Interest Clustering Coefficient: a New Metric for Directed Networks like Twitter

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ABSTRACT

We study here the clustering of directed social graphs. The clustering coefficient has been introduced to capture the social phenomena that a friend of a friend tends to be my friend. This metric has been widely studied and has shown to be of great interest to describe the characteristics of a social graph. In fact, the clustering coefficient is adapted for a graph in which the links are undirected, such as friendship links (Facebook) or professional links (LinkedIn). For a graph in which links are directed from a source of information to a consumer of information, it is no more adequate. We show that former studies have missed much of the information contained in the directed part of such graphs. We thus introduce a new metric to measure the clustering of a directed social graph with interest links, namely the interest clustering coefficient. We compute it (exactly and using sampling methods) on a very large social graph, Twitter, a person can follow someone she is interested in; the resulting graph, where there is a link $u \rightarrow v$ if the account associated to the node $u$ followed the account associated to the node $v$, is thus directed. In this study, we used as main dataset the snapshot of Twitter (TS in short) extracted by Gabriélov et al. as explained in [14] and made available by the authors. The TS has around 505 million nodes and 23 billion arcs, making it one of the biggest snapshots of a social network available today.

The classic definition of the clustering coefficient cannot be directly applied on directed graphs. This is why most of the studies computed it on the so-called mutual graph, as defined by Myers & al. in [27], i.e., on the subgraph built with only the bidirectional links. We call mutual clustering coefficient (mcc for short) the clustering coefficient associated with this graph. We computed this coefficient in the TS, using both exact and approximated methods. We find a value for the mcc of 10.7%. This is a high value, of the same order than the ones found in other social networks.

However, classical way to operate leaves out 2/3 of the graph! Indeed, the bidirectional edges only represents 35% of the edges of the TS. A way to avoid it is to consider all links as undirected and to compute the clustering coefficient of the obtained undirected graph. We call undirected clustering coefficient (ucc for short) the corresponding computed coefficient. Such a computation in the TS gives a value of ucc of only 0.11%. This is way lower than what was found in most undirected social networks. It is thus a necessity to introduce specific clustering coefficients for the directed graphs. More generally, when analyzing any directed datasets, it is of crucial importance to take into account the information contained in its directed part in the most adequate way.

A first way to do that is to look at the different ways to form triangles with directed edges. Fagiolo computed the expected values of clustering coefficients considering directed triangles for random graphs in [11] and illustrated his method on empirical data on world-trade flows. There are two possible orientations of triangles:
We computed both coefficients for the snapshot, obtaining \( tcc \) where a K22 is defined as a set of four nodes in which two of them (e.g., Bob → Carol and Alice → Carol). Indeed, when discussing interests, consider two nodes having similar interests. Apart from being friends, these two nodes do not have any reason to be directly connected. However, they would tend to be connected to the same out-neighbors. We exploit this to study a new notion of connections in directed networks and the new naturally associated clustering coefficient, which we name interest clustering coefficient, or icc in short, and define as follows:

\[
icc = \frac{4 \cdot \# \text{K22s}}{\# \text{open K22s}},
\]

where a K22 is defined as a set of four nodes in which two of them follow the two others, and an open K22 is a K22 with a missing link, see Figure 1d. We computed the icc on the Twitter snapshot, obtaining icc = 3.6% (3.1% when removing the bidirectional structures). This value, an order of magnitude higher than the previous clustering coefficients computed on the non bidirectional directed graph, confirm the interest of this metric. If the clustering coefficient of triangles are good metrics to capture the social aspect of a graph, the interest clustering coefficient is a good metric to capture the informational aspect.

In summary, our contributions are the following:

- We define a new clustering coefficient for graphs with interest links.
- We succeeded in computing it, both exactly and using sampling methods, for a snapshot of Twitter with 505 million nodes and 23 billion edges.
- We additionally provide the values of the directed and undirected clustering coefficients previously defined in the literature. We believe this is the first time that such coefficients are computed exactly for a large directed online social network.

We computed both coefficients for the snapshot, obtaining \( tcc = 1.9\% \) and \( ccc = 1.7\% \). However, note that a large part of the transitive and cyclic triangles comes from bidirectional triangles. When removing them, we arrive to values of \( tcc = 0.51\% \) and \( ccc = 0.24\% \).

We believe those metrics miss an essential aspect of the Twitter graph: while the clustering coefficient were defined to represent the social cliques between people, it is not adequate to capture the information aspect of Twitter, known to be both a social and information media [18, 27]. In this work, we go one step further in the way directed relationships are modeled. We argue that in directed networks, the best way to define a relation or similarity between two individuals (Bob and Alice) is not always by a direct link, but by a common interest, that is, two links towards the same node (e.g., Bob → Carol and Alice → Carol). Indeed, when discussing interests, consider two nodes having similar interests. Apart from being friends, these two nodes do not have any reason to be directly connected. However, they would tend to be connected to the same out-neighbors. We exploit this to study a new notion of connections in directed networks and the new naturally associated clustering coefficient, which we name interest clustering coefficient, or icc in short, and define as follows:

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We compute this new metric as much as the previous ones on other directed datasets to highlight the differences and interests of the different metrics.

We then propose a new random graph model to obtain random directed graphs with a high interest clustering coefficient. We prove this model follows power-law in- and out-degree distributions, and analyse the interest clustering coefficient value by simulation.

Lastly, we discuss the usage of this new metric for link recommendation. The principle is to recommend links closing a large number of K22s (instead, classically, of triangles). We discuss the strengths/weaknesses of this method for a set of Twitter users.

The paper is organized as follows. We first discuss related work in Section 2. In Section 3, we present the algorithms we used to compute the values of the interest clustering coefficient, both exactly and by sampling. We discuss the results on the clustering coefficients of Twitter in Section 4, and of other directed datasets in Section 5. In section 6, we propose and study a preferential attachment model providing a high interest clustering coefficient. Lastly, we discuss the use of interest clustering coefficient for link recommendation in Section 7.

2 RELATED WORK

Complex networks. Even if the study of complex networks is an old field [36], it keeps receiving a lot of attention from the research community. The reason for this is twofold. First, a great number of very large practical systems emerged recently can be seen as complex networks, in particular online social media networks, see [24] for a survey. Second, with the development of big data analysis, entrepreneurs, analysts or researchers have new tools to study those huge amounts of data. Complex networks often exhibit common properties, like small diameter [1], small average distance [3, 21, 40], heavy tail degree distributions [8, 21], high clustering [40], communities [37], etc.

Clustering coefficient. Among those properties, the clustering coefficient shows that, when two people know each other, there is a high probability that those people have common friends. The clustering coefficient has numerous important applications, such as spam detection [6], link recommendation [7, 35], information spread [15], study of biased network samples [28], performance of some neural networks [16], etc. There are different definitions of the clustering coefficient. The local clustering coefficient of a node i, first introduced by Watts and Strogatz [40], is defined as the probability that two neighbors of i are also connected together. This probability can be computed as

\[
CC(i) = \frac{\# \text{triangles with the node } i}{\# \text{connected triplets centered on } i},
\]

where (\# connected triplets centered on i) = \( \frac{\deg(i)^2}{2} \). From here can be defined for the whole graph a clustering coefficient as the mean of the local clustering coefficients over all the nodes of the graph:

\[
CC_g = \frac{1}{n} \sum_{i \in V} CC(i)
\]
Another definition was first introduced by Barrat and Weigt in [4], and is called the global clustering coefficient, or transitivity. It is defined as

$$CC_g = 3 \times \frac{\# \text{triangles in the graph}}{\# \text{connected triplets of vertices in the graph}}.$$ 

We use the global clustering coefficient in this paper. The clustering coefficient has also been defined for weighted graphs [29, 32].

