Classical and quantum polyhedra: A fusion graph algebra point of view

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Abstract

Representation theory, for the classical binary polyhedral groups $\tilde{T}$, $\tilde{C}$ and $\tilde{I}$, is encoded by the affine Dynkin diagrams $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ (McKay correspondence). The quantum versions of these classical geometries are associated with representation theories described by the usual Dynkin diagrams $E_6$, $E_7$ and $E_8$. The purpose of these notes is to compare several chosen aspects of the classical and quantum geometries by using the study of spaces of paths and spaces of essential paths (Ocneanu theory) on these diagrams. To keep the size of this contribution small enough, most of our discussion will be limited to the cases of diagrams $E_6$ and $E_6^{(1)}$, i.e., to the classical and quantum tetrahedra. We shall in particular interpret the $A_{11}$ labelling of $E_6$ vertices as a quantum analogue of the usual decomposition of spaces of sections for vector bundles above homogeneous spaces. We also show how to recover Klein invariants of polyhedra by paths algebra techniques and discuss their quantum generalizations.
1 Introduction

One way to understand the classification of modular invariant partition functions, for instance the \( ADE \) classification of \( SU(2) \) conformal theories \([1], [7]\) or its generalizations \([8], [7]\), should be understood, following A. Ocneanu (many talks since 1995, for instance \([13], [14]\)) through the study of “quantum symmetries” on graphs, and in particular, on Dynkin \( ADE \) diagrams. In turn, this study leads, in each case, to new kinds of partition functions which are not modular invariant but are nevertheless quite remarkable: the concept of “torus structure” of \( ADE \) graphs is due to A.Ocneanu (unpublished), the corresponding “twisted partition functions” have been recently discussed by \([19]\), the expressions of the “toric matrices” for the quantum tetrahedron (\( E_6 \) model) have been presented in \([8]\) and explicit calculations for the other models, using the techniques of this last paper should appear in \([3]\). The needed mathematical material is unfortunately not really standard and often not even available in published form. It happens, however, that many of these algebraic (“quantum”) manipulations can be seen as a quantum analogue of finite group constructions. The purpose of these notes is to give a fresh look to the old-fasion representation theory of groups of symmetries of regular polyhedra, and to do it in such a way that generalization to the quantum case becomes (almost) straightforward. However, we shall not cover all the way, from \( ADE \) diagrams to Ocneanu quantum symmetries, even when discussing the classical analogue of these constructions: in particular the study of Racah-Wigner bi-algebras for Platonic groups will be left aside (this should be done within \([4]\)) and we shall not even introduce the classical analogue of Ocneanu graphs … In particular, the twisted partition functions that we can associate to the different vertices of the Ocneanu graph will not be described. We hope nevertheless that this set of notes will provide a useful introduction to this fascinating subject.

The major part of what is going to be explained below is certainly known, at least in some circles. We believe, however, that our geometrical interpretation, for the “essential” labelling of vertices of exceptional Dynkin diagrams (like \( E_6 \)) by vertices belonging to an appropriate \( A_N \) graphs (which is \( A_{11} \) in the case of \( E_6 \)), as the quantum analogue of the decomposition of a space of sections of a homogeneous vector bundle, is probably new, and may be of interest for the expert. Also, explicit calculations of projectors decomposing the representations \([2]^p\) of binary polyhedral groups into irreps have been probably carried out by several group theorists, chemists, or solid state physics practitioners, but we do not think that a description of the method using paths on graphs together with the data provided by spectral decomposition of the (quantum) \( R \) matrices was given before. One should be aware, however, that in several cases, like for instance the calculation of Klein invariants, the results themselves have been known for more than a century!
2 Classical and quantum Platonic bodies

As it is well known, geometry of classical Platonic bodies is encoded by their symmetry groups that we shall call respectively $T$ (for the tetrahedron), $C$ (for the cube and its dual, the octahedron) and $I$ (for the icosahedron and its dual, the dodecahedron). These are subgroups of $SO(3)$. By adding reflections, we may also consider the “full” polyhedral groups, which are subgroups of $O(3)$, but the objects of interest, for us, are the so-called binary polyhedral groups $\hat{T}, \hat{C}, \hat{I}$, that are non abelian subgroups of $SU(2)$, the two-fold cover of $SO(3)$. They are defined as pre-image of the corresponding $SO(3)$ subgroups by using the sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow SU(2) \rightarrow SO(3) \rightarrow 0$$

We concentrate our attention mostly on the particular example given by the binary tetrahedral group (of order $2 \times 12 = 24$) since we want to compare it, or better its representation theory, with a kind of quantum analogue. One reason to limit our study to $\hat{T}$ is lack of space, the other is that the quantum geometry corresponding to $C$ is somehow much more difficult to study that the one corresponding to $T$ and $I$.

2.1 Classical geometry

2.1.1 Several realisations of the group $\hat{T}$

Elementary geometric considerations show that $T$ itself is isomorphic with the group of even permutations on four elements (label the vertices of the tetrahedron by $(1, 2, 3, 4)$). Therefore

$$\# \hat{T} = 2 \times \# T = 2 \times \frac{4!}{2} = 2 \times 12 = 24$$

Since $\hat{T}$ is, by definition, a finite subgroup of $Spin(3) \simeq SU(2)$, we can express all its elements in terms of $Cliff(\mathbb{R}^3)$, i.e., in terms of the “gamma matrices” of $\mathbb{R}^3$, namely the Pauli matrices $\tau_i$. Setting $\gamma_x = -\tau_2$, $\gamma_y = \tau_1$ and $\gamma_z = \tau_3$, we get $\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij}$ with

$$\gamma_x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \gamma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easy to see that $\hat{T}$ is generated, as a group, by

$$t = \gamma_x \gamma_y \quad s = \frac{1}{2}(\gamma_x \gamma_y + \gamma_y \gamma_z + \gamma_z \gamma_x - 1)$$

Notice that $t^2 = -1, t^4 = -t, t^4 = 1$ and $s^2 = -(s + 1)$, so that $s^3 = 1$. Moreover $(st)^3 = 1$. Explicitly:

$$s = \begin{pmatrix} (1 + i)/2 & (1 + i)/2 \\ (1 + i)/2 & -(1 + i)/2 \end{pmatrix} \quad t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
The reader may be interested in knowing that $\tilde{T}$ (resp. $\tilde{C}$ and $\tilde{I}$) is isomorphic with the group $(2,3,n)$ (Threlfall notation), when $n = 3$ (resp. $n = 4$ and $n = 5$). This notation refers, by definition, to the group generated by two elements $A$ and $B$, with relations $A^3 = B^n = (AB)^2$. These groups, when $n = 3$ or $n = 5$ are also isomorphic with the groups $SL(2, F_n)$ or $SL(2, F_3)$, here $F_n$ is the field with $n$ elements; there is no such isomorphism when $n = 4$. It may be also nice to remember that the “extended” triplets $\langle 3,3,3 \rangle$, $\langle 2,4,4 \rangle$ and $\langle 2,3,6 \rangle$ are the only positive integer solutions of the equation:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

These extended triplets encode the affine Dynkin diagrams of type $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ (lengths of the three legs counted from the triple point). Substracting 1 from a maximal element of these extended triplets give the triplets $\langle 2,3,3 \rangle$, $\langle 2,3,4 \rangle$ and $\langle 2,3,5 \rangle$ which satisfy the inequality\footnote{The other solutions of this inequality, if we exclude $a = 1, b \neq c$, are $\langle 1,n,n \rangle$ and $\langle 2,2,n \rangle$, corresponding to the $A_{2n-1}$ and $D_{n+2}$ Dynkin diagrams.}

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$$

and encode the usual Dynkin diagrams for exceptional Lie groups. The same triplets also characterize the binary polyhedral groups since they give the relations obeyed by their generators in Threlfall notation.

### 2.1.2 Representations of the group $T$

One way to get the representations of $T$ is to remember that $T \simeq A_4 \subset S_4$ and use the fact that representation theory of permutation groups (like $S_4$) is well known (use Young tableaux, for instance).

