The Mathematical and Geometrical Structure of the Spacetime and the Concept of Unification, Matter and Energy

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Abstract—Geometrical analysis of a new type of Unified Field Theoretical models follow the guidelines of previous works of the authors is presented. These new unified theoretical models are characterized by an underlying hypercomplex structure, zero non-metricity and the geometrical action is determined fundamentally by the curvature provenient of the breaking of symmetry of a group manifold in higher dimensions. This mechanism of Cartan–MacDowell–Mansouri type, permits us to construct geometrical actions of determinant type leading a non topological physical Lagrangian due the splitting of a reductive geometry. Our goal is to take advantage of the geometrical and topological properties of this theory in order to determine the minimal group structure of the resultant spacetime Manifold able to support a fermionic structure. From this fact, the relation between antisymmetric torsion and Dirac structure of the spacetime is determined and the existence of an important contribution of the torsion to the gizromagnetic factor of the fermions, shown. Also we resume and analyze previous cosmological solutions in this new UFT where, as in our work [Class. Quantum Grav. 22 (2005) 4987–5004] for the non abelian Born-Infeld model, the Hosoya and Ogura ansatz is introduced for the important cases of tratorial, totally antisymmetric and general torsion fields. In the case of spacetimes with torsion the real meaning of the spin-frame alignment is find and the question of the minimal coupling is discussed.

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1. MOTIVATION AND SUMMARY OF THE RESULTS

From long time ago in the history of the modern theoretical physics the possibility of the unification of all fundamental forces have been treated from the mathematical and theoretical point of view. Several models, formulations and sophisticated mathematical tools were used in order to solve the intricate puzzle of to conciliate gravity with the other fundamental forces of the nature: electromagnetic, weak and strong.
Although many attempts appear, this issue is still without concrete solution, as is the string theory the typical case. In string theory is common the claiming about to be the consistent solution of the unification trouble but, beside particular formulations, the theoretical and conceptual environment joined with an obscure mathematical basis put certainly in doubt the affirmative acceptance of such claim.

As was pointed out by us in later works [1, 2], the cornerstone of the problem is where to start conceptually to reformulate the theoretical arena where the fundamental unified theory will be placed, and where the geometry is the unifying essence. According to Mach spacetime doesn’t exist without matter. Then, two basic ideas immediately arise to fulfill the observation given by Mach: the concept of dualistic or non-dualistic theories. In the first one the simplest and economical description can be formulated in terms of the gravitational field without torsion plus the energy momentum tensor that, however, is added “by hand” in order to cover the lack of knowledge of a fundamental structure of the space time giving the matter plus energy distribution. In the second one there are not prescriptions for the interaction of gravity with the “matter” fields because they are arising from the same fundamental geometrical structure.

In previous works of the authors we present a new model of a non-dualistic Unified Theory. The goal that we introduce firstly in our preliminary model in [1], absolutely consistent from the mathematical and geometrical point of view, is that was based in a manifold equipped with an underlying hypercomplex structure and zero non-metricity, that lead the important fact that the Torsion of the space-time structure turns to be totally antisymmetric. As is well known in the particular case of totally antisymmetric torsion tensor this type of affine geometrical frameworks have the geodesics and the minimal length equations equivalent, and the most important is that is the only case that the equivalence principle is fulfilled as was shown in [9, 10] and we demonstrate also here.

The other goal that we introduce as main ingredient in [1, 2] and here, is that the specific form of our action is determined by the curvature from the breaking of symmetry of a group manifold in higher dimensions via a Cartan–MacDowell–Mansouri mechanism [1, 2]. This mechanism permits to construct geometrical actions of determinantal type that, due the splitting of a reductive geometry (as is the case of the group manifold treated here) via the breaking to the higher dimensional group (i.e.: as is the typical case so(1, 4) → so(1, 3) ⊗ \mathbb{M}_{11}) leads a non topological physical Lagrangian.

Following the guidelines of our last works [1–3], in this paper we complete the previous analysis considering the same fundamental model of UFT. The organization of the paper with the corresponding results is as follows: in Section 2 the geometrical framework is introduced and the theoretical basis of the model, based in a geometrical action that takes physical meaning through a breaking of symmetry, is described. In Section 3 the dynamic equations are analyzed and the geometrical and physical meaning are elucidated.

In Section 4 we resume and analyze previous cosmological solutions in the New UFT: as in our work [3] for the non abelian Born–Infeld model, the Hosoya and Ogura ansatz is introduced for the important cases of tratorial and totally antisymmetric torsion. The real meaning of the spin-frame alignment in the case with torsion is find. Also, we explicitly show that, contrarily to the case of the Poincare theory of gravitation (see reference [4]), the possibility in our Theory of the co-existence of both types of torsion in cosmological spacetimes certainly exists.

Section 5 is the most important in the sense that the fermionic structure of the spacetime is described and the possibility of geometrical unification realized: a unified theory of QED and GR can be derived from P(G,M), the Principal Fiber Bundle of frames over the 4D spacetime manifold with G as its structure group. In the subsections, the action of the UFT is analyzed from the group-theoretical point of view considering the G-symmetry of the model. In Section 6 the derivation of the Dirac equation from the G-manifold, the relation between the electromagnetic field/fermionic structure of the spacetime and the contribution of the torsion to the gyromagnetic factor are explicitly shown. However, the physical consequences are explained. Finally, Section 7 is devoted to discuss the cohomological interplay between the fields involved in the spacetime structure and in 8 the concluding remarks are given.

2. THE SPACE-TIME MANIFOLD AND THE GEOMETRICAL ACTION

The starting point is an hypercomplex construction of the (metric compatible) space-time manifold [1].

\[ M_g_{\mu\nu} \equiv e_\mu \cdot e_\nu, \]

where for each point \( M \) of a local space affine A. The connection over A, \( \Gamma \), define a generalized affine connection \( \Gamma \) on \( M \) specified by (\( \nabla, \kappa \)) where \( \kappa \) is an invertible (1,1) tensor over \( M \). We will demand that the connection is compatible and rectilinear

\[ \nabla \kappa = KT, \nabla g = 0, \]

where \( T \) is the torsion, and \( g \) (the space-time metric, used for to raise and to low indices and determines the geodesics) is preserved under parallel transport. This generalized compatibility condition ensures that the affine generalized connection \( \Gamma \) maps autoparallels of \( \Gamma \) on \( M \) in straight lines over the affine space A (locally). The first equation is equal to the condition determining the connection in terms of the fundamental field in the UFT non-symmetric. For instance, \( K \) can be identified with the fundamental tensor in the
non-symmetric fundamental theory. This fact give us
the possibility to restrict the connection to an (anti)
Hermitian theory.

The covariant derivative of a vector with respect to
the generalized affine connection is given by
\[ A^\mu\cdot_v = A^\mu\nu + \Gamma^\mu_{\alpha\nu} A^\alpha, \]
\[ A^\mu\nu = A^\mu\nu - \Gamma^\alpha_{\mu\nu} A_\alpha. \] (3)

The generalized compatibility condition (2) determines
the 64 components of the connection by the 64
equations as follows
\[ K_{\mu\nu;\alpha}^\ell = K_{\mu\nu}^\ell T^\rho_{\nu\alpha} \text{ where } T^\rho_{\nu\alpha} \equiv 2\Gamma^\rho_{[\nu\alpha]} \] (4)
Notice that contraction of indices \( \nu \) and \( \alpha \) above in
the first equation (4), an additional condition over this
hypotential fundamental (nonsymmetric) tensor \( K \) is
obtained
\[ K_{\mu\nu;\alpha}^\ell = 0, \]
that, geometrically speaking is
\[ d^K b K = 0 \]
this is a current free condition over the tensor \( K \) that
can be exemplified nicely with the prototype of non-
symmetric fundamental tensor: \( K_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu} \)
\[ d^K b K = d^K g + d^K f \implies d^K f = 0 \]
( current free e.o.m.)
where, however, playing \( g_{\mu\nu} \) the role of spacetime met-
ic and \( f_{\mu\nu} \) the role of electromagnetic field.

The metric is univocally determined by the metrizability
condition that puts 40 restrictions on the partial derivatives
of the metric
\[ g_{\mu\nu,\rho} = 2\Gamma^\rho_{[\mu\nu]} \] (5)
The space-time curvature tensor, that is defined in the
usual way, has two possible contractions: the Ricci
tensor \( R^\lambda_{\mu\nu} = R_{\mu\nu} \) and the second contraction
\( R^\lambda_{\mu\nu} = 2\Gamma^\lambda_{[\nu,\mu]} \) is identically zero due
the metrizability condition (2). In order to find a symmetry of the torsion
tensor if we denote the inverse of \( K \) by \( \tilde{K} \), \( \tilde{K} \) is uniquely
specified by \( \tilde{K}_{\mu\nu}^\alpha K_{\alpha\sigma} = K_{\mu\sigma} \tilde{K}_{\alpha\sigma} = \delta^\mu_{\sigma}. \) As was pointed
out in [1], inserting explicitly the torsion tensor as the
antisymetric part of the connection in (4) and mul-
tiplying by \( \frac{\tilde{K}_{\mu\nu}}{2} \) this results after straightforward
computations in
\[ (\ln \sqrt{-K})_{\mu\nu} - \Gamma^\nu_{\nu\nu} = 0, \] (6)
where \( K = \det(K_{\mu\nu}). \) Notice that from expression (6)
we arrive to the following condition between the deter-
minants \( K \) and \( g: K/g = \text{constant}. \) Now we can write
\[ \Gamma^\nu_{\nu\nu,\nu} - \Gamma^\nu_{\nu\nu,\alpha} = \Gamma^\nu_{\nu\nu,\alpha} - \Gamma^\nu_{\nu\nu,\beta}, \] (7)
due the fact that the first term of is the derivative of an
scalar. Then, the torsion tensor has the symmetry
\[ T^\nu_{\nu;[\alpha,\beta]} = T^\nu_{\nu;[\alpha,\beta]} = 0. \] (8)
That means that the trace of the torsion tensor
defined as \( T^\nu_{\nu\nu} \), is the gradient of a scalar
\[ T_{\alpha} = \nabla_{\alpha}\phi. \]

