Structure of relative genus fields of cubic Kummer extensions

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Abstract
Let $N = K(\sqrt[3]{D})$ be a cubic Kummer extension of the cyclotomic field $K = \mathbb{Q}(\zeta_3)$, containing a primitive cube root of unity $\zeta_3$, with cube free integer radicand $D > 1$. Denote by $f$ the conductor of the abelian extension $N/K$, and by $N^*$ the relative genus field of $N/K$. The aim of the present work is to find out all positive integers $D$ and conductors $f$ such that the genus group $\mathrm{Gal}(N^*/N) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is elementary bicyclic.

Keywords Pure cubic field · Cubic Kummer extension · Relative genus field · Group of ambiguous ideal classes · 3-Rank · Primitive ambiguous principal ideals · Multiplicity of conductors

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1 Introduction

Let $K = \mathbb{Q}(\zeta_3)$, where $\zeta_3$ is a primitive cube root of unity, $N = \mathbb{Q}(\sqrt[3]{D}, \zeta_3)$, where $D > 1$ is a cube free integer radicand, $\text{Gal}(N/K) = \langle \sigma \rangle$ the cyclic relative Galois group of $N/K$, $\text{Cl}_3(N)$ be the 3-class group of $N$, $N_3^{(1)}$ be the maximal abelian unramified 3-extension of $N$, $f$ the conductor of the abelian extension $N/K$ and $m$ its multiplicity. Let $N^*$ be the maximal abelian extension of $K$ contained in $N_3^{(1)}$, which is called the relative genus field of $N/K$ (see [11, 12] or [15]).

One tries to determine the unramified 3-sub-extensions of $N_3^{(1)}/N$ and then, according to class field theory, extract information about $\text{Cl}_3(N)$, its rank, and the 3-class field tower of $N$. One way to do this is by asking for the structure of the relative genus field $N^*$ of $N/K$. In our recent work [1], we implemented Gerth’s methods [11, 12] to determine the rank of the group $\text{Cl}_3^{(\sigma)}(N)$ of ambiguous ideal classes of $N/K$ and obtained all integers $D > 1$ for which $\text{Gal}(N^*/N) \cong \mathbb{Z}/3\mathbb{Z}$ is cyclic of order 3. The purpose of the present work is to find all positive integers $D$ and conductors $f$ such that $\text{Gal}(N^*/N) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is elementary bicyclic. In fact, we shall prove the following Main Theorem:

**Theorem 1.1** Let $N = \mathbb{Q}(\sqrt[3]{D}, \zeta_3)$, where $D$ is a cube free positive integer, $K = \mathbb{Q}(\zeta_3)$, and $N^*$ the relative genus field of $N/K$. Then $\text{Gal}(N^*/N) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ if and only if the integer $D$ can be written in one of the following thirteen forms:

$$D = \begin{cases} 3^e p_1^{e_1} & \text{with } p \equiv 1 \pmod{9}, \\ p_1^{e_1} q_1^{f_1} & \text{with } p \equiv -q \equiv 1 \pmod{9}, \\ p_1^{e_1} q_1^{f_1} f_1 & \text{with } p \not\equiv 0 \pmod{9} \text{ or } q \not\equiv -1 \pmod{9}, \\ 3^e p_2^{e_2} & \text{with } p \equiv 1 \pmod{9} \text{ or } q \equiv -1 \pmod{9}, \\ p_2^{e_2} f_2 & \text{with } p \equiv 1 \pmod{9} \text{ or } q \equiv -1 \pmod{9}, \\ p_2^{e_2} q_1^{f_1} q_2^{f_2} f_2 & \text{with } p \equiv 1 \pmod{9} \text{ and } q_1, q_2 \equiv 2 \text{ or } 5 \pmod{9}, \\ p_2^{e_2} q_1^{f_1} q_2^{f_2} f_2 & \text{with } p, -q_1, -q_2 \equiv 4 \text{ or } 7 \pmod{9}, \\ p_2^{e_2} q_1^{f_1} q_2^{f_2} q_3^{f_3} & \text{with } p, -q_1, -q_2 \equiv 4 \text{ or } 7 \pmod{9} \text{ and } q_1 \equiv -1 \pmod{9}, \\ 3^e q_1^{f_1} & \text{such that } \exists i \in \{1, 2\} \mid p_i \not\equiv 1 \pmod{9}, \\ q_1^{f_1} q_2^{f_2} q_3^{f_3} & \text{with } q_1 \equiv q_2 \equiv -1 \pmod{9}, \\ q_1^{f_1} q_2^{f_2} q_3^{f_3} & \text{with } q_1 \equiv q_2 \equiv q_3 \equiv -1 \pmod{9}, \\ q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} & \text{such that } \exists i \in \{1, 2, 3, 4\} \mid q_i \equiv -1 \pmod{9}, \\ 3^e q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} & \text{such that } \exists i \in \{1, 2, 3, 4\} \mid q_i \equiv -1 \pmod{9}, \end{cases}$$

where $e, e_1, e_2, f_1, f_2, f_3$ and $f_4$ are positive integers equal to 1 or 2.

As opposed to the radicands $D$ of the shape in [1, Thm. 1.1, p 251], the possible prime factorizations of $D$ in our Main Theorem 1.1 are more complicated. For background see Sect. 2.

In Sect. 3, we give Theorems 3.1 and 3.2 which show that pure cubic fields $L = \mathbb{Q}(\sqrt[3]{D})$ can be collected in multiplets $(L_1, \ldots, L_m)$ sharing a common conductor $f$ with multiplicity $m$ and a common type of ambiguous class group $\text{Cl}_3^{(\sigma)}(N)$ of $N$. At the beginning of Sect. 4, where Theorem 1.1 is proved, we restrict ourselves to those results that will be needed in this paper. More information on 3-class groups can be found in [10–12, 20–23]. For the prime decomposition in a pure cubic field $\mathbb{Q}(\sqrt[3]{D})$,

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we refer the reader to the papers [5, 6, 9, 24], and for the prime ideal factorization rules of $\mathbb{Q}(\zeta_3)$, we refer to [17]. In Sect. 3.1, we present numerical examples concerning the structure of the 3-class groups of pure cubic fields $L$ and of their Galois closures $N$ in the case where conductors $f$ contain splitting prime divisors.

**Notations:**

- $p$ and $q$, $\ell$ are prime numbers such that $p \equiv 1 \pmod{3}$ and $q$, $\ell \equiv -1 \pmod{3}$;
- $L = \mathbb{Q}(\sqrt[3]{D})$ is a pure cubic field, where $D > 1$ is a cube free positive integer;
- $K = \mathbb{Q}(\zeta_3)$, where $\zeta_3 = e^{2\pi i/3}$ denotes a primitive third root of unity;
- $N = \mathbb{Q}(\sqrt[3]{D}, \zeta_3)$ is the normal closure of $L$;
- $f$ is the conductor of the abelian extension $N/K$;
- $\langle \tau \rangle = \text{Gal}(N/L)$ such that $\tau^2 = id$, $\tau(\zeta_3) = \zeta_3^2$ and $\tau(\sqrt[3]{D}) = \sqrt[3]{D}$;
- $\langle \sigma \rangle = \text{Gal}(N/K)$ such that $\sigma^3 = id$, $\sigma(\zeta_3) = \zeta_3$, $\sigma(\sqrt[3]{D}) = \zeta_3^2\sqrt[3]{D}$ and $\tau \sigma = \sigma^2 \tau$;
- $\lambda = 1 - \zeta_3$ and $\pi, \rho$ are prime elements of $K$;
- $q^* = 1$ or 0 according to whether $\zeta_3$ is or is not norm of an element of $N \setminus \{0\}$;
- $t$ denotes the number of prime ideals ramified in $N/K$.

For an algebraic number field $F$:

- $O_F, E_F$: the ring of integers and the group of units of $F$;
- $\text{Cl}_3(F), F_3^{(1)}$: the 3-class group and the Hilbert 3-class field of $F$.

### 2 Background

For the convenience of the reader, we collect all formulas for pure cubic fields $L = \mathbb{Q}(\sqrt[3]{D})$ and their normal closures $N = \mathbb{Q}(\sqrt[3]{D}, \zeta_3)$ on which subsequent proofs and exposition build up. Generally, we adopt the notation of Gerth [11]. In particular, the prime decomposition of the cube free integer *radicand* $D$ is written in the following form (Equation (3.2) of [11, p. 55]):

$$D = 3^e \cdot p_1^{e_1} \cdot p_v^{e_v} \cdot p_{v+1}^{e_{v+1}} \cdots p_w^{e_w} \cdot q_1^{f_1} \cdots q_I^{f_I} \cdot q_{I+1}^{f_{I+1}} \cdots q_J^{f_J},$$  \hspace{1cm} (2)

where $p_i$ and $q_i$ are rational prime numbers such that:

\[
\begin{align*}
p_i &\equiv 1 \pmod{9}, \quad \text{for } 1 \leq i \leq v, \\
p_i &\equiv 4 \text{ or } 7 \pmod{9}, \quad \text{for } v + 1 \leq i \leq w, \\
q_i &\equiv 8 \pmod{9}, \quad \text{for } 1 \leq i \leq I, \\
q_i &\equiv 2 \text{ or } 5 \pmod{9}, \quad \text{for } I + 1 \leq i \leq J, \\
e_i &\equiv 1 \text{ or } 2, \quad \text{for } 1 \leq i \leq w, \\
f_i &\equiv 1 \text{ or } 2, \quad \text{for } 1 \leq i \leq J, \\
e &\equiv 0 \text{ or } 1 \text{ or } 2.
\end{align*}
\]

According to [1, Eqn. (2.5)–(2.6), p. 255] the corresponding prime decomposition of the class field theoretic *conductor* $f$ of the abelian relative extension $N/K$ [14] is given by
Here the exponent $\varepsilon = v_3(f)$ of the distinguished prime 3 characterizes the Dedekind species of $L$:

\[ f = 3^\varepsilon \cdot p_1 \cdots p_v \cdot p_{v+1} \cdots p_w \cdot q_1 \cdots q_1 \cdot q_{l+1} \cdots q_J. \] (3)

Additionally, we use the following conventions for conjugate prime elements in the cyclotomic field $K = \mathbb{Q}(\zeta_3)$ which divide splitting rational prime numbers: $p = \rho \rho^T$, if $p \equiv 1 \pmod{9}$, but $p = \pi \pi^T$, if $p \equiv 4, 7 \pmod{9}$. For inert rational prime numbers, we replace $q$ by $\ell$, if $q \equiv 8 \pmod{9}$, but we keep $q$, if $q \equiv 2, 5 \pmod{9}$.

In order to determine the 3-rank $r$ of the ambiguous 3-class group $\text{Cl}_3^\sigma(N)$ of $N/K$, Gerth used the general formula of Hasse [13, Thm. 13] in [12, Eqn. (2.1), pp. 86–87] and, adapted to the particular situation $K = \mathbb{Q}(\zeta_3)$ in [12, Prop. 5.1, pp. 92–93], in the implicit form $r = t + q^* - 2$, $t = J + 2w(+1)$. However, we only need the following entirely explicit version [11, Lem. 3.1, p. 55] for the normal closure $N$ of a pure cubic field $L$, in dependence on the Dedekind species $v_3(f)$:

\[
\begin{align*}
J + 2w & \quad \text{for } (v = w \text{ and } I = J) \text{ and } v_3(f) > 0 \text{ (Species 1)}, \\
J + 2w - 1 & \quad \text{for } (v < w \text{ or } I < J) \text{ and } v_3(f) > 0 \text{ (Species 1)}, \\
J + w - 1 & \quad \text{for } (v = w \text{ and } I = J) \text{ and } v_3(f) = 0 \text{ (Species 2)}, \\
J + 2w - 2 & \quad \text{for } (v < w \text{ or } I < J) \text{ and } v_3(f) = 0 \text{ (Species 2)}.
\end{align*}
\] (5)

According to [25, Thm. 2.1, p. 833] and [26, Cor. 3.2, p. 2219], the multiplicity $m(f)$ of the conductor $f$, that is, the number of pairwise non-isomorphic pure cubic fields $L = \mathbb{Q}(\sqrt[3]{3D})$ sharing the common discriminant $d_L = -3 \cdot f^2$, is given in terms of all, respectively free (good) and restrictive (bad), prime divisors of $f$, i.e. $T = w + J$, respectively $G = v + I$, $B = T - G$, in dependence on the Dedekind species $v_3(f)$ by

\[
m = m(f) = \begin{cases} 2^T, & \text{if } v_3(f) = 2 \text{ (Species 1a)}, \\ 2^G \cdot X_B, & \text{if } v_3(f) = 1 \text{ (Species 1b)}, \\ 2^G \cdot X_B^{-1}, & \text{if } v_3(f) = 0 \text{ (Species 2)}, \end{cases}
\] (6)

where the sequence $(X_k)_{k \geq -1}$, $X_k = \frac{1}{3}[2^k - (-1)^k]$ starts with $\frac{1}{2}, 0, 1, 1, 3, 5, 11$. In particular, the multiplicity is zero, either if $B = 0$ for a field of species 1b or if $B = 1$ for a field of species 2.

