Existence of flips and minimal models for 3-folds in char $p$

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Abstract. We will prove the following results for 3-fold pairs $(X, B)$ over an algebraically closed field $k$ of characteristic $p > 5$: log flips exist for $\mathbb{Q}$-factorial dlt pairs $(X, B)$; log minimal models exist for projective klt pairs $(X, B)$ with pseudo-effective $K_X + B$; the log canonical ring $R(K_X + B)$ is finitely generated for projective klt pairs $(X, B)$ when $K_X + B$ is a big $\mathbb{Q}$-divisor; semi-ampleness holds for a nef and big $\mathbb{Q}$-divisor $D$ if $D - (K_X + B)$ is nef and big and $(X, B)$ is projective klt; $\mathbb{Q}$-factorial dlt models exist for lc pairs $(X, B)$; terminal models exist for klt pairs $(X, B)$; Kollár-Shokurov connectedness holds for birational contractions $X \to Z$ when $-(K_X + B)$ is ample/$Z$; ACC holds for lc thresholds; etc.

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1. Introduction

We work over an algebraically closed field $k$ of characteristic (char) $p > 0$. The pairs $(X, B)$ we consider in this paper always have $\mathbb{R}$-boundaries $B$ unless otherwise stated.

Higher dimensional birational geometry in char $p$ is still largely conjectural. Even the most basic problems such as base point freeness are not solved in general. Ironically though Mori’s work on existence of rational curves which plays an important role in char zero uses reduction mod $p$ techniques. There are two reasons among others which have held back progress in char $p$: resolution of singularities is not known and Kawamata-Viehweg vanishing fails. However, it was expected that one can work out most components of the minimal model program in dimension 3. This is because resolution of singularities is known in dimension 3 and many problems can be reduced to dimension 2 hence one can use special features of surface geometry.

On the positive side there has been some good progress toward understanding birational geometry in char $p$. People have tried to replace the char zero tools that fail in char $p$. For example, Keel [14] developed techniques for dealing with the base point free problem and semi-ampleness questions in general
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without relying on Kawamata-Viehweg vanishing type theorems. On the other hand, motivated by questions in commutative algebra, people have introduced Frobenius-singularities whose definition do not require resolution of singularities and they are very similar to singularities in char zero (cf. [20]).

More recently Hacon-Xu [11] proved the existence of flips in dimension 3 for pairs $(X, B)$ with $B$ having standard coefficients, that is, coefficients in $S = \{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \cup \{ \infty \} \}$, and char $p > 5$. From this they could derive existence of minimal models for 3-folds with canonical singularities. In this paper, we rely on their results and ideas. The requirement $p > 5$ has to do with the behavior of singularities on surfaces, e.g., a klt surface singularity over $k$ of char $p > 5$ is strongly $F$-regular.

Log flips. Our first result is on the existence of flips.

**Theorem 1.1.** Let $(X, B)$ be a $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of char $p > 5$. Let $X \to Z$ be a $K_X + B$-negative extremal flipping projective contraction. Then its flip exists.

The conclusion also holds if $(X, B)$ is klt but not necessarily $\mathbb{Q}$-factorial. This follows from the finite generation below (1.3). The theorem is proved in Section 5 when $X$ is projective. The quasi-projective case is proved in Section 7. We reduce the theorem to the case when $X$ is projective, $B$ has standard coefficients, and some component of $[B]$ is negative on the extremal ray: this case is [11, Theorem 4.12] which is one of the main results of that paper. Paolo Cascini informed us that he together with Gongyo and Schwede have proved 1.1 independently when $B$ has hyperstandard coefficients and $p \gg 0$ (these coefficients are of the form $\frac{n-1}{n} + \sum \frac{lb_i}{n}$ where $n \in \mathbb{N} \cup \{ \infty \}$, $l_i \in \mathbb{Z} \geq 0$ and $b_i$ are in some fixed DCC set).

To prove Theorem 1.1 we actually first prove the existence of *generalized flips* [11, after Theorem 5.6]. See Section 5 for more details.

Log minimal models. In [11, after Theorem 5.6], using generalized flips, a *generalized LMMP* is defined which is used to show the existence of minimal models for varieties with canonical singularities (or for pairs with canonical singularities and “good” boundaries). Using weak Zariski decompositions as in [2], we construct log minimal models for klt pairs in general.

**Theorem 1.2.** Let $(X, B)$ be a klt pair of dimension 3 over $k$ of char $p > 5$ and let $X \to Z$ be a projective contraction. If $K_X + B$ is pseudo-effective/$\mathbb{Z}$, then $(X, B)$ has a log minimal model over $Z$.

The theorem is proved in Section 7. Alternatively, one can apply the methods of [3] to construct log minimal models for lc pairs $(X, B)$ such that $K_X + B \equiv M/\mathbb{Z}$ for some $M \geq 0$. Note that when $X \to Z$ is a semi-stable fibration over a curve and $B = 0$, the theorem was proved much earlier by Kawamata [12].

Remark on Mori fibre spaces. Let $(X, B)$ be a projective klt pair of dimension 3 over $k$ of char $p > 5$ such that $K_X + B$ is not pseudo-effective. An
important question is whether \((X, B)\) has a Mori fibre space. There is an ample \(\mathbb{R}\)-divisor \(A \geq 0\) such that \(K_X + B + A\) is pseudo-effective but \(K_X + B + (1-\epsilon)A\) is not pseudo-effective for any \(\epsilon > 0\). Moreover, we may assume that \((X, B + A)\) is klt as well (8.2). By Theorem 1.2, \((X, B + A)\) has a log minimal model \((Y, B_Y + A_Y)\). Since \(K_Y + B_Y + A_Y\) is not big, \(K_Y + B_Y + A_Y\) is numerically trivial on some covering family of curves by [7] (see also 1.11 below). Again by [7], there is a nef reduction map \(Y \to T\) for \(K_Y + B_Y + A_Y\) which is projective over the generic point of \(T\). Although \(Y \to T\) is not necessarily a Mori fibre space but in some sense it is similar.

**Finite generation, base point freeness, and contractions.** We will prove finite generation in the big case from which we can derive base point freeness and contractions of extremal rays in many cases. These are proved in Section 9.

**Theorem 1.3.** Let \((X, B)\) be a klt pair of dimension 3 over \(k\) of char \(p > 5\) and \(X \to Z\) a projective contraction. Assume that \(K_X + B\) is a \(\mathbb{Q}\)-divisor which is big/Z. Then the relative log canonical algebra \(\mathcal{R}(K_X + B/Z)\) is finitely generated over \(\mathcal{O}_Z\).

Assume that \(Z\) is a point. If \(K_X + B\) is not big, then \(\mathcal{R}(K_X + B/Z)\) is still finitely generated if \(\kappa(K_X + B) \leq 1\). It remains to show the finite generation when \(\kappa(K_X + B) = 2\): this can probably be reduced to dimension 2 using an appropriate canonical bundle formula, for example as in [7].

A more or less immediate consequence of the above finite generation is the following base point freeness.

**Theorem 1.4.** Let \((X, B)\) be a projective klt pair of dimension 3 over \(k\) of char \(p > 5\) and \(X \to Z\) a projective contraction where \(B\) is a \(\mathbb{Q}\)-divisor. Assume that \(D\) is a \(\mathbb{Q}\)-divisor such that \(D\) and \(D - (K_X + B)\) are both nef and big/Z. Then \(D\) is semi-ample/Z.

Assume that \(Z\) is a point. When \(D - (K_X + B)\) is nef and big but \(D\) is nef with numerical dimension \(\nu(D)\) one or two, semi-ampleness of \(D\) is proved in [7] under some mild restrictions.

**Theorem 1.5.** Let \((X, B)\) be a projective \(\mathbb{Q}\)-factorial dlt pair of dimension 3 over \(k\) of char \(p > 5\), and \(X \to Z\) a projective contraction. Let \(R\) be a \(K_X + B\)-negative extremal ray/Z. Assume that there is a nef and big/Z \(\mathbb{Q}\)-divisor \(N\) such that \(N \cdot R = 0\). Then \(R\) can be contracted by a projective morphism.

Note that if \(K_X + B\) is pseudo-effective/Z, then for every \(K_X + B\)-negative extremal ray \(R/Z\) there exists \(N\) as in the theorem (see 3.3). Therefore such extremal rays can be contracted by projective morphisms.

Chenyang Xu informed us that he has proved Theorems 1.4 and 1.5 independently, more or less at the same time but using a different approach. His proof also rely on our results on flips and minimal models.
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**Dlt and terminal models.** The next two results are standard consequences of the LMMP (more precisely, of special termination). They are proved in Section 6.

**Theorem 1.6.** Let \((X, B)\) be an lc pair of dimension 3 over \(k\) of char \(p > 5\). Then \((X, B)\) has a (crepant) \(\mathbb{Q}\)-factorial dlt model. In particular, if \((X, B)\) is klt, then \(X\) has a \(\mathbb{Q}\)-factorialization by a small morphism.

The theorem was proved in [11, Theorem 6.1] for pairs with standard coefficients.

**Theorem 1.7.** Let \((X, B)\) be a klt pair of dimension 3 over \(k\) of char \(p > 5\). Then \((X, B)\) has a (crepant) \(\mathbb{Q}\)-factorial terminal model.

The theorem was proved in [11, Theorem 6.1] for pairs with standard coefficients and canonical singularities.

**The connectedness principle with applications to semi-ampleness.** The next result concerns the Kollár-Shokurov connectedness principle. In char zero, the surface case was proved by Shokurov by taking a resolution and then calculating intersection numbers [22, Lemma 5.7] but the higher dimensional case was proved by Kollár by deriving it from the Kawamata-Viehweg vanishing theorem [18, Theorem 17.4].

**Theorem 1.8.** Let \((X, B)\) be a projective \(\mathbb{Q}\)-factorial pair of dimension 3 over \(k\) of char \(p > 5\). Let \(f: X \to Z\) be a birational contraction such that \((-K_X + B)\) is ample/Z. Then for any closed point \(z \in Z\), the non-klt locus of \((X, B)\) is connected in any neighborhood of the fibre \(X_z\).

The theorem is proved in Section 8. To prove it we use the LMMP rather than vanishing theorems. When \(\text{dim } X = 2\), the theorem holds in a stronger form (see 8.3).

We will use the connectedness principle on surfaces to prove some semi-ampleness results on surfaces and 3-folds. Here is one of them:

**Theorem 1.9.** Let \((X, B + A)\) be a projective \(\mathbb{Q}\)-factorial dlt pair of dimension 3 over \(k\) of char \(p > 5\). Assume that \(A, B \geq 0\) are \(\mathbb{Q}\)-divisors such that \(A\) is ample and \((K_X + B + A)|_{[B]}\) is nef. Then \((K_X + B + A)|_{[B]}\) is semi-ample.

Note that if one could show that \([B]\) is semi-lc, then the result would follow from Tanaka [25]. In order to show that \([B]\) is semi-lc one needs to check that it satisfies the Serre condition \(S_2\). In char zero this is a consequence of Kawamata-Viehweg vanishing (see Kollár [18, Corollary 17.5]). The \(S_2\) condition can be used to glue sections on the various irreducible components of \([B]\). To prove the above semi-ampleness we instead use a result of Keel [14, Corollary 2.9] to glue sections.

**Log canonical thresholds.** As in char zero, we will derive the following result from existence of \(\mathbb{Q}\)-factorial dlt models and boundedness results on Fano surfaces.
Theorem 1.10. Suppose that $\Lambda \subseteq [0, 1]$ and $\Gamma \subseteq \mathbb{R}$ are DCC sets. Then the set 
\[ \{ \text{lct}(M, X, B) \mid (X, B) \text{ is lc of dimension } \leq 3 \} \]
satisfies the ACC where $X$ is over $k$ with $\text{char } p > 5$, the coefficients of $B$ belong to $\Lambda$, $M \geq 0$ is an $\mathbb{R}$-Cartier divisor with coefficients in $\Gamma$, and $\text{lct}(M, X, B)$ is the lc threshold of $M$ with respect to $(X, B)$.

With some work it seems that using the above ACC one can actually prove termination for those lc pairs $(X, B)$ of dimension 3 such that $K_X + B \equiv M$ for some $M \geq 0$ following the ideas in [4]. But we will not pursue this here.

**Numerically trivial family of curves in the non-big case.** We will also give a somewhat different proof of the following theorem of Cascini-Tanaka-Xu [7]. This was also proved independently by McKernan much earlier but unpublished. He informed us that his proof was inspired by [15].

Theorem 1.11. Assume that $X$ is a normal projective variety of dimension $d$ over an algebraically closed field (of any characteristic), and that $B, A \geq 0$ are $\mathbb{R}$-divisors. Moreover, suppose $A$ is nef and big and $D = K_X + B + A$ is nef. If $D^d = 0$, then for each general closed point $x \in X$ there is a rational curve $L_x$ passing through $x$ with $D \cdot L_x = 0$.

The theorem is independent of the rest of this paper. Its proof is an application of the bend and break theorem.

**Some remarks about this paper.** In writing this paper we have tried to give as much details as possible even if the arguments are very similar to the char zero case. This is for convenience, future reference, and to avoid any unpleasant surprises having to do with positive characteristic. The main results are proved in the following order: 1.1 in the projective case, 1.6, 1.7, 1.2, 1.1 in general, 1.8, 1.9, 1.3, 1.4, 1.5, 1.10, and 1.11.

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2. Preliminaries

We work over an algebraically closed field $k$ of characteristic $p > 0$ fixed throughout the paper.

2.1. **Contractions.** A contraction $f: X \rightarrow Z$ of algebraic spaces over $k$ is a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. When $X, Z$ are quasi-projective
Varieties over $k$ and $f$ is projective, we refer to $f$ as a projective contraction to avoid confusion.

Let $f: X \to Z$ be a projective contraction of normal varieties. We say $f$ is extremal if the relative Kleiman-Mori cone of curves $\overline{NE}(X/Z)$ is one-dimensional. Such a contraction is a divisorial contraction if it is birational and it contracts some divisor. It is called a small contraction if it is birational and it contracts some subvariety of codimension $\geq 2$ but no divisors.

Let $f: X \to Z$ be a small contraction and $D$ an $\mathbb{R}$-Cartier divisor such that $-D$ is ample$/Z$. We refer to $f$ as a $D$-flipping contraction or just a flipping contraction for short. We say the $D$-flip of $f$ exists if there is a small contraction $X^+ \to Z$ such that the birational transform $D^+$ is ample$/Z$.

2.2. Some notions related to divisors. Let $X$ be a normal projective variety over $k$ and $L$ a nef $\mathbb{R}$-Cartier divisor. We define $L^+ := \{ \alpha \in \overline{NE}(X) \mid L \cdot \alpha = 0 \}$. This is an extremal face of $\overline{NE}(X)$ cut out by $L$.

Let $f: X \to Z$ be a projective morphism of normal varieties over $k$, and let $D$ be an $\mathbb{R}$-divisor on $X$. We define the algebra of $D$ over $Z$ as $R(D/Z) = \bigoplus_{m \in \mathbb{Z} \geq 0} f_* \mathcal{O}_X([mD])$. When $Z$ is a point we denote the algebra by $R(D)$. When $D = K_X + B$ for a pair $(X,B)$ (see below) we call the algebra the log canonical algebra of $(X,B)$ over $Z$.

Now let $\phi: X \dashrightarrow Y$ be a birational map of normal projective varieties over $k$ whose inverse does not contract divisors. Let $D$ be an $\mathbb{R}$-Cartier divisor on $X$ such that $D_Y := \phi_* D$ is $\mathbb{R}$-Cartier too. We say that $\phi$ is $D$-negative if there is a common resolution $f: W \to X$ and $g: W \to Y$ such that $f^* D - g^* D_Y$ is effective and its support contains the birational transform of all the prime divisors on $X$ which are contracted$/Y$.

2.3. The negativity lemma. The negativity lemma states that if $f: Y \to X$ is a projective birational contraction of normal quasi-projective varieties over $k$ and $D$ is an $\mathbb{R}$-Cartier divisor on $Y$ such that $-D$ is nef$/X$ and $f_* D \geq 0$, then $D \geq 0$. See [22, Lemma 1.1] for the char zero case. The proof there also works in char $p > 0$ and we reproduce it for convenience. Assume that the lemma does not hold. We reduce the problem to the surface case. Let $P$ be the image of the negative components of $D$. If dim $P > 0$, we take a general hypersurface section $H$ on $X$, let $G$ be the normalization of the birational transform of $H$ on $Y$ and reduce the problem to the contraction $G \to H$ and the divisor $D|_G$. But if dim $X > 2$ and dim $P = 0$, we take a general hypersurface section $G$ on $Y$, let $H$ be the normalization of $f(G)$, and reduce the problem to the induced contraction $G \to H$ and divisor $D|_G$. So we can reduce the problem to the case when $X,Y$ are surfaces, $P$ is just one point, and $f$ is an isomorphism over $X \setminus \{ P \}$. Taking a resolution enable us to assume $Y$ is smooth. Now let $E \geq 0$ be a divisor whose support is equal to the exceptional locus of $f$ and such that $-E$ is nef$/X$: pick a Cartier divisor $L \geq 0$ passing through $P$ and write $f^* L = L^+ + E$ where $L^+$ is the birational transform of $L$; then $E$ satisfies the requirements. Let $e$ be the smallest number such that $D + eE \geq 0$. Now
there is a component $C$ of $E$ whose coefficient in $D + eE$ is zero and that $C$
intersects $\text{Supp}(D + eE)$. But then $(D + eE) \cdot C > 0$, a contradiction.

2.4. Resolution of singularities. Let $X$ be a quasi-projective variety of
dimension $\leq 3$ over $k$ and $P \subset X$ a closed subset. Assume that there is an
open set $U \subset X$ such that $P \cap U$ is a divisor with simple normal crossing (snc)
singularities. Then there is a log resolution of $X, P$ which is an isomorphism
over $U$, that is, there is a projective birational morphism $f: Y \to X$ such that
the union of the exceptional locus of $f$ and the birational transform of $P$ is an
snc divisor, and $f$ is an isomorphism over $U$. This follows from Cutkosky [10,
Theorems 1.1, 1.2, 1.3] when $k$ has char $p > 5$, and from Cossart-Piltant [9,
Theorems 4.1, 4.2][8, Theorem] in general (see also [11, Theorem 2.1]).

2.5. Pairs. A pair $(X, B)$ consists of a normal quasi-projective variety $X$ over
$k$ and an $\mathbb{R}$-boundary $B$, that is an $\mathbb{R}$-divisor $B$ on $X$ with coefficients in $[0, 1]$,
such that $K_X + B$ is $\mathbb{R}$-Cartier. When $B$ has rational coefficients we say $B$ is
a $\mathbb{Q}$-boundary or say $B$ is rational. We say that $(X, B)$ is log smooth if $X$
is smooth and $\text{Supp} B$ has simple normal crossing singularities.

Let $(X, B)$ be a pair. For a prime divisor $D$ on some birational model of $X$
with a nonempty centre on $X$, $a(D, X, B)$ denotes the log discrepancy which is
defined by taking a projective birational morphism $f: Y \to X$ from a normal
variety containing $D$ as a prime divisor and putting $a(D, X, B) = 1 - b$ where
$b$ is the coefficient of $D$ in $B_Y$ and $K_Y + B_Y = f^*(K_X + B)$.