The second important point is the following: let us
consider [1] the extended curvature
\[ \mathcal{R}^{\mu\nu}_{\alpha\beta} = R^{\mu\nu}_{\alpha\beta} + \Sigma^{\mu\nu}_{\alpha\beta}, \] (9)
with
\[ R^{\mu\nu}_{\alpha\beta} = \partial^\mu \omega^\nu_{\alpha\beta} - \partial^\nu \omega^\mu_{\alpha\beta} + \omega^\mu_{\alpha\epsilon} \omega^\nu_{\epsilon\beta} - \omega^\mu_{\epsilon\beta} \omega^\nu_{\alpha\epsilon}, \]
\[ \Sigma^{\mu\nu}_{\alpha\beta} = -\varepsilon^\nu_{\mu\beta} \varepsilon^\mu_{\alpha\beta} \varepsilon^\alpha_{\mu\beta}. \] (10)
We assume here \( \omega^\mu_{\alpha\beta} \) a \( SO(d - 1, 1) \) connection and \( e^\mu_{\alpha\beta} \)
is a vierbein field. The eqs. (9, 10) can be obtained, for
example, using the formulation that was pioneering
introduced in seminal works by E. Cartan long time
ago [1]. Is well known that in such an formalism the
gravitational field is represented as a connection one
form associated with some group which contains the
Lorentz group as subgroup. The typical example is
provided by the \( SO(d, 1) \) de Sitter gauge theory of
gravity. In this specific case, the \( SO(d, 1) \) the gravita-
tional gauge field \( \omega^\mu_{\alpha\beta} = -\omega^\beta_{\alpha\mu} \) is broken into the \( SO(d - 1, 1) \) connection
\( \omega^\mu_{\alpha\beta} \) and the \( \omega^\mu_{\alpha\beta} = e^\mu_{\alpha\beta} \) vierbein field,
with the dimension \( d \) fixed. Then, the de Sitter (anti-
de Sitter) curvature
\[ \mathcal{R}^{\mu\nu}_{\alpha\beta} = \partial^\mu \omega^\nu_{\alpha\beta} - \partial^\nu \omega^\mu_{\alpha\beta} + \omega^\mu_{\alpha\epsilon} \omega^\nu_{\epsilon\beta} - \omega^\mu_{\epsilon\beta} \omega^\nu_{\alpha\epsilon}, \] (11)
splits in the curvature (9). At this point, our goal is to
enlarge the group structure of the spacetime Manifold
of such manner that the curvature (11), obviously after
the breaking of symmetry, permits us to define the geo-
metrical Lagrangian of the theory as
\[ L_g = \sqrt{-\det \mathcal{R}_{\mu\nu}^\alpha \mathcal{R}_{\nu\sigma}^\alpha} = \sqrt{-\det G_{\mu\nu}}, \] (12)
where we have been defined the following geometrical
object
\[ \mathcal{R}^a_{\mu} = \lambda(e^a_{\mu} + f^a_{\mu} + R^a_{\mu}), \]
\[ M^a_{\mu} = e^a_{\mu} M_{\nu\mu}, \] (12)
where \( f^a_{\mu} \) (in sharp contrast to \( e^a_{\mu} \)) carry the follow-
ing symmetry:
\[ e_{\alpha\mu} f^a_{\mu} = f^a_{\mu} = -f^a_{\nu}. \]
The action will contains, as usual, \( \mathcal{R} = \det(\mathcal{R}^a_{\mu}) \) as
the geometrical object that defines the dynamics of the
theory. The particularly convenient definition of $\mathcal{R}^\alpha_{\mu}$ makes easy to establish the equivalent expression in the spirit of the Unified theories developed time ago by Eddington, Einstein and Born and Infeld for example:

$$\sqrt{\det \mathcal{R}^\alpha_{\mu}} \mathcal{R}^\alpha_{av} = \sqrt{\det \left[ \lambda^2 (g_{\mu\nu} + f_{\mu a} f_{\nu}^a) + 2\lambda R_{\mu \nu a} + 2\lambda f_{\mu a} R_{\nu a} + R_{\nu \mu} R_{\nu a} \right]},$$

where $R_{\mu \nu} = R_{\mu \nu}^a + R_{\mu \nu}$.

The important point to consider in this simple Cartan inspired model is that, although a cosmological constant $\lambda$ is inevitably required due the symmetry breaking ($\lambda = 1 - 4$), the expansion of the action in four dimensions lead automatically the Hilbert-Einstein part when $f_{\mu a} = 0$. Explicitly ($R = g^{\alpha \beta} R_{\alpha \beta}$)

$$S = \int d^4 x (e + f)^2 \left[ \lambda^2 - \lambda^2 \left( R - f_{\mu a} R^a_{\mu} \right) + \frac{\lambda^2}{2} \left( R^2 - R_{\mu \nu a} R_{\mu \nu a} \right) + \left( f_{\mu a} R^a_{\mu} \right)^2 - 3\lambda R_{\mu \nu a} R_{\mu \nu a} \right].$$

Notice that the tetrad property was used here. In the remaining part of the work, this property will be used or not, wherever the case.

3. THE DYNAMICAL EQUATIONS

In this case, the variation with respect to the metric remains the same as in previous works (see [1]) eq. (9)): e.g.:

$$\delta_g \sqrt{G} = \sqrt{G}^{-1} \nabla G \delta G = 0.$$  

The variation respect to the connection gives immediately

$$\frac{\delta \sqrt{G}}{\delta \gamma^a_{\mu \nu}} = \{ - \nabla_a \left[ \sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu} \right] \delta^a_{\mu \nu} + \nabla_a \left[ \sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu} \right] \delta^a_{\mu \nu} \} \equiv \nabla_a \left[ \sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu} \right] \delta^a_{\mu \nu} \}$$

where the general form of the Palatini’s identity have been used and

$$G_{\mu \nu} \equiv \mathcal{R}^a_{\mu \nu} \mathcal{R}_{av}$$

with the $\mathcal{R}^a_{\mu \nu}$ from eq. (12). Defining $\Sigma^a_{\mu \nu} \equiv \sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu}$ the above equation can be written in a more suggestive form but due the variation with respect to the metric it is identically zero (due the lack of energy momentum tensor) and the only information, till now, to our disposal is through the antisymmetric part of the variation with respect to the metric (see (12) of ref. [1])

$$R_{\mu \nu} = -\lambda (g_{\mu \nu} + f_{\mu \nu}) \Rightarrow \delta G_{\mu \nu} = \nabla \left[ \frac{\delta \sqrt{G}}{\delta \gamma^a_{\mu \nu}} \right] \equiv \nabla \left[ \delta \gamma^a_{\mu \nu} \right] = \nabla a \left[ G \mathcal{R}^a_{\mu \nu} \right] = 0.$$  

Explicitly

$$\nabla \left[ \frac{\delta \sqrt{G}}{\delta \gamma^a_{\mu \nu}} \right] = \nabla a \left[ \frac{\sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu}}{2 \mathcal{R}} \right] = 0,$$  

where $N_{\mu \nu}$ is given by expression (32) of ref. [1]. The set of equations to solve for this particular case is

$$R_{\mu \nu} = R^a_{\mu \nu} \Rightarrow \delta G_{\mu \nu} = \nabla a \left[ \frac{\delta \sqrt{G}}{\delta \gamma^a_{\mu \nu}} \right] \equiv \nabla a \left[ \frac{\sqrt{G}^{-1} \nabla a \mathcal{R}^a_{\mu \nu}}{2 \mathcal{R}} \right] = 0,$$

where the quantities with a little circle “o” are defined from the Christoffel connection (as in General Relativity). From this set eqs. (19), the link between $T$ and $f$ will be determined.

ii) $f_{\mu \nu}$ has only the role to be the antisymmetric part of a fundamental (non-symmetric) tensor $K$: i.e. $f_{\mu \nu}$ closed but not necessarily exact. Then, the variation of the geometrical Lagrangian $\delta \sqrt{G}$ gives the same information that $\delta \sqrt{G}$ that means that the remaining equations are
\[
R_{(\mu\nu)} = \hat{\nabla}_{\nu} - T_{\mu\rho}^{\kappa} T_{\alpha\nu}^{\rho} = -\lambda g_{\mu\nu}, \quad (20a)
\]
\[
R_{[\mu\nu]} = (\nabla_{\alpha} + 2T_{\alpha})
\]
\[
\times (T_{\alpha\nu}^{\mu} + T_{\nu\alpha}^{\mu} - T_{\nu\mu}^{\alpha} - T_{\mu\nu}^{\alpha}) = -\lambda f_{\mu\nu}, \quad (20b)
\]

3.1. Analysis and Reduction of the Dynamical Equations

One important equation, that appears into the two sets recently described (independently on the specific role of the antisymmetric tensor \(f_{\mu\nu}\), bring us a lot of information about the link between \(T\) and \(f\) are (19b) and (20b). Precisely, this equation \(R_{[\mu\nu]} = -\lambda f_{\mu\nu}\) plus the condition \(\nabla_{\alpha} T_{\mu\nu}^{\alpha} = 0\) lead immediately
\[
\nabla_{\mu} T_{\nu} - \nabla_{\nu} T_{\mu} = - (\lambda f_{\mu\nu} + 2T_{\alpha} T_{\mu\nu}^{\alpha}),
\]
then, the quantity that naturally appears in the RHS is the “definition” in the current literature of the minimal coupling electromagnetic tensor \(\vec{F}_{\mu\nu}\), in a spacetime with torsion. Notice the important fact that \(\nabla_{\alpha} T_{\mu\nu}^{\alpha} = 0\) is equivalent to
\[
d^{*} T = 0,
\]
the torsion is current free. Two cases naturally arise:

i) if we assume the existence of the potential vector we have
\[
\nabla_{\mu} T_{\nu} - \nabla_{\nu} T_{\mu} = \frac{f_{\mu\nu}}{2T_{\alpha} T_{\mu\nu}^{\alpha}},
\]
a link between \(a_{\nu}\) and \(T_{\nu}\) clearly appears: \(T_{\nu} = -\lambda a_{\nu}\). The important fact to remark here is that, although in references [11] the link between the trace of the torsion and the vector field was proposed, but in the theory presented in this paper this relation is derived automatically from its geometrical basis. Beside this point, is notable the suggestive aspect of \(\vec{F}_{\mu\nu}\) as \(F_{\mu\nu} + B_{\mu\nu}\) with \(B_{\mu\nu}\), such type of “background” field generated by the spacetime torsion.

ii) if \(f_{\mu\nu}\) has only the role to be the antisymmetric part of a fundamental (non-symmetric) tensor \(K\), it acquires a potential automatically, being of this manner an exact form were \(T_{\nu}\) takes the role of potential vector. Clearly, now \(f_{\mu\nu}\) cannot be potential for the torsion from this point of view (in a non-trivial topology, it can be, of course).

