Some physical applications of generalized Lambert functions

István Mező\textsuperscript{1,3,4} and Grant Keady\textsuperscript{2}

\textsuperscript{1} School of Mathematics & Statistics, Nanjing University of Information Science & Technology, Nanjing, People’s Republic of China
\textsuperscript{2} Department of Mathematics, Curtin University, Perth, Australia

E-mail: istvanmezo81@gmail.com

Received 23 March 2016, revised 17 July 2016
Accepted for publication 10 August 2016
Published 7 September 2016

Abstract
In this paper we show two applications for a generalization of the Lambert $W$ function. Explicit calculations are given for the inverse Langevin function that plays an important role in the study of paramagnetic materials, and for the dispersion equations for water waves. After these examples we provide some additional knowledge on the generalized Lambert function as well as a review of former studies made towards this direction by other authors.

Keywords: Lambert $W$ function, fluid mechanics, inverse Langevin function

1. Introduction

1.1. The definition of the Lambert $W$ function

The solutions of the transcendental equation

$$xe^x = a$$

were studied more than 250 years ago by Euler and by Lambert; see the fundamental paper \cite{12} for historical details. The inverse of the function on the left hand side is called the Lambert function and is denoted by $W$. Hence the solution(s) for (1) is given by $W(a)$. If $-\frac{1}{e} < a < 0$, there are two real solutions, and thus two real branches of $W$ \cite{42}. If we allow complex values of $a$ and complex solutions, we get infinitely many solutions (except when $a = 0$), and $W$ has infinitely many complex branches \cite{12–14, 22}. These questions are

\textsuperscript{3} Author to whom any correspondence should be addressed.

\textsuperscript{4} The research of István Mező was supported by the Scientific Research Foundation of Nanjing University of Information Science and Technology, the Startup Foundation for Introducing Talent of NUIST, Project no.: SS113062001, and the National Natural Science Foundation for China. Grant no. 11501299.
comprehensively discussed by Corless et al [12]. The question why we use the letter ‘W’ is discussed by Hayes [17]; see also [14].

1.2. Applications of W

The survey paper of Corless et al [12] describes a large number of applications of W: this function appears in the combinatorial enumeration of trees, in the jet fuel problem, in enzyme kinetics, or in the solution of delay differential equations, just to mention a few areas (see [12] for the citations, further examples and deeper explanations). Some specific applications of W in the study of solar wind comes from Cranmer [15], some applications in electromagnetic behavior of materials are given by Houari [19], other appearances in electromagnetics are investigated by Jenn [23]. It is known that Wien’s displacement constant can be expressed by W, and in the discussion of capacitor fields W also appears [41]. The Lambert function has applications in quantum statistics, [40], and in the description of projectile motions [2, 8], too. W is also useful in the determination of the critical thickness of crystalline overlayers on thick substrates [5, 6] and also in the nucleation of misfit dislocations [4].

A simple mathematical application connects W to the distribution of primes via the Prime Number theorem [43].

Although the Lambert function has proven to be very useful, still there are some areas where it is insufficient. In what follows we provide two physical examples—one from modern physics, the other from classical hydrodynamics—where some generalized Lambert function is necessary. Then we review the properties of this new function, and provide some more examples from the recent literature.

2. The generalized Lambert function

2.1. Definition

For the remainder of this paper, all parameters will be assumed to be real, and our concern will be real-valued functions of real variables.

In 2006, motivated by physical applications, T. C. Scott [34–37] and his co-workers defined the generalized W function as the solution(s) of the equation

\[ e^{x(x-t_1)(x-t_2)\cdots(x-t_n)}(x-s_1)(x-s_2)\cdots(x-s_m) = a. \]  

(2)

In [29], where some particular cases of this generalization were studied in great detail mathematically, this generalized Lambert function is denoted by

\[ W(t_1, t_2, \ldots, t_n; s_1, s_2, \ldots, s_m; a). \]

So, in particular

\[ W(a) = \log(a), \quad W(0; a) = W(a), \quad W(1; a) = -W\left(-\frac{1}{a}\right) \]

\[ W(t; a) = t + W(ae^{-t}), \quad W(s; a) = s - W\left(-\frac{e^s}{a}\right). \]

These can be verified easily by the reader.

