Violation of Thermal Conductivity Bound in Horndeski Theory

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ABSTRACT

We consider AdS black planar black hole in the four-dimensional Einstein-Maxwell-Horndeski theory with two free axions and analyse the thermodynamics of the black hole. We calculate the holographic thermoelectric conductivities of the dual field theory and determine the ratio of thermal conductivity over the temperature. At low temperature with zero electric current, we find that the ratio is proportional to temperature squared and hence can arbitrarily approach zero, providing an example that violates the thermal conductivity bound proposed in [arXiv:1511.05970].

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1 Introduction

The gauge/gravity duality has brought many remarkable insights to the dynamics of some strongly coupled systems [1–4]. Several universal bounds of transport coefficients have been conjectured based on the holographic “bottom-up” models. For example, the famous bound of the ratio of the shear viscosity to the entropy density was proposed in [5, 6]. Many of these bounds were later violated through various ways.

In this paper, we are interested in a recently proposed bound for thermal conductivity [7]:

\[
\frac{\kappa_{DC}}{T} \geq C, \tag{1.1}
\]

where \( \kappa_{DC} \) is the thermal conductivity where there is zero electric current and \( C \) is some finite number. The thermal bound (1.1) was formulated by using the holographic model of Einstein-Maxwell-Dilaton theory plus some other matter fields assuming that the scalar potential is bounded from below. Remarkably, this bound was tested positive against a variety of holographic models, and up till now, there is no counterexample.

Motivated by searching for an counterexample, we study the holographic thermoelectric properties of black holes in Horndeski models. We indeed find a counterexample of the bound (1.1).

The Horndeski theories were constructed in 1970s [10], and they were rediscovered and have received much attention through their application in cosmology [11]. A particular property of Horndeski theories is that although their Lagrangians involve terms which are more than two derivatives, the equations of motion for each field involve at most two derivatives of the fields. This feature is similar to that of Lovelock gravities [12].

Black hole solutions that are asymptotic to AdS spacetime have been constructed in Horndeski gravity theories in [13, 14] and the thermodynamics of these AdS black holes were analyzed in [15, 16]. The stability and causality were studied in [17, 19]. Holographic properties in Horndeski theories were deeply investigated in [20, 22]. Further applications and properties were discussed in [23, 20]. Here, we shall study the thermoelectric conductivity of a kind of Horndeski model, which was first introduced in [20].

In this paper, we shall consider Einstein-Maxwell-Horndeski with a cosmological constant plus two free axions which are used for momentum dissipation in four dimensions in section 2. In section 3, we review thermodynamics of the black holes in the theory. We analytically study the holographic DC electrothermal conductivities in section 4. Finally,
we conclude our results in section 5.

2 The theory and AdS black holes

The theory we consider is Einstein-Maxwell-Horndeski theory with two free axions which are responsible for momentum dissipation

\[
\mathcal{L} = \sqrt{g} \left[ \kappa (R - 2\Lambda - \frac{1}{4} F^2) - \frac{1}{2} (\alpha g^{\mu\nu} - \gamma G^{\mu\nu}) \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} \sum_{i=1}^{2} (\partial \phi_i)^2 \right],
\]

(2.1)

where \( F = dA \) and \( G_{\mu\nu} \) is Einstein tensor.

The equations of motion with respect to \( g_{\mu\nu}, \chi, A_\mu \) and \( \phi_i \) are respectively given by

