A SPARSE APPROACH TO MIXED WEAK TYPE INEQUALITIES

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Abstract. In this paper we provide some quantitative mixed-type estimates assuming conditions that imply that \( w \in A_\infty \) for Calderón-Zygmund operators, rough singular integrals and commutators. The main novelty of this paper lies in the fact that we rely upon sparse domination results, pushing an approach to endpoint estimates that was introduced in [8] and extended in [23] and [25].

1. Introduction and Main Results

In [27], Muckenhoupt and Wheeden introduced a new type of weak type inequality, that we call mixed type inequality, that consists in considering a perturbation of the Hardy-Littlewood maximal operator with an \( A_p \) weight. Their result was the following

**Theorem A.** Let \( w \in A_1 \) then

\[
\{ x \in \mathbb{R} : w(x) Mf(x) > t \} \leq \frac{1}{t} \int_{\mathbb{R}} |f(x)| w(x) dx.
\]

Although this kind of estimate may seem not very different to the standard one, the perturbation caused by having the weight inside the level set makes it way harder to be settled, in contrast with analogous case of strong type estimates. Furthermore, \( w \in A_1 \) is no longer a necessary condition for this endpoint estimate to hold (see [27, Section 5]).

Later on, Sawyer [33], motivated by the possibility of providing a new proof for the Muckenhoupt’s theorem, obtained the following result.

**Theorem B.** Let \( u, v \in A_1 \) then

\[
uv \left( \left\{ x \in \mathbb{R} : \frac{M(v)(x)}{v(x)} > t \right\} \right) \leq \frac{1}{t} \int_{\mathbb{R}} |f(x)| u(x)v(x) dx.
\]

Sawyer also conjectured that (1.1) should hold as well for the Hilbert transform. Cruz-Uribe, Martell and Pérez [7] generalized (1.1) to higher dimensions and actually proved that Sawyer’s conjecture holds for Calderón-Zygmund operators via the following extrapolation argument.

**Theorem C.** Assume that for every \( w \in A_\infty \) and some \( 0 < p < \infty \),

\[
\| Tf \|_{L^p(w)} \leq c_w \| Gf \|_{L^p(w)}.
\]

Then for every \( u \in A_1 \) and every \( v \in A_\infty \)

\[
\| Tf \|_{L^1,\infty (uv)} \leq \| Gf \|_{L^1,\infty (uv)}.
\]

The conditions on the weights in that extrapolation result lead them to conjecture that (1.1), and consequently the corresponding estimate for Calderón-Zygmund operators should hold as well with \( u \in A_1 \) and \( v \in A_\infty \). That conjecture was positively answered recently in [24] where several quantitative estimates were provided as well. At this point we would like to mention, as well, a recent generalization provided for Orlicz maximal operators in [2].

In [7], besides the aforementioned results, it was shown that (1.1) holds if \( u \in A_1 \) and \( v \in A_\infty (u) \) (see Section 2.2 for the precise definition of \( A_p (u) \)). The advantage of that condition is that the product \( uv \) is an \( A_\infty \) weight. Over the past few years, there have been new contributions under those assumptions such as [3] for the case of fractional integrals and related operators, [28, 29] for related quantitative estimates and [26] for multilinear extensions.

The second author is supported by CONICET PIP 11220130100329CO.
The case of commutators of Calderón-Zygmund operators was settled in [4]. Recall that given \( T \) a Calderón-Zygmund operator, \( b \in \text{Osc}_{\exp}L^{r} \subset \text{BMO} \) (see Section 2.2 for the precise definition) and a positive integer \( m \), we define the higher order commutator \( T_{b}^{m}f \) by

\[
T_{b}^{m}f(x) = b(x)T_{b}^{m-1}f(x) - T_{b}^{m-1}(bf)(x)
\]

where \( T_{b}^{1}f(x) = b(x)Tf(x) - T(bf)(x) \).

Now we turn to our contribution. Our approach exploits sparse domination and ideas from [25] that can be traced back to [8]. In the case of commutators our approach is inspired by [23] as well. The main novelty of our proofs is precisely that, in contrast with the techniques used up until now to deal with this kind of questions, we heavily rely upon sparse domination. Our first result is the following.

**Theorem 1.1.** Let \( u \in A_{1} \) and \( v \in A_{p}(u) \) for some \( 1 < p < \infty \).

1. If \( T \) is a Calderón-Zygmund operator,

\[
\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(uv)} \leq c_{n,p}[uv]_{A_{\infty}}[u]_{A_{1}} \log \left( e + [uv]_{A_{\infty}}[u]_{A_{1}}[v]_{A_{p}(u)} \right) \| f \|_{L^{1}(uv)}
\]

and if \( m \) is a positive integer, \( r > 1 \) and \( b \in \text{Osc}_{\exp}L^{r} \) then

\[
(1.2) \quad uv\left( \left\{ x \in \mathbb{R}^{n} : \frac{T_{b}^{m}(fv)}{v} > t \right\} \right) \leq c_{n,p}^{m} \Gamma_{u,v}^{m} \int_{\mathbb{R}^{n}} \Phi_{m} \left( \frac{|f| \|b\|_{\text{Osc}_{\exp}L^{r}}} {t} \right) \]

where

\[
\Gamma_{u,v}^{m} = \sum_{h=0}^{m} \left[ u \right]_{A_{1}} [uv]_{A_{\infty}}^{1+\frac{h}{r}} [u]_{A_{\infty}}^{m-h} \log \left( e + [uv]_{A_{\infty}}^{1+\frac{h}{r}} [u]_{A_{\infty}}^{m-h} [v]_{A_{p}(u)} \right) \]

and \( \Phi_{m}(t) = t (1 + \log^{+}(t))^{m} \).

2. If \( \Omega \in L^{\infty}(\mathbb{S}^{n-1}) \) then

\[
\left\| \frac{T_{b}^{m}(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(uv)} \leq c_{n,p}[uv]_{A_{\infty}}[u]_{A_{1}}[v]_{A_{\infty}} \log \left( e + [uv]_{A_{\infty}}[u]_{A_{1}}[v]_{A_{p}(u)} \right) \| f \|_{L^{1}(uv)}.
\]

We would like to note that in the case \( u = 1 \), in the case of Calderón-Zygmund operators, the estimate above reduces to

\[
\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(v)} \leq c_{n,p}[v]_{A_{\infty}} \log \left( e + [v]_{A_{p}} \right) \| f \|_{L^{1}(v)} \quad p \geq 1.
\]

That estimate improves the bound provided in [29, Theorems 1.16 and 1.17], namely,

\[
\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(v)} \leq c_{n,p}[v]_{A_{p}} \log \left( e + [v]_{A_{p}} \right) \| f \|_{L^{1}(v)} \quad p \geq 1.
\]

In the case of the commutator our approach provides a new proof of [4, Theorem 2] obtaining a quantitative estimate as well. An arguable drawback of the estimates above is that in neither of them we recover the best known dependence in the case \( v = 1 \). We wonder whether the factor \([uv]_{A_{\infty}}\) in each of them can be removed.

In our following result we assume that \( v \in A_{1} \) and \( u \in A_{1}(v) \). It is not hard to check that those conditions are equivalent to assume that \( u \in A_{1} \) and \( v \in A_{1}(u) \), so there is no gain in terms of the size of the class of weights considered. However, in this case, if \( v = 1 \) we recover the best known estimates for \( u \in A_{1} \).

**Theorem 1.2.** Let \( v \in A_{1} \) and \( u \in A_{1}(v) \).
1. If $T$ is a Calderón-Zygmund operator

$$\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1\infty}(uv)} \leq c_n, T[v]_A [v]_{A_\infty} [u]_{A_1(v)} \log (e + [uv]_{A_\infty} [v]_{A_1}) \left\| f \right\|_{L^{1}(uv)}$$

and if $m$ is a positive integer, $r > 1$ and $b \in Osc_{exp} L^r$ then

$$(1.3) \quad u(b) \left( \left\{ x \in \mathbb{R}^n : \left| \frac{T_b^m(fv)}{v} \right| > t \right\} \right) \leq c_{n,p} \Gamma_{uv}^m \left( \frac{\left\| f \right\|_{Osc_{exp} L^r}}{t} \right)$$

where

$$\Gamma_{uv}^m = \sum_{h=0}^{m} [v]_A [v]_{A_\infty} [uv]^{m-h} [u]_{A_1(v)} [v]_{A_\infty} \log \left( e + [v]_A [v]_{A_\infty} [uv]^{-1} \right)^{1 + \frac{h}{2}}$$

and $\Phi_p(t) = t \left( 1 + \log^+(t) \right)^{\rho}$. 

