APPROXIMATING THE PARTITION FUNCTION OF THE FERROMAGNETIC POTTS MODEL

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ABSTRACT. We provide evidence that it is computationally difficult to approximate the partition function of the ferromagnetic \( q \)-state Potts model when \( q > 2 \). Specifically we show that the partition function is hard for the complexity class \( \#\text{RHII}_1 \) under approximation-preserving reducibility. Thus, it is as hard to approximate the partition function as it is to find approximate solutions to a wide range of counting problems, including that of determining the number of independent sets in a bipartite graph. Our proof exploits the first order phase transition of the “random cluster” model, which is a probability distribution on graphs that is closely related to the \( q \)-state Potts model.

1. Introduction

Let \( q \) be a positive integer. The \( q \)-state Potts partition function of a graph \( G = (V, E) \), with uniform interactions of strength \( \gamma \geq -1 \) along the edges, is defined as

\[
Z_{\text{Potts}}(G; q, \gamma) = \sum_{\sigma : V \to [q]} \prod_{e = \{u, v\} \in E} (1 + \gamma \delta(\sigma(u), \sigma(v))),
\]

where \([q] = \{1, \ldots, q\}\) is a set of \( q \) spins or colours, and \( \delta(s, s') \) is 1 if \( s = s' \), and 0 otherwise. The partition function is a sum over “configurations” \( \sigma \) which assign spins to vertices in all possible ways. Later, we shall widen this definition to allow the interaction strength \( \gamma \) to be a function of the edge \( e \), and allow \( G \) to be a hypergraph (a generalisation of graph in which edges contain an arbitrary number of vertices), but the above restricted case is sufficient for this overview. Mostly we shall concentrate in this paper on the ferromagnetic situation, characterised by \( \gamma > 0 \). In the ferromagnetic Potts model, configurations \( \sigma \) with many adjacent like spins make a greater contribution to the partition function \( Z_{\text{Potts}}(G; q, \gamma) \) than those with few.

The statistical mechanical model just described was introduced by Potts \cite{Potts1952} and generalises the classical Ising model from two to \( q \) spins.

Definition (1) applies only when \( q \) is a positive integer. However, it transpires that, regarding \( q \) as an indeterminate, (1) defines a polynomial in \( q \), and in this way we can make sense of the Potts partition function for non-integer \( q \), even though the underlying physical model has no meaning. An equivalent, but more
concrete way of approaching the partition function when $q$ is non-integer is via the Tutte polynomial, which in its “random cluster” formulation is defined as follows:

$$Z_{\text{Tutte}}(G; q, \gamma) = \sum_{F \subseteq E} q^{\kappa(V,F)} \gamma^{|F|},$$

where $\kappa(V,F)$ denotes the number of connected components in the graph $(V,F)$. The notation is as before, except that now $q$ is an arbitrary real number. Again, for simplicity, we are assuming that $G$ is a usual graph (not a hypergraph) and that the edge weight $\gamma$ is uniform over edges. The multivariate definition can be guessed at and will in any case appear later in the paper. For readers who are familiar with the classical $(x,y)$-parameterisation of the Tutte polynomial, the transformation between that and the one here is given by $\gamma = y - 1$ and $q = (x - 1)(y - 1)$.

Although (1) and (2) are formally very different, they define the same polynomial in $q$: see Observation 2. We continue the discussion now in terms of the Tutte polynomial (2), remembering all along that we include as a special case the Potts partition function, and as an even more special case that of the Ising model. We denote by $\text{Tutte}(q, \gamma)$ the computational task of computing $Z_{\text{Tutte}}(G; q, \gamma)$ given a graph $G$ as problem instance. Then each pair $(q, \gamma)$ defines a separate computational problem, and we can study the computational complexity of this problem as $q$ and $\gamma$ vary. It is important to note that $q$ and $\gamma$ do not form part of the problem instance, which consists simply of the graph $G$. For the purposes of this discussion, we may assume that $q$ and $\gamma$ are rational, in order to avoid representation issues, but in the main body of the paper we work in the wider class of “efficiently approximable” real numbers.

In a seminal paper, Jaeger, Vertigan and Welsh [18] examined the problem of computing $Z_{\text{Tutte}}(G; q, \gamma)$ exactly. In the exact setting, they completely classified the complexity of $\text{Tutte}(q, \gamma)$ for all $q, \gamma$ (in fact for all complex $q, \gamma$). It transpires that $\text{Tutte}(q, \gamma)$ is #P-hard (i.e., as hard as determining the number of satisfying assignments to a CNF Boolean formula), except when $q = 1$, or when $(q, \gamma)$ is one of a finite number of “special points”; in these cases $\text{Tutte}(q, \gamma)$ is polynomial-time computable.

In light of Jaeger et al.’s strong negative result, attention turned to the question of whether $Z_{\text{Tutte}}(G; q, \gamma)$ could be approximated with arbitrarily small specified relative error. In the context of computing partition functions, the appropriate notion of efficient approximate computation is the “Fully polynomial randomised approximation scheme” or FPRAS, which is rigorously defined in §5. An early positive result was provided by Jerrum and Sinclair [20], who presented an FPRAS for the case $q = 2$ and $\gamma > 0$, that is to say, for the ferromagnetic Ising model. Sadly, no further generally applicable positive results have appeared since then, though FPRAS’s have been proposed for restricted classes of graphs, e.g., dense or degree-bounded [11].
Greater progress has been made in the negative direction. Goldberg and Jerrum [12] showed, under the reasonable complexity-theoretic assumption RP \( \neq \) NP, that no FPRAS exists for \( \text{Tutte}(q, \gamma) \) for a wide range of values for the parameters \((q, \gamma)\). Stated informally, RP \( \neq \) NP is the assumption that there are problems in NP that cannot be decided by a polynomial-time randomised algorithm. Intu-
tively, it is only slightly stronger than the more usual P \( \neq \) NP assumption. As an indicative example of what is known, the intractability result of [12] covers the entire half-plane \( \gamma < -2 \) except for the tractable case \( q = 1 \) and the case \( q = 2 \) where the problem is equivalent to approximately counting perfect matchings. Similar results apply when \( q/\gamma < -2 \). The restriction to planar graphs was treated in a follow-up paper [13]. However none of the existing intractability results apply to the region \( q > 0 \) and \( \gamma > 0 \) that concerns us here, and which is perhaps the one of greatest physical interest.

Our goal here is to present the first evidence that \( \text{Tutte}(q, \gamma) \) is computationally hard in the region \( q > 2 \) and \( \gamma > 0 \), i.e., the region corresponding to the ferromagnetic Potts model with \( q > 2 \) states. We achieve this, but under a stronger complexity-theoretic assumption than RP \( \neq \) NP. To explain this assumption, a digression into computational complexity is required.

The complexity class \#RH\(^{1}\) of counting problems was introduced by Dyer, Goldberg, Greenhill and Jerrum [8] as a means to classify a wide class of approximate counting problems that were previously of indeterminate computational complexity. The problems in \#RH\(^{1}\) are those that can be expressed in terms of counting the number of models of a logical formula from a certain syntactically restricted class. (Although the authors were not aware of it at the time, this syntactically restricted class had already been studied under the title “restricted Krom SNP” [7]. Yet another terminological variation is to say that problems in \#RH\(^{1}\) enumerate solutions to a linear Datalog program.) The complexity class \#RH\(^{1}\) has a completeness class (with respect to approximation-preserving “AP-
reductions”) which includes a wide and ever-increasing range of natural counting problems, including: independent sets in a bipartite graph, downsets in a partial order, configurations in the Widom-Rowlinson model (all [8]), the partition function of the ferromagnetic Ising model with mixed external field (i.e., not consistently favouring one or other spin) [11], and stable matchings [6]. Either all of these problems admit an FPRAS (i.e., are efficiently approximable), or none do. No FPRAS is known for any of them at the time of writing, despite much effort having been expended on finding one.

All the problems in the completeness class mentioned above are inter-reducible via AP-reductions, so any of them could be said to exemplify the completeness class. However, mainly for historical reasons, the particular problem \#BIS, of counting independent sets in a bipartite graph, tends to be taken as the exemplar of the class, much in the same way that SAT has a privileged status in the theory on NP-completeness. Our main result is
Theorem 1. Suppose that \( q > 2 \) and \( \gamma > 0 \) are efficiently approximable. Then \( \#\text{BIS} \leq_{\text{AP}} \text{Tutte}(q, \gamma) \).

Here, \( \leq_{\text{AP}} \) is the symbol for “is AP-reducible to”, and “efficiently approximable” is a concept defined in §5; suffice it to say for now that the rational numbers are trivially efficiently approximable.

One limitation of our result is that the resulting inapproximability of the problem \( \text{Tutte}(q, \gamma) \) is conditional on there being no FPRAS for \( \#\text{BIS} \) (and the rest of the completeness class), rather than on the weaker assumption \( \text{NP} \neq \text{RP} \). In fact, we conjecture that \( \#\text{BIS} \) does not admit an FPRAS. The basis for our conjecture is empirical — namely that the collection of known \#BIS-equivalent problems is growing and that the problem itself has survived its first decade despite considerable efforts to find an FPRAS. For example, Ge and Štefankovič [9] recently proposed an interesting new MCMC algorithm for sampling independent sets in bipartite graphs. Unfortunately, however, the relevant Markov chain mixes slowly [10] so even this interesting new idea does not give an FPRAS.

Despite the fact that our results are limited by a strong complexity-theoretic assumption, we feel there are counterbalancing strengths that fully justify this investigation. One is the range and intrinsic interest of the problem under consideration. Whether in the guise of the Potts partition function, or of the Tutte plane, the computational complexity of \( \text{Tutte}(q, \gamma) \) has received considerable attention since it was first studied by Jaeger et al. [18]; see, for example, [1, 27, 28, 29]. So it seems worth striving for a complexity classification even under a strong assumption such as the one we are making. The situation is similar to working with the Unique Games Conjecture in the area of approximation algorithms for optimisation problems, or employing the class PPAD in analysing the complexity of Nash equilibria. Furthermore, Theorem 1 has a wide range of applicability, covering as it does the whole region \( q > 2, \gamma > 0 \), which, in the classical parameterisation of the Tutte polynomial, equates to the entire upper quadrant of the Tutte plane above the positive branch of the hyperbola \( H_2 = \{(x, y) : (x - 1)(y - 1) = 2\} \). Note that the \#BIS-hard region extends right to the tractable hyperbola \( H_2 \).

Another potential strength of the work is that the reduction introduces a novel technique that may have wider applicability. The idea is conceptually simple and can be sketched informally here. In the first step, we reduce \#BIS to a hypergraph version of the Tutte polynomial. (The conventional Tutte polynomial may be recovered as the specialisation to 2-uniform hypergraphs.) This step, if not routine, is at least standard in its techniques. After this, we show how to simulate each hyperedge containing \( t \) vertices by a graph gadget with \( t \) distinguished vertices or terminals.

At this point we exploit the first order phase transition that is a feature of the so-called random cluster model when \( q > 2 \). The configurations of the random cluster model on a graph \( G \) are spanning subgraphs of \( G \), which are weighted according to the numbers of edges and connected components they contain. As formulated
in [2], the Tutte polynomial is the partition function of this model. The gadget is designed and carefully tuned so that it has two coexisting “phases”: one in which the random cluster configurations (spanning subgraphs) have a large connected (or “giant”) component, and one in which they don’t. We show that it is possible to arrange for the $t$ terminals to be (with high probability) in a single component in one phase and in $t$ distinct components in the other. This provides us with a bistable gadget that simulates a potentially large hyperedge using many 2-vertex edges. Note that AP-reductions often exploit phase transitions, playing one class of configurations off against another. See the examples in [8] and [23]. What is new here is that, as far as we are aware, this is the first time anyone managed to derive stronger complexity results using the complex phase transitions that arise in actual models studied in statistical physics. Unfortunately, the delicate nature of the gadgets needed to exploit this kind of phase transition does lead to significant technical complexity in our analysis.

Non-trivial phase transitions have been used in the past to rule out certain natural Markov chain Monte Carlo (MCMC) approaches to approximate counting problems. Indeed, the first-order phase transition in the random cluster model was already exploited by Gore and Jerrum [14] to demonstrate that the Swendsen-Wang algorithm is not always effective for the ferromagnetic Potts model. (See also Borgs et al. [4] for a more thorough working out of this idea.) In those applications the aim was to rule out a certain algorithmic approach, namely MCMC, whereas here our goal is to show inherent intractability.

Finally, note that Theorem 1 establishes \#BIS-hardness of $\text{Tutte}(q, \gamma)$ but not \#BIS-equivalence. It would be very interesting to know whether there is an AP-reduction from $\text{Tutte}(q, \gamma)$ to \#BIS. Note that the complexity of approximate counting is complicated. Bordewich [3] has shown that if any problem in \#P fails to have an FPRAS, then there is an infinite approximation hierarchy within \#P.

The rest of the paper is laid out as follows. Sections 2 and 3 introduce the models we shall be working with. Section 4 is the technical heart of the paper. It analyses the random cluster model on some families of graphs, building up to the construction of the hyperedge simulation gadget. Sections 5 and 6 cover the computational framework we work in, while §7–10 present and analyse the reductions themselves. The final §11 turns the hypergraph intractability result on its head by focusing on the 3-uniform case. By reduction to the graphic case, we obtain an FPRAS for the partition function.

2. The Tutte polynomial of a hypergraph and the Potts partition function

Let $H = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and hyperedge (multi)set $\mathcal{E}$. Following the usual convention for the Tutte polynomial [26], a hypergraph is allowed to have parallel edges. This is why $\mathcal{E}$ is a multiset. Also, the vertices in a particular hyperedge need not be distinct (so each hyperedge can be viewed as a
The multivariate Tutte polynomial of $H$ is defined as follows
\[ Z_{\text{Tutte}}(H; q, \gamma) = \sum_{\mathcal{F} \subseteq \mathcal{E}} q^{\kappa(\mathcal{V}, \mathcal{F})} \prod_{f \in \mathcal{F}} \gamma_f, \]
where $q$ and $\gamma = \{\gamma_f\}_{f \in \mathcal{E}}$ are commuting indeterminates and $\kappa(\mathcal{V}, \mathcal{F})$ denotes the number of connected components in the subhypergraph $(\mathcal{V}, \mathcal{F})$. (Two vertices $u, v$, are in the same component of $(\mathcal{V}, \mathcal{F})$ if $u = v$, or there is a sequence $f_1, \ldots, f_\ell \in \mathcal{F}$ of hyperedges with $u \in f_1$, $v \in f_\ell$ and $f_i \cap f_{i+1} \neq \emptyset$ for $1 \leq i < \ell$.) This partition function was studied (under a different name) by Grimmett [15]. An undirected graph $G$ can be viewed as a 2-uniform hypergraph (a hypergraph in which every hyperedge has size 2). In this case, $Z_{\text{Tutte}}(G; q, \gamma)$ coincides with the usual definition of the multivariate Tutte polynomial [26].

