EXACT SOLUTION OF THE SCHWINGER MODEL WITH COMPACT U(1)

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The exact solution of the Schwinger model with compact gauge group $U(1)$ is presented. The compactification is imposed by demanding that the only surviving true electromagnetic degree of freedom has angular character. Not surprisingly, this topological condition defines a version of the Schwinger model which is different from the standard one, where $c$ takes values on the line. The main consequences are: the spectra of the zero modes is not degenerated and does not correspond to the equally spaced harmonic oscillator, both the electric charge and a modified gauge invariant chiral charge are conserved (nevertheless, the axial-current anomaly is still present) and, finally, there is no need to introduce a $\theta$-vacuum. A comparison with the results of the standard Schwinger model is pointed out along the text.

1. Introduction

Using the Hamiltonian approach we solve the Schwinger model, with compact gauge group $U(1)$, which we call the compact Schwinger model (CSM) in the sequel. The standard Schwinger model has been solved in many ways and we have not attempted here to provide a complete list of all the related references. We will refer to the latter as the non-compact Schwinger model (NCSM).

The compactification of the gauge group $U(1)$ is realized by demanding that the only surviving electromagnetic degree of freedom, called $c$ in the sequel, behaves as an angular variable living in a circle of length $\frac{2\pi}{eL}$, i.e. $-\pi/eL \leq c \leq +\pi/eL$. Not surprisingly, the compactification prescription leads to a model which drastically differs from the NCSM, as will be seen along the text.

Many solutions of the NCSM, where the electromagnetic degree of freedom $c$ lives in the line $\{-\infty, +\infty\}$, start from considering $c$ as an angular variable. Nevertheless, using appropriate boundary conditions, the corresponding authors manage to unfold the circle into the line, i.e. to go from $U(1)$ to its universal covering.

Here we maintain the angular character of $c$ and fully explore the consequences of this choice. It is important to emphasize that our results follow uniquely from the
compactification condition, together with the standard definitions of both a scalar product and the hermiticity requirements in the corresponding Hilbert space.

A partial solution of the CSM was found in Ref. 11, using the loop approach to this problem, and served as a motivation for the work presented here. These partial results coincide with those obtained in this work. Previous progress towards the complete solution of the CSM were reported in 13.

Since both models, the CSM and the NCSM, differ only in the topology of the gauge group, it is neither surprising that the Hamiltonian method employed in the solution of the latter would be also effective for the former. For this reason and with the necessary modifications, we relay in the work of Refs. 8, 9, which provide a complete Hamiltonian solution of the NCSM.

2. CSM, gauge invariant degrees of freedom and commutator algebra

The model is described by the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu (i\partial_\mu - e A_\mu) \psi \]  

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \) \( \bar{\psi} = \psi^\dagger \gamma^0 \) is a Grassmann valued fermionic field and we are using units such that \( \hbar = c = 1. \) We consider the coordinate space to be \( S^1 \) and we will require periodic (antiperiodic) boundary conditions for the fields \( A_\mu(x) \) \( (\psi(x)) \), where \( L \) is the length of the circle. The gamma matrices are: \( \gamma^0 = \sigma_1, \gamma^1 = i\sigma_2, \gamma^5 = -\gamma^0 \gamma^1 = \sigma_3, \) where \( \sigma_i \) are the standard Pauli matrices. We use the signature \((+,−)\), i.e. \( \eta_{00} = -\eta_{11} = 1. \)

After the standard canonical analysis of the Lagrangian density (1), describing the configuration space variables \( A_0, A_1 \) and \( \psi \), we obtain

\[ \mathcal{H} = \frac{1}{2} E^2 + i \psi^\dagger \sigma_3 (\partial_1 + i e A_1) \psi - A_0 \left( \partial_1 E - e \psi^\dagger \psi \right), \quad \Pi_0 \approx 0, \]  

where the corresponding canonical momenta are \( \Pi_0, \Pi_1 = F_{01} = E \) and \( \Pi_\psi = -i \psi^* \). Conservation in time of the primary constraint \( \Pi_0 \approx 0 \) leads to the the Gauss law constraint

\[ \mathcal{G} = \partial_1 E - e \psi^\dagger \psi \approx 0. \]  

There are no additional constraints. At this stage we partially fix the gauge in the electromagnetic potential by choosing \( A_0 = 0, \Pi_0 = 0. \) The only remaining constraint \( \mathcal{G} \) is first class and it will be imposed strongly upon the physical states of the system.

