ALBANESE VARIETIES OF ABELIAN COVERS

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Abstract. We show that Albanese varieties of abelian covers of projective plane are isogenous to product of isogeny components of abelian varieties associated with singularities of the ramification locus. In particular Albanese varieties of abelian covers of $\mathbb{P}^2$ ramified over arrangements of lines and uniformized by unit ball in $\mathbb{C}^2$ are isogenous to a product of Jacobians of Fermat curves. Periodicity of the sequence of (semi-abelian) Albanese varieties of unramified cyclic covers of complements to a plane singular curve is shown.

1. Introduction

Albanese varieties of cyclic branched covers of $\mathbb{P}^2$ ramified over singular curves are rather special. If singularities of ramification locus are no worse than ordinary nodes and cusps then (cf. [7]) the Albanese variety of a cyclic cover is isogenous to a product of elliptic curves $E_0$ with $j$-invariant zero. More generally, in [25] it was shown that Albanese variety of a cyclic cover with ramification locus having arbitrary singularities is isogenous to a product of isogeny components of local Albanese varieties i.e. the abelian varieties canonically associated with the local singularities of the ramification locus. In particular, Albanese varieties of cyclic covers are isogenous to a product of Jacobians of curves.

In this paper we shall describe Albanese varieties of abelian covers of $\mathbb{P}^2$. The main result is that the class of abelian varieties which are Albanese varieties of ramified abelian covers (with possible non reduced ramification locus) also built from the isogeny components of local Albanese varieties but in abelian case one need to allow local Albanese varieties of non reduced singularities having the same reduced structure as the germs a singularity of ramification locus of the covering.

One of the steps in our proof of this result involves description of Jacobians of abelian covers of projective line having independent interest. In this case we show that all isogeny components of Jacobians of abelian covers of $\mathbb{P}^1$ with arbitrary ramification are components of Jacobians of explicitly described cyclic covers. If the abelian cover is ramified only at 3 points the has the Galois group isomorphic to $\mathbb{Z}_n^2$ then it is biholomorphic to Fermat curve $x^n + y^n = z^n$. In this case, such results are going back to works of Gross, Rohrlich and Coleman where isogeny components of Jacobians of Fermat curves were studied.

The proof of isogeny decomposition of abelian covers is constructive and, as an application, we obtain isogeny classes of Albanese varieties of discovered by Hirzebruch (cf.[19]) abelian covers of $\mathbb{P}^2$ having the unit ball as the universal cover. These Albanese varieties are isogenous to products of Jacobians of Fermat curves described explicitly. Another interesting abelian cover of $\mathbb{P}^2$ ramified over an arrangement of lines is the Fano surface of lines on Fermat cubic threefold. The
Albanese variety of this Fano surface (i.e., according to [6], the intermediate Jacobian of Fermat cubic threefold) is isogenous to product of five copies of $E_0$. This result was recently independently obtained in [27] and [5] (in the former work the isomorphism class of Albanese variety of Fano surfaces was found).

Another application considers behavior of the Albanese varieties in the towers of cyclic and abelian covers. It is known for some time that Betti and Hodge numbers of cyclic (resp. abelian) covers are periodic (resp. polynomially periodic cf. [17]) if they turn out that the sequence of isogeny classes of Albanese varieties of cyclic covers with given ramification locus is periodic but periodicity fails in abelian towers. Moreover, we show similar periodicity for sequence of semi-abelian varieties which are Albanese varieties of quasi-projective surfaces which are unramified covers of $\mathbb{P}^2 \setminus \mathcal{E}$.

The content of the paper is the following. In section 2 we recall several key definitions and results used later, in particular, the characteristic varieties, Albanese varieties in quasi-projective and local cases. Section 3 considers Jacobians of abelian covers of $\mathbb{P}^1$ and the main result is that isogeny components of such Jacobians are all the isogeny components of Jacobians of cyclic covers of $\mathbb{P}^1$. This section also contains calculation of multiplicities of characters of representation of the covering group on the space of holomorphic 1-forms. In the case of cyclic covers, such multiplicities were calculated in [2]. The main result of the paper, showing that Albanese varieties of abelian covers are isogenous to a product of isogeny components of local Albanese varieties of singularities, proven in section 4. The case of covering ramified over arrangements of lines considered in section 5. This includes, mentioned already, the case of Fano surface (of lines) on Fermat cubic threefold. The last section contains applications to calculation of Mordell Weil ranks of isotrivial abelian varieties and periodicity properties of Albanese varieties in towers of abelian covers. Finally, the prime field of all varieties, maps between them and function fields considered in this paper is $\mathbb{C}$.

2. Preliminaries

2.1. Characteristic varieties. We recall construction of invariants of the fundamental group of the complement playing the key role in description of the Albanese varieties of abelian covers. We follow [23] (cf. also [3]).

Let $X$ be a quasi-projective smooth manifold such that $H_1(X, \mathbb{Z}) \neq 0$. The exact sequence

$$0 \to \pi_1(X)'/\pi_1(X)'' \to \pi_1(X)/\pi_1(X)'' \to \pi_1(X)/\pi_1(X)' \to 0$$

(where $G'$ denotes the commutator subgroup of a group $G$) can be used to define the action of $H_1(X, \mathbb{Z}) = \pi_1(X)/\pi_1(X)'$ on the left term in (1). This action allows to view $C(X) = \pi_1(X)'/\pi_1(X)'' \otimes \mathbb{C}$ as a $\mathbb{C}[H_1(X, \mathbb{Z})]$-module. Recall that support of a module $M$ over a commutative noetherian ring $R$ is sub-variety $\text{Supp}(M) \subset \text{Spec}(R)$ consisting of prime ideals $\mathfrak{p}$ for which localization $M_\mathfrak{p} \neq 0$

**Definition 2.1.** Characteristic variety $V_i(X)$ is (reduced) sub-variety of $\text{Spec}\mathbb{C}[H_1(X)]$ which is the support $\text{Supp}(\Lambda^i(C(X)))$ of the $i$-th exterior power of the module $C(X)$. Depth of $\chi \in \text{Spec}\mathbb{C}[H_1(X)]$ is an integer given by

$$d(\chi) = \{\max i | \chi \in V_i(X)\}$$
Using canonical identification $\text{Spec} \mathbb{C}[H_1(X, \mathbb{Z})]$ and $\text{Char}(\pi_1(X))$ one can interpret points of characteristic varieties as rank one local systems on $X$ in terms of which one has the following alternative description of $V_i(X)$ (cf. [15], [23]):

\begin{equation}
V_i(X) \setminus 1 = \{ \chi \in \text{Char}\pi_1(X), \chi \neq 1, dim H^1(X, \chi) \geq i \}
\end{equation}

It follows from [1] that if a smooth projective closure $\overline{X}$ of $X$ satisfies $H_1(\overline{X}, \mathbb{Q}) = 0$ then each $V_i(X)$ is a finite union of translated subgroups of the affine torus $\text{Char}(\pi_1(X))$ i.e. subset of the form $\psi \cdot H$ where $H$ is a subgroup of $\text{Char}(\pi_1(X))$ and $\psi$ is a character of $\pi_1(X)$. Moreover, (cf. [24]) character $\psi$ can be chosen to have a finite order. It also follows from [1] that each irreducible component $V$ of characteristic variety determines a holomorphic map: $\nu : X \to P$ where $P$ is a hyperbolic curve.

In the case when $X = \mathbb{P}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is a plane curve with arbitrary singularities, $P$ is biholomorphic to $\mathbb{P}^1 \setminus D$ where $D$ is a finite subset.

Returning to the case of arbitrary $X$, the component corresponding to a map $\nu : X \to P$ consists of the characters $\nu i(\chi)$ where $\chi \in \text{Char}\pi_1(P)$ (here, for a map $\phi : X \to Y$ between topological spaces $X, Y$, we denote by $\phi^*$ the induced map $\text{Char}(H_1(Y, \mathbb{Z})) = H^1(Y, \mathbb{C}^*) \to H^1(X, \mathbb{C}^*) = \text{Char}(H_1(X, \mathbb{Z}))$).

At the intersection of components the depth of characters is bigger then the depth of generic character in either of components the depth is jumping. More precisely, if $\chi \in V_k(X) \cap V_l(X)$ where both $V_k(X)$ and $V_l(X)$ have positive dimensions then the depth of $\chi$ is at least $k + l$ (cf. [4]). We shall make following assumption which is an inequality in the opposite direction:

**Condition 2.2.** Depth of each character $\chi$ in intersection of several positive dimensional irreducible components $V_1, \ldots, V_s$ of characteristic variety does not exceed the sum of depths of the generic character in each component $V_i$.

