Pulsar radio emission mechanism I : On the amplification of Langmuir waves in the linear regime

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ABSTRACT
Observations suggest that in normal period radio pulsars, coherent curvature radiation is excited within 10% of the light cylinder. The coherence is attributed to Langmuir mode instability in a relativistically streaming one-dimensional plasma flow along the open magnetic field lines. In this work, we use a hot plasma treatment to solve the hydrodynamic dispersion relation of Langmuir mode for realistic pulsar parameters. The solution involves three scenarios of two-stream instability viz., driven by high energy beams, due to longitudinal drift that leads to a separation of electron-positron distribution functions in the secondary plasma and due to cloud-cloud interaction causing spatial overlap of two successive secondary plasma clouds. We find that sufficient amplification can be obtained only for the latter two scenarios. Our analysis shows that longitudinal drift is characterized by high growth rates only for certain multi-polar surface field geometry. For these configurations, very high growth rates are obtained starting from a few tens of km from the neutron star surface, which then falls monotonically with increasing distance. For cloud-cloud overlap, growth rates become high starting only after a few hundred km from the surface, which first increases and then decreases with increasing distance. A spatial window of up to around a 1000 km above the neutron star surface has been found where large amplitude Langmuir waves can be excited while the pair plasma is dense enough to account for high brightness temperature.

Key words: pulsars – radiation mechanism – relativistic plasma – Langmuir mode

1 INTRODUCTION

Observations of radio emission from normal period pulsars (with periods $P$ longer than $\sim 0.1$ seconds) suggest that: a) The radio emission has exceedingly high brightness temperature $T_b \sim 10^{25} - 10^{27}$ K, which is at least 12 orders of magnitude higher than the incoherent synchrotron limit of $10^{12}$ K (see Kellermann & Pauliny-Toth 1969). This necessarily requires a coherent radio emission mechanism (e.g. Ginzburg et al. 1969; Ginzburg & Zhelezniakov 1975; Cordes 1979; Melrose 1993; Mitra 2017); b) The radio emission is highly polarized, which is consistent with coherent curvature radiation (hereafter CCR) (e.g. Mitra et al. 2009; Melikidze et al. 2014); c) The radio emission detaches from the pulsar magnetosphere a few hundred km away from the surface (e.g. Rankin 1993; Mitra 2017).

These observations require plasma processes where stable charge bunches can form and excite CCR in relativistically streaming pair plasma, which can eventually escape from the plasma to reach the observer (see Melikidze et al. 2014; Gil et al. 2004; Mitra et al. 2009). For a general non-zero angle between the propagation vector and the ambient magnetic field, the pulsar pair plasma consists of two eigen-modes viz., the purely transverse X-mode and the quasi-transverse O-mode (see Arons & Barnard 1986). The quasi-transverse O-mode has a sub-Luminal Alfvén branch and super-Luminal LO branch. A number of works show that cyclotron instabilities of the X and O modes can be excited close to the light-cylinder (see for e.g. Kazbegi et al. 1991; Lyutikov 1999). However, several works have shown that cyclotron instabilities of the X and O modes can be excited close to the light-cylinder (see for e.g. Kazbegi et al. 1991; Lyutikov 1999). For the special case when the angle between the propagation vector and the ambient magnetic field is zero, the O-mode becomes purely longitudinal and is referred to as the Lang-
muir mode (see fig. 2 of Arons & Barnard 1986). It has been shown in several studies (e.g. Usov 2002) that closer to the neutron star surface this longitudinal Langmuir mode can become unstable. Langmuir mode instability is a popular candidate for these CCR charge bunches. Theoretically, a combination of linear and non-linear plasma theory is needed to form stable charge bunch (see Melikidze et al. 2000). The linear part of the theory involves development of two stream instability in the plasma that leads to the growth of the amplitude of the longitudinal and electrostatic Langmuir wave mode. While the oscillating electric field of the Langmuir mode can form longitudinal concentrations of charges, it is well known that such linear Langmuir bunches are not capable of radiating coherently (see e.g. Lominadze et al. 1986; Melikidze et al. 2000). Analytical studies show that under certain approximations stable bunches viz., relativistic Langmuir charge soliton can form when non-linear effects are taken into account (see Patarai & Melikidze 1980; Melikidze et al. 2000). Recent numerical analysis has also found such stable charge bunches, when all non-linear interactions are properly taken into account (see Lakoba et al. 2018). However there are several gaps in the theory that remains to be addressed. The non-linear theory requires a priori very large amplitude for the electrostatic waves. The crucial question of quantitative estimates of linear growth rates of Langmuir waves for realistic pulsar plasma parameters, and if the growth rate is sufficient to drive the system beyond the linear regime requires thorough investigation.

The radio emission is excited in relativistically streaming pulsar plasma that consists of a dense secondary positron-electron ($e^+/e^-$) pair plasma, a tenuous high energy primary positron or electron ($e^+/e^-$) beam and a tenuous high energy ion beam. The growth of Langmuir instability requires a two-stream condition to be established in this plasma. Some early works on Langmuir mode in pulsar plasma in fact concluded that Langmuir mode cannot become unstable (e.g. Suvorov & Chugunov 1975). However Lominadze & Mikhailovskii (1979), discussed that in relativistic plasma, particles close to the velocity of light can be in resonance with the Langmuir mode. The authors also discussed two regimes of growth viz., the kinetic and the hydrodynamic regime. There are three ways (referred to as case $C_1$, $C_2$, $C_3$ hereafter) by which the two-stream instability can develop in this flow: first for $C_1$ between the high energy beams and secondary plasma system, second for $C_2$ between the electrons and positrons in the secondary plasma itself due to longitudinal drift, and third for $C_3$ between the overlapping fast and slow particles overlap of successive secondary plasma clouds due to intermittent discharges at the polar gap.

Previous studies of the growth of Langmuir wave in pulsar plasma for the three aforementioned cases of two-stream instability can be briefly summarized as follows:

$C_1$: Initial studies of pulsar radio emission mechanism (e.g. Ruderman & Sutherland 1975 hereafter RS75) appealed to two-stream instability driven by high energy cold $e^+/e^-$ beam. Subsequent works (e.g. Benford & Buschauer 1977) found very small growth rates for such $e^+/e^-$ cold beam. Egorenkov et al. (1983) presented a hot plasma treatment of the high energy $e^+/e^-$ beam and showed that kinetic regime is suppressed and only the hydrodynamic regime survives. Gedalin et al. (2002) explored beam-driven hydrodynamic instability of a low frequency longitudinal beam mode rather than the high-frequency Langmuir mode. Most of the subsequent works (see Lyutikov 1999; Melrose & Gedalin 1999; Rafat et al. 2019) have focussed on this $e^+/e^-$ beam and found the growth rates to be negligible.

$C_2$: The study by Cheng & Ruderman (1977, hereafter CR77) showed that as the secondary pair plasma moves along the curved magnetic field line, longitudinal drift causes the electron and positron distribution function to separate. This can lead to two-stream instability in the secondary plasma. However, they did not consider a hot plasma treatment of the secondary plasma and obtained order of magnitude estimates of growth rate using simple assumptions. Asseo & Melikidze (1998, hereafter AM98) revisited the problem where they presented a hot plasma treatment of the shifted electron-positron distribution function within the secondary plasma cloud.

$C_3$: Usov (1987) showed that in non-stationary plasma flow models, slow and fast moving particles of two successive plasma clouds can overlap within a few hundred km from the surface leading to the development of a two-stream instability in the overlapping region. Usov & Usov (1988) revisited the problem and tried to estimate growth rates by approximating the distribution function of the fast and slow particles in the overlapping region by delta-function. AM98 extended and presented a more realistic analytical way of constructing the form of the hot plasma distribution function in the overlapping region.

The aforementioned studies suggest that large amplitude Langmuir wave cannot exist due to $e^+/e^-$ beam in $C_1$. AM98 showed that two-stream instability in $C_2$ and $C_3$ can result in high growth rates of Langmuir wave. They also found that the growth rate for $C_3$ to be significantly larger than $C_2$. Thus AM98 provided the necessary justification, that in principle large amplitude Langmuir waves can be triggered for both $C_2$ and $C_3$.

However AM98 obtained growth rates in the hydrodynamic regime for $C_2$ and $C_3$ using many simplifying assumptions. For example, they assumed the surface magnetic field to be dipolar while observations suggest the existence of a strong multipolar magnetic field at the surface. Also, they estimated growth rates as a function of the distance from the neutron star, using coarse spatial (and temporal) resolution. In their numerical scheme, AM98 did not obtain the complete solution of the dispersion relation at a given height and estimated growth rates only for some representative wave numbers. The coarse resolution in their analysis can wash away many important features of the evolution of the growth rate as a function of the distance from the neutron star. Hence it is necessary to undertake an updated study of AM98 where these shortcomings should be addressed appropriately. This is the primary focus of this work. Further keeping in view that a tenous high energy beam of ions can exist, we study the effect of the same on Langmuir mode instability in $C_1$ and compare it with the $e^+/e^-$ beam.
It must be noted that the amplification of the Langmuir wave for a given frequency $\omega$ depends on the gain $G$ which is a product of the growth rate ($\omega_\delta$) and the time available for growth ($\Delta t$) as the amplitude is $\propto e^{G\omega \Delta t}$. If time $\Delta t$ is small, even with a high $\omega_\delta$, the amplification factor $G$ will be small and one cannot use these waves to participate in the coherent emission mechanism. In this work, we go beyond just the estimation of growth rate and present a method that employs the complete bandwidth of the growing waves to estimate $\Delta t$ and thereby the maximum gain possible for a given frequency. We present an exhaustive treatment of Langmuir mode instability for C1, C2 and C3 to examine the existence of large amplitude Langmuir waves in the pulsar radio emission region. The outline of the paper is as follows: In sections 2 we discuss physical constraints for the hot plasma description and models of plasma flow. In sections 3 and 4 we describe the analysis method for the linear Langmuir instability and study the growth rates and gain factors for these cases. In sections 5 and 6 we discuss the results and state our conclusions.

2 INPUTS TO THE PULSAR PLASMA PARAMETERS

2.1 Constraints from radio emission height

A number of studies: Blaskiewicz et al. (1991), von Hoensbroech & Xilouris (1997), Mitra & Li (2004), Mitra & Rankin (2011), Weltevrede & Johnston (2008) has consistently found the emission region to be below 10 $\%$ of LC across pulsar period (see fig. 3 of Mitra 2017). As discussed by Mitra & Li (2004), the various methods employed to find radio emission heights can be affected due to measurement as well as systematic errors, however for normal pulsars average estimates of a few hundred kilometers above the neutron star surface is considered reasonable. Specific studies (e.g. Mitra & Rankin 2002) also, focus on estimating the range of emission heights as a function of frequency, and it is found that a certain radius to frequency mapping exists in pulsars where progressively higher frequencies arise closer and closer to the neutron star surface. These studies reveal that the broad-band pulsar emission range from about few ten to hundred km at the highest frequency ~ 5 GHz and to several hundred km at the lowest frequency ~ 100 MHz. Kazbegi et al. (1991) showed that cyclotron resonances can be excited only near the light cylinder. At the radio emission heights all cyclotron resonances are suppressed and only the Cherenkov resonance condition can operate.

