On cosmic inflation in vector field theories

Alexey Golovnev

Saint Petersburg State University, High Energy Physics Department, Ulyanovskaya ul., d. 1, 198504 Saint Petersburg, Petrodvoretz, Russia

E-mail: agolovnev@yandex.ru

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Abstract
We investigate the longitudinal ghost issue in Abelian vector inflation. It turns out that, within the class of Lorentz-invariant vector field theories with three degrees of freedom and without any extra (scalar) fields, the possibilities are essentially exhausted by the classical solution due to Larry Ford with an extremely flat potential which does not feel the fast roll of its argument. And, moreover, one needs to fulfill an extra condition on that potential in order to avoid severe gradient instability. At the same time, some Lorentz-violating modifications are worth exploring.

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1. Introduction

Inflation is one of the most successful attempts to understand the reasons why our Universe is so flat, and large, and homogeneous. Inflation is usually driven by a scalar field called inflaton. Not only does this picture solve the problems of the standard Big Bang cosmology, but it actually provides us with a very good origin of tiny inhomogeneities seen in the cosmic microwave background radiation which presumably gave rise to all the structures in the Universe. However, despite this great success, the nature of the inflaton remains largely unknown. It is therefore interesting to explore some other conceivable types of inflationary models.

Probably, the simplest alternative idea is to use the usual (massive) vector fields. However, it is easy to see that vector fields have two major problems: generically they do not satisfy the slow-roll conditions and induce too large an anisotropy of expansion. In the pioneering work by Ford [1], we find two possible solutions of the slow-roll problem. First, in a version of new inflationary scenario, the effect of tachyonic vector field mass can balance (if fine-tuned) the inflationary dilution of the vector field potential energy (see also [2, 3]). Second, the vector field potential in chaotic inflation can be taken so extremely flat that it would be practically insensitive to the fast roll of its argument, $A_{\mu}A^{\mu}$. The anisotropy problem can be cured by some specific configurations of vector fields (vector triples) [2, 4–7] or with a large number of randomly oriented independent non-interacting fields [7]. It does not actually
quite work for large fields models, and strong anisotropies generically develop in the chaotic inflationary regimes [8, 9]. Nevertheless, we would ignore the anisotropy problem in this paper and concentrate on enforcing the slow-roll conditions in a reasonable way. The standard approach [7] (apart from explicit tachyonic mass) is to prevent the vector fields from decaying with a non-minimal coupling to gravity of the $RA_\mu A^\mu$ form, see also [10, 11]. It perfectly works at the background level. However, the problem is that effectively this coupling acts, of course, as nothing else but tachyonic mass. And the tachyonic mass for vector fields means that the longitudinal modes are ghosts (at sub-Hubble scales) and lead to violent instabilities [9, 12–15].

Let us summarize the known stability problems in vector inflation. Probably, the first to be reported was the problem of anisotropic instability in chaotic-type models. Even at the background level, it was clear [7] that one cannot start vector inflation from arbitrarily high values of $N$ randomly oriented vector fields because the isotropic solution requires a statistical cancellation of pretty large anisotropic terms; and for the large vector field values, the anisotropic statistical fluctuation dominates over the mean isotropic quantity. This simple observation led to conclusion [7] that, for the mass-term potential, one can generically have $\sim 2\pi\sqrt{N}$ e-folds of nearly isotropic inflation. It provides a first hint that it would be quite expectable if even very small anisotropic vector field fluctuations around the background solution are to become (linearly) unstable. And this is actually the case. In [8], it was claimed that gravitational waves generically possess a bad tachyonic instability. This analysis was incomplete as all the couplings with other modes were unsubstantiatedly neglected (recall that the so-called decomposition theorem is not valid in vector inflation [2]). One could also suspect that the exponential growth of anisotropies is just an artifact of the linear perturbation analysis in the Jordan frame. However, in the long-wavelength limit, this effect can be seen at the fully nonlinear level too [9].

Another, and the most worrisome, stability problem is that of the longitudinal vector field fluctuations. For tachyonic masses, the longitudinal modes are ghosts at large momenta. We will see that, in the class of Lorentz-invariant vector field theories, the chaotic inflationary model by L Ford is virtually unique as a candidate for stable realization of vector inflation. But we will also see that even among these fine-tuned ghost-free potentials, very dangerous gradient instabilities do sometimes occur around inflationary backgrounds.

Finally, the number of degrees of freedom is ill-defined in the model with $RA_\mu A^\mu$-coupling [9]. The scalar curvature contains second spacetime derivatives of the metric field, and therefore the temporal components $A_0$ (or more precisely, one combination of $A_0$-components of all the vector fields in the model, see [9]) become dynamical, but not around the homogeneous background for which $A_0 = 0$. Spatial homogeneity amounts to the strong coupling regime for the temporal vector field modes.

