Extension of Busch’s Theorem to Particle Beams

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Phys. Rev. Accel. Beams 21 014201 (2018)
Outline

• Hamiltonian, conjugate variables (short & basic)
• Busch theorem on single particle
• Applications:
  • electron cooling
  • magnetic bottle
• Projected rms-emittances, eigen emittances
• Extension of Busch theorem to beams
• Applications:
  • increase of FEL gain & collider luminosity
  • increase of injection efficiency into ion rings
Hamiltonian, conjugate variables

Hamiltonian is total energy:

\[
H := E_{\text{kin}} + E_{\text{pot}}
\]

\[
H(\hat{p}, \hat{r}) := \frac{\hat{p}^2}{2m} + V(\hat{r})
\]

\( \hat{r} \) := position

\( \hat{p} \) := mechanical momentum = \( m\hat{v} := m\hat{r} \)

equations of motion through derivatives of \( H \):

\[
\dot{\hat{p}} := -\frac{\partial H}{\partial \hat{r}} = -\frac{\partial V}{\partial \hat{r}} = \hat{F}
\]

\[
\dot{\hat{r}} := \frac{\partial H}{\partial \hat{p}} = \frac{\hat{p}}{m}
\]

\( \hat{r} \) and \( \hat{p} \) are conjugate variables
Force from magnetic field

force on particle by constant magnetic field \( \vec{B} := \begin{bmatrix} 0 \\ 0 \\ B_s \end{bmatrix} = \vec{v} \times \vec{A} \)

\( \vec{F} = eq(\vec{v} \times \vec{B}) \)

\( F_x = \dot{p}_x = eqv_yB_s \)

\( F_y = \dot{p}_y = -eqv_xB_s \)

to obtain equations of motion from Hamiltonian mechanics \( \rightarrow \) use "generalized" momentum

\( \tilde{p} := \vec{P} + eq\vec{A} \) in the Hamiltonian:

\[ H(\vec{r}, \vec{P}) = \tilde{H}(\vec{r}, \vec{p}) = \frac{(\vec{p} - eq\vec{A})^2}{2m} \]

and apply previous formalism:

\( \dot{\vec{p}} = \frac{\partial \tilde{H}}{\partial \vec{r}} \) together with \( \tilde{A} = \frac{1}{2} \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix} \) and \( \dot{\vec{p}} = \vec{P} + eq\dot{\vec{A}} \)
Generalized angular momentum

• analogue to generalized momentum, the generalized angular momentum is defined
  \[ \mathbf{\tilde{L}} := \mathbf{\hat{r}} \times \mathbf{\hat{p}} \]

• if there is just magnetic field, the generalized angular momentum is preserved
  \[ \mathbf{\tilde{L}} = \mathbf{\hat{r}} \times (\mathbf{\hat{P}} + eq\mathbf{\hat{A}}) = \text{const} \]

• Busch theorem is special case for:
  • cylindrical symmetric magnetic field \[ \mathbf{\hat{B}} = \mathbf{\hat{B}}(s) \]
  • s-component of \[ \mathbf{\tilde{L}}: L_s = \text{const} \]
Busch Theorem

cylindrically symmetric magnetic field with $\mathbf{v} \times \mathbf{B} = 0$

$$\mathbf{B} = \frac{1}{2} \begin{bmatrix} -x B_s' \\ -y B_s' \\ 2 B_s \end{bmatrix} = \mathbf{v} \times \mathbf{A} = \frac{1}{2} \mathbf{v} \times \begin{bmatrix} -y B_s \\ x B_s \\ 0 \end{bmatrix}$$

$$L_s = [\mathbf{r} \times (\mathbf{P} + eq\mathbf{A})] \cdot \mathbf{e}_s = \text{const}$$