Let $q$ be a positive integer. The $q$-state Potts partition function of $H$ is defined as follows:
\[ Z_{\text{Potts}}(H; q, \gamma) = \sum_{\sigma: \mathcal{V} \rightarrow [q]} \prod_{f \in \mathcal{E}} (1 + \gamma_f \delta(\{\sigma(v) | v \in f\})), \]
where $[q] = \{1, \ldots, q\}$ is a set of $q$ spins or colours, and $\delta(S)$ is 1 if its argument is a singleton and 0 otherwise. The partition function is a sum ranging over objects, in this case assignments of spins to vertices, which are often referred to as “configurations”.

The following observation is due to Fortuin and Kastelyn.

**Observation 2.** If $q$ is a positive integer then $Z_{\text{Potts}}(H; q, \gamma) = Z_{\text{Tutte}}(H; q, \gamma)$.

**Proof.** A proof can be found in [5, Proposition 3.1] or in [15], but we include it here for completeness. The argument is a straightforward generalisation of the standard proof for the graph case. See [26, Theorem 2.3]. Consider
\[ \sum_{\sigma: \mathcal{V} \rightarrow [q]} \sum_{\mathcal{F} \subseteq \mathcal{E}} \prod_{f \in \mathcal{F}} (\gamma_f \delta(\{\sigma(v) | v \in f\})). \]
Now “integrate out” (i.e., explicitly sum over) $\mathcal{F}$ to see that this quantity is equal to $Z_{\text{Potts}}(H; q, \gamma)$ and integrate out $\sigma$ to see that this quantity is equal to $Z_{\text{Tutte}}(H; q, \gamma)$. \qed

The ferromagnetic Potts model corresponds to the Potts model in the special case in which the edge weights $\gamma_f$ are non-negative. In this case, a mono-chromatic edge contributes more weight than an edge with multiple spins.

For a subset $\mathcal{F} \subseteq \mathcal{E}$ of the hyperedges of a graph, we use $\gamma(\mathcal{F})$ to denote $\prod_{f \in \mathcal{F}} \gamma_f$. 
3. The Random Cluster Model

Consider a graph $G = (V, E)$. Every edge $e \in E$ is associated with a quantity $p(e) \in [0, 1]$. Then for a set of edges $A \subseteq E$ define
\[
\tilde{P}(G; A, q, p) = q^{\omega(V, A)} \prod_{e \in A} p(e) \prod_{e \in E \setminus A} (1 - p(e)).
\]
Let
\[
Z_{rc}(G; q, p) = \sum_{A \subseteq E} \tilde{P}(G; A, q, p) = Z_{Tutte}(G; q, \gamma) \prod_{e \in E} (1 - p(e)),
\]
where $\gamma_e = p(e)/(1 - p(e))$. Then the probability of edge-set $A$ in the random cluster model is given by
\[
P(G; A, q, p) = \frac{\tilde{P}(G; A, q, p)}{Z_{rc}(G; q, p)}.
\]

The *random cluster model* refers to the distribution $RC(G; q, p)$, in which a subset $A$ of edges is chosen with probability $P(G; A, q, p)$. The difference between $Z_{Tutte}$ and $Z_{rc}$ is simply one of parameterisation. Nevertheless, the change of parameter is useful, as it allows us to employ probabilistic terminology and exploit existing results from the random graph literature.

A central ingredient which helps us to understand the random cluster model is to compare it to the (multivariate) Erdős-Rényi model of a random graph. Consider a graph $G = (V, E)$. Every edge $e \in E$ is associated with a quantity $p'(e) \in [0, 1]$. In the Erdős-Rényi model $ER(G; p')$, a subset $A$ of edges is chosen with probability
\[
\prod_{e \in A} p'(e) \prod_{e \in E \setminus A} (1 - p'(e)).
\]
Thus, to choose a configuration $A$ in the $ER(G; p')$ model, each edge $e \in E$ is included in $A$ independently with probability $p'(e)$.

3.1. Stochastic domination results. We will use several simple stochastic domination results, which are close in spirit, and indeed in their proofs, to results of Holley [16]; see, in particular, Theorem (6) of that article and its proof.

If $A^+$ and $A^-$ are disjoint subsets of $E$, let RC($G; q, p; A^+, A^-$) be the random cluster model conditioned on the fact that the chosen subset $A$ contains every edge in $A^+$ and no edges in $A^-$. (To avoid trivialities, we assume that no edge $e \in A^+$ has $p(e) = 0$ and that no edge $e \in A^-$ has $p(e) = 1$.) Similarly, let ER($G; p'; A^+, A^-$) be the Erdős-Rényi model with this conditioning. We give conditioned versions of the stochastic domination results.

**Lemma 3.** Consider a graph $G = (V, E)$ in which each edge $e \in E$ is associated with a quantity $p(e) \in [0, 1]$. Let $A^+$ and $A^-$ be disjoint subsets of $E$ such that $p(e) > 0$ for $e \in A^+$ and $p(e) < 1$ for $e \in A^-$. Suppose $q \geq 1$. Then the random cluster model RC($G; q, p; A^+, A^-$) is stochastically dominated by the Erdős-Rényi
model \( ER(G; p; A^+, A^-) \) in the sense that we can select a pair \((A, A')\) such that \( A \) is drawn from \( RC(G; q, p; A^+, A^-) \) and \( A' \) is drawn from \( ER(G; p; A^+, A^-) \) and \( A \subseteq A' \).

**Proof.** Let \( \mathcal{M} \) be the “heat-bath on an edge” Markov chain for \( RC(G; q, p; A^+, A^-) \) and let \( \mathcal{M}' \) be the “heat-bath on an edge” Markov chain for \( ER(G; p; A^+, A^-) \). Start \( \mathcal{M} \) in state \( A_0 \) and \( \mathcal{M}' \) in state \( A_0' \) where \( A_0 \subseteq A_0' \) and \( A_0 \) and \( A_0' \) contain every edge in \( A^+ \) and no edges in \( A^- \). We can then couple the evolution of the chains from the \( i \)th pair of states \((A_i, A_i')\) as follows, guaranteeing that \( A_{i+1} \subseteq A_{i+1}' \). Choose the same edge \( e \in E \setminus A^+ \cup A^- \) in both chains. In \( \mathcal{M} \) the probability of putting \( e \) in \( A_{i+1} \) is either \( p(e) \) or \( p(e)/(p(e) + q(1 - p(e))) \), each of which is at most \( p(e) \), which is the probability of putting it in \( A_{i+1}' \).

**Lemma 4.** Consider a graph \( G = (V, E) \) in which each edge \( e \in E \) is associated with a quantity \( p(e) \in [0, 1] \). For each edge \( e \in E \), let \( p'(e) = p(e)/q \). Let \( A^+ \) and \( A^- \) be disjoint subsets of \( E \) such that \( p(e) > 0 \) for \( e \in A^+ \) and \( p(e) < 1 \) for \( e \in A^- \). Suppose \( q \geq 1 \). Then the Erdős-Rényi model \( ER(G; q; p; A^+, A^-) \) is stochastically dominated by the random cluster model \( RC(G; q, p; A^+, A^-) \) in the sense that we can select a pair \((A, A')\) such that \( A \) is drawn from \( ER(G; q, p; A^+, A^-) \) and \( A' \) is drawn from \( RC(G; q, p; A^+, A^-) \) and \( A \subseteq A' \).

**Proof.** Let \( \mathcal{M} \) be the “heat-bath on an edge” Markov chain for \( ER(G; q, p; A^+, A^-) \) and let \( \mathcal{M}' \) be the “heat-bath on an edge” Markov chain for \( RC(G; q, p; A^+, A^-) \). Start \( \mathcal{M} \) in state \( A_0 \) and \( \mathcal{M}' \) in state \( A_0' \) where \( A_0 \subseteq A_0' \) and \( A_0 \) and \( A_0' \) contain every edge in \( A^+ \) and no edges in \( A^- \). We can then couple the evolution of the chains from the \( i \)th pair of states \((A_i, A_i')\) as follows, guaranteeing that \( A_{i+1} \subseteq A_{i+1}' \). Choose the same edge \( e \in E \setminus A^+ \cup A^- \) in both chains. In \( \mathcal{M} \) the probability of putting \( e \) in \( A_{i+1} \) is either \( p(e) \) or \( p(e)/(p(e) + q(1 - p(e))) \), each of which is at least \( p'(e) \), which is the probability of putting it in \( A_{i+1}' \).

**Lemma 5.** Consider a graph \( G = (V, E) \) in which each edge \( e \in E \) is associated with a quantity \( p(e) \in [0, 1] \). Suppose that, for each edge \( e \in E \), \( p'(e) \geq p(e) \). Let \( A^+ \) and \( A^- \) be disjoint subsets of \( E \) such that \( p(e) > 0 \) for \( e \in A^+ \) and \( p(e) < 1 \) for \( e \in A^- \). Suppose \( q \geq 1 \). Then the random cluster model \( RC(G; q, p; A^+, A^-) \) is stochastically dominated by the random cluster model \( RC(G; q, p'; A^+, A^-) \) in the sense that we can select a pair \((A, A')\) such that \( A \) is drawn from the distribution \( RC(G; q, p; A^+, A^-) \) and \( A' \) is drawn from \( RC(G; q, p'; A^+, A^-) \) and \( A \subseteq A' \).

**Proof.** Let \( \mathcal{M} \) be the “heat-bath on an edge” Markov chain for \( RC(G; q, p; A^+, A^-) \) and let \( \mathcal{M}' \) be the “heat-bath on an edge” Markov chain for \( RC(G; q, p'; A^+, A^-) \). Start \( \mathcal{M} \) in state \( A_0 \) and \( \mathcal{M}' \) in state \( A_0' \) where \( A_0 \subseteq A_0' \) and \( A_0 \) and \( A_0' \) contain every edge in \( A^+ \) and no edges in \( A^- \). We can then couple the evolution of the chains from the \( i \)th pair of states \((A_i, A_i')\) as follows, guaranteeing that \( A_{i+1} \subseteq A_{i+1}' \). Choose the same edge \( e \in E \setminus A^+ \cup A^- \) in both chains. If the endpoints of the edge are in the same component of \( A_i \) then they are in the same component of
Given a subset \( U \) of vertices, let \( G[U] \) be the subgraph of \( G \) induced by \( U \)

\[ A'_i - e. \]  The probability of putting \( e \) in \( A_{i+1} \) is \( p(e) \) and the probability of putting \( e \) in \( A'_{i+1} \) is \( p'(e) \), which is at least as big. Otherwise, the probability of putting \( e \) in \( A_{i+1} \) is \( p(e)/(p(e) + q(1 - p(e))) \). This is at most \( p'(e)/(p'(e) + q(1 - p'(e))) \) and the probability of putting \( e \) in \( A'_{i+1} \) is at least \( p'(e)/(p'(e) + q(1 - p'(e))) \). \( \Box \)

3.2. The Fundamental Lemma of Bollobás, Grimmett and Jansen. Another way to compare the random cluster model to the Erdős-Rényi model is to use the following lemma, which is a multivariate version of Bollobás, Grimmett and Jansen’s “Fundamental Lemma”, [2, Lemma 3.1]. To do the comparison, the lemma should be applied with \( r = 1/q \) so that the distribution \( RC(G[V_1]; rq, p) \) in the statement of the lemma below is identical to the distribution \( ER(G[V_1]; p) \).

Let \( 0 \leq r \leq 1 \) be fixed. Given a subset \( A \) of edges chosen from \( RC(G; q, p) \), colour each component of \( (V, A) \) either red, with probability \( r \), or green, with probability \( 1 - r \): different components are coloured independently of one another. The union of the red components is the red subgraph and the union of the green components is the green subgraph. Let \( R \) be the set of vertices in the red subgraph. Given a subset \( U \) of vertices, let \( G[U] \) be the subgraph of \( G \) induced by \( U \).

**Lemma 6.** (Bollobás, Grimmett and Jansen [2] Lemma 3.1.) Let \( V_1 \subseteq V \). Conditioned on \( R = V_1 \), the red subgraph is distributed according to \( RC(G[V_1]; rq, p) \) and the green subgraph is distributed according to \( RC(G[V \setminus V_1]; (1 - r)q, p) \). Conditioned on \( R = V_1 \), the red subgraph and green subgraph are independent of each other.

**Proof.** Let \( V_2 = V \setminus V_1 \). Let \( E_1 = E \cap V_1^{(2)} \) and \( E_2 = E \cap V_2^{(2)} \), where the notation \( V_i^{(2)} \) denotes the set of unordered pairs of vertices from \( V_i \). Now consider \( A_1 \subseteq E_1 \) and \( A_2 \subseteq E_2 \). The (unconditional) probability that the red subgraph is \( (V_1, A_1) \) and the green subgraph is \( (V_2, A_2) \) is

\[
\frac{\tilde{P}(G; A_1 \cup A_2, q, p) r^{\kappa(V_1, A_1)}(1 - r)^{\kappa(V_2, A_2)}}{Z_{rc}(G; q, p)}
\]

so the conditional probability, conditioned on \( R = V_1 \) is

\[
\frac{\tilde{P}(G; A_1 \cup A_2, q, p) r^{\kappa(V_1, A_1)}(1 - r)^{\kappa(V_2, A_2)}}{\sum_{A'_1 \subseteq E_1, A'_2 \subseteq E_2} \tilde{P}(G; A'_1 \cup A'_2, q, p) r^{\kappa(V_1, A'_1)}(1 - r)^{\kappa(V_2, A'_2)}}
\]

which is

\[
P(G[V_1]; A_1, rq, p) P(G[V \setminus V_1]; A_2, (1 - r)q, p).
\]

\( \Box \)

4. The Random Cluster Model on Some Natural Graphs

In this section we consider the random cluster model on a clique, and also on a pair of connected cliques. The latter is used as a gadget in our constructions. First, however, we need a technical lemma,
4.1. **A technical lemma.** The following lemma is not about the random cluster model, but we will use it in our analysis of the random cluster model on a pair of connected cliques.

**Lemma 7.** Suppose we have a partition of the set $[\nu]$ into $s$ blocks of size at most $\nu_{\text{max}}$. Randomly colour a subset of the elements of $[\nu]$ yellow according to a Bernoulli process with success probability $\xi$. Independently, randomly colour a subset of the elements of $[\nu]$ blue according to a Bernoulli process with success probability $\xi$ (so an element can be coloured yellow, blue, both or neither). We say that a block of the partition is bicoloured if it contains both yellow and blue elements. Then

$$\Pr(\text{no block is bicoloured}) \leq [(1 - \xi)^{\nu/s}(2 - (1 - \xi)^{\nu/s})]^s$$

(which is increasing with $s$ and decreasing with $\nu$) and

$$\Pr(\text{some block is bicoloured}) \leq \nu[1 - (1 - \xi)^{\nu_{\text{max}}}]^2$$

(which is increasing in both $\nu$ and $\nu_{\text{max}}$).