From now on we use the notation \( A_1 = A \) for the surviving electromagnetic degree of freedom. Also we have \( \psi = (\psi_+, \psi_-)^T \), where \( ^T \) denotes transposition. Applying the standard canonical quantization procedure to the resulting Poisson brackets algebra, we obtain the well known commutator (anticommutator) algebra for the involved fields.

The previous choice of gauge does not completely fix the electromagnetic degrees of freedom, leaving the Lagrangian density (1) still invariant under the gauge
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transformations
\[ \psi \rightarrow e^{ie^{i\alpha(x)}} \psi, \quad A_{\mu} \rightarrow A_{\mu} - \partial_{\mu} \alpha(x), \]
generated by \( G \). The constant piece \( \alpha_0 \) of the function \( \alpha(x) \) is irrelevant in the above transformation, leading to an overall phase in the fermionic field. In the sequel we consider \( \tilde{\alpha}(x) = \alpha(x) - \alpha_0 \) as the function generating the gauge transformations. As it is well known, there are two families of gauge transformations: (1) those continuously connected to the identity, called small gauge transformations (SGT), characterized by the functions
\[ \tilde{\alpha}_S(x) = b \left( e^{i2\pi n x/L} - 1 \right), \quad \tilde{\alpha}_S(0) = \tilde{\alpha}_S(L) = 0, \]
which are periodic in \( x \). (2) The second family corresponds to the so-called large gauge transformations (LGT), which are generated by the non-periodic functions
\[ \tilde{\alpha}_L(x) = \frac{2\pi n}{eL} x = 2n\tilde{c} x, \quad n = \pm 1, \pm 2, \ldots, \quad \tilde{c} = \frac{\pi}{eL}, \quad \tilde{\alpha}_L(0) = 0. \]
Let us emphasize that in both cases we have
\[ \tilde{\alpha}(0) = 0. \]

At this stage we define the CSM by demanding that the only true degree of freedom arising from the electromagnetic potential in one dimension, which is the zero mode \( c \), be restricted to the interval
\[ -\tilde{c} \leq c \leq \frac{1}{L} \int_0^L A(z) \, dz \leq \tilde{c}. \]

The compactification condition (8) implies that two values of \( c \) differing by \( 2\tilde{c}N = \frac{2\pi N}{eL} \) must be identified, corresponding to one point in such configuration space.

Next we show that the basic degrees of freedom in the CSM are in fact fully gauge invariant. Let us consider the following Fourier decomposition for the electromagnetic potential \( A \), the field strength \( E \) and the gauge transformation function \( \tilde{\alpha} \)
\[ A(x, t) = c(t) + \sum_{m \neq 0} A_m(t) \, e^{2\pi im x}, \quad E(x, t) = E_0(t) + \sum_{m \neq 0} E_m(t) \, e^{-2\pi im x}, \]
\[ \tilde{\alpha}(x) = \sum_{m \neq 0} \tilde{\alpha}_m \, e^{2\pi im x}, \quad \text{where} \quad 0 = \tilde{\alpha}(0) = \sum_{m \neq 0} \tilde{\alpha}_m. \]

Under a general gauge transformation \( A(x) \rightarrow A(x) - \frac{\partial \tilde{\alpha}(x)}{\partial x} \), the corresponding modes change as
\[ c \rightarrow c - \frac{1}{L} (\tilde{\alpha}(L) - \tilde{\alpha}(0)), \quad A_m \rightarrow A_m - \frac{2\pi im}{L} \tilde{\alpha}_m, \quad m \neq 0. \]
Clearly, the zero mode \( c \) is invariant under SGT. As for LGT, \( c \rightarrow c - \frac{2\pi N}{eL} \), but these points must be identified according to the compactification condition (8). In other
words, $c$ is also invariant under LGT. Summarizing, the zero mode $c$ is fully gauge invariant. This is a consequence of our choice of topology for $c$ and provides the fundamental difference with the NCSM, leading to all the remaining non-standard features of the CSM.

