Operator product expansion on the lattice: analytic Wilson coefficients

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Scheme of deep inelastic scattering

\[ k \rightarrow k' \]

\[ e \rightarrow \text{u, d} \]

\[ N \rightarrow h, \pi^+ \]

\[ C(q^2) \]

\[ A(p) \]
Lattice OPE

Moments of structure functions ($\mathcal{M}_n(q^2)$) are related to operator product expansion (OPE) of the product of two conserved vector currents between states of particles

$$\mathcal{M}_n(q^2) = C_n^{(2)}(a, q) A_n^{(2)}(a) + \frac{C_n^{(4)}(a, q)}{q^2} A_n^{(4)}(a) + \ldots ,$$

$A_n^{(2)}, A_n^{(4)}$: reduced nucleon matrix elements of local operators of twist-two and four

$C_n^{(2)}, C_n^{(4)}$: Wilson coefficients

$A_n^{(i)}$: computed on the lattice

$C_n^{(i)}$: usually computed perturbatively in $\overline{MS}$-scheme
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$A_n^{(2), 4}$: reduced nucleon matrix elements of local operators of twist-two and four

$C_n^{(2), 4}$: Wilson coefficients

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Wilson coefficients

Problems:

- Operator mixing (higher and leading twist) on the lattice
- Renormalon contributions: coefficients of leading twist operators and expectation values of higher twist operators → cancel in the complete OPE sum if calculated in the same framework: lattice → it is recommended to compute the Wilson coefficients on the lattice as well → need to control possible lattice artefacts ($O(a^2)$ - effects)
Wilson coefficients

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  → it is recommended to compute the Wilson coefficients on the lattice as well
  → need to control possible lattice artefacts ($\mathcal{O}(a^2)$ - effects)
Wilson coefficients and $O(a^2)$ - effects

Reduced Wilson coefficients $c(q^2)$:

$$C(a, q) = c(q^2) \, C_{BORN}(a, q),$$

On the lattice both the non-perturbative $C(a, q)$ and $C_{BORN}(a, q)$ have corrections of $O(a^2)$:

$$C_{BORN}(a, q) = C_{BORN}^{(0)} + (aq)^2 \, C_{BORN}^{(2)} + \ldots$$

$$C(a, q) = c^{(0)}(q^2) \, C_{BORN}^{(0)} + (aq)^2 \, c^{(2)}(q^2) \, C_{BORN}^{(2)} + \ldots,$$

Taking the the ratio

$$\frac{C(a, q)}{C_{BORN}(a, q)} = c^{(0)}(q^2) + \left( c^{(2)}(q^2) - c^{(0)}(q^2) \right) \frac{C_{BORN}^{(2)}}{C_{BORN}^{(0)}} (aq)^2 + \ldots,$$
Wilson coefficients and $O(a^2)$ - effects

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$$ C_{BORN}(a, q) = C_{BORN}^{(0)} + (aq)^2 C_{BORN}^{(2)} + \ldots $$

$$ C(a, q) = c^{(0)}(q^2) C_{BORN}^{(0)} + (aq)^2 c^{(2)}(q^2) C_{BORN}^{(2)} + \ldots, $$

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$$ \frac{C(a, q)}{C_{BORN}(a, q)} = c^{(0)}(q^2) + (c^{(2)}(q^2) - c^{(0)}(q^2)) \frac{C_{BORN}^{(2)}}{C_{BORN}^{(0)}} (aq)^2 + \ldots, $$
Wilson coefficients and $\mathcal{O}(a^2)$ - effects

Reduced Wilson coefficients $c(q^2)$:

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$$ C_{\text{BORN}}(a, q) = C^{(0)}_{\text{BORN}} + (aq)^2 \, C^{(2)}_{\text{BORN}} + \ldots $$

$$ C(a, q) = c^{(0)}(q^2) \, C^{(0)}_{\text{BORN}} + (aq)^2 \, c^{(2)}(q^2) \, C^{(2)}_{\text{BORN}} + \ldots , $$

Taking the the ratio

$$ \frac{C(a, q)}{C_{\text{BORN}}(a, q)} = c^{(0)}(q^2) + \left( c^{(2)}(q^2) - c^{(0)}(q^2) \right) \frac{C^{(2)}_{\text{BORN}}}{C^{(0)}_{\text{BORN}}} (aq)^2 + \ldots , $$
Symbolic calculation of $C^{(2)}_{\text{BORN}}$

