Supersymmetry on a Lattice
and
Dirac Fermions in a Random Vector Potential

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Abstract

We study two-dimensional Dirac fermions in a random non-Abelian vector potential by using lattice regularization. We consider $U(N)$ random vector potential for large $N$. The ensemble average with respect to random vector potential is taken by using lattice supersymmetry which we introduced before in order to investigate phase structure of supersymmetric gauge theory. We show that a phase transition occurs at a certain critical disorder strength. The ground state and low-energy excitations are studied in detail in the strong-disorder phase. Correlation function of the fermion local density of states decays algebraically at the band center because of a quasi-long-range order of chiral symmetry and the chiral anomaly cancellation in the lattice regularization (the species doubling). In the present study, we use the lattice regularization and also the Haar measure of $U(N)$ for the average over the random vector potential. Therefore topologically nontrivial configurations of the vector potential are all included in the average. Implication of the present results for the system of Dirac fermions in a random vector potential with noncompact Gaussian distribution is discussed.

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1 Introduction

Random disordered systems are one of the most important problems in condensed matter physics. Especially in one and two dimensions effects of random disorders are so strong that almost all states are localized by random potentials. Nonperturbative methods are required in order to investigate random disordered systems in low dimensions. Recently there appeared important studies on Dirac fermions with random-varying mass and/or in a random vector potential.

Study of the Dirac fermions in a random vector potential was revived in Ref.[1] in the context of quantum Hall plateau transition. Non-Abelian generalization was introduced in Ref.[2] in the context of a d-wave superconductor. After that, there appeared a number of interesting papers[3, 4, 5, 6, 7, 8].

There are two technical problems for studying the system;
(i) Normalization of states before the ensemble average over random variables.
(ii) Integration over the white-noise random vector potential.
For the first one, replica trick and supersymmetric(SUSY) methods are often used. The second problem is how to regularize the integral over the vector potential which has no spatial correlations. Sometimes specific parameterization of vector potential is used especially for the non-Abelian case[6, 8]. There only topologically trivial configurations are integrated over. By using those “technologies”, critical lines in the system are observed.

Among various interesting properties of the random disordered systems, one of the most important problems is a disorder-induced phase transition. For the Dirac fermions in an Abelian random vector potential, a weak-strong disorder transition was found in Ref.[4] by studying multifractal scaling exponents of the critical wave function which is obtained exactly (see also Ref.[7]). Instability of the critical line is also seen by the existence of an infinite set of operators with negative scaling dimensions[1, 4, 6] and very recently solution to the negative dimension problem was suggested by Gurarie[9]. For the case of non-Abelian random vector potential, existence of
operators with negative scaling dimensions was discussed in Ref.\[5\] and a termination mechanism was given in Ref.\[8\] for the $SU(2)$ case. As a related system with the Dirac fermion in a random vector potential, random XY model in two dimensions was studied in Refs.\[10, 11\].

In all the above previous studies, integral region of the vector potential $A_\mu$ is noncompact, i.e., $A_\mu \in (-\infty, +\infty)$, and the probability distribution for it is taken to be Gaussian. However in the network models by Chalker and Coddington\[12\] and their field-theory representation\[13\], the random vector potential corresponds to the random Aharonov-Bohm phase for electron which moves in a random potential. Therefore the random variables are $e^{i A_\mu} \in U(1)$, and the range of the vector potential is compact, i.e., $A_\mu \in [-\pi, +\pi]$. In the weak-disorder case, the above compact distribution of the vector potential might be approximated by the noncompact Gaussian distribution, but at the strong disorder substantial differences in the properties of the system are expected to appear.

In this paper, we shall study random Dirac fermions by employing a lattice regularization to define the system without any ambiguity. The two-dimensional (2D) Dirac fermions in a random vector field is formulated on the lattice and the integral measure of the vector potential is compact as in the original network models\[12, 13\]. Therefore topologically nontrivial configurations of the vector potential are all included. We consider $U(N)$ vector potential for large $N$. Because of the compactness of the group $U(N)$, the one-link integral is evaluated exactly for large $N$ in a closed form.

Concerning with the above problem (i), we shall use the lattice SUSY (LSUSY) methods which we introduced before for the investigation on the SUSY gauge theory\[14\]. Formulation of the LSUSY is an important but still unsolved problem. However the LSUSY given in Ref.\[14\] is suitable for the present study.

For the one-link integral over $U(N)$ for large $N$, it is known that there are two regimes or “phases”, the weak-coupling and strong-coupling regimes. More precisely in the present context, the one-link integral exhibits a third-order phase transition as
the disorder strength is increased. Then one can expect that there is a genuine phase transition corresponding to these two regimes in the thermodynamic limit. In this paper we study the system in both the weak and strong-disorder cases and investigate the properties of both “phases”. Especially as the strong-disorder limit acquires lots of interest recently, we study the strong-disorder phase in detail.

This paper is organized as follows. In Sect. 2, the model and the LSUSY are explained. In Sect. 3, the weak-disorder regime is studied and an effective action is obtained by integrating over the random vector potential. In Sect. 4, the strong-disorder regime is considered. Properties of the ground state and low-energy excitations are clarified. It is shown that there exists a quasi-long-range order and correlation of the local density of states decays algebraically. Section 5 is devoted for discussion. Implication of the result for other interesting cases is discussed. Especially relation between the properties of the present model and results of the previous studies is examined. Physical picture of the result is explained.

