Employing Kuratowski Measure of Non-compactness for Positive Solutions of System of Singular Fractional q-Differential Equations with Numerical Effects

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Abstract. In this work, we investigate the existence of solutions for the system of two singular fractional q-differential equations under integral boundary conditions via the concept of Caputo fractional q-derivative and fractional Riemann-Liouville type q-integral. Some new existence results are obtained by applying Krattowski measure of non-compactness. Also, the Darbo’s fixed point theorem and the Lebesgue dominated convergence theorem are the main tools in deriving our proofs. Lastly, we present an example illustrating the primary effects.

1. Introduction

Fractional calculus and Fractional q-calculus are the significant branches in mathematical analysis. The field of fractional calculus has countless applications (for instance, see [1–4]). Similarly, the subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions (for more details, consider [5–16]). Likewise, some researchers have been investigated the existence of solutions for some singular fractional differential equations (for example, see [17–25]).

In this article, motivated by among these achievements, we will stretch out the positive solutions for the singular system of q-differential equations

\[
\begin{align*}
D_q^\alpha_1 u(t) + g_1(t, u(t), v(t)) &= 0, \\
D_q^\alpha_2 v(t) + g_2(t, u(t), v(t)) &= 0,
\end{align*}
\]

under boundary conditions

\[
\begin{align*}
\alpha_1 &\in (n, n+1] \quad \text{with} \quad n \geq 3, \quad \gamma_1 \geq 1, \quad g_1 \in C(E), \quad g_2 \text{ are singular at } t = 0 \text{ which satisfy the local Carathéodory condition on } E = (0, 1] \times (0, \infty) \times (0, \infty), \quad \text{and } w_1, w_2 \in L^1[0, 1] \text{ are non-negative somehow that}
\end{align*}
\]

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We recall some of the previous works briefly. In 1910, the subject of q-difference equations was introduced by Jackson [26]. After that, at the beginning of the last century, studies on q-difference equations appeared in so many works, especially in Carmichael [27], Mason [28], Adams [29], Trjitzinsky [30], Agarwal [31]. An excellent account in the study of fractional differential and q-differential equations can be found in [32–34]. In 2012, Liu et al. [35] discussed the singular equation \( D^x x(t) + h(t, x(t)) = 0 \) under boundary conditions \( x(1) = 0 \) and \( \{I^2 x(t)\}_{n=0}^\infty = 0 \), where \( t \) belongs to \([0, 1]\) and \( D^\alpha \) is the Riemann-Liouville fractional derivative. In 2013, Zhai et al. [36] discussed about positive solutions for the fractional differential equation with conditions

\[
\begin{cases}
  -D^\alpha x(t) = g(t, x(t)) + h(t, x(t)), \\
x(0) = x'(0) = x''(0) = x'''(1) = 0, \\
or x(0) = x'(0) = x''(0) = 0, \quad x'(1) = \beta x''(\eta),
\end{cases}
\]

where \( t \) and \( \alpha \) belong to \((0, 1), (3, 4), \) respectively, and \( D^\alpha \) is the Riemann-Liouville fractional derivative. In the same year, the singular problem

\[
D^\alpha x(t) = g(t, x(t), D^\beta x(t), D^{\gamma} x(t)) + h(t, x(t), D^\beta x(t), D^{\gamma} x(t)),
\]

under boundary conditions \( x(0) = x'(0) = x''(0) = x'''(0) = 0 \) is reviewed, where \( \alpha, \beta, \gamma \) belong to \((3, 4), (0, 1), (1, 2), \) respectively, \( D^\alpha \) is the Caputo fractional derivative and function \( g \) is a Carathéodory on \([0, 1] \times (0, \infty)^2 \). Also, Wang in [37] investigated the existence of positive solution for the system

\[
D^\alpha x_i(t) + h_i(t, x_i(t), x_2(t)) = 0,
\]

for \( i = 1, 2, \) under boundary conditions \( x_i(0) = x'_i(0) = 0, x_2(0) = x'_2(0) = 0 \) and

\[
x_1(1) = \int_0^1 x_1(t) dt, \quad x_2(1) = \int_0^1 x_2(t) dt,
\]

where \( t \in [0, 1], \alpha \in (2, 3), h_1, h_2 \in C([0, 1] \times [0, \infty) \times [0, \infty), \mathbb{R}), D^\alpha \) is the Riemann-Liouville fractional derivative and \( \int_0^1 x_i(t) dt \) denotes the Riemann-Stieltjes integral. In 2014, Yan et al. [38] studied the fractional boundary value problems

\[
^c D^\alpha_0 u(t) = f(t, u(t), \int_0^t u(t) dt),
\]

for \( t \in [0, 1] \) with boundary conditions \( u(0) = u'(0) = u''(0) = 0 \) and \( u^{(k)}(0) = 0 \) for \( 2 \leq k \leq n - 1 \), where \( ^c D^\alpha_0, ^c D^\beta_0 \) are the Caputo fractional derivatives, \( f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function, \( y : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R} \) is a continuous function and \( m \in \mathbb{R}, n - 1 < \alpha < n, (n \geq 2), 0 < \beta < 1 \) is real number. In 2016, Jleli et al. [8] by using a measure of non-compactness argument combined with a generalized version of Darbo’s theorem, provided sufficient conditions for the existence of at least one solution of the functional equation

\[
u(t) = F\left(t, u(t), \int_0^t u(t) dt, \frac{\int_0^{t} (t - qs)^{(\alpha-1)} g(s, u(s)) ds}{\Gamma(\alpha)}\right),
\]

\( t \in I = [0, 1], \) where \( \alpha \in (1, \infty), q \in (0, 1), f, g : I \times \mathbb{R} \rightarrow \mathbb{R}, \mu, \gamma : I \rightarrow I \) and \( F : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \). In 2019, Samei et al. [7] discussed the fractional hybrid q-differential inclusions

\[
^c D^\alpha_q F\left(t, x, l_q^\alpha x, \ldots, l_q^\alpha x\right) \in F\left(t, x, l_q^\alpha x, \ldots, l_q^\alpha x\right),
\]

with the boundary conditions \( x(0) = x_0 \) and \( x(1) = x_1 \), where \( 1 < \alpha \leq 2, q \in (0, 1), x_0, x_1 \in \mathbb{R}, \alpha_i > 0, \) for \( i = 1, 2, \ldots, n, \beta_i > 0, \) for \( j = 1, 2, \ldots, k, n, k \in \mathbb{N}, \) \( ^c D^\alpha_q \) denotes Caputo type q-derivative of order \( \alpha, l_q^\alpha \) denotes
Riemann-Liouville type q-integral of order \( \beta \), \( f : [0, \infty) \rightarrow [0, \infty) \) is continuous and \( F : [0, \infty) \rightarrow \mathbb{R} \) is multifunction. Also, Ntouyas et al. [13] by applying definition of the fractional q-derivative of the Caputo type and the fractional q-integral of the Riemann–Liouville type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional q-integro-differential equations under some boundary conditions

\[ ^cD_q^nx(t) = w\left(t, x(t), (q_1)x(t), (q_2)x(t), \ldots, (q_n)x(t)\right). \]

In 2020, Liang et al. [14] investigated the existence of solutions for a nonlinear problems regular and singular fractional q-differential equation

\[ ^cD_q^nf(t) = w\left(t, f(t), f'(t), \ldots, D_q^nf(t)\right), \]

with conditions \( f(0) = c_1 f(1), f'(0) = c_2 D_q^β f(1) \) and \( f(0)(0) = 0 \) for \( 2 \leq k \leq n + 1 \), here \( n + 1 < α < n + 3 \), \( β, q, c_1 \in (0, 1) \), \( c_2 \in (0, \Gamma_q(2 - β)) \), function \( w \) is a \( L^\infty \)-Carathéodory, \( w(t, x_1, x_2, x_3) \) may be singular and \( ^cD_q^α \) the fractional Caputo type q-derivative. Similar results have been presented in other studies [9, 10, 12, 15, 16].

The rest of the paper is arranged as follows: in Section 2, we recall some preliminary concepts and fundamental results of q-calculus. Section 3 is devoted to the main results, while example illustrating the obtained results and algorithm for the problems are presented in Section 4.

2. Preliminaries

First, we point out some of the materials on the fractional q-calculus and fundamental results of it which needed in the next sections (for more information, consider [2, 3, 26]). Then, some well-known theorems of fixed point theorem and definition are expressed.

