \ \lambda \in \mathfrak{h}^*. \quad (2.3)$$

We remark that the dot action of $W_0$ on $\mathfrak{h}^*$ is defined by $w \cdot \lambda = w(\lambda + \rho_0) - \rho_0$, where $\rho_0 = \rho + \rho_1$ is the half sum of positive even roots. Since $\rho_1$ is $W_0$-invariant, this dot action coincides with (2.3). We denote

$$\lambda^\sigma = \sigma \cdot \lambda = (-\lambda_0 + 2(m - s) \mid \lambda_1, ..., \lambda_m). \quad (2.4)$$

Let $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be a weight module over $\mathfrak{g}$, where

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v, \ \forall h \in \mathfrak{h}\} \quad \text{with} \quad \dim V_\lambda < \infty,$$

is the weight space of weight $\lambda$. The character $\text{ch} V$ is defined to be

$$\text{ch} V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda,$$

where $e^\lambda$ is the formal exponential, which will be regarded as an element of an additive group isomorphic to $\mathfrak{h}^*$ under $\lambda \mapsto e^\lambda$. Then $\text{ch} V$ is an element of the completed group algebra

$$\varepsilon = \{ \sum_{\lambda \in \mathfrak{h}^*} a_\lambda e^\lambda \mid a_\lambda \in \mathbb{C}, \ a_\lambda = 0 \text{ except} \ \lambda \text{ is in a finite union of } \mathcal{Q}_\Lambda \},$$

where for $\Lambda \in \mathfrak{h}^*$,

$$\mathcal{Q}_\Lambda = \{ \Lambda - \sum_{a \in \Delta^+} i_a \alpha \mid i_a \in \mathbb{Z}_+ \}. \quad (2.5)$$
Sometimes, we may also work with elements in the group $\mathcal{E}$ which is defined as above with $Q_\Lambda$ replaced by

$$Q_\Lambda = \{ \Lambda + \sum_{\alpha \in \Delta^+} i_\alpha \alpha \in \mathfrak{h}^* \mid i_\alpha \in \mathbb{Z}_+ \}. \quad (2.5)$$

### 2.2. Generalised Verma modules and Kac modules

A weight $\lambda \in \mathfrak{h}^*$ is called integral if

$$\lambda_0 \in \mathbb{Z} \quad \text{and} \quad \lambda_i \in \mathbb{Z} + s \mathbb{Z}. \quad (2.6)$$

Denote by $P$ the set of integral weights. An integral weight $\lambda \in P$ is regular if $|\tilde{\lambda}_1|, \ldots, |\tilde{\lambda}_m|$ are distinct number; $\mathfrak{g}_0$-dominant if

$$\lambda_0 \geq 0, \lambda_1 \geq \ldots \geq \lambda_{m-1} \geq |\lambda_m|, \quad \text{and further}, \quad \lambda_m \geq 0 \text{ if } k = 2m + 1. \quad (2.7)$$

We denote by $P^{0+}$ the set of integral $\mathfrak{g}_0$-dominant weights. Then every regular weight $\lambda$ is $W_0$-conjugate under the dot action $[2.3]$ to a unique integral $\mathfrak{g}_0$-dominant weight, which will be denoted by $\lambda^+$ throughout the paper.

For $\lambda \in P^{0+}$, let $L^{(0)}_\lambda$ be the finite-dimensional irreducible $\mathfrak{g}_0$-module with highest weight $\lambda$. Extend it to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$-module by putting $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)L^{(0)}_\lambda = 0$. Then the generalised Verma module $V_\lambda$ is the induced module $[14, 15]$

$$V_\lambda = \text{Ind}^0_{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2} L^{(0)}_\lambda \cong U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \otimes_c L^{(0)}_\lambda. \quad (2.8)$$

One has the following easy result.

**Lemma 2.2.** If $\lambda$ is an integral $\mathfrak{g}_0$-dominant weight, then

$$\text{ch} V_\lambda = \frac{R_1}{R_0} \sum_{w \in W_0} \text{sign}(w) e^{w(\lambda + \rho)} = \frac{1}{R_0} \sum_{w \in W_0} \text{sign}(w) \left( e^{\lambda + \rho_0} \prod_{\beta \in \Delta^+_1} (1 + e^{-\beta}) \right), \quad (2.9)$$

where sign$(w)$ is the signature of $w \in W$, and

$$R_0 = \prod_{\alpha \in \Delta^+_0} (e^{\alpha/2} - e^{-\alpha/2}), \quad R_1 = \prod_{\beta \in \Delta^+_1} (e^{\beta/2} + e^{-\beta/2}).$$

Denote by $L_\lambda$ the unique irreducible quotient module of $V_\lambda$. For convenience we shall always fix a highest weight vector $v_\lambda$ in $V_\lambda$ or $L_\lambda$. Obviously, for any $\lambda \in P^{0+}$, the generalised Verma module $V_\lambda = \oplus_{i \in \mathbb{Z}_+} V^{(i)}_\lambda$ is $\mathbb{Z}$-graded with respect to the eigenvalue of $\frac{1}{2}h_{2\beta}$ such that each $V^{(i)}_\lambda$ is a finite-dimensional $\mathfrak{g}_0$-module.

A weight $\lambda$ is called integral $\mathfrak{g}$-dominant if it satisfies the conditions $[2.10]$, $[2.7]$ and

$$\ell = \lambda_0 \in \mathbb{Z}_+, \quad \text{and if } 0 \leq \ell \leq m - 1 \text{ then } \lambda_{\ell+1} = \lambda_{\ell+2} = \ldots = \lambda_m = 0. \quad (2.10)$$

Let $P^+$ denote the set of integral $\mathfrak{g}$-dominant weights. It was shown in $[13]$ that $\dim L_\lambda < \infty$ if and only if $\lambda \in P^+$. In this case, we can define the quotient module

$$K_\lambda = V_\lambda / U(\mathfrak{g}) f^\lambda_{2\delta} v_\lambda, \quad (2.11)$$

which is usually referred to as the Kac module $[14, 15]$. Obviously, by $[2.8]$ and the PBW Theorem,

$$\dim K_\lambda \leq (\lambda_0 + 1)2^{\dim \theta - 1} \dim L^{(0)}_\lambda < \infty.$$

2.3. Atypicality and central characters. An integral $g_0$-dominant weight $\lambda$ is called **atypical with atypical root** $\gamma = \delta + \varepsilon_\ell$ (resp. $\gamma = \delta - \varepsilon_\ell$) if $\tilde{\lambda}_0 = \tilde{\lambda}_\ell$ (resp. $\tilde{\lambda}_0 = -\tilde{\lambda}_\ell$) for some $1 \leq \ell \leq m$. A weight that is not atypical is called **typical**.

**Remark 2.3.** In case $\tilde{\lambda}_0 = \tilde{\lambda}_\ell = 0$ (this can only occur when $k = 2m$), both roots $\gamma_{\pm} = \delta \pm \varepsilon_\ell$ are atypical roots of $\lambda$. In this case, we always choose $\gamma = \delta - \varepsilon_\ell$.

**Definition 2.4.** (cf. [23]) In case $\tilde{\lambda}_0 = -\tilde{\lambda}_\ell$, the atypical root $\gamma = \delta - \varepsilon_\ell$ is called a **tail atypical root** (cf. Example 3.5).

Assume $\lambda \in P^{0+}$ is atypical with atypical root $\gamma = \delta + \varepsilon_\ell$ or $\delta - \varepsilon_\ell$. Then conditions $\lambda_0 \in \mathbb{Z}$ and $\tilde{\lambda}_0 = \pm \tilde{\lambda}_\ell$ force $\lambda_i \in \mathbb{Z}$ for all $i$. Denote

$$\tilde{\lambda} = (|\tilde{\lambda}_1|, \ldots, |\tilde{\lambda}_{\ell-1}|, |\tilde{\lambda}_{\ell-1}|, \ldots, |\tilde{\lambda}_m|), \quad S(\tilde{\lambda}) = \{|\tilde{\lambda}_1|, \ldots, |\tilde{\lambda}_{\ell-1}|, |\tilde{\lambda}_{\ell-1}|, \ldots, |\tilde{\lambda}_m|\}. \quad (2.12)$$

(Note from (2.7) that $\tilde{\lambda}_m$ is the only possible negative number in (2.12)). We call $\tilde{\lambda}$ the atypicality type of $\lambda$.

Denote by $Z(g)$ the center of the universal enveloping algebra $U(g)$. An element $z \in Z(g)$ can be uniquely written in the form $z = u_z + \sum_i u_i^+ u_i^{-1}$, where $u_z, u_i^0 \in U(h), u_i^\pm \in g_\pm U(g_\pm)$, and $g_\pm$ is the subalgebra of $g$ spanned by the positive/negative root vectors. Then the map $z \mapsto u_z$ gives rise to the Harish-Chandra homomorphism

$$HC : Z(g) \to \mathbb{C}[h^*], \quad HC(z)(\lambda) = u_z(\lambda). \quad (2.13)$$

Recall that a central character is a homomorphism $Z(g) \to \mathbb{C}$. Thus any $\lambda \in h^*$ defines a central character $\chi_\lambda$ by the rule $\chi_\lambda(z) = HC(z)(\lambda)$, such that all element $z \in Z(g)$ acts on $L_\lambda$ as the scalar $\chi_\lambda(z)$. One has [23]

$$\chi_\lambda = \chi_\mu \implies \tilde{\lambda} = \tilde{\mu} \quad \text{for all atypical } \lambda, \mu \in P^+. \quad (2.14)$$

The following result is due to Kac [14, 15]:

**Proposition 2.5.** If $\lambda$ is an integral $g$-dominant typical weight, then $L_\lambda = K_\lambda$, and

$$\text{ch}L_\lambda = \text{ch}K_\lambda = \frac{R_1}{R_0} \sum_{w \in W} \text{sign}(w) e^{w(\lambda + \rho)}. \quad (2.15)$$

3. Structure of generalised Verma modules

In this section, we shall completely determine the structure of generalised Verma modules $V_\lambda$ for all atypical $\lambda \in P^+$.

3.1. Primitive weight graphs. For a $g$-module $V$, a nonzero $g_0$-highest weight vector $v \in V$ is called a **primitive vector** if $v$ generates an indecomposable $g$-submodule and there exists a $g$-submodule $W$ of $V$ such that $v \notin W$ but $g_+ v \in W$. If we can take $W = 0$, then $v$ is called a **strongly primitive vector** or a **$g$-highest weight vector**. The weight of a primitive vector is called a **primitive weight**, and the weight of a strongly primitive vector a **strongly primitive weight** or a **$g$-highest weight**. For a primitive weight $\lambda$ of $V$, we often use $v_\lambda$ to denote a primitive vector of weight $\lambda$.

**Notation 3.1.** Denote by $P(V)$ the set of primitive weights with multiplicities. For $\mu, \nu \in P(V)$, if $\mu \neq \nu$ and $v_\nu \in U(g) v_\mu$, we say that $\nu$ is derived from $\mu$ and write

$$\nu \leftrightarrow \mu \quad \text{or} \quad \mu \rightarrow \nu.$$ 

If $\mu \rightarrow \nu$ and there exists no $\lambda \in P(V)$ such that $\mu \rightarrow \lambda \rightarrow \nu$, then we say that $\nu$ is **directly derived from** $\mu$ and write

$$\mu \rightarrow \nu \quad \text{or} \quad \nu \rightarrow \mu.$$
Sometimes for convenience, we also use symbols $\mu \leftrightarrow \lambda$ to denote either $\mu \rightarrow \lambda$ or $\mu \leftarrow \lambda$. Suppose $V'$ is a submodule of $V$ and $\mu \in P(V) \setminus P(V')$, we use $V' \leftrightarrow \mu$ to indicate $\nu \leftrightarrow \mu$ for some $\nu \in P(V')$.

