BUCHSBAUM* COMPLEXES

CHRISTOS A. ATHANASIADIS AND VOLKMAR WELKER

Abstract. A class of simplicial complexes, which we call Buchsbaum* over a field, is introduced. Buchsbaum* complexes generalize triangulations of orientable homology manifolds as well as doubly Cohen-Macaulay complexes. By definition, the Buchsbaum* property depends only on the geometric realization and the field. Characterizations in terms of simplicial and local cohomology are given. It is proved that Buchsbaum* complexes are doubly Buchsbaum. Enumerative and graph theoretic properties of Buchsbaum* complexes are investigated. It is shown that various constructions, among them one which generalizes convex ear decompositions, yield Buchsbaum* simplicial complexes.

1. Introduction

A major theme in the study of simplicial complexes in the past few decades has been the interplay between their algebraic, combinatorial, homological and topological properties. Several classes of simplicial complexes, such as Buchsbaum, Cohen-Macaulay or Gorenstein complexes, have been introduced and studied in order to isolate important features of triangulations of fundamental geometric objects, such as balls, spheres and various other manifolds. We refer the reader to [18] for a comprehensive introduction to the subject.

The objective of this paper is to introduce and develop the basic properties of a new class of simplicial complexes, named Buchsbaum* complexes, which generalize triangulations of orientable homology manifolds.

In this introductory section we motivate our main definition and outline the remainder of the paper (we refer the reader to [18, Chapter II] [3, Chapter 5] [19, Chapter II] for any undefined terminology). A simplicial complex over the ground set \(\Omega\) is a (finite) collection \(\Delta\) of subsets of \(\Omega\) such that \(\sigma \subseteq \tau \in \Delta\) implies \(\sigma \in \Delta\). Given a field \(k\), the face ring or Stanley-Reisner ring \(k[\Delta]\) of \(\Delta\) over \(k\) is the quotient of the polynomial ring \(k[x_\omega : \omega \in \Omega]\) by the ideal generated by the monomials \(\prod_{\omega \in N} x_\omega\) for all subsets \(N\) of \(\Omega\) not in \(\Delta\). Recall (see [18]) that \(\Delta\) is called Buchsbaum (respectively, Cohen-Macaulay, Gorenstein) over \(k\) if the face ring \(k[\Delta]\) is a Buchsbaum (respectively, Cohen-Macaulay, Gorenstein) ring. Such a complex \(\Delta\) is called doubly Buchsbaum [10] (respectively, doubly Cohen-Macaulay [2, 18, p. 71]) over \(k\) if for every vertex \(v\) of \(\Delta\), the complex \(\Delta \setminus v\), obtained from \(\Delta\) by removing all faces which contain \(v\), is Buchsbaum (respectively, Cohen-Macaulay) over \(k\) of the same dimension as \(\Delta\).

It is known that Buchsbaumness [16, 18, Theorem 8.1]; see also Theorem 2.1 (respectively, Cohen-Macaulayness; see [11, 18, Proposition 4.3]) of \(\Delta\) is a topological property, meaning that it depends only on the homeomorphism type of the geometric realization \(|\Delta|\).
Section 9] of $\Delta$. For instance, all triangulations of manifolds (with or without boundary) are Buchsbaum and all triangulations of balls and spheres are Cohen-Macaulay over all fields. It was conjectured by Baclawski \[2\] and proved by Walker \[21\] as an immediate consequence of the following theorem, that double Cohen-Macaulayness is a topological property as well. In the formulation of the theorem and in the sequel we write $\widetilde{H}_i(X;k)$ and $\widetilde{H}_i(X,A;k)$ for the reduced singular homology of the space $X$ and the pair of spaces $(X,A)$, respectively. We also write $\widetilde{H}_i(\Delta;k)$ and $\widetilde{H}_i(\Delta,\Gamma;k)$ for the reduced simplicial homology of the simplicial complex $\Delta$ and the pair of simplicial complexes $(\Delta,\Gamma)$.

**Theorem 1.1.** (Walker, \[21\, Theorem 9.8\]) Let $\Delta$ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex over a field $k$. The following conditions are equivalent:

(i) $\Delta$ is doubly Cohen-Macaulay over $k$.

(ii) $\dim_k \widetilde{H}_{d-2}(|\Delta| - p; k) = 0$ holds for every $p \in |\Delta|$.

Examples of complexes which are doubly Cohen-Macaulay over all fields are all triangulations of spheres. In contrast, no triangulation of a ball is doubly Cohen-Macaulay over any field.

Double Buchsbaumness of a simplicial complex is also a topological property \[10\] and thus doubly Buchsbaum complexes generalize homology manifolds (without boundary) in a way analogous to the way doubly Cohen-Macaulay complexes generalize homology spheres. However, in certain respects double Buchsbaumness turns out to be too weak of an analogue of double Cohen-Macaulayness. For instance, it is known \[18\, p. 71\] that every doubly Cohen-Macaulay complex $\Delta$ has non-vanishing top-dimensional homology, whereas this is not true for every doubly Buchsbaum complex (since it is not true for every homology manifold without boundary). These considerations and condition (ii) in Theorem 1.1 motivate the following definition.

**Definition 1.2.** Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum simplicial complex over a field $k$. The complex $\Delta$ is called Buchsbaum* over $k$ if

$$\dim_k \widetilde{H}_{d-2}(|\Delta| - p; k) = \dim_k \widetilde{H}_{d-2}(|\Delta|; k)$$

holds for every $p \in |\Delta|$.

The results of this paper show that the notion of a Buchsbaum* complex provides a well behaved manifold analogue to that of a doubly Cohen-Macaulay complex in terms of various enumerative, homological and graph theoretic properties. We summarize some of these results as follows.

The class of Buchsbaum* complexes is shown to be included in the class of doubly Buchsbaum complexes (Corollary 2.9) with non-vanishing top-dimensional homology (Corollary 2.4), to include all triangulations of orientable homology manifolds (Proposition 2.7) and to reduce to the class of doubly Cohen-Macaulay complexes, when restricted to the class of all Cohen-Macaulay complexes (Proposition 2.5). The $h'$-vector of a Buchsbaum complex is a natural analogue for the $h$-vector of a Cohen-Macaulay complex. A formula for the $h'$-vector of a Buchsbaum* complex, in terms of that of the deletion and the link of a vertex, is shown to hold (Proposition 3.1) and an application to the class of flag Buchsbaum*
complexes is given (Corollary 3.3). Partially extending results of Kalai [9] on homology manifolds and Nevo [12] on doubly Cohen-Macaulay complexes, the graph of a connected Buchsbaum* complex of dimension \( d - 1 \geq 2 \) is shown to be generically \( d \)-rigid (Theorem 4.1). This implies that the graph of such a complex is \( d \)-connected (Corollary 4.2), that the \( j \)-vector of a Buchsbaum* complex satisfies the inequalities of Barnette’s lower bound theorem (Proposition 5.3) and that the first three entries of the \( g \)-vector of a Buchsbaum* complex of dimension three or higher satisfy the conditions predicted by the \( g \)-conjecture (Proposition 3.6).

This paper is structured as follows. Section 2 gives several characterizations of Buchsbaum* complexes in terms of homomorphisms of homology groups and one in terms of local cohomology modules (Propositions 2.3 and 2.8), deduces their basic properties and lists some examples. Sections 3 and 4 discuss enumerative and graph theoretic properties. Section 5 constructs a large class of Buchsbaum* complexes by gluing orientable homology manifolds with boundary to an orientable homology manifold without boundary and investigates the behavior of the Buchsbaum* property under standard operations on topological spaces, such as joins and products. A notion of higher Buchsbaum* connectivity for simplicial complexes is also introduced in Section 5.4 where it is shown that passing to skeleta increases the degree of connectivity. Section 6 lists some questions arising from the results of this paper.

2. Characterizations and elementary properties

This section provides characterizations and discusses basic properties of Buchsbaum* complexes. Throughout this paper, if not specified otherwise, \( k \) is an arbitrary field. We recall the following characterization of Buchsbaum complexes.

**Theorem 2.1** (Schenzel [16]). For a \((d - 1)\)-dimensional simplicial complex \( \Delta \), the following conditions are equivalent:

(i) \( \Delta \) is Buchsbaum over \( k \).
(ii) \( \Delta \) is pure and \( \text{lk}_\Delta(\sigma) \) is Cohen-Macaulay over \( k \) for every \( \sigma \in \Delta \setminus \{\emptyset\} \).
(iii) \( \tilde{H}_i(|\Delta|, |\Delta| - p; k) = 0 \) holds for all \( i < d - 1 \) and \( p \in |\Delta| \).

