Thermal entanglement in a two-qubit Heisenberg XXZ spin chain under an inhomogeneous magnetic field

Guo-Feng Zhang

State Key Laboratory for Superlattices and Microstructures,
Institute of Semiconductors, Chinese Academy of Sciences,
P. O. Box 912, Beijing 100083, P. R. China

Shu-Shen Li

China Center of Advanced Science and Technology (CCAST)
(World Laboratory), P.O. Box 8730, Beijing 100080,
China and State Key Laboratory for Superlattices and Microstructures,
Institute of Semiconductors, Chinese Academy of Sciences,
P.O. Box 912, Beijing 100083, China

Abstract

The thermal entanglement in a two-qubit Heisenberg XXZ spin chain is investigated under an inhomogeneous magnetic field $b$. We show that the ground-state entanglement is independent of the interaction of $z$-component $J_z$. The thermal entanglement at the fixed temperature can be enhanced when $J_z$ increases. We strictly show that for any temperature $T$ and $J_z$ the entanglement is symmetric with respect to zero inhomogeneous magnetic field, and the critical inhomogeneous magnetic field $b_c$ is independent of $J_z$. The critical magnetic field $B_c$ increases with the increasing $|b|$ but the maximum entanglement value that the system can arrive becomes smaller.

PACS numbers: 03.67.Lx; 03.65.Ud; 75.10.Jm; 05.50.+q

* Corresponding author; electronic address: gf1978zhang2001@yahoo.com
I. INTRODUCTION

Entanglement is the most fascinating features of quantum mechanics and plays a central role in quantum information processing. In recent years, there has been an ongoing effort to characterize qualitatively and quantitatively the entanglement properties of condensed matter systems and apply them in quantum communication and information. An important emerging field is the quantum entanglement in solid state systems such as spin chains [1, 2, 3, 4, 5, 6, 7, 8]. Spin chains are the natural candidates for the realization of the entanglement compared with the other physics systems. The spin chains not only have useful applications such as the quantum state transfer, but also display rich entanglement features [9]. The Heisenberg chain, the simplest spin chain, has been used to construct a quantum computer and quantum dots [10]. By suitable coding, the Heisenberg interaction alone can be used for quantum computation [11, 12, 13]. The thermal entanglement, which differs from the other kinds of entanglements by its advantages of stability for the reduction in entanglement of an entangled state due to various sources of decoherence and in entanglement in time due to thermal interactions are absent as the entanglement at finite temperature takes thermal decoherence into account implicitly, requires neither measurement nor controlled switching of interactions in the preparing process, and hence becomes an important quantity of systems for the purpose of quantum computing. In the studies on the entanglement of Heisenberg spin model, a lot of interesting work have been done [14, 15, 16]. It is turned out that the critical magnetic field \( B_c \) is increased by introducing the interaction of the \( z \)-component of two neighboring spin in Ref [17]. But only the uniform field case is carefully studied in the above-mentioned papers. The nonuniform case is rarely taken into account. We know that in any solid state construction of qubits, there is always the possibility of inhomogeneous zeeman coupling [18, 19]. So it is necessary to consider the entanglement for a nonuniform field case. M.Asoudeh and V.Karimipour [20] studied the effect of inhomogeneous in the magnetic field on the thermal entanglement of an isotropic two-qubit \( XXX \) spin system. We noticed that the entanglement for a \( XXZ \) spin model in an nonuniform field has not been discussed. Although M.Asoudeh [20] states that the different types of anisotropic interactions may not be of much practical relevance to concrete physical realization of qubits, in the theoretical analysis we think it is very interesting and should be included in the studies of spin chain entanglement. This is the main motivation of this paper.
For a system in equilibrium at temperature $T$, the density matrix is \( \rho = \frac{1}{Z} e^{-\beta H} \), where \( \beta = \frac{1}{kT} \), \( k \) is the Boltzman constant and \( Z = tr e^{-\beta H} \) is the partition function. For simplicity, we write \( k = 1 \). The entanglement of two qubits can be measured by the concurrence \( C \) which is written as \( C = \max[0, 2 \max[\lambda_i] - \sum_i^4 \lambda_i] \), where \( \lambda_i \) are the square roots of the eigenvalues of the matrix \( R = \rho S \rho^* S \), \( \rho \) is the density matrix, \( S = \sigma_y \otimes \sigma_y \) and \( * \) stands for the complex conjugate. The concurrence is available, no matter whether \( \rho \) is pure or mixed. In case that the state is pure \( \rho = |\psi\rangle \langle \psi| \) with

