On the complexity of structure and substructure connectivity of graphs *

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Abstract

The connectivity of a graph is an important parameter to measure its reliability. Structure and substructure connectivity are two novel generalizations of the connectivity. In this paper, we characterize the complexity of determining structure and substructure connectivity of graphs, showing that they are both NP-complete.

Key words: Connectivity; Structure connectivity; Substructure connectivity; NP-complete

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1. Introduction

The graph we considered throughout this paper is simple and undirected. Let $G = (V, E)$ be a graph, where $V$ is the vertex-set of $G$ and $E$ is the edge-set of $G$.

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The degree of a vertex $v$ is the number of incident edges, written $d_G(v)$ or $d(v)$ when the context is clear. The minimum degree of $G$ is $\delta(G)$ and the maximum degree is $\Delta(G)$.

For any subset $X \subset V$, the closed neighborhood of $X$ is defined to be all neighbors of any vertex $x \in X$ together with $X$, denoted by $N[X]$, while the open neighborhood of $X$ is $N[X] \setminus X$, denoted by $N(X)$. If $X = \{x\}$, then we write $N[x]$ and $N(x)$, respectively. The subgraph induced by $X$ is denoted by $G[X]$. A matching of $G$ is a set of independent edges of $G$. For other standard graph notations not defined here please refer to [1].

Lin et al. [9] introduced structure and substructure connectivity to evaluate the fault tolerance of a network from the perspective of a single vertex, as well as some special structures of the network. Let $F = \{F_1, F_2, \ldots, F_t\}$ be a set of pairwise disjoint connected subgraphs of $G$ and let $V(F) = \bigcup_{i=1}^{t} V(F_i)$. Then $F$ is a subgraph-cut of $G$ provided that $G - V(F)$ is disconnected or trivial. Let $H$ be a connected subgraph of $G$, then $F$ is an $H$-structure-cut if $F$ is a subgraph-cut, and each element in $F$ is isomorphic to $H$. The $H$-structure connectivity of $G$, written $\kappa(G; H)$, is the minimum cardinality over all $H$-structure-cuts of $G$. Similarly, if $F$ is a subgraph-cut and each element of $F$ is isomorphic to a connected subgraph of $H$, then $F$ is called an $H$-substructure-cut. The $H$-substructure connectivity of $G$, written $\kappa^s(G; H)$, is the minimum cardinality over all $H$-substructure-cuts of $G$.

Structure and substructure connectivity of some famous interconnection networks have been determined, such as hypercube [9], $k$-ary $n$-cube [11], folded hypercube [12], balanced hypercube [10], arrangement graph [7], alternating group graphs [8]. A natural question arise: what is computational complexity of structure and substructure connectivity in general graphs? In this paper, we study this problem.

2. NP-complete of structure connectivity

3-dimensional matching, 3DM for short, is one of the most standard NP-complete problems to prove NP-complete results. An instance of 3DM consists of three disjoint sets $R$, $B$ and $Y$ with equal cardinality $n$, and a set of triples $T \subseteq R \times B \times Y$. For convenience, let $W = R \cup B \cup Y$. The question is to decide whether there is a subset $T_1 \subseteq T$ covering $W$, that is, $|T_1| = q$ and each element of $W$ occurs in exactly one triple of $T_1$. This instance can be associated with a bipartite graph $G_b$ as follows.
Each element of $W$ and each triple of $T$ is represented by a vertex of $G_b$. There is an edge $wt$ between an element $w \in W$ and a triple $t \in T$ if and only if the element is a member of the triple.

It has been proved in [3,4] that 3DM is NP-complete when each element of $W$ appears in only two or three triples of $T$, i.e., each vertex in the partite set $W$ of $G_b$ has degree two or three only. We shall show that the decision problem of structure connectivity is NP-complete by reducing from 3DM stated previously.

To this end, we state the following decision problem.

**Problem:** The $H$-structure connectivity of an arbitrary graph.

**Instance:** Given a nonempty graph $G = (V, E)$, a subgraph $H$ of $G$ and a positive integer $q < |V|$.

**Question:** Is $\kappa(G; H) \leq q$?

Now we are ready to prove the following theorem.

**Theorem 1.** The $H$-structure connectivity is NP-complete when $H = K_{1,M}$ for any integer $M \geq 5$.

**Proof.** Obviously, the structure connectivity problem is in NP, because we can check in polynomial time whether a set of disjoint $K_{1,M}$s is a structure cut or not. It remains to show that the structure connectivity is NP-hard when $H = K_{1,M}$ for any integer $M \geq 5$. We prove this argument by reducing 3DM to it.

Let $G_b$ be an instance of 3DM defined previously. For convenience, let $T = \{t_k | 1 \leq k \leq |T|\}$ and $W = \{w_l | 1 \leq l \leq 3q\}$. We make a further assumption that each vertex in the partite set $W$ of $G_b$ has degree two or three only.

Now we construct a graph $G = (V, E)$ from $G_b$ as follows (see Fig. 1).

