The Cauchy problem for two dimensional generalized Kadomtsev-Petviashvili-I equation in anisotropic Sobolev spaces

Wei Yan\textsuperscript{a,d}, Yongsheng Li \textsuperscript{b}, Jianhua Huang \textsuperscript{c}, Jinqiao Duan\textsuperscript{d}

\textsuperscript{a}School of Mathematics and Information Science and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, Xinxiang, Henan 453007, P. R. China

\textsuperscript{b}School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China

\textsuperscript{c}College of Science, National University of Defense Technology, Changsha, Hunan 410073, China

\textsuperscript{d}Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA

Abstract. The goal of this paper is three-fold. Firstly, we prove that the Cauchy problem for generalized KP-I equation

\[ u_t + |D_x|^{\alpha} \partial_x u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \alpha \geq 4 \]

is locally well-posed in the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{\alpha-1}{4}$ and $s_2 \geq 0$. Secondly, we prove that the problem is globally well-posed in $H^{s_1,0}(\mathbb{R}^2)$ with $s_1 > -\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)}$ if $4 \leq \alpha \leq 5$. Finally, we prove that the problem is globally well-posed in $H^{s_1,0}(\mathbb{R}^2)$ with $s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+4)}$ if $\alpha > 5$. Our result improves the result of Saut and Tzvetkov (J. Math. Pures Appl. 79(2000), 307-338.) and Li and Xiao (J. Math. Pures Appl. 90(2008), 338-352.).
1. Introduction

In this paper, we consider the Cauchy problem for the fifth-order KP-I equation

\[ u_t + |D_x|^\alpha \partial_x u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \]  
\[ u(x, y, 0) = u_0(x, y) \]

in anisotropic Sobolev space \( H^{s_1, s_2}(\mathbb{R}^2) \) defined in page 6. (1.1) occurs in the modeling of certain long dispersive waves [1, 29, 30]. (1.1) is the higher-order version of the KP equation

\[ u_t + \beta \partial_x^3 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \alpha \neq 0. \]

When \( \beta < 0 \), (1.4) is the KP-I equation. When \( \beta > 0 \), (1.4) is the KP-II equation. The KP-I and KP-II equations arise in physical contexts as models for the propagation of dispersive long waves with weak transverse effects [28], which are two-dimensional extensions of the Korteweg-de-Vries equation [28].

Many people have investigated the Cauchy problem for KP equation, for instance, see [3, 4, 7, 8, 14–19, 21, 22, 24–27, 33, 38, 40–42, 44, 48–54, 56] and the references therein. Bourgain [4] established the global well-posedness of the Cauchy problem for the KP-II equation in \( L^2(\mathbb{R}^2) \) and \( L^2(\mathbb{T}^2) \). Takaokao and Tzvetkov [51] and Isaza and Mejía [21] established the local well-posedness of KP-II equation in \( H^{s_1, s_2}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{3} \) and \( s_2 \geq 0 \). Takaoka [49] established the local well-posedness of KP-II equation in \( H^{s_1, 0}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{2} \) under the assumption that \( D_x^{-\frac{1}{2}+\epsilon} u_0 \in L^2 \) with the suitable chosen \( \epsilon \), where \( D_x^{-\frac{1}{2}+\epsilon} \) is Fourier multiplier operator with multiplier \( |\xi|^{-\frac{1}{2}+\epsilon} \). Hadac et al. [16] established the small data global well-posedness and scattering result of KP-II equation in the homogeneous anisotropic Sobolev space \( \dot{H}^{-\frac{1}{2}, 0}(\mathbb{R}^2) \) which can be seen in [16] and arbitrary large initial data local well-posedness in both homogeneous Sobolev space \( \dot{H}^{-\frac{1}{2}, 0}(\mathbb{R}^2) \) and inhomogeneous anisotropic Sobolev space \( H^{-\frac{1}{2}, 0}(\mathbb{R}^2) \). It is proved that the Cauchy problem for KP-I equation is globally well-posed in the second energy spaces on both \( \mathbb{R}^2 \) and \( \mathbb{T}^2 \) [33, 41, 42]. For KP-I equation, Molinet et al. [40] proved that the Picard iterative methods fails in standard Sobolev space and in anisotropic Sobolev space since the flow map fails to be real-analytic at the origin in these spaces. By introducing
some resolution spaces and bootstrap inequality as well as the energy estimate, Ionescu et al. [18] established the global well-posedness of KP-I in the natural energy space

\[ E^1 = \{ u_0 \in L^2(\mathbb{R}^2), \partial_x u_0 \in L^2(\mathbb{R}^2), \partial_x^{-1} \partial_y u_0 \in L^2(\mathbb{R}^2) \}. \]

Molinet et al. [43] proved that the Cauchy problem for the KP-I equation is locally well-posed in \( H^{s,0}(\mathbb{R}^2) \) with \( s > \frac{3}{2} \). Guo et al. [14] proved that the Cauchy problem for the KP-I equation is locally well-posed in \( H^{1,0}(\mathbb{R}^2) \). Zhang [56] proved that periodic KP-I initial value problem is locally well-posed in the Besov type space \( B_{2,1}^{s}(\mathbb{T}^2) \). It is worth noticing that the resonant function of KP-I equation does not possess the good property as its of KP-II equation.

When \( \alpha = 4 \), (1.1) reduces to the fifth order KP-I equation

\[ u_t + \partial_5^5 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0. \]  

Some people have studied the Cauchy problem for (1.4), for instance, see [6, 13, 36, 47]. Saut and Tzvetkov [47] proved that the Cauchy problem for (1.1) is globally well-posed for initial data \( u_0 \in L^2(\mathbb{R}^2) \) with finite energy. Chen et al. [6] proved that the problem for (1.1) is locally well-posed in \( E^s \) with \( 0 < s \leq 1 \), where

\[ E^s = \left\{ u_0 \in E^s : \| u_0 \|_{E^s} = \left\| \left( 1 + |\xi|^2 + \left| \frac{\mu}{\xi} \right| \right)^s \mathcal{F}_{xy} u_0 (\xi, \mu) \right\|_{L^2} < \infty \right\}. \]

By using the Fourier restriction norm method and sufficiently exploiting the geometric structure of the resonant set of (1.1) to dispose the high-high frequency interaction, Li and Xiao [36] proved that the Cauchy problem for (1.1) is globally well-posed in \( L^2(\mathbb{R}^2) \). Recently, Guo et al. [13] proved that the Cauchy problem for (1.4) is locally well-posed in \( H^{s_1,s_2}(\mathbb{R}^2) \) with \( s_1 \geq -\frac{3}{4}, s_2 \geq 0 \). Saut and Tzvetkov [46] proved that the fifth-order KP-II equation

\[ u_t - \partial_x^5 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0 \]

is locally well-posed in \( H^{s_1,s_2}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{4}, s_2 \geq 0 \). Isaza et al. [20] proved that the Cauchy problem for (1.5) is locally well-posed in \( H^{s_1,s_2}(\mathbb{R}^2) \) with \( s_1 > -\frac{5}{4}, s_2 \geq 0 \) and globally well-posed in \( H^{s_1,0}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{4} \). Recently, Li and Shi [37] proved that the Cauchy problem for (1.5) is locally well-posed in \( H^{s_1,s_2}(\mathbb{R}^2) \) with \( s_1 \geq -\frac{5}{4}, s_2 \geq 0 \).
Recently, Linares et al. [35] proved various ill-posedness and well-posedness results on the Cauchy problem
\begin{equation}
    u_t + uu_x - D_x^\alpha u_x + \gamma \partial_x^{-1} u_{yy} = 0, \gamma \in \mathbb{R}, 0 < \alpha \leq 1.
\end{equation}
To the best of our knowledge, the Cauchy problem for (1.1) in low regularity space is yet to be answered with \( \alpha \geq 4 \). The main reason is that the resonant function of KP-I type equation does not enjoy the same good property as its of KP-II type equation.

In this paper, inspired by [7, 20, 36, 47], by using the Fourier restriction norm method introduced in [2, 5, 34, 45] and developed in [31, 32], the Cauchy-Schwartz inequality and Strichartz estimates as well as suitable splitting of domains, we prove that the Cauchy problem for (1.1) is locally well-posed in the anisotropic Sobolev spaces \( H^{s_1, s_2}(\mathbb{R}^2) \) with \( s_1 > -\frac{\alpha-1}{4} \) and \( s_2 \geq 0 \); combining the local well-posness result of this paper with the I-method introduced in [9, 10], we also prove that the problem is globally well-posed in \( H^{s_1, 0}(\mathbb{R}^2) \) with \( s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+3)} \) if \( 4 \leq \alpha \leq 5 \) and prove that the problem is globally well-posed in \( H^{s_1, 0}(\mathbb{R}^2) \) with \( s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+4)} \) if \( \alpha > 5 \). Thus, our result improves the result of [36, 47].

We introduce some notations before giving the main results. Throughout this paper, we assume that \( C \) is a positive constant which may depend upon \( \alpha \) and vary from line to line. \( a \sim b \) means that there exist constants \( C_j > 0 (j = 1, 2) \) such that \( C_1 |b| \leq |a| \leq C_2 |b| \). \( a \gg b \) means that there exist a positive constant \( C' \) such that \( |a| \geq C' |b| \). \( 0 < \epsilon \ll 1 \) means that \( 0 < \epsilon < \frac{1}{100\alpha} \). We define
\begin{align*}
    \langle \cdot \rangle &:= 1 + |\cdot|, \\
    \phi(\xi, \mu) &:= \xi|\xi|^\alpha + \frac{\mu^2}{\xi}, \\
    \sigma &:= \tau + \phi(\xi, \mu), \sigma_j = \tau_j + \phi(\xi_j, \mu_j) (j = 1, 2), \\
    \mathcal{F} u(\xi, \mu, \tau) &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-ix\xi - iy\mu - it\sigma} u(x, y, t) dx dy dt, \\
    \mathcal{F}_{xy} f(\xi, \mu) &:= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi - iy\mu} f(x, y) dx dy, \\
    \mathcal{F}^{-1} u(\xi, \mu, \tau) &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{ix\xi + iy\mu + it\sigma} u(x, y, t) dx dy dt, \\
    D_x^\alpha u(x, y, t) &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^2} |\xi|^\alpha \mathcal{F} u(\xi, \mu, \tau) e^{ix\xi + iy\mu + it\sigma} d\xi d\mu d\tau, \\
    W(t) f &:= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi + iy\mu + it\phi(\xi, \mu)} \mathcal{F}_{xy} f(\xi, \mu) d\xi d\mu.
\end{align*}
Let $\eta$ be a bump function with compact support in $[-2, 2] \subset \mathbb{R}$ and $\eta = 1$ on $(-1, 1) \subset \mathbb{R}$. For each integer $j \geq 1$, we define $\eta_j(\xi) = \eta(2^{-j}\xi) - \eta(2^{1-j}\xi)$, $\eta_0(\xi) = \eta(\xi)$, $\eta_j(\xi, \mu, \tau) = \eta_j(\sigma)$, thus, $\sum_{j \geq 0} \eta_j(\sigma) = 1$. $\psi(t)$ is a smooth function supported in $[0, 2]$ and equals 1 in $[0, 1]$. Let $I \subset \mathbb{R}$, $\chi_I(x) = 1$ if $x \in I$; $\chi_I(x) = 0$ if $x$ does not belong to $I$. We denote by $\text{mes}(E)$ the Lebesgue measure of a set $E$. We define $|\xi_{\min}| := \min \{||\xi||, |\xi_1|, |\xi_2|\}$ and $|\xi_{\max}| := \max \{||\xi||, |\xi_1|, |\xi_2|\}$. We define

$$
\|f\|_{L^p_t L^q_x} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |f|^p dxdy \right)^{\frac{q}{p}} dt \right)^\frac{1}{q}.
$$

The anisotropic Sobolev space $H^{s_1, s_2}$ is defined as follows:

$$
H^{s_1, s_2}(\mathbb{R}^2) := \left\{ u_0 \in \mathscr{S}(\mathbb{R}^2) : \|u_0\|_{H^{s_1, s_2}(\mathbb{R}^2)} = \|\langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \mathcal{F} u_0(\xi, \mu)\|_{L^2_{\xi \mu}(\mathbb{R}^3)} \right\}.
$$

Space $X^{s_1, s_2}_b$ is defined by

$$
X^{s_1, s_2}_b := \left\{ u \in \mathscr{S}(\mathbb{R}^3) : \|u\|_{X^{s_1, s_2}_b} = \left\| \langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \mathcal{F} u(\xi, \mu, \tau) \right\|_{L^2_{\xi \mu}(\mathbb{R}^3)} < \infty \right\}.
$$

The space $X^{s_1, s_2}_b([0, T])$ denotes the restriction of $X^{s_1, s_2}_b$ onto the finite time interval $[0, T]$ and is equipped with the norm

$$
\|u\|_{X^{s_1, s_2}_b([0, T])} = \inf \left\{ \|g\|_{X^{s_1, s_2}_b} : g \in X^{s_1, s_2}_b, u(t) = g(t) \text{ for } t \in [0, T] \right\}.
$$

For $s < 0$ and $N \in \mathbb{N}^+, N \geq 100$, inspired by $[9, 10]$, we define an operator $I_N$ by

$$
\mathcal{F} I_N u(\xi, \mu, \tau) = M(\xi) \mathcal{F} u(\xi, \mu, \tau), \text{ where } M(\xi) = 1 \text{ if } |\xi| < N; M(\xi) = \left( \frac{|\xi|}{N} \right)^s \text{ if } |\xi| \geq N.
$$

The main results of this paper are as follows.

**Theorem 1.1. (Local well-posedness)**

Let $|\xi|^{-1} \mathcal{F} u_0(\xi, \mu) \in \mathscr{S}(\mathbb{R}^2)$. Then, the Cauchy problem for (1.1) are locally well-posed in the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{\alpha - 1}{2}, s_2 \geq 0$.

**Remark 1.** Note that the resonant function of generalized KP-II equation is

$$
R_{II}(\xi_1, \xi_2, \mu_1, \mu_2) := \left| \xi_1^{\alpha} \xi - \xi_1|\xi_1|^{\alpha} - \xi_2|\xi_2|^{\alpha} + \frac{\xi_1\xi_2}{\xi} \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)^2 \right|^{2}. \quad (1.7)
$$

Combining Lemma 2.8 of [55] with (1.7), we have that

$$
|R_{II}(\xi_1, \xi_2, \mu_1, \mu_2)| \geq ||\xi|^{\alpha} \xi - \xi_1|\xi_1|^{\alpha} - \xi_2|\xi_2|^{\alpha}|. \quad (1.8)
$$
However, the resonant function of the generalized KP-I equation is

\[
\sigma - \sigma_1 - \sigma_2 = R_I(\xi_1, \xi_2, \mu_1, \mu_2) := \phi(\xi, \mu) - \phi(\xi_1, \mu_1) - \phi(\xi_2, \mu_2)
= \left[|\xi|^{\alpha} \xi - |\xi_1|^{\alpha} - |\xi_2|^{\alpha} - \frac{\xi_1 \xi_2}{\xi} \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}\right)^2\right],
\]

(1.9)

thus, we consider

\[
|R_I(\xi_1, \xi_2, \mu_1, \mu_2)| \geq \frac{|\xi|^{\alpha} \xi - |\xi_1|^{\alpha} - |\xi_2|^{\alpha}|}{\alpha},
\]

(1.10)

\[
|R_I(\xi_1, \xi_2, \mu_1, \mu_2)| < \frac{|\xi|^{\alpha} \xi - |\xi_1|^{\alpha} - |\xi_2|^{\alpha}|}{\alpha},
\]

(1.11)

respectively. When (1.10) is valid, we follow the procedure of Lemma 3.1 in [55] to obtain Lemma 3.1 of this paper. When (1.11) is valid, inspired by [7, 36], we sufficiently exploit the geometric structure of (1.11) to Lemma 3.1 of this paper. When \(\alpha = 4\), we improve the local well-posedness result of [36].

**Theorem 1.2.** (Global well-posedness when \(4 \leq \alpha \leq 5\))

Let \(4 \leq \alpha \leq 5\) and \(|\xi|^{-1}F_{xy}u_0(\xi, \mu) \in \mathcal{S}'(\mathbb{R}^2)\). Then, the Cauchy problem for (1.1) are globally well-posed in \(H^{s_1,0}(\mathbb{R}^2)\) with \(s_1 > -\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)}\).

**Remark 2.** In proving Theorem 1.2, we only present the proof of case \(-\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)} < s_1 < 0\) since the global well-posedness of the Cauchy problem for (1.1) in \(H^{s_1,s_2}(\mathbb{R}^2)\) with \(s_1 \geq 0, s_2 \geq 0\) can be easily proved with the aid of \(L^2\) conservation law of (1.1). When \(\alpha = 4\), we improve the global well-posedness result of [36]. The establishment of Lemma 3.2 plays the crucial role in proving Theorem 1.2. When (1.10) is valid, we follow the method of Lemma 3.2 of [55] to deal with case (1.10). When (1.11) is valid, we use the technique used in Lemma 3.1 of this paper to deal with case (1.11).

**Theorem 1.3.** (Global well-posedness when \(\alpha > 5\))

Let \(\alpha > 5\) and \(|\xi|^{-1}F_{xy}u_0(\xi, \mu) \in \mathcal{S}'(\mathbb{R}^2)\). Then, the Cauchy problem for (1.1) are globally well-posed in \(H^{s_1,0}(\mathbb{R}^2)\) with \(s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+4)}\).

In proving Theorem 1.3, we only present the proof of case \(-\frac{\alpha(3\alpha-4)}{4(5\alpha+4)} < s_1 < 0\) since the global well-posedness of the Cauchy problem for (1.1) in \(H^{s_1,s_2}(\mathbb{R}^2)\) with \(s_1 \geq 0, s_2 \geq 0\) can be easily proved with the aid of \(L^2\) conservation law of (1.1).
The rest of the paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we establish two crucial bilinear estimates. In Section 4, we prove the Theorem 1.1. In Section 5, we prove the Theorem 1.2. In Section 6, we prove the Theorem 1.3.

2. Preliminaries

In this section, motivated by [4, 44], we give Lemmas 2.1-2.6 which play a significant role in establishing Lemmas 3.1, 3.2. Lemma 2.2 in combination with Lemma 3.1 yields Theorem 1.1. Lemma 2.7 in combination with Lemmas 3.1, 3.2 yields Lemma 5.1.

