Increasing the smoothness of vector and Hermite subdivision schemes

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Abstract

In this paper we suggest a method for transforming a vector subdivision scheme generating $C^\ell$ limits to another such scheme of the same dimension, generating $C^\ell+1$ limits. In scalar subdivision, it is well known that a scheme generating $C^\ell$ limit curves can be transformed to a new scheme producing $C^\ell+1$ limit curves by multiplying the scheme’s symbol with the smoothing factor $z^{\ell+1}$. We extend this approach to vector and Hermite subdivision schemes, by manipulating symbols. The algorithms presented in this paper allow to construct vector (Hermite) subdivision schemes of arbitrarily high regularity from a convergent vector scheme (from a Hermite scheme whose Taylor scheme is convergent with limit functions of vanishing first component).

Keywords: Vector subdivision schemes; Hermite subdivision schemes; symbol of a subdivision scheme; smoothness; analysis of limit functions

1 Introduction

Subdivision schemes are algorithms which iteratively refine discrete input data and produce smooth curves or surfaces in the limit. The regularity of the limit curve resp. surface is a topic of high interest.

In this paper we are concerned with the stationary and univariate case, i.e. with subdivision schemes using the same set of coefficients (called mask) in every refinement step and which have curves as limits. We study two types of such schemes: vector and Hermite subdivision schemes.

The mostly studied schemes are scalar subdivision schemes with real-valued sequences as masks. These schemes are in fact a special case of vector subdivision, with matrix-valued sequences as masks which refine sequences of vectors. For vector subdivision schemes many results concerning convergence and smoothness are available. An incomplete list of references is Cavaretta et al. (1991); Charina et al. (2005); Dyn (1992); Dyn et al. (1991); Dyn and Levin (2002); Micchelli and Sauer (1998); Sauer (2002).
In Hermite subdivision the refined data is also a sequence of vectors interpreted as function and derivatives values. This results in level-dependent vector subdivision, where the convergence of a scheme already includes the regularity of the limit curve. Corresponding literature can be found in Dubuc (2006); Dubuc and Merrien (2005); Dyn and Levin (1995, 1999); Guglielmi et al. (2011); Han et al. (2005); Merrien and Sauer (2012) and references therein. Note that here we consider inherently stationary Hermite schemes (Conti et al., 2014), where the level-dependence arises only from the specific interpretation of the input data. Inherently non-stationary Hermite schemes are discussed e.g. in Conti et al. (2016).

The convergence and smoothness analysis of subdivision schemes is strongly connected to the existence of the derived scheme or in the Hermite case to the Taylor scheme. The derived scheme (the Taylor scheme) are obtained by an appropriate factorization of the symbols (Dyn and Levin, 2002; Charina et al., 2005) (Merrien and Sauer (2012)). In the scalar and vector case we have the following result: If the derived scheme produces \( C^\ell \) \( (\ell \geq 0) \) limit curves, then the original scheme produces \( C^{\ell+1} \) limit curves, see Dyn and Levin (2002); Charina et al. (2005). In the Hermite case, in addition to the assumption that the Taylor scheme is \( C^\ell \), we also need that its limit functions have vanishing first component (Merrien and Sauer (2012)). These results are an essential tool in our approach for obtaining schemes with increased smoothness.

We start from a scheme which is known to have a certain regularity as the derived scheme (the Taylor scheme) of a new, to be computed scheme. By the above result, the regularity of the new scheme is increased by 1. This idea comes from univariate scalar subdivision, where it is well known that a scheme with symbol \( \alpha^*(z) \) is the derived scheme of \( \beta^*(z) = \frac{1+z^2}{z} \alpha^*(z) \) (Dyn and Levin, 2002), and thus if \( S_\alpha \) generates \( C^\ell \) limits, \( S_\beta \) generates limits which are \( C^{\ell+1} \).

It is possible to generalize this process to obtain vector (Hermite) subdivision schemes of arbitrarily high smoothness from a convergent vector scheme (a Hermite scheme, whose Taylor scheme is convergent with limit functions of vanishing first component).

We would like to mention other approaches which increase the regularity of subdivision schemes: It is known that the de Rham transform (Dubuc and Merrien, 2008) of some Hermite schemes increases the regularity by 1, see Conti et al. (2014). In contrast to our approach, it is not clear if this procedure can be iterated to obtain schemes of higher regularity. Nevertheless, in the examples listed in Conti et al. (2014), the de Rham approach increases the support only by 1, whereas our procedure for increasing the smoothness has the drawback of producing Hermite schemes with large supports (see Corollary 42, Example 14 and Example 45). Also, the authors of Dubuc and Merrien (2008) use geometric ideas, such as corner cutting. Our approach, on the other hand, is of an algebraic nature as it manipulates symbols.

A recent result which increases the regularity of a Hermite scheme, but not vector schemes, is presented in Merrien and Sauer (2017). This is different from our approach, as it also increases the dimension of the matrices of the mask and the dimension of the refined data.

We would also like to mention the paper Sauer (2003), which gives a detailed dis-
discussion of how to generalize the procedure for increasing the smoothness of a scalar subdivision scheme from the univariate to the multivariate scalar setting. Although vector subdivision schemes appear naturally in the analysis of smoothness of multivariate scalar schemes, yet the aim in [Sauer 2003] is to increase the smoothness of scalar schemes.

Our paper is organized as follows. In Section 2 we introduce the notation used throughout this text and recall some definitions concerning subdivision schemes. Section 3 presents the well known procedure for increasing the smoothness of univariate scalar subdivision schemes [Dyn and Levin 2002]. However, we introduce new notation, to emphasize the analogy to the procedures we presented in Sections 4 and 5 for vector and Hermite schemes, respectively. We conclude by two examples, applying our procedure to an interpolatory Hermite scheme of Merrien (1992) and to a Hermite scheme of de Rham-type [Dubuc and Merrien 2008], and obtain schemes with limit curves of regularity $C^2$ and $C^3$, respectively.

2 Notation and background

In this section we introduce the notation which is used throughout this paper and recall some known facts about scalar, vector and Hermite subdivision schemes.

Vectors in $\mathbb{R}^p$ will be labeled by lowercase letters $c$. The standard basis is denoted by $e_1, \ldots, e_p$. Sequences of elements in $\mathbb{R}^p$ are denoted by boldface letters $c = \{c_i \in \mathbb{R}^p : i \in \mathbb{Z}\}$. The space of all such sequences is $\ell(\mathbb{R}^p)$.

We define a subdivision operator $S_\alpha : \ell(\mathbb{R}^p) \to \ell(\mathbb{R}^p)$ with a scalar mask $\alpha \in \ell(\mathbb{R})$ by

$$(S_\alpha c)_i = \sum_{j \in \mathbb{Z}} \alpha_{i-2j} c_j, \quad i \in \mathbb{Z}, \ c \in \ell(\mathbb{R}^p).$$

(1)

We study the case of finitely supported masks, with support contained in $[-N,N]$. In this case the sum in eq. (1) is finite and the scheme is local.

We also consider matrix-valued masks. To distinguish them from the scalar case, we denote matrices in $\mathbb{R}^{p \times p}$ by uppercase letters. Sequences of matrices are denoted by boldface letters $A = \{A_i \in \mathbb{R}^{p \times p} : i \in \mathbb{Z}\}$.

We define a vector subdivision operator $S_A : \ell(\mathbb{R}^p) \to \ell(\mathbb{R}^p)$ with a finitely supported matrix mask $A \in \ell(\mathbb{R}^{p \times p})$ by

$$(S_A c)_i = \sum_{j \in \mathbb{Z}} A_{i-2j} c_j, \quad i \in \mathbb{Z}, \ c \in \ell(\mathbb{R}^p).$$

(2)

We define three kinds of subdivision schemes:

Definition 1.

1. A scalar subdivision scheme is the procedure of constructing $c^n (n \geq 1)$ from input data $c^0 \in \ell(\mathbb{R}^p)$ by the rule $c^n = S_\alpha c^{n-1}$, where $\alpha \in \ell(\mathbb{R})$ is a scalar mask.
2. A **vector subdivision scheme** (VSS) is the procedure of constructing \( c^n (n \geq 1) \) from input data \( c^0 \in \ell(\mathbb{R}^p) \) by the rule \( c^n = S_A c^{n-1} \), where \( A \) is a matrix-valued mask.

3. A **Hermite subdivision scheme** (HSS) is the procedure of constructing \( c^n (n \geq 1) \) from \( c^0 \in \ell(\mathbb{R}^p) \) by the rule \( D^n c^n = S_A D^{n-1} c^{n-1} \), where \( A \) is a matrix-valued mask and \( D \) is the dilation matrix

\[
D = \begin{pmatrix}
1 & \frac{1}{2} & \cdots & \frac{1}{2^{p-1}} \\
\end{pmatrix}.
\]

The difference between scalar and vector subdivision lies in the dimension of the mask. In scalar subdivision the components of \( c \) are refined *independently* of each other. This is not the case in vector subdivision. Note also that scalar schemes are a special case of vector schemes with mask \( A_i = \alpha_i I_p \), where \( I_p \) is the \((p \times p)\) unit matrix. In Hermite subdivision, on the other hand, the components of \( c \) are interpreted as function and derivatives values up to order \( p-1 \). This is represented by the matrix \( D \). In particular, Hermite subdivision is a level-dependent case of vector subdivision: \( c^n = S_{\hat{A}_n} c^{n-1} \) with \( \hat{A}_n = \{ D^{-n} A_i D^{n-1} : i \in \mathbb{Z} \} \).