$T \simeq A_4$ has four irreducible inequivalent representations of respective dimensions $1, 1, 1, 3$. We call them $\sigma_1$, $\sigma_1'$, $\sigma_1''$, $\sigma_3$. We check that the dimensions divide 12 (as they should) and that $1^2 + 1^2 + 1^2 + 3^2 = 12$, of course.

### 2.1.3 Representations of the group $\tilde{T}$

The previous irreps are also irreducible representations for the corresponding binary group $\tilde{T} \subset SU(2)$, but the latter also possesses other irreducible representations. Altogether, $\tilde{T}$ has seven irreducible inequivalent representations; their dimensions are

$$1, 1, 1, 3, 2, 2, 2.$$
2.1.4 Tensor products of representations and McKay correspondence

In the case of $SU(2)$, it is well known that the tensor product of the representation of spin $1/2$ (i.e., dimension 2) by a representation of spin $j$ (dimension $n = 2j + 1$) is equivalent to the sum of two representations of respective spin $j - 1/2$ and $j + 1/2$. In terms of dimensions, we have $2(2j + 1) = (2j) + (2j + 2)$; in terms of the representations themselves (working up to equivalence) we have:

$$\sigma_2 \otimes \sigma_n = \sigma_{n-1} \oplus \sigma_{n+1}$$

The irreps into which a given representation $\sigma_n$ decomposes, upon tensorial multiplication by the fundamental representation $\sigma_2$ are given by the neighbours of $\sigma_n$ on the following semi-infinite diagram called the $A_\infty$ diagram.

![Diagram A_\infty](image)

Figure 1: The diagram $A_\infty$

Returning to the binary tetrahedral group $\tilde{T}$, we decide to encode in the same way the tensor product of the various irreps by the fundamental (the 2-dimensional). The calculation itself is a simple exercise in finite group theory and we shall not dwell on the matter... The point is that, if we decide to encode the results in terms of a graph with seven vertices (the seven irreps), this graph is nothing else than the Dynkin diagram of the exceptional affine Lie algebra $E_6^{(1)}$.

What comes as a surprise is that, if we perform the same construction with the binary groups of the cube and of the icosahedron, we obtain respectively the Dynkin diagrams of the affine Lie algebras $E_7^{(1)}$ and $E_8^{(1)}$. This observation is known as “McKay correspondence” ([14]).

The diagrams $E_6^{(1)}$ and $E_8^{(1)}$, labelled by the seven (nine) irreducible representations of the binary tetrahedral (icosahedral) group are displayed below.

![Diagrams E_6^{(1)} and E_8^{(1)}](image)

Figure 2: The diagrams $E_6^{(1)}$ and $E_8^{(1)}$

As explained before, this reads, for instance in the case of $E_6^{(1)}$, $\sigma'_1 \otimes \sigma_2 = \sigma'_2$, or $\sigma_2 \otimes \sigma_3 = \sigma_2 \oplus \sigma'_2 \oplus \sigma''_2$. The summands appearing on the right hand side are...
the neighbours, on the graph, of the chosen vertex. Of course, the dimensions should match (so, for instance \(2 \times 3 = 2 + 2 + 2\)).

For the above reason we decide to denote by \(\mathcal{H}_{E_6^{(1)}}\), \(\mathcal{H}_{E_7^{(1)}}\) and \(\mathcal{H}_{E_8^{(1)}}\) the group algebras of these three binary groups. These are co-commutative finite dimensional semi-simple Hopf algebras. The corresponding dual objects (algebras of complex valued functions of these groups) are denoted by \(\mathcal{F}_{E_6^{(1)}}\), \(\mathcal{F}_{E_7^{(1)}}\) and \(\mathcal{F}_{E_8^{(1)}}\).

### 2.1.5 Structure of the Grothendieck ring

We already know how to multiply representations of \(\tilde{T}\) by \(\sigma_1\) (the trivial representation) and by \(\sigma_2\) (this is given by the graph \(E_6^{(1)}\)). In other words, we know the first two rows and the first two columns of the table of multiplication of characters. Another useful identity that one should get first is \(1'' = 1\); this is a consequence of the fact that \(1''\) is one dimensional (so it is either \(1\) or \(1''\)) but it is real, since \(\overline{1''} = 1''\). This data is sufficient to reconstruct the whole table of multiplication. We only have to use associativity of \(\otimes\) and perform the calculations in the ring of virtual representations (hence allowing minus signs at intermediate steps in our calculations.)

Let us for instance compute \(\sigma_3 \otimes \sigma_3\) (we shall just write \(3 \otimes 3\) in the sequel):

\[
3 \otimes 3 = (22 - 1) = 322 - 3 = (32)2 - 3 = (2 + 2' + 2'')2 - 3 = 1 + 3 + 1' + 3 + 1'' + 3 - 3 = 3 + 3 + 1 + 1' + 1'' = (3)_2 + 1 + 1' + 1''
\]

Here, the subindex 2 in \((3)_2\) means that \(3\) appears with multiplicity 2.

After some work, all other entries of the multiplication table can be worked out by simple manipulations analogous to the above calculation of \(3 \otimes 3\) and we obtain the following table (that we shall sometimes call the fusion table); it gives all the coupling constants \(C_{ijk}\) of the Grothendieck ring \(Ch(G) = \mathbb{Z}(Irr(\tilde{T}))\); these constants are defined by \(\sigma_i \otimes \sigma_j = C_{ijk}\sigma_k\).

```
|   | 1 | 1' | 2 | 2' | 1'' | 2'' | 3 |
|---|---|----|---|----|-----|-----|---|
| 1 | 1 | 1' | 2 | 2' | 1'' | 2'' | 3 |
| 2 | 2 | 1' | 1' | 1'' | 1'' | 3   | 22''|
| 1' | 1'' | 1'' | 2'' | 1 | 1' | 2  | 3 |
| 2' | 1 | 1' | 1'' | 1'' | 13 | 3   | 22''|
| 1'' | 1'| 2'' | 1 | 2 | 1' | 2' | 3 |
| 2'' | 1 | 1'' | 13 | 2 | 1' | 2' | 3 |
| 3 | 3 | 22'' | 3 | 22'' | 3 | 22'' | 11''3 |
```

Notice that the final table only involves sums (no minus signs, of course!) of irreducible representations.

We should maybe write this table in the order \((1, 1', 1'', 3; 2, 2', 2'')\) to reflect the fact that the subset \(1, 1', 1'', 3\) (irreps of \(T\)) is stable under tensorial multiplication; this corresponds to the fact that the group \(T\) is a quotient of \(\tilde{T}\).
2.1.6 The fusion matrices \( N_i \)

The (non necessarily symmetric) matrices \( N_i \) are defined by

\[
(N_i)_{jk} = C_{ijk}
\]

Still using the order \( \{1, 1', 1'', 2, 2', 2'', 3\} \), we have:

\[
N_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
N_2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

\[
N_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

It is easy to show that we obtain an isomorphism between the – commutative – ring of characters \( Ch(T) \) (or, equivalently, the ring generated by tensor powers of the irreducible representations) and the ring generated (over the integers \( \mathbb{Z} \)) by the matrices \( N_i \). For instance, we have

\[
\sigma_3 \otimes \sigma_3 = \sigma_3 + \sigma_3 + \sigma_1 + \sigma_1'
\]

and, at the same time,

\[
N_3 \cdot N_3 = 2N_3 + N_1 + N_1' + N_1''
\]

In particular, the Dynkin diagram of \( E_6^{(1)} \), considered as the fusion graph of the fundamental representation of the binary tetrahedral group (Mc Kay correspondence) can also be read in terms of the fusion matrices \( N \):

\[
N_2 \cdot N_1 = N_2 \quad N_2 \cdot N_2 = N_1 + N_3 \quad N_2 \cdot N_3 = N_2 + N_2' + N_2''
\]

\[
N_2 \cdot N_1' = N_2''
\]
2.1.7 From the Grothendieck ring to the character table

The seven commuting $7 \times 7$ matrices $N_i$ can be simultaneously diagonalized with a common similarity matrix $X$ (the rows of which give the eigenvalues of the $N_i$).

$$X = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \omega^2 & \omega^2 & \omega & \omega \\
1 & 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\
2 & 0 & -2 & 1 & -1 & 1 & -1 \\
2 & 0 & -2 & \omega^2 & -\omega^2 & \omega & -\omega \\
2 & 0 & -2 & \omega & -\omega & \omega^2 & -\omega^2 \\
3 & -1 & 3 & 0 & 0 & 0 & 0
\end{pmatrix}$$

A nice observation is that the character table is given by the same matrix $X$, i.e., by the list (properly ordered!) of eigenvalues of the matrices $N_i$: each line corresponds to an irreducible representation and each column to a conjugacy class. The point is that we did not have to work out the conjugacy classes themselves: the structure constants of the Grothendieck ring (which are themselves, in the present case, encoded by the graph $E_6^{(1)}$) provide enough data to reconstruct the whole character table. This not so well-known result is true for any finite group.

