Abstract. The analysis of classical consensus algorithms relies on contraction properties of adjoints of Markov operators, with respect to Hilbert’s projective metric or to a related family of seminorms (Hopf’s oscillation or Hilbert’s seminorm). We generalize these properties to abstract consensus operators over normal cones, which include the unital completely positive maps (Kraus operators) arising in quantum information theory. In particular, we show that the contraction rate of such operators, with respect to the Hopf oscillation seminorm, is given by an analogue of Dobrushin’s ergodicity coefficient. We derive from this result a characterization of the contraction rate of a non-linear flow, with respect to Hopf’s oscillation seminorm and to Hilbert’s projective metric.

1. Introduction

1.1. Motivation: from Birkhoff’s theorem to consensus dynamics. The Hilbert projective metric $d_H$ on the interior of a (closed, convex, and pointed) cone $C$ in a Banach space $X$ can be defined by:

$$d_H(x, y) := \log \inf \{ \frac{\beta}{\alpha} : \alpha, \beta > 0, \alpha x \leq y \leq \beta x \},$$

where $\leq$ is the partial order induced by $C$, so that $x \leq y$ if $y - x \in C$. Birkhoff [Bir57] characterized the contraction ratio with respect to $d_H$ of a linear map $T$ preserving the interior $C^0$ of the cone $C$,

$$\sup_{x,y \in C^0} \frac{d_H(Tx, Ty)}{d_H(x, y)} = \tanh \left( \frac{\text{diam } T(C^0)}{4} \right), \quad \text{diam } T(C^0) := \sup_{x,y \in C^0} d_H(Tx, Ty).$$

This fundamental result, which implies that a linear map sending the cone $C$ into its interior is a strict contraction in Hilbert’s metric, can be used to derive the Perron-Frobenius theorem from the Banach contraction mapping theorem, see [Bus73, KP82, EN95] for more information.

Hilbert’s projective metric is related to the following family of seminorms. To any point $e \in C^0$ is associated the seminorm

$$x \mapsto \omega(x/e) := \inf \{ \beta - \alpha : \alpha e \leq x \leq \beta e \},$$

which is sometimes called Hopf’s oscillation [Hop63, Bus73] or Hilbert’s seminorm [GG04]. Nussbaum [Nus94] showed that $d_H$ is precisely the weak Finsler metric obtained when taking $\omega(\cdot/e)$ to be the infinitesimal distance at point $e$. In other words,

$$d_H(x, y) = \inf_{\gamma} \int_0^1 \omega(\gamma(s)/\gamma(s))ds$$

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where the infimum is taken over piecewise \( C^1 \) paths \( \gamma : [0, 1] \to C^0 \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). He deduced that the contraction ratio, with respect to Hilbert’s projective metric, of a non linear map \( f : C^0 \to C^0 \) that is positively homogeneous of degree 1 (i.e. \( f(\lambda x) = \lambda f(x) \) for all \( \lambda > 0 \)), can be expressed in terms of the Lipschitz constants of the linear maps \( Df(x) \) with respect to a family of Hopf’s oscillation seminorms:

\[
(1) \quad \sup_{x,y \in U} \frac{d_H(f(x), f(y))}{d_H(x, y)} = \sup_{x \in U} \frac{\omega(Df(x)z/f(x))}{\sup_{z \in X, \omega(z/x) \neq 0} \omega(z/x)}.
\]

Hence, to arrive at an explicit formula for the contraction rate in Hilbert’s projective metric of non-linear maps, a basic issue is to determine the Lipschitz constant \( \kappa(T, e) \) of linear map \( T \) with respect to Hopf’s oscillation seminorm, i.e.,

\[
(2) \quad \kappa(T, e) := \sup_{z \in X, \omega(z/e) \neq 0} \frac{\omega(T(z)/T(e))}{\omega(z/e)}.
\]

The problem of computing the contraction rate \( \kappa(T, e) \) also arises in the study of consensus algorithms. A consensus operator is a linear map \( T \) which preserves the positive cone \( C \) and fixes a unit element \( e \in C^0 \): \( T(e) = e \). A discrete time consensus system can be described by

\[
(3) \quad x_{k+1} = T_{k+1}(x_k), \quad k \in \mathbb{N},
\]

where \( T_1, T_2, \ldots \) is a sequence of consensus operators. This model includes in particular the case in which \( X = \mathbb{R}^n \), \( C = \mathbb{R}^n_+ \), \( e = (1, \ldots, 1)^\top \) and \( T_k(x) = T(x) := Ax \), for all \( k \), where \( A \) is a stochastic matrix. This has been studied in the field of communication networks, control theory and parallel computation \cite{hir89, bts99, bgps06, mor05, vja05, ot09, ab09}. Consensus operators also arise in non-linear potential theory \cite{del03}. Other interesting consensus operators are the unital completely positive maps acting on the cone of positive semidefinite matrices, corresponding to quantum channel maps \cite{ssr10, rkw11}. The term noncommutative consensus is coined in \cite{ssr10} for the corresponding class of dynamical systems.

The main concern of consensus theory is the convergence of the orbit \( x_k \) to a consensus state, which is nothing but a scalar multiple of the unit element. When \( X = \mathbb{R}^n \), \( C = \mathbb{R}^n_+ \) and \( e = (1, \ldots, 1)^\top \), a widely used Lyapunov function for the consensus dynamics, first considered by Tsitsiklis (see \cite{tba86}), is the “diameter” of the state \( x \) defined as

\[
\Delta(x) = \max_{1 \leq i,j \leq n} (x_i - x_j),
\]

which is precisely Hopf’s oscillation seminorm \( \omega(x/e) \). It turns out that the latter seminorm can still be considered as a Lyapunov function for a consensus operator \( T \), with respect to an arbitrary cone. When \( C = \mathbb{R}^n_+ \), it is well known that if the contraction ratio of \( T \) with respect to the Hopf oscillation seminorm is strictly less than one, and if \( T_k = T \) for all \( k \), then, the orbits of the consensus dynamics converge exponentially to a consensus state. We shall see here that the same remains true in general (Theorem 4.7). For time-dependent consensus systems, a common approach is to bound the contraction ratio of every product of \( p \) consecutive operators \( T_{i+p} \circ \cdots \circ T_{i+1}, i = 1, 2, \ldots \), for a fixed \( p \), see for example \cite{mor05}. Moreover, if \( \{T_k : k \geq 1\} \) is a stationary ergodic random process, then the almost sure convergence of the orbits of \( (3) \) to a consensus state can be deduced by showing that \( \mathbb{E}[\log \|T_{i+p} \cdots T_1\|_H] < 0 \) for some \( p > 0 \), see Bougerol \cite{bou93}. Hence, in consensus applications, a central issue is again to compute the contraction ratio \( \kappa(T, e) \).
1.2. Main results. Our first result characterizes the contraction ratio \([2]\), in a slightly more general setting. We consider a bounded linear map \(T\) from a Banach space \(X_1\) to a Banach space \(X_2\). The latter are equipped with normal cones \(C_i \subset X_i\), and unit elements \(e_i \in C_i^0\).

**Theorem 1.1** (Contraction rate in Hopf’s oscillation seminorm). Let \(T : X_1 \to X_2\) be a bounded linear map such that \(T(e_1) \in \mathbb{R} e_2\). Then

\[
\sup_{z \in X_1, \omega(z/e_1) \neq 0} \frac{\omega(T(z)/e_2)}{\omega(z/e_1)} = \frac{1}{2} \sup_{\nu, \pi \in \text{extr} P(e_2)} \| T^*(\nu) - T^*(\pi) \|_{\ell_1}^* = \sup_{\nu, \pi \in \text{extr} P(e_2)} \sup_{x \in [0, e_1]} \langle \nu - \pi, T(x) \rangle.
\]

The notation and notions used in this theorem are detailed in Section 3. In particular, we denote by the same symbol \(\leq\) the order relations induced by the two cones \(C_i\), \(i = 1, 2\); \(P(e_2) = \{\mu \in C_2^* : \langle \mu, e_2 \rangle = 1\}\) denotes the abstract simplex of the dual Banach space \(X_2^*\) of \(X_2\), where \(C_2^* := \{\mu \in X_2^* : \langle \mu, x \rangle \geq 0, \forall x \in C_2\}\) is the dual cone of \(C_2\); \(\text{extr}\) denotes the extreme points of a set; \(\perp\) denotes a certain disjointness relation, which will be seen to generalize the condition that two measures have disjoint supports; \([u, v] := \{x \in X_1 : u \leq x \leq v\}\), for all \(u, v \in X_1\), and \(T^*\) denotes the adjoint of \(T\). We shall make use of the following norm, which we call Thompson’s norm,

\[
\|z\|_T = \inf\{\alpha > 0 : -\alpha e_1 \leq z \leq \alpha e_1\}
\]
on the space \(X_1\), and denote by \(\| \cdot \|_T^*\) the dual norm.

When \(C = \mathbb{R}^n_+\), and \(T(z) = Az\) for some stochastic matrix \(A\), we shall see that the second supremum in Theorem 1.1 is simply

\[
\frac{1}{2} \max_{i < j} \sum_{1 \leq k \leq n} |A_{ik} - A_{jk}| = \frac{1}{2} \max_{i < j} \| A_i - A_j \|_{\ell_1},
\]
where \(A_i\) denotes the \(i\)th row of the matrix \(A\). This quantity is called Doeblin contraction coefficient in the theory of Markov chains; it is known to determine the contraction rate of the adjoint \(T^*\) with respect to the \(\ell_1\) (or total variation) metric, see [LPW09]. Moreover, the last supremum in Theorem 1.1 can be rewritten more explicitly as

\[
1 - \min_{i < j} \sum_{s=1}^n \min(A_{is}, A_{js}),
\]
a term which is known as Dobrushin’s ergodicity coefficient [Dob56]. Note that in general, the norm \(\| \cdot \|_T^*\) can be thought of as an abstract version of the \(\ell_1\) or total variation norm.

When specializing to a unital completely positive map \(T\) on the cone of positive semidefinite matrices, representing a quantum channel [SSR10, RKW11], we shall see that the last supremum in Theorem 1.1 coincides with the following expression, which provides a noncommutative analogue of Dobrushin’s ergodicity coefficient (see Corollary 1.6):

\[
1 - \min_{X = \{x_1, \ldots, x_n\}} \min_{u, v, u^*v = 0} \sum_{i=1}^n \min\{u^*T(x_i x_i^*) u, v^*T(x_i x_i^*) v\}
\]

Theorem 1.1 shows in particular, when \(C = \mathbb{R}_+^n\), that the contraction rate of \(T\) with respect to Hopf’s oscillation seminorm is the same as the contraction rate of \(T^*\) with respect to the \(\ell_1\) norm, given by the classical formulas of Doeblin and Dobrushin.

Theorem 1.1 can be thought of as the dual of a result of Reeb, Kastoryano, and Wolf [RKW11, Prop. 12], who gave a closely related formula, without the disjointness restriction (and assuming that the dimension is finite) for the contraction rate of \(T^*\) with respect to a certain “base norm”, which is the dual of Thompson’s norm. Thus,
Theorem 1.1 characterizes the contraction rate of $T$, whereas Proposition 12 of \cite{RKW11} characterizes the contraction rate of the adjoint $T^*$. We shall derive here the equality of both contraction rates from general duality considerations, exploiting the observation that Hopf’s oscillation seminorm coincides with the quotient norm of Thompson’s norm (Lemma 2.7). Then, we deduce Theorem 1.1 from a characterization of the extreme points of the unit ball in the dual space of the quotient normed space (Theorem 2.12). The duality between both approaches is discussed more precisely in Remarks 2.6 and 3.3.

Then, we derive analogous results for flows. In particular, some consensus systems are driven by non-linear ordinary differential equations \cite{Str00, SM03}:

$$\dot{x} = \phi(x)$$

where $\phi(x + \lambda e) = \phi(x)$ for all $\lambda \in \mathbb{R}$. The subclass of maps $\phi$ that yield an order preserving flow is of interest in non-linear potential theory. In this context, the opposite of the map $\phi$ has been called a derivator by Dellacherie \cite{Del03}.

For such systems, it is interesting to consider the contraction rate of the flow with respect to Hopf’s oscillation seminorm. In particular, if the contraction rate is negative, we deduce an exponential convergence of the orbits of the system to a consensus state. For simplicity, we only consider here a flow on finite dimensional space (then, a -closed, convex, and pointed- cone is automatically normal).

**Theorem 1.2 (Contraction rate of flows with respect to Hopf’s oscillation).** The contraction rate $\alpha(U)$ with respect to Hopf’s oscillation seminorm, of the flow of the differential equation $\dot{x} = \phi(x)$, restricted to a convex open subset $U \subset \mathcal{X}$, is given by:

$$\alpha(U) = \sup_{x \in U} h(D\phi(x))$$

Here $h(L)$ is defined to be the contraction rate in Hopf’s oscillation seminorm of the linear differential equation $\dot{x} = L(x)$. It is given explicitly by (Proposition 5.1):

$$h(L) := -\inf_{\nu, \pi \in \text{extr } P(e)} \inf_{x \in \text{extr } ([0,e])} \langle \nu, L(x) \rangle + \langle \pi, L(e - x) \rangle.$$ 

For illustration we apply this result to some equations in $\mathbb{R}^n$.

