Abstract

Let $\Gamma$ be a connected, locally finite graph of finite tree width and $G$ be a group acting on it with finitely many orbits and finite node stabilizers. We provide an elementary and direct construction of a tree $T$ on which $G$ acts with finitely many orbits and finite vertex stabilizers. Moreover, the tree is defined directly in terms of the structure tree of optimally nested cuts of $\Gamma$.

Once the tree is constructed, standard Bass-Serre theory yields that $G$ is virtually free. This approach simplifies the existing proofs for the fundamental result of Muller and Schupp that characterizes context-free groups as f.g. virtually free groups. Our construction avoids the explicit use of Stallings’ structure theorem and it is self-contained.

We also give a simplified proof for an important consequence of the structure tree theory by Dicks and Dunwoody which has been stated by Thomassen and Woess. It says that a f.g. group is accessible if and only if its Cayley graph is accessible.

Keywords. Combinatorial group theory, context-free group, structure tree, finite treewidth, accessible graph.

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1 Introduction

A seminal paper of Muller and Schupp [20] showed that a group $G$ is context-free if and only if it is a finitely generated virtually free group. A group $G$ is context-free, if there is some finite set $\Sigma$ and a surjective homomorphism $\varphi : \Sigma^* \rightarrow G$ such that the associated group language $L_G = \varphi^{-1}(1)$ is context-free in the sense of formal language theory. A group $G$ is virtually free, if it has a free subgroup of finite index. Finitely generated (f.g.) virtually free groups were the basic examples for context-free groups, because the standard algorithm to solve their word problem runs on a deterministic pushdown automaton and these automata recognize a proper subfamily of context-free languages. The deep insight by Muller and Schupp is that amazingly the converse holds: If $G$ is
context-free, then $G$ is a finitely generated virtually free group. Over the past decades a wide range of other characterizations of context-free (or f.g. virtually free) groups have been found showing the importance of this class.

The various equivalent characterizations include: (1) fundamental groups of finite graphs of groups with finite vertex groups [15], (2) f.g. groups having a Cayley graph which can be $k$-triangulated [20] (3) f.g. groups having a Cayley graph with finite treewidth [18], (4) universal groups of finite pregroups [21], (5) groups having a finite presentation by some geodesic string rewriting system [12], and (6) f.g. groups having a Cayley graph with decidable monadic second-order theory [18]. For some other related results see the recent survey [2].

The result of Muller and Schupp was stated in [20] as a conjecture, since there it was shown under the assumption that all finitely presented groups are accessible. This assumption was used in order to apply Stallings’ structure theorem [25]. Accessibility means that the process of splitting the group with Stallings’ structure theorem eventually comes to an end. The accessibility of finitely presented groups was proved later by Dunwoody [10]. In subsequent proofs the result [10] could be replaced showing explicit upper bounds on the number how often the splitting with Stallings’ structure theorem can be performed, see e.g. [29].

However the reference to [25] remained. Indeed, almost all proofs in the literature showing that a context-free group is virtually free use the structure theorem by Stallings. Recently, in [2] another proof was given, which instead of Stallings’ structure theorem and a separate result for accessibility uses a more general, but rather technical result due to Dunwoody [8].

The starting point for our contribution has been as follows: Circumvent the deep theorems of Dunwoody and Stallings by starting with a f.g. group $G$ having a Cayley graph with finite treewidth. Construct from these data a tree on which $G$ acts with finite node stabilizers and with finitely many orbits. Apply Bass-Serre theory [24] to see that $G$ can be realized as a fundamental group of a finite graph of groups with finite vertex groups. It is known by [15] that these groups are f.g. and virtually free.

To follow this roadmap became possible due to a recent paper by Krön [16], which presents a simplified version of Dunwoody’s cut construction [9]. We realized that Krön’s proof of Stallings’ structure theorem can be modified such that it yields the tree we were looking for, when we start with a f.g. group $G$ having a Cayley graph with finite treewidth. We could not use Krön’s result as a black box, because in his paper he deals with cuts of globally minimal weight, only. Thus all cuts have the same weight whereas we need to consider cuts of different weight in order to get a non-refinable decomposition as fundamental group of a graph of groups.

Our approach leads to the following result: Let $\Gamma$ be a connected, locally finite graph with finite treewidth, and let $G$ be a group acting on $\Gamma$ such that $G\setminus\Gamma$ is finite and each node stabilizer $G_v$ is finite. Then $G$ is finitely generated and virtually free.

This is the essence of Corollary 5.10. To the best of our knowledge this result has not been formulated elsewhere. On the other hand, it is also clear that Corollary 5.10 can be derived rather easily from existing results in the literature. So, the main new contribution of the present paper is the new construction of optimally nested cuts and a direct self-contained combinatorial proof of Theorem 5.9 which implies Corollary 5.10 by Bass-Serre theory.
In Theorem 7.5 we also give a new elegant self-contained proof for another important result in this area by Thomassen and Woess which is a consequence of [5, Thm. II 2.20]: Let $\Gamma$ be a locally finite, connected, accessible graph, and let a f.g. group $G$ act on $\Gamma$ such that $G\setminus \Gamma$ is finite and each node stabilizer $G_v$ is finite. Then the group $G$ is accessible.

The outline of the paper is as follows: Section 2 fixes some notation. In Section 3 we follow [16] introducing the necessary modifications. The focus in this section is on accessible graphs c.f. Definition 3.6. We work with bi-infinite simple paths rather than with ends. This avoids some technical definitions and is more intuitive when drawing pictures as in Figure 3 or Figure 4. A key observation in Section 3 is Lemma 3.4 which is valid for minimal cuts of different weight, and not only for cuts with globally minimal weight as it was proved before in [9] and [16]. Together with another important – although not new – observation in Lemma 3.7 this leads to Proposition 3.14 saying that the set of optimally nested cuts forms a tree set in the sense of [8].

Section 4 deals with blocks. In the definition of a block we deviate from [16] and we use the one as in [26]. The central result is Proposition 4.6. It says that blocks have at most one end, which finally leads to Theorem 5.9 and Corollary 7.6.

Section 5 recalls the notion of finite treewidth. The results of Section 3 and Section 4 yield the desired proof of Theorem 5.9.

Section 6.1 shows how to derive the result of Muller and Schupp [20] using our approach. This section does not contain any new material, but we tried to have a concise presentation. In particular, we omit the technical notion of a $k$-triangulation of a graph by showing directly that the Cayley graph of a context-free group has finite treewidth. This can be done with the very same ideas which are present in [20]. Then, we can apply Corollary 5.10 to show that a context-free group is virtually free.

2 Preliminaries

2.1 Preliminaries on graphs

A directed graph $\Gamma$ is given by the following data: A set of vertices $V = V(\Gamma)$, a set of edges $E = E(\Gamma)$ together with two mappings $s : E \to V$ and $t : E \to V$. The vertex $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$. A vertex $u$ and an edge $e$ are incident, if $u \in \{s(e), t(e)\}$. The degree of $u$ is the number of incident edges, and $\Gamma$ is called locally finite, if the degree of all vertices is finite.

An undirected graph $\Gamma$ is a directed graph such that the set of edges $E$ is equipped with a fixed point free involution $e \mapsto \overline{e}$. (i.e. a map such that $e = \overline{\overline{e}}$ and $e \neq \overline{e}$ for all $e \in E$). Furthermore we demand $s(e) = t(\overline{e})$. An undirected edge is the set $\{e, \overline{e}\}$. By abuse of language we denote an undirected edge simply by $e$, too.

If we speak about a graph, then we always mean an undirected graph, otherwise we say specifically directed graph. Most of the time we only consider (undirected) graphs without loops and multi-edges. In this case we identify $E$ with two-element sets of incident vertices $\{u, v\}$ and write $e = uv$ if either $s(e) = u$ and $t(e) = v$ or $s(\overline{e}) = u$ and $t(\overline{e}) = v$.
For \( S \subseteq V(\Gamma) \) and \( v \in V(\Gamma) \) define as usual in graph theory \( \Gamma(S) \) (resp. \( \Gamma - S \)) to be the subgraph of \( \Gamma \) which is induced by the vertex set \( S \) (resp. \( V(\Gamma) \setminus S \)) and \( \Gamma - v = \Gamma - \{v\} \). We also write \( \overline{S} \) for the complement of \( S \), i.e., \( \overline{S} = V(\Gamma) \setminus S \). Likewise for \( e \in E(\Gamma) \) we let \( \Gamma - e = (V(\Gamma), E(\Gamma) \setminus \{e\}) \).

A path is a subgraph \( \{(v_0, \ldots, v_n), \{e_1, \ldots, e_n\}\} \) such that \( s(e_i) = v_{i-1} \) and \( t(e_i) = v_i \) for all \( 1 \leq i \leq n \). It is simple, if the vertices are pairwise disjoint. It is closed, if \( v_0 = v_n \). A cycle is a closed path with \( n \geq 3 \) such that \( v_1, \ldots, v_n \) is a simple path.

The distance \( d(u, v) \) between \( u \) and \( v \) is defined as the length (i.e., the number of edges) of the shortest path connecting \( u \) and \( v \). We let \( d(u, v) = \infty \), if there is no such path. A path \( v_0, \ldots, v_n \) is called geodesic, if \( n = d(v_0, v_n) \). An infinite path is defined as geodesic, if all its finite subpaths are geodesic. For \( A, B \subseteq V(\Gamma) \) the distance is defined as \( d(A, B) = \min\{d(u, v) \mid u \in A, v \in B\} \).

A graph \( \Gamma \) is called connected, if \( d(u, v) < \infty \) for all vertices \( u \) and \( v \). A tree is a connected graph without any cycle. If \( T = (V, E) \) is a tree, we may fix a root \( r \in V \). This gives an orientation \( E^+ \subseteq E \) by directing all edges “away from the root”. In this way a rooted tree becomes a directed graph \((V, E^+)\) which refers to the tree \( T = (V, E^+ \cup E^-) \), where \( E^- = E \setminus E^+ \).

In the following, when we write \( \Gamma \) we always mean a locally finite and connected graph. Graphs which do not meet these conditions are denoted with different letters. In particular, the capital letter \( T \) refers to a tree.

### 2.2 Preliminaries on groups

The paper is mainly concerned with finitely generated groups. Let \( G \) be a group. The Cayley graph \( \Gamma \) of \( G \) depends on \( G \) and on a generating set \( X \subseteq G \). It is defined by \( V(\Gamma) = G \) and \( E(\Gamma) = \{(g, ga) \mid g \in G \text{ and } 1 \neq a \in X\} \), with the obvious incidence functions and involution. For an edge \((g, ga)\) we call \( a \) the label of \((g, ga)\) and extend this definition also to paths. The Cayley graph is without loops and without multi-edges. It is connected because \( X \) generates \( G \). The Cayley graph \( \Gamma \) is locally finite if and only if \( X \) is finite. Sometimes we suppress \( X \) if there is a standard choice for the generating set. For example, if \( G = F(X) \) is the free group over \( X \), then the Cayley graph of \( G \) refers to \( X \) and it is a tree. By the infinite grid we mean the Cayley graph of \( \mathbb{Z} \times \mathbb{Z} \) with generators \((1, 0)\) and \((0, 1)\).

A group \( G \) acts on a graph \( \Gamma = (V, E) \), if there is a (left-)action of \( G \) on \( V \) and a (left-)action on \( E \) such that \( s(g \cdot e) = g \cdot s(e) \) and \( t(g \cdot e) = g \cdot t(e) \) for all \( g \in G \) and \( e \in E \). If \( G \) acts on \( \Gamma \), then we can define its quotient graph \( G \backslash \Gamma \). Its vertices (resp. edges) are the orbits \( Gu \) for \( u \in V \) (resp. \( Ge \) for \( e \in E \)). We say that \( G \) acts with finitely many orbits, if \( G \backslash \Gamma \) is finite.

Let \( \mathcal{G} \) denote some class of groups. A group \( G \) is called virtually \( \mathcal{G} \), if it has a subgroup of finite index, which is in \( \mathcal{G} \). Virtually finite groups are finite. The focus in this paper is on virtually free groups.

