Abstract. Suppose that we are interested in the average causal effect of a binary treatment on an outcome when this relationship is confounded by a binary confounder. Suppose that the confounder is unobserved but a nondifferential proxy of it is observed. We show that, under certain monotonicity assumption that is empirically verifiable, adjusting for the proxy produces a measure of the effect that is between the unadjusted and the true measures.

1. Introduction

Suppose that we are interested in the average causal effect of a binary treatment \( A \) on an outcome \( Y \) when this relationship is confounded by a binary confounder \( C \). Suppose also that \( C \) is nondifferentially mismeasured, meaning that (i) \( C \) is not observed and, instead, a binary proxy \( D \) of \( C \) is observed, and (ii) \( D \) is conditionally independent of \( A \) and \( Y \) given \( C \). The causal graph to the left in Figure 1 represents the relationships between the random variables.

Greenland (1980) argues that adjusting for \( D \) produces a partially adjusted measure of the average causal effect of \( A \) on \( Y \) that is between the crude (i.e., unadjusted) and true (i.e., adjusted for \( C \)) measures. Ogburn and VanderWeele (2012) show that, although this result does not always hold, it does hold under some monotonicity condition in \( C \). Specifically, \( E[Y|A,C] \) must be nondecreasing or nonincreasing in \( C \). Since this condition can be interpreted as that the average causal effect of \( C \) on \( Y \) must be in the same direction among the treated (\( A = 1 \)) and the untreated (\( A = 0 \)), Ogburn and VanderWeele (2012) argue that the condition is likely to hold in most applications in epidemiology. Unfortunately, the condition cannot be verified empirically because \( C \) is unobserved. Therefore, one has to rely on substantive knowledge to verify it. Moreover, the condition is sufficient but not necessary. Ogburn and VanderWeele (2013) extend these results to the case where \( C \) takes more than two values. If there are at least two independent
Figure 1. Causal graphs, where $Y$ is a discrete or continuous random variable, and $A$, $C$ and $D$ are binary random variables. Moreover, $C$ is unobserved.

proxies of $C$, then [Miao et al. (2018)] show that the causal effect of $A$ on $Y$ can be identified under certain rank condition.

In this paper, we prove that if the monotonicity condition holds in $D$, then it holds in $C$ as well. Since $D$ is observed, the monotonicity condition in $D$ can be verified empirically. Therefore, if no substantive knowledge is available but data are, then combining our result with that by [Ogburn and VanderWeele (2012)] may allow us to conclude that the partially adjusted effect is between the crude and the true ones and, thus, that the partially adjusted effect is a better approximation to the true effect than the crude one. We also report experiments showing that most random parameterizations of the causal graph to the left in Figure 1 result in a partially adjusted effect that lies between the crude and the true ones, although only half of them satisfy the monotonicity condition in $D$. This confirms that the condition is sufficient but not necessary. This result should be interpreted with caution because, in fields like epidemiology, one is not typically concerned with a random parameterization but, rather, with one carefully engineered by evolution. We also prove that if the monotonicity condition holds in $D$, then it also holds in $C$ when $D$ is a driver of $C$ rather than a proxy, i.e. $D$ causes $C$. We illustrate the relevance of this result with an example on transportability of causal inference across populations.

The rest of the paper is organized as follows. Sections 2 and 3 present our results when $D$ is a proxy and a driver of $C$, respectively. Section 4 closes with some discussion.

2. ON A PROXY OF THE CONFOUNDER

Consider the causal graph to the left in Figure 1, where $Y$ is a discrete or continuous random variable, and $A$, $C$ and $D$ are binary random variables. The graph entails the following factorization:

$$p(A, C, D, Y) = p(C)p(D|C)p(A|C)p(Y|A, C).$$

(1)

Let $A$ take values $a$ and $\bar{a}$, and similarly for $C$ and $D$. Let $A$, $D$ and $Y$ be observed and let $C$ be unobserved. Let $Y_a$ and $Y_{\bar{a}}$ denote the counterfactual outcomes under treatments $A = a$ and $A = \bar{a}$, respectively. The average causal effect of $A$ on $Y$ or true risk difference ($RD_{true}$) is defined as $RD_{true} = E[Y_a] - E[Y_{\bar{a}}]$. It can be rewritten as follows
(Pearl 2009, Theorem 3.3.2):

\[
RD_{true} = E[Y|a,c]p(c) + E[Y|a,\bar{c}]p(\bar{c}) - E[Y|\bar{a},c]p(c) - E[Y|\bar{a},\bar{c}]p(\bar{c}).
\]

