One estimator, many estimands: fine-grained quantification of uncertainty using conditional inference

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Abstract

Statistical uncertainty has many components, such as measurement errors, temporal variation, or sampling. Not all of these sources are relevant when considering a specific application, since practitioners might view some attributes of observations as fixed.

We study the statistical inference problem arising when data is drawn conditionally on some attributes. These attributes are assumed to be sampled from a super-population but viewed as fixed when conducting uncertainty quantification. The estimand is thus defined as the parameter of a conditional distribution. We propose methods to construct conditionally valid p-values and confidence intervals for these conditional estimands based on asymptotically linear estimators.

In this setting, a given estimator is conditionally unbiased for potentially many conditional estimands, which can be seen as parameters of different populations. Testing different populations raises questions of multiple testing. We discuss simple procedures that control novel conditional error rates. In addition, we introduce a bias correction technique that enables transfer of estimators across conditional distributions arising from the same super-population. This can be used to infer parameters and estimators on future datasets based on some new data.

The validity and applicability of the proposed methods are demonstrated on simulated and real-world data.

1 Introduction

Statistical uncertainty has many components, such as variation induced by measurement errors, temporal variation, sampling, or by randomizing treatment in causal inference. When considering a specific application, not all of these sources are relevant, since practitioners might view some attributes as fixed. For example, in finite-sample causal inference (Splawa-Neyman et al., 1990; Hinkelmann and Kempthorne, 1994; Freedman et al., 2008; Rosenbaum, 2010; Imbens and Rubin, 2015), the subjects under study are viewed as fixed while the uncertainty comes from the treatment assignment. When data are collected from different times and locations, the data scientist might want to consider the locations as fixed while averaging over temporal variation. In those cases, we can model observations as drawn from a conditional distribution given a set of attributes and only part of the overall uncertainty remains.

In this paper, we study statistical inference problems when some attributes of the data are fixed, that means, conditioned on. We propose a set of methods for conditional inference based on an asymptotically linear estimator with different sets of conditioning attributes. The quantities of interest, or estimands, are conditional parameters that depend on the conditioning sets, for which we develop conditionally valid procedures for the construction of conditional confidence intervals and p-values.
We discuss several applications of the proposed framework. When the conditioning sets are arranged in a hierarchical fashion, one can conduct simultaneous inference for several conditional parameters, which helps to gauge the generalizability of a statistical finding to different populations that are close to the super-population. Furthermore, we propose a method for predicting estimators and parameters on future datasets, conditionally on some new data. This may help investigate whether a statistical finding replicates on new data.

1.1 Conditional distributions and conditional parameters

In this section, we introduce conditional parameters as functionals of conditional distributions. Furthermore, we discuss examples in which conditional parameters differ from unconditional parameters.

We start from a conventional super-population setting. Assume we are given \( \mathbf{D} \) independently and identically distributed (i.i.d.) data \((D_1, Z_1), \ldots, (D_n, Z_n)\) from an unknown distribution \( \mathbb{P} \), where \( D_i \in D \) are the observations, and \( Z_i \in Z \) are the attributes we condition on. We consider the unknown parameter \( \theta^0 \in \Omega \) defined as a solution to

\[
\mathbb{E}[s(D, \theta)] = 0,
\]

(1.1)

where \( s : D \times \Omega \rightarrow \mathbb{R} \) is a function. The function \( s \) might not be known a priori as \( s \) might depend on unknown nuisance parameters or unobserved quantities which we suppress in the notation. Here and in the following, we assume that the solution to equation (1.1) is unique. In this case, the parameter \( \theta^0 \) is some characteristic of the super-population distribution \( \mathbb{P} \). In many classical statistical problems, \( \theta^0 \) is a deterministic quantity of interest, and inference of \( \theta^0 \) is based on data from the super-population \( \mathbb{P} \).

When attributes \( Z_i \in Z \) are fixed, \( D_i \) are from the conditional distribution given \((Z_1, \ldots, Z_n)\). In that case, an estimand can be seen as a functional of the conditional distribution. Following Buja et al. (2016, 2019), for a random variable or random vector \( Z \), we define the \( Z \)-conditional parameter \( \theta_{\text{cond}} = \theta_{\text{cond}}(Z) \) as the (unique) solution to

\[
\sum_{i=1}^{n} \mathbb{E}[s(D_i, \theta) \mid Z_i] = 0.
\]

(1.2)

Note that the conditional parameter depends on the observed \( Z_1, \ldots, Z_n \), hence the conditional parameter is random and varies with the sample size \( n \). The \( Z \)-conditional parameter \( \theta_{\text{cond}} \) characterizes the distribution of the \( D_i \) given that the \((Z_1, \ldots, Z_n)\) are fixed at their observed values, as opposed to \( \theta^0 \) that characterizes the super-population. We also remark that conditional parameter can be seen as a generalization of the population parameter in the sense that \( \theta^0 = \theta_{\text{cond}}(\emptyset) \), i.e., the unconditioned super-population parameter \( \theta^0 \) is also a conditional parameter for the empty set. Throughout the paper, we assume that the \( Z_i \)'s are i.i.d. from the super-population to enable simultaneous inference for different populations corresponding to parameters of different conditional distributions.

In the following, we consider some examples to gain intuition on how conditional parameters differ from unconditional parameters.

**Model misspecification.** In linear regression under model misspecification, the conditional parameter is usually different from the unconditional parameter as discussed in Example 1.1. If the model is misspecified, the data scientist can add additional terms (such as higher order interactions and additional variables) to the model in order to better approximate the regression surface. In practice however, adding additional terms to the model will usually not completely remove the issue of model misspecification. Thus, conditional parameters are generally expected to differ from marginal parameters.

**Example 1.1 (Conditional least-squares).** Assume that data \( D = (X, Y) \) consists of a target variable \( Y \) and predictors \( X \in \mathbb{R}^p \). We consider the ordinary least square (OLS) parameter, where the
population parameter is $\theta^0 = \arg \min \mathbb{E}[(Y - X^\top \theta)^2]$, the least-square projection of $Y$ on $X$. Taking derivatives, the estimating function is $s(D, \theta) = 2X(Y - X^\top \theta)$. The conditional parameter is the solution to (1.2), i.e.,

$$\theta^\text{cond}_n = \arg \min_b \frac{1}{n} \sum_{i=1}^n \mathbb{E}[(Y_i - X_i^\top b)^2 \mid Z_i].$$

Thus, $\theta^\text{cond}_n$ is the least-square projection of $Y$ on $X$ when the observations are sampled from the conditional distribution given $(Z_1, \ldots, Z_n)$. If $Z_i = X_i$, the parameter $\theta^\text{cond}_n$ can be thought of as the regression coefficient for a set of subjects with fixed regressors, averaging over measurement noise. In model-based statistical inference, if $Z_i = X_i$ and $Y_i = X_i^\top \theta^0 + \epsilon_i$ for $\epsilon_i$ independent of $X_i$, the conditional parameters are identical to the super-population parameter. In practice, however, this will usually not hold. Thus the conditional parameter might be different for different realizations of conditioning sets $X_i$.

**Conditioning on variables that are not in the model.** In the example above, $Z$ might be a variable that is not included in the set of predictors (for example, one might want to learn about the relationship between input parameters and emissions for a specific set of industrial plants). If $Z$ is correlated with both the predictors and the residuals, conditioning on the variable can change parameters in the sense that $\theta^0 \neq \theta^\text{cond}_n$.

**Finite-sample inference and treatment effect heterogeneity.** Finite sample causal effects are a common target of inference in causality (Imbens and Rubin, 2015). In the social sciences, for example, it is expected that individuals react differently to treatments. Thus, treatment effects are very specific to the population under study. As we discuss below, conditional inference can be used to understand how different populations (characterized by different conditioning sets) react to the treatment.

**Example 1.2** (Conditional treatment effect). Assume that we are given i.i.d. data $D = (T, X, Y(1), Y(0))$ sampled from a super-population $\mathbb{P}$, with treatment indicator $T \in \{0, 1\}$, covariates $X$ and potential outcomes $Y(1), Y(0)$. The data scientist observes $(T, X, Y)$, where $Y = T Y(1) + (1 - T) Y(0)$.

For unit $i$, the (unobserved) individual treatment effect is $Y_i(1) - Y_i(0)$. The population treatment effect $\theta^0 = \mathbb{E}[Y(1) - Y(0)]$ can be written as a solution to $\mathbb{E}[s(D, \theta)] = 0$, where $s(D, \theta) = Y(1) - Y(0) - \theta$.

There are many choices of conditioning variables $Z$. The finite-population perspective is equivalent to conditioning on the (unobserved) potential outcomes $Z_i = (Y_i(1), Y_i(0))$, in which case the conditional parameter is the finite-sample average treatment effect

$$\theta^\text{cond}_n = \frac{1}{n} \sum_{i=1}^n (Y_i(1) - Y_i(0)).$$

(1.3)

a characterization of the population where subjects have fixed (but only partially observed) potential outcomes. This is commonly the target of inference in finite-sample causal inference (Speraw-Neyman et al., 1990; Hinkelmann and Kempthorne, 1994; Freedman et al., 2008; Rosenbaum, 2010; Imbens and Rubin, 2015). By conditioning on potential outcomes, we also condition on measurement noise and temporal variation and only account for the randomness in treatment assignment. Note that this example is more involved than the preceding example as we are conditioning on variables that are unobserved. In some cases, it can be more meaningful to condition on covariates and average over measurement noise and temporal variation, leading to

$$\theta^\text{cond}_n = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i(1) - Y_i(0) \mid X_i],$$

3
which corresponds to the population characterized by the covariates \(\{X_i\}_{i=1}^n\). Finally, conditioning the empty set, we obtain the super-population causal effect \(\theta^0 = E[Y(1) - Y(0)]\), which represents the super-population the units are sampled from.

### 1.2 Our contributions

In classical statistics, the estimand is usually the parameter \(\theta^0\), where statisticians often construct a \((1 - \alpha)\)-confidence interval \(\hat{C}(D_1, \ldots, D_n)\) for \(\theta^0\) and desire

\[
P(\theta^0 \in \hat{C}(D_1, \ldots, D_n)) \approx 1 - \alpha,
\]

for data \(D_1, \ldots, D_n\) from the super-population. However, when data are sampled from the conditional distribution with some attributes fixed, the estimand is a conditional parameter corresponding to the conditional distribution, and an inference procedure is ideally conditionally valid. Formally, a \((1 - \alpha)\)-confidence interval \(\hat{C}(D_1, \ldots, D_n)\) for \(\theta^\text{cond} = \theta^\text{cond}(Z)\) should satisfy

\[
P(\theta^\text{cond} \in \hat{C}(D_1, \ldots, D_n) \mid Z_1, \ldots, Z_n) \approx 1 - \alpha,
\]

i.e., the data \(D_1, \ldots, D_n\) are from the conditional distribution given \((Z_1, \ldots, Z_n)\). This formulates the conditional inference problem, where both the estimand and inferential guarantee are conditional. Thus, our goal is to analyze the uncertainty of the estimator when \((Z_1, \ldots, Z_n)\) are fixed. With different conditioning attributes, this leads to a fine-grained analysis of various sources of uncertainty.

In the following, we summarize the contributions of the paper with a brief touch on two application settings to be discussed in subsequent sections.

**Conditional inference.** In this paper, we build a framework of conditional inference for statistical estimation problems where certain attributes are sampled from a super-population but viewed as fixed or conditioned on. The target of inference is a conditional parameter that relates to the conditional attributes, as opposed to a super-population parameter. Similarly, statistical inference is valid conditionally when the data are drawn from the conditional distribution, as opposed to inferential guarantees that hold only marginally. We establish conditionally valid confidence intervals for conditional parameters, and derive conditionally valid \(p\)-values for (random) hypotheses that certain conditional parameters are zero.

We investigate several problems as applications of the conditional inference framework.

**Building a hierarchy of statistical evidence.** We study issues of multiple testing for conditioning sets that are arranged in a hierarchical fashion. As an example, data might be collected from patients at various hospitals in various counties from various states. We can ask the following questions of generalizability, in a hierarchical fashion: if the data of patients are drawn conditionally on the observed states, does the statistical finding generalize to other patients from the observed hospitals? Does the finding generalize to patients from the observed counties? Will it generalize across states?

We propose a multiple hypothesis testing procedure for a hierarchy of conditional parameters. Specifically, when the corresponding conditioning sets are nested in a hierarchical fashion, we show that a naive procedure controls a conditional family-wise-error-rate. In that way, statistical inference for a collection of conditioning sets is guaranteed to be valid simultaneously. Since conditional parameters characterize populations, this also allows a fine-grained evaluation of generalizability to populations characterized by conditioning sets. We also propose a multiple hypothesis testing procedure to combine findings across independent studies that controls conditional false-discovery rate when data are drawn with some attributes fixed.
Transfer across conditional distributions. Another application studied in this paper is transferring conditional parameters across data sets (termed "transductive inference"). To be concrete, assume we have a dataset \( \{(D_i, Z_i)\}_{i=1}^n \) and would like to conduct inference for a new dataset \( \{(D_j^{\text{new}}, Z_j^{\text{new}})\}_{j=1}^m \) where only the attributes \( \{Z_j^{\text{new}}\}_{j=1}^m \) are observed. The inferential target is the new conditional parameter \( \theta_n^{\text{cond}}(Z^{\text{new}}) \) as a solution to

\[
\sum_{j=1}^m \mathbb{E}[s(D_j^{\text{new}}, \theta) \mid Z_j^{\text{new}}] = 0,
\]

which characterizes the population of \( D_j^{\text{new}} \) given the observed new attributes.

We apply the conditional inference framework to the transductive inference problem and develop a conditionally valid inference procedure for \( \theta_n^{\text{cond}}(Z^{\text{new}}) \) based on observations \( \{(D_i, Z_i)\}_{i=1}^n \) where the new data are from the same marginal distribution as the original data. This can be used to gauge whether a statistical finding will replicate on the new dataset that is drawn from the same marginal distribution, but a different conditional distribution. We also discuss an example that allows for conditional inference across different marginal distributions.

Finally, we apply the proposed methods to simulated and real-world data, which shows solid empirical performance and applicability of the proposed methods.

1.3 Related work

Several strands of literature have touched conditional estimation or inference, usually with different guarantees or targets from ours. We discuss related literature as follows.

Finite-sample causal inference. In causal inference, there is a long tradition of finite-sample inference (Splawa-Neyman et al., 1990; Hinkelmann and Kempthorne, 1994; Freedman et al., 2008; Rosenbaum, 2010; Imbens and Rubin, 2015). In this literature, it is common to condition on potential outcomes and derive bounds for the asymptotic variance of estimators of causal effects. Conditioning on potential outcomes can be seen as a special case of our setting. However, simultaneous testing of different conditional parameters necessitates assuming a super-population. Methods for finite-sample causal inference usually do not make super-population assumptions and conditional inference results are usually derived on a case-by-case basis.

Classical conditional inference. A classical line of work (Hinkley, 1980; Cox and Reid, 1987) tackles inference problems in a conditional fashion by conditioning on ancillary statistics or estimators of nuisance parameters, stemming from the ideas of Fisher (Fisher, 1935a,b). For example, see a review in Casella (1992). One strand uses conditional inference to reduce the effect of nuisance parameters, including Cox and Reid (1987). Our framework directly conditions on some attributes of the data instead of summary statistics, which leads to a different parameter, different interpretation and different inferential guarantees than the traditional ones. Another strand under the name of conditional inference uses conditioning to induce relevance of probabilistic analysis to the data at hand, including permutation tests (Fisher, 1936; Ernst et al., 2004; Edgington and Onghena, 2007) and methods in testing categorical data (Agresti, 2003), where inference relies on the conditional distribution of test statistic given the observations. Our framework also works by conditioning on some attributes hence share similar spirits as the classical work in that the inference is closely related to the data at hand. However, we condition on sets of variables that make the observations non-exchangeable, thus permutation tests do not apply in our setting. In addition, we consider a large class of semi-parametric and parametric estimators that do not have a structure that makes permutation inference feasible, even in the marginal case.

Conditional parameter with random covariates. Abadie et al. (2014) quantify the asymptotic deviation of estimators from conditional parameters for maximum likelihood and method of moment
estimators. However, compared to the literature above, the authors focus on marginal inference. Also related is Buja et al. (2016) and Buja et al. (2019) who argue that models should be seen as approximations. The authors define conditional parameters and derive marginally valid asymptotics for the deviation of the estimator from the conditional parameter. The approach outlined in Buja et al. (2016) and Buja et al. (2019) argues to treat the covariates as random and focuses on marginal inference.

**Fixed-design regression.** Kuchibhotla et al. (2018) investigate linear regression in settings where all regressors are fixed. Abadie et al. (2020) derive central limit theorems for the deviation of regression estimators from finite-sample parameters in the case where some regressors are fixed. Compared to our setting, the authors do not assume that the data is sampled from a super-population but make a linear model on potential outcomes that can be restrictive. Similarly, Andrews et al. (2019) derive conditionally valid confidence intervals for linear moment models. Our setting is more general in the sense that we discuss conditional inference based on any asymptotically linear estimator and more restrictive in that we assume that the data is drawn from a super-population.

**Mixed effect models and generalizability theory.** The discussed approach has similarities to mixed effect models and generalizability theory (Pinheiro and Bates, 2006; Brennan, 2001), since we also obtain variance decompositions. However, compared to this literature, we do not make restrictive modelling assumptions about the relationship between variables or stringent assumptions about the error distribution. The proposed conditional inference procedures can be applied to a large variety of parametric and semi-parametric estimators.

**Missing data literature.** The transductive inference part of the paper is related to the missing data literature, see for example Tsiatis (2007). To be specific, in our paper, inference for the new conditional parameter under covariate shifts is based on a well-known efficient estimator from the missing data literature (Robins et al., 1994). Compared to the usual marginal guarantees in the literature, the proposed conditional inference framework allows us to establish conditional guarantee when transferring to a partially observed dataset with a refined uncertainty quantification under partial observations.

### 1.4 Outline of the paper

The rest of the paper is organized as follows.

- Section 2 establishes the inference procedure for conditional parameters based on estimable quantities from data. In particular, Section 2.1 proposes conditionally valid confidence interval and p-value for random hypothesis testing. Section 2.2 discusses estimation of the conditional variance, which is crucial for conditional inference.

- In Section 3 we discuss applications of the conditional inference framework. In Section 3.1 we describe how the proposed framework can be used to gauge the generalizability of a statistical parameter across conditional distributions, with rigorous control of a conditional family-wise error rate. Section 3.2 discusses how the proposed framework can be used to conduct transfer learning of conditional parameters. In Section 3.3 we consider the multiple testing problem with false discovery rate, while Section 3.4 discusses the transfer learning task with distributional shifts.

- In Section 4, we demonstrate the performance of our methods in simulations, including a hierarchy example in Section 4.1 and a transductive inference example in Section 4.2.

- In Section 5, we apply the conditional inference methods to a hierarchical setting in Section 5.1 and a prediction task in Section 5.2.
1.5 Notation

In this paper, we use the conventional notations $o_p(1)$, $O_p(1)$, etc. We say $n \asymp m$ if there exists some constants $c, C > 0$ such that $cn \leq m \leq Cn$ when $m, n$ are sufficiently large. We say that a sequence of random variables $X_n \leq Y_n + o_p(1)$ if for any $\varepsilon > 0$, $\lim_{n \to \infty} P(X_n > Y_n + \varepsilon) = 0$, and similarly $Y_n \geq X_n - o_p(1)$ if for any $\varepsilon > 0$, $\lim_{n \to \infty} P(Y_n < X_n - \varepsilon) = 0$. We use $Z$ to denote a set of random variables, which, in most cases, is the set of the random variables we condition on. We use $Z$ to denote the corresponding random variable or random vector. For any positive integer $n$, we denote $[n] = \{1, \ldots, n\}$.

2 Conditional Inference

In this and the following section, we present our main theoretical results. In Section 2.1, we derive asymptotically valid conditional confidence intervals and $p$-values based on quantities that can be estimated from data. In Section 2.2 we give recommendations on how to construct conditional confidence intervals and $p$-values in practice.

2.1 Conditionally valid inference

In the following, we assume that the data scientist has access to an estimator $\hat{\theta}_n$, where

$$\sqrt{n}(\hat{\theta}_n - \theta^0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(D_i) + o_p(1), \tag{2.1}$$

for some $\phi \in L_2(P)$ with mean zero. Many parametric and semi-parametric estimators are asymptotically linear in standard asymptotic settings, see for example van der Vaart (1998) or Tsiatis (2007). Under regularity conditions (Buja et al., 2016), conditional parameters (1.2) are asymptotically linear with expansion

$$\sqrt{n}(\theta^\text{cond}_n - \theta^0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[\phi(D_i) | Z_i] + o_p(1). \tag{2.2}$$

We refer the readers to Appendix C.1 for detailed discussions on the technical conditions for the asymptotic linearity (2.2). In particular, the two examples proposed in Section 1.1 are concrete instances for the linearity in equation (2.2). Let us revisit the examples to gain more intuition.

Example 2.1 Revisited. In the least-squares example with $s(D, \theta) = 2X(Y - X^\top \theta)$,

$$\theta^0 = \left(\mathbb{E}[X_iX_i^\top]\right)^{-1}\mathbb{E}[X_iY_i], \quad \hat{\theta}_n = \left(\sum_{i=1}^{n} X_iX_i^\top\right)^{-1}\sum_{i=1}^{n} X_iY_i,$$

$$\theta_n^\text{cond} = \left(\sum_{i=1}^{n} \mathbb{E}[X_iX_i^\top | Z_i]\right)^{-1}\sum_{i=1}^{n} \mathbb{E}[X_iY_i | Z_i].$$

Therefore, if $D$ has finite fourth moments and if $\mathbb{E}[XX^\top]$ is positive definite, the asymptotic linearity (2.1) and (2.2) hold with influence function

$$\phi(D_i) = \left(\mathbb{E}[X_iX_i^\top]\right)^{-1}X_i(Y_i - X_i^\top \theta^0).$$

Actually, under regularity conditions, a modification to classical $Z$-estimator analysis shows that general conditional $Z$-estimators defined in equations (1.1) and (1.2) admit the expansions (2.1) and (2.2). See Appendix C.1 for a discussion.
Example 2.2 Revisited. In the conditional treatment effect case, for simplicity we assume the treatment assignment is completely randomized with $P(T = 1) = 0.5$. Thus $\theta^0 = E[Y_i(1) - Y_i(0)] = 2E[T_iY_i - (1 - T_i)Y_i]$. The difference-in-mean estimator $\hat{\theta}_n = \frac{2}{n} \sum_{i=1}^{n} [T_iY_i - (1 - T_i)Y_i]$ satisfies (2.1) with influence function

$$\phi(D_i) = 2(T_iY_i - (1 - T_i)Y_i) - 2E[T_iY_i - (1 - T_i)Y_i].$$

The conditional parameter (1.3) admits the linear expansion (2.2), as

$$\sqrt{n}(\theta_n^c - \theta^0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i(1) - Y_i(0)) - 2E[T_iY_i - (1 - T_i)Y_i] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[\phi(D_i) | Y_i(1), Y_i(0)].$$

Under the linearity in equations (2.1) and (2.2), the random variable $\sqrt{n}(\hat{\theta}_n - \theta_n^c)$ converges in distribution to a centered Gaussian random variable with variance

$$\sigma^2 = E \left[ (\phi(D) - E[\phi(D) | Z])^2 \right]$$

under the marginal distribution $P$. For conditional inference, we need a more refined statement involving distributions conditional on $Z$. Specifically, we have to characterize the behavior of $\sqrt{n}(\hat{\theta}_n - \theta_n^c)$ under $P[\cdot | Z]$, where $Z = (Z_1, \ldots, Z_n)$. We begin with the following two mild regularity conditions.

**Assumption 2.1** (Asymptotic linearity). The estimator $\hat{\theta}_n$ and conditional parameter $\theta_n^c$ satisfy equations (2.1) and (2.2), respectively.

**Assumption 2.2** (Moment condition). The influence function satisfies $E[|\phi(D)|^4] < \infty$.

The following proposition shows that asymptotic normality also holds in the conditional sense under the above regularity conditions, whose proof is deferred to Appendix C.2.

**Proposition 2.3.** Under Assumptions 2.1 and 2.2, for any $x \in \mathbb{R}$, it holds that

$$P \left( \sqrt{n}(\hat{\theta}_n - \theta_n^c) \leq x | Z \right) = \Phi(x/\sigma) + o_P(1),$$

where $\sigma^2 = E \left[ (\phi(D) - E[\phi(D) | Z])^2 \right]$ and $\Phi$ is the cumulative distribution function of a standard Gaussian random variable.

