A q-Deformation of the Harmonic Oscillator

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Abstract

The q-deformed harmonic oscillator is studied in the light of q-deformed phase space variables. This allows a formulation of the corresponding Hamiltonian in terms of the ordinary canonical variables $x$ and $p$. The spectrum shows unexpected features such as degeneracy and an additional part that cannot be reached from the ground state by creation operators. The eigenfunctions show lattice structure, as expected.

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Introduction

In a previous paper [1] we have introduced the q-deformed phase space and we
have seen that it can be embedded in ordinary phase space. Thus the dynamics
of a Hamiltonian expressed in terms of q-deformed phase space variables can be
interpreted as the dynamics of a quantum mechanical Hamiltonian with compli-
cated interactions but in the standard Hilbert space of quantum mechanics.
A. Macfarlane and L. Biedenharn [2, 3] have studied the q-deformed harmonic
oscillator based on an algebra of q-deformed creation and annihilation operators.
They have found the spectrum and eigenvalues of such a harmonic oscillator under
the assumption that there is a state with a lowest energy eigenvalue.
In this paper we are going to express the q-deformed creation and annihilation
operators in terms of q-deformed phase space variables and we obtain this way
an ordinary Hamiltonian that can be diagonalized due to our knowledge of the
q-deformed system. The spectrum, however, has a richer and more complicated
structure than one would have expected. There is the part with a lowest eigenvalue
belonging to the groundstate of the system:

\[ E_m = \frac{1 - q^{-2m}}{1 - q^{-2}} \quad m = 0, 1, \ldots, \infty \]

These eigenvalues are bounded from above by

\[ E_{[\infty]} = \frac{\omega}{1 - q^{-2}} \]

Above \( E_{[\infty]} \) there is an unbounded spectrum which has no lowest eigenvalue:

\[ E_n = \omega \left( \frac{1}{1 - q^{-2}} + \beta \bar{\beta} q^{2n} \right) \quad n = -\infty, \ldots, \infty \]

\( \beta \) is related to the mass and the frequency of the undeformed oscillator. The
spectrum is twofold degenerate. In the limit \( q \to 1 \) the unbounded part of the
spectrum disappears. The degeneracy of the bounded part is lifted as the support
of the eigenfunctions is shifted away for half of the eigenstates, the other half
tends to the eigenfunctions of the ordinary oscillator. This will be explained in
detail in this paper.
The model shows a general feature of q-deformed dynamical systems. In the
neighbourhood (\( q \neq 1 \)) of ordinary quantum mechanical systems we can expect
a class of Hamiltonians that can be diagonalized if the ordinary Hamiltonian can be
diagonalized. The parameter \( h \) of \( q = e^h \) can be interpreted as an interaction
constant, however, the quantum mechanical problem is solved non-perturbatively.
The interaction produces a richer spectrum and the eigenfunctions exhibit lattice-
like structures.

In chapter 1 we define the model. In chapter 2 and 3 we diagonalize the Hamil-
tonian and find its eigenfunctions for the bounded and unbounded part of the
spectrum. In chapter 4 we relate the model to a quantum mechanical model and show that it is a q-deformation of the undeformed harmonic oscillator. In chapter 5 we study the transition of the wave functions to those of the ordinary harmonic oscillator in the limit \( q \to 1 \). In chapter 6 we show that q-deformed Hermite polynomials arise in a natural way.

1 The model

The q-deformed harmonic oscillator has first been studied by A. Macfarlane \[2\] and L. Biedenharn \[3\]. They introduced the algebra of q-deformed creation and annihilation operators and they computed the spectrum of the corresponding Hamiltonian, \( H = \omega a^+ a \), assuming the existence of a ground state.

We shall express the creation and annihilation operators in terms of q-deformed phase space variables \[1\]:

\[
q^{\frac{1}{2}} X P - q^{-\frac{1}{2}} P X = i U
\]

\[
U X = q^{-1} X U, \quad U P = q P U
\]

We assume \( q \) to be real and \( q > 1 \).

The algebra (1.1) allows \( X \) and \( P \) to be hermitean and \( U \) unitary:

\[
X^+ = X, \quad P^+ = P, \quad U^+ = U^{-1}
\]

The operators

\[
a = \alpha U^{-2M} + \beta U^{-M} P
\]

\[
a^+ = \bar{\alpha} U^{2M} + \bar{\beta} P U^M
\]

with \( M \in \mathbb{N}, \alpha, \beta \in \mathbb{C} \), satisfy the algebra

\[
aa^+ - q^{-2M} a^+ a = (1 - q^{-2M}) \alpha \bar{\alpha} = 1
\]

The right hand side can be normalized to 1 for \( M > 0, q > 1 \). This determines \( \alpha \) up to a phase. In chapter 4 we shall relate \( \beta \) to the mass and the frequency of the undeformed harmonic oscillator and we shall see that \( \alpha \bar{\beta} = \bar{\alpha} \beta \).

The operators \( a \) and \( a^+ \) play the role of annihilation and creation operators (for \( M < 0 \), \( a \) and \( a^+ \) would change role and we would deal with the same algebra).

A more general expression for \( a \) and \( a^+ \) in terms of \( P, X \) and \( U \) has been studied in ref. \[4\] and by J. Seifert \[5\].

From (1.4) follows that \( a \) lowers an energy eigenvalue \( E \) of the Hamiltonian \( H = \omega a^+ a \) as long as

\[
E < \frac{\omega}{1 - q^{-2M}} \equiv E_{[\infty]}
\]
For $E > E_{[\infty]}$, $a$ and $a^+$ change their role, $a$ raises and $a^+$ lowers the energy. Starting from an arbitrary eigenvalue $E_0 > E_{[\infty]}$, we find a spectrum

$$E_m = \omega \left(q^{2mM}E_0 + \frac{1 - q^{2mM}}{1 - q^{-2M}}\right)$$

$$m = -\infty, \ldots, +\infty$$

and

$$|E_{m-1}\rangle = \sqrt{\frac{\omega}{E_{m-1}}} a^+ |E_m\rangle$$

$$|E_{m+1}\rangle = \sqrt{\frac{\omega}{E_m}} a |E_m\rangle$$

For $m \to -\infty$, the eigenvalue approaches $E_{[\infty]}$, independent of $E_0$. This leads to a representation that does not have a "lowest weight" state. Such representations have been studied in ref. [6] as well.

