Correlation Scales of the Turbulent Cascade at 1 AU

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Abstract. We collect and review the conclusions of several recent papers that address the transport of energy within the inertial range of interplanetary turbulence at 1 AU. Techniques have been developed that provide a means of measuring energy transport within the spectrum that do not rely on an assumption of specific dynamics other than the adoption of the complete single-fluid, incompressible magnetohydrodynamic equations. These techniques lead to expressions that do not depend on the form of the power spectrum other than requiring scale separation such that there exists an inertial range where the dynamics are energy conserving. We compute the correlation function for the derived expressions and find that the dynamics of turbulent transport decorrelate on the scale size of interest. We also show that the rate of energy transport at any given scale and any given time or location can be either positive or negative and that the typical rate of energy transport is $\sim 10\times$ the average rate. It is the average rate that agrees well with the computed rate of thermal proton heating in the solar wind. This leads to a model of the turbulence where local transport is stronger than normally considered.

1. Introduction

There are numerous complimentary and conflicting theories that attempt to describe inertial-range dynamics governing the spectral transport of energy in the solar wind. They include magnetohydrodynamic (MHD) extensions of traditional Navier-Stokes (NS) theory [14, 38], MHD weak turbulence theory [13, 16, 12], hybrid models [10, 6], and other theories [3]. See reviews [25, 19, 4, 5]. All these theories possess the common traits that they predict, or someday will predict, (1) an asymptotic spectral anisotropy for both fluctuations and the underlying wave vector orientations, (2) a power spectral form that can be compared against observations, and (3) an average rate of energy transport that can be compared against the heating of the plasma. These average energy transport rates are most often cited as the nonlinear transport of turbulence.

Kolmogorov’s prediction for the omnidirectional power spectrum of the total energy fluctuations for isotropic hydrodynamics (HD) is [14]:

$$P(k) = A_K \epsilon^{2/3} k^{-5/3}$$

(1)

where $A_K = 1.6$ [2, 30], $\epsilon$ is the average energy transport in the inertial range, and $k$ is the wave number. Comparison of the above extended to an MHD formalism and early weak
turbulence theory \[13, 16\] against the average thermal proton heating rates at 1 AU found that the Kolmogorov prediction provided the best fit to the observed heating rate once the constant $A_K$ was modified \[35\].

It is possible to perform direct measurement of the energy transport at each scale in the inertial range. Third-moment theory provides this formalism \[15, 23, 24\] without the need to apply any particular theory for the specific and relevant nonlinear dynamics. The derived third-moment expressions require only the MHD or NS equations and a short list of statistical properties: homogeneity, incompressibility, stationarity, isotropy or another geometry, and scale separation. Each of these assumptions have been extended in more advanced derivations \[15, 18, 36, 37, 31, 33, 11, 1\]. The measured average energy transport rate derived from third-moment theory has been shown to provide good agreement with the average thermal plasma heating rate \[18, 31, 7, 20, 11\].

We are now learning that the average energy transport rate is a poor measure of the overall nonlinearity of the inertial range \[8, 9, 27\]. While the average rate of energy transport agrees well with the average rate of plasma heating, the inertial range is not well described as the transport of energy with a uniform and fixed rate at all scales. The inertial range is not a pipeline for energy moving steadily from large to small scales. It is, instead, a highly variable nonlinear system where each scale appears to act independently in moving energy either to large or small scales at any given time and location with the net average transport of energy in agreement with theories of the type described above.

We will review below the results of several papers \[8, 9, 27\]. The correlation functions and correlation scales for the nonlinear dynamics are computed. Third-moment theory will be used to show that the correlation scale for inertial range dynamics is scale dependent with the implication that dynamics at different scales cannot remain correlated. A view of MHD turbulence dynamics will develop that suggests the true measure of nonlinearity within the inertial range is an order of magnitude greater than the above average transport rates would suggest.

