ON THE MODULI SPACE OF POSITIVE RICCI CURVATURE METRICS ON HOMOTOPY SPHERES

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Abstract: We show that the moduli space of Ricci positive metrics on a certain family of homotopy spheres has infinitely many components.

§1 Introduction

In [C], R. Carr proved that the space of positive scalar curvature metrics on the spheres $S^{4k-1}$ has infinitely many path-components. It is not difficult to see that Carr’s arguments apply equally well - with the same conclusion - to any exotic sphere in these dimensions which bounds a parallelisable manifold.

In [KS], M. Kreck and S. Stolz strengthened this result considerably. They showed that for any closed spin manifold $M^{4k-1}$ ($k > 1$) which admits a positive scalar curvature metric and for which $H^1(M; \mathbb{Z}/2) = 0$, the moduli space of positive scalar curvature metrics has infinitely many components. They then use this result to prove that the moduli space of Ricci positive metrics on certain homogeneous 7-manifolds with $SU(3) \times SU(2) \times U(1)$ symmetry has infinitely many components. To the best of our knowledge, these are the only examples of Ricci positive manifolds in the literature for which the moduli space of Ricci positive metrics is shown to have infinitely many components.

The aim of this paper is to establish further examples of this type, and in particular to show that this phenomenon occurs not just in dimension seven, but through an infinite range of dimensions. Specifically, we study the moduli space of positive Ricci curvature metrics on those homotopy spheres in dimensions $4k - 1$ which bound a parallelisable manifold. Note that in each case, this set of Ricci positive metrics is non-empty [W1], [W2]. Our main goal is to establish

Theorem A. Let $\Sigma^{4k-1}$ be any homotopy sphere bounding a parallelisable manifold, where $k > 1$. Let $\mathcal{M}(\Sigma)$ denote the space of all smooth Riemannian metrics on $\Sigma$, equipped with the $C^\infty$ topology, and denote by $\mathcal{R}(\Sigma)$ the subset consisting of all Ricci positive metrics. Give $\mathcal{R}(\Sigma)$ the induced topology from $\mathcal{M}(\Sigma)$. Then the moduli space $\mathcal{R}(\Sigma)/\text{Diff}(\Sigma)$ has infinitely many components.

Possibly the earliest result on the connectedness of spaces of metrics displaying some form of positive curvature was due to Hitchin [H]. He proved that if a closed spin manifold admits a positive scalar curvature metric and has dimension a multiple of eight, then its space of positive scalar curvature metrics has more than one path component and has non-trivial fundamental group. Shortly after [KS], Botvinnik and Gilkey [BG] showed

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that in some cases, the space of positive scalar curvature metrics for odd dimensional spin manifolds with non-trivial finite fundamental group can also have infinitely many path components. More recent results due to Kapovitch, Petrunin and Tuschmann [KPT] show that there are examples of non-compact manifolds in every dimension ≥ 22 for which the moduli space of non-negatively curved metrics has infinitely many components. Let us also mention that in [KS], examples are given of Wallach spaces for which the moduli space of non-negatively curved metrics is not connected. It is observed however in [KPT] that the $SU(3) \times SU(2) \times U(1)$-homogeneous spaces studied in [KS] actually have moduli spaces of non-negatively curved metrics with infinitely many components.

The proof of Theorem A involves a blend of metric and topological construction techniques based on those in [W1] and [W2], together with some ideas and arguments from [KS]. Note that Theorem A does not follow automatically from the fact that the corresponding moduli space of positive scalar curvature metrics has infinitely many components, as it is not clear whether a given component contains a Ricci positive metric. Our basic idea is to construct explicit Ricci positive metrics on $\Sigma$ in infinitely many different ways, and show that the metrics we produce all lie in different components of the moduli space of positive scalar curvature metrics.

The principal technical difficulty addressed by this paper is the extension of certain metrics used for positive scalar curvature surgery across a handlebody. In particular, one has to deal with the corners which occur when attaching handles. The smoothing out of such corners whilst keeping control of the scalar curvature is a delicate issue, which is explored in (the proof of) Proposition 9.

As some of the constructions in this paper are quite involved, we give a broad overview and explain the motivation behind the various elements.

For the exotic spheres under consideration in this paper, we construct in [W1] and [W2] reasonably explicit Ricci positive metrics using a Ricci positive surgery process, beginning with a sphere bundle over a sphere equipped with a Ricci positive submersion metric. Each exotic sphere can be constructed from such a bundle by a sequence of surgeries in infinitely many different ways. In this paper we show that the metrics resulting from these different constructions all belong to different components of the moduli space of positive scalar curvature metrics, using the Kreck-Stolz invariant described in §2. To utilise this invariant we need to extend a given Ricci positive metric across a bounding manifold determined by the sequence of surgeries, in such a way that the resulting metric has positive scalar curvature and is a product near the boundary.

A procedure is given in the proof of Proposition 9 for such a ‘filling-in’ of the exotic sphere metric to obtain a metric on the bounding manifold. However, this metric is not a product near the boundary.

To remedy this, we can try to deform the metric in a neighbourhood of the boundary to a product in some smaller neighbourhood, whilst preserving positive scalar curvature. This deformation is addressed in §6. However, for such a deformation to work, a certain metric condition must hold, which is a kind of mean curvature condition at the boundary (see Lemma 11). Unfortunately, the metric produced by using Proposition 9 does not have this property.

Our approach to this particular problem is to side-step it by starting with a different
metric on the exotic sphere. This new metric has positive scalar curvature (not positive Ricci curvature) and is linked to the original metric by a path of positive scalar curvature metrics (thus staying within the original moduli space of positive scalar curvature metrics). This path is produced by an explicit deformation of the original metric, which is performed on each surgery ‘piece’ in turn. The details of this deformation for a single such ‘piece’ are covered in Lemma 4 and Proposition 7. The discussion after Proposition 7 shows how these individual deformations can be fitted together to produce a global deformation on the whole manifold. When this deformed metric is ‘filled-in’ according to the method of Proposition 9, the bounding manifold has both positive scalar curvature and the right ‘mean curvature’ condition at the boundary (Lemma 11). This metric can then be deformed within positive scalar curvature to a product near the boundary (Corollary 15). From this point, Theorem A can then be proved using the Kreck-Stolz invariant.

This paper is organised as follows. In §2 we discuss the construction of homotopy spheres as boundaries of plumbed manifolds and introduce the s-invariant of Kreck and Stolz. In §3 we look at surgery on Ricci positive manifolds. In §4 we show how to deform a positive Ricci curvature metric on an exotic sphere through positive scalar curvature metrics into a form suitable for use in later constructions. In §5 we consider handlebodies, and address the problem of extending a metric on a surgery across the corresponding handle. In §6 we consider paths of positive scalar curvature metrics, concluding with the proof of Theorem A.

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§2 Homotopy Spheres, Plumbing and the s-invariant

The diffeomorphism classes of homotopy spheres bounding parallelisable manifolds of dimension $n$ form an abelian group under the connected sum operation. This group is denoted $bP_n$. It was shown in [KM] that for $n$ odd, $bP_n = 0$, for $n = 4k + 2$, $bP_n$ is either 0 or $\mathbb{Z}_2$, and for $n = 4k$, $bP_n$ is cyclic.

Let $\Sigma \in bP_{4k}$ for some $k > 1$. It was shown in [W1] that $\Sigma$ arises as the boundary of a plumbed manifold. (For details of the plumbing construction, and in particular for its relevance to homotopy spheres, see [Br]..) For this plumbed manifold, the disc-bundles involved are all $D^{2k}$-bundles over $S^{2k}$, and the plumbing graph is simply-connected. Moreover, for fixed $\Sigma$, this can be achieved in infinitely many different ways. More precisely, a bounding manifold can be constructed having unimodular intersection form and signature $8(p|bP_{4k}|q)$ where $p$ is any non-negative integer, and $q \in \{1, ..., |bP_{4k}|\}$ is a number determined by $\Sigma$ (see Propositions 1.4 and 1.5 in [W1]). Let us denote such a bounding manifold by $X_p$.

In [KS], Kreck and Stolz introduced the s-invariant of a spin manifold $M^{4k-1}$ which admits a positive scalar curvature metric $g$ and has vanishing real Pontrjagin classes. This is a $\mathbb{Q}$-valued invariant with the property that $|s|$ is constant on each component of the moduli space of positive scalar curvature metrics, provided $H^1(M;\mathbb{Z}/2) = 0$ (see [KS;
Proposition 2.14]). The actual definition of the s-invariant is somewhat complicated and
we refer the reader to [KS] for details. However, for the manifolds under consideration in
this paper the s-invariant takes a simple form, as described in the following lemma.

**Lemma 1.** With \((M, g)\) as above, suppose that \(W\) is a parallelisable manifold with \(\partial W = M\), equipped with a positive scalar curvature metric \(\bar{g}\) which takes the form \(dt^2 + g\) in a
neighbourhood of the boundary. Then the s-invariant of \((M, g)\) is given by

\[ s(M, g) = \frac{1}{2^{2k+1}(2^{2k-1} - 1)} \sigma(W) \]

where \(\sigma(W)\) is the signature of \(W\).