2.1.8 Perron-Frobenius data for the graph $E_6^{(1)}$

We want now “reverse” the machine, i.e., forget everything we know about groups $\tilde{T}$ (or $T$) and try to reconstruct as much as we can from the combinatorial data provided by the $E_6^{(1)}$ diagram. The adjacency matrix $\mathcal{G}$ of an oriented graph is a matrix labelled by the vertices of $\mathcal{G}$ whose $(i, j)$ element is equal to $n$ whenever there are $n$ edges from $i$ to $j$. When the graph is not oriented, each edge is considered as carrying both orientations, so that the matrix of the graph is symmetric.

In our case, we label the vertices of the graph $E_6^{(1)}$ by $1, 1', 1'', 2, 2', 2'', 3$, in this order; the corresponding adjacency matrix is clearly

$$\mathcal{G} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

This matrix is nothing else than the matrix $N_2$, i.e., the fundamental generator of the ring of $N$-matrices and it is associated with the fundamental representaion $\sigma_2$. The eigenvalues are $-2, -1, -1, 0, 1, 1, 2$. The biggest eigenvalue (also called “norm of the graph” or “Perron-Frobenius eigenvalue”) is equal to 2. This is also true for the graphs $E_7^{(1)}$ and $E_8^{(1)}$. The corresponding eigenvector (we normalize...
the first entry to 1), also called “Perron-Frobenius vector of the graph” has components $D$ that are positive integers. $D = 1, 1, 1, 2, 2, 2, 3$. We recognize the dimensions of the irreps of $T$.

The conclusion is that, from the graph alone, we can recover the dimension of the irreps. This is also true for the binary cubic and icosahedral groups. From the same graph, we then recover, as already explained, the multiplication of representations by the fundamental (which is 2-dimensional) as well as the whole fusion algebra, by imposing associativity of the Grothendieck ring; the character table itself is obtained by a simultaneous diagonalisation of the $N_i$ matrices encoding the structure constants $C_{ijk}$ of this ring.

2.1.9 Structure of centralizer algebras, tower of commutants for $\tilde{T}$

Using the structure of the Grothendieck ring for $\tilde{T}$, encoded by the graph $E_6^{(1)}$, we see immediately, by taking tensor powers of the fundamental, that

\[
\begin{align*}
[2]^1 &= 1[2] \\
[2]^2 &= 1[1] + 1[3] \\
[2]^3 &= 2[2] + 1[2'] + 1[2''] \\
[2]^4 &= 2[1] + 4[3] + 1[1'] + 1[1''] \\
[2]^5 &= 6[2] + 5[2'] + 5[2''] \\
[2]^6 &= 6[1] + 16[3] + 5[1'] + 5[1''] \\
\ldots &= \text{etc.}
\end{align*}
\]

We call $C_p$ the centralizer algebras of the group $\tilde{T}$ in the representation $[2]^p$. It is clear that these algebras are not isomorphic with the usual Temperley-Lieb algebras $T_p$ (which are isomorphic with the Schur centralizer algebras for $SU(2)$) as soon as $p \geq 3$. For instance $T_4 = M(2, \mathbb{C}) \oplus M(3, \mathbb{C}) \oplus \mathbb{C}$ since, in $SU(2)$, $[2]^4 = 2[1] + 3[3] + 1[5]$, but $C_4 = M(2, \mathbb{C}) \oplus M(4, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ since, in $\tilde{T}$, $[2]^4 = 2[1] + 4[3] + 1[1'] + 1[1'']$. The above results leads immediately to the following structure (Fig. 2.1.9) for the tower of commutants $C_p$’s. As usual, inclusions are defined by the edges of this graph and an appropriate Pascal rule gives the dimensions. Notice that after a few steps, we get the folded $E_6^{(1)}$ diagram of reflected and repeated down to infinity. This picture – paths emanating from the endpoint vertex – can also be generated very simply by considering successive powers of the adjacency matrix $N_2$ of the Dynkin diagram $E_6^{(1)}$ acting on the (transpose) of the vector $\langle 1, 0, 0, 0, 0, 0, 0 \rangle$ characterizing the leftmost vertex (identity representation).
2.1.10 Essential paths for the graph $E_6^{(1)}$

We first define elementary paths on a graph as a sequence $\{\sigma_{a_1}, \sigma_{a_2}, \ldots, \sigma_{a_p}\}$ of consecutive vertices (here, for simplicity, we suppose that edges carry both orientations, i.e., no orientation at all). Elementary paths can therefore backtrack. Then we consider the Hilbert space $\text{Paths}$ obtained by taking arbitrary linear combinations of elementary paths. The scalar product is defined by declaring that the basis of elementary paths is orthonormal. Since every elementary path has a length $p$, the vector space $\text{Paths}$ is graded. In the case of $SU(2)$ (with graph $A_\infty$) or $\tilde{T}$ (with graph $E_6^{(1)}$), irreducible representations appearing in the decomposition of $[2]^p$ can be characterized by paths on those graphs, emanating from the origin; they are also associated with particular projectors, that are, $2^p \times 2^p$ matrices. We need now to introduce special paths called essential paths.

The notion of essential paths on a graph is due to Ocneanu (16). Essential paths may start from any vertex but we shall be mostly interested in those starting from the origin.

Let us begin with the case of $SU(2)$. A given irreducible representation of dimension $d$ appears for the first time in the decomposition of $[2]^{d-1}$ and corresponds to a particular projector in the vector space $(\mathbb{C}^2)^{\otimes d-1}$ which is totally symmetric and therefore projects on the space of symmetric tensors. These symmetric tensors provide a basis of this particular representation space and are, of course, in one to one correspondence with symmetric polynomials in two complex variables $u, v$ (representations of given degree). Paths corresponding to irreducible symmetric representations are essential paths starting at the origin.

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2Warning: the length of $[\sigma_a \sigma_b \sigma_c \sigma_d \sigma_e \sigma_f]$ is 5, not 3.
However, irreducible representations of dimension $d$ appear not only in the reduction of $[2]^{d-1}$ but also in the reduction of $[2]^f$, when $f = d + 1, d + 3, \ldots$. Such representations are equivalent with the symmetric representations previously described but they are nevertheless distinct, as explicit given representations and their associated projectors are not symmetric. For instance the representation $[3]$ that appears in $[2]^2$ corresponds to an essential path starting from the origin, but the three representations $[3]$ that appear in the reduction of $[2]^4$ do not correspond to essential paths.\footnote{The notion of “essential path” on a graph $G$ formalizes and generalizes the above remarks. In the present paper, we shall only need to count these particular paths, so that we do not need to give their precise definition. The interested reader will find this information in the appendix.}

When we move from the case of $SU(2)$ to the case of finite subgroups of $SU(2)$, in particular the binary polyhedral groups whose representation theories are described by the affine Dynkin diagrams $E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$, the notion of essential paths can be obtained very simply by declaring that a path on the corresponding diagram is essential if it describes an irreducible representation that appears in the reduction with respect to the chosen finite subgroup of an irreducible symmetric representation of $SU(2)$. For instance, the $[4]$ dimensional representation of $SU(2)$, obtained in the decomposition of $[2][3] = [2] + [4]$ is symmetric, and is associated with a Wenzl projector $p_4$ of the algebra $T_4$. In the case of the finite subgroup $\tilde{T}$, the corresponding projector of the centralizer algebra $C_4$ splits, and this corresponds to the reduction $[4] \to [2'] + [2'']$ into a sum of two inequivalent irreps. In $SU(2)$ we have therefore one essential path of length 3, emanating from the origin (it ends on $[4]$), but in $\tilde{T}$, this gives two essential paths, one ending on $[2']$ and the other on $[2'']$. In general, essential paths are linear combinations of elementary paths.

