A New Scaling Law for Activity Detection in Massive MIMO Systems

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Abstract—In this paper, we study the problem of activity detection (AD) in a massive MIMO setup, where the Base Station (BS) has $M \gg 1$ antennas. We consider a block fading channel model where the $M$-dim channel vector of each user remains almost constant over a coherence block (CB) containing $D_c$ signal dimensions. We study a setting in which the number of potential users $K_c$ assigned to a specific CB is much larger than the dimension of the CB $D_c$ ($K_c \gg D_c$) but at each time slot only $A_c < K_c$ of them are active. Most of the previous results, based on compressed sensing, require that $A_c \leq D_c$, which is a bottleneck in massive deployment scenarios such as Internet-of-Things (IoT) and Device-to-Device (D2D) communication. In this paper, we show that one can overcome this fundamental limitation when the number of BS antennas $M$ is sufficiently large. More specifically, we derive a scaling law on the parameters $(M, D_c, K_c, A_c)$ and also Signal-to-Noise Ratio (SNR) under which our proposed AD scheme succeeds. Our analysis indicates that with a CB of dimension $D_c$ containing $A_c$ active users among a set consisting of $K_c$ potentially active users $K_c$, which makes it ideally suited for AD in IoT setups. We propose low-complexity algorithms for AD and provide numerical simulations to illustrate our results. We also compare the performance of our proposed AD algorithms with that of other competitive algorithms in the literature.

Index Terms—Activity detection, Internet of Things (IoT), Device-to-Device Communications, Massive MIMO.

I. INTRODUCTION

Massive connectivity is predicted to play a crucial role in future generation of wireless cellular networks that support Internet-of-Things (IoT) and Device-to-Device (D2D) communication. In such scenarios, a Base Station (BS) should be able to connect to a large number of devices and the underlying shared communication resources (time, bandwidth, etc.) are dramatically overloaded. However, a key feature of wireless traffic in those systems (especially, in IoT) is that the device activity patterns are typically sporadic such that over any communication resource only a small fraction of all potential devices are active. A feasible communication in those scenarios typically consists of two phases: i) identifying the set of active users by spending a fraction of communication resources, ii) serving the set of active users via a suitable scheduling over the remaining communication resources.

In this paper, we mainly investigate the first phase, namely, identifying the activity pattern of users, known as activity detection (AD). We consider a generic block fading wireless communication channel between devices and the BS [2]. We assume that the channel can be decomposed into a set of coherence blocks (CBs), where each CB consists of $D_c$ signal dimensions over which the channel fading coefficients of each user remain almost constant, whereas the channel might vary independently across different CBs [2]. AD is a fundamental challenge in massive deployments and random access scenarios to be expected for IoT and D2D (see, e.g., [3–5] for some recent works). A fundamental limitation when considering solely a single-antenna setting is that the required signal dimension $D_c$ to identify reliably a subset of $A_c$ active users among a set consisting of $K_c$ potentially active users scales as $D_c = O(A_c \log(K_c))$, thus, almost linearly with $A_c$. To keep up with the scaling requirements in IoT and D2D setup where $A_c$ and $K_c$ are dramatically large, it is crucial to overcome this limitation in an efficient way that does not require devoting too many CBs to AD. One of the recent works along this line is [6, 7], where the authors proposed using multiple antennas at the BS to overcome this fundamental problem. More specifically, they showed that by assigning random Gaussian pilot sequences to the users, one can identify any subset of active users with a vanishing error probability in a massive MIMO setup (when the number of BS antennas $M \to \infty$) [8] provided that $D_c$ is large. In contrast, in [9] the authors studied another variant of AD and showed that over a CB of dimension $D_c$, one is able to identify up to $O(D_c^2)$ active users provided that the Signal-to-Noise Ratio (SNR) of the active users is sufficiently large and the number of antennas $M \to \infty$. However, [6, 7, 9] do not specify the finite-length scaling law on the number of antennas, pilot dimension, the number of users, and the number of active users $(M, D_c, K_c, A_c)$ required for a reliable AD.

In this paper, we bridge the gap by deriving a finite-length scaling law on $(M, D_c, K_c, A_c)$ and SNR for the scheme proposed in [9]. In particular, we show that: i) with a sufficient number of BS antennas, with $A_c/M = o(1)$, one can identify reliably the activity of $A_c = O(D_c^2 \log^2(K_c))$ active users among a set of $K_c$ users, which is orders of magnitude better the bound $A_c = O(D_c/\log(K_c))$ obtained previously via traditional compressed sensing techniques (see, e.g., [3–5]). ii) one needs to pay only a logarithmic penalty $O(\log^2(K_c))$ for increasing the total number of users $K_c$. Both features make the proposed scheme very attractive for IoT setups, in
which the number of active users $A_c$ as well as the total number of users $K_c$ can be extremely large.

We propose several efficient and low-complexity algorithms for AD and show via theoretical analysis as well as numerical simulation that they fulfill our proposed scaling law. Our proposed AD algorithms depend only on the sample covariance of the observations obtained at the BS and are robust to statistical variations of the users channels.

A. Notation

We represent scalar constants by non-boldface letters (e.g., $x$ or $X$), sets by calligraphic letters (e.g., $X$), vectors by boldface small letters (e.g., $x$), and matrices by boldface capital letters (e.g., $X$). We denote the $i$-th row and the $j$-th column of a matrix $X$ with the row-vector $X_{i,:}$ and the column-vector $X_{:,j}$ respectively. We denote a diagonal matrix with elements $(s_1, s_2, ..., s_k)$ by $\text{diag}(s_1, s_2, ..., s_k)$. We denote the $p$-norm a vector $x$ and the Frobenius norm of a matrix $X$ by $\|x\|_p$ and $\|X\|_F$ respectively. The $k \times k$ identity matrix is represented by $I_k$. For an integer $k > 0$, we use the shorthand notation $[k]$ for $\{1, 2, ..., k\}$.

II. Problem Formulation

A. Signal Model

We consider a generic block fading wireless channel between each user and the BS consisting of several CBs with each CB containing $D_c$ signal dimensions over which the channel is almost flat [2], whereas it may change smoothly or almost independently across adjacent CBs. Without loss of generality, we assume that the BS devotes an individual CB to AD of a specific set of users $K_c$ consisting of $K_c := |K_c|$ users. To perform AD, the BS assigns a specific pilot sequence to each user in $K_c$, where a pilot sequence for a generic user $k \in K_c$ is simply a vector $a_k = (a_{k,1}, ..., a_{k,D_c})^T \in \mathbb{C}^{D_c}$ of length $D_c$, which is transmitted by the user $k$ over $D_c$ signal dimensions in the CB devoted to the AD if it is active. Denoting by $\mathbf{h}_k$ the $M$-dim channel vector of the user $k \in K_c$ to $M$ antennas at the BS, we can write the received signal at the BS over the signal dimension $i \in [D_c]$ inside the CB as

$$y[i] = \sum_{k \in K_c} b_k \sqrt{g_k} a_{k,i} \mathbf{h}_k + z[i],$$

where $a_{k,i}$ denotes the $i$-th element of the pilots sequence $a_k$, where $[D_c] := \{1, ..., D_c\}$, where $g_k \in \mathbb{R}_+$ denotes the large-scale fading coefficient (channel strength) of the user $k \in K_c$, where $b_k \in \{0, 1\}$ is a binary variable with $b_k = 1$ for active and $b_k = 0$ for inactive users, where $z[i] \sim \mathcal{CN}(0, \sigma^2 I_{M})$ denotes the additive white Gaussian noise (AWGN) at the $i$-th signal dimension, and where we used the block fading channel model [2] where the channel vector $\mathbf{h}_k$ of each user $k \in K_c$ is almost constant over the signal dimensions $i \in [D_c]$ inside the CB. Denoting by $\mathbf{Y} = [y[1], ..., y[D_c]]^T$ the $D_c \times M$ received signal over $D_c$ signal dimensions and $M$ BS antennas, we can write (1) more compactly as

$$\mathbf{Y} = \mathbf{A} \Gamma^\frac{1}{2} \mathbf{H} + \mathbf{Z},$$

where $\mathbf{A} = [a_1, ..., a_{K_c}]$ denotes the $D_c \times K_c$ matrix of pilot sequences of the users in $K_c$, where $\Gamma = \text{diag}(\gamma)$ with $\gamma := (\gamma_1, ..., \gamma_K)$, $\gamma_k \in \mathbb{R}_+$ denotes the channel strengths of the users ($\gamma_k = 0$ for inactive ones), and where $\mathbf{H} = [\mathbf{h}_1, ..., \mathbf{h}_{K_c}]^T$ denotes $K_c \times M$ matrix containing the $M$-dim normalized channel vectors of the users. We assume that the channel vectors $\{\mathbf{h}_k : k \in K_c\}$ are independent from each other and are spatially white (i.e., uncorrelated along the antennas), that is, $\mathbf{h}_k \sim \mathcal{CN}(0, I_{M})$. We assume that the user pilots are also normalized to $\|a_k\|^2 = D_c$ and define the average SNR of a generic active user $k \in K_c$ over $D_c$ pilot dimensions by

$$\text{snr}_k = \frac{\|a_k\|^2 \gamma_k}{\mathbb{E}[\|\mathbf{Z}\|^2]} = \frac{\sum_{i=1}^{K_c} \gamma_i a_i a_i^H}{I_M \sigma^2} = \frac{\gamma_k}{\sigma^2}, \quad k \in A_c.$$