Let us also note that vector inflation has been later generalized to higher p-form fields [16, 17] too. With special couplings to Ricci scalar and Ricci tensor, these theories can also exactly mimic the usual slow-roll dynamics of a scalar inflaton. Moreover, in [16] it was found that there exist duality transformations which relate 2-form models to vector inflationary ones, and 3 forms to scalars. The latter allowed for a detailed analysis of linear perturbations in the 3-form (isotropic) inflation [18]. Note though that these transformations involve both the curvature scalar $R$ and $R_{\mu\nu}$ tensor in a non-trivial way, and therefore they contain second time derivatives of metric which leads to the higher (third) derivative terms in the energy–momentum tensor in the dual picture. The troublesome terms vanish in the homogeneous background and correspond, of course, to the aforementioned problem with extra ill-defined degrees of freedom. However, it is important to understand that there is no need to mimic the scalar field evolution precisely, even in slow-roll scenarios. We cannot
afford the usual $\sim H^2$ contributions to the effective mass since they render the Hubble friction ineffective. But any corrections of order $\dot{H}$ make no harm in quasi-de-Sitter stage since the Hubble constant is almost constant. Therefore, one could for example couple the vector inflaton not only to $R$ but also to $R_{\mu\nu}$. This fact is of particular importance for 3-form models which actually admit the isotropic minimal-coupling slow-roll scenario [19, 20] without the usual instability problems. At the same time, the 2-form inflation shares all of the problems typical for vector inflation including both the ghost issues [16] and the catastrophic growth of anisotropies [21].

In this paper, we restrict our attention to the case of Abelian vector inflation, and in particular to its ghost problem. Our major interest is to find out whether it is possible to overcome the ghost problem of vector inflation without introducing extra (scalar) fields into the model. (Note that there is at least one model on the market which produces a stable anisotropic inflation with vector fields coupled to a scalar (dilaton) field via the $f(\phi)F_{\mu\nu}F^{\mu\nu}$ term, see [22–24].) Other stability problems are beyond the scope of this work. Therefore, we consider vector fields in a fixed background geometry. It is enough for the discussion of the vector field longitudinal mode behavior. In this approximation, the curvature scalar and tensor can be treated just as (time-dependent) parameters in a minimally coupled vector field Lagrangian. Even apart from the time dependence, this effective theory may appear to be Lorentz-violating even for Lorentz-invariant models due to possible $R_{\mu\nu}A^\mu A^\nu$ coupling. However, both the Lorentz-violating effects and the time dependence of effective parameters are small in quasi-de-Sitter stages. We will first discuss only Lorentz-invariant vector field theories and then make some comments on Lorentz-violating models. Note also that throughout the paper we assume the spatially flat FRW spacetime, while it is known to be quite hard to start vector inflation in spatially curved backgrounds [25].

2. Canonical vector fields

In vector inflation, we assume that at the background level the fields are homogeneous. For canonical massive (Proca) vector fields, the equations of motion are [7] $\nabla_\mu F^{\mu\nu} + m^2 A^\nu = 0$. Under the spatial homogeneity assumption, the temporal component of these relation yields $A_0 = 0$ [1, 7]. And this is generically true of many more complicated vector models too. At first glance, we have a slow-roll regime for the spatial vector field components due to remaining equations of motion, $\ddot{A}_i + 3H\dot{A}_i + m^2 A_i = 0$, in the FRW spacetime with metric element

$$\text{d}s^2 = \text{d}t^2 - a^2(t) \text{d}^3x^2,$$

where $a$ is the scale factor and $H \equiv \frac{\dot{a}}{a}$ is the Hubble constant. However, it is not what we need because the potential energy, $-\frac{1}{2}A_\mu A^\mu$, decays a bit faster than $a^{-2}(t)$. One can introduce a somewhat more physical variable $B_i \equiv \frac{A_i}{a}$. Unfortunately, it gets a large effective mass:

$$\ddot{B}_i + 3H\dot{B}_i + (m^2 + 2H^2 + \dot{H})B_i = 0,$$

which makes the Hubble friction ineffective and a slow-roll regime impossible. It is really bad news for vector inflation; however, this issue has also received much attention due to a more mundane reason. In particular, we mean the problem of the seeds of primordial magnetic fields in the Universe. Magnetic fields are known to exist in galaxies and in clusters of galaxies, and at the same time it is very hard to imagine how any considerable magnetic seeds could survive after inflation for use in, for example, subsequent dynamo mechanisms. We will not discuss this topic any further, and refer the interested reader to the existing literature [26–28].
In the pioneering paper [1], we can find two possible realizations of the slow-roll dynamics in vector inflation. The first idea is that one can introduce a tachyonic vector field mass in such a way that it will almost compensate the unwanted geometrical contribution, $2H^2 + \dot{H}$. A possible way to introduce such an effective mass without too much evident fine-tuning is to invoke a particular non-minimal coupling term, $\frac{R_{12}}{2} A_\mu A^\mu$, in the Lagrangian. (Recall that in the FRW spacetime, we have precisely $\frac{R}{6} = -2H^2 - \dot{H}$.) In any case, for a test vector field it is just a tachyonic mass term, and one can make a Størkelsh decomposition of the vector field, $A_\mu = \tilde{A}_\mu + \partial_\mu \lambda$, where $\partial_\mu \tilde{A}^\mu \equiv 0$, in order to check that the longitudinal component becomes a ghost [12]. One could object [29] against a change of variables with time derivatives. For if we make a change of variables $x(t) \equiv \dot{y}(t)$ for a free non-relativistic particle, we obtain an equation of motion $\frac{d^4}{dt^4} y = \frac{d^3}{dt^3} x = 0$ which clearly has more solutions than the initial theory admits. However, in the latter example, it is crucial that we have restricted the class of variations for our Lagrangian. Not only $\delta x \equiv \delta \dot{y}$ but also $\delta y$ should be equal to zero at the boundary of the time interval. We make only such variations of the function $x(t)$ which integrate to zero over the given time interval, $\int x(t) dt = 0$. Under this restriction, any paths with $\ddot{x} = \text{const}$ bring the action to its extremal value just as well as all the standard solutions with zero constant would do. Note that nothing like this may happen in the Størkelsh trick where the class of variations remains intact. Therefore, this type of critique is unsubstantiated; and indeed, a careful analysis [13, 30] shows that the longitudinal mode is a ghost for short wavelengths.

