A sharp recovery condition for block sparse signals by block orthogonal multi-matching pursuit

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Abstract We consider the block orthogonal multi-matching pursuit (BOMMP) algorithm for the recovery of block sparse signals. A sharp condition is obtained for the exact reconstruction of block K-sparse signals via the BOMMP algorithm in the noiseless case, based on the block restricted isometry constant (block-RIC). Moreover, we show that the sharp condition combining with an extra condition on the minimum $\ell_2$ norm of nonzero blocks of block K-sparse signals is sufficient to ensure the BOMMP algorithm selects at least one true block index at each iteration until all true block indices are selected in the noisy case. The significance of the results we obtain in this paper lies in the fact that making explicit use of block sparsity of block sparse signals can achieve better recovery performance than ignoring the additional structure in the problem as being in the conventional sense.

Keywords compressed sensing, block sparse signal, block restricted isometry property, block orthogonal multi-matching pursuit

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1 Introduction

The framework of compressed sensing is concerned with the reconstruction of unknown sparse signals from an underdetermined linear system in [3, 4, 15, 16, 39–41]. More concretely, this can be described as

$$y = Ax + e,$$  \hspace{1cm} (1.1)

where $y \in \mathbb{R}^m$ is a vector of measurements, the matrix $A \in \mathbb{R}^{m \times n}$ with $m \ll n$ is a known sensing matrix, the vector $x \in \mathbb{R}^n$ is an unknown $K$-sparse signal ($K \ll n$) and $e \in \mathbb{R}^m$ is a vector of measurement errors. The goal is to recover the unknown signal $x$ based on $y$ and $A$. It has triggered different efficient methods which can be proved to recover unknown $K$-sparse signals $x$ under a variety of different conditions on the sensing matrix $A$ [1, 8, 30–32, 34, 36, 46].

In this paper, we consider the unknown signal $x$ of (1.1) that exhibits additional structure in the form of the nonzero coefficients occurring in blocks. Such a signal is called block sparse signal [18,
19]. We explicitly take this block structure into account to recover block sparse signals through the BOMMP algorithm. Block sparse signals arise naturally in many fields including DNA microarrays [33], equalization of sparse communication [10], multi-band signals [27, 28] and the multiple measurement vector (MMV) problem [5, 11, 18, 26, 38].

Following [17, 18], a block sparse signal $x \in \mathbb{R}^n$ over $\mathcal{I} = \{d_1, d_2, \ldots, d_l\}$ is a concatenation of $l$ blocks of length $d_i$ ($i = 1, 2, \ldots, l$), i.e.,

$$x = [x_1 \cdots x_{d_1} \ x_{d_1+1} \cdots x_{d_1+d_2} \cdots x_{n-d_i+1} \cdots x_n]',$$

where $x[i]$ denotes the $i$-th block of $x$ and $n = \sum_{i=1}^l d_i$. $x$ is called block $K$-sparse if $x[i]$ has nonzero $\ell_2$ norm for at most $K$ indices $i$, i.e., $\sum_{i=1}^l I(||x[i]||_2 > 0) \leq K$, where $I(\cdot)$ is an indicator function. Denote $\|x\|_{2,0} = \sum_{i=1}^l I(||x[i]||_2 > 0)$ and $T = \text{block-supp}(x) = \{i : \|x[i]\|_2 > 0, i = 1, 2, \ldots, l\}$. Then a block $K$-sparse signal $x$ satisfies $\|x\|_{2,0} \leq \delta$ and $|T| \leq \delta$. If $d_i = 1$ ($i = 1, 2, \ldots, l$), the block sparse signal reduces to the conventional sparse signal [2, 6, 9, 12–14, 48, 49]. Similar to (1.2), the sensing matrix $A$ can be expressed as a concatenation of $l$ column blocks, i.e.,

$$A = [A_1 \cdots A_{d_1} \ A_{d_1+1} \cdots A_{d_1+d_2} \cdots A_{n-d_i+1} \cdots A_n],$$

where $A_i$ is the $i$-th column of $A$ for $i = 1, 2, \ldots, n$.

To recover block sparse signals $x$, one approach to exploiting block sparsity is the mixed $\ell_2/\ell_0$ norm minimization problem given by

$$\min_x \|x\|_2 \text{subject to } \|Ax - y\|_2 \leq \varepsilon,$$

where $\varepsilon$ is the noise level. In the noiseless case, $\varepsilon = 0$. The minimization problem is a suitable extension of the standard $\ell_0$-minimization problem. This minimization problem is also NP-hard. Instead, some efficient methods making explicit use of block sparsity to imply the recovery of block sparse signals include the mixed $\ell_2/\ell_0$ norm minimization [17, 18, 22, 23, 35], the mixed $\ell_2/\ell_p$ ($0 < p < 1$) norm minimization [25, 42, 43], the block matching pursuit (BMP) algorithm [17], the block orthogonal matching pursuit (BOMP) algorithm [17, 20, 37, 45], the sparsity adaptive regularized orthogonal matching pursuit algorithm [50] and the block version of StOMP algorithm [21].

To investigate the recovery of block sparse signals, Eldar and Mishali introduced the notion of the block restricted isometry property (block-RIP) and also demonstrated that the block-RIP has advantages over standard RIP in [18]. The sensing matrix $A$ satisfies the block-RIP of order $K$ if there exist parameters $\delta_K |_{\mathcal{I}} \in [0, 1)$ such that $(1 - \delta_K |_{\mathcal{I}})\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_K |_{\mathcal{I}})\|x\|_2^2$ holds for all block $K$-sparse signals $x$ over $\mathcal{I}$, where the smallest constant $\delta_K |_{\mathcal{I}}$ is called as the block restricted isometry constant (block-RIC) of $A$. By abuse of notation, we use $\delta_K$ for the block-RIC $\delta_K |_{\mathcal{I}}$ when it is clear from the context.

This paper focuses on the BOMMP algorithm firstly proposed in [47] and described in Table 1, which is a natural extension of the BOMP algorithm. The BOMP algorithm only selects one correct block index at each iteration. However, the BOMMP algorithm identifies $N$ ($N \geq 1$) block indices which contain at least one correct block index from the block support of the block sparse signal $x$ per iteration. In [47], the block-RIC

$$\delta_{K+(N-2)k+N} < \frac{1}{1 + \sqrt{\frac{K-k+1}{N}}}, \quad 1 \leq k \leq K$$

is proved to be sufficient for the BOMMP to recover block $K$-sparse signals and simulations demonstrate the recovery performance of the BOMMP overtaking those of the BOMP and BMP.