Assume that \( q \in (0, 1) \) and \( a \in \mathbb{R} \). Define \([a]_q = \frac{1-q^a}{1-q}\) [26]. The power function \((x - y)^n_q) \) with \( n \in \mathbb{N}_0 \) is defined by \((x - y)^n_q) = \prod_{k=0}^{n-1}(x - yq^k)\) for \( n \geq 1 \) and \((x - y)^n_q) = 1\), where \( x \) and \( y \) are real numbers and \( \mathbb{N}_0 := [0] \cup \mathbb{N} \) [29]. Also, for \( x > 0 \) and \( a \neq 0 \), we have

\[ (x - y)^n_q) = x^n \prod_{k=0}^{\infty}(x - yq^k)/(x - yq^{n+k}). \]

If \( y = 0 \), then it is clear that \( x^n_q) = x^n \) (Algorithm 1). The q-Gamma function is given by

\[ \Gamma_q(z) = (1 - q)z^{-1} / (1 - q)z^{-1}, \]

where \( z \in \mathbb{R} \setminus [0, -1, -2, \cdots] \) [26]. Note that, \( \Gamma_q(z + 1) = [z]_q \Gamma_q(z) \). The value of q-Gamma function, \( \Gamma_q(z) \), for input values \( q \) and \( z \) with counting the number of sentences \( n \) in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order \( n \) which show in Algorithm 2. For any positive numbers \( \alpha \) and \( β \), the q-Beta function is defined by [34],

\[ B_q(\alpha, \beta) = \int_0^1 (1 - q^α)(1 - q^{β-1}) d_q s. \]

The q-derivative of function \( f \), is defined by \((D_q f)(x) = \frac{f(x+q) - f(x)}{(1-q)x}\) and \((D_q f)(0) = \lim_{x \to 0}(D_q f)(x)\) which is shown in Algorithm 3 [29]. Also, the higher order q-derivative of a function \( f \) is defined by \((D_q^n f)(x) = D_q(D_q^{n-1} f)(x)\) for all \( n \geq 1 \), and \((D_q^n f)(x) = f(x)\) [1, 29]. The q-integral of a function \( f \) defined on \([0, b]\) is defined by

\[ I_q f(x) = \int_0^x f(s)d_q s = x(1 - q) \sum_{k=0}^{n} q^k f(xq^k) \]
for \(0 \leq x \leq b\), provided the series is absolutely converges \([1, 29]\). The q-derivative of function \(f\), is defined by \((D_q f)(x) = \frac{(I_q^a - f)(x)}{(1 - q^a)}\) and \((D_q f)(0) = \lim_{x \to 0}(D_q f)(x)\) which is shown in Algorithm 3 \([1, 29]\). If \(a \in [0, b]\), then

\[
\int_a^b f(u)d_q u = I_q f(b) - I_q f(a) = (1 - q) \sum_{k=0}^{\infty} q^k \left[ b f(bq^k) - a f(aq^k) \right],
\]

whenever the series exists. The operator \(I_q^a\) is given by \((I_q^a f)(x) = h(x)\) and \((I_q^a f)(x) = (I_q^a(I_q^{a-1} h))(x)\) for \(n \geq 1\) and \(g \in C([0,b])\) \([1, 29]\). It has been proved that \((D_q(I_q f))(x) = f(x)\) and \((I_q(D_q f))(x) = f(x) - f(0)\) whenever \(f\) is continuous at \(x = 0\) \([1, 29]\). The fractional Riemann-Liouville type q-integral of the function \(f\) on \(J\) for \(a \geq 0\) is defined by \((I_q^a f)(t) = f(t)\) and

\[
(I_q^a f)(t) = \frac{1}{\Gamma_q(a)} \int_0^t (t - qs)^{a-1} f(s)d_q s,
\]

for \(t \in J\) and \(a > 0\) \([39]\). Also, the Caputo fractional q-derivative of a function \(f\) is defined by

\[
\left( {}^{(C)}D_q^a f \right)(t) = \left( I_q^{a-\alpha} (D_q^a f) \right)(t) = \frac{1}{\Gamma_q(a-\alpha)} \int_0^t (t - qs)^{a-\alpha-1} (D_q^a f)(s)d_q s,
\]

(3)

where \(t \in J\) and \(a > 0\) \([39]\). It has been proved that \((I_q^a(I_q^a f))(x) = (I_q^{a+\alpha} f)(x)\) and \((D_q^a(I_q^a f))(x) = f(x)\), where \(a, \beta \geq 0\) \([39]\). By using Algorithm 2, we can calculate \((I_q^{\alpha} f)(x)\) which is shown in Algorithm 4.

Now, we present some necessary notations. Let \(\overline{J} = [0, 1]\). We denote \(L^1(\overline{J}), C_R(\overline{J}), C_k(\overline{J})\) by \(\mathcal{L}, \mathcal{R}, \mathcal{B}\), respectively. We say that \(h\) satisfies the local Carathéodory condition on \(\overline{J} \times (0, \infty) \times (0, \infty)\) and denote it by \(\text{Car}(\overline{J} \times (0, \infty) \times (0, \infty))\) whenever has the following properties.

- C1) For all \((x_1, x_2) \in (0, \infty) \times (0, \infty), h(\cdot, x_1, x_2) : \overline{J} \to \mathbb{R}\) is measurable.
- C2) For almost all \(t \in \overline{J}, h(t, \cdot, \cdot) : (0, \infty) \times (0, \infty) \to \mathbb{R}\) is continuous.
- C3) For each compact subset \(C\) of \((0, \infty) \times (0, \infty)\) there exists a function \(\psi_C \in \mathcal{L}\) such that \(|h(t, x_1, x_2)| \leq \psi_C(t)\) for each \(t \in \overline{J}\) and all \((x_1, x_2) \in C\).

We denote the set of all bounded subsets of Banach space \(A\) by \(\mathcal{F}_A\).

**Definition 2.1.** \([40]\) The positive real-valued function \(\mu\) define on \(\mathcal{F}_A\) is measure of non-compactness whenever \(\mu(C) = 0\) if and only if \(C\) is relatively compact and satisfies the following conditions:

1. If \(C_1 \subset C_2\) then \(\mu(C_1) \leq \mu(C_2)\).
2. \(\mu(\text{Conv}(C)) = \mu(C)\).
3. \(\mu(C_1 \cup C_2) = \max\{\mu(C_1), \mu(C_2)\}\).
4. \(\mu(C_1 + C_2) \leq \mu(C_1) + \mu(C_2)\).
5. \(\mu(\lambda C) = |\lambda| \mu(C)\) for all scalar \(\lambda\).

Assume that the sets \(S_1, S_2, \ldots, S_n\) be a cover for \(C \in \mathcal{F}_A\). The Kuratowski measure of non-compactness of \(C\) is defined by

\[
K(C) = \inf_{\text{diam}(S_i) < c} c
\]

and denoted by \(K(C)\) \([40]\). Take \(K(C) = \infty, K(C) = 0\) whenever \(C\) is unbounded, \(C\) is empty set, respectively \([40]\). Also, for all \(C \in \mathcal{F}_A\), we have \(K(C) \leq \text{diam}(C)\) \([40]\). We need next results.
Lemma 2.2. [41] If \( x \in [A \cap L] \) with \( D^q_x \in [A \cap L] \), then \( I^q_x D^q_x x(t) = x(t) + \sum_{i=1}^{n} c_i t^{\alpha_i} \), where \([\alpha] \leq u < [\alpha] + 1\) and \( c_i \) is some real number.

Theorem 2.3. [40] Let a nonempty subset \( C \) of a Banach space \( A \) is bounded, closed and convex. The self-continuous operator \( \Theta \) define on \( C \) has a fixed point whenever there exists a constant \( 0 \leq \lambda < 1 \) such that \( K(\Theta(Q)) \leq rK(Q) \) for all \( Q \subset C \), where \( K \) is the Kuratowski measure of non-compactness on \( A \).

3. Main results

In this part, first we provide some lemmas.

