Steady and Oscillatory Side-Band Instabilities in Marangoni Convection with Deformable Interface

†A.A.Golovin, ‡A.A.Nepomnyashchy, †L.M.Pismen, and *H.Riecke

†Department of Chemical Engineering,
‡Department of Mathematics,
Minerva Centre for Nonlinear Physics of Complex Systems,
Technion, Haifa 32000, Israel;
*Department of Engineering Sciences and Applied Mathematics,
Northwestern University, Evanston, USA.

January 2, 2022

Abstract

The stability of Marangoni roll convection in a liquid-gas system with deformable interface is studied in the case when there is a nonlinear interaction between two modes of Marangoni instability: long-scale surface deformations and short-scale convection. Within the framework of a model derived in [1], it is shown that the nonlinear interaction between the two modes substantially changes the width of the band of stable wave numbers of the short-scale convection pattern as well as the type of the instability limiting the band. Depending on the parameters of the system, the instability can be either long- or short-wave, either monotonic or oscillatory. The stability boundaries strongly differ from the standard ones and sometimes exclude the band center. The long-wave limit of the side-band instability is studied in detail within the framework of the phase approximation. It is shown
that the monotonic instability is always subcritical, while the long-wave oscillatory instability can be supercritical, leading to the formation of either travelling or standing waves modulating the pattern wavelength and the surface deformation.
1 Introduction.

An important class of instabilities of one-dimensional patterns are side-band instabilities. The most important one among them is the Eckhaus instability [2] which is a monotonic long-wave instability that renders the pattern unstable with respect to infinitesimal compressions and dilatations. Its onset can be characterized by the vanishing phase diffusion coefficient [3]. Near the threshold, where the pattern can be described by a real (dissipative) Ginzburg–Landau (GL) equation, the Eckhaus instability arises generically if the deviation $q$ of the wave-number of the pattern from that corresponding to the bifurcation point exceeds a certain value, i.e. if

$$|q| > \epsilon/\sqrt{3},$$

(1)

where $\epsilon^2$ is the deviation of the control parameter from the threshold. A similar instability of wave patterns (e.g. described by the complex GL equation) is usually referred to as the Benjamin-Feir instability [4]. Typically, the Eckhaus instability restricts the range of allowed wave numbers of the pattern, i.e. its nonlinear evolution eventually brings the system back to the stable range of wave numbers.

The Eckhaus instability of periodic patterns has been studied extensively, both theoretically and experimentally, in various pattern-forming systems, such as Rayleigh-Bénard convection [5, 6], convection in binary mixtures [7], Taylor vortex flow [8, 9], electrohydrodynamic convection in nematics [10], directional solidification [11]. Even quite close to the threshold, the boundaries of the Eckhaus instability of stationary periodic patterns can differ from (1) due to various factors, such as the system confinement [3, 12], subcritical character of the instability [13], the shape of the neutral stability curve [14, 15], the effect of nonlinearity [16] and resonances [9], the presence of random fluctuations [17], chiral symmetry breaking (e.g. due to rotation) [18], etc.

Although the Eckhaus instability is the most common one-dimensional side-band instability, others are possible as well. In the present paper we show that spatially periodic patterns arising due to Marangoni convection in a liquid layer with a deformable free surface heated from below [1] can possess additional side-band instabilities: a short-scale steady side-band instability and a long-scale oscillatory side-band instability. An important feature of this system is the presence of two primary modes of instability (generating the Marangoni convection itself): a long-scale instability (with zero wave-number) connected with the deformation of the free surface, and short-scale convection (with a finite wave-number) governed by surface tension gradients. A typical neutral stability curve for this system has two minima, corresponding to two critical Marangoni numbers [19]. The one minimum, $M_{\alpha}$, is at a zero wave-number and corresponds to a long-scale instability driven by deformation of the free surface coupled with surface tension gradients. The other minimum, $M_{\alpha}$, is at a non-zero wave-number, and corresponds to a

\[\text{This type of convection can be also driven by mass rather than heat transfer.}\]
short-scale instability caused by surface tension gradients only (surface deformations are not important in this regime).

