QED Interference in Charge Asymmetry
Near the Z Resonance at Future Electron-Positron Colliders

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Abstract

The measurement of the charge asymmetry $A_{FB}(e^-e^+ \rightarrow \mu^-\mu^+)$ will play an important role at the high-luminosity circular electron-positron collider FCCee considered for construction at CERN. In particular, near the $Z$ resonance, $\sqrt{s} \simeq M_Z \pm 3.5$ GeV, $A_{FB}$ will provide a very precise value of the pure electromagnetic coupling constant $\alpha_{\text{QED}}(M_Z)$, which is vitally important for overall tests of the Standard Model. For this purpose, $A_{FB}$ will be measured at the FCCee with an experimental error better than $\delta A_{FB} \simeq 3 \cdot 10^{-5}$, at least a factor 100 more precisely than at past LEP experiments! The important question is whether the effect of interference between photon emission in the initial and final state can be removed from the $A_{FB}$ data at the same precision level using perturbative QED calculations. A first quantitative study of this problem is presented here, with the help of the \texttt{KKMC} program and a newly developed calculation based on soft photon resummation, matched with NLO and NNLO fixed-order calculations. It is concluded that a factor of 10 improvement with respect to the LEP era is obtained. We also present a clear indication that reducing the uncertainty of charge asymmetry near the $Z$ peak due to IFI down to $\delta A_{FB} \simeq 3 \cdot 10^{-5}$, \textit{i.e.} the expected experimental precision at FCCee, is feasible.
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1 Introduction

In past measurements of the charge asymmetry $A_{FB}$ (forward-backward angular asymmetry) at electron-positron colliders, QED interference between photons emitted from the initial and final charged leptons was of interest because this effect had to be subtracted from the measurement of $\gamma \otimes Z$ interference. In overall tests of the SM, the measurement of $A_{FB}$ contributed mainly to knowledge of the $Z$ couplings and/or the electroweak mixing angle.

At the future high-energy high-luminosity circular electron-positron collider FCCee [1] proposed for construction at CERN, the measurement of $A_{FB}(e^-e^+ \rightarrow \mu^-\mu^+)$ will play an additional important role. Near the $Z$ resonance, $\sqrt{s}_\pm \simeq M_Z \pm 3.5$ GeV, the measurement of $A_{FB}$ will provide a very precise value of the pure electromagnetic coupling constant $\alpha_{\text{QED}}(M_Z)$, which is vitally important for overall tests of the SM, at a precision level at least a factor of 10 better than in the past. This kind of the measurement was proposed and analyzed in Ref. [2].

Thanks to the very high luminosity of the FCCee, the charge asymmetry $A_{FB}(M_Z \pm 3.5\text{GeV})$ will possibly be measured with an error $\delta A_{FB}/A_{FB} \simeq 3 \cdot 10^{-5}$ or even better. This immediately poses the question of whether the effect of QED initial-final interference (IFI) can be removed from the data at the same precision level. How big is the effect of IFI in $A_{FB}$? Far from the resonance, it is about $2 - 3\%$, and it is even bigger for a tight cut-off on the total energy of the emitted photons. At the top of the $Z$ resonance, where $A_{FB}$ was measured very precisely in LEP experiments, the IFI effect is suppressed by the ratio $\Gamma_Z/M_Z$ to the level of $\delta A_{FB} \sim 0.1\%$, due to the long time separation between the creation and the decay stages of the $Z$ resonance. As we shall see in our analysis, at $\sqrt{s} \simeq M_Z \pm 3.5$ GeV, the same $\Gamma_Z/M_Z$ suppression of IFI in $A_{FB}$ still works to some extent, but the IFI effect is nevertheless at the $\delta A_{FB} \sim 1\%$ level, and growing for tight cut-offs.

This effect is huge with respect to the planned experimental precision at FCCee, and it would render measurement of the $A_{FB}(M_Z \pm 3.5\text{GeV})$ completely useless unless the theoretical evaluation of IFI is equally precise!

How precise are the theoretical evaluations of the effect of IFI in $A_{FB}$ presently available within perturbative QED? In the pre-LEP era, $\mathcal{O}(\alpha^1)$ fixed-order calculations were quoted to provide $\sim 0.3 - 0.5\%$ precision; see the review of Ref. [3]. In the LEP1 phase, thanks to $\Gamma_Z/M_Z$ suppression, the IFI effect in $A_{FB}$ at the $Z$ peak was not a burning issue – what was added to the QED theoretical toolbox was the resummation of the soft photon effects, including $\Gamma_Z/M_Z$ suppression and the resummation of $\ln(\Gamma_Z/M_Z)$, taking advantage of $\Gamma_Z \ll M_Z$. Pioneering work on the resummation of soft photon effects near a narrow resonance was done by the Frascati group; see Refs. [4–6]. Their calculations were analytical and did not treat hard photon effects in a realistic way, while Monte-Carlo (MC) programs used to analyze LEP1 data [7] were still at the $\mathcal{O}(\alpha^1)$ fixed-order level, without soft photon resumation.

Significant progress was made before the end of the LEP era with the advent of coherent exclusive exponentiation [8,9] (CEEX) and its implementation in the KKMC pro-
Soft photon resummation in CEEX was implemented at the amplitude level, including all the advances of Refs. [4–6] relevant for narrow resonances. Moreover, the SM predictions of KKMC for $A_{FB}$ and other experimental observables were possible for arbitrary event selections (cuts). Correct matching of the $O(\alpha^1)$ IFI contributions with other non-IFI corrections up to complete $O(\alpha^2)$ was also implemented along with soft-photon-resummed amplitudes, throughout the multi-photon phase space, including any number of soft and hard photons. The CEEX/KKMC calculation was instrumental in the analysis of LEP2 data above the $Z$ peak and near the $WW$ threshold, and helped to consolidate data analysis of $e^-e^+ \rightarrow f\bar{f}$ processes near the $Z$ peak. The precision of the QED IFI calculations quoted at the end of the LEP era was $\delta A_{FB} \sim 0.1\%$ at the $Z$ peak and $\delta A_{FB} \sim 0.3\%$ far away from the $Z$ resonance; see Refs. [8], [11], [12] and [13]. These papers represent the state of the art in the perturbative QED calculation of the IFI contributions to $A_{FB}$ until the present day.

The KKMC precision tag on QED IFI calculations was sufficient for analyzing LEP experimental data at the end of the LEP era. However, it was quite clearly underestimated, i.e. most likely it was far better. It was difficult to better quantify the theoretical uncertainty of the IFI prediction of KKMC simply because there was no other calculation at a similar level of sophistication to compare with. One of the main aims of this work will be to develop a new alternative numerical calculation of the IFI contribution, in order to compare with KKMC and quantify theoretical uncertainties of the IFI component in $A_{FB}$ at a higher precision level.

Generally, one may be quite skeptical whether an improvement of the QED calculation of IFI in $A_{FB}$ from the LEP-era $\delta A_{FB} \sim 10^{-3}$ down to $\delta A_{FB} \sim 10^{-5}$, i.e. by a factor of 100, is feasible at all! However, the prediction of perturbative QED for the $Z$ line shape also progressed by a similar factor from the time before LEP started until the end of the LEP era. This was possible mainly due to soft photon resummation techniques. The use of these techniques is again critical for the present task of improving the QED calculation of IFI in $A_{FB}$. The aim of this paper is to check how far we can advance on the road to the precision required for FCCee.

## 2 Physics of IFI

Any efficient evaluation of IFI in perturbative QED must be based on a good understanding of the basic physics governing this phenomenon. Let us consider the process $e^-e^+ \rightarrow \mu^-\mu^+$ accompanied with any number of real and virtual photons, illustrated schematically in Fig. 1. In the case of final fermions emitted at wide angles, IFI can be neglected, and in the case of total photon energy (ISR+FSR) limited to $K < E = \sqrt{s}/2$,

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1In particular, resummation of $\ln(\Gamma_Z/M_Z)$ was included.

2The IFI contribution to the total cross section is also discussed in these works; see also Ref. [14].
Figure 1: Multiple photon emission at a wide scattering angle.

Figure 2: Multiple photon emission at a forward scattering angle.

the angular distribution is uniformly lowered by a $\theta$-independent Sudakov form factor

$$\frac{d\sigma}{d\cos\theta}(K) \simeq \frac{d\sigma_{\text{Born}}}{d\cos\theta} \exp \left[ -\int_K^E \frac{dk^0}{k^0} \left( 2\frac{\alpha}{\pi} \ln \frac{s}{m_e^2} + 2\frac{\alpha}{\pi} \ln \frac{s}{m_\mu^2} \right) \right] = \frac{d\sigma_{\text{Born}}}{d\cos\theta} e^{-\Delta(K/E)}.$$

The above relation is illustrated schematically in Fig. 1.

Photon emission is, however, suppressed in the small-angle limit $\theta \to 0$, as illustrated schematically in Fig. 2, simply because outgoing muon inherits most of the electromagnetic field accompanying the incoming electron of the same charge as the muon, hence there is no need for the compensating action of the bremsstrahlung. In fact, bremsstrahlung dies out completely at $\theta = 0$, and it is the IFI contribution which kills both ISR and FSR.

The virtual form factor in the angular distribution at $t \to 0$, $s - |t| - |u| = 0$, $|u| \to s$ becomes

$$\Delta = \int_K^E \frac{dk^0}{k^0} \left( 2\frac{\alpha}{\pi} \ln \frac{s}{m_e^2} + 2\frac{\alpha}{\pi} \ln \frac{s}{m_\mu^2} - 4\frac{\alpha}{\pi} \ln \frac{|t|}{|u|} \right) \to \int_K^E \frac{dk^0}{k^0} \left( 2\frac{\alpha}{\pi} \ln \frac{t}{m_e^2} + 2\frac{\alpha}{\pi} \ln \frac{t}{m_\mu^2} \right) \simeq 0.$$

The subscript “virt” appears because virtual corrections feature $-\int_0^E$, while real emissions add $+\int_0^K$, so that the uncompensated remnant $-\int_K^E$ is of the virtual origin.
On the other hand, in backward scattering, illustrated schematically in Fig. 3, the situation is completely different. The electromagnetic field accompanying $e^{-}$ has to be replaced by that of $\mu^{+}$, hence the violent compensating action of the bremsstrahlung is much stronger than for wide angles. Here we have $u \to 0$ ($c \to -1$ side), $s - |t| - |u| = 0$, $|t| \to s$. Thus, IFI enhances the total QED correction by a factor of 2:

$$
\Delta = \int_{K}^{E} \frac{dk^{0}}{k^{0}} \left( 2\frac{\alpha}{\pi} \ln \frac{s}{m_{e}^{2}} + 2\frac{\alpha}{\pi} \ln \frac{s}{m_{\mu}^{2}} - 4\frac{\alpha}{\pi} \ln \frac{|t|}{|u|} \right) \to \int_{K}^{E} \frac{dk^{0}}{k^{0}} \left( 4\frac{\alpha}{\pi} \ln \frac{s}{m_{e}^{2}} + 4\frac{\alpha}{\pi} \ln \frac{s}{m_{\mu}^{2}} \right),
$$

creating a dip in the muon angular distribution for backward scattering (in the presence of a cut-off on the total photon energy, as previously).

In reality, the distribution of $\cos \theta$ far from the resonance appears as shown in Fig. 4 for a relatively strong cut-off on total photon energy (2% of the beam energy).

The presence of a narrow resonance significantly changes the pattern of QED cancellations. Let us analyze briefly how the real and virtual corrections combine at a resonance
position $\sqrt{s} = M_Z$.

- For pure ISR, the virtual correction is $\sim -\frac{2\alpha}{\pi} \ln \frac{s}{m_\mu^2} \ln \frac{E}{\lambda}$, as without a resonance, while the real contribution is cut by the resonance profile $\sim +\frac{2\alpha}{\pi} \ln \frac{s}{m_\mu^2} \ln \frac{\Gamma_Z}{\lambda}$. The resulting cross section $\sigma(K)$ is suppressed by the remnant virtual factor $\left[1 - \frac{2\alpha}{\pi} \ln \frac{M_Z^2}{K^2} \right]_{\text{virt}}$ for any cut above the resonance width, $K > \Gamma_Z$.

- The effect of FSR is the same as in the case without a resonance, i.e. $\sigma(K)$ is suppressed by the remnant virtual factor $\left[1 - \frac{2\alpha}{\pi} \ln \frac{s}{m_\mu^2} \ln \frac{E}{\lambda} \right]_{\text{virt}}$.

- The case of IFI is most complicated. The virtual correction $\sim -\frac{4\alpha}{\pi} \ln \frac{t}{u} \ln \frac{\Gamma_Z}{\lambda}$ is cut by the resonance (contrary to the ISR case). The real correction $\sim +\frac{4\alpha}{\pi} \ln \frac{t}{u} \ln \frac{\Gamma_Z}{\lambda}$ is also cut by the resonance (similar to the ISR case). The resulting $d\sigma(K)/d\Omega$ factor is strongly power-suppressed by the $\Gamma_Z/M_Z$ factor for any cut above the resonance width, $K > \Gamma_Z$! For an energy cut below the resonance width, $K < \Gamma_Z$, IFI starts to rise logarithmically, i.e. the suppression factor is $\sim 1 - \frac{2\alpha}{\pi} \ln \frac{t}{u} \ln \frac{\Gamma_Z}{K}$.

Away from the resonance, IFI gradually changes to the previous non-resonant case, and in the entire neighborhood of the resonance, a QED calculation including photon resummation at the amplitude level (CEEX) is mandatory.

The above mechanism is clearly illustrated in Fig. 5, where the IFI contribution to $A_{FB}$ is shown as a function of $v_{\text{max}}$, the cut-off on the total photon energy in units of the beam energy.\[4\] As we see, far from the $Z$ resonance and for a loose photon energy cut-off,

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\[4\]More precisely, $v = 1 - M_{\mu\mu}^2/s$. Here, we have temporarily used the ALEPH definition of $v$.
\[ A_{FB} \simeq 2\% \] and grows for stronger cut-offs. In the middle of the resonance, it is strongly suppressed, \( A_{FB} < 0.1\% \), and starts to grow below \( \nu_{\text{max}} \simeq \Gamma_{Z}/M_{Z} \simeq 0.02 \). Remarkably, at the other two energies \( \sqrt{s} \simeq M_{Z} \pm 3.5 \text{ GeV} \), \( \Gamma_{Z}/M_{Z} \) suppression is still quite strong, more than factor 1/5.