We call the vector $\gamma$ or equivalently the diagonal matrix $\Gamma = \text{diag}(\gamma)$ the activity pattern of the users in $K_c$. We always assume that at each AD slot only a subset $A_c \subseteq K_c$ of the users of size $A_c := |A_c|$ are active, thus, $\gamma$ is a positive sparse vector with only $A_c$ nonzero elements. The goal of AD is to identify the subset of all active users $A_c$, or a subset thereof consisting of users with sufficiently strong channels $A_c(\nu) := \{k \in K_c : \gamma_k > \nu \sigma^2\}$, for a pre-specified threshold $\nu > 0$, from the noisy observations as in (2).

Since we assume that the channel vectors are spatially white and Gaussian, the columns of $\mathbf{Y}$ in (2) are i.i.d. Gaussian vectors with $\mathbf{Y}_{:,i} \sim \mathcal{CN}(0, \mathbf{I}_M)$ where

$$\Sigma_y = \mathbf{A} \Gamma \mathbf{A}^H + \sigma^2 I_M = \sum_{k=1}^{K_c} \gamma_k a_k a_k^H + \sigma^2 I_M$$

the covariance matrix, which is common among all the columns $\mathbf{Y}_{:,i}, i \in [M]$. We also define the empirical/sample covariance of the columns of the observation $\mathbf{Y}$ in (2) as

$$\hat{\Sigma}_y = \frac{1}{M} \mathbf{Y} \mathbf{Y}^H = \frac{1}{M} \sum_{i=1}^{M} \mathbf{Y}_{:,i} \mathbf{Y}_{:,i}^H.$$

B. Generalized Signal Model for Activity Detection

Before we proceed, it is worthwhile to mention that the signal model for AD in (1) can be generalized in several directions:

1) Only a fraction of signal dimensions in a CB is devoted to AD while the remaining signal dimensions are kept for communication: This reduces pilot dimension $D_c$ in (1), thus, the length/dimension of the pilots sequences $\{a_k : k \in K_c\}$ assigned to the users but preserves the number of antennas $M$. This is beneficial when the dimension of the CB $D_c$ is significantly large and the number of active users $A_c \subseteq K_c$ is not so large.

2) More than one, say $\kappa > 1$, CBs are devoted to AD of a specific set $K_c$ of users: As the simplest scheme, one can assume that the same length-$D_c$ pilot sequence $a_k \in \mathbb{C}^{D_c}$ of each user $k \in K_c$ is just repeated across the signal dimensions of all $\kappa$ CBs. Due to spatially white and Gaussian assumption of the channel vectors $\mathbf{h}_k$ in (1), this has the same effect as having only a single CB consisting of $D_c$ signal dimensions while effectively increasing the number of antennas to $M' = \ldots$
\( \kappa M \), i.e., by a factor of \( \kappa \). In general, instead of repeating the pilot sequence of each user over \( \kappa \) CBs, one can vary the pilot sequence of each user over different CBs. This yields more well-conditioned pilot sequences (sufficient randomness and better averaging over different CBs) but does not change the underlying scaling law asymptotically for large \( D_c \), namely, this is still equivalent to having the same pilot dimension \( D_c \) but increasing the number of antennas to \( M' = \kappa M \).

For simplicity, in the rest of the paper, we always assume that the AD is done over an individual CB by using the whole \( D_c \) signal dimensions in the CB.

III. PROPOSED ALGORITHMS FOR ACTIVITY DETECTION

We will propose now three different estimators to solve the activity problem with different assumptions on the underlying statistics of the channel vectors and the sparsity of the activity pattern of the users \( \gamma \).

A. Maximum Likelihood Estimate and Identifiability Condition

We first consider the Maximum Likelihood (ML) estimator of \( \gamma \) by making explicit use of Gaussianity, where after normalization and simplification we have

\[
f(\gamma) = -\frac{1}{M} \log p(Y|\gamma) = -\frac{1}{M} \sum_{i=1}^{M} \log p(Y_i;|\gamma) = \log |A\Gamma A^H + \sigma^2 I_M| + \text{tr} \left( (A\Gamma A^H + \sigma^2 I_{D_c})^{-1} \hat{\Sigma}_Y \right),
\]

where (a) follows from the fact that the columns of \( Y \) are i.i.d. (due to the spatially white user channel vectors), and where \( \hat{\Sigma}_Y \) denotes the sample covariance matrix of the columns of \( Y \) as in (4). Note that for spatially white channel vectors considered here, \( \hat{\Sigma}_Y \rightarrow \Sigma_Y \) as the number of antennas \( M \rightarrow \infty \). We also have the following result.

**Theorem 1:** Consider the signal model (2) for AD. Then, the empirical covariance matrix \( \hat{\Sigma}_Y \) is a sufficient statistics for the activity pattern of the users \( \gamma \).

**Proof:** From (7), it is seen that the log-likelihood of \( Y \) depends on \( Y \) only through the sample covariance matrix \( \hat{\Sigma}_Y \). Thus, from the Fischer-Neyman factorization theorem [10, 11], \( \hat{\Sigma}_Y \) is a sufficient statistics for estimating \( \gamma \).

**Remark 1:** Although we derived (7) for spatially white channel vectors, the sample covariance matrix \( \hat{\Sigma}_Y \) still provides an almost sufficient statistics for estimation of \( \gamma \) as far as the channel vectors are not highly correlated as happens, for example, in line-of-sight (LoS) propagation scenarios. Overall, correlation among the components of a user channel vector can be seen as reducing the effective number of antennas \( M \) (extreme case of \( M = 1 \) antenna in the LoS scenario), which degrades the AD performance accordingly.

We define the maximum likelihood (ML) estimate of \( \gamma \) as

\[
\gamma^* = \arg \min_{\gamma \in \mathbb{R}^{K_c}} f(\gamma).
\]

Note that due to the invariance of the ML estimator with respect to transformation of the parameters [12], \( \hat{\Sigma}_Y \) is also a sufficient statistics for identifying the set of active users \( A_c(\nu) := \{ i : \gamma_i^* > \nu D^2 \} \), where \( \nu > 0 \) is a suitable threshold. Before we proceed, it is worthwhile to investigate conditions under which the activity pattern \( \gamma \) is identifiable. We need some notation first. We define the convex cone produced by the rank-1 positive semi-definite (PSD) matrices \( \{ a_k a_k^H : k \in K_c \} \) generated by pilot sequences as

\[
\text{cone}(a_{K_c}) = \left\{ \sum_{k \in K_c} \beta_k a_k a_k^H : \beta_k \in \mathbb{R}_+ \right\}.
\]

We also use a similar notation \( \text{cone}(a_{V}) \) for the cone generated the pilot sequences of a subset of users \( V \subseteq K_c \). Note that for any \( V \), the cone \( \text{cone}(a_{V}) \) is a sub-cone of the cone of PSD matrices. With this notation, we can also write the ML estimation in (8) equivalently as estimating the cone corresponding to the set of active users. To make sure that the set of active users are identifiable, we need to impose the following identifiability condition on the pilot sequences:

\[
\text{cone}(a_{V}) \neq \text{cone}(a_{V'}) \quad \text{for all } V \neq V', \quad |V|, |V'| \leq \vartheta,
\]

where \( \vartheta \) is set to a sufficiently small number to make sure that condition (10) is fulfilled. Note that since the space of PSD matrices, denoted by \( \mathcal{S}_{D_c}^+ \), has (affine) dimension \( D_c(D_c+1)/2 \), the number of active \( \vartheta \) users should be less than \( D_c(D_c+1)/2 \) for (10) to be fulfilled. This bound can be also seen to be tight from Carathéodory theorem [13]. However, interestingly, this does not restrict the number of potential users \( K_c = |K_c| \). For example, when the pilot sequences \( a_k \), \( k \in K_c \), are sampled i.i.d. from an arbitrary continuous distribution, no matter how large \( K_c \) is, with probability 1, all the sub-cones in (10) would be different and the identifiability condition would be fulfilled provided that \( \vartheta < D_c(D_c+1)/2 \) (as also stated in Theorem 1 in [9]). This implies that, at least in theory, one can identify the activity of \( A_c = D_c(D_c+1)/2 \) users in a set of users of arbitrary size \( K_c \). In practice, however, due to the presence of the noise, one should also limit \( K_c \) in order to guarantee a stable AD.