When the two thresholds are close to each other, the nonlinear interaction between the two modes of Marangoni instability with different scales can lead to various secondary instabilities of the short-scale convection generating various stationary and temporally modulated patterns [1, 20]. The nonlinear evolution and interaction of the two modes is described by a system of coupled equations which can be written in the following scaled form [1]:

\[
\begin{align*}
A_t &= A + A_{xx} - |A|^2 A + Ah, \\
h_t &= -mh_{xx} - wh_{xxx} + s|A|^2_{xx}.
\end{align*}
\]

Here \( A \) is the amplitude of the short-scale convection, \( h \) is the deformation of the free surface, \( x \) and \( t \) are long-scale space coordinate and slow time. The form of the equation for \( h \) is determined by the fact that \( h \) is conserved due to mass conservation. In the non-conserved case certain aspects of the interaction between short- and long-scale structures has been discussed in [21]. The parameter \( m \) is proportional to the difference between the two thresholds, \( Ma_s - Ma_l \), \( w > 0 \) characterizes the effect of the Laplace pressure, and \( s > 0 \) is the computable coupling parameter which was shown to be positive [1]. Note that for the terms in the second equation in (2) to be of the same order, the capillary number, which parameter \( w \) is proportional to, has to be large (which is indeed the case in most of experimental conditions). The coupling between \( A \) and \( h \) reflects the following mechanism of the modes interaction: as expressed by the coupling term in the first equation, the short-scale convection is more intensive beneath surface elevations; since \( s > 0 \), the convection, in turn, suppresses the long-scale deformational mode by changing the average surface temperature or concentration [1]. This stabilization effect of the short-scale convection mode allows the difference between the critical Marangoni numbers \( Ma_l \) and \( Ma_s \) to be \( O(1) \) and still the deformational instability to be restricted to long scales and have small growth rate. However, this stabilization fails if \( Ma_s - Ma_l \) is too large (see [20]).

In Ref. [1], the effect of the mode interaction on the stability of the short-scale convection was studied only in the vicinity of the band center \((A = 1, h = 0)\). In the present paper we extend the analysis to the full wave-number band of the short-scale convection and show that the interaction of the two modes strongly affects the width of the band and can even lead to a change of the type of the instability.

### 2 Linear Stability Analysis of Short-Scale Convection Rolls

We consider the effect of surface deformations on the stability of short-scale convection rolls with wave-numbers lying within an \( O(\epsilon) \) range at small supercriticality \((Ma_s - Ma_l)/Ma_s \approx \epsilon^2\). This convection pattern is described by the following stationary solution of eq. (3):

\[
A_0 = \sqrt{1 - q^2} e^{iqx}, \quad h = 0,
\]

(3)
where \( q < 1 \) is the dimensionless (in units of the characteristic width \( \epsilon \) of the band of excited wave-numbers) deviation from the critical wave-number corresponding to the threshold \( \text{Ma} = \text{Ma}_s \) of the short-scale instability.

Consider a perturbed solution in the form

\[
A = A_0 (1 + a e^{\sigma t + i k x} + b e^{\sigma^* t - i k x}),
\]

\[
h = c e^{\sigma t + i k x} + c^* e^{\sigma^* t - i k x},
\]

where asterisks denote complex conjugates. Substituting (4) in system (2) yields the following implicit dispersion relation for the side-band instability:

\[
\sigma^3 + \alpha \sigma^2 + \beta \sigma + \gamma = 0,
\]

where

\[
\alpha = -m \kappa^2 + w \kappa^4 + 2(1 - q^2 + \kappa^2),
\]

\[
\beta = 2(-m \kappa^2 + w \kappa^4)(1 - q^2 + \kappa^2) + \kappa^4 + 2 \kappa^2(1 - 3q^2) + 2 s \kappa^2(1 - q^2),
\]

\[
\gamma = (-m \kappa^2 + w \kappa^4)(\kappa^4 + 2 \kappa^2(1 - 3q^2)) + 2 s \kappa^4(1 - q^2).
\]

We see that the interaction of the two modes leads to a cubic equation for the eigenvalues defining the limits of the side-band instability (similar, say, to the case of the Eckhaus instability of a hexagonal pattern when there is a coupling between three waves \[22, 23\]). In our case, we have the coupling between two side-band modes, which correspond to longitudinal compression-dilatation waves propagating in opposite directions if \( \sigma \) is complex, and the surface deformation.

According to the Routh-Hurwitz criterion, it follows from (5) that the stationary solution (3) is stable if

\[
\alpha > 0, \quad \gamma > 0, \quad \alpha \beta - \gamma > 0.
\]

The analysis of conditions (6) shows that, depending on the values of the parameters \( m, w \) and \( s \), the side-band instability in this system can be either long-wave or short-wave, monotonic or oscillatory, and the range of \( q \) where the solution (3) is stable can be either wider or narrower than the standard Eckhaus interval defined by (1).