**Computations for social graphs.** The undirected clustering coefficient of some social networks has been provided in the literature. It has been computed on very large snapshots for Facebook [37], Microsoft Messenger [21], Flickr, and YouTube [26]. The local clustering coefficient has also been studied in the undirected mutual graph of Twitter [27]. We can also cite the values given by the Network Repository project [30], providing a large comprehensive collection of network graph data available for which it lists some basic properties. The undirected clustering coefficient is usually much higher in social networks than in random models.

**Directed graphs.** All these studies only consider the undirected clustering coefficient, even for directed graphs like Twitter. Fagiolo introduced definitions of directed clustering coefficients, that we named tcc and ccc [11], but those definitions had never been computed and discussed on large datasets to our knowledge, as we do in this paper. Moreover, we believe that these metrics are not the most relevant ones for directed graphs with interest links.

**Computing substructures.** Researchers studied methods to efficiently compute the number of triangles in a graph, as naive methods are computationally very expensive on large graphs. Two families of methods have been proposed: triangle exact counting or enumeration and estimations. In the first family, the fastest algorithm is due to Alon, Yuster, and Zwick [2] and runs in $O(m^{3/2})$, with $m$ the number of edges and $\omega$ the best known exponent for the fast matrix multiplication. Its current value is 2.3728, due to an algorithm of [9] improved by [20], giving a complexity of $O(m^{1.41})$ for the AYZ algorithm. However, methods using matrix multiplication cannot be used for large graphs because of their memory requirements. In practice, enumeration methods are often used, see e.g., [19, 33]. A large number of methods for approximate counting were proposed, see for example [17] and its references. The authors obtain a running time of $O(m + \frac{m^{3/2} \log n}{\epsilon^2})$ and a $(1 \pm \epsilon)$ approximation. Methods to count rectangles and butterfly structures in undirected bipartite networks were also proposed in [39] and in [31]. In this paper, we propose an efficient enumeration algorithm to count the number of K22s and open K22s in a very large graph. We focused on the case in which only one adjacency can be stored, as this was our case for the TS. To the best of our knowledge, we are the first to consider this setting.

## 3 Computing Clustering Coefficients in Twitter

We computed the interest clustering coefficient and the triangle clustering coefficients on a directed Twitter snapshot (TS in short) that we use as a typical example of a directed social network with interest links. We used two different methods: an exact count and an estimation using sampling techniques, either with a Monte Carlo algorithm or with a sampling of the graph.

### 3.1 The Twitter Snapshot

In order to compute the different clustering coefficients of a real graph, the authors of [13] gave us access to a snapshot of the graph of the followings of Twitter. The snapshot was collected between March 2012 and July 2012. With $n = 505$ million nodes and $m = 24$ billion links collected, this graph is the largest directed social network graph available today, to the best of our knowledge. Each node of the graph represents an account of Twitter, and there is a link between two nodes $u$ and $v$, if the account $u$ follows the account $v$. All account IDs have been anonymized. The snapshot is a perfect case study as Twitter is a directed social network used both as a social and an information network [18, 27]. It allows to study directed/undirected social/interest clustering coefficients.

**Degree distributions of the Twitter Snapshot.** We provide in Figure 2 the degree distributions of the TS. We fitted their tails to power law distributions. We obtained $P^-(i) = C_i^{-2.17}$ and $P^+(i) = C_i^{-2.76}$, with $P^-(i)$ (respectively $P^+(i)$) the probability that a node has in-degree (resp. out-degree) $i$. In the following, we use the obtained values to compute the practical complexity of the algorithms.

Other references of the literature have also provided a power law fit for both distributions, see e.g., [27]. In this work, the authors obtained exponents of values 1.35 and 1.28. However, we believe that the authors did not fit on the complete distributions and not on
We computed the exact numbers of K22s and open K22s in the Trivial algorithm.

Twitter Snapshot. Recall that we are discussing a dataset with Thibaud Trolliet, Nathann Cohen, Frédéric Giroire, Luc Hogie, and Stéphane Pérennes it would check the existence of a K22 and of open K22s. There are quadruplets of vertices with 2 upper vertices. Then, for each quadruplet, x the same vertex. We say that a fork has top (or bottom) vertex K22 edges. We call a fork top vertices and bottom vertices.

In the rest of this paper, we call improved algorithm. The practical complexity can be greatly improved by only considering connected quadruplets, and by mutualizing the computations of the common neighbors of the in-neighbors of a vertex, as explained below. The pseudo-code is given in Algorithm 1.

The algorithm’s main loop iterates on the vertices of the graph. For each vertex x, we consider its in-neighborhood N−(x). We then compute how many times a vertex w (with w < x to avoid counting a K22 twice) appears in the out-neighborhoods of the vertices of N−(x). We denote it #occ(w). We use a hash table to store the value of #occ(w) in order to be able to do a single pass on each out-neighbor.

For a vertex w, any pair of its #occ(w) in-neighbors common with x forms a K22 with x and w as bottom vertices. There are hence \( \binom{\#occ(w)}{2} \) K22s with x and w as bottom vertices. The number of K22s with x as a top vertex is then

\[
#K22(x) = \sum_{w:\#occ(w)\geq 2} \left( \binom{\#occ(w)}{2} \right).
\]

The number of open K22s with x as the top vertex is computed by noticing that, for any pair of vertices u and v of N−(x), we have d+(u) − 1 + d+(v) − 1 − \(1_{v\in N^+(u)} \cdot 1_{u\in N^+(v)}\) open K22s containing this fork (ux, vx). We can count the number of open K22s with x as a top vertex, u as the bottom vertex of out-degree 2 (and thus another vertex v as the bottom vertex of out-degree 1). A vertex u ∈ N−(x) is thus in d+(u) − 1 \(\sum_{v\in N^+(x)} \cdot 1_{v\in N^+(u)}\) d−(x) − 1) such open K22s. The only subtlety is that we count the number of arcs, which are between two vertices of N−(x), during the loop on the out-neighborhoods of the vertices of N−(x). We note this number #internalArcs. We then have:

\[
#openK22(x) = \sum_{u\in N^+(x)} (d+(u) − 1)(d−(x) − 1)) #internalArcs.
\]

Lastly, the global number of K22s (resp. open K22s) in the digraph is just the sum of the number of K22s (resp. open K22s) with a vertex x as a top vertex, as, since we only consider K22s formed with a vertex w such that x < w, we only count each K22 once.

Complexity of the used algorithm. The complexity thus is m + \(\sum_{u} d+(u)\cdot d+(u) \cdot 1\). Indeed, each edge is only considered once as an in-arc and d+ − 1 times as an out-arc. Note that, in the Twitter Snapshot, the sum of the squares of the degrees is equal to 8 \times 10^{13}. The order of the number of iterations needed to compute the number of K22s was thus massively decreased from the 6.4 \times 10^{33} iterations of the trivial algorithm.