Set

$$V_j = \{v_{ij} | 1 \leq i \leq (M + 1)|T|\} \text{ for } j = 1, 2, \ldots, M - 3,$$

$$\hat{V} = \bigcup_{j=1}^{M-3} V_j,$$

$$U = \{u_1, u_2, \ldots, u_{3qM}\}, \text{ and}$$

$$U' = \{u'_1, u'_2, \ldots, u'_{3qM}\}.$$

The vertex set of $G$ is $V = V(G_b) \cup \hat{V} \cup U \cup U'$. The subgraph of $G$ induced by $V_j$ is $K_{(M+1)|T|}$ for each $j = 1, 2, \ldots, M - 3$. Similarly, The subgraph induced by $U$ is $3q$ vertex disjoint $K_M$ and the subgraph induced by $U'$ is $K_{3qM}$. 

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So the edge set of $G$ is

$$E = E(G_b) \cup (M - 3)E(K_{(M+1)|T|}) \cup 3qE(K_M) \cup E(K_{3qM}) \cup E_t \cup E_w \cup E_z,$$

where $E_t = \{\{t_k, v_{kj}\}|1 \leq k \leq |T|, 1 \leq j \leq M - 3\}$, $E_w = \{\{w_l, u_{lM}\}|1 \leq l \leq 3q\}$ and $E_z = \{\{u_i, u'_i\}|1 \leq i \leq 3qM\}$.

We show that $G_b$ has a 3DM $T_1 \subseteq T$ covering $W$ if and only if $\kappa(G; K_{1,M}) \leq q$. First suppose that $G_b$ has a subset $T_1 \subseteq T$ covering $W$, that is, $|T_1| = q$ and each element of $W$ occurs in exactly one triple of $T_1$. We show that $\kappa(G; K_{1,M}) \leq q$. Clearly, $d_G(x) = M$ for each vertex $x \in T_1$ and hence the subgraph of $G$ induced by $N[x]$ is isomorphic to $K_{1,M}$. Thus, $\mathcal{F} = \{G[N[x]]|x \in T_1\}$ is a structure cut of $G$ with $|\mathcal{F}| \leq q$.

Next suppose that $\mathcal{F}$ is a structure cut of $G$ with $|\mathcal{F}| \leq q$. We shall show that $G_b$ has a subset of $T$ covering $W$. Recall that each vertex in the partite set $W$ of $G_b$ has degree two or three only, we may assume that the number of vertices with degree two (resp. three) in $W$ of $G_b$ is $m$ and (resp. $n$), and consequently, $2m + 3n = 3|T|$, which implies that $3q = |W| \geq |T| > q$.

Since each element of $\mathcal{F}$ is a graph isomorphic to $K_{1,M}$, we focus on the center
vertex of $K_{1,M} \in \mathcal{F}$. Let $S$ be the the set of center vertices of all $K_{1,M} \in \mathcal{F}$, and let $S \cap G_b = S'$ and $S'' = S \cap (\tilde{V} \cup U \cup U')$. Since each vertex in $W$ has degree less than $M$, any vertex in $W$ can not be center vertices of $K_{1,M} \in \mathcal{F}$ that is, $S \cap W = \emptyset$. We claim that $\mathcal{F}$ covers $W$. Suppose on the contrary that $\mathcal{F}$ does not cover $W$, we shall show that $G - V(\mathcal{F})$ is connected. Thus, two cases arise.

**Case 1.** $S'' = \emptyset$. So $S = S'$. If there exists an edge $wt \in E(G - V(\mathcal{F}))$ such that $w \in W$ and $t \in T$, then it is not hard to see that $G - V(\mathcal{F})$ is connected, contradicting that $\mathcal{F}$ is a structure cut of $G$. Hence, all vertices in $W$ are covered by $V(\mathcal{F})$. Clearly, components in $\mathcal{F}$ restricted on $G_b$ form a 3-dimensional matching of $G_b$.

**Case 2.** $S'' \neq \emptyset$. Note that $d_{G_b}(t_i) = 3$ for any vertex $t_i \in T$ ($1 \leq i \leq |T|$) and $d_{G_b}(w_j) = 2$ or 3 for any vertex $w_j \in W$ ($1 \leq j \leq 3q$). By the structure of $G$, each vertex in $S''$ (as the center vertex of a $K_{1,M} \in \mathcal{F}$) can subvert at most one vertex in $G_b$. Similarly, each vertex in $S'$ can subvert precisely one vertex in $T$ together with three vertices in $W$.

Since $S'' \neq \emptyset$, we have $|S'| < q < |T|$. This implies that there exists an edge $ab \in E(G - V(\mathcal{F}))$ such that $a \in T$ and $b \in W$. Obviously, after subverting vertices in $\tilde{V}$, each clique $K_{(M+1)|T}$ either joins to $a$ or disappears, and similarly clique $K_{3qM}$ is decidedly joined to $b$ via one clique $K_M$ that $b$ joins. So $G - V(\mathcal{F})$ is connected, a contradiction again.