2. Mean Energy Transport Rates

While various theories based on specific transport dynamics yield predictions for the turbulent spectrum and the average rate of energy transport through the inertial range, a direct measure of the transport rate can be obtained using third-moment theory \[15, 23, 24\]. For incompressible, homogeneous, stationary, isotropic MHD turbulence we can write:

$$D^+ (n, \lambda) = \left( \Delta Z^+ (n, \lambda) \right)^2 \sum_{i=1}^{\lambda^3} \left( \Delta Z^i (n, \lambda) \right)^2.$$  \hspace{1cm} (2)

where $\Delta Z^+ (n, \lambda) = Z^+ (n + \lambda) - Z^+ (n)$ and $Z^\pm \equiv V \pm B / \sqrt{4\pi\rho}$. This yields the local energy transport rate when averaged over a suitable ensemble:

$$\langle D^+ (\lambda) \rangle = \frac{1}{(N-\lambda)} \sum_{n=1}^{N-\lambda} \left( \Delta Z^+_L (n, \lambda) \sum_{i=1}^{\lambda^3} \left( \Delta Z^i (n, \lambda) \right)^2 \right)$$

$$= -\frac{4}{3}\epsilon^+ |L|$$ \hspace{1cm} (3)

where $L = V_{SW} \lambda \tau$ and $\tau$ is the time resolution of the data. Note that $\langle D^+ (\lambda) \rangle$ scales linearly with lag $L$ indicating uniform transport at all inertial range scales. The total energy transport rate is given by $\epsilon^T = (\epsilon^+ + \epsilon^-) / 2$ from which it follows that $\langle D^+ (\lambda) \rangle = \langle (D^+_T + D^-_T) / 2 \rangle = -\frac{4}{3}\epsilon^T L$.

Tests of third-moment theory against the measured average rate of thermal proton heating at 1 AU using data from the ACE spacecraft have provided good agreement \[31, 7\]. Figure 1
shows results from that analysis using an ensemble of 3094 12 hr samples. The sample duration was chosen because it is long compared to the published values of the correlation scale for the primitive variables, thereby allowing Gaussian statistics to be used. Note that $D_3(L)/L$ yields a constant function of lag in Figure 1 (left). Figure 1 (right) compares the average heating rate [35] to the computed energy transport rate derived from hydrodynamic theory, isotropic MHD and a hybrid MHD model that includes both field-aligned and perpendicular wave vectors. The computed rates of energy cascade is in good agreement with the local proton heating rates as shown in Figure 1 (right) when the ensemble is divided according to the parameter $V_{SW}T$ where $V_{SW}$ is the solar wind speed and $T$ is the proton thermal temperature.

3. Local Energy Transport Rates
What Figure 1 omits is the underlying distribution of values that are averaged in order to obtain the means shown. We can reduce the sample size to 1 hr, which remains a reasonable estimate for the correlation scale of the primitive variables and subset the data to a limited range of fluctuation energy and cross helicity. We can then evaluate eq. 3 for each hourly sample. Both fluctuation energy and cross helicity values are limited so as to create a distribution of values from within a common ensemble of expected mean heating rates. The average proton heating rate depends on both [35, 32, 17]. Data intervals from 12 hrs upstream to 36 hrs downstream of interplanetary shocks are omitted to avoid including transient dynamics.

If we subset the ACE data to include only those samples with mean fluctuation energy in the range $300 < E < 700 \text{ km}^2 \text{s}^{-2}$ and mean normalized cross helicity $|\sigma^C| \leq 0.75$, we obtain 7075 1 hr samples. We compute $D_3^T$ across a range of lags 2 – 20 points, corresponding to 128 – 1280 s, and fit the function $D_3^T$ with a straight line for each hourly sample. Since 1 hr is comparable to the nominal correlation length of primitive variables, we adopted this sample size as a proxy for the unknown correlation scale of the underlying dynamics. This made it possible to employ Gaussian statistics in those analyses. We obtain the distribution of $R^2$ values for the straight line fit shown in Figure 2. A total of 4787 samples (approx. 68%) had $R^2 \geq 0.7$. The implication is that a good linear fit can be obtained when the sample size is comparable to the correlation
scale of the primitive variables, and in so doing we obtain the necessary averaging to produce a viable estimate for the local rate of cascade.