A detailed analysis of various possibilities is very complicated since the stability of the system is governed by four parameters, \( m, w, s, \) and \( q \). We shall dwell therefore mostly on the case when the solution (3) becomes unstable with respect to perturbations with infinitesimal wave-number. The corresponding stability limits can be determined from (5) in the limit \( \kappa \rightarrow 0 \).

It is, however, more instructive to obtain this result using the so-called phase approximation which is based on the observation that in the long-wave limit the evolution of the amplitude of the convection pattern is slaved to the evolution of its phase \[24, 25\]. Thus, we set \( A = (\rho_0 + \epsilon \rho) \exp (i q x + \phi), \ h = \mathcal{O}(\epsilon), \ \partial_x = \mathcal{O}(\epsilon), \ \partial_t = \mathcal{O}(\epsilon^2) \), and obtain from eq.(2) for the amplitude correction \( \rho \)

\[
\rho = \frac{q}{\rho_0} \phi_x + \frac{h}{2 \rho_0}, \quad \rho_0 = \sqrt{1 - q^2}.
\]
In our case, the amplitude near the instability threshold is slaved to both the phase \( \phi \) and the surface deformation \( h \). The evolution of these two variables is described by the following system of coupled equations:

\[
\begin{align*}
\phi_t &= \frac{1 - 3q^2}{1 - q^2} \phi_{xx} + \frac{q}{1 - q^2} h_x, \\
h_t &= (s - m) h_{xx} - 2qs \phi_{xxx}.
\end{align*}
\]

Introducing the wave-number deviation, \( k \equiv \phi_x \), one can rewrite system (8) in the form of the coupled diffusion equations:

\[
\begin{align*}
a_t &= -S a_{xx},
\end{align*}
\]

where

\[
a = \begin{pmatrix}
k \\
h
\end{pmatrix}, \quad S = \begin{pmatrix}
\frac{3q^2 - 1}{1 - q^2} & -\frac{q}{1 - q^2} \\
2qs & m - s
\end{pmatrix}.
\]

The system is stable if all eigenvalues of matrix \( S \) have negative real parts. The characteristic equation leads to the following stability conditions:

\[
\begin{align*}
3q^2 - 1 + (m - s)(1 - q^2) &< 0, \\
(3q^2 - 1)(m - s) + 2q^2 s &> 0.
\end{align*}
\]

It should be noted that this system obtained in the long-wave limit does not give the complete stability information. It can be shown that there always exists an interval of \( m \) where the relevant instability is a short-wave instability that can be either monotonic or oscillatory.

We shall dwell on the case when the instability is long-wave in the band center, and no short-wave oscillatory instability arises. A typical stability diagram is shown in Fig.1 for \( s = 3, \ w = 15 \). Fig.1 is true provided \( \{ s < 2w, \ s < 1 + \sqrt{s}w \} \) (see Ref.[1]) and \( w \) is larger than a certain value \( w^*(s) \), respectively (for \( s = 3, \ w > w^* \approx 8 \)). Fig.1 presents the region in the plane \((m, q^2)\) in which the solution (3) is stable. This region is bounded by the lines AB, BC, CD and DE. The lines AB and DE are the loci of \( q^2 = (m - s)/(3m - s) \) and correspond to a monotonic long-wave instability (cf. [1]), while the line CD is the locus of \( q^2 = (1 + s - m)/(3 + s - m) \) and corresponds to an oscillatory long-wave instability (cf. [1]). The line BC corresponds to a monotonic short-wave instability and can be found analytically from the condition \( \gamma = 0 \) (see Eq.(6)) (it is, however, too cumbersome to be reproduced here). The horizontal dashed line corresponds to the threshold of the standard Eckhaus instability, \( q^2 = 1/3 \).

The coordinates of the points A, B, C, D, and E can be found analytically: A \( \rightarrow \{ -\infty, 1/3 \} \), B = \( \{(s - s_w)/6, (5s + s_w)/(3(s + s_w))\} \), \( s_w = \sqrt{3^2 + 48sw} \), D = \( \{s(1 + \sqrt{1 + 4/s})/2, (\sqrt{1 + 4/s - 1})/(3\sqrt{1 + 4/s} + 1)\} \), E = \{s, 0\}; the expressions for the coordinates of the point C are very lengthy (for \( s = 3, \ w = 15 \) in Fig.1, C= \{1.97, 0.5\}).

