The regions of dynamic instability of a plate, the material of which obeys the hereditary low of viscoelasticity, at the second resonance

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Abstract. The problem of determining the side regions of dynamic instability of a rectangular plate, the material of which obeys the hereditary law of deformation, is considered. The solution of the integro-differential equation of vibrations of a plate loaded in its plane with constant and variable loads is considered. The solution used is a series with separated variables. An equation is obtained to determine the critical frequencies of even regions of instability. The influence of the parameters of the nucleus on the position of the second region of dynamic instability is investigated.

1. Introduction
In [1], the main regions of dynamic instability of a rectangular plate loaded in its plane by forces uniformly distributed along the edges were investigated

\[ N_x = N_{x0} + N_{x1} \cos \theta , \quad N_y = N_{y0} + N_{y1} \cos \theta \]  

(1)

The viscoelastic properties of the material will obey the hereditary Boltzmann-Volterra law with a weakly singular creep kernel [2]

\[ R(t - s) = A \exp \left[ - \beta (t - s) \right] (t - s)^{\alpha - 1} (0 < \alpha < 1) \]  

(2)

Three kernel parameters allow you to describe the behavior of many polymeric materials. This paper considers the solution to the problem of determining even domains of dynamic instability and investigates the influence of the kernel parameters on them.

2. Problem setting
We will consider a rectangular plate loaded with forces (1). The viscoelastic properties of the material will obey the hereditary law (2). Taking \( v \) constant, we have the relationship between stresses and deformations

\[
\sigma_x = \frac{E}{1-v^2} (\varepsilon_x + v \varepsilon_y) - \frac{E}{1-v^2} \int_0^t R(t-\theta) (\varepsilon_x + v \varepsilon_y) \, d\theta
\]

\[
\sigma_y = \frac{E}{1-v^2} (\varepsilon_y + v \varepsilon_x) - \frac{E}{1-v^2} \int_0^t R(t-\theta) (\varepsilon_y + v \varepsilon_x) \, d\theta
\]

(3)
\[ \tau_{\theta\theta} = \frac{E}{2(1+\nu)} \gamma_{\theta\theta} - \frac{E}{2(1+\nu)} \int_0^t R(t-\tau) \gamma_{\theta\theta} \, d\theta \]

To describe the transverse vibrations of the plate, we will use the equation [3]

\[ D \left( V^4 w - \int_0^t R(t-s) V^4 w \, ds \right) + \left( N_{x0} + N_{y0} \cos \theta \right) \frac{\partial^2 w}{\partial x^2} + \left( N_{y0} + N_{y0} \cos \theta \right) \frac{\partial^2 w}{\partial y^2} + m \frac{\partial^2 w}{\partial t^2} = 0 \quad (4) \]

where \( D = \frac{Eh^3}{12(1-\nu^2)} \) - cylindrical stiffness, \( E \) - instant modulus of elasticity.

The article [1] considered the problem of finding the boundaries of the regions of instability, for which equation (4) has periodic solutions with a period of \( 2T \), including the main regions of dynamic instability. In this paper, we consider the problem of finding solutions with period \( T \). This will allow us to construct even regions of dynamic instability.

3. Solution technique

The solution to equation (4) will be sought by the Bubnov-Galerkin method, separating the variables in the form [1]

\[ w = \sum f_i(t) \left( x \right) Y_i(y) \quad (5) \]

We have the equation in dimensionless form:

\[ f'' - \varphi \int_0^t R(t-s)f(s) \, ds + \varphi(1-2\mu \cos \tau)f = 0 \quad (6) \]

where the notation is used [3]

dimensionless time \( \tau = \theta t \), \( \omega \) - natural frequency of unloaded plate,

\[ \varphi = \Omega^2, \quad \varepsilon = \frac{\omega^2}{\Omega^2}, \quad \Omega^2 = \omega^2 \left( 1 - \frac{N_{x0} N_{2s} + N_{y0} N_{1s}}{N_{1s} N_{2s}} \right), \quad \mu = \frac{N_{x0} N_{2s} + N_{y0} N_{1s}}{2 N_{1s} N_{2s} - N_{x0} N_{2s} - N_{y0} N_{1s}} \]

\( \Omega \) - the frequency of natural vibrations of the plate loaded \( N_{x0} \) and \( N_{y0} \), \( \mu \) - excitation factor, \( N_{1s} \), \( N_{2s} \) - critical values of efforts \( N_{x0} \) and \( N_{y0} \) at their independent static action,

\[ I_1 = \int_0^b \int_0^a \left( \frac{d^4 X}{dx^4} \right)^2 Y \, dxdy, \quad I_2 = \int_0^b \int_0^a \left( \frac{d^2 Y}{dy^2} \right)^2 X \, dydx, \quad I_3 = \int_0^b \int_0^a X \frac{d^2 X}{dx^2} \frac{d^2 Y}{dy^2} \, dydx, \quad I_4 = -\int_0^b \int_0^a \left( \frac{d^2 Y}{dy^2} \right)^2 X \, dydx \]