### 3 Cuts and structure trees

The constructions in this section follow the paper by Krön [16], which itself gives a simplified approach to Dunwoody’s constructions of cuts ([9]). The main difference lies in the definition of minimal cuts.
3.1 Cuts and optimally nested cuts

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. Recall that we assume $\Gamma$ to be connected and locally finite. For a subset $C \subseteq V(\Gamma)$ we define the edge- and vertex-boundaries of $C$ as follows:

$$\delta C = \{ uv \in E(\Gamma) \mid u \in C, v \in \overline{C} \} = \text{edge-boundary.}$$

$$\beta C = \bigcup \{ \{ u, v \} \mid uv \in \delta C \} = \text{vertex-boundary.}$$

**Definition 3.1** A cut is a subset $C \subseteq V(\Gamma)$ such that the following conditions hold.

1. $C$ and $\overline{C}$ are non-empty and connected.
2. $\delta C$ is non-empty and finite.

The weight of a cut is defined by $|\delta C|$. If $|\delta C| \leq k$, then $C$ is called a $k$-cut.

According to our definition cuts exist only in connected graphs, but connectedness is not enough as cuts have finite weight. Consider the infinite grid $\mathbb{Z} \times \mathbb{Z}$, i.e. the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ where $(i, j)$ is adjacent to the four vertices $(i, j \pm 1)$ and $(i \pm 1, j)$. It is connected and locally finite, but there are no cuts of finite weight splitting the grid into two parts.

In a connected, locally finite graph, cuts can be constructed by some ping-pong. Start with some $S \subseteq V(\Gamma)$ such that $\Gamma - S$ splits into several connected components and let $D$ be one of them. Then choose $C$ to be one connected component in $\Gamma - D$. It turns out that $C$ is connected and non-empty. Moreover, $\beta C \subseteq \beta S$ is finite. Note that all cuts arise this way.

The following well-known observation is crucial. It can be found e.g. in [26] in a slightly different formulation:

**Lemma 3.2** Let $S \subseteq V(\Gamma)$ be finite and $k \geq 1$. There are only finitely many $k$-cuts $C$ with $\beta C \cap S \neq \emptyset$.

**Proof.** We prove the statement by induction on $k$. For $k = 1$ the statement holds because $\Gamma$ is locally finite. Now, let $k > 1$. If $C$ is a $k$-cut such that $\beta C \cap S \neq \emptyset$ and $|\delta C| = k$, then one edge $e \in \delta C$ has an endpoint in $S$. Since $|\delta C| \geq 2$, the graph $\Gamma - e$ is still connected. Let $e = uv$, then there is a path $\gamma$ from $u$ to $v$ in $\Gamma - e$. Since $C$ separates the vertices $u$ and $v$, one vertex of $\gamma$ must be contained in $\beta C$. Therefore $C$ is a $(k-1)$-cut in $\Gamma - e$ with $\beta C \cap \gamma \neq \emptyset$. The vertex set of $\gamma$ is finite, and by the induction hypothesis there are only finitely many $(k-1)$-cuts intersecting $\gamma$. Because of the local finiteness of $\Gamma$ there are only finitely many choices for edges $e$ with an endpoint in $S$. Thus there are only finitely many $k$-cuts $C$ with $\beta C \cap S \neq \emptyset$. □

**Definition 3.3** Two cuts $C$ and $D$ are called nested, if one of the four inclusions $C \subseteq D$, $C \subseteq \overline{D}$, $\overline{C} \subseteq D$ or $\overline{C} \subseteq \overline{D}$ holds.

The set $\{ C \cap D, \overline{C} \cap \overline{D}, C \cap \overline{D}, \overline{C} \cap D \}$ is called the set of corners of $C$ and $D$, see Figure 1. Two corners $E, E'$ of $C$ and $D$ are called opposite, if either $\{E, E'\} = \{C \cap D, \overline{C} \cap \overline{D}\}$ or $\{E, E'\} = \{\overline{C} \cap D, C \cap \overline{D}\}$. Two different
corners are called adjacent, if they are not opposite. Note that two cuts $C,D$ are nested, if and only if one of the four corners of $C$ and $D$ is empty.

We are interested in bi-infinite simple paths which can be split into two infinite pieces with a cut of finite weight. For a bi-infinite simple path $\alpha$ denote:

$$C(\alpha) = \{ C \subseteq V(\Gamma) \mid C \text{ is a cut and } |\alpha \cap C| = \infty = |\alpha \cap \overline{C}| \}.$$  
$$C_{\min}(\alpha) = \{ C \in C(\alpha) \mid |\delta C| \text{ is minimal in } C(\alpha) \}.$$  

Thus, $C(\alpha) \neq \emptyset$ if there is a cut of finite weight such that the graph $\alpha - \delta C$ has exactly two infinite components each of these two being a one-sided infinite subpath of $\alpha$. We define the set of minimal cuts $C_{\min}$ by

$$C_{\min} = \bigcup \{ C_{\min}(\alpha) \mid \alpha \text{ is a bi-infinite simple path} \}.$$  

In the infinite grid $\mathbb{Z} \times \mathbb{Z}$ we have $C_{\min} = \emptyset$. Note that the set of minimal cuts may contain cuts of very different weight. Actually we might have $C,D \in C(\alpha) \cap C_{\min}$ with $C \in C_{\min}(\alpha)$, but $D \notin C_{\min}(\alpha)$. In such a case there must be another bi-infinite simple path $\beta$ with $D \in C(\alpha) \cap C_{\min}(\beta)$ and $|\delta C| < |\delta D|$.

For example, let $\Gamma$ be the subgraph of the infinite grid $\mathbb{Z} \times \mathbb{Z}$ which is induced by the pairs $(i,j)$ satisfying $j \in \{0,1\} \text{ or } i = 0 \text{ and } j \geq 0$. Let $\alpha$ be the bi-infinite simple path with $i = 0 \text{ or } j = 1$ and $i \geq 0$ and let $\beta$ be the bi-infinite simple path defined by $j = 0$. Then there are such cuts with $|\delta C| = 1$ and $|\delta D| = 2$, as depicted in Figure 2.

**Lemma 3.4** Let $C \in C_{\min}(\alpha)$ and $D \in C_{\min}(\beta)$. There are two opposite corners $E,E'$ of $C$ and $D$ such that $E \in C_{\min}(\alpha)$ and $E' \in C_{\min}(\alpha) \cup C_{\min}(\beta)$.

**Proof.** We distinguish two cases. First, let $D \in C_{\min}(\alpha)$. Hence, $C,D \in C_{\min}(\alpha)$. In this case we may assume $\alpha = \beta$. Hence, there are opposite corners $E$ and $E'$ such that $|\alpha \cap E| = |\alpha \cap E'| = \infty$.

In the other case we have $D \notin C_{\min}(\alpha)$. We claim that there must be one corner $K$ of $C$ and $D$ such that $|\alpha \cap K| < \infty$ and $|\beta \cap K| < \infty$. Indeed, if there is no such corner $K$, then infinite parts of $\alpha$ and $\beta$ are in opposite corners. In particular, $\alpha$ and $\beta$ are split by both by $C$ as well as by $D$ in two infinite pieces. This implies $|\delta C| = |\delta D|$, and hence $D \in C_{\min}(\alpha)$. Thus such a corner
Figure 2: The subgraph of the grid \( \mathbb{Z} \times \mathbb{Z} \) induced by the pairs \((i,j)\) satisfying \( j \in \{0,1\} \) or \( i = 0 \) and \( j \geq 0 \). Here we have \( D \in \mathcal{C}(\alpha) \cap \mathcal{C}_{\min} \) but \( D \notin \mathcal{C}_{\min}(\alpha) \).

The graph \( \Gamma(E) \) contains an infinite connected component \( F \subseteq E \) such that \(|\alpha \cap F| = \infty\). Let us show that \( F \) is non-empty and connected. The set \( \overline{F} \) is non-empty and infinite, because \( E' \subseteq \overline{F} \). Now fix a vertex \( v \in \overline{E'} \) and let \( u \in \overline{F} \).

There is a path \( \gamma \) from \( u \) to \( v \) in \( \Gamma \) and on this path there is a first vertex \( w \) with \( w \in \overline{C} \cup \overline{D} \). If the initial path from \( u \) to \( w \) was using a point of \( F \), then it would be a path in \( E \), and \( u \) would be in the connected component \( F \), which was excluded. Hence we can connect \( u \) to \( w \) in \( \overline{\Gamma - F} \). Now, by symmetry \( w \in \overline{C} \). But then \( w, v \in \overline{C} \subseteq \overline{\Gamma - F} \) and \( \overline{C} \) is connected. Hence, \( F \) is a cut.

In a symmetric way we find a cut \( F' \subseteq E' \) such that \(|\beta \cap F'| = \infty\). It remains to show that \( F = E \in \mathcal{C}_{\min}(\alpha) \) and \( F' = E' \in \mathcal{C}_{\min}(\beta) \).

Thus, \( E \) and \( E' \) are defined in both cases. Moreover the notation is such that \(|\alpha \cap E| = |\beta \cap E'| = \infty\). By renaming we may assume in addition that \( E = C \cap D \) and \( E' = \overline{C \cap D} \). It is however not yet clear that \( E \) and \( E' \) are cuts.

Figure 3: For all four corners \( K \) we have \( \max\{|K \cap \alpha|, |K \cap \beta|\} = \infty \).
We can write $\delta E = a + b + c + d$, where

- $a = |\{xy \mid x \in F \land y \in C \cap D\}|$
- $b = |\{xy \mid x \in F \land y \in E'\}|$
- $c = |\{xy \mid x \in F \land y \in C \cap D\}|$
- $d = |\{xy \mid x \in E \setminus F \land y \notin E\}|$

Likewise, we have $\delta E' = a' + b' + c' + d'$, where

- $a' = |\{xy \mid x \in F' \land y \in \overline{C} \cap D\}|$
- $b' = |\{xy \mid x \in F' \land y \in E\}|$
- $c' = |\{xy \mid x \in F' \land y \in C \cap \overline{D}\}|$
- $d' = |\{xy \mid x \in E' \setminus F' \land y \notin E'\}|$

With the minimality of $|\delta C|$ and $|\delta D|$ we derive the following:

- $a + b + c \leq |\delta C| \leq |\delta F| = a + b + c$
- $a' + b' + c \leq |\delta D| \leq |\delta F'| = a' + b' + c'$

We conclude $|\delta C| = |\delta F|$ and $|\delta D| = |\delta F'|$. This implies $F \in \mathcal{C}_{\min}(\alpha)$ and $F' \in \mathcal{C}_{\min}(\beta)$.

We still have to show $E = F$ and $E' = F'$. For this it is enough to show that $d = d' = 0$. Assume by contradiction that $d + d' \geq 1$. Say, $d \geq 1$. Then we have $|\delta C| + |\delta D| > a + b + c + a' + b' + c'$, because $a + b + c$ counts only edges of $\delta C$ and $\delta D$ which have one endpoint in $F$. This contradicts the assertion $|\delta C| = |\delta F|$ and $|\delta D| = |\delta F'|$.

We define for every cut $C$ and $k \geq 1$ a cardinality $m_k(C)$ as follows:

$$m_k(C) = |\{ D \mid C \text{ and } D \text{ are not nested and } D \text{ is a } k\text{-cut.} \}|.$$

**Lemma 3.5** Let $k \in \mathbb{N}$ and $C$ be a cut, then $m_k(C)$ is finite.
Proof. If $C$ and $D$ are not nested, then there are paths from $C \cap D$ to $\overline{C} \cap D$ inside $D$ and from $C \cap \overline{D}$ to $\overline{C} \cap \overline{D}$ inside $\overline{D}$. The first path yields a vertex $u \in \beta C \cap D$ and the second one yields a vertex $v \in \beta C \cap \overline{D}$. Now, let $S$ be a finite connected subgraph of $\Gamma$ containing all the vertices of $\beta C$. Hence, $u,v \in S$ and we see that $\beta D \cap S \neq \emptyset$. Since $S$ is finite, Lemma 3.2 shows that only finitely many $k$-cuts can be not nested with $C$. □

We are mainly interested in graphs $\Gamma$ where the weight over all cuts in $C_{\min}$ can be bounded by some constant. This leads to the notion of accessible graph due to [26]:

**Definition 3.6** A graph is called accessible, if there exists a constant $k \in \mathbb{N}$, such that for every bi-infinite simple path $\alpha$ either $\mathcal{C}(\alpha)$ is empty or $\mathcal{C}(\alpha)$ contains some $k$-cut; i.e., the weight of every cut in $C_{\min}$ is at most $k$.