We now describe various applications requiring the generalized Lambert function.
2.2. Applications

2.2.1. The inverse Langevin function. The Langevin function $L$ plays an important role in the study of paramagnetic materials and of polymers like rubber. More precisely, $L$ describes the magnetization of a paramagnet under the presence of outer classical (i.e., non quantical) magnetic field [24]. The expression of $L$ is

$$L(x) = \coth(x) - \frac{1}{x}.$$ 

The inverse of $L$ is widely studied, and good methods of approximating it are available. A comprehensive article on the approximations of $L^{-1}$ is [20].

We point out here that the inverse Langevin function $L^{-1}$ is also a special case of the generalized Lambert function. As $\coth(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$, the transcendental equation $L(x) = a$ can be rewritten as

$$L(x) = \frac{e^{2x} + 1}{e^{2x} - 1} - \frac{1}{x} = a.$$ 

Now we are going to express the solution(s) of this equation in terms of the generalized $W$ function. First let us note that the equation is equivalent to

$$(e^{2x} + 1) - \left(\frac{1}{x} + a\right)(e^{2x} - 1) = 0.$$ 

Then

$$e^{2x}\left[1 - \left(\frac{1}{x} + a\right)\right] = -1 + \left(\frac{1}{x} + a\right),$$ 

from where it easily comes that

$$e^{2x} = \frac{x(1 + a) + 1}{x(a - 1) + 1}.$$ 

In order to have no factor in the exponential we substitute $y = -x/2$ in place of $x$ so that

$$e^{-y} = \frac{-y/2(1 + a) + 1}{-y/2(a - 1) + 1}.$$ 

Multiply the right hand side with $-2$

$$e^{-y} = \frac{y(1 + a) - 2}{y(a - 1) - 2}.$$ 

Then we put the right hand side into a more suitable form to be able to use (2)

$$e^{-y} = \frac{a + 1}{a - 1} \cdot \frac{y - \frac{2}{a + 1}}{y - \frac{2}{a - 1}},$$ 

from there it follows that

$$\frac{a - 1}{a + 1} = e^{y - \frac{2}{a + 1}}, \quad \frac{y - \frac{2}{a - 1}}{a - 1} = e^{y - \frac{2}{a - 1}}.$$ \quad (3)
Its solution is then given by
\[ y = W \left( \frac{2}{a+1} \frac{a-1}{a+1} \right). \]

Since \( x = -2y \), we finally get that the inverse Langevin function is given as
\[ L^{-1}(a) = -2W \left( \frac{2}{a+1} \frac{a-1}{a+1} \right). \]

2.3. Applications to water waves

Consider unidirectional periodic water waves propagating over a horizontal bottom. Let \( \omega \) denote the wave’s frequency, \( k \) its wave number, \( g \) the acceleration due to gravity and \( h \) the mean water depth. The phase speed \( c \) is defined as \( \omega/k \). There are two distinct applications.

In the first, section 2.3.1, \( g, h \) and the frequency \( \omega \) are given and one wishes to determine the wave number \( k \). This arises, e.g. in connection with computations using the ‘mild slope equation’. The numerical approximations for doing this are satisfactory, but it seems worth noting that exact inversion is possible here, and in more general settings.

In the second, section 2.3.2, \( g, h \) and the phase speed \( c \) are given and one wishes to determine the wave number \( k \). This arises when determining the wave length in the steady wave train set up behind an obstacle towed at speed \( c \).