Lemma 2.1. Let \( b > |a| \geq 0 \). Then, we have that

\[
\int_{-b}^{b} \frac{dx}{(x + a)^{\frac{1}{2}}} \leq C b^{\frac{1}{2}}, \tag{2.1}
\]

\[
\int_{R} \frac{dt}{(t(\gamma (t - a)^{\gamma}) \leq C \langle a \rangle^{-\gamma}, \gamma > 1, \tag{2.2}
\]

\[
\int_{R} \frac{dt}{(t)^{\gamma} |t - a|^{\frac{1}{2}}} \leq C \langle a \rangle^{-\frac{1}{2}}, \gamma \geq 1, \tag{2.3}
\]

\[
\int_{-K}^{K} \frac{dx}{|x|^{\frac{1}{2}} |a - x|^{\frac{1}{2}}} \leq \frac{C K^{\frac{1}{2}}}{|a|^{\frac{1}{2}}}. \tag{2.4}
\]

Proof. The conclusion of (2.1) is given in (2.4) of Lemma 2.1 in [20]. (2.2)-(2.3) can be seen in Proposition 2.2 of [47]. (2.4) can be seen in line 24 of page 6562 in [15].

This completes the proof of Lemma 2.1.

Lemma 2.2. Let \( T \in (0, 1) \) and \( s_{1}, s_{2} \in R \) and \(- \frac{1}{2} < b' \leq 0 \leq b \leq b' + 1 \). Then, for \( h \in X_{b'}^{s_{1}, s_{2}} \), we have that

\[
\left\| \psi(t) S(t) \phi \right\|_{X_{b}^{s_{1}, s_{2}}} \leq C \left\| \phi \right\|_{H^{s_{1}, s_{2}}}, \tag{2.5}
\]

\[
\left\| \psi \left( \frac{t}{T} \right) \int_{0}^{t} S(t - \tau) h(\tau) d\tau \right\|_{X_{b}^{s_{1}, s_{2}}} \leq C T^{1 + b' - b} \left\| h \right\|_{X_{b'}^{s_{1}, s_{2}}}. \tag{2.6}
\]

For the proof of Lemma 2.2, we refer the readers to [5, 11, 31] and Lemmas 1.7, 1.9 of [12].

Lemma 2.3. Let \( b > \frac{1}{2} \). Then, we have

\[
\left\| D_{x}^{\frac{a - 2}{8}} u \right\|_{L_{t}^{2} L_{x}^{\frac{4}{3}} (R^{3})} \leq C \left\| u \right\|_{X_{b}^{0, 0}}. \tag{2.7}
\]
Proof. Combining Lemma 3.1 of [15] with Lemma 3.3 of [11], we have that Lemma 2.3 is valid.

This completes the proof of Lemma 2.3.

Motivated by [21, 49–53] and Theorem 3.3 of [15], we present the proof of Lemma 2.4.

**Lemma 2.4.** Let

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \xi|\xi|^{a} - \xi_1|\xi_1|^{a} - \xi_2|\xi_2|^{a} - \frac{\xi_1\xi_2}{\xi} \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right|^2 \\
\geq \frac{||\xi|\xi - \xi_1|\xi_1|^{a} - \xi_2|\xi_2|^{a}||}{\alpha}
\]

and

\[
\mathcal{F}P_{q}^{4}(u_1, u_2)(\xi, \mu, \tau) = \int_{\mathbb{R}^3} X_{|\xi_1| \leq \frac{|\xi|}{4}} (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \prod_{j=1}^{2} \mathcal{F}u_j(\xi_j, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1.
\]

For \( b > \frac{1}{2} \), we have

\[
\left\| P_{q}^{4}(u_1, u_2) \right\|_{L_{q}^{\mu}L_{\tau}^{\xi}} \leq C \left\| D_{x} f_{2}^{4} \right\|_{X_{0,0}^{0}} \left\| D_{x} f_{2}^{-4} u_{2} \right\|_{X_{0,0}^{0}}.
\] (2.8)

**Proof.** Let

\[
f_1(\xi_1, \mu_1, \tau_1) = |\xi_1|^{\frac{5}{4}} (\sigma_1)^b \mathcal{F}u_1(\xi_1, \mu_1, \tau_1), f_2(\xi_2, \mu_2, \tau_2) = |\xi_2|^{-\frac{5}{4}} (\sigma_2)^b \mathcal{F}u_2(\xi_2, \mu_2, \tau_2).
\]

To obtain (2.8), it suffices to prove that

\[
\left\| \int_{\mathbb{R}^3} X_{|\xi_1| \leq \frac{|\xi|}{4}} \frac{|\xi_1|^{-\frac{5}{4}} |\xi_2|^{\frac{4}{5}} f_1(\xi_1, \mu_1, \tau_1) f_2(\xi_2, \mu_2, \tau_2)}{\prod_{j=1}^{2} (\sigma_j)^b} d\xi_1 d\mu_1 d\tau_1 \right\|_{L_{2}^{\mu}L_{2}^{\xi}} \leq C \prod_{j=1}^{2} \left\| f_j \right\|_{L_{2}^{\mu}L_{2}^{\xi}}. \] (2.9)

To obtain (2.9), by duality, it suffices to prove that

\[
\left\| \int_{\mathbb{R}^6} X_{|\xi_1| \leq \frac{|\xi|}{4}} \frac{|\xi_1|^{-\frac{5}{4}} |\xi_2|^{\frac{4}{5}} f(\xi, \mu, \tau) \prod_{j=1}^{2} f_j(\xi_j, \mu_j, \tau_j)}{\prod_{j=1}^{2} (\sigma_j)^b} d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau \right\|_{L_{2}^{\mu}L_{2}^{\xi}} \leq C \left\| f \right\|_{L_{2}^{\mu}L_{2}^{\xi}} \prod_{j=1}^{2} \left\| f_j \right\|_{L_{2}^{\mu}L_{2}^{\xi}}. \] (2.10)
We define
\[
I(\xi, \mu, \tau) := \int_{\mathbb{R}^3} \chi_{|\xi_1| \leq \frac{|\xi_2|}{\alpha}} \frac{|\xi_1|^{-1}|\xi_2|^2}{\prod_{j=1}^{b} (\sigma_j)^{2b}} d\xi_1 d\mu_1 d\tau_1.
\] (2.11)

For fixed \((\xi, \mu, \tau)\), we make the change of variables \(L : (\xi_1, \mu_1, \tau_1) \rightarrow (\Delta, \sigma_1, \sigma_2)\), where
\[
\Delta := |\xi|^{\alpha} - |\xi_1|^{\alpha} - |\xi_2|^{\alpha}, \sigma_1 := \tau_1 + \phi(\xi_1, \mu_1), \sigma_2 := \tau_2 + \phi(\xi_2, \mu_2).
\]

By using a direct computation, since \(\sigma = \tau + \phi(\xi, \mu)\), we have that
\[
\sigma_1 + \sigma_2 - \sigma = -\Delta + \frac{(\xi_1 \mu_2 - \mu_1 \xi_2)^2}{\xi_1 \xi_2}.
\] (2.12)

Thus, we have that the Jacobian determinant equals
\[
\frac{\partial(\Delta, \sigma_1, \sigma_2)}{\partial(\xi_1, \mu_1, \tau_1)} = -2(\alpha + 1) (\xi_1^\alpha - \xi_2^\alpha) \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)
= -2(\alpha + 1) (\xi_1^\alpha - \xi_2^\alpha) (\sigma_1 + \sigma_2 - \sigma + \Delta)^{\frac{1}{2}} \left( \frac{\xi}{\xi_1 \xi_2} \right)^{\frac{1}{2}}.
\] (2.13)

Notice that it is possible to divide the integration into a finite number of open subsets \(W_i\) such that \(L\) is an injective \(C^1\)-function in \(W_i\) with non-zero Jacobian determinant.

From (2.13), since \(\frac{|\xi_2|}{\alpha} \geq |\xi_1|\) and \(|\Delta| \sim |\xi_1||\xi_2|^\alpha\), we have that
\[
\left| \frac{\partial(\Delta, \sigma_1, \sigma_2)}{\partial(\xi_1, \mu_1, \tau_1)} \right| = 2(\alpha + 1) \left| (\xi_1^\alpha - \xi_2^\alpha) (\xi_1^\alpha + \xi_2^\alpha) \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right) \right|
= \left| (\xi_1^\alpha - \xi_2^\alpha) (\sigma_1 + \sigma_2 - \sigma + \Delta)^{\frac{1}{2}} \left( \frac{\xi}{\xi_1 \xi_2} \right)^{\frac{1}{2}} \right|
\sim |\xi_1|^{-1}|\xi_2|^\alpha |\Delta|^\frac{1}{2} |\sigma_1 + \sigma_2 - \sigma + \Delta|^\frac{1}{2}.
\] (2.14)

Since \(|\sigma_1 + \sigma_2 - \sigma| \geq \frac{|\Delta|}{\alpha}\), by using the change of variables \((\xi_1, \mu_1, \tau_1) \rightarrow (\Delta, \sigma_1, \sigma_2)\) and (2.4), we have that
\[
I(\xi, \mu, \tau) := \int_{\mathbb{R}^3} \chi_{|\xi_1| \leq \frac{|\xi_2|}{\alpha}} \frac{|\xi_2|^2 |\xi_1|^{-1}}{\prod_{j=1}^{b} (\sigma_j)^{2b}} d\xi_1 d\mu_1 d\tau_1
\leq C \int_{\mathbb{R}^3} \chi_{|\Delta| \leq |\sigma_1 + \sigma_2 - \sigma| \Delta d\sigma_1 d\sigma_2} \frac{1}{|\Delta|^\frac{1}{2} |\sigma_1 + \sigma_2 - \sigma + \Delta|^\frac{1}{2} \prod_{j=1}^{b} (\sigma_j)^{2b}}
= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} \chi_{|\Delta| \leq |\sigma_1 + \sigma_2 - \sigma| \Delta d\sigma_1 d\sigma_2} \frac{1}{\prod_{j=1}^{b} (\sigma_j)^{2b}} \right) d\sigma_1 d\sigma_2
\leq C \int_{\mathbb{R}^2} \frac{d\sigma_1 d\sigma_2}{\prod_{j=1}^{b} (\sigma_j)^{2b}} \leq C. (2.15)
\]
Lemma 2.5. Assume 

\[ |\sigma - \sigma_1 - \sigma_2| = \left| \xi_1 \xi_1^\alpha - \xi_1 \xi_1^\alpha - \xi_2 \xi_2^\alpha \right| - \frac{\xi_1 \xi_2}{\xi} - \frac{\mu_1 - \mu_2}{\xi_2} \right|^2 \]

and \( b > \frac{1}{2} \), then, we have that

\[ \left| \left| \xi_1 \xi_1^\alpha - \xi_1 \xi_1^\alpha - \xi_2 \xi_2^\alpha \right| \right| \leq C \left( \frac{2}{\| f \|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{\| f \|_{L^2_{\xi \mu}}} \right) } \right) \] (2.17)

and

\[ \left| \left| \frac{\xi_1 \xi_1^\alpha - \xi_1 \xi_1^\alpha - \xi_2 \xi_2^\alpha}{\alpha} \right| \right| \leq \frac{C}{\| f \|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{\| f \|_{L^2_{\xi \mu}}} \right) } \] (2.18)

and

\[ \left| \left| \frac{\xi_1 \xi_1^\alpha - \xi_1 \xi_1^\alpha - \xi_2 \xi_2^\alpha}{\alpha} \right| \right| \leq \frac{C}{\| f \|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{\| f \|_{L^2_{\xi \mu}}} \right) } \] (2.19)

and

\[ \left| \left| \frac{\xi_1 \xi_1^\alpha - \xi_1 \xi_1^\alpha - \xi_2 \xi_2^\alpha}{\alpha} \right| \right| \leq \frac{C}{\| f \|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{\| f \|_{L^2_{\xi \mu}}} \right) } \] (2.20)

where \( dV = d\xi_1 d\mu_1 d\sigma_1 d\xi_2 d\mu_2 d\sigma_2 \).
Lemma 2.8. Let \(|\frac{|\xi_2|}{4} \geq |\xi_1|\), from Lemma 2.4, we have that (2.17) is valid. When \(|\frac{|\xi_1|}{4} < |\xi_2|\), since \(|\frac{|\xi_1|}{4}| \leq C|\xi_1|^\frac{1}{4}|\xi_2|^\frac{1}{4}\), from Lemma 2.3, we know that (2.17) is valid. Let \(\xi_1 = \xi_1', \mu_1 = \mu_1', \tau_1 = \tau_1'\) and \(-\xi_2 = \xi_4', -\tau_2 = \tau_1'\) and \(-\xi = \xi_4' - \xi_1', -\mu = \mu_4' - \mu_1', -\tau = \tau_1' - \tau_1\) and \(\sigma_2 = \tau_2 - \phi'(\xi_2, \mu_2')\), \(\sigma_1 = \tau_1' - \phi'(\xi_1', \mu_1')\). Thus, \(-\sigma = \sigma'_2, \sigma_1 = \sigma'_1\). Let

\[
H(\xi_1', \mu_1', \tau_1', \xi_4', \mu_4', \tau_4') = f_1(\xi_1', \mu_1', \tau_1')f_2(-\xi_4', -\mu_4', -\tau_4')f(-\xi_2', -\mu_2', -\tau_2').
\]

To obtain (2.25), it suffices to prove that

\[
\left| \int_{\mathbb{R}^2} |\xi_1'|^{-\frac{1}{2}}|\xi_2'|^{\frac{1}{2}} \left| H(\xi_1', \mu_1', \tau_1', \xi_4', \mu_4', \tau_4') \right| \right|_{b(\sigma_1')b(\sigma_2')} d\xi_1'd\mu_1'd\tau_1'd\xi_4'd\mu_4'd\tau_4' \leq C\|f\|_{L_{\tau\mu}^2} \left( \prod_{j=1}^2 \|f_j\|_{L_{\tau_\mu}^2} \right).
\]

(2.21)

Obviously, (2.21) follows from (2.17). By using a proof similar to (2.25), we obtain that (2.19)-(2.20) are valid.

This ends the proof of Lemma 2.5.

Lemma 2.6. Let \(I, J\) be two intervals on the real line and \(f : J \rightarrow \mathbb{R}\) be a smooth function. Then

\[
\inf_{\xi \in J} \|f'(\xi)\| \leq \frac{\mes \{x \in J, f(x) \in I\}}{\mes I}.
\]

(2.22)

Lemma 2.6 can be seen in Lemma 3.8 of [39].

Lemma 2.7. Let \(0 < b_1 < b_2 < \frac{1}{2}\). Then, we have that

\[
\|\chi_I(\cdot)u\|_{X_{-b_2}^{0,0}} \leq C\|u\|_{X_{-b_1}^{0,0}},
\]

(2.23)

\[
\|\chi_I(\cdot)u\|_{X_{b_1}^{0,0}} \leq C\|u\|_{X_{b_2}^{0,0}}.
\]

(2.24)

For the proof of Lemma 2.7, we refer the readers to Lemma 3.1. of [22].

Lemma 2.8. Let \(\phi_\alpha(\xi) = \xi|\xi|^{\alpha}\), \(\xi = \xi_1 + \xi_2\) and \(\alpha \geq 4\) and

\[
\ell_\alpha(\xi, \xi_1) = \phi_\alpha(\xi) - \phi_\alpha(\xi_1) - \phi_\alpha(\xi_2).
\]

Then \(|r_\alpha(\xi, \xi_1)| \sim |\xi_\min||\xi_\max|^{\alpha}\).

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For the proof of Lemma 2.8, we refer the readers to Lemma 3.4 of [15].

3. Bilinear estimates

In this section, we give the proof of Lemmas 3.1, 3.2. Lemma 3.1 is used to prove Theorems 1.1. Lemma 3.2 in combination with $I$-method yields Theorems 1.2. Lemma 3.3 is used to prove Lemma 5.1.

**Lemma 3.1.** Let $s_1 \geq -\frac{\alpha - 1}{4} + 4\alpha\epsilon, s_2 \geq 0, b = \frac{1}{2} + \epsilon, b' = -\frac{1}{2} + 2\epsilon$ and $u_j \in X^{s_1, s_2}_{\frac{1}{2} + \epsilon}(j = 1, 2)$. Then, we have that

$$\|\partial_x (u_1 u_2)\|_{X^{s_1, s_2}_{b'}} \leq C_2 \prod_{j=1}^{2} \|u_j\|_{X^{s_1, s_2}_b}.$$  \hspace{1cm} (3.1)

**Proof.** To prove (3.1), by duality, it suffices to prove that

$$\left| \int_{\mathbb{R}^3} \bar{u} \partial_x (u_1 u_2) dxdydt \right| \leq C \|u\|_{X^{s_1, s_2}_{-b'}} \left( \prod_{j=1}^{2} \|u_j\|_{X^{s_1, s_2}_b} \right).$$  \hspace{1cm} (3.2)

for $u \in X^{s_1, s_2}_{-b'}$. We define

$$F(\xi, \mu, \tau) := \langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \langle \sigma \rangle^{-b'} \mathcal{F} u(\xi, \mu, \tau),$$

$$F_j(\xi_j, \mu_j, \tau_j) := \langle \xi_j \rangle^{s_1} \langle \mu_j \rangle^{s_2} \langle \sigma_j \rangle^{b} \mathcal{F} u_j(\xi_j, \mu, \tau_j)(j = 1, 2),$$

$$dV = d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau$$  \hspace{1cm} (3.3)

and

$$D := \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in \mathbb{R}^6, \xi = \sum_{j=1}^{2} \xi_j, \mu = \sum_{j=1}^{2} \mu_j, \tau = \sum_{j=1}^{2} \tau_j \right\}.$$  \hspace{1cm} (3.4)

To obtain (3.2), from (3.3), it suffices to prove that

$$\int_{D} \frac{|\xi| \langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j)}{\langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \mu_j \rangle^{s_2} \langle \sigma_j \rangle^{b}} dV \leq C \|F\|_{L^2_{r_\xi \mu}} \prod_{j=1}^{2} \|F_j\|_{L^2_{r_\xi \mu}}.$$  \hspace{1cm} (3.5)

Without loss of generality, by using the symmetry, we assume that $|\xi_1| \geq |\xi_2|$ and $F(\xi, \mu, \tau) \geq 0, F_j(\xi_j, \mu_j, \tau_j) \geq 0(j = 1, 2)$ and

$$D^* := \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D, |\xi_2| \leq |\xi_1| \}.$$
We define

\[ \Omega_1 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_2| \leq |\xi_1| \leq 2A \}, \]

\[ \Omega_2 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, |\xi_1| \gg |\xi_2|, |\xi_2| \leq 2A \}, \]

\[ \Omega_3 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, |\xi_1| \gg |\xi_2|, |\xi_2| > 2A \}, \]

\[ \Omega_4 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, 4|\xi| \leq |\xi_2| \sim |\xi_1|, |\xi| \leq 2A, \xi_1 \xi_2 < 0 \}, \]

\[ \Omega_5 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, 4|\xi| \leq |\xi_2| \sim |\xi_1|, |\xi| > 2A, \xi_1 \xi_2 < 0 \}, \]

\[ \Omega_6 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, |\xi_1| \sim |\xi_2|, \xi_1 \xi_2 < 0, |\xi| \geq \frac{|\xi_2|}{4} \}, \]

\[ \Omega_7 = \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 2A, |\xi_2| \sim |\xi_1|, \xi_1 \xi_2 > 0 \}. \]

Obviously, \( D^* \subset \bigcup_{j=1}^{7} \Omega_j \). We define

\[ K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) := \frac{|\xi| (\xi)^{s_1} (\mu)^{s_2}}{\langle \sigma_j \rangle^{-b} \prod_{j=1}^{2} (\xi_j)^{s_1} (\mu_j)^{s_2} (\sigma_j)^{b}} \]  \quad (3.6) \]

and

\[ \text{Int}_j := \int_{\Omega_j} K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) dV \]

where \( 1 \leq j \leq 7, j \in N \) and \( dV \) is defined as in (3.3). Since \( s_2 \geq 0 \) and \( \mu = \sum_{j=1}^{2} \mu_j \), we have that \( (\mu)^{s_2} \leq \prod_{j=1}^{2} (\mu_j)^{s_2} \), thus, we have that

\[ K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| (\xi)^{s_1}}{\langle \sigma_j \rangle^{-b} \prod_{j=1}^{2} (\xi_j)^{s_1} (\sigma_j)^{b}}. \]  \quad (3.7) \]

We only prove (3.1) with \( -\frac{\nu-1}{4} + 16\alpha \epsilon \leq s_1 < 0 \) since case \( s \geq 0 \) can be easily proved.