Lemma 3.1. The solution of the problem \( D^q_x u(t) + v(t) = 0 \) for \( \alpha \geq 3 \) and \( g \in J \), under boundary conditions

\[
    u(0) = u^{(2)}(0) = \cdots = u^{(n-1)}(0) = 0
\]

and \( u(1) = [I^q_x(v(t)u(t))]_{t=1} \) is \( u(t) = \int_0^1 G(t,qs)v(s) \, ds \), where \( v, w \in [L] \), \( \gamma \geq 1 \) and

\[
G(t,qs) = a_1(t,s,\alpha) + \frac{t}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma-1)}w(t) a_1(t,s,\alpha) \, ds, \tag{4}
\]

whenever \( t \leq s \),

\[
G(t,qs) = a_2(t,s,\alpha) + \frac{t}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma-1)}w(t) a_2(t,s,\alpha) \, ds, \tag{5}
\]

whenever \( s \leq t \) for \( s, t \in J \), here \( \mu(\gamma) = \Gamma_q(\gamma) - \int_0^1 t(1 - qt)^{(\gamma-1)}w(t) \, dt \), and

\[
a_1(t,s,\alpha) = \frac{t(1 - qs)^{\alpha-1}}{\Gamma_q(\alpha)}, \tag{6}
\]

\[
a_2(t,s,\alpha) = \frac{t(1 - qs)^{\alpha-1} - (t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}. \tag{7}
\]

Proof. First, note that Lemma 2.2 implies \( u(t) = -I^q_x v(t) + \sum_{i=1}^{n} c_i t^i \), for some real constants \( c_i \). Also, By using the condition \( u(0) = u^{(2)}(0) = 0 \) for \( i \geq 2 \), we obtain \( c_i = 0 \) for \( 0 \leq i \leq n \). Thus, \( u(t) = -I^q_x v(t) + c_1 t \). Since

\[
\left[ I^q_x (v(t)u(t)) \right]_{t=1} = \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qs)^{(\gamma-1)}w(s) \, ds,
\]

by using the boundary condition at \( t = 1 \) we have \( -I^q_x v(1) + c_1 = I^q_x w(1) \). Therefore \( c_1 = I^q_x v(1) + I^q_x w(1) \). Hence \( u(t) = -I^q_x v(t) + I^q_x v(1) + I^q_x w(1) \) and so \( u(t) \) is equal to

\[
\int_0^1 a_1(t,s,\alpha)v(s) \, ds + t \left[ I^q_x (v(t)u(t)) \right]_{t=1}
\]

and

\[
\int_0^1 a_2(t,s,\alpha)v(s) \, ds + t \left[ I^q_x (v(t)u(t)) \right]_{t=1},
\]
when \( t \leq s \) and \( s \leq t \), respectively. This implies that
\[
\left[ I_q^t(w(t)u(t)) \right]_{i=1} = \frac{1}{\Gamma_q(\gamma)} \int_0^1 \left( \int_0^{s_i} (1 - qt)^{(\gamma-1)}w(t)a_i(t, s, \alpha)\varphi(s)ds \right) dt.
\]

for \( t \leq s, s \leq t \), respectively. On the other hand
\[
\left[ I_q^t(w(t)u(t)) \right]_{i=1} = \int_0^1 \left[ I_q^t(w(t)u(t)) \right]_{i=1} dt,
\]
then we have
\[
\int_0^1 \left(1 - \frac{1}{\Gamma_q(\gamma)}(1 - qt)^{(\gamma-1)}tw(t)\right) \left[ I_q^t(w(t)u(t)) \right]_{i=1} dt = I_q^t(w(1) \int_0^1 a_1(t, s, \alpha)\varphi(s)ds),
\]
for \( t \leq s, s \leq t \), respectively. Hence,
\[
\left[ I_q^t(w(t)u(t)) \right]_{i=1} = \frac{1}{\Gamma_q(\gamma)} \int_0^1 \left(1 - qt\right)^{(\gamma-1)}tw(t) dt = I_q^t(w(1) \int_0^1 a_2(t, s, \alpha)\varphi(s)ds),
\]
whenever \( t \leq s, s \leq t \), respectively, and so
\[
\left[ I_q^t(w(t)u(t)) \right]_{i=1} = \frac{1}{\Gamma_q(\gamma)} \int_0^1 \left(1 - qt\right)^{(\gamma-1)}tw(t) dt = I_q^t(w(1) \int_0^1 a_2(t, s, \alpha)\varphi(s)ds),
\]
for \( t \leq s, s \leq t \), respectively. This implies that \( u(t) \) is equal to
\[
\int_0^t a_1(t, s, \alpha)\varphi(s)ds + \frac{t}{\Gamma_q(\gamma)} \left[ I_q^t(w(1) \int_0^1 a_1(t, s, \alpha)\varphi(s)ds) \right],
\]
\[
\int_0^t a_2(t, s, \alpha)\varphi(s)ds + \frac{t}{\Gamma_q(\gamma)} \left[ I_q^t(w(1) \int_0^1 a_2(t, s, \alpha)\varphi(s)ds) \right],
\]
for \( t \leq s, s \leq t \), respectively, which are same as (4) and (5), respectively. So the proof is complete. \( \square \)
By employing simple calculations for \( G(t, qs) \) in (4) and (5), we conclude that
\[
G(t, qs) \in \left[ 0, \frac{(1 - qs)^{(\alpha - 1)}}{G_q(\alpha - 1)} \left( 1 + \frac{1}{\mu(q)} \int_0^1 (1 - qt)^{(\gamma - 1)} w(t) \, dt \right) \right],
\]
for all \( t, s \in \mathcal{J} \). At present, for \( n \geq 1 \), consider the map \( g_{i,n}(t, u, v) = g_i(t, \chi_n(u), \chi_n(v)) \), where
\[
\chi_n(x) = \left\{ \begin{array}{ll}
 x, & x \geq \frac{1}{n}, \\
 \frac{1}{n}, & x < \frac{1}{n}.
\end{array} \right.
\]
Here, we first investigate the regular system
\[
\begin{cases}
 D^\alpha_q u + g_{1,n}(t, u, v) = 0, \\
 D^\alpha_q u + g_{2,n}(t, u, v) = 0,
\end{cases}
\]
under some conditions in the problem (1). For \( i = 1, 2 \) and each \( n \) belongs to \( \mathbb{N} \), define the function
\[
F_{n,i}(u, v)(t) = \int_0^1 G_{n,i}(t, qs) g_{a,i} (s, u(s), v(s)) \, ds,
\]
where \( G_{n,i}(t, qs) \) is the \( q \)-Green function in Lemma 3.1 which replaced \( \alpha \) and \( \gamma \) by \( \alpha_i \) and \( \gamma_i \), respectively. Also, we take \( \Theta_{n}(u, v)(t) = (F_{n,1}(u, v)(t), F_{n,2}(u, v)(t)) \) and
\[
\|\Theta_{n}(u, v)(t)\| = \max \{ F_{n,1}(u, v)(t), F_{n,2}(u, v)(t) \}.
\]
Since \( g_1 \) and \( g_2 \) belong to \( C(\mathcal{J} \times \mathbb{R}^2) \), by simple review we conclude that \( g_{n,1}, g_{n,2} \in C(\mathcal{J} \times \mathbb{R}^2) \) and so there exist \( \psi_1 \) and \( \psi_2 \in \mathcal{K} \) such that \( |g_{n,i}(t, u(t), v(t))| \leq \psi_i(t) \) for \( n \in \mathbb{N} \), \( t \) belongs to \( \mathcal{J} \) and \( i = 1, 2 \). We denote the set of all \( (u, v) \in \mathcal{K}^2 \) such that \( ||(u, v)|| \leq ||\psi_i||_\infty \) by \( \mathcal{D} \), where \( ||\psi||_\infty = \max \{ ||\psi_1||_\infty, ||\psi_2||_\infty \} \). One can check that, \( \mathcal{D} \) is closed, bounded and convex.

**Lemma 3.2.** Let \( n \in \mathbb{N} \). For each bounded subset of \( C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R}) \), the self-map \( \Theta_{n} \) defined on \( \mathcal{D} \) is equi-continuous.