As discussed in Section 1.3, many results in the literature are close to Proposition 2.3. However, these results either focus on marginal inference or special cases. The asymptotic conditional result in Proposition 2.3 relies on a conditional central limit theorem, which has been established in literature (Dedecker and Merlevède, 2003; Grzenda and Zieba, 2008). We include Lemma G.2 in the Appendix for completeness.

Using this result, we construct asymptotically valid confidence intervals for the conditional parameter $\theta_n^c$ as follows. The proof is deferred to Appendix C.2.

**Theorem 2.4** (Asymptotic validity of conditional confidence intervals). Suppose Assumptions 2.1 and 2.2 hold. If $\hat{\sigma} \xrightarrow{P} \sigma > 0$, then for any $\alpha \in (0, 1)$, it holds that

$$P \left( \theta_n^c \in \left[ \hat{\theta}_n - z_{1-\alpha/2}/\sqrt{n}, \hat{\theta}_n + z_{1-\alpha/2}/\sqrt{n} \right] | Z \right) \xrightarrow{P} 1 - \alpha.$$
Remark 2.5. In unconditional inference, a similar protocol is carried out with an estimator of the (unconditional) asymptotic variance, usually of the form $\sigma^2_0 := \text{Var}(\phi(D))$. The asymptotic variance (2.3) we utilize here is always no greater than $\sigma^2_0$, as

$$
\sigma^2 = \text{E} \left[ (\phi(D) - \text{E}[\phi(D) | Z])^2 \right] = \text{Var} \ (\phi(D)) - \text{Var} \ (\text{E}[\phi(D) | Z]) \leq \text{Var} \ (\phi(D)).
$$

Take the least-square example as a special case. The linear expansion in equations (2.1) and (2.2) hold instead for notational simplicity. Define the two-sided (conditional)

$$
\text{P} (H_0(Z) \text{ true}, p < \alpha \mid Z) \leq \alpha + o_p(1),
$$

as $n \to \infty$, where $Z = (Z_1, \ldots, Z_n)$ and $\sigma^2$ is given by equation (2.3).

Theorem 2.7 (Asymptotic validity of conditional $p$-value). Suppose Assumptions 2.1 and 2.2 hold. If $\tilde{\sigma} \to \sigma > 0$, then for any $\alpha \in (0, 1)$, it holds that

$$
P \left( H_0(Z) \text{ true}, p < \alpha \mid Z \right) \leq \alpha + o_p(1),
$$

Remark 2.6. In the well-specified linear regression case, for the first entry we have

$$
\sigma^2 = \sigma^2_0 = (\text{Var} \ (X \epsilon))_{11},
$$

the estimation of which is well established. However, we may hesitate to assume the well-specification of the linear model. Meanwhile, perfect knowledge of conditional distributions or even the formula of the influence function might not be available. Thus, in practice we have to estimate the asymptotic variance (2.3), which will be discussed in Section 2.2.

Now let us turn to hypothesis testing. We consider the (random) hypothesis

$$
H_0(Z) : \theta_n^\text{cond}(Z) = 0.
$$

We use the notation $\theta_n^\text{cond}(Z)$ here to highlight that $\theta_n^\text{cond}$ depends on the random variable (vector) $Z$, and so does the hypothesis $H_0(Z)$. Thus, whether $H_0(Z)$ is true or not is random as well. We will have to take this randomness into account when defining error rates. Hereafter we use $\theta_n^\text{cond}$ and $H_0$ instead for notational simplicity. Define the two-sided (conditional) $p$-value

$$
p = 2 \left( 1 - \Phi \left( \frac{\sqrt{n} \hat{\theta}_n}{\tilde{\sigma}} \right) \right),
$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution, and $\hat{\sigma}$ is an estimator of $\sigma > 0$. For any $\alpha \in (0, 1)$, we reject $H_0$ at level $\alpha$ if $p < \alpha$.

2.2 Estimation of variance

Theorem 2.7 allows us to construct asymptotically valid $p$-values for conditional parameters. A consistent estimator $\tilde{\sigma}$ for the conditional asymptotic variance (2.3) is crucial for the construction of confidence intervals and $p$-values. However, the explicit formula of $\phi(D)$, if available, often involves the parameter $\theta^0$. Meanwhile, the conditional distribution of $D$ given $Z$ is also generally unknown. In the sequel, we address these challenges by approximating influence functions and conditional expectations. We propose a generic estimation procedure consisting of two steps: estimating influence functions and estimating the conditional variance.
The first step is to estimate the influence function for each datapoint, \( \hat{\phi}_i := \phi(D_i) \), where \( \phi \) is the influence function in linear expansions (2.1) and (2.2). For now we assume the explicit formula \( \phi(x; \theta^0) \) of the influence function is known, where \( \theta^0 \) is the unconditional parameter. In that case, one may employ a plug-in estimator of the influence function

\[
\hat{\phi}(x) = \phi(x; \hat{\theta}_n),
\]

where \( \hat{\theta}_n \) is the estimator from observed data.

**Remark 2.8.** When the explicit formula is not available, one may estimate the influence function via a leave-one-out technique

\[
\hat{\phi}_i = \hat{\phi}(X_i) = n(\hat{\theta}_n - \hat{\theta}_{(-i)}),
\]

where \( \hat{\theta}_n \) is the Z-estimator with the whole sample, and \( \hat{\theta}_{(-i)} \) is the Z-estimator using sample excluding \( X_i \). In the literature of robust statistics (Hampel et al., 2011), we have \( \phi(X_i) = \hat{\phi}_i + o_P(1/\sqrt{n}) \) when the parameter, viewed as a functional of distributions, is Gâteaux differentiable at the empirical distribution \( \hat{F}_n \). The leave-one-out technique is also used in the celebrated Jackknife method (Efron and Stein, 1981) to estimate the variance of estimators. The analysis of the leave-one-out approximation (2.8) is beyond the scope of this paper and we suggest it as an heuristic alternative for now. The performance of variance estimation with this approach will be empirically evaluated in the simulations, for details see Section 4.1 and Appendix A.1.2.

In the second step, we propose to estimate the variance \( \sigma^2 \) by regressing \( \hat{\phi}_i \) on \( Z_i \) using machine-learning tools such as random forest or smoothing splines. To be specific, let \( \hat{\phi}_i = \hat{\phi}(D_i) \) be the estimated influence functions from the first step. We obtain \( \hat{\varphi}(z) = \hat{E}[\hat{\phi} \mid Z = z] \) for \( \varphi(Z) = E[\phi(D) \mid Z] \). In this step, we treat the function \( \hat{\phi} \) as fixed and the estimation of the regression surface is independent of the estimation of \( \hat{\phi} \). This can be achieved via data-splitting, which will be discussed in more detail in Algorithm 1. Then on another hold-out data set \( \{D_i, Z_i\}_{i=1}^n \) we estimate the conditional variance via

\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (\hat{\phi}(D_i) - \hat{\varphi}(Z_i))^2.
\]

We summarize the algorithm as follows.

**Algorithm 1** Estimation of Conditional Variance

1. **Input:** Three-fold datasets \( D^{(j)} = \{(D^{(j)}_i, Z^{(j)}_i)\}_{i=1}^n, j = 1, 2, 3 \).
2. Estimate \( \hat{\phi}(\cdot) \) by equation (2.7) using \( D^{(1)} \).
3. Estimate \( \hat{\varphi}(z) = \hat{E}[\hat{\phi} \mid Z = z] \) by regressing \( \{\hat{\phi}^{(2)}_i = \hat{\phi}(D^{(2)}_i)\}_{i=1}^n \) on \( \{Z^{(2)}_i\}_{i=1}^n \) in \( D^{(2)} \).
4. Estimate \( \hat{\sigma}^2 \) by equation (2.9) on \( D^{(3)} \).
5. **Output:** \( \hat{\sigma}^2 \).

**Remark 2.9.** In practice, one might repeat this algorithm over all permutations of \( j = 1, 2, 3 \) and average the resulting variance estimates. Under mild consistency conditions to be specified later, \( \hat{\sigma}^2 \) is asymptotically equivalent to the sample average of \( (\phi(D_i) - \varphi(Z_i))^2 \) in \( D^{(3)} \), as shown in equation (2.12). This allows us to employ each fold as \( D^{(3)} \) then average to further reduce variance in the estimation.

The following proposition shows that Algorithm 1 yields consistent estimators of \( \sigma^2 \) under consistency conditions on the approximation of influence functions and regression. The proof is deferred to Appendix D.
Proposition 2.10 (Consistency of $\hat{\sigma}^2$). Suppose the estimation of influence functions is consistent in the sense that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (\hat{\phi}(D_i^{(2)}) - \phi(D_i^{(2)}))^2 \mid D^{(1)} \right] = o_P(1), \quad (2.10)$$

and the regression step is consistent such that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \hat{\phi}(Z_i^{(3)}) - \mathbb{E}[\hat{\phi}(D) \mid Z = Z_i^{(3)}] \right)^2 \mid D^{(1)}, D^{(2)} \right] = o_P(1), \quad (2.11)$$

where $\phi(\cdot)$ is the true conditional mean function, and $\hat{\phi}(\cdot)$ is viewed as fixed conditioning on $D^{(1)}$. Then the output $\hat{\sigma}^2$ of Algorithm 1 satisfies

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\phi(D_i^{(3)}) - \varphi(Z_i^{(3)}))^2 + o_P(1) \quad (2.12)$$
as $n \to \infty$, implying $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$.

Remark 2.11. Proposition 2.10 completes the picture of conditional inference by showing how to obtain a consistent estimator of $\sigma^2$. We highlight that the only additional condition here is the consistency of the variance estimation procedure. Equation (2.10) holds for the plug-in estimator (2.7) with $\hat{\theta}$ estimated from $D^{(1)}$ if the influence function is sufficiently smooth. Equation (2.11) holds for a couple of nonparametric regression methods if $\hat{\phi}(\cdot)$, viewed as a fixed function, is sufficiently smooth, for example, localized nonparametric methods like kernel regression (Nadaraya, 1964; Watson, 1964), local polynomial regression (Cleveland, 1979; Cleveland and Devlin, 1988), smoothing spline (Green and Silverman, 1993) and modern machine learning methods including regression trees (Breiman et al., 1984) and random forests (Ho, 1995), to name a few.

3 Applications

In this section, we demonstrate several applications of the conditional inference framework. In Section 3.1 we will discuss how conditional inference can be used to construct hierarchies of statistical evidence, which evaluates the generalizability of statistical findings in a disciplined way. Testing such hierarchies raises questions of multiple testing, for which we propose methods with rigorous error control. In Section 3.2, we apply the conditional inference framework for conditional transfer learning, i.e. transfer of parameters across conditional distributions, with conditionally valid confidence statements.

In Section 3.3 we consider the multiple hierarchy setting, where an adaptive multiple testing procedure with controlled conditional FDR is proposed. In Section 3.4 we tackle the transductive inference with covariate shifts.

3.1 A hierarchy of evidence

As introduced in Section 1.2, beyond inference for a single conditional parameter, one might be interested in a handful of characteristics to condition on, which represents different populations for generalization. In this section, we study the inference for a hierarchy of conditional parameters with increasing conditioning sets and establish a simple procedure with asymptotic conditional FWER control. We also consider simultaneous inference for populations across independent studies with asymptotic conditional FDR control, which is deferred to Appendix E.3.

We instantiate the scenario by formulating the hierarchy of conditional parameters. Table 1 provides an illustration of the hierarchical structure.
where we condition on everything observed. The lowest level is $Z_{m}^m$ parameter, where we condition on nothing. From higher level ($Z_{m}^{k}$ up to state. For each $\sigma_j$ state, $Z_{m}^k$ we use the convention $Z_{m}^0 = \varnothing$, $Z_{m}^1 \supseteq \ldots \supseteq Z_{m}^m$ parameter linearity of $\theta_k^m - \theta_0$ decreases. Between each two adjacent levels $Z_{j}^j, Z_{j+1}^{j+1}$, the difference is asymptotically normal distributed and independent of the lower level, with variance equal to the gap $\sigma_j^2 - \sigma_{j+1}^2 = \mathbb{E}[(\mathbb{E}[\phi(D_i) | Z_{j+1}^{j+1}] - \mathbb{E}[\phi(D_i) | Z_{j}^{j}])^2].$

Let $Z_{m}^0 \subseteq Z_{m}^1 \subseteq \ldots \subseteq Z_{m}^m$ be a sequence of nested sets of random variables of interest, where we use the convention $Z_{m}^0 = \varnothing$. Also, we denote $Z_{m}^k$ as the random vector of all variables in the set $Z_{m}^k$ for each $k \in [m]$. In the example mentioned in Section 1.2, we may set $Z_{1}^1$ as the indicator of state, $Z_{2}^2$ as the indicators of state and county, and $Z_{m}^m$ as the largest set, the indicators of hospital up to state. For each $k \in [m]$, we denote the conditional parameter as $\theta_k^m = \theta_k^{\text{cond}}(Z_{m}^k)$ where, with a slight abuse of notation, $Z_{m}^k = (Z_{1}^k, \ldots, Z_{m}^k)$ are the realizations. Thus the conditional parameters characterize different populations of interest. The variances for each conditioning set is denoted in analogy to equation (2.3) as

$$\sigma_k^2 = \mathbb{E}[(\mathbb{E}[\phi(X) | \sigma_k] - \mathbb{E}[\phi(X) | Z_{m}^k])^2] > 0, \quad k = 1, \ldots, m.$$ (3.1)

We assume that the data scientist has access to consistent estimators $\{\hat{\sigma}_k^2\}_{k=0}^m$, for example, via the approach discussed in the previous section. Since the conditioning sets $\{Z_{m}^k\}_{k=0}^m$ are increasing, we have $\hat{\sigma}_0^2 \geq \hat{\sigma}_1^2 \geq \ldots \geq \hat{\sigma}_m^2$. Without loss of generality, we also sort the estimators so that $\hat{\sigma}_0^2 \geq \hat{\sigma}_1^2 \geq \ldots \geq \hat{\sigma}_m^2.$ Correspondingly, for each level $k \in [m]$, the random hypothesis is defined as

$$H_k^0 = H_0(Z^k) : \theta_k^n = 0, \quad k = 0, \ldots, m.$$ (3.2)

In this way, for each $k \in [m]$, $H_k^0$ is a hypothesis for the population given $Z^k$, which varies from the finite sample at hand to the super-population the units are sampled from. The hierarchy of hypotheses $\{H_0^k\}_{k=0}^m$ thus evaluates how the statistical findings from $\hat{\theta}_n$ generalizes to various populations. We define $p$-values $p_k$ and test statistics $T_k$ as

$$p_k = 2(1 - \Phi(T_k)), \quad T_k = \frac{\sqrt{n} |\hat{\theta}_n(X)|}{\hat{\sigma}_k}, \quad k = 0, \ldots, m.$$ (3.3)

For any $\alpha \in (0, 1)$, we consider the criterion conditional FWER of the multiple testing procedure at level $1 - \alpha$, defined as

$$\text{FWER} = \max_{k \in [m]} \mathbb{P} \left( \bigcup_{j \geq k} \{ H_0^j \text{ is true, reject } H_0^j \} \mid Z^k \right).$$ (3.4)

| $p$-value | Conditioning set | Conditional parameter | Linearity of $\theta_k^m - \theta_0$ |
|-----------|------------------|-----------------------|-----------------------------------|
| $p_m$     | $Z_m = D$        | $\theta_n^{\text{cond}}(Z_m) = \theta_n(D)$ | $\frac{1}{n} \sum_{i=1}^n \phi(D_i)$ + $Z(\sigma_{m-1}^2 - \sigma_m^2)/n$ |
| $p_{m-1}$ | $Z_{m-1}$       | $\theta_n^{\text{cond}}(Z_{m-1})$ | $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(D_i) | Z_{m-1}^{m-1}]$ |
| $\vdots$  | $\vdots$        | $\vdots$              | $\vdots$ |
| $p_{j+1}$ | $Z_{j+1}$       | $\theta_n^{\text{cond}}(Z_{j+1})$ | $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(D_i) | Z_{j+1}^{j+1}]$ + $Z(\sigma_{j}^2 - \sigma_{j+1}^2)/n$ |
| $p_j$     | $Z_{j}$         | $\theta_n^{\text{cond}}(Z_{j})$ | $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(D_i) | Z_{j}^{j}]$ |
| $\vdots$  | $\vdots$        | $\vdots$              | $\vdots$ |
| $p_1$     | $Z_{1}$         | $\theta_n^{\text{cond}}(Z_{1})$ | $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(D_i) | Z_{1}^{1}]$ |
| $p_0$     | $Z_{0} = \varnothing$ | $\theta_n^{\text{cond}}(Z_{0}) = \theta^0$ | $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(D_i)]$ |

Table 1: An illustration of the hierarchical structure. The highest level is $Z_m = X$ and $\theta_n^m = \hat{\theta}_n(X)$, where we condition on everything observed. The lowest level is $Z_0 = \varnothing$ and $\theta_0 = \theta^0$, the unconditional parameter, where we condition on nothing. From higher level ($Z_m$) to lower level ($Z_0$), as we condition on fewer characteristics, the variance of conditional parameter $\theta_k^m$ decreases.
One could also aim to control the marginal rate \( \text{FWER}^{\text{marginal}} := \mathbb{P}[H_j^0 \text{ is true}, \text{ reject } H_j^0 \text{ for any } j] \). However, this would make it hard to interpret rejections of conditional hypotheses since the marginal criterion does not yield any conditional guarantees. The conditional FWER (3.4) uniformly controls the conditional probability of rejecting any higher-level false random hypothesis conditional on all levels \( k \in [m] \). Roughly speaking, it characterizes the probability of identifying false effects at higher levels given a certain level of information.

We consider the following multiple testing procedure:

\[
\text{Reject } H_0^k, \ldots, H_m^k, \quad \text{where } k = \min\{i : p_i \leq \alpha\}.
\]

As before, we assume the following asymptotic linearity and moment condition.

**Assumption 3.1.** The estimator \( \hat{\theta}_n \) satisfies (2.1). The conditional parameters \( \{\theta_n^k\}_{k=1}^m \) satisfy (2.2) with variances \( \{\sigma^2_n\}_{k=1}^m \). The influence function satisfies \( \mathbb{E}[|\phi(D_i)|^2] < \infty \).

The following theorem confirms the asymptotic FWER control of this procedure, whose proof is deferred to Appendix E.2.

**Theorem 3.2 (Asymptotic conditional FWER control).** Under Assumption 3.1, if \( \hat{\sigma}_k \overset{p}{\to} \sigma_k \), then FWER \( \leq \alpha + o_{\text{p}}(1) \). Specifically, for any constant \( \epsilon > 0 \), it holds that

\[
\lim_{n \to \infty} \mathbb{P}(\text{FWER} > \alpha + \epsilon) = 0. \tag{3.5}
\]

**Remark 3.3 (Multiple hierarchies).** Our framework also applies to multiple hierarchies. For example, suppose there are multiple effects of interest, where for each effect (parameter), we have an independent estimator \( \hat{\theta}_n \) and a hierarchy of population and sets to condition on. When the conditioning variables are independent across different hierarchies, we propose an adaptive multiple testing procedure that
controls a notion of conditional false-discovery rate (FDR). Another example of multiple hierarchies is several nested collection of conditioning sets, associated with several hierarchies of populations, for one estimator. In that case, the same multiple testing procedure controls marginal FDR, where the conditional validity of \( p \)-values might not hold conditional on a larger set of interdependent random variables. We defer the detailed discussions to Appendix 3.3.

### 3.2 Transductive estimation and inference

In this part, we consider a setting touched in Section 1.2 in which the statistician has a fully observed dataset from one place and some observed covariates for another dataset, and wants to conduct inference for a parameter on the new dataset. We will consider a special case in which the sampling distributions of the old and new data are the same, but conditional distributions differ. If the marginal distributions differ, one may use techniques from the missing data literature (Tsiatis, 2007) to estimate parameters on the new data set. An example is discussed in Appendix 3.4.

The takeaway from this section is that, perhaps surprisingly, transductive inference can be done at the same asymptotic variance at which we can do conditional inference.

Formally, we assume the statistician observes an i.i.d. dataset \( \{(D_i, Z_i)\}_{i=1}^n \) drawn from \( \mathbb{P} \) and would like to conduct inference for new i.i.d. data \( \{(D_{j}^{\text{new}}, Z_{j}^{\text{new}})\}_{j=1}^m \) based on an estimator \( \hat{\theta}_n \) computed with \( \{(D_i, Z_i)\}_{i=1}^n \). The statistician observes \( \{Z_{j}^{\text{new}}\}_{j=1}^m \), but not \( \{D_{j}^{\text{new}}\}_{j=1}^m \). For now we assume \( (D_{j}^{\text{new}}, Z_{j}^{\text{new}}) \overset{\text{i.i.d.}}{\sim} \mathbb{P} \) comes from the same distribution as the fully observed dataset. The inferential target of the statistician is the conditional parameter \( \theta_{\text{cond,new}} \), which is the solution to the equation

\[
\sum_{j=1}^m \mathbb{E}\left[s(D_{j}^{\text{new}}, \theta) \mid Z_{j}^{\text{new}}\right] = 0.
\]

We propose the following procedure employing cross-fitting, see for example Chernozhukov et al. (2018) and references therein. First, split the training data into two halves \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) and let \( \hat{\varphi}^{\mathcal{I}_1}(z) \) and \( \hat{\varphi}^{\mathcal{I}_2}(z) \) be estimates of \( E[\phi(D) \mid Z = z] \), where \( \hat{\varphi}^{\mathcal{I}_1}(z) \) uses only the half-sample \( \mathcal{I}_1 \) and \( \hat{\varphi}^{\mathcal{I}_2}(z) \) uses only the half-sample \( \mathcal{I}_2 \). Now we define the one-step corrected estimator

\[
\hat{\theta}_{n,m}^{\text{transfer}} := \hat{\theta}_n - \hat{c},
\]  

where the estimated correction term \( \hat{c} \) is defined as

\[
\hat{c} := \frac{1}{2|\mathcal{I}_1|} \sum_{i \in \mathcal{I}_1} \hat{\varphi}^{\mathcal{I}_2}(Z_i) + \frac{1}{2|\mathcal{I}_2|} \sum_{i \in \mathcal{I}_2} \hat{\varphi}^{\mathcal{I}_1}(Z_i) - \frac{1}{2m} \sum_{j=1}^m \left( \hat{\varphi}^{\mathcal{I}_1}(Z_{j}^{\text{new}}) + \hat{\varphi}^{\mathcal{I}_2}(Z_{j}^{\text{new}}) \right).
\]

We make the following assumption on asymptotic linearity and consistency.

**Assumption 3.4.** Assume (2.1) and (2.2) hold for \( \hat{\theta}_n \) and \( \theta_{\text{cond,new}}^{\text{m}} \), and

\[
\sup_z \left| \hat{\varphi}^{\mathcal{I}_1}(z) - \mathbb{E}[\phi(D) \mid Z = z] \right| \xrightarrow{p} 0, \quad \text{and}
\]

\[
\sup_z \left| \hat{\varphi}^{\mathcal{I}_2}(z) - \mathbb{E}[\phi(D) \mid Z = z] \right| \xrightarrow{p} 0.
\]

Then we have the following theorem on the asymptotic behavior of the transfer estimator, whose proof can be found in Appendix F.

**Theorem 3.5.** Suppose Assumption 3.4 holds and assume that \( m \geq cn \) for some constant \( c > 0 \). Let \( Z^{\text{new}} = (Z_1^{\text{new}}, \ldots, Z_m^{\text{new}}) \). Then it holds that

\[
\mathbb{P}\left( \sqrt{n} (\hat{\theta}_{n,m}^{\text{transfer}} - \theta_{\text{cond,new}}^{\text{m}}) \geq x \mid Z^{\text{new}} \right) = \Phi\left( \frac{x}{\sigma} \right) + o_{\mathbb{P}}(1),
\]

where \( \sigma^2 = \text{Var}(\phi(D) - \mathbb{E}[\phi(D) \mid Z]) \) is given by equation (2.3).
Theorem 3.5 states that with a refined quantification of uncertainty for partial observations, it is possible to estimate conditional parameters for conditional parameters of a partially observed new dataset \((D^\text{new}_i, Z^\text{new}_i)\) with the same asymptotic variance at which we can estimate conditional parameters for \((D_i, Z_i)\). For this to hold, it is important that both samples are drawn from the same distribution and that the size of the new sample is not too small. The practical implication of Theorem 3.5 is that, when the data are partially observed, we can conduct inference for the new conditional parameter which represents the population given these observed attributes, hence gauge whether the statistical finding would replicate on the new population.

We let \(\hat{\theta}^\text{new}_n\) be the estimator based on the fully observed new data \(\{D^\text{new}_j, Z^\text{new}_j\}\). The transductive estimator (3.6) can also be used to predict the estimator \(\hat{\theta}^\text{new}_m\), as indicated by the following theorem. The proof is deferred to Appendix F.

