Boundary Superpotentials

Eric Sharpe
Physics Department
Princeton University
Princeton, NJ 08544
ersharp@puhep1.princeton.edu

In this paper we work out explicit lagrangians describing superpotential coupling to the boundary of a 5D orientifold, as relevant to a number of quasi-realistic models of nature. We also make a number of general comments on orientifold compactifications of M theory.

November 1996
1. Introduction

Historically heterotic string theory has been the starting point for phenomenologically promising string vacua, with \( N=1 \) supersymmetry in 4D. Although recent advances in the understanding of string duality have put other theories on an equal footing, it is still gives the most realistic concrete string vacua and is a convenient starting point for many studies.

The heterotic string theory compactified to four dimensions has a variety of possible strong coupling limits. One of them, which is inherited from the strong coupling limit in ten dimensions, involves the appearance of a fifth large dimension. In this particular limit, the heterotic string theory compactified to \( R^4 \) (on a compact six-manifold \( X \)) is equivalent to eleven-dimensional M theory compactified (on the same \( X \)) to \( R^4 \times S^1/Z_2 \). Taking account of actual numerical values of coupling constants, as explained in detail in [6, 7], it appears that the \( S^1/Z_2 \) can be much larger than \( X \), so that in this regime, which actually has some possible phenomenological virtues, the universe is quasi-five-dimensional.

The purpose of this paper is to study physics in this quasi-five-dimensional regime. We will in fact focus on the question of the superpotential. There is no superpotential in supersymmetric theories in five dimensions, but superpotentials can definitely be present in \( N=1 \) supersymmetric dynamics in four dimensions, and play a crucial role. How can a superpotential be present in a world that macroscopically looks like \( R^4 \times S^1/Z_2 \)? The answer must be that as the superpotential is impossible in bulk – where the world looks five-dimensional – it must appear as a boundary interaction. That explains the title and theme of this paper: we will consider supersymmetric field theories on \( R^4 \times S^1/Z_2 \) and search for boundary interactions that generate an effective superpotential for the massless four-dimensional modes.

The paper is organized as follows. In section two, we describe several classes of models to which the discussion applies, of which the strong coupling limit of the heterotic string is actually only one. In sections three and four we describe in rigid supersymmetry the boundary couplings that gives superpotentials (the extension to supergravity will be discussed elsewhere [4]). We begin in section three with just hyper plets in the 5D bulk, and demonstrate how to couple the boundary chiral plets descending from hyper plets to a superpotential. We also discuss coupling boundary-confined fields to the same superpotential. In section four we generalize to theories with both vector plets and hyper plets in the bulk. In this case there are two physically distinct possible orientifolds, and we discuss superpotential couplings for both cases. In section five, we discuss anomalies and the compactification of the Green-Schwarz mechanism. In section six we conclude, with a discussion of some novel supersymmetry-breaking mechanisms possible in these orientifold theories, and some novel insights in nonperturbative effects in heterotic string theory.

2. Examples of orientifold compactifications

Actually, the discussion here has certain applications that go beyond the motivation that was stated above. Our framework is relevant to any \( N=1 \) supersymmetric model on a
space-time that looks macroscopically like $R^4 \times S^1 / Z_2$. The strong coupling limit of the heterotic string is an important example, but there are several other examples.

To describe some of these systematically, start with M theory on $Y = R^4 \times S^1 \times X$, where $X$ is a Calabi-Yau six-manifold. To obtain an $N = 1$ model in four dimensions, we divide $Y$ by a $Z_2$ symmetry $\tau$ that acts trivially on $R^4$, acts as $-1$ on $S^1$, and acts as some involution $\tau'$ (or $Z_2$ symmetry) of $X$. Supersymmetry will hold under several possible conditions on $\tau'$:

1. If $\tau'$ acts trivially on $X$, we get M theory on $R^4 \times S^1 / Z_2 \times X$, which is the strong coupling limit of the heterotic string, discussed above as a motivating example.

2. If $\tau'$ acts on $X$ in a way that preserves the complex structure and the holomorphic three-form, we get a more general supersymmetric orientifold. The fixed points of such a $\tau'$ on $X$ will be curves of $A_1$ singularities, so the fixed points of $\tau$ on $Y$ will look locally like $R^6 / Z_2$. Such a model may also have five-branes wrapped around two-cycles in $(S^1 \times X) / Z_2$ in analogy to [16].

3. If $\tau'$ is an anti-holomorphic isometry that reverses the complex structure of $X$, then $(S^1 \times X) / Z_2$ is a seven-dimensional orbifold whose structure group – an extension of $SU(3)$ by $Z_2$ – is a subgroup of $G_2$. Such an orbifold can plausibly be deformed to a smooth manifold of $G_2$ holonomy – as in the work of Joyce [13] – and such orbifolds would appear to make sense in M theory even if they cannot be so deformed.

Let us discuss the three cases in more detail.

1. First, recall the compactification of M theory on $X$ [3]. In 5D we get $h_{1,1}$ vectors ($h_{1,1} - 1$ vector-plets), and $h_{2,1} + 1$ hyper-plets. The vectors all descend from the 11D 3-form potential. There are $h_{2,1}$ hyper-plets each consisting of a complex boson obtained from zero modes of the 11D metric along complex structure deformations, and a pair of real bosons obtained from the 11D 3-form on elements of $H^2,1$ and $H^{1,2}$. In addition, there is one hyper-plet consisting of the volume of $X$, the real scalar dual to a 4D 3-form potential, and two scalars obtained from the 11D 3-form on the holomorphic and antiholomorphic 3-forms of $X$.

Under the orientifold action [4, 2, 3], the 11D 3-form potential is odd, so we can immediately read off the fact that each of the $h_{1,1}$ 5D vectors is odd under the orientifold, and half of each of the $h_{2,1} + 1$ hyper-plets are projected out by the orientifold, yielding a total of $h_{1,1} + h_{2,1} + 1$ chiral-plets on the boundary. This yields a new derivation of one of the old spacetime superpotential nonrenormalization theorems [13], as will be discussed in more detail later.

On each boundary of the 11D theory, one gets an $E_8$ vector. In compactification, a gauge

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1 Consider the normal bundle to the fixed-point set. As $\tau'$ preserves both the complex structure and the holomorphic three-form, it must be the case that the normal bundle has even rank, so as $X$ has complex dimension three, the fixed point set must have dimension one or three. The latter corresponds to case (1).
sheaf is embedded in each $E_8$, and the resulting matter spectrum is well-known. Matter fields are in one-to-one correspondence with sheaf cohomology groups, with Serre duality exchanging particles and antiparticles.

(2) Examples of this sort in six dimensions are well-documented in the literature. For example, in [17] Sen considers M theory on $(K3 \times S^1)/Z_2$, with $\tau'$ acting on $K3$ so as to preserve the complex structure and holomorphic two-form. In the compactification of M theory to 7D on $K3$, we get 7D vector pllets corresponding to elements of $H^2$; $\tau'$ leaves 14 elements of $H^2$ invariant and flips the sign of the other 8. As a result, 14 of the 7D vector pllets are projected to 6D hyper pllets on the boundary, and 8 are projected to 6D vector pllets on the boundary. In addition, Sen assigns magnetic three-form charge $-1/2$ to each of the fixed points of $\tau'$, and notes that half of the fixed points of $\tau'$ have five-branes, essentially just as was done in [16].