- particle entering into region with $\mathbf{B}$ (hence $\mathbf{A}$) acquires orbital angular momentum
- $\mathbf{r} \times \mathbf{A}$ has angular momentum as well
- sum of both acquired angular momenta is zero
Busch theorem from 1926 states

\[ L_s = \left[ \vec{r} \times (\vec{P} + e\vec{qA}) \right] \cdot \vec{e}_s = \text{const} \]

using cylindrical coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta \]

\[ \vec{P} = m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \beta c \end{bmatrix} \]

\[ L_s = mr^2 \dot{\theta} + \frac{1}{2}eqBsr^2 = \text{const} \]

\[ L_s = mr^2 \dot{\theta} + \frac{eq}{2\pi} \Psi = \text{const} \]

\[ L_s = \text{orbital angular momentum} + \text{flux through area of cyclotron motion} = \text{const} \]

H. Busch, Z. Phys. 81 (5) 924 (1926)
Busch Theorem

\[ mr_0^2 \dot{\theta}_0 + \frac{eq}{2} B_{s0} r_0^2 = mr^2 \dot{\theta} + \frac{eq}{2} B_s r^2 = \text{const} \]

preservation of magn. flux:

\[ B_{s0} r_0^2 = B_s r^2 = \text{const} \]

\[ r = r_0 \sqrt{\frac{B_0}{B_s}} \]
Applications:
Electron beam size shaping

\[ r = r_0 \sqrt{\frac{B_{s0}}{B_s}} \]

- cyclotron radius much smaller than beam size → beam size shaped by magnetic field
- applied at low energy electron beams

![Diagram showing electron beam size shaping](image)
mean transv. velocity is measure for transv. beam temperature

\[ T_\perp \sim \langle v_\perp^2 \rangle \]

using \( v_\perp = \frac{reqB_s}{m} \) and \( r = r_0 \sqrt{\frac{B_{s0}}{B_s}} \)

results into \( v_\perp = v_{\perp0} \frac{B_s}{B_{s0}} \)

electron beam expansion by magn. field expansion \( B_s < B_{s0} \):

- increases beam radius
- lowers beam temperature
- lowers electron density

technique applied at many e-coolers: 2*IMP, LEIR, TSR, CRYRING, SIS-18
Applications: Magnetic bottle

from Busch theorem:
\[ B_0 s r_0^2 = B_s r^2, \quad \dot{\theta} = \frac{e q B_s}{m} = \frac{v_\perp}{r} \]

preservation of total kin. energy:
\[ v_{\|0}^2 + v_{\perp0}^2 = v_{\|}^2 + v_{\perp}^2 \]

\[ \rightarrow v_{\|}^2(B_s) = v_{\|0}^2 + v_{\perp0}^2 - \left( \frac{e q r_0}{m} \right)^2 B_{s0} B_s \]

→ beam confinement by strong \( B_s \) in magn. bottles:
  • traps
  • ECR sources
Busch theorem can be further generalized to:

\[ \oint_{\mathcal{C}} \vec{v} \cdot d\vec{C} + \frac{eq}{m\gamma} \psi = \text{const} \]

\( C_i \) enclose possible single particle trajectories

\[ C_i := \text{circles with constant } r_i \rightarrow m\gamma r^2 \dot{\theta} + \frac{eq}{2\pi} \Psi = \text{const} \]
Projected rms emittance

rms emittances defined through beam's second moments:

- \( a_i, b_i \) : two coordinates of particle \( i \)
- \( \langle ab \rangle \): mean of product \( a_i b_i \)
- \( C \) is moment matrix (symmetric)

Projected rms emittance

\[
\varepsilon_x^2 = \langle xx \rangle \langle x'x' \rangle - \langle xx' \rangle^2
\]

\[
C_x = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle \\
\langle x'x \rangle & \langle x'x' \rangle \end{bmatrix}, \quad \varepsilon_x^2 = \det C_x
\]

\[
C_y = \begin{bmatrix} \langle yy \rangle & \langle yy' \rangle \\
\langle y'y \rangle & \langle y'y' \rangle \end{bmatrix}, \quad \varepsilon_y^2 = \det C_y
\]