Now we give an elementary simultaneous proof of Honda’s Theorem [16, Thm., § 1, p. 8], our Main Theorem in [1, Thm. 1.1, p. 251], and our Main Theorem 1.1 in the present article. The proof consists of a systematic combinatorial construction of all possible conductors $f$, ordered firstly by increasing number $t$ of ramified prime ideals of $N/K$, and, for fixed $t$, secondly by decreasing number $w$ of primes splitting in $K$. 

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For each conductor $f$, the rank $r$ and the multiplicity $m$ are determined with the aid of our collection of formulas. On principle, this could be done by a computer search up to a given maximal $t$, if all deterministic rules concerning the 3-rank $r$ of ambiguous class groups (5) and the multiplicity $m$ of conductors $f$ (6) are implemented in the program script. The principal factorization type is abbreviated by PFT [1, Thm. 2.1, p. 254].

The simultaneous proof of [16, Thm., § 1, p. 8], [1, Thm. 1.1, p. 251], and Theorem 1.1 in the Tables 1, 2 and 3, which are already purged from nilets, i.e. multiplets with $m = 0$, namely $f \in \{3\ell, 3\rho\ell, 3\ell_1\ell_2, 3\rho\ell^3\ell, 3\ell_1\ell_2\ell_3\}$ of species 1b, and $f \in \{q, \pi\ell, q\ell, \pi\ell, \rho\ell, q\ell, q\ell_1\ell_2\}$ of species 2, establishes the rank $r$ of the ambiguous
Table 3  Arithmetic invariants for conductors of Dedekind species 2

| J | w | I | v | f | t | q* | r | G | B | T | m | PFT | References |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | ℓ | 1 | 1 | 0 | 1 | 1 | 1 | γ | [1, Thm. 2.3 (2)] |
| 0 | 1 | 0 | 1 | ρρ^ς | 2 | 1 | 1 | 1 | 0 | 1 | 1 | α, γ | [1, Thm. 2.4 (1)] |
| 2 | 0 | 0 | 0 | q_1q_2 | 2 | 0 | 0 | 0 | 2 | 2 | 1 | β | [1, Thm. 2.3 (5)] |
| 2 | 0 | 2 | 0 | ℓ_1ℓ_2 | 2 | 1 | 1 | 2 | 0 | 2 | 2 | β, γ | [1, Thm. 2.5 (2)] |
| 1 | 1 | 0 | 0 | ππ^τq | 3 | 0 | 1 | 0 | 2 | 2 | 1 | α, β | [1, Thm. 2.4 (4)] |
| 1 | 1 | 1 | 1 | ρρ^ςℓ | 3 | 1 | 2 | 2 | 0 | 2 | 2 | α, β, γ | Thm. 3.1 (2) |
| 3 | 0 | 0 | 0 | q_1q_2q_3 | 3 | 0 | 1 | 0 | 3 | 3 | 1 | β | [1, Thm. 2.5 (6)] |
| 3 | 0 | 1 | 0 | q_1q_2ℓ | 3 | 0 | 1 | 1 | 2 | 3 | 2 | β | [1, Thm. 2.5 (7)] |
| 3 | 0 | 3 | 0 | ℓ_1ℓ_2ℓ_3 | 3 | 1 | 2 | 3 | 0 | 3 | 4 | β, γ | Thm. 3.2 (2) |
| 0 | 2 | 0 | 0 | π_1π_2^τπ_2π_3^τ | 4 | 0 | 2 | 0 | 2 | 2 | 1 | α, β | Thm. 3.1 (9) |
| 2 | 1 | 0 | 0 | ππ^τq_1q_2 | 4 | 0 | 2 | 0 | 3 | 3 | 1 | α, β | Thm. 3.1 (7) |
| 2 | 1 | 1 | 0 | ππ^τq_1q_2 | 4 | 0 | 2 | 1 | 2 | 3 | 2 | α, β | Thm. 3.1 (8) |
| 2 | 1 | 0 | 1 | ρρ^ςq_1q_2 | 4 | 0 | 2 | 1 | 2 | 3 | 2 | α, β | Thm. 3.1 (6) |
| 4 | 0 | 0 | 0 | q_1q_2q_3q_4 | 4 | 0 | 2 | 0 | 4 | 4 | 3 | β | Thm. 3.2 (6) |
| 4 | 0 | 1 | 0 | q_1q_2q_3ℓ | 4 | 0 | 2 | 1 | 3 | 4 | 2 | β | Thm. 3.2 (7) |
| 4 | 0 | 2 | 0 | q_1q_2ℓ_1ℓ_2 | 4 | 0 | 2 | 2 | 2 | 4 | 4 | β | Thm. 3.2 (8) |

3-class group Cl_3^{(σ)}(N) which is only an approximation of the entire 3-class group Cl_3(N).

The systematic investigation of the full 3-class group Cl_3(N) of the normal closure N of arbitrary non-Galois cubic number fields L began in 1933 with conclusions from the class number relation h_N = ϱ^3 · h_2^L · h_K of Scholz [32, p. 213, p. 216], involving the quadratic subfield K of N, the torsion free Dirichlet unit rank ϱ of L, and the index u = (E_N : E_0) of the subgroup E_0 generated by the units of all subfields of N in the unit group E_N of N. For the situation of a pure cubic field L = Q(3√D) with ϱ = 1 and h_K = 1, the index u can only take two values, u = 1 for the differential principal factorization type (briefly PFT) α in the sense of [1, Thm. 2.1, p. 254], and u = 3 for the types β and γ. In particular, the isomorphism Cl_3(N) ≅ Cl_3(L) × Cl_3(L^σ) to the direct product is only possible for the types β and γ [32, p. 219], since, due to the relative principal factorization [6] in a field N of type α, the direct product shrinks to a quotient (Cl_3(L) × Cl_3(L^σ))/(Z/3Z). This was confirmed later by Gerth [11, Thm. 5.1, p. 61] in the important case w = 0, that is, if the conductor f is only divisible by non-split primes q_i, 1 ≤ i ≤ J.

The need for a detailed description of the 3-class group Cl_3(N) of the normal closure N of a pure cubic field L, in terms of power products of prime ideals whose classes generate the group, has its origin in the 1992 Doctoral Thesis of Ismaili [18], where he studied the capitulation kernels ker(T_i) in an elementary bicyclic 3-class group Cl_3(N) with respect to the transfers T_i : Cl_3(N) → Cl_3(K_i) from N to the four unramified cyclic cubic extensions K_1, . . . , K_4 of N. Together with El Mesaoudi [19], he determined necessary and sufficient conditions for N to possess an elementary bi-
homo-cyclic 3-class group $\text{Cl}_3(N) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, isomorphic to the direct product of two copies of $\text{Cl}_3(L) \cong \mathbb{Z}/3\mathbb{Z}$, which enforces $u = 3$, i.e. PFT $\beta$ or $\gamma$.

In [3], we studied the capitulation in a non-elementary bi-hetero-cyclic 3-class group $\text{Cl}_3(N) \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, based on a general theory of such class groups in [27]. This scenario occurs for the conductor $f = \rho \rho^z$, i.e. a prime $p \equiv 1 \pmod{9}$ [1, Thm. 2.4 (1)], with $u = 1$ and $\text{Cl}_3(L) \cong \mathbb{Z}/9\mathbb{Z}$.

In a forthcoming work [4], we shall investigate the capitulation in an elementary tri-homo-cyclic 3-class group $\text{Cl}_3(N) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, which enforces $u = 1$, i.e. PFT $\alpha$, for parity reasons in the class number formula of Scholz.

## 3 Conductors and their multiplicities

Before we give the proof of Theorem 1.1 in Sect. 4, we devote the present section to an overview of the special properties of pure cubic fields $L = \mathbb{Q}({\sqrt{\mathcal{D}}})$ possessing the radicands $\mathcal{D}$ of the shapes listed in Eq. (1) of the introduction Sect. 1.

### 3.1 Conductors with splitting prime divisors

Let us start with giving more details concerning the leading eight lines of Eq. (1) in our Theorem 1.1, where $\mathcal{D}$ is divisible by a prime $p_1 \equiv 1 \pmod{3}$ which splits in $K$.

**Theorem 3.1** Let the conductor of $N/K$ be $f = 3^\varepsilon \cdot p_1 \cdots p_w \cdot q_1 \cdots q_j$ as in Eq. (3) with $0 \leq \varepsilon \leq 2$, $T = w + J \geq 1$, and pairwise distinct primes $p_i \equiv 1 \pmod{3}$ for $1 \leq i \leq w$, and $q_i \equiv -1 \pmod{3}$ for $1 \leq i \leq J$. Briefly denote the multiplicity of $f$ by $m := m(f)$. Assume that $w \geq 1$. Then, rank $(\text{Cl}_3^{(\sigma)}(N)) = 2 \iff L$ belongs to one of the following multiplets:

1. **doublets**, $m = 2$, of type $(\alpha x, \beta y, \gamma z)$, $x + y + z = 2$, such that $f = 9p_1$ with $p_1 \equiv 1 \pmod{9}$,
2. **doublets**, $m = 2$, of type $(\alpha x, \beta y, \gamma z)$, $x + y + z = 2$, such that $f = p_1q_1$ with $p_1 \equiv 1 \pmod{9}$ and $q_1 \equiv 8 \pmod{9}$,
3. **singlets**, $m = 1$, of type $\alpha$ or $\beta$ such that $f = 3p_1q_1$ with $p_1 \equiv 4, 7 \pmod{9}$, $q_1 \equiv 2, 5 \pmod{9}$,
4. **doublets**, $m = 2$, of type $(\alpha x, \beta y)$, $x + y = 2$, such that $f = 3p_1q_1$ with either $(p_1 \equiv 1 \pmod{9} \text{ and } q_1 \equiv 2, 5 \pmod{9})$ or $(p_1 \equiv 4, 7 \pmod{9} \text{ and } q_1 \equiv 8 \pmod{9})$,
5. **quartets**, $m = 4$, of type $(\alpha x, \beta y)$, $x + y = 4$, such that $f = 9p_1q_1$ with $p_1 \not\equiv 1 \pmod{9}$ or $q_1 \not\equiv 8 \pmod{9}$,
6. **doublets**, $m = 2$, of type $(\alpha x, \beta y)$, $x + y = 2$, such that $f = p_1q_1q_2$ with $p_1 \equiv 1 \pmod{9}$ and $q_1, q_2 \equiv 2, 5 \pmod{9}$,
7. **singlets**, $m = 1$, of type $\alpha$ or $\beta$ such that $f = p_1q_1q_2$ with $p_1 \equiv 4, 7 \pmod{9}$ and $q_1, q_2 \equiv 2, 5 \pmod{9}$,
8. **doublets**, $m = 2$, of type $(\alpha x, \beta y)$, $x + y = 2$, such that $f = p_1q_1q_2$ with $p_1 \equiv 4, 7 \pmod{9}$, $q_1 \equiv 2, 5 \pmod{9}$ and $q_2 \equiv 8 \pmod{9}$,
9. **singlets**, $m = 1$, of type $\alpha$ or $\beta$ such that $f = p_1p_2$ with $p_1, p_2 \equiv 4, 7 \pmod{9}$.
There exist infinitely many multiplets with conductors of all these shapes (1)–(9).

