Gravity waves in parity-violating Copernican Universes

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In recent work minimal theories allowing the variation of the cosmological constant, $\Lambda$, by means of a balancing torsion, have been proposed. It was found that such theories contain parity violating homogeneous and isotropic solutions, due to a torsion structure called the Cartan spiral staircase. Their dynamics are controlled by Euler and Pontryagin quasi-topological terms in the action. Here we show that such theories predict a dramatically different picture for gravitational wave fluctuations in the parity violating branch. If the dynamics are ruled solely by the Euler-type term, then linear tensor mode perturbations are entirely undetermined, hinting at a new type of gauge invariance. The Pontryagin term not only permits for phenomenologically sounder background solutions (as found in previous literature), but for realistic propagation of gravitational wave modes. We discuss the observational constraints and predictions of these theories.

I. INTRODUCTION

The cosmological constant, $\Lambda$, and the Copernican principle are two cornerstones of modern cosmology. In this paper we explore the implications of the fact that their story may be more intricate than it is usually assumed. That the cosmological “constant” does not actually need to be constant in theories with torsion has been noted, for example, in [1][2]. It is not new that torsion can change dramatically the perspective of many problems (for a selection of examples see [3][4]). It has also been noted [3][5] that under the shadow of torsion, homogeneity and isotropy do not imply parity invariance. The Copernican principle therefore has a choice between incorporating parity invariance or not. Parity odd solutions in homogeneous and isotropic models employ a geometrical structure which has been known since the inception of General Relativity: Cartan’s spiral staircase [5][6]. Thus, a varying Lambda may go hand in hand with parity violating Copernican models, creating an interesting synergy.

Within the theories considered in [1][5] the inverse of Lambda becomes canonically conjugate to the Chern-Simons invariant [20][21]. The radical implications of this fact in quantum cosmology were examined in [21] (see [22][23] for the background problem). In the context of classical solutions, the dynamics are then ruled by two topological invariants, of which the Chern-Simons functional is the density. Depending on whether one considers the real or imaginary parts of the Chern-Simons term, these are the Pontryagin and the Euler (or Gauss-Bonnet) invariants. Since these terms appear in the action multiplied by $\Lambda^{-1}$, they are only topological invariants if $\Lambda$ is a constant. The variability of Lambda disrupts their topological nature, and so they are quasi-topological terms (to use the terminology of [1]).

The theories considered in [1][2][15] have the virtue that they do not add new parameters to gravity with respect to Einstein’s theory with a cosmological constant. The coefficient of the Euler term is fully fixed by the Bianchi identities (from solutions without matter), so that the only true new parameter is the numerical coefficient of the Pontryagin term, should we consider it. However, for these theories $\Lambda$ is longer a free parameter, as it is in Einstein’s theory. Hence, a theory with the Euler term alone would have fewer free parameters than General Relativity, as argued in [1]. As explained in [2] such a theory conflicts dramatically with basic Hot Big Bang cosmology (it refuses to accept a radiation epoch). The introduction of the Pontryagin term allows for a viable expansion history (as studied in [13]), leaving the working theory with the same number of free parameters as General Relativity.

It was found in [15] that the parity-even and parity-odd Copernican solutions belong to separate branches of the dynamics. Indeed, a Hamiltonian analysis revealed a different structure of constraints and consequently a different number of degrees of freedom. We are therefore talking about different phases of the same non-perturbative theory. The underlying gauge symmetry associated with the new constraint of the theory is a form of conformal invariance (generalized for theories with torsion). Lambda appears to be pure gauge with regards to this symmetry in the parity-even branch (in the absence of matter). The parity-odd branch breaks conformal invariance even in the absence of matter, giving a varying Lambda a physical meaning. Non-conformal matter does the same in the parity-even branch, but then Lambda becomes a slave to matter (much in the spirit of [24]). It is interesting to note that the (odd parity) Pontryagin term is only relevant for the homogeneous and isotropic dynamics in the parity-odd branch of the solutions.

1 Models where $\Lambda$ is directly conjugate to the Chern-Simons invariant (or similar quantities) have been considered within the context of unimodular gravity [17][19].
A preliminary investigation \cite{15} revealed that phenomenology in these theories (which, we stress, often have fewer free parameters than General Relativity, and rarely can be made to have more) shows a preference for the parity-odd branch in the presence of Pontryagin dynamics. These considerations concerned only the background solution, which is already very rich in the parity-odd branch. The next obvious step is to investigate the propagation of gravitational waves in the same branch. Such is the purpose of the current investigation.

The plan of this paper is as follows. In Section \textbf{II} we start by reviewing previous results that will be needed in this paper, translating them into the notation we found most useful for establishing a perturbation calculation. In Section \textbf{III} we set up the tensor perturbation variables and work out the linearized equations in various forms (tetrads index and space-time index forms, and then decomposed in Fourier and helicity modes). The equations in general look ominous: we have to contend with first order equations in three variables – metric, and parity even and odd components of the connection – but in subsection \textbf{III} we condense them in a more aesthetically pleasing form, and lay out a strategy for their solution.

The rest of the paper is spent on working out solutions for various parameter settings. In Section \textbf{IV} we briefly discuss general properties of the perturbed equations. Next we discuss a number of limiting cases of interest. As a sanity check we find the General Relativity limit in Section \textbf{V}, with reassuring results. In Section \textbf{VI} we consider the case where the dynamics are ruled purely by an Euler pseudo-topological term. We unveil our first surprise: the tensor mode perturbation is left entirely undetermined by the equations of motion. This could well signify that they have become a gauge degree of freedom in this case.

The introduction of the Pontryagin term changes the picture. Physical propagating tensor modes now do exist, but they are endowed with chiral modified dispersion relations. We concentrate on two limiting forms - in Section \textbf{VII} the propagation of gravitational waves in the late universe is discussed, whilst in Section \textbf{VIII} their propagation at earlier stages when the evolution is dominated by matter and radiation components is discussed. Finally in Section \textbf{IX} we summarize our results and discuss prospects for further development.

