A NOTE ON THE ZEROES OF THE FREDHOLM SERIES

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With an appendix by F. Veneziano

Dedicated to the memory of Edoardo Vesentini

Abstract. The issue had been raised whether the so-called Fredholm series $z + z^2 + \cdots + z^{2^n} + \cdots$ has infinitely many zeroes in the unit disk. We provide an affirmative answer, proving that in fact every complex number occurs as a value infinitely many times.

1. Introduction

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ denote the complex unit disk and let

$$f(z) = z + z^2 + z^4 + \cdots + z^{2^n} + \cdots, \quad z \in D.$$

This function is proposed in several books as a simple instance of a holomorphic function on $D$ which cannot be holomorphically extended to any connected open domain of $\mathbb{C}$ containing $D$ strictly (see e.g. [13], p. 159, or [14], p. 98). It is often called the Fredholm series, though apparently Fredholm had considered a different series; the terminology was introduced by A. J. Kempner in the belief that Fredholm studied it, and then this was followed by several authors: see J. Shallit’s paper [12], especially pp. 161–162, for an accurate description. (Fredholm is also mentioned in R. Remmert’s book [10], p. 254 but concerning indeed a different series sharing the said property.) We shall not change here this convention, though perhaps the function should be called Kempner series.

It satisfies the functional equation $f(z) = z + f(z^2)$.

The function became better known also through the work of K. Mahler [5], who considered its values at (nonzero) algebraic points in $D$ and proved their transcendency (together with similar results for a much wider class of functions). In fact, concerning the function itself, the transcendency of the values at points $1/n, n \in \mathbb{N}^+$, had been already established by Kempner [4]; these values are interesting and related to iterated paperfolding (see [12], pp. 162–163). All of this gave rise to many-sided generalisations and several works, both of arithmetical and functional nature, by a number of authors. See also D. Masser’s book [8] for an account of Mahler’s method in particular, and for some references, and see [9] p. 132 for Mahler’s reminiscences on this topic.

In the book [8] the issue is raised (see Ex. 3.15) whether the function has infinitely many zeroes in $D$, and it is remarked that Mahler found 16 zeroes (apparently using a pocket calculator). Actually already Mahler enquired about the location of the zeroes in the paper [4], commenting that there are probably zeroes “in every neighborhood of the unit circle”. It is easy to locate the (unique) real nontrivial zero $-0.658626\ldots$ on looking at signs, but otherwise the location of zeroes seems not obvious to detect, especially due to the fairly complicated oscillatory nature of $f(z)$ near the boundary of $D$ (which is related to exponential sums with widely growing arguments - see [7] and see Remarks 3.1, 3.2 below). Of course in any disk $rD$ of fixed radius $r < 1$ there are only finitely many zeroes of $f$, so Mahler’s expectation literally amounts to the existence of an infinity of zeroes.

Note that several ‘general’ methods for studying the zeroes (and the values) of holomorphic functions (e.g. Picard Theorems, Nevanlinna Theory) usually apply to functions with singularities, or defined on the whole complex plane, and do not seem to be immediately helpful here since $f(z)$ is regular on the whole $D$ and has the unit circle as a natural boundary. (See

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however W. Hayman’s book \[2\], especially Chs. 5 and 6, and also W. Rudin’s book \[11\], Chs.
15, 17, for some results about the distribution of zeroes of functions holomorphic in \(D\). See Remark 3.1 below for some comments on this.)

It is the main purpose of this note to prove that indeed there are infinitely many zeroes of
\(f(z)\) in \(D\), confirming the expectation of Mahler. Actually we shall prove a sharpening of this
result, namely that the function attains every complex number as a value infinitely often, and
even arbitrarily near 1.

By the results of Mahler any of the zeroes, apart from 0, is a transcendental number. Masser
asked if for instance one can prove sharper results, e.g. of algebraic independence, or if one can
study more accurately the distribution of zeroes. We have no answer to the former issue; the
latter maybe admits some kind of answer through the present method, but we haven’t carried
out any analysis in this direction (see the final remarks for a few comments).

This note contains also an appendix by Francesco Veneziano, who kindly followed some
requests of mine, on performing several computations concerning this problem; in the Appendix
he lists some approximate zeroes of the function and locates a zero of a related function which
appears in the proof. Also, he represented in Figure 1 below some zeroes of a truncated series,
rescaled with respect to the Poincaré distance in \(D\), from the origin.