We examine the underlying distribution of $\epsilon$ values by defining $\epsilon^{in}$ and $\epsilon^{out}$ to be either $\epsilon^\pm$ according to the mean magnetic field direction such that $\epsilon^{in}$ ($\epsilon^{out}$) is the transport of energy associated with inward (outward) propagating fluctuations. We make this division because at 1 AU it is typical to find the energy associated with outward-propagating fluctuations exceeds the energy associated with the inward propagating fluctuations. We also define $\epsilon^C = (\epsilon^+ - \epsilon^-)/2$ to be the difference between the two cascade rates. The parameter $\epsilon^C$ provides a measure of the relative cascade rates of the two components. Figure 3 shows the distribution functions of $\epsilon^{in}$, $\epsilon^{out}$, $\epsilon^T$ and $\epsilon^C$ values derived from the same subset of ACE data under the condition that $R^2 \geq 0.7$. In so doing, we use only those intervals for which $D^2_T$ is well-fit by a linear function of lag. Note that the standard deviation for each panel is $10 \times$ the mean value and that the mean value is positive. Not shown here, there is a general anti-correlation between the transport (cascade) of $\epsilon^{in}$ and $\epsilon^{out}$ [9].

If our interpretation of this result is correct, then several questions must be addressed:

(i) What is the energy reservoir that supports energy transport from small to large scales (negative values of $\epsilon^T$)?
(ii) Does the inertial range act as a coordinated dynamical process that moves energy in the same sense at all scales at a given time?
(iii) What is the minimum sample size that will provide statistically independent samples of $\epsilon$ at a prescribed scale?

The mean rate of energy transport through the inertial range must be positive, indicating transport from large to small scales, in order to accomplish a net, average heating of the background plasma. However, average negative cascade rates have been reported for high cross-helicity (imbalance), shear, and expanding flows [26, 32, 33, 28]. We believe we now understand this as the neglect of an error term that arises when the gradient in the mean flow that provides a significant contribution to the third-moment analysis in some circumstances. Efforts to demonstrate this are ongoing at this time. It appears at this time that this explains away the need for an energy reservoir at small scales so long as negative energy cascade rates are short-lived. Locally negative energy transfer rates have been seen in hybrid Vlasov-Maxwell simulations and shell-shell energy transfer Hall-MHD simulations [21, 29]. The rest of this paper and [27] address questions 2 and 3 above.

Figure 2. Taken from Figure 1 of [9]. Analysis of 7075 1 hr samples using 12 yr of ACE data showing the distribution of $R^2$, the square of the correlation coefficient for $D^2_T(L)$ vs. $L$, for lags 2 – 20 corresponding to 128 – 1280 s. The data has $\tau = 64$ s cadence and transients are removed. The analysis is limited to samples with mean fluctuation energy $300 < E < 700$ km$^2$s$^{-2}$ and mean normalized cross helicity $|\sigma^C| \leq 0.75$. The selection reduces possible distribution spread due to a too broadly defined sample population.
Figure 3. Taken from Figure 2 of [9]. PDF for $\epsilon^{in}$, $\epsilon^{out}$, $\epsilon^T$ and $\epsilon^C$ computed from 4787 1 hr samples that had $R^2 \geq 0.7$. Solid red line marks the mean. Dashed lines are the standard deviation. In each case, the standard deviation is $\sim 10 \times$ the mean value with almost half of the results being negative.

4. Correlation Function of $D_3$

Single-spacecraft measurements in the solar wind are unable to provide detailed measurements of the time scale for solar wind dynamics such as the duration and variability of the local energy cascade rate in the plasma frame. What they can provide is an assessment of spatial scales. Using single-spacecraft measurements and the Taylor frozen-in-flow assumption [34], we can show that specific values of the energy transport are confined to limited regions of space and that the energy transport at different scales is unlikely to be correlated. This creates a model for the turbulence where each scale in the inertial range acts independently with large variations in energy transport. We obtain this result by measuring the correlation function for the energy transport terms $D_3$.

The autocorrelation function for the fluctuation variable $\delta F$ as a function of lag $\delta n$ can be written

$$A(\delta F, \delta n) = \frac{1}{(N-\delta n)} \sum_{n=1}^{N-\delta n} \delta F(n)\delta F(n + \delta n)$$  \hspace{1cm} (4)$$

where the sum is over points in the data set of length $N$. This creates a correlation function $A$ for a variable $\delta F$ that is a function of the separation variable $\delta n$.