**Remark 2.2.** Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex. The following statements are immediate consequences of Theorem 2.1 and Definition 1.2.

(i) \( \Delta \) is Buchsbaum over \( k \) if and only if every connected component of \( \Delta \) is Buchsbaum over \( k \) of dimension \( d - 1 \).
(ii) Assume that \( d \geq 2 \). Then \( \Delta \) is Buchsbaum* over \( k \) if and only if every connected component of \( \Delta \) is Buchsbaum* over \( k \) of dimension \( d - 1 \).

The following proposition provides equivalent versions of Definition 1.2.

**Proposition 2.3.** For a \((d - 1)\)-dimensional simplicial complex \( \Delta \) which is Buchsbaum over \( k \), the following conditions are equivalent:

(i) \( \Delta \) is Buchsbaum* over \( k \).
(ii) For every $p \in |\Delta|$, the inclusion map $\iota : |\Delta| - p \hookrightarrow |\Delta|$ induces an injection
\[ \iota_* : \tilde{H}_{d-2}(|\Delta| - p; k) \to \tilde{H}_{d-2}(|\Delta|; k). \]

(iii) For every $p \in |\Delta|$, the inclusion map $\iota : |\Delta| - p \hookrightarrow |\Delta|$ induces an isomorphism
\[ \iota_* : \tilde{H}_{d-2}(|\Delta| - p; k) \to \tilde{H}_{d-2}(|\Delta|; k). \]

(iv) For every $p \in |\Delta|$, the canonical map
\[ \rho_* : \tilde{H}_{d-1}(|\Delta|; k) \to \tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \]
is surjective.

**Proof.** Since $\Delta$ is Buchsbaum over $k$, we have $\tilde{H}_{d-2}(|\Delta|, |\Delta| - p; k) = 0$ by condition (iii) of Theorem 2.1. Hence, the long exact sequence of the pair $(|\Delta|, |\Delta| - p)$ gives the exact sequence
\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{H}_{d-1}(|\Delta| - p; k) & \longrightarrow & \tilde{H}_{d-1}(|\Delta|; k) & \longrightarrow & \tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \\
& & \downarrow & & \downarrow & & \\
& & \tilde{H}_{d-2}(|\Delta| - p; k) & \longrightarrow & \tilde{H}_{d-2}(|\Delta|; k) & \longrightarrow & 0.
\end{array}
\]

It follows that $\iota_*$ is surjective. This proves that (ii) $\Leftrightarrow$ (iii). Assuming that $\Delta$ is Buchsbaum* over $k$, surjectivity of $\iota_*$ and (1.1) imply that $\iota_*$ is an isomorphism. This proves that (i) $\Rightarrow$ (iii). The reverse implication is trivial. The same exact sequence proves the equivalence (iii) $\Leftrightarrow$ (iv). \qed

**Corollary 2.4.** If $\Delta$ is a $(d-1)$-dimensional Buchsbaum* simplicial complex over $k$, then
\[ \tilde{H}_{d-1}(\Delta; k) \neq 0. \]

**Proof.** Let us choose $p \in |\Delta|$ in the relative interior of a $(d-1)$-dimensional face of $\Delta$. Clearly, we have $\tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \cong k$. The desired statement follows by applying condition (iv) of Proposition 2.3 to such a point $p$. \qed

We note that part (i) of the next proposition fails if Buchsbaum* is replaced by doubly Buchsbaum (see, for instance, part (i) of Example 2.10).

**Proposition 2.5.** Let $\Delta$ be a simplicial complex.

(i) Assume that $\Delta$ is Cohen-Macaulay over $k$. Then $\Delta$ is Buchsbaum* over $k$ if and only if $\Delta$ is doubly Cohen-Macaulay over $k$.

(ii) Assume that $\Delta$ is Gorenstein over $k$. Then $\Delta$ is Buchsbaum* over $k$ if and only if $\Delta$ is Gorenstein* over $k$.

**Proof.** The assumption that $\Delta$ is Cohen-Macaulay over $k$ implies that $\Delta$ is Buchsbaum over $k$ and that $\tilde{H}_{d-2}(\Delta; k) = 0$, where $d - 1$ is the dimension of $\Delta$. Therefore, under this assumption, Definition 1.2 implies that $\Delta$ is Buchsbaum* over $k$ if and only if we have $\tilde{H}_{d-2}(\Delta - p; k) = 0$ for every $p \in |\Delta|$. Thus, part (i) follows from Theorem 1.1.
Assume that $\Delta$ is Gorenstein over $k$. This assumption also implies that $\Delta$ is Buchsbaum over $k$. A Gorenstein simplicial complex $\Gamma$ of dimension $d - 1$ is Gorenstein* if and only if $\tilde{H}_{d-1}(\Gamma; k) \neq 0$. Thus if $\Delta$ is Buchsbaum* over $k$, then $\Delta$ is Gorenstein* over $k$ by Corollary 2.4. Conversely, if $\Delta$ is Gorenstein* over $k$, then $\Delta$ is doubly Cohen-Macaulay over $k$ and hence it is Buchsbaum* over $k$ by part (i). This proves part (ii).

Example 2.6. A zero-dimensional simplicial complex is Buchsbaum* over $k$ if and only if it has at least two vertices. Suppose $\Delta$ is one-dimensional, so that $\Delta$ is a graph. Then by Remark 2.2 (ii), $\Delta$ is Buchsbaum* over $k$ if and only if so is each connected component of $\Delta$. Since a graph regarded as a one-dimensional simplicial complex is Cohen-Macaulay over $k$ if and only if it is connected, we conclude from Proposition 2.5 (i) that $\Delta$ is Buchsbaum* over $k$ if and only if each connected component of $\Delta$ is doubly connected as a graph.

By the term *homology manifold* (without further specification) in this paper, we will always mean one without boundary.

Proposition 2.7. Let $\Delta$ be a triangulation of a homology manifold $X$ over $k$. Then $\Delta$ is Buchsbaum* over $k$ if and only if $X$ is orientable over $k$.

Proof. In view of Remark 2.2 (ii), we may assume that $|\Delta|$ is connected. Let $d - 1$ be the dimension of $\Delta$ and let $p \in |\Delta|$. Our assumptions on $\Delta$ imply that $\Delta$ is Buchsbaum over $k$, that $\tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \cong k$ and $\tilde{H}_{d-2}(|\Delta|, |\Delta| - p; k) = 0$. Assuming that $|\Delta|$ is orientable over $k$, we further have $\tilde{H}_{d-1}(\Delta; k) \cong k$. Thus, the long exact sequence of the pair $(|\Delta|, |\Delta| - p)$ shows that

$$\iota_* : \tilde{H}_{d-2}(|\Delta| - p; k) \to \tilde{H}_{d-2}(|\Delta|; k)$$

is an isomorphism and hence $\Delta$ is Buchsbaum* over $k$ by Proposition 2.3. Conversely, assuming that $\Delta$ is Buchsbaum* over $k$, we have that $\iota_*$ is an isomorphism and the previous argument can be reversed to show that $\tilde{H}_{d-1}(\Delta; k) \cong k$. This implies that $|\Delta|$ is orientable over $k$.

The following proposition provides two characterizations of Buchsbaum* complexes. Recall that the *contrastar* of a face $\sigma$ of a simplicial complex $\Delta$ is defined as the subcomplex $\text{cost}_{\Delta}(\sigma) = \{ \tau \in \Delta : \sigma \not\subseteq \tau \}$ of $\Delta$.

Proposition 2.8. For a $(d - 1)$-dimensional simplicial complex $\Delta$ which is Buchsbaum over $k$, the following conditions are equivalent:

(i) $\Delta$ is Buchsbaum* over $k$.
(ii) For every pair of faces $\sigma \subseteq \tau$ of $\Delta$, the map

$$(2.1) \quad j_* : \tilde{H}_{d-1}(\Delta, \text{cost}_{\Delta}(\sigma); k) \to \tilde{H}_{d-1}(\Delta, \text{cost}_{\Delta}(\tau); k),$$

induced by inclusion, is surjective.
(iii) The socle of the local cohomology module of $k[\Delta]$ in homological dimension $d$ with respect to the irrelevant ideal satisfies $\left( \text{Soc} \; \tilde{H}^d(k[\Delta]) \right)_i = 0$ for all $i \neq 0$. 
Proof. Recall that for $p \in \partial \Delta$ there is a deformation retraction of $\partial \Delta - p$ onto $\partial \Delta(\tau)$, where $\tau$ is the unique face of $\Delta$ such that $p$ lies in the relative interior of $\partial \tau$. As a result, condition (iv) of Proposition 2.3 is equivalent to the condition that for each $\tau \in \Delta$, the canonical maps

$$\rho^\tau_\ast : \widetilde{H}_{d-1}(\Delta; k) \to \widetilde{H}_{d-1}(\Delta, \partial \Delta(\tau); k)$$

is surjective. The commutative diagram of canonical maps

for pairs $\sigma \subseteq \tau$ of faces of $\Delta$ shows that the latter condition is equivalent to (ii). We have shown that (i) $\iff$ (ii). As noted in [15, Section 2], the equivalence (ii) $\iff$ (iii) follows from [8, Theorem 2].