On the methodology side, although we are interested mainly in the IFI effect outside the \( Z \) peak, at \( \sqrt{s} \simeq M_{Z} \pm 3.5 \text{ GeV} \), it is worth also keeping an eye on \( \sqrt{s} = M_{Z} \) and energies far away from the resonance. This is simply because any technical problem or mistreatment of physics which may cause a small effect at \( \sqrt{s} \simeq M_{Z} \pm 3.5 \text{ GeV} \) could be magnified there, making it easier to trace it back and eliminate. This is why we shall often compare our principal results with the results at \( \sqrt{s} = M_{Z} \) and \( \sqrt{s} = 10 \text{ GeV} \).

### 3 Matrix element of multi-soft-photon emission in the semi-soft approximation

Let us consider the matrix element of the process

\[ e^{-}(p_{1}) + e^{+}(p_{2}) \rightarrow \mu^{-}(q_{1}) + \mu^{+}(q_{2}) + \gamma(k_{1}) + \cdots + \gamma(k_{n}) \]

near the \( Z \) resonance in the soft photon limit. The standard kinematic variables \( s = (p_{1} + p_{2})^{2}, \ t = (q_{1} - p_{1})^{2}, \ u = (q_{2} - p_{1})^{2} \) will be used. Around any narrow resonance, the notion of the soft photon limit has to be refined. In the framework of the standard Yennie-Frautschi-Suura (YFS) \[ 15 \] soft photon resummation, one starts with all photons being very soft, i.e. \( k_{0}^{0} \ll \Gamma_{Z} \ll \sqrt{s}/2 \). Near the resonance, however, it is worth considering a wider soft photon range, with \( k_{0}^{0} \ll \sqrt{s}/2 \), but allowing photons energies comparable to or even greater than the resonance width \( \Gamma_{Z} \). In the following, we shall refer to this regime as the semi-soft approximation. Following the notation of Ref. \[ 8 \], in the semi-soft regime, the matrix element of our process reads as follows:

\[
\mathcal{M}^{\mu_{1}\mu_{2}\cdots\mu_{n}}(p_{i},q_{j},k_{l}) = \sum_{\nu=\gamma,Z} \sum_{\nu'} e^{\alpha B_{V}^{\nu}(s_{I},t,m_{r})} \prod_{i\in I} j_{I}^{\mu_{i}}(k_{i}) \prod_{r\in F} j_{F}^{\mu_{r}}(k_{r}) \mathcal{M}_{0}(s_{I},t),
\]

\[
s_{I} = P_{I}^{2}, \quad P_{I} = p_{1} + p_{2} - \sum_{i\in I} k_{j},
\]

\[
j_{I}^{\mu}(k) = e Q_{e} \left( \frac{p_{1}^{\mu}}{kp_{1}} - \frac{p_{2}^{\mu}}{kp_{2}} \right), \quad j_{F}^{\mu}(k) = e Q_{f} \left( \frac{q_{1}^{\mu}}{kq_{1}} - \frac{q_{2}^{\mu}}{kq_{2}} \right),
\]

\[
\alpha B_{V}^{\nu}(s,t,m_{\gamma}) = \alpha B_{4}(s,t,m_{\gamma}) + \alpha \Delta B_{V}^{\nu}(s,t,M_{V}^{2}).
\]

The above formula involves a sum over the set of \( 2^{n} \) partitions \( \{\mathcal{P}\} = \{I,F\}^{n} \),

\[
\{\mathcal{P}\} = \{(I,I,I,...,I),(F,I,I,...,I),(I,F,I,...,I),(F,F,I,...,I),...,(F,F,I,...,F)\},
\]

of photon among the initial and final state. The meaning of the shorthand notation \( i \in I \) is that \( \prod_{i\in I} \) includes all photons with \( \mathcal{P}_{i} = I \) and similarly \( \mathcal{P}_{r} = F \) for \( r \in F \).
The form factor $B_4(p_i, q_i, m_\gamma)$ is the standard one appearing in YFS resummation \cite{15} for four charged particles in the scattering process. As stressed in refs. \cite{4,6}, in the semi-soft regime, an additional term in the form factor

$$\alpha \Delta B_i^2(s, t, \overline{M}^2) = -2Q_e Q_F \frac{\alpha}{\pi} \ln \left( \frac{t}{u} \right) \ln \left( \frac{\overline{M}^2_Z - s}{\overline{M}^2_Z} \right), \quad \overline{M}^2 = M^2_Z - i M_Z \Gamma_Z, \quad \Delta B_i^2 \equiv 0,$$

must be included, but only in the resonant component of the amplitude. For $\gamma$ exchange, only the standard $\alpha B_4$ of the YFS scheme is needed, and $\alpha B^R_4$ is not present. Most important is that, in the semi-soft approximation, the energy argument of the resonance propagator in the Born matrix element $\mathcal{M}_0$ must be shifted by the total energy lost to initial state photons $j \in I$ because of its strong energy dependence. The same additional dependence on $s_I$ also enters into the form factor $\alpha B^R_4$. The summation over all partitions of $n$ photons between the initial and final state $\{I, F\}$ is mandatory in order to obey Bose-Einstein symmetry and gauge invariance. Fermion spinor indices are implicit in $\mathcal{M}_0$. The standard YFS virtual form factor $B_4$ is usually regularized with a photon mass $m_\gamma$. The mass regulator can be removed once the real and virtual calculations are combined.

In the framework of coherent exclusive exponentiation (CEEX) \cite{8,10}, the above matrix element represents a zeroth-order CEEX matrix element defined throughout the entire phase space, including hard photons. Higher orders are also defined in the CEEX scheme, and implemented for a finite number of the hard photons; see Ref. \cite{8}.

The same matrix element can be rewritten in a compact form using a generating functional formulation (Mellin-Fourier transform):

$$\mathcal{M}^{\mu_1, \mu_2, \ldots, \mu_n}(p_i, q_j, k_1, \ldots, k_n) = \sum_{Y=\gamma, Z} \int \frac{d^4 Q d^4 x}{(2\pi)^4} e^{i x \cdot (P - Q)} e^{\alpha B_4}(Q^2, t, m_\gamma) \left[ \prod_{i=1}^n J^{\mu_i}(x, k_i) \right] \mathcal{M}_0(Q^2, t) \tag{3.2}$$

$$J^{\mu}(x, k) = e^{-ik \cdot x} j^{\mu}_I(k) + j^{\mu}_F(k).$$

The corresponding total cross section reads:

$$\sigma(s, E_{\text{max}}) = \frac{1}{\text{flux}(s)} \sum_{n=0}^\infty \frac{1}{n!} \int \frac{d^3 q_1}{q_1^0} \frac{d^3 q_2}{q_2^0} \prod_{i=1}^n \int \frac{d^3 k_i}{k_i^0} \delta(P - q_1 - q_2 - \sum_{i=1}^n k_i) \times \theta\left( \sum \frac{k_i^0}{\sqrt{s}} \leq E_{\text{max}} \right) \mathcal{M}^{\mu_1, \mu_2, \ldots, \mu_n}(p, q, k_1, \ldots, k_n) \mathcal{M}_0^{\mu_1, \mu_2, \ldots, \mu_n}(p, q, k_1, \ldots, k_n)^* \tag{3.3}$$

where $P = p_1 + p_2$ and $E_{\text{max}} \ll \sqrt{s}/2$, but $E_{\text{max}}$ can be comparable to or bigger than $\Gamma_Z$. In the Monte Carlo implementation of KKMC, the cut-off $E_{\text{max}}$ is not present in the phase space integral, because energy conservation naturally limits photon energies from above.\footnote{In the strict YFS soft limit this energy shift may be neglected. In the semi-soft regime it could also be neglected for the $\gamma$-exchange part. For the sake of a better treatment of the collinear (mass) singularities, it is best to keep it everywhere.}

\footnote{One may impose such a cut-off on the generated MC events anyway, if needed.}
This cut-off is, however, useful in order to simplify the analytical exponentiation in the following section.

4 Analytical exponentiation

In the semi-soft approximation, the matrix element of eq. (3.1) is simple enough that in the absence of experimental cuts, and apart from the upper limit $E_{\text{max}}$ on the total photon energy, one can perform an analytical integration over photon angles and energies and sum explicitly over photon multiplicities. This is what we refer to as an analytical exponentiation.

One may perform analytical exponentiation by means of a combinatorial reorganization of the sum over photons without using the generating functional (Mellin-Fourier transform) formulation of eq. (3.2). This method was developed in Ref. [8], albeit for the resonant component only. Another alternative method of integration/summation over semi-soft photons would employ a coherent states technique. This method was used, for instance, in refs. [4,5]. In the following, we shall follow a methodology based on the generating functional form of eq. (3.2), which was used in the original YFS paper [15], although it was applied there for the simpler non-resonant case. Of course, all three methods lead to an identical final result.

In the first step, let us introduce the usual eikonal factors for photon emission from the initial state, final state, and their interference:

\[ S_I(k) = -j_I(k) \cdot j_I(k), \quad S_F(k) = -j_F(k) \cdot j_F(k), \quad S_X(k) = -j_I(k) \cdot j_F(k), \quad (4.1) \]

and Fourier representations of the $\delta$-functions of the phase space:

\[
\begin{align*}
\sigma(s, E_{\text{max}}) &= \\
&= \frac{1}{\text{flux}(s)} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{V,V'} \int \frac{d^3q_1}{q_1^0} \frac{d^3q_2}{q_2^0} \frac{d^4Q}{(2\pi)^4} \frac{d^4Q'}{Q^{'0}} \frac{d^4x}{(2\pi)^4} \frac{d^4x'}{(2\pi)^4} e^{i\langle P-Q \rangle - i\langle P-Q' \rangle} e^{i\langle P-q_1-q_2 \rangle} \\
&\times \prod_{i=1}^{n} \int \frac{d^3k_i}{k_i^0} \left[ e^{-ik_i \cdot (y-x')} S_I(k_i) + e^{-ik_i \cdot (y+x)} S_X(k_i) + e^{-ik_i \cdot (y-x')} S_X(k_i) + e^{-ik_i \cdot y} S_F(k_i) \right] \\
&\times M_V(Q,t)M_{V'}(Q',t) e^{\alpha B_{V}^Y(Q^2,t,m_{\gamma})} e^{\alpha B_{V'}^{Y'}(Q'^2,t,m_{\gamma})} \theta \left( \sum k_i^0 \leq E_{\text{max}} \right).
\end{align*} \tag{4.2}
\]

In the above functional representation, the summation over photon multiplicities (ex-
integrals. At any step, we can go back to standard phase space; for instance eq. (4.4)  
merely to provide compact bookkeeping of the complicated sums in the multiphoton  
omit the cut-off $E$.  
see also the illustration in Fig. 6. Having in mind a MC implementation, we temporarily  
ponentiation) is trivial.\footnote{Both the virtual functions $B_4^V$ and real emission integrals over $S$-factors are regularized temporarily using a small photon mass $m_\gamma$, which will cancel in the final result.}

$$
\sigma(s, E_{\text{max}}) = \frac{1}{\text{flux}(s)} \sum_{V,V'} \int \frac{d^3 q_1}{q_1^0} \frac{d^3 q_2}{q_2^0} \ d^4 Q \int d^4 Q' \ d^4 x' \ d^4 x \ d^4 y \ d^4 y' \ e^{i x'(P-Q) - i x'(P-Q')} \ d^4 y \ e^{i y'(P - q_1 - q_2)} 
\times \exp \left\{ \int \frac{d^3 k}{k_0} \left[ e^{-i k_x (y + x - x')} S_I(k) + e^{-i k_x (y + x)} S_X(k) + e^{-i k_y (y - y')} S_Y(k) + e^{-i k_y y} S_F(k) \right] \right\} 
\times \exp \left\{ \alpha B_4^V(Q^2, s, t) + \alpha(B_4^V(Q^2, s, t))^* \right\} \mathcal{M}_V(Q, t) \mathcal{M}_{V'}(Q', t) \theta \left( \sum k_i^0 \leq E_{\text{max}} \right).
$$

\hspace{1cm} (4.3)

The integrations over $x, x'$ and $y$ can be reorganized in order to achieve a clear factorization  
into ISR, FSR, and IFI parts, as shown in Appendix A.  
A slightly reorganized form of eq. (A.2) with $U^I$ representing the total photon momentum of pure FSR emission, $K^\mu$ representing the total momentum of pure ISR emission, and with $R^\mu$ and $R'^\mu$ aggregating IFI photons present in $\mathcal{M}_0$ and $\mathcal{M}_0^*$ correspondingly, reads as follows

$$
\sigma(s) = \frac{1}{\text{flux}(s)} \int \frac{d^3 q_1}{2 q_1^0} \frac{d^3 q_2}{2 q_2^0} \ d^4 K \ d^4 R \ d^4 R' \ d^4 U \delta^4(P - q_1 - q_2 - K - R - R' - U) 
\times \int \frac{d^4 z}{(2\pi)^4} e^{i z \cdot K} + \frac{\partial k}{\partial z} e^{-i k \cdot z} S_I(k) \int \frac{d^4 u}{(2\pi)^4} e^{i u \cdot R} + \frac{\partial k}{\partial u} e^{-i k \cdot u} S_X(k) 
\times \int \frac{d^4 u'}{(2\pi)^4} e^{i u' \cdot R'} + \frac{\partial k}{\partial u'} e^{-i k \cdot u'} S_Y(k) \int \frac{d^4 y}{(2\pi)^4} e^{i y \cdot U} + \frac{\partial k}{\partial y} e^{-i k \cdot y} S_F(k) 
\times \sum_{V,V' = \gamma,Z} \mathcal{M}_V(P - K - R) \mathcal{M}_{V'}^*(P - K - R') 
\times \exp \left\{ 2\alpha \delta B_4(s, t, m_\gamma) + \alpha \Delta B_4^V((P - K - R)^2) + (\alpha \Delta B_4^V((P - K - R')^2))^* \right\};
$$

\hspace{1cm} (4.4)

see also the illustration in Fig. C. Having in mind a MC implementation, we temporarily  
omit the cut-off $E_{\text{max}}$.  
It should be stressed that the role of the Mellin transform in the above formula is  
merely to provide compact bookkeeping of the complicated sums in the multiphoton integrals. At any step, we can go back to standard phase space; for instance eq. (4.4) can
Figure 6: Exponentiated multiple photon emission from initial and final fermions including ISR, FSR and IFI in the resonant process, as in eq. (4.4). Dashed lines represent multiple real and/or virtual photons.

