Kinetics, Hydrodynamics and Stochastodynamics of Cellular Structure Coarsening

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Abstract

For the first time the phenomenon of cellular structure coarsening are consistently analysed from the positions of kinetic, hydrodynamic and stochastodynamic theories of nonequilibrium statistical systems. Thereby micro-, meso- and macroscopic levels of approach are distinguished. At the microscopic level the cellular structure is describe by a probability distribution function in a phase space of cell coordinates and of cell sizes. A kinetic equation for the function is written and a development to a hydrodynamic equation of a mesoscopic cell medium is realised. It has the form of a diffusion-reaction equation with a negative "diffusion" coefficient and with a cell interface density playing the role of concentration. Its analysis reveals a new effect of macroscopic patterning in the cell medium: existence of space-correlated stochastic fluctuations of the cell interface density.

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1 Introduction

There are many physical systems consisting of homogeneous domains separated by distinct boundaries, so called cellular structures: ordered domains in alloys, magnetic domains, gas bubbles in soap froth and in lipid monolayers, grains in polycrystals, etc. [1, 2, 3, 4]. We have here a simple physical situation (Fig. 1), when the system volume $V$ being the sum of volumes of all cells, the volume specific energy and the interface specific energy $E$ are constant. The internal energy of the system $E$ is completely defined by the cell interface area $\Sigma$. It is easy to see that for any two neighbouring cells the volume decrease of the smaller one (down to complete disappearance) and the simultaneous volume increase of the greater one diminish $\Sigma$. This permanent topological possibility to decrease $E = E \Sigma$ makes the system intrinsically unstable. The mean cell size increases during this statistical process (Fig. 1), named "coarsening" (large cells "eat up" small ones). The changes of $E$ related to the elementary statistical event of cell disappearance are so large in comparison with the energy of thermostuctuations,

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that the coarsening cannot be described by ways of usual thermodynamics [5]. Still this challenging statistical problem can be elegantly solved by methods of physical kinetics, as shown by Lifshitz, Slyozov and Hillert (LSH) [6, 7, 8], if we consider the cellular structure as some dynamic system. To get cell dynamics equation let us following LSH approximately consider each cell as an elastic sphere shell embedded in an elastic environment which represents the average effect of cell-cell interaction. For a cell of radius $a$ we write an interface tension force as

$$F = -E(1/a - \kappa),$$

where the term $1/a$ is related to the effect of interface surface tension of the cell itself and the term $\kappa$ - to the effect of interface surface tension of adjoining cells (the elastic environment). The quantity $\kappa$ is nothing that a self-consistent mean of cell curvature throughout the system. Believing that the cell interface velocity $w$ affected by $F$ is determined by viscous law $w = MF$, where $M$ is the cell interface mobility, we have [6, 7, 8]

$$\frac{da}{dt} = -D(1/a - \kappa) \quad (1)$$

with $D = ME$. In the coarsening process the cells move in a size space $\{a\}$ (some phase space) with a velocity (1). We can write a continuity equation in this space as

$$\frac{\partial f}{\partial t} + \frac{\partial (fda/\partial t) \partial a}{\partial a} = 0 \quad (2)$$

for a cell size distribution function $f(a, t)$ [3, 4, 5]. The function $f(a, t)$ is normalised so that $n(t) = \int_0 a fda$ is the mean number of cells per unit volume:

$$\Omega_d \int_0 a^d fda = 1; \quad (3)$$

$\bar{a}(t) = \int_0 a fda/n$ is the mean cell size and $\rho(t) = \Sigma/V = (\Omega_d d/2) \int_0 a^{d-1} fda$ is the mean cell interface density. Here $d = 2, 3$ is the cellular structure dimensionality and $\Omega_d = 2\pi(d - 1)/d$. Condition (3) is nothing that the conservation law of the total volume of all cells in unit volume. A remarkable property of this statistical dynamic system, discovered by LSH, is the occurrence of a stable steady-state regime of evolution. This regime is observed if and only if the mean curvature $\kappa(t)$ take the unique ”self-consistent” value $\kappa(t) = \sqrt{2/(Dt)}$. All other macroscopic system variables obey also simple scaling laws: $\bar{a}(t) \propto \rho^{-1}(t) \propto t^{1/2}$, $n(t) \propto t^{d/2}$ and the solution of (1)-(3) has a universal scaling form