This assumption is satisfied in the examples considered in section [5].

2.2. Abelian covers. Given a surjection $\pi_\Gamma : \pi_1(X) \to \Gamma$, there are unique quasi-projective manifold $\overline{X}_\Gamma$ and the map $\overline{\pi}_\Gamma : \overline{X}_\Gamma \to X$, which is unramified cover with the covering group $\Gamma$. $\overline{X}_\Gamma$ is characterized by the property that $\Gamma$ acts freely on $\overline{X}_\Gamma$ and $\overline{X}_\Gamma/\Gamma = X$. Let $\overline{X}_\Gamma$ denote a smooth model of a compactification of $\overline{X}_\Gamma$ such that $\overline{\pi}_\Gamma$ extends to a regular map $\overline{\pi}_\Gamma : \overline{X}_\Gamma \to \overline{X}$ ($\overline{X}$ as above). The fundamental group $X_\Gamma$, being birational invariant, depend only on $X$ and $\Gamma$.

Let $\mathcal{C} = X \setminus X$ be the “divisor at infinity”. and let $\overline{\mathcal{C}}$ is a divisor on $\overline{X}$ which is a union of components of $\mathcal{C}$. If $\chi \in \text{Char}\pi_1(X)$ is trivial on components of $\mathcal{C}$ not in $\overline{\mathcal{C}}$ then $\chi$ is the pullback of a character of $\pi_1(X \setminus \overline{\mathcal{C}})$ via the inclusion $X \to X \setminus \overline{\mathcal{C}}$. We shall denote by corresponding character of $\pi_1(X \setminus \overline{\mathcal{C}})$ as $\chi$ as well but (since the depth of $\chi$ depends on the underlying space) corresponding depths will be denoted $d(\chi, \mathcal{C})$ and $d(\chi, \overline{\mathcal{C}})$ respectively.

The homology groups of unramified and ramified covers can be found in terms of characteristic varieties as follows (cf. [24]).

**Theorem 2.3.** (cf. [23]) With above notations:

\begin{equation}
\text{rk} H_1(X_\Gamma, \mathbb{Q}) = \sum_{\chi \in \text{Char} \Gamma} d(\pi_\Gamma^*(\chi), \mathcal{C})
\end{equation}

\footnote{This condition is independent of a choice of smooth compactification $\overline{X}$.
2. (cf. [28]) Let $I(\chi)$ be collection of components of $C$ such that $\chi(\gamma_{C_i}) \neq 1$ ($\gamma_{C_i}$ is the meridian of component $C_i$) and let $\mathcal{E}_X = \bigcup_{i \in I(\chi)} C_i$. Then rank

$$H_1(V\Gamma, \mathbb{Q}) = \sum_{\chi \in \text{Char} \Gamma} d(\pi^*_\Gamma(\chi), \mathcal{E}_{\pi^*_\Gamma(\chi)})$$

The following special case of theorem 2.3 will be used in section 3.

Corollary 2.4. Let

$$\pi_{\Gamma}(a_1, \ldots, a_i) : \pi_1(\mathbb{P}^1 \setminus (a_i, \ldots, a_i)) \to H_1(\mathbb{P}^1 \setminus (a_i, \ldots, a_i)), \mathbb{Z}/n\mathbb{Z}) = \Gamma, 0 \leq i_1, \ldots, i_l \leq k$$

be composition of Hurewicz map with the reduction modulo $n$ and let $X_n(a_i, \ldots, a_i)$ be the corresponding ramified abelian cover of $\mathbb{P}^1$. Then

$$H^1(X_n(a_0, \ldots, a_k), \mathbb{C})\chi = \oplus H^1(X_n(a_i, \ldots, a_i), \mathbb{C})\chi^r(a_i, \ldots, a_i) \quad 3 \leq l \leq k, 0 \leq i_j \leq k$$

where the summation is over the characters $\chi^r(a_i, \ldots, a_i)$ which are restricted in the sense that they do not take value 1 on a cycle which is the boundary of a small disk about any point $a_i, \ldots, a_i$.

2.3. Albanese varieties of quasi-projective manifolds. Let $X$ be a smooth quasi-projective manifold. As above, denote by $\overline{X}$ a smooth compactification of $X$ and by $\mathcal{E} = \overline{X} \setminus C$ which we shall assume in this section to be a divisor with normal crossings. One associates to $X$ a semi-abelian variety, call the Albanese variety of $X$ i.e. the extension:

$$0 \to T \to Alb(X) \to A \to 0.$$ 

In (7) $T$ is a torus and $A$ is an abelian variety (abelian part of $Alb(X)$). Such semi-abelian variety can be obtained as

$$H^0(\overline{X}, \Omega^1(\log(\mathcal{E}))^*/H_1(X, \mathbb{Z})$$

where embedding $H_1(X, \mathbb{Z}) \to H^0(\overline{X}, \Omega^1(\log(\mathcal{E}))^*$ is given by $\gamma \in H_1(X, \mathbb{Z}) \to (\omega \to \int_{\gamma} \omega)$ (and polarization of abelian part is coming from the Hodge form on $H_1(\overline{X}, \mathbb{Z})$ given by $(\gamma_1, \gamma_2) = \int_{\overline{X}} \gamma_1^* \wedge \gamma_2^* \wedge h^{\dim X-1}$ where $h \in H^2(\overline{X}, \mathbb{Z})$ is the class of hyperplane section).

One can also view $AlbX$ as the semi-abelian part of the 1-motif associated to the (level one) mixed Hodge structure on $H_1(X, \mathbb{Z})$ (cf. [9]). The abelian part of $Alb(X)$ is the Albanese variety of a smooth projective compactification of $X$. It clearly is independent of a choice of the latter.

In this paper we shall consider Albanese varieties of abelian covers of quasi-projective surfaces but note that the Albanese variety of an abelian covers of quasi-projective manifold of any dimension can be obtained as the Albanese variety of the corresponding abelian cover of a surface due to the following Lefschetz type result:

Proposition 2.5. Let $X$ be a quasi-projective manifold and $H \cap X$ a generic 2-dimension section by a linear space $H$. Then $\pi_1(X) = \pi_1(X \cap H)$ and the $\Gamma$ covers $\overline{X}_\Gamma$ and $(\overline{X} \cap H)_\Gamma$ corresponding to the surjection $\pi_\Gamma$ of this groups have isomorphic as semi-abelian varieties Albanese varieties.

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2 i.e. the universal cover for the covers having an abelian n-group as the covering group
2.4. Local Albanese varieties of plane curve singularities. For details of the material of this section we refer to [25]. Let \( f(x, y) \) be an analytic germ of a reduced isolated curve singularity in \( \mathbb{C}^2 \). One associates with it the Milnor fiber \( M_f = B \cap f^{-1}(t) \). The latter supports canonical level one limit Mixed Hodge structure on \( H^1(M_f, \mathbb{Z}) \) (cf. [29]). Again one can apply Deligne’s construction [9] 10.3.1 which leads to the following.

**Definition 2.6.** Local Albanese variety of a germ \( f \) is the abelian part of the 1-motif of the limit Mixed Hodge structure on \( H^1(M_f, \mathbb{Z}) \). Equivalently, this is quotient of \( F_0 \text{Gr}^W_{-1} H^1(M_f, \mathbb{Z}) \) where \( F \) and \( W \) are respectively the Hodge and weight filtrations. Canonical polarization is coming from the form induced by the intersection form of \( H^1(M_f, \mathbb{Z}) \) on \( \text{Gr}^W_1 H^1(M_f, \mathbb{Z}) \).

The local Albanese has a description in terms of the Mixed Hodge structure on the cohomology of the link of the surface singularity associated to \( f \).

**Proposition 2.7.** (cf. [25], Prop.3.1) Let \( f(x, y) \) be a germ of a plane curve with Milnor fiber \( M_f \) and for which the semi-simple part of monodromy has an order \( N \).

Let \( L_{f,N} \) the the link of the corresponding surface singularity

\[ z^N = f(x, y) \]

Then there is the isomorphism of the mixed Hodge structures:

\[ \text{Gr}^W_3 H^2(L_{f,N})(1) = \text{Gr}^W_1 H^1(M_f) \]

where the mixed Hodge structure on the left is the Tate twist of the mixed Hodge structure constructed in [12] and the one on the right is the mixed Hodge structure on vanishing cohomology constructed in [29].