The number of essential paths starting from the origin and ending at a given vertex are readily obtained from the tower of centralizers by using a kind of “moderated” Pascal rule: the number of essential paths (with fixed origin) of length $p$ reaching a particular vertex is obtained from the sum of the number of essential paths of length $p - 1$ reaching the neighbouring points (as in Pascal rule) by substracting the number of paths of length $p - 2$ reaching the chosen vertex. This observation was made by [16] in a general setting, and by J.B. Zuber [20] in the context of boundary conformal field theories.

The following picture – essential paths starting from the endpoint vertex – can be generated very simply as follows: Define a rectangular matrix $E_1$, with seven columns and infinitely many rows, whose $j$-th row is $E_1(j) = N_2 E_1(j - 1) - E_1(j - 2)$, with $E_1(1) = (1, 0, 0, 0, 0, 0, 0)$, $E_1(2) = N_2 E_1(1)$, and where $N_2$ is the adjacency matrix of the graph $E_6^{(1)}$. The entries of $E_1(j)$ give the number of paths of length $j - 1$ ending on the different vertices. Essential paths starting from an arbitrary vertex of the Dynkin diagram can be constructed in the same way, by replacing $E_1$ by $E_2(1) = (0, 1, 0, 0, 0, 0, 0)$, $E_3(1) = (0, 0, 1, 0, 0, 0, 0)$ etc. Information about essential paths (in particular their number) is therefore encoded by the set of seven rectangular matrices (in the case of $E_6^{(1)}$) which have
seven rows and infinitely many rows\textsuperscript{3} essential paths for the finite subgroups of $SU(2)$ can be of arbitrary length since symmetric representations of $SU(2)$ can be of arbitrary degree (horizontal Young diagrams with an arbitrary number of boxes). Their quantum analogues, however have only a finite number of rows.

These matrices, introduced in \cite{3}, will be called “essential matrices\textsuperscript{4}.”

\[
\begin{array}{cccccccc}
[1] & [2] & [3] & [2''] & [2'] & [1''] & [1'] & \text{p} \\
\end{array}
\]

\begin{center}
\text{\includegraphics[width=\textwidth]{diagram.png}}
\end{center}

2.1.11 Projectors on irreducible representations

Our purpose, here, is to explain, in a nutshell, how to obtain explicitly a matrix expression for the projectors $\varpi_p[s]$ mapping the reducible representation space $[2]^p$ to one of its irreducible subrepresentations $[s]$, for $SU(2)$ or one of its finite subgroups. These are explicit $2^p \times 2^p$ matrices. We do it for $SU(2)$ first. Here are the steps:

\begin{itemize}
\item Find an explicit matrix realization for the Jones’ projectors $e_i$’s in the appropriate Jones-Temperley-Lieb algebra $T_p$.
\end{itemize}

\textsuperscript{4}In the following picture, we decided arbitrarily to cut the graph at level $p = 8$
• Express the minimal central projectors associated with the various blocks appearing in the algebra $T_p$ in terms of the Temperley-Lieb' generators $\epsilon_i$.

• Call $A$ the classical antisymmetrizer of $SU(2)$ in the representation $[2]^2$ (it is a $4 \times 4$ matrix). We obtain the projectors $\varpi[s]$ from the minimal central projectors by replacing $\epsilon_1, \epsilon_2, \ldots$'s by $\epsilon_1 = A \otimes I_2 \otimes I_2 \otimes I_2 \otimes \ldots$, $\epsilon_2 = I_2 \otimes A \otimes I_2 \otimes I_2 \otimes \ldots$, etc.

The sub-representation $[p + 1]$ of $[2]^p$ is special since it is totally symmetric; the expression of the correspondig central projector of $T_p$ is easy to obtain in this case since a recurrence formula exists for Wenzl projectors (see for instance [10]).

Now, we do it for the binary tetrahedral group.

• We first compute the projector decomposition of $[2]^p$ for $SU(2)$ as above.

• We use the fact that the binary tetrahedral group is generated by two explicit generators $s$ and $t$. Considered as (group-like) elements of the group algebra, i.e., $\Delta s = s \otimes s$ and $\Delta t = t \otimes t$, we can compute their iterated coproduct in representation $[2]^p$, they are explicit $2^p \times 2^p$ matrices. The projectors of $SU(2)$, that we have already obtained, commute with them.

To simplify the discussion, we take $p = 3$ and write $[2]^3 = [2_a] + [2_b] + [4]$ for $SU(2)$, and $[4] \mapsto [2''']$ for the reduction to $\tilde{T}$. We want to obtain $\varpi_3[\sigma_{2'''}]$ and $\varpi_3[\sigma_{2''}']$. Taking an arbitrary $8 \times 8$ matrix $\varpi$, we first impose that it belongs to the centralizer algebra, so it should commute with the iterated coproduct of $s$ and $t$ already calculated (linear equations for the matrix coefficients). This already restricts the number of unknown coefficients.

• We impose that the matrix $\varpi$ should be orthogonal to the known $\varpi_3[\sigma_{2a}]$ and $\varpi_3[\sigma_{2b}]$ (again linear equations). This further restrict the number of coefficients.

• Finally we impose that $\varpi$ should be a projector ($\varpi, \varpi = \varpi$). We find three solutions : one is of rank $4$ (it is $\varpi_3[\sigma_4]$ itself), the other two are of rank $2$ and add up to $\varpi_3[\sigma_4]$. These are the projectors $\varpi_3[\sigma_{2'''}]$ and $\varpi_3[\sigma_{2''}']$ that we were looking for.

Writing even a simple example in full details requires a lot of room, but the procedure should be clear.

2.1.12 Klein invariants

Take a classical polyhedron, put its vertices $V$ on the sphere; from the centroid of the polyhedron draw rays in direction of the points located at the center of the faces and at the middle of the edges. These rays intersect the sphere at points $F$, and $E$. Notice (Euler) that $\#F - \#E + \#V = 2$. Now make a stereographical projection and build a complex polynomial that vanishes precisely at the location of the projected vertices (or center of faces, or mid-edges): this polynomial is, by construction, invariant under the symmetry group of the polyhedron (at least projectively) since group elements only
permute the roots. This is the historical method – see in particular the famous little book [13]. In the case of the tetrahedron, for instance, you get the three polynomials (in homogeneous coordinates): \( V = u^4 + 2i\sqrt{3}u^2v^2 + v^4 \), \( E = uv(u^4 - v^4) \) and \( F = u^4 - 2i\sqrt{3}u^2v^2 + v^4 \). Actually \( V \) and \( F \) are only projectively invariant, but \( X = 108^{1/4}E, \ Y = -VF = -(u^8 + v^8 + 14u^4v^4) \) and \( Z = Y^3 - iX^2 = (u^{12} + v^{12}) - 33(u^8v^4 + u^4v^8) \) are (absolute) invariants, of degrees 6, 8, 12. Together with the relation \( X^4 + Y^3 + Z^2 = 0 \), they generate the whole set of invariants. Alternatively you can build the \( p \)-th power of the fundamental representation of the symmetry group of the chosen binary polyhedral group, and choose \( p \) such that there exists one essential path of length \( p \) starting at the origin of the graph of tensorisation by the fundamental representation (one of the affine ADE diagrams) that returns to the origin. Therefore you get a symmetric tensor (since the path is essential), hence a homogeneous polynomial of degree \( p \); moreover this polynomial is invariant since the path goes back to the origin (the identity representation). By explicitly calculating the projectors corresponding to the (unique) essential path of \([2]^6\), \([2]^8\) and \([2]^{12}\) on the affine \( E_6^{(1)} \) graph, we can recover the polynomials \( X, Y, Z \).