Complexity on graphs following a power-law degree distribution. The complexity of the algorithm on a graph built with preferential attachment can be computed as follows. We consider without loss of generality that the sum of the square of the degrees is minimum for the out-degrees (and not the in-degrees). The maximum degree is \(d_{\text{max}} = O(n^{1/(\alpha+1)})\), with \(\alpha^+\) the exponent of the out-degree power law distribution. Thus, the sum of the squares of the degrees, when \(2 \leq \alpha^- < 3\), is \(\sum_{v\in V} (d^+(v))^2 = C^+ n \sum_{i=1}^{d_{\text{max}}} \frac{i^2}{i^{\alpha^+}} \sim \frac{\alpha^-}{n^{\alpha^-}}\).
Algorithm 1 Enumeration of K22s and open K22s

1:  
2: Input: Digraph(V, A)
3: #occ ← 0  // hash table
4: for x ∈ V do
5:     #internalArcs ← 0  // We count the number of arcs internal
to N+(x) as these arcs do not form open K22s
6:     for u ∈ N+(x) do
7:         #openK22s ← (d+(u) − 1)(d+(x) − 1)
8:         for w ∈ N+(v) \ {x} do
9:         #occ[w] ← 1
10:     if w ∈ N+(x) then  // We use a second hash table to
11:         #internalArcs+ ← 1
12:     for w with #occ[w] ≥ 2 do
13:         #k22+ ← (#occ[w])
14:     #openK22s ← #internalArcs
15:     #occ ← 0  // Done with a double loop
16:     icc ← 4#K22
17:     openK22

\[
C^{+}_n \left[ \frac{d^{max}}{\sum_{i=1}^{p+1} x_i^{p+1}} \right]_1 \left[ \frac{d^{max}}{(3-a^*)^{p+1}} \right]_1 \left[ \frac{C^{+}_n}{(3-a^*)^{p+1}} \right]_1 = \frac{C^{+}_n}{(3-a^*)^{p+1}}
\]

where \( C^{+} = \frac{1}{|A|^{p+1}} \). The complexity is thus in \( O(m + n^{1+\frac{\alpha}{p+1}}) \).

3.3 Approximate Counts

As discussed later in Section 4, the exact count of the number of K22s and open K22s in Twitter implies massive computations. This number can be estimated using Monte Carlo Method and/or computations on a sample of the graph. We discuss both methods below. One of our goals was to see how good computations made in the literature using smaller Twitter snapshots were.

3.3.1 Exact icc on Twitter Samples. We built samples of the
TS to estimate the interest clustering coefficient. Several choices can be made to build the samples. To avoid missing nodes of high degrees (which would lead to a high variance), we sampled the arcs (and not the nodes). Given a sampling probability \( p \), we keep an arc in the sample with probability \( p \). We generated samples of different sizes corresponding to sampling probabilities from \( p = 1/100 \) to \( p = 1/16000 \).

Estimator of the number of K22 and open K22s. Let us call \( \mathcal{A} \) the set of occurrences of a specific pattern (in our case, either a K22 or an open K22). The number of occurrences of the pattern in a sample, \( X \), is given by \( X = \sum_{A \in \mathcal{A}} X_A \), where \( X_A \) is the random variable which is equal to 1 if all the arcs of pattern \( A \) are selected in the sample and 0 otherwise.

If we note \( l \) the number of arcs of the pattern (4 for a K22 and 3 for an open K22), we have that \( \mathbb{P}[X_A = 1] = p^l \). By linearity of the expectation, we get \( E[X] = p^l |\mathcal{A}| \). Thus, \( Y = p^{-l}X \) is an unbiased estimator of \( |\mathcal{A}| \).

Variance. Note that the random variables \( X_A \) are not independent, i.e., two K22s can share a common link. Otherwise, the variance would simply be \( \mathbb{V}[X] = \sum_{A \in \mathcal{A}} \mathbb{V}[X_A] = |\mathcal{A}|p^l(1 - p^l) \leq |\mathcal{A}|p^l \).

However, we can argue that (and we will verify that), in practice, most of the K22s and open K22s do not share any link. It can be used in the analysis as follows.

\[
\mathbb{V}[X] = E[X^2] - E[X]^2 = E[\sum_{A \in \mathcal{A}} X_A^2] - E[X]^2
\]

We now distinguish the couples of dependent patterns, which we note \( \Delta = \{(A, B) | A \cap B \neq \emptyset \} \), from the ones of independent ones, \( \Delta = \{(A, B) | A \cap B = \emptyset \} \).

\[
\mathbb{V}[X] = \sum_{(A, B) \in \Delta} E[X_A X_B] + \sum_{(A, B) \in \Delta} E[X_A X_B] - E[X]^2
\]

When \( A \) and \( B \) are independent, we have

\[
E[X_A X_B] = \mathbb{E}[X_A]\mathbb{E}[X_B] = p^{2l}.
\]

As \( E[X]^2 = p^{2l} |\mathcal{A}|^2 \), we get

\[
\mathbb{V}[X] = \sum_{(A, B) \in \Delta} E[X_A] E[X_B] + \sum_{(A, B) \in \Delta} E[X_A X_B] - E[X]^2
= \sum_{(A, B) \in \Delta} (E[X_A X_B] - p^{2l})
\]

Let us now distinguish different cases. We note \( \Delta_i \) the set of couples of patterns sharing \( 1 \leq i \leq l \) arcs. For a couple \( (A, B) \in \Delta_i \), we
have that \(\mathbb{E}[X A B] = p^{2I_i} - 1\), giving that
\[
\mathbb{E}[X] \leq \sum_{i=1}^{l} \sum_{(A, B) \in \Delta_i} (p^{2I_i} - p^2).
\]
Since \(p < 1\), we get
\[
\mathbb{E}[X] \leq \sum_{i=1}^{l} p^{2I_i - 1} |\Delta_i|.
\]
Note that, when all patterns are independent, \(|\Delta| = |\Delta| = |\mathcal{A}|\) (couples \((A, A) \in \mathcal{A}\)), giving back the variance of the independent case, \(p^2 |\mathcal{A}|\). Chebycheff’s inequality tells us that:
\[
\text{Prob}[|Y - \mu| \geq k\sigma] \leq \frac{1}{k^2},
\]
where \(\mu\) is the expectation and \(\sigma\) is the standard deviation of \(X\). In our case, if we want an accuracy of \(\varepsilon\) with a probability \(q\), we should have \(\frac{1}{k^2} \leq 1 - q\) and \(k\sigma \leq c\varepsilon |\mathcal{A}|\), which can be rewritten as:
\[
\frac{k^2}{\varepsilon^2} \sum_{i=1}^{l} p^{2I_i - 1} |\Delta_i| \leq p^2.
\]
Lastly, to estimate the icc, we use as an estimator
\[
Z = \frac{4Y}{Y_0},
\]
with \(Y\) and \(Y_0\) the estimators of the number of K22s and open K22s, respectively. As \(\lim_{n \to \infty} Y = \#K22s\) and \(\lim_{n \to \infty} Y = \#openK22s\), we have that \(\lim_{n \to \infty} Z = 1\). For the precision, if \(Y\) and \(Y_0\) have an accuracy of \(\varepsilon\) and \(\varepsilon_0\) respectively, then with a probability \(q = 0.99\), \(Z\) has at least an accuracy of \(1 + \varepsilon + \varepsilon_0\) with a probability \(q^2 = 0.98\).**

**Numerical application.** We now consider the K22s of the TS. Note that we know that \(\frac{|\Delta_1|}{|\mathcal{A}|} = \frac{1}{4K22s} = 3.8 \times 10^{-17}\). We also can notice that \(\frac{|\Delta_3|}{|\mathcal{A}|} = 1\). In the TS, an edge is shared by \(\#K22s / \#K22\) K22s on average, with \(m\) the number of links of the TS. Thus, the average number of K22s sharing at least an edge with a K22 is between \(\#K22s / \#K22\) and \(4 \#K22s / \#m\). It gives \(\frac{1}{m} |\mathcal{A}|^2 \leq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \leq \frac{4}{m} |\mathcal{A}|^2\).