2. If $\Omega \in L^\infty(\mathbb{S}^{n-1})$ then

$$\left\| \frac{T_{\Omega}(fv)(x)}{v(x)} \right\|_{L^{1\infty}(uv)} \leq c_n, \Omega [uv]_{A_\infty} [v]_{A_1} [u]_{A_1(v)} [v]_{A_\infty} \log (e + [uv]_{A_\infty} [v]_{A_1}) \left\| f \right\|_{L^{1}(uv)}$$

As we pointed out above, notice that this result recovers the best dependences known obtained in [21, 22, 25, 23, 14] in the case, $v = 1$. Furthermore, in case of the commutator we obtain the following estimate

$$(1.4) \quad u \left( \left\{ x \in \mathbb{R}^n : \left| T_b^m(f) \right| > t \right\} \right) \leq c_{n,p} [u]_A [v]_{A_\infty} \log (e + [uv]_{A_\infty}) \int_{\mathbb{R}^n} \Phi_{uv}^m \left( \frac{\left\| b \right\|_{Osc_{exp} L^r}}{t} \right) u.$$ 

Observe that (1.4) contains as a particular case the endpoint estimate obtained in [14] and provides precise quantitative bound for the case in which the symbol has better local decay properties than BMO functions. We recall that in [1], it was shown that if a commutator of a certain singular integral satisfies a weak-type (1, 1) estimate then $b \in L^\infty$ and that the $L \log L$ estimate, first settled in [31], implies that $b \in BMO$. Bearing those results in mind we wonder whether $b \in Osc_{exp} L^r$ should be a neccessary condition for (1.4), at least in the case $u = 1$, to hold.

The rest of the paper is organized as follows. Section 2 is devoted to provide some basic results and to fix notation that will be used throughout the remainder of the paper and in Section 3 we provide the proofs of the main results.

2. Preliminaries

2.1. Sparse domination results. In this section we begin borrowing some definitions from [20].

Given a cube $Q$ we denote by $D(Q)$ the standard dyadic grid relative to $Q$.

We say that a family of cubes $D$ is a dyadic lattice if it satisfies the following conditions.

1. If $Q \in D$ then $D(Q) \subset D$.
2. If $P, Q \in D$ then there exists $R \in D$ such that $P, Q \in D(R)$.
3. For every compact set $K \subset \mathbb{R}^d$ there exists some $Q \in D$ such that $K \subset Q$.

We recall that $S$ is a $\eta$-sparse family if for every $Q \in S$ there exists $E_Q \subset Q$ such that

1. $|\eta|Q| \leq |E_Q|$.
2. The sets $E_Q$ are pairwise disjoint.

In some situations it is useful to approximate arbitrary cubes by dyadic cubes. For that purpose, one dyadic lattice is not enough, however $3^n$ are. That fact follows from the following Lemma that we borrow from [20].
Lemma 2.1. For every dyadic lattice $\mathcal{D}$ there exist $3^n$ dyadic lattices $\mathcal{D}_j$ such that

$$\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$$

and for every cube $Q \in \mathcal{D}$ and $j = 1, \ldots, 3^n$, there exists a unique cube $R \in \mathcal{D}_j$ of sidelengh $l_R = 3l_Q$ containing $Q$.

In the last years, and after Lerner’s simplification the proof of the $A_2$ theorem [18] that had been settled earlier by Hytönen [10], the sparse domination approach has been widely and success fully applied in the theory of weights. The philosophy behind that approach consists in controlling, in some sense, the operator that we want to study by suitable sparse operators and providing estimates for the latter ones, which are in general easier to settle.

In the following Theorem we gather the sparse domination results that we will rely upon in the main results of the paper.

Theorem 2.2. Let $f \in \mathcal{C}^\infty_c$. 

[6, 20, 16, 13, 19]: If $T$ is a Calderón-Zygmund operator there exist $3^n \varepsilon$-sparse families contained in $3^n$ dyadic lattices $\mathcal{D}_j$ such that

$$|Tf(x)| \leq c_{n,T,\varepsilon} \sum_{j=1}^{3^n} A_{\mathcal{S}}(|f|)(x)$$

where $A_{\mathcal{S}}f(x) = \sum_{Q \in \mathcal{S}} 1_{Q} f(y) \chi_{Q}(x)$.

[14, 23]: If $T$ is a Calderón-Zygmund operator and $b \in \text{BMO}$ then there exist $3^n \varepsilon$-sparse families contained in $3^n$ dyadic lattices $\mathcal{D}_j$ such that

$$|T^m_b f(x)| \leq c_{n,T,\varepsilon} \sum_{j=1}^{3^n} \sum_{h=0}^{m} A_{\mathcal{S}}^{m,h}(b, f)(x)$$

where $h = 0, \ldots, m$ and

$$A_{\mathcal{S}}^{m,h}(b, f)(x) = \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-h} \frac{1}{|Q|} \int_{Q} |b - b_Q|^h f \chi_{Q}(x).$$

[5, 17]: If $\Omega \in L^\infty(S^{n-1})$ then there exists a sparse family $\mathcal{S}$ such that

$$\left( \int_{\mathbb{R}^n} T_\Omega fg \right) \leq c_{n,\Omega} r^{r} \Lambda_{\mathcal{S}}(f, g) \quad r > 1$$

where $f \in L^r$ and $g \in L^1_{\text{loc}}$

$$\Lambda_{\mathcal{S}}(f, g) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|} \int_{Q} |f| \left( \frac{1}{|Q|} \int_{Q} |g|^r \right)^{\frac{1}{r}} |Q|.$$ 

Remark 2.3. Notice that the $3^n$-dyadic lattices trick (Lemma 2.1) allows us to show that for every dyadic lattice $\mathcal{D}$,

$$\Lambda_{\mathcal{S}}(f, g) \leq \sum_{j=1}^{3^n} \Lambda_{\mathcal{S}_j}(f, g)$$

where each $\mathcal{S}_j \subset \mathcal{D}_j$ and the choice of the dyadic lattices $\mathcal{D}_j$ is independent of $f, g$. 

2.2. $A_p$ weights and Orlicz maximal functions. We recall that given a weight $u$, $v \in A_p(u)$ if
\[ [v]_{A_p(u)} = \sup_Q \frac{1}{u(Q)} \int_Q vu \left( \frac{1}{u(Q)} \int_Q v^{-1/p} u \right)^{p-1} < \infty \]
in the case $1 < p < \infty$ and
\[ [v]_{A_1(u)} = \left\| \frac{M_u v}{v} \right\|_{L^\infty} < \infty \]
where $M_u v = \sup_Q \frac{1}{u(Q)} \int_Q vu$. Analogously if $u = 1$ we recover the classical Muckenhoupt’s condition.

We would like also to recall that
\[ A_\infty = \bigcup_{p \geq 1} A_p. \]
This class of weights is characterized in terms of the following condition
\[ w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) < \infty. \]
This characterization was discovered by Fujii [9] and rediscovered by Wilson [34]. Up until now that $[w]_{A_\infty}$ is the smallest constant characterizing the $A_\infty$ class (see Pérez and Hytönen [11]). A result that we will use as well is the following reverse Hölder inequality that was obtained in [11] (see [12] for another proof).

**Lemma 2.4.** There exists $\tau_n$ such that for every $w \in A_\infty$
\[ \left( \frac{1}{|Q|} \int_Q w^{r_w} \right)^{1/n} \leq 2 \frac{1}{|Q|} \int_Q w \]
where $r_w = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$.

We recall that given a Young function $A : [0, \infty) \to [0, \infty)$, namely a convex, non-decreasing function such that $A(0) = 0$ and $\frac{A(t)}{t} \to \infty$ when $t \to \infty$ we can define
\[ \|f\|_{A(u),Q} = \|f\|_{A(L)(u),Q} = \inf \left\{ \lambda > 0 : \frac{1}{u(Q)} \int_Q A \left( \frac{|f(x)|}{\lambda} \right) u(x) dx \leq 1 \right\}. \]
It is possible to provide a definition of the norm equivalent to the latter (see [15]), namely
\[ \|f\|_{A(u),Q} \simeq \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{u(Q)} \int_Q A \left( \frac{|f(x)|}{\mu} \right) u(x) dx \right\}. \]
Associated to each Young $A$ function there exists another Young function $\overline{A}$ such that
\[ \frac{1}{u(Q)} \int_Q |fg| u \leq 2 \|f\|_{A(u),Q} \|g\|_{\overline{A}(u),Q}. \]
We shall drop $u$ in the notation in the case of Lebesgue measure. Some particular cases of interest for us will be $A(t) = t \log(e + t)^{1/2}$ and $\overline{A}(t) = \exp(t^r) - 1$ for $r > 1$.

Let $u$ a weight and $A$ a Young function. We define the maximal operator $M^F_{A(u)}$ by
\[ M^F_{A(u)} f(x) = \sup_{x \in Q \in \mathcal{F}} \|f\|_{A(u),Q}. \]
where the supremum is taken over all the cubes in the family $\mathcal{F}$. We shall drop the superscript in case the context makes clear the family of cubes considered. If we choose $A(t) = t$ and $u = 1$ and $\mathcal{F}$ is the family of all cubes we recover the classical Hardy-Littlewood operator.

Now we recall if $b \in BMO$, then
\[ \sup_Q \|b - b_Q\|_{L,Q} \leq c_n \|b\|_{BMO}. \]
It is possible to define classes of symbols with even better properties of integrability than BMO symbols. Given $r > 1$ we say that $b \in \text{Osc}_{\exp L^r}(w)$ if
\[
\|b\|_{\text{Osc}_{\exp L^r}(w)} = \sup_{Q} \|b - b_Q\|_{\exp L^r(w),Q} < \infty.
\]
Note that $\text{Osc}_{\exp L^r} \subseteq \text{BMO}$ for every $r > 1$. It is not hard to prove that for those classes of functions the following estimates hold.