Before stating our results, we introduce some notation. We denote as usual \(e(w) = \exp(2\pi i w)\)
and we let \(H = \{w \in \mathbb{C} : \Im w > 0\}\) denote the upper-half plane, with which we shall work in
place of \(D\). Indeed, for \(w \in H\) we set

\[
F(w) := f(e(w)) = e(w) + e(2w) + e(4w) + \ldots, \quad w \in H.
\]

This function is holomorphic in \(H\) and satisfies the functional equations

\[
F(w + 1) = F(w), \quad F(w) = e(w) + F(2w).
\]

Consider now the function

\[
G(w) := \sum_{l=1}^{\infty} e\left(-\frac{1}{2^l}\right)\left(e\left(\frac{w}{2^l}\right) - 1\right), \quad w \in H.
\]

Note that for \(w\) in a given compact set \(K \subset \mathbb{C}\) we have \(|e(w/2^l) - 1| \ll_K 2^{-l}\), hence the
series defining \(G\) converges uniformly and boundedly on compact sets of \(\mathbb{C}\), so in fact \(G\) is in
particular holomorphic on the whole \(\mathbb{C}\).

We further define

\[
S(w) = F(w) + G(w), \quad w \in H.
\]

We have

**Proposition 1.1.** Let \(V = S(H) \subset \mathbb{C}\) be the image of \(S\) on \(H\). Then, for every open disk \(U\)
centered at 1 and for every \(v \in V\), the Fredholm function \(f(z)\) assumes on \(D \cap U\) infinitely
many times the value \(v\).

The following result also shows that the set \(V\) of Proposition 1.1 is stable by certain trans-
lations, thus is in a sense ‘large’.

**Proposition 1.2.** Let \(V = S(H) \subset \mathbb{C}\) be as above the image of \(S\) on \(H\). Then \(V + \mathbb{Z} = V\) and
moreover for each \(m \in \mathbb{Z}\) we have \(V + c_m \subset V\), where \(c_m\) is as in \([12]\) below.

A study of the numbers \(c_m\) carried out in \([3, 2]\) will in fact imply the following sharpening
of the proposition (which essentially then becomes a lemma):

**Theorem 1.3.** We have \(V = \mathbb{C}\), that is, the function \(S(w)\) attains every complex value.

In particular, combining these results we obtain the following consequence, which contains
as a corollary the main motivation for this note:

**Theorem 1.4.** For every open disk \(U\) centered at 1, the Fredholm series attains on \(D \cap U\)
every complex number as a value infinitely many times.
So in particular \( f(z) \) has infinitely many zeroes in \( D \) and even in \( D \cap U \). It may be that the methods allow to replace \( U \) with any disk centered on the unit circle but we have not verified this.

Acknowledgements. I thank David Masser for raising several issues and for remarks and references; similarly I thank Pietro Corvaja. I thank Francesco Veneziano for some computer calculations, especially helpful when writing the paper, and for providing the Appendix. I further thank Jeffrey Shallit for pointing out to me a correct historical account of the terminology, and some references.

2. Some formulae

We now let \( \theta \in \mathbb{R} \) be real and \( n > 0 \) be a positive integer, and seek an approximation formula for \( F(w2^{-n} + \theta) \). On iterating (3) we have

\[
F\left(\frac{w}{2^n} + \theta\right) = e\left(\frac{w}{2^n} + \theta\right) + e\left(\frac{w}{2^{n-1}} + 2\theta\right) + \cdots + e\left(\frac{w}{2} + 2^{n-1}\theta\right) + F(w + 2^n\theta)
\]

(6)

\[
= e\left(\frac{w}{2^n}\right)e(\theta) + e\left(\frac{w}{2^{n-1}}\right)e(2\theta) + \cdots + e\left(\frac{w}{2}\right)e(2^{n-1}\theta) + F(w + 2^n\theta).
\]

We shall use this formula, with suitable choices for the involved quantities, to approximate \( F \) with convenient functions.

2.0.1. A Ramanujan sum and related ones. As remarked by Mahler in [2] the behaviour of \( F(w) \) (for rational \( Rw \)) is linked with certain Gaussian periods. Here we adopt a related but different viewpoint, constructing vanishing Ramanujan sums.

We let \( q = 3^k \), where \( k \geq 2 \) is an integer. Note the Euler function value \( \phi(q) = 2 \cdot 3^{k-1} \).