As in char zero, we can define various types of singularities using log discrepancies.
Let $(X, B)$ be a pair. We say that the pair is log canonical or lc for short (resp. Kawamata log terminal or klt for short) if $a(D, X, B) \geq 0$ (resp.
$a(D, X, B) > 0$) for any prime divisor $D$ on birational models of $X$. An lc
centre of $(X, B)$ is the image in $X$ of a $D$ with $a(D, X, B) = 0$. The pair $(X, B)$
is terminal if $a(D, X, B) > 1$ for any prime divisor $D$ on birational models of $X$
which is exceptional/$X$ (such pairs are sometime called terminal in codimension
$\geq 2$). On the other hand, we say that $(X, B)$ is dlt if there is a closed subset
$P \subset X$ such that $(X, B)$ is log smooth outside $P$ and no lc centre of $(X, B)$ is
inside $P$. In particular, the lc centres of $(X, B)$ are exactly the components of
$S_1 \cap \cdots \cap S_r$ where $S_i$ are among the components of $[B]$. Moreover, there is
a log resolution $f: Y \to X$ of $(X, B)$ such that $a(D, X, B) > 0$ for any prime
divisor $D$ on $Y$ which is exceptional/$X$, eg take a log resolution $f$ which is an
isomorphism over $X \setminus P$. Finally, we say that $(X, B)$ is plt if it is dlt and each
connected component of $[B]$ is irreducible. In particular, the only lc centres of
$(X, B)$ are the components of $[B]$.

2.6. Ample divisors on log smooth pairs. Let $(X, B)$ be a projective log
smooth pair over $k$ and let $A$ be an ample $\mathbb{Q}$-divisor. We will argue that there
is $A' \sim_{\mathbb{Q}} A$ such that $A' \geq 0$ and that $(X, B + A')$ is log smooth. The argument
was suggested to us by several people independently. We may assume that $B$
is reduced. Let $S_1, \ldots, S_r$ be the components of $B$ and let $\mathcal{S}$ be the set of the
components of $S_{i_1} \cap \cdots \cap S_{i_n}$ for all the choices $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, r\}$. By Bertini’s theorem, there is a sufficiently divisible integer $l > 0$ such that for any
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$T \in S$, a general element of $\lvert IA \rvert_T$ is smooth. Since $IA$ is sufficiently ample, such general elements are restrictions of general elements of $\lvert IA \rvert$. Therefore, we can choose a general $G \sim IA$ such that $G$ is smooth and $G|_T$ is smooth for any $T \in S$. This means that $(X, B + G)$ is log smooth. Now let $A' = \frac{1}{t}G$.

2.7. Models of pairs. Let $(X, B)$ be a pair and $X \to Z$ a projective contraction over $k$.

A pair $(Y, B_Y)$ with a projective contraction $Y \to Z$ over $k$ is a log birational model of $(X, B)$ if we have a birational map $\phi: X \dasharrow Y/Z$, $B_Y$ on $Y$ which is the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi^{-1}$. We say that $(Y, B_Y)$ is a weak lc model of $(X, B)$ over $Z$ if in addition

1. $K_Y + B_Y$ is nef$/Z$.
2. for any prime divisor $D$ on $X$ which is exceptional$/Y$, we have
   
   $$a(D, X, B) \leq a(D, Y, B_Y)$$

   And we call $(Y, B_Y)$ a log minimal model of $(X, B)$ over $Z$ if in addition

3. $(Y, B_Y)$ is $\mathbb{Q}$-factorial dlt,
4. the inequality in (2) is strict.

When $K_X + B$ is big$/Z$, the lc model of $(X, B)$ over $Z$ is a weak lc model $(Y, B_Y)$ over $Z$ with $K_Y + B_Y$ ample$/Z$.

On the other hand, a log birational model $(Y, B_Y)$ of $(X, B)$ is called a Mori fibre space of $(X, B)$ over $Z$ if there is a $K_Y + B_Y$-negative extremal projective contraction $Y \to T/Z$, and if for any prime divisor $D$ on birational models of $X$ we have

$$a(D, X, B) \leq a(D, Y, B_Y)$$

with strict inequality if $D \subset X$ and if it is exceptional$/Y$.

Note that the above definitions are slightly different from the traditional definitions. However, if $(X, B)$ is plt (hence also klt) the definitions coincide.

Let $(X, B)$ be an lc pair over $k$. A $\mathbb{Q}$-factorial dlt pair $(Y, B_Y)$ is a $\mathbb{Q}$-factorial dlt model of $(X, B)$ if there is a projective birational morphism $f: Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$. On the other hand, when $(X, B)$ is klt, a pair $(Y, B_Y)$ with terminal singularities is a terminal model of $(X, B)$ if there is a projective birational morphism $f: Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$.

2.8. Keel’s results. We recall some of the results of Keel which will be used in this paper. For a nef $\mathbb{Q}$-Cartier divisor $L$ on a projective scheme $X$ over $k$, the exceptional locus $\mathbb{E}(L)$ is the union of those positive-dimensional integral subschemes $Y \subseteq X$ such that $L|_Y$ is not big, i.e. $(L|_Y)^{\dim Y} = 0$. By [6], $\mathbb{E}(L)$ coincides with the augmented base locus $\mathbb{B}_+(L)$. We say $L$ is endowed with a map $f: X \to V$, where $V$ is an algebraic space over $k$, if: an integral subscheme $Y$ is contracted by $f$ (i.e. $\dim Y > \dim f(Y)$) if and only if $L|_Y$ is not big.

**Theorem 2.9** ([14, 1.9]). Let $X$ be a projective scheme over $k$ and $L$ a nef $\mathbb{Q}$-Cartier divisor on $X$. Then
• $L$ is semi-ample if and only if $L|_{E(L)}$ is semi-ample;
• $L$ is endowed with a map if and only if $L|_{E(L)}$ is endowed with a map.

The theorem does not hold if $k$ is of char zero. When $L|_{E(L)} \equiv 0$, then $L|_{E(L)}$ is automatically endowed with the constant map $E(L) \to \text{pt}$ hence $L$ is endowed with a map. This is particularly useful for studying 3-folds because it is often not difficult to show that $L|_{E(L)}$ is endowed with a map, eg when $X$ is a 3-fold and $\dim E(L) = 1$.

**Theorem 2.10** ([14, 0.5]). Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 over $k$ with $B$ a $\mathbb{Q}$-divisor. Assume that $A$ is an ample $\mathbb{Q}$-divisor such that $L = K_X + B + A$ is nef and big. Then $L$ is endowed with a map.

In particular, when $L^\perp$ is an extremal ray, then we can contract $R$ to an algebraic space by the map associated to $L$. In particular, such an extremal ray is generated by the class of some curve.

We also recall the following cone theorems which we will use repeatedly in Section 3. Note that these theorems (as well as 2.10) do not assume singularities to be lc.

**Theorem 2.11** ([14, 0.6]). Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 over $k$ with $B$ a $\mathbb{Q}$-divisor. Assume that $K_X + B \sim_{\mathbb{Q}} M$ for some $M \geq 0$. Then there is a countable number of curves $\Gamma_i$ such that
• $\text{NE}(X) = \text{NE}(X)_{K_X + B \geq 0} + \sum_i \mathbb{R}[\Gamma_i]$,
• all but finitely many of the $\Gamma_i$ are rational curves satisfying $-3 \leq (K_X + B) \cdot \Gamma_i < 0$, and
• the rays $\mathbb{R}[\Gamma_i]$ do not accumulate inside $\text{NE}(X)_{K_X + B < 0}$.

**Theorem 2.12** ([14, 5.5.2]). Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 over $k$. Assume that

$$L = K_X + B + H \sim_{\mathbb{R}} A + M$$

is nef where $H, A$ are ample $\mathbb{R}$-divisors, and $M \geq 0$. Then any extremal ray of $L^\perp$ is generated by some curve $\Gamma$ such that either
• $\Gamma$ is a component of the singular locus of $B + M$ union with the singular locus of $X$, or
• $\Gamma$ is a rational curve satisfying $-3 \leq (K_X + B) \cdot \Gamma < 0$.

**Remark 2.13** Let $(X, B)$ a projective lc pair of dimension 3 over $k$ with $B$ a $\mathbb{Q}$-boundary, and $H$ an ample $\mathbb{Q}$-divisor. Assume that $L = K_X + B + H$ is nef and big. Moreover, suppose that each connected component of $E(L)$ is inside some normal irreducible component $S$ of $\lfloor B \rfloor$. Then $L|_S$ is semi-ample for such components (cf. [24]) hence $L|_{E(L)}$ is semi-ample and this in turn implies that $L$ is semi-ample by Theorem 2.9.
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3. Extremal rays and special kinds of LMMP

As usual the varieties and algebraic spaces in this section are defined over $k$ of char $p > 0$.

3.1. Extremal curve of a ray. Let $X$ be a projective variety and $H$ a fixed ample Cartier divisor. Let $R$ be a ray of $\overline{NE}(X)$ which is generated by some curve $\Gamma$. Assume that $H \cdot \Gamma = \min \{ H \cdot C \mid C \text{ generates } R \}$

In this case, we say $\Gamma$ is an extremal curve of $R$ (in practice we do not mention $H$ and assume that it is already fixed). Let $C$ be any other curve generating $R$.

Since $\Gamma$ and $C$ both generate $R$, $D \cdot R < 0$ for some $\mathbb{R}$-Cartier divisor $D$. Hence

$$D \cdot \Gamma = D \cdot C \left( \frac{H \cdot \Gamma}{H \cdot C} \right) \geq D \cdot C$$

which implies that

$$D \cdot \Gamma = \max \{ D \cdot C \mid C \text{ generates } R \}$$

3.2. Negative extremal rays. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3. Let $R$ be a $K_X + B$-negative extremal ray. Assume that there is a boundary $\Delta$ such that $K_X + \Delta$ is pseudo-effective and $(K_X + \Delta) \cdot R < 0$. By adding a small ample divisor and perturbing the coefficients we can assume that $\Delta$ is rational and that $K_X + \Delta$ is big. Then by Theorem 2.11, $R$ is generated by some extremal curve and $R$ is an isolated extremal ray of $\overline{NE}(X)$.

Now assume that $K_X + B$ is pseudo-effective and let $A$ be an ample $\mathbb{R}$-divisor. Then for any $\epsilon > 0$, there are only finitely many $K_X + B + \epsilon A$-negative extremal rays: assume that this is not the case; then we can find a $\mathbb{Q}$-boundary $\Delta$ such that $K_X + \Delta$ is big and $K_X + \Delta$ is big and

$$K_X + B + \epsilon A \sim_{\mathbb{R}} K_X + \Delta + G$$

where $G$ is ample; so there are also infinitely many $K_X + \Delta$-negative extremal rays; but $K_X + \Delta$ is big hence by Theorem 2.11 all but finitely many of the $K_X + \Delta$-negative extremal rays are generated by extremal curves $\Gamma$ with $-3 \leq (K_X + \Delta) \cdot \Gamma < 0$; if $(K_X + B + \epsilon A) \cdot \Gamma < 0$, then $G \cdot \Gamma \leq 3$; since $G$ is ample, there can be only finitely such $\Gamma$ up to numerical equivalence.

Let $R$ be a $K_X + B$-negative extremal ray where $K_X + B$ is not necessarily pseudo-effective. But assume that there is a pseudo-effective $K_X + \Delta$ with $(K_X + \Delta) \cdot R < 0$. By the remarks above we may assume $\Delta$ is rational, $K_X + \Delta$ big, and that there are only finitely many $K_X + \Delta$-negative extremal rays. Therefore, we can find an ample $\mathbb{Q}$-divisor $H$ such that $L = K_X + \Delta + H$ is nef and big and $L^\perp = R$. That is, $L$ is a supporting divisor of $R$. Moreover, $R$ can be contracted to an algebraic space, by Theorem 2.10. More precisely, there is
a contraction \( X \to V \) to an algebraic space such that it contracts a curve \( C \) if and only if \( L \cdot C = 0 \) if and only if the class \([C]\) \( \in R \).

### 3.3. More on negative extremal rays.

Let \((X, B)\) be a projective \( \mathbb{Q} \)-factorial pair of dimension 3. Let \( C \subset \overline{NE}(X) \) be one of the following:

1. \( \mathcal{C} = \overline{NE}(X/Z) \) for a given projective contraction \( X \to Z \) such that \( K_X + B = P + M/Z \) where \( P \) is nef/\( Z \) and \( M \geq 0 \) (this is a weak Zariski decomposition; see 7.1); or
2. \( \mathcal{C} = N^\perp \) for some nef and big \( \mathbb{Q} \)-divisor \( N \);

We will show that in both cases, each \( K_X + B \)-negative extremal ray \( R \) of \( C \) is generated by an extremal curve \( \Gamma \), and for all but finitely many of those rays we have \(-3 \leq (K_X + B) \cdot \Gamma < 0\).

We first deal with case (1). Fix a \( K_X + B \)-negative extremal ray \( R \) of \( C \). By replacing \( P \) we can assume that \( K_X + B = P + M \). Let \( A \) be an ample \( \mathbb{R} \)-divisor and \( T \) be the pullback of a sufficiently ample divisor on \( Z \) so that \( K_X + B + A + T \) is big and \( (K_X + B + A + T) \cdot R < 0 \). By 3.2, there is a nef and big \( \mathbb{Q} \)-divisor \( L \) with \( L^\perp = R \). Moreover, we may assume that if \( l \gg 0 \), then

\[
Q_1 := K_X + B + T + lL + A
\]

is nef and big and \( Q_1^\perp = R \). By construction, \( T + lL + A \) is ample, \( P + T + lL + A \) is also ample, and

\[
K_X + B + T + lL + A = P + T + lL + A + M
\]

Therefore, by Theorem 2.12, \( R \) is generated by some curve \( \Gamma \) satisfying \(-3 \leq (K_X + B) \cdot \Gamma < 0 \) or \( R \) is generated by some curve in the singular locus of \( B + M \) or \( X \). There are only finitely many possibilities in the latter case. The claim then follows.

Now we deal with case (2). Fix a \( K_X + B \)-negative extremal ray \( R \) of \( C \). Since \( N \) is nef and big, for some \( n > 0 \),

\[
K_X + B + nN \sim_\mathbb{R} G + S
\]

where \( G \) is ample and \( S \geq 0 \). By 3.2, there is a nef and big \( \mathbb{Q} \)-divisor \( L \) with \( L^\perp = R \). Moreover, for some \( l \gg 0 \) and some ample \( \mathbb{R} \)-divisor \( A \),

\[
Q_2 := K_X + B + nN + lL + A
\]

is nef and big with \( Q_2^\perp = R \). Now, \( nN + lL + A \) is ample, \( G + lL + A \) is ample, and

\[
K_X + B + nN + lL + A \sim_\mathbb{R} G + lL + A + S
\]

Therefore, by Theorem 2.12, \( R \) is generated by some curve \( \Gamma \) satisfying \(-3 \leq (K_X + B) \cdot \Gamma < 0 \) or \( R \) is generated by some curve in the singular locus of \( B + S \) or \( X \). There are only finitely many possibilities in the latter case. The claim then follows.

Assume that \( R \) is a \( K_X + B \)-negative extremal ray of \( C \), in either case. Then the above arguments show that there is a \( \mathbb{Q} \)-boundary \( \Delta \) and an ample \( \mathbb{Q} \)-divisor \( H \) such that \( K_X + \Delta \) is big, \( (K_X + \Delta) \cdot R < 0 \), and \( L = K_X + \Delta + H \) is nef and big with \( L^\perp = R \). Therefore, as in 3.2, \( R \) can be contracted via a contraction \( X \to V \) to an algebraic space. Moreover, if \( B \) is rational, then we
can find an ample $\mathbb{Q}$-divisor $H'$ such that $L' = K_X + B + H'$ is nef and big and again $L'/N = R$.

### 3.4. Extremal rays given by scaling.

Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3. Assume that either $\mathcal{C} = \overline{NE}(X/Z)$ for some projective contraction $X \to Z$ such that $K_X + B \equiv M/Z$ for some $M \geq 0$, or $\mathcal{C} = N_{\perp}$ for some nef and big $\mathbb{Q}$-divisor $N$. In addition assume that $(X, B + C)$ is a pair for some $C \geq 0$ and that $K_X + B + C$ is nef on $\mathcal{C}$, that is, $(K_X + B + C) \cdot R \geq 0$ for every extremal ray $R$ of $\mathcal{C}$. Let

$$\lambda = \inf\{ t \geq 0 \mid (K_X + B + tC) \text{ is nef on } \mathcal{C} \}$$

Then we will see that either $\lambda = 0$ or there is an extremal ray $R$ of $\mathcal{C}$ such that $(K_X + B + \lambda C) \cdot R = 0$ and $(K_X + B) \cdot R < 0$. Assume $\lambda > 0$. If the claim is not true, then there exist a sequence of numbers $t_1 < t_2 < \cdots$ approaching $\lambda$ and extremal rays $R_i$ of $\mathcal{C}$ such that $(K_X + B + t_i C) \cdot R_i = 0$ and $(K_X + B) \cdot R_i < 0$.

First assume that $\mathcal{C} = N_{\perp}$ for some nef and big $\mathbb{Q}$-divisor $N$. We can write a finite sum $K_X + B = \sum_j r_j(K_X + B_j)$ where $r_j \in (0, 1]$, $\sum r_j = 1$, and $(X, B_j)$ are pairs with $B_j$ being rational. By 3.3, we may assume that each $R_i$ is generated by some extremal curve $\Gamma_i$ with $-3 \leq (K_X + B_j) \cdot \Gamma_i$ for each $j$. This implies that there are only finitely many possibilities for the numbers $(K_X + B) \cdot \Gamma_i$. A similar reasoning shows that there are only finitely many possibilities for the numbers $(K_X + B + \frac{1}{\lambda} C) \cdot \Gamma_i$ hence there are also only finitely many possibilities for the numbers $C \cdot \Gamma_i$. But then this implies that there are finitely many $t_i$, a contradiction.

Now assume that $\mathcal{C} = \overline{NE}(X/Z)$ for some projective contraction $X \to Z$ such that $K_X + B \equiv M/Z$ for some $M \geq 0$. Then we can write $K_X + B = \sum_j r_j(K_X + B_j)$ and $M = \sum_j r_j M_j$ where $r_j \in (0, 1]$, $\sum r_j = 1$, $(X, B_j)$ are pairs with $B_j$ being rational, $K_X + B_j \equiv M_j/Z$, and $M_j \geq 0$. To find such a decomposition we argue as in \cite[pages 96-97]{5}. Let $V$ and $W$ be the $\mathbb{R}$-vector spaces generated by the components of $B$ and $M$ respectively. For a vector $v \in V$ (resp. $w \in W$) we denote the corresponding $\mathbb{R}$-divisor by $B_v$ (resp. $M_w$). Let $F$ be the set of those $(v, w) \in V \times W$ such that $(X, B_v)$ is a pair, $M_w \geq 0$, and $K_X + B_v \equiv M_w/Z$. Then $F$ is defined by a finite number of linear equalities and inequalities with rational coefficients. If $B = B_{v_0}$ and $M = M_{w_0}$ are the given divisors, then $(v_0, w_0) \in F$ hence it belongs to some polytope in $F$ with rational vertices. The vertices of the polytope give the $B_j, M_j$. The rest of the proof is as in the last paragraph.

### 3.5. LMMP with scaling.

Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3. Assume that either $\mathcal{C} = \overline{NE}(X/Z)$ for some projective contraction $X \to Z$ such that $K_X + B \equiv M/Z$ for some $M \geq 0$, or $\mathcal{C} = N_{\perp}$ for some nef and big $\mathbb{Q}$-divisor $N$. In addition assume that $(X, B + C)$ is a pair for some $C \geq 0$ and that $K_X + B + C$ is nef on $\mathcal{C}$.