**Theorem 3.6.** Suppose Assumption 3.4 holds and the estimators \(\hat{\theta}_n, \hat{\theta}^\text{new}_m\) satisfy asymptotic linearity. Assume that \(m \asymp n\). Then it holds that

\[
P\left( \frac{1}{\sqrt{1/n + 1/m}} (\hat{\theta}^\text{transfer}_{m,n} - \hat{\theta}^\text{new}_m) \geq x \bigg| Z^\text{new}_n \right) = \Phi(x/\sigma) + o_{\text{P}}(1),
\]

where \(\sigma^2 = \text{Var}(\phi(D) - \mathbb{E} [\phi(D) | Z])\) is given by equation (2.3).

Theorem 3.6 may be useful to gauge the replicability of a statistical finding, since it allows to use some data of the new populations to form a prediction intervals for new estimators. Note that Theorem 3.5 relies on asymptotic consistency of the non-parametric regression step — we discuss on finite sample performance in the following two remarks, which will be verified in simulations.

**Remark 3.7** (Finite-sample issue: marginal variance and conditional bias). The asymptotic validity of confidence intervals in Theorems 3.5 and 3.6 relies on the consistency condition in Assumption 3.4. To be specific, Assumption 3.4 ensures the terms (i) and (ii) vanish as \(n \to \infty\). In practice, when the sup-norm of \(\hat{\varphi}^I_1 - \varphi\) and \(\hat{\varphi}^I_2 - \varphi\) is not sufficiently close to zero, the residual terms (i) and (ii) in equation (F.2) are **marginally** unbiased with marginal variance \(\approx [\text{Var}(\hat{\varphi}^I_1 - \varphi) + \text{Var}(\hat{\varphi}^I_2 - \varphi)]/(4n)\) if we view the two estimators \(\hat{\varphi}^I_1\) and \(\hat{\varphi}^I_2\) as fixed, which might add to the variance marginally in finite sample.

**Conditionally**, the residual terms (i) and (ii) in equation (F.2) are biased when we condition on \(Z^\text{new}_n\). Under Assumption 3.4, such bias is conditionally \(o_{\text{P}}(1/\sqrt{n})\) as \(n \to \infty\) indicated by Lemma G.1, which guarantees the asymptotic conditional validity in Theorems 3.5 and 3.6. Still, if in finite sample the sup-norm of \(\hat{\varphi}^I_1 - \varphi\) and \(\hat{\varphi}^I_2 - \varphi\) is not sufficiently small, the term might not be sufficiently close to zero as well. In that case, the confidence interval \([\hat{\theta}^\text{transfer}_{m,n} + \hat{\sigma} z_{\alpha/2}/\sqrt{n}, \hat{\theta}^\text{transfer}_{m,n} + \hat{\sigma} z_{1-\alpha/2}/\sqrt{n}]\) actually covers the linear proxy

\[
\hat{\theta}^\text{linear} := \theta^0 + \frac{1}{2m} \sum_{j=1}^m \left( \hat{\varphi}^I_1(Z^\text{new}_j) + \hat{\varphi}^I_2(Z^\text{new}_j) - \mathbb{E} [\hat{\varphi}^I_1(Z^\text{new}_j) | I_1] - \mathbb{E} [\hat{\varphi}^I_2(Z^\text{new}_j) | I_2] \right),
\]

where \(\hat{\sigma}^2\) is the estimated variance via

\[
\hat{\sigma}^2 := \frac{1}{2} \text{Var} \left( \phi(D_i) - \hat{\varphi}^I_2(D_i); i \in D_1 \right) + \frac{1}{2} \text{Var} \left( \phi(D_i) - \hat{\varphi}^I_1(D_i); i \in D_2 \right).
\]

To see this, note that

\[
\hat{\theta}^\text{transfer}_{m,n} - \hat{\theta}^\text{linear} = \frac{1}{n} \sum_{i \in I_1} \left( \phi(D_i) - \hat{\varphi}^I_2(D_i) + \mathbb{E} [\hat{\varphi}^I_2(Z_i) | I_2] \right)
\]

\[
+ \frac{1}{n} \sum_{i \in I_2} \left( \phi(D_i) - \hat{\varphi}^I_1(D_i) + \mathbb{E} [\hat{\varphi}^I_1(Z_i) | I_1] \right) + o_{\text{P}}(1/\sqrt{n}),
\]
where each term in the summation are independent with mean zero. This phenomenon will be verified in simulations, where the coverage of $\hat{\theta}^{\text{linear}}$ is better than $\theta^{\text{cond,new}}_m$ for relatively small sample sizes. In the following we will give an interpretation of the projected parameter $\hat{\theta}^{\text{linear}}$.

**Remark 3.8** (Projection interpretation of linear proxy). The linear proxy (3.8) can be viewed as the projection of $\hat{\theta}_n$ onto a function space $\mathcal{F}$ employed in the nonparametric regression, especially when the function class in the regression procedure does not cover the true $\varphi(\cdot)$. Specifically, if $\bar{\varphi}_1, \bar{\varphi}_2$ converges in sup-norm to a function $\bar{\varphi} \neq \varphi$, where

$$\bar{\varphi}(\cdot) = \arg \min_{f \in \mathcal{F}} \mathbb{E}\left[\left(\phi(D_i) - f(Z_i)\right)^2\right]$$

and $\mathbb{E}[\bar{\varphi}(Z)] = 0$, then the linear proxy converges to

$$\theta^0 + \frac{1}{2m} \sum_{j=1}^{m} \bar{\varphi}(Z_{j}^{\text{new}}),$$

which is the projection of $\theta^{\text{cond,new}}_m = \theta^0 + \frac{1}{m} \sum_{j=1}^{m} \phi(D_{j}^{\text{new}}) + o_p(1)$ to the function space $\mathcal{F}$.

### 3.3 Multiple hierarchies

Our framework also applies to multiple effects of interest. We now consider hierarchies of conditional parameters from several independent datasets, which is further beyond the case in Section 3.1.

Specifically, we assume there are $L \in \mathbb{N}^+$ parameters of interest, so that for each $\ell \in [L]$, there is an estimator $\hat{\theta}_n$ that could be computed from data. The estimators $\{\hat{\theta}_n\}_{\ell=1}^{L}$ are mutually independent, and admits the asymptotic linearity (2.1) for influence function $\{\phi^{\ell}\}_{\ell=1}^{L}$. Meanwhile, for each $\ell \in [L]$, there is an increasing series of conditioning sets

$$Z^{0,\ell} \subseteq Z^{1,\ell} \subseteq \cdots \subseteq Z^{m,\ell},$$

where $Z^{0,\ell} = \emptyset$. We assume that the corresponding random variables $Z^{k,\ell}$ and $Z^{k',\ell'}$ are independent if $\ell \neq \ell'$. Without loss of generality we assume all hierarchies have the same number of levels $m \in \mathbb{N}^+$. Correspondingly, for each $\ell \in [L]$, there is a hierarchy of conditional parameters

$$\{\theta^{k,\ell}\}_{k=1}^{m} = \{\theta^{\text{cond}}(Z^{k,\ell})\}_{k=1}^{m}.$$

We similarly define sets of (conditional) hypotheses $\{H^{k,\ell}_{0}\}_{k=1}^{m}$ as in equation (3.2) and sets of (conditional) p-values $\{p_{k,\ell}\}_{k=1}^{m}$ as

$$p_{k,\ell} = 2(1 - \Phi(T_{k,\ell})), \quad T_{k,\ell} = \frac{\sqrt{n} \hat{\theta}^{k,\ell}_{n}}{\sigma^{k,\ell}_{n}}, \quad \forall k \in [m], \ell \in [L],$$

which is similar to the one hierarchy case in equation (3.3) with consistent estimators $\{\hat{\sigma}^{2,\ell}_{k}\}_{k=1}^{m}$ for

$$\sigma^{2,\ell}_{k} = \mathbb{E}\left[\left(\phi^{\ell}(D_{k}^{\ell}) - \mathbb{E}[\phi^{\ell}(D_{k}^{\ell}) \mid Z^{k,\ell}]\right)^2\right].$$

Since each hierarchy stands for several levels of conditional parameters corresponding to one effect, one might ask which is the lowest level for which the test is significant, or the largest population to which the statistical finding generalizes. The extreme case is that the test for the superpopulation parameter is significant. Another possible case is that the test for the superpopulation parameter is not significant, yet moving up the hierarchy, at some level the test may become significantly nonzero. We refer the readers to Appendix B.2 for a formal statement of asymptotic behavior of hypotheses.
To this end, a testing procedure should make rejections within each hierarchy for all levels above a certain level, while an error occurs if any true hypothesis within that hierarchy is rejected in this procedure. Equivalently, we could reject exactly one hypothesis at the lowest level within each hierarchy, while an error occurs if this hypothesis is actually true. We consider the following criterion of *conditional FDR at level* $k$

$$
\text{FDR}_k = \mathbb{E} \left[ \frac{|V_k|}{1 \vee |R_k|} \mid Z^k \right],
$$

(3.9)

where

$$
V_k = \{ (j, \ell) : j \geq k, \ell \in [L], H_0^{j,\ell} \text{ is true and rejected} \},
$$

$$
R_k = \{ (j, \ell) : j \geq k, \ell \in [L], H_0^{j,\ell} \text{ is rejected} \},
$$

and $|V_k|$, $|R_k|$ is the cardinality of $V_k$, $R_k$, respectively. Controlling the conditional FDR (3.9) means controlling the expected proportion of all false rejections above any specific level, conditional on that level.

The conditional FDR criterion differs from traditional marginal FDR in that we focus on different sets of discoveries depending on the conditioning set. Also, conditional FDR control corresponds to statistical findings based on data sampled from the conditional distribution, thus providing guarantee on any specific level of population at hand. To this end, we need a quantification of conditional uncertainties for the behavior of $p$-values, in contrast to the traditional setting.

Meanwhile, the hypotheses of interest are random where special structure is necessary for validity of multiple testing procedures. The validity of $p$-values is in a conditional and asymptotic fashion, which calls for specific theoretical analysis for FDR control.

As before, we assume the linear expansion like equations (2.1) and (2.2) hold with bounded moments of influence functions.

**Assumption 3.9.** The estimators $\{ \hat{\theta}_n^\ell \}_{\ell \in [L]}$ and conditional parameters $\{ \hat{\theta}_n^{k,\ell} \}_{k \in [m], \ell \in [L]}$ satisfy equations (2.1) and (2.2) with variances $\{ \sigma_\ell^2 \}_{k \in [m]}$. Also, the influence functions $\{ \phi_\ell \}_{\ell \in [L]}$ satisfy $\mathbb{E}[|\phi_\ell(D)|^4] < \infty$.

For any $\alpha \in (0, 1)$, we consider the following adaptive multiple testing procedure. First of all, we rule out all hypotheses with $Z = \mathcal{D}$ (the largest conditioning set of the whole data), so that the smallest $p$-value is not necessarily rejected.

To begin with, we fix a level $k \in [m]$ we are interested in. Given a $p$-value threshold $t$, the rejection set $U_k(t)$ as

$$
U_k(t) = \{ (j, \ell) : j > k, p_{j,\ell} \leq t, p_{j-1,\ell} > t \} \cup \{ (k, \ell) : p_{k,\ell} \leq t \}.
$$

In other words, we essentially ‘reject’ in each column the lowest level above $k$ whose $p$-value is below threshold $t$. So each rejection $H_0^{j,\ell} \in U_k(t)$ in this criterion can be interpreted as ‘the conditional effect is significant up to level $j$ in hierarchy $\ell$’. This adaptive procedure can be viewed as a modified version of Benjamini–Hochberg procedure (Benjamini and Hochberg, 1995; Benjamini et al., 2006), and is also related to a line of adaptive procedures for multiple testing especially with special structures (Katsevich et al., 2020; Blanchard and Roquain, 2008). We construct the FDP estimate with threshold $t$ via

$$
\hat{\text{FDP}}(t) = \frac{L \cdot t}{|U(t)|}.
$$

With an FDR target $\alpha \in (0, 1)$, we define the final rejection set $R_k := U(t^*_k)$ with the data-dependent threshold

$$
t^*_k = \max \{ t \in (0, p_{j,\ell} : j \geq k) : \hat{\text{FDP}}(t) \leq \alpha \}.
$$

(3.10)

The following theorem establishes the asymptotic conditional FDR guarantee of the above adaptive multiple testing procedure.
Theorem 3.10. Under Assumption 3.9, for a fixed level $k \in [m]$, the conditional FDR of the above procedure satisfies $\text{FDR}_k \leq \alpha + o(1)$.

With mutually independent hierarchies, the conditional validity of null $p$-values still holds when conditioning on the union of all hierarchies at a particular level. The conditional FDR control of the above procedure relies on such validity, combined with the joint asymptotic conditional PRDS of all $p$-values. These properties distinguish our theoretical results from classical literature. We defer the detailed proof of Theorem 3.10 to Appendix E.3.

Remark 3.11 (Other types of hierarchies). In practice, one might also be interested in multiple hierarchies built upon one single parameter associated with one single estimator from data. Concretely, assume there is one estimator $\hat{\theta}_n$ from data $\{D_i\}_{i=1}^n$ for a population parameter $\theta^0$. We define multiple hierarchies based on different nested sets of covariates

$$Z^{0,\ell} \subset Z^{1,\ell} \subset \cdots \subset Z^{m,\ell} \subset \mathcal{X},$$

where $\mathcal{X}$ is the set of all covariates in $D_i$. Therefore $Z^{j,\ell}$ is not necessarily independent of $Z^{k,\ell'}$ when $\ell \neq \ell'$. We remark that the marginal FWER is controlled if we simply reject all $H_{0,j,\ell}^\ell$ for which $p_{j,\ell} \leq \alpha$ similar to Section 3.1. Since $Z^{j,\ell}$ is not necessarily independent of $Z^{k,\ell'}$ for $j \geq k, \ell' \neq \ell$, the conditional validity of null $p$-values may not hold conditioning on a union of variables at some level $k \in [m]$. However, in this case, the marginal validity of the null $p$-values still holds, hence the marginal FWER control can be proved with a slight modification of the proof of Theorem 3.2.

Meanwhile, the marginal FDR is still controlled if we run the aforementioned procedure under this multiple-hierarchy setting. With a slight modification of the proof of Lemma G.7, it can be shown that since all $p$-values are based on the same estimator $\hat{\theta}_n$, they are (asymptotically) linearly dependent, hence asymptotically (both marginally and conditionally) PRDS. In this case, the null $p$-values are still valid, thus the above procedure still controls marginal FDR. Such marginal validity could also be generalized to the setting with several independent $\hat{\theta}^\ell_n$, and each of them is associated with multiple hierarchies.

3.4 Predicting mean outcomes for small populations

The transductive inference setting discussed in Section 3.2 can be generalized to the prediction task with covariate shifts. As an example, we might be interested in predicting the average health outcome of a subset of patients. Intuitively, this task is less challenging than predicting health outcomes of individual patients, but more challenging than predicting the mean health outcome of all patients.

In the prediction task, one observes $(Z_i, Y_i), i = 1, \ldots, n$, where $Z_i$ is the covariate and $Y_i$ is the response. The fully observed i.i.d. dataset $\{(Z_i, Y_i)\}_{i=1}^n$ is drawn from $\mathbb{P}$ and would like to conduct conditional inference for the mean conditional outcome

$$\frac{1}{m} \sum_{j=1}^m \mathbb{E}[Y_{j}^{\text{new}} | Z_{j}^{\text{new}}]$$

of new i.i.d. data $(Z_j^{\text{new}}, Y_j^{\text{new}})_{j=1}^m$. The new dataset is shifted in covariates, so that the distribution of $Z_j^{\text{new}} \sim \mathbb{P}^{\text{new}}_Z$ is different from $Z_i \sim \mathbb{P}_Z$, while the conditional distribution $\mathbb{P}_{Y_j^{\text{new}} | Z_j^{\text{new}}} \sim \mathbb{P}_{Y_j | Z_j}$ is the same as $\mathbb{P}_{Y_j | Z_i} \sim \mathbb{P}_{Y_j | Z}$ in the fully observed dataset. The invariance of conditional distribution guarantees the invariance of $\mathbb{E}[D_i | Z_i]$ across the two datasets, and is necessary to ensure that the transfer learning is meaningful.

Under covariate shift, we employ a AIPW-type estimator, which is widely used in causal inference (Hahn, 1998; Chen et al., 2008) the and missing data literature (Robins et al., 1994; Tsiatis, 2007). We are to show that a well-known efficient estimator for a missing data problem can be used for conditional inference (i.e. we can derive conditionally valid confidence intervals) with a modified variance formula.
To begin with, we view the two datasets as one dataset \( \{(D_i, Z_i)\}_{i=1}^{N} \), where \( D_i = (Z_i, Y_i, T_i) \) with indicator \( T_i \in \{0, 1\} \), \( i = 1, \ldots, N \). Here \( N = n + m \) is the total sample size, and \( T_i = 1 \) indicates the data comes from the new dataset, while \( T_i = 0 \) indicates the data from the original dataset. Therefore, we have

\[
(Z_i, Y_i) \mid T_i = 1 \sim \mathbb{P}_{Z}^\text{new} \times \mathbb{P}_D \mid Z, \\
(Z_i, Y_i) \mid T_i = 0 \sim \mathbb{P}_Z \times \mathbb{P}_D \mid Z.
\]

Such a joint relationship of \( (Y_i, Z_i, T_i) \) is related to missing data problems (Tsiatis, 2007), and is also related to the potential outcome framework in causal inference (Imbens and Rubin, 2015), where one might view \( T_i \) as the treatment, \( Z_i \) as the covariate and \( Y_i \) as the outcome. The invariance of conditional distribution \( \mathbb{P}_D \mid Z \) implies the unconfoundedness assumption as a special case.

Switching back to the conditional inference framework, the aggregated dataset \( \{(D_i, Z_i)\}_{i=1}^{N} \) can be viewed as i.i.d. sample from the aforementioned joint distribution of \( (Y_i, Z_i, T_i) \), and we condition on the random variable \( Z_i^* \), where

\[
Z_i^* = \begin{cases} 
(T_i, Z_i) & \text{if } T_i = 1; \\
T_i & \text{otherwise.}
\end{cases}
\] (3.11)

Conditioning on \( \{Z_i^*\}_{i=1}^{N} \), the conditional parameter is

\[
\hat{g}_{\text{cond. new}} = g_{\text{cond. new}}(Z^*) = \frac{1}{m} \sum_{i: T_i = 1} \mathbb{E}[Y_i \mid Z_i] := \frac{1}{m} \sum_{i: T_i = 1} g(Z_i),
\] (3.12)

where we denote the conditional mean as \( g(Z_i) = \mathbb{E}[Y_i \mid Z_i] \), and the size of the new dataset \( m = \sum_{i=1}^{N} T_i \) is a random variable measurable with respect to \( \{Z_i^*\}_{i=1}^{N} \). Under the above framework, we consider the following AIPW-type transductive estimator

\[
\hat{\theta}_{\text{shift}} = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\hat{e}(Z_i)}{1 - \hat{e}(Z_i)} \cdot \frac{1 - T_i}{\hat{p}_1} \cdot \left( Y_i - \hat{g}(Z_i) \right) + \frac{T_i}{\hat{p}_1} \cdot \hat{g}(Z_i) \right),
\] (3.13)

where \( \hat{e}(Z_i) \) is an estimator of \( \mathbb{P}(T_i = 1 \mid Z_i) \), and \( \hat{p}_1 = m/N \) is an estimate of \( \mathbb{P}(T_i = 1) \).

**Remark 3.12.** The estimator (3.13) is asymptotically equivalent to the estimator proposed in Shu and Tan (2018) for \( \theta^0 := \mathbb{E}[Y(0) \mid T = 1] \) as part of the ATT estimator. In fact, the classical result of efficient influence function (Hahn, 1998; Chen et al., 2008) is stated in Proposition 1 of Shu and Tan (2018) to be

\[
\xi(D_i) := \frac{1}{\hat{p}_1} \cdot \left( \frac{e(Z_i)}{1 - e(Z_i)} \cdot (1 - T_i) \cdot Y_i - \left( \frac{1 - T_i}{1 - e(Z_i)} - 1 \right) \cdot g(Z_i) - T_i \cdot \theta^0 \right),
\]

where \( g(Z_i) = \mathbb{E}[Y_i \mid T_i = 0, Z_i] \) and \( \theta^0 = \mathbb{E}[Y(0) \mid T = 1] = \mathbb{E}[g(Z_i)] \). It can be shown that

\[
\frac{1}{N} \sum_{i=1}^{N} \xi(D_i) = \frac{1}{N} \cdot \frac{1}{m/N} \sum_{i=1}^{N} \left( \frac{e(Z_i)}{1 - e(Z_i)} \cdot (1 - T_i) \cdot (Y_i - g(Z_i)) + T_i \cdot (g(Z_i) - \theta^0) \right)
\]

\[+ o_P\left(1/\sqrt{\min(n,m)}\right),\]

which would be shown to be asymptotically equivalent to the shifted estimator (3.13).

We impose the following assumptions for the shifted estimator to behave well, which are common for those types of estimators (Chernozhukov et al., 2018; Newey and Robins, 2018).

**Assumption 3.13.** The following conditions hold.

1. **Assumption 3.13.**
(i) Consistency: \( \sup_z |\hat{e}(z) - e(z)| = o_P(1) \).

(ii) Overlap: \( \eta \leq e(z) \leq 1 - \eta \) for some constant \( \eta > 0 \).

(iii) Concentration rates: \( \mathbb{E}\left[ (e(Z_i) - \hat{e}(Z_i))^2 \right] \cdot \mathbb{E}\left[ (g(Z_i) - \hat{g}(Z_i))^2 \right] = o_P(1/N) \).

(iv) Moment condition: \( \mathbb{E}[Y_i^4] < \infty \).

The following result, as a corollary of Proposition 2.3, characterizes the estimator \( \hat{\theta}_{\text{shift}} \) and conditional parameter \( \theta_{\text{cond., new}} \) within the conditional inference framework. The linear expansion is specified, which satisfies the conditions of Proposition 2.3 thus facilitates the inference procedure as in Theorems 2.4 and 2.7.

**Corollary 3.14.** Under the aforementioned generative model with \( p_1 = \mathbb{E}[T_i] \), suppose Assumption 3.13 holds. Then we have the asymptotic linear expansions

\[
\hat{\theta}_{\text{shift}} = \frac{1}{N} \sum_{i=1}^{N} \phi(D_i) + o_P(1/\sqrt{N}), \\
\theta_{\text{cond., new}} = \frac{1}{N} \sum_{i=1}^{N} \varphi(Z_i^*) + o_P(1/\sqrt{N}),
\]

where the influence functions take the form

\[
\phi(D_i) = \frac{e(Z_i)}{1 - e(Z_i)} \cdot \frac{1 - T_i}{p_1} \cdot (Y_i - g(Z_i)) + \frac{T_i}{p_1} \cdot (g(Z_i) - \theta_0),
\]

\[
\varphi(Z_i^*) = \mathbb{E}\left[ \phi(D_i) \mid Z_i^* \right] = \frac{T_i}{p_1} \cdot (g(Z_i) - \theta_0),
\]

and \( \theta_0 = \mathbb{E}[Y_j^{\text{new}}] \) is the marginal mean outcome of shifted data. Marginally we have

\[
\sqrt{N}(\hat{\theta}_{\text{shift}} - \theta_{\text{cond., new}}) \xrightarrow{d} N(0, \sigma_{\text{shift}}^2),
\]

and conditional on \( \{Z_i^*\}_{i=1}^N \) defined in equation (3.11), it holds for any \( x \in \mathbb{R} \) that

\[
P\left[ \sqrt{N}(\hat{\theta}_{\text{shift}} - \theta_{\text{cond., new}}) \leq x \mid Z^* \right] = \Phi(x / \sigma_{\text{shift}}) + o_P(1),
\]

where the asymptotic conditional variance is given by

\[
\sigma_{\text{shift}}^2 = \mathbb{E}\left[ (\phi(D_i) - \varphi(Z_i^*))^2 \right] = \text{Var}\left( \frac{e(Z_i)}{1 - e(Z_i)} \cdot \frac{1 - T_i}{p_1} \cdot (Y_i - g(Z_i)) \right).
\]

**Remark 3.15.** Here we treat the functions \( \hat{e}(\cdot) \) and \( \hat{g}(\cdot) \) as fixed. In practice, one might perform sample splitting, train \( \hat{e}(\cdot) \) and \( \hat{g}(\cdot) \) on each fold and compute \( \hat{e}(Z_i) \) and \( \hat{g}(Z_i) \) on the remaining sample.

