Quantum Entanglement and Entropy

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Abstract

Entanglement is the fundamental quantum property behind the now popular field of quantum transport of information. This quantum property is incompatible with the separation of a single system into two uncorrelated subsystems. Consequently, it does not require the use of an additive form of entropy. We discuss the problem of the choice of the most convenient entropy indicator, focusing our attention on a system of 2 qubits, and on a special set, denoted by $\mathcal{I}$. This set contains both the maximally and the partially entangled states that are described by density matrices diagonal in the Bell basis set. We select this set for the main purpose of making more straightforward our work of analysis. As a matter of fact, we find that in general the conventional von Neumann entropy is not a monotonic function of the entanglement strength. This means that the von Neumann entropy is not a reliable indicator of the departure from the condition of maximum entanglement. We study the behavior of a form of non-additive entropy, made popular by the 1988 work by Tsallis. We show that in the set $\mathcal{I}$, implying the key condition of non-vanishing entanglement, this non-additive entropy indicator turns out
to be a strictly monotonic function of the strength of the entanglement, if entropy indexes $q$ larger than a critical value $Q$ are adopted. We argue that this might be a consequence of the non-additive nature of the Tsallis entropy, implying that the world is quantum and that uncorrelated subsystems do not exist.

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I. INTRODUCTION

The entanglement is the fundamental quantum property behind the interesting process of quantum teleportation proposed some years ago by Bennett et al. [1]. For this reason it is important to quantify entanglement [2]. It has also been found that the fidelity of the quantum teleportation is always larger than that of any classical communication protocol even in the noisy environment [3]. However, for the teleportation to take place perfectly, it is necessary for them to share a maximally entangled state. This generates the need for special purifications protocols. On the other hand, the statistical analysis of these protocols shares deep similarities with the second principle of thermodynamics [4–6].

In spite of the plausible conjecture that there exists a deep connection between quantum teleportation and thermodynamics [4–6], the entanglement is expressed by means of an entropic structure, the conventional von Neumann entropy, only in the case of pure states. In general the definitions of entanglement dictated by the purification protocol are not directly related to entropy indicators. Cerf and Adami [7] show that the conditional von Neumann entropy can become negative, thereby pointing out the non-ordinary information aspects of quantum entanglement. However, the reason for the lack of a direct connection between quantum entanglement of mixed states and entropy indicator is probably that, as shown in this paper, the von Neumann entropy is not a reliable indicator of entanglement [7]. The inadequacy of the Shannon information, and consequently of the von Neumann entropy as an appropriate quantum generalization of Shannon entropy [8], has already been pointed out by Brukner and Zeilinger [9,10]. It is notable that according to these authors a new kind of entropy indicator is made necessary by the following deep difference between quantum and classical information. In classical measurement the Shannon information is a natural measure of our ignorance about property of a system, whose existence is independent of measurement. Quantum measurement, on the contrary, cannot be claimed to reveal system’s properties existing before the measurement is done.

In this paper we focus on the earlier mentioned quantum property, essential for teleport-
tation of information: entanglement. Entanglement implies that a system cannot be divided into two uncorrelated subsystems, and this, in turn, makes useless the ordinary request for an additive form of entropy. Thus, in this paper we explore and discuss the possible benefits stemming from the adoption of the non-additive entropy indicator advocated years ago by Tsallis [11]. The Tsallis entropy is applied to a large number of physical conditions, characterized by the existence of extended correlation [12]. The use of this form of non-additive indicator in the field of quantum teleportation, to the best of our knowledge, has been discussed in only a few papers [13,14]. The paper of Ref. [13] claims to the realization of a greater sensitivity to the occurrence of a dephasing process resulting in the annihilation of any form of entanglement. The papers of Refs. [14,16] point out the efficiency of Tsallis entropy for the detection of the breakdown of local realism. The paper of Ref. [15] aims at proving that the Jaynes principle [17,18], applied to the non-extensive entropy, yields naturally the entangled state. The present paper, although based on the adoption of the Tsallis non-extensive entropy as the paper of Ref. [15], adopts a quite different perspective, which does not rest on the adoption of the Jaynes principle.

The outline of the paper is as follows. Section II is devoted to a concise illustration of the main properties of the non-extensive entropy at work in this paper. Section III, devoted to the entanglement of formation, shows that the von Neumann entropy, in general, is not a monotonic function of entanglement, while the non-extensive entropy is a monotonically increasing function of the entanglement for suitably large values of the entropy index. In Section IV we study the entanglement of a de-phasing process, and we extend the monotonic properties of non-extensive entropy to a condition more general than that of Section III. Section IV is devoted to concluding remarks.

II. THE NON-EXTENSIVE ENTROPY

The entropy indicator applied in this paper has the form

$$S_q(\rho) \equiv Tr \frac{\rho - \rho^q}{q - 1}. \quad (1)$$
This form has been originally proposed by Tsallis \[11\] for the purpose of establishing the most convenient thermodynamic perspective for fractal processes. It is worth remarking that this is a generalization of the conventional Gibbs-Shannon entropy indicator, whose explicit form is recovered from Eq. (1) in the limiting case \( q \to 1 \). This entropy indicator does not fit the additivity condition, namely, the requirement that the entropy of a system \( A + B \), consisting of two statistically independent subsystems, \( A \) and \( B \), be the sum of the entropies of these two subsystems. In fact, the definition of Eq. (1) yields, in this case, the following equality

\[
S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B),
\]

making it evident that the additive property is recovered only in the case \( q = 1 \), which, as earlier noted, makes the non-extensive entropy of Eq. (1) become equivalent to the usual Shannon entropy.

According to Tsallis and to the advocates of this non-extensive entropy indicator (for a review, see Ref. \[12\]), the violation of the additive condition turns into a benefit when this entropy indicator is used to the study cases where the ideal condition of statistical independence is prevented by the nature itself of the processes under study \[12\]. Notable examples are the processes with the long-range correlation \[12\]. We think that quantum entanglement, which is the basic property for teleportation \[1\], is probably the most evident example of a condition incompatible with the existence of uncorrelated subsystems. For this reason, the additivity condition can be safely renounced, and the adoption of an entropy index \( q \neq 1 \) might turn out to be beneficial. This paper is devoted to discussing to what an extent this conjecture proves to be correct.