Starting point:

Tree level Compton scattering amplitude $\mathcal{W}_{\mu \nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

$$
\mathcal{W}_{\mu \nu}(a, p, q) = \langle p | \hat{J}_\mu(q) \hat{J}^\dagger_\nu(q) | p \rangle = \langle p | T_{\mu \nu} | p \rangle = \sum_{m,n} C^m_{\mu \nu, \mu_1, \ldots, \mu_n}(aq) \langle p | \mathcal{O}(a)^m_{\mu_1, \ldots, \mu_n} | p \rangle.
$$

- Expansion into powers of $p_\nu (\rightarrow \overleftrightarrow{D}_\nu)$ up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^m_{\mu_1, \ldots, \mu_n}$ contributing to $\mathcal{W}_{\mu \nu}(a, p, q)$
- Unpolarised case: $\bar{\psi} \gamma_\mu \gamma_5 \psi$, $\bar{\psi} \gamma_\mu \gamma_\nu \hat{\psi}$, $\bar{\psi} \gamma_\mu \hat{D}_\nu \psi$
  - Polarised case: $\bar{\psi} \gamma_\mu \gamma_5 \gamma_5 \psi$, $\bar{\psi} \sigma_{\mu \nu} \hat{D}_\nu \psi$
- Expansion of the operators into the bases of irreducible representations of hypercubic group $H(4)$
- Projection of the corresponding coefficients → Wilson coefficients
- This program has been carried out with Mathematica completely
Symbolic calculation of $C^{(2)}_{\text{BORN}}$

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Tree level Compton scattering amplitude $\mathcal{W}_{\mu \nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

$$\mathcal{W}_{\mu \nu}(a, p, q) = \langle p|\hat{J}_\mu(q)\hat{J}^\dagger_\nu(q)|p\rangle$$

$$= \langle p|\mathcal{T}_{\mu \nu}|p\rangle$$

$$= \sum_{m,n} C^m_{\mu \nu, \mu_1, ..., \mu_n}(aq) \langle p|\mathcal{O}(a)^m_{\mu_1, ..., \mu_n}|p\rangle .$$

- Expansion into powers of $p_\nu (\leftrightarrow \vec{D}_\nu)$ up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^m_{\mu_1, ..., \mu_n}$ contributing to $\mathcal{W}_{\mu \nu}(a, p, q)$
- Unpolarised case: $\bar{\psi}\psi, \bar{\psi}\gamma_\mu \vec{D}_\nu \psi, \bar{\psi}\vec{D}_\mu \vec{D}_\nu \psi, \bar{\psi}\gamma_\mu \vec{D}_\nu \vec{D}_\omega \vec{D}_\rho \psi$
- Polarised case: $\bar{\psi}\gamma_\mu \gamma_5 \psi, \bar{\psi}\sigma_{\mu \nu} \vec{D}_\omega \psi, \bar{\psi}\gamma_\mu \gamma_5 \vec{D}_\nu \vec{D}_\omega \psi, \bar{\psi}\sigma_{\mu \nu} \vec{D}_\omega \vec{D}_\rho \vec{D}_\lambda \psi$
- Expansion of the operators into the bases of irreducible representations of hypercubic group H(4)
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Symbolic calculation of $C^{(2)}_{\text{BORN}}$

Starting point:

Tree level Compton scattering amplitude $\mathcal{W}_{\mu\nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

$$\mathcal{W}_{\mu\nu}(a, p, q) = \langle p|\hat{J}_\mu(q)\hat{J}_\nu^\dagger(q)|p\rangle$$

$$= \langle p|T_{\mu\nu}|p\rangle$$

$$= \sum_{m,n} C^m_{\mu\nu,\mu_1,...,\mu_n} (aq) \langle p|\mathcal{O}(a)^m_{\mu_1,...,\mu_n}|p\rangle.$$  

