Dynamics of Mobile Manipulators using Dual Quaternion Algebra

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ABSTRACT

This paper presents two approaches to obtain the dynamical equations of mobile manipulators using dual quaternion algebra. The first one is based on a general recursive Newton-Euler formulation and uses twists and wrenches, which are propagated through high-level algebraic operations and works for any type of joints and arbitrary parameterizations. The second approach is based on Gauss’s Principle of Least Constraint (GPLC) and includes arbitrary equality constraints. In addition to showing the connections of GPLC with Gibbs-Appell and Kane’s equations, we use it to model a nonholonomic mobile manipulator. Our current formulations are more general than their counterparts in the state of the art, although GPLC is more computationally expensive, and simulation results show that they are as accurate as the classic recursive Newton-Euler algorithm.

Keywords: Mobile Manipulator Dynamics, Dual Quaternion Algebra, Newton-Euler Model, Gauss’s Principle of Least Constraint, Euler-Lagrange Equations, Gibbs-Appell Equations, Kane’s Equations.

1 INTRODUCTION

In the last thirty years, there have been an expressible amount of papers dealing with different representations for robot modeling. Notorious examples can be found in the works of Featherstone [1–3], McCarthy [4–6], Selig [7,8], and Bayro-Corrochano [9], among many others.

One of the reasons for such investigations is that the complexity of a robotic system goes far beyond the complexity of the mechanism itself. A typical robotic system involves motion/force/impedance control, path planning, task planning, and many more higher-level layers. Therefore, representations that are very useful for robot modeling, such as homogeneous transformation matrices, not necessarily are easy to use when performing pose control or impedance control, for example [10].

This is precisely the reason why it is common to use homogeneous transformation matrices to obtain the robot kinematics but then indirectly find the geometric Jacobian and, finally, to use quaternions and position vectors to perform pose control in the task-space [11]. There are several drawbacks in using the aforementioned strategy. The mix of different representations unnecessarily complicates the overall representation and the mapping between those different representations usually introduces mathematical artifacts, such as algorithmic singularities and discontinuities.

In contrast, elements of dual quaternion algebra have strong geometrical meaning, such as in screw theory, and are also represented as coupled entities within single elements. In kinematics, this representation has been extensively explored to obtain the robot kinematics and differential kinematics [6,12–17]. Furthermore, in recent works, dual quaternions have been used to perform admittance control [18], which is fundamental in physical human-robot interaction; constrained motion control [19,20], which takes into account geometrical constraints imposed by the workspace; hybrid control, which takes into account the topology of the space of rigid motions [21] and optimal control, which uses a linear-quadratic optimal tracking controller for robotic manipulators [22], distributed pose formation control [23] and cooperative manipulation [24–25], including the ones that involve human-robot collaboration [26]; and to define high-level geometrical tasks [27].

Furthermore, elements such as unit dual quaternions and pure dual quaternions, when equipped with standard multiplication and addition operations, form Lie groups with associated Lie algebras. Therefore, a formulation based on dual quaternion algebra offers the geometrical insights of screw theory, the rigor of Lie Algebra, and a simple algebraic treatment of the dynamical model as in the spatial algebra [1], often reducing the necessity of an extensive geometric analysis of the mechanism, which contrasts with approaches based on the matrix representation of
screw theory [28, 29].

Some works have used dual quaternion algebra to describe rigid body dynamics over the last decades [30, 33], although not necessarily creating a general formalism for multibody system analysis. Among the works that have sought such formalism, most are based on three-dimensional dual vectors and demand some mapping to higher dimensional vectors to obtain the system dynamic equations [5, 34, 36], therefore losing the elegance and compactness of an analysis based only on dual quaternion algebra and, at times, incurring in abuses of notation [5] or demanding artificial swaps on the vectors [37] to deal with the mixing of representations. Other works focused on the propagation of dual quaternions [38] or the computational aspects of algorithms based on dual quaternion algebra [39], rather than on the algebraic and geometrical insights that the algebra provides when dealing with more complex robots (e.g., nonholonomic mobile manipulators) and more general types of joints (e.g., helical, cylindrical, 6-DoF, etc.).

In conclusion, since there is no method based on dual quaternion algebra that adequately encompasses the dynamic model of mobile manipulators and general types of joints, there is still a theoretical gap that creates an unnecessary need for intermediate mappings when using higher-level algorithms based on dual quaternions to connect them to the low-level dynamic model. The purpose of this paper is to fill that gap by proposing a suitable dynamic model of mobile manipulators with arbitrary joints using dual quaternion algebra.

### 1.1 Statement of contributions

This paper presents two approaches to obtain the dynamical equations of serial manipulators using dual quaternion algebra. The first one is based on the recursive Newton-Euler formulation and the second one applies the Gauss’s Principle of Least Constraint to obtain the dynamical model of a serial mobile manipulator subject to nonholonomic constraints. The contributions of this paper to the state-of-the-art are the following:

1. A systematic procedure to obtain the recursive equations for the dynamic model of mobile manipulators using dual quaternion algebra and the Newton-Euler formalism, which has linear cost on the number of links. This approach simplifies the classic procedure by removing the necessity of exhaustive geometrical analyses because wrenches and twists are propagated through high-level algebraic operations. Compared to previous works, our approach is more general because it works for arbitrary types of joints and we do not impose any particular parameterization convention for the propagation of twists;

2. A closed-form for the dynamic model of serial manipulators based on the Gauss’s Principle of Least Constraint (GPLC) and dual quaternion algebra. We impose additional constraints in the GPLC formulation to model nonholonomic mobile manipulators. We apply the fundamental equation proposed by Udwadia-Kalaba [40], which employs a simpler method, albeit equivalent, than Lagrange multipliers to enforce the equality constraints. In addition, we present the skew symmetry property related to the inertia and Coriolis matrices, which is paramount when designing passivity-based controllers. Finally, we show the connections of the Gauss’s Principle of Least Constraint with the Gibbs-Appell and Kane’s equations using our formulation based on dual quaternion algebra.

We validate the proposed algorithms in simulation using three different robots: a fixed-base 50-DoF serial manipulator, a 9-DoF holonomic mobile manipulator, and an 8-DoF nonholonomic mobile manipulator. Moreover, we compare our results with the ones provided by a realistic simulator, and with an implementation of the state of the art. Furthermore, we present the computational costs of the proposed methodologies.

This paper is organized as follows: Section 2 presents a brief mathematical background on dual quaternion algebra; Section 3 introduces a general Newton-Euler formulation based on dual quaternion algebra, whereas Section 4 introduces the dual quaternion formulation based on the Gauss’s Principle of Least Constraints; Section 5 presents both the validation of the proposed methodologies through simulations and their computational costs; finally, Section 6 gives the final remarks and points to further research directions.

### 2 MATHEMATICAL PRELIMINARIES

Dual quaternions \( \mathbb{H} \) are elements of the set

\[
\mathcal{H} = \{ h_P + \varepsilon h_D : h_P, h_D \in \mathbb{H}, \varepsilon \neq 0, \varepsilon^2 = 0 \};
\]

where

\[
\mathbb{H} = \{ h_1 + ih_2 + jh_3 + kh_4 : h_1, h_2, h_3, h_4 \in \mathbb{R} \}
\]

is the set of quaternions, in which \( i, j, k \) are imaginary units with the properties \( i^2 = j^2 = k^2 = ijk = -1 \) [41]. Addition and multiplication of dual quaternions are analogous to their counterparts of real and complex numbers. One must only respect the properties of the dual unit \( \varepsilon \) and imaginary units \( i, j, k \).

Given \( h \in \mathcal{H} \) such that

\[
h = h_P + \varepsilon h_D = h_1 + ih_2 + jh_3 + kh_4 + \varepsilon (h_5 + ih_6 + jh_7 + kh_8),
\]

the operators \( \mathcal{P}(h) = h_P \) and \( \mathcal{D}(h) = h_D \) provide the primary part and dual part of \( h \), respectively, whereas the operators \( \text{Re}(h) = h_1 + \varepsilon h_5 \) and \( \text{Im}(h) = ih_2 + jh_3 + kh_4 + \varepsilon (h_6 + ih_7 + jh_8) \)


The subset \( \mathcal{S} = \{ h \in H : \| h \| = 1 \} \) of unit dual quaternions is used to represent poses (position and orientation) in the three-dimensional space and form the group \( \text{Spin}(3) \times \mathbb{R}^3 \) under the multiplication operation.\(^1\) Any \( \bar{x} \in \mathcal{S} \) can always be written as \( \bar{x} = r + \varepsilon (1/2) p \) where \( p = i x + j y + k z \) represents the position \((x, y, z) \) in the three-dimensional space and \( r = \cos (\phi/2) + n \sin (\phi/2) \) represents a rotation, with \( \phi \in [0, 2\pi) \) the rotation angle around the rotation axis \( n \in \mathbb{H}_p \cap S^3 \), with \( \mathbb{H}_p \triangleq \{ h \in \mathbb{H} : \text{Re}(h) = 0 \} \) and \( S^3 = \{ h \in \mathbb{H} : \| h \| = 1 \} \) \(^2\).