Since 2 is a primitive root modulo 3 \([1] \) and since \( 2^2 \not\equiv 1 \pmod{9} \), it follows easily (e.g. by induction on \( k \)) that 2 is a primitive root modulo \( q \), namely the powers \( 2^n, 0 \leq m \leq \phi(q) - 1 \) form a complete set of residues modulo \( q \) and coprime to \( q \). Since \( q \) is not squarefree, it follows (see e.g. [3], §16.6, especially Thm. 271) that, for any integer \( s \) coprime with \( q \), the Ramanujan sum, i.e. of the primitive \( q \)-th roots of unity, vanishes, which amounts to:

\[
\sum_{m=0}^{\phi(q)-1} e\left(\frac{2^ms}{q}\right) = 0, \quad \gcd(s, q) = 1,
\]

the same equation actually holding when the summation for \( m \) runs through a complete set of residues modulo \( \phi(q) \).

We now choose an integer \( a \geq 1 \) and \( k = 2^a, \) a power of 2. We put

\[
\theta_0 = \frac{1}{q}, \quad n_0 = \phi(q) = 2 \cdot 3^{a-1}.
\]

We also note that \( q \equiv 1 \pmod{2^{a+2}} \), as follows easily by induction on \( a \geq 1 \). Hence for \( l \leq a + 2 \) the ratio \( (1 - q)/2^l \) is an integer. Also, we have for any integer \( n \geq l, \)

\[
2^{n-l}\theta_0 \equiv 2^n\left(\frac{1-q}{2^l}\right)\theta_0 \pmod{\mathbb{Z}}, \quad 0 \leq l \leq a.
\]

In particular, let \( n = n_0 \), so \( 2^n \equiv 1 \pmod{q} \). Then for \( l \leq a \leq n_0 \), on writing \( 2^{n_0} = 1 + q \cdot \text{integer} \), this congruence yields

\[
2^{n_0-l}\theta_0 \equiv \frac{(1 - q)}{2^q} \theta_0 - \frac{1}{2^l} \pmod{\mathbb{Z}}, \quad 0 \leq l \leq a.
\]

\footnote{In what follows this choice is immaterial and there is wide possibility of changing it.}
2.0.2. Preparing an approximate formula. Setting \( s = 1 \) into (7) we obtain
\[
\sum_{l=0}^{n_0-1} e(2^l \theta_0) = 0,
\]
whence, setting \( \theta = \theta_0, n = n_0 \) into (8) we derive, taking into account that \( 2^n \theta_0 \equiv \theta_0 \mod \mathbb{Z} \) and that \( F \) is \( \mathbb{Z} \)-periodic,
\[
F\left(\frac{w}{2^n} + \theta_0\right) = \sum_{l=0}^{n_0-1} e(2^l \theta_0) \left( e\left(\frac{w}{2^n}\right) - 1 \right) + F(w + \theta_0).
\]
Further, on writing \( n_0 - l \) in place of \( l \) in the sum and using (8) for \( 1 \leq l \leq a \), this becomes
\[
(9) \quad F\left(\frac{w}{2^n} + \theta_0\right) = \sum_{l=1}^{n_0} e\left(\frac{\theta_0 - 1}{2^l}\right) \left( e\left(\frac{w}{2^n}\right) - 1 \right) + \sum_{l=a+1}^{n_0} e(2^n - l) \left( e\left(\frac{w}{2^n}\right) - 1 \right) + F(w + \theta_0).
\]