By fibering $K3$ over a $P^1$, we can construct ($K3$-fibration) Calabi-Yau 3-folds with an orbifold symmetry $\tau'$ of the desired form. As above, if $\tau'$ leaves an element of $H^{1,1}$ invariant, then the corresponding 5D vector pllet will project to a chiral pllet on the boundary; if $\tau'$ flips the sign of an element of $H^{1,1}$, then the corresponding 5D vector pllet will project to a vector pllet on the boundary. Note that symmetries which act on elements of $H^{1,1}$ are only present on subsets of Kahler moduli space, just as symmetries which act on elements of $H^{2,1}$ are only present on subsets of complex structure moduli space; this fact will be reflected in constraints on intersection numbers, as will be discussed later in this paper.

Here is a related example of case (2), following [28, 29]. Recall the action of elementary transformations on 3-folds, which act on elements $D$ of $H^2$ as

$$D \rightarrow D + (D \cdot C) E$$

where $E$ is a divisor swept out by a family of holomorphic 2-spheres $C$. As $E \cdot C = -2$, this acts as a reflection on $E$. (For additional mathematical details see [30].) As is well-known, at the fixed point of a symmetry that acts on the Teichmuller space, one gets an enhanced symmetry – in this case, when the divisor $E$ collapses to a curve. For example, the Calabi-Yau $P^1_{1,1,2,2,2}$[8] has a genus 3 curve of $A_1$ singularities at $z_1 = z_2 = 0$. At generic points in Kahler moduli space this curve is blown up into a divisor. As is well-known from mirror symmetry [12] the Kahler moduli space of this 3-fold has a $Z_2$ quotient singularity, precisely

2 Recall from [1] that even after resolving the base space of a heterotic (0,2) compactification, it still may not be possible to resolve the singularities of the “bundle,” so we shall use more nearly correct terminology and refrain from calling the gauge sheaves “bundles.”

3 Recall from [1], that at least in certain (non-geometric) heterotic (0,2) compactifications, moduli associated with bundle deformations and moduli associated with complex moduli of the base are in some sense intermingled. Here we see that in M theory, these moduli are clearly distinguished: complex structure and Kahler moduli of the Calabi-Yau 3-fold propagate in the 11D bulk, whereas moduli associated with bundle deformations (corresponding to elements of $H^m(X, \text{End} V)$) only propagate on the boundary. It is also true that in F theory compactifications [14], the base space moduli and the bundle moduli are easily distinguished.
as expected at the enhanced symmetry point. Returning to our orientifold, if we orientifold M theory by the action defining the $E_8 \times E_8$ heterotic string and simultaneously orbifold the Calabi-Yau by this enhanced symmetry, then we not only will freeze the Calabi-Yau Kahler moduli at a singular point (yielding enhanced gauge symmetry in 5D) but now some of the 5D vector plets will be projected to boundary vectors (whereas in the standard heterotic orientifold, all 5D vectors are projected to chiral plets on the boundary).

(3) As $\tau'$ is an antiholomorphic isometry, some elements of $H^{1,1}$ are odd under the involution, so corresponding 5D vector plets will be even and will project to vectors on the boundary. If any elements of $H^{1,1}$ are even under $\tau'$, then the corresponding 5D vector plets will project to chiral plets on the boundary, as in the previous cases. In addition, there will be boundary-confined chiral plets reflecting modes that in perturbative string theory would be associated with twisted sectors.

3. Pure hyper plet action in bulk

To warm up, we will first consider the case that the bulk theory contains only hyper plets, no vectors. The bulk 5D N=2 rigid matter action is

$$S = \int_M \left[ -\frac{1}{2} \eta^{\mu \nu} g_{x'y'} \partial_\mu \sigma^{x'} \partial_\nu \sigma^{y'} - \frac{1}{4} \overline{\lambda}' \Gamma^\mu D_\mu \lambda_{a'} + 4 \text{ fermi} \right]$$

where

$$D_\mu \lambda_{a'} = \partial_\mu \lambda_{a'} + (\partial_\mu \sigma^{x'}) \omega_{x'a'b'} \lambda_{b'}$$
$$\partial_{y'} f_{a'x'}^{i} = -\omega_{y'a'b'} f_{b'}^{i} + \Gamma_{x'y'}^{i} f_{x'}^{a'}$$
$$\delta \sigma^{x'} = \frac{i}{2} f_{a'b'}^{x'} \epsilon \lambda^{a'}$$
$$\delta \lambda_{a'} = -i f_{a'x'}^{i} \Gamma^\mu \partial_\mu \sigma^{x'} \epsilon_i$$

Conventions closely follow those of [20, 21, 22], and are also outlined in an appendix.

$x'$ labels coordinates $\sigma$ on the hyperKahler manifold

$a'$ is an index in the fundamental of Sp(n); it labels the fermions $\lambda$ of the n hyperplets

$i$ is an index in the fundamental of Sp(1)

The orientifold symmetry is as follows. Under $x^\mu \rightarrow + x^\mu$, for $\mu < 5$, and $x^5 \rightarrow - x^5$, the fermions transform as:

$$\lambda_{a'} \rightarrow + i \Gamma_5 \lambda_{a'}$$
$$\lambda_{a'} \rightarrow - i \Gamma_5 \lambda_{a'}$$
We assume the hyperKahler manifold has an orientifold symmetry which preserves one of the complex structures (but not the others), so that in a coordinate patch containing the boundary of the orientifolded hyperKahler manifold, half the coordinates are invariant (call these $\sigma^x_1$ and $\sigma^x_2$) and the other half get a sign flip (call these $\sigma^x_3$ and $\sigma^x_4$). In other words, in a basis adapted to the complex structure preserved by the orientifold, in each hyperplet one chiralplet is even under the involution and one is odd. [After coupling to supergravity \cite{4}, we will assume that the quaternionic manifold of scalars has such an involution, whose fixed point set is also Kahler.]

Under the orientifold the “veirbein” $f$ transforms as

$$f^{x'a}_{x'i} \rightarrow f_{x'i}^{x'a}$$

The supersymmetries that commute with the orientifold are given by

$$\epsilon_i = +i \Gamma_5 \epsilon^i$$
$$\epsilon^i = -i \Gamma_5 \epsilon_i$$

The complete action describing the coupling of an $N=1$ superpotential $W$ on the boundary is

$$S_{bulk} = \int_M \left[ -\frac{1}{2} f^{\mu'\nu'} g_{x'y'} \partial_\mu \sigma^{x'} \partial_\nu \sigma^{y'} - \frac{1}{4} \chi' \Gamma^\mu D_\mu \lambda_{a'} + 4 \text{fermi} \right]$$

$$S_{boundary} = \int_{\partial M} \left[ g^{xy} (\partial_x W) (\partial_y W) \delta(0) - \frac{1}{8} (D_x D_y W) \bar{\chi}^x \chi^y + \frac{1}{8} (D_\sigma D_{\bar{\sigma}} W) \bar{\chi}^{\sigma} \chi_{\bar{\sigma}} + 4 \text{fermi} \right]$$

$$S = S_{bulk} + S_{boundary}$$

where, in the bulk theory,

$$D_x D_y W = \partial_x \partial_y W - \Gamma_{xy} \partial_z W$$
$$D_\mu \lambda_{a'} = \partial_\mu \lambda_{a'} + (\partial_\mu \sigma^{x'}) \omega_{x'ab'} \lambda^{b'}$$
$$\delta \sigma^{x'} = \frac{i}{2} f_{x'a'}^{x'i} \epsilon^i \lambda^{a'}$$
$$\delta \lambda^{a'} = -i f_{x'a'}^{x'i} \Gamma^\mu \partial_\mu \sigma^{x'} \epsilon^i + \delta(x_5) F^{i'a'} \epsilon^i$$
$$F^{1A} = + (\partial_x W) f_{1A}^{x} + (\partial_\sigma W) f_{1A}^{\sigma}$$
$$F^{2A} = - (\partial_x W) f_{2A}^{x} - (\partial_\sigma W) f_{2A}^{\sigma}$$
$$F^{i'a'} = - F_{i'a'}$$

Factors of $\delta(0)$ in this paper appear for essentially the reasons discussed by Horava and Witten in \cite{4, 2, 3}.
We have defined the boundary fields

\[ A^x = \sigma^{x1} + i \sigma^{x2} \]
\[ \chi^A = \lambda^A - i \lambda^{A'} \]
\[ \chi^x = 2 \left[ f^x_{1A} + if^x_{2A} \right] \lambda^A \]
\[ = 2 f^x_{1A} \lambda^A \]
\[ \epsilon = \epsilon^1 + i \epsilon^2 \]
\[ \epsilon^* = \epsilon^1 - i \epsilon^2 \]

In other words, \( A^x \) is the complex boson that is even under the involution.