Given the set \( \mathcal{H}_p = \{ h \in H : \text{Re}(h) = 0 \} \) of pure dual quaternions, which are used to represent twists and wrenches, the operator \( \text{Ad} : \mathcal{S} \times \mathcal{H}_p \to \mathcal{H}_p \) performs rigid motions on those entities. For instance, given a twist expressed in frame \( F_a \), namely \( \xi^a \in \mathcal{H}_p \), and the unit dual quaternion \( \bar{x}^a \) that gives the pose of \( F_b \) with respect to \( F_a \), the same twist is expressed in frame \( F_b \) as\(^3\):

\[
\xi^b = \text{Ad}(\bar{x}_a) \xi^a = \bar{x}_a^b \xi^a(\bar{x}_a^b)^*.
\] (3)

The time derivative of \( \bar{x}_a^b \) is given by \(^4\)

\[
\dot{\bar{x}}_a^b = \frac{1}{2} \xi^a_{ab} \dot{x}_a^b = \frac{1}{2} x_{ab}^b \dot{x}_a^b.
\] (4)

where

\[
\xi_{ab}^a = \omega_{ab}^a + \varepsilon (p_{ab}^a + p_{ab}^b \times \omega_{ab}^a)
\] (5)

is the twist of frame \( F_b \) with respect to frame \( F_a \), expressed in frame \( F_a \) with \( \omega_{ab}^a \in \mathbb{H}_p \) being the angular velocity, and

\[
\xi^b_{ab} = \text{Ad}(\bar{x}_a^b) \xi^a_{ab} = \omega_{ab}^b + \varepsilon p_{ab}^b
\] (6)

is the twist expressed in \( F_b \). Furthermore, \( \xi^a_{ab} \) and \( \xi^b_{ab} \) are elements of the Lie algebra associated with \( \text{Spin}(3) \times \mathbb{R}^3 \). Additionally,

\[
p \times \omega = \frac{p \omega - \omega p}{2},
\] (7)

\( p, \omega \in \mathbb{H}_p \), is the cross-product between pure quaternions, which is analogous to the cross product between vectors in \( \mathbb{R}^3 \) \(^4\).

The cross-product between \( l, s \in \mathbb{H}_p \), where \( l = l + s t \) and \( s = s + \varepsilon s' \), is analogous to \( (7) \) and given by

\[
l \times s \triangleq \frac{ls - sl}{2} = l \times s + \varepsilon (l \times s' + t' \times s).
\] (8)

**Lemma 2.1.** If \( \bar{x} \in \mathcal{S} \), such that \( \dot{\bar{x}} = (1/2) \xi \bar{x} \) and \( \xi' \in \mathcal{H}_p \), then

\[
\frac{d}{dt} \left( \text{Ad}(\bar{x}) \xi' \right) = \text{Ad}(\bar{x}) \xi' + \xi \times (\text{Ad}(\bar{x}) \xi').
\] (9)

**Proof.** Using \( (3), (4) \), and the fact that \( (\bar{x} \xi)^* = -\bar{x}^* \xi \), we obtain

\[
\frac{d}{dt} \left( \text{Ad}(\bar{x}) \xi' \right) = \dot{x} _{ab} \xi^a + \dot{x} _{ab} \xi^a + \xi \times \dot{x} _{ab} \xi^a
\]

\[
= \frac{1}{2} \xi \left( \bar{x} \xi \bar{x}^* \right) + \xi \bar{x} \bar{x}^* - \frac{1}{2} \left( \bar{x} \xi \bar{x}^* \right) \xi.
\] (10)

Finally, using \( (8) \) in \( (10) \) yields \( (9) \).

The quaternionic inertia tensor is defined as

\[
\mathbb{I}(i_x, i_y, i_z) \in \mathbb{H}_p \subset \mathbb{H}^n,
\] (11)

where \( i_x = I_{xx} \hat{i} + I_{xy} \hat{j} + I_{xz} \hat{k} \), \( i_y = I_{yx} \hat{i} + I_{yy} \hat{j} + I_{yz} \hat{k} \), and \( i_z = I_{zx} \hat{i} + I_{zy} \hat{j} + I_{zz} \hat{k} \), in which \( I_{n,n} \), with \( n \in \{ x, y, z \} \), are elements of the rigid body’s inertia tensor.

**Definition 2.2.** Given \( A = (a_x, a_y, a_z) \in \mathbb{H}_p^3 \) and \( b \in \mathbb{H}_p \), the operator \( L_3 : \mathbb{H}_p^3 \times \mathbb{H}_p \to \mathbb{H}_p \), is defined as

\[
L_3 (A) b = \hat{i}(a_x, b) + j(a_y, b) + k(a_z, b),
\] (12)

where \( \langle \cdot, \cdot \rangle : \mathbb{H}_p \to \mathbb{R} \) is the inner product between quaternions\(^4\) that is, given \( a, b \in \mathbb{H}_p \), then \( \langle a, b \rangle \triangleq -(ab + ba)/2 \).

\(^1\)The symbol \( \times \) represents the semi-direct product between groups \( [6, p. 22] \).

\(^2\)Notice that superscripts represent the original frame, and subscripts represent the modified frame. This convention of subscripts and superscripts is maintained throughout this paper. If no superscript is used, we assume the global inertial frame.

\(^3\)It is important to use three indices here because the twist between two frames can be seen from a third frame. So, for example, \( \xi^b_{ab} \) is the twist of frame \( F_b \) with respect to frame \( F_a \), expressed in frame \( F_c \). The same interpretation is used for wrenches, which will be properly introduced in Section 2.2.

\(^4\)The inner product in \( \mathbb{H}_p \) is equivalent to the inner product in \( \mathbb{R}^3 \).
From Definition 2.2 it follows that the angular momentum \( \ell \) in (dual) quaternion algebra is given by
\[
\ell = \mathcal{L}_0(\mathbb{I}) \omega,
\]
where \( \omega \in \mathbb{H}_p \) is the angular velocity. Direct calculation shows that (13) is equivalent to its counterpart in vector algebra.

Given the quaternionic inertia tensor \( I' \in \mathbb{H}_p^3 \) of a rigid body expressed in frame \( \mathcal{F}' \), and the rigid body’s angular velocity \( \omega \in \mathbb{H}_p \) expressed in frame \( \mathcal{F} \), the angular momentum expressed in frame \( \mathcal{F} \) is given by
\[
\ell = \text{Ad}(r^* ) \mathcal{L}_0(\mathbb{I}') \text{Ad}(r) \omega,
\]
where \( r \) is the rotation quaternion from \( \mathcal{F}' \) to \( \mathcal{F} \). Eq. (14) is analogous to the equation that one obtains when using similarity transformations of rotation matrices and vectors in \( \mathbb{R}^3 \).

3 DUAL QUATERNION NEWTON-EULER MODEL

This section presents the recurrence relations of the Newton-Euler model using dual quaternion algebra for mobile manipulators with arbitrary joints, assuming that the full kinematic model is available using dual quaternion representation [12].

For illustrative purposes, and without loss of generality, consider the mobile manipulator shown in Fig. 1 composed of an \( n_\ell \)-DoF serial manipulator attached to a 3-DoF mobile base. The goal is to find the wrenches \( \Gamma \in \mathbb{H}_p^n \) acting on the \( n = n_\ell + 1 \) centers of mass (CoM) of the robot’s mobile base and the \( n_\ell \) links, given the corresponding robot configuration, generalized velocities, and generalized accelerations. This can be seen as a function \( \mathcal{N} : \mathbb{R}^{n_\ell+3} \times \mathbb{R}^{n_\ell+3} \times \mathbb{R}^{n_\ell+3} \rightarrow \mathbb{H}_p^n \), where \( n_\ell + 3 \) is the dimension of the configuration space and \( n \) is the number of rigid bodies in the kinematic chain (e.g., the mobile base and links), such that
\[
\Gamma = \mathcal{N}(q, \dot{q}, \ddot{q}).
\]

3.1 Forward Recursion

The first process of the algorithm consists of a serial sweeping of the robot kinematic structure to calculate the twist of each CoM. The objective is to find the forward recurrence relations that will then be used to iteratively obtain the wrenches acting on the robot’s mobile base and joints.

5Henceforth, we will use the expression “twist of the CoM” as a shorthand for “the twist of the frame attached to the CoM.”
respectively. Again, when using the DH convention, \( \omega = \dot{\theta} \) for a revolute joint and \( v = \dot{\theta} \) for a prismatic joint. For helical joints, the constant \( h \in \mathbb{R} \) is called the pitch.

