Graph Drawing by Weighted Constraint Relaxation

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Abstract—A popular method of force-directed graph drawing is multidimensional scaling using graph-theoretic distances as input. We present an algorithm to minimize its energy function, known as stress, by using a relaxation method that considers a single pair of vertices at a time. Our results show that relaxation can reach lower stress levels faster and more consistently than majorization, without needing help from a good initialization. We then present various real-world applications to show how the unique properties of relaxation make it easier to produce constrained layouts than previous approaches. We also show how relaxation can be directly applied within the sparse stress approximation of Ortmann et al. [1], making the algorithm scalable up to large graphs.

Index Terms—Graph drawing, multidimensional scaling, constraints, relaxation

1 INTRODUCTION

Graphs are a common data structure, used to describe everything from social networks to food webs, from metabolic pathways to internet traffic. Any set of pairwise relationships between entities can be described by a graph, and the ever increasing amount of data being collected means that visualizing graphs for human analysis has become an important task.

Node-link diagrams are an intuitive representation of graphs, where vertices are represented by dots, and edges by lines connecting them. A primary task is then to find suitable coordinates for these dots that present the data faithfully. However this is far from trivial, and the difficulty behind finding a good layout can be illustrated through a simple example. If we consider the problem of drawing a tetrahedron in 2D space, it is easy to see that no ideal layout exists where all edges have equal lengths. Even for such a small graph with only four vertices, there are too few dimensions available to provide sufficient degrees of freedom. The next logical question is: what layout gets as close as possible to this ideal?

Multidimensional scaling (MDS) is a technique to solve exactly this type of problem, that attempts to minimize the disparity between ideal and low-dimensional distances. This is done by defining an equation to quantify the error in a layout, and then minimizing it. While this equation comes exactly this type of problem, that attempts to minimize

A weighting factor $w$ is also used to either emphasize or dampen the importance of certain pairs. Within the context of graph layout, the most common approach for the constants is to set $d_{ij}$ to the shortest path distance between vertices $i$ and $j$, with $w_{ij} = d_{ij}^2$ to offset the extra weight given to longer paths due to squaring the difference [3].

This definition was first used in the context of general graph layout by Kamada and Kawai [4], who minimized the function using a localized 2D Newton-Raphson process. Within MDS, Kruskal [5] originally used gradient descent, later improved upon by De Leeuw [6] with a method known as majorization. This method minimizes a complicated function by iteratively finding the true minima of a series of simpler functions, each of which touches the original function and is an upper bound for it [2]. This was applied to graph layout by Gansner et al. [7] and is still the state-of-the-art method used when minimizing the full stress equation. A review of approximation methods used for large graphs is described, along with a brief comparison with majorization; some real-world applications are shown that make use of the unique properties of relaxation; a method of making relaxation scalable up to large graphs is described, by adapting the sparse approximation of Ortmann et al. [1]; and finally, we end with a discussion of the results and ideas for future work.

This paper describes a method of minimizing stress by using a relaxation method, here defined as an iterative process that considers the stress between a single pair of vertices at a time. This is similar to relaxation in the context of Jacobi or Gauss-Seidel [8], but instead of solving for one scalar variable at a time, we solve for two vectors.

The structure of this paper is as follows: the algorithm is described, along with a brief comparison with majorization; some real-world applications are shown that make use of the unique properties of relaxation; a method of making relaxation scalable up to large graphs is described, by adapting the sparse approximation of Ortmann et al. [1]; and finally, we end with a discussion of the results and ideas for future work.
 feasibly model through a full physics simulation, which would represent each edge as a stiff spring, summing up and integrating over the resulting forces. To avoid this bottleneck, Jakobsen [9] introduced the idea of considering each edge independently, moving a single pair of vertices at a time. While this is a rather simple and perhaps naive idea, in practice the solution converges in very few iterations.

This was expanded upon by Dwyer [10], who used the method in conjunction with a force-directed layout to achieve effects such as making edges point downwards, or fixing cyclic paths around the edge of a wheel. To define the method in conjunction with a force-directed layout to achieve effects such as making edges point downwards, or fixing cyclic paths around the edge of a wheel. To define each edge independently, moving a single pair of vertices at a time. While this is a rather simple and perhaps naive idea, in practice the solution converges in very few iterations.

