Superpotentials for \( M \)-theory on a \( G_2 \) holonomy manifold and Triality symmetry

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For \( M \)-theory on the \( G_2 \) holonomy manifold given by the cone on \( S^3 \times S^3 \) we consider the superpotential generated by membrane instantons and study its transformations properties, especially under monodromy transformations and triality symmetry. We find that the latter symmetry is, essentially, even a symmetry of the superpotential. As in Seiberg/Witten theory, where a flat bundle given by the periods of an universal elliptic curve over the \( u \)-plane occurs, here a flat bundle related to the Heisenberg group appears and the relevant universal object over the moduli space is related to hyperbolic geometry.
1 Introduction

The conifold transition among Calabi-Yau manifolds in type II string theory has an asymmetrical character: an $S^3$ is exchanged with an $S^2$. When the situation is lifted to $M$-theory the resulting geometries become completely symmetrical [1], [2]: the two small resolutions given by the $S^2$'s (related by a IIA flop) on the one side of the transition become $S^3$'s as well (Hopf fibred by the $M$-theory circle $S^1_{11}$; this concerns, in type IIA language, the situation where on this side one unit of RR flux on the $S^2$ is turned on and one has a $D6$ brane on the $S^3$ of the deformed conifold side; cf. also [3]). This symmetry of the three $S^3$'s is the triality symmetry $\Sigma_3$ of $M$-theory on the corresponding non-compact $G_2$ manifold which is a suitable deformation $X_7$ of a cone over $S^3 \times S^3 = SU(2)^3/SU(2)_D$ and comes naturally in three equivalent versions $X_1, X_2, X_3$.

In [2] it was pointed out that for general reasons the superpotential should have an anti-invariant behaviour under the triality symmetry, i.e. it should transform with the sign-character of $\Sigma_3$ (cf. app. A). For this recall the action of the order two element $\alpha$ coming from the interchange $g_2 \leftrightarrow g_3$ of $SU(2)$ elements in $(g_1, g_2, g_3)$, which on $X_1 = \mathbb{R}^4 \times S^3$ (when gauging $g_3$ to 1) is given by $(g_1, g_2) \rightarrow (g_1 g_2^{-1}, g_2^{-1})$ and induces an orientation reversing endomorphism on the tangent space (at a fixed point with $g_2 = 1$). It acts as an $R$-symmetry under which the superpotential transforms odd. Then a triality symmetric superpotential was considered [2] which was suggested by global symmetry considerations on the moduli space $\mathcal{N} = \mathbb{P}_1^1$ (triality symmetric behaviour meaning here that it transforms anti-invariantly, i.e. with the sign character). The simplest possibility was (with $\Sigma_3$ operating by $t \rightarrow \omega t$ where $\omega = e^{2\pi i/3}$ and $t \rightarrow 1/t$)

$$W(t) = \frac{t^3 - 1}{t^3 + 1}$$ (1.1)

As we will show one can actually arrive at a closely related result on a different route by considering the actual non-perturbative superpotential generated by membrane instantons. The analytic continuation of these local (on $\mathcal{N}$) contributions gives an essentially symmetric superpotential (cf. below). Crucial will be a non-linear realization of the triality symmetry. Under the following action of the triality group $\Sigma_3$

$$\begin{align*}
z \\
\alpha z = \frac{1}{z} \\
\beta z = \frac{1}{1 - z} \\
\beta^2 z = \frac{z - 1}{z} \\
\alpha \beta z = \frac{z - 1}{z - 1} \\
\alpha \beta^2 z = \frac{z - 1}{z - 1}
\end{align*}$$ (1.2)

the holomorphic observables given by the variables $\eta_i$ form a $\mathbb{Z}_3$ orbit: $\eta_{i-1} = \beta \eta_i$, $\eta_{i+1} = \beta^2 \eta_i$. The variables $u_i$, given by the membrane instanton amplitudes and constituting
local coordinates at the semiclassical ends of the global moduli space, are (when globally analytically continued; they are properly only first order variables) holomorphically related to the $\eta_i$. One has a relation $\eta_3 = \frac{1}{1-\eta_1}$ then also for the (global) $u_i$; note the corresponding map in [1] ($t$ and $V$ the sizes of $S^2$ and $S^3$ in type IIA; for $N = 1$ there)

$$\frac{1}{1-e^t} \sim e^V \quad (1.3)$$

We argue (in the framework of [2]) that the full multi-cover membrane instanton superpotential is given by the dilogarithm (cf. [14])

$$W(u) = \text{Li}(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^2} \quad (1.4)$$

To accomplish this we reinterpret the treatment of the one-instanton amplitude in [2]. There the evaluation of the vev $\langle \int_{D_i} C \cdot \int_{Q_i} *G \rangle$ via an auxiliary classical four-form field $G$ was done in connection with the derivation of an ordinary interaction from the superpotential. The scalar potential computation for the superpotential can be argued to describe not only the one-instanton contribution but the full instanton series.

Remarkably, the actual superpotential given by the multi-cover membrane instantons knows 'by itself', via its analytic continuation, that it entails a triality symmetry. For the function $W(u)$ satisfies the following symmetry relations which will ensure that the dilogarithm superpotential is compatible with triality symmetry (i.e. that the local (on $N$) membrane instanton contributions fit together globally in this sense)

$$W\left(\frac{1}{u}\right) = -W(u) - \zeta(2) - \frac{1}{2} \log^2(-u)$$
$$W(1-u) = -W(u) + \zeta(2) - \log u \log(1-u) \quad (1.5)$$

The symmetry relations (given here for the transformation under $\alpha$ and $\alpha\beta$; from these all others are derived) have the consequence that, up to the elementary corrections provided by the products of two log’s and $\zeta(2)$, the $W(u)$ superpotential is invariant under the transformations in the first line of (3.3) and transforms with a minus sign under the mappings of the second line. That is the 'local' superpotential transforms (under the $SL(2)$ action) up to the elementary corrections with the sign character just as the global superpotential did (under the linear action) and as it should a priori. In other words triality symmetry is in this sense 'dynamical': it holds on the level of the superpotential.

Similarly, and much more trivially, the geometric world-sheet instanton series 'knows' about the $S^2$-flop transition. Note the analogous behaviour of the instanton sums
\[ I_{\text{ws}}(\frac{1}{q}) = -I_{\text{ws}}(q) - 1 \] and (1.5) describing the multi-coverings of the supersymmetric cycles provided by the holomorphic \( S^2 \) in the string world-sheet case and the associative \( S^3 \) in the membrane case, respectively.

Note that when (1.4) is globally analytically continued over the critical circle (the boundary of its convergence disk) one gets monodromy contributions. The monodromy representation of the fundamental group \( \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \) will describe the multi-valuedness \( \text{Li}(z) \xrightarrow{t_1} \text{Li}(z) - 2\pi i \log z \). The relevant local system is described by a bundle, flat with respect to a suitable connection. Like in the case of the logarithm where the monodromy of \( \log z \) around \( z = 0 \) is captured by the monodromy matrix

\[
M(z_0) = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}
\]

and the monodromy group is given by \( \mathcal{U}_\mathbb{Z} \hookrightarrow \mathcal{U}_\mathbb{C} \) where \( \mathcal{U} \) denotes the upper triangular group \( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \text{SL}(2) \) (the embedding of \( \mathcal{U}_\mathbb{Z} \) in \( \mathcal{U}_\mathbb{C} \) may include here the factor of \( 2\pi i \)), the corresponding generalisation in the case of the dilogarithm involves upper triangular \( 3 \times 3 \) matrices, i.e. one gets again admixtures from 'lower' components when one considers constants, ordinary logarithms and the dilogarithm all at the same time. One then finds a function

\[
\mathcal{L}(z) = \text{Im} \text{Li}(z) - \text{Im} \log \beta z \text{ Re} \log z
\]  

which because of its \( \pi_1 \)-invariance is \textit{single-valued}. Furthermore the quantity \( \mathcal{L} \) now transforms \textit{precisely} anti-invariantly under \( \Sigma_3 \), i.e. without any correction terms (just as the analogous single-valued cousin \( \text{Re} \log z \) of the logarithm has anti-invariant transformation behaviour under the duality group \( \mathbb{Z}_2 \) with non-trivial element \( \alpha : z \to \frac{1}{z} \)). So both deviations from the expected transformation properties are cured at the same time.

We will describe a number of ways to understand this anti-invariant transformation behaviour of \( \mathcal{L}(z) \). Most importantly for the interpretation via a string theory duality we propose in the outlook is the geometrical interpretation as the hyperbolic volume

\[
\text{vol} \triangle(z) = \mathcal{L}(z)
\]
of an ideal tetrahedron in hyperbolic three space \( \mathbb{H}_3 \) with vertices \( z_1, z_2, z_3, z_4 \) lying on the boundary \( \mathbb{P}^1_C \) of \( \mathbb{H}_3 \) which is manifestly independent of the numbering of the vertices except that the orientation changes under odd renumberings, showing the anti-invariant transformation behaviour (with \( z \) the cross ratio and \( \Sigma_4 \to \Sigma_3^{\text{SL}(2)} \) after the gauging \( (z_1, z_2, z_3, z_4) \to (0, 1, \infty, z) \)). Corresponding to this 3-volume interpretation one has a 1-volume interpretation (in the upper half-space model for \( \mathbb{H}_3 \))

\[
\text{length}(\gamma_z) = \text{Re} \log z
\]
Here $\gamma_z$ is the path on the $j$-axis from $j$ to $|z|j$. As described in the outlook it is natural to consider (1.7) and (1.8) together (cf. (6.33) and (6.34)).

It may be worth mentioning that by our description of the global relations on the quantum moduli space we get two simple reinterpretations of the membrane anomaly

$$\int_{D_1} C + \int_{D_2} C + \int_{D_3} C = \pi$$  \hspace{1cm} (1.9)

First, the non-linear $SL(2,\mathbb{Z})$ realisation (1.2) of $\Sigma_3$, which connects the three different quantities in question by a 'global' relation, makes (1.9) manifest (for $z$ being some $\eta_j$)

$$z \cdot \beta z \cdot \beta^2 z = -1$$  \hspace{1cm} (1.10)

Moreover (1.10) is already a consequence of having a global $\Sigma_3$ symmetry at all (cf. (3.2)). And secondly, in the dual hyperbolic model a relation corresponding to (1.9) (cf. (6.20)) becomes just the angle sum in an euclidean triangle (the two explanations are related, cf. (C.10))

$$\alpha + \beta + \gamma = \pi$$  \hspace{1cm} (1.11)

We also compare $W_{mem}$ to a superpotential induced by $G$-flux

$$W_G = \int_X (C + i\Upsilon) \wedge G$$  \hspace{1cm} (1.12)

A useful analogy is provided by the mass breaking of $N = 2$ $SU(2)$ gauge theory by the tree-level term $W_{tree} = m_tr\Phi^2$ whose quantum corrected version is $W_{qu} = mu$ occuring in Seiberg/Witten theory: it is given by a flux induced superpotential $W_H = \int_{\tilde{Z}} \Omega \wedge H_3$ on the Calabi-Yau $\tilde{Z}$ in type IIB (mirror dual to the type IIA Calabi-Yau $Z$ which describes the string embedding of the Seiberg/Witten theory) in the double scaling limit [5], [8] (crucial for this reinterpretation is that the field theory quantity $u$ occurs, in the appropriate limit, among the Calabi-Yau periods).