Moreover, \( \xi_{c_1,j_1} = 0 \) because \( \dot{\xi}_{j_1} = 0 \). Therefore,
\[
\xi_{0,c_2} = \text{Ad} (\mathbf{x}_{j_2}) \xi_{0,j_2} + \text{Ad} (\mathbf{x}_{j_1}) \xi_{j_1,c_2}.
\]

Furthermore, expanding \( \text{Ad} (\mathbf{x}_{j_2}) \xi_{j_1,c_2} \), we obtain
\[
\text{Ad} (\mathbf{x}_{j_2}) \xi_{j_1,c_2} = \omega_{j_2} + \varepsilon (\omega_{j_2} \times \mathbf{p}_{j_2}^{c_2} - \mathbf{p}_{j_2}^{c_2} \times \omega_{j_2}),
\]
where the linear velocity due to the application of an angular velocity in a point displaced from the CoM (i.e., at \( F_{j_1} \)) arises algebraically. Fig. 2 illustrates this phenomenon when a pure rotational joint is used (i.e., \( \xi_{j_1,c_1} = \omega_{j_1} n_{j_1,c_1} \), where \( n_1 \in \mathbb{R} \) is an arbitrary unit-norm rotation axis).

More generally, the twist in \( F_{c_i} \) that provides the motion of \( F_{c_i} \) with respect to \( F_0 \), which arises from the movement of the first \( i \) rigid bodies in the kinematic chain, is given by
\[
\xi_{c_i} = \xi_{c_i,0,j_1} + \xi_{0,j_1,c_i}
\]
(18)
where \( \xi_{c_i,0,j_1} \) is the twist related to the motion of the first \( i-1 \) rigid bodies and \( \xi_{0,j_1,c_i} \) is the twist related to the motion of the \( i \)th rigid body. Also, \( \xi_{c_i}^{j_0} = 0 \) for any \( a \).

Analyzing (16), (17), and (18) we find, by induction, the recurrence relation for the total twist of the \( i \)th CoM, which has the contribution of all rigid bodies up to the \( i \)th rigid body, expressed in \( F_{c_i} \), as
\[
\xi_{c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1}^{-1}) \xi_{c_i,0,j_1} + \xi_{c_i,j_1-1,c_i} + \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i},
\]
where \( c_0 = j_0 = 0 \), and \( \xi_{c_i-1,j_1} = 0 \) because \( \dot{\xi}_{j_1} = 0 \) for all \( i \). Therefore,
\[
\xi_{c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1}^{-1}) \xi_{c_i,0,j_1} + \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i}.
\]
(19)
Since the twist \( \xi_{c_i,j_1-1,c_i} \) is generated by the \( i \)th joint, its expression depends on which type the \( i \)th joint is (see Table 1). Similarly, twist \( \xi_{c_i,0,c_i} \) depends on which type the mobile base is (for holonomic mobile bases, for instance, it is equivalent to the one given by a planar joint). The transformation \( \mathbf{x}_{c_i,j_1-1} \) is calculated as \( \mathbf{x}_{c_i,j_1-1} = (\mathbf{p}_{c_i,j_1}^{-1}) \mathbf{x}_{c_i,j_1} \), where \( \mathbf{p}_{c_i,j_1} = \mathbf{p}_{c_i,j_1-1} \mathbf{x}_{c_i,j_1-1} \mathbf{x}_{c_i,j_1} \mathbf{x}_{c_i,j_1-1} \) with \( \mathbf{p}_{c_i,j_1} = 1 \), the transformation \( \xi_{j_1-1,c_i} \) is constant, and \( \xi_{j_1-1,c_i} \) is a function of the parameters of the \( (i-1) \)th joint (or the mobile base when \( i = 1 \)).

3.1.2 Time Derivative of the Twists

Taking the time derivative of (19), we use (9) to obtain
\[
\dot{\xi}_{c_i,j_1-1,c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \dot{\xi}_{c_i,j_1-1,c_i} + \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} + \xi_{j_1-1,c_i} \times \left( \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} \right).
\]
Since \( \xi_{c_i,j_1-1,c_i} = -\dot{\xi}_{c_i,j_1-1,c_i} \) then
\[
\xi_{c_i,j_1-1} = -\xi_{c_i,j_1-1,c_i},
\]
and
\[
\dot{\xi}_{c_i,j_1-1} = -\dot{\xi}_{c_i,j_1-1,c_i} = -\xi_{c_i,j_1-1,c_i} \times \xi_{c_i,j_1-1,c_i} = 0.
\]
Therefore,
\[
\dot{\xi}_{c_i,j_1-1,c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{c_i,j_1-1,c_i} + \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} + \xi_{j_1-1,c_i} \times \left( \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} \right),
\]
(20)
where \( \xi_{0,c_0} = 0 \). Also, since \( \xi_{j_1-1,c_i} = \xi_{c_i,j_1-1} + \xi_{j_1-1,c_i-1} \) and \( \xi_{j_1-1,c_i-1} = 0 \), then
\[
\xi_{c_i,j_1-1,c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{c_i,j_1-1,c_i} = -\text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{c_i,j_1-1,c_i}.
\]
(21)
As shown in Section 3.1.1, the twist \( \xi_{j_1-1,c_i} \) depends on the type of the \( i \)th joint and, therefore, so does the term \( \xi_{j_1-1,c_i} \). For instance, if the \( i \)th joint is revolute, then \( \xi_{j_1-1,c_i} = \omega_{j_1} \xi_{j_1} \). If it is prismatic, then \( \xi_{j_1-1,c_i} = \mathbf{v}_{j_1} \xi_{j_1} \). Analogously, if it is helical, then \( \xi_{j_1-1,c_i} = \xi_{j_1} \mathbf{f}_{j_1} \mathbf{f}_{j_1} \). The same reasoning applies to the twist \( \xi_{0,c_0} \) of the mobile base.

Remark 1. Although (20) is written in recursive form, we can always write twists as in (5) and (6). Therefore, as \( \xi_{c_i,j_1-1,c_i} = \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{c_i,j_1-1,c_i} + \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} + \xi_{j_1-1,c_i} \times \left( \text{Ad} (\mathbf{x}_{c_i,j_1-1}) \xi_{j_1-1,c_i} \right) \)
TABLE 1: Twists of some of the most commonly used joints in robotics, where $l \in \mathbb{H}_p \cap S^3$ and $\omega, \omega_x, \omega_y, \omega_z, v, v_x, v_y, v_z, h \in \mathbb{R}$.

| 6-DoF | Revolute | Spherical | Cylindrical | Planar | Prismatic | Helical |
|-------|----------|-----------|-------------|--------|-----------|---------|
| ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) |

\[
\xi = \omega x i + \omega y j + \omega z k + \varepsilon (v_x i + v_y j + v_z k)
\]

\[
\xi = \omega i
\]

\[
\xi = \omega x i + \omega y j + \omega z k
\]

\[
\xi = (\omega + \varepsilon v) l
\]

\[
\xi = \varepsilon v l
\]

\[
\xi = (\omega + \varepsilon h \omega) l
\]

\[
v_i = \omega^{c+1}_{ji,c+1} \times p^{c+1}_{ji,c+1}
\]

\[
\xi^{c+1}_{ji,c+1} = \omega^{c+1}_{ji,c+1} + \varepsilon \left( \omega^{c+1}_{ji,c+1} \times p^{c+1}_{ji,c+1} \right)
\]

\[
\xi^{0}_{0,c_i} = \omega^{0}_{0,c_i} + \varepsilon (\mathbf{p}^{0}_{0,c_i} + \mathbf{p}^{0}_{0,c_i} \times \omega^{0}_{0,c_i}) \\
\xi^{0}_{0,c_i} = \omega^{0}_{0,c_i} + \varepsilon \left( \mathbf{p}^{0}_{0,c_i} + \mathbf{p}^{0}_{0,c_i} \times \omega^{0}_{0,c_i} \right)
\]

\[
\xi^{0}_{0,c_i} \times \text{Ad} (\mathbf{x}^{0}_{0,c_i}) = \xi^{c+1}_{0,c_i} = \omega^{c+1}_{0,c_i} + \varepsilon \left( \mathbf{p}^{c+1}_{0,c_i} + \mathbf{p}^{c+1}_{0,c_i} \times \omega^{c+1}_{0,c_i} \right)
\]

\[
\xi^{0}_{0,c_i} = \omega^{0}_{0,c_i} + \varepsilon (\mathbf{p}^{0}_{0,c_i} + \mathbf{p}^{0}_{0,c_i} \times \omega^{0}_{0,c_i})
\]

\[
\xi^{c+1}_{0,c_i} = \omega^{c+1}_{0,c_i} + \varepsilon \left( \omega^{c+1}_{0,c_i} \times \mathbf{p}^{c+1}_{0,c_i} \right)
\]

3.2 Backward Recursion

The second process of the iterative algorithm consists in sweeping the serial robot from the last to the first rigid body to calculate the wrenches applied at each one of them. For the robotic arm, we are interested in the wrenches at each joint, whereas for the mobile base we want to find the wrench at its CoM. To that aim, we use the twists obtained in Section 3.1 and their time derivatives.