3. Proofs of main assertions

Proof of Proposition (11). Let \( K \subset \mathcal{H} \) be a given compact subset of \( \mathcal{H} \), to which we shall refer by a subscript to indicate dependence of the implicit constants which shall appear. For \( w \in K \) we have
\[
|F(w + \theta_0) - F(w)| \ll_K \frac{1}{q}, \quad \left| e\left(\frac{w}{2^n}\right) - 1 \right| \ll_K \frac{1}{2^n}, \quad \left| e\left(\frac{\theta_0 - 1}{2^l}\right) - e\left(-\frac{1}{2^l}\right) \right| \ll K \frac{1}{2^l} \quad (l \geq 1),
\]
whence, by (9), again for \( w \in K \),
\[
(10) \quad \left| F\left(\frac{w}{2^n} + \theta_0\right) - S(w) \right| \ll_K \sum_{l=a+1}^{\infty} \left| e\left(\frac{w}{2^n}\right) - 1 \right| + \frac{1}{q} \ll_K \frac{1}{2^n}.
\]
The proof is now easily completed by a quite standard argument. Let \( v \in V \) be any value of \( S \) on \( \mathcal{H} \), say \( v = S(w_0) \) for some \( w_0 \in \mathcal{H} \). Let \( C \) be a circle inside \( \mathcal{H} \) and centred at \( w_0 \) such that \( S(w) \neq v \) for \( w \in C \), and choose \( K \) to be a closed disk inside \( \mathcal{H} \), centred at \( w_0 \), and of radius larger than that of \( C \).
The function \( S(w) - v \) is holomorphic in \( K \) and never zero on \( C \), hence its absolute value attains on \( C \) a minimum \( \delta > 0 \).

Put now, for any integer \( a \geq 1 \), \( F_a(w) = F(2^{-na}w + \theta_0) \), where \( n_0, \theta_0 \) are as above, so by (10), \( |F_a(w) - S(w)| \leq c_K 2^{-a} \) for \( w \in K \), where \( c_K \) is a constant, i.e. independent of \( a \). Now suppose that \( a \) is so large that \( 2c_K < \delta 2^a \). Then on \( C \) we have
\[
|F_a(w) - S(w)| \leq \frac{\delta}{2} < \min_{w \in C} |S(w) - v|.
\]
By Rouche’s theorem (see e.g. [13]) the functions \( S(w) - v, F_a(w) - v \) have the same number of zeroes inside \( C \), so since the former vanishes at \( w_0 \) (which is the centre of \( C \)) the latter has at least a zero \( \zeta_a \) inside \( C \).

This means that \( F(2^{-na} \zeta_a + \theta_0) = v \). Now, the points \( 2^{-na} \zeta_a + \theta_0 \in \mathcal{H} \), for varying \( a \), tend to 0 (because \( \zeta_a \in K \)), so the corresponding images through \( e(w) \) tend to 1 (but are different from 1 as they lie in \( D \)). This proves that, if \( U \subset C \) is any neighbourhood of 1, \( f(z) \) attains the value \( v \) at infinitely many points of \( C \cap U \), concluding the argument.

3.1. About the function \( S(w) \). In this subsection we develop some properties of the function \( S(w) \), in particular of its image on \( \mathcal{H} \), above denoted \( V \).

It will be equivalent and notationally convenient to work with the function \( S_1(w) := S(w+1) \).

By definition we have
\[
(11) \quad S_1(w) = F(w+1) + G(w+1) = F(w) + \sum_{l=1}^{\infty} \left( e\left(\frac{w}{2^n}\right) - e\left(-\frac{1}{2^l}\right) \right).
\]
For integers $m \in \mathbb{Z}$, also define constants $c_m$ by

$$(12) \quad c_m := \sum_{l=1}^{\infty} \left( e \left( \frac{m}{2^l} \right) - 1 \right).$$

Hence, setting $H(w) := \sum_{l \geq 1} \left( e \left( \frac{w}{2^l} \right) - 1 \right)$, we may rewrite (11) as

$$(13) \quad S_1(w) = F(w) + \sum_{l=1}^{\infty} \left( e \left( \frac{w}{2^l} \right) - 1 \right) + \sum_{l=1}^{\infty} \left( 1 - e \left( \frac{-1}{2^l} \right) \right) = F(w) + H(w) - c_{-1}.$$  

Note that $H$ is holomorphic in the whole $\mathbb{C}$ and $c_m = H(m)$.

As Masser pointed out to me, the function $F(w) + H(w)$ appears already in S. Ramanujan’s work (see [14], p. 39); it also appears (as a function on $D$) in Exercise 3.11, p. 44 of Masser’s book [16] and in the previous article of lectures by Masser in [17] (see Lecture 3). It is observed that it satisfies a functional equation when (in the present notation) $w$ is changed into $2w$. Indeed, we easily find that

$$(14) \quad S_1(2w) = S_1(w) - 1,$$

This already shows that $V + Z = V$. Note also that this yields (Masser’s observation) that the function $S_1(w) + (\log w/\log 2)$ (which is well-defined in $\mathcal{H}$ on agreeing for instance on the value $\log i = \pi/2$) is holomorphic on $\mathcal{H}$ and invariant under $w \to 2w$, as pointed out in the above quoted sources.