2.2 Quantum geometry

The main interest of the previous section was to show that a good deal of the geometry associated with symmetry groups of platonic solids could be carried out without using the groups themselves but only the exceptionnal affine Dynkin diagrams. Going to the quantum geometry will now be relatively easy: we just replace the affine Dynkin diagrams by the usual Dynkin diagrams. This present section will be rather short. One reason is the limited amount of space available for these proceedings, another reason is that the techniques have already been presented in the previous section, and they can be translated directly without further ado. We shall mention only the differences with the previous (classical) situation.

2.2.1 Realisations of the quantum algebras

The quantum algebra analogues of the group algebras \( \mathcal{H}_{E_6^{(1)}} \) and \( \mathcal{H}_{E_8^{(1)}} \) should be objects called \( \mathcal{H}_{E_6} \) and \( \mathcal{H}_{E_8} \). The quantum analogue of \( \mathcal{H}_{E_7^{(1)}} \) does not exist, for a reason explained later (but the \( E_7 \) diagram leads nevertheless to an interesting quantum geometry). Actually we shall not introduce such quantum algebras at all, but we proceed as if they had been constructed. Besides the basic reference [13], let us mention also the following papers: [19], that uses the formalism of boundary conformal field theories, [2], for a discussion in terms of nets of subfactors and [12], for a discussion in terms of braided modular categories. Let us finally mention the paper [11], that introduces planar algebras, a concept that probably allows one to accommodate many of these constructions.
2.2.2 Representations of these quantum algebras

Since we did not give any definition of these quantum tetrahedral or icosahedral algebras, we shall define their irreducible representations \( \sigma_p \) as mere symbols associated with vertices of the diagrams \( E_6 \) or \( E_8 \).

The norms of the adjacency matrix \( G \) of the three exceptional Dynkin diagram are no longer the integer 2 but the quantum integers \( [2] \) i.e., \( 2 \cos(\pi/\kappa) \) for \( \kappa = 12, 18, 30 \). \( \kappa \) is also the Coxeter number of the diagram\(^5\).

The components of the corresponding (normalized) eigenvector \( D \) (Perron-Frobenius) with the first entry normalized to 1 (extremity of the longest leg), are not positive integers but they provide a definition for the (quantum) dimensions of the irreducible representations. For \( E_6 \), \( D = qdim(\sigma_0, \sigma_1, \sigma_2, \sigma_5, \sigma_4, \sigma_3) = ([1], [2], [3], [2], [1], [3]/[2] = \sqrt{2}) \).

To keep the size of this paper small enough and specify our conventions for the labelling of vertices, we only give the graphs \( E_6 \) and \( E_8 \). Much more information concerning the “quantum tetrahedron” can be found in the paper \(^3\). The diagrams \( E_6 \) and \( E_8 \), with our labelling for vertices (increasing labels starting from the tips) are given as follows

\[
\begin{array}{cccccc}
[\sigma_0] & [\sigma_1] & [\sigma_2] & [\sigma_5] & [\sigma_4] & [\sigma_3] \\
\hline
[\sigma_0] & [\sigma_1] & [\sigma_2] & [\sigma_5] & [\sigma_4] & [\sigma_7] & [\sigma_6]
\end{array}
\]

Figure 3: The diagrams \( E_6 \) and \( E_8 \)

2.2.3 Tensor products of representations and structure of the Grothendieck ring

The representation \( \sigma_0 \) associated with the end-point of the longest leg is the unit. The Dynkin diagram defines multiplication by the algebraic generator \( \sigma_1 \). The only task is to complete the table (which is 6 \( \times \) 6 in the case of the \( E_6 \) diagram) as we did in the classical case. This is rather straightforward and works perfectly, but for the fact that the obtained structure constants are positive integers for \( E_6 \) and \( E_8 \), but not for \( E_7 \). This was observed long ago by \(^{17}\). Here is the “fusion table” we get for \( E_6 \):

\[^{5}\text{The quantum numbers are } [n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \text{ For } E_6, q = \exp(i\pi/12).\]
2.2.4 The fusion matrices $N_i$

These six $6 \times 6$ matrices can be worked out easily, as we did in the case of the binary tetrahedral group. They can be found in [3].

2.2.5 From the Grothendieck ring to the character table

Again, there is no drastic difference with the classical situation and we get a quantum character table... but for the fact that conjugacy classes are not even defined! This defines a kind of Fourier transform which is both finite and quantum; its relation with the action of the modular group should be further investigated.

2.2.6 Structure of centralizer algebras and tower of commutants for $\mathcal{H}_{E_6}$

We can take the powers of $\sigma_1$ and decompose them into irreducibles, as we did in the classical case, just by using the adjacency matrix and the above fusion table. The coefficients – multiplicities – define the would-be centralizer algebras; they could be defined as multi-towers algebras associated with the chosen Dynkin diagram thought of as a Bratelli diagram, see [10]. The next picture displays the first eight rows (one can go down to infinity).

2.2.7 Essential paths

Here we need to know the precise general definition (Ocneanu) of essential paths if we want to find them explicitly (see the appendix or [12]), but if we want only to count them, the method explained in the classical case works. The only difference with the classical case is that essential matrices, which are rectangular, with infinitely many rows in the classical case, are defined as matrices with only $\kappa - 1$ rows in the quantum situation (if the matrices were not truncated at that level, the line $\kappa$ would be filled with 0 and the next ones would contain negative integers). This reflects the fact that essential paths having a bigger length do not exist. For the $E_6$ diagram, $\kappa = 12$.

The six rectangular matrices $E_\alpha$ of size $11 \times 6$ describing essential paths on $E_6$ are explicitly given in [3]. Columns of the essential matrices are labelled by the length $p$ of the paths, so $p$ runs from 0 to 10. This can be seen as a kind of labelling by the vertices of the Dynkin diagram $A_{11}$. Essential matrices have

$$
\begin{array}{cccccc}
0 & 0 & 3 & 4 & 1 & 2 \\
3 & 3 & 0 & 4 & 3 & 2 \\
4 & 4 & 3 & 0 & 5 & 2 \\
1 & 1 & 2 & 5 & 0 & 2 \\
2 & 2 & 15 & 2 & 135 & 0224 \\
5 & 5 & 2 & 1 & 24 & 135 & 02
\end{array}
$$
therefore columns labelled by $E_6$ and rows labelled by $A_{11}$. This is actually more than a simple remark since the algebra generated by the eleven $6 \times 6$ square matrices $F_i(a, b) \cong E_6(i, b)$ provides a representation of the fusion algebra of the graph $A_{11}$. A pictorial description of essential paths for all $ADE$ Dynkin diagrams can be found in the appendix of [16]. The essential matrix describing (essential) paths leaving the origin leads to the next picture.

### 2.2.8 Projectors on irreducible representations

The method is essentially the same as in the classical situation, but for the fact that symmetrizer and antisymmetrizer of $SU(2)$ have to be replaced by their quantum analogues, which can be obtained from the spectral decomposition of the flipped $R$-matrix of the quantum $SU(2)$ Hopf algebra (the one that obeys the braid group relation). However, when $p$ increases (when we reach the triple point) one needs to know how to split the Wenzl projector. This was done, in the classical situation, by explicitly using the generators of the binary polyhedral groups and the expression of their iterated coproducts. Here, the problem is still open, since we do not have an explicit realization for the quantum algebras $\mathcal{H}$.

### 2.2.9 Klein invariants

In the quantum situation, Klein invariants are defined as essential paths starting from the origin and coming back to the origin. From the first essential matrix of $E_6$ or from the above equivalent corresponding graph of essential paths, we see that such an invariant exist (its length $n$ is equal to 6). We can actually compute it explicitly as a path, i.e., as a linear combination of
elementary paths on the graph $E_6$. It would be nice to exhibit also a kind of homogeneous (but non-commutative) polynomial that implements it. This has not be obtained so far.

3 Classical and quantum induction-restriction

The purpose of these two next subsections is to investigate induction-restriction theory of representations in two classical situations (one finite, and the other infinite dimensional); namely we take the group $\tilde{T}$ as a subgroup of $SU(2)$ or as a subgroup of $I$. As we know, representation theory of these three classical objects is encoded by the Dynkin diagrams $E_6^{(1)}$, $E_8^{(1)}$ and $A_\infty$.