\[
|\psi\rangle = a|0, 0\rangle + b|0, 1\rangle + c|1, 0\rangle + d|1, 1\rangle,
\]

the concurrence is simplified to

\[
C(\psi) = 2|ad - bc|.
\]

II. THE MODEL AND THE GROUND-STATE ENTANGLEMENT

The Hamiltonian of the \( N \)-qubit anisotropic Heisenberg XXZ model under an inhomogeneous magnetic field is

\[
H = \frac{1}{2} \sum_{i=1}^{N} [J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z + (B + b)\sigma_i^z + (B - b)\sigma_{i+1}^z],
\]

where \( J \) and \( J_z \) are the real coupling coefficients. The coupling constant \( J > 0 \) and \( J_z > 0 \) corresponds to the antiferromagnetic case, \( J < 0 \) and \( J_z < 0 \) the ferromagnetic case. \( B \geq 0 \) is restricted, and the magnetic fields on the two spins have been so parameterized that \( b \) controls the degree of inhomogeneity. Now, we consider the Hamiltonian for \( N = 2 \) case. Note that we are working in units so that \( B, b, J \) and \( J_z \) are dimensionless.

In the standard basis \( \{|1, 1\}, |1, 0\}, |0, 1\}, |0, 0\}\), the Hamiltonian can be expressed as

\[
H = \begin{pmatrix}
\frac{J+2B}{2} & 0 & 0 & 0 \\
0 & -J_z+2b & J & 0 \\
0 & J & -J_z-2b & 0 \\
0 & 0 & 0 & \frac{J_z-2B}{2}
\end{pmatrix},
\]

A straightforward calculation gives the following eigenstates:

\[
|\phi_1\rangle = |0, 0\rangle.
\]
\[ | \phi_2 \rangle = |1,1 \rangle, \]
\[ | \phi_3 \rangle = \frac{1}{\sqrt{1 + \xi^2/J^2}} \left( \frac{\xi}{J} |1,0 \rangle + |0,1 \rangle \right), \]
\[ | \phi_4 \rangle = \frac{1}{\sqrt{1 + \zeta^2/J^2}} \left( \frac{\zeta}{J} |1,0 \rangle + |0,1 \rangle \right), \]
\[ \quad \text{(5)} \]

with corresponding energies:
\[ E_1 = \frac{1}{2}(J_z - 2B), \]
\[ E_2 = \frac{1}{2}(J_z + 2B), \]
\[ E_3 = -\frac{J_z}{2} - \eta, \]
\[ E_4 = -\frac{J_z}{2} + \eta. \]
\[ \quad \text{(6)} \]

where \( \eta = \sqrt{b^2 + J^2} \), \( \xi = b - \eta \) and \( \zeta = b + \eta \). Note that when \( b \to 0 \) and \( J > 0 \), the two states \( | \phi_3 \rangle \) and \( | \phi_4 \rangle \) respectively go to the maximally entangled state \( \frac{1}{\sqrt{2}}(|0,1 \rangle - |1,0 \rangle) \) and \( \frac{1}{\sqrt{2}}(|0,1 \rangle + |1,0 \rangle) \). For \( J < 0 \), they respectively go to \( \frac{1}{\sqrt{2}}(|0,1 \rangle + |1,0 \rangle) \) and \( \frac{1}{\sqrt{2}}(|0,1 \rangle - |1,0 \rangle) \).