Before obtaining the general expression for the backward recursion, let us consider the mobile manipulator shown in Fig 1.

The wrench at the CoM of the $n$th link (i.e., the $n$th CoM in the kinematic chain, in which $n = n_x + 1$), expressed in $F_{c_n}$, is given by the pure quaternion

\[
\xi_{0,c_n}^c = \xi_{0,c_n}^c - m_n \mathbf{g}^c_n,
\]
being the gravity vector expressed in $\mathcal{F}_{cn}$, and $\zeta_{0,0,cn} = f_{0,cn} + \varepsilon \tau_{0,cn}$, in which $f_{0,cn} = f_x i + f_y j + f_z k$ is the force at the CoM of the $n$th rigid body (i.e., the $n$th link), given by Newton’s second law $f_{0,cn} = m_n \ddot{p}_{0,cn}$.

Therefore, we use (23) to obtain

$$f_{0,cn} = m_n \left( D \left( \zeta_{0,0,cn} \right) + P \left( \zeta_{0,0,cn} \right) \times D \left( \zeta_{0,0,cn} \right) \right).$$

(25)

Furthermore, $\tau_{0,cn}$ is the torque about the $n$th rigid body’s CoM due to the change of its angular momentum, given by the Euler’s rotation equation

$$\tau_{0,cn} = L_3 \left( \mathbb{I}_{n} \right) P \left( \zeta_{0,0,cn} \right) + P \left( \zeta_{0,0,cn} \right) \times \left( L_3 \left( \mathbb{I}_{n} \right) P \left( \zeta_{0,0,cn} \right) \right),$$

(26)

where $L_3$ is given by [12] and $\mathbb{I}_{n}$ is the quaternionic inertia tensor of the $n$th rigid body, expressed at its CoM, given by [11]. Because (26) is calculated with respect to the CoM, the gravity acceleration does not contribute to the torque.

Using the adjoint transformation as in (32) in (24), the wrench at the $n$th joint, resulting from the wrench at the CoM of the $n$th rigid body, is given by

$$\zeta_{0,jn} = \operatorname{Ad} \left( \mathbf{x}_{0,jn}^{-1} \right) \zeta_{0,0,cn}.$$  

(27)

The resultant wrench at the $(n-1)$th rigid body (i.e., at $\mathcal{F}_{cn-1}$) also includes the effects of the wrench from the $n$th rigid body as they are rigidly attached to each other. Therefore, the resultant wrench at the $(n-1)$th joint (i.e., at $\mathcal{F}_{cn-1}$) is given by

$$\zeta_{0,jn-1} = \operatorname{Ad} \left( \mathbf{x}_{0,jn-1}^{-1} \right) \zeta_{0,cn-1} + \operatorname{Ad} \left( \mathbf{x}_{jn,jn-1}^{-1} \right) \zeta_{0,jn-1},$$

(28)

where $\zeta_{0,jn-1} = \zeta_{0,cn-1} - m_{n-1} g_{c_{n-1}}$, with $\zeta_{0,cn-1} = f_{0,0,cn-1} + \varepsilon \tau_{0,0,cn-1}$, is the wrench at the CoM of the $(n-1)$th rigid body expressed in $\mathcal{F}_{cn-1}$.

Thus, analyzing (24), (27), and (28), we find the backward recurrence relation for the total wrench at the $i$th rigid body, which includes the contribution of all wrenches starting at the CoM of the $i$th rigid body up to the wrench at the CoM of the last one, expressed in $\mathcal{F}_{j_{i-1}}$, as

$$\zeta_{0,j_{i-1}} = \operatorname{Ad} \left( \mathbf{x}_{0,j_{i-1}}^{-1} \right) \zeta_{0,i} + \operatorname{Ad} \left( \mathbf{x}_{j_{i-1},j_{i-1}}^{-1} \right) \zeta_{0,j_{i-1}},$$

(29)

with $i \in \{1, \ldots, n\}$ and $c_0 = j_0 = 0$, where $\zeta_{0,j_{n+1}}^{j_{n+1}} = \zeta_{0,j_{n+1}}^{j_n+1} = 0$ (recall that $n = n_{i+1} + 1$) and $\zeta_{0,i} = \zeta_{0,i}^0 - m_i g_{c_i}$, with $\zeta_{0,i}^0 = f_{0,i} + \varepsilon \tau_{0,i}$, is the wrench at the $i$th CoM $f_{0,i} = m_i \ddot{p}_{0,i}$, and $\tau_{0,i}$ is $L_3 \left( \mathbb{I}_i \right) P \left( \zeta_{0,i}^0 \right) + P \left( \zeta_{0,i}^i \right) \times \left( L_3 \left( \mathbb{I}_i \right) P \left( \zeta_{0,i}^i \right) \right)$. For the mobile base (i.e., the first CoM in the serial kinematic chain; hence, $i = 1$), notice that (29) results in $\zeta_{0,j_1} = \operatorname{Ad} \left( \mathbf{x}_{j_1}^0 \right) \zeta_{0,1}^0 + \operatorname{Ad} \left( \mathbf{x}_{j_1}^j \right) \zeta_{0,j_2}$, which implies $\zeta_{0,j_1} = \operatorname{Ad} \left( \mathbf{x}_{j_1}^0 \right) \zeta_{0,1}^0 + \operatorname{Ad} \left( \mathbf{x}_{j_1}^0 \right) \zeta_{0,j_2}^j$.

Moreover, the transformation $\mathbf{x}_{j_{i-1}}^{-1}$ is a function of the joint or mobile base coordinates. For example, the transformation of the mobile base, $\zeta_{0,j}^0 = \zeta_{0,j}^0$ depends on the its coordinates and rotation angle, namely $(\mathbf{x}_{j_1}^0, \mathbf{y}_{j_1}, \varphi_{j_1})$, therefore $\mathbf{x}_{j_1}^0 \triangleq \left[ \mathbf{x}_{j_1}^0 (\mathbf{x}_{j_1}^0, \mathbf{y}_{j_1}, \varphi_{j_1}) \right]$. For manipulators with revolute, prismatic, or helicoidal joints, the transformation $\mathbf{x}_{j_{i-1}}^{-1}$, with $i \geq 2$, is a function of just one parameter, that is $\mathbf{x}_{j_{i-1}}^{-1} \triangleq \mathbf{x}_{j_{i-1}}^{-1} \left( q_{j_{i-1}} \right)$. In that case, $q_{j_{i-1}} = q_{i+2}$. Analogously, spherical and planar joints depend on three parameters whereas helicoidal joints depend on six parameters. Therefore, one must be careful when defining the index for each parameter within the configuration vector $q$.

### 3.2.1 Particular cases: prismatic and revolute joints

In the case of manipulator robots with revolute and/or prismatic joints, which are the most common ones, the wrenches given by (29) must be projected onto the joints motion axes through

$$\langle \zeta_{0,j_1}, l_{j_1} \rangle = f_{l_1} + \varepsilon n_{l_1},$$

(30)

where $f_{l_1}, n_{l_1} \in \mathbb{R}$ and $\langle \mathbf{x}_{0,j_1}, l_{j_1} \rangle$ is the inner product between the wrench $\mathbf{x}_{0,j_1} = f_{0,j_1} + \varepsilon \tau_{0,j_1}$ and the motion axis $l_{j_1} \in \mathbb{H}_p \cap \mathbb{S}^3$ of the $i$th joint, given by [42]

$$\langle \zeta_{0,j_1}, l_{j_1} \rangle = -\frac{\langle \zeta_{0,j_1} f_{0,j_1} + \varepsilon \tau_{0,j_1}, l_{j_1} \rangle}{2} = \langle f_{0,j_1} + \varepsilon \tau_{0,j_1}, l_{j_1} \rangle = f_{l_1} + \varepsilon n_{l_1}.$$

Therefore, if the $i$th joint is revolute, then the corresponding torque is given by $n_{l_1} = \mathcal{D} \left( \zeta_{0,j_1}^{-1}, l_{j_1}^{-1} \right)$. If it is prismatic, then the corresponding force along along the axis $l_{j_1}$ is given by $f_{l_1} = \mathcal{P} \left( \zeta_{0,j_1}^{-1}, l_{j_1}^{-1} \right)$.  