Now let us focus on the ML estimation in (7). It is not difficult to check that \( f(\gamma) \) in (7) is the sum of the concave function \( \gamma \mapsto \log |A\Gamma A^H + \sigma^2 I_{D_c}| \) and the convex function \( \gamma \mapsto \text{tr} \left( (A\Gamma A^H + \sigma^2 I_{D_c})^{-1} \hat{\Sigma}_Y \right) \), so it is not convex in general. However, the following theorem implies that under suitable conditions, the global minimum of \( f(\gamma) \) can be calculated exactly.

**Theorem 2:** Let \( f(\gamma) \) be as in (7). Suppose that the pilot sequences \( \{ a_k : k \in K_c \} \) are such that \( \text{cone}(a_{K_c}) \) coincides with the cone of \( D_c \times D_c \) PSD matrices \( \mathcal{S}_{D_c}^+ \). Then, \( f(\gamma) \) has only global minimizers.

**Proof:** Proof in Appendix A-B

Note that in order for \( \text{cone}(a_{K_c}) \) to be a good approximation of the cone of \( D_c \times D_c \) PSD matrices as in Theorem 2, \( K_c \) should be larger than the dimension of the space of PSD matrices, i.e., \( K_c \gtrsim O(D_c^2) \). Interestingly, as we will see later, this holds in the desired scaling regime that we derive for AD problem and is the highly desirable setup for
Algorithm 1 Activity Detection via Coordinate-wise Descend

1: **Input:** The sample covariance matrix \( \Sigma_Y = \frac{1}{M} YY^H \) of the \( D_c \times M \) matrix of samples \( Y \).
2: **Initialize:** \( \Sigma = \sigma^2 I_{D_c}, \gamma = 0 \).
3: for \( i = 1, 2, \ldots \) do
4: Select an index \( k \in [K_c] \) corresponding to the \( k \)-th component of \( \gamma = (\gamma_1, \ldots, \gamma_{K_c}) \) randomly or according to a specific schedule.
5: ML: Set \( d^* = \max \left\{ \frac{\|a_k^H \Sigma^{-1} Y - a_k \Sigma^{-1} a_k^H \|}{\|a_k\|}, \gamma_k \right\} \)
6: MMV: Set \( d^* = \max \left\{ \frac{\|a_k^H \Sigma^{-1} Y - a_k \Sigma^{-1} a_k^H \|}{\|a_k\|}, \gamma_k \right\} \)
7: NNLS: Set \( d^* = \max \left\{ \frac{\|a_k^H \Sigma^{-1} Y - a_k \Sigma^{-1} a_k^H \|}{\|a_k\|}, \gamma_k \right\} \)
8: Update \( \gamma_k \leftarrow \gamma_k + d^* \).
9: Update \( \Sigma \leftarrow \Sigma + d^*(a_k a_k^H) \).
10: end for
11: **Output:** The resulting estimate \( \gamma \).

IoT applications. Finding the ML estimate \( \gamma^* \) in (8) requires optimization of \( f(\gamma) \) over the positive orthant \( \mathbb{R}_{+}^{K_c} \). In view of Theorem 2, we can apply simple off-the-shelf algorithms such as gradient descent followed by projection onto \( \mathbb{R}_{+}^{K_c} \) to find \( \gamma^* \).

In Appendix A-A, we derive such a coordinate-wise descend algorithm, where at each step we optimize \( f(\gamma) \) with respect to only one of its arguments \( \gamma_k, k \in [K_c] \), and we iterate several times over the whole set of variables until convergence. We can also include the noise variance \( \sigma^2 \) as an optimization parameter and estimate it along with the parameters \( \gamma_k, k \in [K_c] \). As \( f(\gamma) \) has no local minimizers from Theorem 2 (in the regime we consider in this paper), the coordinate-wise optimization will converge to the unique global minimizer of \( f(\gamma) \) with sufficiently many iterations. This coordinate-wise optimization has a closed form expression as derived in Appendix A-A and is summarized in Algorithm 1.

B. Multiple Measurement Vector Approach

Let us consider the signal model \( Y = \mathbf{A} \Gamma^\frac{1}{2} \mathbf{H} + \mathbf{Z} \) in (2) and let us define \( \mathbf{X} := \Gamma^\frac{1}{2} \mathbf{H} \). Then, we can write (2) as

\[
\mathbf{Y} = \mathbf{A} \mathbf{X} + \mathbf{Z}.
\]

This signal model also arises in a compressed sensing setup (14), (15), where one tries to recover a signal vector from the noisy measurements \( Y \), where the \( D_c \times K_c \) pilot matrix \( \mathbf{A} = [a_1, \ldots, a_{K_c}] \) plays the role of the measurement matrix. Note that since the activity patterns \( \gamma \) is a sparse vector, it induces a common (joint) sparsity among the columns of \( \mathbf{X} := \Gamma^\frac{1}{2} \mathbf{H} \). Recovery of \( \mathbf{X} \) from measurements \( \mathbf{Y} \) in (11) is typically known as the Multiple Measurement Vector (MMV) problem in compressed sensing (16) (17). A quite well-known technique for recovering the row-sparse matrix \( \mathbf{X} \) in the MMV setting is \( \ell_{2,1}\)-norm Least Squares (\( \ell_{2,1}\)-LS) as in

\[
\mathbf{X}^* = \arg \min_{\mathbf{X}} \frac{1}{2} \| \mathbf{A} \mathbf{X} - \mathbf{Y} \|^2_F + \varrho \sqrt{M} \| \mathbf{X} \|_{2,1},
\]

where \( \varrho > 0 \) is a regularization parameter and where \( \| \mathbf{X} \|_{2,1} = \sum_{i=1}^{K_c} \| X_{i,:} \| \) denotes the \( \ell_{2,1}\)-norm of the matrix \( \mathbf{X} \) given by the sum of \( \ell_{2,1}\)-norm of its \( K_c \) rows. It is well-known that \( \ell_{2,1}\)-norm regularization in (12) promotes the sparsity of the rows of the resulting estimate \( \mathbf{X}^* \) and seems to be a good regularizer for detecting the user activity pattern \( \gamma \) in our setup. Intuitively speaking, we expect that we should obtain an estimate of activity pattern of the users \( \gamma \), i.e., the strength of the their channels, by looking at the \( \ell_2\)-norm of the rows of the estimate \( \mathbf{X}^* \). We have the following result.

Theorem 3: Let \( \mathbf{X}^* \) be the optimal solution of \( \ell_{2,1}\)-LS as in (12). Let \( \gamma^* = (\gamma_1^*, \ldots, \gamma_{K_c}^*) \in \mathbb{R}_{+}^{K_c} \), where \( \gamma_i^* = \| X_{i,:} \|_{2,1} \).

Then, \( \gamma^* \) is the optimal solution of the convex optimization problem \( \gamma^* = \arg \min_{\gamma \in \mathbb{R}_{+}^{K_c}} g(\gamma) \), where \( g(\gamma) \) is defined by

\[
g(\gamma) := \text{tr}(\Gamma) + \text{tr} \left( (\mathbf{A} \mathbf{X}^* + \varrho \mathbf{I}_{D_c} \Delta)^{-1} \Sigma_Y \right),
\]

where \( \Gamma = \text{diag}(\gamma) \) and \( \Sigma_Y \) is the sample covariance (5).

Proof: The proof follows from Theorem 1 in (13) and is included in Appendix A-C for the sake of completeness.

We will prove that in terms of AD we can aim for a scaling regime \( A_c = O(D_c^2) \), where indeed \( A_c \gg D_c \). However, when \( A_c \gg D_c \), the number of active (nonzero) rows in \( \mathbf{X} \) is much larger than the number of available measurements \( D_c \) via the matrix \( \mathbf{A} \) in (12). In this regime, any algorithm (including \( \ell_{2,1}\)-LS) would have a considerable distortion while decoding \( \mathbf{X} \).