\((x,y,x',y')\) are laboratory coordinates which can be measured
Transport of moments

linear transport from point_1 → point_2 through matrices:

\[
\begin{bmatrix}
  x \\
  x'
\end{bmatrix}_2 = M_x \begin{bmatrix}
  x \\
  x'
\end{bmatrix}_1
\]

\[M_x = \begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix}, \quad \text{det } M_x = 1\]

beam moments transport by matrix equation:

\[C_{x2} = M_x C_{x1} M_x^T\]

analogue in y
4d linear beam dynamics

\[ \varepsilon_{4d}^2 = \det \begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix} \]

transport of moments from 1 → 2 as usual:

\[
\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_1, \quad \det M = 1
\]

\[ C_2 = MC_1M^T \]

if x & y planes are not coupled

\[ \varepsilon_{4d}^2 = \det \begin{bmatrix} <xx> & <xx'> & 0 & 0 \\ <x'x> & <x'x'> & 0 & 0 \\ 0 & 0 & <yy> & <yy'> \\ 0 & 0 & <y'y> & <y'y'> \end{bmatrix} = (\varepsilon_x \cdot \varepsilon_y)^2 \]

transport of moments from 1 → 2 as usual:

\[
M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix}, \quad \det M = \det M_x \cdot \det M_y = 1 \cdot 1 = 1
\]

\[ C_2 = MC_1M^T \]
Eigen-emittances

- linear (4d), Hamiltonian beam line elements preserve:

\[ \varepsilon_{4d}^2 = \det \begin{bmatrix}
<cxx> & <xx'> & <xy> & <xy'> \\
<xx'> & <xx'x'> & <x'y> & <x'y'> \\
<xy> & <yy'> & <yy> & <yy'> \\
<y'y> & <yy'x'> & <y'y'> & <y'y'> \\
\end{bmatrix} \]

- rms emittance \( \varepsilon_{4d}^2 \)

- the two eigen-emittances

\[
\begin{align*}
\varepsilon_1 &= \frac{1}{2} \sqrt{-\text{tr}[(CJ)^2] + \sqrt{\text{tr}^2[(CJ)^2] - 16\text{det}(C)}} \\
\varepsilon_2 &= \frac{1}{2} \sqrt{-\text{tr}[(CJ)^2] - \sqrt{\text{tr}^2[(CJ)^2] - 16\text{det}(C)}}
\end{align*}
\]

\[
C = \begin{bmatrix}
<cxx> & <xx'> & <xy> & <xy'> \\
<xx'> & <xx'x'> & <x'y> & <x'y'> \\
<xy> & <yy'> & <yy> & <yy'> \\
<y'y> & <yy'x'> & <y'y'> & <y'y'> \\
\end{bmatrix} \quad J := \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

- the formulas are pretty ugly and their application consumes much time & paper

A.J. Dragt, Phys. Rev. A 45 4 (1992)
if, and only if there is no $x\leftrightarrow y$ coupling, i.e. $C = \begin{bmatrix} <xx> & <xx'> & 0 & 0 \\ <x'x> & <x'x'> & 0 & 0 \\ 0 & 0 & <yy> & <yy'> \\ 0 & 0 & <y'y> & <y'y'> \end{bmatrix}$

- rms emittances = eigen-emittances

if there is any coupling

- rms emittances ≠ eigen-emittances
- coupling parameter $t = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_1 \varepsilon_2} - 1 \geq 0$

term „eigen-emittance“ is quite unknown, since generally coupling is just ignored
rms vs eigen-emittances: Example

4d distribution behind ECR source

\[ \varepsilon_x = 123 \text{ mm mrad} \]
\[ \varepsilon_y = 125 \text{ mm mrad} \]
\[ \varepsilon_1 = 17 \text{ mm mrad} \]
\[ \varepsilon_2 = 231 \text{ mm mrad} \]

\[ \varepsilon_{4d} = \varepsilon_1 \cdot \varepsilon_2 = 3927 \text{ (mm mrad)}^2 \]
\[ \varepsilon_x \cdot \varepsilon_y = 15375 \text{ (mm mrad)}^2 \]
\[ \varepsilon_x \cdot \varepsilon_y = 3.9 \varepsilon_{4d} \]
Coupling linear elements:
Skew quadrupole

normal quadrupole
no x-y coupling  skew  tilted by 45° (clockwise)
x-y coupling
Coupling linear elements:

**Solenoid**

\[ \kappa := \frac{B}{2(B\rho)} \]

\[ \alpha(L) = -2\kappa L \]