2.2 Signature of Coherent Curvature Radiation

Several lines of evidence (see Lai et al. 2001; Johnston et al. 2005; Rankin 2007; Noutsos et al. 2012, 2013; Force et al. 2015) have revealed that the polarization of the emergent pulsar radiation are directed either perpendicular or parallel to the magnetic field line plane. These polarization modes are commonly referred to as the extraordinary and Ordinary mode respectively which are defined with their electric field vector being perpendicular and parallel to the magnetic field plane respectively. The eigenmodes of the pulsar plasma viz., the X-mode and the O-mode is perpendicular and parallel to the $\hat{k} - \hat{B}$ plane, where $\hat{B}$ is the ambient magnetic field and $\hat{k}$ is the propagation vector of the wave. If the underlying excitation mechanism is due to CCR, then these two planes need to be co-incident. For any other form of excitation, these two planes can maintain arbitrary orientation to each other. This implies that the polarization of emergent radiation carries information about the underlying excitation mechanism. This idea was applied by Mitra et al. (2009) to a sample of nearly 100 $\%$ linearly polarized single pulses which established CCR as the underlying emission mechanism.

2.3 Multi-polar surface magnetic fields and particle flows

It is well known (see e.g. Mitra & Li 2004) that at a few hundred kilometers above the neutron star surface, from regions where the radio emission originates, the underlying magnetic field structure is dipolar. However, in recent years there are several pieces of evidence for the presence of surface multipolar fields. For example Gil & Mitra (2001) and Mitra et al. (2020) suggested that the radio-loud nature of the extremal long period 8.5 s pulsar J2144-3933 (Young et al. 1999) can only be explained if surface magnetic fields have a radius of curvature $\rho_c \sim 10^5$ cm at the surface, which is only possible due to presence of strong multipolar surface magnetic field. The X-ray observations have also confirmed the presence of multipolar fields on the surface (see e.g. Arumugasamy & Mitra 2019).

The presence of multipolar surface magnetic field significantly affects the description of the plasma. At the polar cap magnetically induced pair creation processes are triggered. The presence of multipolar surface magnetic field decreases the radius of curvature at the surface thereby increasing the efficiency of the pair creation process. As a result the number density of the pair plasma exceeds the co-rotational value $\kappa_{GJ}$ by a multiplicity factor $\kappa_{GJ}$. Observations of PWNe has revealed $\kappa_{GJ} \sim 10^4 - 10^5$ (see de Jager 2007; Blasi & Amato 2011). To get this high value of $\kappa_{GJ}$ estimations show that to get $\kappa_{GJ} \sim 10^4 - 10^5$ multi-polar fields are required (see Medin & Lai 2010; Szary et al. 2015; Timokhin & Harding 2019) whereas for purely dipolar fields $\kappa_{GJ}$ is about the order of a few tens to hundred (see Hibschman & Arons 2001; Arendt & Eilek 2002).

2.3.1 Need for multipolar surface magnetic field for CCR

The limiting brightness temperature for incoherent curvature radiation is $T_{lim}^{\text{CR}} \approx 10^{13}$ K (see Melrose 1978). In the Rayleigh-Jeans regime, the brightness temperature is proportional to power. CCR is an ‘$N^2$’ process meaning if ‘$N’ particles are involved, the power is boosted by a factor ‘$N’ compared to what would be achieved if the
charged particles were emitting independently (or incoherently). The number of particles participating in CCR to explain the observed high brightness temperature is given by \( N_{\text{CCR}} = T_{\text{obs}}/T_{\text{CR}}^{\text{sec}} \approx 10^{12} \). Radio emission from pulsars are received from 10 MHz to 10 GHz. The length of the bunch should satisfy the constraint \( L \ll c/\sqrt{\text{High}} \sim 3 \, \text{cm} \) for coherence to be maintained for all frequencies. Assuming \( L \sim 1 \, \text{cm} \), the corresponding number density required \( n_{\text{CCR}} \sim 10^{12} \, \text{cm}^{-3} \). At an emission height of \( r_{\text{em}} \sim 50 \, \text{RNS} \), the Goldreich-Julian value is given by \( n_{\text{GJ}} = 5.32 \times 10^{8} \, \text{cm}^{-3} \). (see Goldreich & Julian 1969). Thus, CCR requires number density in excess of the Goldreich-Julian value by a factor of \( 10^{4} \).

### 2.3.2 Description of particle flow and secondary plasma distribution functions

The models of plasma flow can be divided into two classes: a) The steady flow model (also known as SCLF model) given by Arons & Scharlemann (1979) where when condition above the polar cap is such that \( \Omega_{\text{rot}}/\hat{B} > 0 \) (here \( \Omega_{\text{rot}} = 2\pi f / \text{P} \) is the pulsar rotational frequency), the electrons can be easily pulled out from the star and a stationary flow of electron beam-plasma can be maintained and; b) The non-stationary spark discharge model (also referred to Inner acceleration gap model or the pure vacuum gap model) by RS75 for pulsars with \( \Omega_{\text{rot}}/\hat{B} < 0 \) giving rise to an intermittent plasma flow due to sparking discharges at the polar gap. In both these models the beam-plasma system is established.

The vacuum gap model of RS75 is more successful in explaining pulsar radio observations like sub-pulse drift phenomenon, however the original model required certain modifications. Gil et al. (2003) noticed that the sub-pulse drift rates and the temperature of the thermal X-ray emitting polar cap are both lower than that predicted by the pure vacuum gap model of RS75. They suggested that the pure vacuum gap is untenable and must be partially screened by vacuum gap model of RS75. They suggested that the potential is \( \phi_{\text{vac}} \approx 12 \, \text{volts.} \) The authors constrained \( \eta = 0.1 \), which gives the Lorentz factor of the high energy primary beams of \( e^+/e^- \) and ions to be given by \( \gamma_{\text{b}}(e^+)/e^- \sim 10^{6} \) and \( \gamma_{\text{b,ions}} \sim 10^{3} \) respectively. Assuming CCR we can find an order of magnitude estimate of the bulk Lorentz factor of the secondary pair plasma. Most of the power in CCR for charge bunch with Lorentz factor \( \gamma \) is concentrated near the critical frequency \( \omega_c = 1.5 \gamma^3 c / \rho_c \). (see Jackson 1962, where \( c \) is the velocity of light). Assuming observing frequency \( \nu_{\text{obs}} = 1.4 \, \text{GHz} \) to be close to the critical frequency at \( r_{\text{em}} \sim 50 \, \text{RNS} \) where \( \rho_c \approx 10^{8} \, \text{cm} \), we have \( \gamma \approx 200 - 300 \).

For our work we assume the distribution functions of all the species to be relativistically streaming Gaussians. For secondary plasma, the mean and the width are assumed to be \( -200 - 300 \) and \( 40 - 60 \) respectively. Note that the two stream-condition can be established in non-stationary flow by all three cases of C1, C2 and C3 whereas for stationary flow only the cases C1 and C2.

To summarize CCR needs to be excited by large amplitude Langmuir waves in a hot relativistically streaming dense secondary pair plasma. At the radio emission heights, the wave-particle interaction is mediated by the Cherenkov resonance condition. In subsequent sections we address how large amplitude Langmuir waves can be triggered for the three cases, C1, C2, C3 of two-stream instability discussed earlier.

### 3 ANALYSIS OF LANGMIUR INSTABILITY

In the following subsections, we establish the methodology for studying Langmuir instability. To do this we define a threshold gain for a wave of a particular frequency that can be used as a proxy for the breakdown of the linear theory. This, in turn, is achieved by solving the complex frequencies using the appropriate dispersion relation. For this analysis, the following aspects need to be considered.

#### 3.1 The Dispersion relation in the observer frame of reference

The dispersion relation of the Langmuir mode for a strictly one-dimensional relativistic flow in the observer frame of reference is given by (see section 4 of AM98)

\[
\epsilon(\omega, k) = k c + \sum_{a} \omega_{a}^{2} \int_{-\infty}^{\infty} dp_{a} \frac{\partial f_{a}^{0}}{\partial p_{a}} \frac{1}{\omega - \beta_{a} k c} = 0 \tag{1}
\]

where \( \omega_{a}^{2} = 4\pi n_{a} q_{a}^{2} / m_{a} ; \gamma = \sqrt{1 + p_{a}^{2} / m_{a} c^{2}} ; \beta_{a} = p_{a} / \sqrt{1 + p_{a}^{2} / m_{a} c^{2}} \). Here \( n_{a} , q_{a} , m_{a} , \rho_{a} \) and \( f_{a}^{0}(0) \) is the number density, charge, mass, dimensionless momenta and the equilibrium distribution function of the \( \alpha \)-th species in the plasma such that \( n_{a} = \kappa_{GJ, \alpha} n_{GJ, \alpha} \). \( \rho_{a} = \gamma_{a} m_{a} \gamma / \beta_{a} \) and \( \int_{-\infty}^{\infty} dp_{a} f_{a}^{0}(0) = 1 \).

We assume \( f_{a}^{0}(0) = (1 / \sqrt{2\pi \sigma_{a}^{2}}) \exp \left[-(p_{a} - \mu_{a})^{2} / 2\sigma_{a}^{2}\right] \), with mean \( \mu_{a} \) and width \( \sigma_{a} \) for all \( \alpha \) species. In the super-Luminal region the Cherenkov resonance condition cannot be satisfied and hence there is no singularity in the integral of Eq. 1. The integral can be integrated by parts to give the dispersion relation as

\[
1 - \sum_{a} \omega_{a}^{2} \int_{-\infty}^{\infty} dp_{a} \frac{1}{\gamma^{3}} \frac{f_{a}^{0}(0)}{(\omega - \beta_{a} k c)^{2}} = 0 \tag{2}
\]

At \( k = 0 \) the cut-off \( \omega_{0} \) is given by \( \omega_{0}^{2} = \sum_{a} \omega_{a}^{2} \int_{-\infty}^{\infty} dp_{a} f_{a}^{0}(0) / \gamma^{3} \) while the frequency \( \omega_{0} \) at which the Langmuir mode touches the \( \omega = \kappa c \) line is given by

\[
\omega_{0}^{2} = \sum_{a} \omega_{a}^{2} \int_{-\infty}^{\infty} dp_{a} \frac{1}{\gamma^{3}} \frac{f_{a}^{0}(0)}{1 - \beta_{a}^{2}} \tag{3}
\]

The dispersion relation can be cast in the dimensionless

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3.1.1 Growth rates in the sub-luminal regime.

The Cherenkov resonance condition $\omega - \beta_n kc$ in the denominator of Eq. 1 is satisfied in the sub-Luminal regime, and produces a singularity in the integral for Langmuir wave frequencies ($\geq \Omega$). The pole $p_{pole} = \omega/\sqrt{(kc)^2 - \omega^2}$ of the dispersion function needs to be treated using Landau prescription. For the growth of Langmuir waves, the Landau prescription allows for two regimes of growth (see Appendix A for discussion) viz., the kinetic regime, and the hydrodynamic regime.

In the kinetic regime the pole lies very close to the real axis contour such that the Landau contour has to be analytically continued to the lower half plane. In this regime the dispersion relation is broken into a principal value integral and a residue at the pole. The dimensionless growth rate in the kinetic regime is given by (see Eq. A13 of Appendix A)

$$\Gamma_\text{kin} = \frac{\pi}{2K^2} \chi_b \left( \frac{\partial f_b^{(0)}}{\partial \gamma} \right) \bigg|_{p_b = \langle p_{beam} \rangle}$$

such that $\Gamma_\text{kin} = \omega_{kin}/\omega_1$ and $\langle (...) \rangle = \int_{-\infty}^{+\infty} dp_\alpha (...) f_\alpha^{(0)}$.