\section{Review of Previous Results}

Here we shall review some results, adapting the notation in previous literature to the notation that shall be more useful in this paper. Specifically, we shall use the following conventions for indices:

- $A, B, C, D$: $SO(1, 3)$ gauge indices.
- $I, J, K, L$: $SO(3)$ gauge indices.
- $\mu, \nu, \alpha, \beta$: spacetime coordinate indices.
- $t$: time coordinate index.
- $i, j, k, l$: spatial coordinate indices.

\subsection{The Full Theory and its Equations}

The theories we analyze can be written as:

\begin{equation}
S^g[e, \omega, \Lambda] = -\int \frac{3}{2\Lambda} \left( \epsilon_{ABCD} + \frac{2}{\gamma} \eta_{ACD} \eta_B \right) \left( R^{AB} - \frac{\Lambda}{3} e^A e^B \right) \left( R^{CD} - \frac{\Lambda}{3} e^C e^D \right) - \frac{2}{\gamma} \int T^A T_A. \tag{1}
\end{equation}

where $R^{AB} \equiv d\omega^{AB} + \omega^A \omega^{CB} + T^A = de^A + \omega^A e^B$ and unless otherwise stated, multiplication of differential forms is via the wedge product.\footnote{If the parameter $\gamma \to \infty$ and $\Lambda$ is constrained to be a constant, the resulting theory is the Einstein-Cartan theory alongside an Euler boundary term; the particular coefficient of this boundary term has been found to be associated with interesting properties of Noether charges in gravity \cite{24,25}.}

The action can be rewritten as

S^g = S_{Pal} + S_{Eul} + S_{NY} + S_{Pont},

with

\begin{align*}
S_{Pal} &= \int \epsilon_{ABCD} \left( e^A e^B R^{CD} - \frac{\Lambda}{6} e^A e^B e^C e^D \right), \\
S_{Eul} &= -\frac{3}{2} \int \frac{1}{\Lambda} \epsilon_{ABCD} R^{AB} R^{CD}, \\
S_{NY} &= \frac{2}{\gamma} \int e^A e^B R_{AB} - T^A T_A, \\
S_{Pont} &= -\frac{3}{\gamma} \int \frac{1}{\Lambda} R^{AB} R_{AB}.
\end{align*}

The first term is the Palatini action, though differs from that of the Einstein-Cartan theory in that we allow $\Lambda$ vary as a dynamical field rather than fixing it to be a constant. The second term is the quasi-Euler term of \cite{11}. The third term is the Nieh-Yan topological invariant (replacing the Holst term should there be torsion). The last term is the quasi-Pontryagin term studied in \cite{15}. We stress that the connection proposed here between $\gamma$ and the pre-factor of the quasi-Pontryagin term can be broken, and is not strictly needed. More generally, we could also
look at theories with arbitrary numerical factors in front of the quasi-Euler and quasi-Pontryagin terms.

As usual, matter can be added to the gravitational action, to yield a total action:

$$S = \frac{1}{2\kappa} S^\rho(\epsilon, \omega, \Lambda) + S_M(\Phi, \epsilon, \omega, \Lambda),$$  \hspace{1cm} (6)

where $$\kappa \equiv 8\pi G$$ and we have defined energy momentum 3-form $$\tau_A = \frac{1}{2} \delta S_M/\delta \epsilon^A$$, the spin-current 3-form $$\epsilon_{ABC} = -(1/2)\kappa \omega^{ABC}$$, and the $$\Lambda$$-source 4-form $$J \equiv (2/3)\delta S/\delta \omega$$. They are obtained by varying (1) together with the action for matter with respect to $$\epsilon$$, $$\omega$$ and $$\Lambda$$, respectively. A key property of these models is that Einstein’s equation (7) takes the same form in the Einstein-Cartan formulation of gravity (where $$\Lambda = \text{cst.}$$). Any dynamics for $$\Lambda$$ will arise from the gravitational field $$\omega^{AB}$$ rather than via the addition of explicit kinetic terms for $$\Lambda$$ in the Lagrangian.

In this paper we will confine ourselves to situations where the quantities $$S^{AB}$$ and $$J$$ both are negligible. For standard ‘minimal’ coupling between fermions and the spin connection, the quantity $$\tau^{AB}$$ is sourced by the axial spinor current; is negligible. The assumption that $$J$$ is negligible must be regarded as a simplifying assumption and more detailed analysis is needed to determine its expected coupling to matter. For the particular cosmological consequences of the theory examined in this paper, it will suffice to that the matter content is describable in terms of perfect fluids. By way of example, a perfect fluid with density $$\rho$$, pressure $$p$$ and four-velocity $$U^\mu = e^\mu_A U_A$$ will have stress-energy 3-form:

$$\tau^A = -\frac{1}{8} (\rho + p) U^A \epsilon_{BCDE} U^B e^C e^D e^{-\frac{1}{6} \rho} e^A_{BCDE} U^B e^C e^D$$ \hspace{1cm} (10)

B. The background solution

We now look at the behaviour of the theory in situations where spacetime has Friedmann-Robertson-Walker (FRW) symmetry. This symmetry is widely considered to well approximate the geometry of the universe on large scales and there exist strong constraints on the evolution of the universe within this framework. We will henceforth refer to possible solutions with this symmetry as ‘background’ solutions as later we will consider the behaviour of small perturbations around them. It is important then to demonstrate that the combined action (6) yields solutions that are consistent with these constraints.

