Parametric Denotational Semantics for Extensible Language Definition and Program Analysis

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Abstract

We present a novel approach to construction of a formal semantics for a programming language. Our approach, using a parametric denotational semantics, allows the semantics to be easily extended to support new language features, and abstracted to define program analyses. We apply this in analysing a duck-typed, reflective, curried dynamic language. The benefits of this approach include its terseness and modularity, and the ease with which one can gradually build language features and analyses on top of a previous incarnation of a semantics.
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1 Introduction

Programming language semantics is a sub-field within theoretical computer science where researchers develop formal descriptions for the meaning of computer programs. Over the years, we have seen the development of denotational semantics, where we mathematically model the effect of an execution of a language construct. Operational semantics formalise mechanical steps that transform program states given a particular program. As we shall argue in this thesis, the challenge of analysing dynamic languages, in both concrete and abstract manner, necessitates a semantics that bridges the gap between the two different semantics in order for such task to be feasible.

Abstract interpretation is a unifying theory for program analysis and verification with which we ascertain run-time properties of a program by approximating its semantics. The properties of interest are almost always undecidable. The task of abstraction interpretation can be thought of as over-approximating a set of concrete states in a finite number of steps. The usual semantic domain is replaced by an abstract domain whose elements describe a set of run-time states. Mathematically, such an abstract domain is a partially ordered set (forming a lattice), the ordering corresponding to subset ordering of the powerset of concrete states.

Two distinct needs motivated the development of the present work. First, there is the theoretician’s need for a simple, concise, elegant way of presenting a formal semantics for a programming language and of developing that into various static analyses. We base our approach on a parametric denotational semantics that is modularised to allow the concrete semantics and the abstract interpretation to share a common framework that uniformly handles most aspects of the programming language. Use of denotational semantics provides a strong foundation in proving correctness of an abstract interpretation, and allows us to focus on algorithmic details of analysis.

Second, there is a practical need for program analyses suitable for the dynamic languages that have been growing in popularity in recent years. Traditionally these languages were called “scripting” languages, as they were mainly used for automating tasks and processing strings. However, with the advent of web applications, languages such as Perl and PHP gained popularity as languages for web application development. On the client side, web pages make heavy use of JavaScript, a dynamically typed language, to deliver dynamic contents to the browser. Recent years have seen an increasing use of JavaScript on the server side, as well.

What these languages provide is an ability to rapidly prototype and validate application models in a real time read-eval-print loop. Another strength comes from the fact that programmers do not need to have a class structure defined upfront. Rather, class structures and types of variables in general are dynamically built. This reduces the initial overhead of software design.

However, these features come at a cost. The lack of a formal, static definition of type information makes dynamically typed languages harder to analyse. This difficulty causes several practical problems.

- As applications become more mature, more effort is devoted to program unit testing and writing assertions to ensure type safety of systems. This extra effort can sometimes outweigh the benefit of having a dynamically typed language.
- Whereas programmers using statically typed languages enjoy an abundance of development tools, the choice of tools for development in dynamically typed languages is limited, and the tools that do exist lack much of the power of the tools for statically typed languages, owing largely to the difficulty or infeasibility of type analysis for such languages.
- Lack of static type structure has a significant impact on the performance of dynamically typed languages.

With these problems in mind, we have designed a model language that has a dynamism comparable to that of the aforementioned scripting languages, such as duck typing, reflection, and partial function application. A notable omission is closure scoping. However, allowing function currying gives expressive power to the language comparable to that of languages with closure or lexical scoping.

The two concerns are not distinct ones, but an interconnected dialectic. The theoretical need is there because of the difficulty of describing the abstract and concrete meaning of dynamic languages, which often allow side-effect causing, type-altering functions. With such complexity, duck-typed languages are interesting test cases for which we formulate concrete semantics, abstract interpretation and the proof of correctness.
Our Haskell implementation of both concrete and abstract analysis, appearing in the appendix to this thesis, illustrates the practicality of the proposed programming language semantics.

This work is inspired by Haskell’s use of monads, and we assume the reader’s familiarity with monadic style Haskell programming. We also assume knowledge of lambda notation, denotational semantics, order and fixed point theory at the level of the textbook of Nielson and Nielson’s [1].

In the following section, we discuss other works in the field of language semantics and differentiate our work from them. In section 2, we give a general overview of the proposed language semantics framework and analysis. In sections 3 and 4 we formally introduce our framework. In section 5 we develop a model language with features gradually added on. We also present concrete and abstract analysis of the language in each stage of development in parallel. In sections 7 and 8 we argue formal properties of the language analysis. Finally, in section 9 we conclude this thesis and discuss future direction.

2 Related work

Denotational semantics is the starting point of our development of a formal framework. The idea of incorporating monads into denotational definitions was developed by Liang and Hudak [2, 3]. Whereas these works modularise an analytic framework by having multiple layers of monadic transformations, we instead parametrise the definition of a program state.

Action semantics, as advanced by Mosses [4], shares the motivation that semantics ought to be pragmatic, yet expressive enough to deal with non-trivial, feature-rich languages. While action semantics endeavours to devise a new meta-language for describing semantics, we constrain ourself to the language of denotational semantics, and seek to devise a formalism largely compatible with denotational semantics.

The idea of constructing formulae with parametric types can be found in Wadler’s work [5]. The present work is a special application of the parametricity in the field of language semantics and analysis.

Regarding the type analysis of dynamic languages, there have been numerous studies [6, 7, 8, 9] that consider simple toy languages and their semantics for the purpose of static analysis of dynamic languages. A major difference between those languages and the model language presented in this paper is that our language is designed to capture the critical feature of real world languages which allows functions to alter types through side-effect causing statements. We point out similarities and differences of this work compared to the cited works as we encounter them in this thesis.

Type analysis plays a crucial part in compiling scripting languages, mainly to improve performance. Ancona et al [10] and Dufour [11] design restricted versions of scripting languages so that static inference of types can be performed. We adopt several techniques employed in those projects, such as the use of named memory allocation sites as static references.

An important use case of functions in dynamically typed languages is “mixin” functions [12]. By passing arguments to a mixin function, objects can be extended with extra methods; that is, functionality can be added dynamically. There are model languages and formalisations of mixin functions, such as the works of Anderson et al [6] and Mens et al [13]. Where those works seek to find functional models for mixins, we define instead a language (with side-effect causing functions) that is expressive enough to program mixin inheritance.

Jensen et al [14] describe a feature-complete analyser for the JavaScript language. Our work can be extended further to provide the semantic foundation for such an analyser. Such an attempt to formalise the analysis might pave the way for further refinement and improvement.

3 Overview

Our semantic framework is comprised of two components: one for the syntactic structure, and the other for giving meanings to the primitive operations. What divides the two is the following separation of concerns:

1. What are the semantic operations entailed in a particular syntactic structure? For example, syntactic structure \([a = 30]\) entails a primitive operation \(\text{asg}(a, 30)\).
2. How do we interpret such semantic operations in a particular point of view? If we were to give a concrete interpretation, we would interpret \texttt{asg} \((a, 30)\) as updating an environment with a newly defined variable e.g., \(Env[a = 30]\).

Observe that an interpretation of syntactic structure can remain agnostic of the structure of a program state at a given point. Therefore, once we remove the actual interpretation of primitive operations, what remains in a semantics can be re-used for multiple interpretations of the language. Hence, not only are the primitive operations parametrised, but so is the whole definition of the domain of the program state. Such a separation of concerns also helps to define an extensible semantics, to which adding a new feature takes as little effort as possible.

Now we give a formal definition of our framework.

\textbf{Definition 1 (Parametric semantics).} A parametric semantics \(Q\) is a quintuple \((X, \text{State}, \text{Value}, s, P)\) where \(X\) is a collection of semantic functions for syntactic structures, as outlined below; \(\text{State}\) is a set of representations of computation state, which can be anything to suit a particular analysis; \(\text{Value}\) is a set of all possible values that an expression can be evaluated to be; \(s\) is an initial program state; and \(P\) is the set of primitive operations of the semantics. We assume throughout that different states are incomparable. In other words, \(\text{State}\) is ordered by identity.

Throughout the analysis, these primitive operations are the parameters of our analysis:

- \texttt{esc} takes a state and reports whether it is escaping (i.e., whether or not control flow reaches the successor statement)
- \texttt{cond} interprets the meaning of a branching point when a value and two transformations (one for true and another for false) are given
- \texttt{asg} takes an identifier and a value, and performs assignment
- \texttt{val} takes an identifier and produces its meaning
- \texttt{conval} takes a constant and produces its meaning
- \texttt{getinput} and \texttt{dooutput} define the meanings of console I/O operations
- \texttt{bin} defines the meaning of all binary operations given two values
- \texttt{ret} defines the meaning of a return statement given the value to be returned
- \texttt{fundecl} defines the meaning of dynamic execution of a function declaration
- \texttt{apply} defines the meaning of (possibly partially) applying a function to a list of values
- \texttt{get} and \texttt{set} define the meaning of getting or setting a member of an object
- \texttt{getglobal} and \texttt{getthis} define the meanings of keywords \texttt{global} and \texttt{this}, respectively
- \texttt{newobj} defines the meaning of instantiating a new object from a particular allocation site

Types of these operations are given in section \ref{parametric_semantics} as we introduce them.

Our model language, as we let it evolve through this thesis, has a set of features found commonly in scripting languages. In the remainder of the thesis we provide the semantics for a language with many different features. We introduce the components of the language step by step. The aim is to demonstrate that the semantic formalism enables such a stepwise development, each step being incremental in the sense that it does not require revision of the semantic equations developed in earlier steps.

Figure \ref{model_language} is an example of a program written in the model language. We call this model language Simple Duck-Typed Language (SDTL). A locally-scoped procedural language with support for higher order functions (lines 1 to 5) is introduced in Section \ref{higher_order_functions}. Function currying (lines 7, 8 and 20) is introduced in Section \ref{function_currying}. Object oriented features, including duck-typing and reflection, are introduced in Section \ref{object_oriented_features}. Finally, exception handling (lines 38 to 42) is introduced in Section \ref{exception_handling}.
function fact(f, x) {
    if (x < 2) { return 1; }
    return x * f(f, x-1);
}
output fact(input);
fa=fact(input);
output fa(input);

function Fruit(v) {
    this.value = v;
}
global.answer = 0;

function juicible(fruit, juice) {
    function juiceMe(j, x) {
        return this.value + j + x;
    }
    fruit.juice = juiceMe(juice); #currying
    global.answer=42;
}

# Juicibles
apple = new Fruit(15);
juicible(apple, 20);
grape = new Fruit(30);
juicible(grape, 50);

# Non-juicibles
banana = new Fruit(20);
watermelon = new Fruit(25);

output apple.juice(10); # 15 + 20 + 10
output grape.juice(10); # 30 + 50 + 10
output global.answer; # 42

try {
    if (input > 42) { throw 42;}
} catch(e) {
    output e;
}

Figure 3.1: Example SDTL program

4 Analytic framework

In this section we introduce a monadic construct specifically designed for the purpose of program analysis. We then introduce polymorphic auxiliary functions that are useful in extending theories in a modular manner.

First we define the monadic constructions. We define a type constructor $M$ and a bind operator $\triangleright_Y$.

**Definition 2** (Type constructor). The type constructor $M$ has the following polymorphic definition. $F$ is the set of program semantics. It is necessary to have this as an input to the state transformation in order to give the fixed point characterisation of semantics. $F$ is given a formal definition in section 5. The $a$ parameter to the type is used in different context to extract different information from the semantics.
\[ M \ a = F \rightarrow \text{State} \rightarrow \varphi (\text{State} \times a \cup \\{\text{Null}\}) \]

Observe that a single state can give rise to multiple corresponding successor states. We are essentially modelling a non-deterministic state transformation. This gives us the flexibility to handle both concrete and abstract semantics within a single framework.