The relevance of the ghost problem for models with stable explicit potential and non-minimal coupling is arguable because at small scales, we are not really to trust the approximation in which the scalar curvature is just a fixed (slightly time dependent) quantity and acts as a contribution to the effective mass-squared. However, this observation does not give a recipe for how to treat the quantum production of the vector field fluctuations at sub-horizon scales. Recall that the ghost appears just right below the horizon length scale, at momentum-squared $\approx 2H^2$. And the fluctuations of the metric should produce regions with both signs of the scalar curvature. It was claimed in [30, 31] that the theory is under control, partly because the total energy in the cosmological background with small occupation numbers of quantum fluctuation modes is positive. However, the ghosts are extremely dangerous because they are produced with divergent cross-section in Lorentz-invariant theories, and it is unclear whether a deep sub-horizon UV-cutoff could be helpful in achieving a viable cosmological evolution. Nevertheless, for super-horizon evolution the $\delta N$ formalism was properly generalized and applied to vector inflationary models [30]. And ignoring the short-scale problems, even non-Gaussianity from vector field perturbations can be calculated [32].

The full quantum problem of longitudinal modes still requires a thorough investigation. It is remarkable though that the classical evolution of vector fields is smooth, at least in the test field approximation [9, 14]. Unfortunately, once the gravitational backreaction is taken into account, numerical integration of the full set of equations of motion for linear perturbations leads to divergences [14]. This effect is a bit counter-intuitive, see for example discussion in [9], and a good analytical understanding is needed. However, taken as is, it throws a big shadow on any attempts to refer to the short-scale dynamics of the scalar curvature as a means of resolving the ghost problem because it is presumably the metric fluctuations which have destabilized the longitudinal vector field dynamics in [14]. One can argue that we just have to understand nonlinear dynamics of the theory [31], and that can well be the case but then for now, we have no tractable model even at the level of classical dynamics. And even if we resolve this issue, we may still want to obtain the normal sign of the mass term after inflation which requires a transition through a highly singular point [9, 14] of $m_{\text{eff}}^2 = 0$. 

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Let us now turn to the second model of [1]. It makes use of a very flat potential with the right sign of $V'$ where the derivative is taken with respect to $A_\mu A^\mu = A_0^2 - a^{-2} \vec{A}^2$. In the $B$-variables, $B_0 = A_0$, $\vec{B} = a^{-1} \vec{A}$, we have a function of $B^2 \equiv B_0^2 - \vec{B}^2$. The argument rolls very fast but the function changes only slightly.

One possible example of such a potential is $V = -\frac{B^2}{(B^2)^{2/3}}$ for small values of $\epsilon$. This is non-analytic but we do not worry about the neighborhood of zero and work only with negative values of $B^2$. In this case, $V' < 0$, and therefore there is no ghost. Let us however choose a background solution as follows: $B_\mu = \delta_\mu^0 B(t)$, and expand the potential in terms of the longitudinal vector field fluctuation $\delta B_\mu = \partial_\mu \lambda$ up to the second-order terms. We have in Lagrangian,

$$-V = \frac{B^2 - 2B \partial_1 \lambda + (\partial_\mu \lambda)(\partial^\mu \lambda)}{(B^2 + 2B \partial_1 \lambda - (\partial_\mu \lambda)(\partial^\mu \lambda))^{1+\epsilon}}$$

$$= B^{2\epsilon} \left( -1 - 2\epsilon \frac{\partial_1 \lambda}{B} + \epsilon \frac{(\partial_\mu \lambda)(\partial^\mu \lambda)}{B^2} + (2\epsilon - 2\epsilon^2) \frac{(\partial_1 \lambda)^2}{B^2} \right).$$

Although the coefficients are strongly time dependent, we see that for small values of $\epsilon$, the longitudinal mode exhibits a gradient instability. Only the models with $\epsilon \geq \frac{1}{2}$ are free of this problem but they have potential energy which decays faster than $\frac{1}{B^2}$.

A working example of a very flat potential is $V = C(e^{e^{B}} - 1)$. It has $V' < 0$ for $B^2 < 0$, and in principle can be smoothed around $B^2 = 0$ and continued to $B^2 > 0$ in a healthy way by changing the sign in front of the square root. In this model, the gradient instability does not occur because after Taylor expanding

$$-V = C(1 - e^{B}) + C e^{B} \left( -\kappa \partial_1 \lambda + \frac{\kappa}{2B} (\partial_\mu \lambda)(\partial^\mu \lambda) - \left( \frac{\kappa^2}{2} - \frac{\kappa}{2B} \right) (\partial_1 \lambda)^2 \right) + O(\lambda^3),$$

we always have the correct sign in front of $(\partial_1 \lambda)^2$.

In general, we obtain $-\delta V = -V'(2B \partial_1 \lambda + (\partial_\mu \lambda)(\partial^\mu \lambda)) - 2V''B^2(\partial_1 \lambda)^2 + O(\lambda^3)$. Therefore, we see that for stability we need $V' < 0$ and an additional inequality

$$V'' \geq \frac{V'}{2B^2}$$

(1)

to be satisfied. In particular, one can check that the potential considered by Ford [1], $V = C(1 - e^{B^2})$, is stable with respect to the gradient instability whenever $B^2 \geq \frac{1}{2\epsilon}$. We see that the best we could do without introducing the ghosts is to choose a potential insensitive to the fast roll of the $B$-field, but not to preserve $B_t$ from decaying. However, we should point out that in constructing the most general vector field models, the problem is not to make $A_t$ growing or $B_t$ stable. It can be achieved by a simple field redefinition. The real problem is to have a slowly changing potential energy for a more natural model than that with such an extremely flat potential. The latter task will be discussed later, and now we would like to describe a simple vector field redefinition which gives the growth of the field variable but makes the Hamiltonian analysis significantly harder without changing the physical content of the theory. The results are not surprising of course but we find this calculation instructive and useful.