**Proof:** The parallelisability of \(W\) means in particular that \(W\) is spin, so the following
formula [KS; Proposition 2.13] applies:

\[ s(M, g) = \text{ind} D^+(W, \bar{g}) + t(W). \]

The positive scalar curvature metric on \(W\) means that the term \(\text{ind} D^+(W, \bar{g})\) vanishes (see
Remark 2.2(ii) in [KS]). The parallelisability of \(W\) also means that the Pontrjagin classes
\(p_1(W), \ldots, p_n(W)\) vanish, and by equation (2.11) of [KS] this means that the term \(t(W)\) is
equal to \(1/(2^{2k+1}(2^{2k-1} - 1))\sigma(W)\).

Note that the homotopy sphere \(\Sigma \in bP_{4k}\) is a spin manifold with vanishing real
Pontrjagin classes, and that the bounding manifold \(X_p\) is shown to be parallelisable in
[W1].

§3 Surgery

The main aim of this section is to give a synopsis of the paper [W2], which studies
Ricci positive surgery. (See [R] for a general introduction to surgery.) There are no new
results in this section. Note that the notation being used here differs slightly from that
used in [W2]. The arguments in the current paper have the constructions of [W2] as their
starting point, and for this reason we will highlight the details from [W2] that we will need
later on.

All the exotic spheres under consideration in this paper can be constructed easily
using a sequence of surgery operations. The details of this construction (see [W1]) can
be deduced from the plumbing description mentioned in §2, as plumbing and surgery are
closely related procedures.

Consider a single plumbing between a disc bundle \(\bar{M}\) and \(\bar{E}\), a disc bundle over a
sphere. It is easy to see that the boundary of this plumbing is precisely the result of
performing a surgery on a fibre sphere of \(\partial \bar{M}\). Thus the exotic spheres described in §2
can be constructed, starting with an \(S^{2k-1}\)-bundle over \(S^{2k}\), by performing successive
surgeries. After performing a single surgery, the global \(S^{2k-1}\)-bundle structure will be
destroyed. However, *locally* this structure will be retained. Each of the successive surgeries
required to construct the exotic sphere is a surgery on a ‘fibre’ \(S^{2k-1}\) arising from such a
local bundle structure. In fact, this will always mean performing surgeries on the site of previous surgeries in an iterative manner.

Suppose we want to perform surgery on an embedded sphere $S^{n-1} \subset M^{n+m}$ with trivial normal bundle. In general, different trivialisations of the normal bundle lead to topologically different manifolds on completing the surgery. We will take the following approach to describing such surgeries.

Fix an annulus $A^n = S^{n-1} \times I$ for some closed interval $I$. Consider the product $A^n \times S^m$. View this as an $S^m$-bundle over $A^n$, and extend over the inside of the annulus to obtain an $S^m$-bundle over the disc $D^n$, which we will denote by $E$. Topologically, $E$ must be trivial since its base is contractible. However, for the purposes of surgery, we will only regard $E$ as trivial if there is a bundle map $E \to D^n \times S^m$ which extends the identity map $A^n \times S^m \to A^n \times S^m$. Given an embedding $S^{n-1} \times D^{m+1} \subset M$, we form

$$\hat{M} := M - \text{int}(S^{n-1} \times D^{m+1}) \cup E.$$ 

$\hat{M}$ is said to be the result of performing a surgery on the embedded sphere $S^{n-1} \times \{0\}$. It is clear that performing surgeries using different trivialisations of the normal bundle corresponds to using different bundles $E$ in the above description.

The following is a re-formulation of Theorem 0.3 in [W2]:

**Theorem 2.** Let $n \geq m + 1 \geq 3$. Suppose that $S^{n-1} \times D^{m+1} \subset M$ has an induced metric of the form $\rho^2 ds_{n-1}^2 + dt^2 + N^2 \sin^2(t/N) ds_m^2$ for constants $\rho$ and $N$, and where $t \in [0, R]$ for some $R \in (0, N\pi/2)$. Then there exists $\kappa = \kappa(n, m, \cos(R/N), E) > 0$ such that if $\rho/N < \kappa$, there exists a submersion metric on $E$ which gives a smooth Ricci positive metric on $\hat{M}$. Moreover, this submersion metric on $E$ can be chosen so that in a very small ball at the centre of $E$ it takes the form $\tilde{\rho}^2 ds_m^2 + dr^2 + \tilde{N}^2 \sin(r/\tilde{N}) ds_{n-1}^2$ for values of $\tilde{\rho}$ and $\tilde{N}$ allowing subsequent Ricci positive surgeries on the fibre spheres of $S^m$ of $E$.

Note that to perform this subsequent surgery we need $n = m + 1$, which is precisely the condition arising in the exotic sphere construction discussed above. Thus iteration of the Ricci positive surgery procedure is possible in this situation, and can be used to construct Ricci positive metrics on these exotic spheres.

In the proof of Theorem 2 it suffices to consider the case where $E$ is trivial. The reason for this is explained in detail in [W2], but is essentially due to the fact that the connection (horizontal distribution) which together with the fibre and base metrics gives the submersion metric on $E$, can be chosen to be flat over the complement of any small central annulus in the base disc $D^n$. (Non-triviality of $E$ prevents this connection from being globally flat.) Assuming that all fibre spheres are round and that fibre radii vary according to a rotationally symmetric function on $D^n$, the contribution to the Ricci curvature of the non-flat part of the connection can be rendered insignificant by arranging for the fibre scaling function, restricted to the region of non-flatness, to be sufficiently close to 0 in a $C^2$-sense. Thus for the purposes of Ricci curvature estimation, we are free to assume that the connection is flat (and so the metric is isometric to a warped product over the base), provided a suitable choice can be made for the fibre scaling function near the centre of $D^n$. 

The proof of Theorem 2 involves a metric of the form

\[ G := dr^2 + h^2(r)ds_{n-1}^2 + k^2(f(r) + \alpha)^2ds_m^2 \]

on \( D^n \times S^m \), where \( r \in [0, a] \) for some \( a > 0 \), \( h(r) \) and \( f(r) \) are (modifications of) solutions to certain initial value problems, and where \( k, \alpha \in \mathbb{R}^+ \). These metric scaling functions have the profiles displayed in Figure 1.

![Figure 1](image)

The Ricci curvatures of \( G \) are easy to calculate (see for example [SY]). If \( X \) and \( U \) denote \( G \)-unit vectors tangent to \( S^{n-1} \) respectively \( S^m \), we have:

\[
\begin{align*}
\text{Ric}(\partial/\partial r) &= -(n-1)h^{-1}h'' - m(f + \alpha)^{-1}f'' \\
\text{Ric}(X) &= -h^{-1}h'' + (n-2)h^{-2}[1 - (h')^2] - mh^{-1}h'(f + \alpha)^{-1}f' \\
\text{Ric}(U) &= -(f + \alpha)^{-1}f'' + (m-1)k^{-2}(f + \alpha)^{-2}[1 - (kf')^2] - (n-1)h^{-1}h'(f + \alpha)^{-1}f' \\
\text{Ric}(\partial/\partial r, X) &= \text{Ric}(\partial/\partial r, U) = \text{Ric}(X, U) = 0.
\end{align*}
\]

Let \( G_{\nabla} \) denote the submersion metric on \( E \) corresponding to \( G \) in the case of a non-flat horizontal distribution \( \nabla \). (That is, the submersion metric determined by \( \nabla \) together with the fibre metrics \( k^2(f(r) + \alpha)^2ds_m^2 \) and the base metric \( dr^2 + h^2(r)ds_{n-1}^2 \).)

We will now summarise the construction of the functions \( f(r) \) and \( h(r) \) in the expression above. For each function there is a sequence of steps in the construction, but we will use the same label \( f \) or \( h \) at each stage in the process, with \( G \) always denoting the corresponding warped product metric.
We begin by defining $f(r)$ and $h(r)$ to be the solutions of a certain initial value problem:

$$f'' = f^{-1}, \quad f'(0) = 0, \quad f(0) = 1; \quad h = f'. $$

It is easy to show that $f''(r) > 0$ and $h''(r) < 0$ for all $r > 0$. We then modify $f$ over the interval $r \in [0,1/2]$ keeping the second derivative non-negative, so that $f(r) \equiv 1$ for $r \in [0,1/4]$. We also modify $h$ over $[0,1/2]$ keeping its second derivative strictly negative, so that for $r \in [0,R')$ we have $h(r) = \sin r$, where $0 < R' < 1/4$ is a fixed constant independent of all other parameters. The reason for these first modifications is to ensure the resulting metric will be of a suitable form near the centre of the base $D^n$ to facilitate further Ricci positive surgeries. The values taken by $f$ and $h$ are now fixed for $r \in [0,1/2]$ and unaffected by future considerations.

The role of the constant $\alpha$ in $G$ is to compensate for this modification of $f$: the deformation will reduce the Ricci curvature, and $\alpha$ can be chosen to ensure that the Ricci curvature of the modified metric will remain positive. In fact $\alpha = \alpha(n,m)$ and as a result, if multiple surgeries are to be performed a single value for $\alpha$ can be chosen in advance for all surgeries. We will assume such a value has been fixed for our constructions.

