WELL-POSEDNESS FOR THE CAUCHY PROBLEM FOR A FRACTIONAL POROUS MEDIUM EQUATION WITH VARIABLE DENSITY IN ONE SPACE DIMENSION

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Abstract. We study existence and uniqueness of bounded solutions to a fractional nonlinear porous medium equation with a variable density, in one space dimension.

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AMS subject classification: 35A01, 35A02, 35E15, 35K55, 35R11.

1 Introduction

In this paper we study existence and uniqueness of bounded solutions to nonlocal nonlinear initial-value problems of the following type:

\[
\begin{cases}
\rho \frac{\partial u}{\partial t} + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} [u^m] = 0 & x \in \mathbb{R}, \ t > 0 \\
u = u_0 & x \in \mathbb{R}, \ t = 0;
\end{cases}
\]  

(1.1)

where \( \rho = \rho(x) \), usually referred to as a variable density, is a positive function only depending on the spatial variable \( x \), \( \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \) denotes the one-dimensional fractional Laplacian of order \( 1/2 \), \( m \geq 1 \) and \( u_0 \) is a nonnegative and bounded function.

The interest in the study of nonlocal problems has grown significantly in the last years and the analysis of nonlinear equations involving fractional powers of the Laplacian has become an area of intense research. Such problems in fact arise in many physical situations (see, e.g. \[1\], \[18\], \[19\]) in particular in the analysis of long-range or anomalous diffusions. From a probabilistic point of view, the fractional Laplacian is the infinitesimal generator

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of a Lévy process (see [2]). Then, loosely speaking, uniqueness for problem (1.1) with \( m = 1 \) corresponds to the fact that the Lévy process associated to the operator \( \frac{1}{\rho} \left( -\frac{\partial^2}{\partial x^2} \right)^{1/2} \), starting from any point in \( \mathbb{R} \), does not attain infinity.

If the nonlocal operator \( \left( -\frac{\partial^2}{\partial x^2} \right)^{1/2} \) in (1.1) is replaced by the classical second order partial derivatives \( -\frac{\partial^2}{\partial x^2} \), we get

\[
\begin{aligned}
\rho \partial_t u - \frac{\partial^2 u^m}{\partial x^2} &= 0 & x \in \mathbb{R}, t > 0 \\
\quad u &= u_0 & x \in \mathbb{R}, t = 0,
\end{aligned}
\]  

which is the initial value problem for the porous medium equation with variable density in one space dimension. Problem (1.2) and its counterpart in \( \mathbb{R}^N \), that is,

\[
\begin{aligned}
\rho \partial_t u + \Delta u^m &= 0 & x \in \mathbb{R}^N, t > 0 \\
\quad u &= u_0 & x \in \mathbb{R}^N, t = 0,
\end{aligned}
\]

occurs in various situations of physical interest (see, e.g., [23]) and have been largely investigated in the literature during the last two decades (see [14], [15], [16], [20], [21], [24], [28]-[29]). More recently, similar problems on Riemannian manifolds have been addressed in [25]-[27].

It is well-known that the asymptotic behavior of the varying density \( \rho \) may influence uniqueness of solutions for problem (1.3). Briefly, for \( N = 1 \) or \( N = 2 \) problem (1.3) is well-posed in the class of bounded solutions not satisfying any additional conditions at infinity (see [17]), provided \( \rho \in L^\infty(\mathbb{R}^N) \). The situation is different if \( N \geq 3 \). Indeed, in this case uniqueness prevails when the density \( \rho \) goes to zero slowly as \( |x| \to \infty \). If instead \( \rho \) goes to zero fast as \( |x| \) diverges, then nonuniqueness of bounded solutions occurs (see [15], [16], [21], [24], [27], [28]-[29]).

Concerning the regularity assumed for initial data and solutions we point out that in [20], [28]-[29] weak energy solutions are dealt with, and \( u_0 \) is supposed to belong to \( L^1(\mathbb{R}^N) \), the space of measurable nonnegative functions \( f \) satisfying \( \int_{\mathbb{R}^N} f \rho \, dx < \infty \). In [15], [16], [21], and [24], instead, bounded initial data and so-called very weak solutions are treated. One of the main issues in dealing with very weak solutions lies in the fact that, in general, these solutions do not have finite energy in the whole \( \mathbb{R}^N \).

Very recently, in [11], the nonlocal nonlinear initial-value problem

\[
\begin{aligned}
\partial_t u + \left( (-\Delta)^{\sigma/2} \right) [u^m] &= 0 & x \in \mathbb{R}^N, t > 0 \\
\quad u &= u_0 & x \in \mathbb{R}^N, t = 0
\end{aligned}
\]

(\( 0 < \sigma < 2 \)) has been studied. The particular case \( \sigma = 1 \) has been considered in [10]; more precisely, existence, uniqueness and properties of weak solutions
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to \((1.3)\) have been established assuming \(u_0 \in L^1(\mathbb{R}^N)\). Furthermore, problem

\[
\begin{cases}
\partial_t u + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{m}{2}} [\log(1 + u)] = 0 & x \in \mathbb{R}, \ t > 0 \\
u = u_0 & x \in \mathbb{R}, \ t = 0
\end{cases}
\]

has been addressed in \([12]\).

The motivation in studying problem \((1.1)\) is twofold. In fact, on one hand, when \(\rho \equiv 1\), \((1.1)\) becomes problem \((1.4)\) with \(\sigma = N = 1\), thus problem \((1.1)\) can be regarded as a generalization of problem \((1.4)\) to the case in which a variable density \(\rho\) is taken into account. On the other hand, problem \((1.1)\) can also be considered as nonlocal version of problem \((1.2)\).

The aim of this paper is to prove well-posedness of problem \((1.1)\) in the class of bounded solutions not satisfying any additional conditions at infinity. We shall consider bounded initial data \(u_0\) and, consequently, very weak bounded solutions (see Definition 2.1). Let us mention that our results differ from those in \([10]\) where \(\rho\) is assumed to be identically equal to 1. Moreover, also in the case \(\rho \equiv 1\) considered in \([10]\) uniqueness of very weak solutions is not proved, while uniqueness of weak energy solutions is shown (see \([10\), Lemma 4.1\]). Very weak solutions to problem \((1.4)\) in the space \(C([0, \infty); L^1(\mathbb{R}^N))\) (\(\varphi\) being a suitable weight decaying at infinity) have been considered in \([3]\) where uniqueness of such type of solutions is also shown provided \(0 < m < 1\).

To the best of our knowledge, the uniqueness result presented in this paper is new also in the linear case \(m = 1\), that is for problem

\[
\begin{cases}
\rho \partial_t u + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} [u] = 0 & x \in \mathbb{R}, \ t > 0 \\
u = u_0 & x \in \mathbb{R}, \ t = 0.
\end{cases}
\]  

(1.5)

Some results for nonlocal linear parabolic equation with a variable density have been established in \([7]\), but not for problem \((1.5)\).