**Definition 3.2.** (cf. [26, Definition 6.2]) Suppose every composition factor of $V$ is a highest weight module. Then we can associate $P(V)$ with a directed graph, still denoted by $P(V)$, in the following way: the vertices of the graph are elements of $P(V)$. Two elements $\lambda$ and $\mu$ are connected by a single directed edge (i.e., the two weights are linked) pointing toward $\mu$ if and only if $\mu$ is directly derived from $\lambda$. We shall call this graph the *primitive weight graph* of $V$.

A full subgraph $S$ of $P(V)$ is a subset of $P(V)$ which contains all the edges linking vertices of $S$. We call a full subgraph $S$ closed if for any $\eta \in P(V)$ and $\mu, \nu \in S$,

$$\mu \rightarrow \eta \rightarrow \nu \implies \eta \in S.$$  

It is clear that a module is indecomposable if and only if its primitive weight graph is connected (in the usual sense). It is also clear that a full subgraph of $P(V)$ corresponds to a subquotient of $V$ if and only if it is closed. Thus a full subgraph with only 2 weights is always closed.

For a directed graph $\Gamma$, we denote by $M(\Gamma)$ any module with primitive weight graph $\Gamma$ if such a module exists. If $\Gamma$ is a closed full subgraph of $P(V)$, then $M(\Gamma)$ always exists, which is a subquotient of $V$.

### 3.2. Some technical lemmas.

Let $P_{\lambda}^{0+}$ be the set of integral $g_{0}$-dominant atypical weights with atypical type $\lambda$, and set $P_{\lambda}^+ = P^+ \cap P_{\lambda}^{0+}$. If $\lambda \in P_{\lambda}^{0+}$ has atypical root $\gamma \in \Delta_{\lambda}^+$, we let $a_{\pm} \in \mathbb{N} = \{1, 2, \ldots\}$ be the smallest such that $\lambda + a_{\pm} \gamma$ and $\lambda - a_{-} \gamma$ are $g_{0}$-regular in the sense that $(\lambda + \rho \pm a_{\pm} \gamma, \alpha) \neq 0$ for all roots $\alpha$ of $g_{0}$. If $\gamma = \delta + \varepsilon_{\ell}$ or $\delta - \varepsilon_{\ell}$, then $a_{\pm}$ are the smallest positive integers such that $|\lambda_{\ell} \pm a_{\pm}| \notin S(\bar{\lambda})$ (cf. (2.12)). Now we defined $\chi, \tilde{\chi} \in P_{\lambda}^{0+}$ by

$$\chi = (\lambda + a_{+} \gamma)^+, \; \tilde{\chi} = (\lambda - a_{-} \gamma)^+,$$

where we recall that given an integral weight $\mu$, we denote by $\mu^{\pm}$ the integral $g_{0}$-dominant weight $W_{0}$-conjugate to $\mu$ under the dot action (2.3). We call $\overset{\wedge}{\!}$ and $\overset{\vee}{\!}$ the raising and lowering operators. Note that

$$\overset{\wedge}{\!}(\lambda^{\sigma}) = (\lambda)^{\sigma}, \; \overset{\vee}{\!}(\lambda^{\sigma}) = (\lambda)^{\sigma} \text{ if } \tilde{\lambda}_{0} \neq 0.$$  

**Definition 3.3.** Keep notation introduced above. If $\tilde{\lambda}_{0} \neq 0$, we let $\chi_{+} = \chi$ and $\chi_{+} = \chi$. If $\tilde{\lambda}_{0} = 0$, we take the atypical root $\gamma = \delta - \varepsilon_{m}$ (cf. Remark 2.3) and let

$$\chi_{-} = \chi, \; \chi_{+} = (\lambda + a_{+} \gamma_{+})^{+}, \; \chi_{-} = (\lambda - a_{-} \gamma_{+})^{+}, \; \chi_{+} = \chi,$$

where $\gamma_{+} = \delta + \varepsilon_{m}$, which is another atypical root.

**Convention 3.4.** If an undefined symbol appear in an expression, we regard it as nothing; for instance, in case $\tilde{\lambda}_{0} = \tilde{\lambda}_{\ell} \neq 0$, symbols $\chi_{-}, \chi_{-}$ mean nothing.

**Example 3.5.** (1) $\lambda = (1 \mid 2, 1, -1)$ when $k = 6$. We shall always use $\square$ to indicate a tail atypical root (cf. Definition 2.4), and use $\square$ to indicate a non-tail atypical root. Then

$$\tilde{\lambda} = (-1 \mid 4, 2, -1),$$

and $a_{+} = 1, a_{-} = 2$. So

$$\tilde{\chi} = (0 \mid 4, 2, 0), \; \tilde{\chi} = (-3 \mid 4, 3, -2),$$
and \( \lambda = (2 \mid 2, 1, 0), \lambda = (-1 \mid 2, 2, -2) \).

(2) If \( k = 7 \) and \( \lambda = (2 \mid 2, 0, 0) \), then
\[
\tilde{\lambda} = \left( -\frac{1}{2} \mid \frac{9}{2}, \frac{3}{2}, \frac{1}{2} \right),
\]
and \( a_+ = 2, a_- = 1 \). Thus
\[
\tilde{\lambda} = \left( \frac{1}{2} \mid \frac{9}{2}, \frac{3}{2}, \frac{1}{2} \right), \quad \tilde{\lambda} = \left( -\frac{5}{2} \mid \frac{9}{2}, \frac{3}{2} \right),
\]
and \( \lambda = (3 \mid 2, 0, 0), \lambda = (0 \mid 2, 1, 1) \).

**Remark 3.6.** In the case \( k = 2m \), there exists an outer automorphism \( \tau \) of \( g \) induced by the symmetry automorphism
\[
\tau(\varepsilon_{m-1} - \varepsilon_m) = \varepsilon_{m-1} + \varepsilon_m,
\]
of the Dynkin diagram, which changes the sign of the \( m \)-th coordinate of a weight. Thus the structure of a generalised Verma module \( V_{\lambda} \) with \( \lambda_m < 0 \) is the same as the structure of \( V_{\lambda} \) with \( \lambda_m > 0 \). Therefore, when considering the generalised Verma module \( V_{\lambda} \) (or Kac module \( K_{\lambda} \), or irreducible module \( L_{\lambda} \)), we can always suppose
\[
\lambda_m \geq 0.
\]
However, it should be pointed out that \( V_{\lambda} \) may contain some primitive weight \( \mu \) with \( \mu_m < 0 \).

In the following, we always assume that \( \tilde{\lambda} \) is a fixed atypical type such that its coordinates \( \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m-1} \) satisfy \( \tilde{\lambda}_1 > \cdots > \tilde{\lambda}_{m-1} \geq 0 \) (cf. (2.7), (2.12) and (3.4)).

**Definition 3.7.** Let \( j \) be the smallest non-negative integer such that \( a := j + 1 - s \notin S(\hat{\lambda}) \) (cf. (2.12)). We define \( \lambda^{(0)} \) by
\[
\tilde{\lambda}(0) = \lambda^{(0)} + \rho = (-a \mid \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m-1}, a, \tilde{\lambda}_{m+1-j}, \ldots, \tilde{\lambda}_{m}),
\]
where we have re-labeled \( \tilde{\lambda}_{m-j}, \ldots, \tilde{\lambda}_{m-1} \) by \( \tilde{\lambda}_{m+1-j}, \ldots, \tilde{\lambda}_m \) for convenience.

1. If \( k = 2m + 1 \) or \( k = 2m \) with \( 0 \in S(\bar{\lambda}) \), we define \( \lambda^{(\pm i)} \) (for \( i > 0 \)) inductively by
\[
\lambda^{(i)} = (\lambda^{(i-1)})^\gamma_+, \quad \lambda^{(-i)} = (\lambda^{(1-i)})^\gamma_-,
\]
and set \( \lambda^{(\pm i)} = \lambda^{(\pm \bar{i})} \) for all \( i > 0 \).

2. If \( k = 2m \) and \( 0 \notin S(\bar{\lambda}) \), we have \( a = 0 \). Take the atypical root \( \gamma = \delta - \varepsilon_m \) for the corresponding \( \lambda^{(0)} \) defined by (3.5) as in Definition 3.3, and define \( \lambda^{(\pm i)} \) for \( i > 0 \) by
\[
\lambda^{(1)} = (\lambda^{(0)})^\gamma_+, \quad \lambda^{(i)} = (\lambda^{(i-1)})^\gamma_+ = (\lambda^{(1-i)})^\gamma_-.
\]
[Recall that \( \gamma_+ = \delta + \varepsilon_m \) is also an atypical root of \( \lambda^{(0)} \).] Let \( \lambda^{(\pm i)} = \lambda^{(\pm \bar{i})} \) for \( i > 0 \), and further define \( \lambda_\pm^{(\pm i)} \) by
\[
\lambda^{(-1)} = (\lambda^{(0)})^\gamma_- = (\lambda^{(i)})^\gamma_+, \quad \lambda^{(-i)} = (\lambda^{(0)})^\gamma_- = (\lambda^{(1-i)})^\gamma_+.
\]

**Example 3.8.** If \( k = 4, \bar{\lambda} = (1) \), then \( \tilde{\lambda}(0) = (0 \mid 1, 0) \), and
\[
\tilde{\lambda}(1) = (2 \mid 2, 1), \quad \tilde{\lambda}(2) = (3 \mid 3, 1), \quad \tilde{\lambda}(-1) = (-2 \mid 2, 1), \quad \tilde{\lambda}(-2) = (-3 \mid 3, 1),
\]
\[
\tilde{\lambda}_-(1) = (2 \mid 2, -1), \quad \tilde{\lambda}_-(2) = (3 \mid 3, -1), \quad \tilde{\lambda}_-(1) = (-2 \mid 2, -1), \quad \tilde{\lambda}_-(2) = (-3 \mid 3, -1).
\]
From the definition and condition (2.10), we immediately obtain the following.
Lemma 3.9.  
(1) $P_{\lambda}^{0+} = \{\lambda^{(i)} \mid i \in \mathbb{Z}\}$, and $P_{\lambda}^+ = \{\lambda^{(i)} \mid i \in \mathbb{Z}_+\}$ (cf. Convention 3.4).

(2) $\lambda^{(0)}$ is $\mathfrak{g}$-dominant and is the unique $\mathfrak{g}$-dominant weight among the $\lambda^{(i)}$ which has a tail atypical root.