3. A change of variables

We start with a very simple action

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right)$$

(2)
for a (Proca) vector field in Minkowski space with the field strength \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) and the metric signature \(+, -, -, -\). Let us perform a change of variables such that
\[
A_\mu = f(\mathcal{A}^2) \cdot \mathcal{A}_\mu, \quad (3)
\]
where \( f \) is a function of the scalar argument \( \mathcal{A}^2 \equiv \mathcal{A}_\mu \mathcal{A}^\mu \). This is just a non-derivative one-to-one change of variables whenever \( f \neq 0 \) and \( f + 2f\mathcal{A}^2 \neq 0 \), and we expect of course that the physical Hamiltonian in terms of the new variables can be obtained just by the change of variables (3) in the physical Hamiltonian for action (2). However, a straightforward Hamiltonian analysis turns out to be remarkably complicated for this very simple theory in unusual variables.

We outline the main steps of the canonical analysis for the action
\[
S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} (\partial_\mu (f(\mathcal{A}^2) \mathcal{A}_\nu) - \partial_\nu (f(\mathcal{A}^2) \mathcal{A}_\mu)) (\partial^\mu (f(\mathcal{A}^2) \mathcal{A}^\nu) + \partial^\nu (f(\mathcal{A}^2) \mathcal{A}_\mu)) + \frac{m^2}{2} f^2 (\mathcal{A}^2)^2 \mathcal{A}_\mu \mathcal{A}^\mu \right), \quad (4)
\]
of the standard massive vector field in the new variables. We note first that there is a simple relation for the velocities:
\[
A_\mu = f\mathcal{A}_\mu + 2f'\mathcal{A}_\nu (\mathcal{A}_\mu \mathcal{A}_\nu - \mathcal{A}_\nu \mathcal{A}_\mu),
\]
where, for the sake of brevity, we have omitted the argument \( \mathcal{A}^2 \) of the function \( f \) and its derivative. It gives for canonical momenta \( P^{\mu} \equiv \frac{\partial L}{\partial \dot{A}^\mu} = \frac{\partial L}{\partial \dot{A}_\mu} = P^{\mu}_{\nu} \frac{\partial A^\nu}{\partial A_\mu} \) the following expressions:
\[
P^0 = 2f'\mathcal{A}_0 (\dot{A}_k - \dot{\mathcal{A}}_k); \quad P^i = (f\delta_{ik} - 2f'\mathcal{A}_k)(\dot{A}_k - \dot{\mathcal{A}}_k). \quad (5)
\]

We easily observe that \( \mathcal{A}_i P^i = (f - 2f'\mathcal{A}_k^2) \mathcal{A}_i (\dot{A}_i - \dot{\mathcal{A}}_i) \), and deduce the primary constraint,
\[
P^0 = -\frac{2f'\mathcal{A}_0}{f - 2f'\mathcal{A}_k^2} \mathcal{A}_i P^i = 0. \quad (6)
\]
Of course, in the old variables it can be expressed just as \( P^0 = 0 \). To check this fact, we have to invert the Jacobian matrix \( \frac{\partial \mathcal{A}_i}{\partial A_\mu} = f\delta^\mu_{\nu} + 2f'\mathcal{A}_\nu \mathcal{A}^\mu \). The answer is
\[
\frac{\partial \mathcal{A}_i}{\partial A_\mu} = \frac{1}{f} \left( \delta^\mu_{\nu} - \frac{2f'\mathcal{A}_\nu \mathcal{A}^\mu}{f + 2f'\mathcal{A}^2} \right). \quad (7)
\]
And we readily see that
\[
P^0 = \frac{\partial \mathcal{A}_0}{\partial A_0} P^0 = \frac{1}{f (f + 2f'\mathcal{A}^2)} (P^0 (f - 2f'\mathcal{A}_k^2) - 2f'\mathcal{A}_0 \mathcal{A}_i P^i)
\]
is proportional to constraint (6).