This complete the proof.

\[ \Box \]

### 3. NP-complete of substructure connectivity

A *vertex cover* of $G$ is a subset $V' \subseteq V$ such that for each edge $uv \in E$, at least one of $u$ and $v$ belongs to $V'$. The decision version of the vertex cover problem is one of Karp’s 21 NP-complete problems \[6\] and is therefore a classical NP-complete problem.

We present the decision problem of the substructure connectivity as follows.

**Problem:** The substructure connectivity of an arbitrary graph.

**Instance:** Given a nonempty graph $G = (V, E)$ with $\Delta(G) = M$, a subgraph $H$ of $G$ and a positive integer $k < |V|$.

**Question:** Is $\kappa^s(G, H) \leq k$?

The following lemma will be used later.

**Theorem 2.** The $H$-substructure connectivity is NP-complete when $H = K_{1,M}$.
Proof. Obviously, the substructure connectivity problem is in NP, because we can check in polynomial time whether a set of disjoint subgraphs of $K_{1,M}$ is a substructure cut. It remains to show that the substructure connectivity is NP-hard when $H = K_{1,M}$. We prove this argument by reducing vertex cover to this problem.

Given a graph $G = (V, E)$ with $\Delta(G) = M$, we construct a graph $G' = (V', E')$ from $G$ as follows (see Fig. 2).

Set $V_j = \{v_{ij} | 1 \leq i \leq |V|\}$ for $j = 1, \ldots, k + 2,$ $\hat{V} = \bigcup_{j=1}^{k+2} V_j$, and $V = \{v_1, v_2, \ldots, v_{|V|}\}$.

The vertex set of $G'$ is $V' = V \cup \hat{V}$. The subgraph of $G$ induced by $V_j$ is a complete graph $K_{|V|}$ for each $j = 1, 2, \ldots, k + 2$.

So the edge set of $G$ is

$$E' = E \cup (k + 2)E(K_{|V|}) \cup E_t$$

where $E_t = \{\{v_i, v_{ij}\} | 1 \leq i \leq |V|, 1 \leq j \leq k + 2\}$.

We show that $G$ has a vertex cover of size at most $k$ if and only if $\kappa^*(G', K_{1,M}) \leq k$.

First suppose that $G$ has a vertex cover $K$ with $|K| \leq k$. We show that there exists a substructure cut $\mathcal{F}$ of $G'$ with $|\mathcal{F}| \leq k$. For any vertex $x \in K$, let $V_x = (N_G(x) \setminus K) \cup \{x\}$. Then $G[V_x]$ consists of a spanning subgraph of $K_{1,M}$ with center

Fig. 2. NP-completeness of the substructure connectivity.
vertex $x$. Let $\mathcal{H} = \{G[V_x] | x \in K\}$. Clearly, $V(\mathcal{H}) = V$ and $|\mathcal{H}| = k$. Thus, $G' - V(\mathcal{H})$ is disconnected with $k + 2$ independent cliques $K_{|V|}$, indicating that $\mathcal{H}$ is a $K_{1,M}$-substructure-cut of size at most $k$. So $\kappa_s(G', K_{1,M}) \leq k$.

Next suppose that $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$ is a $K_{1,M}$-substructure-cut of $G'$ with $|\mathcal{F}| \leq k$. We show that $G$ has a vertex cover of size at most $k$.

Since each element of $\mathcal{F}$ is a subgraph of $K_{1,M}$, we focus on the center vertex of $F_i \in \mathcal{F}$ for $1 \leq i \leq k$ (since each center vertex of $K_{1,M}$ covers all its neighbors). Let $S$ be the set of center vertices of $F_i \in \mathcal{F}$, $1 \leq i \leq k$, and let $S \cap V = S'$. Thus, two cases arise.

**Case 1.** $V \subseteq V(\mathcal{F})$. Then there are vertices of $S \setminus S'$ that are adjacent to the vertices of $V$ not covered by $S'$. Since each vertex of $S \setminus S'$ is adjacent exactly one vertex in $V$, the number of vertices of $V$ not covered by $S'$ is not greater than the number of vertices in $S \setminus S'$. Therefore, there exists a vertex cover of $G$ of size at most $k$.

**Case 2.** $V \not\subseteq V(\mathcal{F})$. Then $V' = V \setminus V(\mathcal{F}) \neq \emptyset$. Since there are $k + 2$ disjoint cliques of size $k + 2$ in $G'$, each $F_i$ ($1 \leq i \leq k$) with center vertices in $S'$ could subvert at most $|S'|$ vertices in each $K_{|V|}$. Thus, the remaining subgraph of $G' - V(\mathcal{F})$ consists of at least two cliques of size at least $|V| - |S'|$ and the vertices in separate cliques are not adjacent. It implies that $G' - V(\mathcal{F})$ is connected and not a complete graph, which is a contradiction.

This completes the proof. \hfill $\Box$

**Remark.** The authors [2] constructed the similar graph $G'$ as in Theorem 2 to prove NP-completeness of neighbor connectivity by reducing from dominating set problem. While we show NP-completeness of the substructure connectivity by reducing from vertex cover problem.

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