Every statement is understood as a state transformer. We distinguish between “normal” and “escaping” statements, the latter yielding an “escape” state. For example, when a function returns, the return statement transforms the current state into an escape state. Our “bind” operator relies on a parametric operation \( \text{esc} \) to spell out the precise mechanism for escaping the current program execution flow. The \( \text{esc} \) function returns true if a state does not continue to the next expression or statement (having encountered a return statement, for example). This provides a flexible and general formalisation of a control flow, and it allows the handling of exceptions as well as function return statements.

In such escaping cases, there is no appropriate value of the type \( a \) to be associated with the successor states. Hence, we introduce \( \bot \) to be assigned to successor states of the escaping states.

**Definition 3** (Bind operator). We define a bind operator \( \triangleright \).

\[
(a \triangleright U) : M a \rightarrow (a \rightarrow M b) \rightarrow M b
\]

\[
T \triangleright U = \lambda f \lambda s. \text{let } S = T f s \\text{ in } \bigcup_{(s',a) \in S} (\text{if } \text{esc} (s') \text{ then } \{ (s', \text{Null}) \} \text{ else } U a f s')
\]

**Definition 4** (Point-wise ordering of state transformations). Given \( T, U \in \text{State} \rightarrow \varphi (\text{State} \times a \cup \{\text{Null}\}) \), \( T \sqsubseteq U \text{ iff } \forall x \in \text{State}, (T x) \subseteq (U x) \)

**Definition 5** (Point-wise ordering of monadic functions). Given \( T, U \in M a, T \sqsubseteq U \text{ iff } \forall f \in F, (T f) \subseteq (U f) \)

**Theorem 6** (Preservation of monotonicity). Given monads \( T, T' \) and \( U \), \( T \sqsubseteq T' \implies T \triangleright U \sqsubseteq T' \triangleright U \)

**Proof.** When a state \( x \) is a member of \( T \triangleright U \) for some \( f \in F \) and an initial state \( s \), there exists an intermediate state \( x' \in (T f s) \) from which \( x \) is derived by \( U \). Clearly, such intermediate state is also a member of \( T' f s \) by definition of point-wise ordering.

Formally, \( \forall f \in F \forall s \in \text{State}, \exists x, x' \in ((T \triangleright U) f s) \iff \exists x', x' \in (T f s) \land x \in (U f x') \) by the definition of bind operation. Now, \( T \sqsubseteq T' \implies x' \in (T' f s) \). Hence, \( x \in ((T' \triangleright U) f s) \).

Having a monadic structure helps provide modularity. For example, if a particular parametrised operation takes a state but only produces a value, it would be redundant to include a state as a part of returning type, to match the definition of monadic binding. In such a case, we take a function that returns only a value, then lift it to be used in the monadic context.

**Definition 7** (Monadic functions). We define the following auxiliary functions to incorporate non-monadic functions as a part of monadic transformation:

- (return for \( M \)) \( I_A \) is an identity state transformer that takes a constant and lifts it to an identity state transformer with the constant as a return value
  \[
  I_A : a \rightarrow M a
  \]
  \[
  I_A v = \lambda f \lambda s. \{ (s, v) \}
  \]

- (lift for \( M \)) \( I_V \) lifts a function that takes a state and returns a value to a monadic function
  \[
  I_V : (\text{State} \rightarrow a) \rightarrow M a
  \]
  \[
  I_V t = \lambda f \lambda s. \{ (s, t s) \}, \emptyset
  \]
• \( I_S \) takes a non-deterministic transformation and lifts it to a monadic function

\[
I_S : (\text{State} \to \phi(\text{State})) \to M a
\]

\[
I_S T = \lambda f \lambda s. \{ \langle s', \text{Unit} \rangle \mid s' \in T s \}
\]

**Definition 8 (Record updater).** We model a state as a record with named fields. In this way, an update operation written for a particular set of fields can be reused without redefining it when we add extra dimensions to a \( \text{State} \) domain to accommodate features that are orthogonal to the features of the previous version.

When we have a record \( \rho \) with named fields \( \langle c_1, \ldots, c_k \rangle \), and when an updater function \( U \) updates fields \( c_i, \ldots, c_j \), we define a function \( \Upsilon \) that takes a record, projects its fields into an n-tuple corresponding to the selected fields \( (P_{c_1, \ldots, c_j}) \), lets \( U \) update the tuple, and finally updates the whole record with the updated tuple \( (J_{c_i, \ldots, c_j}) \).

\[
P_{c_i, \ldots, c_j} = \lambda \rho. \langle a_i, \ldots, a_j \rangle
\]

\[
J_{c_i, \ldots, c_j} = \lambda \langle a_i, \ldots, a_j \rangle \lambda \rho. \rho \left[ c_n = a_n, n \in \{i, \ldots, j\} \right]
\]

\[
\Upsilon_{c_i, \ldots, c_j} U = J_{c_i, \ldots, c_j} \circ U \circ P_{c_i, \ldots, c_j}
\]

where \( a_i \) is a value for the field \( c_i \) of a record \( \rho \)

Similarly, we define an operation to update a record and return a value.

\[
\Psi_{c_i, \ldots, c_j} U = \lambda \rho. \langle \rho', a \rangle
\]

where \( \langle v, a \rangle = U (P_{c_i, \ldots, c_j} \rho) \) and \( \rho' = J_{c_i, \ldots, c_j} v \)

Finally, we define a value extractor, that takes a record and selects a value from it.

\[
\Theta_{c_i, \ldots, c_j} U = U \circ P_{c_i, \ldots, c_j}
\]

When \( c \) is an n-tuple space for chosen fields and \( r \) is a domain of a \( \text{State} \) record, the functions defined here have the following type signatures:

\[
\Upsilon : (c \to c) \to (r \to r)
\]

\[
\Psi : (c \to c \times a) \to (r \to r \times a)
\]

\[
\Theta : (c \to a) \to (r \to a)
\]

**Example 9 (Record updater example).** To see these functions in use, suppose we have a simple record structure for personal contacts.

\[
\begin{align*}
\text{Name} &= \text{String} \\
\text{Address} &= \text{String} \\
\text{Age} &= \mathbb{N} \\
\text{Contact} &= \text{Name} \times \text{Address} \times \text{Age}
\end{align*}
\]

• \( \text{addAge} \) updates age field of a contact record.

\[
\text{addAge}(n) = \Upsilon_{\text{Age}} \lambda a. (a + n)
\]

• \( \text{getAgeAndAdd} \) returns the previous age field value while updating the age field.

\[
\text{getAgeAndAdd}(n) = \Psi_{\text{Age}} \lambda a. (a + n, a)
\]
• getAge extracts age information from a contact record.

\[
\text{getAge} = \Theta_{\text{Age}} \lambda a.a
\]

**Definition 10** (Singleton lifting). Another commonly occurring pattern is that functions often return a singleton set. We define a function that takes a function returning a value and lifts it to be a function that returns a singleton set.

\[
\begin{align*}
\Gamma : (a \to b) & \to (a \to \wp(b)) \\
\Gamma F x &= \{F x\}
\end{align*}
\]

For simplicity of notation, we compose this function with the other functions from Definition 8:

\[
\begin{align*}
\bar{\Upsilon} &= \Gamma \circ \Upsilon \\
\bar{\Psi} &= \Gamma \circ \Psi \\
\bar{\Theta} &= \Gamma \circ \Theta
\end{align*}
\]

We now have monadic constructs and auxiliary functions to describe the semantic functions of the model language. We can now define the semantic functions of the language.

## 5 Semantic functions

We use syntax nodes as references to various items constituting program environment. To all statements and expressions in a program, we designate unique identifiers in order to reference them. For that purpose, we define the following syntactic nodes and unique identifier spaces.

- \( Stm \): is the set of statement nodes.
- \( Exp \): is the set of expression nodes.
- \( Lexp \): is the set of left-expression nodes.
- \( Sid \): is the set of statement identifiers.
- \( Eid \): is the set of expression identifiers.
- \( Id \): is the set of alphanumeric identifiers.

Note that we use an sid of a function declaration statement as a reference point for the function defined. The \texttt{param} and \texttt{arity} functions take such an sid and return a list of parameter names, and the arity of the function, respectively.

Where we specifically refer to an identifier to a syntactic construct, we write \([X]_x\) to mean a statement or expression \(X\) with an id \(x\). In cases where such identifiers are not directly referenced, we omit them for simplicity.

**Definition 11** (Semantic functions). The analytic framework contains the following semantic functions:

\[
\begin{align*}
F &= \text{Sid} \to Stm \times (State \to \wp(\text{State})) \\
S &= Stm \to F \to State \to \wp(\text{State} \times \{\text{Unit, Null}\}) \\
E &= Exp \to F \to State \to \wp(\text{State} \times \text{Value} \cup \{\text{Null}\}) \\
L &= Lexp \to F \to State \to \wp(\text{State} \times \text{Value} \cup \{\text{Null}\})
\end{align*}
\]

\(S, E\) and \(L\) are semantic functions for statements, expressions and left expressions, respectively. \(F\) is a function space to model the collection of functions in a program. Given the sid of a function declaration
site, it gives a statement node for function declaration and a state transformer. Note that in this picture a
function “returns” a value by giving a state transformation. Incorporating such a concept as a return value in
a State itself provides a greater flexibility in describing the effects of executing a statement or an expression
at a particular program point.

We define following auxiliary functions to describe the use of the references to syntax nodes.

\[
\begin{align*}
\text{stm} & : F \rightarrow \text{Sid} \rightarrow \text{Stm} \\
\text{param} & : F \rightarrow \text{Sid} \rightarrow [\text{Id}] \\
\text{arity} & : F \rightarrow \text{Sid} \rightarrow \mathbb{N} \cup \{0\} \\
\text{run} & : F \rightarrow \text{Sid} \rightarrow \text{State} \rightarrow \wp(\text{State})
\end{align*}
\]

6 The language under study

We define the model language, the SDTL (Simple Duck-Typed Language).

6.1 The procedural core language

We start off with a procedural language with C-like syntax.

Here \text{Num} and \text{Bool} are the syntactic categories for integers and boolean values.

SDTL does not have a separate category for function and variable declarations. Variables are declared
\textit{ad hoc} whenever such variables appear as a left expression to assign statements. Function declarations
are statements themselves, which allow them to appear anywhere in the program.

SDTL supports higher-order functions, which allows functions to be recursively referenced. For example,
we can define a factorial function in a recursive manner.

\textbf{Example 12} (Recursively defined factorial function). In SDTL, the factorial function can be implemented
in a recursive way.

1. \textbf{function} fact(f,n) {
2. \hspace{1em} \textbf{if}(n>1) \{ \textbf{return} f(f,n-1) \ast n; \} \textbf{else} \{ \textbf{return} 1; \}
3. \}
4. \}
5. \textbf{x=fact (fact ,input);}
6. \textbf{output} x;

8
In this example, the function fact takes two arguments. The first is the function pointer to recursively invoke, and the second is the usual argument to the function. This example illustrates that recursive functions are possible even in the absence of lexical scoping or other special scoping rules to allow a function body to refer to the function itself.