In the case of a single Ricci positive surgery, the role of the constant $k$ in the definition of $G$ is twofold. (We will look at the case of multiple, iterative surgeries of the kind described at the start of this section later on.) Firstly, it is used to ensure that the contribution of $\nabla$ to the Ricci curvature of $G\nabla$ is rendered insignificant. Precisely, as $k \to 0$, the Ricci curvatures of $G\nabla$ tend to those of $G$ (see [W2; Lemma 1.2]), so that the Ricci positivity of $G$ guarantees the Ricci positivity of $G\nabla$. For a fixed $\nabla$ on $E$, all sufficiently small $k$ will do this. The second role of $k$ is that it guarantees that the boundary conditions required for smooth gluing in the surgery process can be achieved within Ricci positivity. (Specifically, it ensures that the derivative of $k(f(r) + \alpha)$ attains the value $\cos(R/N)$. We will say more about this below.) Overall, we have $k = k(\alpha, \nabla, n, m, \cos(R/N)).$

The Ricci curvature of the metric $G$ is not positive for all $r$. For given $k$, there is a number $z_k > 1/4$ such that some Ricci curvatures are zero when $r = z_k$, but all Ricci curvatures are strictly positive for $r < z_k$. It can be shown ([W2; Lemma 2.9]) that $z_k \to \infty$ as $k \to 0$ and $k f'(z_k) \to 1$ as $k \to 0$. Thus, as remarked above, a slope of $\cos(R/N)$ can be achieved by $k(f(r) + \alpha)$ within Ricci positivity provided $k$ is chosen sufficiently small. (In fact a slight excess of slope beyond the value $\cos(R/N)$ is desirable to facilitate smooth Ricci positive gluing.)

We now modify $f$ again near $r = z_k$ to ‘straighten’ its graph: that is, we cause the second derivative of $f$ to drop quickly and smoothly to the constant value zero. As soon as constant slope for $f$ has been achieved, we ‘flatten’ $h$ in a similar fashion, by quickly making $h''$ more negative to lower $h'$, followed by a sharp increase of $h''$ to 0 in such a way that $h$ becomes constant. Let us label the point beyond which $h' \equiv 0$ by $R''$, and the point at which our (modified) $f$ ceases to be strictly concave up by $R'''$. We have $R'' = R''(k, \cos(R/N))$ and $R''' = R'''(k, \cos(R/N))$, and also $R''' < R''$. This modification of $f$ and $h$ can be achieved within Ricci positivity, however note that for $r \geq R'''$, the Ricci curvature is only non negative (though the scalar curvature is strictly positive). By [E], our metric can be deformed smoothly to a Ricci positive metric. This is in fact easy to see directly, as introducing some slight concavity in $f(r)$ for $r > R'''$ is enough to keep the
Ricci curvature strictly positive, but the size of this concavity can be controlled so that the boundary conditions for \( f \) can still be met.

The value of \( \kappa \) in the statement of Theorem 2 can be taken to be

\[
\frac{h(R'')}{k(f(R'') + \alpha)}.
\]

It is easy to show that

\[
\lim_{r \to \infty} \frac{h(r)}{k(f(r) + \alpha)} = 0,
\]

so that if \( \rho/N < \kappa \), there exists \( a > R'' \) for which

\[
\frac{h(a)}{k(f(a) + \alpha)} = \frac{\rho}{N}.
\]

We have \( a = a(k, \cos(R/N), \rho/N) \). We fix this value of \( a \) and let \([0, a]\) be the range of the parameter \( r \) for the metric \( G \). Similarly for \( G_{\nabla} \).

The metric \( G \), or rather its corresponding metric \( G_{\nabla} \), is now nearly ready to use for Ricci positive surgery. There are still, however, two final modifications. The metric must be globally rescaled so as to be the right size to fit in the surgery ‘hole’. (In general, the metric \( G_{\nabla} \) will be too big.) Note that a global rescale has no effect on the positivity of either the Ricci or the scalar curvatures. Explicitly, the global rescale factor should be \( \rho/h(a) \). Prior to this rescale, a small concave down adjustment should be made to \( f \) in a neighbourhood of the boundary so that following the rescale, the surgery gluing is smooth.

Note that after the global rescale, a parameter change can be effected to bring the metric \( G \) back into its original form.

We will denote the rescaled metrics by \( G^* \) respectively \( G_{\nabla}^* \).

The following is clear, and will be needed in §4:

**Observation 3.** The dependence of \( a \) on \( k \) is smooth. Moreover, the dependence of the rescaled, adjusted function \( f \) on \( a \) can be arranged to be smooth.

To conclude this section, we must consider the situation where multiple surgeries are to be performed in an iterative manner as discussed at the start of §3, as it is by this process that the exotic spheres we wish to consider are constructed.

We begin by noting that as an automatic consequence of the way in which we defined the functions \( f \) and \( h \) above, the neighbourhood of \( D^n \times S^m \) corresponding to \( r \in [0, R'] \) when equipped with the metric \( G \) has the form required to perform a (further) surgery on \( \{0\} \times S^m \) according to Theorem 2, provided an appropriate ratio condition \( \rho_1/N_1 < \kappa_1 \) is met. Here, \( \rho_1 = k(1 + \alpha) \) and \( N_1 = 1 \). Moreover, [W2; Proposition 0.4] shows that for this subsequent surgery, we can take the ratio \( R_1/N_1 \) to be some fixed constant which is independent of all other parameters. In fact, for our purposes in the current paper, we will assume \( R_1/N_1 = \delta/4 \), where \( \delta \) is a constant introduced in §4 depending only on the dimension of the surgery, and which can be assumed fixed for all surgeries in a given exotic sphere construction process.
Consider an iterative sequence of surgeries. We have a ratio $R/N$ arising from the first surgery, and the corresponding ratio for all subsequent surgeries is fixed at $\delta/4$. In order to construct a metric for the whole sequence, we proceed surgery by surgery, in reverse order. For the last surgery, we choose $k_n = k_n(\nabla_n, \cos(\delta/4))$ in order to produce the Ricci positive metric $G_\nabla$ on the bundle $E_n$. This then determines the quantity $\kappa_{n-1}$. For this final surgery to be successful, we need $\rho_{n-1}/N_{n-1} = k_{n-1}(1 + \alpha) < \kappa_{n-1}$. Thus for the penultimate surgery we need to choose

$$k_{n-1} = k_{n-1}(\nabla_{n-1}, \cos(\delta/4), \kappa_{n-1}).$$

Continuing in similar fashion allows a metric $G_{\nabla,i}$ to be constructed for the $i$-th surgery ‘piece’, so that after rescaling, all pieces fit together to give a global Ricci positive metric.

§4 A metric deformation

In this section we show how to deform the Ricci positive surgery metric of Theorem 2, through positive scalar curvature metrics, to a metric with a suitable form for later constructions. (See the overview given in §1.) We do this in two stages. Firstly, in Lemma 4 we consider the case of flat horizontal distribution on $E$; that is, we show how to deform the metric $G$ of §3. We then extend the deformation to the case where the horizontal distribution is not flat (Proposition 7); in other words we show how to deform $G_\nabla$. Following this, we show how these deformations can be fitted together in the case of iterative surgeries, and deduce that a global deformation of the Ricci positive exotic sphere metric is possible.

Lemma 4. Given a suitably small number $c > 0$, the metric $G$ of §3 can be deformed smoothly through positive scalar curvature metrics to a metric of the form

$$dr^2 + h_\infty(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m,$$

where $h_\infty(r)$ is a concave down function satisfying $h_\infty(r) = \sin r$ for $r \in [0, \delta]$ where $\delta = \delta(c) < R'$, and $h_\infty(r) \equiv c$ for $r > R'$. (See Figure 2). Moreover, this deformation can be performed keeping the metric constant for $r \in [0, \delta]$.

Proof. The deformation will be parameterised using a ‘time’ parameter $\tau \in [0, 1]$. Set $h_\tau := (1 - \tau)h + \tau h_\infty$. Notice that $h_\tau$ is a concave down function for each $\tau$. We need to check that provided $c$ is chosen sufficiently small in the definition of $h_\infty$, then $dr^2 + h_\tau(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m$ has positive scalar curvature for each $\tau$. 

9
An elementary calculation shows that the scalar curvature of this metric is given by

\[-2(n - 1) \frac{h''}{h_\tau} - 2m \frac{\psi''}{\psi} + (n - 1)(n - 2) \left(1 - \frac{(h'_\tau)^2}{h^2_\tau}\right) + m(m - 1) \frac{(\psi')^2}{\psi^2} - 2(n - 1)m \frac{h'_\tau \psi'}{h_\tau \psi}\]

where \(\psi(r) := k(f(r) + \alpha)\), and all the derivatives are taken with respect to \(r\).

First note that there is nothing to prove for \(r \in [0, 1/4]\) as \(\psi(r)\) is constant at these values of \(r\), and by concavity, the sum of the terms involving only \(h_\tau\) and its derivatives is strictly positive for all \(r\) and for each \(\tau \in [0, 1]\). From now on we therefore assume that \(r \geq 1/4\).

Next, recall that \(f(r)\), and hence \(\psi(r)\), is concave down for \(r > R''\) (see §3), and therefore also for \(r \geq R''\). For each \(\tau \in [0, 1]\), \(h_\tau(r)\) is a constant function for \(r \geq R''\). This means that each term in the scalar curvature expression is non-negative (and in some cases strictly positive) for these values of \(r\). In the remainder of the proof, we can therefore restrict our attention to the compact interval \(r \in [1/4, R'']\).