$$f_0(a, t) \propto (1/\bar{a}(t))^{d+1} \varphi(\mu_1 a/\bar{a}(t)) \quad (4)$$

where $\varphi(z)$ is a peaked function, depending on $d$ only, and $\mu_1 \approx 1$ [3, 4, 5].
There is a great number of publications devoted to the question of "improvement" of the individual cell dynamics equation (1) (we will say microscopic dynamics equation) to get a good agreement of the cell size distribution $f_0(a, t)$ with experimental data, essentially in the metallurgic literature related to normal grain growth [6, 7, 8, 9]. Surprisingly small attention has been paid to mesoscopic and macroscopic behaviours of the cellular structures as some continuous elastic cellular medium with viscous motion law, although the existence of interesting self-organisation effects is almost evident here [11, 12]. Indeed, if we have a good look at Fig. 1 we can see a strong size correlation of adjacent cells. Near a small cell the probability to find another small cell is greater then a large one, and vice versa, near a large cell the probability to find another large cell is greater then a small one. In other words we observe here space-correlated fluctuations of the mesoscopic cell interface density $\rho(x, t)$. Qualitatively this effect can be easy analysed. Let us consider a nonhomogeneous cellular structure with the macroscopic mean interface density $\rho(t)$ and the mesoscopic one $\rho(x, t) = \rho(t) + \delta\rho(x, t)$, weakly varying in space. To be more precise, let us assume that $\delta\rho(x, t) << \rho(t)$ and the wavelength $\lambda$ of spatial variations of $\rho(x, t)$ is much greater than $a \approx 1/\rho(t)$. Let us cut out a medium volume $V_L$ having the size $L << \lambda$, but still including many cells. Each element $d\Sigma$ of the surface $\Sigma_L$, enclosing $V_L$, is affected by the interface tension of the cells, surrounding $V_L$. The total interface tension force acting on $V_L$ is $\int E\rho d\Sigma = \int E\nabla\rho dV$. In a homogeneous cell medium this force equals zero. In the nonhomogeneous medium the force is non zero. The total cell interface in $V_L$, which moves as the whole under the force effect, is $\int \rho dV$. Hence the force acting on cell interface unit is $E\nabla\rho/\rho$. Supposing the viscous law for the cell interface movement, we find a cell interface flux $j = \rho \nu \propto D\nabla\rho$. We see that the cells make a collective motion in the direction $\nabla\rho$. This is an "uphill diffusion" of the cell medium. Taking into account the annihilation law $d\rho(t) \propto -D\rho^3$, which follows from $\rho(t) \propto t^{-1/2}$, we get

$$\frac{\partial \rho}{\partial t} = -\gamma_1 D\rho^3 - \gamma_2 D\Delta \rho. \quad (5)$$

Here $\gamma_1$ and $\gamma_2$ are dimensionless constants, depending on the scaling distribution function $\varphi(z)$, only. Equation (5) has the form of a diffusion-reaction equation with the cell interface density $\rho$ playing the role of concentration and with a negative "diffusion" coefficient. We see that, the amplitude of any small spatial fluctuation $\delta\rho(x, t)$ with wavelength $\lambda << \xi$, where $\xi = 2\pi \sqrt{\beta/3\xi_1 \rho}$ is the correlation length of the fluctuations, grows without bound. It follows that competing stochastic processes of cell disappearance and cell medium "uphill diffusion" can give rise to the formation of large-scale domains of the size $\lambda \leq \xi$ having the density $\rho(x, t)$ greater or smaller than the mean value $\rho(t)$ (Fig. 2 [13]). It also means that space derivatives of higher order of $\rho$ must be taken into account in right side of (5) for detailed analysis of the system stability and more accurate estimation of $\xi$.

