EXPANSIVE MAPS ARE ISOMETRIES
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Abstract. In this short note we prove that any map $f$ from a dense subset $Y$ of a compact metric space $X$ into $X$ that does not decrease the distance is an isometry.

1. Preliminaries

It is a folklore result (but see [2]) that any map $f$ from a compact metric space $X$ to itself that does not decrease the distance ($d(f(x), f(y)) \geq d(x, y)$) – also called an expansive map, is in fact an isometry. After giving a proof of this well known statement we inquired in what measure the hypothesis $X$ complete is necessary, and we saw that in fact it is true for totally bounded metric spaces. A further modification of the proof works with maps defined on a dense subset of a totally bounded metric space. We are aware that the result may well be known, with a published but possibly different proof. Feedback would be appreciated.

2. Some definitions

Let $(X, d)$ a metric space. We say that $X$ is totally bounded if for every $\epsilon > 0$ there exists a covering of $Y$ with finitely many subsets of diameter $\leq \epsilon$. It is well known that a metric space is totally bounded if and only if it is a dense subset in a compact metric space (see [1]).

Let $X$ a totally bounded metric space. Let $\epsilon > 0$. There exists a covering of $X$ with $m_\epsilon$ subsets of diameter $\leq \epsilon$. Hence there exists at most $m_\epsilon$ points of $X$ with pairwise distances $> \epsilon$. Let’s define $n_\epsilon = n_\epsilon(X)$ to be the largest size of a subset of $X$ such that the distance between any two points is $> \epsilon$. It is easy to see that if $Y \subset X$ then $n_\epsilon(Y) \leq n_\epsilon(X)$ with equality if $Y$ is dense in $X$. Let’s define

$$N_\epsilon(X) = \{(x_1, \ldots, x_n) | x_i \in X, d(x_i, x_j) > \epsilon \text{ for all } i \neq j\}$$

(1)

Let’s call the elements of $N_\epsilon(X)$ $\epsilon$-nets of $X$. It is clear that for every $\epsilon$-net $x = (x_1, \ldots, x_n)$ of $X$ and $y$ in $X$ there exists $i$ so that $d(y, x_i) \leq \epsilon$.

For an $\epsilon$-net $x = (x_1, \ldots, x_n)$ define its gauge $G(x)$ to be

$$G(x) = \prod_{i<j} d(x_i, x_j)$$

(2)

the gauge $G(x)$ being a measure of the spread of $x$. It it clear that the function $G: N_\epsilon(X) \rightarrow (0, \infty)$ is bounded above by $\text{diam}(X)^n\epsilon$. We define $g_\epsilon(X)$ to be the supremum of $G$ on $N_\epsilon(X)$. Again, $Y \subset X$ implies $g_\epsilon(Y) \leq g_\epsilon(X)$ with equality if $Y$ is dense in $X$. 

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3. Statement and proof of the main result

Let $X$ a totally bounded metric space, $Y$ a dense subset of $X$ and $f: Y \to X$ such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y$ in $Y$. Then $f$ is an isometry, that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y$ in $Y$.

Proof: Let $\epsilon > 0$. There exists $y \in N_{\epsilon}(Y)$ so that

\[ G(y) > \frac{1}{1 + \epsilon} g_{\epsilon}(X) \]

We conclude that $f(y) \in N_{\epsilon}(X)$ and moreover

\[ d(x_i, x_j) \leq d(f(x_i), f(x_j)) < (1 + \epsilon)d(x_i, x_j) \quad (3) \]

Let now $y, z$ be in $Y$. There exist $i, j$ so that

\[ d(f(y), f(x_i)) \leq \epsilon \]
\[ d(f(z), f(x_j)) \leq \epsilon \]

We conclude that

\[ d(f(y), f(z)) \leq d(f(x_i), f(x_j)) + 2\epsilon \quad (4) \]

Now, we also have

\[ d(y, x_i) \leq \epsilon \]
\[ d(z, x_j) \leq \epsilon \]

and so $d(x_i, x_j) \leq d(y, z) + 2\epsilon$.

From the above we conclude

\[ d(f(y), f(z)) \leq d(f(x_i), f(x_j)) + 2\epsilon < (1 + \epsilon)d(x_i, x_j) + 2\epsilon \leq (1 + \epsilon)(d(y, z) + 2\epsilon) + 2\epsilon \]

Since $\epsilon > 0$ was arbitrary we conclude $d(f(y), f(z)) \leq d(y, z)$.

References

[1] John L. Kelley General Topology D. Van Nostrand Company Inc. 1955
[2] Freudenthal, H., and Hurewicz, Witold. Dehnungen, Verkrümmungen, Isometrien Fundamenta Mathematicae 26.1 (1936): 120-122.