Further we extend the theory to the case of singularities of codimension four, describing four-dimensional non-abelian gauge theories in different phases. For some further relations to type IIA string theory and to five-dimensional Seiberg/Witten theory see [35].

For the speculative global interpretation developed in the outlook think of $\Delta(u)$, or its generalisation to a hyperbolic 3-manifold, as playing the role of the Seiberg/Witten elliptic curve $E_u$ over the $u$-plane\(^1\). The relevant non-perturbative quantity is in both

\(^1\)think of a different copy of $H_3$ over each point $u$ in $\mathbb{P}^1$ as ambient space for $\Delta(u)$ just as one has copies of the Weierstrass embedding plane $\mathbb{P}^2_{x,y,z}$ for $E_u$. 

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cases computed by a geometric period on the object varying over the moduli space. One might understand this as a computation of a $M$-theory superpotential dual to the original membrane instanton sum (analogous to the mentioned mass braking $W = mu$ in Seiberg/Witten, computed in the stringy embedding from a flux superpotential [5] where $u$ is a period of the type IIB mirror Calabi-Yau). If the Calabi-Yau’s are $K3$ fibered over a base $\mathbb{P}^1$ then the (single) Seiberg/Witten curve can be understood as being fibered over the same $\mathbb{P}^1$ (with discrete fibre a spectral set related to $H^2(K3)$). So in the general situation of $M$-theory on a $G_2$ holonomy manifold $X_7$, $K3$ fibered over $\mathbb{S}^3$, one may try to compare the quantum expression given by the membrane instanton sum to a period coming from a ($G$-flux ?) superpotential on a dual manifold $Y_7$ (a different $K3$ fibration over $B_3$, similar to some CY situations), respectively to a period in the ‘thinned out’ (spectral) version of $Y_7$ given by a hyperbolic 3-manifold $M_3$ (where the complexified Chern-Simons invariant is computed; this will be described in more detail elsewhere [41]).

The paper is organised as follows. In Section 2 we recall, following [2], the $G_2$ holonomy manifold $X_7 = \mathbb{S}^3 \times \mathbb{R}^4$, the quantum moduli space $\mathcal{N}$ and the membrane anomaly. In Section 3 we describe the crucial non-linear realization of the triality symmetry. In Section 4 we recall the treatment of the superpotential $W(t)$ by global arguments on $\mathcal{N}$ and then describe the local approach to the superpotential by summing up the membrane instantons and investigate its deviations from strict anti-invariance with respect to the triality symmetry. Using the study of the monodromy representation (describing the Heisenberg bundle and the superpotential as its section) we describe how one can, at the cost of introducing some non-holomorphy, replace the notion of section by a function $\mathcal{L}$. $\mathcal{L}$ is shown in four ways to transform anti-invariantly; one of them uses hyperbolic geometry by giving $\mathcal{L}$ an interpretation as a hyperbolic volume. We also compare with a flux-induced superpotential. In Section 5 we extend to the case of singularities of codimension four, describing four-dimensional non-abelian gauge theories in different phases. In the Outlook we compare the hyperbolic deformation moduli space with the Seiberg/Witten set-up and interprete all findings as describing a dual superpotential computation with the hyperbolic 3-simplex playing the role of the Seiberg/Witten curve. We indicate that the theory should extend to cover general $K3$ fibered compact $G_2$ manifolds (and global hyperbolic 3-manifolds). In the appendix we study the representation of $\Sigma_3$ and give two proofs of the anti-invariance of $\mathcal{L}$. Furthermore we give some background concerning the monodromy representation of $L\hat{i}$, the hyperbolic geometry (including the volume computation of an ideal tetrahedron), and the cohomological interpretation.
2 The $G_2$ manifold over $S^3 \times S^3$ and its moduli space

The three manifolds $X_i$ (cf. [2]) are cones over $Y = S^3 \times S^3 = SU(2)^3/SU(2)_D$ where $Y$ carries the (up to scaling) unique (Einstein) metric with $SU(2)^3$ (acting from the left) and $\Sigma_3$ symmetry $(da^2$ stands for $-Tr(a^{-1}da)^2$ where $a = g_2g_3^{-1}$, $b = g_3g_1^{-1}$, $c = g_1g_2^{-1}$)

$$d\Omega^2 = \frac{1}{36}(da^2 + db^2 + dc^2)$$

(2.1)

The images $D_i$ in $Y$ of the three $SU(2)$ factors fulfill the triality symmetric relation

$$D_1 + D_2 + D_3 = 0$$

(2.2)

(as three-cycles in homology) indicating that there are actually only two $S^3$'s. The singular manifold $X^{\text{sing}}$ is a cone over $Y$ ($r$ the radial coordinate)

$$ds^2 = dr^2 + r^2 d\Omega^2$$

(2.3)

When embedded in one of the $X_i \cong \mathbb{R}^4 \times S^3$ where the $i$'th $SU(2)$ is filled in to a $B_i = \mathbb{R}^4$ one has $D_i \simeq 0$.

The remaining three-sphere which sits at the center of $X_i$, corresponding to the value 0 in the $\mathbb{R}^4$ or $r = r_0$ in (2.4), is called $Q_i$. It is homologous to $\pm D_{i-1} \simeq \mp D_{i+1}$.

The $G_2$ manifold $X$ has a covariant constant three-form $\Upsilon$ (resp. four-form $*\Upsilon$). The modulus $\text{vol}(Q_i) \sim r_0^3$ is not dynamical but more like a coupling constant specified at infinity. The deformed manifold $X = \mathbb{R}^4 \times S^3$ (which near infinity is asymptotic to (and for $r_0 \to 0$ reduces to) the cone (2.3)) has the $G_2$ holonomy metric ($r \in [r_0, \infty)$)

$$ds^2 = \frac{dr^2}{1 - (\frac{r_0}{r})^3} + \frac{r^2}{36}(da^2 + db^2 + dc^2 - \frac{r_0}{r})^3(da^2 - \frac{1}{2}db^2 + dc^2)$$

(2.4)

Let us examine the metric perturbations which preserve $G_2$ holonomy. Using a new radial coordinate $y$ with $\frac{dy^2}{1 - (\frac{r_0}{r})^3} = dy^2$, provided at large $r$, to the accuracy needed, by

$$y = r \left(1 - \frac{1}{4}(\frac{r_0}{r})^3 + \mathcal{O}\left(\frac{r_0}{r}\right)^6\right)$$

(2.5)

one gets for the metric (with $(f_1, f_2, f_3) = (1, -2, 1)$ and up to terms $y^2\mathcal{O}\left(\frac{r_0}{y}\right)^6$)

$$ds^2 = dy^2 + \frac{y^2}{36}(da^2 + db^2 + dc^2 - \frac{1}{2}(\frac{r_0}{y})^3(f_1 da^2 + f_2 db^2 + f_3 dc^2))$$

(2.6)

At small $r_0$ or large $y$ one finds the conical metric with the full $\Sigma_3$ symmetry; the first correction in the expansion of powers of $r_0/y$ (at third order) is parametrized by the $f_i$. So
for \((f_i)\) a positive multiple of \((1, -2, 1)\) or its cyclic permutations or linear combinations\(^2\) of them one gets \(G_2\) holonomy, i.e. for (the negative \(f_i\) indicates which \(D_i\) is filled in)

\[
\sum f_i = 0 \tag{2.7}
\]

One has for the volume of \(Q_i\) and the \(y\)-dependent volume of \(D_i\) embedded in \(X_i\) (at large \(y\); with \(\text{vol} \ D_i\) given up to higher order terms in \((r_0/y)^3\))

\[
\text{vol} \ Q_i = 2\pi^2 r_o^3 \tag{2.8}
\]

\[
\text{vol} \ D_i = \frac{2\pi^2}{27} y^3 \left(1 + \frac{3}{8} f_i \left(\frac{r_o}{y}\right)^3 + \mathcal{O}\left(\frac{r_0}{y}\right)^6\right) \tag{2.9}
\]

\[
\approx \frac{2\pi^2}{27} y^3 + \frac{1}{72} f_i \text{vol} Q_i \tag{2.10}
\]

Here, the first correction to the divergent piece is the finite volume defect \(\frac{1}{72} f_i \text{vol}(S^3_{r_0})\).

Note that semiclassically the volume defects are \(\rho (1, -2, 1)\) or a permutation (with \(\rho \to \infty\); one could choose \(\rho = r_0^3\) or absorb\(^3\) \(\rho\) giving \((f_i) \sim (1, -2, 1))\).

Quantum moduli space and observables

In the quantum domain there is actually [1], [2] a smooth curve \(N = \mathbb{P}_C^1\) (when compactified) of theories interpolating between the three classical limits (large \(r_0\)) given by the \(X_i\) (given by three points \(P_i\) of \(t = \omega^{i+1}\) with \(t\) the global coordinate, \(\omega = e^{2\pi i/3}\)).

A holomorphic observable on \(N\) must combine as SUSY partners the \(C\)-field period\(^4\)

\[
\alpha_i = \int_{D_i} C \tag{2.11}
\]

with an order \(1/r^3\) metric perturbation (w.r.t. the conical metric), as in

\[
y_i = \exp \left( k f_i + i(\alpha_{i+1} - \alpha_{i-1}) \right) \tag{2.12}
\]

(with \(\prod y_i = 1\) by (2.7)). Actually one works with the quantity

\[
\eta_i = \exp \left( \frac{k}{3} (f_{i-1} - f_{i+1}) + i\alpha_i \right) \tag{2.13}
\]

\(^2\)because regarding only the lowest order term amounts to linearization of the theory; this refers thus to the situation at infinity; note that classical reasoning at infinity would expect the \(f_i\) nethertheless to be a positive multiple of a cyclic permutation of \((1, -2, 1)\) to fulfill the non-linear Einstein equations in the interior; together with the other classical expectation \(\int_{D_i} C = 0\) (for \(D_i\) filled in) this would fix \(\eta_i\) to its classical value 1 at \(P_i\); but this behaviour is modified by quantum corrections [2]

\(^3\)In a measurement at infinity the parameter \(r_0\) will not be known; one refers [2] also to \(f_i\) as the volume defect, when stating that the \(f_i\) go to \(\infty\) (in ratio \((1, -2, 1)\) or a permutation) for \(r_0 \to \infty\).