(1). Region \( \Omega_1 \). In this region \( |\xi| \leq |\xi_1| + |\xi_2| \leq 4A \), this case can be proved similarly to case \( \text{low} + \text{low} \rightarrow \text{low} \) of pages 344-345 of Theorem 3.1 in [36].

(2). Region \( \Omega_2 \). In this region, we have that \( |\xi| \sim |\xi_1| \).

By using the Cauchy-Schwartz inequality with respect to \( \xi_1, \mu_1, \tau_1 \), from (3.6), we have
When

By using (2.2), we have that

\[ \nu = \int \frac{\xi}{\langle \sigma \rangle - b'} \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right) \leq \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}}. \] (3.9)

Let \( \nu = \tau + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) \) and \( \Delta = |\xi|\xi_1 - \xi_1|\xi_1 - \xi_2|\xi_2|, \) since \( |\xi_1| \gg |\xi_2| \), then we have that the absolute value of Jacobian determinant equals

\[ \left| \frac{\partial(\Delta, \nu)}{\partial(\xi_1, \mu_1)} \right| = 2 \left| \frac{\mu_1 - \mu_2}{\xi_1 - \xi_2} \right| (\alpha + 1)(\xi_1 - \xi_2) \]

\[ = 2(\alpha + 1)|\sigma - \nu - \Delta|^{\frac{1}{2}} \left| \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2} \right| \left| (\xi_1 - \xi_2) \right| \sim |\sigma - \nu + \Delta|^{\frac{1}{2}} \left| \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2} \right| |\xi_1|^{\alpha}. \] (3.10)

Inserting (3.10) into (3.9), by using (2.3), we have that

\[ \left| \frac{\xi}{\langle \sigma \rangle - b'} \right| \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}} \leq C \left| \frac{\xi}{\langle \sigma \rangle - b'} \right| \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}} \]

\[ \leq \frac{C}{|\xi|^{\frac{1}{2} - 1} \langle \sigma \rangle^{-b'}} \left( \int_{|\Delta| < 20a|\xi|^{\alpha}} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}} \]

\[ \leq \frac{C}{|\xi|^{\frac{1}{2} - 1} \langle \sigma \rangle^{-b'}} \left( \int_{|\Delta| < 20a|\xi|^{\alpha}} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}}. \] (3.11)

When \( |\sigma| < 20a|\xi|^{\alpha} \), combining (3.11) with (2.1), since \( \alpha \geq 4 \), we have that

\[ \frac{C}{|\xi|^{\frac{1}{2} - 1} \langle \sigma \rangle^{-b'}} \left( \int_{|\Delta| < 20a|\xi|^{\alpha}} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}} \leq \frac{C}{|\xi|^{\alpha - 1} \langle \sigma \rangle^{-b'}} \leq C. \] (3.12)

When \( |\sigma| \geq 20a|\xi|^{\alpha} \), from (3.11), since \( \alpha \geq 4 \), we have that

\[ \frac{C}{|\xi|^{\frac{1}{2} - 1} \langle \sigma \rangle^{-b'}} \left( \int_{|\Delta| < 20a|\xi|^{\alpha}} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}} \leq \frac{|\xi|}{\langle \sigma \rangle^{-b'}} \leq C. \] (3.13)
Combining (3.9) with (3.10)-(3.13), we have that

\[
\frac{|\xi|}{\langle \sigma \rangle^{-b'}} \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}} \leq C. \tag{3.14}
\]

Inserting (3.14) into (3.8), by using the Cauchy-Schwartz inequality with respect to \(\xi, \mu, \tau\), we have that

\[
\text{Int}_2 \leq C \int_{\mathbb{R}^3} \frac{|\xi|}{\langle \sigma \rangle^{-b'}} \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}} \times \left( \int_{\mathbb{R}^3} \frac{2}{\prod_{j=1}^{2} |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1} \right)^{\frac{1}{2}} F(\xi, \mu, \tau) d\xi d\mu d\tau \\
\leq C \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{2}{\prod_{j=1}^{2} |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1} \right)^{\frac{1}{2}} F(\xi, \mu, \tau) d\xi d\mu d\tau \\
\leq C \|F\|_{L^2_L \sigma, \mu, \tau} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_L \sigma, \mu, \tau} \right). \tag{3.15}
\]

(3). Region \(\Omega_3\). In this region, we have that \(|\xi| \sim |\xi_1|\). In this region, we consider

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \xi |\xi|^\alpha - \xi_1 |\xi_1|^\alpha - \xi_2 |\xi_2|^\alpha - \frac{\xi_1 \xi_2}{\xi} \frac{\mu_1 - \mu_2}{\xi_1 - \xi_2} \right| \\
\geq \left( \frac{\|\xi \|^\alpha |\xi - \xi_1 |^{\alpha} - |\xi_1 |^{\alpha} - |\xi_2 |^{\alpha}}{\alpha} \right), \tag{3.16}
\]

and

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \xi |\xi|^\alpha - \xi_1 |\xi_1|^\alpha - \xi_2 |\xi_2|^\alpha - \frac{\xi_1 \xi_2}{\xi} \frac{\mu_1 - \mu_2}{\xi_1 - \xi_2} \right| \\
\leq \left( \frac{\|\xi \|^\alpha |\xi - \xi_1 |^{\alpha} - |\xi_1 |^{\alpha} - |\xi_2 |^{\alpha}}{\alpha} \right), \tag{3.17}
\]

respectively.

When (3.16) is valid, we have that one of the following three cases must occur:

\[
|\sigma| := \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|\xi_{\min}|^\alpha |\xi_{\max}|^\alpha, \tag{3.18}
\]

\[
|\sigma_1| := \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|\xi_{\min}|^\alpha |\xi_{\max}|^\alpha, \tag{3.19}
\]

\[
|\sigma_2| := \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|\xi_{\min}|^\alpha |\xi_{\max}|^\alpha. \tag{3.20}
\]
When (3.18) is valid, since $s_1 \geq -\alpha - \frac{1}{4} + 4\alpha\epsilon$, we have that
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| |\xi|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^{-\frac{\alpha}{2} + 2\alpha\epsilon} |\xi_2|^{-s_1 + b'}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^{\frac{\alpha}{2}} |\xi_2|^{\frac{\alpha}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\]
Thus, combining (2.17) with (3.21), we have that
\[
|\mathrm{Int}_3| \leq C \|F\|_{L^2_{r\xi\mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).
\]
When (3.19) is valid, since $s_1 \geq -\alpha - \frac{1}{4} + 4\alpha\epsilon$ and $\langle \sigma \rangle^{-b'} \langle \sigma_1 \rangle^{b} \leq \langle \sigma_1 \rangle^{-b} \langle \sigma_1 \rangle^{b}$, we have that
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| |\xi|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^{\frac{\alpha}{2}} |\xi_2|^{-s_1 + b'}}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} \leq C \frac{|\xi|^{\frac{\alpha}{2}} |\xi_2|^{\frac{\alpha}{2}}}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b}.
\]
Thus, combining (2.19) with (3.22), we have that
\[
|\mathrm{Int}_3| \leq C \|F\|_{L^2_{r\xi\mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).
\]
When (3.20) is valid, since $s_1 \geq -\alpha - \frac{1}{4} + 4\alpha\epsilon$ and $\langle \sigma \rangle^{-b'} \langle \sigma_1 \rangle^{-b} \leq \langle \sigma_1 \rangle^{-b} \langle \sigma_1 \rangle^{b}$, we have that
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| |\xi|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^{\frac{\alpha}{2}} |\xi_1|^{-s_1 + b'}}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} \leq C \frac{|\xi|^{\frac{\alpha}{2}} |\xi_2|^{\frac{\alpha}{2}}}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b}.
\]
Thus, combining (2.25) with (3.23), we have that
\[
|\mathrm{Int}_3| \leq C \|F\|_{L^2_{r\xi\mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).
\]
When (3.17) is valid, from Lemma 2.8, we have that
\[
\left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim |\xi|^\frac{\alpha}{2}.
\]
We consider $|\sigma| \geq |\xi|^{-\frac{2\alpha - 1}{2}}$ and $|\sigma| < |\xi|^{-\frac{2\alpha - 1}{2}}$, respectively.

When $|\sigma| \geq |\xi|^{-\frac{2\alpha - 1}{2}}$, since $-\alpha - \frac{1}{4} + 4\alpha\epsilon \leq s_1 < 0$ and $\alpha \geq 4$, we have that
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi| |\xi|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^{\frac{5 - 2\alpha}{4} + (2\alpha - 1)\epsilon} |\xi_2|^{\frac{\alpha - 1}{4} - 16\alpha\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1|^{\frac{\alpha}{2}} |\xi_2|^{\frac{\alpha}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\]
Thus, combining (2.17) with (3.25), we have that
\[
|\text{Int}_3| \leq C \|F\|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi \mu}} \right).
\]

Now we consider case $|\sigma| < |\xi|^{\frac{2a-1}{2}}$. We dyadically decompose with respect to
\[
\langle \sigma \rangle \sim 2^j, (\sigma_1) \sim 2^{j_1}, (\sigma_2) \sim 2^{j_2}, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}.
\]

Let $D_{j_1,j_2,m_1,m_2}$ be the image of set of all points $(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*$ satisfying
\[
|\xi_1| \geq 2A, |\xi_1| \gg |\xi_2|, |\xi_2| > 2A, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2},
\]
\[
\langle \sigma \rangle \sim 2^j \leq 2^{\frac{2a-1}{2}m}, (\sigma_1) \sim 2^{j_1}, (\sigma_2) \sim 2^{j_2}
\]
under the transformation $(\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2)$. We define
\[
f_{m,j} := |\eta_m(\xi)\eta_j(\sigma)F(\xi, \mu, \sigma_1 - \phi(\xi_1, \mu_1) + \sigma_2 - \phi(\xi_2, \mu_2))|,
\]
\[
P := \left| \sum_{k=1}^{2} f_{m,k} \right|, dV = d\xi d\mu_1 d\sigma_1 d\xi_2 d\mu_2 d\sigma_2.
\]

Thus, we have that
\[
\text{Int}_3 \leq C \sum_{m,m_1,m_2>0} \sum_{j_1,j_2\geq 0, 0<j<\frac{2a-1}{2}m} \int_{D_{j_1,j_2,m_1,m_2}} 2^{j(b-(j_1+j_2)b-m_1s_1+m)} PdV.
\]

In this case, we consider
\[
\left| (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right| > 2^{j+\frac{(\alpha-1)m}{2}},
\]
\[
\left| (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right| \leq 2^{j+\frac{(\alpha-1)m}{2}},
\]
respectively.

Now we consider case (3.31). We make the change of variables
\[
u = \xi_1 + \xi_2, v = \mu_1 + \mu_2, w = \sigma_1 - \phi(\xi_1, \mu_1) + \sigma_2 - \phi(\xi_2, \mu_2), \mu_2 = \mu_2,
\]
thus the Jacobian determinant equals
\[
\frac{\partial(u,v,w,\mu_2)}{\partial(\xi_1,\xi_2,\mu_1,\mu_2)} = (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2.
\]
We assume that \( D_{j,j_1,j_2,m,m_1,m_2}^{(1)} \) is the image of the subset of all points
\[
(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_{j,j_1,j_2,m,m_1,m_2},
\]
which satisfies (3.31) under the transformation (3.33). Combining (3.31) with (3.34), we have that
\[
\left| \frac{\partial (u,v,w,\mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right| > 2^{j + \frac{(\alpha-1)m_1}{2}}. \tag{3.35}
\]
Let \( G_1(u,v,w,\mu_2,\sigma_1,\sigma_2) \) be \( \eta_m(\xi)\eta_j(\sigma) \prod_{k=1}^2 f_{m_k,j_k} \) under the change of the variables (3.33) and
\[
M_1 = |F(u,v,w)G_1(u,v,w,\mu_2,\sigma_1,\sigma_2)|, dV^{(1)} = dudvdwd\mu_2d\sigma_1d\sigma_2. \tag{3.36}
\]
Thus, (3.30) can be controlled by
\[
C \sum_{m,m_1,m_2>0} \sum_{j_1,j_2 \geq 0, 0 < j \leq \frac{7m}{2}} \int_{D_{j,j_1,j_2,m,m_1,m_2}^{(1)}} 2^{j' - (j_1 + j_2) b - m_2 s_1 + m} \frac{M_1 dV^{(1)}}{\left| \frac{\partial (u,v,w,\mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right|}. \tag{3.37}
\]
Inspired by [7, 36], we define
\[
f(\mu) := \sigma_1 + \sigma_2 - (\xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha) + \frac{\xi_1 \xi_2}{\xi} \left[ \frac{\mu_1}{\xi_1} - \frac{\mu}{\xi_2} \right]^2. \tag{3.38}
\]
From (3.38) and (3.24), for fixed \( \sigma_1, \sigma_2, \xi_1, \xi_2, \mu_1, \) we have that
\[
|f(\mu_2)| := |\sigma_1 + \sigma_2 + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) - \phi(\xi, \mu)| = |\tau - \phi(\xi, \mu)| \sim 2^j, \tag{3.39}
\]
\[
|f'(\mu_2)| \sim \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{am}. \tag{3.40}
\]
For fixed \( \sigma_1, \sigma_2, \xi_1, \xi_2, \mu_1, \) combining (3.39), (3.40) with Lemma 2.6, we have that the Lebesgue measure of \( \mu_2 \) can be controlled by \( C2^{j - \frac{am}{2}} \). By using the Cauchy-Schwartz inequality with respect to \( \mu_2 \) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the inverse change of variables related to (3.33) and the Cauchy-Schwartz inequality
with respect to $\sigma_1$ and $\sigma_2$, since $s \geq -\frac{\alpha - 1}{4} + 4\alpha \epsilon$, we have that (3.37) can be bounded by

$$
C \sum \int_{D_{j_1,j_2,m_1,m_2}^{(1)}} 2^{j_2 - (j_1 + j_2) m - m_2} s_1 + m \frac{M_1 dV^{(1)}}{\partial (u,v,w,\mu_2)} \frac{d\mu_2}{\partial (\xi_1,\xi_2,\mu_1,\mu_2)}
$$

$$
\leq C \sum 2^{j_2 - (j_1 + j_2) m - m_2 (s_1 + \frac{2\alpha - 5}{4})} \int F(u,v,w) \left( \int \frac{G^2(u,v,w,\mu_2,\sigma_1,\sigma_2)}{\partial (u,v,w,\mu_2)} \frac{d\mu_2}{\partial (\xi_1,\xi_2,\mu_1,\mu_2)} \right) \frac{d\sigma_1 d\sigma_2}{dV^{(2)}}
$$

$$
\leq C \sum 2^{j_2 - (j_1 + j_2) m - 16\alpha m_1 \epsilon} \|F\|_{L^2} \int \left( \prod_{k=1}^2 \int f_{m_k,j_k}^2 \frac{d\xi_1 d\mu_1 d\xi_2 d\mu_2}{dV^{(3)}} \right) d\sigma_1 d\sigma_2
$$

$$
\leq C \sum 2^{j_2 - (j_1 + j_2) m - 16\alpha m_1 \epsilon} \|F\|_{L^2} \left( \prod_{k=1}^2 \|f_{j_k}^{m_k} dV\right) \leq C \|F\|_{L^2} \left( \prod_{j=1}^2 \|F_j\|_{L^2} \right),
$$

where

$$
dV^{(2)} = dudvdwds_1d\sigma_2, dV^{(3)} = dudvdwd\mu_2, \sum = \sum_{m,m_1,m_2 > 0, j_1,j_2 \geq 0} \sum_{0 < j \leq (2\alpha - 1)m/2} . \quad \text{(3.41)}
$$

Now we consider (3.32). We make the change of variables

$$
u = \xi_1 + \xi_2, v = \mu_1 + \mu_2, w = \sigma_1 + \phi(\xi_1,\mu_1) + \sigma_2 + \phi(\xi_2,\mu_2), \xi_1 = \xi_1. \quad \text{(3.42)}
$$

From (3.42) and (3.24), we have that the absolute value of the Jacobian determinant equals

$$
\left| \frac{\partial (u,v,w,\xi_1)}{\partial (\xi_1,\xi_2,\mu_1,\mu_2)} \right| = 2 \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{\alpha m} . \quad \text{(3.43)}
$$