\[
\begin{align*}
\kappa (G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} F_{\mu\nu}^2 + \frac{1}{8} F^2 g_{\mu\nu}) - \frac{1}{2} \sum_{i=1}^{2} \left( \partial_\mu \phi_i \partial_\nu \phi_i - \frac{1}{2} g_{\mu\nu} (\partial \phi_i)^2 \right) & \\
- \frac{1}{\alpha} \left( \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} (\partial \chi)^2 \right) - \frac{1}{2} \gamma \left( \frac{1}{2} \partial_\mu \chi \partial_\nu \chi R - 2 \partial_\rho \chi \partial_\mu \chi R_\rho^\rho \right) & \\
- \partial_\rho \chi \partial_\sigma \chi R_{\rho\sigma}^{\rho\sigma} - (\nabla_\mu \nabla^\rho \chi)(\nabla_\nu \nabla_\rho \chi) - (\nabla_\mu \nabla_\nu \chi) \Box \chi + \frac{1}{2} G_{\mu\nu} (\partial \chi)^2 & \\
- g_{\mu\nu} \left[ - \frac{1}{2} (\nabla_\rho \nabla_\sigma \chi)(\nabla_\nu \nabla_\sigma \chi) + \frac{1}{2} (\Box \chi)^2 - \partial_\rho \chi \partial_\sigma \chi R^{\rho\sigma} \right] & = 0, \\
\nabla_\mu \left( (\alpha g^{\mu\nu} - \gamma G^{\mu\nu}) \nabla_\nu \chi \right) & = 0, \\
\nabla_\nu F^{\nu\mu} & = 0, \\
\Box \phi_i & = 0.
\end{align*}
\]

(2.2)

For static planar ansatz

\[
\begin{align*}
\text{ds}^2 & = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 dx_1 dx_1, \\
A & = a(r) dt, \quad \chi = \chi(r), \quad \phi_1 = \lambda x_1, \quad \phi_2 = \lambda x_2,
\end{align*}
\]

(2.3)

the theory admits a black hole solution \[20\]

\[
\begin{align*}
h & = g^2 r^2 - \frac{2\kappa \lambda^2}{\beta \gamma + 4\kappa} - \frac{m_0}{r} + \frac{\kappa (3g^2 q^2 (\beta \gamma + 4\kappa) - \kappa \lambda^4)}{3g^2 r^2 (\beta \gamma + 4\kappa)^2} - \frac{\kappa^2 q^4}{60g^2 r^6 (\beta \gamma + 4\kappa)^2} - \frac{\kappa^2 \lambda^2 q^2}{9g^2 r^4 (\beta \gamma + 4\kappa)^2}, \\
f & = \frac{(6g^2 r^4 (\beta \gamma + 4\kappa) - \kappa (q^2 + 2\lambda^2 r^2))^2 h}{6g^2 r^4 (\beta \gamma + 4\kappa)^2}, \\
\chi' & = \sqrt{\frac{6\beta \gamma g^2 r^4 - \kappa (q^2 + 2\lambda^2 r^2)}{6\gamma g^2 r^4} \frac{1}{f}}, \\
a & = a_0 - \frac{q}{r} + \frac{q\kappa \lambda^2}{9g^2 r^3 (\beta \gamma + 4\kappa)} + \frac{\kappa q^3}{30g^2 r^3 (\beta \gamma + 4\kappa)},
\end{align*}
\]

(2.4)
with constraints
\[ \Lambda = -\frac{3g^2(\beta \gamma + 2\kappa)}{2\kappa}, \quad \alpha = 3\gamma g^2. \] (2.5)

The solution with \( \lambda = 0 \) was constructed in \([14]\). The parameters \( a_0, q, m_0 \) are integration constants and \( \lambda \) is a symmetry breaking parameter. Note that the solution has no \( \gamma = 0 \) limit and cannot be reduced to the solution of Einstein theory. It is also worth pointing out that the above solution lives in the critical point \( \alpha = 3\gamma g^2 \), where the theory admits a holographic a-theorem \([26]\).

