THE THIRD HOMOTOPY GROUP AS A $\pi_1$–MODULE

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Abstract. It is well–known how to compute the structure of the second homotopy group of a space, $X$, as a module over the fundamental group, $\pi_1 X$, using the homology of the universal cover and the Hurewicz isomorphism. We describe a new method to compute the third homotopy group, $\pi_3 X$, as a module over $\pi_1 X$. Moreover, we determine $\pi_3 X$ as an extension of $\pi_1 X$–modules derived from Whitehead’s Certain Exact Sequence. Our method is based on the theory of quadratic modules. Explicit computations are carried out for pseudo–projective 3–spaces $X = S^1 \cup e^2 \cup e^3$ consisting of exactly one cell in each dimension $\leq 3$.

1. Introduction

Given a connected 3–dimensional CW–complex, $X$, with universal cover, $\hat{X}$, Whitehead’s Certain Exact Sequence \cite{W2} yields the short exact sequence

$$(1.1) \quad \Gamma \pi_2 X \rightarrow \pi_3 X \rightarrow H_3 \hat{X}$$

of $\pi_1$–modules, where $\pi_1 = \pi_1(X)$. As a group, the homology $H_3 \hat{X}$ is a subgroup of the free abelian group of cellular 3–chains of $\hat{X}$, and thus itself free abelian. Hence the sequence splits as a sequence of abelian groups. This raises the question whether \eqref{1.1} splits as a sequence of $\pi_1$–modules – there are no examples known in the literature.

It is well–known how to compute $\pi_2(X) \cong H_2 \hat{X}$ as a $\pi_1$–module, using the Hurewicz isomorphism, and how to compute $H_3 \hat{X}$ using the cellular chains of the universal cover. In this paper we compute $\pi_3(X)$ as $\pi_1$–module and \eqref{1.1} as an extension over $\pi_1$. We answer the question above by providing an infinite family of examples where \eqref{1.1} does not split over $\pi_1$, as well as an infinite family of examples where it does split over $\pi_1$. As a first surprising example we obtain

Theorem 1.1. There is a connected 3–dimensional CW–complex $X$ with fundamental group $\pi_1 = \pi_1 X = \mathbb{Z}/2\mathbb{Z}$, such that $\pi_1$ acts trivially on both $\Gamma \pi_2 X$ and $H_3 \hat{X}$, but non–trivially on $\pi_3 X$. Hence

$$(1.1) \quad \Gamma \pi_2 X \rightarrow \pi_3 X \rightarrow H_3 \hat{X}$$

does not split as a sequence of $\pi_1$–modules.

Below we describe examples for all finite cyclic fundamental groups, $\pi_1$, of even order, where \eqref{1.1} does not split over $\pi_1$. The examples we consider are CW–complexes,

$$X = S^1 \cup e^2 \cup e^3,$$

with precisely one cell, $e^i$, in every dimension $i = 0, 1, 2, 3$. In general, we obtain such a CW–complex, $X$, by first attaching the 2–cell $e_2$ to $S^1$ via $f \in \pi_1 S^1 = \mathbb{Z}$. We assume $f > 0$. This yields the 2–skeleton of $X$, $X^2 = P_f$, which is a pseudo–projective plane, see \cite{O}. Then $\pi_1 = \pi_1 X = \pi_1 P_f = \mathbb{Z}/f\mathbb{Z}$ is a cyclic group of order $f$. We write $R = \mathbb{Z} / \pi_1$ for the integral group ring of $\pi_1$ and $K$ for the kernel of the augmentation $\varepsilon : R \rightarrow \mathbb{Z}$. Then the pseudo–projective 3–space, $X = P_f \cup e^3$, is determined by the pair, $(f, x)$, of attaching maps, where $x \in \pi_2 P_f = K$ is the attaching map of the 3–cell $e_3$. In this case

$$\pi_2(X) = H_2(\hat{X}) = K/R,$$

and

$$H_3 \hat{X} = \ker(d_x : R \rightarrow R, x \mapsto xy),$$

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where \( xy \) is the product of \( x, y \in R \).

A splitting function \( u \) for the exact sequence \((\ref{eq:1})\) is a function between sets, \( u : H_3\hat{X} \to \pi_3X \), such that \( u(0) = 0 \) and the composite of \( u \) and the projection \( \pi_3X \to H_3\hat{X} \) is the identity. Such a splitting function determines maps

\[
A = A_u : H_3\hat{X} \times H_3\hat{X} \to \Gamma(\pi_2X) \quad \text{and} \quad B = B_u : H_3\hat{X} \to \Gamma(\pi_2X),
\]

by the cross–effect formulæ

\[
A(y, z) = u(y + z) − (u(y) + u(z)) \quad \text{and} \quad B(y) = (u(y))^1 − u(y^1).
\]

Here \( B \) is determined by the action of the generator 1 in the cyclic group \( \pi_1 \), denoted by \( y \mapsto y^1 \).

**Remark 1.2.** The functions \( A \) and \( B \) determine \( \pi_3X \) as a \( \pi_1 \–\)module. In fact, the bijection

\[
H_3\hat{X} \times \Gamma(\pi_2X) = \pi_3(P_{f,x}),
\]

which assigns to \((y, v)\) the element \( u(y) + v \) is an isomorphism of \( \pi_1 \–\)modules, where the left hand side is an abelian group by

\[
(y, v) + (z, w) = (y + z, v + w + A(y, z))
\]

and a \( \pi_1 \–\)module by

\[
(y, v)^1 = (y^1, v^1 + B(y)).
\]

The cross–effect of \( B \) satisfies

\[
B(y + z) − (B(y) + B(z)) = (A(y, z))^1 - A(y^1, z^1),
\]

such that \( B \) is a homomorphism of abelian groups if \( A = 0 \).

In this paper we describe a method to determine a splitting function \( u = u_x \), which, a priori, is not a homomorphism of abelian groups. We investigate the corresponding functions \( A \) and \( B \) and compute them for a family of examples.

**Theorem 1.3.** Let \( X = P_{f,x} \) be a pseudo–projective 3–space with \( x = \hat{x}(\overline{[1]} - \overline{[0]}) \in K, \hat{x} \in Z, \hat{x} \neq 0 \) and \( f > 1 \). Let \( N = \sum_{i=0}^{f-1} [i] \) be the norm element in \( R \). Then

\[
H_3\hat{P}_{f,x} = \{\hat{y}N | \hat{y} \in Z\} \cong Z
\]

is a \( \pi_1 \–\)module with trivial action of \( \pi_1 \), and

\[
\pi_2(P_{f,x}) = (Z/\hat{x}Z) \otimes_Z K,
\]

with the action of \( \pi_1 \) induced by the \( \pi_1 \–\)module \( K \). There is a splitting function \( u = u_x \) such that, for \( y = \hat{y}N \) and \( z \in H_3\hat{P}_{f,x} \), the functions \( A \) and \( B \) are given by

\[
A(y, z) = 0 \quad \text{and} \quad B(y) = -\hat{x}\hat{y}q(\overline{1} - \overline{0}),
\]

where \( \gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x})) \) is the universal quadratic map for the Whitehead functor \( \Gamma \) and \( q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k \). As in \((\ref{eq:2})\), the pair \( A, B \) computes \( \pi_3X \) as a \( \pi_1 \–\)module.

As \( H_3(X) \) is free abelian, the exact sequence \((\ref{eq:1})\) always allows a splitting function which is a homomorphism of abelian groups. This leads, for \( X = P_{f,x}, \) to the injective function

\[
\tau : \text{Ext}_{\pi_1}(H_3\hat{X}, \Gamma(\pi_2X)) \hookrightarrow \text{coker}(\beta),
\]

with

\[
\beta : \text{Hom}_Z(H_3\hat{X}, \Gamma(\pi_2X)) \to \text{Hom}_Z(H_3\hat{X}, \Gamma(\pi_2X)), t \mapsto \beta_t,
\]

given by

\[
\beta_t(\ell) = -t(\ell^1) + (t(\ell))^1.
\]

The function \( \tau \) maps the equivalence class of an extension to the element in \( \text{coker}\beta \) represented by \( B = B_u \), where \( u \) is a \( Z \–\)homomorphic splitting function for the extension. Hence the equivalence class, \( \{\pi_3X\} \), of the extension \( \pi_3X \) in \((\ref{eq:1})\) is determined by the image \( \tau\{\pi_3X\} \in \text{coker}(\beta) \). For the family of examples in \((\ref{eq:2})\), we show
Theorem 1.4. Let $X = P_{f,x}$ be a pseudo-projective 3–space with $x = \tilde{x}([1] - [0]), \tilde{x} \in \mathbb{Z}, \tilde{x} \neq 0$ and $f > 1$. Then $\beta : \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_2 K) \to \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_2 K)$ maps $\ell$ to $-\ell + \ell^1$ and $\tau\{\pi_3X\} \in \text{coker}(\beta)$ is represented by $\tilde{x} \gamma(q([1] - [0])) \in \Gamma(\pi_3)$. Hence $\tau\{\pi_3X\} = 0$ if $\tilde{x}$ is odd, so that, in this case, $\pi_3X$ in (1.1) is a split extension over $\pi_1$. If both $\tilde{x}$ and $f$ are even, then $\tau\{\pi_3X\}$ is a non-trivial element of order 2, and the extension $\pi_3X$ in (1.1) does not split over $\pi_1$. Moreover, $\tau\{\pi_3X\}$ is represented by $B$ in (1.3) if $\tilde{x}$ is even and $f$ is odd, then $\tau\{\pi_3X\}$ is trivial and the extension $\pi_3X$ in (1.1) does split over $\pi_1$.