Our other source of inspiration comes from the force-directed graph layout algorithm of Fruchterman and Reingold [11], who combine gradient descent with a cooling schedule that descends rapidly at first to better escape local minima, and then gradually reduces the ‘temperature’ by descending by smaller and smaller amounts. The corresponding effect for us is to start by overshooting, and then gradually shift towards undershooting until the constraints are only moved a little on each satisfaction.

The last step is to notice that our previous comparison between Equations (1) and (2) leaves out the weighting factor $w_{ij}$. This results in an overall weighting of:

$$\omega = w_{ij} c, \quad c \to 0$$

where $w_{ij}$ is taken from (1), and $c$ comes from the cooling schedule, which should tend toward 0 over the course of the iteration process. This ensures that as $c$ decreases, vertices are moved increasingly smaller amounts towards an eventual point of convergence, with $w_{ij}$ balancing the relative strengths of each constraint. Plots of stress compared to majorization are presented in Figure 2, and pseudocode can be seen in Algorithm 1. Graph data used for timing experiments is from the SuiteSparse Matrix Collection [12].

An important consideration is the order in which constraints are satisfied, which should be randomized on each iteration. Otherwise, the relaxation becomes biased towards the order of satisfaction, and fails to find good minima. Unfortunately this can have a negative effect on performance due to reduced data cache prefetching. Full timing results comparing relaxation with multiple implementations of majorization can be seen in Figure 11.

One last thing to note is that the dimensionality of initialization determines the dimensionality of the output. For example, if we initialize along a line, then because the satisfaction step moves pairs of vertices along the path between them, they will only ever be distributed on this one dimension. This means that scaling to a higher number of dimensions is as simple as initializing within that space.

2.1 Weighted Relaxation

The original relaxation method’s similarity to the Gauss-Seidel method is the inspiration behind our work. A variant of Gauss-Seidel is successive over/under-relaxation (SOR), where the variables over/undershoot towards their target values in each iteration. Overshooting is used to speed up convergence, while undershooting can be used to aid convergence in diverging systems [8].

Our method is simple. Instead of satisfying constraints by moving by a vector $m$, we instead move by:

$$m' = \omega m, \quad 0 < \omega < 2$$

An intuitive reason for the bounds is that if a vertex overshoots towards a destination, as long as it does not overshoot by more than double, it will still end up closer to its target than before. The relaxation factor in SOR similarly should never exceed 2 to guarantee convergence [8].

Looking again at Equation (1), it can be seen that if every term in the summation is satisfied as a constraint (2), then the stress in (1) is zero, corresponding to an ideal layout. This is exactly how our method works; we abandon the force-directed component, and instead place a constraint on every possible pair of vertices, satisfying them one by one as above. However this is clearly almost always impossible, for the same reasons that the aforementioned tetrahedron cannot be embedded in 2D. In such situations, just satisfying constraints as in Figure 1 does not lead to convergence, but we will now describe an extension of this relaxation that does.

Fig. 1. Satisfaction of the distance constraint described by Equation (2). Adapted from [10].
Algorithm 1: Weighted Constraint Relaxation

1. **WCR** \((G)\):
2. \textbf{inputs:} graph \(G = (V, E)\)
3. \textbf{output:} \(k\)-dimensional layout \(X\) with \(n\) vertices
4. \(d_{(i,j)} \leftarrow \text{ShortestPaths}(G)\)
5. \(X \leftarrow \text{RandomMatrix}(n, k)\)
6. \(c \leftarrow 1000\)
7. \textbf{for} 20 iterations:
8. \textbf{foreach} \(\{i, j: j < i\}\) \textbf{in random order}:
9. \(\omega \leftarrow w_{ij} c\)
10. \textbf{if} \(\omega > 2\):
11. \(\omega \leftarrow 2\)
12. \(m \leftarrow d_{ij} - ||X_i - X_j||\)
13. \(X_i \leftarrow X_i + \omega m\)
14. \(X_j \leftarrow X_j - \omega m\)
15. \(c \leftarrow c/2\)

Fig. 3. Pseudocode for the relaxation process described in Section 2.1. The values shown for \(c\) were used to produce Figure 2, with vertex positions initialized uniformly randomly within a \(1 \times 1\) square on line 3.

3. **APPLICATIONS**

Some of the properties of relaxation, in particular the fact that each edge is considered separately along with the ability to consistently avoid local minima well, make relaxation well suited to finding additionally constrained graph layouts. We will now describe some recipes for examples of this, each applied to various real-world graphs in order to show the merits of their use.