Our main results also include analogues of Theorem 1.1 and Theorem 1.2 concerning the contraction rate in Hilbert’s projective metric of a non-linear map (Corollary 3.9), as well as a characterization of the contraction rate in the same metric of a non-linear flow (Theorem 7.1).

The paper is organized as follows. In Section 2 we give preliminary results on Thompson’s metric, Hilbert’s metric, and characterize the extreme points of the dual unit ball. In Section 3 we prove Theorem 1.1 and derive as corollary the analogous result with respect to Hilbert’s projective metric (Corollary 3.9). In Section 4 we apply Theorem 1.1 to discrete time consensus operators. We determine the contraction rate of a linear flow in Hopf’s oscillation seminorm in Section 5. In Section 6 we show Theorem 1.2 and discuss some applications to non-linear consensus dynamics. In Section 7 we prove the analogue of Theorem 1.2 with respect to Hilbert’s projective metric and show its applications.

## 2. Preliminaries

### 2.1. Thompson’s norm and Hopf’s oscillation seminorm.

We consider a real Banach space $(\mathcal{X}, \| \cdot \|)$ and its dual space $\mathcal{X}^*$. Let $C \subset \mathcal{X}$ be a closed pointed convex cone with non empty interior $C^0$, i.e., $\alpha C \subset C$ for $\alpha \in \mathbb{R}^+$, $C + C \subset C$ and $C \cap (-C) = 0$. We
define the partial order \( \leq \) induced by \( C \) on \( \mathcal{X} \) by
\[
x \leq y \iff y - x \in C.
\]
The dual cone of \( C \) is:
\[
C^* := \{ z \in \mathcal{X}^* | \langle z, x \rangle \geq 0, \forall x \in C \}.
\]
Since \( C \) is a closed convex cone, it follows from the strong separation theorem that
\begin{equation}
(4) \quad x \in C \iff \langle z, x \rangle \geq 0, \forall z \in C^*.
\end{equation}
For all \( x \leq y \) we define the order interval:
\[
[x, y] := \{ z \in \mathcal{X} | x \leq z \leq y \}.
\]
(5) For all \( x \leq y \) we define the order interval:
\[
[x, y] := \{ z \in \mathcal{X} | x \leq z \leq y \}.
\]
For \( x \in \mathcal{X} \) and \( y \in C^0 \), following [Nus88], we define
\[
M(x/y) := \inf \{ t \in \mathbb{R} : x \leq ty \},
\]
\[
m(x/y) := \sup \{ t \in \mathbb{R} : x \geq ty \}.
\]
Observe that since \( y \in C^0 \), and since \( C \) is closed and pointed, the two sets in (5) are non-empty, closed, and bounded from below and from above, respectively. In particular, \( m \) and \( M \) take finite values. The difference between \( M \) and \( m \) is called oscillation [Bus73]:
\[
\omega(x/y) := M(x/y) - m(x/y).
\]
Let \( e \) denote a distinguished element in the interior of \( C \), which we shall call a unit. We define
\[
\| x \|_T := \max(M(x/e), -m(x/e))
\]
which we call Thompson’s norm, with respect to the element \( e \), and
\[
\| x \|_H := \omega(x/e)
\]
which we call Hopf’s oscillation seminorm with respect to the element \( e \).

We assume that the cone is normal, meaning that there exists a constant \( K > 0 \) such that
\[
0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|.
\]
It is known that under this assumption the two norms \( \| \cdot \| \) and \( \| \cdot \|_T \) are equivalent, see [Nus94]. Therefore the space \( \mathcal{X} \) equipped with the norm \( \| \cdot \|_T \) is a Banach space.

By the definition and (4), Thompson’s norm with respect to \( e \) can be calculated by:
\begin{equation}
(6) \quad \| x \|_T = \sup_{z \in C^*} \frac{|\langle z, x \rangle|}{\langle z, e \rangle}.
\end{equation}

Example 2.1. We consider the space \( \mathcal{X} = \mathbb{R}^n \), the closed convex cone \( C = \mathbb{R}^n_+ \) and the unit element \( e = 1 := (1, \ldots, 1)^T \). It can be checked that Thompson’s norm with respect to \( 1 \) is nothing but the sup norm
\[
\| x \|_T = \max_i |x_i| = \| x \|_\infty,
\]
whereas Hopf’s oscillation seminorm with respect to \( 1 \) is the so called diameter:
\[
\| x \|_H = \max_{1 \leq i,j \leq n} (x_i - x_j) = \Delta(x).
\]
Example 2.2. Let $\mathcal{X} = S_n$, the space of Hermitian matrices of dimension $n$ and $\mathcal{C} = S_n^+ \subset S_n$, the cone of positive semi-definite matrices. Consider the unit element $e = I_n$, the identity matrix of dimension $n$. Then Thompson’s norm with respect to $I_n$ is nothing but the sup norm of the spectrum of $X$, i.e.,

$$\|X\|_T = \max_{1 \leq i \leq n} \lambda_i(X) = \|\lambda(X)\|_{\infty},$$

where $\lambda(X) := (\lambda_1(X), \ldots, \lambda_n(X))$, $\lambda_1(X) \leq \ldots \leq \lambda_n(X)$, is the vector of ordered eigenvalues of $X$, counted with multiplicities, whereas Hopf’s oscillation seminorm with respect to $I_n$ is the diameter of the spectrum:

$$\|X\|_H = \max_{1 \leq i,j \leq n} (\lambda_i(X) - \lambda_j(X)) = \Delta(\lambda(X)).$$

2.2. Simplex in the dual space and dual unit ball. We denote by $(\mathcal{X}^*, e, \| \cdot \|_T)$ the dual normed space of $(\mathcal{X}, e, \| \cdot \|)$ where the dual norm $\| \cdot \|_T$ of a continuous linear functional $z \in \mathcal{X}^*$ is defined by:

$$\|z\|_T := \sup_{\|x\|_T = 1} \langle z, x \rangle.$$

We define:

$$\mathcal{P}(e) := \{ \mu \in C^* \mid \langle \mu, e \rangle = 1 \}$$

the simplex with respect to $e$ of the dual Banach space $(\mathcal{X}^*, e, \| \cdot \|_T)$.

Remark 2.3. When $\mathcal{X} = \mathbb{R}^n$, $\mathcal{C} = \mathbb{R}_+^n$ and $e = 1$ (Example 2.1), the dual space $\mathcal{X}^*$ is $\mathcal{X} = \mathbb{R}^n$ itself and the dual norm $\| \cdot \|_T$ is the $\ell_1$ norm:

$$\|x\|_T = \sum_i |x_i| = \|x\|_1.$$  

The simplex $\mathcal{P}(1)$ defined in (7) is the simplex in $\mathbb{R}^n$ in the usual sense:

$$\mathcal{P}(1) = \{ \nu \in \mathbb{R}_+^n : \sum_i \nu_i = 1 \},$$

i.e., the set of probability measures on the discrete space $\{1, \ldots, n\}$.

Remark 2.4. In the case of $\mathcal{X} = S_n$, $\mathcal{C} = S_n^+$ and $e = I_n$ (Example 2.2), the dual space $\mathcal{X}^*$ is $\mathcal{X} = S_n$ itself and the dual norm $\| \cdot \|_T$ is the trace norm:

$$\|X\|_T = \sum_{1 \leq i \leq n} |\lambda_i(X)| = \|X\|_1, \quad X \in S_n$$

The simplex $\mathcal{P}(I_n)$ defined in (7) is the set of positive semi-definite matrices with trace 1:

$$\mathcal{P}(I_n) = \{ \rho \in S_n^+ : \text{trace}(\rho) = 1 \}.$$  

The elements of this set are called density matrices in quantum physics, in which they are thought of as noncommutative analogues of probabilities measure.

The next lemma relates $\mathcal{P}(e)$ and the unit ball $B^*_T(e)$ of the space $(\mathcal{X}^*, e, \| \cdot \|_T)$. We denote by conv$(S)$ the convex hull of a set $S$.

Lemma 2.5. The unit ball $B^*_T(e)$ of the space $(\mathcal{X}^*, e, \| \cdot \|_T)$, satisfies

$$B^*_T(e) = \text{conv}(\mathcal{P}(e) \cup -\mathcal{P}(e))$$
Proof. For simplicity we write $\mathcal{P}$ instead of $\mathcal{P}(\mathbf{e})$ and $B_T^\ast$ instead of $B_T^\ast(\mathbf{e})$ in the proof. It follows from (6) that

\[ \|x\|_T = \sup_{\mu \in \mathcal{P}} |\langle \mu, x \rangle| = \sup_{\mu \in \mathcal{P} \cup -\mathcal{P}} \langle \mu, x \rangle. \]

Hence $\|z\|_T^\ast \leq 1$ if and only if, for all $x \in \mathcal{X}$,

\[ \langle z, x \rangle \leq \|x\|_T = \sup_{\mu \in \mathcal{P} \cup -\mathcal{P}} \langle \mu, x \rangle. \]

By the strong separation theorem \cite[Thm 3.18]{FHH01}, if $z$ did not belong to the closed convex hull $\overline{\text{conv}}(\mathcal{P} \cup -\mathcal{P})$, the closure being understood in the weak star topology of $\mathcal{X}^*$, there would exist a vector $x \in \mathcal{X}$ and a scalar $\gamma$ such that $\langle z, x \rangle > \gamma \geq \langle \mu, x \rangle$, for all $\mu \in \mathcal{P} \cup -\mathcal{P}$, contradicting (10).

\[ B_T^\ast = \overline{\text{conv}}(\mathcal{P} \cup -\mathcal{P}). \]

We claim that the latter closure operation can be dispensed with. Indeed, by the Banach Alaoglu theorem, $B_T^\ast$ is weak-star compact. Hence, its subset $\mathcal{P}$, which is weak-star closed, is also weak-star compact. If $\mu \in B_T^\ast$, by the characterization of $B_T^\ast$ above, $\mu$ is a limit, in the weak star topology, of a net $\mu_a = s_a \nu_a - t_a \pi_a$ with $s_a + t_a = 1$, $s_a, t_a \geq 0$ and $\nu_a, \pi_a \in \mathcal{P}$ for $a \in A$. By passing to a subnet we can assume that $s_a, t_a : a \in A$ converge respectively to $s, t \in [0, 1]$ such that $s + t = 1$ and $\nu_a, \pi_a : a \in A$ converge respectively in the weak-star topology to $\nu, \pi \in \mathcal{P}$. It follows that $\mu = s \nu - t \pi \in \overline{\text{conv}}(\mathcal{P} \cup -\mathcal{P})$. □

Remark 2.6. We make a comparison with the paper \cite{RKW11}. In a finite dimensional setting, Reeb, Kastoryano, and Wolf defined a base $\mathcal{B}$ of a proper cone $\mathcal{K}$ in a vector space $\mathcal{V}$ to be a cross section of this cone, i.e., they take $\mathcal{B}$ to be the intersection of the cone $\mathcal{K}$ with a hyperplane given by a linear functional in the interior of the dual cone. So, $\mathcal{V}$ corresponds to $\mathcal{X}^*$ here, and, since $\mathcal{V}$ is of finite dimension, we can identify the dual of $\mathcal{V}$ to $\mathcal{X}$, and consider the dual cone $C \simeq \mathcal{K}^* \subset \mathcal{V}^* \simeq \mathcal{X}$. Modulo this identification, the base $\mathcal{B}$ can be written precisely as $\mathcal{B} = \{ \mu \in \mathcal{K} : \langle \mu, \mathbf{e} \rangle = 1 \}$ for some $\mathbf{e}$ in the interior of $\mathcal{K}^*$, so that the base $\mathcal{B}$ coincides with the simplex $\mathcal{P}(\mathbf{e})$ considered here. They defined the base norm of $\mu \in \mathcal{V}$ with respect to $\mathcal{B}$ by:

\[ \|\mu\|_{\mathcal{B}} = \inf\{ \lambda \geq 0 | \mu \in \lambda \overline{\text{conv}}(\mathcal{B} \cup -\mathcal{B}) \}. \]

Lemma 2.5 shows that the base norm coincides with the dual norm of Thompson’s norm: for $\nu \in \mathcal{X}^*$,

\[ \|\nu\|_{\mathcal{B}} = \inf\{ \lambda \geq 0 | \nu \in \lambda \overline{\text{conv}}(\mathcal{P}(\mathbf{e}) \cup -\mathcal{P}(\mathbf{e})) \} = \inf\{ \lambda \geq 0 | \nu \in \lambda B_T^\ast \} = \|\nu\|_T^\ast. \]

The set

\[ \tilde{M} = \{ x \in \mathcal{V}^* | 0 \leq x \leq \mathbf{e} \}. \]

is also considered in \cite{RKW11}; leading to define the distinguishability norm of $\mu \in \mathcal{V}$ by:

\[ \|\mu\|_{\tilde{M}} = \sup_{0 \leq x \leq \mathbf{e}} \langle \mu, 2x - \mathbf{e} \rangle. \]

It is shown there that

\[ \|\mu\|_{\tilde{M}} = \|\mu\|_{\mathcal{B}}. \]
2.3. Extreme points of the dual unit ball. We first show that Hopf’s oscillation seminorm coincides with the norm on the quotient Banach space of \((X, \| \cdot \|_T)\) by the closed subspace \(\mathbb{R}e\).