- Expansion into powers of $p_\nu (\leftrightarrow \hat{D}_\nu)$ up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^m_{\mu_1,...,\mu_n}$ contributing to $\mathcal{W}_{\mu\nu}(a, p, q)$
  - Unpolarised case: $\bar{\psi}\psi, \bar{\psi}\gamma_\mu \hat{D}_\nu \psi, \bar{\psi} \hat{D}_\mu \hat{D}_\nu \psi, \bar{\psi}\gamma_\mu \hat{D}_\nu \hat{D}_\omega \hat{D}_\rho \psi$
  - Polarised case: $\bar{\psi}\gamma_\mu \gamma_5 \psi, \bar{\psi}\sigma_{\mu\nu} \hat{D}_\omega \psi, \bar{\psi}\gamma_\mu \gamma_5 \hat{D}_\nu \hat{D}_\omega \hat{D}_\rho \psi, \bar{\psi}\sigma_{\mu\nu} \hat{D}_\omega \hat{D}_\rho \hat{D}_\lambda \psi$
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Symbolic calculation of $C^{(2)}_{BORN}$

Starting point:
Tree level Compton scattering amplitude $\mathcal{W}_{\mu\nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

$$\mathcal{W}_{\mu\nu}(a, p, q) = \langle p | \hat{J}_{\mu}(q) \hat{J}^{\dagger}_{\nu}(q) | p \rangle = \langle p | \mathcal{T}_{\mu\nu} | p \rangle = \sum_{m,n} C^{m}_{\mu\nu, \mu_1, \ldots, \mu_n}(aq) \langle p | \mathcal{O}(a)^{m}_{\mu_1, \ldots, \mu_n} | p \rangle .$$

- Expansion into powers of $p\nu$ ($\leftrightarrow \vec{D} \nu$) up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^{m}_{\mu_1, \ldots, \mu_n}$ contributing to $\mathcal{W}_{\mu\nu}(a, p, q)$
- Unpolarised case: $\bar{\psi} \gamma_\mu \gamma_5 \psi$, $\bar{\psi} \gamma_\mu \vec{D} \nu \psi$, $\bar{\psi} \gamma_\mu \vec{D} \nu \vec{D} \omega \vec{D} \rho \psi$
  Polarised case: $\bar{\psi} \gamma_\mu \gamma_5 \gamma_\sigma \psi$, $\bar{\psi} \sigma_{\mu\nu} \vec{D} \omega \psi$, $\bar{\psi} \gamma_\mu \gamma_5 \vec{D} \nu \vec{D} \omega \psi$, $\bar{\psi} \sigma_{\mu\nu} \vec{D} \omega \vec{D} \rho \vec{D} \lambda \psi$
- Expansion of the operators into the bases of irreducible representations of hypercubic group H(4)
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Symbolic calculation of $C_{\text{BORN}}^{(2)}$

Starting point:

Tree level Compton scattering amplitude $\mathcal{W}_{\mu\nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

$$
\mathcal{W}_{\mu\nu}(a, p, q) = \langle p | \hat{J}_\mu(q) \hat{J}_\nu^\dagger(q) | p \rangle \\
= \langle p | T_{\mu\nu} | p \rangle \\
= \sum_{m,n} C_{\mu\nu,\mu_1,\ldots,\mu_n}^m(aq) \langle p | \mathcal{O}(a)^m_{\mu_1,\ldots,\mu_n} | p \rangle.
$$

- Expansion into powers of $p_\nu (\leftrightarrow \vec{D}_\nu)$ up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^m_{\mu_1,\ldots,\mu_n}$ contributing to $\mathcal{W}_{\mu\nu}(a, p, q)$
- Unpolarised case: $\bar{\psi} \gamma^\mu \nu \psi, \bar{\psi} \gamma^\mu D_\nu \psi, \bar{\psi} D^\mu \nu \psi, \bar{\psi} \gamma^\mu D_\nu D_\omega \psi, \bar{\psi} \gamma^5 \gamma^\mu D_\nu \psi, \bar{\psi} \gamma^5 \gamma^\mu D_\nu D_\omega \psi$
- Polarised case: $\bar{\psi} \gamma^\mu \gamma^5 \psi, \bar{\psi} \sigma_{\mu\nu} \vec{D}_\omega \psi, \bar{\psi} \gamma^\mu \gamma_5 \gamma^\mu D_\nu \psi, \bar{\psi} \gamma^\mu \gamma_5 \gamma^\mu D_\nu D_\omega \psi, \bar{\psi} \sigma_{\mu\nu} \vec{D}_\omega \psi, \bar{\psi} \sigma_{\mu\nu} \vec{D}_\omega \vec{D}_\rho \vec{D}_{\lambda} \psi$
- Expansion of the operators into the bases of irreducible representations of hypercubic group $H(4)$
- Projection of the corresponding coefficients $\rightarrow$ Wilson coefficients
- This program has been carried out with *Mathematica* completely
## Symbolic calculation of \( C^{(2)}_{\text{BORN}} \)