We shall denote all background quantities by a bar over the respective variable. For simplicity we assume that the background spatial curvature is zero, so that we can use Cartesian coordinates with

$$\bar{e}^0 = N(t)dt$$
$$\bar{e}^i = a(t)\delta^i_0 dx^i$$

where $$N(t)$$ is the lapse function ($$N = 1$$ for proper time) and $$a(t)$$ is the expansion factor. Note that $$\bar{e}^i = a\delta^i_0$$ and $$\bar{e}^i = a^{-1}\delta^i_0$$. Then, the spin connection will be given by:

$$\bar{\omega}^{0l} = g(t)a(t)\delta^l_i dx^i$$
$$\bar{\omega}^{ij} = -P(t)a(t)e^{lJK}\delta^K k_i dx^k$$

where $$g$$ and $$P$$ are its parity even and odd components, respectively. A connection of the form (14) was considered by Cartan as an extension to Riemannian geometry, with parallel transport according to this connection yielding a rotation of vectors with a ‘handedness’ dictated by the sign of $$P$$. This effect has been termed Cartan’s spiral staircase and we will see that all parity violating effects in this gravitational model appear only when $$P \neq 0$$. The torsion associated with (13) and (14) is given by:

$$\bar{T}^0 = 0$$
$$\bar{T}^l = \bar{T}^l e^0 + P_{e^{lJK}e^j e^K}$$

with the parity even component $$T$$ related to $$g$$ by:

$$T = \left(g - \frac{1}{N} \frac{\dot{a}}{a}\right).$$
The field strength is:

\[
\bar{R}^{I0} = \frac{1}{N} \left( \dot{g} + \frac{\dot{a}}{a} g \right) \bar{e}^0 \bar{e}^I + g P \epsilon^{IJK} \bar{e}_J \bar{e}_K \tag{18}
\]

\[
\bar{R}^{IJ} = \frac{1}{N} \left( \dot{P} + P \frac{\dot{a}}{a} \right) \epsilon^{IJ} \bar{e}^K \bar{e}^0 + \left( g^2 - P^2 \right) \bar{e}^I \bar{e}^J \tag{19}
\]

It can be shown that with this “Copernican” ansatz, equations (7) to (9) become:

\[
g^2 - P^2 = \frac{\Lambda + \kappa \rho}{3} \tag{20}
\]

\[
\frac{(ag)}{a} = \frac{\Lambda}{3} - \frac{\kappa}{6} (\rho + 3p) \tag{21}
\]

\[
T = \frac{\dot{A}}{2 \Lambda^2} \left( \Lambda + \kappa \rho - \frac{6}{\gamma} g P \right) \tag{22}
\]

\[
P = \frac{3 \Lambda}{\Lambda^2} \left( g P + \frac{\Lambda + \kappa \rho}{6 \gamma} \right) \tag{23}
\]

\[
(\Lambda + \kappa \rho) \left( \Lambda - \frac{\kappa}{2} (\rho + 3p) \right) - \Lambda^2 = 18 g P \frac{(aP)}{a} + \frac{9}{\gamma} \left( \frac{\Lambda + \kappa \rho (aP)}{3 a} + \frac{2}{3} \left( \Lambda - \frac{\kappa \rho + 3p}{2} \right) g P \right) \tag{24}
\]

As shown in Appendix [3] this system can be cast in the form of a first-order system of evolution equations for \( \{a, g, \Lambda, P\} \) plus a constraint (the Hamiltonian constraint/Friedmann’s equation). Reference to these background equations will be made at several points in this paper, to simplify the perturbation equations.

C. Background evolution

We now discuss solutions to equations (20)-(24) with an emphasis on solutions that appear likely to be most consistent with the observed expansion history of the universe. Care must be taken here as many probes of background quantities are additionally sensitive to details of cosmological perturbations. For example, the position of the first peak of temperature anisotropies in the cosmic microwave background (CMB) is sensitive to both the distance to last scattering (a background quantity) and the sound horizon at last scattering (a quantity which additionally depends on the form of equations describing cosmological perturbations) [27].

The system of equations (20)-(24) is rather complicated and must be solved numerically. However, relevant approximate solutions do exist, which we will now discuss.

1. Early times

There is strong evidence that the universe has undergone an early period (‘the radiation era’) where relativistic species (such as photons and relativistic neutrinos) dominate the evolution of the universe for a time before the universe cools down enough such that the gravitational effect of near-pressureless/dustlike matter (baryons and dark matter) dominates (‘the matter era’), before eventually a new source of energy - typically termed dark energy - begins to dominate and cause the expansion of the universe to accelerate [28]. We will look to see whether the theory [6] permits this kind of cosmological history, whilst ascribing the recent cosmological acceleration to - now dynamical - \( \Lambda \). An important part of this is that the gravitational effect of new degrees of freedom quantities such as \( \Lambda \) and the torsion \( P \) do not contradict the above picture.

It can be shown that when \( |\gamma| \ll 1 \), to first order in \( \gamma \) there exists a solution for the field \( P \) in the limit \( \Lambda \to 0 \)

\[
P = P_{(\rho)} = \frac{\gamma}{3} \sqrt{\frac{\kappa \rho}{3}} \tag{25}
\]

We see then that when this solution holds, the torsion field \( P \) is proportional to \( \gamma \) and so a smaller value of \( \gamma \) suppresses torsion in the cosmological background. Neglecting the contribution of \( \Lambda \) is expected to be a good approximation in the earlier universe where the ‘dark energy’ is a sub-dominant contributor to the universe’s expansion. When (25) holds it may be shown that the Friedmann equation can be recovered in approximation:

\[
3 \left( \frac{\dot{a}}{a} \right)^2 = \left( 1 - \frac{\gamma^2}{9} \right) \kappa \rho + \mathcal{O}(\gamma^3) \tag{26}
\]

where we have adopted the \( N = 1 \) spacetime gauge. Hence the solution (25) acts to rescale the bare Newton’s constant \( G \) during times when the effect of \( \Lambda \) is negligible. The degree to which this effect is observable depends on how the value of Newton’s constant \( G_N \) measured in tabletop experiments is related to \( G \). If \( G \neq G_N \), then the rate of expansion \( \dot{a}/a \) due to a given \( \rho \) will be different from as is the case in General Relativity and so in principle \( \gamma \) could be constrained by probes of the expansion rate during big bang nucleosynthesis [29].
However, importantly, the solution (25) is not stable. By way of illustration, we may consider the evolution of small, homogeneous perturbations $P = P_0(1 + \delta_P(t))$. It can be shown that deep in the radiation era where $\kappa \rho / 3 \sim H_0^2 \Omega_r / a^4$ - where $H_0$ is the Hubble constant today - that

$$P_0 = \frac{\gamma}{3a^2} H_0 \sqrt{\Omega_r},$$

(27)

$$\delta_P = C a^3$$

(28)

Therefore $\delta_P$ grows as $a$ increases. By way of example, if $\delta_P \ll 1$ at $a \sim 10^{-15}$ then for it to remain smaller than unity at $a \sim 10^{-5}$ we must have $\delta_P(a = 10^{-15}) < 10^{-30}$. This indicates that significant fine-tuning of initial data is required for the spiral staircase field $P$ to find itself following the solution (25).