On the space \( \ell(\mathbb{R}^p) \) we define a norm by

\[
\|c\|_\infty = \sup_{i \in \mathbb{Z}} \|c_i\|,
\]

where \( \|\cdot\| \) is a norm on \( \mathbb{R}^p \). The Banach space of all bounded sequences is denoted by \( \ell^\infty(\mathbb{R}^p) \). A subdivision operator \( S_\alpha \) with finitely supported mask, restricted to a map \( \ell^\infty(\mathbb{R}^p) \to \ell^\infty(\mathbb{R}^p) \) has an induced operator norm:

\[
\|S_\alpha\|_\infty = \sup\{\|S_\alpha c\|_\infty : c \in \ell^\infty(\mathbb{R}^p) \text{ and } \|c\|_\infty = 1\}.
\]

This is also true for subdivision operators with matrix masks.

Next we define convergence of scalar, vector and Hermite subdivision schemes. We start with scalar and vector schemes:

**Definition 2.** A scalar (vector) subdivision scheme associated with the mask \( \alpha \) (\( A \)) is **convergent in** \( \ell^\infty(\mathbb{R}^p) \), also called \( C^0 \), if for all input data \( c^0 \in \ell^\infty(\mathbb{R}^p) \) there exists a function \( \Psi \in C(\mathbb{R},\mathbb{R}^p) \), such that the sequences \( c^n = S^n_\alpha c^0 \) (\( c^n = S^n_A c^0 \)) satisfy

\[
\sup_{i \in \mathbb{Z}} \|c^n_i - \Psi(i/2^n)\| \to 0, \quad \text{as } n \to \infty,
\]

and \( \Psi \neq 0 \) for some \( c^0 \in \ell^\infty(\mathbb{R}^p) \). We say that the scheme is \( C^\ell \), if in addition \( \Psi \) is \( \ell \)-times continuously differentiable for any initial data.
In Section 5 we consider HSSs which refine function and first derivative values. The case of point-tangent data is treated componentwise. With this approach it is sufficient to consider convergence for data in \( \ell(\mathbb{R}^2) \).

In order to distinguish between the convergence of VSSs and the convergence of HSSs, we use the notation introduced in Conti et al. (2014):

**Definition 3.** A HSS associated with the mask \( A \) is said to be \( HC_\ell \) convergent with \( \ell \geq 1 \), if for any input data \( c_0 \in \ell_\infty(\mathbb{R}^2) \), there exists a function \( \Psi = (\psi^0, \psi^1) \) with \( \psi^0 \in C^\ell(\mathbb{R}, \mathbb{R}) \) and \( \psi^1 \) being the derivative of \( \psi^0 \), such that the sequences \( c_n = D^{-n}_S A c_0 \), \( n \geq 1 \), satisfy

\[
\sup_{i \in \mathbb{Z}} \| c_i^n - \Psi(i) \| \to 0, \quad \text{as } n \to \infty.
\]

Note that in contrast to the vector case, a HSS is convergent only if the limit already possesses a certain degree of smoothness.

We conclude by recalling some facts about the generating function of a sequence \( c \), which is the formal Laurent series

\[
c^*(z) = \sum_{i \in \mathbb{Z}} c_i z^i.
\]

The generating function of a mask of a subdivision scheme is called the symbol of the scheme. It is easy to see (e.g. in Dyn and Levin (2002)) that \( c^*(z) \) has the following properties:

**Lemma 4.** Let \( c \) be a sequence and let \( \alpha \) be a scalar or a matrix mask. By \( \Delta \) we denote the forward-difference operator \( ( \Delta c)_i = c_{i+1} - c_i \). Then we have:

\[
(\Delta c)^*(z) = (z^{-1} - 1)c^*(z) \quad \text{and} \quad (S_\alpha c)^*(z) = \alpha^*(z)c^*(z^2).
\]

Furthermore, for finite sequences we have the equalities

\[
c^*(1) = \sum_{i \in \mathbb{Z}} c_{2i} + \sum_{i \in \mathbb{Z}} c_{2i+1} \quad \text{and} \quad c^*(-1) = \sum_{i \in \mathbb{Z}} c_{2i} - \sum_{i \in \mathbb{Z}} c_{2i+1},
\]

\[
c^*(1) = \sum_{i \in \mathbb{Z}} c_{2i}(2i) + \sum_{i \in \mathbb{Z}} c_{2i+1}(2i + 1) \quad \text{and} \quad c^*(-1) = \sum_{i \in \mathbb{Z}} c_{2i+1}(2i + 1) - \sum_{i \in \mathbb{Z}} c_{2i}(2i).
\]

### 3 Increasing the smoothness of scalar subdivision schemes

In this section we recall a procedure increasing the smoothness of scalar subdivision schemes, which is realized by the smoothing factor \( z + 1 \). The results of this section are taken from Section 4 in Dyn and Levin (2002). We introduce notation in order to illustrate the analogy to the procedures we present in Section 4 for VSSs.

The condition \( \sum_{i \in \mathbb{Z}} \alpha_{2i} = \sum_{i \in \mathbb{Z}} \alpha_{2i+1} = 1 \) on the mask \( \alpha \) is necessary for the convergence of \( S_\alpha \). In this case \( \alpha^*(-1) = 0 \), implying that \( \alpha^*(z) \) has a factor \((z + 1)\) and there exists a mask \( \partial \alpha \) such that

\[
\Delta S_\alpha = \frac{1}{2} S_{\partial \alpha} \Delta.
\]
The scalar scheme associated with $\partial \alpha$ is called the derived scheme. It is easy to see that
\[(\partial \alpha)^*(z) = 2z\frac{\alpha^+(z)}{z+1}\] (4) and that $(\partial \alpha)^*$ is a Laurent polynomial. The convergence and smoothness analysis of a scalar subdivision scheme associated with $\alpha$ depends on the properties of $\partial \alpha$:

**Theorem 5.** Let $\alpha$ be a mask which satisfies $\alpha^+(1) = 2$ and $\alpha^*(-1) = 0$.

1. The scalar scheme associated with $\alpha$ is convergent if and only if the scalar scheme associated with $\frac{1}{2} \partial \alpha$ is contractive, namely $\|\left(\frac{1}{2} S_{\partial \alpha}\right)^L\|_\infty < 1$ for some $L \in \mathbb{N}$.

2. If the scalar scheme associated with $\partial \alpha$ is $C^\ell$ ($\ell \geq 0$) then the scalar subdivision scheme associated with $\alpha$ is $C^{\ell+1}$.

Theorem 5 allows us to define a procedure for increasing the smoothness of a scalar subdivision scheme: For a mask $\alpha$, define a new mask $I\alpha$ by $(I\alpha)^*(z) = \left(\frac{1+z}{2}z^{-1}\right)\alpha^*(z)$. Then $(I\alpha)^*(-1) = 0$ and from eq. (4) we get $\partial (I\alpha) = \alpha$ (Note that if $\partial \alpha$ exists, then also $I(\partial \alpha) = \alpha$).

**Corollary 6.** Let $\alpha$ be a mask associated with a $C^\ell$ ($\ell \geq 0$) scalar subdivision scheme. Then the mask $I\alpha$ gives rise to a $C^{\ell+1}$ scheme.

Therefore, by a repeated application of $I$, a scalar subdivision scheme which is at least convergent, can be transformed to a new scheme of arbitrarily high regularity. We call $I$ a smoothing operator and $\frac{z+1}{2}$ a smoothing factor. Note that the factor $z^{-1}$ in $I$ is an index shift.

**Example 7** (B-Spline schemes). The symbol of the scheme generating B-Spline curves of degree $\ell \geq 1$ and smoothness $C^{\ell-1}$ is
\[\alpha^*_\ell(z) = \left(\frac{(z+1)}{2}z^{-1}\right)^\ell(z+1).\] Obviously $\alpha^*_\ell(z) = \left(\frac{z+1}{2}z^{-1}\right)^\ell \alpha^*_{\ell-1}(z) = (I\alpha_{\ell-1})^*(z)$.

## 4 Increasing the smoothness of vector subdivision schemes

In this section we describe a procedure for increasing the smoothness of VSSs, which is similar to the scalar case. It is more involved since we consider masks consisting of matrix sequences.

### 4.1 Convergence and smoothness analysis

First we present results concerning the convergence and smoothness of VSSs. Their proofs can be found in Charina et al. (2005); Cohen et al. (1996); Micchelli and Sauer (1998); Sauer (2002).
For a mask $A$ of a VSS we define

$$A^0 = \sum_{i \in \mathbb{Z}} A_{2i}, \quad A^1 = \sum_{i \in \mathbb{Z}} A_{2i+1}. \quad (5)$$

Following Micchelli and Sauer (1998), let

$$E_A = \{ v \in \mathbb{R}^p : A^0 v = v \text{ and } A^1 v = v \} \quad (6)$$

and $k = \dim E_A$. A priori, $0 \leq k \leq p$. However, for a convergent VSS, $E_A \neq \{0\}$, i.e. $1 \leq k \leq p$. Therefore, the existence of a common eigenvector of $A^0$ and $A^1$ w.r.t. the eigenvalue 1 is a necessary condition for convergence.

The next lemma reduces the convergence analysis to the case $E_A = \text{span}\{e_1, \ldots, e_k\}$.

**Lemma 8.** Let $S_A$ be a $C^\ell$ ($\ell \geq 0$) convergent VSS. Given an invertible matrix $R \in \mathbb{R}^{p \times p}$, define a new mask $\hat{A}$ by $\hat{A}_i = R^{-1}A_iR$ for $i \in \mathbb{Z}$.