As is typical in 4D N=1 supersymmetry, the superpotential \( W \) is holomorphic in the chiral superfields.

Note that in these conventions, \( \epsilon \) and \( \chi^x \) both have positive chirality.

Note that \( \epsilon^* \) is not the complex conjugate of \( \epsilon \), but in fact is merely another linear combination of spinors, of opposite four-dimensional chirality, convenient for our purposes here.

By restricting the bulk supersymmetry transformations to the boundary we find

\[ \delta A^x = \frac{i}{4} \bar{\sigma} \chi^x \]
\[ \delta A^\overline{A} = \frac{i}{4} \epsilon^* \chi^\overline{A} \]
\[ \delta \chi^x = -\Gamma^\mu (\partial_\mu A^x) \epsilon^* + i\delta(0) g^{\overline{A} \overline{B}} (\partial_\overline{B} W) \epsilon + 3 \text{ fermi} \]
\[ \delta \chi^\overline{A} = +\Gamma^\mu (\partial_\mu A^\overline{A}) \epsilon - i\delta(0) g^{\overline{A} \overline{B}} (\partial_\overline{B} W) \epsilon^* + 3 \text{ fermi} \]

Recall that \( x' \) in the bulk theory denotes a coordinate on the hyperKahler space; here, \( x \) and \( \overline{x} \) denote holomorphic and antiholomorphic coordinates on the Kahler manifold. Also, recall that \( a' \) was in the fundamental of \( \text{Sp}(n) \); on the boundary, we’ve split \( a' = 1...2n \) into \( A = 1...n \) and \( A' = n+1...2n \).

There is a constraint on the coordinates \( \sigma^{x1}, \sigma^{x2} \):

\[ f^{x1}_{1A} = + f^{x2}_{2A} \]
\[ f^{x2}_{1A} = - f^{x1}_{2A} \]

This constraint insures that in the coordinates \( A^x = \sigma^{x1} + i \sigma^{x2} \) on the boundary of the hyperKahler manifold, the metric is hermitian, which need not be true for an arbitrary set of coordinates. Note that this constraint is also necessary in order to recover a well-defined chiral plet.
After coupling to supergravity \[4\] we will recover analogous results. For example, the purely bosonic terms in the superpotential coupling will also contain a \(\delta(0)\) factor, as here.

In addition to coupling chiral plets \((A^\mu, \chi^\nu)\) descending from bulk hyper plets, we can also couple chiral plets \((B^\mu, \psi^\nu)\) that are confined to the boundary to the same superpotential. The complete lagrangian is

\[
S = \int_M \left[ -\frac{1}{2} g_{x'x} \partial_\mu \sigma^y \partial^\mu \sigma^{y'} - \frac{1}{4} \chi^x \Gamma_\mu D_\mu \lambda_w \right] \\
+ \int_{\partial M} \left[ -g_{x\bar{w}} \partial_\mu B^x \partial^\mu \bar{B}^w - \frac{i}{4} g_{x\bar{w}} \bar{\psi}^x \Gamma_\mu D_\mu \psi^\bar{w} \right] \\
+ \int_{\partial M} \left[ g^{x\bar{w}} \frac{\partial W}{\partial B^x} \frac{\partial W}{\partial B^w} + \delta(0) g^{x\bar{w}} \frac{\partial W}{\partial A^x} \frac{\partial W}{\partial A^w} \\
- \frac{1}{8} D^2 W \chi^x \chi^y + \frac{1}{8} D^2 W \chi^x \chi^y \\
- \frac{1}{8} D^2 W \psi_x \psi^y + \frac{1}{8} D^2 W \psi_x \psi^y \\
- \frac{1}{4} D^2 W \chi^x \psi^y + \frac{1}{4} D^2 W \chi^x \psi^y + 4 \text{ fermi} \right]
\]

where

\[
D_\mu \psi^\bar{w} = \partial_\mu \psi^\bar{w} + (\partial_\mu B^w) \Gamma^\bar{w}_{xw} \psi^\bar{w} \\
D_x D_y W = \partial_x \partial_y W - \Gamma^z_{xy} \partial_z W
\]

We have assumed the Kahler metric \(g_{x\bar{w}}\) factorizes between the chiral plets descending from bulk hypers and the chiral plets confined to the boundary. (More generally, we would expect that the space of all chiral plets on the boundary has the structure of the total space of a bundle over the space of chiral plets descending from bulk \((A^\mu)\), as we should be able to consistently forget the boundary-confined chiral plets \(B^\mu\).) Supersymmetry transformations for the \((A^\mu, \chi^\nu)\) chiral plet are as before; supersymmetry transformations for the chiral plets confined to the boundary \((B^\mu, \psi^\nu)\) are

\[
\delta B^x = \frac{i}{4} \tau \psi^x \\
\delta B^\bar{w} = \frac{i}{4} \bar{\epsilon} \psi^\bar{w} \\
\delta \psi^x = -\Gamma^\mu (\partial_\mu B^x) \epsilon^* + i g^{x\bar{w}} \left( \frac{\partial W}{\partial B^w} \right) \epsilon + 3 \text{ fermi} \\
\delta \psi^\bar{w} = +\Gamma^\mu (\partial_\mu B^w) \epsilon - i g_{x\bar{w}} \left( \frac{\partial W}{\partial B^x} \right) \epsilon^* + 3 \text{ fermi}
\]
Note that these supersymmetry transformations are identical to those of the \((A^x, \chi^x)\) multiplet, except that \(\delta(0)\) factors have been dropped.

Just to re-emphasize the point, we have, above, explicitly coupled both chiral pllets descending from bulk and chiral pllets confined to the boundary to the same superpotential.

How should the \(\delta(0)\) factor in the bosonic potential terms be interpreted? As mentioned earlier, in general factors of \(\delta(0)\) in this paper occur for essentially the same reasons discussed by Horava and Witten in [1], [2], [3]. More specifically, in the lagrangian above the bosonic potential term with the \(\delta(0)\) factor corresponds to superpotential couplings among chiral pllets descending from bulk hypers – which, in a standard heterotic string compactification, correspond to complex and Kahler moduli. The fact that these superpotential couplings have a bosonic potential weighted by \(\delta(0)\) suggests that they are suppressed. Indeed, for heterotic compactifications with the standard embedding in the gauge sheaf, complex and Kahler deformations \(((2,2)\) moduli) are moduli of the theory, as was shown by Dixon in [18]. A generalization of this lagrangian to supergravity low-energy effective actions [4] would naively appear to imply a more general result than Dixon’s, but a word of caution is in order. As will be discussed later, worldsheet instantons can not be described in the present framework, and as is well-known \((0,2)\) heterotic string compactifications are sometimes destabilized by worldsheet instantons [15].

