Quantum Teleportation:
from Pure to Mixed States and Standard to Optimal

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Teleportation for pure states, mixed states with standard and optimal protocols are introduced and investigated systematically. An explicit equation governing the teleportation of finite dimensional quantum pure states by a generally given non-local entangled state is presented. For the teleportation of a mixed state with an arbitrary mixed state resource, an explicit expression is obtained for the quantum channel associated with the standard teleportation protocol. The corresponding transmission fidelity is calculated. It is shown that the standard teleportation protocol is not optimal. The optimal quantum teleportation is further studied, its fidelity is given and shown to be related to the fully entangled fraction of the quantum resource, rather than the single fraction as in the standard teleportation protocol.

1 Introduction

One of the most profound results of quantum information theory is the discovery of a quantum teleportation protocol\(^1\). By means of a classical communication channel and a quantum resource realized by a nonlocal entangled state such as an EPR-pair of particles, the teleportation process allows to transmit an unknown quantum state from a sender traditionally named “Alice” to a receiver “Bob” which are spatially separated. Quantum teleportation has been introduced in [1] and discussed by a number of authors for both spin-\(\frac{1}{2}\) states and arbitrary quantum states, see e.g. [2-11].

For teleportation of \(N\)-dimensional quantum states, the teleportation problem has been discussed in [7] in the case where the dimensions of the Hilbert spaces associated with the sender, receiver and the auxiliary space are all equal.
to \( N = 2^m \), for a given \( m \in \mathbb{N} \). The relations among quantum teleportation, dense coding, orthonormal bases of maximally entangled vectors and unitary operators with respect to the Hilbert-Schmidt scalar product, and depolarizing operations are investigated in [8].

These teleportation processes can be viewed as quantum channels. The nature of a quantum channel is determined by the particular protocol and the state used as a teleportation resource\(^1,12,13\). The standard teleportation protocol \( T_0 \) proposed in [1] uses Bell measurements and Pauli rotations. When the maximally entangled pure state \( |\Phi\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |ii\rangle \) is used as the quantum resource, it provides an ideal noiseless quantum channel \( \Lambda_{T_0}^{(\Phi)}(\rho) = \rho \). However in realistic situations, instead of pure maximally entangled states, Alice and Bob usually share a mixed entangled state due to the decoherence. Teleportation using a mixed state as an entangled resource is, in general, equivalent to having a noisy quantum channel. Recently, an explicit expression for the output state of the quantum channel associated with the standard teleportation protocol \( T_0 \) with an arbitrary mixed state resource has been obtained \(^14,15\).

In this article we give a systematic description of quantum teleportations for general quantum states\(^15,16,17\). We first discuss the general properties of teleportation for finite dimensional quantum pure states without the assumption on equality for the dimensions of the Hilbert spaces involved. We give a teleportation protocol for generally given entangled states and a constraint equation that governs the teleportation. The solutions of the constraint equation give the unitary transformations of teleportation protocols. Detailed examples and the roles played by the dimensions of the Hilbert spaces are discussed.

We then consider the teleportation of mixed states. Using a different approach as compared with [14] we derive an explicit expression of the protocol for the quantum channel associated with the standard teleportation protocol \( T_0 \) and an arbitrary mixed state resource. We further calculate the transmission fidelity and show that the transmission fidelity of the standard teleportation protocol with an arbitrary mixed state \( \chi \) as a resource depends on the singlet fraction of the resource \( \chi \).

To investigate the optimal teleportation, we consider the following problem. Alice and Bob previously only share a pair of particles in an arbitrary mixed entangled state \( \chi \). In order to teleport an unknown state \( \rho \) to Bob, Alice first performs a joint Bell measurement on her particles (particle 1 and particle 2) and tell her result to Bob by the classical communication channel. Then Bob, instead of the Pauli rotation like in the standard teleportation protocol\(^8\), tries his best to choose a particular unitary transformation which depends on
the quantum resource $\chi$, so as to get the maximal transmission fidelity. We derive an explicit expression for the quantum channel associated with the optimal teleportation with an arbitrary mixed state resource. The transmission fidelity of the corresponding quantum channel is given in terms of the fully entangled fraction of the quantum resource.

2 A most simple example – teleportation of a qubit state

We address the quantum teleportation problem from a most simple example: the teleportation of a quantum bit (qubit) state. The state of a qubit is mathematically a vector in two dimensional complex Hilbert space $\mathcal{H}$. Taking two bases of $\mathcal{H}$ to be $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, a general state of a qubit can be written as

$$|\alpha\rangle = a|1\rangle + b|0\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in \mathbb{C}$$

(1)

with normalization $|a|^2 + |b|^2 = 1$. A quantum measurement on $\phi$ would projects the state to $|1\rangle$ (resp. $|0\rangle$) with probability $|a|^2$ (resp. $|b|^2$).

The states of multiple qubits are then the vectors on the space $\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$. A vector on $\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ that can not be written as $(a_1|1\rangle + b_1|0\rangle) \otimes (a_2|1\rangle + b_2|0\rangle) \otimes \ldots \otimes (a_n|1\rangle + b_n|0\rangle)$ for some $a_i, b_i \in \mathbb{C}$ is called entangled. For example, the EPR (Einstein, Podolsky and Rosen) pair

$$\beta = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(2)

is a maximally entangled state.

The transformations on the vector states are called quantum gates. As a quantum system evolves unitarily, these quantum gates are just unitary transformations given by matrices $M$ such that $MM^\dagger = I$, where $\dagger$ stands for conjugation and transpose, $I$ is the identity matrix.