The number of overlapping K22s with \(i\) arcs is a non-increasing function of \(i\). To make a numerical evaluation, we suppose that most overlapping K22s share one edge and not 2 edges in the TS.

We set that \(|\Delta_1| = \frac{1}{m} = 4.3 \times 10^{-8} |\mathcal{A}|^2\), and \(|\Delta_2| = 10^{-16} |\mathcal{A}|^2\). Now, if we want a precision of \(\varepsilon = 0.1\) with a probability 0.99 (that is \(k = 10\)), we need to take a sampling probability \(p\) such that
\[
p^8 \geq \frac{10^2}{10^{-4}} (4.3 \times 10^{-8} \times 10^{-16} + 10^{-17}) + 6 \times 10^{-17} + p^4 3 \times 10^{-17}.
\]
That is \(p \geq 2.5 \times 10^{-4}\). Thus, under these hypotheses, a sample with sampling probability 1/2500 and larger, e.g., our 1/2000 sample, allows to estimate the number of K22s with a precision of 10%. The number of open K22s is larger and thus, the precision is better. It gives a precision of at least \((\frac{1}{100})^4 = 0.20\) for the estimation of icc. In practice, the Chebycheff inequality and our hypothesis are pessimistic as shown below.

**Results.** We present in Figure 3 the results of the algorithm for different sample sizes, corresponding to sampling probabilities from \(p = 1/100\) to \(p = 1/16,000\). For each sample size, we generated 30 samples. The distribution over the samples of the interest clustering coefficient, K22s and open K22s are provided by a boxplot for each value of \(p\). Note that a K22 of the TS appears in a sample with a probability of only \(p^4\), and of \(p^3\) for an open K22. The clustering coefficient of a sample is thus an estimate of \(p \cdot \text{icc}\). We observe that the clustering coefficient is well estimated using any sample for a sampling probability of 1/1000 or larger. Indeed, for this range of probabilities, the distribution over all samples is very concentrated and around the exact value of the icc. Note that, for \(p = 1/1000\), a K22 is present in the sample with a probability of only \(10^{-12}\). The expectation of the number of nodes with an edge is only 23 million nodes (over 500 million) and the number of edges also around 23 million. Thus, a small sample (5% of the nodes and 0.1% of edges) allows to do an efficient estimation of the icc. For smaller values of \(p\), the variance increases. The median estimates well the icc for a range of \(p\) between 1/8000 and 1/1000, but samples of these sizes may have error of 100% of the value. Lastly, for \(p = 1/16,000\), only the number of open K22s (and not the K22s or the icc) is approximated by the median.

In conclusion, a sample with sampling probability 1/1000 is enough to efficiently estimate the interest clustering coefficient, with a computation time of around 1 minute (instead of days for the whole TS) on a machine of the cluster.
Approximating the number of undirected triangles, we have the wanted precision (and we are not doing many more with $Z$ variable with probability $p$ is large, in Twitter is the number of triangles divided by the number of problem with this method is that the probability to form a triangle direct method would be to select three vertices uniformly at random when $p$ is $\sqrt{n}$. We can thus perform the sampling by picking a vertex $v$ with probability $\left(\frac{d(v)}{2}\right) / \sum_{v \in V} \left(\frac{d(v)}{2}\right)$ and then select two random edges adjacent to $v$.

**Directed triangles.** The method is the same in the case of directed triangles. We select an open triangle uniformly at random. The number of open triangles rooted at a vertex $v$ is $d^-(u) d^+(u)$. We thus select a node $u$ with probability $d^-(u) d^+(u) / \sum_{v \in V} d^-(v) d^+(v)$. We then select uniformly at random an incoming arc and an outgoing arc. Lastly, we check if the triangle is closed (which is the case with a probability equal to tcc and to ccc respectively for transitive and cyclic triangles).

**Precision of the estimation and number of iterations.** Each trial is a Bernoulli variable with a probability $p = tcc = 0.019$. To get an accuracy of 1%, with probability 0.99, we should thus do $n = 4 \times 10^6$ iterations.

**Interest clustering coefficient.** For triangles, we were able to select uniformly at random open triangles using the node degrees. In the case of K22s, node degrees is not sufficient to select an open K22 uniformly at random. In fact, achieving this goal is very costly, but we present a method in which, by picking only forks (as we do for triangles), we can compute the interest clustering coefficient.

**Preliminary: Precision of Monte Carlo Method.** Precision of the estimation and number of iterations. Each trial is a Bernoulli variable with probability $p$. We use as an estimate $Y$ the mean of the random sample. Its expectation is $p$ and its standard deviation is $\sqrt{p(1-p)/n}$. Due to the central limit theorem, we get that, when $n$ is large,

$$\text{Prob} \left[ |Y - p| \leq Z_{\alpha/2} \frac{\sqrt{p(1-p)/n}}{\sqrt{n}} \right] = \alpha,$$

with $Z_{\alpha/2}$ the value giving the $\alpha$ confidence interval a standard normal distribution. To get with probability $\alpha$ an accuracy of $\epsilon$ of the empirical mean $\bar{p}$ (which is not known), we should have $Z_{\alpha/2} \frac{\sqrt{p(1-p)/n}}{\sqrt{n}} \leq \epsilon p$. That is $n \geq \frac{Z_{\alpha/2}^2 p(1-p)}{\epsilon^2 p^2}$. If we take $n \geq \frac{Z_{0.01}^2}{0.01^2}$, we have the wanted precision (and we are not doing many more iterations when $p$ is small). For example, to get an accuracy of 99% ($\epsilon = 0.01$), with probability $\alpha = 0.99$, we should have a number of iterations such that $n \geq \frac{75.625}{p}$.

**Approximating the number of undirected triangles.** A first direct method would be to select three vertices uniformly at random and check if they form a triangle and open triangles. The problem with this method is that the probability to form a triangle in Twitter is the number of triangles divided by the number of triplet of nodes, i.e., $6.23 \times 10^{11} / (5 \times 10^7)^3 = 5 \times 10^{-15}$. Thus the number of needed iterations would be astronomical, $5.5 \times 10^{19}$ for an accuracy of 1%, with probability $\alpha = 0.99$. We thus have to use methods selecting open triangles directly.

To estimate the undirected clustering coefficient, we need to select open (undirected) triangles uniformly at random. We then test if the selected triangle is closed or not (which is the case with probability $\text{tcc}$). The number of open triangles rooted at vertex $v$ is equal to $\frac{d(v) (d(v) - 1)}{2}$. We can thus perform the sampling by picking a vertex $v$ with probability $\left(\frac{d(v)}{2}\right) / \sum_{v \in V} \left(\frac{d(v)}{2}\right)$ and then select two random edges adjacent to $v$.

![Figure 4: Estimation of the clustering coefficient with Monte Carlo Method.](image-url)
We may thus define two efficient unbiased estimates for $K_{22}$ and \#open$K_{22}$s:

$$Y = \frac{\#forks}{2n} \sum_{i=1}^{n} X_i,$$

and

$$Y_o = \frac{\#forks}{n} \sum_{i=1}^{n} X_{oi}.$$

We have $\mathbb{E}[Y] = \#K_{22}$s and $\mathbb{E}[Y_o] = \#openK_{22}$s. The number of forks with a vertex $v$ as a root is given by $\binom{d(v)}{2}$. The total number of forks in the TS is thus $\sum_{v \in V} \binom{d(v)}{2}$. Lastly, as we are interested by the interest clustering coefficient, we define

$$Z = \frac{4Y}{Y_o}.$$

As $\lim_{n \to \infty} Y = \#K_{22}$s and $\lim_{n \to \infty} Y = \#openK_{22}$s, we have that $\lim_{n \to \infty} Z = ICC$.