Thus, due to the coupling between the phase evolution and the surface deformation, both the interval of wave-numbers corresponding to stable stationary convection described by eq.3

\[
\begin{align*}
\end{align*}
\]
and the type of its instability depend on the parameter $m$, i.e., on the difference between the two thresholds of the Marangoni instability, $Ma_s - Ma_l$. There are four particular values of $m$ ($m_1(s, w)$, $m_2(s, w)$, $m_3(s)$, $m_4(s)$), corresponding to the points B, C, E, D in Fig.1, respectively, which define four regions where qualitatively different instabilities are observed. These regions in the $(s, m)$-parameter plane are shown in Fig.2 for $w = 15$.

If $m < m_1(s, w)$, solution (3) is stable when $q^2 < (m - s)/(3m - s)$. Since $s > 0$ in our case, the stability interval for $q$ is larger than in the case of the ordinary Eckhaus instability, $q^2 < 1/3$; part of the energy of phase perturbations is transferred to the damped long-scale deformational mode. This suppresses the Eckhaus instability and makes the stability interval for $q$ larger. Outside this interval of $q$, solution (3) is unstable and the instability is monotonic and long-wave, like in the standard case with no surface deformation. In the limit $m \to -\infty$, surface deformations are damped very strongly, and the Eckhaus interval tends to its ordinary value.

When $m_1(s, w) < m < m_2(s, w)$, the stability threshold for $q$ is determined by a cumbersome function of $w$ and $s$ which we do not represent here. This threshold is still larger than $1/3$. The instability in this case is also monotonic, but short-wave. At the point B the modulation wave number $\kappa$ goes to zero and the short-wave instability continuously changes over to a long-wave instability. This shows that the short-wave instability is not due to a subharmonic resonance as it has been found, e.g., in Taylor vortex flow [26].

When $m_2(s, w) < m < m_3(s) = s$, the stability interval for $q$ is given by $q^2 < (1 + s - m)/(3 + s - m)$. The instability in this case is oscillatory and long-wave and can lead to slow oscillations of the wave-number of the short-scale roll pattern coupled with oscillations of the surface deformation (see sec.3.2 below).

At $m = m_3(s) = s$, the stability region coincides with the ordinary Eckhaus interval, $q^2 < 1/3$.

If $m > m_3(s) = s$, the roll pattern corresponding to the band center, $q = 0$, becomes unstable. This coincides with the results in [1]. Nevertheless, as long as $m_3(s) < m < m_4(s)$, a stable pattern still exists under these conditions, but with a modified wave-number lying in the interval $(m - s)/(3m - s) < q^2 < (1 + s - m)/(3 + s - m)$. The lower boundary of this interval corresponds to the long-wave monotonic instability (i.e., the usual Eckhaus instability) while the upper one to the long-wave oscillatory instability. This splitting of the stable band into two sub-bands is reminiscent of a situation encountered in parametrically excited standing waves: under certain conditions the wave numbers in the center of the band become Eckhaus-unstable while off-center wave numbers remain stable. Such a splitting was found to lead to the formation of stable kinks in the wave number separating domains with different wave numbers [27]. The nonlinear evolution of the present system in this regime will be discussed briefly below.

For $m > m_4(s)$, solution (3) is unstable at any wave number due to the deformational instability of the surface.
Thus, depending on $m$, i.e. on the difference between the threshold $Ma_s$ of the short-scale convection and that of the long-scale deformational instability ($Ma_l$), the band of stable wave numbers can take on two qualitatively different forms. For $m < s$ the band is contiguous (hashed area in Fig.3a) while for $s < m < m_4(s)$ convection is stable within two disconnected wave-number bands (hashed area in Fig.3b).

In conclusion of this section let us note that if the parameter $w$ is not large enough, the oscillatory short-wave side-band instability comes into play. In this case, the stability region shown in Fig.1 slightly changes: there appears a new boundary corresponding to this short-wave oscillatory side-band instability which cuts off the corner at the point C and connects two boundaries BC and CD corresponding to the short-wave monotonic and to the long-scale oscillatory side-band instabilities, respectively.

### 3 Nonlinear evolution of the long-wave instabilities

In this section we shall study the nonlinear evolution of the long-wave instability near the thresholds given by the curves AB, CD and DE in Fig.1.