**Lemma 2.5.** Let $w \in A_\infty$ and $b \in \text{Osc}_{\exp L^r}$. Then
\[
\|b - b_Q\|_{\exp L^r(w)} \leq c[w]_{A_\infty}^{\frac{1}{r}} \|b\|_{\text{Osc}_{\exp L^r}}.
\]
Furthermore, if $j > 0$ then
\[
\left\|\frac{b - b_Q}{j}\right\|_{\exp L^r(w)} \leq c[w]_{A_\infty}^{\frac{1}{r}} \|b\|_{\text{Osc}_{\exp L^r}}.
\]

We end up this section with a result that allows us to change the underlying weight of Orlicz averages.

**Lemma 2.6.** Let $u$ a weight, $v \in A_p(u)$, and $\Phi$ a Young function. Then, for every cube $Q$,
\[
\|f\|_{\Phi(u),Q} \leq \|f\|_{[v]_{A_p(u)}} \Phi_p(L)(uv),Q.
\]

We remit to [30, 32] for more information about Young functions and Orlicz spaces.

### 3. Proofs of the main results

#### 3.1. Scheme of the proofs

Before we provide the needed lemmata and the proofs of the main results we would like to briefly outline the scheme that we are going to follow for each of the proofs of the estimates in the main results that, as we mentioned in the introduction, can be traced back to [8, 23, 25]. Let $T$ a linear operator, possibly a sparse operator and let $\tilde{M}_{uv} f$ a dyadic, in some sense, maximal operator such that
\[
uv \left( \left\{ x \in \mathbb{R}^d : |\tilde{M}_{uv} f (x)| > t \right\} \right) \leq \frac{1}{t} \int A \left( \frac{|f|}{t} \right) uv
\]
where $A$ is a Young function. First, notice that
\[
uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1 \right\} \right) = uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1, |\tilde{M}_{uv} f (x)| \leq \frac{1}{2} \right\} \right) + uv \left( \left\{ x \in \mathbb{R}^d : |\tilde{M}_{uv} f (x)| > \frac{1}{2} \right\} \right).
\]
Since the desired estimate holds for the second term it suffices to control the first one. Let us call
\[
G = \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1, |\tilde{M}_{uv} f (x)| \leq \frac{1}{2} \right\}.
\]
Then it suffices to prove
\[
(3.1) \quad uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1, |\tilde{M}_{uv} f (x)| \leq \frac{1}{2} \right\} \right) \leq c_{n,T} \kappa_{u,v} \int A (|f|) uv + \frac{1}{2} uv(G).
\]
This yields
\[
uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1, |\tilde{M}_{uv} f (x)| \leq \frac{1}{2} \right\} \right) \leq 2c_{n,T} \kappa_{u,v} \int A (|f|) uv
\]
and consequently
\[
uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T (fv) (x)|}{v} > 1 \right\} \right) \leq 2c_{n,T} \kappa_{u,v} \int A (|f|) uv
\]
which by homogeneity allows us to end up the proof.
The purpose of the following sections we will be settling (3.1) for the operators in the main theorems. To achieve in that task we will rely upon sparse domination results, and more in particular we will use suitable splittings of the sparse families involved in the spirit of [8, 23, 25].

3.2. Lemmata. Before starting with the proofs of the main results we provide some technical lemmas.

**Lemma 3.1.** Let \( \gamma_1, \gamma_2 > 1 \). For every \( j, k \) non negative integers let
\[
\alpha_{k,j} = \min \{ \gamma_1 2^{-j} \rho_1, \beta \gamma_2 2^{-j} 2^{-k} \rho_2 \},
\]
where \( \rho_1, \rho_2, \delta \geq 0 \). Then
\[
\sum_{j,k \geq 0} \alpha_{k,j} \leq c_{\rho_1, \rho_2, \gamma_1, \delta} \gamma_1 \log_2 (e + \gamma_2)^{1+\rho_1} + \frac{1}{2^{\gamma}} \beta,
\]
where \( \gamma \geq 1 \).

**Proof.** We start writing
\[
\sum_{j,k \geq 0} \alpha_{k,j} = \sum_{j \geq \lceil \log_2 ((e + \gamma_2) 8 \gamma) \rceil + ([\delta + \rho_2] + 1) k} \alpha_{k,j} + \sum_{j < \lceil \log_2 ((e + \gamma_2) 8 \gamma) \rceil + ([\delta + \rho_2] + 1) k} \alpha_{k,j}
\]
For the first term, notice that
\[
\beta \gamma_2 \sum_{k=0}^{\infty} 2^{-k} 2^{\delta k} \rho_2 \rho_1 \sum_{j \geq \lceil \log_2 ((e + \gamma_2) 8 \gamma) \rceil + ([\delta + \rho_2] + 1) k} 2^{-j} = \beta \gamma_2 \sum_{k=0}^{\infty} 2^{-k} 2^{\delta k} \rho_2 \rho_1 (2^{\log_2 ((e + \gamma_2) 8 \gamma) + ([\delta + \rho_2] + 1) k} - 2^{\log_2 ((e + \gamma_2) 8 \gamma) + ([\delta + \rho_2] + 1) k})
\]
\[
\leq \frac{\beta \gamma_2}{(e + \gamma_2) 8 \gamma} \sum_{k=0}^{\infty} 2^{-k - \rho_2 \rho_1} \leq \frac{\beta \gamma_2}{(e + \gamma_2) 8 \gamma} \sum_{k=0}^{\infty} 2^{-k - (1+\rho_2) k \rho_1 \log \rho_2}
\]
\[
\leq \frac{2 \gamma_2 \beta}{(e + \gamma_2) 8 \gamma} \leq \frac{\gamma_1}{(e + \gamma_2) 4 \gamma} \beta \leq \frac{1}{2^{\gamma}} \beta.
\]
For the second term, we observe that
\[
\sum_{j < \lceil \log_2 ((e + \gamma_2) 8 \gamma) \rceil + ([\delta + \rho_2] + 1) k} \alpha_{k,j}
\]
\[
\leq \gamma_1 \sum_{k=0}^{\infty} 2^{-k} \sum_{1 \leq j < \lceil \log_2 ((e + \gamma_2) 8 \gamma) \rceil + ([\delta + \rho_2] + 1) k} j^{\rho_1}
\]
\[
\leq \gamma_1 \sum_{k=0}^{\infty} ([\log_2 ((e + \gamma_2) 8 \gamma)] + ([\delta + \rho_2] + 1) k)^{1+\rho_1} 2^{-k}
\]
\[
\leq c 2 (\delta + \rho_2) \gamma_1 \log_2 ((e + \gamma_2) 8 \gamma)^{1+\rho_1}
\]
\[
\leq c_{\rho_1, \rho_2, \gamma, \delta} \gamma_1 \log (e + \gamma_2)^{1+\rho_1}
\]
and we are done. □

The second result we will rely upon is the following.

**Lemma 3.2.** Let $A$ a submultiplicative Young function and $S$ a $\frac{A(8)}{1 + A(8)}$-sparse family. Let $f \in C_\infty$ and $w \in A_\infty$ and assume that for every $Q \in S$

$$2^{-j-1} \leq \langle f \rangle_{A(L)(w)Q} \leq 2^{-j}.$$

Then for every $Q \in S$ there exists $\tilde{E}_Q \subseteq Q$ such that

$$\sum_{Q \in S} \chi_{\tilde{E}_Q}(x) \leq c_n[w]_{A_\infty}$$

and

$$w(Q)\|f\|_{A(w),Q} \leq \frac{A(2j+2)}{2j+2} \int_{\tilde{E}_Q} A(|f|) w.$$

**Proof.** We split the family $S$ in the following way

$$S^0 = \{ \text{Maximal in } S \}$$

$$S^1 = \{ \text{Maximal in } S \setminus S^0 \}$$

$$\ldots$$

$$S^i = \{ \text{Maximal in } S \setminus \bigcup_{r=0}^{i-1} S^r \}$$

Note that since $w \in A_\infty$ we have that, for each cube $Q$ and each measurable subset $E \subset Q$,

$$w(E) \leq 2 \left( \frac{|E|}{|Q|} \right)^{\frac{1}{c_n[w]_{A_\infty}}} w(Q)$$

In particular if $Q \in S^i$ and $J_1 = \bigcup_{P \in S^{i+1}, P \subset Q} P$ then

$$|J_1| = \left| \bigcup_{P \in S^{i+1}, P \subset Q} P \right| \leq \left( \frac{1 + A(4)}{A(4)} - 1 \right) |Q| = \frac{1}{A(4)} |Q|.$$

And this yields

$$w(J_1) \leq \left( \frac{1}{A(8)} \right)^{\frac{1}{c_n[w]_{A_\infty}}} w(Q).$$