In passing, it should also be noted that remaining terms in the superpotential which do not have \(\delta(0)\) factors in the bosonic potential, are often necessarily nonvanishing. For example, in a heterotic string compactification to 4D \(N=1\), recall from [19], [9] that if we embed a gauge sheaf \(V\) of rank 3 in an \(E_8\), then the superpotential on the corresponding boundary will contain \(27^3\) and \(27^2\) couplings of \(E_6\) proportional to \(H^1(X, V)^3\) and \(H^1(X, \Lambda^2 V)^3\), at sigma model tree level. These couplings are couplings of chiral pllets confined to the boundary – \((B^x, \psi^x)\) – and so the bosonic potential for these pllets has no \(\delta(0)\) factor.

Finally, we would also like to mention that the lagrangian above can be generalized to include coupling the boundary-confined chiral pllets to boundary-confined vectors, in a very straightforward fashion. As these fields are all defined only on the boundary, we can do standard 4D field theory, and recover standard results such as the Konishi anomaly [23]:

\[
\{ \overline{Q}^* , tr \psi^x B^y \} = -ig \overline{\tau}^x tr \partial_z W B^y + c \delta \tau^y tr \overline{\lambda} \lambda
\]

where \(c\) is a constant we’ve not been careful about, \(\lambda\) is the gaugino superpartner of the boundary-confined vector, and \(Q^*\) is one of the supersymmetry generators:

\[
\delta_{\text{susy}} = \overline{\tau} Q + \tau^* Q^*
\]

### 4. Vectors in bulk
The bulk 5D action for nonabelian vectors is

$$S_{\text{bulk}} = f_M \left[ \frac{1}{2} g_{xy} D_\mu \phi^x D^\mu \phi^y - \frac{1}{4} a_{IJ} F^I_{\mu \nu} F^{J \mu \nu} - \frac{1}{2} \bar{\lambda}^a \Gamma^\mu D_\mu \lambda^a \right. \right.$$ 

$$- \frac{1}{2} \sqrt{6} C_{IJK} h_0^a h_b^a (\bar{\lambda}^a \Gamma^{\mu \nu} \lambda_b^b) F^I_{\mu \nu} - i (\bar{\lambda}^a \lambda_b^b) K_I^x f_a^x h_b^l \left. + \frac{1}{6} \epsilon^{\mu \nu \rho \lambda} C_{IJK} ((dA)^J_{\mu \nu} (dA)^J_{\rho \sigma} A^K_{\lambda}) + \frac{3}{2} (dA)^J_{\mu \nu} A^J_{\rho \sigma} [A_{\sigma}, A_{\lambda}]^K \right.$$

$$+ \frac{3}{5} A^J_{\mu} [A_{\nu}, A_{\rho}]^J [A_{\sigma}, A_{\lambda}]^K + 4 \text{ fermi } \right]$$

$$= f_M \left[ \left. \frac{1}{2} a_{IJ} D_\mu h_I^J D^\mu h_J^I - \frac{1}{4} a_{IJ} F^I_{\mu \nu} F^{J \mu \nu} - \frac{1}{2} a_{IJ} \bar{\lambda}^i \Gamma^\mu D_\mu \lambda^J_i \right. \right.$$ 

$$- \frac{1}{2} \sqrt{6} C_{IJK} (\bar{\lambda}^i \Gamma^{\mu \nu} \lambda^K_i) F^I_{\mu \nu} - i a_{IJ} \bar{\lambda}^i \Gamma^J \left. [h_i, \lambda^K_\mu] \right)^J$$

$$+ \frac{1}{6} \epsilon^{\mu \nu \rho \lambda} C_{IJK} ((dA)^J_{\mu \nu} (dA)^J_{\rho \sigma} A^K_{\lambda}) + \frac{3}{2} (dA)^J_{\mu \nu} A^J_{\rho \sigma} [A_{\sigma}, A_{\lambda}]^K \right.$$

$$+ \frac{3}{5} A^J_{\mu} [A_{\nu}, A_{\rho}]^J [A_{\sigma}, A_{\lambda}]^K + 4 \text{ fermi } \right]$$

and its supersymmetry transformations are

$$\delta A^I_{\mu} = - \frac{1}{2} h_a^I (\bar{\sigma}^i \Gamma_\mu \lambda^a_i)$$

$$\delta \lambda^a_i = - \frac{i}{2} f_a^i \Gamma^\mu D_\mu \phi^x \epsilon_i + \frac{1}{4} \Gamma^{\mu \nu} \epsilon_i F^I_{\nu \mu} h_i^a$$

$$\delta \phi^x = + \frac{i}{2} f_a^x (\bar{\epsilon} \lambda^a_i)$$

with conventions

$$a_{IJ} = C_{IJ} + 2i \sqrt{\frac{2}{3}} C_{IJK} h^K$$

$$C_{IJK} \propto \text{Tr} \, T_I \{ T_J, T_K \}$$

$$(dA)^I_{\mu \nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$$

$$F^I_{\mu \nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + [A_\mu, A_\nu]$$

$$\partial_y f_a^x = \Gamma^y_{xy} f_a^x - \Omega_{ab}^x f_a^b$$

$$h_f^x = \partial_x h^f$$

$$h_{h_a^I} = h_f^x f_a^x$$

$$h_{\lambda_i^I} = a_{IJ} h_{\mu_i^J}$$

$$g_{xy} = a_{IJ} h_f^x h_{h_a^I}$$

$$\lambda_{\lambda_i^I} = h_{\lambda_i^I}$$

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where $C_{IJ}$ and $C_{IJK}$ are constant and completely symmetric. Again, we are closely following the conventions of [20, 21, 22].

Covariant derivatives are given by

\[
D_\mu \phi^x = \partial_\mu \phi^x - A^I_\mu K^x_I
\]
\[
D_\mu h^I = \partial_\mu h^I + [A_\mu, h]^I
\]
\[
D_\mu \lambda^a_i = \partial_\mu \lambda^a_i + (D_\mu \phi^x) \Omega^b_a \lambda^b_i - A^I_\mu L^a_{I,ab} \lambda^b_i
\]

and gauge transformations are

\[
\delta A^I_\mu = \partial_\mu \alpha^I + [A_\mu, \alpha]^I
\]
\[
\delta h^I = [h, \alpha]^I
\]
\[
\delta \phi^x = \alpha^I K^x_I
\]
\[
\delta \lambda^a_i = \alpha^I L^a_{I,ab} \lambda^b_i
\]

The $h^I$ are real adjoint-valued scalars, so under the action of the nonabelian group they transform just as one would expect. The (equivalent) scalars $\phi^x$ transform nonlinearly under the gauge group via the action of a Killing vector $K^x_I$. The Killing vectors form a representation of the gauge group:

\[
[K_I^x, K_J^x] = f^K_{IJ} K^K_x
\]

where $f^K_{IJ} = [T_I, T_J]^K$.

By demanding self-consistency of these gauge transformations one can derive several identities. We will mention only two:

\[
K_I^x h^I_x = h^K f^K_{IJ}
\]
\[
K_I^x L^a_{I,ab} - K_J^x L^a_{J,ab} + [L_I, L_J]^a = f^K_{IJ} L^K_x
\]

Note that upon dimensional reduction to four dimensions, this yields an N=2 theory with prepotential of the general form

\[
\mathcal{F} = C_0 + C_I A^I + \frac{1}{2} C_{IJ} A^I A^J + \frac{1}{3!} C_{IJK} A^I A^J A^K
\]

precisely as expected for very special geometry.