Whereas the physical ideas explaining the possible macroscopic self-organisation effects are simple, the consecutive derivation of the mesoscopic evolution equation (5) and the calculation of kinetic coefficients $\gamma_1$ and $\gamma_2$, defining the effect quantitatively, turns out to be a difficult statistical task [14]. Therefore the "technique" of the problem solution asks for some commentaries. Let us consider an ideal gas as a well-known example for comparison. The consistent description of its evolution include three classic scale levels of consideration:
micro-, meso- and macroscopic [5, 6, 15, 16]. On the *microscopic* level the evolution is effectively described by methods of physical kinetics, i.e. by Boltzmann *kinetic equation* [5, 6] for the system probability distribution function $f$ in a phase space, taking into account interaction and movement laws of atoms. Having the equilibrium solution $f_0$ for the homogeneous gas (Maxwell-Boltzmann (MB) distribution), it is possible to move to the *mesoscopic level* of the system description by introducing mesoscopic quantities (density, pressure, etc.) as moments of the distribution function. In the case of small spatial gradients of the mesoscopic quantities, integration of kinetic equation by Enskog method gives a *hydrodynamic equation* [6, 15]. If the hydrodynamic equation has steady-state nonhomogeneous solutions, it is possible to talk about a *macroscopic* description of the system by means of spatial patterns of size $\xi$ much greater than the mesoscopic size. To study the patterns formation it is convenient to transform the hydrodynamic equation into Langevin *stochastic equation* [16] for an *order parameter*. As we see above LSH distribution (4) plays the same outstanding role for the coarsening cellular structures in steady-state, as MB distribution plays for ideal gases in equilibrium state [6, 15, 16, 8]. All difference between MB and LSH distributions is that the first is determined by thermofluctuations related to collisions of gas atoms and the second is determined by stochastofluctuations related to the process of cells annihilation. Hence the main methodical idea of this paper is to use LSH distribution as a basis for the multilevel description of cellular structure, like MB distribution is used in order to go from the kinetic equation to the hydrodynamic one and then to the stochastic one. This idea has interesting applications for other physical systems, where the scaling regime is reached. For example, its application allows to generalise Lifshitz-Slyozov theory of homogeneous coalescence [15] for the nonhomogeneous case [17, 8], to describe the relaxation of nonhomogeneous dislocation structure during annealing [12, 10] etc.
2 Kinetic theory (microscopic level)

Let us consider as in Sec. 1 a nonhomogeneous cellular structure with the interface density $\rho(x,t)$, weakly varying in space. We divide the volume $V$ of the system into small coarse-graining elements $V_L$ with $L << \lambda$, but still including many cells. In the first approximation the cell dynamics of the structure in each $V_L$ will be defined by the dynamics equation (1), with the curvature $\kappa$ becoming a local characteristic - an interface curvature self-consistent mean field $\kappa(x,t)$. The field $\kappa(x,t)$ in $V_L$ at point $x$ equals to the self-consistent mean of cell curvature $\kappa(t)$ determined by (1)-(4) for the cell structure in $V_L$ if we approximately consider it as homogeneous. The field $\kappa(x,t)$ is a weakly varying function in space as $\rho(x,t)$ is. In the second approximation the dynamics equation (1) will contain gradients of the field $\kappa(x,t)$. It is easy to establish this fact if we analyze an one-dimensional cellular structure ($d = 1$). Let us cell interfaces are located at the points $x_i(i = 0, \pm 1, \pm 2, \ldots)$ of $x$-axe. We have the cell radii $a^{(i)} = (x_{i+1} - x_i)/2$, the cell curvatures $\kappa^{(i)} = 1/a^{(i)}$ and the cell centres coordinates $x^{(i)} = (x_{i+1} + x_i)/2$. Each interface is acted upon by attraction of adjacent ones (the interface tension of the cells) according to the law $F_i = E(\kappa^{(i)} - \kappa^{(i-1)})$ with the result that the cell changes its position $dx^{(i)}/dt = D(\kappa^{(i+1)} - \kappa^{(i-1)})/2$ and its size $da^{(i)}/dt = D(\kappa^{(i)} + \kappa^{(i-1)} - 2\kappa^{(i)})/2$. If we now introduce a curvature exact field $\kappa_e(x,t)$ as a step function, which is $\kappa_e^{(i)}$ if $x_i < x < x_{i+1}$, we can rewrite $dx^{(i)}/dt = D[\kappa_e^{(i)} - a^{(i)}, t] - \kappa_e(x^{(i)} + a^{(i)}, t)]/2$ and $da^{(i)}/dt = D[\kappa_e(x^{(i)} + a^{(i)}, t) + \kappa_e(x^{(i)} - a^{(i)}, t) - 2/a^{(i)}]/2$. Replacing the exact field $\kappa_e(x,t)$ by the mean field $\kappa(x,t)$ and taking into account that $\bar{a} << L << \lambda$ we get respectively the cell velocity and the cell size change rate