\[III. \] THE TSALLIS ENTROPY AT WORK: ENTANGLEMENT OF FORMATION

In this section we derive the central result of this paper. We show that in the case of initial and final states, both with non-vanishing entanglement, and described by a statistical
density matrix diagonal in the Bell basis, the Tsallis entropy decreases upon increase of
the entanglement of formation. The results of this section refer to the entanglement of
formation [19] of a system of two qubits. Thus, to make this paper as self-contained as
possible, we give here a concise illustration of this key measure of entanglement. The main
point is that the entanglement property is defined without any ambiguity for pure states
[1]. The entanglement of formation extends to the statistical case the ordinary definition as
follows. Let us denote with $\rho$ the statistical density matrix for the mixed state of a space
$S_1^{(1)} \times S_1^{(2)}$ of two spin-1/2 particles. The entanglement of formation [20], denoted by the
symbol $E_F$ throughout, is defined as the minimum average entanglement of every ensemble
of pure states that represents $\rho$:

$$E_F(\rho) = \min_{\rho=\sum_i P_i |\alpha_i\rangle \langle \alpha_i|} \sum_i P_i E(|\alpha_i\rangle),$$  \hspace{2cm} (3)

where $E(|\alpha_i\rangle)$ denotes in fact the entanglement of a pure state, which is defined according
to the usual prescription [1] by the expression

$$E(|\alpha\rangle) = -Tr(\rho_A \log_2 \rho_A) = -Tr(\rho_B \log_2 \rho_B)$$  \hspace{2cm} (4)

with

$$\rho_A \equiv Tr_2 \rho, \quad \rho_B \equiv Tr_1 \rho.$$  \hspace{2cm} (5)

Wootters, in an enlightening article [19], derives an explicit formula for the entanglement
of formation of any arbitrary mixed state of a system of 2 qubits. This formula reads:

$$E_F(\rho) = h \left( \frac{1 + \sqrt{1 - C^2(\rho)}}{2} \right),$$  \hspace{2cm} (6)

where

$$h(x) \equiv -x \log_2 x - (1 - x) \log_2 (1 - x).$$  \hspace{2cm} (7)

The quantity $C(\rho)$, referred to by Wootters as concurrence, is defined by

$$C(\rho) \equiv \max \left\{ 0, \lambda_m - \lambda_1 - \lambda_2 - \lambda_3 \right\}.$$  \hspace{2cm} (8)
where $\lambda_m, \lambda_1, \lambda_2$ and $\lambda_3$ are the square roots of eigenvalues of the matrix $\rho \cdot \tilde{\rho}$, set in a decreasing order, with $\lambda_m$ being the maximum eigenvalue. The matrix $\tilde{\rho}$ denotes the spin-flipped state:

$$\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y).$$

The values of the entanglement of formation range from 0 to 1. Furthermore, $E_F$ is a monotonic increasing function of $C$. The values of the concurrence $C$, in turn, range from 0 to 1, as the values of $E_F$ do. Consequently, the concurrence itself can be considered as a proper measure of the entanglement of formation. Note that the entanglement of formation is the only kind of entanglement studied in the present paper. For simplicity, we shall often refer to it simply as entanglement.

**A. On $\mathcal{I}$, the working basis set for a system of 2 qubits**

The most general expression [21] of a mixed state $\rho$ of the space $S_{1/2}^{(1)} \otimes S_{1/2}^{(2)}$ is:

$$\rho = \frac{1}{16} \left\{ 1 \otimes 1 + \left( \sum_{i=1}^{3} r_i \sigma_i \right) \otimes 1 + 1 \otimes \left( \sum_{i=1}^{3} s_i \sigma_i \right) + \sum_{i=1}^{3} \sum_{j=1}^{3} t_{ij} \sigma_i \otimes \sigma_j \right\},$$

where $r_i, s_i$ and $t_{ij}$ are real parameters. Due to the excessive number of involved parameters, the direct use of this expression would make the calculation too complicated. For this reason we decided to limit our investigation to the set defined by:

$$\mathcal{I} \equiv \{ \rho : C(\rho) = 2P_m - 1 > 0 \},$$

where $P_m$ denotes the greatest eigenvalue of the density matrix $\rho$ describing the quantum state. On behalf of future developments, we remark that the condition $P_m > 1/2$ makes this maximum value unique. The adoption of the set $\mathcal{I}$, as our working set, does not rule out the possibility of considering physical conditions of interest for the field of quantum teleportation. In fact, the set $\mathcal{I}$ contains the states with positive entanglement of formation ($E_F(\rho) > 0$) that are described by density matrices corresponding to the solution of the equation.
\[
\rho \cdot \tilde{\rho} = \rho^2. \tag{12}
\]
These density matrices, recently used by Bennett and co-workers [20] (see also [22]), become diagonal when expressed in the Bell basis set. The set \( \mathcal{I} \) contains the entangled Werner states [23] as well as the maximally entangled states. We also note that any mixed state can be brought into a diagonal form in the Bell basis set by random bilateral rotations [20]. This implies that, in spite of the simplification made, our discussion is still rather general.

The problem here under discussion is the significance of the Tsallis entropy as a measure of entanglement. The working set of states that we select, \( \mathcal{I} \), does not conflict with the possibility of discussing this issue with reference to one of the relevant physical conditions that have been recently examined by the investigators of this field of research [20,22]. In fact, we shall study the change of the Tsallis entropy indicator upon the entanglement change of a given pair of particles. This change might be the result of a purification process such as that studied by Bennett and co-workers [20,22]. These authors have discussed the purification of a couple of Werner states, showing how to increase the entanglement of formation of a pair at the price of decreasing that of the other. Here, we discuss the problem of detecting this increased entanglement with the Tsallis entropy. We found to be relatively easy to establish this important result within the set \( \mathcal{I} \). However, we cannot rule out the possibility that this property is shared by any other set of states resulting in the concurrence being an increasing function of \( P_m \). In all these cases we might find the same property of the entropy being proportional to the inverse of the entanglement.

It is worth to anticipate an aspect of fundamental importance. The plausible reason why, as we shall see, the inverse of the Tsallis entropy indicator is a successful measure of entanglement, in the set \( \mathcal{I} \), is the following fundamental fact. The non-extensive nature of Tsallis entropy would make sense only in a world where it were impossible to create uncorrelated systems, and consequently, a vanishing entanglement. In fact, the Tsallis entropy is not the sum of the entropies of the parts, not even when these parts are uncorrelated. The condition that we set, of non vanishing entanglement, as earlier said, is equivalent to ruling
out the occurrence of the splitting of the system into two uncorrelated parts. Therefore, it corresponds to an ideal condition for the application of the Tsallis entropy. We shall come back to this important issue in Section IV.

