ON PRESIMPLIFIABLE GROUP RINGS

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Abstract

A ring $A$ is called presimplifiable if whenever $a, b \in A$ and $a = ab$, then either $a = 0$ or $b$ is a unit in $A$. Let $A$ be a commutative ring and $G$ be an abelian torsion group. For the group ring $A[G]$, we prove that $A[G]$ is presimplifiable if and only if $A$ is presimplifiable and $G$ is a p-group with $p$ belongs to the jacobson radical of $A$, and it is shown that $A[G]$ is domainlike (i.e all zero divisors are nilpotents) if and only if $A$ is domainlike and $G$ is a p-group and $p$ is a nilpotent in $A$. Furthermore, whenever the group ring $A[G]$ is presimplifiable we prove that $A[H]$ is presimplifiable for any subgroup $H$ of $G$. Also, for a torsion free group $G$ we prove that $A[G]$ is domainlike if and only if $A[G]$ is integral domain.

1 Introduction

Throughout this paper all rings are assumed to be commutative have a unity and all groups are abelian and nontrivial unless indicated otherwise. Also, we write $J(A)$, $Z(A)$, $U(A)$, and $nil(A)$, to denote the Jacobson radical, the set of all zero divisors (with the zero element), the set of units of $A$, the nil radical of $A$, respectively. Let $p$ be a prime number, then a group $G$ is called a p-group if the order of each element of $G$ is a power of the fixed prime $p$. A group $G$ is said to be an locally (normal) finite p-group if any finite subset of $G$ generate a finite (normal) p-subgrup of $G$, But if $G$ is abelian, then torsion is equivalent to locally finite.

Let $A$ be a ring with unity and let $G$ be a group. We denote by $A[G]$ the group ring of $G$ over $A$. Let us recall some concepts and notations needed. A typical element of $A[G]$ is a finite formal sum $\alpha = \sum_{g \in G} a_g g$, $a_g \in A$, and we
denote \( suup(\alpha) = \{ g \in G : a_g \neq 0 \} \). The elements of \( A \) commute with those of \( G \). Addition and multiplication are defined in \( A[G] \) in the obvious way, making \( A[G] \) a ring; \( A \) is a subring of \( A[G] \) under the identification \( a = a \cdot 1 \). Consider the function \( \varepsilon : A[G] \to A \) defined by \( \varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \), this function is called the augmentation mapping, \( \varepsilon \) is a ring homomorphism maps \( A[G] \) onto \( A \), whose kernel is the set \( \text{Ker}(\varepsilon) = \{ \alpha = \sum_{g \in G} a_g g : \varepsilon(\alpha) = \sum_{g \in G} a_g = 0 \} \), which is an ideal of \( A[G] \), and it is called the augmentation (fundamental) ideal of \( A[G] \), and it is denoted by \( \triangle \), and for any subgroup \( H \) of \( G \) denote that \( \triangle_H \) the ideal generated by \( \{ 1 - g : g \in H \} \), which is augmentation ideal of \( A[H] \). A complete characterization of the group rings can be found in [10], [12].

A ring \( A \) is called presimplifiable if for every \( a, b \in A \) with the property \( a = ab \) implies that either \( a = 0 \) or \( b \) is a unit in \( A \), many conditions equivalent to this condition are introduced by D. D. Anderson et al., [3], one of those is that all zero divisors are in the jacobson radical. Domain-like rings (i.e all zero divisors are nilpotents) and local rings are examples of presimplifiable rings, we will recall some known results in presimplifiable rings in the next section, many authors investigated this class of rings as Bouvier (he had introduced it), [6], [8], [9], Anderson et al., [1], [2], [3].

