SD-Regular Transducer Expressions for Aperiodic Transformations

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Abstract

FO transductions, aperiodic deterministic two-way transducers, as well as aperiodic streaming string transducers are all equivalent models for first order definable functions. In this paper, we solve the long standing open problem of expressions capturing first order definable functions, thereby generalizing the seminal SF=AP (star free expressions = aperiodic languages) result of Schützenberger. Our result also generalizes a lesser known characterization by Schützenberger of aperiodic languages by SD-regular expressions (SD=AP). We show that every first order definable function over finite words captured by an aperiodic deterministic two-way transducer can be described with an SD-regular transducer expression (SDRTE). An SDRTE is a regular expression where Kleene stars are used in a restricted way: they can appear only on aperiodic languages which are prefix codes of bounded synchronization delay. SDRTEs are constructed from simple functions using the combiners unambiguous sum (deterministic choice), Hadamard product, and unambiguous versions of the Cauchy product and the -chained Kleene-star, where the star is restricted as mentioned. In order to construct an SDRTE associated with an aperiodic deterministic two-way transducer, (i) we concretize Schützenberger’s SD=AP result, by proving that aperiodic languages are captured by SD-regular expressions which are unambiguous and stabilising; (ii) by structural induction on the unambiguous, stabilising SD-regular expressions describing the domain of the transducer, we construct SDRTEs. Finally, we also look at various formalisms equivalent to SDRTEs which use the function composition, allowing to trade the -chained star for a 1-star.

1 Introduction

The seminal result of Kleene, which proves the equivalence of regular expressions and regular languages, is among the cornerstones of formal language theory. The Büchi, Elgot, Trakhtenbrot theorem which proved the equivalence of regular languages with MSO definable languages, and the equivalence of regular languages with the class of languages having a finite syntactic monoid, established the synergy between machines, logic and algebra. The fundamental correspondence between machines and logic at the language level has been generalized to transformations by Engelfreit and Hoogeboom [13], where regular transformations are defined by two way transducers (2DFTs) as well as by the MSO transductions of Courcelle [9]. A generalization of Kleene’s theorem to transformations can be found in [3, 4] and [11].

In [3], regular transformations were described using additive cost register automata
(ACRA) over finite words. ACRAs are a generalization of streaming string transducers (SSTs) [1] which make a single left to right pass over the input and use a finite set of variables over strings from the output alphabet. ACRAs compute partial functions from finite words over a finite input alphabet to a monoid \((\mathbb{D}, +, 0)\). The main contribution of [3] was to provide a set of combinators, analogous to the operators used in regular expressions, which help in forming combinator expressions computing the output of the ACRA over finite words. The result of [3] was generalized to infinite words in [11]. The proof technique in [11] is completely different from [3], and, being algebraic, allows a uniform treatment of the result for transductions over finite and infinite words. Subsequently, an alternative proof for the result of [3] appeared in [4].

The class of star-free languages form a strict subset of regular languages. In 1965, Schützenberger [21] proved his famous result that star-free languages (SF) and languages having an aperiodic syntactic monoid (or aperiodic languages AP) coincide over finite words (SF=AP). This equivalence gives an effective characterization of star-free languages, since one can decide if a syntactic monoid is aperiodic. This was followed by a result [18] of McNaughton and Papert proving the equivalence of star-free languages with counter-free automata as well as first order logic, thereby completing the machine-logic-algebra connection once again. The generalization of this result to transformations has been investigated in [8], [16], proving the equivalence of aperiodic two way transducers and FO transductions a la Courcelle for finite and infinite words. The counter part of regular expressions for aperiodic languages are star-free expressions, which are obtained by using the complementation operation instead of Kleene-star. The generalization of this result to transformations has been an open problem, as mentioned in [3], [4] and [11]. One of the main challenges in generalizing this result to transformations is the difficulty in finding an analogue for the complementation operation on sets in the setting of transformations.

**Our Contributions.** The following central problem remained open till now: Given an aperiodic 2DFT \(A\), does there exist a class of expressions over basic functions and regular combinators such that, one can effectively compute from \(A\), an expression \(E\) in this class, and conversely, such that \([A](w) = [E](w)\) for each \(w \in \text{dom}(A)\)? We solve this open problem, by providing a characterization by means of expressions for aperiodic two way transducers. In the following, we describe the main steps leading to the solution of the problem.

**Concretizing Schützenberger’s characterization.** In 1973, Schützenberger [22] presented a characterization of aperiodic languages in terms of rational expressions where the star operation is restricted to prefix codes with bounded synchronization delay and no complementation is used. This class of languages is denoted by SD, and this result is known as SD=AP. To circumvent the difficulty of using complementation in star-free expressions, we use this SD=AP characterization of aperiodic languages by SD-expressions. An SD-expression is a regular expression where the Kleene stars are restricted to appear only on prefix codes of bounded synchronization delay. Our first contribution is to concretize Schützenberger’s result to more specific SD-expressions. We show that aperiodic languages can be captured by unambiguous, stabilising, SD-expressions. The unambiguity of an expression refers to the unique way in which it can be parsed, while stabilising expressions is a new notion introduced in this paper. Our concretization, (Theorem 10) shows that, given a morphism \(\varphi\) from the free monoid \(\Sigma^*\) to a finite aperiodic monoid \(M\), for each \(s \in M\), \(\varphi^{-1}(s)\) can be expressed by an unambiguous, stabilising SD-expression. The two notions of unambiguity and stabilising help us to capture the runs of an aperiodic two way transducer. These two notions will be described in detail in Section 5.
The Combinators. Our second contribution is the definition of SD-regular transducer expressions (SDRTE). These are built from basic constant functions using combinators such as unambiguous sum, unambiguous Cauchy product, Hadamard product. In addition, we use $k$-chained Kleene star $[L, C]^{k\star}$ (and its reverse) when the parsing language $L$ is restricted to be aperiodic and a prefix code with bounded synchronisation delay. It should be noticed that, contrary to the case of regular transducer expressions (RTE) which define all regular functions, the 2-aperiodic Kleene star $[L, C]^{2\star}$ does not seem sufficient to define all aperiodic functions (see Section 4.7), as well as Figure 2, and $k$-chained Kleene stars for arbitrary large $k$ seem necessary to capture all aperiodic functions.

The semantics of an SDRTE $C$ is a partial function $[C] : \Sigma^* \rightarrow \Gamma^*$ with domain denoted $\text{dom}(C)$. An SDRTE of the form $L \triangleright v$ where $L \subseteq \Sigma^*$ is an aperiodic language and $v \in \Gamma^*$ is such that $[L \triangleright v]$ is a constant function with value $v$ and domain $L$. The Hadamard product $C_1 \odot C_2$ when applied to $w \in \text{dom}(C_1) \cap \text{dom}(C_2)$ produces $[C_1](w) \cdot [C_2](w)$. The unambiguous Cauchy product $C_1 \circ C_2$ when applied on $w \in \Sigma^*$ produces $[C_1](w) \ast [C_2](w)$ if $w$ can be unambiguously decomposed as $u \ast v$, with $u \in \text{dom}(C_1)$ and $v \in \text{dom}(C_2)$. The Kleene star $C^\star$ is defined when $L = \text{dom}(C)$ is an aperiodic language which is a prefix code with bounded synchronisation delay. Then $\text{dom}(C^\star) = L^\star$, and, for $w = u_1u_2 \cdots u_n$ with $u_i \in L$, we have $[C^\star](w) = [C](u_1)[C](u_2) \cdots [C](u_n)$.

As an example, consider the SDRTEs $C = C_1 \cdot C_2$, $C' = C'_1 \cdot C'_2$ and $D = C \odot C'$ with $C_1 = (a\#) \triangleright \varepsilon$, $C_2 = (a \triangleright b)^\star \cdot (\# \triangleright \varepsilon)$, and $C'_1 = (a \triangleright a)^\star \cdot (\# \triangleright \varepsilon)$, $C'_2 = (a^4\# \triangleright \varepsilon) \triangleright \varepsilon$. Then $\text{dom}(C_1) = a^\star \# = \text{dom}(C'_1)$, $\text{dom}(C_2) = a^4\# = \text{dom}(C'_2)$, and $\text{dom}(C) = a^\star \# a^4\# = \text{dom}(C') = \text{dom}(D)$. Further, $[C_1](a^m \#) = \varepsilon$, $[C_2](a^4\#) = b^m$, $[C'_1](a^m \#) = a^m$, and $[C'_2](a^4\#) = \varepsilon$. Also, $[D](a^m \# a^4\#) = b^m a^m$. Notice that $\text{dom}(D)$ is a prefix code with synchronisation delay 1. Hence, we can define the SDRTE $D^\star$ which has domain the aperiodic language $\text{dom}(D^\star) = (a^\star \# a^4\#)^\star$, and $[D^\star](a^m \# a^4\# a^5\#) = b^m a^m b^5 a^4 \cdot$ The SDRTE $D' = (\# \triangleright \varepsilon) \cdot D^\star$ corresponds to the aperiodic 2DFT $A$ in Figure 1. $[A] = [D']$.

SDRTE $\leftrightarrow$ Aperiodic 2DFT. Our third main contribution solves the open problem by proving the effective equivalence between aperiodic two way transducers and SDRTEs over finite words:

- **Theorem 1.** (1) Given an SDRTE, we can effectively construct an equivalent aperiodic 2DFT. (2) Given an aperiodic 2DFT, we can effectively construct an equivalent SDRTE.

The proof of (1) is by structural induction on the SDRTE. All cases except the $k$-chained Kleene star are reasonably simple, and it is easy to see how to construct the equivalent 2DFT. The case of the $k$-chained Kleene star is more involved. We write $[L, C]^{k\star}$ as the
composition of 3 aperiodic functions $f_1, f_2, f_3$, where, (i) $f_1$ takes as input $u_1 u_2 \cdots u_n \in L^*$ with $u_i \in L$ and produces as output $\# u_1 \# u_2 \# \cdots \# u_n \#$, (ii) $f_2$ takes $\# v_1 \# v_2 \# \cdots \# v_m \#$ as input, and produces $\# v_1 \# v_2 \# v_3 \cdots v_{k+1} \# \cdots \# v_{m-k+1} \# \cdots \# v_m \#$ as output, (iii) finally, $f_3$ takes $\# w_1 \# w_2 \# \cdots \# w_l \#$ as input with $w_i \in \Sigma^*$ and produces as output $f(w_1) f(w_2) \cdots f(w_l)$. We produce aperiodic 2DFTs for $f_1, f_2, f_3$, and compose them, obtaining the required aperiodic 2DFT.

The construction of SDRTE from an aperiodic 2DFT $A$ is much more involved, and is based on the transition monoid $TrM$ of the 2DFT $A$. The translation of $A$ to SDRTE is guided by an unambiguous, stabilising, SD-regular expression induced by $TrM$. These expressions are obtained thanks to Theorem 10 applied to the canonical morphism $\varphi : \Sigma \to TrM$ where the transition monoid $TrM$ of $A$ is aperiodic. This construction is illustrated in detail via Examples 23, 26, 27 and 29.

Related Work. A natural operation on functions is that of composition. The composition operation can be used in place of the chained-sum operator of [3], and also in place of the unambiguous 2-chained iteration of [11], preserving expressiveness. In yet another recent paper, [17] proposes simple functions like copy, duplicate and reverse along with function composition to capture regular word transductions.

A closely related paper to our work is [6], where first-order and regular list functions were introduced. Using the basic functions reverse, append, co-append, map, block on lists, and combining them with the function combinator of disjoint union, map, pairing and composition, these were shown to be equivalent (after a suitable encoding) to FO transductions a la Courcelle (extendible to MSO transductions by adding to the basic functions, the prefix multiplication operation on groups). [6] provides an equivalent characterization (modulo an encoding) for FO transductions with basic list functions and combinators.

Contrary to [6] where expressions crucially rely on function composition, we focus on concatenation and iteration as first class combinators, in the spirit of Kleene’s theorem and of Schützenberger’s characterisation $AP=SD$. We are able to characterise 2DFTs with such SD-regular expressions without using composition. Hence, our result is fully independent and complementary to the work in [6]: both formalisms, SDRTEs and list functions are natural choices for describing first order transductions. Our basic functions and combinators are inspired from the back and forth traversal of a two way automaton, and the restrictions on the usage of the Kleene star comes from the unambiguous, stabilising nature of the expressions capturing the aperiodic domain of the 2DFT. We also study in Section 6 how composition may be used to simplify our SDRTEs (Theorem 30). With composition, $k$-chained Kleene star ($k > 1$) is no more necessary, resulting in an equivalent formalism, namely, SDRTE where we only use 1-star. Yet another equivalent formalism is obtained by restricting SDRTE to simple functions, unambiguous sum, Cauchy product and 1-star, but adding the functions duplicate and reverse along with composition.

Structure of the paper. In Section 2 we introduce preliminary notions used throughout the paper. In Section 3 we give a procedure to construct complement-free expressions for aperiodic languages that suits our approach. This is a generic result on languages, independent of two-way transducers. Section 4 presents the combinators and the chain-star operators for our characterization. The main theorem and technical proofs, which is constructing SD-regular transducer expressions from a two-way aperiodic transducer, are in Section 5.
2 Preliminaries

2.1 Rational languages and monoids

We call a finite set \( \Sigma \) an alphabet and its elements letters. A finite sequence of letters of \( \Sigma \) is called a word, and a set of words is a language. The empty word is denoted \( \varepsilon \), and we denote by \( \Sigma^* \) the set of all words over the alphabet \( \Sigma \). More generally, given any language \( L \subseteq \Sigma^* \), we write \( L^* \) for the Kleene star of \( L \), i.e., the set of words which can be written as a (possibly empty) sequence of words of \( L \). Given a word \( u \), we write \(|u|\) for the length of \( u \), i.e., its number of letters, and we denote by \( u_i \) its \( i \)th letter.

A monoid \( M \) is a set equipped with a binary associative law, usually denoted \( \cdot \) or omitted when clear from context, and a neutral element \( 1_M \) for this law, meaning that for any \( s \in M \), \( 1_M \cdot s = s \cdot 1_M = s \). The set of words \( \Sigma^* \) can be seen as the free monoid generated by \( \Sigma \) using the concatenation of words as binary law. Given a morphism \( \varphi : \Sigma^* \to M \), i.e., a function between monoids that satisfies \( \varphi(\varepsilon) = 1_M \) and \( \varphi(xy) = \varphi(x)\varphi(y) \) for any \( x, y \), we say that \( \varphi \) recognizes a language \( L \subseteq \Sigma^* \) if \( M \) is finite and \( L = \varphi^{-1}(P) \) for some \( P \subseteq M \). A monoid is called aperiodic when there exists an integer \( n \) such that for any element \( s \) of \( M \), \( s^n = s^{n+1} \).

Example 2. We define the monoids \( \tilde{U}_n \) for \( n \geq 0 \), as the set of elements \( \{1, s_1, \ldots, s_n\} \), with \( 1 \) being the neutral element, and for any \( 1 \leq i, j \leq n \), \( s_i \cdot s_j = s_i \). Clearly, \( \tilde{U}_n \) is aperiodic, actually idempotent, as \( s_i \cdot s_i = s_i \) for any \( 1 \leq i \leq n \). For instance, the monoid \( \tilde{U}_2 \) is the transition monoid (defined below) of the automaton below with \( \varphi(a) = s_1 \), \( \varphi(b) = s_2 \) and \( \varphi(c) = 1 \).

\[
\begin{array}{c}
\text{a, c} \\
1 \\
\text{a} \\
2 \\
\text{b, c}
\end{array}
\]

Rational languages are languages that can be described by rational expressions, i.e., sets of words constructed from finite sets using the operations of concatenation, union and Kleene star. It is well-known that rational languages are equivalent to regular languages, i.e., languages accepted by finite automata, and to languages recognized by finite monoids (and Monadic Second-order logic \([7]\)). Star-free rational expressions are built from finite sets using the operations of concatenation, union and complement (instead of Kleene star). They have the same expressive power as finite aperiodic monoids \([21]\) (as well as counter-free automata and first-order logic \([13]\)).

2.2 Two-way transducers

Definition 3 (Two-way transducer). A (deterministic) two-way transducer (2DFT) is a tuple \( \mathcal{A} = (Q, \Sigma, \Gamma, \delta, \gamma, q_0, F) \) defined as follows:
- \( Q \) is a finite set of states.
- \( \Sigma \) and \( \Gamma \) are the finite input and output alphabets.
- \( \delta : Q \times (\Sigma \uplus \{\triangleleft, \triangleright\}) \to Q \times \{-1,+1\} \) is the partial transition function. Contrary to one-way machines, the transition function also outputs an integer, indicating the move of the reading head. The alphabet is enriched with two new symbols \( \triangleleft \) and \( \triangleright \), which are endmarkers that are added respectively at the beginning and at the end of the input word, such that for all \( q \in Q \), we have \( \delta(q, \triangleleft) \in Q \times \{+1\} \) (if defined), \( \delta(q, \triangleright) \in Q \times \{-1\} \) (if defined) and \( \delta(q, \triangleleft) \) is undefined for \( q \in F \).
- \( \gamma : Q \times (\Sigma \uplus \{\triangleleft, \triangleright\}) \to \Gamma^* \) is the partial production function with same domain as \( \delta \).
- \( q_0 \in Q \) is the initial state.
6 SD-Regular Transducer Expressions for Aperiodic Transformations

Let $\mathcal{A}$ be a deterministic two-way automaton (2DFA) with set of states $Q$. When computing the transition monoid of a two-way automaton, we are interested in the behaviour of the partial runs, i.e., how these partial runs can be concatenated. Thus we abstract a given $(d, p, q)$-run $\rho$ over a word $w$ to a step $(d, p, q) \in \{\to, \succ, \ll, \leftarrow\} \times Q^2$ and we say that $w$ realises the step $(d, p, q)$. The transition monoid $\text{TrM}$ of $\mathcal{A}$ is a subset of the powerset of steps: $\text{TrM} \subseteq \mathcal{P}(\{\to, \succ, \ll, \leftarrow\} \times Q^2)$. The canonical surjective morphism $\varphi: (\Sigma \cup \{\top, \bot\})^* \to \text{TrM} = \varphi((\Sigma \cup \{\top, \bot\})^*)$ is defined for a word $w \in (\Sigma \cup \{\top, \bot\})^*$ as the set of steps realised by $w$, i.e., $\varphi(w) = \{(d, p, q) \mid$ there is a $(d, p, q)$-run on $w\} \subseteq \{\to, \succ, \ll, \leftarrow\} \times Q^2$. As an example, in Figure 1 we have

$$\varphi(a\#) = \{(\to, q_1, q_2), (\ll, q_1, q_2), (\succ, q_3, q_3), (\leftarrow, q_3, q_3), (\to, q_4, q_4), (\ll, q_4, q_4), (\to, q_5, q_5), (\ll, q_5, q_5)\}.$$ 

The unit of $\text{TrM}$ is $1 = \{(\to, p, p), (\leftarrow, p, p) \mid p \in Q\}$ and $\varphi(\varepsilon) = 1$. 