**Proof.** Assume that \( (u, v) \in \mathcal{D} \) be given, \( i = 1, 2 \) and \( n \geq 1 \). We can see that,
\[
F_{n,i}(u, v)(t) \leq \int_0^1 \left( \frac{(1 - qs)^{(-\alpha_i) + 1}}{G_q(\alpha_i - 1)} \right) \left[ 1 + \frac{1}{\mu(q)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} w(t) \, dt \right] g_{a,i} (s, u(s), v(s)) \, ds.
\]
Thus,
\[
F_{n,i}(u, v)(t) \leq \int_0^1 \left( \frac{(1 - qs)^{(-\alpha_i) + 1}}{G_q(\alpha_i - 1)} \right) \left[ 1 + \frac{1}{\mu(q)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} w(t) \, dt \right] \varphi_i (s) \, ds.
\]
(7)
On the other hand, \( \left[ \frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} t \, dt \right] \in [0, \frac{1}{2}] \), then \( \frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} t \, dt \leq \frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} w(t) \, dt \). Also, we get
\[
\frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} t w(t) \, dt \leq \frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} w(t) \, dt.
\]
Therefore,
\[
\frac{1}{G_q(\gamma_i)} \int_0^1 (1 - qt)^{(\gamma_i - 1)} t w(t) \, dt \in \left[ 0, \frac{1}{2} \right].
\]
and so \(1 - \frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq \, t \in [0, \frac{1}{2})\). Indeed,

\[
\frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq = \frac{1}{\Gamma(\gamma) - \int_0^1 (1 - qt)^{(\gamma - 1)} \, dq \, t} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq t
\]

= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq t

= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq t

\in [0, 1)

and so \(1 + \frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w_i(t) \, dq \, t \leq 2\). By applying the previous inequality and (7), we obtain

\[
F_n(u, v)(t) \leq \frac{2}{\Gamma(\gamma) - \int_0^1 (1 - qs)^{(\alpha - 2)} \, dq \, s} \int_0^1 (1 - qs)^{(\alpha - 2)} d_q s
\]

\leq \frac{2\|\Phi\|_\infty}{\Gamma(\gamma) - \int_0^1 (1 - qs)^{(\alpha - 2)} d_q s}

= \frac{2}{\Gamma(\gamma)} \|\Phi\|_\infty \leq \|\Phi\|_\infty \leq \|\Phi\|_\infty

and so \(\|\Theta_n(u, v)\|_\infty \leq \|\Phi\|_\infty\). Hence, \(\Theta_n\) maps \(\mathcal{D}\) into \(\mathcal{D}\). Assume that \(B \subset C(\bar{J}, \mathbb{R}) \times C(\bar{J}, \mathbb{R})\) is bounded. Also, let \((u_n, v_n)_{n=1}^\infty\) be a bounded sequence in \(B\) and \(t_1, t_2 \in \bar{J}\) with \(t_1 < t_2\). Then, we have

\[
|F_{i,n}(u_k, v_k)(t_2) - F_{i,n}(u_k, v_k)(t_1)|
\]

\[
\leq \frac{1}{\gamma(t_1)} \left[ \int_0^{t_1} \left[ (t_2 - qs)^{(\alpha - 1)} - (t_1 - qs)^{(\alpha - 1)} \right] g_{n,i}(s, u_k(s), v_k(s)) \, dq \, s
\]

\[
+ \int_0^{t_1} \left[ (t_2 - qs)^{(\alpha - 1)} g_{a,i}(s, u_k(s), v_k(s)) \, dq \, s
\]

\[
+ (t_2 - t_1) \int_0^{t_1} \left[ (1 - qs)^{(\alpha - 1)} \right] g_{a,i}(s, u_k(s), v_k(s)) \, dq \, s
\]

\[
\leq \frac{1}{\gamma(t_1)} \left[ \int_0^{t_1} \left[ (t_2 - qs)^{(\alpha - 1)} - (t_1 - qs)^{(\alpha - 1)} \right] \, dq \, s
\]

\[
+ (t_2 - t_1) \int_0^{t_1} \left[ (1 - qs)^{(\alpha - 1)} \right] \, dq \, s
\]

\[
\times \left[ \frac{1}{\gamma(t_1)} + \frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} \, dq \, t \right]
\]

where for \(i = 1, 2\), \(G_{1,i}(t, s)\) is equal to \(a_1(t, s, \alpha), a_2(t, s, \alpha)\) whenever \(t \leq s \leq t\), respectively which is obtained by replacing \(a_i\) by \(a\) in (4) and (5), respectively, and \(G_{2,i}(s)\) is equal to

\[
\frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} \, dq \, t
\]

for \(t \leq s \leq t\), respectively. Let \(e \in \mathcal{F}\) be given, \(t_1, t_2 \in \bar{J}\) such that \(t_1 < t_2\) and \(s \in [0, t_1]\). We choose \(\delta > 0\) such that \(t_1 - t_2 < \delta\) implies \((t_2 - s)^{(\alpha - 1)} - (t_1 - s)^{(\alpha - 1)} < e\). Also, suppose that \(k \in [1, \infty)\) and \(0 \leq t_1 < t_2 \leq 1\) with \(t_1 - t_2 < \min(\delta, e)\) be given. Then we get

\[
|F_{i,n}(u_k, v_k)(t_2) - F_{i,n}(u_k, v_k)(t_1)| \leq e\|\Phi\|_1 \left( \frac{3}{\gamma(t_1)} + \frac{1}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} \, dq \, t \right)
\]
and so \( \lim_{t_n \to t_0} \|\Theta_n(u_k, v_k)(t_2) - \Theta_n(u_k, v_k)(t_1)\| = 0 \). Also, we have

\[
\|\Theta_n(u_k, v_k)(t)\| \leq \max \left\{ \int_0^1 (1 - qs)^{(\alpha - 1)} \frac{1}{1 + G_2(s)} 1 + G_2(s) \phi(s) d_\phi s, \right. \\
\left. \int_0^1 (1 - qs)^{(\alpha - 1)} \frac{1}{1 + G_2(s)} \rho_2(s) d_\rho s \right\}
\]

\[
\leq \max \left\{ \frac{\|\phi\|_1}{\Gamma(\alpha - 1)} \left( 1 + \frac{1}{\mu(\gamma_1)} \right) \int_0^1 (1 - qt)^{\gamma_1 - 1} w_1(t) d_\phi t \right\},
\]

\[
\frac{\|\rho\|_1}{\Gamma(\alpha - 1)} \left( 1 + \frac{1}{\mu(\gamma_2)} \right) \int_0^1 (1 - qt)^{\gamma_2 - 1} w_2(t) d_\rho t.
\]

Let \( (u_k, v_k) \) be sequence in \( B \) and \( (u_k, v_k) \to (u, v) \). Hence, \( u_k \to u, v_k \to v \). Note that,

\[
\|\Theta_n(u_k, v_k)(t) - \Theta_n(u, v)(t)\| \leq \max \left\{ \int_0^1 G_{1,n}(t, qs) |g_{1,n}(s, u_k(s), v_k(s)) - g_{1,n}(s, u(s), v(s))| d_\phi s, \right. \\
\left. \int_0^1 G_{2,n}(t, qs) |g_{2,n}(s, u_k(s), v_k(s)) - g_{2,n}(s, u(s), v(s))| d_\rho s \right\}
\]

\[
\leq 2\|\phi\| \left( 1 + \frac{\Lambda_M}{\Gamma(\alpha - 1)} \right) \int_0^1 (1 - qt)^{\gamma_1 - 1} w_1(t) d_\phi t.
\]

where \( \alpha_m = \min\{\alpha_1, \alpha_2\} \) and

\[
\Lambda_M = \max_{i=1,2} \left\{ \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma_1 - 1} w_i(t) d_\phi t \right\}.
\]

Since for \( i = 1, 2 \), \( |g_{1,n}(s, u(s), v(s)) - g_{2,n}(s, u(s), v(s))| \to 0 \) and by employing the theorem of Lebesgue dominated convergence, we conclude that \( \Theta_n \) is equi-continuous on \( B \) for each \( n \in \mathbb{N} \).

**Theorem 3.3.** Assume that \( g_1, g_2 \in \text{Car} \left( \mathcal{I} \times (0, \infty)^2 \right) \) and for \( n \geq 3 \), \( \alpha_1, \alpha_2 \in (n, n + 1) \). Then for each \( n \geq 1 \) the system

\[
\begin{align*}
D^\gamma_q u + g_{1,n}(t, u, v) &= 0, \\
D^\gamma_q v + g_{2,n}(t, u, v) &= 0
\end{align*}
\]

(8)

under conditions \( u(0) = v(0) = 0 \), \( u^{(i)}(0) = v^{(i)}(0) = 0 \) for \( i = 2, \ldots, n - 1 \), \( u(1) = [I_q^{\gamma_1}(w_1(t)u(t))]_{t=1} \) and \( v(1) = [I_q^{\gamma_2}(w_2(t)v(t))]_{t=1} \) has a solution, whenever the following assumptions hold.