This is a generalization of eq. (88) in Ref. [8], which was obtained there using pure combinatorics, without any use of the Mellin-Fourier transform.

Another advantage of eq. (4.4) is that by means of adding and subtracting

\[ \int_{k^0 \leq E} \frac{d^3k}{k^0} [S_I(k) + 2S_X(k) + S_F(k)], \quad E = \frac{\sqrt{s}}{2}, \]

This follows closely the YFS paper [15].
in the form factor exponent, we obtain a manifestly IR-finite expression:

\[
\sigma(s) = \frac{1}{\text{flux}(s)} \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0} d^4K d^4R d^4R' d^4U \delta^4(P - q_1 - q_2 - K - R - R' - U) \\
\times \int \frac{d^4z}{(2\pi)^4} e^{i z \cdot K + \int \frac{dz^0}{\sqrt{2}} S_I(k)} \int \frac{d^4u}{(2\pi)^4} e^{i u \cdot R + \int \frac{du^0}{\sqrt{2}} [e^{-i k \cdot z - \theta(k^0 < E)]} S_X(k) \\
\times \int \frac{d^4u'}{(2\pi)^4} e^{i u' \cdot R' + \int \frac{du'^0}{\sqrt{2}} [e^{-i k' \cdot z - \theta(k'^0 < E)]} S_F(k) \tag{4.6} \\
\times \sum_{V,V'=\gamma,Z} \mathcal{M}_V(P - K - R) \mathcal{M}_{V'}(P - K - R') \times \exp \left\{ 2\alpha \Re B_4(s, t, m_\gamma) + \alpha \Delta B_4^{V'}((P - K - R^2)) + (\alpha \Delta B_4^{V'}((P - K - R')^2)) \right\},
\]

where

\[
Y(s, t) = 2\alpha \Re B_4(s, t, m_\gamma) + \int \frac{d^3k}{k^0} \left[ S_I(k) + 2S_X(k) + S_F(k) \right] \tag{4.7}
\]

is the classic YFS form factor, finite in the \(m_\gamma \to 0\) limit.

5 Analytical integration over photon momenta

In the next step, we shall integrate over the photon angles in eq. (4.6), restoring the cut-off \(E_{\text{max}}\) on the total photon energy and taking advantage of the semi-soft approximation to reorganize the phase space integral.

Let us show how it is done for the initial state part of this multiphoton integral. In the semi-soft photon limit, the integrand of eq. (4.6) has no dependence on the spatial components of \(K\) outside of the \(e^{i z \cdot K}\) factor. Typically, the Born matrix element and the resonant form factor have a dependence on \(K^0\) through

\[
(P - K - R)^2 = P^2 - 2P \cdot (K + R) + (K + R)^2 \simeq s - 2\sqrt{s}(K^0 + R^0),
\]

but no dependence on the spatial components \(\vec{K}\). Thus, the integral over \(\vec{K}\) yields a factor \(\delta^3(\vec{z})\) leads to

\[
\int \frac{d^4z}{(2\pi)^4} e^{i z \cdot K + \int \frac{dz^0}{\sqrt{2}} S_I(k)[e^{-i k \cdot z - \theta(k^0 < E)]} = \int dK^0 \int \frac{dz^0}{2\pi} e^{i z^0 K^0 + \int \frac{dz^0}{\sqrt{2}} \gamma_I[e^{-i k \cdot z^0 - \theta(k^0 < E)]} = \int \frac{dK^0}{K^0} F(\gamma_I) \frac{K^0}{E} \gamma_I, \tag{5.1}
\]

where the integration over photon angles resulted in

\[
\gamma_I = \gamma_I(s) = \int \frac{d^3k}{k^0} S_I(k) \delta(2k^0/\sqrt{s} - 1). \tag{5.2}
\]

The subtle point is that the elimination of \(\int d^3 \vec{K} \delta^3(\vec{K} - \sum_{i=1}^n \vec{k}_i)\) implies that we keep \(\vec{K} = \sum_{i=1}^n \vec{k}_i\) everywhere in the entire integrand. Note that in KKMC, the above “recoil
effect” in the Born matrix element and phase space integral is taken into account correctly for hard photons as well. The function

$$F(\gamma) \equiv \frac{\exp(-\gamma C_E)}{1 + \gamma}$$

(5.3)

is well known from YFS work (eq. (2.44) in Ref. [15]) and is due to the competition of soft real photons for the available fixed total energy.

Similarly, we are able to integrate over FSR and IFI photons:

$$\int d^4 U \frac{d^4 y}{(2\pi)^4} e^{iy \cdot U} + \int \frac{d^3 p}{2\pi} S_F(k) [e^{-ik \cdot y - \theta_{<0}} - \theta_{<0}] = \int \frac{dU^0}{U^0} \gamma_F \left( \frac{K^0}{E} \right) \gamma_F F(\gamma_F),$$

$$\int d^4 R \frac{d^4 u}{(2\pi)^4} e^{iu \cdot R} + \int \frac{d^3 p}{2\pi} S_X(k) [e^{-ik \cdot u - \theta_{<0}} - \theta_{<0}] = \int \frac{dR^0}{R^0} \gamma_X \left( \frac{K^0}{E} \right) \gamma_X F(\gamma_X),$$

(5.4)

where

$$\gamma_F = \gamma_F(s) = \int \frac{d^3 k}{k^0} S_F(k) \delta(2k^0/\sqrt{s} - 1),$$

$$\gamma_X = \gamma_X(\cos \theta) = \int \frac{d^3 k}{k^0} S_X(k) \delta(2k^0/\sqrt{s} - 1),$$

and $\theta$ is the angle the between the momenta $p_1$ of $e^-$ and $q_1$ of $\mu^-$.

Inserting all the above into eq. (4.6), we finally obtain a compact elegant formula:

$$\sigma(s, v_{\max}) = \frac{3\sigma_0(s)}{8} \sum_{V,V'} \int_0^1 dv_I \, dv_F \, dr \, dr' \int \frac{d\cos \theta d\phi}{2} \theta(v_{\max} - v_I - r - r' - v_F)$$

$$\times \rho(\gamma_I, v_I) \rho(\gamma_X, r) \rho(\gamma_X, r') \rho(\gamma_F, v_F) e^{Y(v_I, q_I)}$$

$$\times \frac{1}{4} \sum_{\sigma \tau} \text{Re} \{ e^{i\Delta B_{\tau}^V(s(1-v_I-v_F))} \mathcal{M}_{\sigma \tau}^V(v_I + r, c) \} e^{i\Delta B_{\tau}^V(s(1-v_I-r'))} \mathcal{M}_{\sigma \tau}^{V'}(v_I + r', c) \},$$

(5.6)

where the Born spin amplitudes of Appendix C are used and we define

$$\rho(\gamma, v) = F(\gamma) e^{\gamma v^{\gamma - 1}}, \quad v_I = \frac{2K^0}{\sqrt{s}}, \quad r = \ln \frac{2R^0}{\sqrt{s}}, \quad r' = \ln \frac{2R^0}{\sqrt{s}}, \quad v_F = \frac{2U^0}{\sqrt{s}}.$$  

(5.7)

The appearance of the real part $\text{Re}[\mathcal{M}_B \mathcal{M}^*_B]$ has resulted from symmetrization over $r$ and $r'$. The overall structure of the above integral is illustrated in Fig. 6.

Note that the YFS function $\rho(\gamma, v)$ obeys the following nice convolution rule (related to the fact that it represents a Markovian process):

$$\int dv_1 dv_2 \delta(v - v_1 - v_2) \rho(\gamma_1, v_1) \rho(\gamma_2, v_2) = \rho(\gamma_1 + \gamma_2, v),$$

(5.8)
but this feature cannot be exploited to simplify the integral of eq. (5.6), because of the peculiar dependence of the matrix element on $r$ and $r'$. Let us stress that the double convolution over ISR photons, separately for the Born amplitude and its conjugate seen in eq. (5.6), is the landmark feature of the semi-soft exponentiation pioneered in refs. [4,5] and implemented in KKMC.

5.1 Matching of analytic exponentiation with fixed orders

Any matching of analytic exponentiation with fixed-order calculations must address the inclusion of the entire phase space beyond the semi-soft regime.

In order to compare analytical exponentiation with KKMC over the entire phase space, let us extrapolate the formula of eq. (5.6) beyond the semi-soft regime to the entire range of the variable

$$v = 1 - M_{\mu\mu}^2, \quad v \in (0, 1),$$

replacing soft photon approximation

$$v = v_I + v_F + r + r',$$

with a multiplicative ansatz guided by the collinear kinematics,

$$1 - v = (1 - v_I)(1 - v_F)(1 - r)(1 - r').$$

The resulting $O(\alpha^0)$ ISR+FSR+IFI formula with resummation of $\ln(\Gamma_Z/M_Z)$ is ready for MC implementation. Employing the Born spin amplitudes of Appendix C it reads as follows:

$$\sigma^{(0)}(s, v_{\text{max}}) = \frac{3\sigma_0(s)}{8} \sum_{V, V'} \int dv dv_I dv_F dr \, \delta_{1-v=(1-v_I)(1-v_F)}, \theta_{v_{\text{max}}>v}$$

$$\times \int \frac{d\cos \theta d\phi}{2} \, \rho(\gamma_I(s, v_I) \rho(\gamma_F(s(1-v_I)(1-v_F)), v_F)) \rho(\gamma_X(c, r) \rho(\gamma_X(c, r') \, e^{Y(p_i, q_i)}}$$

$$\times \frac{1}{4} \sum_{\varepsilon \tau} \mathcal{R}\{e^{\Delta B_I^V(s(1-v_I)(1-r))} \mathcal{M}_{\varepsilon \tau}^V (1 - (1 - v_I)(1 - r), c)$$

$$\times [e^{\Delta B_I^{V'}(s(1-v_I)(1-r'))} \mathcal{M}_{\varepsilon \tau}^{V'} (1 - (1 - v_I)(1 - r'), c)]^*\},$$

(5.9)

where $c = \cos \theta$.

We are also going to implement the following $O(\alpha^0)$ ISR+FSR formula in which IFI is completely neglected:

$$\sigma^{(0)}_{\text{noIFI}}(s, v_{\text{max}}) = \frac{3\sigma_0(s)}{8} \int dv dv_I dv_F \, \delta_{1-v=(1-v_I)(1-v_F)}, \theta_{v_{\text{max}}>v}$$

$$\times \int \frac{d\cos \theta d\phi}{2} \, \rho(\gamma_I(s, v_I) \rho(\gamma_F(s(1-v_I)(1-v_F)), v_F)) \rho(\gamma_X(c, r) \rho(\gamma_X(c, r') \, e^{Y(p_i, q_i)}}$$

$$\times \frac{1}{4} \sum_{\varepsilon \tau} |\mathcal{M}_{\varepsilon \tau}(v_I, c)|^2,$$

(5.10)
The following simplified variant of eq. (5.9) implements IFI in the approximate form:

\[ \sigma^{(0)}(s, v_{\text{max}}) = \frac{3\sigma_0(s)}{8} \int dv \, dv_I \, dv_F \, du \, \delta_{1-v=(1-v_I)(1-v_F)(1-u)} \, \theta_{v_{\text{max}} > v} \]

\[ \times \int \frac{d\cos \theta d\phi}{2} \rho(\gamma_I, v_I) \rho(\gamma_F(s(1-v_I)(1-v_F)), v_F) \rho(2\gamma_X(c), u) \, e^{Y(s, c)} \]

\[ \times \frac{1}{4} \sum_{\varepsilon r} \left| \sum_{\nu} e^{2\alpha \Delta B^I_{\nu}(s(1-v_I)(1-u))} \mathcal{M}^\nu_{\varepsilon r}(c, 1 - (1-v_I)(1-u), c) \right|^2. \]  

(5.11)

The simplification is due to neglecting \( r \) and \( r' \) dependence in the Born matrix element and keeping the integration over \( u = r + r' \). The quality of the above \textit{ad hoc} approximation can only be judged using numerical implementation of the 4-dimesional integration.

