Flat-space Limit of Extremal Curves

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Abstract

According to Ryu-Takayanagi proposal, the entanglement entropy of subsystems in the boundary conformal field theory (CFT) is proportional to the area of extremal surfaces in bulk asymptotically AdS spacetimes. The flat-space limit of these surfaces is not well-defined in the generic case. We introduce a new curve in the three dimensional asymptotically AdS spacetimes with well-defined flat-space limit. We find this curve by using a new vector, which is zero on it and is perpendicular to the bulk modular flow of the original interval in the two-dimensional CFT. The flat limit of this new vector is well-defined and gives rise to the bulk modular flow of the corresponding asymptotically flat spacetime. Moreover, after the Rindler transformation, this new vector is the normal Killing vector of the BTZ inner horizon. We reproduce all known results about the holographic entanglement entropy of Bondi-Metzner-Sachs invariant field theories (BMSFT), which are dual to the asymptotically flat spacetimes.
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1 Introduction

One of the proposals for the holographic dual of asymptotically flat spacetimes is given by [1, 2].

According to this proposal, the flat-space limit in the gravity side in the context of AdS/CFT

 corresponds to taking the ultra-relativistic limit in the field theory side. For the two dimensional CFTs, the ultra-relativistic limit of conformal algebra is performed using the Inonu-Wigner con-

 traction and imposing the vanishing speed of light limit. The resultant algebra, in this case,

 is infinite-dimensional, and it is known as Carroll algebra [3, 4]. This two-dimensional ultra-

 relativistic algebra is isomorphic to a three-dimensional relativistic algebra, which is asymptotic symmetry of asymptotically flat spacetimes at null infinity [5]. The asymptotic symmetry of four-dimensional asymptotically flat spacetimes is known from a long time ago by works of Bondi, Metzner, and also Sachs, and this symmetry is known as BMS symmetry [6]. Recent progress in this way by Barnich et al. [5, 7, 8] shows that by imposing non-globally invariance of symmetry algebra (which was absent in the first formulation of $bms_4$ algebra in [6] and $bms_3$ algebra in [9]), one can obtain infinite-dimensional algebras in both of three and four-dimensional asymptotically
flat spacetimes which are an extension of translation and rotation of Poincare algebra. In this view, both of $bms_3$ and $bms_4$ consist of super-translations and super-rotations.

The isomorphism between two-dimensional Carroll algebra and three dimensional BMS algebra was the motivation of [1, 2] to propose that the holographic dual of asymptotically flat spacetimes are BMS-invariant field theories (BMSFT) and BMSFTs are ultra-relativistic theories in one dimensional lower spacetimes characterized by their infinite-dimensional symmetry algebra. In this paper, we call this correspondence flat/BMSFT. The infinite-dimensional symmetry of BMSFTs provides some universal properties which are independent of the detail of these theories. In this view, one can study the flat-space holography by just using the universal properties of BMSFTs For recent progress in studying the details of BMSFTs, see [10, 11].

There are two approaches to completing the dictionary of flat/BMSFT. One approach is to study BMSFTs directly and relates its observable to the properties of asymptotically flat spacetimes. Another approach is to start from AdS/CFT and take a limit from its dictionary. In the bulk side, this limit is simply the flat-space limit or zero cosmological constant limit, which is performed by taking the limit of AdS radius, $\ell$, to infinity.

In this paper, we focus on the second approach to study the holographic entanglement entropy of BMSFTs. Due to the infinite-dimensional symmetry of two-dimensional BMSFTs, a universal formula has been proposed for the entanglement entropy of intervals in these theories [12]. Later, a holographic description for the BMSFTs entanglement entropy was proposed by [13]. This holographic description of entanglement entropy is very similar to Ryu and Takayanagi’s (RT) proposal in the context of AdS/CFT [14], which relates the dual field theory entanglement entropy to the length of some extremal curves inside the gravitational theory. In the original RT proposal, this curve in the asymptotically AdS spacetime is a geodesic with extremized length connected to the two ends of the interval at the boundary. In the flat/BMSFT correspondence, this curve is also a geodesic with extremized length, but it is connected to two null rays emitted from null infinity where two ends of the interval are located on. Thus in the flat-case, the extremal curves are not connected to the dual field theory intervals directly, and this makes finding them a bit tedious and ambiguous.

The method which is used in [13] for constructing extremal curves in asymptotically flat spacetimes is based on Rindler transformation [15]. In the field theory side, this unitary transformation is part of the symmetry of the theory, which relates the original entanglement entropy to a thermal entropy. The thermal entropy corresponds to the entropy (area) of an event horizon in the gravitational theory. The spacetime in which this event horizon is located is given by the bulk Rindler transformation of original spacetime. The extremal curve in the original bulk spacetime
is nothing but the inverse Rindler transformation of the final event horizon. Thus in this method, one needs to find Rindler transformation and its bulk extension for finding extremal curves. In this view, we do not need to extremize any length, and the extremal curve is a result of Rindler transformation. Although this method is applicable to all field theories, finding corresponding Rindler transformation is not straightforward.

In the original RT proposal in the context of AdS/CFT, it is not necessary to know Rindler transformation. Imposing extremality condition for the length of bulk geodesics anchored to the two ends of the boundary interval is enough to determine it. One can interpret the flat-space limit of these curves as the holographic entanglement entropy related to BMSFTs. However, the flat-space limit of these curves is not well-defined. The simplicity of the extremization procedure, which is used in the AdS/CFT case, is tedious enough to make someone think about the extremal curves with a well-defined flat-space limit. In this paper, we focus on this problem and introduce a new curve in the asymptotically AdS spacetimes whose flat-space limit yields the results of [13].

The main idea in the current calculation is related to the observation of [16] and [17] that the entropy of flat-space cosmological solution (FSC) which is given by taking the flat-space limit from the BTZ black hole [18]-[21], is a result of flat-space limit performed on the entropy of BTZ inner horizon. After the Rindler transformation, the RT extremal curves transform to the outer horizon of the corresponding black hole, the curve with a well-defined flat-space limit is that one which is related to the inner horizon after the Rindler transformation.

In order to bypass the Rindler transformation and introduce a method that just uses the extremality condition, we use the bulk modular flow of the corresponding interval in the AdS case. It is zero on the RT extremal curve and transforms into the Killing vector normal to the outer horizon after the Rindler transformation. The fact that our new curve is related to the inner horizon shows that the Killing vector normal to the inner horizon can be meaningful in the original coordinate if we use the inverse Rindler transformation. The interesting point is that the Killing vectors normal to inner and outer horizons are perpendicular to each other, and the sum of their norms is constant. Thus, besides the Killing equation, we have two other equations that relate two vector fields to each other in any coordinate system. One of these vector fields is the bulk modular flow of the original coordinate in the AdS case, and the second one is a new vector field that has a well-defined flat-space limit. The flat-space limit of our new vector field in the asymptotically AdS spacetime yields the bulk modular flow of the corresponding interval introduced in [13] for BMSFT. The interesting point is that the components of new vector field are zero on the new extremal curve. In other words, this new curve consists the fixed points of the new vector. Thus we can bypass the Rindler transformation in this way that using the bulk
modular flow of original interval in the AdS case; we construct our new vector field just by using Killing equations and normality and norm conditions. These equations determine our new vector field. Then we can take the flat-space limit and find the bulk modular flow of asymptotically flat case and finally find the extremal curve introduced in [13] which these flat bulk modular flow is zero on it.