The followings are typical quantum gates on a single qubit:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $X$ (resp. $Z$) functions as negation (resp. phase shift), $Y = ZX$. A useful gate on two qubits called Controlled-NOT gate is given by

$$C_{not} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
It is easily checked that $C_{\text{not}}$ maps $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, and $|11\rangle \to |10\rangle$. Another useful gate is called Hadamard Transformation $H$

$$ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. $$

It maps $|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Now Alice has a general qubit state (1) which is unknown to her (she does not know the parameters $a$ and $b$). But she would like Bob to have a qubit state of the form (1). In addition they have another pair of qubits in EPR state (2). Alice (resp. Bob) has the first (resp. second) qubit of the EPR pair. Therefore the initial state is given by

$$ |\alpha\rangle \otimes |\beta\rangle = \frac{1}{\sqrt{2}}(a |0\rangle \otimes (|00\rangle + |11\rangle) + b |1\rangle \otimes (|00\rangle + |11\rangle)) $$

$$ = \frac{1}{\sqrt{2}}(a |000\rangle + a |011\rangle + b |100\rangle + b |111\rangle). $$

Alice applies first the gate $C_{\text{not}}$ to her two qubits, then the gate $H$ to the qubit whose state is to be teleported. After these transformations the initial state becomes

$$ (H \otimes I \otimes I)(C_{\text{not}} \otimes I)(|\alpha\rangle \otimes |\beta\rangle) $$

$$ = (H \otimes I \otimes I) \frac{1}{\sqrt{2}}(a |000\rangle + a |011\rangle + b |100\rangle + b |101\rangle) $$

$$ = \frac{1}{2}( |00\rangle (a |0\rangle + b |1\rangle) + |01\rangle (a |1\rangle + b |0\rangle) $$

$$ + |10\rangle (a |0\rangle - b |1\rangle) + |11\rangle (a |1\rangle - b |0\rangle)). $$

Alice measures her two qubits. With equal probabilities she will get one of the states: $|00\rangle$, $|01\rangle$, $|10\rangle$ or $|11\rangle$. Accordingly Bob’s qubit is projected to one of the states: $a |0\rangle + b |1\rangle$, $a |1\rangle + b |0\rangle$, $a |0\rangle - b |1\rangle$ or $a |1\rangle - b |0\rangle$ respectively. Alice tells Bob the results of her measurement by two classical bits information on classical channel. Bob applies one of the gates: $I$, $X$, $Z$, or $Y$ with respect to the classical bits he received.

| bits received | state          | decoding |
|---------------|----------------|----------|
| 00            | $a |0\rangle + b |1\rangle$ | $I$      |
| 01            | $a |1\rangle + b |0\rangle$ | $X$      |
| 10            | $a |0\rangle - b |1\rangle$ | $Z$      |
| 11            | $a |1\rangle - b |0\rangle$ | $Y$      |

The state of Bob’s qubit is then transformed into $|\alpha\rangle$ exactly.
3 Teleportation of general finite-dimensional quantum pure states

The example in the preceding section is about the teleportation of a pure two-dimensional (qubit) state. We consider now the teleportation of a general $N$-dimensional quantum state. We denote by $\{|i\rangle, i = 0, \ldots, N_0 - 1\}$ an orthogonal normalized basis of an $N_0$-dimensional Hilbert space $H_0$, $\theta = 1, 2, 3$. The spaces $H_1$ and $H_2$ are associated with Alice, while $H_3$ is associated with Bob. Alice has particles in general quantum states on the Hilbert space $H_1$ of the form

$$|\Psi_0\rangle = \left(\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_{N_1} \end{array}\right) = \sum_{i=0}^{N_1-1} \alpha_i |i\rangle, \quad |\Psi_0\rangle \in H_1,$$

where $\alpha_i \in \mathbb{C}$, $\sum_{i=0}^{N_1-1} |\alpha_i|^2 = 1$.

A general entangled state of two particles in the Hilbert spaces $H_2$ and $H_3$ is of the form

$$|\Psi_1\rangle = \sum_{i=0}^{N_2-1} \sum_{j=0}^{N_3-1} a_{ij} |ij\rangle, \quad \sum_{i=0}^{N_2-1} \sum_{j=0}^{N_3-1} |a_{ij}|^2 = 1$$

for some complex coefficients $a_{ij} \in \mathbb{C}$. The degree of entanglement depends on the coefficients $a_{ij}$, $i = 0, \ldots, N_2 - 1$, $j = 0, \ldots, N_3 - 1$. To send the state $|\Psi_0\rangle$ to Bob’s hand, it is necessary that $N_3 \geq N_1$. In the following we take $N_3 = N_1$.

The initial state Alice and Bob have is then given by

$$|\Psi_0\rangle \otimes |\Psi_1\rangle = \sum_{i,k=0}^{N_1-1} \sum_{j=0}^{N_2-1} \alpha_i a_{jk} |ijk\rangle \quad \in H_1 \otimes H_2 \otimes H_3.$$  

Alice has the first and the second particles and Bob has the third one. To transform the state of Bob’s particle to be $|\Psi_0\rangle$, similar to the qubit case, one has to do some unitary transformation $U$ and measurements. Let $U$ be the unitary transformation acting on the tensor product of two quantum states in the Hilbert spaces $H_1$ and $H_2$ such that

$$U (|ij\rangle) = \sum_{s,t=0}^{N_1-1 N_2-1} b_{ijst} |st\rangle,$$

with $\sum_{s=0}^{N_1-1} \sum_{t=0}^{N_2-1} b_{ijst} b_{ij's't'} = \delta_{ss'} \delta_{tt'}, \forall i = 0, \ldots, N_1 - 1, j = 0, 1, \ldots, N_2 - 1$. 

5
Theorem 1 If \( b_{ijst} \) satisfies the following relation
\[
\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \alpha_i a_{jk} b_{ijst} = \frac{1}{\sqrt{N_1 N_2}} \alpha_{k-t+1} c_{s k-t+1} t \tag{7}
\]
for some \( c_{ijk} \in \mathbb{C} \) such that \( c_{ijk} c_{ijk}^* = 1 \), \( U \) is the unitary transformation that fulfills the quantum teleportation.