If $K_X + B$ is not nef on $\mathcal{C}$, by 3.4, there is an extremal ray $R$ of $\mathcal{C}$ such that $(K_X + B + \lambda C) \cdot R = 0$ and $(K_X + B) \cdot R < 0$ where $\lambda$ is the smallest number such that $K_X + B + \lambda C$ is nef on $\mathcal{C}$. Assume that $R$ can be contracted by a projective
morphism. The contraction is birational because \( L \cdot R = 0 \) for some nef and big \( \mathbb{Q} \)-Cartier divisor \( L \) (see 3.3). Assume that \( X \to X' \) is the corresponding divisorial contraction or flip, and assume that \( X' \) is \( \mathbb{Q} \)-factorial. Let \( C' \) be the cone given by \( C' = \overline{NE}(X'/Z) \) or \( C' = (N')^+ \) corresponding to the above cases. Let \( \lambda' \) be the smallest nonnegative number such that \( K_{X'} + B' + \lambda'C' \) is nef on \( C' \). If \( \lambda' > 0 \), then there is an extremal ray \( R' \) of \( C' \) such that \( (K_{X'} + B' + \lambda'C') \cdot R' = 0 \) and \( (K_{X'} + B') \cdot R' < 0 \). Assume that \( R' \) can be contracted and so on. Assuming that all the necessary ingredients exist, the process gives a special kind of LMMP which we may refer to as \( \text{LMMP}/C \) on \( K_X + B \) with scaling of \( C \). Note that \( \lambda \geq \lambda' \geq \cdots \)

If \( C = \overline{NE}(X/Z) \), we also refer to the above LMMP as the LMMP/Z on \( K_X + B \) with scaling of \( C \). If \( C = (N')^+ \), and if \( N \) is endowed with a map \( X \to V \) to an algebraic space, we refer to the above LMMP as the LMMP/V on \( K_X + B \) with scaling of \( C \).

In practice, when we run an LMMP with scaling, \( (X, B) \) is \( \mathbb{Q} \)-factorial dlt and each extremal ray in the process intersects some component of \( [B] \) negatively. In particular, such rays can be contracted by projective morphisms and the \( \mathbb{Q} \)-factorial property is preserved by the LMMP (see 4.5). If the required flips exist then the LMMP terminates by special termination (see 4.6).

3.6. Extremal rays given by a weak Zariski decomposition. Let \( (X, B) \) be a projective \( \mathbb{Q} \)-factorial pair of dimension 3 and \( X \to Z \) a projective contraction such that

\[
\begin{align*}
(1) & \quad K_X + B \equiv P + M/Z, \ P \text{ is nef}/Z, \ M \geq 0, \text{ and} \\
(2) & \quad \text{Supp} \ M \subseteq [B].
\end{align*}
\]

Let
\[
\mu = \sup \{ t \in [0, 1] \mid P + tM \text{ is nef}/Z \}.
\]

Assume that \( \mu < 1 \). We will show that there is an extremal ray \( R/Z \) such that
\[
(K_X + B) \cdot R < 0 \quad \text{and} \quad (P + \mu M) \cdot R = 0.
\]

Replacing \( P \) with \( P + \mu M \) we may assume that \( \mu = 0 \). Then by definition of \( \mu \), \( P + \epsilon M \) is not nef/Z for any \( \epsilon > 0 \). In particular, for any \( \epsilon' > 0 \) there is a \( K_X + B \)-negative extremal ray \( R/Z \) such that \( (P + \epsilon' M) \cdot R < 0 \) but \( (P + \epsilon M) \cdot R = 0 \) for some \( \epsilon \in [0, \epsilon') \). If there is no \( K_X + B \)-negative extremal ray \( R/Z \) such that \( P \cdot R = 0 \), then there is an infinite strictly decreasing sequence of sufficiently small positive real numbers \( \epsilon_i \) and \( K_X + B \)-negative extremal rays \( R_i/Z \) such that \( \lim_{i \to \infty} \epsilon_i = 0 \) and \( (P + \epsilon_i M) \cdot R_i = 0 \).

We may assume that for each \( i, \) there is an extremal curve \( \Gamma_i \) generating \( R_i \) such that \( -3 \leq (K_X + B) \cdot \Gamma_i < 0 \) (see 3.3). Since \( \text{Supp} \ M \subseteq [B] \), there is a small \( \delta > 0 \) such that \( (K_X + B - \delta M) \cdot \Gamma_i < 0 \) for each \( i \), \( B - \delta M \geq 0 \), and \( \text{Supp}(B - \delta M) = \text{Supp} \ B \). We have
\[
K_X + B - \delta M \equiv P + (1 - \delta)M/Z
\]

By replacing the sequence of extremal rays with a subsequence, we can assume that each component \( S \) of \( M \) satisfies: either \( S \cdot R_i \geq 0 \) for every \( i \), or \( S \cdot R_i < 0 \)
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for every $i$. Pick a component $S$. If $S \cdot R_i \geq 0$ for each $i$, then by 3.3, we may assume that
\[-3 \leq (K_X + B - \delta M) \cdot \Gamma_i < 0\]
and
\[-3 \leq (K_X + B - \delta M - \tau S) \cdot \Gamma_i < 0\]
for every $i$ where $\tau > 0$ is a small number. In particular, this means that $S \cdot \Gamma_i$ is bounded from below and above. On the other hand, if $S \cdot R_i < 0$ for each $i$, then by considering $K_X + B - \delta M + \tau S$ and arguing similarly we can show that again $S \cdot \Gamma_i$ is bounded from below and above. In particular, there are only finitely many possibilities for the numbers $M \cdot \Gamma_i$. Therefore,
\[
\lim_{i \to \infty} P \cdot \Gamma_i = \lim_{i \to \infty} -\epsilon_i M \cdot \Gamma_i = 0
\]

Write $K_X + B = \sum_j r_j(K_X + B_j)$ where $r_j \in (0, 1]$, $\sum r_j = 1$, and $(X, B_j)$ are pairs with $B_j$ being rational. We can assume that each component of $B - B_j$ has irrational coefficient in $B$ hence $B - B_j$ and $M$ have no common components because $\text{Supp } M \subseteq [B]$. Assume $(K_X + B_j) \cdot \Gamma_i < 0$ for some $i, j$. Let $S$ be a component of $M$ such that $S \cdot \Gamma_i < 0$, and let $S'$ be its normalization. Let $K_{S'} + B_{j,S'} = (K_X + B_j)|_{S'}$ (see 4.1 for adjunction formulas of this type). On the other hand, by 3.3, there is an ample $\mathbb{Q}$-divisor $H$ such that $Q = K_X + B_j + H$ is nef and big and $R_i = Q^+$. Now the face $(Q|_S)^{\perp}$ of $NE(S/Z)$ is generated by finitely many curves $\Lambda_1, \ldots, \Lambda_s$ such that $\alpha_j \leq (K_S + B_{j,S}) \cdot \Lambda_l < 0$ where $\alpha_j$ depends on $(S', B_{j,S'})$ but does not depend on $i$, by Tanaka [24, Theorem 4.4, Remark 4.5]. Since $R_i = Q^+$, each $\Lambda_l$ also generates $R_i$. But as $\Gamma_i$ is extremal, by 3.1, we get
\[
\alpha_j \leq (K_S + B_{j,S}) \cdot \Lambda_l = (K_X + B_j) \cdot \Lambda_l \leq (K_X + B_j) \cdot \Gamma_i < 0
\]

On the other hand, since
\[-3 \leq (K_X + B) \cdot \Gamma_i = \sum_j r_j(K_X + B_j) \cdot \Gamma_i < 0\]
for each $i$, we deduce that $(K_X + B_j) \cdot \Gamma_i$ is bounded from below and above for each $i, j$ which in turn implies that there are only finitely many possibilities for $(K_X + B) \cdot \Gamma_i$. Recalling that there are also finitely many possibilities for $M \cdot \Gamma_i$, we get a contradiction as
\[0 < P \cdot \Gamma_i = (K_X + B) \cdot \Gamma_i - M \cdot \Gamma_i\]
but $\lim_{i \to \infty} P \cdot \Gamma_i = 0$.

3.7. LMMP using a weak Zariski decomposition. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 and $X \to Z$ a projective contraction such that $K_X + B \equiv P + M/Z$ where $P$ is nef$/Z$, $M \geq 0$, and $\text{Supp } M \subseteq [B]$. Let $\mu$ be the largest number such that $P + \mu M$ is nef$/Z$. Assume $\mu < 1$. Then, by 3.6, there is an extremal ray $R/Z$ such that $(K_X + B) \cdot R < 0$ and $(P + \mu M) \cdot R = 0$. By replacing $P$ with $P + \mu M$ we may assume that $P \cdot R = 0$. Assume that $R$ can be contracted by a projective morphism and that it gives a divisorial contraction or a log flip $X \dashrightarrow X'/Z$ with $X'$ being $\mathbb{Q}$-factorial. Obviously,
$K_X + B' \equiv P' + M'/Z$ where $P'$ is nef/Z, $M' \geq 0$, and $\text{Supp} \, M' \subseteq [B']$. Continuing this process we obtain a particular kind of LMMP which we will refer to as the \textit{LMMP using a weak Zariski decomposition} or more specifically the \textit{LMMP/Z on $K_X + B$ using $P + M$}. When we need this LMMP below we will make sure that all the necessary ingredients exist.

4. \textbf{Adjunction and special termination}

All the varieties and algebraic spaces in this section are over $k$ of char $p > 0$.

4.1. \textbf{Adjunction}. In this subsection, we will use some of the results of Kollár [16]. Let $\Lambda$ be a DCC set of numbers in $[0, 1]$. Then the hyperstandard set

$$\mathcal{S}_\Lambda = \left\{ \frac{m-1}{m} + \sum \frac{l_i b_i}{m} \leq 1 \mid m \in \mathbb{N} \cup \{\infty\}, l_i \in \mathbb{Z}_{\geq 0}, b_i \in \Lambda \right\}$$

also satisfies DCC.

Now let $(X, B)$ be an lc pair of dimension 3 and $S$ a component of $[B]$. Let $S^\nu \to S$ be the normalization. Then the pullback of $K_X + B$ to $S^\nu$ can be canonically written as $K_{S^\nu} + B_{S^\nu}$ for some boundary $B_{S^\nu}$ which is called the \textit{different}. Indeed, take a log resolution $f : W \to X$, let $K_W + B_W = f^*(K_X + B)$, and $K_T + B_T = (K_W + B_W)|_T$ where $T$ is the birational transform of $S$. Next, let $B_{S^\nu}$ be the pushdown of $B_T$ via $T \to S^\nu$. This is independent of the resolution. The different can be defined directly, at least when $B$ is rational, without using resolution of singularities as in [16, Section 4.1] (see also [16, 4.7]).

In char zero: if the coefficients of $B$ belong to $\Lambda$ then the coefficients of $B_{S^\nu}$ belong to $\mathcal{S}_\Lambda$. This is a result of Shokurov [22, Corollary 3.10]. The idea is to cut by appropriate hyperplane sections and reduce the problem to the case when $X$ is a surface. If the index of $K_X + S$ is 1 one proves the claim by direct calculations on a resolution. If the index is more than 1 one then uses the index 1 cover.

In char $p > 0$, we will prove a weak version of Shokurov’s result which is enough for our purposes (see 4.4).

\textbf{Lemma 4.2.} Let $(X, B)$ be an lc pair of dimension 3 and $S$ a component of $[B]$. Then we have:

(1) if the coefficients of $B$ are standard, then the coefficients of $B_{S^\nu}$ are also standard;

(2) if $k$ has char $p > 5$ and $(X, B)$ is $\mathbb{Q}$-factorial dlt, then $S$ is normal.

\textbf{Proof.} (1) This follows from Kollár [16, Corollary 3.45] (see also [16, 4.4]).

(2) We may assume $B = S$ by discarding all the other components, in particular, $(X, B)$ is plt hence $(S^\nu, B_{S^\nu})$ is klt. By (1), $B_{S^\nu}$ has standard coefficients. By [11, Theorem 3.1], $(S^\nu, B_{S^\nu})$ is actually strongly $F$-regular. Therefore, $S$ is normal by [11, Theorem 4.1].

$$\square$$
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**Proposition 4.3.** Let $\Lambda \subseteq [0, 1]$ be a DCC set of real numbers. Then there is a DCC set $\Gamma \subseteq [0, 1]$ of real numbers such that the following holds: let $(X, B)$ be a pair such that

- $(X, B)$ is lc of dimension 2,
- the coefficients of $B$ are in $\Lambda$,
- $S$ is a component of $[B]$.

Then the coefficients of $B_{S'}$ are in $\Gamma$.

**Proof.** Assume that the claim does not hold. Then we can find a sequence $(X_i, B_i)$ of pairs satisfying the assumptions of the proposition such that the set of the coefficients of the $B_{S'_i}$ does not satisfy the DCC; here $K_{S'_i} + B_{S'_i} := (K_{X_i} + B_i)|_{S'_i}$. By replacing $(X_i, B_i)$ with a $\mathbb{Q}$-factorial dlt model (we only need LMMP in dimension 2 to construct such a model), we may assume that $(X_i, B_i)$ are $\mathbb{Q}$-factorial dlt. Thus $S_i$ are normal and $S'_i = S_i$ (this follows from a 2-dimensional version of Lemma 4.2 or more directly from [16, 3.35]). By shrinking $X_i$ if necessary we may assume that $\text{Supp} B_{S_i}$ consists of a single point $P_i$, $(X_i, B_i)$ is log smooth outside $P_i$, and that the coefficient of $P_i$ in $B_{S_i}$, say $e_i$, gives a strictly decreasing sequence of numbers. In particular, we can assume that $e_i < 1$ for every $i$.

Let $B'_i := S_i$ and let $e'_i$ be the coefficient of $P_i$ in $B'_{S_i}$. Then there is some natural number $m_0$ such that $e'_i \leq e_i < 1 - \frac{1}{m_0}$ for every $i$. Moreover, by [16, 3.35 and 3.36],

$$e'_i = 1 - \frac{1}{d_i}$$

where $d_i$ is the determinant of the matrix of the dual graph of a log minimal resolution $f_i: Y_i \to X_i$ of $(X_i, B'_i)$ (see [16, 2.26 and 3.33] for definitions of dual graphs, etc.). In particular, $d_i \leq m_0$ hence there are only a finite number of possibilities, depending only on $m_0$, for the dual graph by [16, 3.33]. This implies that there is a number $n > 0$ depending only on $m_0$ such that if $M_i$ is any prime divisor on $X_i$, then $nf_i^*M_i$ is Cartier (the coefficients of $f_i^*M_i$ are calculated using the system of equations given by the dual graph; this bounds the denominators as there are only finitely many possibilities for the dual graph). In particular, if $M_i \neq S_i$, then $nM_i|_{S_i}$ is Cartier.

Assume that $B_i - B'_i = \sum c_{i,j}D_{i,j}$. Then the coefficient of $P_i$ in

$$(B_i - B'_i)|_{S_i} = \sum c_{i,j}D_{i,j}|_{S_i}$$

is of the form $\sum \frac{c_{i,j}l_{i,j}}{n}$ for certain non-negative integers $l_{i,j}$. Since the $c_{i,j}$ satisfy the DCC by assumptions, $\sum \frac{c_{i,j}l_{i,j}}{n}$ also satisfy the DCC. This gives a contradiction because $B_{S_i} = B'_{S_i} + (B_i - B'_i)|_{S_i}$, there are finitely many possibilities for the coefficient $e'_i$ of $P_i$ in $B'_{S_i}$, and the coefficients of $P_i$ in $(B_i - B'_i)|_{S_i}$ satisfy the DCC. 

□

Next we show that a similar statement holds in dimension 3.
Proposition 4.4. Let \( \Lambda \subseteq [0, 1] \) be a DCC set of real numbers. Then there is a DCC set \( \Gamma \subseteq [0, 1] \) of real numbers such that the following holds: let \((X, B)\) be a pair such that

- \((X, B)\) is Q-factorial lc of dimension 3,
- the coefficients of \(B\) are in \(\Lambda\),
- \(S\) is a normal component of \([B]\).

Then the coefficients of \(B_S\) are in \(\Gamma\).

Proof. It is enough to treat the case when \(B\) has rational coefficients: indeed, otherwise there would be a sequence \((X_i, B_i), S_i\) satisfying the assumptions such that the coefficients of the \(B_{S_i}\) do not satisfy the DCC; but then we can decrease the coefficients of the \(B_i\) slightly to get a rational \(B_i'\) and such that the coefficients of the \(B_{S_i}'\) still do not satisfy the DCC (of course the coefficients of \(B_i'\) may not be in \(\Lambda\) but we can make sure that they would be in some DCC set \(\Lambda'\)).

Assume that the proposition does not hold for rational boundaries. Then there is a sequence \((X_i, B_i), S_i\) satisfying the assumptions with the \(B_i\) being rational such that the coefficients of the \(B_{S_i}\) do not satisfy the DCC. Let \(V_i\) be a general hypersurface section of \(X_i\) so that \(V_i\) is normal and \(C_i := V_i|_{S_i}\) is smooth. We can write

\[
(K_{X_i} + B_i + V_i)|_{C_i} = (K_{S_i} + B_{S_i} + C_i)|_{C_i} = K_{C_i} + B_{S_i}|_{C_i} =: K_{C_i} + B_{C_i}
\]

In particular, each coefficient of \(B_{S_i}\) appears as a coefficient of \(B_{C_i}\). Moreover, since \((S_i, B_{S_i})\) is lc and \(C_i\) is a smooth general sufficiently ample divisor, \((C_i, B_{C_i})\) is lc, that is, each coefficient of \(B_{C_i}\) belongs to \([0, 1]\).

On the other hand, by [16, Proposition 4.5(4)], we can write

\[
(K_{X_i} + B_i + V_i)|_{C_i} = (K_{V_i} + B_i|_{V_i})|_{C_i} =: K_{C_i} + \Delta_{C_i}
\]

But then \(B_{C_i} = \Delta_{C_i}\) by functorial properties of residue maps. Moreover, since \((C_i, B_{C_i}) = (C_i, \Delta_{C_i})\) is lc, \((V_i, B_i|_{V_i})\) is lc near \(C_i\) by [16, Proposition 4.5(2)]. Now apply Proposition 4.3 to \((V_i, B_i|_{V_i})\) near \(C_i\), recalling that the coefficients of \(B_i|_{V_i}\) belong to a fixed DCC set and \(C_i\) is a component of \([B_i|_{V_i}]\), to deduce that the coefficients of \(B_{C_i}\) belong to some fixed DCC set. This implies that the coefficients of the \(B_{S_i}\) belong to a fixed DCC set, a contradiction.

\[\square\]

4.5. Pl-extremal rays. Let \((X, B)\) be a projective Q-factorial dlt pair of dimension 3. A \(K_X + B\)-negative extremal ray \(R\) is called a pl-extremal ray if \(S \cdot R < 0\) for some component \(S\) of \([B]\). This is named after Shokurov’s pl-flips.