Below we shall use Albanese varieties for non reduced germs and those can be de\-fine using the abelian part of the 1-motif of mixed Hodge structure \( \text{Gr}^W_3 H^2(L_{f,N})(1) \).

Recall finally that the local Albanese can be described in terms of resolution of singularity [3].

**Theorem 2.8.** (cf. [25] th. 3.11) Let \( f(x, y) = 0 \) be a singularity let \( N \) be the order of the semi-simple part of its monodromy operator. The local Albanese variety of germ \( f(x, y) = 0 \) is isogenous to a product of Jacobians of the exceptional curves of positive genus for a resolution of singularity [3].

3. Jacobians of abelian covers of a line

The following will be used in the proof of the theorem [11]

**Theorem 3.1.** Let \( X_n \) be the abelian cover of \( \mathbb{P}^1 \) ramified at \( A = \{a_0, a_1, ..., a_k \} \subset \mathbb{P}^1 \) corresponding to surjection \( \pi_1(\mathbb{P}^1 \setminus A) \to H_1(\mathbb{P}^1 \setminus A, \mathbb{Z}_n) \). Let \( A_i \in \mathbb{N}, i = 0, ..., k \) be collection of integers such that

\[ \sum_{i=1}^{i=k} A_i = 0 \text{ (mod } n), 1 \leq A_i < n \quad \text{gcd}(n, A_0, ..., A_k) = 1 \]

\[ \text{this assumption is somewhat weaker than the one in [29] but the argument works in this case with no change} \]
Denote by $X_{n|A_0,\ldots,A_k}$ smooth and by \[\text{irreducible model of the cyclic cover of } \mathbb{P}^1\text{ which affine portion is given by}\]
\[y^n = (x - a_0)^{A_0} \cdots (x - a_k)^{A_k}\]

Then the Jacobian of $X_n$ is isogenous to a product of isogeny components of Jacobians of the curves $X_{n|A_0,\ldots,A_k}$.

**Remark 3.2.** If $k = 2$ then the curve $X_n$ is biholomorphic to Fermat curve $x^n + y^n = z^n$ in $\mathbb{P}^2$, since as affine model of abelian cover one can take curve in $\mathbb{C}^4$ given by $x^n = u$, $y^n = 1 - u$, and above theorem follows from calculation \cite{14} which contains explicit formulas for simple isogeny components of Fermat curves.

**Corollary 3.3.** Let $X_\Gamma$ be a covering of $\mathbb{P}^1$ with abelian Galois group $\Gamma$ ramified at $a_0,\ldots,a_k \in \mathbb{P}^1$. Then there exist a collection of cyclic covers \cite{17} such that the Jacobian of $X_\Gamma$ is isogenous to a product of isogeny components of Jacobians of the curves in this collection.

**Proof.** Let $\pi_\Gamma : H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{\lfloor \frac{n}{k} \rfloor} a_i, \mathbb{Z}) \to \Gamma$ be the surjection corresponding to the covering $X_\Gamma$, $\delta_i \in H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{\lfloor \frac{n}{k} \rfloor} a_i, \mathbb{Z})$, $i = 0,\ldots,k$ be the boundary of a small disk about $a_i$, $i = 0,\ldots,k$ and let $n_i$ be the order of the element $\pi_\Gamma(\delta_i) \in \Gamma$. Then for $n = \text{lcm}(n_0,\ldots,n_k)$ one has surjection $H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{\lfloor \frac{n}{k} \rfloor} a_i, \mathbb{Z}/n\mathbb{Z}) \to \Gamma$ and hence a dominant map $X_n \to X_\Gamma$. In particular Jacobian of $X_\Gamma$ is a quotient of the Jacobian of $X_n$ and the claim follows. \[\square\]

**Proof.** (of the theorem \cite{3.1}) We shall assume below that one of ramification points, say $a_0$, is the point of $\mathbb{P}^1$ at infinity.

A projective model of $X_n$ the projective closure in $\mathbb{P}^{k+1}$ (which homogeneous coordinates we shall denote $x, z_1,\ldots,z_k, w$) can be given by complete intersection in $\mathbb{C}^{k+1}$
\[z_0^n = x - a_1,\ldots,z_k^n = x - a_k\]

The Galois covering map $X_n \to \mathbb{P}^1$ is given by the restriction of projection of $\mathbb{P}^{k+1}$ from the subspace $x = w = 0$.

For any $(A_0, A_1,\ldots,A_k)$ as above consider the map
\[\Phi_{n|A_0,\ldots,A_k} : X_n \to X_{n|A_0,A_1,\ldots,A_k}\]
in the chart $w \neq 0$ is restriction on $X_n$ of the map $\mathbb{C}^{k+1} \to \mathbb{C}^2$
\[\Phi_{A_1,\ldots,A_k} : (z_1,\ldots,z_k, x) \to (y, x) = (z_1^{A_1},\ldots,z_k^{A_k}, x)\]

$\Phi_{n|A_0,\ldots,A_k}$ is the map of the covering spaces of $\mathbb{P}^1$ corresponding to the surjection of the Galois groups of these coverings
\[H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{\lfloor \frac{n}{k} \rfloor} a_i, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}\]
given by
\[(i_0, i_1,\ldots,i_k) \to \sum_j i_j A_j \mod n\]

The maps $\Phi_{n|A_0,\ldots,A_k}$ induce the map of Jacobians:
\[\oplus_{A_0,\ldots,A_k,0 \leq A_i < n-1} \left( \Phi_{n|A_0,\ldots,A_k} \right)_* : \text{Jac}(X_n) \to \oplus \text{Jac}(X_{n|A_0,\ldots,A_k})\]

Hence, in fact, one has inequalities:

\[
d(\chi) = m^{1,0}_\chi(n|A_0, ..., A_k) + m^{1,0}_\chi(n|A_0, ..., A_k) = m^{1,0}_{\Phi^*_n|A_0, ..., A_k}(\chi)(n) + m^{1,0}_{\Phi^*|A_0, ..., A_k}(\chi)(n)
\]

Moreover, one has inequalities:

\[
m^{0,1}_\chi(n|A_0, ..., A_k) \leq m^{0,1}_{\Phi^*_n|A_0, ..., A_k}(\chi)(n)
\]
\[
m^{1,0}_\chi(n|A_0, ..., A_k) \leq m^{1,0}_{\Phi^*|A_0, ..., A_k}(\chi)(n)
\]

Hence, in fact,

\[
m^{0,1}_\chi(n|A_0, ..., A_k) = m^{0,1}_{\Phi^*_n|A_0, ..., A_k}(\chi)(n)
\]
\[
m^{1,0}_\chi(n|A_0, ..., A_k) = m^{1,0}_{\Phi^*|A_0, ..., A_k}(\chi)(n)
\]

Now let us fix \( \chi \in \text{Char}(H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})) \) a character of the Galois group of the cover \( X_n \to \mathbb{P}^1 \) such that its value on the cycle \( \delta_i \in H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z}) \) corresponding to \( a_i \in \mathbb{P}^1, i = 0, ..., m \) satisfies:

\[
\chi(\delta_i) = \text{exp}(\frac{2\pi \sqrt{-1} j_i}{n}) \neq 1, (1 \leq j_i < n)
\]

and let \( J = \text{gcd}(j_0, ..., j_k) \). The collection of integers \( A_i = \frac{j_i}{J} \) satisfies condition (10). Denote by \( \Gamma_0 \) the cyclic group \( \chi(H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}) \subset \mathbb{C}^* \). Then \( \chi \) can be considered as a character \( \chi' \in \text{Char}(\Gamma_0) \) and then \( \chi = \pi'(\chi') \) where \( \pi \) is projection of the abelian cover with covering group \( \Gamma \) onto the cyclic cover with the covering group \( \Gamma_0 \). It follows from (20) that any isotypical component in \( H^{1,0}(X_n, \mathbb{C}) \) \( \chi \) is the image of the isotypical component of a cyclic covers and hence the map (17) is surjective which concludes the proof.

\[\square\]

We shall finish this section with an explicit formula for \( \text{dim} H^0(X_n, \Omega^1_{X_n})_\chi \) i.e. the multiplicity of the isotypical component of the covering group of abelian cover acting on the space of holomorphic 1-forms.