In this paper, we provide a sharp sufficient condition of the reconstruction of block $K$-sparse signals through the BOMMP. This is the uniform recovery. We consider nonuniform support recovery in the block sparse signal setting when the sensing matrix is a random matrix or a structure random matrix in future work, as typically done in [24]. In the noiseless case, we prove that the condition with the
we also show $\delta$ is sufficient to perfectly recover any block $K$-sparse signals via the BOMMP. Moreover, we also prove that the sufficient condition (1.4) is optimal, in the sense that for any given $K \in \mathbb{N}^+$, there exists a matrix $A$ satisfying $\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{N}+1}}$ such that BOMMP may fail to recover some block $K$-sparse signals $x$. Lastly, we also show $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N}+1}}$ together with a condition on the minimum $\ell_2$ norm of nonzero blocks of the $K$-sparse signal $x$ can ensure the BOMMP algorithm selects at least one true block index at each iteration until all true block indices are selected in the noisy case. If $N = 1$, then (1.4) is a sharp sufficient condition for the recovery of block sparse signals by the BOMP [45]. When $d_i = 1$ ($i = 1, 2, \ldots, l$), (1.4) ensures that the gOMP or OMPM stably recovers the sparse signal [7,44] and is also sharp [7]. As $N = 1$ and $d_i = 1$ ($i = 1, 2, \ldots, l$), the condition (1.4) turns to be a sharp sufficient condition for sparse signal recovery through OMP [29].

We begin, in Section 2, by giving some notation and some basic lemmas that are used. The main results and their proofs are given in Section 3.

### 2 Notation and lemmas

Throughout this paper, let $\Gamma \subseteq \{1, 2, \ldots, l\}$ be a block index set and $\Gamma^c$ be the complementary set of $\Gamma$. Define a mixed $\ell_2/\ell_p$ norm with $p = 1, 2, \infty$ as $\|x\|_2,\,p = \|w\|_p$, where $w \in \mathbb{R}_+^l$ with $w_i = \|x[i]\|_2$ for $i = 1, 2, \ldots, l$. Note that $\|x\|_{2,2} = \|x\|_2$. Let $I_\Gamma = \{d_i : i \in \Gamma\}$ and the block vector $x_\Gamma \in \mathbb{R}^{\sum_{i \in I_d} d_i}$ over $I_\Gamma$ be a concatenation of $\|\Gamma\|$ blocks of length $d_i$ ($i \in \Gamma$). In addition, let the block vector $\tilde{x}_\Gamma$ over $\mathcal{I}$ be a concatenation of $l$ blocks of length $d_i$ ($d_i \in \mathbb{Z}$) satisfying

$$\tilde{x}_\Gamma[i] = \begin{cases} x_\Gamma[i], & i \in \Gamma, \\ 0 & \text{else, } i \in \{1, 2, \ldots, l\} - \Gamma, \end{cases}$$

where $i = 1, 2, \ldots, l$. Similarly, let $A_\Gamma$ over $\mathcal{I}_d$ be the submatrix of $A$, which is a concatenation of $\|\Gamma\|$ column blocks of length $d_i$ ($i \in \Gamma$). Let $e_i \in \mathbb{R}^n$ be the $i$-th coordinate unit vector and $I_d$ be the $d$-dimensional identity matrix, where $d$ is a positive integer.

Let $\alpha_{NK+1}^k$ be the $N$-th largest value of the set $\{\|A'[i]p_{r}\|_2 : i \in (T \cup \Lambda_\Gamma)^c\}$ and $\beta_{k+1}^1$ be the largest value of the set $\{\|A[i]p_{r}\|_2 : i \in T - \Lambda_\Gamma\}$ in the $(k+1)$-th iteration of the BOMMP algorithm. Let $W_{k+1} \subseteq (T \cup \Lambda_\Gamma)^c$ be a set of $N$ block indices which correspond to $N$ largest values of the set $\{\|A'[i]p_{r}\|_2 : i \in (T \cup \Lambda_\Gamma)^c\}$.

$A_\Lambda^k$ represents the pseudo-inverse of $A_\Lambda$. In particular, when $A_\Lambda$ is of full column rank, $A_\Lambda^k = (A_\Lambda^t A_\Lambda)^{-1} A_\Lambda^t$. Moreover, $P_{\Lambda k} = A_\Lambda A_\Lambda^k$ and $P_{\Lambda k}^k = I - P_{\Lambda k}$ denote two orthogonal projection functions.
operators which project a given vector orthogonally onto the spanned space by all column blocks of $A_k$ and onto its orthogonal complement respectively.

First, we recall some useful lemmas in [45].

**Lemma 2.1.** For any $K_1 \leq K_2$, if the sensing matrix $A$ satisfies the block-RIP of order $K_2$, then $\delta_{K_1} \leq \delta_{K_2}$.

**Lemma 2.2.** Let the sensing matrix $A \in \mathbb{R}^{m \times n}$ satisfy the block-RIP of order $K$ and $\Gamma$ be a block index set with $|\Gamma| \leq K$. Then there is $\|A^*_x\|^2 \leq (1 + \delta_K)x^2$ for any $x \in \mathbb{R}^m$.

Next, we prove the following lemma that plays an important role during our analysis. It is rooted in [7, 29].

**Lemma 2.3.** For any nonempty index subset $W$ and any constants $S, C > 0$, let $t = \pm \sqrt{S + 1} - 1$ and
ti = \begin{cases} 
C/2 (1 - t^2) & \forall i \in W \end{cases}
(2.1)

Then for any vector $h_i \in \mathbb{R}^n$, we have $t^2 < 1$ and

$$
\left\| A\left(x + \sum_{i \in W} t_i h_i \right) \right\|_2^2 - \left\| A\left(t^2 x - \sum_{i \in W} t_i h_i \right) \right\|_2^2 = (1 - t^4) \left( \langle Ax, Ax \rangle - C \sum_{i \in W} \langle Ax, Ah_i \rangle \right).
$$

**Proof.** For $t = \pm \sqrt{S + 1} - 1$, it follows that

$$
t^2 = \frac{(\sqrt{S + 1} - 1)^2}{S} < 1.
$$

By the following chain of equalities and the definition of $t_i (i \in W)$, we have that

$$
\left\| A\left(x + \sum_{i \in W} t_i h_i \right) \right\|_2^2 - \left\| A\left(t^2 x - \sum_{i \in W} t_i h_i \right) \right\|_2^2 = (1 - t^4) \langle Ax, Ax \rangle + 2(1 + t^2) \sum_{i \in W} t_i \langle Ax, Ah_i \rangle
$$

$$
= (1 - t^4) \langle Ax, Ax \rangle - \frac{2}{1 - t^2} (1 - t^2) C \sum_{i \in W} \langle Ax, Ah_i \rangle
$$

$$
= (1 - t^4) \langle Ax, Ax \rangle - C \sum_{i \in W} \langle Ax, Ah_i \rangle.
$$

This completes the proof of Lemma 2.3.

\[\square\]

### 3 Main results

In this section, we consider the reconstruction of block $K$-sparse signals through the BOMMP algorithm. In Subsection 3.1, we provide a sharp sufficient recovery condition of block $K$-sparse signals through the BOMMP in the noiseless case. The block support recovery in the noisy case is discussed in Subsection 3.2.