Starting point:

Tree level Compton scattering amplitude \( \mathcal{W}_{\mu \nu}(a, p, q) \) with off-shell quark states of momentum \( p \) in Wilson’s formulation:

\[
\mathcal{W}_{\mu \nu}(a, p, q) = \langle p|\hat{J}_\mu(q)\hat{J}_\nu^+(q)|p\rangle \\
= \langle p|T_{\mu \nu}|p\rangle \\
= \sum_{m,n} C_{\mu \nu, \mu_1,...,\mu_n}^m(aq) \langle p|\mathcal{O}(a)^m_{\mu_1,...,\mu_n}|p\rangle.
\]

- Expansion into powers of \( p_{\mu} \leftrightarrow \hat{D}_{\nu} \) up to third order (restriction due to numerical simulations)
- Identification of the possible local operators \( \mathcal{O}(a)^m_{\mu_1,...,\mu_n} \) contributing to \( \mathcal{W}_{\mu \nu}(a, p, q) \)
- Unpolarised case:
  - \( \bar{\psi} \gamma_\mu \gamma_5 \psi \), \( \bar{\psi} \gamma_\mu \hat{D}_\nu \psi \), \( \bar{\psi} \hat{D}_\mu \hat{D}_\nu \psi \), \( \bar{\psi} \gamma_\mu \hat{D}_\nu \hat{D}_\omega \hat{D}_\rho \psi \)
- Polarised case:
  - \( \bar{\psi} \gamma_\mu \gamma_5 \psi \), \( \bar{\psi} \sigma_{\mu \nu} \hat{D}_\omega \psi \), \( \bar{\psi} \gamma_\mu \gamma_5 \hat{D}_\nu \hat{D}_\omega \psi \), \( \bar{\psi} \sigma_{\mu \nu} \hat{D}_\omega \hat{D}_\rho \hat{D}_\lambda \psi \)
- Expansion of the operators into the bases of irreducible representations of hypercubic group \( H(4) \)
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Symbolic calculation of $C^{(2)}_{\text{BORN}}$

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tree level Compton scattering amplitude $\mathcal{W}_{\mu\nu}(a, p, q)$ with off-shell quark states of momentum $p$ in Wilson’s formulation:

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\mathcal{W}_{\mu\nu}(a, p, q) = \langle p | \hat{J}_\mu(q) \hat{J}^{\dagger}_\nu(q) | p \rangle
= \langle p | T_{\mu\nu} | p \rangle
= \sum_{m, n} C_{\mu\nu, \mu_1, \ldots, \mu_n}^m(aq) \langle p | \mathcal{O}(a)^m_{\mu_1, \ldots, \mu_n} | p \rangle.
$$

- Expansion into powers of $p_\nu (\leftrightarrow \overrightarrow{D}_\nu)$ up to third order (restriction due to numerical simulations)
- Identification of the possible local operators $\mathcal{O}(a)^m_{\mu_1, \ldots, \mu_n}$ contributing to $\mathcal{W}_{\mu\nu}(a, p, q)$
- Unpolarised case: $\bar{\psi} \gamma_\mu \gamma_5 \psi$, $\bar{\psi} \gamma_\mu \gamma_5 \psi$, $\bar{\psi} \gamma_\mu \gamma_5 \psi$
- Polarised case: $\bar{\psi} \gamma_\mu \gamma_5 \psi$, $\bar{\psi} \gamma_\mu \gamma_5 \psi$
- Expansion of the operators into the bases of irreducible representations of hypercubic group H(4)
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Scattering amplitude in $0^{th}$ order

The Compton scattering amplitude can be given in general form:

$$
T_{\mu\nu}^{(0)}(a, q)/e_\gamma^2 = -a r \delta_{\mu\nu} \bar{\psi}_1 \psi - \frac{8 a r \cos(a q / 2)^2}{Q^2} \delta_{\mu\nu} \bar{\psi}_1 \psi + \\
\sum_\tau \frac{2 a r \cos(a q / 2)^2 \cos(a q_\tau)}{Q^2} \delta_{\mu\nu} \bar{\psi}_1 \psi + \\
\frac{8 a r^3 \sin(a q / 2) \sin(a q / 2)}{Q^2} \bar{\psi}_1 \psi - \\
\sum_\tau \frac{2 a r^3 \cos(a q_\tau) \sin(a q / 2) \sin(a q / 2)}{Q^2} \bar{\psi}_1 \psi + \\
\frac{2 a r \cos(a q / 2) \sin(a q / 2) \sin(a q_\mu)}{Q^2} \bar{\psi}_1 \psi + \\
\frac{2 a r \cos(a q / 2) \sin(a q / 2) \sin(a q_\nu)}{Q^2} \bar{\psi}_1 \psi + \\
\sum_{\tau, \sigma} \frac{2 i a \cos(a q / 2) \cos(a q / 2) \sin(a q_\sigma)}{Q^2} \bar{\psi} \gamma_\tau \gamma_5 \epsilon_{\mu\sigma\nu\tau} \psi ,
$$

with

$$
Q^2 = Q^2(a, q) = \sum_\tau \sin(a q_\tau)^2 + r^2 \left( \sum_\tau (1 - \cos(a q_\tau)) \right)^2 .
$$
Wilson coefficients: $0^{th}$ order

In order to save space we give the following results for the special momentum transfer $q = (f, f, f, f)$, $s = \sin f$, $c = \cos f$, $Q_f^2 = 4s^2 + 16(1 - c)^2$.

(If not shown explicitly we set $a = 1$ and $r = 1$.)

Diagonal part ($\mathcal{I}^{(0)}_{11}$):

| operator       | representation | Wilson coefficient             | $a$ expansion                  |
|----------------|----------------|--------------------------------|--------------------------------|
| $\bar{\psi}1\psi$ | $\tau_1^{(1)}$ | $-\frac{6(3-c)(1-c)}{Q_f^2}$ | $-\frac{3a}{2}(1 - \frac{5}{12}(af)^2)$ |

Off-diagonal part ($\mathcal{I}^{(0)}_{12}$):

| operator       | representation | Wilson coefficient             | $a$ expansion                  |
|----------------|----------------|--------------------------------|--------------------------------|
| $\bar{\psi}1\psi$ | $\tau_1^{(1)}$ | $\frac{2(3-c)(1-c)}{Q_f^2}$   | $\frac{a}{2}(1 - \frac{5}{12}(af)^2)$ |
| $\bar{\psi}\gamma_3\gamma_5\psi$, $-\bar{\psi}\gamma_4\gamma_5\psi$ | $\tau_4^{(4)}$ | $\frac{i(1+c)s}{Q_f^2}$       | $\frac{i}{27}(1 - \frac{13}{12}(af)^2)$ |
Wilson coefficients: $1^{st}$ order, diagonal part

The expansion up to one covariant derivative cannot be given in readable form. Here, I present the Wilson coefficients only.

\[ \mathcal{T}_{11}^{(1)}: \]

Defining the combinations

\[
\begin{align*}
    b_1 &= 4i(1 - c)(74 - 126c + 63c^2 - 9c^3)/Q_f^4 \\
    b_2 &= -4i(6 - 8c + 3c^2)s^2/Q_f^4 \\
    b_3 &= 4i(4 - 3c)s^2/Q_f^4 \\
    b_4 &= -4i(1 - c)(4 - 9c + 3c^2)/Q_f^4
\end{align*}
\]