We can also find that the eigenenergies are even function of the coupling constant \( J \). So we can think the ground-state entanglement exists for both antiferromagnetic and ferromagnetic cases and should be symmetric with respect to the coupling constant \( J \). The ground state depends on the value of the magnetic field \( B \), the coupling constant \( J_z \) and \( \eta \). It is readily found that the ground-state energy is equal to
\[ \begin{cases} 
E_1 = \frac{1}{2}(J_z - 2B), & \text{if } \eta < B - J_z; \\
E_3 = -\frac{J_z}{2} - \eta, & \text{if } \eta > B - J_z. 
\end{cases} \]
\[ \quad \text{(7)} \]

So when \( \eta < B - J_z \), the ground state is the distangled state \( | \phi_1 \rangle \) and when \( \eta > B - J_z \), the ground state is the entangled state \( | \phi_3 \rangle \). For each value of the magnetic field \( B \), there is a threshold parameter \( J^*_f = B - \eta \) above which the ground state will become entangled. Accordingly for each value of inhomogeneity \( \eta \) there is a value of magnetic field \( B^f = \eta + J_z \) above which the ground state will loose its entanglement. In the entangled phase the entanglement of the ground state is found from (2) and (5) to be
\[ C(| \phi_3 \rangle) = \frac{2|\lambda|}{1 + \lambda^2}, \]
\[ \quad \text{(8)} \]

where \( \lambda = \xi/J \). \( \lambda = \pm 1 \), i.e. \( b = 0 \), the system enters the maximally entangled phase \( | \phi_3 \rangle \) with entanglement \( C(| \phi_3 \rangle) = 1 \). This result accords with that in Ref.\[20\]. Here we can also
know that the ground-state entanglement is independent of the interaction of $z$-component $J_z$.

III. THE THERMAL ENTANGLEMENT

As the thermal fluctuation is introducing into the system, the entangled ground states will be mixed with the untangled excited state. This effect will make the entanglement decreases. At the same time, the distangled ground state mixes with entangled excited states. To see the change of the entanglement, we calculate the entanglement of the thermal state $\rho = \frac{1}{Z} e^{-\beta H}$. In the standard basis $\{ |1, 1\rangle, |1, 0\rangle, |0, 1\rangle, |0, 0\rangle \}$, the density matrix of the system can be written as

$$\rho_{12} = \frac{1}{Z} \begin{pmatrix} e^{-\frac{E_1}{kT}} & 0 & 0 & 0 \\ 0 & e^{\frac{J_z}{2kT}}(m - n) & -s & 0 \\ 0 & -s & e^{\frac{J_z}{2kT}}(m + n) & 0 \\ 0 & 0 & 0 & e^{-\frac{E_1}{kT}} \end{pmatrix}$$

(9)

where $Z = e^{-\frac{E_1}{kT}} (1 + e^{\frac{2B}{kT}}) + 2e^{\frac{J_z+B}{kT}} \cosh \frac{\eta}{kT}$, $m = \cosh \frac{\eta}{kT}$, $n = \frac{b \sinh \frac{\eta}{kT}}{\eta}$, $s = \frac{e^{\frac{J_z}{2kT}} J \sinh \frac{\eta}{kT}}{\eta}$. In the following calculation, we will write the Boltzman constant $k = 1$. From Eq.(9) and the definition of concurrence, we can obtain the concurrence at the finite temperature.

Case1: $J_z = 0$. Our model corresponds to a XX spin model. The eigenvalues and eigenvectors can be easily obtained. In Fig.1, we give the results at different temperature for the nonuniform magnetic field ($B = 0$) and the uniform magnetic ($b = 0$). From the figure, we can know that the entanglement is symmetric with respect to zero magnetic field, the nonuniform magnetic field can lead to higher entanglement and double-peak structure. This results accord with those seen from Ref[5].