2.3.1. Inverting dispersion equations for water waves. The dispersion relation for periodic water waves propagating into still water is
\[ \omega^2 = gk \tanh(kh). \]

With \( x = kh \) and \( y = \omega^2 h/g \) the preceding dispersion relation is \( y = x \tanh(x) \). As \( x \tanh(x) \) is increasing in \( x \) for \( x > 0 \), for each \( y > 0 \) there is a unique \( x > 0 \) solving this equation. Some simple bounds on the positive solution \( x(y) \) are immediate. As \( x \tanh(x) < \min(x, x^2) \), \( x > \max(y, \sqrt{y}) \). The equation rewrites to
\[ e^{2x} = \frac{x+y}{x-y}. \quad (4) \]

and by the same steps as for the inverse Langevin function one can easily see that the solution is
\[ x = \frac{1}{2} W \left( \frac{2y}{-2y}, 1 \right). \quad (5) \]

Waves in flows with two layers of different densities have been studied and in [30, p 421]
\[ \omega^2 = gk \frac{\rho_1 - \rho_2}{\rho_1 \coth(kh) + \rho_2}. \]

Using the same definitions of \( x \) and \( y \) as in the preceding paragraph the equation above becomes
which rearranges to

$$e^{2x} = \frac{x + y}{x - \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} y}.$$  

This becomes equation (4) when $\rho_2 = 0$. Once again one can find an expression for $x(y)$ in terms of generalized Lambert functions.

2.3.2. Inverting the phase speed for water waves. The phase speed $c$ for periodic water waves satisfies

$$c^2 = \frac{g}{k} \tanh(kh).$$

With $x = kh$ and $\chi = c^2 g/gh$ the preceding equation is $\chi = \frac{\tanh(x)}{x}$. The equation rewrites to

$$e^{2x} = \frac{x + \frac{1}{\chi}}{x - \frac{1}{\chi}}$$

and its solution is

$$x = \frac{1}{2} W\left(\frac{2/\chi}{-2/\chi}; -1\right).$$  

2.4. A simple functional relation for these generalized Lambert $W$ functions

Since $\chi = y/x^2$ equations (5) and (7) rewrite to

$$x = \frac{1}{2} W\left(\frac{2y}{-2y}; 1\right) = \frac{1}{2} W\left(\frac{2x^2/y}{-2x^2/y}; -1\right).$$

On eliminating $x$ this leads to the curious functional equation

$$W\left(\frac{2y}{-2y}; 1\right) = W\left(\frac{1}{2y} W\left(\frac{2y}{-2y}; 1\right)^2; -1\right).$$

We suggest that there are many functional relations—and other properties—pertaining to generalized Lambert functions remaining to be found.

3. Further remarks on the generalized Lambert function

We now show an interesting variant of the $W\left(t; a\right)$ function.

Because of some combinatorial motivation the first author [29] studied the solution of the equation
This can be rewritten as

\[-r = e^r \frac{x}{x - \frac{n}{r}}\] (9)

from where it is apparent that the solution is

\[W_r(n) := W\left(0 \mid \frac{n}{r}; -r\right).\] (10)

This particular (generally multivalued) function is called \(r\)-Lambert function by the authors in [29] and was denoted by \(W_r(n)\).

Depending on the parameter \(r\), the \(r\)-Lambert function has one, two or three real branches and so the above equations can have one, two or three solutions (we restrict our investigation to the real line). For the details see [29].

Since the \(r\)-Lambert function is expressible as

\[W_r(n) = W\left(0 \mid \frac{n}{r}; -r\right)\]

and here the upper parameter is simply zero, we might think that the \(r\)-Lambert function is just a particular case of \(W(t; a)\). But no, it is in fact equivalent to it. Indeed, the reader is challenged to show that the upper parameter in \(W(t; a)\) can always be canceled by the identity

\[W\left(t; a\right) = t + W\left(0 \mid s-t; ae^{-t}\right).\]

Now equating \(n/r = s - t\) and \(-r = ae^{-t}\) and taking (10) into account it comes that

\[W\left(t; a\right) = t + W\left(0 \mid s-t; ae^{-t}\right)\]

Hence the \(r\)-Lambert function and the \(W(t; a)\) functions are interchangeable. This is extremely useful, because the defining equation of \(W(t; a)\) has a singularity around \(x = s\), and the usual numerical equation solving method can hardly be applied around singularities. In contrary, the equation (8) does not contain singularities at all.