What about Fayet-Iliopoulos terms? Under gauge transformations, $D^I_{ij}$ transforms as

\[
D^I_{ij} \rightarrow -[\alpha, D_{ij}]^I
\]

so if $D_{Iij}$ contains a constant term $r_{Iij}$ then $r_{Iij}$ can only be nonzero when it lies in the center of the gauge group.
Suppose, for example, that the gauge group is $U(1)^n$, so that we can add Fayet-Iliopoulos terms, in principle. To proceed, simply add the term

$$-\frac{1}{4} a_{IJ} D^I_{ij} D^{Jij}$$

to the lagrangian, where

$$D^I_{ij} = a^{IJ} \left[ r_{,ij} - \sqrt{\frac{2}{3}} (\lambda^i \lambda^j) h^K_a h^L_b C_{JKL} \right]$$

and $r_{,ij}$ is the Fayet-Iliopoulos $\text{Sp}(1)$-triplet of constants, and also add a term to $\delta \lambda^i$:

$$\frac{1}{2} D^I_{ij} h^a_I \epsilon^j$$

We can couple the bulk theory to charged hyperplets by adding the terms

$$S_{\text{bulk, hypers}} = \int_M \left[ -\frac{1}{2} D_\mu \sigma^{ia'} D^\mu \sigma_{ia'} - \frac{1}{4} \lambda^{ia'} \Gamma^{ij} D_\mu \lambda_{ia'} - \frac{1}{2} F^{ia'} F_{ia'} - \frac{1}{4} a_{IJ} D^I_{ij} D^{Jij} \right. + i \sigma^{ia'} h^I_a (T_I)_{a'b'} (\lambda^b_{ia'} \lambda^a_i) + \left. \frac{i}{4} (\lambda^{ia'} \lambda_{ib'}) h^I_I (T_I)_{a'b'} + 4 \text{ fermi} \right]$$

with auxiliary fields,

$$F^{ia'} = h^I (T_I)_{a'b'} \sigma^{ib'}$$

$$D^I_{ij} = a^{IJ} \left[ r_{,ij} + \sigma_{ia'} (T_I)_{a'b'} \sigma_{jb'} - \sqrt{\frac{2}{3}} (\lambda^i \lambda^j) h^K_a h^L_b C_{JKL} \right]$$

gauge transformations

$$\delta \sigma^{ia'} = \alpha^I (T_I)_{a'b'} \sigma^{ib'}$$

$$\delta \lambda^{ia'} = \alpha^I (T_I)_{a'b'} \lambda^{ib'}$$

and covariant derivatives

$$D_\mu \sigma^{ia'} = \partial_\mu \sigma^{ia'} - A^I_\mu (T_I)_{a'b'} \sigma^{ib'}$$

$$D_\mu \lambda^{ia'} = \partial_\mu \lambda^{ia'} - A^I_\mu (T_I)_{a'b'} \lambda^{ib'}$$

Note that the hyperplet moduli space is assumed flat, in order to considerably simplify notation. For more general treatments see [24].
The \((T_I)^{a'b'}\) are constant matrices giving the action of the gauge group on the hyperplets. We’ve assumed the gauge group acts only on \(\text{Sp}(n)\) indices, so that, for example, the supersymmetry transformation parameters are neutral under the gauge group. In conventions used here,
\[
(T_I)^{a'b'} (T_J)^{c'd'} (T_I)^{b'c'} = f_{IJK}^L (T_K)^{a'd'}
\]
\[(T_I)^{a'b'} = + (T_I)^{b'a'}\]

The supersymmetry transformations are
\[
\delta A_I^\mu = -\frac{1}{2} h^{Ia} (\sigma^a \Gamma_\mu \lambda_i^a)
\]
\[
\delta \lambda_i^a = -i f^a_{x \mu} (D_\mu \phi^x) \epsilon_i + \frac{1}{4} \Gamma^{\mu\nu} \epsilon_i F^{I\mu}_I h^a_i + \frac{1}{2} D^{Ij}_I h^a_i \epsilon^j + 3 \text{ fermi}
\]
\[
\delta \phi^x = +i f^a_{x \mu} (\sigma^a \lambda^a_I)
\]
\[
\delta \sigma^{ia'} = \frac{i}{2} (\lambda^a I) 
\]
\[
\delta \lambda^{a'} = -i \Gamma^I (D_\mu \sigma^{ia'}) \epsilon_i + F^{ia'} \epsilon_i + 3 \text{ fermi}
\]

Ordinarily [in 4D N=1, for example] one expects that the Fayet-Iliopoulos term can be shifted by quantum corrections [proportional to the sum of the charges of the chiral plets in 4D N=1]. Here, however, it is easy to show that the renormalization of the Fayet-Iliopoulos term is
\[
\delta D^{ij}_I \propto \epsilon^{ij} (T_I)^{a'b'} \Omega_{a'b'} \int \frac{d^5 k}{k^2} = 0
\]
so the shift vanishes by symmetry. In retrospect this is not surprising: consider 4D N=2 QED. Each hyperplet is composed of a pair of chiral plets, with equal and opposite charges, so that the sum of the charges of the chiral plets always vanishes, so the N=1 D term does not get shifted by quantum corrections.

Note that at a generic point in the low energy theory, the gauge group action on the hyperplets will be Higgsed away, so the hyperplets will live on a hyperKahler reduction implicit in the above, with hyperKahler moment maps the D term triplets \(D^{ij}_I\). For example, we can construct \(A_n\) surface singularities as classical Higgs moduli spaces, following [27]. Such a theory has gauge group \(U(1)^n\), and \(n+1\) charged hyperplets, the \(i\)th hyperplet charged under the \(i\)th and \((i+1)\)th \(U(1)\)s. The Fayet-Iliopoulos parameters \(r_{Iij}\) are the Kahler classes of the exceptional divisors with respect to each of the three complex structures. Precisely this situation arises in type II compactifications near conifolds [25, 26]. Consider for example type IIB near a conifold singularity at which some number of \(S^3\)s are collapsing, then as shown in [26] in the low-energy effective field theory the hyperplet moduli space can be locally approximated as having an \(A_n\) singularity.
How do we orientifold this theory? As suggested in section two, there are two possible orientifolds of a theory with bulk vectors. One preserves vectors on the boundary, the other projects the bulk vector plets to boundary chiral plets. We will first consider orientifolding this theory so as to preserve vectors on the boundary. The orientifold symmetry is given by

\[
\begin{align*}
\phi^x & \rightarrow -\phi^x \\
h^I & \rightarrow -h^I \\
A_I^\mu & \rightarrow A_I^\mu \text{ for } \mu < 5 \\
C_{IJ} & \rightarrow +C_{IJ} \\
\lambda^i & \rightarrow +i \Gamma_5 \lambda^i \\
\lambda_i & \rightarrow -i \Gamma_5 \lambda^i \\
f^a & \rightarrow -f^a \\
\sigma^{ia'} & \rightarrow \sigma^{ia'} \\
\lambda^{a'} & \rightarrow +i \Gamma_5 \lambda^{a'} \\
\lambda_{a'} & \rightarrow -i \Gamma_5 \lambda^{a'} \\
(T_I)^{a'b'} & \rightarrow (T_I)^{a'b'} \\
D_{ij}^I & \rightarrow +D_{ij}^I \\
F^{ia'} & \rightarrow -F^{ia'}
\end{align*}
\]

and note this can only be a symmetry of the action when \(C_{IJK} = 0\). Recall from section two that 5D vector plets project to boundary vectors consistently only on subsets of extended Kahler moduli space, precisely as is clear here from constraints on the intersection numbers \(\mathbb{F} C_{IJK}\) in the low-energy effective action.