**Lemma 2.7.** For all \(x \in X\), we have:

\[
\|x\|_H = 2 \inf_{\lambda \in \mathbb{R}} \|x + \lambda e\|_T
\]

*Proof.* \(\|x + \lambda e\|_T = (M(x/e) + \lambda) \vee (-m(x/e) - \lambda)\) is minimal when \((M(x/e) + \lambda) = (-m(x/e) - \lambda)\). Substituting the value of \(\lambda\) obtained in this way in \(\|x + \lambda e\|_T\), we arrive at the announced formula. \(\Box\)

A standard result [Con90, P.88] of functional analysis shows that if \(W\) is a closed subspace of a Banach space \((X, \| \cdot \|)\), then the quotient space \(X/W\) is complete. Besides, the dual of the quotient space \(X/W\) can be identified isometrically to the space of continuous linear forms on \(X\) that vanish on \(W\), equipped with the dual norm \(\| \cdot \|_\star\) of \(X\). Specializing this result to \(W = \mathbb{R}e\), we get:

**Lemma 2.8.** The quotient normed space \((X/\mathbb{R}e, \| \cdot \|_H)\) is a Banach space. Its dual is \((\mathcal{M}(e), \| \cdot \|_{\star H})\) where

\[
\mathcal{M}(e) := \{ \mu \in X^*| \langle \mu, e \rangle = 0 \},
\]

and

(12)

\[
\|\mu\|_{\star H} := \frac{1}{2} \|\mu\|_{\star T}, \ \forall \mu \in \mathcal{M}(e).
\]

The above lemma implies that the unit ball of the space \((\mathcal{M}(e), \| \cdot \|_{\star H})\), denoted by \(B_{\star H}(e)\), satisfies:

(13)

\[
B_{\star H}^\circ(e) = 2B_{\star T}(e) \cap \mathcal{M}(e).
\]

**Remark 2.9.** In the case of \(X = \mathbb{R}^n\), \(C = \mathbb{R}^n\) and \(e = 1\) (Example 2.1 and Remark 2.3), Lemma 2.8 implies that for any two probability measures \(\mu, \nu \in \mathcal{P}(1)\), the dual norm \(\|\mu - \nu\|_{\star H}\) is the total variation distance between \(\mu\) and \(\nu\):

\[
\|\mu - \nu\|_{\star H} = \frac{1}{2} \|\mu - \nu\|_1 = \|\mu - \nu\|_{TV}
\]

Before giving a representation of the extreme points of \(B_{\star H}^\circ(e)\), we define the disjointness relation \(\perp\) on \(\mathcal{P}(e)\).

**Definition 2.10.** For all \(\nu, \pi \in \mathcal{P}(e)\), we say that \(\nu\) and \(\pi\) are disjoint, denoted by \(\nu \perp \pi\), if

\[
\mu = \frac{\nu + \pi}{2}
\]

for all \(\mu \in \mathcal{P}(e)\) such that \(\mu \geq \frac{\nu}{2}\) and \(\mu \geq \frac{\pi}{2}\).

In particular, we remark the following property:

**Lemma 2.11.** Let \(\nu, \pi \in \mathcal{P}(e)\). The following assertions are equivalent:

(a) \(\nu \perp \pi\).

(b) The only elements \(\rho, \sigma \in \mathcal{P}(e)\) such that

\[
\nu - \pi = \rho - \sigma
\]

are \(\rho = \nu\) and \(\sigma = \pi\).
Theorem 2.12. The set of extreme points of $B^*_H(e)$, denoted by $\text{extr} B^*_H(e)$, is characterized by:
\[
\text{extr} B^*_H(e) = \{ \nu - \pi : \nu, \pi \in \text{extr} \mathcal{P}(e), \nu \perp \pi \}.
\]

Proof. It follows from (8) that every point $\nu \in B^*_H(e)$ can be written as $\mu = s\nu - t\pi$ with $s + t = 1$, $s, t \geq 0$, $\nu, \pi \in \mathcal{P}$. Moreover, if $\mu \in \mathcal{M}(e)$, $0 = \langle \mu, e \rangle = s\langle \nu, e \rangle - t\langle \pi, e \rangle = s - t$, and so $s = t = \frac{1}{2}$. Thus every $\mu \in B^*_H(e) \cap \mathcal{M}(e)$ can be written as
\[
\mu = \frac{\nu - \pi}{2}, \; \nu, \pi \in \mathcal{P}(e).
\]

Therefore by (13) we proved that
\[
(14) \quad B^*_H(e) = \{ \nu - \pi : \nu, \pi \in \mathcal{P}(e) \}.
\]

Now let $\nu, \pi \in \text{extr} \mathcal{P}(e)$ and $\nu \perp \pi$. We are going to prove that $\nu - \pi \in \text{extr} B^*_H(e)$. Let $\nu_1, \pi_1, \nu_2, \pi_2 \in \mathcal{P}(e)$ such that
\[
\nu - \pi = \frac{\nu_1 - \pi_1}{2} + \frac{\nu_2 - \pi_2}{2}.
\]

Then
\[
\nu - \pi = \frac{\nu_1 + \nu_2}{2} - \frac{\pi_1 + \pi_2}{2}.
\]

By Lemma 2.11 the only possibility is $2\nu = \nu_1 + \nu_2$ and $2\pi = \pi_1 + \pi_2$. Since $\nu, \pi \in \text{extr} \mathcal{P}(e)$ we obtain that $\nu_1 = \nu_2 = \nu$ and $\pi_1 = \pi_2 = \pi$. Therefore $\nu - \pi \in \text{extr} B^*_H(e)$.

Now let $\nu, \pi \in \mathcal{P}(e)$ such that $\nu - \pi \in \text{extr} B^*_H(e)$. Assume by contradiction that $\nu$ is not extreme in $\mathcal{P}(e)$ (the case in which $\pi$ is not extreme can be dealt with similarly). Then, we can find $\nu_1, \nu_2 \in \mathcal{P}(e)$, $\nu_1 \neq \nu_2$, such that $\nu = \frac{\nu_1 + \nu_2}{2}$. It follows that
\[
\mu = \frac{\nu_1 - \pi}{2} + \frac{\nu_2 - \pi}{2},
\]

where $\nu_1 - \pi, \nu_2 - \pi$ are distinct elements of $B^*_H(e)$, which is a contradiction. Next we show that $\nu \perp \pi$. To this end, let any $\rho, \sigma \in \mathcal{P}(e)$ such that
\[
\nu - \pi = \rho - \sigma.
\]

Then
\[
\nu - \pi = \nu - \pi + \rho - \sigma = \nu - \sigma + \rho - \pi.
\]

If $\sigma \neq \pi$, then $\nu - \sigma \neq \nu - \pi$ and this contradicts the fact that $\nu - \pi$ is extremal. Therefore $\sigma = \pi$ and $\rho = \nu$. From Lemma 2.11 we deduce that $\nu \perp \pi$. 

\[\square\]
Remark 2.13. When $\mathcal{X} = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and $e = 1$ (Example 2.1 and Remark 2.3), the set of extreme points of $\mathcal{P}(1)$ is the set of standard basis vectors $\{e_i\}_{i=1}^n$. The extreme points are pairwise disjoint.

Remark 2.14. When $\mathcal{X} = S_n$, $C = S^+_n$ and $e = I_n$ (Example 2.2 and Remark 2.4), the set of extreme points of $\mathcal{P}(I_n)$ is:

$$\text{extr } \mathcal{P}(I_n) = \{xx^*: x \in \mathbb{C}^n, x^*x = 1\}.$$ 

Two extreme points $xx^*$ and $yy^*$ are disjoint if and only if $x^*y = 0$. To see this, note that if $x^*y = 0$ then any Hermitian matrix $X$ such that $X \geq xx^*$ and $X \geq yy^*$ should satisfy $X \geq xx^* + yy^*$. Hence by definition $xx^*$ and $yy^*$ are disjoint. Inversely, suppose that $xx^*$ and $yy^*$ are disjoint and consider the spectral decomposition of the matrix $xx^* - yy^*$, i.e., there is $\lambda \leq 1$ and two orthonormal vectors $u, v$ such that $xx^* - yy^* = \lambda(uu^* - vv^*)$. It follows that $xx^* - yy^* = uu^* - (1 - \lambda)uu^* + \lambda vv^*$ and $xx^* = uu^*$ thus $\lambda = 1$, $u = x$ and $v = y$. Therefore $x^*y = 0$.

3. The operator norm induced by Hopf's oscillation

Consider two real Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. Let $C_1 \subseteq \mathcal{X}_1$ and $C_2 \subseteq \mathcal{X}_2$ be respectively two closed pointed convex normal cones with non empty interiors $C^0_1$ and $C^0_2$. Let $e_1 \in C^0_1$ and $e_2 \in C^0_2$. Then, we know from Section 2 that the two quotient spaces $(\mathcal{X}_1/\mathbb{R}e_1, \| \cdot \|_H)$ and $(\mathcal{X}_2/\mathbb{R}e_2, \| \cdot \|_H)$ equipped with the Hopf’s oscillation seminorms associated respectively to $e_1$ and $e_2$ are Banach spaces. The dual spaces of $(\mathcal{X}_1/\mathbb{R}e_1, \| \cdot \|_H)$ and $(\mathcal{X}_2/\mathbb{R}e_2, \| \cdot \|_H)$ are respectively the spaces $(\mathcal{M}(e_1), \| \cdot \|_H^1)$ and $(\mathcal{M}(e_2), \| \cdot \|_H^2)$ (see Lemma 2.8).

Let $T$ denote a continuous linear map from $(\mathcal{X}_1/\mathbb{R}e_1, \| \cdot \|_H)$ to $(\mathcal{X}_2/\mathbb{R}e_2, \| \cdot \|_H)$. The operator norm of $T$, denoted by $\| T \|_H$, is given by:

$$\| T \|_H := \sup_{x \in B_H(e_1)} \| T(x) \|_H$$

The adjoint operator $T^*: (\mathcal{M}(e_2), \| \cdot \|_H^1) \to (\mathcal{M}(e_1), \| \cdot \|_H^2)$ of $T$ is by definition:

$$\langle T^*(\mu), x \rangle = \langle \mu, T(x) \rangle, \quad \forall \mu \in \mathcal{M}(e_2), x \in \mathcal{X}_1.$$ 

The operator norm of $T^*$, denoted by $\| T^* \|_{H^1}$, is then:

$$\| T^* \|_{H^1}^* := \sup_{\mu \in B_{H^1}(e_2)} \| T^*(\mu) \|_{H^2}.$$ 

A classical duality result (see [AB99, § 6.8]) shows that an operator and its adjoint have the same operator norm. In particular,

$$\| T \|_H = \| T^* \|_{H^1}^*.$$ 

**Theorem 3.1.** Let $T: \mathcal{X}_1 \to \mathcal{X}_2$ be a bounded linear map such that $T(e_1) \in \mathbb{R}e_2$. Then,

$$\| T \|_H = \frac{1}{2} \sup_{\nu, \pi \in \mathcal{P}(e_2)} \| T^*(\nu) - T^*(\pi) \|_{H^1}^* = \sup_{\nu, \pi \in \mathcal{P}(e_2)} \sup_{x \in [0, e_1]} \langle \nu - \pi, T(x) \rangle.$$ 

Moreover, the supremum can be restricted to the set of extreme points:

$$\| T \|_H = \frac{1}{2} \sup_{\nu, \pi \in \text{extr } \mathcal{P}(e_2)} \| T^*(\nu) - T^*(\pi) \|_{H^1}^* = \sup_{\nu, \pi \in \text{extr } \mathcal{P}(e_2)} \sup_{x \in [0, e_1]} \langle \nu - \pi, T(x) \rangle.$$
Proof. We already noted that $\|T\|_H = \|T^*\|_B^*$. Moreover,

$$\|T^*\|_B^* = \sup_{\mu \in B_H^*(e_2)} \|T^*(\mu)\|_H^*. $$

By the characterization of $B_H^*(e_2)$ in (14) and the characterization of the norm $\|\cdot\|_H^*$ in Lemma 2.8, we get

$$\sup_{\mu \in B_H^*(e_2)} \|T^*(\mu)\|_H^* = \sup_{\nu, \pi \in P(e_2)} \|T^*(\nu) - T^*(\pi)\|_H^* = \frac{1}{2} \sup_{\nu, \pi \in P(e_2)} \|T^*(\nu) - T^*(\pi)\|_T^*.$$

For the second equality, note that

$$\|T^*(\nu) - T^*(\pi)\|_T^* = \sup_{x \in [0, e_1]} \langle T^*(\nu) - T^*(\pi), 2x - e_1 \rangle = 2 \sup_{x \in [0, e_1]} \langle T^*(\nu) - T^*(\pi), x \rangle.$$

We next show that the supremum can be restricted to the set of extreme points. By the Banach-Alaoglu theorem, $B_H^*$ is weak-star compact, and it is obviously convex. The dual space $M$ endowed with the weak-star topology is a locally convex topological space. Thus by the Krein-Milman theorem, the unit ball $B_H^*$, which is a compact convex set in $M$ with respect to the weak-star topology, is the closed convex hull of its extreme points. So every element $\rho$ of $B_H^*(e_2)$ is the limit of a net $(\rho_\alpha)_\alpha$ of elements of convex $\text{extr} \, B_H^*(e_2)$.