Proof. From (6) and (7) we have, with \(|\Psi_0\rangle\) (resp. \(|\Psi_1\rangle\)) as in (3) (resp. (4)):
\[
(U \otimes 1)(|\Psi_0\rangle \otimes |\Psi_1\rangle) \equiv |\psi\rangle = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \alpha_i a_{jk} b_{ijst} |st\k\rangle
\]
\[
= \frac{1}{\sqrt{N_1 N_2}} \sum_{s=0}^{N_1-1} \sum_{t=0}^{N_2-1} \alpha_{k-t+1} c_{s k-t+1} t |st\k\rangle
\]
\[
= \frac{1}{\sqrt{N_1 N_2}} \sum_{i,j=0}^{N_1-1} \sum_{k=0}^{N_2-1} c_{ijk} \alpha_j |ik k+j-1\rangle,
\]
where the indices \( i, j, k \) in the basis vector \(|ijk\rangle\) are understood to be taken modulo by \( N_1-1, N_2-1 \) and \( N_1-1 \) respectively.

Now Alice measures her two qubits in the state \(|\psi\rangle \in H_1 \otimes H_2\). If \(|ik\rangle\) is the state obtained after the measurement, i.e.,
\[
|\psi\rangle \rightarrow |ik\rangle \otimes \left( \sum_{j=0}^{N_1-1} c_{ijk} \alpha_j |k+j-1\rangle \right),
\]
then in order to recover the original state \(|\Psi_0\rangle\), the unitary operator that Bob should use to act on his qubit is
\[
O_{ik} = P_k C_{ik}, \quad i = 0, 1, \ldots, N_1-1, \quad k = 0, 1, \ldots, N_2-1, \tag{8}
\]
where \( P_k \) is the \((k-1)\)-th power of the permutation operator, \( P_k = \Pi^{k-1} \),
\[
\Pi = \begin{pmatrix}
1 & & & \\
1 & & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
1 & & & \\
\end{pmatrix}
\]
(where the places without integer 1 are zero) and \( C_{ik} = \text{diag}(c_{i1k}^*, c_{i2k}^*, \ldots, c_{iN_1k}^*) \).

After this transformation, one gets \(|\psi\rangle \rightarrow |ik\rangle \otimes |\Psi_0\rangle\) and the state \(|\Psi_0\rangle\) given by (3) is teleported from Alice to Bob. \(\square\)
As a consequence of the above Theorem whenever an entangled state in the sense of (4) is given, i.e. the $a_{ij}$ are given, if there are solutions of $b_{jst}$ to equation (7), we have a unitary transformation $U$ that fulfills the teleportation. The condition (7) can also be rewritten as

$$\sqrt{N_1N_2} \sum_{i=0}^{N_2-1} a_{i+t+j-1}b_{jst} = c_{sjt}.$$  \hspace{1cm} (9)

The unitary transformation (6) given by the quantities $b_{jst}$ used in our teleportation protocol depends on the initially given entangled state (4) and the dimensions of the Hilbert spaces $H_1, H_2, H_3$.

For general $N = N_1 = N_2 = N_3$, if we take

$$a_{ij} = \frac{\delta_{ij}}{\sqrt{N}},$$  \hspace{1cm} (10)

the entangled state (4) is given by

$$|\Psi_1\rangle_{\text{max}} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |ii\rangle.$$  

From equation (9), we obtain the unitary transformation (6) used in the teleportation protocol with

$$b_{i+t+i-1 st} = \frac{c_{sit}}{\sqrt{N}},$$  \hspace{1cm} (11)

with $c_{sit}$ as in (7), the other coefficients $b$ in (6) being zero. It is easily checked that the transformation (6) given by (11) is a unitary one. The teleportation is accomplished by applying the unitary operation (8) according to the result of Bob's measurement. For some particular values of the coefficients $c_{sit}$, this result coincide with the one in [11].

According to the Schmidt decomposition, see e.g. [18], in this case the entangled state (4) on the Hilbert spaces $H_2$ and $H_3$ can be written as

$$\sum_{i=0}^{N-1} \sqrt{\lambda_i} |ii\rangle$$

in a suitable basis, where $\lambda_i \geq 0$, $\sum_{i=0}^{N-1} \lambda_i = 1$. That is, we can take $a_{ij} = \sqrt{\lambda_i} \delta_{ij}$. Substituting this into equation (9), we have

$$\sqrt{\lambda_{t+j-1}}b_{j t+j-1 st} = c_{sit}.$$
According to the unitarity of the transformation $U$ and the condition $c_{ijk}c_{ijk}^* = 1$, one gets $\lambda_i = 1/N$, $i = 0, 1, ..., N - 1$, and the state (4) is maximally entangled, which shows that with a less than maximally entangled state it is impossible to give a unitary transformation that fulfills perfect teleportations.

For $N = 2$, taking $c_{111} = c_{211} = c_{112} = c_{212} = c_{122} = -c_{221} = -c_{222} = 1$ (this choice satisfies the condition $c_{ijk}c_{ijk}^* = 1$), we have $O_{11} = I$, $O_{12} = \sigma_x$, $O_{21} = \sigma_z$, $O_{22} = i\sigma_y$, where $\sigma_x,y,z$ are Pauli matrices and $I$ is the $2 \times 2$ identity matrix. The unitary transformation $U$ is then equal to the joint actions of the controlled-not gate $CNOT$ and the Walsh-Hadamard transformation $H$, as defined, e.g., in [20,21]. This recovers the usual protocol for teleporting two level quantum states, the most simple example of teleportation given in section 2.