3.1 **Focusing on a Vertex**

It is often the case that a user will want to examine specific vertices in a graph, especially in an interactive setting. It is therefore important to be able to emphasize distances involving certain vertices. Brandes and Pich [13] presented a general method of doing this in the context of majorization, by interpolating between two stress summations representing general and constrained weights separately.

For relaxation, emphasizing specific distances is as simple as weighting their corresponding constraints more heavily. For example, to focus on vertex number 3, we would change lines 8 and 9 in Algorithm 1 to ensure that \(\omega\) also never drops below 1 if \(i == 3\) or \(j == 3\). This causes only the remaining constraints to continue cooling towards zero, but the system still converges in this case as there are no conflicts between the ones emphasized. These emphasized distances can also be modified from their graph-theoretic values assigned on line 2 if a specific separation is desired, for example to constrain in a circle using the distances introduced by Dwyer [10].

3.2 **Sugiyama Layout**

A common class of graph is the Directed Acyclic Graph (DAG), where there are no cycles and so all edges can be drawn to point in the same direction. A popular representation of a DAG is the Sugiyama-style layout, where vertices are arranged in layers such that all edges point downwards.

Sorting vertices into layers is usually a deterministic process, with the examples shown in Figure 5 placed into layers at a height proportional to their longest path distance to any leaf node. This minimizes the height of the graph, and is done in linear time using topological sorting [15]. A further benefit to this style of layout is therefore that the longest path to the bottom for any vertex is immediately apparent from its height.

Finding the horizontal position of vertices is then usually a more difficult process, but here we use relaxation to find horizontal placements that minimize stress. In the context of Algorithm 1, this was done by initializing vertices at the desired height on line 3, and then only changing \(x\)-coordinates when satisfying constraints on lines 11 and 12. The same concept could encode other data too; for instance, a paper citation network could place vertices at a height proportional to their publication date.
3.3 Unidimensional Scaling

Unidimensional scaling (UDS) involves the scaling of distances to a single dimension. This is usually difficult due to the 1D stress function being “plagued with a plethora of local minima” [2]. Hubert et al. [17] discussed a method of performing UDS based on a quadratic programming problem, and also developed the idea of circular UDS, which projects points around a circle instead of along a line by using the minimum distance either clockwise or anti-clockwise around the circumference.

The concept of relaxation is flexible enough to perform circular UDS, by introducing constraints on the angles between vertices rather than euclidean distance. The only difference is that instead of moving in opposite directions along the line connecting them, we rotate them both by the smallest amount required to reach a desired angle. Figure 6 shows an example of this applied only to the leaves of a phylogenetic tree, where the ordering of the leaves around the circle is determined from circular UDS, with remaining vertices laid out using normal MDS. Since angles are bounded between 0 and π, the target angle between vertices is set to:

\[ \theta_{ij} = \pi \sqrt{d_{ij}/d_{\text{max}}} \]  

Looking closely at Figure 6 also reveals imperfections in the ordering, shown by a small number of edge crossings. This means that the result achieved is not ideal, but such an order is actually a solution to the traveling salesman problem, which is known to be a difficult problem to solve optimally [18].

3.4 Color as a Dimensional Space

Highly connected and small-world graphs, such as social networks, can often produce dense, entangled layouts colloquially termed ‘hairballs’. In this case, it is often useful to try to uncover some other form of information, such as revealing clusters of similar vertices. Since color is simply a linear mix of red, green, and blue, it can be thought of as a three-dimensional space which MDS can scale down to. Figure 7 shows an example of vertices colored by their Jaccard similarity index, defined as:

\[ d_{ij} = 1 - \frac{|N(i) \cap N(j)|}{|N(i) \cup N(j)|} \]

where \(N(i)\) are the neighbors of vertex \(i\). Since \(d_{ij}\) is bounded between 0 and 1, embedded distances fit perfectly within the similarly bounded axes of color.

This process can help to reveal groupings, but can also produce ambiguity when applied to larger graphs due to the lack of distinct color combinations, again a problem caused by a lack of output dimensions. One possibility in this case would be to use an interactive form of visualization in which the user selects a smaller group of vertices at a time, and the algorithm ‘colors in’ their selection by only considering dissimilarities between vertices in the group.