\(^4\)which at large radial coordinate \(r\) is independent of \(r\) for a \(C\)-field flat near infinity (to keep the energy finite), entailing that the components of \(C\) are of order \(1/r^3\)
so \( y_i = (y_{i-1}^2 y_i)^{1/3} \), \( y_i = \eta_{i+1}/\eta_{i-1} \), cf. (A.11)). On the other hand one identifies \( \eta_i \) by global considerations on the genus zero moduli space \( \mathcal{N} \) as the following rational function

\[
\eta_i = \left( \frac{t - \omega^i}{t - \omega^{i-1}} \right)
\]

We have also the local coordinate \( u_i \) at \( P_i \) given by the membrane instanton amplitudes

\[
\begin{align*}
  u_i &= \exp\left(-T \text{vol}(Q_i) + i \int_{Q_i} C\right), \\
  \eta_i &= \exp\left(k \frac{f_i + 2 f_{i-1}}{3} + i \int_{D_i} C\right)
\end{align*}
\]

The local parameter \( u_i \) vanishes at \( P_i \) due to the large volume of the manifold \( X_i \). We denote by \( \Phi_i \) the physical modulus to which it is related via \( u_j = \exp(\omega_j) = e^{i \Phi_j} \), i.e. (where we have set \( T = 1 \); \( Q_i \) is an (isolated) supersymmetric cycle so \( \Upsilon|_{Q_i} \) is the volume form)

\[
\Phi_j = \int_{Q_j} C + i \Upsilon = \phi_j + i \text{vol}(Q_j)
\]

The membrane anomaly

To be well-defined the phase of the \( \eta_i \) variable must be modified \[2\] to

\[
e^{i \alpha_i} = \text{sign Pf}(D) e^{i \int_{D_i} C}\]

where \( D \) is the Dirac operator on \( S^3 \) with values in the positive spinor bundle of the normal bundle and Pf denotes its Pfaffian (square root of the determinant) which occurs in the fermion path integral and must be combined with the classical phase factor \( e^{i \int_{D_i} C} \) in the worldvolume path integral for a membrane wrapping \( S^3 \). Now for a three-manifold \( X_3 \) which is the boundary of a (spin) four-manifold \( B \) one has \[2\] (with \( D_B \) the \( S(N_B) \) valued Dirac operator on \( B \) with Atiyah-Patodi-Singer boundary conditions along \( X_3 \))

\[
\text{sign Pf}(D) =: e^{i \pi \mu(S^3)} = e^{i \pi \text{ind}(D_B)/2} = e^{i \pi w_3(N_B)} = e^{i \pi \chi(B)}
\]

If \( B \) could be chosen to be smooth (as for a single \( D_i \)) the correction would be ineffective, but for the union (relevant for \( \sum_i \alpha_i = \pi \)) of the intersecting \( D_i \) this cannot be the case. Now one gets the result \( \sum_i \alpha_i = \pi \) either from a union of \( B_i \)’s respectively bounded by the \( D_i \) or more directly from slightly perturbing the \( D_i \) as follows. For this let us recall that \( P_H^2 = (H^3 \setminus \{0\})/H(x) = S^{11}/S^{3} \) has \[2\] \( Y = S^{3} \times S^{3} = SU(2)^3/SU(2)_D \) fibres over a triangle \( \Delta = \{[\lambda, \mu, \nu]|\lambda, \mu, \nu \geq 0\} \subset P_H^2 \) from the quaternionic norm \( P_H^2 \xrightarrow{p} \Delta = \)
\[ P^2_\mathbb{R} / Z^2 \] (where \( Z^2 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2^\mu \times \mathbb{Z}_2^\nu) / Z^2_\text{diag} \)). For a line \( B = S^4 = P^1_H \subset P^2_\mathbb{R} \) and \( B = B - \cup_{i=1}^3 D^4_{\text{open}}(p_i) \) for \( p_i = B \cap L_i (L_i \subset P^2_\mathbb{H}) \) the coordinate lines, \( D^4_{\text{closed}}(p_i) \subset B \) small 4-discs around \( p_i \) of resp. boundary \( S^3 \sim D_i \) one has

\[ \partial B \simeq D_1 + D_2 + D_3 \]  \hspace{1cm} (2.19)

and finds 1 for the self-intersection number (the Euler class of the normal bundle, the mod 2 relevant number). We will refer then to the membrane anomaly as

\[ \sum_i \alpha_i = \pi \pmod{2\pi} \quad \text{or} \quad \prod_i \eta_i = -1 \]  \hspace{1cm} (2.20)

The conifold transition in type IIA

In a type IIA reinterpretation (cf. [2]) one divides by the circle \( S^1_{\text{11}} = U(1) \subset SU(2)_1 \) giving for \( X_1 = \mathbb{R}^4 \times S^3 = (SU(2)_1 \times \mathbb{R}^{\geq 0}) \times S^3 \) the type IIA manifold \((S^2 \times \mathbb{R}^{\geq 0}) \times S^3 = \mathbb{R}^3 \times S^3\) with fixed point at the origin, i.e. the deformed conifold \( T^*S^3 \) with a D6-brane wrapping the zero-section. For \( X_2 \) or \( X_3 \) one gets \( \mathbb{R}^4 \times S^3/U(1) = \mathbb{R}^4 \times S^2 \), the two small resolutions of the conifold together with a unit of RR two-form flux on \( S^2 \) (as \( S^3 \) is Hopf fibered by \( S^1_{\text{11}} \) over \( S^2 \)). One may compare with the special Lagrangian deformations [37] of the cone over \( T^2 \) with different \( S^1 \)'s killed in homology; the fixed point set under the \( U(1)_D \) is \( L = S^1 \times \mathbb{R}^2 \subset X = S^3 \times \mathbb{R}^4 \) (from \( C \subset H \)), the \( S^1 \) being the boundary (where the fibre shrinks) of the disc \( D^2 = S^3/U(1) \).

The deviation of the metric from the conical form being of order \((r_0/r)^3\) for large \( r \) (so not square-integrable in seven dimensions), \( r_0 \) is not free to fluctuate (the kinetic energy of the fluctuation would be divergent). So in the four-dimensional low energy theory it is rather a coupling constant than a modulus. To have a normalizable (or at least log normalizable) mode, one of the circles at infinity should approach a constant size (which can happen in many ways related to the Chern-Simons framing ambiguity [15])\(^5\) which is not the case for the \( Z_3 \) symmetric point discussed in [2] and here. In this sense the expression ‘superpotential’ has to be qualified; one gets the actual superpotential for an ordinary modulus if the local geometry \( X_7 \) is embedded in a compact \( G_2 \) manifold (cf. sect. 6.2). The metric (2.4) describes an \( M \)-theory lift of a type IIA model with the string coupling infinite far form the \( D6 \) brane; to have at infinity an \( M \)-theory circle of finite radius one of the three \( SU(2) \) symmetries of (2.4) must be broken to \( U(1) \) [4].

\(^5\)On \( H_3(Y) = Z \oplus Z \) one has a full \( \text{SL}(2, \mathbb{Z}) \) operating but for the 'filled in' versions only the three spaces \( X_i \) are allowed as the closed and co-closed three-form \( \Upsilon \) of class \((p, q) \in Z \oplus Z \) corresponds to a regular metric just for the three cases \((p, q) = (0, 1), (-1, 0) \) or \((1, -1) \) (where the unbroken \( Z_2 \subset \Sigma_3 \) exchanges the \( S^3 \) factors) [16]. But for the \( S^1 \times S^3 = T^2 \) (relating to \( L \) as \( Y \) to \( X \)) not only the \( \Sigma_3 \) but the full \( \text{SL}(2, \mathbb{Z}) \) is allowed which expresses the framing ambiguity [15]. Cf. also the case \( S^5 \times S^5 \) in [17].
3 The non-linear symmetry action

We will consider two actions of $\Sigma_3$ on $\mathbf{P}_1^1$. In the first (‘linear’) action the $\mathbb{Z}_3$ sector acted by multiplication with an element of $\mathbb{C}^*$; in the second case this sector will act non-linearly (the operation of $\alpha$ will be given in both cases by $z \to 1/z$).

The linear action of $\Sigma_3$

Here the ‘rotation’ subgroup $\mathbb{Z}_3$ is generated by the action $t \to \omega t$ on $\mathcal{N} = \mathbb{P}_t^1$ and $\alpha$ acts by $t \to 1/t$, giving as images of $t$ under $\Sigma_3 \left( \begin{array}{ccc} t & \omega t & \omega^2 t \\ t^{-1} & \omega^2 t^{-1} & \omega t^{-1} \end{array} \right)$ (cf. (A.2)). The involution $\iota : t \to -t$ is an automorphism of $(\mathbb{P}_t^1, \Sigma_3)$, i.e. $\Sigma_3$ compatible: $\iota \gamma = \gamma \iota$.

Degenerate orbits in the $t$-plane

We treat the question of fixpoints or degenerate orbits. The structure of $\Sigma_3$ leads one to look for two-element and three-element orbits. The two-element orbit, whose elements are then fixed respectively by the $\mathbb{Z}_3$ cosets, is $\left( \begin{array}{ccc} 0 & 0 & 0 \\ \infty & \infty & \infty \end{array} \right)$. In the other case the full $\Sigma_3$ orbit is already covered by the $\mathbb{Z}_3$ orbit; the transforms under the remaining (order two) elements from the non-trivial $\mathbb{Z}_3$ coset will then just repeat the $\mathbb{Z}_3$ orbit in some order. This gives the two possibilities $\left( \begin{array}{ccc} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{array} \right)$ and $\left( \begin{array}{ccc} -1 & -\omega & -\omega^2 \\ -1 & -\omega^2 & -\omega \end{array} \right)$. $\iota$ exchanges these two orbits and fixes the elements of the two-element orbit.

The non-linear action of $\Sigma_3$

Note that the action induced on the $\eta_i$ is as follows. The cyclic permutation of the points $P_i$ ($i = 1, 2, 3$), which is described by the rotation transformation $t \to \omega t$, produces the corresponding cyclic permutation $\eta_1 \to \eta_3 \to \eta_2 \to \eta_1$ on the $\eta_i$, as seen from (2.14). Furthermore the inversion induces $\eta_1 \to \eta_3^{-1}$, $\eta_2 \to \eta_2^{-1}$. The $\eta_i$ which fulfill the relation $\eta_1 \eta_2 \eta_3 = -1$ (reflecting the membrane anomaly) are actually related by

$$\eta_3 = \frac{1}{1 - \eta_1} \quad , \quad \eta_1 = \frac{1}{1 - \eta_2} \quad , \quad \eta_2 = \frac{1}{1 - \eta_3} \quad (3.1)$$

So consider now instead of the linear action of $\mathbb{Z}_3$ the non-linear action of it resp. of the full symmetry group $\Sigma_3$ as $\text{Sl}(2, \mathbb{Z})/\Gamma(2)$. As is well known from the theory of the Legendre $\lambda$ function, the elements are given then as the fractional linear transformations displayed below. $\Sigma_3$ occurs not only as a quotient but also as a subgroup. For this recall that the holomorphic automorphism group of $\mathbb{P}^1$ is given by $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C}) = \text{GL}(2, \mathbb{C})/\mathbb{C}^* = \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1_2\}$ and that for two triples of points of $\mathbb{P}^1$ there exists an automorphism mapping these two sets of elements onto each other. In
particular we will consider the elements permuting the set \( \{0, 1, \infty\} \) which are then given by transformations \( z, \beta z, \beta^2 z \) and \( \alpha z, \alpha \beta z, \alpha \beta^2 z \) understood as mappings \( \mathbb{P}^1 \to \mathbb{P}^1 \), i.e. \( \text{Aut}(\mathbb{P}^1, \{0, 1, \infty\}) = \Sigma_3 \). This leads just formally to the relation (which restates (2.20))

\[
\prod_{i \in \mathbb{Z}_3} \beta^i z = -1 \tag{3.2}
\]

That this product is a constant follows already from the divisor relations\(^7\) (note that the \( \beta^i \) are permutations on the set \( \{0, 1, \infty\} \)): \( (e z) = 0 - \infty \), \( (\beta z) = \infty - 1 \), \( (\beta^2 z) = 1 - 0 \) imply that their product is a nowhere vanishing globally holomorphic function, so a constant \( x \neq 0 \). Now, in the \( \mathbb{Z}_2 \) sector given by \( \{e, \alpha\} \), the transformation \( \alpha \), mapping \( 0 \leftrightarrow \infty \) and 1 to itself, will operate as multiplicative inversion (i.e. \( z \cdot \alpha z = 1 \)). So \( x^2 = \prod_{\gamma \in \Sigma_3} \gamma z = 1 \) as \( \prod_{i \in \mathbb{Z}_3} \beta^i z = x = \prod_{i \in \mathbb{Z}_3} \beta^i \alpha z = \prod_{i \in \mathbb{Z}_3} \alpha \beta^i z \). Then \( x = (\alpha z) \cdot \beta(\alpha z) \cdot \beta^2(\alpha z) = z \cdot \alpha \beta^2 z \cdot \beta^2 z \) for an \( \alpha \)-fixpoint (so \( z = +1 \) or \( -1 \)) shows that \( x = -1 \).