**Proof.** Let the block sizes be $\nu_1, \ldots, \nu_s$, so that $\max_j \nu_j \leq \nu_{\text{max}}$. Observe that

$$\Pr(\text{some block contains a yellow element}) = 1 - (1 - \xi)^{\nu_j} = \xi_j \text{ (say)},$$

and that the same bound applies, of course, to blue elements. Thus, using the convention that the index $j$ always ranges over $1 \leq j \leq s$,

$$\Pr(\text{some block is bicoloured}) \leq \sum_j \xi_j^2 \leq \sum_j [1 - (1 - \xi)^{\nu_{\text{max}}}]^2,$$

which gives the second bound. Clearly, this bound is monotonically increasing in $\nu$ and $\nu_{\text{max}}$ as claimed.

Now for the first bound.

$$\Pr(\text{no block is bicoloured}) = \prod_j (1 - \xi_j^2) = \prod_j (1 - \xi_j) \prod_j (1 + \xi_j)$$

$$\quad = \prod_j (1 - \xi)^{\nu_j} \prod_j (1 + \xi_j)$$

$$\quad = (1 - \xi)^\nu \prod_j (1 + \xi_j).$$

(3)

To get an upper bound we are interested in evaluating the supremum of

$$\prod_j (1 + \xi_j) = \prod_j [2 - (1 - \xi)^{\nu_j}]$$

over the domain defined by the linear inequalities $0 \leq \nu_j \leq \nu$ and $\sum_j \nu_j = \nu$. We consider this as an optimisation problem over $\mathbb{R}^s$ even though the $\nu_j$ are all integers; of course, this will if anything only increase the supremum. We are considering a continuous function over a closed, bounded set, so the supremum is achieved at
some point; we claim that this (unique) point is $\nu_1 = \cdots = \nu_s = \nu/s$. For if not, then at least one pair, say $\nu_1$ and $\nu_2$ would be unequal. But then it is easily checked that replacing $\nu_1$ and $\nu_2$ by their average would increase $[2 - (1 - \xi)\nu_1][2 - (1 - \xi)\nu_2]$, and hence increase the right hand side of (4): a contradiction. Substituting $\nu_j = \nu/s$ into equations (4) and then (3), we obtain

$$\Pr(\text{no block is bicoloured}) \leq (1 - \xi)^\nu[2 - (1 - \xi)^{\nu/s}]^s$$

as desired.

In only remains to verify the monotonicity claims about (6). Let $u(\nu) = (1 - \xi)^{\nu/s} \in (0, 1]$. Regarding $s > 0$ as fixed, $u(\nu)$ decreases monotonically with $\nu$. Also, $[u(2 - u)]^s$ increases monotonically as a function of $u$ in the range $(0, 1)$. Thus, expression (6) decreases monotonically with $\nu$. Now regard $\nu$ as fixed and make the change of variable $s = ax$, where $a = \nu \ln((1 - \xi)^{-1})$, and note that $x > 0$. Then expression (5) becomes $[e^{-1}(2 - e^{-1/x})]^{ax}$. Thus, it is enough to show that $f(x) = x \ln(2 - e^{-1/x})$ increases monotonically with $x$. Now

$$f''(x) = -2x^{-3}e^{-1/x}(2 - e^{-1/x})^{-2} < 0,$$

and

$$f'(x) = \ln(2 - e^{-1/x}) - x^{-1}e^{-1/x}(2 - e^{-1/x})^{-1} \to 0 \text{ as } x \to \infty.$$

These two facts imply $f'(x) > 0$ for $x > 0$. □

4.2. The random cluster model on a clique. Bollobás, Grimmett and Jansen [2] studied the random cluster model on the complete $N$-vertex graph $K_N$. More detailed analyses have since been performed, for example by Luczak and Luczak [21], but the approach of the earlier paper is easier to adapt to our needs.

For fixed $q$ and a fixed constant $\lambda$, they studied the distribution $RC(K_N, q, p)$ where $p$ is the constant function which assigns every edge $e$ of $K_N$ the value $p(e) = \lambda/N$. They show that there is a critical value $\lambda_c$, depending on $q$, so that, if $\lambda > \lambda_c$ then, as $N \to \infty$, with high probability a configuration $A$ drawn from $RC(K_N, q, p)$ will have a large component (of size linear in $N$) and otherwise, with high probability the largest component will be much smaller (of size logarithmic in $N$).

For $q > 2$, the critical value $\lambda_c$ is defined as follows.

$$\lambda_c = 2 \left(\frac{q - 1}{q - 2}\right) \ln(q - 1).$$

It is important for our analysis that $\lambda_c < q$ (see [2] p.16), or by calculus). We define $\delta = (q - \lambda_c)/2 > 0$ and $\lambda = \lambda_c + \delta$. Let $\theta = (q - 2)/(q - 1)$.

We will use the following lemma, which follows from Theorem 2.2 and Equation (5) of [2].
Lemma 8. Fix $q > 2$ and define $\lambda$ and $\theta$ as above. Let $p$ be the constant function which assigns every edge $e$ of $K_{N}$ the value $p(e) = \lambda/N$. Let $A$ be drawn from $RC(K_{N}, q, p)$. The probability that $A$ has a connected component of size at least $\theta N$ tends to 1 as $N \to \infty$.

4.3. The random cluster model on a pair of connected cliques. Let $\Gamma$ be the complete graph with vertex set $V_{\Gamma} = K \cup T$. Let $E_{\Gamma}$ denote the edge set of $\Gamma$ and let $N = |K|$ and $t = |T|$. Let $K^{(2)}$ denote the set of unordered pairs of distinct elements in $K$ and define $T^{(2)}$ similarly. Let $\varrho$ be a value in $[0, 1]$. Define $p$ as follows.

$$p(e) = \begin{cases} 
\varrho, & \text{if } e \in K^{(2)}, \\
N^{-3/4}, & \text{if } e \in K \times T, \text{ and} \\
1, & \text{if } e \in T^{(2)}. 
\end{cases}$$

Ultimately, we will use the graph $\Gamma$ (or, more precisely, $\Gamma$ with the edges $T^{(2)}$ deleted) as a gadget to simulate the contribution of a hyperedge on the set $T$ to the multivariate Tutte polynomial of $\Gamma$. Thus, we refer to vertices in $T$ as “terminals” of the graph $\Gamma$. For a subset $A \subseteq E_{\Gamma}$, let $Y(A)$ be the number of connected components in the graph $(V_{\Gamma}, A \setminus T^{(2)})$ that contain terminals.

A remark about the gadget $\Gamma$ and its eventual use. When we come to use the gadget, the edges in $T^{(2)}$ will not be present. It is for this reason that we are interested in the structure of connected components in $\Gamma$ in the absence of these edges, and specifically the random variable $Y(A)$. However, it turns out that the key properties of the gadget are easier to verify if we work with a random cluster distribution associated with $\Gamma$, exactly as given above, with the edges $T^{(2)}$ present. Informally, the appropriate “boundary condition” for the gadget is the one in which the terminals are joined with probability 1.

The following two lemmas establish some useful properties of the gadget. The second shows that distribution of $Y(A)$ is concentrated at the extremes of its range, i.e., $Y = 1$ or $Y = t$. This concentration property holds for a wide range of values for the edge probability $\varrho$. The first lemma, which is easier, shows that we can tune $\varrho$ so that the balance of probability between those extremes is the desired value $\gamma$. Later, in Lemmas 12 and 13, we shall show that a sufficiently close approximation to this $\varrho$ can be efficiently computed.

Lemma 9. Fix $q > 2$ and let $\lambda = \lambda_{c} + (q - \lambda_{c})/2$. Fix a weight $\gamma > 0$ and let $N_{0}$ be a sufficiently large quantity depending on $q$ and $\gamma$. Suppose a number of terminals $t > 1$ is given and fix $N \geq \max\{t^{16}, N_{0}\}$. Then there is a parameter $\varrho$ satisfying $N^{-3} \leq \varrho \leq \lambda/N \leq \frac{1}{2}$ such that, if $A$ is drawn from $RC(\Gamma; q, p)$ then

$$\Pr(Y(A) = 1) = \gamma \Pr(Y(A) = t). \quad (7)$$

Proof. Define $\theta$ (depending on $q$) as in Section 4.2. Let $\psi(\varrho) = \Pr(Y(A) = t)/\Pr(Y(A) = 1)$. We will use stochastic domination to show
Now the probability that $\emptyset$ is at least $\theta N$ connected component of size at least $\theta N$ connected to the large component is at most $(1 - \gamma / 1)$.

Next, to show that $\psi(\varrho)$ is monotonically decreasing as a function of $\varrho$, we will again use stochastic domination to compare the random cluster model to the Erdős-Rényi model, but this time we need some conditioning.

First, we will show that the probability that the graph $(K, A \cap K^{(2)})$ has a connected component of size at least $\theta N$ tends to 1 as $N \to \infty$. To do this, let $A^*$ be drawn from the distribution $\text{RC}(\Gamma, q, \hat{p})$ where

$$\hat{p}(e) = \begin{cases} 0, & \text{if } e \in K^{(2)}, \\ \varrho, & \text{otherwise}. \end{cases}$$

Since $\hat{p}(e) \leq p(e)$, Lemma 5 guarantees that the probability in question is at least the probability that the graph $(K, A^*)$ has a connected component of size at least $\theta N$. By Lemma 5, this probability tends to 1 as $N \to \infty$.

Next, we will consider the generation of configuration $A$ from the distribution $\text{RC}(\Gamma; q, p)$ as follows. First, we will select a set $A^+ \subseteq K^{(2)}$ from the appropriate induced distribution. Letting $A^- = K^{(2)} \setminus A^+$, we will select $A$ from the distribution $\text{RC}(\Gamma; q, p; A^+, A^-)$. We will finish by showing that, as long as $(K, A^+)$ has a connected component of size at least $\theta N$, $\text{Pr}(Y(A) = 1 | A^+ \subseteq A, A \cap A^- = \emptyset)$ is greater than $\gamma/(1 + \gamma)$. To do this, let $p'(e) = p(e)/q$ and let $A$ be a random variable drawn from $\text{ER}(\Gamma; p', A^+, A^-)$. By Lemma 4, $\text{Pr}(Y(A) = 1 | A^+ \subseteq A, A \cap A^- = \emptyset) \geq \text{Pr}(Y(A) = 1)$. Now in $A$, the probability that a particular terminal is not connected to the large component is at most $(1 - q^{-1}N^{-3/4})^{\theta N} \leq \exp(-\theta q^{-1}N^{1/4})$, and the probability that there exists a terminal that is not connected to this component is at most $t \exp(-\theta q^{-1}N^{1/4})$. So $\text{Pr}(Y(A) = 1) > \gamma/(1 + \gamma)$. \(\square\)
Lemma 10. Fix $q > 2$ and let $\lambda = \lambda_c + (q - \lambda_c)/2$. Fix a weight $\gamma > 0$ and let $N_0$ be a sufficiently large quantity depending on $q$ and $\gamma$. Suppose a number of terminals $t > 1$ and a tolerance $0 < \eta < 1$ are given and fix $N \geq \max\{t^{16}, \eta^{-1/8}, N_0\}$. For every value of $\rho$ in the range $[N^{-3}, \lambda/N]$, if $A$ is drawn from $\text{RC}(\Gamma; q, p)$ then

$$\text{Pr}(1 < Y(A) < t) < \eta.$$  

(8)

**Proof.** Let $C_1, C_2, \ldots$ be the connected components of $A \cap K^{(2)}$, ordered in non-increasing size. We are going to be relying on the phase transition of the random cluster model. The main fact that we will use is that $|C_1|$ is likely to either be very small (around order $\log(N)$) or very large (a constant fraction of $N$). Thus, it is unlikely that $|C_1|$ is close to $N^{1/8}$. We will not need much detail about the phase transition. We will show

$$\text{Pr}(\{|C_1| \leq N^{1/8}\} \land (Y(A) < t)) \leq \eta/2,$$

(9)

and

$$\text{Pr}(\{|C_1| > N^{1/8}\} \land (Y(A) > 1)) \leq \eta/2.$$  

(10)

(Actually, the event mentioned in (10) holds with all but exponentially small probability.) The required inequality (8) follows directly from (9) and (10).

Inequality (9) is easier. To generate a configuration $A$ from $\text{RC}(\Gamma; q, p)$ we first select a set $A^+ \subseteq K^{(2)}$ from the appropriate induced distribution. Then, letting $A^- = K^{(2)} \setminus A^+$, we select $A$ from the distribution $\text{RC}(\Gamma; q, p; A^+, A^-)$. The size of $C_1$ is entirely determined by $A^+$. If $A^+$ has a component of size greater than $N^{1/8}$ then

$$\text{Pr}(\{|C_1| \leq N^{1/8}\} \land (Y(A) < t) \mid A^+ \subseteq A, A \cap A^- = \emptyset) = 0.$$  

Otherwise, this probability is equal to $\text{Pr}(Y(A) < t \mid A^+ \subseteq A, A \cap A^- = \emptyset)$. By Lemma 10 this is at most $\text{Pr}(Y(\overline{A}) < t)$ where $\overline{A}$ is generated from $\text{ER}(\Gamma; p; A^+, A^-)$.

So now consider $\overline{A}$. Fix attention on two terminals, and colour the vertices in $K$ that are adjacent to the first terminal yellow, and those adjacent to the second terminal blue. We are in the situation of Lemma 7 with $\nu = N$, $\nu_{\text{max}} \leq N^{1/8}$ and $\xi = N^{-3/4}$. Thus the probability that there exists a bicoloured block (i.e., that the terminals are connected via some connected component $C_i$) is at most $N[1 - (1 - N^{-3/4})^{N^{1/8}}]^2 \leq N[1 - 1 + N^{-5/8}]^2 = N^{-1/4}$. Thus the probability that there exists a pair of connected terminals is at most $(\frac{t}{2}) N^{-1/4} \leq N^{-1/8}/2 < \eta/2$. Note that the event “there exists a pair of connected terminals” is the same as the event $Y(\overline{A}) < t$. 
At a high level, the path to establishing inequality (10) is as follows. We define events $\mathcal{L}$ ("large component") and $W$ ("weighty components") and establish

\begin{align}
\Pr\left(\left(\left|C_1\right| > N^{1/8}\right) \land \neg \mathcal{L}\right) &< \eta/6, \\
\Pr(\mathcal{L} \land \neg W) &< \eta/6, \quad \text{and} \\
\Pr\left(W \land (Y(A) > 1)\right) &< \eta/6.
\end{align}

Then the required inequality (10) follows by elementary algebra of sets (events).

We will say that an event holds "with high probability" (abbreviated whp) if the probability that it holds is $o(1/Q(n))$ for any fixed polynomial $Q$.