### 4 Simulations

In this section, we demonstrate the performance of our methods via simulation studies. The conditional inference procedure in Section 2 with a hierarchical structure in Section 3.1 and the transductive inference procedure in Section 3.2 are evaluated. Additional results are deferred to Appendix A.
4.1 Conditional inference

In this part, the estimation accuracy and coverage probability of conditional inference procedure in Section 2 are evaluated on simulated data with a hierarchical structure as in Section 3.1. In a nutshell, the conditional confidence interval achieves the desired coverage for the unobserved conditional parameters for various conditioning sets. The estimation of variances is sometimes conservative yet the estimated conditional variances are considerably smaller than the marginal ones.

4.1.1 Simulation settings

We consider a setting with strong model misspecification, to evaluate whether the proposed methods have the desired coverage in a challenging setting. We generate data \( D = (X_i, Y_i) \) with covariates \( X \in \mathbb{R}^{10} \) and response \( Y \in \mathbb{R} \) according to

\[
X_1, X_2, X_3, \ldots, X_{10} \overset{i.i.d.}{\sim} N(0, 1), \quad X_3 = X_1 + \varepsilon_1, \quad X_4 = X_1 + \varepsilon_2,
\]

\[
(\varepsilon_1, \varepsilon_2)^T \sim N(0, \Sigma), \quad \Sigma_{11} = \Sigma_{22} = 1, \quad \Sigma_{12} = \Sigma_{21} = 1/2,
\]

\[
Y = X_1 + |X_1| + X_3 + \varepsilon', \quad \varepsilon' \sim N(0, \nu^2), \quad \nu \in \{0.1, 0.2, 0.5\}.
\]

The hierarchy is \( Z^k, k = 1, \ldots, m \) for \( m = 4 \) so that \( Z^0 = \emptyset, Z^1 = \{X_1\}, Z^2 = \{X_1, X_2\}, Z^3 = \{X_1, X_2, X_3\} \) and \( Z^4 = \{X_1, \ldots, X_p, Y\} \). The parameters of interest are the first two entries of the ordinary least square coefficient \( \theta = \arg\min_{\beta \in \mathbb{R}^p} E[(Y - \beta^T X)^2] \), which admits an influence function

\[
\phi(d; \theta) = (E[XX^T])^{-1}x(y - \theta^T x), \quad \text{where} \quad d = (x, y) \in \mathbb{R}^p \times \mathbb{R}. \tag{4.1}
\]

We write \( \theta_n^k = \sigma_n^{cond}(Z^k) \). For covariate set \( Z^k \) and \( n \) observations \( Z^k = (Z_1^k, \ldots, Z_n^k) \), the conditional parameter is

\[
\hat{\theta}_n^k = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n E[(Y_i - \beta^T X_i)^2 | Z^k] = \left( \sum_{i=1}^n E[X_iX_i^T | Z^k] \right)^{-1} \sum_{i=1}^n E[X_iY_i | Z^k]. \tag{4.2}
\]

Here the linear model \( Y = X^T \beta + \epsilon \) is misspecified. We consider two methods to estimate the influence functions \( \phi_i = \phi(X_i, Y; \theta) \). Let \( X \in \mathbb{R}^{n \times p} \) and \( Y \in \mathbb{R}^n \) be the observed data, where \( X_i \in \mathbb{R}^p \) is the \( i \)-th observation. Let \( \hat{\theta}_n = \hat{\theta}_n(X, Y) \) be the estimator. The first method is via the explicit formula as in Algorithm 1. Based on equation (4.1), we estimate the influence functions via

\[
\hat{\phi}_i = \phi(X_i, Y_i, \hat{\theta}_n) = (X^T X/n)^{-1} X_i (Y_i - X_i^T \hat{\theta}_n).
\]

The second method is the leave-one-out approach. For each \( i \in [n] \), we compute the LOO estimator \( \hat{\theta}_{(-i)} = \hat{\theta}_n(X_{(-i)}, Y_{(-i)}) \), and estimate the influence function by \( \hat{\phi}_i = n(\hat{\theta}_n - \hat{\theta}_{(-i)}) \).

Our method is carried out for sample sizes \( n = 200, 1000, 2000, 5000 \), each with \( N = 1000 \) replicates. In each replicate, we generate \( n \) i.i.d. observations from the distribution, estimate the influence functions via the two methods to construct sorted estimators \( \{\hat{\sigma}_k\}_{k=1}^m \), the sorted p-values \( \{p_k\}_{k=1}^m \), as well as confidence intervals for \( \{\hat{\theta}_n^k\}_{k=1}^m \), respectively. Then we evaluate the accuracy of \( \{\hat{\sigma}_k\}_{k=1}^m \) compared to ground truth \( \{\sigma_k\}_{k=1}^m \), as well as marginal and conditional coverages of the conditional confidence intervals for conditional parameters computed in each replicate via equation (4.2). The procedures are carried out for the two entries and the corresponding two sets of conditional parameters we focus on.

4.1.2 Simulation results

Accuracy of \( \hat{\sigma}_k \). The estimation accuracy of \( \hat{\sigma}_k \) is illustrated in Figure 1. The numerical results are summarized in Table 3, where we report the mean and standard deviation of our estimator in the
Figure 1: Accuracy of conditional variance estimation: Mean (standard deviation) of the estimated variances $\hat{\sigma}_k$ for two parameters with sample sizes $n \in \{200, 1000, 2000, 5000\}$, where influence functions are approximated with explicit-formula approach. The left panel is for variance estimators of $\hat{\theta}_1$. The right panel is for those of $\hat{\theta}_2$. The red dashed lines are the true values of $\sigma_k$. The columns correspond to different noise levels $\nu \in \{0.1, 0.2, 0.5\}$, while the rows correspond to conditional parameters at different levels $k = 0, \ldots, 3$. The corresponding numerical results are summarized in Table 3.
estimation, as well as the truths of $\sigma_k$. To stabilize the estimation, for each replicate, we obtain all four estimators $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_4$, then take $\hat{\sigma}_j = \min_{i \leq j} \tilde{\sigma}_i$ as the final estimator.

From the results in Figure 1 one sees that sometimes the estimated variances are a bit conservative when sample size is relatively small and noise is relatively large. This might be due to the bias in non-parameteric regression with finite sample. As mentioned before, it does not hurt the coverage of true conditional parameter, though the confidence intervals might be longer than those constructed with the ground truth of $\sigma_k^2$.

**Marginal coverage of conditional parameters.** In Figure 2, we report the coverage probability of the conditional parameters $\theta_{1}^{\text{cond}}(Z^k)$ and $\theta_{2}^{\text{cond}}(Z^k)$ for the first two OLS coefficients. Specifically, we report the proportion of replicates where $\theta_j^k$ actually falls in the conditional interval for $j = 1, 2$. It can be seen that the marginal coverage of conditional parameters approaches the desired 0.95 level as the sample size increases, which is consistent with our theory. The numerical results can be found in Table 4 in the Appendix.

![Figure 2: Marginal coverage of conditional parameters with sample sizes $n \in \{200, 1000, 2000, 5000\}$, where influence functions are approximated with explicit-formula approach. The left panel is for marginal coverage of $\theta_{1}^{\text{cond}}(Z^k)$. The right panel is for that of $\theta_{2}^{\text{cond}}(Z^k)$. The red dashed lines indicate the nominal coverage 0.95. The columns correspond to noise levels $\nu \in \{0.1, 0.2, 0.5\}$, while the rows correspond to conditional parameters at different levels $k = 0, \ldots, 3$.](image)

**Conditional coverage of conditional parameters.** For each $k \in [m]$, conditional on a fixed sample $Z_n^k$, we sample the full dataset for $N = 1000$ times. Then we compute the conditional coverage of 95% confidence intervals of conditional parameters of all higher levels, reported in Figure 3. The coverage approaches the nominal level 0.95 as sample size increases, which is consistent with our theoretical results. Additional simulation results of estimation accuracy and marginal coverage can be found in Appendix A.1.
4.2 Transductive inference

In this part, the performance of transductive inference in Section 3.2 is demonstrated on simulated data. We evaluate the coverage of the new conditional parameter $\theta_{\text{cond},\text{new}}^m$. In words, the simulation results verify the validity of transductive inference outlined by our theory. The conditional coverage of conditional parameters is close to the nominal level for large sample sizes, and finite-sample conditional coverage of a linear proxy is also illustrated.

4.2.1 Simulation settings

The parameter of interest is the OLS coefficient, which is the same as in the simulations on conditional inference in Section 4.1. We adopt two methods of estimating the influence functions $\phi_i = \phi(X_i, Y_i; \theta)$, the explicit-formula approach and the leave-one-out approach. The distribution of the data $D = (X_1, \ldots, X_p) \in \mathbb{R}^p$ with $p = 4$ is the same as the first four entries in Section 4.1. The conditioning set is $Z = X_1$.

We work with sample sizes $n \in \{200, 1000, 2000, 5000, 10000\}$ and $m = n \cdot \epsilon$, where $\epsilon \in \{0.5, 1, 2\}$, each with $N = 1000$ replicates. In each replicate, we generate $n$ i.i.d. observations $\{(D_i, Z_i)\}_{i=1}^n$ as the observations and $m$ i.i.d. data $\{(D_{\text{new},i}, Z_{\text{new},i})\}_{i=1}^m$ as the new dataset. For marginal coverage, we draw the new dataset from the population distribution for $N = 1000$ replicates. For conditional coverage, we draw $\{Z_{\text{new},i}\}_{i=1}^m$ once, then draw other components from the conditional distribution given $\{Z_{\text{new},i}\}_{i=1}^m$ for $N = 1000$ replicates. Then we estimate the influence functions via two methods as in Section 4.1 to construct estimators $\hat{\sigma}^2$ and the corrected transfer estimator $\hat{\theta}_{\text{transfer},m,n}$. Finally we construct confidence intervals for $\theta_{\text{cond},\text{new}}^m$ and $\hat{\theta}_{\text{linear}}$. We evaluate the performance of our method via marginal coverages and conditional coverages of $(1 - \alpha)$-confidence intervals for $\theta_{\text{cond},\text{new}}^m$ and $\hat{\theta}_{\text{linear}}$. As in Section 4.1, we focus on the first two of the OLS coefficients: one with ground truth $\theta_1 = 1$, the other with ground truth $\theta_2 = 0$. 

Figure 3: Conditional coverage of conditional parameters for different choices of simulation settings, where influence functions are approximated with the explicit-formula approach. The left panel shows the conditional coverage of $\theta_{\text{cond},Z_1}^1$. The right panel depicts the coverage of $\theta_{\text{cond},Z_2}^2$. The red dashed lines indicate the nominal coverage 0.95.
4.2.2 Simulation results

In the following, we present the simulation results of the transductive estimator, including marginal and conditional coverages of $\theta_{\text{cond,new}}^m$ and the linear proxy

$$\hat{\theta}_{\text{linear}} = \theta^0 + \frac{1}{2m} \sum_{j=1}^{m} \left( \hat{\varphi}_I \left( Z_{j}^{\text{new}} \right) + \hat{\varphi}_{I_2} \left( Z_{j}^{\text{new}} \right) - E\left[ \hat{\varphi}_I \left( Z_{j}^{\text{new}} \right) \mid I_1 \right] - E\left[ \hat{\varphi}_{I_2} \left( Z_{j}^{\text{new}} \right) \mid I_2 \right] \right),$$

for various configurations of sample size and noise level. Additional results are deferred to Appendix A.2.

**Marginal coverage.** The marginal coverage (among 1000 replicates) of the conditional parameter $\theta_{\text{cond,new}}^m$ and the proxy linear expansion $\hat{\theta}_{\text{linear}}$ of the new dataset for the two coefficients $\theta_1$, $\theta_2$ are in Figure 4. We see that the marginal coverage of transductive confidence interval for the new conditional parameter is close to the nominal level 95%, in particular when sample size is relatively large. When sample size is moderate, as discussed in Remark 3.7, the coverage of the proxy linear expansion is higher than the new conditional parameter and closer to 95%.

![Figure 4: Marginal coverage of $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ with sample sizes $n \in \{200, 1000, 2000, 5000, 10000\}$, where influence functions are approximated with explicit-formula approach. The left panel is the coverage of $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ for $\theta_1$. The right panel is the coverage of those for $\theta_2$. The red dashed lines are the nominal level 0.95. The columns correspond to noise levels $\nu \in \{0.1, 0.2, 0.5\}$, while the rows correspond to size ratios $m/n \in \{0.2, 0.5, 1\}$. The corresponding numerical results are in Table 9.](image)

**Conditional coverage.** The conditional coverage (among 1000 replicates) of $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ of the new dataset for the two coefficients $\theta_1$, $\theta_2$ are in Figure 5. The results shows the conditional validity of our method, where the conditional coverage is close to the nominal level 95%. In some cases, due to the difficulty of the transfer task, the proposed procedure covers a projection of the conditional parameter as discussed in Remark 3.7.
Figure 5: Conditional coverage of $\theta_{\text{cond}, \text{new}}$ and $\hat{\theta}_{\text{linear}}$ for different simulation settings, where influence functions are approximated with the explicit-formula approach. The left panel is the coverage of $\theta_{\text{cond}, \text{new}}$ and $\hat{\theta}_{\text{linear}}$ for $\theta_1$. The right panel depicts the coverage of conditional parameters corresponding to $\theta_2$. The red dashed lines indicate the nominal level 0.95.

5 Real Data Analysis

In this section, we demonstrate how our method performs on a real data set, where the parameter of interest is an OLS coefficient. We employ the method in Section 2 to construct $p$-values and confidence intervals for a hierarchy of random hypotheses for conditional parameters, illustrating a hierarchy of statistical evidence. We also investigate a prediction task with covariate shift using the method proposed in Appendix 3.4, for which the real-data results are deferred to Appendix 5.2.

5.1 OLS parameter for multivariate data

We first apply the conditional inference procedure to the estimation of the OLS parameter $\theta = \arg \min_\alpha \mathbb{E}[(Y - \alpha^\top X)^2]$. The goal is to investigate whether the conditional inference approach leads to different inferential conclusions, depending on the conditioning set. We focus on the OLS coefficient for the attribute `high_cal_food`, denoted as $\theta_\text{h}$, which is an indicator of whether the participant eats high caloric food frequently. The conditional parameter is correspondingly $\theta_{\text{cond}}^{\text{h}}(Z)$ for conditioning set $Z$ to be specified later.

The dataset (Palechor and de la Hoz Manotas, 2019) was collected for the estimation of obesity levels in individuals from the countries of Mexico, Peru and Colombia, based on their eating habits and physical condition. The dataset is a mixture of original survey data and synthetic data generated from the survey data. It is available at the UCI machine learning repository (Dua and Graff, 2017). We adopt 13 covariates in the dataset, where the categorical variables on frequency is turned into numerical based on the reported frequency. The response $Y$ is the body mass index (BMI). There are 2111 samples in total.

A hierarchy of conditional sets. We consider conditioning on a hierarchy of subsets $\{Z^k\}_{k=1}^m$, which yields a sequence of $p$-values for conditional parameters $\{\theta_{\text{cond}}^{\text{h}}(Z^k)\}_{k=1}^m$. Starting from $Z^0 = \emptyset$, we adopt an increasing set of variables as $Z^1 = \{\text{Gender}\}, Z^2 = \{\text{Gender, Age}\}, Z^3 = \{\text{Gender, Age, Caloric Level}\},$
family_history} and $Z^4 = \{\text{Gender}, \text{Age}, \text{family_history}, \text{water}\}$. For each $k \in [4]$, we compute the conditional variances $\sigma_k^2$ and conditional confidence intervals. The influence functions are estimated via explicit formula, and the OLS estimator $\hat{\theta}_n$ is plugged in. For binary variables we use the groupwise mean as the regression function; otherwise we employ random forest to fit the regression function.

A hierarchy of random hypotheses and $p$-values. The random hypotheses related to $\theta_h^{\text{cond}}(Z^k)$ can be flexible to include one-sided hypotheses like $\theta_h^{\text{cond}}(Z^k) \leq \theta_0$ for some fixed $\theta_0 \in \mathbb{R}$. In the following, we consider the random hypotheses $H_0^k : \theta_h^{\text{cond}}(Z^k) \leq 2$, with $p$-values constructed via $p_k = 1 - \Phi\left(\frac{\sqrt{n}(\hat{\theta}_h - 2)}{\hat{\sigma}_k}\right)$. Here $\hat{\theta}_h$ is the fitted OLS coefficient, and $\hat{\sigma}_k^2$ is an estimator for the conditional variance $\sigma_k^2$.

The estimated standard deviations (divided by $\sqrt{n}$) and the $p$-values for testing the random hypotheses $\theta_h^{\text{cond}}(Z^k) \leq 2$ are summarized in Table 2. We see a decreasing sequence of conditional variances and $p$-values. The unconditional OLS coefficient of family_history is not significant at level .05. We can reject the random hypothesis $H_0^k$ for $k = 2, 3, 4$ in our example.

### 5.2 Prediction with covariate shifts: a real-data example

In this section, we carry out the method proposed in Section 3.4 on a real-world dataset for predicting car prices. The dataset is from Ebay-Kleinanzeigen (German Ebay listings) and consists of about 50,000 observations. Features include continuous ones like registration year and discrete ones like brand and make. The dataset has been studied in Kuenzel (2019), where the reliable prediction of car prices is found to be challenging because of the existence of discrete features. In particular, it is difficult to predict the individual prices of some ‘usual’ cars, some examples of which are old cars (registered before 2000), vintage cars and race cars.

Instead of predicting individual prices or estimating the overall mean price, our conditional inference framework provides an approach in-between. To be more specific, we can form conditionally valid prediction intervals for the mean price of a subset of cars, based on the theory outlined in Section 3.4. In the following, we conduct conditional inference for the conditional sample mean of a subpopulation of old cars and evaluate the performance of the procedure by the average coverage.

To facilitate the evaluation where true conditional means are needed, we first fit a random forest model $\hat{m}(\cdot)$ for the conditional mean $m(x) = \mathbb{E}[Y_i | X_i = x]$ on the whole dataset, and view the fitted values $\hat{m}(X_i)$ as the conditional mean. We compute the residuals $\epsilon_i = Y_i - \hat{m}(X_i)$, which will be used to create synthetic datasets later.

Specifically, we randomly sample (without replacement) a population of size $N \in \{2, 5, 10, 20, 50\} \times 10^3$ from the original dataset as the whole population. The features are $\{X_i\}_{i=1}^N$ and the outcome is $\{Y_i^*\}_{i=1}^N$, where $Y_i^* = \hat{m}(X_i) + \epsilon_i^*$ with $\{\epsilon_i^*\}_{i=1}^N$ resampled from $\{\epsilon_i\}_{i=1}^n$ with replacement. We choose the old cars with registration year earlier than 2000, and further take a subsample of proportion $r \in \{0.1, 0.2, \ldots, 0.9\}$ as the new (shifted) dataset of size $m$. The rest of the old cars as well as new cars with registration year later than 2000 form the original dataset of size $n$, so that $m + n = N$.

Then the conditional inference procedure discussed in Section 3.4 is applied to the synthetic dataset, where the confidence interval is constructed as

$$\left[\hat{\theta}_{\text{shift}} + z_{0.025} \cdot \hat{\sigma}_{\text{shift}} / \sqrt{N}, \hat{\theta}_{\text{shift}} + z_{0.975} \cdot \hat{\sigma}_{\text{shift}} / \sqrt{N}\right],$$
with the estimated conditional variance

\[ \hat{\sigma}^2_{\text{shift}} = \frac{1}{N} \sum_{i=1}^{N} \left( \hat{d}_i - \frac{1}{N} \sum_{i=1}^{N} \hat{d}_i \right)^2, \]

where \( \hat{d}_i = \frac{\hat{c}(Z_i)}{1 - \hat{e}(Z_i)} \left( 1 - \frac{1 - T_i}{\hat{p}_1} \right) (Y_i - \hat{g}(Z_i)) \).

Here \( \hat{c}(\cdot) \) is the estimated propensity score and \( \hat{g}(\cdot) \) is the estimated conditional mean, both computed with the synthetic dataset. The coverage for the conditional parameter

\[ \theta^{\text{cond, new}} = \frac{1}{m} \sum_{i=1}^{N} T_i \cdot \hat{m}(X_i) \]

is evaluated over 1000 replicates, where \( T_i = 1 \) indicates \((X_i, Y^*_i)\) is in the new (shifted) dataset.

The average coverages for different configurations of total sample size \( N \) and proportion of shifted population \( r \) are summarized in Figure 6. The largest sample size \( N = 50000 \) exhibits a drop in coverage, which might be due to some extreme outliers that are often included in the resampled dataset when \( N \) is large.

![Figure 6: Average coverage of conditional parameter within 1000 replicates for proportions \( r \) of shifted data. Each plot corresponds to total sample size \( N \in \{2000, 5000, 10000, 20000, 50000\} \). The red dashed lines indicate the nominal coverage 0.95.](image)

From the results we see that the transductive prediction procedure generally works well for reasonably large original dataset and shifted dataset, with coverage close to the nominal level 0.95.

There are also interesting patterns in the coverages depending on how large the two datasets are. The coverage improves as the whole sample size gets larger, especially when the proportion of shifted data is not too large or too small. We observe that the coverage might be deteriorated when the proportion of shifted data is large (like \( r = 0.9 \)), in which case there are fewer representative sample of old cars in the original data and training a model for those conditional means gets harder. On another hand, when the sample size is relatively small (for example \( N = 2000 \)) and the proportion of shifted data is small (like \( r = 0.1 \)), the random noise in the shifted data also manifest itself through the undercoverage in the first plot in Figure 6.

### 6 Discussion

In this paper, we propose a conditional inference framework for statistical inference problems when some attributes of the data are fixed. To be more specific, we assume that the data is drawn from a conditional distribution given these attributes, and that the attributes are drawn from a super-population model. Based on a fine-grained quantification of uncertainty, we derive conditionally valid confidence intervals and \( p \)-values for inference of conditional parameters in parametric and semi-parametric models.
If conditioning sets are arranged in a hierarchical fashion, we show that simply rejecting all hypothesis corresponding to p-values below $\alpha$ controls the family-wise error rate at level $\alpha$. This allows to gauge the generalizability of a statistical finding across conditional distributions, without loss of power compared to only testing the population parameter. An adaptive multiple testing procedure with asymptotic false discovery rate control is also introduced, which allows to combine statistical findings from independent studies.

In addition, we discuss how to conduct a form of transfer learning of conditional parameters for partially observed new datasets, with conditionally valid asymptotic confidence intervals.

To summarize, we believe that conditional inference is conceptually attractive, widely applicable, and may play an important role for fine-grained uncertainty quantification when evaluating replicability and generalizability across conditional distributions.

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Table 3: Performance of estimator $\hat{\sigma}^k$ for parameters $\theta_1$, $\theta_2$ and hierarchy $k = 3, \ldots, 0$.

| $\nu$ | $\hat{\sigma}_3$ (truth) | $\hat{\sigma}_3$ (mean) | $\hat{\sigma}_3$ (std) | $\hat{\sigma}_2$ (mean) | $\hat{\sigma}_2$ (std) | $\hat{\sigma}_0$ (mean) | $\hat{\sigma}_0$ (std) |
|-------|-----------------------------|--------------------------|-------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $n = 200$ | $1.173 (0.160)$ | $1.204 (0.163)$ | $1.398 (0.178)$ | $1.173 (0.160)$ | $1.204 (0.163)$ | $1.398 (0.178)$ |
| $n = 1000$ | $0.898 (0.110)$ | $0.969 (0.110)$ | $1.318 (0.088)$ | $1.153 (0.062)$ | $1.183 (0.063)$ | $1.373 (0.069)$ |
| $n = 2000$ | $0.761 (0.062)$ | $0.822 (0.062)$ | $1.162 (0.074)$ | $1.157 (0.045)$ | $1.187 (0.046)$ | $1.376 (0.050)$ |
| $n = 5000$ | $0.703 (0.062)$ | $0.758 (0.062)$ | $1.162 (0.074)$ | $1.192 (0.028)$ | $1.382 (0.031)$ | $1.376 (0.050)$ |

A Additional Simulation Results

In this section, we present additional numerical results from simulations.

A.1 Simulation results of conditional inference

The detailed results of the simulations in Section 4.1 on conditional inference are summarized in this section. Appendix A.1.1 presents the additional simulation results where influence functions are approximated via the explicit-formula approach. Appendix A.1.2 presents the simulation results for influence functions approximated via leave-one-out approach.

A.1.1 Performance of explicit formula approach

In this part, we present additional results on simulations of conditional inference, when influence functions are approximated via the explicit formula.