Next we consider the expansion of the fermionic variables in a background electromagnetic field $A(x)$. According to Ref. [9], these can be written as

$$
\psi_+(x, t) = \sum_n a_n \phi_n(x) e^{-i \epsilon_n t}, \quad \psi_-(x, t) = \sum_n b_n \phi_n(x) e^{i \epsilon_n t},
$$

(11)

where $a_n, b_n$ are independent fermionic annihilation operators satisfying the standard anticommutators. The states $\psi_+$ ($\psi_-$) describe the positive (negative) chirality sectors of the model. The functions $\phi_n$, together with the energy eigenvalues $\epsilon_n$ are given by

$$
\phi_n(x) = \frac{1}{\sqrt{L}} e^{-i \epsilon_n x - i \int_0^x A(z) dz}, \quad \epsilon_n = \frac{2\pi}{L} \left( n + \frac{1}{2} - \frac{eL}{2\pi c} \right).
$$

(12)

Rewriting the fermionic sector of the Hamiltonian density (2) as $H_F = \psi^\dagger h_F \psi$, we observe that the corresponding eigenvalues of $h_F$ are $+\epsilon_n$ and $-\epsilon_n$ for the positive and negative chirality sectors, respectively.

Since $c$ is invariant under both LGT and SGT, each energy eigenvalue $\epsilon_n$ is fully gauge invariant. Furthermore, according to the definition (12), we obtain

$$
\phi_n(x) \rightarrow e^{ie\bar{\alpha}(x)} \phi_n(x),
$$

(13)

under gauge transformations. Here we have used the condition $\bar{\alpha}(0) = 0$, which is valid for both LGT and SGT. As a consequence of the above properties and in order to recover the transformation law (4) of the fermionic field $\psi$, we are led to

$$
a_n \rightarrow a_n, \quad b_n \rightarrow b_n.
$$

(14)

for the gauge transformation of the fermionic operators $a_n$ and $b_n$.

In other words, consistency among the compactification condition (8), the transformation law (4) and the definition (12) for $\phi_n$ demands that the basic fermionic operators $a_n$ and $b_n$ are fully gauge invariant in the compact case. Let us emphasize that the above property establishes the main difference between the CSM and the NCSM.

Following the same steps in the NCSM we obtain that the change $c \rightarrow c - \frac{2\pi n}{eL}$, under LGT and with these points not identified, implies that the individual energy eigenvalues are not gauge invariant, i.e. $\epsilon_n \rightarrow \epsilon_{n+1}$. This leads to $\phi_n(x) \rightarrow e^{-i e\bar{\alpha}(x)} \phi_{n+1}$. Again, in order to satisfy the transformation property (4) of the fermionic field, we must have now that $a_n \rightarrow a_{n+1}, b_n \rightarrow b_{n+1}$, under LGT, which is the well known result in the NCSM.

In this way, it is transparent that the topological behavior of $c$, i.e. compact versus non-compact case, implies completely different transformation laws for the operators $c, a_n$ and $b_n$ under gauge transformations.
Using the Fourier expansions (9) and (11), we rewrite the commutators for the fields in terms of the corresponding modes. In particular, the commutator 
\[ [E(x), \psi_n(y)] = 0 \]
leads to
\[ [E_m, a_n] = \frac{ie}{2\pi im} (a_n - a_{n+m}), \quad m \neq 0, \quad [E_0, a_n] = 0, \]
with analogous relation for \( [E_m, b_n] \). The remaining commutators are
\[ [A_k, A_l] = 0 = [E_k, E_l], \quad [A_k, E_l] = \frac{i}{L} \delta_{kl}, \quad [A_k, a_n] = 0, \quad [A_k, b_m] = 0. \]

Next, we concentrate on the commutator algebra of the Fourier modes. To this end, it is convenient to introduce the following operators
\[ j_{nm}^{++} = a_n^\dagger a_m, \quad j_{nm}^{--} = b_n^\dagger b_m, \quad j_{nm}^{-+} = a_n^\dagger b_m, \quad j_{nm}^{+-} = b_n^\dagger a_m. \]

Another useful combinations of the above fermionic operators are the currents
\[ j_\pm (x) = \psi_\pm^\dagger (x) \psi_\pm (x) = \frac{1}{L} \sum_{n=-\infty}^{+\infty} e^{\pm \frac{2\pi in}{L}} j_\pm^n, \]
where
\[ j_+^n = \sum_{m=-\infty}^{\infty} j_{nm}^{++}, \quad j_-^n = \sum_{m=-\infty}^{\infty} j_{nm}^{--}. \]