$$dx/dt = Da\nabla\kappa/d,$$  

$$da/dt = -D(1/a - \kappa) + Da^2\Delta\kappa/(2d).$$

It is easy to understand that the abovementioned definition of the curvature mean field $\kappa_e(x,t)$ holds for nonhomogeneous cellular structures in two ($d = 2$) and three dimensions ($d = 3$). Expressions (6) and (7) are also valid. Really, each cell of the structure may be approximately thought of as having the form of sphere. Then the all cellular structure configuration may be described through the cell centre coordinates and the cell radii. Let us choose a cell of radius $a$ with centre in a point $x$ and an interface point with coordinate $x' = x + a(x',t)$ of the cell. The velocity $w(x',t)$ of the interface point affected by the interface tension force $F(x',t) = -E[1/a(t) - \kappa_e(x',t)]a(x',t)/a(t)$, is determined by the viscous law $w(x',t) = M\nabla\kappa(x',t)$. Replacing $\kappa_e(x,t)$ by $\kappa(x,t)$ we can approximately calculate the cell velocity $dx/dt = \oint w(x',t)ds'/\Omega_{da^{d-1}} \approx a\nabla w/d$ and the cell size change rate $da/dt = \oint w(x',t)ds'/\Omega_{da^{d-1}} \approx w + a^2\Delta w/(2d)$ and find expressions (6) and (7).

We see that the cells move in a coordinate-size space $\{x,a\}$ in the process of coarsening. These movements are conveniently described by kinetic equation

$$\partial f/\partial t + \partial (f dx/dt)/\partial x + \partial (f da/dt)/\partial a = 0$$

for the structure probability distribution function $f(x,a,t)$ in the cell coordinate-size space with normalisation condition

$$(\Omega_d/V) \int dx \int_0^\infty da a^d f = 1.$$
Equations (8) and (9) have the physical sense of continuity equation in the coordinate-size space and of conservation law of the total volume of all cells of the system. They are natural generalisations of (2) and (3). For the macroscopically homogeneous cell structure the set of equations (6)-(9) comes to (1)-(3) and has the solution (4) with $\varphi(z) = [dz/(2 - z)^{d+2}]/(2e)^2 \exp[2d/(z - 2)]$ for $0 < z < 2$ and $\varphi(z) = 0$ for $z \geq 2$; $\mu = \int_0^1 z^i \varphi(z) dz$. The mean structure variables obey simple scaling laws: $\bar{a}(t) = \mu_1 \sqrt{Dt}$; $\kappa(t) = (\mu_{a-2}/\mu_{a-1})/\sqrt{Dt}$; $n(t) = 1/(\Omega_d \mu_d (\sqrt{Dt})^d)$ and $\rho(t) = d \mu_{a-1}/(2 \mu_d \sqrt{Dt})$.

To find $f_1(x,a,t) = f_0(a,t) + \delta f(x,a,t)$ in the first-order approximation for small parameters $|\nabla \rho|/\rho^2 << 1$, it is necessary, as in well-known Enskog method [6], to substitute $f_1(x,a,t)$ in (6)-(9) and linearize them, taking into account that $\delta f(x,a,t) << f_0(a,t)$. I hope in a future work to realise this possibility, but this is proving at present to be a difficult task [14].

3 Hydrodynamic theory (mesoscopic level)

Equation (8) provides the kinetic (microscopic) description of the cellular structure coarsening. To get the hydrodynamic equation let us rewrite (4) as

$$f_0(a,t) = \alpha \rho^{d+1} \varphi(\beta \rho),$$

where $\alpha = (\mu_d)^d/\Omega_d (d \mu_{d-1})^{d+1}$ and $\beta = (d \mu_{d-1})/d \mu_{d-1}$ are constants, and use again as in Sec. 1 and 2 the standard coarse-grain averaging procedure [3]. Namely, let us introduce a mesoscopic coarse-graining length $L(t) >> \bar{a}(t)$, so that a mesoscopic coarse-graining volume $V_L = \Omega_d L^d$ includes many cells. Let us divide the volume $V$ of cellular structure into elements of volume $V_L$. For small gradients of mesoscopic variables $\kappa(x,t), a(x,t), \rho(x,t)$, and $n(x,t)$ we may suppose that a local homogeneous steady-state is reached in each separate element $V_L$ after a time $\tau_L = L^2/D$ much greater than the cell annihilation time $\tau_a = \bar{a}^2/D$. For all this, the cellular structure as the whole (macroscopically) is not in the homogeneous state. Then the distribution function $f_0(x,a,t)$ in $V_L$ at point $x$ can be assumed to be a local steady-state function, equal to LSH distribution (10) for the homogeneous cellular structure with density $\rho(x,t)$, that prevails at the point $x$. This supposition is completely analogous to the local system equilibrium one in linear thermodynamics [4]. Multiplying (8) by $\Omega_d da^{d-1}/2$, (6) by $\Omega_d da^{d-1} f_0/2$, (8) by $\Omega d a^d$, integrating them by $a$ and using conservation law (9), we make the averaging procedure. After sufficiently heavy calculations we obtain, respectively, the equation of cell medium continuity, an analogue of Darcy’s law in hydrodynamics and an equation similar to the state equation in gas dynamics [13]:

$$\partial \rho/\partial t + \nabla (\rho \mathbf{v}) = -D[\alpha_1 \rho^3 - \alpha_2 \rho^2 \kappa - \alpha_3 \triangle \kappa],$$

$$\mathbf{v} = (D/2) \rho^{-1} \nabla \kappa,$$

$$\rho^3 \kappa = \beta_1 \rho^4 - \beta_2 \rho \Delta \kappa + \beta_3 \nabla \kappa \nabla \rho.$$  

The set of three hydrodynamic equations (11)-(13), containing three hydrodynamic quantities of the cell medium: the density $\rho(x,t)$, the velocity $\mathbf{v}(x,t)$, and the curvature $\kappa(x,t)$, forms a closed set. The role of the quantities $\rho(x,t)$, $\kappa(x,t)$ and $\mathbf{v}(x,t)$ are completely analogous to the
role of gas density, pressure and velocity in viscous gas dynamics. If initial and boundary conditions are known, the cell medium movement can be determined by traditional hydro-gas dynamics methods [15] for any continuous distribution of $\rho(x, t)$, $\kappa(x, t)$ and $v(x, t)$. By substituting (12)-(13) in (11), the set can be reduced to evolution equation (5). In (5), (11)-(13) $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are calculated constants. For $d = 2$: $\alpha_1 = 1.54, \alpha_2 = 1.11, \alpha_3 = 0.25, \beta_1 = 1.11, \beta_2 = 0, \beta_3 = 0.267, \gamma_1 = 0.308, \gamma_2 = 0.277$. For $d = 3$: $\alpha_1 = 1.16, \alpha_2 = 1.44, \alpha_3 = 0.5, \beta_1 = 0.718, \beta_2 = 0.132, \beta_3 = 0.263, \gamma_1 = 0.129, \gamma_2 = 0.136$. Full expressions are sufficiently bulky (e.g. $\alpha_1 = 4(d - 1)\mu^5_{d-1} \mu_{d-3}/(d^2 \mu^3_{d-1})$) and will be published elsewhere [14].

4 Stochastodynamic theory (macroscopic level)

To analyse the dynamics of macroscopic spatial perturbations of the cell medium, let us introduce a relative value of deviation of the cell interface density from its mean macroscopic value $\psi(x, t) = \delta \rho(x, t)/\rho(t) << 1$, where $\rho(t)$ is the uniform solution of (5). Substituting $\rho(x, t) = \rho(t) + \delta \rho(x, t)$ in (5) and coming to new time and space variables $\partial t = 3\gamma_1 D^2 \rho^2(t) \partial t = -\partial \ln \rho^3(t)$ and $\partial \mathbf{x} = \rho(t) \partial \mathbf{x}$ (intrinsic system variables), we obtain

$$\partial \psi/\partial t = -\delta \mathcal{F}/\delta \psi$$

with $\mathcal{F}\{\psi\} = \int \partial \mathbf{x} [A \psi^2 + C(\nabla \psi)^2]/2, A = 1$ and $C = -\gamma_2/(3\gamma_1)$ ($C = -0.300$ for $d = 2$ and $C = -0.351$ for $d = 3$). Taking $\psi(\mathbf{x}, t)$ in the form $\psi(\mathbf{x}, t) = \psi(\mathbf{k}, \tilde{t})e^{i(k\mathbf{x} - \ell_{\tilde{t}})}$, we get the dispersion relation $\tilde{\omega}(\mathbf{k}) = -i\tilde{\sigma}(\mathbf{k})$, where

$$\tilde{\sigma}(\mathbf{k}) = A + C\tilde{k}^2$$

is the damping coefficient. Including in $\mathcal{F}\{\psi\}$ the term $-h \psi$ describing the effect of an external field $h(\mathbf{x}, \tilde{t}) \propto e^{i(k\mathbf{x} - \ell_{\tilde{t}})}$, we find the generalised susceptibility $\gamma(\mathbf{k}, \tilde{\omega}) = 1/|\tilde{\sigma}(\mathbf{k}) - i\tilde{\omega}|$.

