Interaction between two gravitationally polarizable objects
induced by thermal bath of gravitons

Puxun Wu\textsuperscript{1,3}, Jiawei Hu\textsuperscript{1,2,*} and Hongwei Yu\textsuperscript{1,†}

\textsuperscript{1}Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha, Hunan 410081, China
\textsuperscript{2}Center for Nonlinear Science and Department of Physics, Ningbo University, Ningbo, Zhejiang 315211, China
\textsuperscript{3}Center for High Energy Physics, Peking University, Beijing 100080, China

Abstract

The quadrupole-quadrupole interaction between a pair of gravitationally polarizable objects induced by vacuum fluctuations of the quantum linearized gravitational field is first obtained with a relatively simple method, which is then used to investigate the contribution of thermal fluctuations of a bath of gravitons to the interaction at temperature $T$. Our result shows that, in the high temperature limit, the contribution of thermal fluctuations dominates over that of vacuum fluctuations and the interaction potential behaves like $T/r^{10}$, where $r$ is the separation between the objects, and in the low temperature limit, the contribution of thermal fluctuations is proportional to $T^{10}/r$, which only provides a small correction to the interaction induced by zero-point fluctuations.

PACS numbers: 04.60.Bc, 03.70.+k, 04.30.-w, 42.50.Lc

\* Corresponding author at jwhu@hunnu.edu.cn
\ † Corresponding author at hwyu@hunnu.edu.cn
I. INTRODUCTION

Fluctuating electromagnetic fields in vacuum induce electric dipole moments in neutral atoms and thus generate between them the famous Casimir-Polder (CP) force which does not exist in classical electrodynamics [1]. The CP force behaves like \( r^{-7} \) in the retarded region and \( r^{-6} \) in the non-retarded region, where \( r \) is the separation between two atoms, and it has been extensively studied since its discovery (for recent reviews, see Refs. [2–4]). In particular, it has been generalized to include thermal corrections that arise from thermal fluctuations of a bath of thermal photons at nonzero temperature [5], where it has been found that in the high temperature limit, the contribution of thermal fluctuations dominates over that of the zero-point fluctuations and gives rise to a characteristic temperature \( (T) \) and distance dependence of \( T/r^{-6} \). In the low temperature limit, the thermal fluctuations only produce a small correction to the CP force dominated by vacuum fluctuations.

Likewise, one would expect a CP-like quantum correction to the classical Newtonian force between gravitationally polarizable objects due to vacuum fluctuations of gravitational fields when gravity is quantized. In this regard, the quantum correction has been computed between a pair of gravitationally polarizable objects that arises from the induced quadrupole moments due to two-graviton exchange [6] in close analogy with the calculation of the CP force between a pair of atoms from their induced dipole moments due to two-photon exchange [7], and the quantum gravitational potential is found to behaves as \( r^{-11} \) and \( r^{-10} \) in the far and near regimes respectively. It is also worth noting that the quantum gravitational correction between two mass monopoles has been worked out by summing one-loop Feynman diagrams with off-shell gravitons in the framework where general relativity is treated as an effective field theory [8].

Recently, based on the linearized quantum gravity and leading-order perturbation theory, we give an alternative derivation of this quantum correction due to the vacuum fluctuation induced quadruple-quadruple interactions by treating two objects as two-level harmonic oscillators from a quantum field theoretic prospect [9]. This approach is parallel to that used by Casimir and Polder in studying the quantum electromagnetic vacuum.
fluctuation induced electric dipole-dipole interaction between two neutral atoms in a quantum theory of electromagnetism [1]. Remarkably, this quantum correction can also been obtained by a calculation of the scattering amplitude of two-graviton exchange [10].

As a natural step forward, in this paper, we plan to investigate what happens to the gravitational interaction between gravitationally polarizable objects when they are in a thermal bath of gravitons rather than in a vacuum. Let us note that a thermal bath of gravitons may be created by the Hawking effect or cosmological particle production.