Even a Minimum Mean Squared Error (MMSE) estimator that knows the exact location of active rows (the support of \( \gamma \)) will have a significant MSE distortion while decoding the active rows (although it can perfectly identify the inactive ones from the knowledge of the support of \( \gamma \)). It is interesting and highly non-trivial that when only estimation of the activity pattern \( \gamma \) is concerned, due to the implicit underlying averaging effect of \( \ell_{2,1}\)-LS stated in Theorem 3, namely, the fact that \( \gamma^* = \| X_{i,:} \|_{2,1} \) is given by \( \ell_{2,1}\)-norm of the \( i \)-th row of \( \mathbf{X}^* \), even a noisy estimate \( \mathbf{X}^* \) is sufficient to yield a reliable estimate of \( \gamma \).

Theorem 3 also implies that as far as the strengths (i.e., \( \ell_{2,1}\)-norm) of the rows of \( \mathbf{X}^* \) are concerned, \( \ell_{2,1}\)-LS in (12) can be simplified to the convex optimization (13), which depends on the observations \( \mathbf{Y} \) through their empirical covariance, which is a sufficient statistics for identifying \( \gamma \) from Theorem 4.

Note that the function \( g(\gamma) \) in (13) is a convex function, thus, finding the optimal solution \( \gamma^* \) can be posed as a convex optimization problem over the positive orthant \( \gamma \in \mathbb{R}_{+}^{K_c} \), which can be solved by conventional convex optimization techniques. Using the well-known Schur complement condition for positive semi-definiteness (see (19) page 28), one can also pose the optimization in (13) as the following semi-definite program (SDP):

\[
(\gamma^*, \Delta^*) = \arg \min \text{tr}(\Gamma) + \text{tr}(\Lambda)
\text{ s.t. } \left( \mathbf{A} \mathbf{X}^* + \varrho \mathbf{I}_{D_c} \Delta \right) \geq 0,
\]

where \( \Lambda \) denotes the auxiliary \( D_c \times D_c \) optimization variable, where \( \Gamma = \text{diag}(\gamma) \) and where \( \Delta \) is the square root of \( \Sigma_Y \) (i.e., \( \Sigma_Y = \Delta \Delta^H \)). In contrast with the MMV optimization in (12), the complexity of (14) does not grow by increasing the number of antennas \( M \).

Remark 2: In (14) one can treat also the variable \( \varrho \) as a free optimization variable to directly estimate also the noise power (note that the resulting cost function is still a jointly
convex function of both variables $(\gamma, \varrho)$ and has a unique global optimal solution). As a result, the algorithm does not require the knowledge of the noise power at the BS (although, in practice, the noise power can be easily estimated at the BS from the received noise at un-allocated signal dimensions). °

In this paper, we propose a simple coordinate-wise descend algorithm for minimizing the MMV cost function [13]. The derivation is very similar to that of the coordinate-wise descend algorithm for the ML cost function in Appendix A-A and has the closed-form expression summarized in Algorithm 1. 

C. Non-Negative Least Squares

The last algorithm that we introduce and also theoretically analyze here is based on Non-Negative Least Squares (NNLS) proposed in [9]. We consider the signal model (2) and $\hat{\Sigma}_y$ as in (5). In the NNLS, we match $\hat{\Sigma}_y$ to the true $\Sigma_y$ in (4) in $\ell_2$-distance:

$$\gamma^* = \arg \min_{\gamma \in \mathbb{R}^{K_c}} \| \hat{\Sigma}_y - \text{Adiag}(\gamma) \mathbf{A}^H - \sigma^2 I_{D_c} \|^2_F.$$  \hspace{1cm} (15)

Let $\hat{\sigma}_y = \text{vec}(\hat{\Sigma}_y)$ denotes the $D_c^2 \times 1$ vector obtained by stacking the columns of $\hat{\Sigma}_y$ and let $\mathbb{K}$ be an $D_c^2 \times K_c$ matrix whose $k$-th column, $k \in [K_c]$, is given by $\text{vec}(a_k a_k^H)$. Then, we can write (15) in the convenient form

$$\gamma^* = \arg \min_{\gamma \in \mathbb{R}^{K_c}} \| \hat{\sigma}_y - \mathbf{A} \gamma - \sigma^2 \text{vec}(I_{D_c}) \|^2_F, \hspace{1cm} (16)$$

as a least squares problem with non-negativity constraint, known as nonnegative least squares (NNLS). If the row span of $\mathbb{K}$ intersects the positive orthant, NNLS implicitly also performs $\ell_1$-regularization, as discussed for example in [20] and called $\mathcal{M}^+$-criterion in [5]. Because of these features, NNLS has recently gained interest in many applications in signal processing [21], compressed sensing [5], and machine learning. In our case the $\mathcal{M}^+$-criterion is fulfilled in an optimally-conditioned manner and allows us to establish the following result:

**Theorem 4:** Let $\{a_k\}_{k=1}^{K_c} \subset \mathbb{C}^{D_c}$ independent copies of a random vector with iid. unit-magnitude entries and $K_c \geq 16$. Fix some $\delta \in (8/K_c, 4/\sqrt{41})$. If

$$D_c(D_c - 1) \geq c' \delta^{-2} s \log^2 (e K_c/s) \hspace{1cm} (17)$$

then with overwhelming probability the following holds: For all activity pattern vectors $\gamma^0$, the solution $\gamma^*$ of (16) is guaranteed to fulfill for $1 \leq p \leq 2$:

$$\| \gamma^0 - \gamma^* \|_p \leq \frac{2C}{s} \frac{\sigma_s(\gamma^0)}{s} + \frac{2D_c}{s^2} \left( 1 + \frac{\lambda \gamma^0}{\sqrt{D_c}} \right) \frac{\| \mathbf{d} \|_2}{D_c} \hspace{1cm} (18)$$

where $\sigma_s(\gamma^0)$ denotes the $\ell_1$-norm of $\gamma^0$ after removing its $s$ largest components, where $\mathbf{d} = \text{vec}(\Sigma_y - \sum_{k=1}^{K_c} \gamma_k a_k a_k^H)$, and where $C$, $D_c$ and $\lambda^*$ only depend on $\delta$, and $\lambda$ is a global constant.

The proof, given in the appendix A-D is based on a combination of the NNLS results of [5] and an extension of RIP-results for the heavy-tailed column-independent model [22, 23]. We like to mention that the theorem holds even for more general random models and the constant can be computed explicitly. For example, $\delta = 0.5$ gives $\gamma^* \approx 1.5$, $C \approx 8.6$ and $D \approx 11.3$. The result is uniform meaning that with high probability (on a draw of $\mathbb{K}$) it holds for all $\gamma^0$. For $s = A_c = \| \gamma^0 \|_0$ it implies (up to $\| \mathbf{d} \|_2$-term) exact recovery since then $\sigma_s(\gamma^0) = 0$. A relevant extension of this result to the case $p \to \infty$ would be important but, in this generality, it is known that then one can not hope for a linear scaling in $s$ (see here for example [24, Theorem 3.2]). Since $\| \gamma^0 \|_p \leq \mathbb{E} \| \gamma^0 \|_p$ our result (18) also implies an estimate for the communication relevant $\ell_2$-case but with sub-optimal scaling (we will discuss this below). Furthermore improvements for this particular case may be possible in the non-uniform or averaged case, as it has been investigate for the subgaussian case in [20].

A straightforward analysis in Appendix A-E shows that for Gaussian and spatially white user channel vectors, the statistical fluctuation term $\mathbf{d}$ in Theorem 4 concentrates very well around its mean given by

$$\| \mathbf{d} \| = \| \text{vec}(\hat{\Sigma}_y - \sum_{k=1}^{K_c} \gamma_k a_k a_k^H - \sigma^2 I_{D_c}) \| = \| \text{vec}(\hat{\Sigma}_y - \Sigma_y) \| = \| \hat{\Sigma}_y - \Sigma_y \|_F \approx \sqrt{\mathbb{E} \| \hat{\Sigma}_y - \Sigma_y \|_F^2} \approx \frac{\text{tr} (\Sigma_y)}{\sqrt{M}} \hspace{1cm} (19)$$

Assuming that at a specific AD slot only $A_c$ of the users are active and setting $s = A_c$ in Theorem 4 using $\sigma_s(\gamma^0) = 0$ for $s = A_c$, and the well-known inequality $\| \gamma^0 \|_p \leq \sqrt{A_c} \| \gamma^0 \|_2$, we obtain the following scaling law on the performance of NNLS

$$\| \gamma^0 - \gamma^* \|_p \leq c_3 \left( \frac{(D_c^2)}{s} \frac{\sigma_s(\gamma^0)}{s} + \frac{D_c}{s^2} \sqrt{\frac{A_c}{M}} \| \gamma^0 \|_2 \right) \hspace{1cm} (20)$$

with a constant $c_3 = 2c_2 \left( 1 + \frac{r^* \sqrt{s}}{\sqrt{A_c M}} \right) = O(1)$ provided that

$$A_c \log^2 \left( \frac{K_c}{A_c} \right) \leq O(D_c^2). \hspace{1cm} (21)$$

It is important to note that: i) from (20) the error term only depends on the number of active users $A_c$ and vanishes if the number of antennas $M$ is slightly larger than $A_c$ (i.e., $A_c/M = o(1)$) so that sufficient statistical averaging happens, ii) from (21) NNLS can identify as large as $O(D_c^2)$ active users by paying only a poly-logarithmic penalty $O(\log^2 (K_c))$ for increasing the number of potential users $K_c$. As stated before, this is a very appealing scaling law in setups such as IoT and D2D, where $K_c$ can be dramatically large.