We assume that $D_{j_1,j_2,m_1,m_2}^{(2)}$ is the image of the subset of all points

$$(\xi_1,\mu_1,\sigma_1,\xi_2,\mu_2,\sigma_2) \in D_{j_1,j_2,m_1,m_2},$$

which satisfies (3.32) under the transformation (3.42). Let

$$H_1(u,v,w,\xi_1,\sigma_1,\sigma_2)$$

be $\eta_m(\xi) \eta_j(\sigma) \prod_{k=1}^2 f_{m_k,j_k}$ under the change of the variables as in (3.42) and

$$M_2 = |F(u,v,w)H_1(u,v,w,\xi_1,\sigma_1,\sigma_2)|, dV^{(4)} = dudvdwds_1d\sigma_1d\sigma_2. \quad \text{(3.44)}
$$
Thus, (3.30) can be controlled by

\[ C \sum_{m,m_1,m_2 > 0} \sum_{j_1,j_2 \geq 0} \int_{D_{j_1,j_2,m,m_1,m_2}^{(2)}} 2^{j_1 j_2} \frac{M_2 dV^{(4)}}{\partial (u,v,w,\xi_1) \partial (\xi_2,\mu_1,\mu_2)}. \]  \hspace{1cm} (3.45)

Inspired by [7, 36], we define

\[ h(\xi) := (\alpha + 1)(|\xi|^{\alpha} - |\xi_1|^{\alpha}) - \left( \left( \frac{\mu_1}{\xi} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right), \]  \hspace{1cm} (3.46)

for fixed \( \xi_2, \mu_1, \mu_2 \), from (3.46), we have that

\[ |h'(\xi_1)| = \left| \alpha(\alpha + 1)\xi_1|\xi_1|^{\alpha-2} + 2 \left( \frac{\mu_1}{\xi_1} \right)^2 \xi_1 \right| \geq \alpha(\alpha + 1)|\xi_1|^{\alpha-1} \geq C2^{(\alpha-1)m_1}, \]

\[ |h(\xi_1)| \leq C2^{j_1 (\alpha-1)m_1}. \]  \hspace{1cm} (3.47)

For fixed \( \xi_2, \mu_1, \mu_2 \), combining (3.47) with Lemma 2.6, we have that the Lebesgue measure of \( \xi_1 \) can be controlled by \( C2^{j_1 (\alpha-1)m_1} \). By using the Cauchy-Schwartz inequality with respect to \( \xi_1 \) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the inverse change of variables related to (3.42) and the Cauchy-Schwartz inequality with respect to \( \sigma_1 \) and \( \sigma_2 \), since \( s \geq -\frac{\alpha-1}{4} + 4\alpha \), we have that (3.45) can be bounded by

\[ C \sum_{m,m_1,m_2 > 0} \sum_{j_1,j_2 \geq 0} \int_{D_{j_1,j_2,m,m_1,m_2}^{(2)}} 2^{j_1 j_2} \frac{M_2 dV^{(4)}}{\partial (u,v,w,\xi_1) \partial (\xi_2,\mu_1,\mu_2)} \]

\[ \leq C \sum_{m,m_1,m_2 > 0} 2^{2j_1 (\alpha-1)m_1} \int |F| \left( \int \frac{H^2(u,v,w,\xi_1,\xi_2,\sigma_1,\sigma_2)}{\partial (u,v,w,\xi_1) \partial (\xi_2,\mu_1,\mu_2)} d\xi_1 \right)^{\frac{1}{2}} dV^{(2)} \]

\[ \leq C \sum_{m,m_1,m_2 > 0} 2^{2j_1 (\alpha-1)m_1} \| F \|_{L^2} \left( \int \| H^2(u,v,w,\xi_1,\xi_2,\sigma_1,\sigma_2) \|_{L^2} d\xi_1 d\sigma_1 d\sigma_2 \right)^{\frac{1}{2}} \]

\[ \leq C \sum_{m,m_1,m_2 > 0} 2^{2j_1 (\alpha-1)m_1} \| F \|_{L^2} \left( \int \prod_{k=1}^{2} \left( \int f_{m_k,j_k}^{2} d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right)^{\frac{1}{2}} \right) \]

\[ \leq C \sum_{m,m_1,m_2 > 0} 2^{2j_1 (\alpha-1)m_1} \| F \|_{L^2} \left( \int \prod_{k=1}^{2} \left( \int f_{m_k,j_k}^{2} dV \right)^{\frac{1}{2}} \right) \leq C \| F \|_{L^2} \left( \prod_{j=1}^{2} \| F_j \|_{L^2} \right)^{\frac{1}{2}}, \]  \hspace{1cm} (3.48)

where \( \sum \) is defined as in (3.41).

(4). Region \( \Omega_4 \). In this case, we consider (3.16), (3.17), respectively.

When (3.16) is valid, one of (3.18)-(3.20) must occur.
When (3.18) is valid, since $s_1 \geq -\frac{a-1}{4} + 4\alpha\epsilon$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\|\xi\|\|\xi\|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} (\xi_j)^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi\|^{2+2\epsilon} \|\xi_2\|^{-2s_1-2+8\epsilon}}{\langle \sigma \rangle^b \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi_1\|^{-\frac{1}{2}} \|\xi_2\|}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}. \quad (3.49)$$

Thus, combining (2.17) with (3.49), we have that

$$|\text{Int}_4| \leq C \|F\|_{L^2_{\sigma\xi \mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{\sigma_j \xi_j \mu}} \right).$$

When (3.19) is valid, since $s_1 \geq -\frac{a-1}{4} + 4\alpha\epsilon$ and $\langle \sigma \rangle^b \langle \sigma_1 \rangle^{b} \leq \langle \sigma \rangle^b \langle \sigma_1 \rangle^{b}$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\|\xi\|\|\xi\|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} (\xi_j)^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi\|^{2+2\epsilon} \|\xi_2\|^{-2s_1-2+8\epsilon}}{\langle \sigma \rangle^b \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi_1\|^{-\frac{1}{2}} \|\xi_2\|}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}. \quad (3.50)$$

Thus, combining (2.19) with (3.50), we have that

$$|\text{Int}_4| \leq C \|F\|_{L^2_{\sigma\xi \mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{\sigma_j \xi_j \mu}} \right).$$

When (3.20) is valid, this case can be proved similarly to (3.19).

When (3.17) is valid, from Lemma 2.8, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\|\mu_1 - \mu_2\|}{\xi_1} \sim \frac{\|\xi\|\|\xi_1\|^{2\alpha \frac{1}{2}}}{\xi_2}. \quad (3.51)$$

We consider $|\sigma| \geq |\xi_1|^{\frac{2\alpha+1}{2}}$ and $|\sigma| < |\xi_1|^{\frac{2\alpha-1}{2}}$, respectively.

When $|\sigma| \geq |\xi_1|^{\frac{2\alpha-1}{2}}$, since $s_1 \geq -\frac{a-1}{4} + 4\alpha\epsilon$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\|\xi\|\|\xi\|^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} (\xi_j)^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi\|^{2+2\epsilon} \|\xi_2\|^{-2s_1-2+8\epsilon}}{\langle \sigma \rangle^b \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{\|\xi_1\|^{-\frac{1}{2}} \|\xi_2\|}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.$$

This case can be proved similarly to (3.49).

We consider case $|\sigma| < |\xi_1|^{\frac{2\alpha-1}{2}}$. In this case, we consider

$$\left| (\alpha + 1)(|\xi_1|^{\alpha} - |\xi_2|^{\alpha}) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right] \right| > 2^{j + \frac{2\alpha - 1}{2}}, \quad (3.52)$$

$$\left| (\alpha + 1)(|\xi_1|^{\alpha} - |\xi_2|^{\alpha}) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right] \right| \leq 2^{j + \frac{2\alpha - 1}{2}}, \quad (3.53)$$
Thus the Jacobian determinant equals

\[ \langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}. \]

Let \( D^{(3)}_{j_1,j_2,m,m_1,m_2} \) be the image of set of all points \((\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*\) satisfying

\[ |\xi_1| \geq 2A, 4|\xi| \leq |\xi_2|, |\xi| \leq 2A, \xi_1 \xi_2 < 0, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}, \]

under the transformation \( (\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \).

Thus, we have

\[ \text{Int}_4 \leq C \sum_{m_1,m_2>0} \sum_{m,j_1,j_2=0}^{(2a-1)m_1} \int_{D^{(3)}_{j_1,j_2,m,m_1,m_2}} 2^{j'-(j_1+j_2)b-2m_1s_1+m} P \, dV, \quad (3.54) \]

where \( P \) and \( dV \) are defined as in (3.29).

In this case, we consider (3.52), (3.53), respectively.

When (3.52) is valid, we make the change of variables as in (3.33).

Thus the Jacobian determinant equals

\[ \frac{\partial (u,v,w,\mu_2)}{\partial (\xi_1,\xi_2,\mu_1,\mu_2)} = (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right]. \quad (3.56) \]

We assume that \( D^{(4)}_{j_1,j_2,m,m_1,m_2} \) is the image of the subset of all points

\[ (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D^{(3)}_{j_1,j_2,m,m_1,m_2}, \]

which satisfies (3.52) under the transformation (3.33). Combining (3.56) with (3.52), we have that

\[ \left| \frac{\partial (u,v,w,\mu_2)}{\partial (\xi_1,\xi_2,\mu_1,\mu_2)} \right| > 2^{j + (\alpha - 2)m_1}. \quad (3.57) \]

Let \( G_2(u,v,\mu_2,\sigma_1,\sigma_2) \) be \( \eta_m(\xi)\eta_j(\sigma) \prod_{k=1}^2 f_{m_k,j_k} \) under the change of the variables (3.33) and

\[ M_3 = F(u,v)G_2(u,v,\mu_2,\sigma_1,\sigma_2), dV^{(1)} = dudvdwd\mu_2d\sigma_1d\sigma_2. \quad (3.58) \]

Thus, (3.55) can be controlled by

\[ C \sum_{m_1,m_2>0} \sum_{m,j_1,j_2=0}^{(2a-1)m_1} \int_{D^{(4)}_{j_1,j_2,m,m_1,m_2}} 2^{j'-(j_1+j_2)b-2m_1s_1+m} \frac{M_3dV^{(1)}}{\partial (u,v,w,\mu_2) / \partial (\xi_1,\xi_2,\mu_1,\mu_2)}. \quad (3.59) \]
Inspired by [7, 36], we define
\[ f(\mu) := \sigma_1 + \sigma_2 - (\xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha) + \frac{\xi_1 \xi_2}{\xi} \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]^2. \] (3.60)

From (3.60) and (3.51), for fixed \( \xi_1, \xi_2, \mu_1, \sigma_1, \sigma_2 \), we have that
\[ |f(\mu_2)| = |\sigma_1 + \sigma_2 + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) - \phi(\xi, \mu)| \sim 2^j, \] (3.61)
\[ |f'(\mu_2)| \sim \left| \frac{\xi_1}{\xi} \right| \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^k \frac{m_1}{m_2}. \] (3.62)

Combining (3.61), (3.62) with Lemma 2.6, for fixed \( \xi_1, \xi_2, \mu_1, \sigma_1, \sigma_2 \), we have that the Lebesgue measure of \( \mu_2 \) can be controlled by \( C2^j \frac{m_1}{m_2} \). By using the Cauchy-Schwartz inequality with respect to \( \mu_2 \) and the inverse change of variables related to (3.29) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the Cauchy-Schwartz inequality with respect to \( \sigma_1 \) and \( \sigma_2 \), we have that (3.59) can be controlled by
\[ C \sum \int_{D_{j,j_1,j_2,m,m_1,m_2}^{(4)}} 2^{j b - (j_1 + j_2) b - 2m_1 s_1 + m} M_3 dV^{(1)} \frac{M_3 dV^{(1)}}{\partial(\xi_1, \xi_2, m_1, m_2)} \]
\[ \leq C \sum 2^{j b - (j_1 + j_2) b - (4s_1 + s_1)m_1 + m} \int F \left( \int \frac{G_2^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{\partial(\xi_1, \xi_2, m_1, m_2)^2} d\mu_2 \right)^{1/2} dV^{(2)} \]
\[ \leq C \sum 2^{j b - (j_1 + j_2) b - (4s_1 + s_1)m_1 + m} \| F \|_{L^2} \int \left( \int \frac{G_2^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{\partial(\xi_1, \xi_2, m_1, m_2)^2} d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2 \]
\[ \leq C \sum 2^{j b - (j_1 + j_2) b - 32m_1 + m} \| F \|_{L^2} \int \left( \int \prod_{k=1}^{2} f_{m_k,j_k}^2 d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2 \]
\[ \leq C \sum 2^{j b - (j_1 + j_2) b - 32m_1 + m} \| F \|_{L^2} \left( \int \prod_{k=1}^{2} f_{m_k,j_k}^2 dV \right)^{1/2} \leq C \| F \|_{L^2} \prod_{j=1}^{2} \| F_j \|_{L^2}. \] (3.63)

Here \( \sum = \sum_{m_1, m_2 > 0, m_1, m_2 > 0} \sum_{j_1, j_2 > 0, j \leq (2s_1 + 1)m_1} \).

Now we consider (3.53). We make the change of variables (3.42).

Thus the Jacobian determinant equals
\[ \frac{\partial(u, v, w)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = 2 \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]. \] (3.64)

We assume that \( D_{j,j_1,j_2,m,m_1,m_2}^{(5)} \) is the image of the subset of all points
\[ (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_{j,j_1,j_2,m,m_1,m_2}^{(3)}. \]
which satisfies (3.52) under the transformation (3.42). Combining (3.51) with (3.64), we have that

\[ \left| \frac{\partial (u, v, w, \xi_1)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right| \sim 2^{m + \frac{(\alpha - 2)m_1}{2}}. \tag{3.65} \]

Let \( f_{m_k, j_k} = \eta_{m_k}(\xi_k) \eta_{j_k}(\sigma_k) f_k(\xi_k, \mu_k, \tau_k) (k = 1, 2) \) and

\[ H_2(u, v, w, \xi_1, \sigma_1, \sigma_2) \]

be \( \eta_m(\xi) \eta_j(\sigma) \prod_{k=1}^{2} f_{m_k, j_k} \) under the change of the variables (3.42) and

\[ M_4 = F(u, v, w) H_2(u, v, w, \xi_1, \sigma_1, \sigma_2), dV^{(4)} = dudvdw\xi_1d\sigma_1d\sigma_2. \tag{3.66} \]

Thus, (3.55) can be controlled by

\[ C \sum_{\min\{j, j_1, j_2, m_1, m_2\} \geq 0, m} \int_{D^{(2)}_{j, j_1, j_2, m, m_1, m_2}} 2^{j' - (j_1 + j_2)b - 2m_1s_1 + m} \frac{M_4 dV^{(4)}}{|\partial (u, v, w, \xi_1)|}, \tag{3.67} \]

We assume that \( h(\xi) \) is defined as in (3.46), for fixed \( \xi_2, \mu_1, \mu_2 \), from (3.53), we have that

\[ |h'(\xi_1)| = \left| \alpha (\alpha + 1) \xi_1 |\xi_1|^{\alpha - 2} + 2 \left( \frac{\mu_1}{\xi_1} \right)^2 \xi_1 \right| \geq \alpha (\alpha + 1)|\xi_1|^{\alpha - 1} \geq C 2^{(\alpha - 1)m_1}, \]

\[ |h(\xi_1)| \leq C 2^{j + \frac{(\alpha - 2)m_1}{2}}. \tag{3.68} \]

Combining (3.68) with Lemma 2.6, for fixed \( \xi_2, \mu_1, \mu_2 \), we have that the measure of \( \xi_1 \) can be controlled by \( C 2^{j - \frac{\alpha m_1}{2}} \). By using the Cauchy-Schwartz inequality with respect to \( \xi_1 \) and the inverse change of variables and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the Cauchy-Schwartz inequality with respect to \( \sigma_1 \) and \( \sigma_2 \), we have that
(3.67) can be bounded by

\[
C \sum_{j_1,j_2,m_1,m_2} 2^{j_1-2m_1} \frac{M_4 dV}{(\xi_1,\xi_2,\mu_1,\mu_2)}
\]

\[
\leq C \sum_{j_1,j_2,m_1,m_2} 2^{2j_1-2m_1} \|F\|_L^2 \int \left( \int \prod_{k=1}^2 f_{m_k,j_k}^2 d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right) \frac{1}{2} d\sigma_1 d\sigma_2
\]

\[
\leq C \sum_{m_1,m_2>0, m_1,j_1,j_2>0, j_1 \leq \frac{(2\alpha-1)m_1}{2}} \|F\|_L^2 \int \left( \int \prod_{k=1}^2 f_{m_k,j_k}^2 \right) \frac{1}{2} d\sigma_1 d\sigma_2
\]

where \( \sum = \sum_{m_1,m_2>0, m_1,j_1,j_2>0, j_1 \leq \frac{(2\alpha-1)m_1}{2}} \) and \( dV = dudvdw d\xi_1 \).

(5). Region \( \Omega_5 \).

In this region, we consider (3.16), (3.17), respectively.

When (3.16) is valid, one of (3.18)-(3.20) must occur.

When (3.18) is valid, since \( s_1 \geq -\frac{\alpha-1}{4} + 4\alpha \epsilon \), we have that

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\|F\|_L^2}{\prod_{j=1}^2 \langle \sigma \rangle^{b_j}} \leq C \left( \frac{2^{j_1-2m_1} \|F\|_L^2}{(\xi,\mu,\tau)} \right) \frac{1}{2} \langle \sigma \rangle^{b_j} \prod_{j=1}^2 \langle \sigma \rangle^{s_j} \langle \sigma \rangle^b
\]

\[
\leq C \left( \frac{2^{j_1-2m_1} \|F\|_L^2}{(\xi,\mu,\tau)} \right) \frac{1}{2} \langle \sigma \rangle^{b_j} \prod_{j=1}^2 \langle \sigma \rangle^{s_j} \langle \sigma \rangle^b
\]

Thus, combining (2.17) with (3.70), we have that

\[
|\text{Int}_5| \leq C \left( \prod_{j=1}^2 \|F\|_L^2 \right) \left( \prod_{j=1}^2 \|F\|_L^2 \right)
\]
When (3.17) is valid, since $s_1 \geq -\frac{\alpha - 1}{4} + 4\alpha \epsilon$ and $\langle \sigma \rangle^{b'}(\sigma_1)^b \leq \langle \sigma \rangle^{-b}(\sigma_1)^{b'}$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi_1(\langle \xi \rangle^{s_1})}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1(\langle \xi \rangle)^{\frac{\alpha}{2} + s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C |\xi_1 \xi_2| \frac{\langle \xi \rangle^{\frac{\alpha}{2} + 2\epsilon}}{\langle \sigma \rangle^{b'}(\sigma_2)^b} \leq C |\xi_1 \xi_2|^\frac{\alpha}{2}. \quad (3.71)$$

Thus, combining (2.19) with (3.71), we have that

$$|\text{Int}_5| \leq C \|F\|_{L^2_{\xi \mu}} \left( \sum_{j=1}^{2} \|F_j\|_{L^2_{\xi \mu}} \right).$$

When (3.20) is valid, this case can be proved similarly to (3.19) with the aid of (2.25).