- $F \subseteq Q$ is the set of final states.

A configuration $c$ of $\mathcal{A}$ over an input word $w = w_1 \cdots w_{|w|}$ is simply a pair $(p, i)$ where $p \in Q$ is the current state and $0 \leq i \leq |w| + 1$ is the position of the head on the input tape containing $\top$ at $i$. Two configurations $c = (p, i)$ and $c' = (q, j)$ are successive if we have $\delta(p, w_i) = (q, d)$ and $i + d = j$, with $w_0 = \top$ and $w_{|w|+1} = \bot$. In this case, they produce an output $v = \gamma(p, w_i)$. Abusing notations we will sometime write $\gamma(c)$ when the input word $w$ is clear. A run $\rho$ is a sequence of successive configurations $c_0 \cdots c_n$. The run $\rho$ is initial if $c_0 = (q_0, 0)$ and is final if $c_n = (q, |w| + 1)$ for some $q \in F$. It is accepting if it is both initial and final.

The output of a run $\rho = c_0 \cdots c_n$ is the concatenation of the output of the configurations, and will be denoted $[\rho] = \gamma(c_0) \cdots \gamma(c_{n-1})$. Given a deterministic two-way transducer $\mathcal{A}$ and an input word $w$, there is at most one accepting run of $\mathcal{A}$ over $\top$-$w$-$\bot$, which we will denote $\rho(w)$. The output of $\mathcal{A}$ over $w$ is then $[\mathcal{A}](w) = [\rho(w)]$. The domain of $\mathcal{A}$ is the set $\text{dom}(\mathcal{A})$ of words $w$ such that there exists an accepting run of $\mathcal{A}$ over $w$. Finally, the semantics of $\mathcal{A}$ is the partial function $[\mathcal{A}] : \Sigma^* \to \Gamma^*$ defined on $\text{dom}(\mathcal{A})$ by $w \mapsto [\mathcal{A}](w)$.

Let $\rho = (p_0, i_0) \cdots (p_n, i_n)$ be a run over a nonempty word $w \in \Sigma^*$ such that $1 \leq i_j \leq |w|$ for all $0 \leq j < n$. It is a left-right run if $i_0 = 1$ and $i_n = |w| + 1$. If this is the case, we say that $\rho$ is a $(\to, p_0, p_n)$-run. Similarly, it is a left-left $(\ll, p_0, p_n)$-run if $i_0 = 1$ and $i_n = 0$. It is a right-left $(\to, p_0, p_n)$-run if $i_0 = |w|$ and $i_n = 0$. It is a right-right $(\ll, p_0, p_n)$-run if $i_0 = |w|$ and $i_n = |w| + 1$. Notice that if $|w| = 1$, then left-right runs and right-right runs coincide, also right-left runs and left-left runs coincide.

Remark 4. Given our semantics of two-way transducers, a run associates states to each position, whereas the classical semantics of one-way automata keeps the states between two positions. Then, if we consider a word $w = w$ and a left-left run $(\ll, p, q, o)$ on $v$, we begin on the first position of $v$ in state $p$, and the state $q$ is reached at the end of the run on the last position of $u$. This allows for easy sequential composition of partial runs when concatenating non empty words, as the end of a partial run is the start of the next words.

However, in order to keep our figures as readable as possible, we will represent these states between words. A state $q$ between two words $u$ and $v$ is to be placed on the first position of $v$ if it is the start of a run going to the right, and on the last position of $w$ otherwise. For instance, in Figure 4 state $q_1$ is on the first position of $u_{i+1}$ and state $q_3$ is on the last position of $u_i$. 

Transition monoid of a two-way automaton
A 2DFA is aperiodic if its transition monoid $\text{TrM}$ is aperiodic. Also, a 2DFT is aperiodic if its underlying input 2DFA is aperiodic.

When talking about a given step $(d, p, q)$ belonging to an element of $\text{TrM}$, we will sometimes forget $p$ and $q$ and talk about a $d$-step, for $d \in \{>, <, \rightarrow, \leftarrow\}$ if the states $p, q$ are clear from the context, or are immaterial for the discussion. In this case we also refer to a step $(d, p, q)$ as a $d$-step having $p$ as the starting state and $q$ as the final state.

### 3 Complement-free expressions for aperiodic languages

As the aim of the paper is to obtain rational expressions corresponding to transformations computed by aperiodic two-way transducers, we cannot rely on extending the classical (SF=AP) star-free characterization of aperiodic languages, since the complement of a function is not a function. We solve this problem by considering the SD=AP characterization of aperiodic languages, namely prefix codes with bounded synchronisation delay, introduced by Schützenberger [22].

A language $L$ is called a code if for any word $u \in L^*$, there is a unique decomposition $u = v_1 \cdots v_n$ such that $v_i \in L$ for $1 \leq i \leq n$. For example, the language $W = \{a, ab, ba, bba\}$ is not a code: the words $abba, aba \in W^*$ have decompositions $a \cdot bba = ab \cdot ba$ and $a \cdot ba = ab \cdot a$ respectively. A prefix code is a language $L$ such that for any pair of words $u, v$, if $u, w \in L$, then $v = \varepsilon$. $W$ is not a prefix code, while $W_1 = W \setminus \{ab\}$ and $W_2 = W \setminus \{a\}$ are prefix codes. Prefix codes play a particular role in the sense that the unique decomposition can be obtained on the fly while reading the word from left to right.

**Definition 5.** Let $d$ be a positive integer. A prefix code $C$ over an alphabet $\Sigma$ has a synchronisation delay $d$ (denoted $d$-SD) if for all $u, v, w \in \Sigma^*$, $uw \in C^*$ and $v \in C^d$ implies $wv \in C^*$ (hence also $w \in C^*$). An SD prefix code is a prefix code with a bounded synchronisation delay.

As an example, consider the prefix code $C = \{aa, ba\}$ and the word $ba(aa)^d \in C^*$. We have $ba(aa)^d = uwv$ with $u = b$, $v = (aa)^d \in C^d$ and $w = a$. Since $w \notin C^*$, the prefix code $C$ is not of bounded synchronisation delay. Likewise, $C = \{aa\}$ is also not of bounded synchronisation delay. On the other hand, the prefix code $C = \{ba\}$ is 1-SD.

The syntax of regular expressions over the alphabet $\Sigma$ is given by the grammar

$$E ::= \emptyset \mid \varepsilon \mid a \mid E \cup E \mid E \cdot E \mid E^*$$

where $a \in \Sigma$. We say that an expression is $\varepsilon$-free (resp. $\emptyset$-free) if it does not use $\varepsilon$ (resp. $\emptyset$) as subexpressions. The semantics of a regular expression $E$ is a regular language over $\Sigma^*$ denoted $\mathcal{L}(E)$.

An SD-regular expression is a regular expression where Kleene-stars are restricted to SD prefix codes: If $E^*$ is a sub-expression then $\mathcal{L}(E)$ is a prefix code with bounded synchronization delay. Thus, the regular expression $(ba)^*$ is a SD-regular expression while $(aa)^*$ is not.

The relevance of SD-regular expressions comes from the fact that they are a complement-free characterization of aperiodic languages.

**Theorem 6.** [22] A language $L$ is recognized by an aperiodic monoid if, and only if, there exists an SD-regular expression $E$ such that $L = \mathcal{L}(E)$.

Theorem 6 concretizes this result, and extends it to get more specific expressions which are (i) unambiguous, a property required for the regular combinators expressing functions over words, and (ii) stabilising, which is a new notion introduced below that suits our need.
for characterizing runs of aperiodic two-way transducers. Our proof technique follows the local divisor technique, which was notably used by Diekert and Kufleitner to lift the result of Schützenberger to infinite words [12, 13].

A regular expression \( E \) is unambiguous, if it satisfies the following:
1. for each subexpression \( E_1 \cup E_2 \) we have \( \mathcal{L}(E_1) \cap \mathcal{L}(E_2) = \emptyset \),
2. for each subexpression \( E_1 \cdot E_2 \), each word \( w \in \mathcal{L}(E_1 \cdot E_2) \) has a unique factorisation \( w = uv \) with \( u \in \mathcal{L}(E_1) \) and \( v \in \mathcal{L}(E_2) \),
3. for each subexpression \( E_1^* \), the language \( \mathcal{L}(E_1) \) is a code, i.e., each word \( w \in \mathcal{L}(E_1^*) \) has a unique factorisation \( w = v_1 \cdots v_n \) with \( v_i \in \mathcal{L}(E_1) \) for \( 1 \leq i \leq n \).

**Definition 7.** Given an aperiodic monoid \( M \) and \( X \subseteq M \), we say that \( X \) is \( n \)-stabilising if \( xy = x \) for all \( x \in X^n \) and \( y \in X \). We say that \( X \) is stabilising if it is \( n \)-stabilising for some \( n \geq 1 \).

**Remark.** Stabilisation generalizes aperiodicity in some sense. For aperiodicity, we require \( x^n = x^{n+1} \) for each element \( x \in M \) and some \( n \in \mathbb{N} \), i.e., all singleton subsets of \( M \) should be stabilising.

**Example 8.** Continuing Example 2 any subset \( X \subseteq \{s_1, \ldots, s_n \} \subseteq \widetilde{U}_n \) is 1-stabilising.

As another example, consider the aperiodic 2DFT \( \mathcal{A} \) in Figure 1 and consider its transition monoid \( \mathcal{Tr} M \). Clearly, \( \mathcal{Tr} M \) is an aperiodic monoid. Let \( \varphi \) be the morphism from \( (\Sigma \cup \{\uparrow, \downarrow\})^* \) to \( \mathcal{Tr} M \). Consider the subset \( Z = \{Y, Y^2\} \) of \( \mathcal{Tr} M \) where \( Y = \varphi(a \#a \#a) \):

\[
Y = \{ (\uparrow, q_1, q_4), (\downarrow, q_3, q_2), (\uparrow, q_4, q_1), (\rightarrow, q_2, q_1), (\zeta, q_0, q_1), (\zeta, q_0, q_1) \}
\]

\[
Y^2 = \{ (\uparrow, q_1, q_4), (\downarrow, q_3, q_2), (\uparrow, q_4, q_1), (\rightarrow, q_2, q_1), (\zeta, q_0, q_1), (\zeta, q_0, q_1) \}
\]

It can be seen that \( Y^3 = Y^2 \), hence \( Z \) is 2-stabilising.

Let \( \varphi : \Sigma^* \to M \) be a morphism. We say that a regular expression \( E \) is \( \varphi \)-stabilising (or simply stabilising when \( \varphi \) is clear from the context) if for each subexpression \( F^* \) of \( E \), the set \( \varphi(\mathcal{L}(F)) \) is stabilising.

Continuing Example 8 we can easily see that \( \varphi(a) \) is idempotent and we get \( \varphi(a^+ \#a^+ \#a^+) = \{Y\} \). Since \( Y^3 = Y^2 \), we deduce that \( (aa^+ \#aa^+ \#a)^* \) is a stabilising expression. Notice also that, by definition, expressions without a Kleene-star are stabilising vacuously.

**Example 9.** As a more non trivial example to illustrate stabilising expressions, consider the 2DFT \( \mathcal{A} \) in Figure 2 whose domain is the language \( b(a^*b)^{\geq 2}a^* \). Consider \( b(a^*b)^{\geq 2} \subseteq \text{dom}(\mathcal{A}) \).

Note that \( a^*b \) is a prefix code with synchronisation delay 1. Let \( X = \varphi(a^*b) \), where \( \varphi \) is the morphism from \( (\Sigma \cup \{\uparrow, \downarrow\})^* \) to \( \mathcal{Tr} M \). We will see that \( X \) stabilises.
First, we have $X = \{Y_1, Y_2\}$ where

$$Y_1 = \varphi(b) = \{(\rightarrow, s, q_0), (\rightarrow, q_0, q_1), (\rightarrow, q_1, q_2), (\rightarrow, q_2, q_3), (\rightarrow, q_3, q_4), (\rightarrow, q_4, q_5), (\rightarrow, q_5, q_6), (\rightarrow, q_6, q_0)\},$$

$$Y_2 = \varphi(a^+b) = \{(\rightarrow, q_0, q_1), (\rightarrow, q_1, q_2), (\rightarrow, q_2, q_3), (\rightarrow, q_3, q_4), (\rightarrow, q_4, q_5), (\rightarrow, q_5, q_6), (\rightarrow, q_6, q_0)\}$$

Next, we can check that $X^2 = \{Y_3, Y_4\}$ where

$$Y_3 = Y_1 Y_1 = Y_1 Y_2 = \{(\rightarrow, s, q_1), (\rightarrow, q_0, q_2), (\rightarrow, q_1, q_3), (\rightarrow, q_2, q_3), (\rightarrow, q_3, q_4), (\rightarrow, q_4, q_5), (\rightarrow, q_5, q_6), (\rightarrow, q_6, q_1)\},$$

$$Y_4 = Y_2 Y_1 = Y_2 Y_2 = \{(\rightarrow, s, q_0), (\rightarrow, q_0, q_0), (\rightarrow, q_0, q_1), (\rightarrow, q_1, q_2), (\rightarrow, q_2, q_3), (\rightarrow, q_3, q_4), (\rightarrow, q_4, q_5), (\rightarrow, q_5, q_6), (\rightarrow, q_6, q_0)\}$$

Then, we have $X^3 = \{Z_1, Z_2\}$ where

$$Z_1 = Y_3 Y_3 = Y_3 Y_4 = \{(\rightarrow, s, q_2), (\rightarrow, q_0, q_3), (\rightarrow, q_1, q_4), (\rightarrow, q_2, q_4), (\rightarrow, q_3, q_4), (\rightarrow, q_4, q_5), (\rightarrow, q_5, q_6), (\rightarrow, q_6, q_2)\},$$

$$Z_2 = Y_4 Y_3 = Y_4 Y_4 = \{(\rightarrow, s, q_0), (\rightarrow, q_0, q_0), (\rightarrow, q_0, q_1), (\rightarrow, q_0, q_2), (\rightarrow, q_0, q_3), (\rightarrow, q_0, q_4), (\rightarrow, q_0, q_5), (\rightarrow, q_0, q_6)\}$$

Finally, we can easily check that $Z_1 Y_1 = Z_1 Y_2 = Z_1$ and $Z_2 Y_1 = Z_2 Y_2 = Z_2$. Therefore, $X$ is 4-stabilising. Moreover, $b(a^+b)^{\geq 3} \subseteq \varphi^{-1}(Z_1)$ and $a^+b(a^+b)^{\geq 3} \subseteq \varphi^{-1}(Z_2)$.

Given a morphism $\varphi$ from $\Sigma^*$ to some aperiodic monoid $M$, our goal is to build, for each language $\varphi^{-1}(s)$ with $s \in M$, an SD-regular expression which is both unambiguous and stabilising. The proof is by induction on the monoid $M$ via the local divisor technique, similar to Diekert and Kufleitner [12, 13, 14], and to Perrin and Pin [20, Chapter VIII, Section 6.1], with the objective to get stronger forms of SD-regular expressions.

**Theorem 10.** Given a morphism $\varphi$ from the free monoid $\Sigma^*$ to a finite aperiodic monoid $M$, for each $s \in M$ there exists an unambiguous, stabilising, SD-regular expression $E_s$ such that $L(E_s) = \varphi^{-1}(s)$.

The proof of this theorem makes crucial use of marked substitutions (see [20]) that we define and study in the next section.

### 3.1 Marked substitutions

Let $A, B$ be finite alphabets. A map $\alpha: A \rightarrow \mathcal{P}(B^*)$ is called a marked substitution if it satisfies the following two properties:

- There exists a partition $B = B_1 \uplus B_2$ such that for all $a$ in $A$, $\alpha(a) \subseteq B_1^* B_2$,
- For all $a_1$ and $a_2$ in $A$, $a_1 \neq a_2$ implies $\alpha(a_1) \cap \alpha(a_2) = \emptyset$.

A marked substitution $\alpha: A \rightarrow \mathcal{P}(B^*)$ can be naturally extended to words in $A^*$ using concatenation of languages, i.e., to a morphism from the free monoid $A^*$ to $(\mathcal{P}(B^*), \cdot, \{\varepsilon\})$. It is then further lifted to languages $L \subseteq A^*$ by union: $\alpha(L) = \bigcup_{w \in L} \alpha(w)$.

**Lemma 11** [20, Chapter VIII, Proposition 6.2]. Let $\alpha: A \rightarrow \mathcal{P}(B^*)$ be a marked substitution, and $X \subseteq A^*$ be a prefix code with synchronisation delay $d$. Then $Y = \alpha(X) \subseteq (B_1^* B_2)^+$ is a prefix code with synchronisation delay $d + 1$. 


Proof. First, since $B_1$ and $B_2$ are disjoint, $B_1^* B_2 \subseteq B^*$ is a prefix code. Hence, given a word $w \in \alpha(A^*) \subseteq (B_1^* B_2)^*$, there exists a unique decomposition $w = w_1 \cdots w_n$ such that $w_i \in B_1^* B_2$ for $1 \leq i \leq n$. Now since images of different letters from $A$ are disjoint, there exists at most one $a_i$ such that $w_i \in \alpha(a_i)$. We deduce that there is exactly one word $w' \in A^*$ such that $\alpha(w') = w$. This word is denoted $\alpha^{-1}(w)$.

Now, we prove that $Y$ is a prefix code. Let $v, w \in \alpha(A^*) \subseteq (B_1^* B_2)^*$ and assume that $v$ is a prefix of $w$. Write $w = w_1 \cdots w_n$ with $w_i \in B_1^* B_2$. Since $v$ ends with a letter from $B_2$ we deduce that $v = w_1 \cdots w_i$ for some $1 \leq i \leq n$. Let $w' = \alpha^{-1}(w) = a_1 \cdots a_n$. We have $v' = \alpha^{-1}(v) = a_1 \cdots a_i$. Now, if $v, w \in \alpha(X)$ then we get $v', w' \in X$. Since $X$ is a prefix code we get $i = n$. Hence $v = w$, proving that $Y$ is also a prefix code.