One finds for the concrete functional form of the transformations

\[
\begin{align*}
z & \quad \beta z = \frac{1}{1-z} \\
\alpha z & = \frac{1}{z} \\
\alpha \beta z & = 1 - z \\
\alpha \beta^2 z & = \frac{z}{z-1}
\end{align*} \tag{3.3}
\]

Degenerate orbits in the \( \eta \)-plane

Let us consider again the question of degenerate orbits (now under the non-linear \( \text{Sl}(2) \) action; the cases will correspond to the descriptions in the \( t \)-plane under (2.14)). Concerning first the two-element orbits note that their elements are \( \mathbb{Z}_3 \) fixpoints. Therefore the condition (2.20) reads in this case \( \eta^3 = -1 \), so \( \eta = -\omega \) or \( -\omega^2 \) (the solution \( -1 \) leads to another case, cf. below). So up to permutation in the still running factor \( \mathbb{Z}_2 = \{e, \alpha\} \) one finds the case \( \begin{pmatrix} -\omega & -\omega & -\omega \\ -\omega^2 & -\omega^2 & -\omega^2 \end{pmatrix} \). To develop the three-element orbits note that the three possibilities to repeat a value \( \eta = e(\eta) \) from the first line in the second line (which will then, as a set, repeat the first line) lead to the cases \( \eta = 1/\eta \) so \( \eta = \pm 1 \), or \( \eta = 1 - \eta \) so \( \eta = 1/2 \) or \( \infty \), or finally \( \eta = \eta/(\eta - 1) \) so \( \eta = 0 \) or \( \eta = 2 \). If one looks for the corresponding orbits one finds that up to cyclic permutations (in the \( P_t \) with which one starts) one has just the two cases \( \begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} -1 & 1/2 & 2 \\ -1 & 2 & 1/2 \end{pmatrix} \), \( \iota \) exchanges these two orbits and leaves fixed the two elements of the two-element-orbit.\(^8\)

\(^6\)This product of the meromorphic functions \( \beta^i z \) has nothing to do with the group multiplication given by composing mappings which leads to \( \Sigma_3^{\text{Sl}(2)} \). The point of the argument is to fix the sign-prefactors relevant for (3.2); to write down just transformations with the prescribed divisors is of course easy.

\(^7\)so for example \((e z) = 0 - \infty\) indicates that the function \( z \to z \) has a simple zero/pole at 0/\( \infty \).

\(^8\)The \( \pm 1 \) occurring here are \( \alpha \)-fixpoints just as the elements of the two-element-orbit were \( \mathbb{Z}_3 \)-fixpoints.

\(^9\)\( \mathbb{P}_\eta^1 \) has (by transport from the \( \mathbb{P}_t^1 \)) the involution \( \iota : \eta \to \frac{\eta-2}{2\eta-1} \), which is \( \Sigma_3^{\text{Sl}(2)} \) compatible: \( \iota \gamma = \gamma \iota \).
Relations of the variables

Under the non-linear action (3.3) of the triality group $\Sigma_3$ the $\eta_i$ form a $\mathbb{Z}_3$ orbit

$$\eta_{i-1} = \beta \eta_i \quad \eta_{i+1} = \beta^2 \eta_i$$

and the membrane anomaly (2.20) becomes manifest (for $z$ some $\eta_j$)

$$\prod_{i \in \mathbb{Z}_3} \beta^i z = -1$$

As shown after (3.2) this is already a consequence of having a global $\Sigma_3$ symmetry at all.

Recall that in a semiclassical regime with $D_i = 0$ one has

$$Q_i \simeq \pm D_{i-1} = \mp D_{i+1}$$

From the classical fact (3.6) one has $\mp \int_{D_i} C = \int_{Q_{i+1}} C$, what is (with holomorphy) tantamount to saying that $\eta_{i+1} \sim u_i$ to first order (as reflected in the first order zero of $\eta_{i+1}$ at $P_i$ where $u_i$ vanishes to first order; the respective lower sign in (3.6) is fixed at $P_i$). We assume that such a relation persists\(^{10}\) (cf. footn.'s 11 and 13) so that one has a relation $\beta u_i = \eta_i$: the holomorphic relation between the $u_i$ and the $\eta_i$ is fixed (up to a real factor) by the relation between their arguments (imaginary parts of logarithms), so (2.15), (3.6) imply $u_i = \eta_{i+1} = \beta^2 \eta_i$, i.e.

$$\beta u_i = \eta_i$$

Note that the $u_i$ are, in contrast to the $\eta_i$, actually only first order parameters; therefore the assertion made, where we think of them as globally analytically continued, has to be suitably interpreted\(^{11}\). But note that in [14] indeed flat coordinates $-u_i$ are described (the $u, v$ there, being shifted by $\pi i$, being the $w_i = \log u_i$ here).

(Concerning an ensuing relation (3.5) then also for the $u_i$ note the deviation from the relation $q \cdot q^\omega \cdot q^{\omega^2} = +1$ which one would have in classical variables for the linear action on $q = e^s$ (cf. also footn. 18) from $s \rightarrow \omega s$ (and which would be the analogue of $q \cdot q^{-1} = 1$ in the $\mathbb{Z}_2$ case of the usual IIA flop).\(^{12}\)

\(^{10}\)Thereby a relation $\eta_{i-1} = \frac{1}{1-\eta_i}$ holds also for the $u_i$: $u_3 = \beta u_1 = \frac{1}{1-u_1}$; note the corresponding map $e^t - 1 \sim \frac{1}{e^t}$ in [1] (in the case $N = 1$ there)

\(^{11}\)In the approach choosen the relation could practically be taken as a definition of $u_i^{(glob)}$. In principle it would be possible to rephrase the whole discussion on superpotentials to follow w.r.t. the $\eta_i$ instead of $u_i^{(glob)}$ but it would be much less intuitive.

\(^{12}\)This $s$ and the mentioned classical linear action is not to be confused with the $t$ of the true global quantum moduli space and its (quantum) linear action of the same form (which corresponds under $t \leftrightarrow \eta$ to the quantum non-linear action).
We recall below that the crucial argument in [2] for a one-component quantum moduli space was the fact that the classical statement $\alpha_i = \int_{D_i} C = \text{Im} \log \eta_i = 0$ is modified by membrane instantons; such a membrane instanton contribution will be present as soon as $u_i \neq 0$, i.e. away from the infinite volume limit for $Q_i$ which would suppress the latter contribution. Similarly the classical expectation of finding the $f_i$ in (a permutation of) the ratio $(1, -2, 1)$ and therefore having $\text{Re} \log \eta_i = 0$ for one of the $\eta_i$ gets modified (cf. footn. 2). So one expects that actually $u_i \neq 0 \implies \log \eta_i \neq 0$ as is the case in (3.7).

The critical circle and special points

The ‘critical’ circle $|u_i| = 1$ corresponds classically to the case $\text{vol}(Q_i) = 0$ of a shrinking $Q_i$. Let us study the parameter $u_i$ on the critical circle. One has ($\phi \in [0, 2\pi]$)

$$\log(\beta e^{i\phi}) = -\log(2 \sin \frac{\phi}{2}) + i(\pi - \phi)/2$$

(3.8)

So here (3.7) means for $\phi_i = \int_{Q_i} C$ and $\alpha_i = \int_{D_i} C$ just $\alpha_i = (\pi - \phi_i)/2$ or (cf. (2.20))

$$\int_Q C + 2 \int_D C = \pi \pmod{2\pi}$$

(3.9)

Note that by (3.8) the $\beta$-transform of a parameter $z = re^{i\phi} = : e^{f_i + i\phi}$ on the critical circle $f = 0$ stays there in the cases ($z$ being then a $\beta$-fixpoint by (3.9))

$$f = 0 \quad \phi = \pm \pi/3$$

(3.10)

There are some special points in moduli space which are interesting to consider. Let us consider first the phases where the $S^3$ given by $Q_i$ has (seen classically) still physical (non-negative) volume, i.e. the domain $|u_i| \leq 1$ or $f_i \leq 0$ in parameter space$^{13}$: here $|u_i| < 1$ (i.e.$^{13}$ $\text{vol}(Q_i) > 0$) or $f_i < 0$ is the region where $D_i$ is filled in (i.e. $Q_i$ is not shrunken where $D_i$ is shrinkable). So at the (classical) intersection of all three phases, where all the three-spheres have to be shrinkable at the same time, one finds $f_i = 0$, i.e. the $Z_3$ orbit given by the $\eta_i$ triple lies on the unit circle in the $\eta$-plane. This leads via (3.10) to the two possibilities$^{14}$

$$f_i = 0 \ , \ \alpha_i = \pm \pi/3$$

(3.11)

$^{13}$The geometrical interpretation applies strictly only semiclassically, in general the arguments apply just to the global variables. It is nevertheless instructive to keep this interpretation in mind.

$^{14}$ $\eta = -\omega$ and $\eta = -\omega^2$, identified above: the case where the $\eta_i$ constitute a completely degenerate $Z_3$ orbit ($\eta_i = \beta \eta_i = \beta^2 \eta_i$), i.e. the full $\Sigma_3$ orbit degenerates to a two-element orbit.
4 The superpotential: the local approach

The global superpotential

Let us recall first the global approach to the superpotential [2]. One expects the superpotential $W$ to vanish at the $P_i$. If there are no further zeroes then $W$ will have exactly three poles on the genus zero moduli space $\mathcal{N}$. Both of these three element sets will have to be complete $\Sigma_3$ orbits by themselves. This leads to the two degenerate three-element orbits $\omega_i$ and $-\omega_i$. So the minimal solution is [2]

$$W \sim \frac{t^3 - 1}{t^3 + 1}$$ (4.1)

Note that (under the linear action) it transforms with the sign character

$$W(\gamma t) = \text{sign}(\gamma)W(t)$$ (4.2)

i.e. $W(\omega t) = W(t)$, $W(t^{-1}) = -W(t)$ (it transforms ‘anti-invariantly’).

Concerning the zeroes and poles of the global superpotential $W(t)$ for the $\mathbb{Z}_3^{\text{lin}}$ orbits of 1 and $-1$ (in the $t$-plane), respectively, note that one has corresponding $\mathbb{Z}_3^{\text{Skl}}$ orbits of $\pm 1$ (now in the $\eta$-plane!) for the $\eta_i$, and so for the $u_i$: $u_i = +1$ or $-1$, i.e.\(^{16}\) $\text{vol}(Q_i) = 0$ with $\int_{Q_i} C = 0$ or $\int_{Q_i} C = \pi$, so the cases are\(^{17}\): first $u_i = 0$, i.e. $\text{vol}(Q_i) = \infty$ and $\int_{Q_i} C = 0 \implies$ zero for $W_t$; and, secondly, $u_i = -1$, i.e. $\text{vol}(Q_i) = 0$ and $\int_{Q_i} C = \pi \implies$ pole for $W_t$ (this case may be compared with the case of a vanishing $S^2$ with $\int_{S^2} B = \pi$).