#### 3.1 Eckhaus instability

The monotonic long-wave instability threshold corresponds to the curves AB and DE in Fig.1 and is given by

$$ q_m^2 = \frac{m - s}{3m - s}. \quad (12) $$

We use the following scaling: $X = \epsilon x$, $T = \epsilon^4 t$, and apply the following ansatz:

$$ A = \rho \exp i\phi, $$

$$ \rho = \rho_0 + \epsilon^2 \rho_1(X, T) + \epsilon^4(X, T)\rho_2 + \ldots, $$

$$ \phi = qx + \epsilon\phi_1(X, T) + \epsilon^2\phi_2(X, T) + \ldots, $$

$$ h = \epsilon^2h_1(X, T) + \epsilon^3h_2(X, T) + \ldots, $$

$$ q = q_m + \epsilon^2 q_2 + \ldots $$

Substituting (13) in system (2), one obtains in the first order that the change in the local wave-number, $k_1 \equiv \partial_X\phi_1$, is proportional to the surface deformation $h_1$,

$$ k_1 = \frac{s - m}{2sq_m}h_1. \quad (14) $$

Proceeding to the sixth order, we obtain the following nonlinear evolution equation for the surface deformation $h_1$:

$$ \partial_T h_1 + \sigma_1 \partial_{XX} h_1 + \sigma_2 \partial_{XXX} x h_1 + \sigma_3 \partial_{XX} (h_1^2) = 0, \quad (15) $$
where
\[ \sigma_1 = q_2q_m \frac{(4m - s)(s - 3m)}{2(-m^2 + ms + s)}, \]
\[ \sigma_2 = \frac{-3m^2 + ms + 4sw}{4(-m^2 + ms + s)}, \]
\[ \sigma_3 = \frac{m(s - 3m)(5s - 6m)}{8s(-m^2 + ms + s)}. \]

The evolution equation for the long-wave modulation of the wave-number \( k_1 \) can be obtained from eq.(15) by a simple renormalization of the coefficient \( \sigma_3 \to \sigma_3 \frac{2sq_m}{(s - m)} \) (cf. (14)). When \( m \to -\infty \), the surface is undeformable, and the coefficients in the equation for \( k_1 \) tend to those of the nonlinear evolution equation describing an ordinary Eckhaus instability in the real GL equation.

It can be shown that the coefficients \( \sigma_2 \) and \( \sigma_3 \) are positive (in the parameter region where the instability is long-wave), and the instability is subcritical. Therefore, all periodic solutions of (13) blow up beyond the stability limit \( \sigma_1 = 0 \) indicating a breakdown of (13). The same happens in the case of the monotonic short-scale instability, corresponding to the curve BC in Fig.1; our computations show that it is also subcritical. However, numerical simulations of the full amplitude equations (2) show that different physical processes occur when crossing the lines AC and DE in Fig.1, respectively. This is related to the fact that along AC the deformational mode is linearly damped (\( m < s \)), whereas along DE it is excited (\( m > s \)). Figs.4,5 show typical results of numerical simulations of (2).

Fig.4 shows the evolution of the pattern occurring when the line AC is crossed. While the surface deformation and the amplitude of the short-scale convection remain regular for all times, the wave-number goes through a true singularity. It signifies a phase slip, which reduces the total phase in the system by \( 2\pi \). Following this phase slip, the pattern relaxes to a periodic pattern with a wave-number in the stable band.

The evolution below the line DE is shown in Fig.5. Here the deformational mode is excited; it has been shown previously that the instability is subcritical at \( q = 0 \) \([20]\). The simulation of the coupled amplitude equations (2) indicates that the instability does indeed not saturate which results in a singularity in the surface deformation and in the amplitude of the short-scale convection. Thus, even the amplitude equations (2) become invalid. This result suggests a dry-out of the fluid film as has been observed experimentally \([28]\).

Despite the splitting of the wave-number band, no stable wave-number kinks could be found. This indicates that the higher-order contributions to the phase equation (15), which were not calculated, differ from those arising solely from a non-monotonicity of the phase diffusion coefficient \([27]\).