#### 3.1 Noiseless case

First, we define the BOMMP makes a “success” in an iteration if at least one correct block index of $N$ block indices is selected. It is clear that if $\beta^k > \alpha_N^k (1 \leq k \leq K)$, then the BOMMP makes a success in the iteration. The following theorems provide a sufficient condition to guarantee the BOMMP algorithm success.
Theorem 3.1. Suppose \( x \) is a block \( K \)-sparse signal and the sensing matrix \( A \) satisfies the block-RIP of \( K + N \) order with the block-RIC

\[
\delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N} + 1}}.
\] (3.1)

Then the BOMMP algorithm makes a success in the first iteration.

Remark 3.1. As \( N = 2 \), the bound (3.1) is \( \delta_{K+2} < \frac{1}{\sqrt{\frac{K}{2} + 1}} \) for the first iteration of the BOMMP. In this case, (1.3) takes the form \( \delta_{K+2} < \frac{1}{1+\sqrt{\frac{K}{2}}} < \frac{1}{\sqrt{\frac{K}{2}+1}} \), i.e., the sufficient condition (3.1) is weaker than that in [47] for the first iteration of the BOMMP.

Proof. It is clear that we only need to consider the block \( K \)-sparse signal \( x \neq 0 \) in the proof. Recall the definitions of \( W_1 \), \( \alpha_N^1 \) and \( \beta_1 \). \( W_1 \) is the set of \( N \) block indices which correspond to \( N \) largest values of the set \( \{ ||A'[i]r^0||_2 : i \in T^c \} \). \( \alpha_N^1 \) is the \( N \)-th largest value of the set \( \{ ||A'[i]r^0||_2 : i \in T^c \} \). \( \beta_1^1 \) is the largest value of the set \( \{ ||A'[i]r^0||_2 : i \in T \} \).

Firstly, we consider \( \alpha_N^1 > 0 \), then \( ||A'[i]Ax||_2 > 0 \) for \( \forall i \in W_1 \). Hence, we have that

\[
\alpha_N^1 = \min \{ ||A'[i]Ax||_2 : i \in W_1 \} = \min \{ \langle A'[i]Ax, A'[i]Ax \rangle : i \in W_1 \} = \min \{ \langle Ax, A[i]A[i] \rangle : i \in W_1 \} = \min \{ \langle Ax, A\tilde{a}(i) \rangle : i \in W_1 \} \leq \sum_{i \in W_1} \langle Ax, A\tilde{a}(i) \rangle \frac{N}{2},
\] (3.2)

where \( a(i) = \frac{A'[i]Ax}{||A'[i]Ax||_2} \) with \( ||a(i)||_2 = 1 \). It follows from the definition of \( \beta_1^1 \) and \(|T| \leq K\) that

\[
\langle Ax, Ax \rangle = \sum_{i \in T} \langle A[i]x[i], Ax \rangle = \sum_{i \in T} \langle x[i], A'[i]Ax \rangle \leq \sum_{i \in T} ||x[i]||_2 ||A'[i]Ax||_2 \\
\leq \beta_1^1 ||x||_{2,1} \leq \beta_1^1 \sqrt{K} ||x||_2 = \beta_1^1 \sqrt{K} ||x||_2.
\] (3.3)

Let \( t = -\sqrt{\frac{K}{N} + 1} \) and \( t_i = -\sqrt{\frac{K}{2N}} (1-t^2) ||x||_2 \), where \( i \in W_1 \subseteq T^c \) with \( |W_1| = N \). Then we have that

\[
t^2 = \frac{\sqrt{\frac{N}{K} + 1} - \sqrt{\frac{N}{K} + 1}}{\sqrt{\frac{N}{K} + 1} + 1} < 1
\]

and

\[
\sum_{i \in W_1} t_i^2 = \left( \frac{\sqrt{K}}{2N} (1-t^2) ||x||_2 \right)^2 \frac{N}{2} = \frac{K}{4N} (1-t^2)^2 ||x||_2^2 = \frac{K}{4N} \left( 1 - \frac{\sqrt{\frac{N}{K} + 1} - \sqrt{\frac{N}{K} + 1}}{\sqrt{\frac{N}{K} + 1} + 1} \right)^2 ||x||_2^2 \\
= \frac{K}{N} \left( \frac{1}{\sqrt{\frac{N}{K} + 1} + 1} \right)^2 ||x||_2^2 = \frac{\sqrt{\frac{N}{K} + 1} - \sqrt{\frac{N}{K} + 1}}{\sqrt{\frac{N}{K} + 1} + 1} ||x||_2^2 = t^2 ||x||_2^2.
\] (3.4)
From (3.2), (3.3), Lemma 2.3 and \( t^2 < 1 \), it is clear that

\[
(1 - t^4)\sqrt{K}\|x\|_2(\beta_1^1 - \alpha_N^1) \geq (1 - t^4)\left((Ax, Ax) - \sum_{i \in W_1} \sqrt{K}\|x\|_2 \frac{(Ax, A\tilde{a}_{(i)})}{N}\right)
= \left\|A\left(x + \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right)\right\|^2_2 - \left\|A\left(t^2x - \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right)\right\|^2_2. \tag{3.5}
\]

Because the sensing matrix \( A \) satisfies the block-RIP of order \( KN + 1 \) with \( \delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N} + 1}} \), \( x \neq 0 \) with the block-supp(\( x \)) \( \subseteq T \) and \( \|a_{(i)}\|_2 = 1 \) with \( i \in W_1 \subseteq T^c \), it follows from (3.4) that

\[
\left\|A\left(x + \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right)\right\|^2_2 - \left\|A\left(t^2x - \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right)\right\|^2_2 \\
\geq (1 - \delta_{K+N})\left(\left\|x + \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right\|^2_2\right) - (1 + \delta_{K+N})\left(\left\|t^2x - \sum_{i \in W_1} t_i \tilde{a}_{(i)}\right\|^2_2\right) \\
= (1 - \delta_{K+N})\left(\|x\|^2_2 + \sum_{i \in W_1} t_i^2\right) - (1 + \delta_{K+N})\left(\|t^2x\|^2_2 + \sum_{i \in W_1} t_i^2\right) \\
= (1 - \delta_{K+N})(1 + t^2)\|x\|^2_2 - (1 + \delta_{K+N})(t^4 + t^2)\|x\|^2_2 \\
= (1 - t^4)\|x\|^2_2 - \delta_{K+N}(1 + t^2)\|x\|^2_2 \\
\geq (1 + t^2)^2\|x\|^2_2 \frac{1 - t^2}{1 + t^2 - \delta_{K+N}} \\
> 0.
\]

It follows from the above two inequalities that \( \beta_1^1 > \alpha_N^1 \), which represents that the BOMMP algorithm selects at least one block index from the block support \( T \) under \( \alpha_N^1 > 0 \). As the above discussion, we have that \( \beta_1^1 > 0 \). When \( \alpha_N^1 = 0 \), it is clear that \( \beta_1^1 > \alpha_N^1 \).