Here subscript b and s correspond to the beam and secondary plasma respectively. Note that the distribution having the pole correspond to b and the distribution function away from the pole correspond to s. The kinetic growth rate is a local description as it requires only the derivative of the distribution functions at $p_{beam}$, and is referred to be of resonant type where only the set of particles at and around $p_{beam}$ contribute to the growth. It must also be noted that the expression for kinetic growth rate has been derived under the assumption that the slopes are gentle viz., $\alpha_\sigma$ is broad and the distribution functions have no discontinuity.

In the hydrodynamic regime the pole lies above the Landau contour. In this regime the dispersion relation can be integrated by parts along the real axis for complex frequency $\omega = \omega_R + i\omega_I$ where $\omega_R > 0$. The real and imaginary part of the dimensionless dispersion relation (see Eq. A18 of Appendix A) in the hydrodynamic regime are given by

$$1 - \sum_\alpha \chi_\alpha \int_{-\infty}^{+\infty} dp_\alpha \frac{f_\alpha^{(0)}}{\gamma^3} \left( \frac{(\Omega_R - \beta_\alpha K^2 - \Omega_1^2)}{(\Omega_R - \beta_\alpha K^2 + \Omega_1^2)} \right)^2 = 0$$

$$-i 2 \Omega_1 \sum_\alpha \chi_\alpha \int_{-\infty}^{+\infty} dp_\alpha \frac{f_\alpha^{(0)}}{\gamma^3} \left( \frac{(\Omega_R - \beta_\alpha K)}{(\Omega_R - \beta_\alpha K^2 + \Omega_1^2)} \right)^2 = 0$$

where $\Omega_1 = \omega_1/\omega_1; \Omega_R = \omega_R/\omega_1$. The above set of equations have to be solved simultaneously to get a solution for the dimensionless quantities $\Omega_R$ and $\Omega_1$. The dimensional growth rate is a product of the dimensionless growth rate($\Omega_1$) and the scaling factor $\omega_1$. Quantities in the dimensional form will have the following notation $\omega_R = \text{Re}(\omega)$ [in rad/s]; $\omega_I = \text{Im}(\omega)$ [in s$^{-1}$]. Unlike the kinetic regime, the growth rates in the hydrodynamic regime require a complete description of the distribution function for all the species involved. In this sense the hydrodynamic regime represents a non-resonant type of growth where all the particles contribute to the growth. It must be noted that the growth rates in the hydrodynamic regime are necessarily greater than that in the kinetic regime.

Next, we define the equivalent distribution function (hereafter EDF) as the number density weighted summation of the distribution functions of the species that constitute the system. The expression for growth rates in both regimes requires that the EDF satisfy the relativistic generalization of Gardner’s theorem 1 states that if the EDF of a plasma system is single-humped then such a system cannot support a growing set of waves (see Appendix B for proof). Thus two-stream instability cannot be satisfied for a single-humped EDF.

If the distribution functions in the EDF are given by gaussians and the mean of the gaussians are well-separated, the hydrodynamic growth-rate has to satisfy the condition that

$$\Omega_1 \geq \frac{\Delta \gamma_T}{\gamma_T^3}$$

where the quantities $\Delta \gamma_T$ and $\bar{\gamma}_T$ refer to the width and mean of the gaussian distribution function with the higher mean (see e.g. Eq. 49 of AM98). The condition Eq. 7 can be used as a separation between the hydrodynamic and the kinetic regime, where the condition is reversed for the kinetic regime. However, if the means of the gaussians are not-sufficiently separated this threshold is much lower.

In this work apart from a brief discussion of the kinetic regime in C1, we focus exclusively on the hydrodynamic regime for all three scenarios. The algorithm for solving the hydrodynamic equations are presented in Appendix D.

3.2 Constraint on $\omega_1$

The solution of the dispersion relation must have the character that $\text{Re}(\omega) \geq \omega_1$. Combining the number density constraint as shown in 2.3.1, the corresponding value for the scaling $\omega_1$ using Eq. 3 is given by

$$\omega_1^{\text{Th}} \sim \sqrt{\sqrt{\text{CCR}}} \times 10^{1.5} \text{rad/s} \geq 10^{11} \text{rad/s}$$

1 see Gardner (1963) for the original version stated for a non-relativistic plasma system.
3.3 Maximum gain factor for a particular $\text{Re}(\omega)$

Following Gedalin et al. (2002) we introduce a method to estimate the maximum amplification for a given frequency $\text{Re}(\omega)$ using the bandwidth of growing waves as a proxy for the time available ($\Delta t$) for growth. Let us consider the dispersion relation at two points ‘A’ and ‘B’ along a given open magnetic field line. The ratio of the scaling frequencies at these two points is given by $\frac{\omega_B^3}{\omega_A^3} = (r_A/r_B)^{3/2}$. The same frequency corresponds to the frequency $\omega_1^B + \Delta \omega_B$ at point ‘B’ where $\Delta \omega_B$ is the bandwidth of growing waves at ‘B’. Then we have

$$\frac{\Delta \omega_B}{\omega_1^B} = \left(\frac{\omega_A}{\omega_1} \right) \left(\frac{r_B}{r_A} \right)^{3/2} - 1$$

$$\Rightarrow \Delta \Gamma = r_B - r_A = r_A \left(\frac{\Delta \Omega B + 1}{\Omega A} \right)^{2/3} - 1$$

The time for which the growth rate for frequency $\omega_A$ can be sustained is given by $\Delta t = \Delta \Gamma / c$. Assuming that the growth rate remains constant and using Eq. (9) the maximum gain for frequency $\omega_A$ is given by

$$G_{\text{max}} = \Gamma_A \Delta \Gamma = \Gamma_A \Delta \Gamma / c$$

This will be used for the estimation of the maximum gain following the numerical solution of the dispersion relations to get the growth rate ($\Gamma_{\omega_A}$) and the bandwidth for cases C2 and C3.

3.4 Criterion for breakdown of the linear theory

The solution of the dispersion relation does not carry information about the amplitude of the Langmuir wave i.e., we can only calculate $c^2$ given the growth rate and the time available for growth. The amplitude is given by $E(t) = E(t=0) e^{i\theta}$. The initial amplitude $E(t=0)$ of the wave at a particular frequency has to be obtained from a different treatment of the dispersion function as is done in subsection C1 of Appendix C. However, since the Langmuir wave grows at the expense of the particles in the plasma, the maximum energy that the wave can gain is equal to the total energy of all the particles in the plasma. Thus although the linear theory can predict arbitrary gain, in reality, there exists a gain threshold which cannot be exceeded. The next paragraph describes how to get an estimate of this threshold from consideration of maximum energy available in the plasma. If the linear theory predicts a gain close to or higher than this threshold, then it must be taken as a definitive indicator of the breakdown of the linear theory.

To indicate the breakdown of the linear theory we propose the following hypothetical situation: The growth rates are sufficient for breakdown of linear theory if the linear theory predicts the energy density in the field to be equal to the total energy density. Let us consider the dispersion relation for a wave of frequency $\omega_A$ at two points ‘A’ and ‘B’ with point ‘B’ higher up along a given field line. Assuming a constant growth rate the field energy density at point ‘B’ for $\omega_A$ should satisfy the condition $|E_{\omega_A}|^2/8\pi \approx (|E_{\omega_A}|^2/8\pi) e^{2G_{\text{max}}} = W_B$ where $W_A$ and $W_B$ are the total energy density at points ‘A’ and ‘B’. Using Eq. C9 from Appendix C we obtain a threshold gain indicating the breakdown of the linear regime viz.,

$$G_{\text{max}}^{\text{Th}} \approx \ln \left[ \sum \gamma_i^2 \right]/2.$$ 

For high energy beam driven instability this threshold is dictated by the Lorentz factor of the high energy beams. The gain threshold for instability in case C1 driven by $e^b$ beam and the ion beam comes out to be 12 and 6 respectively. The gain threshold for cases C2 and C3 involving only the species in the secondary plasma is given by 5. Thus, in all three cases C1, C2, C3 a representative threshold of gain to indicate the breakdown of linearity can be taken as

$$G_{\text{max}}^{\text{Th}} = 5$$

(11)

4 ESTIMATION OF GROWTH RATES AND GAIN

In what follows all analyses will be done along the last open field line of an aligned rotator with period $P = 1$ second and global dipolar strength $B_d = 10^{12}$ Gauss. For case C1 we get an analytical estimate of the maximum gain. For cases C2 and C3 the hydrodynamic equations Eq. A18 are solved numerically (see Appendix D) to obtain growth rates and maximum gain as a function of $r/R_{NS}$.

4.1 C1: Beam-driven Growth

The beam distribution function is given by

$$f_b^{(0)} = \frac{1}{\sqrt{\pi} \rho_b} e^{-\rho_b - \rho_b^2 / \rho_b^2}$$

such that $\rho_b = \bar{\rho}$ and $\sigma_b = \rho_b / \sqrt{2}$. Further let us introduce the width to mean ratio given by

$$x_b = \frac{\rho_b}{\overline{\rho}_b}$$

(13)

From subsection 3.3 the maximum gain for a particular frequency $\omega_A$ is given by

$$G_{\text{max}}^{\text{b}} = \Gamma_{\omega_A} \left(\frac{r_A}{c} \right) \left[ \left| \frac{\Delta \Omega + 1}{\Omega A} \right|^{2/3} - 1 \right]$$

Combining this with expressions for bandwidth of growing waves (from subsection C2 of Appendix C) and the threshold Eq. 7 we obtain the expression for the maximum gain for Langmuir waves of frequency $\omega_A$ due to beam-driven instability as

$$G_{\text{max}}^{\text{b}} \approx x_b \frac{P_t \rho_b}{\gamma_b^2} \left(\frac{r_A}{c} \right) \left[ \frac{8 \gamma_b}{3 \bar{\gamma}_b} \right]^2 \omega_A$$

(14)

4.1.1 High energy positron/electron beam with $\gamma_b \sim 10^6$

Egorenkov et al. (1983) demonstrated that only the hydrodynamic regime exists for the high energy $e^+ / e^-$ beam even
for a broad distribution function or large $x_b$. To estimate the gain we chose a representative value $x_{min,ion} = 0.01$ and $x_{e^+/e^-} = 0.3$. A multiplicity factor of $κ_{GJ} = 10^4$ and $κ_e = 200$ has been used.