4 Sparse Approximation

To understand how many layout algorithms tackle scaling to larger graphs, it is convenient to rewrite Equation (1) by...
splitting the summation into two parts: paths that traverse one edge, and paths that traverse multiple:

\[
\text{stress}(X) = \sum_{\{i,j\} \in E} w_{ij} (||X_i - X_j|| - d_{ij})^2 + \sum_{\{i,j\} \notin E} w_{ij} (||X_i - X_j|| - d_{ij})^2 \tag{7}
\]

where \( E \) is the set of edges in the graph. Just considering the preprocessing stage for now, it is clear that we can easily compute \( d \) and \( w \) for the first half of the summation directly from the graph. Real-world graphs are also usually sparse, so for a graph with \( n \) vertices and \( m \) edges, \( m \ll n^2 \) making the space required to store these values tolerable. However the second half is not so easy—an all-pairs shortest paths (APSP) calculation takes \( O(mn + n^2) \) time per vertex for an unweighted graph with a breadth-first search, or \( O((m + n \log n) \log n) \) for a weighted graph using Dijkstra's algorithm [20]. Combined with requiring \( O(n^2) \) space to store all the values of \( d_{ij} \), this makes the preprocessing stage alone intractable for large graphs.

The second stage is iteration, where the layout is gradually improved towards a good minimum. Again, computing the first summation is tolerable, but the number of longer distance contributions quickly grows out of control. Many notable attempts have been made at tackling this second half. A common approach is to ignore \( d_{ij} \), and to approximate the summation as an \( n \)-body repulsion problem, which can be efficiently well approximated using \( k \)-d trees [21]. Hu [22] and independently Hachul and Jünger [23] used this in the context of Fruchterman and Reingold's force-directed model, along with a multilevel coarsening scheme to help avoid local minima. Gansner et al. [24] use it with majorization by summing over \(-\alpha \log ||X_i - X_j||\) instead. Brandes and Pich [25] even ignore the second half completely and capture the long-range structure by first initializing with a fast approximation to classical MDS, which minimizes the inner product rather than Euclidean distance.

There are a couple of issues with this idea, one being that treating all long-range forces equally is unfaithful to the original stress model, and another being that the relative strength of these forces depends on an extra parameter that can strongly affect the final layout of the graph. Keeping these dependent on their graph-theoretic distance sidesteps both of these issues, but brings back the problem of computing and storing shortest paths. One approach to maintaining this dependence comes from Khoury et al. [26], who use a low-rank approximation of the distance matrix based on its singular value decomposition. This can work extremely well, especially on graphs that the method we will describe struggles with, but still requires APSP unless \( w_{ij} = 1/d_{ij} \). The approach we use is that of Ortmann et al. [1], who pick a set of pivots whose shortest paths are used as an approximation for the shortest paths of vertices close to them. Since this approach actually reduces the number of terms in the summation, using it in the context of relaxation also reduces the number of constraints satisfied per iteration.

Our implementation uses max/min random sp to select pivots. Non-random max/min sp starts by picking one or more pivots randomly and computing their shortest paths to all other vertices. Subsequent pivots are chosen by picking the vertex with the maximum shortest path to any pivots chosen so far [27]. The random extension instead samples vertices with a probability proportional to this shortest path to any pivot, rather than simply always picking the maximum.

These pivots \( p \in P \) are then each assigned a region \( R(p) \), which is the set of vertices closer to that pivot than any other. The weights \( w_{ij} \) are then adapted depending on the composition of the region, resulting in a new stress equation, with \( \Delta_{ij} = (||X_i - X_j|| - d_{ij}) \):

\[
\text{stress}(X) = \sum_{\{i,j\} \in E} w_{ij} \Delta_{ij}^2 + \sum_{i \in V} \sum_{p \in P \setminus N(i)} w'_p \Delta_{ij}^2 \tag{8}
\]

where \( N(i) \) are the neighbors of \( i \) to prevent overlap with any edges in the first summation. The adapted weight \( w'_p \) is then set to \( s_p w_{ip} \), where \( s_p \) is the number of vertices in \( R(p) \) at least as close to \( p \) as to \( i \):

\[
s_p = |\{ j \in R(p) : d_{jp} \leq d_{pi}/2 \}| \tag{9}
\]

The reason the weight on vertex \( i \) is increased like this is because its contribution acts as an approximation for the stress to all vertices in \( R(p) \), and Equation (9) is required to prevent the weight on closer vertices from being overestimated. This means that \( w'_p \) is not necessarily equal to \( w_{ip} \) and this asymmetry must be reflected in the calculations. Pseudocode for this can be seen in Algorithm 2, and some results can be seen in Figures 9 and 10.