Assume that \(k\) has char \(p > 5\). Now as in 3.3, assume that \(C = N \cdot (X/Z)\) for some projective contraction \(X \to Z\) such that \(K_X + B \equiv P + M/Z\) where \(P\) is nef/Z and \(M \geq 0\), or \(C = N^+\) for some nef and big \(Q\)-divisor \(N\). Let \(R\) be a \(K_X + B\)-negative pl-extremal ray of \(C\). By 3.3, we can find a \(Q\)-boundary \(\Delta\) and an ample \(Q\)-divisor \(H\) such that \([\Delta] = S\), \((K_X + \Delta) \cdot R < 0\) and \(L = K_X + \Delta + H\) is nef and big with \(L^+ = R\). Let \(X \to V\) be the contraction associated to \(L\) which contracts \(R\) to an algebraic space. Every curve contracted by \(X \to V\)
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is inside $S$. So the exceptional locus $E(L)$ of $L$ is inside $S$. Thus $L$ is semi-ample by 2.13. Therefore $X \to V$ is a projective contraction. In other words, pl-extremal rays can be contracted by projective morphisms. This was proved in [11, Theorem 5.4] when $K_X + B$ is pseudo-effective. The extremal rays that appear below are often pl-extremal rays.

If $X \to V$ is a divisorial contraction put $X' = V$ but if it is a flipping contraction assume $X \dashrightarrow X'/V$ is its flip. Then it is not hard to see that in any case $X'$ is $\mathbb{Q}$-factorial, by the following argument which I learned from Chenyang Xu. We treat the divisorial case; the flipping case can be proved similarly. We can assume that $B$ is a $\mathbb{Q}$-boundary and $\Delta = B$. Let $D'$ be a prime divisor on $X'$ and $D$ its birational transform on $X$. There are rational numbers $\epsilon > 0$ and $\delta$ such that $M := K_X + B + H + \epsilon D + \delta S$ is nef and big, $M \equiv 0/V$, $H + \epsilon D + \delta S$ is ample, and $E(M) = E(L) = S$. Since $M|_S$ is semi-ample, $M$ is semi-ample by Theorem 2.9. That is, $M$ is the pullback of some ample divisor $M'$ on $X'$. But then $\epsilon D' = M' - L'$ is $\mathbb{Q}$-Cartier hence $D'$ is $\mathbb{Q}$-Cartier.

4.6. Special termination. The following important result is proved just like in char zero. We include the proof for convenience.

**Proposition 4.7.** Let $(X, B)$ be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of char $p > 5$. Assume that we are given an LMMP on $K_X + B$, say $X_i \dashrightarrow X_{i+1}/Z_i$ where $X_1 = X$ and each $X_i \dashrightarrow X_{i+1}/Z_i$ is a flip, or a divisorial contraction with $X_{i+1} = Z_i$. Then after finitely many steps, each remaining step of the LMMP is an isomorphism near the lc centres of $(X, B)$.

**Proof.** There are only finitely many lc centres and no new one can be created in the process, so we may assume that the LMMP does not contract any lc centre. In particular, we can assume that the LMMP is an isomorphism near each lc centre of dimension zero.

Now let $C$ be an lc centre of dimension one. Since $(X, B)$ is dlt, $C$ is a component of the intersection of two components $S, S'$ of $|B|$. Let $C_i, S_i \subset X_i$ be the birational transforms of $C, S$. Applying Lemma 4.2, we can see that $C_i, S_i$ are normal. By adjunction, we can write $(K_X + B_i)|_{S_i} = K_{S_i} + B_{S_i}$ where the coefficient of $C_i$ in $B_{S_i}$ is one. Applying adjunction once more, we can write the pullback of $K_{S_i} + B_{S_i}$ to $C_i$ as $K_{C_i} + B_{C_i}$ for some boundary $B_{C_i}$. Since $C_i \simeq C_{i+1}$, we will use the notation $(C, B_i, C)$ instead of $(C_i, B_{C_i})$. Since each step of the LMMP makes the divisor $K_X + B$ "smaller",

$$K_C + B_{i,C} \geq K_C + B_{i+1,C}$$

hence $B_{i,C} \geq B_{i+1,C}$ for every $i$. By Propositions 4.4 and 4.3, the coefficients of $B_{S_i}$ and $B_{i,C}$ belong to some fixed DCC set. Therefore $B_{i,C} = B_{i+1,C}$ for every $i \gg 0$ which implies that after finitely many steps, each remaining step of the LMMP is an isomorphism near $C_i$.

From now on we may assume that all the steps of the LMMP are flips. Let $S$ be any lc centre of dimension 2, i.e. a component of $B$ with coefficient one. If $S_i$ intersects the exceptional locus $E_i$ of $X_i \to Z_i$, then no other component
of $|B_i|$ can intersect the exceptional locus: assume that another component $T_i$ intersects the exceptional locus; if either $S_i$ or $T_i$ contains $E_i$, then $S_i \cap T_i$ intersects $E_i$; but $S_i \cap T_i$ is a union of lc centres of dimension one and this contradicts the last paragraph; so none of $S_i, T_i$ contains $E_i$. But then both contain the exceptional locus of $X_{i+1} \to Z_i$ and similar arguments give a contradiction.

Assume $C_i \subset S_i$ is a component of the exceptional locus of $X_i \to Z_{i-1}$ where $i > 1$. Then the log discrepancy of $C_i$ with respect to $(S_1, B_{S_1})$ is less than one. Moreover, we can assume that the generic point of the centre of $C_i$ on $S_1$ is inside the klt locus of $(S_1, B_{S_1})$. But there can be at most finitely many such $C_i$ (as prime divisors on birational models of $S_1$). Since the coefficients of $C_i$ in $B_{S_i}$ belongs to a DCC set, the coefficient of $C_i$ stabilizes. Therefore after finitely many steps, $S_i$ cannot contain any component of the exceptional locus of $X_i \to Z_{i-1}$. So we get a sequence $S_i \to S_{i+1}$ of birational morphisms which are isomorphisms if $i \gg 0$. In particular, $S_i$ is disjoint from $E_i$ for $i \gg 0$.

\[\square\]

5. Existence of log flips

In this section, we first prove that generalized flips exist (5.3). Next we prove Theorem 1.1 in the projective case, that is, when $X$ is projective. The general case of Theorem 1.1 is proved in Section 7 where $X$ is quasi-projective.

5.1. Divisorial and flipping extremal rays. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 over $k$, and let $R$ be a $K_X + B$-negative extremal ray. Assume that there is a nef and big $\mathbb{Q}$-divisor $L$ such that $R = L^+$. We say $R$ is a divisorial extremal ray if $\dim \mathcal{E}(L) = 2$. But we say $R$ is a flipping extremal ray if $\dim \mathcal{E}(L) = 1$. By 3.3, such rays can be contracted to algebraic spaces. By 3.2, when $K_X + B$ is pseudo-effective, each $K_X + B$-negative extremal ray is either a divisorial extremal ray or a flipping extremal ray. We will show below (1.5) that any divisorial or flipping extremal ray can actually be contracted by a projective morphism if $(X, B)$ is dlt. However, we still need contractions to algebraic spaces as an auxiliary tool.

5.2. Existence of generalized flips. We recall the definition of generalized flips which was introduced in [11]. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial pair of dimension 3 over $k$, and let $R$ be a $K_X + B$-negative flipping extremal ray. We say that the generalized flip of $R$ exists (see [11, after Theorem 5.6]) if there is a birational map $X \to X^+/V$ which is an isomorphism in codimension one, $X^+$ is $\mathbb{Q}$-factorial projective, and $K_{X^+} + B^+$ is numerically positive on any curve contracted by $X^+ \to V$. We first prove that generalized flips exist and then treat Theorem 1.1.

**Theorem 5.3.** Let $(X, B)$ be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of char $p > 5$. Let $R$ be a $K_X + B$-negative flipping extremal ray. Then the generalized flip of $R$ exists.
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The theorem was proved in [11, Theorem 5.6] when $B$ has standard coefficients and $K_X + B$ is pseudo-effective.

Proof. This proof (as well as the proof of [11, Theorem 5.6]) is modeled on the proof of Shokurov's reduction theorem [21, Theorem 1.2]. Since $R$ is a flipping extremal ray, by definition, there is a nef and big $\mathbb{Q}$-divisor $L$ such that $R = L^\perp$. Moreover, $L$ is endowed with a map $X \to V$ to an algebraic space which contracts the curves generating $R$. Note that if $B'$ is another boundary such that $(K_X + B') \cdot R < 0$, then the generalized flip exists for $(X, B)$ if and only if it exists for $(X, B')$. This follows from the fact that $K_X + B \equiv t(K_X + B')/V$ for some number $t > 0$ where the numerical equivalence means that $K_X + B - t(K_X + B')$ is numerically trivial on any curve contracted by $X \to V$.

Let $\mathcal{S}$ be the set of standard coefficients as defined in the introduction. Define

$$\zeta(X, B) = \#\{S \mid S \text{ is a component of } B \text{ and its coefficient is not in } \mathcal{S}\}$$

Assume that the generalized flip of $R$ does not exist. We will derive a contradiction. We can assume that $\zeta(X, B)$ is minimal, that is, we may assume that generalized flips always exist for pairs with smaller $\zeta$. We can decrease the coefficients of $[B]$ slightly so that $(X, B)$ becomes klt and $\zeta(X, B)$ is unchanged. In addition, each component $S$ of $B$ whose coefficient is not in $\mathcal{S}$ satisfies $S \cdot R < 0$ otherwise we can discard $S$ and decrease $\zeta(X, B)$ which is not possible by the minimality assumption.

First assume that $\zeta(X, B) > 0$. Choose a component $S$ of $B$ whose coefficient $b$ is not in $\mathcal{S}$. There is a positive number $a$ such that $K_X + B \equiv aS/V$. Let $g: W \to X$ be a log resolution, and let $B_W = B^\sim + E$ and $\Delta_W = B_W + (1 - b)S^\sim$ where $E$ is the reduced exceptional divisor of $g$ and $B^\sim, S^\sim$ are birational transforms. Note that $[B_W] = E$ and $[\Delta_W] = S^\sim + E$. Since $(X, B)$ is klt,

$$K_W + \Delta_W = K_W + B_W + (1 - b)S^\sim = g^*(K_X + B) + G + (1 - b)S^\sim$$

where $G$ is effective and its support is equal to the support of $E$. Thus

$$K_W + \Delta_W \equiv g^*(aS) + G + (1 - b)S^\sim = (a + 1 - b)S^\sim + F/V$$

where $F$ is effective and $\text{Supp } F = \text{Supp } E$. By construction, we have

$$\text{Supp } (S^\sim + F) = [\Delta_W] \text{ and } \zeta(W, \Delta_W) < \zeta(X, B)$$

Run an LMMP$/V$ on $K_W + \Delta_W$ with scaling of some ample divisor, as in 3.5. Recall that this is an LMMP$/\mathcal{C}$ on $K_W + \Delta_W$ where $\mathcal{C} = N^\perp$ and $N$ is the pullback of the nef and big $\mathbb{Q}$-divisor $L$. In each step some component of $[\Delta_W]$ is negative on the corresponding extremal ray. So such extremal rays are pl-extremal rays, they can be contracted by projective morphisms, and the $\mathbb{Q}$-factorial property is preserved (see 4.5). Moreover, if we encounter a flipping contraction, then its generalized flip exists because $\zeta(W, \Delta_W) < \zeta(X, B)$ and because we chose $\zeta(X, B)$ to be minimal; the flip is a usual one since its extremal ray is contracted projectively. By special termination (4.7), the LMMP terminates on some model $Y/V$. 

Now run an LMMP/V on \( K_Y + B_Y \) with scaling of \((1 - b)S_Y\) where \( B_Y \) is the pushdown of \( B_W \) and \( S_Y \) is the pushdown of \( S^- \). Since we have the numerical equivalence \( K_Y + B_Y \equiv aS_Y + F_Y/V \) and \( \text{Supp} \, F_Y = [B_Y] \), in each step of the LMMP the corresponding extremal ray intersects some component of \([B_Y]\) negatively hence they are pl-extremal rays and they can be contracted by projective morphisms (4.5). Moreover, if one of these rays gives a flipping contraction, then its generalized flip exists because \( K_Y + B_Y - bS_Y \) is negative on that ray and \( \zeta(Y, B_Y - bS_Y) < \zeta(X, B) \). Note that again such flips are usual flips. The LMMP terminates on a model \( X^+ \) by special termination.

Let \( h: W' \rightarrow X \) and \( e: W' \rightarrow X^+ \) be a common resolution. Now the negativity lemma (2.3) applied to the divisor \( h^*(K_X + B) - e^*(K_X^+ + B^+) \) over \( X \) implies that

\[ h^*(K_X + B) - e^*(K_X^+ + B^+) \geq 0 \]

Thus every component \( D \) of \( E \) is contracted over \( X^+ \) because

\[ 0 < a(D, X, B) \leq a(D, X^+, B^+) \]

Therefore \( X \rightarrow X^+ \) is an isomorphism in codimension one. It is enough to show that \( K_{X^+} + B^+ \) is numerically positive/V. Let \( H^+ \) be an ample divisor on \( X^+ \) and \( H \) its birational transform on \( X \). There is a positive number \( c \) such that \( K_X + B \equiv cH/V \) hence \( K_{X^+} + B^+ \equiv cH^+/V \) which implies that \( K_{X^+} + B^+ \) is numerically positive/V. So we have constructed the generalized flip and this contradicts our assumptions above.

Now assume that \( \zeta(X, B) = 0 \). If \( K_X + B \) is pseudo-effective, then we can simply apply [11, Theorem 5.6] to get a contradiction. Unfortunately, \( K_X + B \) may not be pseudo-effective (note that even if we originally start with a pseudo-effective log divisor we may end up with a non-pseudo-effective \( K_X + B \) since we decreased some coefficients). However, this is not a problem because the proof of [11, Theorem 5.6] still works. Since there is a nef and big \( \mathbb{Q} \)-divisor \( L \) with \( L \cdot R = 0 \), there is a prime divisor \( S \) with \( S \cdot R < 0 \). There is a number \( a > 0 \) such that \( K_X + B \equiv aS/V \). Now take a log resolution \( g: W \rightarrow X \) and define \( B_W \) and \( \Delta_W \) as above (if \( S \) is not a component of \( B \) simply let \( b = 0 \)). Run an LMMP/V on \( K_W + \Delta_W \). The extremal rays in the process are all pl-extremal rays hence they can be contracted by projective morphisms. Moreover, if we encounter a flipping contraction, then its flip exists by [11, Theorem 4.12] because all the coefficients of \( \Delta_W \) are standard. The LMMP terminates on some model \( Y \) by the special termination. Next, run the LMMP/V on \( K_Y + B_Y \) with scaling of \((1 - b)S_Y\). Again, the extremal rays in the process are all pl-extremal rays hence they can be contracted by projective morphisms. Moreover, if we encounter a flipping contraction, then its flip exists by [11, Theorem 4.12] because all the coefficients of \( B_Y \) are standard. The LMMP terminates on some model \( X^+ \) by the special termination. The rest of the argument goes as before.

\[ \square \]

5.4. Proof of 1.1 in the projective case.
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Proof. (of Theorem 1.1 in the projective case) Assume that X is projective. Then by Theorem 5.3, the generalized flip of the extremal ray of X → Z exists. But since X → Z is a projective contraction, the generalized flip is a usual flip.

If X is only quasi-projective, we postpone the proof to Section 7. Until then we need flips only in the projective case.

□

6. Crepant models

6.1. Divisorial extremal rays. The next lemma is essentially [11, Theorem 5.6(2)].

Lemma 6.2. Let (X, B) be a projective Q-factorial dlt pair of dimension 3 over k of char p > 5. Let R be a KX + B-negative divisorial extremal ray. Then R can be contracted by a projective morphism X → Z where Z is Q-factorial.

Proof. We may assume that (X, B) is klt. Since R is a divisorial extremal ray, by definition, there is a nef and big Q-divisor L such that R = L⊥ and dim E(L) = 2. Moreover, R can be contracted by a map X → V to an algebraic space. There is a prime divisor S with S · R < 0. In particular, E(L) ⊆ S and S is the only prime divisor contracted by X → V. There is a number a > 0 such that KX + B ≡ aS/V. Let g: W → X be a log resolution and define ∆W as in the proof of Theorem 5.3. Run an LMMP/V on KW + ∆W. As in 5.3, the extremal rays in the process are pl-extremal rays hence they are contracted projectively and the LMMP terminates with a model Z. We are done if we show that Z → V is an isomorphism. Assume this is not the case.

Recall that

\[ K_W + \Delta_W \equiv (a + 1 - b)S^\sim + F/V \]

and now \((a + 1 - b)S^\sim + F\) is exceptional/V. In particular, \((a + 1 - b)S_Z + F_Z\) is effective, exceptional and nef/V.

Let HZ be a general ample divisor on Z and H its birational transform on X. There is a number \(t \geq 0\) such that \(H + tS \equiv 0/V\). Therefore there is an effective and exceptional/V divisor PZ such that \(H_Z + P_Z \equiv 0/V\). Let s be the smallest number such that

\[ Q_Z := (a + 1 - b)S_Z + F_Z - sP_Z \leq 0 \]

Then \(Q_Z\) is numerically positive over V and there is some prime exceptional/V divisor D which is not a component of \(Q_Z\). This is not possible since \(Q_Z\) cannot be numerically positive on the general curves of D contracted/V.

□

6.3. Projectivization and dlt models.

Lemma 6.4. Let X be a normal projective variety over k and \(D \neq X\) a closed subset. Then there is a reduced effective Cartier divisor H whose support contains D.
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Proof. We may assume that each irreducible component of $D$ is a prime divisor hence we can think of $D$ as a reduced Weil divisor. Let $A$ be a sufficiently ample divisor. Let $U$ be the smooth locus of $X$. Since $(A-D)|_U$ is sufficiently ample, we can choose a reduced effective divisor $H'$ with no common components with $D$ such that $H'|_U \sim (A-D)|_U$. This extends to $X$ and gives $H' \sim A - D$. Now $H := H' + D \sim A$ is Cartier and satisfies the requirements.

□

The next few results are standard consequences of special termination (cf. [3, Lemma 3.3][11, Theorem 6.1]).

Lemma 6.5. Let $(X, B)$ be an lc pair of dimension 3 over $k$ of char $p > 5$, and let $\overline{X}$ be a projectivization of $X$. Then there is a projective $\mathbb{Q}$-factorial dlt pair $(\overline{Y}, B_{\overline{Y}})$ with a birational morphism $\overline{Y} \rightarrow \overline{X}$ satisfying the following:

- $K_{\overline{Y}} + B_{\overline{Y}}$ is nef$/\overline{X}$,
- let $Y$ be the inverse image of $X$ and $B_Y = B_{\overline{Y}}|_Y$; then $(Y, B_Y)$ is a $\mathbb{Q}$-factorial dlt model of $(X, B)$.

Proof. We may assume that $\overline{X}$ is normal. By Lemma 6.4, there is a reduced effective Cartier divisor $H$ containing the complement of $X$ in $\overline{X}$. We may assume that $H$ has no common components with $B$. Let $f: \overline{W} \rightarrow \overline{X}$ be a log resolution. Now let $B_{\overline{W}}$ be the sum of the reduced exceptional divisor of $f$ and the birational transform of $B$, and let $\Delta_{\overline{W}}$ be the sum of $B_{\overline{W}}$ and the birational transform of $H$.