We next claim that the term \(h'_\tau \psi'/h_\tau \psi\) is decreasing with respect to \(\tau\) for each \(r\). As \(\psi\) is independent of \(\tau\), it suffices to consider the behaviour of

\[
\frac{h'_\tau}{h_\tau} = \frac{(1 - \tau)h' + \tau h'_\infty}{(1 - \tau)h + \tau h_\infty} = \frac{h'}{h + \frac{\tau}{1 - \tau} c}
\]

for these values of \(r\). Now \(\tau/(1 - \tau)\) is strictly increasing for \(\tau \in [0, 1]\), and this establishes the claim.

To complete the proof of Lemma 4, it therefore suffices to check that \(-2h''/h_\tau + (n - 2)(1 - (h'_\tau)^2)/h^2_\tau\) is non-decreasing with \(\tau\). To this end, a straightforward calculation shows that the derivative of this expression with respect to \(\tau\) is equal to

\[
\frac{2}{[(1 - \tau)h + \tau c]^3} \left[ c h''[(1 - \tau)h + \tau c] + (n - 2)(h - c) + (n - 2)c(1 - \tau)(h')^2 \right].
\]

\(\dagger\)
As $2[(1 - \tau)h + \tau c]^{-3} > 0$, we only need check the positivity of the term $ch''[(1 - \tau)h + \tau c] + (n - 2)(h - c) + (n - 2)c(1 - \tau)(h')^2$. Consider the limit of this last expression as $c \to 0$. This limit is simply $(n - 2)h$, which is positive. By the compactness of $[1/4, R''']$, we can clearly choose a value for $c$ which allows the positivity of $(n - 2)h(r)$ to dominate all other terms over this interval.

Having fixed a value for $c$, we finally set $\delta$ to be any number consistent with the requirements for $h_\infty$ specified in the statement of the Lemma. \hfill \Box

**Lemma 5.** An upper bound for the value of $c$ in Lemma 4 can be chosen depending only on $n$ and $h(r)$ for $r \in [1/4, 1/2]$.

**Proof.** From the proof of Lemma 4, it is clear that an upper bound for $c$ depends only on $n$ and the function $h(r)$ over the range $r \in [1/4, R''']$. Recall from the discussion after Theorem 2 that on the interval $[1/2, R''']$, $h''(r)$ is negative and increasing. This means that if $\lambda := \sup |h''(r)|$ for $r \in [1/4, R''']$ then

$$
\lambda = \max\{ \sup_{r \in [1/4, 1/2]} |h''(r)|, h''(1/2) \}.
$$

Consider the expression (5) above. Ignoring the (good) term $(n - 2)c(1 - \tau)(h')^2$, we can certainly keep the expression positive by choosing a value for $c$ such that $(n - 2)(h - c) + ch''[(1 - \tau)h + \tau c] > 0$. We do this first in the case where $r \in [1/4, R''']$. Here, it clearly suffices to keep $(n - 2)(h - c) > c\lambda[(1 - \tau)h + \tau c]$. This rearranges to give the inequality

$$
c^2\tau\lambda + (n - 2)c < h(n - 2 - c\lambda(1 - \tau)).
$$

Now $h$ is an increasing function, so over our interval we have $h(1/4) \leq h(r)$. Setting $\mu = h(1/4)$, it will be sufficient to choose $c$ such that

$$
c^2\lambda + (n - 2)c < \mu(n - 2 - c\lambda),
$$

or equivalently

$$
c^2\lambda + c(n - 2 + \mu\lambda) < (n - 2)\mu.
$$

It is obvious that this inequality will be true for all $c$ sufficiently small, and moreover the upper bound will depend on $n$, $\lambda$ and $\mu$. But $\lambda$ and $\mu$ depend on the values taken by $h$ over $[1/4, 1/2]$, so the result would established if we were considering only $r \leq R'''$.

To complete the proof, we claim that provided $c$ is chosen to produce positive scalar curvature for all $\tau \in [0, 1]$ at $r = R'''$, then we can arrange for the function $h'(r)$ in the interval $r \in [R'', R''']$ to be such that the positivity of the scalar curvature is maintained throughout this interval and for all $\tau$, without the need to re-choose $c$. As explained in §3 (with reference to [W2; Lemma 2.15]), $h$ should bend in a concave down manner to a constant function, starting at $r = R'''$, over a suitably short interval of length $\epsilon$. We choose this bend in the following way. Bend $h$ sharply so that very close to $r = R'''$ we have

$$
|h''| > mh'f'/f.
$$
This in itself guarantees positive scalar curvature for all $\tau$. Maintain this rate of bend so that $h''$ continues to satisfy this inequality, until $h'$ is so small that the inequality

$$2(n-1)\frac{h'f'}{hf} < (m-1)\frac{1-f'^2}{f^2}$$

is then satisfied. This also guarantees the positivity of the scalar curvature for all $\tau$, since $h'/h_\tau$ decreases with $\tau$ as established in the proof of Lemma 4. As the positivity of the scalar curvature does not now depend on the size of $|h''|$, we see that positivity is guaranteed for all subsequent $r$ and for all $\tau$. Finally, make a suitable reduction in $|h''|$ so that $h$ becomes constant.

We deduce:

**Corollary 6.** Uniform values for $c$ and $\delta$ can be chosen in advance for all surgeries.

**Proof.** For $c$, this follows from Lemma 5 together with the fact that $h(r)$ is fixed for all surgeries over the interval $r \in [0,1/2]$. (See §3.) The result for $\delta$ now follows from the fact that $\delta = \delta(c)$. ⊓ ⊔

We now come to the main result of this section.

**Proposition 7.** The submersion metric $G_\nabla$ of §3 can be deformed through positive scalar curvature metrics to the corresponding submersion metric where the original base scaling function $h(r)$ has been replaced by the function $h_\infty(r)$ from Lemma 4.

**Proof.** In the case where the horizontal distribution $\nabla$ is flat, the metric is isometric to the warped product $dr^2 + h(r)ds_{n-1}^2 + k^2(f(r)+\alpha)^2ds_m^2$. The result then follows immediately from Lemma 4.

It therefore remains to deal with the case where the horizontal distribution is not flat. In [W2], any non-flatness is restricted to the region corresponding to $r \in (1,2)$. However, this choice of region is arbitrary, and in the current situation we demand that the horizontal distribution is flat outside the region corresponding to $r \in (\delta/2,\delta)$, where $\delta$ is the constant arising in Lemma 4. In this case, the path of submersion metrics with fixed horizontal distribution corresponding to the path of warped product metrics in Lemma 4 will all have positive scalar curvature, since the metrics are independent of the path parameter for $r \leq \delta$, and at $r > \delta$ they are isometric to the warped products of Lemma 4.

Finally, note that our choice of horizontal distribution depends on $\delta$, and the horizontal distribution in turn influences the choice of $k$ (which counters any ‘bad’ curvature arising from the distribution). If $\delta$ depends in any way on $k$, then we are in danger of falling into a circular argument. However by Corollary 6, $\delta$ can be chosen in advance for all surgeries, and is in particular independent of $k$. The proof is therefore complete. ⊓ ⊔

Having discussed deforming an individual surgery ‘piece’ through positive scalar curvature metrics, we next want to iterate this procedure to create a global metric deformation on an exotic sphere. Below is a summary of the steps which make up this extension.
Fitting deformations together for iterative surgeries

We begin by observing that our deformation can be extended to the following situations. Note that the deformation always destroys positive Ricci curvature: even if the initial metric is Ricci positive, the final metric will not be.

1. **Ricci positive surgery on a fibre** $S^{n-1}$ **of a sphere bundle** $M^{n+m}$ **equipped with a Ricci positive submersion metric having totally geodesic, isometric, round fibres.** To perform surgery we can assume without loss of generality that the metric is locally a product $\rho^2 ds_{n-1}^2 + dt^2 + N^2 \sin^2(t/N) ds_m^2$. We can always arrange for this to be the case by rechoosing the connection and possibly shrinking the fibres if necessary (see [Be; §9G]), and locally deforming the base metric to have constant positive sectional curvature, see [SY; page 134]. According to Theorem 2, there exists a Ricci positive metric on $\hat{M}$, the manifold resulting from the surgery. Recall from §3 that the metric on the ‘glue-in’ piece $E$ is $G_\nabla^*$, the globally rescaled version of $G_\nabla$. After applying this (fixed) global rescale, the deformation of Proposition 7 is clearly extendable to the metric on $\hat{M}$, simply by choosing $\rho$ smaller at each stage in the deformation. Specifically, set $\rho(\tau) = (1 - \tau)h(a) + \tau c$. Note that the positive scalar curvature of the deformation is preserved by the rescaling. (See Figure 3.)