Case2: $J_z = J$. In order to compare with the results in Ref[20], we make the substitution $J \rightarrow 2J$, $B \rightarrow 2B$, $b \rightarrow 2b$. The eigenvalues and eigenvectors can be easily obtained as

$$| \varphi_1 \rangle = |0, 0\rangle,$$

$$| \varphi_2 \rangle = |1, 1\rangle,$$

$$| \varphi_3 \rangle = \frac{1}{\sqrt{1 + x^2/J^2}} (\frac{x}{J} |1, 0\rangle + |0, 1\rangle),$$

$$| \varphi_4 \rangle = \frac{1}{\sqrt{1 + y^2/J^2}} (\frac{y}{J} |1, 0\rangle + |0, 1\rangle),$$

(10)
FIG. 1: The concurrence for $J_z = 0$ and $J = 1$ case. $T = 0.4$ (solid curve) and $T = 1.0$ (dotted curve). The left panel corresponds to the nonuniform case and the right panel corresponds to the uniform case. $T$ is plotted in units of the Boltzmann’s constant $k$. And we work in units where $B$ and $b$ are dimensionless.

with corresponding energies:

$$
e_1 = J - 2B, \quad e_2 = J - 2B, \quad e_3 = -J - 2|J|\sqrt{1 + \delta^2}, \quad e_4 = -J + 2|J|\sqrt{1 + \delta^2}. \tag{11}$$

where $\delta = b/J$, $x = 2b - 2\sqrt{1 + \delta^2}$ and $y = 2b + \sqrt{1 + \delta^2}$. For the ferromagnetic case $J = -1$, the ground-state concurrence is $C(|\varphi_3\rangle) = \frac{1}{\sqrt{1+\delta^2}}$. For the ferromagnetic case $J = 1$, the ground-state concurrence is $C(|\varphi_4\rangle) = \frac{1}{\sqrt{1+\delta^2}}$. These results are same with those obtained in Ref [20]. In Fig.2, we give the plot of the thermal concurrence for $J_z = J = -1$ case. In order to compare our result with that in Ref [20], we let the inhomogeneous magnetic field $b = 0.458$ (this accords to the value of $\xi$ in Ref [20]). We can see that the thermal entanglement develops and is maximized for zero magnetic field $B$ and gets the maximum value at $T = 0$.

**Case3:** For any $J_z$. With $B = 0$, the concurrence as a function of $b$ and $T$ for two value of $J_z$ are given in Fig.3. They show that the concurrence are 1 for different $J_z$ when $b = 0$ and $T = 0$. At the point, the ground state is $|\varphi_3\rangle$ with energy $-\frac{J_z}{2} - 1$, which is the maximally entangled state and the corresponding concurrences are 1. As the temperature increases, the concurrences decreases due to the mixing of other states with the maximally entangled state. We can also know that the concurrence decrease with the increasing of $|b|$. From the two figures in Fig.3, we can find that with the increasing $J_z$, the critical temperature $T_c$ is
FIG. 2: (Color online) The thermal concurrence for $J_z = J = -1$ case. And the inhomogeneous magnetic field $b = 0.458$. $T$ is plotted in units of the Boltzmann’s constant $k$. And we work in units where $B$ and $b$ are dimensionless.

FIG. 3: (Color online) The concurrence in the XXZ spin model is plotted vs $b$ and $T$. Coupling constant $J = 1$, the magnetic field $B = 0$. The left panel corresponds to $J_z = 0$ case and the right panel corresponds to $J_z = 0.9$ case. $T$ is plotted in units of the Boltzmann’s constant $k$. And we work in units where $b$ is dimensionless.

improved (for $J_z = 0$, $T_c$ is about 2, but for $J_z = 0.9$, $T_c$ has a higher value). Thus, we can get the higher entanglement at a fixed temperature when $J_z$ is increased.

Fig. 4 shows the concurrence at a fixed temperature and magnetic field for three values of positive $J_z$. It is shown that the concurrences drop with the increasing value of $b$ and arrive to zero at the same $b$ value, which is called the critical inhomogeneous magnetic field, for various values of $J_z$. This is to say the critical inhomogeneous magnetic field is independent of $J_z$. Moreover, we can see that for a higher value of $J_z$, the system has a
FIG. 4: The concurrence in the XXZ model is plotted vs $b$ for various value of $J_z$, where $J = 1$, $B = 0.8$ and $T = 0.6$. From top to bottom, $J_z$ equals 0.9, 0.4, 0. $T$ is plotted in units of the Boltzmann’s constant $k$. And we work in units where $B$ and $b$ are dimensionless.