1) There exist \( \gamma_1, \gamma_2 \geq 1 \) and nonnegative functions \( w_1, w_2 \in \mathcal{L} \) such that

\[
[I_q^{\gamma_1}(w_1(t))]_{t=1}, [I_q^{\gamma_2}(w_2(t))]_{t=1} \in \left[ 0, \frac{1}{2} \right].
\]

2) There exist \( h_1, h_2 \in \mathcal{L} \) such that \( 2\|h_i\|_1 < \Gamma(\alpha_i - 1) \) for almost all \( t \in \mathcal{I} \) and \( i = 1, 2 \).

3) For any bounded subset \( Q \) of \( \overline{\mathcal{A}}^2 \), \( K(g_i(t, Q)) \leq h_i(t)K(Q) \) for \( i = 1, 2 \).

**Proof.** Let \( Q \) be a bounded subset of \( \overline{\mathcal{A}}^2 \) for \( n \geq 1 \) and \( i = 1 \) or 2. We choose bounded subsets \( A \) and \( B \) of \( \overline{\mathcal{A}} \) such that \( Q = (A, B) \). We take the sets \( A_1 \) and \( B_1 \) of all \( u \in A \) and \( u \in B \), respectively, such that \( u \geq \frac{1}{n} \). Then,
we get
\[ K(g_{\alpha,n}(t, Q)) = K(g_{\alpha,n}(t, A, B)) = K(g(t, \chi_{n}(A), \chi_{n}(B))) \leq K(\chi_{n}(A), \chi_{n}(B)) \]
\[ = K\left(A_1 \cup \left( \frac{1}{n} \right), B_1 \cup \left( \frac{1}{n} \right) \right) \]
\[ = K\left((A_1, B_1) \cup \left( \frac{1}{n} \right), A_1 \cup \left( \frac{1}{n} \right) \right) \]
\[ = \max \left\{ K(A_1, B_1), K\left(A_1, \frac{1}{n} \right), K\left(B_1, \frac{1}{n} \right) \right\}. \]

Let \( K(B_1) = d \). Then there exist \( C_1 \subset \overline{A} \) and \( m \in \mathbb{N} \) such that \( B_1 \subset \bigcup_{m=1}^{m} C_i \) and \( \text{diam} (C_i) < d \). Hence, \( \left( \frac{1}{n}, B_1 \right) \subset \bigcup_{m=1}^{m} \left( \frac{1}{n}, C_i \right) \).
\[ \text{diam} \left( \frac{1}{n}, C_i \right) = \sup_{x,y \in C_i} \left\| \left( \frac{1}{n}, x \right) - \left( \frac{1}{n}, y \right) \right\|_s = \sup_{x,y \in C_i} \|x - y\| = \text{diam} (C_i), \]
and \( K\left( \frac{1}{n}, B_1 \right) \leq K(B_1) \). By employing a similar technique, we will have \( K(B_1) \leq K\left( \frac{1}{n}, B_1 \right) \). Thus, \( K(B_1) = K\left( \frac{1}{n}, B_1 \right) \) and \( K(A_1) = K(A_1, \frac{1}{n}) \). Hence, there exist \( m_0 \geq 1 \) and \( (Y_i, Z_i) \subset \overline{A}^2 \) such that \( (A_1, S(B_1) \subset \bigcup_{m=1}^{m_0} (Y_i, Z_i) \) and \( \text{diam} (Y_i, Z_i) \leq d_0 \) whenever \( K(A_1, B_1) = d_0 \). This implies that
\[ \sup \| (y,z) - (y', z') \| : (y,z), (y', z') \in (Y_i, Z_i) \| \leq d_0 \]
and so
\[ \sup \{ \max \|y - y'\|, |z - z'| : y, y' \in Y_i, z, z' \in Z_i \} \leq d_0. \]
Hence, \( \sup_{Y_i \in Y_i} |y - y'| \leq d_0 \) and
\[ \sup_{z, z' \in Z_i} |z - z'| \leq d_0. \]
Thus, \( A_1 \subset \bigcup_{i=1}^{m_0} Y_i \) with \( \text{diam} (Y_i) \leq d_0 \) and \( B_1 \subset \bigcup_{i=1}^{m_0} Z_i \) with \( \text{diam} (Z_i) \leq d_0 \) for each \( i \). Indeed, \( K(A_1) \leq K(A_1, B_1) \) and \( K(B_1) \leq K(A_1, B_1) \). Hence,
\[ \max \left\{ K(A_1, B_1), K\left(A_1, \frac{1}{n} \right), K\left(B_1, \frac{1}{n} \right) \right\} = K(A_1, B_1) \]
and so for \( i = 1, 2 \), we get \( K(g_{\alpha,n}(t, Q)) \leq h_i(t)K(A_1, B_1) \leq h_i(t)K(Q) \). As well, we obtain
\[ K(\Theta_n(Q)) = K\left( \int_0^1 G_{\alpha}(t, q)s g_{1,n}(s, Q) d_q s, \int_0^1 G_{\alpha}(t, q)s g_{2,n}(s, Q) d_q s \right). \]
For each \( s \in J, n \geq 1 \) and \( i = 1, 2 \), we take \( d_i(s) := K(g_{\alpha,n}(s, Q)) \leq h_i(s)K(Q) \). Choose \( k_0 \in \mathbb{N} \) and bounded subsets \( X_{ij} \) of \( \overline{A}^2 \) for \( i = 1, 2 \) somehow that \( g_{\alpha,n}(s, Q) \subset \bigcup_{j=1}^{k_0} X_{ij} \). Then, we have \( \text{diam} (h_{ij}) \leq d_i(s) \leq h_i(s)K(Q) \) and
\[ G_{\alpha}(t, q)s g_{\alpha,n}(s, Q) \subset \int_0^1 \int_{\gamma(n)}^1 \frac{1}{\Gamma(n)} \left( 1 + \frac{1}{\mu(\gamma)} \right) \int_0^1 (1 - qt)^{y-1} w_i(t) d_q t \right) X_{ij} d_q s \]
\[ = \int_{j=1}^{k_0} \int_0^1 \frac{1}{\Gamma(n)} \left( 1 + \frac{1}{\mu(\gamma)} \right) \int_0^1 (1 - qt)^{y-1} w_i(t) d_q t \right) X_{ij} d_q s \]
for \( i = 1, 2 \), here
\[ \int_0^1 \frac{1}{\Gamma(n)} \left( 1 + \frac{1}{\mu(\gamma)} \right) \int_0^1 (1 - qt)^{y-1} w_i(t) d_q t \right) X_{ij} d_q s \]
is the set of all
\[
\int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) \alpha(s) d_s
\]
where \( x \in X_{i,j} \). Thus,
\[
\text{diam}\left( \int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) X_{i,j} d_s \right)
\]
\[
= \sup_{x, x' \in X_{i,j}} \left| \int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) x(s) d_s - x'(s) d_s \right|
\]
\[
= \sup_{x, x' \in X_{i,j}} \int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) \text{diam}(X_{i,j}) d_s
\]
and so
\[
K\left( \int_0^1 G_{q, t, q, s}(s, Q) d_s \right)
\]
\[
\leq \int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) K(g_{i, s}(s, Q)) d_s
\]
\[
\leq \int_0^1 (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) h_i(s) K(Q) d_s
\]
\[
\leq K(Q) \left\| (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) \right\|_\infty \| h_i \|_1.
\]
By simple review, we can conclude that
\[
\lambda_i = \left\| (1 - qs)^{a_i} \left( \frac{1}{\Gamma_q(a_i - 1)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1 - qt)^{\gamma - 1} w_i(t) d_q t \right) \right\|_\infty \| h_i \|_1 \in [0, 1)
\]
for \( i = 1, 2 \). So, by applying last result, we obtain
\[
\max_{i=1,2} \left\{ K\left( \int_0^1 G_{q, t, q, s}(s, Q) d_s \right) \right\} \leq \lambda K(Q),
\]
here \( \lambda = \max(\lambda_1, \lambda_2) \). At present, consider the space \( \mathcal{A}^2 \) endowed with norm
\[
\|(\cdot, \cdot)\|_\infty \|(y_1, y_2)\|_\infty = \max\|y_1\|_\infty, \|y_2\|_\infty.
\]
It is proved in first part, if \( Y \) and \( Y' \subset \mathcal{A}^2 \) then \( K(Y), K(Y') \leq K(Y, Y') \), where \( Y, Y' \) are bounded sets. We know that \( (\mathcal{A}^2, \|(\cdot, \cdot)\|_\infty) \) is a Banach space. Suppose that \( K(Y), K(Y') \) are equal to \( r, r' \), respectively and \( r := \max(r, r') \). We choose \( n, n' \geq 1 \) such that \( Y \subset \bigcup_{i=1}^n Z_i \) and \( Y' \subset \bigcup_{j=1}^{n'} Z'_j \), where \( Z_i, Z'_j \subset \mathcal{A}^2 \), diam \( (Z_i) \leq r \)
and \( \text{diam} (Z_i) < r' \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, n' \). Let \( n \geq n' \). Put \( Z_{n+1} = Z_{n+2} = \cdots = Z_n := Z_n \). Then, \((Y, Y') \subset \bigcup_{i=1}^{n} (Z_i, Z_i')\) and for each \( i = 1, \ldots, n \), we have