Now we shall investigate the generalised Verma module $V_{\lambda}$ for $\lambda \in P_{\lambda}^{0+}$ satisfying (3.4). By the PBW Theorem, $U(\mathfrak{g}_- \oplus \mathfrak{g}_-)$ has a basis

\[ B = \left\{ f_\Theta \mid \Theta \in \mathbb{Z}_+ \times \{0, 1\}^{2m} \right\}, \]

where each $f_\Theta$ is an ordered product of the form $f_\Theta = f^\Theta_{\theta_1}s_{\delta-\epsilon_1} \cdots f^\Theta_{\theta_m}s_{\delta+\epsilon_i}$ for $\Theta = (\theta, \theta_1, \ldots, \theta_m, \bar{\theta}_m, \ldots, \bar{\theta}_1)$. Define a total order on $B$ by

\[ f_\Theta > f_{\Theta'} \iff |\Theta| > |\Theta'| \quad \text{or} \quad |\Theta| = |\Theta'| \quad \text{but} \quad \Theta > \Theta', \]

where $|\Theta|$ is $\theta + \sum_{i=1}^m (\theta_i + \bar{\theta}_i)$ is the level of $\Theta$, and the order on $\mathbb{Z}_+ \times \{0, 1\}^{2m}$ is defined lexicographically. Recall that a nonzero vector $v \in V_{\lambda}$ can be uniquely written as

\[ v = b_1v_1 + \cdots + b_nv_n, \quad b_i \in B, \quad b_1 > b_2 > \cdots, \quad 0 \neq v_i \in L_{\lambda}^{(i)}. \]  

(3.6)

We call $b_1v_1$ the leading term (cf. [25, §5]). A term $b_iv_i$ is called a prime term if $v_i \in \mathbb{C}v_{\lambda}$. Note that a vector $v$ may have zero or more than one prime terms. One immediately has

Lemma 3.10. Let $v = gu$ for some $u \in V_{\lambda}$ and $g \in U(\mathfrak{g}^-)$. 

(i) If $u$ has no prime term then $v$ has no prime term.

(ii) Let $v' = gu$, $u' \in V_{\lambda}$. If $u, u'$ have the same prime terms then $v, v'$ have the same prime terms.

Lemma 3.11. 

(i) Let $v_\mu \in V_{\lambda}$ be a $\mathfrak{g}_0$-highest weight vector of weight $\mu$. Then 

\[ \lambda - \mu = 2s\theta\delta + \sum_{i=1}^m (\theta_i(\delta - \epsilon_i) + \bar{\theta}_i(\delta + \epsilon_i)), \] 

for some $\theta \in \mathbb{Z}_+$, $\theta_i, \bar{\theta}_i \in \{0, 1\}$. Furthermore the leading term $b_1v_1$ of $v_\mu$ must be a prime term.

(ii) Suppose $v'_\mu = \sum_{i=1}^{n'} b'_iv'_i$ is another $\mathfrak{g}_0$-highest weight vector with weight $\mu$. If all prime terms of $v'_\mu$ are the same as those of $v'_i$, then $v_\mu = v'_\mu$.

Proof. (i) Let $v_\mu$ be as in (3.3). If $v_1 \notin \mathbb{C}v_{\lambda}$, then there exists $e \in \mathfrak{g}_0^+$, $ev_1 \neq 0$. We have 

\[ ev_\mu = b_1(ev_1) + [e, b_1]v_1 + b_2(ev_2) + [e, b_2]v_2 + \cdots. \]

Obviously when writing $[e, b_1]$ in terms of linear combination of $B$, we can see that $b_1(ev_1)$ is the leading term of $ev_\mu$, i.e., $ev_\mu \neq 0$, contradicting that $v_\mu$ is a $\mathfrak{g}_0$-highest weight vector. So, $v_1 \in \mathbb{C}v_{\lambda}$ and $\lambda - \mu$ is the weight of $b_1$, i.e., we have (3.7).

(ii) Let $v = v_\mu - v'_\mu$. If $v \neq 0$ (then it must be a $\mathfrak{g}_0$-highest weight vector), since its prime terms are all cancelled, $v$ has no prime term, therefore by (i), it is not $\mathfrak{g}_0$-highest weight vector, a contradiction. \hfill \Box

Recall from Notation 3.1 that $P(V_{\lambda})$ is the set of primitive weights of $V_{\lambda}$. Let $M_{\lambda}$ be any high weight $\mathfrak{g}$-module with highest weight $\lambda$, we also define

\[ P_0(M_{\lambda}) = \{ \mu \in P_{\lambda}^{0+} \mid \mu \text{ is a } \mathfrak{g}_0\text{-highest weight in } M_{\lambda} \}. \]  

(3.8)

Since $\lambda$ and any weight in $P(V_{\lambda})$ have the same central character, they have the same atypicality type by (2.11). Thus $P(V_{\lambda}) \subset P_0(V_{\lambda})$. 
Let
\[ a_{\lambda, \mu} = [V_\lambda, L_\mu] \text{ is the multiplicity of } L_\mu \text{ in } V_\lambda, \]
\[ b_{\lambda, \mu} = [V_\lambda, L_\mu^{(0)}] \text{ is the multiplicity of } g_0\text{-module } L_\mu^{(0)} \text{ in } V_\lambda. \] (3.9)

In defining \( b_{\lambda, \mu} \) we restrict \( V_\lambda \) to a \( g_0 \text{-module} \). Also, in principle the multiplicity \( [V_\lambda, L_\mu] \) of \( L_\mu \) in \( V_\lambda \) needs to be defined as in [19] \( \S 5 \). However it turns out that \( V_\lambda \) has composition series of finite length, thus the complications dealt with in [19] \( \S 5 \) do not arise in our setting. Clearly \( a_{\lambda, \mu} \leq b_{\lambda, \mu} \) for \( \mu \in P(V_\lambda) \).

We define a partial order on \( \mathfrak{h}^* \) by: \( \lambda > \mu \) if and only if \( \lambda - \mu \) is a sum of positive roots for \( \lambda, \mu \in \mathfrak{h}^* \).

**Lemma 3.12.** Let \( \lambda \in P_\lambda^{0+} \) with \( \lambda_m > 0 \).

1. If \( k \neq 2m \) or \( 0 \notin S(\bar{\lambda}) \), then \( P_0(V_\lambda) \subset \{ \lambda, \lambda, \lambda^\sigma, (\lambda^\sigma)^{\gamma}, (\lambda^\sigma)^{\gamma} \} \) and \( b_{\lambda, \lambda} \leq 1 \). Furthermore, if \( \lambda \) has a tail atypical root, then \( P_0(V_\lambda) \subset \{ \lambda, \lambda \} \).
2. If \( k = 2m \) and \( 0 \notin S(\bar{\lambda}) \), then
   1. \( P_0(V_\lambda^{(0)}) \subset \{ \lambda^{(0)}, \lambda^{(-1)}_1 \} \) and \( b_{\lambda^{(0)}}, \lambda^{(-1)}_1 \leq 1 \);
   2. \( P_0(V_\lambda^{(-i)}) \subset \{ \lambda^{(-i)}, \lambda^{(-i-1)} \} \) and \( b_{\lambda^{(-i)}, \lambda^{(-i-1)}} \leq 1 \) for \( i \geq 1 \);
   3. \( P_0(V_\lambda^{(i)}) \subset \{ \lambda^{(i)}, \lambda^{(i-1)}, \lambda^{(i-1)} \} \) and \( b_{\lambda^{(i)}, \lambda^{(i-1)}} \leq 1 \) for \( i \geq 1 \).

**Proof.** We shall prove (1) only, as the proof for (2) is similar. Let \( \mu \in P_0(V_\lambda) \). Then \( \mu \leq \lambda \). Denote by \( \gamma_\lambda \) and \( \gamma_\mu \) the atypical roots of \( \lambda \) and \( \mu \) respectively.

**Case 1:** Suppose \( \gamma_\lambda = \delta + \varepsilon_\ell, \gamma_\mu = \delta + \varepsilon_\ell \) and \( \ell \leq t \).

Then we can write
\[
\bar{\lambda} = (\Lambda_0 | \Lambda_1, ..., \Lambda_{\ell-1}, \Lambda_\ell, \Lambda_{\ell+1}, ..., \Lambda_{t-1}, \Lambda_\ell, \Lambda_{t+1}, ..., \Lambda_m),
\]
\[
\bar{\mu} = (\tilde{\Lambda}_0 | \tilde{\Lambda}_1, ..., \tilde{\Lambda}_{\ell-1}, \tilde{\Lambda}_{\ell+1}, ..., \tilde{\Lambda}_t, \tilde{\Lambda}_{t+1}, ..., \tilde{\Lambda}_m),
\]
where \( \Lambda_0 = \Lambda_\ell, \tilde{\Lambda}_0 = \tilde{\Lambda}_t \). By (3.7), there exists \( \Theta \in \mathbb{Z}_+ \times \{0, 1\}^{2m} \) such that
\[
\Lambda_0 = \tilde{\Lambda}_0 = M_0 + 2s_\theta + \sum_{i=1}^m (\theta_i + \tilde{\theta}_i), \quad \theta_i = \tilde{\theta}_i (i < \ell \text{ or } i > t),
\]
\[
\Lambda_i = \Lambda_{i+1} - \theta_i + \tilde{\theta}_i (\ell \leq i < t), \quad \Lambda_\ell = \tilde{\Lambda}_t = \Lambda_\ell - \theta_{t+1} + \tilde{\theta}_{t+1},
\]
Note that \( \Lambda_i \geq \Lambda_{i+1} + 1 \) for all \( i \) and \( \Lambda_i \geq \tilde{\Lambda}_t + 1 \geq \Lambda_{t+1} + 2 \) (the last inequality occurs only when \( t < m \)) , and \( \theta_i, \tilde{\theta}_i \in \{0, 1\} \). Thus \( \theta_i = 0, \tilde{\theta}_i = 1 \) for \( \ell \leq i \leq t \) and \( \theta = \tilde{\theta}_j = \tilde{\theta}_{j+1} = 0 \) for \( j < \ell \) or \( j > t \). So
\[
\Lambda_\ell = \Lambda_{\ell+1} + 1 = \Lambda_{\ell+2} + 2 = ... = \Lambda_t + t - \ell = M_t + t + 1 - \ell. \] (3.11)

(If \( \ell = t \), there is no last equality and \( \tilde{\theta}_t \) can be 0 or 1). Thus \( \mu = \lambda \) or \( \lambda \). Since in each case, the solution \( \Theta \) is unique, by Lemma 3.11(2), \( a_{\lambda, \mu} \leq 1 \).

**Case 2:** Suppose \( \gamma_\lambda = \delta + \varepsilon_\ell, \gamma_\mu = \delta - \varepsilon_\ell \) and \( \ell > t \).

Then \( M_0 = M_t > \Lambda_{t+1} \geq \Lambda_\ell = \Lambda_0 \), which contradicts the first equation in (3.10). Thus \( \ell > t \) cannot occur.

**Case 3:** Suppose \( \gamma_\lambda = \delta + \varepsilon_\ell, \gamma_\mu = \delta - \varepsilon_\ell \) and \( \ell \leq t \).

Then \( M_0 = -M_t \). We still have (3.11), thus \( \mu \) has to be \( \lambda^\sigma \) or \( (\lambda^\sigma)^\gamma \) (in this case, the solution of \( \Theta \) might not be unique).
Case 4: Suppose $\gamma_\lambda = \delta + \varepsilon_\ell$, $\gamma_\mu = \delta - \varepsilon_\ell$ and $\ell > t$.
Then again $M_0 = -M_t$. We re-write $\lambda, \mu$ as
\[
\tilde{\lambda} = (\Lambda_0 | \Lambda_1, ..., \Lambda_{t-1}, \Lambda_t, \Lambda_{t+1}, ..., \Lambda_{t-1}, \Lambda_\ell, \Lambda_{\ell+1}, ..., \Lambda_m),
\]
\[
\tilde{\mu} = (M_0 | \Lambda_1, ..., \Lambda_{t-1}, M_t, \Lambda_t, ..., \Lambda_{\ell-2}, \Lambda_{\ell-1}, \Lambda_{\ell+1}, ..., \Lambda_m).
\]
Thus
\[
M_t = \Lambda_t + 1 = \Lambda_{t+2} + 2 = ... = \Lambda_\ell + \ell - t + 1.
\]
In this case, $\mu$ has to be $(\lambda^n)$ (the solution of $\Theta$ might not be unique).