Observe now that the function

$$\varphi : \mu \mapsto \|T^*(\mu)\|_H = \sup_{x \in B_H(e_1)} \langle T^*(\mu), x \rangle = \sup_{x \in B_H(e_1)} \langle \mu, T(x) \rangle$$

which is a sup of weak-star continuous maps is convex and weak-star lower semi-continuous. This implies that $\varphi(\rho) \leq \lim \inf_{\alpha} \varphi(\rho_\alpha) \leq \sup_{\text{conv extr} \, B_H^*(e_2)} \varphi(\mu) = \sup_{\text{extr} \, B_H^*(e_2)} \varphi(\mu)$. Using the characterization of the extreme points in Proposition 2.12, we get:

$$\sup_{\mu \in B_H^*(e_2)} \|T^*(\mu)\|_H^* = \sup_{\mu \in \text{extr} \, B_H^*(e_2)} \|T^*(\mu)\|_H^* = \sup_{\nu, \pi \in \text{extr} \, P(e_2)} \|T^*(\nu) - T^*(\pi)\|_H^*. \quad \Box$$

Remark 3.2. When $X_1$ is of finite dimension, the set $[0, e_1]$ is the convex hull of the set of its extreme points, hence, the supremum over the variable $x \in [0, e_1]$ in (13) is attained at an extreme point. Similarly, if $X_2$ is of finite dimension, the suprema over $(\nu, \pi)$ in the same equation are also attained, because the map $\varphi$ in the proof of the previous theorem, which is a supremum of an equi-Lipschitz family of maps, is continuous (in fact, Lipschitz).

Remark 3.3. Theorem 3.1 should be compared with Proposition 12 of [RKW11] which can be stated as follows.

**Proposition 3.4 (Proposition 12 in [RKW11]).** Let $L : V \to V'$ be a linear map and let $B \subset V$ and $B' \subset V'$ be bases. Then

$$\sup_{v_1 \neq v_2 \in B} \|L(v_1) - L(v_2)\|_{B'} = \frac{1}{2} \sup_{v_1, v_2 \in \text{extr} \, B} \|L(v_1) - L(v_2)\|_{B'}. \quad (16)$$

The first term in (16) is called the *contraction ratio* of the linear map $L$, with respect to base norms. One important applications of this proposition concerns the *base preserving* maps $L$ such that $L(B) \subset B'$. Let us translate this proposition in the present setting. Consider a linear map $T : X/\mathbb{R}e_1 \to X/\mathbb{R}e_2$. Then $T^*(P(e_2)) \subset P(e_2)$ is a base
preserving linear map and so, Proposition 12 of [RKW11] shows that:

\[
(17) \quad \sup_{\nu, \pi \in \mathcal{P}(e_2)} \frac{\|T^*(\nu - \pi)\|_T}{\|\nu - \pi\|_T} = \frac{1}{2} \sup_{\nu, \pi \in \text{extr}\mathcal{P}(e_2)} \|T^*(\nu) - T^*(\pi)\|_T^*.
\]

Hence, by comparison with [RKW11], the additional information here is the equality between the contraction ratio in Hopf’s oscillation seminorm of a unit preserving linear map, and the contraction ratio with respect to the base norms of the dual base preserving map. The latter is the primary object of interest in quantum information theory whereas the former is of interest in the control/consensus literature. We also proved that the supremum in (17) can be restricted to pairs of disjoint extreme points \( \nu, \pi \). Finally, the expression of the contraction rate as the last supremum in Theorem 3.1 leads here to an abstract version of Dobrushin’s ergodic coefficient, see Eqn (20) and Corollary 4.3 below.

Recall that Hilbert’s projective metric between two elements \( x, y \in C^0 \) is defined as:

\[ d_H(x, y) = \log(M(x/y)/m(x/y)). \]

Consider a linear operator \( T : X_1 \to X_2 \) such that \( T(C^0_1) \subset C^0_2 \). Following [Bir57, Bus73], we define the projective diameter of \( T \) as below:

\[ \text{diam} T = \sup\{d_H(T(x), T(y)) : x, y \in C^0_1\}. \]

The Birkhoff’s contraction formula [Bir57, Bus73] states that:

**Theorem 3.5** ([Bir57, Bus73]).

\[ \sup_{x,y \in C^0_1} \frac{\omega(T(x), T(y))}{\omega(x, y)} = \sup_{x,y \in C^0_1} \frac{d_H(T(x), T(y))}{d_H(x, y)} = \tanh(\frac{\text{diam} T}{4}). \]

Following [RKW11], we define the projective diameter of \( T^* \) by:

\[ \text{diam} T^* = \sup\{d_H(T^*(u), T^*(v)) : u, v \in C^*_2 \backslash \{0\}\}. \]

Note that \( \text{diam} T = \text{diam} T^* \). This is because

\[ \sup_{x,y \in C^0_1} \frac{M(T(x)/T(y))}{m(T(x)/T(y))} = \sup_{x,y \in C^0_1} \sup_{u,v \in C^*_2 \backslash \{0\}} \frac{\langle u, T(x) \rangle \langle v, T(y) \rangle}{\langle u, T(y) \rangle \langle v, T(x) \rangle} \frac{M(T^*(u)/T^*(v))}{m(T^*(u)/T^*(v))} \]

**Corollary 3.6** (Compare with [RKW11]). Let \( T : X_1 \to X_2 \) be a bounded linear map such that \( T(e_1) \in \mathbb{R}e_2 \) and \( T(C^0_1) \subset C^0_2 \), then:

\[ \|T^*\|_H^* = \|T\|_H \leq \tanh(\frac{\text{diam} T}{4}) = \tanh(\frac{\text{diam} T^*}{4}). \]

**Proof.** It is sufficient to prove the inequality. For this, note that

\[ \|T\|_H = \sup_{x \in X_1/\mathbb{R}e_1} \omega(T(x), e_2)/\omega(x, e_1) = \sup_{x \in C^0_1} \omega(T(x), e_2)/\omega(x, e_1). \]

Then we apply Birkhoff’s contraction formula. \( \square \)

**Remark 3.7.** Reeb et al [RKW11] showed in a different way that

\[ \|T^*\|_H^* \leq \tanh(\frac{\text{diam} T^*}{4}). \]

The proof above shows that as soon as the duality formula \( \|T^*\|_H^* = \|T\|_H \) has been obtained, the latter inequality follows from Birkhoff contraction formula.
Nussbaum \cite{Nus94} showed that the Lipschitz constant in Hilbert’s projective metric of a non-linear map is determined by the operator norm of its derivative with respect to Hopf’s oscillation seminorm. We first use this result to deduce a characterization of the contraction rate of non-linear maps in Hilbert’s metric. We first quote the result of \cite{Nus94} which we shall use.

**Theorem 3.8 (Coro 2.1, \cite{Nus94}).** Let $U \subset C^0$ be a convex open set such that $tU \subset U$ for all $t > 0$. Let $f : U \to C^0$ be a continuously differentiable map such that $\omega(f(x)/f(y)) = 0$ whenever $x, y \in U$ and $\omega(x/y) = 0$. For each $x \in U$ define $\lambda(x), \lambda_0$ and $k_0$ by:

$$
\lambda(x) := \inf \{c > 0 : \omega(Df(x)v/f(x)) \leq c\omega(v/x) \text{ for all } v \in \mathcal{X} \},
$$

$$
\lambda_0 := \sup \{\lambda(x) : x \in U \},
$$

$$
k_0 := \inf \{c > 0 : d_H(f(x), f(y)) \leq cd_H(x, y) \text{ for all } x, y \in U \}.
$$

Then it follows that $\lambda_0 = k_0$.

Then a direct corollary of Theorem 3.1 and 3.8 yields the Lipschitz constant in Hilbert’s metric of a (non-linear) map.

**Corollary 3.9.** Let $U$ and $f$ be as in Theorem 3.8. Then:

$$
\sup_{x, y \in U} \frac{d_H(f(x), f(y))}{d_H(x, y)} = \sup_{x \in U} \sup_{\nu, \pi \in \text{extr P}(f(x))} \sup_{z \in [0, x]} \langle \nu - \pi, Df(x)z \rangle.
$$

**Remark 3.10.** This corollary generalizes Corollary 2.1 of \cite{Nus94}, which gives a similar characterization in terms of extreme points, when $\mathcal{X} = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$. Note that in the finite dimensional case, the suprema over the variable $z$ and over the variables $\nu, \pi$ are attained (see Remark 3.2). Moreover, the supremum over $z$ is attained at an extreme point of $[0, x]$.

4. **APPLICATION TO DISCRETE CONSENSUS OPERATORS ON CONES**

A classical result, which goes back to Doeblin and Dobrushin, characterizes the Lipschitz constant of a Markov matrix acting on the space of measures (i.e., a row stochastic matrix acting on the left), with respect to the total variation norm (see the discussion in Remark 4.5 below). The same constant characterizes the contraction ratio with respect to the “diameter” (Hopf oscillation seminorm) of the consensus system driven by this Markov matrix (i.e., a row stochastic matrix acting on the right). Consensus operators on cones extend Markov matrices. In this section, we extend to these abstract operators a number of known properties of Markov matrices.

A linear map $T : \mathcal{X} \to \mathcal{X}$ is a consensus operator with respect to a unit vector $e$ in the interior $C^0$ of a closed convex pointed cone $C \subset \mathcal{X}$ if it satisfies the two following properties:

(i) $T$ is positive, i.e., $T(C) \subset C$.

(ii) $T$ preserves the unit element $e$, i.e., $T(e) = e$.

**Example 4.1.** When $\mathcal{X} = \mathbb{R}^n$, $C$ is the standard orthant and $e$ is the standard unit vector $\mathbf{1}$ (Example 2.1), a linear map $T(x) = Ax$ is a consensus operator if and only if $A$ is a row stochastic matrix. The operator norm is the contraction rate of the matrix $A$ with respect to the diameter $\Delta$:

$$
\|T\|_H = \tau(A) := \sup_x \frac{\Delta(Ax)}{\Delta(x)},
$$

13
and the dual operator norm is the Lipschitz constant of $A^\top$ on $\mathcal{P}(1)$ with respect to the total variation distance:

$$
\|T\|_H^* = \delta(A) := \sup_{\mu, \nu \in \mathcal{P}(1)} \frac{\|A^\top \mu - A^\top \nu\|_{TV}}{\|\mu - \nu\|_{TV}}.
$$

The value $\tau(A)$ allows one to bound the convergence rate of the stationary linear consensus system the dynamics of which is given by the matrix $A$, [MDA05, VJA05]. The value $\delta(A)$ is known as the ergodicity coefficient of the Markov chain with transition probability matrix $A^\top$, see [LPW09].

**Example 4.2.** When $\mathcal{X} = S_n$, $\mathcal{C} = S_n^+$ and $e = I_n$ (Example 2.2), the linear map $\Phi : S_n \to S_n$ defined by

$$
\Phi(X) = \sum_{i=1}^m V_i^* X V_i, \quad \sum_{i=1}^m V_i^* V_i = I_n
$$

is a consensus operator. The dual operator is then given by:

$$
\Psi(X) = \sum_{i=1}^m V_i X V_i^*.
$$

Both maps are completely positive. They represent a purely quantum channel [RKW11, SSR10]. The map $\Phi$ is unital and acts between spaces of operators while the adjoint map $\Psi$ is trace-preserving and acts between spaces of states (density matrices). The operator norm of $\Phi : S_n / \mathbb{R} I_n \to S_n / \mathbb{R} I_n$ is the contraction rate of the diameter of the spectrum:

$$
\|\Phi\|_H = \sup_{X \in S_n} \frac{\lambda_{\text{max}}(\Phi(X)) - \lambda_{\text{min}}(\Phi(X))}{\lambda_{\text{max}}(X) - \lambda_{\text{min}}(X)}.
$$

The operator norm of the adjoint map $\Psi : \mathcal{P}(I_n) \to \mathcal{P}(I_n)$ is the contraction rate of the trace distance:

$$
\|\Psi\|_H^* = \sup_{\rho_1, \rho_2 \in \mathcal{P}(I_n)} \frac{\|\Psi(\rho_1) - \Psi(\rho_2)\|_1}{\|\rho_1 - \rho_2\|_1}.
$$

The value $\|\Phi\|_H$ and $\|\Psi\|_H^*$ are the noncommutative counterparts of $\tau(\cdot)$ and $\delta(\cdot)$.