By this interchangeability we can confirm that the complexities of the equations \(xe^x + rx = n\) and, for example, \(\coth(x) - \frac{1}{x} = a\) are equivalent. This is apparent from the fact that the \(L^{-1}\) inverse Langevin function can be expressed via the \(W(t; a)\) Lambert function, and it can be expressed by the \(r\)-Lambert function.

We inform the interested reader that in [29] one can find a host of mathematical properties of the \(r\)-Lambert and \(W(t; a)\) functions.

### 3.1. The generalized Lambert function in the literature

In this short section we review the some known applications and appearances of the generalized Lambert \(W\) function. Each of these examples could be made accessible to sufficiently advanced undergraduate physics students but to keep this paper short we leave the cited papers as the main sources.

**A problem in molecular physics.** Corless and his co-workers [12] mentioned that there was an anomaly in molecular physics: when physicists tried to calculate the eigenstates and
eigenenergies of the hydrogen molecular ion \( (H_2^+) \), the experimental results and the mathematical predictions were not matching \cite{16}. In the mathematical formulation of the problem the physicists tried to solve the Schrödinger equation with a double well Dirac delta potential with radius \( R \). To make the resulting wave function continuous on each well, they had to solve a transcendental equation with respect to the parameters of the system. The problem was that—being unable to solve this equation—the physicists used numerical approximations which were inadequate, especially for small \( R \).

This problem originally emerged in 1956 and was solved by Scott and his coworkers \cite{34} just in 1993. They correctly realized that the Lambert function must be applied in the solution. When \( R \) is small, the previously applied approximations were inadequate but using the correct approximations of \( W \) helped to take exponentially subdominant terms into account in the solution which could explain the anomaly.

If one were to solve a similar eigenvalue problem for ions with unequal charges, \( W \) is no longer sufficient. Instead, one must solve a transcendental equation of the form \cite{34}
\[
e^{-ct} = a_0(x - t_1)(x - t_2).
\]

Here the \( t_1, t_2 \) and \( c \) constants involve physical parameters of the bimolecular system like the atomic numbers and the width \( R \) of the applied interaction potential.

The solutions, as it is easy to see, can be given by the generalized Lambert function
\[
cW \left( c_1, c_2; \frac{c^2}{a_0} \right).
\]

A problem in general relativity. 'One of the oldest and most notoriously vexing problems in gravitational theory is that of determining the (self-consistent) motion of \( N \) bodies and the resultant metric they collectively produce under their mutual gravitational influence...' \cite{27}. Mann and Ohta were considering the equations of motion of two bodies of equal mass in one spatial dimension (to make the field equations tractable and eliminate gravitational waves, this dimensional restriction is very useful). They completely describe the Hamiltonian of the two body system in terms of the Lambert \( W \) function. Also, the explicit trajectory is determined and illustrated in their paper.

When the masses are unequal, one needs to solve a more general equation where the generalized Lambert functions can help (see sections 5 and 6 in \cite{27}). This shows some similarity with the above mentioned molecular physics problem when handling unequal charges lead to a generalization of the Lambert function.

Delay differential equations: The generalized Lambert function helps in the solution of linear constant-coefficient delay differential equations (and stability analysis for delay differential equations).

Consider linear constant-coefficient delay differential equations of the form
\[
\sum_{k=0}^m a_k u^{(k)}(t) = \sum_{k=0}^m b_k u^{(k)}(t - \tau).
\]

Seeking solutions of the form \( u(t) = \exp(\lambda t) \) leads to the characteristic equation
\[
\sum_{k=0}^n a_k \lambda^k = e^{-\lambda \tau} \sum_{k=0}^m b_k \lambda^k.
\]
Next, suppose the polynomials are factorized to give
\[ a_n \prod_{k=1}^{n} (\lambda - t_k) = b_m e^{-\lambda \tau} \prod_{k=1}^{m} (\lambda - s_k). \]
This is an equation solvable by the generalized Lambert W function.