This result is a corollary of (1.3), the computations are contained at the end of Section 8.

Given a pseudo-projective 3–space, $P_{f,x}$, and an element $z \in \pi_3(P_{f,x})$, we obtain a pseudo-projective 4–space, $X = P_{f,x,z} = S^1 \cup e^2 \cup e^3 \cup e^4$, where $z$ is the attaching map of the 4–cell $e^4$. For $n \geq 2$, the attaching map $z$ of an $(n + 1)$–cell in a CW–complex, $X$, is homologically non-trivial if the image of $z$ under the Hurewicz homomorphism is non-trivial in $H_n(X^n)$.

Theorem 1.5. Let $X = S^1 \cup e^2 \cup e^3 \cup e^4$ be a pseudo-projective 4–space with $\pi_1X = \mathbb{Z}/2\mathbb{Z}$ and homologically non-trivial attaching maps of cells in dimension 3 and 4. Then the action of $\pi_1X$ on $\pi_3X$ is trivial.

Theorem 1.5 is a corollary to Theorem 9.1.

2. Crossed Modules

We recall the notions of pre-crossed module, Peiffer commutator, crossed module and nil(2)–module, which are ingredients of algebraic models of 2– and 3–dimensional CW–complexes used in the proofs of our results, see [B] and [BHS]. In particular, Theorem 2.2 provides an exact sequence in the algebraic context of a nil(2)–module equivalent to Whitehead’s Certain Exact Sequence (1.1).

A pre-crossed module is a homomorphism of groups, $\partial : M \to N$, together with an action of $N$ on $M$, such that, for $x \in M$ and $\alpha \in N$,

$$\partial(x^\alpha) = -\alpha + \partial x + \alpha.$$  

Here the action is given by $(\alpha, x) \mapsto x^\alpha$ and we use additive notation for group operations even where the group fails to be abelian. The Peiffer commutator of $x, y \in M$ in such a pre-crossed module is given by

$$\langle x, y \rangle = -x - y + x + y \partial x.$$  

The subgroup of $M$ generated by all iterated Peiffer commutators $\langle x_1, \ldots, x_n \rangle$ of length $n$ is denoted by $P_n(\partial)$ and a nil(n)–module is a pre-crossed module $\partial : M \to N$ with $P_{n+1}(\partial) = 0$. A crossed module is a nil(1)–module, that is, a pre-crossed module in which all Peiffer commutators vanish. We also consider nil(2)–modules, that is, pre-crossed modules for which $P_3(\partial) = 0$.

A morphism or map $(m, n) : \partial \to \partial'$ in the category of pre-crossed modules is given by a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{m} & M' \\
\partial \downarrow & & \partial' \downarrow \\
N & \xrightarrow{n} & N'
\end{array}$$

in the category of groups, where $m$ is $n$–equivariant, that is, $m(x^\alpha) = m(x)^n(\alpha)$, for $x \in M$ and $\alpha \in N$. The categories of crossed modules and nil(2)–modules are full subcategories of the category of pre-crossed modules.

Note that $P_{n+1}(\partial) \subseteq \ker \partial$ for any pre-crossed module, $\partial : M \to N$. Thus we obtain the associated nil(n)–module $r_n(\partial) : M/P_{n+1}(\partial) \to N$, where the action on the quotient is determined by demanding that the quotient map $q : M \to M/P_{n+1}(\partial)$ be equivariant. For $n = 1$ we write $\partial'' = r_1(\partial) : M^{\alpha} = M/P_2(\partial) \to N$ for the crossed module associated to $\partial$.

Given a set, $Z$, let $\langle Z \rangle$ denote the free group generated by $Z$. Now take a group, $N$, and a group homomorphism, $f : F = \langle Z \rangle \to N$. Then the free $N$–group generated by $Z$ is the free
group, \( (Z \times N) \), generated by elements denoted by \( x^\alpha = ((x, \alpha)) \) with \( x \in Z \) and \( \alpha \in N \). These are elements in the product \( Z \times N \) of sets. The action is determined by
\[
((x, \alpha))^\beta = ((x, \alpha + \beta)).
\]
Define the group homomorphism \( \partial_f : (Z \times N) \to N \) by \( ((x, \alpha)) \mapsto -\alpha + f(x) + \alpha \), for generators \( ((x, \alpha)) \in Z \times N \), to obtain the pre-crossed module \( \partial_f \) with associated nil\((n)\)-module \( r_n(\partial_f) : \langle Z \times N / P_{n+1}(\partial_f) \rangle \to N \). Note that \( r_n(\partial_f) \iota = f \), where \( \iota = \iota_E \) is the composition of the inclusion \( \iota_E : F = \langle Z \rangle \to \langle Z \times N \rangle \) and the projection \( p : \langle Z \times N \rangle \to M = \langle Z \times N / P_{n+1}(\partial_f) \rangle \) onto the quotient.

**Remark 2.1.** The nil\((n)\)-module, \( r_n(\partial_f) : M = \langle Z \times N / P_{n+1}(\partial_f) \rangle \to N \), satisfies the following universal property: For every nil\((n)\)-module, \( \partial' : M' \to N' \), and every pair of group homomorphisms, \( m_F : F = \langle Z \rangle \to M' \), and \( n : N \to N' \) with \( \partial' m_F = nf \), there is a unique group homomorphism, \( m : M \to M' \), such that \( m = m_F \), and \( (n, m) : r_n(\partial_f) \to \partial' \) is a map of nil\((n)\)-modules.

Thus \( r_n(\partial_f) \) is called the **free nil\((n)\)-module with basis \( f \)**. A free nil\((n)\)-module is totally free if \( N \) is a free group.

Given a path connected space \( Y \) and a space \( X \) obtained from \( Y \) by attaching 2-cells, let \( Z_2 \) be the set of 2-cells in \( X - Y \), and let \( f : Z_2 \to \pi_1(Y) \) be the attaching map. J.H.C. Whitehead [W1] showed that
\[
(2.2) \quad \partial : \pi_2(X, Y) \to \pi_1(Y)
\]
is a free crossed module with basis \( f \). Then \( \ker \partial = \pi_2(X), \coker \partial = \pi_1(X) \) and \( \partial \) is totally free if \( Y \) is a one-point union of 1-spheres. Whitehead also proved that the abelianisation of the group \( \pi_2(X, Y) \) is the free \( R \)-module \( \langle Z_2 \rangle_R \) generated by the set \( Z_2 \), where \( R = \mathbb{Z}[\pi_1(X)] \) is the group ring [W1].

Now take a totally free nil\((2)\)-module \( \partial : M \to N \) with associated crossed module \( \partial^{cr} : M^{cr} \to N \). Let
\[
M \xrightarrow{q} M^{cr} \xrightarrow{h_2} C = (M^{cr})^{ab}
\]
be the composition of projections. Put \( K = h_2(\ker(\partial^{cr})) \). Further, let \( \Gamma \) be **Whitehead’s quadratic functor** and \( \tau : \Gamma(K) \to K \otimes K \subset C \otimes C \) the composition of the injective homomorphism induced by the quadratic map \( K \to K \otimes K, k \mapsto k \otimes k \) and the inclusion. The **Peiffer commutator map**, \( w : C \otimes C \to M \), is given by \( w(\{x\} \otimes \{y\}) = \langle x, y \rangle \), for \( x, y \in M \) with \( \{x\} = h_2(\langle q(x) \rangle), \{y\} = h_2(\langle q(y) \rangle) \).

**Theorem 2.2.** Let \( \partial : M \to N \) be a totally free nil\((2)\)-module. Then the sequence
\[
\Gamma(K) \xrightarrow{\tau} C \otimes C \xrightarrow{w} M \xrightarrow{q} M^{cr}
\]
is exact and the image of \( w \) is central in \( M \).

### 3. Pseudo–Projective Spaces in Dimensions 2 and 3

Real projective \( n \)-space \( \mathbb{R}P^n \) has a cell structure with precisely one cell in each dimension \( \leq n \). More generally, a CW–complex,
\[
X = S^1 \cup e^2 \cup \ldots \cup e^n,
\]
with precisely one cell in each dimension \( \leq n \), is called a pseudo–projective \( n \)-space. For \( n = 2 \) we obtain pseudo–projective planes, see [Q]. In this section we fix notation and consider pseudo–projective spaces in dimensions 2 and 3. In particular, we determine the totally free crossed module associated with a pseudo–projective plane and begin to investigate the totally free nil(2)–module associated with a pseudo–projective 3–space.