For the rest of this section we assume that $\Gamma$ is accessible. By Lemma 3.5 we can define for each $C \in C_{\min}$ a natural number as follows:

$$m(C) = |\{ D \in C_{\min} \mid C \text{ and } D \text{ are not nested } \}|.$$

Note that by Lemma 3.4 we know that for all $C,D \in C_{\min}$ there exist opposite corners $E,E'$ with $E,E' \in C_{\min}$.

**Lemma 3.7** Let $C,D \in C_{\min}$ be not nested and let $E,E'$ be two opposite corners of $C$ and $D$ with $E,E' \in C_{\min}$. Then the following holds:

$$m(E) + m(E') < m(C) + m(D)$$

**Proof.** First, we show two claims:

1. If a cut $F$ is neither nested with $E$ nor with $E'$, then $F$ is neither nested with $C$ nor with $D$.

   To see this, let $F$ be nested with $C$. We may assume that $C \subseteq F$. It follows that $C \cap D, C \cap \overline{D} \subseteq F$; and for two opposite corners at least one of them is nested with $F$.

2. If $F$ is not nested with $E$ or not nested with $E'$, then $F$ is not nested with $C$ or not nested with $D$.

   To see this, let $F$ be nested with $C$ and with $D$. We may assume that $C \subseteq F$. Moreover, we may assume $D \subseteq F$ or $D \subseteq \overline{F}$ (exchanging, if necessary, $D$ with $\overline{D}$). If there was $D \subseteq \overline{F}$, then we would have $C \cap D = \emptyset$. This is in contradiction to the assumption that $C$ and $D$ are not nested. Hence, $D \subseteq F$ and therefore $C \cap D \subseteq F$. We obtain $C \cap D \subseteq F, \overline{C} \cap D \subseteq F$, and $C \cap \overline{D} \subseteq F$. Moreover, $\overline{F} \subseteq C \cap \overline{D}$. So all the four corners are nested with $F$.

   Counting only those $F$, claims (1) and (2) yield:

   $$m(E) + m(E') \leq m(C) + m(D)$$

Now, $C$ is nested with both corners $E$ and $E'$. Hence, it is not counted on the left-hand side of the inequality. However, $C$ is counted on the right-hand side, because $C$ is not nested with $D$. That means the inequality is strict. □
We use the following notation:

\[ m_\alpha = \min \{ m(C) \mid C \in C_{\min}(\alpha) \} \]

\[ C_{\text{opt}}(\alpha) = \{ C \in C_{\min}(\alpha) \mid m(C) = m_\alpha \} \]

\[ C_{\text{opt}} = \bigcup \{ C_{\text{opt}}(\alpha) \mid \alpha \text{ is a bi-infinite simple path} \} \]

**Definition 3.8** A cut \( C \in C_{\text{opt}} \) is called an **optimally nested cut**.

**Proposition 3.9** Let \( C, D \in C_{\text{opt}} \). Then \( C \) and \( D \) are nested.

**Proof.** Let \( C, D \in C_{\text{opt}} \). We can find simple bi-infinite paths \( \alpha \) and \( \beta \), such that \( C \in C_{\text{opt}}(\alpha) \) and \( D \in C_{\text{opt}}(\beta) \). We may assume that \( m_\beta \leq m_\alpha \). With Lemma 3.3 we obtain that there are two opposite corners \( E \) and \( E' \) of \( C \) and \( D \), such that \( E \in C_{\min}(\alpha) \) and \( E' \in C_{\min}(\alpha) \cup C_{\min}(\beta) \).

Now, let us assume that \( C \) and \( D \) are not nested. Then Lemma 3.7 gives us the following contradiction:

\[ m_\alpha + m_\beta \leq m(E) + m(E') < m(C) + m(D) = m_\alpha + m_\beta. \]

\[ \square \]

Analog results to Proposition 3.9 are Theorem 1.1 in [9] or Theorem 3.3 of [10]. In contrast to these results, Proposition 3.9 allows that \( C_{\text{opt}} \) may contain cuts of different weights. We have to deal with cuts of different weights, because we wish to get a “complete” decomposition of virtually free groups like \((\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ast \mathbb{Z}/2\mathbb{Z})\). Like in the graph in Figure 2 in the Cayley graph of this group cuts with weight 1 and 2 are necessary to split all bi-infinite paths in two infinite pieces, see Figure 5.

![Figure 5: The Cayley graph of the group \((\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ast \mathbb{Z}/2\mathbb{Z})\) moving very fast towards your eyes.](image)

### 3.2 The structure tree

Recall that \( \Gamma \) is assumed to be accessible, hence \( C_{\text{opt}} \) is defined and there is some \( k \in \mathbb{N} \) such that every cut in \( C_{\text{opt}} \) is a \( k \)-cut.
Lemma 3.10  Let $C, D \in C_{\text{opt}}$. Then the set $\{E \in C_{\text{opt}} \mid C \subseteq E \subseteq D\}$ is finite.

Proof. Choose two vertices $u \in C$ and $v \in \overline{D}$, and a path $\gamma$ in $\Gamma$ connecting them. Every cut $E$ with $C \subseteq E \subseteq D$ must separate $u$ and $v$ and thus contain a vertex of $\gamma$. With Lemma 3.2 and the accessibility of $\Gamma$ it follows that there are only finitely many such cuts. □

The set $C_{\text{opt}}$ is partially ordered by $\subseteq$. By Lemma 3.10 we can draw its Hasse diagram. This means we introduce an arc from $C \in C_{\text{opt}}$ to $D \in C_{\text{opt}}$ if and only if $C \nsubseteq D$ and there is no $E \in C_{\text{opt}}$ between them. If there is an arc from $C$ to $D$, then there is also an arc from $D$ to $C$. In such a situation we put $C$ and $D$ in one class:

Definition 3.11  For $C, D \in C_{\text{opt}}$ we define the relation $C \sim D$ by the following condition:

Either $\overline{C} \nsubseteq D$ and for no $E \in C_{\text{opt}}$ we have $\overline{C} \nsubseteq E \nsubseteq D$ or $C = D$. (1)

The intuition behind this definition is as follows: Consider $(C, \overline{C})$ for $C \in C_{\text{opt}}$ as an edge set of some graph. Call edges $(C, \overline{C})$ and $(D, \overline{D})$ to be adjacent, if $C \sim D$. This makes sense due the following property.

Lemma 3.12  The relation $\sim$ is an equivalence relation.

Proof. Reflexivity and symmetry are immediate. Transitivity requires to check all inclusions how the cuts can be nested. A proof can be found e.g. in [8]. In order to keep the paper self-contained we repeat the proof for transitivity. Let $C \sim D \neq C$ and $D \sim E \neq D$. This implies $\emptyset \neq \overline{D} \subseteq C \cap E$. We have to show that $C \sim E$. The cuts $C$ and $E$ are nested due to Proposition 3.9. Hence we have one of the following four inclusions:

- $C \subseteq E$: This implies $\overline{D} \nsubseteq C \subseteq E$. Hence $C = E$, because $D \sim E$.
- $E \subseteq C$: This implies $\overline{C} \nsubseteq E \subseteq C$, Hence $C = E$, because $D \sim C$.
- $E \subseteq \overline{C}$: This contradicts $C \cap E \neq \emptyset$.
- $C \subseteq E$: Since $C \cap E \neq \emptyset$, we see $\overline{C} \nsubseteq E$. Now, let $\overline{C} \nsubseteq F \subseteq E$ for some $F \in C_{\text{opt}}$. Since $F$ and $D$ are nested, we obtain one of the following inclusions:

  - $D \subseteq F$: This implies $D \subseteq E$, in contradiction to $\overline{D} \nsubseteq E$.
  - $F \nsubseteq \overline{D}$: This implies $\overline{C} \nsubseteq F \nsubseteq \overline{D}$, in contradiction to $C \sim D$.
  - $F \subseteq \overline{D}$: This implies $\overline{C} \nsubseteq F \subseteq \overline{D}$, in contradiction to $\overline{C} \subseteq D$.
  - $\overline{D} \nsubseteq F$: This implies $\overline{D} \nsubseteq F \subseteq E$. Hence $F = E$, because $D \sim E$.

□

Definition 3.13  Let $T(C_{\text{opt}})$ denote the following graph:

$$V(T(C_{\text{opt}})) = \{[C] \mid C \in C_{\text{opt}}\}$$

$$E(T(C_{\text{opt}})) = \{(C, \overline{C}) \mid C \in C_{\text{opt}}\}$$

The incidence maps are defined by $s((C, \overline{C})) = [C]$ and $t((C, \overline{C})) = \overline{C}$.  

11
The directed edges are in canonical bijection with the pairs $([C], [\overline{C}])$. Indeed, let $C \sim D$ and $\overline{D} \sim \overline{C}$. It follows $C = D$ because otherwise $C \not\subseteq \overline{D} \not\subseteq C$. Thus, $T(C_{\text{opt}})$ is an undirected graph without self-loops and multi-edges.

The graph $T(C_{\text{opt}})$ is locally finite if and only if the equivalence classes $[C]$ are finite. Hence it is not locally finite, in general: For each $C \in C_{\text{opt}}$ there might be infinitely many $D \in C_{\text{opt}}$ with $D \sim C$. For example, consider the Cayley graph of the group $(\mathbb{Z} \times \mathbb{Z}) \ast \mathbb{Z}/2\mathbb{Z}$. There is one $\mathbb{Z} \times \mathbb{Z}$-plane through the origin and for every $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ there is one edge leaving this plane. Removing this edge defines a unique 1-cut $C_{i,j}$ with $\mathbb{Z} \times \mathbb{Z} \subseteq C_{i,j}$. It is in $C_{\text{opt}}$, because all other minimal cuts are nested with $C_{i,j}$. We have $C_{i,j} \subseteq C_{0,0}$ for $(i, j) \neq (0, 0)$, but there is no $E \in C_{\text{opt}}$ with $C_{i,j} \not\subseteq E \subseteq C_{0,0}$.

**Proposition 3.14** The graph $T(C_{\text{opt}})$ is a tree.

**Proof.** Let $\gamma$ be a simple path in $T(C_{\text{opt}})$ of length at least two. Then $\gamma$ corresponds to a sequence of cuts

$$C_0, \overline{C}_0 \sim C_1, \ldots, \overline{C}_{n-2} \sim C_{n-1}, \overline{C}_{n-1} = C_n$$

with $[C_{i-1}] \neq [C_{i+1}]$, so in particular $\overline{C}_{i-1} \neq C_i$ for $1 \leq i \leq n - 1$ (otherwise we would have $C_{i-1} = \overline{C}_i \sim C_{i+1}$). So we get a sequence

$$C_0 \not\subseteq C_1 \not\subseteq C_2 \not\subseteq \cdots \not\subseteq C_n$$

Therefore we have $C_0 \neq \overline{C}_{n-1}$ and $\overline{C}_0 \not\subseteq \overline{C}_{n-1}$. So $C_0 \not\sim \overline{C}_{n-1} = C_n$ and the original path is not a cycle. So $T(C_{\text{opt}})$ has no cycles.

It remains to show that $T(C_{\text{opt}})$ is connected. Let $[C], [D] \in V(T(C_{\text{opt}}))$. Since $C$ and $D$ are nested and $(C, \overline{C}), (D, \overline{D}) \in E(T(C_{\text{opt}}))$, we can assume $C \subseteq D$. By Lemma 3.10 there are only finitely many cuts $E \in C_{\text{opt}}$, with $C \subseteq E \subseteq D$. Now, let $C_0, C_1, \ldots, C_n$ be a non-refinable sequence of cuts in $C_{\text{opt}}$ such that

$$C = C_0 \not\subseteq C_1 \not\subseteq C_2 \not\subseteq \cdots \not\subseteq C_{n-1} \not\subseteq C_n = D$$

Then we obtain a path from $C$ to $D$:

$$C = C_0, \overline{C}_0 \sim C_1, \overline{C}_1 \sim C_2, \ldots, \overline{C}_{n-1} \sim C_n = D$$

Hence, $T(C_{\text{opt}})$ is connected and therefore a tree. □

**Remark 3.15** Dunwoody (8) introduced the notion of a tree set as a set of pairwise nested cuts, which is closed under complementing and such that for each $C, D \in \mathcal{C}$ the set $\{ E \subseteq C \mid C \subseteq E \subseteq D \}$ is finite. Thus, using this terminology, Proposition 3.14 and Lemma 3.10 show that $C_{\text{opt}}$ is a tree set. Once this is established Proposition 3.13 becomes a general fact due to Dunwoody.