Corollary 2.9. Every Buchsbaum* complex over $k$ is doubly Buchsbaum over $k$.

Proof. This statement follows from the implication (i) $\Rightarrow$ (ii) of Proposition 2.8 and the fact (see [10, Theorem 4.3]) that a $(d - 1)$-dimensional simplicial complex $\Delta$ is doubly Buchsbaum over $k$ if and only if $\Delta$ is Buchsbaum over $k$ and the map (2.1) is surjective for every pair of nonempty faces $\sigma \subseteq \tau$ of $\Delta$.

Example 2.10. Some examples of doubly Buchsbaum complexes which are not Buchsbaum* are the following.

(i) The one-dimensional simplicial complex $\Delta$ on the vertex set $\{a, b, c, d, p\}$ with facets (edges) $\{p, a\}$, $\{p, b\}$, $\{a, b\}$, $\{p, c\}$, $\{p, d\}$, $\{c, d\}$ is doubly Buchsbaum but not Buchsbaum*, since $\widetilde{H}_0(|\Delta|; k) = 0$ and $\widetilde{H}_0(|\Delta| - p; k) \cong k$ (alternatively, since $\Delta$ is not doubly connected as a graph).

(ii) Condition (ii) of Theorem 2.1 and the fact that all homology spheres over $k$ are doubly Cohen-Macaulay over $k$ imply that all homology manifolds over $k$ are doubly Buchsbaum over $k$. This fact and Proposition 2.7 imply that every non-orientable homology manifold over $k$ is doubly Buchsbaum but not Buchsbaum* over $k$.

(iii) Let $\Gamma$ be a triangulation of the two-dimensional torus for which some three edges of $\Gamma$ of the form $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$ are the support of a 1-cycle which represents an element of a basis of $H_1(\Gamma; k)$. Let $\Delta$ be the simplicial complex obtained from $\Gamma$ by adding the two-dimensional face $\sigma = \{a, b, c\}$. It is easy to check that $\Delta$ is doubly Buchsbaum over all fields $k$. However, since $H_1(\Delta; k) \cong k$ and $H_1(|\Delta| - p; k) \cong k^2$
for every point \( p \) in the relative interior of \( |\sigma| \), the complex \( \Delta \) is not Buchsbaum* over \( k \).

**Corollary 2.11.** If \( \Delta \) is a Buchsbaum* simplicial complex over \( k \), then \( \text{lk}_\Delta(\sigma) \) is doubly Cohen-Macaulay over \( k \) for every nonempty face \( \sigma \) of \( \Delta \).

**Proof.** This statement follows from Corollary 2.9 and the fact (see, for instance, [10, Lemma 4.2]) that the link of any nonempty face in a doubly Buchsbaum complex is doubly Cohen-Macaulay. \( \square \)

### 3. Face enumeration

This section is concerned with enumerative properties of Buchsbaum* complexes. Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex. The \( f \)-vector of \( \Delta \) is the sequence \( f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)) \), where \( f_i(\Delta) \) stands for the number of faces of \( \Delta \) of dimension \( i \) (in particular, \( f_{-1}(\Delta) = 1 \) unless \( \Delta = \emptyset \)). The \( h \)-vector of \( \Delta \) is the sequence \( h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta)) \) defined by

\[
h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(\Delta).
\]

(3.1)

The \( h' \)-vector of \( \Delta \) is the sequence \( h'(\Delta) = (h'_0(\Delta), h'_1(\Delta), \ldots, h'_d(\Delta)) \) defined by

\[
h'_j(\Delta) = h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \tilde{\beta}_{i-1}(\Delta),
\]

(3.2)

where \( \tilde{\beta}_{i-1}(\Delta) = \dim_k \tilde{H}_{i-1}(\Delta; k) \). It was proved by Schenzel [16] that if \( \Delta \) is Buchsbaum over an infinite field \( k \), then the numbers \( h'_j(\Delta) \) are nonnegative integers which may depend on the characteristic of \( k \). Note that

\[
h'_d(\Delta) = \tilde{\beta}_{d-1}(\Delta)
\]

and that if \( \Delta \) is Cohen-Macaulay over \( k \), then \( h'(\Delta) = h(\Delta) \). To simplify the notation, in this section we write \( \Delta/v \) for the link \( \text{lk}_\Delta(v) \) of a vertex \( v \) of \( \Delta \).

**Proposition 3.1.** Let \( \Delta \) be a \((d-1)\)-dimensional Buchsbaum* simplicial complex over \( k \). For each vertex \( v \) of \( \Delta \) and \( 0 \leq j \leq d \) we have

\[
h'_j(\Delta) = h'_j(\Delta \setminus v) + h_{j-1}(\Delta/v),
\]

where \( h_{-1}(\Delta \setminus v) = 0 \) by convention.
Proof. Applied to $\Delta \setminus v$, equation (3.2) yields

$$(3.4) \quad h'_j(\Delta \setminus v) = h_j(\Delta \setminus v) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta \setminus v).$$

It is well known (see, for instance, [1, Lemma 4.1]) that

$$(3.5) \quad h_j(\Delta) = h_j(\Delta \setminus v) + h_{j-1}(\Delta/v)$$

holds for $0 \leq j \leq d$. Combining equations (3.4) and (3.5), we get

$$(3.6) \quad h'_j(\Delta \setminus v) + h_{j-1}(\Delta/v) = h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta \setminus v).$$

Since $\Delta$ is Buchsbaum* over $k$, we have $\beta_{i-1}(\Delta \setminus v) = \beta_{i-1}(\Delta)$ for $i \leq d - 1$. Hence the right hand side of (3.2) is equal to that of (3.6) and the result follows.

Recall that a simplicial complex $\Delta$ is called flag if every minimal non-face of $\Delta$ has at most two elements. As an application of Proposition 3.1, we will show (Corollary 3.3) that among all Buchsbaum* flag simplicial complexes of dimension $d - 1$, the simplicial join of $d$ copies of the zero-dimensional sphere has the minimum $h'$-vector. This result generalizes one of [1] on Cohen-Macaulay complexes to the setting of Buchsbaum complexes. It is worth noting that the question of formulating such a generalization provided the initial motivation for introducing the class of Buchsbaum* complexes.

We will use the following proposition, which extends [17, Theorem 2.1] (see also [18, Theorem 9.1]) in the setting of Buchsbaum complexes.

**Proposition 3.2.** Let $\Delta$ be a simplicial complex of dimension $d - 1$ and $\Gamma$ be a subcomplex of dimension $e - 1$. Assume that no set of $e + 1$ vertices of $\Gamma$ is a face of $\Delta$ (this condition holds automatically if $d = e$). If both $\Gamma$ and $\Delta$ are Buchsbaum over $k$, then $h'_i(\Gamma) \leq h'_i(\Delta)$ holds for all $0 \leq i \leq d$.

**Proof.** By equation (3.3), a proof of the proposition can be given by simply replacing the term $h$-vector with $h'$-vector in the proof of [17, Theorem 2.1].

**Corollary 3.3.** If $\Delta$ is a $(d - 1)$-dimensional flag simplicial complex which is Buchsbaum* over $k$, then the inequalities

$$(3.7) \quad h'_i(\Delta) \geq \binom{d}{i}$$

hold for $0 \leq i \leq d$.

**Proof.** In view of Propositions 3.1 and 3.2, this follows by replacing $h$-vectors by $h'$-vectors in the argument of [1, Section 4] and using the fact (Corollary 2.9) that $\Delta$ is doubly Buchsbaum over $k$.

Note that by [14, Theorem 3.4] we have $h'_i(\Delta) \geq \binom{d}{i} \beta_{i-1}(\Delta)$ for every $(d - 1)$-dimensional Buchsbaum complex $\Delta$. 