The fundamental group of a pseudo–projective plane \( P_f = S^1 \cup e^2 \), with attaching map \( f \in \pi_1(S^1) = \mathbb{Z} \), is the cyclic group \( \pi_1 = \pi_1(P_f) = \mathbb{Z}/f\mathbb{Z} \). We obtain \( \pi_1 = \mathbb{Z} \) for \( f = 0 \), \( \pi_1 = \{0\} \) for \( f = 1 \), and the bijection of sets

\[
\{0, 1, 2, \ldots, f - 1\} \to \pi_1 = \mathbb{Z}/f\mathbb{Z}, \quad k \mapsto k + f\mathbb{Z},
\]

for \( 1 < f \). Addition in \( \pi_1 \) is given by

\[
\overline{k + \ell} = \begin{cases} 
\overline{k + \ell} & \text{for } k + \ell < f; \\
\overline{k + \ell - f} & \text{for } k + \ell \geq f.
\end{cases}
\]

Denoting the integral group ring of the cyclic group \( \pi_1 \) by \( R = \mathbb{Z}[\pi_1] \), an element \( x \in R \) is a linear combination

\[
x = \sum_{\alpha \in \pi_1} x_\alpha[\alpha] = \sum_{k=0}^{f-1} x_\overline{k},
\]

with \( x_\alpha, x_\overline{k} \in \mathbb{Z} \). Note that \( 1_R = [1] \) is the neutral element with respect to multiplication in \( R \) and, for \( x = \sum_{\alpha \in \pi_1} x_\alpha[\alpha], y = \sum_{\beta \in \pi_1} y_\beta[\beta] \),

\[
xy = \sum_{\alpha, \beta \in \pi_1} x_\alpha y_\beta [\alpha + \beta] = \sum_{\ell=0}^{f-1} \sum_{k=0}^{\ell} x_\overline{k} y_{\overline{k-\ell}} + \sum_{k=1}^{f-1} x_\overline{k} y_{\overline{k+f-\ell}}.
\]

The augmentation \( \varepsilon = \varepsilon_R : R \to \mathbb{Z} \) maps \( \sum_{\alpha \in \pi_1} x_\alpha[\alpha] \) to \( \sum_{\alpha \in \pi_1} x_\alpha \). The augmentation ideal, \( K \), is the kernel of \( \varepsilon \). For a right \( R \)–module, \( C \), we write the action of \( \alpha \in \pi_1 \) on \( x \in C \) exponentially as \( x^\alpha = x[\alpha] \).

Given a pseudo–projective plane \( P_f = S^1 \cup e^2 \) with attaching map \( f \in \pi_1(S^1) = \mathbb{Z} \), Whitehead’s results on the free crossed module [22] imply that

\[
\partial : \pi_2(P_f, S^1) \to \pi_1(S^1)
\]

is a totally free crossed module with one generator, \( e_i \), in dimensions \( i = 1, 2 \), and basis \( f : Z_2 = \{e_2\} \to \pi_1(S^1) \) given by \( f(e_2) = fe_1 \). Note that \( \partial \) has cokernel \( \pi_1(P_f) = \mathbb{Z}/f\mathbb{Z} = \pi_1 \) and kernel \( \pi_2(P_f) \).

**Lemma 3.1.** The diagram

\[
\begin{array}{ccc}
\pi_2(P_f, S^1) & \xrightarrow{\partial} & \pi_1(S^1) \\
\downarrow \cong & & \downarrow \cong \\
R & \xrightarrow{f_{\varepsilon_R}} & \mathbb{Z}
\end{array}
\]

is an isomorphism of crossed modules, where \( \varepsilon_R : R \to \mathbb{Z} \) is the augmentation.

**Proof.** By Whitehead’s results [W1] on the free crossed module [22], it is enough to show that \( \pi_2(P_f, S^1) \) is abelian. As \( \partial \) is a totally free crossed module with basis \( f \), \( \pi_2(P_f, S^1) \) is generated by elements \( e^n = ((e_2, n)) \), see [21]. Note that we obtain \( e^n \) by the action of \( n \in \mathbb{Z} \) on \( (e_2) = ((e_2, 0)) = e^0 \) and \( \partial(e^n) = -n + \partial e + n = \partial e = f \) as \( \pi_1(S^1) = \mathbb{Z} \) is abelian. We obtain

\[
\langle e^n, e^m \rangle - \langle e^m, e^n \rangle = -e^n - e^m + e^n + (e^m)\partial(e^n) - (-e^m - e^m + e^m + (e^m)\partial(e^m))
\]

\[
= -e^n - e^m + e^n + (e^m)f - (e^m)f + e^m
\]

\[
= (e^n, e^m),
\]

where \( (a, b) = -a - b + a + b \) denotes the commutator of \( a \) and \( b \). Thus commutators of generators are sums of Peiffer commutators which are trivial in a crossed module. \( \square \)
With the notation of Theorem 2.2 and $M = \pi_2(P_f, S^1)$, Lemma 3.1 shows that $M = M^{cr} = (M^{cr})^{ab} = R$ and that $\pi_2(P_f) = \ker \partial = \ker \partial^{cr} = \ker (f : \varepsilon) = K$ is the augmentation ideal of $R$, for $f \neq 0$. Thus the homotopy type of a pseudo–projective 3–space,

\begin{equation}
(3.2) \quad P_{f,x} = S^1 \cup e^2 \cup e^3,
\end{equation}

is determined by the pair $(f, x)$ of attaching maps, $f \in \pi_1(S^1) = \mathbb{Z}$ of the 2-cell $e^2$, and $x \in \pi_2(P_f) = K \subseteq R$ of the 3-cell $e^3$. We obtain the totally free nil(2)–module

\begin{equation}
(3.3) \quad M = \pi_2(P_{f,x}, S^1) \xrightarrow{\partial} N = \pi_1(S^1).
\end{equation}

In the next section we use Theorem 2.2 to describe the group structure of $\pi_2(P_{f,x}, S^1)$, as well as the action of $N$ on $\pi_2(P_{f,x}, S^1)$. The formulæ we derive are required to compute the homotopy group $\pi_3(P_{f,x})$ as a $\pi_1$–module.

4. Computations in nil(2)–modules

In this Section we consider totally free nil(2)–modules, $\partial : M \to N$, generated by one element, $e_i$, in dimensions $i = 1, 2$, with basis $\tilde{f} : \{e_2\} \to N \cong \mathbb{Z}$. Then $\pi_1 = \text{coker} \partial = \mathbb{Z}/f \mathbb{Z}$ and, with $R = \mathbb{Z} / \pi_1$, we obtain $(M^{cr})^{ab} = C = R$. Thus Theorem 2.2 yields the short exact sequence

\begin{equation}
(4.1) \quad (R \otimes R)/\Gamma(K) \longrightarrow M \longrightarrow R
\end{equation}

with the image of $(R \otimes R)/\Gamma(K)$ central in $M$. This allows us to compute the group structure of $M$, as well as the action of $N = \mathbb{Z}$ on $M$, by computing the cross–effects of a set–theoretic splitting $s$ of (4.1) with respect to addition and the action of $N$, even though here $M$ need not be commutative.

The element $x \otimes y \in R \otimes R$ represents an equivalence class in $R \otimes R/\Gamma(K)$, also denoted by $x \otimes y$, so that $w(x \otimes y) = (\tilde{x}, \tilde{y})$ is the Peiffer commutator for $x, y \in R$, with $x = q(\tilde{x})$ and $y = q(\tilde{y})$. As a group, $M$ is generated by elements $e^n = ((e_2, n))$, in particular, $e = e^0 = ((e_2, 0))$, see (2.1). We write

\[ ke^n = \begin{cases} 
  e^n + \ldots + e^n & \text{(}k\text{ summands)} \\
  0 & \text{for } k = 0,
\end{cases} \]

\[ -e^n - \ldots - e^n & \text{(-}k\text{ summands)} \quad \text{for } k < 0,
\]

and define the set-theoretic splitting $s$ of (4.1) by

\[ s : R \longrightarrow M, \quad \sum_{k=0}^{f-1} x_{\varepsilon}^k \longmapsto x_{\varepsilon}^0 + x_{\varepsilon}^1 + \ldots + x_{\varepsilon}^{f-1}.
\]

Then every $m \in M$ can be expressed uniquely as a sum $m = s(x) + w(m^\otimes)$ with $x \in R$ and $m^\otimes \in (R \otimes R)/\Gamma(K)$. The following formulæ for the cross–effects of $s$ with respect to addition and the action provide a complete description of the nil(2)–module $M$ in terms of $R$ and $R \otimes R/\Gamma(K)$.

Given a function, $f : G \to H$, between groups, $G$ and $H$, we write

\begin{equation}
(4.2) \quad f(x|y) = f(x + y) - (f(x) + f(y)), \quad \text{for } x, y \in G.
\end{equation}

Lemma 4.1. Take $x = \sum_{m=0}^{f-1} x_{\varepsilon}^m, y = \sum_{n=0}^{f-1} y_{\varepsilon}^n \in R$. Then

\[ s(x|y) = w(\nabla(x, y)),
\]

where

\[ \nabla(x, y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{\varepsilon}^m y_{\varepsilon}^n w([-m] \otimes [m] - [m] \otimes [m]).
\]

Thus $\nabla(x, y)$ is linear in $x$ and $y$, yielding a homomorphism $\nabla : R \otimes R \to R \otimes R$. 
Proof. First note that, by definition, \( \nabla(k \langle m \rangle, \ell \langle m \rangle) = 0 \) unless \( m > n \). To deal with the latter case, recall that commutators are central in \( M \) and use induction, first on \( k \), then on \( \ell \), to show that

\[
(ke^m, \ell e^n) = k\ell(e^m, e^n),
\]

for \( k, \ell > 0 \). To show equality for negative \( k \) or \( \ell \), replace \( e^m \) or \( e^n \) by \(-e^{-m}\) and \(-e^{-n}\), respectively. Furthermore, note that the equality

\[
(e^n, e^m) = -e^n - e^m + e^n + e^m = \langle e^n, e^m \rangle - \langle e^m, e^m \rangle
\]

for commutators of generators of totally free cyclic crossed modules derived in the proof of Lemma 3.1 holds in any totally free nil(n)–module generated by one element in each dimension. Taking

\[ x = \sum_{m=0}^{f-1} x_{\langle m \rangle} \text{ and } y = \sum_{n=0}^{f-1} y_{\langle n \rangle}, \]

we obtain

\[
s(x + y) = (x_{\langle m \rangle} + y_{\langle m \rangle}) e + \ldots + (x_{\langle m \rangle} + y_{\langle m \rangle}) e^m + \ldots + (x_{\langle f-1 \rangle} + y_{\langle f-1 \rangle}) e^{f-1}
\]

\[
= (x_{\langle m \rangle} + y_{\langle m \rangle}) e + \ldots + (x_{\langle m \rangle} + y_{\langle m \rangle}) e^m + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{\langle m \rangle} y_{\langle n \rangle} (e^n, e^m)
\]

\[
= s(x) + s(y) + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{\langle m \rangle} y_{\langle n \rangle} \langle e^n, e^m \rangle
\]

\[
= s(x) + s(y) + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{\langle m \rangle} y_{\langle n \rangle} \omega(\langle m \rangle \otimes \langle m \rangle - \langle m \rangle \otimes \langle m \rangle).
\]