4 Blocks

In this section, $\Gamma$ denotes a connected, locally finite, and accessible graph such that the group of automorphisms $\text{Aut}(\Gamma)$ acts with finitely many orbits on $\Gamma$. This means $\text{Aut}(\Gamma) \backslash \Gamma$ is finite. For example, if $\Gamma$ is the Cayley graph of a group $G$ with respect to some finite generating set $\Sigma \subseteq G$, then $\Gamma$ is connected, locally
Lemma 4.1 Let $|\text{Aut}(\Gamma)\backslash \Gamma| = 1$. However the extra condition is that $\Gamma$ is accessible\(^1\).

For $S \subseteq V(\Gamma)$ and $k \geq 1$ we let $N^k S = \{ v \in V(\Gamma) \mid d(v, S) \leq k \}$ denote the $k$-th neighborhood of $S$. Now, for a cut $C$ we can choose $k$ large enough such that $N^k C \cap \overline{C}$ is connected, because $\overline{C}$ is connected. (Indeed, all points in $\beta C \cap \overline{C}$ can be connected in $\overline{C}$, hence for some $k$ large enough these points can be connected in $N^k C \cap \overline{C}$. This $k$ suffices to make $N^k C \cap \overline{C}$ connected.) Finally, $\text{Aut}(\Gamma)\backslash \Gamma$ is finite, hence by accessibility and Lemma 4.2 there are only finitely many orbits of minimal cuts (see also proof of Lemma 4.7). Thus we can choose some $\kappa \in \mathbb{N}$ which works for all $C \in C_{\text{opt}} \subseteq C_{\text{min}}$ and fix it for the rest of this section.

Definition 4.1 Let $\text{Aut}(\Gamma)\backslash \Gamma$ be finite and $C_{\text{opt}}$ be the set of optimally nested cuts. Let $\kappa \geq 1$ be defined as above such that $N^\kappa C \cap \overline{C}$ is connected for all $C \in C_{\text{opt}}$. The block assigned to $[C] \in V(\Gamma(C_{\text{opt}}))$ is defined by:

$$B_{[C]} = \bigcap_{D \sim C} N^\kappa D$$

Lemma 4.2 Let $C \in C_{\text{opt}}$ and $g \in \text{Aut}(\Gamma)$ be such that $g(C) \sim C$. Then we have $g(B_{[C]}) = B_{[C]}$.

Proof. Let $v \in B_{[C]} = \bigcap \{ N^\kappa D \mid D \sim C \}$. Hence $gv \in \bigcap \{ N^\kappa gD \mid gD \sim gC \}$ for all $g \in \text{Aut}(\Gamma)$. Now, if $D \sim C$ and $gC \sim C$ for some $g \in G$, then $gD \sim gC \sim C$. Hence $gv \in B_{[C]}$. □

Example 4.3 Figure 6 shows a part of the Cayley graph of the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle a, b \mid a^2 = b^3 \rangle$. Here the minimal cuts cut the edges with label $a$, i.e. they cut through cosets of $\mathbb{Z}/2\mathbb{Z}$. The optimally nested cuts are exactly the minimal cuts. The three cuts depicted with dashed lines belong to the same equivalence class and the bold vertices form the respective block. Here we can choose $\kappa = 1$ for the definition of the blocks.

Lemma 4.4 We have $N^\kappa D \cap \overline{C} \subseteq B_{[C]}$ for all $D \sim C \in C_{\text{opt}}$.

Proof. It is enough to show that we have $N^\kappa C \cap \overline{C} \subseteq B_{[C]}$. Clearly, $N^\kappa C \cap \overline{C} \subseteq N^\kappa C$. Thus is enough to consider $D \sim C$, $D \neq C$ and to show that $N^\kappa C \cap \overline{C} \subseteq N^\kappa D$. This follows from:

$$N^\kappa C \cap \overline{C} \subseteq \overline{C} \subseteq D \subseteq N^\kappa D$$

□

Lemma 4.5 1. $B_{[C]}$ is connected for all $C \in C_{\text{opt}}$.

2. There is a number $\ell \in \mathbb{N}$ such that for all $C \in C_{\text{opt}}$ and all $S \subseteq B_{[C]}$ we have: Whenever two vertices $u, v \in B_{[C]}$ lie in different components of $B_{[C]} - S$, then $u, v \notin N^\ell S$ implies that they are in different connected components of $\Gamma - N^\ell S$, too.

\(^1\)This is always true, if $G$ is finitely presented due to a deep theorem of [10] together with the result of [26]. However, in the present approach we do not need to use these results; and there are examples of finitely generated groups where $\Gamma$ is not accessible by [11].
Figure 6: Block of six vertices in the Cayley graph of $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z}$

**Proof.** Note that $\boxdot$ is a special case of $\boxplus$ by choosing $S = \emptyset$. Let $\ell = \max \{ d(u, v) \mid D \in \mathcal{C}_{\text{opt}}, u, v \in \overline{D} \cap N^\kappa D \}$. Thus $\ell$ is a uniform bound on the diameters for the sets $N^\kappa D \cap \overline{D}$ for $D \in \mathcal{C}_{\text{opt}}$. It exists, because there are only finitely many orbits of optimally nested cuts.

Now let $B_{|C|} - S$ be disconnected and let $u, v \in B_{|C|} - N^\ell S$ be two vertices in different connected components which are connected by some path $\gamma$ in $\Gamma - N^\ell S$. If there is no such path then we have nothing to do. Let $v_m$ be the first vertex of $\gamma$, which does not lie in $B_{|C|}$. Thus for some $D \sim C$ we have $v_m \notin N^\kappa D$. Since $\kappa \geq 1$, we have $v_{m-1} \in N^\kappa D \cap \overline{D}$. For some $n > m$ we find the next vertex $v_n$, which is the first vertex after $v_m$ lying in $N^\kappa D$ again. As $v_n$ is the first one, we have $v_n \in N^\kappa D \cap \overline{D}$, too. Since $N^\kappa D \cap \overline{D}$ is connected, we can choose a path from $v_{m-1}$ to $v_n$ inside $N^\kappa D \cap \overline{D}$. This is a path inside $B_{|C|}$ by Lemma 4.4. Repeating this procedure for all segments of $\gamma$ which originally are not in $B_{|C|}$ yields a path inside $B_{|C|}$. Every such path must use a vertex from $S$. Hence on at least one new segment there is some $s \in S$. We may assume that this is on the segment above between $v_{m-1}$ and $v_n$. This implies $d(s, v_n) \leq \ell$. Thus there is no such path $\gamma$ in $\Gamma - N^\ell S$. \qed

$\Gamma$ is said to have **more than one end** if there is a finite set $S \subseteq V(\Gamma)$ such that $\Gamma - S$ has at least two infinite connected components. Otherwise, it has at most **one end**; and it has no end, if $\Gamma$ is finite. Since we only consider connected and locally finite graphs it follows that $\Gamma$ has more than one end, if and only if there exists a bi-infinite simple path $\alpha$ such that $\mathcal{C}(\alpha) \neq \emptyset$. Note that this observation does not hold for graphs which are not locally finite or not connected. (In the case of non-locally finite graphs there are even several non-equivalent ways to define ends.)
The key property of blocks is that blocks cannot have more than one end:

**Proposition 4.6** For $C \in \mathcal{C}_{\text{opt}}$ the block $B_{|C|}$ has at most one end.

**Proof.** Assume by contradiction that $B_{|C|}$ has more than one end. By Lemma 4.7 $B_{|C|}$ is connected, hence there is a bi-infinite simple path $\alpha$ and a finite subset $S \subseteq B_{|C|}$ such that two different connected components of $B_{|C|} - S$ contain infinitely many elements of $\alpha$. However, for all $D \sim C$ we have $\alpha \subseteq B_{|C|} \subseteq N^eS$ and $N^eS \cap \overline{D}$ is finite. Hence for all $D \sim C$ almost all nodes of $\alpha$ are in $D$ and $|\alpha \cap \overline{D}| < \infty$.

By Lemma 4.5 there are two different connected components of $\Gamma - N^eS$ containing each infinitely many elements of $\alpha$. Thus, the set $\mathcal{C}(\alpha)$ is not empty, hence there is an optimally nested cut $E$ such that $E \in \mathcal{C}(\alpha)$. This means $|\alpha \cap E| = \infty = |\alpha \cap \overline{E}|$. The cuts $C$ and $E$ are nested. We cannot have $E \subseteq \overline{C}$ or $E \subseteq C$. Hence by symmetry $E \not\subseteq C$. By Lemma 4.6 there is some $D \in [C]$ such that $E \subseteq D \not\subseteq C$. But we have just seen that almost all nodes of $\alpha$ belong to $D$. Thus, $|\alpha \cap E| < \infty$. This is a contradiction. $\square$

4.1 Actions on accessible graphs

We continue to assume that $\Gamma$ is a connected, locally finite, and accessible graph such that $\text{Aut}(\Gamma) \setminus \Gamma$ is finite. It is clear that $\text{Aut}(\Gamma)$ acts on $\mathcal{C}_{\text{opt}}$ and on the structure tree $T(\mathcal{C}_{\text{opt}})$.

**Lemma 4.7** Let $k \in \mathbb{N}$. The canonical action of $\text{Aut}(\Gamma)$ on the set of $k$-cuts has finitely many orbits, only. In particular $\text{Aut}(\Gamma)$ acts on $\mathcal{C}_{\text{opt}}$ and on the tree $T(\mathcal{C}_{\text{opt}})$ with finitely many orbits.

**Proof.** Let $\text{Aut}(\Gamma) \setminus V(\Gamma)$ be represented by some finite vertex set $U \subseteq V(\Gamma)$. With Lemma 4.2 it follows that there are only finitely many $k$-cuts $C$ such that $U \cap \beta C \neq \emptyset$. Since every cut is in the same orbit as some cut $C$ with $U \cap \beta C \neq \emptyset$, the group $\text{Aut}(\Gamma)$ acts on the set of $k$-cuts with finitely many orbits.

Since $\Gamma$ is accessible, there is a $k$ such that for all cuts $C \in \mathcal{C}_{\text{opt}}$ holds $|\beta C| \leq k$. For the last statement observe that $\{ (C, \overline{C}) : C \in \mathcal{C}_{\text{opt}} \}$ is the edge set of $T(\mathcal{C}_{\text{opt}})$. Thus, the action of $\text{Aut}(\Gamma)$ on $T(\mathcal{C}_{\text{opt}})$ has only finitely many orbits, too. $\square$

**Proposition 4.8** Let $\mathcal{G}$ be a class of groups which is closed under taking finite-index normal subgroups, and let $G$ be a group which acts on $\Gamma$ such that all vertex stabilizers $G_v = \{ g \in G : gv = v \}$ belong to the class $\mathcal{G}$. Then we have:

1. The group $G$ acts with virtually $\mathcal{G}$ edge stabilizers on the tree $T(\mathcal{C}_{\text{opt}})$.
2. If $B_{|C|}$ is finite for all $C \in \mathcal{C}_{\text{opt}}$, then $G$ acts with virtually $\mathcal{G}$ vertex stabilizers on the tree $T(\mathcal{C}_{\text{opt}})$.

**Proof.** First, let $\emptyset \neq U \subseteq V(\Gamma)$ be any finite set. The action of $G$ induces a homomorphism of the stabilizer $G_U = \{ g \in G : gU \subseteq U \}$ to the finite group of permutations on $U$. Its kernel is $\bigcap_{u \in U} G_u$.

Now fix one vertex $v \in U$. Then for every $k \in \mathbb{N}$ an element $g \in G_v$ defines a permutation on the set of vertices $\{ u \in V(\Gamma) : d(v, u) \leq k \}$. Choose $k$ large.
because $\Gamma$ is locally finite. The action of $G$ together with a family $\mathcal{X}$ closed under forming finite index normal subgroups, so $N$ is $G$. Furthermore $N \leq \bigcap_{u \in U} G_u \leq G_U$ with finite index and so $G_U$ is virtually $G$.