Finally, we prove that $Y$ has synchronization delay $d + 1$. Let $u, v, w \in B^*$ such that $uvw \in Y^*$ and $v \in Y^{d+1}$. We need to prove that $uwv \in Y^*$ and $v \in Y^{d+1}$. It can be written $v = v_0 v_1 \cdots v_d$ with $v_i \in Y$ for $0 \leq i \leq d$. Then, let us remark that $\alpha(A) \subseteq B_1^* B_2$ is a prefix code with synchronisation delay 1. Since $uv_0 \cdots v_d w \in Y^* \subseteq \alpha(A)^*$ and $v_0 \in Y \subseteq \alpha(A)^*$, we deduce that $uw$ belongs to $\alpha(A)^* = \alpha(A^*)$, as well as $v_1 \cdots v_d$ and $w$. Let $r = \alpha^{-1}(uv_0)$, $s = \alpha^{-1}(v_1 \cdots v_d)$ and $t = \alpha^{-1}(w)$. We have $rst = \alpha^{-1}(uvw)$ and since $uvw \in Y^* = \alpha(X)^*$, we deduce that $rst \in X^*$. Similarly, from $v_1 \cdots v_d \in Y^d = \alpha(X)^d$, we get $s \in X^d$. Now, $X$ has synchronization delay $d$. Therefore, $rs \in X^*$, meaning that $uvw = uv_0 \cdots v_d \in \alpha(rs) \subseteq \alpha(X)^* = Y^*$.

Marked substitutions also preserve unambiguity of union, concatenation and Kleene star.

Lemma 12. Let $\alpha: A \to \mathcal{P}(B^*)$ be a marked substitution and let $L_1, L_2 \subseteq A^*$.
1. If the union $L_1 \cup L_2$ is unambiguous then so is $\alpha(L_1) \cup \alpha(L_2)$.
2. If the concatenation $L_1 \cdot L_2$ is unambiguous then so is $\alpha(L_1) \cdot \alpha(L_2)$.
3. If the Kleene star $L_1^*$ is unambiguous then so is $\alpha(L_1)^*$.

Proof. As stated in the previous proof, a marked substitution is one-to-one. We denote by $\alpha^{-1}(w)$ the unique inverse of $w$, for $w \in \alpha(A^*)$.

If $w \in \alpha(L_1) \cup \alpha(L_2)$ then $\alpha^{-1}(w) \in L_1 \cup L_2$. This shows that unambiguity of union is preserved by $\alpha$.

Assume now that the concatenation $L_1 \cdot L_2$ is unambiguous. Let $w \in \alpha(L_1) \cdot \alpha(L_2)$ and consider its unique factorisation $w = v_1 b_1 \cdots v_n b_n$ as above and $\alpha^{-1}(w) = a_1 \cdots a_n$. Since $\alpha(L_1) \subseteq (B_1^* B_2)^*$, a factorisation of $w$ according to $\alpha(L_1) \cdot \alpha(L_2)$ must be of the form $w = (v_1 b_1 \cdots v_i b_i) \cdot (v_{i+1} b_{i+1} \cdots v_n b_n)$ with $a_1 \cdots a_i \in L_1$ and $a_{i+1} \cdots a_n \in L_2$. From unambiguity of the product $L_1 \cdot L_2$ we deduce that such a factorisation of $w$ is unique. Hence, the product $\alpha(L_1) \cdot \alpha(L_2)$ is unambiguous.

We can prove similarly that $\alpha$ preserves unambiguity of Kleene stars.

We will be interested in marked substitutions that are defined by regular expressions. A regular marked substitution (RMS) is a map $\alpha: A \to \operatorname{Reg}(B^*)$ which assigns a regular expression $\alpha(a)$ over $B$ to each letter $a \in A$ such that $\tilde{\alpha}: A \to \mathcal{P}(B^*)$ defined by $\tilde{\alpha}(a) = \mathcal{L}(\alpha(a))$ is a marked substitution.

Let $\alpha: A \to \operatorname{Reg}(B^*)$ be an RMS and let $E$ be a regular expression over $A$. We define $\alpha(E) = E[\alpha(a)/a, \forall a \in A]$ to be the regular expression over $B$ obtained from $E$ by substituting each occurrence of a letter $a \in A$ with the expression $\alpha(a)$. Notice that $\alpha$ is compositional: $\alpha(E_1 \cup E_2) = \alpha(E_1) \cup \alpha(E_2)$, $\alpha(E_1 \cdot E_2) = \alpha(E_1) \cdot \alpha(E_2)$ and $\alpha(E_1^+) = \alpha(E_1)^+$. In particular, we have $\tilde{\alpha}(\mathcal{L}(E)) = \mathcal{L}(\alpha(E))$.

Further, we say that a RMS $\alpha$ is unambiguous (URMS) if $\alpha(a)$ is unambiguous for each $a \in A$. Similarly, an RMS $\alpha$ is SD-regular (SDRMS) if $\alpha(a)$ is an SD-regular expression for each $a \in A$. We obtain:
We now treat the three cases above.

1. Let \( \alpha : X \rightarrow \text{Reg}(B^\ast) \) be an RMS and \( E \) be a regular expression over \( A \).
2. If \( \alpha \) and \( E \) are SD-regular, then \( \alpha(E) \) is SD-regular.
3. If \( \alpha \) and \( E \) are unambiguous, then \( \alpha(E) \) is unambiguous.

Proof. 1. Let \( F^\ast \) be a subexpression of \( \alpha(E) \). If \( F^\ast \) is a subexpression of some \( \alpha(a) \) then, \( \alpha(a) \) being SD-regular we obtain that \( \mathcal{L}(F) \) is an SD prefix code. Otherwise, \( F = \alpha(G) \) where \( G^\ast \) is a subexpression of \( E \). Since \( E \) is SD-regular, \( \mathcal{L}(G) \) is an SD prefix code. By lemma [11] we deduce that \( \mathcal{L}(F) = \bar{\alpha}(\mathcal{L}(G)) \) is an SD prefix code.

2. First, we know that each \( \alpha(a) \) is unambiguous. Next, a subexpression of \( \alpha(E) \) which is not a subexpression of some \( \alpha(a) \) must be of the form \( \alpha(F) \) where \( F \) is a subexpression of \( E \). We conclude easily using unambiguity of \( E \) and Lemma [12].

3.2 Proof of Theorem 10

Proof. We first consider the set of neutral letters, i.e., letters whose image is the neutral element 1 of \( M \). To ease the proof, we first explain how to handle them, and in the rest of the proof, focus on the case where we do not have neutral letters.

Let \( \varphi : \Sigma^\ast \rightarrow M \) be a morphism and \( \Sigma_0 = \{ a \in \Sigma \mid \varphi(a) = 1 \} \) be the set of neutral letters. Further, let \( \Sigma_1 = \Sigma \setminus \Sigma_0 \) and let \( \varphi_1 : \Sigma_1^\ast \rightarrow M \) be the restriction of \( \varphi \) to \( \Sigma_1^\ast \). Let \( \alpha : \Sigma_1^\ast \rightarrow \text{Reg}(\Sigma^\ast) \) be the regular marked substitution defined by \( \alpha(a) = \Sigma_0 a \). Clearly, \( \alpha \) is unambiguous and since \( \Sigma_0 \) is a 1-SD prefix code we get that \( \alpha \) is SD-regular. By Corollary [13] we deduce that \( \alpha \) preserves unambiguity and also SD-expressions. It also preserves stabilising expressions, i.e., if \( E \in \text{Reg}(\Sigma_1^\ast) \) is \( \varphi_1 \)-stabilising then \( \alpha(E) \in \text{Reg}(\Sigma^\ast) \) is \( \varphi \)-stabilising. Indeed, \( \varphi(\Sigma_0) \) is 1-stabilising. Further, if \( G^\ast \) is a subexpression of \( \alpha(E) \) different from \( \Sigma_0^\ast \) then there is a subexpression \( F^\ast \) of \( E \) such that \( G = \alpha(F) \). Hence, \( \mathcal{L}(G) = \bar{\alpha}(\mathcal{L}(F)) \) and \( X = \varphi(\mathcal{L}(G)) = \varphi_1(\mathcal{L}(F)) \) is stabilising.

Now, suppose we have unambiguous, stabilising, SD-expressions \( E_s \) for \( \varphi_1 \) and each \( s \in M : \mathcal{L}(E_s) = \varphi_1^{-1}(s) \). We deduce that \( E_s^\ast = \alpha(E_s) : \Sigma_0^\ast \) is an unambiguous, stabilising, SD-expression. Moreover, we have \( \mathcal{L}(E_s^\ast) = \varphi^{-1}(s) \).

In the rest of the proof, we assume that the morphism \( \varphi \) has no neutral letters. The proof is by induction on the size of \( M \), using a result from Perrin and Pin [20] Chapter XI, Proposition 4.14] stating that if \( \varphi \) is a surjective morphism from \( \Sigma^\ast \) to a finite aperiodic monoid \( M \), then one of the following cases hold:

1. \( M \) is a cyclic monoid, meaning that \( M \) is generated by a single element.
2. \( M \) is isomorphic to \( \tilde{U}_n \) for some \( n \geq 1 \)
3. There is a partition \( \Sigma = A \sqcup B \) such that \( \varphi(A^\ast) \) and \( \varphi((A^\ast B^\ast)^\ast) \) are proper submonoids of \( M \).

We now treat the three cases above.

1. \( M \) is a cyclic monoid. Then \( M \) is of the form \( \{1, s, s^2, \ldots, s^n\} \) with \( s^i s^j = s^{i+j} \) if \( i + j \leq n \) and \( s^i s^j \) otherwise. Notice that since we have no neutral letters, \( \varphi^{-1}(1) = \{ \varepsilon \} \). For \( 1 \leq i \leq n \), we denote by \( \Sigma_i \) the set of letters whose image is \( s^i \). Now, we define inductively stabilising, unambiguous, SD-regular expressions \( E_j \) such that \( \mathcal{L}(E_j) = \varphi^{-1}(s^j) \) for \( 1 \leq j \leq n \). Let \( E_1 = \Sigma_1 \). Then, for \( 1 < j < n \) we let

\[
E_j = \Sigma_j \cup \bigcup_{1 \leq i < j} E_i \Sigma_{j-i}.
\]

Notice that expressions \( E_j \) for \( 1 \leq j < n \) are unambiguous and do not use the Kleene
star, hence are stabilising and SD. Finally, let
\[ E_n = \left( \Sigma_n \cup \bigcup_{1 \leq i, j < n, |i| \leq |j|} E_i \Sigma_j \right) \Sigma^*. \]

Notice that the separation between the first part of \( E_n \) and \( \Sigma^* \) is done at the letter at which the image of the prefix reaches \( s^n \). Then as above, \( E_n \) is unambiguous, and \( \Sigma \) is an \( n \)-stabilising, 1-SD prefix code. Moreover, \( \mathcal{L}(E_j) = \varphi^{-1}(s^j) \) for \( 1 \leq j \leq n \), which concludes the proof in the case of cyclic monoids.

2. \( M \) is isomorphic to \( \hat{U}_n \) for some \( n \). Then \( M \) is of the form \( \{1, s_1, \ldots, s_n\} \) where \( s_i s_j = s_i \) for all \( 1 \leq i, j \leq n \). As above, since we have no neutral letters, we deduce that \( \varphi^{-1}(1) = \{\varepsilon\} \). We similarly define \( \Sigma_i = \varphi^{-1}(s_i) \cap \Sigma \). Clearly, \( \varphi^{-1}(s_i) = \mathcal{L}(\Sigma_i \Sigma^*) \), and \( \Sigma, \Sigma^* \) is an unambiguous, 1-stabilising, 1-SD regular expression.

3. There is a partition \( \Sigma = A \sqcup B \) such that \( M_A = \varphi(A^*) \) and \( M_B = \varphi((A^* B)^*) \) are proper submonoids of \( M \). We set \( C = \varphi(A^* B) \subseteq M_B \) and view \( C \) as a new alphabet. Note that \( C \) generates \( M_B \). Finally, let \( f : A^* \to M_A \) be the restriction of \( \varphi \) to \( A \) and \( g : C^* \to M_B \) be the evaluation morphism defined by \( g(c) = c \) for \( c \in C \).

Then \( f \) and \( g \) are surjective morphisms to monoids \( M_A \) and \( M_B \) whose size are smaller than \( M \). We can thus use the induction hypothesis and get unambiguous, stabilising, SD-expressions for elements of \( M_A \) and \( M_B \) with respect to \( f \) and \( g \) respectively. Given an element \( s \) in \( M_A \), we get an unambiguous, \( f \)-stabilising, SD-expression \( E_s \) over \( A \) for \( f^{-1}(s) \). Similarly for an element \( t \) in \( M_B \), we get an unambiguous, \( g \)-stabilising, SD-expression \( F_t \) over the alphabet \( C \) for \( g^{-1}(t) \). Notice that we have in particular expressions \( E_1 \) for \( f^{-1}(1) \) and \( F_1 \) for \( g^{-1}(1) \).

To go back from expressions over \( A, C \) to expressions over \( \Sigma^* \), we first define expressions \( G_c \) for \( \varphi^{-1}(c) \cap A^* B \), for each \( c \in C \):
\[ G_c = \bigcup_{s \in M_A, b \in B, s \varphi(b) = c} E_s \cdot b. \]

Notice that \( \alpha : C \to \mathcal{R}(\Sigma^*) \) defined by \( \alpha(c) = G_c \) is a regular marked substitution with respect to the partition \( \Sigma = A \sqcup B \). Indeed, \( \alpha(c) \subseteq A^* B \) and \( \varphi(\alpha(c)) = c \) which implies that the images are pairwise disjoint. Moreover, each \( G_c \) is unambiguous, \( \varphi \)-stabilising and SD-regular.

Let \( t \in M_B \). By Corollary 13, \( \alpha(F_t) \) is unambiguous and SD-regular. It is also \( \varphi \)-stabilising. Indeed, let \( G^* \) be a subexpression of \( \alpha(F_t) \). If \( G^* \) is a subexpression of some \( \alpha(c) \) we get the result since \( G_c \) is \( \varphi \)-stabilising. Otherwise, there is a subexpression \( F^* \) of \( F_t \) such that \( G = \alpha(F) \). Hence, \( \mathcal{L}(G) = \widehat{\alpha}(\mathcal{L}(F)) \) and \( X = \varphi(\mathcal{L}(G)) = g(\mathcal{L}(F)) \) is stabilising.

For each \( t \in M_B \), we have \( \mathcal{L}(\alpha(F_t)) = \varphi^{-1}(t) \cap (A^* B)^* \). Finally, we need to combine these elements to get an expression for any word over \( \Sigma \). Noticing that \( \Sigma^* = (A^* B)^* A^* \), we define for \( s \in M \)
\[ E'_s = \bigcup_{t \in M_B, r \in M_A, tr = s} \alpha(F_t) \cdot E_r. \]

We can easily show that \( E'_s \) is unambiguous and satisfies \( \mathcal{L}(E_s) = \varphi^{-1}(s) \). It is also clearly a \( \varphi \)-stabilising SD-expression. Hence, we get the result.

\section{Combinator expressions}

In this section, we present our combinators to compute first order definable functions from finite words to finite words. The simpler combinators of unambiguous concatenation and
sum are similar to those in [3,11], but we differ in the equivalent of the Kleene-Star to match the aperiodicity that we tackle.

4.1 Simple Functions

For each \( v \in \Gamma^* \) we have a constant function \( f_v \) defined by \( f_v(u) = v \) for all \( u \in \Sigma^* \). Abusing notations, we simply denote the constant function \( f_v \) by \( v \). We denote by \( \perp : \Sigma^* \rightarrow \Gamma^* \) the function with empty domain. These atomic functions are the most simple ones.

4.2 Unambiguous sums

We will use two equivalent ways of defining a function by cases. First, the if-then-else construct is given by \( h = L ? f : g \) where \( f, g : \Sigma^* \rightarrow \Gamma^* \) are functions and \( L \subseteq \Sigma^* \) is a language. We have \( \text{dom}(h) = (\text{dom}(f) \cap L) \cup (\text{dom}(g) \setminus L) \). Then, for \( w \in \text{dom}(h) \) we have

\[
\begin{cases}
  f(w) & \text{if } w \in L \\
  g(w) & \text{otherwise}.
\end{cases}
\]

We will often use this case definition with \( L = \text{dom}(f) \). To simplify notations we define \( f + g = \text{dom}(f) ? f : g \). Note that \( \text{dom}(f + g) = \text{dom}(f) \cup \text{dom}(g) \) but the sum is not commutative and \( g + f = \text{dom}(g) ? g : f \). For \( w \in \text{dom}(f) \cap \text{dom}(g) \) we have \((f + g)(w) = f(w) + (g + f)(w) = g(w)\). When the domains of \( f \) and \( g \) are disjoint then \( f + g \) and \( g + f \) are equivalent functions with domain \( \text{dom}(f) \uplus \text{dom}(g) \). In all cases the sum is associative and commutative and the sum notation is particularly useful when applied to a sequence \( f_1, \ldots, f_n \) of functions:

\[
\sum_{1 \leq i \leq n} f_i = f_1 + \cdots + f_n = \text{dom}(f_1) ? f_1 : \text{dom}(f_2) ? f_2 : \cdots \text{dom}(f_{n-1}) ? f_{n-1} : f_n
\]

If the domains of the functions are pairwise disjoint then this sum is associative and commutative.

Further, we let \( L \bowtie f = L ? f : \perp \) the function \( f \) restricted to \( L \cap \text{dom}(f) \). When \( L = \{w\} \) is a singleton, we simply write \( w \bowtie f \).

4.3 The Hadamard product

The Hadamard product of two functions \( f, g : \Sigma^* \rightarrow \Gamma^* \) first applies \( f \) and then applies \( g \). It is denoted by \( f \odot g \). Its domain is \( \text{dom}(f) \cap \text{dom}(g) \) and \((f \odot g)(u) = f(u)g(u)\) for each input word \( u \) in its domain.

4.4 The unambiguous Cauchy product

Consider two functions \( f, g : \Sigma^* \rightarrow \Gamma^* \). The unambiguous Cauchy product of \( f \) and \( g \) is the function \( f \cdot g \) whose domain is the set of words \( w \in \Sigma^* \) which admit a unique factorization \( w = uv \) with \( u \in \text{dom}(f) \) and \( v \in \text{dom}(g) \), and in this case, the computed output is \( f(u)g(v) \).

Contrary to the Hadamard product which reads its full input word \( w \) twice, first applying \( f \) and then applying \( g \), the Cauchy product splits unambiguously its input word \( w \) as \( uv \), applies \( f \) on \( u \) and then \( g \) on \( v \).

Sometimes we may want to reverse the output and produce \( g(v)f(u) \). This reversed Cauchy product can be defined using the Hadamard product as

\[
f \cdot_r g = ((\text{dom}(f) \bowtie \varepsilon) \odot (f \odot (\text{dom}(g) \bowtie \varepsilon)))
\]
4.5 The \( k \)-chained Kleene-star and its reverse

Let \( L \subseteq \Sigma^* \) be a code, let \( k \geq 1 \) be a natural number and let \( f : \Sigma^* \to \Gamma^* \) be a partial function. We define the \( k \)-chained Kleene-star \([L, f]^{k\star} : \Sigma^* \to \Gamma^* \) and its reverse \([L, f]^{k\star}_r : \Sigma^* \to \Gamma^* \) as follows.