The supersymmetries that commute with the orientifold are given by

\[
\begin{align*}
\epsilon_i & = +i \Gamma_5 \epsilon^i \\
\epsilon^i & = -i \Gamma_5 \epsilon_i
\end{align*}
\]

We can put a superpotential on the boundary of this theory in almost the same fashion as previously. First, add a term to the supersymmetry transformation of \(\lambda^{a'}\): \(\delta(x_5) G^{ia'} \epsilon_i\), where in the notation of the last section

\[
\begin{align*}
G^{1A} & = \frac{1}{2} \left[ \partial_A W + \partial^*_A \overline{W} \right] \\
G^{2A} & = \frac{1}{2i} \left[ \partial_A W - \partial^*_A \overline{W} \right]
\end{align*}
\]

The superpotential is now a gauge-invariant holomorphic function of the boundary chiral superfields. The fields of the boundary chiral plet are

\[
A^A = \sigma^{1A} + i \sigma^{2A}
\]
\[ \chi^A = \lambda^A - i \lambda^{A'} \]

By restricting the bulk supersymmetry transformations to the boundary we find

\[ \delta A^A = \frac{i}{4} \bar{\epsilon} \chi^A \]
\[ \delta A^{A*} = \frac{i}{4} \bar{\epsilon}^* \chi^{A*} \]
\[ \delta \chi^A = -\Gamma^\mu (D_\mu A^A) \epsilon^* + i \delta(0) (\partial^*_A W) \epsilon + 3 \text{ fermi} \]
\[ \delta \chi^{A*} = +\Gamma^\mu (D_\mu A^{A*}) \epsilon - i \delta(0) (\partial_A W) \epsilon^* + 3 \text{ fermi} \]

with the boundary interaction

\[
S_{\text{boundary}} = \int_{\partial M} \left[ (\partial_A W) (\partial^*_A W) \delta(0) - \frac{1}{8} (\partial_A \partial_B W) \bar{\chi}^A \chi^B + \frac{1}{8} (\partial^*_A \partial^*_B W) \bar{\chi}^{A*} \chi^{B*} \right]
\]

There is a second orientifold of this theory, which projects the bulk vector plets to boundary chiral plets. The relevant orientifold symmetry is

\[
\begin{align*}
\phi^x & \to +\phi^x \\
h^I & \to +h^I \\
A^I_\mu & \to -A^I_\mu \text{ for } \mu < 5 \\
C_{IJ} & \to +C_{IJ} \\
C_{IJK} & \to +C_{IJK} \\
\lambda^{ia} & \to +i\Gamma_5 \lambda^{ia} \\
\lambda^{ia}_i & \to -i\Gamma_5 \lambda^{ia} \\
f^{ia}_x & \to +f^{ia}_x
\end{align*}
\]

and similarly for the hyper plets. The supersymmetries that commute with this orientifold are also as before.

Note that chiral plets obtained from bulk vector plets in this manner can not couple perturbatively to a superpotential: perturbatively, the 5th component of a $U(1)$ vector $A^I_5$ has a gauge symmetry $A^I_5 \to A^I_5 + \partial_5 \epsilon$, so $A^I_5$ behaves like an axion on the boundary. However, this Peccei-Quinn-like symmetry can be broken by nonperturbative effects. For example, in the M theoretic description of the heterotic string, worldsheet instantons are simply 2-branes stretched between the two ends of the world, and these 2-branes will break the gauge symmetry. In fact, this is precisely the M-theoretic understanding of the old sigma model nonrenormalization theorem for the spacetime superpotential [15].
One might be curious about gauge-invariance of the Chern-Simons term in a boundary theory. The boundary terms that are generated upon gauge-transformation of the Chern-Simons term all vanish on the boundary, because of the boundary condition on the gauge parameter, leaving only the usual $\int_M (g^{-1} dg)^5$ term, which fixes the coefficient of the Chern-Simons term to be proportional to an integer.

5. Anomalies

In this section, we will explain the strong coupling version of a familiar fact about the weakly coupled heterotic string. In heterotic string compactification on a six-manifold $X$ with some gauge sheaf $V$, one often obtains at tree level what appears to be – just going by the massless fermion spectrum – an anomalous $U(1)$ [32, 9]. Let $A_\mu^{(1)}$ be the gauge field with apparently anomalous couplings, and let $F = dA$. In this situation, if one looks closely, one finds always a scalar field $a$ with a coupling $\partial_\mu a A_\mu^{(1)}$ and a gauge transformation law just right to cancel the anomaly. All this is ensured by the Green-Schwarz anomaly cancellation mechanism in ten dimensions. The physical consequence is that $A_\mu^{(1)}$ becomes massive by Higgsing of $a$. We want to see how this scenario works out in the strongly coupled region of the $E_8 \times E_8$ heterotic string, related to physics on $R^4 \times S^1/\mathbb{Z}_2$.

How are anomalies cancelled in models seen as 5D orientifolds? The chiral anomalies arising on the 4D boundary of the 5D effective theory are cancelled by bulk 5D Chern-Simons terms that descend from the 11D 3-form potential bosonic interaction, and the $U(1)$ itself is Higgsed by terms descending from Green-Schwarz interactions.

Before demonstrating this more precisely, we will review the Green-Schwarz mechanism in M theory from [2]. The variation of the boundary effective action after gauge transformation of the boundary $E_8$ vector is proportional to

$$\int_{M^{10}} \epsilon^{M_1 M_2 \ldots M_{10}} \text{tr}(\epsilon F_{M_1 M_2}) \text{tr}(F_{M_3 M_4} F_{M_5 M_6}) \text{tr}(F_{M_7 M_8} F_{M_9 M_{10}})$$

This anomaly is cancelled by the variation of a Chern-Simons-like term in the 11D bulk. Specifically, recall the bosonic 3-form potential has the 11D coupling

$$\int_{M^{11}} \left[ -\frac{\sqrt{2}}{3456} \epsilon^{I_1 I_2 \ldots I_{11}} C_{I_1 I_2 I_3} G_{I_4 \ldots I_7} G_{I_8 \ldots I_{11}} \right]$$

and also recall gauge transformations of the boundary vector are accompanied by a gauge transformation of the three-form potential

$$\delta C_{11AB} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2} \delta(x^{11}) \epsilon F_{AB}$$

Note this is in contrast to the usual situation in 3D. Given a 3D Chern-Simons form $I_{CS}$ on a three-manifold $M$ with boundary, $\epsilon^{I_{CS}}$ is not well-defined under gauge transformations but rather picks up a factor due to boundary terms, and so is interpreted as a section of a bundle over the space of connections on $\partial M$. Were it not for the orientifold boundary conditions an analogous phenomenon would occur in the 5D theory being discussed.
so under a gauge transformation of the boundary vector the 11D Chern-Simons-like interaction picks up the variation

$$\int_{M^{10}} \epsilon^{M_1 M_2 \ldots M_{10}} tr(\epsilon F_{M_1 M_2}) G_{M_3 \ldots M_6} G_{M_7 \ldots M_{10}}$$

Finally, recall because of the Bianchi identity we have the result

$$G_{ABCD} = -\frac{3}{\sqrt{2}} \lambda^2 (x^{11}) F^{a}_{[AB} F^{a}_{CD]} + \cdots$$

(ignoring Riemann curvature terms) so the variation of the 11D Chern-Simons-like term is proportional to the variation of the effective action due to the 10D chiral anomaly, and so we cancel the anomaly.