Since \( C \) is unobserved, \( RD_{true} \) cannot be computed. It can be approximated by the unadjusted average causal effect or crude risk difference (\( RD_{crude} \)):

\[
RD_{crude} = E[Y|a] - E[Y|\bar{a}]
\]

and by the partially adjusted average causal effect or observed risk difference (\( RD_{obs} \)):

\[
RD_{obs} = E[Y|a,d]p(d) + E[Y|a,\bar{d}]p(\bar{d}) - E[Y|\bar{a},d]p(d) - E[Y|\bar{a},\bar{d}]p(\bar{d}).
\]

We say that \( E[Y|A,D] \) is nondecreasing in \( D \) if

\[
E[Y|a,d] \geq E[Y|a,\bar{d}] \quad \text{and} \quad E[Y|\bar{a},d] \geq E[Y|\bar{a},\bar{d}]. \tag{2}
\]

Likewise, \( E[Y|A,D] \) is nonincreasing in \( D \) if

\[
E[Y|a,d] \leq E[Y|a,\bar{d}] \quad \text{and} \quad E[Y|\bar{a},d] \leq E[Y|\bar{a},\bar{d}]. \tag{3}
\]

Moreover, \( E[Y|A,D] \) is monotone in \( D \) if it is nondecreasing or nonincreasing in \( D \). Ogburn and VanderWeele (2012, Lemma 1) show that if \( E[Y|A,C] \) is monotone in \( C \), then \( E[Y|A,D] \) is monotone in \( D \). The following theorem proves the converse result. The relevance of this result is as follows. Ogburn and VanderWeele (2012, Result 1) show that if \( E[Y|A,C] \) is monotone in \( C \), then \( RD_{obs} \) lies between \( RD_{true} \) and \( RD_{crude} \). The antecedent of this rule cannot be verified empirically, because \( C \) is unobserved. Therefore, one must rely on substantive knowledge to apply the rule. The following theorem implies that, luckily, the rule also holds for \( D \) and, thus, that the antecedent can be verified empirically.

**Theorem 1.** Consider the following causal graph to the left in Figure [1]. If \( E[Y|A,D] \) is monotone in \( D \), then \( E[Y|A,C] \) is monotone in \( C \).

**Proof.** Assume to the contrary that \( E[Y|A,C] \) is not monotone in \( C \), i.e.

\[
E[Y|a,c] \leq E[Y|a,\bar{c}] \quad \text{and} \quad E[Y|\bar{a},c] \geq E[Y|\bar{a},\bar{c}] \tag{4}
\]

or

\[
E[Y|a,c] \geq E[Y|a,\bar{c}] \quad \text{and} \quad E[Y|\bar{a},c] \leq E[Y|\bar{a},\bar{c}] \tag{5}
\]

This gives four cases to consider: Whether Equation 2 or 3 holds, and whether Equation 4 or 5 holds. Hereinafter, we focus on the first case. The other cases are similar.
Assume that Equations 2 and 4 hold. We show next that the first inequalities in Equations 2 and 4 imply that \( p(c|a, d) \leq p(c|a, \overline{d}) \). Specifically,

\[
E[Y|a, d] \geq E[Y|a, \overline{d}]
\]

\[
E[Y|a, d, c]p(c|a, d) + E[Y|a, d, \overline{c}]p(\overline{c}|a, d) \geq E[Y|a, \overline{d}, c]p(c|a, \overline{d}) + E[Y|a, \overline{d}, \overline{c}]p(\overline{c}|a, \overline{d}) \]

because \( Y \) is conditionally independent of \( D \) given \( A \) and \( C \) due to the causal graph under consideration and, thus,

\[
E[Y|a, c]p(c|a, d) + E[Y|a, \overline{c}](1 - p(c|a, d)) \geq E[Y|a, \overline{d}, c]p(c|a, \overline{d}) + E[Y|a, \overline{d}, \overline{c}](1 - p(c|a, \overline{d}))
\]

\[
(E[Y|a, c] - E[Y|a, \overline{c}])p(c|a, d) \geq (E[Y|a, \overline{d}, c] - E[Y|a, \overline{d}, \overline{c}])p(c|a, \overline{d})
\]

\[
p(c|a, d) \leq p(c|a, \overline{d})
\]

because \( E[Y|a, c] \leq E[Y|a, \overline{c}] \) by Equation 4.

