PACKING 1.35 · 10^{11} RECTANGLES INTO A UNIT SQUARE

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Abstract. It is known that \( \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1. \) In 1968, Meir and Moser asked for finding the smallest \( \epsilon \) such that all the rectangles of sizes \( 1/i \times 1/(i+1) \) for \( i = 1, 2, \ldots \), can be packed into a unit square or a rectangle of area \( 1 + \epsilon \). In this paper, we show that we can pack the first 1.35 · 10^{11} rectangles into the unit square and give an estimate for \( \epsilon \) from this packing.

1. Introduction

Packing of rectangles means that the rectangles have mutually disjoint interiors. In 1968, Meir and Moser [1] asked the following problem: since \( \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1 \), it is reasonable to ask whether the set of rectangles of sizes \( 1/i \times 1/(i+1) \) for \( i = 1, 2, \ldots \), can be packed into a square or rectangle of area \( 1 + \epsilon \). Failing that, find the smallest \( \epsilon \) such that the rectangles can be packed into a rectangle of area \( 1 + \epsilon \).

Meir and Moser proved that \( \epsilon \leq 0.0678 \). Jennings [2], [3] presented packing such that \( \epsilon \leq 0.009877 \). Balint [4], [5] proved that \( \epsilon \leq 0.004004 \). Paulhus [6] made a great progress and showed that \( \epsilon \leq 10^{-9} \). Joó [7] pointed out that the proof of the lemma of Paulhus is not true and proved that \( \epsilon \leq 1.26 \cdot 10^{-9} \) for square container and \( \epsilon \leq 6.878 \cdot 10^{-10} \) for rectangular shape container. Grzegorek and Januszewski [8] fixed the lemma of Paulhus.

Meir and Moser asked a similar question in [1]. Can we pack the squares of side lengths \( 1/2, 1/3, \ldots \) into a rectangle of area \( \sum_{i=2}^{\infty} \frac{1}{i^2} = \pi^2/6 - 1 \)? Failing that, find the smallest \( \epsilon \) such that the reciprocal squares can be packed into a rectangle of area \( \pi^2/6 - 1 + \epsilon \). The answer is not known yet, whether \( \epsilon = 0 \) or \( \epsilon > 0 \), but there are some results. In [1], [2], [9], [6], [8] can be found better and better packings of the squares. Chalcraft [10] generalized this question. He packed the squares of side lengths \( 1, 2^{-t}, 3^{-t}, \ldots \) into a square of area \( \sum_{i=1}^{\infty} i^{-2t} \). He proved that there is a packing of the squares of side lengths \( 1, 2^{-t}, 3^{-t}, \ldots \) into a square of area \( \sum_{i=1}^{\infty} i^{-2t} \) for all \( t \) in the range \( [0.5964, 0.6] \).
Wästlund [11] proved if $1/2 < t < 2/3$, then the squares of side lengths $1, 2^{-t}, 3^{-t}, \ldots$ can be packed into some finite collection of square boxes of the area $\sum_{i=1}^{\infty} i^{-2t}$. Joós [12] packed the squares $1, 2^{-t}, 3^{-t}, \ldots$ into a rectangle of area $\sum_{i=1}^{\infty} i^{-2t}$ for $\log_2 2 < t < 2/3$. Januszewski and Zielonka [13] extended this interval to $(1/2, 2/3]$. In [14] can be found a 3-dimensional generalization of the question of Chalcraft, i.e. a packing of the 3-dimensional cubes of edge lengths $1, 2^{-t}, 3^{-t}, \ldots$ into a 3-box of the right area for all $t$ in the range $[0.36273, 4/11]$. Januszewski and Zielonka [13] extended this interval to $(2/3, 4/11]$. Joós [15] generalized this problem and packed the $d$-cubes of side lengths $1, 2^{-t}, 3^{-t}, \ldots$ into a $d$-box of the right area for all $t$ on an interval. Januszewski and Zielonka [16] extended this interval to $(1/d, 2d^{-1}/(d2d^{-1} - 1)]$.

Tao [17] recently proved that any $1/2 < t < 1$, and any $i_0$ that is sufficiently large depending on $t$, the squares of side length $i^{-t}$ for $i \geq i_0$ can be packed into a square of area $\sum_{i=i_0}^{\infty} i^{-2t}$. Sono [18] generalized the result of Tao and considered the squares of side length $f(i)^{-t}$. He proved that for any $1/2 < t < 1$, there exists a positive integer $i_0$ depending on $t$ such that for any $i \geq i_0$, squares of side length $f(i)^{-t}$ for $i \geq i_0$ can be packed into a square of area $\sum_{i=i_0}^{\infty} f(i)^{-2t}$ if the function $f$ satisfies some suitable conditions. McClenagan [19] proved that if $1/d < t < 1/d^1$, and $i_0$ is sufficiently large depending on $t$, then the $d$-cubes of side length $i^{-t}$ for $i \geq i_0$ can be perfectly packed into a $d$-cube of volume $\sum_{i=i_0}^{\infty} f(i)^{-dt}$.

In the paper of Tao [17] one can read the following for any $1/2 < t < 1$. “We remark that the same argument (with minor notational changes) would also allow one to pack rectangles of dimensions $n^{-t} \times (n+1)^{-t}$ for $n \geq n_0$ perfectly into a square of area $\sum_{n=n_0}^{\infty} \frac{1}{n(n+1)^t}$; we leave the details of this modification to the interested reader.”