Run the LMMP$/\overline{X}$ on $K_{\overline{W}} + \Delta_{\overline{W}}$ inductively as follows. Assume that we have arrived at a model $\overline{Y}$. Let $R$ be a $K_{\overline{Y}} + \Delta_{\overline{Y}}$-negative extremal ray$/\overline{X}$. Let $\overline{Y} \rightarrow \overline{Z}$ be the contraction of $R$ to an algebraic space, and let $L$ be a nef and big $\mathbb{Q}$-divisor with $L^+ = R$. Any curve contracted by $\overline{Y} \rightarrow \overline{Z}$ is also contracted over $\overline{X}$. If $\dim \mathcal{E}(L) = 2$, then $R$ is a divisorial extremal ray hence $\overline{Y} \rightarrow \overline{Z}$ is a projective contraction by Lemma 6.2. In this case, we continue the program with $\overline{Z}$. Now assume that $\dim \mathcal{E}(L) = 1$. Let $C$ be a connected component of $\mathcal{E}(L)$ and $P$ its image in $\overline{X}$ which is just a point. If $P \in \text{Supp} H$, then $C$ is contained in some component of the pullback of $H$ hence it is contained in some component of $[\Delta_{\overline{Y}}]$. In this case, $\overline{Y} \rightarrow \overline{Z}$ is again a projective contraction by 2.13. Now assume that $P$ does not belong to the support of $H$. Since $(X, B)$ is lc, over $X \setminus H$ the divisor

$$K_{\overline{W}} + \Delta_{\overline{W}} - f^*(K_X + B)$$

is effective and exceptional$/\overline{X}$ hence some component of $\Delta_{\overline{Y}}$ intersects $R$ negatively which implies again that the contraction $\overline{Y} \rightarrow \overline{Z}$ is projective. Therefore in any case $R$ can be contracted by a projective morphism and we can continue the LMMP as usual. The required flips exist by the results of Section 5. By special termination (4.7), the LMMP terminates say on $\overline{Y}$.

Next, we run the LMMP$/\overline{X}$ on $K_{\overline{Y}} + B_{\overline{Y}}$ with scaling of $\Delta_{\overline{Y}} - B_{\overline{Y}}$ as in 3.5. Note that $\Delta_{\overline{Y}} - B_{\overline{Y}}$ is nothing but the birational transform of $H$. Since the pullback of $H$ is numerically trivial over $\overline{X}$, each extremal ray in the process intersects some exceptional divisor negatively hence such extremal rays can be
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contracted by projective morphisms. Moreover, the required flips exist and by special termination the LMMP terminates on a model which we may again denote by \( Y \). Now let \( Y \) be the inverse image of \( X \) under \( Y \rightarrow X \) and let \( B_Y \) be the restriction of \( B_Y \) to \( Y \). Then \( (Y, B_Y) \) is a \( Q \)-factorial dlt model of \( (X, B) \) because \( K_Y + B_Y - g^*(K_X + B) \) is effective and exceptional hence zero as it is nef/X.

\[ \square \]

\textit{Proof. (of Theorem 1.6)} This is already proved in Lemma 6.5.

\[ \square \]

6.6. Extraction of divisors and terminal models.

\textbf{Lemma 6.7.} Let \( (X, B) \) be an lc pair of dimension 3 over \( k \) of char \( p > 5 \) and let \( \{D_i\}_{i \in I} \) be a finite set of exceptional/\( X \) prime divisors (on birational models of \( X \)) such that \( a(D_i, X, B) \leq 1 \). Then there is a \( Q \)-factorial dlt pair \( (Y, B_Y) \) with a projective birational morphism \( Y \rightarrow X \) such that

- \( K_Y + B_Y \) is the crepant pullback of \( K_X + B \),
- every exceptional/\( X \) prime divisor \( E \) of \( Y \) is one of the \( D_i \) or \( a(E, X, B) = 0 \),
- the set of exceptional/\( X \) prime divisors of \( Y \) includes \( \{D_i\}_{i \in I} \).

\textit{Proof.} By Lemma 6.5, we can assume that \( (X, B) \) is projective \( Q \)-factorial dlt. Let \( f: W \rightarrow X \) be a log resolution and let \( \{E_j\}_{j \in J} \) be the set of prime exceptional divisors of \( f \). We can assume that for some \( J' \subseteq J \), \( \{E_j\}_{j \in J'} = \{D_i\}_{i \in I} \).

Now define

\[ K_W + B_W := f^*(K_X + B) + \sum_{j \notin J'} a(E_j, X, B)E_j \]

which ensures that if \( j \notin J' \), then \( E_j \) is a component of \( |B_W| \). Run an LMMP/\( X \) on \( K_W + B_W \) which would be an LMMP on \( \sum_{j \notin J'} a(E_j, X, B)E_j \). So each extremal ray in the process intersects some component of \( |B_W| \) negatively hence such rays can be contracted by projective morphisms (4.5), the required flips exists (Section 5), and the LMMP terminates by special termination (4.7), say on a model \( Y \). Now \( (Y, B_Y) \) satisfies all the requirements.

\[ \square \]

\textit{Proof. (of Corollary 1.7)} Apply Lemma 6.7 by taking \( \{D_i\}_{i \in I} \) to be the set of all prime divisors with log discrepancy \( a(D_i, X, B) \leq 1 \).

\[ \square \]

7. Existence of log minimal models

7.1. Weak Zariski decompositions. Let \( D \) be an \( \mathbb{R} \)-Cartier divisor on a normal variety \( X \) and \( X \rightarrow Z \) a projective contraction over \( k \). A weak Zariski decomposition/\( Z \) for \( D \) consists of a projective birational morphism \( f: W \rightarrow X \) from a normal variety, and a numerical equivalence \( f^*D \equiv P + M/\mathbb{Z} \) such that

1. \( P \) and \( M \) are \( \mathbb{R} \)-Cartier divisors,
2. \( P \) is nef/\( Z \), and \( M \geq 0 \).
We then define $\theta(X, B, M)$ to be the number of those components of $f_* M$ which are not components of $\lfloor B \rfloor$.

7.2. From weak Zariski decompositions to minimal models. We use the methods of [2] to prove the following result.

**Proposition 7.3.** Let $(X, B)$ be a projective lc pair of dimension 3 over $k$ of char $p > 5$, and $X \to Z$ a projective contraction. Assume that $K_X + B$ has a weak Zariski decomposition/Z. Then $(X, B)$ has a log minimal model over $Z$.

**Proof.** Assume that $\mathcal{W}$ is the set of pairs $(X, B)$ and projective contractions $X \to Z$ such that

- $\mathbf{L}$: $(X, B)$ is projective, lc of dimension 3 over $k$,
- $\mathbf{Z}$: $K_X + B$ has a weak Zariski decomposition/Z, and
- $\mathbf{N}$: $(X, B)$ has no log minimal model over $Z$.

Clearly, it is enough to show that $\mathcal{W}$ is empty. Assume otherwise and let $(X, B)$ and $X \to Z$ be in $\mathcal{W}$. Let $f : W \to X$, $P$ and $M$ be the data given by a weak Zariski decomposition/Z for $K_X + B$ as in 7.1. Assume in addition that $\theta(X, B, M)$ is minimal. Perhaps after replacing $f$ we can assume that $f$ gives a log resolution of $(X, \operatorname{Supp}(B + f_* M))$. Let $B_W = B^- + E$ where $B^-$ is the birational transform of $B$ and $E$ is the reduced exceptional divisor of $f$. Then

$$K_W + B_W = f^*(K_X + B) + F \equiv P + M + F/Z$$

is a weak Zariski decomposition where $F \geq 0$ is exceptional/X. Moreover,

$$\theta(W, B_W, M + F) = \theta(X, B, M)$$

and any log minimal model of $(W, B_W)$ is also a log minimal model of $(X, B)$ [2, Remark 2.4]. So by replacing $(X, B)$ with $(W, B_W)$ and $M$ with $M + F$ we may assume that $W = X$, $(X, \operatorname{Supp}(B + M))$ is log smooth, and that $K_X + B \equiv P + M/Z$.

First assume that $\theta(X, B, M) = 0$, that is, $\operatorname{Supp} M \subseteq \lfloor B \rfloor$. Run the LMMP/Z on $K_X + B$ using $P + M$ as in 3.7. Obviously, $M$ intersects each extremal ray in the process, and since $\operatorname{Supp} M \subseteq \lfloor B \rfloor$, the rays are pl-extremal rays. Therefore those rays can be contracted by projective morphisms (4.5), the required flips exist (Section 5), and the LMMP terminates by special termination (4.6). Thus we get a log minimal model of $(X, B)$ over $\tilde{Z}$ which contradicts the assumption that $(X, B)$ and $X \to Z$ belong to $\mathcal{W}$. For the rest of the proof we do not use LMMP.

From now on we assume that $\theta(X, B, M) > 0$. Define

$$\alpha := \min \{ t > 0 \mid \lfloor (B + tM)^{\leq 1} \rfloor \neq \lfloor B \rfloor \}$$

where for a divisor $D = \sum d_i D_i$ we define $D^{\leq 1} = \sum d'_i D_i$ with $d'_i = \min \{ d_i, 1 \}$. In particular, $(B + \alpha M)^{\leq 1} = B + C$ for some $C \geq 0$ supported in $\operatorname{Supp} M$, and
\(\alpha M = C + A\) where \(A \geq 0\) is supported in \([B]\) and \(C\) has no common components with \([B]\). Note that \(\theta(X, B, M)\) is equal to the number of components of \(C\). The pair \((X, B + C)\) is lc and the expression

\[K_X + B + C \equiv P + M + C/Z\]

is a weak Zariski decomposition. By construction

\[\theta(X, B + C, M + C) < \theta(X, B, M)\]

so \((X, B + C)\) has a log minimal model over \(Z\) by minimality of \(\theta(X, B, M)\) and the definition of \(\mathfrak{M}\). Let \((Y, (B + C)_Y)\) be the minimal model.

Let \(g: V \to X\) and \(h: V \to Y\) be a common resolution. By definition, \(K_Y + (B + C)_Y\) is nef/\(Z\). In particular, the expression

\[g^*(K_X + B + C) = P' + M'\]

is a weak Zariski decomposition/\(Z\) of \(K_X + B + C\) where \(P' = h^*(K_Y + (B + C)_Y)\) and \(M' \geq 0\) is exceptional/\(Y\) (cf. [2, Remark 2.4 (2)]). Moreover,

\[g^*(K_X + B + C) = P' + M' \equiv g^*P + g^*(M + C)/Z\]

Since \(M'\) is exceptional/\(Y\),

\[h_*(g^*(M + C) - M') \geq 0\]

On the other hand,

\[g^*(M + C) - M' \equiv P' - g^*P/Z\]

is anti-nef/\(Y\) hence by the negativity lemma, \(g^*(M + C) - M' \geq 0\). Therefore \(\text{Supp } M' \subseteq \text{Supp } g^*(M + C) = \text{Supp } g^*M\).

Now,

\[
(1 + \alpha)g^*(K_X + B) \equiv g^*(K_X + B) + \alpha g^*P + \alpha g^*M \\
\equiv g^*(K_X + B) + \alpha g^*P + g^*C + g^*A \\
\equiv P' + \alpha g^*P + M' + g^*A/Z
\]

hence we get a weak Zariski decomposition/\(Z\) as

\[g^*(K_X + B) \equiv P'' + M''/Z\]

where

\[P'' = \frac{1}{1+\alpha}(P' + \alpha g^*P)\quad\text{and}\quad M'' = \frac{1}{1+\alpha}(M' + g^*A)\]

and \(\text{Supp } M'' \subseteq \text{Supp } g^*M\) hence \(\text{Supp } g_*(M'') \subseteq \text{Supp } M\). Since \(\theta(X, B, M)\) is minimal,

\[\theta(X, B, M) = \theta(X, B, M'')\]

So every component of \(C\) is also a component of \(g_*(M'')\) which in turn implies that every component of \(C\) is also a component of \(g_*(M')\). But \(M'\) is exceptional/\(Y\) hence so is \(C\) which means that \((B + C)_Y = B^{\sim} + C^{\sim} + E = B^{\sim} + E = B_Y\) where \(\sim\) stands for birational transform and \(E\) is the reduced exceptional divisor of \(Y \dashrightarrow X\). Thus we have \(P' = h^*(K_Y + B_Y)\). Although \(K_Y + B_Y\) is nef/\(Z\), \((Y, B_Y)\)
is not necessarily a log minimal model of \((X, B)\) over \(Z\) because condition (4) of definition of log minimal models may not be satisfied (see 2.7).

Let \(G\) be the largest \(\mathbb{R}\)-divisor such that \(G \leq g^*C\) and \(G \leq M'\). By letting \(\tilde{C} = g^*C - G\) and \(\tilde{M}' = M' - G\) we get the expression
\[
g^*(K_X + B) + \tilde{C} = P' + \tilde{M}'
\]
where \(\tilde{C}\) and \(\tilde{M}'\) are effective with no common components.

Assume that \(\tilde{C}\) is exceptional/\(X\). Then \(g^*(K_X + B) - P' = \tilde{M}' - \tilde{C}\) is antinef/\(X\) so by the negativity lemma \(\tilde{M}' - \tilde{C} \geq 0\) which implies that \(\tilde{C} = 0\) since \(\tilde{C}\) and \(\tilde{M}'\) have no common components. Thus
\[
g^*(K_X + B) - h^*(K_Y + B_Y) = \sum_D a(D, Y, B_Y)D - a(D, X, B)D = \tilde{M}'
\]
where \(D\) runs over the prime divisors on \(V\). If \(\text{Supp} g_*\tilde{M}' = \text{Supp} g_*M'\), then \(\text{Supp} \tilde{M}'\) contains the birational transform of all the prime exceptional/\(Y\) divisors on \(X\) hence \((Y, B_Y)\) is a log minimal model of \((X, B)\) over \(Z\), a contradiction. Thus
\[
\text{Supp}(g_*M' - g_*G) = \text{Supp} g_*\tilde{M}' \subset \text{Supp} g_*M' \subset \text{Supp} M
\]
so some component of \(C\) is not a component of \(g_*\tilde{M}'\) because \(\text{Supp} g_*G \subset \text{Supp} C\). Therefore
\[
\theta(X, B, M) > \theta(X, B, \tilde{M}')
\]
which gives a contradiction again by minimality of \(\theta(X, B, M)\) and the assumption that \((X, B)\) has no log minimal model over \(Z\).

So we may assume that \(\tilde{C}\) is not exceptional/\(X\). Let \(\beta > 0\) be the smallest number such that \(\tilde{A} := \beta g^*M - \tilde{C}\) satisfies \(g_*\tilde{A} \geq 0\). Then there is a component of \(g_*\tilde{C}\) which is not a component of \(g_*\tilde{A}\). Now
\[
(1 + \beta)g^*(K_X + B) \equiv g^*(K_X + B) + \beta g^*M + \beta g^*P
\]
\[
\equiv g^*(K_X + B) + \tilde{C} + \tilde{A} + \beta g^*P
\]
\[
\equiv P' + \beta g^*P + M' + \tilde{A}/Z
\]
where \(\tilde{M}' + \tilde{A} \geq 0\) by the negativity lemma. Thus we get a weak Zariski decomposition/\(Z\) as \(g^*(K_X + B) \equiv P'' + M'''/Z\) where
\[
P'' = \frac{1}{1 + \beta} (P' + \beta g^*P) \quad \text{and} \quad M''' = \frac{1}{1 + \beta} (\tilde{M}' + \tilde{A})
\]
and \(\text{Supp} g_*M''' \subset \text{Supp} M\). Moreover, by construction, there is a component \(D\) of \(g_*\tilde{C}\) which is not a component of \(g_*\tilde{A}\). Since \(g_*\tilde{C} \leq C\), \(D\) is a component of \(C\) hence of \(M\), and since \(\tilde{C}\) and \(\tilde{M}'\) have no common components, \(D\) is not a component of \(g_*\tilde{M}'\). Therefore \(D\) is not a component of \(g_*M''' = \frac{1}{1 + \beta}(g_*\tilde{M}' + g_*\tilde{A})\) which implies that
\[
\theta(X, B, M) > \theta(X, B, M''')
\]
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giving a contradiction again.

7.4. Proofs of 1.2 and 1.1.

Proof. (of Theorem 1.2) By applying Lemma 6.5, we can reduce the problem to the case when $X, Z$ are projective. By taking a crepant $\mathbb{Q}$-factorial terminal model using 1.7 we may in addition assume that the pair is $\mathbb{Q}$-factorial and terminal. Let

$$
\mathcal{E} = \{ B' \mid K_X + B' \text{ is pseudo-effective}/Z \text{ and } 0 \leq B' \leq B \}
$$

which is a compact subset of the $\mathbb{R}$-vector space $V$ generated by the components of $B$. Let $B'$ be an element in $\mathcal{E}$ which has minimal distance from 0 with respect to the standard metric on $V$. So either $B' = 0$, or $K_X + B''$ is not pseudo-effective$/Z$ for any $0 \leq B'' \leq B'$.

Run the generalized LMMP$/Z$ on $K_X + B'$ as follows [11, proof of Theorem 5.6]: let $R$ be a $K_X + B'$-negative extremal ray$/Z$. By 3.3, $R$ is either a divisorial extremal ray or a flipping extremal ray (see the beginning of Section 5 for definitions), and $R$ can be contracted to an algebraic space. If $R$ is a divisorial extremal ray, then it can actually be contracted by a projective morphism, by Lemma 6.2, and we continue the process. But if $R$ is a flipping extremal ray, then we use the generalized flip, which exists by Theorem 5.3, and then continue the process.

No component of $B'$ is contracted by the LMMP; otherwise let $X_1 \to X_{i+1}$ be the sequence of log flips and divisorial contractions of this LMMP where $X = X_1$. Pick $j$ so that $\phi_j: X_j \to X_{j+1}$ is a divisorial contraction which contracts a component $D_j$ of $B_j'$, the birational transform of $B'$. Now there is $a > 0$ such that

$$K_{X_j} + B_j' = \phi_j^*(K_{X_{j+1}} + B_{j+1}') + aD_j$$

Since $K_{X_{j+1}} + B_{j+1}'$ is pseudo-effective$/Z$, $K_{X_j} + B_j' - aD_j$ is pseudo-effective$/Z$ which implies that $K_X + B' - bD$ is pseudo-effective$/Z$ for some $b > 0$ where $D$ is the birational transform of $D_j$, a contradiction. Therefore every $(X_j, B_j')$ has terminal singularities. The LMMP terminates for reasons similar to the char zero case [23, Corollary 2.17][19, Theorem 6.17] (see also [11, proof of Theorem 1.2]). So we get a log minimal model of $(X, B')$ over $Z$, say $(Y, B_Y')$.

Let $g: V \to X$ and $h: V \to Y$ be a common resolution. By letting $P = h^*(K_Y + B_Y')$ and

$$M = g^*(K_X + B') - h^*(K_Y + B_Y')$$

we get a weak Zariski decomposition$/Z$ as $g^*(K_X + B) = P + M/Z$. Note that $M \geq 0$ because $g^*(K_X + B') - h^*(K_Y + B_Y') \geq 0$. Therefore $(X, B)$ has a log minimal model over $Z$ by Proposition 7.3.

Proof. (of Theorem 1.1 in general case) Recall that we proved the theorem when $X$ is projective, in Section 5. By perturbing the coefficients, we can assume that $(X, B)$ is klt. By Theorem 1.2, $(X, B)$ has a log minimal model over $Z$, say
8. The connectedness principle with applications to semi-ampleness

8.1. Connectedness. In this subsection, we prove the connectedness principle in dimension \( \leq 3 \). The proof is based on LMMP rather than vanishing theorems.

I learned the following lemma from Chenyang Xu although we state in a more general form and the proof we give is different from his.