![Scaling functions for $G_\nabla^*$](image1)

Scaling functions for $G_\nabla^*$

![Metric scaling functions for $M$](image2)

Metric scaling functions for $M$

Figure 3

2. **Ricci positive surgery on the central fibre** $S^m$ **in** $E$, **equipped with the submersion metric** $G_\nabla$. Given the form of this metric near the centre of $E$, by Theorem 2 we can certainly perform the proposed surgery within Ricci positivity, provided the value of $k$ for this metric is sufficiently small. Relabelling this value $k_1$, the constants $\rho_1$, $R_1$, $N_1$ for our surgery corresponding to $\rho$, $R$, $N$ in Theorem 2 are given by $\rho_1 = k_1(1 + \alpha)$,
$N_1 = 1$, and $R_1$ can be taken to be $\delta/4$. In fact, suppose we want to perform an iterative sequence of surgeries using bundles $E_2, E_3, \ldots$ (see case 3 below). For the corresponding (unrescaled) metrics $G_{\nabla,n}$ we can take $\rho_n = k_n(1 + \alpha)$, $N_n = 1$ and $R_n = \delta/4$, as by Corollary 6, $\delta$ can be chosen in advance for all such surgeries and the form of the metric in a neighbourhood of the surgery is independent of all other parameters once $\delta$ has been fixed.

For the ‘glue-in’ piece $E_2$ (topologically $D^{m+1} \times S^{n-1}$), let us call the ‘new’ metric scaling functions $h_2(r)$ and $k_2(f_2(r) + \alpha)$, with the original functions for $E = E_1$ being labelled $h_1(r)$ and $k_1(f_1(r) + \alpha)$. After the appropriate global rescale (so $E_2$ fits the ‘hole’ created in $E$) the deformation of $h_2$ to $h_2,\infty$ forces a re-choosing of $k_1$ in order to keep the gluing between the two pieces smooth. In turn, by Observation 3 we can achieve the boundary conditions for $f_1$ in a smoothly varying way. (See Figure 4.)

**Figure 4**

3. Having performed the surgery on $E = E_1$ in case 2 above, consider performing a further surgery at the centre of $E_2$. Suppose that the new ‘glue-in’ piece $E_3$ has a metric defined by functions $h_3$ and $f_3$, and a constant $k_3$. First deform the metric on $E_3$ as in case 2 and globally rescale, choosing $k_2$ smaller as necessary in the $E_2$ metric. Note that the required global rescale factor for performing surgery between $E_2$ and $E_1$ will vary smoothly with $k_2$ (a consequence of Observation 3 again).

Next perform the deformation on $E_2$. This has no consequences for $E_3$ as the $E_2$ metric is fixed through this deformation at radii $r \in [0, \delta]$, and it is within this region (specifically at $r = \delta/4$) that the surgery with $E_3$ is performed. In order to join
$k_2(f_2 + \alpha)$ with $h_1$, at each stage of the deformation we perform a global rescale of $E_2$ and $E_3$ simultaneously using a smoothly varying factor as discussed in the previous paragraph. If necessary, we choose $k_1$ smaller (in a smoothly varying way) to join $f_1$ to $h_2$. Note that we can assume without loss of generality that $k_1(1 + \alpha)$ (the initial value of $k_1(f_1(r) + \alpha)$) is greater than or equal to the boundary value of $h_{2,\infty}$, by choosing $a_2$ (that is, the base radius of $E_2$) larger if necessary to force a smaller global rescale.

Finally, we deform $h_1$ to $h_{1,\infty}$ in $E_1$. In the same way that the $E_2$ deformation has no effect on $E_3$, this final deformation has no consequences for $E_2$ and $E_3$.

It now follows that a Ricci positive metric constructed using the methods of [W2] on an exotic sphere built according to the surgery description of [W1] can be deformed through positive scalar curvature metrics by successively modifying the metric on each surgery ‘piece’ using the three cases outlined above. If $\Sigma$ is such an exotic sphere, we will refer to the resulting metric as ‘the deformed positive scalar curvature metric on $\Sigma$’.

§5 Handles

In this section we show how to extend certain surgery metrics across the corresponding handlebody.

Let $M^{n+m}$ be a manifold, some neighbourhood of which is diffeomorphic to $S^{n-1} \times D^{m+1}$. Suppose that we want to perform surgery on $S^{n-1} \times \{0\}$. Suppose further that $M = \partial M$. Then the manifold $\hat{M}$ which results from the surgery can be realised as the boundary of a manifold created by attaching a handle to $\hat{M}$. Here, the handle is simply the disc bundle $\hat{E}$ corresponding to the sphere bundle $E$ (see §3).

Notice that the plumbing of two disc-bundles over spheres can be viewed as attaching a handle to either of the bundles. This means, for example, that the plumbed manifolds described in §2 having exotic spheres as boundary can be thought of as being constructed from a disc bundle over a sphere by successively adding handles.

We will prove a metric result about adding handles (Proposition 9), but first we need a lemma concerning metrics on the disc bundle $\hat{E}$. We will say that a smooth Riemannian manifold with boundary has positive scalar (respectively positive Ricci) curvature when the metrics on the interior and the boundary both have positive scalar (respectively Ricci) curvature.

**Lemma 8.** Suppose that $E$ is the total space of a bundle with base $M$, a manifold with a fixed Ricci positive metric, fibre $S^{n-1}$, and structural group $SO(n)$. Let $\theta(s)$, $s \geq 0$, be any function satisfying: $\theta$ is odd at $s = 0$, $\theta'(0) = 1$, $\theta''(s) < 0$ for all $s > 0$, $\theta'(s) > 0$ for all $s$, and $\lim_{s \to +0} \theta''(s)/\theta(s) < 0$. For a given choice of principal connection $\nabla$ on the principal bundle associated to $E$, there exists $R_0 = R_0(\nabla; \theta)$ such that for all $\rho \leq R_0$, the submersion metric on $E$ with fibres isometric to $\theta^2(\rho)ds_{n-1}^2$ and the submersion metric on the corresponding $D^n$-bundle $\hat{E}$ with fibres isometric to $ds^2 + \theta^2(s)ds_{n-1}^2$ where $s \in [0, \rho]$, both have positive Ricci curvature.
Proof: We view the metric on \( \bar{E} \) as \( ds^2 + g_s \), where \( g_s \) is the induced submersion metric on \( E \) with fibre radius \( \theta(s) \). The following Ricci curvature formulas for \( ds^2 + g_s \) are easily computed (for example by adapting the formulas in [W3; Proposition 4.2]).

\[
\begin{align*}
\text{Ric}(\partial/\partial s, \partial/\partial s) = -(n-1) \frac{\theta''}{\theta}; \\
\text{Ric}(X_i, X_i) &= \text{Ric}_M(X_i, X_i) - 2\theta^2 \langle AX_i, AX_i \rangle; \\
\text{Ric}(U_j, U_j) &= \frac{(n-2)}{\theta^2} (1 - \theta'^2) - \frac{\theta''}{\theta} + \theta^2 \langle AV_j, AV_j \rangle; \\
\text{Ric}(X_i, U_j) &= -\theta \langle (\delta \bar{A}) X_i, V_j \rangle.
\end{align*}
\]

Here, \( \{X_i\} \) are (the lifts to \( E \) of) orthonormal vector fields tangent to the base \( M \); \( \{U_j\} \) are orthonormal vector fields tangent to the distance spheres \( S^{n-1} \subset D^n \), and the \( \{V_j\} \) are the corresponding vector fields which would be orthonormal with respect to the metric \( ds_n^2 \); the \( A \)-tensor expressions appearing here are those for the unit sphere bundle diffeomorphic to \( E \) (see [Be; §9G] for definitions), and so are independent of \( s \).

The concavity of \( \theta \) ensures that \( \text{Ric}(\partial/\partial s, \partial/\partial s) > 0 \). The Ricci positivity of \( M \) ensures \( \text{Ric}(X_i, X_i) > 0 \) provided \( s \) (and hence \( \theta(s) \)) is suitably small. Similarly, the positivity of \( \text{Ric}(U_j, U_j) \) is clear.

It remains to check the influence of the ‘mixed’ term \( \text{Ric}(X_i, U_j) \). (Note that the other mixed terms \( \text{Ric}(\partial/\partial s, X_i) \) and \( \text{Ric}(\partial/\partial s, U_j) \) vanish.) To ensure that this term does not upset Ricci positivity we need to check that

\[
\text{Ric}(X_i, X_i)\text{Ric}(U_j, U_j) > (\text{Ric}(X_i, U_j))^2.
\]

Using the above expressions and considering both sides of this inequality as \( s \to 0 \), we arrive at the limiting inequality

\[
\text{Ric}_M(X_i, X_i) \left( \lim_{s \to 0^+} \frac{(n-2)}{\theta^2(s)} (1 - \theta'^2(s)) - \frac{\theta''}{\theta(s)} \right) > 0,
\]

which is true. Thus the bundle metric on \( \bar{E} \) is Ricci positive for \( \rho \) suitably small. It is also clear (for example by [Be; §9G]) that the metrics \( g_s \) on \( E \) have positive Ricci curvature for all suitably small \( s \). \( \square \)

Proposition 9. Let \( M \) be a manifold of dimension at least five, equipped with a positive scalar curvature metric. Suppose that \( M \) is the boundary of \( \bar{M} \), and that the metric on \( M \) can be extended to give a positive scalar curvature metric on \( \bar{M} \). Given an embedding of \( S^{n-1} \times D^{m+1} \) into \( \bar{M} \) where \( n \geq m + 1 \geq 3 \), consider performing surgery using this embedding together with a sphere bundle \( E \), and the associated handle addition to \( \bar{M} \) of the corresponding disc bundle \( E \). Using the outward normal parameter to the boundary, extend the embedding to an embedding of \( [-\epsilon, 0] \times S^{n-1} \times D^{m+1} \) into \( \bar{M} \). Assume that the metric on \( \bar{M} \) pulled back to this product takes the form

\[
ds^2 + \phi^2(s)ds_{n-1}^2 + dt^2 + N^2 \sin^2(t/N)ds_m^2,
\]

16
where \( s \) parametrises \([-\epsilon, 0]\) and \( t \in [0, R] \) is the radial parameter of \( D^{m+1} \). Suppose further that \( \phi'(s) > 0 \) and \( \phi''(s) < 0 \) for all \( s \in [-\epsilon, 0] \). Then there exists \( \kappa = \kappa(n, m, R/N, \tilde{E}) > 0 \) such that if \( \phi(0)/N < \kappa \), we can extend the metric on \( \bar{M} \) to give a smooth positive scalar curvature metric on \( M \cup \tilde{E} \).