First, we prove inequality (11). Let $S_1, S_2, \ldots$ be the connected components of $A$ and let $\hat{C}_1, \hat{C}_2, \ldots$ be the sets $S_1 \cap K^{(2)}, S_2 \cap K^{(2)}, \ldots$ ordered in non-increasing size. (Thus, $\{\hat{C}_j\}$ is a coarsening of $\{C_j\}$.) Event $\mathcal{L}$ is $|\hat{C}_1| > N^{15/16}$. Let $\mathcal{E}_1$ be the event $(|C_1| > N^{1/8}) \land \neg \mathcal{L}$. We are interested showing that the event $\mathcal{E}_1$ is unlikely.

Construct the red subgraph and green subgraph of $(V_R, A)$ as in Lemma 6 with $r = 1/q$. Let $\mathcal{E}_2$ be the event

\[(|R \cap K| \leq N/q + N^{15/16}) \land (C_1 \subseteq R) \land (|C_1| > N^{1/8}).\]

We will show

\[\Pr(\mathcal{E}_2) \geq \Pr(\mathcal{E}_1)/q^2\]

and

\[\Pr(\mathcal{E}_2) \leq \eta/(6q^2),\]

which proves (11). The first of these follows from $\Pr(\mathcal{E}_2 \mid \mathcal{E}_1) \geq q^{-2}$, which follows since $\Pr(C_1 \subseteq R \mid \mathcal{E}_1) = q^{-1}$ (by the definition of the red subgraph) and

\[\Pr\left(|R \cap K| \leq N/q + N^{15/16} \mid \mathcal{E}_1 \land (C_1 \subseteq R)\right) \geq q^{-1}\]

(also by the definition of the red subgraph — with probability $q^{-1}$, red is the rarest colour among vertices in $K \setminus \hat{C}_1$ where $\hat{C}_1$ is the element of $\{\hat{C}_j\}$ containing $C_1$).

Now the probability of event $\mathcal{E}_2$ is at most

\[\Pr\left((C_1 \subseteq R) \land \left(|C_1| > N^{1/8}\right) \mid |R \cap K| \leq N/q + N^{15/16}\right)\]

which is at most the conditional probability that $A[R \cap K]$ has a component of size at least $N^{1/8}$, conditioned on $|R \cap K| \leq N/q + N^{15/16}$. But, by Lemma 6 $A[R \cap K]$ is an Erdős-Rényi random graph with $|R \cap K| \leq (q^{-1} + N^{-1/16}) N$ vertices and subcritical edge probability $\frac{\lambda}{N} < |R \cap K|^{-1}$. So the probability that it has a component of size at least $N^{1/8}$ is at most $\varepsilon(N)$, where $\varepsilon(\cdot)$ is smaller than any inverse polynomial [19, Proof of Theorem 5.4]. Thus, we have shown $\Pr(\mathcal{E}_2) \leq \eta/(6q^2)$ (provided $N$ is sufficiently large), so we have proved (11).

Next we prove inequality (12). The event $W$ is that $|C_1| + \cdots + |C_s| \geq N^{15/16}$, for some $s \leq 2N^{5/16}$. The component $\hat{C}_1$ is in general composed of a number of components from $\{C_j\}$. However, the number of constituent components cannot be larger than the number $|A \cap (K \times T)|$ of edges joining $K$ to $T$. This number can be upper-bounded using a bounding configuration $\overline{A}$ drawn from ER($\Gamma'; p$). We
see from a Chernoff bound that whp the number $|A \cap (K \times T)|$ of edges joining $K$ to $T$ is less than $2tN^{1/4} \leq 2N^{5/16}$, i.e., twice the expected number, so whp, at most $2N^{5/16}$ components from $\{C_j\}$ contribute to $\hat{C}_1$. Thus $\Pr(W | L) = 1 - \varepsilon(N)$ and (12) follows.

Finally we prove inequality (13). As before, generate $A$ by selecting a set $A^+ \subseteq K^{(2)}$ from the appropriate induced distribution, letting $A^- = K^{(2)} \setminus A^+$, and selecting $A$ from the distribution $RC(\Gamma; q, p; A^+, A^-)$. We will condition on the event $W$ (which is entirely determined by $A^+$). As above,

$$\Pr \left( Y(A) = 1 \mid A^+ \subseteq A, A \cap A^- = \emptyset \right) \geq \Pr(Y(\tilde{A}) = 1),$$

where $p'(e) = p(e)/q$ and $\tilde{A}$ is a random variable drawn from $\text{ER}(\Gamma; p'; A^+, A^-)$. Consider two terminals $i, j$ in $T$. We can use Lemma 7 to find an upper bound for the probability that $i$ and $j$ are not connected in $\tilde{A}$ via one of the components $C_1, \ldots, C_s$. In $\tilde{A}$, edges from $T$ to $K$ are selected independently with probability $q^{-1}N^{-3/4}$. If we colour vertices in $K$ adjacent to $i$ (respectively, $j$) yellow (respectively, blue) then we are in the situation of Lemma 7, with $\xi = q^{-1}N^{-3/4}$. From the remarks about monotonicity in $s$ and $\nu$, we may assume for an upper bound that $\nu = N^{15/16}$ and $s = 2N^{5/16}$. The probability we want to bound is that of not having a bicoloured component. Observe

$$(1 - \xi)^{\nu/s} = (1 - q^{-1}N^{-3/4})^{2N^{5/8}} \leq 1 - \frac{1}{2}q^{-1}N^{-1/8},$$

for $N$ sufficiently large, and since the function $u \mapsto u(2 - u)$ is increasing in the range $(0, 1)$,

$$(1 - \xi)^{\nu/s}(2 - (1 - \xi)^{\nu/s}) \leq (1 - \frac{1}{2}q^{-1}N^{-1/8})(1 + \frac{1}{2}q^{-1}N^{-1/8}).$$

Applying Lemma 7

$$\Pr(i \not\sim j \text{ in } \tilde{A}) = (1 - \frac{1}{16}q^{-2}N^{-1/4})^{2N^{5/16}} < \exp(-\frac{1}{8}q^{-2}N^{1/16}).$$

So whp $i \sim j$. It follows that whp all vertices in $T$ are connected to each other, and so $\Pr(Y(\tilde{A}) = 1) \geq 1 - \varepsilon(N)$. This deals with (13) and completes the proof.

4.4. The random cluster model on a clique connected to an independent set. Construct $\Gamma$ as in Section 4.3. Let $\Gamma = (V_\Gamma, E_\Gamma \setminus T^{(2)})$ be the graph derived from $\Gamma$ by deleting edges within $T$. The vertices in $T$ are called the “terminals” of $\Gamma$. Let $\gamma = \{\gamma_e\}$ be the set of edge weights defined by $\gamma_e = p(e)/(1 - p(e))$.

For an edge subset $A' \subseteq E_\Gamma \setminus T^{(2)}$, let $\kappa'(V_\Gamma, A')$ denote the number of connected components that do not contain terminals in the graph $(V_\Gamma, A')$. Let $A^k$ denote the set of edge subsets $A' \subseteq E_\Gamma \setminus T^{(2)}$ for which the terminals of $(V_\Gamma, A')$ are contained in exactly $k$ connected components. Let $A = \bigcup_{k \in [t]} A^k$ (this is the set of all edge subsets of $\Gamma$). Let $Z^k$ be $q^{-k}$ times the contribution to $Z_{\text{Paths}}(\Gamma; q, \gamma)$ from edge subsets $A' \in A^k$. Formally, $Z^k = \sum_{A' \in A^k} q^{\kappa'(V, A')}\gamma(A')$. Let $Z = \sum_{k=1}^t Z^k$.

We will use the following lemma to apply Lemmas 9 and 10 from Section 4.3 in our reductions.
Lemma 11. \( Z^k/Z = \Pr(Y(A) = k) \), where \( A \) is drawn from \( \text{RC}(\Gamma; q, p) \).

Proof. From the definitions of the random cluster model,
\[
\Pr(Y(A) = k) = \frac{\sum_{A' \in A} \tilde{P}(\Gamma; A' \cup T^{(2)}, q, p)}{\sum_{A' \in A} \tilde{P}(\Gamma; A' \cup T^{(2)}, q, p)}.
\]
Plugging in the definition of \( \tilde{P}(\Gamma; A' \cup T^{(2)}, q, p) \), this is
\[
\frac{\sum_{A' \in A} \gamma(A') q^{\kappa(V, A') + 1} \prod_{e \in E_{\Gamma \setminus T^{(2)}}} (1 - p(e))}{\sum_{A' \in A} \gamma(A') q^{\kappa(V, A') + 1} \prod_{e \in E_{\Gamma \setminus T^{(2)}}} (1 - p(e))},
\]
which is what we require, once we cancel a factor of
\[
q \prod_{e \in E_{\Gamma \setminus T^{(2)}}} (1 - p(e))
\]
from the numerator and denominator. \( \square \)

5. Computational problems, fully polynomial randomised approximation schemes and efficiently approximable real numbers

Fix real numbers \( q > 2 \) and \( \gamma > 0 \) and consider the following computational problem, which is parameterised by \( q \) and \( \gamma \).

Problem: \( \text{TUTTE}(q, \gamma) \).

Instance: Graph \( G = (V, E) \).

Output: \( Z_{\text{TUTTE}}(G; q, \gamma) \), where \( \gamma \) is the constant function with \( \gamma_e = \gamma \) for every \( e \in E \).

We will have much more to say about computational approximations of real numbers below. For the moment, it may help the reader to think of \( q \) and \( \gamma \) as being rational. Jaeger, Vertigan and Welsh \[18\] have shown that \( \text{TUTTE}(q, \gamma) \) is \#P-hard for every fixed \( q > 2 \) and \( \gamma > 0 \). Thus, it is unlikely that there is a polynomial-time algorithm for exactly solving this problem. (If there were such an algorithm, this would entail \( P = \#P \), and of course \( P = NP \).)

We are interested in the complexity of approximately solving \( \text{TUTTE}(q, \gamma) \). We start by defining the relevant concepts. A \textit{randomised approximation scheme} is an algorithm for approximately computing the value of a function \( f : \Sigma^* \rightarrow \mathbb{R} \). The approximation scheme has a parameter \( \varepsilon > 0 \) which specifies the error tolerance. A \textit{randomised approximation scheme} for \( f \) is a randomised algorithm that takes as input an instance \( x \in \Sigma^* \) (e.g., for the problem \( \text{TUTTE}(q, \gamma) \), the input would be an encoding of a graph \( G \)) and a rational error tolerance \( \varepsilon > 0 \), and outputs a rational number \( z \) (a random variable of the “coin tosses” made by the algorithm) such that, for every instance \( x \),

\[
\Pr \left[ e^{-\varepsilon} f(x) \leq z \leq e^\varepsilon f(x) \right] \geq \frac{3}{4}.
\]
The randomised approximation scheme is said to be a fully polynomial randomised approximation scheme, or FPRAS, if it runs in time bounded by a polynomial in $|x|$ and $\varepsilon^{-1}$. Note that the quantity $3/4$ in Equation (14) could be changed to any value in the open interval $(1/2, 1)$ without changing the set of problems that have randomised approximation schemes [22, Lemma 6.1].

We say that a real number $z$ is efficiently approximable if there is an FPRAS for the problem which maps any input to the output $z$. Approximations to real numbers are useful. For example, if $\hat{q}$ and $\hat{\gamma}$ are approximations to $q$ and $\gamma$ satisfying
\[ e^{-\varepsilon n + m} q \leq \hat{q} \leq e^{\varepsilon n + m} q \]
and
\[ e^{-\varepsilon n + m} \gamma \leq \hat{\gamma} \leq e^{\varepsilon n + m} \gamma \]
and $\hat{\gamma}_e = \hat{\gamma}$ for every $e \in E$ then
\[ e^{-\varepsilon} Z_{\text{Tutte}}(G; q, \gamma) \leq Z_{\text{Tutte}}(G; \hat{q}, \hat{\gamma}) \leq e^{\varepsilon} Z_{\text{Tutte}}(G; q, \gamma). \]

Thus, to approximate $Z_{\text{Tutte}}(G; q, \gamma)$, it suffices to first compute rational approximations $\hat{q}$ and $\hat{\gamma}$, and then approximate $Z_{\text{Tutte}}(G; \hat{q}, \hat{\gamma})$.

When the parameters are efficiently approximable reals, it is possible to approximate quantities associated with the gadgets $\Gamma$ and $\tilde{\Gamma}$ that we studied in Section 4.3

**Lemma 12.** Suppose $q > 2$ is an efficiently computable real. Consider the gadget $\tilde{\Gamma}$ from Section 4.3 with parameters $t$, $N$ and $\varrho$ where $\varrho \in [0, 1]$ is a rational number and $N^{1/4}$ is an integer. There is an FPRAS for computing $Z^1$ and $Z^t$, given inputs $t$, $N$ and $\varrho$.

**Proof.** In this proof only, the notation $\tilde{\Gamma}_{N,t}$ is used to make explicit the size of the gadget $\tilde{\Gamma}$.

Let $E(\tilde{\Gamma}_{N,t})$ be the set of edges of $\tilde{\Gamma}_{N,t}$. Let $A^{k,\ell}$ denote the set of edge subsets $A \subseteq E(\tilde{\Gamma}_{N,t})$ with $k$ connected components containing vertices in $T$, and $\ell$ other connected components. Let $w(t, N, k, \ell) = \sum_{A \in A^{k,\ell}} \gamma(A)$. Thus $Z^k = \sum_{\ell=0}^N w(t, N, k, \ell) q^\ell$. We exhibit recurrence relations for $w(t, N, j, k)$, from which it follows that these can be computed in polynomial time by dynamic programming. The rest of this proof is straightforward, but provides the details. To reduce the number of boundary cases we need to consider, it is convenient to allow one or other of $k$ and $\ell$ to take on the value $-1$. Of course, we stipulate
\[ w(t, N, -1, \ell) = w(t, N, k, -1) = 0, \quad \text{for all } t, N, k, \ell \geq 0. \]

Another easy-to-verify boundary case is
\[ w(t, 0, k, \ell) = \begin{cases} 1, & \text{when } k = t \text{ and } \ell = 0; \\
0, & \text{otherwise}, \end{cases} \]
which is valid for all \( t, k, \ell \geq 0 \). Also \( w(t, N, k, \ell) = 0 \) if exactly one of \( t \) and \( k \) is 0 and \( w(0, N, 0, 0) \) is 1 is \( N = 0 \) and 0 otherwise. Finally, \( w(0, 0, k, \ell) = 0 \) if \( k + \ell > 0 \).