**Lemma 8.2.** Let \((X, B)\) be a projective pair of dimension \( \leq 3 \) over \( k \) of char \( p > 5 \). Assume that \((X, B)\) is klt (resp. \(\mathbb{Q}\)-factorial dlt) and \(A\) is a nef and big (resp. ample) \(\mathbb{R}\)-divisor. Then there is \(0 \leq A' \sim_{\mathbb{R}} A\) such that \((X, B + A')\) is klt (resp. dlt).

**Proof.** We first treat the \(\mathbb{Q}\)-factorial dlt case. Let \(f : W \to X\) be a log resolution of \((X, B)\) which extracts only prime divisors with positive log discrepancy with respect to \((X, B)\). This exists by the definition of dlt pairs. There is an \(\mathbb{R}\)-divisor \(E'\) exceptional \(X\) with sufficiently small coefficients such that \(f^*A - E'\) is ample. Letting \(A'_W \sim_{\mathbb{R}} f^*A - E'\) be general and letting \(\Delta_W\) be the birational transform of \(B\) plus \(A'_W\) plus the reduced exceptional divisor of \(f\), we obtain a dlt pair \((W, \Delta_W)\) satisfying

\[
K_W + \Delta_W \sim_{\mathbb{R}} f^*(K_X + B + A) + E \equiv E/X
\]

where \(E \geq 0\) is exceptional \(X\) and its support contains every divisor contracted by \(f\). Running the LMMP/\(X\) on \(K_W + \Delta_W\) and using special termination (4.7) we get a log minimal model \((Y, \Delta_Y)\) of \((W, \Delta_W)\) over \(X\). Since \(E_Y \geq 0\) is exceptional and nef \(X, E_Y = 0\) hence \(Y \to X\) is a small morphism. Since \(X\) is \(\mathbb{Q}\)-factorial, \(Y \to X\) is the isomorphism. In particular, \((X, \Delta_X)\) is dlt. Now \(\Delta_X = A + A'\) where \(A' := f_*A'_W \sim_{\mathbb{R}} A\).

Now we deal with the klt case. By Theorem 1.6, we can assume that \(X\) is \(\mathbb{Q}\)-factorial. Since \(A\) is nef and big, by definition, \(A \sim_{\mathbb{R}} G + D\) with \(G \geq 0\) ample and \(D \geq 0\). So by replacing \(A\) with \((1 - \epsilon)A + \epsilon G\) and replacing \(B\) with \(B + \epsilon D\) we can assume that \(A\) is ample. Now apply the dlt case.

**Proof.** (of Theorem 1.8) Assume that the statement does not hold for some \(z\). By Lemma 6.5, there is a \(\mathbb{Q}\)-factorial dlt pair \((Y, B_Y)\) and a birational morphism \(g: Y \to X\) with \(K_Y + B_Y\) nef \(X\), every exceptional divisor of \(g\) is a component of \([B_Y]\), and \(g_*B_Y = B\). Moreover, \(K_Y + B_Y + E_Y = f^*(K_X + B)\) for some \(E_Y \geq 0\) with \(\text{Supp } E_Y \subseteq [B_Y]\). Also the non-klt locus of \((Y, B_Y)\), that is \([B_Y]\), maps surjectively onto the non-klt locus of \((X, B)\) hence \([B_Y]\) is not connected in some neighborhood of \(Y_z\).
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Now by assumptions, $K_Y + B_Y + E_Y + L_Y \sim_{\mathbb{R}} 0/Z$ for some globally nef and big $\mathbb{R}$-divisor $L_Y$. Since $X$ is $\mathbb{Q}$-factorial, we can write $L_Y \sim_{\mathbb{R}} A_Y + D_Y$ where $A_Y$ is ample and $D_Y \geq 0$ is exceptional$/X$. In particular, Supp $D_Y \subset [B_Y]$. By picking a general

$$G_Y \sim_{\mathbb{R}} \epsilon A_Y + (1 - \epsilon)L_Y - \delta [B_Y]$$

for some small $\delta > 0$ and applying Lemma 8.2 we can assume that $(Y, B_Y + G_Y)$ is dlt. By construction, $K_Y + B_Y + G_Y \sim_{\mathbb{R}} P_Y := -\epsilon D_Y - E_Y - \delta [B_Y] /Z$ and Supp $P_Y = [B_Y]$.

Run a generalized LMMP$/Z$ on $K_Y + B_Y + G_Y$ as in the proof of Theorem 1.2. We show that this is actually a usual LMMP hence it terminates by special termination (4.7). Assume that we have arrived at a model $Y'$ and let $R$ be a $K_{Y''} + B_{Y''} + G_{Y''}$-negative extremal ray$/Z$. Since $Y' \to Z$ is birational, $R$ is either a divisorial extremal ray or a flipping extremal ray. In the former case $R$ can be contracted by a projective morphism by Lemma 6.2. So assume $R$ is a flipping extremal ray. Then the generalized flip $Y' \dashrightarrow Y''/V$ exists by Theorem 5.3 where $Y' \to V$ is the contraction of $R$ to the algebraic space $V$. Since $P_{Y''} \cdot R < 0$, some component $S_{Y''}$ of $[B_{Y''}]$ intersects $R$ positively. Now there is a boundary $\Delta_{Y''}$ such that $(Y'', \Delta_{Y''})$ is plt, $S_{Y''} = [\Delta_{Y''}]$, and $(K_{Y''} + \Delta_{Y''}) \cdot R = 0$. But then we can find $N_{Y''} \geq 0$ such that $(Y'', \Delta_{Y''} + N_{Y''})$ is plt and $(K_{Y''} + \Delta_{Y''} + N_{Y''}) \cdot R < 0$. Therefore by 4.5 and 2.13, $Y'' \to V$ is a projective morphism which implies that $Y' \to V$ is also a projective morphism and that the flip is a usual flip.

We claim that the connected components of $[B_Y]$ over $z$ remain disjoint over $z$ in the course of the LMMP: assume not and let $Y'$ be the first model in the process such that there are irreducible components $S_Y, T_Y$ of $[B_Y]$ belonging to disjoint connected components over $z$ such that $S_Y, T_Y$ intersect over $z$. Let $\Delta_Y = B_Y - \tau([B_Y] - S_Y - T_Y)$ for some small $\tau > 0$. Then $(Y, \Delta_Y + G_Y)$ is plt in some neighborhood of $X_z$ because $[\Delta_Y + G_Y] = S_Y + T_Y$ and $S_Y, T_Y$ are disjoint over $z$. Moreover, $Y \dashrightarrow Y'$ is a partial LMMP on $K_Y + \Delta_Y + G_Y$ hence $(Y', \Delta_{Y''} + G_{Y''})$ is also plt over $z$. But since $S_{Y''}, T_{Y''}$ intersect over $z$, $(Y', \Delta_{Y''} + G_{Y''})$ cannot be plt over $z$, a contradiction.

Next we claim that no connected component of $[B_Y]$ over $z$ can be contracted by the LMMP (although some of their irreducible components might be contracted). By construction $-P_Y \geq 0$ and Supp $-P_Y = [B_Y]$, and $-P_Y$ is positive on each extremal ray in the LMMP. Write $-P_Y = \sum -P_Y^i$ where $-P_Y^i$ are the connected components of $-P_Y$ over $z$. By the previous paragraph, $-P_Y^i$ and $-P_Y^j$ remain disjoint during the LMMP if $i \neq j$. Moreover, if we arrive a model $Y'$ in the LMMP on which we contract an extremal ray $R$, then $-P_Y^i \cdot R > 0$ for some $j$ and $-P_Y^i \cdot R = 0$ for $i \neq j$. Therefore the contraction of $R$ cannot contract any of the $-P_Y^i$.

The LMMP ends up with a log minimal model $(Y', B_{Y'} + G_{Y'})$ over $Z$. Then $P_{Y'}$ is nef$/Z$. Assume that $Y'_z \not\subseteq$ Supp $P_{Y'}$ set-theoretically. Since $Y'_z$ intersects Supp $P_{Y'}$, there is some curve $C \subset Y'_z$ not contained in Supp $P_{Y'}$ but intersects
it. Then as \( -P_Y \geq 0 \) we have \( -P_Y \cdot C > 0 \) hence \( P_Y \cdot C < 0 \), a contradiction. Now since \( Y_z' \) is connected, it is contained in exactly one connected component of \( [B_Y'] \) over \( z \). This is a contradiction because by assumptions at least two connected components of \( [B_Y'] \) over \( z \) intersect the fibre \( Y_z' \).

\[ \square \]

We now show that a strong form of the connectedness principle holds on surfaces.

**Theorem 8.3.** Let \((X, B)\) be a \( \mathbb{Q} \)-factorial projective pair of dimension 2 over \( k \). Let \( f: X \to Z \) be a projective contraction (not necessarily birational) such that \(- (K_X + B)\) is ample/\( Z \). Then for any closed point \( z \in Z \), the non-klt locus \( N \) of \((X, B)\) is connected in any neighborhood of the fibre \( X_z \) over \( z \). More strongly, \( N \cap X_z \) is connected.

**Proof.** It is enough to prove the last claim. Assume that \( N \cap X_z \) is not connected for some \( z \). We use the notation and the arguments of the proof of Theorem 1.8. Let \((Y, B_Y)\) be the pair constructed over \( X \) and \( Y \to Y' \) the LMMP/\( Z \) on \( K_Y + B_Y + G_Y \sim_{\mathbb{R}} P_Y \) and \( h: Y' \to Z \) the corresponding map. The same arguments of the proof of Theorem 1.8 show that the connected components of \( P_Y \) over \( z \) remain disjoint in the course of the LMMP and none of them will be contracted.

By assumptions, \([B_Y] \cap Y_z\) is not connected. We claim that the same holds in the course of the LMMP. If not, then at some step of the LMMP we arrive at a model \( W \) with a \( K_W + B_W + G_W \)-negative extremal birational contraction \( \phi: W \to V \) such that \([B_W] \cap W_z\) is not connected but \([B_Y] \cap V_z\) is connected. Let \( C \) be the exceptional curve of \( W \to V \). Now \( \phi([B_W]) = [B_Y] \): the inclusion \( \supseteq \) is clear; the inclusion \( \subseteq \) follows from the fact that if \( C \) is a component of \([B_W] \cap V_z\), then at least one other irreducible component of \([B_Y] \cap V_z\) intersects \( C \). Therefore \( \phi([B_W] \cap W_z) = [B_Y] \cap V_z \). Since \([B_Y] \cap V_z\) is connected but \([B_W] \cap W_z\) is not connected, there exist two connected components of \([B_W] \cap W_z\) whose images under \( \phi \) intersect. So there are closed points \( w, w' \) belonging to different connected components of \([B_W] \cap W_z\) such that \( \phi(w) = \phi(w') \). In particular, \( w, w' \in C \). Note that \( C \) is not a component of \([B_W] \) otherwise \( C \subset [B_W] \cap W_z \) connects \( w, w' \) which contradicts the assumptions. Therefore \([B_W] \cap C\) is a finite set of closed points with more than one element. Now perturbing the coefficients of \( B_W + G_W \) we can find a \( \Gamma_w \leq [B_W] \) such that \((W, \Gamma_w)\) is plt in a neighborhood of \( C \), \((K_W + \Gamma_w) \cdot C < 0 \) and such that \([\Gamma_w] \cap C\) is a finite set of closed points with more than one element. Then in a formal neighborhood of \( \phi(w), [\Gamma_v] \) has at least two branches which implies that \([\Gamma_v] \) is not normal which in turn contradicts the plt property of \((V, \Gamma_v)\).

Since \([B_Y] \cap Y_z'\) is not connected, there is a component \( D \) of \( Y_z' \) not contained in \( \text{Supp } P_Y = [B_Y] \) but intersects it. Thus \( P_Y \) cannot be nef/\( Z \) as \(-P_Y \) \( \geq 0 \). Therefore the LMMP terminates with a Mori fibre space \( Y' \to Z'/Z \). If \( Z' \) is a point, then \([B_Y]\) has at least two disjoint irreducible components which contradicts the fact that the Picard number \( \rho(Y') = 1 \) in this case. So we can assume that \( Z' \) is a curve.
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Assume that \( Z \) is also a curve in which case \( Z' = Z \). Let \( F \) be the reduced variety associated to a general fibre of \( Y' \to Z' \). Then by the adjunction formula we get \( F \cong \mathbb{P}^1 \), \( K_{Y'} \cdot F = -2 \), and \( (B_{Y'} + G_{Y'}) \cdot F < 2 \). On the other hand, since \( [B_{Y'}] \cap Y'_z \) has at least two points, \( [B_{Y'}] \cap F \) also has at least two points hence
\[
(B_{Y'} + G_{Y'}) \cdot F \geq (|B_{Y'}| + G_{Y'}) \cdot F > 2
\]
which is a contradiction. Now assume that \( Z \) is a point. Since \( [B_{Y'}] \cap Y'_z \) is not connected, \( [B_{Y'}] \) has at least two disjoint connected components, say \( M_{Y'} \), \( N_{Y'} \). On the other hand, since \( P_{Y'} \cdot F < 0 \), we may assume that \( M_{Y'} \) intersects \( F \) (hence \( M_{Y'} \) intersects every fibre of \( Y' \to Z' \)). If some component of \( N_{Y'} \) is vertical/\( Z' \), then \( M_{Y'}, N_{Y'} \) intersect a contradiction. Thus each component of \( N_{Y'} \) is horizontal/\( Z' \) hence they intersect each fibre of \( Y' \to Z' \). But then we can get a contradiction as in the \( Z' = Z \) case.

\[ \square \]

8.4. Semi-ampleness. We use the connectedness principle on surfaces to prove some semi-ampleness results in dimension 2 and 3. These are not only interesting on their own but also useful for the proof of the finite generation (1.3).

Proof. (of Theorem 1.9) Let \( S \leq |B| \) be a reduced divisor. Assume that \( (K_X + B + A)|_S \) is not semi-ample. We will derive a contradiction. We can assume that if \( S' \leq S \) is any other reduced divisor, then \( (K_X + B + A)|_{S'} \) is semi-ample. Note that \( S \) cannot be irreducible by abundance for surfaces (cf. [24]). Using the ample divisor \( A \) and applying Lemma 8.2, we can perturb the coefficients of \( B \) so that we can assume \( S = |B| \).

Let \( T \) be an irreducible component of \( S \) and let \( S' = S - T \). By assumptions, \( (K_X + B + A)|_T \) and \( (K_X + B + A)|_{S'} \) are both semi-ample. Let \( g: T \to Z \) be the projective contraction associated to \( (K_X + B + A)|_T \). By adjunction define \( K_T + B_T := (K_X + B)|_T \) and \( A_T = A|_T \). Since \( K_T + B_T + A_T \sim 0/Z \) and since \( A_T \) is ample, \( -(K_T + B_T) \) is ample/\( Z \). Moreover, \( S' \cap T = |B_T| \) as topological spaces. By the connectedness principle for surfaces (8.3), \( |B_T| \to Z \) has connected fibres hence \( S' \cap T \to Z \) also has connected fibres. Now apply Keel [14, Corollary 2.9].

\[ \square \]

Theorem 8.5. Let \( (X, B + A) \) be a projective \( \mathbb{Q} \)-factorial dlt pair of dimension 3 over \( k \) of char \( p > 5 \). Assume that
- \( A, B \geq 0 \) are \( \mathbb{Q} \)-divisors with \( A \) ample,
- \( (Y, B_Y + A_Y) \) is a \( \mathbb{Q} \)-factorial weak lc model of \( (X, B + A) \),
- \( Y \dashrightarrow X \) does not contract any divisor,
- \( \text{Supp} A_Y \) does not contain any lc centre of \( (Y, B_Y + A_Y) \),
- if \( \Sigma \) is a connected component of \( \mathbb{E}(K_Y + B_Y + A_Y) \) and \( \Sigma \not\subseteq |B_Y| \), then \( (K_Y + B_Y + A_Y)|_{\Sigma} \) is semi-ample.

Then \( K_Y + B_Y + A_Y \) is semi-ample.
**Proof.** Note that if $K_X + B + A$ is not big, then $\mathcal{E}(K_Y + B_Y + A_Y) = Y$ hence the statement is trivial. So we can assume that $K_X + B + A$ is big. Let $\phi$ denote the map $X \dashrightarrow Y$ and let $U$ be the largest open set over which $\phi$ is an isomorphism. Then since $A$ is ample and $X$ is $\mathbb{Q}$-factorial, $\text{Supp} A_Y$ contains $Y \setminus \phi(U)$: indeed let $y \in Y \setminus \phi(U)$ be a closed point and let $W$ be the normalization of the graph of $\phi$, and $\alpha: W \to X$ and $\beta: W \to Y$ be the corresponding morphisms; first assume that $\dim \beta^{-1}\{y\} > 0$: then $\alpha^* A$ intersects $\beta^{-1}\{y\}$ because $A$ is ample hence $\text{Supp} A_Y$ contains $y$; now assume that $\dim \beta^{-1}\{y\} = 0$: then $\beta$ is an isomorphism over $y$; on the other hand, $\alpha$ cannot be an isomorphism near $\beta^{-1}\{y\}$ otherwise $\phi$ would be an isomorphism near $\alpha(\beta^{-1}\{y\})$ hence $y \in \phi(U)$, a contradiction; thus as $X$ is $\mathbb{Q}$-factorial, $\alpha$ contracts some prime divisor $E$ containing $\beta^{-1}\{y\}$; but then $Y \dashrightarrow X$ contracts a divisor, a contradiction.

Let $C \geq 0$ be any $\mathbb{Q}$-divisor such that $(X, B + A + C)$ is dlt. Then $(Y, B_Y + A_Y + \epsilon C_Y)$ is dlt for any sufficiently small $\epsilon > 0$ because $(Y, B_Y + A_Y)$ has no lc centre inside $Y \setminus \phi(U) \subset \text{Supp} A_Y$. Now let $G_Y \geq 0$ be a general small ample $\mathbb{Q}$-divisor on $Y$ and $G$ its birational transform on $X$. Since $G$ is small, $A - G$ is ample. Let $C \sim_{\mathbb{Q}} A - G$ be a general $\mathbb{Q}$-divisor. Let

$$\Gamma_Y := B_Y + (1 - \epsilon) A_Y + \epsilon C_Y + \epsilon G_Y$$

Then

$$K_Y + \Gamma_Y \sim_{\mathbb{Q}} K_Y + B_Y + A_Y$$

and $|B_Y| = |\Gamma_Y|$. Moreover, by the above remarks and by Lemma 8.2 we can assume that $(Y, \Gamma_Y)$ is dlt.

Now by Theorem 1.9, $(K_Y + \Gamma_Y)|_{\Gamma_Y}$ is semi-ample hence $(K_Y + B_Y + A_Y)|_{|B_Y|}$ is semi-ample. Therefore $(K_Y + B_Y + A_Y)|_{\Sigma}$ is semi-ample for any connected component of $\mathcal{E}(K_Y + B_Y + A_Y)$ hence we can apply Theorem 2.9.

\[ \square \]

## 9. Finite generation and base point freeness

### 9.1. Finite generation

In this subsection we prove Theorem 1.3.