**Experiments.** We carried out two runs with 10 million iterations. It took about 2min30 for one run (60,000 iterations per second). The value of the estimator of the ICC for the two runs is plotted as a function of the number of iterations in Figure 4. We first see that the estimator converges as expected to the value of the ICC of TS represented by a straight horizontal line (and which was computed exactly in the previous section). We also plotted the estimated standard deviation as a function of the number of iterations. To obtain it, we did one billion iterations. We then estimated the standard deviation $\sigma$, and plotted $\frac{\sigma}{\sqrt{n}}$. We see that large jumps or discontinuity happen, but only at the beginning. They correspond to the draw of a fork with a lot of $K_{22}$s and open $K_{22}$s corresponding to a user who does not have the same ICC as the global network. Then, the convergence is quick. After 300 iterations, the standard deviation is below 10% and after 1000 iterations, we do not experience a value of the runs less precise than 10%.

### 4 RESULTS: CLUSTERING COEFFICIENTS IN TWITTER

To compute the number of $K_{22}$s and open $K_{22}$s, directed triangles, and undirected triangles in the Twitter Snapshot, we used a cluster with a rack of 16 Dell C6420 dual-Xeon 2.20GHz (20 cores), with 192GB RAM, all sharing an NFS Linux partition over Infiniband. It took 51 hours to compute the exact numbers of $K_{22}$s and open $K_{22}$s, corresponding to 265h of cumulative computation times on the cluster. We reported the results in Table 2. If we take off those mutual triangles, both the tcc and the ccc values drop to 0.51% and 0.24%, respectively, while the ICC stays about the same at 3.1%. This tends to confirm the hypothesis that the directed triangle clusterings somehow measure the friendship part of the TS more than the interest part.

We can even go one step further by computing the number of triangles in the graph in which all bidirectional edges have been removed. In that case, the ccc drastically drops to 0 (we found no cyclic triangles without at least a bidirectional arc in the dataset!) while tcc and interest clustering coefficient stay almost the same, 3.6 and 4.2 respectively. This confirms that cyclic triangles are artificially created by friendship relations and that the ccc gives no information about the directed part of the graph.

**Distribution of the ICC and local clustering.** We also provide the distribution of the values of the interest clustering coefficient over all users (having open $K_{22}$s) in Figure 5. We see that the ICC greatly varies between 0 and 1. A large number of nodes have a low value of ICC, e.g., $2.23 \times 10^7$ users (10.2% of the users with open $K_{22}$s) have a value of 0, meaning they are part of open $K_{22}$s but not of $K_{22}$s. At the opposite end, $2.4 \times 10^4$ users (0.011% of the users...
Interest Clustering Coefficient: a New Metric for Directed Networks like Twitter

| #closed | #open | cc  |
|---------|-------|-----|
| icc     | 25,605,832,012,451,571 | 3,138,466,676,914,054,233 | 0.032634831 |
|         | 2.61016 | 3.11018 | 3.3% |
| tcc     | 2,469,018,039,988 | 129,023,573,841,024 | 0.019136178 |
|         | 2.51012 | 1.31014 | 1.9% |
| ccc     | 723,131,368,202 | 129,023,573,841,024 | 0.016813936 |
|         | 7.21011 | 1.31014 | 1.7% |
| ucc     | 623,873,346,660 | 1,631,948,600,661,523 | 0.001146862 |
|         | 6.231011 | 1,631015 | 0.11% |
| mcc     | 317,649,850,664 | 8,924,125,201,234 | 0.106783526 |
|         | 3.21011 | 8.91012 | 10.7% |

Table 1: Clustering coefficients (exact and approximated count) in the TS.

| cc  | icc | tcc | ccc | ucc |
|-----|-----|-----|-----|-----|
|      | 3.1% | 0.51% | 0.24% | 0.057% |

Table 2: Clustering coefficients without the mutual structures.

|        | icc | tcc | ccc | ucc |
|--------|-----|-----|-----|-----|
| icc    | 25,605,832,012,451,571 | 3,138,466,676,914,054,233 | 0.032634831 |
| tcc    | 2,469,018,039,988 | 129,023,573,841,024 | 0.019136178 |
| ccc    | 723,131,368,202 | 129,023,573,841,024 | 0.016813936 |
| ucc    | 623,873,346,660 | 1,631,948,600,661,523 | 0.001146862 |
| mcc    | 317,649,850,664 | 8,924,125,201,234 | 0.106783526 |

The results are quite similar to what we found for Twitter: the icc and tcc are high and of the same order; the ccc is also high because of the bidirectionnal edges (it drastically drops to 0.06% when removing those links). The mcc is the highest value, while the ucc is lower than the others. This confirms that social networks exhibit some common characteristics.

**Flickr**: Flickr is an image and video hosting service, which allows you to follow other people on the platform to see more easily their content. The dataset was collected in 2008 by Mislove et al. [25]. This is once again a graph of followers of a directed social network. The values are similar to the previous one but for the tcc, which is higher. We can notice that Flickr looks more like a social media than Twitter and Instagram, since there is 62% of links implied in bidirectional. This explains why the undirected clustering coefficient is not so different from the mutual one.

**Berkley-Stanford.edu web pages**: The dataset was collected in 2002 by Leskovec et al. [23]. The nodes represent the pages from berkeley.edu and stanford.edu domains and directed edges represent hyperlinks between them. The tcc, icc, and mcc are really high. For the tcc, this is due to the very hierarchical structure of the institution web pages. As an example, a researcher will be linking towards his group, laboratory, and university in its website, while the group website is linking to its laboratory and university... This strong structure translates into a high value of the tcc. As for the icc, research and educational domains form naturally strong communities creating large number of common neighbors for individuals of the same domain, and thus a high icc. Groups/teams/departments also constitutes strong social communities, leading to a high mcc.

**Citations**: Collected by Leskovec et al.[22], it includes all citations made by patents granted between 1975 and 1999. This is a good example of information network, giving a high value of icc of 22.6%, while the tcc value is 9.1%. Indeed, research fields and industry domains are strong communities leading to a high icc. Moreover, it is also not rare to cite a patent and its citations (the patent acting as a survey), explaining the tcc value. Note that there are no cyclic triangles nor bidirectional links, because of the temporal structure of citations - a paper will only cite older papers.
A high value of $\text{icc}$ indicates the presence of clusters of interests such as research communities or interest fields.

- A high value of $\text{tcc}$ is the sign of an important local phenomena of friends’ or acquaintances’ recommendations and/or of a high hierarchical structure in the dataset.

- The $\text{ccc}$ has no real social meaning. If its value can be high in a directed graph, this is only due to the presence of bidirectional arcs and triangles. The closure of a cyclic triangles is very rare in directed networks with no bidirectional edges, confirming the general intuition.

- Directed networks have a high $\text{mcc}$. Indeed, their bidirectional parts (mutual graph) have strong social communities, leading to a high clustering coefficient.

- The $\text{ucc}$ is usually significantly lower, showing that the directed part of the network is better understood using directed clustering coefficients.

- Directed social networks have similar mixes of values of their undirected and directed clustering coefficients, however, with some notable differences, due to their diverse usages and information.

### 6 MODEL WITH ADDITION OF K22S

To model complex networks, a model with a high number of triangles was introduced in [38]. In this section, we introduce a new random graph model in which the number of K22s is higher than classical directed random graphs. The model is based on the model from Bollobás et al. [5] to which we add what we call a K22 event. A K22 event closes an open K22. The principle is that if a user has a common interest with another user, and if this user has another interest, it has an increased chance to be interested and to follow it. We then show that the in-degree and out-degree distributions of the introduced model follow a power law (as many real networks).