**Proof** The multiplicity $m(f)$ of each conductor is calculated by means of Formula (6), using the sequence $(X_k)_{k \geq 1} = (\frac{1}{2}, 0, 1, 1, 3, 5, 11, \ldots)$:

1. For $D = 3^e p_1^f \not\equiv \pm 1 \pmod{9}$ with $p_1 \equiv 1 \pmod{9}$, we have $f = 3^2 p_1$ of species 1a. We must take into consideration that $G = 1$, $B = 0$, $T = 1$, where $G$, $B$ and $T$ are the numbers defined in (6), and we obtain $m(f) = 2^T = 2$, a doublet, independently of $G$ and $B$.

   Since $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{ab})$ for $a, b \in \mathbb{N}$, we can choose $e_1 = 1$, so $D = 3^e p$ with $e \in \{1, 2\}$. As $p_1 \equiv 1 \pmod{3}$, then $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are primes of $K$ such that $\pi_1^3 = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3 \mathcal{O}_K}$, and $p$ is totally ramified in $L$. So $\mathcal{O}_N = \mathcal{P}_1^3$ and $\mathcal{P}_2 \mathcal{O}_N = \mathcal{P}_2^3$ where $\mathcal{P}_1, \mathcal{P}_2$ are two prime ideals of $N$. The fact that $D \not\equiv \pm 1 \pmod{9}$ implies that 3 is totally ramified in $L$, then $\lambda$ is ramified in $N/K$. So the number of ideals which are ramified in $N/K$ is $t = 3$. As $3 = -\zeta_3^2 \lambda^2$, then $N = K(\sqrt[3]{x})$, where $x = \zeta_3^2 \lambda^2 \pi_1 \pi_2$. If $p \equiv 1 \pmod{9}$, then $\pi_1$ and $\pi_2$ are congruent to 1 (mod $\lambda^3$), so by [1, Lem. 3.3, p. 264], $\zeta_3$ is a norm of an element of $N - \{0\}$ and $q^* = 1$, thus rank $(\text{Cl}^{(a)}_3(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}^{(a)}_3(N)) = 2$ to this shape of conductor will be proved in Sect. 4.1.1.

2. For $D = p_1^{e_1} q_1^{f_1}$ with $p_1 \equiv -q_1 \equiv 1 \pmod{9}$, we have $f = p_1 q_1$ of species 2. We must take into consideration that $G = 2$, $B = 0$, and we obtain $m(f) = 2^G \cdot X_{B-1} = 2^2 \cdot \frac{1}{2} = 2$, a doublet.

   Here $e_1, f_1 \in \{1, 2\}$. Since $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{ab})$ for $a, b \in \mathbb{N}$, we can choose $e_1 = 1$, so $D = pq^{f_1}$. As $D \equiv \pm 1 \pmod{9}$, 3 is not ramified in the field $L$, and then $\lambda$ is not ramified in $N/K$. The fact that $p \equiv 1 \pmod{3}$, implies that $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are primes of $K$ such that $\pi_1^3 = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3 \mathcal{O}_K}$, since $p$ is totally ramified in $L$, then $\pi_1$ and $\pi_2$ are totally ramified in $N$. As $q \equiv -1 \pmod{3}$, $q$ is inert in $K$. Then, the primes ramified in $N/K$ are $\pi_1, \pi_2$ and $q$, and $t = 3$. Since $p \equiv -q \equiv 1 \pmod{9}$, then $\pi_1 \equiv \pi_2 \equiv \pi \equiv 1 \pmod{\lambda^3}$, where $-q = \pi$ is a prime number of $K$. Then, $N = K(\sqrt[3]{x})$, where $x = \pi_1^{e_1} \pi_2^{e_1} \pi^{f_1}$. The fact that all primes $\pi$, $\pi_1$ and $\pi_2$ are congruent to 1 (mod $\lambda^3$), implies by [1, Lem. 3.3, p. 264] that $\zeta_3$ is a norm of an element of $N - \{0\}$ and $q^* = 1$. Hence, rank $(\text{Cl}^{(a)}_3(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}^{(a)}_3(N)) = 2$ to this shape of conductor will be proved in Sect. 4.2.1.

3. For $D = p_1^{e_1} q_1^{f_1} \not\equiv \pm 1 \pmod{9}$ with $(p_1 \equiv 4, 7 \pmod{9}$ and $q_1 \equiv 2, 5 \pmod{9})$, we have $f = 3 p_1 q_1$ of species 1b. We must take into consideration that $G = 0$, $B = 2$, and we obtain $m(f) = 2^G \cdot X_B = 1 \cdot 1 = 1$, a singlet.

   Here $e_1, f_1 \in \{1, 2\}$. As $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{ab})$ for $a, b \in \mathbb{N}$, we can choose $e_1 = 1$, i.e $D = pq^{f_1}$. Since $D \not\equiv \pm 1 \pmod{9}$, then 3 is ramified in the field $L$, so $\lambda = 1 - \zeta_3$ is ramified in $N/K$. Since $p \equiv 1 \pmod{3}$, then $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are primes of $K$ such that $\pi_1^3 = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3 \mathcal{O}_K}$,
the prime $p$ is totally ramified in $L$, so $\pi_1$ and $\pi_2$ are totally ramified in $N$. Since $q \equiv -1 \pmod{3}$, then $q$ is inert in $K$. Thus the primes ramified in $N/K$ are $\pi_1, \pi_2$ and $q$, and $t = 4$.

Let $-q = \pi$ be a prime number of $K$, and put $x = \pi_1^{e_1} \pi_2^{e_2} \pi^{f_1}$, then $N = K(\sqrt[3]{x})$. Then:

- If $p \not\equiv 1 \pmod{9}$, then by [1, Lem. 3.1, p. 263], $\pi_1 \not\equiv 1 \pmod{\lambda^3}$ and $\pi_2 \not\equiv 1 \pmod{\lambda^3}$, then according to [1, Lem. 3.3, p. 264], $\zeta_3$ is not a norm of an element of $N - \{0\}$.
- If $q \not\equiv -1 \pmod{9}$, then by [1, Lem. 3.2, p. 264], $\pi \not\equiv 1 \pmod{\lambda^3}$, then according to [1, Lem. 3.3, p. 264], $\zeta_3$ is not a norm of an element of $N - \{0\}$.

In all cases we have $q^* = 0$. We conclude that rank $(\text{Cl}^\sigma_3(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}^\sigma_3(N)) = 2$ to this shape of conductor will be proved in Sect. 4.2.1.

(4) For $D = p_1^{e_1} q_1^{f_1} \not\equiv \pm 1 \pmod{9}$ with either $(p_1 \equiv 1 \pmod{9}$ and $q_1 \equiv 2, 5 \pmod{9})$ or $(p_1 \equiv 4, 7 \pmod{9}$ and $q_1 \equiv 8 \pmod{9})$, we have $f = 3p_1q_1$ of species 1b. We must take into consideration that $G = 1, B = 1$, and we obtain $m(f) = 2^G \cdot X_B = 2 \cdot 1 = 2$, a doublet.

(5) For $D = 3^e p_1^{e_1} q_1^{f_1} \not\equiv \pm 1 \pmod{9}$ with $p_1 \not\equiv 1 \pmod{9}$ or $q_1 \not\equiv 8 \pmod{9}$, we have $f = 3^2 p_1q_1$ of species 1a, $T = 2$, and we get $m(f) = 2^T = 2^2 = 4$, a quartet, independently of $G$ and $B$.

Here $e_1, f_1 \in \{1, 2\}$. As $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b}) \forall a, b \in \mathbb{N}$, we can choose $e_1 = 1$, i.e. $D = 3^e p q^{f_1}$. Since $D \not\equiv \pm 1 \pmod{9}$, we reason as in case (3) and we obtain $t = 4$. Since $3 = -\zeta_3^2 \lambda^2$, then $N = K(\sqrt[3]{x})$, where $x = \zeta_3^2 \lambda^2 \pi_1^{e_1} \pi_2^{e_2} \pi^{f_1}$. As in case (3), if $p \not\equiv 1 \pmod{9}$ or $q \not\equiv -1 \pmod{9}$, then by using [1, Lem. 3.1, Lem. 3.2, Lem. 3.3, p. 263-264], we get $\zeta_3$ is not a norm of an element of $N - \{0\}$ and $q^* = 0$. We conclude that rank $(\text{Cl}^\sigma_3(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}^\sigma_3(N)) = 2$ to this shape of conductor will be proved in Sect. 4.2.1.

(6) For $D = p_1^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 1 \pmod{9}$ with $p_1 \equiv 1 \pmod{9}$ and $q_1, q_2 \equiv 2$ or $5 \pmod{9}$, we have $f = p_1q_1q_2$ of species 2. We must take into consideration that $G = 2, B = 2$, and we obtain $m(f) = 2^G \cdot X_{B-1} = 2 \cdot 1 = 2$, a doublet.

Here $e_1, f_1 \in \{1, 2\}$. As $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b}) \forall a, b \in \mathbb{N}$, we can choose $e_1 = 1$, i.e. $D = pq^{f_1} q_2^{f_2}$. Since $D \equiv \pm 1 \pmod{9}$, then 3 is not ramified in the field $L$, so $\lambda$ is not ramified in $N/K$. Since $p \equiv 1 \pmod{3}$, then $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are primes of $K$ such that $\pi_1^r = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_K}$, the prime $p$ is totally ramified in $L$, so $\pi_1$ and $\pi_2$ are totally ramified in $N$. Since $q_1 \equiv q_2 \equiv -1 \pmod{3}$, then $q_1$ and $q_2$ are inert in $K$. Thus the primes ramified in $N/K$ are $\pi_1, \pi_2, q_1$, and $q_2$. Then, $t = 4$. 
Let \(-q_1 = \pi'\) and \(-q_2 = \pi''\) be two prime numbers of \(K\), and put \(x = \pi_1^{e_1}\pi_2^{e_2}\pi^f_1\pi^f_2\), then \(N = K(\sqrt{x})\). As \(q_1 \equiv -1 (\text{mod } 9)\), then by [1, Lem. 3.2, p. 264], \(\pi' \not\equiv 1 (\text{mod } \lambda^3)\), then according to [1, Lem. 3.3, p. 264], \(\zeta_3\) is not a norm of an element of \(N - \{0\}\), so \(q^* = 0\). Thus, rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\). The opposite implication of the rank condition rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\) to this shape of conductor will be proved in Sect. 4.3.1.

(7) For \(D = p_1^{e_1}q_1^{f_1}q_2^{f_2} \equiv \pm 1 (\text{mod } 9)\) with \(p_1, -q_1, -q_2 \equiv 4 \text{ or } 7 (\text{mod } 9)\), we have \(f = p_1q_1q_2\) of species 2. We must take into consideration that \(G = 0, B = 3\), and we obtain \(m(f) = 2G \cdot X_{B-1} = 1 \cdot 1 = 1\), a singlet.

We reason as in case (6), we get \(t = 4\) and \(q^* = 0\), then rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\). The opposite implication of the rank condition rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\) to this shape of conductor will be proved in Sect. 4.3.1.

(8) For \(D = p_1^{e_1}q_1^{f_1}q_2^{f_2} \equiv \pm 1 (\text{mod } 9)\) with \(p_1, -q_2 \equiv 4 \text{ or } 7 (\text{mod } 9)\) and \(q_1 \equiv 8 (\text{mod } 9)\), we have \(f = p_1q_1q_2\) of species 2. We must take into consideration that \(G = 0, B = 2\), and we obtain \(m(f) = 2G \cdot X_{B-1} = 2 \cdot 1 = 2\), a doublet.

We reason as in case (6), we get \(t = 4\) and \(q^* = 0\), then rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\). The opposite implication of the rank condition rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\) to this shape of conductor will be proved in Sect. 4.3.1.

(9) For \(D = p_1^{e_1} p_2^{e_2} \equiv \pm 1 (\text{mod } 9)\) such that \(p_i \not\equiv 1 (\text{mod } 9)\) for some \(i \in \{1, 2\}\), we have \(f = p_1p_2\) of species 2. We must take into consideration that \(G = 0, B = 2\), and we obtain \(m(f) = 2G \cdot X_{B-1} = 1 \cdot 1 = 1\), a singlet.

Here \(e_1, e_2 \in \{1, 2\}\). We have \(p_1 = \pi_1\pi_2\) and \(p_2 = \pi_3\pi_4\), where \(\pi_1, \pi_2, \pi_3\) and \(\pi_4\) are primes of \(K\) such that \(\pi_2 = \pi_1^1, \pi_4 = \pi_3^3\), and \(\pi_1 \equiv \pi_2 \equiv \pi_3 \equiv \pi_4 \equiv 1 (\text{mod } 3\mathcal{O}_K)\), and since \(p_1\) and \(p_2\) are ramified in \(L\), then \(\pi_1, \pi_2, \pi_3, \text{ and } \pi_4\) are ramified in \(N\). The fact that \(D \equiv \pm 1 (\text{mod } 9)\) implies that 3 is not ramified in \(L\), so \(\lambda\) is not ramified in \(N/K\). Then we get \(t = 4\).