In the holographic description proposed for the entanglement entropy of BMSFT introduced in [13], besides an extremal spacelike curve, there are two null curves starting from null infinity from two ends of the BMSFT interval and intersect the extremal curve. In fact, this is the length of the extremal curve bounded between two intersections by null curves, which is proportional to the BMSFT entanglement entropy. In this paper, we also propose two new null curves in the asymptotically AdS spacetimes, which their flat-space limit results in the null rays of [13]. The fact that our new curve gets to the boundary at points which are different than the two ends of the original interval, makes it possible to find null rays that connect two ends of the interval to the new curve. However, the number of these null geodesics are infinite. We observe that imposing the condition that null curves also pass through two cut off points picks up two distinct null curves that their flat space limit yields the flat null rays of [13].

In section two, we start from preliminaries and review related topics necessary in the rest of this paper. Section three is the main part of our paper and consists of the calculations for introducing the new curves and new vector field with a well-defined flat space limit. We also propose a new recipe for the holographical calculation of BMSFT entanglement entropy, which does not need Rindler transformation. The last section is devoted to the summary and conclusion.

2 Preliminaries

2.1 Entanglement entropy, modular Hamiltonian, modular flow and Rindler transformation

For the subsystem $A$ of a quantum field theory in a pure state $|Ψ⟩$, the reduced density matrix is given by

$$\rho_A = Tr_{A}ρ = Tr_{A} |Ψ⟩⟨Ψ|,$$

(2.1)

where $A$ is the complement of $A$. The entanglement entropy of region $A$ is given by the Von-Neumann entropy of the reduced density matrix,

$$S_A = -Tr(ρ_A \log ρ_A).$$

(2.2)
Since $\rho_A$ is hermitian and positive semi-definite, it can be written in terms of another hermitian operator $\mathcal{H}_A$

$$\rho_A = \frac{e^{-\mathcal{H}_A}}{\text{Tr}(e^{-\mathcal{H}_A})}. \quad (2.3)$$

$\mathcal{H}_A$ is called entanglement Hamiltonian or modular Hamiltonian in the literature and is the conserved charge of a geometric flow $k_t$ which is known as modular flow.

For a generic QFT, the calculation of the entanglement entropy given by (2.2) is not straightforward. However, this calculation is simplified by making use of the symmetry of the QFT.

There are two methods for this calculation, which are known as the replica trick [22, 23] and the Rindler method [15]. Here we briefly introduce the Rindler method since it is simply generalized to the BMSFTs.

The aim of the Rindler method is to find symmetry transformation, $U_R$, that map the density matrix of the entangled region to the thermal one

$$\rho_A = U_R \rho_B U_R^{-1}, \quad (2.4)$$

where tilde entities stand for the thermal system. Since the unitary transformation does not change the entropy of the system, we would expect that entanglement entropy of the region $A$ to be equal to the thermal entropy of the region $\tilde{B}$.

Now in the thermal system $\tilde{B}$ we can identify the partition function and the geometric flow,

$$Z(\tilde{B}) = \text{Tr} \ e^{-\tilde{\beta}_i Q_{\tilde{\beta}_i}}, \quad \rho_{\tilde{B}} = Z(\tilde{B})^{-1} e^{-\tilde{\beta}_i Q_{\tilde{\beta}_i}}. \quad (2.5)$$

Knowing the partition function for a thermal system we can work out the entropy as well,

$$S(A)_{EE} = S(\tilde{B}) = (1 - \tilde{\beta}_i \partial_{\tilde{\beta}_i}) Z(\tilde{B}). \quad (2.6)$$

The modular flow can be written as $k_t = \tilde{\beta}_i \partial_{\tilde{\beta}_i}$. It vanishes at the boundary of the entangled surface, $k_t|_{\partial A} = 0$ [13].

2.2 Holographic entanglement entropy in AdS/CFT and bulk modular flow

In the gauge/gravity duality the entanglement entropy of the boundary sub-systems has a geometric interpretation in terms of extremal curves within the bulk theory introduced by Ryu and Takayanagi (RT) in the context of AdS/CFT [14]. Accordingly, the holographic entanglement entropy is given by

$$S = \frac{\mathcal{A}}{4G_N} \quad (2.7)$$

where $\mathcal{A}$ is an exterimized surface anchored to the interval $A$ on the conformal boundary of asymptotically local $AdS$ spacetime and $G_N$ is Newton constant.
Let us consider AdS$_3$, with line element,
\[
\ell^2 \frac{dz^2}{z^2} (dx^2 + dz^2 - dt^2),
\]
where its boundary is given by $z = 0$. For an interval at the boundary of this spacetime given by
\[
\mathbf{A} : \begin{cases} 
-R < x < R \\
t = 0
\end{cases}
\]
the exterimized surface in the bulk is given by
\[
x^2 + z^2 = R^2, \quad t = 0
\]
So using (2.7) we find that
\[
S = \frac{2}{4G_N} \int_{\epsilon}^{R} \frac{dz}{z\sqrt{R^2 - z^2}} = \frac{1}{2G_N} \log \frac{2R}{\epsilon},
\]
where $\epsilon$ is a cut-off. This is the celebrated entanglement entropy in CFT$_2$ [22, 23].

We can also find the bulk modular flow $k^\text{bulk}$ using the extension of the boundary modular flow to the bulk. The modular flow of the interval (2.9) is given by [15]
\[
k_t = -\frac{2\pi x t}{R} \partial_x + \frac{\pi}{R} (R^2 - t^2 - x^2) \partial_t.
\]
The RT extremal surface, $\mathcal{A}$, can be interpreted as the surface that
\[
k^\text{bulk} \bigg|_{\mathcal{A}} = 0.
\]
Moreover, $k^\text{bulk}$ is a Killing vector of the bulk metric and $k^\text{bulk} \big|_{z=0} = k_t$. Putting all together we find,
\[
k^\text{bulk} \equiv \xi = -\frac{2\pi t}{R} (x \partial_x + z \partial_z) + \frac{\pi}{R} (R^2 - x^2 - t^2 - z^2) \partial_t.
\]
Since the general interval play a crucial role in the rest of this paper, we also calculate modular flow of general interval. To this aim, we boost the boundary coordinate as
\[
t' = t \cosh \eta + x \sinh \eta, \\
x' = t \sinh \eta + x \cosh \eta.
\]
Demanding new interval to be
\[
\Delta' : \begin{cases} 
-x'_\frac{l_x}{2} < x' < \frac{l_x}{2}, \\
-l'_t < t' < \frac{l_t}{2}, \\
R^2 = \frac{l_x^2 - l_t^2}{4}, \quad (l_x > l_t),
\end{cases}
\]
will uniquely determine the boost parameter $\eta$ as,