4.1.2 High energy ion beam with $γ_{ion} \sim 10^3$

Unlike the $e^+/e^-$ beam case, there is no apriori information indicating for what value of $x_{min}$ will the hydrodynamic regime exist exclusively. Thus to proceed we get an estimate of $x_{min,ion}$ for which the kinetic regime gets suppressed completely. We assume the ion beam to be composed of iron ions such that $n_i/n_{ion} = κ_{GJ} \sim 10^4$, $γ_{ion}/γ_e \sim 10$, $n_{ion}/m_e$ ~ 56 and $Q_{ion}/Q_e$ ~ 26. By substituting Eq. 12 in Eq. 5 and estimating it at $p = \sqrt{2}ρ_{NS}$, we have the maximum dimensionless growth rate in the kinetic regime $Γ_{kin}^\max \approx f_{ion,γ}^{0.5}x_{min,ion}^2/8ρ_{NS}^2e^\chi γ_e^2$. $Γ_{kin}^\max$ so obtained must follow the constraint $Γ_{kin}^\max \leq p_{T,ion}/γ_{ion}^2$. This gives $x_{h,ion} \geq \{[\sqrt{2}^0.5(n_{ion}Q_{ion}^2m_eγ_{ion}^3/8e^22n_iQ_{ion}^2m_{ion}^2γ_e^2)]^{0.5} ≈ 0.01$. Substituting $x_{min,ion} = 0.01$ in Eq. 14 we get $Γ_{kin}^\max$ plotted as dashed red line in Fig.1 as a function of $r/NS$. As evident in the figure, the maximum gain for the ion is larger than the $e^+ / e^-$ beam, however still significantly smaller than the gain threshold given by Eq. 11.

It can be seen from Fig. 1 that none of the high energy beams can exceed the gain threshold given by Eq. 11.

4.2 C2: Growth due to longitudinal drift

CR77 suggested that due to the motion of the combined system of “beam + secondary plasma” along curved magnetic field lines, the electron-positrons distribution function in secondary plasma has to separate to provide a steady state current dictated by the local Goldreich-Julian value and the solenoidal nature of current flow. The separation of the bulk velocity $Δβ$ of the species in the secondary plasma at a distance of $r_A$ from the neutron star surface is given by:

$$|Δβ|_A \approx \left(\frac{ρ_b}{ρ_s}\right) \left(\frac{(Ω_{Rot} - \hat{β} \cdot \hat{B}_{rot})_A}{(Ω_{Rot} - \hat{β} \cdot \hat{B}_{rot})_0} - 1\right)$$

(15)

where $ρ_b$ and $ρ_s$ correspond to the charge density of the beam and secondary plasma respectively. The ratio $(ρ_b/ρ_s) = 1/κ_{GJ}$, which we call as the density term. Here the reference point “O” is taken at $r = 1.02 R_{NS}$ where pair creation cascades cease. The correction $'f_{Rot} ≈ 1 + O(Q_{NS}^2r^2/c^2)$ due to rotation can be taken to be 1, as the higher order term $O(Q_{NS}^2r^2/c^2) ≈ 0.01$ at $r = 50R_{NS}$ for a pulsar with $P = 1$ second. It can be seen that the separation of the electron-positron distribution function is a product of two terms viz., the density term $(ρ_b/ρ_s)$. This gives $G_{max} ≈ \sqrt{1 - β_e^2}$. The geometrical factor is zero only for very straight magnetic field lines. Thus curved magnetic field line is a necessary requirement of longitudinal drift/ separation of $e^\pm$ distribution in a secondary plasma.

**Simulating EDF for C2:** Eq. 15 just gives the difference between $β_{+}$ and $β_{-}$. To solve for $β_{+}$ and $β_{-}$ we need an additional constraint. We make the simplifying assumptions that [i] longitudinal drift affects only the mean of the distribution functions i.e, the separation of the bulk velocity is equal to the separation of the mean of the $e^\pm$ distribution functions; and [ii] The $e^\pm$ distribution functions are co-incident at “O” with mean lorentz factor $γ_{O}$ and at any point $r_A$ the mean of the distribution functions separate to attain values that are symmetrical about $γ_{O}$.

The requirement of symmetry translates to the condition that for any other point “A” along the field line

$$|Δγ_{+}| = |Δγ_{-}| = |Δγ| = \left|γ_{A} - γ_{O}\right|$$

(16)

where $γ_{O}$ is the mean of the overlapped distribution function at point “O” and $γ_{A}$ is the mean of the electron-positron distribution function at “A”.

Let the beta value corresponding to the bulk velocity of both the electrons and positrons at “O” be denoted by $β_{O}$. For any other point “A”, let the beta factor corresponding to the bulk velocity of the positrons and electrons be denoted by $β_{+}$ and $β_{-}$, then

$$β_{+} - β_{-} = [β_{O} + Δβ_{+}] - [β_{O} - Δβ_{-}]$$

is given by

$$Δβ = Δβ_{+} + Δβ_{-}$$

(17)

Let us introduce the factor ‘$f_{ratio}$’ given by $Δβ_{+} = f_{ratio} Δβ_{-}$ and perform the following steps to get the separation ($2Δγ$) at any distance $r_A$. For a given mean lorentz factor $γ_{O}^2$ at “O”, and the density term($= 1/κ_{GJ}$) and geometrical factor at “A”, Eq. 17 is solved for $f_{ratio}$ so as to satisfy the symmetry constraint given by Eq. 16. Once $f_{ratio}$ is obtained, the bulk velocity for the separated distribution functions can be estimated as $β_{+} = β_{O} + Δβ_{ratio}(1 + f_{ratio})$ and $β_{-} = β_{O} - Δβ_{ratio}(1 + f_{ratio})$. The bulk velocity so obtained are transformed to the mean lorentz factors $γ_{+}$ and $γ_{-}$ via the transformation $γ_{O} = 1/\sqrt{1 - β_{O}^2}$. The corresponding momenta is given by $p_{O} = γ_{O}$. 

**Figure 1.** Plot for maximum gain that can be obtained for high energy beam driven instability. The dashed red line and the solid blue line gives the maximum gain for an ion beam and positron/electron beam with $x_{min,ion} = 0.01$ and $x_{e^+/e^-} = 0.3$. A multiplicity factor of $κ_{GJ} = 10^4$ and $κ_e = 200$ has been used.
In this case the EDF consists of the summation of the shifted gaussian distribution functions with mean $\gamma(+)\text{ and } \gamma(-)$. After getting the EDF we follow the steps outlined in Appendix D to solve the hydrodynamic Eq. A18.

We proceed to solve growth rates for two surface magnetic field configuration viz., a purely dipolar one and multi-polar field. For both field configurations we consider the last open field line for an aligned rotator and assume secondary plasma distribution function to be a gaussian (with mean $\mu = 250, \text{width } \sigma = 40$) at $r/R_{NS} = 1.02$.

### 4.2.1 Simple Dipolar Geometry

The results are shown in Fig. 2. The top panel shows (A) the separation of the distribution functions as a function of $r/R_{NS}$ for two multiplicity factors ($\kappa_G \sim 50, 500$) and (B) shows the EDF for $\kappa_G = 500$ at a distance of 500 km from the neutron star surface. In the middle panel (C) the solution of the dispersion relation along with the residuals in the dimensionless form is shown while (D) shows the dimensional growth rate along with the group velocity dispersion of the dispersion relation along with the residuals in a unstable wave of a given frequency is shown as a function of $r/R_{NS}$. It must be noted that the $Re(\omega)$ does not satisfy the constraint given in section (2.3.1).

### 4.2.2 Multi-polar Geometry

Simulating the multipolar field configuration: As discussed in section 2.3 any multipolar field configuration must satisfy the following two conditions for CCR: [i] At the radio emission heights the pulsar magnetic field must have a purely dipolar character; [ii] The neutron surface must have a much smaller radius of curvature than the local component is situated at the $\theta_m$ and $\rho_m$ with respect to the global dipole field (see fig.1 of Gil et al. 2002). The strengths of the magnetic moments of the global dipolar field and the local crustal field is given by $|d| = 0.5 B_3 R_{NS}^3$ and $|\vec{b}| = 0.5 B_{\text{rot}}(0.05 R_{NS})^3$ respectively. The boundary condition is chosen such that at the radio emission region $r/R_{NS} = 10$ the composite magnetic configuration should satisfy condition[i]. The middle panel of Fig. 3 shows $\rho_c$ as a function of $r/R_{NS}$ for a purely dipole field (shown in dashed green line) and a composite configuration (shown as a solid red line) for certain model parameters are given in the caption. It can be seen that $\rho_c$ due to the multi-polar configuration satisfies condition [ii] at the surface. Both $\rho_c$ and the strength of the magnetic field $B_{\text{rot}}$ shown in the middle and the lower panel of Fig. 3 resembles that of a purely dipolar configuration within 10 km from the surface. The magnetic field strength for the multipolar configuration differs from that of a purely dipolar configuration by less than 0.8% at $r/R_{NS} = 2$. This means that the superposed field configuration is insensitive to any change in the boundary condition beyond few tens of km from the surface. Since the multi-polar configuration has $\rho_c \sim 10^3 \text{cm}$, we can justifiably use high $\kappa_G$. It must be noted that for $r/R_{NS} \geq 2$ both $\rho_c$ and $B_{\text{rot}}$ attain a purely dipolar character while the geometrical factor quickly attains a boosted steady value compared to a purely dipolar surface geometry as shown in the upper panel of Fig. 3. This simulated geometrical factor and high $\kappa_G$ are then used as inputs for simulating the EDF.

The results of our analysis are shown in Fig. 4 and the plot description are similar to Fig 2 and the parameters for the simulations are described in the caption to the figure. It is important to note that in this case, unlike the dipolar example above, $Re(\omega)$ satisfy the constraint given in section (2.3.1).

To summarize as seen in the bottom panel of figures 4.2.1 and 4.2.2 sufficient growth rates exceeding the threshold limit Eq. 11 can be obtained for both dipolar and multipolar field configuration.

### 4.3 C3: Growth due to cloud-cloud overlap

This model by Usov (1987) later developed by Ursov & Usov (1988) is based on the non-steady sparking discharge model (also referred to Inner acceleration gap model or the pure vacuum gap model) by RS75. In the RS75 model for pulsars with $\Omega_{\text{rot}} \cdot \vec{B} < 0$, positive charges are needed to screen the co-rotational electric field above the polar cap. However due to the high binding energy of the ions supply to positive charges are inhibited, and a vacuum gap with a strong electric field develops above the polar cap. The gap initially grows, however, once it reaches a height $h \approx 60 - 100 \text{ m}$, it discharges via magnetic pair creation. Due to the strong electric field in the gap, the electrons are accelerated towards the stellar surface, while the positron streams relativistically away from the stellar surface. The upstreaming positron has sufficient energy to produce pair cascade, thus creating the secondary plasma cloud. This process continues until the electric field in the gap is screened, and hence for the gap emptying time $h$ is a time $\tau = h/c \sim$ which is about a few hundreds of nanoseconds, the sparking process stops. Once the gap empties, the electric field grows and the sparking process starts again. Hence during steady-state, a non-stationary flow of secondary plasma cloud is generated, with each cloud having a spread in particle velocity. In the original model of Usov & Usov (1988) the overlap of the fastest and slowest particles of these successive secondary plasma clouds leads to two-stream instability.

AM98 extended the cloud-cloud overlap formalism of Usov & Usov (1988) by categorizing the particles in each cloud of the secondary plasma into fast, slow and intermediate particles based on their speeds $\nu$. The authors presented an analytical expression for EDF in the overlapped region
using $\Psi = x - vt$. The integral of motion $\Psi$ kept track of the position of these three categories of particles in each secondary plasma cloud. The distribution function for each cloud is given by $F(p, \Psi) = F(p) F(\Psi)$. The phase function $F(\Psi)$ modulates the shape of the distribution function $F(p)$ as a function of $r/R_{\text{NS}}$. In our scheme of constructing the EDF, we assume that $F(\Psi)$ can be ignored within a single secondary plasma cloud. We justify this assumption based on two considerations. Firstly, in the hydrodynamic regime, the dip in the EDF containing Re$(p_{\text{pole}})$ is of paramount importance. Since the hydrodynamic equations involve integration over the whole distribution functions, the modulated
The position of overlap is given by
\[ x_{\text{OV}} = \frac{v_{\text{max}}^{(2)} v_{\text{arb}}^{(1)} \tau}{v_{\text{max}}^{(2)} - v_{\text{arb}}^{(1)}} \]
which can be represented as a function of \( r/R_{\text{NS}} \).