B. Parametrization of the eigenvalues of the statistical density matrix

Here we adopt a perspective of thermodynamic kind, inspired to the lines proposed by Plenio and Vedral [4]. We assume that a purification protocol yields an entanglement change, $\Delta E_F$. We aim at establishing the change of the non-extensive entropy corresponding to the same "thermodynamic" transformation. To solve this delicate problem we imagine the statistical density matrix $\rho$, belonging to the set $\mathcal{I}$, to be a function of a real parameter $\xi$, belonging to an interval $[\xi_1, \xi_2]$, which can be thought of as playing the same role as that of variables like pressure, temperature and volume in the state transformations in ordinary thermodynamics. We assume that the initial and final conditions correspond to $\rho (\xi_1)$ and $\rho (\xi_2)$, respectively, so that $\Delta E_F \equiv E_F (\rho (\xi_2)) - E_F (\rho (\xi_1))$. We set the dependence of $\rho$ on $\xi$ in such a way as to fulfill the constraint imposed by the norm conservation $Tr \rho = 1$, and among the infinitely large number of possibilities fitting these conditions, we select the form most convenient for the purpose of evaluating $\Delta S_q \equiv S_q (\rho (\xi_2)) - S_q (\rho (\xi_1))$. Let $P_m$ be the largest of the four eigenvalues of the statistical density matrix $\rho$ and let us denote with $P_1, P_2$ and $P_3$ the other three eigenvalues. These eigenvalues are assumed to be functions of the parameter $\xi$ with the following condition: in the whole interval $[\xi_1, \xi_2]$ we have

$$P_m (\xi) > \frac{1}{2}. \quad (13)$$

The derivatives $dP_m/d\xi$ and $dP_j/d\xi$ with $j$ ranging from 1 to 3, are assumed to be finite. The norm conservation constraint enforces the condition

$$\frac{dP_m}{d\xi} = -\frac{dP_1}{d\xi} - \frac{dP_2}{d\xi} - \frac{dP_3}{d\xi}. \quad (14)$$

As earlier pointed out, the concurrence is equivalent to the entanglement of formation and, in the set $\mathcal{I}$, the concurrence is a strictly increasing monotonic function of $P_m$. Thus,
the change of $P_m$ will be either positive or negative according to whether the entanglement change is positive or negative.

We shall assume that in the whole interval $[\xi_1, \xi_2]$ the derivative $dP_m/d\xi$ is either always positive or always negative. As earlier stressed, this convenient choice is legitimated by the fact that, as we do in ordinary thermodynamics, we are considering a state transformation, and consequently both the entanglement and entropy changes depend only on the initial and final states, and are independent of the paths used to connect these two states.

C. Failure of the von Neumann entropy

Here we show that the von Neumann entropy, namely, the quantum version of the Shannon information, turns out to be inadequate as an entanglement indicator. It is so because, as we hereby see, the von Neumann entropy can either increase or decrease, in correspondence of the entanglement change, $\Delta E_F$, regardless of whether the entanglement change is positive or negative. This conclusion agrees with the remarks of the recent work by Brukner and Zeilinger [9].

We note that to prove the inadequacy of the von Neumann’s entropy as an indicator of entanglement it is enough to find a case where the sign of the entropy change is not strictly determined by that of the entanglement change. Thus, let us consider the special physical condition corresponding to:

$$\frac{dP_3}{d\xi} = 0 \quad (15)$$

in the whole interval $[\xi_1, \xi_2]$. The von Neumann’s entropy, corresponding to $q = 1$, and consequently denoted as $S_1$, reads

$$S_1 (\rho) = -P_m \ln P_m - \sum_{j=1}^{3} P_j \ln P_j \quad (16)$$

and its dependence upon the parameter $\xi$ is given by:

$$\frac{dS_1}{d\xi} = -\frac{dP_m}{d\xi} \ln \left( \frac{P_m}{P_1} \right) + \frac{dP_2}{d\xi} \ln \left( \frac{P_2}{P_1} \right) \quad (17)$$
Let us consider, for example, the case $dP_m/d\xi < 0$. As earlier noted, in this case the entanglement of the mixed state is a strictly decreasing function of $\xi$, and the entanglement change $\Delta E_F$ must be negative. However, using Eq. (15) and Eq. (14) we see that the sign of $dS_1/d\xi$ depends on the special values selected for the parameters $P_m, P_1, P_2$ and for the corresponding $\xi$-derivatives. Thus, it is possible to realize either the inequality

$$\frac{dP_m}{d\xi} < \frac{dP_2}{d\xi} \ln \left( \frac{P_1}{P_2} \right) / \ln \left( \frac{P_m}{P_1} \right),$$

which would lead to an entropy increase, or the opposite inequality, which would yield an entropy decrease. This proves that the sign of the von Neumann’s entropy change is not correlated to that of the entanglement, and that consequently the von Neumann’s entropy is not an adequate entanglement indicator.

**D. Search for a critical entropy index**

The theoretical developments of this subsection show that the failure of the von Neumann’s entropy is the consequence of the fact that the von Neumann’s entropy means $q = 1$ and this entropy index is smaller than, or equal to, a critical value $Q(\rho_1, \rho_2)$, where $\rho_1$ and $\rho_2$ denote the initial and the final state, respectively. Hereby we show, in fact, that for $q > Q(\rho_1, \rho_2)$ the non-extensive entropy becomes a monotonic function of the entanglement strength, inversely proportional to it.

For this purpose, let us study the $\xi$-derivative of the non-extensive entropy. This quantity reads:

$$\frac{dS_q}{d\xi} = -\frac{qP_m^{-1} dP_m}{q - 1} \left\{ 1 + \sum_{j=1}^{3} \left( \frac{dP_m}{d\xi} \right)^{-1} \left( \frac{P_j}{P_m} \right)^{q-1} \frac{dP_j}{d\xi} \right\}.$$  

We remind the reader that $P_m$ is the largest eigenvalue of $\rho$, thereby implying the property

$$0 \leq \frac{P_j}{P_m} < 1.$$  

By using these inequalities and the assumption made in Section III B that the derivatives $dP_j/d\xi$ and $dP_m/d\xi$ are finite, we obtain
This simple result implies that \textit{great enough} values of the entropy index yield the important property

\[
\text{sign} \left( \frac{dE_F}{d\xi} \right) = \text{sign} \left( \frac{dP_m}{d\xi} \right) = -\text{sign} \left( \frac{dS_q}{d\xi} \right),
\]

which, in turn, means that increasing entanglement yields a decreasing entropy, and vice-versa.