At this stage we introduce the \( \zeta \)-regularized form of the currents defined in Eq.(19),
\[ j_+^n |_{reg} = \lim_{s \to 0} \sum_{m=-\infty}^{\infty} \frac{1}{\lambda_{m,s}} a_m^\dagger a_{m+n}, \quad j_-^n |_{reg} = \lim_{s \to 0} \sum_{m=-\infty}^{\infty} \frac{1}{\lambda_{m,s}} b_m^\dagger b_{m+n}, \]
where the regulator is given by \( \lambda_{n,s} = |\lambda \epsilon_n|^s \), with \( \lambda \) being a parameter with dimensions of inverse energy. In the sequel we will drop the subindex \( |_{reg} \) from the above currents, but we will always consider their form (20) in calculating any relation involving them. At the end of the calculation we will take the \( s \to 0 \) limit. In other words, we will construct an algebra among regularized objects, which will be further restricted to the action upon the physical Hilbert space of the problem. It can be shown that the regularized current algebra of the operators (21) is given by
\[ [j_+^n, (j_+^m)^\dagger] = n \delta_{m,n}, \quad [j_-^n, (j_-^m)^\dagger] = n \delta_{m,n}, \quad [j_+^n, j_-^m] = 0, \]

together with the hermiticity properties \( (j_\pm^m)^\dagger = j_{\mp}^{-m} \). The above commutation relations are the same as those obtained in the NCSM.

In order to satisfy the commutation relations (15) and (16) we make the ansatz
\[ E_m = \frac{1}{iL} \frac{\partial}{\partial A_m} - \frac{e}{2\pi im} (j_+^m + (j_-^m)^\dagger), \quad m \neq 0, \quad E_0 = \frac{1}{iL} \frac{\partial}{\partial c}. \]
which clearly satisfies the third commutation relation in (14). Substituting the expressions (22) in the corresponding commutators of Eq.(15) we obtain

\[
\frac{\partial a_n}{\partial A_m} = -\frac{e L}{2\pi m} a_n, \quad m \neq 0, \quad \frac{\partial a_n}{\partial c} = 0.
\]

(23)

The above equation leads to the following solution for the fermionic operators with respect to their dependence on the gauge field

\[
a_m = \exp \left( -\frac{e L}{2\pi} \sum_{k \neq 0} \frac{1}{k} A_k \right) \tilde{a}_m,
\]

(24)

where \(\tilde{a}_m\) are new fermionic operators which are independent of the gauge field \(A_k\) and which also satisfy the basic fermionic anticommutation relations. The expression (24) reproduces the fully gauge invariant character of \(a_n\). In fact, under the gauge transformation (10), the exponential in (24) changes by a factor \(\exp(ie \sum_{k \neq 0} \tilde{\alpha}_k)\) which is exactly \(\exp(ie\tilde{\alpha}(0)) = 1\), according to the fourth relation in (9). Analogous results are obtained for the operators \(b_n\).

With the ansatz (22), the Gauss law \(G(x) = -\frac{1}{L} \sum \exp(-2\pi i m x) G_m\) reduces to

\[
G_0 = e \left( j^0_+ + (j^-_0)^\dagger \right) = eQ
\]

\[
G_m = 2\pi im E_m + e \left( j^m_+ + (j^-_m)^\dagger \right) = 2\pi m \frac{\partial}{\partial A_m}, \quad m \neq 0.
\]

(25)

Our expression (25) for the Gauss law constraint is somewhat different from the one obtained in Ref. 9.

Summarizing, we have shown that the compactification condition (8) implies that the basic operators which are used to solve the model: \(c, a_n\) and \(b_n\) are fully gauge invariant. Also, the Gauss constraint implies that the wave function of the system must have zero electric charge, being independent of the electromagnetic modes \(A_m, \quad m \neq 0\).