In this paper, we first present another different but rather simple approach to derive the quadruple-quadruple interaction between a pair of gravitationally polarizable objects. Then, we consider the contribution of thermal fluctuations by immersing two objects in a thermal bath. Throughout this paper, the Latin indices run from 0 to 3, while the Greek letter is from 1 to 3. The Einstein convention is assumed for repeated index and \( h = c = k_B = 1 \) is set. Here, \( h \) is the reduced Planck constant and \( k_B \) is the Boltzmann constant.

II. SIMPLE DERIVATION OF THE QUADRUPLE-QUADRUPLE INTERACTION

We assume that there is a pair of gravitationally polarizable objects, labeled \( A \) and \( B \), at \( r_A \) and \( r_B \) with respect to an arbitrary origin and the metric can be expressed as \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where \( h_{\mu \nu} \) describes the fluctuating vacuum gravitational fields which are quantized [11]. A direct consequence of quantization of gravity is the lightcone fluctuations which have been examined in [11–14]. In the present paper, we are concerned with the correction they induce to the classical Newtonian interactions.

At an arbitrary point \( r \), the vacuum field of linearized gravity has the standard form [15]

\[
h_{ij}(r) = \sum_{k, \lambda} \sqrt{\frac{8\pi G}{(2\pi)^3 \omega}} [a_{ij}(\omega) e_{ij}(k, \lambda) e^{ik \cdot r - i\omega t} + \text{H.c.}] \\
≡ \sum_{k} h_{ij}^k(r),
\]

where the transverse tracefree (TT) gauge is taken, \( H.c. \) denotes the Hermitian conjugate, \( a_{ij}(\omega) \) is the gravitational field operator, which defines the vacuum \( a_{ij}(\omega)\{0\} = 0 \), \( \lambda \) labels
the polarization states, $e_{ij}(k, \lambda)$ are polarization tensors, $\omega = |k| = (k_x^2 + k_y^2 + k_z^2)^{1/2}$, and $G$ is the Newton’s gravitational constant.

The fluctuating vacuum fields induce quadrupole moments in the objects $A$ and $B$, which are written as

$$Q_{i,lm}(\omega) = \alpha_i(\omega) E_{lm}^k(r_i), \quad i = A \text{ or } B,$$

where $\alpha_i(\omega)$ represents the polarizability of object $i$, which we assume to be isotropic for simplicity, and $E_{lm} = \sum_k E_{lm}^k$ is the gravito-electric tensor defined as

$$E_{lm} = -R_{0\ell0m},$$

by an analogy between the linearized Einstein field equations and the Maxwell equations [16]. Here, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor. From the definition of $E_{lm}$, we have

$$E_{lm} = \frac{1}{2} \ddot{h}_{lm},$$

where a dot denotes a derivative with respect to time $t$.

If the object $A$ is polarized by vacuum fluctuations, it will emit the gravitational waves, which generate a field at the position of object $B$: $E_{ij}^k(A \rightarrow B)$. As a result, there exists an inevitable interaction between the quadrupole moment of object $B$ induced by vacuum fluctuations and the field $E_{ij}^k(A \rightarrow B)$. The interaction potential can be obtained perturbatively. It is well known that the energy of a localized mass distribution $\rho(x)$ in the presence of an external potential $\Phi(x)$ is

$$V_t = \int \rho(x)\Phi(x)d^3x.$$

If we assume that the potential $\Phi(x)$ varies slowly over the region where the mass is located, then the external potential can be expanded in a Taylor series as

$$\Phi(x) = \Phi(x_0) + x_i \frac{\partial \Phi(x_0)}{\partial x_i} + \frac{1}{2} x_i x_j \frac{\partial^2 \Phi(x_0)}{\partial x_i \partial x_j} + \cdots.$$

Because there is no mass dipole in gravitation, the leading term of the interaction between two objects after the Newtonian potential is the quadrupole term

$$V = \frac{1}{2} \int d^3x \rho(x)x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j}.$$
Note that $\nabla^2 \Phi = 0$ in an empty space, so we can rewrite the above equation as

$$V = -\frac{1}{2} Q_{ij} E_{ij},$$

where

$$Q_{ij} = \int d^3x \rho(x) \left( x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right),$$

and

$$E_{ij} = -\frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \frac{1}{3} \delta_{ij} \nabla^2 \Phi.$$ (10)

In general relativity, $E_{ij}$ is defined in Eq. (3), which can be shown to coincide with the expression given in Eq. (10) in the Newtonian limit.