At this stage of our path towards the detection of \( Q(\rho_1, \rho_2) \) we explore two distinct cases concerning the behavior of \( E_F(\xi) \) in the whole interval \([\xi_1, \xi_2]\): (a) \( dE_F/d\xi < 0 \); (b) \( dE_F/d\xi > 0 \). As earlier mentioned, the arbitrary choice of a \( \xi \)-derivative, either always positive or always negative, is legitimate since the entanglement and the entropy changes are independent of the path connecting the initial to the final state. In cases (a) and (b) we shall identify the important auxiliary functions \( q^*(\xi) \) and \( q^{**}(\xi) \), respectively. These auxiliary functions will be used first to build up the functions \( Q^*(\xi_1, \xi_2) \) and \( Q^{**}(\xi_1, \xi_2) \), and finally the key quantity \( Q(\rho_1, \rho_2) \).

Let us consider case (a) first. As a consequence of \( dE_F/d\xi < 0 \), from the limit of Eq. (21) we naturally obtain that a critical value \( q^*(\xi) \) exists such that \( q > q^*(\xi) \) yields

\[
1 + \sum_{j=1}^{3} \left( \frac{dP_m}{d\xi} \right)^{-1} \left( \frac{P_j}{P_m} \right)^{q^{-1}} \left( \frac{dP_j}{d\xi} \right) > 0.
\]

The critical value \( q^*(\xi) \) fulfilling the condition of Eq. (23) is not unique. We therefore adopt a criterion to estimate one of the possible critical values. This will imply that the resulting \( Q(\rho_1, \rho_2) \) is not unique either, but, as shown later, we shall be able to find at least one of them fulfilling the earlier mentioned properties of \( Q(\rho_1, \rho_2) \). The choice that we adopt to find one of the possible \( q^*(\xi) \)’s is as follows. We set the inequality

\[
\left| \frac{dP_m}{d\xi} \right|^{-1} \left( \frac{P_j}{P_m} \right)^{q^{-1}} \frac{dP_j}{d\xi} < \frac{1}{3},
\]

with the subscript \( j \) running from 1 to 3. We assume that this property holds true for any \( q > q^*(\xi) \). This set of conditions, after an easy algebra, yields...
\[
q^* (\xi) = \max \{ 1, \alpha_1^* (\xi), \alpha_2^* (\xi), \alpha_3^* (\xi) \} .
\] (25)

As to the definition of \( \alpha_j^* (\xi) \), with the subscript \( j \) running from 1 to 3, we must distinguish two cases. The first is the case when the constraints

\[
P_j (\xi) > 0 \text{ and } dP_j / d\xi > 0
\] (26)

hold true. In this case, we set

\[
\alpha_j^* \equiv 1 + \left( \ln \left( \frac{P_m}{P_j} \right) \right)^{-1} \ln \left( 3 \left| \frac{dP_m}{d\xi} \right|^{-1} \frac{dP_j}{d\xi} \right).
\] (27)

If the constraints of Eq. (26) do not apply, we set \( \alpha_j^* (\xi) = 1 \). This implies that for any \( q > q^* (\xi) \) the condition

\[
\frac{d}{d\xi} S_q (\xi) > 0
\] (28)

yields

\[
\frac{d}{d\xi} S_q (\xi) > 0.
\] (29)

On the basis of this result we define now the function \( Q^* (\xi_1, \xi_2) \) as follows:

\[
Q^* (\xi_1, \xi_2) \equiv \sup_{\xi \in [\xi_1, \xi_2]} \{ q^* (\xi) \} .
\] (30)

Using Eq. (28) and the ensuing inequality for \( dS_q / d\xi \) we conclude immediately that for any \( q \) fulfilling the inequality \( q > Q^* (\xi_1, \xi_2) \) the condition \( \Delta E_F < 0 \) yields \( \Delta S_q > 0 \). In fact, in this case \( \Delta E_F \) and \( \Delta S_q \) can be written under the form of integrals in the interval \([\xi_1, \xi_2]\) with integrands always negative and positive, respectively.

In case (b) we adopt the same procedure which yields, in this case,

\[
q^{**} (\xi) \equiv \max \{ 1, \alpha_1^{**} (\xi), \alpha_2^{**} (\xi), \alpha_3^{**} (\xi) \} .
\] (31)

As to the term \( \alpha_j^{**} (\xi) \), in the case where the constraints

\[
P_j (\xi) > 0 \text{ and } dP_j / d\xi < 0
\] (32)

hold true.
hold true, we set
\[
\alpha_j^{**}(\xi) \equiv 1 + \left( \ln \left( \frac{P_m}{P_j} \right) \right)^{-1} \ln \left( 3 \left( \frac{dP_m}{d\xi} \right)^{-1} \left| \frac{dP_j}{d\xi} \right| \right).
\]
If the constraints of the Eq. (32) do not apply we set \( \alpha_j^{**}(\xi) = 1 \). The counterpart of Eq. (30) becomes
\[
Q^{**}(\xi_1, \xi_2) \equiv \sup_{\xi \in [\xi_1, \xi_2]} \{ q^{**}(\xi) \}.
\]
Of course, the condition \( \Delta E_F > 0 \) yields \( \Delta S_q < 0 \).

Note that we have not discussed the problem of the possible divergence of \( Q^*(\xi_1, \xi_2) \) or \( Q^{**}(\xi_1, \xi_2) \). We shall come back to this issue in the following section where we will consider a special parametrization of the eigenvalues, without losing any generality, within which, as we shall see, this critical index will be proved to be finite.

E. Searching a critical value of the entropy index, beyond which the Tsallis entropy exhibits a correct dependence on the entanglement strength

Now let us see how to use the earlier results to make predictions in the case where the transformation, and the ensuing entanglement change as well, is defined only by the initial and the final states, \( \rho_1 \) and \( \rho_2 \), with the density matrix belonging to the set \( \mathcal{S} \) defined by Eq. (11). The main idea is to build up auxiliary states, \( \rho_B^{(1)} \) and \( \rho_B^{(2)} \), equivalent to \( \rho_1 \) and \( \rho_2 \) as far as their entanglement and entropy are concerned, but fulfilling the condition of being connected the one to the other by the \( \xi \)-transformation of Section III C. This makes these states compatible with the earlier prescriptions, and thus with the earlier results. These two states are defined as follows:
\[
\rho_B^{(1)} \equiv P_m^{(1)}|e_m\rangle\langle e_m| + \sum_{j=1}^{3} P_j^{(1)}|e_j\rangle\langle e_j|
\]
and
\[
\rho_B^{(2)} \equiv P_m^{(2)}|e_m\rangle\langle e_m| + \sum_{j=1}^{3} P_j^{(2)}|e_j\rangle\langle e_j|,
\]
where the set \{\ket{e_m}, \ket{e_j}, j = 1, 2, 3\} is the Bell basis set [20], no matter what the order is. The symbols \(P_j^{(1)}\) and \(P_m^{(1)}\) denote the eigenvalues of the density matrix \(\rho_B^{(1)}\), and, obviously, the symbols \(P_j^{(2)}\) and \(P_m^{(2)}\) denote the eigenvalues of the density matrix \(\rho_B^{(2)}\). These quantum states have the following properties: (i) They belong to the set \(\mathcal{S}\); (ii) \(E_F(\rho_1) = E_F\left(\rho_B^{(1)}\right)\), \(E_F(\rho_2) = E_F\left(\rho_B^{(2)}\right)\); (iii) \(S_q(\rho_1) = S_q\left(\rho_B^{(1)}\right), S_q(\rho_2) = S_q\left(\rho_B^{(2)}\right)\).