A direct application of Theorem 3.1 leads to following characterization of operator norm, which will be seen to extend Dobrushin’s formula (see Remark 4.5 below).

**Corollary 4.3.** Let $T : \mathcal{X} \to \mathcal{X}$ be a consensus operator with respect to $e$. Then,

$$
\|T\|_H = \|T\|_H^* = 1 - \inf_{\nu, \pi \in \text{extr} \mathcal{P}(e)} \inf_{x \in [0, e]} \langle \pi, T(x) \rangle + \langle \nu, T(e - x) \rangle.
$$

**Proof.** Since $T(e) = e$, we have:

$$
\sup_{\nu, \pi \in \text{extr} \mathcal{P}(e)} \sup_{x \in [0, e]} \langle \nu - \pi, T(x) \rangle = \sup_{\nu, \pi \in \text{extr} \mathcal{P}(e)} \sup_{x \in [0, e]} 1 - \langle \pi, T(x) \rangle - \langle \nu, T(e - x) \rangle.
$$

\[\square\]

**Remark 4.4.** In the finite dimensional case, as already noted in Remark 3.2, the supremum is reached at $\text{extr}[0, e]$.

**Remark 4.5.** In the case of a stochastic matrix $A$ (Example 1.1), Corollary 4.3 implies that:

$$
\tau(A) = \delta(A) = \frac{1}{2} \sup_{i \neq j} \|A^\top e_i - A^\top e_j\|_1.
$$
This is a known result in the study of Markov chain [Sen91]. The value \( \tau(A) \) is known under the name of Dobrushin’s ergodic coefficient of the stochastic matrix \( A \) [Dob56]. It is explicitly given by:

\[
(20) \quad \tau(A) = 1 - \min_{i \neq j} \sum_{s=1}^{n} \min(A_{is}, A_{js}).
\]

Indeed, the characterization of \( \tau(A) = \|T\|_{H} \) by the last supremum in Corollary 4.3 yields

\[
\tau(A) = 1 - \min_{i \neq j} \min_{I \subset \{1, \ldots, n\}} \left( \sum_{k \in I} A_{ik} + \sum_{k \notin I} A_{jk} \right)
\]

from which (20) follows.

A simple classical situation in which \( \tau(A) < 1 \) is when there is a Dæblin state, i.e., an element \( j \in \{1, \ldots, n\} \) such that \( A_{ij} > 0 \) holds for all \( i \in \{1, \ldots, n\} \).

Specializing Corollary 4.3 to the case of quantum channels (Example 4.2), we obtain the noncommutative version of Dobrushin’s ergodic coefficient.

**Corollary 4.6.** Let \( \Phi \) be a quantum channel defined in (18). Then,

\[
(21) \quad \|\Phi\|_{H} = \|\Psi\|_{H}^{*} = 1 - \min_{u, v: u^{*}v=1} \min_{X: X^{*}X=I_{n}} \sum_{i=1}^{n} \min\{u^{*}\Phi(x_{i}x_{i}^{*})u, v^{*}\Phi(x_{i}x_{i}^{*})v\}
\]

**Proof.** It can be easily checked that

\[ \text{extr}[0, I_{n}] = \{P \in \mathcal{S}_{n} : P^{2} = P \}. \]

Hence, Corollary 4.3 and Remark 2.14 yield:

\[
\|\Phi\|_{H} = \|\Psi\|_{H}^{*} = 1 - \min_{u, v: u^{*}v=1} \min_{X: X^{*}X=I_{n}} \sum_{i=1}^{n} \min\{u^{*}\Phi(x_{i}x_{i}^{*})u, v^{*}\Phi(x_{i}x_{i}^{*})v\}
\]

from which (21) follows. \( \square \)

We now make the following basic observations for a consensus operator \( T : \mathcal{X} \to \mathcal{X} \):

\[
\mathcal{M}(T(x)/e) \leq \mathcal{M}(x/e), \quad \mathcal{M}(T(x)/e) \geq \mathcal{M}(x/e), \quad \forall x \in \mathcal{X}.
\]

It follows that \( \|T\|_{H} \leq 1 \). The case when \( \|T\|_{H} < 1 \) or equivalently \( \|T^{*}\|_{H} < 1 \) is of special interest, as shown by the following theorem, which shows that the iterates of \( T \) converge to a rank one projector with a rate bounded by \( \|T\|_{H} \).

**Theorem 4.7** (Geometric convergence to consensus). If \( \|T\|_{H} < 1 \) or equivalently \( \|T^{*}\|_{H} < 1 \), then there is \( \pi \in \mathcal{P}(e) \) such that for all \( x \in \mathcal{X} \)

\[
\|T^{n}(x) - \langle \pi, x \rangle e\|_{T} \leq (\|T\|_{H})^{n} \|x\|_{H},
\]

and for all \( \mu \in \mathcal{P}(e) \)

\[
\|(T^{*})^{n}(\mu) - \pi\|_{H} \leq (\|T\|_{H})^{n}.
\]
Proof. The intersection
\[ \cap_n [m(T^n(x)/e), M(T^n(x)/e)] \subset \mathbb{R} \]
is nonempty (as a non-increasing intersection of nonempty compact sets), and since \( \|T\|_H < 1 \) and
\[ \omega(T^n(x)/e) \leq (\|T\|_H)^n \omega(x/e), \]
this intersection must be reduced to a real \( \{c(x)\} \subset \mathbb{R} \) depending on \( x \), i.e.,
\[ c(x) = \cap_n [m(T^n(x)/e), M(T^n(x)/e)]. \]
Thus for all \( n \in \mathbb{N} \),
\[ -\omega(T^n(x)/e)e \leq T^n(x) - c(x)e \leq \omega(T^n(x)/e)e. \]
Therefore by definition:
\[ \|T^n(x) - c(x)e\|_T \leq \omega(T^n(x)/e). \]
Then we get:
\[ \|T^n(x) - c(x)e\|_T \leq (\|T\|_H)^n \|x\|_H. \]
It is immediate that:
\[ c(x)e = \lim_{n \to \infty} T^n(x) \]
from which we deduce that \( c : \mathcal{X} \to \mathbb{R} \) is a continuous linear functional. Thus there is \( \pi \in \mathcal{X}^* \) such that \( c(x) = \langle \pi, x \rangle \). Besides it is immediate that \( \langle \pi, e \rangle = 1 \) and \( \pi \in \mathcal{C}^* \) because
\[ x \in \mathcal{C} \Rightarrow c(x)e \in \mathcal{C} \Rightarrow c(x) \geq 0 \Rightarrow \langle \pi, x \rangle \geq 0. \]
Therefore \( \pi \in \mathcal{P} \). Finally for all \( \mu \in \mathcal{P} \) and all \( x \in \mathcal{X} \) we have
\[ \langle (T^*)^n(\mu) - \pi, x \rangle = \langle \mu, T^n(x) - \langle \pi, x \rangle e \rangle \leq \|\mu\|_{\mathcal{P}} \|T^n(x) - \langle \pi, x \rangle e\|_T \leq (\|T\|_H)^n \|x\|_H. \]
Hence
\[ \|(T^*)^n(\mu) - \pi\|_H \leq (\|T\|_H)^n. \]
\[ \square \]

Remark 4.8. Specializing Theorem 4.7 to the case of \( \mathcal{X} = \mathbb{R}^n \) (Example 2.1) we obtain that if \( \tau(A) = \delta(A) < 1 \), then
\[ A^n \to \mathbf{1}\pi^T, \ n \to +\infty \]
where \( \pi \) is the unique invariant measure of the stochastic matrix \( A \). This is a well-known result in the study of ergodicity property and mixing times of Markov chains, see for example [Sen91] and [LPW09].

Remark 4.9. A time-dependent consensus system is described by
\[ x_{k+1} = T_{k+1}(x_k), \ k \in \mathbb{N} \]
where \( \{T_k : k \geq 1\} \) is a sequence of consensus operators sharing a common unit element \( e \in \mathcal{C}^0 \). Then if there is an integer \( p > 0 \) and a constant \( \alpha < 1 \) such that for all \( i \in \mathbb{N} \)
\[ \|T_{i+p} \ldots T_{i+1}\|_H \leq \alpha, \]
then the same lines of proof of Theorem 4.7 imply the existence of \( \pi \in \mathcal{P}(e) \) such that for all \( \{x_k\} \) satisfying \( 22 \),
\[ \|x_k - \langle \pi, x_0 \rangle e\|_T \leq \alpha^{\lfloor \frac{k}{p} \rfloor} \|x_0\|_H, \ n \in \mathbb{N}. \]
Remark 4.10. In the case of $X = \mathbb{R}^n$ and $T_k(x) = A_kx$ where $A_k$ is a stochastic matrix, Moreau [Mor05] showed that if all the non-zero entries are bounded from below by a positive constant and if there is $p \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ there is a node connected to all other nodes in the graph associated to the matrix $A_{i+p} \ldots A_{i+1}$, then the system (22) is globally uniformly convergent. These two conditions imply exactly that the Dobrushin’s ergodic coefficient (20) of $A_{i+p} \ldots A_{i+1}$, which is also the operator norm $\|T_{i+p} \ldots T_{i+1}\|_H$, is bounded by a constant less than 1.

5. The contraction rate in Hopf’s oscillation of a linear flow

5.1. Abstract formula for the contraction rate. Hereinafter, we only consider a finite dimensional vector space $X$. The set of linear transformations on $X$ is denoted by $\text{End}(X)$. Let $L \in \text{End}(X)$ such that $L(e) = 0$. The next proposition characterizes the contraction rate of the flow associated to the linear differential equation
\[
\dot{x} = L(x),
\]
with respect to Hopf’s oscillation seminorm.

Proposition 5.1. The optimal constant $\alpha$ such that
\[
\|\exp(tL)x\|_H \leq e^{\alpha t}\|x\|_H, \quad \forall t \geq 0, x \in X
\]
is
\[
(23) \quad h(L) := -\inf_{\nu, \pi \in \mathcal{P}(e)} \inf_{x \in \text{extr}([0, e])} \langle \nu, L(x) \rangle + \langle \pi, L(e - x) \rangle.
\]

Proof. Let $I : X \rightarrow X$ denote the identity transformation. We define a functional on $\text{End}(X)$ by:
\[
F(W) = \sup_{\nu, \pi \in \mathcal{P}(e)} \sup_{x \in [0, e]} \langle \pi - \nu, W(x) \rangle
\]
By Theorem 3.1 the optimal constant $\alpha$ is:
\[
(24) \quad \alpha = \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1}(\|\exp(\epsilon L)\|_H - 1)
\]
Recall that a map is said to be semidifferentiable at a point if it has one-sided directional derivatives in all directions, and if the limit defining the one-sided directional derivative is uniform in the direction, see Definition 7.20 of [RW98], to which we refer for information on the different notions used here. The limit in (24) coincides with to the semiderivative of $F$ at point $I$ in the direction $L$ if $F$ is semidifferentiable. We next show that it is so, and compute the limit. Since we assume that $\mathcal{P}(e)$ and $[0, e]$ are compact sets and the function
\[
F_{\nu, \pi, x}(W) = \langle \pi - \nu, W(x) \rangle
\]
is continuously differentiable on $W$ such that $F_{\nu, \pi, x}(W)$ and $DF_{\nu, \pi, x}(W)$ are jointly continuous on $(\nu, \pi, x, W)$, we know that $F : \text{End}(X) \rightarrow \mathbb{R}$ defines a subsmooth function (see [RW98] Def 10.29) therefore $F$ is semidifferentiable and the semiderivative of $F$ at point $I$ in the direction $L$ equals to (see [RW98] Thm 10.30)
\[
DF(I)(L) = \sup_{\nu, \pi, x \in T(I)} \langle \pi - \nu, L(x) \rangle
\]
where
\[
T(I) = \arg \max_{x \in [0, e], \nu, \pi \in \mathcal{P}(e)} F_{\nu, \pi, x}(I).
\]
Therefore we have:

\[
\alpha = DF(I)(L) = \sup_{\nu, \pi \in \mathcal{P}(e)} \sup_{x \in [0, e]} \langle \pi - \nu, L(x) \rangle = -\inf_{\nu, \pi \in \mathcal{P}(e)} \inf_{x \in [0, e]} \langle \nu, L(x) \rangle + \langle \pi, L(0, e - x) \rangle.
\]

Since \( X \) is finite dimensional, the sets \( \mathcal{P}(e) \) and \([0, e]\) are both compact, and they are the convex hull of their extreme points. Henceforth, arguing as in Remark 3.2 above, we can replace \( \mathcal{P}(e) \) and \([0, e]\) by extr\( \mathcal{P}(e) \) and extr\([0, e]\), respectively.

5.2. **Contraction rate in** \( \mathbb{R}^n \). One may specialize Formula (23) to the case \( X = \mathbb{R}^n \), \( C = \mathbb{R}^+_n \) and \( e = 1 \). For \( x \in \mathbb{R}^n \) we denote by \( \delta(x) \) the diagonal matrix with entries \( x \).