**Proposition 3.4.** Let the values of a character \( \chi \in \text{Char}H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z}) \) be given as in (21). Assume that \( J = \text{gcd}(j_0, ..., j_k) = 1 \) and let \( M = \sum_i (n - j_i) \). Then

\[
\text{dim} H^1(X_n) \chi = \left[\frac{M}{n}\right]
\]
Remark 3.5. If $J \neq 1$ then Prop. 3.4 yields an expression for the dimension of isotypical component corresponding to $\chi \in \text{Char} H_1(\mathbb{P}^1 \setminus A, \mathbb{Z}/n\mathbb{Z})$ as well since this dimension coincides with the dimension of isotypical component for $\chi$ considered as the character of $H_1(\mathbb{P}^1 \setminus A, \mathbb{Z}/(4\mathbb{Z})$.

Proof. (of Prop. 3.4) The equations of projective closure of complete intersection (21) are

$$z_i^n = (x - a_i w)w^{n-1}, \quad i = 1, \ldots, k$$

The only singularity of (23) occurs at $w = 0$, $z_i = 0$, $x = 1$. Near it (23) is a complete intersection locally given by $z_i^n = w^{n-1} \gamma_i$ where $\gamma_i$ is a unit. It has $n - k$ branches (corresponding to the orbits of the action $(z_1, \ldots, z_k) \to (\zeta z_1, \ldots, \zeta z_k), \zeta^n = 1$) each equivalent to $z_i = t^{n-1}, w = t^v$. Therefore (23) is a ramified cover of $\mathbb{P}^1$ with $k + 1$ points, branching points $a_1, \ldots, a_k$, over which it has $n^{-1}$ preimages at each ramification point.

The space $H^0(\Omega^1_{X_a})$ (for a smooth model of (23)) is generated by the residues of $k + 1$-forms

$$\frac{z_1^{j_1 - 1} \ldots z_k^{j_k - 1} P(x, w) \Omega}{(z_1^n - (x - a_i w)w^{n-1})} \quad (1 \leq j_i) \text{ where } \sum_{i=1}^k (j_i - 1) + \deg P + k + 2 = nk$$

(cf. [13], here $\Omega = \sum z_i dz_1 \wedge \ldots \wedge dz_i \wedge dx \wedge dw + x dz_1 \wedge \ldots \wedge dz_k \wedge dx \wedge dw + wdz_1 \wedge \ldots \wedge dz_k \wedge dx.$ In the chart $x \neq 0$ such residue (of (24)) is given by:

$$\frac{z_1^{j_1 - 1} \ldots z_k^{j_k - 1} P(w) dw}{(z_1 \ldots z_k)^{n-1}}$$

Using (23), one can reduce powers of $z_i$ i.e. we can assume:

$$1 \leq j_i \leq n - 1$$

and a basis of the eigenspace $H^0(\Omega^1_{X_a})_{\chi}$, with $\chi$ as in (21), can be obtained by selecting $P(w) = w^s$ where $s$ must satisfy:

$$\sum_{i=1}^k (j_i - 1) + s + k + 2 \leq nk$$

The adjunction condition assuring that the residue of (24) will be regular on normalization of (23) is

$$- \sum_{i=1}^k (n - j_i)(n - 1) + sn + n - 1 \geq 0$$

To count the number of solutions of (27) and (28) i.e. $\dim H^0(\Omega^1_{X_a})_{\chi}$ with $\chi$ given by (21), let $\bar{j}_i = n - j_i$. Then $1 \leq \bar{j}_i \leq n - 1$ and (27),(28) have form

$$\sum_{i=1}^k (n - 1 - \bar{j}_i) + s + k + 2 \leq kn, \quad -(\sum_{i=1}^k \bar{j}_i)(n - 1) + sn + n > 0.$$ Hence:

$$s + 2 \leq \sum_{i=1}^k \bar{j}_i < \frac{(s + 1)n}{n - 1} = s + 1 + \frac{s + 1}{n - 1}$$

Notice that from (27) one has $s \leq nk - k - 2$ i.e. $\frac{s + 1}{n - 1} \leq k - \frac{1}{n - 1}$ and hence $\sum_{i=1}^k \bar{j}_i \leq k + s$. In particular possible values of $\sum_{i=1}^k \bar{j}_i$ are $s + 2, s + 3, \ldots, s + k$ and therefore for given $\bar{j}_i$, parameter $s$ can take at most $k - 1$ values $\sum_{i=1}^k \bar{j}_i - 2, \ldots, \sum_{i=1}^k \bar{j}_i - k$. In particular, multiplicities of the $\chi$-eigenspaces do not exceed $k - 1$. 
Let $\sum j_i = M$. Then from (29) one has $M - 1 - \frac{M}{n} < s \leq M - 2$ and hence the number of possible values of $s$ is

$$M - 2 - \left[M - 1 - \frac{M}{n}\right] = -1 - \left[-\frac{M}{n}\right] = \left[\frac{M}{n}\right]$$

as claimed in the Prop. 3.4.

Remark 3.6. One can deduce the theorem 5.1 using Prop. 5.3 and the following:

Proposition 3.7. (2, Prop. 6.5). For $x \in \mathbb{R}$ denote by $< x > = x - \lfloor x \rfloor$ the fractional part of $x$. Assume that $\gcd(i, n) = 1$ and $n$ does not divide either of $A_0, \ldots, A_k$. Then for the curve (17) the dimension of the eigenspace corresponding to the eigenvalue $\exp(\frac{2\pi i}{n})$ of the automorphism of $H^1(X_n, A_0, \ldots, A_k, \mathbb{C})$ induced by the map $(x, y) \to (x, y \exp(\frac{-2\pi i}{n}))$ equals to

$$(30) \quad - < \frac{i \sum_{0}^{k} A_s}{n} > + \sum_{0}^{k} < \frac{i A_s}{n} >$$

Indeed, the equality of multiplicities (20) follows by comparison (22) with (30) since for $i = 1$ (30) yields $-\sum A_s + \left[\frac{\sum A_s}{n}\right] + \sum \frac{A_s}{n} = \left[\sum A_s\right]$.

Remark 3.8. Special case of Prop. 3.7 appears also in [25] (cf. lemma 6.1). The multiplicity of the latter corresponds to the case $j = n - i$ in Prop. 3.7.

4. Decomposition theorem for abelian covers of plane

The main result of this section relates the Albanese variety of ramified covers to the local Albanese varieties of ramification locus as follows.

Theorem 4.1. Let $\mathcal{C}$ be a plane algebraic curve and let $\pi_T : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \Gamma$ be a surjection onto an abelian group. Then the Albanese variety of abelian cover $\bar{X}_T$ ramified over $\mathcal{C}$ and associated with $\pi_T$ is isogenous to a product of isogeny components of local Albanese varieties of possibly non-reduced germs having as reduced singularity a singularity of $C$.

Proof. To each component of characteristic variety of positive dimension corresponds the isogeny component of Albanese variety of $\bar{X}_T$ as follows.

Let $\text{Char}_j$ be an irreducible component of the characteristic variety $V_1(\mathbb{P}^2 \setminus \mathcal{C})$ of $\mathcal{C}$ (cf. (2.1)) and let $\phi_j : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1 \setminus D_j$, where $D_j$ is a subset of $\mathbb{P}^1$ having cardinality $\dim(\text{Char}_j) + 1$, be the corresponding holomorphic map. In particular $\text{Char}_j = \phi_j^*(\text{Char}\pi_1(\mathbb{P}^1 \setminus D_j))$. Let $\Gamma_j$ be the push-out of $\pi_T$ and surjection ($\phi_j^*$, $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1(\mathbb{P}^1 \setminus D_j)$). With these notations we have the universal (for the groups fillings the right left corner of) commutative diagram:

$$\begin{array}{cccc}
\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) & \to & \pi_1(\mathbb{P}^1 \setminus D_j) \\
\downarrow & & \downarrow \\
\Gamma & \to & \Gamma_j \\
\end{array}$$

(31)

A character of $H_1(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{Z})$, which is the image of a character of $\Gamma$ for the map $\text{Char}\Gamma \to \text{Char}\mathbb{H}_1(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{Z})$, is a pullback of a character of $H_1(\mathbb{P}^1 \setminus D_j)$ if and only if it is a pullback of a character of $\Gamma_j$ via maps in diagram (31). Let $D_j \to \mathbb{P}^1$ be ramified over $D_j$ cover with the Galois group $\Gamma_j$ and let $\Phi_j : \text{Alb}(\bar{X}_T) \to \text{Jac}(D_j)$ be corresponding Albanese map. $\text{Jac}(D_j)$ is the isogeny component of $\text{Alb}(\bar{X}_T)$ depending only on $\text{Char}_j$ and $\Gamma$. 


Next let $\chi_k, k = 1, \ldots, N$ be collection of characters of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ which depth is greater than the depth of generic point on the component of characteristic variety to which it belongs. We shall call such characters the\textit{ jumping characters} of $\mathcal{C}$.