The general recurrence, covering all situations other than the boundary cases already mentioned can now be given.

\[
(16) \quad w(t, N, k, \ell) = \sum_{1 \leq i \leq t, 1 \leq j \leq N} \binom{\ell}{i} \binom{N-1}{j-1} w(i, j, 1, 0) w(t-i, N-j, k-1, \ell)
\]

\[
(17) \quad + \sum_{1 \leq j \leq N} \binom{N-1}{j-1} w(0, j, 0, 1) w(t, N-j, k, \ell-1).
\]

(Although valid for all \( k, \ell \geq 0 \), the recurrence becomes trivial when \( k + \ell = 1 \), a point we must return to at the end.) The explanation is as follows. We partition the sum defining \( w(t, N, k, \ell) \) according to the connected component \( C \) in \( A \) containing some distinguished vertex \( v \) in \( K \). (Note that \( N > 0 \), since one of the boundary cases covers \( N = 0 \).) Let \( i = |C \cap T| \) be the number of vertices of \( C \) that lie in \( T \), and \( j = |C \cap K| \) the number that lie in \( K \). Summation (16) deals with the situation \( i > 0 \) and (17) with the situation \( i = 0 \). The binomial coefficients in (16) count the number of connected components \( C \ni v \) that contain the distinguished vertex, and have the correct intersections \( |C \cap T| = i \) and \( |C \cap K| = j \) with \( T \) and \( K \); the factor \( w(i, j, 1, 0) \) counts the weight of connected subgraphs of \( \tilde{\Gamma}[C] \); and \( w(t-i, N-j, k-1, \ell) \) the weight of subgraphs of \( \tilde{\Gamma}[K \cup T \setminus C] \) that have the correct number of connected components (i.e., \( k-1 \) with vertices in \( T \), and \( \ell \) without). The analysis of summation (17) is entirely similar.

Assuming \( k + \ell > 1 \), the recurrence is well founded, in the sense that all occurrences of \( w(t', N', \cdot, \cdot) \) on the right hand side (apart from those that get multiplied by zero) have \( t' + N' < t + N \). When \( k + \ell = 1 \), i.e., \( (k, \ell) \in \{(0,1), (1,0)\} \) the recurrence (16,17) becomes the trivial \( w(t, N, k, \ell) = w(t, N, k, \ell) \). Note that \( (k, \ell) = (0,1) \) entails \( t = 0 \) (otherwise \( w(t, N, k, \ell) = 0 \)) and \( (k, \ell) = (1,0) \) entails \( t > 0 \), so only one of the possibilities \( (k, \ell) \in \{(0,1), (1,0)\} \) occurs for a given pair \((t, N)\). In order to make progress in this situation, we apply a preprocessing step:

\[
w(t, N, 1, 0) = \prod_{e \in E(\tilde{T}_{N,t})} (1 + \gamma_e) - \sum_{0 \leq k \leq 0 \leq \ell \leq N \atop k + \ell > 1} w(t, N, k, \ell).
\]

Exactly the same formal expression applies to \( w(t, N, 0, 1) \). The identity merely expresses complementation: the weight of subgraphs with \( k + \ell = 1 \) equals the total weight of subgraphs, less the weight of subgraphs with \( k + \ell > 1 \). With this modification, the recurrence becomes well founded.

Since only a polynomial number of distinct tuples \((t, N, k, \ell)\) arise, the recurrence can be solved by dynamic programming in polynomial time. Note that the
computation of $\prod_{e \in E(\Gamma_{x,t})} (1 + \gamma_e)$ in the preprocessing step can be done exactly since $\rho$ is a rational and $N^{1/4}$ is an integer.

Once the $w(t, N, k, \ell)$ values are computed, we wish to estimate

$$Z^k = \sum_{\ell=0}^{N} w(t, N, k, \ell) q^\ell.$$ 

Let $\varepsilon$ be the desired accuracy in the approximation-preserving reduction. Compute a rational $\hat{q}$ in the range $e^{-\varepsilon/N} q \leq \hat{q} \leq e^{\varepsilon/N} q$ and return $Z^k = \sum_{\ell=0}^{N} w(t, N, k, \ell) \hat{q}^\ell$. \hfill $\square$

It will also be necessary for us to approximate the critical edge probability $\rho$, so that, with this approximation, the graph $\Gamma$ approximately satisfies Equation (7) in Lemma 9. The following lemma shows that this is possible.

**Lemma 13.** Suppose $q > 2$ is an efficiently computable real. Fix $\gamma > 0$ and let $\lambda = \lambda_c + (q - \lambda_c)/2$. Suppose that $\chi > 0$ is rational. Consider the gadget $\Gamma$ from Section 4.3 with parameters $t$, $N$, and $\rho$. There is a randomised algorithm whose running time is at most a polynomial in $\chi^{-1}$, $N$ and $t$ which takes input $N$ and $t$ (where it is assumed that $N^{1/4}$ is an integer and that $N \geq \max\{t^4, N_0\}$ for the constant $N_0$ from Lemma 7) and, with probability at least $3/4$, computes a rational $\rho$ in the range $[N^{-\delta}, \lambda/N]$ such that, if $A$ is drawn from $\text{RC}(\Gamma; q, p)$, then

$$e^{-\chi \gamma} \leq \frac{\Pr(Y(A) = 1)}{\Pr(Y(A) = t)} \leq e^{\chi \gamma}. \tag{18}$$

**Proof.** First, we establish a useful preliminary fact. If $0 < \rho \leq 1/4$, $0 < \delta \leq 1$ and $1 + \rho \leq \hat{\rho} \leq (1 + \delta) \rho$ then

$$e^{-2\delta} \frac{\rho}{1 - \rho} \leq \frac{\hat{\rho}}{1 - \hat{\rho}} \leq e^{2\delta} \frac{\rho}{1 - \rho}. \tag{19}$$

To see this, note that $e^\delta \rho \leq 3/4$ so $1 - e^\delta \rho \geq \rho$. Then letting $x = e^\delta - 1$,

$$1 - \rho \leq 1 - \rho + x(1 - e^\delta \rho) - x\rho = (1 + x)(1 - (1 + x)\rho) = e^\delta (1 - e^\delta \rho),$$

which, together with $1 + \delta \leq e^\delta$, gives the right-most inequality in (19). Similarly, $\rho e^{-\delta} \leq 1/4$ so $1 - e^{-\delta} \rho \geq \rho$ so letting $x = 1 - e^{-\delta}$,

$$1 - \rho \geq 1 - \rho + x\rho - x(1 - e^{-\delta} \rho) = (1 - x)(1 - (1 - x)\rho) = e^{-\delta} (1 - e^{-\delta} \rho),$$

which, together with $e^{-\delta} \leq \frac{1}{1 + \delta}$, gives the left-most inequality in (19).

Now consider the gadget $\Gamma$ with parameters $t$, $N$, and $\rho$. Let $n$ and $m$ be the number of vertices, and edges, respectively, of the variant $\Gamma$. We will be interested in three quantities which depend upon $\rho$: $Z^1$, $Z^t$ and $\zeta = \frac{\Pr(Y(A) = 1)}{\Pr(Y(A) = t)}$. For this proof only, we will denote these as $Z^1_\rho$, $Z^t_\rho$ and $\zeta_\rho$ to make the dependence on $\rho$ clear. By Lemma 11, $\zeta_\rho = Z^t_\rho/Z^t_\rho$. 


Lemma \([\text{12}]\) provides an FPRAS for computing \(\zeta_\varrho\) for rational parameters \(\varrho\). We will set \(\delta = \chi/(16(n+m))\) and we will use the FPRAS with accuracy parameter \(\chi/4\) to estimate \(\zeta_\varrho\) for every rational \(\varrho\)

\[
\varrho = \frac{1}{N^3(1+\delta)^\mu}
\]

where \(\mu\) is a non-negative integer with \(\frac{1}{N^3}(1+\delta)^\mu \leq \frac{1}{\lambda}\). Note that the number of integers \(\mu\) in this range is \(O(\delta^{-1}\log N)\). We will first “power up” the success probability of the FPRAS using standard techniques \([\text{21}]\) so that the probability that any of these \(O(\delta^{-1}\log N)\) calls to the FPRAS fails is at most \(3/4\). If we find a rational \(\hat{\varrho}\) for which our estimate \(\hat{\zeta}_{\hat{\varrho}}\) satisfies \(e^{-\chi/4}\gamma \leq \hat{\zeta}_{\hat{\varrho}} \leq e^{\chi/2}\gamma\), we will return this value \(\hat{\varrho}\). Clearly, this will be an acceptable answer, since \(e^{-\chi/4}\zeta_{\hat{\varrho}} \leq \hat{\zeta}_{\hat{\varrho}} \leq e^{\chi/4}\zeta_{\hat{\varrho}}\).

To finish the proof, we just need to argue that we will try a \(\hat{\varrho}\) which we accept. Lemma \([\text{4}]\) guarantees that there is a value \(\varrho^* \in [N^{-3}, \lambda N^{-1}]\) such that \(\zeta_{\varrho^*} = \gamma\). In our computations, we will compute \(\zeta_{\hat{\varrho}}\) for some \(\hat{\varrho}\) in the range \(\frac{1}{N^3}\varrho^* \leq \hat{\varrho} \leq (1+\delta)\varrho^*\). Recall that every edge \(e\) of the gadget has edge-weight \(\gamma_e = p(e)/(1 - p(e))\), and that \(p(e) = \varrho\) for edges within the part of the gadget corresponding to the clique \(K\). The weights of the other edges do not depend on \(\varrho\). Thus, by \((\text{19})\) and by analogy to \((\text{15})\), \(e^{-\chi/4}\zeta_{\varrho^*} \leq \zeta_{\hat{\varrho}} \leq e^{\chi/4}\zeta_{\varrho^*}\). Thus, \(\hat{\varrho}\) will be accepted.

\[\square\]

6. APPROXIMATION-PRESERVING REDUCTIONS AND \#BIS

Our main tool for understanding the relative difficulty of approximation counting problems is approximation-preserving reductions. We use Dyer, Goldberg, Greenhill and Jerrum’s notion of approximation-preserving reduction \([\text{8}]\). Suppose that \(f\) and \(g\) are functions from \(\Sigma^*\) to \(\mathbb{R}\). An “approximation-preserving reduction” from \(f\) to \(g\) gives a way to turn an FPRAS for \(g\) into an FPRAS for \(f\). Here is the definition. An approximation-preserving reduction from \(f\) to \(g\) is a randomised algorithm \(A\) for computing \(f\) using an oracle for \(g\). The algorithm \(A\) takes as input a pair \((x, \varepsilon) \in \Sigma^* \times (0, 1)\), and satisfies the following three conditions: (i) every oracle call made by \(A\) is of the form \((w, \delta)\), where \(w \in \Sigma^*\) is an instance of \(g\), and \(0 < \delta < 1\) is an error bound satisfying \(\delta^{-1} \leq \text{poly}(|x|, \varepsilon^{-1})\); (ii) the algorithm \(A\) meets the specification for being a randomised approximation scheme for \(f\) (as described above) whenever the oracle meets the specification for being a randomised approximation scheme for \(g\); and (iii) the run-time of \(A\) is polynomial in \(|x|\) and \(\varepsilon^{-1}\).

If an approximation-preserving reduction from \(f\) to \(g\) exists we write \(f \lesssim_{\text{AP}} g\), and say that \(f\) is AP-reducible to \(g\). Note that if \(f \lesssim_{\text{AP}} g\) and \(g\) has an FPRAS then \(f\) has an FPRAS. (The definition of AP-reduction was chosen to make this true). If \(f \lesssim_{\text{AP}} g\) and \(g \lesssim_{\text{AP}} f\) then we say that \(f\) and \(g\) are AP-interreducible, and write \(f \equiv_{\text{AP}} g\).
The definitions allow us to construct approximation-preserving reductions between problems \( f \) and \( g \) with real parameters without insisting that the parameters themselves be efficiently approximable. Nevertheless, some of our results restrict attention to efficiently approximable parameters. According to the definition, approximation-preserving reductions may use randomisation. Nevertheless, the reductions that we present in this paper are deterministic, except for where they make use of an FPRAS to approximate a real parameter. A word of warning about terminology: Subsequent to [8], the notation \( \leq_{\text{AP}} \) has been used to denote a different type of approximation-preserving reduction which applies to optimisation problems. We will not study optimisation problems in this paper, so hopefully this will not cause confusion.

Dyer et al. [8] studied counting problems in \( \#P \) and identified three classes of counting problems that are interreducible under approximation-preserving reductions. The first class, containing the problems that admit an FPRAS, are trivially AP-interreducible since all the work can be embedded into the reduction (which declines to use the oracle). The second class is the set of problems that are AP-interreducible with \( \#\text{Sat} \), the problem of counting satisfying assignments to a Boolean formula in CNF. Zuckerman [30] has shown that \( \#\text{Sat} \) cannot have an FPRAS unless RP = NP. The same is obviously true of any problem to which \( \#\text{Sat} \) is AP-reducible.

The third class appears to be of intermediate complexity. It contains all of the counting problems expressible in a certain logically-defined complexity class. Typical complete problems include counting the downsets in a partially ordered set [8], computing the partition function of the ferromagnetic Ising model with varying interaction energies and local external magnetic fields [11] and counting the independent sets in a bipartite graph, which is defined as follows.

**Problem:** \#BIS.

**Instance:** A bipartite graph \( B \).

**Output:** The number of independent sets in \( B \).

We showed in [8] that \#BIS is complete for the logically-defined complexity class \( \#\text{RHΠ}_1 \) with respect to approximation-preserving reductions. We presume that there is no FPRAS for \#BIS, but this is not known.

7. Our results

The next three sections of this paper give an approximation-preserving reduction from \#BIS to the problem \( \text{TUTTE}(q, \gamma) \).

We start by defining the problem of computing the Tutte polynomial of a uniform hypergraph \( H \) with fixed positive edge weights. For fixed positive real numbers \( q \) and \( \gamma \) the problem \( \text{UNIFORMHYPERTUTTE}(q, \gamma) \) is defined as follows.

**Problem:** \( \text{UNIFORMHYPERTUTTE}(q, \gamma) \).

**Instance:** A uniform hypergraph \( H = (V, \mathcal{E}) \).
Output: $Z_{\text{Tutte}}(H; q, \gamma)$, where $\gamma$ is the constant function with $\gamma_f = \gamma$ for every $f \in E$.

In Section 8 we show that, for every $q > 1$, there is an approximation-preserving reduction from $\#\text{BIS}$ to $\text{UNIFORMHYPERTUTTE}(q, q - 1)$.

$\text{UNIFORMHYPERTUTTE}(q, \gamma)$ is the problem of computing the Tutte polynomial of a uniform hypergraph. The most difficult part of the paper is reducing this approximation problem to the problem of approximately computing the (multivariate) Tutte polynomial of an undirected graph (a 2-uniform hypergraph). Section 8 shows that for any positive real numbers $q > 2$ and $\gamma > 0$, there is an approximation-preserving reduction from $\text{UNIFORMHYPERTUTTE}(q, \gamma)$ to the following problem.