**Proof:** We will first consider the case where \( \tilde{E} \) is trivial: \( \tilde{E} = D^n \times D^{m+1} \). Recall that under the above conditions, there exists a metric \( G = dr^2 + h^2(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m \) on \( D^n \times S^{m} \) which has positive Ricci curvature, and after a global rescaling to a metric \( G^* \) glues smoothly onto \( M - \text{int}(S^{n-1} \times D^{m+1}) \). Notice that we can still complete this surgery within positive scalar curvature using the deformed metric \( G_{\infty} = dr^2 + h_{\infty}(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m \) of Lemma 4, again after a suitable global rescale. A smaller choice of \( \kappa \) will be required for this, but this new \( \kappa \) (which is determined by \( G_{\infty} \)) still depends on the same parameters, since the upper bound for \( c \) (the constant value taken by \( h_{\infty}(r) \) for suitably large \( r \)) depends only on \( n \), as established in Lemma 5. It is this deformed metric which we extend over \( \bar{E} \).

As noted above, in order to complete the metric surgery, we need to globally rescale \( G \) or \( G_{\infty} \) to \( G^* \) respectively \( G^*_{\infty} \). After such a global rescale, a parameter change can be effected to bring the metric back into the same form as \( G \) or \( G_{\infty} \). In order to simplify the exposition of the construction below, we will abuse notation, and assume without loss of generality from now on that \( dr^2 + h_{\infty}(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m \) denotes the globally rescaled metric. As a consequence, the parameter \( r \), the functions \( h_{\infty} \) and \( f \), the constants \( k \) and \( \alpha \), together with related quantities such as \( a \) and \( \delta \), will not in general be the same as before the rescale. However, this change of notation has no effect on the validity of the arguments below. This is because the only features of this metric that we will need are the derivatives of \( k(f(r) + \alpha) \), which are invariant under rescale/parameter change, and the fact that \( f(r) \) and \( h_{\infty}(r) \) are constant for \( r \) small respectively \( r \) large. Again, these properties are clearly preserved by the rescale.

The deformed metric \( dr^2 + h_{\infty}(r)ds^2_{n-1} + k^2(f(r) + \alpha)^2ds^2_m \) on \([0, a] \times S^{n-1} \times S^{m}\), can be viewed as consisting of a ‘cap’ \([0, a] \times S^{n-1}; dr^2 + h^2_{\infty}(r)ds^2_{n-1}\) which is topologically a disc \( D^n \), and a ‘tube’ \([0, a] \times S^{m}; dr^2 + f^2(r)ds^2_m\), which share the parameter \( r \). Let us consider the issue of smoothly gluing this object into the ‘ambient’ manifold \( M \), after removing the interior of the image of \( S^{n-1} \times D^{m+1} \). By considering a slightly smaller disc \( D^{m+1} \) if necessary, we may assume that for some \( \epsilon' > 0 \), a neighbourhood of the boundary of the ambient manifold takes the form \([|R, R + \epsilon'|] \times S^{n-1} \times S^{m}; dt^2 + \phi^2(0)ds^2_{n-1} + N^2 \sin^2(t/N)ds^2_m\).

We can then split the gluing problem into two: the problem of gluing the ‘cap’ onto its ‘ambient continuation’ \([|R, R + \epsilon'|] \times S^{n-1}; dt^2 + \phi^2(0)ds^2_{n-1}\); and the problem of gluing the ‘tube’ onto its ‘ambient continuation’ \([R, R + \epsilon'] \times S^{m}; dt^2 + N^2 \sin^2(t/N)ds^2_m\). Clearly, what we need is the smooth joining of the functions \( h_{\infty}(r) \) to the constant function with value \( \phi(0) \), and \( k(f(r) + \alpha) \) to \( N \sin(t/N) \), when the \( r \) and \( t \) parameters are suitably concatenated.

We want to go from this surgery picture to adding the handle \( \bar{E} \) to \( \bar{M} \) in such a way that the boundary metric agrees with that for the surgery, and of course whole metric must be smooth and have positive scalar curvature. Topologically, going from \( E \) to the handle \( \bar{E} \) requires the ‘filling-in’ of the fibre spheres \( S^{m} \), that is, replacing the ‘tube’ by a corresponding ‘solid tube’. (See Figure 5.) Again, the smooth gluing problem can be
divided into gluing the ‘cap’ onto its continuation \([-\varepsilon, 0] \times S^{n-1}; ds^2 + \phi^2(s)ds_{n-1}^2\) and the ‘solid tube’ onto its continuation \([-\varepsilon, 0] \times D^{m+1}; ds^2 + dt^2 + N^2 \sin^2(t/N)ds_m^2\). Note that in this handle picture, the sphere radii in the cap are naturally parametrised by \(s\), the parameter running along the central axis in the solid tube. Thus \(s\) will be a shared parameter in the handle metric in the same way that \(r\) was a shared parameter for the surgery metric.

Figure 5: the ‘solid tube’.

We focus first on the ‘solid tube’. Our approach to defining a metric on this is to view it as a subset of a ‘big tube’ \([0, b] \times D^{m+1}; ds^2 + dt^2 + N^2 \sin^2(t/N)ds_m^2\), for some \(b > 0\) to be chosen later. (See Figure 6.)

Figure 6: the ‘big tube’.

For simplicity, we are using the same symbols \(s\) and \(t\) for the parameters here as for the corresponding parameters in the ambient manifold, on the understanding that a (metrically insignificant) adjustment will need to be made to the \(s\) parameter on one side of the join in order for concatenation to take place. This ‘big tube’ will clearly glue onto
([-\epsilon,0] \times D^{m+1}; ds^2 + dt^2 + N^2 \sin^2(t/N)ds_m^2)$ on concatenation, to give a smooth positive scalar curvature metric on the interior. We must therefore show how to cut the ‘solid tube’ from the ‘big tube’ in such a way that the boundary of the resulting object is smooth, and has a metric which agrees with that coming from the corresponding surgery.

We will define a curve $\gamma(r)$ in the $s - t$-plane (with metric $ds^2 + dt^2$) which will define the solid tube. The $s$-coordinate will parameterize the central axis of the solid tube, and the $t$-coordinate will determine the radius of the boundary spheres $S_m^n$. We will write $\gamma(r) = (\gamma_s(r), \gamma_t(r))$.

Recall that the (boundary) tube must have induced metric $dr^2 + k^2(f(r) + \alpha)^2 ds_m^2$. For such a warped product, the parameter $r$ measures distance around the tube (as opposed to along the central axis). For this reason we clearly need $\gamma(r)$ to be a unit speed curve, that is, $\gamma_s'(r) + \gamma_t'(r) \equiv 1$.

We need to choose $\gamma_t(r)$ so that the distance sphere in $(D^{m+1}; dt^2 + N^2 \sin^2(t/N)ds_m^2)$ at a distance of $\gamma_t(r)$ from the centre is round of radius $k(f(r) + \alpha)$. In other words we need $N\sin(\gamma_t(r)/N) = k(f(r) + \alpha)$, and therefore

$$
\gamma_t(r) = N \sin^{-1}(N^{-1}k(f(r) + \alpha)).
$$

Using this together with $\gamma_s'(r) + \gamma_t'(r) \equiv 1$ allows us to solve for $\gamma_s(r)$: namely

$$
\gamma_s(r) = \int_0^r \left(1 - \frac{\psi'^2(u)}{1 - N^2\psi^2(u)}\right)^{\frac{1}{2}} du,
$$

where $\psi(r) = k(f(r) + \alpha)$ as before. Note that the value of $b$ defining the range of $s$ above, is equal to $\gamma_s(a)$, where $r = a$ corresponds to the end of the tube to be glued to the ambient manifold. (See Figure 7.)