From above statements over the “trace” of the torsion, is clearly seen that two ansatz appear as candidates for the torsion tensor structure: the “tratorial” structure \(T_{\mu\nu}^{\alpha} = (\delta_{\mu}^{\alpha} a_{\nu} - \delta_{\nu}^{\alpha} a_{\mu})\); and the “product” structure \(T_{\mu\nu}^{\alpha} = k^{\alpha} f_{\mu\nu}\) where the vector \(k^{\alpha}\) is eigenvector of the antisymmetric tensor \(f_{\mu\nu}\), in general (notice that torsion tensor with this “product structure” also has the possibility to be fully antisymmetric).

The other possibility is to take \(\nabla_{\alpha} T_{\mu\nu}^{\alpha} = -\lambda f_{\mu\nu}\) then \(\nabla_{\mu} T_{\nu} - \nabla_{\nu} T_{\mu} = -2T_{\alpha} T_{\mu\nu}^{\alpha}\), but their interpretation are not so clean as before. Even more, probably carry us to a “product structure” with the torsion tensor not fully antisymmetric, of course.

3.2. A Potential for the Torsion

As was shown in [1], if we impose the restriction \(T_{\alpha\beta\gamma} = T_{[\alpha\beta\gamma]}\) (e.g. totally antisymmetric torsion tensor), from eq. (2) for example, we note that only the antisymmetric part of the fundamental tensor \(K_{\alpha\beta}\) determines fully the torsion tensor. Then, due the assumption of a torsion tensor completely antisymmetric, the potential torsion \(f_{\mu\nu}\) exists and arises in a natural form (the \(V\) for the covariant derivative with respect the full connection \(\Gamma\)). This potential torsion has the following properties
\[
f_{\mu\nu} = f_{\mu\nu} = -f_{\nu\mu} \in \mathbb{H} \mathbb{C},
\]
\[
\nabla_{[\kappa} f_{\mu\nu]} = T_{\mu\nu\rho},
\]
\[
= e_{\mu\nu\rho\sigma} h^{\sigma},
\]
where the last equality coming from the full antisymmetry of the Torsion field. Immediately we can see, as a consequence of the above statements, the following

i) the torsion is the dual of an axial vector \(h^{\sigma}\)

ii) from i), the existence in the spacetime of a completely antisymmetric tensor covariantly constant \(e_{\mu\nu\rho\sigma} (\nabla e = 0)\).

Notice that, the choice for the real nature of the metric and the pure hypercomplex potential tensor coming from the Hermitian nature of the theory: as was clearly explained in [1].

The variational equations (in the Palatini’s sense[10,12], see eqs. (12) and (13) of ref. [1]), despite their simplest and compact form, it is necessary to show what is the deep physical and geometrical meaning inside these eqs.

For expression (13) of ref. [1] we have a highly nonlinear dynamical (propagating) equation for the torsion field, where the variation was performed with respect to their potential \(f_{\mu\nu}\) and having a nonlinear term proportional to \(f_{\mu\nu}\) playing the role of current for the \(T^{\alpha\beta\gamma}\). Then, the potential two form is associated nonlinearly to the torsion field as his source regarding similar association between the electromagnetic field and the spin in particle physics.
For the expression (12) of ref. [1], firstly is useful to split the equation into the symmetric and the antisymmetric parts using $R_{\mu\nu}$ explicitly as before

$$R_{(\mu\nu)} = R_{\mu\nu} - T_{\mu\rho}^\alpha T_{\alpha\nu}^\rho = -2\lambda g_{\mu\nu}, \quad (24)$$

$$R_{[\mu\nu]} = \overset{\circ}{\nabla}_\alpha T_{\mu\nu}^\alpha = -2\lambda f_{\mu\nu} = \nabla_\alpha T_{\mu\nu}^\alpha, \quad (25)$$

(the last equality coming from the totally antisymmetricity of the torsion).

Notice the important fact that $-2\lambda f_{\mu\nu}$ is the “current” for the torsion field as the terms proportional to the 1-form potential vector $a_\mu$ acts as current of the electromagnetic field $f_{\mu\nu}$ in the equation of motion for the electromagnetic field into the standard theory: $\nabla_\alpha a^\alpha_{\mu} = J_\mu$ (constants absorbed into the $J_\mu$).

The symmetric part (24) can be written in a “GR” suggestive fashion

$$R_{\alpha\nu} = -2\lambda g_{\mu\nu} + T_{\mu\rho}^\alpha T_{\alpha\nu}^\rho, \quad (26)$$

we can advertise that the equation has the aspect of the Einstein equations with the cosmological term modified by the torsion symmetric term $T_{\mu\rho}^\alpha T_{\alpha\nu}^\rho$. This can be interpreted, as was shown in [1], by the energy of the gravitational field itself.

The second antisymmetric part (25) is more involved. In order to understand it, will be necessary to use the language of differential forms to rewrite the symbols and conceptual simplicity, permit us to check consistency and covariance step by step.

$$\nabla_\alpha T_{\mu\nu}^\alpha = -2\lambda f_{\mu\nu}, \quad \partial^*T = -2\lambda \ast f, \quad (27)$$

now, using $T = \ast h$

$$dh = -2\lambda \ast f \Rightarrow \ast f = -\frac{1}{2\lambda} dh, \quad (28)$$

in more familiar form

$$\nabla_\mu h_\nu - \nabla_\nu h_\mu = -2\lambda \ast f_{\mu\nu}, \quad (29)$$

then follows using again: $T = df = \ast h$ and eq. (27)

$$d\ast f = 0 \quad (30)$$

and fundamentally

$$df = -\frac{1}{2\lambda} d\ast dh = T = \ast h, \quad (31)$$

$$d\ast dh = -2\lambda \ast h, \quad (32)$$

that we can recognize the Laplace-de Rham operator that help us to write the wave covariant equation

$$[(d\delta + \delta d) + 2\lambda] \ast h = 0 \quad (33)$$

If we start with the potential is not difficult to see that equivalent equation can be find

$$(\Delta + 2\lambda) \ast f = 0. \quad (34)$$

Notice that equation (33) coming from (28) and is consequence of the $Tfh$-relation ($T = df = \ast h$) but (34) comes directly from (27). The geometric interplay between

$$\Gamma \xrightarrow{(-1)^{d/2}} \tilde{\Gamma} \xrightarrow{(-1)^{d/2}} \tilde{\tilde{\Gamma}} \xrightarrow{(-1)^{d/2}} \tilde{\Gamma} \xrightarrow{(-1)^{d/2}} \Gamma \xrightarrow{(-1)^{d/2}} \tilde{\Gamma} \xrightarrow{(-1)^{d/2}} \tilde{\Gamma} \xrightarrow{(-1)^{d/2}} \Gamma \quad (35)$$

4. EXACT SOLUTIONS IN THE NEW UFT THEORY

The main motivation in this Section is clear: we must equip our “theoretical arena” by studying wormhole solutions beyond to Einstein equations coupled to possible matter fields. We know that the many problems appear in the conventional “dualistic” approach even at the classical level, that make that the “dream” of a quantum formulation of the gravity that permit its interaction with other fields becomes practically impossible. Then, let us construct wormhole solutions in the viewpoint of the UFT model introduced here. The action in four dimensions is given by

$$S = -\frac{1}{16\pi G} \int d^4 x \sqrt{|G_{\mu\nu}|}, \quad (36)$$

$$\mathcal{R} = \sqrt{G} \left(-\frac{\Lambda}{2} G - \frac{\Lambda}{3} G^{\Lambda} + \frac{1}{8} (G^{\ast})^2 - \frac{1}{4} G^4 \right). \quad (37)$$

4.1. Totally Antisymmetric Torsion

Scalar curvature $R$ and the torsion 2-form field $T_{\mu\nu}$ with a $SU(2)$—Yang-Mills structure are defined in terms of the affine connection $\Gamma^\lambda_\mu_\nu$ and the SU(2) potential torsion $f^a_\mu$ by

$$R = g^{\mu\nu} R_{\mu\nu} = R^\lambda_\mu_\nu \equiv \partial_\nu \Gamma^\lambda_\mu_\nu - \partial_\mu \Gamma^\lambda_\nu_\nu + \cdots \quad (38)$$

$$T^{a}_{\mu\nu} = \partial_\nu f^a_\mu - \partial_\mu f^a_\nu + \epsilon_{bcd} f^b_\mu f^c_\nu, \quad (39)$$

$\mathcal{G}$ and $\Lambda$ are the Newton gravitational constant and the cosmological constant respectively. Notice the important fact that from the last equation for the Torsion 2-form, the potential $f^a_\mu$ must be proportional with the
antisymmetric part of the affine connection $\Gamma_{\mu\nu}^\lambda$ as in the Strauss-Einstein UFT. As in the case of Einstein-Yang-Mills systems, for our new UFT model it can be interpreted as a prototype of gauge theories interacting with gravity (e.g. QCD, GUTs, etc.). Upon varying the action, we obtain the gravitational “Einstein-Eddington-like” equation

$$R_{\mu
u} = -2\lambda (g_{\mu\nu} + f_{\mu\nu})$$  \hspace{1cm} (39)

and the field equation for the torsion two form in differential form

$$d^*T^a + \frac{1}{2} \varepsilon^{abc} (f_b \wedge *T_c - *T_b \wedge f_c) = F^a,$$  \hspace{1cm} (40)

where we define as usual

$$T^a_{bc} = \frac{\partial L_G}{\partial T_{bc}}, \quad F^a_{bc} = \frac{\partial L_G}{\partial F_{bc}},$$

we are going to seek for a classical solution of eqs. (39) and (40) with the following spherically symmetric ansatz for the metric and gauge connection

$$ds^2 = d\tau^2 + a^2 (\tau) \sigma^i \otimes \sigma^i = d\tau^2 + e^i \otimes e^i, \quad (41)$$

here $\tau$ is the euclidean time and the dreibein is defined by $e^i = a(\tau) \sigma^i$. The gauge connection is

$$f^0 = f^0_a dx^a = h \sigma^0,$$  \hspace{1cm} (42)

for $a = 1, 2, 3$ and for $a = 0$

$$f_0 = f_0^a dx^a = s \sigma^0,$$  \hspace{1cm} (43)

this choice for the potential torsion is the most general and consistent from the physical and mathematical point of view due the symmetries involved in the problem, as we will show soon.