Now let us introduce the transformation \(\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right]\) defined by:

\[
\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right] \left(\rho_B^{(1)}\right) = \rho_B^{(1)}, \quad \Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right] \left(\rho_B^{(2)}\right) = \rho_B^{(2)}.
\]

The transformation \(\Xi_\xi\) has the required properties: (a) It keeps the state \(\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right] \left(\rho_B^{(1)}\right)\) within the set \(\mathcal{S}\) for every value of \(\xi\) belonging to the interval [0, 1]; (b) \(\Xi_0\left[\rho_B^{(1)}, \rho_B^{(2)}\right] \left(\rho_B^{(1)}\right) = \rho_B^{(1)}\); (c) \(\Xi_1\left[\rho_B^{(1)}, \rho_B^{(2)}\right] \left(\rho_B^{(1)}\right) = \rho_B^{(2)}\). Note that the functions \(P_m(\xi), P_1(\xi), P_2(\xi)\) and \(P_3(\xi)\), are defined in the interval \([\xi_1, \xi_2]\).

They fulfill the properties of Eq. (13), the parameter conditions of Section IIB, the relation \(dP_m/d\xi > 0\), in the case \(P_m^{(1)} < P_m^{(2)}\), and the relation \(dP_m/d\xi < 0\), in the case \(P_m^{(1)} > P_m^{(2)}\). This makes it possible for us to use \(Q^*(\xi_1, \xi_2)\) of Eq. (30) and \(Q^{**}(\xi_1, \xi_2)\) of Eq. (31), and the relations on which these quantities rest as well, to derive \(Q(\rho_1, \rho_2)\). This is done as follows. We write the explicit forms that \(Q^*(\xi_1, \xi_2)\) and \(Q^{**}(\xi_1, \xi_2)\) get when \(\xi_1 = 0\) and \(\xi_2 = 1\). Using the transformation of Eq. (38) and Eq. (39), we obtain for \(Q^*(0, 1)\) the following expression:

\[
Q^*(0, 1) \equiv \sup_{\xi \in [0, 1]} \max \{1, \beta_1^*(\xi), \beta_2^*(\xi), \beta_3^*(\xi)\}. \tag{40}
\]

If the constraints
\[ P_j^{(2)}(\xi) > P_j^{(1)}(\xi) \quad \text{and} \quad P_j^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right) > 0 \quad (41) \]

hold true, we set
\[
\beta_j^*(\xi) \equiv 1 + \ln \left( \frac{3 \frac{P_m^{(1)} - P_j^{(2)}}{P_m^{(1)} - P_m^{(2)}}}{\frac{P_m^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right)}{P_j^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right)}} \right). \quad (42)
\]

If the constraints of Eq. (45) do not apply, we set \( \beta_j^*(\xi) = 1 \).

Note that this mathematical definition must be interpreted as follows. First of all, we consider a given value of \( \xi \) belonging to the interval \([0, 1]\). Then we make the index \( j \) run from 1 to 3. We select the indexes \( j \) fulfilling the conditions \( P_j^{(2)}(\xi) > P_j^{(1)}(\xi) \) and \( P_j^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right) > 0 \) and calculate \( \beta_j^*(\xi) \) using the above definitions. Then we take the maximum of the values of a set whose components are given by \( \beta_j^*(\xi) \) and by 1. Then, we make \( \xi \) explore all the possible values of the interval \([0, 1]\). Thus, we get an infinite set of maxima, from which we select the supremum. The resulting number defines the critical index of the left hand side of Eq. (40).

The resulting critical index is finite. To prove this important property we proceed as follows. We note that the term that could make \( Q^{**}(0, 1) \) diverge is \( \frac{P_m^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right)}{P_j^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right)} \). We denote this term by \( \gamma(\xi) \). The special condition resulting in the divergence of the critical index would be given by \( \gamma \rightarrow 1^+ \). We observe that \( \gamma(\xi) \) is either an increasing (decreasing) or a constant function of \( \xi \) depending on whether the quantity \( P_j^{(1)} P_m^{(2)} - P_m^{(1)} P_j^{(2)} \) is positive (negative) or equal to 0. So the minimum value of \( \gamma(\xi) \) is \( \gamma(0) = \frac{P_m^{(1)}}{P_j^{(1)}} \), in the case of \( d\gamma/d\xi > 0 \), and \( \gamma(1) = \frac{P_m^{(2)}}{P_j^{(2)}} \), in the case of \( d\gamma/d\xi < 0 \). In the remaining case \( d\gamma/d\xi = 0 \), the two minima get the same value. From this properties we obtain the following inequality
\[
Q^*(0, 1) \leq \max_{j=1,2,3} \left\{ 1 + \ln \left( \min_{j=1,2,3} \left\{ \frac{P_m^{(1)}}{P_j^{(1)}}, \frac{P_m^{(2)}}{P_j^{(2)}} \right\} \right)^{-1} \ln \left( 3 \left| \frac{P_j^{(2)} - P_j^{(1)}}{P_m^{(1)} - P_m^{(2)}} \right| \right) \right\}, \quad (43)
\]
proving that \( Q^*(0, 1) \) is finite.

As to \( Q^{**}(0, 1) \), we get
\[ Q^{**}(0,1) \equiv \sup_{\xi \in [0,1]} \max \{1, \beta^*_1(\xi), \beta^*_2(\xi), \beta^*_3(\xi)\}. \quad (44) \]

If the constraints
\[ P_j^{(1)}(\xi) > P_j^{(2)}(\xi) \quad \text{and} \quad P_j^{(1)} + \xi \left( P_j^{(2)} - P_j^{(1)} \right) > 0 \quad (45) \]
hold true, we set
\[ \beta^*_j(\xi) \equiv 1 + \frac{\ln \left( \frac{3 P_j^{(1)} - P_j^{(2)}}{P_m^{(1)} - P_m^{(2)}} \right)}{\ln \left( \frac{P_j^{(1)} + \xi (P_j^{(2)} - P_j^{(1)})}{P_j^{(1)} + \xi (P_j^{(2)} - P_j^{(1)})} \right)}. \quad (46) \]

If the constraints of Eq. (45) do not apply, we set \( \beta^*_j(\xi) = 1 \).