Let us now seek other functional equations. Setting $w + m2^s$ in place of $w$, for integers $m \in \mathbb{Z}$, $s \geq 0$, we obtain

$$(15) \quad S_1(w + m2^s) = S_1(w) + \sum_{l=s+1}^{\infty} e \left( \frac{w}{2^l} \right) \left( e \left( \frac{m}{2^{l-s}} \right) - 1 \right).$$

say. Note that for $w$ in a compact subset $K$ of $\mathcal{H}$, we have

$$(16) \quad \Delta_{m,s}(w) := \sum_{l=1}^{\infty} e \left( \frac{w}{2^{l+s}} \right) \left( e \left( \frac{m}{2^l} \right) - 1 \right) = O_K \left( \frac{1}{2^s} \right).$$

**Proof of Proposition 1.2.** We have already noted above that, in view of (14), the image of $S$ is invariant under translation by integers, which yields the first claim of the proposition.

Let now $K$ be any compact disk inside $\mathcal{H}$ and consider formula (15). Note that by (16), for $s \to \infty$ (and fixed $m$ for instance) the functions $\Delta_{m,s}$ converge uniformly and boundedly to 0 for $w \in K$. Then, exactly by the same argument as in the proof of Proposition 1.1 if $v = S_1(w_0)$ is a value of $S_1$ at a point $w_0$ in the interior of $K$, then for large $s$ (and fixed $m$) the function $S_1(w + m2^s) - c_m$ assumes as well that value. Hence $S_1$ (and therefore $S$) assumes the value $v + c_m$, proving the contention.

Before going ahead, we pause to give a proof of the special case of Theorem 1.4 in which $\mathbb{C}$ is replaced by $\mathbb{R}$ as a set of possible values. This proof is a bit simpler than the general case and does not require the considerations of 4.3.2.

**Proof of Theorem 1.4 for real values.** By Proposition 1.1, it suffices to prove that $\mathbb{R} \subset V$ (i.e. that $S_1$ attains all real numbers as values on $\mathcal{H}$). Also, by Proposition 1.2 applied with $m = -1$, it suffices to show that $\mathbb{R} \subset c_{-1} \subset V$; namely, using (13), it suffices to show that $F(w) + H(w)$ assumes any real value in $\mathcal{H}$. Let us look at the values for purely imaginary $w = it$, $t \in \mathbb{R}$. These values are real and by the formula $S_1(2w) = S_1(w) - 1$ this set of real values is invariant by any integer translation. Thus it suffices to prove that it contains an interval of length 1; on the other hand this is clear by the same formula $S_1(2w) = S_1(w) - 1$, since the imaginary line is connected.
3.2. About the constants $c_m$. Note that $c_{2m} = c_m$ and that $c_m = c_{-m}$. As remarked above, $c_m$ is just the value of $H(z)$ at $z = m$. These numbers seem interesting on their own and we wonder whether they have irrationality or transcendency properties. We cannot prove any neat assertion of this type, but some small information in this direction will appear below.

Proposition 1.2 says that $V$ is stable under translation by the semigroup generated by $Z$ and the $c_m$. Let $\sigma_m := c_m + c_{-m}$ noting it is real, and in particular we find that $V$ is stable under translation by the real semigroup generated by $Z$ and the $\sigma_m$. We shall study a bit this semigroup, denoted here $\Gamma$.

Now, it turns out that a related sequence has better approximation properties, useful for our purposes.

Define $k_l := e(1/2^l) - 1$. We have $k_0 = 0$, $k_1 = -2$, $k_2 = i - 1$, $k_3 = (\sqrt{2} - 1 + i\sqrt{2})/2$ and $k_l = k_{l+1}(k_{l+1} + 2)$. Setting $\mu_l := |k_l|^2$ we also have $\mu_1 = 2 - 2\cos(\pi/2^{l-1})$ and $\mu_{l+1} = 2 - \sqrt{4 - \mu_l}$.

Further, $c(m/2^l) = (k_l + 1)^m$ and $c_m = \sum_{l=1}^{\infty} ((k_l + 1)^m - 1)$. For $m \in \mathbb{Z}$ we now set

$$c(m) = \sum_{l=1}^{\infty} k_l^m.$$  \hfill (17)

Note that the $k_l$ tend to 0 exponentially so this formula yields good approximations of the $c(m)$ (by linear recurrence sequences) on taking truncations of the series.