Corollary 4.2. Take \( x \in R \) and \( r \in \mathbb{Z} \). Then

\[
s(rx) = rs(x) + \binom{r}{2} w(\nabla(x, x)), \quad \text{where} \quad \binom{r}{2} = \frac{r(r-1)}{2}.
\]

As \( N = \mathbb{Z} \) is cyclic, the action of \( N \) on \( M \) is determined by the action of the generator, \( 1 \in \mathbb{Z} \). The formula for general \( k \in \mathbb{Z} \) provided in the next lemma is required for the definition of the set–theoretic splitting \( \hat{u} \) of (1.1) and the explicit computation of \( A \) and \( B \) in Theorem 1.3.

Lemma 4.3. Take \( x = \sum_{n=0}^{f-1} x_{\langle n \rangle} \in R \) and \( k \in \pi_1 \). Write \( R = \mathbb{Z}[\overline{0}, \ldots, \overline{f-1}] = R_k \times \hat{R}_k \), where \( R_k = \mathbb{Z}[\overline{0}, \ldots, \overline{f-k-1}] \) and \( \hat{R}_k = \mathbb{Z}[\overline{f-k}, \ldots, \overline{f-1}] \). Then

\[
(s(x))^k = s(x^k) + w(\nabla_k(a, b)),
\]

where \( x = (a, b) \) and

\[
\nabla_k : R_k \times \hat{R}_k \to R \otimes R, \quad (a, b) \mapsto Q_k(a, b) + L_k(b)
\]

with

\[
Q_k(a, b) = \sum_{p=0}^{f-\ell-1} \sum_{q=0}^{\ell-1} x_{\langle p+q+f-k-1 \rangle} \omega(\langle p+q+f-k-1 \rangle \otimes \langle q \rangle - \langle q \rangle \otimes \langle p+q+f-k-1 \rangle)
\]

\[
L_k(b) = \sum_{q=0}^{\ell-1} x_{\langle q+f-k \rangle} \omega(\langle q+f-k \rangle \otimes \langle q \rangle).
\]

Thus \( Q_k \) is linear in \( a \) and \( b \) and \( L_k \) is linear in \( b \).

Proof. For \( f \in \pi_1 \) and \( p \in \mathbb{Z} \),

\[
e^{j+f} = (e^j)^{\delta(e)}
\]

\[
e^j + (e^j, e) + \langle e, e^j \rangle
\]

\[
e^j - ((e, e^j) - \langle e^j, e \rangle) + \langle e, e^j \rangle
\]

\[
e^j + \langle e^j, e \rangle.
\]
Thus, for $\overline{p}, \overline{k} \in \pi_1$, with $\overline{p} + \overline{k} = \overline{f}$,

$$(s(\overline{p}))^k = \begin{cases} e^j, & \text{for } 0 \leq n < f - k, \\ e^j + (e^j, e^j), & \text{for } f - k \leq n < f \\ s(\overline{p})^k, & \text{for } 0 \leq n < f - k, \\ s(\overline{p})^k + w([\overline{f}] \otimes [\overline{f}]), & \text{for } f - k \leq n < f. \end{cases}$$

Hence, for $x = \sum_{p=0}^{f-1} x\overline{p}$,

$$(s(x))^k = x\overline{p} s([0])^k + x\overline{p} s([1])^k + \ldots + x\overline{p} s([f-1])^k$$

$$= x\overline{p} s([\overline{f}])^k + x\overline{p} s([\overline{f}]^k) + \ldots + x\overline{p} s([\overline{f}-1]^k) + \sum_{n=f-k}^{f-1} x\overline{p} w([n + k - f] \otimes [n + k - f])$$

$$= x\overline{f-k} s([\overline{f}-k]^k) + \ldots + x\overline{p} s([\overline{f}-1]^k) + x\overline{p} s([\overline{0}]^k) + \ldots + x\overline{p} s([\overline{f}-k-1]^k)$$

$$+ \sum_{p=0}^{f-k-1} \sum_{n=f-k}^{f-1} x\overline{p} s([\overline{p} + n]^k), x\overline{p} s([\overline{p} + n]^k) + \sum_{q=0}^{k-1} x\overline{p} s([\overline{q}] \otimes [\overline{q}])$$

$$= s(x^k) + \sum_{p=0}^{f-k-1} \sum_{q=0}^{k-1} x\overline{p} x\overline{q} s([\overline{p} + \overline{q}] \otimes [\overline{q}]), x\overline{p} s([\overline{p} + \overline{q}] \otimes [\overline{q}]), x\overline{p} s([\overline{p} + \overline{q}] \otimes [\overline{q}]), x\overline{p} s([\overline{p} + \overline{q}] \otimes [\overline{q}]), x\overline{p} s([\overline{p} + \overline{q}] \otimes [\overline{q}]).$$

\[\square\]

**Remark 4.4.** We use the final results of this section to define and establish the properties of the set-theoretic splitting $u_x$ of (1.1). The next result shows how the cross-effects interact with multiplication in $R$.

**Lemma 4.5.** Take $x, y \in R$. Then

$$\sum_{i=0}^{f-1} y_i (s(x))^i = s(xy) + w(\mu(x, y)),$$

where $\mu : R \times R \to R \otimes R$ is given by

$$\mu(x, y) = -\sum_{i<j} y_i y_j \nabla(x^i, x^j) + \sum_{i=0}^{f-1} (\nabla_i (y^i x) - \left(\nabla_i \left(\frac{y^i}{2}\right) \nabla(x, x)^i\right)).$$

**Proof.** By Lemmas 4.1 and 4.3 and Corollary 4.2, we obtain, for $x, y \in R$,

$$\sum_{i=0}^{f-1} y_i (s(x))^i = \sum_{i=0}^{f-1} (y_i s(x))^i$$

$$= \sum_{i=0}^{f-1} s(y_i x) - \left(\frac{y_i}{2}\right) w(\nabla(x, x))^i$$

$$= \sum_{i=0}^{f-1} s(y_i x^i) + w(\nabla_i (y^i x)) - \left(\frac{y_i}{2}\right) w(\nabla(x, x))^i$$

$$= s(\sum_{i=0}^{f-1} y_i x^i) - \sum_{i<j} w(\nabla(y_i x^i, y_j x^j)) + \sum_{i=0}^{f-1} w(\nabla_i (y^i x)) - \left(\frac{y_i}{2}\right) w(\nabla(x, x)^i).$$

\[\square\]

Finally, the definitions and a simple calculation yield
Lemma 4.6. For \( x, y, z \in R \) and with the notation in \((f, z)\),

\[
\mu(x, y|z) = - \sum_{i<j} (y_i y_j z_{i+j} + z_i y_j y_{i+j}) + 2 \sum_{i=1}^{f-1} y_i Q_i(x) - \sum_{i=0}^{f-1} y_i x_i^3/3 + x_i^3/3.
\]

Hence, for fixed \( x \in R, \mu(x, \cdot) : R \times R \to R \otimes R, (y, z) \mapsto \mu(x, y|z) \) is bilinear.

5. Quadratic Modules

In dimension 3, quadratic modules assume the role played by crossed modules in dimension 2. We recall the notion of quadratic modules and totally free quadratic modules, see \([B]\), which we require for the description of the third homotopy group \( \pi_3(P_f, x) \) of a 3–dimensional pseudo–projective space \( P_f, x \), as in \((f, z)\).

A quadratic module \( (\omega, \delta, \partial) \) consists of a commutative diagram of group homomorphisms

\[
\begin{array}{c}
C \otimes C \\
\downarrow \omega \quad \downarrow \delta \\
L \quad \downarrow \partial \quad \downarrow N,
\end{array}
\]

such that

- \( \partial : M \to N \) is a nil\((2)\)–module with quotient map \( M \to C = (M^c)^{ab}, x \mapsto \{x\} \), and Peiffer commutator map \( w \) given by \( w(\{x\} \otimes \{y\}) = (x, y); \)
- the boundary homomorphisms \( \partial \) and \( \delta \) satisfy \( \partial \delta = 0 \), and the quadratic map \( \omega \) is a lift of \( w \), that is, for \( x, y \in M \),
  \[
  \delta \omega(\{x\} \otimes \{y\}) = (x, y); 
  \]
- \( N \) acts on \( L \), all homomorphisms are equivariant with respect to the action of \( N \) and, for \( a \in L \) and \( x \in M \),
  \[
  a^{\partial(x)} = a + \omega(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\});
  \]
- finally, for \( a, b \in L \),
  \[
  (a, b) = -a - b + a + b = \omega(\{\delta a\} \otimes \{\delta b\}).
  \]

A map \( \varphi : (\omega, \delta, \partial) \to (\omega', \delta', \partial') \) of quadratic modules is given by a commutative diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L \\
\downarrow=\varphi \otimes 1 & & \downarrow \delta \\
C' \otimes C' & \xrightarrow{\omega'} & L'
\end{array}
\begin{array}{ccc}
\downarrow l & & \downarrow m \\
M & \xrightarrow{\partial} & N \\
\downarrow \delta & & \downarrow \partial' \\
M' & \xrightarrow{\partial'} & N'
\end{array}
\]

where \( l \) is \( n \)–equivariant, and \((m, n)\) is a map between pre–crossed modules inducing \( \varphi_* : C \to C' \).