Finiteness of this set of characters follows from the proof of Prop. 1.7 in [1] (i.e. vanishing of the term $E_2^{0,1}$ for Leray spectral sequences with abutment $H^*(\mathbb{P}^1 \setminus D_j, \chi)$ for almost all $\chi$, cf. also [11] and [10]). We also shall include in this set of\textit{ jumping characters} isolated characters of the characteristic variety (i.e. viewing $Char\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ as component of depth zero in the characteristic variety).

To each character $\chi \in \mathbb{P}^2 \setminus \mathcal{C}$ correspond well defined unramified cyclic covering space of $\mathbb{P}^2 \setminus \mathcal{C}$ having $\text{Ker} \chi \subset \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ as the fundamental group. The Galois group of this cover is $\text{Im} \chi \subset \mathbb{C}^\times$. Let $A_k, k = 1, \ldots, N$ be collection of Albanese varieties of ramified cyclic covers of $\mathbb{P}^2$ corresponding to characters $\chi_k$ compactifying just mentioned unramified covers of $\mathbb{P}^2$. Since Galois groups of these covers are quotients of $\Gamma$ and hence the covers corresponding to characters are quotients of $X_\Gamma$, by Albanese functoriality one has the maps $\Phi_k : Alb(X_\Gamma) \rightarrow A_k$. We claim injectivity:

\begin{equation}
0 \rightarrow Alb(X_\Gamma) \oplus \Phi_j \oplus \Phi_k \oplus \text{Jac}(D_j) \bigoplus \oplus k A_k \tag{32}
\end{equation}

To see injectivity of the latter, consider the induced homomorphism

$H_1(Alb(X_\Gamma), \mathbb{C}) \rightarrow H_1(\bigoplus \text{Jac}(D_j) \bigoplus \oplus k A_k, \mathbb{C})$.

$\Gamma$ acts on both vector spaces and this homomorphism is $\Gamma$-equivariant. For a character $\chi$ which depths is equal to the depth of the generic character in its component of positive dimension on has isomorphism $H_1(X_\Gamma, \mathbb{C})_\chi \rightarrow H_1(D_j, \mathbb{C})_\chi$. For a jumping character $\chi$ one has isomorphism $H_1(X_\Gamma, \mathbb{C})_\chi \rightarrow H_1(A_k, \mathbb{C})_\chi$ for appropriate $k$ and moreover, for distinct characters the images of these isomorphisms are independent. Hence asserted injectivity follows.

To finish the proof of theorem 4.1 it suffices to show that each summand in the last term in (32) is isogenous to a product of components of local Albanese varieties of $\mathcal{C}$. Indeed Poincare complete reducibility theorem implies that the image of the middle map is isogenous to a direct sum of irreducible summands of the last term.

Isogeny between Albanese varieties $A_k$ of cyclic covers and a product of components of local Albanese varieties of $\mathcal{C}$ was shown in [25][1]. We need to show that the Jacobians of $D_j$ are isogenous to a product of components of local Albanese varieties of singularities of $\mathcal{C}$.

Denote by the same letter $\phi_j$ the extension of a regular map $\phi_j : \mathbb{P}^2 \setminus \mathcal{C} \rightarrow \mathbb{P}^1 \setminus D_j$ to the map $\mathbb{P}^2 \setminus S_j \rightarrow \mathbb{P}^1$ where $S_j$ is the finite collection of indeterminacy points of extension of $\phi_j$ to $\mathbb{P}^2$. Let $\mathcal{C}_d = \phi_j^{-1}(d), d \in D_j$. Then $\mathcal{C}$ contains the union of the closures $\mathcal{C}_d$ of (which are possibly reducible and non reduced curves). Each $P \in S_j$ belongs to at least $\text{Card}D_j$ irreducible components and since $\text{Card}D_j > 1$, $P$ is a singular point of $\mathcal{C}$. We claim the following:

\begin{itemize}
  \item[4] Sometime use of the term “jumping characters” is a little different than just mentioned, meaning the characters of $\pi_1(X)$ for which the cohomology of corresponding local system satisfy $H^1(X, L_\chi) \neq 0$
  \item[5] Since characters jumping characters may have arbitrary ramification over one may need consider local Albanese varieties corresponding to non-reduced germs as well
\end{itemize}
Claim 4.2. Resolution $\tilde{\mathbb{P}}^2_{\mathcal{C}, P} \to \mathbb{P}^2$ of singularity of $\mathcal{C}$ at $P$ contains exactly one exceptional curve $E_P$ such that the regular extension $\hat{\phi}_j$ of $\phi_j$ to $\tilde{\mathbb{P}}^2_{\mathcal{C}, P} \to \mathbb{P}^1$ induces a finite map $\hat{\phi}_j : E_P \to \mathbb{P}^1$.

To see this, consider sequence of blow ups $\tilde{\mathbb{P}}^2_{\mathcal{C}, P, h}, h = 1, ..., N(\mathcal{C}, P)$ of the plane starting with the blow up of $\mathbb{P}^2$ at $P$ and in which the last one produces the resolution of singularity of $\mathcal{C}$ at $P$. Let, for each $h$, $\phi_{j, h} : \tilde{\mathbb{P}}^2_{\mathcal{C}, P, h} \to D_j$ be the extension of $\phi$ from $\mathbb{P}^2 \setminus \mathcal{C} \tilde{\mathbb{P}}^2_{\mathcal{C}, P, h}$. For every base point $Q$ of the map $\phi_{j, h}$ on $\tilde{\mathbb{P}}^2_{\mathcal{C}, P, h}$ consider the pencil of tangent cones to fibers of the map $\phi_{j, h}$. The fixed (possibly reducible) component of the pencil of tangent cones $T_d, d \in \mathbb{P}^1$ to curves $\hat{\phi}_j^{-1} (d)$ \footnote{i.e. union of lines which are tangent to a component of curve $\phi_{j, h}^{-1} (d)$ for any $d$} either:

a) coincide with the tangent cone $T_d$ to each curve $\hat{\phi}_j^{-1} (d)$, or
b) there exist $d$ such that the tangent cone $T_d$ to $\hat{\phi}_j^{-1} (d)$ at $Q$ contains a line not belonging to the fixed component of the pencil of tangent cones.

Since on $\tilde{\mathbb{P}}^2_{\mathcal{C}, P}$ (i.e. eventually after sufficiently many blow ups) no two fibers of $\phi$ intersect, in a sequence of blow ups desingularizing $\mathcal{C}$ at $P$, there is point $Q$ infinitesimally close to $P$ at which tangent cones satisfied b). At such point $Q \in \mathbb{P}^2_{\mathcal{C}, j, h}$ any two distinct fiber of $\phi_{j, h}$ admit distinct tangents since otherwise, since we have one dimensional linear system, the common tangent to two fibers will belong to the fixed component. Let $E_P \subset \tilde{\mathbb{P}}^2_{\mathcal{C}, j, h+1}$ be the exceptional curve of the blow up of $\tilde{\mathbb{P}}^2_{\mathcal{C}, j, h}$. Exceptional curves preceding or following this one on the resolution tree (which up to this point did not have vertices with valency greater than 2!) belong to one of the fibers of $\phi_j$. Restriction of $\phi_{j, h+1}$ onto $E_P$ is the map claimed in (4.2).

Finally, the ramified $\Gamma$-covering of $\mathbb{P}^2$ lifted to $\tilde{\mathbb{P}}^2_{\mathcal{C}, P}$ and restricted on the proper preimage of the curve $E_P$ in $\tilde{\mathbb{P}}^2_{\mathcal{C}, P}$ induces the map onto $\Gamma_j$ covering of $\mathbb{P}^1$ ramified at $D_j$. Hence the Jacobian of the latter covering is a component of the Jacobian of a covering of $E_P$ which is the direct summand of the local Albanese of the singularity of $\mathcal{C}$ at $P$. Q.E.D.

The following theorem \cite{4.4} allows to describe the isogeny class of Albanese varieties of abelian covers in explicit examples considered in the next section. The Albanian variety of abelian cover with Galois group $\Gamma$ will be obtained as sum of isogeny components of Albanian varieties of cyclic covers of $\mathbb{P}^2$ associated with jumping characters of $\Gamma$ and isogeny components of Jacobians of abelian covers of he line associated with $\Gamma$ and corresponding to the positive dimensional components of the characteristic variety of $\pi_1 (\mathbb{P}^2 \setminus \mathcal{C})$. To state the theorem we shall use the following partial order on the set of mentioned isogeny components.