| operator                                      | repr. | Wilson coeff. | a expansion          |
|-----------------------------------------------|-------|---------------|----------------------|
| $\frac{1}{2}(\mathcal{O}_{11} + \mathcal{O}_{22} + \mathcal{O}_{33} + \mathcal{O}_{44})$ | $\tau_1^{(1)}$ | $\frac{1}{2}(b_1 + b_3)$ | $\frac{i}{2f^2} (1 + \frac{29}{24}(af)^2)$ |
| $\frac{1}{2}(\mathcal{O}_{11} + \mathcal{O}_{22} - \mathcal{O}_{33} - \mathcal{O}_{44})$ | $\tau_1^{(3)}$ | $\frac{1}{2}(b_1 - b_4)$ | $\frac{15i}{16} (1 - \frac{4}{3}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{11} - \mathcal{O}_{22})$ | $\tau_1^{(3)}$ | $\frac{1}{2}(b_1 - b_4)$ | $\frac{15i}{16} (1 - \frac{4}{3}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{12} + \mathcal{O}_{21}), \frac{1}{\sqrt{2}}(\mathcal{O}_{13} + \mathcal{O}_{31})$ | $\tau_3^{(6)}$ | $\frac{1}{\sqrt{2}}(b_2 + b_3)$ | $\frac{ia}{8\sqrt{2}} (1 - \frac{4}{3}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{14} + \mathcal{O}_{41})$ | $\tau_3^{(6)}$ | $\frac{1}{\sqrt{2}}(b_2 + b_3)$ | $\frac{ia}{8\sqrt{2}} (1 - \frac{4}{3}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{23} + \mathcal{O}_{32}), \frac{1}{\sqrt{2}}(\mathcal{O}_{24} + \mathcal{O}_{42})$ | $\tau_3^{(6)}$ | $\sqrt{2}b_3$ | $\frac{i}{2\sqrt{2}f^2} (1 - \frac{1}{6}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{34} + \mathcal{O}_{43})$ | $\tau_3^{(6)}$ | $\sqrt{2}b_3$ | $\frac{i}{2\sqrt{2}f^2} (1 - \frac{1}{6}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{12} - \mathcal{O}_{21}), \frac{1}{\sqrt{2}}(\mathcal{O}_{13} - \mathcal{O}_{31})$ | $\tau_1^{(6)}$ | $\frac{1}{2}(b_2 - b_3)$ | $-\frac{i}{4f^2} (1 - \frac{5}{12}(af)^2)$ |
| $\frac{1}{\sqrt{2}}(\mathcal{O}_{14} - \mathcal{O}_{41})$ | $\tau_1^{(6)}$ | $\frac{1}{2}(b_2 - b_3)$ | $-\frac{i}{4f^2} (1 - \frac{5}{12}(af)^2)$ |

\[ (\mathcal{O}_{\mu\nu} = \bar{\psi} \gamma_{\mu} \overset{\leftrightarrow}{D}_{\nu} \psi) \]
Wilson coefficients: $1^{st}$ order, off-diagonal part

$T_{12}^{(1)}$:

We find the combinations

\begin{align*}
b_1 &= -4i(1 - c)(33 - 52c + 24c^2 - 3c^3)/Q_f^4 \\
b_2 &= 4i(7 - 9c + 3c^2)s^2/Q_f^4 \\
b_3 &= -4i(3 - 2c)s^2/Q_f^4 \\
b_4 &= 6i(1 - c)^2(1 + c)(2 - c)/Q_f^4 \\
b_5 &= 2i(1 - c)^2(4 - 9c + 3c^2)/Q_f^4 \\
b_6 &= 2i(1 - c)^2(1 + c)(4 - 3c)/Q_f^4 \\
b_7 &= -4(1 - c)(19 - 18c + 3c^2)s/Q_f^4 \\
b_8 &= 2(1 - c)(14 - 15c + 3c^2)s/Q_f^4 \\
b_9 &= 12(2 - c)s^3/Q_f^4 \\
b_{10} &= -4s^3/Q_f^4 \\
b_{11} &= -2(1 - c)(4 - 9c + 3c^2)s/Q_f^4 \\
b_{12} &= -2(4 - 3c)s^3/Q_f^4
\end{align*}
Wilson coefficients: 1\textsuperscript{st} order, off-diagonal part