To show an application, we cite S A Campbell: ‘second-order delay differential equations arise in a variety of mechanical, or neuro-mechanical systems in which inertia plays an important role. Many of these systems are regulated by feedback which depends on the state and/or the derivative of the state. In this case the model equations take the form
\[ \ddot{u}(t) + bu(t) + au(t) = f(u(t - \tau), \dot{u}(t - \tau)), \]
where \( a, b \) are positive constants representing physical attributes of the system, \( \tau \) is the time delay, \( u, u(t - \tau) \) are the values of the regulated variable evaluated at, respectively, times \( t \) and \( t - \tau \) and the function, \( f(x, y) \), describes the feedback’ [9]. Linearizing about an equilibrium solution gives a linear constant-coefficient delay differential equations (with \( n = 2, a_2 = 1 \) and \( m = 1 \) in the above) whose characteristic equation is
\[ (\lambda - t_1)(\lambda - t_2) = b_1 e^{-\lambda \tau}(\lambda - s_1). \]

Another application arises in the paper of Chambers [10]. He considered some differential-difference equations with characteristic equations solvable by the generalized Lambert function with one upper and one lower parameter [10], equation (2.1b).

An application in connection with Bose–Fermi mixtures. There is an interesting application in statistical physics where the generalized Lambert function in its full generality is needed. This is the theory of Bose–Fermi mixtures. This theory is too advanced even to write down briefly, but the reader can find the details in [34].

Conclusion

In this paper we gave two new physical applications of the generalized Lambert function which recently emerges in the mathematical and physical literature. After these applications we briefly reviewed the literature where additional applications can be found.

The author wrote C code to calculate the real \( r \)-Lambert function on all the real branches. This can be downloaded from https://sites.google.com/site/istvanmezo81/.

Acknowledgements

The authors are grateful to the referees for their comments which have led to improvements in the exposition in the paper.

References

[1] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (New York: Dover)
[2] Belgacem C H 2014 Range and flight time of quadratic resisted projectile motion using the Lambert W function Eur. J. Phys. 35 055025
[3] Belgacem C H 2016 Analysis of projectile motion with quadratic air resistance from a nonzero height using the Lambert W function J. Taibah Univ. Sci. at press (doi:10.1016/j.jtusci.2016.02.009)
Belgacem C H 2016 Explicit solution for critical thickness of semicircular misfit dislocation loops in strained semiconductors heterostructures *Silicon* 8 397–9

Belgacem C H and Fnaiech M 2010 Exact analytical solution for the critical layer thickness of a lattice-mismatched heteroepitaxial layer *J. Electron. Mater.* 39 2248–50

Belgacem C H and Fnaiech M 2011 Solution for the critical thickness models of dislocation generation in epitaxial thin films using the Lambert W function *J. Mater. Sci.* 46 1913–5

Bell W W 1968 *Special Functions for Scientists and Engineers* (London: Van Nostrand Company)

Bernardo R C, Esguerra J P, Vallejos J D and Canda J J 2015 Wind-influenced projectile motion *Eur. J. Phys.* 36 025016

Campbell S A 1999 Stability and bifurcation in the harmonic oscillator with multiple, delayed feedback loops *Dyn. Contin. Discrete Impuls. Syst.* 5 225–35

Chambers G Li 1989 Solutions of the neutral differential-difference equation \( ax'(t) + b \delta x(t - r) + \gamma x(t) + \delta x(t - r) = f(t) \) *Int. J. Math. Math. Sci.* 15 773–80

Corcino C B and Corcino R B 2013 An asymptotic formula for \( r \)-Bell numbers with real arguments *ISRN Discrete Math.* 2013 7

Corless R M, Donnet G H, Hare D E G and Knuth D E 1996 On the Lambert W function *Adv. Comput. Math.* 5 329–59

Corless R M and Jeffrey D J 1996 The unwinding number *SIGSAM Bull.* 30 28–35

Corless R M, Jeffrey D J and Knuth D E 1997 A sequence of series for the Lambert W function *ISSAC ’97 Proc. 1997 Int. Sym. on Symbolic and Algebraic Computation* pp 197–204