Remark 3.4. The $b''$-vector of $\Delta$ is the sequence $b''(\Delta) = (b''_0(\Delta), b''_1(\Delta), \ldots, b''_d(\Delta))$ defined by $b''_i(\Delta) = b'_i(\Delta) - \binom{d}{i} \hat{b}_{d-1}(\Delta)$ for $0 \leq i \leq d-1$ and $b''_d(\Delta) = \hat{b}_{d-1}(\Delta)$. We note that $b''_0(\Delta) = 1$ and refer to [13] for a history and known results on the $b''$-vector. After the first version of this paper was made public, it was shown by I. Novik (personal communication with the authors) that, under the assumptions of Corollary 3.3, the inequalities (3.7) can be strengthened to

\[ \text{(3.8)} \]

for $0 \leq i \leq d - 2$. The proof uses results from [14] and then follows the general outline of the proof of Corollary 3.3.

We conclude this section with two results on the face enumeration of Buchsbaum* complexes, the proofs of which will be given in Section 4. They both extend to Buchsbaum* complexes results of Nevo [12] on doubly Cohen-Macaulay complexes. The $g$-vector of $\Delta$ is the sequence $g(\Delta) = (g_0(\Delta), g_1(\Delta), \ldots, g_{d/2}(\Delta))$, defined by $g_0(\Delta) = h_0(\Delta) = 1$ and

\[ g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta) \]

for $i \geq 1$. Recall that a sequence $(a_0, \ldots, a_r)$ of nonnegative integers is called an $M$-vector if there is a standard graded $k$-algebra $A = A_0 \oplus \cdots \oplus A_r$ such that $\dim_k A_i = a_i$ for each $i$; see [3] Chapter 1.4 for details.

Proposition 3.5. Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum* simplicial complex over some field and let $n$ be the number of vertices of $\Delta$. If $d \geq 3$, then $f_i(\Delta) \geq f_i(n, d)$ holds for $0 \leq i \leq d - 1$, where

\[ f_i(n, d) = \begin{cases} \binom{d}{i} n - \binom{d+1}{i+1}, & \text{if } 0 \leq i \leq d - 2 \\ (d-1)n - (d+1)(d-2), & \text{if } i = d - 1 \end{cases} \]

is the number of $i$-dimensional faces of a stacked $(d-1)$-dimensional sphere with $n$ vertices.

Proposition 3.6. Let $\Delta$ be a connected $(d-1)$-dimensional simplicial complex which is Buchsbaum* over some field. If $d \geq 4$, then $(g_0(\Delta), g_1(\Delta), g_2(\Delta))$ is an $M$-vector.

Remark 3.7. Let $\Delta$ be as in Proposition 3.6. Since $\Delta$ is connected, we have $h_i(\Delta) = h'_i(\Delta)$ for $i \leq 2$. Hence $(g'_0(\Delta), g'_1(\Delta), g'_2(\Delta))$ is also an $M$-vector, where $g'_0(\Delta) = h'_0(\Delta) = 1$ and $g'_i(\Delta) = h'_i(\Delta) - h'_{i-1}(\Delta)$ for $i \geq 1$. For general Buchsbaum* simplicial complexes, Proposition 3.6 cannot be extended to longer initial segments for either of $g(\Delta)$ and $g'(\Delta)$. We are grateful to Ed Swartz for pointing out to us a counterexample for $g'(\Delta)$; there are simple counterexamples for $g(\Delta)$.

4. The graph of a Buchsbaum* complex

The graph of a simplicial complex $\Delta$ is defined as the abstract graph $G(\Delta)$ whose nodes are the vertices of $\Delta$ and whose edges are the one-dimensional simplices. This section focuses on the graph of a Buchsbaum* complex. More specifically, it is shown that a result of Nevo [12] on the rigidity of graphs of doubly Cohen-Macaulay complexes extends easily
to those of connected Buchsbaum* complexes. Since the proofs in this section follow from those of [12] by minor modifications, we will only indicate those points in the proofs where some modification is actually needed.

Let $G$ be an abstract graph (without loops or multiple edges) on the set of nodes $V$ and let $\|x\|$ denote the Euclidean length of $x \in \mathbb{R}^d$. A map $f : V \to \mathbb{R}^d$ is called $G$-rigid if there exists $\varepsilon > 0$ with the following property: if $g : V \to \mathbb{R}^d$ is a map satisfying $\|f(v) - g(v)\| < \varepsilon$ for every $v \in V$ and $\|g(u) - g(v)\| = \|f(u) - f(v)\|$ for every edge $\{u, v\}$ of $G$, then we have $\|g(u) - g(v)\| = \|f(u) - f(v)\|$ for all $u, v \in V$. The graph $G$ is called generically $d$-rigid if the set of all $G$-rigid maps $f : V \to \mathbb{R}^d$ is open and dense in the topological vector space of all maps $f : V \to \mathbb{R}^d$.

**Theorem 4.1.** Let $\Delta$ be a $(d-1)$-dimensional connected simplicial complex which is Buchsbaum* over some field $k$. If $d \geq 3$, then the graph $G(\Delta)$ is generically $d$-rigid.

**Proof.** As in the proof of [12, Theorem 1.3], it suffices to show that for $d \geq 2$ every such complex $\Delta$ admits a decomposition into minimal $(d-1)$-cycle complexes, as in [12, Theorem 3.4]. The case $d = 2$ is covered by [12, Theorem 3.4], since every one-dimensional connected Buchsbaum* complex is doubly Cohen-Macaulay (see Example 2.6). Thus we may assume that $d \geq 3$ and proceed by induction on $d$. Let $v$ be any vertex of $\Delta$ and note that the induction hypothesis applies to the complex $lk_{\Delta}(v)$, which is doubly Cohen-Macaulay over $k$ by Corollary 2.11. Note also that if $s$ is a minimal $(d-2)$-cycle for $lk_{\Delta}(v)$, then there exists a $(d-1)$-chain $c$ for $\Delta \setminus v$ such that $\partial_{d-1}(c) = s$. Indeed, this follows from condition (ii) of Proposition 2.3, since $s$ is trivial as an element of $H_{d-2}(\Delta; k)$. The remainder of the proof follows that of [12, Theorem 3.4] without change.

We now provide the missing proofs of the results stated at the end of Section 3.

**Proof of Propositions 3.5 and 3.6.** Proposition 3.5 follows from Theorem 4.1 and the discussion in [12, Section 1]. As already mentioned in Remark 3.7, we have $h_i(\Delta) = h'_i(\Delta)$ for $i \leq 2$ and hence (3.3) continues to hold for $i \leq 2$, if $h'_i(\Delta)$ is replaced by $h_i(\Delta)$. Thus Proposition 3.6 follows from Theorem 4.1 as in the discussion in [12, Section 2].

Given a positive integer $m$, an abstract graph $G$ is said to be $m$-connected if $G$ has at least $m+1$ nodes and any graph obtained from $G$ by deleting $m-1$ or fewer nodes and their incident edges is connected (necessarily with at least one edge). Part (i) of the following corollary is a special case of a result independently found by Björner [5]. Part (ii) is not valid if Buchsbaum* is replaced by doubly Buchsbaum; see Example 2.10 (i).

**Corollary 4.2.** Let $\Delta$ be a $(d-1)$-dimensional, connected simplicial complex.

(i) If $\Delta$ is Buchsbaum over some field, then the graph $G(\Delta)$ is $(d-1)$-connected.

(ii) If $\Delta$ is Buchsbaum* over some field, then the graph $G(\Delta)$ is $d$-connected.

**Proof.** Part (ii) follows from the discussion in Example 2.6 for $d \leq 2$ and from Theorem 4.1 for $d \geq 3$, since every generically $d$-rigid graph is $d$-connected. Part (i) can be proved by an elementary combinatorial argument. We omit this argument since both parts also
5. Constructions

5.1. A generalized convex ear decomposition. In the sequel we describe a class of Buchsbaum* complexes significantly larger than that provided by Proposition 2.7. The construction is motivated by and generalizes the convex ear decomposition of simplicial complexes, introduced by Chari [7].

**Theorem 5.1.** Suppose that $\Delta$ is a $(d-1)$-dimensional simplicial complex and that there exist subcomplexes $\Delta_1, \Delta_2, \ldots, \Delta_m$ such that:

(i) $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$.

(ii) $\Delta_1$ is a $(d-1)$-dimensional orientable homology manifold over $k$.

(iii) For $2 \leq i \leq m$, $\Delta_i$ is a $(d-1)$-dimensional connected orientable homology manifold over $k$ with boundary $\partial \Delta_i$ which has the following properties:

(a) $\partial \Delta_i$ is a $(d-2)$-dimensional connected orientable homology manifold over $k$.

(b) $\partial \Delta_i = \Delta_i \cap (\Delta_1 \cup \cdots \cup \Delta_{i-1})$.