Recently, in 2012, M. Ghanem et al., [11], raised the following question: Is it true that, if \( A \) is presimplifiable and \( C_n \) is the finite cyclic group of order \( n = p^m, p \in J(A) \) then \( A[G] \) is presimplifiable? we have answered this question, and we give a characterization of presimplifiable group rings. Moreover, we consider the question that what are sufficient and necessary conditions in a ring \( A \) and a group \( G \) so that the group ring \( A[G] \) is domainlike. We prove several facts that establish when the group ring is domainlike or presimplifiable.

### 2 Preliminaries

Before we deal with deeper results on the presimplifiable group rings, we devote this section to provide elementary definitions and constructions.

**Definition 2.1.** Let \( A \) be a ring and \( I \trianglelefteq A \).

1. The ring \( A \) is called presimplifiable if whenever \( a, b \in A \) and \( a = ab \), then either \( a = 0 \) or \( b \in U(A) \).
The ideal \( I \) is an presimplifiable ideal of \( A \) if whenever \( a \in I, b \in A \) and \( a = ab \), then either \( a = 0 \) or \( b \in U(A) \).

**Definition 2.2.** A ring \( A \) is domainlike if \( Z(A) \subseteq \text{nil}(A) \).

Examples of presimpliable rings include integral domains and quasilocal rings. For a ring \( A \), \( A \) is presimpliable iff \( A[[X]] \) is presimpliable. It is known that \( Z_n \) is presimplifiable if and only if \( n = p^m \) where \( p \) is a prime number. It is easily seen that a presimpliable ring is indecomposable, that it has only the trivial idempotents. However, a direct product of presimplifiable rings is never presimplifiable, and a direct product of domainlike rings is never domainlike. These facts due to M. Axtel, [4], page 152.

**Lemma 2.3.** [4, Theorem 4.] If \( A \) is domainlike, then \( Z(A) \) is the minimal unique prime ideal of \( A \).

**Lemma 2.4.** [3, Theorem 1.] For a commutative ring \( A \), the following conditions are equivalent.

(i) \( A \) is presimpliable.

(ii) \( Z(A) \subseteq \{ 1 - u : u \in U(A) \} \).

(iii) \( Z(A) \subseteq J(A) \).

(iv) If \( 0 \neq r \in A, sAr = Ar \), then \( s \in U(A) \).

Recall that an ideal \( I \) of a ring \( A \) is a strongly p-nilary ideal if \( \sqrt{I} \) is a prime ideal of \( A \), and that \( A \) is a strongly p-nilary ring if the zero ideal of \( A \) is strongly p-nilary. A strongly p-nilary ring was introduced by G. Birkenmeier et al., [5]. Let us write his important result.

**Lemma 2.5.** [5, Proposition 2.3.] For a ring \( A \) (not necessary to be commutative), the following conditions are equivalent.

(i) \( A \) is strongly p-nilary.

(ii) Let \( r, s \in A \). If \( < r > < s > \subseteq P(A) \), then \( < r >^n = 0 \) or \( < s >^m = 0 \) for some positive integers \( m \) and \( n \).

(iii) Every nilpotent ideal is a strongly p-nilary ideal of \( R \).

(iv) \( \sqrt{0} \) is the unique minimal prime ideal of \( R \).
In commutative ring we can add another condition to these in the above lemma that $A$ is a p-nilary ring (i.e. for any two principle ideals $H, K$ of $A$ with $HK = 0$ we have $H^m = 0$ or $K^n = 0$ where $m, n$ are positive integers). Since a domainlike ring has a minimal unique prime ideal, by using Lemma 2.5, then every domainlike ring is p-nilary. To summarize, we have the following implications.

$A$ is integral domain $\Rightarrow$ $A$ is domainlike $\Rightarrow$ $A$ is strongly p-nilary $\iff$ $A$ is p-nilary.

The following four lemmas were proved by I. Connell, remark that the ring $A$ is not necessary to be a commutative ring also the group $G$ is any group.