Thus, from Eq. (8), one can see that the potential from quadrupole-quadrupole interaction between objects $A$ and $B$ takes the form

$$V(r) = -\frac{1}{2} \sum_k \langle \{0\}|Q_{B,ij}(\omega) E_{ij}^k(A \rightarrow B)|\{0\}\rangle,$$ (11)

where $r = |r_A - r_B|$ is the separation between objects $A$ and $B$. At the same time, the interaction potential can also be expressed as

$$V(r) = -\frac{1}{2} \sum_k \langle \{0\}|Q_{A,ij}(\omega) E_{ij}^k(B \rightarrow A)|\{0\}\rangle$$ (12)

if we exchange roles of objects $A$ and $B$ in the above physical picture. Using Eq. (2), the above expression can be rewritten as

$$V(r) = -\frac{1}{2} \sum_k \langle 0|\alpha_A(\omega) E_{ij}^k(A) E_{ij}(B \rightarrow A)|0\rangle.$$ (13)

With $\hat{n} = r/r$, the field $E_{ij}^k(B \rightarrow A)$ is given by [6]

$$E_{ij}^k(B \rightarrow A) = \text{Re} \left\{ \frac{G e^{i\omega r}}{r^5} \Lambda_{ij}^{lm}(\omega r, \hat{n}) Q_{B,lm}(\omega) \right\},$$ (14)

where

$$\Lambda_{ij}^{lm}(\gamma, \hat{n}) = \frac{1}{2} \left[ (6 - 6\gamma^2 + 2\gamma^4 - 6i\gamma + 4i\gamma^3)\delta_i^l \delta_j^m \\
+ (-15 + 9\gamma^2 - \gamma^4 + 15i\gamma - 4i\gamma^3)(n^i n^j \delta_l^m + n^j n^l \delta_i^m + n^i n^m \delta_j^l + n^j n^m \delta_i^l) \\
+ (-15 + 3\gamma^2 + \gamma^4 + 15i\gamma + 2i\gamma^3)n^lm \delta_{ij} \\
+ (105 - 45\gamma^2 + \gamma^4 - 105i\gamma + 10i\gamma^3)n^i n^j n^l n^m) \right].$$ (15)
Since $Q_{B,lm}(\omega)$ is induced by vacuum fluctuations, it can also be expressed as $Q_{B,lm}(\omega) = \alpha_B(\omega) E_{B,lm}^k(r_B)$. Thus, to obtain the interaction potential given in Eq. (13), we need to calculate the following vacuum expectation value
\begin{equation}
\langle\{0\}| E_{A,ij}^k(r_A) E_{B,lm}^k(r_B)|\{0\}\rangle = \frac{\omega^4}{4} \langle\{0\}| h_{ij}^k(r_A) h_{lm}^k(r_B)|\{0\}\rangle
= \frac{G\omega^3}{(2\pi)^2} \sum_\lambda e_{ij}(k, \lambda) e_{lm}(k, \lambda)e^{ik(r_A-r_B)}
= \frac{G\omega^3}{(2\pi)^2 g_{ijlm}(\hat{k})} e^{i\hat{k} \cdot \hat{r}},
\end{equation}
where $\hat{k} = k/\omega$, and the summation of polarization state gives [11]
\begin{equation}
g_{ijlm}(\hat{k}) = \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{ij}\delta_{lm} + \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m + \hat{k}_l\hat{k}_m\delta_{ij} - \hat{k}_i\hat{k}_l\hat{k}_m\delta_{jm} - \hat{k}_i\hat{k}_m\delta_{il} - \hat{k}_j\hat{k}_l\delta_{im}.
\end{equation}
Substituting Eqs. (14, 16) into Eq. (13) leads to
\begin{equation}
V(r) = -\frac{G^2}{8\pi^2} \text{Re} \left\{ \sum_k \alpha_A(\omega)\alpha_B(\omega) \frac{\omega^3 e^{i(\hat{k} \cdot \hat{r}+\omega r)}}{r^5} \Lambda_{ij}^{lm}(\omega r, \hat{n}) g_{ijlm}(\hat{k}) \right\}.
\end{equation}
Using $\hat{k} \cdot \hat{n} = \cos(\theta)$, we obtain
\begin{equation}
\Lambda_{ij}^{lm}(\gamma, \hat{n}) g_{ijlm}(\hat{k}) = \frac{1}{16} \left[ -27 + 27 i\gamma + 39\gamma^2 - 30 i\gamma^3 - 35\gamma^4 
-4(15 - 15 i\gamma - 27\gamma^2 + 22 i\gamma^3 + 7\gamma^4) \cos(2\theta) 
-(105 - 105 i\gamma - 45\gamma^2 + 10 i\gamma^3 + \gamma^4) \cos(4\theta) \right]\end{equation}
Now we replace the summation in Eq. (18) by integral and then change into the spherical coordinate
\begin{equation}
\sum_k \rightarrow \int d^3k \rightarrow \int_0^\infty k^2 dk \int_0^\pi d\theta 2\pi .
\end{equation}
Making a further replacement $e^{ik \cdot r} = e^{i\omega r \cos(\theta)} \rightarrow \cos(\omega r \cos(\theta))$ and taking the real part in Eq. (18), we obtain
\begin{equation}
V(r) = -\frac{G^2}{\pi r^{10}} \lim_{\eta \rightarrow 0} \int_0^\infty d\omega \ e^{-\eta \omega} \alpha_A(\omega)\alpha_B(\omega)f(\omega r),
\end{equation}
where a convergence factor is introduced to avoid divergence and