As mentioned, if \( \delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N} + 1}} \), then the BOMMP algorithm makes a success in the first iteration. \( \square \)

**Theorem 3.2.** Suppose the BOMMP algorithm has performed \( k \) iterations successfully, where \( 1 \leq k < K \). Then the BOMMP algorithm will be successful for the \( (k + 1) \)-th iteration if the sensing matrix \( A \) satisfies the block-RIP of order \( NK + 1 \) with the block-RIC \( \delta_{NK+1} \) fulfilling

\[
\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}.
\]

**Proof.** For the BOMMP algorithm, \( r^k = P_{A^k}^\perp y \) is orthogonal to each block of \( A_{\lambda^k} \) then

\[
r^k = P_{A^k}^\perp y = P_{A^k}^\perp A_T x_T \\
= P_{A^k}^\perp (A_{T \setminus \lambda^k} x_{T \setminus \lambda^k} + A_{T \cap \lambda^k} x_{T \cap \lambda^k}) \\
= P_{A^k}^\perp A_{T \setminus \lambda^k} x_{T \setminus \lambda^k} \\
= A_{T \setminus \lambda^k} x_{T \setminus \lambda^k} - P_{A^k} A_{T \setminus \lambda^k} x_{T \setminus \lambda^k} \\
= A_{T \setminus \lambda^k} x_{T \setminus \lambda^k} - A_{\lambda^k} z_{\lambda^k} \\
= A_{T \cup \lambda^k} \omega_{T \cup \lambda^k},
\]
where we used the fact that $P_{A_k^*}A_{T-A_k^*}x_{T-A_k^*} \in \text{span}(A_{A_k^*})$, so $P_{A_k^*}A_{T-A_k^*}x_{T-A_k^*}$ can be written as $A_{A_k^*}z_{A_k^*}$ for some $z_{A_k^*} \in \mathbb{R}^\sum_{i \in A_k^*}d_i$ and $\omega_{T \cup A_k^*}$ is given by

$$\omega_{T \cup A_k^*} = \begin{pmatrix} x_{T-A_k^*} \\ -z_{A_k^*} \end{pmatrix}.$$ 

For the $(k+1)$-th iteration, if $T - A_k^* = \emptyset$, then $T \subseteq A_k^*$. Hence, the original block $K$-sparse signal $x$ has already been recovered exactly. Now assuming $T - A_k^* \neq \emptyset$, then $\omega_{T \cup A_k^*} \neq \emptyset$. In the remainder of the proof, we consider firstly $\alpha_N^{k+1} > 0$, then $\|A'[i]r_k\|_2 > 0$ for all $i \in W_{k+1}$. We take $a_{i} = \frac{A'[i]r_k}{\|A'[i]r_k\|_2}$, then $\|a_{i}\|_2 = 1$. In view of the definition of $\alpha_N^{k+1}$, we have that

$$\alpha_N^{k+1} = \min\{\|A'[i]r_k\|_2 : i \in W_{k+1}\}$$

$$= \min\left\{\left\langle A'[i]r_k, \frac{A'[i]r_k}{\|A'[i]r_k\|_2} \right\rangle : i \in W_{k+1}\right\}$$

$$= \min\{\langle r_k, A[i]a_{i}\rangle : i \in W_{k+1}\}$$

$$= \min\{\langle A_{T \cup A_k^*} \omega_{T \cup A_k^*}, A a_{i}\rangle : i \in W_{k+1}\}$$

$$\leq \frac{\sum_{i \in W_{k+1}} \langle A a_{i} \omega_{T \cup A_k^*}, A a_{i}\rangle}{N}. \quad (3.6)$$

Combining the definition of $\beta_1^{k+1}$ with $A_{A_k^*}r_k = 0$, we derive that

$$\beta_1^{k+1} = \max\{\|A'[i]r_k\|_2 : i \in T - A_k^*\}$$

$$= \|A'_{T-A_k^*}r_k\|_{2,\infty}$$

$$= \|A'_{T \cup A_k^*}r_k\|_{2,\infty}$$

$$= \|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2,\infty}. \quad (3.7)$$

Notice the fact that

$$\|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2,\infty} \geq \frac{1}{\sqrt{K}} \|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2,2}$$

$$= \frac{1}{\sqrt{K}} \|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2,2}$$

$$= \frac{1}{\sqrt{K}} \|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2}. \quad (3.8)$$

From (3.7) and (3.8), it follows that

$$\langle A \omega_{T \cup A_k^*}, A \omega_{T \cup A_k^*} \rangle = \langle A_{T \cup A_k^*} \omega_{T \cup A_k^*}, A_{T \cup A_k^*} \omega_{T \cup A_k^*} \rangle$$

$$= \langle A_{T \cup A_k^*} A_{T \cup A_k^*} \omega_{T \cup A_k^*}, \omega_{T \cup A_k^*} \rangle$$

$$\leq \|A'_{T \cup A_k^*}A_{T \cup A_k^*}\omega_{T \cup A_k^*}\|_{2} \|\omega_{T \cup A_k^*}\|_2$$

$$\leq \sqrt{K} \beta_1^{k+1} \|\omega_{T \cup A_k^*}\|_2$$

$$= \sqrt{K} \beta_1^{k+1} \|\omega_{T \cup A_k^*}\|_2. \quad (3.9)$$

Similar to the proof of Theorem 3.1, let $t = -\frac{\sqrt{K} + 1}{\sqrt{K}^2}$ and

$$t_i = -\frac{\sqrt{K}}{2N} (1 - t^2) \|\omega_{T \cup A_k^*}\|_2, \quad i \in W_{k+1} \subseteq (A_k^* \cup T)^c.$$ 

By (3.6), (3.9) and Lemma 2.3, we have that

$$(1 - t^4) \sqrt{K} \|\omega_{T \cup A_k^*}\|_2 (\beta_1^{k+1} - \alpha_N^{k+1})$$
\[
\frac{(1-t^4)\left(\langle A\tilde{\omega}_{T,UA^k}, A\tilde{\omega}_{T,UA^k} \rangle - \sqrt{K}\|\tilde{\omega}_{T,UA^k}\|_2 \sum_{i \in W_{k+1}} \langle A\tilde{\omega}_{T,UA^k}, A\tilde{a}_i \rangle \right)}{N} \\
= \left\| A\left(\tilde{\omega}_{T,UA^k} + \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right) \right\|^2_2 - \left\| A\left(t^2\tilde{\omega}_{T,UA^k} - \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right) \right\|^2_2.
\] (3.10)