For $E = E_{[\infty]}$, $a$ and $a^+$ do not change the eigenvalue.

That we have to encounter eigenvalues $E > E_{[\infty]}$ follows from the following short argument.

The representations of the algebra (1.1) have been studied in ref. [1] and we know that $P$ is an unbounded operator:

$$P|n, \sigma\rangle = \sigma q^n |n, \sigma\rangle$$

$$\sigma = +, -, \quad n = -\infty, \ldots, +\infty$$

From (1.3) follows that $a^+a$ cannot be bounded either:

$$\langle n, \sigma | a^+a | n, \sigma \rangle = \alpha \bar{\alpha} + \beta \bar{\beta} q^{2n}$$

2 The bounded spectrum

By an explicit construction we show that the Hamiltonian $H = \omega a^+a$ has eigenvalues $E < E_{[\infty]}$. In this energy range the operator $a$ lowers the energy eigenvalue. As the eigenvalues of $H$ cannot become negative we have to assume the existence of a "ground" state:

$$a |0\rangle^{(M)} = 0 \quad (M)\langle 0 \mid 0\rangle^{(M)} = 1$$

The spectrum follows immediately:

$$a^+ a |n\rangle^{(M)} = \frac{1 - q^{-2nM}}{1 - q^{-2M}} |n\rangle^{(M)} = [n]_M |n\rangle^{(M)}$$
\[ n = 0, \ldots, +\infty \]

where we have defined the q-number \([n]_M\). The normalized states are:

\[
|n\rangle^{(M)} = \frac{1}{\sqrt{[n]_M!}} (a^+)^n |0\rangle^{(M)}
\]

\[ (M) \langle n|m\rangle^{(M)} = \delta_{n,m} \]

This spectrum is bounded from above:

\[
[n]_M \rightarrow [\infty]_M =: \frac{1}{1 - q^{-2M}} =: [\infty]_M
\]

and explains our notation \(E_{[\infty]}\).

We now construct a ground state by first expanding it in terms of the "momentum" eigenstates of (1.7):

\[
|0\rangle^{(M)} = \sum_{m=-\infty}^{+\infty} c^{(\sigma)}_m |m, \sigma\rangle
\]

From (2.1) follows the recursion formula

\[
c^{(\sigma)}_m = -\sigma \frac{\alpha}{\beta} q^{-m} c^{(\sigma)}_{m-M}
\]

with 2\(M\) independent solutions. We define

\[
\tilde{c}^{(\sigma,\mu)}_m = c^{(\sigma)}_{m+\mu} \quad 0 \leq \mu < M, \ \mu \in \mathbb{N}_0
\]

and find from (2.6)

\[
\tilde{c}^{(\sigma,\mu)}_m = (-\sigma \frac{\alpha}{\beta})^m q^{-m^2+Mm+2\mu m} c^{(\mu)}_0
\]

The 2\(M\) independent ground states are

\[
|0\rangle^{(M)} = \sum_{m=-\infty}^{+\infty} \tilde{c}^{(\sigma,\mu)}_m |mM + \mu, \sigma\rangle
\]

\[ 0 \leq \mu < M, \ \sigma = +, - . \]

Due to the factor \(q^{-\frac{1}{2}Mm^2}\) they are all of finite norm. On each of these ground states we can act with the creation operator \(a^+\) to obtain exited states. The spectrum will be 2\(M\) times degenerate.

It is easy to compute the ground state expectation value of the momentum operator:

\[
\langle \sigma, \mu | P | \sigma, \mu \rangle^{(M)} = \sigma \sum_{m=-\infty}^{+\infty} \left( \frac{\alpha}{\beta} \right)^m \left( \frac{\bar{\alpha}}{\bar{\beta}} \right)^{m} q^{-Mm^2-2\mu m+\mu} |c^{(\mu)}_0|^2
\]

4
It is different from zero and the sign depends on $\sigma$. The exited states can be obtained by acting with $a^+$ on the ground states. For the first exited state we obtain:

$$\langle 1 | P | 1 \rangle^{(M)}_{(\sigma, \mu)} = \frac{2}{q^M (1 + q^M)} \langle 0 | P | 0 \rangle^{(M)}_{(\sigma, \mu)}$$

The expectation value of the momentum operator decreases in absolute value.

In analogy to ordinary quantum mechanics the eigenstates can be found by solving the recursion formula associated with the eigenvalue problem:

$$H | E \rangle = E | E \rangle$$

where we used $\bar{\alpha} \beta = \alpha \bar{\beta}$.

The solution (2.9) with (2.8) for $M = 1$, $\omega = 1$. The respective recursion formula is:

$$E c_n = (\alpha \bar{\alpha} + \beta \bar{\beta} q^{2n}) c_n + \alpha \bar{\beta} \sigma (q^n c_{n-1} + q^{n+1} c_{n+1})$$

Following our knowledge of quantum mechanics we try a polynomial ansatz:

$$f_n^{(K)} = \sum_{r=0}^{K} \eta_r q^{-2nr}$$

From (2.13) follows:

$$\frac{\beta \bar{\beta}}{\alpha \bar{\alpha}} (1 - q^{2(l+1)}) \eta_{l+1} = \left( q^{-2l} + \frac{E - \alpha \bar{\alpha}}{\alpha \bar{\alpha}} \right) \eta_l$$

For $l = -1, \ldots, K$ we learn

$$E = (1 - q^{-2K}) \alpha \bar{\alpha} = \frac{1 - q^{-2K}}{1 - q^{-2}}$$

These are our energy eigenvalues for $E < E_{[\infty]} = \alpha \bar{\alpha}$. They are determined from the requirement that the series (2.16) terminates. The other values of $l$ determine $\eta_{l+1}$ in terms of $\eta_l$. We find $\eta_{-1} = 0$ in agreement with the ansatz (2.16).
If we did not make a polynomial ansatz, eqn. (2.17) would tell us the asymptotic behaviour of the series (2.16):

\[
\eta_{l+1} \rightarrow_{l \to \infty} \frac{\alpha \bar{\alpha} - E}{q^{2l}\beta \bar{\beta}q^2}
\]

(2.19)

This is the behaviour of a series of the form \( \eta_l \sim q^{-l^2} \). Such a series can be summed and behaves like \( q^{n^2} \). This would lead to non-normalizable states. So our ansatz (2.14) with (2.16) does not lead to the eigenstates for \( E \geq E[\infty] \).