Let us use the primary constraint (6) in the Hamiltonian \( H = \mathcal{P}^i \mathcal{A}_i - \mathcal{L} \) to obtain
\[
H = \mathcal{P}^i \left( \mathcal{A}_i + \frac{2f'\mathcal{A}_0 \mathcal{A}_i}{f - 2f'\mathcal{A}_k^2} \right) - \frac{1}{2} (\dot{A}_i - \dot{\mathcal{A}}_i)^2 + \frac{1}{4} (\partial_i (f\mathcal{A}_k) - \partial_k (f\mathcal{A}_i))^2 - \frac{1}{2} m^2 f^2 \mathcal{A}^2.
\]
We need to get rid of velocities. Using the definition of momenta, we have
\[
(\mathcal{P}^i)^2 = f^2 (\dot{A}_i - \dot{\mathcal{A}}_i)^2 + 4 f^2 \mathcal{A}_k^2 (f - f') (\dot{\mathcal{A}}_i (\dot{A}_i - \dot{\mathcal{A}}_i))^2
\]
and then
\[
\frac{1}{2} (\dot{A}_i - \dot{\mathcal{A}}_i)^2 = \frac{1}{2 f^2} (\mathcal{P}^i)^2 - 2 \frac{f^2 \mathcal{A}_k^2 - f f'}{f^2 (f - 2f'\mathcal{A}_k^2)^2} [\mathcal{A}_i \mathcal{P}^i] = \frac{1}{2 f^2} \left( (\delta_{ik} + \frac{2f'\mathcal{A}_k \mathcal{A}^i}{f - 2f'\mathcal{A}^2}) \mathcal{P}^k \right)^2.
\]
With some simple algebra, we also transform the first term in the Hamiltonian
\[ \mathcal{H} \left( \partial_t + \frac{2 f' \mathcal{A}_0 \mathcal{A}_0}{f - 2 f' \mathcal{A}_m^2} \right) = \mathcal{H} \left( \partial_t + \frac{2 f' \mathcal{A}_0 \mathcal{A}_k \mathcal{A}_k}{f - 2 f' \mathcal{A}_m^2} \right) \]
\[ = \mathcal{H} \left( \frac{2 f' \mathcal{A}_0 \mathcal{A}_k (\mathcal{A}_k - \partial_t \mathcal{A}_0)}{f - 2 f' \mathcal{A}_m^2} \right) + \mathcal{H} \left( \frac{2 f' \mathcal{A}_0 \partial_t \mathcal{A}_0}{f - 2 f' \mathcal{A}_m^2} \right), \]
of which the first part yields the expression we already know (we use (5)):
\[ \mathcal{H} \left( \partial_t + \frac{2 f' \mathcal{A}_0 \mathcal{A}_k (\mathcal{A}_k - \partial_t \mathcal{A}_0)}{f - 2 f' \mathcal{A}_m^2} \right) \]
\[ = \left( \partial_t - \partial_t \mathcal{A}_0 \right) - \frac{2 f'}{f - 2 f' \mathcal{A}_m^2} \mathcal{A}_0 \mathcal{A}_k (\mathcal{A}_k - \partial_t \mathcal{A}_0) \]
\[ = (\mathcal{A}_k - \partial_t \mathcal{A}_0)^2. \]
This is twice the contribution from the Lagrangian density and has the opposite sign, precisely as it should be for the quadratic in momenta part of the Hamiltonian given that the Lagrangian was quadratic in velocities. Finally, we combine everything together and obtain the Hamiltonian density:
\[ H = \frac{1}{2 f^2} \left( \frac{\delta_{ik} + \frac{2 f' \mathcal{A}_0 \mathcal{A}_k}{f - 2 f' \mathcal{A}_m^2}}{2} \right) \mathcal{H}^2 + \mathcal{H} \left( \frac{\partial_t (f' \mathcal{A}_0)}{f - 2 f' \mathcal{A}_m^2} \right) \]
\[ + \frac{1}{4} \left( \partial_t (f' \mathcal{A}_0) - \partial_t (f' \mathcal{A}_0) \right)^2 + \frac{1}{2} 2m^2 f' \mathcal{A}_m^2 - \frac{1}{2} 2m^2 f' \mathcal{A}_m^2. \]
This is actually the usual Hamiltonian \( H = \frac{1}{2} (P_\nu)^2 - A_0 \partial_t P_\nu + \frac{1}{2} F_{ik}^2 + \frac{1}{2} m^2 (A_i^2 - A_0^2) \) in the new variables. Indeed, the first terms in the Hamiltonians do coincide because
\[ P^\mu = \frac{\partial \mathcal{A}_\nu}{\partial A^\mu} \mathcal{H}^\nu = \frac{1}{f} \left( \delta_{ik} + \frac{2 f' \mathcal{A}_0 \mathcal{A}_k}{f - 2 f' \mathcal{A}_m^2} \right) \mathcal{H}^k, \]
where we have used equation (7) and the primary constraint (6). And now we also see that after integration by parts, the second term in (8) becomes
\[ -f' \mathcal{A}_0 \partial_t P_\nu = -A_0 \partial_t P_\nu. \]
The next step is to find the secondary constraint. We know that after canonical transformation, the Poisson brackets should not have changed, and therefore the secondary constraint must acquire a form equivalent to \( \partial_t P_\nu + m^2 A_0 = 0 \). But it is not an easy task to establish this result with a straightforward computation of the Poisson bracket of Hamiltonian (8) with the primary constraint (6). We can however explicitly check that for \( P^\nu = \frac{1}{f - 2 f' \mathcal{A}_m^2} \mathcal{H}^\nu \) and \( A_\mu = f (\mathcal{A}^k) \cdot \mathcal{A}_\mu \), the Poisson brackets are \( [P^\nu, A_\mu] = \frac{\delta^\nu_{\mu}}{\partial \mathcal{A}_\nu} \cdot \frac{\partial \mathcal{A}_\mu}{\partial A^\nu} = \delta^\nu_{\mu} \). To calculate the \( \{H, P^\nu\} \) quantity, we also need the \( \{P^\mu, P^\nu\} \) bracket which can be computed using the obvious relation \( P^\mu = \frac{\partial \mathcal{A}_\nu}{\partial A^\mu} \mathcal{H}^\nu \) together with formula (7). It is quite a bulky endeavor which can be simplified by observing that the right-hand side of (7) contains two scalar functions of \( \mathcal{A}_i^2 \): \( h = \frac{1}{f} \) and \( g = \frac{1}{f - 2 f' \mathcal{A}_m^2} \). The Poisson bracket appears to be \( \{P^\mu, P^\nu\} = (2h' + h g - 2 g g') (\mathcal{A}_0 \mathcal{A}_\nu - \mathcal{A}_i \mathcal{A}_i^\nu) = 0 \). And hence we have
\[ \{H, P^\nu\} = \partial_t P^\nu = m^2 A_0 = 0 \] which gives the secondary constraint
\[ \partial_t \left( \delta_{ik} + \frac{2 f' \mathcal{A}_0 \mathcal{A}_k}{f - 2 f' \mathcal{A}_m^2} \right) \mathcal{H}^k + m^2 f' \mathcal{A}_0 = 0. \]
We have a pair of second-class constraints, and the canonical analysis stops. It is however not possible to exclude the unphysical variable \( \mathfrak{A}_0 \) from the Hamiltonian explicitly due to a very complicated form of equation (9). Nevertheless, it is obvious that the Hamiltonian density is positive definite and equals