**Corollary 5.2.** Let \( A \) be a square matrix such that \( A1 = 0 \). Then

\[
(25) \quad h(A) = -\min_{i \neq j} (A_{ij} + A_{ji} + \sum_{k \notin \{i, j\}} \min(A_{ik}, A_{jk})).
\]

**Proof.** Recall that

\[
\text{extr}(\mathcal{P}(1)) = \{e_i : i = 1, \ldots, n\}, \quad \text{extr}[0, 1] = \{e_i : I \subset \{1, \ldots, n\}\}.
\]

Therefore we have:

\[
\begin{align*}
    h(A) &= -\min_{i \neq j} \min_{I \subseteq \{1, \ldots, n\}} \sum_{i \in I} A_{ik} + \sum_{k \notin I} A_{jk} \\
    &= -\min_{i \neq j} A_{ij} + A_{ji} + \min_{I \subseteq \{1, \ldots, n\}} \sum_{i \notin I} A_{ik} + \sum_{k \notin I \cup \{i\}} A_{jk} \\
    &= -\min_{i \neq j} A_{ij} + A_{ji} + \sum_{k \notin \{i, j\}} \min(A_{ik}, A_{jk}).
\end{align*}
\]

**Remark 5.3.** Consider the order-preserving case, i.e. \( A_{ij} \geq 0 \) for \( i \neq j \). Such situation was studied extensively in the context of consensus dynamics. In particular, let \( G = (V, E) \) be a graph and equip each arc \((i, j) \in E\) a weight \( C_{ij} > 0 \) (the node \( j \) is connected to \( i \)). One of the consensus systems that Moreau [Mor05] studied is:

\[
\dot{x}_i = \sum_{(i, j) \in E} C_{ij}(x_j - x_i), \quad i = 1, \ldots, n.
\]

This can be written as \( \dot{x} = Ax \), where \( A_{ij} = C_{ij} \) for \( i \neq j \) and \( A_{ii} = \sum_j C_{ij} \) is a **discrete Laplacian**. A general result of Moreau implies that if there is a node connected by path to all other nodes in the graph \( G \), then the system is globally convergent. Our results show that if \( h(C) < 0 \) then the system converges exponentially to consensus with rate \( h(C) \). The condition \( h(C) = 0 \) means that there are two nodes disconnected with each other \((C_{ij} + C_{ji} = 0)\) and all other nodes are connected by arc to at most one of them \((\sum_{k \notin \{i, j\}} \min(C_{ik}, C_{jk}) = 0)\). The condition \( h(C) < 0 \), though more strict than Moreau’s connectivity condition, gives an explicit contraction rate.
Remark 5.4. In addition, our result applies to not necessarily order-preserving flows. For example, consider the matrix
\[
A = \begin{pmatrix}
-3 & 1 & 2 \\
1 & 0 & -1 \\
1 & 1 & -2
\end{pmatrix}.
\]
A basic calculus shows that \(h(A) = -1\). Therefore, every orbit of the linear system 
\(\dot{x} = Ax\) converges exponentially with rate \(-1\) to a multiple of the unit vector.

Remark 5.5. We point out that as a contraction constant, \(h(A)\) makes sense only when \(A1 = 0\). However, as a functional \(h\) is well defined on the space of square matrices. Moreover, since the diagonal elements do not account in the formula (25), it is clear that for any square matrix \(B \in \mathbb{M}_n(\mathbb{R})\) and \(x \in \mathbb{R}^n\)
\[
h(B) = h(B - \delta(x)).
\]

5.3. Contraction in the space of Hermitian matrices. We now specialize Formula (23) to the case \(X = S_n, \mathcal{C} = S_n^+\) and \(e = I_n\):

Corollary 5.6. Let \(\Phi : S_n \to S_n\) be a linear application such that \(\Phi(I_n) = 0\). Then

\[
(26)\quad h(\Phi) = -\inf_{X = (x_1, \ldots, x_n)} \inf_{XX^* = I_n} \left( x_1^* \Phi(x_2x_2^*)x_1 + x_2^* \Phi(x_1x_1^*)x_2 + \sum_{k=3}^n \min(x_k^* \Phi(x_kx_k^*)x_1, x_k^* \Phi(x_kx_k^*)x_2) \right),
\]
where \(x_i\) is the \(i\)-th column vector of each unitary matrix \(X\).

Proof. Recall that
\[
\text{extr}(\mathcal{P}(I_n)) = \{xx^* : x \in \mathbb{C}^n, x^*x = 1\}, \quad \text{extr}[0, I_n] = \{P \in S_n : P^2 = P\}.
\]
Then,
\[
h(\Phi) = -\inf_{x_1^*x_1 = x_2^*x_2 = 1} \inf_{P^2 = P, Px_1 = 0, Px_2 = x_2} x_1^* \Phi(P)x_1 + x_2^* \Phi(I_n - P)x_2
\]
\[
= -\inf_{x_1^*x_1 = x_2^*x_2 = 1} \inf_{P^2 = P, Px_1 = 0} \sum_{i=2}^k x_i^* \Phi(x_ix_i^*)x_1 + x_2^* \Phi(I_n - P)x_2
\]
\[
= -\left( \inf_{x_1^*x_1 = x_2^*x_2 = 1} x_1^* \Phi(x_2x_2^*)x_1 + x_2^* \Phi(x_1x_1^*)x_2 \right)
\]
\[
+ \inf_{X = (x_1, x_2, \ldots, x_n)} \sum_{i=3}^k x_i^* \Phi(x_ix_i^*)x_1 + \sum_{i=k+1}^n x_i^* \Phi(x_ix_i^*)x_2. \quad \square
\]

As pointed out in Remark 5.5, \(h\) is a functional well defined for all linear applications from \(S_n\) to \(S_n\). It is interesting to remark that for any linear application \(\Psi\) and any square matrix \(Z\),
\[
h(\Psi) = h(\Phi)
\]
where \(\Phi(X) = \Psi(X) - ZX - XZ\) for all \(X \in S_n\).
5.4. Contraction rate of time-dependent linear flows. We now state the result analogous to Proposition \ref{5.1}, which applies to time dependent linear flows. Let \( t_0 > 0 \) and \( L(t) : [0, t_0) \times \mathcal{X} \to \mathcal{X} \) be a continuous application linear in the second variable such that \( L_t(e) = 0 \) for all \( t \in [0, t_0) \). We denote by \( U(s, t) \) the evolution operator of the following linear time-varying differential equation:

\[
\dot{x}(t) = L_t(x), \quad t \in [0, t_0).
\]

Then a slight modification of the proof of Proposition \ref{5.1} leads to the following result.

**Proposition 5.7.** The optimal constant \( \alpha \) such that

\[
\|U(s, t)x\|_H \leq e^{\alpha(t-s)}\|x\|_H, \quad \forall s, t \in [0, t_0), x \in \mathcal{X}.
\]

is defined on \( t \in [0, t_0) \)

\[
\sup_{t \in [0, t_0)} h(L_t) = -\inf_{t \in [0, t_0)} \inf_{\nu \in \text{extr} \mathcal{P}(e)} \inf_{x \in \text{extr}([0, e])} \langle \nu, L_t(x) \rangle + \langle \pi, L_t(e - x) \rangle.
\]

6. Contraction rate in Hopf’s oscillation seminorm of nonlinear flows

Let us consider a differentiable application \( \phi : \mathcal{X} \to \mathcal{X} \). Since \( \phi \) is locally Lipschitz, we know that for all \( x_0 \in \mathcal{X} \), there is a maximal interval \( J(x_0) \) such that a unique solution \( x(t; x_0) \) of

\[
\dot{x}(t) = \phi(x(t)), \quad x(0) = x_0
\]

is defined on \( J(x_0) \). We define an application \( M(\cdot) : \mathbb{R} \times \mathcal{X} \to \mathcal{X} \) by:

\[
M_t(x_0) = x(t; x_0), \quad t \in J(x_0).
\]

The application \( M \) is the flow of the equation \( (28) \) and it may not be everywhere defined on \( \mathbb{R} \times \mathcal{X} \). Since \( \phi \) is continuously differentiable, the flow is differentiable with respect to the second variable. We denote by \( DM_t(x) \) the derivative of the application \( M \) with respect to the second variable at point \( (t, x) \). Recall that

\[
DM_t(x)z = D\phi(M_t(x))(DM_t(x)z), \quad t \in J(x), x \in \mathcal{X}.
\]

Let \( U \subset \mathcal{X} \) be a convex open set. For \( x_0 \in U \) define:

\[
t_U(x_0) := \sup\{t_0 \leq J(x_0) : x(t; x_0) \in U, \forall t \in [0, t_0)\}
\]

the time when the solution of \( (28) \) leaves \( U \).

Suppose that \( \phi \) satisfies \( \phi(x + \lambda e) = \phi(x) \) for all \( \lambda \in \mathbb{R} \) and \( x \in \mathcal{X} \). By uniqueness of the solution, it is clear that for all \( x_0 \in \mathcal{X} \) and \( \lambda \in \mathbb{R} \),

\[
M_t(x_0 + \lambda e) = M_t(x_0) + \lambda e, \quad t \in J(x_0).
\]

We define the contraction rate of the flow on \( U \) with respect to Hopf's oscillation seminorm:

\[
\alpha(U) := \inf_{t \in U} \{t \in \mathbb{R} : \|M_t(x) - M_t(y)\|_H \leq e^{\beta t}\|x - y\|_H, x, y \in U, t \leq t_U(x) \land t_U(y)\}. \tag{29}
\]

**Theorem 6.1.** Let \( \phi \) satisfy the above conditions. Then we have

\[
\alpha(U) = \sup_{x \in U} h(D\phi(x))
\]

where \( h \) is defined in \( (23) \).
Proof. Denote

\[ \beta = \sup_{x \in U} h(D\phi(x)). \]

For any \( x \in U \), define

\[ L_t = D\phi(M_t(x)), \ t \in [0, t_U(x)]. \]

Let any \( z \in \mathcal{X} \). Then \( DM_t(x)z : t \in [0, t_U(x)) \) is the solution of the following linear time-varying differential equation:

\[
\begin{cases}
\dot{x} = L_t(x), \ t \in [0, t_U(x)), \\
x(0) = z.
\end{cases}
\]

By Proposition 5.7 it is immediate that for all \( z \in \mathcal{X} \),

\[ \omega(DM_t(x)z/e) \leq e^{3t} \omega(z/e), \ t \in [0, t_U(x)). \]

Let \( x, y \in U \) and \( h < t_U(x) \land t_U(y) \). Denote \( \gamma(s) = sx + (1 - s)y : s \in [0, 1] \). Then,

\[ \omega(M_h(x) - M_h(y)/e) \leq \int_0^1 \omega(DM_h(\gamma(s))(x - y)/e)ds \leq e^{3h} \omega(x - y/e). \]

Therefore, for all \( x, y \in U \),

\[ \lim_{h \to 0^+} \sup_{x \in U} \frac{\|M_h(x) - M_h(y)\|_H}{h} \leq \beta \|x - y\|_H. \]

We deduce that for all \( x, y \in U \) and \( t < t_U(x) \land t_U(y) \),

\[ \lim_{h \to 0^+} \sup_{x \in U} \frac{\|M_{t+h}(x) - M_{t+h}(y)\|_H}{h} \leq \beta \|M_t(x) - M_t(y)\|_H. \]

Therefore,

\[ \|M_t(x) - M_t(y)\|_H \leq e^{3t} \|x - y\|_H, \ t < t_U(x) \land t_U(y). \]

This implies that

\[ \alpha(U) \leq \beta. \]

Inversely, for all \( x \in U \), there is \( t_0 > 0 \) such that for all \( h \leq t_0, z \in \mathcal{X} \),

\[ \|M_h(x + z) - M_h(x)\|_H \leq e^{\alpha(U)h} \|z\|_H. \]

Therefore,

\[ \|DM_h(x)(z)\|_H = \lim_{t \to 0^+} \frac{M_h(x + tz) - M_h(x)}{t} = \lim_{t \to 0^+} \frac{\|M_h(x + tz) - M_h(x)\|_H}{t} \leq e^{\alpha(U)h} \|z\|_H. \]

By Theorem 3.1 we obtain that for \( h \leq t_0 \),

\[ \sup_{\nu, \pi \in P(e)} \sup_{z \in [0, e]} \langle \nu - \pi, DM_h(x)z \rangle \leq e^{\alpha(U)h}. \]

It is then immediate that for \( h \leq t_0 \),

\[ \sup_{\nu, \pi \in \text{extr } P(e)} \sup_{z \in \text{extr } [0, e]} -\langle \nu, DM_h(x)(e - z) \rangle - \langle \pi, DM_h(x)z \rangle \leq e^{\alpha(U)h} - 1. \]

Dividing the two sides by \( h \) and passing to the limit as \( h \to 0 \) we get:

\[ h(D\phi(x)) \leq \alpha(U). \]

Therefore \( \beta \leq \alpha(U). \)
6.1. Applications to non-linear consensus in \( \mathbb{R}^n \). Let \( G = (V, E) \) denote a directed graph. Let us equip every arc \((i, j) \in E\) with a weight \( C_{ij} > 0 \). For \((i, j) \notin E\), we set \( C_{ij} = 0 \).