3. Fermionic Fock space

We now construct the fermionic Fock space in a background electromagnetic field. Starting from the vacuum \(|0\rangle\) annihilated by the operators \(a_n\) (positive chirality sector), \(b_n\) (negative chirality sector), the Dirac vacuum \(|\text{vac}\rangle\) is constructed in such a way that all negative energy levels are filled. Our compactification condition (8) for the electromagnetic variable \(c\) implies that all levels with \(n \leq -1 (n \geq 0)\) have negative energies for the positive (negative) chirality sectors, respectively. In this way, the Dirac vacuum is

\[
|\text{vac}\rangle = \prod_{n=-\infty}^{-1} a_n^\dagger |0\rangle \otimes \prod_{n=0}^{\infty} b_n^\dagger |0\rangle.
\]

(26)
Using the \( \zeta \)-regularized expressions for the operators electric charge \( Q \), chiral charge \( Q_5 \) and energy \( H_F \), we obtain the following eigenvalues on the Dirac vacuum state:

\[
Q|\text{vac}\rangle = 0, \quad Q_5|\text{vac}\rangle = -\frac{ecl}{\pi}|\text{vac}\rangle, \\
H_F|\text{vac}\rangle = \frac{2\pi}{L} \left( \left( \frac{ecl}{2\pi} \right)^2 - \frac{1}{12} \right)^2|\text{vac}\rangle \equiv \varepsilon_0|\text{vac}\rangle, 
\]

At this level it is already convenient to introduce the modified chiral charge \( \bar{Q}_5 = \lim_{s \to +\infty} \sum_{n = -\infty}^{+\infty} \frac{1}{|\lambda\varepsilon_n|^s} (a_n^+ a_n - b_n^+ b_n) + \frac{ecl}{\pi}, \)

which is fully gauge invariant in the CSM. This is not the case in NCSM, where \( \bar{Q}_5 \to \bar{Q}_5 + 2n \) under LGT. The eigenvalues of \( \bar{Q}_5 \) are even numbers in the fermionic Hilbert space. In the sequel we will refer to \( \bar{Q}_5 \) as the modified chiral charge of the system. Also, a given eigenvalue \( 2M \) of \( \bar{Q}_5 \) will be referred to as the \( M \)-chiral sector of the theory. In this way, the second equation (27) reads \( \bar{Q}_5|\text{vac}\rangle = 0 \), which assigns zero modified chiral charge to the Dirac vacuum. Without writing the explicit label of zero electric charge, which will be understood for all physical states in the sequel, we denote

\[
|\text{vac}\rangle = |\varepsilon_0, 0\rangle. 
\]

It can be shown that the current operators \( j_+^n \) and \( j_-^n \) do not change either the electric or the chiral charge. Also, the action of any linear combination of the operators \( j_+^{pq} \) \( (j_-^{pq}) \), which leaves invariant the electric charge, will change the chiral charge by \( +2 \) (\( -2 \)) units.

Next, using the operators (17) we can construct additional states with minimum energy, but different chirality. The are

\[
|\varepsilon, 2N\rangle = \prod_{n = -\infty}^{N-1} a_n^+ |0\rangle \otimes \prod_{m = N}^{\infty} b_m^+ |0\rangle, \quad \varepsilon_{N+}\left(\varepsilon_{N}\right) = \frac{2\pi}{L} \left( \left( N - \frac{ecl}{2\pi} \right)^2 - \frac{1}{12} \right) \]

All the states in (30) have zero electric charge. They satisfy the recursions

\[
j_{+}^{N-1}|\varepsilon, 2N\rangle = |\varepsilon, 2(N + 1)\rangle, \quad j_{-}^{N-1}|\varepsilon, 2N\rangle = |\varepsilon, 2(N - 1)\rangle. \]

Summarizing, from the Dirac vacuum we have so far constructed states with minimum energy for each possible chirality. Each one of these states can be considered as a local vacuum in the corresponding chirality sector.

The properties

\[
j_{\pm}^{n}|\varepsilon, 2N\rangle = 0, \quad n \geq 1, \quad [H_F, (j_{\pm}^n)^\dagger] = \frac{2\pi n}{L} (j_{\pm}^n)^\dagger. \]
allow us to construct the complete fermionic Fock space in the background electromagnetic field. It will consist of all the local vacuums (30), together with all possible states constructed from them by the application of an arbitrary number of the current operators \((j_{\pm}^n)^\dagger, n = 1, 2, \ldots\) defined in Eq. (20). The spectrum of this fermionic Fock space is
\[
\{\varepsilon_N + \frac{2\pi}{L} M, M = 1, 2, \ldots, N = 0, \pm 1, \pm 2, \ldots\}.
\] (33)