$$\eta = \log \sqrt{\frac{l_x^2 + l_t^2}{l_x^2 - l_t^2}}. \quad (2.17)$$

The modular flow for this boosted interval can be found using (2.15) as

$$\xi = \frac{\pi}{2} \left( \frac{l_x^2 l_t - l_t^3}{l_t^2 - l_t^2} x' l_t - 4l_t (x'^2 + t'^2 - z^2) \right) \partial x'$$

$$+ \frac{\pi (l_x^3 + 8l_x x' t' - l_x l_t^2 - 4l_x (x'^2 + t'^2 + z^2))}{2(l_x^2 - l_t^2)} \partial t' - \frac{4\pi (-l_t x' + l_x t')z}{l_x^2 - l_t^2} \partial z. \quad (2.18)$$

For the boosted interval, the holographic entanglement entropy is given by the following extremal surface

$$x'^2 = \frac{l_x^2(l_x^2 - l_t^2 - 4z^2)}{4(l_x^2 - l_t^2)},$$

$$y'^2 = \frac{l_t^2(l_t^2 - l_x^2 - 4z^2)}{4(l_x^2 - l_t^2)}. \quad (2.19)$$

### 2.3 A brief review of Flat/BMSFT correspondence

Asymptotically flat spacetimes are given by taking the flat-space limit (zero cosmological constant limit or large AdS radius limit) from the asymptotically AdS spacetimes. According to the proposal of [2], the flat-space limit in the asymptotically local AdS spacetime corresponds to the ultra-relativistic limit in the boundary CFT. The resultant ultra-relativistic field theory is known as BMSFT, and the correspondence between asymptotically flat spacetimes and BMSFTs is called flat/BMSFT. BMSFTs are BMS-invariant field theories. For the $d$-dimensional BMSFTs, BMS symmetry is the asymptotic symmetry of $(d + 1)$-dimensional asymptotically flat spacetimes at null-infinity. The BMS algebra as the asymptotic symmetry algebra is infinite-dimensional. In three dimensions, $\text{bms}_3$ is given by

$$[L_m, L_n] = (m - n)L_{m+n},$$

$$[L_m, M_n] = (m - n)M_{m+n}, \quad (2.20)$$

$$[M_m, M_n] = 0, \quad m, n \in \mathbb{Z}.$$ 

where $L_n$ and $M_n$ are respectively the generators of super-rotation and super translation. For $n = 0, -1, 1$ the resultant sub-algebra is Poincare algebra. The algebra (2.20) is the asymptotic symmetry algebra of three dimensional spacetimes given by

$$ds^2 = M du^2 - 2dudr + 2N dud\phi + r^2 d\phi^2, \quad (2.21)$$

where $M$ and $N$ are functions of $u$ and $\phi$ and they satisfy

$$\partial_u M = 0, \quad 2\partial_u N = \partial_{\phi} M. \quad (2.22)$$
u is retarded time and coordinate \( u, r, \phi \) in (2.21) is known as BMS coordinate [24]. The line element (2.21) is given by taking the flat-space limit from the asymptotically AdS metric,

\[
ds^2 = \left( -\frac{r^2}{\ell^2} + \mathcal{M} \right) du^2 - 2du dr + 2\mathcal{N} dud\phi + r^2 d\phi^2,
\]

(2.23)

where \( \mathcal{M} \) and \( \mathcal{N} \) are functions of \( u \) and \( \phi \) and are constrained by using the equations of motion as

\[
\partial_u \mathcal{M} = \frac{2}{\ell^2} \partial_\phi \mathcal{N}, \quad 2\partial_u \mathcal{N} = \partial_\phi \mathcal{M}.
\]

(2.24)

The functions \( \mathcal{M} \) and \( \mathcal{N} \) are the resultant functions of taking the flat-space limit from the functions \( \mathcal{M} \) and \( \mathcal{N} \).

The generators of (2.20) can be obtained by taking the flat-space limit from the generators of conformal algebra [24],

\[
L_m = \lim_{G \ell \to 0} (L_m - \bar{L}_m), \quad M_m = \lim_{G \ell \to 0} \frac{G}{\ell}(L_m + \bar{L}_m).
\]

(2.25)

where \( L_m \) and \( \bar{L}_m \) are the generators of conformal algebra,

\[
[L_m, L_n] = (m-n)L_{m+n},
\]

\[
[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n},
\]

\[
[L_m, \bar{L}_n] = 0, \quad m, n \in \mathbb{Z}.
\]

(2.26)

It was proposed in [2] that the limit (2.25), which is taken in the gravity side, corresponds to the ultra-relativistic limit in the field theory side. The algebra of conserved charges in both of (2.20) and (2.26) are centrally extended.

Similar to other field theories, it is possible to define the entanglement entropy for the subsystems of BMSFT. The infinite-dimensional symmetry of BMSFTs admits to find universal formulas for the entanglement entropy of sub-regions [12]. Moreover, using the flat/BMSFT correspondence, one can find a holographic description for the BMSFT entanglement entropy. Recently, a prescription (similar to the Ryu-Takayanagi’s proposal for the CFT entanglement entropy [14]) has been proposed for the BMSFT entanglement entropy [13] that relates it to the area of some particular curves in the asymptotically flat bulk spacetimes.

To be precise, let us consider the null-orbifold or Poincare patch (with \( M = N = 0 \) in (2.21)) which is given by

\[
ds^2 = -2du dr + r^2 d\phi^2.
\]

(2.27)

For the interval \( B \) in the dual BMSFT which is determined by \(-\frac{l_u}{2} < u < \frac{l_u}{2}\) and \(-\frac{l_{\phi}}{2} < \phi < \frac{l_{\phi}}{2}\) where \( l_u \) and \( l_{\phi} \) are constants, the entanglement entropy is [13]

\[
S_{EE} = \frac{C_{LL}}{6} \log \frac{l_{\phi}}{\epsilon_{\phi}} + \frac{C_{LM}}{6} \left( \frac{l_u}{l_{\phi}} - \epsilon_u \right).
\]

(2.28)
where $\epsilon_\phi$ and $\epsilon_u$ are cut-offs in $\phi$ and $u$ directions and $C_{LL}$ and $C_{LM}$ are central charges of (2.20) related to the central charges $c$ and $\bar{c}$ of conformal algebra (2.26) as

$$C_{LL} = \lim_{\ell \to 0} \frac{G}{\ell}(c - \bar{c}), \quad C_{LM} = \lim_{\ell \to 0} \frac{G}{\ell}(c + \bar{c}).$$

(2.29)

For the BMSFT dual to Einstein gravity, $C_{LL} = 0$ and $C_{LM} = 3$.