The local superpotential

The actual superpotential arises from the sum of all the multi-cover membrane instantons

$$W(u_i) = \sum_{a=1}^{\infty} a_n u_i^n$$ (4.3)

How do the non-perturbative contributions from the local semiclassical informations near the $P_i$ fit together over the whole quantum moduli space $\mathcal{N}$? Is $W(u_i^{(\text{glob})}(t)) = W(u_{i-1}^{(\text{glob})}(t))$, i.e. $W(u) = W(\beta u)$: is $W$ (at least $\mathbb{Z}_3$) triality symmetric?

First note that the membrane instantons make the deviation from the classical result $\alpha_i = 0$ possible ([2] and recalled below), just as sums of world-sheet instanton contributions in the case of the type IIA string on a Calabi-Yau manifold give quantum corrections

\(^{15}\)In $W(t) = \prod_{i \in \mathbb{Z}_3} \frac{t - \omega_i}{t + 1}$ an underlying anti-invariant projection (A.4) is made manifest by using the relation

$$W(t) \sim \prod_{i \in \mathbb{Z}_3} \frac{t - \omega_i}{t + 1} = \frac{1}{3} \sum_{i \in \mathbb{Z}_3} \frac{t - \omega_i}{t + 1} \omega_i, \text{ giving } W(t) \sim \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma) \frac{1}{t + 1} |\gamma t|.$$

\(^{16}\)but note here the issue of $u_i^{(\text{glob})}$ vs. $u_i^{(\text{loc})}$, cf. footn. 13 and remark after (3.7)

\(^{17}\)Note that in case the independent variables $\gamma_i$ relevant for $W$ build a $\mathbb{Z}_3$ orbit (like is the case for the $\eta_i$) the first case, $\gamma_i = 0$ or $\gamma_{i-1} = \beta \gamma_i = 1$, differs only in the $C$ field period, from the second one.
to a classical (complexified) Kähler volume. In the IIA case of \( N = 2 \) supersymmetry one can answer the analogue of our question above by considering the resummation of the geometric instanton series \( I_{ws}(q) = \sum_{n \geq 1} \frac{q^n}{n} = \frac{q}{1-q} \) for a flop [7] where the Kähler parameter \( t = \int_{\mathbb{P}^1} B + i \text{area}(\mathbb{P}^1) \) in \( q = e^{2\pi it} \) is reflected as \( t \to -t \):

\[
I_{ws}\left(\frac{1}{q}\right) = -I_{ws}(q) - 1 \tag{4.4}
\]

The deviation from anti-invariance stems from change in classical intersection numbers.

Now let us ask what the corresponding ‘reflection’ is on the modulus \( \Phi_j = \int_{Q_j} C + i \Upsilon = \phi_j + i \text{vol}(Q_j) \) in (2.16) under which we should look for reasonable transformation behaviour of the quantum corrections (reasonable meaning a way of transformation so that the three contributions fit together in a triality symmetric way over \( \mathcal{N} \)).

In the case of \( M\)-theory on our \( G_2 \) holonomy manifold the question of flopping an \( S^3 \), instead of an \( S^2 \) in type IIA on a Calabi-Yau manifold, is more complicated as the Kähler moduli no longer fit together naturally at the classical level [2]. Rather the metric moduli \( \text{vol}(S^3_i) \) of the \( X_i \), running classically over a half-line \([0, \infty)\), are at angles \( 2\pi/3 \) to one another (in a copy of \( \mathbb{R}^2 \) containing the root lattice \( \Lambda \) of \( SU(3) \)); the \( C \)-field periods measured at infinity on the different \( X_i \) take values in different subgroups \( E_i \cong H^3(X_i, U(1)) \cong U(1) \) of \( H^3(Y, U(1)) \cong U(1) \times U(1) \) (when restricted to \( Y \)).

So in view of the problems with the three rays in \( \mathbb{R}^2 \) we will not consider the transformation\(^{18} \) given by rotation with \( 2\pi/3 \) around the origin in this \( \mathbb{R}^2 \), i.e. multiplication of the modulus with \( \omega \). Rather one should now consider instead of this linear action the non-linear action (cf. (3.3)) of \( \Sigma_3 \) represented as \( Sl(2, \mathbb{Z})/\Gamma(2) \).

**The one-membrane instanton contribution**

Actually there is a one-component moduli space comprising all the \( P_i \) \([2]\) as quantum effects given by membrane instantons cause a deviation from the classical result \( \alpha_i = 0 \). For this recall that to convert the interaction given by \( u \), which is like a superpotential, to an ordinary interaction one has to integrate over the fermionic collective coordinates of the membrane instanton, i.e. to evaluate the chiral superspace integral \( \int d^2\theta u \) (there is also an integration \( \int d^4y \) over the membrane position in \( \mathbb{R}^4 \) to be made). As the fermion integral has the properties of a derivation with respect to \( w \) one gets\(^{19} \) \((u = e^w; T = 1)\)

\[
\int d^2\theta u = u \int d^2\theta w = -2u \int d^2\theta \int_Q \Upsilon \tag{4.5}
\]

\(^{18}\)generating \( \mathbb{Z}_3 \), or \( \Sigma_3 \) when combined with complex conjugation on \( \mathbb{R}^2 \cong \mathbb{C} \) (or with inversion)

\(^{19}\)the contribution to \( \int_D C \) of a second term \( 2u \int d^4w \int d^2w \) occurring here is subleading for large \( r \).
For this one gets the evaluation \((w = i\Phi, \text{ cf. (2.16)})\)

\[
\int d^2\theta w \sim \int_{Q_i} *G \tag{4.6}
\]

In a second step one finds that the contribution \(\int_{\mathbb{R}^4 \times Q_i} *G\) to the effective action induces a non-zero value of \(\alpha_i = \int_{D_i} C\) as one has \(^{20}\)

\[
< \int_{\mathbb{R}^4 \times Q_i} *_{11} G \cdot \int_{D_i} C > \neq 0 \tag{4.7}
\]

because the 'linking number' of the two three-spheres \(Q_i\) and \(D_i\) is one (effectively given by the intersection number of \(Q_i\) with \(B_i\)).

**Derivation of the dilogarithm superpotential**

The evaluation (4.7) occurred in the transition from the superpotential \(u\) to an ordinary interaction \(\int d^4yd^2\theta u\): enhancing this argument we want to argue that this determines already the complete scalar potential (also suggested by the form of a \(G\) flux induced superpotential, cf. subsect. 4.5), so that one gets thereby the *full* membrane instanton amplitude including all the higher wrappings, i.e. the full quantum corrections.

Note first that more generally than in (4.5) the derivative nature (w.r.t. \(w = \log u\), cf.[2] p.60) of the fermion integral gives

\[
\int d^2\theta W(u) \approx \frac{dW}{d\log u} \int d^2\theta w \tag{4.8}
\]

Now to determine the actual sum \(W(u)\) of the multi-cover membrane instantons we interpret (4.7) as representing actually a relation (with \(T = 1\)) between the full \(\int d^2\theta W(u)\) (cf. footn. 48) and \(\int_{D} C \cdot \int_{Q} *_{7} G = \int_{D} C \cdot \int d^2\theta w\)

\[
\int d^2\theta W(u) \sim i \int_{D} C \cdot \int d^2\theta w \tag{4.9}
\]

For this note that classically one has \(\int_{D} C = 0\) and of course also \(\int d^2\theta W = 0\) as \(W = 0\). A shift \(\Delta \int_{D} C \neq 0\) away from the classical vanishing value was argued [2] to occur via a contribution \(\int d^2\theta W \neq 0\) from the membrane instantons. Here we argue that actually the relation between \(\Delta \int_{D} C\) and \(\Delta \int d^2\theta W\) should be used to show that \(\int d^2\theta W = \frac{dW}{d\log u} \int d^2\theta w = \frac{dW}{d\log u} \int_{Q} *_{7} G = \frac{dW}{d\log u}\) is (proportional to) \(\int_{D} C = \text{Im} \log \eta\) as in (4.9). This leads with (3.7) to the differential equation\(^{21}\) for \(W\) (cf. (4.60))

\[
\frac{dW}{d\log u} = i \int_{D} C = i \text{Im} \log \eta \frac{\partial}{\partial \zeta} \zeta = \log \eta = \log \beta u \tag{4.10}
\]

\(^{20}\) with a classical \(G\)-field generated by a source \(\int_{D} C\) as a means to evaluate (4.7)

\(^{21}\)here we made a holomorphic completion on the rhs which the holomorphic lhs suggests
The two crucial inputs to get this were the two interpretations (3.7) and (4.9).

With the differential equation (4.10) we get for the full superpotential (cf. [13], [4])
\[ W(u_i) = -\int_0^{u_i} \frac{dt}{t} \log(1 - t) = \sum_{n=1}^{\infty} \frac{u_i^n}{n^2} = Li_2(u_i) \] (4.11)
(cf. (B.1) for the polylogarithm). Note that the integral representation (4.11) defines
\( W \) on the complex plane, cut along the part \((1, \infty)\) of the positive real axis. In the
series representation for large volume \( \text{vol}(Q_i) \approx \infty \) the instanton contributions vanish,
i.e. \( W(0) = 0 \). The function of the modulus \( \Phi = -iw = -i \log u \) has, by (4.10), a critical
point exactly at \( u = 0 \), the large volume point \( P_i \) (cf. (4.59)). So in total\(^{22}\)
\[ \frac{\partial W}{\partial \Phi} = 0 \iff u = 0 \quad , \quad W(\Phi) = 0 \iff u = 0 \] (4.12)
giving no proper supersymmetric vacuum but the common decompactification runaway.

4.1 The triality symmetry relations of the local superpotential

Anti-invariance-with-correction-terms of the superpotential

Now, remarkably, in the case of the actual membrane instanton superpotential, the
function \( W(u) \) satisfies\(^{23}\) the following symmetry relations which will ensure that the
local superpotential is compatible with triality symmetry (almost)
\[ W\left(\frac{1}{u}\right) = -W(u) - \zeta(2) - \frac{1}{2} \log^2(-u) \] (4.13)
\[ W(1 - u) = -W(u) + \zeta(2) - \log u \log(1 - u) \] (4.14)
The symmetry relations entail that, up to\(^{24}\) the elementary corrections provided by the
products of two log’s and \( \zeta(2) \), the \( W(u) \) superpotential is invariant under the transforma-
tions in the first line of (3.3) and transforms with a minus sign under the mappings
of the second line. That is the ‘local’ superpotential transforms (under the \( \text{Sl}(2) \) action)
up to the elementary corrections with the sign character just as the global superpotential
did (under the linear action) and as it should a priori. The behaviour under \( \mathbb{Z}_3 \) shows
how the local (on \( \mathcal{N} \)) membrane instanton contributions fit together globally.