**Lemma 2.6.** [110], Proposition 5. Let $g \in G$. Then $1 - g$ is a divisor of 0 in $A[G]$ iff $g$ has finite order.

**Lemma 2.7.** [110], Proposition 15. 

(i) If $\triangle \subseteq J(A[G])$ then, $G$ is p-group and $p \in J(A)$.

(ii) If $A$ is commutative and $\triangle$ is nil then, $G$ is locally normal p-group and $p$ is nilpotent.

**Lemma 2.8.** [110], Theorem 9. $\triangle$ is (locally) nilpotent if and only if

(i) $G$ is a (locally) finite p-group, and 

(ii) $p$ is nilpotent in $A$.

**Lemma 2.9.** [110], Proposition 9. Let $A$ be a ring, and $G$ be an abelian p-group with $p \in J(A)$, then $J(A) \subseteq J(A[G])$.

Notice that, since $J(A[G])$ is an ideal in $A[G]$, then we have $J(A)[G] \subseteq J(A[G])$.

**Lemma 2.10.** Let $A$ be a ring and $G$ be any group, if $I \triangleleft A$ and $I \cap \triangle \neq 0$, then there exists $g \in G$ such that $1 - g \in I \cap \triangle$.

**Proof.** For any two sided ideal $I$ of $A[G]$, we have the normal subgroup of $G$ defined as, $\Omega(I) = \langle g : 1 - g \in I \rangle$, [110], page651. Since $I \cap \triangle$ non trivial ideal, then $\Omega(I \cap \triangle)$ is nontrivial normal subgroup of $G$, hence there exists $g \in G$ such that $1 - g \in I \cap \triangle$. \qed
3 The Main Result

The main goal of this section is to obtain several results related to presimplifiable and domainlike group rings. We will determine the necessary and sufficient conditions so that the group ring is presimplifiable or domainlike.

Theorem 3.1. Let \( A \) be a ring, \( G \) be an abelian \( p \)-group with \( p \in J(A) \). Then

\[
x = \sum_{g \in G} a_g g \in U(A[G]) \iff \varepsilon(x) = \sum_{g \in G} a_g \in U(A)
\]

Proof. \((\Leftarrow)\) : Let \( x = \sum_{g \in G} a_g g \in A[G] \) and assume that \( \varepsilon(x) = \sum_{g \in G} a_g \in U(A) \). Since \( A \) is commutative and \( G \) is abelian, it is clear that \( A[G] \) is commutative. Since \( G \) is a \( p \)-group, we have that for any \( g \in G \) there exists a nonnegative integer \( k \) such that \( o(g) = p^k \). Now, let \( m = \text{Min}\{k \in \mathbb{Z}^+ : g^{p^k} = 1 \text{ for all } g \in \text{supp}(x)\} \), we can rewrite \( x \) as

\[
x = \sum_{i=1}^{i=n} a_i g_i.
\]

Thus

\[
x^{p^m} = \left[ \sum_{i=1}^{i=n} a_i g_i \right]^{p^m}.
\]

By using the multinomial theorem, we find

\[
x^{p^m} = \sum_{i=1}^{i=n} a_i^{p^m} g_i^{p^m} + pY.
\]

For some \( Y \) in \( A[G] \). Thus

\[
x^{p^m} = \sum_{i=1}^{i=n} a_i^{p^m} + pY \quad \ldots(1)
\]

\[
\left[ \sum_{i=1}^{i=n} a_i \right]^{p^m} = \sum_{i=1}^{i=n} a_i^{p^m} + pY' \quad \ldots(2)
\]

For some \( Y' \) in \( A \). Since \( G \) is an abelian \( p \)-group and \( p \in J(A) \), by using Lemma 2.9, we get that \( J(A) \subseteq J(A[G]) \), thus \( p \in J(A[G]) \), and because that
the jacobson radical is an ideal, we get that \( pY \in J(A[G]) \) and \( pY' \in J(A) \). Since \( \sum_{i=1}^{n} a_i \) is a unit in \( A \) we get that \( [\sum_{i=1}^{n} a_i]^{p^m} \) is a unit, therefore \( \sum_{i=1}^{n} a_i^{p^m} + pY' \) is a unit in \( A \), since \( pY' \in J(A) \), we get \( \sum_{i=1}^{n} a_i^{p^m} \) is a unit in \( A \). Similarly, by using statement (1), we find that \( x^{p^m} \) is a unit in \( A[G] \), hence \( x \) is a unit in \( A[G] \).