Given a nil\((2)\)–module \( \partial : M \to N \), a free group \( F \) and a homomorphism \( f : F \to M \) with \( \partial f = 0 \), a quadratic module \( (\omega, \delta, \partial) \) is free with basis \( f \), if there is a homomorphism \( i : F \to L \) with \( \delta i = f \), such that the following universal property is satisfied: For every quadratic module \( (\omega', \delta', \partial') \) and map \( (m, n) : \partial \to \partial' \) of nil\((2)\)–modules and every homomorphism \( l_F : F \to L' \) with \( mf = \delta' l_F \), there is a unique map \( (l, m, n) \) of quadratic modules with \( li = l_F \).
For $F = \langle Z \rangle$, the homomorphism $\tilde{f}$ is determined by its restriction $\tilde{f}|_Z$ which is then called a basis for $(\omega, \delta, \partial)$. A quadratic module $(\omega, \delta, \partial)$ is totally free if it is free, if $\partial$ is a free nil(2)–module and if $N$ is a free group.

6. The Homotopy Group $\pi_3$ of a Pseudo–Projective 3–Space and the Associated Splitting Function $u_x$

In this section we return to pseudo–projective 3–spaces

$$P_{f,x} = S^1 \cup e^2 \cup e^3,$$

determined by the pair $(f, x)$ of attaching maps, $f \in \pi_1(S^1) = Z$ and $x \in \pi_2(P_f) = K \subseteq R$, as in (6.1). Using results on totally free quadratic modules in [B], we investigate the structure of the third homotopy group $\pi_3(P_{f,x})$ as a $\pi_1$–module by defining a set–theoretic splitting $u_x$ of J.H.C. Whitehead’s Certain Exact Sequence of the universal cover, $\tilde{P}_{f,x}$,

$$\Gamma(\pi_2(P_{f,x})) \rightarrowtail \pi_3(P_{f,x}) \twoheadrightarrow H_3(\tilde{P}_{f,x}). \tag{6.1}$$

Recall that $\pi_1 = \pi_1(P_f) = Z/fZ$ with augmentation ideal $K = ker f_{\varepsilon}$, and let $B$ be the image of $d_x : R \rightarrow R, y \mapsto xy$. Then

$$\pi_2(P_{f,x}) = H_2(\tilde{P}_{f,x}) = K/B = (ker f_{\varepsilon})/xR. \tag{6.2}$$

The functor $\sigma$ in (IV 6.8) in [B] assigns a totally free quadratic module $(\omega, \delta, \partial)$ to the pseudo–projective 3–space $P_{f,x}$ and we obtain the commutative diagram

$$\begin{array}{ccc}
\Gamma(\pi_2(P_{f,x})) & \rightarrowtail & \pi_3(P_{f,x}) \twoheadrightarrow \pi_3(\tilde{P}_{f,x}) \\
\downarrow \downarrow & & \downarrow \downarrow \\
R \otimes R/\Delta_B & \rightarrowtail & R \otimes R/\Gamma(K) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\pi_3(P_{f,x}) & \twoheadrightarrow & \pi_3(\tilde{P}_{f,x}) \\
\downarrow u_x & & \downarrow t_x \\
H_3(P_{f,x}) & \twoheadrightarrow & H_3(\tilde{P}_{f,x}) \\
\downarrow d_x & & \downarrow f_{\varepsilon} \\
R & \twoheadrightarrow & R \\
\downarrow \downarrow & & \downarrow \downarrow \\
\Delta_B & = & \Gamma(B) + [K, B].
\end{array}$$

By Corollary (IV 2.14) in [B], taking kernels yields Whitehead’s short exact sequence (6.1) in the left hand column of the diagram, that is, $ker q = \Gamma(\pi_2(\tilde{P}_{f,x}))$, $ker \delta = \pi_3(P_{f,x})$ and $ker d_x = H_3(\tilde{P}_{f,x})$. As $(\omega, \delta, \partial)$ is a quadratic module associated to $P_{f,x}$, we may assume that $\delta(\varepsilon_3) = s(x)$.

In Section 4 we determined the structure of $M$ as an $N$–module by computing the cross–effects of the set–theoretic splitting $s$ with respect to addition and the action. Analogously to the definition of $s$, we now define a set-theoretical splitting of the short exact sequence in the second column of this diagram by

$$t_x : R \rightarrowtail L, \quad \sum_{k=0}^{f-1} y^k [e_3] \rightarrowtail y^0 \varepsilon_3^0 + \cdots + y^{f-1} \varepsilon_3^f.$$

The cross–effects of $t_x$ with respect to addition and the action determine the $N$–module structure of $L$, but we want to determine the module structure of $\pi_3(P_{f,x})$. To obtain a set–theoretic splitting of the first column which will allow us to do so, we must adjust $t_x$, such that the image of $H_3(\tilde{P}_{f,x})$ under the new splitting is contained in $ker \delta = \pi_3(P_{f,x})$. Recall that $\delta$ is a homomorphism which
is equivariant with respect to the action of $N$ and $\delta(e_3) = s(x)$. Thus Lemma 4.5 yields, for $y \in H_3(\tilde{P}_{f,x}) = \ker d_x$, that is, for $d_x(y) = xy = 0$,

$$
\delta(t_x(y)) = \delta\left(\sum_{i=0}^{f-1} y_i e_3^i\right) = \sum_{i=0}^{f-1} y_i \delta(e_3^i) = \sum_{i=0}^{f-1} y_i (s(x))^{i}
$$

$$
= s(xy) + w(\mu(x, y))
$$

$$
= \delta \mu(x, y).
$$

Hence $t_x(y) - \mu(x, y) \in \ker \delta = \pi_3(\tilde{P}_{f,x})$, giving rise to the set theoretic splitting

$$
u_x : H_3(\tilde{P}_{f,x}) \longrightarrow \pi_3(\tilde{P}_{f,x}), \quad y \longrightarrow t_x(y) - \mu(x, y)
$$

of the Hurewicz map $\pi_3 \to H_3$. The cross–effects of $u_x$ with respect to addition and the action determine 5.4 as a short exact sequence of $\pi_1$–modules. In Section 4 we determine the cross–effects of $t_x$ and investigate the properties of the functions $A$ and $B$ describing the cross–effects of $u_x$.

7. Computations in Free Quadratic Modules

The first two results of this Section describe the cross–effects of $t_x$ with respect to addition and the action, respectively. We then turn to the properties of the cross–effects of $u_x$.

**Lemma 7.1.** Take $z, y \in R$. Then, with the notation in (4.2),

$$
t_x(z|y) = \omega(\Psi(z, y)),
$$

where

$$
\Psi(z, y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} \omega x[\bar{m}] \otimes x[\bar{n}] = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z_{\bar{m}} y_{\bar{n}} x[\bar{m}] \otimes x[\bar{n}] = \omega(x[\bar{m}] \otimes x[\bar{n}]).
$$

Thus $\Psi(z, y)$ is linear in $z$ and $y$, yielding a homomorphism $\Psi : R \otimes R \to R \otimes R$.

**Proof.** As in the proof of Lemma 4.1 we obtain

$$
t_x(z|y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z_{\bar{m}} y_{\bar{n}} e_3^m e_3^n = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} \omega(\delta(e_3^m)) \otimes \delta(e_3^n).$$

Note that $\{\delta(e_3^m)\} = \{\delta(t_x(\bar{m}))\} = d_x(\bar{m}) = x[\bar{m}]$. Thus (5.2) yields

$$
t_x(z|y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z_{\bar{m}} y_{\bar{n}} \omega(\delta(e_3^m)) \otimes \delta(e_3^n) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z_{\bar{m}} y_{\bar{n}} \omega(x[\bar{m}] \otimes x[\bar{n}]).
$$

As $N = Z$ is cyclic, the action of $N$ on $L$ is determined by the generator $1 \in Z$.