\(^{22}\)because of the deviations (cf. below) from strict anti-invariance this captures just the \( u = 0 \) end
\(^{23}\)Integrating the relation \( \frac{du}{u} W\left(\frac{1}{u}\right) = -\log(1 - \frac{1}{u}) \cdot \frac{1}{u} = \log(1 - u) \log(-u) \) gives (4.13) (for the integration
constant compare at \( u = 1 \)). Partial integration gives (4.14): \(- \int_0^u \frac{dt}{t} \log(1 - t) = - \log u \log(1 - u) - \int_0^u \frac{dt}{1 - t} \log t \), the last integral being \( W(1 - u) - W(1) \) (by the substitution \( s = 1 - t \) in \( W(1 - u) \)).
\(^{24}\)The differential equation (4.10) makes it technically clear that \( W \) is not precisely anti-invariant (cf.
remark after (A.29)); what is remarkable is that it is almost anti-invariant.
Let us compare the analogous behaviour of the instanton sums $L_{i0}$ and $L_{i2}$, describing the multi-coverings of SUSY-cycles provided by the holomorphic $S^2$ in the string world-sheet case and the associative $S^3$ in the membrane case, respectively (where the 'lower terms' $-\zeta(2) - \frac{1}{2}(-\pi^2 \pm 2\pi iw)$ are at most linear in $w = \log u$)

\[ W_{\text{mem}}\left(\frac{1}{e^w}\right) = -W_{\text{mem}}(e^w) - \frac{1}{2}w^2 + \text{lower terms} \quad (4.15) \]

\[ I_{\text{ws}}\left(\frac{1}{q}\right) = -I_{\text{ws}}(q) - 1 \quad (4.16) \]

Note then that (4.15) corresponds after taking $\partial^2/\partial w^2$ via (B.1) to (4.16).

Let us look on a related example concerning the issue of corrections of polylogarithms. By (4.4) $L_{i0}(q) = \sum_{n \geq 1} q^n = \frac{q}{1-q}$ had the anti-invariant transformation behaviour under $\mathbb{Z}_2$ up to mentioned correction. Said differently, when one considers the full expression which includes the classical contribution and the quantum corrections one finds a smooth behaviour\(^{25}\). The change in the classical intersection number will then be balanced exactly by the change in the quantum contribution.

Now if a curve $C = \mathbb{P}^1$ is flopped at a point $x_0$ along the Horava/Witten intervall this is argued in [31] to cause a $G = (\pm)\delta_C$ contribution (from $dG = (\pm)\delta_C \delta(x_{11} - x_0)dx_{11}$). This comes as one has the usual anomaly balance $dG = (tr F_{\text{obs/hid}} \wedge F_{\text{obs/hid}} - \frac{1}{2}tr R \wedge R) \cdot \delta(x_{11} - x_{\text{obs/hid}})dx_{11}$ at the boundaries, but along the intervall, when crossing the flop point, the gravitational contribution will have changed\(^{26}\) [32], with $\delta_C = \Delta_{\text{flop}} \frac{2}{C}$. How can one have a jump between the endpoints of the flop transition if these can also be smoothly related (when one does not go through the singular point but encircles it by rotating the $B$-field) ? As the latter process is not just classical geometry (as would be comparing just $c_2$'s) one has to look at the quantum corrected quantities where a classical-quantum balance now takes place at well. The relevant expression to look at is

\[ 12 F_1 = \left(\frac{c_2}{2} \cdot J\right)t + L_{i1}(q) \quad (4.17) \]

and $\partial_t F_1 \sim \frac{c_2}{2} \cdot J + L_{i0}(q)$ brings us effectively back to the previous balancing argument.

\(^{25}\)i.e. start with the prepotential as given by a cubic polynomial with the intersection numbers as coefficients (possible lower polynomial terms are not relevant here) plus the $L_{i3}$ term (including the instanton coefficients, i.e. the number of rational curves in specific cohomology classes) then take the third derivative (w.r.t. $t = \log q$) and find the classical intersection number plus $L_{i0}(q)$ (again by (B.1))

\(^{26}\)The reason for the $1/2$ is that $c_2(CY) \cdot S = c_2(S) - c_1^2(S)$ changes by 2 as an blow-up increases the Euler number of $S$ by one and the canonical class gets a contribution from the exceptional divisor.
Anti-invariance of the superpotential-with-correction-terms?

One might ask whether one could make the superpotential anti-invariant by adding suitable terms to $W$ so that the troublesome remainder terms27 $R_\gamma(z)$ (for $\gamma \in \Sigma_3$) in

$$W(\gamma z) = \text{sign}(\gamma)W(z) + R_\gamma(z) \quad (4.18)$$

(the log’s and constants interfering with the precise anti-invariant transformation behaviour) would be canceled. For example the deviation in (4.16) can be rectified that way, the modified $\tilde{I}_{ws} := I_{ws} + 1/2$ is strictly anti-invariant: $\tilde{I}_{ws}(1/q) = -\tilde{I}_{ws}(q)$. As for $W$ the deviations consist of quadratic polynomials in the log’s of $u$ and its $\mathbb{Z}_3^{Sl}$ transforms one may try to adjust $W$ by adding a term of this type (as suggested by the fact that $I_{ws}$ and $W_{mem}$ are related by taking a two-fold derivative w.r.t. log $u$, cf. above).

So one would make an ansatz to correct the superpotential by additional terms28 $C(z)$ which cancel the unwanted terms. Including $C(z)$ one gets a modified superpotential

$$\tilde{W}(z) = W(z) + C(z) \quad (4.19)$$

such that $\tilde{W}$ transforms just with the sign character. One expects $\tilde{W}$ to have the structure making manifest an underlying anti-invariant projection (A.4) (cf. app., subsect. A.3)

$$\frac{1}{6} \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)W(\gamma z) = \frac{1}{6} \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)[\text{sign}(\gamma)W + R_\gamma] = W + \frac{1}{6} \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)R_\gamma \quad (4.20)$$

(From this one has for the sought-after correction term29 $C = \frac{1}{6} \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)R_\gamma$

$$C = \frac{1}{6} \left(-\frac{1}{2} \log^2 z - 2 \log z \log \beta z + \frac{3}{2} \log^2 \beta z + 2 \log \beta z \log \beta^2 z + \frac{1}{2} \log^2 \beta^2 z \right) \quad (4.21)$$

One finds also (with a subtle sign and remote similarity to the Rogers modification (B.16))

$$\tilde{W}(z) = Li(z) - \frac{1}{2} \log \beta z \log z - \frac{\pi^2}{6} \pm \frac{i\pi}{6}(\log z - \log \beta z) \quad (4.22)$$

But a physical motivation for $C$ is unclear; one wants to see the local individual superpotentials in the three different phases not added like in (4.20) but rather naturally patching together (cf. remark after (4.3)).

27From $R_\alpha$ and $R_{\alpha\beta}$ in (4.13) and (4.14) one derives iteratively that $R_\beta = -2\zeta(2) - \log z \log \beta z - \frac{1}{2} \log^2(-\beta z)$, $R_{\beta\mu} = -\zeta(2) - \log z \log \beta z - \frac{1}{2} \log^2(-\beta z)$, $R_{\alpha\beta} = -\frac{1}{2} \log^2(1 - z)$; $R_{\alpha\beta\mu}$ follows also directly from integrating $\frac{d}{dz}W(\frac{z}{z-1}) = -\log(1 - \frac{z}{z-1}) = \log(1 - z) \left(\frac{1}{2} + \frac{1}{1-z}\right)$.

28Reasonably one has to demand that $C(z)$ is 'more elementary' than $W(z)$ itself (like a polynomial in log terms) as one has always the anti-invariant projection (A.4) with $C = \sum_{\gamma \in \Sigma_3, \gamma \neq e} \text{sign}(\gamma)W$. From $C \sim \sum \text{sign}(\gamma)R_\gamma$ as used above one finds that for this it suffices that the $R_\gamma$ are 'more elementary'.

29Coming with (A.15) from $6C = -2 \log z \log \beta z + \log z \log \beta^2 z + \log^2(-z) + \frac{1}{2} \log^2 \beta z - \frac{1}{2} \log^2(-\beta z)$. 
Behaviour of the local superpotential on the critical circle

Let us study $W$ on $^{30}$ the critical circle $|u_i| = 1$ (the boundary of the domain of convergence of the series (4.11)), so just as a function of $\phi_i = \int_{Q_i} C$. Then $W(e^{i\phi}) - W(1) = -i \int_0^\phi d\chi \log(1 - e^{i\chi})$ and (3.8) give the elementary evaluation

$$\text{Re } W(e^{i\phi}) = \sum_{n \geq 1} \frac{\cos n\phi}{n^2} = \zeta(2) - \frac{1}{4} \phi(2\pi - \phi) \quad (4.23)$$

for the real part and the non-elementary odd function $I(\phi)$ for the imaginary part

$$I(\phi) = \sum_{n \geq 1} \frac{\sin n\phi}{n^2} = -\int_0^\phi \log(2\sin \frac{\psi}{2}) d\psi \quad (4.24)$$

which is of period $^{31} 2\pi$. As we will have opportunity to consider terms like

$$\Pi(\phi) := \frac{1}{2} I(2\phi) = -\int_0^\phi \log(2\sin \psi) d\psi \quad (4.25)$$

let us note$^{32}$

$$\frac{1}{2} I(2\phi) = I(\phi) + I(\pi + \phi) = I(\phi) - I(\pi - \phi) \quad (4.26)$$

In view of the membrane anomaly relation $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ (for the $D_i$) with symmetry for $\alpha_i = \pi/3 = \arg(-\omega^2)$ or $\alpha_i = -\pi/3 = \arg(-\omega)$, note$^{33}$ that $I(\phi)$ becomes maximal at $\phi = \pi/3$ (as seen from solving $I'(\phi) = -\log(2\sin \frac{\phi}{2}) = 0$; cf. (3.11)) and that (by (4.26))

$$I(\pi/3) = \frac{3}{2} I(2\pi/3) \quad (4.27)$$

So for the six roots $e^{k\frac{2\pi i}{6}}$ ($k = 1, \ldots, 6$) one has ($I := I(\pi/3), \zeta := \zeta(2) = \pi^2/6$)

$$W(e^{k\frac{2\pi i}{6}}) = \begin{vmatrix}
-\omega^2 & \omega & -1 & \omega^2 & -\omega & 1 \\
-\omega & \frac{1}{2}\zeta + iI & -2\frac{1}{6}\zeta & -\frac{3}{2}\zeta & -\frac{2}{3}iI & \frac{1}{6}\zeta - iI \\
\end{vmatrix} \quad (4.28)$$

A $\mathbb{Z}_N$ symmetry property

The result that $\sum_{\mathbb{Z}_N} \text{Li}(e^{k2\pi i/6}) = \zeta/6 = \text{Li}(1)/6$ leads to a more general observation concerning the angular degree of freedom $\phi$ of $\text{Li}(z)$, more precisely on the interrelation between entries equidistributed with respect to $\phi$. Generally one has$^{34}$ (with $\omega_N = e^{2\pi i/N}$)

$$\frac{1}{N} \text{Li}(z^N) = \sum_{k \in \mathbb{Z}_N} \text{Li}(\omega_N^k z) \quad (4.29)$$

$^{30}$which would correspond classically$^{13}$ to a shrinking $Q_i$ of $\text{vol}(Q_i) = 0$

$^{31}$If $\phi \notin [0, 2\pi]$ one has to take the absolute value inside the logarithm.

$^{32}$As $I(2\phi) = -2 \int_0^\phi d\chi \log(2\sin \chi) = -2 \int_0^\phi d\chi \log(2 \sin \frac{\chi}{2}) = 2I(\phi) + 2 \int_{\pi - \phi}^\pi d\xi \log(2\sin \frac{\xi}{2}) = 2I(\phi) + 2I(\pi) - 2I(\pi - \phi)$.