\((\Rightarrow)\) : Because \( \varepsilon \) is epimorphism, then the proof is straightforward. 

\[ \square \]

In the next proof, we will use the facts that in commutative rings, sum of nilpotents is nilpotent, and the product by any nilpotent by any other element is nilpotent. and that the nil radical is an ideal which is exactly the prime radical.

**Theorem 3.2.** Let \( A \) be a ring, \( G \) be an abelian \( p \)-group with \( p \in \text{nil}(A) \). then

\[ x = \sum_{g \in G} a_g g \in \text{nil}(A[G]) \iff \varepsilon(x) = \sum_{g \in G} a_g \in \text{nil}(A) \]

**Proof.** Assume that \( x = \sum_{g \in G} a_g g \in \text{nil}(A[G]) \) then \( x^m = 0 \) where \( m \) is nonnegative integer, therefore we have that \( \varepsilon(x)^m = 0 \), hence \( \varepsilon(x) \in \text{nil}(A) \).

Now assume that \( \varepsilon(x) = \sum_{g \in G} a_g \in \text{nil}(A) \), as we done in the proof of the above theorem. Since \( G \) is a \( p \)-group, we have that for any \( g \in G \) there exists a nonnegative integer \( k \) such that \( o(g) = p^k \). Now, let \( m = \text{Min}\{k \in \mathbb{Z}^+ : g^{p^k} = 1 \ \forall \ g \in \text{supp}(x)\} \), we can rewrite \( x \) as

\[ x = \sum_{i=1}^{i=n} a_i g_i. \]

Thus

\[ x^{p^m} = [\sum_{i=1}^{i=n} a_i g_i]^{p^m}. \]

By using the multinomial theorem, we find

\[ x^{p^m} = \sum_{i=1}^{i=n} a_i g_i^{p^m} + pY \]
For some $Y$ in $A[G]$. Thus
\[ x^{p^m} = \sum_{i=1}^{i=n} a_i^{p^m} + pY \quad \cdots (1) \]
\[ \left( \sum_{i=1}^{i=n} a_i \right)^{p^m} = \sum_{i=1}^{i=n} a_i^{p^m} + pY' \quad \cdots (2) \]
For some $Y'$ in $A$. It is clear that $\text{nil}(A) \subseteq \text{nil}(A[G])$, thus $p \in \text{nil}(A[G])$ and $pY' \in \text{nil}(A)$. Since $\sum_{i=1}^{i=n} a_i$ is nilpotent, then $[\sum_{i=1}^{i=n} a_i]^{p^m}$ is nilpotent, therefore $\sum_{i=1}^{i=n} a_i^{p^m} + pY'$ is nilpotent, since $pY' \in \text{nil}(A)$, we get $\sum_{i=1}^{i=n} a_i^{p^m}$ is nilpotent. Similarly, by using statement (1), we find that $x^{p^m}$ is a nilpotent in $A[G]$, hence $x$ is a nilpotent in $A[G]$.

\[ \square \]

For the next two theorems, the group $G$ not necessary to be a torsion group.

**Theorem 3.3.** Let $H$ be a subgroup of $G$. Then $A[G]$ is domainlike ring $\Rightarrow A[H]$ is domainlike.

**Proof.** Assume that $A[G]$ is domainlike ring, thus we have $Z(A[G]) \subseteq \text{nil}(A[G])$.

It follows that $Z(A[H]) \subseteq Z(A[G]) \subseteq \text{nil}(A[G])$.