The estimation accuracy of $\hat{\sigma}^k$ corresponding to Figure 1 is summarized in Table 3, where we report the mean and standard deviation of our estimator in the estimation, as well as the truths of $\sigma^k$.

Marginal coverage of conditional parameters. In Table 4, we report the coverage probability of the conditional parameters $\theta_1^{cond}(Z^k)$ and $\theta_2^{cond}(Z^k)$ for the first two OLS coefficients. We report the proportion of replicates where $\theta_j^k$ actually falls in the conditional interval for $j = 1, 2$. The marginal coverage of conditional parameters approaches the desired 0.95 level as the sample size increases, which validates our theory.

Conditional coverage of conditional parameters. The numerical results corresponding to Figure 3 of conditional coverage are summarized in Table 5, where we report the proportion among $N = 1000$ replicates where the 95% conditional confidence interval actually covers the conditional
parameter. The results are coverages for conditional parameters $\theta^n_j$ for $j \geq k$, when the data are sampled conditional on $Z^n_k$.

### A.1.2 Performance of leave-one-out approach

In this part, we present additional results on simulations of conditional inference, when influence functions are approximated via the leave-one-out approach discussed in Remark 2.8. These are supplementary to Section 4.1.

**Accuracy of $\hat{\sigma}_k$.** The estimation accuracy of $\hat{\sigma}_k$ when influence functions are approximated via the leave-one-out approach is summarized in Table 6, where we report the mean and standard deviation of our estimator in the estimation, as well as the truths of $\sigma_k$.

**Marginal coverage of conditional parameters.** In Table 7, we report the coverage probability of the conditional parameters for $\theta_1$ and $\theta_2$. Specifically, we report the proportion of replicates where $\theta_k^n_j$ actually falls in the conditional interval for $j = 1, 2$.

**Conditional coverage of conditional parameters.** The conditional coverages of confidence intervals via Leave-One-Out estimation of influence functions for various hierarchies are summarized in Table 8. We report the proportion among $N = 1000$ replicates where the 95% conditional confidence interval actually covers the conditional parameter. The results are coverages for conditional parameters $\theta^n_j$ for $j \geq k$, when the data are sampled conditional on $Z^n_k$.

### A.2 Simulation results of transductive inference

The detailed numerical results of simulations on transductive inference are summarized in this section, which are supplementary to Section 4.2. Appendix A.2.1 presents the numerical results correspond to Figures 4 and 5, where influence functions are approximated via explicit-formula approach. Appendix A.2.2 summarizes the performances of transductive inference with influence functions approximated via the leave-one-out approach.
| Table 5: Conditional coverage of 95% confidence intervals for conditional parameters. |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
|                | coverage of \( \hat{\theta}^3_n \) | coverage of \( \hat{\theta}^3_n \) | coverage of \( \theta^2_n \) | coverage of \( \theta^2_n \) | coverage of \( \theta^2_n \) | coverage of \( \theta^2_n \) | coverage of \( \theta^2_n \) |
|                | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) |
| \( \theta_1 \) | \( n = 200 \) | \( n = 1000 \) | \( n = 2000 \) | \( n = 5000 \) | \( n = 200 \) | \( n = 1000 \) | \( n = 2000 \) | \( n = 5000 \) | \( n = 200 \) | \( n = 1000 \) | \( n = 2000 \) | \( n = 5000 \) |
|                | 1.000 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|                | 0.996 | 0.986 | 0.991 | 0.993 | 0.978 | 0.965 | 0.993 | 0.977 | 0.964 |
|                | 0.995 | 0.972 | 0.948 | 0.996 | 0.979 | 0.963 | 0.992 | 0.980 | 0.961 |
|                | 0.990 | 0.954 | 0.941 | 0.991 | 0.970 | 0.954 | 0.991 | 0.968 | 0.952 |

| Table 6: Performance of estimator \( \hat{\sigma}^k \) for parameters \( \theta_1, \theta_2 \) and hierarchy \( k = 3, \ldots, 0 \). |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
|                | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) |
| \( \theta_1 \) | \( n = 200 \) | \( n = 1000 \) | \( n = 2000 \) | \( n = 5000 \) | \( n = 200 \) | \( n = 1000 \) | \( n = 2000 \) | \( n = 5000 \) |
|                | 1.166 | 1.194 | 1.384 | 2.087 | 2.014 | 2.104 | 2.217 |
|                | 1.165 | 1.194 | 1.384 | 2.087 | 2.014 | 2.104 | 2.217 |
|                | 1.165 | 1.194 | 1.384 | 2.087 | 2.014 | 2.104 | 2.217 |
|                | 1.165 | 1.194 | 1.384 | 2.087 | 2.014 | 2.104 | 2.217 |
|                | 1.165 | 1.194 | 1.384 | 2.087 | 2.014 | 2.104 | 2.217 |

35
Table 7: Marginal coverage of 95% confidence intervals for conditional parameters.

|          | coverage of $\theta_1^i$ |          | coverage of $\theta_2^i$ |
|----------|--------------------------|----------|--------------------------|
|          | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ |
| $\theta_1$ | n = 200 | 0.999 | 0.999 | 0.996 | 0.945 | 0.951 | 0.944 |
|           | n = 1000 | 0.998 | 0.996 | 0.994 | 0.936 | 0.939 | 0.941 |
|           | n = 2000 | 0.986 | 0.982 | 0.977 | 0.948 | 0.954 | 0.942 |
|           | n = 5000 | 0.973 | 0.968 | 0.970 | 0.947 | 0.949 | 0.945 |
| $\theta_2$ | n = 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|           | n = 1000 | 0.996 | 0.982 | 0.996 | 0.984 | 0.981 | 0.963 |
|           | n = 2000 | 0.993 | 0.976 | 0.963 | 0.994 | 0.976 | 0.962 |
|           | n = 5000 | 0.984 | 0.967 | 0.955 | 0.985 | 0.968 | 0.956 |

|          | coverage of $\theta_1^i$ |          | coverage of $\theta_2^i$ |
|----------|--------------------------|----------|--------------------------|
|          | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ |
| $\theta_1$ | n = 200 | 0.954 | 0.956 | 0.956 | 0.957 | 0.940 | 0.945 |
|           | n = 1000 | 0.941 | 0.937 | 0.945 | 0.955 | 0.952 | 0.950 |
|           | n = 2000 | 0.950 | 0.944 | 0.945 | 0.954 | 0.954 | 0.949 |
|           | n = 5000 | 0.948 | 0.948 | 0.946 | 0.954 | 0.953 | 0.952 |
| $\theta_2$ | n = 200 | 0.944 | 0.945 | 0.952 | 0.947 | 0.949 | 0.955 |
|           | n = 1000 | 0.948 | 0.946 | 0.933 | 0.949 | 0.946 | 0.936 |
|           | n = 2000 | 0.946 | 0.945 | 0.939 | 0.947 | 0.945 | 0.940 |
|           | n = 5000 | 0.946 | 0.950 | 0.945 | 0.947 | 0.954 | 0.946 |

Table 8: Conditional coverage of 95% confidence intervals for conditional parameters.

|          | coverage of $\theta_1^i$ |          | coverage of $\theta_2^i$ |
|----------|--------------------------|----------|--------------------------|
|          | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ |
| $\theta_1$ | n = 200 | 0.999 | 0.999 | 0.994 | 0.946 | 0.945 | 0.945 |
|           | n = 1000 | 0.996 | 0.997 | 0.997 | 0.987 | 0.986 | 0.976 |
|           | n = 2000 | 0.986 | 0.984 | 0.983 | 0.986 | 0.986 | 0.981 |
|           | n = 5000 | 0.968 | 0.972 | 0.964 | 0.986 | 0.986 | 0.981 |
| $\theta_2$ | n = 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|           | n = 1000 | 0.996 | 0.987 | 0.973 | 0.993 | 0.979 | 0.967 |
|           | n = 2000 | 0.996 | 0.973 | 0.948 | 0.995 | 0.979 | 0.966 |
|           | n = 5000 | 0.990 | 0.954 | 0.941 | 0.991 | 0.970 | 0.954 |

|          | coverage of $\theta_1^i$ |          | coverage of $\theta_2^i$ |
|----------|--------------------------|----------|--------------------------|
|          | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ |
| $\theta_1$ | n = 200 | 0.999 | 0.999 | 0.996 | 0.946 | 0.945 | 0.945 |
|           | n = 1000 | 0.996 | 0.996 | 0.994 | 0.937 | 0.944 | 0.948 |
|           | n = 2000 | 0.991 | 0.988 | 0.982 | 0.955 | 0.958 | 0.956 |
|           | n = 5000 | 0.981 | 0.981 | 0.969 | 0.949 | 0.948 | 0.956 |
| $\theta_2$ | n = 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|           | n = 1000 | 0.994 | 0.983 | 0.965 | 0.991 | 0.979 | 0.964 |
|           | n = 2000 | 0.994 | 0.971 | 0.965 | 0.993 | 0.972 | 0.966 |
|           | n = 5000 | 0.992 | 0.974 | 0.961 | 0.988 | 0.973 | 0.962 |

condition on $Z^3 = \{X_1, X_2, X_3\}$

condition on $Z^2 = \{X_1, X_2\}$

condition on $Z^1 = \{X_1\}$

Table 9: Conditional coverage of 95% confidence intervals for conditional parameters.
A.2.1 Performance of explicit formula approach

**Marginal coverage.** The marginal coverages of the corresponding 95% confidence interval for $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ with various sample sizes and noise levels are summarized in Table 9, where influence functions are approximated with explicit formula approach. We report the proportion among $N = 1000$ replicates when the conditional confidence interval actually covers the new conditional parameter.

**Conditional coverage.** The conditional coverages of the corresponding 95% confidence interval for $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ with various sample sizes and noise levels are summarized in Table 10, where influence functions are approximated with explicit formula approach. Similar to the marginal coverage, we report the proportion among $N = 1000$ replicates when the new conditional parameter is covered.

| $n$ (sample size) | $m/n$ | Coverage of $\theta_{\text{cond,new}}^m$ | Coverage of $\hat{\theta}_{\text{linear}}$ |
|-------------------|-------|----------------------------------------|----------------------------------------|
| 200               | 0.5   | $0.711$ $0.771$ $0.749$                | $0.866$ $0.870$ $0.895$                |
|                   | 1     | $0.771$ $0.779$ $0.813$                | $0.843$ $0.854$ $0.879$                |
|                   | 2     | $0.810$ $0.817$ $0.845$                | $0.861$ $0.868$ $0.876$                |
| 1000              | 0.5   | $0.812$ $0.820$ $0.841$                | $0.915$ $0.908$ $0.932$                |
|                   | 1     | $0.871$ $0.876$ $0.894$                | $0.927$ $0.927$ $0.926$                |
|                   | 2     | $0.883$ $0.892$ $0.906$                | $0.912$ $0.918$ $0.927$                |
| 2000              | 0.5   | $0.867$ $0.871$ $0.896$                | $0.932$ $0.935$ $0.942$                |
|                   | 1     | $0.888$ $0.893$ $0.915$                | $0.921$ $0.924$ $0.928$                |
|                   | 2     | $0.916$ $0.915$ $0.913$                | $0.931$ $0.934$ $0.938$                |
| 5000              | 0.5   | $0.900$ $0.898$ $0.911$                | $0.940$ $0.940$ $0.941$                |
|                   | 1     | $0.920$ $0.924$ $0.932$                | $0.939$ $0.935$ $0.94$                |
|                   | 2     | $0.923$ $0.93$ $0.931$                 | $0.949$ $0.944$ $0.941$                |
| 10000             | 0.5   | $0.932$ $0.930$ $0.933$                | $0.939$ $0.940$ $0.946$                |
|                   | 1     | $0.927$ $0.932$ $0.935$                | $0.942$ $0.943$ $0.941$                |
|                   | 2     | $0.922$ $0.925$ $0.927$                | $0.935$ $0.932$ $0.930$                |

Table 9: Marginal coverage of 95% confidence intervals for $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ when influence functions are approximated with explicit formula approach.

A.2.2 Performance of leave-one-out approach

**Marginal coverage.** The marginal coverages of the corresponding 95% confidence interval for $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ with various sample sizes and noise levels are summarized in Table 11, where influence functions are approximated with leave-one-out approach. We report the proportion among $N = 1000$ replicates when the conditional confidence interval actually covers the new conditional parameter.

**Conditional coverage.** The conditional coverages of the corresponding 95% confidence interval for $\theta_{\text{cond,new}}^m$ and $\hat{\theta}_{\text{linear}}$ with various sample sizes and noise levels are summarized in Table 10, where influence functions are approximated with leave-one-out approach. Similar to the marginal coverage, we report the proportion among $N = 1000$ replicates when the new conditional parameter is covered.
| n/n | \( \theta_1 \) | \( \theta_2 \) |
|-----|----------------|----------------|
| 200 | m/n = 0.5      | 0.762 0.765 0.798 | 0.773 0.784 0.811 |
|     | m/n = 1        | 0.841 0.845 0.863 | 0.861 0.871 0.883 |
|     | m/n = 2        | 0.865 0.863 0.886 | 0.906 0.908 0.921 |
| 1000| m/n = 0.5      | 0.881 0.890 0.898 | 0.907 0.910 0.919 |
|     | m/n = 1        | 0.869 0.861 0.875 | 0.915 0.918 0.918 |
|     | m/n = 2        | 0.891 0.895 0.903 | 0.936 0.935 0.931 |
| 2000| m/n = 0.5      | 0.881 0.877 0.896 | 0.927 0.923 0.921 |
|     | m/n = 1        | 0.818 0.832 0.937 | 0.951 0.949 0.951 |
|     | m/n = 2        | 0.918 0.922 0.932 | 0.931 0.936 0.932 |
| 5000| m/n = 0.5      | 0.940 0.939 0.941 | 0.956 0.954 0.956 |
|     | m/n = 1        | 0.936 0.935 0.936 | 0.938 0.939 0.943 |
|     | m/n = 2        | 0.942 0.944 0.957 | 0.941 0.946 0.956 |
| 10000| m/n = 0.5      | 0.945 0.942 0.941 | 0.934 0.933 0.936 |
|   | m/n = 1        | 0.933 0.934 0.928 | 0.913 0.915 0.918 |
|   | m/n = 2        | 0.931 0.934 0.94  | 0.922 0.924 0.929 |
| 2000| m/n = 0.5      | 0.950 0.946 0.950 | 0.939 0.942 0.943 |
|   | m/n = 1        | 0.922 0.925 0.930 | 0.946 0.946 0.942 |
|   | m/n = 2        | 0.929 0.932 0.938 | 0.948 0.949 0.944 |
| 5000| m/n = 0.5      | 0.923 0.922 0.930 | 0.941 0.945 0.944 |
|   | m/n = 1        | 0.935 0.942 0.936 | 0.958 0.960 0.953 |
|   | m/n = 2        | 0.928 0.935 0.935 | 0.945 0.946 0.948 |
| 10000| m/n = 0.5      | 0.926 0.919 0.928 | 0.940 0.937 0.950 |
|    | m/n = 1        | 0.937 0.938 0.939 | 0.948 0.95 0.95 |
|    | m/n = 2        | 0.946 0.952 0.954 | 0.959 0.958 0.957 |

Table 10: Conditional coverage of 95% confidence intervals for \( \theta_{m,\text{new}}^\text{cond} \) and \( \hat{\theta}^\text{linear} \) when influence functions are approximated with explicit formula approach.
|          | coverage of $\theta_{\text{cond, new}}^{\text{m}}$ | coverage of $\hat{\theta}^{\text{linear}}$ |
|----------|-----------------------------------------------|---------------------------------|
|          | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ | $\nu = 0.1$ | $\nu = 0.2$ | $\nu = 0.5$ |
| $n = 200$ | 0.735 0.726 0.762 | 0.884 0.885 0.906 | 0.787 0.799 0.821 | 0.860 0.864 0.888 |
|          | 0.810 0.817 0.845 | 0.881 0.868 0.876 | 0.819 0.824 0.845 | 0.916 0.910 0.932 |
| $n = 1000$ | 0.873 0.880 0.898 | 0.930 0.929 0.929 | 0.883 0.892 0.906 | 0.912 0.918 0.927 |
|          | 0.870 0.871 0.897 | 0.933 0.936 0.943 | 0.890 0.894 0.915 | 0.921 0.924 0.928 |
| $n = 2000$ | 0.917 0.917 0.914 | 0.933 0.933 0.938 | 0.900 0.899 0.911 | 0.940 0.940 0.941 |
|          | 0.920 0.924 0.932 | 0.939 0.935 0.94 | 0.923 0.930 0.931 | 0.949 0.944 0.941 |
| $n = 5000$ | 0.952 0.930 0.933 | 0.939 0.940 0.946 | 0.927 0.932 0.935 | 0.942 0.944 0.942 |
|          | 0.922 0.925 0.928 | 0.936 0.932 0.930 | 0.908 0.910 0.901 | 0.939 0.937 0.928 |
| $n = 10000$ | 0.945 0.941 0.944 | 0.947 0.946 0.948 | 0.919 0.922 0.925 | 0.939 0.934 0.934 |
|          | 0.951 0.952 0.953 | 0.952 0.953 0.958 | 0.935 0.934 0.945 | 0.939 0.944 0.949 |
| $n = 200$ | 0.938 0.939 0.947 | 0.945 0.944 0.944 | 0.94 0.943 0.952 | 0.939 0.940 0.948 |
|          | 0.946 0.943 0.952 | 0.944 0.943 0.949 | 0.939 0.935 0.934 | 0.939 0.939 0.933 |
| $n = 1000$ | 0.959 0.944 0.950 | 0.940 0.940 0.948 | 0.955 0.954 0.943 | 0.947 0.952 0.945 |
|          | 0.949 0.947 0.953 | 0.949 0.949 0.954 | 0.949 0.945 0.933 | 0.946 0.946 0.944 |
| $n = 5000$ | 0.947 0.945 0.953 | 0.946 0.946 0.954 | 0.952 0.947 0.942 | 0.951 0.950 0.946 |
|          | 0.948 0.945 0.943 | 0.951 0.949 0.947 | 0.946 0.945 0.944 | 0.946 0.945 0.940 |

Table 11: Marginal coverage of 95% confidence intervals for $\theta_{\text{cond, new}}^{\text{m}}$ and $\hat{\theta}^{\text{linear}}$ when influence functions are approximated with the leave-one-out approach.
### Conditional coverage

|       | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) | \( \nu = 0.1 \) | \( \nu = 0.2 \) | \( \nu = 0.5 \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( n = 200 \) | \( m/n = 0.5 \) | 0.778 | 0.784 | 0.818 | 0.853 | 0.847 | 0.879 |
|       | \( m/n = 1 \) | 0.783 | 0.799 | 0.824 | 0.881 | 0.879 | 0.896 |
|       | \( m/n = 2 \) | 0.856 | 0.854 | 0.874 | 0.873 | 0.874 | 0.884 |
| \( n = 1000 \) | \( m/n = 0.5 \) | 0.908 | 0.913 | 0.926 | 0.925 | 0.921 | 0.931 |
|       | \( m/n = 1 \) | 0.87 | 0.863 | 0.89 | 0.908 | 0.908 | 0.922 |
|       | \( m/n = 2 \) | 0.884 | 0.891 | 0.903 | 0.908 | 0.913 | 0.921 |
| \( \theta_1 \) | \( n = 2000 \) | \( m/n = 0.5 \) | 0.869 | 0.865 | 0.878 | 0.916 | 0.918 | 0.921 |
|       | \( m/n = 1 \) | 0.892 | 0.895 | 0.905 | 0.936 | 0.936 | 0.934 |
|       | \( m/n = 2 \) | 0.882 | 0.879 | 0.899 | 0.928 | 0.925 | 0.923 |
|       | \( n = 5000 \) | \( m/n = 0.5 \) | 0.881 | 0.884 | 0.908 | 0.930 | 0.930 | 0.931 |
|       | \( m/n = 1 \) | 0.919 | 0.932 | 0.937 | 0.954 | 0.949 | 0.951 |
|       | \( m/n = 2 \) | 0.919 | 0.922 | 0.932 | 0.934 | 0.936 | 0.933 |
| \( n = 10000 \) | \( m/n = 0.5 \) | 0.942 | 0.939 | 0.941 | 0.956 | 0.954 | 0.956 |
|       | \( m/n = 1 \) | 0.936 | 0.935 | 0.936 | 0.938 | 0.939 | 0.943 |
|       | \( m/n = 2 \) | 0.942 | 0.945 | 0.957 | 0.941 | 0.946 | 0.955 |

Table 12: Conditional coverage of 95% confidence intervals for \( \theta_{\text{cond,new}}^m \) and \( \hat{\theta}_{\text{linear}} \) when influence functions are approximated with the leave-one-out approach.

### Conditional coverage

The conditional coverages of 95% confidence interval for \( \theta_{\text{cond,new}}^m \) and \( \hat{\theta}_{\text{linear}} \) with various sample sizes and noise levels are summarized in Table 12, where influence functions are approximated with leave-one-out approach.

### B Distributional Results

In this section, we establish the distributional results on conditional parameters and random hypotheses. We consider the most general case of the hierarchy structure described in Section 3.1, with a set of conditional parameters \( \{\theta_k^m\}_{k=0}^n \), increasing conditioning sets \( \{Z_k^m\}_{k=0}^n \) and decreasing variances \( \{\sigma_k^2\}_{k=0}^n \). Taking any piece \( k \in [m] \) from the hierarchy applies to the inference of one single conditional parameter in Section 2.1. The results apply directly to the conditional FWER control for one hierarchy in Section 3.1. They also apply to conditional FDR control in Section 3.3 when considering multiple hierarchies.

#### B.1 Asymptotic behaviour of conditional laws

We first establish the asymptotic conditional laws of the conditional parameters in the hierarchical setting, which act as the foundation of theoretical results in the paper.
Lemma B.1. Suppose Assumption 3.1 holds. For levels $k > j \geq s$, we denote the variance gap as
\[
\sigma_{k,j}^2 = \sigma_j^2 - \sigma_k^2 = \text{Var}(\mathbb{E}[\phi(D) \mid Z^j]) - \text{Var}(\mathbb{E}[\phi(D) \mid Z^j]).
\]
If $\sigma_{k,j} > 0$, then for any $x \in \mathbb{R}$, it holds that
\[
\mathbb{P}\left(\sqrt{n}(\theta_n^k - \theta_n^j) \leq x \mid Z^s\right) = \Phi\left(\frac{x}{\sigma_{k,j}}\right) + o_p(1).
\]
If $\sigma_{k,j} = 0$, then for any constant $\epsilon > 0$, it holds that
\[
\mathbb{P}\left(\sqrt{n}|\theta_n^k - \theta_n^j| > \epsilon \mid Z^s\right) = o_p(1).
\]

Proof of Lemma B.1. By Assumption 3.1, we have
\[
\sqrt{n}(\theta_n^k - \theta_n^j) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{E}[\phi(D_i) \mid Z_i^k] - \mathbb{E}[\phi(D_i) \mid Z_i^j]) + o_p(1).
\]
If $\sigma_{k,j} = 0$, we have $\mathbb{E}[\phi(D_i) \mid Z_i^k] \overset{a.s.}{=} \mathbb{E}[\phi(D_i) \mid Z_i^j]$, hence $\sqrt{n}(\theta_n^k - \theta_n^j) = o_p(1)$. Applying Lemma G.1 to $\sqrt{n}(\theta_n^k - \theta_n^j)$ and the $\sigma$-fields $Z^s = (Z_1^s, \ldots, Z_n^s)$, we have
\[
\mathbb{P}\left(\sqrt{n}|\theta_n^k - \theta_n^j| > \epsilon \mid Z^s\right) = o_p(1).
\]
Hereafter, we assume $\sigma_{k,j} > 0$. For notational simplicity, we write
\[
d_n = \sqrt{n}(\theta_n^k - \theta_n^j) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i, \quad \text{where} \quad \zeta_i = \mathbb{E}[\phi(D_i) \mid Z_i^k] - \mathbb{E}[\phi(D_i) \mid Z_i^j], \ i = 1, \ldots, n.
\]
By Assumption 3.1, we know $d_n = o_p(1)$. Hence by Lemma G.1, for any fixed $\epsilon > 0$, we have
\[
\mathbb{P}(|d_n| > \epsilon \mid Z^s) = o_p(1). \quad (B.1)
\]
On the other hand, we consider the conditional law of the other term, denoted as
\[
\mathcal{L}_n = \mathcal{L}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{E}[\phi(D_i) \mid Z_i^k] - \mathbb{E}[\phi(D_i) \mid Z_i^j]) \mid Z^s\right).
\]
By the conditional CLT in Lemma G.2, taking $g(X_i) = \mathbb{E}[\phi(D_i) \mid Z_i^k]$, $Z_i = Z_i^j$ and the filtration $\mathcal{G}_n = \sigma\{Z_i^1 \mid i = 1\} \subset \sigma\{Z_i^j \mid i = 1\}$, we know that the conditional law $\mathcal{L}_n$ converges almost surely to $\mathcal{N}(0, \sigma_{k,j}^2)$. That is, for any $x \in \mathbb{R}$, we have
\[
\mathbb{P}(\sqrt{n}(\theta_n^k - \theta_n^j) + d_n \leq x \mid Z^s) \overset{a.s.}{\rightarrow} \Phi\left(\frac{x}{\sigma_{k,j}}\right),
\]
where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. By equation (B.1), for any constant $\epsilon > 0$, it holds that
\[
\mathbb{P}(\sqrt{n}(\theta_n^k - \theta_n^j) \leq x \mid Z^s) = \mathbb{P}(\sqrt{n}(\theta_n^k - \theta_n^j) \leq x, |d_n| \leq \epsilon \mid Z^s) + \mathbb{P}(\sqrt{n}(\theta_n^k - \theta_n^j) \leq x, |d_n| > \epsilon \mid Z^s)
\]
\[
\leq \mathbb{P}(\sqrt{n}(\theta_n^k - \theta_n^j) + d_n \leq x + \epsilon \mid Z^s) + \mathbb{P}(|d_n| > \epsilon \mid Z^s) = \Phi\left(\frac{x + \epsilon}{\sigma_{k,j}}\right) + o_p(1). \quad (B.2)
\]
On the other hand, we have
\[
\mathbb{P}(\sqrt{n}(\theta^k_n - \hat{\theta}^j_n) \leq x \mid Z^s) \\
\geq \mathbb{P}(\sqrt{n}(\theta^k_n - \hat{\theta}^j_n) + d_n \leq x - \epsilon, |d_n| \leq \epsilon \mid Z^s) \\
\geq \mathbb{P}(\sqrt{n}(\theta^k_n - \hat{\theta}^j_n) + d_n \leq x - \epsilon \mid Z^s) - \mathbb{P}(|d_n| > \epsilon \mid Z^s) = \Phi\left(\frac{x - \epsilon}{\sigma_{k,j}}\right) + o_{\mathbb{P}}(1). \quad (\text{B.3})
\]

By the arbitrariness of \( \epsilon > 0 \) in equations (B.2) and (B.3), for any fixed \( x \in \mathbb{R} \), it holds that
\[
\mathbb{P}(\sqrt{n}(\theta^k_n - \hat{\theta}^j_n) \leq x \mid Z^s) = \Phi\left(\frac{x}{\sigma_{k,j}}\right) + o_{\mathbb{P}}(1).
\]

Therefore, we conclude the proof of Lemma B.1. \( \square \)

### B.2 Asymptotic behaviour of random hypotheses

In this section, we discuss on a crucial observation that within a hierarchy, the random hypotheses \( \{H^k_0\}_{k=0}^m \) defined in equation (3.2) correspond to a set of deterministic hypotheses \( \{H^k_0\}_{k=0}^m \) to be specified later, so that a random hypothesis cannot be true asymptotically if its deterministic counterpart is not. Even though the random hypotheses \( \{H^k_0\}_{k=0}^m \) are not necessarily nested, the deterministic ones have a nested structure, which is crucial for multiple testing.