---

6 If an external wrench is applied at the end-effector, then $\zeta_{0,j_{n+1}}^{j_{n+1}} \neq 0$.  

---
3.2.2 Particular case: planar joints/holonomic base For robots with holonomic mobile bases and/or planar joints (which are kinematically equivalent), we must project the wrenches onto all the three axes of motion of the joint/mobile base\(^7\). That is, the corresponding forces along the \(x\)-axis and \(y\)-axis of the joint/mobile base are given by

\[
\begin{align*}
\mathbf{f}_{tx} &= \mathcal{P} \left( \left( \mathbf{C}^{j_{i-1}}_{\text{D},j_i}, \text{Ad} \left( \mathbf{r}^{j_{i-1}}_0 \right) \hat{i} \right) \right), \\
\mathbf{f}_{ty} &= \mathcal{P} \left( \left( \mathbf{C}^{j_{i-1}}_{\text{D},j_i}, \text{Ad} \left( \mathbf{r}^{j_{i-1}}_0 \right) \hat{j} \right) \right),
\end{align*}
\]

whereas the corresponding torque about the \(z\)-axis of the joint/mobile base is given by

\[
\tau_z = \mathcal{D} \left( \left( \mathbf{C}^{j_{i-1}}_{\text{D},j_i}, \text{Ad} \left( \mathbf{r}^{j_{i-1}}_0 \right) \hat{k} \right) \right).
\]

4 GAUSS’S PRINCIPLE OF LEAST CONSTRAINT

The GPLC\(^8\) is a differential variational principle, equivalent to the D’Alembert one, that is based on the variation of the acceleration. For a system composed of \(n\) bodies, it can be stated as the least-squares minimization problem

\[
\min \sum_{i=1}^{n} \frac{1}{2} \left( \mathbf{a}_{ci} - \mathbf{a}_{ci} \right)^T \mathbf{P}_{ci} \left( \mathbf{a}_{ci} - \mathbf{a}_{ci} \right), \tag{31}
\]

where \(\mathbf{a}_{ci}\) and \(\mathbf{a}_{ci}\) are the accelerations of the center of mass of the \(i\)th rigid body under constraints and without constraints, respectively. Furthermore, \(\mathbf{P}_{ci} \triangleq \mathbf{P}_{ci} (\mathbf{I}, m_i)\) encapsulates the inertial parameters of the \(i\)th rigid body, such as the inertia tensor \(\mathbf{I} \in \mathbb{R}^{3 \times 3}\) and the mass \(m_i\).

This principle has been used in robotics to describe the dynamics of robot manipulators\(^{44}\) and rigid body simulations\(^{45}\). Wieber\(^{46}\) uses the GPLC to derive the analytic expression of the Lagrangian dynamics of a humanoid robot. Bouyarmane and Kheddar\(^{47}\) extend Wieber’s work by handling arbitrary parameterization of free-floating-base mechanisms. This allows using rotation matrices or unit quaternions to represent the free-floating-base orientations. In this section, we rewrite the GPLC for articulated bodies, similar to Wieber’s formulation\(^{46}\), but using dual quaternion algebra. This allows a more compact and unified representation than the one by Bouyarmane and Kheddar\(^{47}\).

First, we rewrite the constrained accelerations\(^{20}\) as a linear function of the vector of joints velocities and joints accelerations. This allows solving\(^{51}\) for the joints accelerations and, therefore, additional constraints can be included in the optimization formulation. We then define constraints to model nonholonomic mobile manipulators. Different from\(^{47}\), we do not use Lagrange multipliers. Instead, we apply Udwadia-Kalaba’s fundamental equation\(^{40}\), which is a simpler method for solving quadratic optimization problems such as\(^{51}\).

4.1 Constrained acceleration \(\mathbf{a}_{ci}\)

Consider the robotic system in Fig. 1. The robot is composed of rigid bodies that are constrained\(^7\) to one another by joints. To express the twist \(\mathbf{ξ}_{ci}\) of the \(i\)th center of mass explicitly as a linear combination between its Jacobian \(J_{\mathbf{ξ}_{ci}}\) and the vector of joints velocities \(\dot{\mathbf{q}} \in \mathbb{R}^n\), we use the operators \(\mathbf{vec}_{\mathbf{I}}: \mathcal{H} \rightarrow \mathbb{R}^8\), which maps the coefficients of a dual quaternion into an eight-dimensional vector\(^9\) and \(\mathbf{H}_8: \mathcal{H} \rightarrow \mathbb{R}^{8 \times 8}\), such that \(\mathbf{vec}_{\mathbf{I}} \mathbf{h} = \mathbf{vec}_{\mathbf{H}_8} \mathbf{h}\). Therefore, from\(^{43}\) we obtain \(\mathbf{ξ}_{ci} = 2 (\mathbf{vec}_{\mathbf{I}} \mathbf{r}^{j_{i-1}}_0) \mathbf{vec}_{\mathbf{H}_8} \mathbf{r}^{j_{i-1}}_0\), which implies \(\mathbf{vec}_{\mathbf{H}_8} \mathbf{ξ}_{ci} = 2 \mathbf{H}_8 (\mathbf{vec}_{\mathbf{I}} \mathbf{r}^{j_{i-1}}_0) \mathbf{vec}_{\mathbf{H}_8} \mathbf{r}^{j_{i-1}}_0\).

Because \(\mathbf{ξ}_{ci} \in \mathcal{H}_{p}, \) the first and fifth elements of \(\mathbf{vec}_{\mathbf{I}} \mathbf{ξ}_{ci}\) equal zero, thus we also use the operator \(\mathbf{vec}_{\mathbf{I}}: \mathcal{H}_{p} \rightarrow \mathbb{R}^6\) such that \(\mathbf{vec}_{\mathbf{I}} \mathbf{ξ}_{ci} = \mathbf{I} \mathbf{vec}_{\mathbf{I}} \mathbf{ξ}_{ci}\), where

\[
\mathbf{I} \triangleq \begin{bmatrix} 0_{3 \times 1} & I_3 & 0_{3 \times 1} & 0_{3 \times 3} \\ 0_{3 \times 1} & 0_{3 \times 3} & 0_{3 \times 1} & I_3 \end{bmatrix},
\]

with \(I_3 \in \mathbb{R}^{3 \times 3}\) being the identity matrix and \(0_{m \times n} \in \mathbb{R}^{m \times n}\) being a matrix of zeros. Moreover, \(\mathbf{vec}_{\mathbf{I}} \mathbf{z}^{j_{i-1}}_0 = J_{\mathbf{z}^{j_{i-1}}_0} \dot{\mathbf{q}}\), with \(\dot{\mathbf{q}} = [\dot{q}_1 \cdots \dot{q}_3]^T\), and \(J_{\mathbf{z}^{j_{i-1}}_0} \in \mathbb{R}^{8 \times 3}\) is the Jacobian matrix that is obtained algebraically\(^{12}\). Hence,

\[
\mathbf{ν}_{ci} \triangleq \mathbf{vec}_{\mathbf{I}} \mathbf{ξ}_{ci} = \begin{bmatrix} J_{\mathbf{ξ}^{j_{i-1}}_0} & 0_{6 \times (n-i)} \end{bmatrix} \dot{\mathbf{q}}, \tag{32}
\]

where \(J_{\mathbf{ξ}^{j_{i-1}}_0} = 2 \mathbf{I} \mathbf{H}_8 (\mathbf{vec}_{\mathbf{I}} \mathbf{r}^{j_{i-1}}_0) J_{\mathbf{z}^{j_{i-1}}_0} \in \mathbb{R}^{6 \times 3}\).

Finally, the constrained acceleration of the \(i\)th center of mass is given by

\[
\mathbf{a}_{ci} \triangleq \mathbf{vec}_{\mathbf{I}} \mathbf{ξ}_{ci} = J_{\mathbf{ξ}^{j_{i-1}}_0} \dot{\mathbf{q}} + J_{\mathbf{ξ}^{j_{i-1}}_0} \dot{\mathbf{q}}. \tag{33}
\]

\(^{7}\)Notice that this procedure applies for all joints with more than one axis of movement (e.g., spherical joints, planar joints, etc.).

\(^{8}\)In this case, there are both holonomic and nonholonomic constraints. The former are constraints between adjacent links in the kinematic chain. The latter is the constraint of the mobile base.

\(^{9}\)Given \(\mathbf{h} = h_1 + ih_2 + jh_3 + kh_4 + \varepsilon \left( h_5 + ih_6 + jh_7 + kh_8 \right)\), \(\mathbf{vec}_{\mathbf{I}} \mathbf{h} = \begin{bmatrix} h_1 & \cdots & h_8 \end{bmatrix}^T\).
We recall that (33) is equivalent to (20) as the Newton-Euler formalism implicitly considers the linkage constraints of the bodies.