Let \(l = |T \cap \Lambda^k|\). Then \(k \leq l \leq K - 1\) and \(Nk + K - l + N \leq NK + 1\). Since \(A\) satisfies the block-RIP of order \(NK + 1\) with the block-RIC \(\delta_{NK+1} \neq 0\) with the block-sup\(\supp(\tilde{\omega}_{T,UA^k}) \subseteq T \cup \Lambda^k\) and \(\|a_i\|_2 = 1\) with \(i \in W_{k+1} \subseteq (T \cup \Lambda^k)^c\), it follows from Lemma 2.1 and \(\sum_{i \in W_{k+1}} t_i^2 = t^2\|\tilde{\omega}_{T,UA^k}\|_2^2\) that

\[
\left\| A\left(\tilde{\omega}_{T,UA^k} + \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right) \right\|^2_2 - \left\| A\left(t^2\tilde{\omega}_{T,UA^k} - \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right) \right\|^2_2 \\
\geq (1 - \delta_{NK+K-l+N})\left(\left\|\tilde{\omega}_{T,UA^k} + \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right\|^2_2 \right) \\
-(1 + \delta_{NK+K-l+N})\left(\left\|t^2\tilde{\omega}_{T,UA^k} - \sum_{i \in W_{k+1}} t_i\tilde{a}_i \right\|^2_2 \right) \\
= (1 - \delta_{NK+K-l+N})\left(\|	ilde{\omega}_{T,UA^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right) \\
-(1 + \delta_{NK+K-l+N})\left(t^4 \|	ilde{\omega}_{T,UA^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right) \\
= (1 - \delta_{NK+K-l+N})\left(1 + t^2\right)\|	ilde{\omega}_{T,UA^k}\|_2^2 - (1 + \delta_{NK+K-l+N})\left(t^4 + t^2\right)\|	ilde{\omega}_{T,UA^k}\|_2^2 \\
= (1 + t^2)^2\|	ilde{\omega}_{T,UA^k}\|_2^2 \left(\frac{1 - t^2}{1 + t^2} - \delta_{NK+K-l+N} \right) \\
\geq (1 + t^2)^2\|	ilde{\omega}_{T,UA^k}\|_2^2 \left(\frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right).
\]

Combining the fact that

\[
\frac{1 - t^2}{1 + t^2} = \frac{1}{\sqrt{\frac{1}{N} + 1}}
\]

with the condition \(\delta_{NK+1} < \frac{1}{\sqrt{\frac{1}{N} + 1}}\), it follows from \(t^2 < 1\) and \(\tilde{\omega}_{T,UA^k} \neq 0\) that

\[
\left(1 - t^4\right)\sqrt{K}(\beta_1^{k+1} - \alpha_N^{k+1}) \geq (1 + t^2)^2 \left(\frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right)\|	ilde{\omega}_{T,UA^k}\|_2 \\
\geq (1 + t^2)^2 \left(\frac{1}{\sqrt{\frac{1}{N} + 1}} - \delta_{NK+1} \right)\|	ilde{\omega}_{T,UA^k}\|_2 \\
> 0,
\]

i.e., \(\beta_1^{k+1} > \alpha_N^{k+1}\), which ensures that the set \(T^k\) contains at least one correct block index in the \((k+1)\)-th iteration of the BOMMP algorithm under \(\alpha_N^{k+1} > 0\). For \(\alpha_N^{k+1} = 0\), it is obvious that \(\beta_1^{k+1} > \alpha_N^{k+1}\) based on \(\omega_{T,UA^k} \neq 0\). We have completed the proof of the theorem.

Now combining the conditions for success in the first iteration in Theorem 3.1 with that in non-initial iterations in Theorem 3.2, we obtain overall sufficient condition to guarantee the perfect recovery of block \(K\)-sparse signals via the BOMMP algorithm in the following theorem.

**Theorem 3.3.** Suppose \(x\) is a block \(K\)-sparse signal and the sensing matrix \(A\) satisfies the block-RIP of order \(NK + 1\) with the block-RIC \(\delta_{NK+1}\) fulfilling \(\delta_{NK+1} < \frac{1}{\sqrt{\frac{1}{N} + 1}}\). Then the BOMMP algorithm can recover the block sparse signal \(x\) exactly from \(y = Ax\).
Proof. For \( N \geq 1, K \geq 1 \) and \( N \leq \min\{K, \frac{d}{K}\} \), then \( K + N \leq NK + 1 \). It follows from Lemma 2.1 that
\[
\delta_{K+N} \leq \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}.
\]
Therefore, under the sufficient condition \( \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}} \), the BOMMP algorithm can recover perfectly any block \( K \)-sparse signals from \( y = Ax \) based on Theorems 3.1 and 3.2.

\[\blacksquare\]

Next, we prove that the condition \( \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}} \) is optimal.

**Theorem 3.4.** For any given \( K \in \mathbb{N}^+ \), there are a block \( K \)-sparse signal \( x \) and a matrix \( A \) satisfying
\[
\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{N} + 1}}
\]
such that the BOMMP may fail.

In order to prove Theorem 3.4, for a positive integer \( d \), we firstly investigate the following matrix \( A(d) \in \mathbb{R}^{d(NK+1) \times d(NK+1)} \):
\[
A(d) = \begin{pmatrix}
0 & \cdots & 0 & \frac{1}{b}I_d & \cdots & \frac{1}{b}I_d \\
\sqrt{\frac{K}{K+N}}I_{dK} & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \frac{1}{b}I_d & \cdots & \frac{1}{b}I_d \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & I_{dN}
\end{pmatrix},
\]
where \( b = \sqrt{K(K+N)} \). Then we have that
\[
A'(d)A(d) = \begin{pmatrix}
0 & \cdots & 0 & sI_d & \cdots & sI_d \\
\frac{K}{K+N}I_{dK} & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & sI_d & \cdots & sI_d \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
sI_d & \cdots & sI_d & 0 & \cdots & 0 & (1+s)I_d & \cdots & sI_d \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
sI_d & \cdots & sI_d & 0 & \cdots & 0 & sI_d & \cdots & (1+s)I_d
\end{pmatrix},
\]
where \( s = \frac{1}{K+N} \). By elementary transformation of determinant, we have that
\[
|A'(d)A(d) - \lambda I_{d(NK+1)}|
\]
By the first row of (3.11), we prove (3.12). As for (3.11)

\[
\begin{pmatrix}
0 & \cdots & 0 & sI_d & \cdots & NsI_d \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
s_1I_{dK} & \cdots & \cdots & \cdots & s_2I_{d(NK+1-N-K)} & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \cdots & \cdots & \vdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \cdots & \cdots & \vdots & \cdots & 0 \\
KsI_d & \cdots & sI_d & 0 & \cdots & 0 & sI_d & \cdots & s_3I_d
\end{pmatrix},
\]

where \( s_1 = \frac{K}{K+N} - \lambda, \) \( s_2 = 1 - \lambda \) and \( s_3 = 1 + \frac{N}{K+N} - \lambda \). Next, we claim that

\[
A'(d)A(d) - \lambda I_{d(NK+1)} = (1 - \lambda)^{d(NK-K)} \left( \frac{K}{K+N} - \lambda \right)^{d(K-1)} \left( \lambda^2 - 2\lambda + \frac{K}{K+N} \right)^d.
\]