\[
f(\gamma) = (-630 \gamma + 330 \gamma^3 - 42 \gamma^5 + 4 \gamma^7) \cos(2\gamma)
\]
\[
+ (315 - 585 \gamma^2 + 129 \gamma^4 - 14 \gamma^6 + \gamma^8) \sin(2\gamma). \tag{22}
\]

Assuming an approximate static polarizability \( \alpha_A(0) \) and performing the integration in Eq. (21), one can obtain

\[
V(r) = -\frac{3987 G^2}{4 \pi r^{11}} \alpha_A(0) \alpha_B(0), \tag{23}
\]

which is the same as what was obtained in \([6, 9, 10]\) with different methods.

III. THE INTERACTION FROM THERMAL FLUCTUATIONS

Now we begin to study the contribution from thermal fluctuations with the method we just introduced in the proceeding section. We assume that two objects are in a thermal bath of gravitons at a temperature \( T \). Then, the interaction potential takes the form

\[
V(r) = -\frac{1}{2} \sum_k \langle \{\beta\} | \alpha_A(\omega) E_{ij}^k E_{ij}^k B \rightarrow A \rangle | \{\beta\} \rangle,
\]

where \( \beta = 1/T \). For a thermal state of gravitons \( |\{\beta\}\rangle \), one has the following relations

\[
\langle \{\beta\} | a_{\lambda}(\omega) a_{\lambda}(\omega) | \{\beta\} \rangle = N(\beta, \omega), \quad \langle \{\beta\} | a_{\lambda}(\omega) a_{\lambda}^\dagger(\omega) | \{\beta\} \rangle = 1 + N(\beta, \omega), \tag{25}
\]

where

\[
N(\omega, \beta) = \frac{1}{e^{\beta \omega} - 1}.
\]

Following the same procedure as in the proceeding section, one can obtain

\[
V(r) = -\frac{G^2}{\pi r^{10}} \lim_{\eta \to 0} \int_0^{\infty} d\omega \ e^{-\eta \omega} \alpha_A(\omega) \alpha_B(\omega) f(\omega r) [1 + 2N(\beta, \omega)]
\]
\[
\equiv V_0(r) + V_T(r), \tag{27}
\]

where \( V_0(r) \) and \( V_T(r) \) represent the contributions from vacuum fluctuations and thermal fluctuations, respectively.
Taking the high-temperature limit, which means that $\beta \ll r$, we find that

$$V_T(r) = -\frac{315G^2}{\beta r^{10}}\alpha_A(0)\alpha_B(0)$$ \hspace{1cm} (28)

Since the interaction potential is determined by both the zero-point fluctuations and the thermal ones, we have

$$V(r) = -\frac{G^2}{r^{11}}\alpha_A(0)\alpha_B(0) \left[\frac{3987}{4\pi} + \frac{315r}{\beta}\right].$$ \hspace{1cm} (29)

It is easy to see that since $\beta \ll r$ the thermal fluctuations play a dominating contribution on the potential and this potential is proportional to $r^{-10}\beta^{-1}$.