The position of the minimum velocity particles of cloud ‘1’ at time \( t_{\text{OV}} \) is given by
\[ x_{\text{min}}^{(1)} = \frac{v_{\text{min}}^{(1)} v_{\text{arb}}^{(2)} \tau}{v_{\text{max}}^{(2)} - v_{\text{min}}^{(1)}} = v_{\text{lower}}^{(1)}(T + t_{\text{OV}}) \]

The equality is used to solve for \( v_{\text{lower}}^{(2)} \) and the solution transformed to dimensionless momenta \( p_{\text{lower}}^{(2)} \) via the transformation \( p = \beta / \sqrt{1 - \beta^2} \).

The EDF \( f^{\text{OV}} \) in the overlapped region is given by
\[ f^{\text{OV}} = f_1^{(1)} \left[ v_{\text{min}}^{(1)} : p_{\text{upper}}^{(1)} \right] + f_2^{(2)} \left[ p_{\text{lower}}^{(2)} : p_{\text{max}}^{(2)} \right] \]
where the notation \( f_\alpha[a : b] \) refers to the portion of distribution function \( f_\alpha \) from \( a \) to \( b \) for cloud with index ‘\( \alpha \)’.

The spatial extent of the overlapped region is given by the expression
\[ \Delta_{\text{OV}} = x_{\text{max}}^{(2)} - x_{\text{min}}^{(1)} = \left( \frac{v_{\text{max}}^{(1)} v_{\text{min}}^{(1)} - v_{\text{min}}^{(2)} v_{\text{max}}^{(2)}}{v_{\text{max}}^{(2)} - v_{\text{min}}^{(1)}} \right) \tau \]

The dispersion relation in the overlapped region is given by the expression
\[ 1 - \chi \int_{-\infty}^{+\infty} dp \frac{f^{\Delta_{\text{OV}}}}{1/\gamma^2} = 0 \quad (18) \]

In this case, the EDF is being determined by the lower and higher momenta cut-off \( p_{\text{min}} \) and \( p_{\text{max}} \) of the secondary plasma distribution function and the gap closing time \( \tau \). Here we assume a gaussian distribution function with \( \mu = 200 \), \( \sigma = 60 \), \( p_{\text{min}} = 5 \), \( p_{\text{max}} = 400 \) for the secondary plasma clouds. After getting the EDF we follow the steps outlined in Appendix D to solve the hydrodynamic equations. The results of our numerical solution are shown in Fig. 5. As seen from the third subplot of (F) sufficient growth rates can be obtained exceeding the gain threshold defined in section 3.4.

4.3.1 Effect due to longitudinal drift

In the previous numerical simulation we have taken the value of \( p_{\text{min}} = 5 \). However, if this value were to be higher the contribution to the EDF at a given height due to the leading cloud \( f_1^{(1)} : p_{\text{upper}}^{(1)} \) becomes smaller which decreases the dimensionless growth rate drastically. We consider a situation where for some orientation of the crust-anchored dipole and high \( \kappa_{\text{GJ}} \), the longitudinal drift can lead to splitting in the electron-positron distribution function in the secondary plasma but does not produce minima in the EDF for \( C2 \). However when combined with \( C3 \) this separation lowers \( p_{\text{min}}^{(1)} \) of the distribution functions as the cloud flows outward along the field line. We perform the next numerical simulation to
study the effect of longitudinal drift on the cloud-cloud overlap for the aforementioned scenario. The results are shown in Fig. 6. As seen from the third subplot of (F) even in this hybrid of cases C2 and C3 the maximum gain exceeds the gain threshold defined in section 3.4.

We find that in the absence of C2 the dimensionless growth rate (Ω < 10^{-8}) and comparable to the residuals of the hydrodynamic equations.

5 DISCUSSION AND COMPARISONS WITH PREVIOUS STUDIES

In sections 3 and 4 we provided a hot plasma treatment of two-stream instability and estimated growth rates of Langmuir mode for various models of one-dimensional plasma flow in pulsars. Based on our analysis our final aim is to examine under what conditions excitation of CCR is possible in pulsars. There are at least three conditions, namely, (I), (II) and (III) that need to be fulfilled. The first condition (I) is that for two-stream instability to occur in one-dimensional plasma flow, the EDF should not be single-humped (Gardner’s theorem). If condition (I) is satisfied, excitation of CCR further requires the following two constraints to be satisfied simultaneously viz., (II) The amplification criteria which re-
Figure 5. Plots for C3 for $\tau = 100$ nanoseconds and $T = 30\, \tau$ as discussed in section 5. In the top left panel (A) the upper and the lower subplot shows the momenta of the particles in cloud 1 to be overtaken by the maximum velocity particles in cloud 2 and the extent of the overlapped region as a function of $r/R_{\text{NS}}$. The top right panel (B) shows the EDF at $r = 50\, R_{\text{NS}}$. The vertical black dot-dashed line shows $\text{Re}(p_{\text{pole}})$. In the middle left panel (C) the first and second subplot shows the dimensionless real (in dot-dashed blue line) and imaginary parts (in solid blue line) of the dispersion relation as a function of the dimensionless wavenumber $K$ for the EDF shown in (B). The third subplot shows the residuals of the real (in dashed black line) and imaginary (in dashed orange line) parts of the dispersion relation as a function of the dimensionless wavenumber $K$. In the middle right panel (D) the first and second subplots show the corresponding dimensional dispersion relation to (C) for $\kappa_{\text{GR}} = 10^4$. The third subplot shows the group velocity dispersion as a function of wavenumber $k$. In the lower left panel (E) the first subplot shows the maximum dimensionless growth rate while the second subplot shows the residues of the real (in dashed black line) and the imaginary part (in orange line) of the dispersion relation as a function of $r/R_{\text{NS}}$. In the lower right panel (F) the first and second subplot shows the solution of the dispersion relation for the maximum growth rate for $\kappa_{\text{GR}} = 10^4$ and $\kappa_{\text{GR}} = 10^5$ shown as blue and red solid line respectively as a function of $r/R_{\text{NS}}$. The third subplot shows the maximum gain calculated from Eq. 10 with the dashed black line showing the threshold Eq. 11.
Figure 6. Plots for C3 aided by C2 along the last open field line for multi-polar configuration as used in Fig. 3 and discussed in section 4.3.1. Here \( r = 100 \) nanoseconds and \( T = 30 \) r. A gaussian with \( \mu = 240, \sigma = 50, p_{\text{min}} = 25, p_{\text{max}} = 400 \) has been assumed for the distribution function for the secondary plasma cloud at \( r = 1.02 R_{\text{NS}} \). Multiplicity factor of \( k_{\text{PG}} = 2 \times 10^4 \) has been used. For left panel (A) the upper subplot shows the maximum dimensionless growth rate as a function of \( r/R_{\text{NS}} \). The lower subplot shows the residuals of the real(in dashed black line) and imaginary(in dashed orange line) parts of the dispersion relation respectively as a function of \( r/R_{\text{NS}} \). For the right panel (B) the upper and the middle subplot shows real and imaginary part of the dispersion relation in its dimensional form as a function of \( r/R_{\text{NS}} \). The lower subplot shows the maximum gain obtained using Eq. 10 as a function of \( r/R_{\text{NS}} \). The dashed black line in the second subplot of (B) represents the threshold Eq.11.

Figure 7. Color map showing the separation (\( \Delta \theta \)) of the electron-positron distribution function due to longitudinal drift at \( r/R_{\text{NS}} = 50 \) due to various orientation (\( \theta_m, \theta_i \)) of the crust-anchored field (\( b = 10, B_3 = 10^{12} \text{gauss} \)) as shown in fig. 1 of Gil et al. 2002. The mean value of the gaussian is taken to be \( \gamma_s = 250 \) and the ratio of dipole moments of the crust-anchored dipole to global dipole star centred dipole has been fixed to \( |m/d| = 1.25 \times 10^{-3} \). For \( \sigma = 40 \), the separation is said to be sufficient only if \( \Delta \gamma_s >= 1.5 \sigma \).

Note that the estimation of gain requires the description of EDF and the scaling \( \omega_1 \). The solutions of the dimensionless hydrodynamic equation are determined by the EDF and subsequently one obtains the dimensionless growth rate (\( \Omega_1 \)) and the bandwidth of growing waves (\( \Delta \Omega_1 \)). The dimensionless growth rate (\( \omega_1 \)) is a product of \( \Omega_1 \) and \( \omega_1 \). The scaling factor varies as \( \omega_1 \propto \sqrt{k_{\text{PG}}/R_{\text{NS}}} \) and falls monotonically with distance. Thus to satisfy condition (III) high \( k_{\text{PG}} \) is necessary, which requires multi-polar surface magnetic field geometry. In what follows we check how these conditions (II) and (III) are fulfilled for cases C1, C2 and C3 respectively. We compare our results with previous studies and discuss further implications.

5.1 Results for C1

Observations suggest the presence of an ion component in pulsar plasma (see Gil et al. 2003) along with the \( e^+/e^+ \) beam. We assume the ion component to be composed of iron and characterized by a bulk Lorentz factor of \( \gamma_{\text{ion}} \approx 10^3 \). While hot plasma treatment for \( e^+/e^+ \) beam exists in the literature, as far as we know that such treatment for an ion beam does not exist in the literature. We analyze the ion component similarly as Egorenkov et al. (1983) did for the
high energy $e^+/e^-$ beam. We find that for width to mean ratio of 1% the kinetic regime is completely suppressed. Using this width we estimate the maximum gain for the ion beam in the hydrodynamic regime. For the sake of comparison, we also estimate the maximum gain for the high energy $e^+/e^-$ beam. We find that although the gain for the ion beam is $\sim 5$ orders of magnitude higher than $e^+/e^-$ beam yet it cannot satisfy condition (II). Since none of the beams can satisfy condition (II) the beam driven Langmuir instabilities are excluded as candidates for pulsar radio emission.

5.2 Results for $C2$ and $C3$

AM98 found that growth rates in $C3$ exceed that of $C2$ for the same $\kappa_{GJ}$ at the radio emission region (see panels (a) and (b) of fig. 6 in AM98). AM98 further asserts that $C3$ dominates $C2$ below $r/R_{NS} = 50$ and that the role changes beyond this distance. In our work we find these assertions to be inconsistent. We find the exact opposite result as can be seen from the panel (F) in Fig. 4 and 5 for $\kappa_{GJ} = 10^4$. These conflicting results can be understood by comparing the methodology for the construction of EDF in this work and AM98.

5.2.1 EDF for $C2$ As discussed in section 4.2 the separation of the bulk velocities $\Delta \beta$ is equal to the product of the geometrical term and the density term. However, both CR77 and AM98 assume the geometrical term to be equal to unity. This assumption is not valid at the radio emission region. For both dipolar and multipolar surface magnetic field geometries, the geometrical term is much less than unity (see the upper subplot in panel A of Fig. 2 and Fig. 4).