As \(p_1 \not\equiv 1 (\text{mod } 9)\) or \(p_2 \not\equiv 1 (\text{mod } 9)\), then by [1, Lem. 3.1, p. 263], \(\exists i \in \{1, 2, 3, 4\}\) such that \(\pi_i\) is not congruent to 1 (mod \(\lambda^3\)), then according to [1, Lem. 3.3, p. 264], \(\zeta_3\) is not norm of an element of \(N - \{0\}\) and \(q^* = 0\).

Hence, rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\). The opposite implication of the rank condition rank \((\text{Cl}_3^{(\sigma)}(N)) = 2\) to this shape of conductor will be proved in Sect. 4.3.2.

The principal factorization type is a consequence of the estimates in [1, Thm. 2.1, p. 254]. Since \(s = 1\), we have \(0 \leq R \leq s = 1\) and type \(\alpha\) with relative PF may generally occur. For all items (1)–(9), we have \(3 \leq t \leq 4\) and thus \(1 \leq A \leq \min(2, t - s) = 2\) with \(2 = 3 - 1 \leq t - s \leq 4 - 1 = 3\), which generally enables type \(\beta\) with absolute PF, but type \(\gamma\) may occur. For items (3)–(9), there exists either a prime factor \(p_1 \equiv 4, 7 (\text{mod } 9)\) or a prime factor \(q_1 \equiv 2, 5 (\text{mod } 9)\), whence type \(\gamma\) is impossible, because \(\zeta_3\) can be norm of a unit in \(N\) only if the prime factors of \(f\) are 3 or \(\ell_j \equiv 1, 8 (\text{mod } 9)\).

All claims on the infinitude of the various sets of conductors \(f\) are a consequence of Dirichlet’s theorem on primes in arithmetic progressions. □
Table 4  Radicands $D = 3^e p_1$ of species 1a with $m = 2$

| $D$  | $3^e$ | $p_1$ | $f$  | $\left(\frac{3}{p_1}\right)_3$ | $u$ | PFT | $\text{Cl}_3(L)$ | $\text{Cl}_3(N)$ |
|------|-------|-------|------|-------------------------------|-----|-----|-----------------|-----------------|
| 57   | 3     | 19    | 171  | $\neq 1$                      | 3   | $\gamma$ | (3)             | (3, 3)          |
| 111  | 3     | 37    | 333  | $\neq 1$                      | 3   | $\gamma$ | (3)             | (3, 3)          |
| 219  | 3     | 73    | 657  | $= 1$                         | 3   | $\beta$  | (3, 3)          | (9, 3, 3)       |
| 327  | 3     | 109   | 981  | $\neq 1$                      | 3   | $\gamma$ | (3)             | (3, 3)          |
| 813  | 3     | 271   | 2439 | $= 1$                         | 1   | $\alpha$ | (27, 3)         | (27, 27, 3)     |
| 921  | 3     | 307   | 2763 | $= 1$                         | 3   | $\beta$  | (3, 3)          | (9, 3, 3)       |
| 1569 | 3     | 523   | 4707 | $= 1$                         | 1   | $\alpha$ | (9, 3)          | (9, 9, 3)       |
| 1629 | $3^2$ | 181   | 1629 | $\neq 1$                      | 3   | $\gamma$ | (3)             | (3, 3)          |
| 1791 | $3^2$ | 199   | 1791 | $\neq 1$                      | 3   | $\gamma$ | (3)             | (3, 3)          |

**Example 1**  In all of our applications, we present the structure of the 3-class groups of pure cubic fields $L$ and of their Galois closures $N$ in the case where the conductor $f$ contains splitting prime divisors. Computations were performed with Magma [7, 8, 30] and Pari/GP [31].

Let $\text{Cl}_3(N)$ (respectively $\text{Cl}_3(L)$) be the 3-class group of $N$ (respectively of $L$), and $N_3^{(1)}$ be the maximal abelian unramified 3-extension of $N$. Let $u$ be the index of the subgroup generated by the units of intermediate fields of the extension $N/Q$ in the group of units of $N$. According to [6, § 12, Theorem 12.1, p. 229], there are two possibilities, either $u = 1$ or $u = 3$.

In Table 4, we start with conductors $f = 9 p_1$ of species 1a, where $p_1 \equiv 1 \pmod{9}$. See Theorem 3.1, item (1), and the first line of Eq. (1).

**Remark 3.1**  In Table 4, if the 3-class group $\text{Cl}_3(N)$ is of type $(3, 3)$, then $N^* = N_3^{(1)}$. In this case, the cubic residue symbol $\left(\frac{3}{p_1}\right)_3$ is different from 1 if and only if 3 divides exactly the class number of $L$, which implies that $u = 3$. Consequently, Table 4 confirms the previous results by Ismaili and El Mesaoudi [19, Theorem 1, case (i), p. 157; Theorem 2 and Corollary 1, pp. 161–162].

Table 5 deals with conductors $f = p_1 q_1$ of species 2, where $p_1 \equiv -q_1 \equiv 1 \pmod{9}$. See Theorem 3.1, item (2), and the second line of Eq. (1).

**Remark 3.2**  In Table 5, if the 3-class group $\text{Cl}_3(N)$ is of type $(3, 3)$, then $N^* = N_3^{(1)}$. In this case, the structure of the 3-class group $\text{Cl}_3(N)$ is independent of the cubic residue symbol $\left(\frac{3}{p_1}\right)_3$. However, the cubic residue symbol $\left(\frac{q_1}{p_1}\right)_3$ is different from 1 if and only if 3 divides exactly the class number of $L$, which implies that $u = 3$. Consequently, Table 5 confirms the previous results by Ismaili and El Mesaoudi [19, Theorem 1, case (ii), p. 157; Theorem 3 and Corollary 2, pp. 162–163].

Table 6 concerns conductors $f = 3 p_1 q_1$ of species 1b, where $p_1 \equiv 1 \pmod{9}$ and $q_1 \equiv 2, 5 \pmod{9}$. See Theorem 3.1, item (4), and the third line of Eq. (1).
Table 5 Radicands \(D = p_1 q_1\) of species 2 with \(m = 2\)

| \(D\) | \(p_1\) | \(q_1\) | \(f\) | \(\left(\frac{3}{p_1}\right)_3\) | \(\left(\frac{q_1}{p_1}\right)_3\) | \(\gamma\) | \(\beta\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) |
|-------|-------|-------|-------|----------------|----------------|-------|-------|-------|-------|-------|-------|-------|
| 323   | 19    | 17    | 323   | ≠ 1            | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     |
| 1007  | 19    | 53    | 1007  | ≠ 1            | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     |
| 1241  | 73    | 17    | 1241  | = 1            | = 1            | 1     | 1     | 3     | 3     | 1     | 1     | 3     |
| 1349  | 19    | 71    | 1349  | ≠ 1            | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     |
| 2033  | 19    | 107   | 2033  | ≠ 1            | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     |
| 3401  | 19    | 179   | 3401  | ≠ 1            | = 1            | 1     | 1     | 3     | 3     | 1     | 1     | 3     |
| 3869  | 73    | 53    | 3869  | = 1            | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     |

Table 6 Radicands \(D = p_1 q_1\) of species 1b with \(m = 2\)

| \(D\) | \(p_1\) | \(q_1\) | \(f\) | \(\left(\frac{q_1}{p_1}\right)_3\) | \(\gamma\) | \(\beta\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) |
|-------|-------|-------|-------|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 38    | 19    | 2     | 114   | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     |
| 95    | 19    | 5     | 285   | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     |
| 146   | 73    | 2     | 438   | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     |
| 209   | 19    | 11    | 627   | = 1            | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     |
| 1577  | 19    | 83    | 4731  | = 1            | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     |
| 2147  | 19    | 113   | 6441  | = 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     |
| 3287  | 19    | 173   | 9861  | ≠ 1            | 3     | 3     | 1     | 1     | 3     | 3     | 1     | 1     | 3     | 3     |

Other examples are presented in [2, Exm. 7.4, pp. 116–117]: 14801 = 19^2 \cdot 41 and 56129 = 37^2 \cdot 41 contain \(p_1 \equiv 1 \pmod{9}\) whereas 833 = 7^2 \cdot 17, 8959 = 17^2 \cdot 31, 97997 = 43^2 \cdot 53 contain \(q_1 \equiv 8 \pmod{9}\), also with \(m = 2\). However, 1573 = 11^2 \cdot 13, 4901 = 13^2 \cdot 29, 22747 = 23^2 \cdot 43, 32269 = 23^2 \cdot 61 are singlets \((m = 1)\) belonging to Theorem 3.1, item (3). All of them are M0-fields. (Cfr. [28, Thm. 4–6].)

**Remark 3.3** In Table 6, if the 3-class group \(\text{Cl}_3(N)\) is of type \((3, 3)\), then \(N^* = N_3^{(1)}\). In this case, the cubic residue symbol \(\left(\frac{q_1}{p_1}\right)_3\) is different from 1 if and only if 3 divides exactly the class number of \(L\), which implies that \(u = 3\). Consequently, Table 6 confirms the previous results by Ismaili and El Mesaoudi [19, Theorem 1, case (ii), p. 157; Theorem 5 and Corollary 4, p. 165].

Table 7 treats conductors \(f = 9p_1q_1\) of species 1a, where \(p_1 \equiv 4, 7 \pmod{9}\) and \(q_1 \equiv -1 \pmod{3}\). See Theorem 3.1, item (5), and the fourth line of Eq. (1).

**Remark 3.4** In Table 7, if the 3-class group \(\text{Cl}_3(N)\) is of type \((3, 3)\), then \(N^* = N_3^{(1)}\). In this case, the cubic residue symbol \(\left(\frac{3e q_1}{p_1}\right)_3\) is different from 1 if and only if 3 divides exactly the class number of \(L\), which implies that \(u = 3\). Consequently, Table 7 confirms the previous results by Ismaili and El Mesaoudi [19, Theorem 1, case (ii), p. 157; Theorem 8 and Corollary 7, p. 174].

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Table 7 Radicands $D = 3^e p_1 q_1$ of species 1a with $m = 4$

| $D$ | $3^e$ | $p_1$ | $q_1$ | $f$ | $\left( \frac{3}{p_1} \right)_3$ | $\left( \frac{q_1}{p_1} \right)_3$ | $\left( \frac{3^e q_1}{p_1} \right)_3$ | $u$ | PFT | $\text{Cl}_3(L)$ | $\text{Cl}_3(N)$ |
|-----|-------|-------|-------|-----|-------------------------------|-------------------------------|-------------------------------|-----|-----|----------------|----------------|
| 342 | $3^2$ | 19    | 2     | 342 | $\neq 1$                      | $\neq 1$                      | $= 1$                          | 1   | $\alpha$ | (9, 3)         | (9, 9, 3)     |
| 555 | 3     | 37    | 5     | 1665 | $\neq 1$                      | $\neq 1$                      | $\neq 1$                      | 3   | $\beta$ | (3)            | (3, 3)        |
| 3663| $3^2$ | 37    | 11    | 3663 | $\neq 1$                      | $= 1$                         | $\neq 1$                      | 3   | $\beta$ | (3)            | (3, 3)        |
| 5757| 3     | 19    | 101   | 17,271 | $\neq 1$                     | $\neq 1$                      | $= 1$                          | 3   | $\beta$ | (3, 3)        | (9, 3, 3)     |
| 6441| 3     | 19    | 113   | 19,323 | $\neq 1$                     | $= 1$                         | $\neq 1$                      | 3   | $\beta$ | (3)            | (3, 3)        |
| 9519| 3     | 19    | 167   | 28,557 | $\neq 1$                      | $\neq 1$                      | $= 1$                          | 1   | $\alpha$ | (27, 3)        | (27, 27, 3)   |
Table 8 deals with conductors \( f = p_1 q_1 q_2 \) of species 2, where \( p_1 \equiv 1 \pmod{9} \) and \( q_1, q_2 \equiv 2, 5 \pmod{9} \). See Theorem 3.1, item (6), and the fifth line of Eq. (1). Other examples are given in [2, Exm. 7.1, p. 113], namely the M0-fields \( D = 12673 = 19 \cdot 23 \cdot 29, 20539 = 19 \cdot 23 \cdot 47 \).