Furthermore, arguing by induction, if we denote $J_\nu = \bigcup_{P \in S^{i+\nu}, P \subset Q} P$

$$w(J_\nu) \leq \left( \frac{1}{A(8)} \right)^{\nu} w(Q)$$

And in particular if we choose $\nu = \lceil c_n[w]_{A_\infty} \rceil$, then

$$w(J_\nu) \leq \frac{1}{A(8)} w(Q).$$
Let \( Q \in S_i \) and let \( \tilde{E}_Q = Q \setminus \bigcup_{P \in S_{\text{sp}}^{+, \left[ c_n \right] \times A_\infty}} P \).

\[
w(Q)\|f\|_{A(w), Q} \leq w(Q) \left\{ 2^{-j-2} + \frac{2^{-j-2}}{w(Q)} \int_Q A(2^{j+2}|f|) w \right\} \]

\[
\leq w(Q)2^{-j-2} + \frac{1}{2^{j+2}} \int_{E_Q} A(2^{j+2}|f|) w
\]

\[
\leq w(Q)2^{-j-1} + \frac{1}{2^{j+2}} \int_{E_Q} A(2^{j+2}|f|) w + \frac{1}{2^{j+2}} \sum_{P \in S_{j,k}^{+, \left[ c_n \right] \times A_\infty}} \int_P A(2^{j+2}|f|) w
\]

\[
\leq w(Q)2^{-j-2} + \frac{A(2^{j+2})}{2^{j+2}} \int_{E_Q} A(|f|) w + \frac{1}{2^{j+2}} \sum_{P \in S_{j,k}^{+, \left[ c_n \right] \times A_\infty}} \int_P A(2^{j+2}|f|) w
\]

Observe that we can bound the last term as follows

\[
\sum_{P \in S_{j,k}^{+, \left[ c_n \right] \times A_\infty}} \int_P A(2^{j+2}|f|) w
\]

\[
\leq A(4) \sum_{P \in S_{j,k}^{+, \left[ c_n \right] \times A_\infty}} w(P) \frac{1}{w(P)} \int_P A(2^{j}|f|) w
\]

\[
\leq A(4) \sum_{P \in S_{j,k}^{+, \left[ c_n \right] \times A_\infty}} w(P)
\]

\[
\leq A(4) \frac{1}{A(8)} w(Q) \leq \frac{1}{4} w(Q)
\]

Hence

\[
w(Q)\|f\|_{A(w), Q} \leq \frac{1}{2^{j+2}} w(Q) + \frac{A(2^{j+2})}{2^{j+2}} \int_{E_Q} A(|f|) w + \frac{1}{2^{j+2}} \frac{1}{4} w(Q)
\]

\[
\leq \left( \frac{1}{2} + \frac{1}{4} \right) w(Q)\|f\|_{A(w), Q} + \frac{A(2^{j+2})}{2^{j+2}} \int_{E_Q} A(|f|) w
\]

\[
= \frac{3}{4} w(Q)\|f\|_{A(w), Q} + \frac{A(2^{j+2})}{2^{j+2}} \int_{E_Q} A(|f|) w,
\]

from which readily follows the desired conclusion. \(\square\)

The following lemma will be also used repeatedly.

**Lemma 3.3.** Let \( w \in A_\infty \) and \( S \) a \( \eta \)-sparse family of cubes. Then

\[
\sum_{Q \in S} w(Q) \leq c_n [w]_{A_\infty} w \left( \bigcup_{Q \in S} Q \right).
\]

**Proof.** We can assume that \( S = \bigcup_{k=0}^\infty S_k \) where \( \{S_k\} \) is an increasing sequence of finite sparse families. Now we fix \( k \) and consider \( S_k^{\ast} \) the family of maximal cubes of \( S_k \) with respect to the inclusion. Then

\[
\sum_{Q \in S_k} w(Q) = \sum_{Q \in S_k^{\ast}} \sum_{P \in S_k, P \subset Q} w(P).
\]
Notice that
\[
\sum_{P \in S_k, P \subset Q} w(P) \leq \frac{1}{\eta} \sum_{P \in S_k, P \subset Q} \frac{1}{|P|} w(P) |E_P| \leq \frac{1}{\eta} \sum_{P \in S_k, P \subset Q} \frac{1}{|P|} w(P) |E_P|
\]
\[
\leq \frac{1}{\eta} \sum_{P \in S_k, P \subset Q} \inf_{z \in P} M(w \chi_Q)(z) |E_P| \leq \frac{1}{\eta} \sum_{P \in S_k, P \subset Q} \int_{E_P} M(w \chi_Q)
\]
\[
\leq \frac{1}{\eta} \int_Q M(w \chi_Q) = \frac{1}{\eta w(Q)} \int_Q M(w \chi_Q) w(Q) \leq \frac{1}{\eta} w(A_\infty) w(Q).
\]

Hence
\[
\sum_{Q \in S^*} \sum_{P \in S_k, P \subset Q} w(P) \leq \frac{1}{\eta} [w]_{A_\infty} \sum_{Q \in S_k^*} w(Q) = \frac{1}{\eta} [w]_{A_\infty} w \left( \bigcup_{Q \in S_k} Q \right)
\]
\[
= \frac{1}{\eta} [w]_{A_\infty} w \left( \bigcup_{Q \in S_k} Q \right) = \frac{1}{\eta} [w]_{A_\infty} w \left( \bigcup_{Q \in S} Q \right)
\]
\[
\leq \frac{1}{\eta} [w]_{A_\infty} w \left( \bigcup_{Q \in S} Q \right).
\]

Consequently
\[
\sum_{Q \in S_k} w(Q) \leq \frac{1}{\eta} [w]_{A_\infty} w \left( \bigcup_{Q \in S} Q \right)
\]
and letting \( k \to \infty \) we are done. \( \square \)

To end the section we provide some results related to singular weighted maximal functions.

**Lemma 3.4.** Let \( A \) a Young function such that \( A(st) \leq \kappa A(s) A(t) \). Let \( D_j, j = 1, \ldots, k \) be dyadic grids and let \( w \) a weight. Then
\[
w \left( \left\{ x \in \mathbb{R}^n : M^F_{A(w)} f(x) > t \right\} \right) \leq \kappa c_n \int_{\mathbb{R}^d} A \left( \frac{|f(x)|}{t} \right) w(x) dx
\]
where \( F = \bigcup_{j=1}^{3^n} D_j \).

**Proof.** Let \( t > 0 \). Notice that
\[
M^F_{w} f(x) \leq \sum_{j=1}^{3^n} M^D_{w} f(x).
\]

Then taking into account that
\[
w \left( \left\{ x \in \mathbb{R}^n : M^D_{w} f(x) > \lambda \right\} \right) \leq \int_{\mathbb{R}^d} A \left( \frac{|f(x)|}{\lambda} \right) w(x) dx
\]
(see [20, Section 15]) we have that
\[ w \left( \{ x \in \mathbb{R}^n : M_w^F f(x) > t \} \right) \]
\[ \leq w \left( \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{3^n} M_{w,j}^D f(x) > t \right\} \right) \]
\[ \leq \sum_{j=1}^{3^n} w \left( \left\{ x \in \mathbb{R}^n : M_{w,j}^D f(x) > \frac{t}{3^n} \right\} \right) \]
\[ \leq \sum_{j=1}^{3^n} \int_{\mathbb{R}^d} A \left( \frac{3^n |f(x)|}{t} \right) w(x) dx \]
\[ \leq c_n \kappa \int_{\mathbb{R}^d} A \left( \frac{|f(x)|}{t} \right) w(x) dx \]
and we are done. \( \square \)

3.3. Proof of Theorem 1.1.

3.3.1. Calderón-Zygmund operators. Using pointwise sparse domination it suffices to settle the result for a sparse operator \( A_S \) where \( S \) is a \( \delta \)-sparse family contained in a dyadic lattice \( D \).

Let \( G = \{ \frac{A_S(fv)}{v(x)} > 1 \} \setminus \{ M_{w}^D(f) > \frac{1}{2} \} \) and assume that \( \|f\|_{L^1(w)} = 1 \). Then it suffices to prove that
\[ uv(G) \leq c_{n,p}[uv]_{A_1} \left( e + [uv]_{A_\infty} [u]_{A_1} [v]_{A_p(u)} \right) + \frac{1}{2} uv(G). \]
If we denote \( g = \chi_G \) then
\[ uv(G) \leq c_n c_T \sum_{Q \in S} \langle fv \rangle_{Q,1} \langle g \rangle_{Q,1} w(Q). \]
\[ \leq c_n c_T [u]_{A_1} \sum_{Q \in S} \langle fv \rangle_{Q,1} \langle g \rangle_{Q,1} uv(Q) \]
and it suffices to prove that
\[ c_n c_T [u]_{A_1} \sum_{Q \in S} \langle fv \rangle_{Q,1} \langle g \rangle_{Q,1} uv(Q) \]
\[ \leq c_{n,p}[uv]_{A_\infty} [u]_{A_1} \left( e + [uv]_{A_\infty} [u]_{A_1} [v]_{A_p(u)} \right) + \frac{1}{2} uv(G). \]
We split the sparse family as follows. Let \( Q \in S_{k,j} \), \( k, j \geq 0 \) if
\[ 2^{-j} - 1 < \langle f \rangle_{Q,1} \leq 2^{-j}, \]
\[ 2^{-k} - 1 < \langle g \rangle_{Q,1} \leq 2^{-k}. \]
Let us call
\[ s_{k,j} = \sum_{Q \in S_{k,j}} \langle f \rangle_{Q,1} \langle g \rangle_{Q,1} uv(Q). \]
We claim that
\[ s_{k,j} \leq \begin{cases} c_n 2^{-k}[uv]_{A_\infty}, & \text{if } k \geq 0, \\ c_{n,p}[uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^k(p-1) uv(G). & \text{if } k < 0. \end{cases} \]
For the top estimate we argue as follows. Using Lemma 3.2 we have that there exist sets \( \tilde{E}_Q \subset Q \) such that
\[ \sum_{Q \in S_{k,j}} \chi_{\tilde{E}_Q}(x) \leq [c_n[uv]_{A_\infty}] \]
and
\[ \int_Q fuv \leq 4 \int_{E_Q} fuv. \]

