TRIANGULATIONS INTO GROUPS

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Abstract. If a (cusped) surface \( S \) admits an ideal triangulation \( T \) with no shears, we show an efficient algorithm to give \( S \) as a quotient of hyperbolic plane \( \mathbb{H}^2 \) by a subgroup of \( \text{PSL}(2, \mathbb{Z}) \). The algorithm runs in time \( O(n \log n) \), where \( n \) is the number of triangles in the triangulation \( T \). The algorithm generalizes to producing fundamental groups of general surfaces and geometric manifolds of higher dimension.

1. Introduction

Let \( S \) be a cusped hyperbolic surface admitting an ideal triangulation with the following property:

Property 1. For any pair of adjacent ideal triangles \( ABC \) and \( ABD \), the cross ratio of the four points \( A, B, C, D \) equals 1, where the cross ratio is defined as:

\[
[A, B, C, D] = \frac{(A - C)(B - D)}{(B - C)(A - D)},
\]

and we have implicitly identified the hyperbolic plane with the upper halfplane \( \mathbb{H} \subset \mathbb{C} \).

This property has a number of equivalent formulations. One is geometric:

Property 2. We can choose a family of horocycles \( h_1, \ldots, h_n \), where each \( h_i \) is center on the \( i \)-th cusp of of \( S \), and \( h_i \) is tangent to \( h_j \) whenever \( c_i \) is adjacent to \( c_j \) in \( T \).

Another is algebraic:
Property 3. $S$ is the quotient of $\mathbb{H}$ by a subgroup $\Gamma$ of $PSL(2, \mathbb{Z})$.

In this note we will prove that the three properties are equivalent, and also give an algorithm to express $S = \Gamma \backslash \mathbb{H}$. The algorithm runs in time bounded by $O(n \log n)$, and produces a set $G$ of independent matrix generators for $\Gamma$. Since $S$ is cusped, $\Gamma$ is a free group, and so this is a complete description of $\Gamma$. In fact, we construct the generators as words in the two linear fractional transformations $L$ and $R$, where:

$$L(z) = z + 1, \quad R(z) = \frac{-1}{z - 1},$$

and the matrix generators are obtained by multiplying the words out.

The algorithm consists of a number of steps:

Step 1. Construct the Poincaré dual $\mathcal{T}^*$ of the triangulation. This will have a vertex for each face of $\mathcal{T}$ and a face for each vertex of $\mathcal{T}$. This is an oriented complex, and thus we can cyclically order the edges at each vertex.

Step 2. Construct a maximal spanning tree $M$ of the 1-skeleton $\mathcal{T}_1^*$ of $\mathcal{T}^*$. The edges of $\mathcal{T}_1^*$ fall into two types. The edges of the first type are the edges of $M$, the edges of the second type are not.

Step 3. Split each edge of the second type. By “split”, I mean that we replace the edge $AB$ by a pair of edges $AC_1, BC_2$. We will henceforth refer to $C_1$ and $C_2$ as twins.

After we split all the edges of the second type in the graph $\mathcal{T}_1^*$, we obtain a graph $B$, which is a tree where every non-leaf node has degree three. In addition, every leaf node is annotated with a cyclic ordering of the three edges. We are ready for:

Step 4. Construct the shortest path from each leaf node to its twin. This path will look like $C_1v_1 \ldots v_kC_2$. At each vertex $v_k$ we have a fork in the road, and we annotate $v_k$ with an $L$ or an $R$ depending on whether we go left or right at the fork.

Now we are done: each path from $C_1$ to $C_2$ gives a generator of the fundamental group of $S$, if we replace $L$ and $R$ by the linear fractional transformations with the same names (this should be done as we are constructing the paths, doing it after will bring us back to $O(n^2)$ running time).

The correctness of the algorithm above follows immediately from the Poincaré Polygon Theorem (see, eg, [1]).

1.1. **Crossratios are all 1 if and only if there is a horodisk packing.**

. This follows from the observation that there is a unique horodisk
packing of an ideal triangle. Indeed, if represent the ideal triangle $ABC$ as one whose vertices are the three roots of unity in the Poincaré disk model, the symmetric arrangement of horodisks obviously works. Let the points of tangency of the horocycles (which are on the sides of $ABC$) be $p_{AB}, p_{AC},$ and $p_{BC}$. Now, suppose that there is another arrangement, with points of tangency $q_{AB}, q_{AC}, q_{BC},$ and let $d(p_{AB}, q_{AB}) = r_{AB}$, and similarly for the other two sides. Suppose $p_{AB}$ lies between $A$ and $q_{AB}$. Then the same is true of $p_{AC}$ and $q_{AC}$.

But the last two assertions would imply there are not actually tangent along $BC$. To show the result we now need the following easy lemma:

**Lemma 1.** Let $T_1 = ABC$ and $T_2 = ABD$ be two adjacent ideal triangles, and let $\gamma$ be a horocycle centered on $A$. Let $\gamma_1 = \gamma \cap T_1$, and $\gamma_2 = \gamma \cap T_2$. Then

$$\frac{|\gamma_1|}{|\gamma_2|} = \exp([A, B, C, D]).$$

**Proof.** Let $C = -1, A = \infty, B = 0, D = z$, and compute. \qed

1.2. **All crossratios are 1 implies that the surface is a quotient of the upper halfplane by a subgroup of the modular group.** This is not hard to see, especially if one looks at the modular figure: each adjacent pair of colored and white triangles forms a fundamental domain for the action of $PSL(2, \mathbb{Z})$ on the hyperbolic plane. Since the crossratios are all 1, the baricentric subdivisions of pairs of adjacent ideal triangles agree, and so we see that our surface covers the modular orbifold.
2. **Complexity**