In this way, the fermionic Hamiltonian in the external field can be rewritten in the Sugawara form
\[
H_F = \varepsilon_N(c) + \frac{2\pi}{L} \sum_{n>0} ((j_+^n)^\dagger j_+^n + (j_-^n)^\dagger j_-^n).
\] (34)

4. Complete solution and comments

The next step is to write the complete Hamiltonian \(H = H_{EM} + H_F\) in terms the currents operators, together with the electromagnetic degrees of freedom, which are the zero mode of the electric field \(\partial/\partial c\), and the zero mode of the gauge potential \(c\). The result is
\[
H = H_{EM} + H_F = H_0 + \sum_{n>0} H_n - \frac{2\pi}{12L},
\] (35)

where
\[
H_0 = \frac{\pi}{2L} \left( Q^2 + \left( \bar{Q}_5 - \frac{ecL}{\pi} \right)^2 \right) - \frac{1}{2L} \left( \frac{\partial}{\partial c} \right)^2,
\]
\[
H_n = \frac{2\pi}{L} ((j_+^n)^\dagger j_+^n + (j_-^n)^\dagger j_-^n) + \frac{e^2L}{4\pi^2n^2}((j_+^n)^\dagger + j_-^n)(j_+^n + j_-^n)^\dagger.
\] (36)

Following Refs. 8, 9, we have explicitly used the Gauss law constraint (27) to express the electric field modes \(E_n\) in terms of the fermionic currents.

In order to diagonalize the expression (36) for the Hamiltonian, we use the Bogoliubov transformations \(\tilde{j}_+^n = U^\dagger_n (j_+^n) U_n\), given in (27), for the currents operators. The Bogoliubov transformation affects only the modes \(n \geq 1\) of the system and, in particular, the currents \(j_0^\pm\), or equivalently \(Q\) and \(\bar{Q}_5\), remain unchanged. In this way, the fully rotated Hamiltonian \(H_B = U^\dagger H (j_+^n, j_-^n) U = H(\tilde{j}_+^n, \tilde{j}_-^n)\), is
\[
H_B = \frac{\pi}{2L} \left( Q^2 + \left( \bar{Q}_5 - \frac{ecL}{\pi} \right)^2 \right) - \frac{1}{2L} \left( \frac{\partial}{\partial c} \right)^2 + \sum_{n>0} \frac{\varepsilon_n}{n} ((j_+^n)^\dagger j_+^n + (j_-^n)^\dagger j_-^n),
\] (37)

up to an infinite constant.

The general structure of the states in the full Hilbert space of the model will be of the type
\[
|\text{state}\rangle = F(c) \times |\text{fermionic}\rangle.
\] (38)
The whole wave function will have zero electric charge and definite chiral charge, which really implies a condition only upon fermionic piece. The strategy to construct the Hilbert space will be to start from the zero modes $F_N(c) \times |\nu, 2N\rangle$ and to subsequently apply all possible combinations of the raising operators $(j^\pm_m)^\dagger$.

First we consider the zero modes. They correspond to the case of zero fermionic excitations above the corresponding Dirac vacuum and can be written as

$$|N\rangle_B = F_N(c) \times |\nu, 2N\rangle.$$  \hspace{1cm} (39)

The subscript $B$ in any ket is to remind us that such vector is written in the Bogoliubov rotated frame, where the Hamiltonian has the form (37). Its action upon the above wave functions reduces to the following Schrödinger equation for the zero mode wave functions $F_N(c)$

$$-\frac{1}{2L} \left( \frac{\partial}{\partial c} \right)^2 + \frac{e^2L}{2\pi} \left( \frac{2\pi N}{eL} - e \right)^2 F_N(c) = \mathcal{E}_{N,0} F_N(c),$$  \hspace{1cm} (40)

which corresponds to a piecewise harmonic oscillator. Each sector, labeled by $N$, is defined in the interval $-\bar{e} \leq c \leq \bar{e}$.

For arbitrary functions $F(c)$ and $G(c)$, we define the inner product in the standard way as

$$\langle F|G \rangle = \int_{-\bar{e}}^{\bar{e}} dc \ F^*(c)G(c).$$  \hspace{1cm} (41)

Next we demand the hermiticity of the zero mode electric field operator $E_0 = \frac{1}{iL} \frac{\partial}{\partial c}$, together with the Hamiltonian (40). The above requirements lead to the boundary conditions

$$F_N|_{c=-\bar{e}} = F_N|_{c=+\bar{e}}, \quad \frac{\partial F_N}{\partial c}|_{c=-\bar{e}} = \frac{\partial F_N}{\partial c}|_{c=+\bar{e}},$$  \hspace{1cm} (42)

for the wave function $F_N$ and their derivatives.