As all mesoscopic quantities, $\psi$ is a fluctuating quantity. But on writing (14) we have neglected these fluctuations. The quantity $\psi$ is a sum of a number of microscopic random quantities. Each of them corresponds to the contribution of a cell being in $V_L$. In view of central limit theorem of statistics, $\psi$ has Gaussian distribution function with the mean value $<\psi >_L = 0$ and the dispersion $<\psi^2 >_L = \nu/V_L N$, where $\nu = \mu_{2(d-1)}/\mu_{d-1}^2 - 1$ ($\nu = 0.109$ for $d = 2$ and $\nu = 0.373$ for $d = 3$) [15]. Equation (14), after taking into account the fluctuations, becomes Langevin stochastic equation with $\mathcal{F}\{\psi\}$ being a stochastodynamic potential [19]. The noise $\zeta$ is Gaussian noise, with $<\zeta> = 0$ and the correlation function $<\zeta(\mathbf{x}, \tilde{t})\zeta(\mathbf{x}', \tilde{t}'> > = 2\theta(\mathbf{x} - \mathbf{x}')\delta(\tilde{t} - \tilde{t}')$, defined by the fluctuations at scales $a << L$ and $\tau_a << \tau_L$. Here $\theta = \Omega(d/2)(\mu_d^{d-1}/\mu_{d-1}^{d-1})(\mu_{d-1}^{d-1}/\mu_{d-2}^{d-1})(\theta = 0.619$ for $d = 2$ and $\theta = 8.09$ for $d = 3$) [14]. The forms of evolution equation (14) and of the noise $\zeta$ guarantee that the steady-state probability distribution functional is given by Boltzmann formula $P(\psi) \propto exp[-\mathcal{F}\{\psi\}/\theta]$ [14]. We see that our cell system is rated in the class of so-called potential systems with potential $\mathcal{F}\{\psi\}$ having Landau-Ginzburg form [13] and the quantity $\psi$ plays a role of an order parameter. It is not surprising because in general sense the system is in perfect analogy to a thermodynamic system with a nonconserved order parameter $\psi$. So the forms
of evolution equation (14) and of potential $\mathcal{F}\{\psi\}$ are dictated by simple symmetry macroscopic considerations. Instead of the thermal noise in thermodynamic systems we have here the stochastic noise, created by the continuous process of cells annihilation. In virtue of reasoning, fluctuation-dissipation theorem may be applied, according to which the generalised susceptibility $\tilde{\chi}(\vec{k}, \tilde{\omega})$ determines the spectral correlation function of fluctuations as $\tilde{S}(\vec{k}, \tilde{\omega}) = (2\tilde{\theta}/\tilde{\omega})\text{Im} \tilde{\chi}(\vec{k}, \tilde{\omega})$. The structure function has usual Ornstein-Zernike form $\tilde{S}(\vec{k}) = \tilde{\theta}/\tilde{\sigma}(\vec{k})$.

From (15) we see, that the order parameter modes $\psi(\vec{k}, \tilde{t})$ with $\tilde{k} > \sqrt{-A/C}$ are unstable. This is every indication that highest order terms of the expansion of $\mathcal{F}\{\psi\}$ in powers of $\nabla \psi$ must be considered for detailed analysis of the system stability. The derivation of the exact expression for these terms by Enskog method is a very heavy task [6, 14] not realized to the present. We proceed now differently. Let us use the fact that our system is potential with the nonconserved order parameter $\psi$. Then we may consider $\mathcal{F}\{\psi\}$ as some phenomenological potential in the abovementioned Landau sense. This potential have necessarily the general form $\mathcal{F}\{\psi\} = \int \, d\vec{x}[A\psi^2 + C(\nabla \psi)^2 + D(\Delta \psi)^2]/2$ with $D > 0$ [6, 14]. The phenomenological coefficients $A, C$ and $D$ can be found from experiments, computer simulations or by a direct calculation as we have made above for $A$ and $\mathcal{C}$. As a consequence we get the damping coefficient in the form