IV. Simulation Results

In this section, we evaluate the performance of our algorithms via numerical simulations.

A. Performance Metric

We assume that the output of each algorithm is an estimate $\gamma^*$ of the activity pattern of the users. We define $A_c(\nu) := \{ i : \gamma^*_i > \nu \sigma^2 \}$, with $\nu > 0$, as the estimate of the
set of active users. We also define the detection/false-alarm probability averaged over the active/inactive users as
\[ p_D(\nu) = \frac{\mathbb{E}[|A_c \cap \hat{A}_c|]}{A_c}, \quad p_{FA}(\nu) = \frac{\mathbb{E}[|\hat{A}_c \setminus A_c|]}{K_c - A_c} \]

(22)

where \( A_c \) and \( K_c \) denote the number of active and the number of potential users, respectively. By varying \( \nu \in \mathbb{R}^+ \), we plot the Receiver Operating Characteristic (ROC) as a measure of performance of our proposed algorithms.

B. Comparison with the Literature

1) Vector Approximate Message Passing

We compare the performance of our proposed algorithm with that of Vector Approximate Message Passing (VAMP) algorithm proposed for AD in [6, 7]. Given the noisy observation \( Y = AX + Z \) as in (11), VAMP recovers an estimate of the row-sparse signal \( X \) as follows
\[
X^{t+1} = \eta(\mu^{t} + X^{t}, g, t),
\]
\[
R^{t+1} = Y - AX^{t+1} + \frac{K_c}{\mu^{t}}R^{t}(\nu(\mu^{t} + X^{t}, g, t))
\]

(23)

(24)

where \( \eta \) is the MMSE vector denoising function applied row-wise, where \( \nu(\cdot) \) denotes the component-wise derivative of \( \eta \), where \( g = (g_1, \ldots, g_{K_c})^T \) denotes the large-scale fading coefficients of the users, where \( \langle \cdot \rangle \) denotes the averaging operator, and \( R^{t}(\nu(\mu^{t} + X^{t}, g, t)) \) is the well-known Onsager’s term. It was shown in [26, 27] that, due to the Onsager correction, in the asymptotic regime of \( D_c, A_c, K_c \to \infty \), the noisy input \( X^t := A\mu^{t} + X^t \) to the MMSE vector denoiser decouples to (for spatially white user channel vectors)
\[
\tilde{X}^t_{i,:} = X^t_{i,:} + \tau^t N_{i,:}, \quad i \in [K_c],
\]

(25)

where \( N \) is a \( K_c \times M \) matrix consisting of i.i.d. \( \mathcal{CN}(0, 1) \) elements, and where \( \tau^t \) is obtained by a simple state evolution (SE) equation \( \tau^t = \xi(\tau^{t-1}, g) \). We refer to [6, 7] for further explanation of VAMP and the derivation of MMSE denoiser and the SE equation. For simulations, we consider an ideal scenario for VAMP where the large-scale fading coefficients \( g \) are known at the BS and are used in the VAMP algorithm and also in the derivation of the SE equation.

2) Empirical Genie-aided tuning of VAMP (g-VAMP)

We have observed empirically that, for large number of BS antennas \( M \), VAMP is quite sensitive and numerically unstable when the number of active users \( A_c \) is much larger than the pilot dimension \( D_c \). To avoid this numerical instability, we apply a genie-aided tuning of the VAMP, where at each iteration we use the rows of \( \tilde{X}^t := A\mu^{t} + X^t \) (the noisy input to the MMSE vector denoiser as in (23)) corresponding to the inactive users to estimate the noise variance \( \tau^t \) in the decoupled model (25). We call the resulting algorithm g-VAMP and use it for comparison.

3) Comparison with our proposed algorithms

Our proposed algorithms have the following advantages compared with VAMP:

1) they do not require the knowledge of large-scale fading coefficients of the channel \( g = (g_1, \ldots, g_{K_c}) \) (or its statistics) and the number of active users \( A_c \), which is required for deriving (and tuning) the VAMP algorithm but is difficult to have in IoT setups due to the sporadic nature of the traffic of the users (especially \( A_c \)).

2) they require only the sample covariance of channel observations, thus, they are quite robust to variation in the statistics (i.e., Gaussianity, spatial correlation, etc.) of the user channel vectors, whereas having the knowledge of the statistics of the channel vectors is quite critical for deriving the MMSE vector denoiser in VAMP and obtaining the appropriate SE equation.

3) they do not require any tuning and are numerically stable.

C. Scaling with Number of Antennas \( M \)

In this section, we compare the performance of our proposed algorithm for different number of BS antennas \( M \). We consider a CB of dimension \( D_c = 100 \), and a total number of \( K_c = 2000 \) users assigned to the CB. We assume that at each slot only \( A_c = 200 \) out of \( K_c = 2000 \) users are active. We also assume a symmetric scenario where all the active users have the same large-scale fading coefficients \( g_k, k \in [K_c] \), and an SNR of \( 10^3 \) dB. It is worthwhile to note that our algorithms do not require the knowledge of channel strengths \( g_k, k \in [K_c] \). Fig.1 illustrated the simulation results. It is seen that

1) NNLS does not perform well when the number of antennas is less the the number of active users \( (M = 100, A_c = 200) \) but its performance dramatically improves by increasing the number of antennas \( (M = A_c = 200) \).

2) As reported in [6, 7], VAMP performs very well when the pilot dimensions is close to the number of active users \( (D_c \lesssim A_c) \). Our results also confirm this and illustrate that in the more interesting regime of \( D_c \ll A_c \) (here \( D_c = 100, A_c = 200 \)) suited for IoT, VAMP performance is comparable to that of MMV for \( M = 100 \) while MMV works much better for \( M = 200 \). Also, note that MMV as well as NNLS and ML do not require the knowledge of the large-scale fading coefficient of the
channel, thus, are quite robust.

3) ML performs much better than all the other algorithms and requires much less number of antennas $M$.

D. Scaling with the total Number of Users $K_c$

In this section, we investigate the dependence of the performance of the algorithms on the total number of users $K_c$. We again assume a CB of dimension $D_c = 100$, and $A_c = 200$ active users. We set the number of antennas $M = A_c = 200$ to guarantee a good performance for all the algorithms. We vary the total number of users in $K_c = \{2000, 4000\}$. Fig. 2 illustrates the simulation results. It is seen that the performance of NNLS (and that of the other algorithms) is not sensitive to the number of users $K_c$, as expected from the scaling law (the poly-logarithmic dependence on $K_c$) claimed in Theorem 4.

V. CORRELATED USER CHANNEL VECTORS

In this paper, we always assumed that the channel vector of all the users are spatially white (see also Remark 1). In this section, we repeat the simulations for spatially correlated channel vectors. We consider a simple case where the BS is equipped with a Uniform Linear Array (ULA) with $M$ antennas. It is well-known that for the ULA, the spatial covariance matrix of the channel vectors becomes a Toeplitz matrix, which can be diagonalized approximately with the Discrete Fourier Transform (DFT) matrix such that the correlated channel vector $h \in \mathbb{C}^M$ of a generic user can be written as $h = F_M w$, where $F_M$ denotes the DFT matrix of order $M$ with normalized columns and where $w = (w_1, \ldots, w_M)^T$ denotes a complex Gaussian vector with independent components $w_i \sim \mathcal{CN}(0, \beta_i)$, where $\sum_{i=1}^M \beta_i = \mathbb{E}[\|w\|^2] = \mathbb{E}[\|h\|^2] = M$.

In a ULA, the $i$-th column of $F_M$ corresponds to the array response for a planar wave with the angle of arrival (AoA) $\theta_i = -\theta_{\max} + \frac{2(i-1)\theta_{\max}}{M}$, $i \in [M]$, where $\theta_{\max}$ denotes the maximum angle covered by the BS antennas, and where $w_i$ denotes the contribution to the channel vector $h$ coming from those scatterers lying in the interval $\theta \in [\theta_i, \theta_{i+1})$. We refer to [28] [29] for further details.

Fig. 2: Scaling of performance of the algorithms vs. $K_c$.