The domain of both these functions is contained in \( L^* \), the set of words having a (unique) factorization over the code \( L \). Let \( w \in L^* \) and consider its unique factorization \( w = u_1 u_2 \cdots u_n \) with \( n \geq 0 \) and \( u_i \in L \) for all \( 1 \leq i \leq n \). Then, \( w \in \text{dom}([L, f]^{k\star}) \) if \( u_{i+1} \cdots u_{i+k} \in \text{dom}(f) \) for all \( 0 \leq i \leq n - k \) and in this case we set

\[
[L, f]^{k\star}(w) = f(u_1 \cdots u_k) \cdot f(u_2 \cdots u_{k+1}) \cdots f(u_{n-k+1} \cdots u_n)
\]

\[
[L, f]^{k\star}_r(w) = f(u_{n-k+1} \cdots u_n) \cdots f(u_2 \cdots u_{k+1}) \cdot f(u_1 \cdots u_k).
\]

Notice that when \( n < k \), the right-hand side is an empty product and we get \([L, f]^{k\star}(w) = \varepsilon\) and \([L, f]^{k\star}_r(w) = \varepsilon\). When \( k = 1 \) and \( L = \text{dom}(f) \) is a code then we simply write \( f^* = [\text{dom}(f), f]^{1\star} \) and \( f^*_r = [\text{dom}(f), f]^{1\star}_r \). We have \( \text{dom}(f^*) = \text{dom}(f^*_r) = L^* \).

The \( k \)-chained Kleene star was also defined in \( [3, 11] \); however as we will see below, we use it in a restricted way for aperiodic functions.

4.6 SD-regular transducer expressions (SDRTE)

SD-regular transducer expressions (SDRTEs) are obtained from classical regular transducer expressions (RTEs) \([3, 11]\) by restricting the \( k \)-chained Kleene-star \([L, f]^{k\star} \) and its reverse \([L, f]^{k\star}_r \) to aperiodic languages \( L \) that are prefix codes of bounded synchronisation delay. The if-then-else choice \( L ? f : g \) is also restricted to aperiodic languages \( L \). Hence, the syntax of SDRTEs is given by the grammar:

\[
C ::= \bot \mid v \mid L ? C : C \mid C \circ C \mid C \cdot C \mid [L, C]^{k\star} \mid [L, C]^{k\star}_r
\]

where \( v \in \Gamma^* \), and \( L \subseteq \Sigma^* \) ranges over aperiodic languages (or equivalently SD-regular expressions), which are also prefix codes with bounded synchronisation delay for \([L, C]^{k\star}\) and \([L, C]^{k\star}_r\).

The semantics of SDRTEs is defined inductively. \([\bot] \) is the function which is nowhere defined, \([v] \) is the constant function such as \([v](u) = v \) for all \( u \in \Sigma^* \), and the semantics of the other combinators has been defined in the above sections.

As discussed in Section 4.2, we will use binary sums \( C + C' = \text{dom}(C)?C : C' \) and generalised sums \( \sum_i C_i \). Also, we use the abbreviation \( L \triangleright C = L ? C : \bot \) and the reversed Cauchy product \( C \cdot C' = (\text{dom}(C) \triangleright \varepsilon) \cdot (C' \circ (\text{dom}(C') \triangleright \varepsilon)) \).

\[\textbf{Lemma 14.}\] If \( C \) is an SDRTE, then \( \text{dom}(C) \) is an aperiodic language.

\[\textbf{Proof.}\] We prove the statement by induction on the syntax of SDRTEs. We recall that aperiodic languages are closed under concatenation, union, intersection and complement.

\[\text{dom}(L) = \emptyset \text{ and } \text{dom}(v) = \Sigma^* \text{ are aperiodic languages.}\]

\[C = L ? C_1 : C_2. \text{ By induction, } \text{dom}(C_1) \text{ and } \text{dom}(C_2) \text{ are aperiodic. We have } \text{dom}(C) = (L \cap \text{dom}(C_1)) \cup (\text{dom}(C_2) \setminus L), \text{ which is aperiodic thanks to the closure properties of aperiodic languages.}\]

\[C = C_1 \circ C_2. \text{ By induction, } \text{dom}(C_1) \text{ and } \text{dom}(C_2) \text{ are aperiodic. We deduce that } \text{dom}(C) = \text{dom}(C_1) \cap \text{dom}(C_2) \text{ is aperiodic.}\]

\[C = C_1 \cdot C_2. \text{ By induction, } L_1 = \text{dom}(C_1) \text{ and } L_2 = \text{dom}(C_2) \text{ are aperiodic. We have } \text{dom}(C) \subseteq \text{dom}(C_1) \cdot \text{dom}(C_2). \text{ However, } C \text{ is undefined on words having more than one}\]
Hadamard product. A word which admits at least two decompositions can be written \(ww\) with \(v \neq \varepsilon, u, w \in L_1\) and \(vw, w \in L_2\). Let \(\varphi: \Sigma^* \to M\) be a morphism to a finite aperiodic monoid recognising both \(L_1\) and \(L_2\). We have \(L_1 = \varphi^{-1}(P_1)\) and \(L_2 = \varphi^{-1}(P_2)\) for some \(P_1, P_2 \subseteq M\). The set \(L_3\) of words having at least two decompositions is precisely

\[
L_3 = \bigcup_{r,s,t \mid r,s \in P_1 \land s,t \in P_2} \varphi^{-1}(r)(\varphi^{-1}(s) \setminus \{\varepsilon\})\varphi^{-1}(t)
\]

which is aperiodic. We deduce that \(\text{dom}(C) = (L_1 \cdot L_2) \setminus L_3\) is aperiodic.

\[\begin{align*}
C &= [L, C']^{k*}.
\end{align*}\]

By induction, \(\text{dom}(C')\) is aperiodic and by definition \(L\) is an aperiodic SD prefix code. Hence \(L^*\) is aperiodic. Notice that \(\text{dom}(C) \subseteq L^*\) but \(C\) is undefined on words \(w = u_1 \cdots u_n\) with \(u_i \in L\) if there is a factor \(u_{i+1} \cdots u_{i+k}\) which is not in \(\text{dom}(C')\). We deduce that \(\text{dom}(C) = L^* \setminus (L^*(L^k \setminus \text{dom}(C'))L^*)\) which is aperiodic thanks to the closure properties given above.

Notice that \(\text{dom}([L, C']^{k*}) = \text{dom}([L, C']^{k*})\), which is aperiodic as proved above.

\begin{proposition}
Given an SDRTE \(C\) and a letter \(a \in \Sigma\),
\begin{enumerate}
\item we can construct an SDRTE \(a^{-1}C\) such that \(\text{dom}(a^{-1}C) = a^{-1}\text{dom}(C)\) and \([a^{-1}C](w) = [C](aw)\) for all \(w \in a^{-1}\text{dom}(C)\),
\item we can construct an SDRTE \(Ca^{-1}\) such that \(\text{dom}(Ca^{-1}) = \text{dom}(C)a^{-1}\) and \([Ca^{-1}](w) = [C](wa)\) for all \(w \in \text{dom}(C)a^{-1}\).
\end{enumerate}
\end{proposition}

\textbf{Proof.} We recall that aperiodic languages are closed under left and right quotients. The proof is by structural induction on the given SDRTE \(C\) over alphabet \(\Sigma\). We only construct below the SDRTEs for the left quotient. Formulas for the right quotient can be obtained similarly. A point to note is that, unlike the left quotient, the right quotient of a language might break its prefix code property, which could be a problem if applied to a parsing language \(L\) used for \(k\)-star or its reverse. However, the quotient by a letter only modifies the first or last copy of \(L\), which can be decoupled so that the remaining iterations are still performed with the same parsing language \(L\).

\textbf{Basic cases.} We define \(a^{-1}_{\bot} = \bot\) and \(a^{-1}v = v\) for \(v \in \Gamma^*\).

\textbf{If-then-else.} Let \(C = L? C_1 : C_2\). We define \(a^{-1}_C = a^{-1}L? a^{-1}C_1 : a^{-1}C_2\).

Recall that \(\text{dom}(C) = (\text{dom}(C_1) \cap L) \cup (\text{dom}(C_2) \setminus L)\). We deduce that

\[
a^{-1}\text{dom}(C) = ((a^{-1}\text{dom}(C_1)) \cap (a^{-1}L)) \cup ((a^{-1}\text{dom}(C_2)) \setminus (a^{-1}L))
\]

Moreover, for \(w \in a^{-1}\text{dom}(C)\), we have

\[
[C](aw) = \begin{cases}
[C_1](aw) & \text{if } aw \in L \\
[C_2](aw) & \text{otherwise.}
\end{cases}
= \begin{cases}
[a^{-1}C_1](w) & \text{if } w \in a^{-1}L \\
[a^{-1}C_2](w) & \text{otherwise.}
\end{cases}
= [a^{-1}L ? a^{-1}C_1 : a^{-1}C_2](w)
\]

\textbf{Hadamard product.} Let \(C = C_1 \odot C_2\). We define \(a^{-1}_C = a^{-1}C_1 \odot a^{-1}C_2\).

Recall that \(\text{dom}(C) = \text{dom}(C_1) \cap \text{dom}(C_2)\). We deduce that

\[
a^{-1}\text{dom}(C) = (a^{-1}\text{dom}(C_1)) \cap (a^{-1}\text{dom}(C_2)) = \text{dom}(a^{-1}C_1 \odot a^{-1}C_2)
\]

Moreover, for \(w \in a^{-1}\text{dom}(C)\), we have

\[
[C](aw) = [C_1](aw)[C_2](aw) = [a^{-1}C_1](w)[a^{-1}C_2](w) = [a^{-1}C_1 \odot a^{-1}C_2](w)
\]
Cauchy product. Let $C = C_1 \cdot C_2$. The SDRE $a^{-1}C$ is the unambiguous sum of two expressions depending on whether the letter $a$ is removed from $C_1$ or from $C_2$. Hence, we let $C' = (a^{-1}C_1) \cdot C_2$ and $C'' = (\varepsilon \bowtie [C_1][\varepsilon]) \cdot (a^{-1}C_2)$. Notice that $\text{dom}(C'') = \emptyset$ when $\varepsilon \not\in \text{dom}(C_1)$, i.e., $[C_1][\varepsilon] = \emptyset$. Now, we define $a^{-1}C = (a^{-1}\text{dom}(C)) \bowtie (C' + C'')$.

Let $w \in a^{-1}\text{dom}(C)$. Then $aw$ admits a unique factorization $aw = wu$ with $u \in \text{dom}(C_1)$ and $v \in \text{dom}(C_2)$. There are two exclusive cases.

If $u \neq \varepsilon$ then $w = au'$ with $u' \in a^{-1}\text{dom}(C_1)$. The word $w$ admits a unique factorization according to $\text{dom}(a^{-1}C_1)\text{dom}(C_2)$ which is $w = au'v$. Hence, $w \in \text{dom}(C')$ and

$$[C](aw) = [C_1](u)[C_2](v) = [a^{-1}C_1](u')[C_2](v) = [C'](w).$$

If $u = \varepsilon$ then $v = av'$ and $v' \in a^{-1}\text{dom}(C_2)$. The word $w = v'$ admits a unique factorization according to $\{\varepsilon\} \cdot \text{dom}(a^{-1}C_2)$ which is $w = \varepsilon : w$. Hence, $w \in \text{dom}(C'')$ and

$$[C](aw) = [C_1](\varepsilon)[C_2](v) = [C_1](\varepsilon)[a^{-1}C_2](w) = [C''](w).$$

We deduce that $a^{-1}\text{dom}(C) \subseteq \text{dom}(C') \cup \text{dom}(C'') = \text{dom}(C' + C'')$ and $\text{dom}(a^{-1}C) = a^{-1}\text{dom}(C)$ as desired.

Finally, assume that $w \in \text{dom}(C') \cap \text{dom}(C'')$. Then, $w$ admits two factorizations $w = u'v = uv'$ with $u' \in \text{dom}(a^{-1}C_1)$, $v \in \text{dom}(C_2)$, $\varepsilon \in \text{dom}(C_1)$ and $v' \in \text{dom}(a^{-1}C_2)$. We deduce that $aw$ admits two distinct factorizations $aw = (aw')v = \varepsilon(aw')$ with $aw', \varepsilon \in \text{dom}(C_1)$ and $v, aw' \in \text{dom}(C_2)$. This is a contradiction with $aw \in \text{dom}(C)$.

We deduce that in both cases above, we have

$$[C](aw) = [[a^{-1}\text{dom}(C)) \bowtie (C' + C'')](w).$$

$k$-star. Let $L \subseteq \Sigma^*$ be an aperiodic prefix code with bounded synchronisation delay and let $C$ an SDRE. Notice that, since $L$ is a code, $\varepsilon \notin L$. Also, $a^{-1}\text{dom}([L, C]^{k}) \subseteq a^{-1}L^* = (a^{-1}L)L^*$. Let $w \in a^{-1}L^*$. It admits a unique factorization $w = u_1u_2\cdots u_n$ with $u_1 = au'_1 \in L$ and $u_2, \ldots, u_n \in L$. The unique factorization of $aw$ according to the code $L$ is $aw = u_1u_2\cdots u_n$.

Now, by definition of $k$-star, when $n < k$ we have $[[L, C]^{k}](aw) = \varepsilon$. Hence, we let $C' = ((a^{-1}L) \cdot L^{k-1}) \bowtie \varepsilon$ so that in the case $n < k$ we get

$$[[L, C]^{k}](aw) = \varepsilon = [C'](w).$$

Next we assume that $n \geq k$. We define two SDREs:

$$C'' = ((a^{-1}L) \cdot L^{k-1}) \bowtie a^{-1}C \cdot (L^* \bowtie \varepsilon)$$

$$C''' = (a^{-1}L \bowtie \varepsilon) \cdot [L, C]^{k}$$

Notice that $L^*$ is aperiodic since $L$ is an aperiodic prefix code with bounded synchronisation delay. Hence, $C''$ is indeed an SDRE. We get

$$[C''](w) = [a^{-1}C](u_1u_2\cdots u_k) = [C](u_1u_2\cdots u_k)$$

$$[C'''](w) = [C](u_2\cdots u_{k+1})[C](u_3\cdots u_{k+2})\cdots [C](u_{n-k+1}\cdots u_n)$$

$$[[L, C]^{k} \bowtie C'''](w) = [[L, C]^{k}](aw).$$

Therefore, we define the SDRE

$$a^{-1}([[L, C]^{k} \bowtie \varepsilon]) = (a^{-1}L^*) \bowtie (C' + (C'' \bowtie C''')).$$

Notice that $\text{dom}(C') = a^{-1}L^{k}$ and $\text{dom}(C'' \bowtie C''') \subseteq a^{-1}L^{2k}$ are disjoint.
Reverse \( k\)-star. This case is similar. Let \( L \subseteq \Sigma^* \) be an aperiodic prefix code with bounded synchronisation delay and let \( C \) an SDRT\(E\). We define
\[
\alpha^{-1}([L,C]_r^{k*}) = (\alpha^{-1}L^*) \triangleright (C' + (C'^m \odot C''))
\]
where \( C', C'' \) are as above and \( C'^m = (\alpha^{-1}L \triangleright \varepsilon) \cdot [L,C]_r^{k*} \).

▶ Lemma 16. Given an SDRT\(E\) \( C \) over an alphabet \( \Sigma \) and a sub-alphabet \( \Sigma' \subseteq \Sigma \), we can construct an SDRT\(E\) \( C' \) over alphabet \( \Sigma' \) such that \( \text{dom}(C') \subseteq \Sigma'^* \) and for any word \( w \) in \( \Sigma'^* \), \( [C]_r(w) = [C']_r(w) \).

Proof. The proof itself is rather straightforward, and simply amounts to get rid of letters that do not appear in \( \Sigma' \). We first construct \( C' \) by structural induction from \( C \), and then prove that it is indeed an SDRT\(E\). Thus \( C' \) is defined as follows:
- if \( C = \perp \) then \( C' = \perp \),
- if \( C = v \) then \( C' = v \), with \( \text{dom}(v) = \Sigma'^* \) here since \( C' \) is over \( \Sigma' \),
- if \( C = L \cdot C_1 ; C_2 \) then \( C' = (L \cap \Sigma'^*) \cdot C'_1 ; C'_2 \),
- if \( C = C_1 \circ C_2 \) then \( C' = C'_1 \odot C'_2 \),
- if \( C = C_1 \cdot C_2 \) then \( C' = C'_1 \odot C'_2 \),
- if \( C = [L,C]_r^{k*} \) then \( C' = [L \cap \Sigma'^*,C'_1]_r^{k*} \),
- if \( C = [L,C]_r^{k*} \) then \( C' = [L \cap \Sigma'^*,C'_1]_r^{k*} \).

To prove that \( C' \) is SD-regular, we construct, given an SD-expression \( E \) for \( L \) over \( \Sigma \), an SD-expression \( E' \) over \( \Sigma' \) for \( L \cap \Sigma'^* \). Again, the proof is an easy structural induction:
- if \( E = \emptyset \) then \( E' = \emptyset \),
- if \( E = a \in \Sigma' \) then \( E' = a \),
- if \( E = a \in \Sigma \setminus \Sigma' \) then \( E' = \emptyset \),
- if \( E = E_1 + E_2 \) then \( E' = E'_1 + E'_2 \),
- if \( E = E_1 \cdot E_2 \) then \( E' = E'_1 \cdot E'_2 \),
- if \( E = E_1^* \) then \( E' = E'_1^* \).

We conclude by stating that being a prefix code with bounded synchronisation delay is a property preserved by subsets, hence \( E' \) is an SD-expression.

4.7 Can the 2-chained Kleene star suffice for all aperiodic functions?

It is known [11] that the 2-chained Kleene star can simulate the \( k \)-chained Kleene-star for regular functions. However, we believe that, contrary to the case of regular functions, the \( k \)-chained Kleene-star operator cannot be simulated by the 2-chained Kleene-star while preserving the aperiodicity of the expression. The key idea is that, in order to simulate a \( k \)-chained Kleene-star on a SD prefix code \( L \) using a 2-chained Kleene-star, one needs to use \( L^{[k/2]} \) as a parser. However, for any given prefix code \( L \), the language \( L^n \) for \( n \geq 1 \), while still a prefix code, is not of bounded synchronisation delay (for the same reason that \( \{aa\} \) is not, i.e., for \( v = (aa)^d \) that we consider, \( ava \) belongs to \( (aa)^* \) but \( av \) does not). Intuitively, parsing \( L^n \) reduces to counting factors of \( L \) modulo \( n \), which is a classical example of non-aperiodicity.