An element in an edge stabilizer $\beta C$ maps $\beta C$ to itself. Since $\beta C$ is finite, $G_{\langle \beta C \rangle}$ is virtually $G$.

Now, let $g \in G_{\langle C \rangle}$, then we have $g(B_{\langle C \rangle}) = B_{\langle C \rangle}$ by Lemma 5.2. If $B_{\langle C \rangle}$ is finite, $G_{\langle C \rangle}$ is virtually $G$. □

5 Finite treewidth

In this section we do not assume that, a priori, $\Gamma$ is accessible. We give a sufficient condition for accessibility in Proposition 5.9.

Definition 5.1 Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. A tree $T = (V(T), E(T))$ together with a family $\mathcal{X} = \{X_t\}_{t \in V(T)}$ of subsets of $V(\Gamma)$ is called tree decomposition of $\Gamma$, if the following conditions are fulfilled:

(T1) For every node $v \in V(\Gamma)$ there is a $t \in V(T)$, such that $v \in X_t$, i.e. $V(\Gamma) = \bigcup_{t \in V(T)} X_t$.

(T2) For every edge $uv \in E(\Gamma)$ there is a $t \in V(T)$, such that $u, v \in X_t$.

(T3) If $v \in X_t \cap X_s$, then we have $v \in X_r$ for all vertices $r$ of the tree which are on the unique geodesic path from $s$ to $t$, i.e., the set $\{ t \in V(T) \mid v \in X_t \}$ forms a subtree of $T$.

The sets $X_t$ are called buckets or bags. $\Gamma$ is said to have finite treewidth, if there exists a constant $k \in \mathbb{N}$ such that $|X_t| \leq k$ for all $X_t \in \mathcal{X}$.

Tree decompositions were introduced by Robertson and Seymour in [22]. For some basic properties of tree decompositions see [7].

Lemma 5.2 If $\Gamma$ has finite treewidth, then all subgraphs of $\Gamma$ have finite treewidth, too.

Proof. Trivial. □

Lemma 5.3 If $\Gamma$ is locally finite with finite treewidth, then there is finite tree decomposition where each vertex $v \in \Gamma$ occurs in finitely many bags, only.

Proof. Choose and fix for every edge $uv \in E(\Gamma)$ some vertex $t = t_{uv} \in V(T)$ with $u, v \in X_t$. Now, for each vertex $u$ let $T_u$ be the subtree spanned by the $t_{uv}$ for $v \in V(\Gamma)$. It is finite because $\Gamma$ is locally finite. Remove $u$ from all bags which do not belong to $T_u$. This yields still a tree decomposition. □

Lemma 5.4 Let $\Gamma$ be a graph with finite treewidth and uniformly bounded degree. Then there exists a $k \in \mathbb{N}$, such that: For every one-sided infinite simple path $\gamma$, every $v_0 \in V(\Gamma)$ and every $n \in \mathbb{N}$ there is a $k$-cut $D$ with $d(v_0, D) \geq n$, such that $v_0 \in D$ and $|D \cap \gamma| = \infty$. 

16
Remark 5.5 It follows from the following proofs that in the case of \( \Gamma \) being a locally finite Cayley graph also the converse of the lemma holds. Thus, when restricting to Cayley graphs of f.g. groups, the statement of Lemma 5.4 gives a characterization of Cayley graphs of context-free groups by its own. A very similar result is due to Woess [27]. It states that a group is context-free if and only if the ends of its Cayley graph have uniformly bounded diameter.

Proof. Let \( d \) be the maximal degree of \( \Gamma \) and let \( m = \max \{|X_t| \mid t \in V(T)\} \) be the maximal size of a bag in the tree decomposition \((T, \mathcal{X})\). We let \( k = dm \).

Let \( t_0 \in V(T) \) such that \( v_0 \in X_{t_0} \). Consider vertices \( u, v \) which are in bags of two different connected components of \( T - t_0 \), then every path from \( u \) to \( v \) has a vertex in \( X_{t_0} \), so \( u \) and \( v \) are not in the same connected component of \( \Gamma - X_{t_0} \). Now let \( C_{t_0, \gamma} \) be the connected component of \( \Gamma - X_{t_0} \) which contains infinitely many vertices of \( \gamma \). Then the set \( C_{t_0, \gamma} \) is contained in the union of the bags of one connected component of \( T - t_0 \). Let \( t_1 \) be the neighbor of \( t_0 \) in this connected component, which is uniquely defined because \( T \) is a tree.

Repeating this procedure yields a sequence of connected sets \( C_{t_0, \gamma}, C_{t_1, \gamma}, C_{t_2, \gamma}, \ldots \) and a simple path \( t_0, t_1, t_2, \ldots \) in \( T \). By Lemma 5.3 we may assume that every node \( v \in \Gamma \) is contained in only finitely many bags. Hence, we can choose \( \ell \) large enough, that \( X_{t_\ell} \) does not contain any \( v \in \Gamma \) with \( d(v_0, v) \leq n \). Since \( \gamma \) is a one-sided infinite simple path, \( X_{t_\ell} \) separates all but finitely many vertices of \( \gamma \) from \( v_0 \).

Now, let \( D \) be the connected component of \( C_{t_\ell, \gamma} \) which contains \( v_0 \). Then \( D \) is connected, because every vertex in another connected component of \( C_{t_\ell, \gamma} \) is connected inside \( T \) with \( C_{t_\ell, \gamma} \).

Since every edge of \( \delta D \) has one node in \( X_{t_\ell} \), we have \( |\delta D| \leq dm = k \). Thus, \( D \) is a \( k \)-cut with \( v_0 \in D \) and \( |\delta D \cap \gamma| = \infty \). Furthermore every path from \( v_0 \) to a vertex \( v \in \delta D \) goes through \( X_{t_\ell} \), so we have \( d(v_0, \delta D) \geq n \).

Proposition 5.6 Let \( \Gamma \) be a graph with finite treewidth and uniformly bounded degree. Then \( \Gamma \) is accessible.

Proof. Let \( \alpha \) be a bi-infinite simple path such that \( C(\alpha) \neq \emptyset \) and let \( C \in C(\alpha) \). We fix a vertex \( v_0 \in \beta C \) and we let \( n = \max \{ d(v_0, w) \mid w \in \beta C \} \). Let \( k \in \mathbb{N} \) be according to Lemma 5.3. It follows that there is a \( k \)-cut \( D \) with \( |\alpha \cap D| = \infty \) and \( v_0 \in D \). Because of the choice of \( n \), we also have \( \beta C \subseteq D \) what means that either \( C \subseteq D \) or \( C \subseteq D \). In either case \( D \) splits \( \alpha \) in two infinite pieces.

Lemma 5.7 Let \( \Gamma \) be a connected, locally finite, and infinite graph such that \( \text{Aut}(\Gamma) \setminus \Gamma \) is finite. Then there is a bi-infinite geodesic.

Proof. There are arbitrarily long geodesics, hence geodesics of every length. For each geodesics \( \gamma \) with an odd number of vertices let \( m(\gamma) \) be the vertex in the middle. Because \( \text{Aut}(\Gamma) \setminus \Gamma \) is finite, there exists a single vertex \( v_0 \) such that infinitely many geodesics \( \gamma \) satisfy \( m(\gamma) = v_0 \). These geodesics form the vertices of a tree as follows: The root is the \( v_0 \) (viewed as a geodesic of length 0). The parent of a geodesic \( (v_{-k}, v_{-k+1}, \ldots, v_0, \ldots, v_{k-1}, v_k) \) is defined as \( (v_{-k+1}, \ldots, v_0, \ldots, v_{k-1}) \). Since \( \Gamma \) is locally finite we obtain an infinite tree where each node has finite degree. By Königs Lemma there is an infinite path, which defines a bi-infinite geodesic through \( v_0 \).
Note that we cannot remove any of the requirements in Lemma 5.7. In particular, we cannot remove that $\text{Aut}(\Gamma) \backslash \Gamma$ is finite. For example consider the graph $\Gamma$ with $V(\Gamma) = \mathbb{Z}$ and $E(\Gamma) = \{ (n, n \pm 1), (n, -n) \mid n \in \mathbb{Z} \}$. This graph is connected, locally finite, and infinite. It has a bi-infinite simple path, but there is no bi-infinite geodesic.

**Lemma 5.8** Let $\Gamma$ be connected, locally finite, and infinite such that $\text{Aut}(\Gamma) \backslash \Gamma$ is finite and let $\Gamma$ have finite treewidth. Then $\Gamma$ has more than one end.

**Proof.** The graph $\Gamma$ has uniformly bounded degree, because it is locally finite and $\text{Aut}(\Gamma) \backslash \Gamma$ is finite. By Lemma 5.3 there is some $k$ such that for every $n \in \mathbb{N}$, $v_0 \in V(\Gamma)$ and every one-sided infinite simple path $\alpha$ there is a $k$-cut $C$ with $v_0 \in C$, $d(v_0, C) \geq n$, and $|C \cap \alpha| = \infty$.

By Lemma 4.7 there are only finitely many orbits of $k$-cuts under the action of $\text{Aut}(\Gamma)$. Therefore there is $m \in \mathbb{N}$, such that $\max \{ d(u, v) \mid u, v \in \beta C \} \leq m$ for all $k$-cuts $C$.

Assume that $\Gamma(G)$ has only one end. Now by Lemma 5.7 there is a bi-infinite geodesic $\alpha = \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$. Let $C$ be a $k$-cut with $d(v_0, C) > m$ such that $v_0 \in C$ and $|C \cap \alpha| = \infty$. Then $|C \cap \alpha| < \infty$, for otherwise $C(\alpha) \not= \emptyset$.

Hence there are $i, j > m$ with $v_{-i}, v_j \in \beta C \cap \alpha$. But this implies $d(v_{-i}, v_j) = d(v_{-i}, v_0) + d(v_0, v_j) > 2m$ in contradiction to $d(u, v) \leq m$ for all $u, v \in \beta C$. □

Now we have all the tools to state and prove our main theorem.

**Theorem 5.9** Let $\mathcal{G}$ be a class of groups which is closed under taking finite-index normal subgroups. Let $\Gamma$ be a connected, locally finite graph with finite treewidth. Let a group $G$ act on $\Gamma$ such that $G \backslash \Gamma$ is finite and each node stabilizer $G_v$ is in $\mathcal{G}$.

Then $G$ acts on the tree $T(\mathcal{C}_{\text{opt}})$, such that all vertex and edge stabilizers are virtually $\mathcal{G}$ and $G \backslash T(\mathcal{C}_{\text{opt}})$ is finite.

**Proof.** Since the blocks $B_C$ have finite treewidth by Lemma 5.2, it follows from Lemma 5.8 that they are finite or have more than one end. The latter case is excluded by Proposition 1.6 which states that they have at most one end. This means that the blocks are finite. The theorem then follows with Lemma 4.7 and Proposition 1.8. □

**Corollary 5.10** Let a group $G$ act on a connected, locally finite graph $\Gamma$ with finite treewidth such that $G \backslash \Gamma$ is finite and each node stabilizer $G_v$ is finite. Then $G$ is the fundamental group of a finite graph of finite groups.

**Proof.** By Theorem 5.9 $G$ acts on a tree $T$ with finite vertex stabilizers such that $G \backslash T$ is finite. Bass-Serre theory (21) yields the result. □

Note that if we know that $G$ is finitely generated, then the condition $|G \backslash \Gamma| < \infty$ in Theorem 5.9 and Corollary 5.10 is no real restriction, since in this case we always can construct a subgraph of $\Gamma$ on which $G$ acts with finitely many orbits. To do that we proceed as follows: Let $\Sigma$ be a finite generating set of $G$ and let $v_0 \in V(\Gamma)$ be some arbitrary vertex. For all $\alpha \in \Sigma$ we fix paths $\gamma_\alpha$ from $v_0$ to $\alpha v_0$. Let $\Delta$ be the subgraph of $\Gamma$ induced by the vertex set $G \cdot \bigcup_{\alpha \in \Sigma} \gamma_\alpha$. This graph is connected, locally finite and has finite treewidth by Lemma 5.2.