2 Model and LSUSY

We shall study 2D Dirac fermions in a $U(N)$ random vector potential by employing the lattice regularization. Action of the Dirac fermion $\psi^a \ (a = 1, \ldots, N)$ on the square lattice is given by

$$S_D = \frac{1}{2} \sum \left[ \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + \mu) - \bar{\psi}(x + \mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) \right], \quad (2.1)$$

where $x = (x_0, x_1)$ denotes lattice site, $\mu = (0, 1)$ is the direction index, $U_\mu(x)$ is $U(N)$ field on the link $(x, x + \mu)$ $U_\mu(x) = \left(U_\mu(x)\right)_b^a \in U(N)$ and we set the lattice spacing $a_L = 1$. The random $U(N)$ vector potential $A_{\mu, \alpha}(x)$ ($\alpha$ is the $U(N)$ index) is related with $U_\mu(x)$ as follows,

$$U_\mu(x) = e^{ia_L \sum \alpha T^a A_{\mu, \alpha}(x)}, \quad (2.2)$$
where $T^\alpha$'s are generators of the $U(N) = U(1) \times SU(N)$ Lie algebra. In the (naive) continuum limit $a_L \to 0$, we can expand $U_\mu(x)$ as $U_\mu(x) = 1 + i a_L \sum_\alpha T^\alpha A_{\mu,\alpha}(x) + \cdots$ and recover the usual action of the Dirac fermion in the continuum. The two-dimensional $\gamma$-matrices are explicitly given by the Pauli matrices as $\gamma_0 = \sigma_x$, $\gamma_1 = \sigma_y$ and $\gamma_5 = \sigma_z$.

In order to take the ensemble average over the vector potential as random variables, we shall introduce boson field in a SUSY manner. To this end we shall slightly rewrite $S_D$ in Eq.\,(2.1). By the following transformation,

$$
\psi(x) = T(x)\chi(x), \quad \bar{\psi}(x) = \bar{\chi}(x)T^\dagger(x), \tag{2.3}
$$

with $T(x) = (\gamma_0)^x_0(\gamma_1)^x_1$ and using the identities like $(\gamma_\mu)^2 = 1$ and $(\gamma_0)^n\gamma_1 = (-)^n\gamma_1(\gamma_0)^n$ ($n$ is an integer),

$$
S_D = \frac{1}{2} \sum \left[ \bar{\chi}(x)\eta_\mu(x)U_\mu(x)\chi(x + \mu) - \bar{\chi}(x + \mu)\eta_\mu(x)U^\dagger_\mu(x)\chi(x) \right]
= \sum \bar{\chi}(x)\hat{D}\chi(x), \tag{2.4}
$$

where $\eta_0(x) = 1$, $\eta_1(x) = (-)^x_0$, and

$$
\hat{D}\chi(x) = \frac{1}{2} \sum_\mu \left[ \eta_\mu(x)U_\mu(x)\chi(x + \mu) - \eta_\mu(x - \mu)U^\dagger_\mu(x - \mu)\chi(x - \mu) \right]. \tag{2.5}
$$

The fields $\chi$ and $\bar{\chi}$ are two-component spinors but their spinor indices are diagonal in the action \,(2.4). We add the following “mass term” to the action which measures deviation from the band center or critical line,

$$
S_M = M \sum \bar{\chi}(x)\gamma_5\chi(x)
= M \sum \left[ \bar{\chi}_-\chi_+ - \bar{\chi}_+\chi_- \right], \tag{2.6}
$$

where $\chi = (\chi_+ , \chi_-)^t$ and $\bar{\chi} = (\bar{\chi}_- , \bar{\chi}_+)^t$. The specific form of the above mass term comes from merely a technical reason which becomes clear shortly. We are interested in the limit $M \to 0$.

We introduce a complex scalar field $\phi(x)$ whose action is given by

$$
S_\phi = \sum \hat{D}\phi^\dagger(x)\hat{D}\phi(x) + m^2 \sum \phi^\dagger(x)\phi(x). \tag{2.7}
$$
As the modified “Dirac” operator $\hat{D}$ does not contain the $\gamma$-matrices, the same $\hat{D}$ can be applied for the scalar field $\phi$. This is an essential point in the present construction of the SUSY lattice model. It can be shown that in the classical continuum limit the action $S_\phi$ in Eq. (2.7) reduces to the usual action of the scalar field in the continuum, and the integral over the scalar fields is well-defined because the action is positive-definite.