For \( E = E[\infty] = \alpha \bar{\alpha} \), the recursion formula (2.13) is identical to the one of ref. [7] (eqn. (15)) for coefficients \( d_n = (-iq)^n c_n \). There it was shown that the recursion formula leads to a unique solution with the corresponding state normalizable. This solution can be expressed in terms of q-deformed cosine and sine functions.

For \( E \neq \alpha \bar{\alpha} \) the recursion formula (2.13) has terms tending to infinity for \( n \to +\infty \) and \( n \to -\infty \). For \( n \to +\infty \), (2.13) tends asymptotically to the recursion formula of ref. [7] with the above substitution \( d_n = (-iq)^n c_n \). The same is true for \( n \to -\infty \) with the identification \( d_n = i^n c_{-n} \). This determines the asymptotic behaviour of the exact solution of (2.13) for \( n \to \pm \infty \) which will only match for certain values of \( E \).

That these solutions of the recursion formula exist will be shown in the following chapter.

### 3 The unbounded spectrum

To compute the spectrum above \( E[\infty] \) we retreat to perturbation theory. The simple form of the interaction will allow us to draw exact conclusions. In the following we set \( M = 1 \) and \( \omega = 1 \). We will only treat the case \( \sigma = +1 \) (1.7), the case \( \sigma = -1 \) is analogous.

The complete Hamiltonian is

\[
H = \alpha \bar{\alpha} + \beta \bar{\beta} P^2 + \alpha \bar{\beta} \left( UP + PU^{-1} \right)
\]

(3.1)

of which we will treat the second part as a perturbation:

\[
H_I = \alpha \bar{\beta} \left( UP + PU^{-1} \right)
\]

(3.2)

\[
H_0 = \alpha \bar{\alpha} + \beta \bar{\beta} P^2
\]

\( H_0 \) is diagonal in the momentum representation:

\[
H_0 |m\rangle = \left( \alpha \bar{\alpha} + \beta \bar{\beta} q^{2m} \right) |m\rangle \equiv E_m^{(0)} |m\rangle
\]

(3.3)

As \( \alpha \bar{\alpha} = \frac{1}{q^{2M}} = E[\infty] \) we see that all the unperturbed energy eigenvalues are larger than \( E[\infty] \).
It is obvious from (3.2) that the first order correction $E^{(1)}_m$ to the energy eigenvalue is zero. An explicit calculation shows that this is also true for the second order correction $E^{(2)}_m$. We shall prove by induction that the corrections to the energy eigenvalues are zero to all orders.

The standard expression in perturbation theory [8] for $E^{(l)}_m$, with the assumption that $E^{(r)}_m = 0$ for $r < l$, becomes:

$$E^{(l)}_m = \sum_{n_1, n_2, \ldots, n_{l-1}} \frac{\langle m | H_I | n_{l-1} \rangle \langle n_{l-1} | H_I | n_{l-2} \rangle \cdots \langle n_1 | H_I | m \rangle}{(E_m - E_{n_{l-1}})(E_m - E_{n_{l-2}})\cdots(E_m - E_{n_1})} \quad (3.4)$$

This sum is restricted to $n_r \neq m$.

The nonvanishing matrix elements of $H_I$ are

$$\langle n + 1 | H_I | n \rangle = \alpha \beta q^{n+1}$$

$$\langle n - 1 | H_I | n \rangle = \alpha \beta q^n \quad (3.5)$$

As this restricts the jumps in $n$ to $\Delta n = \pm 1$ we conclude from the restriction $n_r \neq m$ that the sum (3.4) splits in a natural way into two parts $n_r > m$ for all $r$ and $n_r < m$. We shall show that these two contributions are equal in magnitude and opposite in sign.

For any "path" $n_r = m + k_r$ ($r = 1, \ldots, l - 1$) with all $k_r > 0$ there is a path $n_r' = m - k_r$. We compare their contributions to (3.4). First we notice that, due to (3.3), $l$ has to be even. This leads to an odd number of factors in the denominator and an even number of factors in the numerator. From the contributions to the path we separate the first and the last step in the two cases:

$$\frac{\langle m | H_I | m + 1 \rangle \langle m + 1 | H_I | m \rangle}{E_m - E_{m+1}} = \alpha \beta \frac{q^2}{1 - q^2} \quad (3.6)$$

and

$$\frac{\langle m | H_I | m - 1 \rangle \langle m - 1 | H_I | m \rangle}{E_m - E_{m-1}} = -\alpha \beta \frac{q^2}{1 - q^2} \quad (3.7)$$

The rest of the factors will give the same contribution in both cases:

$$\frac{\langle n_{l-1} | H_I | n_{l-2} \rangle \cdots \langle n_2 | H_I | n_1 \rangle}{(E_m - E_{n_{l-2}})\cdots(E_m - E_{n_1})} = \frac{\alpha^{l-2}}{\beta} \frac{q_{n_1+n_2+\cdots+n_{l-2}-\frac{1}{2}(l-2)}}{q^{m(l-2)}(1 - q^{2(n_{l-1} - m)})\cdots(1 - q^{2(n_{l-2} - m)})} \quad (3.8)$$

The factor $q^{n_1+n_2+\cdots+n_{l-2}+\frac{1}{2}(l-2)}$ results from the following consideration. Looking at (3.5), we see that starting from a state $|n_j\rangle$ at the right hand side we get factors $q^{n_j+1}$ or $q^{n_j}$ depending whether the state on the left hand side is $|n_j + 1\rangle$.
or $|n_j - 1\rangle$. In our path we have to have as many "increasing" as "decreasing" matrix elements. For each of the $\frac{1}{2} (l-2)$ "increasing" steps (the first and last step we treated separately) we have a factor of $q$ in addition to the factor $q^{n_j}$. Now we look at the corresponding contribution from the path $n'_r = m - k_r$:

$$\frac{\langle n'_{l-1}|H_I|n'_{l-2}\rangle \cdots \langle n'_2|H_I|n'_1\rangle}{(E_m - E_{n'_l}) \cdots (E_m - E_{n'_1})} = \left(\frac{\alpha}{\beta}\right)^{l-2} \frac{q^{n'_1 + n'_2 + \cdots + n'_{l-2} + \frac{1}{2}(l-2)}}{q^{2m(l-2)}(1 - q^{2(n'_1 - m)}) \cdots (1 - q^{2(n'_l - m)})}$$  (3.9)