According to [13], the entanglement entropy of sub-region $B$ of BMSFT$_2$ is given by

$$S_{HEE} = \frac{\text{Length}(\gamma)}{4G} = \frac{\text{Length}(\gamma \cup \gamma_+ \cup \gamma_-)}{4G}$$

(2.30)

where $\gamma$ is a spacelike geodesic and $\gamma_+$ and $\gamma_-$ are null rays from $\partial \gamma$ to $\partial B$. In order to find these curves, [13] uses a Rindler transformation as

$$\tilde{r} = \sqrt{\frac{M}{16l_\phi^2}} \left(8u - 4l_u + r(l_\phi - 2\phi)^2\right) \left(8u + 4l_u + r(l_\phi + 2\phi)^2\right) + \frac{J^2}{4M},$$

$$\tilde{\phi} = -\frac{1}{\sqrt{M}} \log \frac{\sqrt{M}}{4l_\phi} \left(8u - 4l_u + r(l_\phi - 2\phi)^2\right) + \frac{J}{2\sqrt{M}},$$

$$\tilde{u} = \frac{\tilde{r}}{M} + \frac{1}{4l_\phi \sqrt{M}} (8u + 4r\phi^2 - rl_\phi^2) - \frac{J}{2M} \tilde{\phi}.$$  

(2.31)

which transforms (2.27) to

$$ds^2 = \tilde{M} d\tilde{u}^2 - 2d\tilde{u} d\tilde{r} + \tilde{J} d\tilde{u} d\tilde{\phi} + \tilde{r}^2 d\phi^2$$

(2.32)

where $\tilde{M}$ and $\tilde{J}$ are constants. The metric (2.32) is known as flat-space cosmological solution (FSC) [18, 19] and has a cosmological horizon located at

$$\tilde{r}_C = \frac{\tilde{J}}{2\sqrt{M}}.$$  

(2.33)

If we assume that $\gamma$ and $\gamma_\pm$ are mapped to the cosmological horizon after the bulk Rindler transformation (2.32), one can start with the condition $\tilde{r} = \tilde{r}_C$ and use (2.31) to find them [13]. In the next section, we find these curves by making a flat-space limit from particular curves in the asymptotically AdS spacetimes.

3 Holographic BMSFT entanglement entropy using flat-space limit

In order to study the holographic entanglement entropy of BMSFTs by using the method of [13], one needs to find appropriate Rindler transformation. On the other hand, taking the flat-space limit from the known results in the AdS/CFT correspondence is another method for studying
flat-space holography. In this section, we propose proper curves in the asymptotically AdS spacetimes, which their flat-space limit results in $\gamma$ and $\gamma_\pm$. We restrict our study to the AdS$^3$ in the Poincare coordinate and in the appendix argue that our results are simply generalizable to other coordinates by making use of coordinate transformation. In other words, we propose an alternative method for studying the holographic entanglement entropy of BMSFTs, which does not use Rindler transformation.

### 3.1 Initial setup in 3d asymptotically AdS spacetime

The main goal is to take the flat-space limit from the holographic calculation in bulk, which is AdS$^3$ spacetime in Poincare coordinate. However, in order to have a well-defined flat-space limit, we need to write our metric in an appropriate gauge. By well-defined gauge, we mean a set of the coordinate system where taking the flat-space limit $\ell \to \infty$ of the AdS$^3$ metric ends up with well-defined 3d flat-space metric. This appropriate gauge was introduced in [24] and is called BMS-gauge.

The AdS$^3$ in Pincare-BMS coordinate is written as

$$ds^2 = r^2 \left( d\phi^2 - \frac{dr^2}{\ell^2} \right) - 2drdu. \tag{3.1}$$

This metric is given by the following transformation from the metric (2.8),

$$x = \ell \phi,$$

$$t = u - \frac{\ell^2}{r},$$

$$z = \frac{\ell^2}{r}. \tag{3.2}$$

This coordinate transformation gives rise to a well-defined flat limit of the interval as well as the metric. Using (2.18) and (3.2) we can find the components of the bulk modular flow,

$$\xi^r = -\frac{4\pi \ell (l_\phi (\ell^2 - ru) + r\phi l_u)}{l_u^2 - \ell^2 l_\phi^2},$$

$$\xi^u = \frac{\pi \ell \left( -l_\phi \left( l_u^2 + 4 \left( u^2 + \ell^2 \phi^2 \right) \right) + 8u\phi l_u + \ell^2 l_\phi^3 \right)}{2 \left( l_u^2 - \ell^2 l_\phi^2 \right)},$$

$$\xi^\phi = \frac{\pi \left( l_u \left( r\ell^2 l_\phi^2 + 4r \left( u^2 + \ell^2 \phi^2 \right) - 8u \ell^2 \right) + 8\ell^2 \phi l_\phi \left( \ell^2 - ru \right) - r l_u^3 \right)}{2r \ell \left( l_u^2 - \ell^2 l_\phi^2 \right)}, \tag{3.3}$$

where $l_u$ and $l_\phi$ are

$$l_u = l_t,$$

$$l_\phi = \frac{l_x}{\ell}. \tag{3.4}$$
Taking $r \to \infty$ limit from $\xi^u$ and $\xi^\phi$ in (3.3) results in the components of the modular flow for an interval in the boundary CFT which is determined by $-\frac{l_2}{2} < u < \frac{l_2}{2}$ and $-\frac{l_2}{2} < \phi < \frac{l_2}{2}$.

### 3.2 A new vector normal to Bulk modular flow

Although the flat-space limit of (3.1) is well defined and gives rise to (2.27), it is not difficult to check that the $\ell \to \infty$ limit is not well-defined for (3.3). This means that we can not find the modular flow of similar interval in BMSFT by taking the flat-space limit from the modular flow of a corresponding interval in CFT. However, starting from (3.3), we introduce a new vector field, $\lambda^\mu$, with a well-defined flat-space limit. To do so, we first use the Rindler transformation; however, in the end, we introduce a recipe for the calculation of $\lambda^\mu$ from $\xi^\mu$ directly.