4.2 Unconstrained acceleration \( \dot{a}_{ci} \)

Consider \( \dot{x}^0_{ci} = p^0_{ci} + (1/2)\ddot{r}^0_{0,ci} r^0_{ci} \), which represents the rigid motion from \( F_0 \) to \( F_{ci} \), and the twist \( \xi^0_{0,ci} \) at \( F_{ci} \) of the ith body under no constraints. From (22), the unconstrained acceleration is given explicitly as

\[
\dot{a}_{ci} \triangleq \mathop{vec}_6 \xi^0_{0,ci} = \left[ \begin{array}{c}
\mathop{vec}_3 \dot{\omega}^0_{0,ci} \\
\mathop{vec}_3 \left( \dot{p}^0_{0,ci} + \dot{p}^0_{0,ci} \times \omega^0_{0,ci} \right)
\end{array} \right],
\]

(34)

where \( \mathop{vec}_3 : \mathbb{R}^6 \to \mathbb{R}^3 \) such that \( \mathop{vec}_3 (a\hat{i} + b\hat{j} + c\hat{k}) = [a \ b \ c]^T \). Whereas (33) depends on \( q, \dot{q} \), and \( \ddot{q} \), Eq. (34) does not because it is unconstrained.

4.3 Euler Lagrange equations

Let \( \mathcal{G}(q, \dot{q}, \ddot{q}) = \sum_{i=1}^n \left( G_{ai}(q, \dot{q}, \ddot{q}) + G_{bi}(q, \dot{q}) \right) \), in which \( \dot{a}_{ci} \) and \( \ddot{a}_{ci} \) are given by (33) and (34), where \( \Psi_{ci} \) is \( \mathop{vec}_3 \mathop{vec}_3 \dot{\omega}^0_{0,ci} \), \( \mathop{vec}_3 \left( \dot{p}^0_{0,ci} + \dot{p}^0_{0,ci} \times \omega^0_{0,ci} \right) \).

Expanding \( \mathcal{G}(q, \dot{q}, \ddot{q}) \), we obtain

\[
\mathcal{G}(q, \dot{q}, \ddot{q}) = \sum_{i=1}^n \left( G_{ai}(q, \dot{q}, \ddot{q}) + G_{bi}(q, \dot{q}) \right),
\]

(35)

where

\[
G_{ai}(q, \dot{q}, \ddot{q}) \triangleq \frac{1}{2} \dot{q}^T \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \mathcal{J}_{\xi^0_{0,ci}} \dot{q} + \dot{q}^T \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \ddot{q} - \ddot{q}^T \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \dot{q} + \mathcal{G}_b(q, \dot{q}),
\]

(36)

where \( G_{ai} \) is \( \mathop{vec}_3 \mathop{vec}_3 \dot{\omega}^0_{0,ci} \), \( \mathop{vec}_3 \left( \dot{p}^0_{0,ci} + \dot{p}^0_{0,ci} \times \omega^0_{0,ci} \right) \).

From the optimality condition, the solution of (31) is computed as

\[
\frac{\partial \mathcal{G}(q, \dot{q}, \ddot{q})}{\partial \dot{q}} = \frac{\partial}{\partial \ddot{q}} \left( \sum_{i=1}^n G_{ai}(q, \dot{q}, \ddot{q}) \right) = 0_{1 \times n}.
\]

(37)

Using (35) in (36), we obtain

\[
0_{n \times 1} = \sum_{i=1}^n \left( \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \mathcal{J}_{\xi^0_{0,ci}} \ddot{q} + \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \ddot{q} + \Phi \right),
\]

(38)

where \( \Phi = -\mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \mathcal{J}_{\xi^0_{0,ci}} \dot{q} \).

Since \( \mathcal{J}^T_{\xi^0_{0,ci}} \left[ \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \mathcal{J}_{\xi^0_{0,ci}} \right] \), using (34) and the elements \( \mathcal{J}^T_{\xi^0_{0,ci}} \), the term \( \Phi \) from (37) can be rewritten as

\[
\Phi = -\mathcal{J}^T_{\xi^0_{0,ci}} \mathop{vec}_3 \mathcal{J}^T_{\xi^0_{0,ci}} \Psi_{ci} \mathcal{J}_{\xi^0_{0,ci}} \mathop{vec}_3 \dot{q} - m_i \mathcal{T}^T_{\xi^0_{0,ci}} \mathop{vec}_3 \left( \dot{p}^0_{0,ci} \times \omega^0_{0,ci} \right),
\]

(39)

where \( \mathop{vec}_3 \left( \dot{p}^0_{0,ci} \times \omega^0_{0,ci} \right) = -S \left( \omega^0_{0,ci} \right) \mathop{vec}_3 \dot{p}^0_{0,ci} \) with \( \mathop{vec}_3 \dot{p}^0_{0,ci} = \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q}, \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q} = m_i \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q} \) and \( S (\cdot) \in \text{so}(3) \) is the skew-symmetric matrix used as an operator that performs the cross-product (48).

Furthermore, as \( \mathop{vec}_3 \omega^0_{0,ci} = \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q} \) for convenience’s sake we use the \( \mathop{vec}_3 \) operator to rewrite (36) as

\[
\mathop{vec}_3 \cdot \mathop{vec}_3 \ddot{q} = \mathop{vec}_3 \dot{q} + \mathcal{S}(s_{ci}) \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q},
\]

(40)

where \( s_{ci} \triangleq \mathop{vec}_3 \omega^0_{0,ci} \), and use it in (35) to obtain

\[
\Phi = -\mathcal{J}^T_{\xi^0_{0,ci}} \mathop{vec}_3 \mathcal{T}_{\xi^0_{0,ci}} - \mathcal{J}^T_{\xi^0_{0,ci}} \mathcal{J}^T_{\xi^0_{0,ci}} \mathop{vec}_3 \mathcal{J}_{\xi^0_{0,ci}} + \mathcal{J}^T_{\xi^0_{0,ci}} \mathcal{S}(s_{ci}) \mathcal{J}^T_{\xi^0_{0,ci}} \dot{q},
\]

(41)

Finally, using (40) in (37) yields

\[
\mathcal{M}_{GP} \ddot{q} + \mathcal{C}_{GP} \dot{q} = \tau_{GP},
\]

(42)

where \( \mathcal{M}_{GP} \triangleq \sum_{i=1}^n J^T_{\xi^0_{0,ci}} \Psi_{ci} J_{\xi^0_{0,ci}} \),

(43)

\[
\mathcal{C}_{GP} \triangleq \sum_{i=1}^n J^T_{\xi^0_{0,ci}} \mathcal{S}(s_{ci}) J_{\xi^0_{0,ci}} + \Psi_{ci} J_{\xi^0_{0,ci}},
\]

(44)

\[
\tau_{GP} \triangleq \sum_{i=1}^n J^T_{\xi^0_{0,ci}} \xi_{ci},
\]

(45)
where \( \xi_{ci} \) is the wrench at the \( i \)th center of mass, defined as

\[
\xi_{ci} = \left[ \begin{array}{c} 0_{1 \times 3} \\ I_{3 \times 3} \end{array} \right] \vec{c}_{0,ci}.
\]  

(46)

with \( \xi_{ci} = f_{0,ci}^c + \varepsilon \tau_{0,ci} \).

Furthermore, since the gravity does not generate any resultant moment at the center of mass of a link, the vector of gravitational forces \( \bar{\tau} \), and the gravitational forces \( \tau \) at the center of mass of a link, the vector of gravitational forces and gravitational acceleration, respectively, both expressed in the inertial frame. Hence, \( \bar{\tau} = \tau = 0 \).

By considering the generalized forces \( \tau_{GP} \) applied at the joints and the gravitational forces \( \tau_g \), the resultant forces acting on the system are \( \tau_{GP} = \tau_{GP} + \tau_g \). Let \( g_{GP} = -\tau_g \), then (42) is rewritten in the canonical form as

\[
M_{GP} \ddot{q} + C_{GP} q + g_{GP} = \tau_{GP}.
\]  

(48)

In this way, solving (31) leads to the Euler-Lagrange dynamic description of a mechanical system by means of dual quaternion algebra. Once again, we assume that the robot forward kinematics and differential kinematics are available in dual quaternion space [12].

**Remark 2.** Let \( A \equiv (1/2) \dot{M}_{GP} - C_{GP} \), then

\[
A = -\sum_{i=1}^{n} J^T \xi_{ci}^c \vec{S} \left( \omega_{0,ci}, \Psi_c \right) J \xi_{ci}^c.
\]  

Since \( \vec{S} \left( \omega_{0,ci}, \Psi_c \right) \) is skew-symmetric by construction, then \( A^T = -A \), which implies

\[
u^T \left( 1/2 M_{GP} (q) - C_{GP} (q, \dot{q}) \right) u = 0
\]  

(49)

for all \( q, \dot{q}, u \in \mathbb{R}^n \). Property (49) is useful to show formal closed-loop stability in robot dynamic control using strategies based on Lyapunov functions [49].

