A Generalization of the Böcher-Grace Theorem

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Abstract

The Böcher-Grace Theorem can be stated as follows: Let \( p \) be a third degree complex polynomial. Then there is a unique inscribed ellipse interpolating the midpoints of the triangle formed from the roots of \( p \), and the foci of the ellipse are the critical points of \( p \). Here, we prove the following generalization: Let \( p \) be an \( n \)th degree complex polynomial and let its critical points take the form \( \alpha + \beta \cos\frac{k\pi}{n}, \quad k = 1, \ldots, n - 1, \quad \beta \neq 0 \).

Then there is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of \( p \), and the foci of this ellipse are the two most extreme critical points of \( p \): \( \alpha \pm \beta \cos\frac{\pi}{n} \).

1 Introduction

The Böcher-Grace Theorem has been discovered independently by many mathematicians. Recently, proofs have been given by Kalman [3], and Minda and Phelps [6]. Maxime Böcher proved the theorem in 1892 and then John H. Grace proved it in 1902. Surprisingly, a significant generalization was proved prior to both Böcher and Grace by Siebeck in 1864. He showed that the critical points of an \( n \)th degree polynomial are the foci of the curve of class \( n - 1 \) which touches each line segment, joining the roots of the polynomial, at its midpoints. The Böcher-Grace Theorem is the \( n = 3 \) case of Siebeck’s Theorem. Marden’s book [5] gives a wonderful introduction to this material and it has an extensive bibliography.

The purpose of this paper is to give a new and different generalization the Böcher-Grace Theorem.

2 Background

A conformal similarity transformation in the complex plane \( \mathbb{C} \) is a complex function that takes the form

\[ S(z) = \alpha + \beta z \] (1)

for complex numbers \( \alpha \) and \( \beta \neq 0 \). Upon writing \( \beta = re^{i\theta} \) it is easy to see what action is taken upon \( z \) when applying \( S \): a rotation, a uniform scaling of the real and imaginary parts of \( z \), and a translation in the plane. It is understood that conformal similarity transformations preserve the
e concentricity of ellipses, and for this reason any ellipse in the complex plane can be mapped via a conformal similarity transformation to an ellipse with foci ±1.

When studying relationships between the critical points of a polynomial and its roots, conformal similarity transformations play a very important standardizing role. Specifically, if $S$ is a conformal similarity transformation as defined in equation (1) and a complex polynomial $p(z) = \prod_{k=1}^{n} (z - z_k)$ has critical points $r_k, k = 1, \ldots, n - 1$, then the polynomial $P(z) = \prod_{k=1}^{n} (z - S(z_k))$ has critical points $S(r_k), k = 1, \ldots, n - 1$. A proof of this fact can be found in [3].

An affine transformation in $\mathbb{C}$ is a complex function that takes the form

$$
\Phi(z) = \alpha z + \beta \overline{z} + \gamma
$$

(2)

for complex numbers $\gamma, \alpha \neq 0$ and $\beta \neq 0$. Unlike the conformal similarity transformation it is less clear what action is taken when applying $\Phi$ to $z$. However, if $|\alpha| \neq |\beta|$ then we can introduce real parameters $a > |b| > 0$ and $\varphi, \theta \in (-\pi, \pi]$ so that

$$
\alpha = \frac{1}{2}(a + b)e^{i(\varphi + \theta)} \quad \text{and} \quad \beta = \frac{1}{2}(a - b)e^{i(\varphi - \theta)}.
$$

With these new parameters $\Phi$ can be decomposed into three distinct components: a rotation, a purely affine transformation, and a conformal similarity transformation. That is,

$$
\Phi(z) = (S \circ A \circ R)(z)
$$

(3)

where

$$
R(z) = e^{i\theta} z, \quad A(z) = \frac{a + b}{2c} z + \frac{a - b}{2c} \overline{z}, \quad S(z) = c e^{i\varphi} z + \gamma,
$$

and so the image of the unit circle, parametrized by $e^{it}$, under the affine transformation $A$ is an ellipse with eccentricity $c/a$ and foci ±1. Additionally, we note that the images of the rotated roots of unity $e^{i(\theta + 2k\pi/n)}, k = 1, \ldots, n$, under $A$ are

$$
a \cos \left( \theta + \frac{2k\pi}{n} \right) + i b \sin \left( \theta + \frac{2k\pi}{n} \right), \quad k = 1, \ldots, n.
$$

(5)

We require several properties of affine transformations, properties we summarize here, without proof. Affine transformations are invertible if $|\alpha| \neq |\beta|$. Thus not only is the image of the unit circle under an affine transformation an ellipse, but ellipses in $\mathbb{C}$ can be mapped onto the unit circle via an affine transformation. Affine transformations preserve parallel lines, and preserve the midpoints of line segments.