When (3.17) is valid, from Lemma 2.8, we have that

$$\left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim |\xi_1||\xi|^{\frac{\alpha - 2}{2}}. \quad (3.72)$$

We consider $|\sigma| \geq |\xi_1|^{\frac{2\alpha - 1}{2}}$ and $|\sigma| < |\xi_1|^\frac{2\alpha - 1}{2}$, respectively.

When $|\sigma| \geq |\xi_1|^{\frac{2\alpha - 1}{2}}$, since $s_1 \geq -\frac{\alpha - 1}{4} + 4\alpha \epsilon$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi_1(\langle \xi \rangle^{s_1})}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1(\langle \xi \rangle)^{\frac{\alpha}{2} + s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^b} \leq C |\xi_1 \xi_2| \frac{\langle \xi \rangle^{\frac{\alpha}{2} + (2\alpha - 1)\epsilon}}{\langle \sigma \rangle^b(\sigma_2)^b} \leq C |\xi_1 \xi_2|^\frac{\alpha}{2}, \quad (3.74)$$

respectively. We dyadically decompose with respect to

$$\langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^j, \langle \sigma_2 \rangle \sim 2^j, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}.$$

Let $D_{j,j_1,j_2,m,m_1,m_2}^{(6)}$ be the image of set of all points $(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*$ satisfying

$$|\xi_1| \geq 2A, 4|\xi| \leq |\xi_2|, |\xi| > 2A, \xi_1 \xi_2 < 0, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}$$

$$\langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^j, \langle \sigma_2 \rangle \sim 2^j \quad (3.75)$$
under the transformation \((\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2)\).

Thus, we have that

\[
\text{Int}_5 \leq C \sum_{m_1, m_2, m > 0} \sum_{j_1, j_2 \geq 0, 0 < j \leq \frac{(2\alpha - 1)m_1}{2}} \left( \int_{D^{(6)}_{j_1, j_2, m, m_1, m_2}} 2^{j b' + j_1 + j_2 - 2m s_1 + m} PdV \right), \tag{3.76}
\]

where \(P\) and \(dV\) are defined as in (3.29).

In this case, we consider (3.73), (3.74), respectively.

When (3.73) is valid, we make the change of variables as in (3.33).

Thus the Jacobian determinant equals

\[
\frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} = (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right]. \tag{3.77}
\]

We assume that \(D^{(7)}_{j_1, j_2, m, m_1, m_2}\) is the image of the subset of all points

\[(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D^{(6)}_{j_1, j_2, m, m_1, m_2},\]

which satisfies (3.73) under the transformation (3.33). Combining (3.73) with (3.77), we have that

\[
\left| \frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right| > 2^{j + \frac{m_2}{2} + \frac{(\alpha - 2)m_1}{2}}. \tag{3.78}
\]

Let \(G_3(u, v, w, \mu_2, \sigma_1, \sigma_2)\) be \(\eta_m(\xi)\eta_j(\sigma) \prod_{k=1}^{2} f_{m_k, j_k}\) under the change of the variables (3.33) and

\[
M_5 = F(u, v, w)G_3(u, v, w, \mu_2, \sigma_1, \sigma_2), \quad dV^{(1)} = dudvdwd\mu_2d\sigma_1d\sigma_2. \tag{3.79}
\]

Thus, (3.76) can be controlled by

\[
C \sum_{m_1, m_2, m > 0} \sum_{j_1, j_2 \geq 0, 0 < j \leq \frac{(2\alpha - 1)m_1}{2}} \int_{D^{(7)}_{j_1, j_2, m, m_1, m_2}} 2^{j b' - j_1 - j_2 - s_1 m_1 + m} M_5dV^{(1)} \left| \frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right|. \tag{3.80}
\]

Combining (3.60)-(3.62) with Lemma 2.6, for fixed \(\xi_1, \xi_2, \mu_1, \sigma_1, \sigma_2\), we have that the Lebesgue measure of \(\mu_2\) can be controlled by \(C 2^{j + \frac{2m_1}{2}}\). By using the Cauchy-Schwartz inequality with respect to \(\mu_2\) and the inverse change of variables related to (3.33) and the Cauchy-Schwartz inequality with respect to \(u, v, w\) and the Cauchy-Schwartz inequality.
with respect to \( \sigma_1 \) and \( \sigma_2 \), we have that (3.80) can be controlled by

\[
C \sum \int_{D_j^{(7)}} 2^{j^*-(j_1+j_2)b+(-1/4-s_1)m_1+m} M_5 dV^{(1)}
\]

\[
\leq C \sum 2^{j^*-(j_1+j_2)b+(-1/4-s_1)m_1+m} \int F(u, v, w) \left( \int \frac{G_3^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} d\mu_2 \right)^{1/2} dV^{(2)}
\]

\[
\leq C \sum 2^{j^*-(j_1+j_2)b+(-1/4-s_1)m_1+m} \| F \|_{L^2} \int \left( \int \prod_{k=1}^{2} f_{m_k,j_k}^2 d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2
\]

\[
\leq C \sum 2^{j^*-(j_1+j_2)b+(-1/4-s_1)m_1+m} \| F \|_{L^2} \left( \int \prod_{k=1}^{2} f_{m_k,j_k}^2 dV \right)^{1/2} \leq C \| F \|_{L^2} \left( \prod_{j=1}^{2} \| F \|_{L^2} \right), \quad (3.81)
\]

where \( \sum = \sum_{m_1,m_2,m>0} \sum_{j_1,j_2 \geq 0, 0< j \leq (2\alpha-1)m_1} \).

Now we consider (3.74). We make the change of variables as in (3.42).

Thus the Jacobian determinant equals

\[
\frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = 2 \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]. \quad (3.82)
\]

We assume that \( D_j^{(8)} \) is the image of the subset of all points

\[
(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_j^{(6)}
\]

which satisfies (3.74) under the transformation (3.42). Combining (3.72) with (3.82), we have that

\[
\left| \frac{\partial(u, v, w, \xi_1)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right| \sim 2^{m+\frac{(\alpha-2)m_1}{2}}. \quad (3.83)
\]

Let \( f_{m_k,j_k} = \eta_m(\xi) \eta_j(\sigma)f_k(\xi, \mu_k, \tau_k)(k = 1,2) \) and

\[
H_3(u, v, w, \xi_1, \sigma_1, \sigma_2)
\]

be \( \eta_m(\xi)\eta_j(\sigma) \prod_{k=1}^{2} f_{m_k,j_k} \) under the change of the variables (3.42) and

\[
M_6 = F(u, v, w)H_3(u, v, w, \xi_1, \sigma_1, \sigma_2), dV^{(4)} = dudvdwd\xi_1 d\sigma_1 d\sigma_2. \quad (3.84)
\]
Thus, (3.76) can be controlled by

$$
C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b-s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

We assume that $h(\xi)$ is defined as in (3.46), for fixed $\xi_2, \mu_1, \mu_2$, from (3.74), we have that

$$
|h'(\xi_1)| = \alpha(\alpha+1)|\xi_1|^{\alpha-2} + 2 \frac{\mu_1}{\xi_1} |\xi_1| \geq \alpha(\alpha+1)|\xi_1|^{\alpha-1} \geq C2^{(\alpha-1)m_1},
$$

$$
|h(\xi_1)| \leq C2^{j^{\frac{(\alpha-2)m_1}{2}} + \frac{m_1}{2}}. \tag{3.86}
$$

For fixed $\xi_2, \mu_1, \mu_2$, combining (3.86) with Lemma 2.6, we have that the Lebesgue measure of $\xi_1$ can be controlled by $C2^{j^{\frac{(\alpha-2)m_1}{2}} + \frac{m_1}{2}}$. By using the Cauchy-Schwartz inequality with respect to $\xi_1$ and the inverse change of variables related to (3.42) and the Cauchy-Schwartz inequality with respect to $u, v, w$ and the Cauchy-Schwartz inequality with respect to $\sigma_1$ and $\sigma_2$, we have that (3.85) can be bounded by

$$
C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b-s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

$$
\leq C \sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(8)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j^2-(j_1+j_2)b+s_1m_1+m} \frac{M_6dV^{(4)}}{\partial(u,v,w,\xi_1)} \tag{3.85}
$$

where $\sum_{m_1,m_2,m>0} \sum_{j_1,j_2\geq 0, 0<j\leq \frac{(2\alpha-1)m_1}{2}}$ and $dV^{(5)} := dudvdw d\xi_1$.

(6). Region $\Omega_6$.

In this region, we consider (3.16), (3.17), respectively.

When (3.16) is valid, one of (3.18)-(3.20) must occur.

When (3.18) is valid, since $s_1 \geq -\frac{\alpha-1}{4} + 4\alpha\epsilon$, we have that

$$
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi_1|^{1-s_1}}{|\sigma|^{-b} \prod_{j=1}^{2} |\sigma_j|^b} \leq C \frac{|\xi_2|^{s_1-s_1+\frac{4\alpha\epsilon}{2} + 2(\alpha+1)\epsilon}}{|\sigma|^{-b} \prod_{j=1}^{2} |\sigma_j|^b} \leq C \frac{|\xi_1|^{1-\frac{b}{2}} |\xi_2|^b}{|\sigma|^{-b} \prod_{j=1}^{2} |\sigma_j|^b}. \tag{3.88}
$$
Thus, combining (2.17) with (3.88), we have that

$$|\text{Int}_6| \leq C\|F\|_{L^2_{r\xi\mu}}^2 \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).$$

When (3.19) is valid, since $s_1 \geq -\frac{\alpha - 1}{4} + 4\alpha\epsilon$ and $\langle \sigma \rangle^b \langle \sigma_1 \rangle^{-b} \leq \langle \sigma \rangle^{-b} \langle \sigma_1 \rangle^b$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi|^{1-s_1}}{\langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_2|^{-1/2}}{\langle \sigma \rangle^{b} \langle \sigma_2 \rangle^{-b}} \leq C \frac{|\xi_2|^{-1/2} \langle \sigma \rangle^{-b} \langle \sigma_2 \rangle^{-b}}{\langle \sigma \rangle^{b} \langle \sigma_2 \rangle^b}.$$  \hspace{1cm} (3.89)

Thus, combining (2.20) with (3.89), we have that

$$|\text{Int}_6| \leq C\|F\|_{L^2_{r\xi\mu}}^2 \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).$$

When (3.20) is valid, this case can be proved similarly to (3.19).

When (3.17) is valid, from Lemma 2.8, we have that

$$\left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim |\xi_1|^{\frac{3}{2}}.$$ \hspace{1cm} (3.90)

We consider $|\sigma| \geq |\xi_1|^{\frac{2\alpha - 1}{2}}$ and $|\sigma| < |\xi_1|^{\frac{2\alpha - 1}{2}}$, respectively.

When $|\sigma| \geq |\xi_1|^{\frac{2\alpha - 1}{2}}$, since $s_1 \geq -\frac{\alpha - 1}{4} + 4\alpha\epsilon$, we have that

$$K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi|^{1-s_1}}{\langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1|^{-\frac{\alpha + 4(14\alpha - 2\epsilon)}{4}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1|^{-\frac{\alpha}{2}} \langle \sigma \rangle^{-b} \langle \sigma_2 \rangle^{-b}}{\langle \sigma \rangle^{b} \langle \sigma_2 \rangle^b}.$$ \hspace{1cm} (3.91)

Thus, combining (2.17) with (3.91), we have that

$$|\text{Int}_6| \leq C\|F\|_{L^2_{r\xi\mu}}^2 \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{r\xi\mu}} \right).$$

When $|\sigma| < |\xi_1|^{\frac{2\alpha - 1}{2}}$ is valid, we dyadically decompose with respect to

$$\langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}.$$

Let $D^{(8)}_{j,j_1,j_2,m,m_1,m_2}$ be the image of set of all points $(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*$ satisfying

$$|\xi_1| \geq 2A, |\xi_1| \sim |\xi_2|, \xi_1 \xi_2 < 0, |\xi| \geq \frac{|\xi_2|}{4}, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2},$$

$$\langle \sigma \rangle \sim 2^j \leq C2^{3m_2}, \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}$$ \hspace{1cm} (3.92)

under the transformation $(\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2)$.
Thus, we have that
\[ \text{Int}_6 \leq C \sum_{m_1, m_2, m > 0} \sum_{j_1, j_2 \geq 0, j \leq (2a - 1)m} \int_{D_{j_1, j_2, m, m_1, m_2}^{(6)}} 2^{j\nu - (j_1 + j_2)b - m_2 s_1 + m} P dV, \]  
(3.93)
where \( P \) and \( dV \) are defined in (3.93).

In this case, we consider (3.31), (3.32), respectively.

When (3.31) is valid, we make the change of variables as in (3.33).

Thus the Jacobian determinant equals
\[ \frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right]. \]  
(3.94)

We assume that \( D_{j_1, j_2, m, m_1, m_2}^{(9)} \) is the image of the subset of all points
\[ (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_{j_1, j_2, m, m_1, m_2}^{(8)}, \]
which satisfies (3.31) under the transformation (3.33). Combining (3.94) with (3.31), we have that
\[ \left| \frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right| > 2^{j + (a - 1)m_1}. \]  
(3.95)

Let \( G_4(u, v, w, \mu_2, \sigma_1, \sigma_2) \) be \( \eta_m|\eta_j(\sigma) \prod_{k=1}^{2} f_{m_k, j_k} \) under the change of the variables (3.33) and
\[ M_7 = F(u, v, w)G_4(u, v, w, \mu_2, \sigma_1, \sigma_2), dV^{(1)} = du dv dw d\mu_2 d\sigma_1 d\sigma_2. \]  
(3.96)

Thus, (3.93) can be controlled by
\[ C \sum_{m_1, m_2 > 0} \sum_{j_1, j_2 \geq 0, j \leq (2a - 1)m_1} \int_{D_{j_1, j_2, m, m_1, m_2}^{(9)}} 2^{j\nu - (j_1 + j_2)b + (1 - s_1)m_1} \frac{M_7 dV^{(1)}}{\left| \frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right|}. \]  
(3.97)

Inspired by [7, 36], we define
\[ f(\mu) := \sigma_1 + \sigma_2 - (\xi_1|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha) + \frac{\xi_1 \xi_2}{\xi} \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]^2. \]  
(3.98)

From (3.98) and (3.90), we have that
\[ |f(\mu_2)| = |\sigma_1 + \sigma_2 + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) - \phi(\xi, \mu)| = |\tau - \phi(\xi, \mu)| \sim 2^j, \]  
(3.99)
\[ |f'(\mu_2)| \sim \left| \frac{\xi_1}{\xi} \right| \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{a_1}. \]  
(3.100)
Combining (3.99), (3.100) with Lemma 2.6, for fixed $\xi_1, \xi_2, \mu_1, \sigma_1, \sigma_2$, the Lebesgue measure of $\mu_2$ can be controlled by $C2^{a_1\frac{m_1}{2}}$. By using the Cauchy-Schwartz inequality with respect to $\mu_2$ and the inverse change of variables related to (3.33) and the Cauchy-Schwartz inequality with respect to $u, v, w$ and the Cauchy-Schwartz inequality with respect to $\sigma_1$ and $\sigma_2$, we have that (3.98) can be bounded by

$$\sum_{m, m_1, m_2 > 0 \atop j_1, j_2 \geq 0, 0 < j \leq \frac{(2a_1 - 1)m_1}{2}} \sum_{i=1}^{2m+1} \sum_{j=1}^{2m_1} \sum_{\Sigma} \mu_2 \int \left( \int \frac{G^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{d\mu_2} \right)^{\frac{1}{2}} d\sigma_1 d\sigma_2$$

(3.101)

where $\sum = \sum_{m, m_1, m_2 > 0 \atop j_1, j_2 \geq 0, 0 < j \leq \frac{(2a_1 - 1)m_1}{2}}$.