By inductive method, we prove (3.12). As for \( d = 1 \), by a direct calculation, it follows from (3.11) that

\[
A'(1)A(1) - \lambda I_{NK+1} = (1 - \lambda)^{NK-K} \left( \frac{K}{K+N} - \lambda \right)^{K-1} \left( \lambda^2 - 2\lambda + \frac{K}{K+N} \right).
\]

For \( d - 1 \) (\( d \geq 2 \)), suppose

\[
A'(d-1)A(d-1) - \lambda I_{(d-1)(NK+1)} = (1 - \lambda)^{(d-1)(NK-K)} \left( \frac{K}{K+N} - \lambda \right)^{(d-1)(K-1)} \left( \lambda^2 - 2\lambda + \frac{K}{K+N} \right)^{(d-1)}.
\]

For \( d \geq 2 \), we expand the determinant (3.11) by the first column, then expand the remaining determinant by the first row of \( s_1I_d, s_2I_d \) and \( s_3I_d \). Hence, we have that

\[
A'(d)A(d) - \lambda I_{d(NK+1)}
= (-1)^{1+1} \left( \frac{K}{K+N} - \lambda \right) \left( -1 \right)^{(d-1)+1+(d-1)+1} \left( \frac{K}{K+N} - \lambda \right)
\]

\[
\cdots (-1)^{(d-1)(d-1)+1+(K-1)(d-1)+1} \left( \frac{K}{K+N} - \lambda \right)^{(d-1)(K-1)} (-1)^{K(d-1)+1+K(d-1)+1} (1 - \lambda)
\]

\[
\cdots (-1)^{(NK-1)(d-1)+1+(NK-1)(d-1)+1} (1 - \lambda) (-1)^{NK(d-1)+1+NK(d-1)+1} \left( 1 + \frac{N}{K+N} - \lambda \right)
\]

\[
A'(d-1)A(d-1) - \lambda I_{(d-1)(NK+1)}
= \left( \frac{K}{K+N} - \lambda \right)^{K-1} (1 - \lambda)^{NK+1-K-N+1} \left( 1 + \frac{N}{K+N} - \lambda \right).
\]
\[ A'(d - 1)A(d - 1) - \lambda I_{(d-1)(NK+1)} \]
\[- \frac{K}{K + N} \left( \left( \frac{K}{K + N} - \lambda \right)^{K-1} (1 - \lambda)^{NK+1-K-N+N-1} \frac{N}{K + N} \right) \times A'(d - 1)A(d - 1) - \lambda I_{(d-1)(NK+1)} \]
\[= (1 - \lambda)^{d(NK-K)} \left( \frac{K}{K + N} - \lambda \right)^{d(K-1)} \left( \lambda^2 - 2\lambda + \frac{K}{K + N} \right)^d. \]

Therefore, we have completed the proof of (3.12).

Now, we present the proof of Theorem 3.4.

**Proof of Theorem 3.4.** For convenience, we assume that the block \( K \)-sparse signal \( x \) consists of \( NK + 1 \) blocks each having identical length of \( d \), i.e., \( n = d(NK + 1) \). For any given positive integer \( K \), let \( A = A(d) \). By (3.12), it is clear that \( \frac{K}{K + N}, 1, 1 - \frac{1}{\sqrt{K + 1}} \) and \( 1 + \frac{1}{\sqrt{K + 1}} \) are eigenvalues of \( A'A \) with multiplicity of \( d(K - 1) \), \( d(NK - K) \), \( d \) and \( d \), respectively. Moreover, \( 1 - \frac{1}{\sqrt{K + 1}} \) and \( 1 + \frac{1}{\sqrt{K + 1}} \) are the minimum and maximum eigenvalues of \( A'A \), respectively.

So for \( \forall x \in \mathbb{R}^{d(NK+1)} \), we easily derive that
\[
\left( 1 - \frac{1}{\sqrt{K + 1}} \right) \|x\|_2^2 \leq x'A'Ax \leq \left( 1 + \frac{1}{\sqrt{K + 1}} \right) \|x\|_2^2,
\]
i.e.,
\[
\left( 1 - \frac{1}{\sqrt{K + 1}} \right) \|x\|_2^2 \leq \|Ax\|_2^2 \leq \left( 1 + \frac{1}{\sqrt{K + 1}} \right) \|x\|_2^2.
\]

Therefore, we have that \( \delta_{NK+1} \leq \frac{1}{\sqrt{K + 1}} \). Next, we claim that the matrix \( A \) satisfies the block-RIP of order \( NK + 1 \) with the block-RIC \( \delta_{NK+1} = \frac{1}{\sqrt{K + 1}} \). Let \( h \in \mathbb{R}^{NK+1} \) be the eigenvector of \( A'(1)A(1) \) corresponding to the eigenvalue \( 1 + \frac{1}{\sqrt{K + 1}} \) and \( x \in \mathbb{R}^{d(NK+1)} \) with \( x[i] = h_i e_1 \) \((e_1 \in \mathbb{R}^d \) is the first coordinate unit vector) for \( 1 \leq i \leq NK + 1 \). Then we obtain that
\[
x'A'Ax = h'A'(1)A(1)h = \left( 1 + \frac{1}{\sqrt{K + 1}} \right) \|h\|_2^2 = \left( 1 + \frac{1}{\sqrt{K + 1}} \right) \|x\|_2^2.
\]

Therefore, \( A \) satisfies the block-RIC \( \delta_{NK+1} = \frac{1}{\sqrt{K + 1}} \).

Consider the block \( K \)-sparse signal \( x = (e'_1, e'_1, \ldots, e'_1, 0, \ldots, 0)' \in \mathbb{R}^{d(NK+1)} \), i.e., \( T = \text{block-sup}(x) = \{1, 2, \ldots, K\} \). For the first iteration, there are
\[
\|A'[i]v^0\|_2 = \|A'[i]Ax\|_2 = \begin{cases} 
\frac{K}{K + N}, & i \in T, \\
0, & i \in \{K + 1, \ldots, NK + 1 - N\}, \\
\frac{K}{K + N}, & i \in \{NK + 2 - N, \ldots, NK + 1\}.
\end{cases} \tag{3.13}
\]

Therefore, it follows from the definitions of \( \beta_i^1 \) and \( \alpha_i^1 \) and (3.13) that \( \beta_i^1 = \frac{K}{K + N} \) and \( \alpha_i^1 = \frac{K}{K + N} \), i.e., \( \beta_i^1 = \alpha_i^1 \). This implies the BOMMMP may fail to identify at least one correct index in the first iteration. So the BOMMMP algorithm may fail for the given matrix \( A \) and the block \( K \)-sparse signal \( x \). \( \blacksquare \)
3.2 Noisy case

In this subsection, we show that a high order block-RIP condition combining with an extra condition on the minimum $\ell_2$ norm of nonzero blocks of block $K$-sparse signals can guarantee the BOMMP algorithm selects at least one true block index at each iteration until all true block indices are selected in bounded $\ell_2$ noisy setting from $y = Ax + e$. A sufficient condition in terms of the block-RIC $\delta_{NK+1}$ and the minimum $\ell_2$ norm of nonzero blocks of block $K$-sparse signals $x$ is described as follow.