**Example 6.2.** (Non linear consensus) Consider the following nonlinear consensus protocol \cite{SM03}:

\[
\dot{x}_k = \sum_{(i,k) \in E} C_{ik} \gamma_{ik}(x_i - x_k), \ k = 1, \ldots, n, \tag{30}
\]

where we suppose that every map \( \gamma_{ik} : \mathbb{R}^n \to \mathbb{R} \) is differentiable. When every \( \gamma_{ik} \) is the identity map, the operator at the right hand-side of (30) is the discrete Laplacian of the digraph \( G \), in which \( C_{ik} \) is the conductivity of arc \((i, k)\).

**Proposition 6.3.** Let \( w > 0 \). Suppose that

\[
\alpha := \inf \{ \gamma'_{ik}(t) : t \in [-w, w], (i, k) \in E \} \geq 0. \tag{31}
\]

Consider the convex open set

\[
U(w) = \{ x : \| x \|_H < w \}.
\]

For \( x(0) \in U(w) \), the solution of (30) satisfies:

\[
\| x(t) \|_H \leq e^{h(C) \alpha t} \| x(0) \|_H, \ \forall t \geq 0.
\]

**Proof.** For all \( x \in U \),

\[
h(D\phi(x)) = -\min_{i \neq j} \frac{\partial \phi_i(x)}{\partial x_j} + \frac{\partial \phi_j(x)}{\partial x_i} + \sum_{k \neq i, j} \min \left( \frac{\partial \phi_i(x)}{\partial x_k}, \frac{\partial \phi_j(x)}{\partial x_k} \right),
\]

where

\[
\frac{\partial \phi_i(x)}{\partial x_j} = C_{ij} \gamma'_{ij}(x_j - x_i), \ i \neq j.
\]

Hence for all \( x \in U \),

\[
h(D\phi(x)) \leq -\min_{i \neq j} C_{ij} \alpha + C_{ji} \alpha + \sum_{k \neq i, j} \min(C_{ik} \alpha, C_{jk} \alpha) = \alpha h(C).
\]

We apply Theorem 6.1 and consider \( y = 1 \) in the formula (29). Since \( \alpha \geq 0 \) and \( h(C) \leq 0 \), we deduce that the set \( U(w) \) is invariant. Therefore we conclude. \( \square \)

The Kuramoto equation \cite{Str00} is a special case of the protocol (30).

\[
\dot{\theta}_i = \sum_{j: (i,j) \in E} C_{ij} \sin(\theta_j - \theta_i), \ i = 1, \ldots, n. \tag{32}
\]

Let \( w < \pi/2 \). Then

\[
\inf \{ \cos(t) : t \in [-w, w] \} \geq \cos w > 0.
\]

We apply Proposition 6.3 to obtain that for all \( \theta(0) \) such that \( \| \theta(0) \|_H < w \), the solution of (32) satisfies:

\[
\| \theta(t) \|_H \leq e^{h(C) \cos(w) t} \| \theta(0) \|_H, \ \forall t \geq 0.
\]

In particular, for all \( \theta(0) \in (-\pi/4, \pi/4)^n \), the solution of equation (32) satisfies:

\[
\| \theta(t) \|_H \leq e^{h(C) \cos(\| \theta(0) \|_H) t} \| \theta(0) \|_H, \ \forall t \geq 0.
\]
Remark 6.4. Moreau [Mor05] showed that if there is a node connected by path to all other nodes in the graph \((V,E)\), then the systems \((30)\) is globally convergent and \((32)\) is globally convergent on the set \((-\pi/2, \pi/2)^n\). Compared to his results (see Remark 5.3), our condition for convergence is more strict but we obtain an explicit exponential contraction rate.

Another class of maps satisfying \((31)\) is \(\gamma_{ik}(t) = \arctan(t)\). Consider the following system

\[
\dot{x}_i = \sum_{j: (i,j) \in E} C_{ij} \arctan(x_j - x_i), \quad i = 1, \ldots, n.
\]

Then we obtain in the same way that for all \(x(0) \in \mathbb{R}^n\), the solution of \((33)\) satisfies:

\[
\|x(t)\|_H \leq e^{\frac{\|h(C)\|}{2} t} \|x(0)\|_H, \quad \forall t \geq 0.
\]

Example 6.5. (Discrete \(p\)-Laplacian) We now analyze the degenerate case of the \(p\)-Laplacian consensus dynamics for \(p \in (1,2) \cup (2, +\infty)\). Then latter can be described by the dynamical system in \(\mathbb{R}^n\):

\[
\dot{v}_i = \sum_{j: (i,j) \in E} C_{ij}(v_j - v_i) |C_{ij}(v_i - v_j)|^{p-2}, \quad i = 1, \ldots, n.
\]

Let \(\alpha > \beta > 0\) and consider the convex open sets

\[
V(\beta) := \{ v : \min_{i \neq j} |v_i - v_j| > \beta \}, \quad U(\alpha) := \{ v : \max_{i \neq j} |v_i - v_j| < \alpha \}.
\]

A basic calculus shows that for \(v \in V(\beta)\),

\[
\frac{\partial \phi_i(v)}{\partial v_j} = \begin{cases} 0, & (i,j) \notin E \\ (p-1)|v_i - v_j|^{p-2}C_{ij}^{-1}, & (i,j) \in E \end{cases}
\]

Let \(C^{p-1}\) denote the matrix with entries \(C_{ij}^{p-1}\). Recall that \(h(C^{p-1}) \leq 0\). We have:

\[
h(D\phi(x)) \leq \begin{cases} (p-1)h(C^{p-1})\beta^{p-2}, & p > 2, x \in V(\beta) \\ (p-1)h(C^{p-1})\alpha^{p-2}, & 1 < p < 2, x \in V(\beta) \cap U(\alpha) \end{cases}
\]

When \(1 < p < 2\), the contraction rate on \(V(\beta) \cap U(\alpha)\) tends to \(-\infty\) while \(\alpha\) tends to \(0\). When \(p > 2\), the contraction rate on \(V(\beta)\) tends to \(0\) while \(\beta\) tends to \(0\). If we fix some \(\beta > \min_{(i,j) \in E} C_{ij}^{-1}\), it can be checked that the contraction rate on \(V(\beta)\) tends to \(-\infty\) when \(p\) tends to \(+\infty\).

7. Contraction rate in Hilbert’s metric of non-linear flows

In this section, we apply Theorem 3.1 to determine the contraction rate in Hilbert’s metric of the flow of an ordinary differential equations, still in the finite dimensional case.

7.1. Contraction rate formula in Hilbert’s metric. In the following, we consider a continuously differentiable application \(\phi : \mathcal{C}^0 \to \mathcal{X}\) such that \(\phi(\lambda x) = \lambda \phi(x)\), for all \(\lambda > 0\) and \(x \in \mathcal{C}^0\). Note that the later property implies that

\[
D\phi(x)x = \phi(x).
\]

We denote by \(M\) the flow associated to the differential equation (see Section 6 for notations):

\[
\dot{x} = \phi(x).
\]
Then for the same reason as in the proof of Proposition 5.1, it follows that

\begin{equation}
\alpha(x) := \inf \{ \alpha \in \mathbb{R} : d_H(M_t(x_1), M_t(x_2)) \leq e^{\alpha t} d_H(x_1, x_2), x_1, x_2 \in U, t \leq t_U(x_1) \wedge t_U(x_2) \}. 
\end{equation}

For \( x \in \mathcal{C}^0 \), define:

\begin{equation}
c(x) := - \inf_{z \in [0,x]} \inf_{\nu, \pi \in \mathcal{P}(x)} \langle \pi, D\phi(x)z \rangle + \langle \nu, D\phi(x)(x - z) \rangle
\end{equation}

For the same reason as in the proof of Proposition 5.1, it follows that

\begin{equation}
c(x) = - \inf_{z \in \text{extr}[0,x]} \inf_{\nu, \pi \in \mathcal{P}(x)} \langle \pi, D\phi(x)z \rangle + \langle \nu, D\phi(x)(x - z) \rangle
\end{equation}

**Theorem 7.1.** Let \( U \subset \mathcal{C}^0 \) denote a convex open set such that \( \lambda U = U \) for all \( \lambda > 0 \). Then

\begin{equation}
\alpha(U) = \sup_{x \in U} c(x).
\end{equation}

**Proof.** First we prove that for all \( x \in U \),

\begin{equation}
c(x) = \lim_{t \to 0^+} t^{-1} (\sup_z \frac{\omega(DM_t(x)z/M_t(x))}{\omega(z/x)} - 1).
\end{equation}

For this, fix \( x \in U \) and define a functional on a neighborhood of \( I \):

\begin{equation}
F(W) = \sup_{z \in [0,x]} \sup_{\nu, \pi \in \mathcal{P}(x)} \frac{\nu}{\langle \nu, W(x) \rangle} - \frac{\pi}{\langle \nu, W(x) \rangle} W(z).
\end{equation}

By Theorem 3.1, for \( t \in [0, t_U(x)) \),

\begin{equation}
\|DM_t(x)\|_H = \sup_{z \in [0,x]} \sup_{\nu, \pi \in \mathcal{P}(DM_t(x)x)} \langle \nu - \pi, DM_t(x)z \rangle
= \sup_{z \in [0,x]} \sup_{\nu, \pi \in \mathcal{P}(x)} \langle \nu, DM_t(x)x \rangle - \langle \pi, DM_t(x)x \rangle, DM_t(x)z
= F(DM_t(x)).
\end{equation}

Therefore,

\begin{equation}
\lim_{t \to 0^+} t^{-1} (\sup_z \frac{\omega(DM_t(x)z/M_t(x))}{\omega(z/x)} - 1)
= \lim_{t \to 0^+} t^{-1} (\|DM_t(x)\|_H - 1)
= \lim_{t \to 0^+} t^{-1} (F(DM_t(x)) - F(I))
\end{equation}

Recall that \( DM_t(x) : [0, t_U(x)) \to \text{End}(\mathcal{X}) \) satisfies:

\begin{equation}
\lim_{t \to 0^+} t^{-1} (DM_t(x) - I) = D\phi(x).
\end{equation}

The following reasoning is similar to that in the proof of Proposition 5.1. The limit in (39) equals to the semiderivative of \( F \) at \( I \) in the direction \( D\phi(x) \), if this semiderivative exists.

Since \([0, x]\) and \( \mathcal{P}(x) \) are compact sets and the function

\begin{equation}
F_\nu,\pi,\zeta(W) = \langle \nu, W(x) \rangle - \langle \pi, W(x) \rangle W(z)
\end{equation}
is continuously differentiable on $W$ such that $F_{\nu, \pi, z}$ and the derivative $DF_{\nu, \pi, z}$ are jointly continuous on $(\nu, \pi, z, W)$, we know that $F$ is semidifferentiable. The derivative of $F_{\nu, \pi, z}$ at point $I$ in the direction $D\phi(x)$ is:

\[
DF_{\nu, \pi, z}(I)(D\phi(x)) = \frac{\langle \nu, D\phi(x)z \rangle (\nu, x) - \langle \nu, z \rangle (\nu, D\phi(x)x) - \langle \pi, D\phi(x)z \rangle (\pi, x) - \langle \pi, z \rangle (\pi, D\phi(x)x)}{\langle \pi, x \rangle^2}
\]

Denote

\[
T(W) = \arg \max_{\nu, \pi, z \in P(e)} F_{\nu, \pi, z}(W).
\]

Then

\[
T(I) = \{ \nu, \pi \in P(e), z \in [0, x] : \langle \frac{\nu}{\langle \nu, x \rangle}, \frac{\pi}{\langle \pi, x \rangle}, z \rangle = 1 \}.
\]

The semiderivative of $F$ at point $I$ in the direction $D\phi(x)$ is then:

\[
\lim_{t \to 0^+} t^{-1}(F(DM_t(x)) - F(I)) = \sup_{\nu, \pi, z \in T(I)} DF_{\nu, \pi, z}(W)(D\phi(x)) = \sup_{z \in [0, x]} \sup_{\nu, \pi \in P(x)} (\nu, D\phi(x)z) - (\nu, D\phi(x)x) - (\pi, D\phi(x)z) = c(x).
\]

Now fix $x_0 \in U$. By Cauchy-Lipschitz, there is $r > 0$ and $t_0 > 0$ such that the flow is well-defined on $[0, t_0] \times B(x_0; r)$ where $B(x_0; r)$ is the open ball of radius $r$ centered at $x_0$. We assume that $B(x_0; r) \subset U$ and consider the set $G := \cup_{\lambda > 0} \lambda B(x_0; r)$. For every $t \leq t_0$, the application $M_t$ is well defined on $G$ such that

\[
d_H(M_t(x), M_t(y)) \leq e^{\alpha(U)t} d_H(x, y), \quad \forall x, y \in G.
\]