The $\sigma^i$ one-form satisfies the $SU(2)$ Maurer-Cartan structure equation

$$d\sigma^a + e_a^{\ bc} \sigma^b \wedge \sigma^c = 0.$$  \hspace{1cm} (44)

Notice that in the ansatz the frame and isospin indexes are identified as for the case with the NBI Lagrangian of ref. [3]. The torsion two-form

$$T^i = \frac{1}{2} T^i_{\mu\nu} dx^\mu \wedge dx^\nu,$$  \hspace{1cm} (45)

becomes

$$T^a = df^a + \frac{1}{2} \varepsilon^{abc} f^b \wedge f^c = \left( -h + \frac{1}{2} h^2 \right) \varepsilon^{abc} g^b \wedge \sigma^c.$$  \hspace{1cm} (46)

Notice that $f^0$ plays no role here because we take simply $ds = 0$ (the $U(1)$ component of $SU(2)$, in principle), does not form part of the space spherical symmetry), and the expression for the torsion is analogous to the non abelian two form strength field of [3]. Is important to note that, when we goes from the Lorentzian to Euclidean gravitational regime, $\tau \rightarrow \tau$ and the torsion pass from the field of the Hypercomplex to the complex numbers, for invariance reasons (geometrically, multiplication of hypercomplex numbers preserves the (square) Minkowski norm $(x^2 - y^2)$ in the same way that multiplication of complex numbers preserves the (square) Euclidean norm $(x^2 + y^2)$). Inserting $T^a$ from eq. (46) into the dynamical equation (40) we obtain

$$d^*T^a + \frac{1}{2} \varepsilon^{abc} (f_b \wedge *T_c - *T_b \wedge f_c) = *F^a,$$  \hspace{1cm} (47)

$$(-2h + h^2)(1 - h) d\tau \wedge e^b \wedge e^c = -2\lambda d\tau \wedge e^b \wedge e^c,$$

where

$$*T^a = \frac{1}{\sqrt{3}} \frac{\lambda}{\sqrt{1 - 2h}} h A(-2h + h^2) d\tau \wedge e^b \wedge e^c,$$

$$*F^a = -\frac{2\lambda^2}{\sqrt{3}} \frac{\sqrt{1 - 2h}}{h} A d\tau \wedge e^b \wedge e^c,$$

$$A = \lambda^4 [(1 + \alpha)^2 + \alpha/2]$$  \hspace{1cm} (49)

and

$$\alpha = \frac{1}{2} (s^2 + 3h^2),$$  \hspace{1cm} (50)

from expression (47) we have an algebraic cubic equation for $h$

$$(-2h + h^2)(1 - h) + 2\lambda = 0.$$  \hspace{1cm} (52)

We can see that, in contrast with our previous work with a dualistic theory [3] where the energy-momentum tensor of Born-Infeld was considered, for $h$ there exist three non trivial solutions depending on the cosmological constant $\lambda$. But, at this preliminary analysis of the problem, only the values of $h$ that make the quantity $[-h + 1/2h^2] \in \mathbb{R}$ are relevant for our proposes: due the pure imaginary character of $T$ in the euclidean framework and mainly to compare with the NABI wormhole solution of our previous work (the question of the $h \in \mathbb{C}$ will be the focus of a further paper [5]). As the value of $h \in \mathbb{R}$ is $-1$ and in 4 space-time dimensions $\lambda = 1 - d = 3$, then

$$T^a_{bc} \bigg|_{h=1} = \frac{3 \epsilon_{abc}}{2 a^2}; \quad T^a_{0c} = 0.$$  \hspace{1cm} (53)

Namely, only the magnetic field is non vanishing while the electric field vanishes. An analogous feature can be seen in the solution of Giddings and Strominger and in our previous paper [3]. Substituting the expression for the Torsion two form (53) into the symmetric part of the variational equation, namely
we reduce the equation (24) to an ordinary differential equation for the scale factor $a$,

\[ R_{\mu\nu} = \frac{\ddot{a}}{a} - T_{\mu\rho} T_{\nu}^\rho = -2\lambda g_{\mu\nu}, \]  

we reduce the equation (24) to an ordinary differential equation for the scale factor $a$.

\[ \left[ \frac{\dot{a}}{a} \right]^2 - \frac{1}{a^2} = \frac{2\lambda}{3} - \frac{9}{2a^4}, \]  

\[ \ln \left[ 1 + 4a^2 + 2\sqrt{-9 + 2a^2 + 4a^4} \right] = \tau - \tau_0, \]  

\[ T_{\mu\rho} T_{\nu}^\rho = \left( -\frac{h + \frac{1}{2}h^2} {a^4} \right)^2 2\delta_{\mu\nu} \]  

\[ = \frac{9}{2a^4} \delta_{\mu\nu}. \]  

There are 2 values for the scale factor $a$: max. and min. respectively, namely

\[ a = \pm e^{-2\sqrt{2\lambda(\tau - \tau_0)} \sqrt{37 - 2e^{2\sqrt{2\lambda(\tau - \tau_0)}} + e^{4\sqrt{2\lambda(\tau - \tau_0)}}}}. \]  

Expression (58) for the scale factor $a$ is described in the Fig. 1 for the real value of $h$.

As is easily seen from (58), the scale factor has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the “dualistic” non-abelian BI theory Fig. 4. Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of the Coleman’s mechanism) is fulfilled. Now will need to see what happens with the equation (27) in this particular case under consideration: equation (27) takes the following form

\[ d^\ast T^\alpha + \frac{1}{2} e^{ab\gamma} (f_{\alpha} \wedge T_c - \ast T_b \wedge f_{\gamma}) = -2\lambda \ast f^a, \]  

\[ (-2h + h^2)(1 - h)d\tau \wedge e^b \wedge e^c = -2\lambda d\tau \wedge e^b \wedge e^c, \]  

\[ \ast T^\alpha \equiv h(-2h + h^2)d\tau \wedge \frac{e^a}{a^2}, \]  

\[ \ast f^a = -h d\tau \wedge e^b \wedge e^c. \]  

Then we arrived to the same equation for $\lambda$ as (52) corroborating the self-consistency of the procedure.

4.2. “Tratorial” Torsion

For begin with, let us consider the problem involving the set of eq. (19) with the usual definition for the SU(2) electromagnetic field strength

\[ f' = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu \]  

and as before, we are going to seek for a classical solution of eqs. (19) with the following spherically symmetric ansatz for the metric and gauge connection

\[ ds^2 = d\tau^2 + a^2(\tau) \sigma^i \otimes \sigma^i \equiv d\tau^2 + e' \otimes e', \]  

here $\tau$ is the euclidean time and the dreibein is defined by $e' \equiv a(\tau) \sigma^i$. However, in the case of the set (19) we have been assume that the two form $f'$ comes from a 1-
form potential $A$ where, as in the non-abelian Born-Infeld model of ref. [3], is defined as $A \equiv A^\alpha_\mu dx^\alpha = h \sigma^\alpha$.

The extremely important fact in this case is that we know that one-form satisfies the $SU(2)$ Maurer-Cartan structure equation, as fundamental geometrical structure of the non-abelian electromagnetic field

$$d_{SU(2)} \sigma^a + e^a_{bc} \sigma^b \wedge \sigma^c = 0,$$

but now due the identification assumed in (63):

$$e^a_i = \alpha(a(\tau) \sigma^i).$$

$\Rightarrow de^a = T^a - e^a_b \wedge \sigma^b,$

here we make the difference between the exterior derivatives in the spacetime with torsion and in the $SU(2)$ group manifold. Is clearly seen that a question of compatibility involving the identification of the gauge group with the geometrical structure of the space-time with torsion certainly exists. From (64–66) we see that

$$\partial_a ad \tau \vee \sigma^a - \alpha e^a_{bc} \sigma^b \wedge \sigma^c = T^a - e^a_b \wedge \sigma^b.$$  (67)

If

$$e^a_b = -e^a_{bc} \sigma^c$$  (68)

and

$$T^a = \delta^a_b (\partial_a \alpha) d \tau \wedge \sigma^b,$$  (69)

the space-time and gauge group are fully compatible then

$$d \sigma^a + e^a_{bc} \sigma^b \wedge \sigma^c = 0$$

is restored. Hence, the general form assumed for the torsion field, due the symmetry conditions prescribed above, is

$$T^\mu_\nu = \xi (\partial^\mu u^\nu - \partial^\nu u^\mu) + \xi h^\nu \delta^\mu_\nu \quad (\xi, \xi : \text{const}.)$$

Notice that the condition of compatibility that impose such type of “trator” form for the torsion tensor in order to restore the behaviour of the volume form of the space-time with respect to the covariant derivative, here appear in a natural manner without introduce any extra scalar field (dilaton) or to pass to other frame (i.e.: Jordan, Einstein, etc.). Moreover, if we have been continue without make the correspondences (68–69), the equations of motion for the electromagnetic field itself bring automatically these conditions (see in the next paragraph).

Notice that in the HO ansatz the frame and isospin indexes are identified as for the case with the NBI Lagrangian of ref. [3]. The electromagnetic field two-form

$$f^a = dA^a + \frac{1}{2} e^a_{bc} A^b \wedge A^c = h^a_b (\partial_\tau a) d \tau \wedge \sigma^b + h T^a_\tau$$

$$- \left( h + \frac{1}{2} h^2 \right) e^a_{bc} \sigma^b \wedge \sigma^c$$

(72)

where in the last equality conditions (68–69) have been assumed. The dynamical eqs.

$$d^a_{\nu} \equiv 2 \frac{\partial L}{\partial F^a_\nu} \Rightarrow a^a = \frac{\sqrt{\lambda}}{\sqrt{3}} h \delta_a ( - 2 h + h^2 )$$

$$\times d \tau \wedge e^a_\nu \equiv M h ( - 2 h + h^2 ) d \tau \wedge e^a_\nu,$$  (73)

$\frac{1}{a^2} = \frac{\sqrt{\lambda}}{\sqrt{3}} h \delta_a ( - 2 h + h^2 )$
Inserting it in the Yang-Mills type field equation (19c) we obtain
\[ d^a \mathcal{F}^a = \frac{1}{2} e^{a b c} (A_b \wedge \mathcal{F}_c - \mathcal{F}_b \wedge A_c) = 0 \]
\[ = Mh d \tau \wedge a^b \wedge \sigma^c (-2h + h^2)(h - 1) \]
\[ \equiv \lambda^4 [(1 + \alpha)^2 + \alpha/2]. \]

Then, there exists a non trivial solution: \( h = 1 \), with \( s = 0 \) in \( \mathcal{A}_0 \) as before in \([1]\). The electromagnetic field is immediately determined, and is as in the non abelian Born-Infeld model of our previous reference and in the result of Giddings and Strominger, namely
\[ f^a_{b c} = \frac{e^{a b c}}{\lambda^2} f^a_{b c} = 0, \]
only we have magnetic field.