(c) The inclusion maps induce the zero homomorphisms

$$
\tilde{H}_{d-2}(\partial \Delta_i; k) \to \tilde{H}_{d-2}(\Delta_1 \cup \cdots \cup \Delta_{i-1}; k)
$$

and

$$
\tilde{H}_{d-3}(\partial \Delta_i; k) \to \tilde{H}_{d-3}(\Delta_1 \cup \cdots \cup \Delta_{i-1}; k).
$$

Then $\Delta$ is Buchsbaum* over $k$.

The following lemma takes care of the crucial special case $m = 2$.

**Lemma 5.2.** Let $\Gamma$ and $\Delta$ be two simplicial complexes such that:

(i) $\Gamma \cup \Delta$ is a simplicial complex.

(ii) $\Gamma$ is a $(d-1)$-dimensional Buchsbaum* complex over $k$.

(iii) $\Delta$ is a $(d-1)$-dimensional connected orientable homology manifold over $k$ with boundary $\partial \Delta$ which has the following properties:

(a) $\partial \Delta$ is a $(d-2)$-dimensional connected orientable homology manifold over $k$.

(b) $\partial \Delta = \Gamma \cap \Delta$.

(c) The inclusion maps induce the zero homomorphisms

$$
\tilde{H}_{d-2}(\partial \Delta; k) \to \tilde{H}_{d-2}(\Gamma; k)
$$

and

$$
\tilde{H}_{d-3}(\partial \Delta; k) \to \tilde{H}_{d-3}(\Gamma; k).
$$

Then $\Gamma \cup \Delta$ is Buchsbaum* over $k$.

**Proof.** Since $\Gamma$ and $\Delta$ are Buchsbaum over $k$ of dimension $d - 1$ and $\Gamma \cap \Delta$ is Buchsbaum over $k$ of dimension $d - 2$, it follows by a standard argument (used, for instance, in the proof of [6, Lemma 1]) that $\Gamma \cup \Delta$ is also Buchsbaum over $k$. To show that (1.1) holds for the complex $\Gamma \cup \Delta$, we consider a point $p \in |\Gamma \cup \Delta|$ and distinguish three cases.
Case 1: $p \in |\Gamma| \setminus |\Delta|$. Since $\Gamma$ and $\Gamma \cup \Delta$ are Buchsbaum over $k$, the naturality of the long exact homology sequence for pairs gives the commutative diagram

$$
\begin{array}{cc}
\tilde{H}_{d-1}(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p; k) & \to \tilde{H}_{d-2}(|\Gamma \cup \Delta| - p; k) \to \tilde{H}_{d-2}(|\Gamma \cup \Delta|; k) \to 0 \\
\downarrow & \downarrow & \downarrow \\
\tilde{H}_{d-1}(|\Gamma|, |\Gamma| - p; k) & \to \tilde{H}_{d-2}(|\Gamma| - p; k) \to \tilde{H}_{d-2}(|\Gamma|; k) \to 0.
\end{array}
$$

Since $\Gamma$ is Buchsbaum over $k$, the map $\delta_*$ is an isomorphism, so $\partial_*$ has to be the zero map. The map $\epsilon_*$ on the left is an excision map and hence an isomorphism. The commutativity of the square on the left implies that $\tilde{\partial}_*$ has to be the zero map. Hence $\tilde{\epsilon}_*$ is an isomorphism as well.

Case 2: $p \in |\Delta| \setminus |\Gamma|$. An application of the long exact homology sequence for the triple $(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p, \partial|\Delta|)$ yields the exact sequence

$$
- \to \tilde{H}_{d-1}(|\Gamma \cup \Delta|, \partial|\Delta|; k) \to \tilde{H}_{d-1}(|\Gamma \cup \Delta| - p; k)
$$

$$
\begin{array}{cc}
\tilde{H}_{d-2}(|\Gamma \cup \Delta| - p, \partial|\Delta|; k) & \to \tilde{H}_{d-2}(|\Gamma \cup \Delta|, \partial|\Delta|; k) \to \tilde{H}_{d-2}(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p; k) = 0,
\end{array}
$$

where $\tilde{H}_{d-2}(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p; k)$ vanishes since $\Gamma \cup \Delta$ is Buchsbaum over $k$.

We claim that the map $\delta_*$ is surjective, so that $\tilde{\epsilon}_*$ is an isomorphism and hence

$$
\dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta| - p, \partial|\Delta|; k) = \dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta|, \partial|\Delta|; k).
$$

To prove the claim, consider the following commutative diagram:

$$
\begin{array}{cc}
\tilde{H}_{d-1}(|\Delta| - p, \partial|\Delta|; k) & \to \tilde{H}_{d-1}(|\Delta|, \partial|\Delta|; k) \to \tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \\
\downarrow & \downarrow & \downarrow \\
\tilde{H}_{d-1}(|\Gamma \cup \Delta|, \partial|\Delta|; k) & \to \tilde{H}_{d-1}(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p; k).
\end{array}
$$

The top part comes from the long exact homology sequence for the triple $(|\Delta|, |\Delta| - p, \partial|\Delta|)$. Since $\Delta$ triangulates a connected orientable homology manifold with boundary $\partial\Delta$ and $p \in |\Delta|$ is an interior point, we have $\tilde{H}_{d-1}(|\Delta| - p, \partial|\Delta|; k) = 0$ and $\tilde{H}_{d-1}(|\Delta|, \partial|\Delta|; k) \cong \tilde{H}_{d-1}(|\Delta|, |\Delta| - p; k) \cong k$. Hence $\delta_*$ is an isomorphism. Since $\epsilon_*$ is an isomorphism by
excision, the commutativity of the square implies that the map \( \tilde{\delta} \) is surjective, as claimed. Our assumption (c) and the long exact homology sequence for the pair \((|\Gamma \cup \Delta|, \partial|\Delta|)\) yield an exact sequence of the form

\[
0 \to \tilde{H}_{d-2}(|\Gamma \cup \Delta|; k) \to \tilde{H}_{d-2}(|\Gamma \cup \Delta|, \partial|\Delta|; k) \to \tilde{H}_{d-3}(\partial|\Delta|; k) \to 0.
\]

It follows that

\[
\dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta|, \partial|\Delta|; k) = \dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta|; k) + \dim_k \tilde{H}_{d-3}(\partial|\Delta|; k).
\]

The same argument proves that

\[
\dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta| - p, \partial|\Delta|; k) = \dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta| - p; k) + \dim_k \tilde{H}_{d-3}(\partial|\Delta|; k).
\]

Equations (5.1), (5.2) and (5.3) imply that \( \dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta|; k) = \dim_k \tilde{H}_{d-2}(|\Gamma \cup \Delta| - p; k) \), which was the desired equality.

**Case 3**: \( p \in |\Gamma \cap \Delta| \). From the Mayer-Vietoris sequence for the pairs \((|\Delta| - p, |\Gamma| - p)\) and \((|\Delta|, |\Gamma|)\) and the naturality of such sequences, we get a commutative diagram as follows:

\[
\begin{array}{ccc}
\tilde{H}_{d-2}(|\Gamma \cap \Delta|; k) & \xrightarrow{\tilde{\alpha}_*} & \tilde{H}_{d-2}(|\Gamma|; k) \\
& & \oplus \\
& & \tilde{H}_{d-2}(|\Delta|; k)
\end{array} \begin{array}{ccc}
\tilde{H}_{d-2}(|\Gamma \cup \Delta|; k) & \xrightarrow{\tilde{\beta}_*} & \tilde{H}_{d-2}(\partial|\Delta|; k) \\
& & \oplus \\
& & \tilde{H}_{d-3}(|\Gamma \cap \Delta|; k)
\end{array}
\]

By condition (a) of our assumption (iii) on \( \partial|\Delta| = |\Gamma \cap \Delta| \) and Propositions 2.3 (iii) and 2.7, the map \( j_* \) is an isomorphism.