It should be noted that at the point D the coefficients \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) tend to infinity, while at the point E the coefficient \( \sigma_1 \) vanishes regardless of the value of \( q_2 \); the system dynamics in the vicinity of these points is more complex; it cannot be described by eq.(15) and requires separate analysis.
3.2 Oscillatory long-wave instability

Along line CD in Fig.1 an oscillatory long-wave instability arises. Its threshold is given by

\[ q_o^2 = 1 + s - m / 3 + s - m. \] (16)

To study its nonlinear behavior we set \( A = \rho \exp i\phi \), introduce the slow coordinates \( X = \varepsilon x \), \( T_0 = \varepsilon^2 t \), \( T = \varepsilon^4 t \), and expand

\[
\begin{align*}
\rho &= \rho_0 + \varepsilon \rho_1(X,T_0,T) + \varepsilon^2 \rho_2(X,T_0,T) + \ldots, \\
\phi &= qx + \phi_0(X,T_0,T) + \varepsilon \phi_1(X,T_0,T) + \varepsilon^2 \phi_2(X,T_0,T) + \ldots, \\
h &= \varepsilon h_1(X,T_0,T) + \varepsilon^2 h_2(X,T_0,T) + \varepsilon^3 h_3(X,T_0,T) + \ldots, \\
q &= q_o + \varepsilon^2 q_2 + \ldots,
\end{align*}
\] (17)

with

\[ \rho_0 = \sqrt{1 - q_o^2}. \]

Substituting (17) in system (3), we obtain in the first order a linear system for \( h_1 \) and \( k_0 = \partial_X \phi_0 \) which, after appropriate diagonalization, can be reduced to a linear free Schrödinger equation

\[ i\partial_{T_0} \psi = -\omega_0 \partial_X \psi, \] (18)

where

\[
\psi = ((m - s + i\omega_0)h_1 - k_0)/(2i\omega_0), \quad \omega_0 = \sqrt{-m^2 + s(m + 1)}.
\]

Eq. (18) describes waves propagating along the convection pattern and modulating its wave-number and the surface deformation. The phase velocity of the waves \( v_p \) depends on the wave number \( \kappa \) of the modulation, as \( v_p = \omega_0 \kappa \). The wave-number \( k_0 \) and the surface deformation \( h_1 \) are proportional to each other

\[ k_0 = \delta h_1, \quad \delta = \frac{s - m + i\omega_0}{2sq_0}, \] (19)

and there is a phase shift between oscillations of the wave-number and the surface deformation equal to \( \arg(\delta) \).

At the onset of the oscillatory long-wave instability, a whole band of wave-numbers \( \sim O(\varepsilon) \) is excited. However, a strong (quadratic) dispersion of the waves described by (18) allows only quartic interactions between waves with different wave-numbers within the excited interval. The evolution of the wave amplitudes can thus be described in the third order of the perturbation theory by a system of Landau equations. A similar situation occured in a study of long-scale oscillatory double-diffusive convection [29].

In order to study the wave interaction, we consider two pairs of oppositely propagating waves having wave-numbers \( \kappa_1 \) and \( \kappa_2 \), and present \( k_0 \) and \( h_1 \) in the following form:

\[
\begin{pmatrix} k_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} \delta \\ 1 \end{pmatrix} \left[ A_1(T)e^{i\theta_1} + B_1(T)e^{i\theta_1} + A_2(T)e^{i\theta_2} + B_2(T)e^{i\theta_2} \right] + c.c., \] (20)
where the amplitudes \( A_{1,2} \) and \( B_{1,2} \) undergo slow evolution on the time scale \( T \equiv e^{4t} \).

Using (20) and (17), we obtain from (4), as solvability conditions in the fourth and in the fifth orders, the following system of Landau equations for the amplitudes \( A_{1,2} \) and \( B_{1,2} \):

\[
\begin{aligned}
\dot{A}_1 &= \gamma_1 A_1 + \lambda_{11}^a |A_1|^2 A_1 + \lambda_{11}^b |B_1|^2 A_1 + \lambda_{12}^a |A_2|^2 A_1 + \lambda_{12}^b |B_2|^2 A_1, \\
\dot{B}_1 &= \gamma_1 B_1 + \lambda_{11}^a |B_1|^2 B_1 + \lambda_{11}^b |A_1|^2 B_1 + \lambda_{12}^a |A_2|^2 B_1 + \lambda_{12}^b |B_2|^2 B_1, \\
\dot{A}_2 &= \gamma_2 A_2 + \lambda_{22}^a |A_2|^2 A_2 + \lambda_{22}^b |B_2|^2 A_2 + \lambda_{21}^a |A_1|^2 A_2 + \lambda_{21}^b |B_1|^2 A_2, \\
\dot{B}_2 &= \gamma_2 B_2 + \lambda_{22}^a |B_2|^2 B_2 + \lambda_{22}^b |A_2|^2 B_2 + \lambda_{21}^a |A_1|^2 B_2 + \lambda_{21}^b |B_1|^2 B_2.
\end{aligned}
\]  