2.1. **Constructing the oriented dual (Step 1).** The complexity of Step 1 (constructing the oriented dual) depends on how one is given the triangulation. The most natural way is for it to be given as a rotation system, which is simply the graph with a cyclic ordering of the edges at every vertex. It is easy to see that in this case the dual graph can be constructed in time linear in the number of edges (the algorithm is simple: maintain a list of edges. Each edge is marked by 0 or 1. Initially, all the edges have label 0. We pick the first edge $e$, and construct a list of edges obtained by always picking the edge which precedes $e$ in the cyclic order. A closed cycle gives us a face (already equipped with the cyclic ordering of boundary edges). Every time an edge is seen we increase the label by 1. If the label is 2, we delete the edge from the list. Since each edge is seen at most twice, and we do constant work per edge, the algorithm is linear).

The spanning tree (Step 2) can be done in time linear in the number of edges (see, eg, [2]), and Step 3 can obviously be done in time linear in the number of vertices. This leaves us with Step 4, which we analyze below.

2.2. **Constructing the generators.** At this point we have a tree (with every interior node of degree 3) $M$ and a collection of pairs of leaves of $M$, and we need to construct paths between the two vertices in each pair. Since a shortest path between two vertices of a tree can be constructed in time $O(V(M))$, (see [2]) and the number of pairs is half the number of all leaf nodes (so $O(V(M))$ as well), this gives an $O(V^2(M))$ algorithm for computing all the generators. We can do better, however, by first showing the following:

**Lemma 2.** Let $G$ be a tree with every non-leaf node having degree three. For every non-leaf node $v$, removing $v$ separates $G$ into three subgraphs $G_{\text{max}}(v), G_{\text{med}}(v), G_{\text{min}}(v)$, with $|V(G_{\text{max}}(v))| \geq |V(G_{\text{med}}(v))| \geq |V(G_{\text{min}}(v))|$. Let $\tilde{v}$ be the vertex which minimizes $|V(G_{\text{max}}(v))|$. Then

$$\frac{V(G) - 1}{3} \leq |V(G_{\text{max}}(\tilde{v}))| \leq \frac{2}{3} V(G) + 1.$$  

*Proof.* Denote the three orders by $N_{\text{max}}, N_{\text{med}}, N_{\text{min}}$. The first inequality is true at any vertex, since

$$N_{\text{max}} + N_{\text{med}} + N_{\text{med}} + 1 = |V(G)|.$$  

To show the second inequality, let $v_1$ be the vertex in $G_{\text{max}}(\tilde{v})$ adjacent to $\tilde{v}$. At $v_1$ the orders of the three components into which $v_1$ separates $G$ are $N_1, N_2, N_3$, where $N_3 = N_{\text{min}} + N_{\text{med}} + 1$, and $N_1 + N_2 + 1 = N_{\text{max}}$. 

Assume that $N_1 \geq N_2$. We then have two possibilities. The first is that $N_3 \geq N_1$. In that case, $N_{\text{med}} + N_{\text{min}} + 1 \geq (N_{\text{max}} - 1)/2$. Adding $N_{\text{max}}$ to both sides, we get the second side of the inequality. The second possibility is that $N_1 \geq N_3$. However, since $N_1 < N_{\text{max}}$, this contradicts the defining property of $\tilde{v}$. □

The next result we will need is the result of [3]: Given a rooted tree $T$ with a positive weight associated with every node, there is a linear time algorithm to partition the tree into a minimal collection of subtrees such that the weight of no subtree exceeds $k$. In our application, all the weights are equal to 1, and $k = \frac{2}{3}V(T) + 2$. By Lemma 2, the tree will be broken up into exactly two components.

The algorithm is then simple: We partition our tree into two rooted subtrees (the roots will be the two endpoints of the edge we delete to partition). For each of the two pieces we compute the pair distances, and all the distances from the leaves to the root (recursively), then use the distances we had computed to compute all the distances in the original tree. To make the first step the same as all the others, we pick an arbitrary leaf and call it the root.

It is clear that at each step we have the following recurrence inequality for the running time:

$$T \leq cV(G) + T(V_1) + T(V_2),$$

where $T_1 + T_2 = V(G) - 1$, and $\max V_1, V_2 \leq \frac{2}{3}V(G) + 2$.

This clearly implies an $O(n \log n)$ running time.
3. Extensions

The algorithm described above easily extends to other cases. The simplest extension is where the ideally triangulated surface does not have all cross ratios equal to one. In that case, we simply replace the linear fractional transformations $L$ and $R$ by the appropriate conjugates of the transformation $L_{a,b}(z) = az + b$.

For a surface (or a higher dimensional manifold) equipped with a triangulation by finite triangles, we simply develop the fundamental domain (as given by the spanning tree) into the model space, and then use the side-pairing information to produce the generators of the fundamental group. It should be noted that if a surface is finely triangulated, the generating set will contain many instances of the identity element, but this is obviously not a serious problem.

It should be noted that Step 4 of our algorithm can be replaced by using the spanning tree information to embed the triangulation in $\mathbb{H}^2$, and then computing the relevant isometries. This would, however, necessitate running a version of the continued fraction algorithm for each side pairing, and thus will be much less efficient ($O(n^2)$ vs our $O(n \log n)$.)

References

[1] Alan Beardon. Geometry of Discrete Groups, volume 91 of Graduate Texts in Mathematics. Springer-Verlag, 1995.
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[3] Sukhamay Kundu and Jayadev Misra. A linear tree partitioning algorithm. SIAM Journal on Computing, 6(1):151–154, 1977.