Furthermore,

\[
p(c|a, d) = \frac{p(a, d|c)p(c)}{p(a, d|c)p(c) + p(a, d|\overline{c})p(\overline{c})} = \frac{1}{1 + \exp(-\delta(a, d))} = \sigma(\delta(a, d))
\]

where

\[
\delta(ad) = \ln \frac{p(a, d|c)p(c)}{p(a, d|\overline{c})p(\overline{c})}
\]

is known as the log odds, and \( \sigma() \) is known as the logistic sigmoid function \({\text{Bishop}}^{2006}, \text{Section 4.2}\). Note that \( \sigma() \) is an increasing function. Then,

\[
p(c|a, d) \leq p(c|a, \overline{d})
\]

\[
\delta(a, d) \leq \delta(a, \overline{d})
\]

\[
\ln p(a|c) + \ln p(d|c) + \ln p(c) - \\
\ln p(a|\overline{c}) - \ln p(d|\overline{c}) - \ln p(\overline{c}) \leq \\
\ln p(a|c) + \ln p(\overline{d}|c) + \ln p(c) - \\
\ln p(a|\overline{c}) - \ln p(\overline{d}|\overline{c}) - \ln p(\overline{c})
\]
Table 1. Results of 10000 random parameterizations of
the causal graph to the left in Figure [1]

| In-between | Nondec. in D | Noninc. in D | Neither |
|------------|-------------|-------------|---------|
| 2430       | 1175        | 1255        | 0       |
| 2461       | 1225        | 1236        | 0       |
| 4460       | 0           | 0           | 5109    |

because $A$ is conditionally independent of $D$ given $C$ due to the causal
graph under consideration and, thus,

$$
\ln p(d|c) - \ln p(d|\overline{c}) \leq \ln p(\overline{d}|c) - \ln p(\overline{d}|\overline{c})
$$

Likewise, the second inequalities in Equations 2 and 4 imply that

$$
\frac{p(d|c)}{p(d|\overline{c})} \leq \frac{p(\overline{d}|c)}{p(\overline{d}|\overline{c})} \leq \frac{p(d|\overline{c})}{p(\overline{d}|\overline{c})}
$$

which contradicts Equation 6 unless equality holds. However, equality
only occurs if $p(d|c) = p(d|\overline{c})$, which implies that $C$ and $D$ are indepen-
dent and, thus, that $D$ is not a mismeasured confounder.

Corollary 2. Consider the causal graph to the left in Figure [1]. If

$E(Y|A,D)$ is monotone in $D$, then $RD_{\text{obs}}$ lies between $RD_{\text{true}}$ and $RD_{\text{crude}}$.

Proof. The result follows directly from Theorem 1 and [Ogburn and VanderWeele (2012, Result 1)]

2.1. Experiments. In this section, we report some experiments that
shed additional light on the relationships between the various risk dif-
fences. Specifically, we randomly parameterized 10000 times the
causal graph to the left in Figure [1] by parameterizing the terms in
the right-hand side of Equation 1 with parameter values drawn from
a uniform distribution. For each parameterization, we then computed
$RD_{\text{true}}$, $RD_{\text{obs}}$ and $RD_{\text{crude}}$. The results are reported in Table 1. Of
the 10000 runs, 4891 were monotone in $C$ and also in $D$, as expected
from [Ogburn and VanderWeele (2012, Lemma 1)]. There were no other
runs that were monotone in $D$, as expected from Theorem 1. In all
these 4891 runs, $RD_{\text{obs}}$ was between $RD_{\text{true}}$ and $RD_{\text{crude}}$, as expected
from Corollary 2 and [Ogburn and VanderWeele (2012, Result 1)]. It is