If $P$ deviates considerably from the tracking solution, the tendency is for $P$ to evolve to dominate the evolution of the universe. In this case it may be shown that $P = P_0/a$ where $P_0$ is a constant and $a \sim (1 + \gamma)(t - t_0)$ - here the evolution of the universe due to $P$ resembles a General-Relativistic empty universe with negative spatial curvature. It is hard to see how such a universe could be consistent with experiment. This is the case even if $P$ is initially negligible. Therefore, the phenomenological viability of the model rests on $P$ being able to initially find itself sufficiently close to the form (25) to avoid dominating the evolution of the universe.

2. Late times

During late time cosmological evolution for realistic cosmologies we expect that universe to begin accelerating and we look to ascribe this to $\Lambda$ and $P$ beginning to dominate cosmic evolution. Again assuming $|\gamma| \ll 1$ and now assuming $P^2 \ll \Lambda$ and taking the limit $\rho \to 0$ we have the following evolution equations for $\Lambda$ and $P$:

$$\frac{dP}{d\ln a} = -3P, \quad \frac{d\Lambda}{d\ln a} = 2\sqrt{3} \gamma P \sqrt{\Lambda},$$

(29)

which possess solutions

$$P = \frac{P_i}{a^3},$$

(30)

$$\Lambda = \Lambda_0 - \frac{2\gamma}{\sqrt{3a^3}} P_i \sqrt{\Lambda_0},$$

(31)

So, asymptotically for large $a$, $\Lambda \to \Lambda_0$ and $P \to 0$, leading to a confluence with the current standard cosmological picture of the late-time universe’s evolution being dominated by a cosmological constant of magnitude $\Lambda_0$. The contribution of $P$ to the Hamiltonian constraint goes as $\sim P^2$ so we see that in this regime $P$ evolves like a shear component, its energy density diluting as $a^{-6}$. For realistic cosmologies a typical value for $P_i$ will be given by its value when $\Lambda$ begins to dominate the evolution of the universe at a scale factor $a \sim a_i$ following a period of matter domination during which $P \sim (\gamma/3) H_0 a_i^{-3/2}$ (from the solution (25)). We expect then $P_i \sim (\gamma/3) H_0 a_i^{3/2}$. This has important implications: if phenomenologically viable cosmologies involve $P$ staying on the tracking solution (25) for an appreciable amount of time, this means that fixing $\gamma$ fixes the size of $P$ during matter domination, and the size of $P_i$ as the cosmological constant begins to dominate.

We now discuss the evolution of $\Lambda$. It can be seen from (25) that the tracking solution can exist if $\text{sign}(P) = \text{sign}(\gamma)$. Recall that the $\Lambda$ equation of motion is $\ddot{\Lambda} = 2\gamma \Lambda^2 P / (6 \gamma q P + \Lambda + \kappa \rho)$ and therefore in the earlier universe if $\kappa \rho$ is initially greater than $\Lambda$ it will tend to suppress time variation of $\Lambda$. Furthermore, we will have $\dot{\Lambda} > 0$ throughout, meaning that $\Lambda$ must be of smaller magnitude in the past than today. A typical evolution of $\Lambda$ and $P$ are shown in Figure 1.

In summary then, numerical exploration suggests that unless $P$ finds itself on the tracking solution (25) for much of cosmic history, it will tend to dominate the evolution of the universe and therefore very likely in conflict with cosmological data. This requires fine tuning of the initial value of $P$ so that it begins close to the tracking solution. During the tracking stage, the effect of $P$ is to rescale Newton’s constant $G$. We will find later that deviations of gravitational wave speed from unity tend to be of order $\gamma^2$. This justifies our assumption that $|\gamma| \ll 1$ and - in conjunction with recent constraints on the speed of gravity - restricts the fractional rescal-
of $G$ to be $\mathcal{O}(10^{-15})$, which is well within bounds that will be placed by BBN constraints for the foreseeable future \cite{29}. Additionally we see that a smaller value of $\gamma$ tends to lower the total time variation of $\Lambda$ over cosmic history, making it more difficult to distinguish from a genuine cosmological constant.

### III. THE PERTURBED EQUATIONS OF MOTION FOR TENSOR MODES

We now look at the evolution of small perturbations to the cosmological background. We perturb the tetrad and connection as:

\begin{align}
\delta e^0 &= 0 \quad (32) \\
\delta e^I &= \frac{1}{2} H^{IJ} \dot{\bar{e}}_J \quad (33) \\
\delta \omega^{0I} &= \frac{1}{2} \bar{\epsilon}_{IJ} \dot{e}_J \quad (34) \\
\delta \omega^{IJ} &= \frac{1}{2} \epsilon^{IJ} \dot{\bar{e}}_L \quad (35)
\end{align}

where $H^{[IJ]} = E^{[IJ]} = B^{[IJ]} = 0$ and $H^{IJ} = E^{IJ} = B^{IJ} = 0$. In addition we apply the restriction of looking at tensor (transverse, traceless) modes, so that we impose:

\begin{equation}
\dddot{D}_I H^{IJ} = \dddot{D}_I E^{IJ} = \dddot{D}_I B^{IJ} = 0. \quad (36)
\end{equation}

where $\dddot{D}_I \equiv \dddot{e}_i^I \dddot{D}_i$ and $\dddot{D}_i$ is the covariant derivative according to $\dddot{e}_i^I$. Note the field $P$ does not contribute to the expressions \cite{30} and so the equations are equivalent to $v^i_1 \partial_i H^{IJ} = \dot{v}^i_1 \partial_i \dot{E}^{IJ} = \dot{v}^i_1 \partial_i \dot{B}^{IJ} = 0$.