1. The VSS associated with $\hat{A}$ is also $C^\ell$.
2. There exist invertible matrices such that $\hat{A}$ satisfies $E_{\hat{A}} = \text{span}\{e_1, \ldots, e_k\}$, where $k = \dim E_A$.

In Cohen et al. (1996), Sauer (2002) the following generalization of the forward-difference operator $\Delta$ is introduced:

$$\Delta_k = \begin{pmatrix} \Delta I_k & 0 \\ 0 & I_{p-k} \end{pmatrix}, \quad (7)$$

where $I_k$ is the $(k \times k)$ unit matrix. It is shown there that if $E_A = \text{span}\{e_1, \ldots, e_k\}$, then in analogy to eq. (3), there exists a matrix mask $\partial_k A$ such that

$$\Delta_k S_A = \frac{1}{2} S_{\partial_k A} \Delta_k. \quad (9)$$

Algebraic conditions guaranteeing eq. (9) are stated and proved in the next subsection.

We denote by $\partial_k A$ any mask satisfying eq. (9). The vector scheme associated with $\partial_k A$ is called the *derived scheme* of $A$ with respect to $\Delta_k$. Furthermore, we have the following result concerning the convergence of $S_A$ in terms of $S_{\partial_k A}$:

**Theorem 9.** Let $A$ be a mask such that $E_A = \text{span}\{e_1, \ldots, e_k\}$. If $\|(\frac{1}{2} S_{\partial_k A})^L\| < 1$ for some $L \in \mathbb{N}$ (that is, $\frac{1}{2} S_{\partial_k A}$ is contractive), then the vector scheme associated with $A$ is convergent.

In fact there is a stronger result in Charina et al. (2005); Cohen et al. (1996), but we only need this special case. Two important results for the analysis of smoothness of VSSs are
Theorem 10 (Micchelli and Sauer (1998)). Let $A$ be a mask of a convergent VSS, such that $E_A = \text{span}\{e_1, \ldots, e_k\}$ for $k \leq p$, then
\[
\dim E_{\partial_k A} = \dim E_A.
\] (10)

Theorem 11 (Charina et al. (2005)). Let $A$ be a mask such that $E_A = \text{span}\{e_1, \ldots, e_k\}$. If the VSS associated with $\partial_k A$ is $C^\ell$ for $\ell \geq 0$, then the VSS associated with $A$ is $C^{\ell+1}$.

Remark 12. In the last theorem we omitted the assumption that $S_A$ is convergent required in Charina et al. (2005). This is possible because if $S_{\partial_k A}$ is $C^\ell$, then $\frac{1}{2} S_{\partial_k A}$ is contractive implying that $S_A$ is convergent in view of Theorem 9.

A useful observation for our analysis is

Lemma 13. Let $A$ be a matrix mask. Then
\[
E_A = \{v \in \mathbb{R}^p : A^*(1)v = 2v \text{ and } A^*(-1)v = 0\}.
\]

Proof. It follows immediately from eq. (5) and the definition of a symbol that $A^0 = \frac{1}{2} \left( A^*(1) + A^*(-1) \right)$ and $A^1 = \frac{1}{2} \left( A^*(1) - A^*(-1) \right)$. This, together with eq. (6), implies the claim of the lemma. \hfill \Box

4.2 Algebraic conditions

We would like to modify a given mask $B$ of a $C^\ell$ VSS to obtain a new scheme $S_A$ which is $C^{\ell+1}$. The idea is to define $A$ such that $\partial_k A = B$, i.e. such that eq. (9) is satisfied for some $k$. If we can prove that $E_A = \text{span}\{e_1, \ldots, e_k\}$, then by Theorem 11, the scheme $S_A$ is $C^{\ell+1}$. There are some immediate questions:

1. Under what conditions on a mask $B$ can we define a mask $A$ such that $\partial_k A = B$?

2. How to choose $k$?

In order to answer these questions, we have to study in more details the mask of the derived scheme $\partial_k A$ and its relation to the mask $A$.

Definition 14. For a mask $A$ of dimension $p$, i.e. $A_i \in \mathbb{R}^{p \times p}$ for $i \in \mathbb{Z}$, and a fixed $k \in \{1, \ldots, p\}$, we introduce the block notation
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]
with $A_{11}$ of size $(k \times k)$.

In the next lemma, we present algebraic conditions on a symbol $A^*(z)$ guaranteeing the existence of $\partial_k A$ for a fixed $k \in \{1, \ldots, p\}$, and also show that if $E_A = \text{span}\{e_1, \ldots, e_k\}$ these conditions hold.
Lemma 15. Let \( A, B \) be masks of dimension \( p \). With the notation of Definition 14 we have

1. If there exists \( k \in \{1, \ldots, p\} \) such that \( A_{11}^*(-1) = 0, A_{21}^*(-1) = 0 \) and \( A_{21}^*(1) = 0 \), then there exists a mask \( \partial_k A \) satisfying eq. (9).

2. If \( \mathcal{E}_A = \text{span}\{e_1, \ldots, e_k\} \), then \( A^*(z) \) the conditions of (1) are satisfied.

Proof. Under the assumptions of (1), the matrix

\[
2 \begin{pmatrix}
\frac{A_{11}^*(z)}{(z^{-1} + 1)} & (z^{-1} - 1)A_{12}^*(z) \\
A_{21}^*(z)/(z^{-2} - 1) & A_{22}^*(z)
\end{pmatrix},
\]

(11)
is a matrix Laurent polynomial. If we denote it by \((\partial_k A)^*(z)\), then the equation \( \Delta_k S_A = \frac{1}{2} S_{\partial_k A} \Delta_k \) is satisfied. Indeed, if we write this last equation in terms of symbols, we get

\[
\begin{pmatrix}
(z^{-1} - 1)I_k & 0 \\
0 & I_{p-k}
\end{pmatrix}
\begin{pmatrix}
A_{11}^*(z) & A_{12}^*(z) \\
A_{21}^*(z)/(z^{-2} - 1) & A_{22}^*(z)
\end{pmatrix}
= \begin{pmatrix}
A_{11}^*(z)/(z^{-1} + 1) & (z^{-1} - 1)A_{12}^*(z) \\
A_{21}^*(z)/(z^{-2} - 1) & A_{22}^*(z)
\end{pmatrix}
\begin{pmatrix}
(z^{-2} - 1)I_k & 0 \\
0 & I_{p-k}
\end{pmatrix}.
\]

(12)

It is easy to verify the validity of eq. (12).

In order to prove (2), we deduce from Lemma 13 that \( \mathcal{E}_A = \text{span}\{e_1, \ldots, e_k\} \) implies the properties of \( A \) required in (1).

Remark 17. If \( k = p \) in Lemma 16 then \((I_p B)^*(z) = \frac{z^{-1} + 1}{2} B^*(z)\), where \( \frac{z^{-1} + 1}{2} \) is the smoothing factor in the scalar case.
In Lemma 15 and Lemma 16 we constructed two operators $\partial_k$ and $I_k$ operating on masks, which (under some conditions) are inverse to each other. Denote by $\ell^k_a$ the set of all masks satisfying the conditions (1) of Lemma 15 and by $\ell^k_b$ the set of all masks satisfying the condition of Lemma 16. Then it is easy to show that

$$\partial_k : \ell^k_a \to \ell^k_b$$
$$I_k : \ell^k_b \to \ell^k_a$$

(15)

and that

$$\partial_k(I_k B) = B$$
$$I_k(\partial_k A) = A.$$  

(16)

This shows that the condition of Lemma 16 on a mask $B$ allows to define a mask $A = I_k B$ such that $\partial_k A = B$. This answers question (1). Still we need to deal with question (2).

Remark 18. It follows from Lemma 15 and Lemma 16 that the existence of $\partial_k A$ and $I_k B$ depends only on algebraic conditions. Yet this is not sufficient to define a procedure for changing the mask of a VSS in order to get a mask associated with a smoother VSS. Even if $I_k B$ exists for some $k$, the application of Theorem 11, in view of Lemma 8, to $A = I_k B$ is based on the dimension of $E_A$ which is not necessarily $k$. But if $E_A = \text{span}\{e_1, \ldots, e_k\}$, we can conclude from Theorem 11 that $S_A$ has smoothness increased by 1 compared to the smoothness of $S_B$.

In the next section we show that if for $B$ associated with a converging VSS $\dim E_A = k$, then there exists a canonical transformation $R$ such that $B = R^{-1} R B$ satisfies the algebraic conditions of Lemma 16 and $E_{I_k B} = \text{span}\{e_1, \ldots, e_k\}$. Therefore by Theorem 11 if $S_B$ is $C^\ell$, then $S_{I_k B}$ is $C^{\ell+1}$.

4.3 The canonical transformations to the standard basis

Let $B$ be a mask of a convergent VSS $S_B$. Denote by $k = \dim E_B$. We define a new mask $\overline{B}$ such that

$$E_B = \text{span}\{e_1, \ldots, e_k\}, \overline{B} \in \ell^k_b$$

This is achieved by considering the matrix $M_B = \frac{1}{2}(B^0 + B^1)$. First we state a result of importance to our analysis, which follows from Theorem 2.2 in Cohen et al. (1996) and from its proof.

Theorem 19. Let $B$ be a mask of a convergent VSS. A basis of $E_B$ is also a basis of the eigenspace of $M_B = \frac{1}{2}(B^0 + B^1)$ corresponding to the eigenvalue 1. Moreover $\lim_{n \to \infty} M^n_B$ exists.