By Theorem 5.8 we have

\[
\omega(DM_t(x)v/M_t(x)) \leq e^{\alpha(U)t}\omega(v/x) \quad \forall x \in G, v \in \mathcal{X}.
\]

Therefore,

\[
c(x_0) = \lim_{t \to 0^+} \frac{1}{t} \left( \sup_z \omega(DM_t(x_0)z/M_t(x_0)) \right) - 1) \leq \alpha(U).
\]

It follows that

\[
\alpha(U) \geq \sup_{x \in U} c(x).
\]

Finally, denote

\[
c = \sup_{x \in U} c(x).
\]

Then for all $x \in U$, $v \in \mathcal{X}$ and $t \in U(x)$,

\[
\lim_{h \to 0^+} \frac{\omega(DM_{t+h}(x)v/M_{t+h}(x)) - \omega(DM_t(x)v/M_t(x))}{h}
\]

\[
= \lim_{h \to 0^+} \frac{\omega(DM_h(M_t(x))(DM_t(x)v)/M_t(M_t(x))) - \omega(DM_t(x)v/M_t(x))}{h}
\]

\[
= \lim_{h \to 0^+} \frac{\omega(DM_t(x)v/M_t(x))}{h} \left( \frac{\omega(DM_h(M_t(x))(DM_t(x)v)/M_h(M_t(x))) - \omega(DM_t(x)v/M_t(x))}{\omega(DM_t(x)v/M_t(x))} - 1 \right)
\]

\[
\leq c(M_t(x)) \omega(DM_t(x)v/M_t(x)) \leq \omega(DM_t(x)v/M_t(x)).
\]
Therefore, for all \( x \in U \), \( v \in \mathcal{X} \) and \( t \in t_U(x) \) we have that,
\[
\omega(DM_t(x)v/M_t(x)) \leq e^{ct}\omega(v/x).
\]
Let \( x, y \in U \) and define \( \gamma(s) = (1-s)x + sy \), \( 0 \leq s \leq 1 \). By the compacity of the set \( \{\gamma(s) : s \in [0,1]\} \), we know that
\[
t_0 := \inf \{ t_U(\gamma(s)) : s \in [0,1] \} > 0.
\]
Therefore, using the Finsler structure of Hilbert’s metric ([Nus94 Thm 2.1]), we get that for every \( t \leq t_0 \),
\[
d_H(M_t(x), M_t(y)) \leq \int_0^1 \omega(DM_t(\gamma(s))(y-x)/M_t(\gamma(s)))ds \leq \int_0^1 e^{ct}\omega(y-x/\gamma(s))ds = e^{ct}d_H(x,y).
\]
Consequently we proved that for all \( x, y \in U \)
\[
limit_{h \to 0^+} \frac{d_H(M_{h+t}(x), M_{h+t}(y)) - d_H(x,y)}{h} \leq cd_H(x,y).
\]
This implies that for all \( x, y \in U \) and \( t < t_U(x) \land t_U(y) \):
\[
limit_{h \to 0^+} \frac{d_H(M_{t+h}(x), M_{t+h}(y)) - d_H(M_t(x), M_t(y))}{h} = \limsup_{h \to 0^+} \frac{d_H(M_{h}(M_t(x)), M_{h}(M_t(y))) - d_H(M_t(x), M_t(y))}{h} \leq cd_H(M_t(x), M_t(y)).
\]
It follows that
\[
d_H(M_t(x), M_t(y)) \leq e^{ct}d_H(x,y), \ \forall x, y \in U, t < t_U(x) \land t_U(y).
\]
Therefore
\[
\alpha(U) \leq c.
\]
\[\square\]

7.2. Contraction rate in Hilbert’s projective metric of a non-linear flow on the standard positive cone. We specialize the contraction formula (38) to the case \( \mathcal{X} = \mathbb{R} \) and \( \mathcal{C} = \mathbb{R}^+ \) under the same notations and assumptions.

Corollary 7.2. When \( \mathcal{X} = \mathbb{R}^n \) and \( \mathcal{C} = \mathbb{R}^+_n \), the contraction rate formula (38) can be specified as below:
\[
\alpha(U) = \sup_{x \in U} c(x) = \sup_{x \in U} h(A(x)), \ \forall x \in U
\]
where
\[
A(x) = \delta(x)^{-1}D\phi(x)\delta(x)
\]
and \( h \) is defined in (25).

Proof. It is sufficient to remark that in this special case:
\[
\extr P(x) = \delta(x)^{-1} \extr P,
\]
and
\[
\extr [0, x] = \delta(x) \extr ([0, 1])
\]
Therefore,
\[
c(x) = - \inf_{z \in \text{extr}[0, x]} \inf_{\pi, \nu \in \text{extr} P} \langle \nu, D\phi(x)z \rangle + \langle \pi, D\phi(x)(x - z) \rangle
\]
\[
= - \inf_{z \in \text{extr}[0, 1]} \inf_{\pi, \nu \in \text{extr} P} \langle \delta(x)^{-1}\nu, D\phi(x)\delta(x)z \rangle + \langle \delta(x)^{-1}\pi, D\phi(x)\delta(x)(1 - z) \rangle
\]
\[
= h(A(x)).
\]

\[\square\]

Remark 7.3. Consider the linear flow in \(\mathbb{R}^n\) of the following equation:
\[
\dot{x} = Ax,
\]
where \(A_{ij} \geq 0\), for all \(i \neq j\), so that the flow is order preserving. Let \(x\) be in the interior of \(\mathbb{R}^n_+\). Then we have
\[
\delta(x)^{-1}A\delta(x)_{ij} = A_{ij}\frac{x_j}{x_i}, \quad i, j = 1, \ldots, n.
\]
Therefore,
\[
h(\delta(x)^{-1}A\delta(x)) = - \min_{i \neq j} A_{ij}\frac{x_j}{x_i} + A_{ij}\frac{x_j}{x_i} + \sum_{k \notin \{i,j\}} \min(A_{ik}\frac{x_k}{x_i}, A_{jk}\frac{x_k}{x_j}).
\]
The global contraction rate (restricted to \(C^0\)) is then
\[
\sup_{x \in C^0} h(\delta(x)^{-1}A\delta(x)) = - \min_{i \neq j} 2\sqrt{A_{ij}A_{ji}}.
\]
Such contraction rate can be alternatively obtained by differentiating with respect to \(t\) at 0 the contraction ratio of \(I + tA\), using Birkhoff’s theorem. Hence a positive global contraction rate exists if and only if \(A_{ij} > 0\) for all \(i \neq j\). However, strict local contraction may occur even if there is \(A_{ij} = 0\) for some \(i \neq j\). Let \(K > 1\) and consider the convex open set
\[
U(K) = \{ x \in \mathbb{R}^n : \frac{1}{K} < \frac{x_i}{x_j} < K \}.
\]
Then the local contraction rate with respect to \(U\) is
\[
\sup_{x \in U(K)} h(\delta(x)^{-1}A\delta(x)) \leq \frac{h(A)}{K}.
\]
Therefore, \(h(A) < 0\) is sufficient to have a strict local contraction. Moreover, the above bound on the contraction rate decreases (faster convergence) as the orbit approaches to consensus, i.e., a multiple of \(1\).

7.3. Application to the space of Hermitian matrices. We specialize the contraction formula \[38\] to the case \(X = S_n\) and \(C = S_n^+\) under the same notations and assumptions.

Corollary 7.4. When \(X = S_n\) and \(C = S_n^+\), the contraction rate formula \[38\] can be specified as below:
\[
\alpha(U) = \sup_{P \in U} c(P) = \sup_{P \in U} h(\Phi(P))
\]
where \(\Phi(P) : S_n^+ \to S_n^+\) is a linear application given by:
\[
\Phi(P)(Z) = P^{-\frac{1}{2}}D\phi(P)(P^\frac{1}{2}ZP^\frac{1}{2})P^{-\frac{1}{2}}
\]
and \(h\) is defined in \[26\].
Proof. Remark that in this special case,

\[ \text{extr}[0, P] = P^{\frac{1}{2}}(\text{extr}[0, I_n])P^{\frac{1}{2}}, \]

and

\[ \text{extr}(P(P)) = P^{-\frac{1}{2}}(\text{extr} P)P^{-\frac{1}{2}}. \]

The desired formula is obtained the same way as in the proof of Corollary 7.2.

\[ \square \]

Example 7.5. As an example, let us show a calculus of contraction rate using Corollary 7.4 for the following differential equation in \( S_n \):

\[ \dot{P} = \phi(P) := \frac{-PB P}{\text{trace}(CP)} + AP + PA' \]

where \( B, C \in \hat{S}_n^+ \). Let \( \dot{P} \in S_n^+ \). Then the application \( \Phi(P) : S_n \to S_n \) defined in Corollary 7.2 is given by:

\[ \Phi(P)(Z) = P^{-\frac{1}{2}}D\phi(P)(ZP^{\frac{1}{2}})P^{-\frac{1}{2}} \]

\[ = (-ZP^{\frac{1}{2}}BP^{\frac{1}{2}} - P^{\frac{1}{2}}BP^{\frac{1}{2}}Z) \text{trace}(CP)^{-1} \]

\[ + P^{\frac{1}{2}}BP^{\frac{1}{2}} \text{trace}(CP)^{-2} \text{trace}(CP^{\frac{1}{2}}ZP^{\frac{1}{2}}) + P^{-\frac{1}{2}}AP^{\frac{1}{2}}Z + ZA'P^{-\frac{1}{2}} \]

Therefore let \( x, y \in \mathbb{C}^n \) such that \( x^*y = 0 \) then

\[ y^*\Phi(P)(x x^*)y = y^*P^{\frac{1}{2}}BP^{\frac{1}{2}}y \text{trace}(CP^{\frac{1}{2}}x x^*P^{\frac{1}{2}}) \text{trace}(CP)^{-2}. \]

Let \( \{x_1, \ldots, x_n\} \) be an orthonormal basis. Denote \( \alpha_1 = x_1^*P^{\frac{1}{2}}BP^{\frac{1}{2}}x_1, \alpha_2 = x_2^*P^{\frac{1}{2}}BP^{\frac{1}{2}}x_2, \beta_1 = x_1^*P^{\frac{1}{2}}CP^{\frac{1}{2}}x_1 \) and \( \beta_2 = x_1^*P^{\frac{1}{2}}CP^{\frac{1}{2}}x_1 \). Suppose that \( \alpha_1 \leq \alpha_2 \). Then

\[ x_1^*\Phi(P)(x_2 x_2^*)x_1 + x_2^*\Phi(P)(x_1 x_1^*)x_2 + \sum_{k=3}^{n} \min(x_k^*\Phi(P)(x_k x_k^*)x_1, x_k^*\Phi(P)(x_k x_k^*)x_2) \]

\[ = (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1 \sum_{k=3}^{n} x_k^*P^{\frac{1}{2}}CP^{\frac{1}{2}}x_k) \text{trace}(CP)^{-2} \]

\[ = (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1(\text{trace}(CP) - \beta_1 - \beta_2)) \text{trace}(CP)^{-2} \]

\[ = (\alpha_1\beta_1 + \alpha_2(\alpha_2 - \alpha_1)) \text{trace}(CP)^{-2} \]

\[ \geq \lambda_{\min}(BP) \text{trace}(CP)^{-1}. \]

Therefore by the definition in (20),

\[ h(\Phi(P)) \leq -\lambda_{\min}(BP) \text{trace}(CP)^{-1}. \]

Let us consider the convex open set

\[ U = \{P \in \hat{S}_n^+ : d_H(P, I_n) < K \}. \]

Then,

\[ \sup_{P \in U} h(\Phi(P)) \leq \sup_{P \in U} -\lambda_{\min}(BP) \text{trace}(CP)^{-1} \]

\[ \leq -\frac{\lambda_{\min}(BP)}{n\lambda_{\max}(CP)} \leq -\frac{\lambda_{\min}(B)\lambda_{\min}(P)}{n\lambda_{\max}(C)\lambda_{\max}(P)} \]

\[ \leq -\frac{\lambda_{\min}(B)}{n\lambda_{\max}(C)e^K}. \]

Let \( \alpha = -\frac{\lambda_{\min}(B)}{n\lambda_{\max}(C)e^K} \). Then by Corollary 7.4 for all \( P_1, P_2 \in U \) we have:

\[ d_H(M_t(P_1), M_t(P_2)) \leq e^{\alpha t}d_H(P_1, P_2), \quad 0 \leq t < t_U(P_1) \wedge t_U(P_2). \]
If $A, B, C$ are matrices such that
\[ \phi(I_n) = -B \text{trace}(C)^{-1} + A + A' = \lambda_0 I_n, \]
then we know that
\[ M_t(I_n) = e^{\lambda_0 t} I_n. \]
In that case, for $P \in U$ we have:
\[ d_H(M_t(P), e^{\lambda_0 t} I_n) \leq e^{\alpha t} d_H(P, I_n), \quad 0 \leq t < t_U(P). \]
It follows that $t_U(P) = +\infty$ and therefore every solution of equation converges exponentially to a scalar multiplication of $I_n$.

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