When $N = 2^l$ for some $l \in \mathbb{N}$, a case discussed in [7], $|\Psi_1\rangle$ can be rewritten as

$$|\Psi_1\rangle = \prod_{i=1}^{l} |\beta_i\rangle = \prod_{i=1}^{l} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_i,$$

$|\beta_i\rangle$ stands for the $i$-th EPR pair with the first (resp. second) qubit attached to the Hilbert space $H_2$ (resp. $H_3$). Therefore instead of a fully entangled state of two $N$-level qubits, we only need $l$ pairs of entangled two-level qubits. This conforms with the discussions in [7].

Generally, the dimension $N_2$ of $H_2$ can be greater than $N_1$. As long as one prepares the entangled state of two qubits in the Hilbert spaces $H_3$ and the sub Hilbert space $H_2 \subset H_2$, with dim($H_2$) = $N_1$, the above results are still valid.

We consider now some special cases of teleportations when some components of the initial state are zero. Without losing generality, let $\alpha_i \neq 0$ for $i = 1, ..., n_1$, $n_1 < N_1$, and $\alpha_i = 0$ for $i = n_1 + 1, ..., N_1$ (We remark that for a given $N_1$-dimensional vector it is always possible to make some of its components to be zero by changing the basis. But such a basis transformation depends of course on the components of the given vector, hence for an unknown quantum state this kind of transformation has no practical use).

The initial state to be teleported under the above hypothesis can be written as

$$|\Psi_0\rangle = \sum_{i=1}^{n_1} \alpha_i |i\rangle.$$

We take the dimension of $H_2$ to be $N_2 = n_1 < N_1$. The entangled state used
to teleport $|\Psi_0\rangle$ can be prepared in the following way:

$$a_{ij} = \begin{cases} 
\frac{1}{\sqrt{n}} \delta_{ij}, & j = 1, \ldots, n_1 \\
0, & j = n_1 + 1, \ldots, N_1 
\end{cases} \quad (12)$$

for $i = 1, 2, \ldots, n_1$. From (9) we get the unitary transformation (6) with

$$b_{it+i-1\mathrm{st}} = \frac{c_{\text{sit}}}{\sqrt{N_1}}$$

for $t, t+i - 1 \pmod{n_1} = 1, \ldots, n_1$, $i, s = 0, 1, \ldots, N_1 - 1$, the other coefficients $b_{jist}$ in (9) being zero.

An example is the teleportation of an EPR pair $|\Psi_0\rangle = a|01\rangle + b|10\rangle$, $|a|^2 + |b|^2 = 1$, as discussed in [4]. In this case we have $N_1 = 4$. $|\Psi_0\rangle$ can be written as $\alpha|3\rangle + \beta|2\rangle \equiv \alpha|1'\rangle + \beta|2'\rangle$. The dimension of the auxiliary Hilbert space $H_2$ is only needed to be $n_1 = 2$. The entangled state is given by

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) = \frac{1}{\sqrt{2}}(|13\rangle + |22\rangle) = \frac{1}{\sqrt{2}}(|101\rangle + |010\rangle).$$

Here as $n_1 = N_1/2 = 2$, instead of (12), we may alternatively take $a_{ij} = \frac{1}{\sqrt{N_1}} \delta_{ij}$ for $j = n_1 + 1, \ldots, N_1$ and $a_{ij} = 0$ for $j = 1, \ldots, n_1$. Then the entangled state becomes $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |24\rangle) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, which is called a GHZ triplet consisting of three two-level qubits and can be realized experimentally [22,23]. The unitary transformation is given by $b_{it+i-1\mathrm{st}} = c_{\text{sit}}/\sqrt{N_1}$ for $t = 1, \ldots, n_1$, $t+i - 1 \pmod{n_1} = n_1 + 1, \ldots, N_1$, $i, s = 0, 1, \ldots, N_1 - 1$, and the other coefficients $b_{jist}$ in (9) being zero. For a suitable choice of the sign of $c_{\text{sit}}$, this recovers the result in [4].

4 Standard teleportation of mixed states

In realistic situations, due to the interactions with the environment, instead of a pure state, Alice may have a mixed state $\rho$. And instead of the pure maximally entangled states, Alice and Bob usually share a mixed entangled state $\chi$. Before considering the the teleportation of a mixed state by a mixed entangled state as an entangled resource, we first reformulate the teleportation protocol for pure states.

Any linear operator $A : H_\alpha \to H_\beta$ can be represented by an $N_\beta \times N_\alpha$-matrix as follow:

$$A(|\alpha\rangle) = \sum_{i=0}^{N_\beta-1} A_{\alpha i} |i\rangle, \quad |\alpha\rangle \in H_\alpha.$$
Take $N_1 = N_3 = n$, $N_2 = m$. To transform the state $|\Psi_0\rangle$ in (3) to Bob by using the entangled state (4) and the unitary transformation $U$ in (6), we introduce a number $mn$ of $m \times n$-matrices with matrix elements $(B_{st})_{ij} = b_{ij,st}$. Let $A$ be an $n \times m$ matrix with matrix element $(A)_{ij} = a_{ij}$. We have

**Theorem 2** If $\{B_{st}\}$ and $A$ satisfy the following relation

$$
\begin{cases}
    \text{tr} \left( B_{st} B_{st}' \right) = \delta_{tt'} \delta_{ss'}, \\
    B_{st}^+ A^+ A B_{st} = |\lambda_{st}|^2 I_{n \times n}, \\
    A A^+ = \frac{1}{n} I_{n \times n},
\end{cases}
$$

(13)

for some nonzero complex number $\lambda_{st}$ such that $\sum_{s,t} |\lambda_{st}|^2 = 1$, then $U$ is a unitary transformation that fulfills perfect quantum teleportation.