The purpose of the last subsection is to investigate a similar situation in a quantum (but finite) case, encoded by the two graphs $E_6$ and $A_{11}$.

3.1 Classical induction-restriction: $E_8^{(1)}$ versus $E_6^{(1)}$, i.e., $\tilde{I}$ versus $\tilde{T}$
3.1.1 Classical branching rules $\tilde{I} \rightarrow \tilde{T}$

Both groups are finite subgroups of $SU(2)$ and we have inclusions: $\tilde{I} \subset \tilde{T} \subset SU(2)$. In both cases (diagrams $E_6^{(1)}$ and $E_8^{(1)}$) we have given the dimensions of the corresponding representations. Representations of $SU(2)$ can be restricted to its subgroups, and, in the same way, irreducible representations of $\tilde{I}$ can be restricted to $\tilde{T}$, and decomposed into irreducible representations of the latter.

The following table gives the branching rules $\tilde{I} \rightarrow \tilde{T}$:

| $\tilde{I}$ | $\tilde{T}$ |
|-------------|-------------|
| 1           | 1           |
| 2           | 2           |
| 3           | 3           |
| 4           | 2' + 2''    |
| 5           | 3 + 1 + 1'' |
| 6           | 2 + 2' + 2'' |
| 4'          | 3 + 1      |
| 3'          | 3          |
| 2'          | 2          |

It is easy to obtain the above table: one should just compare multiplications of irreps of $SU(2)$, $\tilde{I}$, or $\tilde{T}$ by the 2-dimensional fundamental representation. For instance, $2 \times 3 = 2 + 4$ in $SU(2)$, but $2 + 2' + 2''$ in $\tilde{T}$ and $2 + 4$ in $\tilde{I}$; therefore, with respect to the branching $\tilde{I} \rightarrow \tilde{T}$ we have $4 \rightarrow 2' + 2''$. Then we have $2 \times 4 = 3 + 5$ in $SU(2)$, but $2 \times (2' + 2'') = 1 + 3 + 1' + 3 + 1'' + 3$ in $\tilde{T}$, whereas $2 \times 4 = 3 + 5$ in $\tilde{I}$; therefore, we get $5 \rightarrow 3 + 1' + 1''$ for the branching $\tilde{I} \rightarrow \tilde{T}$. In the same way we get $2 \times 5 = 4 + 6$ both in $SU(2)$, and in $\tilde{I}$, the restriction to $\tilde{T}$ reads $2 \times (3 + 1' + 1'') = 2 + 2' + 2'' + 2' + 2''$, but we already know that $4 \rightarrow 2' + 2''$ therefore, we find $6 \rightarrow 2 + 2' + 2''$. Next we have $2 \times 6 = 5 + 4' + 3'$ in $\tilde{I}$, so that restriction of both sides to $\tilde{T}$ gives $1 + 3 + 1' + 3 + 1'' + 3 = 3 + 1' + 1'' + (4' + 3')_T$ and we find $4' + 3' \rightarrow 3 + 3 + 1$, so that the only possibility is $4' \rightarrow 3 + 1$ and $3' \rightarrow 3$. The last branching rules can be obtained by restricting $2 \times 4' = 6 + 2'$ to $\tilde{T}$; we get $2 \times (3 + 1) = 2 + 2' + 2'' + (2')_T$. i.e., $2 + 2' + 2'' + 2 = 2 + 2' + 2'' + (2')_T$ and therefore the restriction $2' \rightarrow 2$.

3.1.2 Sections of classical vector bundles over $\tilde{I}/\tilde{T}$

Since $\tilde{T}$ is a subgroup of $\tilde{I}$, we can write the binary icosahedral group as a principal bundle over the quotient $\tilde{I}/\tilde{T}$, with structure group (typical fiber) $\tilde{T}$, the binary tetrahedral group. For each representation of the structure group, in particular for each irreducible representation $\rho$ (and carrier space $V_\rho$) of $\tilde{T}$ (we know that there are 7 of them), we can build an associated vector bundle $\tilde{I} \times_{\tilde{T}} V_\rho$, with basis $\tilde{I}/\tilde{T}$ and typical fiber the vector space $V_\rho$. Now we may consider the spaces of sections $\Gamma_\rho$ of those bundles, which are functions on the finite homogenous space $\tilde{I}/\tilde{T}$ and valued in the corresponding vector spaces. In the particular case of the trivial representation of $\tilde{T}$ (called 1), the carrier vector space is $\mathbb{C}$ and the sections of $\Gamma_1$ just coincide with the space of complex valued functions on the finite set $\tilde{I}/\tilde{T}$, whose cardinality is $5 = 120/24$.

Here we are in a finite dimensional situation, but Peter Weyl theory of induced representations still applies. Let us take the example of $\Gamma_3$, the space of sections of $\tilde{I} \times_{\tilde{T}} V_3$; this vector bundle is a collection of five vector spaces of dimension 3 (one above each of the five points of the coset); the dimension of $\Gamma_3$ is therefore $3 \times 5 = 15$. This fifteen dimensional space is the carrier space of a natural representation of $\tilde{I}$ (the one induced by this particular vector bundle),
but this representation is not irreducible: its decomposition, in irreps of $\tilde{I}$, can be obtained from the previously given table of branching rules: 3 (of $\tilde{T}$) appears on the right-hand side of the branching rules corresponding to the irreps 3, 5, 4’ and 3’ of $\tilde{I}$, from this information, one deduces that $\Gamma_3 = [3] \oplus [5] \oplus [4'] \oplus [3']$ (whose sum is indeed $5 \times 3 = 15$, as it should). This induction process leads to the following table, for the decomposition of the various spaces of sections (the vector spaces $\Gamma_\rho$) in irreducible representations of $\tilde{I}$:

| $\rho$ | dim($\Gamma_\rho$) | $\Gamma_\rho$ |
|--------|---------------------|--------------|
| 1      | 5 x 1 = 5          | 1 + 4'       |
| 2      | 5 x 2 = 10         | 2 + 6 + 2'   |
| 3      | 5 x 3 = 15         | 3 + 5 + 4' + 3' |
| 2'     | 5 x 2 = 10         | 4 + 6        |
| 1'     | 5 x 1 = 5          | 5            |
| 2''    | 5 x 2 = 10         | 4 + 6        |
| 1''    | 5 x 1 = 5          | 5            |

In particular, the space of functions on the finite set (five points) $\tilde{I}/\tilde{T}$, that we may call $\text{Fun}(\tilde{I}/\tilde{T}) \equiv \Gamma_1$ decomposes into irreps of $\tilde{I}$ as $4'+1$.

In our case (the space of sections of vector bundles above the finite left homogeneous space $\tilde{I}/\tilde{T}$) we see that $\sum \text{dim}(\Gamma_\rho) = 5 + 10 + 15 + 10 + 5 + 10 + 5 = 60$, which is one half of the order of the group $\tilde{I}$ (by considering both left and right bundles, we would get $120 = \#\tilde{I}$).

For us, the main interest of the previous remarks, is that, knowing only the dimensions of the spaces of sections, we could recover the order of $\tilde{I}$ and the order of $\tilde{T}$. The order of $\tilde{I}$, namely 120, is obtained by by taking the double of the sum of the dimensions of the spaces of sections. The cardinality of the quotient (namely $\text{dim}\Gamma_1 = 5$) is then obtained by summing the dimensions appearing in the decomposition of the space of sections associated with the trivial representation. The order of $\tilde{T}$, namely 24, is finally obtained by dividing the order of $\tilde{I}$ by the the cardinality of the quotient.

Once $\text{dim}\Gamma_1 = 5$ is known, we can recover the dimensions of the irreducible representations $\rho$ themselves by taking the ratio $\text{dim}(\Gamma_\rho)/\text{dim}(\Gamma_1)$.