$^{33}$for the $Q_i$, but in view of the identification$^{11,13}$ (3.7) of the $D_i$ and the $Q_i$ in a $\mathbb{Z}_3$ rotated phase

$^{34}$as $1 - y^N = \prod_k (\omega_N^k - y) = \prod_k (1 - \omega_N^{-k} y)$ gives $-\frac{1}{N} \int_0^{z^N} \log(1 - t) dt \log t = - \int_0^z \log(1 - y^N) \log y = -\sum_k \int_0^z \log(1 - \omega_N^{-k} y) \log y = -\sum_k \int_0^z \log(1 - \omega_N^{-k} x) \log x = -\sum_k \int_0^z \int_0^{\omega_N^{-k} t} \log(1 - t) dt \log t$
4.2 The monodromy representation

The multi-valuedness of \( \log z \) and \( \text{Li}(z) \) around \( z = 0,1 \) and \( \infty \) is described by the monodromy representation of the fundamental group \( \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \). This describes for the generator loops \( l_i(t) \) (\( i = 0,1 \), \( t \in [0,1] \)) which encircle (in the mathematically positively oriented sense) \( z = 0 \) and \( z = 1 \), respectively (then \( l_\infty \circ l_1 \circ l_0 = 1 \)), the increments (cf. also app. B)

\[
\log z \xrightarrow{l_0} \log z + 2\pi i \tag{4.30}
\]

\[
\log \beta z \xrightarrow{l_1} \log \beta z - 2\pi i, \quad \text{Li}(z) \xrightarrow{l_1} \text{Li}(z) - 2\pi i \log z \tag{4.31}
\]

The relevant local system is described by a bundle, flat with respect to a suitable connection. In the case of the logarithm the monodromy (4.30) is captured by the matrix

\[
M(l_0) = \begin{pmatrix} 1 & 0 & 2\pi i \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

acting on the two-vector \((\log z, 1)^t\) and the monodromy group is given by \( \mathcal{U}_\mathbb{Z} \hookrightarrow \mathcal{U}_\mathbb{C} \) where \( \mathcal{U} \) denotes the upper triangular group \( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \text{Sl}(2) \) (the embedding of \( \mathcal{U}_\mathbb{Z} \) in \( \mathcal{U}_\mathbb{C} \) may include the factor of \( 2\pi i \)). The generalisation in the case of the dilogarithm involves the upper triangular \( 3 \times 3 \) matrices \[18\], i.e. one gets again admixtures from 'lower' components: the hierarchical structure of the poly-logarithm \( \text{Li} = \text{Li}_2 \) with respect to its predecessor \( \log \beta z = \text{Li}_1(z) \) entails that its monodromy is not any longer given just by the addition of integers (multiplied by \( 2\pi i \)); rather one has to consider constants, ordinary logarithms and the dilogarithm all at the same time and to consider the lower ones as monodromy contributions of the next higher one. One can organize this as follows. Analytic continuation about a loop \( l \) in \( \mathbb{P}^1 \setminus \{0,1,\infty\} \) (based at \( 1/2 \), say) leads to the monodromy representation

\[
M : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \rightarrow \text{Gl}(3, \mathbb{C}) \tag{4.32}
\]

There are two equivalent ways to express this. In a \textit{vector picture} one assembles \( \text{Li} \), the ordinary logarithm and the constants to a three-vector \( c_3 \) and finds for the images of the generator loops \( l_i(t) \) (\( i = 0,1 \)) the matrices \( M(l_i) \) representing (4.30), (4.31) for \( c_3 \)

\[
c_3 = \begin{pmatrix} \text{Li}(z) \\ \log z \\ 1 \end{pmatrix} : \quad M(l_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\pi i \\ 0 & 0 & 1 \end{pmatrix}, \quad M(l_1) = \begin{pmatrix} 1 & -2\pi i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.33}
\]

Alternatively, in a \textit{Heisenberg picture}, consider the complexified Heisenberg group \( \mathcal{H}_\mathbb{C} \)

\[
\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mathbb{Z}} (a,b|c) \in \mathcal{H}_\mathbb{C} \tag{4.34}
\]
Instead of \( c_3 \) one considers here the expression (a flat section of a suitable connection)
\[
\Lambda(z) = \left( - \log \beta z, \log z \mid - Li(z) \right)
\] (4.35)
and left operation with \( \mathcal{H}_Z \) expresses the multi-valuedness (4.31). More precisely one finds for the monodromy along the loops \( l_i \) the representing left multipliers \( h_i \)
\[
h_0 = (0, 1 \mid 0) \quad , \quad h_1 = (1, 0 \mid 0) \quad , \quad h_\infty = (-1, -1 \mid 0)
\] (4.36)
So \( h_0 \cdot (u, v \mid w) = (u, v + 1 \mid w) \) and \( h_1 \cdot (u, v \mid w) = (u + 1, v \mid w + v) \) give\(^{35}\) (4.31).
One has from \( (a, b \mid c) \rightarrow (x, y) = (e^a, e^b) \) a bundle projection with fibre \((2\pi i)^2 \mathbb{Z} \setminus \mathbb{C}_c \overset{\cong}{\rightarrow} \mathbb{C}^* \) (via \( c \rightarrow S := e^{c/2\pi i} \)); the entries of \( \mathcal{H}_Z \) are actually from \((2\pi i)^2 \mathbb{Z}, (2\pi i) \mathbb{Z} \mid (2\pi i)^2 \mathbb{Z})\)
\[
\mathcal{H}_Z / \mathcal{H}_C
\]
\[
\downarrow
\]
\[
\mathbb{C}_x^* \times \mathbb{C}_y^* = (2\pi i \mathbb{Z})^2 \setminus \mathbb{C}_{a,b}^2
\] (4.37)
This carries a connection of curvature \( \frac{1}{2\pi i} d \log x \wedge d \log y \) coming from the connection
\[
\nabla S = \frac{1}{2\pi i} S (2\pi i d \log S - u \, dv) = dS - S \, u \, dv / 2\pi i
\] (4.38)
on the pullback\(^{36}\) of the bundle (4.37) to \( \mathbb{C} \times \mathbb{C} \) along \( (a, b) \rightarrow (e^a, e^b) \). The latter trivialises the bundle so that a section can be understood as a map \( S : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^* \).

Now consider the pullback (along the base map \( z \rightarrow (1 - z, z) \)) of the bundle \( \mathcal{H}_Z / \mathcal{H}_C \) lying over \( \mathbb{C}^* \times \mathbb{C}^* \) to what we will call the Heisenberg bundle \( \mathcal{H} \) over \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \)
\[
\mathcal{H}
\]
\[
\downarrow
\]
\[
\mathcal{H}_Z / \mathcal{H}_C
\]
\[
\downarrow
\]
\[
\mathbb{P}^1 \setminus \{0, 1, \infty\} \xrightarrow{(1-z,z)} \mathbb{C}^* \times \mathbb{C}^*
\] (4.39)
As the first two entries of a section \( s \) of \( \mathcal{H} \) are fixed by the construction (up to the indeterminacy caused by the coset) \( s \) has the form \( s(z) = \mathcal{H}_Z(- \log \beta z, \log z \mid c) \). Asking even for a flat section one finds (undoing the fibre identification \( c \rightarrow e^{c/2\pi i} = S \)) that the flatness condition \( dc = u \, dv \) (from (4.38)) just expresses (4.10), the \( Li \) integral, and that the coset takes into account the multi-valuedness (4.31). So \( \mathcal{H} \) possesses the flat section (4.35) and the Heisenberg bundle just encodes the fact that the ’function’ \( Li \) is a section.

To gain information about \( Li \) itself by somehow projecting to it is not straightforward as the immediate extraction of \( Li \) is obstructed by the \( \mathcal{H}_Z \) coset. What actually can be extracted is the (suitably adjusted) imaginary part of it as we describe now.

\(^{35}\) Note that actually we consider \( (a, b, c) \in \mathbb{Z}^3 \) embedded in \( \mathcal{H}_Z \) via \( (2\pi ia, 2\pi ib \mid (2\pi i)^2 c) \).

\(^{36}\) The bundle \( (2\pi i)^2 \mathbb{Z} \setminus \mathcal{H} \rightarrow \mathbb{C}_a \times \mathbb{C}_b \) of fibre \( (2\pi i)^2 \mathbb{Z} \setminus \mathbb{C}_c \); so this is the complex analogue of (B.7).
4.3 Anti-invariance of the adjusted imaginary part $L$ of $W$

To motivate this note that the integral representation (4.11) defines $W$ on the complex plane, cut along the part $(1, \infty)$ of the positive real axis where $W$ jumps by $2\pi i \log z$ when crossing the cut; so the expression $W(z) + i \arg(1 - z) \log z$ is continuous; its imaginary part coincides with $L$ below.

Now note that the complex-valued 'function' $Li(z)$ of the complex variable $z$ is not well-defined as a function according to the multi-valuedness expressed by the increments $\Delta_i$ around the $l_i$ which follow from (4.31), i.e. $\Delta_0 = 0$, $\Delta_1 = -2\pi i \log z$. Note that if we restrict the values by considering just $\text{Im} Li(z)$ then this real-valued 'function' of the complex variable $z$ has still $\Delta_0 = 0$, $\Delta_1 = -2\pi \text{Re} \log z$. Therefore, if we go one step further and consider the real-valued 'function' of the real degree of freedom $z = e^{i\phi}$ living on the critical circle $|z| = 1$, we get indeed a well-defined function.

Now it is interesting to see that, with a slight modification, we can actually do better. Namely, to extrapolate this property beyond the critical circle, consider the expression $\psi = \log \beta z \text{Re} \log z$ (vanishing on $|z| = 1$). One finds that the real-valued combination of a complex degree of freedom $\mathcal{L}(z) = \text{Im} Li(z) - \text{Im} \psi(z)$ is actually not only a well-defined function i.e. $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$-invariant, but at the same time also $\Sigma_3$ anti-invariant, so it transforms precisely with the sign-character, i.e. without correction terms. Furthermore, it is even expressible by a function depending just on a real degree of freedom: the critical circle.

This is the case although $\mathcal{L}$ is not just depending only on the angular part of the complex variable $z$; rather it depends on the value of $I(\phi) = \text{Im} Li_{e^{i\phi}} = \mathcal{L}_{e^{i\phi}}$ on the angular parts of the different $\mathbb{Z}_3$ transforms of $z$, which themselves are not depending on the angular part of $z$ alone, cf. (4.42).

So note first that the function one finds (which also satisfies $\mathcal{L}(\bar{z}) = -\mathcal{L}(z)$)

$$\mathcal{L}(z) = \text{Im} Li(z) - \text{Im} \log \beta z \text{Re} \log z$$

(cf. (B.14)) is $\pi_1$-invariant, i.e. single-valued as is also easily checked from (4.31).

Now, just as the single-valued cousin $\text{Re} \log z$ of the logarithm has anti-invariant transformation behavior under the duality group $\mathbb{Z}_2$ (with $\alpha : z \to 1/z$), we will see that $\mathcal{L}$ transforms anti-invariantly under $\Sigma_3$. We give four arguments for this: the direct computational check, an argument using representation theory, a manifestly invariant rewriting and finally a geometric interpretation.
Anti-invariance of \( L \) (first argument): explicit evaluation (appendix A.1)

This is the brute force procedure given by the explicit check.

Anti-invariance of \( L \) (second argument): representation theory (appendix A.2)

More conceptually one has a representation theoretic argument (cf. (A.24), (A.27)).