A fundamental difference between the CSM and the NCSM arises in the energy spectrum $\{\mathcal{E}_{\alpha,N,0}\}$ of zero mode sector. Here $\alpha = 0, 1, 2, \ldots$ labels the eigenvalues of the zero mode 0 in the $N$-chiral sector of the model. The solution corresponding to $N = 0$ has been already discussed in Ref.\textsuperscript{11}, together with the corresponding wave functions. Here we extend the calculation for arbitrary $N \neq 0$. The general solution of the above Schrödinger equation can be expressed in terms of cylindrical parabolic functions \textsuperscript{15}. The energy eigenvalues are parametrized as $\mathcal{E}_{\alpha,N,0} = -\frac{e^2}{\pi a_{\alpha,N}}$, where $a_{\alpha,N}$ is determined by a complicated transcendental equation arising from the boundary conditions \textsuperscript{13}. As in the $N = 0$ case \textsuperscript{13}, this function can only be determined numerically for arbitrary $l = \frac{eL}{\pi^{3/2}}$. In Figs.1 and 2 we show the results for $a_{\alpha,N}$ versus $l$, for the choices $\alpha = 0, 1, 2$ and $N = 0, 1, 2, 3$.

Among the zero modes, we now focus on the local minimum ($\alpha = 0$) energy states: $|0, N, 0\rangle_B$, for each chiral sector $N$ of the theory. They have energies $\mathcal{E}_{0,N,0}$. An important consequence of the compactification prescription is that these states are not fully degenerated as they are in the NCSM. Most importantly, from the
numerical calculation we find that the absolute minimum value of $\mathcal{E}_{0,N,0}$ correspond to $N = 0$. Thus, in the compact case the physical, non-degenerated, vacuum of the theory is $|0,0,0\rangle_B$. This means that we do not need to introduce the $\theta$-vacua in the CSM.

Fig. 1. The numerical solution for the parameter $a_{0,N}(l)$, for $N = 0, 1, 2, 3$ and a given value of $l$. The energies are $\mathcal{E}_{0,N,0} = -(e/\pi^{1/2})a_{0,N}$.

The excited states are obtained by applying the creation operators $(j_{\pm m})^\dagger$ to the zero modes constructed previously. Each individual action raises the energy by $\mathcal{E}_m$, as can be seen from Eq.(37). The excited states will be labeled by

$$|\alpha, N, N_1, \ldots, N_k, \ldots\rangle_B,$$

where $N_k$ is the the total number of times that the operators $(j_{\pm k})^\dagger$ have been applied to the corresponding minimum energy state. The total energy of the state (43) is given by

$$\mathcal{E}_{\alpha,N,N_1,N_2,\ldots,N_k,\ldots} = \mathcal{E}_{\alpha,N,0} + \sum_{k>0} N_k \mathcal{E}_k.$$

(44)
Fig. 2. The numerical solution of $a_{\alpha,N}(l)$, with $N = 0, 1$ and $\alpha = 1, 2$, for a given value of $l$, is given. The energies are $E_{\alpha,N,0} = -(e/\pi 1/2) a_{\alpha,N}$. 
The fact that the chiral charge \( \bar{Q}_5 = j_+^0 - j_-^0 + e c L/\pi \) is conserved in the full Hilbert space of the model deserves a separate discussion. We make the calculation of the commutator \([\bar{Q}_5, H_B]\) in the Bogoliubov rotated frame, where \( \bar{Q}_5 \) preserves the above expression and \( H_B \) is given by (37). It is a direct matter to verify that \( \bar{Q}_5 \) commutes with all the terms in the full Hamiltonian (37), with the exception of the derivative term. We analyze this piece in the sequel. Let us begin with

\[
C_+^n = \left[ \frac{\partial}{\partial c}, \bar{j}_+^n \right] = \lim_{s \to 0} \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial c} \left( \frac{1}{|\epsilon_n|^s} \right) a_m^\dagger a_{m+n} \tag{45}
\]