(3.8) and (3.9) give the same contribution. There is no change in sign as in (3.6) and (3.7) because there is an even number of factors in the denominator. This completes our proof by induction and we get the surprising result

$$E_m = E_m^{(0)} = \alpha \bar{\alpha} + \beta \bar{\beta} q^{2m}$$  (3.10)

It determines $E_0$ in (1.3):

$$E_0 = \alpha \bar{\alpha} + \beta \bar{\beta}$$  (3.11)

After having found the exact energy eigenvalues we will now proceed to calculate the corresponding eigenstates $|E_m\rangle$

$$|E_m\rangle = \sum_{n=-\infty}^{+\infty} c_n^{(m)} |n\rangle$$  (3.12)

where $|n\rangle$ are the momentum eigenstates and the eigenstates of the unperturbed Hamiltonian as well. As usual in perturbation theory, the coefficients $c_n^{(m)}$ are expanded

$$c_n^{(m)} = \sum_k a_{l,k}^{(m)}$$  \text{for } k \neq 0 \text{ , } c_m^{(m)} = 1$$  (3.13)

After having shown that the energy corrections are zero we have the following expression for the expansion:

$$a_{l,k}^{(m)} = \sum_{n_1,n_2,\ldots,n_{l-1}}^{|n|} \frac{\langle m+k|H_I|n_{l-1}\rangle \langle n_{l-1}|H_I|n_{l-2}\rangle \cdots \langle n_1|H_I|m\rangle}{(E_m - E_{m+k})(E_m - E_{n_{l-1}}) \cdots (E_m - E_{n_1})}$$  (3.14)

For $k > 0$ we have $n_r > m$ for all $r$, for $k < 0$ we have $n_r < m$. Each term of the sum is composed of $l$ factors of the type

$$\frac{\langle n|H_I|n'\rangle}{E_m - E_n} = \frac{\alpha \bar{\beta} (q^n \delta_{n,n'+1} + q^{n+1} \delta_{n,n'-1})}{\beta \bar{\beta} q^{2m}(1 - q^{2(n-m)})}$$  (3.15)
We first take care of the case $k > 0$. There we have the following estimate for the factors:

$$\left| \frac{\langle n | H_I | n' \rangle}{E_m - E_n} \right| \leq \left| \frac{\alpha}{\beta} \right| q^{-m} \frac{1}{1 - q^{-2}}$$  \hspace{1cm} (3.16)

There are $l$ factors of this type in each term of the sum (3.14). There certainly cannot be more than $2^l$ terms in the sum. Remember that at each of the $l$ steps in the "path" we can at most go one step up or one step down.

This now leads to the estimate

$$\left| \alpha_{l,k}^{(m)} \right| \leq \left| \frac{\alpha}{\beta} \right| q^{-ml} \frac{1}{(1 - q^{-2})^l} 2^l$$  \hspace{1cm} (3.17)

Furthermore we see from (3.14) that

$$\alpha_{l,k}^{(m)} = 0 \quad \text{for} \quad l < k$$  \hspace{1cm} (3.18)

Now we get an upper bound for $c_{m+k}^{(m)}$ for $k > 0$:

$$\left| c_{m+k}^{(m)} \right| \leq (2 \left| \frac{\alpha}{\beta} \right| q^{-m} \frac{1}{1 - q^{-2}})^k \sum_{r=0}^{\infty} \left( 2 \left| \frac{\alpha}{\beta} \right| q^{-m} \frac{1}{1 - q^{-2}} \right)^r$$  \hspace{1cm} (3.19)

For fixed value of $\alpha$ and $\beta$ we can always find an $m$ such that

$$2 \left| \frac{\alpha}{\beta} \right| q^{-m} \frac{1}{1 - q^{-2}} < 1$$  \hspace{1cm} (3.20)

and the sum in (3.19) converges.

We could make a similar estimate for the case $k < 0$. By a general analysis of the recursion formula (2.13) we will rather relate this case to the case $k > 0$.

We substitute (3.10) for the energy eigenvalues in (2.13) and obtain a recursion formula for the coefficients $c_n^{(m)}$:

$$\beta \bar{\beta} \left( q^{2m} - q^{2n} \right) c_n^{(m)} = \alpha \bar{\beta} \left( q^n c_{n-1}^{(m)} + q^{n+1} c_{n+1}^{(m)} \right)$$  \hspace{1cm} (3.21)

This can be seen as a recursion for decreasing or increasing indices:

$$c_{m+k}^{(m)} = \frac{\beta}{\alpha} q^{m-k} \left( 1 - q^{2(k-1)} \right) c_{m+k-1}^{(m)} - q c_{m+k-2}^{(m)}$$  \hspace{1cm} (3.22)

$$c_{m-k}^{(m)} = \frac{\beta}{\alpha} q^{m-k-1} \left( 1 - q^{-2(k-1)} \right) c_{m-(k-1)}^{(m)} - q c_{m-(k-2)}^{(m)}$$

If we substitute $c_{m-k}^{(m)} = (-q)^k \bar{c}_{m+k}^{(m)}$ in the second relation we see that $\bar{c}_{m+k}^{(m)}$ satisfies the first relation.

Our perturbation expansion (3.13) is normalized by $c_m^{(m)} = 1$. From (3.22) follows

$$c_{m+1}^{(m)} = -q^{-1} c_{m-1}^{(m)}$$  \hspace{1cm} (3.23)
Thus \(c_{m+k}^{(m)}\) and \(c_{m+k}^{(m)}\) not only satisfy the same recursion relation but also have the same initial values. Therefore we obtain:

\[
c_{m-k}^{(m)} = (-q)^k c_{m+k}^{(m)} \tag{3.24}
\]

The previous estimate (3.19) now tells us that

\[
|c_{m-k}^{(m)}| \leq (2q^{\left|\frac{\alpha}{\beta}\right|} \frac{q^{-m}}{1-q^2})^k \sum_{r=0}^{\infty} (2q^{\left|\frac{\alpha}{\beta}\right|} \frac{q^{-m}}{1-q^2})^r \tag{3.25}
\]

We see that the state \(|E_m\rangle\) is normalizable for

\[
2q^{\left|\frac{\alpha}{\beta}\right|} \frac{q^{-m}}{1-q^2} < 1 \tag{3.26}
\]

This of course is a very rough estimate but it proves the existence of normalizable states for \(m\) large enough. The other energy eigenstates of the unbounded spectrum can be reached by applying the operator \(a^+\).