Another interesting observation about the tree $T(\mathcal{C}_{\text{opt}})$ is, that together with the blocks $B_C$ it forms a tree decomposition of $\Gamma$ with finite width.
6 Context-free groups

A *formal language* is a subset $L$ of the free monoid $\Sigma^*$ over some alphabet $\Sigma$. Here, an *alphabet* simply means any finite set. We say that a class $K$ of formal languages is closed under inverse homomorphism if $L \in K$ implies $\varphi^{-1}(L) \in K$ for all homomorphisms $\varphi: \Sigma^* \to \Sigma^*$. Almost all classes investigated in formal language theory or complexity theory are closed under inverse homomorphism, see e.g. [14]. For example, all classes in the Chomsky hierarchy have this property. Other examples are the classes of deterministic context-free languages, the class of languages where the word problem can be solved in polynomial time or the class of recursive languages. We say that $L \subseteq \Sigma^*$ is a *group language*, if there is homomorphism $\varphi: \Sigma^* \to G$ onto a group $G$ such that that $L = \varphi^{-1}(1)$.

If $L$ belongs to a class $K$ and $K$ is closed under inverse homomorphism, then we also say that $G$ belongs to the class $K$. This is a property of $G$ and does not depend on the presentation $\varphi: \Sigma^* \to G$: Indeed, let $\varphi': \Sigma^* \to G$ be another presentation of $G$. Since $\Sigma^*$ is free we find a homomorphism $\psi: \Sigma^* \to \Sigma^*$ such that $\varphi' = \varphi \circ \psi$. Hence, $\varphi'^{-1}(1) = \psi^{-1}(\varphi^{-1}(1))$.

It is very easy to verify the classical results that a group $G$ is regular if and only if $G$ is finite and that all context-free groups are finitely presented, see [1].

6.0.1 Finitely generated virtually free groups are deterministic context-free

Let $G$ be a finitely generated virtually free group and $F(X)$ be a free subgroup of finite index. Choose a set $R$ with $1 \in R \subseteq G$ such that the canonical projection $G \to F(X)\backslash G$ induces a bijection between $R$ and the finite quotient $F(X)\backslash G$.

We use the disjoint union $\Sigma = X^\pm \cup R$ as a finite generating alphabet, where $X^\pm = X \cup X^{-1}$. For all letters $a, b \in \Sigma$ we can define rewrite rules as follows:

$$ab \to x_{ab}r \quad \text{if } x_{ab} \text{ is a word over } X^\pm \text{ and } r \in R \text{ such that } ab = x_{ab}r \in G.$$ 

This system can be used by a deterministic pushdown automaton transforming an input word $w \in \Sigma^*$ into its normal form $w = x_0r$ with $x_0 \in (X^\pm)^*$ and $r \in R$ as follows. First, we choose $k \in \mathbb{N}$ such that $k \geq |x_{ab}r|$ for all rules $ab \to x_{ab}r$. The pushdown stack contains freely reduced words over $X^\pm$, the set of states are the words $yr \in F(X) \cdot R$ of length at most $k$. We start with an empty stack in state $1 \in R$ and with the input word $w$. We perform the following instructions:

- If the input is empty and the state is a letter $r \in R$, then stop.
- If the state is a letter $s \in R$, but the input is not empty, then read the next input letter $b$ and change the state to $x_{ab}r$ according to the rule $sb \to x_{ab}r$.
- If the state is a word $ys \in F(X) \cdot R$ with $1 \neq y \in F(X)$ and the stack content is a freely reduced word $z$ over $X^\pm$, then replace (within less than $k$ steps) $z$ by the freely reduced word corresponding the group element $zy \in F(X)$, and after that switch to the state $s \in R$.

The description how the pushdown automaton works is just standard way how to compute normal forms in linear time. Indeed, if we start with an input word $w$ then we stop in a configuration where $x$ is a freely reduced word on the stack and we are in state $r \in R$. It is clear that $w = xr \in G$. 

19
6.0.2 Finitely generated virtually free groups are context-free

The statement itself is trivial from the precedent subsection and standard facts how to transform a pushdown automaton into a context-free grammar, see any textbook on formal languages like [14]. Let us recall however that, a priori, the class of context-free groups could be larger than the class of deterministic context-free groups.

It is well-known that there are context-free languages which are not deterministic context-free. Indeed, consider the group $\mathbb{Z} \times \mathbb{Z}$ with generators $a = (1, 0)$, $b = (0, 1)$, and $c = (-1, -1)$. A standard exercise shows that set of the words $w \in \{a, b, c\}^*$ which are equal to $(0, 0)$ is not context-free, but its complement is context-free. It cannot be deterministic context-free, because deterministic context-free languages are closed under complementation, [14]. Thus, $\mathbb{Z} \times \mathbb{Z}$ is co-context-free in the sense of [13]. The class of co-context-free groups is very interesting in its own, for example it includes the Higman-Thompson group [19].

6.0.3 Context-free groups are finitely presented

This is a classical result due to Anisimov. He showed it by using the so-called $uvwxy$-Theorem [1]. We obtain a more concise finite presentation by using a reduced context-free grammar. To be more precise, let $\varphi : \Sigma^* \to G$ a surjective homomorphism such that $L_G = \{ w \in \Sigma^* \mid \varphi(w) = 1 \}$ is context-free. Let $(V, \Sigma, P, S)$ be a reduced context-free grammar which generates $L_G$ according to the notation of [14]: This means $V \cap \Sigma = \emptyset$ and all production rules of $P$ have the form $A \to \alpha$ where $A \in V$ is a variable and $\alpha \in (V \cup \Sigma)^*$ is a word.

The grammar is reduced, if every variable $A \in V$ appears in some derivation $S \overset{\gamma A \delta}{\longrightarrow}^P w \in \Sigma^*$. Now, the canonical homomorphisms $\Sigma^* \to F(\Sigma) \to F(V \cup \Sigma) \to F(V \cup \Sigma)/P$ yield an isomorphism:

$$G = \Sigma^* / \{ u = v \mid u, v \in L_G \} \to F(V \cup \Sigma)/P.$$  

This fact has a straightforward verification. It has been generalized to other languages and grammar types leading to the notion of Hotz-isomorphism. We refer to [6] for details and some open problems in this area of formal language theory.

6.0.4 Quasi-isometric sections

This section is not needed in order to understand the result of Muller and Schupp. It yields a direct construction of a context-free grammar (in Chomsky normal form) associated to a f.g. virtually free group which uses only the fact that virtually free groups have a Cayley graph with a quasi-isometric section. In [3] Bridson and Gilman introduced quasi-isometric sections as broomlike combings and proved that the groups with quasi-isometric sections are exactly the virtually free groups.

Throughout this section we assume that $G$ is finitely generated and that its Cayley graph refers to a monoid presentation. This means we start with
a surjective homomorphism $\varphi : \Sigma^* \to G$, where $\Sigma$ is a finite alphabet. For a moment we let $\Gamma = G(\Sigma, \Sigma)$ denote the directed Cayley graph with $V(\Gamma) = G$ and $E(\Gamma) = G \times \Sigma$. For a directed edge $e = (g, a)$ we let $s(g, a) = g$ its source and $t(g, a) = ga$ its target, where, by abuse of language, we simply write $ga$ for $g\varphi(a)$. The letter $a$ is the label of the edge $e$. Since $\varphi$ is a monoid presentation, $\Gamma$ is strongly connected, and for every edge $e = (g, a)$ there is a directed path from $ga$ to $g$ whose length is bounded by some constant.

The set of words $\Sigma^*$ forms a tree. The empty word $\varepsilon$ is the root and a word $u$ has the children $ua$ for letters $a \in \Sigma$. The geodesic distance $d(u, v)$ in the tree $\Sigma^*$ yields a natural metric on $\Sigma^*$. That means, we have $d(u, v) = d$ if and only if $d = |u'| + |v'|$ where $u = pu'$ and $v = pv'$ and $p$ is the longest common prefix of $u$ and $v$.

A quasi-isometric section of $\Gamma$ is a mapping $\sigma : G \to \Sigma^*$ such that

1. we have $\sigma(1) = \varepsilon$,
2. we have $\varphi(\sigma(g)) = g$ for all $g \in G$,
3. there is some $1 \leq k \in \mathbb{N}$ such that $d(\sigma(g), \sigma(ga)) \leq k$ for all edges $(g, a) \in E(\Gamma)$.

Note that $\sigma(G)$ yields a set of normal forms with $\varepsilon \in \sigma(G)$. The important property is however that vertices $g, h$ of distance $d$ in the Cayley graph have representing words of distance at most $kd$ in the tree $\Sigma^*$.

The existence of a quasi-isometric section depends only on the group $G$ and not on its presentation $\varphi : \Sigma^* \to G$: Indeed, let $\sigma : G \to \Sigma^*$ be a quasi-isometric section of $\Gamma$ and $\psi : \Sigma^* \to G$ be another monoid presentation. Then we find a homomorphism $\tau : \Sigma^* \to \Sigma^*$ such that $\varphi(w) = \psi(\tau(w))$ for all words $w \in \Sigma^*$. Now, the set of normal forms $\sigma(G)$ is mapped onto the set of normal forms $\tau(\sigma(G))$ satisfying (1) and (2). Moreover, consider $u = pu'$ and $v = pv'$ with $|u'| + |v'| \leq k$. Then there is constant $\ell$ (depending on $\tau$) such that $|\tau(u')| + |\tau(v')| \leq k\ell$. This shows (3) for $\psi : \Sigma^* \to G$. Thus we can say that $G$ has quasi-isometric section.

Choosing $\Sigma = X \cup X^{-1} \cup R$ and $k$ as above, it follows from Section 6.0.1 that f.g. virtually groups have quasi-isometric sections.

Now, let $G$ have a quasi-isometric section $\sigma : G \to \Sigma^*$ for the monoid presentation $\varphi : \Sigma^* \to G$. We let $k \geq 1$ such that $d(\sigma(g), \sigma(ga)) \leq k$ for all edges $(g, a) \in E(\Gamma)$. We are going to define a context-free grammar for the language $L_G = \{ w \in \Sigma^* \mid \varphi(w) = 1 \}$. The set of variables is a finite subset of $G$. We let $V = \{ \varphi(w) \in G \mid w \in \Sigma^*, |w| \leq k \}$.

The axiom is $S = 1 \in G$ and we define a single $\varepsilon$-rule $S \to \varepsilon$. All other rules are of the form $A \to BC$ or $A \to a$ with $A, B, C \in V$ and $a \in \Sigma$. Thus, the grammar will be automatically in Chomsky normal form. We add the following set of productions:

$$\{ A \to BC \mid AB = C \in G \} \cup \{ A \to a \mid A = \varphi(a) \in G \}.$$

All words generated by the grammar represent the identity in $G$, since derivation rules are identities in $G$. Thus it remains to show, that for every word $w \in \Sigma^*$ with $\varphi(w) = 1$ there is some derivation $S \xrightarrow{p} w$.  

21
Let \( w = a_1 \cdots a_n \) with \( a_i \in \Sigma \) and \( \varphi(w) = 1 \). For \( 1 \leq i \leq n \) we let 
\( A_i = \varphi(a_i) \).

We have \( A_i \in V \) for all \( 1 \leq i \leq n \), and the grammar has rules \( A_i \rightarrow a_i \). Therefore it is enough to find a derivation \( S \xrightarrow{p} A_1 \cdots A_n \).

We do something more: Let \( \varepsilon = u_0, u_1, \ldots, u_n \) be any sequence of words \( u_i \in \Sigma^* \) for \( 0 \leq i \leq n \) such that \( \varphi(u_i) = 1 \) and \( d(\varphi(u_{i-1}), \varphi(u_i)) \leq k \) for \( 1 \leq i \leq n \). Then we have \( A_i = \varphi(u_{i-1})^{-1}\varphi(u_i) \in V \) and we prove by induction on \( n \) that there is a derivation \( S \xrightarrow{p} A_1 \cdots A_n \).

We may assume \( u_i = \sigma(\varphi(u_i)) \) for all \( 1 \leq i \leq n \). This simplifies the notation. The result is true, if \( \varphi(u_i) = 1 \), i.e. \( u_i = \varepsilon \) for all \( 1 \leq i \leq n \) due to the rule \( S \rightarrow SS \), no matter how large \( n \) is. Hence \( \varphi(u_i) \neq 1 \) for some \( i \) and \( n \geq 2 \).