What happens after compactification? To be specific, consider embedding a gauge sheaf $V$ of the form $E \oplus L$ in one of the $E_8$s, where $c_1(E) = -c_1(L)$ and $L$ is rank 1, $E$ is rank 4, then $E_8$ is broken to $SU(5) \times U(1)$. In particular, consider the $U(1)^3$ anomaly in 4D. Let $F_\gamma$ be the de Rham image of $c_1(L)$. Then the 4D axion obtained from compactifying $C_{11AB}$ on $F_\gamma$ transforms under gauge transformations of the low-energy $U(1)$. If we denote by $A_\mu^{(1)}$ the low-energy boundary $U(1)$, then because of the $U(1)^3$ anomaly under gauge transformations we pick up a contribution to the boundary effective action

$$\int \epsilon^{\mu \nu \rho \sigma} \epsilon F^{(1)}_{\mu \nu} F^{(1)}_{\rho \sigma}$$

The 5D bulk theory contains a Chern-Simons term of the form

$$\int \epsilon^{\mu \nu \rho \sigma \delta} A_I^{(1)} F^{(1)}_{\mu \nu} F^{(1)}_{\rho \sigma}$$

where each 5D vector is obtained by compactifying the 11D 3-form potential. Because gauge variations of 10D vectors are coupled to gauge variations of the 11D 3-form potential, we can read off that gauge variations of 4D boundary vectors are coupled to gauge variations of these 5D vectors. Thus, under a gauge transformation of $A_\mu^{(1)}$, the 5D Chern-Simons term transforms as

$$\int \epsilon^{\mu \nu \rho \sigma \delta} A_I^{(1)} F^{(1)}_{\mu \nu} F^{(1)}_{\rho \sigma}$$

and by compactifying $G_{ABCD}$ on $F_\gamma$ to get the remaining 5D vectors, we can read off the 5D boundary conditions from the 11D boundary conditions on $G_{ABCD}$, to find that the variation of the 5D Chern-Simons term under a gauge variation of $A_\mu^{(1)}$ is proportional to

$$\int \epsilon^{\mu \nu \rho \sigma} F^{(1)}_{\mu \nu} F^{(1)}_{\rho \sigma}$$

which is exactly as needed to cancel out the 4D boundary chiral anomaly.
By compactifying the $G_{ABCD}$ on other field strengths, we can cancel out more general non-abelian anomalies and even gauge-gravitational-gravitational anomalies (by using the Riemann curvature terms that have been suppressed so far). For example, by compactifying one of the 11D $G_{ABCD}$ factors on $c_2(E)$, then the boundary condition on the other $G_{ABCD}$ factor makes it proportional to $tr F \wedge F$, where the trace runs over both $U(1)$ and $SU(5)$ indices, thereby yielding a contribution to both the $U(1)^3$ and $U(1)-SU(5)^2$ anomalies.

In the analysis sketched above, the $C_{IJK}$ factor in the 5D Chern-Simons term was suppressed. As noted in [5], this factor is proportional to the obvious intersection form on the 3-fold. For example, the first case studied (which contributes to $U(1)^3$) would have $C_{IJK}$ proportional to $< X | c_1(E)^3 >$. More to the point, for the anomaly cancellation mechanism outlined above to function, it must be the case that all contributions to the anomaly factorize, i.e., must be of the form $< X | c_1(E) \cup \cdots >$. This is precisely the compactification of the 10D Green-Schwarz factorization condition.

Let’s work through the $SU(5) \times U(1)$ example in detail, following [9]. As our gauge sheaf $V$ has rank 5, we can expect 10s of $SU(5)$ corresponding to elements of $H^m(X, V)$, 5s of $SU(5)$ corresponding to elements of $H^m(X, \wedge^2 V)$, and some $SU(5)$ singlets corresponding to elements of $H^m(X, \text{End} V)$. Let the $U(1)$ subgroup of the structure group that yields the low-energy $U(1)$ be defined by $\text{diag}(1, 1, 1, 1, -4)$, then we have the following chiralplets charged under the low-energy $U(1)$:

| $SU(5)$ representation | Sheaf cohomology group | $U(1)$ charge |
|------------------------|------------------------|--------------|
| 10                     | $H^m(X, E)$            | 1            |
|                        | $H^m(X, L)$            | -4           |
| 5                      | $H^m(X, \wedge^2 E)$   | 2            |
|                        | $H^m(X, E \otimes L)$  | -3           |
| Singlets               | $H^m(X, E \otimes L^{-1})$ | 5           |

Given this information we can rapidly compute some anomalies. For example, the $U(1)$ trace anomaly is

$$(10) \left[ \chi(E) - 4 \chi(L) \right] + (5) \left[ 2 \chi(\wedge^2 E) - 3 \chi(E \otimes L) \right] + \left[ 5 \chi(E \otimes L^{-1}) \right]$$

where

$$\chi(V) = \sum_i (-1)^i \dim H^i(X, V)$$

Using the Hirzebruch-Riemann-Roch theorem

$$\chi(V) = < X | ch(V) \cup td(TX) >$$

For simplicity, we are assuming $E, L$ are well-defined bundles.
we can rewrite the $U(1)$ trace anomaly as

$$< X | c_1(E) \cup \left[ \frac{65}{3} c_1(E)^2 - 30 c_2(E) + \frac{15}{2} c_2(TX) \right] >$$

This $U(1)$ trace anomaly is proportional to the $U(1) - \text{graviton}^2$ triangle diagram. If it is nonzero, then we get a Fayet-Iliopoulos term generated at 1-loop. Such a term would spontaneously break supersymmetry and thereby destabilize the vacuum in a Coulomb phase – but as will be shown later, the $U(1)$ is Higgsed, so the Fayet-Iliopoulos term merely shifts some scalar vevs, rather than breaking supersymmetry.

Proceeding similarly, the $U(1)^3$ anomaly is

$$(10) \left[ \chi(E) - (64) \chi(L) \right] + (5) \left[ (8) \chi(E^2) - (27) \chi(E \otimes L) \right] + \left[ (125) \chi(E \otimes L^{-1}) \right]$$

$$= < X | c_1(E) \cup \left[ \frac{770}{3} c_1(E)^2 - 300 c_2(E) + 85 c_2(TX) \right] >$$

and the $U(1) - SU(5)^2$ anomaly is

$$[ \chi(E) - 4 \chi(L) ] d^{(2)}(10) + \left[ (2) \chi(E^2) - 3 \chi(E \otimes L) \right] d^{(2)}(5)$$

$$= < X | c_1(E) \cup \left[ \frac{9}{2} c_1(E)^2 - 5 c_2(E) + \frac{7}{4} c_2(TX) \right] > d^{(2)}(5)$$

using $d^{(2)}(10) = 3 d^{(2)}(5)$. As anticipated earlier, in each case the $< X | c_3(E) >$ contribution cancels, so the anomaly factorizes.

In addition we can also easily see how anomalous $U(1)$s are Higgsed, closely following [32, 9]. Let’s first review how this works in standard heterotic compactifications. Recall that the torsion picks up a couple Chern-Simons terms, so in 10D the kinetic term for the antisymmetric tensor has the form

$$(H_{ABC} + \omega(YM)_{ABC} + \omega(R)_{ABC})^2$$

By compactifying on $F_7$ (and ignoring Riemann curvature terms) one component of this becomes

$$< F >^2 \left( \partial_\mu a + A^{(1)}_\mu \right)^2$$

(Note if we did not know in advance that the axion $a$ had a translation symmetry gauged by $A^{(1)}_\mu$, we could have deduced it from the coupling above.) This has precisely the effect of Higgsing the $U(1)$, as desired.