Perturbation theory can be used much more efficiently to give the exact coefficients \(c_n^{(m)}\) in the expansion of the eigenstates (3.12). This will be shown now. We first exhibit the \(m\)-dependence of the coefficients \(c_n^{(m)}\) by having a closer look at (3.14) and (3.15). As in the study of the energy eigenvalues it is useful to label the path by \(n_r = m + k_r\) as in (3.8). Equation (3.15) becomes

\[
\frac{\langle m+k_r+1|H|m+k_r \rangle}{E_m - E_{m+k_{r+1}}} = (\frac{\alpha}{\beta} q^{-m}) q^{k_r+1} \left( \delta_{k_{r+1},k_{r+1}} + q \delta_{k_{r+1},k_{r}-1} \right) \frac{1-q^{-2}}{1-q^{2k_{r+1}}} \tag{3.27}
\]

The \(m\)-dependence is entirely contained in the factor \(\alpha/\beta q^{-m}\). We obtain from (3.14)

\[
d_{l,k}^{(m)} = (\frac{\alpha}{\beta} q^{-m}) f_{l,k}(q) \tag{3.28}
\]

From (3.13) follows

\[
c_{m+k}^{(m)} = F_k(\frac{\alpha}{\beta} q^{-m}) \quad , \quad c_m^{(m)} = F_0(\frac{\alpha}{\beta} q^{-m}) = 1 \tag{3.29}
\]

or

\[
|E_m\rangle = \sum_{k=-\infty}^{\infty} F_k(\frac{\alpha}{\beta} q^{-m}) |m+k\rangle \tag{3.30}
\]

We now use the fact that the operator \(a\) applied to this state leads to a state with eigenvalue \(E_{m+1}\). The normalization of our states is fixed by the condition \(F_0(\alpha/\beta q^{-m}) = c_m^{(m)} = 1\).

\[
a |E_m\rangle = \sum_{n=-\infty}^{\infty} \left( \alpha c_{n-2}^{(m)} + \beta q^{n-1} c_{n-1}^{(m)} \right) |n\rangle \tag{3.31}
\]

\[
= \gamma_m |E_{m+1}\rangle = \gamma_m \sum_{n=-\infty}^{+\infty} c_{n}^{(m+1)} |n\rangle
\]
where $\gamma_m$ is a relative normalization factor.

From (3.31) follows

$$\gamma_m c_n^{(m+1)} = \alpha c_n^{(m)} + \beta q^{n-1} c_{n-1}^{(m)}$$  \hspace{1cm} (3.32)

As we know that $c_{m+1}^{(m+1)} = 1$ we find

$$\gamma_m = \alpha c_{m-1}^{(m)} + \beta q^m$$  \hspace{1cm} (3.33)

We are now going to combine the relations (3.21), (3.32) and (3.33). The recursion relation (3.21) can be written in the form

$$\alpha c_{n-2}^{(m)} + \beta q^{n-1} c_{n-1}^{(m)} = \beta q^{2m-(n-1)} c_{n-1}^{(m)} - \alpha q c_n^{(m)}$$  \hspace{1cm} (3.34)

From equation (3.32) we find

$$\gamma_m c_n^{(m+1)} = \beta q^{2m-(n-1)} c_{n-1}^{(m)} - \alpha q c_n^{(m)}$$  \hspace{1cm} (3.35)

Setting $n = m$ and using the explicit $m$-dependence of the $c_n^{(m+l)}$ when expressed through the functions $F_k$ (3.29) we obtain an expression for $F_{-1}$

$$\left( \alpha F_{-1} \left( \frac{\alpha}{\beta} q^{-m} \right) + \beta q^m \right) F_{-1} \left( \frac{\alpha}{\beta} q^{-(m+1)} \right) = \beta q^{m+1} F_{-1} \left( \frac{\alpha}{\beta} q^{-m} \right) - \alpha q$$  \hspace{1cm} (3.36)

This is an equation that can be solved for $F_{-1}$, at least perturbatively. We abbreviate $z = \frac{\alpha}{\beta} q^{-m}$:

$$z F_{-1}(z) F_{-1}(q^{-1}z) = q \left( F_{-1}(z) - q^{-1} F_{-1}(q^{-1}z) \right) - qz$$  \hspace{1cm} (3.37)

If we make an ansatz

$$F_{-1}(z) = \sum_{k=0}^{\infty} d_k z^k,$$  \hspace{1cm} (3.38)

equation (3.37) allows us to compute the coefficients $d_k$. We find that $d_{2k} = 0$ and the first $d$’s for odd $k$ are

$$d_1 = \frac{1}{1-q^{-2}}$$
$$d_3 = \frac{q^{-2}}{(1-q^{-2})^2(1-q^{-4})}$$
$$d_5 = \frac{q^{-3}(q^{-1}+q^{-3})}{(1-q^{-2})^3(1-q^{-4})(1-q^{-6})}$$  \hspace{1cm} (3.39)

This reproduces in a very compact way the result one obtains for the coefficient $c_{m-1}^{(m)}$ from (3.13) and (3.14). The connection is

$$F_{-1} \left( \frac{\alpha}{\beta} q^{-m} \right) = c_{m-1}^{(m)} = \sum_{t=0}^{+\infty} a_{-1,t}^{(m)} = a_{-1,0}^{(m)} + a_{-1,1}^{(m)} + a_{-1,2}^{(m)} + \cdots$$  \hspace{1cm} (3.40)

$$a_{-1,k}^{(m)} = d_k \left( \frac{\alpha}{\beta} q^{-m} \right)^k$$

The other coefficients $c_n^{(m)}$ can now be calculated from the recursion formula (3.21).
4 The model and its relation to the undeformed harmonic oscillator

In this chapter we are going to show that the Hamiltonian of the previous chapters can be interpreted as a q-deformation of the usual harmonic oscillator. We shall express the q-deformed phase space variables $X, P$ and $U$ in terms of the usual canonical variables as we did in ref. [1]:

$$P = \hat{p}, \quad U = q^{-\frac{i}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})} \quad (4.1)$$

with

$$[\hat{x}, \hat{p}] = i, \quad \hat{x}^+ = \hat{x}, \quad \hat{p}^+ = \hat{p} \quad (4.2)$$

As a consequence of (4.2), $P$ and $U$ will satisfy (1.1) and (1.2).