Then
\[ s_{k,j} \leq 2^{-k} \sum_{Q \in S_{j,k}} \int_Q fuv \]
\[ \leq 4 \cdot 2^{-k} \sum_{Q \in S_{j,k}} \int_{E_Q} fuv \]
\[ \leq c_n[uv]_{A_\infty} 2^{-k} \int_{\mathbb{R}^n} fuv = c_n[uv]_{A_\infty} 2^{-k}. \]

For the lower estimate, using Lemma 3.3,
\[ s_{k,j} \leq 2^{-j} 2^{-k} \sum_{Q \in S_{j,k}} uv(Q) \]
\[ \leq c_n[uv]_{A_\infty} 2^{-j} 2^{-k} uv \left( \bigcup_{Q \in S_{j,k}} Q \right) \]
\[ \leq c_n[uv]_{A_\infty} 2^{-j} 2^{-k} uv \left( \left\{ x \in \mathbb{R}^n : M_u g > 2^{-k-1} \right\} \right). \]

Now notice that since \( v \in A_p(u), \) Lemma 2.6 yields
\[ \frac{1}{u(Q)} \int_Q gu \leq \left( \frac{[v]_{A_p(u)}}{uv(Q)} \int_Q guv \right)^{\frac{1}{p}}. \]

Taking that into account, by Lemma 3.4
\[ \leq c_n[uv]_{A_\infty} 2^{-j} 2^{-k} uv \left( \left\{ x \in \mathbb{R}^n : M_u g > 2^{-k-1} \right\} \right) \]
\[ \leq c_n[uv]_{A_\infty} 2^{-j} 2^{-k} uv \left( \left\{ x \in \mathbb{R}^n : M_{uv} g > 2^{-k-p} [v]^{-1}_{A_p(u)} \right\} \right) \]
\[ \leq c_{n,p}[uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{(p-1)uv(G)}. \]

Combining the estimates above
\[ uv(G) \leq c_n c_T[u]_{A_1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \]
\[ \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ c_n c_T 2^{-k}[u]_{A_1} [uv]_{A_\infty}, c_{n,p}[u]_{A_1} [uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{-k} 2^{kp} uv(G) \right\}. \]

Now we are left with estimating the double sum. Applying Lemma 3.1 with
\[ \gamma_1 = c_n c_T[u]_{A_1} [uv]_{A_\infty} \]
\[ \gamma_2 = c_{n,p}[u]_{A_1} [uv]_{A_\infty} [v]_{A_p(u)} \]
\[ \delta = p, \ \beta = uv(G), \ \rho_1 = \rho_2 = 0 \] and \( \gamma = 1 \) we are done.

3.3.2. Commutators. Using pointwise sparse domination it suffices to settle the result for suitable dyadic operators. Let
\[ G = \left\{ \frac{\sum_{Q \in S} |b - b_Q|^{m-h} \chi_{Q} \frac{1}{v(x)} \int_Q |b - b_Q|^h f v}{v(x)} > 1 \right\} \setminus \left\{ M^D_{L(\log L)} f_{uv}(f) > \frac{1}{2} \right\}. \]
Assume that \( \|b\|_{Osc^{L^r}} = 1 \). It suffices to prove that

\[
uv(G) \leq c_0 \varphi_{m,h}(u,v) \int_{\mathbb{R}^n} \Phi_{\frac{h}{r}}(|f(x)|)dx + \frac{1}{2} uv(G)
\]

where

\[
\varphi_{m,h}(u,v) = [u]_{A_1} [uv]_{A_\infty}^{1 + \frac{h}{r}} [u]_{A_\infty}^{m-h} \log \left( e + [u]_{A_1} [uv]_{A_\infty}^{1 + \frac{h}{r}} [u]_{A_\infty}^{m-h} [v]_{A_{p(u)}} \right)^{1 + \frac{h}{r}}.
\]

If we denote \( g = \chi_G \) then

\[
uv(G) \leq \sum_{Q \in S} \frac{1}{|Q|} \int_Q |b - b_Q|^h f v \int_Q |b - b_Q|^{m-h} u
\]

\[
\leq \sum_{Q \in S} \frac{u(Q)}{|Q|} \int_Q |b - b_Q|^h f v \frac{1}{u(Q)} \int_Q |b - b_Q|^{m-h} u
\]

\[
\leq [u]_{A_1} \sum_{Q \in S} \frac{1}{uv(Q)} \int_Q |b - b_Q|^h f uv \frac{1}{u(Q)} \int_Q |b - b_Q|^{m-h} uv(Q)
\]

\[
\leq [u]_{A_1} \sum_{Q \in S} (\|b - b_Q|^h \|_{exp L^r/(uv),Q} \|f\|_{L(\log L)^{1+(h/r)}(uv),Q} \\
\quad \times \|b - b_Q|^{m-h} \|_{exp L^r/m-(h),Q} \|g\|_{L(\log L)^{m-h}/(u),Q} uv(Q))
\]

\[
\leq c \|b\|^{m-h}_{Osc^{L^r}} [uv]_{A_\infty}^{\frac{h}{r}} [u]_{A_\infty}^{m-h} \sum_{Q \in S} \|f\|_{L(\log L)^{1+(h/r)}(uv),Q} \|g\|_{L(\log L)^{m-h}/(u),Q} uv(Q)
\]

\[
= c [uv]_{A_\infty}^{\frac{h}{r}} [u]_{A_\infty}^{m-h} \sum_{Q \in S} \|f\|_{L(\log L)^{1+(h/r)}(uv),Q} \|g\|_{L(\log L)^{m-h}/(u),Q} uv(Q) .
\]

We split the sparse family as follows \( Q \in \mathcal{S}_{k,j}, k, j \geq 0 \) if

\[
2^{-j-1} < \|f\|_{L(\log L)^{1+(h/r)}(uv),Q} \leq 2^{-j},
\]

\[
2^{-k-1} < \|g\|_{L(\log L)^{m-h}/(u),Q} \leq 2^{-k}.
\]

Then

\[
\sum_{Q \in S} \|f\|_{L(\log L)^{1+(h/r)}(uv),Q} \|g\|_{L(\log L)^{m-h}/(u),Q} uv(Q) = \sum_{k,j \geq 0} s_{k,j}.
\]

Now we observe that

\[
s_{k,j} \leq \left\{ \begin{array}{ll}
c_n [uv]_{A_\infty} 2^{-k} j^\frac{h}{r}, \\
c_{n,p,m} [uv]_{A_\infty} [v]_{A_{p(u)}} 2^{-j} 2^{k(p-1)} j^\frac{m-h}{r} uv(Q) uv(G).
\end{array} \right.
\]

For the top estimate we use Lemma 3.2 with \( w = uv \) and \( A(t) = \Phi_{\frac{h}{r}}(t) \), and we have that

\[
uv(Q) ||f||_{L(\log L)^{1+(h/r)}(uv),Q} \leq cj^{\frac{h}{r}} \int_{E_Q} \Phi_{\frac{h}{r}}(|f|) uv.
\]

with

\[
\sum_{Q \in \mathcal{S}_{k,j}} \chi_{E_Q}(x) \leq [c_n [uv]_{A_\infty}] .
\]
Then
\[ s_{k,j} \leq 2^{-k} j^b \sum_{Q \in S_{k,j}} \int_{E_Q} \Phi_{\frac{h}{r}} (|f|) \, u(v) \, dv \]
\[ \leq 2 \cdot 2^{-k} j^b \sum_{Q \in S_{k,j}} \int_{E_Q} \Phi_{\frac{h}{r}} (|f|) \, u(v) \, dv \]
\[ \leq c_n |u|_{A_{\infty}} 2^{-k} j^b \int_{\mathbb{R}^n} \Phi_{\frac{h}{r}} (|f|) \, u(v) \, dv. \]

For the lower estimate, by Lemma 3.3
\[ s_{k,j} \leq 2^{-j} 2^{-k} \sum_{Q \in S_{k,j}} \int_{u(Q)} \Phi_{\frac{h}{r}} (|f|) \, u(v) \, dv \]
\[ = c |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^n} \Phi_{\frac{h}{r}} (2^{k+1} g) \, u(v) \, dv \]
\[ \leq c_n |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^n} \Phi_{\frac{h}{r}} (2^{k+1} g) \, u(v) \, dv. \]