Problem: $\text{TWOWEIGHTFERROTUTTE}(q)$.
Instance: Graph $G = (V, E)$ with an edge-weight function $\gamma' : E \to \{\gamma', \gamma''\}$ where $\gamma'$ and $\gamma''$ are rationals in the interval $[|V|^{-3}, 1]$.
Output: $Z_{\text{Tutte}}(G; q, \gamma')$.

Finally, Section 10 gives an approximation-preserving reduction from the problem $\text{TWOWEIGHTFERROTUTTE}(q)$ to the problem $\text{TUTTE}(q, \gamma)$ for any $q > 2$ and $\gamma > 0$.

8. APPROXIMATELY COMPUTING THE TUTTE POLYNOMIAL OF A UNIFORM HYPERGRAPH

We first define a parameterised version of $\#\text{BIS}$, and also a restricted version of this in which vertices on the right-hand side are required to have the same degree.

Problem: $\#\text{BIS}(\mu)$.
Instance: Bipartite graph $B$.
Output: $Z_{\text{IS}}(B; \mu) = \sum I \mu^{|I|}$, where the sum is over all independent sets $I$ in $B$.

Problem: $\#\text{SEMIREGULARBIS}(\mu)$.
Instance: Bipartite graph $B = (U, V, E)$ in which every vertex in $V$ has the same degree.
Output: $Z_{\text{IS}}(B; \mu) = \sum I \mu^{|I|}$, where the sum is over all independent sets $I$ in $B$.

Lemma 14. Suppose that $\mu > 0$ is efficiently approximable. Then $\#\text{BIS} \leq_{\text{AP}} \#\text{SEMIREGULARBIS}(\mu)$.

Proof. In \cite[Lemma 15]{8}, we gave an AP-reduction from $\#\text{BIS}$ to $\#\text{BIPARTITEMAXIS}$, the problem of counting maximum independent sets in a bipartite graph. Here, we give two AP-reductions — first, a reduction from $\#\text{BIPARTITEMAXIS}$ to $\#\text{BIS}(\mu)$, and then a reduction from the intermediate problem $\#\text{BIS}(\mu)$ to $\#\text{SEMIREGULARBIS}(\mu)$. 


Let $B$ be an instance of \texttt{#BIS} with $n$ vertices and $m$ edges and let $\varepsilon$ be the desired accuracy of the approximation-preserving reduction. Let $\xi$ be the size of a maximum independent set of $B$ and let $Y$ be the number of maximum independent sets. To do the construction, we first need a rough estimate of $\mu$, so compute a rational value $\tilde{\mu}$ in the range $\frac{3}{4}\mu \leq \tilde{\mu} \leq \frac{5}{2}\mu$. Let $s$ be an integer satisfying

$$s - 1 \leq \left\lceil \frac{n + 3}{\lg(1 + 2\tilde{\mu}/3)} \right\rceil \leq s.$$ 

Note that $s \geq \frac{n + 3}{\lg(1 + \mu/2)}$. Let $B'$ be the graph with vertex set $\{(u, i) \mid u \in V(B), i \in [s]\}$ and edge set $\{((u, i), (v, j)) \mid (u, v) \in E(B)\}$. Next, we need a more accurate estimate of $\mu$. Compute a rational number $\hat{\mu}$ in the range $e^{-\varepsilon/(60ns)}\mu \leq \hat{\mu} \leq e^{\varepsilon/(60ns)}\mu$. Now since there are $ns$ vertices in $B'$,

$$e^{-\varepsilon/60}Z_{IS}(B'; \mu) \leq Z_{IS}(B'; \tilde{\mu}) \leq e^{\varepsilon/60}Z_{IS}(B'; \mu).$$

Thus, by using our oracle for \texttt{#BIS}(\mu) with accuracy parameter $\varepsilon/60$, we can compute a value $Z$ satisfying

$$e^{-\varepsilon/30}Z_{IS}(B'; \tilde{\mu}) \leq Z \leq e^{\varepsilon/30}Z_{IS}(B'; \hat{\mu}).$$

Now note that every independent set of $B'$ points out an independent set of $B$. The vertex $u$ of $B$ is in the independent set of $B'$ if there is at least one vertex $(u, i)$ that is in the independent set of $B'$. A size-$k$ independent set of $B$ thus makes a contribution of $((1 + \hat{\mu})^s - 1)^k$ to $Z_{IS}(B'; \hat{\mu})$. Since the number of independent sets of $B'$ is at most $2^n$, we have

$$Y \leq \frac{Z_{IS}(B'; \hat{\mu})}{((1 + \hat{\mu})^s - 1)^\xi} \leq Y + \frac{2^n}{((1 + \hat{\mu})^s - 1)} \leq Y + \frac{1}{4},$$

where the final inequality follows from the definition of $s$ since $\hat{\mu} \geq \mu/2$. The following simple procedure now gives a sufficiently accurate estimate to $Y$. Take the value $Z$ from Equation (20), divide it by $((1 + \hat{\mu})^s - 1)^\xi$ and round down to the nearest integer. The fact that the accuracy is sufficient follows from (20) and (21). See [8, Theorem 3].

Finally, we present an AP-reduction from $\texttt{#BIS}(\mu)$ to $\texttt{SemiRegularBIS}(\mu)$. Let $B = (U, V, E)$ be an $n$-vertex instance of $\texttt{#BIS}(\mu)$ in which the maximum degree of a vertex in $V$ is $d > 1$. We’ll construct $B' = (U', V', E')$ – an instance of $\texttt{SemiRegularBIS}(\mu)$ in which every vertex in $V'$ has degree $d$. Let $\varepsilon$ be the desired accuracy of the approximation-preserving reduction. As above, compute a rational value $\mu$ such that $\frac{3}{4}\mu \leq \mu \leq \frac{5}{2}\mu$. Let $\mu^-$ denote the computed lower bound $\frac{4}{5}\mu$ and let $\mu^+$ denote the computed upper bound $\frac{5}{3}\mu$. Let $D(x) = (1 + x)^{d - 1}$, $U(x) = xD(x)$, $L(s, x) = (1 + x)^s$, and $Y(s, x) = L(s, x) + D(x) - 1$. We start by computing an integer $s$ such that

$$\frac{1}{L(s, \mu^-)}\max(D(\mu^+) - 1, U(\mu^+)) \leq \frac{\varepsilon}{6dn}.$$
Note that $s = O(n\epsilon^{-1})$ so we can efficiently compute such a value $s$ by starting with $s = 1$, and increasing $s$ one-by-one until we find a value for which (22) holds.

Now let $\Psi$ be a complete bipartite graph with vertex sets $\{z_1, \ldots, z_d\}$ and $\{y_1, \ldots, y_s\}$. For each vertex $v \in V$ of degree $\delta$, take $d - \delta$ new copies of $\Psi$, and attach $v$ to vertex $z_1$ of each copy. Let $g < dn$ be the number of copies of $\Psi$ that get included in $B'$.

Now let $\Psi$ be a complete bipartite graph with vertex sets $\{z_1, \ldots, z_d\}$ and $\{y_1, \ldots, y_s\}$. For each vertex $v \in V$ of degree $\delta$, take $d - \delta$ new copies of $\Psi$, and attach $v$ to vertex $z_1$ of each copy. Let $g < dn$ be the number of copies of $\Psi$ that get included in $B'$.

Now $Z_{IS}(\Psi; \mu) = L(s, \mu) + (1 + \mu)D(\mu) - 1$ and the total contribution to $Z_{IS}(\Psi; \mu)$ from independent sets including the vertex $z_1$ is $U(\mu)$. Let $Y = Z_{IS}(\Psi; \mu) - U(\mu) = Y(s, \mu)$. Clearly, $Y$ is the total contribution to $Z_{IS}(\Psi; \mu)$ from independent sets not including the vertex $z_1$.

Then $Z_{IS}(B; \mu) \leq Z_{IS}(B'; \mu) \leq Z_{IS}(B; \mu)Z_{IS}(\Psi; \mu)^g$ so

$Z_{IS}(B; \mu) \leq \frac{Z_{IS}(B'; \mu)}{Y^g} \leq \frac{Z_{IS}(\Psi; \mu)}{Y^g} \leq Z_{IS}(B; \mu)$

Now

- $Z_{IS}(B'; \mu)$ may be estimated with accuracy parameter $\epsilon/3$ using the oracle.
- $Y^g$ may be estimated directly (to the same accuracy) by computing a value $\hat{\mu}$ satisfying $e^{-\epsilon/(6dn)} \mu \leq \hat{\mu} \leq e^{\epsilon/(6dn)} \mu$ which ensures that $e^{-\epsilon/(6dn)} L(s, \mu) \leq L(s, \hat{\mu}) \leq e^{\epsilon/(6dn)} L(s, \mu)$

And noting from (22) that

$L(s, \mu) \leq Y = L(s, \mu) \left(1 + \frac{D(\mu) - 1}{L(s, \mu)}\right) \leq L(s, \mu) e^{\epsilon/(6dn)}$.

- Finally, (22) gives

$\left(\frac{Z_{IS}(\Psi; \mu)}{Y}\right)^g = \left(1 + \frac{U(\mu)}{Y(s, \mu)}\right)^g \leq e^{\epsilon/3}$,

which finishes the reduction.

We are interested in the situation $\mu > 1$ so that we have $q = \mu + 1 > 2$ in the parameters of $\text{UniformHyperTutte}(q, \gamma)$ in Lemma 15 below.

**Lemma 15.** Suppose that $\mu > 0$ is efficiently approximable. Then

$\#\text{SemiRegularBIS}(\mu) \leq AP \text{UniformHyperTutte}(\mu + 1, \mu)$.

**Proof.** Let $B = (U, V, E)$ be an instance of $\#\text{SemiRegularBIS}(\mu)$. Let $H = (V, \mathcal{E})$ be an instance of $\text{UniformHyperTutte}(\mu + 1, \mu)$ constructed as follows. Let $s$ be a new vertex that is not in $U \cup V$ and let $V = U \cup \{s\}$. For $v \in V$, let
\( \Gamma(v) = \{u \in U : (u, v) \in E\} \) and let \( F_v = \Gamma(v) \cup \{s\} \). Let \( \mathcal{E} = \bigcup_{v \in V} F_v \). We will show below that

\[
Z_{IS}(B, \mu) = (\mu + 1)^{-1}Z_{Tutte}(H; \mu + 1, \mu).
\]

(23)

Now let \( \varepsilon \) be the desired accuracy in the approximation-preserving reduction. To complete the reduction, we first compute a value \( x \) in the range

\[
e^{-\varepsilon/2}(\mu + 1)^{-1} \leq x \leq e^{\varepsilon/2}(\mu + 1)^{-1}.
\]

This is easy to do since \( \mu \) is efficiently approximable and

\[
e^{-\varepsilon/2}(\mu + 1)^{-1} \leq \frac{1}{e^{\varepsilon/2}\mu + 1} \quad \text{and} \quad \frac{1}{e^{-\varepsilon/2}\mu + 1} \leq \frac{e^{\varepsilon/2}}{\mu + 1}.
\]

Then we use the oracle to estimate \( Z_{Tutte}(H; \mu + 1, \mu) \) with accuracy parameter \( \varepsilon/2 \).

We finish the proof by establishing (23). Let \( S \subseteq V \). The contribution to \( Z_{IS}(B, \mu) \) from independent sets \( I \) with \( I \cap V = S \) is

\[
\mu^{|S|}(\mu + 1)^{|U| - |\Gamma(S)|},
\]

where \( \Gamma(S) = \bigcup_{v \in S} \Gamma(v) \). That is,

\[
Z_{IS}(B, \mu) = \sum_{S \subseteq V} \mu^{|S|}(\mu + 1)^{|U| - |\Gamma(S)|}.
\]

On the other hand, the contribution to \( Z_{Tutte}(H; \mu + 1, \mu) \) from the hyperedge set \( \mathcal{E} = \{F_v : v \in S\} \) is

\[
\mu^{|S|}(\mu + 1)^{|\mathcal{V}, \mathcal{E}|} = \mu^{|S|}(\mu + 1)^{|U| - |\Gamma(S)| + 1},
\]

since the vertices in \( \Gamma(S) \) together with \( s \) form one connected component, and all other vertices are isolated. Thus

\[
Z_{Tutte}(H; \mu + 1, \mu) = \sum_{S \subseteq V} \mu^{|S|}(\mu + 1)^{|U| - |\Gamma(S)| + 1}.
\]

\[\square\]

9. **Approximately computing the multivariate Tutte polynomial of a graph**

**Lemma 16.** Suppose that \( q > 2 \) and \( \gamma > 0 \) are efficiently approximable. Then \( \text{UniformHyperTutte}(q, \gamma) \preceq_{\text{AP}} \text{TwoWeightFerroTutte}(q) \).

**Proof.** Start with a \( t \)-uniform hypergraph \( H = (\mathcal{V}, \mathcal{E}) \). \( H \) is an instance of the problem \( \text{UniformHyperTutte}(q, \gamma) \). For convenience, let \( \mathcal{V} = \{v_1, \ldots, v_n\} \) and \( \mathcal{E} = \{f_1, \ldots, f_m\} \).

The basic idea is to simulate each hyperedge with a copy of the gadget \( \Gamma \) from Section 4.3 (actually with the variant \( \tilde{\Gamma} \) from Section 4.4). We start by setting the parameters. Let \( \varepsilon \) be the desired accuracy in the approximation-preserving reduction (see the definition of an AP-reduction in Section 8). Let \( \chi = \varepsilon/(4m) \).
The tolerance $\eta$, which is a parameter to Lemma \ref{lem:two-weight-ferro-tutte}, may be chosen to be any value such that
\[
1 + \frac{\eta(1 + e^x\gamma)}{1 - \eta} \leq e^x.
\]
Note that this inequality is achieved for $\eta = O(\varepsilon/m)$.

The reduction will construct an instance $\widehat{G} = (V, E)$ of the target problem \textsc{TwoWeightFerroTutte}(q) along with an appropriate edge-weight function $\gamma' : E \to \{\gamma', \gamma''\}$. We will show that using an oracle to approximate the solution to this instance with accuracy parameter $\delta = \varepsilon/2$ enables us to get within $\exp(\pm \varepsilon)$ of $Z_{\text{Tutte}}(H; q, \gamma)$.

Let $\lambda = \lambda_c + (q - \lambda_c)/2$ and let $N_0$ be the quantity from Lemmas \ref{lem:gamma-equal-0} and \ref{lem:two-weight-ferro-tutte}. Let $N$ be the smallest integer greater than $\max\{t^{16}, \eta^{-1/8}, N_0\}$ for which $N^{1/4}$ is an integer. Using the algorithm from Lemma \ref{lem:gamma-equal-0} (suitably powered up so that its failure probability is at most $3^{-8}$) we can compute a rational $q$ in the range $[N^{-3}, \lambda/N]$ such that, if $A$ is drawn from $\text{RC}(\Gamma; q, p)$, then Equation \eqref{eq:gamma-equal-0} holds. Note from Lemma \ref{lem:two-weight-ferro-tutte} that Equation \eqref{eq:gamma-equal-0} holds.