Further CR77 assumed the radio emission region to be sufficiently far from the surface. Assuming a delta-function for $e^+/e^-$ distribution functions they estimated $\text{Im}(\omega)$ such that $\text{Re}(\omega) \leq \omega_{\text{CCR}}$. This is incompatible with the condition (III). Improving upon the cold plasma approximation of CR77, AM98 presented a hot plasma treatment. But they incorrectly assumed (in addition to neglecting the geometrical term), the same $\Delta \beta$ for different $\kappa_{GJ}$, whereas a self-consistent treatment requires different $\Delta \beta = 1/\kappa_{GJ}$.

In their case $\Omega_I$ is constant since the EDF has been assumed to be the same for different $\kappa_{GJ}$. Consequently $\omega_I$ at any height scales only as $\omega_I \propto \sqrt{\Delta \beta}$. (shown as vertical shift of $\omega_I$ in logarithmic scale in panel (a) of fig. 6 of AM98). Thus, AM98 made two inconsistent assumptions for $C2$. Our work
Linear amplification of Langmuir waves

5.2.2 EDF for C3

AM98 constructed the EDF at a few \( r/R_{NS} \) (see Table 2 of AM98) for which the shape of the EDF does not change and extrapolated the results in between. In their analysis, \( \Omega \) remains fixed and the growth rate \( (\omega_I) \) falls monotonically as the scaling \( \omega_I \) (see panel (b) of fig.6 in AM98). Panel (b) of fig.6 in AM98 also shows \( \omega_I \) to be high even for moderately low \( \kappa_{GI} \sim 10^{-2} \). However, both these assertions are incomplete and invalid as is discussed below. Many important features have been missed in AM98 due to the coarse resolution of their numerical simulations.

We find that the variation of the growth rate \( \omega_I \) as a function of \( r/R_{NS} \) in C2 is not monotonic. It can be divided into two distinct spatial regions - the first part is dominated by the EDF and the later part by the number density via \( \omega_I \). Although the overlap of the distribution functions begins very close to the neutron star at \( r \approx 2p_{\text{min}}c \tau \sim 1.5 \, \text{km} \), \( \omega_I \) remains very low until a few hundred km from the surface. It is in this very regime that AM98 incorrectly asserts that C3 will dominate C2. We find that \( \Omega_I \) remains very low \( \lesssim 10^{-8} \) until a substantial contribution \( (f_i r_{\text{min}}^{(4)} - f_i^{(4)}) \) from the leading cloud “1” gives rise to a prominent low-momenta tail in the EDF (see panel B of Fig. 5). As seen from panel (A) of Fig. 5, \( p^{(4)} \) changes slowly beyond a few hundred km. This implies the shape of the EDF changes rapidly closer to the neutron star and vice versa. Thus as a prominent tail starts developing \( \Omega_I \) first increases rapidly and then attains a steady value. Consequently for the first few hundred km \( \omega_I \) remains very low, followed by a subsequent increase and then a decline following a turnover. The panel (F) of Fig. 5 shows \( \omega_I \) before the turnover. However even after the development of a prominent tail \( \Omega_I \) does not exceed \( 10^{-7} \) (see panel E of Fig. 4.3). This necessarily requires very high \( \kappa_{GI} = 10^{-2} \) to give rise to high growth rates. This is again opposite to what AM98 obtained. To conclude we find conditions (II) and (III) are satisfied simultaneously for C3 only beyond a few hundred km from the surface and for very high \( \kappa_{GI} \). This is an essentially new result that has been obtained due to the finer resolution of our numerical simulations.

To summarize, for C2 and C3 the coarseness of the numerical resolution coupled with many simplifying assumptions led AM98 to general conclusions which are not valid for realistic pulsar parameters. In this study, we have addressed the inconsistencies of AM98.

5.3 Window of Opportunity (WoU)

We are now interested to find the spatial region along the magnetic field where both conditions (II) and (III) are fulfilled simultaneously, and call this the “Window of Opportunity” (hereafter WoU). WoU is shown as a shaded yellow region for C2 and C3 in panels (A) and (B) of Fig.8. It can be seen that for C2, the gain \( G_{\text{max}} \) decreases monotonically

\[ \text{WoU} \]
while for C3 the gain shows a turnover. As discussed previously the EDF for C2 retains the same shape beyond few tens of km from the surface. This means the gain curve just reflects the scaling $\omega_1$ which decreases monotonically. On the other hand gain curve for C3 is dominated by EDF ($f^{\Omega_2}$) before turnover and by scaling $\omega_1$ beyond it.

However there is an additional difference between the gain curves of C2 and C3. The EDF for C2 follows condition (I) for all $r/R_{NS} \geq 2$, while EDF for C3 follows condition (I) only till $p_{\text{lower}}^{(1)} < p_{\text{mean}}^{(2)}$. For $p_{\text{lower}}^{(1)} \geq p_{\text{mean}}^{(2)}$, the EDF is not single humped and consequently $\omega_2$ ceases. The gain for C3 can go to zero abruptly. Lower the $p_{\text{mean}}^{(2)}$, closer to the neutron star surface can this termination happen. In this scenario abrupt termination in gain can violate condition (II) much closer to the neutron star. For our chosen parameters this termination occurs outside the distance where both conditions (II) and (III) are violated. Panels (C) and (D) in Fig. 8 shows EDF for C2 and C3 at $r/R_{NS} = 180$, where it can be seen ( compared to panel (B) for Fig. 4 and Fig. 5 ) that far from the neutron star surface the dip in the EDF decreases for C3 and remains the same for C2.

In a realistic scenario both C2 and C3 are at work. However, in such a combination the EDF with the greater growth rate dominates. As discussed in section 4.3 the EDF for the condition (I). For these situations exclusively C3 can operate. As discussed in section 4.3 the EDF for C3 depends on $\tau$ and the momenta cut-offs in the distribution functions. If $p_{\text{min}}^{(1)}$ were to be lower, the contribution $f_1 [p_{\text{min}}^{(1)} : p_{\text{upper}}^{(2)}]$ from the leading cloud becomes smaller at a given height. This then decreases $\Omega_1$. However when combined with C2 $p_{\text{min}}^{(1)}$ decreases with height which then increases the contribution due to $f_1$. The results for such a hybrid scenario are shown in Fig. 6 where it is seen that both conditions (II) and (III) are satisfied. Thus, a variety of combinations of C2 and C3 can satisfy conditions (II) and (III) simultaneously, thereby giving the “WoU”. It must also be noted that our conclusions are valid only for a strictly one-dimensional plasma flow where only the Cherenkov resonance condition operates. This assumption is not valid very far away from the neutron star surface where curvature drift (viz., the flow of particles perpendicular to the magnetic field plane $u_3 \approx \gamma^2 r^2 / \omega_B, \rho_c$ ; see Kazbegi et al. 1991) will set in.

We suggest the presence of WoU can be applied to explain the phenomenon of Radius-to-Frequency mapping where radio emission appears to arise from a range of emission heights. The variation of the gain curve can be used to explain time-variable features in pulsar radio emission like moding and nulling. These aspects will be addressed in future work.

5.4 Choice of the distribution functions

We have adopted a semi-numerical approach in this work where we have assumed the distribution functions of the species in plasma to be gaussians. The mean values of secondary plasma distribution functions reflect the order of magnitude estimate derived from considerations of CCR. Recently the choice of relativistically streaming gaussians has been criticized by Rafat et al. (2019). For C2 the symmetric nature of the gaussian distribution function allowed the bulk velocity to be associated with the mean. This allowed us to construct EDF for C2 by only shifting the means without changing the shape. For an asymmetric distribution function, no such prescription exists and the shape of the $e^+ - e^-$ distribution functions can get distorted while separating. This can affect the dip in the EDF and consequently the growth rate (see the first paragraph in Appendix D). However, for C3 even for a non-gaussian distribution function the most relevant parameters would still be the location of the peak and $p_{\text{min}}$. The generic features of the amplification curve for C2 and C3 as shown in Fig. 8 would not change with the change in the distribution function. The detailed aspects of WoU for different distribution functions will be addressed in a future study.

6 CONCLUSION

In this study, we find that both C2 and C3 can lead to excitation of large amplitude Langmuir waves, while the secondary plasma should be dense enough to account for high $T_b$ in CCR. Contrary to the results obtained by AM98, we find that for certain multipolar surface magnetic field configurations, the amplification gain for C2 vastly exceeds that of C3 for the same $\kappa_{GJ}$. For these special configurations, very high amplification can be achieved very close to the neutron star and the spatial extent over which C2 operates vastly exceeds that due to C3. A generic feature for C3 is that the gain becomes high only for $\kappa_{GJ} \geq 10^4$ after a few hundred km from the neutron star surface. For $\kappa_{GJ} \sim 10^5$ C2 operates exclusively as growth rates in C2 are suppressed completely. For surface field configurations and high $\kappa_{GJ}$ wherein the EDF for C2 is single-humped, the separation nevertheless aids C3 by enhancing the low momenta tail in the EDF for C3. We find a window of opportunity (WoU) of Cherenkov resonance around $r/R_{NS} \sim 100$ where any combination of C2 and C3 can account for CCR. The presence of large amplitude Langmuir waves in WoU provides an impetus for a higher-order plasma theory.

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The following appendix is being provided for an expanded discussion and derivations that has been used extensively in the main paper.

APPENDIX A: MAKING USE OF THE LANDAU PRESCRIPTION FOR ELECTROSTATIC MODE FOR A ONE-DIMENSIONAL RELATIVISTIC PLASMA

In the sub-Luminal region we make use of the Landau Prescription where we allow \( \omega \) to be complex and hence the integral in the dispersion relation can be replaced by a contour integral to give

\[
\kappa c + \sum_{\alpha} \sum_{\beta} \omega_{\alpha, \beta} I_{L}^{\beta} \beta_{\alpha} \frac{\partial \beta_{\alpha}}{\partial \alpha} \frac{1}{(\omega - \beta_{\alpha} c)} = 0
\]

where \( L \) stands for the Landau Contour.

Thus we have a pole at a value of particle three momenta

\[
P_{pole} = \frac{\omega}{\sqrt{(\kappa c)^2 - \omega^2}}
\]

Since \( \omega \) can now be complex we can write

\[
\omega = \omega_R + i \omega_I
\]

Thus the pole can be written as

\[
P_{pole} = \frac{\sqrt{|U|^2 + b^2} + \sqrt{|U|^2 - b^2}}{Q}
\]

where

\[
|U| = \sqrt{a^2 + b^2}
\]

\[
a = k^2 c^2 (\omega_R^2 - \omega_I^2) - (\omega_R^2 + \omega_I^2)^2
\]

\[
b = 2 \omega_R \omega_I (k^2 c^2)
\]

\[
Q = \sqrt{(\kappa c)^2 + 2(\omega_R^2 - \omega_I^2)} + (\omega_R^2 + \omega_I^2)^2
\]
Let us consider the case when $\epsilon \ll \omega_R$.

### A1 Resonant Landau/kinetic growth: $\omega_1 \ll \omega_R$

First we consider the case when $\omega_1 = 0$.

The pole is at

$$p_{\text{pole}} = \frac{\omega_R}{\sqrt{\kappa c^2 - \omega_1^2}} = \Delta_2$$

The dispersion relation $A1$ in this case given by

$$\kappa c + \sum_a \omega_{k,a}^2 \mathcal{P} \int_{-\infty}^{\infty} dp_a \frac{\partial \epsilon}{\partial p_a} \frac{1}{(\omega - \beta_a, K)}$$

$$= i \pi \sum_a \omega_{k,a}^2 \frac{\partial \epsilon}{\partial p_a} \bigg|_{p_a = \Delta_2} = 0$$

where $\mathcal{P}$ stands for the Principal Value Integral.