A direct consequence of the last theorem, concluded from the existence of $\lim_{n \to \infty} M^n_B$, is

Corollary 20. Let $B$ be a mask associated with a converging VSS. Then the algebraic multiplicity of the eigenvalue 1 of $M_B$ equals its geometric multiplicity, and all its other eigenvalues have modulus less than 1.
In particular, since $M_B = \frac{1}{2}B^*(1)$, Theorem 19 implies that if $S_B$ is a convergent VSS, then $E_B$ is the eigenspace of $B^*(1)$ w.r.t. to the eigenvalue 2.

We proceed to define from a mask $B$ associated with a convergent VSS, a new mask $\overline{B}$ satisfying eq. (17).

Let $B$ be a mask associated with a convergent VSS and let $V = \{v_1, \ldots, v_k\}$ be a basis of $E_B$ (and therefore also a basis of the eigenspace w.r.t. 1 of $M_B$). We define a real matrix

$$\overline{R} = [v_1, \ldots, v_k|Q],$$

where the columns of $Q$ span the invariant space of $M_B$ corresponding to the eigenvalues different from 1 of $M_B$. $Q$ completes $V$ to a basis of $\mathbb{R}^p$ and $\overline{R}$ is an invertible matrix. We call $\overline{R}$ defined by eq. (18) a canonical transformation. There are many canonical transformations, since $Q$ is not unique. Our smoothing procedure is independent of the choice of a canonical transformation. Define a modified mask $\overline{B}$ by

$$\overline{B}_i = \overline{R}^{-1}B_i\overline{R}, \quad i \in \mathbb{Z}.$$  

Then by eq. (18) and Theorem 19 we have that $E_{\overline{B}} = \text{span}\{e_1, \ldots, e_k\}$. This proves the first claim in eq. (17). Also by Lemma 8, $S_{\overline{B}}$ is convergent and has the same smoothness as $S_B$.

Furthermore, by eq. (18),

$$M_{\overline{B}} = \frac{1}{2}(\overline{B}^0 + \overline{B}^1) = \overline{R}^{-1}M_B\overline{R} = \begin{pmatrix} I_k & 0 \\ 0 & J \end{pmatrix}$$

is the Jordan form of $M_B$. By Corollary 20, $J$ has eigenvalues with modulus less than 1. Transformations $\overline{R}$ which result in representations of $M_B$ similar to the one in eq. (20) have already been considered in e.g. Cohen et al. (1996); Sauer (2002). The special structure of $M_B$ is the key to our smoothing procedure. The next theorem follows from eq. (20) and proves the remaining claims of eq. (17).

**Theorem 21.** Let $S_B$ be a convergent VSS and let $k = \dim E_B$. Define $\overline{B}$ by eq. (19) with $\overline{R}$ a canonical transformation. Then $\overline{B}$ has the following properties:

1. $\overline{B} \in \ell^k_b$,
2. $E_{\overline{B}} = \text{span}\{e_1, \ldots, e_k\}$.

**Proof.** We start by proving (1). Since

$$\overline{B}^*(1) = \overline{B}^0 + \overline{B}^1 = 2M_B = 2\begin{pmatrix} I_k & 0 \\ 0 & J \end{pmatrix},$$

it follows that $\overline{B}^*_{12}(1) = 0$. Thus by Lemma 16, $\overline{B} \in \ell^k_b$ and therefore $I_k\overline{B}$ exists.

In order to prove (2), we use Lemma 13 and show that $E_{\overline{B}} = \{v \in \mathbb{R}^p : (I_k\overline{B})^*(1)v = 2v \text{ and } (I_k\overline{B})^*(-1)v = 0\}$ is spanned by $e_1, \ldots, e_k$. Indeed by eq. (21) it follows that
$B_{12}(1) = 2I_k$ and $B_{22}(1) = 2J$. Since by eq. (21) $B_{12}^*(1) = 0$, there exists a symbol $C^*(z)$ such that $B_{12}^*(z) = (z^{-1} - 1)C^*(z)$, and therefore eq. (14) implies the block form:

$$
(I_kB)^*(1) = \begin{pmatrix} 2I_k & \frac{1}{2}C^*(1) \\ 0 & J \end{pmatrix}, \quad (I_kB)^*(-1) = \begin{pmatrix} 0 & \frac{1}{2}C^*(-1) \\ 0 & \frac{1}{2}B_{22}^*(-1) \end{pmatrix},
$$

(22)

Equation (22), in view of Lemma 13, implies that span\{e_1, \ldots, e_k\} = E_{I_kB}, since the eigenspace of $(I_kB)^*(1)$ w.r.t. the eigenvalue 2 is exactly span\{e_1, \ldots, e_k\} (the matrix $J$ only contributes eigenvalues with modulus less than 1), and these vectors are in the kernel of $(I_kB)^*(-1)$.

Summarizing the above results, we arrive at

**Corollary 22.** Let $B$ be a mask of a convergent VSS, let $k = \dim E_B$, and let $B$ be as in Theorem 21. Then $I_kB$ exists and

$$
E_{I_kB} = E_B = \text{span}\{e_1, \ldots, e_k\}.
$$

### 4.4 A procedure for increasing the smoothness

Theorem 21 allows us to define the following procedure which generates VSSs of higher smoothness from given convergent VSSs:

**Procedure 23.** The input data is a mask $B$ associated with a $C^\ell$ VSS, $\ell \geq 0$, and the output is a mask $A$ associated with a $C^{\ell+1}$ VSS.

1. Choose a basis $V$ of $E_B$ and define $R$, a canonical transformation, as in eq. (18).
2. Define $\overline{B} = R^{-1}BR$.
3. Define $k = \dim(E_B)$.
4. Define $\overline{A} = I_k\overline{B}$ as in eq. (14).
5. Define $A = \overline{R}\overline{A}\overline{R}^{-1}$.

A schematic representation of Procedure 23 is given in Figure 1.

**Remark 24.** Step 5 in Procedure 23 is not essential. The scheme $S_A$ is already $C^{\ell+1}$. Step 5 guarantees that $E_A = E_B$. In both cases to apply another smoothing procedure to get a $C^{\ell+2}$ VSS, a new canonical transformation has to be applied.

In the notation of Procedure 23, we define the smoothing operator $I_k$ applied to a mask $B$ of a convergent VSS as

$$
I_kB = \overline{R}(I_k\overline{B})\overline{R}^{-1}.
$$

(23)

This is a generalization of the smoothing operator in the case of scalar subdivision schemes.

An important property of Procedure 23, which is easily seen from eq. (14) is,
Corollary 25. Assume that B and A are masks as in Procedure 23. If the support of B is contained in \([-N_1,N_2]\) with \(N_1, N_2 \in \mathbb{N}\), then the support of A is contained in \([-N_1-2,N_2]\).

Therefore Procedure 23 increases the support length by at most 2, independently of the dimension of the mask. Recall that in the scalar case the support size is increased by 1.

An interesting observation follows from Procedure 23, eq. (21) and eq. (22).

Corollary 26. Assume that A, B are masks as in Procedure 23. Then \(A^*(1)\) and \(B^*(1)\) share the eigenvalue 2 and the corresponding eigenspace. To each eigenvalue \(\lambda \neq 2\) of \(B^*(1)\) there is an eigenvalue \(\frac{1}{2}\lambda\) of \(A^*(1)\).

Note that a similar result to that in Corollary 26 is in general not true for \(B^*(-1)\) and \(A^*(-1)\). However, Example 27 shows that this can well be the case.

Example 27 (Double-knot cubic spline subdivision). We consider the VSS with symbol

\[
B^*(z) = \frac{1}{8} \left( \begin{array}{cc} 2+6z+z^2 & 2z+5z^2 \\ 5+2z & 1+6z+2z^2 \end{array} \right).
\]

(24)

It is known that this scheme produces \(C^1\) limit curves (see e.g. Dyn and Levin (2002)). We apply Procedure 23 to B to obtain a VSS \(S_A\) of regularity \(C^2\):

1. First we find a basis of \(E_B\) in order to compute a canonical transformation \(R\). The matrices \(B^*(1)\) and \(B^*(-1)\) are given by

\[
B^*(1) = \frac{1}{8} \left( \begin{array}{cc} 9 & 7 \\ 7 & 9 \end{array} \right), \quad B^*(-1) = \frac{1}{8} \left( \begin{array}{cc} -3 & 3 \\ 3 & -3 \end{array} \right)
\]

and have the following eigenvalues and eigenvectors

For \(B^*(1)\) : eigenvalues : \(2, \frac{1}{4}\), eigenvectors: \(\left( \begin{array}{c} 1 \\ 1 \end{array} \right)\), \(\left( \begin{array}{c} -1 \\ 1 \end{array} \right)\), resp. (25)

For \(B^*(-1)\) : eigenvalues : \(0, -\frac{3}{4}\), eigenvectors: \(\left( \begin{array}{c} 1 \\ 1 \end{array} \right)\), \(\left( \begin{array}{c} -1 \\ 1 \end{array} \right)\), resp.

13
Therefore $\mathcal{E}_B$ is spanned by $(1, 1)$. The transformation $\overline{R}$ is determined by the eigenvectors of $B^*(1)$:

$$ R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. $$

2. We continue by computing $\overline{B} = R^{-1}B\overline{R}$ from the symbol of $B$ in eq. (24), and get

$$ B^*(z) = \frac{1}{8} \begin{pmatrix} 4(1 + z)^2 & 3(z^2 - 1) \\ -2(z^2 - 1) & -1 + 4z - z^2 \end{pmatrix}. $$

3. From Step 1 we see that $k = \dim \mathcal{E}_B = 1$.

4. We compute $A = I_1B$ by computing its symbol.

$$ A^*(z) = \frac{1}{16} \begin{pmatrix} 4z^{-1}(1 + z)^3 & -3z^{-1}(z + 1) \\ 2z^{-2}(z^2 - 1)^2 & -1 + 4z - z^2 \end{pmatrix}. $$

5. In this step we transform back to the original basis $A = \overline{R}A\overline{R}^{-1}$, by deriving $A^*(z)$.

$$ A^*(z) = \frac{1}{32} z^{-2} \begin{pmatrix} z^4 + 16z^3 + 18z^2 + 7z - 2 & 3z^4 + 8z^3 + 14z^2 + z - 2 \\ 7z^4 + 8z^3 + 12z^2 + 7z + 2 & 5z^4 + 16z^3 + 4z^2 + z + 2 \end{pmatrix}. $$

It follows from the analysis preceeding Procedure 23 that $S_A$ is $C^2$.