The construction is as follows. For every $j \in [m]$, let $K_j$ be a set of $N$ vertices and let $V_j = K_j \cup f_j$. Let $E_j = K_j^{(2)} \cup K_j \times f_j$ and let $G_j$ be the graph $(V_j, E_j)$. Note that $G_j$ is a copy of the graph $F$ from Section 4.4 — $f_j$ is the set of terminals. Now we construct the graph $\widehat{G} = (V, E)$ where $V = V \cup \bigcup_{j \in [m]} K_j$ and $E = \bigcup_{j \in [m]} E_j$. Define
\[
\hat{p}(e) = \begin{cases} q_j & \text{if } e \in K_j^{(2)} \text{ for some } j \in [m], \\ N^{-3/4} & \text{if } e \in K_j \times f_j \text{ for some } j \in [m]. \end{cases}
\]
Let $\gamma_e = \hat{p}(e)/(1 - \hat{p}(e))$, and let $\gamma' = \{\gamma_e\}_{e \in E}$. Note that $\gamma' = q/(1 - q)$ and $\gamma'' = N^{-3/4}/(1 - N^{-3/4})$. These are both rational values in the interval $[|V|^{-3}, 1]$, as required, since $\lambda/N \leq 1/2$ by Lemma \ref{lem:gamma-equal-0}.

Denote by $\Pi_j$ the set of all partitions of $f_j$. Given $\pi_j \in \Pi_j$, let $A_j^{\pi_j}$ be the collection of all edge subsets $A_j \subseteq E_j$ that induce the partition $\pi_j$ (into connected components) on $f_j$. For $A_j \subseteq E_j$, let $Y(A_j)$ denote the number of connected components that contain terminals (vertices in $f_j$) in the graph $(V_j, A_j)$. Let $\kappa'(V_j, A_j) = \kappa(V_j, A_j) - Y(A_j)$ be the number of remaining connected components (that do not contain terminals). For partitions $\pi$ and $\pi'$, $\pi \vee \pi'$ denotes the finest partition that is a common coarsening of $\pi$ and $\pi'$. If two elements are together in $\pi$ or in $\pi'$ then they are together in the coarsening. The “coarsest” partition has one block and the “finest” partition consists of singleton blocks. Technically, $\pi_j \in \Pi_j$ is a partition of $f_j$ but we consider it as a partition of the entire vertex set $V$ by extending it with singleton blocks.

Now
\[
Z_{\text{Tutte}}(\widehat{G}; q, \gamma') = \sum_{A \subseteq E} q^{\kappa'(V, A)} \prod_{e \in E} \gamma_e.
\]
Let $A_j = A \cap E_j$. Let $\gamma(A_j) = \prod_{e \in A_j} \gamma_e$. Then

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') = \sum_{\pi_j \in \Pi_j} \sum_{A_j \in A_j^{\pi_j}} q^{\kappa(V \cup \bigcup_{j \in [m]} A_j)} \prod_{j \in [m]} \gamma(A_j)$$

$$= \sum_{\pi_j \in \Pi_j} q^{\kappa(\pi_1 \lor \cdots \lor \pi_m)} \sum_{A_j \in A_j^{\pi_j}} \prod_{j \in [m]} \gamma(A_j) q^{\kappa(V_j \setminus A_j)},$$

where $\kappa(\pi_1 \lor \cdots \lor \pi_m)$ denotes the number of blocks in the partition $\pi_1 \lor \cdots \lor \pi_m$.

Now let

$$Z_j(\pi_j) = \sum_{A_j \in A_j^{\pi_j}} \gamma(A_j) q^{\kappa(V_j \setminus A_j)}$$

and pull out the contribution of each $j$ to get

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') = \sum_{\pi_j \in \Pi_j} q^{\kappa(\pi_1 \lor \cdots \lor \pi_m)} \prod_{j=1}^m Z_j(\pi_j).$$

At this point, we make some connections to Section 4.4. Let $\Pi^k_j$ denote the set of all partitions of $f_j$ into $k$ blocks and let $Z^k_j = \sum_{\pi_j \in \Pi^k_j} Z_j(\pi_j)$. From Lemma [11] we deduce that

$$\frac{Z^k_j}{Z_j^k} = \frac{\Pr(Y(A) = k)}{\Pr(Y(A) = k')}.$$  

where $A$ is drawn from $\text{RC}(\Gamma; q, p)$ for the function $p$ from Section 4.3.

Denote by $\bot_j$ the finest partition (with $t$ blocks) of $f_j$ and by $\top_j$ the coarsest partition (with one block) on the same set. By Lemma [13] Equation (18) and Equation (25) we have

$$e^{-x} \gamma \leq \frac{Z_j(\top_j)}{Z_j(\bot_j)} \leq e^x \gamma.$$  

Use the FPRAS from Lemma [12] (again, suitably powered up so that its failure probability is at most $3/8$) to compute a value $\alpha$ which satisfies

$$e^{-x} Z_j(\bot_j) \leq \alpha \leq e^x Z_j(\top_j).$$

Now, define $b_j(0) = \bot_j$ and $b_j(1) = \top_j$. Then, by restricting the sum in (24), to terms satisfying $\pi_j \in \{\bot_j, \top_j\}$ for all $j$, we get

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') \geq \sum_{z \in \{0, 1\}^m} q^{\kappa(b_j(z_1) \lor \cdots \lor b_m(z_m))}.$$
Letting $\|z\|$ denote the Hamming weight of $z$ and using Equation (26), the right-hand side is at least

$$Z(\perp_j)^m e^{-\chi m} \sum_{z \in \{0,1\}^m} q^{\kappa(b_1(z_1) \lor \cdots \lor b_m(z_m)) \gamma \|z\|}.$$  

Using Equation (27), we get

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') \geq \alpha^m e^{-2\chi m} \sum_{z \in \{0,1\}^m} q^{\kappa(b_1(z_1) \lor \cdots \lor b_m(z_m)) \gamma \|z\|} \quad (28)$$

$$= \alpha^m e^{-2\chi m} Z_{\text{Tutte}}(H; q, \gamma).$$

To finish, we just have to show that this lower-bound for $Z_{\text{Tutte}}(\hat{G}; q, \gamma')$ is actually a good estimate because the terms that we threw away in inequality (28) don’t amount to much. For $j \in [m]$, let $\Pi^j_{\perp} = \{\top_j\}$ and let $\Pi^j_0 = \Pi_j \setminus \{\top_j\}$.

Then, starting from (24),

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') = \sum_{z \in \{0,1\}^m} \sum_{\pi_j \in \Pi^j_z} \prod_{j=1}^m Z_j(\pi_j).$$

Now note the partition $\pi_j \in \Pi^j_z$ is refined by $b_j(z_j)$. Thus, $b_1(z_1) \lor \cdots \lor b_m(z_m)$ has at least as many connected components as $\pi_1 \lor \cdots \lor \pi_m$. So we get

$$Z_{\text{Tutte}}(\hat{G}; q, \gamma') \leq \sum_{z \in \{0,1\}^m} \sum_{\pi_j \in \Pi^j_z} \prod_{j=1}^m Z_j(\pi_j)$$

$$= \sum_{z \in \{0,1\}^m} q^{\kappa(b_1(z_1) \lor \cdots \lor b_m(z_m))} \sum_{\pi_j \in \Pi^j_z} \prod_{j=1}^m Z_j(\pi_j)$$

$$= \sum_{z \in \{0,1\}^m} q^{\kappa(b_1(z_1) \lor \cdots \lor b_m(z_m))} \prod_{j=1}^m Z_j(\Pi^j_z),$$

where $Z_j(\Pi^j_z) = \sum_{\pi_j \in \Pi^j_z} Z_j(\pi_j)$. Now, $Z_j(\Pi^j_1) = Z_j(\top_j) \leq e^{2\chi \gamma \alpha}$. Also, $Z_j(\Pi^j_0) = \sum_{k=2}^t Z_j^k$. By Equation (23), this is

$$Z_j^t \left(1 + \sum_{k=2}^{t-1} \frac{\Pr(Y(A) = k)}{\Pr(Y(A) = t)}\right),$$
where \( A \) is drawn from \( RC(\Gamma; q, p) \) for the function \( p \) from Section 4.3. Since \( Z_j^t \leq e^x \alpha \), we have

\[
Z_j(\Pi_j^0) \leq e^x \alpha \left( 1 + \sum_{k=2}^{t-1} \frac{\Pr(Y(A) = k)}{\Pr(Y(A) = t)} \right).
\]

Now by Lemma 10 Equation (8),

\[
\sum_{t-1}^{k=2} \Pr(Y(A) = k) \leq \eta.
\]

Thus, \( \Pr(Y(A) = 1) + \Pr(Y(A) = t) \geq 1 - \eta \) and by Equation (18),

\[
\Pr(Y(A) = t) \geq \frac{1 - \eta}{1 + e^x \gamma}.
\]

Thus,

\[
Z_j(\Pi_j^0) \leq e^x \alpha \left( 1 + \frac{\eta(1 + e^x \gamma)}{1 - \eta} \right) \leq e^{2x} \alpha.
\]

Thus,

\[
Z_{\text{Tutte}}(\hat{G}; q, \gamma') \leq e^{2x} m \alpha^m \sum_{z \in \{0, 1\}^m} q^{\kappa(h_1(z_1) \cdots \kappa(m(z_m)))} \gamma_{\|z\|} = e^{2x} m \alpha^m Z_{\text{Tutte}}(H; q, \gamma).
\]

Thus, if we knew a quantity \( \psi \) in the range

\[
e^{-\varepsilon/2} \leq \frac{\psi}{Z_{\text{Tutte}}(\hat{G}; q, \gamma')} \leq e^{\varepsilon/2},
\]

we would have

\[
e^{-\varepsilon} \leq \frac{\psi \alpha^{-m}}{Z_{\text{Tutte}}(H; q, \gamma)} \leq e^\varepsilon,
\]

so we have completed the AP-reduction. \( \square \)

10. **APPROXIMATELY COMPUTING THE TUTTE POLYNOMIAL OF A GRAPH**

In this section, we complete the approximation-preserving reduction from \#BIS to \( \text{Tutte}(q, \gamma) \) by giving an approximation-preserving reduction from the problem \( \text{TwoWeightFerroTutte}(q) \) to the problem \( \text{Tutte}(q, \gamma) \).

For this, we first need to define what it means to “implement” an edge weight. This description is mainly taken from [13, Section 1.6]. Fix \( q > 2 \). Let \( W \) be a set of edge weights. For example, \( W \) might contain the edge weight \( \gamma \) (a parameter of the problem \( \text{Tutte}(q, \gamma) \)) or it might contain a selection of edge weights that we have already implemented using \( \gamma \). Let \( \gamma^* \) be a weight (which may not be in \( W \)) which we want to “implement”. Suppose that there is a graph \( \Upsilon \), with distinguished vertices \( s \) and \( t \) and an edge-weight function \( \hat{\gamma} : E(\Upsilon) \to W \) such that

\[
\gamma^* = \frac{qZ_{st}(\Upsilon)}{Z_{\text{st}}(\Upsilon)},
\]

where \( Z_{\text{st}}(\Upsilon) \) denotes the contribution to \( Z_{\text{Tutte}}(\Upsilon; q, \hat{\gamma}) \) arising from edge-sets \( A \) in which \( s \) and \( t \) are in the same component. That is, \( Z_{\text{st}}(\Upsilon) = \sum_A \hat{\gamma}(A) q^{\kappa(V, A)} \), where
the sum is over subsets $A \subseteq E(T)$ in which $s$ and $t$ are in the same component. Similarly, $Z_{st}$ denotes the contribution to $Z_{Tutte}(T; q, \hat{\gamma})$ arising from edge-sets $A$ in which $s$ and $t$ are in different components. In this case, we say that $T$ and $\hat{\gamma}$ implement $\gamma^*$ (or even that $W$ implements $\gamma^*$).

The purpose of “implementing” edge weights is this. Let $G$ be a graph with edge-weight function $\gamma$. Let $f$ be some edge of $G$ with edge weight $\gamma_f = \gamma^*$. Suppose that $W$ implements $\gamma^*$. Let $T$ be a graph with distinguished vertices $s$ and $t$ with a weight function $\hat{\gamma}$ satisfying (29). Construct the weighted graph $\tilde{G}$ by replacing edge $f$ with a copy of $T$ (identify $s$ with either endpoint of $f$ (it doesn’t matter which one) and identify $t$ with the other endpoint of $f$ and remove edge $f$). Define the weight function $\tilde{\gamma}$ as follows.

$$\tilde{\gamma}_e = \begin{cases} \hat{\gamma}_e, & \text{if } e \in E(T), \text{ and} \\ \gamma_e, & \text{otherwise.} \end{cases}$$

Then the definition of the multivariate Tutte polynomial gives

$$Z_{Tutte}(\tilde{G}; q, \tilde{\gamma}) = \frac{Z_{st}(T)}{q^2} Z_{Tutte}(G; q, \gamma). \tag{30}$$

So, as long as $Z_{st}(T)$ is easy to evaluate and $q$ is efficiently approximable, approximating the multivariate Tutte polynomial of $\tilde{G}$ with weight function $\tilde{\gamma}$ is essentially the same as approximating the multivariate Tutte polynomial of $G$ with weight function $\gamma$.

Two especially useful implementations are series and parallel compositions. These are explained in detail in [17, Section 2.3]. So we will be brief here. Parallel composition is the case in which $T$ consists of two parallel edges $e_1$ and $e_2$ with endpoints $s$ and $t$ and $\hat{\gamma}_{e_1} = \gamma_1$ and $\hat{\gamma}_{e_2} = \gamma_2$. It is easily checked from Equation (29) that $\gamma^* = (1 + \gamma_1)(1 + \gamma_2) - 1$. Also, the extra factor in Equation (30) cancels, so in this case $Z_{Tutte}(\tilde{G}; q, \tilde{\gamma}) = Z_{Tutte}(G; q, \gamma)$.

Series composition is the case in which $T$ is a length-2 path from $s$ to $t$ consisting of edges $e_1$ and $e_2$ with $\hat{\gamma}_{e_1} = \gamma_1$ and $\hat{\gamma}_{e_2} = \gamma_2$. It is easily checked from Equation (29) that $w^* = \gamma_1 \gamma_2 / (q + \gamma_1 + \gamma_2)$. Also, the extra factor in Equation (30) is $q + \gamma_1 + \gamma_2$, so in this case $Z_{Tutte}(\tilde{G}; q, \tilde{\gamma}) = (q + \gamma_1 + \gamma_2) Z_{Tutte}(G; q, \gamma)$. It is helpful to note that $\gamma^*$ satisfies

$$\left(1 + \frac{q}{\gamma^*}\right) = \left(1 + \frac{q}{\gamma_1}\right) \left(1 + \frac{q}{\gamma_2}\right).$$

We are now ready to prove this lemma.