Given the availability of higher-order functions, we formulate the meaning of a function as a fixed point (see the definition of $F$). The semantic functions for SDTL are defined in figure 6.1. We use auxiliary functions evalParams and call to describe a function call. (Note that we use $\epsilon$ for the empty sequence, $[a]$ for the set of sequences of any number of values of type $a$, and the notation $a \dashv b$ to denote concatenation of sequences $a$ and $b$.)

$$
call : \text{Sid} \to [\text{Value}] \to M \text{Value}
$$

$$
call(n,p) = \lambda f \lambda \rho. \{ \text{leave } \rho \rho' \mid \rho' \in S \} \text{ where } S = \text{run}(f,n) \ (\text{enter } \rho \ n \ p \ \text{param}(f,n))
$$

$$
evalParams : [\text{Exp}] \times [\text{Value}] \to F \to \text{State} \to \phi(\text{State} \times [\text{Value}])
$$

$$
evalParams \epsilon \ ps \ f \ \rho = \lambda f \lambda \rho. \{ (\rho, ps) \}
$$

$$
evalParams [E] \sim \text{Exps} \ ps = \lambda f \lambda \rho. \bigcup_{(\rho', \text{con}) \in S} \{ \text{evalParams } \text{Exps} \ (ps \sim \text{con}) \ f \rho' \}
$$

$$
\text{where } S = E \ [E] \ f \ \rho
$$

$\text{enter}$ is a parametrised function that takes a caller’s state at the time of function invocation and an id-to-constant value mapping, and constructs an initial state for a callee. $\text{leave}$ takes both the caller’s state and the resulting states of callee’s, and constructs the caller’s states after the function call. These functions are parametrised so as to allow each interpretation to define the exact shape of a program state and its manipulation during a function call and return. These functions have the following types:

$$
\text{enter} : \text{State} \to \text{Sid} \to [\text{Value}] \to [\text{Id}] \to \text{State}
$$

$$
\text{leave} : \text{State} \to \text{State} \to \text{State}
$$

The types of primitive operations are as follows.

$$
\text{ret} : \text{Value} \to \text{State} \to \phi(\text{State})
$$

$$
\text{cond} : \text{Value} \to M \text{Value} \to M \text{Value} \to M \text{Value}
$$

$$
\text{asg} : \text{Id} \to \text{Value} \to \text{State} \to \phi(\text{State})
$$

$$
\text{doutput} : \text{Value} \to \text{State} \to \phi(\text{State})
$$

$$
\text{fundecl} : \text{Id} \to \text{Sid} \to \text{State} \to \phi(\text{State})
$$

$$
\text{conval} : \mathbb{Z} \cup \{ \text{true, false} \} \to \text{Value}
$$

$$
\text{getinput} : M \text{Value}
$$

$$
\text{bin} : \{+,-,\ast,\div\} \to \mathbb{Z} \cup \{ \text{true, false} \} \to \mathbb{Z} \cup \{ \text{true, false} \} \to \text{Value}
$$

$$
\text{val} : \text{Id} \to \text{State} \to \text{Value}
$$

6.1.1 Concrete interpretation

Domain

$$
\text{FunPointer} = \text{Sid}
$$

$$
\text{Value} = \mathbb{Z} \cup \{ \text{true, false} \} \cup \text{FunPointer}
$$

$$
\text{Env} = \text{Id} \to \text{Value}
$$

$$
\text{Return} = \text{Value} \cup \{ \text{Void} \}
$$

$$
\text{CState} = \text{Env} \times \text{IO} \times \text{Return}
$$
Figure 6.1: Semantic equations for a procedural core of SDTL

$CState$ is a Cartesian product of environment, input/output state and a return value. The return value is set to be a value when a function is returning any value inside a function body. This has been incorporated as a part of a program state so that we can signal escaping from a program flow. The initial program state is $\langle \emptyset, \text{io}, \text{Void} \rangle$, where $\text{io}$ is an initial IO state.

Functions

\[
\begin{align*}
\text{enter} & = \lambda \langle _, \text{IO}, _, \rangle, n, P, p. \langle [p_k \mapsto P_k], \text{IO}, \text{Void} \rangle \\
\text{leave} & = \lambda \langle V, _, _, \rangle, \langle _, \text{IO}', R \rangle. \langle \langle V, \text{IO}', \text{Void} \rangle, R \rangle \\
\text{esc} & = \Theta_{\text{Return}} (\lambda R. (R \neq \text{Void})) \\
\text{cond}(v, s_1, s_2) & = \begin{cases} 
  s_1 & \text{if } v = \text{true} \\
  s_2 & \text{if } v = \text{false}
\end{cases} \\
\text{asg}(id, v) & = \tilde{T}_{\text{Env}} (\lambda V.V [id \mapsto v]) \\
\text{val}(id) & = \tilde{T}_{\text{Env}} (\lambda V.V (id)) \\
\text{conval}(\text{con}) & = \text{\{con\}} \\
\text{getinput}(p) & = \tilde{\Psi}_{\text{IO}} \lambda O. (O', \text{user input integer}) \text{ see note} \\
\text{doutput}(v) & = \tilde{T}_{\text{IO}} \lambda O. O' \text{ see note} \\
\text{fundecl}(id, n) & = \tilde{T}_{\text{Env}} (\lambda V.V [id \mapsto n]) \\
\text{ret}(v) & = \tilde{T}_{\text{Return}} (\lambda R. v) \\
\text{bin}(op, c_1, c_2) & = c_1 \text{ op } c_2 \text{ (perform binary operation between two constants)}
\end{align*}
\]
We omit a detailed description of the IO environment. Normally, IO can be modelled as a queue of inputs and outputs as they are given and produced during the execution of a program.

**Example 13** (Concrete interpretation of recursive factorial function). The program in example 12 is concretely interpreted as follows.

- At line 1, `fundecl(fact, 1)` updates environment to be `fact ↦→ Function 1` assuming the function declaration has a unique id of 1.
- At line 5, `getinput` gives a user input. Assume that the input was 2. `evalParam` gives `[(Function 1), 2]`. In a function call `call(1, [(Function 1), 2])`, we first construct the initial state of a function call. `enter` gives `⟨[f ↦→ (Function 1), n ↦→ 2], IO, Void⟩`.
- At line 2, evaluating expression `n > 1` yields true. `cond` invokes another function call, with initial state `⟨[f ↦→ (Function 1), n ↦→ 1], IO, Void⟩`.
- On the second call `fact(f, 1), n > 1` yields false. Hence, `cond` invokes `ret(1)` which gives final state of `⟨[f ↦→ (Function 1), n ↦→ 1], IO, 1⟩`.
- On the first call, this state is first evaluated to yield value 1 by `leave`. Then, `f(f, 1) * 2` evaluates to 2, which becomes the ultimate return value.
- At line 5, after the function call `leave` gives `⟨[fact ↦→ Function 1], IO, Void⟩` as the final state. `asg` adds symbol `z` to the environment: `z ↦→ 2`
- At line 6, `val(z)` evaluates to 2, which is the final output of the program.

### 6.1.2 Abstract interpretation

At this stage, abstract interpretation looks largely similar to concrete interpretation. Notable differences are that we approximate each constant by its type, and that `cond` is a non-deterministic transformation where it collects effects of both branches at a branching point.

**Domain**

| FunPointer    | = Sid         |
|---------------|--------------|
| AVal          | = {Num, Bool} ∪ FunPointer |
| AEnv          | = Id → AVal  |
| AReturn       | = AVal ∪ {Void} |
| AState        | = AEnv × AReturn |

Composition of an abstract domain is similar to that of the concrete counterpart, except that it does not include an IO state. ⊥ is an undetermined value, which is used to approximate unknown function calls at an initial stage. The initial program state is `⟨∅, Void⟩`.

**Functions**

| enter       | = λ ⟨_, _⟩, n, P, p. ⟨[pk ↦→ Pk], Void⟩ |
| leave       | = λ ⟨V, _⟩, ⟨_, R⟩. ⟨⟨V, Void⟩, R⟩ |
esc = Θ_{AReturn} (λR. (R ≠ Void))
cond (v, s₁, s₂) = λfλη. (s₁ f η) ∪ (s₂ f η)
асg (id, v) = Τ_{AEnv} (λσ.σ [id = v])
val (id, v) = Τ_{AEnv} (λσ.σ (id))
conval (con) = \begin{cases} Γ \text{ Num} & \text{if con} \in \mathbb{N} \\ Γ \text{ Bool} & \text{if con} \in \mathbb{B} \end{cases}
getinput = Γ (λη. ⟨η, \text{ Num}⟩)
dooutput (v) = Γ (λη.η)
fundecl (id, n) = Τ_{AEnv} (λσ.σ [id ↦ n])
ret (v) = Υ_{AReturn} (λR : AReturn.v)
bin (op, c₁, c₂) = \begin{cases} Γ \text{ Num} & \text{if op} ∈ \{+,−,∗,/\} \\ Γ \text{ Bool} & \text{otherwise} \end{cases}

Function definitions are largely similar to that of concrete definition.

Example 14 (Abstract interpretation of recursive factorial function). The program in example 12 is abstractly interpreted as follows.

- At line 1, fundecl (fact, 1) updates environment to be [fact ↦ Function 1] assuming the function declaration has a unique id of 1.
- At line 5, getinput gives an abstract value Num. evalParam gives [(Function 1), Num]. In a function call call (1, [(Function 1), Num]), we first construct initial state of a function call. enter gives

  ⟨[f ↦ (Function 1), n ↦ Num], IO, Void⟩

- The meaning of this function call is determined via a fixed point iteration by progressively updating the current approximation of the meaning of the function call, starting from a null hypothesis that the function call does not return any state.

| Current approximation | Meaning of function call | Note |
|-----------------------|--------------------------|------|
| ∅                     | [(fact ↦ (Function 1), Num)] |      |
| [(fact ↦ (Function 1), Num)] | [(fact ↦ (Function 1), Num)] | Fixed point |

Implementation of this fixed point iteration is found in the fun function in appendix A.3

- This yields [(fact ↦ (Function 1), Num)]
- After leave, asg and dooutput, we have the final state of the program: [fact ↦ Function 1, z ↦ Num]

Example 15 (Abstract interpretation of a while loop). The following example illustrates an interpretation of a while loop through a fixed point iteration.

The program calculates sum of a sequence. For the purpose of illustration, we have added variable x that changes its type inside a while loop.

```plaintext
sum = 0;
z = input;
x = 50;
while (z > 0) {
  sum = sum + z;
z = z - 1;
x = true;
}
output sum;
```
• At line 4, we have environment \[
\text{sum} \mapsto \text{Num}, \ z \mapsto \text{Num}, \ x \mapsto \text{Num}.
\]

• As an initial hypothesis, we assume that the statement body of a while loop does not cause any change in the program state for any given initial state. We progressively update this approximation until we meet a fixed point.

| Current Approximation | Init \rightarrow\text{Final State} |
|-----------------------|----------------------------------|
| \emptyset             | \left\{ \begin{array}{l}
\text{sum} \mapsto \text{Num} \\
\text{z} \mapsto \text{Num} \\
\text{x} \mapsto \text{Num}
\end{array} \right\} |

We take a simple add function, and curry one argument to produce different adders.

\begin{verbatim}
function add(x, y) {
  return x+y;
}

add5 = add(5);
add7 = add(7);

output add5(input) + add7(input);
\end{verbatim}

If we were to write this in JavaScript, we could have written the following for the same effect.

6.2 Function currying

We now introduce function currying to the SDTL language. Introduction of this language feature allows the language to be flexible enough to express what JavaScript programmers would do with lexical scoping.

Example 16 (Function currying). We take a simple add function, and curry one argument to produce different adders.

```javascript
function add(x, y) {
  return x+y;
}

add5 = add(5);
add7 = add(7);

output add5(input) + add7(input);
```
```javascript
function adder(toadd) {
    return function(y) {
        return toadd + y;
    }
}

add5 = adder(5);
add7 = adder(7);

// suppose input is a platform-specific console input function
console.log(add5(input()) + add7(input()));
```

The introduction of function currying does not change the syntax of the language. Therefore, there is no inherent reason for changing semantic functions. However, we do redefine the semantics of function calls to include an eid as an input, for the reason explained below.

\[
\mathcal{E}[L(\vec{E})]_e = \mathcal{L} [L] \circ \lambda n.\text{evalParams } \vec{E} \emptyset \circ \lambda p.\text{apply}(n,p,e)
\]

Here we introduce the apply function. It invokes the call function when the function arguments are saturated, or it returns a pointer to a curried function otherwise. Its type is as follows.

\[
\text{apply} : \text{Value} \to [\text{Value}] \to \text{Eid} \to \text{M Value}
\]

6.2.1 Concrete interpretation

We need to extend the definition of FunPointer to hold curried parameters. This means that the fundecl function also needs to be modified to match the new type signature.

Domain FunPointer = Sid × [Value]

The initial program state is unchanged.