**Proof.** The first condition in (13) is equal to the unitary condition for the transformation (6). From (5) and (6), after Alice measures her two particles with outcome state $|st\rangle$, Bob’s particle will become

$$
|\Psi_0\rangle \rightarrow T_{st} |\Psi_0\rangle = AB_{st} |\Psi_0\rangle.
$$

If the condition (13) is satisfied, then we have $T_{st}^+ T_{st} = T_{st} T_{st}^+ = |\lambda_{st}|^2 I_{n \times n}$, and perfect quantum teleportation is fulfilled. \qed

For the $m = n$ case, introducing $U_{st} = \sqrt{n} B_{st}$, condition (13) is equivalent to

$$
\begin{cases}
    \text{tr} \left( U_{st} U_{st}' \right) = n \delta_{tt'} \delta_{ss'}, \\
    U_{st}^+ U_{st}^+ = I_{n \times n},
\end{cases}
$$

(14)

with

$$
A A^+ = A^+ A = \frac{1}{n} I_{n \times n} \text{ and } |\lambda_{st}|^2 = \frac{1}{n^2}.
$$

(15)

Hence, only the maximally entangled state (4) (satisfying $A A^+ = A^+ A = \frac{1}{n} I_{n \times n}$) can fulfill the perfect quantum teleportation. For the maximally entangled state shared by Alice and Bob, in order to teleport an unknown state perfectly, one should find a number $n^2$ of $n \times n$-matrices which satisfy (14) (these constitute, according to Werner’s definition \cite{8}, a basis for the unitary operators). The classification of the general solution of (14) is an open problem \cite{8}. Fortunately, we can construct a special solution up to a global unitary transformation in the sense of [8] as follows.

Let the $n \times n$ matrices $h$, $g$ be such that $h|j\rangle = |(j + 1) \mod n\rangle$, $g|j\rangle = \omega^j |j\rangle$, with $\omega = \exp \left( -\frac{2\pi i}{n} \right)$. Define $n^2$ linear-independent $n \times n$-matrices

$$
U_{st} = h^t g^s
$$

(16)
which satisfy

\[ U_{st} U_{s't'} = \omega^{s't'-ts'} U_{s't'} U_{st}, \quad tr(U_{st}) = n \delta_{s0} \delta_{t0}. \]  

(17)

One can check that such \( U_{st} \) in (17) satisfy equation (14) and form a complete basis of \( n \times n \)-matrices. They can be used to complete the perfect teleportation through a maximally entangled state\(^{1,19} \).

In the following part of this paper, we focus on the case \( m = n \). Set

\[ |\phi\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |ii\rangle, \]

and let the \( \{U_{st}\} \) be as in (17). The standard teleportation protocol \( T_0 \)\(^1 \) can be written as the following linear transformation

\[ T_{T_0}^{(|\Phi\rangle)}(|\phi\rangle) = \sum_{s,t} \left( \frac{1}{\sqrt{n}} B_{st}^+ \right)^+ \left( P_{st} (U \otimes 1 (|\phi\rangle \otimes |\Phi\rangle)) \right) \]

\[ = \sum_{s,t} \frac{1}{\sqrt{n}} B_{st}^+ \frac{1}{\sqrt{n}} B_{st} |\phi\rangle = |\phi\rangle, \]

(19)

where \( P_{st} \) is the projection \( (|st\rangle \langle st|) \otimes 1 \). Namely, the noiseless quantum channel is

\[ \Lambda_{T_0}^{(|\Phi\rangle)}(\rho) = \rho. \]

Although in principle one can create pure and maximally entangled states for teleportation, in a realistic situation any pure state will be evolved into a mixed state due to its interaction with the environment (decoherence). These unwanted interactions show up as noise in quantum information processing systems. Teleportation using a mixed state as an entangled resource is, in general, equivalent to having a noisy quantum channel.

We shall derive an explicit expression for the quantum channel associated with the standard teleportation protocol in terms of an arbitrary mixed state resource. From the complete basis (17), one can introduce a complete orthogonal normalized basis \( \{ |\Phi_{st}\rangle \} \) of the Hilbert spaces associated with Alice and Bob,

\[ |\Phi_{st}\rangle = (1 \otimes U_{st}) |\phi\rangle = \frac{1}{\sqrt{n}} \sum_{i,j} (U_{st})_{ij} |ij\rangle, \quad \text{and} \quad |\Phi_{00}\rangle = |\Phi\rangle. \]

(21)

As locally unitary transformations do not change the degree of the entanglement, each \( |\Phi_{st}\rangle \) is a maximally entangled state.
Theorem 3  The standard teleportation protocol $T_0$, when used with an arbitrary mixed state with density matrix $\chi$, as a resource, acts as a general trace-preserving quantum channel,

$$\Lambda_{T_0}^{(\chi)}(\rho) = \sum_{s,t=0}^{n-1} \langle \Phi_{st} | \chi | \Phi_{st} \rangle U_{st} \rho U_{st}^+. \quad (22)$$

Proof. If Alice and Bob share with a general pure state (4) as their quantum resource, after the measurement based on the Bell-basis $|\Phi_{st}\rangle$, Bob’s particle becomes an (unnormalized) state

$$|\phi\rangle \rightarrow (P_{st} (U \otimes 1 (|\phi\rangle \otimes |\Psi\rangle))) = AB_{st} |\phi\rangle.$$  

In terms of the density matrix, we have

$$\rho \rightarrow AB_{st} \rho B_{st}^+ A^+.$$  

After some local unitary operations by Bob, according to the results of the measurement made by Alice, the final state can be given as

$$\Lambda^{(|\Psi\rangle\langle\Psi|)}(\rho) = \sum_{t,\beta} U_{t\beta}^{+} AB_{t\beta} \rho B_{t\beta}^+ A^+ U_{t\beta}.$$  

Suppose that $\{p_{\alpha}, |\Psi_{\alpha}\rangle\}$ is one of the ensemble constituting the arbitrary mixed state $\chi$, i.e.