The bulk Rindler transformation, which changes (3.1) to the BTZ black hole written in the BMS gauge, is given in the Appendix. The final metric is

$$ds^2 = \left(-\frac{\hat{r}^2}{\ell^2} + \hat{M}\right) d\hat{u}^2 - 2 d\hat{u} d\hat{r} + 2 \hat{N} d\hat{u} d\hat{\phi} + \hat{r}^2 d\hat{\phi}^2,$$

(3.5)

where $\hat{M}$ and $\hat{N}$ are given in terms of the BTZ inner and outer horizons $\hat{r}_\pm$ as

$$\hat{M} = \frac{\hat{r}_+^2 + \hat{r}_-^2}{\ell^2}, \quad \hat{N} = \frac{\hat{r}_+ \hat{r}_-}{\ell}.$$  

(3.6)

After the Rindler transformation, the components of $\xi^\mu$ in (3.3) transform as

$$\hat{\xi}^\phi = \pi \ell \left(\frac{2 \hat{r}_-}{\hat{r}_+^2 - \hat{r}_-^2}\right), \quad \hat{\xi}^\hat{r} = 0, \quad \hat{\xi}^\hat{u} = -\pi \ell^2 \left(\frac{2 \hat{r}_+}{\hat{r}_+^2 - \hat{r}_-^2}\right).$$

(3.7)

The interesting point is that $\hat{\xi}^\mu$ is the Killing vector normal to the outer horizon and can be written as

$$\hat{\xi} = \frac{2\pi}{\hat{\kappa}_+} \left(\partial_{\hat{u}} - \hat{\Omega}_+ \partial_{\hat{\phi}}\right),$$

(3.8)

where $\hat{\kappa}_+$ and $\hat{\Omega}_+$ which are respectively the surface gravity and angular velocity of the outer horizon, are given as

$$\hat{\kappa}_+ = \frac{|\hat{r}_+^2 - \hat{r}_-^2|}{\ell^2 \hat{r}_+^2}, \quad \hat{\Omega}_+ = \frac{\hat{r}_-}{\ell \hat{r}_+}.$$  

(3.9)

In all of the previous works, which develop the flat/BMSFT correspondence by taking the flat-space limit from the AdS/CFT calculations, the corresponding quantity in the asymptotically AdS spacetime with well-defined flat space limit is related to the inner horizon of the BTZ black hole [16, 17]. We want to continue this idea for the current problem and introduce $\lambda^\mu$ as the vector field, which is given by taking the inverse Rindler transformation from the Killing vector field normal to the inner horizon. This vector field which is denoted by $\hat{\lambda}^\mu$, is given by

$$\hat{\lambda} = \frac{2\pi}{\hat{\kappa}_-} \left(\partial_{\hat{u}} - \hat{\Omega}_- \partial_{\hat{\phi}}\right),$$

(3.10)
where \( \hat{\kappa}_- \) and \( \hat{\Omega}_- \) are the surface gravity and angular velocity of the inner horizon,

\[
\hat{\kappa}_- = \frac{|\hat{r}_+^2 - \hat{r}_-^2|}{\ell^2 \hat{r}_-}, \quad \hat{\Omega}_- = \frac{\hat{r}_+}{\ell \hat{r}_-}. \tag{3.11}
\]

Comparing (3.8) and (3.10) shows that they are deducible from each other if we use the following transformation:

\[
\hat{r}_+ \iff \hat{r}_-. \tag{3.12}
\]

Using the inverse Rindler transformation given in the Appendix, we find \( \lambda^\mu \) as

\[
\lambda^r = \frac{4\pi (l_u (\ell^2 - ru) + r \ell^2 \phi l_\phi)}{l_u^2 - \ell^2 l_\phi^2},
\]

\[
\lambda^u = \frac{\pi (l_u (\ell^2 l_\phi^2 + 4 (u^2 + \ell^2 \phi^2)) - 8u \ell^2 \phi l_\phi - l_u^3)}{2 (l_u^2 - \ell^2 l_\phi^2)},
\]

\[
\lambda^\phi = \frac{\pi (-l_\phi (rl_u^2 + 4r (u^2 + \ell^2 \phi^2) - 8u \ell^2)) + 8 \phi l_u (ru - \ell^2) + r \ell^2 l_\phi^3)}{2r (l_u^2 - \ell^2 l_\phi^2)}. \tag{3.13}
\]

Now, \( \ell \to \infty \) is well-defined and results in the bulk modular flow corresponding to the interval in the BMSFT introduced in [13]:

\[
\xi_{flat}^u = -\frac{\pi (-8 \phi l_u \phi + l_u (4 \phi^2 + l_\phi^2))}{2l_\phi^2}, \quad \xi_{flat}^r = -\frac{4\pi (l_u + l_\phi r \phi)}{l_\phi^2},
\]

\[
\xi_{flat}^\phi = \frac{-\pi (rl_\phi^2 - 8l_u \phi + 8u \phi - 4r l_\phi \phi^2)}{2r l_\phi^2}. \tag{3.14}
\]

All of \( \xi^\mu, \lambda^\mu, \hat{\xi}^\mu \) and \( \hat{\lambda}^\mu \) are Killing vectors of the related spacetimes. Since \( \hat{\xi}^\mu \) and \( \hat{\lambda}^\mu \) are Killing vectors normal to the horizons, we find that

\[
\hat{\xi}^\mu \hat{\lambda}_\mu = 0, \quad \hat{\xi}^\mu \hat{\xi}_\mu + \hat{\lambda}^\mu \hat{\lambda}_\mu = 4\pi^2 \ell^2. \tag{3.15}
\]

Using the fact that \( \xi^\mu \) and \( \lambda^\mu \) are given by coordinate transformation from \( \hat{\xi}^\mu \) and \( \hat{\lambda}^\mu \), we can conclude that

\[
\xi^\mu \lambda_\mu = 0, \quad \xi^\mu \xi_\mu + \lambda^\mu \lambda_\mu = 4\pi^2 \ell^2. \tag{3.16}
\]

For a given \( \xi \), equations (3.16) beside the fact that \( \lambda \) is a Killing vector are enough to determine it without using the Rindler transformation.
Comparing (3.3) and (3.13), one can find an interesting relation between the components of $\xi^\mu$ and $\lambda^\mu$. The similar components are changed to each other if we make the following transformation,

$$ l_u \Rightarrow \ell l_\phi, $$
$$ l_\phi \Rightarrow l_u / \ell. \tag{3.17} $$

One can use this simple transformation to find $\lambda^\mu$ of more complicated cases from $\xi^\mu$. In this view, the bulk modular flow of asymptotically flat spacetimes is simply achievable from the bulk modular flow of corresponding asymptotically AdS spacetimes by firstly using the transformation (3.17) and then taking the flat space limit. Using the Rindler transformation introduced in Appendix, one can deduce (3.17) from (3.12).

### 3.3 New extremal curve with well-defined flat-space limit

In this subsection, we perform the final step and find the corresponding curves in the asymptotically AdS spacetimes, which their flat-space limit results in extremal curves $\gamma$ and $\gamma_\pm$ in the asymptotically flat spacetimes. To this end, let us first look at $\xi^\mu$ and extremal curves in the asymptotically AdS spacetimes that, according to the proposal of RT, their length give rise to the entanglement entropy of the corresponding interval in the boundary CFT.