### 3.2 Classical induction-restriction: $A_\infty$ versus $E_6^{(1)}$ (i.e., $SU(2)$ versus $\tilde{T}$)

We restrict the representations of $SU(2)$ to irreps of $\tilde{T}$. We build the principal bundle $SU(2)$ as a $\tilde{T}$ bundle over the quotient $SU(2)/\tilde{T}$ (which is a three dimensional manifold) and consider the (seven) associated vector bundles relative to the seven irreps of $\tilde{T}$. We have therefore one such vector bundle for every point of the extended Dynkin diagram $E_6^{(1)}$. The only difficulty is to compute the branching rules. One method is to proceed step by step, i.e., to use the information provided by the two Dynkin diagrams encoding tensor multiplication by the fundamental representation, computing tensor products of irreps both for
SU(2) and its finite subgroup and comparing the results. The easiest method is to use essential matrices (they have infinitely many rows, in the present case); another technique – which amounts to the same but is aesthetically more appealing – is to draw the essential paths on $E_{6}^{(1)}$. The only relevant essential matrix (for our present purpose) is the one labelled by the trivial representation of $	ilde{T}$, i.e., by the space of essential paths emanating from the leftmost point of the Dynkin diagram $E_{6}^{(1)}$. For instance the line $p = 8$ of that graph (referring to $[2]^{8}$ i.e., to the representation of dimension 9 of SU(2), and associated with an essential path of length 8 on the graph $A_{\infty}$) tells us that $[9] \rightarrow [1]+2[3]+[1']+[1'']$ in the branching SU(2) versus $\tilde{T}$. In other words, in order to perform reduction, we read the first essential matrix “horizontally”, i.e., we look at representations of SU(2) given by symmetric polynomials of degree $n$ and branch them to this particular subgroup. In order to perform induction, we look at the same essential matrix, but “vertically”, i.e., we choose a particular irrep of $\tilde{T}$ (a particular column) and see for which values it appears in branching rules of $[p+1]$.

### 3.2.1 Sections of vector bundles over SU(2)/$\tilde{T}$

Call $V_\rho$ the vector space carrying the irreducible representation $\rho$ of $\tilde{T}$ and $\Gamma_\rho$ the space of sections of the homogeneous vector bundle $SU(2) \times \tilde{T} V_\rho$. The spaces of sections of these vector bundles can be decomposed as follows into irreducible representations of SU(2) (subscript give the multiplicity):

$$
\begin{align*}
\Gamma_1 &= 1 + 7 + 9 + 13 + 15 + 17 + \ldots = \text{Fun}(SU(2)/\tilde{T}) \\
\Gamma_2 &= 2 + 6 + 8 + 10 + 12 + 14 + 16 + \ldots \\
\Gamma_3 &= 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + \ldots \\
\Gamma_2' &= 4 + 6 + 8 + 10 + 12 + 14 + 16 + \ldots \\
\Gamma_1' &= 5 + 9 + 11 + 13 + 15 + 17 + \ldots \\
\Gamma_2'' &= 4 + 6 + 8 + 10 + 12 + 14 + 16 + \ldots \\
\Gamma_1'' &= 5 + 9 + 11 + 13 + 15 + 17 + \ldots
\end{align*}
$$

The degree of the homogenous polynomials providing a basis for a representation space of dimension $d$ is $d - 1$, so that representations of degree 0, 6, 8, 12, 14, 16, \ldots appear in $\Gamma_1$, as it should: we recover the fact that these representations of SU(2) indeed contain $\tilde{T}$-invariant subspaces (Klein polynomials for the tetrahedron).

We can make the same kind of comments as in the previous section for instance $\text{dim}(\Gamma_3)/\text{dim}(\Gamma_1) = 3$, but we now have to manoeuvre infinite sums and we should use generating functions (we shall not do it here). In the case of $\Gamma_3$, for instance, we can also write

$$
[3] \otimes ([1] \oplus [7] \oplus [9] \oplus 2[13] \oplus \ldots) = [3] \oplus [5] \oplus 2[7] \oplus \ldots
$$

The reader may wonder why we did not introduce also “essential matrices” in the previous subsection (with columns labelled by irreps of the binary tetra-
hedral group and rows labelled by irreps of the binary icosahedral group). There is no reason: we could have done it as well.

3.3 Quantum induction-restriction: $A_{11}$ versus $E_6$

We now replace the diagram $A_\infty$ that describe irreps of $SU(2)$ by the diagram $A_{11}$ and the classical binary tetrahedral group by its would-be quantum counterpart described by the diagram $E_6$.

Both examples studied in the corresponding classical two sections actually provide interesting – and complementary – classical analogues: the first ($E_8^{(1)} \to E_6^{(1)}$) because it is finite dimensional, and the other $A_\infty \to E_6^{(1)}$ because $A_{11}$ looks indeed as a “truncated” $A_\infty$.

From the embedding $\tilde{T} \subset \tilde{I}$, we can deduce an embedding of the corresponding group algebras (finite dimensional Hopf algebras) $\mathcal{H}_{\tilde{T}} = \mathbb{C}\tilde{T} \subset \mathcal{H}_{\tilde{I}} = \mathbb{C}\tilde{I}$ but although we do not plan to give a construction here of the “would-be groups” (or would-be group algebras) that we could associate with the two genuine Dynkin diagrams $E_6$ and $A_{11}$, we want nevertheless to consider the first as a kind of sub-object of the next. We proceed as if we had an embedding $\mathcal{H}_{E_6} \subset \mathcal{H}_{A_{11}}$.

We take $\hat{q} = \exp(i\pi/12)$ (so that if $q = \hat{q}^2$, then $q^{12} = 1$). Irreps of $A_{11}$ are representations called $\tau_0, \tau_1, \ldots, \tau_{10}$. They have $q-$dimension respectively equal to $[1]$, $[2]$, $[3]$, $[4]$, $[5]$, $[6]$, $[7] = [5]$, $[8] = [4]$, $[9] = [3]$, $[10] = [2]$, $[11] = [1]$. There actually is a non semi-simple Hopf algebra defined as a finite dimensional quotient of the enveloping quantum algebra of $SU(2)$, when $q$ is a twelfth root of unity, and which is such that the above list of $\tau_i$ indeed labels its irreducible representations of non-zero quantum dimension; however this knowledge will not be used here. The “representations” $\tau_i$ are therefore just abstract symbols that we can associate with the various points of the diagram $A_{11}$, whose own fusion table could have been worked out as discussed previously, and the “$q$-dimensions” is just a name for the entries of the normalized eigenvector associated with the norm of the adjacency matrix of this diagram (the norm being, by definition, its biggest eigenvalue). Remember that both graph $A_{11}$ and $E_6$ have same norm. A priori, the ring of representations that we are considering here have a “dimension function” valued in a $\mathbb{Z}$-ring linearly generated by the $q$-integers $[1]$, $[2]$, $[3]$, $[4]$, $[5]$, $[6]$. This is a clearly the case both for $A_{11}$ and $E_6$.

It may be useful to note that

$$[1] = 1, [2] = \frac{\sqrt{2}}{\sqrt{3} - 1}, [3] = \frac{2}{\sqrt{3} - 1}, [4] = \frac{\sqrt{6}}{\sqrt{3} - 1}, [5] = \frac{1 + \sqrt{3}}{\sqrt{3} - 1}, [6] = \frac{2\sqrt{2}}{\sqrt{3} - 1}$$

3.3.1 Quantum branching rules $A_{11} \to E_6$

The branching rules from $A_{11}$ to $E_6$ (that gave restriction in one direction and induction in the other) are gotten from the “$q$-symmetric” representations, or, equivalently, from the essential matrices of the graph $E_6$.