Given a quantity $Y_{IJ}$ that represents a small perturbation, it can be converted into a tensor $\bar{Y}_{ij}$ in the spatial coordinate basis via $\bar{Y}_{ij} \equiv \epsilon^{kl}_{ij} Y_{kl}$. Our assumption of vanishing spatial curvature, a ‘co-moving’ tensor $\bar{Y}_{ij} = Y_{ij}/a^2$ can further be constructed.

The linearly-perurbed form of equations \cite{7, 8} can be written as a system of linear partial differential equations. For simplicity we decompose these perturbations into plane-wave Fourier components labelled by wave vector $k$ and as a further simplification we decompose each co-moving tensor mode Fourier mode into helicity eigenstates i.e.:

\begin{equation}
\bar{Y}_{ij}(x, t) = \frac{1}{(2\pi)^3} \int d^3 k \sum_{\pm} Y_{\pm}(\bar{k}, t) e^{ik \cdot x} \bar{T}^{\pm}_{ij}. \quad (37)
\end{equation}

Here $\bar{T}^{\pm}_{ij}$ are co-moving polarization tensors for + and − helicity components. Without loss of generality we will focus the remainder of the analysis on modes that propagate along the z-direction of the coordinate system i.e. $\bar{k} = (0, 0, k)$. We then have the following important identities:

\begin{equation}
\pm k_{\lambda} \bar{P}_{\lambda ij} = \pm k^{\pm} \bar{P}_{ij} \quad \text{and} \quad \bar{P}_{ij} \bar{P}^{\lambda \nu ij} = \delta^{\lambda \nu} \quad \text{where} \quad \lambda = +, -, \quad \text{and} \quad \epsilon_{ijk} \text{ is the co-moving three dimensional Levi-Civita symbol.} \quad \text{Indices of co-moving tensors are taken to be raised and lowered with the Kronecker delta symbol.}
\end{equation}

After some algebra, it can be shown that the spin connection equations of motion yield the following equations:

\begin{equation}
\left( B^\pm \right) = \frac{1}{A^2 + B^2} \left( \begin{array}{cc}
A & B \\
-B & A \end{array} \right) \left( \dot{H}^\pm + \left( \frac{2a}{a} - g \right) H^\pm \right) \quad (38)
\end{equation}

where here and subsequently we choose the spacetime gauge $N = 1$ (proper time) and where

\begin{align}
A &= 1 - \frac{3\Lambda}{A^2} (g + k^\pm P^{-1}) \quad (39) \\
B &= \frac{3\Lambda}{A^2} (k^\pm - g\gamma^{-1}) \quad (40)
\end{align}

We have introduced the polarization-dependent, torsion-adjusted proper wavenumber $k_P$ according to:

\begin{equation}
k^\pm_P \equiv \frac{k}{a} - P \quad (41)
\end{equation}

For reference, in the usual Einstein-Cartan theory we have $A = 1, B = 0$; in that case, $B^\pm$ is related to spatial derivatives of $H^\pm$ and $\dot{E}^\pm$ is related to time variations of $H^\pm$. All modifications to the relation between $\{E^\pm, B^\pm\}$ and $\dot{H}^\pm$ stem from non-constancy of $\Lambda$. Hence the connection equations imply that in general the parity even and odd components of the connection $\{E, B\}$ can be obtained from their Einstein-Cartan expressions via a rotation, with an angle $\theta$ satisfying:

\begin{equation}
\tan \theta = \frac{B}{A} = \frac{3\Lambda}{A^2} (g - k^\pm P^{-1}) \quad (42)
\end{equation}

followed by a dilatation by $1/\sqrt{A^2 + B^2}$.

Then we may look to find the “second order” evolution equation for $\dot{H}^\pm$ by inserting the solution \cite{38} for $B^\pm$ and $\dot{E}^\pm$ into the Einstein equation:

\begin{equation}
0 = \dot{E}^\pm + \left( \frac{a}{a} + g \right) E^\pm - k^\pm_P B^\pm - \left( \frac{2}{3} \Lambda + \kappa \rho (\rho - 3p) \right) H^\pm \quad (43)
\end{equation}

We now look at solutions to the system \cite{38} and \cite{43}.

### IV. GENERAL FEATURES

Generally, if the solution \cite{38} is inserted into \cite{43} then the resulting coefficient of $\dot{H}^\pm$ is proportional to:

\begin{equation}
\left( 1 - \frac{6P}{A + \kappa \rho} \right) k^\pm_P + \ldots \quad (44)
\end{equation}

Where dots denote terms of higher order in $|\gamma| < 1$. We see that the coefficient is not positive-definite and hits zero when $k = k^\pm_s$:

\begin{equation}
k^\pm_s = \pm \frac{a}{6P} \left( A + \kappa \rho + 6P^2 \right) \quad (45)
\end{equation}
For example in the very late universe we may expect \( \Lambda \sim \Lambda_0 = \text{cst.} \) to dominate the evolution of the universe hence then:

\[
k^\pm_k \sim \pm \frac{a}{6P} \Lambda_0
\]

(46)

where we’ve assumed that \( P^2/\Lambda_0 \ll 1 \).

Following the arguments proposed in Subsections II C 1 and II C 2 we have that \( k^\pm_k \sim \pm 2a^2 \Lambda_0/\gamma \) for realistic cosmologies. Reaching \( k^\pm_k \) likely corresponds to the coefficient of the kinetic term of one of the gravity wave polarizations in the perturbed Hamiltonian passing through zero, signalling that the mode is becoming ghostlike. If the theory is to be experimentally viable then the scales \( k^\pm_k \) should be of far greater magnitude than those for which the approximation of gravitational waves as small perturbations obeying linear equations of motion is expected to hold well. We will see that these observations from the LIGO experiment severely constrain the value of \( |\gamma| \) because of this.

As for the case of \( P(t) \), we see that a key parameter for the size of \( k^\pm_k \) is \( \gamma \).