**Lemma 7.2.** Take $x \in R$. Then

$$
(t_x(y))^1 = t_x(y^\top) + \omega(\overline{\Psi_1(a, b)}),
$$

where

$$
\overline{\Psi_1} = \sum_{p=0}^{f-2} y_{\bar{p}} y_{\bar{f}-1} x[\bar{p}+1] \otimes x[\bar{0}] + y_{\bar{f}-1} (x \otimes [\bar{0}] + [\bar{0}] \otimes x).
$$

**Proof.** With $\{\delta(e_3^m)\} = x[\bar{m}]$ from above and (5.1), we obtain

$$
e_3^1 f = (e_3^1)^f = (e_3^1)^{\delta(e)} = e^1 + \omega(\{\delta(e_3^1)\} \otimes \{e\} + \{e\} \otimes \{\delta(e_3^1)\})
$$

$$
= t_x(\overline{x[\bar{1}]}) + \omega(x[\bar{1}] \otimes [\bar{0}] + [\bar{0}] \otimes x[\bar{1}]).
$$

Thus, for $\overline{\pi} \in \pi$,

$$
(t_x(\overline{\pi}))^1 = \begin{cases}
\omega(t_x(\overline{\pi}^\top)) & \text{for } 0 \leq n < f - 1,
\omega(t_x(\overline{\pi}^\top) + x[\bar{1}] \otimes [\bar{0}] + [\bar{0}] \otimes x[\bar{1}]) & \text{for } n = f - \ell.
\end{cases}
$$
With (5.2), we obtain, for \( y = \sum_{n=0}^{f-1} y_{3n}[n] \),
\[
(t_x(y))^1 = y_0^f e_0^1 + y_1^f e_1^2 + \ldots + y_{f-1}^f e_{f-1}^1 + y_f^f e_3^f
\]
\[
= y_0^f t_x(\overline{0}) + \ldots + y_{f-1}^f t_x([f-1]^1) + y_f^f t_x([f-1]^1) + y_f^f \omega(x \otimes \overline{0} + \overline{0} \otimes x)
\]
\[
= t_x(y^1) + \sum_{p=0}^{f-2} y_p^f y_f^f t_x([f-p]^1, e_3) + y_f^f \omega(x \otimes \overline{0} + \overline{0} \otimes x)
\]
\[
= t_x(y^1) + \sum_{p=0}^{f-2} y_p^f y_f^f x[\overline{p+1}] \otimes x[\overline{0}] + y_f^f(x \otimes \overline{0} + \overline{0} \otimes x)
\]

\[\square\]

The next two results concern the properties of the maps \( A \) and \( B \) which describe the cross–effects of \( u_x \) with respect to addition and the action, respectively.

**Lemma 7.3.** For \( x \in K \) the map
\[
A : H_3\hat{P}_{f,x} \times H_3\hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y, z) \mapsto u_x(yz)
\]
is bilinear.

**Proof.** Take \( x \in K \) and \( y, z \in H_3\hat{P}_{f,x} \). By definition
\[
A(y, z) = u_x(yz) = t_x(yz) - \omega \mu(x, yz) = \omega(\Psi(y, z) - \mu(x, yz)).
\]
Thus Lemmata 4.6 and 7.1 imply that \( A \) is bilinear. \(\square\)

**Lemma 7.4.** For \( x \in K \) define
\[
B : H_3\hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), y \mapsto (u_x(y))^1 - u_x(y^1)
\]
Then
\[
H_3\hat{P}_{f,x} \times H_3\hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y, z) \mapsto B(yz)
\]
is bilinear.

**Proof.** Take \( x \in K \) and \( y, z \in H_3\hat{P}_{f,x} \). Then
\[
(A(y, z))^1 = (u_x(y + z) - (u_x(y) + u_x(z)))^1
\]
\[
= (u_x(y + z))^1 - (u_x(y))^1 - (u_x(z))^1
\]
\[
= B(y + z) + u_x((y + z)^1) - (B(y) + u_x(y^1) + B(z) + u_x(z^1)).
\]
\[
= B(y^1) + A(y^1, z^1)
\]

Thus
\[
(7.1) \quad B(y^1) = (A(y^1))^1 - A(y^1, z^1)
\]
and bilinearity follows from that of \( A \) and the properties of an action. \(\square\)

8. **Examples of Pseudo–Projective 3–Spaces**

In this Section we provide explicit computations for examples of pseudo–projective 3–spaces, including proofs for Theorem 1.1 and Theorem 1.2.

Note that, as abelian group, the augmentation ideal \( K \) of a pseudo–projective 3–space \( P_{f,x} \), as in [3.2], is freely generated by \( \{[\overline{1}] - \overline{0}, \ldots, [\overline{f-1}] - \overline{0} \} \). We consider pseudo–projective 3–spaces, \( P_{f,x} \), with \( x = \bar{x}(\overline{1} - \overline{0}) \) and \( \bar{x} \in \mathbb{Z} \). We compute \( \pi_2(P_{f,x}), H_3(\hat{P}_{f,x}) \), as well as the cross–effects of \( u_x \) for this special case. For \( f = 2 \), the general case coincides with the special case and provides an example where \( \pi_1 \) acts trivially on \( \Gamma\pi_2(P_{2,\bar{x}}) \) and on \( H_3(\hat{P}_{2,\bar{x}}) \), but non–trivially on \( \pi_3(P_{2,\bar{x}}) \).
Lemma 8.1. For \( x = \hat{x}(\overline{1} - \overline{0}) \) with \( \hat{x} \in \mathbb{Z} \),
\[
H_3(\hat{P}_{f,x}) = \{ \hat{y}N \mid \hat{y} \in \mathbb{Z} \} \cong \mathbb{Z},
\]
is generated by the norm element \( N = \sum_{k=0}^{f-1} [\mathbb{K}] \). Hence \( \pi_1 \) acts trivially on \( H_3(\hat{P}_{f,x}) \). Furthermore,
\[
\pi_2(P_{f,x}) = (\mathbb{Z}/\hat{x}\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}.
\]
Hence \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \).

Proof. Take \( x = \hat{x}(\overline{1} - \overline{0}) \) with \( \hat{x} \in \mathbb{Z} \) and \( y = \sum_{k=0}^{f-1} y_k[\mathbb{K}] \in \ker d_x \). Then
\[
d_x(y) = xy = 0 \iff \hat{x} \sum_{k=0}^{f-1} y_k([\mathbb{K} + \overline{1}] - [\mathbb{K}]) = 0
\]
\[
\iff y_{f-1} = y_0 = y_1 = \cdots = y_{f-2} = \hat{y}.
\]
for some \( \hat{y} \in \mathbb{Z} \). Hence \( y = \hat{y}N \).

By (6.2), \( \pi_2(P_{f,x}) = K/xR \). As abelian group, \( K = \ker \varepsilon \) is freely generated by \( \{ [\mathbb{K}] - [\overline{0}] \}_{1 \leq k \leq f-1} \) and hence also by \( \{ [\mathbb{K}] - [k-1] \}_{1 \leq k \leq f-1} \). For \( y = \sum_{k=0}^{f-1} y_k[\mathbb{K}] \in R \) we obtain
\[
xy = \hat{x} \sum_{i=1}^{f-1} y_i([\mathbb{K} - (i-1)]) + \hat{x} y_{f-1}([\overline{1}] - [f-1])
\]
\[
= \hat{x} \sum_{i=1}^{f-1} y_i([\mathbb{K} - (i-1)]) - \hat{x} y_{f-1} \sum_{i=1}^{f-1} ([\mathbb{K} - (i-1)])
\]
\[
= \hat{x} \sum_{i=1}^{f-1} (y_i - y_{f-1}([\mathbb{K} - (i-1)]).
\]
As \( \hat{x}K \subseteq xR \), we obtain \( xR = \hat{x}K \) and hence
\[
\pi_2(P_{f,x}) = K/xR = K/\hat{x}K = (\mathbb{Z}/\hat{x}\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}.
\]
If \( \hat{x} \) is odd, then every element \( \ell \in \Gamma(\pi_2(P_{f,x})) \) has order \( \hat{x} \). If \( \hat{x} \) is even, an element \( \ell \in \Gamma(\pi_2(P_{f,x})) \) has order \( 2\hat{x} \) or \( \hat{x} \). In either case, \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \).

Lemma 8.2. Take \( x = \hat{x}(\overline{1} - \overline{0}) \) and \( y, z \in H_3(\hat{P}_{f,x}) \). Then
\[
A(y, z) = 0.
\]

Proof. By definition,
\[
A(y, z) = u_x(y|z) = t_x(y|z) - \omega \mu(x, y|z) = \omega(\Psi(y, z) - \mu(x, y|z)).
\]
The definition of \( \Psi \) and Lemma 4.4 yield
\[
\Psi(y, z) - \mu(x, y|z) = \hat{y}\hat{z} \sum_{p=1}^{f-1} \sum_{q=0}^{p-1} x[\mathbb{Q}] \otimes x[\mathbb{Q}] + \sum_{q=1}^{f-1} \sum_{p=0}^{f-1} \nabla(x^r, x^s) - \sum_{p=1}^{f-1} Q_p(x) + \sum_{p=0}^{f-1} \nabla(x, x) \mathbb{P}).
\]
Recall that \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \) and note that, by the properties of \( Q \) and \( \nabla \), each summand in the above sum has a factor of \( \hat{x}^2 \).

Lemma 8.3. Let \( \gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x})) \) be the universal quadratic map for the Whitehead functor \( \Gamma \). Take \( q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k, x = \hat{x}(\overline{1} - \overline{0}) \) and \( y = \hat{y}N \). Then
\[
B(y) = -\hat{x}\hat{y}\gamma q(\overline{1} - \overline{0}).
\]
Proof. Note that \(y^\beta = y\) for \(\beta \in \pi_1\). As \(\hat{x}^2 \ell = 0\) for every \(\ell \in \Gamma(\pi_2(P_{f,x}))\), any summand with a factor \(\hat{x}^2\) is equal to 0. By Lemma 7.2,

\[
\Psi_1(y) = \sum_{p=0}^{f-2} \bar{\gamma}^2(\hat{x}(\mathbb{I} - \mathbb{0}) + \bar{\gamma}(\mathbb{I} - \mathbb{0})) + \bar{\gamma}(\mathbb{I} - \mathbb{0}) \otimes [0] + [0] \otimes \hat{x}(\mathbb{I} - \mathbb{0}))
\]

so that \(\bar{\gamma}(\mathbb{I} - \mathbb{0}) \otimes [0] + [0] \otimes (\mathbb{I} - \mathbb{0}))\).