The total action of the system is given by

$$S = S_\chi + S_\phi,$$

$$S_\chi = S_D + S_M$$

$$= \sum [\bar{\chi}_+ \hat{D} \chi_- + \bar{\chi}_- \hat{D} \chi_+] + M \sum [\bar{\chi}_- \chi_+ - \bar{\chi}_+ \chi_-].$$

(2.8)

It is seen that the partition function is just unity if $M = m$ for an arbitrary fixed configuration of the vector potential because of the cancellation of the fermion and boson determinants. Actually

$$\int [D\phi^\dagger D\phi] e^{-S_\phi} = \text{det}^{-1}(-\hat{D}^2 + m^2),$$

$$\int [D\bar{\chi}D\chi] e^{-S_\chi} = \text{Det} \left( \begin{array}{cc} \hat{D} + M & 0 \\ 0 & \hat{D} - M \end{array} \right)$$

$$= \text{det}(\hat{D}^2 - M^2),$$

(2.9)

where Det is the determinant of the spinor and real spaces and det is that of the real space.

Moreover the action $S$ is invariant under the following LSUSY transformation for $M = m$,

$$\delta \phi = \bar{\epsilon}_+ \chi_- + \bar{\epsilon}_- \chi_+,$$

$$\delta \phi^\dagger = \bar{\chi}_- \epsilon_+ + \bar{\chi}_+ \epsilon_-,$$

$$\delta \chi_\pm = \pm M \phi \epsilon_\pm - \hat{D} \phi \epsilon_\pm,$$

$$\delta \bar{\chi}_\pm = \pm M \phi^\dagger \bar{\epsilon}_\pm - \hat{D} \phi^\dagger \bar{\epsilon}_\pm,$$

(2.10)
where $\epsilon_\pm$ are anticommuting spinor variables with chirality $\gamma_5 \epsilon_\pm = \pm \epsilon_\pm$.

The bosonic part of the action can be rewritten into more symmetric form

$$S_\phi \Rightarrow S_\omega \phi = \sum [\omega^\dagger \hat{D} \phi + \phi^\dagger \hat{D} \omega] + m \sum \phi^\dagger \phi + m \sum \omega^\dagger \omega, \quad (2.11)$$

where $\omega$ and $\phi$ are complex boson fields. By integrating over $\omega(x)$ (or $\phi(x)$), one can easily verify the equivalence of $S_\phi$ and $S_\omega \phi$.

The lattice Dirac action (2.1) has exact chiral symmetry for $M = 0$. This means that there appear multi-flavour Dirac fermions with opposite chirality in the continuum limit, i.e., the species doubling. $S_\chi$ with $M = 0$ is therefore invariant under the following chiral $U(1) \times U(1)$ symmetry on the lattice,

$$\begin{align*}
\chi_+(x) &\rightarrow V_\epsilon(x) \chi_+(x), \quad \bar{\chi}_-(x) \rightarrow \bar{\chi}_-(x) V^*_\epsilon(x), \\
\chi_-(x) &\rightarrow W_\epsilon(x) \chi_-(x), \quad \bar{\chi}_+(x) \rightarrow \bar{\chi}_+(x) W^*_\epsilon(x),
\end{align*} \quad (2.12)$$

where $\epsilon(x) = (-)^{x_0 + x_1}$ and $V_\pm, W_\pm \in U(1)$. This symmetry plays a very important role in the discussion on the phase structure as we shall see later on.

Expectation value of physical quantity $X$ is given by the following functional integral,

$$\langle X \rangle = \int [D Ud \bar{\chi} D \chi D \phi^\dagger D \phi] P[U] e^{-S} X, \quad (2.13)$$

where the probability distribution for the random vector potential is given by

$$P[U] = \exp \left( \frac{N}{g} \sum_{x,\mu} Tr(U_\mu(x) + U_\mu^\dagger(x)) \right), \quad (2.14)$$

$[DU] = \prod_{\text{link}} dU_\mu(x)$ is the Haar measure of $U(N)$ and $g$ is a parameter which controls the disorder strength. As we explained in the introduction, the random vector potentials are taken to be compact random variables. This is in contrast with the previous studies where the vector potential is taken to be noncompact and the probability distribution is taken to be Gaussian. Since in the weak-disorder case, i.e., the case of small $g$, the compact distribution (2.14) can be approximated as (here we
show the Abelian case for notational simplicity)
\[
e^{\frac{2N}{g} \cos A_\mu} \quad A_\mu \in [-\pi, \pi] \quad \rightarrow \quad e^{-\frac{N}{g} A_\mu^2 + \cdots} \quad A_\mu \in (-\infty, +\infty)
\]
(2.15)

one may expect that the both compact and noncompact systems give similar results at least qualitatively.\(^1\) However in the strong-disorder case, substantial difference will appear. Because of the regularization by the lattice, topologically nontrivial configurations are all included in the integral. This is in contrast with the discussion in terms of the conformal field theory (CFT)\(^3\)\(^8\).

In the subsequent sections we shall perform \(U(N)\)-integral in (2.13). It is known that there are two “phases” for this one-link integral, i.e., weak-coupling “phase” for small \(g\) and strong-coupling “phase” for large \(g\). Result of the integral exhibits a third-order phase transition at a certain critical value of \(g\)\(^13\). This is merely a matter of kinematics of the integral over the group \(U(N)\) for large \(N\). However we think that there exists a genuine phase transition in the system of the random Dirac fermions from weak to strong-disorder phases, i.e., disorder-induced phase transition.