We shall call a convex $n$-gon $P$ affinely regular if it is the affine image of the regular convex polygon formed by the $n$ roots of unity. In Figure 1 we illustrate an affinely regular polygon in the case when $n = 5$. At left is the application of $R(z) = e^{i\theta} z$ to the roots of unity $e^{i2k\pi/n}, k = 1, \ldots, n$; at right is the application of $A$ to these rotated roots of unity; here $S(z) = z$. We also draw attention to the circumscribing and inscribing circles at left, and the corresponding circumscribing and inscribing ellipses at right. The circles have radii 1 and $\cos \pi/n$, while the ellipses have foci ±1 and $\pm \cos \pi/n$, respectively. In light of the summarized properties of affine transformations, we note that since the inscribing circle is tangent at the midpoints of the regular polygon pictured, the inscribed ellipse is also tangent at the midpoints of the corresponding affinely regular polygon.

In fact the existence of an inscribed ellipse interpolating the midpoints of a convex polygon can be shown to characterize affinely regular polygons. To see this, let $P$ be a polygon with an inscribed
ellipse interpolating its midpoints, and let $\mathcal{P}'$ denote the convex polygon formed by the midpoints. Let $\Phi$ be an affine transformation that maps the inscribed ellipse to the unit circle. The images of the vertices of $\mathcal{P}'$ under $\Phi$ lie on the unit circle, and the sides of $\Phi(\mathcal{P})$ are tangent to the unit circle. There are $2n$ right triangles that can be formed using the origin, a vertex from $\Phi(\mathcal{P}')$, and an adjacent vertex from $\Phi(\mathcal{P})$. It is not difficult to see that each of these triangles is congruent to all the other such triangles. Hence, the vertices of $\Phi(\mathcal{P})$ all lie on a common circle, implying that $\mathcal{P}$ is an affinely regular polygon.

Here we list some useful properties of the Chebyshev polynomials that can be found in [4]. The Chebyshev polynomials $T_n(x)$ of the first kind and $U_n(x)$ of the second kind can be defined by

$$T_n(\cos \theta) = \cos(n\theta), \quad U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad n \geq 0.$$  \hfill (6)

From equation (6) we see that $T'_n(x) = nU_{n-1}(x)$, and that the $n-1$ roots of $U_{n-1}(x)$ are \{cos $k\pi/n : k = 1, \ldots, n-1$\}. For complex $z$ the formula for $T_n$ is given by

$$T_n(z) = \frac{1}{2} \left( (z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n \right).$$  \hfill (7)

From our decomposition of affine transformations in equation (3), a general affine transformation can be thought of first as a mapping of the unit circle to an ellipse with foci $\pm 1$, followed by a conformal similarity transformation. There is effectively a one-to-one correspondence between the functions $A$ and the family of ellipses with foci $\pm 1$. This family, parametrized by $s > 0$, is given by

$$\cosh s \cos t + i \sinh s \sin t, \quad 0 \leq t < 2\pi.$$  \hfill (8)

The foci of each ellipse is $\pm 1$ since $\cosh^2 s - \sinh^2 s = 1$. The eccentricity of each ellipse is $0 < \text{sech} s < 1$. Since conformal similarities preserve the eccentricity of an ellipse, it follows that any ellipse in the plane can be mapped by a conformal similarity transformation to a member of this family.

A property of the Chebyshev polynomials $T_n(z)$ that deserves to be better known is that they are periodic on the family of ellipses parametrized in equation (8). We explain this assertion in the following lemma.
Lemma: Let \( a > b > 0 \) with \( c^2 = a^2 - b^2 \). The complex function

\[
f(t) = T_n\left(\frac{a}{c} \cos t + i \frac{b}{c} \sin t\right)
\]

is periodic for real \( t \), with period \( 2\pi/n \). Said another way, the Chebysev polynomial \( T_n(z) \) is periodic on the ellipse

\[
\frac{a}{c} \cos t + i \frac{b}{c} \sin t,
\]

taking on the value \( T_n\left(\frac{a}{c} \cos \theta + i \frac{b}{c} \sin \theta\right) \) exactly \( n \) times.

Proof: Key to this result is the observation that if \( a^2 - b^2 = 1 \) and if \( z = a \cos t + ib \sin t \), then

\[
\sqrt{z^2 - 1} = b \cos t + ia \sin t
\]
(see exercise 5, page 17 in [4]). Equation (11) implies that

\[
z \pm \sqrt{z^2 - 1} = (a \pm b) \cos t + i(a \pm b) \sin t.
\]

Substituting equation (12) into the formula (7) yields

\[
T_n(a \cos t + ib \sin t) = \frac{1}{2} \left( (a + b)^n e^{int} + (a - b)^n e^{-int} \right).
\]

Of course, \( e^{int} \) is periodic with period \( 2\pi/n \).