$$
\tilde{\sigma}(\vec{k}) = A + C\tilde{k}^2 + \mathcal{D}\tilde{k}^4.
$$

We do not know the real value of $\mathcal{D}$, but we are able to forecast the possible scenarios of system evolution [20]. If $A > 0, C < 0$ and $D > C^2/(4A)$ the evolution equation (14) has the single stable solution $\psi_0(\vec{x}) = 0$, and modes $\psi(\vec{k}, \tilde{t})$ with any $\vec{k}$ decay with time. Therefore the system is stable with respect to large-scale fluctuations of $\psi(\vec{x}, \tilde{t})$ near $\psi_0(\vec{x}) = 0$. The modes $\psi(\vec{k}, \tilde{t})$ with $\tilde{k} = kC = \sqrt{-C/(2D)} = 2\pi/\tilde{\xi}$, where $\tilde{\xi}$ is an intrinsic correlation length, have the lowest decay rate. That is why under the action of the stochastic internal noise $\zeta$, created by a continuous process of cells annihilation, space-correlated fluctuations arise in the system. The system is in a modulated state (short range ordering) and the structure function

$$
\tilde{S}(\tilde{k}) = \tilde{\theta}/(A + C\tilde{k}^2 + \mathcal{D}\tilde{k}^4).
$$

has a maximum at $\tilde{k} = \tilde{k}_C$ (Fig. 3), reflecting the abovementioned fact that the modes $\psi(\vec{k}_C, \tilde{t})$ decay with the lowest rate. If $A > 0, C < 0$ and $0 < D < C^2/(4A)$, the evolution equation (14) has the unstable solution $\psi_0(\vec{x}) = 0$ and stationary one-, two-, ... $d$-dimensional space-periodical solutions with wavelength $\lambda = \tilde{\xi}$. Under the action of the stochastic internal noise $\zeta$ the system comes from the homogeneous state into one of these patterned state (long range ordering). The question of existence of the first or of the second type of ordering depends of the value of coefficient $\mathcal{D}$.

5 Discussion

We see that in the process of structure coarsening the macroscopic cell interface density $\rho(t)$ decreases with time as $\rho(t) \propto t^{-1/2}$ and the mesoscopic density $\rho(\vec{x}, t) = \rho(t)[1 + \delta \rho(\vec{x}, t)/\rho(t)]$ also decreases (Fig. 1). The competition of two stochastic processes of the cells annihilation and of the cells collective motion
creates space-correlated fluctuations of the cell interface density. These fluctuations are the adjacent domains of size $\xi$ with $\rho(x, t)$ greater or smaller than $\rho(t)$ (Fig. 1, 2). In other words, they are the domains in which the cells have the mean size $\bar{a}(x, t)$ greater or smaller than the macroscopic mean value $\bar{a}(t)$. The amplitude of relative fluctuations $\delta\rho(x, t)/\rho(t)$ and the domain size (characteristic wavelength of fluctuations), divided by the mean cell size $\xi/\bar{a}(t) \propto \xi\rho(t) = \bar{\xi}$ stay constant. These domains can be short-range ordered (Fig. 1) or long-range (Fig. 2) ordered in space. The short-range ordering of cell interface density fluctuations resemble the fluid density fluctuations ordering in the well-known phenomenon of critical opalescence. The long-range ordering of cell interface density fluctuations resemble to the effect of spinodal decomposition in thermodynamic systems.

This is just a moment to return to the question discussed in Sec. 1 and 2 about basing of the self-consistent mean field approximation for description of the cellular systems. The condition of its validity is the condition that the relative value of field fluctuations $\delta\kappa(x, t)/\kappa(t) \approx \delta\rho(x, t)/\rho(t) \approx \psi(x, t)$ is small. More precisely, the fluctuations averaged over the coarse-graining volume $V_L$ must be small: $<\psi^2(x, t)>_L = \nu/[V_L N(x, t)] << 1$. Choosing the volume of a first coordination sphere as a minimum possible value of $V_L$ and taking into consideration that $n = V_L N$ is the number of cells in $V_L$, we see that this condition is well satisfied both for $d = 2(\nu = 0.109, n \approx 7)$ and $d = 3(\nu = 0.373, n \approx 13)$. The general condition of validity of mean-field theory for the systems with Landau-Ginzburg potential $F\{\psi\}$ is well-known Ginzburg criterion: fluctuations of $\psi$ averaged over the correlation volume must be small $<\psi^2(x, t)>_\xi = \nu/[V_\xi N(x, t)] << 1$. This criterion is obviously fulfilled if $L \leq \xi$. Uniting this condition with the foregoing condition of existence of stochastodynamic potential $F\{\psi\}$, we have finally $\bar{a} << L \leq \xi$. This is just the familiar condition of the choice of the coarse graining size $L$, which is also
the condition of validity of the self-consistent mean field approximation in phase transformation theory. It can be satisfied if $\bar{a} << L \leq \xi$. In coarsening cellular systems this is always the case for the short range ordering ($D > C^2/(4A)$) and this is so indeed if $D >> -C/(8\pi^2)$ for the long-range ordering ($0 < D < C^2/(4A)$).