We claim that the map \( \tilde{\alpha}_* \), induced by the inclusions of \( \partial|\Delta| = |\Gamma \cap \Delta| \) into \( |\Gamma| \) and \( |\Delta| \), is the zero map. Indeed, for the map \( \tilde{H}_{d-2}(\partial|\Delta|; k) \to \tilde{H}_{d-2}(|\Gamma|; k) \) induced by inclusion this holds by condition (c). For the other map, consider the exact sequence

\[
0 \to \tilde{H}_{d-2}(|\Delta|; k) \to \tilde{H}_{d-2}(|\Delta|, \partial|\Delta|; k) \to \tilde{H}_{d-2}(\partial|\Delta|; k) \to 0.
\]

obtained from the long exact homology sequence of the pair \((|\Delta|, \partial|\Delta|)\). Since \( \Delta \) triangulates a connected orientable homology manifold with nonempty boundary, we have \( \tilde{H}_{d-2}(|\Delta|; k) = 0 \). Since \( \partial\Delta \) triangulates another connected homology manifold without boundary of dimension \( d-2 \), we have \( \tilde{H}_{d-2}(|\Delta|, \partial|\Delta|; k) \cong \tilde{H}_{d-2}(\partial|\Delta|; k) \cong k \). Hence the map \( \partial_* \) must be an isomorphism and therefore \( \gamma_* : H_{d-2}(\partial|\Delta|; k) \to \tilde{H}_{d-2}(|\Delta|; k) \) has to be the zero map. Finally, we note that the vertical maps \( \tilde{H}_{d-2}(|\Gamma| - p; k) \to \tilde{H}_{d-2}(|\Gamma|; k) \),
\[\tilde{H}_{d-2}(|\Delta| - p; k) \rightarrow \tilde{H}_{d-2}(|\Delta|; k)\] and \[\tilde{H}_{d-3}(\partial|\Delta| - p; k) \rightarrow \tilde{H}_{d-3}(\partial|\Delta|; k)\], induced by inclusions, are isomorphisms. This follows by our assumption (ii) in the first case, by condition (a) of our assumption (iii) and Proposition 2.7 in the third case and by the long exact homology sequence of the pair \((|\Delta|, |\Delta| - p)\) in the second case, if one observes that \(p \in \partial|\Delta|\) implies \(H_i(|\Delta|, |\Delta| - p; k) = 0\) for all \(i\). Thus \(\iota_*\) is an isomorphism.

Given the above, it follows by simple diagram chasing that the map \(\tilde{H}_{d-2}(|\Gamma \cup \Delta| - p; k) \rightarrow \tilde{H}_{d-2}(|\Gamma \cup \Delta|; k)\) is injective and hence an isomorphism. Thus (1.1) holds in this case as well. \(\square\)

**Proof of Theorem 5.3.** The complex \(\Delta_1\) is Buchsbaum* over \(k\) by Proposition 2.7. The theorem follows from this statement and Lemma 5.2 by induction on \(m\). \(\square\)

### 5.2. Products

Next we show that the Buchsbaum* property is preserved under direct products of simplicial complexes.

**Theorem 5.3.** Let \(\Gamma\) be a \((d - 1)\)-dimensional simplicial complex and \(\Delta\) be an \((e - 1)\)-dimensional simplicial complex.

(i) If \(\Gamma\) and \(\Delta\) are Buchsbaum over \(k\), then every simplicial complex triangulating \(|\Gamma| \times |\Delta|\) is Buchsbaum over \(k\).

(ii) If \(\Gamma\) and \(\Delta\) are Buchsbaum* over \(k\), then every simplicial complex triangulating \(|\Gamma| \times |\Delta|\) is Buchsbaum* over \(k\).

**Proof.** Let \(p \in |\Gamma| \times |\Delta|\). There are unique faces \(\sigma \in \Gamma\) and \(\tau \in \Delta\) such that \(p\) lies in the relative interior of \(|\sigma| \times |\tau|\). Then \(|\text{cost}_\Gamma(\sigma)| \times |\Delta| \cup |\Gamma| \times |\text{cost}_\Delta(\tau)|\) is a deformation retract of \(|\Gamma| \times |\Delta| - p\) and hence

\[(5.4) \quad (|\Gamma|, |\text{cost}_\Gamma(\sigma)|) \times (|\Delta|, |\text{cost}_\Delta(\tau)|) \cong (|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p)\]

is a deformation retraction.

(i) By (5.4) and the Künneth formula we have:

\[\tilde{H}_i(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k) \cong \bigoplus_{j=0}^{i} \tilde{H}_j(\Gamma, \text{cost}_\Gamma(\sigma); k) \otimes \tilde{H}_{i-j}(\Delta, \text{cost}_\Delta(\tau); k).\]

For \(i < d + e - 2\) we have either \(j < d - 1\) or \(i - j < e - 1\). Thus by Buchsbaumness of \(\Gamma\) and \(\Delta\), either \(\tilde{H}_j(\Gamma, \text{cost}_\Gamma(\sigma); k) = 0\) or \(\tilde{H}_{i-j}(\Delta, \text{cost}_\Delta(\tau); k) = 0\). Thus

\[\tilde{H}_i(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k) = 0\]

for \(i < d + e - 2\). Hence every simplicial complex triangulating \(|\Gamma| \times |\Delta|\) is Buchsbaum over \(k\).

(ii) By (5.4), the map

\[\tilde{H}_{d+e-2}(|\Gamma| \times |\Delta|; k) \xrightarrow{\rho_*} \tilde{H}_{d+e-2}(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k)\]
from the exact sequence of the pair \( (|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p) \) equals the map
\[
\tilde{H}_{d+e-2}(|\Gamma| \times |\Delta|; k) \xrightarrow{\rho^*} \tilde{H}_{d+e-2}(|\Gamma|, |\text{cost}_\Gamma(\sigma)| \times |\Delta|, |\text{cost}_\Delta(\tau)|; k).
\]

In turn, by the Künneth formula, this map can be written as
\[
\tilde{H}_{d-1}(\Gamma; k) \otimes \tilde{H}_{e-1}(\Delta; k) \xrightarrow{\rho^*} \tilde{H}_{d-1}(\Gamma, \text{cost}_\Gamma(\sigma); k) \otimes \tilde{H}_{e-1}(\Delta, \text{cost}_\Delta(\tau); k).
\]

From the fact that \( \Gamma \) is Buchsbaum* over \( k \) and Proposition 2.3, we deduce that the projection from \( \tilde{H}_{d-1}(\Gamma; k) \) to \( \tilde{H}_{d-1}(\Gamma, \text{cost}_\Gamma(\sigma); k) \) is surjective. Analogously, the projection from \( \tilde{H}_{e-1}(\Delta; k) \) to \( \tilde{H}_{e-1}(\Delta, \text{cost}_\Delta(\tau); k) \) is surjective. Hence \( \rho^* \) is surjective as well. This fact and Proposition 2.3 imply that every triangulation of \(|\Gamma| \times |\Delta|\) is Buchsbaum* over \( k \).

5.3. Joins. We recall that the join \( \Gamma \ast \Delta \) of two simplicial complexes \( \Gamma \) and \( \Delta \) on disjoint ground sets has as its faces the sets of the form \( \sigma \cup \tau \), where \( \sigma \in \Gamma \) and \( \tau \in \Delta \). The Buchsbaum and Buchsbaum* properties are in general not inherited by simplicial joins of Buchsbaum or Buchsbaum* complexes, respectively. For example, let \( \Gamma \) be nonempty and Buchsbaum (respectively, Buchsbaum*) and \( \Delta \) be Buchsbaum (respectively, Buchsbaum*) but not Cohen-Macaulay. Then the link in \( \Gamma \ast \Delta \) of any maximal simplex of \( \Gamma \) is equal to \( \Delta \) and hence it is not Cohen-Macaulay. Thus \( \Gamma \ast \Delta \) is not Buchsbaum. Indeed we can classify the situations in which \( \Gamma \ast \Delta \) is Buchsbaum*.

**Proposition 5.4.** Let \( \Gamma \) and \( \Delta \) be simplicial complexes, each having at least one vertex. The following are equivalent:

(i) \( \Gamma \ast \Delta \) is Buchsbaum* over \( k \).
(ii) \( \Gamma \ast \Delta \) is doubly Cohen-Macaulay over \( k \).
(iii) \( \Gamma \) and \( \Delta \) are doubly Cohen-Macaulay over \( k \).

**Proof.** (i) \( \Rightarrow \) (iii): Since \( \Delta \) contains at least one vertex, there exists a nonempty maximal simplex \( \sigma \in \Delta \). Since \( \Gamma \ast \Delta \) is Buchsbaum* over \( k \) and the link of \( \sigma \) in \( \Gamma \ast \Delta \) is equal to \( \Gamma \), it follows from Corollary 2.11 that \( \Gamma \) is doubly Cohen-Macaulay over \( k \). It follows in a similar way that \( \Delta \) is doubly Cohen-Macaulay over \( k \).