As an example, consider the prefix code \( L = (a + b)^* c \) which has synchronisation delay 1. Define a function \( f \) with domain \( L^3 \) by \( f(u_1 u_2 u_3) = u_3 u_1 \) when \( u_1, u_2, u_3 \in L \), which can be written using combinators as \( (L^2 \triangleright \varepsilon) \cdot (L \triangleright \text{id}) \circ ((L \triangleright \text{id}) \cdot (L^2 \triangleright \varepsilon)) \). The identity function \( \text{id} \) can itself be written as \( (a + a + b + b + c + c)^* \) (see also Figure 2 which is a simplification of the same function, but nevertheless has the same inexpressiveness with 2 chained star). Then
we believe that the function \([L, f]^{k*}\), which associates to a word \(u_1 \cdots u_n \in L^*\) the word \(u_1 u_2 u_3 \cdots u_n u_{n-2}\) is not definable using only 2-chained Kleene-stars. While not a proof, the intuition behind this is that, in order to construct \(u_1 u_{n-1} u_{n-2}\), we need to highlight words from \(L^3\). In order to do this with a 2-chained Kleene-star, it seems necessary to apply a chained star with parser \(L^2\), which is a prefix code but not of bounded synchronisation delay. A similar argument would hold for any \([L, f]^{k*}\), \(k \geq 3\) with a function \(f(u_1 u_2 \cdots u_k) = u_k u_1\).

## 5 The Equivalence of SDRTE and Aperiodic 2DFT

In this section, we prove the main result of the paper, namely the equivalence between SDRTE and aperiodic 2DFT stated in Theorem 1. The first direction, given an SDRTE \(C\), constructing an equivalent aperiodic 2DFT \(A\) is given by Theorem 17, while Theorem 24 handles the converse.

### 5.1 Aperiodic 2DFTs for SD-regular transducer expressions

> **Theorem 17.** Given an SDRTE \(C\), we can construct an equivalent aperiodic 2DFT \(A\) with \([C] = [A]\).

**Proof.** We construct \(A\) by induction on the structure of the SDRTE \(C\). In the suitable cases, we will suppose thanks to induction that we have aperiodic transducers \(A_i\) for expressions \(C_i\), \(i \leq 2\). We also have a deterministic and complete aperiodic automaton \(A_L\) for any aperiodic language \(L\).

- \(C = \perp\). Then \(A\) is a single state transducer with no final state so that its domain is empty.
- \(C = v\). Then \(A\) is a single state transducer which produces \(v\) and accepts any input word. Clearly, \(A\) is aperiodic.
- \(C = L ? C_1 : C_2\). The transducer \(A\) first reads its input, simulating \(A_L\). Upon reaching the end of the input word, it goes back to the beginning of the word, and either executes \(A_1\) if the word was accepted by \(A_L\), or executes \(A_2\) otherwise. Since every machine was aperiodic, so is \(A\).
- \(C = C_1 \circ C_2\). The transducer \(A\) does a first pass executing \(A_1\), then resets to the beginning of the word and simulates \(A_2\). Since both transducers are aperiodic, so is \(A\).
- \(C = C_1 \cdot C_2\). We express \(A\) as the composition of three functions \(f_1, f_2, f_3\), each aperiodic. Since aperiodic functions are closed under composition, we get the result. The first function \(f_1\) associates to each word \(w \in \Sigma^*\) the word \(u_1 \# u_2 \# \cdots \# u_n\), such that \(w = u_1 u_2 \cdots u_n\) and for any prefix \(u\) of \(w\), \(u\) belongs to the domain of \(C_1\) if, and only if, \(u = u_1 \cdots u_i\), for some \(1 \leq i < n\). Notice that \(u_1 = \varepsilon\) iff \(\varepsilon \in \text{dom}(C_1)\) and \(u_n = \varepsilon\) iff \(w \in \text{dom}(C_1)\). The other \(u_i\)'s must be nonempty. The second function \(f_2\) takes as input a word in \((\Sigma \cup \{\#\})^*\), reads it from right to left, and suppresses all \# symbols except for the ones whose corresponding suffix belongs to the domain of \(C_2\). Then, \(f_2(f_1(w))\) contains exactly one \# symbol if and only if \(w\) has a unique factorisation \(w = u \# v\) with \(u \in \text{dom}(C_1)\) and \(v \in \text{dom}(C_2)\). In this case, \(f_2(f_1(w)) = u \# v\).

Finally, the function \(f_3\) has domain \(\Sigma^* \# \Sigma^*\) and first executes \(A_1\) on the prefix of its input up to the \# symbol, treating it as the right endmarker \(\dagger\), and then executes \(A_2\) on the second part, treating \# as the left endmarker \(*\). The functions \(f_1\) and \(f_2\) can be realised by aperiodic transducers as they only simulate automata for the aperiodic domains of \(C_1\) and the reverse of \(C_2\) respectively, and the function \(f_3\) executes \(A_1\) and \(A_2\) one after the other, and hence is also aperiodic.
\( C = |L, C_1|^{k*} \) or \( C = [L, C_1]^{k*} \). Here \( L \subseteq \Sigma^* \) is an aperiodic language which is also a prefix code with bounded synchronisation delay, and \( k \geq 1 \) is a natural number. Let \( f = |C_1| : \Sigma^* \to \Gamma^* \) be the aperiodic function defined by \( C_1 \). We write \( [L, f]^{k*} = L^{ck} ? (\Sigma^* \triangleright \varepsilon) : (f_3 \circ f_2 \circ f_1) \), where \( \varepsilon \) is the output produced when the input has less than \( k \) \( L \) factors; otherwise the output is produced by the composition of 3 aperiodic functions. As aperiodic functions are closed under composition, this gives the result. The first one, \( f_1 : \Sigma^* \to (\Sigma \cup \{\#\})^* \) splits an input word \( w \in L^* \) according to the unique factorization \( w = u_1 u_2 \cdots u_n \) with \( n \geq 0 \) and \( u_i \in L \) for all \( 1 \leq i \leq n \) and inserts \( \# \) symbols: \( f_1(w) = \#u_1 \#u_2 \cdots \#u_n \# \). The domain of \( f_1 \) is \( L^* \).

The second function \( f_2 \) constructs the sequence of \( k \) factors. Its domain is \( (\Sigma^* \#)^{\geq k} \) and it is defined as follows, with \( u_i \in \Sigma^* \):

\[
f_2(\#u_1 \#u_2 \cdots \#u_n \#) = \#u_1 u_2 \cdots u_k \#u_{k+1} \cdots \#u_{n-k+1} \cdots u_n \#.
\]

Finally, the third function simply applies \( f \) and erases the \( \# \) symbols:

\[
f_3((\#v_1 \#v_2 \cdots \#v_m \#)) = f(v_1) f(v_2) \cdots f(v_m).
\]

In particular, \( f_3(\#) = \varepsilon \). We have \( \text{dom}(f_3) = \#(\text{dom}(f) \#)^* \).

For the reverse iteration, we simply change the last function and use instead

\[
f_4((\#v_1 \#v_2 \cdots \#v_m \#)) = f(v_m) \cdots f(v_2) f(v_1).
\]

Lemma 18 below proves that the functions \( f_i \) for \( i \leq 4 \) are aperiodic.

\[ \blacktriangleleft \]

**Lemma 18.** The functions \( f_1, f_2, f_3, f_4 \) are realised by aperiodic 2DFTs.

**Proof.** The function \( f_1 \). First, since \( L \) is an aperiodic language which is a prefix code with bounded synchronisation delay, \( L^* \) is aperiodic. Let \( A_1 \) be a deterministic automaton that recognizes \( L^* \). Let \( w \) be a word in \( L^* \) and \( w = u_1 \cdots u_n \) with \( u_i \in L \). Since \( L \) is a code, this decomposition is unique. Notice that \( \varepsilon \notin L \). We claim that the run of \( A_1 \) over \( w \) reaches final states exactly at the end of each \( u_i \). Should this hold, then we can easily construct a (one-way) aperiodic transducer \( T_1 \) realising \( f_1 \) by simply simulating \( A_1 \) and copying its input, adding \( \# \) symbols each time \( A_1 \) reaches a final state.

It remains to prove the claim. First, since for any \( 1 \leq i \leq n \), \( u_1 \cdots u_i \) belongs to \( L^* \), \( A_1 \) reaches a final state after reading \( u_i \). Conversely, suppose \( A_1 \) reaches a final state after reading some nonempty prefix \( v \) of \( w \). Then \( v \) can be written \( u_i \cdots u_i' \) for some index \( 0 \leq i < n \) and some nonempty prefix \( u_i' \) of \( u_{i+1} \). But since \( A_1 \) reaches a final state on \( v \), we have \( v \in L^* \). Hence, there is a unique decomposition \( v = v_1 \cdots v_m \) with \( v_j \in L \). Since
we get

Then we solve non-determinism by duplicating the problematic states and 

endmarker. Then we solve non-determinism by duplicating the problematic states 

endmarker can only be reached while moving to the left, and symmetrically for the right 

reading a symbol, or at least \( \# \) symbol, the only traversals it 

to state \( \text{end} \) to check if it was the last \( \# \) symbol, and otherwise reads 

back the last \( k - 1 \# \) symbols and starts again.

Let us prove the aperiodicity of \( T_2 \). First, notice that the \( \triangleright \) and \( \subset \) are always aperiodic 

relations, for any finite 2DFT. This is due to the fact that if a \( \triangleright \) step exists in some \( u \neq \varepsilon \), it 

also appears in \( uv \). So for \( (v^n)_{n \geq 0} \), the \( \triangleright \) and \( \subset \) relations are monotone, and since we consider 

finite state machines, they eventually stabilize. So we turn to traversal steps. These traversal 

steps only depend on the number of \# symbols in the word, as well as the starting and 

ending symbols. In particular, if a word \( v \) has \( k + 1 \) or more \# symbols, the only traversals it 

realises are in \( \{ \rightarrow, 0, k \}, \{ \rightarrow, 0, \text{end} \}, \{ \rightarrow, 1, k \}, \{ \rightarrow, 1, \text{end} \} \), starting from 0 is possible only if 

\( v \) starts with \#, the target state is \( \text{end} \) if the last letter of \( v \) is \#, otherwise it is \( k \). Notice 

that both \( \{ \rightarrow, 0, \text{end} \} \) and \( \{ \rightarrow, 1, \text{end} \} \) are possible if \( v \in \#(\Sigma^*)^{k} \Sigma^+ \). Similarly, both \( \{ \rightarrow, 0, k \} \) and 

\( \{ \rightarrow, 1, k \} \) are possible if \( v \in \#(\Sigma^*)^{k+1} \Sigma^+ \). Then given any word \( v \in (\Sigma \cup \{ \# \})^+ \), both 

\( v^{k+1} \) and \( v^{k+2} \) have either no \# symbol, or at least \( k + 1 \# \) symbols; further they have the 

same starting and ending letters. Thus they realise the same steps.

The function \( f_3 \). The goal of \( f_3 \) is to iteratively simulate \( f \) on each factor appearing 

between \# symbols. To this end, \( T_3 \) is defined as the transducer \( T \) realising \( f \), with the 

exception that it reads the \# symbols as endmarkers, and upon reaching a final state while 

reading a \# symbol, it first checks if the next symbol is \( \varepsilon \), and in this case ends the run, or 

simulates the move of \( T \) reading \( \top \) from the initial state. Note that \# being used for both 

endmarkers could generate some non-determinism, however this can be avoided as the left 

endmarker can only be reached while moving to the left, and symmetrically for the right 

endmarker. Then we solve non-determinism by duplicating the problematic states \( q \) to states 

\( q_0 \) (where \# is seen as the left endmarker) and \( q_r \) (where \# is seen as a right endmarker), which 

can only be reached while moving to the left or the right respectively.

We now turn to the aperiodicity of \( T_3 \). If the input word \( v \) does not contain any \# symbol, 

then the \( (\triangleright, \subset, \rightarrow, \leftarrow) \)-runs of \( v^n \) are the same as the ones in \( T \), and since \( T \) is aperiodic then 

we get \( \varphi(v^n) = \varphi(v^{n + 1}) \) for some \( n \), where \( \varphi \) is the syntactic morphism of \( T \).
Otherwise, let us remark that by design, once the reading head has gone right of a given # symbol, it never goes back to its left, and secondly the behavior of $T_3$ when going from left to right of a # symbol is always the same since it simulates the initial transition of $T$. So given a word $v$ with at least one # symbol, let $u_1$ and $u_2$ be the prefix and suffix of $v$ up to the first and from the last # respectively, i.e., $v = u_1#u_1\cdots u_m#u_2$ with $m \geq 0$ and $u_1, u_2, w_1, \ldots, w_m \in \Sigma^*$. Then there exists no $\leftarrow$ traversal of $v^n$ for $n \geq 2$ since the reading head cannot move from right to left of a # symbol. The $\rightarrow$ traversals of $v^n$, for $n \geq 2$, exist if and only if $u_2u_1, w_1, \ldots, w_m$ belong to the domain of $T$, and consist of all $(\rightarrow, p, q)$, where $\varphi(u_1)$ contains $(\rightarrow, p, f)$ for some final state $f$, and $\varphi(u_2)$ contains $(\rightarrow, \iota, q)$ where $\iota$ is the initial state. These traversals are then the same for $v^2$ and $v^3$, which concludes the proof of aperiodicity of $T_3$.

**The function** $f_4$. The transducer $T_4$ realising $f_4$ is similar to $T_3$. The main difference is that it starts by reaching the end of the word, then goes back to the previous # symbol to simulate $T$. On reaching the end of the run in $T$ (in a final state of $T$ when reading #) it treats # as $\perp$ and then enters into a special state which moves the reading head to the left, till the time it has finished reading two # symbols, while outputting $\varepsilon$ all along. When it reads the second #, it moves right entering the state $T$ would, on reading $\perp$ from its initial state, and continues simulating $T$. This goes on until it reaches the start symbol $\perp$, and then it goes to the final state of $T_3$ that only moves to the right outputting $\varepsilon$ all along until the end of the input to $\perp$.

The arguments for the aperiodicity of $T_4$ are similar to the ones for $T_3$. ▶

### 5.2 SD-regular transducer expressions for aperiodic 2DFTs

In this section, we show that the runs of an aperiodic 2DFT have a “stabilising” property. This property crucially distinguishes aperiodic 2DFTs from non aperiodic ones, and we use this in our proof to obtain SDRTEs from aperiodic 2DFTs. In the remainder of this section, we fix an aperiodic 2DFT $A = (Q, \Sigma, \Gamma, \delta, \gamma, q_0, F)$. Let $\varphi: (\Sigma \cup \{\perp, \perp\})^* \rightarrow \text{TrM}$ be the canonical surjective morphism to the transition monoid of $A$.

#### 5.2.1 Stabilising runs in two-way automata

Consider a code $L \subseteq \Sigma^*$ such that $X = \varphi(L)$ is $k$-stabilizing for some $k > 0$. We will see that a run of $A$ over a word $w \in L^*$ has some nice properties. Intuitively, if it moves forward through $k$ factors from $L$ then it never moves backward through more than $k$ factors.

More precisely, let $w = u_1u_2\cdots u_n$ be the unique factorisation of $w \in L^*$ with $u_i \in L$ for $1 \leq i \leq n$. We assume that $n \geq k$. We start with the easiest fact.

► **Lemma 19.** If $(\iota,p,q) \in \varphi(w)$ then the run of $A$ over $w$ starting on the left in state $p$ only visits the first $k$ factors $u_1\cdots u_k$ of $w$.

**Proof.** Since $X$ is $k$-stabilising, we have $\varphi(w) = \varphi(u_1\cdots u_k)$. Hence, $(\iota,p,q) \in \varphi(u_1\cdots u_k)$ and the result follows since $A$ is deterministic. ▶

Notice that the right-right $(\subseteq)$ runs of $A$ over $w$ need not visit the last $k$ factors only (see Lemma 22 below). This is due to the fact that stabilising is not a symmetric notion.

Next, we consider the left-right runs of $A$ over $w$.

► **Lemma 20.** Assume that $(\rightarrow,p,q) \in \varphi(w)$. Then the run $\rho$ of $A$ over $w$ starting on the left in state $p$ has the following property, that we call $k$-forward-progressing: for each $1 \leq i < n-k$, after reaching the suffix $u_{i+k+1}\cdots u_n$ of $w$, the run $\rho$ will never visit again the prefix $u_1\cdots u_i$. See Figure 4 for a non-example and Figure 10 for an example.
Proof. Towards a contradiction, assume that for some $1 \leq i < n-k$, the run $\rho$ visits $u_1 \cdots u_i$ after visiting $u_{i+k+1} \cdots u_n$ (See Figure 4). Then, there exists a subrun $\rho'$ of $\rho$ making some $(\Rightarrow, q_1, q_3)$-step on $u_{i+1} \cdots u_n$ and visiting $u_{i+k+1}$ (on Figure 4 we have $\rho' = \rho_2\rho_3$). Hence $(\Rightarrow, q_1, q_3) \in \varphi(u_{i+1} \cdots u_n)$ and by Lemma 19 we deduce that $\rho'$ visits $u_{i+1} \cdots u_{i+k}$ only, a contradiction.

Lemma 21. Assume that $(\leftarrow, p, q) \in \varphi(w)$. Then the run $\rho$ of $A$ over $w$ starting on the right in state $p$ has the following property, that we call $k$-backward-progressing: for each $1 \leq i < n-k$, after reaching the prefix $u_1 \cdots u_i$ of $w$, the run $\rho$ will never visit again the suffix $u_{i+k+1} \cdots u_n$.

Proof. This Lemma is a consequence of Lemma 19. Indeed, consider any part of $\rho$ that visits $u_{i+1}$ again (in some state $q_1$) after visiting $u_i$, for some $1 \leq i < n-k$. As $\rho$ is a $\leftarrow$-run, it will later cross from $u_{i+1}$ to $u_i$ (reaching some state $q_3$). Then $(\Rightarrow, q_1, q_3)$ is a run on $u_{i+1} \cdots u_n$. By Lemma 19 it does not visit $u_{i+k+1} \cdots u_n$, which concludes the proof (See Figure 5 for a non-example).

Lemma 22. Assume that $(\subseteq, p, q) \in \varphi(w)$ and let $\rho$ be the run of $A$ over $w$ starting on the right in state $p$. Then, either $\rho$ visits only the last $k$ factors $u_{n-k+1} \cdots u_n$, or for some $1 \leq i \leq n-k$ the run $\rho$ is the concatenation $\rho_1\rho_2\rho_3$ of a $k$-backward-progressing run $\rho_1$ over $u_1 \cdots u_i$ followed by a run $\rho_2$ staying inside some $u_i \cdots u_{i+k}$, followed by some $k$-forward-progressing run $\rho_3$ over $u_{i+1} \cdots u_n$. See Figure 6.

Proof. Assume that $\rho$ visits $u_1 \cdots u_{n-k}$ and let $u_i$ ($1 \leq i \leq n-k$) be the left-most factor visited by $\rho$. We split $\rho$ in $\rho_1\rho_2\rho_3$ (see Figure 6) where $\rho_1$ is the prefix of $\rho$, starting on the right of $w$ in state $p$ and going until the first time $\rho$ crosses from $u_{i+1}$ to $u_i$. Hence, $\rho_1$ is a run over $u_{i+1} \cdots u_n$ starting on the right in...
state \( p \) and exiting on the left in some state \( q_1 \). We have \( ([→, p, q_1]) \in \varphi(u_{i+1} \cdots u_n) \). By Lemma 21 we deduce that \( ρ_1 \) is \( k \)-backward-progressing.