Choose \( 1 \leq m \leq n \) such that \( |u_m| \geq |u_i| \) for all \( i \). Then we have \( m < n \) and \( d(u_{m-1}, u_{m+1}) \leq k \) because \( |u_m| \geq \max\{|u_{m-1}|, |u_{m+1}|\} \). Hence we can apply the induction hypothesis to the sequence \( u_0, \ldots, u_{m-1}, u_{m+1}, \ldots, u_n \). This yields a derivation
\[
S \xrightarrow{p} A_1 \cdots A_{m-1} A A_{m+2} \cdots A_n
\]
where \( A = \varphi(u_{m-1})^{-1}\varphi(u_{m+1}) \in V \). We need one further derivation step using the rule \( A \rightarrow A_m A_{m+1} \) where \( A_m = \varphi(u_{m-1})^{-1}\varphi(u_m) \) and \( A_{m+1} = \varphi(u_{m})^{-1}\varphi(u_{m+1}) \).

### 6.0.5 Cayley graphs of context-free groups have finite treewidth

Muller and Schupp have shown that a Cayley graph of a context-free group has a \( k \)-triangulation \([20]\). The definition of a \( k \)-triangulation is technical. We skip it here because the proof in \([20]\) can also be used to show directly that a Cayley graph of a context-free group has finite treewidth. This suffices for our purposes.

**Proposition 6.1** Let \( \Gamma \) be a Cayley graph of a context-free group \( G \) with respect to a finite generating set \( X \). Then \( \Gamma \) has finite treewidth.

**Proof.** If \( G \) is finite, then the assertion is trivial. Hence let \( G \) be infinite. We may assume that \( 1 \notin X \subseteq G \).

The vertex set of \( \Gamma \) is the group \( G \), by \( B_n \) we denote the ball of radius \( n \) around the origin \( 1 \in G \). Hence \( B_n = \{ g \in G \mid d(1, g) \leq n \} \). We are heading for a tree decomposition where certain finite subsets of \( G \) become nodes in a tree. For \( n \in \mathbb{N} \) we define sets \( V_n \) of level \( n \) such that \( V_0 = \{ \Gamma - 1 \} \) and \( V_n = \{ C \mid C \) is a connected component of \( \Gamma - B_n \} \) for \( n \geq 1 \). This defines a tree \( T \) with root \( B_1 \) as follows:

\[
\begin{align*}
V(T) = \{ \beta C \mid C \in V_n, n \in \mathbb{N} \}, \\
E(T) = \{ \{ \beta C, \beta D \} \mid D \subseteq C \in V_n, D \in V_{n+1}, n \in \mathbb{N} \}
\end{align*}
\]

The nodes are subsets of \( G \) hence we can identify nodes \( t \in T \) with their bags \( X_t \subseteq G \). If \( \{ g, h \} \) is an edge is the Cayley graph \( \Gamma \), then there are essentially two cases; either \( d(1, g) = n \) and \( d(1, h) = n+1 \) or \( d(1, g) = d(1, h) = n+1 \) for some \( n \). In both cases the elements \( g, h \) are in some bag \( \beta C \) for some \( C \in V_n \) and \( n \in \mathbb{N} \).
It remains to show that $|\beta C|$ is bounded by some constant for all $C \in V_n, \, n \in \mathbb{N}$. It is here where the context-freeness comes into the play. We denote $\Sigma = X \cup X^{-1}$. This is a set of monoid generators of $G$. We let $L_G = \{ w \in \Sigma^* \mid w = 1 \in G \}$ its associated group language. By hypothesis $L_G$ is generated by some context-free grammar $(V, \Sigma, P, S)$, and we may assume that it is in Chomsky normal form. This means all rules are either of the form $A \rightarrow BC$ with $A, B, C \in V$ or of the form $A \rightarrow a$ with $A \in V$ and $a \in \Sigma^*$ such that $|a| \leq 1$. We write $A \xrightarrow{P}^* w$, if we can derive $w \in (V \cup \Sigma)^*$ with production rules from $P$. We define a constant $k \in \mathbb{N}$, $k \geq 1$ such that $k \geq \min \{|w| \mid A \xrightarrow{P}^* w \in \Sigma^* \}$ for all $A \in V$.

Consider $C \in V_n$ and $n \in \mathbb{N}$. Let $g, h \in \beta C$. We are going to show that $d(g, h) \leq 3k$. For $n = 0$ we have $\beta C = B_1$. Hence, we may assume $n \geq 1$.

Let $\alpha$ be a geodesic path from $1$ to $g$ with label $u \in \Sigma^*$, $\gamma$ a geodesic path from $h$ to $1$ with label $w \in \Sigma^*$, and $\beta$ some path from $g$ to $h$ with label $v \in \Sigma^*$, which is entirely contained in $C$. Such a path exists, since $C$ is connected. The composition of these paths forms a closed path $\alpha \beta \gamma$ with label $uvw$. We have $uvw \in L_G$ and there is a derivation $S \xrightarrow{P}^* uvw$. We may assume that $|v| \geq 2$, because otherwise there is nothing to do.

Since the grammar is in Chomsky normal form we can find a rule $A \rightarrow BC$ and derivations as follows:

$$S \xrightarrow{P} u'Aw' \xrightarrow{P} u'BCw' \xrightarrow{P} u'v'v''w' = uvw$$

such that $B \xrightarrow{P} v', \, C \xrightarrow{P} v''$, and $|u'| \leq |u| < |u'v'| < |uv| \leq |u'v'v''|$.

This yields three nodes $x \in \alpha$, $y \in \beta$, and $z \in \gamma$ such that $d(x, y)$, $d(y, z)$, $d(x, z) \leq k$. (These three nodes correspond exactly to a triangle with endpoints $x, y, z$ in the $k$-triangulation of the cycle $\alpha \beta \gamma$ in \cite{20}.)

![Diagram](image-url)

Figure 7: The distance between $g$ and $h$ is bounded by $3k$.

Now we have:

$$d(x, g) = d(1, g) - d(1, x) \leq d(1, y) - d(1, x) \leq d(x, y).$$
The first equality holds, because \( \alpha \) is geodesic and \( x \) lies on \( \alpha \); the second one because \( d(1,g) \leq n + 1 \leq d(1,y) \). Likewise we obtain \( d(z,h) \leq d(z,y) \). Thus, it follows

\[
d(g,h) \leq d(g,x) + d(x,z) + d(z,h) \\
\leq d(y,x) + d(x,z) + d(z,y) \leq 3k.
\]

This implies that the size of the bags is uniformly bounded by some constant, since \( \Gamma \) has uniformly bounded degree. \( \square \)

6.1 The result of Muller and Schupp revisited

To date various equivalent characterizations of context-free groups are known. The following theorem mentions those characterizations which we met in this paper for proving the fundamental result of Muller and Schupp that context-free groups are virtually free.

**Theorem 6.2** Let \( G \) be a finitely generated group and \( \Gamma \) be its Cayley graph with respect to some finite set of generators. The following assertions are equivalent.

1. \( G \) is virtually free.
2. \( G \) is deterministic context-free.
3. \( \Gamma \) has a quasi-isometric section.
4. \( G \) is context-free.
5. \( \Gamma \) has finite treewidth.
6. The group \( G \) is the fundamental group of a finite graph of finite groups.

**Proof.** A review on the implications \( 1 \Rightarrow 2, 1 \Rightarrow 3, 2 \Rightarrow 4, \) and \( 3 \Rightarrow 4 \Rightarrow 5 \) has been given in this section. The implication \( 5 \Rightarrow 3 \) is a direct consequence of Corollary 5.10. The last implication \( 6 \Rightarrow 1 \) follows from [15]. \( \square \)

7 Accessibility of groups

In this section we assume all groups to be finitely generated. As another application of the construction in Section 3 and Section 4 we give a proof of a theorem of Thomassen and Woess [26, Thm. 1.1]. It is an important corollary of [7, Thm. II 2.20] where Dicks and Dunwoody develop their the structure tree theory. This result allows us to consider all groups which act on a locally finite, connected, accessible graph with finite stabilizers and finitely many orbits, and not only those which act on graphs with finite treewidth. The result in [26] gave birth to the notion of accessibility for graphs.

We need some standard facts of Bass-Serre theory. The following lemma is well-known, see e.g. [4]. For convenience of the reader, we give a proof.

**Lemma 7.1** Let \( G \) be a f.g. fundamental group of a finite graph of groups with finite edge groups. Then every vertex group is finitely generated.
Proof. We give a sketch only. Let \( V \) be the set of vertices, \( Y \) be the set of edges of the finite graph, and \( Z \) be the union over all edge groups. For each vertex \( v \in V \) let \( X_v \) be some generating set of the vertex group \( G_v \). Then there is a finite generating set \( X \) inside \( \bigcup \{ X_v \mid v \in V \} \cup Y \cup Z \) such that \( Y \cup Z \subseteq X \).

Now consider any \( x \in X_v \), it is enough to show that \( x \) can be expressed as a product over \( X \cap X_v \). To see this, write \( x \) as shortest word in \( X \). Assume this word contained a factor \( yzy^{-1} \) with \( y \in Y \) and where \( z \) belongs to edge group of \( y \) sitting in \( G_t(y) \), then we could perform a “Britton reduction” replacing \( yzy^{-1} \) by some \( z' \) in the edge group of \( y \) sitting in \( G_s(y) \). This would lead to a shorter word, since \( Y \cup Z \subseteq X \). Hence, this is impossible; and the word representing \( x_v \) is “Britton reduced”. This implies that the word uses letters from \( X \cap X_v \), only.

\[ \square \]

Definition 7.2

1. A group is called more than one ended (resp. at most one ended), if its Cayley graph has more than one end (resp. at most one end).

   (This definition does not depend on the choice of the finite generating set for the Cayley graph.)

2. A group \( G \) is called accessible, if it acts on a tree with finitely many orbits, finite edge stabilizers, and vertex stabilizers with at most one end.

If a group \( G \) is accessible, then Bass-Serre theory yields an upper bound on the number how often \( G \) can be split properly as an HNN-extension or amalgamated product over finite subgroups. This observation is also another definition of accessibility used frequently in literature. The link to accessibility of the corresponding Cayley graphs is due to the next proposition.

Proposition 7.3 Let \( G \) be a f.g. group which acts on a tree with finitely many orbits, finite edge stabilizers and no vertex stabilizer having more than one end. Then the Cayley graph \( \Gamma \) of \( G \) is accessible.

Proof. Again, we give only a sketch. Bass-Serre theory tells us that \( G \) is the fundamental group of a finite graph of groups with finite edge groups. By Lemma 7.1, every vertex group \( G_v \) is finitely generated. We only consider the case where \( G = A \ast_H B \) is an amalgamated product of two f.g. groups \( A \) and \( B \) over a common finite subgroup. The case of HNN-extensions follows analogously and is left to the reader.

We assume that \( A \) and \( B \) have accessible Cayley graphs and show that this implies that \( G = A \ast_H B \) has an accessible Cayley graph. Then Proposition follows by induction.

Let \( A \) be generated by \( X_A \) and \( B \) be generated by \( X_B \), where \( X_A \) and \( X_B \) are finite. As a generating set for \( G \) we use \( X = H \cup HX_A H \cup HX_B H \) and we may assume that \( \Gamma \) is the Cayley graph of \( G \) w.r.t. \( X \). We may regard the Cayley graphs of \( A \) and \( B \) as subgraphs of \( \Gamma \) and refer to them as \( A \) or \( B \). Now, consider any bi-infinite simple path \( \alpha \) in \( \Gamma \) such that there is a cut \( C \) (of finite weight) with \( |C \cap \alpha| = |C \cap \alpha| = \infty \). We can assume that \( |\delta C| \) is minimal among all such cuts. In order to show that \( \Gamma \) is accessible we need a uniform bound on \( |\delta C| \). The path \( \alpha \) gives us a bi-infinite sequence of labels in \( X \). We may assume the origin \( 1 \in G \) is a vertex of \( \alpha \). If all the labels belong to \( H \cup HX_A H \), then the path is entirely in \( A \). So by hypothesis there is an upper bound on \( |\delta C| \). Thus we may assume that there is at least one label in \( A \setminus H \)
and one label in $B \setminus H$ and that 1 is sitting between two such labels of minimal distance. Without restriction the label on the right of 1 belongs to $A \setminus H$ and on the left it belongs to $B$. Let $1 = x_0, x_1, x_2, \ldots$ be the one-sided infinite sequence of vertices from $\alpha$ going to the right of 1 and $\ldots, y_2, y_1, y_0 = 1$ the corresponding one on the left. For every $x \in G$ the set $H x H$ is finite. Hence, switching to infinite subsequences of $x_0, x_1, x_2, \ldots$ and $\ldots, y_2, y_1, y_0$ we may assume that no $x_i^{-1} x_j$ or $y_j y_i^{-1}$ belongs to $H$ for $i < j$. Grouping consecutive labels from $A \setminus H$ (resp. $B \setminus H$) into blocks we obtain sequences

$$h_1, \ldots, h_2, h_1, g_1, g_2 \ldots g_i$$

such that $g_1 \in A \setminus H$, $h_1 \in B \setminus H$, and the $g$- and the $h$-vertices alternate between $A \setminus H$ and $B \setminus H$. It might happen that $i$ or $j$ remains bounded, but there are sequences with $1 \leq i, j$. The final step is to observe that every path connecting $g_1 \cdots g_i$ to $(h_j \cdots h_1)^{-1}$ must use a vertex from $H$. This is due to the normal form theorem for amalgamated products. \hfill \Box

Proposition 7.3 yields one direction for the result of Thomassen and Woess. For the other direction let us state first another lemma concerning the blocks $B[C]$. We use the notation of Section 4.