To be precise, let us consider an interval in CFT dual to (3.1), characterized by $-\frac{l_u}{\ell} < u < \frac{l_u}{\ell}$ and $-\frac{l_\phi}{\ell} < \phi < \frac{l_\phi}{\ell}$. According to the proposal of [14], the entanglement entropy of this interval is proportional to the length of an extremal curve in the bulk anchored to the two ends of interval located at $(u = -l_u/2, \phi = -l_\phi/2)$ and $(u = l_u/2, \phi = l_\phi/2)$. Thus using the metric of the bulk, one can find this extremal curve. Moreover, the components of the bulk modular flow $\xi^\mu$ are zero on this curve and it consists the fixed points of the bulk modular flow. Using (3.3), we find that this extreme curve satisfies the following equation:

$$ \phi l_u - u l_\phi + \ell^2 l_\phi / r = 0, $$
$$ l_\phi l_u^2 + 4u^2 l_\phi + 4\ell^2 l_\phi \phi^2 - 8u l_u l_\phi - \ell^2 l_\phi^3 = 0. \tag{3.18} $$

One can also look for the curve in bulk on which the components of $\lambda^\mu$ are zero. For $\lambda^\mu$ given by (3.13), we find the following curve:

$$ \ell^2 l_\phi l_\phi - u l_u + \ell^2 l_u / r = 0, $$
$$ 4l_u \ell^2 l_\phi l_\phi^2 - 8\ell^2 l_\phi u l_u + \ell^2 l_\phi l_u^2 + 4l_u u^2 - l_u^3 = 0. \tag{3.19} $$

\(^1\text{Since finally we want to take the } \ell \to \infty \text{ limit, it is assumed that } \ell^2 l_\phi > l_u^2.\)
It is clear that (3.18) and (3.19) are changed to each other by using (3.17). Moreover, the new curve which we show it by $\gamma_{AdS}$ in this paper, gets to the boundary at the points \((u = -\frac{l_u}{2}, \phi = -\frac{l_\phi}{2\ell})\) and \((u = \frac{l_\phi}{2}, \phi = \frac{l_u}{2\ell})\). These points can be assumed as the two ends of a new timelike interval on the boundary, which is given by transformation (3.17) from the original spacelike interval. The curve $\gamma_{AdS}$ is exactly the same extremal curve, which is given by the RT proposal for the new interval on the boundary. The interesting point is that both of $\gamma_{AdS}$ and RT extremal curve have the same length. This means that their length is invariant under the transformation (3.17).

Now let us look at (3.19) and keep terms which have $\ell^2$. This yields the following equations:

\[
\begin{align*}
    l_\phi \phi + \frac{l_u}{r} &= 0, \\
    4l_u \phi^2 - 8l_\phi \phi u + l_\phi^2 l_u &= 0
\end{align*}
\]

(3.20)

These are exactly the same equations which are resulted in from $\xi^\mu_{flat} = 0$ where $\xi_{flat}$ is given by (3.14). One can use (3.20) and write $\phi$ and $u$ in terms of $r$ as

\[
\begin{align*}
    \phi &= -\frac{l_u}{l_\phi r}, \\
    u &= -\frac{l_\phi^2}{8r} - \frac{l_u^2}{2l_\phi^2 r}.
\end{align*}
\]

(3.21)

These are the equations that describe curve $\gamma$ in [13]. However, $\gamma$ is not the whole of this curve and located on a portion of it, which is determined by the intersection of $\gamma^+$ and $\gamma^-$. Our next task is to determine new curves $\gamma_{AdS}^\pm$ in (3.1) which their flat-space limit yields $\gamma^\pm$.

To do so, we use the fact that $\gamma^\pm$ are null curves that connect two ends of the interval at null infinity to $\gamma$. Thus we propose $\gamma_{AdS}^\pm$ as those null geodesics which connect two ends of the interval at the boundary to $\gamma_{AdS}$ (figure 1).

Starting from geodesic equations of metric (3.1), we look for null geodesics that connects \((u = -l_u/2, \phi = -l_\phi/2)\) and \((u = l_u/2, \phi = l_\phi/2)\) at $r = \infty$ to the new curve (3.19). Solving the geodesic equations with this conditions yields the following results:

- $\gamma_{AdS}$ is a null ray which connects \((u = -l_u/2, \phi = -l_\phi/2)\) to $\gamma_{AdS}$. It is given by

\[
\begin{align*}
    u &= -\frac{l_u}{2} + \frac{\beta_-}{r}, \\
    \phi &= -\frac{l_\phi}{2} + \frac{\alpha_-}{r},
\end{align*}
\]

(3.22)

where $\alpha_-$ and $\beta_-$ are two constants satisfy the following equation,

\[
\alpha_-^2 + 2\beta_- - \frac{\beta_-^2}{\ell^2} = 0.
\]

(3.23)

\footnote{Only two equations of $\xi^\mu_{flat} = 0$ are independent.}
Figure 1: Curve $A$ is the boundary interval and $A'$ is the extremal curve given by RT prescription. $A'$ is the new interval and $\gamma_{AdS}$ is the new extremal curve. $\gamma^\pm_{AdS}$ connects two ends of $A$ to $\gamma_{AdS}$.

- $\gamma^+_{AdS}$ is a null geodesic which connects $(u = l_u/2, \phi = l_\phi/2)$ to $\gamma_{AdS}$. It is given by

$$u = \frac{l_u}{2} + \frac{\beta_+}{r}, \quad \phi = \frac{l_\phi}{2} + \frac{\alpha_+}{r},$$

where $\alpha_+$ and $\beta_+$ are two constants satisfy the following equation,

$$\alpha_+^2 + 2\beta_+ - \frac{\beta_+^2}{\ell^2} = 0. \quad (3.25)$$

Thus instead of a unique curve we find a bunch of curves for $\gamma^\pm_{AdS}$. The interesting point is that $\lambda^\mu$ is tangent to all of these curves. In order to determine unique null rays emanating from the two ends of the boundary interval, we demand that these curves intersects some cut-off points. Precisely, we demand that $\gamma^+_{AdS}$ passes through $(u = \frac{l_u}{2} + \epsilon_u, \phi = \frac{l_\phi}{2} - \epsilon_\phi)$ and $\gamma^-_{AdS}$ crosses $(u = -\frac{l_u}{2} + \epsilon_u, \phi = \frac{-l_\phi}{2} + \epsilon_\phi)$ at $r = \frac{1}{\epsilon_-}$ where $\epsilon_u$, $\epsilon_\phi$ and $\epsilon_\pm$ are infinitesimal positive constants and $\epsilon_u/\epsilon_\phi$ is finite. With these conditions, we find that $\beta_\pm = \frac{\epsilon_u}{\epsilon_\phi} \alpha_\pm$ and there are to set of solutions for $\alpha_\pm$:

$$\begin{cases} 
\alpha_+ = \alpha_- = 0 \\
\text{or} \\
\alpha_+ = \alpha_- = -\frac{2\epsilon_u}{\epsilon_\phi} \left( 1 - \frac{\epsilon_u^2}{\ell^2 \epsilon_\phi^2} \right).
\end{cases} \quad (3.26)$$

For the second case, after taking the flat-space limit, one can deduce that $\epsilon_\pm < 0$, which is opposed to our earlier assumption. Thus we need to choose $\alpha_+ = \alpha_- = \beta_+ = \beta_- = 0$ which its flat-space limit results in $\gamma_\pm$ of [13].
3.4 Our proposal for calculation of holographic BMSFT entanglement entropy

In the previous subsections, we proposed a holographic method for the calculation of BMSFT entanglement entropy, which does not use the Rindler method and takes a flat-space limit from the calculations of the AdS/CFT correspondence. In this subsection, we summarize our method and give a generic recipe:

- The first step is finding an asymptotically AdS spacetime that its flat-space limit yields our asymptotically flat metric. For this purpose, we need to write our metric and the interval at null infinity in the BMS gauge.