**Remark 3.5** In Table 8, if the 3-class group \( \text{Cl}_3(N) \) is of type (3, 3), then \( N^* = N_3^{(1)}. \) In this case, the cubic residue symbol \( \left( \frac{q_1 q_2}{p_1} \right)_3 \) is different from 1 if and only if 3 divide exactly the class number of \( L \), and imply that \( u = 3 \). Consequently, Table 8 confirms the previous results by Ismaili and El Mesoudi [19, Theorem 1, case (iii)], p. 157; Theorem 4 and Corollary 3, pp. 164–165].

An example for Theorem 3.1, item (8), and the seventh line of Eq. (1) is given by the M0-field \( D = 52417 = 23 \cdot 43 \cdot 53 \) in [2, Exm. 7.3, p. 115], which belongs to a doublet \( (m = 2) \).

In the outlook of the paper [1, §6, Outlook, Figure 3, Scenario III, p. 273], we presented a scenario for the relative 3-genus field \( N^* \) in the case where the 3-class group \( \text{Cl}_3(N) \) is of type (3, 3), and which is relevant for Remarks 3.1, 3.2, 3.3, 3.4 and 3.5.

**Remark 3.6** It should be pointed out that Ismaili [18] has investigated another related scenario for the relative 3-genus field \( N^* \). If the conductor \( f \) is divisible by exactly one prime \( p \equiv 1 \pmod{3} \), that is \( s = 1 \), but is not contained in Theorem 3.1, and if \( C_{k,3} = (3,3) \) is elementary bicyclic, then the genus field coincides with the Hilbert 3-class field, \( N^* = N_3^{(1)}, \) and the composite \( N \cdot L_3^{(1)} = N \cdot (L_3^{(1)})^3 \) coincide with one of the four unramified cyclic cubic extensions \( K_1, \ldots, K_4 \) of \( N \) within \( N_3^{(1)} \), as illustrated in Fig. 1 and studied in detail by Ismaili and El Mesoudi [19].

### 3.2 Conductors without splitting prime divisors

Now we give more details concerning the trailing five lines of Eq. (1) in our Theorem 1.1, where \( D \) is only divisible by primes \( q_j \equiv -1 \pmod{3} \) which do not split in \( K \).

**Theorem 3.2** Let the conductor of \( N / K \) be \( f = 3^\varepsilon \cdot p_1 \cdots p_w \cdot q_1 \cdots q_J \) as in Eq. (3) with \( 0 \leq \varepsilon \leq 2, T = w + J \geq 1, \) and pairwise distinct primes \( p_i \equiv 1 \pmod{3} \) for \( 1 \leq i \leq w, \) and \( q_i \equiv -1 \pmod{3} \) for \( 1 \leq i \leq J \). Briefly denote the multiplicity of \( f \) by \( m := m(f). \) Assume that \( w = 0. \) Then, \( \text{rank} \left( Cl_3^\sigma(N) \right) = 2 \iff L \) belongs to one of the following multiplets:

1. quartets, \( m = 4, \) of type \( (\beta^x, \gamma^y), x + y = 4, \) such that \( f = 9q_1 q_2 \) with \( q_1 \equiv q_2 \equiv 8 \pmod{9}, \)
2. quartets, \( m = 4, \) of type \( (\beta^x, \gamma^y), x + y = 4, \) such that \( f = q_1 q_2 q_3 \) with \( q_1 \equiv q_2 \equiv q_3 \equiv 8 \pmod{9}, \)
3. triplets, \( m = 3, \) of type \( (\beta, \beta, \beta) \) such that \( f = 3q_1 q_2 q_3 \) with \( q_1, q_2, q_3 \equiv 2, 5 \pmod{9}, \)
### Table 8  Radicands $D = p_1 q_1 q_2$ of species 2 with $m = 2$

| $D$  | $p_1$ | $q_1$ | $q_2$ | $f$     | $\left(\frac{q_1}{p_1}\right)_3$ | $\left(\frac{q_2}{p_1}\right)_3$ | $\left(\frac{q_1 q_2}{p_1}\right)_3$ | $u$ | PFT | Cl$_3$(L) | Cl$_3$(N) |
|------|-------|-------|-------|--------|-------------------------------|-------------------------------|---------------------------------|----|-----|-----------|-----------|
| 190  | 19    | 2     | 5     | 190    | $\neq 1$                       | $\neq 1$                       | $\neq 1$                        | 3  | $\beta$ | (3)       | (3, 3)    |
| 874  | 19    | 2     | 23    | 874    | $\neq 1$                       | $\neq 1$                       | $\neq 1$                        | 3  | $\beta$ | (3, 3)    | (9, 3, 3) |
| 1558 | 19    | 2     | 41    | 1558   | $\neq 1$                       | $\neq 1$                       | $\neq 1$                        | 3  | $\beta$ | (3)       | (3, 3)    |
| 9361 | 37    | 11    | 23    | 9361   | = 1                            | = 1                            | = 1                             | 1  | $\alpha$| (27, 3)   | (27, 9, 3, 3) |
| 24679| 37    | 23    | 29    | 24,679 | = 1                            | = 1                            | = 1                             | 3  | $\beta$ | (9, 3)    | (9, 9, 3, 3) |
| 43993| 37    | 29    | 41    | 43,993 | = 1                            | $\neq 1$                       | $\neq 1$                        | 3  | $\beta$ | (3)       | (3, 3)    |
Fig. 1 Case where $N^* = N_3^{(1)}$

(4) doublets, $m = 2$, of type $(\beta, \beta)$ such that $f = 3q_1q_2q_3$ with $q_1, q_2 \equiv 2, 5 \pmod{9}$ and $q_3 \equiv 8 \pmod{9}$,
(5) quartets, $m = 4$, of type $(\beta, \beta, \beta, \beta)$ such that $f = 3q_1q_2q_3$ with $q_1 \equiv 2, 5 \pmod{9}$ and $q_2 \equiv q_3 \equiv 8 \pmod{9}$,
(6) triplets, $m = 3$, of type $(\beta, \beta, \beta)$ such that $f = q_1q_2q_3q_4$ with $q_1, q_2, q_3, q_4 \equiv 2, 5 \pmod{9}$,
(7) doublets, $m = 2$, of type $(\beta, \beta)$ such that $f = q_1q_2q_3q_4$ with $q_1, q_2, q_3 \equiv 2, 5 \pmod{9}$ and $q_4 \equiv 8 \pmod{9}$,
(8) quartets, $m = 4$, of type $(\beta, \beta, \beta, \beta)$ such that $f = q_1q_2q_3q_4$ with $q_1, q_2 \equiv 2, 5 \pmod{9}$ and $q_3 \equiv q_4 \equiv 8 \pmod{9}$,
(9) octets, $m = 8$, of type $(\beta, \beta, \beta, \beta, \beta, \beta, \beta, \beta)$ such that $f = 9q_1q_2q_3$ with $q_i \not\equiv 8 \pmod{9}$ for some $i \in \{1, 2, 3\}$.

There exist infinitely many multiplets with conductors of all these shapes (1)–(9).

**Proof** The multiplicity $m(f)$ of each conductor is calculated by means of Formula (6), using the sequence $(X_k)_{k \geq -1} = (\frac{1}{2}, 0, 1, 1, 3, 5, 11, \ldots)$:

(1) For $D = 3^e q_1^{f_1} q_2^{f_2} \not\equiv \pm 1 \pmod{9}$ with $q_1 \equiv q_2 \equiv -1 \pmod{9}$, we have $f = 3^2 q_1q_2$ of species 1a. We must take into consideration that $G = 2$, $B = 0$, $T = 2$, and we obtain $m(f) = 2^T = 2^2 = 4$, a quartet, independently of $G$ and $B$. Here $e, f_1$, and $f_2 \in \{1, 2\}$. Since $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b})$ for $a, b \in \mathbb{N}$, we can choose $e = 1$, so $D = 3q_1^{f_1} q_2^{f_2}$. The prime $q_i$ is inert in $K$ for each $i \in \{1, 2\}$, and $q_i$ is ramified in $L$. The fact that $D \not\equiv \pm 1 \pmod{9}$ implies that 3 is ramified.
in $L$, then $\lambda$ is ramified in $N/K$. Hence $t = 3$. As $3 = -\varepsilon_3^2 \lambda^2$, then $N = K(\sqrt[3]{\lambda})$
where $x = \zeta_3^2 \lambda^2 \pi_1 f_i \pi_2 f_2$, and for $i \in \{1, 2\}$, $-q_i = \pi_i$ is a prime number of $K$.
If $q_1 \equiv q_2 \equiv -1 \pmod{9}$, then all primes $\pi_1, \pi_2$ are congruent to $1 \pmod{\lambda^3}$,
so by [1, Lem. 3.3, p. 264] we have $\zeta_3$ is norm of an element of $N - \{0\}$ and
$q^* = 1$. Thus $\text{rank} (\text{Cl}_{3}^{(\sigma)}(N)) = 2$. The opposite implication of the rank condition
rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$ to this shape of conductor will be proved in Sect. 4.1.2.

(2) For $D = q_1 f_i q_2 f_2 q_3 f_3$ with $q_1 \equiv q_2 \equiv q_3 \equiv 8 \pmod{9}$, we have $f = q_1 q_2 q_3$ of
species $2$. We must take into consideration that $G = 3$, $B = 0$, and we obtain
$m(f) = 2^G \cdot X_{B-1} = 2^3 \cdot \frac{1}{2} = 4$, a quartet.

The prime $q_j$ is inert in $K$ for each $i \in \{1, 2, 3\}$, because $q_i \equiv 2 \pmod{3}$, and
$q_j$ is ramified in $L$. Since $D \equiv \pm 1 \pmod{9}$, then $3$ is not ramified in $L$,
and then $\lambda$ is not ramified in $N/K$. So $t = 3$. Since $q_i \equiv -1 \pmod{9}$ (mod 9), then
$\pi_i \equiv 1 \pmod{3\mathcal{O}_K}$, where $-q_i = \pi_i$ is a prime number of $K$. Thus $N = K(\sqrt[3]{\lambda})$
with $x = \pi_1 f_i \pi_2 f_2 \pi_3 f_3$. The fact that all primes $\pi_1, \pi_2$ and $\pi_3$ are congruent to
$1 \pmod{\lambda^3}$ implies by [1, Lem. 3.3, p. 264] that $\zeta_3$ is a norm of an element of
$N - \{0\}$ and $q^* = 1$. Hence, rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$. The opposite implication of the rank condition
rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$ to this shape of conductor will be proved in § 4.2.2.

(3) For $D = q_1 f_i q_2 f_2 q_3 f_3 \not\equiv \pm 1 \pmod{9}$ with $q_1, q_2, q_3 \equiv 2, 5 \pmod{9}$, we have
$f = 3q_1 q_2 q_3$ of species $1b$. We must take into consideration that $G = 0, B = 3$,
and we obtain $m(f) = 2^G \cdot X_B = 1 \cdot 3 = 3$, a triplet.

For the equivalence of the rank condition rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$ to this shape of conductor, we reason as in case (4) and we will prove the opposite implication in § 4.2.2.

(4) For $D = q_1 f_i q_2 f_2 q_3 f_3 \not\equiv \pm 1 \pmod{9}$ such that $(q_1, q_2 \equiv 2, 5 \pmod{9}$ and
$q_3 \equiv 8 \pmod{9}$), we have $f = 3q_1 q_2 q_3$ of species $1b$. We must take into consideration that $G = 1$ and $B = 2$, and we obtain $m(f) = 2^G \cdot X_B = 2 \cdot 1 = 2$, a doublet.

For each $i \in \{1, 2, 3\}$, $q_i$ is inert in $K$ because $q_i \equiv 2 \pmod{3}$, and $q_i$ is ramified in $L$. As $D \not\equiv \pm 1 \pmod{9}$, $3$ is ramified in $L$, so $\lambda$ is ramified in $N/K$. Then
$t = 4$.