### 4.4 Connections with the Gibbs-Appell and Kane’s equations

The Gibbs-Appell and Kane’s equations have proven to be a powerful mathematical tool to describe both unconstrained and constrained mechanical systems without the use of Lagrange multipliers [50, 51]. Both are different ways to get the equations of motion, but equivalent in the sense that a set of equations implies the other [52–54].

The Gibbs-Appell method is closely related with the Gauss’s Principle of Least Constraint, since both approaches use scalar quadratic functions in terms of accelerations. The former can be seen as a generalization of the latter [55, 56]. However, they are equivalent and both can be derived from the other [55, 57, 58]. Nonetheless, different from the Gibbs-Appell and Kane’s equations, the Gauss’s principle strategy does not require setting up quasi-velocities and allows taking into account additional constraints directly in the optimization formulation.

Now, we rewrite the Gibbs-Appell and Kane’s equations using the equations derived in Sections 4.1–4.3. Furthermore, we show that the Euler-Lagrange dynamic description of a mechanical system can be shown to be a particular case of the Gibbs-Appell and Kane’s equations. This is done by selecting the quasi-velocities to be the same as the generalized velocities.

For \( n \) rigid bodies, the Gibbs-Appell equations are given by [51]

\[
\frac{\partial S (q, \dot{q}, \ddot{q})}{\partial u} = \sum_{i=1}^{n} \left( \frac{\partial}{\partial u} \nu_{ci} \right) ^T \xi_{ci},
\]  

(50)

where

\[
S (q, \dot{q}, \ddot{q}) \equiv \sum_{i=1}^{n} \left( \begin{array}{c} \dot{a}_{ci}^T \Psi_c, \dot{a}_{ci} \vec{S} \left( \omega_{0,ci}, \Psi_c \right) \nu_{ci} \end{array} \right)
\]  

(51)

is a scalar function of the configuration \( q \), configuration velocity \( \dot{q} \), and configuration acceleration \( \ddot{q} \). The vector \( \nu_{GA} \) contains generalized forces associated with the quasi-velocities \( u \). Furthermore, \( \nu_{ci} \) and \( \xi_{ci} \) are the twist and the generalized forces of the \( i \)th body, given by (32) and (46), respectively.

We let \( u \equiv \dot{q} \), and use (33) in (51). We take the result and apply the partial derivative \( \partial S/\partial \ddot{q} \), and then compare with (43) and (44) to obtain

\[
\frac{\partial S (q, \dot{q}, \ddot{q})}{\partial \ddot{q}} = M_{GP} \ddot{q} + C_{GP} \ddot{q}.
\]  

(52)

Using (32), we obtain

\[
\frac{\partial}{\partial u} \nu_{ci} = \frac{\partial}{\partial \ddot{q}} \left( J \xi_{ci}^c \dot{q} \right) = J \xi_{ci}^c.
\]  

(53)
Therefore, we compute the generalized forces as $\tau_{GA} = \sum_{i=1}^{n} J_{c_i}^{T} \vec{\xi}_{c_i} = \tau_{GP}$ (see (45)). In this way, solving (50) leads to the Euler-Lagrange equation, which is identical to the one obtained by using the Gauss’s principle (42).

On the other hand, the Kane’s equation of motion is given by

$$\varphi - \varphi = 0,$$

(54)

where $\varphi$ contains the generalized active forces and $\varphi$ contains the generalized inertia forces.

We obtain the equation of motion using Kane’s method by grouping both the Newton’s and Euler’s equations for $n$ rigid bodies as follows: \[^{11}\]

$$\sum_{i=1}^{n} \left[ (\bar{c}_i \vec{c} \omega_{0,c_i} - \bar{S}(s_{c_i}) \vec{c} \omega_{0,c_i}) \right] = \sum_{i=1}^{n} \vec{c} \omega_{0,c_i}.$$

Using (41), and the fact that $\nu_{c_i} = \vec{c} \omega_{0,c_i}$, with $\xi_{0,c_i} = \vec{c} \omega_{0,c_i} = \xi_{0,c_i}$, we rewrite (55) as

$$\sum_{i=1}^{n} \left[ \vec{\mu}_{c_i} + \bar{S} \left( \omega_{0,c_i}, \Psi_{c_i} \right) \nu_{c_i} \right] = \sum_{i=1}^{n} \xi_{c_i}. \quad (56)$$

Since $\kappa_{c_i} = \xi_{c_i}$ for $i \in \{1, \ldots, n\}$, we multiply each of the $n$ terms $\kappa_{c_i}$ from (56) by $\left( \vec{c} \omega_{0,c_i} \right)^{T} = J_{c_i}^{T} \xi_{c_i}$, to obtain the Kane’s equations (59), which yields

$$\sum_{i=1}^{n} \left[ J_{c_i}^{T} \vec{\mu}_{c_i} + \bar{S} \left( \omega_{0,c_i}, \Psi_{c_i} \right) \nu_{c_i} \right] = \sum_{i=1}^{n} J_{c_i}^{T} \xi_{c_i}.$$

(57)

Finally, using (32), (33), (43), (44) and (45) in (57) we obtain $\varphi = M_{GP} \dot{q} + C_{GP} \ddot{q}$ and $\varphi = \bar{\tau}_{GP}$, which is the same dynamic equation as the one obtained by the Gauss’s principle (42), as expected.

\[\text{Therefore, when considering the quasi-velocities to be the same as the generalized velocities (i.e., } \ddot{q} = \dot{q} \text{ the relations between Gauss’s principle, Gibbs-Appell equations and Kane’s method are given by}\]

$$\frac{\partial \mathcal{G}(\dot{q}, \ddot{q})}{\partial \dot{q}} = \frac{\partial \mathcal{S}(\dot{q}, \ddot{q})}{\partial \dot{q}} - \tau_{GA} = \varphi - \varphi = 0_{n \times 1}. \quad (58)$$

### 4.5 Constrained Robotic Systems using the Gauss’s Principle of Least Constraint

Additional constraints can be imposed in the GPLC formulation. This can be done by means of Lagrange multipliers [47] or using the Udwadia-Kalaba formulation [40]. The former requires the computation of the Lagrange multipliers, whereas the latter employs a simpler method, albeit equivalent, which is based on generalized inverses as the solution to a constrained quadratic program.

Using the Udwadia-Kalaba formulation [40], additional constraints in the form $A \ddot{q} = b$, with $A \triangleq A(q, \dot{q})$ and $b \triangleq b(q, q)$, are taken into account in (42) as follows

$$M_{GP} \ddot{q} = Q(q, \dot{q}) + Q_c(q, \dot{q}), \quad (59)$$

where $Q(q, \dot{q}) \triangleq -\tau_{GP} - C_{GP} \ddot{q}$, and the additional term $Q_c(q, \dot{q}) \triangleq M_{GP}^{1/2} D^{+} (b - A M_{GP}^{-1} Q(q, \dot{q}))$ represents the constraint force due to the additional constraints. Furthermore, $D^{+}$ is the Moore-Penrose generalized inverse [48] of $D$, with $D \triangleq A M_{GP}^{-1}/2$.

Finally, letting $\Omega \triangleq \left( I - M_{GP}^{1/2} D^{+} A M_{GP}^{-1} \right)$, we rewrite (59) as

$$M_{GP} \ddot{q} + \Omega C_{GP} \ddot{q} - M_{GP}^{1/2} D^{+} b = \Omega \bar{\tau}_{GP}. \quad (60)$$

For example, consider the well-known differential-drive mobile robot, in which the nonholonomic constraint ensures the conditions of pure rolling and non-slipping movements [60]. The robot configuration is specified by the vector $q = [x y \phi]^{T}$, where $x$ and $y$ are the position coordinates and $\phi$ is the orientation of the robot on the plane. The nonholonomic constraint is given by

$$\begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix} \dot{q} = 0, \quad (61)$$

which can be enforced in (60) by taking the time derivative of (61) such that $A \dot{q} + A \ddot{q} = 0$. Therefore, $b = -A \dot{q}$.
5 RESULTS
To assess the dual quaternion Newton-Euler formalism (dqNE) and the Euler-Lagrange model obtained using the dual quaternion Gauss’s Principle of Least Constraint (dqGP), we performed simulations using three different robots; namely, a fixed-base 50-DoF serial manipulator, a 9-DoF holonomic mobile manipulator, and an 8-DoF nonholonomic mobile manipulator.