In the numerical comparison of the above \( \mathcal{O}(\alpha^2) \) \textit{exp} formulas with KKMC, it is worth including numerically significant \( \mathcal{O}(\alpha^2) \) contributions from the trivial phase integration. It was shown in Ref. [8] (see eq. (206) there) that the following substitution does the job:

\[ \rho(\gamma_I, v_I) \rightarrow \rho_I^{(0)}(\gamma_I, v_I) = \rho(\gamma_I, v_I) \exp \left[ \frac{1}{4} \gamma_I + \frac{\alpha}{\pi} \left( -\frac{1}{2} + \frac{\pi^2}{3} \right) \right] \left[ 1 - \frac{1}{4} \gamma_I \ln(1 - v_I) \right], \]

\[ \rho(\gamma_F, v_F) \rightarrow \rho_F^{(0)}(\gamma_F, v_F) = \rho(\gamma_F, v_F) \exp \left[ \frac{1}{4} \gamma_F + \frac{\alpha}{\pi} \left( -\frac{1}{2} + \frac{\pi^2}{3} \right) - \frac{\gamma_F}{2} \ln(1 - v_F) \right] \left[ 1 - \frac{1}{4} \gamma_F \ln(1 - v_F) \right], \]

(5.12)

where \( \gamma_F = \gamma_F(s(1-v_I)(1-v_F)). \)

In order to compare with \( \mathcal{O}(\alpha^2) \) KKMC calculations (including non-IR contributions of IFI up to \( \mathcal{O}(\alpha^4) \)) it is also quite easy to upgrade the ISR and FSR radiator functions in eqs. (5.10) to \( \mathcal{O}(\alpha^2) \):

\[ \rho(\gamma_I, v_I) \rightarrow \rho_I^{(2)}(\gamma_I, v_I) = \rho(\gamma_I, v_I) \exp \left[ \frac{1}{4} \gamma_I + \frac{\alpha}{\pi} \left( -\frac{1}{2} + \frac{\pi^2}{3} \right) \right] \left[ 1 + \frac{\gamma_I}{4} + \frac{\gamma_I^2}{8} + v_I \left( -\frac{1}{2} + \frac{v_I^2}{2} \right) + \gamma_F \left( -\frac{1}{2} + \frac{\gamma_F^2}{8} \right) \right], \]

(5.13)

\[ \rho(\gamma_F, v_F) \rightarrow \rho_F^{(2)}(\gamma_F, v_F) = \rho(\gamma_F, v_F) \exp \left[ \frac{1}{4} \gamma_F + \frac{\alpha}{\pi} \left( -\frac{1}{2} + \frac{\pi^2}{3} \right) - \frac{\gamma_F}{2} \ln(1 - v_F) \right] \left[ 1 + \frac{\gamma_F}{4} + \frac{\gamma_F^2}{8} + v_F \left( -\frac{1}{2} + \frac{v_F^2}{2} \right) + \gamma_F \left( -\frac{1}{2} + \frac{\gamma_F^2}{8} \right) \right] \left[ 1 + \gamma_F \left( 1 - \frac{v_F}{2} \right) + \gamma_F \left( -\frac{v_F}{2} + \frac{v_F^2 - v_F}{8} \right) \right]. \]

see Tables I and II in Ref. [8].

In the final push towards inclusion of as many known fixed order results as possible into the analytical exponentiation formula, we include the complete \( \mathcal{O}(\alpha^4) \) virtual IFI

\footnote{There are more variants of this formula, for instance setting \( u = 0 \) in Born matrix element and form factor, \textit{etc.}}

\footnote{We could also use \( \gamma_F = \gamma_F(s(1-v)) \), but we have checked that it leads to the same numerical results.}
contributions. This amounts to adding the non-IR parts of the $\gamma\gamma$ and $\gamma Z$ box diagrams of Appendix C to the Born spin amplitudes:

$$M_{\epsilon \tau}(s,t) \rightarrow M_{\epsilon \tau}(s,t) + M_{\epsilon \tau}^{\{\gamma\gamma\}}(s,t,m_\gamma) + M_{\epsilon \tau}^{\{\gamma Z\}}(s,t,m_\gamma) - 2\alpha B_4(s,t,m_\gamma) M_{\epsilon \tau}(s,t) - \alpha \Delta B_4^Z(s,t) M_{\epsilon \tau}^Z(s,t).$$ (5.14)

This is done in the framework of the standard YFS reorganization of the IR singularities, without any danger of double counting. The additional subtraction of $\alpha \Delta B_4^Y$ prevents double counting with the resummation/exponentiation of this term in the semi-soft regime.

With all the above changes due to matching with $O(\alpha^1)$ and $O(\alpha^2)$ known fixed-order corrections, we are now ready to implement the results of analytical exponentiation of eqs. (5.9) and (5.10) with radiator functions of eq. (5.12,5.13) and box insertions, using the Monte Carlo method.

Note that in the case where IFI is switched off, this kind of comparison of KKMC with the numerical tool KKsem based on analytical exponentiation was already done in Ref. [8]. What is new thing here is the inclusion of the IFI into the game.

6 Numerical integration methodology

Our aim is to perform numerically the 5-dimensional and 3-dimensional integrals in eqs. (5.9) and (5.10) using the Monte Carlo integrator FOAM [16,17]. This is not quite trivial because the integrands of eqs. (5.9) and (5.10) are singular and non-positive. Singularities due to $\rho_I$ of ISR and $\rho_F$ of FSR can be easily eliminated with the following simple mapping of variables:\footnote{11FOAM can cope with these singularities even without such a mapping.}

$$v = x_{\max} y_1^{1/\gamma_I}, \quad u = x_{\max} y_2^{1/\gamma_I}, \quad y_i \in (0,1).$$

The main problem is the integration over the two more strongly singular and non-positive $\rho_X$ factors. This occurs when

$$\gamma_X(\theta) = 2Q_e Q_f \frac{\alpha}{\pi} \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

becomes negative: $\gamma_X(\theta) = -\beta < 0$ in the forward hemisphere, where $\cos \theta > 0$.

In fact, one may think that in such a case the integral of eq. (5.6) does not make sense at all, because the singularity $r^{\gamma_X-1} = r^{-\beta-1}$ from $\rho_X$ is even not integrable! However, a closer examination of the multiphoton integral which has led to $\rho_X(r,\gamma_X(\theta))$ reveals that the original distribution is in fact regularized with the familiar plus-prescription $(...)_+$. 

11FOAM can cope with these singularities even without such a mapping.
In order to understand the problem better, it is worth examining the generic YFS multiphoton integral

\[
\int_{K_0 < E} \frac{d^4 K}{(2\pi)^4} e^{i z \cdot K} + f \frac{d^4 k}{(2\pi)^4} S(k) [e^{-i k \cdot z} - \theta_{k,0 < E}]
\]

\[
= \int_{K_0 < E} d^4 K e^{-f \frac{d^4 k}{(2\pi)^4} S(k) \theta_{k,0 < E}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\varepsilon < k_0 < E} \frac{d^3 k_i}{(2\pi)^3} S(k_i) \delta^4 \left( K - \sum_{i=1}^{n} k_i \right)
\]

\[
= \int_{0}^{E} dK^0 \int \frac{dz}{2\pi} e^{izK^0} + f \frac{d^4 k}{(2\pi)^4} \gamma \int_{\varepsilon < k_0 < K_0} \frac{\gamma}{k_i^0} \delta^4 \left( K_0 - \sum_{i=1}^{n} k_i^0 \right)
\]

(6.1)

\[
= \int_{0}^{E} \frac{dK^0}{K^0} \gamma F(\gamma) \left( \frac{K^0}{E} \right)^{\gamma} = \int_{0}^{1} dv \gamma v^{\gamma-1} F(\gamma) = \int_{0}^{1} dv \rho(\gamma, v) = F(\gamma).
\]

It is easy to check that the above integral is always finite and well-defined for any choice of \( S = S_I, S_F, S_X \), even for negative \( S \) and for negative \( \gamma \). Obviously, for \( \gamma > 0 \), the singularity \( v^{\gamma-1} \) is integrable and does not require any regulation. Closer inspection of eq. (6.1) with an explicit IR regulator \( \varepsilon \ll 1 \) reveals that for any \( \gamma \), including \( \gamma = -\beta < 0 \), the following holds:

\[
\rho(\gamma, v) = e^{-f \frac{d^4 k}{(2\pi)^4} \theta_{k,0 < E}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\varepsilon < k_0 < K_0} \frac{d^3 k_i}{(2\pi)^3} \delta^4 \left( K^0 - \sum_{i=1}^{n} k_i^0 \right)
\]

\[
= \delta(v) F(\gamma) \left[ 1 - \int_{\varepsilon}^{1} dv' \gamma v'^{\gamma-1} \right] + \theta(v - \varepsilon) F(\gamma) \gamma v^{\gamma-1} = F(\gamma) \left[ \delta(v) + (v v^{\gamma-1})_+ \right].
\]

(6.2)

The standard plus prescription can be formulated either in a regulator-independent way

\[
\int_{0}^{1} dv \phi(v) (\gamma v^{\gamma-1})_+ = \int_{0}^{1} dv [\phi(v) - \phi(0)] v^{\gamma-1},
\]

(6.3)

or with an explicit regulator \( \varepsilon \ll 1 \)

\[
(\gamma v^{\gamma-1})_+ = \gamma v^{\gamma-1} \theta(v - \varepsilon) - \delta(v) \int_{\varepsilon}^{1} dv' \gamma v'^{\gamma-1} = \gamma v^{\gamma-1} \theta(v - \varepsilon) - \delta(v) [1 - \varepsilon^\gamma].
\]

(6.4)

Of course, for \( \gamma > 0 \), it becomes simpler, because for \( \varepsilon \to 0 \), we get \( (\gamma v^{\gamma-1})_+ \to \gamma v^{\gamma-1} - \delta(v) \) and \( \rho(\gamma, v) \to F(\gamma) \gamma v^{\gamma-1} \). However, the explicit IR regulator remains mandatory for \( \gamma < 0 \).

As a closing cross-check, let us verify that for \( \gamma < 0 \) and regularized

\[
\rho(-\beta, v) = F(-\beta) \left[ \delta(v) \varepsilon^{-\beta} - \theta(v > \varepsilon) \beta v^{-1-\beta} \right],
\]

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Figure 7: The role of IFI. In forward scattering, the upward arrow (IFI) counteracts partly the action of the downward arrow (ISR+FSR).

the basic convolution rule of eq. \ref{eq:5.8} still holds: \[\int dv_1dv_2\delta(v-v_1-v_2)\rho(-\beta,v_1)\rho(\gamma,v_2) = \rho(\gamma-\beta,v). \] (6.5)

In terms of the Markovian process, the function \(\rho(\gamma,v)\) for \(\gamma > 0\) represents adding more (soft) photons. The other \(\rho(-\beta,v)\) function is undoing that (backward evolution).

In the context of the explanation of the physics of IFI in Sect. 2, the presence of \(\rho(\gamma,v)\) with negative \(\gamma\) in the forward hemisphere in eq. \ref{eq:5.6} is now perfectly understandable: \(\rho(\gamma_X,r)\rho(\gamma_X,r')\) is undoing part of the ISR and FSR photon emission coming from \(\rho(\gamma_I,v)\rho(\gamma_F,u)!\) See the upward (blue) arrow in Fig. 7 for the corresponding graphical illustration.

In the numerical MC integration, it is not difficult to introduce a small IR regulator \(\varepsilon\) into \(\rho(\gamma_X,r)\) when \(\gamma_X < 0\). In the integrand for FOAM, this is done as a part of the mapping of the integration variables \(r\) and \(r'\) into internal variables of FOAM.

Another issue is that the integrand becomes negative for \(\gamma_X < 0\), for \(r > \varepsilon\), or for \(r' > \varepsilon\). This is handled in a standard way using weighted MC events with a non-positive weight. In the actual integration by means of FOAM, the modulus of the integrand is used during the exploration stage, while in the following MC calculation of the integral, the MC events are weighted with the true signed distribution. The distribution of the MC weights in the second stage has two peaks:\[13] the bigger one close to +1 and smaller one near −1; see Fig. 8. More details on the mappings used in the construction of the integrand for FOAM in \textsc{KkFoam} program are given in Appendix B.

\[\text{In the } \varepsilon \rightarrow 0 \text{ limit, of course.}\]
\[\text{This entails a certain loss of integration precision, but it turns out to be affordable.}\]
Weight distribution Foam

Figure 8: The right-hand plot is an example of the MC weight distribution for calculating the total cross section using FOAM according to eq. (5.9). The left-hand plot presents the MC weight distribution without IFI, see eq. (5.10).

7 Numerical results

In this section, we present results from the updated v4.22 of KKMC, also referred to as KKMCee, the non-MC integrator KKsem,\(^\text{14}\) and the newly developed MC integrator (simulator) program KKFoam, based on the C++ version of FOAM,\(^\text{17}\) which implements the 5-dimensional integral of eq. (5.9) including IFI, together with its 3-dimensional variant without IFI of eq. (5.10). They will be often nicknamed in the following as KKFoam5 and KKFoam3 correspondingly.

Another sub-generator in KKFoam taking care of 2-dimensional integration over \(v\) and \(\cos \theta\) will be used for reproducing and/or implementing old pure \(\mathcal{O}(\alpha^1)\) results without resummation.

In KKFoam5 and KKFoam3 one may choose ISR and FSR structure functions with soft photon exponentiation and QED corrections up to \(\mathcal{O}(\alpha^0), \mathcal{O}(\alpha^1)\) and \(\mathcal{O}(\alpha^2)\), as defined in Tables I and II in Ref. [8]. Pure QED non-logarithmic \(\mathcal{O}(\alpha^2)\) corrections are \(< 10^{-5}\), hence are neglected for ISR, FSR and IFI. They should be included and evaluated more precisely in the future.

The Born cross section in both KKFoam5 and KKFoam3 is implemented using two types of subprograms of KKMC, each using spin amplitudes: either calculated in the scheme of Ref. [8] and labeled with GPS or CEEX, or using spin amplitudes of KORALZ\(^\text{19}\) and labeled as EEX. Note that it is not possible to use EEX Born for IFI implementation; hence in KKFoam5 only GPS/CEEX Born amplitudes are implemented. Electroweak and QCD corrections are included in the rescaled coupling constants of Born amplitudes, as

\(^{14}\)KKsem uses Gauss quadrature programs to integrate analytical formulas up to 3 dimensions. Its was developed at the time of preparing Ref. [8].

\(^{15}\)This is a variant of Kleiss-Stirling method of Ref. [18].
in KKMC, both for CEEX/GPS and EEX Born, the same way as in KKMC.

It should be kept in mind that in KKFoam hard photon corrections are included in the integrated form in the structure functions up to $O(\alpha^2)$ for ISR and FSR, while for IFI they are not included – only the finite part of the virtual $O(\alpha^1)$ IFI corrections (boxes) is included there. (In KKFoam3 IFI is completely absent.)

The immediate short-term aim in this section is to prove that these programs correctly calculate $\sigma(v < v_{\text{max}})$ and $A_{FB}(v_{\text{max}})$ with physical and technical precision $\delta A_{FB} \sim 10^{-4}$ and $\delta \sigma/\sigma \sim 3 \cdot 10^{-4}$. This is a factor of 10 better than at LEP, but still a factor of 10 short of what needed for FCCee near the $Z$ resonance. An additional cut-off $|\cos \theta| < c_{\text{max}}$ will sometimes be imposed. An analysis for more realistic cuts will be presented in a separate publication. The IFI effect in $A_{FB}$ depends strongly on the cut-off on the total photon energy $v_{\text{max}}$, which will typically be varied between $v_{\text{max}} = 0.002$ and $v_{\text{max}} = 0.200$.