\[
\text{diam} (Z_i, Z_i') = \sup_{z_1, z_2 \in Z_i, z_1', z_2' \in Z_i'} \| (z_1, z_1') - (z_2, z_2') \|_*
\]

\[
= \sup_{z_1, z_2 \in Z_i, z_1', z_2' \in Z_i'} \| (z_1 - z_1', z_2 - z_2') \|_*
\]

\[
= \sup_{z_1, z_2 \in Z_i, z_1', z_2' \in Z_i'} \{ \max \{ \| (z_1 - z_2) \|_*, \| (z_1' - z_2') \|_* \} \}
\]

\[
\leq \max \{ r, r' \} = r.
\]

Hence, \( K(Y, Y') \leq \max \{ K(Y), K(Y') \} \) and so \( K(Y, Y') = \max \{ K(Y), K(Y') \} \). Thus,

\[
K(\Theta_n(Q)) = K \left( \int_0^1 G_{n,1}(t, q_s) g_{1,n}(s, Q) d_q s, \int_0^1 G_{n,2}(t, q_s) g_{2,n}(s, Q) d_q s \right)
\]

\[
= \max_{i=1,2} \left\{ \int_0^1 G_{n,1}(t, q_s) g_{1,n}(s, Q) d_q s \right\}
\]

\[
\leq \lambda K(Q).
\]

Therefore, by using the Darbo’s fixed point theorem, \( \Theta_n \) has a fixed point in \( D \) for all \( n \). This implies that the system has a solution \((u_n, v_n) \in D\), that is,

\[
u_n(t) = \int_0^1 G_{n,1}(t, q_s) g_{1,n}(s, u_n(s), v_n(s)) d_q s, \quad v_n(t) = \int_0^1 G_{n,2}(t, q_s) g_{2,n}(s, u_n(s), v_n(s)) d_q s.
\]

Then the proof is complete. □

Now, we provide result for the singular system.

**Theorem 3.4.** Let \( g_1, g_2 \in \text{Car} (\bar{J} \times (0, \infty)^2), \alpha_1, \alpha_2 \in (n, n + 1] \) with \( n \geq 3 \). Then the singular system

\[
\begin{cases}
D^q_{\alpha_1} u + g_1(t, u, v) = 0 \\
D^q_{\alpha_2} v + g_2(t, u, v) = 0
\end{cases}
\]

(9)

with boundary conditions \( u(0) = v(0) = 0, u(t)(0) = v(t)(0) = 0 \) for \( i = 2, \ldots, n - 1 \), \( u(1) = [I^\gamma_q (w_1(t)u(t))]_{t=1} \) and \( v(1) = [I^\gamma_q (w_2(t)v(t))]_{t=1} \) has a solution, whenever the following assumptions hold.

1) There exist \( \gamma_1, \gamma_2 \geq 1 \) and non-negative functions \( w_1, w_2 \in \bar{I} \) such that

\[
[I^\gamma_q (w_1(t))]_{t=1}, [I^\gamma_q (w_2(t))]_{t=1} \in [0, \frac{1}{2}].
\]

2) There exist \( h_1, h_2 \in \bar{I} \) such that \( 2 \| h_i \| < \Gamma_q (\alpha_i - 1) \) for each \( t \in \bar{J} \) and \( i = 1, 2 \).

3) For any bounded subset \( Q \) of \( \bar{I}^2 \), \( K(g_i(t, Q)) \leq h_i(t) K(Q) \) where \( i = 1, 2 \) and \( K \) is the Kuratowski measure of non-compactness.

Proof. By applying Theorem 3.3, we conclude that the problem (1) has a solution \((u_n, v_n) \in D\) for all \( n \). Also, there is \((u, v) \in D\) such that \( \lim_{n \to \infty} (u_n, v_n) = (u, v) \), because \( D \) is closed. By simple check, we conclude that \((u, v)\) satisfies the boundary condition of the problem (1). On the other hand, we observe that \( \lim_{n \to \infty} g_{1,n}(t, u_n(t), v_n(t)) = g_1(t, u(t), v(t)) \) for almost all \( t \in \bar{J} \) and \( i = 1, 2 \). Thus, we obtain

\[
G_{n,1}(t, q_s) g_{1,n}(s, u_n(s), v_n(s)) \leq \frac{1}{\Gamma (\alpha_1 - 1)} \left( \frac{1}{\mu (\gamma_1)} \int_0^1 (1 - qt)^{\gamma_1 - 1} w_1(t) d_q t \right) g_1(s),
\]
for $i = 1, 2$, each $n$ and all $(t, s) \in \mathbb{I}^2$. Now, by applying the Lebesgue dominated convergence theorem, we get

$$
u(t) = \int_0^1 G_{\alpha, t}(1, q) g_{1, v}(s, u(s), v(s)) \, ds, \quad \nu(t) = \int_0^1 G_{\alpha, t}(1, q) g_{2, u}(s, u(s), v(s)) \, ds.$$

This implies that, $(u, v)$ is a solution for the problem (1). \hfill \Box

### 4. Example illustrative for the problem with algorithms

Here, we provide an example to illustrate our main result. In this way, we give a computational technique for checking the problem (1) in Theorem 3.4. We need to present a simplified analysis could be executed values of the q-Gamma function. To this aim, we consider a pseudo-code description of the method for calculation of the q-Gamma function of order $n$ in Algorithm 2 (for more details, see the link [https://en.wikipedia.org/wiki/Q-gamma_function](https://en.wikipedia.org/wiki/Q-gamma_function)).

Table 1 shows that when $q$ is constant, the q-Gamma function is an increasing function. Also, for smaller values of $x$, an approximate result is obtained with less values of $n$. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values $q$ near to one is obtained with more values of $n$ in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but $x$ values increase in 3. Similarly, the q-Gamma function for values $q$ near to one is obtained with more values of $n$ in comparison with other columns. Furthermore, we provided algorithms 3 and 5 which calculated $D_q^x f(x)$ and $I_q^x f(x)$, respectively.