To verify Remark 24 we show that $\mathcal{E}_A$ has the same basis as $\mathcal{E}_B$. We compute

$$ A^*(1) = \frac{1}{8} \begin{pmatrix} 10 & 6 \\ 9 & 7 \end{pmatrix}, \quad A^*(-1) = \frac{1}{16} \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} $$

and their eigenvalues and eigenvectors:

For $A^*(1)$: eigenvalues : $2, \frac{1}{5}$, eigenvectors: $\left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} -2 \\ 3 \end{array} \right)$, resp.

For $A^*(-1)$: eigenvalues : $0, -\frac{3}{8}$, eigenvectors: $\left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} -1 \\ 1 \end{array} \right)$, resp.

Therefore by eq. (25), eq. (27) and Lemma 13, $\mathcal{E}_A$ and $\mathcal{E}_B$ are spanned by $(1, 1)$.

Note that the eigenvectors corresponding to the eigenvalues which have modulus less than 1, of $M_A$ and $M_B$ are different. Thus in order to generate a $C^3$ scheme from $S_A$, a new canonical transformation has to be computed.

Also, comparing the eigenvalues of $A^*(1)$ and $B^*(1)$ we see that Corollary 26 is satisfied. In fact in this example, also the eigenvalues of $A^*(-1)$ and $B^*(-1)$ have the same property.

It is easy to see from eq. (26) that the support of the mask $A$ is 4, and from eq. (24) that the support of $B$ is 2, in accordance with Corollary 25.
5 Increasing the smoothness of Hermite subdivision schemes

In this section we describe a procedure for increasing the smoothness of HSSs refining function and first derivative values, based on the procedure for the vector case described in Section 4. We consider HSSs which operate on data \( c \in \ell(R^2) \), using the notation of Section 2.

5.1 Algebraic conditions

As in the vector case, HSSs use matrix-valued masks \( A = \{ A_i \in R^{2 \times 2} : i \in Z \} \) and subdivision operators \( S_A \) as defined in eq. (2). The input data \( c^0 \in \ell(R^2) \) is refined via

\[
D_n c^n = S_n A c^0, 
\]

where \( D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \).

A HSS is called interpolatory if its mask \( A \) satisfies \( A_0 = D \) and \( A_{2i} = 0 \) for all \( i \in Z \setminus \{0\} \).

We always assume that a HSS satisfies the spectral condition [Dubuc and Merrien, 2009]. This condition requires that there is \( \varphi \in R \) such that both the constant sequence \( k = \{(1,0) : i \in Z\} \) and the linear sequence \( \ell = \{(i+\varphi) : i \in Z\} \) obey the rule

\[
S_A k = k, \quad S_A \ell = \frac{1}{2} \ell. \tag{28}
\]

The spectral condition is crucial for the convergence and smoothness analysis of linear HSSs. If the HSS is interpolatory we can choose \( \varphi = 0 \).

We now characterize the spectral condition in terms of the symbol of the mask \( A \). We introduce the notation

\[
A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \tag{29}
\]

where \( \alpha_{ij} \in \ell(R) \) for \( i, j \in \{1,2\} \). It is easy to verify that the spectral condition in eq. (28) is equivalent to the algebraic conditions in the next lemma.

**Lemma 28.** A mask \( A \) satisfies the spectral condition given by eq. (28) with \( \varphi \in R \) if and only if its symbol \( A^*(z) \) satisfies

1. \( \alpha_{11}^*(1) = 2, \quad \alpha_{11}^*(-1) = 0 \).
2. \( \alpha_{21}^*(1) = 0, \quad \alpha_{21}^*(-1) = 0 \).
3. \( \alpha_{11}''(1) - 2\alpha_{12}'(1) = 2\varphi, \quad \alpha_{11}''(-1) + 2\alpha_{12}'(-1) = 0 \).
4. \( \alpha_{21}''(1) - 2\alpha_{22}'(1) = -2, \quad \alpha_{21}''(-1) + 2\alpha_{22}'(-1) = 0 \).

Parts (1) and (2) relate to the reproduction of constants, whereas parts (3) and (4) are related to the reproduction of linear functions.
Next we cite results on $HC^\ell$ smoothness of HSS. Consider the Taylor operator $T$, first introduced in Merrien and Sauer (2012):

$$T = \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}.$$ 

The Taylor operator is a natural analogue of the operator $\Delta_k$ for VSSs and the forward difference operator $\Delta$ in scalar subdivision. We have the following result analogous to eq. (9):

**Lemma 29** (Merrien and Sauer (2012)). If the HSS associated with a mask $A$ satisfies the spectral condition of eq. (28), then there exists a matrix mask of dimension $2$, $\partial_t A$, such that

$$TS_A = \frac{1}{2}S_{\partial_t T}.$$ 

(30)

The mask $\partial_t A$ determines a VSS called the Taylor scheme associated with $A$.

### 5.2 Properties of the Taylor scheme

In order to increase the smoothness of a HSS, the obvious idea is to pass to its Taylor scheme defined in eq. (30), increase the smoothness of this VSS by Procedure 23 and then use the resulting VSS as the Taylor scheme of a new HSS. The first question which arises in this process is if the last step is always possible, i.e., if the smoothing operator $I_k$ of eq. (23) maps Taylor schemes to Taylor schemes. To answer this question depicted in Figure 2, we state algebraic conditions on a mask $B$ of a VSS guaranteeing that $S_B$ is a Taylor scheme.

**Definition 30.** The algebraic conditions on a mask $B$,

1. $\beta^*_{12}(1) = 0, \beta^*_{12}(-1) = 0, $
2. $\beta^*_{22}(1) = 2, \beta^*_{22}(-1) = 0, $
3. $\beta^*_{11}(1) + \beta^*_{21}(1) = 2,$

are called Taylor conditions. (Here we use the notation of eq. (29)).

We prove in Lemma 32 that the mask $\partial_t A$ obtained via eq. (30) satisfies the Taylor conditions. This justifies the name Taylor conditions.

**Remark 31.** It is easy to verify that conditions (1) and (2) of Definition 30 are equivalent to $e_2 \in \mathcal{E}_B$.

The next lemmas are concerned with the connection between masks satisfying the spectral condition of eq. (28) and masks satisfying the Taylor conditions of Definition 30.

**Lemma 32.** Let $A$ be a mask satisfying the spectral condition. Then we can define a mask $\partial_t A$ such that eq. (30) is satisfied, and $\partial_t A$ satisfies the Taylor conditions.
Note that the existence of $\partial_t A$ in Lemma 32 is a result of Merrien and Sauer (2012) (see Lemma 29). We prove it here because its proof is used in our analysis.

**Proof.** By solving eq. (30) in terms of symbols for $\partial_t A$, it is easy to see that

\[
(\partial_t A)_{11}^* = 2\left(\frac{\alpha_{11}^*(z)}{z-1} + \frac{\alpha_{21}^*(z)}{z-2-1}\right),
\]

(31)

\[
(\partial_t A)_{12}^* = 2\left(\frac{(z^{-1} - 1)\alpha_{12}^*(z) - \alpha_{22}^*(z)}{z-1} + \frac{\alpha_{11}^*(z)}{z-2-1} - \frac{\alpha_{21}^*(z)}{z-2-1}\right),
\]

(32)

\[
(\partial_t A)_{21}^* = 2\frac{\alpha_{21}^*(z)}{z-2-1},
\]

(33)

\[
(\partial_t A)_{22}^* = 2\left(\frac{\alpha_{22}^*(z)}{z-2-1} + \frac{\alpha_{21}^*(z)}{z-2-1}\right).
\]

(34)

By the algebraic conditions of Lemma 28, $(\partial_t A)^*(z)$ defined by eqs. (31) – (34) is a Laurent polynomial. Note that we only need the first two conditions of Lemma 28 equivalent to the reproduction of constants to define $\partial_t A$.

We now show that $\partial_t A$ satisfies the Taylor conditions. Multiplying eq. (32) with the factor $(z^{-2} - 1)$, differentiating with respect to $z$, substituting $z = 1$ and $z = -1$, and applying Lemma 28, we obtain:

\[
(\partial_t A)_{12}^*(1) = -2\alpha_{22}^*(1) + \alpha_{11}^*(1) + \alpha_{21}^*(1) = 0,
\]

\[
(\partial_t A)_{12}^*(-1) = -4\alpha_{12}^*(-1) - 2\alpha_{22}^*(-1) - 2\alpha_{11}^*(-1) - 2\alpha_{11}^*(-1) - \alpha_{21}^*(-1) = 0.
\]

This proves that part (1) of Definition 30 is satisfied.