The expectation is that semi-soft photon resummation employed in KKFoam5 (taking into account the energy shift due to ISR in the $Z$ propagator) will work fairly well in this cut-off range near the $Z$ pole.\footnote{On the other hand, strict YFS soft photon approximation neglecting the ISR energy shift in the $Z$ propagator is expected to be adequate for our precision requirements for only $v_{\text{max}} \leq 10^{-4}$.}

In the following analysis, event selection will be examined in terms of two variables only, $\cos \theta$ for the angle between $e^-$ and $\mu^-$ and $v = 1 - M_{\mu\mu}^2$. The variable $v$ represents approximately the total energy of all ISR and FSR photons, in units of the beam energy. (More results for realistic selection cuts will be shown in the next paper.) Of course, once harder photons are allowed, the definition of $\cos \theta$ is no longer unique. For the KKMC results, we will use the $\cos \theta$ definition of Ref.\textsuperscript{[20]} unless otherwise stated – see the following section for more discussion of other choices and their definitions.

### 7.1 On the choice of the scattering angle $\theta$

In the limit when all photons are very soft, the momenta of the final muons are back to back and the scattering angle $\theta$ between $e^-$ and $\mu^-$ is unique. Once at least one photon becomes energetic, the final muons are not back to back and there are many possible definitions of the effective $\theta$. Using $\theta^{(1)} = \angle(e^-, \mu^-)$ or $\theta^{(2)} = \angle(e^+, \mu^+)$ is not a favorable choice experimentally, because it does not exploit fully the power of the tracker detector, which detects both $\mu^\pm$ equally well.

An example of a choice favorable for experiments, taking full advantage of the very good angular resolution of the muon detectors (trackers), which is much higher than the energy resolution, is that of ref\textsuperscript{[21]}

$$
\cos \theta^* = y_1 \cos \theta_1 - y_2 \cos \theta_2,
$$

$$
y_1 = \frac{\sin \theta_2}{\sin \theta_1 + \sin \theta_2}, \quad y_2 = \frac{\sin \theta_1}{\sin \theta_1 + \sin \theta_2}, \quad y_1 + y_2 = 1.
$$

(7.1)

However, it was shown in Ref.\textsuperscript{[20]} that analytical evaluation of the IFI effect according to the $O(\alpha^1)$ QED matrix element can be easily done using

$$
\cos \theta^* = x_1 \cos \theta^{(1)} + x_2 \cos \theta^{(2)}, \quad x_i = q_i^0/(q_1^0 + q_2^0), \quad x_1 + x_2 = 1.
$$

(7.2)
Figure 9: The difference between $A_{FB}^*$ and $A_{FB}^{\star}$. From KKMC at 10 GeV with IFI on.

We use this choice for most of the numerical results presented in this work, unless otherwise stated. Moreover, in Ref. [20] compact analytical results were obtained for a charge asymmetry defined using the first moment

$$
\tilde{A}_{FB}^* = \frac{3}{2} \int_{-1}^{1} \cos \theta^* \frac{d\sigma}{\sigma}
$$

instead of the conventional forward-backward asymmetry $A_{FB} = (\sigma_F - \sigma_B)/\sigma$.

For KKFoam, the choice of $\cos \theta$ is irrelevant as long as all photons are sufficiently soft. Once at least one photon becomes energetic, the $O(\alpha^1)$ contribution calculated for a well-defined choice of $\cos \theta$ should be included in KKFoam. So far, this is not yet done – it should be done in the next version. Most likely, the preferred choice for KKFoam will be $\cos \theta^\star$. On the other hand, KKMC is a regular MC event generator providing four-momenta of both muons (and all photons), hence it provides a prediction for $A_{FB}$ with any definition of $\cos \theta$. Let us examine, using KKMC, how different the QED predictions for $A_{FB}$ are for the above two choices of $\theta$ when $v < 0.2$. Fig. 9 shows that the difference between $A_{FB}^*$ and $A_{FB}^{\star}$ is below expected FCC experimenta precision of $\delta A_{FB} \sim 3 \cdot 10^{-5}$, i.e. all of our analysis for $\cos \theta^\star$ is valid for $\cos \theta^\star$ and vice versa.

Using KKMC, it is easy to examine the difference between $\tilde{A}_{FB}^*$ and $A_{FB}^{\star}$. Fig. 10 shows that such a difference might be sizeable, up to $\sim 1\%$. However, the difference in the IFI component could cancel between two calculations – for instance, we have checked that it does cancel in the difference between KKMC and KKsem, for IFI switched on.

---

17Sufficiently from the point of view of the FCC precision
Figure 10: The difference between $A_{FB}$ and $\tilde{A}_{FB}$ at 94.3 GeV with IFI on.

### 7.2 Baseline calibration, ISR+FSR without IFI

Let us start with the baseline calibration of the MC tools at the precision level $\sim 10^{-5}$ at $\sqrt{s^-} = 87.9$ GeV and $\sqrt{s^+} = 94.3$ GeV. Although KKsem does not include IFI, it is still useful for checking the normalization of both KKMC and KKFoam. Of course, normalization is irrelevant for our main observable, $A_{FB}(v_{\text{max}})$, but it is still profitable to test it, simply because some technical problem that would be evident in $\sigma(v_{\text{max}})$ could produce a small annoying effect in $A_{FB}$ as well. Thus, it is better to keep an eye on both of these.

As already underlined, our main aim in the present study is a precise prediction for $A_{FB}$ at two energies $\sqrt{s^\pm} = 87.9$ near $Z$ resonance. However, in order to get better confidence in the implementation of the QED matrix element, we are also going to check $A_{FB}$ at $\sqrt{s} = 10$ GeV, where the $Z$ resonance is negligible, and at $\sqrt{s} = M_Z$, where the suppression of IFI due to the long life time of the $Z$ is maximal.

Let us start with a purely technical test with IFI off at $\sqrt{s} = 94.3$, 87.9 and 10 GeV, presented in Fig. [11]. In the LHS plot, all cross sections $\sigma(v_{\text{max}})$ are divided by the reference cross section from KKsem. All calculations are at simplistic exponentiated $O(\alpha^0)$ QED including ISR and FSR, but without IFI. Different types of Born matrix element, EEX or GPS are used. Very good agreement is seen, up to statistical error $\delta \sigma / \sigma \sim 3 \cdot 10^{-5}$. The agreement for $A_{FB}$ is also very good, essentially up to statistical error $\delta A_{FB} \sim 1 \cdot 10^{-5}$ at $\sqrt{s^\pm}$. The above equality of the KKFoam and KKsem results is very important, because it illustrates/proves the quality of the MC integrators – it should be kept in mind that for IFI off they integrate exactly the same 3-dimensional integrand. Even more significant is the agreement of the KKMC with two other programs for simplified EEX0 matrix element to within $\delta \sigma / \sigma \sim 1 \cdot 10^{-5}$ near $Z$ resonance. This is because for MC statistic of $2 \cdot 10^{10}$ events one may expect problems with rounding errors in the accumulation of the weights in the
The slightly bigger discrepancy beyond statistical error of $\delta A_{FB} \sim 3 \cdot 10^{-5}$ for 10 GeV is not yet statistically significant and not so important for our aims.

We conclude that the technical precision of the MC numerical integration in all three programs, KKsem, KKFoam and KKsem, is satisfactory for our needs. Moreover, the above test is also important due to the fact that the IFI effect is added in KKMC by reweighting MC events generated without IFI. Hence, the technical precision established for the non-IFI mode persists when IFI is switched on.

In Fig. 12, we continue baseline testing without IFI, now with $O(\alpha^2)$ exponentiated ISR and FSR. The relative differences $\delta \sigma / \sigma$ between KKMC and KKFoam versus KKsem are examined. It is done for the CEEX/GPS and EEX Born matrix element. The relative difference $\delta \sigma / \sigma \sim 3 \cdot 10^{-4}$ for KKMC confirms all older tests in Ref. [8], rated at the $\sim 1 \cdot 10^{-3}$ level. On the other hand, the differences in $A_{FB}$ between KKMC and KKFoam or KKsem are again of the order of the statistical error, which is $\sim 3 \cdot 10^{-5}$, except $\sqrt{s} = 10$ GeV, where it is slightly bigger.

The main result of the tests presented in Figs. 11 and 12 is that the basic technical precision (in the MC integration) of KKMC and KKFoam near the $Z$ resonance is generally better than $\delta A_{FB} \sim 3 \cdot 10^{-5}$. The implementation of QED photonic corrections for ISR and FSR (no IFI) up to $O(\alpha^2)$ was also tested at this precision level.

### 7.3 IFI contribution to $A_{FB}$ from KKMC and KKFoam

Let us now increase the level of sophistication by one important step – including IFI. This will be done first in the simpler case (A) for ISR, FSR and IFI at the level $O_{\text{exp}}(\alpha^0)$ with exponentiation, and next in the case (B) for exponentiated IFI at the level $O_{\text{exp}}(\alpha^1)$, accompanied by ISR and FSR up to exponentiated $O_{\text{exp}}(\alpha^2)$.

In case (A), results for $A_{FB}(v_{\text{max}})$ from KKMC and KKFoam are shown in Fig. 13 while in case (B), the results are shown in Fig. 14 for energies $\sqrt{s} = 87.9, 94.3, 10$ GeV in both cases. The absolute predictions for $A_{FB}$ from KKMC and KKFoam are seen in the LHS plots of the these figures. The differences in $A_{FB}$ due to switching on the IFI contribution are quite sizeable and rising quickly for $v_{\text{max}} \leq 0.06$, up to 5% for $v_{\text{max}} \leq 0.002$.

The IFI contribution to $A_{FB}$ is shown more clearly in the RHS plots of Figs. 13 and 14, where the absolute predictions for the IFI effect in $A_{FB}$ from KKMC and KKFoam are presented. The most important result is the one represented by the red curve (c) in the RHS figures in Fig. 14. It represents the difference between KKMC and KKFoam for the IFI contribution. This crucial difference is up to $\delta A_{FB} \sim 5 \cdot 10^{-4}$.

How can we understand the above result? In the case of Fig. 13 where both KKMC and KKFoam...
Figure 11: Technical test, $\mathcal{O}_{\exp}(\alpha^0)$ ISR+FSR without IFI at 94.3, 87.9 and 10 GeV.
Figure 12: Results with $O_{\exp}(\alpha^2)$ ISR+FSR without IFI at 94.3, 87.9 and 10 GeV.
Figure 13: Results with $O_{\text{exp}}(\alpha^0)$ ISR+FSR and IFI at 94.3 GeV, 87.9 GeV and 10 GeV.
Figure 14: Results with $O_{\exp}(\alpha^2)$ ISR+FSR and $O_{\exp}(\alpha^1)$ IFI at 94.3, 87.9 and 10 GeV.
KKFoam are at the same $\mathcal{O}_{\exp}(\alpha^0)$ level for ISR, FSR and IFI, with semi-soft resummation of IFI, the source of the difference is a different treatment of the matrix element far away from the infrared point $v_{\text{max}} = 0$. Remembering that the energy shift in the Z-resonance propagator is properly taken into account in the semi-soft approximation, the difference between KKMC and KKFoam (curve (c)) should be proportional to $v_{\text{max}}$ and should vanish for $v_{\text{max}} \to 0$. This is what we see in Fig. 13.

In the case of Fig. 14, the difference between KKMC and KKFoam should reflect the fact that in KKMC the entire $\mathcal{O}_{\exp}(\alpha^1)$ real and virtual contributions are included, while in KKFoam, the $\mathcal{O}_{\exp}(\alpha^1)$ real contribution is incomplete. This could increase the difference between KKMC and KKFoam. In fact it changes sign and increases the difference by at most a factor of 2. This can be seen as unexpected. In order to have an idea how big the $\mathcal{O}_{\exp}(\alpha^1)$ real photon IFI contribution can be, we have also included this contribution (curve (e)) in Fig. 14 subtracting the soft component in an ad-hoc manner. As we see, curve (e) typically has the same sign as the difference between KKMC and KKFoam shown in curve (c), but is a factor of 3 − 4 bigger. Apparently, KKFoam includes most of the $\mathcal{O}(\alpha^1)$ hard photon IFI contribution.

The inclusion of QED $\mathcal{O}_{\exp}(\alpha^1)$ virtual corrections and box diagrams was done in KKFoam following the prescription of eq. (5.14). The pure $\mathcal{O}(\alpha^1)$ numerical results in Fig. 14 were reproduced using analytical formulas of refs. [20,21], which are collected and tested numerically one more time in Appendix C.

In spite of the incompleteness of the $\mathcal{O}(\alpha^1)$ IFI in KKFoam, the above result makes us confident that we are quite close to reaching our first intermediate goal of controlling the IFI effect in $A_{FB}$ at the level of $\delta A_{FB} \sim 10^{-4}$ in the semi-soft resumation regime ($v_{\text{max}} \leq 0.06$).

Let us also finally show just one example of the entire angular distribution $d\sigma/d\cos\theta$ from KKFoam and KKMC, simply because agreement in $A_{FB}$ does not necessarily imply agreement in the angular distributions. In Fig. 15, such a comparison is done for a relatively mild cut-off $v_{\text{max}} = 0.02$ on the total photon energy. The angular distributions agree to within 0.005% as expected.

### 7.4 $A_{FB}(s_{\pm})$ from KKMC and KKFoam in presence of IFI.

As explained in Ref. [2], the QED coupling constant $\alpha_{\text{QED}}(M_Z)$ is closely related to $A_{FB}(s_{\pm})$, but the exact relation is not straightforward and we are not trying to reproduce it. We limit our interest to the propagation of errors from $A_{FB}(s_{\pm})$ to $\alpha_{\text{QED}}(M_Z)$, which is simpler and can be read from eq. (4.9) in Ref. [2]. For our purpose, it will be enough to use a simplified version of this equation,

$$\frac{\delta \alpha_{\text{QED}}}{\alpha_{\text{QED}}} \bigg|_{M_Z} \approx \frac{\delta A_{FB}(s_+)}{A_{FB}(s_+)} - \frac{\delta A_{FB}(s_-)}{A_{FB}(s_-)}. \quad (7.4)$$

---

21 The slight difference at $v_{\text{max}} \to 0$ for $\sqrt{s} = 10$ GeV can be traced to small spikes in the $0.99 < |\cos\theta| < 1$ range, to be examined separately. It goes away for realistic experimental cut-offs.