### 3 Weak-disorder regime

Expectation values of physical quantities are given by (2.13) and (2.14). We shall first perform the functional integral of the vector potential \(U_\mu\). Then let us consider the following one-link integral,
\[
e^{W(\bar{D}, D)} = \int dU_\mu \exp \left[ \text{Tr} (\bar{D}_\mu U_\mu + U_\mu^\dagger D_\mu) \right].
\]
(3.1)

In the present case,
\[
D_\mu(x)_b^a = A_\mu(x)_b^a + \frac{N}{g} \delta_b^a,
\]
\(^1\)Actually this expectation is too naive. The compactness of the vector potential plays a very important role for the correlation functions in which singular configurations of \(A_\mu\) give dominant contribution. See discussion in Sect.5.
\[ \bar{D}_\mu(x)^a_b = \bar{A}_\mu(x)^a_b + \frac{N}{g} \delta_b^a, \]
\[ A_\mu(x)^a_b = \frac{\eta_\mu(x)}{2} [\bar{\chi}_b(x + \mu) \chi^a(x) + \omega_b^\dagger(x + \mu) \varphi^a(x) + \varphi_b^\dagger(x + \mu) \omega^a(x)], \]
\[ \bar{A}_\mu(x)^a_b = -\frac{\eta_\mu(x)}{2} [\bar{\chi}_b(x) \chi^a(x + \mu) + \omega_b^\dagger(x + \mu) \varphi^a(x + \mu) + \varphi_b^\dagger(x) \omega^a(x + \mu)]. \quad (3.2) \]

Let us introduce a parameter \( s \) by

\[ s = \frac{1}{N} \sum_{a=1}^N x_a^{-1/2} = \text{Tr}(\bar{D}D)^{-1/2}, \quad (3.3) \]

where the \( x_a \)'s are eigenvalues of \( \frac{1}{N} \bar{D}D \). In Ref. [15], it is shown that there are two regimes for the above integral \( (3.1) \), i.e., weak-coupling regime for \( s < 2 \) and strong-coupling regime for \( s > 2 \). For \( D^a_b = \frac{N}{g} \delta^a_b \), we can estimate the critical value of \( g \) as \( g_c = 2 \) from \( (3.3) \). In the present case there are extra factors \( A \) and \( \bar{A} \) in \( D \) and \( \bar{D} \) and then precise value of \( g_c \) cannot be determined. However we can expect that for sufficiently small(large) \( g \) the system is in the weak(strong)-coupling regime.

Let us consider the weak-disorder phase first, i.e., the phase of small \( g \). In this case the result of the \( U(N) \)-integral is given as [15]

\[ W(\bar{D}, D) = N \left\{ 2 \sum_a x_a^{1/2} - \frac{1}{2N} \sum_{a,b} \log(x_a^{1/2} + x_b^{1/2}) \right\}. \quad (3.4) \]

From \( (3.2) \)

\[ (\bar{D}_\mu D_\mu(x))^a_b = \left( \frac{N}{g} \right)^2 \delta^a_b + \frac{N}{g} A^a_{\mu b} + \frac{N}{g} \bar{A}^a_{\mu b} + \bar{A}^a_{\mu c} A^c_{\mu b}. \quad (3.5) \]

It is straightforward to obtain \( (\bar{D}D)^{1/2} \) in powers of \( \frac{g}{N} \),

\[ (\bar{D}D)^{1/2,a}_b = \frac{N}{g} \delta^a_b + \frac{1}{2} (A^a_{\mu b} + \bar{A}^a_{\mu b}) - \frac{g}{8N} (A^a_{\mu c} + \bar{A}^a_{\mu c})(A^c_{\mu b} + \bar{A}^c_{\mu b}) + \frac{g}{2N} A^a_{\mu c} A^c_{\mu b} + O((g/N)^2). \quad (3.6) \]

Evaluation of the second term of the formula \( (3.4) \) is also not so difficult. As matrices \( D \) and \( \bar{D} \) have both large diagonal matrix elements \( N/g \), we can use the Taylor expansion for a regular function \( f(x, y) \),

\[ \sum_{a,b} f(x_a, x_b) = \sum_{a,b} f \left( \frac{1}{g^2} + w_a, \frac{1}{g^2} + w_b \right) \]
\[
\sum_{n,m} \frac{1}{n!m!} f^{(n,m)}(\frac{1}{g^2}, \frac{1}{g^2}) \text{Tr}\left(\frac{\bar{D}D}{N^2} - \frac{1}{g^2}\right)^n \text{Tr}\left(\frac{\bar{D}D}{N^2} - \frac{1}{g^2}\right)^m,
\]
where \(w_a\)'s are eigenvalues of \((\frac{\bar{D}D}{N^2} - \frac{1}{g^2})\). Leading-order terms are obtained as,

\[
\sum_{a,b} \log\left(x_1^{a/2} + x_2^{a/2}\right) = \frac{g}{2} (A_{\mu a}^a + \bar{A}_{\mu a}^a) + \frac{g}{2} \frac{g}{N} \bar{A}_{\mu b}^a A_{\mu a}^b + O((g/N)^2) \text{.}
\]

From (3.6) and (3.7), the effective theory which appears after the integration over the \(U(N)\) field for small \(g\) is a SUSY extension of the Gross-Neveu model. Detailed study of this model will be given elsewhere[16]. However here we mention that the four-Fermi coupling in the interaction term \(\bar{A}_{\mu b}^a A_{\mu a}^b\) has the sign which indicates instability of the Dirac fermion at the band center or on criticality. By the usual \(1/N\) expansion, it is expected that the chiral condensation \(\langle \bar{\psi}\psi \rangle \neq 0\) occurs in the present system. However the system under study is invariant under the chiral \(U(1) \times U(1)\) transformation (2.12), and therefore it is expected that only a quasi-long-range order exists as in the Kosterlitz-Thouless phase, though careful study is required because of the existence of the bosons. If this expectation is correct, then excitations are massless “pions”, their SUSY partners and the Dirac fermions[16].

