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Abstract. Consider the system with self-excited oscillations under the condition that the existence of oscillations depends on the slowly changing parameter. The dependence of the self-excited oscillations on stability of the steady-state is studied.

1. Introduction
Mathematical models of complex nonlinear systems naturally depend on many different parameters. The usual condition for study is that the parameters of the system are constants. However, in real-life models the parameters can change in time. So the system’s dimension increases, and behavior can vary dramatically. Therefore it is necessary to study the dependence of the behavior of the system on changing parameters. In the paper at hand we study the appearance of the self-excited oscillations in case of slowly increasing parameter responsible for the existence of oscillations.

The notion 'self-excited oscillations' was introduced by A.A. Andronov[1]. Nowadays, by self-excited oscillations we mean generation and maintenance of a periodic motion by a source of power that lacks any corresponding periodicity. The oscillator itself controls the phase with which the external power acts on it. As an example of such a system we consider the Van der Pol generator and study dependence of oscillations on slowly changing parameter of the system.

2. System with self-excited oscillations
Consider the Van der Pol generator. It describes a simple electric circuit with self-excited oscillations. The mathematical model for the generator is the following [1]:

\[ \ddot{x} - (a - x^2)\dot{x} + x = 0. \]

Here \( x \) is a dimensionless voltage, \( a \) is a parameter, describing main properties of the electric circuit. The usual proposal is that \( a \) is small. It is well-known that for small positive values of \( a \) the system has self-excited oscillations [1].

We rewrite the Van der Pol equation as a system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= a(1 - x^2)y - x.
\end{align*}
\]

The system has a trivial equilibrium. We consider the system in a small neighborhood of the equilibrium and study its stability. For this we linearize the system. The linearization matrix \( B \) is the following

\[
B = \begin{pmatrix} 0 & -1 \\ -1 & a \end{pmatrix}.
\]
The linearization matrix $B$ has eigenvalues defined by

$$\lambda = (a \pm \sqrt{a^2 - 4})/2.$$  

Since $a$ is small, the eigenvalues are complex conjugated. It means that in the small neighborhood of the equilibrium the system possesses oscillations. The amplitude variation depends on the value of the parameter $a$. The real part of eigenvalues is negative for $a<0$ and positive for $a>0$. Thus, the trivial solution is asymptotically stable focal point for negative $a$, and unstable focal point for positive $a$. Also, it is known that for positive value of the parameter $a$ there exists an attracting limit cycle. The parameter $a$ can be considered as a control parameter for the existence of oscillations.

On picture 1 it is shown the behavior of the system for different values of the parameter $a$. The solutions are shown for small negative $a$ (left figure) and for small positive $a$ (central and right figures). Also, for $a>0$ we have two cases – with initial values in a small neighborhood of the trivial solution and not. It is clearly, that for negative value of $a$ the system possesses damped oscillations, and for positive value of $a$ the amplitude of oscillation tends to some non-trivial value. So, we can say that the solutions wrap themselves around the limit cycle.

![Figure 1](image.png)

**Figure 1.** Dependence of the voltage on time for different values of the control parameter in case of its constancy.

In real-life models it is a usual condition that the parameters, describing different properties of the system, are not constant. This can be due to different reasons: aging of the system, changes of the environment. In what follows we suppose that control parameter $a$, starting with some negative initial value, slowly changing in time. So, we have the following system

$$\dot{a} = \varepsilon,$$

$$\dot{x} = y,$$

$$\dot{y} = a(1-x^2)y - x,$$

with $0 < \varepsilon \ll 1$. Thus we get a slow-fast system with slow variable $a$ and fast variables $x, y$. In our study we use asymptotical and geometrical methods of analysis of multi-speed systems (see e.g. [2]). The fast subsystem has a trivial equilibrium. In the small neighborhood of this equilibrium the linearization matrix has a pair of complex conjugated eigenvalues depending on control variable $a$. The main difficulty consists in the fact that the stability of the equilibrium depends on the slowly changing parameter. For $a<0$ the eigenvalues have negative real part, and the real part grows slowly with increasing of the control variable. For $a>0$ the real part becomes positive. So, the trivial solution is attracting for negative $a$ and repelling for positive $a$, or we can say that the trivial solution changes its attractivity in time.

The behavior of other solutions depends on the initial point. Starting with $a>0$ we get the dramatical growth of the amplitude of the oscillations. But for $a<0$ we get a very interesting behavior. Solutions starting for $a < 0$ in a small neighborhood of the trivial solution approach the attracting part of it and follow it until $a=0$. But after crossing the point $a=0$, solutions do not leave the small
neighborhood of the repelling part of the trivial solution. They follow it for some time of order $O(1)$ and then jump away. So the system possesses a delayed loss of stability [3].

Thus we get that for slow increase of the control variable the system possesses a self-excited oscillations. But due to the delayed loss of stability effect the growth of the amplitude is jump-like. This phenomenon is important to know in consequence of the fast growth of the amplitude of oscillation, so this could be the reason of faulty operations.

On the picture 2 it is shown the behavior of the unperturbed system with different initial conditions for $\alpha$. It is shown dependence of voltage on time with the same initial value and for $\alpha = -0.4$, $\alpha = -0.8$, $\alpha = -1.2$, correspondingly. The dotted line shows the dependence of the real part of the eigenvalues on time. Notice that the amplitude of the oscillations grows not immediately after point of change of stability and is jump-like. Also, we get that the more time solutions spend in the small neighborhood of the attracting part of the trivial solution, the longer time they spend in the neighborhood of the repelling part [4].

![Figure 2](image)

**Figure 2.** Dependence of the voltage on time for different initial values of slowly changing control parameter.

3. Conclusions

In the work we study the influence of the aging of the system with self-excited oscillations on the amplitude and type of oscillations. It is shown that the slowly changing parameters can be the reason of dramatical changes of behavior. As an example of such a system we study the Van der Pol generator. It is shown the various working regimes depending on the different initial values of the control parameter responsible for the existence of oscillations under the assumptions of constancy or slow changes.

4. References

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