- Then, \( ρ_2 \) goes until the last crossing from \( u_i \) to \( u_{i+1} \).

- Finally, \( ρ_3 \) is the remaining suffix of \( ρ \). Hence, \( ρ_3 \) is a run over \( u_{i+1} \cdots u_n \) starting on the left in some state \( q_2 \) and exiting on the right in state \( q \). We have \( (→, q_2, q) \in \varphi(u_{i+1} \cdots u_n) \).

By Lemma 20 we deduce that \( ρ_3 \) is \( k \)-forward-progressing.

It remains to show that \( ρ_2 \) stays inside \( u_i \cdots u_{i+k} \). Since \( u_i \) is the left-most factor visited by \( ρ \), we already know that \( ρ_2 \) does not visit \( u_1 \cdots u_{i-1} \). Similarly to Lemma 21 any maximal subrun \( ρ'_2 \) of \( ρ_2 \) that does not visit \( u_i \) is a \( \vartriangleright \) run on \( u_{i+1} \cdots u_n \). Since \( ρ_2 \) starts and ends at the frontier between \( u_i \) and \( u_{i+1} \). By Lemma 19 the subrun \( ρ'_2 \) does not visit \( u_{i+k+1} \cdots u_n \) and thus \( ρ_2 \) stays inside \( u_i \cdots u_{i+k} \).

**Example 23.** We illustrate the stabilising runs of an aperiodic 2DFT using the aperiodic 2DFT \( \mathcal{A} \) in Figure 2. Figure 6 depicts the run of \( \mathcal{A} \) on words in \( b(a\ast b)^{\geq 3} \). We use the set \( Z_1 \) computed in Example 9. Notice that a run of \( \mathcal{A} \) on such words is \( 4 \)-forward-progressing, as seen below. For each \( w = u_1u_2 \cdots u_n \) with \( n > 3 \), \( u_1 = b \) and \( u_i \in a^+b \) for \( 2 \leq i \leq n \), we have \( \varphi(w) = Z_1 \) and one can see that

- each \( (\vartriangleright, p, q) \in Z_1 \), is such that, whenever the run of \( \mathcal{A} \) starts at the left of \( w \) in state \( p \), it stays within \( u_1 \cdots u_4 \) and never visits \( u_5 \cdots u_n \) (as in Lemma 19).

- each \( (→, p, q) \in Z_1 \), is such that, whenever the run of \( \mathcal{A} \) starts at the left of \( w \) in state \( p \) and reaches \( u_{i+5} \), for \( i \geq 1 \), it no longer visits any of \( u_1 \cdots u_i \) (4-forward-progressing as in Lemma 20).

- each \( (\varleftarrow, p, q) \in Z_1 \), is such that, whenever the run of \( \mathcal{A} \) starts at the right of \( w \) in state \( p \), it never visits \( u_1 \cdots u_{n-4} \) (the easy case of Lemma 22).

### 5.2.2 Computing SDRTE

In this section, we show how to construct SDRTEs which are equivalent to aperiodic 2DFTs. Recall that \( \varphi : (Σ \cup \{+, −\})^* \to \text{TrM} \) is the canonical surjective morphism to the transition monoid of the 2DFT \( \mathcal{A} = (Q, Σ, Γ, δ, γ, q_0, F) \). Given a regular expression \( E \) and a monoid element \( s \in \text{TrM} \), we let \( \mathcal{L}(E, s) = \mathcal{L}(E) \cap \varphi^{-1}(s) \). The main construction of this section is given by Theorem 24.

Recall that \( \text{TrM} \) represents the transition monoid of a 2DFT, and consists of elements \( \varphi(w) \) for all \( w \in Σ^* \), where each \( \varphi(w) = \{(d, p, q) \mid \text{there is a } (d, p, q)\text{-run on } w \} \subseteq \{→, ≥, ≤\} \times Q^2 \). The elements of \( \varphi(w) \) are called steps, since any run of \( w \) is obtained by a sequence of such steps. If the states \( p, q \) in a step \( (d, p, q) \) are clear from the context, or is
immaterial for the discussion we also refer to a step as a $d$ step, $d \in \{\rightarrow, \leftarrow\}$. In this case we also refer to a step $(d, p, q)$ as a $d$ step having $p$ as the starting state and $q$ as the final state.

**Theorem 24.** Let $E$ be an unambiguous, stabilising, SD-regular expression over $\Sigma \uplus \{\uparrow, \downarrow\}$ and let $s \in \text{TrM}$. For each step $x \in \{\rightarrow, \leftarrow\} \times Q^2$, we can construct an SDRTE $C_{E,s}(x)$ such that:

1. $C_{E,s}(x) = \perp$ when $x \not\in s$, and otherwise
2. $\text{dom}([C_{E,s}(x)]) = \mathcal{L}(E,s)$ and for all words $w \in \mathcal{L}(E,s)$, $[C_{E,s}(x)](w)$ is the output produced by $A$ running over $w$ according to step $x$.

When $w = \epsilon$ and $s = 1 = \varphi(\epsilon)$ with $x \in 1$, this means $[C_{E,s}(x)](\epsilon) = \epsilon$.

**Proof.** The construction is by structural induction on $E$.

**Atomic expressions**

We first define $C_{E,s}(x)$ when $E$ is an atomic expression, i.e., $\emptyset$, $\epsilon$ or $a$ for $a \in \Sigma$.

- $E = \emptyset$: we simply set $C_{\emptyset,s}(x) = \perp$, which is the nowhere defined function.
- $E = \epsilon$: when $s = 1$ and $x \in s$ then we set $C_{\epsilon,s}(x) = \epsilon \uparrow \epsilon$ and otherwise we set $C_{\epsilon,s}(x) = \perp$.
- $E = a \in \Sigma \uplus \{\uparrow, \downarrow\}$: again, we set $C_{a,s}(x) = \perp$ if $s \neq \varphi(a)$ or $x \not\in s$. Otherwise, there are two cases. Either $x \in \{(\rightarrow, p, q), (\leftarrow, p, q)\}$ for some states $p, q$ such that $\delta(p, a) = (q, +1)$, or $x \in \{(\leftarrow, p, q), (\rightarrow, p, q)\}$ for some states $p, q$ with $\delta(p, a) = (q, -1)$. In both cases the output produced is $\gamma(p, a)$ and we set $C_{a,s}(x) = a \rightarrow \gamma(p, a)$.

**Disjoint union**

If the expression is $E \cup F$ with $\mathcal{L}(E)$ and $\mathcal{L}(F)$ disjoint, then we simply set $C_{E \cup F,s}(x) = C_{E,s}(x) + C_{F,s}(x)$.

**Unambiguous concatenation** $E \cdot F$

Here, we suppose that we have SDRTEs for $C_{E,s}(x)$ and $C_{F,s}(x)$ for all $s$ in $\text{TrM}$ and all steps $x \in \{\rightarrow, \leftarrow\} \times Q^2$. We show how to construct SDRTEs for $C_{E \cdot F,s}(x)$, assuming that the
concatenation $\mathcal{L}(E) \cdot \mathcal{L}(F)$ is unambiguous.

A word $w \in \mathcal{L}(E \cdot F)$ has a unique factorization $w = uv$ with $u \in \mathcal{L}(E)$ and $v \in \mathcal{L}(F)$. Let $s = \varphi(u)$ and $t = \varphi(v)$. A run $\rho$ over $w$ is obtained by stitching together runs over $u$ and runs over $v$ as shown in Figure 8. In the left figure, the run over $w$ follows step $x = (\rightarrow, p, q)$ starting on the left in state $p$ and exiting on the right in state $q$. The run $\rho$ splits as $\rho_0 \rho_1 \rho_2 \rho_3 \rho_4 \rho_5$ as shown in the figure. The output of the initial part $\rho_0$ is computed by $C_{E,s}(\rightarrow, p_1)$ over $u$ and the output of the final part $\rho_5$ is computed by $C_{F,t}(\rightarrow, p_5, q)$ over $v$. We focus now on the internal part $\rho_1 \rho_2 \rho_3 \rho_4$ which consists of an alternate sequence of left-left runs over $u$ and right-right runs over $u$. The corresponding sequence of steps $x_1 = (\rightarrow, p_1, p_2) \in t$, $x_2 = (\leftarrow, p_2, p_3) \in s$, $x_3 = (\rightarrow, p_3, p_4) \in t$ and $x_4 = (\leftarrow, p_4, p_5) \in s$ depends only on $s = \varphi(u)$ and $t = \varphi(v)$.

These internal zigzag runs will be frequently used when dealing with concatenation or Kleene star. They alternate left-left (>) steps on the right word $v$ and right-right (<) steps on the left word $w$. They may start with a >-step or a <-step. The sequence of steps in a maximal zigzag run is entirely determined by the monoid elements $s = \varphi(u)$, $t = \varphi(v)$, the starting step $d \in \{>,<\}$ and the starting state $p'$ of step $d$. The final step of this maximal sequence is some $d' \in \{>,<\}$ and reaches some state $q'$. We write $Z_{s,t}(p', d) = (d', q')$. For instance, on the left of Figure 8, we get $Z_{s,t}(p_1, >) = (\leftarrow, p_2)$ whereas on the right of Figure 8 we get $Z_{s,t}(p_2, <) = (\rightarrow, p_3)$. By convention, if the sequence of zigzag steps is empty then we define $Z_{s,t}(p, >) = (\leftarrow, p)$ and $Z_{s,t}(p, <) = (\rightarrow, p)$.

**Lemma 25.** We use the above notation. We can construct SDRTEs $ZC_{E,s}^{F,t}(p, d)$ for $p \in Q$ and $d \in \{>, <\}$ such that $\text{dom}([ZC_{E,s}^{F,t}(p, d)]) = \mathcal{L}(E, s) \mathcal{L}(F, t)$ and for all $u \in \mathcal{L}(E, s)$ and $v \in \mathcal{L}(F, t)$ the value $[ZC_{E,s}^{F,t}(p, d)](uv)$ is the output produced by the internal zigzag run of $A$ over $(u, v)$ following the maximal sequence of steps starting in state $p$ with a $d$-step.

**Proof.** We first consider the case $d = >$ and $Z_{s,t}(p, >) = (\leftarrow, q)$ for some $q \in Q$ which is illustrated on the left of Figure 8. Since $A$ is deterministic, there is a unique maximal sequence of steps (with $n \geq 0$, $p_1 = p$ and $p_{2n+1} = q$): $x_1 = (\rightarrow, p_1, p_2) \in t$, $x_2 = (\leftarrow, p_2, p_3) \in s$, $x_{2n-1} = (\rightarrow, p_{2n-1}, p_{2n}) \in t$, $x_{2n} = (\leftarrow, p_{2n}, p_{2n+1}) \in s$. The zigzag run $\rho$ following this sequence of steps over $uv$ splits as $\rho_1 \rho_2 \cdots \rho_{2n}$ where $\rho_{2i}$ is the unique run on $u$ following step $x_{2i}$ and $\rho_{2i+1}$ is the unique run on $v$ following step $x_{2i+1}$. The output of these runs are given by $[C_{E,s}(x_{2i})](u)$ and $[C_{F,t}(x_{2i+1})](v)$. When $n = 0$ the zigzag run $\rho$ is empty and we simply set $ZC_{E,s}^{F,t}(p, >) = (\mathcal{L}(E, s) \mathcal{L}(F, t)) \circ \varepsilon$. Assume now that $n > 0$. The required SDRTE computing the output of $\rho$ can be defined as

$$ZC_{E,s}^{F,t}(p, >) = ((\mathcal{L}(E, s) \circ \varepsilon) \cdot C_{F,t}(x_1)) \circ (C_{E,s}(x_2) \cdot C_{F,t}(x_3)) \circ \cdots \circ (C_{E,s}(x_{2n-2}) \cdot C_{F,t}(x_{2n-1})) \circ (C_{E,s}(x_{2n}) \cdot (\mathcal{L}(F, t) \circ \varepsilon))$$.
Notice that each Cauchy product in this expression is unambiguous since the product \( \mathcal{L}(E) \cdot \mathcal{L}(F) \) is unambiguous.

The other cases can be handled similarly. For instance, when \( Z_{s,t}(p,\triangleright) = (\triangleright, q) \) as on the right of Figure 8, the sequence of steps ends with \( x_{2n-1} = (\triangleright, p_{2n-1}, p_{2n}) \in t \) with \( n > 0 \) and \( p_{2n} = q \) and the zigzag run \( \rho = \rho_1 \rho_2 \cdots \rho_{2n-1} \). The SDRTE \( Z_{E,s}^{F,t}(p,\triangleright) \) is given by

\[
\left( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(x_1) \right) \circ \left( C_{E,s}(x_2) \cdot C_{F,t}(x_3) \right) \circ \cdots \circ \left( C_{E,s}(x_{2n-2}) \cdot C_{F,t}(x_{2n-1}) \right).
\]

The situation is symmetric for \( Z_{E,s}^{F,t}(p,\triangleleft) \): the sequence starts with a right-right step \( x_2 = (\triangleleft, p_2, p_3) \in s \) with \( p = p_2 \) and we obtain the SDRTE simply by removing the first factor \( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(x_1) \) in the Hadamard products above.

We come back to the definition of the SDRTEs for \( C_{E,F,r}(x) \) with \( r \in TrM \) and \( x \in r \). As explained above, the output produced by a run \( \rho \) following step \( x \) over a word \( w = uv \) with \( u \in \mathcal{L}(E, s), v \in \mathcal{L}(F, t) \) and \( r = st \) consists of an initial part, a zigzag internal part, and a final part. There are four cases depending on the step \( x \).

- \( x = (\triangleright, p, q) \). Either the run \( \rho \) stays inside \( u \) (zigzag part empty) or there is a zigzag internal part starting with \( (p', \triangleright) \) such that \( (\rightarrow, p, p') \in \varphi(u) \) and ending with \( (\triangleright, q') \) such that \( (\leftarrow, q', q) \in \varphi(u) \). Thus we define the SDRTE \( C_{E,F,r}(x) \) as

\[
\sum_{st=r|\exists x \in s} C_{E,s}(x) \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) + \sum_{st=r|\exists x \in s} \left( C_{E,s}(\rightarrow, p, p') \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) \right) \circ Z_{E,s}^{F,t}(p',\triangleright) \circ \left( C_{E,s}(\leftarrow, q', q) \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) \right).
\]

Notice that all Cauchy products are unambiguous since the concatenation \( \mathcal{L}(E) \cdot \mathcal{L}(F) \) is unambiguous. The sums are also unambiguous. Indeed, a word \( w \in \mathcal{L}(E \cdot F, r) \) has a unique factorization \( w = uv \) with \( u \in \mathcal{L}(E) \) and \( v \in \mathcal{L}(F) \). Hence \( s = \varphi(u) \) and \( t = \varphi(v) \) are uniquely determined and satisfy \( st = r \). Then, either \( x \in s \) and \( w \) is only in the domain of \( C_{E,s}(x) \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) \) or \( x \) is a unique \( p' \) with \( (\rightarrow, p, p') \in s \) and a unique \( q' \) with \( Z_{s,t}(p',\triangleright) = (\triangleright, q') \) and \( (\leftarrow, q', q) \in s \). Notice that if \( (\rightarrow, p, p') \notin s \) then \( C_{E,s}(\rightarrow, p, p') = \varepsilon \) and similarly if \( (\leftarrow, q', q) \notin s \). Hence we could have added the condition \( (\rightarrow, p, p'), (\leftarrow, q', q) \in s \) to the second sum, but do not, to reduce clutter.

- \( x = (\rightarrow, p, q) \). Here the run must cross from left to right. Thus we define the SDRTE \( C_{E,F,r}(x) \) as

\[
\sum_{st=r|\exists x \in s} \left( C_{E,s}(\rightarrow, p, p') \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) \right) \circ Z_{E,s}^{F,t}(p',\triangleright) \circ \left( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(\leftarrow, q', q) \right)
\]

- \( x = (\leftarrow, p, q) \). This case is similar. The SDRTE \( C_{E,F,r}(x) \) is

\[
\sum_{st=r|\exists x \in s} \left( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(\leftarrow, p, p') \right) \circ Z_{E,s}^{F,t}(p',\triangleleft) \circ \left( C_{E,s}(\leftarrow, q', q) \cdot (\mathcal{L}(F, t) \triangleright \varepsilon) \right)
\]

- \( x = (\triangleright, p, q) \). Finally, for right-right runs, the SDRTE \( C_{E,F,r}(x) \) is

\[
\sum_{st=r|\exists x \in t} (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(x) + \sum_{st=r|\exists x \in t} \left( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(\leftarrow, p, p') \right) \circ Z_{E,s}^{F,t}(p',\triangleleft) \circ \left( (\mathcal{L}(E, s) \triangleright \varepsilon) \cdot C_{F,t}(\rightarrow, q', q) \right)
\]
Example 26. We go back to our running example of the aperiodic 2DFT $A$ in Figure 2 and illustrate the unambiguous concatenation. Consider $F = a^+ b$, $G = F^2$ and $E = F^4 = G^2$. We know from Example 9 that $\varphi(F) = Y_2$, $\varphi(G) = Y_4$ and $\varphi(E) = Z_2$. We compute below some steps of $Y_2$, $Y_4$ and $Z_2$.