Functions

\[
\text{apply}((n,C),p,\_ ) = \lambda f. \begin{cases} \text{call}((n,C \sim p)) f & \text{if } \|C \sim p\| = \text{arity}(f,n) \\ \lambda \rho. \{(\rho, (n,C \sim p))\} & \text{otherwise} \end{cases}
\]

\[
\text{fundecl}(id, n) = \Upsilon_{env} \lambda V. V[id \mapsto \langle n, \emptyset \rangle]
\]

Here, \(\|s\|\) is the length of a sequence \(s\).

Example 17 (Concrete interpretation of a function currying). Consider the program shown in example 16.

- At line 1, we are presented with a function declaration. fundecl adds the identifier as a reference to a function pointer with no curried value. Assuming add has given a unique id of 1 during the parsing of the program, we have in environment \(\text{add} \mapsto \langle 1, \emptyset \rangle\).

- At lines 5 and 7, we partially apply the add function. Since the number of arguments is not saturated, apply \(\langle (1, \emptyset), [5], \_ \rangle\) gives \(\langle 1, [5] \rangle\) for add5. Similarly, add7 gets \(\langle 1, [7] \rangle\).

- At line 8, we saturate the parameters, apply \(\langle (1, [5]), [x], \_ \rangle\) give call \(\langle 1, [5, x] \rangle\), where \(x\) is an arbitrary number given from user input. Hence we have the addition done. It works similarly for add7.
\[ \text{apply}(⟨n, c, e⟩, p, e') = \begin{cases} Θ_{\text{Curried}} & \lambda ν. \bigcup_{C ∈ ν(n,c,e)} \text{call}(n, C ⊲ p) \text{ if } c + \|p\| = \text{arity}(n) \\ Ψ_{\text{Curried}} & \lambda ν. \bigcup_{C ∈ ν(n,c,e)} \langle ν(\text{arity}(n) + \|p\|, e') \mapsto (C ⊲ p), ⟨n, c + \|p\|, e'⟩) \rangle \text{ otherwise} \end{cases} \]

\[ \text{fundecl}(id, n) = \tilde{Y}_{\text{AEnv}} \lambda σ.σ[\text{id} ↦ \langle n, 0, 0 \rangle] \]

Figure 6.2: Abstract functions for curried functions

### 6.2.2 Abstract interpretation

Note that curried functions introduce a possibility of creating closures requiring an infinite number of arguments.

**Example 18** (Currying loop). Consider the following program.

```plaintext
1 function foo(a, b) {  
2 return a;  
3 }  
4  
5 x = 0;  
6 while (true) {  
7 x = foo(x);  
8 }
```

If we naively interpret this program, we would not be able to reach a fixed point in analysis. Instead, we would have:

\[
\begin{align*}
  x &= \text{foo}(\text{Num}) \\
  x &= \text{foo}(\text{foo}(\text{Num})) \\
  x &= \text{foo}(\text{foo}(\text{foo}(\text{Num}))) \\
  \vdots
\end{align*}
\]

A solution to this problem is to have a curried function anchored to a particular language construct. In this case, we can use an eid of a curried expression as a point of reference (or ‘0’ if not curried).

**Domain**

- \( \text{FunPointer} = \text{Sid} × c × (\text{Eid} ∪ \{0\}) \)
- \( \text{Curried} = \text{Sid} × c × \text{Eid} → [\text{AVal}] \) where \( c \) is a number of parameters given
- \( \text{AState} = \text{AEnv} × \text{Curried} × \text{AReturn} \)

**Curried** takes a function id, number of argument curried and the eid of a calling site, and gives the list of curried values. The initial program state is \( ⟨∅, ∅, \text{Void}⟩ \).

**Functions** Definitions are given in figure 6.2

**Example 19** (Abstract interpretation of a currying loop). The program shown in example 18 can be analysed as follows:

- At line 1, \( \text{fundecl}(\text{foo}, 1) \) gives \( [\text{foo} ↦ \langle 1, 0, 0 \rangle] \), assuming 1 is a unique id given to the function.
\[
S[E_1.id = E_2] = \mathcal{E} [E_1] \times \lambda_r. \mathcal{E} [E_2] \times \lambda v. I_S \text{ set}(r, id, v)
\]
\[
\mathcal{E}[\text{global}] = I_V \text{ getglobal}
\]
\[
\mathcal{E}[\text{this}] = I_V \text{ getthis}
\]
\[
\mathcal{E}[\mid L \left( \vec{E} \right) \mid_c ] = \left( \begin{array}{c}
\mathcal{L} [L] \times \lambda n. \\
\text{evalParams } \vec{E} \emptyset \times \lambda p. \\
I_V \text{ getthis } \times \lambda t. \\
\text{apply} (n, p, t, e)
\end{array} \right)
\]
\[
\mathcal{E}[\mid \text{new } L \left( \vec{E} \right) \mid_c ] = \left( \begin{array}{c}
\mathcal{L} [L] \times \lambda n. \\
\text{evalParams } \vec{E} \emptyset \times \lambda p. \\
\text{newobj} (e) \times \lambda m. \\
\text{apply} (n, p, m, e) \times \lambda _.
\end{array} \right)
\]
\[
\mathcal{E}[\mid E_1.id \left( \vec{E} \right) \mid_c ] = \left( \begin{array}{c}
\mathcal{E} [E_1] \times \lambda t. \\
I_V \text{ get}(t, id) \times \lambda n. \\
\text{evalParams } \vec{E} \emptyset \times \lambda p. \\
\text{apply} (n, p, t, e)
\end{array} \right)
\]
\[
\mathcal{L}[E_1.id] = \mathcal{E} [E_1] \times \lambda v. I_V \text{ get}(v, id)
\]

Figure 6.3: Semantic equations for the object-oriented extension of the language

- At line 7, assuming the partial application expression has a unique id of 7, apply \((1, 0, 0), \text{[Num]}, 7)\) gives \([x \mapsto (1, 1, 7)]\) for environment and \([(1, 1, 7) \mapsto \text{[Num]}]\) for curried in the first iteration. Subsequently, we have \(\text{apply} ((1, 0, 0), (1, 1, 7), 7)\) gives \([x \mapsto (1, 1, 7)]\) for environment and \([(1, 1, 7) \mapsto [(1, 1, 7)]\) for the list of curried value. Here we reach a fixed point.

- All possible final states calculated by the interpretation are:

| Environment   | Curried      |
|---------------|-------------|
| \(x \mapsto \text{Num}\) | None        |
| \(x \mapsto (1, 1, 7)\) | \((1, 1, 7) \mapsto \text{[Num]}\) |
| \(x \mapsto (1, 1, 7)\) | \((1, 1, 7) \mapsto [(1, 1, 7)]\) |

### 6.3 Object oriented features

We now extend our language to support objects.

\[\langle Lexp \rangle ::= \langle Exp \rangle \text{ '.' } \text{ID} \]

\[\langle Exp \rangle ::= \text{global' } | \text{'this'} | \text{'new'} \langle Lexp \rangle \text{ '(' } \langle Lexp \rangle \text{ ',' } \langle Lexp \rangle \text{ ')' }\]

global is a reference to a global object, and the reference to global object is program invariant (akin to ‘window’ in Client-side JavaScript or ‘global’ in Node.js). this is the usual reference to the receiver object of a method call.

As this extension introduces new syntactic structure, extra definitions for semantic functions are given in figure 6.3. Note, however, that existing semantic equations are not impacted by this.

We redefine the call function to include receiver object reference, and to account for side-effects that functions can have on object memories.
\( \text{get}(n, id) = \mathcal{O}_\text{ObjMem}(\lambda \Omega \cdot \Omega(n)(id)) \)

\( \text{set}(n, id, v) = \mathcal{Y}_\text{ObjMem}(\lambda \Omega \cdot \Omega(\{n \mapsto \Omega(n)[id \mapsto v]\})) \)

\( \text{getglobal} = \mathcal{O}_\text{ObjMem}(\lambda \Omega \cdot \Omega(0)) \)

\( \text{getthis} = \mathcal{T}_\text{This}(\lambda T \cdot T) \)

\( \text{apply}(\langle n, C \rangle, p, t, _) = \begin{cases} 
\text{call}(n, C \triangleright p, t) & \text{if } \|C \triangleright p\| = \text{arity}(n) \\
\Gamma(\lambda \rho . \langle n, C \triangleright p \rangle) & \text{otherwise}
\end{cases} \)

\( \text{newobj}(_) = \mathcal{O}_\text{ObjMem}(\lambda \Omega . \Omega(\{n \mapsto \emptyset\}, n)) \text{ where } n = \|\Omega\| \)

Figure 6.4: Concrete functions for the object-oriented extension

\[ \text{call} : \text{Sid} \to [\text{Value}] \to \text{Value} \to M \text{Value} \]

\[ \text{call}(n, p, t) = \lambda f \lambda \rho, \gamma . \Theta(\{\text{leave } \rho \mapsto \rho' \mid \rho' \in S\}, \gamma') \text{ where } \langle S, \gamma' \rangle = f(n)(\text{enter } \rho \mapsto n \mapsto p \mapsto t \mapsto \text{param}(f, n)) \gamma \]

\[ \text{enter} : \text{State} \to \text{Sid} \to [\text{Value}] \to \text{Value} \to [\text{Id}] \to \text{State} \]

\[ \text{leave} : \text{State} \to \text{State} \to \text{State} \]

Note that by modelling function calls as an effect on a state, we can readily model such concepts as mixins. In the JS0 language \[6, 7\], mixin creating behaviour is modelled in a functional language. A similar approach has been taken in the discussion of the \(\lambda_S\) language by Guha et al \[8\]. However, with languages like JavaScript, it is in the nature of such a language that functions are side-effect causing, and much of the type operations are being done by the side-effects. Hence our deviation from a purely functional approach to devise a model language that resembles a real-life language.

### 6.3.1 Concrete interpretation

We extend the domain to include object memory (where object references are mapped to a symbol mapping), and a reference to this object.

**Domain**

\[
\begin{align*}
\text{Value} & = \mathbb{Z} \cup \{\text{true}, \text{false}\} \cup \text{Object} \cup \text{FunPointer} \\
\text{Object} & = \mathbb{N} \\
\text{ObjMem} & = n \to \text{Env} \\
\text{This} & = \text{Object} \\
\text{CState} & = \text{Env} \times \text{ObjMem} \times \text{This} \times \text{Return}
\end{align*}
\]

The initial program state is \(\langle\emptyset, \{0 \mapsto \emptyset\}, 0, \text{Void}\rangle\). 0 is a unique id referring to the global object.

**Functions** We redefine \text{enter} and \text{leave} functions to allow passing of a receiver object.

\[
\begin{align*}
\text{enter} & = \lambda (_, \Omega, _, _) \mapsto n, P, T'. (\langle \text{param}(n)_k \mapsto P_k \rangle, \Omega, T', \text{Void}) \\
\text{leave} & = \lambda (V, _, T, _), (_, \Omega', r) \mapsto (\langle V, \Omega', T, \text{Void} \rangle, r)
\end{align*}
\]

The other functions are given in figure \[6.4\]

**Example 20** (Concrete analysis of objects). Consider the following program.
function Fruit(v) {
    this.value = v;
}

function juicible(fruit, juice) {
    function juiceMe(j, x) {
        return this.value + j + x;
    }
    fruit.juice = juiceMe(juice);
}

apple = new Fruit(15);
juicible(apple, 20);

output apple.juice(10); # 15 + 20 + 10

- Assume that Fruit has an id of 1, juicible has 2, and juiceMe has 3.
- Right before line 12, we have $[\text{Fruit} \mapsto \langle 1, \emptyset \rangle, \text{juicible} \mapsto \langle 2, \emptyset \rangle]$. At line 12, the new expression creates an object in object memory through $\text{newobj}(\_),$ then passes it on as a receiver of a method call to Fruit. Inside the Fruit function, the new object gets the member value. After the line, we have $\text{Env} = [\text{Fruit} \mapsto \langle 1, \emptyset \rangle, \text{juicible} \mapsto \langle 2, \emptyset \rangle, \text{apple} \mapsto \text{Obj 1}]$ and $\text{ObjMem} = [1 \mapsto \langle \text{value} \mapsto 15 \rangle]$.
- At line 13, $\text{apply}(\langle 2, \emptyset \rangle, [\text{Obj 1, 20}], \text{global}, \_, \_)$ gives call $(2, [\text{Obj 1, 20}], \text{global})$. Inside the juicible function, object memory is manipulated in line 9 to be $[1 \mapsto \langle \text{value} \mapsto 15, \text{juice} \mapsto \langle 3, [20] \rangle \rangle]$.
- At line 15, by the method call semantics, $\text{apply}(\langle 3, [20] \rangle, [10], \text{Obj 1, \_})$ is invoked, which gives 45 as a final result.