1Code available at https://www.dropbox.com/s/pa8y2sausib6hc/montonicity.R?dl=0.
also worth noticing from the table that the 10000 runs are rather evenly distributed among the different entries. Finally, 4460 of the 5109 runs where the monotonicity assumption did not hold still resulted in that $RD_{obs}$ was between $RD_{true}$ and $RD_{crude}$. In other words, although half of the runs violated the monotonicity assumption, few of them resulted in $RD_{obs}$ being outside the range of $RD_{true}$ and $RD_{crude}$. In total, $RD_{obs}$ was between $RD_{true}$ and $RD_{crude}$ in 94% of the runs. A surprisingly large percentage. Therefore, $RD_{obs}$ was a better approximation to $RD_{true}$ than $RD_{crude}$ in most of the runs.
The plots in Figure 2 show some additional descriptive statistics for the runs where \( \text{RD}_{\text{obs}} \) belonged to the interval between \( \text{RD}_{\text{true}} \) and \( \text{RD}_{\text{crude}} \). The top left plot shows that most intervals were quite small and, thus, that \( \text{RD}_{\text{obs}} \) was a good approximation to \( \text{RD}_{\text{true}} \) in most cases. However, the top right plot shows that \( \text{RD}_{\text{obs}} \) was typically closer to \( \text{RD}_{\text{crude}} \) than to \( \text{RD}_{\text{true}} \). The bottom left plot is a zoom of the previous plot at the smallest intervals. Finally, the bottom right plot shows that the lower the correlation between \( C \) and \( D \) when measured by the Youden index, the closer \( \text{RD}_{\text{obs}} \) was to \( \text{RD}_{\text{crude}} \). In summary, \( \text{RD}_{\text{obs}} \) seems to be a good approximation to \( \text{RD}_{\text{true}} \), but it seems to be biased towards \( \text{RD}_{\text{crude}} \). This is a problem when the interval between \( \text{RD}_{\text{crude}} \) and \( \text{RD}_{\text{true}} \) is large. However, the length of the interval is unknown in practice, and we doubt substantive knowledge may provide hints on it. The bias seems to decrease with increasing correlation between \( C \) and \( D \).

2.2. Transitivity. Consider the causal graph \( A \rightarrow B \rightarrow C \). Let \( E[B|A] \) and \( E[C|B] \) be nondecreasing in \( A \) and \( B \), respectively. Unfortunately, there is no guarantee that \( E[C|A] \) is nondecreasing in \( A \), i.e. the nondecreasing property is not transitive in general (VanderWeele and Robins, 2010, Example 3.2). However, transitivity does hold when \( A, B \) and \( C \) are binary random variables (VanderWeele and Robins, 2010, p. 119). For binary random variables, Ogburn and VanderWeele (2012, Lemma 1) also implies a sort of transitivity result: If \( E[C|B] \) is monotone in \( B \), then \( E[C|A] \) is monotone in \( A \). Theorem 1 implies then a sort of inverse transitivity result: If \( E[C|A] \) is monotone in \( A \), then \( E[C|B] \) is monotone in \( B \).

3. On a Driver of the Confounder

Consider the causal graph to the right in Figure 1, where \( Y \) is a discrete or continuous random variable, and \( A, C \) and \( D \) are binary random variables. Note that \( D \) is now a driver rather than a proxy of \( C \), i.e. \( D \) causes \( C \). The graph entails the following factorization:

\[
p(A, C, D, Y) = p(D)p(C|D)p(A|C)p(Y|A, C). \tag{7}
\]

Let \( A \) take values \( a \) and \( \bar{a} \), and similarly for \( C \) and \( D \). Let \( A, D \) and \( Y \) be observed and let \( C \) be unobserved. We show next that our previous results also apply to the new causal graph under consideration.

**Theorem 3.** Consider the causal graph to the right in Figure 1. If \( E[Y|A, D] \) is monotone in \( D \), then \( E[Y|A, C] \) is monotone in \( C \).

**Proof.** The proof of Theorem 1 also applies when \( D \) is a driver of \( C \). ∎

**Corollary 4.** Consider the causal graph to the right in Figure 1. If \( E[Y|A, D] \) is monotone in \( D \), then \( \text{RD}_{\text{obs}} \) lies between \( \text{RD}_{\text{true}} \) and \( \text{RD}_{\text{crude}} \).
Proof. Note that every probability distribution that is representable by the causal graph to the right in Figure 1 can be represented by the causal graph to the left in Figure 1. Simply, let \( p_L(A|C) = p_R(A|C) \) and \( p_L(Y|A,C) = p_R(Y|A,C) \) where the subscript \( L \) or \( R \) indicates whether we refer to Equation 1 or 7, respectively. Moreover, let

\[
    p_L(C) = p_R(C) = p_R(C|d)p_R(d) + p_R(C|\overline{d})p_R(\overline{d})
\]

and

\[
    p_L(D|C) = p_R(D|C) = \frac{p_R(C|D)p_R(D)}{p_R(C|d)p_R(d) + p_R(C|\overline{d})p_R(\overline{d})}.
\]