One can expect that if \(N \geq 2\) the nonlocal problem

\[
\begin{cases}
\rho \partial_t u + (-\Delta)^{\frac{1}{2}} [u^m] = 0 & x \in \mathbb{R}^N, \ t > 0 \\
u = u_0 & x \in \mathbb{R}^N, \ t = 0
\end{cases}
\]

may exhibit uniqueness or nonuniqueness of solutions depending on the behavior at infinity of the density \(\rho\). This would reflect in the nonlocal case the the situation described above for problem \((1.3)\). However, some technical difficulties prevent us from adapting the methods used in this paper to investigate uniqueness of very weak solutions in any space dimension; see Remark 4.3. We postpone to a forthcoming paper the proof of existence and
uniqueness of solutions to problem
\[
\begin{cases}
\rho \partial_t u + (-\Delta)^{\frac{\sigma}{2}} [u^m] = 0 & \text{if } x \in \mathbb{R}^N, \ t > 0 \\
u = u_0 & \text{if } x \in \mathbb{R}^N, \ t = 0
\end{cases}
\] (1.6)
for any $0 < \sigma < 2$ and any $N \geq 1$. In this case, according with [10]-[11], we shall consider weak energy solutions and completely different arguments.

The paper is organized as follows. In Section 2 we introduce the setting of the problem and give the precise definition of solutions we are dealing with. We also recall some basic facts about fractional Laplacian, mainly its realization through the harmonic extension, which permits to look at problem (1.1) in a “local way” as a quasi-stationary problem with dynamical boundary conditions. In Section 3 we prove existence of solutions to problem (1.1); see Theorem 3.1. Even if the general strategy of the proofs in this Section goes along the same lines as in [10]-[11], there are some differences here that will be expedient as we will point out in Remark 3.8. We should remark that all results in Section 3 are valid in general for any $N \geq 1$. In subsequent Section 4 we establish uniqueness of solutions; see Theorem 4.1. The proof exploits the above mentioned realization of the fractional Laplacian combined with properties of a suitable family of test functions we construct in Lemma 4.2. Here the restriction $N = 1$ will be crucial as explained in Remarks 4.4 and 4.5.

To conclude, let us mention that our techniques also applies to more general fractional nonlinear diffusion equations with variable density, such as
\[
\rho \partial_t u + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\sigma}{2}} [G(u)] = 0 \quad x \in \mathbb{R}, \ t > 0,
\]
provided $G$ satisfies suitable conditions as in [15], [24]. Moreover, changing sign bounded solutions could also be considered. However, for sake of simplicity, we limit ourselves to the case of nonnegative solutions and $G(u) = u^m$ ($m \geq 1$).

2 Problem setting and assumptions

We recall that the nonlocal operator $\left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\sigma}{2}}$ is defined for any function $\varphi$ belonging to the Schwartz class by
\[
\left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\sigma}{2}} \varphi = \frac{1}{\pi} \text{ P.V.} \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(y)}{|x - y|^2} dy.
\]
As is well-known, $\left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\sigma}{2}}$ can be also defined in many other ways; we refer the reader to [4], [6], [13], for a comprehensive account on the subject. In
particular, if \( \varphi \) is a smooth and bounded function defined in \( \mathbb{R} \), we can consider its harmonic extension \( v \) to the upper half-space

\[
\Omega := \mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, y > 0\}.
\]

Hence (see [4], [6], [10], [12]),

\[
- \frac{\partial v(x, 0)}{\partial y} = \left( - \frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \varphi(x) \quad \text{for all } x \in \mathbb{R}.
\]

In the sequel, we always make the following assumption:

\[
\begin{cases}
(i) & \rho \in C(\mathbb{R}), \rho > 0 \text{ in } \mathbb{R}; \\
(ii) & u_0 \text{ is nonnegative, continuous and bounded in } \mathbb{R}.
\end{cases} \quad \text{(A_0)}
\]

Following [10] and [11], using the harmonic extension, we shall rewrite the problem (1.1) in terms of local differential operators. More precisely, solving problem (1.1) is equivalent to solve the quasi-stationary problem with a dynamical boundary conditions:

\[
\begin{cases}
\Delta w = 0 \quad (x, y) \in \Omega, \ t > 0 \\
\frac{\partial w}{\partial y} = \rho \frac{\partial \left[ w^{\frac{m}{2}} \right]}{\partial t} \quad x \in \Gamma, \ t > 0 \\
w = u_0^{\frac{m}{2}} \quad x \in \Gamma, \ t = 0;
\end{cases} \quad (2.1)
\]

here \( \Gamma := \overline{\Omega} \cap \{ y = 0 \} = \mathbb{R} \).

Now, take any open bounded subset \( D \subset \Omega \) with \( \partial D \cap \Omega \) smooth and \( \partial D \cap \Gamma \neq \emptyset \). Then the exterior normal to \( \partial D \) exists almost everywhere; we denote such vector by \( \vec{\nu} \). Let \( T > 0, \psi \in C^{2,1}_{x,t}(\overline{D} \times [0, T]), \psi \geq 0, \psi = 0 \) in \( (\partial D \setminus \Gamma) \times [0, T], \psi \neq 0 \) in \( (\partial D \cap \Gamma) \times [0, T] \). Formally, multiplying the differential equation in (2.1) by \( \psi \), integrating by parts and considering the initial and the dynamical boundary condition we get:

\[
0 = \int_0^T \int_D \psi \Delta w \, dx \, dy \, dt
\]

\[
= - \int_0^T \int_D \langle \nabla \psi, \nabla w \rangle \, dx \, dy \, dt + \int_0^T \int_{\partial D} \psi \frac{\partial w}{\partial \vec{\nu}} \, dx \, dy \, dt
\]

\[
= - \int_0^T \int_D \langle \nabla \psi, \nabla w \rangle \, dx \, dy \, dt - \int_0^T \int_{\partial D \cap \Gamma} \psi \frac{\partial w}{\partial y} \, dx \, dt
\]

\[
= \int_0^T \int_D w \Delta \psi \, dx \, dy \, dt - \int_0^T \int_{\partial D \cap \Omega} w \frac{\partial \psi}{\partial S} \, dS \, dt + \int_0^T \int_{\partial D \cap \Gamma} w \frac{\partial \psi}{\partial y} \, dx \, dt
\]

\[
- \int_0^T \int_{\partial D \cap \Gamma} \rho \psi \, \partial_y u \, dx \, dt
\]
$$\int_{0}^{T} \int_{D} w \Delta \psi \, dx \, dy \, dt - \int_{0}^{T} \int_{\partial D \cap \Omega} w \frac{\partial \psi}{\partial v} \, dS \, dt + \int_{0}^{T} \int_{\partial D \cap \Gamma} w \frac{\partial \psi}{\partial y} \, dx \, dt$$

$$= \int_{0}^{T} \int_{D} w \Delta \psi \, dx \, dy \, dt - \int_{0}^{T} \int_{\partial D \cap \Omega} w \frac{\partial \psi}{\partial v} \, dS \, dt + \int_{0}^{T} \int_{\partial D \cap \Gamma} \rho u \frac{\partial \psi}{\partial y} \, dx \, dt$$

$$- \int_{\partial D \cap \Gamma} \rho(x) \left[ u(x,T) \psi(x,0,T) - u_0(x) \psi(x,0,0) \right] \, dx.$$
If \( f \in W^{1,2}(\Omega_R) \), we denote by \( f|_{\Gamma_R}, f|_{\Sigma_R} \) the trace of \( f \) on \( \Gamma_R \) and \( \Sigma_R \), respectively.