If $\exists i \in \{1, 2, 3\}$ such that $q_i \not\equiv -1 \pmod{9}$, then we have $-q_i = \pi_i$ is a prime
number of $K$. Put $x = \pi_1 f_i \pi_2 f_2 \pi_3 f_3$, then $N = K(\sqrt[3]{\lambda})$. According to [1, Lem.
3.2, p. 264], there exists a prime $\pi_i$ not congruent to $1 \pmod{\lambda^3}$, then by [1, Lem.
3.3, p. 264], $\zeta_3$ is not a norm of an element of $N - \{0\}$ and $q^* = 0$. We conclude that rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$. The opposite implication of the rank condition
rank $(\text{Cl}_{3}^{(\sigma)}(N)) = 2$ to this shape of conductor will be proved in Sect. 4.1.2.

(5) For $D = q_1 f_i q_2 f_2 q_3 f_3 \not\equiv \pm 1 \pmod{9}$ such that $(q_1 \equiv 2, 5 \pmod{9}$ and $q_2, q_3 \equiv 8 \pmod{9})$, we have $f = 3q_1 q_2 q_3$ of species $1b$. We must take into consideration that $G = 2$ and $B = 1$, and in the two cases we obtain $m(f) = 2^G \cdot X_B = 4 \cdot 1 = 4$, a quartet.
(6) For $D = q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} \equiv \pm 1 \pmod{9}$ such that $q_1, q_2, q_3, q_4 \equiv 2, 5 \pmod{9}$, we have $f = q_1 q_2 q_3 q_4$ of species 2. We must take into consideration that $G = 0, B = 4$, and we obtain $m(f) = 2^G \cdot X_{B-1} = 1 \cdot 3 = 3$, a triplet.

For the equivalence of the rank condition rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$ to this shape of conductor, we reason as in case (8) and we will prove the opposite implication in Sect. 4.3.3.

(7) For $D = q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} \equiv \pm 1 \pmod{9}$ such that $q_1, q_2, q_3 \equiv 2, 5 \pmod{9}$ and $q_4 \equiv 8 \pmod{9}$, we have $f = q_1 q_2 q_3 q_4$ of species 2. We must take into consideration that $G = 1, B = 3$, and we obtain $m(f) = 2^G \cdot X_{B-1} = 2 \cdot 1 = 2$, a doublet.

For the equivalence of the rank condition rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$ to this shape of conductor, we reason as in case (6) and we will prove the opposite implication in Sect. 4.3.3.

(8) For $D = q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} \equiv \pm 1 \pmod{9}$ with $q_1, q_2 \equiv 2, 5 \pmod{9}$ and $q_3 \equiv q_4 \equiv 8 \pmod{9}$, we have $f = q_1 q_2 q_3 q_4$ of species 2. We must take into consideration that $G = 2, B = 2$, and we obtain $m(f) = 2^G \cdot X_{B-1} = 2^2 \cdot 1 = 4$, a quartet.

For each $i \in \{1, 2, 3, 4\}$, $q_i$ is inert in $K$ because $q_i \equiv 2 \pmod{3}$, and $q_i$ is ramified in $L$. As $D \equiv \pm 1 \pmod{9}$, 3 is not ramified in $L$, so $\lambda$ is not ramified in $N/K$. Then $t = 4$. Put $x = \pi_1^{f_1} \pi_2^{f_2} \pi_3^{f_3} \pi_4^{f_4}$, then $N = K(\sqrt{x})$. If there exists $i \in \{1, 2, 3, 4\}$ such that $q_i \equiv -1 \pmod{9}$, then by [1, Lem. 3.2, p. 264], $\pi_i \equiv 1 \pmod{3(\mathcal{O}_K)}$, where $-q_i = \pi_i$ is a prime number of $K$, then according to [1, Lem. 3.3, p. 264], $\zeta_3$ is not a norm of an element of $N - \{0\}$ and $q^* = 0$. We conclude that rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$ to this shape of conductor will be proved in Sect. 4.3.3.

(9) For $D = 3^e q_1^{f_1} q_2^{f_2} q_3^{f_3} \not\equiv \pm 1 \pmod{9}$ such that $\exists i \in \{1, 2, 3\} \mid q_i \equiv 8 \pmod{9}$, we have $f = 9 q_1 q_2 q_3$ of species 1a, with $T = 3$, and we obtain $m(f) = 2^T = 2^3 = 8$, an octet, independently of $G$ and $B$.

We reason as in case (3), we get $t = 4$ and $q^* = 0$, then rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$. The opposite implication of the rank condition rank $(\text{Cl}_3^{(\sigma)}(N)) = 2$ to this shape of conductor will be proved in §4.2.2.

The principal factorization type is a consequence of the estimates in [1, Thm. 2.1, p. 254]. Since $s = 0$, we have $0 \leq R \leq 0$ and type $\alpha$ with relative PF is generally forbidden. For all cases, we have $3 \leq t \leq 4$ and thus $1 \leq A \leq \min(2, t-s) = 2$ with $3 = 3 - 0 \leq t - s \leq 4 - 0 = 4$, which enables type $\beta$ with absolute PF. For items (3)–(8), there exists a prime factor $q \equiv 2, 5 \pmod{9}$, and type $\gamma$ is impossible, because $\zeta_3$ can be norm of a unit in $N$ only if the prime factors of $f$ are 3 or $\ell_j \equiv 1, 8 \pmod{9}$. The latter condition is satisfied by items (1)–(2), whence type $\gamma$ may occur.

All claims on the infinitude of the various sets of conductors $f$ are consequences of Dirichlet’s theorem on prime numbers arising from invertible residue classes. \hfill $\Box$

**Example 2** Several M0-fields [2, Dfn. 4.3, p. 105] have radicands $D$ of the shape in the eleventh line of Equation (1): $9922 = 2 \cdot 11^2 \cdot 41$, $38686 = 2 \cdot 23 \cdot 59^2$ belong to
Fig. 2  Galois correspondence between principal genus 
\( Cl_3^{-\sigma}(N) \) and relative genus field \( N^* \)

\[
\begin{array}{ccc}
N_3^{(1)} & \{1\} & \\
\downarrow & \downarrow & \\
N^* & Gal(N_3^{(1)}/N^*) \cong Cl_3^{1-\sigma}(N) & \\
\downarrow & & \\
N & Gal(N_3^{(1)}/N) \cong Cl_3(N) & \\
\end{array}
\]

Theorem 3.2, item (3), \(850 = 2 \cdot 5^2 \cdot 17, 6358 = 2 \cdot 11 \cdot 17^2, 17986 = 2 \cdot 17 \cdot 23^2, 94162 = 2 \cdot 23^2 \cdot 89\) belong to Theorem 3.2, item (4), and \(61268 = 2^2 \cdot 17^2 \cdot 53\) belongs to Theorem 3.2, item (5). The \(M_0\)-field \(D = 55522 = 2 \cdot 17 \cdot 23 \cdot 71\) realizes Theorem 3.2, item (8) and line thirteen of Eq. (1). So Eq. (1) indeed gives rise to numerous \(M_0\)-fields \([2]\). See sequence A363699 in OEIS \([29]\).

4 Proof of the Main Theorem 1.1

Let \(N = \mathbb{Q}(\sqrt[3]{D}, \zeta_3)\) be the normal closure of the pure cubic field \(L = \mathbb{Q}(\sqrt[3]{D})\), where \(D > 1\) is a cube free positive integer, \(K = \mathbb{Q}(\zeta_3)\), and \(Cl_3(N)\) be the 3-class group of \(N\). Let \(N_3^{(1)}\) be the maximal abelian unramified 3-extension of \(N\). It is known that \(N_3^{(1)}/K\) is Galois, and according to class field theory:

\[
Gal\left(N_3^{(1)}/N\right) \cong Cl_3(N).
\]

We denote by \(N^*\) the maximal abelian extension of \(K\) contained in \(N_3^{(1)}\), which is called the relative genus field of \(N/K\) (see [11, 12] or [15]).

It is known that the commutator subgroup of \(Gal\left(N_3^{(1)}/K\right)\) coincides with \(Gal\left(N_3^{(1)}/N^*\right)\) and then:

\[
Gal\left(N^*/K\right) \cong Gal\left(N_3^{(1)}/K\right)/Gal\left(N_3^{(1)}/N^*\right).
\]

Let \(\sigma\) be a generator of \(Gal\left(N/K\right)\), and let \(Cl_3^{1-\sigma}(N)\) be the subgroup of \(Cl_3(N)\) defined by \(Cl_3^{1-\sigma}(N) = \{A^{1-\sigma} \mid A \in Cl_3(N)\}\), which is called the principal genus of \(Cl_3(N)\). The fact that \(N/K\) is abelian and that \(N \subseteq N^*\) implies that \(Gal\left(N_3^{(1)}/N^*\right)\) coincides with \(Cl_3^{1-\sigma}(N)\), by the aid of the isomorphism (7) above, and by Artin’s reciprocity law. See Fig. 2.

Then \(Gal\left(N^*/N\right) \cong Cl_3(N)/Cl_3^{1-\sigma}(N)\). Let \(Cl_3^{(\sigma)}(N) = \{A \in Cl_3(N) \mid A^\sigma = A\}\) be the 3-group of ambiguous ideal classes of \(N/K\). Since the Sylow 3-subgroup of the ideal class group of \(K\) is reduced to \(\{1\}\), and by the exact sequence:

\[
1 \longrightarrow Cl_3^{(\sigma)}(N) \longrightarrow Cl_3(N) \overset{1-\sigma}{\longrightarrow} Cl_3(N) \longrightarrow Cl_3(N)/Cl_3^{1-\sigma}(N) \longrightarrow 1
\]
we see that $\text{Gal} \left( N^*/N \right) \cong \text{Cl}_3^{(\sigma)}(N)$.

We assume $\text{Gal} \left( N^*/N \right) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. In Equation (3.2) of [11, p. 55], the integer $D$ is written in the following form:

$$D = 3^e \cdot p_1^{e_1} \cdots p_v^{e_v} \cdot p_{v+1}^{e_{v+1}} \cdots p_w^{e_w} \cdot q_1^{f_1} \cdots q_I^{f_I} \cdot q_{I+1}^{f_{I+1}} \cdots q_J^{f_J}, \quad (8)$$

where $p_i$ and $q_i$ are positive rational primes such that:

\[
\begin{align*}
p_i &\equiv 1 \pmod{9}, & & \text{for } 1 \leq i \leq v, \\
p_i &\equiv 4 \text{ or } 7 \pmod{9}, & & \text{for } v + 1 \leq i \leq w, \\
q_i &\equiv 8 \pmod{9}, & & \text{for } 1 \leq i \leq I, \\
q_i &\equiv 2 \text{ or } 5 \pmod{9}, & & \text{for } I + 1 \leq i \leq J, \\
e_i &\equiv 1 \text{ or } 2, & & \text{for } 1 \leq i \leq w, \\
f_i &\equiv 1 \text{ or } 2, & & \text{for } 1 \leq i \leq J, \\
e &\equiv 0, 1 \text{ or } 2.
\end{align*}
\]

Since $\text{rank} \left( \text{Cl}_3^{(\sigma)}(N) \right) = 2$, then Lemma 3.1 of [11, p. 55] gives the following cases:

- Case 1: $2w + J = 2$;
- Case 2: $2w + J = 3$;
- Case 3: $2w + J = 4$;

where $w$ and $J$ are defined in Eq. (8). We shall treat the three cases above as follows:

### 4.1 Case 1

In the case $1$, $2w + J = 2$, we either have $(w = 1$ and $J = 0)$ treated in Sect. 4.1.1 or $(w = 0$ and $J = 2)$ treated in Sect. 4.1.2.

#### 4.1.1 Radicands divisible by one splitting rational prime

If $w = 1$ and $J = 0$, then

$$D = 3^e p^{e_1},$$

where $p$ is a prime number such that $p \equiv 1 \pmod{3}$, $e \in \{0, 1, 2\}$ and $e_1 \in \{1, 2\}$. Then:

Species 2: If $D \equiv \pm 1 \pmod{9}$, then we necessary have $p \equiv 1 \pmod{9}$ and $e = 0$. So the integer $D$ can be written in the form $D = p^{e_1}$, with $p \equiv 1 \pmod{9}$ and $e_1 \in \{1, 2\}$. According to [1, Thm.1.1, p. 251], rank $\left( \text{Cl}_3^{(\sigma)}(N) \right) = 1$, which contradicts our hypothesis.