Therefore, \( RD_{crude} \), \( RD_{obs} \) and \( RD_{true} \) are the same whether they are computed from the graph to the right or to the left in Figure 1. Likewise, if \( E[Y|A,D] \) is monotone in \( D \) for the graph to the right in Figure 1 then it is also monotone in \( D \) for the graph to the left, which implies that \( RD_{obs} \) lies between \( RD_{true} \) and \( RD_{crude} \) by Corollary 2.

\[\blacksquare\]

VanderWeele et al. (2008, Result 1) prove that (i) if \( E[Y|A,C] \) and \( E[A|C] \) are both nondecreasing or both nonincreasing in \( C \), then \( RD_{obs} \geq RD_{true} \), and (ii) if \( E[Y|A,C] \) and \( E[A|C] \) are one nondecreasing and the other nonincreasing in \( C \), then \( RD_{obs} \leq RD_{true} \). The antecedents of these rules cannot be verified empirically, because \( C \) is unobserved. Therefore, one must rely on substantive knowledge to apply the rules. Luckily, the rules also hold for \( D \) and, thus, the antecedents can be verified empirically. The following theorem proves it.

**Theorem 5.** Consider the causal graph to the right in Figure 1. If \( E[Y|A,D] \) and \( E[A|D] \) are both nondecreasing or both nonincreasing in \( D \), then \( E[Y|A,C] \) and \( E[A|C] \) are both nondecreasing or both nonincreasing in \( C \). If \( E[Y|A,D] \) and \( E[A|D] \) are one nondecreasing and the other nonincreasing in \( D \), then \( E[Y|A,C] \) and \( E[A|C] \) are one nondecreasing and the other nonincreasing in \( C \).

Proof. We prove the result when \( E[Y|A,D] \) and \( E[A|D] \) are both nondecreasing in \( D \). The proofs for the rest of the cases are similar. Then, we have that (i) \( E[Y|a,d] \geq E[Y|a,\overline{d}] \), and (ii) \( E[Y|d] \geq E[Y|\overline{d}] \). Assume to the contrary that (iii) \( E[Y|a,c] \leq E[Y|a,\overline{c}] \), and (iv) \( E[Y|c] \geq E[Y|\overline{c}] \). As shown in the proof of Theorem 4 (i) and (iii) imply that \( p(c|a,d) \leq p(c|a,\overline{d}) \), which implies that

\[
    \frac{p(d|c)}{p(d|\overline{c})} \leq \frac{p(\overline{d}|c)}{p(\overline{d}|\overline{c})}.
\]

Likewise, (ii) and (iv) imply that \( p(c|d) \geq p(c|\overline{d}) \), which implies that

\[
    \frac{p(d|c)}{p(d|\overline{c})} \geq \frac{p(\overline{d}|c)}{p(\overline{d}|\overline{c})}.
\]
As shown in the proof of Theorem 1, this contradicts the fact that \( C \) and \( D \) are dependent. Therefore, either the assumption (iii) or (iv) or both are false. In the latter case, we get a similar contradiction. So, either the assumption (iii) or (iv) is false. We reach a similar contradiction if replace \( a \) with \( \overline{a} \) in the assumptions (i) and (iii). This together with the fact that \( E[Y|A,C] \) and \( E[A|C] \) are both monotone in \( C \) by Theorem 1 prove the result.

\[ \square \]

**Corollary 6.** Consider the causal graph to the right in Figure 4. If \( E[Y|A,D] \) and \( E[A|D] \) are both nondecreasing or both nonincreasing in \( D \), then \( RD_{obs} \geq RD_{true} \). If \( E[Y|A,D] \) and \( E[A|D] \) are one nondecreasing and the other nonincreasing in \( D \), then \( RD_{obs} \leq RD_{true} \).

**Proof.** The result follows directly from Theorem 5 and VanderWeele et al. (2008, Result 1).

For completeness, we show below that the converse of Theorem 5 also holds.

**Theorem 7.** Consider the causal graph to the right in Figure 4. If \( E[Y|A,C] \) and \( E[A|C] \) are both nondecreasing or both nonincreasing in \( C \), then \( E[Y|A,D] \) and \( E[A|D] \) are both nondecreasing or both nonincreasing in \( D \). If \( E[Y|A,C] \) and \( E[A|C] \) are one nondecreasing and the other nonincreasing in \( C \), then \( E[Y|A,D] \) and \( E[A|D] \) are one nondecreasing and the other nonincreasing in \( D \).