The only relevant essential matrix, for our present purpose, is the one labelled by the “trivial representation” (leftmost point of the graph $E_6$). The
following table summarizes the results for the reduction $A_{11} \to E_6$. One should remember that the $q$-dimension corresponding to irreps $\tau_p$ of $A_{11}$, (i.e., vertices of $A_{11}$) is $[p + 1]$.

\[
\begin{array}{c|c|c|c}
\tau_0 & \sigma_0 & \tau_1 & \sigma_1 \\
\tau_3 & \sigma_3 + \sigma_5 & \tau_4 & \sigma_2 + \sigma_4 \\
\tau_5 & \sigma_1 + \sigma_5 & \tau_6 & \sigma_0 + \sigma_2 \\
\tau_7 & \sigma_2 & \tau_9 & \sigma_5 \\
\tau_8 & \sigma_0 + \sigma_2 & \tau_10 & \sigma_4 \\
\end{array}
\]

3.3.2 Sections of quantum vector bundles over $A_{11}/E_6$

Using the previous table, and using a formal analogy, we associate a quantum vector bundle to each point of the $E_6$ graph and decompose its spaces of sections $\Gamma_{\sigma_p}$, using induction, exactly as we did in the classical case (for instance we see that $\sigma_0$ can be obtained from the reduction of $\tau_0$ and $\tau_6$). We obtain:

\[
\begin{align*}
\Gamma_{\sigma_0} & = \tau_0 + \tau_6 \\
\Gamma_{\sigma_1} & = \tau_1 + \tau_5 + \tau_7 \\
\Gamma_{\sigma_2} & = \tau_2 + \tau_4 + \tau_6 + \tau_8 \\
\Gamma_{\sigma_5} & = \tau_3 + \tau_5 + \tau_9 \\
\Gamma_{\sigma_4} & = \tau_4 + \tau_10 \\
\Gamma_{\sigma_3} & = \tau_3 + \tau_7 \\
\end{align*}
\]

This information can also be displayed as

```
0  6  1  5  2  4  3  5  4  10

*      \quad : 3, 7
```

The quantum dimension of $\Gamma_{\sigma_0}$ is

\[
[1] + [7] = [1] + [5] = 1 + \frac{\sin(5\pi/12)}{\sin(\pi/12)} = \frac{2\sqrt{3}}{\sqrt{3} - 1}
\]

Morally this is the quantum dimension of the space of “functions” on the quantum space $A_{11}/E_6$. We may then check that, by dividing the $q$-dimension of each space of sections $\Gamma_{\sigma_p}$ by the above $q$-dimension of $\Gamma_{\sigma_0}$, we recover exactly the $q$-dimensions of the “typical fibres”, i.e., the $q$-dimensions of the irreducible

\footnote{There are several ways to define quantum principal bundles and associated quantum vector bundles, (see for instance \[\text{[3]}\]), but we do not use these technical definitions here.}
representations $\sigma_p$, already obtained from the normalized Perron-Frobenius vector associated with the graph $E_6$. We have therefore a perfect analogy with the classical situation.

4 Appendix: The general notion of essential paths on a graph $G$

The following definitions are not needed if we only want to count the number of essential paths on a graph. They are necessary if we want to obtain explicit expressions for them. The definitions are adapted from [16].

Call $\beta$ the norm of the graph $G$ (the biggest eigenvalue of its adjacency matrix $G$) and $D_i$ the components of the (normalized) Perron-Frobenius eigenvector. Call $\sigma_i$ the vertices of $G$ and, if $\sigma_j$ is a neighbour of $\sigma_i$, call $\xi_{ij}$ the oriented edge from $\sigma_i$ to $\sigma_j$. If $G$ is unoriented (the case for $ADE$ and affine $ADE$ diagrams), each edge should be considered as carrying both orientations.

An elementary path can be written either as a finite sequence of consecutive (i.e., neighbours on the graph) vertices, $[\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\ldots]$, or, better, as a sequence $(\xi(1)\xi(2)\ldots)$ of consecutive edges, with $\xi(1) = \sigma_{a_1}\sigma_{a_2} = \sigma_{a_2}\sigma_{a_3} = \sigma_{a_3}\sigma_{a_4}$, etc. Vertices are considered as paths of length 0.

The length of the (possibly backtracking) path $(\xi(1)\xi(2)\ldots\xi(p))$ is $p$. We call $r(\xi_{ij}) = \sigma_j$, the range of $\xi_{ij}$ and $s(\xi_{ij}) = \sigma_i$, the source of $\xi_{ij}$.

For all edges $\xi(n+1) = \xi_{ij}$ that appear in an elementary path, we set $\xi(n+1)^{-1} = \xi_{ji}$.

For every integer $n > 0$, the annihilation operator $C_n$, acting on elementary paths of length $p$ is defined as follows: if $p \leq n$, $C_n$ vanishes, whereas if $p \geq n+1$ then

$$C_n(\xi(1)\xi(2)\ldots\xi(n)\xi(n+1)\ldots) = \sqrt{\frac{D_r(\xi(n))}{D_s(\xi(n))}} \delta_{\xi(n),\xi(n+1)^{-1}}(\xi(1)\xi(2)\ldots\xi(n)\xi(n+1)\ldots)$$

Here, the symbol “hat” (like in $\hat{\xi}$) denotes omission. The result is therefore either 0 or a linear combination of paths of length $p - 2$. Intuitively, $C_n$ chops the round trip that possibly appears at positions $n$ and $n + 1$.

A path is called essential if it belongs to the intersection of the kernels of the annihilators $C_n$’s.

In the case of the diagram $E_6^{(1)}$, for instance

$$C_3(\xi_{12}\xi_{23}\xi_{32}\xi_{2'3}\xi_{2''3}) = C_3(\xi_{12}\xi_{23}\xi_{32}\xi_{2'3}\xi_{2''3}) = \sqrt{\frac{2}{3}}(\xi_{12}\xi_{23}\xi_{32})$$

The following difference of non essential paths of length 5 starting at $\sigma_1$ and ending at $\sigma_2$ is an essential path of length 5 on $E_6^{(1)}$:

$$(\xi_{12}\xi_{23}\xi_{32}\xi_{2'3}\xi_{2''3}) - (\xi_{12}\xi_{23}\xi_{32}\xi_{2'3}\xi_{2''3}) = [1, 2, 3, 2', 3, 2] - [1, 2, 3, 2'', 3, 2]$$
Another example: in the case of the diagram $E_6$ (square brackets enclose $q$-numbers),
\[
C_3(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) = \sqrt{\left[\begin{array}{c} 1 \\ 2 \end{array}\right]}(\xi_{01}\xi_{12})
\]
\[
C_3(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) = \sqrt{\left[\begin{array}{c} 2 \\ 3 \end{array}\right]}(\xi_{01}\xi_{12})
\]

The following difference of non essential paths of length 4 starting at $\sigma_0$ and ending at $\sigma_2$ is an essential path of length 4 on $E_6$:
\[
\sqrt{\left[\begin{array}{c} 2 \\ \end{array}\right]}(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) - \sqrt{\left[\begin{array}{c} 3 \\ 2 \end{array}\right]}(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) = \sqrt{\left[\begin{array}{c} 2 \\ \end{array}\right]}[0,1,2,3,2] - \sqrt{\left[\begin{array}{c} 3 \\ 2 \end{array}\right]}[0,1,2,5,2]
\]

Remember the values of the $q$-numbers: $[2] = \frac{\sqrt{2}}{\sqrt{3} - 1}$ and $[3] = \frac{\sqrt{2}}{\sqrt{3} - 1}$.

Acting on elementary path of length $p$, the creating operators $C_n^\dagger$ are defined as follows: if $n > p+1$, $C_n^\dagger$ vanishes and, if $n \leq p+1$ then, setting $j = r(\xi(n-1))$,
\[
C_n^\dagger(\xi(1)\ldots\xi(n-1)\ldots) = \sum_{d(j,k)=1} \sqrt{D_k}D_j(\xi(1)\ldots\xi(n-1)\xi_{jk}\xi_{kj}\ldots)
\]

The above sum is taken over the neighbours $\sigma_k$ of $\sigma_j$ on the graph. Intuitively, this operator adds one (or several) small round trip(s) at position $n$. The result is therefore either 0 or a linear combination of paths of length $p + 2$.

For instance, on paths of length zero (i.e., vertices),
\[
C_1^\dagger(\sigma) = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}}\xi_{jk}\xi_{kj} = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}}[\sigma_j\sigma_k\sigma_j]
\]

Jones’ projectors $e_k$ are defined (as endomorphisms of Path$^p$) by
\[
e_k \equiv \frac{1}{\beta}C_k^\dagger C_k
\]

The reader can check that all Jones-Temperley-Lieb relations between the $e_i$ are satisfied. Essential paths can also be defined as elements of the intersection of the kernels of the Jones projectors $e_i$’s.

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