First, we look at some steps in $F$ for which the SDRTE are obtained directly by looking at the automaton $A$ in Figure 2 (we cannot give more details here since we have not explained yet how to deal with Kleene-plus, hence we rely on intuition for these steps).

$$C_{F,Y_2}(\langle -\rangle, q_0, q_1) = (a^+ b \triangleright \varepsilon)$$
$$C_{F,Y_2}(\langle \triangleright, q_2, q_3 \rangle) = (a \triangleright a) \cdot (b \triangleright b)$$
$$C_{F,Y_2}(\langle \leftarrow, q_3, q_4 \rangle) = (a^+ b \triangleright \varepsilon)$$
$$C_{F,Y_2}(\langle \leftarrow, q_5, q_0 \rangle) = (a^+ b \triangleright \varepsilon).$$

Next, we compute some steps using the unambiguous concatenation $G = F \cdot F$. We start with step $(\langle -\rangle, q_0, q_2)$ for which the zigzag part is empty: $ZC_{F,Y_2}^{E,Y_2}(\langle -\rangle, q_0, q_2) = F^2 \triangleright \varepsilon$. Hence, we get using the formula in the proof above

$$C_{G,Y_2}(\langle -\rangle, q_0, q_2) = (C_{F,Y_2}(\langle -\rangle, q_0, q_1) \cdot (F \triangleright \varepsilon)) \circ (F^2 \triangleright \varepsilon) \circ ((F \triangleright \varepsilon) \cdot C_{F,Y_2}(\langle -\rangle, q_1, q_2))$$

and after some simplifications

$$C_{G,Y_2}(\langle -\rangle, q_0, q_2) = C_{F,Y_2}(\langle -\rangle, q_0, q_1) \cdot C_{F,Y_2}(\langle -\rangle, q_1, q_2) = ((a^+ b)^2 \triangleright \varepsilon).$$

Similarly, we can compute the following steps

$$C_{G,Y_2}(\langle -\rangle, q_0, q_2) = C_{F,Y_2}(\langle -\rangle, q_0, q_1) \cdot C_{F,Y_2}(\langle -\rangle, q_1, q_0) = (a \triangleright a) \cdot (b \triangleright b) \cdot (a^+ b \triangleright \varepsilon).$$

For step $(\triangleright, q_2, q_3)$, the run only visits the first factor since $(\triangleright, q_2, q_3) \in Y_2$:

$$C_{G,Y_2}(\langle \triangleright, q_2, q_3 \rangle) = C_{F,Y_2}(\langle \triangleright, q_2, q_3 \rangle) \cdot (F \triangleright \varepsilon) = (a \triangleright a) \cdot (b \triangleright b) \cdot (a^+ b \triangleright \varepsilon).$$

Now, for step $(\triangleright, q_1, q_4)$, the zigzag part is reduced to step $(\triangleright, q_2, q_3)$ and we get:

$$C_{G,Y_2}(\langle \triangleright, q_1, q_4 \rangle) = (C_{F,Y_2}(\langle \triangleright, q_1, q_2 \rangle) \cdot (F \triangleright \varepsilon)) \circ ((F \triangleright \varepsilon) \cdot C_{F,Y_2}(\langle \triangleright, q_2, q_3 \rangle))$$

$$\circ (C_{F,Y_2}(\langle \leftarrow, q_3, q_4 \rangle) \cdot (F \triangleright \varepsilon))$$

$$= (a^+ b \triangleright \varepsilon) \cdot (a \triangleright a) \cdot (b \triangleright b).$$

Similarly, we compute

$$C_{G,Y_2}(\langle \leftarrow, q_4, q_0 \rangle) = ((G \triangleright \varepsilon) \cdot C_{F,Y_2}(\langle \leftarrow, q_4, q_5 \rangle)) \circ (C_{F,Y_2}(\langle \leftarrow, q_5, q_0 \rangle) \cdot (F \triangleright \varepsilon))$$

$$\circ ((F \triangleright \varepsilon) \cdot C_{F,Y_2}(\langle -\rangle, q_6, q_0)))$$

$$= (a^+ b \triangleright \varepsilon) \cdot (a \triangleright a) \cdot (b \triangleright b).$$

Finally, we consider the unambiguous decomposition $E = G \cdot G$ in order to compute $C_{E,Z_2}(x)$ where $x = (\langle -\rangle, q_0, q_2)$. Notice that $(\langle -\rangle, q_0, q_1), (\triangleright, q_1, q_4), (\leftarrow, q_4, q_0), (\langle -\rangle, q_0, q_2) \in Y_4 = \varphi(G)$. Hence, $Z_{Y_4,Y_2}(q_1, \triangleright) = (\leftarrow, q_0)$ and applying the formulas in the proof above we obtain

$$C_{E,Z_2}(x) = (C_{G,Y_2}(\langle -\rangle, q_0, q_1) \cdot (G \triangleright \varepsilon)) \circ ZC_{G,Y_2}^{E,Y_2}(q_1, \triangleright) \circ ((G \triangleright \varepsilon) \cdot C_{G,Y_2}(\langle -\rangle, q_0, q_2)))$$

$$ZC_{G,Y_2}^{G,Y_2}(q_1, \triangleright) = ((G \triangleright \varepsilon) \cdot C_{G,Y_2}(\langle \triangleright, q_1, q_4 \rangle)) \circ (C_{G,Y_2}(\langle \leftarrow, q_4, q_0 \rangle) \cdot (G \triangleright \varepsilon)).$$
Putting everything together and still after some simplifications, we get

\[
C_{E,Z_2}(x) = \left( (a^+a) \cdot (b \cdot (a^+b)^3 \cdot \varepsilon) \right) \circ \left( (a^+b)^3 \cdot \varepsilon \right) \cdot (a^+a) \cdot (b \cdot b) \cdot (a^+b)^2 \cdot \varepsilon
\]

For instance, applying \( C_{E,Z_2}(x) \) to \( w = aba^2ba^3ba^4b \) we obtain \( aba^4ba^2b \).

**SD-Kleene Star**

The most interesting case is when \( E = F^* \). Let \( L = \mathcal{L}(F) \subseteq \Sigma^* \). Since \( E \) is a stabilising SD-regular expression, \( L \) is an aperiodic prefix code of bounded synchronisation delay, and \( X = \varphi(L) \) is \( k \)-stabilising for some \( k > 0 \). Hence, we may apply the results of Section 5.2.1.

By induction, we suppose that we have SDRTEs \( C_{F,s}(x) \) for all \( s \) in \( \text{TrM} \) and steps \( x \). Since \( L = \mathcal{L}(F) \) is a code, for each fixed \( \ell > 0 \), the expression \( F^\ell = F \cdot F \cdot \ldots \cdot F \) is an unambiguous concatenation. Hence, from the proof above for the unambiguous concatenation, we may also assume that we have SDRTEs \( C_{F^\ell,s}(x) \) for all \( s \) in \( \text{TrM} \) and steps \( x \). Similarly, we have SDRTEs for \( Z_{F^\ell,s}(\cdot) \) and \( Z_{E,s}(\cdot) \). Notice that \( F^0 \) is equivalent to \( \varepsilon \) hence we have \( C_{F_0,s}(x) = C_{\varepsilon,s}(x) \).

We show how to construct SDRTEs \( C_{E,s}(x) \) for \( E = F^* \). There are four cases, which are dealt with below, depending on the step \( x \).

We fix some notation common to all four cases. Fix some \( w \in \mathcal{L}(E, s) = L^* \cap \varphi^{-1}(s) \) and let \( w = u_1 \cdots u_n \) be the unique factorization of \( w \) with \( n \geq 0 \) and \( u_i \in L \) for \( 1 \leq i \leq n \). For a step \( x \in s \), we denote by \( \rho \) the unique run of \( A \) over \( w \) following step \( x \).

\[
x = (\rho, p, q) \in s
\]

The easiest case is for left-left steps. If \( n < k \) then the output of \( \rho \) is \( [C_{F^n,s}(x)](w) \). Notice that here, \( C_{F^n,s}(x) = \perp \) since \( x \notin 1 = \varphi(\varepsilon) \). Now, if \( n \geq k \) then, by Lemma 19, the run \( \rho \) stays inside \( u_1 \cdots u_k \). We deduce that the output of \( \rho \) is \( [C_{F^n,s}(x)](u_1 \cdots u_k) \). Therefore, we define

\[
C_{E,s}(x) = \left( \sum_{n < k} C_{F^n,s}(x) \right) + \left( C_{F^n,s}(x) \cdot (F^* \cdot \varepsilon) \right)
\]
Notice that the sums run which is 2-forward-progressing.

\( x = (\rightarrow, p, q) \in s \)

We turn now to the more interesting left-right steps. Again, if \( n < k \) then the output of \( \rho \) is \([C_{F^n, s}(x)](w)\). Assume now that \( n \geq k \). We apply Lemma 20 to deduce that the run \( \rho \) is \( k \)-forward-progressing. See Figure 10 for a sample run which is 2-forward-progressing. We split \( \rho \) in \( \rho_0 \rho_1 \cdots \rho_\i \) where \( \rho_0 \) is the prefix of \( \rho \) going until the first crossing from \( u_k \) to \( u_k+1 \). Then, \( \rho_1 \) goes until the first crossing from \( u_k+1 \) to \( u_k+2 \). Continuing in the same way, for \( 1 \leq i < n-k \), \( \rho_i \) goes until the first crossing from \( u_k+i \) to \( u_k+i+1 \). Finally, \( \rho_{n-k} \) is the remaining suffix, going until the run exits from \( w \) on the right. Since the run \( \rho \) is \( k \)-forward-progressing, we deduce that \( \rho_i \) does not go back to \( u_1 \cdots u_{i-1} \), hence it stays inside \( u_i \cdots u_{i+k} \), starting on the left of \( u_{i+k} \) and exiting on the right of \( u_{i+k} \).

Since \( X = \varphi(L) \) is \( k \)-stabilising, we have \( \varphi(u_1 \cdots u_{i+k}) = \varphi(w) \) for all \( 0 \leq i \leq n-k \). Now, \( \rho_0 \cdots \rho_i \) is a run on \( u_1 \cdots u_{i+k} \), starting on the left in state \( p \) and exiting on the right. Since \( A \) is deterministic and \( x = (\rightarrow, p, q) \in \varphi(w) = \varphi(u_1 \cdots u_{i+k}) \) we deduce that \( \rho_i \) exits on the right of \( u_{i+k} \) in state \( q \). In particular, \( \rho_0 \) is a run on \( u_1 \cdots u_k \) starting on the left in state \( p \) and exiting on the right in state \( q \). Moreover, for each \( 1 \leq i < n-k \), \( \rho_i \) is the concatenation of a zigzag internal run over \( (u_i \cdots u_{i+k-1}, u_{i+k}) \) starting with \((q, \varphi)\) ending with \((c, q_i) = Z_{s', s''}(q, \varphi)\) where \( s' = \varphi(u_i \cdots u_{i+k-1}) \), \( s'' = \varphi(u_{i+k}) \) and a \((\rightarrow, q_i, q)\) run over \( u_{i+k} \).

Let \( v_1 \) be the output produced by \( \rho_i \) for \( 0 \leq i \leq n-k \). Then, using Lemma 25, the productions \( v_i \) with \( 0 < i \leq n-k \) are given by the SDRTE \( f \) defined as

\[
 f = \sum_{(q', s', q'' | (c, q') = Z_{s', s''}(q, \varphi))} ZC_{F^{s'', s''}(q, \varphi)} \odot ((\varphi(F^k, s') \circ \varepsilon') \cdot C_{F^{s'', s''}((\rightarrow, q', q))})
\]

Then the product \( v_1 \cdots v_{n-k} \) is produced by the \((k+1)\)-chained Kleene-star \([L, f]^{(k+1)*}(w)\). From the above discussion, we also deduce that \( v_0 = [C_{F^n, s}(x)](u_1 \cdots u_k) \). Therefore, we define

\[
 C_{E, s}(x) = \left( \sum_{n < k} C_{F^n, s}(x) \right) + \left( (C_{F^n, s}(x) \cdot (F^* \circ \varepsilon)) \odot [F, f]^{(k+1)*} \right)
\]

\( x = (\leftarrow, p, q) \in s \)

The case of right-left runs is almost symmetric to the case of left-right runs. Again, if \( n < k \) then the output of \( \rho \) is \([C_{F^n, s}(x)](w)\). Assume now that \( n \geq k \). We apply Lemma 21 to deduce that the run \( \rho \) is \( k \)-backward-progressing. As illustrated in Figure 11.
we split ρ in ρn−k+1 · · · ρ2ρ1 where ρn−k+1 is the prefix of ρ going until the first crossing from u_{n−k+1} to u_{n−k}. Then, ρn−k goes until the first crossing from u_{n−k} to u_{n−k−1}. Continuing in the same way, for n − k > i > 1, ρi goes until the first crossing from u_i to u_{i−1}. Finally, ρ1 is the remaining suffix, going until the run exits from u_1 on the left. Since the run ρ is k-backward-progressing, we deduce that, for 1 ≤ i ≤ n − k, the run ρi does not go back to u_{i+k+1} · · · u_n. Hence it is the concatenation of a zigzag internal run over (u_i, u_{i+1} · · · u_k), starting with some (q_i, q_i') and exiting with (q_n, q_n') where s' = ϕ(u_i), s'' = ϕ(u_{i+1} · · · u_k), and a (←, q_{i−1})-run over u_i (see again Figure 11). Let v_i be the output produced by ρi for 1 ≤ i ≤ n − k + 1. The output produced by ρ is v = v_n · · · v_1. Now, the situation is slightly more complicated than for left-right runs where we could prove that q_i = q for each i. Instead, let us remark that q_i is the unique state (by determinacy of A) such that there is a run over u_{i+1} · · · u_n following step (←, p, q_i). But since X is k-stabilising, we know that ϕ(u_{i+1} · · · u_n) = ϕ(u_{i+1} · · · u_k). Then, given s' = ϕ(u_i) and s'' = ϕ(u_{i+1} · · · u_k), we get that q_i is the unique state such that (←, p, q_i−1) ∈ s's''. Thus we define the function g generating the v_i with 1 ≤ i ≤ n − k by

\[
g = \sum_{s,s',q,q',p,p' \in s'} ZC_{F,s}^{p,s'}(p',q') \odot \left( C_{F,s'}((←, q', q'')) \cdot (\mathcal{L}(F^k, s'') \triangleright \varepsilon) \right)
\]

Finally, a right-left run for F* is either a right-left run over F^n for n < k, or the concatenation of a right-left run ρn−k+1 over the k rightmost iterations of F, and a sequence of runs ρi with 1 ≤ i ≤ n − k as previously and whose outputs are computed by g. Therefore, we define

\[
C_{E,s}(x) = \sum_{n<k} C_{F^n,s}(x) + \left( \sum_{s=s', q', p', \varepsilon} (\mathcal{L}(F^*, s') \triangleright \varepsilon) \cdot C_{F^*, s'}((←, p, p')) \right) \odot \left[ F, g \right]^{(k+1)*}_r
\]

x = (s, p, q) ∈ s

We finally deal with right-right runs, thus completing the case of E = F*. If n < k, then the output of ρ is \([C_{F^n,s}(x)](w)\). Assume now that n ≥ k. If ρ only visits the last k factors, then w can be decomposed as uv where u ∈ \(\mathcal{L}(F^k, s')\) and v ∈ \(\mathcal{L}(F^k, s'')\) with s = s's'' and (s, p, q) ∈ s''. The output of ρ is \([\mathcal{L}(F^*, s') \triangleright \varepsilon] \cdot C_{F^*, s'}(x)\](w). Otherwise, by Lemma 22, we know that there is a unique integer 1 ≤ i ≤ n − k such that ρ only visits u_i · · · u_n, and is the concatenation of 3 runs ρ1, ρ2 and ρ3, where ρ1 is a k-backward-progressing run over u_{i+1} · · · u_n, ρ2 is a zigzag internal run over

\[
\text{Figure 11 A right-left run which is 2-backward-progressing.}
\]
\((u_i, u_{i+1} \cdots u_{i+k})\) and \(\rho_3\) is a \(k\)-forward-progressing run over \(u_{i+1} \cdots u_n\), as depicted in Figure 3.

Let \(r = \varphi(u_1 \cdots u_{i-1})\), \(s' = \varphi(u_i)\) and \(s'' = \varphi(u_{i+1} \cdots u_{i+k})\). Since \(i\) is uniquely determined, the tuple \((r, s', s'')\) is also unique. Notice that \(s'' = \varphi(u_{i+1} \cdots u_n)\) since \(X\) is \(k\)-stabilising and \(s'' \in X^k\), hence we have \(s = rs's''\). Moreover, the starting and ending states of \(\rho_2\) are the unique states \(p', q'\) such that \((\leftarrow, p, p') \in s''\), \(Z_{s', s''}(p', \zeta) = (\zeta, q')\), \((\rightarrow, q', q) \in s'\'). Once the tuple \((r, s', s'', p', q')\) is fixed, using the previous points, we have SDRTEs computing the outputs of \(\rho_1\) and \(\rho_2\):

\[
\[\mathcal{C}_{E,s''}((\leftarrow, p, p'))\]((u_{i+1} \cdots u_n)) \quad \quad \[\mathcal{C}_{E,s''}((\rightarrow, q', q))\]((u_{i+1} \cdots u_n))
\]

and from Lemma 25, the output of \(\rho_2\) is \([\mathcal{ZC}_{F,s'}(p', \zeta)]((u_i, u_{i+1} \cdots u_{i+k}))\). This leads to the following SDRTE:

\[
h = \sum_{s=rs's''} (\mathcal{L}(F^*, r) \triangleright \varepsilon) \cdot \left( (\mathcal{L}(F, s') \triangleright \varepsilon) \cdot \mathcal{C}_{E,s''}((\leftarrow, p, p')) \right) \quad \mathcal{ZC}_{F,s'}(p', \zeta) \cdot (F^* \triangleright \varepsilon) \quad \mathcal{C}_{E,s''}((\rightarrow, q', q))
\]

Notice that if \(r, s', s'', p', q'\) satisfy the conditions of the sum, then there is a unique index \(i\) with \(\varphi(u_i) = s'\) and \(\varphi(u_{i+1} \cdots u_n) = s''\): there is a \((\leftarrow, p, p')\)-run on \(u_{i+1} \cdots u_n\) but not on \(u_1 \cdots u_{i-1}\). Hence all the Cauchy products in \(h\) are unambiguous and uniquely decompose \(w\) in \(u_1 \cdots u_{i-1}\) matched by \(\mathcal{L}(F^*, r)\), \(u_i\) matched by \(\mathcal{L}(F, s')\) and \(u_{i+1} \cdots u_n\) matched by \(\mathcal{C}_{E,s''}((\leftarrow, p, p'))\) and \(\mathcal{C}_{E,s''}((\rightarrow, q', q))\). Also, \(u_{i+1} \cdots u_{i+k}\) is matched by \(\mathcal{ZC}_{F,s'}(p', \zeta)\) and \(n \geq i + k\).

Finally, for the right-right step \(x = (\zeta, p, q)\) we define

\[
\mathcal{C}_{E,s}(x) = \left( \sum_{n<k} \mathcal{C}_{F^n,s}(x) \right) + \left( \sum_{n=s} \mathcal{L}(F^*, s') \triangleright \varepsilon \right) \cdot \mathcal{C}_{F^n,s'}(x) + h.
\]

The sums above are unambiguous. Indeed, the first case happens exactly when \(w \in F^n\) for \(n < k\). The second happens exactly when \(n \geq k\) and the right-right run does not visit \(u_{n-k}\). Otherwise, the third case happens.

Example 27. Let us continue with our example in Figure 2. Here we illustrate computing an SDRTE for some \(E = F^*\) and some left-right step. We consider \(F = a^+ b\) so that \(\varphi(F) = Y_2\) as computed in Example 9 where we also computed \(Z_2 = Y_2^4\). Consider \(x = (\rightarrow, q_6, q_2)\) \(\in Z_2\).

We explain how to compute \(\mathcal{C}_{E,Z_2}(x)\). Since \(\varphi(F) = Y_2\) is \(4\)-stabilising, the runs following step \(x\) over words in \(\mathcal{L}(E, Z_2) = (a^+ b)^{2^4}\) are \(4\)-forward-progressing, and the construction in the proof above uses a \(5\)-chained star.