As is known [4], domains of unboundedly increasing solutions are separated from stability domains by periodic solutions with periods \( T \) and \( 2T \). Moreover, two solutions of the same period limit the region of instability, two solutions of different periods - the region of stability. We seek periodic solutions with period \( T \) in the form of a series:

\[ f(t) = \sum_{k=2,4,6} a_k \sin \frac{k\tau}{2} + b_k \cos \frac{k\tau}{2} + b_0 \quad (7) \]

Substitute series (7) into equation (6):
\[
\sum_{k=2,4}^{\infty} \left[ -a_k \frac{k^2}{4} \sin \frac{k\tau}{2} - b_k \frac{k^2}{4} \cos \frac{k\tau}{2} \right] - \varepsilon \phi \sum_{k=2,4}^{\infty} \left[ \sin \frac{k\tau}{2} (a_k B_k + b_k A_k) + \cos \frac{k\tau}{2} (b_k B_k - a_k A_k) \right] \\
- \varepsilon \phi b_0 R_0 + \varepsilon \phi \left( 1 - 2\mu \cos \tau \right) + \phi \sum_{k=2,4}^{\infty} \left( a_k \sin \frac{k\tau}{2} + b_k \cos \frac{k\tau}{2} \right) - \\
\mu \phi \sum_{k=2,4}^{\infty} a_k \left( \sin \frac{k\tau}{2} - \sin \frac{k\tau}{2} \right) + \mu \phi \sum_{k=2,4}^{\infty} b_k \left( \cos \frac{k\tau}{2} + \cos \frac{k\tau}{2} \right) = 0
\] 

(8)

where the following notation is introduced

\[
R_0 = \int_0^\tau R(x) \, dx \quad A_k = \int_0^\tau R(x) \sin \frac{kx}{2} \, dx \quad B_k = \int_0^\tau R(x) \cos \frac{kx}{2} \, dx, \quad x = \tau - s
\]

(9)

Let us equate the coefficients at the same \( \sin k\tau/2 \) and \( \cos k\tau/2 \) equations (8). As a result, we obtain a system of linear algebraic equations with respect to \( a_k \) and \( b_k \) with constant coefficients

\[
\varepsilon \phi b_0 R_0 - b_0 \phi + \mu \phi b_0 = 0 \\
a_k + \varepsilon \phi a_k B_k + \varepsilon \phi b_k A_k - \phi a_k + \mu \phi a_k = 0 \\
b_k + \varepsilon \phi b_k B_k - \varepsilon \phi a_k A_k + 2\mu b_k \phi \phi b_k + \mu \phi b_k = 0 \\
\frac{k^2}{4} a_k + \varepsilon \phi a_k B_k + \varepsilon \phi b_k A_k - \phi a_k + \mu (a_{k+2} + a_{k-2}) = 0 \\
\frac{k^2}{4} b_k + \varepsilon \phi b_k B_k - \varepsilon \phi a_k A_k - \phi b_k + \mu (b_{k+2} + b_{k-2}) = 0
\]

(10)

We divide the equations of system (10) by \( \varphi \) and perform the replacement \( 1/\varphi = \Omega^2 / \omega^2 = 4 \rho^2 \). Let's compose the determinant of the system and equate it to zero. We obtain the equation of critical frequencies

\[
\begin{vmatrix}
\varepsilon R_0 & 0 & \mu & 0 & 0 & \ldots \\
0 & 4 \rho^2 & \varepsilon A_2 & \mu & 0 & \ldots \\
2\mu & -\varepsilon A_2 & 4 \rho^2 + \varepsilon B_2 - 1 & 0 & \mu & \ldots \\
0 & \mu & 0 & 16 \rho^2 + \varepsilon B_4 - 1 & \varepsilon A_4 & \ldots \\
0 & 0 & \mu & -\varepsilon A_4 & 16 \rho^2 + \varepsilon B_4 - 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0
\]

(11)

Solving equation (11), we find the boundaries of even domains of dynamic instability. To calculate in the first approximation the second domain of dynamic instability, we will store in the determinant nine terms

\[
\begin{vmatrix}
\varepsilon R_0 & 0 & \mu & 0 & 0 & \varepsilon A_2 \\
0 & 4 \rho^2 + \varepsilon A_2 & \varepsilon A_2 & \mu & 0 & \ldots \\
2\mu & -\varepsilon A_2 & 4 \rho^2 + \varepsilon B_2 - 1 & 0 & \mu & \ldots \\
0 & \mu & 0 & 16 \rho^2 + \varepsilon B_4 - 1 & 0 & \ldots \\
0 & 0 & \mu & -\varepsilon A_4 & 16 \rho^2 + \varepsilon B_4 - 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0
\]

(12)

After disclosing the determinant from the obtained equation, we obtain the formula for determining the critical frequencies

\[
\varepsilon R_0 = 0
\]

(13)
\[ r = 0.5 \left[ 1 - \varepsilon B_k + \left( \mu^2 - \varepsilon^2 A_k^2 (\varepsilon R_0 - 1)^2 \right)^{1/2} (\varepsilon R_0 - 1) \right]^{1/2} \]  

(13)