Case 5: Suppose $\gamma_\lambda = \delta - \varepsilon_\ell$, $\gamma_\mu = \delta \pm \varepsilon_\ell$.
Then $M_0 \leq \Lambda_0 \leq 0$. As in Case 2, we have $t \leq \ell$. Similar to Case 1, we obtain $\mu = \lambda$ or $\chi$. The atypical root of $\mu$ is $\gamma_\mu = \delta - \varepsilon_\ell$ in both situations, and the solution for $\Theta$ is unique. Thus $a_{\lambda,\mu} \leq 1$. This proves the lemma. $\square$

For $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda - \mu$ is a sum of distinct positive odd roots, i.e., $\lambda - \mu = \sum_{\beta \in \Sigma} \beta$ for some $\Sigma \in \Delta_1^+$, we introduce the relative level $|\lambda - \mu|$ to be the cardinality $|\Sigma|$, which is also equal to $\lambda_0 - \mu_0$. For any integral (not necessarily $g_0$-dominant) weight $\lambda$ with the atypical root $\gamma_\lambda$ (cf. Remark 2.3), we introduce the Bernstein-Leites formula [2]
\[
\chi^{BL}_\lambda = \frac{1}{P_0} \sum_{w \in W} \text{sign}(w) \left( e^{\lambda + \rho_0} \prod_{\beta \in \Delta_1^+ \setminus \{\gamma_\lambda\}} (1 + e^{-\beta}) \right).	ag{3.12}
\]
For convenience, we introduce the following notation:
\[
\chi^{BL}_{0,\lambda} \text{ denotes the right-hand side of } (3.12) \text{ with } W \text{ replaced by } W_0,
\]
\[
\chi^V_{\lambda} \text{ denotes the right-hand side of } (2.15) \text{ for } \lambda \in P.
\]
Then by recalling the definition of regular weights (immediately before (2.7)), one easily sees that
\[
\chi^V_{\lambda} \neq 0 \iff \lambda \text{ is regular.} \tag{3.14}
\]

Lemma 3.13. Assume that $\lambda \in P_+^{\chi}$ with $\lambda_m \geq 0$ has a tail atypical root $\gamma_\lambda = \delta - \varepsilon_\ell$.
\begin{enumerate}
\item If $\lambda \neq \lambda^{(0)}$ or $0 \in S(\lambda)$, then $P(V_\lambda) = \{\lambda, \lambda\}$, and the primitive weight graph of $V_\lambda$ is $\lambda \to \chi$. Furthermore,
\[
\text{ch } L_\lambda = \chi^{BL}_{0,\lambda}, \quad \text{ch } L_{\lambda^{(0)}} = \frac{1}{2} \chi^{BL}_{\lambda^{(0)}}.
\]
\item If $\lambda = \lambda^{(0)}$ and $0 \notin S(\lambda)$, then $P(V_{\lambda^{(0)}}) = \{\lambda^{(0)}, \lambda^{(-1)}\}$ with the primitive weight graph
\[
\lambda^{(0)} \overset{\lambda^{(-1)}}{\longrightarrow}.
\]
Furthermore,
\[
\text{ch } L_{\lambda^{(0)}} = \chi^{BL}_{\lambda^{(0)}}.
\]
\end{enumerate}

Proof. (1) Since $\lambda$ has a tail atypical root, we have $\lambda = \lambda^{(i)}$ for some $i \leq 0$. As $L_{\lambda^{(0)}}$ is finite dimensional, we have $L_{\lambda^{(0)}} \neq V_{\lambda^{(0)}}$, i.e., $\mu \in P(V_{\lambda^{(0)}})$ for some $\mu \neq \lambda^{(0)}$. By Lemma 3.12, $\mu = \chi = \lambda^{(-1)}$. Now one can use the arguments in the paragraph after (3.35) to prove $\lambda^{(i-1)} \in P(V_{\lambda^{(i)}})$ for $i < 0$. For the purpose of the next lemma and other purposes, we provides below a way to explicitly construct the primitive vector $v_\mu$ (cf. (3.17) and (3.23)) for $\mu = \chi$. 

By Case 5 in the proof of Lemma 3.12, we have
\[ \lambda = (\lambda_0 | \Lambda_1, \ldots, \Lambda_{t-1}, \Lambda_t + \ell - t, \Lambda_t + \ell - t - 1, \ldots, \Lambda_t + 1, \lambda_{t+1}, \ldots, \Lambda_m), \]
\[ \mu = (M_0 | \Lambda_1, \ldots, \Lambda_{t-1}, \Lambda_t + \ell - t + 1, \Lambda_t + \ell - t, \ldots, \Lambda_t + 2, \Lambda_t + 1, \Lambda_{t+1}, \ldots, \Lambda_m), \]
where \( t \leq \ell \). We define \( \mu^{(0)} = \lambda \) and \( \mu^{(i)} = \mu^{(i-1)} - (\delta - \varepsilon_{t+1-i}) \). Then \( \mu = \mu^{(\ell-t+1)} \).

**Case 1: \( t = \ell \).**

We want to construct, from the highest weight vector \( v_\lambda \) of \( V_\lambda \), a \( g_0 \)-highest weight vector \( v_\mu \) with weight \( \mu \). By Lemma 3.11(1), \( v_\mu \) should contain a leading term which has to be \( cf_\delta v_\lambda \) for some nonzero \( c \in \mathbb{C} \). Set \( I = \{1, \ldots, \ell - 1\} \), and for \( S = \{i_1 < \cdots < i_p\} \subset I \), we define \( f_S \) by
\[ f_S = \begin{cases} f_{\delta - \varepsilon_1} f_{\varepsilon_1} - f_{\varepsilon_2} f_{\varepsilon_2} - \cdots - f_{\varepsilon_p} f_{\varepsilon_p} - f_\delta - \varepsilon_\ell & \text{if } S \neq \emptyset, \\ f_\delta - \varepsilon_\ell & \text{if } S = \emptyset. \end{cases} \]  
(3.16)

In order for \( v_\mu \) to have weight \( \mu \), we can suppose that \( v_\mu \) has the form
\[ v_\mu = \sum_{S \subseteq I} c_S f_S v_\lambda, \]  
(3.17)
for some \( c_S \in \mathbb{C} \). Now if we define \( c_S \) uniquely by
\[ c_S = \begin{cases} 1 & \text{if } S = I, \\ -(\delta - \varepsilon_1 - \varepsilon_{t+1} - q - 1) c_{S'} & \text{if } S \nsubseteq I, \; S' = S \cup \{q\}, \; q = \max(I \setminus S), \end{cases} \]  
(3.18)
then we can prove

**Claim 1.** \( v_\mu \) is a \( g_0 \)-highest weight vector.

First note that if \( t = 1 \) then (3.17) is simply reduced to \( v_\mu = f_{\delta - \varepsilon_1} v_\lambda \). In general, note from \( t = \ell \) that \( \lambda_{t-1} > \lambda_\ell + 1 \), i.e., \( \lambda_{t-1} > \lambda_\ell \), and that \( q \leq \ell - 1 \), we have
\[ \lambda_q - \lambda_\ell + \ell - q - 1 \geq \lambda_{t-1} - \lambda_\ell + \ell - q - 1 > \ell - q - 1 \geq 0, \]
and so \( c_S \neq 0 \) for all \( S \subset I \). In particular, the leading term of \( v_\mu \) is \( c_0 f_{\delta - \varepsilon_\ell} v_\lambda \neq 0 \), i.e., \( v_\mu \neq 0 \).

To prove the claim, we only need to prove \( ev_\mu = 0 \) for
\[ e \in \{e_{\varepsilon_a - \varepsilon_a + 1} \; (1 \leq a < m), \; e_{\varepsilon_m - 1 + \varepsilon_m} \; (if \; k = 2m), \; e_{\varepsilon_m} \; (if \; k = 2m + 1)\}. \]  
(3.19)
Note from Notation 2.1 that (in the following we denote \( \varepsilon_0 = \delta \) so that \( b \) can be 0)
\[ [e_{\varepsilon_a - \varepsilon_a + 1}, f_{b - \varepsilon_c}] = \begin{cases} h_{\varepsilon_a - \varepsilon_a + 1} & \text{if } b = a, \; c = a + 1, \\ -f_{b + 1 - \varepsilon_c} & \text{if } b = a, \; c > a + 1, \\ f_{b - \varepsilon_a} & \text{if } b < a, \; c = a + 1, \\ 0 & \text{otherwise}, \end{cases} \]  
(3.20)
where \( b < c \). We see that \( e_{\varepsilon_a - \varepsilon_a + 1} \) (\( \ell \leq a \leq m - 1 \)), \( e_{\varepsilon_m - 1 + \varepsilon_m} \) (if \( k = 2m \)), \( e_{\varepsilon_m} \) (if \( k = 2m + 1 \)) commute with \( f_S \) for all \( S \subset I \). Thus it suffices to consider \( e = e_{\varepsilon_a - \varepsilon_a + 1} \) for \( 1 \leq a < \ell \).

For any \( S_0 = \{i_1 < \cdots < i_x < j_1 < \cdots < j_p\} \subset I \) with \( i_x < a \), \( a + 1 < j_1 \) (where \( x \) or \( p \) can be zero). Let \( S_1 = S_0 \cup \{a, a + 1\} \), \( S_2 = S_0 \cup \{a\} \), \( S_3 = S_0 \cup \{a + 1\} \). We want to prove
\[ c_{S_2} = -(\lambda_a - \lambda_\ell + \ell - a - 1)c_{S_1}, \quad c_{S_3} = -(\lambda_{a+1} - \lambda_\ell + \ell - a - 2)c_{S_1}. \]  
(3.21)
Set $b = \max(I \setminus S_0)$. If $b = a + 1$, then we have (3.21) by definition (3.18). If $b > a + 1$, then (3.18) shows $c_{S_i} = - (\lambda_b - \lambda_{\ell + b - 1})c_{S_i}'$ for $S_i' = S_i \cup \{b\}$ and $i = 1, 2, 3$. Thus (3.21) can be proved by induction on $\max(I \setminus S)$. From (3.20) and (3.21), we obtain

$$
[e_{\varepsilon_{a-\varepsilon_{a+1}}} f_{S_i}]v_\lambda = ((\lambda_a - \lambda_{a+1} + 1)c_{S_1} + c_{S_2} - c_{S_3}) \hat{f}_{S_i} v_\lambda = 0,
$$

(3.22)

where $\hat{f}_{S_i}$ is defined as $f_{S_i}$ in (3.16) but with the factor $f_{\varepsilon_{a-\varepsilon_{a+1}}}$ removed. Note that the family $\{S I S \subset I\}$ of subsets of $I$ can be divided into a disjoint union of blocks, each block has the form $\{S_0, S_1, S_2, S_3\}$ defined as above. Thus (3.22) together with $[e_{\varepsilon_{a-\varepsilon_{a+1}}} f_{S_0}] = 0$ implies $e_{\varepsilon_{a-\varepsilon_{a+1}}} v_\mu = 0$, and so the claim is proved.