But we have that $A[H] \cap Z(A[G]) \subseteq A[H] \cap \text{nil}(A[G])$. Because $A[G], A, G$ are commutative, then the nil radical and the prime radical are equals. by using [10, Proposition 9.], we get that

$A[H] \cap \text{nil}(A[G]) = \text{nil}(A[H])$.

but we have $Z(A[H]) \subseteq \text{nil}(A[G]) \cap A[H]$. thus $Z(A[H]) \subseteq \text{nil}(A[H])$. It follows that $A[H]$ is domainlike.

\[ \square \]
Theorem 3.4. Let $H$ be a subgroup of $G$. Then

$A[G]$ is a presimplifiable ring $\Rightarrow$ $A[H]$ is presimplifiable.

Proof. Let $a, b \in A[H]$ such that $ab = a$, and assume that $a \neq 0$. Since $A[H]$ is subring of $A[G]$, and that $A[G]$ is a presimplifiable ring, it follows that $b$ is a unit in $A[G]$, thus there exists $t \in A[G]$ such that $bt = 1$, by using [10, Proposition 4(i)], then there exists $t' \in A[G]$ with the property that $bt' = 1$ and

$$\langle supp(t') \rangle \subseteq \langle supp(b) \rangle \subseteq H.$$ 

Therefor $t'$ is a unit in $A[H]$, hence $A[H]$ is a presimplifiable ring.

Corollary 3.5. Let $A$ be a ring and $G$ be a group. Then

$A[G]$ is (presimplifiable)domainlike ring $\Rightarrow$ $A$ is (presimplifiable)domainlike.

Proof. By using Theorem 3.3 and Theorem 3.4, putting $H = \{1\}$, as required.

Theorem 3.6. Let $A$ be a ring, and $G$ be a torsion group.

If $A[G]$ is presimplifiable, then $G$ is a p-group and $p$ in $J(A)$.

Proof. Assume that $A[G]$ is presimplifiable, by using Lemma 2.4 we get $Z(A[G]) \subseteq J(A[G])$. Let $g \in G$, since $G$ is torsion then $g$ has finite order, by using Lemma 2.6 we follow that $1 - g$ is zero divisor, thus $1 - g \in Z(A[G]) \subseteq J(A[G])$. Because $J(A[G])$ is an ideal, we get $\langle 1 - g : g \in G \rangle \subseteq J(A[G])$, i.e $\Delta \subseteq J(A[G])$, by using Lemma 2.7 it follows that $G$ is a p-group and $p$ in $J(A)$.

Theorem 3.7. Let $A$ be a ring, $G$ be a p-group with $p \in J(A)$. then

if $A$ is presimplifiable then $A[G]$ is presimplifiable.

Proof. Assume that $A$ is presimplifiable. Let $a, b \in A[G]$ such that $a = ab$, assume that $a \neq 0$. Let $H$ be the subgroup generated by $supp(a) \cup supp(b)$. Since $G$ is abelian and torsion, and $supp(a) \cup supp(b)$ is finite set, then $H$ is finite a p-group. Thus we can write $H = \{g_1, g_2, ..., g_n\}$ and $|H| = p^l = n$ for some nonnegative integers $l, n$. It is clear that $a, b \in A[H]$. Now, it is sufficient to show that $b$ is a unit in $A[H]$. We can write $a = \sum_{i=1}^{n} a_i g_i$ and $b = \sum_{i=1}^{n} b_i g_i$ where $a_i, b_i \in A$. Now, by using the augmentation mapping of $A[H]$, we get that $\varepsilon(a) = \varepsilon(a)\varepsilon(b)$. Now we have two cases:
Case(1) If \( \varepsilon(a) \neq 0 \), since \( A \) is presimplifiable, then \( \varepsilon(b) \) is a unit in \( A \), because \( H \) is finite a p-group, by using Theorem 3.1 we find that \( b \) is a unit in \( A[H] \).