6.3.2 Abstract interpretation

Similarly, we extend the abstract interpretation. Note that, whereas in a concrete interpretation we generate a unique id for each of the objects instantiated at run-time, in abstract interpretation we use a reference to the allocation site of an object as a reference to a particular object, hence we contain object memory in a finite domain.

Domain

\[
\begin{align*}
AObj & = \{0\} \cup \text{Eid} \\
AVal & = \{\text{Num, Bool}\} \cup AObj \cup \text{FunPointer} \\
AObjMem & = AObj \to AEnv \\
AThis & = AObj \\
AState & = AEnv \times AObjMem \times AThis \times \text{Curried} \times AReturn
\end{align*}
\]

The initial program state is $\langle \emptyset, \{0 \mapsto \emptyset\}, 0, \emptyset, \text{Void} \rangle$. 0 is a unique id referring to the global object.

Functions

\[
\begin{align*}
\text{enter} & = \lambda (\_, \alpha, \_, \nu, \_), n, P, \tau'. ((\text{param}(n))_k \mapsto P_k], \alpha, \tau', \nu, \text{Void}) \\
\text{leave} & = \lambda (\sigma, \_, \tau, \_, \_), \langle \_, \alpha', \_, \nu', r \rangle. \langle \langle \sigma, \alpha', \tau, \nu', \text{Void} \rangle, r \rangle
\end{align*}
\]

Other functions are defined in figure 6.5.
\[ \begin{align*}
\text{get}(n, id) &= \Theta_{AObjMem} \lambda \alpha. \alpha(n) \\
\text{set}(n, id, v) &= \Upsilon_{AObjMem} \lambda \alpha. \alpha(n)[id \mapsto v] \\
\text{obj}(j) &= \Theta_{AObjMem} \lambda \alpha. \alpha(j) \\
\text{getthis} &= \Theta_{AThis} \lambda \tau. \tau \\
\text{newobj}(n) &= \Psi_{AObjMem} \lambda \alpha. \langle \alpha[n \mapsto \emptyset], n \rangle \\
\text{apply}(\langle n, c, e \rangle, p, t, e) &= \begin{cases} \\
\Theta_{Curried} \lambda \nu. \bigcup_{C \in \nu} \text{call}(n, C \prec p, t, e) & \text{if } c + \|p\| = \text{arity}(n) \\
\Psi_{Curried} \lambda \nu. \bigcup_{C \in \nu} \langle \nu[\langle n, c + \|p\|, e \rangle \mapsto C \prec p], \langle n, c + \|p\|, e \rangle \rangle & \text{otherwise}
\end{cases}
\end{align*} \]

Figure 6.5: Abstract functions for the object-oriented extension

Example 21 (Abstract interpretation of objects). The program in example 20 results in the following final state.

| Environment | Object Memory | Curried |
|-------------|---------------|---------|
| Fruit \mapsto \langle 1, 0, 0 \rangle | Obj 1 \mapsto \langle \text{value} \mapsto \text{Num}, \text{juice} \mapsto \langle 3, 1, 9 \rangle \rangle | \langle 3, 1, 9 \rangle \mapsto \text{[Num]} |
| juicible \mapsto \langle 2, 0, 0 \rangle | | |
| juiceMe \mapsto \langle 3, 0, 0 \rangle | | |
| apple \mapsto Obj 1 | | |

Notice that as we add object oriented programming to the programming language, the functions are now capable of causing side-effects. The following modified example illustrates how the meaning of recursive function call is calculated when the function causes side-effects.

Example 22 (Side-effect causing factorial function). Consider the following modification of program analysed in example 14.

```plaintext
1 function fact(f, n) {
2    if(n>1) { global.x=5; return f(f, n-1) * n; } else { return 1; }
3 }  
4 z=fact(fact,input);  
5 output z;  
```

- At line 5, the meaning of the function call is calculated as follows. For simplicity, we only record \(AEnv, AObjMem\) and \(AReturn\). The rest of the components remain unchanged.
### 6.4 Exception handling

As a final extension to SDTL, we introduce exception throwing and handling. First, we introduce a familiar syntax for exception handling.

\[
\langle Stm \rangle ::= \text{try } \langle Stm \rangle \text{ catch } \langle Stm \rangle \text{ throw } \langle Exp \rangle
\]

Following functions are the semantic functions for the newly introduced syntax.

\[
S[\text{try } S_1 \text{ catch } (id) \ S_2] = S[S_1]_{noesc} S_2 \\
S[\text{throw } E] = E[E]_{\lambda v. \text{throw } (v)}
\]

- **throw**: \( Value \rightarrow M Value \)
- **catch**: \( Id \rightarrow M Value \rightarrow M Value \)

Note that we cannot use \(\triangleright\) to combine the try code body and catch, since the \(\triangleright\) operator has \(\text{esc}\) built into its definition, which will terminate the program flow when the exception is raised. In order to allow exception-raising transformations to flow through, we introduce a non-escaping bind operator, \(\triangleright_{noesc}\). **catch** primitive operator takes an exception variable name and the meaning of a catch code block, and produces the meaning of a language construct in which an exception is caught and handled.

**Definition 23** (Non-escaping bind operator). We define a non-escaping bind operator, \(\triangleright_{noesc}\)

\[
\triangleright_{noesc} : M a \rightarrow (a \cup \{\text{Null}\} \rightarrow M a) \rightarrow M a
\]

\[
T \triangleright_{noesc} U = \lambda f \lambda s. \text{let } S = T f s \text{ in } \bigcup_{(s', a) \in S} (U a f s')
\]
6.4.1 Concrete interpretation

As the control flow of the execution is abstracted out and parametrised as a function $\text{esc}$, it is not very surprising that introducing exceptions to a program does not necessitate a heavy modification to the interpretation of the language.

Domain

\[
Ex = Value \cup \{\text{Void}\}
\]

\[
CState = Env \times ObjMem \times This \times Return \times Ex
\]

The initial program state is $\langle \emptyset, \{0 \mapsto \emptyset\}, 0, \text{Void}, \text{Void}\rangle$.

Functions

\[
\text{enter} = \lambda \langle _, \Omega, _, _, _\rangle, \langle P, T' \rangle, n. ((\text{param}(n))_k \mapsto P_k), \Omega, T', \text{Void}, \text{Void}\)
\]

\[
\text{leave} = \lambda \langle V, _, T, _, _\rangle, \langle _, \Omega', _, r, e \rangle. \langle \langle V, \Omega', T, \text{Void}, e \rangle, r \rangle
\]

\[
\text{esc} = \Theta_{\text{Return}, Ex} \lambda R, E. (R \neq \text{Void}) \land (E \neq \text{Void})
\]

\[
\text{throw}(v) = \tilde{\gamma}_{Ex} \lambda E. v
\]

\[
\text{catch}(id, s) = \Theta_{Ex} \lambda E. \begin{cases} I_A \text{Unit} & \text{if } E = \text{Void} \\ \lambda f, r, s \cdot \text{exs}(\rho, id) & \text{otherwise} \end{cases}
\]

\[
\text{exs}(id) = \gamma_{\text{Env}, Ex} \lambda V, E. \langle V[id \mapsto E], \text{Void} \rangle
\]

$\text{exs}$ helps constructing an entering state to a catch block if an exception was caught. As we do not support lexical scoping, an exception variable enters a variable mapping as we enter the block.

Example 24 (Concrete interpretation of exception handling). The following program illustrates exception handling in SDTL.

```
1 x = input;
2 try {
3  if (x<0) {
4    throw 0;
5  }
6  output x;
7  j = 3;
8  } catch(e) {
9    output x;
10   output e;
11 }
```

- At line 3, we have $[x \mapsto a]$ where $a$ is a user input. Suppose $a$ is less than 0, then if leads to $\text{throw}(0)$ which gives $Ex: 0$.
- The resulting state is deemed to be an escaping state by $\text{esc}$. Hence, the code block of the try clause gives the resulting program state of $\langle [x \mapsto a], \{0 \mapsto \emptyset\}, 0, \text{Void}, 0 \rangle$.
- With non-escaping binding to a $\text{catch}$ primitive function afterwards, $\text{catch}$ yields the state with environment of $[x \mapsto a, e \mapsto 0]$ then it evaluates the catch clause, which outputs $a$ and 0.

It is important that this exception handling does not interfere with other control flow. The following example shows how it interacts with return state.
Example 25 (Function call, return and exception handling control flow). Consider the following example.

```plaintext
1 function tryorerror(func, error, a) {
2     try {
3         return func(a);
4     } catch(e) {
5         return error;
6     }
7 }
8 function positive(a) {
9     if(a>0) { return a; }
10    throw 0;
11 }
12 gracefulpositive = tryorerror(positive, -1);
13 function doandprint(func,a) {
14     output func(a);
15 }
16 try {
17     doandprint(positive,50);
18     doandprint(gracefulpositive,-50);
19     doandprint(positive,-50);
20     } catch (e) {
21         output e;
22 }
```

- At line 3, when `tryorerror` is called from line 21, it sets a return state with Void exception state. As `Ex` is set to Void, the `catch` primitive operation does not interfere, and the control flow proceeds to return the value of 50.

- When called from line 22, the call to `positive` function sets an exception state. Observe that, during the evaluation of the return statement, if the exception is set while evaluating the returning expression, the control flow does not proceed to set a return state. The `catch` primitive operation notices exception state being set, and proceeds to evaluate the exception handling block, which returns -1. The meaning of the whole try-catch block, then, is evaluated to be setting a return state of -1.

- Finally, when the exception throwing function `positive` is directly called, at line 23, the exception state is set to be -50, and `catch` primitive operation proceeds to line 25, printing 0 to the output.

### 6.4.2 Abstract interpretation

Abstract definitions are largely similar to that of concrete ones. Notice that we carry on modified exception state returned from a function call, thereby allowing exception propagation.

**Domain**

\[
AEx = AVal \cup \{Void\} \\
AState = AEnv \times AObjMem \times AThisCurried \times AReturn \times AEx
\]

The initial program state is \(\langle \emptyset, \{0 \mapsto \emptyset\}, 0, \emptyset, Void, Void\rangle\).
Functions

\[
\begin{align*}
\text{enter} &= \lambda (\_, \alpha, \_, \nu, \_, \_), (P, \tau'), n. ([\text{param}(n)]_k \mapsto P_k], \alpha, \tau', \nu, \text{Void}, \text{Void}) \\
\text{leave} &= \lambda (\sigma, \_, \_, \_, \_, \_), (\_, \_, \_, \_, \_, \_, \_), (\sigma, \alpha', \tau', \nu, \_, \_, \_, \_, \_, \_, \_)
\end{align*}
\]

\[
\begin{align*}
\text{esc} &= \Theta_{A\text{Ret},A\text{Ex}} \lambda R, E. (R \neq \text{Void}) \land (E \neq \text{Void}) \\
\text{throw}(v) &= \Upsilon_{A\text{Ex}} \lambda E. v \\
\text{catch}(id, s) &= \Theta_{A\text{Ex}} \lambda E. \begin{cases} I_A \text{Unit} & \text{if } E = \text{Void} \\
\lambda f \lambda \eta. s \ \text{exs}(\eta, id) & \text{otherwise} \end{cases} \\
\text{exs}(id) &= \Upsilon_{A\text{Env}, A\text{Ex}} \lambda \sigma, E. (\sigma [id \mapsto E], \text{Void})
\end{align*}
\]

Example 26 (Abstract interpretation of exception handling). The program in example 24 can be abstractly interpreted as follows.