**Example 4.1.** We define the singular fractional system similar to the problem (1) by

$$
\begin{align*}
\begin{cases}
D_q^x u(t) + \frac{1}{3 \sqrt{t}} (\frac{1}{2} u(t) + \frac{1}{3} v(t)) = 0, \\
D_q^x u(t) + \frac{3}{10 \sqrt{t}} (\frac{1}{2} u(t) + \frac{3}{2} v(t)) = 0,
\end{cases}
\end{align*}
$$

under boundary conditions $u(0) = v(0) = u'(0) = v'(0) = u''(0) = v''(0) = 0$ and

$$u(1) = \left[ I_q^0 (tu(t)) \right]_{i=1}, \quad v(1) = \left[ I_q^0 (t^2 v(t)) \right]_{i=1}. $$

By comparison with problem (1), we can consider the maps

$$
g_1(t, u, v) = \frac{1}{5 \sqrt{t}} \left( \frac{1}{2} u + \frac{1}{3} v \right),
$$

$$
g_2(t, u, v) = \frac{3}{10 \sqrt{t}} \left( \frac{1}{2} u + \frac{3}{2} v \right).$$

Also, by definition of functions $g_1$ and $g_2$, we consider $h_1(t) = \frac{1}{5 \sqrt{t}},$ $h_2(t) = \frac{3}{10 \sqrt{t}},$ $x(u, v) = \frac{1}{2} u + \frac{1}{3} v$ and $y(u, v) = \frac{1}{2} u + \frac{3}{2} v.$ Put $a_1 = \frac{1}{2},$ $a_2 = \frac{3}{2},$ $\gamma_1 = \frac{1}{2},$ $\gamma_2 = \frac{3}{2},$ $w_1(t) = t,$ $w_2(t) = \sqrt{t}.$ It can be seen that $g_1, g_2 \in Car (\mathbb{I} \times (0, \infty)^2), h_1, h_2 \in \mathcal{L}$ are non-negative and $w_1, w_2 \in \mathcal{L}.$ Also, we have

$$
\begin{align*}
I_q^0 (\gamma_1 (w_1(t)))_{i=1} = \left[ I_q^0 \left( t \right) \right]_{i=1} = \frac{1}{\Gamma_q \left( \frac{1}{2} \right)} \int_0^1 (1 - qs)^{(\frac{1}{2}))} s \, ds = \frac{1}{\Gamma_q \left( \frac{1}{2} \right)} \frac{\Gamma_q \left( 2 \right) \Gamma_q \left( 2 + \frac{1}{2} \right)}{\Gamma_q \left( 2 + \frac{3}{2} \right)} \in \left[ 0, \frac{1}{2} \right),
\end{align*}
$$

$$
\begin{align*}
I_q^0 (\gamma_2 (w_2(t)))_{i=1} = \left[ I_q^0 (\sqrt{t}) \right]_{i=1} = \frac{1}{\Gamma_q \left( \frac{1}{2} \right)} \int_0^1 (1 - qs)^{(\frac{1}{2}))} \sqrt{s} \, ds = \frac{1}{\Gamma_q \left( \frac{1}{2} \right)} \frac{\Gamma_q \left( \frac{3}{2} \right) \Gamma_q \left( 2 + \frac{1}{2} \right)}{\Gamma_q \left( 2 + \frac{3}{2} \right)} \in \left[ 0, \frac{1}{2} \right)
\end{align*}
$$

(11)
and

\[ \|h_1\|_1 = \int_0^1 \frac{1}{5} t^{0.4} dt = 0.4 < \frac{1}{2} \Gamma(\frac{7}{2} - 1) = \frac{1}{2} \Gamma(\alpha_1 - 1), \]

\[ \|h_2\|_1 = \int_0^1 \frac{3}{10} t^{0.18} dt = 0.18 < \frac{1}{2} \Gamma(\frac{10}{3} - 1) = \frac{1}{2} \Gamma(\alpha_2 - 1). \]  

(12)

Tables 4 and 5 show the values of \( I_1^{\gamma_1}(w_1(t)) \) and \( I_2^{\gamma_2}(w_2(t)) \) in inequalities (11) for some different values of \( q \), respectively. Also, we get

\[ K(x(Q)) = K(x((M, N))) = K\left( \frac{1}{2} M + \frac{1}{3} N \right) = \max\{K(M), K(N)\} \left( \frac{5}{6} \right) = K(Q) \left( \frac{5}{6} \right) \leq K(Q). \]

for each \( Q = (M, N) \subset \mathbb{R}^2 \). Since \( g_1(t, u, v) = h(t)x(u, v) \), we conclude that

\[ K(g_1(t, Q)) = K(h_1(t)x(Q)) = h_1(t)K(x(Q)) \leq h_1(t)K(Q). \]

Therefore, by employing a similar technique, we have

\[ K(g_2(t, Q)) = K(h_2(t)y(Q)) = h_1(t)K(y(Q)) \leq h_1(t)K(Q). \]

Theorem 3.4 implies that the system (1) has a solution.

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Algorithm 1 The proposed method for calculated \((a - b)^\alpha_q\)

**Input:** \(a, b, \alpha, n, q\)
1: \(s \leftarrow 1\)
2: if \(n = 0\) then
3: \(p \leftarrow 1\)
4: else
5: for \(k = 0\) to \(n\) do
6: \(s \leftarrow s \cdot (a - b \cdot s^k)/(a - b \cdot q^\alpha)\)
7: end for
8: \(p \leftarrow a^{\alpha} \cdot s\)
9: end if

**Output:** \((a - b)^\alpha_q\)

Algorithm 2 The proposed method for calculated \(\Gamma_q(x)\)

**Input:** \(n, q \in (0, 1), x \in \mathbb{R}\backslash\{0, -1, 2, \cdots\}\)
1: \(p \leftarrow 1\)
2: for \(k = 0\) to \(n\) do
3: \(p \leftarrow p(1 - q^\alpha)(1 - q^k)\)
4: end for
5: \(\Gamma_q(x) \leftarrow p/(1 - q)^{n+1}\)

**Output:** \(\Gamma_q(x)\)

Algorithm 3 The proposed method for calculated \((D_q f)(x)\)

**Input:** \(q \in (0, 1), f(x), x\)
1: syms \(z\)
2: if \(x = 0\) then
3: \(g \leftarrow \lim((f(z) - f(q \cdot z))/((1 - q)z), z, 0)\)
4: else
5: \(g \leftarrow (f(x) - f(q \cdot x))/((1 - q)x)\)
6: end if

**Output:** \((D_q f)(x)\)

Table 1: Some numerical results for calculation of \(\Gamma_q(s)\) with \(q = \frac{1}{2}\) that is constant, \(x = 4.5, 8.4, 12.7\) and \(n = 1, 2, \ldots, 15\) of Algorithm 2.

| \(n\) | \(x = 4.5\)   | \(x = 8.4\)   | \(x = 12.7\)  |
|------|---------------|---------------|---------------|
| 1    | 2.472950      | 11.909360     | 68.080769     |
| 2    | 2.383247      | 11.468397     | 65.559266     |
| 3    | 2.354446      | 11.326853     | 64.749894     |
| 4    | 2.344963      | 11.280255     | 64.483434     |
| 5    | 2.341815      | 11.264786     | 64.394880     |
| 6    | 2.340767      | 11.259636     | 64.365536     |
| 7    | 2.340418      | 11.257921     | 64.355725     |
| 8    | 2.340301      | 11.257349     | 64.352456     |
Algorithm 4 The proposed method for calculated \((L^0_q f)(x)\)

**Input:** \(g \in (0, 1), a, n, f(x), x\)
1: \(s \leftarrow 0\)
2: \(\text{for } i = 0 \text{ to } n \text{ do}\)
3: \(\frac{pf}{(1 - q^i)^{i-1}}\)
4: \(s \leftarrow s + pf \cdot q^i \cdot f(x \cdot q^i)\)
5: \(\text{end for}\)
6: \(g \leftarrow l^0 \cdot (1 - q) \cdot s\)
**Output:** \((L^0_q f)(x)\)

Algorithm 5 The proposed method for calculated \(\int_a^b f(r) dr\)

**Input:** \(g \in (0, 1), a, n, f(x), a, b\)
1: \(s \leftarrow 0\)
2: \(\text{for } i = 0 \text{ to } n \text{ do}\)
3: \(s \leftarrow s + q^i \cdot (b \cdot f(b \cdot q^i) - a \cdot f(a \cdot q^i))\)
4: \(\text{end for}\)
5: \(g \leftarrow (1 - q) \cdot s\)
**Output:** \(\int_a^b f(r) dr\)

Table 2: Some numerical results for calculation of \(\Gamma_q(x)\) with \(q = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, x = 5\) and \(n = 1, 2, \ldots, 35\) of Algorithm 2.