Lastly, we exhibit the increase of the interest clustering coefficient of the generated graphs with the probability of a K22 event.

### Takeaways

The following takeaways summarize the variety of informations given by the different clustering coefficients:

#### 6.1 Presentation of the model

We recall here the events defining the classic preferential attachment model of [5] and define the K22 event. We start with an initial graph $G_0 = (V_0, E_0)$. Then, at each time step $t$:

- With a probability $1-p$ (Bollobás et al. event):
  - With a probability $\alpha$, we add a node $u$ and a link leaving this node and reaching an existing node $v$ chosen with a probability proportional to $d_{in}(v) + \delta_{in}$;
  - With a probability $\beta$, we add a node $v$ and a link reaching this node and leaving an existing node $u$ chosen with a probability proportional to $d_{out}(u) + \delta_{out}$;
  - With a probability $1 - \alpha - \beta$, we add an edge between two existing nodes, chosen with probability proportional to $d_{out}(u) + \delta_{out}$ for the leaving node $u$ and $d_{in}(v) + \delta_{in}$ for the reached node $v$.

- With a probability $p$ (K22 event):
  1. We choose a random node (called $u_1$) with a probability proportional to its out-degree $d_{out}(u_1)$;
  2. We pick uniformly at random an out-neighbor of the node $u_1$ (called $v_1$);
  3. We pick uniformly at random an in-neighbor of the node $v_1$ (called $u_2$);
  4. We pick uniformly at random pick an out-neighbor of the node $u_2$ (called $v_2$);
  5. We add a link from $u_1$ to $v_2$.

The idea of the K22 event is to close an open K22; since $u_2$ follows $v_1$ and $v_2$ at the same time, $v_1$ and $v_2$ have a higher probability to be similar, and a person $u_4$ following $v_1$ has a higher chance to be interested in $v_2$. Note that it is possible to introduce multiedges with the K22 events. Indeed, to make the problem tractable, we allow $u_1 = u_2$ in Step 3, or $v_2 = v_1$ in Step 4. In the empirical study, we construct the random graphs with the multiedges and we get rid of them at the end of the constructions. We empirically verify that the multiedges do not impact the results in the end of the section. Indeed, most of them appear for low degree nodes and, thus, they do not affect the tail of the degree distributions.

| Is a Social Network | $N$ | $|E|$ | $|E|/m$ | $|E|/E$ |
|---------------------|-----|------|---------|--------|
| Instagram            | Yes | $4.5 \times 10^4$ | $6.7 \times 10^5$ | 11%    |
| Flickr               | Yes | $2.3 \times 10^6$ | $3.3 \times 10^7$ | 62%    |
| Web (.edu)           | No  | $6.9 \times 10^5$ | $7.6 \times 10^6$ | 25%    |
| Citations            | No  | $3.8 \times 10^6$ | $1.7 \times 10^7$ | 0%     |

Table 3: Datasets information. $N$ is the number of nodes, $|E|$ the number of edges, and $|E|/m$ the fraction of edges implied in a bidirectional link.

| icc | tcc | ccc | mcc | ucc |
|-----|-----|-----|-----|-----|
| Instagram | 12.0% | 15.4% | 3.7% | 22.6% | 4.1% |
| Flickr | 12.4% | 12.2% | 9.3% | 13.9% | 10.8% |
| Web (.edu) | 46.3% | 59.6% | 18.8% | 78.5% | 0.69% |
| Citations | 22.3% | 9.1% | 0% (none) | 6.7% |

Table 4: Clustering coefficients of the directed datasets.
6.2 In-degree and out-degree distributions

We show in what follows that the in- and out-degree distributions of the introduced model follow power-laws, as most real networks. More precisely:

**Theorem 6.1.** The probability \( P(i) \) (resp. \( P(o) \)) for a node to have in-degree \( i \) (resp. out-degree \( o \)) in the new model is:

\[
P(i) \sim i^{-(1+\beta)} \quad \text{and} \quad P(o) \sim o^{-(1+\alpha)},
\]

where \( A = p + \frac{(1-p)(1-\beta)}{1+(1-p)(\alpha+\beta)\delta_{in}} \) and \( B = p + \frac{(1-p)(1-\alpha)}{1+(1-p)(\alpha+\beta)\delta_{out}} \).

**Proof.** We first focus on the in-degree distribution. This result is derived from the equation giving the evolution of the number of nodes of in-degree \( i \) as a function of time, sometimes called Master Equation.

Let \( G(t) = (V(t), E(t)) \) be the graph obtained at time \( t \), and \( N(t) = |V(t)| \). The number of edges at time \( t \) is \( |E(t)| = t + |E| = t \), while the number of nodes is \( N(t) = (1-p)(\alpha+\beta)(t+|V_0|) = (1-p)(\alpha+\beta)t \) when \( t \) is high enough. Hence, the mean in-degree (and out-degree) of the network is \( m = \frac{1}{t(1-p)(\alpha+\beta)} \).

Let us compute the in-degree distribution. Calling \( N(i, t) \) the number of nodes of in-degree \( i \) at time \( t \), we can write the Master Equation:

\[
N(i, t+1) - N(i, t) = (1-p)\alpha\delta_{in,i} + (1-p)\beta\delta_{in,i} + (1-p)(1-\beta)\sum_{i=0}^{\infty} N(i, t)(i + \delta_{in}) - (1-p)(1-\beta)\sum_{i=0}^{\infty} iN(i, t)(i + \delta_{in}) + p\sum_{i=0}^{\infty} N(i, t)i - p\sum_{i=0}^{\infty} iN(i, t)i,
\]

where \( \delta_{in,i} \) is the Kronecker delta.

The Master Equation formulates the variation of the number of nodes with degree \( i \) between time \( i \) and time \( i+1 \). The two first terms on the right-hand side correspond to the addition of a new node, with degree 0 or 1 (depending on if we are in the first or second case of the Bollobás et al. event). The third and fourth terms are the probabilities that, during the Bollobás et al. event, an edge is connected to a node of degree \( i-1 \) or \( i \). This would lead to the arrival of a new node of degree \( i \), or the loss of one of them. Those events occur with probability \((1-p)(\alpha+1-\alpha-\beta)\). Finally, the last two terms correspond to the probability that an edge connects a node of degree \( i-1 \) or \( i \) during the K22 event.

We now show that the probability to connect to a node \( (v_2) \) of a given degree after following an open K22 is proportional to the degree of this node. Indeed, the probability to connect to a node \( (v_2) \) of a given degree after following an open K22 is

\[
P(x = v_2) = \sum_{y \in N^+(v_2)} P(y = u_2) \times \frac{1}{d_{out}(y)},
\]

where \( N^+(v_2) \) is the set of in-neighbors of \( v_2 \), and \( u_2 \) is defined in the model. Using the same reasoning, we have

\[
P(x = u_2) = \sum_{y \in N^-(u_2)} P(y = v_1) \times \frac{1}{d_{in}(y)}
\]

and

\[
P(x = v_1) = \sum_{y \in N^+(v_1)} P(y = u_1) \times \frac{1}{d_{out}(y)}.
\]

Since \( P(y = u_1) = \frac{d_{out}(y)}{t} \), we deduce that

\[
P(x = v_2) = \frac{d_{in}(x)}{t},
\]

which gives us the expected result.