These numbers are linear combinations of the $c_m$ with integer coefficients and conversely. In fact $c(m) = \sum_{l=1}^{\infty} (c(m/2^l) - \binom{m}{l+1} c((m-1)/2^l) + \ldots = c_m - \binom{m}{1} c_{m-1} + \ldots + (-1)^{m-1} c_1$.

Similarly, $c_m = \sum_{l=0}^{\infty} (k_l^m + \binom{m}{l} k_l^{m-1} + \ldots + \binom{m}{1} k_l) = c(m) + \binom{m}{1} c(m-1) + \ldots + \binom{m}{1} c(1)$.

For the sake of curiosity, we also note the recurrence $c(m) = \sum_{l=0}^{\infty} \binom{l}{l+1} c(k_{l+1} + 2)^m = c(2m) + 2\binom{m}{1} c(2m-1) + 4\binom{m}{2} c(2m-2) + \ldots + 2^{m-1} \binom{m}{1} c(m+1) + 2^m c(m)$, whence

$$(1 - 2^m) c(m) = \sum_{h=1}^{m} 2^{m-h} \binom{m}{h} c(m+h),$$

so the $c(m)$ satisfy several linear relations with rational coefficients. These correspond in fact to the more evident relations $c_{2m} = c_m$. We wonder whether there are other such linear relations holding among them.

Now, for our purposes it is more useful to consider the numbers

$$\sigma(m) := \sum_{l=1}^{\infty} (k_l + \bar{k}_l)^m = \sum_{l=1}^{\infty} \left( e\left(\frac{1}{2^l}\right) + e\left(-\frac{1}{2^l}\right) - 2\right)^m.$$  

Observe that, for $2h \neq m$, $k_l^{h} k_l^{m-h} + k_l^{m-h} k_l^{h}$ equals an integer linear combination of terms $e(s/2^l) + e(-s/2^l) - 2 = \sigma_s$: this is seen on expanding $k_l^{h} = (e(1/2^l) - 1)^h$ with the binomial theorem (and similarly for the complex conjugate). The same holds for a term $k_l^{h} k_l^{h}$ in case $2h = m$. On summing over $l$ we deduce that $\sigma(m)$ is an integer linear combination of $\sigma_1, \ldots, \sigma_m$. (One can also write down an explicit formula in terms of Chebyshev polynomials.)

Now, $(-1)^n \sigma(m)$ is of the shape $4^m + 2^m + O((2 - \sqrt{2})^m)$. We conclude that for growing $m$, either one among $\sigma_1, \ldots, \sigma_m$ is irrational, or the least common denominator of these numbers grows to infinity. In fact, if all $\sigma_j$ would be rationals with a bounded denominator, the said relation would imply the same for $\sigma(m)$, whence, since $0 < 2 - \sqrt{2} < 1$, for large $m$ we would have $\sigma(m) = (k_l + \bar{k}_l)^m + (k_l + \bar{k}_l)^m$, which is plainly false.

We in turn deduce that the semigroup $\Gamma$ generated by $Z$ and the $\sigma_m$ is dense in $\mathbb{R}$. In fact, if some $\sigma_m$ is irrational this follows from well-known (easy) theorems, and otherwise it follows from the deduction just obtained.

3.3. Concluding arguments. We can now use the results just obtained to complete the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3 and Theorem 1.4. In view of Proposition 1.1 it suffices to prove Theorem 1.3...
horizontal strip around the line \( \Im w = \Im z_0 \). In particular, \( V \) contains the line \( \Im w = \Im z_0 \) itself. Hence \( V \) contains any line \( \Im w = t_0 \) as soon as it contains a point in such a line. But since \( V \) is invariant under translation by \( c_{\pm 1} \neq 0 \), thus by any multiple \( sc_{\pm 1} \) (any integer \( s > 0 \), any sign), the imaginary parts of elements of \( V \) are unbounded both from above and from below. But the set of such imaginary parts is connected, hence is the whole \( \mathbb{R} \). Combining this with the above deduction, we conclude that \( V = \mathbb{C} \), as required.