In spite of that the ordering effects of cell interface density fluctuations can be evidently established by visual analysis of different cellular structures in experiments (Fig. 1) and computer simulations (Fig. 2), I am not aware of publications, where the structure function (17), describing the effect quantitatively, has been directly obtained. I feel that it is related to two reasons. Firstly, it must be noted that hydrodynamic and stochastodynamic theories developed here are valid in the long wavelength limit $\tilde{k} = k\rho(t) < 1$, when we can guarantee the possibility of application of the cell medium conception and the self-consistent mean field approximation. In the long wavelength limit $\tilde{k} < 1$ the maximum of $S(\tilde{k})$ is hardly observed: if $\tilde{k}_C \approx 1$ it is hardly separated from the other maximum $\tilde{k}_0 \approx 2$ corresponding to the trivial effect that the mean distance between neighbouring interfaces is $2\bar{a}$. This later maximum cannot be described within our long wavelength approach and is not shown in Fig. 3. Therefore the more strong confirmation of existence of the large-scale fluctuations of cell interface density is the observation of the negative curvature of $S(\tilde{k})$ in the macroscopic $\tilde{k} \to 0$ limit (Fig. 3). However it demands of a laborious statistical analysis or a scattering study of very large cellular systems. In actual practice the finite-size induced effects stays too essential \[^{21}\]. Secondly, in computer experiments, using usually "$q$-states Potts model" \[^{10,13}\], the structure function $S(k)$ is calculated as a Fourier transform of the correlation function of a phases volume density. If all phases have the same equilibrium energy, and the boundaries separating the phase domains, are sufficiently distinct, the system state is completely defined by the total interfacial area. In consequence, the cell interface density $\rho(x,t)$ and the corresponding structure function (17) is a much more sensible characteristic of the system than the phases volume density and the corresponding to it structure function. Quite recently \[^{22}\] it has been demonstrated by computer simulation of normal grain growth that a size correlation between neighbouring grains exist in grain structure. Unfortunately the spectral correlation analysis of the cell interface density fluctuations by means of the structure function (17) has not been made and direct quantitative comparison of these results with present theory results is difficult. Still there is no doubt that we are dealing with the first observation of the short-range ordering of cell interface density fluctuations in cellular structures. New experiments or computer simulations with careful space spectral analysis of cell interface density fluctuations are needed to confirm the existence of dynamic patterning in cellular structures.

6 Summary

In summary, the problems of multilevel description of cellular structure evolution are considered from positions of kinetic, hydrodynamic and stochastodynamic theories of nonequilibrium statistical systems. It is demonstrated that the effective method to solve the problem is to describe the cellular structure on the microscopic level by the probability distribution function in the coordinate-
size space, i.e. in the space of the cell coordinates and the cell sizes. The kinetic equation for the weakly nonhomogeneous cellular structure is written and the development to hydrodynamic equations for a continuous elastic cellular medium with viscous motion law is realised. These non-linear differential equations contain three mesoscopic quantities of the cell medium (the cell interface density, the cell medium velocity and the cell interface curvature) and are analogous to the continuity equation, Darcy’s law and the equation of state in gas dynamics. For continuous flow of the cell medium the hydrodynamic equation set is reduced to a single evolution equation for the cell interface density. It has the form of a diffusion-reaction equation with a negative ”diffusion” coefficient and with the cell interface density playing the role of ”concentration”. It is shown that the relative value of deviation of the cell interface density from its mean macroscopic value is a suitable order parameter of the system. The order parameter dynamics are described by Langevin stochastic equation with a stochastodynamic potential having Ginzburg-Landau form and with a stochastic noise, created by the continuous process of cells annihilation. Analysis of the order parameter dynamics in coarsening reveals a new effect of macroscopic patterning in the cell medium: competition between two stochastic processes of the small cells ”eating up” by large ones and of the ”uphill cells diffusion” leads to the creation of space-correlated fluctuations of the cell interface density.

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