As long as the expression under the inner radical is positive, this expression gives two real roots for the value of \( r \), corresponding to the two boundaries of the second instability region. In the extreme case

\[ \mu^2 - \varepsilon^2 A_k^2 (\varepsilon R_0 - 1)^2 = 0 \]  

(14)

We determine from (14) the minimum value of the excitation coefficient \( \mu_* \), at which the occurrence of sustained oscillations is still possible, and the corresponding value \( p_* \)

\[ \mu_* = \left[ (\varepsilon A_k (\varepsilon R_0 - 1))^2 \right]^{1/4}, \quad p_* = 0.5 \left[ 1 - \varepsilon B_k + \mu_* (\varepsilon R_0 - 1) \right]^{1/2} \]  

(15)

It should be noted that the coefficients of the equation of critical frequencies (11) are variable, since they depend on time. As time changes, they quickly approach the values at which the integrals have an infinite upper limit. In this case, the regions do not change their position, starting from some time. To calculate \( R_0, B_k, A_k \) you can use the relations

\[ R_0 = A \beta^{-\alpha} \Gamma(\alpha), \]
\[ A_k = A \Gamma(\alpha) \left( \beta^2 + \frac{k^2}{4} \right)^{-\alpha/2} \sin \left( \alpha \arctg \frac{k}{2\beta} \right) \]  

(16)
\[ B_k = A \Gamma(\alpha) \left( \beta^2 + \frac{k^2}{4} \right)^{-\alpha/2} \cos \left( \alpha \arctg \frac{k}{2\beta} \right) \]

4. Results

Let us investigate the influence of the parameters of the nucleus on the position of the even regions of instability. Consider the change in the values of the excitation coefficient \( \mu_* \) and \( p_* \) in the case of constant kernel parameters \( \alpha, \beta \), and variable \( A \). We will use formulas (15) obtained in the first approximation. Let's accept the following values of the kernel parameters [1]: \( \alpha = 0.15, \beta = 0.05, A \) in the range from 0.019 to 0.103, \( \varepsilon = 1.25 \). To calculate the coefficients, we use formulas (16), substituting the kernel parameters and \( k = 2 \):

\[ R_0 = A \beta^{-\alpha} \Gamma(\alpha) = A(0.05)^{-0.15} \Gamma(015) = 3.968 A \]
\[ A_2 = A \Gamma(\alpha) (\beta^2 + 1)^{-\alpha/2} \sin \left( \alpha \arctg \frac{1}{\beta} \right) = 1.4067 A \]  

(17)
\[ B_2 = A \Gamma(\alpha) (\beta^2 + 1)^{-\alpha/2} \cos \left( \alpha \arctg \frac{1}{\beta} \right) = 6.058 A \]

The calculation results are shown in table 1.

| \( A \) | 0.019 | 0.032 | 0.051 | 0.064 | 0.083 | 0.103 |
|-------|-------|-------|-------|-------|-------|-------|
| \( \mu_* \) | 0.174 | 0.217 | 0.259 | 0.277 | 0.293 | 0.294 |
| \( p_* \) | 0.453 | 0.419 | 0.362 | 0.317 | 0.237 | 0.104 |

An increase in \( A \) leads to an increase in the value of the excitation coefficient \( \mu_* \). This indicates an increase in the amplitudes of the periodic components of the forces, which can cause a loss of dynamic stability. On the plane \( \mu, p \) the boundaries of the dynamic instability region will move towards increasing values to the right. The results presented in the bottom row of table 1 show a decrease \( p_* \) with increasing \( A \). This allows us to conclude that the frequency of the periodic component of the load decreases. In this case, the boundary of the dynamic instability region on the plane \( \mu, p \) will shift downward toward the
axis $\mu$. Consider the stability regions at constant $\alpha=0.25$ and variables $\beta$ and $A$. Retaining twenty-five elements in equation (11), we obtain the equation for the second approximation. The calculation results are shown in figure 1. At a constant value $\alpha$, an increase in $\beta$ and $A$ leads to a decrease in the frequency of the periodic load, causing a loss of dynamic stability, with first an increasing and then decreasing minimum excitation coefficient $\mu$, and amplitudes of forces

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Dynamic area boundaries instability of the second order at $\alpha=0.25$: 1 - $\beta=0.0054$, $A=0.05$, 2 - $\beta=0.046$, $A=0.085$, 3 - $\beta=0.086$, $A=0.1$, 4 - $\beta=3.37$, $A=0.25$, 5 - $\beta=54$, $A=0.5$.}
\end{figure}

5. **Conclusions**

In this work, an equation of critical frequencies for even regions of dynamic instability is obtained and the influence of the parameters of the nucleus on the position of the second region of dynamic instability. It was revealed that an increase in parameter $A$ with constant other parameters leads to an increase in the amplitudes and a decrease in the frequencies of the periodic components of the forces, which can cause a loss of dynamic stability. It should be noted that a simultaneous increase in $A$ and $\beta$ with unchanged $\alpha$, also leads to a decrease in the frequency of the periodic component of the load at first an increasing and then decreasing minimum excitation coefficient. In [1], similar results were obtained for the main region of instability.

**References**

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