Closely related model appears in the case of 2D Dirac fermion in a random non-compact Abelian vector potential, i.e., a SUSY Thirring model. This is not surprising because in the weak-disorder limit the compact measure of the vector potential is well approximated by the noncompact Gaussian distribution, as we explained before. In Ref.[5], it is shown that there exist an infinite number of relevant operators with negative scaling dimensions. Very recently Gurarie suggested a possible solution to this problem[9]. In this argument, some ad hoc cutoff regularization is used for the functional integral over the noncompact vector potential. On the other hand in the present system, the compact Haar measure is used for the integration over the vector potential and therefore it is expected that a cutoff appears in a natural way. We shall

\[\text{More precisely, Dirac fermions in this theory has “flavour” degrees of freedom as a result of the species doubling.}\]
show that this is the case. Relation between Gurarie’s argument and the present lattice model will be discussed later on.

Physical picture of the above phenomenon will be discussed rather in detail in Sect.5 after investigation of the strong-coupling regime. In the following section, we shall study the strong-disorder phase which is the main subject of the present paper.

4 Strong-disorder regime

In the strong-coupling regime of the $U(N)$-integral, $W(D, D)$ is given by the following formula [15],

$$W(D, D) = N^2 \left\{ -\frac{3}{4} - c + \frac{2}{N} \sum_a (c + x_a)^{1/2} \right. $$

$$- \frac{1}{2N^2} \sum_{a,b} \log((c + x_a)^{1/2} + (c + x_b)^{1/2}) \left. \right\},$$

(4.1)

where $x_a$’s are again eigenvalues of $\frac{1}{N^2}DD$ and a constant $c$ is given by

$$1 = \frac{1}{2N} \sum a (c + x_a)^{-1/2}.$$  \hspace{4cm}  (4.2)

As we explained above, the formula (4.1) is suitable for large $g$. The limit $g \to +\infty$ is nothing but the strong-coupling limit of the SUSY lattice gauge theory which was studied in Ref. [14]. There we showed that the condensations like $\langle \bar{\chi}\chi \rangle$, $\langle \phi^\dagger\phi \rangle$, etc. occur, whereas $\langle \phi^\dagger\omega \rangle = \langle \omega^\dagger\phi \rangle = 0$. Here we assume a similar pattern of condensations for large $g$. Then it is not so difficult to calculate the effective action from (4.1) by the $1/g$-expansion. Here again the Taylor expansion is useful to convert the summation over the eigenvalues $x_a$ into the trace of the matrix $\bar{D}D$.

After some calculation, we obtain [17]

$$\frac{1}{N^2}W(D, D) = \frac{1}{N} \left[ \sum_{\pm} F(\lambda_{\pm}) - F(\xi) - F(\zeta) \right] + \frac{\eta_\mu(x)}{gN^2} \left[ \chi^a(x)\bar{\chi}_a(x + \mu)G(\lambda) \right. $$

$$- \varphi^a(x)\omega_\lambda^\dagger(x + \mu)G(\xi) - \omega^a(x)\varphi_\lambda^\dagger(x + \mu)G(\zeta) \left. \right]\left[ \chi^a(x)\bar{\chi}_a(x + \mu)G(\lambda) \right. $$

$$- \varphi^a(x)\omega_\lambda^\dagger(x + \mu)G(\xi) - \omega^a(x)\varphi_\lambda^\dagger(x + \mu)G(\zeta) \left. \right] + O(1/(g^2N)),$$

(4.3)

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where $\lambda_{\pm}$ etc are composite fields of $\chi_{\pm}$ etc, and they are explicitly given by

$$
\lambda_{\pm} = \lambda_{\mu}(x) = m_{\pm}(x)m_{\pm}(x + \mu), \quad m_{\pm}(x) = \frac{1}{N} \sum_{a} \chi_{\pm}(x) \bar{\chi}_{a\pm}(x),
$$

$$
\xi = \xi_{\mu}(x) = \alpha(x)\beta(x + \mu), \quad \zeta = \zeta_{\mu}(x) = \beta(x)\alpha(x + \mu),
$$

$$
\alpha(x) = \frac{1}{N} \sum_{a} \varphi_{a}(x)\varphi_{a}^{\dagger}(x), \quad \beta(x) = \frac{1}{N} \sum_{a} \omega_{a}(x)\omega_{a}^{\dagger}(x). \quad (4.4)
$$

Functions $F(x)$ and $G(x)$ are given by

$$
F(x) = 1 - (1 - x)^{1/2} + \log\left[\frac{1}{2}(1 + (1 - x)^{1/2})\right],
$$

$$
G(x) = (1 + (1 - x)^{1/2})^{-1}. \quad (4.5)
$$

Actually there are additional terms of composites like $\chi_{a}(x)\varphi_{a}^{\dagger}(x)$, but they do not have nonvanishing expectation values and give only higher-order corrections in $1/N$ to the effective action of $m_{\pm}(x)$ etc.