**Lemma 17.** Suppose that $q > 2$ and $\gamma > 0$ are efficiently approximable. Then $\text{TwoWeightFerroTutte}(q) \leq_{AP} \text{Tutte}(q, \gamma)$.

**Proof.** Let $G = (V, E)$ be an instance of $\text{TwoWeightFerroTutte}(q)$ with edge-weight function $\gamma' : E \rightarrow \{\gamma', \gamma''\}$ where $\gamma'$ and $\gamma''$ are rationals in the interval
We will assume without loss of generality that $|E|$ is sufficiently large with respect to the fixed parameters $q$ and $\gamma$.

Let $\epsilon$ be the desired accuracy in the approximation-preserving reduction. Let $\chi = \epsilon/(4(|V| + |E|^2))$ Let $\tilde{q}$ be a rational in the range $e^{-\chi}q \leq \tilde{q} \leq e^{\chi}q$ and let $\tilde{\gamma}$ be a rational in the range $e^{-\chi}\gamma \leq \tilde{\gamma} \leq e^{\chi}\gamma$. Since $q$ and $\gamma$ are efficiently approximable, the amount of time that it takes to compute $\tilde{q}$ and $\tilde{\gamma}$ is at most a polynomial in $|V|$, $|E|$ and $\epsilon^{-1}$.

The idea of the proof is to show how to use series and parallel compositions from the set $W = \{\tilde{\gamma}\}$ to implement edge-weights $\gamma'^*$ and $\gamma''^*$ satisfying

\begin{equation}
 e^{-\chi}\gamma' \leq \gamma'^* \leq e^{\chi}\gamma',
\end{equation}

and

\begin{equation}
 e^{-\chi}\gamma'' \leq \gamma''^* \leq e^{\chi}\gamma''.
\end{equation}

Letting $\gamma^*$ be the edge-weight function derived from $\gamma'$ by replacing $\gamma'$ with $\gamma'^*$ and $\gamma''$ with $\gamma''^*$, Equation (15) gives

\begin{equation}
 e^{-\epsilon/4}Z_{\text{Tutte}}(G; q, \gamma') \leq Z_{\text{Tutte}}(\tilde{G}; \tilde{q}, \gamma^*) \leq e^{\epsilon/4}Z_{\text{Tutte}}(G; q, \gamma').
\end{equation}

Let $\tilde{\gamma}$ be the edge-weight function which assigns every edge weight $\tilde{\gamma}$. We can think of our implementations as constructing a graph $\tilde{G}$ such that $Z_{\text{Tutte}}(\tilde{G}; \tilde{q}, \tilde{\gamma})$ is equal to the product of $Z_{\text{Tutte}}(\tilde{G}; \tilde{q}, \tilde{\gamma})$ and an easily-computed function of $\tilde{q}$ and $\tilde{\gamma}$. We will ensure that each implementation uses at most $|E|$ edges, so the total number of edges in $\tilde{G}$ is at most $|E|^2$. To finish, we note (from (15)) that

\begin{equation}
 e^{-\epsilon/4}Z_{\text{Tutte}}(\tilde{G}; q, \gamma) \leq Z_{\text{Tutte}}(\tilde{G}; \tilde{q}, \tilde{\gamma}) \leq e^{\epsilon/4}Z_{\text{Tutte}}(\tilde{G}; q, \gamma),
\end{equation}

where $\gamma$ is the constant edge-weight function which assigns every edge weight $\gamma$. We finish the approximation of $Z_{\text{Tutte}}(\tilde{G}; \tilde{q}, \tilde{\gamma})$ by using the oracle to approximate $Z_{\text{Tutte}}(\tilde{G}; q, \gamma)$ using accuracy parameter $\delta = \epsilon/2$.

It remains to show how to do the implementations. Taking

\begin{equation}
 \pi = \frac{\chi}{2|V|^3} \leq \frac{\gamma'\chi}{2} \leq \gamma'(1 - e^{-\chi}),
\end{equation}

We show how to use $W = \{\tilde{\gamma}\}$ to implement an edge-weight $\gamma'^*$ which satisfies $\gamma' - \pi \leq \gamma'^* \leq \gamma'$. This ensures that Equation (31) holds. The implementation of $\gamma''^*$ is similar.

Our implementation is taken from Section 2.1 of our paper [13]. First, we can implement a weight $\gamma_1 \leq \frac{1}{4}$ by taking a series composition of $k$ edges of weight $\tilde{\gamma}$ for sufficiently large $k$. It suffices to take

\begin{equation}
 k = \left\lceil \frac{\log(1 + 4\tilde{q})}{\log(1 + \tilde{q}/\tilde{\gamma})} \right\rceil.
\end{equation}

\footnote{This easily-computed function arises from the extra factor $Z_{\text{st}}(\Upsilon)$ in Equation (29). It is easy to compute because our implementations use only series and parallel composition.
Then implement a weight \( \gamma_j \) by taking a series composition of \( j \) copies of \( \gamma_1 \). The following (recursive) definitions are from [13, Section 2.1] for integers \( j \geq 1 \).

\[
d_j = \left\lfloor \frac{\log((1 + \gamma') \prod_{\ell=1}^{j-1} (1 + \gamma_{\ell})^{-d_{\ell}})}{\log(1 + \gamma_j)} \right\rfloor, \quad \text{and} \quad m = \left\lceil \frac{\log(q(1 + \gamma')/\pi + 1)}{\log(\hat{q}/\gamma_1 + 1)} \right\rceil.
\]

Then the implementation combines, in parallel, \( d_j \) edges with edge-weight \( \gamma_j \), for all \( j \in [m] \).

The calculation in [13, Section 2.1] shows that the implemented value \( \gamma'^* \) satisfies \( \gamma' - \pi \leq \gamma'^* \leq \gamma' \), as required.

Now to finish we need to show that \( d_1 + \cdots + d_m \leq |E| \), which we used above. First, note that the fixed parameters \( q \) and \( \gamma \) give fixed upper and lower bounds on \( \hat{q} \) and \( \hat{\gamma} \). Using the upper bound \( \gamma' \leq 1 \), we see that the value \( m \) is at most logarithmic in \( \pi^{-1} \) which is at most logarithmic in \( |V|, |E|, \) and \( \varepsilon^{-1} \). The calculation in [13, Section 2.1] shows that the same is true of \( d_1, \ldots, d_m \). In fact, there is a fixed upper bound for \( d_j \) depending only on \( \hat{q} \) and \( \hat{\gamma} \). The proof of this fact uses the fact that \( 0 < \gamma_j \leq \gamma_1 \leq \frac{1}{4} \). Since we assumed, without loss of generality, that \( |E| \) is sufficiently large with respect to the fixed parameters \( q \) and \( \gamma \), we conclude that \( d_1 + \cdots + d_m \leq |E| \), as required.

Proof of Theorem 1. Theorem 1 follows from Lemmas 14, 15, 16 and 17.

11. 3-Uniform Hypergraphs

Lemmas 14 and 15 have the following corollary.

Corollary 18. Suppose that \( q > 0 \) is efficiently approximable. Then \( \#\text{BIS} \leq \text{AP\textsc{-UniformHyperTutte}}(q, q-1) \).

Thus, assuming that there is no FPRAS for \#BIS, we can conclude that there is no FPRAS for computing the Tutte polynomial of a uniform hypergraph when the edge-weights are set to \( \gamma = q - 1 \). The Ising model corresponds to the \( q = 2 \) case of the Potts model. Thus, we conclude that there is no FPRAS for computing the partition function of the Ising model on a uniform hypergraph in which every edge has weight 1.

We conclude this paper with a contrasting positive result for 3-uniform hypergraphs. Consider the following problem.

**Problem:** 3-UniformHyperTutte\((q, \gamma)\)

**Instance:** A 3-uniform hypergraph \( H = (\mathcal{V}, \mathcal{E}) \).

**Output:** \( Z_{\text{Tutte}}(H; q, \gamma) \), where \( \gamma \) is the constant function with \( \gamma_f = \gamma \) for every \( f \in \mathcal{E} \).

Lemma 19. Suppose that \( \gamma > 0 \) is efficiently approximable. There is an FPRAS for 3-UniformHyperTutte\((2, \gamma)\).
Proof. Jerrum and Sinclair [20] have given an FPRAS for Tutte$(2, \gamma')$ for every $\gamma' > 0$. We will give a reduction from 3-UNIFORMHYPERTUTTE$(2, \gamma)$ to Tutte$(2, \gamma')$ where $\gamma' = (1 + \gamma)^{1/2} - 1$.

Let $H = (\mathcal{V}, \mathcal{E})$ be a 3-uniform hypergraph, an instance of the target problem 3-UNIFORMHYPERTUTTE$(2, \gamma)$. Let $\gamma$ be the constant function with $\gamma_f = \gamma$ for every $f \in \mathcal{E}$. Let $y = \gamma + 1$. Now, by Observation 2

$$Z_{\text{Tutte}}(H; 2, \gamma) = Z_{\text{Potts}}(H; 2, \gamma) = \sum_{\sigma: \mathcal{V} \to \{0, 1\}} y^{\text{mono}(\sigma)},$$

where mono$(\sigma)$ denotes the number of hyperedges $f \in \mathcal{E}$ that are monochromatic in configuration $\sigma$.

Construct a (multi-)graph $G$ with vertex set $\mathcal{V}$ and edge set

$$E = \bigcup_{(u, v, w) \in \mathcal{E}} \{(u, v), (v, w), (u, w)\}.$$

Let $\gamma'$ be the constant function with $\gamma'_j = \gamma'$ for every $j \in E$. Let $y' = \gamma' + 1 = y^{1/2}$. Now if a hyperedge $f \in \mathcal{E}$ is monochromatic in $\sigma$, it contributes $y'^3$ to the corresponding term in $Z_{\text{Potts}}(G; 2, \gamma')$. Otherwise, it contributes $y'$ to the term. Thus,

$$Z_{\text{Tutte}}(G; 2, \gamma) = Z_{\text{Potts}}(G; 2, \gamma') = y'^{|E|} Z_{\text{Potts}}(H; 2, \gamma),$$

which completes the proof. □

References

[1] Noga Alon, Alan Frieze, and Dominic Welsh. Polynomial time randomized approximation schemes for Tutte-Gröthendieck invariants: the dense case. Random Structures Algorithms, 6(4):459–478, 1995.
[2] B. Bollobás, G. Grimmett, and S. Janson. The random-cluster model on the complete graph. Probab. Theory Related Fields, 104(3):283–317, 1996.
[3] Magnus Bordewich. On the approximation complexity hierarchy. In preparation, 2010.
[4] Christian Borgs, Jennifer T. Chayes, Alan M. Frieze, Jeong Han Kim, Prasad Tetali, Eric Vigoda, and Van H. Vu. Torpid mixing of some Monte Carlo Markov chain algorithms in statistical physics. In FOCS, pages 218–229, 1999.
[5] Sergio Caracciolo, Alan D. Sokal, and Andrea Sportiello. Grassmann integral representation for spanning hyperforests. J. Phys. A, 40(46):13799–13835, 2007.
[6] Prasad Chebolu, Leslie Ann Goldberg, and Russell Martin. Approximately counting stable matchings. In preparation, 2010.
[7] Victor Dalmau. Linear datalog and bounded path duality of relational structures. Logical Methods in Computer Science, 1(1), 2005.
[8] Martin E. Dyer, Leslie Ann Goldberg, Catherine S. Greenhill, and Mark Jerrum. The relative complexity of approximate counting problems. Algorithmica, 38(3):471–500, 2003.
[9] Qi Ge and Daniel Stefankovic. A graph polynomial for independent sets of bipartite graphs. CoRR, abs/0911.4732, 2009.
[10] L.A. Goldberg and M. Jerrum. Counterexample to rapid mixing of the GS Process. Technical note, 2010.
[11] Leslie Ann Goldberg and Mark Jerrum. The complexity of ferromagnetic Ising with local fields. *Combinatorics, Probability & Computing*, 16(1):43–61, 2007.
[12] Leslie Ann Goldberg and Mark Jerrum. Inapproximability of the Tutte polynomial. *Inform. and Comput.*, 206(7):908–929, 2008.
[13] Leslie Ann Goldberg and Mark Jerrum. Inapproximability of the Tutte polynomial of a planar graph. *CoRR*, abs/0907.1724, 2009.
[14] Vivek K. Gore and Mark R. Jerrum. The Swendsen-Wang process does not always mix rapidly. *J. Statist. Phys.*, 97(1-2):67–86, 1999.
[15] Geoffrey Grimmett. Potts models and random-cluster processes with many-body interactions. *J. Statist. Phys.*, 75(1-2):67–121, 1994.
[16] Richard Holley. Remarks on the FKG inequalities. *Comm. Math. Phys.*, 36:227–231, 1974.
[17] Bill Jackson and Alan D. Sokal. Zero-free regions for multivariate Tutte polynomials (alias Potts-model partition functions) of graphs and matroids, 2008.
[18] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Cambridge Philos. Soc.*, 108(1):35–53, 1990.
[19] Svante Janson, Tomasz Łuczak, and Andrzej Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[20] Mark Jerrum and Alistair Sinclair. Polynomial-time approximation algorithms for the Ising model. *SIAM J. Comput.*, 22(5):1087–1116, 1993.
[21] Mark Jerrum, Leslie G. Valiant, and Vijay V. Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theor. Comput. Sci.*, 43:169–188, 1986.
[22] Mark R. Jerrum, Leslie G. Valiant, and Vijay V. Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theoret. Comput. Sci.*, 43(2-3):169–188, 1986.
[23] Steven Kelk. *On the relative complexity of approximately counting H-colourings*. PhD thesis, University of Warwick, University of Warwick, Coventry, UK, July 2004.
[24] Malwina Luczak and Tomasz Łuczak. The phase transition in the cluster-scaled model of a random graph. *Random Structures Algorithms*, 28(2):215–246, 2006.
[25] R. B. Potts. Some generalized order-disorder transformations. *Proc. Cambridge Philos. Soc.*, 48:106–109, 1952.
[26] Alan Sokal. The multivariate Tutte polynomial. In *Surveys in Combinatorics*. Cambridge University Press, 2005.
[27] D. L. Vertigan and D. J. A. Welsh. The computational complexity of the Tutte plane: the bipartite case. *Combin. Probab. Comput.*, 1(2):181–187, 1992.
[28] Dirk Vertigan. The computational complexity of Tutte invariants for planar graphs. *SIAM J. Comput.*, 35(3):690–712 (electronic), 2005.
[29] D. J. A. Welsh. *Complexity: knots, colourings and counting*, volume 186 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.
[30] David Zuckerman. On unapproximable versions of NP-Complete problems. *SIAM Journal on Computing*, 25(6):1293–1304, 1996.

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