We can apply this formalism to the expressions for $D_3^\pm$ and $D_3^T$ to compute the correlation function at lag $\lambda$ as a function of separation $\delta n$:

$$A \left(D_3^{\pm, T}(\lambda), \delta n \right) \equiv \langle D_3^{\pm, T}(n, \lambda)D_3^{\pm, T}(n + \delta n, \lambda) \rangle_\lambda$$

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\[ = \frac{1}{(N - \lambda - \delta n)} \sum_{n=1}^{N-\lambda-\delta n} D^{\pm T}_3(n, \lambda)D^{\pm T}_3(n + \delta n, \lambda). \]  

The anticipated ensemble average becomes \( \langle \ldots \rangle_t \), which denotes an average over time, which becomes \( \sum_{n=1}^{N-\lambda-\delta n} \) when applied to the data.

There are two techniques we have used to define the correlation scale for \( A(\delta n) \). The first is based on the normalized integral of the correlation function:

\[ \lambda^i_c \equiv \left\{ \int_0^\infty A(\delta n)d(\delta n) \right\} / A(0) \]  

and the second is \( \lambda^e_c \) defined to be that scale at which the correlation function falls to 1/e of \( A(0) \). Both give comparable values in this analysis [27].

Figure 4 shows our analysis of ACE magnetic field data for Carrington solar rotation 2303 (DOYs 101-127 of 2002). The correlation function of \( D^T_3(\delta n) \) is shown for four values of lag \( \lambda = 5, 10, 25, \) and 100 pts (\( \lambda \tau = 320, 640, 1600, \) and 6400 s). The correlation scale as determined from the exponential definition is consistently \( \sim 20\% \) of the lag value. The definition of the correlation scale that is derived from integration of the correlation function yields comparable results. If both large and small scales within the inertial range each decorrelate differently, it is difficult to see how the associated dynamics could remain correlated across those scales.

5. Discussion and Summary

We have examined the functions that yield the rate of energy transport through the inertial range for incompressible MHD when averaged over a suitably defined ensemble. That average is critical to the derivation and we must stress that in the formal sense we have not computed the correlation scale of the nonlinear dynamics. We have only computed the correlation scale \( \lambda^i_c \) and \( \lambda^e_c \) of the transport dynamics decorrelate over relatively short distances and that the dynamics of two different scales within the inertial range are generally uncorrelated. However, this is only an implication as computation of the nonlinear dynamics as determined by third-moment theory requires an ensemble average of \( D^T_3 \) to represent energy transport. This is not possible in the present analysis. \( D^T_3 \) possess comparable correlation functions and scales.
of the function that, when suitably averaged, yields the energy transport rate of the nonlinear dynamics. Still, it is an indication of the potential correlation scale for the nonlinear dynamics and it indicates that the correlation scale is $\sim 20\%$ of the lag value. This means that the inertial range dynamics decorrelate over relatively short distances. When the nonlinear dynamics at two different scales decorrelate differently it is difficult to impossible to argue they may remain correlated across those two scales. The view of turbulence that derives from this result is that the inertial range nonlinear dynamics are composed of seemingly random variations in sign and magnitude that transport energy at $\sim 10\times$ the average rate of cascade. That average rate, when a suitably large ensemble of samples is employed to produce convergence, is in good agreement with the average rate of thermal proton heating in the solar wind. However, heating is not the result of a systematic dynamic that moves energy from large to small scales throughout the inertial range. Rather, it is the result of many random dynamics moving energy at a much faster rate that, only on average, moves energy at a rate that is consistent with plasma heating. The degree of uncorrelated transport within the inertial range is much greater than traditional theory would suggest.

Last, we should note that the application of eq. 3 to the data interval used here yields a negative value of $\epsilon^+$ and $\epsilon^{out}$. As this stands in contradiction to the published results using larger ensembles [31, 7], we conclude that the sample is insufficient to obtain convergence [22]. This is not a concern to this analysis as our only interest here is to obtain insights into the underlying distribution of values and the scale on which those values decorrelate. It appears from our analyzes [8, 9, 27] that the third-moment expression converges to a linear scaling with lag with much less data than is required for convergence to the mean of the ensemble.

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