We expect that the $1/g$-expansion in (4.3) has a finite convergence radius. Then it is easily verified that the effective action can be written in terms of the composites $m(x), \alpha(x)$ and $\beta(x)$, or more precisely $\lambda_{\mu}(x), \xi_{\mu}(x)$ and $\zeta_{\mu}(x)$. Then we can introduce elementary fields corresponding to the composite fields in the path-integral formalism. First for the composite “meson” field $m_{\pm}(x)$, we have identity like (up to an irrelevant constant),

$$
Z_{0}^{F}(J) \equiv \int d\bar{\chi}d\chi e^{Jm} = J^{N} = \int_{0}^{2\pi} \frac{d\theta}{2\pi} (\rho e^{i\theta})^{-N} \exp(J \rho e^{i\theta}) \equiv \int d\mathcal{M} \mathcal{M}^{-N} \exp(J\mathcal{M}). \quad (4.6)
$$

Equation (4.6) means that the path integral of the elementary meson fields $\mathcal{M}_{\pm}$ is defined by the above contour integral and the radius $\rho$ should be taken for the angle integral to be well-defined, i.e., $\rho$ should be a maximum or saddle point of the effective potential.

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3Strictly speaking, here we assume that the $U(N)$ symmetry is not spontaneously broken.
On the other hand for the boson-composite field $\alpha(x)$, we can prove the following identity,

$$Z_0^B(J) \equiv \int d\bar{\phi} d\phi e^{-J\alpha} = J^{-N} = \int_{-\infty}^{+\infty} d(\ln \Phi) \Phi^N \exp(-J\Phi). \quad (4.7)$$

In a similar way we introduce elementary field $\Psi(x)$ for the composite field $\beta(x)$.

From (4.1), (4.3) and (4.7), the effective action in the strong-disorder phase is obtained as

$$\frac{1}{N} S_{eff} = -\sum_{x,\mu,\pm} \left[ F(\lambda_{\mu\pm}(x)) - \frac{1}{4} \log \lambda_{\mu\pm}(x) \right] - M \sum_x (M_+ - M_-)$$

$$+ \sum_{x,\mu} \left[ F(\xi_{\mu}(x)) - \frac{1}{4} \log \xi_{\mu}(x) \right] + m \sum_x \alpha$$

$$+ \sum_{x,\mu} \left[ F(\zeta_{\mu}(x)) - \frac{1}{4} \log \zeta_{\mu}(x) \right] + m \sum_x \beta + O(1/g^2), \quad (4.8)$$

where $\lambda_{\mu\pm}(x) = M_{\pm}(x)M_{\pm}(x+\mu)$, $\xi_{\mu}(x) = \Phi(x)\Psi(x+\mu)$ and $\zeta_{\mu}(x) = \Psi(x)\Phi(x+\mu)$. Terms of $O(1/g^2)$ have a similar structure to the leading-order terms.

Then it is straightforward to study the structure of the ground state and low-energy excitations. For vanishing masses $M = m = 0$, the ground state is parameterized as follows,

$$\langle M_\pm(x) \rangle = \begin{cases} v U_0 \pm, & \text{at even sites} \\ v U_0^*, & \text{at odd sites} \end{cases} \quad (4.9)$$

$$\langle \Phi(x) \rangle = \begin{cases} v e^{\sigma_1}, & \text{at even sites} \\ v e^{-\sigma_2}, & \text{at odd sites} \end{cases} \quad (4.10)$$

$$\langle \Psi(x) \rangle = \begin{cases} v e^{\sigma_2}, & \text{at even sites} \\ v e^{-\sigma_1}, & \text{at odd sites} \end{cases} \quad (4.11)$$

where $U_{0\pm} \in U(1)$, $\sigma_i$’s $(i = 1, 2)$ are real numbers and $v$ is obtained from the stationary condition of the effective potential,

$$\frac{dV(v^2)}{dv^2} = \frac{dF(v^2)}{dv^2} - \frac{1}{4v^2} = 0, \quad (4.12)$$

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with the following solution

\[ v^2 = 3/4. \]  

(4.13)

The mass terms lift the above degeneracy and determine the expectation values as \( \langle M_+ \rangle = -\langle M_- \rangle \) and \( \sigma_1 = \sigma_2 = 0 \). Obviously the ground state preserves the SUSY\textsuperscript{14}.