Case(2) If \( \varepsilon(a) = 0 \), hence \( a \in \triangle \) the augmentation ideal of \( A[H] \), and we have \( a(1 - b) = 0 \), let \( J = aA[G] \) the principal ideal generated by \( a \), thus \( 1 - b \in \text{ann}(J) \). But we have that \( J \cap \triangle \neq 0 \), therefore there exists \( g \in H \) such that \( 1 - g \in J \cap \triangle \), and assume that order of \( g \) is \( s \) which is some power of \( p \), hence \( 1 - b \in \text{ann}(J) \subseteq \text{ann}(1 - g) \). By using [10], Proposition 4(ii)], we find that \( \text{ann}(1 - g) = A(1 + g + \ldots + g^{s-1}) \), thus \( 1 - b \in J \) and \( 1 - b \in \text{ann}(J) \). But we have that \( J \cap \triangle \neq 0 \), therefore there exists \( g \in H \) such that \( 1 - g \in J \cap \triangle \), and assume that order of \( g \) is \( s \) which is some power of \( p \), hence \( 1 - b \in \text{ann}(J) \subseteq \text{ann}(1 - g) \). By using [10], Proposition 4(ii)], we find that \( \text{ann}(1 - g) = A(1 + g + \ldots + g^{s-1}) \), thus \( 1 - b \in A[G] \) for some \( t \in A \). Notice that \( g^r \sum_{j=0}^{s-1} g^j = \sum_{j=0}^{s-1} g^j \) for all \( r = 0, \ldots, n-1 \) thus

\[
(1 - b)^2 = t^2[ \sum_{j=0}^{n-1} g^j]^2 = t^2[ \sum_{j=0}^{s-1} (\sum_{j=0}^{s-1} g^j=g^j)]
\]

and it is easy to check that \( (1 - b)^2 = st^2 \sum_{j=0}^{s-1} g^j \), but we have that \( p \in J(A) \) which is an ideal, since \( p \) divides \( s \), then \( s \in J(A) \), therefore \( (1 - b)^2 = 1 - b(2 - b) \in J(A) \), this implies that \( b(2 - b) \) are units in \( A[H] \), therefore \( b \) is a unit in \( A[H] \) and so \( b \) is a unit in \( A[G] \). Hence \( A[G] \) is a presimplifiable ring.

**Theorem 3.8.** Let \( A \) be a ring, \( G \) be a torsion group, then,

\( A[G] \) is presimplifiable if and only if \( A \) is presimplifiable and \( G \) is a p-group with \( p \in J(A) \).

**Proof.** By using Corrolary 3.1, Theorem 3.6, and Theorem 3.7.

**Theorem 3.9.** If the following conditions are satisfied, then \( A[G] \) is a (p-nilary) domainlike ring.

(i) \( A \) is a (p-nilary) domainlike ring.

(ii) \( G \) is a locally finite p-group.

(iii) \( p \) is nilpotent in \( A \).

**Proof.** (i) For \( A[G] \) is domainlike. Let \( a \) be a zero divisor in \( A[G] \), it follows that there exists a nonzero element \( b \in A[G] \) such that \( ab = 0 \). Let \( H \) be the subgroup generated by \( \text{supp}(a) \cup \text{supp}(b) \). Since \( G \) is a locally finite p-group,
and $\text{supp}(a) \cup \text{supp}(b)$ is finite set, then $H$ is a finite $p$-group. It sufficient to show that $a$ is a nilpotent in $A[H]$, consider $\varepsilon$ is the augmentation mapping of $A[H]$, and $\Delta$ is the augmentation ideal of $A[H]$. Now we have either $\varepsilon(a) = 0$ or $\varepsilon(a) \neq 0$.

Case (1) If $\varepsilon(a) = 0$ then $a \in \Delta$, under our assumption, and by using Lemma 2.8, we find that $\Delta$ is nilpotent, by using Theorem 3.2, we get that $a$ is nilpotent.