Now we consider (3.32). We make the change of variables as in (3.42). Thus the Jacobian determinant equals

$$\frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = 2 \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]. \quad (3.102)$$

We assume that $D_{j_1, j_2, m, m_1, m_2}^{(10)}$ is the image of the subset of all points

$$(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_{j_1, j_2, m, m_1, m_2}^{(8)},$$

which satisfies (3.32) under the transformation (3.42). Combining (3.32) with (3.102), we have that

$$\left| \frac{\partial(u, v, w, \xi_1)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right| \sim 2^{a_1\frac{m_1}{2}}. \quad (3.103)$$
Let
\[ f_{m_k,j_k} = \eta_{m_k}(\xi_k) \eta_{j_k}(\sigma_k) f_k(\xi_k, \mu_k, \tau_k) (k = 1, 2) \]
and \( H_4(u, v, w, \xi_1, \sigma_1, \sigma_2) \) be \( \eta_m(\xi) \eta_j(\sigma) \prod_{k=1}^2 f_{m_k,j_k} \) under the change of the variables (3.42) and
\[
M_8 = F(u, v, w) H_4(u, v, w, \xi_1, \sigma_1, \sigma_2), dV^{(4)} = dudvdw\xi_1d\sigma_1d\sigma_2.
\]
Thus, we have that
\[
\text{Int}_6 \leq C \sum_{m,m_1,m_2>0,j_1,j_2\geq 0,0<\eta_j<\frac{\eta_1}{2}} \int_{L_{j_1,j_2,m_1,m_2}} 2^{j_1+j_2+b+1-s_1} M_8 dV^{(4)} \frac{\partial(u,v,w,\xi_1)}{\partial(\xi_1,\xi_2,\mu_1,\mu_2)}. \tag{3.105}
\]
We assume that \( h(\xi) \) is defined as in (3.46), from (3.32), for fixed \( \xi_2, \mu_1, \mu_2 \), we have that
\[
|h'(\xi_1)| = \alpha(\alpha + 1)|\xi_1|^\alpha - 2\left(\frac{\mu_1}{\xi_1}\right)^2 |\xi_1| \geq \alpha(\alpha + 1)|\xi_1|^{\alpha - 1} \geq C2^{(\alpha - 1)m_1},
\]
and
\[
|h(\xi_1)| \leq C2^{j_1 + \frac{(\alpha - 1)m_1}{2}}. \tag{3.106}
\]
Combining (3.106) with Lemma 2.6, for fixed \( \xi_2, \mu_1, \mu_2 \), we have that the Lebesgue measure of \( \xi_1 \) can be controlled by \( C2^{j_1 + \frac{(\alpha - 1)m_1}{2}} \). By using the Cauchy-Schwartz inequality with respect to \( \xi_1 \) and the inverse change of variables related to (3.42) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the Cauchy-Schwartz inequality with respect to \( \sigma_1 \) and \( \sigma_2 \), we have that (3.105) can be bounded by
\[
C \sum_{m,m_1,m_2>0,j_1,j_2\geq 0,0<\eta_j<\frac{\eta_1}{2}} \int_{L_{j_1,j_2,m_1,m_2}} 2^{j_1+j_2+b+1-s_1} M_8 dV^{(4)} \frac{\partial(u,v,w,\xi_1)}{\partial(\xi_1,\xi_2,\mu_1,\mu_2)} \]
\[
\leq C \sum_{m,m_1,m_2>0,j_1,j_2\geq 0,0<\eta_j<\frac{\eta_1}{2}} 2^{j_1+j_2+b+\frac{2\alpha - s_1}{4}m_1} \|F\|_L^2 \int \left( \int H_4^2(u, v, w, \xi_1, \sigma_1, \sigma_2) \right)^{1/2} d\sigma_1 d\sigma_2 \]
\[
\leq C \sum_{m,m_1,m_2>0,j_1,j_2\geq 0,0<\eta_j<\frac{\eta_1}{2}} 2^{j_1+j_2+b-4\alpha m_1} \|F\|_L^2 \left( \int \prod_{k=1}^2 f_{m_k,j_k} d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2 \]
\[
\leq C \sum_{m,m_1,m_2>0,j_1,j_2\geq 0,0<\eta_j<\frac{\eta_1}{2}} 2^{j_1+j_2-4\alpha m_1} \|F\|_L^2 \left( \int \prod_{k=1}^2 f_{m_k,j_k} dV \right)^{1/2} \leq C \|F\|_L^2 \left( \prod_{j=1}^{2} \|F_j\|_L^2 \right). \]
where \( \sum = \sum_{m,m_1,m_2>0, j_1,j_2\geq 0, 0<j<1} \sum_{0<j<1} \) and \( dV^{(5)} = du dv dw d\xi_1 \).

(7) Region \( \Omega_7 \). This case can be proved similarly to Region \( \Omega_6 \).

This ends the proof of Lemma 3.1.

**Lemma 3.2.** Let \( -\frac{2a-5}{8} + 2\alpha \epsilon \leq s < 0 \) and \( b = \frac{1}{2} + \epsilon \) and \( b' = -\frac{1}{2} + 2\epsilon \). Then, we have that

\[
\| \partial_x [I_N(u_1u_2) - I_Nu_1I_Nu_2] \|_{X^{0,0}} \leq CN^{3\alpha \epsilon} \max \left\{ N^{-\frac{a}{4}}, N^{-\frac{2a-5}{4}} \right\} \prod_{j=1}^{2} \| I_Nu_j \|_{X^{0,0}}. 
\]

**Proof.** To prove (3.107), by duality, it suffices to prove that

\[
\left| \int_{\mathbb{R}^3} h \partial_x [I_N(u_1u_2) - I_Nu_1I_Nu_2] \, dx dy dt \right|
\leq CN^{3\alpha \epsilon} \max \left\{ N^{-\frac{a}{4}}, N^{-\frac{2a-5}{4}} \right\} \| h \|_{X^{-\alpha \epsilon}_b} \left( \prod_{j=1}^{2} \| I_Nu_j \|_{X^{0,0}} \right).
\]

for \( h \in X_{-\alpha \epsilon}^0 \). Let

\[
F(\xi, \mu, \tau) = \langle \sigma \rangle^{-b'} M(\xi) \mathcal{F} h(\xi, \mu, \tau),
\]

\[
F_j(\xi_j, \mu_j, \tau_j) = M(\xi_j) \langle \sigma_j \rangle^b \mathcal{F} u_j(\xi_j, \mu, \tau_j) (j = 1, 2).
\]

To obtain (3.108), from (3.109), it suffices to prove that

\[
\int_{D} |\xi| G(\xi_1, \xi_2) F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) \langle \sigma_j \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^b \, d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau
\leq CN^{2 \alpha \epsilon} \max \left\{ N^{-\frac{a}{4}}, N^{-\frac{2a-5}{4}} \right\} \| F \|_{L^2_{\xi_1\mu_1\tau_1}} \left( \prod_{j=1}^{2} \| F_j \|_{L^2_{\xi_j\mu_j\tau_j}} \right),
\]

where \( G(\xi_1, \xi_2) = \frac{M(\xi_1)M(\xi_2) - M(\xi)}{M(\xi_1)M(\xi_2)} \) and \( D \) is defined as in Lemma 3.1.

Without loss of generality, we assume that \( F(\xi, \mu, \tau) \geq 0, F_j(\xi_j, \mu_j, \tau_j) \geq 0 (j = 1, 2) \). By symmetry, we can assume that \(|\xi_1| \geq |\xi_2|\).

We define

\[
A_1 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_2| \leq |\xi_1| \leq \frac{N}{2} \right\},
\]

\[
A_2 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, |\xi_2| \leq 2 \right\},
\]

\[
A_3 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, 2 < |\xi_2| \leq N \right\},
\]

\[
A_4 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, |\xi_2| > N \right\}.
\]
Here $D^*$ is defined as in Lemma 3.1. Obviously, $D^* \subset \bigcup_{j=1}^{4} A_j$. We define

$$K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) := \frac{|\xi|G(\xi_1, \xi_2)}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b}$$

(3.111)

and

$$J_k := \int_{A_j} K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau$$

with $1 \leq k \leq 4, k \in \mathbb{N}$.

We consider (3.16) and (3.17), respectively.

When (3.16) is valid, one of (3.18)-(3.20) must occur, from Lemma 3.2 of [55], we have that

$$\sum_{k=1}^{4} J_k \leq CN^{-\frac{\alpha}{2}+1+(2\alpha+2)\varepsilon} \|F\|_{L^2_{\xi,\mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi,\mu}} \right).$$

Thus, we only consider the case (3.17).

(1) Region $A_1$. In this case, since $M(\xi_1, \xi_2) = 0$, thus we have that $J_1 = 0$.

(2) Region $A_2$. From page 902 of [20], we have that

$$G(\xi_1, \xi_2) \leq C \frac{|\xi_2|}{|\xi_1|}.$$  

(3.112)

Inserting (3.112) into (3.111) yields

$$K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi|G(\xi_1, \xi_2)}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq \frac{C}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b}.$$  

(3.113)

By using the Cauchy-Schwartz inequality with respect to $\xi_1, \mu_1, \tau_1$, from (3.113) and (3.15), we have that

$$J_2 \leq C \int_{\mathbb{R}^3} \frac{|\xi|G(\xi_1, \xi_2)}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \left( \int_{\mathbb{R}^3} \frac{d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{2b}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{\mathbb{R}^3} \prod_{j=1}^{2} |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} \frac{F(\xi, \mu, \tau)}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} F(\xi, \mu, \tau) d\xi d\mu d\tau$$

$$\leq C N^{-\frac{\alpha}{4}} \|F\|_{L^2_{\xi,\mu}} \left( \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi,\mu}} \right).$$

(3.114)
(3) Region $A_3$. From page 902 of [20], we have that (3.112) is valid. Combining (3.111) with (3.112), we have that

$$K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \min \{ |\xi|, |\xi_1|, |\xi_2| \} \langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^{b}$$

(3.115)

When (3.17) is valid, from Lemma 2.8, we have that

$$\left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim |\xi||\xi_1|^{\frac{1}{2}}.$$  

(3.116)

We consider $|\sigma| \geq |\xi_1|^{\frac{2a-1}{2}}$ and $|\sigma| < |\xi_1|^{\frac{2a-1}{2}}$, respectively.

When $|\sigma| \geq |\xi_1|^{\frac{2a-1}{2}}$, since $s_1 \geq -\frac{a-1}{4} + 4\alpha$, from (3.115), we have that

$$K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C |\xi_1|^{-\frac{2a-1}{2}+(2a-1)\epsilon} \min \{ |\xi|, |\xi_1|, |\xi_2| \} \prod_{j=1}^{2} \langle \sigma_j \rangle^{b}$$

$$\leq CN^{-\frac{2a-3}{4}+(2a-1)\epsilon} |\xi_1|^{-\frac{1}{2}} |\xi_2| \prod_{j=1}^{2} \langle \sigma_j \rangle^{b}$$

(3.117)

Thus, combining (2.17) with (3.117), we have that

$$|J_3| \leq CN^{-\frac{2a-3}{4}+(2a-1)\epsilon} \| F \|_{L^2_{\xi \mu}} \left( \prod_{j=1}^{2} \| F_j \|_{L^2_{\xi \mu}} \right).$$

Now we consider case $|\sigma| < |\xi_1|^{\frac{2a-1}{2}}$. We dyadically decompose with respect to

$$\langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^j, \langle \sigma_2 \rangle \sim 2^j, |\xi| \sim 2^m, |\xi_1| \sim 2^m, |\xi_2| \sim 2^m.$$

Let $D_{j,j_1,j_2,m,m_1,m_2}^{(11)}$ be the image of set of all points $(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*$ satisfy

$$|\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, 2A < |\xi_2| \leq N, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2},$$

$$\langle \sigma \rangle \sim 2^j \leq 2^{\frac{2a-1}{2}m}, \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}$$

(3.118)

under the transformation $(\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2)$. We define

$$g_{m,k,jk} := \eta_{m_k} (\xi_k) \eta_{j_k} (\sigma_k) F_k (\xi_k, \mu_k, \tau_k) (k = 1, 2),$$

(3.119)

$$g_{m,j} := \eta_m (\xi) \eta_j (\sigma) |F(\xi, \mu, \sigma_1 + \sigma_2 - \phi(\xi_1, \mu_1) - \phi(\xi_2, \mu_2))|,$$

(3.120)

$$Q := \sum_{k=1}^{2} g_{m,k,jk}, dV = d\xi_1 d\mu_1 d\sigma_1 d\xi_2 d\mu_2 d\sigma_2.$$  

(3.121)
Thus, we have that \( J_3 \) can be bounded by

\[
C \sum_{m_1, m_2 > 0, m_1, m_2 > 0, 0 < j \leq \frac{(2 \alpha - 1) m_1}{2}} \left( \int_{D_{j,j_1,j_2,m,m_1,m_2}^{(11)}} 2^{j \nu - (j_1 + j_2) b + \min\{m, m_1, m_2\}} Q dV \right). \tag{3.122}
\]

In this case, we consider (3.31), (3.32), respectively. When (3.31) is valid, we make the change of variables (3.32), thus the Jacobian determinant equals

\[
\frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = (\alpha + 1)(|\xi_1|^{\alpha} - |\xi_2|^{\alpha}) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right]. \tag{3.123}
\]

We assume that \( D_{j,j_1,j_2,m,m_1,m_2}^{(12)} \) is the image of the subset of all points \((\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D_{j,j_1,j_2,m,m_1,m_2}^{(11)}\) which satisfies (3.31) under the transformation as in (3.33). Combining (3.123) with (3.31), we have that

\[
|\frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)}| > 2^{j + \frac{\alpha - 1}{m_1}} \tag{3.124}
\]

Let \( G_5(u, v, w, \mu_2, \sigma_1, \sigma_2) \) be \( \eta_m(\xi) \eta_j(\sigma) \prod_{k=1}^{2} g_{m_k,j_k} \) under the change of the variables (3.33) and

\[
M_9 = F(u, v, w) G_5(u, v, w, \mu_2, \sigma_1, \sigma_2), dV(1) = dudvdw\mu_2d\sigma_1d\sigma_2. \tag{3.125}
\]

Thus, (3.122) can be controlled by

\[
C \sum_{m_1, m_2 > 0, m_1, m_2 > 0, 0 < j \leq \frac{(2 \alpha - 1) m_1}{2}} \int_{D_{j,j_1,j_2,m,m_1,m_2}^{(12)}} 2^{j \nu - (j_1 + j_2) b + \min\{m, m_1, m_2\}} \frac{M_5 dV(1)}{|\partial(u, v, w, \mu_2)/\partial(\xi_1, \xi_2, \mu_1, \mu_2)|}. \tag{3.126}
\]

Inspired by [7, 36], we define

\[
f(\mu) := \sigma_1 + \sigma_2 - (|\xi_1|^{\alpha} - |\xi_1|^{\alpha} - |\xi_2|^{\alpha}) + \frac{\xi_1 \xi_2}{\xi} \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]^2. \tag{3.127}
\]

For fixed \( \sigma_1, \sigma_2, \xi_1, \xi_2, \mu_1, \) from (3.127), we have that

\[
|f(\mu_2)| = |\sigma_1 + \sigma_2 + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) - \phi(\xi, \mu)| = |\tau - \phi(\xi, \mu)| \sim 2^j, \tag{3.128}
\]

\[
|f'(\mu_2)| \sim \left| \frac{\xi_1}{\xi} \right| \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{\frac{\alpha - 1}{m_1}}. \tag{3.129}
\]
Combining (3.128), (3.129) with Lemma 2.6, for fixed \( \sigma_1, \sigma_2, \xi_1, \xi_2, \mu_1 \), we have that the Lebesgue measure of \( \mu_2 \) can be controlled by \( C 2^{j - \frac{am}{2}} \). By using the Cauchy-Schwartz inequality with respect to \( \mu_2 \) and the inverse change of variables related to (3.33) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the Cauchy-Schwartz inequality with respect to \( \sigma_1 \) and \( \sigma_2 \), we have that (3.126) can be bounded by

\[
C \sum \int_{D^{(12)}_{j_1, j_2, m_1, m_2}} 2^{j' - (j_1 + j_2) + \alpha m} M_9 dV^{(1)} \left( \frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right) ^{\frac{1}{2}} \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \int F(u, v, w) \left( \int \frac{G_2^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} d\mu_2 \right) ^{\frac{1}{2}} dV^{(2)} \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \int \left( \int \frac{G_2^2(u, v, w, \mu_2, \sigma_1, \sigma_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} d\sigma_1 d\sigma_2 \right) dV^{(3)} \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \int \left( \int \prod_{k=1}^2 \frac{g_{m_k, j_k}^2 \mu_1 d\mu_1 d\mu_2}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right) ^{\frac{1}{2}} d\sigma_1 d\sigma_2 \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \int \left( \int \prod_{k=1}^2 \frac{g_{m_k, j_k}^2 d\xi_k d\mu_1 d\mu_2}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right) ^{\frac{1}{2}} d\sigma_1 d\sigma_2 \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \int \left( \int \prod_{k=1}^2 \frac{g_{m_k, j_k}^2 dV}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right) ^{\frac{1}{2}} \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \left( \prod_{j=1}^2 \| F_j \| _{L^2} \right) ^{\frac{1}{2}} \\
\leq C \sum 2^{j' - (j_1 + j_2) + \alpha m} \left( \prod_{j=1}^2 \| F_j \| _{L^2} \right) ^{\frac{1}{2}} \\
\leq C N ^{\frac{5 - 2a}{4} + \epsilon} \left( \prod_{j=1}^2 \| F_j \| _{L^2} \right) ^{\frac{1}{2}}, \quad (3.130)
\]

where \( \sum = \sum _{j_1, j_2 \geq 0, 0 < j \leq \frac{(2a - 1)m_1}{2} \} \) .

When (3.32) is valid, we make the change of variables (3.42), thus the Jacobian determinant equals

\[
\frac{\partial (u, v, w, \xi_1)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} = 2 \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right]. \quad (3.131)
\]

We assume that \( D^{(13)}_{j_1, j_2, m_1, m_2} \) is the image of the subset of all points

\[
(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D^{(11)}_{j_1, j_2, m_1, m_2},
\]

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which satisfies (3.32) under the transformation (3.42). Combining (3.116) with (3.131), we have that
\[ \left| \frac{\partial(u, v, w, \xi_1)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right| \sim 2^{m+\frac{(\alpha-2)m_1}{2}}. \] (3.132)

Let \( H_5(u, v, w, \xi_1, \sigma_1, \sigma_2) \) be \( \eta_m(\xi) \eta_j(\sigma) \prod_{k=1}^2 g_{m_k,j_k} \) under the change of the variables (3.42) and
\[ M_{10} = F(u, v, w)H_5(u, v, w, \xi_1, \sigma_1, \sigma_2), dV^{(4)} = dudvdw\xi_1d\sigma_1d\sigma_2. \] (3.133)

Thus, (3.122) can be controlled by
\[ C \sum_{m_1, m_2 > 0, j_1, j_2 \geq 0, 0 < j \leq \frac{(2\alpha-1)m_1}{2}} \int_{D^{(13)}_{j_1,j_2,m_1,m_2}} 2^{j\beta-(j_1+j_2)\beta+\min\{m,m_1,m_2\}} \frac{M_{10}dV^{(4)}}{\left| \frac{\partial(u,v,w,\xi_1)}{\partial(\xi_1,\xi_2,\mu_1,\mu_2)} \right|} \] (3.134)

We assume that \( h(\xi) \) is defined as in (3.46), from (3.32), for fixed \( \xi_2, \mu_1, \mu_2 \), we have that
\[ |h'(\xi_1)| = \left| \alpha(\alpha+1)|\xi_1|^{\alpha-2} + 2 \left( \frac{\mu_1}{\xi_1} \right)^2 \xi_1 \right| \geq \alpha(\alpha+1)|\xi_1|^{\alpha-1} \geq C2^{(\alpha-1)m_1}, \]
\[ |h(\xi_1)| \leq C2^{\frac{(\alpha-1)m_1}{2}}, \] (3.135)

for fixed \( \xi_2, \mu_1, \mu_2 \), combining (3.136) with Lemma 2.6, we have that the Lebesgue measure of \( \xi_1 \) can be controlled by \( C2^{\frac{(\alpha-1)m_1}{2}} \). By using the Cauchy-Schwartz inequality with respect to \( \xi_1 \) and the inverse change of variables related to (3.42) and the Cauchy-Schwartz inequality with respect to \( u, v, w \) and the Cauchy-Schwartz inequality with
respect to $\sigma_1$ and $\sigma_2$, we have that (3.134) can be bounded by

$$C \sum_{E^{(13)}} \sum_{j_1, j_2, m_1, m_2} 2^{j_1 \epsilon} (j_1 + j_2) b + m \frac{M_{10} dV^{(4)}}{\sigma (u, v, w, \xi_1)}$$

$$\leq C \sum_{E^{(13)}} \sum_{j_1, j_2, m_1, m_2} 2^{j_1 \epsilon} (j_1 + j_2) b + m \frac{m}{2} F \|L^2 \int \left( \int \frac{H^2_N(u, v, w, \xi_1, \sigma_1, \sigma_2)}{\sigma (u, v, w, \xi_1)} dV^{(2)} \right)^{1/2}$$

$$\leq C \sum_{E^{(13)}} \sum_{j_1, j_2, m_1, m_2} 2^{j_1 \epsilon} (j_1 + j_2) b + m \frac{m}{2} \|F\|_{L^2} \int \left( \int \prod_{k=1}^{2} g_{m_k, j_k} d\xi_1 d\mu_1 d\mu_2 \right)^{1/2}$$

$$\leq C \sum_{m_1, m_2 > 0} N^{-2\alpha - 5 + (\alpha - 1) m_1 \epsilon} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right)^{1/2}$$

where $\sum = \sum_{m_1, m_2 > 0} \sum_{j_1, j_2, 0 < j_2 \leq (2\alpha - 1) m_1}$ and $dV^{(5)} := dudvwd\xi_1$.