22 It would be interesting to include this missing $\mathcal{O}_{\exp}(\alpha^1)$ real photon IFI contribution in KKFoam.
which is valid for small $\delta A_{FB}(s_{\pm})$ and/or when there are no strong cancellations between them. This will be true in the following numerical examples, and we shall show typically the numerator $\delta A_{FB}(s_{+}) - \delta A_{FB}(s_{-})$ along with the uncertainties $\delta A_{FB}(s_{\pm})$.

Having the above in mind we reexamine the comparisons between KKMC and KKFoam of the previous section for this $\Delta A_{FB}$.

From now on, we impose a realistic cut-off $|\cos \theta| < 0.9$ in the tests; however the cut-off has little influence on the resulting $A_{FB}$. To start with, in Fig. 16 we show $A_{FB}(v_{\text{max}})$ from KKMC at $\sqrt{s_{\pm}}$ with IFI switched on/off and with the best QED matrix element in KKMC. The $A_{FB}$ changes sign between these two energies. On the other hand, IFI keeps the same sign, hence we expect partial cancellation of the IFI effect in the $\alpha_{\text{QED}}(M_{Z})$. We do not pursue the reconstruction of $\alpha_{\text{QED}}(M_{Z})$ and only plot the difference of IFI effect between two energies in the LHS of Fig. 16 as a guide.

We have produced the same figure for KKFoam, but we do not show it here, because it looks essentially the same as Fig. 16. What is more interesting is to reexamine the difference between KKMC and KKFoam

$$\delta A_{FB}(s_{\pm}) = A_{FB}(s_{\pm})\bigg|_{\text{KKMC}} - A_{FB}(s_{\pm})\bigg|_{\text{KKFoam}},$$

already shown in curve (c) of Fig. 14, and its difference between two energies $\sqrt{s_{\pm}}$

$$\Delta \delta A_{FB} = \delta A_{FB}(s_{+}) - \delta A_{FB}(s_{+})$$

relevant for the uncertainty in the measurement of $\alpha_{\text{QED}}(M_{Z})$. We are interested in the above quantity primarily for IFI switched on. This quantity is shown in Fig. 17, see curve

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23We thank P. Janot for pointing this out to us.
(c) there. It turns out that $\Delta \delta A_{FB} \leq 2 \cdot 10^{-4}$ within the interesting range of photon energy cut-off $v_{max} \leq 0.1$. In Fig. [17] we have also included two dashed lines marking the band of the present uncertainty $\delta \alpha_{QED}/\alpha_{QED}(M_Z) = 1.1 \cdot 10^{-4}$ according to ref. [22].

The aim of FCCee is of course to get substantially smaller error than that.

The main contribution to $\Delta \delta A_{FB}$ in curve (c) comes from the uncertainty in the IFI implementation (most likely in KKFoam), as can be seen from curve (d) in Fig. [17] which represents $\Delta \delta A_{FB}$ for IFI switched off. The aim of future work will be to get $\Delta \delta A_{FB} \leq 3 \cdot 10^{-5}$ for IFI on, that is to the same level as for IFI off, in the semi-soft regime $v_{max} \leq 0.06$.

The above $\Delta \delta A_{FB} \leq 2 \cdot 10^{-4}$ can be treated as an (over) conservative estimate of the uncertainty of the IFI prediction for KKMC in the semi-soft regime, which is much better than the LEP-era estimate but still not up to the needs of FCCee. A less conservative estimate will be provided in the next section.

### 7.5 On $A_{FB}$ for $O_{\exp}(\alpha^i), i = 0, 1, 2$ in KKMC

The differences between KKMC and KKFoam provide much valuable information, because the two programs differ quite a lot technically (MC soft photon phase space integration versus analytical integration), while implementing the same physics of QED corrections. However, KKMC alone offers interesting insight into missing higher order QED corrections related to IFI.

24Provided we trust the smallness of the technical precision error of KKMC.
Figure 17: Difference $\delta A_{FB}(v_{\text{max}})$ between $A_{FB}$ from KKMC and KKFoam and their difference between two energies $\sqrt{s}$ . IFI is switched on/off in both KKMC and KKFoam. The band marked with dashed line corresponds to the precision estimate of the $\alpha_{\text{QED}}(M_Z)$ of ref. [22].

In KKMC, one may choose three types of the QED multiphoton matrix element with resummation at increasing sophistication levels, $O_{\text{exp}}(\alpha^i)$, $i = 0, 1, 2$. In Fig. 18 we examine differences in the IFI contribution $A_{FB}^{\text{IFI}}(v_{\text{max}})$ between $O(\alpha^2)$ and $O(\alpha^1)$ and also between $O(\alpha^1)$ and $O(\alpha^0)$. In all of them, IFI may be switched on or off. Complete non-IR $O(\alpha^1)$ corrections are included in the $O(\alpha^i)$, $i = 1, 2$ case while in the $O(\alpha^0)$ case, only the IR part of exponentiated IFI is implemented. In the most sophisticated case of the $O_{\text{exp}}(\alpha^2)$ QED matrix element in KKMC, only pure non-log photonic corrections are missing.

In the LHS of Fig. 18, we show plots of the IFI component for all three cases $O_{\text{exp}}(\alpha^i)$, $i = 0, 1, 2$, while in the RHS we see the differences, for the two energy points $\sqrt{s}$. The most important difference in Fig. 18 between $A_{FB}$ for $O(\alpha^2)$ and $O(\alpha^0)$, is below the statistical error of $10^{-4}$. This can be treated as a measure of the missing QED photonic higher order corrections in the KKMC predictions for $A_{FB}$ for this particular type of experimental cut-offs, $v_{\text{max}} < 0.2$ and $|\cos \theta| < 0.9$, near $Z$ resonance, $|M_Z - \sqrt{s}| \leq 3.5$ GeV.

Finally, let us remark that the MC results for $A_{FB}$ presented here with a statistical precision of $10^{-4}$ were obtained using $\sim 10^{10}$ MC events generated in parallel runs on PC farms. Reducing the statistical error to $10^{-5}$ will be feasible, but not trivial. However, higher precision may be also feasible with less MC events using the technique of recording

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26 In particular, the non-IR parts of QED penta-boxes are missing; see Fig. 5 in Ref. [8].

27 Differences in Fig. 18 are obtained using MC weights event per event, so statistical errors are grossly overestimated. This explains the lack of fluctuations among bins.
7.6 More on the uncertainty of the ISR effect in $A_{FB}$.

In this section, we will present a few results from KKMC which in particular will give us more insight on the ISR effects in $A_{FB}$ when IFI is switched on and off.

In Fig. 19, we show differences between $O_{\text{exp}}(\alpha^i)$, $i = 1, 2$ results from KKMC with the CEEX matrix element in the case of IFI switched off – that is pure ISR and FSR effects. In fact, the ISR effect is dominant here. The variation is smaller than $3 \cdot 10^{-5}$ and cancels between the two energies $\sqrt{s_{\pm}}$.

The same phenomenon is seen in Fig. 20, albeit the differences are smaller, as expected. Note also that in both of the above cases, the effect of ISR is completely negligible for $v_{\text{max}} \leq 0.05$, that is for cutoffs on photon energy interesting experimentally!

Finally, we switch on IFI and examine again the differences between $O_{\text{exp}}(\alpha^i)$, $i = 1, 2$ results from KKMC with the CEEX matrix element in the case of IFI switched on. The results are shown in Fig. 21. This is the most interesting result, because it shows the indirect influence of ISR on the IFI contribution to $A_{FB}$. Curve (c) shows that for the difference in $A_{FB}$ between the two energies $\sqrt{s_{\pm}}$, the first and second order results agree to $\leq 2 \cdot 10^{-5}$. The disagreement is larger than was seen in the previous graph with IFI off in the semi-soft region $v_{\text{max}} \leq 0.06$.

\footnote{Such a cancellation of the ISR effect was already noticed in Ref. [2].}
One may conclude that the above result provides us strong indication that the QED uncertainty in $A_{FB}$ from KKMC is of the order of the expected FCC experimental error $\delta A_{FB} \simeq 3 \cdot 10^{-5}$. In the above plots, statistical MC errors are negligible, because all
Figure 21: Differences between $A_{FB}$ calculated using the CEEX matrix element $\mathcal{O}_{\exp}(\alpha^i)$, $i = 1, 2$ with IFI switched on.

differences between the various QED matrix elements are calculated using differences between the weights for the same sets of weighted MC events.
8 Summary and outlook

The extensive numerical results presented in this work allow us to conclude that the uncertainty of the prediction of KKMC for the IFI component of $A_{FB}$ near the $Z$ resonance is below $\sim 10^{-4}$. This is definitely better than the state of art in the LEP era, $\sim 10^{-3}$. Some of the results presented here indicate that the precision of $A_{FB}$ near the $Z$ resonance from KKMC is in fact at the level $\sim 3 \cdot 10^{-5}$, i.e. what is needed in the FCCee experiment proposed to measure the QED coupling constant at the scale $M_Z$ with this precision. This would allow $\alpha_{\text{QED}}(M_Z)$ to be determined to a precision significantly better than the present estimate of ref. [22], which is $\frac{\delta \alpha_{\text{QED}}}{\alpha_{\text{QED}}} \simeq 1.1 \cdot 10^{-4}$. However, more work is needed for better confidence in the technical precision and higher order photonic QED corrections in the KKMC results. More work is also needed to estimate other missing non-photonic QED corrections (e.g. pair emission) and electroweak corrections. Extension of the presented analysis to more realistic experimental selections (cuts) is also desirable.

The newly developed auxiliary MC program KKFoam was instrumental in the above achievement. For more precise tests of KKMC, it would be profitable to include in the phase space of KKFoam the exact contribution from non-soft real $O(\alpha^1)$ emission matched with semi-soft analytical resummation.

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APPENDIX

A  Factorizing the exponentiated formula

Starting from eq. (4.3), let us introduce \( \int d^4z \, \delta^4(z-y-x+x') = 1 \), \( \int d^4u \, \delta^4(u-y+x) = 1 \) and \( \int d^4u' \, \delta^4(u-y+x') = 1 \) in the Fourier expression, obtaining

\[
\sigma(s, E_{\text{max}}) = \frac{1}{\text{flux}(s)} \sum_{V,V'} \int \frac{d^3q_1}{q_1^0} \frac{d^3q_2}{q_2^0} \frac{d^4Qd^4x}{(2\pi)^4} \frac{d^4Q'd^4x'}{(2\pi)^4} e^{ix\cdot(P-Q)-ix'\cdot(P-Q')} \frac{d^4y}{(2\pi)^4} e^{iy\cdot(P-q_1-q_2)}
\times \int \frac{d^4Kd^4z}{(2\pi)^4} \frac{d^4Rd^4u}{(2\pi)^4} \frac{d^4R'd^4u'}{(2\pi)^4} e^{i(z-y-x+x'):K+i(u-y-x):R+i(u'-y-x'):R'}
\times \exp \left\{ \int \frac{d^3k}{k^0} \left[ e^{-ik\cdot(y+x)} S_I(k) + e^{-ik\cdot(y+x)} S_X(k) + e^{-ik\cdot(y+x')} S_X(k) + e^{-ik\cdot y} S_F(k) \right] \right\}
\times \exp \left\{ \alpha B_4^V(Q^2, t, \gamma) + \alpha(B_4^{V'}(Q'^2, t, \gamma))^* \right\} M_V(Q, t) M_{V'}^{*}(Q', t) \theta \left( \sum k_i^0 \leq E_{\text{max}} \right).
\]

(A.1)

The lowest-order spin amplitudes \( M_V, V = \gamma, Z \) are, up to a normalization constant, equal to the amplitudes \( M^{\nu}_{V}\) defined in Appendix C, but fermion helicities are temporarily suppressed.

Thanks to the above reorganization, we may clearly factorize the result into contributions due to the ISR, FSR, and IFI components of multiphoton emission:

\[
\sigma(s) = \frac{1}{\text{flux}(s)} \int \frac{d^3q_1}{q_1^0} \frac{d^3q_2}{q_2^0} \frac{d^4Qd^4x}{(2\pi)^4} \frac{d^4Q'd^4x'}{(2\pi)^4} e^{ix\cdot(P-Q)-ix'\cdot(P-Q')} \sum_{V,V'}
\times \int d^4Kd^4Rd^4R' e^i(-x+x'):K+i(-x):R+i(+x'):R'
\times \int \frac{d^4z}{(2\pi)^4} e^{iz\cdot K+j\frac{d^3k}{k^0} e^{-ik\cdot S_I(k)}} \int \frac{d^4u}{(2\pi)^4} e^{iu\cdot R+j\frac{d^3k}{k^0} e^{-ik\cdot S_X(k)}}
\times \int \frac{d^4u'}{(2\pi)^4} e^{iu'\cdot R'+j\frac{d^3k}{k^0} e^{-ik\cdot u' S_X(k)}} \int \frac{d^4y}{(2\pi)^4} e^{iy\cdot(P-q_1-q_2-K-R-R')} + j\frac{d^3k}{k^0} e^{-ik\cdot y S_F(k)}
\times \exp \left\{ \alpha B_4^V(Q^2, t, \gamma) + \alpha(B_4^{V'}(Q'^2, t, \gamma))^* \right\} M_V(Q) M_{V'}^{*}(Q')
\]

(A.2)

= \frac{1}{\text{flux}(s)} \sum_{V,V'} \int \frac{d^3q_1}{q_1^0} \frac{d^3q_2}{q_2^0} \frac{d^4Q}{(2\pi)^4} \frac{d^4Q'}{(2\pi)^4} e^{iz\cdot K+j\frac{d^3k}{k^0} e^{-ik\cdot S_I(k)}} \int \frac{d^4u}{(2\pi)^4} e^{iu\cdot R+j\frac{d^3k}{k^0} e^{-ik\cdot S_X(k)}}
\times \int \frac{d^4u'}{(2\pi)^4} e^{iu'\cdot R'+j\frac{d^3k}{k^0} e^{-ik\cdot u' S_X(k)}} \int \frac{d^4y}{(2\pi)^4} e^{iy\cdot(P-q_1-q_2-K-R-R')} + j\frac{d^3k}{k^0} e^{-ik\cdot y S_F(k)}
\times \exp \left\{ \alpha B_4^V(Q^2, t, \gamma) + \alpha(B_4^{V'}(Q'^2, t, \gamma))^* \right\} M_V(Q) M_{V'}^{*}(Q').