(iii) \( \Rightarrow \) (ii): It is well known that the join of two Cohen-Macaulay simplicial complexes over \( k \) is Cohen-Macaulay over \( k \). The implication follows from this statement, the definition of double Cohen-Macaulayness and the fact that for every vertex \( v \), say of \( \Gamma \), the complex \( (\Gamma \ast \Delta) \setminus v \) is equal to the simplicial join \( (\Gamma \setminus v) \ast \Delta \).

(ii) \( \Rightarrow \) (i): This follows from Proposition 2.5 (i). \( \square \)

5.4. Higher Buchsbaum* connectivity and skeleta. Given a subset \( \tau \) of the set of vertices of \( \Delta \), we denote by \( \Delta \setminus \tau \) the subcomplex \( \{ \sigma \in \Delta : \sigma \cap \tau = \emptyset \} \) of \( \Delta \), consisting of all faces of \( \Delta \) which do not contain any element of \( \tau \). We define a notion of higher Buchsbaum* connectivity for simplicial complexes as follows.
Definition 5.5. Let $\Delta$ be a simplicial complex and let $m$ be a nonnegative integer. We call $\Delta$ $m$-Buchsbaum* over $k$ if $m = 0$ and $\Delta$ is Buchsbaum over $k$ or $m \geq 1$ and $\Delta \setminus \tau$ is Buchsbaum* over $k$ of the same dimension as $\Delta$ for every set $\tau$ of vertices of $\Delta$ of cardinality less than $m$.

Thus the class of 0-Buchsbaum* complexes coincides with that of Buchsbaum complexes and the class of 1-Buchsbaum* complexes coincides with that of Buchsbaum* complexes. Our notion of higher connectivity for Buchsbaum* complexes is analogous to that already existing for Buchsbaum and Cohen-Macaulay complexes: Given a positive integer $m$, a simplicial complex $\Delta$ is called $m$-Buchsbaum* over $k$ if $\Delta \setminus \tau$ is Buchsbaum over $k$ (respectively, Cohen-Macaulay over $k$) of the same dimension as $\Delta$ for every set $\tau$ of vertices of $\Delta$ of cardinality less than $m$.

The following two statements generalize Proposition 2.5 (i) and Corollary 2.9, respectively.

Proposition 5.6. For a Cohen-Macaulay simplicial complex $\Delta$ over $k$ and a nonnegative integer $m$, the following conditions are equivalent:

(i) $\Delta$ is $m$-Buchsbaum* over $k$.
(ii) $\Delta$ is $(m + 1)$-Cohen-Macaulay over $k$.

Proof. (i) $\Rightarrow$ (ii): The implication is trivial for $m = 0$ and follows from Proposition 2.5 for $m = 1$. We assume that $m \geq 2$ and proceed by induction on $m$. Suppose that $\Delta$ is $m$-Buchsbaum* over $k$. To verify (ii), it suffices to show that $\Delta \setminus v$ is $m$-Cohen-Macaulay over $k$ of the same dimension as $\Delta$ for every vertex $v$ of $\Delta$. Indeed, $\Delta$ is doubly Cohen-Macaulay over $k$ by the special case $m = 1$ already treated and hence $\Delta \setminus v$ is Cohen-Macaulay over $k$ of the same dimension as $\Delta$. Since $\Delta \setminus v$ is $(m - 1)$-Buchsbaum* over $k$ by Definition 5.5, the desired statement follows from the induction hypothesis.

(ii) $\Rightarrow$ (i): This follows from part (i) of Proposition 2.5 and the relevant definitions.

Proposition 5.7. Let $m$ be a nonnegative integer and $\Delta$ be simplicial complex. If $\Delta$ is $m$-Buchsbaum* over $k$, then $\Delta$ is $(m + 1)$-Buchsbaum over $k$.

Proof. Let $d - 1$ be the dimension of $\Delta$. The statement is a tautology for $m = 0$. Assume that $m \geq 1$ and let $\tau$ be a set of vertices of $\Delta$ of cardinality at most $m$. We need to show that $\Delta \setminus \tau$ is Buchsbaum over $k$ of dimension $d - 1$. This is clear if $\tau = \emptyset$. Otherwise, let $v$ be an element of $\tau$ and let $\sigma = \tau \setminus \{v\}$ and $\Gamma = \Delta \setminus \sigma$. The complex $\Gamma$ is Buchsbaum* over $k$ by Definition 5.5 and hence it is doubly Buchsbaum over $k$ by Corollary 2.9. This implies that $\Gamma \setminus v$ is Buchsbaum over $k$ of dimension $d - 1$. Since $\Gamma \setminus v = \Delta \setminus \tau$, the latter complex is Buchsbaum over $k$ of dimension $d - 1$. This completes the proof.

Next we show that Buchsbaum* connectivity increases when passing to skeleta. Recall that the $i$-skeleton of a simplicial complex $\Delta$ is defined as the simplicial complex $\Delta^{(i)}$ of all faces of $\Delta$ of dimension $\leq i$. It is known [10] Corollary 7.6] that if $\Delta$ is $(d - 1)$-dimensional Buchsbaum over $k$, then the $i$-skeleton of $\Delta$ is doubly Buchsbaum over $k$ for every $i \leq d - 2$. In view of Corollary 2.9, the following is a stronger statement.
Proposition 5.8. Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex which is Buchsbaum* over \( k \). Then the \( i \)-skeleton \( \Delta^{(i)} \) of \( \Delta \) is Buchsbaum* over \( k \) for every \( 0 \leq i \leq d - 2 \).

Proof. It suffices to prove the assertion for \( i = d - 2 \). We set \( \Gamma = \Delta^{(d - 2)} \). For \( d = 2 \), the 0-skeleton of any \((d - 1)\)-dimensional simplicial complex consists of at least two points and therefore it is Buchsbaum* over all fields. Assume that \( d \geq 3 \). It is known (and follows, for instance, from condition (iii) of Theorem 2.1) that \( \Gamma \) is Buchsbaum over \( k \). Thus we only need to check that \( \tilde{H}_{d-3}(\Gamma) \cong \tilde{H}_{d-3}(\Delta) \) for every \( p \in |\Gamma| \). By condition (iii) of Theorem 2.1 and the long exact homology sequence for the pair \((|\Delta|, |\Delta| - p)\), we know that \( \tilde{H}_{d-3}(|\Delta| - p; k) \cong \tilde{H}_{d-3}(|\Delta|; k) \) holds for every \( p \in |\Delta| \). Since \( p \in |\Gamma| \subseteq |\Delta| \), it follows from the fact that the chains groups of \( \Gamma \) and \( \Delta \) in simplicial homology coincide in dimensions \( \leq d - 2 \) that \( \tilde{H}_{d-3}(|\Gamma| - p; k) \cong \tilde{H}_{d-3}(|\Delta| - p; k) \) and \( \tilde{H}_{d-3}(|\Gamma|; k) \cong \tilde{H}_{d-3}(|\Delta|; k) \). This completes the proof. \( \square \)

Theorem 5.9. Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex which is \( m \)-Buchsbaum* over \( k \). Then the \( i \)-skeleton \( \Delta^{(i)} \) is \((m + d - i - 1)\)-Buchsbaum* over \( k \) for every \( 0 \leq i \leq d - 1 \).

Proof. The statement is a tautology for \( i = d - 1 \). By induction on \( d - i - 1 \), it suffices to show that \( \Delta^{(d - 2)} \) is \((m + 1)\)-Buchsbaum* over \( k \). Let \( \tau \) be any set of vertices of \( \Delta \) of cardinality at most \( m \) and set \( \Gamma = \Delta \setminus \tau \). Since \( \Delta \) is \((m + 1)\)-Buchsbaum over \( k \) by Proposition 5.7, the complex \( \Gamma \) is Buchsbaum over \( k \) of dimension \( d - 1 \). It follows from Proposition 5.8 that \( \Gamma^{(d - 2)} \) is Buchsbaum* over \( k \). Since this skeleton is equal to \( \Delta^{(d - 2)} \setminus \tau \), we conclude that the latter complex is Buchsbaum* over \( k \) of dimension \( d - 2 \). Since \( \tau \) was arbitrary of cardinality at most \( m \), the desired statement follows. \( \square \)

We continue with the proof of Corollary 4.2 promised in Section 4.

Proof of Corollary 4.2. In view of the discussion in Example 2.6, part (i) (respectively, part (ii)) of the corollary follows from the special case \( m = 0 \) (respectively, \( m = 1 \)) and \( i = 1 \) of Theorem 5.9. \( \square \)

Remark 5.10. It is known by the results of Miyazaki [10] and Walker [21] that double Buchsbaumness and double Cohen-Macaulayness (over a fixed field) are topological properties. It is also known that for \( m \geq 3 \), neither \( m \)-Buchsbaumness nor \( m \)-Cohen-Macaulayness is a topological property. In view of Proposition 5.6, it follows that for \( m \geq 2 \), the \( m \)-Buchsbaum* condition is not a topological property either (it is obvious from Definition 1.2 that the Buchsbaum* condition is a topological property).