**Proof.** As shown in the proof of Corollary 4, every probability distribution that is representable by the causal graph to the right in Figure 4 can be represented by the causal graph to the left in Figure 4. Therefore, if \( E[Y|A,C] \) and \( E[A|C] \) are monotone in \( C \) for the right graph, then they are so for the left graph as well. Then, \( E[Y|A,D] \) and \( E[A|D] \) are monotone in \( D \) for the left graph (Ogburn and VanderWeele, 2012, Lemma 1) and, thus, they are so for the right graph as well. The result follows now from the contrapositive formulation of Theorem 5.

Given a sufficiently large sample from \( p(A,D,Y) \), we may conclude from it that \( E[Y|A,D] \) is monotone in \( D \), which implies that \( RD_{obs} \) lies between \( RD_{true} \) and \( RD_{crude} \) by Corollary 4. We can also estimate \( RD_{obs} \) and \( RD_{crude} \) from the sample, which implies that (i) if \( RD_{crude} \leq RD_{obs} \) then \( RD_{obs} \leq RD_{true} \), and (ii) if \( RD_{crude} \geq RD_{obs} \) then \( RD_{obs} \geq RD_{true} \). Consequently, Corollary 6 is superfluous when data over \( (A,D,Y) \) are available. The following example illustrates that the corollary may be useful when no such data are available.

**Example 8.** Let \( A \) and \( Y \) represent a treatment and a disease, respectively. Let \( D \) and \( C \) represent pre-treatment covariates such as socioeconomic and health status, respectively. Say that we have a sample from \( p_1(A,D,Y) \) and a sample from \( p_2(A,D,Y) \), i.e. we have two...
samples from two different populations. We are interested in drawing conclusions about \( RD_{\text{true}} \) for a third population, from which we have no data. We make the following assumptions:

- \( p_1(D) \neq p_2(D) \) because the socio-economic profile of the third population differs from the other populations’ profiles.
- \( p_1(C|D) = p_2(C|D) = p_3(C|D) \) because this distribution represents psychological and physiological processes shared by the three populations.
- \( p_1(Y|A,C) = p_2(Y|A,C) \neq p_3(Y|A,C) \) because these distributions represent psychological and physiological processes shared by the first and third populations but not by the second. Then, \( E_3[Y|A,D] = E_1[Y|A,D] \) which can be estimated from the sample from \( p_1(A,D,Y) \).
- \( p_1(A|C) \neq p_2(A|C) = p_3(A|C) \) because the second and third populations share the same treatment policy but the first does not. Then, \( E_3[A|D] = E_2[A|D] \) which can be estimated from the sample from \( p_2(A,D,Y) \).

Then, we cannot estimate \( RD_{\text{crude}} \) for the third population and, thus, we cannot use Corollary 4 as we did before to bound \( RD_{\text{true}} \). Corollary 6 may, on the other hand, be useful in drawing conclusions. For instance, assume that \( E_3[Y|A,D] \) and \( E_3[A|D] \) are both nondecreasing or both nonincreasing in \( D \). Then, \( RD_{\text{obs}} \geq RD_{\text{true}} \) by the corollary. If we are interested in testing whether \( k \geq RD_{\text{true}} \) for a given constant \( k \), then it may be worth assuming the cost of collecting data from the third population in order to compute \( RD_{\text{obs}} \), in the hope that \( k \geq RD_{\text{obs}} \) which confirms the hypothesis. If we are interested in testing whether \( RD_{\text{true}} \geq k \), then we may also be willing to assume the cost, in the hope that \( k \geq RD_{\text{obs}} \) which allows us to reject the hypothesis. In the latter case, we may instead decide to not assume the cost because we can never confirm the hypothesis. Such a seemingly negative result may save us time and money. Similar conclusions can be drawn when \( E[Y|A,D] \) and \( E[A|D] \) are one nondecreasing and the other nonincreasing in \( D \).

On the other hand, no such conclusions can be drawn from Corollary 4 before collecting data.