### Algorithm 2: Sparse Relaxation

```
1 SparseWCR \((G, h)\):
inputs: graph \( G = (V, E) \), number of pivots \( h \)
output: \( k \)-dimensional layout \( X \) with \( n \) vertices
2 \( P \leftarrow \text{MaxMinRandomSP}(G, h) \)
3 \( d_{p,i,j} \leftarrow \text{SparseShortestPaths}(G, P) \)
4 foreach \( \{p, i : p \notin N(i)\} \in (P \times V) \):
5 \( s \leftarrow |\{ j \in R(p) : d_{pj} \leq d_{pi}/2 \}| \)
6 \( w'_{pi} \leftarrow w_{pi} \)
7 \( w'_{ip} \leftarrow s w_{pi} \)
8 foreach \( \{i, j\} \in E \):
9 \( w'_{ij} \leftarrow w_{ji} \leftarrow w_{ij} \)
10 \( X \leftarrow \text{RandomMatrix}(n, k) \)
11 \( c \leftarrow 1000 \)
for 30 iterations:
13 foreach \( \{i, j\} \in E \cup (V \times P) \) in random order:
14 \( \omega_i \leftarrow \text{Min}(w'_{ij} c, 2) \)
15 \( \omega_j \leftarrow \text{Min}(w'_{ji} c, 2) \)
16 \( m \leftarrow \frac{d_{ij} - ||X_i - X_j||}{2} \)
17 \( X_i \leftarrow X_i + \omega_i m \)
18 \( X_j \leftarrow X_j - \omega_j m \)
19 \( c \leftarrow 0.75 c \)
```

Fig. 8. Pseudocode for sparse relaxation. The values shown for \( c \) were used for Figures 9 and 10. The value 0.75 on line 19 was chosen to give a slower rate of cooling, which is usually necessary in the sparse case. Note that all \( R(p) \) and \( w'_{ip} \) can be constructed over the course of shortest path calculations without increasing the asymptotic complexity.
Fig. 9. Examples of sparse relaxation on the graphs 3elt, EVA, USPowerGrid. From left to right: 10 pivots, 50 pivots, 200 pivots, full stress. EVA clearly causes the most problems, likely due to its low diameter and high degree distribution. Data from [12].

Fig. 10. Some larger graphs, each approximated with 200 pivots. From left to right, top row then bottom: 3elt (11,738 vertices), bcsstk_31 (35,588), commanche_dual (7,920, weighted) and its original layout, finance256 (37,376), bcsstk_32 (44,609), luxembourg_osm (114,599, weighted) and its original layout. For luxembourg_osm, the variable $c$ in Algorithm 2 was initialized to 1,000,000 on line 11 and run for 100 iterations, which was required due to the large variance in weighted path lengths. Data from [12].
5 Discussion

We have presented a method of performing MDS, shown within the context of graph layout, and have described various extensions that permit both easy constrained layouts and scalability up to large graphs. A clear question at this point is whether or not relaxation should be used instead of the current state-of-the-art. With this in mind, we will now discuss the benefits and downsides of weighted relaxation compared to majorization.

5.1 Quality

Figures 2 and 11 show that relaxation reaches the same low stress levels on every run. While majorization is proven to monotonically decrease stress [7], it can often struggle with local minima and only move tiny amounts per iteration. This can be seen through the larger variance in its stress trajectories from different starting configurations. Perhaps the most important benefit of relaxation is its consistency regardless of initialization, despite being non-deterministic due to the randomized order of satisfaction.

The fact that constraints selfishly ignore other constraints when satisfying themselves is what we believe lets relaxation more easily avoid local minima; collective stress is never on the algorithm’s agenda, but unintentional collaboration imposed by the cooling schedule ends up being the best compromise for the summation. Overshooting in early iterations also likely contributes to avoiding these pitfalls, even causing stress to rise at first. This means that smart initialization is unnecessary, as any starting configuration, good or bad, is immediately clobbered.