$$\chi = \sum_{\alpha} p_{\alpha} |\Psi_{\alpha}\rangle\langle\Psi_{\alpha}|, \quad 0 \leq p_{\alpha} \leq 1 \quad (23)$$

and

$$\sum_{\alpha} p_{\alpha} = 1, \quad |\Psi_{\alpha}\rangle = \sum_{i,j} a_{ij}^{(\alpha)} |ij\rangle.$$  

From the standard teleportation protocol $T_0$, with the resource state $\chi$, the final Bob’s state becomes

$$\Lambda_{T_0}^{(\chi)}(\rho) = \sum_{t,\beta} \sum_{\alpha} p_{\alpha} U_{t\beta}^{+} A^{(\alpha)} B_{t\beta} \rho B_{t\beta}^+ (A^{(\alpha)})^+ U_{t\beta}. \quad (24)$$  

Since each matrix $A^{(\alpha)}$ can be decomposed in the basis $\{U_{st}\}$,

$$(A^{(\alpha)})_{ij} = \sum_{s,t} a_{ij}^{(\alpha)} (U_{st})_{ij}, \quad (25)$$
using $B_{t\beta} = \frac{1}{\sqrt{n}} U_{t\beta}$, (24) becomes

$$\Lambda^{(\chi)}(\rho) = \frac{1}{n} \sum_{s,t} \sum_{s',t'} \left( \sum_{\alpha} p_{\alpha} a_{st}^{(\alpha)} a_{s't'}^{(\alpha)*} \right) \sum_{\gamma,\beta} U_{\gamma\beta}^{+} U_{st} \rho U_{\gamma\beta}^{+} U_{s't'}^{+} U_{\gamma\beta}$$

$$= \frac{1}{n} \sum_{s,t} \sum_{s',t'} \left( \sum_{\alpha} p_{\alpha} a_{st}^{(\alpha)} a_{s't'}^{(\alpha)*} \right) U_{st} \rho U_{s't'}^{+} \sum_{\gamma,\beta} \omega_{s\beta - t\gamma - s'\beta + t'\gamma}$$

$$= \frac{1}{n} \sum_{s,t} \sum_{s',t'} \left( \sum_{\alpha} p_{\alpha} a_{st}^{(\alpha)} a_{s't'}^{(\alpha)*} \right) U_{st} \rho U_{s't'}^{+} n^{2} \delta_{ss'} \delta_{tt'}, \quad (26)$$

where we have used the equations (17) and the identity $\sum_{k=1}^{n} \omega^{mk} = n \delta_{m0}$. Using the definition of the generalized Bell-states $\{|\Phi_{st}\rangle\}$ given in (21), after a lengthy calculation, we arrive at

$$n \sum_{\alpha} p_{\alpha} a_{st}^{(\alpha)} a_{s't'}^{(\alpha)*} = \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle.$$ 

Substituting the above results into (26), one obtains (22). Using (14), the trace-preserving property of the quantum channel can be derived from the following useful identity

$$\sum_{s,t} U_{st} \rho U_{st}^{+} = n \text{tr}(\rho) I_{n \times n}. \quad (27)$$

The most important consequence of our result is that in order to fulfill the perfect noise-free teleportation, the entangled resource should not only be a maximally entangled state but also a pure state $^8$, otherwise the outgoing state is always mixed. The success of teleportation can be quantified by the transmission fidelity between outstate and instate defined by

$$f(\chi) = \frac{\langle \Phi_{in} | \Lambda^{(\chi)}(\rho) | \Phi_{in} \rangle}{\langle \Phi_{in} | \Phi_{in} \rangle} \langle \Phi_{in} | \Phi_{in} \rangle,$$ 

averaged over all pure input states $\phi_{in}$.

In order to calculate the transmission fidelity (28), we need an elementarily irreducible representation $G$ of the unitary group $U(n)$. Let $U(g)$ be the unitary matrix representation of any element of $G$. Recalling Schur’s Lemma, one has the identity

$$\int_{G} dg \, (U^{+}(g) \otimes U^{+}(g)) \sigma (U(g) \otimes U(g)) = \alpha_{1} I \otimes I + \alpha_{2} P, \quad (29)$$

$$\alpha_{1} = \frac{n^{2} \text{tr}(\sigma) - n \text{tr}(\sigma P)}{n^{2}(n^{2} - 1)}, \quad \alpha_{2} = \frac{n^{2} \text{tr}(\sigma P) - n \text{tr}(\sigma)}{n^{2}(n^{2} - 1)},$$
for any operator \( \sigma \) acting on the tensor space, where \( P \) is the flip operator such that \( P|ij\rangle = |ji\rangle \). The invariant (Haar) measure \( dg \) on \( G \) is normalized such that \( \int_G dg = 1 \).

**Theorem 4** The transmission fidelity of the standard teleportation protocol with arbitrary mixed state \( \chi \) as a resource is given by

\[
f(\chi) = \frac{n}{n+1} F(\chi) + \frac{1}{n+1},
\]

(30)

where \( F(\chi) = \langle \Phi | \chi | \Phi \rangle \) is the singlet fraction of the resource \( \chi \).