Taking into account Lemma 2.6
\[ \|g\|_{L(\log L)^{\frac{m-h}{2}}} \leq \|g\|_{[v]_{A_{p}(\log L)^{\frac{m-h}{2}}}^{A_{p}(\log L)^{\frac{m-h}{2}}}}. \]

That estimate combined with Lemma 3.4 allows us to argue as follows
\[ \leq c_{n,p} |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^d} [v]_{A_{p}(\log L)^{\frac{m-h}{2}}} (2^{k+1} g)^p \, u(v) \, dv. \]
\[ \leq c_{n,p} |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^d} [v]_{A_{p}(\log L)^{\frac{m-h}{2}}} (2^{k+1} g)^p \, u(G). \]
\[ \leq c_{n,p} |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^d} [v]_{A_{p}(\log L)^{\frac{m-h}{2}}} (2^{k+1} g)^p \, u(G). \]
\[ \leq c_{n,p,m} |u|_{A_{\infty}} 2^{-j} 2^{k(p-1)} \int_{\mathbb{R}^d} [v]_{A_{p}(\log L)^{\frac{m-h}{2}}} (2^{k+1} g)^p \, u(G). \]

Combining the estimates above
\[ u(G) \leq c |u|_{A_{\infty}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \]
\[ \leq c_{n,p,m} |u|_{A_{\infty}} 2^{-j} 2^{-k} \int_{\mathbb{R}^d} \Phi_{\frac{h}{r}} (|f|) \, u(v) \, dv. \]

We end up the proof applying Lemma 3.1, with \( \gamma_1 = c_{n} |u|_{A_{1}} [u]_{A_{\infty}}^{1+} [u]_{A_{\infty}}^{1+} \int_{\mathbb{R}^d} \Phi_{\frac{h}{r}} (|f|) \, u(v) \), \( \gamma_2 = c_{n,p,m} |u|_{A_{1}} [u]_{A_{\infty}}^{1+} [u]_{A_{\infty}}^{1+} \int_{\mathbb{R}^d} \Phi_{\frac{h}{r}} (|f|) \, u(v) \), \( \beta = u(G) \), \( \delta = p-1 \), \( \gamma = 1 \), \( \rho_1 = \frac{h}{r} \) and \( \rho_2 = \frac{m-h}{2} \).

3.3.3. Rough singular integrals. Let us fix a dyadic lattice \( D \) and let \( D_j \) with \( j = 1, \ldots, 3^n \) obtained using the 3^\text{n} dyadic lattices trick (Lemma 2.1). Now let
\[ G = \left\{ \frac{T_{1}(f v)(x)}{v(x)} > 1 \right\} \setminus \left\{ M_{u v}(f) > \frac{1}{2} \right\} \]
where \( F = \bigcup_{j=1}^{3^n} D_j \) and assume that \( \|f\|_{L^1(u v)} = 1. \)
Then it suffices to prove that

\[ uv(G) \leq c_{n,p}[uv]_{A_\infty}[u]_{A_\infty}[A_1 \log (e + [uv]_{A_\infty}[u]_{A_\infty}[u]_{A_\infty}[v]_{A_p(u)}) + \frac{1}{2}uv(G). \]

Note that

\[ uv(G) \leq \left| \int \frac{T_{\Omega}(fv)}{v}uvg \right| = \left| \int T_{\Omega}(fv)ug \right| \]

where \( g \simeq \chi_G \). Then for \( s = 1 + \frac{1}{2n[u]_{A_\infty}} \), notice that, arguing as in [25]

\[ \frac{v^s(G \cap Q)}{u^s(Q)} \lesssim \left( \frac{u(G \cap Q)}{u(Q)} \right)^{\frac{1}{2}}. \]

Taking into account (2.1) and Remark 2.3 we have that

\[
uv(G) \lesssim c_{n,T} s \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle fv \rangle_{Q,1} \langle \chi_G uv \rangle_{Q,s} \\
= c_{n,T} s \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle fv \rangle_{Q,1} \langle \chi_G \rangle_{Q,s}^{uv} \langle u \rangle_{Q,s} \\
\leq c_{n,T} [u]_{A_\infty} [u]_{A_1} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle f \rangle_{Q,1} \langle g \rangle_{Q,2s} uv(Q)
\]

where \( g = \chi_G \) and each \( S_j \subset D_j \).

At this point one remark is in order. Notice that in this case, since we don’t have pointwise domination, we need to remove the cubes where the maximal function is large from the sparse family using just one maximal function. On the other hand if we choose the standard maximal function instead of some dyadic version that would lead to some dependence on the doubling constant of the measure \( uvdx \), which is something that we avoid with our choice (see Lemma 3.4).

After that remark we continue with the proof. Notice that it suffices to prove that for each \( j \),

\[
c_{n,T} [u]_{A_\infty} [u]_{A_1} \sum_{Q \in S_j} \langle f \rangle_{Q,1}^{uv} \langle g \rangle_{Q,2s} uv(Q) \\
\leq c_{n,p}[uv]_{A_\infty} [u]_{A_1} \log (e + [uv]_{A_\infty}[u]_{A_\infty}[u]_{A_1}[v]_{A_p(u)}) + \frac{1}{2}3^n uv(G). \]

Taking into account the definition of \( G \), since we remove the set where \( M_{uv}^K(f) > \frac{1}{2} \) we can split sparse family as follows \( Q \in S_{k,j}, k, j \geq 0 \) if

\[
2^{-j-1} < \langle f \rangle_{Q,1}^{uv} \leq 2^{-j} \\
2^{-k-1} < \langle g \rangle_{Q,2s}^{uv} \leq 2^{-k}. 
\]

Let us call

\[
s_{k,j} = \sum_{Q \in S_{k,j}} \langle f \rangle_{Q,1}^{uv} \langle g \rangle_{Q,2s} uv(Q). 
\]

Now we observe that

\[
s_{k,j} \leq \begin{cases} 
2^{-k}[uv]_{A_\infty} \\
2^{-j}2^{(2ps-1)k}uv(G).
\end{cases}
\]
For the top estimate we argue as we did in (3.2). For the lower estimate, using Lemma 3.3,

\[ s_{k,j} \leq 2^{-j}2^{-k} \sum_{Q \in S_{j,k}} uv(Q) \]

\[ \leq c_n[uv]_{A_\infty} 2^{-j}2^{-k}uv \left( \bigcup_{Q \in S_{j,k}} Q \right) \]

\[ \leq c_n[uv]_{A_\infty} 2^{-j}2^{-k}uv \left( \left\{ x \in \mathbb{R}^n : (Mu)_\mathbb{R} > 2^{-k-1} \right\} \right) . \]

Since \( v \in A_p(u) \), taking into account Lemmas 2.6 and 3.4

\[ \leq c_n[uv]_{A_\infty} 2^{-j}2^{-k}uv \left( \left\{ x \in \mathbb{R}^n : (v)_{A_p(u)}M_{uv}g > 2^{-k-1} \right\} \right) \]

\[ \leq c_n[uv]_{A_\infty} 2^{-j}2^{-k}uv \left( \left\{ x \in \mathbb{R}^n : M_{uv}g > 2^{-2spk-2sp[v]^{-1}} \right\} \right) \]

\[ \leq c_{n,p}[uv]_{A_\infty} [v]_{A_p(u)} 2^{-j}2(2ps-1)kuv(G) . \]

Combining the estimates above

\[ uv(G) \leq c_n c_T \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \]

\[ \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ c_n c_T \beta_{u,v} 2^{-k}, c_{n,p} \beta_{u,v}[v]_{A_p(u)} 2^{-j}2(2ps-1)kuv(G) \right\} \]

where \( \beta_{u,v} = [uv]_{A_\infty} [u]_{A_\infty} [u]_{A_1} \). We end the proof using Lemma 3.1, with \( \gamma_1 = c_n c_T \beta_{u,v} \), \( \gamma_2 = c_{n,p} \beta_{u,v}[v]_{A_p(u)} \), \( \beta = uv(G) \), \( \delta = 2ps - 1 \), \( \gamma = 3^n \) and \( \rho_1 = \rho_2 = 0 . \)

3.4. Proof of Theorem 1.2.

3.4.1. Calderón-Zygmund operators. Using pointwise sparse domination it suffices to settle the result for a sparse operator \( A_S \) where \( S \) is a \( \frac{s}{9} \)-sparse family.