\[
\mathcal{H} = \frac{1}{2} (p')^2 + \frac{1}{4} F_{ik}^2 + \frac{1}{2} m^2 (A_i^2 + A_0^2) = \frac{1}{2} (p')^2 + \frac{1}{2m^2} (\partial_i p')^2 + \frac{1}{4} F_{ik}^2 + \frac{1}{2} m^2 A_i^2.
\]

Note also that the apparent singularity at \( f = 2\sqrt{\mathfrak{A}} \) is unphysical and only reflects the fact that the momentum \( \mathfrak{P}^0 \) cannot always be assumed unphysical. Near this locus we should have excluded \( \mathfrak{A}, \mathfrak{P}^0 \) rather than \( \mathfrak{P}^0 \).

Of course, we could perform the same analysis with a general potential. And, clearly, we could search for such a potential which will acquire the form of a mass term in the new variables. In this case, due to the growth of the new vector field variables, the potential energy may become slowly rolling. However, it will require a tachyonic potential in the initial model. And now, separating the longitudinal modes, we will make a St"uckelberg decomposition for the \( f(\mathfrak{A})^2 \mathfrak{A}_i \) field which would of course reveal the ghosty kinetic energy again. In other words, what we have seen in this section is not a modification of the vector field theory. And we have to test something different.

### 4. Lorentz-invariant modifications

Probably, the first idea after discussion in the previous section is to take a combination of two kinetic terms,

\[
\mathcal{L} = -\frac{1}{4} (c F_{\mu\nu} F^{\mu\nu} + \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}) - V
\]

where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \tilde{F}_{\mu\nu} \equiv \partial_\mu (f(A^2)A_\nu) - \partial_\nu (f(A^2)A_\mu) \). However, in this case we have the same momentum \( p^0 \) as in equation (5) and the spatial momenta \( p' \) are shifted by \( c F_{0i} \); therefore, we obtain the following relation instead of the primary constraint (6):

\[
2f A_0 A_i p' - (f - 2f A_0^2) p^0 = 2f^2 c A_{0i} (\dot{A}_i - \partial_i A_0).
\]

If \( A_0 \neq 0 \), it allows us to solve for \( A_i \dot{A}_i \) in terms of fields and momenta. The temporal momentum contains the time derivative of \( f(\mathfrak{A})^2 \mathfrak{A}_i \) (see (5) and recall that the Latin letters are used to denote the variables which correspond to the Gothic fields of the previous section), and therefore now we can determine \( \dot{A}_0 \). Finally, we have enough equations from the definition of \( p' \)'s to solve for the two remaining velocities. Therefore, a model with different kinetic terms has an ill-defined number of degrees of freedom which equals 4 almost everywhere. We would however prefer a vector field with strictly three degrees of freedom.

The next possibility is to have a kinetic self-coupling, \( f(A^2)F^2 \). Let us take a mass-term potential for simplicity

\[
\mathcal{L} = -\frac{1}{4} f(A^2) F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2
\]

and perform the Hamiltonian analysis. The canonical momenta are simply given by \( p' = f F_{0i} \) and \( p^0 = 0 \). The Hamiltonian density

\[
\mathcal{H} = \frac{(p')^2}{2f} - A_0 \partial_i p' + \frac{1}{4} F_{ik} F_{ik} - \frac{1}{2} m^2 A^2
\]

leads to the secondary constraint \(-\partial_i p' = m^2 A_0 + \frac{\omega^2}{f^2} - A_0 - \frac{1}{2} f A_0 F_{ik} F_{ik} \) which cannot be explicitly solved but allows us to write the Hamiltonian in the following form:

\[
\mathcal{H} = \frac{(p')^2}{2f} \left( 1 + \frac{2f}{f} \mathfrak{A}_0^2 \right) + \frac{1}{4} F_{ik} F_{ik} \left( 1 - 2 \frac{f}{f} \mathfrak{A}_0^2 \right) + \frac{1}{2} m^2 (\mathfrak{A}_0^2 + \mathfrak{A}_i^2),
\]
which is bounded neither from below nor from above for any non-constant function \( f \). This is a short-wavelength problem, and therefore can be dangerous. (A little thought shows that this result is generic also for nonlinear non-trivial functions of two arguments, \( A^2 \) and \( F^2 \).) And let us look at the equations of motion

\[
\nabla_\mu (f F^{\mu \nu}) - \frac{1}{2} f F^{2} A^\nu + m^2 A^\nu = 0.
\]

If we are searching for a slow-roll solution with negligible time dependence of the function \( f \), then we effectively have a massive vector field of mass \( m^2 = \frac{\mathcal{F}}{F} - \frac{\mathcal{F}^2}{2 F^2} \) with \( F^2 \approx H^2 B^2 \). As \( F^2 < 0 \) and \( f > 0 \), we must have \( f' < 0 \) and \( |f' F^2| > 2m^2 \) for the mass to be negative (let alone being close to \(-2F^2\)). Having obtained these inequalities, we check the quadratic term in the longitudinal mode action \( \mathcal{L} = \left( -\frac{1}{2} f F^2 + \frac{1}{2} m^2 \right) (\mathcal{D}^\mu \lambda) (\mathcal{D}_\mu \lambda) \), and find a ghost again.