For any fixed \( \epsilon > 0 \), \( R > 0 \) and \( g \in L^\infty(\Gamma_R) \) we consider the following auxiliary elliptic problem in \( \Omega_R \):

\[
\begin{aligned}
\Delta v_R &= 0 \quad (x, y) \in \Omega_R \\
-\epsilon \frac{\partial v_R}{\partial y} + \rho (v_R) \frac{1}{\sqrt{v_R}} &= \rho g \quad x \in \Gamma_R \\
v_R &= 0 \quad (x, y) \in \Sigma_R.
\end{aligned}
\]

(3.1)

**Definition 3.2.** A solution to problem (3.1) is a pair of functions \((z_R, v_R)\) such that

- \( z_R \in L^\infty(\Gamma_R) \);
- \( v_R \in W^{1,2}(\Omega_R) \cap L^\infty(\Omega_R) \);
- \( v_R|_{\Gamma_R} = z_R^m, v_R|_{\Sigma_R} = 0 \);
- for any \( \varphi \in C^1(\Omega_R), \varphi = 0 \) on \( \Sigma_R \) there holds

\[
\int_{\Omega_R} (\nabla v_R, \nabla \varphi) \, dx\, dy + \frac{1}{\epsilon} \int_{\Gamma_R} \rho (z_R - g) \, \varphi \, dx = 0.
\]

(3.2)

In the following Lemma we establish existence, uniqueness and some properties of solutions to problem (3.1).

**Lemma 3.3.** Let \( R > 0 \) and \( \epsilon > 0 \) be fixed. For any \( g \in L^\infty(\Gamma) \), there exists a unique solution \((z_R, v_R)\) to the problem (3.1). Furthermore the following properties hold:

i. **Contractivity:** the mapping

\[
g \mapsto \frac{1}{\rho} z_R
\]

is a contraction in the norm of \( L^1(\Gamma_R) \), namely, if \( v_R \) and \( \tilde{v}_R \) are solutions of (3.1) corresponding to \( g \) and \( \tilde{g} \) respectively, then

\[
\int_{\Gamma_R} \rho (z_R - \tilde{z}_R)_+ \, dx \leq \int_{\Gamma_R} \rho [g - \tilde{g}]_+ \, dx;
\]

(3.3)

in particular, if \( g \geq 0 \) then \( z_R \geq 0 \) in \( \Omega_R \);

ii. **Uniform boundedness:** for any \( \epsilon > 0 \) and any \( R > 0 \)

\[
0 \leq v_R \leq \|g\|^m_{L^\infty(\Gamma)} \quad \text{in } \Omega_R,
\]

(3.4)

provided \( g \geq 0 \);
iii. Monotonicity: if \( R' > R \) then \( v_{R'} \geq v_R \) in \( \Omega_R \), provided \( g \geq 0 \).

Proof. Let \( \tilde{H}^1(\Omega_R) := \{ f \in H^1(\Omega_R) : f|_{\Sigma_R} = 0 \} \). Existence of solutions easily follows by standard arguments, since the functional \( J : \tilde{H}^1(\Omega_R) \to \mathbb{R} \) defined by

\[
J(v) := \frac{1}{2} \int_{\Omega_R} |\nabla v|^2 \, dx \, dy + \frac{m}{m+1} \int_{\Gamma_R} \rho \frac{v^{m+1}}{m+1} \, dx - \int_{\Gamma_R} \rho v g \, dx
\]

is coercive in \( \tilde{H}^1(\Omega_R) \); in fact, for suitable constants \( C_1 = C_1(R) > 0 \) and \( C_2 = C_2(R) > 0 \), we have

\[
J(v) \geq \frac{1}{2} \int_{\Omega_R} |\nabla v|^2 \, dx \, dy + \left( \min_{\Gamma_R} \rho \right) \frac{m}{m+1} \int_{\Gamma_R} v^{m+1} \, dx - \left( \max_{\Gamma_R} \rho \right) \int_{\Gamma_R} v g \, dx \geq C_1 \|v\|_{H^1(\Omega_R)}^2 - C_2 \|v\|_{H^1(\Omega_R)}. \]

Now, to show contractivity, take a smooth monotone approximation \( p \) of the sign function, and set \( \varphi(x) := p[v_R(x) - \tilde{v}_R(x)] \), \( x \in \Omega_R \). Clearly, such a \( \varphi \) can be used as a test function in the weak formulation of Definition 3.2.

We get

\[
0 = \int_{\Omega_R} p'(v_R - \tilde{v}_R)|\nabla (v_R - \tilde{v}_R)|^2 \, dx \, dy
+ \frac{1}{\epsilon} \int_{\Gamma_R} \rho (z_R - \tilde{z}_R) p[(z_R)^m - (\tilde{z}_R)^m] \, dx - \frac{1}{\epsilon} \int_{\Gamma_R} \rho (g - \tilde{g}) p[(z_R)^m - (\tilde{z}_R)^m] \, dx.
\]

Passing to the limit in the approximation \( p \), we discover

\[
\int_{\Gamma_R} \rho (z_R - \tilde{z}_R) \, dx \leq \int_{\Gamma_R} \rho (g - \tilde{g}) \text{sgn}(z_R^m - \tilde{z}_R^m) \, dx
\]

which immediately gives (3.3). Furthermore, if \( g \geq 0 \), (3.3) implies \( z_R \geq 0 \) in \( \Gamma_R \). From (3.3), uniqueness can be easily deduced.

To prove uniform boundedness, set \( \tilde{g} := \|g\|_{L^\infty(\Gamma)} \); thus the function \( \tilde{v}_R := (\tilde{g})^m \) verifies:

\[
\begin{align*}
\Delta \tilde{v}_R &= 0 \quad (x, y) \in \Omega_R \\
- \frac{\partial \tilde{v}_R}{\partial y} + \rho (\tilde{v}_R)^{\frac{m}{m-1}} &= \rho \tilde{g} \geq \rho g \quad x \in \Gamma_R \\
\tilde{v}_R &\geq 0 \quad (x, y) \in \Sigma_R.
\end{align*}
\]

Then, by (3.3), \( z_R \leq \tilde{z}_R \) in \( \Gamma_R \). Thus, the function \( \zeta_R := \tilde{v}_R - v_R \) satisfies the problem

\[
\begin{align*}
\Delta \zeta_R &= 0 \quad (x, y) \in \Omega_R \\
\zeta_R &\geq 0 \quad (x, y) \in \Sigma_R \cup \Gamma_R.
\end{align*}
\]
The comparison principle then implies \( \zeta_R \geq 0 \) in \( \Omega_R \), that is
\[
v_R \leq \| g\|_{L^\infty(\Gamma)}, \quad \text{in } \Omega_R.
\]
On the other hand, since \( z \equiv 0 \) is a subsolution to problem (3.5), \( v_R \geq 0 \); (3.4) is then proved.