Specifically, for each \( k \in [m] \), we define the deterministic hypothesis
\[
\tilde{H}^k_0 : \theta^0 = 0 \quad \text{and} \quad \text{Var}(\mathbb{E}[\phi(D) \mid Z^k]) = 0. \quad (\text{B.4})
\]

Here \( \theta^0 \) is the unconditional parameter corresponding to the lowest level \( Z^0 = \emptyset \). We denote the random null set \( I_0 \) and the deterministic null set \( I^*_0 \) as
\[
I_0 = \{ k : \tilde{H}^k_0 \text{ is true} \}, \quad I^*_0 = \{ k : \tilde{H}^k_0 \text{ is true} \}. \quad (\text{B.5})
\]

The following lemma shows the asymptotic inclusion of \( I_0 \) in \( I^*_0 \).

**Lemma B.2.** Let the sets \( I_0, I^*_0 \) be as in equation (B.5). Then for any \( k \notin I^*_0 \), we have \( \lim_{n \to \infty} \mathbb{P}(k \in I_0) = 0 \). Furthermore, we have \( \lim_{n \to \infty} \mathbb{P}(I_0 \setminus I^*_0 \neq \emptyset) = 0 \), or equivalently, \( \lim_{n \to \infty} \mathbb{P}(I_0 \subseteq I^*_0) = 1 \).

**Proof of Lemma B.2.** We first show that for any \( k \notin I^*_0 \), we have \( \lim_{n \to \infty} \mathbb{P}(k \in I_0) = 0 \).

If the deterministic parameter \( \theta^0 \neq 0 \), then \( I^*_0 = \emptyset \). Note that the variance gap is \( \sigma^2_{k,0} = \text{Var}(\mathbb{E}[\phi(D) \mid Z^k]) \). If \( \sigma_{k,0} \), taking \( j = s = 0 \) in Lemma B.1 we have \( \sqrt{n}(\theta^k_n - \theta^0) = o_{\mathbb{P}}(1) \), hence \( \mathbb{P}(k \in I_0) = \mathbb{P}(\theta^k_n = 0) = o_{\mathbb{P}}(1) \). If \( \sigma_{k,0} = 0 \), taking \( j = s = 0 \) in Lemma B.1, for any \( x \in \mathbb{R} \) we have
\[
\mathbb{P}(\sqrt{n}(\theta^k_n - \theta^0) \leq x) = \Phi(x/\sigma_{k,0}) + o_{\mathbb{P}}(1),
\]
which also leads to \( \mathbb{P}(k \in I_0) = \mathbb{P}(\theta^k_n = 0) = o_{\mathbb{P}}(1) \). So the desired conclusion holds.

Hereafter we assume \( \theta^0 = 0 \), then for any \( i \notin I^*_0 \), since \( \sigma^2_{k,0} = \text{Var}(\mathbb{E}[\phi(X) \mid Z^k]) > 0 \), by Lemma B.1 applied to \( j = s = 0 \), for any \( x \in \mathbb{R} \) it holds that
\[
\mathbb{P}(\sqrt{n}\theta^k_n \leq x) = \Phi(x/\sigma_{k,0}) + o_{\mathbb{P}}(1).
\]
This indicates
\[
\mathbb{P}(i \in I_0) = \mathbb{P}(\sqrt{n}|\theta^k_n - \theta^0| = 0) = o(1),
\]
which concludes the proof of the first conclusion.

Furthermore, by union bounds on a finite number of events, we have
\[
\mathbb{P}(I_0 \setminus I^*_0 \neq \emptyset) \leq \sum_{j=0}^m \mathbb{P}(j \notin I^*_0, j \in I_0) = \sum_{j \notin I^*_0} \mathbb{P}(j \in I_0) \to 0,
\]
which proves the second assertion. For the last one, note that
\[
\mathbb{P}(I_0 \subseteq I^*_0) = \mathbb{P}(I_0 \setminus I^*_0 = \emptyset) = 1 - \mathbb{P}(I_0 \setminus I^*_0 \neq \emptyset) \to 1.
\]
Therefore, we conclude the proof of Lemma B.2. \( \square \)
B.3 Asymptotics of test statistics and p-values

In one hierarchy, we proceed to apply the asymptotic conditional laws established in Appendix B.1 to establish the conditional validity of the p-values \( p_k \) for any \( k \in [m] \) in the following lemma.

**Lemma B.3.** Suppose Assumption 3.1 holds. For any level 0 \( 0 \leq j < k < m \), let \( H_0^k \) be the random hypothesis defined in equation (3.2) and let \( p_k \) be the p-value defined in equation (3.3), where \( \hat{\sigma}_k \xrightarrow{p} \sigma_k \). Then we have \( \lim_{n \to \infty} \Pr(H_0^k \text{ true}, p_k < \alpha) \leq \alpha \). Moreover, we have \( \Pr(H_0^k \text{ true}, p_k < \alpha \mid Z^j) \leq \alpha + o_p(1) \).

**Proof of Lemma B.3.** Firstly, note that \( \sqrt{n}(\hat{\theta}_n - \theta_n^k) \xrightarrow{d} N(0, \sigma_n^2) \). Then we have

\[
\Pr(H_0^k \text{ true}, p_k < \alpha) = \Pr\left( |\theta_n^k| = 0, \frac{\sqrt{n}|\hat{\theta}_n|}{\hat{\sigma}_k} > z_{1-\alpha/2} \right)
\leq \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n|}{\hat{\sigma}_k} > z_{1-\alpha/2} \right).
\]

Since \( \hat{\sigma}_k \xrightarrow{p} \sigma_k \), we know

\[
\lim_{n \to \infty} \Pr(H_0^k \text{ true}, p_k < \alpha) \leq \Pr(|Z| > z_{1-\alpha/2}) = \alpha.
\]

Similarly, conditional on \( Z^j \), we have

\[
\Pr(H_0^k \text{ true}, p_k < \alpha \mid Z^j) = \Pr\left( |\theta_n^k| = 0, \frac{\sqrt{n}|\hat{\theta}_n|}{\hat{\sigma}_k} > z_{1-\alpha/2} \mid Z^j \right)
\leq \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n|}{\hat{\sigma}_k} > z_{1-\alpha/2} \mid Z^j \right).
\]

Since \( \hat{\sigma}_k \xrightarrow{p} \sigma_k \) and \( c_n = o(1) \), we know \( \hat{\sigma}_k z_{1-\alpha/2} = \sigma_k z_{1-\alpha/2} + o(1) \). Denoting \( r_n = \hat{\sigma}_k z_{1-\alpha/2} - \sigma^k z_{1-\alpha/2} \), we have \( r_n = o(1) \), thus by Lemma G.1, we have \( \Pr(|r_n| > \epsilon \mid Z^k) = o(1) \) for all fixed \( \epsilon > 0 \). Then for any \( \epsilon > 0 \),

\[
\Pr\left( \frac{\sqrt{n}|\hat{\theta}_n - \theta_n^k|}{\hat{\sigma}_k} > z_{1-\alpha/2} \mid Z^j \right)
= \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n - \theta_n^k|}{\sigma_k} > \sigma_k z_{1-\alpha/2} + r_n \mid Z^j \right)
= \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n - \theta_n^k|}{\sigma_k} > \sigma_k z_{1-\alpha/2} + r_n, |r_n| \leq \epsilon \mid Z^j \right)
+ \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n - \theta_n^k|}{\sigma_k} > \sigma_k z_{1-\alpha/2} + r_n, |r_n| > \epsilon \mid Z^j \right)
\leq \Pr\left( \frac{\sqrt{n}|\hat{\theta}_n - \theta_n^k|}{\sigma_k} > \sigma_k z_{1-\alpha/2} - \epsilon \mid Z^j \right) + o_p(1)
= 2 - 2\Phi\left( z_{1-\alpha/2} - \frac{\epsilon}{\sigma_k} \right) + o_p(1).
\]

B the arbitrariness of \( \epsilon > 0 \), we have \( \Pr(H_0^k \text{ true}, p_k < \alpha \mid Z^k) \leq \alpha + o_p(1) \), which completes the proof.

\[ \square \]

**C Proofs of Results in Section 2.1**

In this section, we provide the proofs of inference results in Section 2.1. In Appendix C.1, we discuss the linearity of conditional Z-estimators. In Appendix C.2, we show the validity of conditional inference in Section 2.1, which are based on the distributional results in Appendix B.
C.1 Linearity of Z-estimators

In the following, we show that under regularity conditions, general Z-estimators defined in equations (1.1) and (1.2) admit the expansion in equations (2.1) and (2.2). We tackle the one-dimensional case, where multidimensional case could be generalized with similar regularity conditions as in the literature.

Recall the definition that

\[ \theta_{n, \text{cond}} = \arg \min_{\theta} \sum_{i=1}^{n} \mathbb{E}[s(D_i, \theta) \mid Z_i]. \]

By independence of \( D_i \) given \( Z = (Z_1, \ldots, Z_n) \), we can write \( \mathbb{E}[s(D_i, \theta) \mid Z_i] = s_i(Z_i, \theta) \) for some function \( s^* \), so that

\[ \theta_{n, \text{cond}} = \arg \min_{\theta} \sum_{i=1}^{n} s_i(Z_i, \theta). \]

Under standard regularity conditions (i.e., smoothness of \( s^* \) and compactness of the parameter space) we have \( \hat{\theta}_n \xrightarrow{p} \theta^0 \) and \( \theta_{n, \text{cond}} \xrightarrow{p} \theta^0 \). By the Taylor expansion around \( \theta^0 \), we have

\[ 0 = \sum_{i=1}^{n} s_i(Z_i, \theta_{n, \text{cond}}) = \sum_{i=1}^{n} s_i(Z_i, \theta^0) + \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) \cdot (\theta_{n, \text{cond}}^{\text{cond}} - \theta^0) + \sum_{i=1}^{n} \hat{s}_i(Z_i, \tilde{\theta}_i) \cdot (\theta_{n, \text{cond}}^{\text{cond}} - \theta^0)^2, \]

where \( \hat{s}_i(z, \theta) = \frac{\partial}{\partial \theta} s(z, \theta) \) is the gradient and \( \hat{s}_i(z, \theta) = \frac{\partial^2}{\partial \theta^2} s(z, \theta) \) is the Hessian, and \( \tilde{\theta}_i \) lies between \( \theta_{n, \text{cond}}^k \) and \( \theta^0 \) for all \( i = 1, \ldots, n \). Therefore

\[ \sqrt{n}(\theta_{n, \text{cond}}^{\text{cond}} - \theta^0) = \left( \frac{1}{n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) + \frac{1}{2n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \tilde{\theta}_i) (\theta_{n, \text{cond}}^{\text{cond}} - \theta^0) \right)^{-1} \left( - \frac{1}{n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) \right). \]  

(C.1)

Additionally assume \( |\hat{s}_i(x, \theta)| \leq m(x) \) for some function \( m(\cdot) \) and all \( \theta \) such that \( \mathbb{E}[m(Z)] < \infty \). Then we have

\[ \left| \frac{1}{2n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) (\theta_{n, \text{cond}}^{\text{cond}} - \theta) \right| \leq \frac{1}{2n} \sum_{i=1}^{n} m(Z_i) \cdot |\theta_{n, \text{cond}}^{\text{cond}} - \theta| = o_p(1) \]

by the law of large numbers. Therefore,

\[ \frac{1}{n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) + \frac{1}{2n} \sum_{i=1}^{n} \hat{s}_i(Z_i, \tilde{\theta}_i) (\theta_{n, \text{cond}}^{\text{cond}} - \theta) = \mathbb{E}[\hat{s}_i(Z_i, \theta^0)] + o_p(1) = \mathbb{E}[\hat{s}_i(D_i, \theta^0)] + o_p(1), \]

where the last equality follows from the tower property of conditional expectations. Similar arguments holds for the estimator \( \tilde{\theta}_n \) by replacing \( Z_i \) with \( X_i \). Thus we have the asymptotic linearity

\[ \sqrt{n}(\theta_{n, \text{cond}}^{\text{cond}} - \theta^0) = \frac{-1}{\mathbb{E}[\hat{s}(D_i, \theta^0)] + o_p(1)} \sum_{i=1}^{n} \hat{s}_i(Z_i, \theta^0) \]

\[ = - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{E}[\hat{s}(D_i, \theta^0)] \right)^{-1} s(Z_i, \theta^0) + o_p(1) \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}[\hat{s}(D_i, \theta^0) \mid Z_i] + o_p(1), \]

as long as \( \text{Var}(s(Z_i, \theta^0)) \neq 0 \), where

\[ \phi(D_i) = (\mathbb{E}[\hat{s}(D_i, \theta^0)])^{-1} s(D_i, \theta^0), \]
\[ \sqrt{n}(\hat{\theta}_n - \theta^0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(D_i) + o_p(1). \]

C.2 Proofs of validity of conditional inference

**Proof of Proposition 2.3.** Take \( k = m \) and \( j = s \) in Lemma B.1 so that \( Z^j = Z^s = Z \). Under the condition that \( \sigma = \sigma_{k,j} > 0 \), we arrive at equation (2.4).

**Proof of Theorem 2.4.** Consider a three-level hierarchy with \( m = 2 \), \( Z^0 = \emptyset \), \( Z^1 = Z \), \( Z^2 = D \) the whole data and \( \sigma^2 = \sigma^2_n > 0 \). Taking \( j = s = 1 \) and \( k = 2 \), we have \( \theta^k_n = \hat{\theta}_n \) (the estimated quantity) and \( \theta^i_n = \theta_n^{\text{cond}} \) (the conditional parameter). By Lemma B.1, we have

\[ P\left( \sqrt{n}(\hat{\theta}_n - \theta_n^{\text{cond}}) \leq x \mid Z \right) = \Phi(x/\sigma) + o_p(1). \]

Since \( \hat{\sigma} \overset{p}{\to} \sigma \), for any constant \( \epsilon > 0 \), we have

\[ P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) \]
\[ \geq P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot (\sigma - \epsilon) \mid Z \right) + P(\hat{\sigma} < \sigma - \epsilon \mid Z), \]
where \( P(\hat{\sigma} < \sigma - \epsilon \mid Z) = o_p(1) \) by Lemma G.1. Therefore, for any \( \epsilon > 0 \), we have

\[ P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) \geq 2\Phi(z_{1-\alpha/2}(1 - \epsilon/\sigma)) - 1 = o_p(1). \] (C.2)

On the other hand, for any \( \epsilon > 0 \), we have

\[ P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) \]
\[ \leq P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot (\sigma + \epsilon) \mid Z \right) + P(\hat{\sigma} > \sigma + \epsilon \mid Z), \]
where \( P(\hat{\sigma} > \sigma + \epsilon \mid Z) = o_p(1) \) by Lemma G.1. Therefore, for any \( \epsilon > 0 \), we have

\[ P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) \leq 2\Phi(z_{1-\alpha/2}(1 + \epsilon/\sigma)) - 1 + o_p(1). \] (C.3)

Combining equations (C.2) and (C.3), by the arbitrariness of \( \epsilon > 0 \), we have

\[ P\left( \theta_n^{\text{cond}} \in \left[ \hat{\theta}_n - z_{1-\alpha/2}\hat{\sigma}/\sqrt{n}, \hat{\theta}_n + z_{1-\alpha/2}\hat{\sigma}/\sqrt{n} \right] \mid Z \right) \]
\[ = P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| \leq z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) = 1 - \alpha + o_p(1), \]
which completes the proof of Theorem 2.4.

**Proof of Theorem 2.7.** Consider a three-level hierarchy with \( m = 2 \), \( Z^0 = \emptyset \), \( Z^1 = Z \), \( Z^2 = \mathcal{X} \) the whole data and \( \sigma^2 = \sigma^2_n > 0 \). Taking \( j = s = 1 \) and \( k = 2 \), we have \( \theta^k_n = \hat{\theta}_n \) and \( \theta^i_n = \theta_n^{\text{cond}} \). By the proof of Theorem 2.4, we have

\[ P(H_0 \text{ is true, } p < \alpha \mid Z) = P\left( \theta_n^{\text{cond}} = 0, \sqrt{n}|\hat{\theta}_n|/\hat{\sigma} > z_{1-\alpha/2} \mid Z \right) \]
\[ \leq P\left( \sqrt{n}|\hat{\theta}_n - \theta_n^{\text{cond}}| > z_{1-\alpha/2} \cdot \hat{\sigma} \mid Z \right) = \alpha + o_p(1), \]
which completes the proof of Theorem 2.7.
D Proofs of Results in Section 2.2

Proof of Proposition 2.10. By the formula in equation (2.9), the estimator based on the dataset $D^{(3)}$ can be decomposed into

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}(D_i^{(3)}) - \varphi(Z_i^{(3)}) \right)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \phi(D_i^{(3)}) - \varphi(Z_i^{(3)}) + \hat{\phi}(D_i^{(3)}) - \hat{\phi}(Z_i^{(3)}) + \varphi(Z_i^{(3)}) \right)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \phi(D_i^{(3)}) - \varphi(Z_i^{(3)}) \right)^2 + \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}(D_i^{(3)}) - \hat{\phi}(Z_i^{(3)}) + \varphi(Z_i^{(3)}) \right)^2$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \left( \phi(D_i^{(3)}) - \varphi(Z_i^{(3)}) \right) \left( \hat{\phi}(D_i^{(3)}) - \hat{\phi}(D_i^{(3)}) - \hat{\phi}(Z_i^{(3)}) + \varphi(Z_i^{(3)}) \right). \quad (D.1)$$

By the law of large numbers, the first term (i) satisfies

$$\text{(i)} = \frac{1}{n} \sum_{i=1}^{n} \left( \phi(D_i^{(3)}) - \varphi(Z_i^{(3)}) \right)^2 \xrightarrow{p} \sigma^2.$$

On the other hand, by the Cauchy-Schwarz inequality, the second term (ii) is bounded as

$$\text{(ii)} = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}(D_i^{(3)}) - \phi(D_i^{(3)}) \right)^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \left( \hat{\phi}(D_i^{(3)}) - \phi(D_i^{(3)}) \right)^2 + \frac{2}{n} \sum_{i=1}^{n} \left( \hat{\phi}(Z_i^{(3)}) - \varphi(Z_i^{(3)}) \right)^2.$$
which follows from the consistency of nonparametric regression in equation (2.11). Combining the above results we have

\[
(ii) = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}(D_i^{(3)}) - \phi(D_i^{(3)}) - \hat{\varphi}(Z_i^{(3)}) + \varphi(Z_i^{(3)}) \right)^2 = o_P(1).
\]

By Cauchy-Schwarz inequality, the last two terms in equation (D.1) are both \(o_P(1)\), hence \(\hat{\sigma}^2\) is consistent for the conditional variance \(\sigma^2\). Therefore, we conclude the proof of Proposition 2.10. \(\square\)

## E  Proofs of Results in Section 3

In this section, we provide proofs of results in Section 3.

**E.1 Proof of Corollary 3.14**

*Proof of Corollary 3.14.* Under conditions (i)-(iii) in Assumption 3.13, the linear expansion of \(\hat{\theta}_{\text{shift}}\) (3.14) has been shown in Shu and Tan (2018) to hold. We do not include the details for the nuisance estimation analysis and focus on the conditional inference implications here.

Recall the definition of conditioning variable \(Z_i^*\) in equation (3.11). It holds that

\[
E[\phi(D_i) \mid Z_i^*] = (1 - T_i) \cdot E[\phi(D_i) \mid T_i] + T_i \cdot E[\phi(D_i) \mid T_i, Z_i] = \frac{T_i}{p_1} \cdot g(Z_i).
\]

By the definition of the new conditional parameter in equation (3.12), we have

\[
\theta^\text{cond,new} = \frac{1}{m} \sum_{i=1}^{N} T_i \cdot g(Z_i) = \frac{1}{N} \sum_{i=1}^{N} T_i \cdot \frac{1 - T_i}{p_1} \cdot g(Z_i)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} E[\phi(D_i) \mid Z_i^*] + \left( \frac{1}{p_1} - \frac{1}{p_1} \right) \cdot \frac{1}{N} \sum_{i=1}^{N} T_i \cdot g(Z_i) = \frac{1}{N} \sum_{i=1}^{N} E[\phi(D_i) \mid Z_i^*] + o_P(1/\sqrt{N}),
\]

where the last equality follows from the law of large numbers so that \(\hat{p}_1 \xrightarrow{p} p_1\) and the central limit theorem so that \(\frac{1}{N} \sum_{i=1}^{N} T_i \cdot g(Z_i) = O_P(1/\sqrt{N})\). This proves the linear expansion (3.15).