The criterion adopted to define this critical index is the same as that earlier illustrated to properly define the critical index of Eq. (40). Thus, we can prove that \( Q^{**}(0,1) \) is finite adopting a procedure analogous to that used for \( Q^{*}(0,1) \). In this case we arrive at the following inequality:
\[ Q^{**}(0,1) \leq \max_{j=1,2,3} \left\{ 1 + \left( \min_{j=1,2,3} \left( \frac{P_j^{(1)}}{P_j^{(2)}}, \frac{P_j^{(2)}}{P_j^{(1)}} \right) \right)^{-1} \ln \left( 3 \left| \frac{P_j^{(1)} - P_j^{(2)}}{P_m^{(1)} - P_m^{(2)}} \right| \right) \right\}, \quad (47) \]
which shows in fact that also \( Q^{**}(0,1) \) is finite.

At this stage we can finally define the critical value \( Q(\rho_1, \rho_2) \). This is given by
\[ Q(\rho_1, \rho_2) \equiv \max \{Q^*(0,1), Q^{**}(0,1)\}. \quad (48) \]

On the basis of the earlier described theoretical treatment we can conclude that \( Q(\rho_1, \rho_2) \) is finite and that for any initial and final states, \( \rho_1 \) and \( \rho_2 \), respectively, belonging to the set \( \mathcal{S} \), with different entanglement, \( E_F(\rho_1) \neq E_F(\rho_2) \), the corresponding entropy change \( \Delta S_q \) is positive or negative, according to whether \( \Delta E < 0 \) or \( \Delta E > 0 \). Note that we have found \(+\infty > Q(\rho_1, \rho_2) \geq 1\). As a consequence of the pseudoadditivity of Eq. (2), the adoption of a value of the entropy index larger than the unity makes the entropy of the whole system smaller than the sum of the entropies of the two parts. However, in this paper we never make a direct use of this property, since, as earlier stressed, our treatment is valid only in the case of non vanishing entanglement, which rules out the possibility of realizing the factorized condition behind Eq. (2).
F. From the non-extensive entropy to the entanglement of formation

As a purpose of this subsection, we try to prove a property that is the reverse of that discussed in Section III E. Ideally, the reverse of the property of Section III E should be expressed as follows. Let us focus our attention on a transformation from an initial state $\rho_1$ to a final state $\rho_2$, both belonging to the set $\mathcal{I}$. Let us consider a case where this transformation makes the non-extensive entropy $S_q$ increase (decrease). Then, the entanglement decreases (increases) if an entropy index $q$ larger than the critical value $Q$ is adopted. Unfortunately, we cannot prove this property in this attractive form, but only under weaker conditions. This is so because a transformation resulting in an entropy change does not necessarily imply an entanglement change. We notice that the entanglement, expressed in the set $\mathcal{I}$, is a function of the eigenvalue $P_m$ only, while the non-extensive entropy is a function of all four eigenvalues. Thus, the entropy can change without implying a corresponding entanglement change. The same difficulty is shared by the non-extensive entropy. However, upon increase of the entropy index $q$ the dependence of the non-extensive entropy on the other three eigenvalues becomes weaker and weaker. In the case of enough great values of the entropy index $q$ the non-extensive entropy becomes virtually independent of the other three eigenvalues. This is the reason why in Section III we could find a way to make the non-extensive entropy become a monotonic function of entanglement. We want to remark that in general the entropy critical index is not the same as that used in Section III.

We think that one of the benefits resulting from the adoption of the set $\mathcal{I}$, and of very large entropy indices as well, is that the margin of entanglement dependence on entropy is significantly reduced. Nevertheless, we are forced to make a weaker request for the reverse of the property discussed in Section E. We shall show, in fact, that if the entropy increases (decreases), and the entanglement changes, then the entanglement decreases (increases), for entropy indices $q$ larger than a critical value $Q^{(S)}$, not necessarily equal to $Q$. The conditions emphasized by the adoption of italics make the property weaker than we would wish. Even in this case we have to assume the entropy index to be larger than a critical value. We
denote this critical value with the symbol \( Q^{(S)} \) because, as earlier mentioned, we cannot prove that it is identical to the critical entropy index \( Q \) of Section III.

In the case of entropy increase, by expressing the non-extensive entropy as a function of its four eigenvalues, we get

\[
\left( \frac{P_m^{(2)}}{P_m^{(1)}} \right)^q < 1 + \sum_{j=1}^{3} \left( \left( \frac{P_j^{(1)}}{P_m^{(1)}} \right)^q - \left( \frac{P_j^{(2)}}{P_m^{(1)}} \right)^q \right).
\]

(49)

Since the two eigenstates have different entanglement we have \( P_m^{(2)}/P_m^{(1)} \neq 1 \). Since the inequality of Eq. (43) must hold true in the case of entropy indices arbitrarily larger than \( Q^{(S)} \), and consequently, must hold true also for values much larger than the unity, we reach the conclusion that \( P_m^{(2)} < P_m^{(1)} \), which in the set \( \Im \) is equivalent to \( \Delta E_F < 0 \). The opposite conclusion would be reached in the case of a negative \( \Delta S_q \).

In spite of the earlier restrictions, we can use the obtained results to illustrate one of the most interesting findings of this work. This is as follows. Let us consider a generic subset \( \Im' \), of the set \( \Im \), only fulfilling the request of containing a finite number of states, with different entanglements. Then, we can conclude that these entanglements are equivalent to the inverse of the non-extensive entropy, provided that entropy indices \( q \) are larger than a given value \( Q_{\Im'} \), which is given by the following formula

\[
Q_{\Im'} \equiv \max \{ Q(\rho_i, \rho_j), \forall \rho_i, \rho_j \in \Im', i \neq j \}.
\]

(50)

In the set \( \Im' \) for entropy indices larger than the critical value the ordering in the direction of increasing (decreasing) entanglement is equivalent to ordering in the direction of decreasing (increasing) entropy. A significant consequence of this is that entropy minimization yields the maximally entangled state and the entropy maximization the minimally entangled state.