In the low-temperature limit ($\beta \gg r$), one has

$$V_T(r) = -\frac{83456\pi^9G^2}{10395} \frac{1}{r^{10}}\alpha_A(0)\alpha_B(0).$$ \hspace{1cm} (30)

From Eqs. (23) and (30), the potential is

$$V(r) = -\frac{G^2}{r^{11}}\alpha_A(0)\alpha_B(0) \left[\frac{3987}{4\pi} + \frac{83456\pi^9}{10395} \frac{r^{10}}{\beta^{10}}\right].$$ \hspace{1cm} (31)

The contribution from thermal fluctuations is proportional to $r^{-1}\beta^{-10}$, which provides a correction term to the potential dominated by the vacuum fluctuations.

Now a question arises as to when the thermal correction might become appreciable as compared to the classical Newtonian interaction in the high temperature limit. To address this issue, let us first consider a simple system of mass, i.e., an elastic sphere. If the object has radius $R_i$, mass $M_i$ and frequency $\omega_i$, we have $\alpha_i \sim M_iR_i^2/\omega_i^2$ from the dimensional analysis. For convenience, we introduce the orbital frequency $\Omega_i = \sqrt{GM_i/R_i^3}$ of a gravitationally bound system as the reference frequency, which sets a lower bound on $\omega_i$ for any physical system. Then, in the high temperature limit, we have

$$V(r) \approx -315\frac{T}{\omega_0} \left(\frac{\Omega_A\Omega_B}{\omega_A\omega_B}\right)^2 \frac{R_A^5R_B^5}{r^{10}}.$$ \hspace{1cm} (32)

Assuming that $R_A = R_B = R$, and $M_A = M_B = M$ for simplicity, we can obtain the ratio between $V(r)$ and the classical Newtonian gravitational potential $V_N(r) = -GM^2/r$ as

$$\frac{V(r)}{V_N(r)} = 315\frac{k_B T}{GM^2/R} \left(\frac{\Omega}{\omega_0}\right)^4 \left(\frac{R}{r}\right)^9.$$ \hspace{1cm} (33)
where $\omega_0 = \omega_A = \omega_B$, $\Omega = \Omega_A = \Omega_B$ and we returned to the SI units. For a gravitationally bounded object, we have $\Omega_i \sim \omega_i$. Thus, if the temperature is high enough ($k_B T \approx \frac{GM^2}{R}$), the interaction between two massive objects, which results from thermal fluctuations of gravitons, may become appreciable to the classical Newtonian interaction. However, for a gravitationally bounded object, the gravitational potential energy is of the order of $-GM^2/R$. When the temperature of the environment grows high enough ($k_B T \sim GM^2/R$), such gravitationally bounded objects will break up. Therefore, for a gravitationally bounded object, we have $\frac{V(r)}{V_N(r)} \sim \left(\frac{R}{r}\right)^9$ for the best. For an electrically bounded object, the breakup temperature is higher, i.e. $k_B T \sim \hbar \omega_0$. However, since $\omega_0 \gg \Omega$ for an electrically bounded object, the ratio (33) is suppressed. At the temperature $k_B T = \hbar \omega_0$, Eq. (33) can be rewritten as

$$\frac{V(r)}{V_N(r)} = 315 \left(\frac{l_P}{R}\right)^2 \left(\frac{c}{\omega_0 R}\right)^3 \left(\frac{R}{r}\right)^9,$$

(34)

where $l_P = \sqrt{\hbar G/c^3}$ is the Planck length. The radius $R$ is typically much greater than the Planck length. As a result, the ratio (34) is usually extremely small, although the transition wavelength $c/\omega_0$ may be large compared to $R$. As a concrete example, we consider the interaction between two hydrogen atoms. The temperature needed to ionize hydrogen atoms is of the order of $10^5$ K. At this temperature, the ratio (34) can be estimated as $\frac{V(r)}{V_N(r)} \sim 10^{-36} \left(\frac{R}{r}\right)^9$, where we have taken $R$ as the Bohr radius ($\sim 10^{-11}$ m), and the transition wavelength $c/\omega_0 \sim 10^{-7}$ m. Therefore, although it seems that when the temperature grows high enough, the interaction resulting from thermal fluctuations of gravitons may become appreciable to the classical Newtonian interaction. Realistic physical objects, however, break up well before this temperature is reached.