Case (2) If $\varepsilon(a) \neq 0$, suppose that $\varepsilon(b) \neq 0$, but it is clear $\varepsilon(a) \varepsilon(b) = 0$, thus $\varepsilon(a) \in Z(A)$, since $A$ is domainlike, then $\varepsilon(a) \in \text{nil}(A)$, by using Theorem 3.2, we find that $a$ is nilpotent.

Suppose that $\varepsilon(b) = 0$, now we have $a$ does not belong to $\Delta$ and $b \in \Delta$. Let $J = <b>$, since $ab = 0$, then $a \in \text{ann}(J)$, it is clear that $J \cap \Delta \neq 0$, therefore, by using Lemma 2.10, there exists $g \in G$ such that $1 - g \in J \cap \Delta$, i.e. $1 - g \in J$. But we have $a \in \text{ann}(J) \subseteq \text{ann}(1 - g)$, thus $a \in \text{ann}(1 - g)$.

By using [10, Proposition 4(ii)], we find that $\text{ann}(1 - g) = A(1 + g + \ldots + g^{s-1})$, where $s$ is the order of $g$ which is a power of the prime $p$, thus $a \in A[\sum_{j=0}^{s-1} g^j]$, therefore $a = t[\sum_{j=0}^{s-1} g^j]$ for some $t \in A$. Notice that

$$g^r \sum_{j=0}^{s-1} g^j = \sum_{j=0}^{s-1} g^j \text{ for all } r = 0, \ldots, n - 1.$$ 

Thus

$$a^2 = t^2[\sum_{j=0}^{s-1} g^j]^2 = t^2[\sum_{j=0}^{s-1} (\sum_{j=0}^{s-1} g^j) g^j].$$

and it is easy to check that $a^2 = st^2 \sum_{j=0}^{s-1} g^j$, but we have that $p \in \text{nil}(A)$ which is an ideal, since $p$ divides $s$, then $s \in \text{nil}(A)$, therefore $a^2$ is nilpotent, it follows that $a$ is nilpotent. Hence $A[G]$ is a domainlike ring. (ii)Since $A[G]$ is domainlike then it is a $p$-nilary ring.

**Theorem 3.10.** Let $A$ be a ring, and $G$ be a torsion group.

If $A[G]$ is domainlike if and only if $A$ is domainlike and $G$ is a $p$-group where $p$ is a nilpotent in $A$.

**Proof.** Assume that $A[G]$ is domainlike, by using Lemma 2.3 we have $Z(A[G])$ is an ideal in $A[G]$. Let $g \in G$. Since $G$ is torsion, then $g$ has a finite order. By using lemma 2.6 it follows that $1 - g$ is zero divisor, thus
$1 - g \in Z(A[G]) \subseteq \text{nil}(A[G])$. Because $Z(A[G])$ is an ideal we get that $(1 - g : g \in G) \subseteq Z(A[G]) \subseteq \text{nil}(A[G])$, therefore $\Delta \subseteq \text{nil}(A[G])$, by using Lemma 2.8 we get $G$ is a $p$-group and $p$ is nilpotent. The converse follows directly from Theorem 3.9.

**Theorem 3.11.** Let $A$ be a ring, $G$ be any abelian group. Then

If $A[G]$ is (domainlike) presimplifiable then $A$ is (domainlike) presimplifiable and every finite nontrivial subgroup of $G$ is $p$-group and $p \in J(A)$ ($p$ is nilpotent) for a fixed prime number $p$.

**Proof.** Assume $A[G]$ is presimplifiable. Let $H$ a finite subgroup of $G$, by using Theorem 3.4, we get that $A[H]$ is presimplifiable, and by using Theorem 3.6, we find that $H$ is a $p$-group and $p \in J(A)$. Since $J(A)$ is an proper ideal, then it contains only one prime, that if $p, q$ are different primes in $J(A)$, then for some integers $m, n$ we have $mp + nq = 1$, thus $1 \in J(A)$, a contradiction. For a domainlike part the proof is similarly.