To establish \( iii. \) we consider \( v_R' \) solution of (3.1) in \( \Omega_{R'} \). Since, by \( ii. \), \( v_R' \) is nonnegative in \( \Omega_{R'} \), we have
\[
\begin{cases}
\Delta v_R' = 0 & (x, y) \in \Omega_R \\
-\epsilon \frac{\partial v_R'}{\partial y} + \rho (v_R')^\frac{1}{m} = \rho g & x \in \Gamma_R \\
v_R' \geq 0 & (x, y) \in \Sigma_R.
\end{cases}
\]
Then, by (3.3),
\[
\int_{\Gamma_R} (z_R' - z_R) + \rho \, dx = 0;
\]
this implies \( z_R' \geq z_R \) on \( \Gamma_R \). To conclude, observe that the function \( \eta_R := v_R' - v_R \) satisfies,
\[
\begin{cases}
\Delta \eta_R = 0 & (x, y) \in \Omega_R \\
\eta_R \geq 0 & (x, y) \in \Sigma_R \cup \Gamma_R.
\end{cases}
\]
Hence, by comparison, \( v_R' \geq v_R \) on \( \Omega_R \).

Next step in the proof of Theorem 3.1 consists in studying the following evolutive problem in \( \Omega_R \):
\[
\begin{cases}
\Delta w_R = 0 & (x, y) \in \Omega_R, \quad t > 0 \\
w_R = 0 & (x, y) \in \Sigma_R, \quad t > 0 \\
\frac{\partial w_R}{\partial y} = \rho \left( \frac{(w_R)^{\frac{1}{m}}}{\partial t} \right) & x \in \Gamma_R, \quad t > 0 \\
w_R = u_0^m & x \in \Gamma_R, \quad t = 0.
\end{cases}
\]

**Definition 3.4.** A solution to problem (3.6) is a pair of functions \((u_R, w_R)\) such that
\begin{itemize}
  \item \( u_R \in L^\infty(\Gamma_R \times (0, \infty)) \cap C([0, \infty); L^1(\Gamma_R)), \ u_R \geq 0; \)
  \item \( w_R \in L^\infty(\Omega_R \times (0, \infty)), \ w_R \geq 0, \ |\nabla w_R| \in L^2(\Omega_R \times (0, +\infty)); \)
  \item \( w_R|_{\Gamma_R \times (0, \infty)} = u_R^m, \ w_R|_{\Sigma_R \times (0, \infty)} = 0; \)
  \item for any \( T > 0 \) and any \( \psi \in C_{x,t}^{2,1}(\overline{\Omega_R \times [0, T]}), \ \psi \equiv 0 \) in \( \partial \Sigma_R \times (0, T) \) there holds
\end{itemize}
\[
\begin{align*}
- \int_0^T \int_{\Omega_R} (\nabla w_R, \nabla \psi) \, dx \, dy \, dt + \int_0^T \int_{\Gamma_R} \rho u_R \partial_t \psi \, dx \, dt \\
= \int_{\Gamma_R} \rho (x) \left[ u_R(x, T) \psi(x, 0, T) - u_0(x) \psi(x, 0, 0) \right] \, dx.
\end{align*}
\]
Observe that, by standard results, for any \( g \in L^\infty(\Gamma_R) \) there exists a unique solution \( v \in W^{1,2}(\Omega_R) \cap L^\infty(\Omega_R) \) of problem
\[
\begin{cases}
\Delta v = 0 & \text{in } \Omega_R \\
v = 0 & \text{on } \Sigma_R \\
v(x,0) = g(x) & \text{on } \Gamma_R;
\end{cases}
\]
here boundary conditions are meant in the sense of trace. We shall denote such a solution by \( E_R(g) \) since it can be regarded as the harmonic extension of \( g \) to \( \Omega_R \), completed with homogeneous zero Dirichlet boundary condition on \( \Sigma_R \). Indeed, by standard elliptic regularity results, \( v \in C^2(\Omega_R) \cap C(\Omega_R \cup \Sigma_R) \) and \( v(x) = 0 \) for all \( x \in \Sigma_R \).

**Proposition 3.5.** Let assumption \( \mathcal{A}_0 \) be satisfied. Then for any \( R > 0 \) there exists a solution \((u_R, w_R)\) of problem \( \text{(3.6)} \).

**Proof.** We introduce a time-discretized version of \( \text{(3.6)} \). Fix any \( T > 0 \). For any \( n \in \mathbb{N} \) we divide the time interval \([0, T]\) in \( n \) subintervals of length \( \epsilon = T/n \) and endpoints \([ (k-1)\epsilon, k\epsilon ] \) for \( k = 1, \ldots, n \). For any \( k = 1, \ldots, n \), by Lemma 3.3 with \( g = u_{R,k-1} \), there exists a unique solution \((u_{R,k}^{\epsilon}, w_{R,k}^{\epsilon})\) to the problem
\[
\begin{cases}
\Delta u_{R,k}^{\epsilon} = 0 & (x,y) \in \Omega_R \\
w_{R,k}^{\epsilon} = 0 & (x,y) \in \Sigma_R \\
\epsilon \frac{\partial w_{R,k}^{\epsilon}}{\partial y} = \rho \left( u_{R,k}^{\epsilon} - u_{R,k-1}^{\epsilon} \right) & x \in \Gamma_R \\
\left. u_{R,0}^{\epsilon} \right|_{\Gamma_R} = u_0 & x \in \Gamma_R.
\end{cases}
\]
(3.8)

The solution \((u_{R,k}^{\epsilon}, w_{R,k}^{\epsilon})\) satisfies
\[
u_{R,k-1}^{\epsilon} = \left( \left. \left( \frac{u_{R,k}^{\epsilon}}{w_{R,k}^{\epsilon}} \right) \right|_{\Gamma_R} \right), \quad w_{R,k}^{\epsilon} = E_R \left( \left( u_{R,k}^{\epsilon} \right)^m \right).
\]