An attractive, albeit heuristic way, of illustrating the same conclusions is given by the following formula:

\[
E_q^{(eff)}(\rho) \equiv \frac{2}{\pi} \arctan \left\{ \sum_{i=1}^{4} \Theta \left( P_i - \sum_{k \neq i} P_k \right) \left( 3 - 4S_2(\rho) - 4 \sum_{k \neq i} P_k^2 \right) S_q^{-1} \right\},
\]

(51)

which establishes a direct connection between entanglement and entropy. The quantity \( E_q^{(eff)} \) is “equivalent” to the entanglement, in the sense that it increases or it decreases
upon increasing or decreasing the entanglement strength. Furthermore, it is equal to 1 when the entanglement is 1 and tends to vanish with the entanglement measure tending to zero. The key ingredient of this heuristic formula is the term arctan and the factor \( \sum_{i=1}^{4} \Theta \left( P_i - \sum_{k \neq i} P_k \right) \left( 3 - 4S_2(\rho) - 4 \sum_{k \neq i} P_k^2 \right) \). Without the first one, the condition \( P_m \to 1^- \) would generate divergencies. Furthermore with \( P_m \to (1/2)^+ \) the inverse of the entropy would tend to a minimum, which would be different from 0, which is the right value. With the factor \( \sum_{i=1}^{4} \Theta \left( P_i - \sum_{k \neq i} P_k \right) \left( 3 - 4S_2(\rho) - 4 \sum_{k \neq i} P_k^2 \right) \), we get rid of the divergencies and we do succeed in ensuring that the quantity \( E_q^{(eff)} \) tends to vanish with the entanglement tending to zero. Note that this *ad hoc* factor is nothing but the square of the concurrence. In principle one could express the concurrence in terms of \( S_2 \), but this would not afford the attractive condition of the entanglement being a monotonical increasing function of the inverse of the non-extensive entropy.

In conclusion, we find that entanglement increase implies entropy decrease and vice versa. This property must be compared with the results of the work of Abe and Rajagopal \[14\]. These authors adopt the principle of entropy maximization under suitable constraints to infer a plausible form of physical state, and conclude that the entangled states are the important result of this maximization process. Here we adopt a different perspective, based on the fact that the definition of entanglement of formation is already inspired to statistical mechanics \[19\]. Within this perspective the state of maximum entanglement corresponds to the minimum amount of information necessary to describe the state. Within this same perspective, the amount of information necessary to describe the state becomes increasingly larger upon reducing the entanglement strength. From an intuitive point of view, the occurrence of de-coherence, that is judged by many authors \[24\] to be the key condition to derive classical from quantum physics, implies a significant entropy increase. However, de-coherence, as a form of real wave function collapse \[25\], implies the breakdown, in the long-time limit, of the entanglement condition, and, as a consequence, the breakdown of the theory itself of the present paper. The result of this paper has to be considered within this perspective. As it appears from the literature on this new and exciting subject, the
thermodynamic significance of the processes of quantum teleportation is a very delicate and difficult issue. We are inclined to believe that the adoption of a non-extensive form of entropy might be of some relevance, under specific restrictions. The first is that real wave function collapses are ignored, and the second is that, in a world dominated by quantum entanglement, the condition of maximum entanglement is perceived as that requiring the minimum amount of information. In other words, increasing entanglement means smaller, rather than larger, entropy values.

IV. THE TSALLIS ENTROPY AT WORK: DEPHASING PROCESSES IN THE BELL BASIS SET

Before ending this paper, it is convenient to illustrate another interesting result that does not require a restriction to the set $\mathcal{S}$. This has to do with an important result obtained by Bennett et al [20]. These authors studied the entanglement changes as a function of a dephasing process. More precisely, they focused their attention on the transformation

$$D_B = \frac{1}{4} \sum_{i=0}^{3} U_i^\dagger \rho U_i,$$

which brings the initial condition described by the density matrix $\rho$, expressed in the Bell basis, into the diagonal form

$$[D_B]_{ij} \equiv \delta_{ij} [\rho]_{ij},$$

where the operators $U_i, i = 0,1,2,3$ are respectively $I, B_x B_x, B_y B_y, B_z B_z$ and $B_i$ is the bilateral rotation of $\pi/2$ around the $i$-th axis of the space $S_{1/2}^{(1)} \times S_{1/2}^{(2)}$. This bilateral rotation is defined by these authors [20] as

$$B_i = \frac{1}{2} (I_{2\times2} - i\sigma_{1i}) \times (I_{2\times2} - i\sigma_{2i}).$$

Note that the matrix $D_B$ of Eq. (53) is the "diagonal" of the statistical density matrix $\rho$ expressed in the Bell basis and that it results from a random application of four local unitary
transformations, so that moving from the initial state $\rho^{(1)}$ to the state described by $D_B^{(1)}$ the entanglement cannot increase\cite{26}. Consequently, we have

$$E_F\left(\rho^{(1)}\right) \geq E_F\left(D_B^{(1)}\right).$$

(55)

We shall analyze these theoretical results by means of the non-extensive entropy. The first analysis is made by focusing our attention on the natural values $n > 1$ of the entropy index $q$. In this special case the non-extensive entropy reads as follows

$$S_n(\rho) = \frac{1 - \text{Tr} \left( U \cdot \rho \cdot U^\dagger \right)^n}{n - 1} = \text{Tr} \rho - \rho^n.$$

(56)

Let us define the auxiliary function

$$g_n(x) \equiv \frac{x - x^n}{n - 1}.$$

(57)

We note that this is a concave function. On the other hand, several years ago Wehrl \cite{27} noticed that in that case we can write

$$\text{Tr} g_n(D_B) \geq \text{Tr} g_n(\rho).$$

(58)

We notice that $S_n(\rho) = \text{Tr} g_n(\rho)$ and $S_n(D_B) = \text{Tr} g_n(D_B)$. Consequently, we can write:

$$S_n(D_B) \geq S_n(\rho)$$

(59)

The results of the dephasing process makes it possible for us to generalize the results of Section III. Let us consider a transformation from an initial state described by a generic density matrix. As to the final state, we set the condition that it belongs to the set $\mathcal{S}$. Let $p_m^{(1)}$ be the maximum of the diagonal elements of the initial state $\rho^{(1)}$, expressed in the Bell basis set. Let us suppose also that $p_m^{(1)}$ is larger than the maximum eigenvalue of the density matrix $\rho^{(2)}$, referring to the final state. This condition is expressed by the relation:

$$p_m^{(1)} \equiv \max_{i=1,2,3,4} \left[ \rho^{(1)} \right]_{ii} > P_m^{(2)} > \frac{1}{2}.$$

(60)

As a consequence of this relation we have
\[ E_F (\rho^{(1)}) \geq E_F (D_B^{(1)}) > E_F (\rho^{(2)}) > 0. \] (61)

This is a transformation with a decreasing entanglement. On the basis of the results of Section III and of Eq. (59), we are in a position to find values of the entropy index \( q \) such that the non-extensive entropy of the final state is larger than that of the initial state. This is done as follows. We move from the initial condition \( \rho^{(1)} \) to \( D_B^{(1)} \), through the dephasing process earlier described. As we have seen, with the adoption of natural values, larger than the unity, for the entropy indices, the entropy does not decrease. This means

\[ S_n (D_B^{(1)}) \geq S_n (\rho^{(1)}). \] (62)

According to our assumptions, \( D_B^{(1)} \) and \( \rho^{(2)} \) belong to the set \( \Im \). Thus, we know, on the basis of the results of Section III, that there exists a critical value of the entropy index, \( Q (D_B^{(1)}, \rho^{(2)}) \), beyond which the non-extensive entropy increases. If we choose critical values of the entropy index that are natural numbers larger than

\[ N \equiv \left[ Q (D_B^{(1)}, \rho^{(2)}) \right], \] (63)

we conclude that the non-extensive entropy increases. As earlier anticipated, this has the effect of making more general the results of Section III.