(21)

The linear growth rate coefficients \( \gamma_n \) and the Landau coefficients \( \lambda_{nm}^{a,b} \) \((n = 1, 2)\) have the following form:

\[
\gamma_n = q_2 \tilde{\alpha}(m, s) \kappa_n^2 - \tilde{\beta}(m, s, w) \kappa_n^4, \quad \lambda_{nm}^{a,b} = \tilde{\lambda}_{a,b}(m, s) \kappa_n^2, \quad n = 1, 2,
\]

(22)

where \( \Re(\tilde{\alpha}(m, s)) \geq 0 \), \( \Re(\tilde{\beta}(m, s)) \geq 0 \) if \( m_2 < m < m_4 \). The Landau coefficients \( \lambda_{np}^{a,b} \) \((n \neq p)\) are complicated functions of \( m, s, \kappa_1, \kappa_2 \) satisfying the relationships

\[
\begin{aligned}
\lambda_{12}^{a,b}(\kappa_1, \kappa_2) &= \lambda_{21}^{a,b}(\kappa_2, \kappa_1), \\
\lambda_{12}^{a,b}(\kappa_1, \kappa_1) &= \lambda_{21}^{a,b}(\kappa_1, \kappa_1) = \lambda_{11}^{a,b}, \\
\lambda_{22}^{a,b}(\kappa_1, \kappa_1) &= \lambda_{22}^{a,b}(\kappa_1, \kappa_1) = f(m, s) \lambda_{11}^{a,b}.
\end{aligned}
\]

The real parts \( \Re(\tilde{\lambda}_{a,b}(m, s)) \) of the functions \( \tilde{\lambda}_{a,b}(m, s) \) in Eq.(22) are shown in Fig.6 as functions of \( m \). They become unbounded in the vicinity of the point \( D \) \((m = s(1+\sqrt{1+4/s})/2)\) where system (21) is not valid.

System (21) allows to study the competition between travelling and standing waves. Let us consider two waves with a fixed wave-number propagating in opposite directions. Setting \( A_2 = B_2 = 0 \) in (21), we study the stability of the two solutions of system (21)

\[
\begin{aligned}
1) \quad A_1^2 &= \frac{-\gamma_1}{\lambda_{11}^a}, \quad B_1 = A_2 = B_2 = 0 \quad \text{(travelling wave)}; \\
2) \quad A_2^2 &= \frac{-\gamma_1}{\lambda_{11}^a + \lambda_{11}^b}, \quad A_2 = B_2 = 0 \quad \text{(standing wave)};
\end{aligned}
\]

(23)  (24)

with respect to infinitesimal perturbations of the amplitudes \( A_1 \) and \( B_1 \). The symbols of the Landau coefficients in (23), (24) and below denote now only the real parts of the respective coefficients.

The stability analysis shows that

\( a) \) travelling waves are stable if

\[
\lambda_{11}^a < 0, \quad \lambda_{11}^b < 0, \quad \lambda_{11}^a/\lambda_{11}^b < 1;
\]

(25)

\( b) \) standing waves are stable if

\[
\lambda_{11}^a < 0, \quad |\lambda_{11}^a/\lambda_{11}^b| > 1;
\]

(26)
otherwise, the system undergoes a subcritical instability which may lead to a blow-up and may return the system back to the stable region. We have performed some numerical simulations of (2) which support this expectation.

We have found (see Fig.6) that in our case \( w = 15, s = 3 \), when \( 2.135 < m < 3.361 \) standing waves are stable, and when \( 3.361 < m < 3.745 \) traveling waves are stable. In the rest intervals of \( m \) within the interval \( m_2 < m < m_4 \), close to the points C and D, both types of waves are subcritically unstable.