**Theorem 3.5.** Suppose $\|e\|_2 \leq \varepsilon$ and the sensing matrix $A$ satisfies a high order block-RIP with the block-RIC

$$\delta_{NK+1} < \frac{1}{\sqrt{\frac{k}{N} + 1}}$$

(3.14)

Then the BOMMP algorithm with the stopping rule $\|r^k\|_2 \leq \varepsilon$ selects at least one true block index of block $K$-sparse signals $x$ at each iteration until all true block indices are selected if all the nonzero blocks $x[i]$ satisfy

$$\min_{i \in T} \|x[i]\|_2 > \max \left\{ \frac{\sqrt{2K(1+\delta_{NK+1})\varepsilon}}{\sqrt{\frac{k}{N} + 1} - \delta_{NK+1}}, \frac{2\varepsilon}{\sqrt{1 - \delta_{NK+1}}} \right\}$$

(3.15)

**Proof.** Use mathematical induction method to prove the theorem. Suppose the BOMMP performed $k$ ($1 \leq k \leq K - 1$) iterations successfully. Now considering the $(k+1)$-th iteration, we have that

$$r^k = P_{A^{k}}^\perp y = P_{A^{k}}^\perp A_T x_T + P_{A^{k}}^\perp e = A_{T \cup A^k} \omega_{T \cup A^k} + (I - P_{A^k}) e$$

for some $\omega_{T \cup A^k}$ as in the proof of Theorem 3.2. One consider the following two cases.

**Case 1.** $T = \Lambda^k = \emptyset$

This implies $T \subseteq \Lambda^k$. Then the correct support $T$ of the original block $K$-sparse signal $x$ has already been recovered.

**Case 2.** $T - \Lambda^k \neq \emptyset$, i.e., $|T - \Lambda^k| > 1$.

In this case, it is clear that $\omega_{T \cup A^k} \neq 0$. Without loss of generality, we only consider $\alpha_{NK}^{k+1} > 0$, then $\|A'[i]r^k\|_2 > 0$ for all $i \in W_{k+1} \subseteq (T \cup \Lambda^k)^c$. In the following proof, we take $a_{(i)} = \frac{A'[i]A_{T \cup A^k} \omega_{T \cup A^k}}{\|A'[i]A_{T \cup A^k} \omega_{T \cup A^k}\|_2}$ for all $i \in W_{k+1}$.

Using the definition of $\alpha_{NK}^{k+1}$, we have that

$$\alpha_{NK}^{k+1} = \min \{ \|A'[i]r^k\|_2 : i \in W_{k+1} \}$$

$$\leq \min \left\{ \|A'[i]A_{T \cup A^k} \omega_{T \cup A^k}\|_2 + \|A'[i](I - P_{A^k}) e\|_2 : i \in W_{k+1} \right\}$$

$$= \min \left\{ \left\langle A'[i]A_{T \cup A^k} \omega_{T \cup A^k}, \|A'[i]A_{T \cup A^k} \omega_{T \cup A^k}\|_2 \right\rangle + \|A'[i](I - P_{A^k}) e\|_2 : i \in W_{k+1} \right\}$$

$$= \min \{ A_{T \cup A^k} \omega_{T \cup A^k}, A[S(i)] + \|A'[i](I - P_{A^k}) e\|_2 : i \in W_{k+1} \}$$

$$= \min \{ A_{T \cup A^k} \omega_{T \cup A^k}, A[S(i)] + \|A'[i](I - P_{A^k}) e\|_2 : i \in W_{k+1} \}$$

$$\leq \sum_{i \in W_{k+1}} \left(A[S(i)] + \sum_{i \in W_{k+1}} \|A'[i](I - P_{A^k}) e\|_2 \right).$$

(3.16)

By the definition of $\beta_{1}^{k+1}$ and the fact $A_{A^k} r^k = 0$, it follows from (3.8) and (3.9) that

$$\sqrt{K}\|\omega_{T \cup A^k}\|_2 \beta_{1}^{k+1} = \sqrt{K}\|\omega_{T \cup A^k}\|_2 \max \{ \|A'[i]r^k\|_2 : i \in T - \Lambda^k \}$$

$$= \sqrt{K}\|\omega_{T \cup A^k}\|_2 \|A'[T - \Lambda^k]r^k\|_{2, \infty}.$$
Let $t = -\frac{\sqrt{K} + 1 - 1}{\sqrt{K}}$ and $t_i = -\frac{\sqrt{K}}{\sqrt{N}}(1 - t^2)\|\omega_{T,\Lambda^k}\|_2^2$, $i \in W_{k+1} \subseteq (T \cup \Lambda^k)^c$. Then we have

$$\sum_{i \in W_{k+1}} t_i^2 = t^2 \|\omega_{T,\Lambda^k}\|_2^2.$$  \hspace{1cm} (3.18)

It follows from (3.16), (3.17) and $t^2 < 1$ that

$$(1 - t^4)\sqrt{K} \|\omega_{T,\Lambda^k}\|_2^2 (\beta_1^{k+1} - \alpha_N^{k+1})$$

$$\geq (1 - t^4) \left( \sum_{i \in W_{k+1}} t_i \bar{a}_i(t) \right) \|\omega_{T,\Lambda^k}\|_2^2$$

$$- \frac{\sqrt{K} \|\omega_{T,\Lambda^k}\|_2^2 (\sum_{i \in W_{k+1}} (A \bar{w}_{T,\Lambda^k}, A \bar{a}_i(t)) + \sum_{i \in W_{k+1}} \|A[i](I - P_{A_k})e\|_2^2)}{N}$$

$$= \|A(\bar{w}_{T,\Lambda^k} + \sum_{i \in W_{k+1}} t_i \bar{a}_i(t))\|_2^2 - \|A(t^2 \bar{w}_{T,\Lambda^k} - \sum_{i \in W_{k+1}} t_i \bar{a}_i(t))\|_2^2$$

$$- (1 + \delta_{NK+K-l+N}) \left( \|\bar{w}_{T,\Lambda^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right)$$

$$- (1 + \delta_{NK+K-l+N}) \left( t^4 \|\bar{w}_{T,\Lambda^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right)$$

$$\geq (1 - \delta_{NK+K-l+N}) \left( \|\bar{w}_{T,\Lambda^k}\|_2^2 (1 + t^2) - (1 + \delta_{NK+K-l+N}) \|\bar{w}_{T,\Lambda^k}\|_2^2 (t^4 + t^2) \right)$$

$$= (1 - t^4) \|\bar{w}_{T,\Lambda^k}\|_2^2 - \delta_{NK+K-l+N} \|\bar{w}_{T,\Lambda^k}\|_2^2 (1 + t^2)^2$$

$$= (1 + t^2)^2 \|\bar{w}_{T,\Lambda^k}\|_2^2 \left( t^4 (1 + t^2) - \delta_{NK+K-l+N} \right)$$

$$\geq (1 + t^2)^2 \|\bar{w}_{T,\Lambda^k}\|_2^2 \left( t^4 (1 + t^2) - \delta_{NK+K-l+N} \right).$$  \hspace{1cm} (3.20)