Applying the same procedure to eq. (34), we obtain

\[
(\partial_t A)_{22}^*(1) = 2\alpha_{22}^*(1) - \alpha_{21}^*(1) = 2,
\]

\[
(\partial_t A)_{22}^*(-1) = 2\alpha_{22}^*(-1) + \alpha_{21}^*(-1) = 0.
\]

This concludes part (2) of Definition 30. Similarly eqs. (31) and (33) imply

\[
(\partial_t A)_{11}^*(1) + (\partial_t A)_{21}^*(1) = (2 + \alpha_{21}^*(1)) - \alpha_{21}^*(1) = 2,
\]

which proves (3) of Definition 30.

\[
\begin{array}{c}
S_A \in HC^\ell \\
\xrightarrow{\partial_t} & \xrightarrow{S_{\partial_t A} \in C^{\ell-1}} \xrightarrow{I_k} \xrightarrow{?} S_{C_{\partial_t A}} \in C^{\ell} \xrightarrow{S_A \in HC^{\ell+1}} \xrightarrow{?} S_{\partial_t A} \in C^{\ell}
\end{array}
\]

Figure 2: A schematic representation of the idea for smoothing HSSs.
Lemma 33. Let $\mathbf{B}$ be a mask satisfying the Taylor conditions. Then we can define a mask $\mathcal{I}_t \mathbf{B}$ such that

$$TS_{\mathcal{I}_t \mathbf{B}} = \frac{1}{2} S_B T$$

is satisfied, and $\mathcal{I}_t \mathbf{B}$ satisfies the spectral condition.

Proof. Suppose that $\mathbf{B}$ satisfies the Taylor conditions. We define a mask $\mathcal{I}_t \mathbf{B}$ satisfying the equation $TS_{\mathcal{I}_t \mathbf{B}} = \frac{1}{2} S_B T$ by writing it in terms of symbols. This yields the symbol

$$(\mathcal{I}_t \mathbf{B})_{11}^*(z) = \frac{1}{2}(z^{-1} + 1)(\beta_{11}^*(z) + \beta_{21}^*(z)),$$

$$(\mathcal{I}_t \mathbf{B})_{12}^*(z) = \frac{1}{2}\left(\beta_{12}^*(z) - \beta_{11}^*(z) - \beta_{21}^*(z) + \beta_{22}^*(z)\right)/(z^{-1} - 1),$$

$$(\mathcal{I}_t \mathbf{B})_{21}^*(z) = \frac{1}{2} \beta_{21}^*(z)(z^{-2} - 1),$$

$$(\mathcal{I}_t \mathbf{B})_{22}^*(z) = \frac{1}{2}(\beta_{22}^*(z) - \beta_{21}^*(z)).$$

It follows from the Taylor conditions that $(\mathcal{I}_t \mathbf{B})^*(z)$ is a Laurent polynomial and thus well-defined.

We continue by showing that $\mathcal{I}_t \mathbf{B}$ satisfies the spectral condition. It is immediately clear from the definition of $\mathcal{I}_t \mathbf{B}$ that (1) and (2) of Lemma 28 are satisfied. Furthermore, it is easy to see that

$$(\mathcal{I}_t \mathbf{B})_{21}^*(1) - 2(\mathcal{I}_t \mathbf{B})_{22}^*(1) = -\beta_{21}^*(1) - \beta_{22}^*(1) + \beta_{21}^*(1) = -2,$$

$$(\mathcal{I}_t \mathbf{B})_{21}^*(-1) + 2(\mathcal{I}_t \mathbf{B})_{22}^*(-1) = \beta_{21}^*(1) + \beta_{22}^*(-1) - \beta_{21}^*(-1) = 0,$$

which proves (3) of Lemma 28.

From the definition of $\mathcal{I}_t \mathbf{B}$ we see that

$$(\mathcal{I}_t \mathbf{B})_{11}^*(-1) + 2(\mathcal{I}_t \mathbf{B})_{12}^*(-1) = -\frac{1}{2}\left(\beta_{11}^*(-1) + \beta_{21}^*(-1)\right) - \frac{1}{2}\left(\beta_{12}^*(-1) - \beta_{11}^*(-1) - \beta_{21}^*(-1) + \beta_{22}^*(-1)\right) = 0.$$
eq. (28) and by ℓ, the set of all masks satisfying the Taylor conditions of Definition 30. Then
\[ \partial_t : \ell_s \to \ell_t \quad \mathcal{I}_t : \ell_t \to \ell_s \]  
and it easy to verify that
\[ \partial_t (\mathcal{I}_t B) = B \quad \text{and} \quad \mathcal{I}_t (\partial_t A) = A. \]  

5.3 Relations between converging vector and Hermite schemes

In the previous section we derived a one-to-one correspondence between a mask satisfying the spectral condition and a mask satisfying the Taylor conditions. For masks of converging schemes we formulate a result based on Theorem 21 in Merrien and Sauer (2012), and on the results of Section 5.2.

**Theorem 34.** A $C^\ell$, $\ell \geq 0$, VSS $S_B$ satisfying the Taylor conditions with limit functions with vanishing first component, gives rise to an HC$^{\ell+1}$ Hermite scheme $S_A$ satisfying the spectral condition.

In the next lemma we show that the condition of vanishing first component in the limits generated by $S_B$ can be replaced by a condition on the mask $B$. This also follows from results in Micchelli and Sauer (1998).

**Lemma 35.** Let $S_B$ be a convergent VSS. Denote by $\Psi_c = (\psi_1, \psi_2)$ the limit function generated from the initial data $c \in \ell(\mathbb{R}^2)$. Then
\[ \mathcal{E}_B = \text{span}\{e_2\} \iff \psi_{1,c} = 0 \text{ for all initial data } c. \]

**Proof.** First we show that $\mathcal{E}_B = \text{span}\{e_2\}$ implies $\psi_{1,c} = 0$ for all $c$. This follows from the observation that $\Psi_c(x) \in \mathcal{E}_B$ for all $x \in \mathbb{R}$. The observation follows from the convergence of $S_B$ to a continuous limit and from the basic refinement rules for large $k$
\[ (S_{i+1}^B c)_{2i} = \sum_{j \in \mathbb{Z}} B_{2j}(S^B_{i} c)_{i-j}, \quad (S_{i+1}^B c)_{2i+1} = \sum_{j \in \mathbb{Z}} B_{2j+1}(S^B_{i} c)_{i-j}, \text{ for } i \in \mathbb{Z}. \]

To prove the other direction we use the proof of Theorem 2.2 in Cohen et al. (1996). It shows that
\[ \lim_{n \to \infty} M^B_B = \int_\mathbb{R} \Phi(x)dx, \]  
where $M^B_B$ is defined in Theorem 19 and $\Phi$ is the limit function generated by $S_B$ from the initial data $\delta I_2$. Here $I_2$ is the identity matrix of dimension 2 and $\delta \in \ell(\mathbb{R})$ satisfies $\delta_0 = 1$, $\delta_i = 0$, $i \neq 0$, $i \in \mathbb{Z}$, or equivalently $(\phi_{0j}(x))$ is the limit from the initial data $\delta e_j$ for $j \in \{1, 2\}$. Thus
\[ \phi_{11}(x) = \phi_{12}(x) = 0 \quad \text{for } x \in \mathbb{R}. \]
It follows from eq. (38) that
\[
\lim_{n \to \infty} M^n_B = \begin{pmatrix} 0 & 0 \\ \nu & \theta \end{pmatrix}, \quad \nu, \theta \in \mathbb{R}.
\] (39)

Assume \( \mathcal{E}_B \neq \text{span}\{e_2\} \). Then \( \mathcal{E}_B = \mathbb{R}^2 \), and \( M_B = I_2 \), since by Theorem 19 the eigenspace of \( M_B \) with respect to 1 is exactly \( \mathcal{E}_B \). Thus \( \lim_{n \to \infty} M^n_B = I_2 \) in contradiction to eq. (39).

5.4 Imposing the Taylor conditions

Denote by \( \tilde{\ell}_t \subseteq \ell_t \) the set of masks satisfying \( B \in \ell_t \) and \( \mathcal{E}_B = \text{span}\{e_2\} \). It follows from Theorem 34 and Lemma 35 that for \( B \in \tilde{\ell}_t \), a mask of a \( C^t \) VSS, if also \( \mathcal{I}_1 B \in \tilde{\ell}_t \), then \( \mathcal{I}_1 B \) is a mask of a \( C^{t+1} \) VSS which is the Taylor scheme of a \( HC^{t+2} \) Hermite scheme. The next results show that \( I_1(\tilde{\ell}_t) \subseteq \tilde{\ell}_t \) does not hold in general. Nevertheless, in the following we construct a transformation \( R \) such that \( R^{-1}(\mathcal{I}_1 B) R \in \tilde{\ell}_t \) for \( B \in \tilde{\ell}_t \).

First we look for a canonical transformation of a mask \( B \in \ell_t \) to define \( I_1 B \).

Lemma 36. Let \( B \in \ell_t \). Then \( M_B \) has the eigenvalue 1 with eigenvector \((0, 1)^\top\) and the eigenvalue \( \frac{1}{2} \beta_{11}^*(1) \) with eigenvector \((1, \frac{1}{2})^\top\). A canonical transformation and its inverse are
\[
R = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{with inverse} \quad R^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Proof. From the Taylor conditions we immediately get
\[
M_B = \frac{1}{2}(B^0 + B^1) = \frac{1}{2} B^*(1) = \begin{pmatrix} \frac{1}{2} \beta_{11}^*(1) \\ \frac{1}{2} \beta_{21}^*(1) \end{pmatrix}.
\]
The eigenvalues of \( M_B \) can now be read from the diagonal. Also, it is clear that \((0, 1)^\top\) is an eigenvector with eigenvalue 1. For the other eigenvector we use the Taylor condition (3) (in Definition 30) in the third equality below, and obtain
\[
M_B \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \beta_{11}^*(1) \\ \frac{1}{2} \beta_{21}^*(1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \beta_{11}^*(1) \\ \frac{1}{2} \beta_{21}^*(1) - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \beta_{11}^*(1) \\ -\frac{1}{2} \beta_{11}^*(1) \end{pmatrix}.
\]

The structure of \( R \) follows directly from eq. (18).