We can rewrite the problem on \( \Gamma_R \) in (3.8) as
\[
\begin{cases}
u_{R}^{\epsilon,k} + \epsilon A(u_{R,k}^{\epsilon}) = u_{R,k-1}^{\epsilon}, & x \in \Gamma_R, \quad k = 1, \ldots, n \\
u_{R}^{\epsilon,0} = u_0, & x \in \Gamma_R,
\end{cases}
\]
where \( A : \mathcal{D}(A) \subset L^1(\Gamma_R) \to L^\infty(\Gamma_R) \) is the operator defined
\[
A(v) := -\frac{1}{\rho} \left( \left. \frac{\partial E_R(v^m)}{\partial y} \right|_{\Gamma_R} \right),
\]
with domain
\[
\mathcal{D}(A) := \left\{ v \in L^\infty(\Gamma_R) : \ A(v) \in L^\infty(\Gamma_R), \ ||v||_{L^\infty(\Gamma_R)} \leq ||u_0||_{L^\infty(\Gamma_R)} \right\}.
\]
The operator $A$ satisfies the following properties. For any $\epsilon > 0$ the mapping $(I + \epsilon A)$ is one-to-one from $\mathcal{D}(A)$ onto a subspace $\mathcal{R}_\epsilon(A) \subseteq L^\infty(\Gamma_R)$. In fact, by Lemma 3.3 for any $\epsilon > 0$ there exists a unique solution of

$$
\begin{cases}
\Delta w^{\epsilon,k} = 0 & (x,y) \in \Omega_R \\
u^{\epsilon,k}_R - \frac{1}{\rho} \frac{\partial w^{\epsilon,k}}{\partial y} = u^{\epsilon,k-1}_R & x \in \Gamma_R \\
w^{\epsilon,k}_R = 0 & (x,y) \in \Sigma_R;
\end{cases}
$$

(3.9)
in addition, by (3.3), the inverse mapping $(I + \epsilon A)^{-1} : \mathcal{R}_\epsilon(A) \to L^\infty(\Gamma_R)$ is a contraction with respect to the norm of $L^1(\Gamma_R)$.

A second property satisfied by $A$ is the following rank condition: for any $\epsilon > 0$ $\mathcal{R}_\epsilon(A) = L^\infty(\Gamma_R) \supseteq L^\infty(\Gamma_R) = \mathcal{D}(A)$ for any $\epsilon > 0$.

The validity of the two properties above permits to invoke the Crandall–Liggett theorem (see [9]) and to infer that $u^{\epsilon,k}_R$ converges in $L^1(\Gamma_R)$ to some function $u_R \in C([0,T];L^1(\Gamma_R))$, as $\epsilon \to 0$. Furthermore, by (3.4) and (A0)-(ii),

$$
0 \leq w^{\epsilon,k}_R \leq \|u_0\|^{m+1}_{L^\infty(\mathbb{R})} \text{ in } \Omega_R, \text{ uniformly with respect to } \epsilon, k \text{ and } R.
$$

Then $w^{\epsilon,k}_R$ converges weakly-* to some function $w_R \in L^\infty(\Omega_R \times [0,T])$ as $\epsilon \to 0$.

Using $w^{\epsilon,k}_R$ as a test function for (3.8), integrating by parts, and applying Young’s inequality we obtain

$$
0 \leq \epsilon \int_{\Omega_R} |\nabla w^{\epsilon,k}_R|^2 \, dx \, dy \leq \frac{1}{1 + m} \int_{\Gamma_R} \rho \left[ (u^{\epsilon,k-1}_R)^{m+1} - (u^{\epsilon,k}_R)^{m+1} \right] \, dx.
$$

(3.10)

Then, adding for $k$ from 1 to $n$ and passing to the limit as $\epsilon \to 0$ we discover

$$
\int_0^T \int_{\Omega_R} |\nabla w_R|^2 \, dx \, dy \, dt \leq \frac{1}{1 + m} \int_{\Gamma_R} \rho \left( u_0 \right)^{m+1} \, dx.
$$

Thus, $w_R \in L^2([0,T];H^1(\Omega_R))$. Moreover, it is easily seen that $w_R|_{\Sigma_R \times (0,\infty)} = 0$. Concerning the limit function $u_R$, we observe that, by (3.10),

$$
\int_{\Gamma_R} \rho \left( u^{\epsilon,k}_R \right)^{m+1} \, dx \leq \int_{\Gamma_R} \rho \left( u^{\epsilon,k-1}_R \right)^{m+1} \, dx \leq \int_{\Gamma_R} \rho u_0^{m+1} \, dx,
$$

for any $\epsilon > 0$; then for $\epsilon \to 0$, we get

$$
\int_{\Gamma_R} \rho \left( u_R(x,t) \right)^{m+1} \, dx \leq \int_{\Gamma_R} \rho u_0^{m+1} \, dx, \text{ for any } t \in [0,T].
$$
Now, by choosing appropriate test functions as in [30] and taking into account (3.2) we have for any $k = 1, \ldots, n$,
\[
\int_{\Omega_R} \langle \nabla w^{\epsilon,k}_R, \nabla \psi \rangle \, dx \, dy = \frac{1}{\epsilon} \int_{\Gamma_R} \rho \left[ u^{\epsilon,k-1}_R - u^{\epsilon,k}_R \right] \psi \, dx.
\] (3.11)

Integrating on $[t_{k-1}, t_k]$ and then adding over $k$, we can rewrite the right hand side of equality above as
\[
\frac{1}{\epsilon} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Gamma_R} \rho \left[ u^{\epsilon,k}_R(x,t) - u^{\epsilon,k}_R(x,t) \right] \psi(x,0,t) \, dx \, dt
= \int_{0}^{T} \int_{\Gamma_R} \rho u^\epsilon_R(x,t) \frac{\psi(x,0,t+\epsilon) - \psi(x,0,t)}{\epsilon} \, dx \, dt
+ \frac{1}{\epsilon} \int_{0}^{T} \int_{\Gamma_R} \rho u^\epsilon_0(x) \psi(x,0,t) \, dx \, dt - \frac{1}{\epsilon} \int_{T-\epsilon}^{T} \int_{\Gamma_R} \rho u^\epsilon_R(x,T) \psi(x,0,t) \, dx \, dt,
\] (3.12)

where $u^\epsilon_R := \{u^{\epsilon,1}_R, \ldots, u^{\epsilon,n}_R\}$. By sending $\epsilon$ to 0, (3.11) and (3.12) give (3.7).

In the next Proposition we list some properties satisfied by solutions of (3.6).