5.2 Speed

Our results show that relaxation converges to low stress levels faster than majorization. In fact, the number of iterations can be deterministic from the cooling schedule, so the running time can be predicted and even tuned to match a desired accuracy. This does mean that extra thought needs to go into the composition of this schedule, but all of the results presented here use a simple exponential decay, which was sufficient for the variety of graphs we tested on. On easier, more mesh-like structures the temperature can be cooled even faster because the global minimum is more obvious. A clear area for potential improvement is therefore a method of automatically calculating a good cooling rate, similar to the adaptive heuristic of Hu [22] used in the context of force-directed layout.

The real-world time per iteration must also be considered, even though both share a complexity $O(n^2)$. We adapted the Cholesky factorization routine from Numerical Recipes [28] to C#, and found that iterations are around three times as fast as relaxation. However the initial decomposition before back-substitution requires “$N^3/6$ executions of the inner loop (consisting of one multiply and one subtract)” [28], so the total time quickly tips in favor of relaxation. Conjugate gradient (CG), with tolerance 0.1 and max iterations 10 as in [24], was always slower than Cholesky over 100 iterations, but on easier graphs that many iterations is rarely necessary. CG also benefits from optimized matrix multiplication routines [7] which we did not try here. Localized majorization, which is used to majorize the sparse model in Section 4, iterates faster than CG but converges slower. It is also worth noting that over-relaxation has been used before in the context of majorization to achieve an average of 1.5 times speedup [29]. Full timing results can be seen in Figure 11.

The slow-down due to the hit in cache performance caused by randomizing the order of satisfaction has proven difficult to alleviate. We attempted to address this by dividing the constraints into blocks that often hit the same indices, and randomly iterating between and then within blocks. This resulted in a noticeable increase in iteration speed, but was offset by a reduced ability to find global minima, resulting in an increased number of iterations necessary and no overall improvement in performance.

5.3 Applicability

One of the major reasons why previous force-directed algorithms, such as in [10], [11], [22], have become popular is how simple and intuitive the concept is. The idea of a physical system pushing and pulling vertices, modeled as sets of springs and electrical forces, makes them easy to understand and quick to implement for practical use.

Relaxation shares these qualities, as the concept of moving pairs of vertices one by one towards an ideal distance is just as simple. The implementation also requires no equation solver, and there is no need to consider smart initialization, which can often be just as complex a task [3]. It is also reliable, as shown by the consistency in the stress levels reached, which could be crucial for real-time applications with hard, fixed limits on computation time. Considering only a single pair of vertices at a time also makes further constrained layouts easy to implement, and allows an appropriate sparse approximation to grant scalability up to large graphs.

The method can of course also be used for normal MDS in contexts other than graph layout, and could have the potential to be of further use in more general optimization problems. A more rigorous theoretical analysis of its convergence properties could therefore be a fruitful avenue of exploration.

6 Conclusion

In this paper we have presented a new set of algorithms based on a relaxation method to minimize stress as defined by Equation (1). An investigation comparing the method to majorization shows consistently faster convergence to lower stress levels, and the fact that only a single pair of vertices is considered at a time makes it well suited for the pivot-based approximation of Ortmann et al. [1]. This improved performance—combined with a simplicity that forgoes an equation solver or smart initialization, and the flexibility to make constrained layout easy—makes weighted relaxation a strong candidate for general graph layout applications.

Code used for timing experiments, along with some example Jupyter notebooks, is open source and available at www.github.com/jxz12/wcr.
Fig. 11. Graphs of stress against time for all three implementations of majorization, and relaxation. 10 runs were used for each graph, and the line plots run through the average stress and time per iteration. Graphs were considered as unweighted, and layouts were initialized randomly within a 1×1 square. The graph btree9 is a binary tree of depth 9. All tests were performed using C# running in Visual Studio, on an Intel Core i7-4790 CPU with 16GB of RAM. The cooling schedule in Algorithm 1 is used for the left column, while the right contains more mesh-like graphs with a quicker cooling of $c \leftarrow c/5$ after each iteration instead. The layouts show examples of what the graphs look like at the time when relaxation finishes, in which majorization completed around twice as many iterations on average. The top layout for each plot uses conjugate gradient, and the bottom uses relaxation. Note that the sharp spike in the majorized dwt_2680 is a side-effect of fixing the position of vertex with index 0, which is a requirement for conjugate gradient and Cholesky factorization. Data from [12].
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