More generally, if \( a^2 - b^2 = c^2 \neq 1 \), then

\[
T_n\left(\frac{a}{c} \cos \left(\theta + \frac{2k\pi}{n}\right) + i \frac{b}{c} \sin \left(\theta + \frac{2k\pi}{n}\right)\right) = \frac{1}{2} \left( \left(\frac{a + b}{c}\right)^n e^{int\theta} + \left(\frac{a - b}{c}\right)^n e^{-int\theta} \right),
\]

for \( k = 1, \ldots, n. \) ■

At this stage we invite the reader to compare the formula for the affine image of the rotated roots of unity under the affine transformation \( A \) in equation (5) with the argument of \( T_n \) in equation (13). They are the same. This means that \( T_n \) is constant on the affine images of the rotated roots of unity. This is true for every affine transformation \( A \), as defined in equation (3).

In Figure 2 we illustrate some members of the family (8). In addition, we plot the \( n = 5 \) images of a particular rotation of the \( n \) roots of unity under a particular affine transformation \( A \). The polynomial \( T_n \) is constant on these \( n \) points. Along the real axis we also mark the \( n - 1 \) roots of \( T'_n(z) = nU_{n-1}(z) \)

![Figure 2: Ellipses with foci ±1; Affine image of rotated roots of unity.](image)
3 Main Result

Böcher-Grace Theorem for Polygons: Let \( p \) be an \( n^{th} \) degree complex polynomial and let its critical points take the form

\[
\alpha + \beta \cos k\pi/n, \quad k = 1, \ldots, n - 1, \quad \beta \neq 0.
\]  

(14)

There is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of \( p \), and the foci of this ellipse are the two most extreme critical points of \( p \): \( \alpha \pm \beta \cos \pi/n \).

**Proof:** Assume that the critical points of \( p \) take the form in expression (14). Without loss of generalization, exploiting properties of conformal similarity transformations, we may assume \( \alpha = 0 \) and \( \beta = 1 \). Thus \( p'(z) = \gamma U_{n-1}(z) \), and \( p(z) = \gamma/n T_n(z) + \delta \), for complex constants \( \gamma \neq 0 \) and \( \delta \). If we designate by \( z_k, k = 1, \ldots, n \), the roots of \( p \), then \( T_n(z_k) = -n\delta/\gamma \), \( k = 1, \ldots, n \). That is to say, \( T_n \) is constant on the \( z_k, k = 1, \ldots, n \). As noted in our discussion, \( T_n(z) \) can only be constant for \( n \) distinct points in \( \mathbb{C} \), all of which lie on a single ellipse from the family (8). Thus the \( z_k \) must take the form (5). In turn this implies that the roots of \( p \) form an affinely regular polygon \( P \), and hence admit an inscribed ellipse interpolating the midpoints of \( P \). In our discussion of affinely regular polygons, we noted that the foci of the inscribed ellipse are \( \pm \cos \pi/n \), consistent with the assertion of the theorem. \( \blacksquare \)

When \( n = 3 \) the critical points always take the form \( \alpha \pm \beta \cos \pi/3 \). That is, all triangles are affinely regular; all triangles can be mapped to an equilateral triangle with an affine transformation. This is the original Böcher-Grace theorem. When \( n = 4 \) the critical points take the form \( \alpha \pm \beta \cos k\pi/4 \), \( k = 1, 2, 3 \), if and only if the roots of \( p \) form a parallelogram. Said another way, the only quadrilaterals that are affine images of squares are parallelograms.

The form of the critical points in expression (14) can be used to characterize affinely regular polygons. That is, a polygon \( P \) formed from the roots of a polynomial \( p \) is affinely regular if and only if the critical points of \( p \) take the form expression (14). With this observation we conclude this article with a stronger statement than the generalized Böcher-Grace Theorem above.

Böcher-Grace Characterization Theorem: Let \( p \) be an \( n^{th} \) degree polynomial and let \( P \) denote the convex polygon formed by the roots of \( p \). The polygon \( P \) admits an inscribed ellipse interpolating its midpoints if and only if the critical points of \( p \) take the form of the expression expression (14).

**References**

[1] M. Böcher, *Some propositions concerning the geometric representation of imaginaries*, Ann. of Math., 7(1892), pp 70-76.

[2] J. H. Grace, *The roots of a polynomial*, Proc. Cambridge Philos. Soc., 11(1902), pp 352-76.

[3] D. Kalman, *An Elementary Proof of Marden’s Theorem*, American Mathematical Monthly, Vol 115, n4 April 2008, pp 330-338.

[4] J. C. Mason and David C. Handscomb, *Chebyshev Polynomials*, Taylor & Francis, 2002.

[5] M. Marden, *Geometry of Polynomials*, Mathematical Surveryrs Number 3, AMS, 1966 second edition.

[6] D. Minda and S. Phelps, *Triangles, Ellipses, and Cubic polynomials*, American Mathematical Monthly, Vol 115, n8 October 2008, pp 679-689.

[7] J. Siebeck, *Ueber eine neue analytsche Behandlungweise der Brennpunkte*, J. Reine Angew. Math., 64(1864), p 175.