**Proposition 3.6.** Let $(u_R, w_R)$ be a solution of (3.6). Then the following properties are satisfied:

i. If $u_R$ and $\tilde{u}_R$ are solutions of (3.6) corresponding to $u_0$ and $\tilde{u}_0$ respectively, then
\[
\int_{\Gamma_R} \left[ u_R(x,t) - \tilde{u}_R(x,t) \right] \rho \, dx \leq \int_{\Gamma_R} [u_0 - \tilde{u}_0]_+ \rho \, dx; \quad (3.13)
\]
in particular, if $u_0 \geq 0$ then $u_R(x,t) \geq 0$ for every $x \in \Gamma_R$ and every $t > 0$;

ii. There exists a constant $C$ independent of $R$ such that $0 \leq w_R \leq C$ for every $x \in \Omega_R$ and every $t > 0$;

iii. For any $R' > R$, $w_{R'}(x,t) \geq w_R(x,t)$ for every $x \in \Omega_R$ and every $t > 0$;

iv. The inequality
\[
\rho(x) [(m-1)t \partial_t w_R + mw_R] \geq 0
\]
holds in the sense of distributions in $\Omega_R \times (0, \infty)$;

v. $\partial_t u \in L^1_{loc}((0, \infty); L^1(\Gamma))$;
vi. For any $0 \leq \tau < T$,
\[ \int_{\tau}^{T} \int_{\Omega_R} |\nabla w_R|^2 \, dx \, dy \, dt + \frac{1}{m+1} \int_{\Gamma_R} \rho(u_R)^{m+1}(x, T) \, dx = \frac{1}{m+1} \int_{\Gamma_R} \rho(u_R)^{m+1}(x, \tau) \, dx; \quad (3.14) \]

vii. For any $0 \leq \tau < T$ and any cutoff $\eta \in C^2(\bar{\Omega})$ with $0 \leq \eta \leq 1$ and $\text{supp}(\eta) \subset B_R$,
\[ \int_{\tau}^{T} \int_{\Omega_R} |\nabla w_R|^2 \eta^2 \, dx \, dy \, dt + \frac{2}{m+1} \int_{\Gamma_R} \rho(x)(u_R)^{m+1}(x, T) \eta^2 \, dx = \frac{2}{m+1} \int_{\Gamma_R} \rho(x)(u_R)^{m+1}(x, \tau) \eta^2 \, dx - 2C^2 \int_{0}^{T} \int_{\Omega_R} |\nabla \eta|^2 \, dx \, dy \, dt. \quad (3.15) \]

Proof. Properties i.–iii. are inherited by $(u_R, w_R)$ from those listed in Lemma 3.3 for solutions to the auxiliary elliptic problem. In particular i. is a consequence of (3.3). Property ii. follows taking into account that $w_R$ is null on $\Sigma_R$ and previous property i.; property iii. follows by Lemma 3.3–iii.

Properties iv.–vi. can be obtained arguing as in [10, Sections 3 and 5]. Moreover, it is direct to check that
\[ \int_{\tau}^{T} \int_{\Omega_R} |\nabla w_R|^2 \eta^2 \, dx \, dy \, dt + \frac{1}{m+1} \int_{\Gamma_R} \rho(x)(u_R)^{m+1}(x, T) \eta^2 \, dx = \frac{1}{m+1} \int_{\Gamma_R} \rho(x)(u_R)^{m+1}(x, \tau) \eta^2 \, dx - 2 \int_{\tau}^{T} \int_{\Omega_R} \eta w_R \langle \nabla \eta, \nabla w_R \rangle \, dx \, dy \, dt. \]

Using Young’s inequality, we estimate the last term in the computation above as follows:
\[ \left| 2 \int_{0}^{T} \int_{\Omega_R} \eta w_R \langle \nabla \eta, \nabla w_R \rangle \, dx \, dy \, dt \right| \leq 2 \int_{0}^{T} \int_{\Omega_R} (|\nabla w_R| |\eta|)(w_R ||\nabla \eta||) \, dx \, dy \, dt \leq \frac{1}{2} \int_{0}^{T} \int_{\Omega_R} |\nabla w_R|^2 \eta^2 \, dx \, dy \, dt + 2 \int_{0}^{T} \int_{\Omega_R} |w_R|^2 |\nabla \eta|^2 \, dx \, dy \, dt. \]

The last inequality, together with the fact that, by ii., $w_R^2 \leq C^2$, gives (3.15).

Proof of Theorem 3.1. By Proposition 3.5, for any $R > 0$ there exists a solution $(u_R, w_R)$ of (3.6). Moreover, thanks to the uniform boundedness and
monotonicity of \( w_R \) (properties ii. and iii. of Proposition 3.6), there exist the limits

\[
\lim_{R \to \infty} u_R =: u \quad \text{in } \Gamma \times (0, \infty),
\]

\[
\lim_{R \to \infty} w_R =: w \quad \text{in } \Omega \times (0, \infty);
\]

again by Proposition 3.6 ii., \( w \in L^\infty(\Omega \times (0, \infty)) \) and \( u \in L^\infty(\Gamma \times (0, \infty)) \).

We shall prove that \((u, w)\) is a solution to problem (1.1). To see this, take \( T > 0, D \) and \( \psi \) as in Definition 2.1; then select \( R_0 > 0 \) large enough, such that \( D \subset \Omega_{R_0} \). By Definition 3.4, for every \( R > R_0 \) we have:

\[
- \int_0^T \int_D \langle \nabla w_R, \nabla \psi \rangle \, dx \, dy \, dt + \int_0^T \int_{\partial D \cap \Gamma_R} \rho u_R \partial_t \psi \, dx \, dt \\
= \int_{\partial D \cap \Gamma_R} \rho(x) \left[ u(x, T) \psi(x, 0, T) - u_0(x) \psi(x, 0, 0) \right] \, dx. \tag{3.16}
\]

Furthermore, by Proposition 3.6 viii., since \( u \in L^\infty(\Gamma \times (0, \infty)) \) and \( u_0 \in L^\infty(\Gamma) \), for each compact set \( K \subset \overline{\Omega} \), there exists a positive constant \( C \), depending on \( K \) but independent of \( R \) and \( T > 0 \), such that

\[
\int_0^T \int_K |\nabla w_R|^2 \leq C \quad \text{in } K.
\]

By usual compactness arguments, \( \nabla w \in L^2_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \), so \( w|_\Gamma \) is well-defined; moreover, letting \( R \to \infty \) in (3.16) we get

\[
- \int_0^T \int_D \langle \nabla w, \nabla \psi \rangle \, dx \, dy \, dt + \int_0^T \int_D \rho u \partial_t \psi \, dx \, dt \\
= \int_D \rho(x) \left[ u(x, T) \psi(x, 0, T) - u_0(x) \psi(x, 0, 0) \right] \, dx. \tag{3.17}
\]

Hence, it is direct to check that \((u, w)\) is also a solution in the sense of Definition 2.1.

**Remark 3.7.** By construction, the solution \((u, w)\) given in the proof of Theorem 3.1 is minimal, namely, if \((\tilde{u}, \tilde{w})\) is a solution to (2.1) with \( \tilde{u} \geq 0 \) and \( \tilde{w} \geq 0 \), then \( \tilde{u} \geq u \) and \( \tilde{w} \geq w \).

**Remark 3.8.** (i) As already pointed out in the Introduction, our construction of the solution is slightly different from those in [10] and [11]. In fact, the strategy in [10] consists in passing first to the limit as \( R \to \infty \) in the discretization (3.8) to obtain a solution of an elliptic problem in the whole of \( \Omega \), and then in sending the discretization parameter \( \epsilon \) to zero. Our approach consists instead in passing first to the limit as \( \epsilon \to 0 \) in the discretized elliptic problem, in order to get, for any \( R > 0 \), a solution of the parabolic problem...
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Finally sending $R \to \infty$ we get a solution in the whole space. A similar strategy has been implemented in [11], where unbounded cylinders $B_R^N \times \mathbb{R}^+$, instead of bounded domains $\Omega_R$ considered here, have been used to invade $\Omega$. Note that the use of $\Omega_R$ will be expedient in proving Theorem 4.1 in the following Section. Indeed, we shall construct a suitable family of solutions to elliptic problems in bounded domains $\Omega_R$.