**Claim 2.** $e_{\delta_{-\varepsilon_1}} v_\mu = 0$.

For any $S = \{i_1 < \cdots < i_p\} \subset I$ with $1 < i_1$, let $S_1 = S \cup \{1\}$. Then as in the proof of (3.21), we have $c_{S_i} = - (\lambda_1 - \lambda_{\ell + 2})c_{S_1}$, and

$$
[e_{\delta_{-\varepsilon_1}}, c_S f_S + c_{S_1} f_{S_1}] = (- (\lambda_1 - \lambda_{\ell + 2}) + h_{\delta_{-\varepsilon_1}}) \hat{f}_{S_i} v_\lambda = 0,
$$

where $\hat{f}_{S_i}$ is defined as $f_{S_i}$ in (3.16) but with the factor $f_{\delta_{-\varepsilon_1}}$ removed, and the last equality follows from the fact that $f_{S_i} v_\lambda$ is an eigenvector of $h_{\delta_{-\varepsilon_1}}$ with eigenvalue

$$
\lambda_0 - \lambda_1 - 1 = \Lambda_0 + m - s - \lambda_1 - 1 = -\Lambda_\ell + m - s - \lambda_1 - 1 = \lambda_1 - \lambda_\ell + \ell - 2.
$$

This proves the claim. Thus the lemma is proved in this case.

**Case 2:** $t < \ell$.

We inductively construct $v_{\mu(i)}$ for $i = 1, \ldots, \ell - t + 1$ as follows:

$$
v_{\mu(i)} = \sum_{S(i) \subset I(i)} c_{S(i)} f_{S(i)} v_{\mu(i-1)},
$$

(3.23)

where

$$
I(i) = \begin{cases} \{1, \ldots, \ell - 1 - i\} & \text{if } i \leq \ell - t, \\
\{1, \ldots, t - 1\} & \text{if } i = \ell + 1 - t,
\end{cases}
$$

and $f_{S(i)} c_{S(i)}$ are defined as in (3.16), (3.18) with $\ell$ replaced by $\ell + 1 - i$ and $\lambda$ replaced by $\mu(i-1)$.

First note from the definition of $c_{S(i)}$ in (3.18) that when $i \leq \ell - t$, we have

$$
\mu^{(i-1)}_q - \mu^{(i-1)}_{\ell+1-i} + (\ell + 1 - i) - q - 1 \geq \ell - i - q > 0
$$

since $q \in I(i)$, and when $i = \ell - t + 1$, we have

$$
\mu^{(i-1)}_q - \mu^{(i-1)}_t + t - q - 1 \geq \mu^{(i-1)}_q - \mu^{(i-1)}_t = \lambda_q - \lambda_\ell > 0.
$$

Thus $c_{S(i)} \neq 0$ for all $S(i) \subset I(i)$. Also note that when we write $v_{\mu(i)}$ in terms of (3.6), it produces only one possible leading term, and the coefficient of the leading term is $\prod_{i=1}^{\ell - t + 1} c_{S(i)}^{(i)} \neq 0$. In particular $v_{\mu(i)} \neq 0$.

The same arguments in Case 1 show that $e_{\alpha} v_{\mu(i)} = 0$ for all simple roots $\alpha \in \Pi$ except $\alpha = \varepsilon_{\ell-1} - \varepsilon_{\ell}$. Induction on $i$ shows

$$
e_{\alpha} v_{\mu(i)} = 0 \text{ for all } \alpha \in \Pi \setminus I(i),
$$

(3.24)

where

$$
I(i) = \begin{cases} \{\varepsilon_{j} - \varepsilon_{j+1} \mid \ell - i \leq j \leq \ell - 1\} & \text{if } i \leq \ell - t, \\
\{\varepsilon_{j} - \varepsilon_{j+1} \mid t \leq j \leq \ell - 1\} & \text{if } i = \ell + 1 - t.
\end{cases}
$$

**Claim 3.** $v_{\mu} = v_{\mu(\ell+1-t)}$ is a primitive weight vector.
Otherwise there exists some element $x \in U(\mathfrak{g}^+) = U(\mathfrak{g}_0^+)U(\mathfrak{g}_{+1})$ of the weight, say, $\alpha$ (here $U(\mathfrak{g}_{+1})$ denotes the skew-symmetry tensor space of $\mathfrak{g}_{+1}$), such that $v = xv_\mu \neq 0$ is primitive. Since there is no other primitive weight between $\lambda$ and $\mu = \lambda$ by Lemma 3.12 it follows that $v$ must have the weight $\lambda$, i.e., $\lambda = \alpha + \mu$. Observe from the construction of $v_\mu$ that for all $\beta \in \Delta^+$ with $\beta > \delta - \varepsilon_\ell$, we have $e_\beta v_\mu = 0$. Since the relative level of $\lambda$ and $\mu$ is $|\lambda - \mu| = \ell + 1 - t$, there must be $\ell + 1 - t$ positive odd roots in order to produce $\alpha = \lambda - \mu = \sum_{i=1}^{t} (\delta - \varepsilon_i)$, thus for any term in $x$, there must be a root vector $e_{\beta - \varepsilon_i}$ occurring as a factor for some $j \leq t$. But $e_{\delta - \varepsilon_j} v_\mu = 0$ by (3.24). Thus $v = xv_\mu = 0$, a contradiction. This proves the first statement of Lemma 3.13(1).

Thus the primitive weight graph of $V_{\lambda(i)}$ for $i \leq 0$ is given by $\lambda(i) \to \lambda(i-1)$. Hence, $\operatorname{ch} V_{\lambda(i)} = \operatorname{ch} L_{\lambda(i)} + \operatorname{ch} L_{\lambda(i-1)}$ and it follows that

$$\operatorname{ch} L_{\lambda(i)} = \sum_{j=0}^{\infty} (-1)^j \operatorname{ch} V_{\lambda(i-j)} = \lambda_{BL}^{\operatorname{ch}_{0,\lambda(i)}},$$

where the last equality can be obtained from (3.11) and (3.14), or from the same arguments in [30]. To prove $\operatorname{ch} L_{\lambda(0)} = \frac{1}{2} \chi_{\lambda(0)}^{\operatorname{BL}}$, suppose the atypical root of $\lambda(0)$ is $\gamma = \delta - \varepsilon_\ell$. Note from (2.4) that $(\lambda(0)) = \lambda(\ell) = \lambda(0) + 2a\delta$, where $a$ is defined in (3.5), so $\sigma(\lambda(0) + \rho_0) = \sigma(\lambda(0) + \rho + \rho_1) = \lambda(1) + \rho_0 - 2\rho_1$, and

$$\sigma \left( \prod_{\beta \in \Delta_1 \setminus \{\delta - \varepsilon_\ell\}} (1 + e^{-\beta}) \right) = e^{2\rho_1 - \delta - \varepsilon_\ell} \prod_{\beta \in \Delta_1 \setminus \{\delta + \varepsilon_\ell\}} (1 + e^{-\beta}) = D_1 \sum_{i=1}^{\infty} (-1)^{i-1} e^{2\rho_1 - i(\delta + \varepsilon_\ell)},$$

(3.25)

where $D_1 = \prod_{\beta \in \Delta_1} (1 + e^{-\beta})$ is $W_0$-invariant. Thus

$$\sigma(\chi_{0,\lambda(0)}^{\operatorname{BL}}) = \sum_{i=1}^{\infty} (-1)^{i-1} \chi_{\lambda(i-1)}^{V} = (-1)^{2a-1} \chi_{0,\lambda(i-1)}^{\operatorname{BL}} = -\chi_{\lambda(0)}^{\operatorname{BL}},$$

(3.26)

where the last equality is proved as follows.

If $s = 1$, i.e., $k = 2m$, then $\theta \in \mathbb{Z}$ and $0 \in S(\lambda)$, so $(-1)^{2a-1} = -1$, and $\lambda(0) = \theta \cdot (\lambda(1) - 2a(\delta + \varepsilon_\ell))$, where the action is the dot action defined by (2.3), and $\theta \in W_0$ with $\operatorname{sign}(\theta) = 1$ is the unique element changing the signs of $\ell$-th and $m$-th coordinates. If $s = \frac{3}{2}$, i.e., $k = 2m + 1$, then $2a - 1 \in 2\mathbb{Z}$ and in this case $\theta$ with $\operatorname{sign}(\theta) = -1$ is the unique element changing the sign of $\ell$-th. In any case, we have (3.26), which together with $\operatorname{ch} L_{\lambda(0)} = \chi_{0,\lambda(0)}^{\operatorname{BL}}$ implies $\operatorname{ch} L_{\lambda(0)} = \frac{1}{2} \chi_{\lambda(0)}^{\operatorname{BL}}$.

(2) We can prove (3.13) similarly as in part (1) of the proof. From this we obtain

$$\operatorname{ch} L_{\lambda(0)} = \chi_{\lambda(0)}^{V} - \chi_{0,\lambda(0)}^{\operatorname{BL}} - \chi_{0,\lambda(-1)}^{\operatorname{BL}} = \chi_{0,\lambda(0)}^{\operatorname{BL}} - \chi_{0,\lambda(-1)}^{\operatorname{BL}} = \chi_{\lambda(0)}^{\operatorname{BL}}.$$

This completes the proof. $\square$

**Lemma 3.14.** Let $\lambda \in P_+^\times$ with $\lambda_m \geq 0$. Assume that $\mu = \lambda \in P_+^\times$ and both $\lambda$ and $\mu$ have the same non-tail atypical root $\gamma = \delta + \varepsilon_1$. Then $\mu \in P(V_\lambda)$ and $\alpha_{\lambda,\mu} = b_{\lambda,\mu} = 1$.

**Proof.** In principle we can prove the lemma by formal arguments, but we prefer to construct the corresponding highest weight vectors explicitly, as they give extra information. Now we have $\lambda_0 = \lambda_1 > \lambda_2 + 1$ and $\mu_0 = \mu_1 = \lambda_0 - 1$. From this and the proof of Lemma 3.13, we immediately obtain $b_{\lambda,\mu} \leq 1$. We assume $k = 2m + 1$ as the case $k = 2m$ is similar. We can decompose $\gamma$ as

$$\gamma = \delta + \varepsilon_1 = \tau_{-m-1} + \tau_{-m} + \cdots + \tau_{-2} + \tau_{-1} + \tau_1 + \tau_2 + \cdots + \tau_m,$$
where
\[
\tau_{-m-1} = \delta - \varepsilon_1, \quad \tau_i = \varepsilon_{m+i+1} - \varepsilon_{m+i+2} \quad (-m \leq i \leq -2), \\
\tau_{-1} = \tau_1 = \varepsilon_m, \quad \tau_i = \varepsilon_{m+i+1} - \varepsilon_{m-i+2} \quad (2 \leq i \leq m).
\]

For any \(i, j\) with \(-m-1 \leq i \leq j \leq m\), we denote
\[
\tau_{ij} = \sum_{i \leq p \leq j, p \neq 0} \tau_p.
\]
Then \(\tau_{ij}\) is not a root if and only if \(j = -i\), in this case we set \(f_{\tau_{ij}} = 0\). Set \(I = \{-m, -m+1, \ldots, -1, 1, 2, \ldots, m\}\). For any subset \(S = \{i_1 < i_2 < \cdots < i_p\}\) of \(I\) satisfying (3.27), we define
\[
f_S = f_{\tau_{-m-1,i_{i_1}-1}} f_{\tau_{i_1,i_{i_2}-1}} \cdots f_{\tau_{i_{p-1},i_p-1}} f_{\tau_{i_p,m}}.
\]
Note that if \(S = \emptyset\), then \(f_S = f_{\tau_{-m-1,m}} = f_{\delta+\varepsilon_1}\). Also, \(f_S = 0\) if \(i_{p' + 1} = -i_{p'} + 1\) for some \(p' < p\) or \(i_p = -m\). So we suppose
\[
i_{p' + 1} \neq -i_{p'} + 1 \quad (1 \leq p' < p), \quad i_p \neq -m.
\]
Now we define \(v_\mu = \sum_{S \subseteq I} c_S f_S v_\lambda\) such that \(c_S \in \mathbb{C}\) is defined by
\[
c_S = \begin{cases} 
1 & \text{if } S = I, \\
(\lambda_1 - \lambda_{m+q} + m - q)c_{S'} & \text{if } S \subseteq I, 1 < q \leq m, \\
(\lambda_1 + 1)c_{S'} & \text{if } S \subseteq I, q = 1, \\
-(\lambda_q - \lambda_1 - q)c_{S'} & \text{if } S \subseteq I, q < 0,
\end{cases}
\]
where \(q\) is the maximal integer in \(I \setminus S\) such that \(S' = S \cup \{q\}\) satisfies condition (3.27). Note that in all cases \(c_S \neq 0\) as long as \(c_{S'} \neq 0\). Now as in Case 1 in the proof of Lemma 3.13 we see \(v_\mu\) is a \(\mathfrak{g}\)-highest weight vector. \(\square\)

### 3.3. Main results on generalised Verma modules

Now we can prove the following result on the structure of generalised Verma modules. Fix an atypicality type \(\overline{\lambda}\), and let \(\lambda^{(i)} \in P^0_\overline{\lambda}\) be the weights defined by Definition 3.7.