The case $t = 1$ is unsolved for packing squares of side lengths $1/2, 1/3, \ldots$ into a square of rectangle of area $\sum_{i=2}^{\infty} 1/i^2 = \pi^2/6 - 1$ or for packing rectangles of dimensions $1 \times 1/2, 1/2 \times 1/3, \ldots$ into a square of rectangle of area $\sum_{i=1}^{\infty} 1/(i(i+1)) = 1$. We consider this last question.

2. Result

We follow the algorithm in [7] and write a Julia program to run out the results for different number of packed rectangles. Our packing result for 1000 rectangles is in Fig. 1.
2.1. **Result from computer program.** We call rectangles that we pack and we call (empty) boxes which are the remaining (rectangular shape) empty spaces to avoid confusion. By using computer program, we have packed the first $10^{11}$ rectangles into the unit square. The left largest empty box (let $E$ be this box) has a width and length as $1.888\,883\,876\,3176\,668 \times 10^{-6} \times 1.888\,893\,876\,343\,809\,9 \times 10^{-6}$. From the sum of the area sequence to $10^{11}$, we can estimate the remaining area to be packed is $\frac{1}{10^{11+1}} < 10^{-11}$. Observe, the side lengths of $E$ are less than $\sqrt{10^{-11}} = \sqrt{10} \cdot 10^{-6}$.

We are going to pack rectangles from $10^{11} + 1$ and on, into $E$. Since $E$ is close to a square, there are at least $188\,888^2 > 3.5 \cdot 10^{19}$ following rectangles can be packed into $E$. So we can pack at least $1.35 \cdot 10^{11}$ into the unit square.

2.2. **Ratio of largest rectangle to the total remaining area.** When we calculate the ratio of largest empty box to the total remaining area, we find a very interesting phenomenon: the ratio is near to 0.36 when number of packed rectangles $n$ becomes larger. The values are listed in Table. 1.

During the packing of the first $10^5$ rectangles into the unit square the ratios of largest empty box to the total remaining area can be seen in Fig. 2.
### Table 1. Ratio table

| number of rectangles | ratio  |
|----------------------|--------|
| $10^9$               | 0.4142 |
| $10^4$               | 0.3441 |
| $10^3$               | 0.3577 |
| $10^6$               | 0.3554 |
| $10^5$               | 0.3502 |
| $10^8$               | 0.3400 |
| $10^9$               | 0.3701 |
| $2 \cdot 10^9$       | 0.3648 |
| $4 \cdot 10^9$       | 0.3580 |
| $5 \cdot 10^9$       | 0.3613 |
| $1 \cdot 10^{10}$    | 0.3687 |
| $2 \cdot 10^{10}$    | 0.3677 |
| $5 \cdot 10^{10}$    | 0.3631 |
| $1 \cdot 10^{11}$    | 0.3568 |

**Figure 2.** The ratios of largest empty box to the total remaining area

### 2.3. Theorem for estimating the area.**

It is easy to prove the following theorem, though the actual area is greater than the Balint’s rectangle area $1 + 6/(5n)$ in [5]. We can use a small trick to decrease the total area of $1 + \epsilon$.

**Theorem 2.1.** If $n - 1$ ($n \geq 1000$) rectangles can be packed into the unit square, then all rectangles $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$ can be packed into the square of side length $1 + 1/n$. The actual packing area is less than $1 + 2/n \times (\ln 2 + 1/2n)$
Proof. Denote the rectangle of dimensions \( \frac{1}{n} \times \frac{1}{n+1} \) by \( P_n \). Assume \( P_1, \ldots, P_{n-1} \) are packed into the unit square. Observe, \( P_n \) can be packed into the square of side length \( \frac{1}{n} \). Pack rectangles from \( n \) to infinity in this way of rows and columns into a large rectangle as Fig. 3 showing. The rectangles from \( P_{2^n-1} \) to \( P_{2^n} \) lie in the \( i \)-th row for \( i = 1, 2, \ldots \). The width of the smallest rectangular shape container is

\[
\frac{1}{n} + \frac{1}{2n} + \frac{1}{4n} + \cdots = \frac{2}{n}
\]

and the length of the smallest rectangular shape container is

\[
\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}.
\]

Since \( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} < \int_n^{2n} \frac{1}{x} \, dx = \ln 2 \), the length of the container:

\[
\frac{1}{n} + \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} \right) - \frac{1}{2n} < \frac{1}{n} + \ln 2 - \frac{1}{2n} = \ln 2 + \frac{1}{2n}.
\]

The length of the \( i \)-th row is \( \ln 2 + \frac{1}{2n} \), \( i = 1, 2, \ldots \). Since \( n \geq 1000 \), this length \( \ln 2 + \frac{1}{2n} \) is less than 1.

The rectangles from \( n \) to infinity are put into a box of \( \frac{2}{n} \times (\ln 2 + \frac{1}{2n}) \). Divide this box into two equal smaller boxes of dimensions \( \frac{1}{n} \times (\ln 2 + \frac{1}{2n}) \) and glue to the unit square as in Fig.4.

We proved that all rectangles can be packed into a square of side length \( (1 + 1/n) \), and it is just a rough estimate, although in two
Figure 4. final packing

stripes it is not efficient to pack them, and can be improved by some ways obviously like [4].

From the computer program result of subsection 2.1, we have

**Corollary 2.2.** The rectangles of dimensions $\frac{1}{n} \times \frac{1}{(n+1)}$ for $n \geq 1$ can be packed into a square of side length $1 + \frac{1}{1.35 \cdot 10^{11}}$, which shows that $\epsilon < 1.49 \cdot 10^{-11}$.

3. Conclusion

We hope to pack more rectangles into the unit square by computer program, though the computer memory is the constraint of improvement. A mathematical proof for this problem might be needed.

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