- For the asymptotically AdS metric and its corresponding interval, we can use RT prescription or its generalizations to find an extremal curve in the bulk which its length is proportional to the entanglement entropy of the interval.

- The bulk modular flow is a Killing vector, which is zero on the extremal curve. So our next step is finding bulk modular flow by using the extremal curve of the asymptotically AdS case.

- Using the bulk modular flow, we look for a new Killing vector which is normal to it, and their norms satisfy (3.16). This new vector has well-defined flat-space limit which is the bulk modular flow of the corresponding asymptotically flat spacetimes.

- The points in bulk in which the components of the new vector are zero construct a new extremal curve in the asymptotically AdS spacetime. The flat-space limit of this new curve is well-defined and yields a curve in the asymptotically flat spacetime that $\gamma$ is a part of it.

- As the next step, we look for the null geodesics, which start from two ends of the interval and intersect our new extremal curve. The flat-space limit of these curves results in $\gamma_{\pm}$. However, instead of a unique curve, we encounter a bunch of curves with this property. We remove this ambiguity by demanding that the desired null curve must pass through another point in the spacetime, which we call it the cut-off point.

- Knowing $\gamma$ and $\gamma_{\pm}$, we can calculate the length of $\gamma$, which is proportional to the entanglement entropy of the interval in BMSFT.

4 Summary and Conclusion

We introduced new curves in the three dimensional asymptotically AdS spacetimes whose flat-space limit yields the extremal curves in the asymptotically flat spacetimes. These curves can be
used for the holographic calculation of the dual BMSFT entanglement entropy. In our proposal, a new vector field, which is normal to the bulk modular flow of the corresponding CFT interval, plays an important role. In fact, instead of the BMSFT interval, we considered a similar CFT interval, and using RT proposal, we found the extremal curve and also the bulk modular flow in the asymptotically AdS spacetime. Then using the Killing equation and also normality and norm conditions, we found a new vector field. The points in the spacetime in which this new vector field is zero construct a curve with a well-defined flat-space limit. We also have proposed two null rays with a well-defined flat space limit in the asymptotically AdS spacetime, which connects two ends of the interval to the new extremal curve. The flat-space limit of these curves together, provide a holographic description for the BMSFT entanglement entropy proposed previously in [13].

Our method can be generalized to the higher dimensions. One of the future directions is to use it to find the entanglement entropy of three-dimensional BMSFTs, which are dual to the four-dimensional asymptotically flat spacetimes. Moreover, this method can be used to relate the first law of entanglement entropy in the boundary theory to the linearized equation of motion of the bulk theory [25, 26]. In the context of flat/BMSFT, this relation has been studied earlier in [27, 28] (See also more recent papers [29, 30]. The main advantage of our method is that one can study this problem by taking the flat space limit from the known results in the AdS/CFT. For example, the extremal curves related to perturbed geometry in the asymptotically flat spacetime can be found by taking the flat-space limit from the extremal curves in the asymptotically AdS spacetime. We plan to study this problem in our future works.

There is an interesting observation in the presented calculation that can initiate more interesting future directions. When we consider intervals in the boundary of the BTZ black holes, we find that our new curves are located in the region between the inner and the outer horizons. The length of these curves has information about the entanglement entropy of the boundary theory. This is a new holographic interpretation of entanglement entropy, which may have applications in other problems such as the black holes information paradox.

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A Rindler transformation

We are looking for a Rindler transformation which changes
\[ ds^2 = -\frac{r^2}{\ell^2} du^2 - 2 dudr + r^2 d\phi^2, \]  
(A.1)
to a BTZ black hole written in the BMS coordinate,
\[ ds^2 = \left( -\tilde{r}^2 + \tilde{M} \right) d\tilde{u}^2 - 2d\tilde{u}d\tilde{r} + 2\tilde{N}d\tilde{u}d\tilde{\phi} + \tilde{r}^2 d\tilde{\phi}^2, \]  
(A.2)
where \( \tilde{M} \) and \( \tilde{N} \) are constants and are given in terms of the BTZ inner and outer horizons \( \tilde{r}_\pm \) as
\[ \tilde{M} = \frac{\tilde{r}_+^2 + \tilde{r}_-^2 - \ell^2}{\ell^2}, \quad \tilde{N} = \frac{\tilde{r}_+ - \tilde{r}_-}{\ell}. \]  
(A.3)
In order to use the transformations which were introduced in literature, we use (2.15) and (3.2) to change (A.1) to Poincare coordinate,
\[ ds^2 = \frac{\ell^2}{z^2} \left( dz^2 - dt^2 + dx^2 \right). \]  
(A.4)
Then using the following transformation
\[ \rho = \frac{1}{2z^2}, \quad U = x - t, \quad V = t + x, \]  
(A.5)
we change (A.4) to
\[ ds^2 = \ell^2 \left( \frac{d\rho^2}{4\rho^2} + 2\rho dU dV \right). \]  
(A.6)
The corresponding Rindler transformation is given by (7.2) of [13] as
\[ T_\tilde{u} \tilde{U} = \frac{1}{4} \log \left[ \frac{(1 + \rho(2U + l_U)\tilde{V})^2 - \rho^2 l_U^2 (l_U / 2 + U)^2}{(1 + \rho(2U - l_U)\tilde{V})^2 - \rho^2 l_U^2 (l_U / 2 - U)^2} \right], \]
\[ T_\tilde{V} \tilde{V} = \frac{1}{4} \log \left[ \frac{(1 + \rho(2V + l_V)\tilde{U})^2 - \rho^2 l_V^2 (l_V / 2 + V)^2}{(1 + \rho(2V - l_V)\tilde{U})^2 - \rho^2 l_V^2 (l_V / 2 - V)^2} \right], \]
\[ \tilde{\rho} \frac{T_\tilde{V} T_\tilde{V}}{T_\tilde{U} T_\tilde{U}} = \rho^2 \left( \frac{l_U^2}{l_V} - 4V^2 \right) - 4U^2 l_U^2 + 4(2\rho UV + 1)^2, \]  
(A.7)
and transforms (A.6) to
\[ ds^2 = \ell^2 \left( T_\tilde{U}^2 d\tilde{U}^2 + 2\tilde{\rho} d\tilde{U} d\tilde{V} + T_\tilde{V}^2 d\tilde{V}^2 + \frac{d\rho^2}{4(\tilde{\rho}^2 - T_\tilde{U}^2 T_\tilde{V}^2)} \right). \]  
(A.8)
The final part of transformation is given by
\[ \tilde{U} = \frac{\tanh^{-1} \left( \frac{2\tilde{r}}{\tilde{r} + T_\tilde{U}} \right) - \tanh^{-1} \left( \frac{2\tilde{r}}{\tilde{r} + T_\tilde{U}} \right)}{2T_\tilde{U}} + \frac{\ell \tilde{\phi} + \tilde{u}}{2\ell^2}, \]
\[ \tilde{V} = \frac{\tanh^{-1} \left( \frac{2\tilde{r}}{\tilde{r} + T_\tilde{V}} \right) + \tanh^{-1} \left( \frac{2\tilde{r}}{\tilde{r} + T_\tilde{U}} \right)}{2T_\tilde{V}} + \frac{\ell \tilde{\phi} - \tilde{u}}{2\ell^2}, \]
\[ \tilde{\rho} = \frac{1}{2} \left( 4\tilde{r}^2 - T_\tilde{U}^2 - T_\tilde{V}^2 \right), \]  
(A.9)
which transforms (A.8) to (A.2) and we have