The dimensionless dispersion relation is given by

$$\epsilon(\Omega, K) = 1 + \sum a \chi_a \frac{\partial \epsilon}{\partial p_a} \frac{1}{\Omega - \beta_a, K} = 0$$

where $\Omega = \frac{\omega}{\omega_1}$, $K = \frac{\kappa c}{\omega_1}$, $\chi_a = \frac{\omega_{k,a}}{\omega_1}$.

### A1.1 Derivation of growth rate in the Landau/Kinetic regime

Let us consider the case when $\Omega = \Omega_R \pm i \Gamma_{\text{kin}}$ such that $\Gamma_{\text{kin}} \ll \Omega_R$.

The particle distribution function having the pole is labelled by subscript “$b$” and the distribution function away from the pole is labelled by subscript “$s$”.

Under this approximation the dispersion function can be written as

$$\epsilon(\Omega_R \pm i \Gamma_{\text{kin}}, K) = 0$$

$$\Rightarrow \Re \epsilon(\Omega, K) + i \Im \epsilon(\Omega, K) \equiv \pm i \Gamma_{\text{kin}} \frac{\partial \epsilon}{\partial \Omega} \bigg|_{\Omega = \Omega_R} = 0$$

$$\Rightarrow \Gamma_{\text{kin}} = \mp \frac{\Im \epsilon(\Omega, K)}{\partial \Omega} \bigg|_{\Omega = \Omega_R}$$

(A9)

Again the dispersion function can be written as

$$\epsilon(\Omega, K) = \Re \epsilon(\Omega, K) + i \Im \epsilon(\Omega, K)$$

$$= 1 + \frac{\chi_b}{K} \int_{-\infty}^{\infty} dp_b \frac{\partial \epsilon(0)}{\partial p_b} \frac{1}{(\Omega_R - \beta_b, K)}$$

$$+ \frac{\chi_b}{K} \int_{-\infty}^{\infty} dp_b \frac{\partial \epsilon(0)}{\partial p_b} \frac{1}{(\Omega_1 - \beta_b, K)}$$

$$\mp i \frac{\chi_b}{K} \int_{-\infty}^{\infty} dp_b \frac{\partial \epsilon(0)}{\partial p_b} \delta(\Omega_R - \beta_b, K)$$

(A10)

The root of the argument in the delta function is given by

$$\Omega_R - \beta^\text{res}_b K = 0 \Rightarrow p_{b, res} = \frac{\chi_b}{\sqrt{1 - (\frac{\Delta_2}{\Delta_1})}}$$

We use the identity $\delta(f(x)) = \sum_{a} \delta(x - a) \frac{\partial f(x)}{\partial x}$, where $a_i$ are the roots of $f(x)$.

Using the above identity we get

$$\delta \left( \Omega_R - \beta_b, K \right) = \frac{\delta(p_b - p_{b, res})}{K} \bigg|_{p_b = p_{b, res}}$$

(A11)

Equating the imaginary part we get

$$\Im \epsilon(\Omega, K) = \mp \pi \frac{\chi_b}{K^2} \frac{\partial \epsilon(0)}{\partial p_b} \bigg|_{p_b = p_{b, res}}$$

Now we evaluate the denominator of A9

$$\frac{\partial \epsilon}{\partial \Omega} \bigg|_{\Omega = \Omega_R} = 2 \frac{\chi_b}{\Omega_R} \int_{-\infty}^{\infty} dp_b \frac{1}{\gamma^2 (1 + \beta_b)^2}$$

(A12)

Substituting A12 into A9 we get

$$\Gamma_{\text{kin}} = \frac{\pi}{2 K^2} \frac{\partial \epsilon(0)}{\partial p_b} \bigg|_{p_b = p_{b, res}}$$

(A13)

where

$$\langle (\ldots) \rangle = \int_{-\infty}^{\infty} dp_a \partial \epsilon(0) / \partial p_a (\ldots)$$

### A2 Non-resonant/Hydrodynamic Growth: $\omega_1 > 0$

The pole is at

$$p_{\text{pole}} = \frac{\sqrt{\Omega + i \delta} + i \sqrt{\Omega + i \delta}}{Q} = \Delta_3$$

(A14)
The dispersion relation \( A1 \) in this case given by
\[
\kappa c + \sum_\alpha \omega^2_{\alpha, \alpha} \int_{-\infty}^{+\infty} dp \frac{\partial f_{\alpha}(0)}{\partial p} \frac{1}{\gamma^2 (\omega - \beta_\alpha) K^2} = 0 \quad (A15)
\]

The above equation can be integrated by parts to get
\[
1 - \sum_\alpha \omega^2_{\alpha, \alpha} \int_{-\infty}^{+\infty} dp \frac{f_{\alpha}(0)}{\gamma^2 (\omega - \beta_\alpha) K^2} = 0 \quad (A16)
\]

Normalising by \( \epsilon \), we get
\[
1 - \sum_\alpha \Omega \int_{-\infty}^{+\infty} dp \frac{f_{\alpha}(0)}{\gamma^2 (\Omega - \beta_\alpha) K^2} = 0 \quad (A17)
\]

where \( \Omega = \frac{\omega}{\epsilon}, K = \frac{\kappa c}{\epsilon}, \chi_\alpha = \frac{\omega^2_{\alpha, \alpha}}{\epsilon^2} \)

Substituting \( \Omega = \Omega_0 + \frac{i}{\epsilon} \Omega_1 (\Omega_1 > 0) \) in the above equation and separating it into its real and imaginary part we get
\[
\nu_R = 1 - \sum_\alpha \chi_\alpha \int_{-\infty}^{+\infty} dp \frac{f_{\alpha}(0)}{\gamma^2 (\Omega_R - \beta_\alpha K)^2 + \Omega_1^2} = 0 \quad (B1)
\]
\[
\nu_I = 2 \sum_\alpha \chi_\alpha \int_{-\infty}^{+\infty} dp \frac{f_{\alpha}(0)}{\gamma^2 (\Omega_R - \beta_\alpha K) (\Omega_R - \beta_\alpha K) + \Omega_1^2} = 0 \quad (B2)
\]

\( \nu_R = \frac{\omega^2_{\alpha, ref}}{(k_c)^2} \int_{-\infty}^{+\infty} dp \frac{\partial F_{\alpha}(0)}{\partial p} \frac{1}{\beta - \frac{\omega^2_{\alpha, ref}}{(k_c)^2}} = 0 \quad (B3) \)

\( \nu_I = \frac{\omega^2_{\alpha, ref}}{(k_c)^2} \int_{-\infty}^{+\infty} dp \frac{\partial F_{\alpha}(0)}{\partial p} \frac{1}{\beta - \frac{\omega^2_{\alpha, ref}}{(k_c)^2}} = 0 \quad (B4) \)

From Fig. B1 of the EDF we have the maximum of the equivalent distribution function at \( p = p_0 \) which corresponds to \( \beta_0 = \frac{p_0}{\sqrt{1 + p_0^2}} \) which gives us
\[
\frac{\partial F_{\alpha}(0)}{\partial p} > 0 \quad \text{for} \quad p < p_0 \quad \text{i.e., for} \quad \beta < \beta_0
\]
\[
\frac{\partial F_{\alpha}(0)}{\partial p} < 0 \quad \text{for} \quad p > p_0 \quad \text{i.e., for} \quad \beta > \beta_0
\]
so that
\[
(\beta_0 - \beta) \frac{\partial F_{\alpha}(0)}{\partial p} \geq 0 \quad \text{for} \quad -\infty < p < \infty
\]

Since \( \nu_R = 0 \) and \( \nu_I = 0 \) we have
\[
\nu_R + \left( \frac{\beta_0 k_c - \omega_r}{\omega_r} \right) \nu_I = 0 \quad (B5)
\]

Using B4 in the equation above we get ,
\[
1 + \frac{\omega^2_{\alpha, ref}}{(k_c)^2} \int_{-\infty}^{+\infty} \frac{(\beta_0 - \beta) F_{\alpha}(0)}{\beta - \frac{\omega^2_{\alpha, ref}}{(k_c)^2}} = 0 \quad (B6)
\]
The integrand is non-negative for all values of $p$ as seen from B6. Hence, the left-hand side is always greater than unity so that the equation cannot be satisfied. This proves that our original assumption must be incorrect.

APPENDIX C: USEFUL DERIVATIONS FOR THE LANGMUIR MODE

C1 Energy distribution among Langmuir waves and particles in the plasma

Following “Introduction to Plasma Physics” by Nicholson (1983) we have

$$\frac{d|E|^2}{16\pi} \frac{d\gamma}{d\omega} \frac{d|E|^2}{d\omega} = \frac{1}{2} |E|^2 \sigma_{\text{R}}(\omega_R)$$

(C1)

The above expression can be further simplified as

$$\left(1 - 4\pi \frac{d\gamma}{d\omega} \right) \frac{d|E|^2}{d\omega} = \frac{1}{2} |E|^2 \sigma_{\text{R}}(\omega_R)$$

$$\Rightarrow \frac{dW_{\text{total}}}{dt} = \left(1 - 4\pi \frac{d\gamma}{d\omega} \right) \frac{dW_{\text{field}}}{dt}$$

$$\Rightarrow W_{\text{total}} = \left(1 - 4\pi \frac{d\gamma}{d\omega} \right) W_{\text{field}}$$

(C2)

Using B4 the above expression can be further simplified to finally obtain

$$W_{\text{total}} = \frac{d}{d\omega} \left| \omega_R(\omega) \right| \frac{dW_{\text{field}}}{dt}$$

(C3)

For our case we will evaluate the above expression at $\omega = \omega_1$ where the Langmuir mode touches the $\omega = \kappa c$ line. Please note the quantities for the Langmuir mode has been normalized at this frequency.

The energy distribution between particles and electric field for Langmuir mode for the relativistic pulsar plasma evaluated at the point where the mode touches the $\omega = \kappa c$ line is given as

$$W_{\text{total}} = \frac{d}{d\Omega} \left| \Omega e(\Omega K) \right| \frac{dW_{\text{field}}}{dt}$$

(C3)

Now, we have

$$\Omega e(\Omega K) = \Omega \left(1 + \sum \frac{\chi_2}{K} \int_{-\infty}^{+\infty} dp_a \frac{\partial f^{(0)}_a}{\partial p_a} - \frac{1}{\Omega - \beta_a K} \right)$$

$$\Rightarrow \frac{d}{d\Omega} [\Omega e(\Omega K)]$$

$$= e(\Omega K) + \Omega \left(1 - \sum \frac{\chi_2}{K} \int_{-\infty}^{+\infty} dp_a \frac{\partial f^{(0)}_a}{\partial p_a} \frac{1}{\Omega - \beta_a K} \right)$$

$$\Rightarrow \frac{d}{d\Omega} [\Omega e(\Omega K)]$$

$$= -\sum \frac{\chi_2}{K} \int_{-\infty}^{+\infty} dp_a \left(1 - \beta_a K^2 \right)$$

$$= 2 \sum \frac{\chi_2}{K} \left(\gamma^3 (1 + \beta_a)^3 \right)_a$$

(C4)

Now, $\omega_1$ can be simplified as

$$\omega_1^2 = \sum \omega_{p,a}^2 \left(\gamma(1 + \beta_a)^3 \right)_a$$

(C5)

For the ultra-relativistic case as it true for pulsar plasma $\beta_a = 1$ which further simplifies $\omega_1$ to

$$\omega_1^2 = 4 \sum \omega_{p,a}^2 \left(\gamma \right)_a$$

(C6)

Thus, $\chi_2$ can be simplified to

$$\chi_2 = \frac{1}{4} \sum \omega_{p,a}^2 \left(\gamma \right)_a$$

(C7)

where $\eta_0 = \chi_2 / \omega_{p,\text{ref}}$ where $\omega_{p,\text{ref}}$ is the plasma frequency of some reference species.