Now, how does this work in M theory compactifications? Recall that the 11D low-energy action has a kinetic term for the 3-form potential proportional to $G_{IJKL}G^{IJKL}$. In particular, consider the $G_{11ABC}G^{11ABC}$ component. Recall from [2] that the modified Bianchi identity is solved by modifying $G_{IJKL}$ as

$$G_{11ABC} = (\partial_{11} C_{ABC} \pm \cdots) + \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) \omega_{ABC}$$
(ignoring Riemann curvature terms, once again) so following the $SU(5) \times U(1)$ example above, there is a $G_{11\mu J}G^{11\mu J}$ term which is of the form

$$< F >^2 (F_{(11)\mu} + \delta(x^{11}) A^{(1)}_\mu)^2$$

where $F_{(11)\mu}$ is the field strength of the 5D $U(1)$ that descends from $G_{IJKL}$. This $U(1)$ vector projects to a chiral plet on the boundary, the chiral plet containing the axion $a$ mentioned in the last paragraph. Again, $A^{(1)}_\mu$ gauges a translation symmetry of $a$, effectively, and the boundary-confined $U(1)$ is Higgsed. World-sheet instantons (membranes stretched between the ends of the world) break the remaining global symmetry.

Naively this would appear to Higgs any $U(1)$; why does it only Higgs anomalous $U(1)$s? This is implicit in the existence of a cohomologically nontrivial field strength $F_\Sigma$ on which to compactify the 11D 3-form potential. If it had been the case that $c_1(L) = 0$, then $C_{11\Sigma}$ would have been gauge-trivial, and so we would have gotten neither an axion $a$ nor an interaction term $\delta(x^{11}) A^{(1)}_\mu$ in the 4D theory, so the $U(1)$ vector $A^{(1)}_\mu$ would not have been Higgsed. In such a case the $U(1)$ is not anomalous, so there is no difficulty.

So far we have considered anomalies due (primarily) to boundary-confined vectors. What about boundary vectors that descend from bulk vector plets? The boundary conditions for such an orientifold demand that $C_{IJK} = 0$, so if there are any boundary chiral anomalies in such vectors, they can not be cancelled by a bulk Chern-Simons term.

6. Discussion

In this paper we have worked out explicit lagrangians describing superpotential coupling to the boundary of a 5D orientifold, as relevant to a number of compactifications, and also made some general comments on compactifications of 11D M theory orientifolds, relevant to the strong coupling limit of the $E_8 \times E_8$ heterotic string.

Note that the superpotentials we have discussed in this paper do not include couplings generated by membranes stretched between the ends of the world (worldsheet instantons in standard heterotic compactifications). Such objects can not be described locally in five dimensions. When they are absent (or, when the radius of the fifth dimension is large, so that they are exponentially suppressed), we have a nonrenormalization theorem specifying that chiral plets descending from bulk vectors do not couple to a superpotential. In heterotic string compactifications this is a well-known result [13], but note that this also gives constraints on superpotentials of the models of types (2) and (3) discussed in section two.

Note also that in orientifolds there are new ways to spontaneously break supersymmetry. For example, put some simple O’Raifeartaigh model on a boundary, with chiral plets all descending from bulk vector plets. Then we have spontaneously broken supersymmetry due to a purely boundary interaction! Essentially the same idea has been discussed in [3].

There are additional ways to break supersymmetry in these models. Consider a 5D theory
with a single hyper plet. Put a superpotential on each boundary of the form $\lambda \Phi + m \Phi^2$, with distinct $\lambda$, $m$ on each boundary. This superpotential uniquely fixes a nonzero vev for the boundary chiral plet, but as the couplings are different on the boundaries, the vevs are distinct on the boundaries. It is easy to see that this spontaneously breaks supersymmetry in the 5D bulk. Unlike the case above where supersymmetry was clearly broken locally on the boundary, here supersymmetry is naively unbroken on the boundary, and is only broken by the global 5D topology.

Moreover, in this strong-coupling description of the heterotic string, it may be possible to derive certain heterotic nonperturbative effects. For example, consider compactifying M theory on a singular Calabi-Yau. Suppose for definiteness the Calabi-Yau contains a genus $g$ curve of $A_n$ singularities, then following [29] we should expect to recover SU$(n+1)$ gauge theory with $g$ adjoint hyper plets. [Recall this is the classical result, but following [37] Seiberg-Witten theory in 5D is trivial.] The orientifold action that yields the $E_8 \times E_8$ heterotic string will then project each vector plet to a boundary chiral plet, so in the low energy theory it seems likely that we will find $g+1$ chiral plets that are in the adjoint representation of a global symmetry group, SU$(n+1)$ ($g$ from bulk hyper plets, 1 from bulk vector plet). In addition, as the base space is singular there are no doubt small instanton effects that also need to be taken into account. Thus, ignoring small instanton effects we see that compactifying heterotic $E_8 \times E_8$ on a singular Calabi-Yau 3-fold of this form may yield new massless neutral matter and an enhanced global symmetry group, the projection of bulk 5D vector plets.\[\footnote{Also note that this means the “end-of-the-world” branes in M theory may have many properties expected of brane probes \[\footnote{Such as possessing enhanced global symmetry in the presence of background enhanced local symmetry.} the “end-of-the-world” branes in M theory may have many properties expected of brane probes such as possessing enhanced global symmetry in the presence of background enhanced local symmetry.}

Finally, we will note that this technology may have broader applications than have been discussed in this paper. One such possible application is a generalization of Seiberg duality away from the IR limit. A 5D orientifold such as has been discussed here looks like a 4D N=1 theory at very low energies, where all wavelengths are much longer than the size of the 5th dimension. As one strong coupling limit of the 4D compactification of the $E_8 \times E_8$ heterotic string is five dimensional, perhaps some insight into Seiberg duality can be gained by going to five dimensions.

Acknowledgements

We would like to thank P. Horava and especially E. Witten for useful conversations.

Appendix on Conventions

The metric $\eta = \text{diag}(-, +, +, +, +)$

The index $i$ is in the fundamental of Sp$(1)$, the index $a'$ is in the fundamental of Sp$(n)$.
Sp(1) indices are raised and lowered as

\[ V^i = \epsilon^{ij} V_j \]
\[ V_i = V^j \epsilon_{ji} \]

with \( \epsilon_{12} = \epsilon^{12} = 1 \), so for example \( V_i W^i = -V^i W_i \). The Sp(n) indices behave identically, under the action of the Sp(n)-invariant tensor \( \Omega_{a'b'} \).

Note that the Sp(n) spin connection has the amusing property

\[ \omega_{a'b'} = +\omega_{b'a'} \]

that is, it is symmetric rather than antisymmetric in its “local Lorentz” indices, because the adjoint representation of Sp(n) is a symmetric tensor in its fundamental representations, as opposed to SO(n).

All 5D spinors are symplectic-Majorana. Boundary 4D spinors are, by construction, Weyl. Five-dimensional spinor conventions are such that

\[ \bar{\psi}^i \Gamma_{\nu_1 \cdots \nu_n} \epsilon^{ij} = +\epsilon^{ijkl} \Gamma_{\nu_1 \cdots \nu_n} \psi^j \]

Gamma matrices are defined in “strength one” conventions:

\[ \Gamma_{\nu_1 \cdots \nu_n} = \frac{1}{n!} [\Gamma_{\nu_1} \Gamma_{\nu_2} \cdots \Gamma_{\nu_n} \pm \cdots] \]

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