![Figure 7](image)

It is easy to see that the above expression for $\gamma_s(r)$ is well defined precisely when

$$
\psi'(r) \leq \sqrt{1 - (\psi(r)/N)^2}.
$$

Notice that at the boundary of the handle ($r = a$) we have $\psi'(a) = \sqrt{1 - (\psi(a)/N)^2} = \cos(R/N)$. The condition $\psi'(r) \leq \sqrt{1 - (\psi(r)/N)^2}$ is not considered in [W2] as it is not needed there. In the construction of [W2], $\psi$ is concave up at least as far as the point at
which \( \psi'(r) = \cos(R/N) \) for the first time. At this point, \( \psi < N \sin(R/N) \), so the desired inequality clearly holds. Although \( \psi' \) subsequently increases beyond \( \cos(R/N) \), the excess can be made arbitrarily small, and it is easy to see that the ensuing concave down bending of \( \psi \) can be chosen so as to preserve this inequality.

For notational simplicity, let us denote the solid tube and its metric by \(([0,b] \times D^{m+1}, ds^2 + g_s)\) where \( g_s \) is the metric \( dt^2 + N^2 \sin^2(t/N)ds^2_n \) for \( t \in [0,\gamma_l(\psi(s))] \), where \( \psi : [0,b] \to \mathbb{R} \) is defined as follows. For \( s < b \) the value of \( \psi(s) \) is determined by the equation \( \gamma_s(\psi(s)) = s \). Notice that \( \psi(s) \) is smooth for these values of \( s \) since \( \gamma \) is a smooth curve. For \( s = b \) set \( \psi(b) = a \), which means that \( \gamma_l(\psi(b)) = R \). Note that \( \psi \) will in general not be continuous at \( s = b \).

We turn our attention now to the ‘cap’ \( D^n \). In the surgery picture, we require the distance sphere at radius \( r \) to have the metric \( h^2_\infty(r)ds^2_n \). In the handle picture, these spheres will be parametrised by \( s \), as discussed above. Clearly, the required metric for any given value of \( s \) is \( h^2_\infty(\psi(s))ds^2_n \). Notice that \( h(\psi(s)) \) is smooth on \([0,b] \), despite the fact that \( \psi \) cannot be assumed continuous at \( b \). This is because \( h_\infty(r) \) is constant for \( r \in [R',a] \) (see §4).

A fundamental difference between the handle and surgery cases is that the function \( h(\psi(s)) \) needs to be joined to the function \( \phi(s) \), as opposed to a constant function. However, since \( \phi' > 0 \) and \( \phi'' < 0 \), this joining can easily be achieved by a small concave down deformation of \( h(\psi(s)) \) in a neighbourhood of the joining point. Such a bend can only have a positive effect on the scalar curvature. Note that the second derivative of \( h_\infty(\psi(s)) \) is non-positive. To see this, recall that \( f(r) \equiv 1 \) for small \( r \), and this means that for \( s \) in the same range we have \( \psi(s) = s \). Also, where \( f \) is non-constant we have \( h_\infty(r) \equiv c \). The assertion now follows immediately since the desired second derivative is equivalent to \( h''_\infty(\psi(s))(\psi'(s))^2 + h'(\psi(s))\psi''(s) \). As a result, we deduce that the cap itself has positive scalar curvature (and non-negative Ricci curvature).

Using the above notation, the whole handle and its metric can now be expressed as

\[
([0,b] \times S^{n-1} \times D^{m+1}, ds^2 + h^2_\infty(\theta(s)) + g_s).
\]

As the boundary metric by construction agrees with the surgery metric, it is clear that \( \bar{M} \cup \bar{E} \) admits a smooth positive scalar curvature metric, as claimed.

Finally, we must consider the situation when \( \bar{E} \) is not trivial. Clearly, the technique of cutting out the ‘solid tube’ from the ‘big tube’ explained above requires \( \bar{E} \) to be a product, and so will not work here. To overcome this problem, we recall the assumption made in the proof of Corollary 5, namely that the non-flat part of the horizontal distribution is restricted to the region \( r \in (\delta/2, \delta) \). Accordingly, we are free to assume that \( \bar{E} \) is a product outside this region.

For \( r \in [0,\delta] \), recall that the fibres of \( E \) are all isometric to a round sphere of radius \( k(1 + \alpha) \). There is clearly a submersion metric on \( \bar{E} \) for \( r \in [0,\delta] \), having fibres isometric to \( dt^2 + N^2 \sin^2(t/N)ds^2_n \), for which the induced metric on the boundary agrees with that of \( E \). We choose a value for \( k \) for which guarantees the Ricci positivity of \( E \), and also the Ricci positivity of \( \bar{E} \) restricted to the non-trivial region \( r \in (\delta/2, \delta) \). That such a value of \( k \) exists follows from Lemma 8.

For \( r \geq \delta \) we use the metric constructed for the product case. This clearly glues smoothly with the above metric for \( r \in [0,\delta] \) to create the desired metric on \( \bar{E} \).
It is not clear if a positive Ricci curvature metric on $\bar{M}$ with the same local form as required by Proposition 9 can be extended to give a positive Ricci curvature metric on $\bar{M} \cup \bar{E}$. The construction of Proposition 9 gives some metric on $\bar{M} \cup \bar{E}$, but it is not clear whether this has has positive Ricci curvature. In particular, the solid tube has zero Ricci curvatures in $s$-directions, and the second derivative of $h(\iota(s))$ is not necessarily non-positive, leading to possible negative Ricci curvatures on the cap. This naturally leads to the following

**Open Question.** Given an exotic sphere $\Sigma$ arising as the boundary of a plumbed manifold and equipped with a Ricci positive metric, can the metric be extended within Ricci positivity over the plumbed manifold?

Let $\Sigma$ now be a homotopy sphere representing any element of $bP_{4k}$, and let $X$ be a plumbed manifold as discussed in §2 with $\Sigma = \partial X$.

**Corollary 10.** The (deformed) positive scalar curvature metric on $\Sigma$ can be extended across $X$ to give a positive scalar curvature metric on $X$.

**Proof.** The existence of this positive scalar curvature metric on $X$ follows immediately from repeated application of Proposition 9, starting with the Ricci positive metric on the ‘central’ disc-bundle guaranteed by Lemma 8 and systematically extending the metric over each plumbing in turn.

Fixing this metric on $X$, consider a collar neighbourhood of the boundary with normal parameter $w \in [-\epsilon, 0]$. By the compactness of $X$ there exists an $\epsilon > 0$ such that this neighbourhood is diffeomorphic via the exponential map to $[-\epsilon, 0] \times \partial X$. We will express the metric here as $dw^2 + g(w)$, with $g(w)$ a metric on the equidistant hypersurface at a distance $|w|$ from $\partial X$. By the openness of the positive scalar curvature condition, it follows that $\text{scal}(g(w)) > 0$ for all $w$ suitably close to 0, as well as $\text{scal}(dw^2 + g(w)) > 0$.

Fix an arbitrary local coordinate system on $\partial X$, $\{x_1, \ldots, x_{4k-1}\}$, and extend to a local coordinate system on the collar neighbourhood $\{w, x_1, \ldots, x_{4k-1}\}$ via the identification of the collar with $[-\epsilon, 0] \times \partial X$. In the following Lemma, the subscript $i$ refers to the coordinate system on $\partial X$.

**Lemma 11.** For $w$ suitably close to 0, the metrics $g(w)$ satisfy

$$\frac{\partial g_{ii}(w)}{\partial w} \geq 0.$$

**Proof.** To begin with, we will choose the coordinate system $\{x_1, \ldots, x_{4k-1}\}$ to lie entirely on the boundary of one of the handles $\bar{E}$ used in the construction of $X$.

First assume that $\bar{E}$ is trivial, and that the connection is chosen to be flat. We can identify four mutually orthogonal directions: the $s$-direction with metric $ds^2$; the $t$-direction with metric $dt^2$; the $S^m$ direction with metric $N^2 \sin^2(t/N)ds_m^2$; and the $S^{n-1}$ direction with metric $h^2_\infty(\iota(s))ds_{n-1}^2$. The boundary of solid tube is determined by the function $f$, and in turn this determines the direction of the normal vector $\partial/\partial w$. 

21
Assume the coordinate system lies in the region where $f(\iota(s)) \equiv 1$. Here we have $\partial/\partial w = \partial/\partial t$. Clearly $g_{ii}$ is non-decreasing along $t$ parameter lines, hence the result.

If the coordinate system lies in the region corresponding where $f(\iota(s))$ is increasing, we have that $\partial/\partial w$ can be decomposed into non-negative components in the $-\partial/\partial s$ and $\partial/\partial t$ directions. As $h_\infty(\iota(s))$ is constant at these values of $s$, we see that the metric itself is constant along $s$-parameter lines. Again, we have that $g_{ii}$ is non-decreasing along $t$ parameter lines, hence the result in this case.

Now suppose that $\bar{E}$ is not trivial and the connection is not everywhere flat. Recall from the proof of Proposition 7 that the non-flatness is constrained to lie within the region corresponding to $r \in (\delta/2, \delta)$, which is equivalent to $s \in (\delta/2, \delta)$. Note that changing the principal connection on the associated $SO(m+1)$-bundle only alters the metric on $\bar{E}$ in directions tangent to the distance sphere bundles. In particular, normal directions to the distance sphere bundles are unaffected, so in this region $\partial/\partial w = \partial/\partial t$ as before. The same arguments as for the flat case again yield the result that $g_{ii}$ is non-decreasing in normal directions.