3.1. Bounds. Causal effects are typically defined in terms of distributions of counterfactuals. For instance, the causal effect on \( Y \) of an intervention setting \( A = a \) is defined as \( E[Y_a] \). It can be rewritten as follows (Pearl, 2009, Theorem 3.3.2):

\[
E[Y_a] = E[Y|a,c]p(c) + E[Y|a,c]\overline{p(\overline{c})}.
\]

Since \( C \) is unobserved, this effect cannot be computed. It can be approximated by the following quantity:

\[
S_a = E[Y|a,d]p(d) + E[Y|a,d]\overline{p(\overline{d})}.
\]
Likewise for the causal effect on $Y$ of an intervention setting $A = \overline{a}$. VanderWeele et al. (2008) Result 1) prove that (i) if $E[Y|A,C]$ and $E[A|C]$ are both nondecreasing or both nonincreasing in $C$, then $S_a \geq E[Y_a]$ and $S_{\overline{a}} \leq E[Y_{\overline{a}}]$, and (ii) if $E[Y|A,C]$ and $E[A|C]$ are one nondecreasing and the other nonincreasing in $C$, then $S_a \leq E[Y_a]$ and $S_{\overline{a}} \geq E[Y_{\overline{a}}]$. These results also hold when $E[Y|A,D]$ and $E[A|D]$ are nondecreasing or nonincreasing in $D$ by Theorem 5. The following corollary shows that the results also hold under weaker assumptions: It is not necessary that $E[Y|A,D]$ is nondecreasing or nonincreasing in $D$, it suffices with $E[Y|a,D]$ and $E[Y|\overline{a},D]$ being so, which is always true. Specifically, we say that $E[Y|a,D]$ is nondecreasing in $D$ if
\[ E[Y|a,d] \geq E[Y|a,\overline{d}] \]
and we say that it is nonincreasing in $D$ if
\[ E[Y|a,d] \leq E[Y|a,\overline{d}] \].
Likewise for $E[Y|\overline{a},D]$.

**Corollary 9.** Consider the causal graph to the right in Figure 4. If $E[Y|a,D]$ and $E[A|D]$ are both nondecreasing or both nonincreasing in $D$, then $S_a \geq E[Y_a]$. If $E[Y|a,D]$ and $E[A|D]$ are one nondecreasing and the other nonincreasing in $D$, then $S_a \leq E[Y_a]$. Likewise for $\overline{a}$ instead of $a$.

**Proof.** We prove the result for when $E[Y|a,D]$ and $E[A|D]$ are both nondecreasing in $D$. The proof is similar for the remaining cases. If $E[Y|\overline{a},D]$ is not nondecreasing in $D$, then make it so by parameterizing $p(Y|\overline{a},C)$ appropriately in Equation 7, e.g. by setting $p(Y|\overline{a},c) = p(Y|\overline{a},\overline{c})$ so that $E[Y|\overline{a},d] = E[Y|\overline{a},\overline{d}]$. Then, as discussed previously, $S_a \geq E[Y_a]$ for the new distribution. Finally, note that the expressions for $S_a$ and $E[Y_a]$ do not involve $p(Y|\overline{a},C)$. So, $S_a$ and $E[Y_a]$ are the same for the new and the original distributions. □

Of course, $S_a$ is always an upper or lower bound of $E[Y_a]$. The previous corollary allows us to determine always whether it is one or the other, because $E[Y|a,D]$ and $E[A|D]$ are always nondecreasing or nonincreasing in $D$. Likewise for $\overline{a}$ instead of $a$. On the other hand, given a random parameterization, there is only 50% chance that $E[Y|a,D]$ and $E[Y|a,D]$ are both nondecreasing or both nonincreasing in $D$ and, thus, $E[Y|A,D]$ is nondecreasing or nonincreasing in $D$ and, thus, we can apply the combination of Theorem 5 and the result by VanderWeele et al. (2008 Result 1).

4. Discussion

We have extended the result by Ogburn and VanderWeele (2012) stating that if $E[Y|A,C]$ is monotone in $C$, then $RD_{obs}$ lies between $RD_{true}$ and $RD_{crude}$. We have done so by showing that the result also
holds when $E[Y|A,D]$ is monotone in $D$. This makes the result much more applicable in practice, as the monotonicity condition in $D$ can be verified empirically. We have also extended the result by VanderWeele et al. (2008) along the same lines.

The monotonicity condition in $D$ is, however, sufficient but not necessary. In fact, we have shown through experiments that 94% of the random parameterizations of the causal graph studied resulted in $RD_{obs}$ being inside the range of $RD_{true}$ and $RD_{crude}$. However, the monotonicity condition did not hold for approximately half of them. Therefore, in future work, we plan to study how to relax this condition while keeping it sufficient and empirically testable.

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