| \(n\) | \(q = \frac{1}{4}\) | \(q = \frac{3}{4}\) | \(q = \frac{1}{2}\) | \(q = \frac{3}{4}\) | \(q = \frac{3}{4}\) | \(q = \frac{3}{4}\) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 3.016353        | 6.291859        | 18.932472       | 18              | 2.853224        | 4.921884        | 8.476643        |
| 2    | 2.906140        | 5.548726        | 14.154784       | 19              | 2.853224        | 4.921879        | 8.474597        |
| 3    | 2.870699        | 5.222330        | 11.819974       | 20              | 2.853224        | 4.921877        | 8.473234        |
| 4    | 2.859031        | 5.069033        | 10.537540       | 21              | 2.853224        | 4.921876        | 8.472325        |
| 5    | 2.855157        | 4.994707        | 9.782069        | 22              | 2.853224        | 4.921876        | 8.471719        |
| 6    | 2.853868        | 4.958107        | 9.317265        | 23              | 2.853224        | 4.921875        | 8.471315        |
| 7    | 2.853438        | 4.939945        | 9.023265        | 24              | 2.853224        | 4.921875        | 8.471046        |
| 8    | 2.853295        | 4.930899        | 8.839940        | 25              | 2.853224        | 4.921875        | 8.470866        |
| 9    | 2.853247        | 4.926384        | 8.710584        | 26              | 2.853224        | 4.921875        | 8.470747        |
| 10   | 2.853232        | 4.924129        | 8.629588        | 27              | 2.853224        | 4.921875        | 8.470667        |
| 11   | 2.853226        | 4.923002        | 8.576133        | 28              | 2.853224        | 4.921875        | 8.470614        |
| 12   | 2.853224        | 4.922438        | 8.540736        | 29              | 2.853224        | 4.921875        | 8.470578        |
| 13   | 2.853224        | 4.922157        | 8.517243        | 30              | 2.853224        | 4.921875        | 8.470555        |
| 14   | 2.853224        | 4.922016        | 8.501627        | 31              | 2.853224        | 4.921875        | 8.470539        |
| 15   | 2.853224        | 4.921945        | 8.491237        | 32              | 2.853224        | 4.921875        | 8.470529        |
| 16   | 2.853224        | 4.921910        | 8.484320        | 33              | 2.853224        | 4.921875        | 8.470522        |
| 17   | 2.853224        | 4.921893        | 8.479713        | 34              | 2.853224        | 4.921875        | 8.470517        |
Table 3: Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4, q = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2.

| n   | $q = \frac{1}{3}$ | $q = \frac{2}{3}$ | $q = \frac{1}{2}$ |
|-----|------------------|------------------|------------------|
| 1   | 11.909360        | 62.618630        | 664.267809       |
| 2   | 11.468397        | 55.707508        | 474.800503       |
| 3   | 11.326853        | 52.245122        | 384.795341       |
| 4   | 11.280255        | 50.621828        | 336.326796       |
| 5   | 11.264786        | 49.835472        | 308.146441       |
| 6   | 11.259636        | 49.448420        | 290.958806       |
| 7   | 11.257921        | 49.256401        | 280.150029       |
| 8   | 11.257349        | 49.160766        | 273.216364       |
| 9   | 11.257158        | 49.113041        | 268.710272       |
| 10  | 11.257095        | 49.089202        | 265.756606       |
| 11  | 11.257074        | 49.077288        | 263.809514       |
| 12  | 11.257066        | 49.071333        | 262.521127       |
| 13  | 11.257064        | 49.068355        | 261.666471       |
| 14  | 11.257063        | 49.066867        | 261.098857       |
| 15  | 11.257063        | 49.066123        | 260.720833       |
| 16  | 11.257063        | 49.065751        | 260.469369       |
| 17  | 11.257063        | 49.065364        | 260.301890       |
| 18  | 11.257063        | 49.065471        | 260.105310       |
| 19  | 11.257063        | 49.065425        | 260.119957       |
| 20  | 11.257063        | 49.065402        | 260.066402       |

Table 4: Some numerical results of $\left[\int_0^\infty L_n^q(z)\right]_{n=1}$ inequality (11) in Example 4.1 for $q \in \left\{\frac{1}{3}, \frac{2}{3}, \frac{1}{2}\right\}$. One can check that $\left[\int_0^\infty L_n^q(z)\right]_{n=1} \in [0, 1]$

| n   | $q = \frac{1}{3}$ | $q = \frac{2}{3}$ | $q = \frac{1}{2}$ |
|-----|------------------|------------------|------------------|
| 1   | 1.002            | 2.0979           | 4.4776           |
| 2   | 1.0002           | 2.0938           | 4.7777           |
| 3   | 1                | 2.0933           | 4.7777           |
| 4   | 1                | 2.0932           | 4.7777           |
| 5   | 1                | 2.0932           | 4.7777           |
| 6   | 1                | 2.0932           | 4.7777           |
| 7   | 1                | 2.0932           | 4.7777           |
| 8   | 1                | 2.0932           | 4.7777           |
| 9   | 1                | 2.0932           | 4.7777           |
| 10  | 1                | 2.0932           | 4.7777           |
| 11  | 1                | 2.0932           | 4.7777           |
| 40  | 1                | 2.0932           | 4.7777           |
| 41  | 1                | 2.0932           | 4.7777           |
| 42  | 1                | 2.0932           | 4.7777           |
| 43  | 1                | 2.0932           | 4.7777           |
Table 5: Some numerical results of $I_{\gamma}^{q}(t)$ inequality (11) in Example 4.1 for $q \in \{1, \frac{1}{2}, \frac{3}{4} \}$. One can check that $I_{\gamma}^{q}(t)$ \in [0, 1].

| $n$ | $q = \frac{1}{2}$ | $q = \frac{3}{4}$ | $q = \frac{5}{4}$ |
|-----|------------------|------------------|------------------|
|     | $\Gamma_{q}(2)$ | $\Gamma_{q}(2 + \gamma_{1})$ | $I_{\gamma}^{q}(t)$ \left|_{\gamma = 1}^{\gamma_{1}}$ | $\Gamma_{q}(2)$ | $\Gamma_{q}(2 + \gamma_{1})$ | $I_{\gamma}^{q}(t)$ \left|_{\gamma = 1}^{\gamma_{1}}$ | $\Gamma_{q}(2)$ | $\Gamma_{q}(2 + \gamma_{1})$ | $I_{\gamma}^{q}(t)$ \left|_{\gamma = 1}^{\gamma_{1}}$ |
| 1   | 0.9887           | 1.877            | 0.5418           | 0.9985           | 21.6657          | 0.046            | 1.6936           | 25936.1144        | 0.0001            |
| 2   | 0.9675           | 1.8733           | 0.5165           | 0.9565           | 18.9992          | 0.0503           | 1.4923           | 11873.769         | 0.0001            |
| 3   | 0.9673           | 1.8728           | 0.5165           | 0.9382           | 17.8313          | 0.0526           | 1.3628           | 6503.4478         | 0.0002            |
| 4   | 0.9673           | 1.8728           | 0.5165           | 0.9294           | 17.2835          | 0.0538           | 1.2721           | 4015.4293         | 0.0003            |
| 5   | 0.9673           | 1.8728           | 0.5165           | 0.9251           | 17.0181          | 0.0544           | 1.2053           | 2706.2688         | 0.0004            |
| 6   | 0.9673           | 1.8728           | 0.5165           | 0.923            | 16.8875          | 0.0547           | 1.1535           | 1949.717          | 0.0006            |
| 7   | 0.9673           | 1.8728           | 0.5165           | 0.9219           | 16.8227          | 0.0548           | 1.1128           | 1479.9943         | 0.0008            |
| 8   | 0.9673           | 1.8728           | 0.5165           | 0.9214           | 16.7904          | 0.0549           | 1.0801           | 1171.4155         | 0.0009            |
| 9   | 0.9673           | 1.8728           | 0.5165           | 0.9211           | 16.7743          | 0.0549           | 1.0533           | 959.2878          | 0.0011            |
| 10  | 0.9673           | 1.8728           | 0.5165           | 0.921            | 16.7662          | 0.0549           | 1.0313           | 807.9574          | 0.0013            |
| 11  | 0.9673           | 1.8728           | 0.5165           | 0.9209           | 16.7622          | 0.0549           | 1.0124           | 696.637           | 0.0015            |
| 45  | 0.9673           | 1.8728           | 0.5165           | 0.9209           | 16.7582          | 0.055            | 0.8945           | 245.149           | 0.0036            |
| 46  | 0.9673           | 1.8728           | 0.5165           | 0.9209           | 16.7582          | 0.055            | 0.8943           | 244.6677          | 0.0037            |
| 47  | 0.9673           | 1.8728           | 0.5165           | 0.9209           | 16.7582          | 0.055            | 0.8941           | 244.2408          | 0.0037            |