**Remark 3.1.** Concerning the distribution of zeroes of \( f(z) \), one could also think for instance of using criteria such as Theorem 15.23 of [11], which would predict a convergent sum \( \sum(1 - |\alpha_i|) \) for the zeros \( \alpha_i \), provided \( \int_1^{2\pi} |f(re(\theta))|d\theta \) remained bounded as \( r \to 1^- \).

However one may show that this last condition does not hold for the present function, that is, \( f \) does not belong to the class denoted \( N \) (Nevanlinna) in [11]. We give a very brief sketch for this claim. One first may expand \( f(re(\theta)) \) for \( r = e^{-c/2^n} \) \((1/2 < c \leq 1)\), using the first \( n \) terms and bounding the remainder by a constant (see e.g. [6] above). Approximating in turn \( r^n = 1 + O(m/2^n) \), \( m = 1, 2, \ldots, 2^{n-1} \), for each of these \( n \) terms, one is reduced to estimate the exponential sums \( s(\theta) = s_n(\theta) = e(\theta) + e(2\theta) + \ldots + e(2^{n-1}\theta) \) in absolute value. (See also [7] for such approximate formulae.)

In conclusion, if the said integral would be bounded then it is easy to see that, in particular, the measure of the set \( A = A_n = \{ \theta \in [0, 1]: |s(\theta)| > \sqrt{n}/2 \} \) would tend to zero, hence would be less than \( \epsilon \), for any prescribed \( \epsilon > 0 \) and large enough \( n \). Now, however, we easily find the moments \( \int_0^1 |s(\theta)|^2d\theta = n, \int_0^1 |s(\theta)|^4d\theta = 2n^2 - n \). On the one hand \( \int_A |s|^2 \geq n - (\sqrt{n}/2)^2 = 3n/4 \), on the other hand \( \int_A |s|^4 \leq 2\sqrt{n} \). This is false for \( \epsilon < 9/32 \), proving the claim.

We do not know if nevertheless the series \( \sum(1 - |\alpha_i|) \) converges (though we are inclined to believe it does not).

**Remark 3.2.** Quite possibly there are other methods to prove the results of the paper (and maybe more). For instance, one could apply Ahlfors’ theory (see especially [2], Ch. 5). Also, on approximating \( f(z) \) as indicated in the previous remark (or see [7]), one should obtain that for \( r = e^{-c/4^n}, (1/2 < c \leq 1) \), \( f(re(\theta)) \) is approximated by \( s_{2n}(\theta) \) up to a bounded function. In turn, we have \( s_{2n}(\theta) = s_n(\theta) + s_n(2^n\theta) \), and on changing \( \theta \) to \( \theta_0 := \theta + (a/2^n) \) for an arbitrary integer \( a \), we get \( s_{2n}(\theta_0) = s_n(\theta_0) + s_n(2^n\theta) \). On varying \( a \), the first term essentially varies along the whole set of values of \( s_n \), whereas the second term remains constant.

One nearly reduces to study \( s_n(u) + s_n(v) \) for independent variables \( u, v \), and one can also iterate (and compare with a compatible probabilistic distribution). This possibly leads to a better understanding of the image of \( f(re(\theta)) \) for fixed \( r \) approaching 1, and in turn an understanding of the distribution of the values attained by the function.

**Appendix by Francesco Veneziano**

**Finding a zero of \( S(w) \).** Here we will prove computationally that the holomorphic function \( S(w) \) defined in [5] and expressed by the series

\[
S(w) = \sum_{n=0}^{\infty} \left( e(w2^n) + e\left(\frac{w-1}{2^{n+1}}\right) - e\left(-\frac{1}{2^{n+1}}\right)\right)
\]

has a zero in the upper-half plane. This fact, together with Proposition [1.1], provides an alternative proof of the main assertion that the Fredholm series has infinitely many zeroes, independent from the study of the quantities \( c_m \), carried out in Section 3.2.