**Definition 4.3.** Let $\Psi_i : B \to A_i, i \in I$ be a collection of equivariant morphisms of abelian varieties endowed with the action of a group $\Gamma$. An isotypical isogeny component of collection $A_i$ is an abelian variety of the form $S^m$ where $S$ is $\Gamma$-simple. Define the partial order of the set of isotypical components of $\Pi_{i \in I} A_i$ as follows: $A \geq A'$ satisfy if and only if each $A$ and $A'$ belongs to the image of one of $\Psi_i$ ($i \in I$) and $A = S^m, A' = S^{m'}, m \geq m'$.
We have the following description of the Albanese variety of abelian cover \( \bar{X}_\Gamma \).

**Theorem 4.4.** Let \( \mathcal{C} \) be such that the characteristic variety \( V_1(\mathbb{P}^2 \setminus \mathcal{C}) \) of \( \mathcal{C} \) consists of positive dimensional components \( \Xi_i \) and zero dimensional components \( \Upsilon_j \neq 1 \). Let \( \pi_\Gamma : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \Gamma \) be a surjection onto an abelian group and let \( \Gamma_{\Xi_i} \) be corresponding push-out group given by diagram (31). Let \( \bar{\mathbb{P}}^{\Gamma_{\Xi_i}} \) denotes ramified cover of \( \mathbb{P}^1 \) with the covering group \( \Gamma_{\Xi_i} \) which is compactification of the cover of the target map of \( \mathbb{P}^2 \setminus \mathcal{C} \) corresponding to the component \( \Xi_i \). Let \( \Gamma_{\Upsilon_j} \) be the image of the character \( \Upsilon_j : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \mathbb{C}^* \) and \( \bar{\mathbb{P}}^{\Gamma_{\Upsilon_j}} \) denotes the cyclic ramified cover of \( \mathbb{P}^2 \) with the covering group \( \Gamma_{\Upsilon_j} \) corresponding to the surjection \( \Upsilon_j \).

(1) For any \( j \) and any \( i \) there are \( \Gamma \)-equivariant morphisms

\[
\text{Alb}(\bar{X}_\Gamma) \to \text{Alb}(\bar{\mathbb{P}}^{\Gamma_{\Upsilon_j}}) \quad \text{Alb}(\bar{X}_\Gamma) \to \text{Jac}(\bar{\mathbb{P}}^{\Gamma_{\Xi_i}})
\]

(2) Let \( A_m, m \in M \) be the set of maximal elements in the ordering of isotypical components of collection of morphisms in (1). Then there is an isogeny

\[
\text{Alb}(\bar{X}_\Gamma) \to \bigoplus_{m \in M} A_m
\]

**Remark 4.5.** The maps in (33) corresponding to different characters may coincide (this is always the case for example for conjugate characters). The theorem asserts that selection among jumping characters and component of characteristic varieties can be made so that maximal isotypical components in corresponding covers provide isotypical decomposition of \( \text{Alb}(\bar{X}_\Gamma) \).

**Proof.** Morphisms \( \bar{X}_\Gamma \to \bar{\mathbb{P}}^{1}_{\Gamma_{\Xi_i}} \) were constructed in the beginning of the proof of theorem 4.1. The group \( \Gamma_{\Upsilon_j} \) is the quotient of \( \Gamma \) which allows to construct the morphisms in (33) corresponding to jumping characters \( \Upsilon_j \). This shows (1) and hence we are in situation of definition (4.3).

Let \( A_m, m \in M \) be collection of maximal isotypical components in the Albanese varieties which are targets of the maps (33). Composition of a map (33) with projection on the isogeny components \( A_m, m \in M \) gives the map \( \text{Alb}(\bar{X}_\Gamma) \to A_m \). Each isogeny component of \( \text{Alb}(\bar{X}_\Gamma) \) is an isogeny component in one of varieties \( \bar{\mathbb{P}}^{1}_{\Gamma_{\Xi_i}} \) or \( \bar{\mathbb{P}}^{2}_{\Gamma_{\Upsilon_j}} \) and the dimension of \( \Gamma \)-eigenspace corresponding to any character coincides with the dimension of \( \chi \)-eigenspace of the targets (33). Hence the map (34) has finite kernel.

Let \( \chi \) be a character having non zero eigenspace on \( H^1(A_m) \). Then by theorem 2.3 part (2), \( \dim H^1(A_m) \chi = \dim H^1(A) \chi = \dim H^1(\bar{X}_\Gamma) \chi \) where \( A \) is one of the targets of the maps (33). Since \( H^1(\bar{X}_\Gamma) \) is a direct sum of \( \Gamma \)-eigenspaces and the image of \( H^1(\bar{X}_\Gamma) \chi \) is non-trivial in exactly one summand in (34) one obtains the surjectivity in (34). QED.

**Remark 4.6.** Multiplicities of isotypical components \( A_m \) are poorly understood in general as well as jumping characters (cf. [7] where the problem of bounding the multiplicities of the roots of Alexander polynomials of the complements to plane curves, which are in correspondence with the jumping characters, is discussed). Nevertheless in all known examples, the above theorem is sufficient to completely determine isogeny class of Albanese varieties of abelian covers.
5. Albanese varieties of abelian covers ramified over arrangements of lines.

In the case when ramification set is an arrangement of lines theorems 4.1 and 4.4 yield considerably simpler than in general case results. We shall start with:

Corollary 5.1. Let $A$ be an arrangement of lines in $\mathbb{P}^2$ with double and triple points only. Let $X_n(A)$ be a compactification of the abelian cover of $\mathbb{P}^2 \setminus A$ corresponding to the surjection $H_1(\mathbb{P}^2 \setminus A, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus A, \mathbb{Z}/n\mathbb{Z})$.

$(1)$ Albanese variety of $X_n(A)$ is isogenous to a product of isogeny components of Jacobians of Fermat curves.

$(2)$ If characteristic variety of $\mathbb{P}^2 \setminus A$ has no jumping characters and the pencils corresponding to positive dimensional components have no multiple fibers then $\text{Alb}(X_n(A))$ is isogenous to a product of Jacobians of Fermat curves. The same is the case if non-trivial values of jumping characters on meridians of all lines in the arrangement are the same and $\gcd(3, n) = 1$.

Proof. Each component of characteristic variety having a positive dimension corresponds to the map $\mathbb{P}^2 \setminus A \to \mathbb{P}^1 \setminus D$ where $\text{Card} D = 3$. Those induce maps of $\text{Alb}(X_n(A))$ onto the Jacobians of abelian covers of $\mathbb{P}^1$ ramified along corresponding $D$. Such abelian cover of $\mathbb{P}^1$ is a quotient of the cover of $\mathbb{P}^1$ with the Galois group $H_1(\mathbb{P}^1 \setminus D, \mathbb{Z}/n\mathbb{Z})$ ramified at three points i.e. Fermat curve of degree $n$ (cf. Remark 3.2). Moreover, since the resolution of singularities of a cyclic cover of $\mathbb{C}^2$ having degree $n$ and ramified along a triple of concurrent lines contains only one exceptional curve of positive genus and this exceptional curve is a cyclic cover of $\mathbb{P}^1$ with ramification set consisting of three points, the local Albanese variety will be also a component of Jacobian of Fermat curve. Therefore the maximal isotypical isogeny components are components of Jacobians of Fermat curves and part (1) follows from theorem 4.4.

If characteristic variety does not have jumping characters then $\text{Alb}(X_n(A))$ is just product of Jacobians corresponding to positive dimensional components of characteristic variety (i.e. there is no contribution in $\mathcal{A}_n$ coming from Jacobians of covers corresponding to jumping characters) and assumption about absence of multiple fibers implies that map of $X_n(A)$ corresponding to each positive dimensional component of characteristic variety of $A$ has as target the cover as in Remark 3.2 i.e. Fermat curve. Hence $\text{Alb}(X_n(A))$ is a product of Jacobians of Fermat curve and we obtain the first part of (2).