Fig. 3: Comparison of the performance of the algorithms for spatially white and spatially correlated channel vectors with the same effective number of antennas $M = 200$ (spatially white) and $M = 400$, $M_{eff} = 200$ (spatially correlated).

To create spatially correlated channel vectors, we will assume an angular block-sparse propagation model where $\beta_i = \left\{ \frac{M}{M_{eff}} \text{ for } i \in \{i_0, \ldots, (i_0 + M_{eff} - 1) \mod M\} \right\}$, such that only $M_{eff}$ consecutive AoAs are active with $i_0$ denoting the index of the start of the block. As also mentioned in Remark 1 such a spatial correlation reduces the effective number of antennas from $M$ to $M_{eff} < M$.

For the simulations, we assume that the start of the block $i_0 \in [M]$ is selected uniformly at random for each user. We also assume that $\frac{M_{eff}}{M} = 0.5$ such that the channel vector of each user has the normalized angular spread of 50% (compared with the whole angular spread $2\theta_{\max}$ of the array). Fig. 3 illustrates the simulation results. For the case of correlated channel vectors, we consider $M = 400$ antennas and 50% spatial correlation, which amounts to $M_{eff} = 200$ effective number of antennas. We compare the results with the case of spatially white channel vectors. It is seen from Fig. 3 that the results match very well, which confirms the fact that spatial correlation reduces the effective number of antennas (as stated in Remark 1).

VI. CONCLUSION

In this paper, we studied the problem of user activity detection in a massive MIMO setup, where the BS has $M \gg 1$ antennas. We showed that with a CB containing $D_c$ signal dimensions one can stably estimate the activity of $A_c = O(D_c^2 / \log^2 (\frac{K_c}{A_c}))$ active users in a set of $K_c$ users, which is much larger than the previous bound $A_c = O(D_c)$ obtained via traditional compressed sensing techniques. In particular, in our proposed scheme one needs to pay only a poly-logarithmic penalty $O(\log^2 (\frac{K_c}{A_c}))$ for increasing the number of potential users $K_c$, which makes it ideally suited for activity detection in IoT setups. We proposed low-complexity algorithms for activity
Appendix A
Proofs of the Results

A. Derivation of the coordinate-wise ML Optimization

In this section, we derive a closed-form expression for the coordinate-wise optimization of the ML cost function \( f(\gamma) \) in (7). Let \( k \in [K_c] \) be the index of the selected coordinate and let us define \( f_k(d) = f(\gamma + de_k) \) where \( e_k \) denotes the \( k \)-th canonical basis with a single 1 at its \( k \)-th coordinate and zero elsewhere. Setting \( \Sigma = \Sigma(\gamma) = A \Gamma A^H + \sigma^2 I_{D_c} \) where \( \Gamma = \text{diag}(\gamma) \) and applying the well-known Sherman-Morrison rank-1 update identity \([30]\) we obtain that

\[
(\Sigma + da_k a_k^H)^{-1} = \Sigma^{-1} - \frac{d a_k^H \Sigma^{-1} \Sigma^{-1} a_k}{1 + d a_k^H \Sigma^{-1} a_k}.
\]

(26)

Using (26) and applying the well-known determinant identity

\[
|\Sigma + da_k a_k^H| = (1 + d a_k^H \Sigma^{-1} a_k)|\Sigma|,
\]

(27)

we can simplify \( f_k(d) \) as follows

\[
f_k(d) = c + \log|\Sigma| + \text{tr}(\Sigma^{-1} \Sigma_y) = c + \log(1 + d a_k^H \Sigma^{-1} a_k)
\]

\[
- \frac{a_k^H \Sigma^{-1} \Sigma_y \Sigma^{-1} a_k}{(1 + d a_k^H \Sigma^{-1} a_k)^2}.
\]

(28)

where \( c = \log|\Sigma| + \text{tr}(\Sigma^{-1} \Sigma_y) \) is a constant term independent of \( d \). Note that from (28), \( f_k(d) \) is well-defined only when \( d > d_0 := -\frac{1}{a_k^H \Sigma^{-1} a_k} \). Taking the derivative of \( f_k(d) \) yields

\[
f_k'(d) = \frac{a_k^H \Sigma^{-1} \Sigma_y \Sigma^{-1} a_k}{(1 + d a_k^H \Sigma^{-1} a_k)^2}.
\]

The only solution of \( f_k(d) \) is given by

\[
d^* = \frac{a_k^H \Sigma^{-1} \Sigma_y \Sigma^{-1} a_k}{(a_k^H \Sigma^{-1} a_k)^2}.
\]

(29)

Note that \( d^* \geq d_0 = -\frac{1}{a_k^H \Sigma^{-1} a_k} \), thus, one can check from (28) that \( f_k \) is indeed well-defined at \( d = d^* \). Moreover, we can check from (28) that \( \lim_{a_k \to 0^+} f_k(d_0 + \varepsilon) = \lim_{d \to +\infty} f_k(d) = +\infty \), thus, \( d = d^* \) must be the global minimum of \( f_k(d) \) in \((d_0, +\infty)\). Note that since after the update we have \( \gamma_k \leftarrow \gamma_k + d \), to preserve the positivity of \( \gamma_k \), the optimal update step \( d \) is in fact given by \( \max\{d^*, -\gamma_k\} \) as illustrated in Algorithm 1.

B. Proof of Theorem 2

This follows from the geodesic convexity of the ML cost function \( f(\gamma) \) \([31]\) when \( A \Gamma A^H \) spans the whole set of \( D_c \times D_c \) PSD matrices. Here, we provide another simple proof.

Consider \( \text{cone}(K_c) = \{ \sum_{k=1}^{K_c} \beta_k a_k a_k^H : \beta_k \geq 0 \} \) as a subcone of PSD matrices \( S_{D_c}^+ \) produced by the pilot sequences. By our assumption, \( \text{cone}(K_c) \) approximates the cones of \( D_c \times D_c \) PSD matrices \( S_{D_c}^+ \) very well. Let us first define

\[
P_{D_c} := \{ \sigma^2 I_{D_c} + \sum_{k=1}^{K_c} \beta_k a_k a_k^H : \beta_k \in \mathbb{R}_+ \}
\]

(30)

\[
= \{ \sigma^2 I_{D_c} + C : C \in \text{cone}(K_c) \}
\]

(31)

\[
= \{ \sigma^2 I_{D_c} + S_{D_c}^+ \}
\]

(32)

where in (a) we used the assumption that \( \text{cone}(K_c) \approx S_{D_c}^+ \). We also define \( Q_{D_c} = \mathcal{P}^{-1}_{D_c} := \{ P^{-1} : P \in \mathcal{P}_{D_c} \} \). It is not difficult to check that

\[
Q_{D_c} = \{ Q \in S_{D_c}^+ : \lambda_{\text{max}}(Q) \leq \frac{1}{\sigma^2} \}
\]

(33)

where \( \lambda_{\text{max}} \) denotes the largest singular value of a matrix. Since \( \lambda_{\text{max}} \) is a convex function over the convex set of PSD matrices \( S_{D_c}^+ \), it results that \( Q_{D_c} \) is indeed a convex subset of \( S_{D_c}^+ \). Applying this change of variable, we can, therefore, write the ML estimation of \( \gamma \) equivalently as the following optimization problem

\[
Q^* = \arg\min_{Q \in Q_{D_c}} - \log|Q| + \text{tr}(Q \Sigma_y).
\]

(34)

Note that since \( Q \mapsto - \log|Q| + \text{tr}(Q \Sigma_y) \) is a convex function of \( Q \) and \( Q_{D_c} \) is a convex set, \( (34) \) is a convex optimization problem, whose local minimizers are global minimizers as well. This implies that the ML cost function \( f(\gamma) \) in (7) has only global minimizers.