**Remark 3.15.** Recall from Lemma 3.9 that \(P^0_\overline{\lambda} = \{\lambda^{(i)} \mid i \in \mathbb{Z}\}\) with \(\lambda^{(i)} = \lambda^{(i)}\) for all \(i\). By Remark 3.6, the generalised Verma modules \(V^{(i)}_{\lambda^{(i)}}\) and \(V^{(1-i)}_{\lambda^{(1-i)}}\) have the same structure when \(\lambda^{(i)}\) is defined.

**Theorem 3.16.** For \(\lambda = \lambda^{(i)} \in P^0_\overline{\lambda}\), the composition factors of the generalised Verma module \(V_\lambda\) all have multiplicity 1. Furthermore, the primitive weight graph of \(V_\lambda\) can be described as follows.

1. If \(k = 2m + 1\) or \(0 \in S(\overline{\lambda})\) (in this case, \((\lambda^{(i)})^{(1)} = \lambda^{(1-i)}\)), then

\[
\begin{align*}
\lambda &= \lambda^{(i)} \downarrow \lambda^{(i-1)}, & \lambda &= \lambda^{(1)} \downarrow \lambda^{(-1)}, & \lambda &= \lambda^{(2)} \downarrow \lambda^{(0)} \downarrow \lambda^{(1)}, & \lambda &= \lambda^{(i)} \downarrow \lambda^{(1-i)} \downarrow \lambda^{(1-i)}.
\end{align*}
\]

(3.28)
(2) If \( k = 2m \) and \( 0 \notin S(\bar{\lambda}) \) (in this case, \( (\lambda^{(i)})^\sigma = \lambda^{(-i)} \)), then

\[
\lambda = \lambda^{(i)} \downarrow \lambda^{(i-1)}, \quad \lambda = \lambda^{(0)} \downarrow \lambda^{(-1)} \downarrow \lambda^{(-i)}, \quad \lambda = \lambda^{(i)} \downarrow \lambda^{(i-1)}, \quad \lambda = \lambda^{(0)} \downarrow \lambda^{(-1)} \downarrow \lambda^{(-i)}.
\]

\[ (3.29) \]

**Proof.** The fact that the multiplicity of each composition factor is 1 has already been proven in Lemma 3.12 for some cases, and the remaining cases are treated presently. As for the rest of the theorem, i.e., statements (1) and (2), we note that their proofs are similar, thus we consider (1) only.

First we need the following: for any integral \( \mathfrak{g}_0 \)-dominant weight \( \lambda, \mu \) with \( \mu \leq \lambda \), using the well-known Weyl character formula of a \( \mathfrak{g}_0 \)-module, by (3.9) and (2.9), we have

\[
b_{\lambda, \mu} = \sum_{p \in \mathbb{Z}_+, \mu \in \Delta^+, |S|=|\lambda-\mu|, -\nu \in \lambda-\sum_{\beta \in S} \beta - 2p \delta \text{ is regular}} \text{sign}(w).
\]

We already have the first graph of (3.28) by Lemma 3.13. To prove the second, we first use (3.30) to prove

\[
b_{\lambda^{(1)}, \lambda^{(0)}} = 0.
\]

(3.31)

For convenience, suppose \( k = 2m + 1 \) and \( \lambda^{(0)} \) has the form (3.5). Then \( a = j + \frac{1}{2} \), \( \bar{\lambda}_{m-i} = i + \frac{1}{2} \) for \( j - 1 \geq i \geq 0 \), and \( \lambda^{(1)} \) has the form (3.5) with \(-a\) replaced by \( a \) in the first place. Suppose \( (3.30) \) with \( \lambda = \lambda^{(1)}, \mu = \lambda^{(0)} \) has a term \( \text{sign}(w_0) \) with \( p_0 \in \mathbb{Z}_+, S_0 \subset \Delta^+, w_0 \in W_0 \) such that

\[
\nu := \lambda^{(1)} - \sum_{\beta \in S_0} \beta - 2p_0 \delta \text{ is regular, and } \nu = w_0 \cdot \lambda^{(0)}.
\]

(3.32)

This implies that \( \bar{\nu}_i - \bar{\lambda}_i = 0, \pm 1 \) for \( 1 \leq i \leq m \). First assume \( \bar{\nu}_i = \bar{\lambda}_i - 1 \) for some \( i \neq m \). Then (3.32) implies that

\[
\bar{\lambda}_{i+1} - \bar{\lambda}_i = 1, \quad \bar{\nu}_{i+1} - \bar{\lambda}_i = 1, \quad \text{and } \delta - \varepsilon_i, \delta - \varepsilon_{i+1} \in S_0 \text{ but } \delta - \varepsilon_i, \delta + \varepsilon_{i+1} \notin S_0.
\]

If we choose the subset \( S_1 = (S \cup \{\delta - \varepsilon_i, \delta + \varepsilon_{i+1}\}) \setminus \{\delta + \varepsilon_i, \delta - \varepsilon_{i+1}\} \) of \( \Delta^+ \), and also \( w_1 = (i, i+1)w_0 \in W_0 \), where \( (i, i+1) \) is the permutation exchanging \( i \)-th and \( (i+1) \)-th coordinates, then (3.30) has another term \( \text{sign}(w_1) = -\text{sign}(w_0) \) which cancels \( \text{sign}(w_0) \). Next assume \( \bar{\nu}_m = \bar{\lambda}_{m-1} - \frac{1}{2} = -\frac{1}{2} \). Then \( \delta + \varepsilon_m \in S_0, \delta - \varepsilon_m \notin S_0 \). If we take \( w_1 = (-m)w \) (where \(-m\) is the element in \( W_0 \) which changes the sign of the \( m \)-th coordinate), and

\[
S_1 = (S_0 \cup \{\delta\}) \setminus \{\delta + \varepsilon_m\} \quad \text{if } \delta \notin S_0,
\]

or

\[
S_1 = S_0 \setminus \{\delta, \delta + \varepsilon_m\} \text{ and } p_1 = p_0 + 1 \text{ if } \delta \in S_0,
\]

then again (3.30) has another term \( \text{sign}(w_1) = -\text{sign}(w_0) \) which cancels \( \text{sign}(w_0) \). Thus we obtain (3.31), and \( \lambda^{(0)} \notin \mathcal{P}(V_{\lambda^{(1)}}) \subset \{\lambda^{(1)}, \lambda^{(0)} / \lambda^{(-1)}\} \) by Lemma 3.12(1). Since \( L_{\lambda^{(1)}} \) is finite dimensional, we must have \( \lambda^{(-1)} \in \mathcal{P}(V_{\lambda^{(1)}}) \). Using Lemma 3.13(1) and as in the proof of Lemma 3.13, we can obtain

\[
\text{ch} L_{\lambda^{(1)}} = \lambda^{BL}_{\lambda^{(1)}} - \text{ch} L_{\lambda^{(0)}} - (a_{\lambda^{(1)}, \lambda^{(-1)}} - 1)\text{ch} L_{\lambda^{(-1)}}.
\]

(3.33)
Since all terms except the last term in the right-hand side of (3.33) are $W$-invariant, we obtain $a_{\lambda^{(1)},\lambda^{(-1)}} = 1$, and we have the second graph of (3.28).

Now we prove $\lambda^{(i)} \to \lambda^{(i-1)}$ for $i \geq 2$. This is true when $i \gg 0$ by Lemma 3.14 and Definition 3.7(1). Suppose $\lambda^{(i_0-1)} \notin P(V_{\lambda^{(i_0)}})$ for some $i_0 \geq 2$ and $i_0$ is maximal. This together with $b_{\lambda^{(i_0)},\lambda^{(i_0-1)}} = 1$ (which can be proved as in (3.31)) implies $\lambda^{(i_0-1)} \in P_0(L_{\lambda^{(i_0)}})$ (cf. (3.3)). By the choice of $i_0$, we have $\lambda^{(i_0)} \in P(V_{\lambda^{(i_0+1)}})$, thus $P_0(L_{\lambda^{(i_0)}}) \subset P_0(V_{\lambda^{(i_0+1)}})$, and so $\lambda^{(i_0-1)} \in P_0(V_{\lambda^{(i_0+1)}})$, which contradicts Lemma 3.12(1).

As in the proof of (3.31), we have $b_{\lambda^{(2)},\lambda^{(0)}} = 1$. This together with the arguments in the last paragraph shows $\lambda^{(2)} \to \lambda^{(0)}$. Since $L_{\lambda^{(0)}}, L_{\lambda^{(1)}}$ are finite-dimensional and generalised Verma modules do not contain finite-dimensional submodules, we must have $\lambda^{(1)} \to \mu, \lambda^{(0)} \to \nu$ in $V_{\lambda^{(2)}}$ for some non-dominant weights $\mu, \nu$. But $\lambda^{(1)} \to \mu, \lambda^{(0)} \to \nu$ must be quotients of $V_{\lambda^{(1)}}, V_{\lambda^{(0)}}$ respectively, by the first two graph of (3.28), $\mu, \nu$ have to be $\lambda^{(-1)}$. Thus we have the third graph of (3.28) except the relation concerning $\lambda^{(-2)}$. If we do not have $\lambda^{(2)} \to \lambda^{(-2)}$ in $V_{\lambda^{(2)}}$, then in $V_{\lambda^{(3)}}$, the submodule $U_{\lambda^{(2)}}$ generated by $v_{\lambda^{(2)}}$ (the primitive vector with weight $\lambda^{(2)}$), which is a quotient of $V_{\lambda^{(2)}}$, has to be $L_{\lambda^{(2)}}$, contradicting that $V_{\lambda^{(3)}}$ does not contain a finite-dimensional submodule. Thus we have $\lambda^{(2)} \to \lambda^{(-2)}$ and do not have $\lambda^{(-1)} \to \lambda^{(-2)}$ in $V_{\lambda^{(2)}}$. As in (3.33), we have

$$\text{ch} L_{\lambda^{(2)}} = \chi_{\lambda^{(2)}}^{BL} - (a_{\lambda^{(2)},\lambda^{(-1)}} - 1)\text{ch} L_{\lambda^{(-1)}} - (a_{\lambda^{(2)},\lambda^{(-2)}} - 1)\text{ch} L_{\lambda^{(-2)}},$$

which proves that $a_{\lambda^{(2)},\lambda^{(-1)}} = a_{\lambda^{(2)},\lambda^{(-2)}} = 1$.