Remark 5.11. A different notion of a higher Buchsbaum* property one may try is the following. Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex and let \( m \) be a nonnegative integer. We consider the following condition:

(M) \( \Delta \) is Buchsbaum over \( k \) and \( \dim_k \tilde{H}_{d-2}(|\Delta| - P; k) = \dim_k \tilde{H}_{d-2}(\Delta; k) \) holds for every set \( P \subseteq |\Delta| \) of cardinality at most \( m \).

This condition reduces to Buchsbaumness and to the Buchsbaum* condition for \( m = 0 \) and \( m = 1 \), respectively. We claim, however, that no simplicial complex satisfies (M) for
m ≥ 2. Clearly, it suffices to show that no simplicial complex satisfies (M) for m = 2. Suppose on the contrary that ∆ is such a complex. We choose two points p, q ∈ |∆| which lie in the relative interior of some (d − 1)-dimensional simplex, say τ, of ∆. We triangulate τ by adding a vertex z and faces {z} ∪ σ for σ ⊂ τ. We realize the new complex in such a way that p and q lie in the relative interior of the realization of two distinct (d − 1)-dimensional simplices τ_p = {z} ∪ σ_p and τ_q = {z} ∪ σ_q. We denote by ∆′ the simplicial complex whose simplices are those of ∆ other than τ and the faces triangulating τ in the way just described. In particular, |∆′ ∖ {τ_p, τ_q}| is a deformation retract of |∆| − {p, q}.

The assumption that ∆ is Buchsbaum* over k and excision give

\[ 0 = \tilde{H}_{d-2}(|\Delta|, |\Delta| - q; k) \cong \tilde{H}_{d-2}(|\Delta| - p, |\Delta| - {p, q}; k). \]

The long exact sequence of the triple (|∆|, |∆| − p, |∆| − {p, q}) then yields that \( \tilde{H}_{d-2}(|\Delta|, |\Delta| - q; k) = 0 \). Thus, using the same arguments as in the proof of Proposition 2.3 it follows from (M) that the inclusion map \( \Delta' \setminus \{\tau_p, \tau_q\} \to \Delta' \) induces an isomorphism

\[ \tilde{H}_{d-2}(\Delta' \setminus \{\tau_p, \tau_q\}; k) \to \tilde{H}_{d-2}(\Delta'; k). \]

Clearly, \( \partial_{d-1}(\tau_p) \) is a boundary in \( \Delta' \). Since the map (5.5) is an isomorphism, \( \partial_{d-1}(\tau_p) \) must be a boundary in \( \Delta' \setminus \{\tau_p, \tau_q\} \) as well. However, this is not possible since \( \tau_p \cap \tau_q \) is a (d − 2)-dimensional simplex which lies in the support of \( \partial_{d-1}(\tau_p) \) and which is not contained in any (d − 1)-dimensional simplex of \( \Delta' \) other than \( \tau_p \) and \( \tau_q \). This yields the desired contradiction.

**Remark 5.12.** One of the standard constructions on simplicial complexes is rank selection on balanced complexes; see [18] Section III.4 for an exposition of these concepts. Responding to a question raised in an earlier version of this paper, J. Browder and S. Klee (personal communication with the authors) have shown that if a balanced simplicial complex ∆ is Buchsbaum* over k, then so is every rank selected subcomplex of ∆.

### 6. Questions

The classical \( g \)-theorem by Stanley and Billera & Lee gives a complete characterization of \( f \)-vectors of boundary complexes of simplicial polytopes. It is conjectured that the same set of inequalities appearing in the \( g \)-theorem also characterizes the \( f \)-vectors of triangulations of spheres. More generally, Björner & Swartz [20] Section 4] conjecture that the inequalities for the \( g \)-vector from the \( g \)-theorem hold for every doubly Cohen-Macaulay simplicial complex. Work of Kalai and Novik (see [13]) and more recently of Novik & Swartz [14] makes the \( h'' \)-vector (see Remark 3.4) appear as the most appropriate candidate for a \( g \)-theorem type result. Indeed, the following conjecture (due to Kalai) appears in [13].

**Conjecture 6.1.** (Kalai) Let ∆ be a (d − 1)-dimensional orientable homology manifold over k. Then there is a linear system of parameters \( \Theta \subseteq k[\Delta] \) and a linear form \( \omega \in k[\Delta] \) such that for \( I = \bigoplus_{i=1}^{d-1} \text{Soc}(k[\Delta]/\Theta)_i \) and \( k[\Delta] = (k[\Delta]/\Theta)/I \), we have:

- (i) \( \dim_k k[\Delta]_i = h''_i(\Delta) \) for \( 1 \leq i \leq \lfloor \frac{d}{2} \rfloor \).
(ii) The multiplication map $\omega^{d-2i} : k[\Delta]_i \to k[\Delta]_{d-i}$ is an isomorphism for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Part (i) of Conjecture 6.1 follows from the results of [15]. The equality $h''_i(\Delta) = h''_{d-i}(\Delta)$ for $1 \leq i \leq d-1$ is a consequence of this conjecture, was observed to hold already in [13]. Clearly, Conjecture 6.1 implies the $g$-conjecture for triangulations of spheres.

The equality $h''_i(\Delta) = h''_{d-i}(\Delta)$ for $1 \leq i \leq d-1$ does not hold for a general Buchsbaum* simplicial complex. Simple examples can be found among doubly Cohen-Macaulay simplicial complexes. For such complexes we have $h''_i(\Delta) = h_i(\Delta)$ for $0 \leq i \leq d$. The following conjecture has been proposed by Björner & Swartz (see [20, Problem 4.2]).

**Conjecture 6.2.** Let $\Delta$ be a $(d-1)$-dimensional simplicial complex which is doubly Cohen-Macaulay over $k$. Then

(i) $h_i(\Delta) \leq h_{d-i}(\Delta)$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

(ii) $g(\Delta)$ is an $M$-sequence.

Combining the numerical consequences of Conjecture 6.1 (ii) with Conjecture 6.2, one can ask the following question.

**Question 6.3.** Are the following true for every $(d-1)$-dimensional simplicial complex $\Delta$ which is Buchsbaum* over $k$?

(i) $h''_i(\Delta) \leq h''_{d-i}(\Delta)$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

(ii) $g''(\Delta) = (g''_0(\Delta), \ldots, g''_{\lfloor \frac{d}{2} \rfloor}(\Delta))$ is an $M$-sequence, where $g''_0(\Delta) = 1$ and $g''_i(\Delta) = h''_i(\Delta) - h''_{i-1}(\Delta)$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

As mentioned in Remark 3.7, an example due to Ed Swartz shows that the analogous question for the $h'$-vector has a negative answer.

A conjecture of Kalai states that the $h$-vector of every flag Cohen-Macaulay simplicial complex is equal to the $f$-vector of a simplicial complex; see [18, p. 100]. Theorem 1.3 in [1] is implied by this statement. Affirmative answers to the following two questions, motivated by Corollary 3.3, would similarly give successively stronger statements than Corollary 3.3. Analogous questions can be asked for the $h''$-vector.

**Question 6.4.**

(i) Does the conclusion of Corollary 3.3 hold for every Buchsbaum flag simplicial complex $\Delta$ of dimension $d-1$ which satisfies $\tilde{\beta}_{d-1}(\Delta) \neq 0$?

(ii) Is the $h'$-vector of every Buchsbaum flag simplicial complex equal to the $f$-vector of a simplicial complex?

Two more questions related to Corollary 3.3 are the following.

**Question 6.5.** Assume that $\Delta$ is a simplicial complex as in Corollary 3.3.

(i) If $h'_i(\Delta) = \binom{d}{i}$ holds for some $i \in \{1, 2, \ldots, d-1\}$, is $\Delta$ necessarily isomorphic to the simplicial join of $d$ copies of the zero-dimensional sphere?

(ii) Is it true that

$$h'_i(\Delta) - h'_{i-1}(\Delta) \geq \binom{d}{i} - \binom{d}{i-1}$$

holds for all $1 \leq i \leq \lfloor d/2 \rfloor$?
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Department of Mathematics, Division of Algebra-Geometry, University of Athens, Panepistimiopolis, Athens 15784, Greece

E-mail address: caath@math.uoa.gr

Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, 35032 Marburg, Germany

E-mail address: welker@mathematik.uni-marburg.de