C. Proof of Theorem 3

The proof follows by extending Theorem 1 in \([33]\). The key observation is that for a \( u \in \mathbb{C}^M \), the \( \ell_2 \) norm \( \|u\| \) can be written as the output of the following optimization

\[
\|u\| = \min_{v \in \mathbb{C}^M, \delta \in \mathbb{C}^M : \delta v = u} \frac{1}{2}(\|v\|^2 + \|\delta\|^2).
\]

(35)

In particular, \( \|u\| = \|\delta^*\|^2 \), where \( s^* \) is the optimal solution of \( (35) \). Applying this argument to the rows of \( X \), we can write the \( l_{2,1} \) norm of \( X \) as follows

\[
\|X\|_{2,1} = \min_{V \in \mathbb{C}^{K_c \times M}, \Delta \in \mathbb{D}, \Delta v = x} \frac{1}{2}(\|V\|^2 + \|\Delta\|^2).
\]

(36)

where \( \mathbb{D} \) denotes the space of \( K_c \times K_c \) diagonal matrices with diagonal elements in \( \mathbb{C} \), and where \( \Delta = \text{diag}(\delta_1, \ldots, \delta_{K_c}) \in \mathbb{D} \). In particular, \( \|X_{.:k}\| = \|\delta_k^*\|^2 \), where \( \Delta^* = \text{diag}(\delta_{K_c}^*, \ldots, \delta_1^*) \) is the optimal solution of \( (36) \). Replacing \( \|X\|_{2,1} \) with \( (36) \), we can transform the optimization problem (12) into

\[
(V^*, \Delta^*) = \arg\min_{V \in \mathbb{C}^{K_c \times M}, \Delta \in \mathbb{D}} \frac{1}{M} \sum_{k=1}^{M} \|\Delta V_{.:k} - Y_{.:k}\|^2_2 + \|\Delta\|^2.
\]

(37)

For a fixed \( \Delta \), the minimizing \( V \) as a function of \( \Delta \) can be obtained via a least-square minimization, where after replacing the solution in \( (37) \) and applying the matrix inversion lemma \( [32] \) and further simplifications, we obtain the following optimization in terms of \( \Delta \)

\[
\Delta^* = \arg\min_{\Delta \in \mathbb{D}} \frac{\|\Delta\|^2}{\sqrt{M}} + \frac{1}{M} \sum_{k=1}^{M} \text{tr}(A \frac{\Delta \Delta^H}{\sqrt{M}} \Delta^H + \phi I_m)^{-1} y(t)y(t)^H).
\]

(38)
Note that this optimization can be reparameterized with $P = \sqrt{M} = \text{diag}(\frac{1}{\sqrt{M}}, \ldots, \frac{1}{\sqrt{M}}) \in \mathbb{D}_+$, where $\mathbb{D}_+$ denotes the space of all $K_c \times K_c$ diagonal matrices with positive diagonal elements. Thus, we can write:

$$
P^* = \arg \min_{P \in \mathbb{D}_+} \text{tr}(P) + \frac{1}{M} \sum_{k=1}^{M} \text{tr}\left( (AP^H + g \mathbf{I}_m)^{-1} y(t)(y(t))^H \right)$$  \hspace{1cm} (39)

Denoting by $P^* = (p'_1, \ldots, p'_n)$ the optimal solution of (39), we have the underlying relation $\|X_k\| = \|p'_k\|^2 = p'_k$. This implies that the optimal solution $\gamma$ in the statement of the theorem, is indeed the solution of the following convex optimization

$$
\gamma^* := \arg \min_{\gamma \in \mathbb{D}_+} \sum_{k=1}^{K_c} \gamma_k \left( (\text{Adiag}(\gamma) A^H + g \mathbf{I}_m)^{-1} \mathbf{a}_{d} \right),
$$

where we replaced $\mathbf{a}_{d} = \frac{1}{M} \mathbf{Y} \mathbf{Y}^H = \frac{1}{M} \sum_{k=1}^{M} \mathbf{Y}_k \mathbf{Y}_k^H$ and defined $\Gamma = \text{diag}(\gamma)$. This completes the proof.

D. Proof of the Recovery Guarantee for NNLS, Theorem 4

Let $\{A_k := a_k a_k H\}_{k=1}^{K_c}$ be the rank-1 models generated by the pilots sequences of the users. We consider the noisy model $\tilde{\mathbf{y}} = \sum_{k=1}^{K_c} \gamma_k \mathbf{a}_k + \mathbf{d}$ where $\gamma_k = (\gamma_1, \ldots, \gamma_{K_c}) \in \mathbb{R}_{+}^{K_c}$ denotes the unknown activity pattern of the users to be estimated. We assume that $\gamma$ is potentially s-sparse or compressible (well-approximated by sparse vectors) and $\mathbf{d}$ is a residual error matrix. In particular, the vectorized problem is:

$$\hat{\mathbf{y}} = \text{vec}(\tilde{\mathbf{y}}) = : A\gamma^o + \mathbf{d}.$$  \hspace{1cm} (40)

where $\mathbf{d} = \text{vec}(\mathbf{D})$, $\gamma^o \in \mathbb{R}_{+}^{K_c}$ and $\mathbf{A}$ is a $D^2 \times K_c$ matrix whose $k$-th column is given by vec($a_k a_k H$). We are interested in the behavior of the NNLS solution (16) for a given sparsity $s$ of the users to be estimated. Furthermore, this special structure gives us the inequality:

$$\|\gamma - \gamma^o\|_p \leq \frac{2C}{s^{1-p}} \sigma_s(\gamma^o)^{1-p} + \frac{2D}{s^{1-p}} \left( \frac{1}{D_c} + \frac{1}{p} \right) \|\mathbf{d}\|_2$$  \hspace{1cm} (43)

with $C = \frac{1}{s^{1-p}}$ and $D = \frac{2D}{s^{1-p}}$. This argumentation is already sufficient to replace regular $\ell_1$-minimization with NNLS.

The essential main task here is now to establish that the nullspace property for the random matrix $\mathbf{A}$ holds with high probability for the desired sampling rates. To this end, we will restrict to those measurements which are related to the isotropic part of $\mathbf{A}$. More precisely, define the “centered” matrices $A_i^c := a_i a_i H - D_c$. Now it easy to check that if $a_k$ has independent and unit magnitude entries it follows that for all complex matrices $\mathbf{Z}$ it holds:

$$E(|(A_k^c, Z)|)^2 = E(|(a_k, Z_{sk}) - tr(Z)|)^2 = \|Z\|^2$$  \hspace{1cm} (44)

meaning that vec$(A_k^c)$ is a complex-isotropic random vector. Furthermore, this special structure gives us the inequality:

$$\|\mathbf{A} \mathbf{v}\|^2_2 = \|\mathbf{A}^o \mathbf{v}\|^2_2 + D_c \|\mathbf{1}_{K_c} \mathbf{v}\|^2 \geq \|\mathbf{A}^o \mathbf{v}\|^2_2$$  \hspace{1cm} (45)

where $\mathbf{A}^o$ is the corresponding matrix having the “centered” columns vec$(A_i^c)$. Indeed, the inequality above is tight since already 2-sparse vectors can be orthogonal to $\mathbf{1}_{K_c}$. This implies, that if the matrix $\mathbf{A}^o$ has $\ell_p$-NSP with the same parameters,

To establish the $\ell_p$-NSP of $\mathbf{A}^o$ we shall make use of existing RIP results for the real-valued matrices with independent isotropic and heavy-tailed columns. Therefore, let us consider the equivalent real matrix $\mathbf{A}^o := [\text{Re}(\mathbf{A}^o); \text{Im}(\mathbf{A}^o)] \in \mathbb{R}^{2D_c \times K_c}$ and denote its $k$-th column by vec$(A_k^R) \in \mathbb{R}^{2D_c}$ where $A_k^R := \text{Re}(A_k^o) + i \cdot \text{Im}(A_k^o)$. For any real matrix $\mathbf{Z}$ we have

$$E(|(A_k^R, Z)|)^2 = \|\text{Re}(A_k^o, Z) + \text{Im}(A_k^o, Z)\|^2 = \|A_k^o, Z\|^2$$  \hspace{1cm} (46)
Thus, the real matrix $\mathcal{A}^R$ has (real-)isotropic and independent columns with subexponential marginal norms. We adopt a normalization $\lambda > 0$ such that we have $\mathbb{E} \exp(|(\lambda \mathcal{A}^R_k, Z)|) \leq 2$.