(ii) Introducing a varying density $\rho$ does not bring any additional technical difficulty to the proof of existence of solutions to problem (2.1) compared with those in [10]-[11], in which $\rho$ is assumed identically 1.

(iii) The restriction $N = 1$ does not play any role yet and all results of Section 3 can be easily adapted to show existence of very weak solutions in any space dimension.

(iv) Concerning the regularity required for the initial data, we should remark that in [10] and [11] $u_0$ is assumed to belong to $L^1$, whereas in [28] and [29] - where the (non fractional) porous medium equation with varying density has been studied - it is supposed to belong to $L^1_\rho(\Gamma)$. As a consequence, the solution considered in [10] and [11] satisfies $|\nabla w| \in L^2(\Omega \times (\tau, \infty))$ for any $\tau > 0$. We assume instead $u_0$ nonnegative, continuous and bounded, so $u_0$ is only in $L^1_{\text{loc}}(\Gamma) \equiv L^1_{\rho,\text{loc}}(\Gamma)$, but in general $u_0 \notin L^1_\rho(\Gamma)$. Hence we can infer that $\nabla w \in L^2_{\text{loc}}(\Omega \times (0, \infty))$, as required in our Definition 2.1, but in general $\nabla w \notin L^2(\Omega \times (\tau, \infty))$.

(v) We emphasize that, unlike what have been discussed in (ii) and (iv) above, the presence of a variable density and the assumptions on the initial data shall entail significant differences in the proof of uniqueness of solutions, relative to those in the case in which $\rho \equiv 1$ and $u_0 \in L^1$; see also Remark 4.3. Moreover, the hypothesis $N = 1$ will be essential in our proof of uniqueness; see Remark 4.4.

4 Uniqueness of solutions

The scope of this Section is to establish the following

Theorem 4.1. Let assumption $(A_0)$ be satisfied. Then problem (1.1) admits at most one solution $(u, w)$ with $u \geq 0$, $w \geq 0$.

To prove Theorem 4.1 we need some preliminary materials. To begin with, observe that the function

$$\Theta(x, y) := -\frac{1}{\pi} \log |(x, y)|, \quad (x, y) \in \Omega$$

satisfies

$$\begin{cases}
\Delta \Theta = 0 & \text{in } \Omega \\
-\partial_y \Theta = \delta_0 & \text{in } \Gamma,
\end{cases}$$

(4.2)
where $\delta_0$ is the delta distribution concentrated at the origin. As a consequence, (see, e.g., [4] and [5]) for any $F \in C_0^\infty(\Gamma)$, $F \geq 0$, the function

$$w(\cdot, y) := \Theta(\cdot, y) * F$$

is, up to additive constants, the unique bounded solution of

$$\begin{cases}
\Delta U = 0 & \text{in } \Omega \\
\frac{\partial U}{\partial y} = F & \text{in } \Gamma;
\end{cases}$$

for a detailed proof of the fact that $w$ solves problem (4.3) we refer, e.g., to [8].

The following Lemma will play a crucial role in the proof of Theorem 4.1.

**Lemma 4.2.** Let assumption $(A_0)$ be satisfied, and $R_0 > 0$ be fixed. Let $\psi_R$ be the solution of the problem:

$$\begin{cases}
\Delta U = 0 & (x, y) \in \Omega_R \\
\frac{\partial U}{\partial y} = F & x \in \Gamma_R \\
U = 0 & (x, y) \in \Sigma_R
\end{cases}$$

where $R > R_0$, $F \in C^\infty(\Gamma)$, $F \geq 0$, supp $F \subseteq \Gamma_{R_0}$. Then the following statements hold true:

(i) For any $R_1, R_2 \in (R_0, \infty)$, $R_1 \leq R_2$ we have

$$0 < \psi_{R_1} \leq \psi_{R_2} \text{ in } \Omega_{R_1};$$

(ii) There exists a constant $M > 0$ depending on $R_0$ such that for any $R > 2R_0$ we have

$$- \frac{M}{R(\log R - \log R_0)} \leq \frac{\partial \psi_R}{\partial \nu_R} < 0 \text{ on } \Sigma_R,$$

$\nu_R$ denoting the outer normal to $\Sigma_R$.

**Proof.** (i) By the maximum principle $\psi_R > 0$ in $\Omega_R$ for any $R > R_0$; hence the function $\psi_{R_1} - \psi_{R_2}$ ($R_0 < R_1 < R_2$) is a subsolution of problem

$$\begin{cases}
\Delta U = 0 & (x, y) \in \Omega_{R_1} \\
\frac{\partial U}{\partial y} = 0 & x \in \Gamma_{R_1} \\
U = 0 & (x, y) \in \Sigma_{R_1}.
\end{cases}$$
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Then again by the maximum principle we get (4.5).

(ii) Clearly, by the strong maximum principle it follows that, for any $R > R_0$,
\[
\frac{\partial \psi_R}{\partial \nu_R} < 0 \quad \text{on } \Sigma_R.
\] (4.7)

Put
\[
\Theta(\cdot, y) \ast F =: \psi_{\infty}(x, y), \quad (x, y) \in \bar{\Omega},
\]
where $\Theta = \Theta(|(x, y)|)$ is defined in (4.1). It is easily seen that for any $R > R_0$, $\psi_{\infty}$ is a supersolution (4.4). By comparison principle,
\[\psi_R \leq \psi_{\infty} \quad \text{in } \Omega_R.
\]

Indeed, it could be shown that
\[\psi_{\infty}(x, y) = \lim_{R \to \infty} \psi_R(x, y), \quad (x, y) \in \bar{\Omega}.
\]

Now, for any $R > R_0$, consider the problem
\[
\begin{cases}
\Delta U = 0 & (x, y) \in \Omega_R \setminus \Omega_{R_0} \\
\frac{\partial U}{\partial y} = 0 & x \in \Gamma_R \setminus \Gamma_{R_0} \\
U = 0 & (x, y) \in \Sigma_R \\
U = M & (x, y) \in \Sigma_{R_0},
\end{cases}
\] (4.8)

where $M := \max_{\Sigma_{R_0}} \psi_{\infty}$. Define
\[
Z(x, y) = Z(|(x, y)|) := M \frac{\Theta(|(x, y)|) - \Theta(R)}{\Theta(R_0) - \Theta(R)}, \quad (x, y) \in \Omega_R \setminus \Omega_{R_0}.
\]

Using (4.2) immediately follows that $Z$ is a supersolution to problem (4.8) for any $R > 2R_0$. On the other hand, since in particular $R > R_0$, $\psi_R$ is a subsolution to the same problem. By comparison we get for any $R > 2R_0$
\[Z \geq \psi_R \quad \text{in } \Omega_R \setminus \Omega_{R_0}.
\]

Consequently, since for any $R > 2R_0$
\[Z = \psi_R = 0 \quad \text{on } \Sigma_R,
\]
we can infer that, for any $(x, y) \in \Sigma_R$,
\[
\frac{\partial \psi_R}{\partial \nu_R} \geq \frac{\partial Z}{\partial \nu_R} = \frac{\partial Z(R)}{\partial r} = -\frac{M}{R(-\log R_0 + \log R)};
\]

here $r \equiv r(x, y) = |(x, y)|$. The above inequality, combined with (4.7) gives (4.6). This completes the proof. \qed
Now we can prove Theorem 4.1.