**Complete solenoid matrix**

\[ M_{\text{sol}} = M_{fo} \cdot M_{||} \cdot M_{fi} \]
How elements change emittances

applying ugly formulas →

| element               | $\text{rms}_{x,y}$ | $4d \text{ rms}$ | $\text{eigen}_{1,2}$ |
|-----------------------|--------------------|-------------------|----------------------|
| drift                 | no                 | no                | no                   |
| quadrupole            | no                 | no                | no                   |
| tilted quadrupole     | yes                | no                | no                   |
| dipole                | no                 | no                | no                   |
| tilted dipole         | yes                | no                | no                   |
| solenoid              | yes                | no                | no                   |
| solenoid fringe       | yes                | no                | yes                  |
| solenoid axial field  | yes                | no                | yes                  |
How elements change emittances

| element            | rms$_{x,y}$ | 4d rms | eigen$_{1,2}$ |
|--------------------|-------------|--------|---------------|
| drift              | no          | no     | no            |
| quadrupole         | no          | no     | no            |
| tilted quadrupole  | yes         | no     | no            |
| dipole             | no          | no     | no            |
| tilted dipole      | yes         | no     | no            |
| solenoid           | yes         | no     | no            |
| solenoid fringe    | yes         | no     | yes           |
| solenoid axial field | yes   | no     | yes           |

- eigen-emittances seem to change by magnetic flux through beam surface (note: in front of and behind solenoid flux is zero !)

- in the following this will be proven ...
Preservation of eigen emittances in conjugate coordinates

• reminder: generalized momentum $\tilde{p} := \tilde{P} + eq\tilde{A}$, i.e.,
  
  \[ p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)} \]
  \[ p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)} \]

• original ansatz of Busch for single particle:
  • angular momentum including the contribution from $\tilde{r} \times eq\tilde{A}$ is preserved
  • „generalized angular momentum“ is preserved

• ansatz for extension to beams:
  • eigen-emittances including contribution from $\tilde{r} := \tilde{A}/(B\rho)$ are preserved
  • „generalized“ eigen-emittances are preserved
Preservation of eigen emittances in conjugate coordinates

- calculation of „generalized“ eigen-emittance through replacing \((x', y')\) by \((p_x, p_y)\)

\[
C = \begin{bmatrix}
<xx> & <xx'> & <xy> & <xy'> \\
<xx'> & <xx'’> & <xy'> & <xy'’> \\
<yx> & <yx'> & <yy> & <yy'> \\
<y'x> & <y'x'> & <y'y> & <y'y'>
\end{bmatrix}
\rightarrow \tilde{C} = \begin{bmatrix}
<x^2> & <xp_x> & <xy> & <xp_y> \\
<xp_x> & <p_x^2> & <yp_x> & <p_x p_y> \\
<xy> & <yp_x> & <y^2> & <yp_y> \\
<x p_y> & <p_x p_y> & <yp_y> & <p_y^2>
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

\[
\tilde{\varepsilon}_{1/2} = \frac{1}{2} \sqrt{-tr[(\tilde{C}J)^2] \pm \sqrt{tr^2[(\tilde{C}J)^2] - 16 \det(\tilde{C})}}
\]

- if \(\tilde{\varepsilon}_{1/2}\) are preserved, also \(\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2\) must be preserved

- state \(\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = \text{const}\) by substituting

\[
p_x := x' + \frac{A_x}{B_\rho} = x' - \frac{y B_s}{2 (B_\rho)}
\]

\[
p_y := y' + \frac{A_y}{B_\rho} = y' + \frac{x B_s}{2 (B_\rho)}
\]

- to obtain expression for useful „laboratory“ eigen-emittances
Preservation of eigen emittances in conjugate coordinates

• state $\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = \text{const}$ by substituting

\[
p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}
\]
\[
p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}
\]

with $A := \sqrt{\langle x^2 \rangle \langle y^2 \rangle - \langle xy \rangle^2}$

• this delivers:

\[
(\varepsilon_1 - \varepsilon_2)^2 + \left[ \frac{AB_s}{(B\rho)} \right]^2 + 2\frac{B_s}{(B\rho)} \left[ \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \right] = \text{const}
\]