- After lines 3-5, we have two possible states \(\langle [x \mapsto \text{Num}], \{0 \mapsto \emptyset\}, 0, \emptyset, \text{Void}, \text{Void} \rangle\) and \(\langle [x \mapsto \text{Num}], \{0 \mapsto \emptyset\}, 0, \emptyset, \text{Void}, \text{Num} \rangle\).

- By the bind operation, the latter state becomes the final state of the try clause. The former state progresses further, and updates its environment to \([x \mapsto \text{Num}, j \mapsto \text{Num}]\).

- At line 8, the environment of exception throwing state is updated to be \([x \mapsto \text{Num}, e \mapsto \text{Num}]\), then the catch clause is evaluated, which does not cause any state update.

- The final states are \(\langle [x \mapsto \text{Num}, j \mapsto \text{Num}], \{0 \mapsto \emptyset\}, 0, \emptyset, \text{Void}, \text{Void} \rangle\) and \(\langle [x \mapsto \text{Num}, e \mapsto \text{Num}], \{0 \mapsto \emptyset\}, 0, \emptyset, \text{Void}, \text{Void} \rangle\).

7 Well-definedness

All the functions appearing in both concrete and abstract versions of interpretations are ordered point-wise. Recall that the bind operation preserves the monotonicity when binding two monotonic functions. Observe that all of the primitive operations that yield a state transformer, and the value-returning operations lifted to yield one, are monotonic. This follows from the fact that the domain of such transformers are ordered by identity. Then, it follows that \(S, E\) and \(L\) are monotonic for any given \(f \in F\).

Now, observe the monotonic relationship between \(F\) and monadic functions.

Definition 27 (Point-wise ordering of function approximation). Given \(f, f' \in F\), then \(f \sqsubseteq f'\) iff \(\forall s \in \text{Sid}, \forall \sigma \in \text{State}, (f \ s \ \sigma) \subseteq (f' \ s \ \sigma)\).

Theorem 28. \(\forall f, f' \in F, f \sqsubseteq f' \implies \forall s \in \text{Sid}, t \in \text{Stm}, (s \ t \ f) \subseteq (s \ t \ f')\)

Proof. Clearly, the call auxiliary function is monotonic with regards to the function approximation. The call function is the only function for which the function approximation value is referenced. Recall the preservation of monotonicity of bind operation. Then, this theorem immediately follows.

It follows from this theorem that there exists a least fixed point of the meaning of function \(f \in F\) for a particular program. Hence the well-definedness of the fixed point of \(F\) in a particular analysis, and that of the semantics both in concrete and abstract interpretations.

8 Correctness

We define an abstraction relation \(\triangleright\) between concrete and abstract states.
8.1 Definition of correct abstraction

First, we define an abstraction between values and other components of the states.

**Definition 29.** \( \text{Num} \supset \mathbb{Z}, \text{Bool} \supset \mathbb{B}, \text{Void} \supset \text{Void} \)

**Definition 30 (Abstraction of objects and symbol maps).** \( \alpha, \Omega \vdash n \succ m \) if \( \alpha(n) \succ \Omega(m) \)

\( \sigma : \text{AEnv} \succ V : \text{Env} \) if \( \forall \nu \in \text{dom}(V), \sigma(id) \succ V(id) \)

**Definition 31 (Function pointers).** \( C : \text{Curried} \vdash (n, c, e) \succ (m, P) \) if \( n = m, c = \| P \| \)

and \( \exists A \in C(n, c, e), A_k \succ P_k \) for \( 1 \leq k \leq c \)

Now we are ready to define an abstraction between concrete and abstract states.

**Definition 32 (Abstraction relation).** Given \( \rho = (V : \text{Env}, \Omega : \text{ObjMem}, T : \text{This}, R : \text{Return}, E : \text{Ex}) \in \text{CState} \)

\( \eta = (\sigma : \text{AEnv}, \alpha : \text{AObjMem}, \tau : \text{AThis}, C : \text{Curried}, r : \text{AReturn}, e : \text{AEx}) \in \text{AState} \)

\( \eta \succ \rho \) if \( (\sigma \succ V) \land (\forall m \in \text{dom}(\Omega) \exists n \in \text{dom}(\alpha), \alpha(n) \succ \Omega(m)) \land (\tau \succ T \land r \succ R \land e \succ E) \)

**Definition 33 (Abstraction between powersets).** Given an abstract value \( a \), a concrete value \( c \), \( p \in \varphi(\text{AState} \times a) \) and \( q \in \varphi(\text{CState} \times c) \), \( p \succ q \) if

\( \forall \rho \in q \exists \eta \in p \), \( \eta \succ \rho \) \land \( \rho \vdash a \succ c \)

8.2 Morphism

First, we observe that the bind operation preserves morphisms of the functions

**Definition 34.** A pair of functions \( \langle F, F' \rangle \) shows a morphism over \( \succ \) if the following condition is met

\( a \succ b \implies F'(a) \succ F(b) \)

**Theorem 35.** If \( \langle F, F' \rangle \) and \( \langle G, G' \rangle \) are pairs of functions that show a morphism over \( \succ \), then \( \langle F \succ G, F' \succ G' \rangle \) is also such pair.

Proof. \( \forall \rho \in \text{CState}, \) let \( p = F(\rho), p' = F \succ G(\rho), \eta \in \text{AState} \) such that \( \eta \succ \rho, q = F'(\eta), q' = F' \succ G'(\eta) \).

Then, \( \forall p_1 \in p \),

1. when \( \text{esc}(p_1) \) is true, then the morphism and \( \succ \) relation imply that \( \exists q_1 \in q, \text{esc}(q_1) \).

Given the definition of \( \succ \) operation, \( p_1 \) and \( q_1 \) are also elements of \( p' \) and \( q' \).

2. otherwise, morphism of \( \langle G, G' \rangle \) gives that \( \forall p_2 \in G(p_1) \exists q_2 \in G'(q_1) \) such that \( q_2 \succ p_2 \).

Such entities also exist in \( q' \).

Finally, we observe that operations resulting from abstract functions and their corresponding concrete functions form such a morphism over \( \succ \). We omit function-by-function proof, as such proof would be a mechanical exercise.

9 Conclusion and future direction

As we have noted in the introduction, this work is a snapshot of an ongoing dialogue between theory and practice, positively informing each other to gradually move towards a better theorisation (and practical implementation) of the difficult task of analysing dynamic languages. We have sought to modularise the theoretical framework so that we can take an incremental approach. We anticipate that, as a result of having such a theory, adding new features to the current model language while maintaining formality and rigour will be considerably less laborious than to invent a new incarnation of a more feature-complete model language and produce theory for it.
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A Haskell Implementation

A.1 Syntax parser

Creates an augmented syntax tree with unique id numbers.

```
{%
module Sdtl where
import Data.Char
import Control.Monad.State
import Data.Map
%
%name sdttl
%tokentype { Token }
%
token
  if   {TokenIf}
  else {TokenElse}
  while {TokenWhile}
  output {TokenOutput}
  input {TokenInput}
  new  {TokenNew}
  this {TokenThis}
  function {TokenFunction}
  return {TokenReturn}
  true {TokenTrue}
  false {TokenFalse}
  global {TokenGlobal}
  try  {TokenTry}
  catch {TokenCatch}
  throw {TokenThrow}
  id   {TokenId $$}
  '='  {TokenAsg}
  '+'  {TokenPlus}
  '-'  {TokenMinus}
  '*'  {TokenMult}
  '/'  {TokenDiv}
  gt   {TokenGreaterThan}
  lt   {TokenLessThan}
  eq   {TokenEqual}
  '('  {TokenLP}
  ')'  {TokenRP}
  '{'  {TokenLC}
  '}'  {TokenRC}
  ';'  {TokenScol}
  ','  {TokenComma}
  number {TokenNumber $$}
%
%nonassoc gt lt eq
%left '+' '-',
%left '*', '/'
%
%monad {P}
%
%
Program :: { SdtlProgram }
Program : Stmts (% state $ \(s0, m0)\) -> (SdtlProgram $1 m0, (s0, m0))

Stmts :: { AStmt }
Stmts : Stmts AStmt (% augStmt $ Stmts $1 $2)
   | { % augStmt Empty }
```
AStmt :: { AStmt }
AStmt : Stmt (% augStmt $1)

Cblock :: { AStmt }
Cblock : '{' Stmts '}' { $2 }
| Stmt (% augStmt $1)

Stmt :: { Stmt }
Stmt : AExpr ' ; ' { Expr $1 }
| Lexpr ' = ' AExpr ' ; ' { Asg $1 $3 }
| if ' ( ' AExpr ' ) ' Cblock else Cblock { Ite $3 $5 $7 }
| if ' ( ' AExpr ' ) ' Cblock { If $3 $5 }
| while ' ( ' AExpr ' ) ' Cblock { While $3 $5 }
| output AExpr ' ; ' { Output $2 }
| return AExpr ' ; ' { Return $2 }
| function id ' ( ' Idlist ' ) ' Cblock { Function $2 $4 $6 }
| try Cblock catch ' ( ' id ' ) ' Cblock { TryCatch $2 $5 $7 }
| throw AExpr ' ; ' { Throw $2 }

Params :: {[AExpr]}
Params : {[]}
| AExpr ([$1])
| AExpr ', ' Params ($1:$3)

Idlist :: {[String]}
Idlist : {[]}
| id ([String])
| id ', ' Idlist ($1:$3)

AExpr :: {AExpr}
AExpr : Expr (% augExpr $1)

Expr :: Lexpr (Lexpr $1)
| true {Boolean True}
| false {Boolean False}
| number {Number $1}
| AExpr ' + ' AExpr {Binop Plus $1 $3}
| AExpr ' - ' AExpr {Binop Minus $1 $3}
| AExpr ' * ' AExpr {Binop Mult $1 $3}
| AExpr ' / ' AExpr {Binop Div $1 $3}
| AExpr gt AExpr {Binop GreaterThan $1 $3}
| AExpr lt AExpr {Binop LessThan $1 $3}
| AExpr eq AExpr {Binop Equal $1 $3}
| input {Input}
| new AExpr {New $2}
| Lexpr ' ( ' Params ' ) ' {Call $1 $3}
| '(' Expr ')' {($2)}
| this {This}
| global {Global}

Lexpr :: (Lexpr)
Lexpr : (' Lexpr ')' {($2)}
| AExpr '. ' id {Ref $1 $3}
| id {Id $1}