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The regularized radiator distribution for the IFI component in the semi-soft photon analytical exponentiation

\[ \rho(\gamma, v) = F(\gamma)(\delta(v) \varepsilon^\gamma + \theta(v - \varepsilon) \gamma v^{\gamma - 1}), \quad \int_0^1 dv \rho(\gamma, v) = F(\gamma) \equiv F_\gamma, \]  

(B.1)
is valid for both positive and negative \( \gamma \). The regulator \( \varepsilon \) should be smaller than any scale dependence in the Born cross section times the target precision of the calculation. In our case it should be below \( \Gamma_Z/M_Z \) by a factor of at least \( 10^{-4} \), i.e. \( \varepsilon < 10^{-5} \) is recommended.

The distribution for FOAM should be positive in the exploration phase, hence

\[ \tilde{\rho}(\gamma, v) = |\rho(\gamma, v)| = F_\gamma [\delta(v) \varepsilon^\gamma + \theta(v - \varepsilon) |\gamma| v^{\gamma - 1}] \]  

(B.2)
is used. The mapping from \( v \) to the internal variable \( r \in (0, 1) \) of FOAM is chosen such that its Jacobian compensates exactly \( \tilde{\rho}(v) \). More precisely, \( v(r) \) is the solution of the equation

\[ r \int_0^1 dv' \tilde{\rho}(\gamma, v') = \int_0^v dv' \tilde{\rho}(\gamma, v') = F_\gamma R(v). \]  

(B.3)

Note that for \( \gamma > 0 \) we have \( R(1) = 1 \), while for \( \gamma < 0 \) we get \( R(1) = 2e^\gamma - 1 > 1 \). Differentiating eq. (B.3) we get \( F_\gamma R(1) dr = \tilde{\rho}(\gamma, v) dv \), hence the Jacobian is

\[ J(v) = |dv/dr| = F_\gamma R(1)(\tilde{\rho}(\gamma, v))^{-1}. \]  

For \( \gamma > 0 \), the mapping (with \( R(1) = 1 \) and \( R(\varepsilon) = \varepsilon^\gamma \)) is simply

\begin{align*}
v(r) &= 0, \quad \text{for} \quad r < R(\varepsilon) = \varepsilon^\gamma, \\
v(r) &= r^{1/\gamma}, \quad \text{for} \quad r > R(\varepsilon).
\end{align*}

(B.4)
The corresponding Jacobian is

\[ J(v) = 1/R(\varepsilon) = \varepsilon^{-\gamma} \quad \text{for} \quad v = 0 \quad \text{and} \quad J(v) = F_\gamma (\tilde{\rho}(\gamma, v))^{-1} = \frac{v}{r^{1/\gamma}} \quad \text{for} \quad v > \varepsilon. \]  

(B.5)

For \( \gamma < 0 \) the mapping (with \( R(1) = 2\varepsilon^\gamma - 1 \) and \( R(\varepsilon) = \varepsilon^\gamma \)) is more complicated:

\begin{align*}
v(r) &= 0, \quad \text{for} \quad r < R(\varepsilon) = \frac{\varepsilon^\gamma}{2\varepsilon^\gamma - 1}, \\
v(r) &= \left[2R(\varepsilon) - r R(1)\right]^{1/\gamma} = \left[2\varepsilon^\gamma - r(2\varepsilon^\gamma - 1)\right]^{1/\gamma} \quad \text{for} \quad r > \frac{R(\varepsilon)}{R(1)}.
\end{align*}

(B.6)
The corresponding Jacobian reads

\[ J(v) = \frac{R(1)}{R(\varepsilon)} = \frac{2\varepsilon^\gamma - 1}{\varepsilon^\gamma} \quad \text{for} \quad v = 0 \quad \text{and} \quad J(v) = \frac{F_\gamma R(1)}{\tilde{\rho}(\gamma, v)} \quad \text{for} \quad v > \varepsilon. \]  

(B.7)

\[ ^{29} \text{In the actual MC runs, we use } \varepsilon = 10^{-6}. \]
In the second simulation stage, FOAM generates weighted MC events with the IFI component being \( w = J(v)\rho(\gamma, v) \). In the case of \( \gamma > 0 \), the weight (component) in FOAM will be \( w = 1 \) for any \( v \), while for \( \gamma < 0 \) it will be \( w = R(1) \) for \( v = 0 \) and \( w = -R(1) \) for \( v > \varepsilon \).

In addition, special care has to be taken in the case \( \gamma \to 0 \), that is for \( \cos \theta \simeq 0 \), because in this region the above mappings can be numerically unstable due to the limited range of the exponent in floating-point arithmetic. Because of that, when \( |\gamma \ln \varepsilon| < \Delta \ll 1 \), many of the above formulas have to be expanded accordingly\(^{30}\).

For \( |\gamma \ln \varepsilon| < \Delta \ll 1 \) and \( \gamma > 0 \), the expanded distribution, mapping, and Jacobian read:

\[
\tilde{\rho}(v) = \rho(v) = F_\gamma \left[ \delta(v)(1 + \gamma \ln \varepsilon) + \theta(v > \varepsilon) \frac{\gamma}{v} \right], \quad R(\varepsilon) = (1 + \gamma \ln \varepsilon), \quad R(1) = 1,
\]

\[
v(r) = 0 \quad \text{for} \quad r < R(\varepsilon), \quad v(r) = \exp \left[ -\frac{1}{\gamma}(1 - r) \right] \quad \text{for} \quad r > R(\varepsilon),
\]

\[
J(v) = 1/R(\varepsilon) \quad \text{for} \quad v = 0 \quad \text{and} \quad J(v) = F_\gamma (\tilde{\rho}(\gamma, v))^{-1} \quad \text{for} \quad v > \varepsilon.
\]  

(B.8)

For \( \gamma < 0 \), the expanded expressions with \( R(\varepsilon) = 1 + \gamma \ln \varepsilon, \quad R(1) = 1 + 2\gamma \ln \varepsilon > 1 \), read:

\[
\tilde{\rho}(v) = |\rho(v)| = F_\gamma \left[ \delta(v)(1 + \gamma \ln \varepsilon) - \theta(v > \varepsilon) \frac{\gamma}{v} \right],
\]

\[
v(r) = 0, \quad \text{for} \quad r < \frac{R(\varepsilon)}{R(1)} = \frac{1 + \gamma \ln \varepsilon}{1 + 2\gamma \ln \varepsilon},
\]

\[
v(r) = \exp \left[ \frac{1}{\gamma}(1 - r)R(1) \right], \quad \text{for} \quad r > \frac{R(\varepsilon)}{R(1)},
\]

\[
J(v) = \frac{R(1)}{R(\varepsilon)} = \text{for} \quad v = 0 \quad \text{and} \quad J(v) = \frac{F_\gamma R(1)}{\tilde{\rho}(\gamma, v)} \quad \text{for} \quad v > \varepsilon.
\]  

(B.9)

C Zero and first order amplitudes without resummation

For constructing the semi-soft photon analytical resummation and matching with the fixed-order \( \mathcal{O}(\alpha^1) \) result, we need the zeroth and first order amplitudes and distributions in analytical form. In particular, we will need the differential cross section of the final muons, integrated over photon angles, but keeping control over the photon energy. The relevant results are scattered over several papers \(^{20,21,23}\). See also Refs. \(^{24,26}\), where they are sometimes incomplete, or given in a form not suitable for our purposes; hence it is worth collecting them once more in this appendix.

Following the notation of Ref. \(^{21}\), the Born cross section and charge asymmetry read

\(^{30}\) The value \( \Delta = 10^{-4} \) used now looks OK, as the error of \( \sim \Delta^2 = 10^{-8} \) is more than acceptable.
as follows:

\[
\frac{d\sigma^{(0)}(s(1 - v))}{dc} = 3\sigma_0(s) \frac{1}{8} \sum_{\varepsilon, \tau = \pm} |\mathcal{M}_{\varepsilon\tau}(v, c)|^2 = 3\sigma_0(s) \frac{1}{8} [(1 + c^2) \mathcal{D}(v) + 2c\mathcal{D}(v)],
\]

\[
\mathcal{M}_{\varepsilon\tau}(v, c) = \mathcal{M}_{\varepsilon\tau}^0(v, c) + \mathcal{M}_{\varepsilon\tau}^Z(v, c) = (\varepsilon\tau + c)D_{\varepsilon\tau}(v),
\]

\[
D_{\varepsilon\tau}(v) = D^\gamma_{\varepsilon\tau}(v) + D^Z_{\varepsilon\tau}(v) = \frac{q\tilde{q}}{1 - v} + \frac{g_\varepsilon\tilde{g}_\tau}{\zeta - v},
\]

\[
\zeta = \frac{s - M_Z^2 + i\Gamma Z M_Z}{s}, \quad g_\tau = g_\nu + \tau g_A, \quad \tilde{g}_\tau = \tilde{g}_\nu + \tau\tilde{g}_A, \quad \sigma_0 = \frac{4\alpha\pi^2}{3s},
\]

where \(c = \cos \theta\), \(q = Q_e\), \(\tilde{q} = Q_\mu\) are electric charges, \(\varepsilon, \tau = \pm\) are twice the helicity of \(e^-\) and \(\mu^+\), and

\[
\mathcal{D}(v) = \frac{1}{4} \sum_{\varepsilon\tau} |D_{\varepsilon\tau}(v)|^2 = \frac{c_0}{(1 - v)^2} + \Re \frac{2c_1}{(1 - v)(\zeta - v)} + \frac{c_2}{|\zeta - v|^2},
\]

\[
\mathcal{D}(v) = \frac{1}{4} \sum_{\varepsilon\tau} \varepsilon\tau |D_{\varepsilon\tau}(v)|^2 = \Re \frac{2d_1}{(1 - v)(\zeta - v)} + \frac{d_2}{|\zeta - v|^2},
\]

\[
c_0 = (q\tilde{q})^2, \quad c_1 = q\tilde{q}g_\nu\tilde{g}_\nu, \quad c_2 = (g_\nu^2 + g_\nu^2)(\tilde{g}_\nu^2 + \tilde{g}_\nu^2),
\]

\[
d_1 = q\tilde{q}g_\alpha\tilde{g}_\alpha, \quad d_2 = 4g_\nu g_\nu\tilde{g}_\nu\tilde{g}_\nu.
\]

The integration over \(\cos \theta\) results in

\[
\sigma^{(0)} = \sigma_0 \frac{1}{4} \sum_{\varepsilon\tau} |D_{\varepsilon\tau}(0)|^2, \quad \sigma^{(\Delta)} = \frac{3}{4} \int 2\cos \theta^\Delta d\sigma^{(1)} = \sigma_0 \frac{1}{4} \sum_{\varepsilon\tau} \varepsilon\tau |D_{\varepsilon\tau}(0)|^2
\]

\[
A^{(0)}_{FB} = \frac{3}{4} (2\cos \theta^\Delta)^{(0)} = \frac{3}{4} \frac{\int 2\cos \theta^\Delta d\sigma^{(1)}}{\sigma^{(0)}} = \frac{3}{4} \frac{\sum_{\varepsilon\tau} \varepsilon\tau |D_{\varepsilon\tau}(0)|^2}{\sum_{\varepsilon\tau} |D_{\varepsilon\tau}(0)|^2} = \frac{3}{4} \frac{\mathcal{D}(0)}{\mathcal{D}(0)}.
\]

Following the notation of Ref. [21], the non-interference \(\mathcal{O}(\alpha^4)\) results with implicit integration over photon angles and explicit integration over photon energy up to \(x = v_{\text{max}}\) read:

\[
\overline{A}_{FB}^{(1)}(x) = \frac{3}{4} \frac{\sigma^{(1)}(x)}{\sigma^{(1)}(x)}, \quad \overline{\sigma}^{(1)}(x) = \int_{v < x} 2\cos \theta^\Delta d\sigma^{(1)},
\]

\[
\overline{\sigma}^{(1)}(x) = [1 + F(x)] \mathcal{D}(0) + W(x), \quad \overline{\sigma}^{(1)}(x) = [1 + F^\Delta(x)] \mathcal{D}(0) + W^\Delta(x),
\]

\[
W(x) = \int_0^x dv \left[ \gamma_I(s) P(v) + q^2\frac{\alpha}{\pi} \Delta_s \delta(v) \right] (1 - v) \mathcal{D}(v),
\]

\[
W^\Delta(x) = \int_0^x dv \left[ \gamma_I(s) P(v) + q^2\frac{\alpha}{\pi} \Delta_s \delta(v) - q^2\frac{\alpha}{\pi} v \right] (1 - v) \mathcal{D}(v),
\]
\[
F(x) = \int_0^x dv \left[ \gamma_F(s(1 - v))P(v) + \frac{\alpha}{\pi} \Delta_s \delta(v) \right], \quad \Delta_s = -\frac{1}{2} + \frac{\pi^2}{3},
\]
\[
F^\Delta(x) = \int_0^x dv \left[ \gamma_F(s(1 - v))P(v) + \frac{\alpha}{\pi} \Delta_s \delta(v) - \frac{\alpha}{\pi} \right],
\]
\[
P(v) = \left(1 + (1 - v)^2\right)_{+} = -\delta(v) \frac{3}{4} \ln \frac{1}{\varepsilon} + \theta(v - \varepsilon) \frac{1 + (1 - v)^2}{v},
\]

In Ref. [21], analytical integrations over \(v\) were done, but for the purpose of the present resummation, we are more interested in the above unintegrated version.