If jumping characters are present but values of all jumping characters on meridians of all components of $A$ are the same, then order of each jumping character must be root of the Alexander polynomial of the complement corresponding to the map of $H_1(\mathbb{P}^2 \setminus A)$ sending each meridian to the same generator of the (cyclic) covering group. Since we assume that arrangement has points of multiplicity at most 3, the Alexander polynomial has as its roots different than 1 only roots of unity of degree 3 (cf. [21]). Hence if $\gcd(3, n) = 1$ then no character of $H_1(\mathbb{P}^2 \setminus A, \mathbb{Z}/n\mathbb{Z})$ is a jumping character and hence maximal isotypical isogeny components again are not affected by cyclic covers corresponding to jumping characters. Therefore we obtain second part (2).
Example 5.2. Consider Ceva arrangement \(xyz(x - z)(y - z)(x - y) = 0\) and the universal \(\mathbb{Z}_5\) cover (with the covering group which is the quotient of \(\mathbb{Z}_5^2\) by the cyclic subgroup generated by \((1, 1, 1, 1, 1)\)). Then the irregularity of the corresponding abelian cover is 30 (cf. [10, 23] section 3.3 ex.2). The characteristic variety consists of five 2-dimensional components \(\Xi_i, i = 1, ..., 5\) (cf. [23]), each being the pull back of \(H^1(\mathbb{P}^1 \setminus D, \mathbb{C}^*)\), \(\text{Card} D = 3\) via either a linear projection from one of 4 triple points or via a pencil of quadrics three degenerate fiber of which form the 6 lines of the arrangement. Each of these 5 pencils induces a map on the abelian cover of \(\mathbb{P}^1\) branched at 3 points, which has as the Galois group the quotient of \(\oplus_3 \mu_5\) by the diagonally embedded group of roots of unity \(\mu_5\) of degree 5. This cover, i.e. \(\overline{P}_{\Xi_i}, i = 1, ..., 5\), is Fermat curve of degree 5. The Jacobian of degree 5 Fermat curve is isogenous to a product of Jacobians of three curves \(C_i, i = 1, 2, 3\) of genus 2 each being a cyclic cover of \(\mathbb{P}^1\) ramified at three points. (cf. [8], [20]). Hence the Albanese variety of this abelian cover is isogenous to a product of 15 copies of the Jacobian of ramified at three points cover of \(\mathbb{P}^1\) of degree 5. In this example there are no jumping characters and the isogeny can be derived from Corollary 5.1.

Example 5.3. Consider again Ceva arrangement and calculate the abelian component of (semi-abelian) Albanese variety (cf. section 2.4) of its Milnor fiber \(M\) given by \(w^6 = \Pi I_i\). Notice that the characteristic polynomial of the monodromy is \((t - 1)^3(t^2 + t + 1)\) (cf. [23]). The \(\zeta_3\)-eigenspace of \(H^1(M, \mathbb{C})\) can be identified with the contribution in sum \(\xi\) of the pullback of the character \(\chi\) of \(\mathbb{P}^1 \setminus D\) via the pencil of quadrics formed by lines of the arrangement. Here \(D\) is the triple of points corresponding to the reducible quadrics in the pencil and \(\chi\) is the character taking the same value \(\omega_3\) on standard generators if \(\pi_1(\mathbb{P}^1 \setminus D)\). This pencil can be lifted to the elliptic pencil on a compactification of \(M\) onto 3-fold cyclic cover of \(\mathbb{P}^1\) ramified at \(D\) and corresponding to \(\text{Ker} \chi\). Moreover, above expression for the characteristic polynomial of the monodromy shows that the map induced by this pencil is isogeny i.e. the abelian (i.e. compact) component of the Albanese of \(M\) is the elliptic curve \(E_0\).

Example 5.4. Consider abelian cover of \(\mathbb{P}^2\) ramified along arrangement of lines dual to 9 inflection points of a cubic with Galois group \(\mathbb{Z}_9^n/\mathbb{Z}_n\). It has 9 lines and 12 triple points. The characteristic variety consists of 12 components corresponding to 12 triple points and 4 additional two-dimensional component intersecting along cyclic subgroup of order 3. Characters at the intersection are jumping and have depth 2 (cf. II) while depth of generic character in each positive dimensional component has depth 1. In the case \(n = 5\), in which according to Hirzebruch on obtains a quotient of unit ball, the Albanese variety is isogenous to product of 16 copies of Fermat curve of degree 5, as follows from Corollary 5.1 (2) or equivalently 48 copies of curves of Jacobians of curves of genus 2 with automorphism of order 10 or, what is the the same, the 2-dimensional variety of CM type corresponding to cyclotomic field \(\mathbb{Q}(\zeta_5)\). For arbitrary \(n\) such that \(\text{gcd}(3, n) = 1\) one get several copies of Jacobians of Fermat curves of degree \(n\) corresponding to components of characteristic variety.

If \(n\) is divisible by 3 then the product of Jacobians of Fermat curves which are the Jacobians corresponding to positive dimensional components of characteristic variety must be factored by product \(E_0^{\kappa - \delta}\) where \(\kappa\) is the number of components
containing a jumping character (taking value $\exp(\frac{2\sqrt{-1}}{3})$ or $\exp(\frac{4\sqrt{-1}}{3})$ on all 9 lines of arrangement) and $\delta$ is the depth of the jumping character.

In the case $n = 3$ the abelian cover with the covering group $\mathbb{Z}_3^3/\mathbb{Z}_3$ one obtains from theorem 4.3 or Corollary 4.4

(35) $\text{Alb}(\mathbb{P}^2_{\mathbb{Z}_3}) = E_0^{16}/E_0^2 = E_0^{14}$

Indeed, the in this case $\kappa = 4, \delta = 2$.

In the case $3|n, n > 3$ product of Jacobians corresponding to positive dimensional components has several copies of $E_0$ as isogeny components and $\text{Alb}(X_n)$ is the quotient of this product by $E_0^{\kappa-\delta} = E_0^2$.

Example 5.5. Consider Hesse arrangement $\mathcal{H}$ formed by 12 lines containing 9 inflection points of a smooth cubic. It was shown in [23] (cf. section 3, example 5) that the characteristic variety of the fundamental group of the complement to this arrangement consists of 10 three dimensional components and 54 two dimensional components none of which belongs to a three-dimensional component. As earlier, it is convenient to describe components in terms of corresponding pencils i.e. maps $\mathbb{P}^2 \setminus \mathcal{H} \to \mathbb{P}^1 \setminus h$ where $h$ is a set of points of cardinality 4 or 3 so that the characters in each component formed by pullbacks via these maps. The pencils corresponding to components of dimension 3 are linear projections from each of 9 quadruple points and additional pencil is pencil of curves of degree 3 containing 4 cubic curves each being a union of a triple of lines in the arrangement $\mathcal{H}$. The 54 maps $\mathbb{P}^2 \setminus \mathcal{H} \to \mathbb{P}^1 \setminus h$ (Card $h = 3$) are restrictions of the maps corresponding to the pencil of quadrics union of which are 6-tuples of lines in $\mathcal{H}$ forming a Ceva arrangement.

The pencil corresponding to 3-dimensional component of characteristic variety induces the map of abelian cover of the plane ramified along $\mathcal{H}$ with Galois group $(\mathbb{Z}_3)^{12}/\mathbb{Z}_3$ on the maximal abelian cover $\mathbb{Z}_3$ cover of $\mathbb{P}^1$ ramified at 4 points. In particular the Albanese variety in question maps onto the Jacobian $J_{10}$ of curve of genus 10. Similarly each 2-dimensional component of characteristic variety induces map of Albanese of abelian cover of $\mathbb{P}^2$ onto maximal abelian 3-cover of $\mathbb{P}^1$ ramified at 3 points. The latter is Fermat curve of degree i.e. the elliptic curve with j-invariant zero.

Hence the Albanese variety of the cover considered by Hirzebruch (cf. [19]) is isogenous to

(36) $J_{10}^{10} \times E_0^{54}$

Example 5.6. Varieties of lines on Fermat hypersurfaces Previous results imply immediately the following:

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7cf.[29], Prop. 4.8. This effect of characters in the intersection of several components of characteristic varieties is erroneously omitted in the final formula in Example 3 in section 3.3 of [23].

8This was explained in [29]. Recall that in interpretation of inflection points of the cubic as points in affine plane over field $F_3$, the twelve lines correspond to the full set of lines in this plane and 6 tuples are in one to one correspondence with quadruples of points in this finite plane no three of which are collinear. Counting first ordered quadruples of this type one sees that there are $6 \times 6 \times 5 \times 4$ choices for the first two points, 6 choices for the third point (it cannot be the third point on the line containing first two) and 3 choices for the forth). Therefore one get 54 unordered quadruples of points and hence 54 6-tuples of lines.
Theorem 5.7. Let $F_3$ be variety if lines on Fermat cubic threefold:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

Then there is an isogeny:

$$\text{Alb}(F_3) = E_0^5$$

This isogeny was observed recently [3]. Also, Roulleau cf. [27] obtained the isomorphism class of the Albanese variety of Fermat cubic threefold.