Now considering only a “trator” form for the torsion, eq. (16b) is identically null due the magnetic character of \( f^a \) and the particular form of the symmetric coefficients of the connection. Inserting the torsion eq. (69) into the eq. (19a), as in previous section, we obtain
\[ \frac{d}{d \tau} \left( \left( \frac{\lambda}{3} \right)^2 - \frac{1}{a^2} \right) = \frac{\lambda}{3}. \]  
(76)

Integration of this last expression immediately leads
\[ a(\tau) = \left( \frac{\lambda}{3} \right)^{1/2} \sinh \left( \left( \frac{\lambda}{3} \right)^{1/2} (\tau - \tau_0) \right). \]  
(77)

Then is quite evident that this particular case doesn’t lead wormhole configurations: only eternal expansion with \( a(\tau_0) = 0 \) (the origin of the euclidean time Fig. 2).

Now considering only the product form for the torsion, eq. (19c) doesn’t change but eq. (19b) takes the form of a wave equation for the scalar factor
\[ [\square a + (\partial_0 a) (\partial^0 a)] = \lambda, \]
due \( T^a_{\beta \gamma} = \zeta k^a \mathcal{E}_{\beta \gamma} \rightarrow c_{ab} (\partial^0 a). \) Is not difficult to see that the \( su(2) \) structure of the electro-magnetic tensor is of some manner transferred to the structure of the torsion. But here we enter in conflict because the system of eqs. (19) turns to be over determined: probably we need more freedom in the ansatz for \( f^a_{b c} (s \neq 0, \text{ or } h = h(\tau)) \). This fact will be studied in near future \([5]\).

4.3. General Case

Let to assume the full form (71) for \( T^a \)
\[ \mathcal{F}^a = Mh [\partial_0 a (\partial_1 a)] e^{a \alpha} \sigma^\alpha \wedge \sigma^d \]
\[ + \left[ \frac{h}{a} [\xi (\delta^a_{b c} u_0 - \delta^a_{d c} u_b) + \zeta h \mathcal{E}_{c b}^d] e^{ij} \mathcal{E}_{k l} \omega^k \wedge \omega^l \right] \]
\[ + (2h + h^2) e^{a b c} e^{b c} (d \tau \wedge \sigma^d), \]
(78)

here, in order to avoid the cumbersome expression in the second term due the standard orthonormal splitting, \( ij = 0, a, b, c \) and the \( \omega^k \) are the corresponding 1-forms (\( d \tau, \sigma^a .. \)) wherever the case. The YM type equation can be written as
\[ d^a \mathcal{F}^a = \frac{1}{2} e^{a b c} (A_b \wedge \mathcal{F}_c - \mathcal{F}_b \wedge A_c) = Mh \]
\[ \times \left[ h [\xi (\delta^a_{b c} u_0 - \delta^a_{d c} u_b) + \zeta h \mathcal{E}_{c b}^d] e^{a \alpha} \right] \]
\[ \times e^{b c} e^d (d \tau \wedge \sigma^e \wedge \sigma^f) \]
\[ + \left[ \frac{h}{a} (\delta^a_{b c} u_0 - \delta^a_{d c} u_b) + \zeta h \mathcal{E}_{c b}^d \right] 2 d (\sigma^e \wedge \sigma^f) \]
\[ + M \left[ \frac{h}{a} (\delta^a_{b c} u_0 - \delta^a_{d c} u_b) + \zeta h \mathcal{E}_{c b}^d \right] ( - 2h + h^2 ) \]
\[ \times (h - 1) d \tau \wedge \sigma^d \wedge \sigma^e = 0, \]
(79)

from the above equation we obtain information about the determination of the \( f^a \) field and of the torsion field as in the previous cases: the first term
\[ [h \delta^a_{b c} (\partial_0 a) (\partial_1 a)], \]
\[ \times \left[ \frac{h}{a} (\delta^a_{b c} u_0 - \delta^a_{d c} u_b) + \zeta h \mathcal{E}_{c b}^d \right] = 0, \]
(80)

leads immediately
\[ \left[ \eta_{ab} \partial_0 a + (\xi (\eta_{ab} u_0 - \eta_{ab} u_b) + \zeta h \mathcal{E}_{c b}^d) \right] \]
\[ = \Xi_{ab} + \Xi_{ab}^0 \Rightarrow \zeta h \mathcal{E}_{c b}^d = \Xi_{ab} \Rightarrow \eta_{ab} \partial_0 a \]
\[ + \xi (\eta_{ab} u_0 - \eta_{ab} u_b) = \Xi_{ab}, \]
(81)

where the tensor
\[ \Xi_{ab} = \Xi_{ab} + \Xi_{ab}^0 \]
is independent of the time, and the superscripts \( A \) and \( S \) indicate the totally antisymmetric part of the another non-totally antisymmetric. Then, the second and third equalities above follows. Is not difficult to see, that contracting indices, tracing and considering the symmetries involved, we obtain explicitly
\[ T^a_{b 0} = \delta^a_{b 0} a - a \Xi_{b 0} \]
\[ T^a_{b c} = - a \Xi_{b c} + \zeta h \mathcal{E}_{b c}^0, \]
(82)
\[ T^0_{bc} = -a \Xi^0_{bc} + \zeta h_\nu \epsilon^{\nu}_{bc}, \quad (84) \]

where the integration tensor (independent on time) are related with \( u_c \) and \( \Xi^{S}_{ij} (\tilde{\eta}_{ij} = 0, a, b, c) \) as follows:

\[ u_c = -a \Xi^S_{2c}, \quad u_0 = -\frac{1}{2 \zeta} (3 \tilde{\sigma}_0 a + a \Xi^S_0), \]
\[ \Xi^S_0 = \Xi_{ij}, \quad \Xi^S_0 = \Xi_{ij} \text{ and } \Xi^S_{ij} = -\frac{1}{2} (\tilde{\sigma}_i \Xi^S_j - \tilde{\sigma}_j \Xi^S_i). \]

The last term, however, indicate us that there exist a simplest solution with \( h = 1 \), as the previous case for the non abelian \( f \).

Then

\[ f^a_{bc} = \frac{-K_{bc}}{a^2}, \quad f^a_{bb} = 0, \]
again, and the second is identically cero due the symmetry of the torsion 2-form with respect to the tetrad defined by (63). Now the question is if the system of equations is overdetermined or not: \( k^a \) and \( a \) are without determine. To this end, we carry the information into the expressions (82–84) to the second equation of the set, namely eq. (19b). Again, the symmetry involved both : from the equations

\[ \nabla_i T^i_{ab} + 2T_i T^i_{ab} = -\lambda f^a_{ab} e_c, \quad (85) \]
\[ \nabla_i T^a_{ao} + 2T_i T^a_{i0} = 0, \quad (86) \]
fix the torsion tensor components as

\[ T^a_{b0} = \delta^a_{[a} \tilde{\sigma}_{b]} a, \quad (87) \]
\[ T^0_{bc} = -a \Xi^S_{bc} + \zeta h_0 \epsilon^{0a}_{bc}, \quad (88) \]
\[ T^0_{bc} = 0. \quad (89) \]

Expression (86) turns a null identity, and from (85) only

\[ 4T_i T^i_{ab} = 4a \Xi^S_{ab} (-a \Xi^S_{bc} + \zeta h_0 \epsilon^{0a}_{bc}) = -\lambda f^a_{ab} e_c \]

\[ \Rightarrow 4(a \Xi^S_{ab} \Xi^S_{bc} - a \Xi^S_{bc} \Xi^S_{ab}) = -\lambda f^a_{ab} e_c, \]
\[ a \Xi^S_{ab} \zeta h_0 \epsilon^{0c}_{ab} = -\lambda f^a_{ab} e_c = \frac{\lambda \epsilon^{0c}_{ab}}{a^2} e_c, \quad (90) \]
\[ \frac{\lambda \epsilon^{0c}_{ab}}{a^2} e_c, \]

where in the last line with use the property \( \Xi^S_{ab} = \Xi^S_{ab} = \Xi^S_{bc} (\tilde{\sigma}_0 \Xi^S_j - \tilde{\sigma}_j \Xi^S_0) = 0 \) (see definitions above).

Is easily seen, that squaring both sides of (90) and from (89) we obtain

\[ h_0 = \frac{\lambda \sigma_0}{a^2 |\Xi^S| 2 \zeta}, \quad h_c = \frac{\lambda |\Xi^S| 2 \zeta}{2 \zeta}, \]

and analogically to the previous cases, from the eqs. (19a) the equation to integrate takes the form

\[ \frac{da}{d\tau} = \pm \frac{1}{2} a - \frac{1}{3} a^2 + \frac{2}{3} a^3 d |\Xi^S|^2 + \frac{3}{8} \left( \frac{\lambda}{|\Xi^S| a} \right)^{3/2}. \]

One interesting case when the above equation can be integrated exactly is precisely when \( d = 4 \). This condition, besides improving the integrability condition of the equation, fix \( |\Xi^S|^2 > 3/2 \). The scale factor \( a(\tau) \) takes the following form

\[ a(\tau) = \sqrt{B + (A - B) \tan^2 \left( \frac{\tau - \tau_0}{\sqrt{(A - B)}} \right)}, \]

where \( A \) and \( B \) are nonlinear functions of the norm square \( |\Xi^S|^2 \). The explicit form of these functions are not crucial: only the bound for \( |\Xi^S|^2 > 3/2 \) need to be preserved (also through the normalization of \( A \) and \( B \) into the graphic representation i.e. Fig. 3) Notice that the spacetime is asymptotically Minkowskian with a throat \( a(\tau_0) = \sqrt{B} \) (however the values of the constants have been selected according the previous remarks). Other possibilities not enumerated here, lead spacetimes with cyclic singularities due transcendental functions into the denominator of the expression for the scale factor \( a(\tau) \). This issue is a focus of a future discussion somewhere [5].