In Ref. [20], the contribution of IFI was added to the above charge asymmetry, but in a version that was integrated over \(v\). The unintegrated version\(^{31}\) including ISR+FSR+IFI with complete \(O(\alpha^3)\) for \(v \in (0, 1)\), needed for resummation is as follows:

\[
A^{(1)}_{FB}(x) = 3 \sigma^{(1)}_{\Delta}(x) = 3 \int_{v < x} d\sigma^{(1)} \cos \theta \cos \delta,
\]
\[
\sigma^{(1)}(x) = \sigma^{(1)}(x) + \sigma_0 U(x), \quad \sigma^{(1)}_{\Delta}(x) = \sigma^{(1)}_{\Delta}(x) + \sigma_0 U_{\Delta}(x),
\]
\[
U(x) = \int_0^x dv \rho^{(1)}_X(v) (1 - v) \mathfrak{D}(v, 0) + 3q\frac{\alpha}{\pi} \text{Re} \left\{ A_\gamma \mathfrak{F}^\gamma(0) + A_Z \mathfrak{F}^Z(0) \right\},
\]
\[
U_{\Delta}(x) = \int_0^x dv \rho^{(1)}_{\Delta X}(v) (1 - v) \mathfrak{D}(v, 0) + 2q\frac{\alpha}{\pi} \text{Re} \left\{ A_{\gamma}^\Delta \mathfrak{F}^\gamma(0) + A_{Z}^\Delta \mathfrak{F}^Z(0) \right\},
\]
\[
\rho^{(1)}_X(v) = 2q\frac{\alpha}{\pi} \left\{ \delta(v) \left[ 3 \ln \frac{1}{\delta} + \theta(v - \delta)(-3) \frac{2 - v}{2v} \right] \right\}
\]
\[
\rho^{(1)}_{\Delta X}(v) = 2q\frac{\alpha}{\pi} \left\{ \delta(v) \left[ 5 \ln \frac{1}{\delta} + \theta(v - \delta)(-1) \frac{10(1 - v) + 3v^2}{2(2 - v)v} \right] \right\}.
\]

The combined contributions to the total cross section from real soft emission (interference part) and virtual \(\gamma\gamma\) and \(\gamma Z\) boxes can be deduced from the \(k_{\text{max}} \to 0\) limit of formulas in Ref. [23]:

\[
A_\gamma = -\frac{1}{2}, \quad A_Z = -\ln |1 - | - \zeta + (1 - \zeta)(2 - \zeta) \ln \frac{-\zeta}{1 - \zeta}.
\]
\[
\mathfrak{F}^\gamma(0) = c_0 + \frac{c_1}{\zeta}, \quad \mathfrak{F}^Z(0) = c_1 + \frac{c_2}{\zeta \zeta^*},
\]

The analogous contributions to \(\sigma^{(1)}_{\Delta}\) can be obtained from formulas in Ref. [20]:

\[
A^{\Delta}_{\gamma} = 65 \frac{36}{36} - \frac{2}{3} \pi,
\]
\[
A^{\Delta}_Z = \frac{31}{2} + 9\zeta^2 + 4\zeta^3 - \ln(1 - \zeta) \left( \frac{15}{2} - 14\zeta + 12\zeta^2 - 4\zeta^3 \right)
\]
\[
+ \ln(-\zeta) \left( 5 - \frac{17}{2} \zeta + 2\zeta^2 \right) + 4\zeta(1 - \zeta)^3 \left( \text{Li}_2 \left( \frac{-\zeta}{1 - \zeta} \right) - \frac{\pi^2}{6} \right).
\]
\[
\mathfrak{F}^{\Delta \gamma}(0) = \frac{d_1}{\zeta}, \quad \mathfrak{F}^{\Delta Z}(0) = \frac{d_1}{\zeta} + \frac{d_2}{\zeta \zeta^*}.
\]

\(^{31}\)The unintegrated version of \(U_{\Delta \gamma}(x)\) was obviously used in Ref. [20], but was not explicitly shown there. Also, \(U_{\Delta \gamma}(x)\) was not provided there.
The following combinations of the Born amplitudes are involved:

\[
\mathcal{D}(v, u) = \mathcal{R} \frac{1}{4} \sum_{\varepsilon \tau} (D_{\varepsilon, \tau}(v)^{*} D_{\varepsilon, \tau}(u)) = \\
= \mathcal{R} \left\{ \frac{c_0}{(1-v)(1-u)} + \frac{c_1}{(1-v)(\zeta^*-u)} + \frac{c_1}{(\zeta-v)(1-u)} + \frac{c_2}{(\zeta-v)(\zeta^*-u)} \right\} \\
\mathcal{D}(v, u) = \mathcal{R} \frac{1}{4} \sum_{\varepsilon \tau} \varepsilon \tau (D_{\varepsilon, \tau}(v)^{*} D_{\varepsilon, \tau}(u)) = \\
= \mathcal{R} \left\{ \frac{d_1}{(1-v)(\zeta^*-u)} + \frac{d_1}{(\zeta-v)(1-u)} + \frac{d_2}{(\zeta-v)(\zeta^*-u)} \right\}, \\
\mathcal{B}^\gamma(0) = \frac{1}{4} \sum_{\varepsilon \tau} D_{\varepsilon, \tau}(0)^{*} D_{\varepsilon, \tau}(0), \quad \mathcal{B}^Z(0) = \frac{1}{4} \sum_{\varepsilon \tau} \varepsilon \tau D_{\varepsilon, \tau}^\gamma(0)^{*} D_{\varepsilon, \tau}(0), \quad V = \gamma, Z;
\]

Let us remark that the following relations hold:

\[
\mathcal{D}(v) = \mathcal{D}(v, v), \quad \mathcal{D}(v) = \mathcal{D}(v, v), \\
\mathcal{R} \mathcal{B}^\gamma(0) + \mathcal{R} \mathcal{B}^Z(0) = \mathcal{D}(0), \quad \mathcal{R} \mathcal{B}(0)^\gamma + \mathcal{R} \mathcal{B}(0)^Z = \mathcal{D}(0),
\]

We also need the virtual box and real soft contributions before integration over \(c = \cos \theta\). Spin amplitudes for two \(\gamma\gamma\) box diagram and two \(\gamma Z\) box diagram contributions, normalized the same way as the Born spin amplitudes, read as follows:

\[
\mathcal{M}^{(\gamma\gamma)}_{\varepsilon \tau} = (q^\dagger q)^2 (\varepsilon \tau X_1(c) + X_2(c)), \\
\mathcal{M}^{(\gamma Z)}_{\varepsilon \tau} = q^\dagger q \varepsilon \tau (\varepsilon \tau Z_1(c) + Z_2(c)).
\]

Their interference with Born amplitudes leads to the following contributions:\footnote{We use \((1/4) \sum_{\varepsilon \tau} \varepsilon \tau q^\dagger q \varepsilon \tau = d_1 = q^\dagger q \varepsilon \tau g_a \bar{g}_a \) and \((1/4) \sum_{\varepsilon \tau} \varepsilon \tau q^\dagger q \varepsilon \tau = c_1 = q^\dagger q \varepsilon \tau g_a \bar{g}_a\)}

\[
d\sigma^{\gamma\gamma}_{\varepsilon \tau} \frac{dc}{dc} = \frac{3\sigma_0}{8} \frac{1}{4} \sum_{\varepsilon \tau} 2\mathcal{R} [\mathcal{M}^{(\gamma\gamma)}_{\varepsilon \tau} \mathcal{M}^*_{\varepsilon \tau}(0, c)] = \\
\frac{3\sigma_0}{8} \frac{1}{4} \sum_{\varepsilon \tau} 2\mathcal{R} \left\{ (q^\dagger q)^2 [X_1(c) + c X_2(c) + \varepsilon \tau (c X_1 + X_2(c))] \right\} D^*_{\varepsilon \tau}(0) = \\
\frac{3\sigma_0}{8} q^\dagger q 2\mathcal{R} \left\{ (c_0 + \frac{c_1}{\zeta^*})[X_1(c) + c X_2(c)] + \frac{d_1}{\zeta^*}[c X_1(c) + X_2(c)] \right\} = \\
\frac{3\sigma_0}{8} q^\dagger q 2\mathcal{R} \left\{ (c_0 + \frac{c_1}{\zeta^*})[F^{\gamma\gamma}(c) - F^{\gamma\gamma}(-c)] + \frac{d_1}{\zeta^*}[F^{\gamma\gamma}(c) + F^{\gamma\gamma}(-c)] \right\},
\]

where

\[
X_1(c) + c X_2(c) = F^{\gamma\gamma}(c) - F^{\gamma\gamma}(-c), \\
c X_1(c) + X_2(c) = F^{\gamma\gamma}(c) + F^{\gamma\gamma}(-c), \quad c_\pm = \frac{1 \pm c}{2}, \\
F^{\gamma\gamma}(c) = 2 \frac{\alpha}{\pi} \left\{ 2 \left( \ln \frac{m^2}{s} + i \pi \right) c_+^2 \ln \frac{c_-}{c_+} - \frac{1}{2} c \left( \ln^2 c_- + 2i \pi \right) + c_+ \left( \ln c_- + i \pi \right) \right\}.
\]
Similarly, for the $\gamma Z$ box, we have:
\[
\frac{d\sigma^{\gamma Z}}{dc} = \frac{3\sigma_0}{8} \frac{1}{4} \sum_{\tau} 2\Re\{ \mathcal{M}_{\tau \gamma}^{\gamma Z} \mathcal{M}_{\tau \gamma}^{\gamma Z}(0, c) \}
\]
\[
= \frac{3\sigma_0}{8} \frac{1}{4} \sum_{\tau} 2\Re\{ q\tilde{q}[Z_1(c) + cZ_2(c) + \varepsilon \tau(cZ_1 + Z_2(c))]} D_{\tau \gamma}^{\gamma Z}(0) \}
\]
\[
= \frac{3\sigma_0}{8} q\tilde{q} 2\Re\{ (c_1 + c_2 \tilde{c})[Z_1(c) + cZ_2(c)] + (d_1 + d_2 \tilde{c})[cZ_1(c) + Z_2(c)]\}
\]
\[
= \frac{3\sigma_0}{8} q\tilde{q} 2\Re\{ (c_1 + c_2 \tilde{c})[F^{\gamma Z}(c) - F^{\gamma Z}(-c)] + (d_1 + d_2 \tilde{c})[F^{\gamma Z}(c) + F^{\gamma Z}(-c)]\}
\]
where $F^{\gamma Z}(c)$ is related in a simple way to $f(s, t, u)$ of Ref. [27]:
\[
F^{\gamma Z}(c) = 2\alpha \frac{\pi^2}{c^2} sf(s, t, u), \quad F^{\gamma Z}(-c) = 2\frac{\alpha}{\pi} c^2 sf(s, u, t).
\]

In the KKMC code, the $\gamma Z$ box of Ref. [27] is programmed as follows:
\[
F^{\gamma Z}(c) = \ln \frac{t}{u} \ln \frac{m_{\gamma}^2}{(tu)^{1/2}} - 2\ln \frac{t}{u} \ln \frac{M^2 - s}{M^2} + \text{Li}_2\left(\frac{M^2 + u}{M^2}\right) - \text{Li}_2\left(\frac{M^2 + t}{M^2}\right)
\]
\[
+ \frac{(M^2 - s)(u - t - M^2)}{u^2} \left( \ln \frac{t}{s} \ln \frac{M^2 - s}{M^2} + \text{Li}_2\left(\frac{M^2 + t}{M^2}\right) - \text{Li}_2\left(\frac{M^2 - s}{M^2}\right) \right) \quad (C.14)
\]
\[
+ \frac{(M^2 - s)^2}{us} \ln \frac{M^2 - s}{M^2} + \frac{(M^2 - s)^2}{u} \ln \frac{t}{M^2},
\]
where $M^2 = M_{\gamma}^2 - M_Z^2 \Gamma Z$, $t = -(1 - c)s$ and $u = (1 + c)s$.

Finally, the above box contributions have to be combined with (interference) the corresponding soft real emission contribution
\[
\frac{d\sigma^{\text{soft}}}{dc} = \frac{d\sigma^{(0)}}{dc} q\tilde{q} \frac{\alpha}{\pi} \delta^{\text{soft}}(c),
\]
\[
\delta^{\text{soft}}(c) = 4 \ln \frac{c_+}{c_-} \ln \frac{s^{1/2}}{m_{\gamma}} + \ln^2 c_+ - \ln^2 c_- + 2\text{Li}_2(c_+) - 2\text{Li}_2(c_-)
\]
such that the usual cancellation of the IR regulator $m_{\gamma}$ occurs, leaving out the IR cut-off on photon energy $v \leq \epsilon \ll 1$.

Let us finally define explicit relations between integrated and unintegrated virtual+soft contributions:
\[
3A_{\gamma} = \int dc \left( F^{\gamma\gamma}(c) - F^{\gamma\gamma}(-c) + \delta^{\text{soft}}(c) \right),
\]
\[
2A_{\gamma}^{\Delta} = \int 2dc \left( F^{\gamma\gamma}(c) + F^{\gamma\gamma}(-c) + \delta^{\text{soft}}(c) \right),
\]
\[
3A_{Z} = \int dc \left( F^{\gamma Z}(c) - F^{\gamma Z}(-c) + \delta^{\text{soft}}(c) \right),
\]
\[
2A_{Z}^{\Delta} = \int 2dc \left( F^{\gamma Z}(c) + F^{\gamma Z}(-c) + \delta^{\text{soft}}(c) \right),
\]
\[C.16\]
Finally, in Figs. 22 and 23 we crosscheck the old analytical results with KKFoam, in which the integration over $\cos \theta$ (virtual) and over photon energy $v$ (real photon) is done numerically. As we see, there is perfect agreement between old analytical formulas and new results using KKFoam.

Figure 22: $O(\alpha^1)$ from old papers and KKFoam.

Figure 23: $O(\alpha^1)$ from old papers and KKFoam.
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