There is a curvature singularity located at \( r = r_* \), where \( f \) diverges, and is determined by
\[ F(r_*) \equiv 6g^2r_*^4(\beta \gamma + 4\kappa) - \kappa(q^2 + 2\lambda^2 r_*^2) = 0. \] (2.6)

In order to describe a black hole, we require that the largest root \( r_* \) should be inside the event horizon, that is \( r_* < r_0 \), where the radius of the event horizon \( r_0 \) is the largest root of \( f(r_0) = 0 \). The Hawking temperature can be obtained through standard method
\[ T = \frac{6g^2r_0^4(\beta \gamma + 4\kappa) - \kappa q^2 - 2r_0^2\kappa \lambda^2}{8\pi r_0^3(\beta \gamma + 4\kappa)}. \] (2.7)

It is worthwhile pointing out that the requirement that the curvature singularity \( r_* \) should be inside the event horizon guarantees that the temperature is positive, \( T > 0 \).

Though the linearised equations of motion for Horndeski theory involve only two derivatives, it is still necessary to check if the kinetic term of the Horndeski scalar is positive or not for possible ghost-like behaviour. The kinetic term for the axion perturbation \( \delta \chi \) is given by
\[ P^{00} \delta \chi \delta \chi, \quad \text{where} \quad P^{\mu \nu} = -\frac{1}{2}(\alpha g^{\mu \nu} - \gamma G^{\mu \nu}). \] (2.8)

And the \( P^{00} \) component of \( P^{\mu \nu} \) in our case takes the form
\[ P^{00} = \frac{144\gamma g^4 \kappa r^6(\beta \gamma + 4\kappa)^2 (q^2 + \lambda^2 r^2)}{F(r)^3}, \] (2.9)

where the function \( F(r) \) is defined in \((2.6)\). In order to avoid ghost problem, \( P^{00} \) should be non-negative from horizon to infinity. It was pointed out in \((2.6)\) that there is a curvature singularity \( r_* \), which is the largest root of \( F(r_*) = 0 \). To avoid naked singularity, we require that \( r_* < r_0 \). Thus, \( F(r) \) is always positive from horizon to infinity. Now, we can see from \((2.9)\) that \( P^{00} \) is the product of \( \gamma \) and a positive factor. Then the positivity of \( P^{00} \) requires...
that
\[ \gamma > 0. \] (2.10)

3 Black hole thermodynamics

The thermodynamics of AdS black hole in Horndenski theory have been studied extensively in [15, 16] with the aid of Wald formula [27, 28]. Here, we just review the main result. The variation of Lagrangian \( \mathcal{L} \) gives the equations of motion and total derivative terms

\[ \delta \mathcal{L} = \text{e.o.m} + \sqrt{g} \nabla_\mu J^\mu, \] (3.1)

from which we can define a 1-form \( J^{(1)} = J_\mu dx^\mu \) and its Hodge dual \( \Theta^{(3)} = -1 * J^{(1)} \). Specializing the variation to be induced by an infinitesimal diffeomorphism \( \delta x^\mu = \xi^\mu \), we can define a 3-form and show that

\[ J^{(3)} \equiv \Theta^{(3)} - i_\xi L = \text{e.o.m} - d * J^{(2)}, \] (3.2)

where \( i_\xi \) represents a contraction of \( \xi^\mu \) with the 3-form \( *L \) and \( J^{(2)} = d\xi \).

Then, one can define a 2-form \( Q^{(2)} \equiv *J^{(2)} \), such that \( J^{(3)} = dQ^{(2)} \) on shell. Wald found that the variation of the Hamiltonian is given by

\[ (\delta Q - i_\xi \Theta)_{\text{total}} = -2r \sqrt{h f} \left( \kappa + \frac{\gamma}{4} f \chi'^2 \right) \delta f \Omega^{(2)} \]
\[ -r^2 \sqrt{h f} \kappa \left( \frac{f}{h} \, a \delta a' + \frac{a a'}{2} \left( \frac{\delta f}{h} - \frac{f \delta h}{h^2} \right) \right) \Omega^{(2)}. \] (3.3)