**Proof.** From Theorem 3, we have

\[
f(\chi) = \sum_{s,t} \langle \Phi_{st} | \chi | \Phi_{st} \rangle \langle \phi_{in} | U_{st} | \phi_{in} \rangle \langle \phi_{in} | U_{st}^+ | \phi_{in} \rangle
\]

\[
= \sum_{s,t} \langle \Phi_{st} | \chi | \Phi_{st} \rangle \langle \phi_{in} | (U_{st} \otimes U_{st}^+) | \phi_{in} \rangle |\phi_{in}\rangle
\]

\[
= \sum_{s,t} \langle \Phi_{st} | \chi | \Phi_{st} \rangle \int_G dg \langle 00 | (U_{st}^+ \otimes U_{st}^+) (U_{st} \otimes U_{st}^+) (U(g) \otimes U(g)) |00\rangle
\]

\[
= \frac{1}{n(n+1)} \sum_{s,t} \langle \Phi_{st} | \chi | \Phi_{st} \rangle \left\{ tr(U_{st}) \ tr(U_{st}^+) + tr(U_{st} U_{st}^+) \right\}
\]

\[
= \langle \Phi_{00} | \chi | \Phi_{00} \rangle \frac{n}{n+1} + \frac{1}{n+1},
\]

where we have used the identity (29) and \( tr_{12} ( (A \otimes B) P ) = tr(AB) \) in deriving the fourth equation, and have used the identity (17) in the last equation. \( \square \)

Formula (30) implies that the transmission fidelity of the standard teleportation protocol depends on the maximally entangled fraction only. Hence, in order to improve the transmission fidelity of the standard teleportation protocol with a given entangled resource, one must distill the singlet fraction of the resource in terms of a distillation protocol \(^{11}\).

### 5 Optimal teleportation of mixed quantum states

From above we see that the main operations in the quantum teleportation protocol are: 1) a unitary transformation and a measurement by Alice (Bell measurement); 2) according to the results of the measurement, a unitary transformation by Bob. In the last section the unitary transformation used by Bob
is the same as the one in standard teleportation (i.e., when the entangled state used in teleportation is maximally entangled). A question one would ask is whether the fidelity given by (30) is the best one or not. In this section we consider the following problem: based on the Bell measurement, what kind of unitary transformation should be used by Bob to get an optimal fidelity.

\{U_{st}\} in (16) form a complete basis of \(n \times n\)-matrices, namely, for any \(n \times n\) matrix \(W\), \(W\) can be expressed as

\[
W = \frac{1}{n} \sum_{s,t} \text{tr}(U_{st}^+ W) U_{st}.
\] (32)

In general, all the maximally entangled pure states are equivalent to \(|\Phi\rangle\) in (18): \(|\Psi_{\text{max}}\rangle = 1 \otimes U |\Phi\rangle\), where \(U\) is a unitary transformation. One can define the fully entangled fraction\(^1\) of a state \(\chi\) by

\[
\mathcal{F}(\chi) = \max \left\{ \langle \Phi | (1 \otimes U^+) \chi (1 \otimes U) |\Phi\rangle \right\}
\] (33)

for all \(UU^+ = U^+ U = I_{n \times n}\). Since the group of unitary transformations in \(n\)-dimensions is compact, there exists an unitary matrix \(W_\chi\) such that

\[
\mathcal{F}(\chi) = \langle \Phi | (1 \otimes W_\chi^+) \chi (1 \otimes W_\chi) |\Phi\rangle.
\] (34)

Suppose now again Alice and Bob previously share a pair of particles in an arbitrary mixed entangled state \(\chi\). To transform an unknown state to Bob, Alice first performs a joint Bell measurement based on the generalized Bell-states (21) on her parties. According to the measurement results of Alice, Bob chooses particular unitary transformations \(\{T_{st}\}\) to act on his particle.

**Theorem 5** The teleportation protocol defined by \(\{T_{st}\}\), when used with an arbitrary mixed state with density matrix \(\chi\) as a resource, acts as a general trace-preserving quantum channel

\[
\Lambda_{\{T\}}(\chi) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \sum_{\gamma\beta} \left\{ T_{\gamma\beta}^+ U_{\gamma\beta} \rho U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma\beta} \right\}.
\] (35)

**Proof.** The proof can be given in two steps:

**Step 1. Pure entangled state as a resource.** To transform the state \(|\Psi_0\rangle\) to Bob, Alice performs a joint Bell measurement based on the generalized Bell-states Eq.(21) on her parties. After her measurement with outcomeing in the state \(|\Phi_{st}\rangle\), Bob’s particle gets into an (unnormalized) state

\[
|\phi\rangle \rightarrow \frac{1}{\sqrt{n}} A U_{st} |\phi\rangle.
\]
Once Bob learns from Alice that she has obtained the result $s\ell$, he performs on his previously entangled particle (particle 3) a unitary transformation $T_{st}$. Then the final state becomes $\frac{1}{\sqrt{n}} T_{st}^+ U_{st} |\phi\rangle$. In terms of the density matrix, the teleportation based on the unitary matrices $\{T_{st}\}$, with quantum resource being a pure state $|\Psi_1\rangle$, is a quantum channel with the output

$$\Lambda^{(|\Psi_1\rangle\langle\Psi_1|)}(\rho) = \frac{1}{n} \sum_{st} T_{st}^+ U_{st} \rho U_{st}^+ T_{st}. $$

**Step 2. An arbitrary mixed entangled state as a resource.** Applying the teleportation protocol $T$ with a mixed state $\chi$, Bob’s state becomes

$$\Lambda^{(\chi)}(\rho) = \frac{1}{n} \sum_{s,t} \sum_{s',t'} p_\alpha T_{st}^+ A^{(\alpha)} U_{st} \rho U_{st}^+ (A^{(\alpha)})^+ T_{st}. $$

As $A^{(\alpha)}$ can be decomposed in the basis $\{U_{st}\}$ by formula (25), (36) becomes

$$\Lambda^{(\chi)}(\rho) = \frac{1}{n} \sum_{s,t} \sum_{s',t'} \sum_\alpha p_\alpha T_{st}^+ A^{(\alpha)} U_{st} \rho U_{st}^+ (A^{(\alpha)})^+ T_{st}. $$

Using the definition of generalized Bell-states $\{|\Phi_{st}\rangle\}$ in (21), after a lengthy calculation, we arrive at

$$n \sum_{s,t} p_\alpha A^{(\alpha)}_{st} (A^{(\alpha)})^* = |\Phi_{st}\rangle$.