For vanishing masses, there exist the degeneracies of the ground state parameterized by \( U_{0\pm} \) and \( \sigma_i \) which originate from the chiral symmetry (2.12) and its SUSY counterpart for \( \varphi \) and \( \omega \). By the Coleman-Mermin-Wagner theorem, in two dimensions continuous symmetry is not spontaneously broken and there exists no long-range order. Therefore we cannot expect the condensations \( \langle M_\pm \rangle \neq 0 \), etc. Instead the ground state exists in the Kosterlitz-Thouless phase with gapless excitations. In fact we can explicitly show the existence of massless modes which destroy the off-diagonal long-range order. These excitations are described by the “pion” fields,

\[
M_\pm(x) = \begin{cases} 
vU_\pm(x) = ve^{i\pi_\pm(x)}, & \text{at even sites} \\
vU^*_\pm(x) = ve^{-i\pi_\pm(x)}, & \text{at odd sites}
\end{cases}
\]  

(4.14)

and similar SUSY excitations for \( \Phi(x) \) and \( \Psi(x) \). From (4.10) and (4.11), it is obvious that these low-energy excitations are nothing but “density wave” of the SUSY bosons which is commensurate with the lattice structure.

Effective action of \( \pi_\pm(x) \) is obtained as follows from (4.8) and (4.14),

\[
S_\pi = \frac{N}{2} C \sum [\pi_\pm(x + \mu) - \pi_\pm(x)]^2;
\]

\[
C = F''(v)v^4 + \frac{1}{4}, \quad v^2 = 3/4.
\]  

(4.15)

Therefore the correlator of \( M_\pm(x) \) exhibits a power-law decay

\[
\langle \bar{\psi}\psi(x)\bar{\psi}\psi(0) \rangle = \langle M_\pm(x)M_\pm(0) \rangle \sim |x|^{-1/(2\pi NC)}.
\]  

(4.16)

Corrections of \( O(1/g^2) \) can be calculated systematically and the scaling dimension of \( M_\pm \) acquires correction of \( O(1/g^2) \).
There appear no signs of instability of the ground state. Low-energy excitations in the boson sector are given by local fluctuations of $\sigma_i$ ($i = 1, 2$) in Eqs. (4.10) and (4.11). They are SUSY counterparts of $\pi_{\pm}(x)$ and stable. All correlation functions in the fermion sector have nonsingular behaviour like Eq. (4.16). This is in sharp contrast with the previous results which show that fermion composite operators (as well as boson composite operators) have negative scaling dimensions. We think that the compactness of the functional-integral measure of the vector potential plays a very important role for the stability. This important point will be discussed in the following section.

From the discussion given so far, it is obvious that the algebraic decay of the correlation functions in the strong-disorder phase comes from the exact chiral symmetry on the lattice which is a result of the species doubling. In the single-flavour case in the continuum, there exists anomaly in the chiral symmetry because of the coupling with the vector potential, and therefore the genuine condensation $\langle \bar{\psi}\psi \rangle \neq 0$ is possible even in two dimensions just as in the Schwinger model.

5 Discussion

In this paper we studied Dirac fermions in a random $U(N)$ vector potential. We employed the lattice regularization and the compact Haar measure in order to make the functional integral over $U(N)$ vector potential well-defined. In this formalism topologically nontrivial configurations are all integrated over. This is in sharp contrast with the approaches given so far. The ensemble average over the random vector potential was taken by introducing bosons in a SUSY way. We think that this approach is important because there appeared some evidences that there exists a disorder-induced phase transition in the present system. In order to investigate this problem,

4In other words, the would-be Nambu-Goldstone boson acquires a mass by the chiral anomaly, and it generates no severe infrared singularities.
a well-defined formalism is indispensable.

For the one-link $U(N)$ integral, it is known that there are two regimes, i.e., the weak and strong-coupling regimes, which correspond to the weak and strong-disorder cases, respectively. We obtained effective theory by integrating over the vector potential in both regimes. In the weak-disorder phase, the effective theory is a SUSY extension of the Gross-Neveu model. We call this regime phase A. Detailed studies on the effective field theory of the phase A will be reported elsewhere, but sign of the effective coupling constant of the four-Fermi interaction indicates instability of the ground state to the state with the chiral condensation. From the investigation of the Gross-Neveu model by the $1/N$ expansion, we expect that there appears the chiral condensation with a quasi-long-range order.

Then we studied the strong-disorder phase rather in detail. We call this regime phase B. We showed that in the phase B the density operator of the fermion has the quasi-long-range order and low-energy excitations are the “pions” and density wave of the bosons whereas no Dirac fermions with the original $U(N)$ quantum number appear. This result stems from not only the strong-disorder properties of the vector potential but also the exact chiral symmetry and its SUSY counterpart on the lattice, i.e., anomaly cancellation by the species doubling. Therefore, the result indicates that genuine condensation of the fermion density operator occurs in the single-flavour case with chiral anomaly just as in the Schwinger model. Anyway, we expect the existence of the phase transition at a certain critical value of the disorder strength $g_c$ from the phase A to B.

Let us discuss physical picture of the above phenomena. Coupling with the vector potential reduces the effective hopping of fermions which is simply given by $t_{\text{eff}} = t \cdot \langle U_\mu(x) \rangle$ where $t$ is the original hopping parameter. Obviously as increasing the disorder strength $g$, $t_{\text{eff}}$ decreases and fermionic states tend to localize. Study in this paper shows that in the present system that localization phenomenon occurs.