### 4.4. Coexistence of Both Type of Torsion in Cosmological Spacetimes

Is interesting to note that in reference [4] the field equations of vacuum quadratic Poincare gauge field theory (QPGFT) were solved for purely null tratorial torsion. The author there expressing the contortion tensor for such a case as

\[ K_{h,uv} = -2(g_{hv} a_v - g_{hv} a_u). \]

However, the important thing is that the author have been discussed the relationship between this class (tratorial) and a similar class of solution with null axial vector torsion, arriving to the conclusion that cosmological solutions with different type of torsion are forbidden. The main reason of this situation can have 2 origins: the specific theory and action (QPGFT), or the Newman-Penrose method used in the computations that works, as is well know, with null geometric quantities. Here we shown that this problem not arises in our theory.
5. THE UNDERLYING DIRAC STRUCTURE OF THE SPACETIME MANIFOLD

The real structure of the Dirac equation in the form
\begin{align*}
(γ_0 p_0 - iγ \cdot p)u &= mv, \\
(γ_0 p_0 - iγ \cdot p)v &= mu,
\end{align*}
(91, 92)
with
\begin{align*}
γ_0 &= \begin{pmatrix} σ_0 & 0 \\ 0 & σ_0 \end{pmatrix}, \quad γ = \begin{pmatrix} 0 & -σ \\ σ & 0 \end{pmatrix},
\end{align*}
(93)
where \(σ\) are the Pauli matrices and \(p = (p_1, p_2, p_3)\), determines a 4D real vector space with \(G\) as its automorphism, such that \(G ∈ L(4)\). This real vector space can be made coincides with the tangent space to the spacetime manifold \(M\), being this the idea. The principal fiber bundle (PFB) \(P (G, M)\) with the structural group \(G\) determines the (Dirac) geometry of the spacetime. We suppose now \(G\) with the general form
\begin{align*}
G &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad G^* G = I_4,
\end{align*}
(94)
where \(A, B\) are \(2 \times 2\) matrices. Also there exists a fundamental tensor \(J_{\lambda}^\nu = δ_{\lambda}^\nu\) invariant under \(G\) with structure
\begin{align*}
J &= \begin{pmatrix} 0 & σ_0 \\ -σ_0 & 0 \end{pmatrix},
\end{align*}
(95)
where however, the Lorentz metric \(g_{\mu\nu}\) is also invariant under \(G\), due its general form (94). Finally, a third fundamental tensor \(σ_{\mu\nu}\) is also invariant under \(G\) where the following relations between the fundamental tensors are
\begin{align*}
J_{\lambda}^\nu &= σ_{\mu\nu} J_{\lambda}^\nu, \quad g_{\mu\nu} = σ_{\mu\nu} J_{\lambda}^\nu, \quad σ_{\mu\nu} = J_{\lambda}^\nu g_{\mu\nu},
\end{align*}
(96)
where
\begin{align*}
g_{\lambda\nu} &= \frac{∂g}{∂g_{\lambda\nu}} \quad (g = \text{det}(g_{\mu\nu})).
\end{align*}
(97)
Then, the necessary fundamental structure is given by
\begin{align*}
G &= L(4) ∩ Sp(4) ∩ K(4),
\end{align*}
(98)
which leaves concurrently invariant the three fundamental forms
\begin{align*}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, \\
σ &= σ_{\lambda\nu} dx^\lambda ∧ dx^\nu, \\
φ &= J_{\nu}^\lambda w^\nu v_\lambda,
\end{align*}
(99, 100, 101)
where \(w^\nu\) are components of a vector \(w^\nu \in V^\nu\), the dual vector space. In expression (98) \(L(4)\) is the Lorentz group in 4D. \(Sp(4)\) is the Symplectic group in 4D real vector space and \(K(4)\) denotes the almost complex group that leaves \(φ\) invariant [6].

For instance, \(G\) leaves the geometric product invariant [7]
\begin{align*}
γ_{\mu} γ_{\nu} &= \frac{1}{2} (γ_{\mu} γ_{\nu} - γ_{\nu} γ_{\mu}) + \frac{1}{2} (γ_{\nu} γ_{\mu} + γ_{\mu} γ_{\nu})
\end{align*}
(102)
\begin{align*}
= γ_{\mu} γ_{\nu} - γ_{\nu} γ_{\mu} = g_{\mu\nu} + σ_{\mu\nu},
\end{align*}
where the are now regarded as a set of orthonormal basis vectors, of such a manner that any vector can be represented as \(v = γ^\nu γ_\nu\) and
\begin{align*}
e_{\alpha\beta\gamma\delta} ≡ γ_\alpha γ_\beta γ_\gamma γ_\delta.
\end{align*}
(103)

In resume, the fundamental structure of the spacetime is then represented by \(P(G, M)\), where \(G\) is given by (98), which leaves invariant the fundamental forms (99–101), implying that
\begin{align*}
∇_\lambda g_{\mu\nu} &= 0, \\
∇_\lambda σ_{\mu\nu} &= 0, \\
∇_\lambda J_{\nu}^\lambda &= 0,
\end{align*}
(104, 105, 106)
where \(∇_\lambda\) denotes the covariant derivative of the \(G\) connection. Is interesting to note that it is only necessary to consider two of above three equations: the third follows automatically. Then, we will consider (104) (105) because in some sense they represent the boson and fermion symmetry respectively.

### 5.1. Field Equations and Group Structure

It is necessary to introduce now other antisymmetric tensor \(σ_{\mu\nu}\), which is not heretical, that means that is different of \(σ_{\mu\nu}\) of (102) but also invariant with respect to the generalized connection \(G : ∇_\nu σ_{\lambda\mu} = 0\). For instance, we can construct also the antisymmetric tensor \(σ_{\mu\nu} = σ_{\mu\nu} - σ_{\nu\mu} ≠ 0\), that obeys \(∇_\nu σ_{\mu\nu} = 0\) and obviously \(1/6(∂_\mu σ_{\nu\lambda} + ∂_\nu σ_{\lambda\mu} + ∂_\lambda σ_{\mu\nu}) = T_{\nu\mu\lambda}\) due the completely antisymmetric nature of \(T\).

### 5.2. Antisymmetric Torsion and Fermionic Structure of the Spacetime

We know that [8]
\begin{align*}
Γ_{\rho\lambda\mu}^{\rho} &= \{^\rho_{\mu\lambda}\} + g^{\nu\rho} (T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\nu\mu\lambda}),
\end{align*}
(107)
where \(Γ_{\rho\lambda\mu}^{\rho}\) are the coefficients of the G-connection and \(\{^\rho_{\mu\lambda}\}\) denotes the coefficients of the Levi-Civita connection whose covariant derivative is denoted by \(∇_\lambda\). From (105) we make the link between the fermi-
ionic structure of the fundamental geometry of the manifold and the torsion tensor
\[ \nabla_{[\nu} \sigma_{\mu\rho]} = 0 \Rightarrow \] (108)
\[ \frac{1}{2} \partial_{[\nu} \sigma_{\mu\rho]} = T_{[\nu\mu}^{\rho} \sigma_{\rho\lambda]}. \] (109)

A particular simplest solution for \( T \) arises when the torsion tensor is totally antisymmetric [9]
\[ T_{\mu\lambda\nu} = T_{[\mu\lambda\nu]}, \] (110)
in order that the equivalence principle be obeyed [5, 9, 10]. In this case, as we shown already in [1, 2, 9], we have
\[ T_{\mu\lambda\nu} = \varepsilon_{\mu\lambda\nu p} h^p, \] (111)
where the axial vector \( h^p \) is still to be determined. As will be clear soon, is useful to put for \( d \) dimensions [9]
\[ h^p = \frac{1}{\sqrt{w}} P^p, \] (112)
where \( P^p \) is the generalized momentum vector. If \( d = 4 \), \( w = 6 \).

Expression (109) can be simplified taking account on the symmetries of \( T_{\mu\lambda\nu} \) and the contraction with the fundamental tensor \( J_\tau \)
\[ T_{\mu\nu} = \frac{1}{w} J_\tau^p \partial_{[\nu} \sigma_{\rho\mu]}. \] (113)

5.3. About the Equivalence Principle (EP) and the Antisymmetry of the Torsion Tensor: A Theorem

As is well known, in order that experimental evidence forms the foundation of the theory, the PE has to be imposed as well the foregoing symmetry principles.

Because the \( G \)-connection contains a torsion tensor by specific requirements, is currently suspected that due this fact, the EP can be violated. Then a good question naturally arises: what is the implication of PE that due this fact, the EP can be violated. Then a good sor by specific requirements, is currently suspected

vi) The above relations have tensorial character, for instance they are valid in all coordinate systems (and in all points \( p \)), then
\[ T_{\mu\lambda\alpha} = -T_{\alpha\lambda\mu}, \] (A6)
and
\[ \vec{\partial} \dot{\varepsilon}_{\lambda\mu\alpha} = 0, \] (A7)

These equations show geometrically that the imposition of the PE implies the following equivalence
\[ \nabla g_{\alpha\beta} = 0 \text{ and } PE \Leftrightarrow \text{(eqs. A6 and A7)} \] (A8)

v) The above relations have tensorial character, for instance they are valid in all coordinate systems (and in all points \( p \)), then
\[ T_{\mu\lambda\alpha} = -T_{\alpha\lambda\mu}, \] (A6)

With this Proof we conclude that: the full antisymmetry for the torsion tensor is the result of imposition of the Equivalence Principle (EP) on the spacetime structure. Is not as the result of a priori assumptions concerning the hypotetic or possible physical meaning of the torsion tensor.

5.4. The \( G \)-invariance of the Action

As is well known, the Palatini principle has a twice role that is the determining of the connection required for the spacetime symmetry as the field equations. By means this principle, we were able to construct the action integral \( S \). This action \( S \) necessarily need to yield the \( G \)-invariant conditions (104–106) without prior assumption; and, the Einstein, Dirac and Maxwell equations need to arise from \( S \) as a causally connected closed system. This equations will be generalized inevitably, so that causal connections between them can be established. Our action fulfill the above requirements, having account that the role of \( f_{\mu\nu} \) that enters symmetrically with \( g_{\mu\nu} \) in \( S \), is linked with the fundamental tensor \( g_{\mu\nu} \) of the previous Section denoting the dual of \( g_{\mu\nu} \) by
\[ f_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} g^{\rho\sigma} = \ast g_{\mu\nu}, \]
(where \( g^{\mu\nu} \) is the inverse tensor to \( g_{\mu\nu} \)).