**Lemma 3.12.** [10], Page 675 $A[G]$ is an integral domain if and only if $A$ is an integral domain and $G$ is abelian torsion-free.

**Theorem 3.13.** Let $A$ be a ring, $G$ be a torsion free group. Then $A[G]$ is domainlike if and only if $A[G]$ is integral domain.

**Proof.** ($\Rightarrow$): Assume that $A[G]$ is domainlike. Let $a \in A[G]$ be any zero divisor, consider the ideal $J = \langle a \rangle$. Since $A[G]$ is domainlike, then $a$ is nilpotent, therefore there exist a nonnegative integer $m$ such that $J^m = 0$. If $J = 0$, then $a = 0$, and thus the proof is finished. Suppose that $J \neq 0$, now we have two cases, either $\Delta \cap J \neq 0$ or $\Delta \cap J = 0$ If $\Delta \cap J \neq 0$, then there exists $0 \neq 1 - g \in J$. Therefore $(1 - g)^m \in J^m = 0$. Thus $(1 - g)^m = 0$. Hence $1 - g$ is a divisor of zero. But using Lemma 2.6 we find that $g$ has finite order, a contradiction. If $\Delta \cap J = 0$, then $J \subseteq \text{ann}(\Delta)$, indeed $J \Delta \subseteq J \cap \Delta = 0 \Rightarrow J \Delta = 0 \Rightarrow J \subseteq \text{ann}(\Delta)$. Since $G$ is infinite, by using [10], Proposition 4., we get that $\text{ann}(\Delta) = 0$, thus $J = 0$, therefore $a = 0$. Hence $A[G]$ is a domain. The converse is straightforward.

**Corollary 3.14.** Let $A$ be a ring, $G$ be a torsion free group. Then $A[G]$ is domainlike if and only if $A$ is integral domain.
Proof. For a torsion free group $G$, the proof follows directly from Corollary 3.13 and Lemma 3.12.

From the above results we notice the following corollary.

**Corollary 3.15.** Let $A$ be a ring, $G$ be a torsion free group. Then

If $A$ is integral domain then $A[G]$ is presimplifiable.

*Proof.* The proof is straightforward.

**Corollary 3.16.** Let $A$ be a ring, $G$ be a non torsion group. Then

if $A[G]$ is domainlike then $A$ is integral domain.

*Proof.* Let $g \in G$ be an infinite order element, and let $H$ be the subgroup generated by $g$, hence $H$ is torsion free. Because $A[G]$ is domainlike. By using Theorem 3.3, we find that $A[H]$ is domainlike, by using Corollary 3.14, we get that $A$ is integral domain.

**Example 3.17.** 1. Let $G = C_6 = \langle x \rangle$ which is not a $p$-group, and let $A = \mathbb{Z}_2$. Notice that $A$ is presimplifiable and Domainlike, and order of $G$ is nilpotent and belong to the $J(A) = 0$. We can find $g \in G$ such that order of $g$ is 3. Because 3 is a unit in $A$, then we have nontrivial idempotents as $e = (1 + g + g^2)$. Hence, $A[G]$ is neither presimplifiable nor domainlike.

2. Let $G = C_4 = \langle x \rangle$ which is a 2-group, and let $A$ be any field of characteristic 0, notice that $A$ is presimplifiable and Domainlike. We have $|G| = 2^2$ an 2 is a unit in $A$. Now we find that $A[G]$ have nontrivial idempotents as $e = 2(1 + g + g^2 + g^3)$. Hence, $A[G]$ is neither presimplifiable nor domainlike.

3. Let $G$ be any finite nontrivial group, then $Q[G]$ is never be presimplifiable because we can construct nontrivial idempotents, but we know that $Q$ is a presimplifiable ring.

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