Lemma 36 leads to

Theorem 37. Let \( B \in \tilde{\ell}_t \) and let its associated vector scheme \( \mathcal{S}_B \) be convergent. Let \( \mathcal{I}_1 \) be the smoothing operator for VSSs in eq. (23). Then \( \mathcal{I}_1 B \in \tilde{\ell}_t \) if and only if the Laurent polynomial \( \beta_{11}^*(z) + \beta_{21}^*(z) - \beta_{12}^*(z) - \beta_{22}^*(z) \) has a root at 1 of multiplicity at least 2.
Proof. From Remark 24 we know that $\mathcal{E}_{I_1B} = \mathcal{E}_B = \text{span}\{e_2\}$. Furthermore, recall from eq. (23) that $I_1B = R(I_1B)^{-1}B = R^{-1}BR$. In Lemma 36 a canonical transformation $\overline{R}$ is computed. Therefore $\overline{B}$ is given by
\[ \overline{B} = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{pmatrix} = \begin{pmatrix} \beta_{12} + \beta_{22} & \beta_{11} + \beta_{21} - \beta_{12} - \beta_{22} \\ \beta_{12} & \beta_{11} - \beta_{12} \end{pmatrix}. \quad (40) \]

The parts of the Taylor conditions concerning the elements of $B^*(1)$ imply that the symbol $\overline{B}_{12}(z)$ has a root at 1. Therefore there exists a Laurent polynomial $\kappa^*(z)$ such that $\overline{B}_{12}(z) = (z^{-1} - 1)\kappa^*(z)$. Combining eq. (40) with eq. (14) we obtain
\[ (I_1B)^*(1) = \begin{pmatrix} 2 & \frac{1}{2}\kappa^*(1) \\ 0 & \frac{1}{2}\beta_{11}^* \end{pmatrix} \quad \text{and} \quad (I_1B)^*(-1) = \begin{pmatrix} 0 & \frac{1}{2}\beta_{11}^*(-1) \\ 0 & \frac{1}{2}\beta_{11}^*(-1) \end{pmatrix}. \]

Therefore
\[ (I_1B)^*(1) = \overline{R}(I_1B)^*(1)\overline{R}^{-1} = \begin{pmatrix} 2 + \frac{1}{2}\beta_{11}^*(1) & 0 \\ \frac{1}{2}\beta_{11}^*(-1) & 0 \end{pmatrix} \quad \text{and} \quad (I_1B)^*(-1) = \overline{R}(I_1B)^*(-1)\overline{R}^{-1} = \begin{pmatrix} \frac{1}{2}\beta_{11}^*(-1) \\ \frac{1}{2}((\kappa^*(-1) - \beta_{11}^*(-1)) & 0 \end{pmatrix}. \quad (41) \]

By eq. (41), (1) and (2) of the Taylor conditions in Definition 30 are satisfied by $I_1B$. The mask $I_1B$ satisfies (3) of the Taylor conditions if and only if $\kappa^*(1) = 0$. By the definition of $\kappa$, this is equivalent to the Laurent polynomial $\overline{B}_{12}(z) = \beta_{11}^*(z) + \beta_{21}^*(z) - \beta_{12}^*(z) - \beta_{22}^*(z)$ having a root of multiplicity 2 at 1.

Thus, in general, $I_1(\ell_t) \not\subseteq \ell_t$. In the next two lemmas we solve this problem.

**Lemma 38.** Let $B$ be a mask of a converging VSS satisfying $\mathcal{E}_B = \text{span}\{e_2\}$ and $\beta_{11}^*(1) \neq 2$. Then there exists a transformation $\overline{R}$ such that $B = \overline{R}^{-1}BR \in \ell_t$.

**Proof.** First we note that by Remark 31 the mask $B$ satisfies (1) and (2) of the Taylor conditions and obtain
\[ B^*(1) = \begin{pmatrix} a & 0 \\ b & 2 \end{pmatrix}, \]

with $a, b \in \mathbb{R}$ and $a \neq 2$ by the assumption of the lemma. To impose (3) of the Taylor conditions we take $\overline{R}$ with a second column $e_2$ in order to retain the above second columns. A normalized choice of the first column of $\overline{R}$ yields
\[ \overline{R} = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix}, \quad \overline{R}^{-1} = \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix}, \quad (42) \]

and we obtain
\[ \overline{B}^*(1) = \begin{pmatrix} a \\ (2-a)\eta + b \end{pmatrix}. \]

To satisfy (3) of the Taylor conditions $(2 - a)\eta + b + a = 2$. Therefore we choose $\eta = 1 + \frac{a}{a-2}$. From the form of $\overline{B}^*(1)$ and since $a \neq 2$, we see that $\mathcal{E}_B = \text{span}\{e_2\}$. \qed
Next we show that we can apply the smoothing procedure and transform the resulting mask to a mask in ˜\(\ell\).

**Corollary 39.** Let \(B \in \hat{\ell}_\ell\) such that \(S_B\) is a \(C^\ell\) VSS, for \(\ell \geq 0\). Then \(\hat{I}_1(B) \in \hat{\ell}_\ell\) and \(S_{\hat{I}_1(B)}\) is a \(C^{\ell+1}\) VSS.

**Proof.** It follows from Remark 24 that \(E_{\hat{I}_1B} = E_B = \text{span}\{e_2\}\). Equation (41) implies \((\hat{I}_1B)_{i1}^1(1) = \frac{1}{2} \beta_{i1}^1(1)\). From Lemma 36 we know that \(\frac{1}{2} \beta_{i1}^1(1)\) is an eigenvalue of \(M_B\). By Corollary 20 \(\frac{1}{2} \|\beta_{i1}^1(1)\| \leq 1\). In particular \((\hat{I}_1B)_{i1}^1(1) \neq 2\). Therefore, \(\hat{I}_1B\) satisfies the conditions of Lemma 38 and with the transformation \(R\) in eq. (42), \(R^{-1}(\hat{I}_1B)R \in \hat{\ell}_\ell\). The statement about smoothness follows from the construction of \(\hat{I}_1\) in eq. (23). \(\square\)

### 5.5 A procedure for increasing the smoothness of Hermite schemes

Theorem 37 and Corollary 39 allow to define the following procedure for increasing the smoothness of HSSs:

**Procedure 40.** The input is a mask \(A\) satisfying the spectral condition (Lemma 28). Furthermore we assume that its Taylor scheme is \(C^{\ell-1}\) for \(\ell \geq 1\) and that the limit functions have vanishing first component for all input data (this implies that \(S_A\) is \(HC^\ell\)). The output is a mask \(C\) which satisfies the spectral condition and its associated Hermite scheme \(S_C\) is \(HC^{\ell+1}\).

1. Compute the Taylor scheme \(\partial_t A\) (Lemma 32).
2. Apply Procedure 23 and Lemma 38 to obtain \(B = \hat{I}_1(\partial_t A)\).
3. Define \(C = \hat{I}_\ell(B)\) (Lemma 33).

In the following we execute Procedure 40 for a general mask \(A\) satisfying the assumptions of the procedure, and present explicitly \(C^* (z)\).

From the definition of \(\eta\) in the proof of Lemma 38 it is easy to see that \(\eta = \frac{\alpha_{22}^1(1)}{\tau - \alpha_{22}^1(1)}\). This is well-defined, since \(M_{\partial_t A}\) has \(\alpha_{22}^1(1)\) as an eigenvalue. By Corollary 20 \(\alpha_{22}^1(1) \neq \)
2. Then with \( \zeta = \eta + 1 \) we get

\[
\gamma_{11}^*(z) = \frac{1}{2}(z^{-1} + 1)\left(\alpha_{12}^*(z)\left((\zeta - \zeta^2)z^{-3} + \zeta^2z^{-2} + (\zeta^2 - 1)z^{-1} - (\zeta^2 + \zeta)\right) + \alpha_{11}^*(z)(\zeta(z^{-1} - 1))(1 - \zeta) + \alpha_{22}^*(z)(\zeta^{-2} - 1)(\zeta - 1)\right) + \alpha_{21}^*(z)(\zeta^2 - \zeta),
\]

\[
\gamma_{12}^*(z) = \frac{1}{2}\left(\alpha_{12}^*(z)\left((1 - \zeta)^2z^{-3} + \zeta (1 - \zeta)z^{-2} + (1 - \zeta)z^{-1} + \zeta^2\right) + \alpha_{12}^*(z)\left(-\zeta^2z^{-3} + (\zeta + \zeta^2)(z^{-2} + z^{-1}) - (\zeta + 1)^2\right) + \alpha_{22}^*(z)\zeta(\zeta(z^{-1} - 1)) + \alpha_{22}^*(z)\zeta((1 - \zeta)(z^{-2} - 1)) + \alpha_{21}^*(z)(\zeta - \zeta^2)\right),
\]

\[
\gamma_{21}^*(z) = \frac{1}{2}(z^{-2} - 1)\left(\alpha_{12}^*(z)\left((\zeta^2 - \zeta)z^{-3} + (1 - \zeta^2)z^{-2} - \zeta^2z^{-1} + (\zeta^2 + \zeta)\right) + \alpha_{11}^*(z)(1 - \zeta)(1 - \zeta(z^{-1} - 1)) + \alpha_{22}^*(z)((1 - \zeta)(z^{-2} - 1) + 1) + \alpha_{21}^*(z)(\zeta - \zeta^2)\right).
\]