{ type Sid = Int
type Eid = Int

data SdtlProgram = SdtlProgram AStmt (Map Int AStmt) deriving (Show)

data Lexpr =
| Ref AExpr String
| Id String
deriving Show
\begin{verbatim}
data Op = Plus | Minus | Mult | Div | GreaterThan | LessThan | Equal
    deriving Show

data AExpr = AExpr Expr Eid deriving Show

data Expr =
    Lexpr Lexpr
  | Boolean Bool
  | Number Int
  | Binop Op AExpr AExpr
  | Input
  | New AExpr
  | Call Lexpr [AExpr]
  | This
  | Global
deriving Show

data AStmt =
    AStmt Stmt Sid
deriving Show

data Stmt =
    Asg Lexpr AExpr
  | ITE AExpr AStmt AStmt
  | If AExpr AStmt
  | While AExpr AStmt
  | Output AExpr
  | Expr AExpr
  | Return AExpr
  | Function String [String] AStmt
  | TryCatch AStmt String AStmt
  | Empty
  | Stmts AStmt AStmt
  | Throw AExpr
deriving Show

data Token =
    TokenIf
  | TokenElse
  | TokenWhile
  | TokenInput
  | TokenOutput
  | TokenNew
  | TokenThis
  | TokenFunction
  | TokenReturn
  | TokenTrue
  | TokenFalse
  | TokenGlobal
  | TokenId String
  | TokenAsg
  | TokenPlus
  | TokenMinus
  | TokenMult
  | TokenDiv
  | TokenGreaterThan
  | TokenLessThan
  | TokenEqual
  | TokenLP
  | TokenRP
  | TokenLC
  | TokenRC
  | TokenScol
  | TokenDot
  | TokenComma
\end{verbatim}
type P = State (Int, Map Int AStmt)

augStmt :: Stmt -> P AStmt
augStmt s = state $ \(s0, m0) -> let
    news = AStmt s s0 in
    (news, (s0+1, insert s0 news m0))

augExpr :: Expr -> P AExpr
augExpr e = state $ \(s0, m0) ->
    (AExpr e s0, (s0+1, m0))

lexer :: String -> [Token]
lexer s = lexer1 s

lexer1 :: String -> [Token]
lexer1 [] = []
lexer1 (c:cs)
    | isSpace c = lexer1 cs
    | c == '\'t' = lexer1 cs
    | c == '\'r' = lexer1 cs
    | c == '\'n' = lexer1 cs
    | isAlpha c = lexVar (c:cs)
    | isDigit c = lexNum (c:cs)
    | c == '-' = lexer1 (dropWhile (=) (\'n\') cs)

lexer1 ('=':cs) =
    case cs of
        ('=':ss) -> TokenEqual : lexer1 ss
        _ -> TokenAsg : lexer1 cs

lexer1 ('*':cs) = TokenMult : lexer1 cs
lexer1 ('-':cs) = TokenMinus : lexer1 cs
lexer1 ('(':cs) = TokenLP : lexer1 cs
lexer1 (')':cs) = TokenRP : lexer1 cs
lexer1 ('{':cs) = TokenLC : lexer1 cs
lexer1 ('}':cs) = TokenRC : lexer1 cs
lexer1 ('.':cs) = TokenDot : lexer1 cs
lexer1 (';':cs) = TokenScol : lexer1 cs
lexer1 ('>':cs) = TokenGreaterThan : lexer1 cs
lexer1 ('<':cs) = TokenLessThan : lexer1 cs
lexer1 (',':cs) = TokenComma : lexer1 cs
lexer1 ('\':cs) = TokenEScape : lexer1 cs

lexer1 (c:cs) =
    case span isAlphaNum cs of
        ("if", rest) -> TokenIf : lexer1 rest
        ("else", rest) -> TokenElse : lexer1 rest
        ("while", rest) -> TokenWhile : lexer1 rest
        ("input", rest) -> TokenInput : lexer1 rest
        ("output", rest) -> TokenOutput : lexer1 rest
        ("new", rest) -> TokenNew : lexer1 rest
        ("this", rest) -> TokenThis : lexer1 rest
        ("function", rest) -> TokenFunction : lexer1 rest
        ("return", rest) -> TokenReturn : lexer1 rest

lexNum cs = TokenNumber (read num) : lexer1 rest
    where (num, rest) = span isDigit cs

lexNegativeNum cs = TokenNumber (-1 * (read num)) : lexer1 rest
    where (num, rest) = span isDigit cs

lexVar cs =
    case span isAlphaNum cs of
        ("if", rest) -> TokenIf : lexer1 rest
        ("else", rest) -> TokenElse : lexer1 rest
        ("while", rest) -> TokenWhile : lexer1 rest
        ("input", rest) -> TokenInput : lexer1 rest
        ("output", rest) -> TokenOutput : lexer1 rest
        ("new", rest) -> TokenNew : lexer1 rest
        ("this", rest) -> TokenThis : lexer1 rest
        ("function", rest) -> TokenFunction : lexer1 rest
        ("return", rest) -> TokenReturn : lexer1 rest
A.2 Semantic functions

{-# LANGUAGE GADTs, MultiParamTypeClasses #-}
module SemanticFunctions where
import Data.Maybe as Maybe
import Sdtl
import Data.Map as Map
import Debug.Trace

class (Eq v) => Value v where
  conval :: Expr -> v
  getSid :: v -> Sid
  binop :: Op -> v -> v -> v

class (Eq s) => State s where
  fundecl :: String -> [String] -> Sid -> s -> [s]
  esc :: s -> Bool

class (State s, Value v) => StateValue s v where
  ret :: v -> s -> [s]
  cond :: (Eq a) => v -> (M s v a) -> (M s v a) -> (M s v a)
  asg :: String -> v -> s -> [s]
  dooutput :: v -> s -> [s]
  getinput :: s -> [(s,v)]
  val :: s -> String -> v
  enter :: s -> Sid -> [v] -> v -> [String] -> s
  leave :: s -> s -> (s, Maybe v)
  fix :: ((M s v ())) -> (M s v ()) -> M s v ()
  apply :: v -> [v] -> v -> Eid -> (M s v v)
  set :: v -> String -> v -> s -> [s]
  get :: v -> String -> s -> v
  getglobal :: s -> v
  getthis :: s -> v
  newobj :: Eid -> (M s v v)
  runthrow :: v -> s -> [s]
  runcatch :: (M s v ()) -> (M s v ())

data M s v a where
  M :: (State s, Value v) => ((Sid -> (ASTmt, (M s v ())))) -> s -> [(s,Maybe a)] -> M s v a

--data (State s) => M s a = M (F s -> s -> [(s,Maybe a)])

instance (StateValue s v) => Monad (M s v) where
  (M t) >>= u =
    M $ \
      \f -> \s0 ->
      let ss = t f s0
        in
        concat [ if esc s1 then [(s1, Nothing)]
          else let
            Just a' = a
            (M us) = u a'
            in us f s1
          | (s1, a) <- ss ]

happyError :: [Token] -> a
happyError _ = error "Parse error"
return a = M $ \{ f -> s0 -> \}

>>> (StateValue s v) >> (M s v a) >> (Maybe a -> (M s v a)) >> (M s v a)
(M t) >> u = M $ \{ f -> s0 -> \}
let ss = t f s0 in concat [ let (M us) = u a in us f s1 | (s1, a) <- ss ]

{-# Takes a state transformer and transforms into a monad #-}

liftS :: (StateValue s v) -> s -> [s] -> M s v ()
liftS strans = M $ \{ f -> s0 -> \}
let ss = t f s0 in concat [ let (M us) = u a in us f s1 | (s1, a) <- ss ]

liftV :: (State s, Value v) -> s -> M s v v
liftV stov = M $ \{ f -> s -> \}
[ (s, Just (stov s)) ]

getState :: (State s, Value v) -> M s v s
getState = M $ \{ f -> s0 -> \}
[ (s0, Just s0) ]

sstmt :: (StateValue s v) -> AStmt -> M s v ()
sstmt (AStmt Empty sid) = return ()
sstmt (AStmt (Stmts s1 s2) sid) = do
  traceShow s1 $ sstmt s1
  traceShow s2 $ sstmt s2
sstmt (AStmt (Expr e1) sid) = do
  sexpr e1
  return ()
sstmt (AStmt (Return e1) sid) = do
  v <- sexpr e1
  liftS $ ret v
sstmt (AStmt (If e1 s1 s2) sid) = do
  v <- sexpr e1
  cond v (sstmt s1) (return ())
sstmt (AStmt (Ite e1 s1 s2) sid) = do
  v <- sexpr e1
  cond v (sstmt s1) (sstmt s2)
sstmt (AStmt (Asg (Id i1) e1) sid) = do
  v <- sexpr e1
  liftS $ asg i1 v
sstmt (AStmt (While e1 s1) sid) = fix $ \x -> do
  v <- sexpr e1
  cond v (sstmt s1 >> x) (return ())
sstmt (AStmt (Output e1) sid) = do
  v <- sexpr e1
  liftS $ dooutput v
sstmt (AStmt (Function id1 ids (AStmt _ s1)) sid) = liftS $ fundecl id1 ids sid
sstmt (AStmt (Asg (Ref e1 id1) e2) sid) = do
  r <- sexpr e1
  v <- sexpr e2
  liftS $ set r id1 v
sstmt (AStmt (TryCatch s1 id1 s2) sid) =
  (sstmt s1) >>= \_ -> runcatch id1 (sstmt s2)
sstmt (AStmt (Throw e1) sid) = do
  v1 <- sexpr e1
  liftS $ runthrow v1

sexpr :: (StateValue s v) -> AExpr -> M s v v

sexpr (AExpr (Number n) eid) = return $ conval (Number n)
sexpr (AExpr (Boolean b) eid) = return $ conval (Boolean b)
sexpr (AExpr (Lexpr l1) eid) = slexpr l1
sexpr (AExpr Input eid) = M $ \{ f -> s0 -> \}
(\(s, Just v) | (s, v) <- getinput s0]
sexpr (AExpr (Call (Ref e1 id1) es) eid) = do
  t <- sexpr e1
  n <- liftV $ get t id1
  p <- evalParams es []
  apply n p t eid
sexpr (AExpr (Call l1 es) eid) = do
  n <- slexpr l1
  p <- evalParams es []
  t <- liftV $ getthis

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apply n p t eid
sexpr (AExpr (Binop op e1 e2) eid) = do
  v1 <- sexpr e1
  v2 <- sexpr e2
  return $ binop op v1 v2
sexpr (AExpr This eid) = liftV $ getthis
sexpr (AExpr Global eid) = liftV $ getglobal
sexpr (AExpr (New (AExpr (Call l1 es) eid1)) eid) = do
  n <- slexpr l1
  p <- evalParams es []
  m <- newobj eid
  apply n p m eid
return m
evalParams :: (StateValue s v) => [AExpr] -> [v] -> M s v [v]
evalParams [] vs = return vs
evalParams (e:es) vs = do
  v <- sexpr e
  evalParams es (vs ++ [v])
slexpr :: (StateValue s v) => Lexpr -> M s v v
slexpr (Id id1) = do
  s0 <- getState
  return $ val s0 id1
slexpr (Ref e1 id1) = do
  v <- sexpr e1
  liftV $ get v id1
--putState :: (StateValue s v) => s -> (M s v (v))
--putState s = M $ \f -> \s0 -> [(s,Just ())]  
--getFunc :: (StateValue s v) => M s v (Sid -> (M s v (v)))
--getFunc = M $ \f -> \s -> [(s, Just f)]
call :: (StateValue s v) => Sid -> [v] -> v -> (M s v v)
call n p t = M $ \f -> \s0 ->
  [leave s0 s | (s,_) <- let (AStmt (Function _ ids _) _, M x) = (f n) in x f (enter s0 n p t ids) ]
runMonad :: (StateValue s v) => AStmt -> (Sid -> (AStmt, (M s v ()))) -> s -> v -> [s]
runMonad mainStmt = \f -> \s0 -> \_ -> let M t = sstmt mainStmt \_ (Prelude.map) fst \_ t s0
params :: (Sid -> (AStmt, (M s v ()))) -> Sid -> [String]
params f sid = let (\_ (AStmt (Function _ ids _) _), _ ) = (f sid) in ids
arity :: (Sid -> (AStmt, (M s v ()))) -> Sid -> Int
arity f sid = length (params f sid)

A.3 Concrete interpretation

{-# LANGUAGE MagicHash, UnboxedTuples, GADTs, MultiParamTypeClasses #-}
module Main (main, runString) where
import SemanticFunctions
import Sdtl
import Data.Map
import qualified Control.Exception as E
import GHC.IO
import GHC.Exts
import System.Environment
import qualified Control.Monad.State as S
import System.IO.Unsafe
import Data.ByteString.Internal
import Debug.Trace
{-
fromState :: State# RealWorld -> RealWorld
fromState = unsafeCoerce#}
toState :: RealWorld -> State# RealWorld
toState = unsafeCoerce#

execIO :: IO a -> RealWorld -> (RealWorld, a)
execIO (IO k) = \w -> case k (toState w) of (# s', x #) -> (fromState s', x)