By the moment condition \(E[Y_i^4] < \infty\) and the overlap condition in Assumption 3.13, we have the bounded moment \(E[\phi(D_i)^4] < \infty\). Thus Assumptions 2.1 and 2.2 in Proposition 2.3 are satisfied. By Proposition 2.3, it holds that

\[
\sqrt{N}(\hat{\theta}_{\text{shift}} - \theta^\text{cond,new}) \xrightarrow{d} N(0, \sigma^2_{\text{shift}}),
\]

where the variance takes the form

\[
\sigma^2_{\text{shift}} = \text{Var}\left( \frac{e(Z_i)}{1 - e(Z_i)} \cdot \frac{1 - T_i}{p_1} \cdot (Y_i - \varphi(Z_i)) \right).
\]

Conditioning on \(\{Z_i^*\}_{i=1}^{N}\), still by Proposition 2.3, it holds that

\[
P\left[ \sqrt{N}(\hat{\theta}_{\text{shift}} - \theta^\text{cond,new}) \leq x \mid Z^* \right] = \Phi(x/\sigma_{\text{shift}}) + o_P(1),
\]

which completes the proof of Corollary 3.14. \(\square\)
E.2 Proof of conditional FWER

Proof of Theorem 3.2. For each $k \in [m]$, we define

$$\text{FWER}_k := \mathbb{P}\left( \exists j \geq k, H_0^j \text{ is true, reject } H_0^j \mid Z^k \right).$$

Then by equation (3.4), we have $\text{FWER} = \max_{k \in [m]} \text{FWER}_k$. Therefore, for fixed $m$, it suffices to show $\text{FWER}_k \leq \alpha + o_p(1)$ for any $k \in [m]$. We define the two sets

$$I_0^k := \{ j \geq k : j \in I_0 \}, \quad I_0^{*k} := \{ j \geq k : j \in I_0^* \},$$

where the inequality follows from the definition of our multiple testing procedure. Meanwhile, since $P = 0$, because $V_k = \{ j \geq k : j \in I_k^0 \}$ where the two null sets are defined in equation (B.5). Also we denote the sets

$$V_k := \{ j \in I_0^k : H_0^j \text{ rejected} \}, \quad V_k^* := \{ j \in I_0^{*k} : H_0^j \text{ rejected} \},$$

which are the set of rejected true random hypotheses above level $k$, and the set of rejected true deterministic hypotheses above level $k$, respectively. By the first result in Lemma B.2, we know $\lim_{n \to \infty} \mathbb{P}(I_0^k \subseteq I_0^{*k}) = 1$. Since $V_k, V_k^*$ are the intersections of $I_0^k, I_0^{*k}$ and the rejected indices, respectively, we also have $\lim_{n \to \infty} \mathbb{P}(V_k \subseteq V_k^*) = 1$. Therefore,

$$\mathbb{P}\left( \bigcup_{j \geq k} \{ H_0^j \text{ is true, reject } H_0^j \} \bigg| Z^k \right) = \mathbb{P}(V_k \geq 1 \mid Z^k)$$

$$\leq \mathbb{P}(|V_k^*| \geq 1 \mid Z^k) + \mathbb{P}(\{ V_k \subseteq V_k^* \}^c \mid Z^k)$$

$$\leq \mathbb{P}(|V_k^*| \geq 1 \mid Z^k) + o_p(1).$$

Here the first inequality is because $V_k \subseteq V_k^*$ indicates $|V_k^*| \geq |V_k|$, while the second inequality is because $\mathbb{P}(\{ V_k \subseteq V_k^* \}^c) = o(1)$, hence

$$\mathbb{P}(\{ V_k \subseteq V_k^* \}^c \mid Z^k) = o_p(1)$$

by Lemma G.1. Thus it suffices to show that

$$\mathbb{P}(|V_k^*| \geq 1 \mid Z^k) \leq \alpha + o_p(1).$$

Note that if the deterministic parameter $\theta^0 \neq 0$, then $|I_0^{*k}| = 0$ hence $|V_k^*| = 0$ always, so $\mathbb{P}(\text{FWER}_k = 0) \xrightarrow{a.s.} 1$ for all $k \in [m]$ and equation (3.5) follows.

Hereafter we assume $\theta^0 = 0$. We denote

$$j^* = \min \{ j \geq k : \mathbb{P}(\mathbb{E}[\phi(X) \mid Z^j]) = 0 \},$$

then $I_0 = \{ j^*, j^* + 1, \ldots, m \}$. Therefore,

$$\mathbb{P}(|V_k^*| \geq 1 \mid Z^k) = \mathbb{P}\left( \bigcup_{j \geq j^*} \{ H_0^j \text{ rejected} \} \bigg| Z^k \right) \leq \mathbb{P}(p_{j^*} \leq \alpha \mid Z^k),$$

where the inequality follows from the definition of our multiple testing procedure. Meanwhile, since $\mathbb{Var}(\mathbb{E}[\phi(X) \mid Z^j]) = 0$ and $\theta_0 = 0$, by the linearity in Assumption 3.1, we have $\sqrt{n} \cdot \theta_n^j = o_p(1)$. Therefore,

$$\mathbb{P}(p_{j^*} \leq \alpha \mid Z^k) = \mathbb{P}\left( \sqrt{n} \cdot |\hat{\theta}_n| \geq z_{1-\alpha/2} \cdot \hat{\sigma}_{j^*} \mid Z^k \right)$$

$$\leq \mathbb{P}\left( \sqrt{n} \cdot |\hat{\theta}_n - \theta_n^{j^*}| \geq z_{1-\alpha/2} \cdot \hat{\sigma}_{j^*} \mid Z^k \right) + o_p(1) = \alpha + o_p(1),$$

where the first inequality is due to Lemma G.1 and the last equality is follows from Lemma B.3 as well as the consistency of $\hat{\sigma}_{j^*}$. Therefore, we conclude the proof of Theorem 3.2. \hfill \Box \hfill \Box
E.3 Proof of conditional FDR

Proof of Theorem 3.10. It suffices to show that $\text{FDR}_k \leq \alpha + o_\theta(1)$ for each $k \in [m]$. Hereafter we fix some $k \in [m]$ and run our multiple testing procedure. Similar to equations (B.4) and (B.5), with a slight abuse of notations, we define the deterministic hypotheses

$$\tilde{H}^{k,\ell}_0 : \theta^{0,\ell} = 0 \quad \text{and} \quad \text{Var}(E[\phi^\ell(D) | Z^{k,\ell}]) = 0,$$

and the random and deterministic null sets

$$I_0 = \{(k, \ell) : H^{k,\ell}_0 \text{ is true}\}, \quad I^*_0 = \{(k, \ell) : \tilde{H}^{k,\ell}_0 \text{ is true}\}.$$ 

Meanwhile, we define the set of lowest true deterministic hypotheses above level $k$

$$I^{*,k}_0 = \{H^{j,\ell}_0 : j > k, \ (j, \ell) \in I^*_0, \ (j + 1, \ell) \notin I^*_0 \} \cup \{H^{k,\ell}_0 : (k, \ell) \in I^*_0 \}.$$ 

By the nested structure of deterministic hypotheses, if $(j, \ell) \in I^*_0$, then $(j', \ell) \in I^*_0$ for all $j' \leq j$. Now let

$$V^*_k = V_k \cap I^{*,k}_0 = V_k \cap I^*_0$$

be the corresponding falsely rejected deterministic hypotheses. With exactly the same arguments as in Lemma B.2, we have $\lim_{n \to \infty} \mathbb{P}(I_0 \subseteq I^*_0) = 1$ hence $\lim_{n \to \infty} \mathbb{P}(V_k \subseteq V^*_k) = 1$. Therefore, we have

$$\text{FDR}_k = \mathbb{E}\left[\frac{|V_k|}{1 \vee |R_k|} \mid Z^k\right]$$

$$= \mathbb{E}\left[\frac{|V_k|}{1 \vee |R_k|} I\{V_k \subseteq V^*_k\} \mid Z^k\right] + \mathbb{E}\left[\frac{|V_k|}{1 \vee |R_k|} (1 - I\{V_k \subseteq V^*_k\}) \mid Z^k\right]$$

$$\leq \mathbb{E}\left[\frac{|V^*_k|}{1 \vee |R_k|} \mid Z^k\right] + \mathbb{E}\left[\frac{|V_k|}{1 \vee |R_k|} \mid Z^k\right]$$

$$\leq \mathbb{E}\left[\frac{|V^*_k|}{1 \vee |R_k|} \mid Z^k\right] + o_\theta(1), \quad (E.2)$$

where the last inequality follows from Lemma G.1. Thus it suffices to work with $V^*_k$ instead of $V_k$. For simplicity we define the corresponding FDR as

$$\text{FDR}_k^* = \mathbb{E}\left[\frac{|V^*_k|}{1 \vee |R_k|} \mid Z^k\right].$$

By the definition of $t^*$ in equation (3.10), we have

$$\text{FDR}_k^* = \sum_{(j, \ell) \in I^{*,k}_0} \mathbb{E}\left[\frac{1\{p_{j,\ell} \leq t^*_k\}}{1 \vee |R_k|} \mid Z^k\right] \leq \sum_{(j, \ell) \in I^{*,k}_0} \mathbb{E}\left[\frac{1\{p_{j,\ell} \leq \alpha R_k/L\}}{1 \vee |R_k|} \mid Z^k\right], \quad (E.3)$$

where the last inequality follows from the fact that

$$\frac{L t^*_k}{U(t^*_k)} \leq \alpha \Rightarrow t^*_k \leq \alpha |U(t^*_k)|/L = \alpha R_k/L.$$

In the sequel, we show that for each $(j, \ell) \in I^{*,k}_0$, it holds that

$$\mathbb{E}\left[\frac{1\{p_{j,\ell} \leq \alpha |R_k|/L\}}{1 \vee |R_k|} \mid Z^k\right] \leq \frac{\alpha}{Z} + o_\theta(1). \quad (E.4)$$
To this end, fix any $\epsilon > 0$ and $\rho \in (0, 1)$. Let $N$ be a fixed positive integer such that $\rho^N < \epsilon$. We let $v_0 = 0$ and $v_i = \rho^{N+1-i}$ for all $i \geq 1$. Then we have

$$\mathbb{E} \left[ \frac{1}{1 \lor v_i} \left( \sum_{k=1}^{2N+1} \mathbb{E} \left[ \left[ \frac{1}{1 \lor v_i} \left( |R_k|/L \right) \right] Z^k \right] \right) \right]$$

where the first inequality follows from the fact that $v_{2N+1} = \rho^{-N} > 1/\epsilon$. Meanwhile, it holds that

$$\mathbb{P}(|R_k| \in [v_{i-1}, v_i), p_{j,\ell} \leq \alpha v_i/L \mid Z_k)$$

where the first inequality follows from the fact that $j \geq k$ and $(j, \ell) \in I^*_0$, hence $\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z_k) \leq \alpha v_i/L + o_p(1)$ by Lemma B.3.

For any $v > 0$ and any fixed value of $p_{j,\ell}$, we define the set $D$ of vectors of $p$-values as

$$D(v, p_{j,\ell}) = \{ p_{-(j,\ell)} : |R_k(p)| < v \},$$

where $p = (p^j)_{j \geq k, \ell \in [L]}$ is the vector of all $p$-values above level $k$, and $p_{-(j,\ell)}$ is the vector obtained by excluding $p_{j,\ell}$ from $p$. Here we use the notation $R_k(p)$ to highlight the dependence of rejection set on all the $p$-values above level $k$. Roughly speaking, $D$ is the set of other $p$-values so that running the procedure on $p$ makes at least $v$ rejections.

It’s clear from the definition of the procedure that decreasing the $p$-values does not decrease the number of rejections. Thus $|R_k(p)|$ is nonincreasing in $p_{-(j,\ell)}$, hence $D$ is an increasing set.
By the asymptotic conditional PRDS of the p-values in Lemma G.7, we have

\[
\frac{\mathbb{P}(|R_k| \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} = \frac{\mathbb{P}(|R_k| < v_i, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} - \frac{\mathbb{P}(|R_k| < v_i-1, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} \\
\leq \frac{\mathbb{P}(|R_k| < v_i, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} - \frac{\mathbb{P}(|R_k| < v_i-1, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} + o_p(1)
\]

(E.7)

Combining equations (E.5), (E.6) and (E.7), we have

\[
\mathbb{E}\left[\frac{1(p_{j,\ell} \leq \alpha |R_k|/L)}{1\vee |R_k|} \mid Z^k\right] \leq \frac{\alpha}{\rho L} \sum_{i=1}^{2^{N+1}} \left( \frac{\mathbb{P}(|R_k| < v_i, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} - \frac{\mathbb{P}(|R_k| < v_i-1, p_{j,\ell} \leq \alpha v_i/L \mid Z^k)}{\mathbb{P}(p_{j,\ell} \leq \alpha v_i/L \mid Z^k)} \right) + \epsilon + o_p(1)
\]

\[
\leq \frac{\alpha}{\rho L} + \epsilon + o_p(1).
\]

Here the first inequality follows from the fact that finite sum of \(o_p(1)\) is still \(o_p(1)\), and the second inequality is a direct telescoping. By the arbitrariness of \(\rho \in (0,1)\) and \(\epsilon > 0\), we have the desired result in equation (E.4). Combining equations (E.3) and (E.4), we have

\[
\text{FDR}_k \leq \alpha \cdot |I_0^k|/L + o_p(1) \leq \alpha + o_p(1),
\]

which follows from the fact that \(|I_0^k| \leq L\) since we only count the lowest level of true null hypotheses. Therefore, we conclude the proof of Theorem 3.10.

F Proofs of Transductive Inference

Proof of Theorem 3.5. For simplicity, we write \(\varphi(z) = \mathbb{E}[\phi(D) \mid Z = z]\), so that \(\mathbb{E}[\phi(D_i) \mid Z_i] = \varphi(Z_i)\) and \(\mathbb{E}[\phi(D_j^{\text{new}}) \mid Z_j^{\text{new}}] = \varphi(Z_j^{\text{new}})\) for all \(i \in [n]\) and all \(j \in [m]\) in the following, we are to show that

\[
\hat{\theta}_{n,m} - \theta_m^{\text{cond,new}} = \frac{1}{n} \sum_{i=1}^{n} (\phi(D_i) - \varphi(Z_i)) + o_p(1/\sqrt{\min(n,m)}).
\]

(F.1)

By equation (2.2), we have the asymptotic linearity that

\[
\hat{\theta}_n - \theta_n^{\text{cond}} = \frac{1}{n} \sum_{i=1}^{n} \phi(D_i) - \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}[\phi(D_j^{\text{new}}) \mid Z_j^{\text{new}}] + o_p(1/\sqrt{n} + 1/\sqrt{m}).
\]

51
By the definition of $\hat{\theta}_{n,m}^{\text{transfer}}$ in equation (3.6), we have the decomposition

$$\hat{\theta}_{n,m}^{\text{transfer}} - \hat{\theta}_{n,m}^{\text{cond,new}} = \hat{\theta}_n - \hat{\theta}_n^{\text{cond,new}}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \phi(D_i) - \hat{\theta}_n - \frac{1}{m} \sum_{j=1}^{m} \varphi(Z_{j}^{\text{new}}) + o_p(1/\sqrt{n} + 1/\sqrt{m}).$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \phi(D_i) - \varphi(Z_i) \right) + o_p(1/\sqrt{n} + 1/\sqrt{m}) \quad \text{(F.2)}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \varphi(Z_i) - \frac{1}{2|I_1|} \sum_{i \in I_1} \hat{\varphi}_T^2(Z_i) - \frac{1}{2|I_2|} \sum_{i \in I_2} \hat{\varphi}_T^2(Z_i) + \frac{1}{2} \mathbb{E}[\hat{\varphi}_T^2(Z) \mid I_1] + \frac{1}{2} \mathbb{E}[\hat{\varphi}_T^2(Z) \mid I_2] \quad \text{(i)}$$

$$+ \frac{1}{2m} \sum_{j=1}^{m} \left( \hat{\varphi}_T^2(Z_{j}^{\text{new}}) + \hat{\varphi}_T^2(Z_{j}^{\text{new}}) \right) - \frac{1}{m} \sum_{j=1}^{m} \varphi(Z_{j}^{\text{new}}) - \frac{1}{2} \mathbb{E}[\varphi(Z_{j}^{\text{new}}) \mid I_1] - \frac{1}{2} \mathbb{E}[\varphi(Z_{j}^{\text{new}}) \mid I_2]. \quad \text{(ii)}$$

In the sequel, we bound the terms (i) and (ii) in equation (F.2) separately. Since $I_1$ and $I_2$ are equal-sized with $|I_1| + |I_2| = n$, we have

$$\text{(i)} = \left( \frac{1}{2|I_1|} \sum_{i \in I_1} \varphi(Z_i) - \frac{1}{2|I_1|} \sum_{i \in I_1} \hat{\varphi}_T^2(Z_i) + \frac{1}{2} \mathbb{E}[\hat{\varphi}_T^2(Z) \mid I_1] \right) + \left( \frac{1}{2|I_2|} \sum_{i \in I_2} \varphi(Z_i) - \frac{1}{2|I_2|} \sum_{i \in I_2} \hat{\varphi}_T^2(Z_i) + \frac{1}{2} \mathbb{E}[\hat{\varphi}_T^2(Z) \mid I_2] \right)$$

$$+ \frac{1}{2|I_2|} \sum_{i \in I_2} \varphi(Z_i) - \frac{1}{2|I_2|} \sum_{i \in I_2} \hat{\varphi}_T^2(Z_i) + \frac{1}{2} \mathbb{E}[\varphi(Z_{j}^{\text{new}}) \mid I_1] + o_p(1/n).$$

For the term (i,a), we note that $\mathbb{E}[\varphi(Z_i) \mid I_2] = 0$ for $i \in I_1$, hence

$$(i,a) = \frac{1}{2|I_1|} \sum_{i \in I_1} \xi_i, \quad \text{where} \quad \xi_i = (\varphi(Z_i) - \hat{\varphi}_T^2(Z_i)) - \mathbb{E}[(\varphi(Z_i) - \hat{\varphi}_T^2(Z_i) \mid I_2), \quad \forall i \in I_1.$$}

Conditional on $I_2$, the terms $\{\xi_i\}_{i \in I_1}$ in the above summation are i.i.d. with mean zero and bounded by $\sup_z |\varphi(z) - \hat{\varphi}_T^2(z)|$. By Hoeffding’s inequality, for any $t > 0$, we have

$$\mathbb{P} \left( \left| \sum_{i \in I_1} \xi_i \right| > t \sqrt{|I_1|} \mid I_2 \right) \leq \exp \left( \frac{-2t^2}{\sup_z |\varphi(z) - \hat{\varphi}_T^2(z)|} \right).$$

Hence by tower property, for any $t > 0$ and $\delta > 0$ it holds that

$$\mathbb{P} \left( \left| \sum_{i \in I_1} \xi_i \right| > t \sqrt{|I_1|} \right) \leq \mathbb{E} \left[ \exp \left( \frac{-2t^2}{\sup_z |\varphi(z) - \hat{\varphi}_T^2(z)|} \right) \right]$$

$$\leq \exp \left( -2t^2/\delta \right) + \mathbb{P} \left( \sup_z |\varphi(z) - \hat{\varphi}_T^2(z)| > \delta \right),$$

where the second inequality uses the fact that the exponential term is no grater than 1. By Assumption 3.4, $\sup_z |\varphi(z) - \hat{\varphi}_T^2(z)|$ converges to zero in probability. Taking $n \to \infty$ and by the arbitrariness of $t, \delta > 0$, we have $|(i,a)| = o_p(1/\sqrt{n})$. The same arguments also apply to the term (i,b), which lead to

$$|{(i)}| = o_p(1/\sqrt{n}).$$
Furthermore, the arguments apply similarly to the term (ii) with sample size \( m \), hence
\[
|(ii)| = o_P(1/\sqrt{m}).
\]

Putting them together, we have
\[
\hat{\theta}_{n,m}^{\text{transfer}} - \hat{\theta}^{\text{cond,new}}_m = \frac{1}{n} \sum_{i=1}^{n} (\phi(D_i) - \varphi(Z_i)) + o_P(1/\sqrt{n} + 1/\sqrt{m}).
\]

By the conditional CLT result in Lemma G.2 applied to \( g(X_i) = \phi(D_i) \) and filtrations
\[
G_n = \sigma(\{Z_{ij}^{\text{new}}\}_{j=1}^{m}) \subset \mathcal{F}_n = \sigma(\{Z_i\}_{i=1}^{n}; \{Z_{ij}^{\text{new}}\}_{j=1}^{m}),
\]
we conclude the proof of Theorem 3.5.

\[\square\]

**Proof of Theorem 3.6.** By the asymptotic linearity of \( \hat{\theta}_n \) and \( \hat{\theta}_m^{\text{new}} \), we have
\[
\hat{\theta}_{n,m}^{\text{transfer}} - \hat{\theta}_m^{\text{new}}
= \hat{\theta}_n - \hat{\theta}^{\text{cond,new}}_m
= \frac{1}{n} \sum_{i=1}^{n} (\phi(D_i) - \hat{\theta}^{\text{cond,new}}(Z_i)) + \frac{1}{m} \sum_{j=1}^{m} (\phi(D_j^{\text{new}}) - \varphi(Z_j^{\text{new}})) + o_P(1/\sqrt{n} + 1/\sqrt{m})
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \varphi(Z_i) - \frac{1}{2|I_1|} \sum_{i \in I_1} \hat{\varphi}^{I_1}(Z_i) - \frac{1}{2|I_2|} \sum_{i \in I_2} \hat{\varphi}^{I_2}(Z_i) + \frac{1}{2} \mathbb{E}[\hat{\varphi}^{I_1}(Z) \mid I_1] + \frac{1}{2} \mathbb{E}[\hat{\varphi}^{I_2}(Z) \mid I_2] \]
\[
\text{(i)}
\]
\[
+ \frac{1}{2m} \sum_{j=1}^{m} (\hat{\varphi}^{I_1}(Z_j^{\text{new}}) + \hat{\varphi}^{I_2}(Z_j^{\text{new}})) - \frac{1}{m} \sum_{j=1}^{m} \varphi(Z_j^{\text{new}}) - \frac{1}{2} \mathbb{E}[\hat{\varphi}^{I_1}(Z) \mid I_1] - \frac{1}{2} \mathbb{E}[\hat{\varphi}^{I_2}(Z) \mid I_2]. \]
\[
\text{(ii)}
\]

According to the proof of Theorem 3.5, we have
\[
|(i)| = o_P(1/\sqrt{n}), \quad |(ii)| = o_P(1/\sqrt{m}).
\]

Similar to the proof of Theorem 3.5, by the independence of \( \{(D_i, Z_i)\}_{i=1}^{n} \) and \( \{D_i^{\text{new}}, Z_{ij}^{\text{new}}\}_{j=1}^{m} \) and the conditional CLT in Lemma G.2, conditioning on \( Z^{\text{new}} \), the joint distribution satisfies
\[
\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\phi(D_i) - \varphi(Z_i)) \right. \left. \quad \left( \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (\phi(D_j^{\text{new}}) - \varphi(Z_j^{\text{new}})) \right) \right) 
\overset{d}{\sim} N(0, \sigma^2 I_2).
\]

The desired result follows from the fact that \( m \asymp n \). \[\square\]

**G Auxiliary Results**

In this section, we provide auxiliary technical results for the proofs in preceding sections.
G.1 Auxiliary results for conditional laws

**Lemma G.1.** Suppose a sequence of random variables $E_n$ satisfies $E_n = o_p(1)$ as $n \to \infty$. Then for any $\sigma$-algebras $F_n$ and any constant $\epsilon > 0$, it holds that

$$P\left(|E_n| > \epsilon \mid F_n\right) = o_p(1).$$

**Proof of Lemma G.1.** Note that $E[P\left(|E_n| > \epsilon \mid F_n\right)] = P(|E_n| > \epsilon)$. Thus for any $\delta > 0$, we have

$$P\left(P\left(|E_n| > \epsilon \mid F_n\right) > \delta\right) \leq \frac{1}{\delta} P(|E_n| > \epsilon) \to 0.$$

Therefore we have $P(|E_n| > \epsilon \mid F_n) = o_p(1)$. \qed

**Lemma G.2** (Conditional CLT). Let $g(\cdot)$ be a function such that $E[|g(X_i)|^4] < \infty$, where $\{(X_i, Z_i)\}_{i=1}^n$ are i.i.d. data. Define the filtration $F_n = \sigma(\{Z_i\}_{i=1}^n)$. Then for any $x \in \mathbb{R}$, it holds that

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) - E[g(X_i) \mid Z_i]) \leq x \mid F_n\right) \Rightarrow \Phi(x/\sigma),$$

where $\Phi$ is the cumulative distribution function of standard normal distribution, and

$$\sigma^2 = E\left[(g(X_i) - E[g(X_i) \mid Z_i])^2\right].$$

Moreover, for any filtration $G_n \subset F_n$, we also have

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) - E[g(X_i) \mid Z_i]) \leq x \mid G_n\right) \Rightarrow \Phi(x/\sigma).$$

**Proof of Lemma G.2.** Let $L_n$ denote the conditional law of $\frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i$ given $F_n$, where $\zeta_i := g(X_i) - E[g(X_i) \mid Z_i]$. Since the data are i.i.d., $\{X_i\}_{i=1}^n$ are mutually independent conditional on $F_n = \sigma(\{Z_i\}_{i=1}^n)$. Thus the characteristic function of $L_n$ is

$$\varphi_{L_n}(t) = E[e^{i\frac{t}{\sqrt{n}} \sum_{i=1}^n \zeta_i} \mid F_n] = \prod_{j=1}^n E[e^{i\frac{t}{\sqrt{n}} \zeta_j} \mid F_n]$$

for all $t \in \mathbb{R}$. We have the following lemma on the convergence of conditional characteristic functions.