Species 1: If $D \not\equiv \pm 1 \pmod{9}$, then according to [1, p. 266], the integer $D$ is written in one of the following forms:

$$d = \begin{cases} 
p^{e_1} & \text{with } p \equiv 4 \text{ or } 7 \pmod{9}, \\
3^e p^{e_1} \not\equiv \pm 1 \pmod{9} & \text{with } p \equiv 1 \pmod{3},
\end{cases}$$
where \( e, e_1 \in \{1, 2\} \).

Assume that \( D = p^{e_1} \) or \( D = 3^e p^{e_1} \), with \( p \equiv 4 \) or \( 7 \pmod{9} \), then by [1, Thm. 1.1, p. 251], we have \( \text{rank} (\text{Cl}_{3}^{(\sigma)}(N)) = 1 \), which is a contradiction.

Hence, the possible form of \( D \) in this situation is:

\[
D = 3^e p^{e_1} \not\equiv \pm 1 \pmod{9} \quad \text{with} \quad p \equiv 1 \pmod{9},
\]

where \( e, e_1 \in \{1, 2\} \), which is the first form of \( D \) in Theorem 1.1.

### 4.1.2 Radicands not divisible by a splitting rational prime

If \( w = 0 \) and \( J = 2 \), then

\[
D = 3^e q_1^{f_1} q_2^{f_2},
\]

with \( q_1 \equiv q_2 \equiv 2 \pmod{3} \), \( e \in \{0, 1, 2\} \) and \( f_1, f_2 \in \{1, 2\} \), then:

Species 2: If \( D \equiv \pm 1 \pmod{9} \), then according to [1, p. 266–267], \( e = 0 \) and \( q_1 \equiv q_2 \equiv -1 \pmod{9} \), so \( \text{rank} (\text{Cl}_{3}^{(\sigma)}(N)) = 1 \) by [1, Thm. 1.1, p. 251], which is a contradiction.

Species 1: If \( D \not\equiv \pm 1 \pmod{9} \). Assume that \( \exists i \in \{1, 2\} \mid q_i \not\equiv -1 \pmod{9} \) such that

\[
D = 3^e q_1^{f_1} q_2^{f_2} \not\equiv \pm 1 \pmod{9},
\]

with \( e \in \{0, 1, 2\} \) and \( e_1, f_1, f_2 \in \{1, 2\} \), then according to [1, Thm. 1.1, p. 251] we get \( \text{rank} (\text{Cl}_{3}^{(\sigma)}(N)) = 1 \) which is a contradiction.

Thus, it remains only the form:

\[
D = 3^e q_1^{f_1} q_2^{f_2} \not\equiv \pm 1 \pmod{9}, \quad \text{such that} \quad q_1 \equiv q_2 \equiv -1 \pmod{9},
\]

where \( e, f_1, \) and \( f_2 \in \{1, 2\} \), which is the 9th form of \( D \) in Theorem 1.1.

### 4.2 Case 2

In the case \( 2w + J = 3 \), we either have \( (w = 1 \) and \( J = 1 \)) treated in § 4.2.1 or \( (w = 0 \) and \( J = 3 \)) treated in § 4.2.2.

#### 4.2.1 Radicands divisible by one splitting rational prime

If \( w = 1 \) and \( J = 1 \), then

\[
D = 3^e p^{e_1} q^{f_1},
\]

where \( p \) and \( q \) are prime numbers such that \( p \equiv 1 \pmod{3} \) and \( q \equiv 2 \pmod{3} \), \( e \in \{0, 1, 2\} \) and \( e_1, f_1 \in \{1, 2\} \). Then:

Species 2: If \( D \equiv \pm 1 \pmod{9} \):

1. If \( p \equiv 4 \) or \( 7 \pmod{9} \) and \( q \equiv -1 \pmod{9} \), then according to [1, p. 267] we get a contradiction.
2. If \( p \equiv -q \equiv 4 \) or \( 7 \pmod{9} \), then according to [1, p. 267] we have \( D = p^{e_1} q^{f_1} \equiv \pm 1 \pmod{9} \), where \( e_1, f_1 \in \{1, 2\} \). But in this case, \( \text{rank} (\text{Cl}_{3}^{(\sigma)}(N)) = 1 \) by [1, Thm. 1.1, p. 251]. Rejected case.
(3) If \( p = -q \equiv 1 \pmod{9} \), we get \( D \equiv \pm 3^e \pmod{9} \), so we have necessary \( e = 0 \). Then \( D = p^{e_1}q^{f_1} \), where \( e_1, f_1 \in \{1, 2\} \), which is the second form of \( D \) in Theorem 1.1.

(4) If \( p \equiv 1 \pmod{9} \) and \( q \equiv 2 \) or \( 5 \pmod{9} \), then by [1, p. 268] we get a contradiction.

Species 1: If \( D \neq \pm 1 \pmod{9} \):

According to [12, § 5, p. 92], \( \text{rank}(\text{Cl}_3^\sigma(N)) = t - 2 + q^* \), where \( t \) and \( q^* \) are defined in the notations. Since \( p \equiv 1 \pmod{3} \), then \( p = \pi_1\pi_2 \), where \( \pi_1 \) and \( \pi_2 \) are two primes of \( K \) such that \( \pi_2 = \pi_1^2 \) and \( \pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_K} \), and \( p \) is ramified in \( L \), then \( \pi_1 \) and \( \pi_2 \) are ramified in \( N \). Also \( q \) ramifies in \( L \) and \( -q = \pi \) is a prime in \( K \). Since \( D \neq \pm 1 \pmod{9} \), then \( 3 \) is ramified in \( L \), and we have \( 3\mathcal{O}_K = (\lambda)^2 \), where \( \lambda = 1 - \zeta_3 \). We get \( t = 4 \).

If \( p = -q \equiv 1 \pmod{9} \), then \( \pi \equiv \pi_1 \equiv \pi_2 \equiv 1 \pmod{\lambda^3} \), and according to [1, Lem. 3.3, p. 264], \( \zeta_3 \) is norm of an element of \( N - \{0\} \), so \( q^* = 1 \). Thus, \( \text{rank}(\text{Cl}_3^\sigma(N)) = 3 \), which is a contradiction.

Hence, the forms of \( D \) in this situation are

\[
D = \begin{cases} 
  p^{e_1}q^{f_1} \not\equiv \pm 1 \pmod{9} & \text{with } D \equiv \pm 1 \pmod{9} \text{ or } q \not\equiv -1 \pmod{9} \\
  3^ep^{e_1}q^{f_1} \not\equiv \pm 1 \pmod{9} & \text{with } p \not\equiv 1 \pmod{9} \text{ or } q \not\equiv -1 \pmod{9}
\end{cases}
\]

where \( e, e_1, f_1 \in \{1, 2\} \), which are the 3rd and 5th forms of \( D \) in Theorem 1.1.

### 4.2.2 Radicands not divisible by a splitting rational prime

If \( w = 0 \) and \( J = 3 \), then

\[
D = 3^eq_1^{f_1}q_2^{f_2}q_3^{f_3},
\]

where \( q_i \) is a prime number such that \( q_i \equiv 2 \pmod{3} \), \( e \in \{0, 1, 2\} \) and \( f_i \in \{1, 2\} \) for each \( i \in \{1, 2, 3\} \). Then:

Species 2: If \( D \equiv \pm 1 \pmod{9} \):

(1) If \( q_1 \equiv 2 \) or \( 5 \pmod{9} \) and \( q_2 \equiv q_3 \equiv -1 \pmod{9} \), then according to [1, p. 268], we get a contradiction.

(2) If \( q_1 \equiv q_2 \equiv 2 \) or \( 5 \pmod{9} \) and \( q_3 \equiv -1 \pmod{9} \), then according to [1, p. 268], we have \( e = 0 \), so \( D = q_1^{f_1}q_2^{f_2}q_3^{f_3} \equiv \pm 1 \pmod{9} \), where \( f_1, f_2, f_3 \in \{1, 2\} \). But in this case we have \( \text{rank}(\text{Cl}_3^\sigma(N)) = 1 \) by [1, Thm. 1.1, p. 251] which is absurd.

(3) If \( q_1 \equiv q_2 \equiv q_3 \equiv -1 \pmod{9} \), then we have necessary \( e = 0 \). Then \( D = q_1^{f_1}q_2^{f_2}q_3^{f_3} \), where \( f_1, f_2, f_3 \in \{1, 2\} \), which is the 10th form of \( D \) in Theorem 1.1.

(4) If \( q_1 \equiv q_2 \equiv q_3 \equiv 2 \) or \( 5 \pmod{9} \), then by [1, p. 268] we have \( e = 0 \). So, \( D = q_1^{f_1}q_2^{f_2}q_3^{f_3} \equiv \pm 1 \pmod{9} \), where \( f_1, f_2, f_3 \in \{1, 2\} \), and by [1, Thm. 1.1, p. 251], \( \text{rank}(\text{Cl}_3^\sigma(N)) = 1 \), which is absurd.
Species 1: If \( D \not\equiv \pm 1 \pmod 9 \):

For each \( i \in \{1, 2, 3\} \), \( q_i \) is inert in \( K \) because \( q_i \equiv 2 \pmod 3 \), and \( q_i \) is ramified in \( L \). As \( D \not\equiv \pm 1 \pmod 9 \), 3 is ramified in \( L \), so \( \lambda \) is ramified in \( N/K \). Then \( t = 4 \).

If \( q_1 \equiv q_2 \equiv q_3 \equiv -1 \pmod 9 \):

For each \( i \in \{1, 2, 3\} \), \( \pi_i \equiv 1 \pmod {3\mathcal{O}_K} \), where \( -q_i = \pi_i \) is a prime number of \( K \). Put \( x = \pi_1^{f_1} \pi_2^{f_2} \pi_3^{f_3} \), then \( N = K(\sqrt[3]{x}) \). Since all the primes \( \pi_1, \pi_2 \) and \( \pi_3 \) are congruent to 1 (mod \( \lambda^3 \)), then according to [1, Lem. 3.3, p. 264], \( \xi \) is a norm of an element of \( N - \{0\} \) and \( q^* = 1 \). We conclude that rank \( (\text{Cl}^1_{3}(\sigma)(N)) = 3 \), which is absurd.

Thus, it remains only the case where there exist \( i \in \{1, 2, 3\} \) such that \( q_i \not\equiv -1 \pmod 9 \). Hence, the forms of \( D \) in this situation are \( D = \pm 1 \pmod 9 \), \( \exists i \in \{1, 2, 3\} \mid q_i \not\equiv -1 \pmod 9 \), \( 3^e q_1^{f_1} q_2^{f_2} q_3^{f_3} \not\equiv \pm 1 \pmod 9 \), \( \exists i \in \{1, 2, 3\} \mid q_i \not\equiv -1 \pmod 9 \),

where \( e, f_1, f_2 \) and \( f_3 \in \{1, 2\} \), which are the 11\(^{th}\) and 12\(^{th}\) forms of \( D \) in Theorem 1.1.

4.3 Case 3

In the case \( 2w + J = 4 \), we either have \( (w = 1 \text{ and } J = 2) \) treated in § 4.3.1, or \( (w = 2 \text{ and } J = 0) \) treated in § 4.3.2, or \( (w = 0 \text{ and } J = 4) \) treated in Sect. 4.3.3.