Let us now study the stability of standing and travelling waves with respect to perturbations with different wave-numbers, i.e. the stability of solutions (23), (24) with respect to perturbations of the amplitudes \( A_2 \) and \( B_2 \). The stability analysis shows that

a) **travelling waves** are stable if, besides (25), the following conditions are fulfilled:

\[
\gamma_2 - \gamma_1 \frac{\lambda^2}{\lambda_1^a} < 0, \quad \gamma_2 - \gamma_1 \frac{\lambda^b}{\lambda_1^a} < 0;
\]

b) **standing waves** are stable if, besides (26), the following condition holds:

\[
\gamma_2 - \gamma_1 \frac{\lambda^a}{\lambda_1^a} + \frac{\lambda^b}{\lambda_1^b} < 0;
\]

The analysis of conditions (27) and (28) shows that within the intervals defined by (25) and (26) there exist intervals of wave-numbers where the waves are also stable to perturbations with an arbitrary wave-number. Fig.7 shows how these intervals change with the variation of the parameter \( m \) along curve CD in Fig.1. It can be seen that standing waves are stable at intermediate wave-numbers, while travelling waves can be stable at infinitesimal wave-number as well.

Thus, the oscillatory long-wave Eckhaus instability in Marangoni convection with deformable interface can generate either standing or travelling compression-dilatation waves propagating along the periodic pattern and modulating its wavelength together with the deformation of the free surface. Typical results of numerical simulations of (2) are given in Fig.8a,b. It shows space-time diagrams for typical standing waves and traveling waves, respectively.

**Acknowledgement.**

This research was supported by the Israel Science Foundation. A.A.G. acknowledges the support of the Ministry for Immigrant Absorption. H.R. gratefully acknowledges a travel grant by the Minerva Center and support by DOE through DE-FG02-92ER14303.

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Figure 1. The dependence of the wavenumber $q$ corresponding to the onset of the various side-band instabilities on the parameter $m$ ($w = 15$, $s = 3$). Different sections of the solid line correspond to different types of the instability: AB – long-wave monotonic (Eckhaus); BC – short-wave monotonic; CD – long-wave oscillatory; DE – long-wave monotonic (Eckhaus). The dashed line presents the standard Eckhaus instability threshold $q^2 = 1/3$.

Figure 2. Regions in the $(m, s)$-parameter plane corresponding to different types of the side-band instability (at $w = 15$): ML – monotonic long-wave (Eckhaus), MS – monotonic short-wave, OL – oscillatory long-wave.

Figure 3. Two qualitatively different types of stable wave-number bands for the short-scale Marangoni convection near the instability threshold:

a) $m < m_3$: the stable region (hashed) is larger than in the standard case (dashed line), and its boundary (solid line) can correspond to either a monotonic, long- or short-wave instability, or to an oscillatory long-wave instability;

b) $m_3 < m < m_4$: the band center is unstable, and the stable region (hashed) is split up into two parts; its inner boundary corresponds to a monotonic long-wave instability and the outer one to a long-wave oscillatory instability. The dotted line is the neutral stability curve.

Figure 4. Typical space-time diagrams for the evolution of the long-wave monotonic instability along line AC (see Fig.1) as obtained by numerical simulation of (2): the surface deformation a) and the amplitude of the short-scale convection b) remain regular while the wavenumber passes through a singularity indicating the phase slip.

Figure 5. Typical space-time diagrams for the evolution of the long-wave monotonic instability along line DE (see Fig.1) as obtained by numerical simulation of (2). Here the surface deformation a) and the amplitude of the short-scale convection b) become singular indicating dry-out of the fluid film.

Figure 6. The real parts of the interaction coefficients $\Re(\tilde{\lambda}_a(m))$ (curve a) and $\Re(\tilde{\lambda}_b(m))$ (curve b) from Eq.(22) at $s = 3$, $w = 15$, as functions of $m$. The vertical asymptote corresponds to $m = m_4(s)$ (point D in Fig.1).

Figure 7. Regions of stability of standing and travelling waves with a wavenumber $\kappa_1$ (lying within the range of linearly excited modes) with respect to perturbations with a wavenumber $\kappaappa_2$ (lying within the same region), at different values of the parameter $m$: s – stable, u – unstable.
Figure 8. a) Typical space-time diagram of a standing wave arising from the long-wave oscillatory side-band instability of the periodic short-scale convection.
b) Typical space-time diagram of a traveling wave arising from the long-wave oscillatory side-band instability of the periodic short-scale convection.
Both diagrams were obtained by numerical simulation of (2).
no stable region

ML

OL,ML

OL

m1

m2

m3

m4

MS

ML