**Proof of Theorem 4.1.** Let \((u, w)\) be the solution of problem (1.1) constructed in Theorem 3.1 such solution is minimal as observed in Remark 3.7. Let \((u, w)\) be any other solution of the same problem and set

\[ K := \|w\|_{L^\infty(\Omega \times (0,\infty))}. \]

We claim that

\[ \int_0^T \int_\Gamma [u^m - u^m] F \, dx \, dt = 0 \quad (4.9) \]

for any \(T > 0, F \in C^\infty_0(\Gamma)\).

By the Claim the conclusion follows. In fact, in view of the arbitrariness of \(F\), (4.9) implies

\[ u = \underline{u} \quad \text{in} \quad \Gamma \times (0,T), \quad (4.10) \]

whence the conclusion.

It remains to prove the Claim. To this aim, without loss of generality, we suppose \(\text{supp}(F) \subseteq \Gamma_{R_0}\) for some \(R_0 > 0, F \geq 0, F \neq 0\). Take \(R > 2R_0\) and let \(\psi_R(x, y)\) be the solution of problem (4.4). Using \(\psi_R\) as a test function for the equation (2.1) (see Definition 2.1) we obtain, for any \(\tau > 0,\)

\[ 0 = -\int_0^\tau \int_{\partial\Omega_R} w \frac{\partial \psi_R}{\partial \nu_R} \, dS \, dt - \int_{\Gamma_R} \rho(x) \left[ u(x, \tau) - u_0(x) \right] \psi_R(x, 0) \, dx \]

\[ = \int_0^\tau \int_{\Gamma_R} w \frac{\partial \psi}{\partial y} \, dx \, dt - \int_0^\tau \int_{\Sigma_R} w \frac{\partial \psi_R}{\partial \nu_R} \, dx \, dy \, dt \]

\[ - \int_{\Gamma_R} \rho(x) \left[ u(x, \tau) - u_0(x) \right] \psi_R(x, 0) \, dx \]

\[ = -\int_0^\tau \int_{\Gamma_R} u^m F \, dx \, dt - \int_0^\tau \int_{\Sigma_R} w \frac{\partial \psi}{\partial \nu_R} \, dx \, dy \, dt \]

\[ - \int_{\Gamma_R} \rho(x) \left[ u(x, \tau) - u_0(x) \right] \psi_R(x, 0) \, dx. \quad (4.11) \]

Similarly, using \(\psi_R\) as a test function for \(\underline{u}\), we get

\[ 0 = -\int_0^\tau \int_{\Gamma_R} \underline{u}^m F \, dx \, dt - \int_0^\tau \int_{\Sigma_R} w \frac{\partial \psi}{\partial \nu_R} \, dx \, dy \, dt \]

\[ - \int_{\Gamma_R} \rho(x) \left[ \underline{u}(x, \tau) - u_0(x) \right] \psi_R(x, 0) \, dx. \quad (4.12) \]
By subtracting (4.12) to (4.11) we have

\[
\int_{\Gamma_R} \rho(x) \left[ u(x, \tau) - \underline{u}(x, \tau) \right] \psi_R(x, 0) \, dx + \int_{\Gamma_R} \left[ u^m - \underline{u}^m \right] F(x) \, dx \, dt = - \int_{\tau_0}^{T} \int_{\Sigma_R} \left\{ w - \underline{w} \right\} \frac{\partial \psi_R}{\partial \nu_R} \, dx \, dt.
\]  

(4.13)

Since \( F \geq 0, \psi_R \geq 0 \) and, by minimality of \( u, \underline{u} \geq \underline{u} \), equality (4.13) with \( \tau = T \) gives

\[
\int_{\tau_0}^{T} \int_{\Sigma_R} \left\{ w - \underline{w} \right\} \frac{\partial \psi_R}{\partial \nu_R} \, dx \, dt \leq \liminf_{R \to \infty} \left| \int_{\tau_0}^{T} \int_{\Sigma_R} \left\{ w - \underline{w} \right\} \frac{\partial \psi_R}{\partial \nu_R} \, dx \, dt \right|.
\]  

(4.14)

By recalling that \(|\Sigma_R| = \pi R/2\) and by using (4.6) we obtain

\[
\left| \int_{\tau_0}^{T} \int_{\Sigma_R} \left\{ w - \underline{w} \right\} \frac{\partial \psi_R}{\partial \nu_R} \, dx \, dt \right| \leq K \pi R \sup_{\Sigma_R} \left| \frac{\partial \psi_R}{\partial \nu_R} \right| \leq K \pi \frac{M}{(-\log R + \log R)} \to 0, \quad \text{as } R \to \infty.
\]  

Hence, the right hand side of (4.14) is zero and the claim (4.9) follows. This concludes the proof.

Remark 4.3. Uniqueness results in [10] and [11] cannot be applied to problem (1.1), as they require \( \rho \equiv 1 \). Furthermore, even in the case in which \( \rho \) is identically 1, the results in [10] and [11] cannot be adapted to prove uniqueness of solutions in the sense of Definition 2.1.

Remark 4.4. If \( N \geq 2 \), a result similar to Lemma 4.2 could be proved. In this case, the estimate (4.6) must be replaced by the following:

\[
- \frac{M}{R^{-N}} \leq \frac{\partial \psi_R}{\partial \nu_R} < 0, \quad \text{on } \Sigma_R,
\]  

(4.15)

for some \( M > 0 \), for any \( R > 2R_0 \). Hence we cannot get the conclusion in the previous proof. A similar issue arises for problem (1.3) when \( N \geq 3 \); however, methods used in [24] to get the conclusion in that case cannot be adapted to the present nonlocal situation.

Remark 4.5. Note that the proof of uniqueness for problem (1.3) when \( N = 1, 2 \) is quite different from the previous one (see [17], [27]). However, the estimate (4.6) is crucial in [17] for \( N = 2 \), too. Observe finally that the hypothesis \( \rho \in L^\infty(\mathbb{R}^2) \) used in [17] is not required here. Thus our arguments can be adapted in order to prove uniqueness for problem (1.3) in the case \( N = 2 \) and \( \rho \) possibly unbounded.
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