The usual Euler-Lagrange equations from the action with the explicit computation of the determinant in \( (d = 4) \) of expression (8) that will help us in
order to compare the unitarian model introduced here (in the sense of Eddington [see 1, 2]) with the
dualistic non abelian Born-Infeld model of [3], takes
the familiar form [1–3]

\[ S = \frac{b^2}{4\pi} \int \sqrt{-g} dx^4 \]  

\( \times \left\{ \lambda_2 - 2G^2 - \frac{L^2}{3} + \left( \frac{G^2}{2} - \frac{1}{4G} \right)^2 \right\} \]  

(114)

\[ G_{\mu\nu} = \left[ \lambda_2 (G_{\mu\nu} + f^\alpha_{\mu\nu} + 2\lambda R_{\mu\nu}) \right. \]

\[ \left. + 2\lambda f_{\rho\gamma} R_{\mu\nu} + R_{\mu\nu} R_{\rho\gamma} \right] \]  

(115)

\[ G_\gamma = \left[ \lambda_2 (d + f_{\mu\nu} \mu_{\mu\nu}) + 2\lambda (R_\mu + R_\lambda) \right. \]

\[ \left. + (R_\gamma + R_\lambda)^2 \right] \]  

(116)

with (the upper bar on the tensorial quantities indica-
tes traceless condition)

\[ R_\gamma \equiv g_{\mu\nu} R_{(\mu\nu)}; \quad R_\lambda \equiv f_{\mu\nu} R_{(\mu\nu)} \]  

\[ \gamma = \frac{G_{\gamma}}{G} \]  

(117)

\[ \frac{G_{\mu\nu}}{G} = \frac{G_{\mu\nu} - \frac{G_{\alpha\beta} G_{\gamma\delta}}{4}}{G_{\gamma\delta}}; \quad \nabla_{\rho} \frac{G_{\gamma\delta}}{G_{\gamma\delta}} \equiv \frac{G_{\gamma\delta}}{G_{\gamma\delta}} \]  

where the variation was made with respect to the elec-
tromagnetic potential \( a \), as follows

\[ \frac{\delta \sqrt{G}}{\delta a_{\tau}} = V_{\rho} \left( \frac{\partial \sqrt{G}}{\partial a_{\tau}} \right) = V_{\rho} \sqrt{F_{\rho \tau}} = 0. \]  

(118)

Explicitly

\[ \nabla_{\rho} \left[ \frac{\lambda^2 N^{\mu\nu} (\delta f^\rho_{\mu} + \delta f^\rho_{\nu})}{2} \right] = 0, \]  

(119)

where \( N^{\mu\nu} \) is given by

\[ N^{\mu\nu} = \frac{1}{2} \left[ \gamma^2 - \frac{1}{2} \frac{d^{\mu\nu}}{d} - \frac{1}{2} \frac{d^{\mu\nu}}{d} \right] \]  

(120)

The set of equations to solve for the action (13) in this particular case is

\[ R_{(\mu\nu)} = \frac{\hat{\rho} F_{\mu\nu} - T_{\mu\nu}^\rho - \lambda g_{\mu\nu}}{2} \]  

(19a)

\[ R_{[\mu\nu]} = \frac{\nabla_{\rho} T_{\rho [\mu \nu]} - \lambda f_{\mu\nu}}{2} \]  

(19b)

\[ \nabla_{\rho} \left[ \frac{\lambda^2 N^{\mu\nu} (\delta f^\rho_{\mu} + \delta f^\rho_{\nu})}{2} \right] = 0, \]  

(19c)

from this set, the link between \( T \) and \( f \) will be deter-
mined (\( f \) is not a priori potential for the torsion \( T \))

\[ \hat{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T_{\mu\rho}^\rho T_{\nu\sigma}^\sigma \]  

(121)

\[ = -\lambda g_{\mu\nu} + w h_{\mu\nu} = -\lambda g_{\mu\nu} + P_{\mu} P_{\nu}, \]  

(122)

then we can obtain, as in mass shell condition

\[ P^2 = m^2 \Rightarrow m = \pm \sqrt{\frac{\hat{R} + \lambda f}{4}}. \]  

(123)

The key point now is eq. (112)

\[ \hat{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T_{\mu\rho}^\rho T_{\nu\sigma}^\sigma \]  

(121)

\[ = -\lambda g_{\mu\nu} + w h_{\mu\nu} = -\lambda g_{\mu\nu} + P_{\mu} P_{\nu}, \]  

(122)

then we can obtain, as in mass shell condition

\[ P^2 = m^2 \Rightarrow m = \pm \sqrt{\frac{\hat{R} + \lambda f}{4}}. \]  

(123)

Notice that there exists a link between the dimension of the spacetime and the scalar “Einsteinian” curvature

\( \hat{R} \). Moreover, the curvature is constrained to take de-
finite values \( \in \mathbb{N} \) the natural number characteristic of the dimension. By the other hand, knowing that \( |\lambda| = d - 1 \) and accepting that the parameter \( m \in \mathbb{R} \), the limit-
ing condition on the physical values for the mass is

\[ \hat{R} \geq (1 - d) d \]  

Introducing the geometric product in above equation (e.g., \( \gamma^\mu \gamma^\nu (P_{\mu} - e A_{\mu}) (P_{\nu} - e A_{\nu}) - m^2 ) \Psi = 0 \),

(124)

where \( \Psi = u + iv \) given in (91, 92). That is

\[ [\gamma^\mu (P_{\mu} - e A_{\mu}) + m] [\gamma^\nu (P_{\nu} - e A_{\nu}) - m] u^\lambda = 0, \]  

(125)

which lead the Dirac equation

\[ [\gamma^\mu (P_{\mu} - e A_{\mu}) + m] u^\lambda = 0, \]  

(126)

with \( m \) given by (123). Notice that this condition, in the Dirac case, is not only to pass from classical vari-
ables to quantum operators, but in the case that the ac-
tion does not contains explicitly \( A_{\mu} \), \( h_{\mu} \) remains
without specification due the gauge freedom in the momen-
tum. Applying the geometric product to (124) is not difficult to see that

\[ \left[ (P_{\mu} - e A_{\mu})^2 - m^2 - \frac{1}{2} e \sigma^{\mu\nu} F_{\mu\nu} \right] u^\lambda + \frac{1}{2} e \sigma^{\mu\nu} R_{[\mu\nu]} u^\lambda = 0. \]  

(127)

It is interesting to see that

- i) the above formula is absolutely general for the type of geometrical Lagrangians involved containing the
generalized Ricci tensor inside,

- ii) for instance, the variation of the action will carry the
symmetric contraction of components of the torsion
tensor (i.e., eq.(121)), then the arising of terms as \( h_{\mu} h_{\nu} \),

- iii) the only thing that changes is the mass (123) and
the explicit form of the tensors involved as \( R_{[\mu\nu]} \).
\( F_{\mu\nu} \) etc., without variation of the Dirac general structure of the equation under consideration,

iv) eq. (127) differs from that obtained by Landau and Lifshitz by the appearance of the last two terms: the term involving the curvature tensor is due the spin interaction with the gravitational field (due torsion term in \( R^{\kappa}_{\mu\nu\kappa} \)) and the last term is the spin interaction with the the electromagnetic and mechanical momenta,

v) expression (127) is valid for another vector \( \gamma^{\lambda} \), then is valid for a bispinor of the form

\[ \Psi = u + i\nu, \]

vi) the meaning for a quantum measurement of the spacetime curvature is mainly due by the term in (127) involving explicitly the curvature tensor.

The important point here is that the spin-gravity interaction term is so easily derived as the spinors are represented as spacetime vectors whose covariant derivatives are defined in terms of the G-(affine) connection. In their original form the Dirac equations would have, in curved spacetime, their momentum operators replaced by covariant derivatives in terms of “spin-connection” whose relation is not immediately apparent.

6. DIRAC STRUCTURE, ELECTROMAGNETIC FIELD AND ANOMALOUS GYROMAGNETIC FACTOR

The interesting point now is based in the observation that if we introduce expression (19b) in (127) then

\[
\left[ (\hat{P}_\mu - e\hat{A}_\mu) - m^2 - \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} \right] u^\lambda - \frac{\lambda}{2} \frac{1}{d^2} \sigma^{\mu\nu} f_{\mu\nu} u^\lambda = 0,
\]

\[
\left[ (\hat{P}_\mu - e\hat{A}_\mu)^2 - m^2 - \frac{1}{2} \sigma^{\mu\nu} \left( e F_{\mu\nu} + \frac{\lambda}{d} f_{\mu\nu} \right) \right] u^\lambda - \frac{\epsilon \sigma^{\mu\nu}}{2} (\hat{A}_\mu \hat{P}_\nu - \hat{A}_\nu \hat{P}_\mu) u^\lambda = 0,
\]

we can see clearly that if \( \hat{A}_\mu = j_\alpha \hat{A}_\mu \) (with \( j \) arbitrary constant), \( F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu \) the last expression takes the suggestive form

\[
\left[ (\hat{P}_\mu - e\hat{A}_\mu)^2 - m^2 - \frac{1}{2} \left( e j + \frac{\lambda}{d} \sigma^{\mu\nu} F_{\mu\nu} \right) \right] u^\lambda - \frac{\epsilon \sigma^{\mu\nu}}{2} u^\lambda = 0,
\]

with the result that the gyromagnetic factor have been modified to \( 2/(i + \lambda/ed) \). Notice that in an Unified Theory with the characteristics introduced here is reasonable the identification introduced in the previous step \( (F \leftrightarrow j) \) in order that the fields arise from the same geometrical structure.

The concrete implications about this important contribution of the torsion to the gyro-magnetic factor will be given elsewhere with great detail on the dynamical property of the torsion field. Only we remark the following:

i) there exists an important contribution of the torsion to the gyromagnetic factor that can have implications to the trouble of the anomalous momentum of the fermionic particles,

ii) this contribution appear (taking the second equality of expression 19b), as a modification on the vertex of interaction, almost from the effective point of view;

iii) is quite evident that this contribution will justify probably the little appearance of the torsion at great scale, because we can bounded the torsion due the other well know contributions to the anomalous momenta of the elementary particles (QED, weak, hadronic contribution, etc.),

iv) the form of the coupling spin-geometric structure coming from the first principles, as the Dirac equation, not prescriptions,

v) then, from iii) how the covariant derivative works in presence of torsion is totally determined by the G structure of the spacetime,

vi) the Dirac equation (128) (where was introduced the second part of the equivalence (19b) coming from the equation of motion), said us that the vertex was modified without a dynamical function of propagation. Then, other form to see the problem treated in this paragraph is to introduce the propagator for the torsion corresponding to the first part of the equivalence (19b). This important possibility will be studied elsewhere [5].

7. SPACE-TIME AND STRUCTURAL COHOMOLOGIES

As is well know from the physical and mathematical point of view, the cohomological interplay between the fields involved in any well possessed geometrical and unified theory is crucial. This importance arises as a consequence of the logical (and causal) structure of the physical fields (sources, fields, conserved quantities) and not only as a mathematical play. In the theory presented here, there exist two cohomological structures: Spacetime cohomology and structural cohomology

The difference between them is that in the Spacetime cohomology the Dirac (fermionic) structure of the space time is not involved directly in the relations between the fields involved.