It is worth pointing out that we consider \( \lambda \) as a constant like \( g \) rather than a thermodynamical variable. We choose a gauge that the electric potential is zero on the horizon, then the variation of the Hamiltonian on the horizon has the value

\[ \delta H_+ = 16\pi T (\kappa + \frac{\gamma}{4} f \chi'^2) \delta \left( \frac{r_0^2}{4} \right), \] (3.4)

where \( T \) is given in (2.7), while in the infinity, it gives

\[ \delta H_\infty = \kappa \mu \delta q - \frac{(4\kappa + \beta_\gamma) \delta m_0}{2}, \] (3.5)
And Wald showed that the variation of the Hamiltonian vanishes on the Cauchy surface. For a black hole it implies that

$$\delta H_+ + \delta H_\infty = 0,$$

which gives the first law of the thermodynamics,

$$dM = T dS + \Phi_e dQ_e + \Phi_\lambda^+ dQ_\lambda^+.$$ (3.7)

with

$$M = \frac{1}{2}(4\kappa + \beta\gamma)m_0,$$
$$\Phi_e = \mu,$$
$$Q_e = \kappa q,$$
$$S = \left(\kappa + \frac{\gamma}{4}(f\chi'^2)|_{r_0}\right)4\pi r_0^2 = \frac{16\pi r_0(\beta\gamma + 4\kappa)}{3g^2}T,$$
$$\Phi_\lambda^+ = -\frac{\gamma r_0^2 T}{8}f\chi'^2|_{r_0},$$
$$Q_\lambda^+ = 16\pi \int_{r=r_0} \sqrt{(\partial\chi)^2} = 16\pi \sqrt{f}\chi'|_{r=r_0}. \quad (3.8)$$

$\Phi_\lambda^+$ and $Q_\lambda^+$ are the scalar charge and scalar potential respectively, more details about these can be found in [16]. Note that the temperature $T$ is given in (2.7). It is important to note that although we appear to be able to set $\gamma = 0$ in the above thermodynamical quantities, as we have remarked earlier, there is no smooth limit of $\gamma = 0$ in the full black hole solution.

### 4 Holographic thermoelectric conductivity

There are various ways to obtain the holographic DC thermoelectric conductivities based on “membrane paradigm” [29–31]. The key step is to construct the relevant radially conserved current, which serves as a bridge between the holographic boundary physical properties and the black hole horizon information. Actually, we shall follow the method proposed in [32] to get the thermoelectric conductivity.

We consider the following perturbations around the background solution,

$$\delta g_{tx_1} = tU_1(r) + \Psi_{tx_1}, \quad \delta g_{rx_1} = \Psi_{rx_1}, \quad \delta A_{x_1} = tU_2(r) + a_{x_1}, \quad \delta \phi_1 = \frac{\Phi(r)}{\lambda}. \quad (4.1)$$

The radially conserved electric current can be easily obtained with the help of Maxwell equation $\partial_r(\sqrt{g}F^{rx_1}) = 0$,

$$J = \kappa \sqrt{g} F^{rx_1}.$$ (4.2)

Whilst, the radially conserved holographic heat current is more difficult to construct, since
its conservation involves both Einstein equations and Maxwell equations. Fortunately, a
general formula of deriving the holographic heat current was proposed in \cite{33} by using
Noether symmetry for a rather general class of gravity theories. Applying this formula to
our theory, we can get the holographic heat current

\[
Q = \sqrt{g} \left( \kappa (2 \nabla^r \xi^1 + a F^r x_1) + \frac{\gamma}{2} g^{rr}(\partial_r \chi)^2 \nabla^r \xi^1 \right),
\]  

(4.3)

where \( \xi \) is the time-like Killing vector \( \partial_t \). One can see the Horndeski scalar directly con-
tributes a \( \gamma \) term in the heat current. We found that the electric and heat current can be
time independent by choosing

\[
U_1 = -\zeta h, \quad U_2 = -E + \zeta a,
\]  