{-}

data CValue =
  Intval Int
  | Boolval Bool
  | FunPointer Sid [CValue]
  | Object Int
deriving (Show, Eq)

instance Value CValue where
  conval (Boolean True) = Boolval True
  conval (Boolean False) = Boolval False
  conval (Number i) = Intval i
  getSid (FunPointer n []) = n
  binop Plus (Intval i1) (Intval i2) = Intval $ i1 + i2
  binop Minus (Intval i1) (Intval i2) = Intval $ i1 - i2
  binop Mult (Intval i1) (Intval i2) = Intval $ i1 * i2
  binop Div (Intval i1) (Intval i2) = Intval $ i1 `div` i2
  binop GreaterThan (Intval i1) (Intval i2) = Boolval $ i1 > i2
  binop LessThan (Intval i1) (Intval i2) = Boolval $ i1 < i2
  binop Equal (Intval i1) (Intval i2) = Boolval $ i1 == i2

data CState = CState {
  env :: Map String CValue,
  --io :: RealWorld,
  objmem :: Map Int (Map String CValue),
  returning :: Maybe CValue,
  exception :: Maybe CValue,
  this :: CValue
} deriving (Show)

instance Eq CState where
  (==) _ _ = False

instance State (CState) where
  esc (CState {returning = v, exception = e}) = case v of
    Nothing -> case e of
      Nothing -> False
      _ -> True
    _ -> True

  --fundecl :: String -> [String] -> Sid -> s -> [s]
fundecl fnname fnparams n s =
  [s (env = insert fnname (FunPointer n []) (env s))]

instance StateValue CState CValue where
  ret v s = trace (show s) $ [s {returning = Just v}]
  --cond :: (CValue v) => v -> (M s v a) -> (M s v a) -> (M s v a)
  cond (Boolval True) t u = t
  cond _ t u = u

  --asg :: (CValue v) => String -> v -> s -> [s]
  asg i v s = [s {env = (insert i v (env s))}]
  --dooutput :: (CValue v) => v -> s -> [s]
dooutput (Intval n) s =
    --let (io1, _) = (execIO $ putStrLn (show n)) (io s)
    --in [s {io = io1}]
    let v = ($) inlinePerformIO $ putStrLn (show n) in
      if v == () then [s] else [s]

  --getinput :: (CValue v) => s -> [(s,v)]
getinput s =
  --let (io1, v) = (execIO $ getLine) (io s)
  --in [(s {io = io1}, Intval (read v))]
  [(s, Intval (read (inlinePerformIO $ getLine)))]

--val :: (CValue v) => s -> String -> v
val s i = (env s) ! i

--enter :: (CValue v) => s -> Sid -> [v] -> s
enter s n vs t ids =
  s {env = fromList (zip ids vs), this = t}

--leave :: (CValue v) => s -> s -> (s, Maybe v)
leave s0 s =
  trace (show s) $ (CState {env = (env s0), (-io = (io s), -)}
    returning = Nothing,
    exception = (exception s),
    objmem = (objmem s),
    this = (this s0)), (returning s))

fix x = x (fix x)

apply (FunPointer n c) p t _ =
  M $ \f -> \s ->
    if (arity f n) == (length (c ++ p))
    then let (M tcall) = call n (c ++ p) t in (tcall f s)
    else [(s, Just (FunPointer n (c++p)))]

set (Object n) id1 v = \s ->
  let oldo = (objmem s) ! n
  in [s {objmem = insert n (insert id1 v oldo) (objmem s)}]

get (Object n) id1 = \s ->
  (objmem s) ! n ! id1

getglobal = \s -> (Object 0)

getthis = \s -> (this s)

newobj _ = M $ f -> \s ->
  let newn = size (objmem s)
  in traceShow (Object newn) $
    [(s {objmem = insert newn (fromList []) (objmem s)}, Just (Object newn))]

--runthrow :: v -> s -> [s]
runthrow v = \s -> [s {exception=Just v}]

--runcatch :: String -> (M s v ()) -> (M s v ()
runcatch id1 (M u) = M $ f -> \s0 ->
  case (exception s0) of
    Nothing -> [(s0, Just ())]
    Just v -> (u f (s0 {env = insert id1 v (env s0), exception=Nothing}))

{-
getWorld :: IO RealWorld
getWorld = IO (
  \s -> (# s, fromState s #))

putWorld :: RealWorld -> IO ()
putWorld s' = IO (\_ -> (# toState s', () #))
-}

fun :: (StateValue s v) -> (Map Int AStmt) -> Int -> (AStmt, (M s v ()))
fun sidMap sid = trace ("call\_\_sid\_\_" ++ (show sid)) $
  let Just astmt = Data.Map.lookup sid sidMap
      astmt (Function _ _ fbody) _ = astmt in (astmt, sstmt fbody)

runConcrete :: SdtlProgram -> IO ()
runConcrete (SdtlProgram mainStmt sdtlMap) = do
  --putStrLn $ show mainSid
  putStrLn $ show sdtlMap
let [endState] =
  runMonad mainStmt
  (fun sdtlMap)
  (CState {env = fromList [], {-io = w, -}} returning = Nothing, exception = Nothing,
           objmem = fromList [(0, fromList [])], this = (Object 0))
  (Intval 0)
in do
  --putWorld (io endState)
  putStrLn $ show (env endState)

runString :: String -> IO()
runString program = runConcrete (fst (S.runState (sdtl lexer program)) (1, fromList []))

main = do
  (sdtlfile:_) <- getArgs
  program <- readFile sdtlfile
  runString program

A.4 Abstract interpretation

{-# LANGUAGE MultiParamTypeClasses #-}
module Main (main, runString) where

import SemanticFunctions
import Sdtl
import Data.Map
import qualified Control.Exception as E
import System.Environment
import Data.ByteString.Internal
import Debug.Trace
import qualified Control.Monad.State as S
import qualified Data.List as L

data AValue = Intval
  | Boolval
  | FunPointer Sid Int Eid
  | Object Int
  deriving (Show, Eq, Ord)

instance Value AValue where
  conval (Boolean True) = Boolval
  conval (Boolean False) = Boolval
  conval (Number i) = Intval
  getSid (FunPointer n _) = n
  binop Plus (Intval) (Intval) = Intval
  binop Minus (Intval) (Intval) = Intval
  binop Mult (Intval) (Intval) = Intval
  binop Div (Intval) (Intval) = Intval
  binop GreaterThan (Intval) (Intval) = Boolval
  binop LessThan (Intval) (Intval) = Boolval
  binop Equal (Intval) (Intval) = Boolval

data AState = AState{
  env :: Map String AValue,
  returning :: Maybe AValue,
  exception :: Maybe AValue,
  objmem :: Map Eid (Map String AValue),
  this :: AValue,
  curried :: Map (Sid, Int, Eid) [AValue]
} deriving (Show, Eq, Ord)

instance State (AState) where
  esc (AState {returning = v, exception = e}) = case v of
Nothing -> case e of
  Nothing -> False
  _ -> True
  _ -> True

--fundecl :: String -> [String] -> Sid -> s -> [s]
fundecl fnname fnparams n s =
  [s {env = insert fnname (FunPointer n 0 0) (env s) }]

y x = x (y x)
mapToMonad m = M $ \f -> \s -> if member s m then (m ! s) else [(s, Just ())]

instance StateValue AState AValue where
  fix x = M $ \f -> \s ->
    let (fixedMap) =
      (y $ \x0 -> \m0 ->
       let (M t1) = (x (mapToMonad m0))
           newmap = traceShow m0 $
             union m0 $ fromList
             [(s1, t1 f s1) | s1 <- (L.map fst (L.nub $ concat $ elems m0)) ++ [s] ]
      in
        if m0 == newmap then newmap else x0 newmap) (fromList [])
    in
      trace ("final" ++ (show fixedMap)) $
        L.nub $ concat $ elems fixedMap

  ret v s = trace (show s) $ [s {returning = Just v}]

--cond :: (CValue v) => v -> (M s v a) -> (M s v a) -> (M s v a)
cond (Boolval) (M t) (M u) = M $ \f -> \s0 ->
  let s1 = t f s0
      s2 = u f s0
  in L.nub $ s1 ++ s2

--cond _ t u = u

--asg :: (CValue v) => String -> v -> s -> [s]
asg i v s = [s {env = (insert i v (env s))}]

--dooutput :: (CValue v) => v -> s -> [s]
dooutput _ s = [s]

--getinput :: (CValue v) => s -> [(s, v)]
getinput s = [(s, Intval)]

--val :: (CValue v) => s -> String -> v
val s i = (env s) ! i

--enter :: (CValue v) => s -> Sid -> [v] -> s
enter s n vs t ids = s {env = fromList (zip ids vs), this = t}

--leave :: (CValue v) => s -> s -> (s, Maybe v)
leave s0 s = trace (show s) $
  (AState {env = (env s0), returning = Nothing, exception = (exception s),
           curried = (curried s),
           objmem = (objmem s),
           this = (this s0)}, (returning s))

apply (FunPointer n c ce) p t e =
  M $ \f -> \s ->
    let cv if c > 0 then ((curried s) ! (n, c, ce)) else [] in
      if (arity f n) == c + (length p)
        then let (M tcall) = (call n (cv ++ p) t) in (tcall f s)
          else
            [(s {curried = insert (n, c + length p), e} (cv ++ p) (curried s))
              , Just (FunPointer n (c + (length p)) e)]
set (Object n) id1 v = \s ->
  let oldo = (objmem s) ! n
  in [s {objmem = insert n (insert id1 v oldo) (objmem s)}]

get (Object n) id1 = \s -> (objmem s) ! n ! id1

global = \s -> (Object O)
getthis = \s -> (this s)
newobj eid = M $ \f -> \s ->
  [s {objmem = insert eid (fromList []) (objmem s)}, Just (Object eid)]

runthrow :: v -> s -> [s]
runthrow v = \s -> [s {exception=Just v}]

runcatch :: String -> (M s v ()) -> (M s v ()
  runcatch id1 (M u) = M $ \f -> s0 ->
    case (exception s0) of
      Nothing -> [(s0, Just ())] 
      Just v -> (u $ (s0 {env = insert id1 v (env s0), exception=Nothing}))

getWorld :: IO RealWorld
getWorld = IO (
s -> (# s, fromState s #))

putWorld :: RealWorld -> IO ()
putWorld s' = IO (# toState s', () #)

fun :: (Map Int AStmt) -> (Map (Int, AState) [(AState, Maybe ())]) -> Int
  -> (AStmt, (M AState AValue ()))
  fun sidMap fapprox sid = trace ("call\_<sid>" ++ (show sid)) $ 
    let Just astmt = Data.Map.lookup sid sidMap
        astmt = traceShow astmt $ astmt in
    (astmt, 
     (M $ \f -> \s -> 
      if member (sid, s) fapprox then astmt ! (sid, s) 
      else 
        (y $ \localx -> \prevresult -> 
         let
           newapprox = (insert (sid, s) prevresult fapprox)
           M t = (astmt fbody)
           result = t (fun sidMap newapprox) s
         in 
           if result == prevresult then result 
           else localx result) [])

runAbstract :: SdtlProgram -> IO ()
runAbstract (SdtlProgram mainStmt sdtlMap) = do
  putStrLn $ show mainSid
  putStrLn $ show sdtlMap
  w <- getWorld
  let endStates =
    runMonad mainStmt
    SdtlProgram (fun sdtlMap (fromList []))
    (AState (env = fromList [],
      returning = Nothing,
      exception = Nothing,
      curried = fromList [],
      objmem = fromList [[0, fromList []]],
      this = (Object 0)))
    (Intval)
  in do
    --putWorld (io endState)
    putStrLn "Final States" 
    putStrLn $ show endStates

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runString :: String -> IO()
runString program = runAbstract (fst (S.runState (sdtl (lexer program)) (1, fromList [])))

main = do
       (sdtlfile:_ _) <- getArgs
       program <- readFile sdtlfile
       runString program