4.3.1 Radicands divisible by one splitting rational prime

If \( w = 1 \) and \( J = 2 \), then

\[
D = 3^e p^e q_1^{f_1} q_2^{f_2},
\]

where \( p, q_1 \) and \( q_2 \) are prime numbers such that \( p \equiv 1 \pmod 3 \) and \( q_1 \equiv q_2 \equiv 2 \pmod 3 \), \( e \in \{0, 1, 2\} \) and \( e_1, f_1, f_2 \in \{1, 2\} \). Then:

Species 2: If \( D \equiv \pm 1 \pmod 9 \):

(1) If \( p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod 9 \),

By [12, § 5, p. 92], rank \( (\text{Cl}^1_{3}(\sigma)(N)) = t - 2 + q^* \), where \( t \) and \( q^* \) are defined in the notations. Since \( p \equiv 1 \pmod 3 \), then \( p = \pi_1 \pi_2 \), where \( \pi_1 \) and \( \pi_2 \) are two primes of \( K \) such that \( \pi_2 = \pi_1^2 \) and \( \pi_1 \equiv \pi_2 \equiv 1 \pmod {3\mathcal{O}_K} \), and the prime \( p \) is ramified in \( L \), then \( \pi_1 \) and \( \pi_2 \) are ramified in \( N \). For each \( i \in \{1, 2\} \), \( q_i \) is inert in \( K \) because \( q_i \equiv 2 \pmod 3 \), and \( q_i \) is ramified in \( L \). As \( D \equiv \pm 1 \pmod 9 \), 3 is not ramified in \( L \), so \( \lambda \) is not ramified in \( N/K \). So, \( t = 4 \). As \( q_1 \equiv q_2 \equiv -1 \pmod 9 \), then for each \( i \in \{1, 2\} \), \( \pi_i' \equiv 1 \pmod {3\mathcal{O}_K} \), where \( -q_i = \pi_i' \) is a prime number of \( K \). Put \( x = -\xi^2 \xi^2 \pi_1^{e_1} \pi_2^{e_2} \pi_1^{f_1} \pi_2^{f_2} \), then \( N = K(\sqrt[3]{x}) \). Since
all the primes $\pi_1$, $\pi_2$, $\pi'_1$ and $\pi'_2$ are congruent to 1 (mod $\lambda^3$), then according to
[1, Lem. 3.3, p. 264], $\zeta_3$ is a norm of an element of $N - \{0\}$ and $q^* = 1$. We conclude that rank $(\text{Cl}_3^\sigma(N)) = 3$, which is absurd.

(2) If $p \equiv 1$ (mod 9) and $q_1 \equiv q_2 \equiv 2$ or 5 (mod 9), then $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 3^e, \pm 4 \times 3^e \text{ or } \pm 7 \times 3^e$ (mod 9). Since $D \equiv \pm 1$ (mod 9), then $e = 0$, so the possible form for the integer $D$ is $D = p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 1$ (mod 9), which is the 6th form of $D$ in Theorem 1.1.

(3) If $p \equiv 1$ (mod 9), $q_1 \equiv -1$ (mod 9) and $q_2 \equiv 2$ or 5 (mod 9), then $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 2 \times 3^e \text{ or } \pm 5 \times 3^e$ (mod 9), so $D \neq \pm 1$ (mod 9), which is absurd.

(4) If $p \equiv 4$ or 7 (mod 9) and $q_1 \equiv q_2 \equiv -1$ (mod 9). Then $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 4 \times 3^e \text{ or } \pm 7 \times 3^e$ (mod 9), so $D \neq \pm 1$ (mod 9), which is absurd.

(5) If $p \equiv 4$ or 7 (mod 9) and $q_1 \equiv q_2 \equiv 2$ or 5 (mod 9), then $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 3^e, \pm 4 \times 3^e \text{ or } \pm 7 \times 3^e$ (mod 9). Since $D \equiv \pm 1$ (mod 9), then $e = 0$, so the possible form of $D$ is $D = p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 1$ (mod 9), which is the 7th form of $D$ in Theorem 1.1.

(6) If $p \equiv 4$ or 7 (mod 9), $q_1 \equiv -1$ (mod 9) and $q_2 \equiv 2$ or 5 (mod 9), then $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2} \equiv 3^e, 4 \times 3^e \text{ or } 7 \times 3^e$ (mod 9), since $D \equiv \pm 1$ (mod 9), then $e = 0$, so the form of $D$ is $D = p^{e_1} q_1^{f_1} q_2^{f_2} \equiv \pm 1$ (mod 9), which is the 8th form of $D$ in Theorem 1.1.

Species 1: If $D \neq \pm 1$ (mod 9): The fact that rank $(\text{Cl}_3^\sigma(N)) = 2$ implies that $t \leq 4$. We have $D = 3^e p^{e_1} q_1^{f_1} q_2^{f_2}$, with $p \equiv 1$ (mod 3) and $q \equiv 2$ (mod 3).

We shall calculate the number of prime ideals which are ramified in $N/K$. Since $p \equiv 1$ (mod 3), then $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are two primes of $K$ such that $\pi_2 = \pi_1^r$ and $\pi_1 \equiv \pi_2 \equiv 1$ (mod $3O_K$), and the prime $p$ is ramified in $L$, then $\pi_1$ and $\pi_2$ are ramified in $N$. Also $q_1$ and $q_2$ are inert in $K$, $q_1$ and $q_2$ are ramified in $L$. Since $D \neq \pm 1$ (mod 9), then 3 is ramified in $L$, and we have $3O_K = (\lambda)$ where $\lambda = 1 - \zeta_3$. Hence, $t = 5$ which contradicts the fact that $t \leq 4$. Rejected case.

4.3.2 Radicands divisible by two splitting rational primes

If $w = 2$ and $J = 0$, then

$$D = 3^e p_1^{e_1} p_2^{e_2},$$

where $p_1$ and $p_2$ are prime numbers such that $p_1 \equiv p_2 \equiv 1$ (mod 3), $e \in \{0, 1, 2\}$ and $e_1, e_2 \in \{1, 2\}$. We shall calculate the number of prime ideals which are ramified in $N/K$. Since $p_1 \equiv p_2 \equiv 1$ (mod 3), then $p_1 = \pi_1 \pi_2$ and $p_2 = \pi_3 \pi_4$, where $\pi_1$, $\pi_2$, $\pi_3$ and $\pi_4$ are primes of $K$ such that $\pi_2 = \pi_1^r$, $\pi_4 = \pi_3^r$, and $\pi_1 \equiv \pi_2 \equiv \pi_3 \equiv \pi_4 \equiv 1$ (mod $3O_K$), and the primes $p_1$ and $p_2$ are ramified in $L$, then $\pi_1$, $\pi_2$, $\pi_3$, and $\pi_4$ are ramified in $N$.

Species 2: $D \equiv \pm 1$ (mod 9), 3 is not ramified in $L$, so $\lambda$ is not ramified in $N/K$. Then $t = 4$. 
Assuming $p_1 \equiv p_2 \equiv 1 \pmod{9}$. Since $3 = -\xi_3^2 \lambda^2$, then $N = K(\sqrt[3]{\lambda})$, where $x = -\xi_3^2 \lambda^2 \pi_1^{e_1} \pi_2^{e_2} \pi_3^{e_3} \pi_4^{e_4}$. Since all the primes $\pi_1, \pi_2, \pi_3$ and $\pi_4$ are congruent to $1 \pmod{\lambda^3}$, then according to [1, Lem. 3.3, p. 264], $\xi_3$ is norm of an element of $\mathcal{N} - \{0\}$ and $q^* = 1$. We conclude that rank $(\Cl_3^{(\sigma)}(N)) = 3$, which is an absurd.

It remains the case where $p_1 \not\equiv 1 \pmod{9}$ or $p_2 \not\equiv 1 \pmod{9}$, this case implies that $p_1^{e_1} p_2^{e_2} \equiv 1, 4$ or $7 \pmod{9}$, then $D = 3^e p_1^{e_1} p_2^{e_2} = 3^e, 4 \times 3^e$ or $7 \times 3^e \pmod{9}$. Since $D \equiv \pm 1 \pmod{9}$, we have necessarily $e = 0$, then the possible form of the integer $D$ in this case is:

$$D = p_1^{e_1} p_2^{e_2} \equiv \pm 1 \pmod{9},$$

with $e_1, e_2 \in \{1, 2\}$, which is the $4^{th}$ form of $D$ in Theorem 1.1.

Species 1: $D \not\equiv \pm 1 \pmod{9}$, 3 is ramified in $L$, so $\lambda$ is ramified in $N/K$. Then $t = 5$. In this case, we conclude that rank $(\Cl_3^{(\sigma)}(N)) \geq 3$, which is absurd.

4.3.3 Radicands not divisible by a splitting rational prime

If $w = 0$ and $J = 4$, then

$$D = 3^e q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4},$$

where $q_i$ is a prime number such that $q_i \equiv 2 \pmod{3}$, $e \in \{0, 1, 2\}$ and $f_i \in \{1, 2\}$ for each $i \in \{1, 2, 3, 4\}$. Then:

Species 2: If $D \equiv \pm 1 \pmod{9}$, then:

1. If $q_1 \equiv q_2 \equiv q_3 \equiv q_4 \equiv -1 \pmod{9}$:

   For each $i \in \{1, 2, 3, 4\}$, $q_i$ is inert in $K$ because $q_i \equiv 2 \pmod{3}$, and $q_i$ is ramified in $L$. As $D \equiv \pm 1 \pmod{9}$, 3 is not ramified in $L$, so $\lambda$ is not ramified in $N/K$. Then $t = 4$. Since $q_1 \equiv q_2 \equiv q_3 \equiv q_4 \equiv -1 \pmod{9}$, then for each $i \in \{1, 2, 3, 4\}$, $\pi_i \equiv 1 \pmod{3\mathcal{O}_K}$, where $-q_i = \pi_i$ is a prime number of $K$. Put $x = -\xi_3^2 \lambda^2 \pi_1^{f_1} \pi_2^{f_2} \pi_3^{f_3} \pi_4^{f_4}$, then $N = K(\sqrt[3]{\lambda})$. Since all the primes $\pi_1, \pi_2, \pi_3$ and $\pi_4$ are congruent to $1 \pmod{\lambda^3}$, then according to [1, Lem. 3.3, p. 264], $\xi_3$ is a norm of an element of $\mathcal{N} - \{0\}$ and $q^* = 1$. We conclude that rank $(\Cl_3^{(\sigma)}(N)) = 3$, which is absurd.

2. If $\exists i \in \{1, 2, 3, 4\}$ such that $q_i \not\equiv -1 \pmod{9}$, then $D = 3^e q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} \equiv \pm 3^e, \pm 4 \times 3^e$ or $\pm 7 \times 3^e \pmod{9}$, since $D \equiv \pm 1 \pmod{9}$, then $e = 0$, so the possible form of $D$ is $D = q_1^{f_1} q_2^{f_2} q_3^{f_3} q_4^{f_4} \equiv \pm 1 \pmod{9}$, which is the $13^{th}$ form of $D$ in Theorem 1.1.

Species 1: If $D \not\equiv \pm 1 \pmod{9}$: For each $i \in \{1, 2, 3, 4\}$, $q_i$ is inert in $K$ because $q_i \equiv 2 \pmod{3}$, and $q_i$ is ramified in $L$. As $D \not\equiv \pm 1 \pmod{9}$, 3 is ramified in $L$, so $\lambda$ is ramified in $N/K$. Then $t = 5$. We conclude that rank $(\Cl_3^{(\sigma)}(N)) = 3$ or 4, which is absurd.

Finally, if the integer $D$ takes one of the forms given in Theorem 1.1, then rank $(\Cl_3^{(\sigma)}(N)) = 2$ according to the proof of Theorems 3.1 and 3.2.
5 Conclusion

We have characterized all Kummer extensions $N/K$, which possess a relative 3-genus field $N^*$ with elementary bicyclic Galois group $\text{Gal}(N^*/N)$, by precisely thirteen forms of radicands $D$ of $N = \mathbb{Q}((\sqrt[3]{D}, \zeta_3)$ with up to four prime divisors, at most two among them congruent to 1 modulo 3. However, the genus group $\text{Gal}(N^*/N)$ is a common property of the whole multiplet $N_1, \ldots, N_m$ of $m$ fields sharing the same conductor $f$, as opposed to the property of being an M0-field [2], which sensibly depends on the particular shape of the radicand $D$. In our numerical examples, Tables 4, 5, 6, 7, and 8, $\text{rank } (\text{Cl}_3(N)) = \text{rank } (\text{Cl}_3(\sigma)(N))$ or $1 + \text{rank } (\text{Cl}_3(\sigma)(N))$ or $2 + \text{rank } (\text{Cl}_3(\sigma)(N))$. Consequently, there arises the question whether the ambiguous 3-class group $\text{Cl}_3(\sigma)(N)$ is a good approximation of the complete 3-class group $\text{Cl}_3(N)$ or the difference $\text{rank } (\text{Cl}_3(N)) - \text{rank } (\text{Cl}_3(\sigma)(N))$ may be arbitrarily large, and whether the types $\alpha$ or $\gamma$ impact on 3-class groups $\text{Cl}_3(L)$ and $\text{Cl}_3(N)$ for being elementary or non-elementary groups.

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Data availability Data underlying our paper can be obtained from the authors upon request.

Declarations

Conflict of interest The authors declare that no conflict of interest arises by publication of their paper.

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