**Lemma 9.2.** Let $(X, B)$ be a pair and $M$ a $\mathbb{Q}$-divisor satisfying the following properties:

1. $(X, \text{Supp}(B + M))$ is projective log smooth of dimension $3$ over $k$ of char $p > 5$;
2. $K_X + B$ is a big $\mathbb{Q}$-divisor;
3. $K_X + B \sim_{\mathbb{Q}} M \geq 0$ and $|B| \subset \text{Supp} M \subset \text{Supp} B$;
4. $M = A + D$ where $A$ is an ample $\mathbb{Q}$-divisor and $D \geq 0$;
5. $\alpha M = N + C$ for some rational number $\alpha > 0$ such that $N, C \geq 0$ are $\mathbb{Q}$-divisors, $\text{Supp} N = |B|$, and $(X, B + C)$ is dlt;
6. there is an ample $\mathbb{Q}$-divisor $A' \geq 0$ such that $A' \leq A$ and $A' \leq C$.

If $(X, B + tC)$ has an lc model for some real number $t \in (0, 1]$, then $(X, B + (t - \epsilon)C)$ also has an lc model for any sufficiently small $\epsilon > 0$. 


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Proof. We can assume that $C \neq 0$. If we let $\Delta = B - \delta(N + C)$ for some small rational number $\delta > 0$, then $(X, \Delta)$ is klt and $K_X + \Delta$ is a positive multiple of $K_X + \Delta$ up to $\mathbb{Q}$-linear equivalence. Similarly, for any $s \in (0, 1]$, there is $s' \in (0, s)$ such that $(X, \Delta + s'C)$ is klt and $K_X + B + sC$ is a positive multiple of $K_X + \Delta + s'C$ up to $\mathbb{Q}$-linear equivalence. So if $(Y, \Delta_Y + s'C_Y)$ is a log minimal model of $(X, \Delta + s'C)$, which exists by Theorem 1.2, then $(Y, B_Y + sC_Y)$ is a $\mathbb{Q}$-factorial weak lc model of $(X, B + sC)$ such that $Y \dasharrow X$ does not contract divisors and $X \dasharrow Y$ is $K_X + B + sC$-negative (see 2.2 for this notion). We will make use of this observation below.

Let $T$ be the lc model of $(X, B + tC)$ and let $(Y, B_Y + tC_Y)$ be a $\mathbb{Q}$-factorial weak lc model of $(X, B + tC)$ such that $X \dasharrow Y$ is $K_X + B + tC$-negative and its inverse does not contract divisors. Then the induced map $Y \dasharrow T$ is a morphism and $K_T + B_T + tC_T$ pulls back to $K_Y + B_Y + tC_Y$.

First assume that $t$ is irrational. Then $C_Y \equiv 0/T$. Moreover, $C_T$ is $\mathbb{Q}$-Cartier because the set of those $s \in \mathbb{R}$ such that $K_T + B_T + sC_T$ is $\mathbb{R}$-Cartier forms a rational affine subspace of $\mathbb{R}$ (this can be proved using simple linear algebra similar to 3.4). Since $t$ belongs to this affine subspace and $t$ is not rational, the affine subspace is equal to $\mathbb{R}$ hence $K_T + B_T + sC_T$ is $\mathbb{R}$-Cartier for every $s$ which implies that $C_T$ is $\mathbb{Q}$-Cartier. Thus $C_Y \sim_\mathbb{Q} 0/T$ hence $K_T + B_T + (t - \epsilon)C_T$ pulls back to $K_Y + B_Y + (t - \epsilon)C_Y$ and the former is ample for every sufficiently small $\epsilon > 0$. This means that $T$ is also the lc model of $(X, B + (t - \epsilon)C)$.

From now on we assume that $t$ is rational. Replace $Y$ with a $\mathbb{Q}$-factorial weak lc model of $(X, B + (t - \epsilon)C_Y)$ over $T$ so that $X \dasharrow Y$ is still $K_X + B + (t - \epsilon)C$-negative. Since $K_T + B_T + tC_T$ is ample, by choosing $\epsilon$ to be small enough, we can assume that $K_X + B + (t - \epsilon)C$ is nef globally, by 3.3. Then $(Y, B_Y + (t - \epsilon)C_Y)$ is a weak lc model of $(X, B + (t - \epsilon)C)$ hence it is enough to show that $K_Y + B_Y + (t - \epsilon)C_Y$ is semi-ample. Perhaps after replacing $\epsilon$ with a smaller number we can assume that $K_Y + B_Y + (t - \epsilon')C_Y$ also nef globally for some $\epsilon' > \epsilon$.

Let $Y \dasharrow V$ be the contraction to an algebraic space associated to $K_Y + B_Y + (t - \epsilon)C_Y$. Any curve contracted by $Y \dasharrow V$ is also contracted by $Y \dasharrow T$ because $K_Y + B_Y + tC_Y$ and $K_Y + B_Y + (t - \epsilon')C_Y$ are both nef and $\epsilon' > \epsilon$. Thus we get an induced map $V \dasharrow T$. Moreover, there is a small contraction $Y' \dasharrow V$ from a $\mathbb{Q}$-factorial normal projective variety $Y'$: recall that $(Y, \Lambda_Y := \Delta_Y + t'C_Y)$ is klt where $\Delta$ and $t'$ are as in the first paragraph; now $Y'$ can be obtained by taking a log resolution $W \dasharrow Y$, defining $\Lambda_W$ to be the birational transform of $\Lambda_Y$ plus the reduced exceptional divisor of $W \dasharrow V$, running an L MMP/V on $K_W + \Lambda_W$, and using special termination and the fact that $K_W + \Lambda_W \equiv E/V$ for some $E \geq 0$ whose support is equal to the reduced exceptional divisor of $W \dasharrow V$. Since $K_Y + B_Y + (t - \epsilon)C_Y \equiv 0/V$, $K_{Y'} + B_{Y'} + (t - \epsilon)C_{Y'}$ is also nef and the former is semi-ample if and only if the latter is. So by replacing $Y$ with $Y'$, we can in addition assume that $Y \dasharrow V$ is a small contraction.

Let $\Sigma$ be a connected component of the exceptional set of $Y \dasharrow V$. Since $Y \dasharrow V$ is a small morphism, $\Sigma$ is one-dimensional. On the other hand, since $K_Y + B_Y + (t - \epsilon)C_Y \equiv 0/V$
and
\[ K_Y + B_Y + tC_Y \equiv 0/V \]

we get \( C_Y \equiv 0/V \) hence \( N_Y \equiv 0/V \). Therefore either \( \Sigma \subset \text{Supp} N_Y \) or \( \Sigma \cap \text{Supp} N_Y = \emptyset \). Moreover, if \( \Sigma \cap \text{Supp} N_Y = \emptyset \), then \( (K_Y + B_Y + (t - \epsilon)C_Y)|_\Sigma \) is semi-ample because near \( \Sigma \) the divisor \( K_Y + B_Y + (t - \epsilon)C_Y \) is a multiple of \( K_Y + B_Y + tC_Y \) and the latter is semi-ample.

We can assume that \( A' \) in (6) has small coefficients. Let \( B' = B + (t - \epsilon)C - A' \). Since \( (Y, B_Y' + A_Y' + \epsilon C_Y) \) is lc, \( \text{Supp} C_Y \) (hence also \( \text{Supp} A_Y' \)) does not contain any lc centre of \( (Y, B_Y' + A_Y') \). Now applying Theorem 8.5 to \( (X, B' + A') \) shows that \( K_Y + B_Y + (t - \epsilon)C_Y \) is semi-ample. Note that the exceptional locus of \( Y \to V \) is equal to \( \mathbb{E}(K_Y + B_Y' + A_Y') \).

\[ \square \]

Proof. (of Theorem 1.3 when \( Z \) is a point) Step 1. By taking a \( \mathbb{Q} \)-factorialization we can assume that \( X \) is \( \mathbb{Q} \)-factorial. We follow the proof of [3, Proposition 3.4] but with some twists. There is \( M \geq 0 \) such that \( K_X + B \sim_{\mathbb{Q}} M \). We can choose \( M \) so that \( M = A + D \) where \( A \geq 0 \) is ample and \( D \geq 0 \). By taking a log resolution we can assume that \( (X, \text{Supp}(B + M)) \) is log smooth. Moreover, by adding a small multiple of \( M \) to \( B \) we can also assume that \( \text{Supp} M \subseteq \text{Supp} B \).

Then \( (X, B) \), \( M \) satisfy the properties (1) to (4) listed in Lemma 9.2. Therefore it is enough to show that \( R(K_X + B) \) is finitely generated for any pair \( (X, B) \) satisfying those properties (of course such pairs are not necessarily klt).

Step 2. Let \( (X, B), M \) satisfy the properties (1) to (4) listed in Lemma 9.2. Assume that \( R(K_X + B) \) is not finitely generated. We will derive a contradiction.

By changing \( A \) up to \( \mathbb{Q} \)-linear equivalence we can assume that

(7) \( A = aS \) where \( S \) is a sufficiently ample prime divisor, and \( a \) is a sufficiently small rational number.

In particular, this ensures that \( \theta(X, B, M) > 0 \) otherwise \( K_X + B \) is ample hence \( R(K_X + B) \) is finitely generated, a contradiction.

Define
\[ \alpha := \min \{ t > 0 \mid \lfloor (B + tM)^{\leq 1} \rfloor \neq [B] \} \]

In particular, \( (B + \alpha M)^{\leq 1} = B + C \) for some \( C \geq 0 \) supported in \( \text{Supp} M \), and \( \alpha M = C + N \) where \( N \geq 0 \) is supported in \([B]\) and \( C \) has no common components with \([B]\). Property (3) ensures that \( \text{Supp} N = [B] \). The pair \( (X, B + C) \) is dlt log smooth,

\[ \theta(X, B + C, M + C) < \theta(X, B, M) \]

and \( (X, B + C) \) and \( M + C \) satisfy properties (1) to (4) of the proposition. If \( R(K_X + B + C) \) is not finitely generated we replace \( (X, B) \) with \( (X, B + C) \) and replace \( M \) with \( M + C \) and repeat the process. After finitely many times doing this we get to the situation in which \( R(K_X + B + C) \) is finitely generated, by property (7). Note that properties (5) and (6) of Lemma 9.2 are satisfied since we can assume that \( A \leq C \).
Step 3. Let
\[ \mathcal{T} = \{ t \in [0, 1] \mid (X, B + tC) \text{ has an lc model} \} \]
Since \( 1 \in \mathcal{T} \), \( \mathcal{T} \neq \emptyset \). Moreover, if \( t \in \mathcal{T} \cap (0, 1) \), then by Lemma 9.2, \( [t-\epsilon, t] \subset \mathcal{T} \) for some \( \epsilon > 0 \). Now let \( \tau = \inf \mathcal{T} \). If \( \tau \in \mathcal{T} \), then \( \tau = 0 \) which implies that \( R(K_X + B) \) is finitely generated, a contradiction. So we may assume \( \tau \notin \mathcal{T} \).

There is a sequence \( (t_1, t_2, \ldots) \) of rational numbers in \( \mathcal{T} \) approaching \( \tau \). For each \( i \), there is a \( \mathbb{Q} \)-factorial weak lc model \( (Y_i, B_{Y_i} + t_iC_{Y_i}) \) of \( (X, B + t_iC) \) such that \( Y_i \to X \) does not contract divisors (see the beginning of the proof of Lemma 9.2). By taking a subsequence, we can assume that all the \( Y_i \) are isomorphic in codimension one. In particular, \( N_\sigma(K_{Y_i} + B_{Y_i} + \tau C_{Y_i}) = 0 \).

Arguing as in the proof of Theorem 8.5, we can show that \( (Y_1, B_{Y_1} + \tau C_{Y_1}) \) is dlt because \( A \leq C \) is ample and \( \text{Supp} A_{Y_1} \) does not contain any lc centre of \( (Y_1, B_{Y_1} + \tau C_{Y_1}) \). Run the LMMP on \( K_{Y_1} + B_{Y_1} + \tau C_{Y_1} \) with scaling of \( (t_1 - \tau)C_{Y_1} \) as in 3.5. Since \( \alpha M_{Y_1} = N_{Y_1} + C_{Y_1} \), the LMMP is also an LMMP on \( N_{Y_1} \). Thus each extremal ray in the process is a pl-extremal ray hence they can be contracted by projective morphisms (4.5). Moreover, the required flips exist by Theorem 1.1, and the LMMP terminates with a model \( Y \) on which \( K_Y + B_Y + \tau C_Y \) is nef, by special termination (4.7). Note that the LMMP does not contract any divisors by the \( N_\sigma = 0 \) property. Moreover, \( K_Y + B_Y + (\tau + \delta)C_Y \) is nef for some \( \delta > 0 \). Now, by replacing the sequence we can assume that \( K_Y + B_Y + t_iC_Y \) is nef for every \( i \) and by replacing each \( Y_i \) with \( Y \) we can assume that \( Y_i = Y \) for every \( i \). A simple comparison of discrepancies (cf. [3, Claim 3.5]) shows that \( (Y, B_Y + \tau C_Y) \) is a \( \mathbb{Q} \)-factorial weak lc model of \( (X, B + \tau C) \).

Step 4. Let \( T_i \) be the lc model of \( (X, B + t_iC) \). Then the map \( Y \to T_i \) is a morphism and \( K_Y + B_Y + t_iC_Y \) is the pullback of an ample divisor on \( T_i \). Moreover, for each \( i \), the map \( T_{i+1} \to T_i \) is a morphism because any curve contracted by \( Y \to T_{i+1} \) is also contracted by \( Y \to T_i \). So perhaps after replacing the sequence, we can assume that \( T_i \) is independent of \( i \) so we can drop the subscript and simply use \( T \). Since \( C \sim_Q 0/T \), we can replace \( Y \) with a \( \mathbb{Q} \)-factorialization of \( T \) so that we can assume that \( Y \to T \) is a small morphism (such a \( \mathbb{Q} \)-factorialization exists by the observations in the first paragraph of the proof of Lemma 9.2).

Assume that \( \tau \) is irrational. If \( K_Y + B_Y + (\tau - \epsilon)C_Y \) is nef for some \( \epsilon > 0 \), then \( K_Y + B_Y + \tau C_Y \) is semi-ample because in this case \( K_T + B_T + (\tau - \epsilon)C_T \) is nef and \( K_T + B_T + t_iC_T \) is ample hence \( K_T + B_T + \tau C_T \) is ample. If there is no \( \epsilon \) as above, then by 3.4 and 3.3, there is a curve \( \Gamma \) generating some extremal ray such that \( (K_Y + B_Y + \tau C_Y) \cdot \Gamma = 0 \) and \( C_Y \cdot \Gamma > 0 \). This is not possible since \( \tau \) is assumed to be irrational. So from now on we assume that \( \tau \) is rational.

Step 5. Let \( Y \to V \) be the contraction to an algebraic space associated to \( K_Y + B_Y + \tau C_Y \). This map factors through \( Y \to T \) so we get an induced map \( T \to V \). We can write
\[
K_T + B_T + \tau C_T = a(K_T + B_T + t_iC_T) + bN_T
\]
for some $i$ and some rational numbers $a, b > 0$. Since $K_T + B_T + t_i C_T$ is ample, we get
\[ \mathcal{E}(K_T + B_T + \tau C_T) \subset \text{Supp } N_T = |B_T| \]
Thus since $N_Y \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} C_Y/T$, the locus $\mathcal{E}(K_Y + B_Y + \tau C_Y)$ is a subset of the union of $\text{Supp } N_Y = |B_Y|$ and the exceptional set of $Y \to T$. Let $\Lambda$ be a connected component of the exceptional set of $Y \to T$. Then, since $N_Y \sim_{\mathbb{Q}} 0/T$ and since $\Lambda$ is one-dimensional, either $\Lambda \subset \text{Supp } N_Y$ or $\Lambda \cap \text{Supp } N_Y = \emptyset$. Therefore if $\Sigma$ is a connected component of $\mathcal{E}(K_Y + B_Y + \tau C_Y)$, then either $\Sigma \subset \text{Supp } N_Y$ or $\Sigma \cap \text{Supp } N_Y = \emptyset$. In the latter case, $(K_Y + B_Y + \tau C_Y)|_{Y'}$ is semi-ample because near $\Sigma$ the divisor $K_Y + B_Y + \tau C_Y$ is a multiple of $K_Y + B_Y + t_i C_Y$ and the latter is semi-ample. Finally as in the end of the proof of Lemma 9.2 we can apply Theorem 8.5 to show that $K_Y + B_Y + \tau C_Y$ is semi-ample. This is a contradiction because we assumed $\tau \notin T$.

Proof. (of Theorem 1.3 in general) By taking projectivizations of $X, Z$ and taking a log resolution, we may assume that $X, Z$ are projective and that $(X, B)$ is log smooth. We can also assume that $K_X + B \sim_{\mathbb{Q}} M = A + D/Z$ where $A$ is an ample $\mathbb{Q}$-divisor and $D \geq 0$. By adding some multiple of $M$ to $B$ we may assume $\text{Supp } M \subseteq \text{Supp } B$. Let $(Y, B_Y)$ be a log minimal model of $(X, B)$ over $Z$. Let $H$ be the pullback of an ample divisor on $Z$. Since $A \leq B_i$ for each integer $m \geq 0$, there is $\Delta$ such that $K_X + B + mH \sim_{\mathbb{Q}} K_X + \Delta$ is big globally and that $(X, \Delta)$ is klt. Moreover, $(Y, \Delta_Y)$ is a log minimal model of $(X, \Delta)$ over $Z$. Now by 3.3, if $m \gg 0$, then $K_Y + \Delta_Y$ is big and globally nef. On the other hand, $R(K_Y + \Delta_Y)$ is finitely generated over $k$ which means that $K_Y + \Delta_Y$ is semi-ample. Therefore $K_Y + B_Y$ is semi-ample/Z hence $\mathcal{R}(K_X + B/Z)$ is a finitely generated $O_Z$-algebra.

\[ \square \]

9.3. Base point freeness.

Proof. (of Theorem 1.4) It is enough to show that $\mathcal{R}(D/Z)$ is a finitely generated $O_Z$-algebra. By taking a $\mathbb{Q}$-factorialization using Theorem 1.6, we may assume that $X$ is $\mathbb{Q}$-factorial. Let $A = D - (K_X + B)$ which is nef and big/Z by assumptions. By replacing $A$ we may assume that it is ample globally. By Lemma 8.2, we can change $A$ up to $\mathbb{Q}$-linear equivalence so that $(X, B + A)$ is klt. But then $\mathcal{R}(K_X + B + A/Z)$ is finitely generated by Theorem 1.3 hence $\mathcal{R}(D/Z)$ is also finitely generated.

\[ \square \]

9.4. Constructions.

Proof. (of Theorem 1.5) We may assume that $B$ is a $\mathbb{Q}$-divisor and that $(X, B)$ is klt. We can assume $N = H + D$ where $H$ is ample/Z and $D \geq 0$. Let $G$ be the pullback of an ample divisor on $Z$, and let $N' = mG + nN + \epsilon H + \epsilon D$ where $\epsilon > 0$ is sufficiently small and $m \gg n \gg 0$. Then we can find $A \sim_{\mathbb{Q}} N'$ such that $(X, B + A)$ is klt, $K_X + B + A$ is globally big, and $(K_X + B + A) \cdot R < 0$. By 3.3, we can find an ample divisor $E$ such that $L := (K_X + B + A + E)$ is nef and big globally and $L^\perp = R$. We can also assume that $(X, B + A + E)$ is klt
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hence by Theorem 1.4, $L$ is semi-ample which implies that $R$ can be contracted by a projective morphism.

\[\square\]

10. ACC for lc thresholds

In this section, we prove Theorem 1.10 by a method similar to the char zero case. Let us recall the definition of lc thresholds. Let $(X, B)$ be an lc pair over $k$ and $M \geq 0$ an $\mathbb{R}$-Cartier divisor. The lc threshold of $M$ with respect to $(X, B)$ is defined as

$$lct(M, X, B) = \sup \{t | (X, B + tM) \text{ is lc}\}$$

10.1. Surfaces. We first prove ACC for lc thresholds on surfaces.

**Proposition 10.2.** ACC for lc thresholds holds in dimension 2 (formulated similar to 1.10).

**Proof.** If this is not the case, then there is a sequence $(X_i, B_i)$ of lc pairs of dimension 2 over $k$ and $\mathbb{R}$-Cartier divisors $M_i \geq 0$ such that the coefficients of $B_i$ are in $\Lambda$, the coefficients of $M_i$ are in $\Gamma$ but such that the $t_i := lct(M_i, X_i, B_i)$ form a strictly increasing sequence of numbers. If for infinitely many $i$, $(X_i, \Delta_i := B_i + t_iM_i)$ has an lc centre of dimension one contained in $\text{Supp} M_i$, then it is quite easy to get a contradiction. We may then assume that each $(X_i, \Delta_i)$ has an lc centre $P_i$ of dimension zero contained in $\text{Supp} M_i$. We may also assume that $(X_i, \Delta_i)$ is plt outside $P_i$. Let $(Y_i, \Delta_{Y_i})$ be a $\mathbb{Q}$-factorial dlt model of $(X_i, \Delta_i)$ such that there are some exceptional divisors on $Y_i$ mapping to $P_i$. Such $Y_i$ exist by a version of Lemma 6.7 in dimension 2.