Substituting the above results into (36), we obtain (35). Using (14) and the identity

$$\sum_{s,t} U_{st}^+ A U_{st} = n \text{tr}(A) I_{n \times n}, \text{ for any } n \times n \text{ matrix } A,$n$$

the trace-preserving property of the quantum channel can be proved,

$$\text{tr} \left( \Lambda^{(\chi)}(\rho) \right) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \left( \sum_{\gamma\beta} \text{tr} \left( U_{\gamma\beta}^+ U_{s't'}^+ U_{st} U_{\gamma\beta} \rho \right) \right)$$

$$= \frac{1}{n} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \text{tr} \left( U_{st} U_{s't'}^+ \right) \times \text{tr}(\rho)$$

$$= \sum_{s,t} \langle \Phi_{st} | \chi | \Phi_{st} \rangle = \text{tr}(\chi) = 1.$$
Theorem 6  The transmission fidelity of the teleportation protocol defined by \( \{ T_{st} \} \) with arbitrary mixed state \( \chi \) as a resource is given by

\[
f(\chi) = \frac{1}{n(n+1)} \sum_{\gamma\beta} \langle \Phi | \left( 1 \otimes (T_{\gamma}\beta U_{\gamma\beta}^+) \right) \chi \left( 1 \otimes T_{\gamma}\beta U_{\gamma\beta}^+ \right) | \Phi \rangle \\
+ \frac{1}{n+1}.
\]

(37)

Proof. From (28), Theorem 5 and formula (29), one has

\[
f(\chi) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \\
\sum_{\gamma\beta} \langle \phi_{in} | T_{\gamma}\beta U_{st} U_{\gamma\beta} | \phi_{in} \rangle \langle \phi_{in} | U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma}\beta | \phi_{in} \rangle \\
= \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \\
\sum_{\gamma\beta} \langle \phi_{in} | \otimes \langle \phi_{in} | \left( T_{\gamma}\beta U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma}\beta \right) | \phi_{in} \rangle \otimes | \phi_{in} \rangle \\
= \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} \langle \Phi_{st} | \chi | \Phi_{s't'} \rangle \\
\sum_{\gamma\beta} \left\{ \text{tr} \left( T_{\gamma}\beta U_{st} U_{\gamma\beta} \right) \text{tr} \left( U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma}\beta \right) \\
+ \text{tr} \left( T_{\gamma}\beta U_{st} U_{\gamma\beta} U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma}\beta \right) \right\} \\
= \frac{1}{n(n+1)} \sum_{\gamma\beta} \langle \Phi | \left( 1 \otimes (T_{\gamma}\beta U_{\gamma\beta}^+) \right) \chi \left( 1 \otimes T_{\gamma}\beta U_{\gamma\beta}^+ \right) | \Phi \rangle + \frac{1}{n+1},
\]

where the identity \( \text{tr}_{12} ((A \otimes B)P) = \text{tr}(AB) \), (14) and (32) have been used.

\( \Box \)

Obviously when the term \( \langle \Phi | \left( 1 \otimes (T_{\gamma}\beta U_{\gamma\beta}^+) \right) \chi \left( 1 \otimes T_{\gamma}\beta U_{\gamma\beta}^+ \right) | \Phi \rangle \) is maximized, i.e., \( T_{\gamma}\beta U_{\gamma\beta}^+ = W_\chi \), one gets the maximal fidelity. Recalling the definition of the fully entangled fraction (33) and (34), we arrive at our main result:
Theorem 7 The optimal teleportation based on the Bell measurements, when used with an arbitrary mixed state with density matrix $\chi$ as a resource, acts as a general trace-preserving quantum channel

$$\Lambda^{(\chi)}(\rho) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} (\Phi_{st}|\chi|\Phi_{s't'})$$

$$\left\{ \sum_{\gamma\beta} U_{\gamma\beta}^+ W_X U_{\gamma\beta} \rho U_{\gamma\beta}^+ U_{s't'} W_X U_{\gamma\beta} \right\}.$$

The corresponding transmission fidelity is given by

$$f_{\text{max}}(\chi) = \frac{nF(\chi)}{n+1} + \frac{1}{n+1},$$

where $F(\chi)$ is the fully entangled fraction (33) and $W_\chi$ is the unitary matrix which fulfills such a fully entangled fraction in (34).

We have studied the general properties of teleportation for finite dimensional discrete pure quantum states. The protocol we presented is for generally given entangled states with $N_3 = N_1$. If $N_3 > N_1$, one can always take a subspace $H_3 \subset H_3$ such that $\dim(H_3) = N_1$ and prepare the entangled state in the Hilbert spaces $H_2$ and $H_3$. Accordingly the initial state $|\Psi_0\rangle$ will be sent to the subspace $H_3$.

For teleportation of quantum mixed states, we obtained an explicit expression of the output state of the optimal teleportation, with arbitrary mixed entangled state as resource, in terms of some noisy quantum channel. This allows us to calculate the transmission fidelity of the quantum channel. It is shown that the transmission fidelity depends only on the fully entangled fraction of the quantum resource shared by the sender and the receiver, whereas that of a standard teleportation depends on the singlet fraction. Therefore the fidelity in our optimal teleportation protocol is in general greater than the one in the standard teleportation protocol\(^{1,14,15,16}\).

Acknowledgments Acknowledgments W-L Yang would like to thank Prof. G. von Gehlen for his continuous encouragement during his visiting in Physikalisches Institut der Universität Bonn, where some part of this work was done. He also would like to thank Prof. R. Sasaki and the Yukawa Institute for Theoretical Physics, Kyoto University for their warm hospitality. He is supported by the Japan Society for the Promotion of Science.
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