The fact that $\lambda^{(-2)} \to \lambda^{(-1)}$ is not needed for our purpose of computing characters, nevertheless, we can prove this fact for the case $g = \mathfrak{osp}_{3|2}$ as follows: Let $\lambda = \lambda^{(2)}, \mu = \lambda - 2(\lambda_0 + 1)\delta$ (note that $\mu \notin P_{\lambda}^{0+}$, i.e., $\lambda$ is not the atypical type of $\mu$), then as in the proof of (3.31), $b_{\lambda,\mu} = 1$ and the unique $\mathfrak{g}_0$-highest weight vector with weight $\mu$ (up to a nonzero scalar) is $v_{\mu} := f_{\lambda_0+1}v_{\lambda}$ (cf. definition of Kac-modules in (2.11)). We can also prove $b_{\lambda^{(-2)},\mu} = 1$ as in (3.31), and $b_{\lambda^{(-3)},\mu} = 0$ (in fact $\lambda^{(-3)} < \mu$). Thus $\mu$ is a $\mathfrak{g}_0$-highest weight of $L_{\lambda^{(-2)}}$, and $v_{\mu}$ is in the submodule $U_{\lambda^{(-2)}}$ of $V_{\lambda^{(2)}}$ generated by $v_{\lambda^{(-2)}}$ (primitive vector with weight $\lambda^{(-2)}$). Since the Kac-module $K_{\lambda^{(2)}}$ is finite dimensional and $L_{\lambda^{(-1)}}$ is infinite-dimensional, $v_{\mu}$ must generate $v_{\lambda^{(-1)}}$ (the $\mathfrak{g}$-primitive vector with weight $\lambda^{(-1)}$ in $V_{\lambda^{(2)}}$). This proves the third graph of (3.28).

Now suppose $i \geq 3$ and inductively assume that we have the graph in (3.28) for all $\lambda^{(i_0)}$ with $i_0 < i$. From this, we can deduce as in (3.33) that

$$\text{ch} L_{\lambda^{(i_0)}} = \chi_{\lambda^{(i_0)}}^{BL}, \quad 2 \leq i_0 < i.$$  

From Lemma 3.12(1), we have $P(V_{\lambda^{(i)}}) \subset \{ \lambda^{(i)}, \lambda^{(i-1)}, \lambda^{(2-i)}, \lambda^{(1-i)}, \lambda^{(-i)} \}$. As above, we can then deduce

$$\text{ch} L_{\lambda^{(i)}} = \chi_{\lambda^{(i)}}^{BL} - (a_{\lambda^{(i)},\lambda^{(1-i)}} - 1)\text{ch} L_{\lambda^{(1-i)}} - (a_{\lambda^{(i)},\lambda^{(-i)}} - 1)\text{ch} L_{\lambda^{(-i)}} - a_{\lambda^{(i)},\lambda^{(2-i)}}\text{ch} L_{\lambda^{(2-i)}},$$

which proves that $a_{\lambda^{(i)},\lambda^{(1-i)}} = a_{\lambda^{(i)},\lambda^{(-i)}} = 1$, $a_{\lambda^{(i)},\lambda^{(2-i)}} = 0$ as in (3.33), and we have the last graph of (3.28). \hfill \Box

3.4. Character and dimension formulae for irreducible modules. Now we derive a character formula and a dimension formula for the atypical finite-dimensional irreducible modules following similar reasoning as that in the proof of 27 Theorem
Proposition 3.18. (but the case at hand is much easier). For atypical $\lambda \in P^+$, we define

$$S_\lambda = \{\lambda, \lambda^\sigma\} \cap \{\nu \in P^+ \mid \nu \leq \lambda\}, \quad m_\lambda = \#\left(\{\lambda, \lambda^\sigma\} \cap \{\nu \in P^+ \mid \nu \geq \lambda\}\right),$$

where $\lambda^\sigma$ is defined by $[24]$. Then it is easy to see that

$$S_\lambda = \begin{cases} \{\lambda, \lambda^\sigma\}, & \text{if } \lambda > \lambda^\sigma \in P^+, \\ \{\lambda\}, & \text{otherwise}, \end{cases} \quad m_\lambda = \begin{cases} 2, & \text{if } \mu < \mu^\sigma, \\ 1, & \text{otherwise}. \end{cases}$$

For, $\mu \in S_\lambda$, we denote $\theta_{\lambda, \mu} \in W$ to be the unique element with minimal length such that $\theta \cdot \lambda = \mu$, namely, $\theta_{\lambda, \mu} = 1$ if $\lambda = \mu$ or $\theta_{\lambda, \mu} = \sigma$ otherwise.

By using Theorem 3.16 (3.33) and (3.34), we immediately obtain the following result.

**Theorem 3.17.** Let $L_\lambda$ be the finite-dimensional irreducible $\mathfrak{g}$-module with atypical highest weight $\lambda$.

1. The formal character $\text{ch} L_\lambda$ of $L_\lambda$ is given by

$$\text{ch} L_\lambda = \sum_{\mu \in S_\lambda} \frac{(-1)^{\theta_{\lambda, \mu}}}{m_\mu R_0} \sum_{w \in W} \text{sign}(w) w \left( e^{\mu + \rho_0} \prod_{\beta \in \Delta_1^+ \setminus \{\gamma_\mu\}} (1 + e^{-\beta}) \right),$$

where $\gamma_\mu$ is the atypical root of $\mu \in S_\lambda$, and $|\theta_{\lambda, \mu}|$ is the length of $\theta_{\lambda, \mu}$.

2. The dimension $\dim L_\lambda$ of $L_\lambda$ is given by

$$\dim L_\lambda = \sum_{\mu \in S_\lambda, B \subset \Delta_1^+ \setminus \{\gamma_\mu\}} (-1)^{\theta_{\lambda, \mu}} m_\mu^{-1} \prod_{\alpha \in \Delta_0^+} \frac{\langle \alpha, \rho_0 + \mu - \sum_{\beta \in B} \beta \rangle}{\langle \alpha, \rho_0 \rangle}.$$ (3.35)

The following result on finite dimensional Kac modules can be easily proven.

**Proposition 3.18.** The formal character $\text{ch} K_\lambda$ of the finite-dimensional (typical or atypical) Kac $\mathfrak{g}$-module $K_\lambda$ is $\text{ch} K_\lambda = \chi_\lambda^V$ with the right-hand side given by (2.15) unless $\lambda^\sigma \in P^+$. If $\lambda^\sigma \in P^+$, then $\text{ch} K_\lambda = \text{ch} L_\lambda$ with the right-hand side given by (3.34). In particular,

$$\chi_\lambda^V = \text{ch} L_\lambda^{(i)} + (-1)^{\delta_{i,1}} \text{ch} L_\lambda^{(i-1)} \text{ for } i \geq 1.$$ (3.36)

**Proof.** Note that the graph of $K_\lambda$ is obtained from that of $V_\lambda$ by deleting all weight $\lambda^{(i)}$ with $i < 0$. This together with Theorems 3.16 and 3.17(1) implies the result. $\square$

An application of the character formula (3.34) will be found in Lemma 4.3

4. First and second cohomology groups

In this section we apply results obtained on generalised Verma modules and irreducible modules to determine the first and second cohomology groups of $\mathfrak{osp}_{k|2}$ with coefficients in finite-dimensional irreducible modules.

4.1. Lie superalgebra cohomology. Let us begin by recalling some basic concepts of Lie superalgebra cohomology. The material can be found in many sources, say, [10] [21] [26]. For $p \geq 1$ and a finite-dimensional $\mathfrak{g}$-module $V$, let the space $C^p(\mathfrak{g}, V)$ of $p$-cochains be the $\mathbb{Z}_2$-graded vector space of all $p$-linear maps $\varphi : \mathfrak{g} \times \cdots \times \mathfrak{g} \to V$ satisfying the super skew symmetry condition

$$\varphi(x_1, \ldots, x_i, x_{i+1}, \ldots, x_p) = -(-1)^{|x_i||x_{i+1}|} \varphi(x_1, \ldots, x_{i+1}, x_i, \ldots, x_p)$$

where $|\cdot|$ is the graded degree.
for $1 \leq i \leq p - 1$, where, $[x_i] \in \mathbb{Z}_2$ denotes the parity of the element $x_i$. Set $C^0(\mathfrak{g}, V) = V$. We define the differential operator $d : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$ by

$$(d\varphi)(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^{i+[x_i]+[x_0]+\cdots+[x_{i-1}]} x_i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_p)$$

$$+ \sum_{i<j} (-1)^{i+[x_j]} \varphi(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_p),$$

for $\varphi \in C^p(\mathfrak{g}, V)$ and $x_0, \ldots, x_p \in \mathfrak{g}$, where the sign $\hat{\cdot}$ means that the element under it is omitted. It can be verified that $d^2 = 0$. Set

$$Z^p(\mathfrak{g}, V) = \text{Ker}(d|_{C^p(\mathfrak{g}, V)}),$$

$$B^p(\mathfrak{g}, V) = \text{Im}(d|_{C^{p-1}(\mathfrak{g}, V)}),$$

$$H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)/B^p(\mathfrak{g}, V).$$

The space $H^p(\mathfrak{g}, V)$ is the $p$-th Lie superalgebra cohomology group of $\mathfrak{g}$ with coefficients in the module $V$.

We briefly discuss the long exact sequence of cohomology groups, which is one of the essential tools used in this section. Let $U, V, W$ be three $\mathfrak{g}$-modules such that

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is a short exact sequence, where $f, g$ are homogenous $\mathfrak{g}$-module homomorphisms. Then there exists a long exact sequence

$$\cdots \to H^p(\mathfrak{g}, U) \xrightarrow{f^p} H^p(\mathfrak{g}, V) \xrightarrow{g^p} H^p(\mathfrak{g}, W) \xrightarrow{d^*} H^{p+1}(\mathfrak{g}, U) \to \cdots,$$  \hspace{1cm} (4.1)

where the maps $f^p, g^p$ can easily be defined from $f, g$, and $d^*$ is the connecting homomorphism (cf. [26 (2.50)]).