When \( k \neq k^\pm_k \) and with the important exception of the limit \( |\gamma| \to \infty \) (see Section VI), it is possible to write the Einstein equation (43) in the following form :

\[
\ddot{\mathcal{H}}^\pm = -\omega^2_k(k,t)\mathcal{H}^\pm - f^\pm_k(k,t)\dot{\mathcal{H}}^\pm
\]

(47)

For arbitrary values of \( \gamma \), the form of \( \omega^2_k(k,t) \) and \( f^\pm_k(k,t) \) will be extremely complicated and so we will concentrate in detail on how (47) looks in relevant, limiting cases.

V. THE EINSTEIN-CARTAN LIMIT

We start by finding the Einstein-Cartan limit of these theories, noting that when \( \gamma \) is finite and \( \rho = p = 0 \) there are solutions where \( P = 0 \), \( g = \acute{a}/a \) and \( \Lambda \) is constant [15]. Taking these background solutions we should obtain the Einstein-Cartan limit for our theory, which is equivalent to General Relativity in this situation. Inserting these conditions into the formalism just developed, we find \( \Lambda = 0 \), and so \( A = 1 \) and \( B = 0 \), as already announced in the previous Section. The connection equations are therefore:

\[
B^\pm = -k^\pm_k \mathcal{H}^\pm
\]

(48)

\[
\mathcal{E}^\pm = \left( \frac{d}{dt} + \frac{\acute{a}}{a} \right) \mathcal{H}^\pm.
\]

(49)

Note that since \( P = 0 \) we have \( k^\pm_k = \pm a/k \), and so for gravity waves in Einstein-Cartan theory, the parity-odd connection perturbation, \( B \), is a spatial gradient of the metric, whereas the parity-even component, \( \mathcal{E} \), is a time derivative of the metric (cf. Eqs (35) and (34)). Inserting these expressions into the Einstein equation (43), as prescribed, we find:

\[
\ddot{\mathcal{H}}^\pm + 3\frac{\acute{a}}{a} \dot{\mathcal{H}}^\pm + \left( \dot{g} + 2g^2 - \frac{2}{3} \Lambda \right) \mathcal{H}^\pm + (k^\pm_k)^2 \mathcal{H}^\pm = 0
\]

(50)

where the dot denotes derivative with respect to the background proper time. In the Einstein-Cartan theory we have \( \dot{T} = 0 \) in the absence of background sources of torsion, so \( g = \acute{a}/a \), and the background equations of motion read (see (20) and (21)):

\[
g^2 = \frac{\Lambda}{3}
\]

(51)

\[
\dot{g} + \frac{\acute{a}}{a} g = \dot{g} + g^2 = \frac{\Lambda}{3}.
\]

(52)

Therefore we find:

\[
\ddot{\mathcal{H}}^\pm + 3\frac{\acute{a}}{a} \dot{\mathcal{H}}^\pm = -(k^\pm_k)^2 \mathcal{H}^\pm
\]

(53)

Thus, our formalism for gravity waves reduces to the textbook equations for gravity waves in General Relativity in this limit.

VI. EULER THEORY (\( \gamma \to \infty \)) IN A PARITY-ODD BACKGROUND (\( P \neq 0 \))

Our first surprise arises when we consider a theory with the Euler pseudo-topological term only, by letting \( \gamma \to \infty \), but with a background with \( P \neq 0 \). Then, as the background Equation (23) shows (with \( P 
eq 0 \) and \( \gamma \to \infty \)), we must have \( 3\Lambda \alpha = \alpha^2 \). Therefore, the definitions of \( A \) and \( B \) (Eqns. (39) and (40)) lead to:

\[
A = 0, \quad B = \frac{k^\pm_k}{g}
\]

(54)

These are orthogonal to the Einstein-Cartan values, in the sense that for the latter the matrix (38) is diagonal, whereas here the matrix is purely off-diagonal. Indeed the rotation part of the transformation is now \( \theta = \pi/2 \). This is reflected in the way the connection is related to the metric. the Einstein-Cartan case (cf. Eqs. (48) and (49)) we have:

\[
\mathcal{E}^\pm = g \mathcal{H}^\pm
\]

(55)

\[
k^\pm_k B^\pm = g \left( \ddot{\mathcal{H}}^\pm + \left( \frac{2\ddot{a}}{a} - g \right) \dot{\mathcal{H}}^\pm \right)
\]

(56)

Inserting into the Einstein equation (43) we find that not only does this imply an absence of second order time derivatives for \( \mathcal{H}^\pm \), but the first time derivatives cancel out. In addition the algebraic equation obtained is

\[
\left( \frac{1}{2} \dot{g} + g^2 - \frac{2}{3} \Lambda - \frac{\Lambda}{3} + \frac{1}{12} (3p - \rho) \right) \mathcal{H}^\pm = 0
\]

(57)

The term multiplying \( \mathcal{H}^\pm \) in (57) vanishes due to the background equations of motion, therefore the tensor mode perturbation is \( \mathcal{H}^\pm \) completely undetermined by the perturbed equations of motion.\footnote{This would appear to contradict the result found in [39] which says that tensor modes propagate luminally as in General Relativity in a model with tensor mode perturbation equations that should be mappable to the ones considered here.}
One may wonder to what extent this is a result of the particular choice for our action. For example, if the coefficient $\frac{-3}{2\xi}$ in the term (3) is replaced by $\frac{-3}{2\epsilon_0}$ then it can be shown that Einstein's equation instead becomes:

$$\frac{1}{\xi} \left(1 - \xi\right) \left(4\Lambda + \rho - 3p\right) H^\pm = 0$$

(58)

Thus in the case when $\Lambda \neq 0$, $\rho \neq 3p$ and $\xi \neq 1$, the perturbation $H^\pm$ is not undetermined but fixed to vanish. It appears that the presence of the Euler term in the absence of the Pontryagin term is sufficient to nullify the dynamics of the perturbation $H^\pm$ with the existence of a special case $\xi = 1$, which leaves them undetermined by the perturbed equations of motion. Note that the case of simultaneous vanishing of the Euler and Pontryagin term ($\xi \to \infty$) does not correspond to General Relativity. In fact such limit yields a rather exotic background solution $a = 0$ due to $\Lambda$ being a dynamical field.