Cranmer S R 2004 New views of the solar wind with the Lambert W function *Am. J. Phys.* 72 1397–403

Frost A A 1956 Delta-function model: I. Electronic energies of hydrogen-like atoms and diatomic molecules *J. Chem. Phys.* 25 1150

Hayes B 2005 Why W? *Am. Sci.* 93 104–8

Hoofar A and Hassan M 2008 Inequalities on the Lambert W function and hyperpower function *J. Inequal Pure Appl. Math.* 9 51

Houari A 2013 Additional applications of the Lambert W function in physics *Eur. J. Phys.* 34 695–702

Jedynak R Approximation of the inverse Langevin function revisited *Rheological Acta* 54 29–39

Jeffrey D J, Corless R M, Hare D E G and Knuth D E 1995 Sur l’inversion de \( y^\delta e^y \) au moyen de nombres de stirling associés *C. R. Acad. Sci., Paris* I 320 1449–52

Jeffrey D J, Hare D E G and Corless R M 1996 Unwinding the branches of the Lambert W function *Math. Sci.* 21 1–7

Jenn D C 2002 Applications of the Lambert W function in electromagnetics *IEEE Antennas Propag. Mag.* 44 139–42

Kittel C 2004 *Introduction to Solid State Physics* (New York: Wiley)

Krall H L and Frink O 1949 A new class of orthogonal polynomials: the Bessel polynomials *Trans. Am. Math. Soc.* 65 100–15

Kruchinin V 2011 Derivation of Bell polynomials of the second kind arXiv:1104.5065v1

Mann R B and Ohta T 1997 Exact solution for the metric and the motion of two bodies in \( (1 + 1) \)-dimensional gravity *Phys. Rev. D* 55 4723–27

Mező I 2013 The r-Bell numbers *J. Integer Seq.* 14 Article 11.1.1

Mező I and Baricz Á 2016 On the generalization of the Lambert W function *Trans. Am. Math. Soc.* at press (available online (under a different title) at *http://arxiv.org/abs/1408.3999*)

Milne-Thomson L M 1953 *Theoretical Hydrodynamics* (London: MacMillan)

Moser L and Wyman M 1955 An asymptotic formula for the Bell numbers *Trans. R. Soc. Can., Section III* 49 49–54

Mugnaini G 2015 Generalization of Lambert W-function, Bessel polynomials and transcendental equations arXiv:1501.00138v2

Scott T C, Aubert-Frécon M and Grotendorst J 2006 New approach for the electronic energies of the hydrogen molecular ion *Chem. Phys.* 324 323–38

Scott T C, Babb J F, Daugarno A and Morgan J D III 1993 *J. Chem. Phys.* 99 2841–54

Scott T C, Fee G and Grotendorst J 2013 Asymptotic series of generalized Lambert W function *ACM Commun. Comput. Algebra* 47 75–83

Scott T C, Fee G, Grotendorst J and Zhang W Z 2014 Numerics of the generalized Lambert W function *ACM Commun. Comput. Algebra* 48 42–56
[37] Scott T C, Mann R and Martinez R E 2006 General relativity and quantum mechanics: towards a
generalization of the Lambert W function Appl. Algebra Eng. Commun. Comput. 17 41–7
[38] Stewart S M 2009 On certain inequalities involving the Lambert W function J. Inequal Pure Appl.
Math. 10 96
[39] Szegő G 1975 Orthogonal Polynomials 4th edn (Providence, RI: American Mathematical Society)
[40] Valluri S R, Gil M, Jeffrey D J and Basu S 2009 The Lambert W function and quantum statistics
J. Math. Phys. 50 11
[41] Valluri S R, Jeffrey D J and Corless R M Some applications of the Lambert W function to physics
Can. J. Phys. 78 823–31
[42] Veberič D 2010 Having fun with the Lambert W(x) arxiv:1003.1628
[43] Visser M 2013 Primes and the Lambert W function arXiv:1311.2324v1
[44] Wilf H S 1994 Generatingfunctionology (New York: Academic)