(4) Region $A_4$. When (3.17) is valid, from Lemma 2.8, we have that

$$\left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{m_1 \epsilon - m_2 \epsilon}.$$  \hspace{1cm} (3.137)

In this case, we have that

$$M(\xi_1, \xi_2) \leq C \left( \prod_{j=1}^{2} \frac{|\xi_j|}{N^2} \right)^{-s_1} \cdot$$  \hspace{1cm} (3.138)

In this case, we consider $|\sigma| \geq |\xi_1|^{2\alpha + \frac{1}{2}}$ and $|\sigma| < |\xi_1|^{2\alpha + \frac{1}{2}}$, respectively.
Thus the Jacobian determinant equals

\[ K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C N^{2s} \frac{\| \xi \|_2 \prod_{j=1}^{2} |\xi_j|^{-s}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C N^{2s} \frac{\| \xi_1 \|_2^{-\frac{2a+1}{2} + (2a+1)\epsilon - s} |\xi_2|^s |\xi|}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \]

\[ \leq C N^{-\frac{2a-3}{4} + (2a+1)\epsilon} \frac{|\xi_1|^\frac{2}{j} |\xi_2|^{-\frac{2}{j}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}. \tag{3.139} \]

Thus, combining (2.17) with (3.139), we have that

\[ |J_4| \leq C N^{-\frac{2a-3}{4} + (2a+1)\epsilon} \| F \|_{L_\infty^2} \left( \prod_{j=1}^{2} \| F_j \|_{L_\infty^2} \right). \]

We consider case \(|\sigma| < |\xi|^{\frac{2a+1}{2}}\). We consider (3.31), (3.32), respectively.

We dyadically decompose with respect to

\[ \langle \sigma \rangle \sim 2^j, \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}. \]

Let \( D_{j,j_1,j_2,m,m_1,m_2}^{(14)} \) be the image of set of all points \((\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*\) satisfying

\[ |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, |\xi_2| > N, |\xi| \sim 2^m, |\xi_1| \sim 2^{m_1}, |\xi_2| \sim 2^{m_2}, \]

\[ \langle \sigma_1 \rangle \sim 2^{j_1}, \langle \sigma_2 \rangle \sim 2^{j_2}, \langle \sigma \rangle \sim 2^j \leq 2^{\frac{(2a+1)m_1}{2}}, \tag{3.140} \]

under the transformation \((\xi_1, \mu_1, \tau_1, \xi_2, \mu_2, \tau_2) \rightarrow (\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2)\). Thus, we have that \(J_4\) can be bounded by

\[ C N^{2s} \sum_{m_1,m_2>0} \sum_{m_1,m_2>0} \int_{D_{j,j_1,j_2,m,m_1,m_2}^{(14)}} 2^{j'-(j_1+j_2)b+(m_1+m_2)s|m+m} QdV, \tag{3.141} \]

where \(Q, dV\) are defined as in (3.121).

When (3.31) is valid, we make the change of variables (3.33).

Thus the Jacobian determinant equals

\[ \frac{\partial(u, v, w, \mu_2)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} = (\alpha + 1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \left[ \left( \frac{\mu_1}{\xi_1} \right)^2 - \left( \frac{\mu_2}{\xi_2} \right)^2 \right]. \tag{3.142} \]
We assume that $D^{(14)}_{j,j_1,j_2,m,m_1,m_2}$ is the image of the subset of all points

$$(\xi_1, \mu_1, \sigma_1, \xi_2, \mu_2, \sigma_2) \in D^{(13)}_{j,j_1,j_2,m,m_1,m_2},$$

which satisfies (3.31) under the transformation (3.33). Combining (3.142) with (3.31), we have that

$$\left| \frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right| > 2^{j + m + \frac{(\alpha - 2)m_1}{2}}, \quad (3.143)$$

Let $G_6(u, v, w, \mu_2, \sigma_1, \sigma_2)$ be $\eta_m(\xi)\eta_j(\sigma) \prod_{k=1} g_{m_k,j_k}$ under the change of the variables (3.33) and

$$M_{11} = F(u, v, w)G_6(u, v, w, \mu_2, \sigma_1, \sigma_2), \quad \sum_{m_1,m_2>0,m_{j_1,j_2} \geq 0,j < \frac{(2\alpha+1)m_1}{2}},$$

$$dV^{(1)} = dudvdw\mu_2d\sigma_1d\sigma_2. \quad (3.144)$$

Thus, (3.141) can be controlled by

$$CN^{2s} \sum \int_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j'-(j_1+j_2)b-(m_1+m_2)s+m} M_{11}dV^{(1)} \left| \frac{\partial (u, v, w, \mu_2)}{\partial (\xi_1, \xi_2, \mu_1, \mu_2)} \right|. \quad (3.145)$$

We assume that $f(\mu)$ is defined as in (3.38), for fixed $\sigma_1, \sigma_2, \xi_1, \xi_2, \mu_2$, we have that

$$|f(\mu_2)| = |\sigma_1 + \sigma_2 + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) - \phi(\xi, \mu)| = |\tau - \phi(\xi, \mu)| \sim 2^j, \quad (3.146)$$

$$|f'(\mu_2)| \sim \left| \frac{\xi_1}{\xi} \right| \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| \sim 2^{\frac{\alpha m_1}{2}}, \quad (3.147)$$

for fixed $\sigma_1, \sigma_2, \xi_1, \xi_2, \mu_2$, combining (3.146), (3.147) with Lemma 2.6, we have that the Lebesgue measure of $\mu_2$ can be controlled by $C2^{j-\frac{\alpha m_1}{2}}$. By using the Cauchy-Schwartz inequality with respect to $\mu_2$ and the inverse change of variables related to (3.33) and the Cauchy-Schwartz inequality with respect to $u, v, w$ and the Cauchy-Schwartz inequality with respect to $\sigma_1$ and $\sigma_2$, since $-\frac{2\alpha - 5}{8} + 2\alpha \epsilon \leq s < 0$, we have that (3.145) can be
bounded by

\[ CN^{2s} \sum_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j' - (j_1 + j_2) - b - (m_1 + m_2)s + m} \frac{M_{11} dV(1)}{|\partial(u,v,w,\mu_2)|} \]

\[ \leq CN^{2s} \sum_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j' - (j_1 + j_2) - b - 2m_1s - \frac{\alpha_{m_1}}{4} + m} \int F \left( \int \frac{G^2_{0}(u,v,w,\mu_2,\sigma_1,\sigma_2)}{|\partial(u,v,w,\mu_2)|^2} d\mu_2 \right)^{1/2} dV(2) \]

\[ \leq CN^{2s} \sum_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j' - (j_1 + j_2) - b - 2m_1s - \frac{\alpha_{m_1}}{4} + m} \|F\|_{L^2} \int \left( \int \frac{G^2_{0}(u,v,w,\mu_2,\sigma_1,\sigma_2)}{|\partial(u,v,w,\mu_2)|^2} d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2 \]

\[ \leq CN^{2s} \sum_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j' - (j_1 + j_2) - b - 2m_1s - \frac{\alpha_{m_1}}{4} + m} \|F\|_{L^2} \left( \int \prod_{k=1}^{2} g^2_{m_k,j_k} d\xi_1 d\mu_1 d\xi_2 d\mu_2 \right)^{1/2} d\sigma_1 d\sigma_2 \]

\[ \leq CN^{2s} \sum_{D^{(14)}_{j,j_1,j_2,m,m_1,m_2}} 2^{j' - (j_1 + j_2) - b - 2m_1s - \frac{\alpha_{m_1}}{4} + m} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right) \]

\[ \leq CN^{2s} \sum_{m_1,m_2 > 0} 2^{-m_1(2s+\frac{\alpha_{m_1}}{4})+m} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right) \]

\[ \leq CN^{2s} \sum_{m_1,m_2 > 0} 2^{-m_1(2s+\frac{\alpha_{m_1}}{4})} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right) \]

\[ \leq C \sum_{m_1,m_2 > 0} 2^{-m_1(2s+\frac{\alpha_{m_1}}{4}) - \epsilon} 2^{-m_2 \epsilon} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right) \]

\[ \leq CN^{-\frac{\alpha_{m_1}}{4} - \epsilon} \|F\|_{L^2} \left( \prod_{j=1}^{2} \|F_j\|_{L^2} \right) \]  \hfill (3.148)

Here \( \sum = \sum_{m_1,m_2 > 0} \sum_{m_1,m_2 > 0, m_1,j_1,j_2 \geq 0, 0 < j \leq \frac{\alpha_{m_1} + 1 m_1}{2}} \).

Now we consider (3.32). We make the change of variables (4.42).

Thus the Jacobian determinant equals

\[ \frac{\partial(u,v,w,\mu_2)}{\partial(\xi_1,\xi_2,\mu_1,\xi_1)} = 2 \left[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right] . \]  \hfill (3.149)

We assume that \( D^{(15)}_{j,j_1,j_2,m,m_1,m_2} \) is the image of the subset of all points

\[ (\xi_1,\mu_1,\sigma_1,\xi_2,\mu_2,\sigma_2) \in D^{(13)}_{j,j_1,j_2,m,m_1,m_2}, \]

which satisfies (3.32) under the transformation (3.42). Combining (3.149) with (3.137),
we have that
\[
\left| \frac{\partial(u, v, w, \xi_1)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)} \right| \sim 2^{m+\frac{(\alpha-2)m_1}{2}}. \tag{3.150}
\]

Let \(H_6(u, v, w, \xi_1, \sigma_1, \sigma_2)\) be \(\eta_\mu(\xi)\eta_j(\sigma) \prod_{k=1}^{2} f_{m_k,j_k}\) under the change of the variables (3.42) and

\[
M_{12} = F(u, v, w)H_6(u, v, w, \xi_1, \sigma_1, \sigma_2), \quad dV^{(4)} = dudvdwd\xi_1d\sigma_1d\sigma_2. \tag{3.151}
\]

Thus, (3.141) can be controlled by

\[
CN^{2s} \sum_{\min\{j, j_1, j_2, m_1, m_2\} \geq 0, m} \int_{D_{j, j_1, j_2, m, m_1, m_2}^{(15)}} 2^{j-\frac{(\alpha-1)m_1}{2}} M_{12}dV^{(4)} \frac{\partial(u, v, w, \xi_1)}{\partial(\xi_1, \xi_2, \mu_1, \mu_2)}. \tag{3.152}
\]

We assume that \(h(\xi)\) is defined as in (3.46), from (3.32), for fixed \(\xi_2, \mu_1, \mu_2\), we have that

\[
|h'(\xi_1)| = \left| \alpha(\alpha + 1)|\xi_1|^{\alpha-2} + 2 \left( \frac{\mu_1}{\xi_1} \right)^2 \xi_1 \right| \geq \alpha(\alpha + 1)|\xi_1|^{\alpha-1} \geq C2^{(\alpha-1)m_1},
\]

\[
|h(\xi_1)| \leq C2^{j+\frac{(\alpha-1)m_1}{2}}, \tag{3.153}
\]

for fixed \(\xi_2, \mu_1, \mu_2\), combining (3.153) with Lemma 2.6, we have that the Lebesgue measure of \(\xi_1\) can be controlled by \(C2^{j-\frac{(\alpha-1)m_1}{2}}\). By using the Cauchy-Schwartz inequality with respect to \(\xi_1\) and the inverse change of variables related to (3.42) and the Cauchy-Schwartz inequality with respect to \(u, v, w\) and the Cauchy-Schwartz inequality with respect to \(\sigma_1, \sigma_2\), since \(-\frac{2\alpha-5}{8} + 2\alpha \epsilon \leq s < 0\), we have that (3.151) can be bounded
Proof. To prove (3.155), by duality, it suffices to prove that
\begin{equation}
CN^{2s} \sum_{j_1,j_2,m_1,m_2} 2^{j'-j_1-j_2} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} M_2 dV^{(4)} \left( \int \frac{H_0^2(u,v,w,\xi_1,\sigma_1,\sigma_2)}{\partial(u,v,w,\xi_1)} \right)^{\frac{1}{2}} dV^{(2)}
\end{equation}

\begin{equation}
\leq CN^{2s} \sum_{j_1,j_2,m_1,m_2} 2^{j'-j_1-j_2} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} \|F\|_L^2 \left( \int \frac{H_0^2(u,v,w,\xi_1,\sigma_1,\sigma_2)}{\partial(u,v,w,\xi_1)} d\xi_1 \right)^{\frac{1}{2}} d\sigma_1 d\sigma_2
\end{equation}

\begin{equation}
\leq CN^{2s} \sum_{j_1,j_2,m_1,m_2} 2^{j'-j_1-j_2} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} \|F\|_L^2 \left( \int \frac{2^{2} \alpha^{2} g_{m_1,j_1} d\xi_1 d\mu_1 d\xi_2 d\mu_2}{\partial(u,v,w,\xi_1)} \right)^{\frac{1}{2}} d\sigma_1 d\sigma_2
\end{equation}

\begin{equation}
\leq CN^{2s} \sum_{m_1,m_2>0} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} \|F\|_L^2 \left( \prod_{j_1,j_2=1}^{2} \|F_j\|_L^2 \right)
\end{equation}

\begin{equation}
\leq CN^{2s} \sum_{m_1,m_2>0} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} \|F\|_L^2 \left( \prod_{j_1,j_2=1}^{2} \|F_j\|_L^2 \right)
\end{equation}

\begin{equation}
\leq CN^{2s} \sum_{m_1,m_2>0} 2^{-m_1(2s+\frac{\sigma_1-1}{4})-m} \|F\|_L^2 \left( \prod_{j_1,j_2=1}^{2} \|F_j\|_L^2 \right)
\end{equation}

\begin{equation}
\leq CN^{-\sigma_1+3\alpha} \|F\|_L^2 \left( \prod_{j_1,j_2=1}^{2} \|F_j\|_L^2 \right).
\end{equation}

Here \( \sum = \sum \sum \sum \). This completes the proof of Lemma 3.2.

Lemma 3.3. Let \( s \geq -\frac{\alpha-1}{4} + 4\alpha, s_2 \geq 0 \) and \( u_j \in X_b^{s_1,s_2}(j = 1, 2) \) and \( b = \frac{1}{2} + \epsilon \) and \( b' = -\frac{1}{2} + 2\epsilon \). Then, we have that
\begin{equation}
\|\partial_\sigma I(u_1 u_2)\|_{X_b^{0,0}} \leq C \prod_{j=1}^{2} \|I u_j\|_{X_b^{0,0}}.
\end{equation}

Proof. To prove (3.155), by duality, it suffices to prove that
\begin{equation}
\left| \int_{R^3} \tilde{u} \partial_\sigma I(u_1 u_2) dx dy dt \right| \leq C \|u\|_{X_b^{0,0}} \prod_{j=1}^{2} \|I u_j\|_{X_b^{0,0}}.
\end{equation}

for \( u \in X_{-b}^{0,0} \). Let
\begin{equation}
F(\xi, \mu, \tau) = \langle \sigma \rangle^{-b'} F u(\xi, \mu, \tau), F_j(\xi_j, \mu_j, \tau_j) = M(\xi_j) \langle \sigma_j \rangle^b F u_j(\xi_j, \mu, \tau_j)(j = 1, 2),
\end{equation}

(3.157)
$D$ is defined as in Lemma 3.1. To obtain (3.155), from (3.156) and (3.157), it suffices to prove that
\[ \int_D |\xi| M(\xi) F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) d\xi d\mu_1 d\tau_1 \leq C \| F \|_{L^2_{\tau \xi \mu}} \prod_{j=1}^{2} \| F_j \|_{L^2_{\tau \xi \mu}}. \quad (3.158) \]

From (2.4) of [22], we have that
\[ M(\xi) \prod_{j=1}^{2} M(\xi_j) \leq C \| \xi \|_s \prod_{j=1}^{2} \| \xi_j \|_s. \quad (3.159) \]

By using (3.159), we have that the left hand side of (3.158) can be bounded by
\[ \int_D |\xi| \langle \xi \rangle^s F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) \prod_{j=1}^{2} \langle \xi_j \rangle^s \langle \sigma_j \rangle^b d\xi d\mu_1 d\tau_1. \quad (3.160) \]

By using (3.5), we have that (3.160) can be bounded by
\[ C \| F \|_{L^2_{\tau \xi \mu}} \left( \prod_{j=1}^{2} \| F_j \|_{L^2_{\tau \xi \mu}} \right). \]

This completes the proof of Lemma 3.3.

4. Proof of Theorem 1.1

In this section, combining Lemmas 2.2, 3.1 with the fixed point theorem, we present the proof of Theorem 1.1. Let $b, b'$ be defined as in Lemma 3.1.