VII. THE LEADING ORDER SOLUTION FOR THE GENERAL CASE IN THE LATE UNIVERSE

We now consider a general finite value of $\gamma$ and look at the perturbed equations in a regime where the evolution of the universe is dominated by $\Lambda$. We define a dimensionless parameter $\epsilon_p \equiv P/\sqrt{\Lambda}$ which is expected to be of magnitude much smaller than unity in the late universe. Furthermore we assume that $|\gamma| \ll 1$. Inserting the solutions for $\mathcal{E}^\pm$ and $\mathcal{B}^\pm$ from (38) into the Einstein equations and keeping only terms up to second order in $\{\epsilon_p, \gamma\}$ we find:

$$\dot{H}^\pm = -\omega_{\pm}^2 (k, t) H^\pm - f_{\pm} (k, t) \dot{H}^\pm$$

$$\omega_{\pm}^2 (k, t) = \left[1 - \kappa \left(\rho (-\Lambda + \kappa \rho) + 3p (\Lambda + \kappa \rho)\right) \right] \gamma^2$$

$$- \frac{8\sqrt{3}}{(\kappa \rho + \Lambda)^{5/2}} \left(\Lambda^{5/2} + \kappa \Lambda^{3/2} - \kappa \rho^2 - 3\kappa^2 \rho \Lambda - 3\kappa \Lambda^{3/2} p\right)$$

$$+ \mathcal{O} (\epsilon_p, \gamma)^3$$

$$f_{\pm} (k, t) = \sqrt{3 (\kappa \rho + \Lambda)} \left[\pm \frac{1}{(\Lambda + \kappa \rho)^2} \left(\rho (-\Lambda + \kappa \rho) + 3p (\Lambda + \kappa \rho)\right) \pm 4\sqrt{\frac{3}{\epsilon_p}} \sqrt{\Lambda} \frac{1}{(\Lambda + \kappa \rho)^{3/2}} (3p \kappa - 2\Lambda + \kappa \rho)\right]$$

(59)

(60)

Roughly speaking, positivity of both $\omega_{\pm}^2 (k, t)$ and $f_{\pm} (k, t)$ imply that $H^\pm$ evolves in a stable manner.

Following (31) (see also (32) and (34)) the speed of monochromatic tensor modes today $c_T^\pm$ (taking $a = 1$) is given by $c_T^\pm = \omega_{\pm}^T / \xi$. In general our expression for $c_T$ will be rather complicated but it is instructive to detail the order of magnitude of terms appearing in its expressions. Given how we expect $P(t)$ to scale with $\gamma$ from the results of Section II C 2 we find that:

$$c_T^\pm \sim 1 + \mathcal{O} \left(\gamma^2\right) + \ldots$$

(62)

Constraints from the LIGO experiment roughly constrain the deviation of $c_T^\pm$ from unity by approximately $10^{-15}$. The constraint on the speed of gravitational wave speed then places the following restriction on $\gamma$:

$$\gamma^2 \lesssim \mathcal{O} (10^{-15})$$

(63)

Given this constraint and the small value of $H_0 / k_{\text{LIGO}}$, the remaining immediate constraint from $c_T^\pm$ is that

| $k^\pm$ | $\gg k_{\text{LIGO}}$ |

(64)

which is necessary for the consistency of our use of the linearly perturbed equations of motion. We can translate this into a constraint on $\gamma$ by assuming as above that $P \sim (\gamma/3) H_0 a_i^{3/2} a^{-3}$ and so using equation 45 we have $k^\pm \sim 1/\Lambda$ and so

$$|\gamma| \ll \mathcal{O} (10^{-21})$$

(65)

VIII. PERFECT FLUID DOMINATION

In this limit, we consider the evolution of perturbations on a background where the evolution is dominated by a combination of perfect fluids. It was shown in II C that there exist solutions where $\Lambda \sim 0$ and $P^2 \sim \gamma^2 k \rho / 2T$ with $\gamma \ll 1$ and that these seem to be the solutions that yield a realistic cosmology. Assuming that these solutions hold then to quartic order in the small parameter $\gamma$ we have that:
\[
\hat{H}^\pm = -\omega^2_\pm(k, t)\mathcal{H}^\pm - f^\pm(k, t)\dot{\mathcal{H}}^\pm
\]

\[
\omega^2_\pm(k, t) \equiv \left[ 1 + \frac{1}{9} \left( 1 + \frac{3p}{\rho} \right) \gamma^2 \right] (\frac{k}{a})^2 \pm \frac{2}{3\sqrt{3} \rho} \gamma^3 (\frac{k}{a})^3 + O(\gamma^5)
\]

\[
f^\pm(k, t) \equiv \left[ \frac{(18 - \gamma^2)}{6\sqrt{3}} \sqrt{\kappa \rho} \pm \frac{(3p + \rho) \gamma(9 + 8\gamma^2)}{24\rho} (\frac{k}{a})^3 \right] + O(\gamma^5)
\]

\[
\pm \frac{4}{9\kappa \rho^2} \kappa(3p + \rho) \gamma^3 (\frac{k}{a})^3
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freedom. The non-perturbative Hamiltonian analysis of this model and various generalizations of it have been performed in [52] and it would be interesting to explore the appearance of this branch structure within the more general results there. We have discovered that tensor mode fluctuations about certain background solutions additionally display interesting properties; in particular, the under-determination of the perturbed tensor equations of motion for a theory with a pure Euler term hints that a novel type of gauge symmetry may exist in restricted situations.