\[ T_{\hat{\phi}} = \hat{r}_+ + \hat{r}_- , \quad T_{\hat{\phi}} = \hat{r}_+ - \hat{r}_- . \]  

(A.10)

B Holographic entanglement entropy of BMSFT dual to Global Minkowski

Let us consider an interval in BMSFT\(_2\) dual to the three dimensional global Minkowski spacetime given by the following metric,

\[ ds^2 = -d\hat{u}^2 - 2d\hat{u}d\hat{r} + \hat{r}^2d\hat{\phi}^2 . \]  

(B.1)

The interval is determined by

\[ \hat{u} = \frac{L_u \sin \hat{\phi}}{2 \sin \frac{L\phi}{2}} , \]  

(B.2)

and two ends of it are at \((\hat{u} = -L_u/2, \hat{\phi} = -L\phi/2)\) and \((\hat{u} = +L_u/2, \hat{\phi} = +L\phi/2)\).

The metric (B.1) is given by taking the flat-space limit from the global AdS written in the BMS gauge,

\[ ds^2 = -\left(1 + \frac{\hat{r}^2}{\ell^2}\right)d\hat{u}^2 - 2d\hat{u}d\hat{r} + \hat{r}^2d\hat{\phi}^2 . \]  

(B.3)

In order to find the holographic entanglement of the interval (B.2) in BMSFT, we start with the same interval in the dual CFT\(_2\) of (B.3). The bulk modular flow, \(\xi^\mu\), and the new vector \(\lambda^\mu\) are given by using the following transformation from (3.3) and (3.13):

\[ r = \frac{1}{2}\left(\ell \sin \frac{\hat{u}}{\ell} + \hat{r} \cos \frac{\hat{u}}{\ell} + \hat{r} \cos \hat{\phi}\right) , \]

\[ \phi = \frac{\hat{r} \sin \hat{\phi}}{r} , \]

\[ u = \frac{\ell}{r}\left(\ell - \ell \cos \frac{\hat{u}}{\ell} + \hat{r} \sin \frac{\hat{u}}{\ell}\right) . \]  

(B.4)

The transformation (B.4) determines \(l_\phi\) and \(l_u\) in terms of \(L_\phi\) and \(L_u\),

\[ l_u = \frac{4\ell \sin\left(\frac{L_u}{2}\right)}{\cos\left(\frac{L\phi}{2}\right) + \cos\left(\frac{L\phi}{2}\right)} , \]

\[ l_\phi = \frac{4 \sin\left(\frac{L\phi}{2}\right)}{\cos\left(\frac{L\phi}{2}\right) + \cos\left(\frac{L\phi}{2}\right)} . \]  

(B.5)
and gives $\xi^\mu$ and $\lambda^\mu$ of the global AdS as follows:

$$\xi = 4\pi \left( \cos(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \left( \ell \cos\left(\frac{\phi}{\ell}\right) - \ell \sin\left(\frac{\phi}{\ell}\right) \right) + \sin(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \left( \ell \cos\left(\frac{\phi}{\ell}\right) + \ell \sin\left(\frac{\phi}{\ell}\right) \right) \right) \partial_\ell +$$

$$\frac{2\pi \ell \left( -2 \cos(\phi) \sin\left(\frac{L_\phi}{\ell}\right) \cos\left(\frac{\phi}{\ell}\right) - 2 \sin(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \sin\left(\frac{\phi}{\ell}\right) + \sin(L_\phi) \right)}{\cos(L_\phi) - \cos\left(\frac{L_\phi}{\ell}\right)} \partial_\ell +$$

$$\lambda = -\frac{4\pi \left( \cos(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \left( \ell \cos\left(\frac{\phi}{\ell}\right) - \ell \sin\left(\frac{\phi}{\ell}\right) \right) + \sin(\phi) \sin\left(\frac{L_\phi}{\ell}\right) \cos\left(\frac{L_\phi}{\ell}\right) \left( \ell \cos\left(\frac{\phi}{\ell}\right) + \ell \sin\left(\frac{\phi}{\ell}\right) \right) \right) \partial_\ell +$$

$$\frac{2\pi \ell \left( 2 \cos(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \cos\left(\frac{\phi}{\ell}\right) + 2 \sin(\phi) \sin\left(\frac{L_\phi}{\ell}\right) \sin\left(\frac{\phi}{\ell}\right) \cos\left(\frac{L_\phi}{\ell}\right) - \sin(L_\phi) \right)}{\cos(L_\phi) - \cos\left(\frac{L_\phi}{\ell}\right)} \partial_\ell +$$

$$\frac{2\pi \left( -2 \cos(\phi) \sin\left(\frac{L_\phi}{\ell}\right) \cos\left(\frac{\phi}{\ell}\right) + 2 \sin(\phi) \cos\left(\frac{L_\phi}{\ell}\right) \sin\left(\frac{\phi}{\ell}\right) \cos\left(\frac{L_\phi}{\ell}\right) + \ell \sin(L_\phi) \right)}{\ell \cos\left(\frac{L_\phi}{\ell}\right) - \cos\left(\frac{L_\phi}{\ell}\right)} \partial_\ell.$$

It is clear that these two vector fields transform to each other by using the transformation

$$L_\ell \Rightarrow \ell L_\phi,$$

$$L_\phi \Rightarrow L_\ell/\ell.$$  \hspace{1cm} (B.8)

The rest of the calculation is straightforward and one can find $\gamma_{AdS}$ and also $\gamma_\pm$ and take the flat-space limit from them to find $\gamma$ and $\gamma_\pm$ in the global Minkowski spacetime. The final results are the same as [13].

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