Then we have

$$\frac{d}{d\Omega} \left| \Omega e(\Omega K) \right| = -\sum \frac{\eta_0}{\sum \eta_0} \left(\gamma \right)_a \left(\gamma^3 (1 + \beta_a)^3 \right)_a$$

(C8)

Making use of the ultra-relativistic approximation again we have

$$\frac{d}{d\Omega} [\Omega e(\Omega K)] \frac{dW_{\text{field}}}{dt} = 4 \sum \frac{\eta_0}{\sum \eta_0} \left(\gamma \right)_a \left(\gamma^3 \right)_a$$

(C9)

For a one-component plasma $\alpha = 1$ we have

$$W_{\text{total}} = 4 \left(\frac{\gamma^3}{\gamma} \right) W_{\text{field}}$$

$$\Rightarrow W_{\text{field}} = \frac{\gamma^3}{4 \gamma^3} W_{\text{total}}$$

Thus, in an ultra-relativistic plasma the electric field energy density of an electrostatic wave is much lower than the energy of oscillation of the particles. As the Lorentz factor of the particles becomes very high, the electric field energy tends to be very very small.
C2 Bandwidth of the growing waves

We have
\[
\frac{\partial \epsilon}{\partial K} = \frac{\partial}{\partial K} \left( 1 + \sum_{a} \frac{x_a}{K} \int_{-\infty}^{\infty} dp \frac{\partial^{(0)}}{\partial p} \frac{1}{(\Omega - p - iK)} \right)
\]
\[
= -\sum_{a} x_a \left( \gamma(1 + \beta_a)^3 \right) \chi_a
\]
(C10)

Similarly we have
\[
\frac{\partial \epsilon}{\partial \Omega} = 2 \sum_{a} x_a \int_{-\infty}^{\infty} dp \frac{\partial^{(0)}}{\partial p} \frac{1}{\gamma(1 - \beta_a)^3}
\]
\[
= 2 \sum_{a} \frac{x_a}{\gamma} \left( \gamma(1 + \beta_a)^3 \right) \chi_a
\]
(C11)

Using the ultra-relativistic approximation \( \beta_a \approx 1 \) we can write
\[
\frac{\partial \epsilon}{\partial K} \Bigg|_{1} = -1 - \Lambda \frac{\partial \epsilon}{\partial \Omega} \Bigg|_{1}
\]
(C12)

where
\[
\Lambda = \frac{1}{2} \sum_{a} x_a \left( \gamma(1 + \beta_a)^3 \right) \chi_a
\]

Comparing both the expressions we get
\[
\frac{\partial \epsilon}{\partial K} \Bigg|_{1} = -(1 - \Lambda) \frac{\partial \epsilon}{\partial \Omega} \Bigg|_{1}
\]
(C13)

Again we have, from the normalised dispersion function
\[
\epsilon(\Omega, K) = 0
\]
\[
= \frac{\Omega}{K} = 1 - \Lambda \left[ 1 - \frac{1}{K} \right]
\]
(C14)

The bandwidth of the growing waves can be obtained using the constraint
\[
|\Omega - K\beta| \leq \frac{\nu_{thp}}{\gamma_{th}^{1/3}}
\]
\[
\Rightarrow \Lambda K \geq \frac{\nu_{thp}}{\gamma_{th}^{1/3}} \frac{1}{\Lambda}
\]
(C15)

We have
\[
\Delta \Omega_K = \Delta K (1 - \Lambda) = \Delta K
\]
(C16)

Thus we have
\[
\Delta \Omega_K \leq \frac{\nu_{thp}}{\gamma_{th}^{1/3}} \frac{1}{\Lambda}
\]
(C17)

where
\[
\Lambda = \frac{1}{2} \sum_{a} x_a \left( \gamma(1 + \beta_a)^3 \right) \chi_a
\]

APPENDIX D: ALGORITHM FOR NUMERICALLY SOLVING FOR THE HYDRODYNAMICAL EQUATIONS AND \( \dot{G}_{\text{MAX}} \) USING PYTHON PACKAGES

Location of the hydrodynamic pole and the EDF: The imaginary part of the hydrodynamic dispersion relation \( \epsilon_1 \) is a corollary of the Gardner’s theorem discussed in section B. The condition \( \epsilon_1 = 0 \) can be satisfied only if \( \text{Re}(p_{\text{pole}}) \) of the hydrodynamic pole lies in the dip of the EDF. As discussed in A2 the hydrodynamic pole \( p_{\text{pole}} \) is necessarily complex with \( \text{Im}(p_{\text{pole}}) > 0 \). As the dip in the EDF decreases \( \text{Im}(p_{\text{pole}}) \) and the growth rate \( \omega_0 \) also decreases until it vanishes altogether for a single humped EDF. Following this generic ideas we have carried out the following steps to solve the hydrodynamic equations.

Step 1: At a given \( \nu/\nu_{th} \) we first check whether EDF is single-humped or not as shown in plot B of Fig.2 and 4 and 5 for \( \nu/\nu_{th} = 50 \); and plots (C) and (D) for \( \nu/\nu_{th} = 100 \) of Fig. 8 of the main paper. If it is not, the three-momenta values \( p \) between the two peaks of the EDF/ dip of the EDF is divided uniformly into \( n = 100 \) grid points to be used as guess values for \( \text{Re}(p_{\text{pole}}, \text{guess}) \). Using these we estimate guess values for \( \beta_{\text{pole}}, \text{guess} = \text{Re}(p_{\text{pole}}, \text{guess}) \sqrt{1 + \text{Re}(p_{\text{pole}}, \text{guess})^2} \).

Step 2: We first need the the solution (\( \Omega_{K}, \Omega_{\Omega} \)) for \( K = 1 \). The guess values for \( \text{Re}(\Omega) \) is taken as \( \text{Re}(p_{\text{pole}}) \text{guess}, \Omega_{\text{guess}} \text{guess} \). The guess values for \( \Omega_{\Omega} \text{guess} \) is taken from \( 10^{-4} \) to \( 10^{-6} \) uniformly divided into \( m = 1000 \) points. This gives a 2D grid with \( n \times m \) points such that each grid represents a guess value \( (\Omega_{\text{guess}}, \Omega_{\text{guess}}) \) for the hydrodynamic dispersion relations for \( K = 1 \). Both \( \epsilon_0 \) and \( \epsilon_1 \) are estimated for each grid point. The grid points for which \( \epsilon_0 \) and \( \epsilon_1 \) \( \leq 10^{-10} \) were filtered to be used as guess values for the python package fsolve.

Step 3: The python package fsolve takes the filtered guess values \( (\Omega_{\text{guess}}, \Omega_{\text{guess}}) \) and arguments \( (K, x_a, \text{EDF}) \) and a tolerance value \( xtol = 10^{-12} \). It then solves for the real and imaginary part of the hydrodynamic dispersion relation \( \epsilon_0 = 0 \) and \( \epsilon_1 = 0 \) simultaneously. It converges to a \( (\Omega_{K}, \Omega_{\Omega}) \). The solution so obtained is then inserted into the expression for \( \epsilon_0 \) and \( \epsilon_1 \) to get the residuals. We take a conservative approach wherein the solution is taken as valid only if the residuals are at least 3 orders of magnitude lower than \( \Omega_{\Omega} \).

After getting a solution we can estimate \( \text{Re}(p_{\text{pole}}) \text{pole} \) by following these steps \( \beta_{\text{pole}} = \Omega_{K} \times K \rightarrow \text{Re}(p_{\text{pole}}) = \beta_{\text{pole}} \sqrt{1 - p_{\text{pole}}^2} \). The \( \text{Re}(p_{\text{pole}}) \text{pole} \) so obtained for \( K = 1 \) is shown as a black dashed line in plot(B) of Fig. 2, 4 and 5 ; and plot (C) and (D) of Fig.8 of the main paper.

Step 4 After getting the solution for \( K = 1 \), the wavenumber is changed in steps of \( \Delta K_{\text{pole}} = 10^{-3} \) and the previous step is repeated with the difference that from now on only one guess value needs to be provided. The guess values \( (\Omega_{\text{guess}}, \Omega_{\text{guess}}) \text{for the} \text{I-th iteration is the solution for (I-1)th iteration. After a solution converges for wavenumber} \text{K = 1 + Ix \Delta K_{pole}, the residuals for} \epsilon_{K} \text{and} \epsilon_{I} \text{are estimated. The process is terminated at a wavenumber} \text{K_{cut-off} where either} \epsilon_{K} \leq 10^{-6} \text{or the residuals do not satisfy the criteria mentioned above, whichever occurs first.} \text{At the end of this stage we have obtained the dimensionless dispersion relation. The first, second and third subplot of (C) in Fig. 2, 4 and 5 shows} \Omega_{K}, \Omega_{\Omega} \text{and the residuals (numerical errors) of} \epsilon_{K} \text{and} \epsilon_{I} \text{as a function of} K. \text{Step 5 : To get the dispersion relation in the dimensionless form we estimate the scaling factor} \omega_{0} \text{[in rad/s]} \text{via Eq. 3 of the main paper. The dispersion relation in the dimensional form is obtained by the following steps:} \omega_{0} = \Omega_{K} \times \omega_{1} \text{[in rad/s],} \omega_{1} = \Omega_{K} \times \omega_{1} \text{[in s^{-1}]} \text{and} K = K / \omega_{0} / c \text{[in cm^{-1}]}. \text{The first and second subplot of (D) in Fig. 2, 4 and 5 of the main paper shows the dimensional dispersion relation (also referred to as spectrum of the growing set of waves). The third subplot of (D) in the Fig. 2, 4 and 5 shows the group velocity dispersion} \text{dv}_{g}/dk \text{[in cm^{2}/s^{-1}] as a function of the wavenumber} K. \text{In all three figures group velocity dispersion is positive for the growing set of waves.}
Step 6: At a given $r/R_{NS}$ the normalized bandwidth of the growing waves is given by $\Delta \Omega_R = (\Omega_R|_{K_{\text{cut\_off}}} - \Omega_R|_{K_{\text{cut}}})$. Using $r$, $\omega_1$, $\Omega_{k,1}$ and $\Delta \Omega_R$ in Eq. 9 and Eq. 10 of the main paper we obtain the maximum gain $G_{\text{max}}$ that can be associated with $\text{Re}(\omega) = \Omega_{k,1} \times \omega_1$. The $\text{Re}(\omega_{R,1})$, $\text{Re}(\omega_{I,1})$ and the gain curve are shown as upper, middle and lower subplots of panel (F) in Fig. 2, 4 and 5.

REFERENCES

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