(4.4)

where \( E \) and \( \zeta \) are constants which parametrize sources for the electric and heat currents,
respectively. Near the black hole horizon, we imposed the in going wave condition

\[
a'_1 = -\frac{E + \zeta a}{\sqrt{h_f}} + \ldots, \quad \Psi_{tx_1} = \Psi_{tx_1}^{(0)} - \zeta h \int \frac{1}{\sqrt{h_f}} + \ldots,
\]  

(4.5)

where \( \Psi_{tx_1}^{(0)} \) is a regular function whose value on the horizon can be determined through the
linear perturbation equation of motion

\[
\Psi_{tx_1}(r_0) = \frac{-48Eg^2 \kappa q r_0^5 (\beta \gamma + 4 \kappa) + \zeta \left( \kappa \left( q^2 + 2 \lambda^2 r_0^2 \right) - 6g^2 r_4 (\beta \gamma + 4 \kappa) \right)^2}{48g^2 \kappa \lambda^2 r_0^3 (\beta \gamma + 4 \kappa)^2}.
\]  

(4.6)

Now, we are in a position to evaluate the radially conserved currents on the horizon

\[
J = (\kappa + \frac{\kappa q^2}{\lambda^2 r_0^2}) E + \frac{4\pi^2 q (\beta \gamma + 4 \kappa)}{3g^2 \lambda^2 r_0} T^2 \zeta,
\]
\[
Q = \frac{4\pi^2 q (\beta \gamma + 4 \kappa)}{3g^2 \lambda^2 r_0} T^2 E + \frac{16\pi^4 (\beta \gamma + 4 \kappa)^2}{9g^4 \kappa \lambda^2} T^4 \zeta.
\]  

(4.7)

The DC conductivity matrix is then given by

\[
\sigma_{DC} = \frac{\partial J}{\partial E} = \kappa (1 + \frac{q^2}{r_0^2 \lambda^2}),
\]
\[
\alpha_{DC} = \frac{1}{T} \frac{\partial J}{\partial \zeta} = \frac{4\pi^2 q (\beta \gamma + 4 \kappa)}{3g^2 r_0 \lambda^2} T,
\]
\[
\bar{\alpha}_{DC} = \frac{1}{T} \frac{\partial Q}{\partial E} = \frac{4\pi^2 q (\beta \gamma + 4 \kappa)}{3g^2 r_0 \lambda^2} T,
\]
\[
\bar{\kappa}_{DC} = \frac{1}{T} \frac{\partial Q}{\partial \zeta} = \frac{16\pi^4 (\beta \gamma + 4 \kappa)^2}{9g^4 \kappa \lambda^2} T^3.
\]  

(4.8)
It is obvious that the electric bound, which was proposed in [35], is satisfied

$$\sigma_{DC} = \kappa (1 + \frac{q^2}{r_0^2 \lambda^2}) \geq 1.$$  \hspace{1cm} (4.9)

The electric conductivity $\sigma_{DC}$ is the same as that of Einstein-Maxwell case [34] while it is expressed in terms of black hole horizon radius $r_0$, as is pointed out in [20]. It is easy to check that $\alpha_{DC} = \tilde{\alpha}_{DC}$, which means the Onsager relation holds. And, we also find that the thermal relation

$$ST \alpha_{DC} - Q_e \tilde{\kappa}_{DC} = 0$$ \hspace{1cm} (4.10)

holds for this system.