IV. CONCLUSION

In this paper, we first introduce a relatively simple method to obtain the quadrupole-quadrupole correction caused by the quantum gravitational vacuum fluctuations to the classical Newtonian potential between a pair of gravitationally polarizable objects. Then, we use it to treat the case where the gravitationally polarizable objects are in a thermal
bath of gravitons rather than in a vacuum. Assuming an approximately static polarizability we find that in the high temperature limit the thermal fluctuations produce a dominant contribution and the potential behaves like \( r^{-10} T \). While, in the low temperature limit, the contribution from thermal fluctuations is proportional to \( r^{-1} T^{10} \), which is much less than that from zero-point fluctuations, and in this case the potential is dominated by the \( r^{-11} \) term.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China under Grants No. 11435006, No. 11375092, No. 11447022, No. 11690034 and No. 11690030; and the Zhejiang Provincial Natural Science Foundation of China under Grant No. LQ15A050001.

[1] H. B. G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948).
[2] S. K. Lamoreaux, Rep. Prog. Phys. 68, 201 (2005).
[3] K. A. Milton, J. Phys.: Conf. Ser. 161, 012001 (2009).
[4] M. Bordag et al., *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).
[5] E. M. Lifshitz, Sov. Phys. JETP 2, 73 (1956); I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Adv. Phys. 10, 165 (1961); P. W. Milonni and A. Smith, Phys. Rev. A 53, 3484 (1996); R. Passante and S. Spagnolo, Phys. Rev. A 76, 042112 (2007).
[6] L. H. Ford, M. P. Hertzberg, and J. Karouby, Phys. Rev. Lett. 116, 151301 (2016).
[7] B. E. Sernelius, *Surface modes in physics*, (Wiley-VCH, Berlin, 1999).
[8] J. F. Donoghue, Phys. Rev. Lett. 72, 2996 (1994); J. F. Donoghue, Phys. Rev. D 50, 3874 (1994); H. W. Hamber and S. Liu, Phys. Lett. B 357, 51 (1995); I. B. Khriplovich and G. G. Kirilin, Zh. Eksp. Teor. Fiz. 95, 1139 (2002) [J. Exp. Theor. Phys. 95, 981 (2002)]; N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D67, 084033 (2003); 71, 069903 (2005).
[9] P. Wu, J. Hu and H. Yu, Phys. Lett. B 763, 40 (2016).
[10] B. R. Holstein, J. Phys. G 44, 01LT01 (2017); arXiv:1610.07957.

[11] H. Yu and L. H. Ford, Phys. Rev. D 60, 084023 (1999).

[12] H. Yu and L. H. Ford, Phys. Lett. B 496, 107 (2000).

[13] H. Yu and P. Wu, Phys. Rev. D 68, 084019 (2003).

[14] H. Yu, N. F. Svaiter, and L. H. Ford, Phys. Rev. D 80, 124019 (2009).

[15] T. Oniga and C. H.-T. Wang, Phys. Rev. D 94, 061501(R) (2016).

[16] W. B. Campbell and T. A. Morgan, Am. J. Phys. 44, 356 (1976); A. Matte, Can. J. Math. 5, 1 (1953); W. Campbell and T. Morgan, Physica (Amsterdam) 53, 264 (1971); P. Szekeres, Ann. Phys. (N.Y.) 64, 599 (1971); R. Maartens and B. A. Bassett, Classical Quant. Grav. 15, 705 (1998); M. L. Ruggiero and A. Tartaglia, Nuovo Cimento B 117, 743 (2002); J. Ramos, M. de Montigny, and F. Khanna, Gen. Relativ. Gravit. 42, 2403 (2010).