The main equations necessary for the construction are

\[
\nabla_a T^a_{\mu\nu} = -\lambda f_{\mu\nu},
\]

\[
d^* T = -\lambda * f = dh,
\]
being the interplay schematically as

\[
\begin{array}{ccc}
A^- & \rightarrow & \mathcal{T} \\
A^+ & \rightarrow & B^+ \\
B^- & \rightarrow & C^+ \\
C^- & \rightarrow & \mathcal{H}
\end{array}
\]

where the operators are

\[
A_+ \equiv (-1)^{d+1} (-\lambda)^* \int & A_- \equiv (-\lambda)^{-1} d^*
B_+ \equiv (-1)^{d+1} & B_- \equiv *
C_+ \equiv -\lambda & C_- \equiv \int [(-1)^{d+1} (-\lambda)^{-1} d
d_+ \equiv (-1)^{d+1} d & d_- \equiv (-1)^{d+1} \int
E_+ \equiv d & E_- \equiv *
G_+ \equiv [(-1)^{d+1} (-\lambda)^{-1} & G_- \equiv -\lambda^*
\end{array}
\]

The Structural cohomology, in contrast, involve directly the fermionic structure of the space-time due that in the basic formulas \( \mathcal{G}_{\mu
u} \) enters directly into the cohomological game, as is easily seen below

\[
\begin{array}{ccc}
E & \rightarrow & \mathcal{A} \\
B & \rightarrow & \mathcal{D} \\
C & \rightarrow & \mathcal{H} \\
G & \rightarrow & \mathcal{A}
\end{array}
\]

Notice the important thing that, in this case clearly the degree of the relations between the quantities involved are more fundamental that in the previous case (jerarchical sense).

8. CONCLUDING REMARKS

In this paper we make an exhaustive analysis of the model based in the theory developed in early references of the authors. The simplest structure of the spacetime described by this new theory make, beside the connection between curvature and matter, the link between the torsion and the spin.

As was well explained through all this paper, the mechanism of rupture of symmetry is the responsible that the geometrical Lagrangian can be written in a suggestive Eddington-Born-Infeld like form. Three cases were treated from the point of view of the solutions, depending on the form of torsion used: totally antisymmetric (with torsion potential), not totally antisymmetric ("tratorial" type), and with a torsion tensor with both characteristics. In all the cases they were compared from the point of view of the obtained solutions with the non dualistic model of reference [3], namely the Non-Abelian Born-Infeld model.

In all these cases the (non-dualistic) unified model proposed here have deep differences with the dualistic non-Abelian Born-Infeld model of our early reference [3].

The first obvious difference come from a conceptual framework: the geometrical action will provide, besides the spacetime structure, the matter-energy spin distribution. This fact is the same basis of the unification: all the (apparently disconnected) theories and interactions of the natural world appears naturally as a consequence of the intrinsic spacetime geometry.

For the case of totally antisymmetric tensor torsion with torsion potential, several points were answered and elucidated:

i) about the Hosoya and Ogura ansatz the natural question arising was:

why the identification of the isospin structure of the Yang-Mills field with the space frame lead a similar physical situation that a non-dualistic unified theory with torsion? The answer is: because at once such identification is implemented, a potential torsion is introduced and the solution of the set of equations is the consistency between the definition of the torsion tensor from the potential and the Cartan structure equations [1, 2].

ii) about the obtained solutions for the scale factor, the difference with our previous work is precisely the particular form of the energy-momentum tensor in the NABI case (in the UFT model presented here, there are not energy-momentum tensor, of course): both solutions describe a wormhole-instanton but the final form of the differential equations for the scale factor are different, then the scale factor here has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the "dualistic" non-Abelian BI theory. Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of the Coleman’s mechanism) is fulfilled.

The contact point between the compared models, however, are the dynamical equations that are very similar although the existence of a “current term” in the UFT model (cf. (45)) that not appears in the NABI case. This fact was pointed out in an slightly different context by N. Chernikov.

For the case of non-totally antisymmetric (tratorial type) the spacetime structure was analyzed from the point of view of the interacting fields arising from the same geometry of the space time and relaxing now the condition of a totally antisymmetric torsion, then, the prior existence of an antisymmetric 2-form potential for it. The precise results can be easily enumerated as:

(i) from its \( SL(2C) \) underlying structure: the notion of minimal coupling has been elucidated and come naturally of the compatibility condition between the
gauge field structure of the antisymmetric part of the fundamental tensor and the \( SL(2,\mathbb{C}) \) structure of the base manifold,

(ii) trough exact cosmological solutions from this model, where the geometry is Euclidean \( R \otimes O(3) \sim R \otimes SU(2) \), the relation between the space-time geometry and the structure of the gauge group was explicitly shown,

(iii) this relation is directly connected with the relation of the spin and torsion fields.

From the point of view of the obtained solutions, a solution of this model was explicitly compared with our previous ones and we find that:

(i) the torsion is not identified directly with the Yang Mills type strength field,

(ii) there exists a compatibility condition connected with the identification of the gauge group with the geometric structure of the space-time: this fact lead the identification between derivatives of the scale factor \( a \) with the components of the torsion in order to allows the Hosoya-Ogura ansatz (namely, the alignment of the isospin with the frame geometry of the space-time),

(iii) this compatibility condition precisely mark the fact that local gauge covariance, coordinate independence and arbitrary space time geometries are harmonious concepts and

(iv) of two possible structures of the torsion the “tratorial” form forbids wormhole configurations, leading only, cosmological instanton space-time in eternal expansion.

For the general case, i.e. with torsion with totally antisymmetric and tratorial parts, the full analysis was given in a clear manner in Section 6. Here we point out that the Hosoya and Ogura anzats can be implemented as in the previous cases, and, the most important, the fact that wormhole solutions can be obtained for some particular cases. The solutions are asymptotically flat, where appear vector and tensor integration constants that are constrained in norm to bring physical consistency to the solution.

About the problem of the possibility of coexistence of the trace of the torsion due the tratorial part and the axial vector from the totally antisymmetric part of the torsion, we saw here that there are no problem in the new theory: there are tratorial and antisymmetric torsion fields without contradictions.

The fact that in reference [4] the field equations of vacuum quadratic Poincare gauge field theory (QPGFT) were solved for purely null tratorial torsion, if well permit to express the contortion tensor for such a case as (tratorial form, with notation of ref. [4])

\[
K_{\mu \nu} = -2 (g_{\lambda \rho} a_{\lambda} - g_{\lambda \nu} a_{\lambda} a_{\mu}),
\]

does not permit the coexistence with an axial torsion vector, as was clearly shown by Singh in the beautiful paper [4]. The two points that lead such discrepancy are:

- i) the different theories described, not only in foundations but also because one is unitarian and the other of [4] dualistic
- ii) and the fact that the Newman-Penrose formulation was used in [4], that as is well known such method works in a null tetrad.

8.1. On the Geometrical Structure

From the point of view of the concrete structure able to explain the content of the bosonic and fermionic matter of the universe, the present paper is left open-ended as many physical consequences need to be explored. Some words concerning to the realization and the choice of the correct group structure of the tangent space to \( M \) is that \( G = L(4) \cap Sp(4) \cap K(4) \) preserves the boson and fermion symmetry simultaneously without imply supersymmetry of the model. As we like to show in a future work, the supergravitational extension of the model will be discussed joint with the problem of it quantization, where the key point will be precisely the group structure of the tangent space to the spacetime manifold \( M \). Here we conclude enumerating the main results concerning to the basic structure of the Manifold supporting an Unified Field Theoretical model:

- i) the simplest geometrical structure able to support the fermionic fields was constructed based in a tangent space with a group structure \( G = L(4) \cap Sp(4) \cap K(4) \)
- ii) then, the explicitly link of the fermionic structure with the torsion field was realized and the Dirac type equation was obtained from the same spacetime manifold
- iii) notice that the matter was not included on the Geometrical Lagrangian of the Unified theory presented here: only symmetry arguments (that will lead the correct dynamical equations for the material fields arising from the same manifold) need to allow the appearance of matter and this fact is not the essence of the unification, of course (several references trying to include matter into the Eddington “type” theories by hand without physical and symmetry principles).

8.2. On the Energy Concept

1) On the equation

\[
\hat{R}^\alpha_{\mu \nu} = -\lambda g_{\mu \nu} + T^\alpha_{\mu \rho} T^\rho_{\alpha \nu},
\]

notice that the concept here of the terms that arise as “energy-momentum” part coming from the symmetric contraction of the torsion components is different in essence to the concept coming from the inclusion of the energy-momentum tensor in the Einstein theory. The conceptual framework that “matter and energy curve the spacetime” implicitly carry the idea of some “embedding-like” situation where the matter and energy are putted on some Minkowskian flexible carpet and you see how it is curved under the “weight” of
the “ball” (matter+energy). Here, in the theory presented, the situation is that the torsion terms (contributing as “energy momentum in above equation) arise from the same geometry, then we have the picture as an unique entity: the interplay fields-spacetime. the idea is the same as the solitonic vortex in the water.

This fact can be also interpreted as that the concept of force is introduced due the torsion in the unified model, thing that is lost in the Einstein theory [10] where the concept is that there are not force, but curvature only.

2) Some remarks on the general Hodge-de Rham decomposition of \( h = h_\alpha dx^\alpha \).

**Theorem 1.** if \( h = h_\alpha dx^\alpha \in F(M) \) is a 1-form on \( M \), then there exist a zero-form \( \Omega \), a 2-form \( \alpha = A_\mu \gamma^\beta dx^\mu \wedge dx^\gamma \) and an harmonic 1-form \( q = q_\alpha dx^\alpha \) on \( M \) that

\[
 h = d\Omega + q + q_\alpha = \nabla_\alpha \Omega + e_\alpha^\beta \gamma^\delta \nabla_\beta A_\gamma \delta + q_\alpha.
\]

Notice that if even is not harmonic and assuming that \( q_\alpha \) is a polar vector, an axial vector can be added such that above expression takes the form

\[
 h_\alpha = \nabla_\alpha \Omega + e_\alpha^\beta \gamma^\delta \nabla_\beta A_\gamma \delta + e_\alpha^\mu \eta^\nu \eta^\mu N_\beta \gamma \delta + q_\alpha.
\]

where \( M_{\beta \gamma \delta} \) is a completely antisymmetric tensor.

3) Notice the important fact that when the torsion is totally antisymmetric tensor field, \(-2\lambda f_{\mu \nu}\) takes the role of “current” for the torsion field, as normally the terms proportional to the 1-form potential vector \( a_\mu \) acts as current of the electromagnetic field \( f_{\mu \nu} \) in the equation of motion for the electromagnetic field into the standard theory: \( \nabla_\alpha f^\alpha_\mu = J_\mu \) (constants absorbed into the \( J_\mu \)). The interpretation and implications of this question will be analyzed concretely in [5].

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