We define
\[ \Phi_1(u) := \psi(t) W(t) u_0 + \frac{1}{2} \psi \left( \frac{t}{\tau} \right) \int_0^t W(t - \tau) \partial_x(u^2) d\tau, \quad (4.1) \]
\[ B_1(0, 2C\| u_0 \|_{H^{1,1}}) := \left\{ u : \| u \|_{X^0_{b,1,1}} \leq 2C \| u_0 \|_{H^{1,1}} \right\}. \quad (4.2) \]

Combining Lemmas 2.2, 3.1 with (4.1), (4.2), we have that
\[ \| \Phi_1(u) \|_{X^0_{b,1,1}} \leq \| \psi(t) W(t) u_0 \|_{X^0_{b,1,1}} + \left\| \frac{1}{2} \psi \left( \frac{t}{\tau} \right) \int_0^t W(t - \tau) \partial_x(u^2) d\tau \right\|_{X^0_{b,1,1}} \]
\[ \leq C \| u_0 \|_{H^{1,1}} + CT^e \| \partial_x(u^2) \|_{X^0_{b,1,1}} \]
\[ \leq C \| u_0 \|_{H^{1,1}} + CT^e \| u \|_{X^0_{b,1,1}}^2 \leq C \| u_0 \|_{H^{1,1}} \| u \|_{H^{1,1}}^2 + 4C^3 T^e \| u_0 \|_{H^{1,1}}^2. \quad (4.3) \]

We choose $T \in (0, 1)$ such that
\[ T^e = \left[ 16C^2(\| u_0 \|_{H^{1,1}} + 1) \right]^{-1}. \quad (4.4) \]
Combining (4.3) with (4.4), we have that
\[
\| \Phi_1(u) \|_{X^{s_1+s_2}} \leq C \| u_0 \|_{H^{s_1+s_2}} + C \| u_0 \|_{H^{s_1+s_2}} = 2C \| u_0 \|_{H^{s_1+s_2}} .
\] (4.5)

Thus, \( \Phi_1 \) maps \( B_1(0,2C\|u_0\|_{H^{s_1+s_2}}) \) into \( B_1(0,2C\|u_0\|_{H^{s_1+s_2}}) \). By using Lemmas 2.2, 3.1, (4.4)-(4.5), we have that
\[
\| \Phi_1(u_1) - \Phi_1(u_2) \|_{X^{s_1+s_2}} \leq C \left\| \frac{1}{2} \psi \left( \frac{t}{\tau} \right) \int_0^t W(t-\tau) \partial_x (u_1^2 - u_2^2) d\tau \right\|_{X^{s_1+s_2}} \\
\leq CT^\epsilon \| u_1 - u_2 \|_{X^{s_1+s_2}} \left[ \| u_1 \|_{X^{s_1+s_2}} + \| u_2 \|_{X^{s_1+s_2}} \right] \\
\leq 4C^2T^\epsilon \| u_0 \|_{H^{s_1+s_2}} \| u_1 - u_2 \|_{X^{s_1+s_2}} \leq \frac{1}{2} \| u_1 - u_2 \|_{X^{s_1+s_2}} .
\] (4.6)

Thus, \( \Phi_1 \) is a contraction in the closed ball \( B_1(0,2C\|u_0\|_{H^{s_1+s_2}}) \). Consequently, \( u \) is the fixed point of \( \Phi \) in the closed ball \( B_1(0,2C\|u_0\|_{H^{s_1+s_2}}) \). Then \( v := u|_{[0,T]} \in X^{s_1,s_2}_b([0,T]) \) is a solution in the interval \([0,T]\) of the Cauchy problem for (1.1) with the initial data \( u_0 \). For the facts that uniqueness of the solution and the solution to the Cauchy problem for (1.1) is continuous with respect to the initial data, we refer the readers to Theorems II, III of [23].

This ends the proof of Theorem 1.1.

5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We present the proof of Lemma 5.1 before giving the proof of Theorem 1.2.

**Lemma 5.1.** Let \( s_1 > -\alpha - x \) and \( R := \frac{1}{8(C+1)^x} \), where \( C \) is the largest of those constants which appear in (2.5)-(2.6), (3.155). Then, the Cauchy problem for (1.1) locally well-posed for data satisfying \( I_N u_0 \in L^2(\mathbb{R}^2) \) with
\[
\| I_N u_0 \|_{L^2} \leq R.
\] (5.1)

Moreover, the solution to the Cauchy problem for (1.1) exists on a time interval \([0,1]\).

**Proof.** We define \( v := I_N u \). If \( u \) is the solution to the Cauchy problem for (1.1), then \( v \) satisfies the following equation
\[
v_t + \partial_x^2 v + \partial_x^{-1} \partial_y^2 v + \frac{1}{2} I_N \partial_x (I_N^{-1} v)^2 = 0.
\] (5.2)
Then $v$ is formally equivalent to the following integral equation

$$v = W(t)I_N u_0 + \frac{1}{2} \int_0^t W(t - \tau)I_N \partial_x (I_N^{-1} v)^2. \quad (5.3)$$

We define

$$\Phi_2(v) = \psi(t) W(t) I_N u_0 + \frac{1}{2} \psi(t) \int_0^t W(t - \tau)I_N \partial_x (I_N^{-1} v)^2. \quad (5.4)$$

Let $b, b'$ be defined as in Lemmas 3.1-3.3. By using Lemmas 2.2, 3.3, we have that

$$\|\Phi_2(v)\|_{X_b^{0,0}} \leq \|\psi(t) W(t)I_N u_0\|_{X_b^{0,0}} + C \left\| \psi(t) \int_0^t W(t - \tau)I_N \partial_x (I_N^{-1} v)^2 \right\|_{X_b^{0,0}}$$

$$\leq C \|I_N u_0\|_{L^2} + C \|I_N \partial_x (I_N^{-1} v)^2\|_{X_b^{0,0}}$$

$$\leq C \|I_N u_0\|_{L^2} + C \|I_N \partial_x (I_N^{-1} v)^2\|_{X_b^{0,0}}$$

$$\leq C \|I_N u_0\|_{L^2} + C \|v\|_{X_b^{0,0}}^2 \leq CR + C \|v\|_{X_b^{0,0}}^2. \quad (5.5)$$

We define

$$B_2(0, 2CR) := \left\{ v : \|v\|_{X_b^{0,0}} \leq 2CR \right\}. \quad (5.6)$$

Combining (5.5)-(5.6) with the definition of $R$, we have that

$$\|\Phi_2(v)\|_{X_b^{0,0}} \leq CR + 4C^3 R^2 = 2CR. \quad (5.7)$$

Thus, $\Phi_2$ maps $B_2(0, 2CR)$ into $B_2(0, 2CR)$. We define

$$v_j = I_N u_j (j = 1, 2), w_1 = I_N^{-1} v_1 - I_N^{-1} v_2, w_2 = I_N^{-1} v_1 + I_N^{-1} v_2. \quad (5.8)$$

By using Lemmas 2.2, 3.1, 3.2, (5.5)-(5.6) and the definition of $R$, we have that

$$\|\Phi_2(v_1) - \Phi_2(v_2)\|_{X_b^{0,0}} \leq C \left\| \psi(t) \int_0^t W(t - \tau) \partial_x I_N \left[ (I_N^{-1} v_1)^2 - (I_N^{-1} v_2)^2 \right] d\tau \right\|_{X_b^{0,0}}$$

$$\leq C \|\partial_x I_N (w_1 w_2)\|_{X_b^{0,0}} \leq C \|v_1 - v_2\|_{X_b^{0,0}} \left[ \|v_1\|_{X_b^{0,0}} + \|v_2\|_{X_b^{0,0}} \right]$$

$$\leq 4C^2 R^2 \|v_1 - v_2\|_{X_b^{0,0}} \leq \frac{1}{2} \|v_1 - v_2\|_{X_b^{0,0}}. \quad (5.9)$$

Thus, $\Phi_2$ is a contraction in the closed ball $B_2(0, 2CR)$. Consequently, $u$ is the fixed point of $\Phi_2$ in the closed ball $B_2(0, 2CR)$. Then $v := u|_{[0, 1]} \in X_b^{0,0}([0, 1])$ is a solution in the interval $[0, 1]$ of the Cauchy problem for (5.3) with the initial data $I_N u_0$. For the uniqueness of the solution, we refer the readers to Theorem II of [23]. For the fact that
the solution to the Cauchy problem for (5.3) is continuous with respect to the initial data, we refer the readers to Theorem III of [23]. Since the phase function \( \phi(\xi, \mu) \) is singular at \( \xi = 0 \), to define the derivative of \( W(t)u_0 \), the requirement \( |\xi|-1 \mathcal{F}_{xy}u_0(\xi, \mu) \in \mathcal{S}'(\mathbb{R}^2) \) is necessary.

This ends the proof of Lemma 5.1.

Inspired by [20], we use Lemmas 2.7, 3.2, 5.1 to prove Theorem 1.2.

For \( \lambda > 0 \), we define

\[
u_\lambda(x, y, t) = \lambda^\alpha u(\lambda x, \lambda^{\frac{n}{2}+1}y, \lambda^{\alpha+1}t), \quad u_{0\lambda}(x, y) = \lambda^\alpha u(\lambda x, \lambda^{\frac{n}{2}+1}y). \tag{5.10}\]

Thus, \( u_\lambda(x, y, t) \in X^{s,0}_b([0, T]) \) is the solution to

\[
\partial_t u_\lambda + |D_x|^\alpha \partial_x u_\lambda + \partial_x^{-1} \partial_y^2 u_\lambda + u_\lambda \partial_x u_\lambda = 0, \tag{5.11}
\]

\[
u_\lambda(x, y, 0) = u_{0\lambda}(x, y), \tag{5.12}\]

if and only if \( u(x, y, t) \in X^{s,0}_b([0, T]) \) is the solution to the Cauchy problem for (1.1) in \([0, T]\] with the initial data \( u_0 \). By using a direct computation, for \( \lambda \in (0, 1) \), we have that

\[
\|I_Nu_{0\lambda}\|_{L^2} \leq C N^{-s} \lambda^{\frac{3\alpha-4}{4}+s} \|u_0\|_{H^{s,0}}. \tag{5.13}\]

For \( u_0 \neq 0 \) and \( u_0 \in H^{s,0}(\mathbb{R}^2) \), we choose \( \lambda, N \) such that

\[
\|I_Nu_{0\lambda}\|_{L^2} \leq C N^{-s} \lambda^{\frac{3\alpha-4}{4}+s} \|u_0\|_{H^{s,0}} := \frac{R}{4}. \tag{5.14}\]

Then there exist \( w_3 \) which satisfies that \( \|w_3\|_{X^{s,0}_b} \leq 2CR \) such that \( v := w_3 \mid_{[0,1]} \) is a solution to the Cauchy problem for (5.10) with \( u_{0\lambda} \). Multiplying (5.10) by \( 2I_Nu_\lambda \) and integrating with respect to \( x, y \) and integrating by parts with respect to \( x \) yield

\[
\frac{d}{dt} \int_{\mathbb{R}^2} (I_Nu)^2 \, dx \, dy + \int_{\mathbb{R}^2} I_Nu \partial_x I_N [(u)^2] \, dx \, dy = 0. \tag{5.15}\]

Combining \( \int_{\mathbb{R}^2} I_Nu \partial_x [(I_Nu)^2] \, dx \, dy = 0 \) with (5.15), we have that

\[
\frac{d}{dt} \int_{\mathbb{R}^2} (I_Nu)^2 \, dx \, dy = - \int_{\mathbb{R}^2} I_Nu \partial_x \left[ I_N((u)^2) - (I_Nu)^2 \right] \, dx \, dy. \tag{5.16}\]
From (5.16) and Lemma 2.7, we have that

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \, dx \, dy - \int_{\mathbb{R}^2} (I_N u_\lambda)^2 \, dx \, dy
\]

\[= - \int_0^1 \int_{\mathbb{R}^2} I_N u_\lambda \partial_x [I_N ((u_\lambda)^2) - (I_N u_\lambda)^2] \, dx \, dy \, dt
\]

\[= - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\chi_{[0,1]}(t) I_N u_\lambda) (\partial_x [I_N ((u_\lambda)^2) - (I_N u_\lambda)^2]) \, dx \, dy \, dt
\]

\[\leq C \|\chi_{[0,1]}(t) I_N u_\lambda\|_{X^0_{\alpha, 0}} \|\chi_{[0,1]}(t) \partial_x [I_N ((u_\lambda)^2) - (I_N u_\lambda)^2]\|_{X^0_{\alpha, 0}}
\]

\[\leq C N^{\frac{2\alpha - 5}{4} + 3\alpha} \|I_N u_\lambda\|_{X^3_{\alpha, 0}}. \tag{5.17}
\]

Combining (5.14) with (5.17), we have that

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \, dx \, dy \leq \frac{R^2}{4} + CN^{\frac{2\alpha - 5}{4} + 3\alpha} \|I_N u_\lambda\|_{X^3_{\alpha, 0}}^3 \leq \frac{R^2}{4} + 8C^4 N^{\frac{2\alpha - 5}{4} + 3\alpha} R^{5}. \tag{5.18}
\]

Thus, if we take \( N \) sufficiently large such that such that \( 8C^4 N^{\frac{2\alpha - 5}{4} + 3\alpha} R^{3} \leq \frac{3}{4} R^2 \), then

\[
\left\lfloor \int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \, dx \, dy \right\rfloor^{\frac{1}{2}} \leq R. \tag{5.19}
\]

We consider \( I_N u(x, y, 1) \) as the initial data and repeat the above argument, we obtain that (5.11)-(5.12) possess a solution in \( \mathbb{R}^2 \times [1, 2] \). In this way, we can extend the solution to (5.11)-(5.12) to the time interval \([0, 2]\). The above argument can be repeated \( L \) steps, where \( L \) is the maximal positive integer such that

\[
8C^4 N^{\frac{2\alpha - 5}{4} + 3\alpha} R^3 L \leq \frac{3}{4} R^2. \tag{5.20}
\]

More precisely, the solution to (5.11)-(5.12) can be extended to the time interval \([0, L]\).

Thus, we can prove that (5.11)-(5.12) are globally well-posed in \([0, T_{\alpha+1}]\) if we can choose a number \( N \) such that

\[
L \geq \frac{T}{\lambda}. \tag{5.21}
\]

From (5.20), we know that

\[
L \sim N^{\frac{2\alpha - 5}{4} - 3\alpha}. \tag{5.22}
\]

We know that (5.21) is valid provided that the following inequality is valid

\[
CN^{\frac{2\alpha - 5}{4} - 3\alpha} \geq \frac{T}{\lambda^{\alpha+1}} \sim CTN^{-\frac{4(\alpha+1)}{3\alpha}}. \tag{5.23}
\]

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In fact, (5.23) is valid if \( N^{\frac{2\alpha - 5}{4}} > N^{-\frac{4(\alpha + 1)}{5\alpha + 3}} \) which is equivalent to \(-\frac{(\alpha - 1)(3\alpha - 4)}{4(5\alpha + 3)} < s < 0\). This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3.

From Lemmas 5.1, 3.2, 2.7 and (5.10)-(5.16), we have that

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \ dx \ dy - \int_{\mathbb{R}^2} (I_N u_0)^2 \ dx \ dy \\
= - \int_0^1 \int_{\mathbb{R}^2} I_N u_\lambda \partial_x \left[ I_N \left( (u_\lambda)^2 \right) - (I_N u_\lambda)^2 \right] \ dx \ dy \ dt \\
= - \int_0^1 \int_{\mathbb{R}^2} \chi_{[0, 1]}(t) I_N u_\lambda \left( \chi_{[0, 1]}(t) \partial_x \left[ I_N \left( (u_\lambda)^2 \right) - (I_N u_\lambda)^2 \right] \right) \ dx \ dy \ dt \\
\leq C \left\| \chi_{[0, 1]}(t) I_N u_\lambda \right\|_{X_{b, 0}^{\alpha, 0}} \left\| \chi_{[0, 1]}(t) \partial_x \left[ I_N \left( (u_\lambda)^2 \right) - (I_N u_\lambda)^2 \right] \right\|_{X_{b, 0}^{\alpha, 0}} \\
\leq C \left\| I_N u_\lambda \right\|_{X_{b, 0}^{\alpha, 0}} \left\| \partial_x \left[ I_N \left( (u_\lambda)^2 \right) - (I_N u_\lambda)^2 \right] \right\|_{X_{b, 0}^{\alpha, 0}} \leq C N^{-\frac{2}{3} + 3\alpha} \left\| I_N u_\lambda \right\|_{X_{b, 0}^{\alpha, 0}}^{3}. \tag{6.1}
\]

Combining (5.14) with (6.1), we have that

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \ dx \ dy \leq R^2 + C N^{-\frac{2}{3} + 3\alpha} \left\| I_N u_\lambda \right\|_{X_{b, 0}^{\alpha, 0}}^{3} \leq R^2 + 8 C^4 N^{-\frac{2}{3} + 3\alpha} R^3. \tag{6.2}
\]

Thus, if we take \( N \) sufficiently large such that such that \( 8 C^4 N^{-\frac{2}{3} + 3\alpha} R^3 \leq \frac{3}{4} R^2 \), then

\[
\left[ \int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 \ dx \ dy \right]^{\frac{1}{2}} \leq R. \tag{6.3}
\]

We consider \( I_N u(x, y, 1) \) as the initial data and repeat the above argument, we obtain that (5.11)-(5.12) possess a solution in \( \mathbb{R}^2 \times [1, 2] \). In this way, we can extend the solution to (5.11)-(5.12) to the time interval \([0, 2]\). The above argument can be repeated \( L \) steps, where \( L \) is the maximal positive integer such that

\[
8 C^4 N^{-\frac{2}{3} + 3\alpha} R^3 L \leq \frac{3}{4} R^2. \tag{6.4}
\]

More precisely, the solution to (5.11)-(5.12) can be extended to the time interval \([0, L]\).

Thus, we can prove that (5.11)-(5.12) are globally well-posed in \([0, \frac{T}{\lambda + 1}]\) if we can choose a number \( N \) such that

\[
L \geq \frac{T}{\lambda}. \tag{6.5}
\]
From (6.5), we know that

\[ L \sim N^{\frac{2}{4} - 3\alpha}. \]  

(6.6)

We know that (6.6) is valid provided that the following inequality is valid

\[ CN^{\frac{2}{4} - 3\alpha} \geq \frac{T}{\lambda^{\alpha+1}} \sim CTN^{\frac{4(\alpha+1)s}{5\alpha+4}}. \]  

(6.7)

In fact, (6.7) is valid if \( N^{\frac{2}{4} - 3\alpha} > N^{\frac{-4(\alpha+1)s}{5\alpha+4}} \), which is equivalent to \( -\frac{\alpha(2\alpha-4)}{4(5\alpha+4)} < s < 0 \).

This completes the proof of Theorem 1.3.

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