As in the proof of Theorem 3.2, $l = |T \cap \Lambda^k|$ then $NK + K - l + N \leq NK + 1$. Because $A$ satisfies the block-RIP with the block-RIC $\delta_{NK+1}$, $\bar{w}_{T,\Lambda^k} \neq 0$ with block-supp($\bar{w}_{T,\Lambda^k}$) $\subseteq T \cup \Lambda^k$, and $\|a_i\|_2 = 1$ for $i \in W_{k+1} \subseteq (T \cup \Lambda^k)^c$, it follows from (3.18) and Lemma 2.1 that

$$\|A(\bar{w}_{T,\Lambda^k} + \sum_{i \in W_{k+1}} t_i \bar{a}_i(t))\|_2^2 - \|A(t^2 \bar{w}_{T,\Lambda^k} - \sum_{i \in W_{k+1}} t_i \bar{a}_i(t))\|_2^2$$

$$\geq (1 - \delta_{NK+K-l+N}) \left( \|\bar{w}_{T,\Lambda^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right)$$

$$- (1 + \delta_{NK+K-l+N}) \left( t^4 \|\bar{w}_{T,\Lambda^k}\|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right)$$

$$\geq (1 - \delta_{NK+K-l+N}) \|\bar{w}_{T,\Lambda^k}\|_2^2 (1 + t^2) - (1 + \delta_{NK+K-l+N}) \|\bar{w}_{T,\Lambda^k}\|_2^2 (t^4 + t^2)$$

$$\geq (1 + t^2)^2 \|\bar{w}_{T,\Lambda^k}\|_2^2 \left( t^4 (1 + t^2) - \delta_{NK+K-l+N} \right)$$

$$\geq (1 + t^2)^2 \|\bar{w}_{T,\Lambda^k}\|_2^2 \left( t^4 (1 + t^2) - \delta_{NK+K-l+N} \right).$$  \hspace{1cm} (3.20)
\[ \|A'(T \cup A^k)(I - P_{A^k})e\|_{2,\infty} = \|A'[j_k](I - P_{A^k})e\|_2. \]

Hence,
\[
\|A'_{T \cup A^k}(I - P_{A^k})e\|_{2,\infty} + \|A'_{T \cup A^k}(I - P_{A^k})e\|_{2,\infty}
\leq \|A'[j_k](I - P_{A^k})e\|_2 + \|A'[j_k](I - P_{A^k})e\|_2
\leq \sqrt{2}\|A'[j_k](I - P_{A^k})e\|_2
\leq \sqrt{2}(1 + \delta_{N})\|e\|_2
\leq \sqrt{2}(1 + \delta_{N})\|e\|_2,
\]
(3.21)

where we use Lemmas 2.1 and 2.2 and the fact
\[
\|e\|_2 \leq \|I - P_{A^k}\|_2 \leq \|e\|_2 \leq \varepsilon.
\]

From (3.19)–(3.21), (3.14) and (3.15), it follows that
\[
(1 - t^4)\sqrt{K} \left( \|\tilde{\omega}_{T \cup A^k}\|_2 \beta_1^{k+1} - \alpha_N^{k+1} \right)
\geq (1 - t^4) \|\tilde{\omega}_{T \cup A^k}\|_2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{N} \right)
- (1 - t^4)\sqrt{K} \|\tilde{\omega}_{T \cup A^k}\|_2
\times \left( \|A'_{T \cup A^k}(I - P_{A^k})e\|_{2,\infty} + \frac{\sum_{i \in W_{A^k}} \|A'[i](I - P_{A^k})e\|_2}{N} \right)
\geq (1 - t^4) \|\tilde{\omega}_{T \cup A^k}\|_2 \left( \frac{1 - t^2}{1 + t^2} - \frac{\delta_{N}}{\sqrt{2}(1 + \delta_{N})} \right)
- (1 - t^4)\sqrt{K} \|\tilde{\omega}_{T \cup A^k}\|_2 \sqrt{2(1 + \delta_{N})}\varepsilon
\geq (1 - t^4) \|\tilde{\omega}_{T \cup A^k}\|_2 \left( \frac{1 - t^2}{1 + t^2} - \frac{\delta_{N}}{\sqrt{2(1 + \delta_{N})}} \right)
\geq (1 - t^4) \|\tilde{\omega}_{T \cup A^k}\|_2 \left( \frac{1 - t^2}{1 + t^2} - \frac{\delta_{N}}{\sqrt{2(1 + \delta_{N})}} \right)
\geq (1 - t^4) \|\tilde{\omega}_{T \cup A^k}\|_2 \left( \frac{1 - t^2}{1 + t^2} - \frac{\delta_{N}}{\sqrt{2(1 + \delta_{N})}} \right)
\geq 0,
\]
i.e., \( \beta_1^{k+1} > \alpha_N^{k+1} \) which guarantees at least one index selected from the correct support in the \((k + 1)\)-th iteration.

It remains to show that the BOMMP exactly stops under the stopping rule \( \|r^k\| \leq \varepsilon \) when all the correct block indices are selected. First, assume that \( T - \Lambda^k = \emptyset \). Then \( T \subseteq \Lambda^k \) and \((I - P_{A^k})Ax = 0\). Therefore, it follows that
\[
\|r^k\|_2 = \|(I - P_{A^k})Ax + (I - P_{A^k})e\|_2 = \|(I - P_{A^k})e\|_2 \leq \|e\|_2 \leq \varepsilon.
\]

Second, assume that \( T - \Lambda^k \neq \emptyset \). Then it follows from the definition of the block-RIP and (3.15) that
\[
\|r^k\|_2 = \|(I - P_{A^k})Ax + (I - P_{A^k})e\|_2
\geq \|(I - P_{A^k})Ax\|_2 - \|(I - P_{A^k})e\|_2
\geq \|A'_{T \cup A^k}(I - P_{A^k})e\|_2
\geq \sqrt{1 - \delta \|\tilde{\omega}_{T \cup A^k}\|_2} - \|e\|_2
\]
\[
\sqrt{1 - \delta_{T \cup A_k}} \| x_{T \setminus A_k} \|_2 - \varepsilon \\
\geq \sqrt{1 - \delta_{T \cup A_k}} \| T - \Lambda^k \| \min_{i \in T} \| x[i] \|_2 - \| e \|_2 \\
\geq \sqrt{1 - \delta_{N+1}} \min_{i \in T} \| x[i] \|_2 - \varepsilon \\
> \varepsilon.
\]

Therefore, the OMMP does not stop early. The proof of Theorem 3.4 is completed.

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