Thanks to uniform convergence on compacts, we can write

\[
S^{(k)}(w) = \sum_{n=0}^{\infty} \left( (2^{n+1}\pi i)^k e(w2^n) + \left(\frac{\pi i}{2^n}\right)^k e\left(\frac{w-1}{2^{n+1}}\right)\right).
\]

Writing \( t = \Im w \) we have

\[
|S^{(k)}(w)| \leq (2\pi)^k \sum_{n=0}^{\infty} 2^{kn} e^{-2^{n+1}\pi t} + \pi^k \sum_{n=0}^{\infty} 2^{-kn} e^{-2\pi t}.
\]
Set now $a = \log_2 \frac{k}{2\pi t}$, and let $n$ be an integer, $n \geq [a] + 1$. By standard calculus we find
\begin{equation}
2^{kn} e^{-2^{n+1} \pi t} = e^{kn} \log 2 - 2^{n+1} \pi t \leq e^{-\frac{1}{2} k e^{-\frac{(4k+2)^2}{2} k}}.
\end{equation}
Splitting the first sum of (18) at the summand $[a]$ we get
\[
\left| S^{(k)}(w) \right| \leq (2\pi)^k \sum_{n=0}^{[a]} 2^{kn} e^{-2^{n+1} \pi t} + (2\pi)^k \sum_{n=[a]+1}^{\infty} 2^{kn} e^{-2^{n+1} \pi t} + \pi k \sum_{n=0}^{\infty} 2^{-kn} e^{-\pi^2 k}.
\]
Estimating the first and third terms with geometrical sums, putting these terms together, and using (19) to estimate the middle term we get, for $k \geq 2$,
\[
\left| S^{(k)}(w) \right| \leq \frac{(2\pi)^k}{2^k - 1} 2^{k([a]+1)} + 3^{k+1} \leq \frac{4}{3} (2^{a+2})^k + 3^{k+1}.
\]
From the definition of $a$ and the elementary inequality $k! \leq \frac{k^k}{e}$, we eventually obtain
\begin{equation}
\left| S^{(k)}(w) \right| \leq \left( \frac{2e}{t} \right)^k k! + 3^{k+1}.
\end{equation}
Consider now a point $w_0$ on the upper-half plane, with imaginary part $t_0$. For a point $w$ at distance $\rho$ from $w_0$ using (20) we have that
\[
\left| S(w) \right| = \left| \sum_{k=0}^{\infty} \frac{S^{(k)}(w_0)}{k!} (w - w_0)^k \right| \geq \left| S'(w_0) \right| \rho - \left| S(w_0) \right| - \sum_{k=2}^{\infty} \frac{\left| S^{(k)}(w_0) \right|}{k!} \rho^k
\geq \left| S'(w_0) \right| \rho - \left| S(w_0) \right| - \sum_{k=2}^{\infty} \frac{2e\rho}{t_0} \rho^k - 3 \sum_{k=2}^{\infty} \frac{(3\rho)^k}{k!}
= \left| S'(w_0) \right| \rho - \left| S(w_0) \right| - \frac{4e^2 \rho^2}{t_0(t_0 - 2e\rho)} - 3e^{3\rho} + 9\rho + 3.
\]
Now we can finally set $w_0 = -0.177323882 + 0.144626388i$ and $\rho = 0.002$ and check numerically that $|S(w)| > |S(w_0)|$ on the circle of centre $w_0$ and radius $\rho$, thus proving that within that disk lies a zero of $S(w)$.

**Some zeroes of the Fredholm series.** As shown, the Fredholm series has infinitely many zeroes. We list here the approximate values of a few of them, other than the zero at $z = 0$ and the only other real zero at $z = -0.65862675\ldots$

$$
\begin{align*}
-0.39753706 \pm 0.33482855i, & \quad -0.92536435 \pm 0.35488765i, & \quad -0.82600557 \pm 0.55112556i, \\
-0.68520628 \pm 0.67053411i, & \quad -0.55421602 \pm 0.82701155i, & \quad -0.39621064 \pm 0.92165325i, \\
0.12031484 \pm 0.93460594i, & \quad 0.14635991 \pm 0.98490932i, & \quad 0.18459117 \pm 0.95835110i, \\
0.39186275 \pm 0.89825762i, & \quad 0.74745993 \pm 0.65768729i, & \quad 0.7754337 \pm 0.61813413i.
\end{align*}
$$

Of course the zeroes tend to accumulate towards the unit circle. Figure 1 shows the zeroes of the 13th partial sum which lie in the interior of the unit circle (there are 1126 of them); in order to visualize them better, we have rescaled their distances from the origin according to the hyperbolic metric in the Poincaré disk model. In other words, each zero $\rho e^{i\theta}$ in the unit circle has been mapped in the picture to the point $\log \left( \frac{1+\rho}{1-\rho} \right) e^{i\theta}$.

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A NOTE ON THE ZEROES OF THE FREDHOLM SERIES

Figure 1.

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