4.2. **Computation of cohomology groups.** If the cohomology group $H^*(\mathfrak{g}, L_\lambda) \neq 0$ for integral $\mathfrak{g}$-dominant $\lambda$, then $\lambda$ and 0 must have the same typical, hence $\lambda = \Lambda^{(i)}$ for some $j \geq 0$, where $\Lambda^{(0)} = (0, 0, \ldots, 0)$ and

$$\Lambda^{(i)} = (2m + i - 1 - 2s \mid i - 1, 0, \ldots, 0) \quad \text{for } i \geq 1. \hspace{1cm} (4.2)$$

For any $\mathfrak{g}$-module $V$, using the proof of [26 Lemma 3.1], we have (cf. Notation 3.1)

$$H^1(\mathfrak{g}, V) \neq 0 \iff \exists M(V \leftarrow \Lambda^{(0)}),$$

where $M(V \leftarrow \mu)$ means an indecomposable module $M$ which contains the submodule $V$ such that the quotient $M/V$ is isomorphic to $L_\mu$. Also,

$$H^2(\mathfrak{g}, V) \neq 0 \implies \exists M(V \leftarrow \mu) \quad \text{for some } \mu \text{ with } H^1(\mathfrak{g}, L_\mu) \neq 0. \hspace{1cm} (4.4)$$

From this and Theorem 3.16 and [26 Lemma 6.7], we immediately obtain
Theorem 4.1. Let $g = \mathfrak{osp}_{k|2}$ with $k > 2$. Let $L_{\lambda}$ and $K_{\lambda}$ respectively denote the finite-dimensional irreducible and Kac $g$-modules with highest weight $\lambda$. Then

$$H^1(g, L_{\lambda}) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^{(2)}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

$$H^1(g, K_{\lambda}) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^{(3)}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

$$H^2(g, L_{\lambda}) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^{(1)}, \Lambda^{(3)}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

$$H^2(g, K_{\lambda}) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^{(1)}, \Lambda^{(4)}. \\ 0 & \text{otherwise.} \end{cases} \quad (4.8)$$

Remark 4.2. The first and second cohomology groups of $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$ for all $m$ and $n$ were computed in [27], and those of $\mathfrak{osp}_{3|2}$ were determined in [11].

Let us first consider the following lemma, the proof of which may be considered as an application of the character formula (3.34). The lemma will be used in the proof of Theorem 4.1.

Lemma 4.3.\hspace{1cm} (1) A module $M(\Lambda^{(0)} \to \Lambda^{(2)} \to \Lambda^{(1)})$ does not exist.

(2) $H^2(g, \mathbb{C}) = 0$.

Proof. Let $\mathbb{C}v_0$ be the trivial $g_0$-module. We define the induced module

$$\tilde{M} := \text{Ind}_{g_0}^g \mathbb{C}v_0 = U(g) \otimes_{U(g_0)} \mathbb{C}v_0 \cong U(g_1) \otimes_{\mathbb{C}} \mathbb{C}v_0,$$

here $U(g_1)$ denotes the skew-symmetry tensor space of $g_1$. Obviously,

$$\text{ch} \tilde{M} = R_1^2 = \frac{R_1}{R_0} \sum_{w \in W} \text{sign}(w)w(e^{\rho_0 + \rho_1} \prod_{\beta \in \Delta_i^+} (1 + e^{-\beta})) = \sum_{B \in \Delta_i^+} \chi_{\lambda_B}^V,$$

where $\lambda_B = 2\rho_1 - \sum_{\beta \in B} \beta$. We only need to consider those $B$’s such that $\lambda_B$ is regular. When $\lambda_B$ is regular, we take the unique $w_B \in W$ such that $w_B(\lambda_B + \rho) := \mu_B + \rho$ is $g$-dominant. Thus $\text{ch} \tilde{M} = \sum_{B \in \Delta_i^+} \text{sign}(w_B)\chi_{\mu_B}^V$. We have (to see how it works, one can take $k = 3, 4, 5, 6$ as examples)

$$\text{ch} \tilde{M} = \chi_{\Lambda^{(2)}}^V - \chi_{\Lambda^{(1)}}^V + \ldots,$$

where the omitted terms are (signed) sum of some $\chi_{\Lambda^i}^V$’s with $\lambda \neq \Lambda^{(i)}$ for $i \in \mathbb{Z}_+$. Thus $\tilde{M}$ is decomposed into a direct sum of two submodules

$$\tilde{M} = \tilde{M}_0 \oplus \tilde{M}_1, \quad (4.9)$$

such that $\tilde{M}_0$ has character $\text{ch} \tilde{M}_0 = \chi_{\Lambda^{(2)}}^V - \chi_{\Lambda^{(1)}}^V = \text{ch} L_{\Lambda^{(2)}} + 2 \text{ch} L_{\Lambda^{(0)}}$ (cf. (3.36)), and all $L_{\Lambda^{(i)}}$’s are not composition factors of $\tilde{M}_1$. In particular, $\tilde{M}$ does not have composition factor $L_{\Lambda^{(i)}}$, this proves (1) (since a module $M(\Lambda^{(0)} \to \Lambda^{(2)} \to \Lambda^{(1)})$ must be a quotient of $\tilde{M}$). From (1), we obtain (cf. Remark 4.4)

$$M(\Lambda^{(1)} \to \Lambda^{(2)} \to \Lambda^{(0)}) \text{ does not exist.} \quad (4.10)$$
Furthermore, \( \widetilde{M}_0 \) as a module generated by \( v'_0 \) (where \( v'_0 \) is the projection of \( v_0 \) onto \( \widetilde{M}_0 \) with respect to decomposition (4.9)) must be indecomposable, thus it has to be the module

\[
M(\Lambda^{(0)} \to \Lambda^{(2)} \to \Lambda^{(0)}).
\]

(4.11)

Now, from the short exact sequence \( 0 \to L_{\Lambda^{(0)}} \to N \to L_{\Lambda^{(2)}} \to 0 \), where \( N = M(\Lambda^{(2)} \to \Lambda^{(0)}) \), we obtain the exact sequence (cf. (4.11))

\[
0 = H^1(\mathfrak{g}, L_{\Lambda^{(0)}}) \to H^1(\mathfrak{g}, N) \xrightarrow{\phi} H^1(\mathfrak{g}, L_{\Lambda^{(2)}}) \to H^2(\mathfrak{g}, \mathbb{C}) \to H^2(\mathfrak{g}, N).
\]

(4.12)

Note from (4.3) and (4.10) that \( H^2(\mathfrak{g}, N) = 0 \), also (4.11) and (4.3) show that \( H^1(\mathfrak{g}, N) \neq 0 \), thus \( \phi \) is a bijection. So (4.12) with (4.3) proves \( H^2(\mathfrak{g}, \mathbb{C}) = 0 \). □

The following remark will be used in the proof of Theorem 4.1.

**Remark 4.4.** Let \( P(V) \) be a primitive weight graph. The dual primitive weight graph \( P^*(V) \) is the graph obtained from \( P(V) \) by reversing the directions of all arrows and changing all weights to their dual weights. Note that \( P^*(V) = P(V^*) \), where \( V^* \) denote the dual module of \( V \). If we change the action of \( \mathfrak{g} \) on \( P(V^*) \) by the automorphism \( \omega \in \text{Aut}(\mathfrak{g}) \) which interchanges \( \mathbb{C}e_\alpha \)’s and \( \mathbb{C}f_\alpha \)’s, then we obtain another module, called the inverse module of \( V \), with graph \( \tilde{P}(V) \) obtained from \( P(V) \) by reversing the directions of all arrows (note that using the automorphism \( \omega \), the module \( L^*_\mu \) becomes \( L_{\mu} \) for all \( \mu \)). In particular, we have

\[
\exists M(\mu \to \nu) \iff \exists M(\mu \leftarrow \nu) \iff \exists M(\mu^* \to \nu^*) \iff \exists M(\mu^* \leftarrow \nu^*).
\]

(4.13)

**Proof of Theorem 4.1.** Note that a module \( M(\Lambda^{(i)} \to \Lambda^{(0)}) \) with \( i \geq 0 \) must be a highest weight module thus a quotient of \( V_{\Lambda^{(i)}} \) and so \( i = 2 \) by (3.28). Thus there is a module \( M(\Lambda^{(i)} \leftarrow \Lambda^{(0)}) \) if and only if \( i = 2 \) (cf. (4.13)). By (4.3), we have (4.1) since a module with structure \( \Lambda^{(2)} \to \Lambda^{(0)} \) is unique.

Using (4.3) and (4.5), we obtain that \( H^1(\mathfrak{g}, K_\Lambda) \neq 0 \) only if \( K_\Lambda \) contains a composition factor \( L_{\Lambda^{(2)}} \), i.e., \( \lambda = \Lambda^{(2)}, \Lambda^{(3)} \). One can prove as in the previous paragraph that \( H^1(\mathfrak{g}, K_{\Lambda^{(2)}}) \cong \mathbb{C} \). Also, Lemma 4.3 shows that \( H^1(\mathfrak{g}, K_{\Lambda^{(3)}}) = 0 \). This proves (4.6).

Analogously, using (4.4), we obtain that \( H^2(\mathfrak{g}, L_\Lambda) \neq 0 \) implies \( \lambda = \Lambda^{(0)}, \Lambda^{(1)}, \Lambda^{(3)} \). Lemma 4.3(2) shows \( H^2(\mathfrak{g}, L_{\Lambda^{(0)}}) = 0 \). For \( \lambda = \Lambda^{(1)}, \) we have the following short exact sequence (where \( M = M(\Lambda^{(2)} \to \Lambda^{(1)}) \), which exists by (3.28))

\[
0 \to L_{\Lambda^{(1)}} \to M \to L_{\Lambda^{(2)}} \to 0.
\]

(4.14)

By (4.1), we have the exact sequence

\[
0 = H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, L_{\Lambda^{(2)}}) \to H^2(\mathfrak{g}, L_{\Lambda^{(1)}}) \to H^2(\mathfrak{g}, M) = 0,
\]

(4.15)

where the first equality follows from (4.3) and that both \( M(\Lambda^{(0)} \to \Lambda^{(2)} \to \Lambda^{(1)}) \) and \( M(\Lambda^{(2)} \to \Lambda^{(0)} \to \Lambda^{(0)}) \) do not exist. The last equality of (4.15) follows from (4.4), (4.5) and that both \( M(\Lambda^{(2)} \to \Lambda^{(2)} \to \Lambda^{(1)}) \) and \( M(\Lambda^{(2)} \to \Lambda^{(1)} \to \Lambda^{(2)}) \) do not exist (one can prove that a module with structure \( \Lambda^{(2)} \to \Lambda^{(1)} \to \Lambda^{(2)} \) must be decomposable). Thus \( H^2(\mathfrak{g}, L_{\Lambda^{(1)}}) \cong H^1(\mathfrak{g}, L_{\Lambda^{(3)}}) \cong \mathbb{C} \) by (4.15). Similarly, we have (4.7) for \( \Lambda^{(3)} \).

Using (4.4), we obtain that \( H^2(\mathfrak{g}, K_\Lambda) \neq 0 \) only if \( \lambda = \Lambda^{(1)}, \Lambda^{(4)} \). Since \( K_{\Lambda^{(1)}} = L_{\Lambda^{(1)}}, \) we have (4.8) for \( \Lambda^{(1)} \). To prove (4.8) for \( \Lambda^{(4)} \), note that there exists an indecomposable module \( M = M(\Lambda^{(4)} \to \Lambda^{(3)} \to \Lambda^{(2)}) \) which can be obtained from the direct sum of two modules \( K_{\Lambda^{(4)}} \) and \( M(\Lambda^{(3)} \to \Lambda^{(2)}) \) by factoring the submodule \( U(\mathfrak{g})(v_{\Lambda^{(3)}} + v'_{\Lambda^{(3)}}) \), where \( v_{\Lambda^{(3)}}, v'_{\Lambda^{(3)}} \) are respectively primitive vectors with weight \( \Lambda^{(3)} \) in the two said
modules. Then for this new defined $M$, we have (4.14) with $L_{\Lambda}^{(1)}$ replaced by $K_{\Lambda}^{(4)}$, and the above arguments now show that we have (4.8) for $\Lambda^{(4)}$. □

Acknowledgements. We thank Shun-Jen Cheng and Ngau Lam for informing us about reference [6]. This work is supported by the Australian Research Council and the National Science Foundation of China (grant no. 10825101).

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