Lemma 6.6 yields

\[
\mu(x, y) = -\sum_{q=0}^{f-1} \sum_{p=0}^{q-1} \hat{x}^2 \bar{\gamma}^2(\mathbb{I} - \mathbb{0}) + \bar{\gamma}(\mathbb{I} - \mathbb{0})) + \sum_{p=0}^{f-1} \nabla_p(\bar{\gamma}\hat{x}(\mathbb{I} - \mathbb{0}))
\]

Thus

\[
B(y) = (u_x(y))^1 - u_x(y\mathbb{1}) = \omega(\Psi_1(y) - (\mu(x, y))^1 + \mu(x, y)) = -\hat{x}y \gamma q(\mathbb{I} - \mathbb{0})).
\]

Together Lemmata 8.1, 8.2 and 8.3 provide a proof of Theorem 1.3.

For \(f = 2\) the special case coincides with the general case and we obtain

**Theorem 8.4.** Let \(X = P_{2,x}\) be a pseudo–projective 3–space with \(x = \hat{x}(\mathbb{I} - \mathbb{0})\), for \(\hat{x} \in \mathbb{Z}\) and \(\hat{x} \neq 0\). Then \(u_x\) is a homomorphism and the fundamental group \(\pi_1 = \mathbb{Z}/2\mathbb{Z}\) acts trivially on \(\Gamma(\pi_2 P_{2,x})\) and on \(H_3 P_{2,x}\). The action of \(\pi_1\) on \(\pi_3 P_{2,x}\) is non–trivial if and only if \(\hat{x}\) is even.

Proof. For \(f = 2\) the augmentation ideal \(K\) is generated by \(k = \mathbb{I} - \mathbb{0}\). Since \(k\mathbb{I} = -k\), the action of \(\pi_1 = \mathbb{Z}/2\mathbb{Z}\) on \(K\) and hence on \(\pi_2 P_{2,x} = K/xR = \mathbb{Z}/\hat{x}\mathbb{Z}\) is multiplication by \(-1\). As the \(\Gamma\)–functor maps multiplication by \(-1\) to the identity morphism, the action on \(\pi_1\) in \(\Gamma(\pi_2 P_{2,x})\) is trivial. The group \(H_3 P_{2,x}\) is generated by the norm element \(N = [0] + [\mathbb{I}]\). As \(N\mathbb{I} = N\), \(\pi_1\) acts trivially on \(H_3 P_{2,x}\). As \(\pi_2 = \mathbb{Z}/\hat{x}\mathbb{Z}\) is cyclic, \(\Gamma_2 = \pi_2\) if \(\hat{x}\) is odd and \(\Gamma_2 = \mathbb{Z}/2\hat{x}\mathbb{Z}\) if \(\hat{x}\) is even, that is,

\[
(8.1) \quad \Gamma_2 = \mathbb{Z}/\gcd(\hat{x}, 2)\hat{x}\mathbb{Z}.
\]

By Lemma 8.3 and 8.1, the action of \(\pi_1\) on \(\pi_3 X\) is non–trivial if and only if \(\hat{x}\) is even. \(\square\)

Theorem 4.1 is a corollary to Theorem 8.3.

**Proof of 7.4**. Note that \(\mathbb{Z}/\hat{x}\mathbb{Z} \otimes K\) is generated by \(\{\alpha_k = q([k] - [k-1])\}_{0 < k < f}\), where \(q : K \to \mathbb{Z}/\hat{x}\mathbb{Z} \otimes \mathbb{Z}\) and \(k \mapsto 1 \otimes k\). Thus \(\Gamma(\pi_2(P_{f,x})) = (\mathbb{Z}/\hat{x}\mathbb{Z} \otimes K) \otimes (\mathbb{Z}/\hat{x}\mathbb{Z} \otimes K)\) is generated by \(\{\gamma q(\alpha_k), [q(\alpha)], q(\alpha_k)\}_{0 < k, 0 < k < f}\). With \(\alpha_0^k = \alpha_{k+1}\) for \(1 < k < f - 1\) and \(\alpha_{f-1}^k = [0] - [f-1] = -\sum_{i=1}^{f-1} \alpha_i\), we obtain, for \(\ell = \sum_{k=1}^{f-1} \ell_k \gamma(\alpha_k) + \sum_{k=2}^{f-1} \sum_{j=1}^{k-1} \ell_{k,j} [\alpha_j, \alpha_k] \in \mathbb{Z}/\hat{x}\mathbb{Z} \otimes K\), that

\[
\sum_{k=1}^{f-1} \ell_k \gamma(\alpha_k) + \sum_{k=2}^{f-1} \sum_{j=1}^{k-1} \ell_{k,j} [\alpha_j, \alpha_k] \in \mathbb{Z}/\hat{x}\mathbb{Z} \otimes K.
\]
\[ \Gamma(\pi_2(P_{f,x})), \]

\[ \ell^2 - \ell = \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k)^2 + \sum_{j=1}^{k-1} \ell_{j,k} [q(\alpha_j), q(\alpha_k)] - \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k) - \sum_{j=1}^{k-1} \ell_{j,k} [q(\alpha_j), q(\alpha_k)] \]

\[ = \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_{k+1}) + \ell_{f-1} \gamma q(-\sum_{i=1}^{f-1} \alpha_i) + \sum_{j=1}^{k-1} \ell_{j,k} [q(\alpha_{j+1}), q(\alpha_{k+1})] \]

\[ + \sum_{j=1}^{f-1} \ell_{j,f-1} [q(\alpha_{j+1}), q(-\sum_{i=1}^{f-1} \alpha_i)] - \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k) - \sum_{j=1}^{k-1} \ell_{j,k} [q(\alpha_j), q(\alpha_k)] \]

\[ = (\ell_{f-1} - \ell_1) \gamma q(\alpha_1) + \sum_{k=2}^{f-1} (\ell_{k-1} - \ell_k + \ell_{f-1} - 2\ell_{k-1,f-1}) \gamma q(\alpha_k) \]

\[ + \sum_{k=2}^{f-1} (\ell_{f-1} - \ell_1,k - \ell_{k-1,f-1}) [q(\alpha_1, q(\alpha_k)] \]

\[ + \sum_{k=3}^{f-1} (\ell_{f-1} + \ell_{j-1,k-1} - \ell_{j,k} - \ell_{j-1,f-1} - \ell_{k-1,f-1}) [q(\alpha_j), q(\alpha_k)]. \]

Thus the sequence (1.1) splits if and only if there is at least one solution of the system of equations

(A) \[ 0 = \ell_{f-1} - \ell_1 \mod 2\hat{x} \]

(Bk) \[ 0 = \ell_{k-1} - \ell_k + \ell_{f-1} - 2\ell_{k-1,f-1} \mod 2\hat{x} \text{ for } 2 \leq k \leq f - 1 \]

(Ck) \[ 0 = \ell_{f-1} - \ell_{1,k} - \ell_{k-1,f-1} \mod \hat{x} \text{ for } 2 \leq j \leq k \leq f - 1 \]

(Dk) \[ 0 = \ell_{f-1} + \ell_{j-1,k-1} - \ell_{j,k} - \ell_{j-1,f-1} - \ell_{k-1,f-1} \mod \hat{x} \text{ for } 2 \leq j \leq k, 2 < k < f - 1. \]

For odd \( f \), a solution of the system is given by \( \ell_{j,k} = 0 \) for \( 1 \leq j \leq k - 1, 1 < k < f - 1, \ell_k = 0 \) for \( k \) odd, and \( \ell_k = \hat{x} \) for \( k \) even. Hence (1.1) splits if \( f \) is odd. It remains to show that there are no solutions for even \( f > 2 \).

For \( 2 \leq j < \frac{1}{2}(f - 2) \), subtract the equation \((D_{i,f-1+i})\) from the equation \((D_{i,f-1+i+1})\) for \( 2 \leq i < j \). Add \((D_{j,f-1})\) and \((C_{f-j})\), then subtract \((C_{f-j+1})\). Adding the resulting equations yields

\[ (E_j) \quad 0 = \ell_{f-1} - \ell_{j,f-1} - \ell_{f-1,j-1,f-1} \mod \hat{x}. \]

Multiplying the equations \((C_{f-1})\) and \((E_j)\), \( 2 \leq j \leq \frac{1}{2}(f - 2) \), by 2 and adding them we obtain

\[ 0 = (f - 2)\ell_{f-1} - \frac{f-2}{2} \sum_{j=1}^{f-2} \ell_{j,f-1} \mod 2\hat{x}. \]

On the other hand, adding the equations \((A)\) and \((B_k)\), \( 1 < k < f - 1 \), the resulting equation is

\[ \hat{x} = (f - 2)\ell_{f-1} - \frac{f-2}{2} \sum_{j=1}^{f-2} \ell_{j,f-1} \mod 2\hat{x}. \]

Hence there are no solutions for \( f \) even. \( \square \)