The thermal conductivity at zero electric current is

$$\kappa_{DC} = \frac{16 \pi^4 (\beta \gamma + 4 \kappa)^2}{9 \kappa g^4 \left(\lambda^2 + \frac{q^2}{r_0^2}\right)^2} T^3.$$ \hspace{1cm} (4.11)

There are two more quantities of interest, Lorentz ratios of the thermal conductivities over the electric conductivities, which are given by

$$\tilde{L} = \frac{\tilde{\kappa}_{DC}}{\sigma_{DC} T} = \frac{S^2}{\kappa^2 \left(q^2 + \lambda^2 r_0^2\right)}.$$ \hspace{1cm} (4.12)

$$L = \frac{\kappa_{DC}}{\sigma_{DC} T} = \frac{\lambda^2 r_0^2 S^2}{\kappa^2 \left(q^2 + \lambda^2 r_0^2\right)^2}.$$ \hspace{1cm} (4.13)

Usually, the Lorentz ratio $L$ is a constant, due to the fact that the heat transport and the electric transport both involve the charge carriers, like free electrons in metal, which is well known as the Wiedemann-Franz law. As we can see, this law was violated in our case, which may be explained in terms of independent transportation of charge and heat in a strongly coupled system.

It was observed that there is a bound for Lorentz ratio $\tilde{L}$ in [32]

$$\tilde{L} \leq \frac{S^2}{Q_e^2}.$$ \hspace{1cm} (4.14)

From (4.12), we can see the above bound is satisfied in our case

$$\tilde{L} = \frac{S^2}{\kappa^2 \left(q^2 + \lambda^2 r_0^2\right)} \leq \frac{S^2}{Q_e^2}.$$ \hspace{1cm} (4.15)

With the electric bound (4.9) and the Lorentz ratio bound (4.15) both satisfied, it is more
interesting to check the thermal conductivity bound (1.1), which hasn’t been violated so far. The thermal conductivity bounds (1.1) states that the ratio of the thermal conductivity with zero electric current over temperature is great than a finite number which depends on the parameters of the theory and is usually of order one. From (4.11), one can obtain the thermal conductivity $\kappa_{\text{DC}}$ over temperature $T$, which is given by

$$\frac{\kappa_{\text{DC}}}{T} = \frac{16\pi^4(\beta\gamma + 4\kappa)^2}{9\kappa g^4 \left(\lambda^2 + \frac{q^2}{r_0}\right)} T^2. \quad (4.16)$$

As discussed in section 2, the temperature of the black hole can approach zero, but cannot reach zero. When the temperature is sufficiently small, the ratio has the form

$$\frac{\kappa_{\text{DC}}}{T} \sim \frac{16\pi^4(\beta\gamma + 4\kappa)^2}{9g^4 \sqrt{\kappa \sqrt{6g^2 q^2 (\beta\gamma + 4\kappa) + \kappa \lambda^4}}} T^2 + O(T^3). \quad (4.17)$$

Thus, the ratio can be arbitrarily small at low temperature and hence our holographic model violates the proposed thermal conductivity bound (1.1).

5 Conclusions

In this paper, we considered the Einstein-Maxwell-Hordeski theory coupled with two additional free axions. The theory admits analytical planar AdS black hole, where the axions span over the two-dimensional plane. We analysed the thermodynamics of the black hole and calculated the holographic thermoelectric conductivities of the dual field theory. The focus of the paper is to examine the universal bounds proposed in literature.

The Horndeski term doesn’t contribute directly to the electric conductivity, which takes the same form as that of Einstein-Maxwell theory. Thus the electric conductivity bound is preserved in this situation. Furthermore, the Onsager relation $\alpha_{\text{DC}} = \bar{\alpha}_{\text{DC}}$ and the thermal relation $ST\alpha_{\text{DC}} - Q_e\bar{\kappa}_{\text{DC}} = 0$ are also both satisfied. The situation for the thermal conductivity, on the other hand, is quite different. The ratio of thermal conductivity with zero electric current over temperature equals to a positive function times the square of the temperature, so it can be arbitrarily small as the temperature is low, violating the conductivity bound (1.1). This rare counterexample indicates that an underlying principal is needed to understand the condition when the bound is valid.
Acknowledgement

We are grateful to Hong Lu for discussions. H-S.L. is supported in part by NSFC grants No. 11305140, No. 11375153, No. 11475148 and No. 11675144.

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