More precisely, from the inequality \[43\] Lemma 2.5] we have that the subexponential norm $|(\lambda \mathcal{A}^R_k, Z)| = \inf \{ K \geq 0 : \mathbb{E} \exp(|x|/K) \leq 2\}$ is controlled by the second moment, i.e., there is a universal constant $\lambda$ such that for all $Z$ with $\|Z\|_F = 1$ we have:

$$\|\langle \lambda \mathcal{A}^R_k, Z \rangle\|_{\psi_1} \leq (\mathbb{E}(\|\mathcal{A}^R_k, Z\|)^2)^{1/2} = 1$$ (47)

Using this normalization, a general $\ell_2$-RIP statement for a matrix $\frac{1}{\sqrt{m} \mathcal{A}^R}$ has been shown in \[22, 23]. Define the restricted isometry constant

$$\delta_{2s} := \sup_{0 \leq \|v\|_2 \leq 2s} \left| \frac{\|\lambda \mathcal{A}^R v \|_2^2}{m \|v\|_2^2} - 1 \right|$$ (48)

If $\delta_{2s} \in [0, 1)$ the matrix $\frac{1}{\sqrt{m} \mathcal{A}^R}$ has $\ell_2$-RIP of order $2s$. Furthermore, if even $\delta_{2s} < 4/\sqrt{11} \approx 0.62$ then $\frac{1}{\sqrt{m} \mathcal{A}^R}$ has the $\ell_2$-robust NSP of order $s$ with parameters with $\rho = \delta_{2s}/(\sqrt{1 - \delta_{2s}^2} - \delta_{2s}/4)$ and $\tau' = \sqrt{1 + \delta_{2s}/(\sqrt{1 - \delta_{2s}^2} - \delta_{2s}/4)}$ \[33\] Theorem 6.13]. We use now \[23, 24\] Theorem 1, case 2, $\alpha = 1$. Assume $8/\delta \leq K_c \leq c \exp(c \sqrt{m}/2)$ (for example take $K_c \geq 16$ to have some feasible $\delta \leq 4/\sqrt{11}$; for the constant $c$, see \[22\]) and set the sparsity parameter to the following integer part:

$$2s = \frac{\delta_{2s}^2}{c'} \log \left( \frac{c' K_c}{m} + 2 \right)$$ (49)

then, according to mention result in \[23\] it holds that $\mathbb{P}\{\delta_{2s} \leq \delta\} \geq 1 - 2^{-\alpha} \delta$ ($c'$ is the constant $C^2$ in \[23\]). Furthermore for $m \geq c' \delta_{2s}/\delta^2$ this can be rearranged to:

$$D_c(D_c - 1) = m \geq c' \delta_{2s}^2 \log^2(K_c/s)$$ (50)

Therefore for all $v \in \mathbb{R}^{K_c}$ and all subsets $S \subset [K_c]$ with $|S| \leq s$ holds:

$$\|v_S\|_2 \leq \frac{\rho}{\sqrt{s}} \|v\|_1 + \frac{\tau'\lambda}{\sqrt{m}} \|\lambda \mathcal{A}^R v\|_2$$

$$= \frac{\rho}{\sqrt{s}} \|v\|_1 + \frac{\tau'\lambda}{\sqrt{m}} \|\lambda \mathcal{A}^R v\|_2$$ (51)

$$\leq \frac{\rho}{\sqrt{s}} \|v\|_1 + \frac{\tau'\lambda}{\sqrt{m}} \|\lambda \mathcal{A}^R v\|_2$$

with $\rho \leq \delta/(\sqrt{1 - \delta^2} - \delta/4)$ and $\tau' \leq \sqrt{1 + \delta/(\sqrt{1 - \delta^2} - \delta/4)}$. Summarizing, $\ell_2$-RIP of order $2s$ for $\frac{1}{\sqrt{m} \mathcal{A}^R}$ implies $\ell_2$-NSP of $\mathcal{A}$ of order $s$ with parameters $\rho$ and $\tau = \tau'\sqrt{m}$.

Finally, let us write \[43\] for the case $q = 2$ (therefore $1 \leq p \leq 2$) in a more convenient form since $m = D_c(D_c - 1)$:

$$\|\gamma^{\frac{p}{2}} - \gamma^\star\|_p \leq \frac{2C}{s \tau'} \sigma_s(\gamma^\star) + \frac{2D}{s \tau'} (1 + D_c \tau) \frac{\|d\|_2}{D_c}$$ (52)

$$= \left( \frac{2C}{s \tau'} \sigma_s(\gamma^\star) + \frac{2D}{s \tau'} \left( 1 + \frac{\lambda \tau' \sqrt{D_c}}{\sqrt{D_c - 1}} \right) \right) \frac{\|d\|_2}{D_c}$$

which is \[18\].

E. Analysis of Error of the Sample Covariance Matrix

We first consider the simple case where $\{y(t) : t \in [M]\}$, are M i.i.d. realization of a $D_c$-dim complex Gaussian vector with zero mean $\mathbb{E}w(t) = 0$ and a diagonal covariance matrix $\Sigma_y = \mathbb{E}[y(t)y(t)^\dagger] = \text{diag}(\beta)$, thus, $y_i(t) \sim CN(0, \beta_i)$, $i \in [D_c]$. Let us denote by $\Delta = \Sigma_y - \Sigma_y$ be the deviation of the sample covariance matrix from its mean. Note that the $(i, j)$ component of $\Delta$ is given by:

$$\Delta_{ij} = \frac{1}{M} \sum_{t \in [M]} y_i(t)y_j^\ast(t) - \beta_i \delta_{ij}$$ (53)

where $\delta_{ij} = \mathbb{1}_{i=j}$ denotes the discrete delta function. Note that $\mathbb{E}[\Delta_{ij}] = 0$ for all $i, j$. Moreover, since $\Delta_{ij}$ is the average of $M$ i.i.d. terms, $\{y_i(t)y_j^\ast(t) - \beta_i \delta_{ij} : t \in [M]\}$, we have

$$\mathbb{E}[\|\Delta_{ij}\|^2] = \mathbb{E} \left[ \mathbb{E}[\|\Delta_{ij}\|^2] - \beta_i \delta_{ij} \right]$$ (54)

For $i \neq j$, we have that

$$\mathbb{E}[\|y_i(t)y_j^\ast(t) - \beta_i \delta_{ij}\|^2] = \mathbb{E}[\|y_i(t)y_j^\ast(t)\|^2]$$

$$= \mathbb{E}[\|y_i(t)^2\|^2 - 2\beta_i \mathbb{E}[\|y_i(t)\|^2] + \beta_i^2]$$

$$= 2\mathbb{E}[\|y_i(t)^2\|^2] - 2\beta_i^2 + \beta_i^2$$

where in (a) we used the independence of the different components of $y(t)$. Also, for $i = j$, we have that

$$\mathbb{E}[\|y_i(t)y_j^\ast(t) - \beta_i \delta_{ij}\|^2] = \mathbb{E}[\|y_i(t)^2\|^2 - 2\beta_i \mathbb{E}[\|y_i(t)\|^2] + \beta_i^2]$$

$$= 2\mathbb{E}[\|y_i(t)^2\|^2] - 2\beta_i^2 + \beta_i^2$$

where in (a) we used the identity $\mathbb{E}[\|y_i(t)^2\|^2 = 2\mathbb{E}[\|y_i(t)\|^2]$. Overall, from \[55\] and \[56\], we can write $\mathbb{E}[\|\Delta_{ij}\|^2] = \frac{\beta_i \delta_{ij}}{M}$. Thus, we have that

$$\mathbb{E}[\|\Delta\|^2] = \sum_{ij} \mathbb{E}[\|\Delta_{ij}\|^2] = \frac{\sum_{ij} \beta_i \delta_{ij}}{M} = \frac{(\sum \beta_i)^2}{M} = \frac{\mathbb{E}[\Sigma_y]^2}{M}$$ (57)

Remark 3: It is worthwhile to mention that although \[57\] was derived under the Gaussianity of the observations $\{y(t) : t \in [M]\}$, the result can be easily modified for general distribution of the components of $y(t)$. More specifically, let us define

$$\max_i \mathbb{E}[\|y_i(t)\|^4] =: \varsigma < \infty.$$ (58)

Then, using \[55\] and applying \[58\] to \[56\], we can obtain the following upper bound

$$\mathbb{E}[\|\Delta\|^2] \leq \max(\varsigma - 1, 1) \times \frac{\sum_{ij} \beta_i \delta_{ij}}{M}$$ (59)

$$\leq \max(\varsigma - 1, 1) \times \frac{\text{tr}(\Sigma_y)^2}{M}$$ (60)

which is equivalent to \[57\] up to the constant multiplicative factor $\max(\varsigma - 1, 1)$. \"
In practice, \( \| \Delta \|^2 \) concentrates very well around its mean \( \mathbb{E}[\| \Delta \|^2] \). Therefore, the deviation between the true and empirical covariance matrix can be approximated by

\[
\| \Sigma - \Sigma_y \|_F \simeq \frac{\text{tr}(\Sigma_y)}{\sqrt{M}}.
\]  

(61)

Now, assume that the covariance matrix \( \Sigma_y \) is not in a diagonal form and let \( \Sigma_y = \text{UDU}^H \) be the Singular Value Decomposition (SVD) of \( \Sigma_y \). By multiplying all the vectors \( y(t) \) by the orthogonal matrix \( \text{U}^H \) to whiten them and noting the fact that multiplying by \( \text{U}^H \) does not change the Frobenius norm of a matrix, we can see that (61) holds true in general also for non-diagonal covariance matrices.

APPENDIX B

ACKNOWLEDGEMENT

The authors would like to thank R. Kueng for inspiring discussions and helpful comments. P.J. is supported by DFG grant JU 2795/3.

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