Let \( G = \{ \frac{A_S(f)(x)}{v(x)} > 1 \} \setminus \{ M^D_v(f) > \frac{1}{2} \} \) and assume that \( f \geq 0 \) and \( \|f\|_{L^1(uv)} = 1 \). Then it suffices to prove that

\[ uv(G) \leq c_{n,p}[v]_{A_1} [v]_{A_\infty} [u]_{A_1(v)} \log (e + [uv]_{A_\infty} [v]_{A_1} [u]_{A_1(v)}) + \frac{1}{2} uv(G) . \]

If we denote \( g = \chi_G \) then

\[ uv(G) \leq c_n c_T \sum_{Q \in S} \langle f \rangle_{Q,1} \int_Q gu \]

\[ = c_n c_T \sum_{Q \in S_j} \langle f \rangle_{Q,1} \frac{v(Q)}{|Q|} \int_Q gu \]

\[ \leq [v]_{A_1} c_n c_T \sum_{Q \in S_j} \langle f \rangle_{Q,1} \int_Q gu . \]

\[ = c_n c_T [v]_{A_1} \sum_{Q \in S_j} \langle f \rangle_{Q,1} (g)_{Q,1} uv(Q) \]
and it suffices to prove that
\[ c_n c_T[v] A_1 \sum_{Q \in S} \langle f \rangle_{Q,1}^v \langle g \rangle_{Q,1}^u \nu v(Q) \leq c_{n,p}[v] A_1[v] A_\infty [u] A_{1(v)} \log (e + \langle u \rangle_{A_\infty}[v] A_1[u] A_{1(v)}) + \frac{1}{2} \nu v(G). \]

We split the sparse family as follows. Let \( Q \in S_{k,j}, \) \( k, j \geq 0 \) if
\[
2^{-j-1} < \langle f \rangle_{Q,1}^v \leq 2^{-j} \quad \text{and} \quad 2^{-k-1} < \langle g \rangle_{Q,1}^u \leq 2^{-k}.
\]

Let us call
\[ s_{k,j} = \sum_{Q \in S_{k,j}} \langle f \rangle_{Q,1}^v \langle g \rangle_{Q,1}^u \nu v(Q). \]

Now we observe that
\[
s_{k,j} \leq \begin{cases} c_n 2^{-k}[u] A_{1(v)}[v] A_\infty \\ c_n[u] A_\infty 2^{-j-k} \nu u v(G). \end{cases}
\]

For the top estimate we argue as follows. Using Lemma 3.2 we have that
\[
\int_Q f v \leq 4 \int_{\tilde{E}_Q} f v
\]
where \( \tilde{E}_Q \subset Q \) and
\[
\sum_{Q \in S_{k,j}} \chi_{\tilde{E}_Q}(x) \leq \lceil c_n[v] A_\infty \rceil.
\]

Then
\[
s_{k,j} \leq 2^{-k} \sum_{Q \in S_{k,j}} \frac{\nu u v(Q)}{v(Q)} \int_Q f v \leq 2 \cdot 2^{-k} \sum_{Q \in S_{k,j}} \frac{\nu u v(Q)}{v(Q)} \int_{\tilde{E}_Q} f v \leq 2 \cdot 2^{-k}[u] A_{1(v)} \sum_{Q \in S_{k,j}} \int_{\tilde{E}_Q} f u v \leq c_n 2^{-k}[u] A_{1(v)}[v] A_\infty \int_{\mathbb{R}^n} f u v.
\]

For the lower estimate, using Lemma 3.3,
\[
s_{k,j} \leq 2^{-j} 2^{-k} \sum_{Q \in S_{k,j}} \nu u v(Q) \leq c_n[u] A_\infty 2^{-j} 2^{-k} \nu u v \left( \bigcup_{Q \in S_{k,j}} Q \right) \leq c_n[u] A_\infty 2^{-j} 2^{-k} \nu u v \left( \left\{ x \in \mathbb{R}^n : M^D_{u v}(g) > 2^{-k-1} \right\} \right).
\]

Now using the weak-type \((1,1)\) of \( M_{u v} \) (Lemma 3.4)
\[
\leq c_n[u] A_\infty 2^{-j} 2^{-k} \nu u v \left( \left\{ x \in \mathbb{R}^n : M^D_{u v}(g) > 2^{-k-1} \right\} \right) \leq c_n[u] A_\infty 2^{-j} 2^{-k} \nu v(G).
\]
Combining the estimates above,

\[ uv(G) \leq c_n c_T[v] A_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \]

\[ \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ c_n c_T 2^{-k} [v] A_1 [v] A_\infty [u] A_1 (v), c_n [v] A_1 [uv] A_\infty 2^{-j} 2^{-k} uv (G) \right\}. \]

An application of Lemma 3.1 with

\[ \gamma_1 = c_n c_T [v] A_1 [v] A_\infty [u] A_1 (v) \]
\[ \gamma_2 = c_n [v] A_1 [uv] A_\infty \]
\[ \delta = 0, \beta = uv(G), \gamma = 1 \text{ and } \rho_1 = \rho_2 = 0 \text{ ends the proof.} \]

3.4.2. **Commutators.** Using pointwise sparse domination it suffices to settle the result for suitable dyadic operators. Let

\[ G = \left\{ \sum_{Q \in S} |b(x) - b_Q|^{m-h} \chi_Q(x) \frac{1}{|Q|} \int_Q |b - b_Q|^h f \right\} \quad \left\{ \frac{\Delta}{L (\log L)^{\frac{1}{p}} (Q) > \frac{1}{2}} \right\} \]

Assume that \( |b| \leq 1 \). It suffices to prove that

\[ uv(G) \leq c \varphi_{m,h} (u, v) \int_{\mathbb{R}^n} \Phi_{\frac{1}{r}} (|f(x)|) dx + \frac{1}{2} uv(G). \]

where \( \gamma_1 = c c_n [u] A_1 [uv]^{1 + \frac{h}{r}} [u] A_\infty \int_{\mathbb{R}^n} \Phi_{\frac{1}{r}} (|f|) \) \( uv \), \( \gamma_2 = c c_n [u] A_1 [uv]^{1 + \frac{h}{r}} [u] A_\infty \) \( c |v| A_1 [uv]^{1 + \frac{h}{r}} [u] A_\infty \) \( c |v| A_1 [uv]^{1 + \frac{h}{r}} [u] A_\infty \)

If we denote \( g = \chi_G \) then

\[ uv(G) \leq \sum_{Q \in S} \frac{1}{|Q|} \int_Q |b - b_Q|^h f \int_Q |b - b_Q|^{m-h} \]

\[ \leq \sum_{Q \in S} \frac{1}{v(Q)} \int_Q |b - b_Q|^h f \int_Q |b - b_Q|^{m-h} \]

\[ \leq |v| A_1 \sum_{Q \in S} \frac{1}{v(Q)} \int_Q |b - b_Q|^h f \int_Q |b - b_Q|^{m-h} \]

\[ \leq |v| A_1 \sum_{Q \in S} \left( ||b - b_Q||^{h}_{L^r (Q)} ||f||_{L (\log L)^{\frac{1}{p}} (Q)} \right) \]

\[ \times \left( ||b - b_Q||^{m-h}_{L^r (Q)} ||g||_{L (\log L)^{\frac{m-h}{r}} (Q)} \right) \]

\[ \leq c ||b||^{m-h}_{L^r (Q)} ||f||_{L (\log L)^{\frac{m-h}{r}} (Q)} ||g||_{L (\log L)^{\frac{m-h}{r}} (Q)} \]

\[ = c |v| A_1 [uv]^{\frac{h}{r}} A_\infty \sum_{Q \in S} \frac{||f||_{L (\log L)^{\frac{1}{p}} (Q)}}{L (\log L)^{\frac{m-h}{r}} (Q)} \]

Let us split the sparse family as follows. Let \( Q \in S_{k,j}, k, j \geq 0 \) if

\[ 2^{-j-1} < \|f\|_{L (\log L)^{\frac{1}{p}} (Q)} \leq 2^{-j} \]

\[ 2^{-k-1} < \|g\|_{L (\log L)^{\frac{m-h}{r}} (Q)} \leq 2^{-k}. \]

Let us call

\[ s_{k,j} = \sum_{Q \in S_{k,j}} \frac{||f||_{L (\log L)^{\frac{1}{p}} (Q)}}{L (\log L)^{\frac{m-h}{r}} (Q)} \]