**Lemma G.3.** Under the same assumption as Lemma G.2, we have

$$\varphi_{L_n}(t) \to \exp\left(-t^2 \sigma^2/2\right),$$

for all $t \in \mathbb{R}$, where $\sigma^2$ is defined in Lemma G.2.

**Proof of Lemma G.3.** We quote the following well-known complex analysis result without proof.

**Lemma G.4.** Suppose $z_{n,k} \in \mathbb{C}$ are such that $z_n = \sum_{k=1}^n z_{n,k} \to z_\infty$ and $\eta_n = \sum_{k=1}^n |z_{n,k}|^2 \to 0$ as $n \to \infty$. Then

$$\varphi_n := \prod_{k=1}^n (1 + z_{n,k}) \to \exp(z_\infty) \text{ as } n \to \infty.$$
We now focus on \( z_{n,j} = \mathbb{E}[e^{\frac{it}{\sqrt{n}} \zeta_j} | Z_j] \). By the tower property of conditional expectations, we have \( \mathbb{E}[\zeta_j | Z_j] = 0 \) for all \( j \in [n] \). Therefore

\[
  z_{n,j} = -\frac{t^2}{2n} \mathbb{E}[\zeta_j^2 | Z_j] + R_{n,j}, \quad \text{where} \quad R_{n,j} = \mathbb{E}\left[ e^{\frac{it}{\sqrt{n}} \zeta_j} - 1 - \frac{it}{\sqrt{n}} \zeta_j + \frac{t^2}{2n} \zeta_j^2 \mid Z_j \right].
\]

Since the random variables \( \{\mathbb{E}[\zeta_j^2 | Z_j]\}_{i=1}^n \) are i.i.d., by the law of large numbers, it holds that

\[
  \sum_{m=1}^{n} \left( -\frac{t^2}{2n} \mathbb{E}[\zeta_j^2 | Z_j] \right) \xrightarrow{a.s.} -\frac{t^2}{2} \mathbb{E}[\zeta_j^2] = -\frac{t^2}{2} \sigma^2,
\]

where \( \sigma^2 \) is defined in Lemma G.2. Note that \( |e^{ix} - 1 - ix + x^2/2| \leq \min\{|x|^2, |x|^3/6\} \) for any \( x \in \mathbb{R} \), thus

\[
  |R_{n,j}| = \left| \mathbb{E}\left[ e^{\frac{it}{\sqrt{n}} \zeta_j} - 1 - \frac{it}{\sqrt{n}} \zeta_j + \frac{t^2}{2n} \zeta_j^2 \mid Z_j \right] \right| \
  \leq \mathbb{E} \left[ \min\left\{ \frac{t^2}{2n} \zeta_j^2, \frac{t^3}{6n^{3/2}} |\zeta_j|^3 \right\} \mid Z_j \right] \leq \frac{t^3}{6n^{3/2}} \mathbb{E}[|\zeta_j|^3 \mid Z_j].
\]

Under the finite third-moment condition in Assumption 3.1, by the law of large numbers we have

\[
  \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[|\zeta_j|^3 \mid Z_j] \xrightarrow{a.s.} \mathbb{E}[|\zeta_j|^3] < \infty,
\]

hence \( \sum_{i=1}^{n} |R_{n,j}| \xrightarrow{a.s.} 0 \). Therefore we have

\[
  \sum_{j=1}^{n} z_{n,j} \xrightarrow{a.s.} -\frac{t^2}{2} \sigma^2, \tag{G.3}
\]

We now show \( \sum_{j=1}^{n} |z_{n,j}|^2 \xrightarrow{a.s.} 0 \). Simply note that \( (x + y)^2 \leq 2x^2 + 2y^2 \), so

\[
  \sum_{j=1}^{n} |z_{n,j}|^2 \leq \frac{t^4}{2n^2} \sum_{j=1}^{n} (\mathbb{E}[\zeta_j^2 | Z_j])^2 + 2 \sum_{j=1}^{n} R_{n,j}^2
  \leq \frac{t^4}{2n^2} \sum_{j=1}^{n} \mathbb{E}[\zeta_j^4 | Z_j] + 2 \sum_{j=1}^{n} R_{n,j}^2 \xrightarrow{a.s.} 0, \tag{G.4}
\]

where the second inequality follows from Jensen’s inequality. The a.s. convergence follows from the strong law of large numbers under the moment condition in Assumption 3.1, as well as the fact that

\[
  \sum_{j=1}^{n} R_{n,j}^2 \leq \sum_{j=1}^{n} |R_{n,j}| \cdot \max_{j} |R_{n,j}| \leq \left( \sum_{j=1}^{n} |R_{n,j}| \right)^2 \xrightarrow{a.s.} 0.
\]

Combining equations (G.3), (G.4) and Lemma G.4, we conclude the proof of Lemma G.3.

By Lemma G.3, we know that the conditional law \( \mathcal{L}_n \) converges almost surely to \( \mathcal{N}(0, \sigma^2) \), which completes the proof of equation (G.1). Since the conditional probabilities are bounded within \([0, 1]\), equation (G.2) follows from dominated convergence theorem. Therefore we conclude the proof of Lemma G.2.
G.2 Auxiliary results for multiple testing

In this section, we provide auxiliary results for the FDR control procedure in Section 3.3. We build upon the property discussed in Benjamini and Yekutieli (2001), called positive regression dependency on each one from a subset (PRDS). In our case, we extend this definition to asymptotic (and) conditional PRDS properties, in analogy to classical ones.

Definition G.5 (Increasing set, PRDS (Benjamini and Yekutieli, 2001)). We say a set C is increasing if for any increasing set C and for each i ∈ I0, it holds that for any ϵ > 0 and a ≤ b, when n is sufficiently large,

\[ \mathbb{P}(X_i | X_i < a) ≤ \mathbb{P}(X_i | X_i ≤ b) + ϵ. \]

Here \( X_i \) is a random vector that implicitly depends on \( n \). In this paper, we often refer to \( Z^k \) as an increasing set (filtration) of independent data.

For hierarchy k, recall the definition of set \( I_{0,k} \) as in equation (E.1). We then have the following lemma on the asymptotic (conditional) PRDS property of the whole set of p-values.

Lemma G.7. Suppose the p-values are obtained from the multiple hierarchical structure discussed in Section 3.3. Reindex them into a vector \( p ∈ \mathbb{R}^K \), then \( p \) has asymptotic PRDS property on the whole set of elements. Meanwhile, for each \( k \in [m] \), \( p \) also has asymptotic conditional PRDS property on \( I_{0,k} \) specified in equation (E.1), conditional on \( Z^k \) the union of all variables at level k.

Proof of Lemma G.7. According to the construction of p-values, for any \( j \in [m] \) and any \( ℓ \in [L] \), we have

\[ p_{j,ℓ} = f(T_{j,ℓ}) := 2(1 - Φ(1/T_{j,ℓ})), \quad \text{where } T_{j,ℓ} = \frac{\hat{σ}_{j,ℓ}}{\sqrt{n} |θ_{j,ℓ}^n(X)|}, \]

and \( \hat{σ}_{j,ℓ} \) is the sorted estimates of \( σ_{j,ℓ} \), \( j = 1, \ldots, m \) calculated separately for each \( ℓ \). Hence \( p_{j,ℓ} \) is strictly increasing in \( T_{j,ℓ} \). We define the function \( f: \mathbb{R}^K → \mathbb{R}^K \), so that

\[ f(t_i) = 2(1 - Φ(1/t_i)), \quad t_i > 0 \quad \text{for each } i \in [K]. \]

Note that each entry of the function \( f \) is a one-to-one map \( t → 1(1 - Φ(1 - t)) \). For an increasing set \( C \), we define its pre-image as \( C^f = f^{-1}(C) \). Since each argument of \( f \) is increasing, \( C^f \) is still an increasing set. Moreover, for any \( a ∈ \mathbb{R} \) and any increasing set \( C \), the sets \( A = \{p_i ∈ C, p_i ≤ a \} \) and \( B = \{p_i ≤ a \} \) are both increasing sets, so are their pre-images \( f^{-1}(A), f^{-1}(B) \). Therefore, to show the asymptotic and conditional PRDS of \( p \), it suffices to show the desired properties for the random vector \( t = f(p) \).
Asymptotic PRDS. We first show the asymptotic PRDS of $\mathbf{p}$ via the asymptotic PRDS of $\mathbf{t} = f(\mathbf{p}) \in \mathbb{R}^K$. For notational simplicity, we still use the double index $(j, \ell) \in [m] \times [L]$ in the re-indexed vertex $\mathbf{t}$ and $\mathbf{p}$. Marginally, by Lemma B.1 the random vector $\mathbf{t}$ converges in distribution to a random vector $\mathbf{t}^*$ obtained from reindexing $\{T_{j,\ell}^*\}_{j \in [m], \ell \in [L]}$, where

$$T_{j,\ell}^* = \frac{\sigma_{j,\ell}}{\tau_{\ell}} \cdot G^\ell, \quad \tau_{\ell}^2 = \text{Var} \left( \phi^\ell(X) \right), \tag{G.5}$$

and $\{G^\ell\}_{\ell=1}^L$ are i.i.d. standard normal random variables. Without loss of generality we assume $\sigma_{j,\ell} > 0$, so the distribution of $T_{j,\ell}^*$ is non-degenerate. Under the indexing of $\mathbf{t}^* = (T_1^*, \ldots, T_K^*)$ and $\mathbf{t} = (T_1, \ldots, T_K)$,

$$P(T_{-i} \in C \mid T_i \leq a) = \frac{P(T_{-i} \in C, T_i \leq a)}{P(T_i \leq a)} = \frac{P(T_{-i}^* \in C, T_i^* \leq a)}{P(T_i^* \leq a)} + o(1)$$

for any $a \in \mathbb{R}$. Note that here $T_i$ implicitly depends on $n$, while $T_i^*$ does not. Therefore, to show the asymptotic PRDS of $\mathbf{t}$, it suffices to show that

$$P(T_{-i}^* \in C \mid T_i^* \leq a) \leq P(T_{-i} \in C \mid T_i \leq b) \tag{G.6}$$

for all real numbers $a \leq b$ and increasing set $C$. Following Benjamini and Yekutieli (2001), equation (G.6) is implied by a stronger result that

$$P(T_{-i}^* \in C \mid T_i^* = x) \tag{G.7}$$

is a non-decreasing function of $x$. Therefore, it suffices to show equation (G.7). To this end, note that any element in $T_{-i}^*$ is either proportional to $T_i^*$ or independent of $T_i^*$ by the definition in equation (G.5). Returning to the index in terms of $(j, \ell)$ so that $T_i^* = T_{j,\ell}^*$, we denote the two vectors

$$\mathbf{t}_1 = (T_{j',\ell'} : j' \neq j), \quad \mathbf{t}_2 = (T_{j',\ell'} : \ell' \neq \ell),$$

with appropriate re-ordering of elements so that they agree with $T_{-i}^*$. Then by tower property of conditional expectations,

$$P(T_{-i}^* \in C \mid T_i^*) = E \left[ P(\mathbf{t}_2 \in C(\mathbf{t}_1) \mid T_i^*) \bigg| T_i^* \right].$$

where for each $t \in \mathbb{R}^+$ we define the set $C(t) = \{x : (t, x) \in D\}$. By the independence of $\mathbf{t}_2$ and $(T_i^*, \mathbf{t}_1)$, letting $g(y) = P(\mathbf{t}_2 \in D(y))$, it holds that

$$P(T_{-i}^* \in C \mid T_i^*) = E \left[ P(\mathbf{t}_2 \in C(\mathbf{t}_1) \mid T_i^*, \mathbf{t}_1) \bigg| T_i^* \right] = E \left[ g(\mathbf{t}_1) \bigg| T_i^* \right].$$

Since $C$ is an increasing set, for $t' \geq t$ and any $x$ with $(t, x) \in C$, it must hold that $(t', x) \in C$, hence $C(t) \subset C(t')$ and $g(t) \leq g(t')$ for all $t \leq t'$. Since $\mathbf{t}_1 = \gamma \cdot T_i^*$ for some fixed vector $\gamma = \left( \frac{\sigma_{j',\ell'}}{\sigma_{j,\ell}} \right)_{j' \neq j}$, we have

$$P(T_{-i}^* \in C \mid T_i^* = x) = g(\gamma \cdot x).$$

For $x' \geq x \in \mathbb{R}$, since elements in $\gamma$ are all positive, we have $\gamma \cdot x' \geq \gamma \cdot x$, which indicates

$$P(T_{-i}^* \in C \mid T_i^* = x') \geq P(T_{-i}^* \in C \mid T_i^* = x).$$

Therefore we conclude the proof of PRDS of $\mathbf{t}^*$ hence asymptotic PRDS of $\mathbf{t}$. As discussed before, this indicates the asymptotic PRDS property of $\mathbf{p}$. 57
Asymptotic conditional PRDS. Now we proceed to show the asymptotic conditional PRDS of \( p \) via that of \( t \), which amounts to show that for any increasing set \( C \) and real numbers \( a \leq b \), conditioning on some hierarchy \( k \),

\[
\frac{\mathbb{P}(T_{-i} \in C, T_i \leq a \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k)} \leq \frac{\mathbb{P}(T_{-i} \in C, T_i \leq b \mid Z^k)}{\mathbb{P}(T_i \leq b \mid Z^k)} + o_p(1).
\]

Here \( T_{-i}, T_i \) and \( Z^k \) all implicitly depend on \( n \), and \( Z^k = \bigcup_{\ell=1}^{L} Z^{k,\ell} \) is the union of independent variables at hierarchy \( k \) across all \( \ell \in [L] \). Meanwhile, since \( T_j > 0 \) for all \( j \), it suffices to consider \( 0 < a \leq b \). Note that

\[
\frac{\mathbb{P}(T_{-i} \in C, T_i \leq b \mid Z^k)}{\mathbb{P}(T_i \leq b \mid Z^k)} = \frac{\mathbb{P}(T_{-i} \in C, T_i \leq a \mid Z^k) + \mathbb{P}(T_{-i} \in C, a < T_i \leq b \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k) + \mathbb{P}(a < T_i \leq b \mid Z^k)},
\]

where \( \mathbb{P}(T_i \leq a \mid Z^k) \) and \( \mathbb{P}(a < T_i \leq b \mid Z^k) \) is non-diminishing since \( T_i \)'s are non-degenerate. Thus it suffices to show that

\[
\frac{\mathbb{P}(T_{-i} \in D, T_i \leq a \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k)} \leq \frac{\mathbb{P}(T_{-i} \in D, a < T_i \leq b \mid Z^k)}{\mathbb{P}(a < T_i \leq b \mid Z^k)} + o_p(1).
\]

Under the double index so that \( T_i = T_{j,\ell} \), we use the notation \( t_1 = (T'_{j,\ell} : j' \neq j) \) and \( t_2 = (T'_{j',\ell} : \ell' \neq \ell) \), then we have \( T_{-i} = (t_1, t_2) \) under appropriate re-organizing of elements. Moreover, by the independence of different studies, \( (T_i, t_1) \) is independent of \( t_2 \) conditioning on \( Z^k \). For any fixed \( a > 0 \), setting \( A = (-\infty, a] \), by the independence of \( t_2 \) and \( (T_i, t_1) \), conditional on \( Z^k \) we have

\[
\mathbb{P}(T_{-i} \in D, T_i \leq a \mid Z^k) = \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} \mid Z^k \right] = \mathbb{E}\left[ f(T_i, t_1, Z^k) \mid Z^k \right], \quad (G.8)
\]

where

\[
f(x, y, Z^k) = \mathbb{P}((x, y, t_2) \in A \times C \mid Z^k) = \mathbb{1}_{\{x \in A\}} \mathbb{P}((y, t_2) \in C \mid Z^k).
\]

Note that since \( C \) is an increasing set, for a.s. \( Z^k \), the function \( g(y, Z^k) := \mathbb{P}((y, t_2) \in C \mid Z^k) \) is also increasing in \( y \). Therefore

\[
\mathbb{P}(T_{-i} \in C, T_i \leq a \mid Z^k) = \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mid Z^k \right].
\]

Fix any \( \delta > 0 \). Similar arguments yield

\[
\mathbb{P}(T_{-i} \in C, a + \delta < T_i \leq b \mid Z^k) = \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_i \leq b\}} g(t_1, Z^k) \mid Z^k \right].
\]

Moreover, by the definition of test-statistics, we have \( T_{j,\ell}/T_{i,\ell} \overset{p}{\to} \sigma_{j,\ell}/\sigma_{i,\ell} \) for any \( i, j \in [m] \). For any fixed \( \epsilon > 0 \), we define the set

\[
E := \left\{ (t_1, t) : \left| \frac{t_{j',\ell} - \frac{\sigma_{j',\ell}}{\sigma_{j,\ell}} t_{i,\ell}}{\sigma_{j',\ell}} \right| \leq \epsilon, \ \forall j' \neq j \right\}. \quad (G.9)
\]

By the convergence in probability, we know

\[
\mathbb{P}(T_1 \leq a, (T_1, t_1) \notin E) = o(1),
\]

hence by Lemma G.1 we have

\[
\mathbb{P}(T_1 \leq a, (T_1, t_1) \notin E \mid Z^k) = o_p(1).
\]

58
Therefore
\[
\mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mid Z^k \right] = \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mathbb{1}_{\{T_i \in E\}} \mid Z^k \right] + \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mathbb{1}_{\{T_i \notin E\}} \mid Z^k \right] \\
\leq \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mid Z^k \right] + o_p(1)
\]
\[
\leq \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mathbb{1}_{\{t_i \leq \gamma \cdot T_i + \epsilon \cdot 1\}} \mid Z^k \right] + o_p(1)
\]
\[
\leq \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(\gamma \cdot T_i + \epsilon \cdot 1, Z^k) \mid Z^k \right] + o_p(1)
\]
\[
\leq \mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(\gamma \cdot a + \epsilon \cdot 1, Z^k) \mid Z^k \right] + o_p(1).
\] (G.10)

Here the first inequality is because \( g(y, Z^k) \in [0, 1] \) for all \( y, Z^k \), hence
\[
\mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(t_1, Z^k) \mathbb{1}_{\{T_i \notin E\}} \mid Z^k \right] \leq \mathbb{E} \left[ \mathbb{1}_{\{T_i \leq a\}} \mathbb{1}_{\{T_i \notin E\}} \mid Z^k \right] = o_p(1).
\]

The second inequality follows from the definition of \( E \) in equation (G.9). The third and fourth inequalities follow from the monotonicity of \( g(y, Z^k) \) in \( y \) for a.s. \( Z^k \).

On the other hand, by similar arguments, it holds for any \( \delta > 0 \) that
\[
\mathbb{P}(a + \delta < T_1 \leq b, (T_1, t_1) \notin E) = o(1),
\]
hence
\[
\mathbb{P}(a + \delta < T_1 \leq b, (T_1, t_1) \notin E \mid Z^k) = o_p(1).
\]

We further have
\[
\mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(t_1, Z^k) \mid Z^k \right]
\geq \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(t_1, Z^k) \mathbb{1}_{\{T_i \in E\}} \mid Z^k \right]
= \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(t_1, Z^k) \mathbb{1}_{\{T_i \in E\}} \mid Z^k \right] \\
\geq \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(\gamma \cdot T_i - \epsilon \cdot 1, Z^k) \mathbb{1}_{\{T_i \in E\}} \mid Z^k \right]
\geq \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(\gamma \cdot (a + \delta) - \epsilon \cdot 1, Z^k) \mid Z^k \right] - \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} \mathbb{1}_{\{T_i \notin E\}} \mid Z^k \right]
\geq \mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(\gamma \cdot (a + \delta) - \epsilon \cdot 1, Z^k) \mid Z^k \right] - o_p(1).
\] (G.11)

Here the first inequality is by monotonicity of conditional expectations. The second inequality is by definition of event \( C \). The third inequality follows from the monotonicity of \( g(y, Z^k) \) in \( y \) for a.s. \( Z^k \).

The fourth inequality is a probability decomposition along with the fact that \( g(y, Z^k) \in [0, 1] \). The last inequality is because
\[
\mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} \mathbb{1}_{\{T_i \notin E\}} \mid Z^k \right] = o_p(1)
\] as discussed before.

Since equations (G.10) and (G.11) hold for all fixed \( \epsilon > 0 \), we can choose \( \epsilon > 0 \) small enough so that \( \gamma \cdot a + \epsilon \cdot 1 < (a + \delta) - \epsilon \cdot 1 \), as \( \gamma > 0 \) elementwisely. For this \( \epsilon > 0 \), again by the monotonicity of \( g(y, Z^k) \) in \( y \), for a.s. \( Z^k \) it holds that
\[
\frac{\mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} g(\gamma \cdot (a + \delta) - \epsilon \cdot 1, Z^k) \mid Z^k \right]}{\mathbb{E}\left[ \mathbb{1}_{\{a + \delta < T_1 \leq b\}} \mid Z^k \right]} \leq \frac{\mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} g(\gamma \cdot a + \epsilon \cdot 1, Z^k) \mid Z^k \right]}{\mathbb{E}\left[ \mathbb{1}_{\{T_i \leq a\}} \mid Z^k \right]},
\]
which indicates that for any \( \delta > 0 \),
\[
\frac{\mathbb{P}(T_i \in D, T_i \leq a \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k)} \leq \frac{\mathbb{P}(T_i \in D, a + \delta < T_i \leq b \mid Z^k)}{\mathbb{P}(a + \delta < T_i \leq b \mid Z^k)} + o_p(1)
\]
\[
\leq \frac{\mathbb{P}(T_i \in D, a < T_i \leq b \mid Z^k)}{\mathbb{P}(a < T_i \leq b \mid Z^k)} - \mathbb{P}(a < T_i \leq a + \delta \mid Z^k) + o_p(1).
\] (G.12)
To bound the final expression in equation (G.12), we note that by Lemma B.1, the conditional law satisfies that for any \( x \in \mathbb{R} \),
\[
\mathbb{P}\left( \frac{\sqrt{n}(\hat{\theta}_n^\ell X) - \theta_n^{j,\ell}}{\sigma_{j,\ell}} \leq x \bigg| Z^k \right) = \Phi(x) + o_P(1),
\]
where \( \sigma_{j,\ell}^2 = \text{Var}(\phi^\ell(X)) - \text{Var}(E[\phi^\ell(X) \mid Z^{j,\ell}]) \), and the conditional parameter \( \theta_n^{j,\ell} \) is measurable with respect to \( Z^k \).

Note that for \( i \in I_0^* \), under the corresponding double index we have \( \sqrt{n} \cdot \theta_n^{j,\ell} = o_P(1) \) and \( \hat{\sigma}_n^\ell \to \sigma_j^\ell \). Therefore
\[
\mathbb{P}\left( \frac{\sqrt{n} \cdot \hat{\theta}_n(X)}{\hat{\sigma}_{j,\ell}} \leq x \bigg| Z^k \right) = \Phi\left( \frac{\sigma_k}{\sigma_{j,\ell}} \cdot x \right) + o_P(1),
\]
which indicates
\[
\mathbb{P}(a < T_i \leq a + \delta \mid Z^k) = \mathbb{P}\left( a < \frac{\hat{\sigma}_{j,\ell}}{\sqrt{n}|\theta_n(X)|} \leq a + \delta \bigg| Z^k \right)
= \Phi\left( \frac{\sigma_k}{\sigma_{j,\ell}} \cdot 1 \right) - \Phi\left( \frac{\tau_k}{\sigma_{j,\ell}} \cdot \frac{1}{a+\delta} \right) + o_P(1).
\]

Plugging in equation (G.12), for any \( \delta > 0 \) and any \( a < b \), it holds that
\[
\frac{\mathbb{P}(T_{-i} \in D, T_i \leq a \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k)} \leq \frac{\mathbb{P}(T_{-i} \in D, a < T_i \leq b \mid Z^k)}{\mathbb{P}(a < T_i \leq b \mid Z^k) - \Phi\left( \frac{\sigma_k}{\sigma_{j,\ell}} \cdot \frac{1}{a} \right) + \Phi\left( \frac{\tau_k}{\sigma_{j,\ell}} \cdot \frac{1}{a+\delta} \right) + o_P(1)} + o_P(1).
\]

By the arbitrariness of \( \delta \), we have
\[
\frac{\mathbb{P}(T_{-i} \in D, T_i \leq a \mid Z^k)}{\mathbb{P}(T_i \leq a \mid Z^k)} \leq \frac{\mathbb{P}(T_{-i} \in D, a < T_i \leq b \mid Z^k)}{\mathbb{P}(a < T_i \leq b \mid Z^k)} + o_P(1),
\]
which completes the proof of asymptotic conditional PRDS.