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Appendix A: Correct equations - no background equations were used

For completeness we provide the position-space form of the perturbed Einstein and spin-connection equations, which take the following form:

\[ \frac{\partial}{\partial t} \mathcal{E}^{IJ} + (H + g) \mathcal{E}^{IJ} + (\dot{\gamma} + H g - \Lambda) H^{IJ} - \epsilon^{IKL} \dot{D}_K B^J_L - 2 P B^{IJ} = 0 \]  

(A1)

\[ \epsilon^{IKL} \dot{D}_K H^I_L + 2 P H^{IJ} + (1 - \frac{3g}{\Lambda^2} \dot{\Lambda} - \frac{6P}{\gamma \Lambda^3} \dot{\Lambda}) B^{IJ} = \frac{3}{\Lambda^2} \dot{\Lambda} (\epsilon^{IKL} \dot{D}_K \mathcal{E}^I_L + 2 P \mathcal{E}^{IJ}) + \frac{3}{\gamma \Lambda^2} \dot{\Lambda} (\epsilon^{IKL} \dot{D}_K B^I_L - \epsilon \mathcal{E}^{IJ}) , \]  

(A2)

\[ - (\frac{\partial}{\partial t} H^{IJ} + (H - T) H^{IJ}) + (1 - \frac{3g}{\Lambda^2} \dot{\Lambda} - \frac{6P}{\gamma \Lambda^3} \dot{\Lambda}) \mathcal{E}^{IJ} = - \frac{3}{\Lambda^2} \dot{\Lambda} (\epsilon^{IKL} \dot{D}_K \mathcal{B}^I_L + 2 P \mathcal{B}^{IJ}) + \frac{3}{\gamma \Lambda^2} \dot{\Lambda} (\epsilon^{IKL} \dot{D}_K \mathcal{E}^I_L + g \mathcal{B}^{IJ}) \]  

(A3)

Appendix B: Alternative form of the background equations of motion

It is possible to write down the field equations (7)-(9) as a system of first-order ordinary differential equations for variables \( \{ P, g, \Lambda, \alpha \} \) along with a constraint equation:

\[ \dot{P} = \frac{-6P \kappa \rho (6\gamma Pg^2 + 3g(\Lambda + 2P^2) - \Lambda \gamma P) + \kappa^2 \rho (6P(3gp + \gamma P) + \gamma \Lambda \kappa (\rho - 3p)) - \gamma \kappa^3 \rho^2 (3p + \rho)}{6\kappa \rho (6\gamma Pg + \Lambda \kappa + \rho)} \]  

(B1)

\[ \dot{g} = \frac{6 (\gamma^2 + 1) g^2 P^2}{6\gamma Pg + \Lambda + \kappa \rho} - g^2 + g \gamma P + \frac{\Lambda}{3} - \frac{\kappa}{6} (\rho + 3P) \]  

(B2)

\[ \dot{\Lambda} = \frac{2 \gamma P \Lambda^2}{6\gamma Pg + \Lambda + \kappa \rho} \]  

(B3)

\[ \dot{\alpha} = a \left( \frac{6 (\gamma^2 + 1) g^2 P^2}{6\gamma Pg + \Lambda + \kappa \rho} + \gamma - \gamma P \right) \]  

(B4)

\[ \frac{\Lambda}{3} = g^2 - P^2 - \kappa \rho \frac{\rho}{3} \]  

(B5)

It can be checked via differentiation of the constraint equation that \( \dot{\rho} = -3 \frac{\alpha}{n} (\rho + p) \) as in the case of a perfect fluid in Gen-
general Relativity. Using the constraint equation we can rewrite the evolution equation as
\[ \Lambda = \left(\frac{2}{3}\gamma\right)P\Lambda^2/(g^2 - P^2 + 2g\gamma P). \]
From this perspective, the dynamics for \( \Lambda \) can be seen as arising from the spin connection components \( g \) and \( P \).

**Appendix C: Linearly perturbed field strength and torsion**

Central objects in the field equations (7)-(9) are the curvature and torsion two-forms \( R^A B \) and \( T^A \). Their linearly-perturbed forms around the cosmological background are found to be:

\[
\delta R^{0I} = -\epsilon^{IKL} \bar{e}_L \bar{e}_K + \left( \frac{1}{\bar{a}} \frac{\partial}{\partial t} \epsilon^{IJ} + \frac{\dot{\bar{a}}}{\bar{a}} \epsilon^{IJ} \right) \bar{e}_J + \bar{D}^K \epsilon^{IJ} \bar{e}_K \bar{e}_J + \bar{P} \epsilon^{IJ} \epsilon^{JKL} \bar{e}_K \bar{e}_L
\]

\[
\delta T^I = \left( \frac{1}{\bar{a}} \frac{\partial}{\partial t} \epsilon^{IM} + \frac{\dot{\bar{a}}}{\bar{a}} \epsilon^{IM} - \epsilon^{IJ} \right) \bar{e}_J + \bar{D}^L \epsilon^{IJ} + \bar{P} \epsilon^{IJ} \epsilon^{KLM} \bar{e}_K \bar{e}_L (C1)
\]

\[
\delta R^{IJ} = 2g\epsilon_L \left[ \epsilon^{[I|L|} \epsilon^J] + \epsilon^{IJ} \left( \frac{1}{\bar{a}} \frac{\partial}{\partial t} \bar{B}^{KL} + \frac{\dot{\bar{a}}}{\bar{a}} \bar{B}^{KL} \right) \bar{e}_L \right] + (\bar{D}_L \bar{B}^{KM}) \bar{e}_L \bar{e}_M + \bar{P} \bar{B}^K \epsilon^{PMN} \bar{e}_M \bar{e}_N (C2)
\]

\[
\delta T^I = \left( \frac{1}{\bar{a}} \frac{\partial}{\partial t} \epsilon^{IM} + \frac{\dot{\bar{a}}}{\bar{a}} \epsilon^{IM} - \epsilon^{IJ} \right) \bar{e}_J + \bar{D}^L \epsilon^{IJ} + \bar{P} \epsilon^{IJ} \epsilon^{KLM} \bar{e}_K \bar{e}_L (C3)
\]

where \( \bar{D}_I \equiv \epsilon^L I \bar{D}_L \) where \( \bar{D}_I \) is the covariant derivative according to \( \omega^{IJ} \).

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