We give some selected examples for the Wilson coefficients

| operator | repr. | Wilson coeff. | a expansion |
|----------|-------|---------------|-------------|
| $\frac{1}{2}(O_{11} + O_{22} + O_{33} + O_{44})$ | $\tau_1^{(1)}$ | $b_1 + b_5$ | $-\frac{i}{4f^2} (1 + \frac{25}{12} (af)^2)$ |
| $\frac{1}{2}(O_{11} + O_{22} - O_{33} - O_{44})$ | $\tau_1^{(3)}$ | $b_1 - b_5$ | $-\frac{i}{4f^2} (1 + \frac{19}{12} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{12} + O_{21})$ | $\tau_3^{(6)}$ | $\sqrt{2}b_2$ | $\frac{i}{2\sqrt{2f^2}} (1 - \frac{1}{6} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{13} + O_{31}), \frac{1}{\sqrt{2}}(O_{14} + O_{41})$ | $\tau_3^{(6)}$ | $\frac{1}{\sqrt{2}}(b_3 + b_4)$ | $\frac{i}{4\sqrt{2f^2}} (1 + \frac{1}{12} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{23} + O_{32}), \frac{1}{\sqrt{2}}(O_{24} + O_{42})$ | $\tau_3^{(6)}$ | $\frac{1}{\sqrt{2}}(b_3 + b_4)$ | $\frac{i}{4\sqrt{2f^2}} (1 + \frac{1}{12} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{34} + O_{43})$ | $\tau_3^{(6)}$ | $\sqrt{2}b_6$ | $\frac{i}{8\sqrt{2f^2}} (1 - \frac{3}{4} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{13} - O_{31}), \frac{1}{\sqrt{2}}(O_{14} - O_{41})$ | $\tau_1^{(6)}$ | $\frac{1}{\sqrt{2}}(b_3 - b_4)$ | $\frac{i}{4\sqrt{2f^2}} (1 - \frac{17}{12} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{23} - O_{32}), \frac{1}{\sqrt{2}}(O_{24} - O_{42})$ | $\tau_1^{(6)}$ | $\frac{1}{\sqrt{2}}(b_3 - b_4)$ | $\frac{i}{4\sqrt{2f^2}} (1 - \frac{17}{12} (af)^2)$ |
| $\sqrt{\frac{2}{3}}(O_{T1_{23}}^T + O_{T2_{13}}^T)$ | $\tau_2^{(8)}$ | $\sqrt{\frac{2}{3}}(2b_{10} + b_9)$ | $\frac{a}{2\sqrt{6f^2}} (1 - \frac{4}{3} (af)^2)$ |
| $\sqrt{\frac{2}{3}}(O_{T1_{34}}^T + O_{T2_{14}}^T)$ | $\tau_2^{(8)}$ | $\sqrt{6}b_{12}$ | $-\frac{\sqrt{3a}}{4\sqrt{2f}} (1 - \frac{3}{4} (af)^2)$ |
| $\sqrt{2}O_{T2_{13}}^T, \sqrt{2}O_{T2_{14}}^T$ | $\tau_2^{(8)}$ | $-\sqrt{6}b_9$ | $-\frac{3\sqrt{3a}}{2\sqrt{2f}} (1 - \frac{3}{4} (af)^2)$ |
| $\sqrt{2}O_{T3_{14}}^T, -\sqrt{2}O_{T3_{24}}^T$ | $\tau_2^{(8)}$ | $-\sqrt{2}b_{12}$ | $\frac{a}{4\sqrt{2f}} (1 - \frac{4}{3} (af)^2)$ |
| $-\frac{1}{\sqrt{6}}(O_{T1_{22}}^T + O_{T133} - 2O_{T144})$ | $\tau_1^{(8)}$ | $\sqrt{2}(b_{11} + b_7)$ | $-\frac{3a}{4\sqrt{2f}} (1 - \frac{4}{3} (af)^2)$ |
| $\frac{1}{\sqrt{2}}(O_{T2_{11}}^T + O_{T233} - 2O_{T244})$ | $\tau_1^{(8)}$ | $\sqrt{2}b_{11}$ | $\frac{a}{4\sqrt{2f}} (1 - \frac{4}{3} (af)^2)$ |
| $\frac{1}{6} \sum_{p \in \{1,2,3\}} \text{sgn}(p)O_{T123}^T$ | $\tau_4^{(4)}$ | $\sqrt{\frac{2}{3}}(b_9 - b_{10})$ | $\sqrt{\frac{2a}{3f}} (1 - \frac{4}{3} (af)^2)$ |
| $\frac{1}{\sqrt{3}}(O_{T122}^T + O_{T133} + O_{T144})$ | $\tau_1^{(4)}$ | $\frac{4}{\sqrt{3}}(4b_{11} - 2b_7)$ | $\frac{2\sqrt{3a}}{f} (1 - \frac{4}{3} (af)^2)$ |
| $\frac{1}{\sqrt{3}}(O_{T211}^T + O_{T233} + O_{T244})$ | $\tau_1^{(4)}$ | $-\frac{4}{\sqrt{3}}b_{11}$ | $-\frac{a}{2\sqrt{3f}} (1 - \frac{4}{3} (af)^2)$ |

\[
(O_{\mu\nu} = \bar{\psi}\gamma_\mu \not{D}_\nu \psi, O_{\mu\nu\omega} = \bar{\psi}\sigma_{\mu\nu} \not{D}_\omega \psi)
\]
Conclusions

- We have developed a program for expanding tree-level scattering amplitudes symbolically.
- We have found analytic expressions for Wilson coefficients corresponding to local operators up to third order in covariant derivatives.
- We have classified them according to irreducible representations.
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