9. Pseudo–Projective Spaces in Dimension 4

In the final section we consider 4-dimensional pseudo–projective spaces and provide a proof of Theorem 1.5. We begin by constructing a 4-dimensional pseudo–projective space associated to given algebraic data. Namely, take \( f \in \mathbb{Z} \) with \( f \geq 0, x, y \in R = \mathbb{Z}/f\mathbb{Z} \) with \( xy = 0 \) and \( f\varepsilon(x) = 0 \), where \( \varepsilon \) is the augmentation of the group ring, \( R \), so that \( xR \subseteq \ker \varepsilon \). Finally, take \( \gamma \in \Gamma((\ker f\varepsilon)/xR) \). Given such data, \( (f, x, y, \alpha) \), take a 3-dimensional pseudo–projective space \( P_{f,x} \) as in [11.2]. Then the set–theoretic splitting \( u_x \) of the short exact sequence

\[ \Gamma(\pi_2(P_{f,x})) \rightarrow \pi_3(P_{f,x}) \rightarrow H_3(\tilde{P}_{f,x}) \]
implies that every element of $\pi_3(P_{f,x})$ may be expressed uniquely as a sum $u_x(v) + \beta$ with $v \in H_3(\hat{P}_{f,x})$, that is, $xv = 0$, and $\beta \in \Gamma(\pi_2(P_{f,x})) = \Gamma((\ker f \xi)/xR)$, see (6.2). Using $u_x(y) + \alpha \in \pi_3(P_{f,x})$ we obtain the 4-dimensional pseudo–projective space,

$$P = P_{f,x,y,\alpha} = S_1 \cup e^2 \cup e^3 \cup e^4.$$ 

Note that the homotopy type of $P = P_{f,x,y,\alpha}$ is determined by $(f, x, y, \alpha)$ and that every 4–dimensional pseudo–projective space is of this form. The cellular chain complex, $C_*(\hat{P})$, of the universal cover, $\hat{P} = \hat{P}_{f,x,y,\alpha}$, is the complex of free $R$–modules,

$$\langle e_4 \rangle_R \xrightarrow{d_4} \langle e_3 \rangle_R \xrightarrow{d_3} \langle e_2 \rangle_R \xrightarrow{d_2} \langle e_1 \rangle_R \xrightarrow{d_1} \langle e_0 \rangle_R,$$

given by $d_1(e_1) = e_0([\bar{1}] - [\bar{0}]), d_2(e_2) = e_1 N$, that is, multiplication by the norm element, $N = \sum_{i=0}^{f-1} [\bar{i}], d_3(e_3) = e_2 x$, and $d_4(e_4) = e_3 y$. Let $b : R \to \pi_3 P_{f,x}$ be the homomorphism of $R$–modules which maps the generator $[\bar{0}] \in R$ to $b([\bar{0}]) = u_x(y) + \alpha$, so that composition with the projection onto $H_3 \hat{P}_{f,x}$ yields the homomorphism of $R$–modules induced by the boundary operator $d_4$. Thus we obtain the commutative diagram

\[
\begin{array}{ccc}
H_4 \hat{P} & \xrightarrow{b} & \Gamma \pi_2 P \\
\downarrow \quad \downarrow b & & \quad \downarrow j \\
R & \xrightarrow{\hat{P}_{f,x}} & \pi_3 P \\
\downarrow d_4 & & \downarrow h \\
H_3 \hat{P}_{f,x} & \xrightarrow{\pi_3} & H_3 \hat{P}
\end{array}
\]

in the category of $R$–modules, where the middle column is the short exact sequence (6.1) and

\[(9.1) \quad H_4 \hat{P} \xrightarrow{b} \Gamma \pi_2 P \xrightarrow{j} \pi_3 P \xrightarrow{h} H_3 \hat{P}
\]

is Whitehead’s Certain Exact Sequence of the universal cover, $\hat{P} = \hat{P}_{f,x,y,\alpha}$.

Now we restrict attention to the case $f = 2$. Then $\pi_1 = \pi_1 P = \mathbb{Z}/2\mathbb{Z}$ and the augmentation ideal, $K$ is generated by $[\bar{1}] - [\bar{0}]$. Thus

$$x = \hat{x}([\bar{1}] - [\bar{0}]) \quad \text{and} \quad y = \hat{y}([\bar{1}] + [\bar{0}]), \quad \text{for some} \ \hat{x}, \hat{y} \in \mathbb{Z}.$$ 

We assume that $x$ and $y$ are non–trivial, that is, $\hat{x}, \hat{y} \neq 0$.

**Theorem 9.1.** For $P = P_{2,x,y,\alpha}$, with $x$ and $y$ as above, $\pi_1 P = \mathbb{Z}/2\mathbb{Z}$ acts on $\pi_2 P = \mathbb{Z}/\hat{x}\mathbb{Z}$ via multiplication by $-1$, trivially on $H_3 \hat{P} = \mathbb{Z}/\hat{y}\mathbb{Z}$ and via multiplication by $-1$ on $H_4 \hat{P} = \mathbb{Z} = ([\bar{1}] - [\bar{0}])$. The exact sequence (9.1) is given by

\[(9.2) \quad H_4 \hat{P} = \mathbb{Z} \xrightarrow{b} \Gamma \pi_2 P = \Gamma(\mathbb{Z}/\hat{x}\mathbb{Z}) \xrightarrow{j} \pi_3 P \xrightarrow{h} H_3 \hat{P} = \mathbb{Z}/\hat{y}\mathbb{Z}.
\]

Denoting the generator of $\Gamma \pi_2 P$ by $\xi$, the boundary $b$ is determined by

$$b([\bar{1}] - [\bar{0}]) = \hat{x}\hat{y}\xi,$$

and the action of $\pi_1 P$ on $\pi_3 P$ is trivial. As abelian group, $\pi_3 P$ is the extension of $H_3 \hat{P}$ by $\ker b$ given by the image of $-\alpha \in \Gamma \pi_2$ under the homomorphism

$$\tau : \Gamma \pi_2 \xrightarrow{\text{coker } b} \text{coker } b \xrightarrow{\text{coker } b} \text{coker } b = \text{Ext}(\mathbb{Z}/\hat{y}\mathbb{Z}, \text{coker } b).$$

Hence the extension $\pi_3 P$ over $\mathbb{Z}$ determines $\alpha$ modulo $\ker \tau$.

Theorem 1.3 is a corollary to Theorem 9.1.
Proof. As the augmentation ideal $K \cong \mathbb{Z}$ is generated by $k = [\overline{1}] - [\overline{0}]$, the action of $\pi_1 = \mathbb{Z} / 2\mathbb{Z}$ on $K = \pi_2 P_2$ and hence on $\pi_2 P = K / xR = \mathbb{Z} / \hat{x} \mathbb{Z}$ is multiplication by $-1$, since $k [\overline{1}] = -k$. But the $\Gamma$–functor maps multiplication by $-1$ to the identity morphism, so that $\pi_1$ acts trivially on $\Gamma (\pi_2 P)$.

As $d_3(e_3) = e_2 x$, we obtain $H_3 \hat{P}_{2,x} \cong \mathbb{Z}$, generated by the norm element $N = [\overline{1}] + [\overline{0}]$. Since $N [\overline{1}] = N$, the action of $\pi_1$ on $H_3 \hat{P}_{2,x}$ is trivial.

As $d_4(e_4) = e_3 y$, we obtain $H_3 P \cong \mathbb{Z} / y \mathbb{Z}$ and $H_4 \hat{P} \cong \mathbb{Z}$, generated by $k = [\overline{1}] - [\overline{0}]$. Hence the action of $\pi_1$ on $H_4 \hat{P}$ is multiplication by $-1$.

Now let $\xi = ([\overline{1}] - [\overline{0}]) \otimes ([\overline{1}] - [\overline{0}])$ be the generator of $\Gamma (K)$. Note that $v [\overline{1}] = v$ and $\beta [\overline{1}] = \beta$, for $v \in H_3 \hat{P}_{2,x}$ and $\beta \in \Gamma (\pi_2 P)$, since $\pi_1$ acts trivially on both $H_3 \hat{P}_{2,x}$ and $\Gamma (\pi_2 P)$. Lemma 8.3 implies

$$\left( u(v) + \beta \right) [\overline{1}] = -\tilde{x} \tilde{y} \omega (\xi) + u(v) [\overline{1}] + \omega (\beta) [\overline{1}] = -\tilde{x} \tilde{y} \omega (\xi) + u(v) + \omega (\beta).$$

We obtain

$$\tilde{b} (e_4 ([\overline{1}] - [\overline{0}])) = (u(y) + \omega (\alpha)) ([\overline{1}] - [\overline{0}]) = -\tilde{x} \tilde{y} \omega (\alpha) + u(y) + \omega (\alpha) - (u(y) + \omega (\alpha)) = -\tilde{x} \tilde{y} \omega (\xi).$$

By definition of $\tilde{b}$,

$$\pi_3 P = \pi_3 P_{2,x} / \text{im} \tilde{b}.$$

Hence $\pi_1$ acts trivially on $\pi_3 (P)$.

Sequence (9.3) yields the short exact sequence

$$G = \text{coker} \tilde{b} \longrightarrow \pi_3 P \xrightarrow{h} H_3 \hat{P} \cong \mathbb{Z} / y \mathbb{Z},$$

which represents $\pi_3 P$ as an extension of $\mathbb{Z} / y \mathbb{Z}$ by $G = \text{coker} \tilde{b}$. Thus the extension $\pi_3 P$ over $\mathbb{Z}$ determines $\gamma$ modulo the kernel of the map

$$\tau : \Gamma \pi_2 \longrightarrow \text{coker} \tilde{b} \longrightarrow \text{coker} \tilde{b} / y \text{coker} \tilde{b} = \text{Ext} (\mathbb{Z} / y \mathbb{Z}, \text{coker} \tilde{b}) .$$

\[\square\]

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