In the special case \( \alpha_{12}^*(1) = 0, \zeta = 1 \), \( C^*(z) \) reduces to

\[
\gamma_{11}(z) = \frac{1}{2}(z^{-1} + 1)\left((z^{-2} - 2)\alpha_{12}(z) + \alpha_{11}(z)\right),
\]

\[
\gamma_{12}(z) = \frac{1}{2}\left(\alpha_{12}^*(z)\right)\left(\frac{1}{z^{-1} - 1}\right),
\]

\[
\gamma_{21}(z) = \frac{1}{2}(z^{-2} - 1)\left(\alpha_{21}^*(z) - \alpha_{11}^*(z)(z^{-1} - 2) + \alpha_{22}^*(z)(z^{-2} - 2) - \alpha_{12}^*(z)(z^{-1} - 2)(z^{-2} - 2)\right),
\]

\[
\gamma_{22}(z) = \frac{1}{2}\left(\alpha_{22}^*(z) - (z^{-1} - 2)\alpha_{12}^*(z)\right).
\]

With the explicit form of \( C \), we can prove

**Lemma 41.** Let \( \varphi_\mathbf{A} \) be the constant corresponding to the spectral condition in eq. \( \text{(28)} \) satisfied by \( \mathbf{A} \). Then the constant corresponding to the spectral condition satisfied by \( \mathbf{C} \) is \( \varphi_\mathbf{C} = \varphi_\mathbf{A} - \frac{1}{2} \).

In particular, the application of Procedure \( \text{[40]} \) to interpolatory HSSs does not result in interpolatory HSSs.

**Proof.** Differentiating \( \gamma_{11}^*(z) \) and \( \gamma_{12}^*(z) \) given in eq. \( \text{(43)} \), and evaluating at \( z = 1 \) we
obtain in view of condition (3) in Lemma 28
\[ 2 \varphi_C = \gamma_{11}'(1) - 2 \gamma_{12}'(1) = \alpha_{11}'(1) - 2 \alpha_{12}'(1) + (\zeta - 1)(\alpha_{21}'(1) - 2 \alpha_{22}'(1)) \
+ 2(\zeta - 1) + \frac{1}{2} \alpha_{12}'(1) - \zeta - \frac{1}{2} \alpha_{22}'(1)(1 - \zeta) = 2 \varphi_A + \frac{1}{2}(\alpha_{12}'(1) - \alpha_{22}'(1)) - \frac{1}{2} \zeta(2 - \alpha_{22}'(1)) = 2(\varphi_A - \frac{1}{2}). \]

From the explicit form of \( C \) we can infer

**Corollary 42.** Let \( A \) and \( C \) be masks as in Procedure 40. If \( A \) has support contained in \([-N_1, N_2]\) with \( N_1, N_2 \in \mathbb{N} \), then the support of \( C \) is contained in \([-N_1 - 5, N_2]\).

Therefore Procedure 40 increases the support length at most by 5.

**Corollary 43.** Let \( A \) be a mask satisfying the spectral condition of eq. (28) and let its associated Taylor scheme be convergent. Assume that \( \alpha_{12}'(1) = 0 \) (i.e. \( \zeta = 1 \)). Denote by \( C \) the mask obtained via Procedure 40. Then \( \gamma_{12}'(1) = 0 \) if and only if \( \alpha_{12}'(1) = 0 \).

**Proof.** From the definition of \( C \) in eq. (43) it is easy to see that \( \gamma_{12}'(1) = -\frac{1}{2} \alpha_{12}'(1) \). Therefore \( \gamma_{12}'(1) = 0 \) iff \( \alpha_{12}'(1) = 0 \).

Let \( r \) be the multiplicity of the root at 1 of \( \alpha_{12}'(z) \). Corollary 43 implies that \( r - 1 \) iterations of the smoothing procedure stay within the special case of \( \zeta = 1 \).

**Example 44.** We consider the Hermite scheme generating \( C^1 \) piecewise cubic polynomials interpolating the inital data (see Merrien (1992)). The mask of the scheme is given by
\[
A_{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{pmatrix}.
\]

It is easy to see that it satisfies the spectral condition of eq. (28) with \( \varphi_A = 0 \). In Merrien and Sauer (2012) it is proved that its Taylor scheme is convergent with limit functions of vanishing first component (and thus the original HSS is \( HC^1 \)).

We apply Procedure 40 to this scheme to obtain a new HSS of regularity \( HC^2 \), using the explicit expressions in eq. (43) and eq. (44). First we compute the symbol:
\[
A^*(z) = \begin{pmatrix} \frac{1}{8}(1 + z)^2z^{-1} & -\frac{1}{8}(1 - z^2)z^{-1} \\ \frac{3}{4}(1 - z^2)z^{-1} & -\frac{1}{8}z^{-1} + \frac{1}{2} - \frac{1}{8}z \end{pmatrix}.
\]

Note that \( \alpha_{12}'(1) = 0 \) with multiplicity 1. Therefore we are in the special case \( \zeta = 1 \).

We apply eq. (44) and obtain the symbol of \( C \):
\[
C^*(z) = \frac{1}{16} \begin{pmatrix} (z^{-1} + 1)^2(-z^{-2} + z^{-1} + 6 + 2z) & -z - 1 \\ (z^{-2} - 1)(z^{-4} - 3z^{-3} - 3z^{-2} + 13z^{-1} + 6) & z^{-2} - 3z^{-1} + 3 + z \end{pmatrix}.
\]
Figure 3: Basic limit functions and their first derivatives of the HSSs of Example \[44\]
First column: interpolatory $HC^1$ scheme $S_A$ with basic limit function $f$. Second column: the smoothed non-interpolatory $HC^2$ scheme $S_C$ with basic limit function $g$.

From Lemma \[41\] we also know that $C$ satisfies the spectral condition with $\varphi_C = -\frac{1}{2}$. Therefore the HSS associated with $C$ is an $HC^2$ scheme which is not interpolatory. A basic limit function of this scheme is depicted in Figure \[3\]. Note that the support of $C$ is $[-6, 1]$ and has thus increased from length of 3 to the length of 8.

If we want to apply another round of Procedure \[40\] we have to use eq. \[43\] with $\zeta = \frac{14}{15}$.

**Example 45.** We consider one of the de Rham-type HSSs of \[44\] obtained from the scheme of Example \[44\]. Its mask is given by

\[
A_{-2} = \frac{1}{8} \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ -\frac{9}{2} & -\frac{5}{4} \end{pmatrix}, \quad A_{-1} = \frac{1}{8} \begin{pmatrix} \frac{27}{4} & -\frac{9}{8} \\ \frac{9}{2} & \frac{3}{4} \end{pmatrix},
\]

\[
A_0 = \frac{1}{8} \begin{pmatrix} \frac{27}{4} & \frac{9}{8} \\ -\frac{9}{2} & \frac{3}{4} \end{pmatrix}, \quad A_1 = \frac{1}{8} \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ -\frac{9}{2} & -\frac{5}{4} \end{pmatrix}.
\]

It is easy to see that it satisfies the spectral condition of eq. \[28\] with $\varphi_A = -\frac{1}{2}$. In \[44\] it is proved that its Taylor scheme is $C^1$ with limit functions of vanishing first component (and thus the original HSS is $HC^2$).

We apply Algorithm \[40\] to this scheme to obtain a new HSS of regularity $HC^3$. First we compute the symbol:

\[
A^*(z) = \frac{1}{16} \begin{pmatrix} \frac{1}{2}(z^{-1} + 1)(5z + 2z + 5z^{-1}) & -\frac{3}{4}(z^{-1} - 1)(z + 4 + z^{-1}) \\ 9(z^{-2} - 1)(z + 1) & \frac{1}{2}(z^{-1} + 1)(-5z + 8 - 5z^{-1}) \end{pmatrix}.
\]
Note that $\alpha_{12}^*(1) = 0$ with multiplicity 1. Therefore, as in Example 44 we are in the special case $\zeta = 1$. We apply eq. (44) and obtain the symbol of $C$:

\[
\gamma_{11}^*(z) = \frac{1}{128}(z^{-1} + 1)(-3z^{-4} - 9z^{-3} + 25z^{-2} + 75z^{-1} + 36 + 4z),
\]
\[
\gamma_{12}^*(z) = -\frac{3}{128}(z + 4 + z^{-1}),
\]
\[
\gamma_{21}^*(z) = \frac{1}{128}(z^{-2} - 1)(3z^{-5} - 7z^{-4} - 37z^{-3} + 37z^{-2} + 128z^{-1} + 20 - 8z),
\]
\[
\gamma_{22}^*(z) = \frac{1}{128}(3z^{-3} - 7z^{-2} - 21z^{-1} + 21 - 4z).
\]

We also know from Lemma 41 that $C$ satisfies the spectral condition with $\varphi_C = -1$. Therefore the HSS associated with $C$ is an $HC^3$ scheme which is not interpolatory. A basic limit function of this scheme is depicted in Figure 4. Note that the support of $C$ is $[-7, 1]$ and has thus increased from length of 4 to the length of 9.

If we want to apply another round of Procedure 40 we have to use eq. (43) with $\zeta = \frac{41}{44}$.

Figure 4: Basic limit functions, their first and second derivatives of the HSSs of Example 45. First column: non-interpolatory $HC^2$ scheme $S_A$ with basic limit function $f$. Second column: smoothed non-interpolatory $HC^3$ scheme $S_C$ with basic limit function $g$.  

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