Using this property and knowing that

\[
\sum_{i=0}^{\infty} i \cdot N(i, t) = |E(t)| = t
\]

and

\[
\sum_{i=0}^{\infty} N(i, t)\delta_{in} = \delta_{in} N(t) = (1-p)(\alpha+\beta)\delta_{in},
\]

we can rewrite the equation as:

\[
N(i, t+1) = \alpha\delta_{in,i} + \beta\delta_{out,i} + (1-p)(1-\beta)\sum_{i=0}^{\infty} N(i, t)(i + \delta_{in}) - (1-p)(1-\beta)\sum_{i=0}^{\infty} iN(i, t)(i + \delta_{in}) + p\sum_{i=0}^{\infty} N(i, t)i - p\sum_{i=0}^{\infty} iN(i, t)i
\]

Let us call

\[
Z \equiv 1 + (1-p)(\alpha+\beta)\delta_{in}.
\]

We need the following lemma from [10]:

**Lemma 6.2 ([10]).** If we have an equation of the form:

\[
N(i, t+1) = \left(1 - \frac{b(t)}{t}\right)N(i, t) + g(t)
\]

where \( b(t) \to b \) and \( g(t) \to g \) as \( t \to +\infty \), then

\[
\frac{N(i, t)}{t} \to \frac{g}{b+1}.
\]

Using Lemma 6.2 and calling \( P(i) = \lim_{t \to +\infty} \frac{N(i, t)}{t} \), we have:

\[
P(i) = \sum_{i=0}^{\infty} \frac{(1-p)(1-\beta)}{t+i+2}\sum_{i=0}^{\infty} P(i-1).
\]

Let us call

\[
A \equiv \frac{(1-p)(1-\beta)}{Z} + p.
\]

We thus have:

\[
P(i) = \frac{i+1}{i^2+2\beta i+\beta}P(i-1)
\]

\[
= P(1) \prod_{k=2}^{i} \frac{k-1+\delta_{in}}{k+\frac{\alpha+\beta}{2} + \frac{1}{2}}
\]

\[
= \frac{\Gamma(i+\delta_{in})\Gamma(\frac{1}{2}+\delta_{out})}{\Gamma(\frac{i}{2}+\frac{\alpha+\beta}{2} + \frac{1}{2})}.
\]

Leading to

\[
P(i) \sim i^{-(1+\frac{\beta}{2})},
\]
The out-degree distribution calculation follows the same method. The master equation is the same, except that $\delta_{in}$ and $\beta$ are replaced by $\delta_{out}$ and $\alpha$. The slope of the out-degree distribution is thus:

$$P_{out}(o) \sim o^{-(1 + \frac{1}{\alpha})}, \text{with } B = \frac{(1 - p)(1 - \alpha)}{1 - (1 - p)(\alpha + \beta)\delta_{out}} + p.$$  

Concentration. We have studied here the mean of the distributions. We now use the Azuma’s inequalities to show the concentration around the mean. We have the following result [10]: Let $X_t$ be a martingale with $|X_s - X_{s-1}| \leq c$ for $1 \leq s \leq t$. Then:

$$P(|X_t - X_0| > x) \leq \exp(-x^2/2t).$$

Let $Z(i, t)$ be the number of vertices of degree $i$ at time $t$ and let $F_s$ denote the $\sigma$-field generated by the choices up to time $s$. We apply the result to $X_s = E(Z(i, t)|F_s)$. We have that $|X_s - X_{s-1}| \leq 2$. Indeed, when we add an edge in the network, we affect only the degrees of its two end-vertices. Since $Z(i, 0) = E(Z(i, t))$, using the result with $x = \sqrt{t}\log(t)$, we have

$$P(|Z(i, t) - E(Z(i, t))| > \sqrt{t}\log(t)) \leq t^{-\frac{1}{2}}.$$  

And hence, $Z(i, t) \rightarrow P(i)$ in probability.

The degree distributions of the model follow power-laws, with exponents between $-2$ and $-\infty$. We notice that, for $p = 0$, we recover the exponents of the Bollobás et al. model $-\left(1 + \frac{1 + (\alpha + \beta)\delta_{in}}{1 - p}\right)$ and $\left(1 + \frac{1 + (\alpha + \beta)\delta_{out}}{1 - p}\right)$ [5], while, when $p$ goes to $1$, the exponent goes to $-2$.

Note that, similarly to the Bollobás et al. model, we cannot generate graphs with any wanted mean-degree and fixed slopes of the power-law. Some constraints exist in order to keep $\delta_{in} > 0$ and $\delta_{out} > 0$. For instance, with $\alpha = \beta = 0.4$ and slopes of $-2.5$ (the values of our experiments), $p$ has to stay in the interval $[\frac{1}{2}, \frac{1}{5}]$.

Validation by simulations. We validate the analysis and the hypothesis by simulation. In Figure 6, we present the in- and out-degree distributions of a network built with our new model as an example. The parameters are fixed to $p = 0.5$, $\alpha = \beta = 0.4$, and $\delta_{in} = \delta_{out} = 2.0$. In this case, the expected slopes are $-2.5$. The fit is almost perfect: $-2.509$ and $-2.498$ for the in- and out-degree distributions.

### 6.3 Interest clustering coefficient of the new model

We show by simulation how the ICC increases as $p$ increases. We compare it with the one of the Bollobás et al. model. Note that, when $p$ increases, the average degree of the model increases. Indeed, the mean degree is $m_{new} = \frac{1}{1 - p(\alpha + \beta)}$. To compare networks with the same characteristics (mean degrees and exponents of the in-degree distribution), we adapt the parameters of the second model with the value of $p$.

Since, in the Bollobás et al. model, the mean degree is $m_{Bol} = \frac{1}{\alpha + \beta}$, we can compare the two models by: choosing the values of $\alpha$, $\beta$, and $p$ for our model. This imposes a value of $m$. We then choose $\alpha$, $\beta$ for the Bollobás et al. model, so that the two networks have the same mean degree. Finally, we choose $\delta_{in}$ so that the exponent of the in-degree distribution stays the same in both networks. In practice, we have fixed the exponent to $-2.5$ and imposed $a_{new} = \beta_{new} = 0.4$. We compare the ICC for both models for different values of $p$ and report the results in Figure 7. We used graphs of size $N = 10^7$ nodes and averaged over 10 networks for each point. We see that the ICC varies from $0.036\%$ to $4.4\%$ when $p$ varies from $0.2$ to $0.6$. 

![Figure 6: In- (Top) and out- (Bottom) degree distributions of a network built with the new model. The obtained distribution is given by the blue points; the black crosses represent the logarithmic binning of the distribution (a mean of a given amount of points on a logarithmic scale). The red straight line is the fit of the logarithmic binning; it has slope of $-2.509$ (resp. $-2.498$) for the in- (resp. out-) degree distribution (expected slopes from analysis are $-2.5$).](image1)

![Figure 7: Interest clustering coefficient of our new model as a function of $p$, the probability of a K22 event. The value is compared with the one of the Bollobás et al. model [5].](image2)
We define the We propose to use the K22s defined for our metric to carry out

We are considering a directed graph and are focusing on interest

These last nodes (which were not already followed by

This is due to the fact that for a user with few outgoing links, it is more probable that the followed users are also following at least one other user (providing a TT recommendation) than they are followed by other users (necessary to provide a K22 recommendation). We do not advocate to use only K22 recommendations, but to use it as a complementary tool. In particular, for users with no TT and K22, recommendations would only be made based on global social network statistics (trending topics for example).