• confirmed: change of laboratory eigen-emittances just through long. magn. field $B_s$

• quadrupoles and dipoles (even) skewed: $B_s = 0 \rightarrow \varepsilon_{1/2} = \text{const}$
Sum of quantities is invariant

\[
(\varepsilon_1 - \varepsilon_2)^2 + \left[ \frac{AB_s}{B\rho} \right]^2 + 2 \frac{B_s}{B\rho} \left[ \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \right] = \text{const}
\]

- sum of three quantities forms an invariant
- difference of eigen-emittances, flux through beam area, ......
- what is the third?
  - has dimension m^3
  - scales with beam rms area as for \( y = ax \) it vanishes
  - vanishes for uncorrelated beams
  - invariant under rotation around beam axis
  - investigate term for some examples...
Understanding third term:
Example objects

\[ W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yy' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \]

rigid object rotating with \( \omega \):

\[ W_A = 2\omega A^2 = 2AL \]

object under shear:

\[ W_A = -\alpha A^2 \neq 2AL \]

solenoid fringe field performs rigid beam rotation
thin skew quad performs shear
Understanding $W_A$: Transformations

\[ W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \]

check, how $W_A$ is changed under transformation through:

- thin reg. quadrupole: \( x' \to x' - qx \) and \( y' \to y' + qy \)
- thin skew quadrupole: ...
- short solenoid: ...

- \( \to \) non of them changes $W_A$

scheme of mutual cancellation of constituents of $W_A$
Understanding $W_A$: Pick idea from gen. Busch theorem

\[ \oint_C \vec{v} \cdot d\vec{C} + \frac{eq}{m\gamma} \psi = \text{const} \]

sort of mean rotation around beam area

try ansatz:

\[ W_A = 2A \int_C \vec{r}'(x, y, s) \cdot d\vec{C} \]

mean angle integrated along curve enclosing beam area ... multiplied with beam area

\[ \vec{r}'(x, y, s) := [x'(x, y, s), y'(x, y, s), 1] \]

mean: average (x',y') at given (x,y)
Calculating $W_A$

$$W_A = 2A \int_C r'(x, y, s) \cdot d\vec{C}$$

as $W_A$ is invariant under rotation
→ calculated for ellipse being turned upright
→ $<xy> = 0$!
Calculating $W_A$

$$W_A = 2A \int_{C} \vec{r}'(x, y, s) \cdot d\vec{c}$$

$$x = \sqrt{\langle x^2 \rangle} \cos \theta,$$

$$y = \sqrt{\langle y^2 \rangle} \sin \theta,$$

$$d\vec{c} = \begin{bmatrix} -x \\ y \end{bmatrix} d\theta$$

Taylor expansion of $\vec{r}'$ to first order:

$$\vec{x}'(x, y) := \vec{x}'(0, 0) + \frac{\partial \vec{x}'}{\partial x} \cdot x + \frac{\partial \vec{x}'}{\partial y} \cdot y.$$

$$\vec{y}'(x, y) := \vec{y}'(0, 0) + \frac{\partial \vec{y}'}{\partial x} \cdot x + \frac{\partial \vec{y}'}{\partial y} \cdot y.$$  

Finally confirms:

$$W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle)$$
Busch theorem for particle beams

using expression for $W_A$ finally delivers:

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[ \frac{AB_s}{(B\rho)} \right]^2 + \frac{4AB_s}{(B\rho)} \oint r'd\vec{C} = const$$

acceleration can be included by initially multiplying with $m\gamma\beta c$ at both sides

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

... resulting in Busch's theorem extended to accelerated particle beams:

$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{ep\psi}{mc\pi} \right]^2 + \frac{4ep\psi\beta\gamma}{mc\pi} \oint r' \cdot d\vec{C} = const$$
original Busch theorem for single particle:

\[
\frac{eq}{m\gamma} \psi + \oint_C \vec{v} \cdot d\vec{C} = \text{const}
\]

theorem extended to accelerated particle beams:

\[
(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{eq\psi}{mc\pi} \right]^2 + \frac{4eq\psi \beta \gamma}{mc\pi} \oint_C \vec{r}' \cdot d\vec{C} = \text{const}
\]