We realize the relations (4.2) as follows:

$$\hat{x} = x, \quad \hat{p} = p + \frac{1}{\sqrt{1 - q^{-2M}}} \gamma, \quad \gamma \in \mathbb{R} \quad (4.3)$$

We now insert (4.1) into (1.3) and study the limit $q \to 1$, which for $q = e^h$ means $h \to 0$.

The constant $\alpha$ of (1.3) is singular

$$\alpha = \frac{e^{i\phi}}{\sqrt{1 - q^{-2M}}} \approx \frac{e^{i\phi}}{\sqrt{2Mh}} \quad (4.4)$$

This shows that $\sqrt{h}$ is a natural expansion parameter. In (4.3) a similar singular factor occurs for $\hat{p}$. The two singularities have to conspire to produce a finite result. We expand:

$$a = \frac{e^{i\phi}}{\sqrt{2Mh}} \left(1 + i\gamma x\sqrt{2Mh}\right) + \beta \left(1 + \frac{1}{2} i\gamma x\sqrt{2Mh}\right) \left(p + \frac{\gamma}{\sqrt{2Mh}}\right) + O(\sqrt{h}) \quad (4.5)$$

The singular terms cancel if:

$$\beta \gamma = -e^{i\phi} \quad (4.6)$$

Recall that $\gamma$ has to be real and therefore $\alpha\bar{\beta} = \bar{\alpha}\beta$.

We find:

$$a \xrightarrow{h \to 0} e^{i\phi} \left(\frac{1}{2} i\gamma x - \frac{1}{\gamma} p\right) = a_0 \quad (4.7)$$

There are two possibilities to identify (4.7) with the usual annihilation operator of the harmonic oscillator:

$$e^{i\phi} = \mp i, \quad \gamma = \pm\sqrt{2m\omega}, \quad \beta = \frac{i}{\sqrt{2m\omega}} \quad (4.8)$$
The relevant difference of these two choices is the sign of $\gamma$. The constants $m, \omega$ are the mass and frequency of the unperturbed oscillator.

The next term in the expansion is easily computed:

$$a = a_0 + \frac{i}{4\sqrt{2M}} \sqrt{\frac{1}{1 - q^{-2M}}} \left(1 - \frac{3}{2} \gamma^2 x^2\right) + \cdots$$

(4.9)

The two signs refer to the two choices in (4.8).

For the Hamiltonian we compute:

$$H = a_0^+ a_0 + \frac{1}{4 \sqrt{M}} \left(1 - 3m\omega x^2 + (1 - 3m\omega x^2) q\right)$$

(4.10)

The Hamiltonian $H = \omega a_0^+ a_0$ can be viewed as a quantized harmonic oscillator with specific interactions where $\sqrt{\hbar}$ appears as a coupling constant. In this sense it represents a $q$-deformation of the harmonic oscillator. It is interesting to write the full Hamiltonian in terms of the variables $x$ and $p$:

$$H = \omega \left(\frac{1}{1 - q^{-2M}} + \frac{1}{2m\omega} \left(p \pm \sqrt{\frac{2m\omega}{1 - q^{-2M}}}\right)^2\right) - \frac{\omega}{2m(1 - q^{-2M})} \left[p \pm \sqrt{\frac{2m\omega}{1 - q^{-2M}}} \frac{\sqrt{q} \sqrt{M}(x + px) \mp iM \sqrt{\frac{2m\omega}{1 - q^{-2M}}} x}{\sqrt{q} \sqrt{M}(x + px) \pm iM \sqrt{\frac{2m\omega}{1 - q^{-2M}}} x}\right]$$

(4.11)

This Hamiltonian has eigenfunctions with the eigenvalues (2.2) and (1.3). For the spectrum (2.2) we know the eigenfunctions explicitly. We discuss the ground states (2.9):

$$|0\rangle^{(M)}_{(\sigma,s,\mu)} = \sum_{n=\infty}^{+\infty} c_n^{(\sigma,s,\mu)} |nM + \mu, \sigma\rangle |s\rangle$$

(4.12)

$$= \sum_{n=-\infty}^{+\infty} \left(\pm \frac{\sigma}{s} \sqrt{\frac{2m\omega}{1 - q^{-2M}}}\right)^n q^{-\frac{1}{2} (Mn^2 + Mn + 2\mu_n)} c_0^{(\mu_n)} |nM + \mu, \sigma\rangle |s\rangle$$

As we are dealing with reducible representations of the algebra (1.1) where we express $P, X$ and $U$ in $p$ and $x$ as in (1.1) and (4.3) we have to label the irreducible parts by the variable $s$. We follow the notation of ref. [1]. The sign factor $\pm \sigma$ again refers to the two choices in (4.8).

The eigenstates of $P$ now have to be expressed in a basis which is labeled by the eigenvalues of $p$ in (4.3):

$$|n, \sigma\rangle |s\rangle = \int dp_0 q^{\frac{1}{2}} \delta\left(p_0 \pm \sqrt{\frac{1}{1 - q^{-2M}} - \sigma q^n}\right) |p_0\rangle$$

(4.13)
and we obtain

$$|0\rangle^{(M)}_{(\sigma,s,\mu)} = \sum_{n=-\infty}^{+\infty} \left( \pm \frac{\sigma}{s} \sqrt{\frac{2m\omega}{1-q^{-2}M}} \right)^n q^{-\frac{1}{2}(Mn^2+2\mu n-\mu)} c_0^{(s,\mu)}$$

$$\cdot \int dp_0 \delta \left( p_0 \pm \sqrt{\frac{2m\omega}{1-q^{-2}M} - \sigma sq^{nM+\mu}} \right) |p_0\rangle$$

These eigenfunctions live on a q-deformed lattice in momentum space. For $\sigma = +1$, $p_0$ ranges from $\mp \sqrt{\frac{2m\omega}{1-q^{-2}M}}$ to $+\infty$, for $\sigma = -1$, $p_0$ ranges from $-\infty$ to $\mp \sqrt{\frac{2m\omega}{1-q^{-2}M}}$.