V. CONCLUSIONS

This paper shows that in the set \( \Im \), enforcing the important condition of a non vanishing entanglement, the Tsallis entropy is a monotonic and decreasing function of the increasing entanglement. The entanglement is, in turn, a monotonic and decreasing function of the increasing entropy under the key restriction of transformations yielding an entanglement change. This conclusion was reached adopting a perspective taking the warning of a recent paper by Horodecki et al [26] into account. As a matter of fact, these authors show that the principle of entropy maximization yields fake entanglement, and consequently becomes questionable. We share the conviction of these authors and adopt in fact an approach that
does not rest on the Jaynes principle \cite{17,18}. Thus we establish a comparison between entanglement and non-extensive entropy without invoking the Jaynes principle. We do not need to maximize entropy after minimizing entanglement, as they do \cite{26}, and the monotonic dependence of entropy on entanglement is a natural consequence of the adoption of suitably large entropy indices.

This means that we share the view of Rajagopal and Abe \cite{14} that a non-extensive form of entropy can prove to be a convenient tool to study quantum teleportation. In this sense, this paper contributes deepening our understanding about the significance of the Tsallis entropy. This entropy indicator does not split into the sum of two independent contributions, when applied to a system consisting of two uncorrelated subsystems. This suggests that this kind of entropy might be a proper theoretical tool only when applied to cases where the re-partition into two uncorrelated systems is impossible. Quantum mechanical systems, in principle, are significant examples of where this condition applies, if environmental decoherence, or other kind of de-coherence processes, are ignored. In this condition the Tsallis entropy, according to the main result of this paper, seems to work properly, provided that the warning of Ref. \cite{26} is taken into account. This is where our procedure departs from the point of view of Rajagopal and Abe \cite{14}. Their approach is still based on the Jaynes principle, supplemented by the choice of a suitable additional constraint, concerning the fluctuations around the average, as well as the ordinary constraint on the mean value (see also Ref. \cite{28}). This procedure yields convincing, although non general conclusions. Our approach, which unfortunately shares the lack of generality of Ref. \cite{14}, is based on a different perspective, aiming at identifying the inverse of entanglement with the non-extensive entropy.

We think that the alternative perspective adopted in the present paper might contribute, as Refs. \cite{14,28} do, to a better understanding of the thermodynamic nature of entanglement. We are afraid that the non-extensive entropy might become inefficient when we leave the physical condition where the no-cloning theorem and the principle of no-increasing entanglement, recently found by Horodecki and Horodecki \cite{29}, is broken. According to these authors the occurrence of real wave function collapses, incompatible with the restriction of
adopting unitary transformations, provokes the breakdown of this equivalence. In our opinion, the occurrence of real wave function collapses is incompatible with the restriction of working on the set $\mathbb{I}$, which enforces the condition of a non-vanishing entanglement. Thus, we expect that in that case the theory of this paper, and with it the non-extensive entropy, does not work. To explore the uncertain border between quantum and classical mechanics we probably need to adopt a still more advanced perspective.
REFERENCES

[1] C.H. Bennett, G. Brassard, C. Crépeau, R. Josza, A. Peres and W.K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).

[2] S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).

[3] S. Popescu, Phys. Rev. Lett. 72, 797 (1994).

[4] M.B. Plenio and V. Vedral, quant-ph/9804075 v2 (1998).

[5] S. Popescu, D. Rohrlich, Phys. Rev. A. 56, R3319 (1997).

[6] P. Horodecki, R. Horodecki, M. Horodecki, Acta Physica Slovaca 48, 141 (1998).

[7] N.J. Cerf and C. Adami, Phys. Rev. Lett. 79, 5194 (1997).

[8] C. Brukner and A. Zeilinger, quant-ph/0008091.

[9] C. Brukner and A. Zeilinger, quant-ph/0006087.

[10] C. Brukner and A. Zeilinger, Phys. Rev. Lett. 83, 3354 (1998).

[11] C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[12] C. Tsallis, Brazilian Journal of Physics, 29 (1999).

[13] A. Vidiella-Barranco, Phys. Lett. A 260, 335 (1999).

[14] S. Abe and A.K. Rajagopal, Phys. Rev. A 60, 3461 (1999).

[15] S. Abe and A.K. Rajagopal, quant-ph/0001085.

[16] C. Tsallis, S. Lloyd, M. Baranger, quant-ph/0007112.

[17] E. Jaynes, Phys. Rev. 108, 171 (1957).

[18] E. Jaynes, Phys. Rev. 108, 620 (1957).

[19] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[20] C.H. Bennet, D.P. DiVincenzo, J.Smolin and W.K.Wootters, Phys.Rev.A 54, 3824 (1996)

[21] R.Horodecki and M. Horodecki, Phys. Rev. A 54 1838 (1996).

[22] C.H. Bennett, G. Brassard, S, Popescu, B. Schumacher, J. A. Smolin, and W.K. Wootters, Phys. Rev. Lett. 76, 722 (1996).

[23] R.F. Werner, Phys. Rev. A 40, 4277 (1989).

[24] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, H. -D. Zeh, Decoherence and the Appearance of a Classical World in Quantum Theory, (Springer Verlag, Berlin, 1996).

[25] L. Tessieri, D. Vitali, P. Grigolini, Phys. Rev. A 51, 4404 (1995).

[26] R.Horodecki, M. Horodecki and P. Horodecki, Phys. Rev. A 59 1799 (1999).

[27] A. Wehrl, Rev. Mod. Phys. 50, 221 (1978).

[28] A.K. Rajagopal, quant-ph/9903083

[29] M. Horodecki, R. Horodecki, Phys. Lett. A 244, 473 (1998).