- both expressions include flux and "vorticity" \( \vec{v} \cdot d\vec{C} \approx (\vec{V} \times \vec{v}) \, d\vec{A} \) (Stoke’s law)
- the extended theorem additionally includes eigen-emittances
- theorem allows very fast modelling of setups for emittance gymnastics
Varification through simulations

- beam tracking through three solenoids
- extended fringe fields from $\vec{B}$-maps
- invariance confirmed
flat electron beams, i.e., $\varepsilon_x \ll \varepsilon_y$ useful for:

- increase of luminosity in $e^-/e^+$ colliders
- production of X-ray pulses with femto seconds in duration

Test accelerator at FERMILAB demonstrated $\varepsilon_y/\varepsilon_x = 100$:

- create beam at photo cathode being immersed into $B_s = B_0$
- reduce $B_s$ to zero, accelerate, and decouple x/y-planes

P. Piot et al., Phys. Rev. ST Accel. Beams 9 031001 (2006)
prior to formulation of extended Busch theorem, the deviation of final beam emittances took several pages ...

\[ \epsilon_{nfx/y} = \pm L \beta \gamma + \sqrt{(L \beta \gamma)^2 + \epsilon_{4d}^2} \]

with \( \epsilon_{4d} = \epsilon_{ni1} \cdot \epsilon_{ni2} \)

and \( L := (eB_0A_0)/(2m\gamma\beta c) \)

Kwang-Je Kim, Phys. Rev. ST Accel. Beams 6 104002 (2006)
Application to electron beams: Flat beam creation at FERMILAB

Applying theorem:

\[(\epsilon_{1} - \epsilon_{2})^2 + \frac{e q \psi}{mc^2} \frac{e q \beta \gamma}{mc^2} \int r' \cdot d\bar{C} = \text{const}\]

At cathode beam is symmetric:

- eigen = rms emittances: \(\epsilon_{1/2} = \epsilon_{x/y}\)
- eigen (rms) emittances are equal: \(\epsilon_{1/x} = \epsilon_{2/y}\)
- immersed into B-flux
- no x/y coupling \(\rightarrow W_{A} = 0\)

Final beam:

- eigen = rms emittances: \(\epsilon_{1/2} = \epsilon_{x/y}\)
- eigen (rms) emittances differ: \(\epsilon_{1/x} \neq \epsilon_{2/y}\)
- no B-flux
- x/y coupling removed \(\rightarrow W_{A} = 0\)
replacing $\varepsilon_{nf_y} = \varepsilon_{4d}/\varepsilon_{nf_x}$

and $eB_0A_0 = 2m\gamma\beta c \cdot L$

results into $\varepsilon_{nf_x} = L\beta\gamma \pm \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2}$

using upper sign gives $\varepsilon_{nf_{x/y}} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2}$
Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

- beams from linacs: $\varepsilon_x \approx \varepsilon_y$
- hor. multi-turn injection into rings profits from $\varepsilon_x < \varepsilon_y$
- EmTEx @ transfer channel:
  - place charge state stripper inside short solenoid
  - x/y-decoupling afterwards
prior to formulation of extended Busch theorem, the deviation of final beam emittances took several pages ...

\[
(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 = (\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + (A_f B_0)^2 \left[ \frac{1}{(B \rho)_{7+}} - \frac{1}{(B \rho)_{3+}} \right]^2
\]

C. Xiao et. al, Phys. Rev. ST Accel. Beams 16 044201 (2013)  
L. Groening, arXiv 1403.6962 (2014)
change of q is a non-Hamiltonian action → „splitting“ into two Hamiltonian actions

extended theorem before stripping (q=3+)

\[
(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[ \frac{A_f B_0}{(B \rho)_{3+}} \right]^2 + \frac{2B_0}{(B \rho)_{3+}} \mathcal{W}_{A_f} = \text{const}
\]

distributed theorem after stripping (q=7+)

\[
(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[ \frac{A_f B_0}{(B \rho)_{7+}} \right]^2 + \frac{2B_0}{(B \rho)_{7+}} \mathcal{W}_{A_f} = \frac{\text{const}}{}
\]
Application to ion beams: EmTEx up to stripping foil

beam line entrance:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- measured: $\varepsilon_{x,3+}$ and $\varepsilon_{y,3+}$
- no B-flux
- no x/y coupling $\rightarrow W_A = 0$

inside solenoid & just before foil:

- eigen ≠ rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- x/y coupling $\rightarrow W_{A_f} = -(B_0A_f^2)/(B\rho)_{3+}$

$A_f :=$ beam area at foil, from measurements, $\approx$ const along short solenoid

$$(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_fB_0}{(B\rho)_{3+}}\right]^2 + \frac{2B_3}{(B\rho)_{3+}}W_{A_f}$$

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Application to ion beams: EmTEx up to stripping foil

\[(\varepsilon_x,3^+ - \varepsilon_y,3^+)^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 + \left(\frac{A_f B_0}{(B \rho)_{3+}}\right)^2 + \frac{2B_0}{(B \rho)_{3+}} W_{Af}\]

plugging in \(W_{Af} = -(B_0 A_f^2)/(B \rho)_{3+}\):

\[(\varepsilon_x,3^+ - \varepsilon_y,3^+)^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 - \left(\frac{A_f B_0}{(B \rho)_{3+}}\right)^2\]

using experiment’s parameters:

\[(\varepsilon_{1f} - \varepsilon_{2f})^2 = 2.755 \text{ (mm mrad)}^2\]
Application to ion beams: EmTEx behind the stripping foil

- foil changes just $q$, i.e., $(B\rho)$
- $\varepsilon_{1,2,x,y}$, B-flux, and $W_{Af}$ do not change

inside solenoid & just after foil:

- eigen ≠ rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- $x/y$ coupling → $W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+}$

beam line exit:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- to be calculated/measured: $\varepsilon_{x,7+}$ and $\varepsilon_{y,7+}$
- no B-flux
- no $x/y$ coupling → $W_A = 0$
Application to ion beams: EmTEx behind the stripping foil

\[
(p_{1f} - p_{2f})^2 + \left(\frac{A_f B_0}{(B_p)_{7+}}\right)^2 + \frac{2B_0}{(B_p)_{7+}}W_{A_f} = (\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 + 0 + 0
\]

plugging in \( W_{A_f} = -(B_0 A_f)^2/(B_p)_{3+} \) and \( (p_{1f} - p_{2f})^2 = 2.755 \text{ (mm mrad)}^2 \)

finally delivers \(|\varepsilon_{x,7+} - \varepsilon_{y,7+}| = 2.21 \text{ mm mrad}\)

the measured values are :

- \( \varepsilon_{x,7+} = 2.76(14) \text{ mm mrad} \)
- \( \varepsilon_{y,7+} = 0.72(4) \text{ mm mrad} \)
- \( |\varepsilon_{x,7+} - \varepsilon_{y,7+}| = 2.04(14) \text{ mm mrad} \) → very good agreement to extended Busch theorem

L. Groening et. al, Phys. Rev. Lett. 113 044201 (2014)
EmTEx increases injection efficiency into SIS18

\[
\left( \varepsilon_{x,7^+} - \varepsilon_{y,7^+} \right)^2 = \left( \varepsilon_{x,3^+} - \varepsilon_{y,3^+} \right)^2 + \left( A_f B_0 \right)^2 \left[ \frac{1}{(B \rho)_{7^+}} - \frac{1}{(B \rho)_{3^+}} \right]^2
\]

emittance shaping
Summary

• Busch’s original theorem for single particle was extended to particle beams

• Original and extended theorem look very similar

\[
\frac{e\gamma}{m} \psi + \oint_C \vec{v} \cdot d\vec{C} = \text{const}
\]

\[
(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{e\gamma \psi}{mc} \right]^2 + \frac{4e\gamma \beta \gamma}{mc} \oint_C \vec{r}' \cdot d\vec{C} = \text{const}
\]

• Extended theorem for very fast modelling of emittance gymnastic exp. as
  • flat electron beams at FERMILAB
  • flat ion beams at GSI

• Its power is through provision of an invariant

• Using invariants is much more convenient than solving equs. of motion