The corresponding wave functions for the ground states in $x$-space are:

$$\Psi_{0}^{(M)}_{(\sigma,s,\mu)} (x) = \sum_{n=-\infty}^{+\infty} \left( \pm \frac{\sigma}{s} \sqrt{\frac{2m\omega}{1-q^{-2}M}} \right)^n q^{-\frac{1}{2}(Mn^2+2\mu n-\mu)} c_0^{(s,\mu)}$$

$$\cdot e^{i x (\sigma sq^{nM+\mu} \mp \sqrt{\frac{2m\omega}{1-q^{-2}M}})}$$

They can be seen to be eigenfunctions of the Hamiltonian (4.11) without ever referring to q-deformation. The excited states can be obtained by applying the $a^+$-operators expressed in terms of $x$ and $p = -i \frac{\partial}{\partial x}$ to the ground state wave functions.

It is interesting to see the limit $q \to 1$ on the wave functions, this will be done in the next chapter.

5 The wave functions in the limit $q \to 1$

The q-deformation of the Hamiltonian has led to a degeneracy of the spectrum that is not encountered for $h = 0$. It is interesting to see how the wave functions of the q-deformed oscillator behave for $h \to 0$. First we realize that for $h \to 0$ $E_{[\infty]} \to \infty$. The part $E > E_{[\infty]}$ disappears from the spectrum.

To study the behaviour of the wave functions for $E < E_{[\infty]}$, we start with the ground state (2.5). To simplify the discussion, we consider the case $M = 1$ first.

$$|0\rangle^{(1)}_{(\sigma)} = \sum_{n=-\infty}^{+\infty} \left( -\sigma \frac{\alpha}{\beta} \right)^n q^{-\frac{1}{2}(n^2+n)} c_0 |n, \sigma\rangle$$

The normalization constant $c_0$ can be computed:

$$|c_0|^2 = \sum_{n=-\infty}^{+\infty} \left( \frac{\alpha}{\beta} \right)^{2n} q^{-n^2-n}$$

$$= \sum_{n=-\infty}^{+\infty} e^{-h(n^2+n)} + n \ln \frac{2m\omega}{1-q^{-2}}$$
This expression will be singular for $h \to 0$ and we want to estimate its behaviour. We complete the square in the exponent of (5.2) and obtain:

$$|c_0|^2 = e^{\frac{4}{\hbar} - \frac{1}{2} \ln\frac{2m\omega}{1-q^{-2}} + \frac{1}{4\hbar} \left( \ln\frac{2m\omega}{1-q^{-2}} \right)^2 \sum_{n=-\infty}^{+\infty} e^{-\hbar \left( n + \frac{1}{2} - \frac{1}{2\hbar} \ln\frac{2m\omega}{1-q^{-2}} \right)^2} (5.3)$$

The sum in (5.3) can be estimated by considering it to approach the upper and lower Riemann integral of a Gaussian. We obtain

$$\sum_{n=-\infty}^{+\infty} e^{-h(n+a)^2} \longrightarrow \sqrt{\frac{\pi}{h}} (5.4)$$

independent of $a$. This now exhibits the behaviour of $c_0$ for $h \to 0$.

For any state

$$|f\rangle = \sum_{n=-\infty}^{+\infty} c_n |n, +\rangle,$$  (5.5)

the norm will approach an integral in the limit $h \to 0$.

$$\langle f | f \rangle = \sum_{n=-\infty}^{+\infty} |c_n|^2 (5.6)$$

We identify the points of the ”lattice”, according to (4.3), with points $s_p$ in $R^1$:

$$\sigma q^n = p + \frac{1}{\sqrt{1-q^{-2}}} \gamma = p + \sqrt{\frac{2m\omega}{1-q^{-2}}} (5.7)$$

For the sake of definiteness we have chosen the positive sign for $\gamma$ in (4.8). We see that the variable $p$ will range from $\frac{\sqrt{2m\omega}}{1-q^{-2}}$ to $+\infty$, if $\sigma = +1$. If $\sigma = -1$, $p$ will range from $-\infty$ to $-\sqrt{\frac{2m\omega}{1-q^{-2}}}$. In the limit $h \to 0$, the support of the wave function for $\sigma = -1$ will disappear, for $\sigma = +1$ it will range from $-\infty$ to $+\infty$.

If we had chosen the other sign for $\gamma$ in (4.8) the roles of the $\sigma = +1, -1$ states would have been interchanged. This shows how the $\sigma$-degeneracy of the states disappears.

Let us now transform (5.6) into an integral with the identifications (5.7) for $\sigma = +1$. We have:

$$q^n = p + \sqrt{\frac{2m\omega}{1-q^{-2}}} , \quad n = \frac{1}{\hbar} \ln \left( p + \sqrt{\frac{2m\omega}{1-q^{-2}}} \right) (5.8)$$

$$\Delta_n p = q^{n+1} - q^n = (q - 1) q^n = (q - 1) \left( p + \sqrt{\frac{2m\omega}{1-q^{-2}}} \right)$$

We find:

$$\langle f | f \rangle = \sum_{n=-\infty}^{+\infty} |c_n|^2 \frac{\Delta_n p}{(q - 1) \left( p + \sqrt{\frac{2m\omega}{1-q^{-2}}} \right)} \longrightarrow \int_{-\infty}^{+\infty} |f(p)|^2 dp (5.9)$$
where
\[ f(p) = \frac{c_n}{\sqrt{(q-1)(p + \sqrt{\frac{2m\omega}{1-q^{-2}}})}} , \quad n = \frac{1}{\hbar} \ln \left( p + \sqrt{\frac{2m\omega}{1-q^{-2}}} \right) \] (5.10)

This is the formula by which we identify the wave function of the ground state in the limit \( h \to 0 \).

We now go back to (5.1) and make the substitutions of (5.10) on the individual parts:

\[ q^{-\frac{1}{2}n} = e^{-\frac{1}{2} \left( \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} + \ln \left( 1 + p\sqrt{\frac{1-q^{-2}}{2m\omega}} \right) \right)} \]
\[ q^{-\frac{1}{2}n^2} = e^{-\frac{1}{2\hbar} \left( \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} + \ln \left( 1 + p\sqrt{\frac{1-q^{-2}}{2m\omega}} \right) \right)^2} \] (5.11)
\[ \left( -\frac{\alpha}{\beta} \right)^n = e^{\frac{1}{2\hbar} \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} \left( \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} + \ln \left( 1 + p\sqrt{\frac{1-q^{-2}}{2m\omega}} \right) \right)} \]
\[ c_0 = \left( \frac{\hbar}{\pi} \right)^{\frac{1}{4}} e^{-\frac{h}{8} + \frac{1}{2} \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} - \frac{1}{2\hbar} \left( \ln \sqrt{\frac{2m\omega}{1-q^{-2}}} \right)^2} \]

Putting all together we find the surprising result:
\[ f(p) = \frac{1}{(m\omega \pi)^{\frac{1}{4}}} e^{-\frac{p^2}{2m\omega}} \] (5.12)

This is the correctly normalized ground state wave function of the undeformed harmonic oscillator.