There is a prime exceptional divisor $E_i$ of $Y_i \to X_i$ such that it intersects the birational transform of $M_i$. Note that $E_i$ is normal and actually isomorphic to $\mathbb{P}^1_k$ since $E_i$ is a component of $|\Delta_{Y_i}|$ and $(K_{Y_i} + \Delta_{Y_i}) \cdot E_i = 0$. Now by adjunction define $K_{E_i} + \Delta_{E_i} = (K_{Y_i} + \Delta_{Y_i})|_{E_i}$. Then by Proposition 4.3 and its proof, the set of all the coefficients of the $\Delta_{E_i}$ is a subset of a fixed DCC set but they do not satisfy ACC. This is a contradiction since $\deg \Delta_{E_i} = 2$.

\[\square\]

**Proposition 10.3.** Let $\Lambda \subset [0, 1]$ be a DCC set of real numbers. Then there is a finite subset $\Gamma \subset \Lambda$ with the following property: let $(X, B)$ be a pair and $X \to Z$ a projective morphism such that

- $(X, B)$ is lc of dimension 2 over $k$,
- the coefficients of $B$ are in $\Lambda$,
- $K_X + B \equiv 0/Z$,
- $\dim X > \dim Z$.

Then the coefficient of each horizontal/Z component of $B$ is in $\Gamma$.

**Proof.** We can assume that $1 \in \Lambda$. If the proposition is not true, then there is a sequence $(X_i, B_i), X_i \to Z_i$ of pairs and morphisms as in the proposition such that the set of the coefficients of the horizontal/Z components of all the $B_i$ put together does not satisfy ACC. By taking $\mathbb{Q}$-factorial dlt models we can assume
that \((X_i, B_i)\) are \(\mathbb{Q}\)-factorial dlt. Write \(B_i = \sum b_{i,j} B_{i,j}\). We may assume that \(B_{i,1}\) is horizontal/\(Z_i\) and that \(b_{1,1} < b_{2,1} < \ldots\).

First assume that \(\dim Z_i = 1\) for every \(i\). Run the LMMP/\(Z_i\) on \(K_{X_i} + B_i - b_{i,1} B_{i,1}\) with scaling of \(b_{i,1} B_{i,1}\). This terminates with a model \(X_i'\) having an extremal contraction \(X_i' \to Z_i'/Z_i\) such that \(K_{X_i'} + B_i' - b_{i,1} B_{i,1}'\) is numerically negative over \(Z_i'\). Let \(F_i'\) be the reduced variety associated to a general fibre of \(X_i' \to Z_i'\). Since \(K_{X_i'} + B_i' \equiv 0/Z_i'\) and \(F_i'^2 = 0\), we get \((K_{X_i'} + B_i' + F_i') \cdot F_i' = 0\) hence the arithmetic genus \(p_a(F_i') < 0\) which implies that \(F_i' \simeq \mathbb{P}^1_k\). We can write

\[
(K_{X_i'} + B_i' + F_i')|_{F_i'} = K_{F_i'} + \sum n_{i,j} b_{i,j} = 0
\]

for certain integers \(n_{i,j} \geq 0\) such that \(n_{i,1} > 0\). Since the \(b_{i,j}\) belong to the DCC set \(\Lambda\), \(n_{i,1}\) is bounded from above and below. Moreover, we can assume that the sums \(\sum_{j \geq 2} n_{i,j} b_{i,j}\) satisfy the DCC hence \(n_{i,1} b_{i,1} = 2 - \sum_{j \geq 2} n_{i,j} b_{i,j}\) satisfies the ACC, a contradiction.

From now on we may assume that \(\dim Z_i = 0\) for every \(i\). Run the LMMP/\(Z_i\) on \(K_{X_i} + B_i - b_{i,1} B_{i,1}\) with scaling of \(b_{i,1} B_{i,1}\). This terminates with a model \(X_i'\) having an extremal contraction \(X_i' \to Z_i'\) such that \(K_{X_i'} + B_i' - b_{i,1} B_{i,1}'\) is numerically negative over \(Z_i'\). If \(\dim Z_i' = 1\) for infinitely many \(i\), then we get a contradiction by the results above. So we assume that \(Z_i'\) are all points hence each \(X_i'\) is a Fano with Picard number one.

Assume that \((X_i', B_i')\) is lc but not klt for every \(i\). Assume that each \((X_i', B_i')\) has an lc centre \(S_i'\) of dimension one. Let \(K_{S_i'} + B_{S_i'} = (K_{X_i'} + B_i')|_{S_i'}\) by adjunction. Note that \(S_i'\) is normal since \((X_i', B_i' - b_{i,1} B_{i,1}')\) is \(\mathbb{Q}\)-factorial dlt. Since \(K_{S_i'} + B_{S_i'} \equiv 0\), \(S_i' \simeq \mathbb{P}^1_k\). If \(\text{Supp } B_{i,1}'\) contains an lc centre for infinitely many \(i\), then we get a contradiction by ACC for lc thresholds in dimension 2. So we can assume that \(\text{Supp } B_{i,1}'\) does not contain any lc centre, in particular, none of the points of \(S_i' \cap B_{i,1}'\) is an lc centre. Now, since \(\{b_{i,j}\}\) does not satisfy ACC, by the proof of Proposition 4.3, the set of the coefficients of all the \(B_{S_i'}\) satisfies DCC but not ACC which gives a contradiction as above (by considering the coefficients of the points in \(S_i' \cap B_{i,1}'\)). So we can assume that each \((X_i', B_i')\) has an lc centre of dimension zero. By a version of Lemma 6.7 in dimension 2, there is a projective birational contraction \(Y_i' \to X_i'\) which extracts only one prime divisor \(E_i'\) and it satisfies \(a(E_i', X_i', B_i') = 0\). Let \(K_{Y_i'} + B_{Y_i'}\) be the pullback of \(K_{X_i'} + B_{i}'\). By running the LMMP on \(K_{Y_i'} + B_{Y_i'} - E_i'\), we arrive on a model on which either the birational transform of \(E_i'\) intersects the birational transform of \(B_{i,1}'\) for infinitely many \(i\), or we get a Mori fibre space over a curve whose general fibre intersects the birational transform of \(B_{i,1}'\) for infinitely many \(i\). In any case, we can apply the arguments above to get a contradiction. So from now on we may assume that \((X_i', B_i')\) are all klt.

If there is \(\epsilon > 0\) such that \(X_i'\) is \(\epsilon\)-lc for every \(i\), then we are done since such \(X_i'\) are bounded by Alexeev [1]. So we can assume that the minimal log discrepancies of the \(X_i'\) forms a strictly decreasing sequence of positive numbers. Since \((X_i', B_i')\) are klt, we can assume that the minimal log discrepancies of the \((X_i', B_i')\) also forms a strictly decreasing sequence of positive numbers.
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By a version of Lemma 6.7 in dimension two, there is a projective birational contraction $Y'_i \to X'_i$ which extracts a only a prime divisor $E'_i$ with minimal log discrepancy $a(E'_i, X'_i, 0)$. Let $K_{Y'_i} + B_{Y'_i}$ be the pullback of $K_{X'_i} + B'_i$. So the coefficients of $E'_i$ in $B_{Y'_i}$, say $e_i$, form a strictly increasing sequence. Run the LMMP on $K_{Y'_i} + B_{Y'_i} - e_i E'_i$ and let $X''_i$ be the resulting model which comes with a Mori fibre structure $X''_i \to Z''_i$. If $\dim Z''_i = 1$ for each $i$, we use the results proved above. So we may assume that $\dim Z''_i = 0$ for each $i$.

Now it is easy to see that the number of prime divisors $D$ on birational models of $X'$ such that $a(D, X''_i, 0) < \epsilon$ is less than the number of prime divisors $D$ such that $a(D, X'_i, 0) < \epsilon$. By continuing this process, we arrive at a model which is $\epsilon$-lc so we can apply the previous paragraph.

10.4. 3-folds.

Proof. (of Theorem 1.10) If the theorem does not hold, then there is a sequence $(X_i, B_i)$ of lc pairs of dimension 3 over $k$ and $\mathbb{R}$-Cartier divisors $M_i \geq 0$ such that the coefficients of $B_i$ are in $\Lambda$, the coefficients of $M_i$ are in $\Gamma$ but such that the $t_i := \text{lt}(M_i, X_i, B_i)$ form a strictly increasing sequence of numbers. We may assume that each $(X_i, \Delta_i := B_i + t_i M_i)$ has an lc centre of dimension $\leq 1$ contained in $\text{Supp } M_i$. Let $(Y_i, \Delta_{Y_i})$ be a $\mathbb{Q}$-factorial dlt model of $(X_i, \Delta_i)$ such that there is an exceptional divisor on $Y_i$ mapping onto an lc centre inside $\text{Supp } M_i$. Such $Y_i$ exist by Lemma 6.7.

There is a prime exceptional divisor $E_i$ of $Y_i \to X_i$ such that $E_i$ intersects the birational transform of $M_i$ and that it maps into $\text{Supp } M_i$. Note that $E_i$ is normal by Lemma 4.2. Let $E_i \to Z_i$ be the contraction induced by $E_i \to X_i$. Now by adjunction define $K_{E_i} + \Delta_{E_i} = (K_{Y_i} + \Delta_{Y_i})|_{E_i}$. Then the set of all the coefficients of the horizontal $Z_i$ components of the $\Delta_{E_i}$ is a subset of a fixed DCC set but it does not satisfy ACC, by Proposition 4.4 and its proof. This contradicts Proposition 10.3.

11. NON-BIG LOG DIVISORS: PROOF OF 1.11

Lemma 11.1. Let $X$ be a normal projective variety of dimension $d$ over an algebraically closed field (of any characteristic). Let $A$ an ample $\mathbb{R}$-divisor and $P$ a nef $\mathbb{R}$-divisor with $P^d = 0$. Then for any $\epsilon > 0$, there exist $\delta \in [0, \epsilon]$ and a very ample divisor $H$ such that $(P - \delta A) \cdot H^{d-1} = 0$.

Proof. First we show that there is an ample divisor $H$ such that $(P - \epsilon A) \cdot H^{d-1} < 0$. Put $r(\tau) := (P - \epsilon A)(P + \tau A)^{d-1}$. Then

$$r(\tau) = (P - \epsilon A)(P^{d-1} + a_{d-2} \tau P^{d-2} A + \ldots + a_1 \tau^{d-2} P A^{d-2} + \tau^{d-1} A^{d-1})$$

where the $a_i > 0$ depend only on $d$. Put $a_{d-1} = a_0 = 1$, $a_{-1} = 0$, and let $n$ be the smallest integer such that $P^{d-n} A^n \neq 0$. Then we can write

$$r(\tau) = \sum_{i=0}^{d-1} (a_{i-1} \tau^{d-i} - \epsilon a_i \tau^{d-i-1}) P^i A^{d-i}$$
from which we get
\[ r(\tau) = \sum_{i=0}^{d-n} (a_{i-1} \tau^{d-i} - \epsilon a_i \tau^{d-i-1}) P_i A^{d-i} \]

hence
\[ \frac{r(\tau)}{\tau^{n-1}} = (a_{d-n-1} \tau - \epsilon a_{d-n}) P^{d-n} A^n + \tau s(\tau) \]

for some polynomial function \( s(\tau) \). Now if \( \tau > 0 \) is sufficiently small it is clear that the right hand side is negative hence \( r(\tau) < 0 \).

Choose \( \tau > 0 \) so that \( r(\tau) < 0 \). Since \( P + \tau A \) is ample and ampleness is an open condition, there is an ample \( \mathbb{Q} \)-divisor \( H \) close to \( P + \tau A \) such that \( (P - \epsilon A) \cdot H^{d-1} < 0 \). By replacing \( H \) with a multiple we can assume that \( H \) is very ample. Since \( P \cdot H^{d-1} \geq 0 \) by the nefness of \( P \), it is then obvious that there is some \( \delta \in [0, \epsilon] \) such that \( (P - \delta A) \cdot H^{d-1} = 0 \).

\[ \square \]

**Proof.** (of Theorem 1.11) Assume that \( D^d = 0 \). By replacing \( A \) we may assume that it is ample. Fix \( \alpha > 0 \). By Lemma 11.1, there exist a number \( t \) sufficiently close to 1 (possibly equal to 1) and a very ample divisor \( H \) such that
\[ (K_X + B + t(A + \alpha D)) \cdot H^{d-1} = 0 \]

Now we can view \( H^{d-1} \) as a 1-cycle on \( X \). For each point \( x \in X \), there is an effective 1-cycle \( C_x \) whose class is the same as \( H^{d-1} \) and such that \( x \in C_x \). Since \( H \) is very ample, we may assume that \( C_x \) is irreducible and that it is inside the smooth locus of \( X \) for general \( x \). In particular, we have
\[ (K_X + B + t(A + \alpha D)) \cdot C_x = 0 \]

Pick a general \( x \in X \) and let \( C_x \) be the curve mentioned above. Since \( B \) is effective and \( A + \alpha D \) is ample, we get \( K_X \cdot C_x < 0 \). Thus by Kollár [17, Chapter II, Theorem 5.8], there is a rational curve \( L_x \) passing through \( x \) such that
\[
0 < A \cdot L_x \leq (A + \alpha D) \cdot L_x \leq (2d) \frac{(A + \alpha D) \cdot C_x}{-K_X \cdot C_x} = \frac{2d}{t} \left(1 + \frac{B \cdot C_x}{K_X \cdot C_x}\right) \leq \frac{2d}{t} < 3d
\]
because \( K_X \cdot C_x < 0 \), \( B \cdot C_x \geq 0 \), and \( t \) is sufficiently close to 1. Note that although \( K_X \) and \( B \) need not be \( \mathbb{R} \)-Cartier, the intersection numbers still make sense since \( C_x \) is inside the smooth locus of \( X \).

As \( A \) is ample and \( A \cdot L_x \leq 3d \), we can assume that such \( L_x \) (for general \( x \)) belong to a bounded family \( L \) of curves on \( X \) (independent of the choice of \( t, \alpha \)). Therefore there are only finitely many possibilities for the intersection numbers \( D \cdot L_x \). If we choose \( \alpha \) sufficiently large, then the inequality \( (A + \alpha D) \cdot L_x \leq 3d \) implies \( D \cdot L_x = 0 \) and so we get the desired family.

\[ \square \]
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**References**

[1] V. Alexeev; *Boundedness and $K^2$ for log surfaces*. Internat. J. Math. 5 (1994), no. 6, 779–810.

[2] C. Birkar; *On existence of log minimal models and weak Zariski decompositions*. Math Annalen, Volume 354 (2012), Number 2, 787-799.

[3] C. Birkar; *On existence of log minimal models*. Compositio Math. volume 146 (2010), 919-928.

[4] C. Birkar; *Ascending chain condition for lc thresholds and termination of log flips*. Duke math. Journal, volume no. 1 (2007) 173-180.

[5] C. Birkar, M. Păun; *Minimal models, flips and finite generation : a tribute to V.V. SHOKUROV and Y.-T. SIU*. In "Classification of algebraic varieties", European Math Society series of congress reports (2010).

[6] P. Cascini, J. McKernan, M. Mustata; *The augmented base locus in positive characteristic*. To appear in Shokurov’s conference proceedings.

[7] P. Cascini, H. Tanaka, C. Xu; *On base point freeness in positive characteristic*. arXiv:1305.3502v1.

[8] V. Cossart and O. Piltant; *Resolution of singularities of threefolds in positive characteristic II*. Journal of Algebra 321 (2009), 1836–1976.

[9] V. Cossart and O. Piltant; *Resolution of singularities of threefolds in positive characteristic I*. J. Alg. 320 (2008), 1051–1082.

[10] S. D. Cutkosky; *Resolution of singularities for 3-folds in positive characteristic*. American Journal of Mathematics, Volume 131, Number 1, (2009), pp. 59-127.

[11] C. Hacon and C. Xu; *On the three dimensional minimal model program in positive characteristic*. arXiv:1302.0298v2.

[12] Y. Kawamata; *Semistable minimal models of threefolds in positive or mixed characteristic*. J. Alg. Geom. 3 (1994), 463–491.

[13] Y. Kawamata, K. Matsuda, K. Matsuki; *Introduction to the minimal model problem*. Algebraic geometry (Sendai, 1985), Adv. Stud. Pure Math., no. 10, North-Holland, Amsterdam (1987) 283-360.

[14] S. Keel; *Basepoint freeness for nef and big line bundles in positive characteristic*. Annals of Math, Second Series, Vol. 149, No. 1 (1999), 253–286.

[15] S. Keel, K. Matsuki, J. McKernan; *Log abundance theorem for threefolds*. Duke Math. J. Volume 75, Number 1 (1994), 99-119.

[16] J. Kollár; *Singularities of the minimal model program*. Cambridge University Press (2013).

[17] J. Kollár; *Rational curves on algebraic varieties*. Springer, 1999 edition.

[18] J. Kollár; *Flips and abundance for algebraic threefolds*. Astérisque 211, Soc. Math. France, 1992.

[19] J. Kollár, S. Mori; *Birational Geometry of Algebraic Varieties*. Cambridge University Press (1998).

[20] K. Schwede; *F-singularities and Frobenius splitting*. Lecture notes. Available on the author's website.

[21] V.V. Shokurov; *Prelimiting flips*. Proc. Steklov Inst. Math. 240 (2003), 75-213.

[22] V.V. Shokurov; *Three-dimensional log flips*. With an appendix in English by Yujiro Kawamata. Russian Acad. Sci. Izv. Math. 40 (1993), no. 1, 95–202.

[23] V. V. Shokurov; *The nonvanishing theorem*. Izv. Akad. Nauk SSSR Ser. Mat., 49:3 (1985), 635–651.

[24] H. Tanaka; *Minimal models and abundance for positive characteristic log surfaces*. arXiv:1201.5699v2.

[25] H. Tanaka; *Abundance theorem for semi log canonical surfaces in positive characteristic*. arXiv:1301.6889v1
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