Proof. It follows from discussion in [30] that Fano surface $F_3$ is abelian cover of degree $3^4$ of $\mathbb{P}^2$ ramified over Ceva arrangement. Hence the isogeny (38) follows as in example 5.2.

6. Applications

6.1. Mordell Weil ranks of isotrivial families of abelian varieties. Recall the following (cf. [25])

Proposition 6.1. Let $\mathcal{A} \to \mathbb{P}^2$ be a regular model of an isotrivial abelian variety over $\mathbb{C}(x,y)$ with smooth fiber $\mathcal{A}$. Assume that there is a ramified abelian cover $X \to \mathbb{P}^2$ such that the pullback of $\mathcal{A}$ to $X$ is trivial over $X$. Let $\Gamma$ be the Galois group of $\mathbb{C}(X)/\mathbb{C}(x,y)$. Then trivialization of $\mathcal{A}$ over $X$ yields the action of $\Gamma$ on $\text{Alb}(X)$ and the Mordell Weil rank of $\mathcal{A}$ is equal to $\dim \text{Hom}_\mathbb{C}(\text{Alb}(X), \mathcal{A}) \otimes \mathbb{Q}$.

Given an abelian cover $X \to \mathbb{P}^2$ with covering group $\Gamma$ and a homomorphism $\Gamma \to \text{Aut}\mathcal{A}$, where $\mathcal{A}$ is an abelian variety over $\mathbb{C}$, an example of isotrivial $\mathcal{A}$ as in Prop. 6.1 can be obtained as (a resolution of):

$$A_X = X \times \mathcal{A}/\Gamma$$

where $\Gamma$ acts on $X \times \mathcal{A}$ diagonally: $(x,a) \to (\gamma \cdot x, \gamma \cdot a), \gamma \in \Gamma, x \in X, a \in \mathcal{A}$.

Moreover, any isotrivial $\mathcal{A}$ as in Prop. 6.1 is obtained as (39).

Calculations of Albanese varieties in examples of previous sections yield values of Mordell Weil ranks of isotrivial abelian varieties in many examples as in Prop. 6.1.

Example 6.2. Let $J_{2,5}$ denote the Jacobian of smooth projective model of $y^5 = x^2(x - 1)^2$ (i.e. one of the curves $C_i$ in Example 5.2). Let the direct sum $\mathbb{Z}_5^5$ act on $C_i$ so that the generator of each summand acts as the multiplication by $\zeta, \zeta = exp(\frac{2\pi i}{5}) : (x,y) \to (x,\zeta y)$ (cf. 5.2). This induces the action of $\mathbb{Z}_5^5$ on $J_{2,5} = \text{Jac}(C_i)$.

Consider isotrivial family of abelian varieties over $\mathbb{P}^2$ with the zero set of discriminant being Ceva arrangement of lines which is the quotient of $X \times J_{2,5}$, where $X$ is the abelian over with the covering group $\mathbb{Z}_5^5$ considered in example 5.2. The action is the diagonal action of $\Gamma = \mathbb{Z}_5^5$ as in 5.2. The Albanese variety of the abelian cover $X$ in example 5.2 is isogenous to $(J_{2,5})^{15}$ (cf. 5.2) and hence rank of Mordell Weil group of the quotient is equal to

$$\text{rk Hom}_{\mathbb{Z}_5^5}(J_{2,5}^{15}, J_{2,5}) \otimes \mathbb{Q}$$

Since for each summands $J_{2,5}^{15}$ at least one of generators of $\Gamma$ acts trivially on it while by our definition its actions is non trivial on the target of the homomorphism in 40 the Mordell-Weil rank in this case is zero.
6.2. Periodicity of Albanese varieties.

Theorem 6.3. Let \( C \) be a curve in \( \mathbb{P}^2 \) such that exist a surjection \( \pi : \pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z} \).

Consider two sequences of cyclic cover composed of ramified and unramified covers corresponding to surjections \( \pi_n : \pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \):

1. Sequence of isogeny classes of Albanese varieties of a tower of cyclic branched covers with given ramification locus \( C \) corresponding to surjections \( \pi_n \) is periodic.

2. Sequence of isogeny classes of semi-abelian varieties which are Albanese varieties of unramified covers a complement to a curve \( C \) corresponding to surjections \( \pi_n \) is periodic.

Proof. Let \( \Delta_n(t) \) be the Alexander polynomial of \( C \) corresponding to the surjection \( \pi \) (cf. [21]). For each root of unity \( \xi \), \( \Delta_n(\xi) = 0 \), let \( n_\xi \) be its order. Orders \( n_\xi \) define a finite set of arithmetic progressions \( N_\xi, \ldots, N_k \subset \Gamma = \mathbb{Z} \) of integers divisible by a fixed square free product of integers \( n_{\xi_1}, \ldots, n_{\xi_k} \). Let \( n_N \) be the minimal positive integer in such progression \( N \). It follows from \( \mathbb{P}^2 \) that, denoting \( X_n \) (resp. \( X_\pi \)) unramified (resp. ramified) cover of \( \mathbb{P}^2 \setminus C \) (resp. \( \mathbb{P}^2 \setminus C \)) \( H_1(X_n, \mathbb{C}) \to H_1(X_{nN}, \mathbb{C}) \) (resp. \( H_1(X_n, \mathbb{C}) \to H_1(X_{nN}, \mathbb{C}) \)) are isomorphisms for all \( n \) belonging to one and only one arithmetic progression \( N_{\xi_1, \ldots, \xi_k} \). For \( n \) not belonging to any of these arithmetic progressions \( H_1(X_n, \mathbb{C}) = H_1(X_n, \mathbb{C}) = 0 \). Moreover the map \( H_1(X_n, \mathbb{C}) \to H_1(X_{nN}, \mathbb{C}) \) (resp. \( H_1(X_n, \mathbb{C}) \to H_1(X_{nN}, \mathbb{C}) \)) is injective (resp. has finite kernel and co-kernel). Hence the isogeny class of Albanese variety of \( X_n \) with \( n \) in one and only one arithmetic progression as above. Hence the claims (1) and (2) follow.

Remark 6.4. This result can be compared with results on periodicity properties of Betti and Hodge numbers. For a curve \( C \in \mathbb{P}^2 \) which a union of \( r \) components, let \( h_{n,0}^1(n) \) (resp. \( h_{n,0}^1(n) \)) denote the sequence of the Hodge numbers of a smooth compactification \( X_\pi(n) \) (resp. \( X_\pi(n) \)) of abelian (resp. cyclic) covers of the complement in the tower of abelian (resp. cyclic) cover of \( \mathbb{P}^2 \). Consider two sequences of cyclic cover composed of ramified and unramified covers corresponding to surjections \( \pi_n : \pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \). It follows from \( \mathbb{P}^2 \) that the sequence \( h_{n,0}^1(n) \) is polynomial periodic (similarly [21] implies that sequence \( h_{n,0}^1(n) \) is periodic). Recall that \( n \to a(n) \in \mathbb{N} \) is polynomial periodic if there are periodic functions \( a_i(n) \) such that \( a(n) = \sum a_i(n)n^i \).

Now let \( K_{A/B} \) be the \( K \)-group of motives of abelian varieties over \( C \) up to isogeny. More precisely this is the \( K \)-group with the category \( AB \) with objects \( A, A' \), ... being abelian variety over \( C \) with the morphisms being \( Hom(A, A') \otimes \mathbb{Q} \). This is a (infinitely generated) \( \mathbb{Z} \)-module with canonical surjection \( dim : K_{A/B} \to \mathbb{Z} \) given by \( A \to dim A \). The theorem [6,3] implies that the sequence \( Alb(X_\pi(n)) \in K_{A/B} \) is periodic. However isogeny components of Albanese varieties of \( abelian \) possibly non-cyclic covers span an infinitely generated subgroup of the \( K \)-group of isogeny classes of abelian varieties.

In particular, there no periodic functions \( a_i(n) \in K_{A/B} \) such that \( Alb(X_\pi(n)) = a_i(n)n^i \) (though as was mentioned \( dim Alb(X_\pi(n)) \) is polynomially periodic). Details of this will be presented elsewhere.

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9For any curve in \( C \) (including irreducible in which case \( H_1(\mathbb{P}^1 \setminus C, \mathbb{Z}) = \mathbb{Z}/(\deg C)\mathbb{Z} \) adding to \( C \) a generic line in \( \mathbb{P}^2 \) yields a curve admitting such surjection cf. [21]
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