First, let us consider the action of \( C_+^n, n \neq 0 \) upon an arbitrary vector \(|\{m_i\}\rangle = \prod_i a_{m_i}^\dagger |0\rangle\) in the positive-chirality fermionic Fock subspace. In general, the subindex \( m_i \) will take values over an infinite subset of integer numbers. The only non-zero result of the action of the \( i \)th-term of (45) upon the above vector, is to replace the \( m_i + n \) fermion by the \( m_i \) fermion, thus leading to a sum of linearly independent states. In this way, the \( s \to 0 \) limit must be taken separately in each term of the series and no infinite summation occurs. Since

\[
\frac{\partial}{\partial c} \left( \frac{1}{|\epsilon_n|^s} \right) \approx -\frac{s}{|\epsilon_n|^{s+1}}, \tag{47}
\]

this limit is zero and the operators commute.

Now, let us consider the \( n = 0 \) case together with the action of \( C_+^0 \) upon the local ground state \( F_N(c) \times |\varepsilon_N, 2N\rangle \) of each chirality sector. We obtain

\[
C_+^0 |N\rangle_B = \lim_{s \to 0} \sum_{m=-\infty}^{N-1} \frac{\partial}{\partial c} \left( \frac{1}{|\epsilon_n|^s} \right) |N\rangle_B
= -\frac{e L}{2\pi} \lim_{s \to 0} s \zeta(s+1, \frac{1}{2} + \frac{e c L}{2\pi} - N) |N\rangle_B = -\frac{e L}{2\pi} |N\rangle_B, \tag{48}
\]

where \( \zeta(s, q) \) is the standard Riemann zeta-function. We have used the property \( \lim_{s \to 0} s \zeta(s+1, q) = 1 \) \(15\). In analogous manner we obtain

\[
C_-^0 |N\rangle_B = \frac{e L}{2\pi} |N\rangle_B. \tag{49}
\]

The above results lead to

\[
\left[ \frac{\partial}{\partial c}, \bar{Q}_5 \right] |N\rangle_B = \left( C_+^0 - C_-^0 + \frac{e L}{\pi} \right) |N\rangle_B = 0. \tag{50}
\]

Besides, any excited state is constructed by applying the raising operators \( (j_+^n)^\dagger, n \geq 1 \) to \(|N\rangle_B\). These operators commute with \( \bar{Q}_5 \) and \( \frac{\partial}{\partial c} \) in such way that the commutator \( \left[ \frac{\partial}{\partial c}, \bar{Q}_5 \right] \) is zero in the full Hilbert space of the problem. This completes our
proof that the fully gauge invariant charge $\bar{Q}_5$ commutes with the total Hamiltonian (37).

Nevertheless, the axial-current anomaly is still present in the CSM, as we now discuss. The charge $Q_5$ arises from the current $J_5 \mu(x) = \bar{\psi}(x)\gamma_\mu \gamma_5 \psi(x)$ which possesses the anomaly

$$\partial_\mu J_5 \mu = -\frac{e}{\pi} E(x),$$

(51)

that can be directly calculated using the mode decomposition of $J_5 \mu$, together with the unrotated Hamiltonian (35) and the Gauss law (22). On the other hand, one can introduce the conserved local current

$$\bar{J}_5 \mu(x) = J_5 \mu(x) - \frac{e}{\pi} \epsilon_{\mu\nu} A^\nu, \quad \epsilon_{01} = +1.$$  

(52)

leading to the charge $\bar{Q}_5$. Nevertheless, the current (52) is not gauge invariant, so that it cannot be restricted to the physical Hilbert space of the problem.

Summarizing this point, the axial current anomaly (51) is also present in the CSM, and it cannot be removed, in spite that it is possible to define the conserved and gauge invariant modified chiral charge $\bar{Q}_5$.

Finally we comment that our boundary conditions (42) are an unavoidable consequence of the compactification of the electromagnetic degree of freedom $c$. These should be contrasted with those appropriate for the NCSM (Eqs. (3.15) of Ref. 5, or Eq. (48) of Ref. 8). The latter are correctly designed to recover the non-compact case, i.e. to go from the compact gauge group $U(1)$ to the corresponding universal covering. Also, for a given $L \neq 0$, the boundary conditions for the CSM and those for the NCSM can not be continuously connected between each other. This emphasizes the fact that the compactification condition (8) has produced a new version of the Schwinger model which is different from the standard one.

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