We could also try to modify the \( F_{\mu \nu} F^{\mu \nu} \) structure of the kinetic function. However, it is easy to see that there are only two quadratic ghost-free possibilities: the standard one and \( (\mathcal{D}_\mu A^\mu)^2 \) which propagates only one degree of freedom. Relaxing the condition of being quadratic, we obtain one more possible structure, \( G_{\mu \nu} = A^\alpha (\partial_\mu A_\alpha - A_\mu \partial_\alpha A_\nu) \), which is naturally produced in a particular combination with \( F_{\mu \nu} \) by the change of variables (3). Let us now take the most general kinetic part of Lagrangian quadratic in \( F \) and \( G \):

\[
\mathcal{L} = -\frac{1}{2} (f (A^2) F_{\mu \nu} F^{\mu \nu} + 2g (A^2) F_{\mu \nu} G^{\mu \nu} + h (A^2) G_{\mu \nu} G^{\mu \nu}).
\]

The canonical momenta are as follows:

\[
p^0 = A_0 A_\alpha (g F_{\alpha \kappa} + h G_{\alpha \kappa}); \quad p^\mu = (f F_\mu + g G_\mu) - A_\mu A_\kappa (g F_{\kappa \alpha} + h G_{\kappa \alpha}). \quad (10)
\]

So that we easily find a simple relation

\[
p^\mu = f F_\mu + g G_\mu - \frac{A_\mu}{A_0} p^0
\]

and see that if

\[
f \frac{g}{A_0} = \frac{A_1}{A_0}
\]

then it can be written entirely in terms of momenta without velocities, and therefore in this case we find a primary constraint. Otherwise we can combine relation (11) with the definition of the temporal momentum in (10) and determine both \( A_1 F_{\alpha \kappa} \) and \( A_1 G_{\alpha \kappa} \) in terms of momenta if \( A_0 \neq 0 \). It gives us \( A_1 A_\lambda \) and \( A_0 \). And remaining two independent equations in (10) allow us to find the two remaining velocities. Hence, equation (12) is the necessary and sufficient condition to have a vector field with three degrees of freedom. However, in this case our Lagrangian is simply \( \mathcal{L} = -\frac{1}{2} (f F + h G)^2 \) and it can always be represented as a change of variables (3) in a Lagrangian of \(-\frac{1}{2} F^2 \) type (probably, with constant \( f \)) which we have studied above. Indeed, we want to convert \( f F + h G \) into \( f \hat{F} + h \hat{G} \) for what it suffices to put \( \hat{h} = \exp \int \frac{h}{F} \) and \( \hat{f} = \frac{f}{F} \).

We also have a possibility of taking nonlinear functions of simple kinetic terms, for example, \( \mathcal{L} = -f (F^2) - V (A^2) \). An accelerating solution for \( f (F^2) = \frac{F^2}{4} - \frac{e}{F^2} \) with negative constant \( e \) was constructed in [33]. However, this model has an ill-defined number of degrees of freedom at \( f' = 0 \), and also it has a Hamiltonian unbounded from below. One can actually check that \( f' > 0 \) and \( V' < 0 \) are necessary conditions for the Hamiltonian to be bounded from below [15]. Unfortunately, those theories of this class which can give an interesting dynamics are necessarily unstable. If a vector field is to play any significant role in the cosmological expansion, then some terms in its Lagrangian should not be diluted. Unless either \( V \) or \( f \) is an extremely flat function (in the latter case, the transverse vector fluctuations would be strongly coupled), it means that either \( \frac{\Delta}{a} \) or \( \frac{\Delta^2}{a^2} \) have to roll slowly. In [15], it was shown that neither
of these options is available whenever \( f' > 0 \) and \( V' < 0 \). Introduction of an additional \( FF \) argument with dual field strength tensor \( \bar{F} \) to the function \( f \) does not change the cosmological dynamics [15], as the dual tensor has only spatial non-vanishing components. An extra \( \nabla_{\mu}A^{\mu} \) argument would generically lead to an extra (fourth) degree of freedom.

In principle, one could consider a nonlinear function of several arguments \( f(F^2, FG, G^2, A^2) \). However, in order for the momenta to satisfy a linear constraint equation we would need a condition analogous to (12) with \( \frac{\partial f}{\partial F^2}, \frac{\partial f}{\partial FG} \) and \( \frac{\partial f}{\partial G^2} \) instead of \( f', 2g \) and \( h \), respectively, and the ratios being independent of kinetic arguments. It follows then that a linear in \( F^2, FG \) and \( G^2 \) increment of the function \( f \) is always proportional to \( (c_1(A^2)f + c_2(A^2)G^2) \), and again we have got no new options.