Let \(w = u_1 u_2 \cdots u_n\) with \(n \geq 4\) and \(u_1, u_2, \ldots, u_n \in a^+ b\). As can be seen on Figure 12 (with \(n = 8\)), the \(4\)-forward-progressing run \(\rho\) over \(w\) following step \(x\) splits as \(\rho = \rho_0 \cdots \rho_4\) where \(\rho_0\) is the prefix from \(u_1\) till first crossing from \(u_4\) to \(u_5\), \(\rho_1\) is the part from \(u_5\) till the first crossing from \(u_5\) to \(u_6\), \(\rho_2\) is the part of the run from \(u_6\) till the first crossing from \(u_6\) to \(u_7\), \(\rho_3\) is the part of the run from \(u_7\) till the first crossing from \(u_7\) to \(u_8\), and \(\rho_4\) is the part from \(u_8\) till exiting at the right of \(u_8\).

\footnote{Actually, the runs over step \(x\) are \(3\)-forward-progressing. Hence, we could simplify the example below by using a \(4\)-chained star only. But we decided to use a \(5\)-star in order to follow the construction described in the proof above.}
We obtain
\[ C_{(a+b)^*Z_2}((\rightarrow, q_0, q_2)) = (C_{(a+b)^*Z_2}((\rightarrow, q_0, q_2)) \cdot ((a+b)^* \triangleright \varepsilon)) \odot [a+b, f]^5 * \]
where
\[ f = ZC_{(a+b)^4Z_2}(q_2, \triangleright) \odot ((a+b)^4 \triangleright \varepsilon) \cdot C_{a+bZ_2}((\rightarrow, q_1, q_2)) \]
and the expressions below where computed in Example 26:
\[ C_{a+bZ_2}((\rightarrow, q_1, q_2)) = a+b \triangleright \varepsilon \]
\[ C_{a+bZ_2}((\triangleright, q_2, q_3)) = (a \triangleright a)^+ \cdot (b \triangleright b) \]
\[ C_{(a+b)^4Z_2}(\triangleleft, q_3, q_1) = ((a+b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot (a+b \triangleright \varepsilon) \]
\[ C_{(a+b)^4Z_2}((\rightarrow, q_0, q_3)) = \left( (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a+b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \right) \odot \left( (a+b \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a+b)^2 \triangleright \varepsilon) \right) \]
After simplifications, we obtain
\[ f = ZC_{(a+b)^4Z_2}(q_2, \triangleright) \]
\[ = \left( ((a+b)^4 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \right) \odot \left( ((a+b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a+b)^2 \triangleright \varepsilon) \right). \]
For instance, consider \( w = u_1u_2 \cdots u_8 \) with \( u_i = a^i b \). Then,
\[ f(u_1 \cdots u_5) = a^5 ba^2 b \]
\[ f(u_2 \cdots u_6) = a^6 ba^4 b \]
\[ f(u_3 \cdots u_7) = a^7 ba^4 b \]
\[ f(u_4 \cdots u_8) = a^8 ba^5 b \]
\[ [C_{(a+b)^*Z_2}((\rightarrow, q_0, q_2))][u_1 \cdots u_4] = aba^3 ba^2 b \]
\[ [C_{(a+b)^*Z_2}((\rightarrow, q_0, q_2))][u_1 \cdots u_8] = aba^4 ba^2 ba^5 ba^6 ba^4 ba^7 ba^4 ba^8 ba^5 b. \]
We conclude the section by showing how to construct SDRTEs equivalent to 2DFTs.

**Theorem 28.** Let \( A = (Q, \Sigma, \Gamma, \delta, \gamma, q_0, F) \) be an aperiodic 2DFT. We can construct an equivalent SDRTE \( C'_A \) over alphabet \( \Sigma \) with \( \text{dom}(C'_A) = \text{dom}(A) \) and \( [A](w) = [C'_A](w) \) for all \( w \in \text{dom}(A) \).

**Proof.** We first construct below an SDRTE \( C'_A \) with \( \text{dom}(C'_A) = \top \text{dom}(A) \top \) and such that \( [A](w) = [C'_A](\top w \top) \) for all \( w \in \text{dom}(A) \). Then, we obtain \( C'_A \) using Proposition 15 by

\[
C''_A = (\top^{-1}C'_A)^{-1}.
\]

Finally, we get rid of lingering end markers in \( C''_A \) using Lemma 16 to obtain \( C_A \) as the projection of \( C''_A \) on \( \Sigma^* \).

Let \( \varphi: (\Sigma \{ \top, \bot \})^* \to \text{TrM} \) be the canonical surjective morphism to the transition monoid of \( A \). Since \( A \) is aperiodic, the monoid \( \text{TrM} \) is also aperiodic. We can apply Theorem 10 to the restriction of \( \varphi \) to \( \Sigma^* \): for each \( s \in \text{TrM} \), we get an unambiguous, stabilising, SD-regular expression \( E_s \) with \( \mathcal{L}(E_s) = \varphi^{-1}(s) \cap \Sigma^* \). Let \( E = \top \cdot (\bigcup_{s \in \text{TrM}} E_s) \) which is an unambiguous, stabilising, SD-regular expression with \( \mathcal{L}(E) = \top \Sigma^* \). Applying Theorem 25 for each monoid element \( s \in \text{TrM} \) and each step \( x \in \{ \rightarrow, \Rightarrow, \leftarrow \} \times Q^2 \), we construct the corresponding SDRTE \( C_{E,s}(x) \). We also apply Lemma 25 and construct for each state \( p \in Q \) an SDRTE \( ZC_E^{a,t}(p, \varnothing) \) where \( t = \varphi(\bot) \).

\[
\text{Figure 13} \quad \text{Removing end markers. On the left when there is a non trivial zig zag until reaching a final state } q; \text{ on the right when we have an empty zigzag. } q_0 \text{ is the initial state and } q \in F. \]

Finally, we define

\[
C'_A = \sum_{s, p, q | q=p} \left( C_{E,s}((\rightarrow, q_0, p)) \cdot (\top \Rightarrow \epsilon) \right) \odot ZC_E^{a,t}(p, \varnothing)
\]

See also Figure 13 illustrating \( C'_A \). We can easily check that \( C_A \) satisfies the requirements stated above.

**Example 29.** We complete the series of examples by giving an SDRTE equivalent with the transducer \( A \) of Figure 2 on words in \( E_1 = b(a^+b)^{\geq 4} \subseteq \text{dom}(A) \). Notice that by Example 9 we have \( \varphi(E_1) = Z_1 = Y_1 Z_2 \). We compute first \( C_{E_1,Z_1}((\rightarrow, s, q_2)) \). We use the unambiguous product \( E_1 = b \cdot E_2 \) with \( E_2 = (a^+b)^{\geq 4} \). From Example 9, we have \( (\rightarrow, s, q_0), (\Rightarrow, q_0, q_0), (\rightarrow, q_0, q_0) \in Y_1 = \varphi(b) \) and \( (\Rightarrow, q_0, q_0), (\rightarrow, q_0, q_0) \in Z_2 = \varphi(E_2) \). We deduce that
in the product, the zigzag part consists of the two steps \((\triangleright, q_0, q_5)\) and \((\subset, q_5, q_0)\). Therefore we obtain:

\[
\begin{align*}
C_{E_1, Z_1}((\rightarrow, s, q_2)) &= (C_{b, Y_1}((\rightarrow, s, q_0)) \cdot (E_2 \triangleright \varepsilon)) \circ ZC_{E_2, Z_2}(q_0, \triangleright) \circ \{(b \triangleright \varepsilon) \cdot C_{E_2, Z_2}((\rightarrow, q_6, q_2))\} \\
ZC_{E_2, Z_2}(q_0, \triangleright) &= ((b \triangleright \varepsilon) \cdot C_{E_2, Z_2}((\triangleright, q_0, q_5)) \circ \{(b \triangleright \varepsilon) \cdot C_{E_2, Z_2}((\rightarrow, q_6, q_2))\}.
\end{align*}
\]

Since \(C_{b, Y_1}((\rightarrow, s, q_0)) = C_{b, Y_1}((\subset, q_5, q_0)) = b \triangleright \varepsilon\), we obtain after simplifications

\[
C_{E_1, Z_1}((\rightarrow, s, q_2)) = ((b \triangleright \varepsilon) \cdot C_{E_2, Z_2}((\triangleright, q_0, q_5)) \circ \{(b \triangleright \varepsilon) \cdot C_{E_2, Z_2}((\rightarrow, q_6, q_2))\}.
\]

Let \(E = (a^+ b)^*\) as in Example 27. Since \(\mathcal{L}(E_2, Z_2) = \mathcal{L}(E, Z_2)\) we have \(C_{E_2, Z_2}(x) = C_{E, Z_2}(x)\) for all steps \(x\). Recall that in Example 27 we have computed \(C_{E_2, Z_2}((\rightarrow, q_6, q_2))\). Moreover, \(C_{E_2, Z_2}((\triangleright, q_0, q_5)) = C_{(a^+ b)^+, Z_2}((\triangleright, q_0, q_5)) \cdot (E \triangleright \varepsilon)\) and a computation similar to Example 26 gives

\[
C_{(a^+ b)^+, Z_2}((\triangleright, q_0, q_5)) = ((a^+ b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot (a^+ b \triangleright \varepsilon).
\]

Finally, with the notations of Example 27 we obtain

\[
C_{E_1, Z_1}((\rightarrow, s, q_2)) = \begin{align*}
&((b(a^+ b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a^+ b)^+ \triangleright \varepsilon)) \\
&\circ ((b \triangleright \varepsilon) \cdot C_{E, Z_2}((\rightarrow, q_6, q_2))) \\
&= ((b(a^+ b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a^+ b)^+ \triangleright \varepsilon)) \\
&\circ ((b \triangleright \varepsilon) \cdot C_{(a^+ b)^+, Z_2}((\triangleright, q_0, q_5))) \cdot (E \triangleright \varepsilon)) \\
&\circ ((b \triangleright \varepsilon) \cdot ([a^+ b, f]_{5^*}) \\
&= ((b(a^+ b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a^+ b)^+ \triangleright \varepsilon)) \\
&\circ ((b \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a^+ b)^2 \triangleright \varepsilon)) \\
&\circ ((b(a^+ b)^2 \triangleright \varepsilon) \cdot (a \triangleright a)^+ \cdot (b \triangleright b) \cdot ((a^+ b)^2 \triangleright \varepsilon)) \\
&\circ ((b \triangleright \varepsilon) \cdot ([a^+ b, f]_{5^*}).
\end{align*}
\]

Note that for our example, we have a simple case of using Theorem 25, since \(\vdash\) is visited only once at state \(s\), and there are no transitions defined on \(\vdash\), i.e., \(\varphi(\vdash) = \emptyset\). So for \(E' = E_1\), we have \((\rightarrow, s, q_2) \in \varphi(E')\) and \(ZC_{E_1, \varphi(E')}((q_2, \vdash)) = \varepsilon\) (zigzag part empty as seen in the right of Figure 13). Thus, an expression equivalent to \(A\) on words in \(E_1\) effectively boils down to \(C_{E_1, Z_1}((\rightarrow, s, q_2))\). For the sake of simplicity, we stop the example here and do not provide the expression on the full domain \(\text{dom}(A) = b(a^+ b)^{\geq 2}a^*\).

\section{Adding composition}

Since SDRTEs are used to define functions over words, it seems natural to consider the composition of functions, as it is an easy to understand but powerful operator. In this section, we discuss other formalisms using composition as a basic operator, and having the same expressive power as SDRTEs.

Theorem 4 gives the equivalence between SDRTEs and aperiodic two-way transducers, the latter being known to be closed under composition. Hence, adding composition to SDRTEs does not add expressiveness, while allowing for easier modelisation of transformations.

Moreover, we prove that, should we add composition of functions, then we can replace the \(k\)-chained star operator and its reverse by the simpler 1-star \([L, f]_1^*\) and its reverse, which in particular are one-way (left-to-right or right-to-left) operator when \(f\) is also one-way.
Finally, we prove that we can furthermore get rid of the reverse operators as well as the Hadamard product by adding two basic functions: \textit{reverse} and \textit{duplicate}. The reverse function is straightforward as it reverses its input. The duplicate function is parameterised by a symbol, say \#, duplicates its input inserting \# between the two copies: $\text{dup}_\#(u) = u\#u$.

\textbf{Theorem 30.} The following families of expressions have the same expressive power:

1. SDRTEs,
2. SDRTEs with composition of functions,
3. SDRTEs with composition and chained star restricted to 1.
4. Expressions with simple functions, unambiguous sum, Cauchy product, 1-star, duplicate, \textit{reverse} and composition.

\textbf{Proof.} It is trivial that $3 \subseteq 2$ as $3$ is obtained as a syntactical restriction of $2$. Although it is not needed in our proof, note that $1 \subseteq 2$ holds for the same reason. Now, thanks to Theorem 1, we know that SDRTEs are equivalent to aperiodic two-way transducers that are closed under composition. Hence, composition does not add expressive power and we have $2 \subseteq 1$.

To prove that $4 \subseteq 3$, we simply have to prove that the duplicate and reverse functions can be expressed with SDRTEs using only the 1-star operator and its reverse. The duplicate function definition relies on the Hadamard and is given by the expression:

$$\text{dup}_\# = (\text{id}_{\Sigma^*} \cdot (\varepsilon \triangleright \#)) \circ \text{id}_{\Sigma^*},$$

where $\text{id}_{\Sigma^*}$ is the identity function and can be written as $[\Sigma, \text{id}_{\Sigma^*}]^{1*}$ where $\text{id}_{\Sigma} = \sum_{a \in \Sigma} a \triangleright a$.

The reverse function is also easy to define using the 1-star reverse:

$$\text{rev} = [\Sigma, \text{id}_{\Sigma^*}]^{1*}.$$

To prove the last inclusion $1 \subseteq 4$ we need to express the Hadamard product and the (reverse) $k$-chained star, using duplicate, reverse and composition.

The Hadamard product $f \odot g$ is easy to define using $\text{dup}_\#$ where \# is a fresh marker:

$$f \odot g = (f \cdot (\# \triangleright \varepsilon) \cdot g) \circ \text{dup}_\#.$$

We show now how to reduce $k$-star to 1-star using duplicate and composition. The proof is by induction on $k$. When $k = 1$ there is nothing to do. Assume that $k > 1$. We show how to express $[L, f]^k$ using $(k - 1)$-star, 1-star and duplicate. The main idea is to use composition to mark each factor in $L$ in order to duplicate them, then use a $(k - 1)$-star to have a reach of $k$ factors of $L$ (with some redundant information), and lastly use composition to prune the input to a form suitable to finally apply $f$.

More formally, let \# and $\$ be two fresh markers and define

$$f_1 = [L, \text{dup}_\$ \cdot (\varepsilon \triangleright \#)]^{1*}$$

with domain $L^*$ and, when applied to a word $u = u_1 \cdots u_n$ with $u_i \in L$, produces $u_1\$u_1\#u_2\$u_2\#u_3\# \cdots u_{n-1}\#u_{n-1}\#u_n\# \in \{\varepsilon\} \cup \Sigma^*\$\Sigma^*\Sigma^*\Sigma^*\$$. Notice that $\Sigma^*\Sigma^*\$ is a 1-SD prefix code and that taking $k - 1$ consecutive factors from this language allows us to have a reach of $k$ factors of $L$. Then we define the function $g$

$$g = (\text{id}_{\Sigma^*} \cdot (\Sigma^* \triangleright \varepsilon))^{k-2} \cdot (\text{id}_{\Sigma^*} \cdot (\# \triangleright \varepsilon) \cdot \text{id}_{\Sigma^*} \cdot (\$ \triangleright \varepsilon)).$$
with domain $(\Sigma^* \# \Sigma^* \$)^{k-1}$ and, when applied to a word $v_1 \# v'_1 \$ v_2 \# v'_2 \$ \cdots v_{k-1} \# v'_{k-1} \$,$ produces $v_1 v_2 \cdots v_{k-1} v'_k$. In particular, \(g(u_{i+1} \# u_{i+2} \$ u_{i+3} \# u_{i+k-1} \# u_{i+k} \$) = u_{i+1} \cdots u_{i+k}\). Finally, we have

\[ [L, f]^{k\star} = ([\varepsilon \& \varepsilon] + (\Sigma^* \$ \& \varepsilon) \cdot [\Sigma^* \# \Sigma^* \$, f \circ g]^{(k-1)\star} \cdot (\Sigma^* \# \& \varepsilon)) \circ f_1. \]

The reverse $k$-star $[L, f]^{k\star}_r$ is not expressed in a straightforward fashion using reverse composed with $k$-star, because while reverse applies on all the input, the reverse $k$-star swaps the applications of function $f$ while keeping the function $f$ itself untouched. In order to express it, we reverse a $k$-star operator not on $f$, but on $f$ reversed. The result is that the applications of $f$ are reversed twice, thus preserving them. Formally, we have:

\[ [L, f]^{k\star}_r = \text{rev} \circ [L, \text{rev} \circ f]^{k\star}. \]

\section{Conclusion}

We conclude with some interesting avenues for future work, arising from the open questions based on this paper.

We begin with complexity questions, and then move on to other directions for future work. The complexity of our procedure, especially when going from the declarative language SDRTE to the computing machine 2DFT, is open. This part relies heavily on the composition of 2DFTs which incurs at least one exponential blowup in the state space. A possibility to reduce the complexity incurred during composition, is to obtain reversible 2FT (2RFT) for each of the intermediate functions used in the composition. 2RFTs are a class of 2DFTs which are both deterministic and co-deterministic, and were introduced in [10], where they prove that composition of 2RFTs results in a 2FT with polynomially many states in the number of states of the input transducers. Provided that the composition of 2RFTs preserves aperiodicity, if we could produce 2RFTs in our procedures in section 5.1, then we would construct a 2RFT which is polynomial in the size of the SDRTE. Another open question is the efficiency of evaluation, i.e., given an SDRTE and an input word, what is the time complexity of obtaining the corresponding output. This is crucial for an implementation, along the lines of DReX [2].

Yet another direction is to extend our result to transformations over infinite words. While Perrin [19] generalized the SF=AP result of Schützenberger to infinite words in the mid 1980s, Diekert and Kufleitner [12, 13] generalized Schützenberger’s SD=AP result to infinite words. One could use this SD=AP over infinite words and check how to adapt our proof to the setting of transformations over infinite words. Finally, a long standing open problem in the theory of transformations is to decide if a function given by a 2DFT is realizable by an aperiodic one. This question has been solved in the one-way case, or in the case when we have origin information [5], but the general case remains open. We believe that our characterisation of stabilising runs provided in Section 5.2.1 could lead to a forbidden pattern criteria to decide this question.

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