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5Order of this phase transition can be of third-order as the one-link integral indicates.
with the chiral condensation of fermions which makes the fermions massive. As $g$

is increased further more, even a local hopping of a single fermion cannot occur

anymore and movement of a fermion always accompanies the same movement of an

anti-fermion for fluctuations of the vector potential in the hopping cancels out with

each other, i.e., $\langle U_\mu(x) \rangle = 0$ but $\langle U_\mu(x) U_\mu^\dagger(x) \rangle = 1$. That is, only a “bound state” of

fermion and anti-fermion pair can move in this phase. Then condensation of fermion
density operator is generated. This phase (the phase B) is more or less similar to
the conventional confinement phase of the strong-coupling gauge theory. From this
picture, disorder-induced phase transition is naturally understood.

Finally let us discuss relation between the results in this paper and previous stud-
ies. For both the Abelian and non-Abelian cases, it is known that there exist an
infinite number of relevant operators with negative dimensions if the noncompact
Gaussian distribution is used for the vector potential. It indicates some instability
of the critical line. On the other hand in the present study, no signs of instability
appear at least in the effective action obtained by integrating over the vector poten-
tial. We think that this stability stems from the compactness of the integral measure
of the vector potential. Actually as recently Gurarie discussed and the study on
the random XY model shows, the instability comes from the noncompactness of
the vector potential. More explicitly in the discussion in the continuum, the vec-
tor potential $A_\mu$ is parameterized as follows (we here consider the Abelian case for
simplicity),

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \theta + \partial_\mu \eta, \quad \epsilon_{01} = -\epsilon_{10}, \quad (5.1)$$

where $\theta(x)$ and $\eta(x)$ are scalar fields and $\theta(x), \eta(x) \in (-\infty, +\infty)$. The instability and

the negative dimensions of the relevant operators essentially come from the following

\footnote{Strictly speaking, the correlation function in the boson sector $\langle \Phi(x) \Psi(0) \rangle$ tends to diverge for

$|x| \to \infty$. However correlators in the fermion sector, which are physical quantities in the present

system, exhibit no singular behaviour.}

\footnote{For the Abelian case, similar discussion of the strong-disorder case is possible. See Ref.
[18].}
correlation function of $\theta(x)$ \[ \langle e^{-c\theta(x_1)}e^{c\theta(x_2)} \rangle, \] where $c$ is a real number and the above expectation value is evaluated with the following probability distribution,

\[ P[\theta] \propto \exp \left\{ -\frac{1}{g} \int d^2x \left( \partial_\mu \theta \right)^2 \right\}. \] (5.3)

Then it is not difficult to show that the operator $e^{c\theta(x)}$ has a negative dimension and the correlator in Eq.(5.2) tends to diverge for large $|x_1 - x_2|$. However it is obvious that if the integral region of $\theta(x)$ is compact, this divergence does not occur. This is an essential point of Gurarie’s argument\[9\].

Gurarie used some ad hoc cutoff regularization for the functional space of $\theta(x)$. On the other hand in this paper, we use the Haar measure which is compact, and therefore we expect that a similar cut off appears naturally in the present formalism. Actually we can parameterize the vector potentials $U_\mu(x)$ in terms of two $U(1)$ fields $u(x)$ and $v(x + \hat{0} + \hat{1}/2) \ (\hat{0}(\hat{1})$ is the unit vector of the 0(1) direction), where $u(x)$ is defined on the sites of the original lattice and $v(x + \hat{0} + \hat{1}/2)$ is on the sites of the dual lattice,

\[ U_0(x) = u(x + \hat{0})u^*(x)v(x + \hat{0} + \hat{1}/2)v^*(x + \hat{0} - \hat{1}/2), \] (5.4)

and similarly for $U_1(x)$ where $u^*(x)$ is the complex conjugate to $u(x)$ etc. It is not difficult to show that if we impose the conditions like $\prod_x u = \prod_x v = 1$, then there is no ambiguity in this parameterization. It is obvious that $\theta(x)(\eta(x))$ in the continuum expression (5.1) is related to $v(x + \hat{0} + \hat{1}/2)(u(x))$ in (5.4) as follows,

\[ \theta \sim \ln v, \quad \eta \sim \ln u. \] (5.5)

Therefore the integral region of $\theta(x)$ is compact in the present formalism, $\theta(x) \in [-\pi, +\pi]$, i.e., there exists the natural cutoff. Please notice that this compact region of $\theta(x)$ remains the same even if we recover the lattice spacing $a_L$. 

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In the sense explained above, the model in the present paper is different from those with noncompact random vector potential which were studied in the previous papers. Our model is close to the network models by Chalker and Coddington\[12\] and their field-theory models\[13\]. Then it is not so surprising that the stable ground state appears and the low-energy excitations are the massless “pions” etc as in the Kosterlitz-Thouless phase even at the strong-disorder limit. Also we can conclude that the disorder-induced phase transition which we found in this paper is a new one. This phase transition is expected to be of a topological nature of the vector potential but it is not definitive at this stage.

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Note added
After submitting this paper, we got acquainted with the paper by Altland and Simons\[19\] which also studies the random flux model on a lattice.
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