FIG. 5: (Color online) The concurrence in the XXZ spin model is plotted vs $T$ and $B$, where $J = 1$, $J_z = 0.4$. The left panel corresponds to $b = 0$ case and the right panel corresponds to $b = 0.8$ case. $T$ is plotted in units of the Boltzmann’s constant $k$. And we work in units where $B$ and $b$ are dimensionless.

stronger entanglement which is consistent with Fig.3.

In Fig.5 we give the plot of concurrence as a function of $T$ and $B$ for $b = 0$ and $b = 0.8$ respectively when $J_z = 0.4$. For $B = 0$ and $b = 0$, the maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$ is the ground state with eigenvalue $-\frac{J_z^2}{2} - |J|$. Then the maximum entanglement is at $T = 0$, i.e. $C = 1$. As $T$ increases, the concurrence decreases as seen from Fig.5 due to the mixing of other states with the maximally entangled state. For a high value of $B$ (in left Fig.5 $B = 1.40$ and in right Fig.5 $B = 1.70$) the state $|\phi_1\rangle$ becomes the ground state, which means there is no entanglement at $T = 0$. However by increasing $T$, the entangled state $|\phi_3\rangle$ and $|\phi_4\rangle$ will mix with the state, which makes the entanglement
increase. From the two figures in Fig.5, we can see that when $b$ is raised, the critical magnetic field $B_c$ increases, but the maximum entanglement value that the system can arrive becomes smaller.

IV. THE CONCLUSIONS

In conclusion, we have investigate the thermal entanglement in two qubit Heisenberg XXZ spin chain under an inhomogeneous magnetic field. The ground-state entanglement and thermal entanglement at a finite temperature are given. We find the entanglement exists for both antiferromagnetic and ferromagnetic cases. And the entanglement is enhanced by increasing the interaction of $z$-component $J_z$. The critical inhomogeneous magnetic field is independent of $J_z$. The critical magnetic field $B_c$ increases with the increasing $|b|$ but the maximum entanglement value that the system can arrive becomes smaller.

[1] M. A. Nielsen, Ph.D thesis, University of Mexico, 1998, quant-ph/0011036.
[2] X. Wang, Phys. Rev. A 64, 012313 (2001).
[3] G. L. Kamta and A. F. Starace, Phys. Rev. Lett. 88, 107901 (2002).
[4] K. M. O'Connor and W. K. Wootters, Phys. Rev. A 63, 052302 (2001).
[5] Y. Sun, Y. Chen, and H. Chen, Phys. Rev. A 68, 044301 (2003).
[6] Y. Yeo, Phys. Rev. A 66, 062312 (2002).
[7] D. V. Khveshchenko, Phys. Rev. B 68, 193307 (2003).
[8] G. Zhang, S. Li and J. Liang, Optical Communication 245, 457-463 (2005).
[9] S. Bose, Phys. Rev. Lett. 91, 207901 (2003); M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. 92, 187902 (2004).
[10] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57, 120(1998). G. Burkard, D. Loss and D. P. DiVincenzo, Phys. Rev. B 59, 2070 (1999).
[11] D. A. Lidar, D. Bacon and K. B. Whaley, Phys. Rev. Lett. 82, 4556 (1999).
[12] D. P. Divincenzo, D. Bacon, J. Kempe, G. Burkard and K. B. Whaley, Nature 408, 339 (2000).
[13] L. F. Santos, Phys. Rev. A 67, 062306 (2003).
[14] X. Wang, Phys. Rev. A 64, 012313 (2002), X. G. Wang, Phys. Lett. A 281, 101 (2001).
[15] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[16] M. C. Arnesen, S. Bose and V. Vedral, Phys. Rev. Lett. 87, 017901 (2001).
[17] L. Zhou, H. S. Song, Y. Q. Guo, and C. Li, Phys. Rev. A 68, 024301 (2003).
[18] X. Hu and S. Das Sarma, Phys. Rev. A 61, 062301 (2000).
[19] X. Hu, R. de Sousa and S. Das Sarma, Phys. Rev. Lett. 86, 918 (2001).
[20] M. Asoudeh and V. Karimipour, Phys. Rev. A 71, 022308 (2005).