It is easy to see that for \( M \neq 1 \) all the different wave functions for the ground state collapse to the same wave function (5.12). Thus all the degeneracy is removed in the limit \( h \to 0 \).

The excited states are obtained from the ground states by applying creation operators. As these operators tend to the undeformed creation operators of the harmonic oscillator we reproduce all the wave functions of the undeformed oscillator. It is not surprising that a direct calculation on the states defined by the coefficients (2.16) leads to the same result.

### 6 q-deformed Hermite polynomials

We recall the definition of the ground state \((M = 1)\)
\[ a |0\rangle = \left( \alpha U^{-2} + \beta U^{-1} P \right) |0\rangle = 0 \] (6.1)
with the immediate consequence:

$$U \ket{0} = -\frac{\alpha}{q^2 \beta} P^{-1} \ket{0} \quad (6.2)$$

This relation together with the definition of the algebra (1.1) can be used to show:

$$a^+ \ket{0} = \frac{i}{q^2 \beta} X \ket{0} \quad (6.3)$$

The ground state as well as the one-particle state can be obtained by applying a polynomial in $X$ to the ground state. We are going to show that this is true for the $n$-particle state as well:

$$\left(a^+\right)^n \ket{0} = \left(\frac{1\sqrt{2}}{q}\right)^n H_n^{(q)} \left(\frac{iX}{\sqrt{2} \beta}\right) \ket{0} \quad (6.4)$$

$H_n^{(q)}$ will turn out to be a polynomial of degree $n$, satisfying a q-deformed recursion formula for Hermite polynomials. The normalization and the scaling of the argument to a dimensionless parameter

$$\xi = \frac{iX}{\sqrt{2} \beta} = \sqrt{m\omega} X \quad (6.5)$$

are chosen to identify $H_n^{(1)}(\xi)$ with the usual Hermite polynomial.

To prove eqn. (6.4) we start from the following relation which can be obtained from the definition of the algebra (1.1)

$$a^+ \xi = q^{-2} \xi a^+ - \frac{1}{\sqrt{2}} q^{-\frac{3}{2}}$$

and we prove by induction:

$$\left(\left(a^+\right)^n + q^{-\frac{1}{2}} q^{-2n} \sqrt{2} \xi \left(a^+\right)^n + q^{-2} \frac{1 - q^{-2n}}{1 - q^{-2}} \left(a^+\right)^{n-1}\right) \ket{0} = 0 \quad (6.7)$$

For $n = 0$, this is just equation (6.3). With the help of (6.6) the step $n$ to $n + 1$ is easily verified.

Eqn. (6.7) shows that $H_n^{(q)}(\xi)$ is a polynomial of degree $n$ in $\xi$. Eqn. (6.7) can now be written in the form

$$\left(H_n^{(q)}(\xi) - q^{-\frac{1}{2}} q^{-2n} 2 \xi H_n^{(q)}(\xi) + 2 q^{-2} \frac{1 - q^{-2n}}{1 - q^{-2}} H_{n-1}^{(q)}(\xi)\right) \ket{0} = 0 \quad (6.8)$$

If a polynomial satisfies a recursion formula

$$H_{n+1}^{(q)}(\xi) - q^{-\frac{1}{2}} q^{-2n} 2 \xi H_n^{(q)}(\xi) + 2 q^{-2} \frac{1 - q^{-2n}}{1 - q^{-2}} H_{n-1}^{(q)}(\xi) = 0 \quad (6.9)$$
and if
\[ H_0^{(q)}(\xi) = 1, \quad H_1^{(q)}(\xi) = 2 q^{-\frac{1}{2}} \xi \] (6.10)
then \( H_{n+1}(\xi), \) obtained from (6.9), will be the correct polynomial in (6.4).

For \( q = 1, \) (6.9) is just the recursion formula for Hermite polynomials and (6.10) are the first two Hermite polynomials.

We found that the states of the spectrum of the q-deformed harmonic oscillator for \( E < E_{[\infty]} = \omega(1 - q^{-2})^{-1} \) can be obtained by applying q-deformed Hermite polynomials to the ground states \( |0\rangle_{(\sigma=+1)} \) and \( |0\rangle_{(\sigma=-1)} \). However, as indicated in (1.8), these states do not form a complete set of states in the Hilbert space where \( P \) and \( X \) are represented. This is in contrast to the case \( q = 1. \)
The orthogonality relations of the eigenstates of the Hamiltonian lead to interesting orthogonality relations for the q-deformed Hermite polynomials. They will involve the q-deformed cosine and sine functions as the ground state in (6.4) will have to be expressed in the \( X \)-basis.

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References

[1] M. Fichtmüller, A. Lorek, J. Wess:
q-deformed Phase Space and its Lattice Structure,
preprint MPI-PhT/95-109, hep-th/9511106
accepted for publication in Z. Phys. C

[2] A. J. Macfarlane:
On q-analogues of the quantum harmonic oscillator and quantum group SU(2)_q,
J. Phys. A 22 (1989) 4581

[3] L. C. Biedenharn:
The quantum group SU_q(2) and a q-analogue of the boson operators,
J. Phys. A 22 (1989) L873

[4] A. Ruffing:
Doctorate thesis, LMU München, 1996

[5] J. Seifert:
private communication
[6] A. Lorek, J. Wess:
Dynamical Symmetries in q-deformed Quantum Mechanics,
Z. Phys. C 67 (1995) 671

[7] A. Hebecker, S. Schreckenberg, J. Schwenk, W. Weich, J. Wess:
Representations of a q-deformed Heisenberg Algebra,
Z. Phys. C 64 (1994) 355

[8] L. I. Schiff:
Quantum Mechanics,
McGraw-Hill 1968 (Third Edition), 244ff