**Lemma 7.4** Let $\Gamma$ be a connected, locally finite, and accessible graph such that a group $G$ acts on $\Gamma$ with finitely many orbits and finite vertex stabilizers $G_v$. Let $C \in C_{\text{opt}}$. Then the stabilizer $G[C]$ acts with finitely many orbits on the block $B[C]$.

**Proof.** Since $G$ acts with finitely many orbits on $\Gamma$ and every $g$ with $g C \sim C$ is contained in $G[C]$, we know that $G[C]$ acts with finitely many orbits on $[C]$. Thus $G[C]$ acts with finitely many orbits on $\bigcup_{D \sim C} \beta D$.

We are going to show that there is some $m \in \mathbb{N}$ such that for every $v \in B[C]$ there is a cut $D \in [C]$ with $d(v, \beta D) \leq m$. This implies the lemma, since $\Gamma$ is locally finite.

So let $v \in B[C]$. If $v \in N^\kappa D \cap \overline{\mathcal{T}}$ for some $D \sim C$, then we have $d(v, \beta D) \leq \kappa$ (recall that $\kappa$ is a fixed constant). Thus it remains to consider the case $v \in D$ for all $D \sim C$.

Let $U$ be a finite subset of $B[C]$ such that $B[C] \subseteq G \cdot U$. There is a constant $m \geq \kappa$ such that $d(v, \beta C) \leq m$ for $v \in U$. We conclude that for the node $v \in B[C]$ there is some $g \in G$ and $E = g C$ such that $d(v, \beta E) \leq m$. Thus, we actually may assume $v \in \beta E$ and show that this implies $v \in \bigcup_{D \sim C} \beta D$.

Because $C$ and $E$ are nested, we can assume (after replacing $E$ with $\overline{E}$, if necessary) that $C \subseteq \overline{E}$ or $\overline{E} \subseteq C$. If $C \subseteq \overline{E}$ (thus $E \subseteq \overline{C}$), then $\beta E \subseteq \beta C \cup \overline{C}$. But $v \in C$, hence $v \in \beta C = \beta C$. On the other hand, if $\overline{E} \subseteq C$, then there is a $D \sim C$, such that $\overline{E} \subseteq \overline{D} \subseteq C$. It follows that $v \in D \cap \beta E \subseteq D \cap (\beta \overline{E} \cup \overline{D}) \subseteq \beta D$. \hfill \Box

**Theorem 7.5** Let $\Gamma$ be a locally finite, connected, accessible graph. Let $G$ act on $\Gamma$ such that $G[\Gamma]$ is finite and each node stabilizer $G_v$ is finite. Then $G$ acts on the tree $T(\mathcal{C}_{\text{opt}})$ with finitely many orbits, finite edge stabilizers, and no vertex stabilizer has more than one end.
Proof. By Lemma 4.7 and Proposition 4.8 we know that \( G \) acts with finitely many orbits and finite edge stabilizers on \( T(\mathcal{C}_{opt}) \). Now consider a vertex stabilizer \( G_{[C]} \) for some \( C \in \mathcal{C}_{opt} \). By Lemma 7.1 we have that \( G_{[C]} \) is finitely generated. Thus, its Cayley graph is locally finite and the number of ends is defined.

The block \( B_{[C]} \) has at most one end by Proposition 4.8. So it suffices to show that if the Cayley graph of \( G_{[C]} \) has more than one end, then \( B_{[C]} \) has more than one end, too.

Because of Lemma 7.4 there is a finite set \( U \subseteq B_{[C]} \) such that \( G_{[C]} \cdot U = B_{[C]} \). More precisely, we can identify \( B_{[C]} \) with \( G_{[C]} \times U \). Let \( Z = \{ g \in G_{[C]} \mid \exists u, v \in U : (u, gv) \in E(\Gamma) \} \). Then we have \( |Z| < \infty \), since \( U \) is finite, \( \Gamma \) is locally finite, and all vertex stabilizers are finite. Thus, we can define \( m = \max \{ d(1, a) \mid a \in Z \} < \infty \) (here, \( d \) denotes the distance in the Cayley graph of \( G_{[C]} \)).

Assume that \( G_{[C]} \) has more than one end. Then the Cayley graph of \( G_{[C]} \) has a cut of finite weight \( D \subseteq G_{[C]} \) with \( |D| = |\mathcal{D}| = \infty \). We claim that there are only finitely many pairs \( g, h \) such that \( g \in D, h \in G_{[C]} - D \) and \( g^{-1}h \in Z \).

Indeed, since \( g^{-1}h \in Z \), there is a path of length at most \( m \) from \( g \) to \( h \) in the Cayley graph of \( G_{[C]} \). Since \( g \in D \) and \( h \in G_{[C]} - D \) this path uses an edge of \( \delta D \). Since \( \delta D \) is finite and the Cayley graph is locally finite there are only finitely many such paths of length at most \( m \), hence there are only finitely many such \( g \) and \( h \).

Now consider \( E = \{ gu \mid g \in D, u \in U \} \subseteq B_{[C]} \). Every edge of the boundary \( \delta E \) inside \( B_{[C]} \) has endpoints \( gu \) and \( hv \) with \( g \in D, h \in G_{[C]} \setminus D \) and \( u, v \in U \). By the above claim there are only finitely many choices for \( g \) and \( h \). Together with the finiteness of \( U \) this implies that \( E \) has finite boundary inside \( B_{[C]} \).

Thus, \( \beta E \subseteq B_{[C]} \) is a finite set of vertices. Since \( |D| = |\mathcal{D}| = \infty \) and \( B_{[C]} = G_{[C]} \times U \), we see that \( |E| = |B_{[C]} \setminus E| = \infty \), too. Since \( B_{[C]} \) is connected, \( B_{[C]} - \beta E \) has more than one infinite connected component. This in turn implies that \( B_{[C]} \) has more than one end. \( \square \)

Corollary 7.6 ([5, 26]) A finitely generated group is accessible if and only if its Cayley graph is accessible.

Proof. If the Cayley graph of \( G \) is accessible, then Theorem 7.5 shows that the group \( G \) is accessible. The converse is stated in Proposition 7.3. \( \square \)

8 Conclusion

Throughout the paper we used Bass-Serre theory as a standard tool in modern combinatorial group theory. If the reader is willing to accept this viewpoint then the paper is entirely self-contained, and it gives direct and simplified proofs for two fundamental results: The Theorem of Muller and Schupp and the accessibility result Corollary 7.6 by Dicks and Dunwoody resp. Thomassen and Woess. This became possible due to the paper of Krön [16] and our intuition that having a Cayley graph with finite tree width should unravel a simplicial tree on which the group acts with finitely many orbits and finite node stabilizers.
A future research program is to investigate whether our constructions can be performed effectively. The problem is to find the minimal cuts, i.e. to decide whether a given cut is minimal with respect to some bi-infinite simple path. If this could be done in elementary time for graphs with finite tree width, it would lead to an elementary time algorithm for the isomorphism problem of context-free groups by first constructing the graphs of groups and then using Krstic’s algorithm ([17]) to check whether the fundamental groups are isomorphic.

References

[1] A. V. Anisimov. Group languages. *Kibernetika*, 4:18–24, 1971. English translation in Cybernetics 4 (1973), 594-601.

[2] Y. Antolin. On Cayley graphs of virtually free groups. *Groups – Complexity – Cryptology*, 3:301–327, 2011.

[3] M. R. Bridson and R. H. Gilman. A remark about combings of groups. *Internat. J. Algebra Comput.*, 3(4):575–581, 1993.

[4] W. Dicks. *Groups, Trees and Projective Modules*. Lecture Notes in Mathematics. Springer, 1980.

[5] W. Dicks and M. J. Dunwoody. *Groups acting on graphs*. Cambridge University Press, 1989.

[6] V. Diekert and A. Möbus. Hotz-isomorphism theorems in Formal Language Theory. *R.A.I.R.O. — Informatique Théorique et Applications*, 23:29–43, 1989. Special issue STACS 88.

[7] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2006.

[8] M. J. Dunwoody. Accessibility and Groups of Cohomological Dimension One. *Proceedings of the London Mathematical Society*, s3-38(2):193–215, 1979.

[9] M. J. Dunwoody. Cutting up graphs. *Combinatorica*, 2:15–23, 1982.

[10] M. J. Dunwoody. The accessibility of finitely presented groups. *Inventiones Mathematicae*, 81:449–457, 1985.

[11] M. J. Dunwoody. An inaccessible group. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 75–78. Cambridge Univ. Press, Cambridge, 1993.

[12] R. H. Gilman, S. Hermiller, D. F. Holt, and S. Rees. A characterisation of virtually free groups. *Arch. Math. (Basel)*, 89(4):289–295, 2007.

[13] D. F. Holt, S. Rees, Röver, C. E., and R. M. Thomas. Groups with context-free co-word problem. *Journal of the London Mathematical Society*, 71:643–657, 2005.

[14] J. E. Hopcroft and J. D. Ulman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley, 1979.
[15] A. Karrass, A. Pietrowski, and D. Solitar. Finite and infinite cyclic extensions of free groups. *Journal of the Australian Mathematical Society*, 16(04):458–466, 1973.

[16] B. Krön. Cutting up graphs revisited – a short proof of Stallings’ structure theorem. *Groups – Complexity – Cryptology*, 2:213–221, 2010.

[17] S. Krstic. Actions of finite groups on graphs and related automorphisms of free groups. *Journal of Algebra*, 124:119 – 138, 1989.

[18] D. Kuske and M. Lohrey. Logical aspects of Cayley-graphs: the group case. *Ann. Pure Appl. Logic*, 131(1-3):263–286, 2005.

[19] J. Lehnert and P. Schweitzer. The co-word problem for the Higman-Thompson group is context-free. *Bull. London Math. Soc.*, 39:235–241, 2007.

[20] D. E. Muller and P. E. Schupp. Groups, the theory of ends, and context-free languages. *Journal of Computer and System Sciences*, 26:295–310, 1983.

[21] F. Rimlinger. Pregroups and Bass-Serre theory. *Mem. Amer. Math. Soc.*, 65(361):viii+73, 1987.

[22] N. Robertson and P. D. Seymour. Graph minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36:49 – 64, 1984.

[23] G. Sénizergues. On the finite subgroups of a context-free group. In G. Baumslag, D. Epstein, R. Gilman, H. Short, and C. Sims, editors, *Geometric and Computational Perspectives on Infinite Groups*, number 25 in DIMACS series in Discrete Mathematics and Theoretical Computer Science, pages 201–212. Amer. Math. Soc., 1996.

[24] J.-P. Serre. *Trees*. Springer, 1980. French original 1977.

[25] J. Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, New Haven, Conn., 1971.

[26] C. Thomassen and W. Woess. Vertex-transitive graphs and accessibility. *J. Comb. Theory Ser. B*, 58(2):248–268, 1993.

[27] W. Woess. Graphs and groups with tree-like properties. *J. Combin. Theory Ser. B*, 47(3):361–371, 1989.