uv(Q).
Now we observe that
\[
 s_{k,j} \leq \begin{cases} 
 c_n [u] A_1 [v] A_\infty 2^{-k} j^\frac{\beta}{\gamma} \int_{\mathbb{R}^n} \Phi_{\frac{\rho}{\gamma}} (|f|) uv \\
 c_{n,p,m} [uv] A_\infty 2^{-j 2^{2(p-1)} k^\frac{m-h}{\rho}} p uv (G).
\end{cases}
\]
For the top estimate we use Lemma 3.2 with \( w = v \) and \( A (t) = \Phi_{\frac{\rho}{\gamma}} (t) \), and we have that
\[
v (Q) \| f \| _{L (\log L) \Phi_{\frac{\rho}{\gamma}} (v), Q} \leq c j^\frac{\beta}{\gamma} \int_{E_Q} \Phi_{\frac{\rho}{\gamma}} (|f|) uv.
\]
with
\[
\sum_{Q \in S_{k,j}} \chi_{E_Q} (x) \leq [c_n [v] A_\infty ].
\]
Then
\[
s_{k,j} \leq 2^{-k} j^\frac{\beta}{\gamma} \sum_{Q \in S_{j,k}} \frac{uv (Q)}{v (Q)} \int_{E_Q} \Phi_{\frac{\rho}{\gamma}} (|f|) v.
\]
\[
\leq 2 [u] A_1 [v] 2^{-k} j^\frac{\beta}{\gamma} \sum_{Q \in S_{j,k}} \int_{E_Q} \Phi_{\frac{\rho}{\gamma}} (|f|) uv.
\]
\[
\leq c_n [u] A_1 [v] A_\infty 2^{-k} j^\frac{\beta}{\gamma} \int_{\mathbb{R}^n} \Phi_{\frac{\rho}{\gamma}} (|f|) uv.
\]
For the lower estimate, by Lemma 3.4
\[
s_{k,j} \leq 2^{-j 2^{-k}} \sum_{Q \in S_{j,k}} uv (Q)
\]
\[
= c [uv] A_\infty 2^{-j 2^{-k}} uv \left( \bigcup_{Q \in S_{j,k}} Q \right)
\]
\[
\leq c_n [uv] A_\infty 2^{-j 2^{-k}} uv \left( \left\{ x \in \mathbb{R}^n : M_{L (\log L) \Phi_{\frac{\rho}{\gamma}} (uv)} g > 2^{-k-1} \right\} \right)
\]
\[
\leq c_{n,p} [uv] A_\infty 2^{-j 2^{-k}} \Phi_{\frac{\rho}{\gamma}} (2^{k+1})^p uv (G).
\]
\[
\leq c_{n,p} [uv] A_\infty 2^{-j 2^{-k}} 2^{kp} p \log \left( e + 2^{k+1} \right) \frac{m-h}{\rho} uv (G).
\]
\[
\leq c_{n,p,m} [uv] A_\infty 2^{-j 2^{2(p-1)} k^\frac{m-h}{\rho}} p uv (G).
\]
Combining the estimates above
\[
uv (G) \leq c [v] A_1 \left[ v \right] A_\infty^\frac{h}{\gamma} uv \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j}
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ cc_{n,p,m} [v] A_1 [v] A_\infty^\frac{h}{\gamma} uv \frac{m-h}{A_\infty} 2^{-j 2^{2(p-1)} k^\frac{m-h}{\rho}} p uv (G),
\right. \\
\left. cc_n [v] A_1 [v] A_\infty^\frac{h}{\gamma} uv \frac{m-h}{A_\infty} [u] A_1 [v] A_\infty 2^{-k} j^\frac{\beta}{\gamma} \int_{\mathbb{R}^n} \Phi_{\frac{\rho}{\gamma}} (|f|) uv \right\}
\]
We end up the proof applying Lemma 3.1, with
\[
\gamma_1 = cc_n [v] A_1 [v] A_\infty^\frac{h}{\gamma} uv \frac{m-h}{A_\infty} [u] A_1 [v] A_\infty 2^{-k} j^\frac{\beta}{\gamma} \int_{\mathbb{R}^n} \Phi_{\frac{\rho}{\gamma}} (|f|) uv,
\]
\[
\gamma_2 = cc_{n,p,m} [v] A_1 [v] A_\infty^\frac{h}{\gamma} uv \frac{m-h}{A_\infty} 2^{-j 2^{2(p-1)} k^\frac{m-h}{\rho}} p.
\]
\[
\beta = uv (G), \delta = p-1, \gamma = 1, \rho_1 = \frac{\beta}{\gamma}, \text{ and } \rho_2 = \frac{m-h}{\rho}.
\]
3.4.3. Rough singular integrals. Let us fix a dyadic lattice $D$ and let $D_j$, $j = 1, \ldots, 3^n$ obtained using the $3^n$ dyadic lattices trick. Now let
\[
G = \left\{ \frac{T_\Omega(fv)(x)}{v(x)} > 1 \right\} \setminus \left\{ M_{uv}(f) > \frac{1}{2} \right\}
\]
where $F = \bigcup_{j=1}^{3^n} D_j$ and assume that $\|f\|_{L^1(uv)} = 1$. Then it suffices to prove that
\[
uv(G) \leq c_{n,p}[uv]_{A_\infty} [v]_{A_1} [u]_{A_1(v)} [v]_{A_\infty} \log (e + [uv]_{A_\infty} [v]_{A_1}) + \frac{1}{2} uv(G).
\]
Note that
\[
uv(G) \leq \int \frac{T_\Omega(fv)(x)}{v(x)} uvg \left| \frac{T_\Omega(fv)(x)}{v(x)} \right| = \int \frac{T_\Omega(fv)(x)}{v(x)} uvg \left| \frac{T_\Omega(fv)(x)}{v(x)} \right|
\]
where $g \simeq \chi_G$. Then for $s = 1 + \frac{1}{2\tau_{[uv]_{A_\infty}}}$, notice that, arguing as in [25]
\[
\left( \frac{uv}\nu(uv) (G \cap Q) \right)^{\frac{1}{2}} \lesssim \left( \frac{uv(G \cap Q)}{uv(Q)} \right)^{\frac{1}{2}}
\]
Taking that into account we have that
\[
uv(G) \leq c_n c_T [uv]_{A_\infty} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle fv \rangle_{Q,1} (\chi_G v)_{Q,s} |Q|
\]
\[
\leq c_n c_T [uv]_{A_\infty} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle f \rangle_{Q,1} \frac{v(Q)}{|Q|} (\chi_G v)_{Q,s} |Q|
\]
\[
\leq c_n c_T [uv]_{A_\infty} [v]_{A_1} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle f \rangle_{Q,1} (\chi_G v)_{Q,s} |Q|
\]
\[
\leq c_n c_T [uv]_{A_\infty} [v]_{A_1} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle f \rangle_{Q,1} (\chi_G v)_{Q,s} \left( \frac{(uv)\nu(uv)}{|Q|} \right)^{\frac{1}{2}} |Q|
\]
\[
\leq c_n c_T [uv]_{A_\infty} [v]_{A_1} \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle f \rangle_{Q,1} \left( \frac{(uv)\nu(uv)}{|Q|} \right)^{\frac{1}{2}} uv(Q).
\]
Hence it suffices to prove that for every sparse family $S$,
\[
c_n c_T [uv]_{A_\infty} [v]_{A_1} \sum_{Q \in S} \langle f \rangle_{Q,1} \left( \frac{(uv)\nu(uv)}{|Q|} \right)^{\frac{1}{2}} uv(Q)
\]
\[
\leq c_n, p [uv]_{A_\infty} [v]_{A_1} [u]_{A_1(v)} [v]_{A_\infty} \log (e + [uv]_{A_\infty} [v]_{A_1}) + \frac{1}{2} uv(G).
\]
We split the sparse family as follows $Q \in S_{k,j}$, $k, j \geq 0$ if
\[
2^{j-k} < \langle f \rangle_{Q,1} \leq 2^j
\]
\[
2^{k-1} < \langle g \rangle_{Q,1} \leq 2^k
\]
Let us call
\[
s_{k,j} = \sum_{Q \in S_{k,j}} \langle f \rangle_{Q,1} \left( \frac{(uv)\nu(uv)}{|Q|} \right)^{\frac{1}{2}} uv(Q)
\]
Now we observe that
\[
s_{k,j} \leq \begin{cases} 
c_n 2^{-k}[u]_{A_1(v)} [v]_{A_\infty} \\
2^{-j} 2^k uv(G)
\end{cases}
\]
For the top estimate we argue as we did to get (3.3). For the lower estimate, using Lemma 3.3,
\[ s_{k,j} \leq 2^{-j}2^{-k} \sum_{Q \in S_{j,k}} uv(Q) \]
\[ = c[uv]_{A_{\infty}} 2^{-j}2^{-k}uv \left( \bigcup_{Q \in S_{j,k}} Q \right) \]
\[ \leq c_n [uv]_{A_{\infty}} 2^{-j}2^{-k}uv \left( \{ x \in \mathbb{R}^n : (M_{uv}g)^{1/2} > 2^{-k-1} \} \right). \]

Since \( v \in A_p(u) \), taking into account Lemmas 2.6 and 3.4,
\[ \leq c_n [uv]_{A_{\infty}} 2^{-j}2^{-k}uv \left( \{ x \in \mathbb{R}^n : M_{uv}g > 2^{-2(k+1)} \} \right) \]
\[ \leq c_n [uv]_{A_{\infty}} 2^{-j}2^{-k}2^{2(k+1)}uv(G) \]
\[ \leq c_n [uv]_{A_{\infty}} 2^{-j}2^{k}uv(G). \]

Combining the estimates above
\[ uv(G) \leq c_n c_T \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \]
\[ \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ c_n c_T [uv]_{A_{\infty}} [v]_{A_{1}} [u]_{A_{1}(u)} [v]_{A_{\infty}} 2^{-k}, \ c_n [uv]_{A_{\infty}}^{2} [v]_{A_{1}} 2^{-j}2^{k}uv(G) \right\}. \]

We end the proof using Lemma 3.1, with
\[ \gamma_1 = c_n c_T [uv]_{A_{\infty}} [v]_{A_{1}} [u]_{A_{1}(u)} [v]_{A_{\infty}}, \quad \gamma_2 = c_n [uv]_{A_{\infty}}^{2} [v]_{A_{1}}, \]
\[ \beta = uv(G), \ \delta = 1 \text{ and } \gamma = 3^p. \]

**ACKNOWLEDGMENT**

The authors would like to thank Sheldy Ombrosi for his comments on an earlier version of this manuscript and for some enlightening discussions on this topic.

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