We implemented the simulations on the robot simulator V-REP PRO EDU V3.6.21 using an interface with Matlab 2020a and the computational library DQ Robotics [61] for dual quaternion algebra on a computer running Ubuntu 18.04 LTS 64 bits equipped with a Intel Core i7 6500U with 8GB RAM.

Furthermore, we present the computational costs of the proposed methods and compare them with their respective classic counterparts.

5.1 Simulation Setup
The comparisons between the generalized accelerations obtained through our proposed models and the values from V-REP were made considering the coefficient of multiple correlation (CMC) [62] between the waveforms. The CMC provides a coefficient ranging between zero and one that indicates how similar two given waveforms are. Identical waveforms have CMC equal to one, whereas completely different waveforms have CMC equal to zero.

The simulator does not allow the direct reading of accelerations. Therefore, to obtain the configuration acceleration vector \( \ddot{q} \), we first read the velocity vector \( \dot{q} \in \mathbb{R}^{n_e+3} \), then filtered all elements \( \dot{q}_1, \ldots, \dot{q}_{n_e+3} \) with a discrete low-pass Butterworth filter, and used those values to obtain the acceleration by means of numerical differentiation based on a second forward finite difference approximation. We then calculated the CMC between each element \( \ddot{q}_i \), with \( i \in \{1, \ldots, n_l + 3\} \), of the generalized acceleration waveform and its counterpart from V-REP. Afterward, we used those CMCs to obtain the mean, minimum, and maximum CMCs for the model, alongside their standard deviation.

Furthermore, for the simulation of the fixed-base 50-DoF serial manipulator, we also used the classic Newton-Euler algorithm (rtNE) implemented on the Robotics Toolbox [63], and calculated the CMC between the joint acceleration waveforms yielded by it and the ones from V-REP. The Robotics Toolbox is a widely used library whose accuracy has been verified throughout the years. Therefore, it is an appropriate baseline for the evaluation of the CMCs obtained by using our models.

5.2 Results
Table 2 presents the CMC between the generalized acceleration waveforms obtained through the different dynamic model strategies (dqNE, dqGP, and rtNE) and the values obtained from V-REP.

The proposed dual quaternion Newton-Euler formalism does not allow the inclusion of equality constraints into the model; therefore, it was not applied to the dynamic modeling of the nonholonomic mobile manipulator. Similarly, the current version of the Robotics Toolbox only supports dynamic modeling of fixed-base robots and was only applied to the 50-DoF serial manipulator. The cases where the model could not be obtained using the listed strategy are indicated in Table 2 by N/A. (i.e., not available). For all other cases, all models presented mean (mean) and minimum (min) CMC close to one, with standard deviation (std) and maximum (max) CMC; thus indicating high similarity between the generalized acceleration waveform obtained from them and the values from V-REP.

For the 50-DoF serial manipulator, both the dqNE and the dqGP are equivalent to the rtNE, which demonstrates the accuracy of our proposed strategies when compared to the classic Newton-Euler approach.

For qualitative analysis, Fig. 3 presents the generalized accelerations obtained using dqGP, alongside the V-REP values, for the minimum, maximum, and intermediate CMCs found during simulations. Even for the smallest value of CMC (i.e., 0.8860), the accelerations obtained using our formulation match closely the V-REP values. The small discrepancies arise from both discretization effects and because the accelerations in V-REP are estimated from noisy velocity values.

5.3 Computational cost
Here we compare the proposed methods with their classic counterparts in terms of the number of multiplications and additions involved in each technique. The results, considering an \( n \)-DoF serial robot, are summarized in Table 3. For the classic

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13 Available at: https://www.coppeliarobotics.com/
14 In this case, we used the Vortex Studio engine (www.cm-labs.com) because it presented better numerical stability for the 50-DoF manipulator.
Newton-Euler algorithm, we consider the version based on three-dimensional vectors proposed by Luh et al. [64], whose mathematical cost was calculated by Balafoutis [65], and is, to the best of our knowledge, one of the most efficient implementations in the literature. Furthermore, for the classic Euler-Lagrange algorithm we consider the version proposed by Hollerbach [66].

The algorithm presented by Luh et al. [64] costs less than our Dual Quaternion Newton-Euler algorithm. The cost we presented for our method is, however, fairly conservative and is given as an upper bound. For instance, our calculations could be further optimized by exploring the fact that several operations involve pure dual quaternions, which have six elements instead of eight. Additionally, the cost presented by Balafoutis [65] does not include the costs of obtaining the robot kinematic model. Also, our method works for any type of joint and we have not optimized the calculations for any particular type of joint, differently from Luh et al. [64], who only consider prismatic and revolute joints, which are exploited to optimize the computational cost. Nonetheless, both our algorithm and the one of Luh et al. have linear costs in the number of DoF, with coefficients of the same order of magnitude.

The Euler-Lagrange method based on the Gauss’s Principle of Least Constraint is, as expected, more expensive than the ones based on the Newton-Euler and classic Euler-Lagrange formalism since it is not based on recursive strategies. However, this strategy allows taking into account additional constraints in the accelerations, which can be exploited, for instance, in nonholonomic robotic systems. For those cases, the Euler-Lagrange dynamic equation is given by (60).

### 6 CONCLUSIONS

This work presented two strategies for the formulation of the dynamics of mobile manipulators based on dual quaternion algebra. The first one is based on the recursive Newton-Euler formulation, and uses twists and wrenches instead of free vectors. This representation removes the necessity of exhaustive geometrical analyses of the kinematic chain since wrenches and twists are propagated through high-level algebraic operations. Furthermore, our formulation works for any type of joint because it takes into account arbitrary twists. Thus, our strategy is more general than the work of Miranda et al. [67], which considered only manipulators with revolute joints.

The second proposed method is based on the Gauss’s Principle of Least Constraint and is also formulated based on twists and wrenches represented using dual quaternion algebra in matrix form. This strategy allows the incorporation of equality constraints directly in the optimization formulation.

The cost comparison performed between the proposed methods and their classic counterparts, in terms of number of multiplications and additions, showed that the use of dual quaternions does not significantly increase the cost of the Newton-Euler formalism, as the algorithm has linear complexity on the number of rigid bodies in the kinematic chain. However, the cost of obtaining the Euler-Lagrange model using the Gauss’s Principle of Least Constraint and dual quaternion algebra is higher than the best classic Euler-Lagrange recursive solution found in the literature. Nonetheless, our method is far more general than its classic counterpart. Also, we made no hard attempt, if any, to optimize our implementation since we are currently more interested in the theoretical aspects of the dynamic modeling using dual quaternion algebra than in ensuring computational efficiency. In our current MATLAB implementation, the dqNE and the dqGP take, in average, 23.17s and 8.73s to generate the joint accelerations for a 50-DoF manipulator robot, respectively. Those values are expected to decrease to around 99 ms and 37 ms in a C++ implementation [61], respectively.

Obtaining the Euler-Lagrange model through the Newton-Euler formalism requires several executions of the algorithm. One execution to obtain the gravitational vector, one to obtain the vector of Coriolis and centrifugal terms and one for each row of the inertia matrix $M$. For a 50-DoF manipulator robot, this results in 52 executions of the dqNE for each simulation step. For control applications, however, we are usually interested in finding the joint torques, which requires only one execution of the dqNE to generate each control input. Therefore, using the dqNE to compute the joint torques for a 50-DoF manipulator robot, the execution time is expected to reduce by a factor of 52, from around 99 ms to 1.9 ms in a C++ implementation.
TABLE 3: Cost comparison between the proposed methods and their classic counterparts for obtaining the dynamical model for an $n$-DoF serial robot.

| Method                                                                 | Mult.               | Add.               |
|-----------------------------------------------------------------------|---------------------|--------------------|
| Dual Quaternion Newton-Euler algorithm (cost for arbitrary joints)    | $882n - 48$         | $724n - 40$        |
| Classic Newton-Euler algorithm [65]                                    | $150n - 48$         | $131n - 48$        |
| Dual Quaternion Euler-Lagrange algorithm using Gauss’s Principle of Least Constraint | $4n^3 + 386n^2 + 401n$ | $\frac{16}{3} n^3 + 326n^2 + \frac{908}{3} n$ |
| Classic Euler-Lagrange algorithm [66]                                 | $412n - 277$        | $320n - 201$       |

Finally, we compared the joints accelerations obtained through the proposed strategies for three different robots, with the values obtained from V-REP PRO EDU V3.6.2, which is a realistic simulator. The results showed that all of our methods are accurate for both fixed-base serial manipulators and mobile manipulators.

Future works will focus on extending the dual quaternion Newton-Euler algorithm to non-serial multibody systems (e.g., humanoids), and in wrench control strategies. Concerning the Euler-Lagrange model obtained using the Gauss’s Principle of Least Constraint and dual quaternion algebra, future works will be focused on exploiting inequality constraints in the optimization formulation.

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