However, when a K22 recommendation exists for a user, it has much more strength than the TT recommendations for her. Indeed, 21% of users have TT recommendations of strengths 0 or 1. This number is just 1.2% for K22 recommendations. A recommendation of strength 1 has very good chance to be of no interest, as it is based on the following of a single user over 500 million ones. Similarly, 28% of users only have TT recommendations of strengths 2 or lower (to be compared with 2.5% for K22 recommendations). This means that, for a very large portion of users, TT recommendations are based on very few links. On the contrary, more than 94% of users have a top K22 recommendation with strength more than 10. We are thus able to carry out a meaningful recommendation for the vast majority of users using K22s.

Top 10 recommendations. When considering a recommendation system proposing a top 10, we see that 25% of users have their 10th TT-recommendation of strength 1 or lower, and 35% of strength 2 or lower. There does not exist a significant top 10 list for more than one third of users. On the contrary, 94% of users have their 10th K22-recommendation with strength higher than 10. Top 10 recommendation systems can thus be implemented for most users using K22s. Moreover, the distribution of recommendation strengths is very flat when using TT (a large number of top recommendations have strength 1), see Figure 9. Thus, it is very hard to discriminate between recommended users and to do a meaningful ranking of recommendations. At the opposite end, the distribution usually is steep for K22. It is thus a lot easier to establish a ranking.

Typical users. We present in Figure 9 the strengths of the top 10 recommendations using K22 (Left) and TT (Right) for two typical users. For the first one (Top), it is implicated in around 200 triangles, representing each a potential recommendation. However, the strength of the recommendations is very low, just 1 for all of them. Recommendations for this user would be very bad for two reasons: first, they are based on the choice of only 1 user. Second, if the recommendation system had to propose a top 10, how would it discriminate between the 200 similar potential ones with similar strength. On the contrary, the K22 recommendations have much more strengths: 72 for the 1st and the 2d ones, and 52 for the 10th

We report in Figure 8 histograms of the cumulative distribution over the 1000 random users of the strengths of the recommendation with maximum strength and of the 10th recommendation. The top plots present K22 recommendations while the bottom ones the TT recommendations. The right plots show the complete cumulative distribution in log scale, while the left plots are a zoom on recommendations with weak strengths (≥ 20). Beware that the y-scale of the K22 zoom left plot which is between 0 and 0.1. Notice also the difference in x-scale for the right plots.

Top/Max recommendation. We remark that a small amount of users have TT recommendations and no K22 recommendation. This is due to the fact that for a user with few outgoing links, it is more probable that the followed users are also following at least one other user (providing a TT recommendation) than they are followed by other users (necessary to provide a K22 recommendation). We do not advocate to use only K22 recommendations, but to use it as a complementary tool. In particular, for users with no TT and K22, recommendations would only be made based on global social network statistics (trending topics for example).

We compared the method with classic recommendations using

We tested our method on the Twitter snapshot. We considered a population of 1000 users selected uniformly at random over the full population of Twitter’s users. Note that we excluded users following no one. Indeed, isolated users are not interesting users per se and for this study and they have no TT or K22 recommendations.

For each node, we computed its open K22s (for a node x, we follow all its out-neighbors, then for each out-neighbor, we follow its in-neighbors, then for each in-neighbor, we follow its out-neighbors. These last nodes (which were not already followed by x) are the recommended nodes. We then count how many times a node is recommended. This gives the link strength.

We compared the method with classic recommendations using triangles. For example, on Facebook, it is frequent to have a message such as “8 of your friends know Bob. Do you know Bob?” Connecting with Bob would close 8 open (undirected) triangles. As we are considering a directed graph and are focusing on interest links, we computed recommendations based on transitive triangles, as they have more social sense than cyclic triangles. For a user x, we recommend the out-neighbors of the out-neighbors of x.

Note that there are a lot more open K22s than open triangles in the graph, 3.1 × 10^{18} compared to 1.3 × 10^{14}. We argue in the following that it allows to make more recommendations and most importantly better recommendations.

Figure 8: Cumulative distribution of the max and 10th recommendation strength over 1000 random Twitter’s users for K22 recommendation (Top) and transitive triangle recommendation (Bottom). The left plots are a zoom on recommendations with weak strengths (≥ 20). The right plots present the complete cumulative distribution in log scale. Beware of the y-scale for the K22 zoom left plot.

7 LINK RECOMMENDATION

We propose to use the K22s defined for our metric to carry out link recommendation, as we advocate that the interest clustering coefficient is a good measure of common user interests. For a neighbor, the principle is to recommend links closing open K22s.

We define the strength of a link as the number of open k22s it would close if added to the graph. Links are then recommended by decreasing strengths. Typical recommendation systems propose the strongest link to a user (e.g., Facebook) or a top 10/top 20 list (e.g., Youtube).

We report in Figure 8 histograms of the cumulative distribution over the 1000 random users of the strengths of the recommendation with maximum strength and of the 10th recommendation. The top plots present K22 recommendations while the bottom ones the TT recommendations. The right plots show the complete cumulative distribution in log scale, while the left plots are a zoom on recommendations with weak strengths (≥ 20). Beware that the y-scale of the K22 zoom left plot which is between 0 and 0.1. Notice also the difference in x-scale for the right plots.

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We tested our method on the Twitter snapshot. We considered a population of 1000 users selected uniformly at random over the full population of Twitter’s users. Note that we excluded users following no one. Indeed, isolated users are not interesting users per se and for this study and they have no TT or K22 recommendations.

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Note that there are a lot more open K22s than open triangles in the graph, 3.1 × 10^{18} compared to 1.3 × 10^{14}. We argue in the following that it allows to make more recommendations and most importantly better recommendations.
We then proposed a new model, building random directed networks, to capture the interest phenomena in a directed graph. Indeed, the classical undirected clustering coefficient apprehends the social phenomena that my friends tend to be connected. However, it is not adequate to take into account directed interest links. The interest clustering coefficient is based on the idea that, if two people are following a common neighbor, they have a higher chance to have other common neighbors, since they have at least one interest in common. We computed this new metric on a network known to be at the same time a social and information media, a snapshot of Twitter from 2012 with 505 million users and 23 billion links. The interest clustering coefficient is indeed used as a social and information media, and that the new metric introduced in this paper captures the interest phenomena.

We then proposed a new model, building random directed networks with a high value of K22s, and a new method for link recommendation using K22s. As a future work, we would like to investigate further link recommendation based on the K22 structure defined for the interest clustering coefficient: in particular, it would be interesting to carry out a real-world user case study to investigate if users are more satisfied by such recommendations.

8 CONCLUSION

In this paper, we introduce a new metric, the interest clustering coefficient, to capture the interest phenomena in a directed graph. Indeed, the classical undirected clustering coefficient apprehends the social phenomena that my friends tend to be connected. However, it is not adequate to take into account directed interest links. The interest clustering coefficient is based on the idea that, if two people are following a common neighbor, they have a higher chance to have other common neighbors, since they have at least one interest in common. We computed this new metric on a network known to be at the same time a social and information media, a snapshot of Twitter from 2012 with 505 million users and 23 billion links. The computation was made on the total graph, giving the exact value of the interest clustering coefficient, and using sampling methods. The value of the interest clustering coefficient of Twitter is around 3.3%, higher than (undirected and directed) clustering coefficients introduced in the literature and based on triangles, which we also computed on the snapshot. This consolidates the idea that Twitter is indeed used as a social and information media, and that the new metric introduced in this paper captures the interest phenomena.

We then proposed a new model, building random directed networks with a high value of K22s, and a new method for link recommendation using K22s. As a future work, we would like to investigate further link recommendation based on the K22 structure defined for the interest clustering coefficient: in particular, it would be interesting to carry out a real-world user case study to investigate if users are more satisfied by such recommendations.

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