Next suppose that our initial coordinate system $\{x_1, ..., x_{4k-1}\}$ lies on the disc-bundle with which we begin our construction of $X$, where the metric takes the form described in Lemma 8. Arguments analogous to the above show that $\partial g_{ii}(w)/\partial w \geq 0$, since $\theta(s)$ is a strictly increasing function.

Finally, if we choose a coordinate system $\{x_1, ..., x_{4k-1}\}$ which crosses the boundary between a handle and the original disc-bundle, or between two handles, the non-negativity of a derivative arbitrarily close to the boundary on either side is clearly sufficient to guarantee non-negativity throughout the region. \qed

§6 Paths of metrics

In this section we will ultimately prove Theorem A. Firstly, however, we must study the problem of deforming a positive scalar curvature metric on a manifold with boundary to a product in some neighbourhood of the boundary, in such a way that the positive scalar curvature is preserved. For the manifolds and metrics we are concerned about in this paper, Lemma 11 (above) is crucial to being able to perform such a deformation. The importance of this Lemma arises from the following result:

**Lemma 12.** Consider a metric of the form $\eta^2(w) dt^2 + g(w)$ on $I \times M$ where $I$ is an interval parametrised by $w$, $M^n$ a manifold, $\eta(w)$ a smooth real-valued function and $g(w)$ a smooth path of metrics on $M$. Given a local coordinate system $\{x_1, ..., x_n\}$ for $M$, consider the local coordinate system $\{w, x_1, ..., x_n\}$ on $I \times M$. With respect to this coordinate system the scalar curvature of the above metric is given by

$$ \text{scal}(\eta^2(w) dw^2 + g(w)) = \text{scal}(g(w)) + \frac{H(w)}{\eta^2(w)} + \frac{\eta'(w)}{\eta^3(w)} \sum_{i=1}^{n} \left( \frac{\partial g_{ii}(w)}{\partial w} / g_{ii}(w) \right) $$

where $H(w)$ is a smooth one-parameter family of real-valued functions on the domain of
the coordinate system \( \{x_1, \ldots, x_n\} \), which is independent of \( \eta(w) \) and given by

\[
H(w) = \frac{1}{2} \sum_{i<j} \left( \sum_{k=1}^{n} \sum_{p=1}^{n} g^{kp} \frac{\partial g_{ij}}{\partial w} g_{ki} - \frac{\partial g_{ij}}{\partial w} \sum_{k=1}^{n} \sum_{p=1}^{n} g^{kp} \frac{\partial g_{ip}}{\partial w} g_{ki} \right) / \left( g_{ii} g_{jj} - g_{ij}^2 \right) + \sum_{i=1}^{n} \frac{1}{2g_{ii}} \left( \sum_{k=1}^{n} \sum_{p=1}^{n} g^{kp} \frac{\partial g_{ik}}{\partial w} \frac{\partial g_{ip}}{\partial w} - 2 \frac{\partial^2 g_{ii}}{\partial w^2} \right).
\]

**Proof:** This result follows from an elementary calculation (for example by computing Christoffel symbols for the coordinate system), and we omit the details. \( \square \)

**Lemma 13.** Assume that the metric \( dw^2 + g(w) \) has positive scalar curvature for \( w \in [-\epsilon, 0] \). Given a local coordinate system on \( X \) as above, suppose that

\[
\sum_{i=1}^{n} \frac{\partial g_{ii}(w)}{\partial w} \geq 0
\]

at all points of \([−\epsilon′, 0] \times \partial X\) covered by the coordinate system, for some \( \epsilon' \leq \epsilon \). Then there exists a smooth positive function \( \beta(w) \) for \( w \in [-\epsilon', 0] \) with

\[
\beta(w) = 1 \text{ for } w \in [-\epsilon', -\epsilon'/2];
\]

\[
\beta(w) = \Lambda \text{ for } w \in [-\epsilon'/4, 0]
\]

for any given large constant \( \Lambda \), such that the metric \( \beta^2 dw^2 + g(w) \) has positive scalar curvature.

**Proof:** By Lemma 12, \( \text{scal}(dw^2 + g(w)) = \text{scal}(g(w)) + H(w) \) for all \( w \in [-\epsilon, 0] \), and this is positive by hypothesis. Since \( \text{scal}(g(w)) > 0 \) it follows that \( \text{scal}(g(w)) + H(w)/\beta > 0 \) for any \( \beta \geq 1 \). It is then clear that for any increasing function \( \beta(w) \) of the required form,

\[
\text{scal}(g(w)) + \frac{1}{\beta^2(w)} H(w) + \frac{\beta'(w)}{\beta^3(w)} \sum_{i=1}^{n} \left( \frac{\partial g_{ii}(w)}{\partial w} / g_{ii}(w) \right) > 0
\]

since the last term is non-negative by hypothesis. \( \square \)

**Lemma 14.** If \( g(w) \) is any smooth path through positive scalar curvature metrics on a compact manifold \( M \), and \( w \) belongs to a compact interval \( I \), then the metric \( \Lambda^2 dw^2 + g(w) \) has positive scalar curvature on \( I \times M \) for all \( \Lambda \) sufficiently large.

**Proof:** By Lemma 12, the scalar curvature of \( \Lambda^2 dw^2 + g(w) \) is \( \text{scal}(g(w)) + H(w)/\Lambda^2 \). By the compactness of \( I \times M \), there exists a constant \( C \geq 0 \) such that

\[
\text{scal}(\Lambda^2 dw^2 + g(w)) \geq \text{scal}(g(w)) - C/\Lambda^2.
\]
Since $\text{scal}(g(w)) > 0$ for all $w \in I$, we can choose $\Lambda$ sufficiently large so that $\text{scal}(g(w)) - C/\Lambda^2 > 0$, and this guarantees the positivity of $\text{scal}(\Lambda^2 dw^2 + g(w))$. 

\[ \square \]

**Corollary 15.** For a given homotopy sphere $\Sigma \in bP_{4k}$, suppose that $X$ is a plumbed manifold as discussed in §2 with $\partial X = \Sigma$. Then $X$ can be equipped with a positive scalar curvature metric which is a product $dw^2 + g$ in a neighbourhood of the boundary, and where the component of the moduli space of positive scalar curvature metrics on $\Sigma$ containing $g$ also contains a Ricci positive metric.

**Proof:** We equip $X$ with the positive scalar curvature metric from Corollary 10. In a neighbourhood of the boundary of $X$, this takes the form $dw^2 + g(w)$ ($w \leq 0$), for some smooth path of metrics $g(w)$ on $\Sigma$, which for $w$ sufficiently close to zero can be assumed to have positive scalar curvature. By Lemma 11 and Lemma 13, we can adjust this metric in a neighbourhood of the boundary to take the form $\Lambda^2 dw^2 + g(w)$ for any large constant $\Lambda$, whilst preserving positive scalar curvature. Next, adjoin a collar $\Sigma \times [0,1]$ to $\partial X$ and choose a smooth path of positive scalar curvature metrics $h(w)$ ($w \in [0,1]$) on $\Sigma$ which smoothly extends the path $g(w)$, and for which $h(w) = h(1)$ for all $w$ sufficiently close to 1. By Lemma 14 it follows that provided $\Lambda$ is chosen sufficiently large, $\Lambda^2 dw^2 + h(w)$ will have positive scalar curvature. Thus we have a metric on $X \cup (\Sigma \times [0,1])$ which has positive scalar curvature and takes the form $\Lambda^2 dw^2 + h(1)$ in a neighbourhood of the boundary. Clearly, a similar statement can be made for $X$ itself, after pulling back this metric via a suitable diffeomorphism $X \cong X \cup (\Sigma \times [0,1])$, and after a global rescale we can assume a positive scalar curvature metric on $X$ taking the required form near the boundary.

It is clear that $h(1)$ belongs to the same component of positive scalar curvature metrics as the Ricci positive metric on the boundary of $X$ arising from Theorem 2. It follows automatically that the same is true when we pass to the moduli space. 

\[ \square \]

**Proof of Theorem A.** Regard a homotopy sphere $\Sigma \in bP_{4k}$ as the boundary of a manifold $X_p$ as described in §2. As noted at the end of §2, $\Sigma$ is a spin manifold with vanishing real Pontrjagin classes and $X_p$ is a parallelisable manifold. Equip $X_p$ with the positive scalar curvature metric guaranteed by Corollary 15. Denote this metric $\tilde{g}_p$, and let $g_p$ be its restriction to $\Sigma = \partial X_p$. Lemma 1 applies to the Riemannian manifolds $(\Sigma, g_p)$ and $(X_p, \tilde{g}_p)$, and in this case asserts that

$$s(\Sigma, g_p) = \frac{1}{2^{2k+1}(2^{2k-1} - 1)} \sigma(X_p)$$

$$= \frac{8(p|bP_{4k}| + q)}{2^{2k+1}(2^{2k-1} - 1)}$$

for some integer $q$ depending on $\Sigma$. Thus if $p \neq p'$, $|s(\Sigma, g_p)| \neq |s(\Sigma, g_{p'})|$, and therefore by [KS; Proposition 2.14] $g_p$ and $g_{p'}$ belong to different components of the moduli space of positive scalar curvature metrics on $\Sigma$. It follows immediately that the moduli space of Ricci positive metrics on $\Sigma$ must have infinitely many components. 

\[ \square \]
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