To summarize, no new viable models of vector inflation have been found in this section. We cannot completely exclude a possibility that there may be some highly nonlinear models of very clever design in the class of Lorentz-invariant vector field theories with three propagating degrees of freedom which would be able to produce a vector inflationary regime. However, it is clear that generically this is not possible, except for vector fields with extremely flat potentials as was proposed in [1]. (We should stress again that, throughout the paper, only Abelian vector fields are being discussed. In [34, 35], an inflationary model is constructed with a special \( F^4 \)-correction to the Yang–Mills action. The model is explicitly gauge invariant and therefore may be free of some problems. However, a full Hamiltonian analysis was not yet performed. And negative values of sound speed squared for some fluctuation modes are reported [35]. It signals a gradient instability of the background solution.)

5. A few remarks on Lorentz-breaking models

It is of course the restrictions of Lorentz invariance which prevent us from constructing a suitable model. And it is also the Lorentz invariance which makes the ghosts so dangerous. The standard argument is as follows. Consider a graviton-mediated creation of a pair of normal particles and a pair of ghosts from nothing. Let us fix a reference frame. One possible kinematic of this decay is the one with compensated spatial momenta inside every pair (a pair of identical particles with precisely opposite momenta, and an analogous pair of ghosts), such that the negative energy of ghosts compensates the positive energy of particles. However, there are lots of other possibilities too. The vacuum decay can proceed with a kinematic which looks in the chosen frame precisely as the pair-momentum-compensated kinematics would have looked like in some other, Lorentz-boosted frame. And therefore, in calculating the amplitude we would have to integrate over all possible momenta and over all possible Lorentz frames. It gives a divergent result regardless of how small the coupling is. The kinematics of the vector ghost is somewhat different. And in the ultraviolet limit even a tiny portion of transverse excitation can compensate the negative longitudinal energy [9]. But nevertheless, at the very least, the infinite Lorentz-group volume is unavoidable. In Lorentz-violating models, the rate of the ghost production could in principle be controlled, but it remains to be understood whether an ultraviolet cutoff deep under the Hubble length scale could be helpful for cosmology.

A natural way to proceed with the Lorentz-breaking scenarios is to invoke Lorentz-breaking vector potentials which could come not only from theories with fundamentally preferred frames but also from couplings to an aether field or some non-trivial background; a small breaking can even occur due to \( R^{\mu\nu}A_\mu A_\nu \) coupling. A simple example is \( V(A^2) = -(m_1^2A_0^2 - m_2^2A_i^2) \) with different masses for temporal and longitudinal components. In [9], it was shown that at the classical level a major analytic problem of the tachyonic vector...
field comes from the temporal component of equations of motion which is normally used to determine the unphysical $A_0$ variable,

$$(- \triangle + m^2)A_0 + \partial_i \dot{A}_i = 0.$$  

With tachyonic mass, the spectrum of the operator in front of $A_0$ contains zero. We can in principle overcome this trouble if we take $m_1^2 > 0$ and $m_2^2 < 0$. (The Hamiltonian is then unbounded from below but only due to the tachyonic effect which can be cured by a nonlinear potential for spatial components.) This is not easy to do with coupling to the Ricci tensor because in the quasi-de-Sitter regime, it is almost proportional to the metric. One can check that in order to have $m_2^2$ of order $-2H^2$ and $m_1^2$ positive, we will need to use couplings to the Ricci scalar and Ricci tensor of order $O(\epsilon)$ where $\epsilon \equiv -\frac{\dot{H}}{H^2}$. This difference of masses may be better achieved with a coupling to an aether field. However, it would solve only the ghost problem, but the gradient instabilities would persist. It is evident from the Stöckelberg analysis, and can also be seen directly from the equations of motion (in Minkowski space):

$$\ddot{A}_i + m_2^2 A_i - \triangle A_i + \left(1 - \frac{m_2^2}{m_1^2}\right) \partial_i \partial_k A_k = 0.$$  

The gradient instability is dangerous because the fluctuation modes grow with an unbounded rate in the ultraviolet, and even the condition of bounded energy of the initial fluctuation does not help to control this process. If however we are dealing with a theory which contains higher spatial derivatives, like in Hořava gravity [36] for example, then the rate of fluctuation growth may become bounded if the higher derivative terms go with proper signs. This issue deserves a further investigation.

There is also a temptation to modify the kinetic term. It is a fairly simple task if there are no restrictions on the choice to be made. A model with no temporal components $L = \frac{1}{2}(\partial_\mu A_i)(\partial^\mu A_i) - \frac{1}{2}m^2 A_i^2$ would be just perfect, but hardly of any physical interest. Note however that there are less radical choices too. For example, a theory with $L = -f^2(\vec{A}^2) F^2 - V(A^2)$ obviously enjoys a Hamiltonian bounded from below if the potential is stable. Note also that a peculiar modification of the kinetic term can be deduced as a dimensional reduction of five-dimensional gravity with four-dimensional Lovelock–(Gauß–Bonnet) invariant, see [15]. It is an open task to explore the cosmological consequences of this model. But any further discussion of Lorentz-violating vector field theories is beyond the scope of this paper.

6. Conclusions

Vector inflation was invented as an alternative to the scalar models and received a considerable interest among cosmologists partly because it could give a good account of possible asymmetries in the CMB [37]. At the background level and at the super-horizon scales, the model is tractable, and allowed for $\delta N$ calculations [30, 32]. However, the full linear perturbation equations are cumbersome [29], and the generic models are known to be badly unstable [9, 12]. In this paper, we have found out that in the class of Lorentz-invariant vector field models with three degrees of freedom, the ghost problem cannot be resolved unless the potential is taken to be an extremely flat function as was proposed in probably the very first paper on the subject [1]. But there are many Lorentz-breaking possibilities which are still waiting for a careful investigation.
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