Anti-invariance of \( L \) (third argument): rewriting to a manifestly invariant expression

The third argument (going in essence back to Kummer) is by an explicit rewriting (4.43). Consider the decomposition of \( \text{Li}(z) \) in real and imaginary parts as we did above for its restriction on the critical circle \(|z| = 1\). One finds with \( z = re^{i\phi} \) that

\[
\text{Li}(z) = -\int_0^r \frac{\log(1 - e^{i\phi}t)}{t} dt = -\frac{1}{2} \int_0^r \frac{\log(1 - 2t \cos \phi + t^2)}{t} dt + \int_0^r \arctan \left( \frac{t \sin \phi}{1 - t \cos \phi} \right) dt
\]  

(4.41)

This gives\(^37\) with \( \arctan \frac{t \sin \phi}{1 - t \cos \phi} =: \chi, \kappa := \chi|_{t=r} = \text{Im} \log \beta z \) and the inversion relation \( t = \frac{\sin \chi}{\sin(\chi + \phi)} \) (considering \( \phi \) as a parameter) the evaluation (using (4.25), (A.13))

\[
\text{Im} \text{Li}(z) = \int_0^r \chi \frac{dt}{t} = \chi \log t|_0^r - \int_0^\kappa \log t \, d\chi = \kappa \log r - \int_0^\kappa \log \frac{\sin \chi}{\sin(\chi + \phi)} \, d\chi
\]

\[
= \kappa \log r + \frac{1}{2} \left(I(2\phi) + I(2\kappa) - I(2\phi + 2\kappa)\right)
\]

\[
= \text{Im} \log \beta z \text{Re} \log z + \frac{1}{2} \left(I(2 \text{Im} \log z) + I(2 \text{Im} \log \beta z) + I(2 \text{Im} \log \beta^2 z)\right)
\]

(4.42)

This shows that the 'function' \( \text{Im} \text{Li}(z) \), which a priori is a non-elementary real 'function' of a complex variable, is actually already determined (up to the elementary logarithmic product term) by the real 'function' \( I(\phi) \) of a real degree of freedom (cf. remark above).

Using the notation \( z \rightarrow e^{i\phi(z)} = z/|z| = \exp\{i \text{Im} \log z\} \) for the (\( \alpha \)-compatible) operation of taking the angular part, one has by (4.42), (4.25)

\[
\mathcal{L}(z) = \sum_{i \in \Sigma_3} \Pi(\phi(\beta^i z))
\]

(4.43)

which shows that \( \mathcal{L}(z) \) is \( \Sigma_3 \) anti-invariant (as \( \phi(\alpha z) = -\phi(z) \) and \( I(\phi) \) is odd). And \( \mathcal{L}(z) = \frac{1}{4\iota} \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma) \text{Li}(e^{2i\phi(\gamma z)}) \) , making a \( \Sigma_3 \) anti-invariant projection in \( \mathcal{L}(z) \) manifest, follows\(^38\) with \( \Pi(\phi) = \frac{1}{2} I(2\phi) = \frac{1}{4\iota} \left( \text{Li}(e^{2i\phi}) - \text{Li}(e^{-2i\phi}) \right) \) (cf. (4.29) for \( N = 2 \)).

\(^37\) Note that \( \text{Im} \log z = \arctan \frac{\text{Im} z}{\text{Re} z} \) and \( \text{Im} \log \beta z = \arctan \frac{\text{Im} \beta z}{\text{Re} \beta z} \).

\(^38\) Note the \( \alpha \)-anti-invariant projection in \( I(\phi) = \text{Im} \text{Li}(e^{i\phi}) = \frac{1}{2\iota} \left( \text{Li}(e^{i\phi}) - \text{Li}(e^{-i\phi}) \right) \) making \( I(\phi) \) odd.
Anti-invariance of \( \mathcal{L} \) (fourth argument): volume of an ideal hyperbolic tetrahedron

This approach uses a geometric interpretation (4.45). The idea is to interpret the transformation behaviour of \( \mathcal{L}(z) \) (under the \( \Sigma_3 \) operation on \( z \)) geometrically in the following sense: \( z \in \mathbb{P}^1 \) is interpreted as being actually a cross ratio (cf. (C.13); the definition is normalized so that \( z = \text{cr}\{\infty, 0, 1, z\} \))

\[
z = \text{cr}\{z_1, z_2, z_3, z_4\} = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_3}{z_2 - z_4} \tag{4.44}
\]

of four points \( z_1, z_2, z_3, z_4 \) in \( \mathbb{P}^1 \) and the operation of \( \Sigma_3 \) as the residual effect of the original \( \Sigma_4 \) on the \( z_i \) (cf. (C.15), (C.16)); then \( \mathcal{L}(z) \in \mathbb{R} \) is understood as a geometrical quantity which transforms under \( \Sigma_4 \) with the sign character (of \( \Sigma_4 \) which induces the corresponding character on \( \Sigma_3 \)). For this geometrical quantity one takes the hyperbolic volume of the ideal tetrahedron in hyperbolic three space \( H_3 \) (cf. app. C) with vertices \( z_1, z_2, z_3, z_4 \) lying on the boundary \( \mathbb{P}_C^1 \) of \( H_3 \). This is then manifestly independent of the numbering of the vertices except that the orientation changes under odd renumberings showing the anti-invariant transformation behaviour.

Here an ideal tetrahedron is a tetrahedron \( \Delta \) (bounded by geodesic faces and geodesic edges)\(^{39}\) with vertices \( z_1, z_2, z_3, z_4 \) on the boundary \( \mathbb{C} \cup \{\infty\} \). One has \( \text{vol} \Delta = \mathcal{L}(z) \) with \( z = \text{cr}\{z_1, z_2, z_3, z_4\} \) (cf. app. C) or, equivalently\(^{40}\)

\[
\text{vol} \Delta(z) = \mathcal{L}(z) \tag{4.45}
\]

for an ideal tetrahedron \( \Delta(z) \) with vertices \((\infty, 0, 1, z)\). As a check note that \( \Delta(z) \) degenerates if one of the faces degenerates, i.e. not only for \( z = 0, 1, \infty \) but also for \( z \) being on a line with 0 and 1, i.e. for \( z \) real; in all these cases (4.40) vanishes as well.

Let us give an example. The symmetric hyperbolic three-simplex \( \Delta_{\text{sym}} \) (with vertices on \( \mathbb{P}_C^1 \) and having all six dihedral angles equal to \( \pi/3 \)) is in the 'circle gauge' (cf. app.) given by the vertices \( \infty, -1, -\omega^2, -\omega, \) so \( z = -\omega = e^{i\pi/3} \) and (cf. (3.11))

\[
\gamma_i = \pi/3 \tag{4.46}
\]

Now the volume (C.11) of a hyperbolic three-simplex becomes maximal\(^{41}\) for the symmetric case (4.46) (cf. the corresponding remark about \( I(\phi) \) in subsection 4.1)

\[
\text{vol}(\Delta_{\text{sym}}) = 3\pi(\pi/3) \tag{4.47}
\]

which, being equal to \( \frac{3}{2} I(2\pi/3) \), equals indeed \( \mathcal{L}(z) = \text{Im}Li(z) = I(\pi/3) \) by (4.27).

\(^{39}\)The geodesics are vertical lines and semi-circles (in vertical planes) with endpoints in the boundary \( \mathbb{C} \cup \{\infty\} \); geodesic planes are vertical planes and hemispheres (over \( \mathbb{C} \) and bounded by geodesics).

\(^{40}\)as the \( z_i \) can be transformed to \((\infty, 0, 1, z)\) by an element of \( \text{Sl}(2, \mathbb{C}) \) on \( \mathbb{P}_C^1 \), an isometry of \( H_3 \).

\(^{41}\)In \( H_2 \) area(\( \Delta_2 \)) = \( \pi - \sum \alpha_i \) becomes maximal (= \( \pi \)) for \( \alpha_i = 0 \) (like for the fundamental domain).
4.4 Linear Modifications

We consider now some possible slight modifications which can occur from different perspectives but all have a somewhat similar flavour. Recall that we found (4.22) for the formal triality symmetric (anti-invariant) modification $\tilde{W}$ of $W$

$$\tilde{W}(z) = R(z) \pm \frac{i\pi}{6} (\log z - \log \beta z) - \frac{\pi^2}{6}$$

(4.48)

So $\tilde{W}$ was a less than quadratic modification of $R$, i.e. up to a constant just a linear modification. We want to point here to other occurrences of such linear modifications.

For this let us recall the differential equation (4.10) for the superpotential before doing the holomorphic completion (which the aim to get a proper superpotential suggested)

$$\frac{dW}{d\log u} = \text{Im } \log \beta u = \frac{1}{2i}(\log \beta u - \log \beta \bar{u})$$

(4.49)

(where we absorbed a factor $i$ into $W$). Considering $z$ and $\bar{z}$ as two degrees of freedom like $\text{Re } z$ and $\text{Im } z$, the antiholomorphic part of the rhs of (4.49) is independent of the differentiation variable and one finds by giving up the holomorphicity demand on $W$ a superpotential $W_{\text{anom}}$ with a holomorphic anomaly

$$W_{\text{anom}}(u) = \frac{1}{2i} \text{Li}(u) - \frac{1}{2i} \log \beta \bar{u} \log u$$

(4.50)

which is a linear modification in the holomorphic coordinate (of course quadratic when considered non-holomorphically). For easier comparison we display some relations

$$\text{Re } W_{\text{anom}}(u) = \frac{1}{2} \text{Im } \text{Li}(u) - \frac{1}{2}(\text{Re } \log \beta u \text{ Im } \log u - \text{Im } \log \beta u \text{ Re } \log u)$$

(4.51)

$$\text{Im } W_{\text{anom}}(u) = -\frac{1}{2} \text{Re } \text{Li}(u) + \frac{1}{2}(\text{Re } \log \beta u \text{ Re } \log u + \text{Im } \log \beta u \text{ Im } \log u)$$

(4.52)

$$\frac{1}{2} \mathcal{L}(u) = \frac{1}{2} \text{Im } \text{Li}(u) - \frac{1}{2} \text{Im } \log \beta u \text{ Re } \log u$$

(4.53)

Further, allowing [14] in (the rhs of) the differential equation (4.10) of $W$ for an additional additive constant $-\log \beta u_*$, one finds again a similar modification but with $u_*$ constant

$$W^{\text{var}}(u) = \text{Li}(u) - \log \beta u_* \log u$$

(4.54)

Finally, including codimension four singularities like in $\mathbb{R}^4 \times S^3 \times \mathbb{R}^4/\mathbb{Z}_N$ leading to non-abelian gauge symmetry on $\mathbb{R}^4 \times S^3$ (cf. sect. 5) one has for the full superpotential

$$W_{Y M, mem} = c W(u_{1,k}) + S\Phi = cN \text{Li}(e^{2\pi ik/N} u^{1/N}) - i S \log u$$

(4.55)

---

42In many respects (cf. sect. 6) the Rogers modification $R(z) = \text{Li}(z) - \frac{1}{2} \log \beta \log z$ (cf. (B.15)), which itself may be described as a quadratic (in the log’s of $z$ and its $\mathbb{Z}_3$ transforms) modification of $\text{Li}$, is a conceptually more natural object to consider.

43cf. footn. 5.10; here $S$ is the superfield $tr W^\alpha W_\alpha$ of highest component $\int d^2 \theta S = tr (F^2 + iF \wedge F)$
4.5 Comparison with a flux superpotential

We now want to compare the membrane instanton superpotential $W \sim Li$ with a flux induced superpotential\(^{44}\) $W_G = \int_{W_7} G \wedge (C + i\mathcal{T})$. A non-trivial $G$-flux turned on (as classical background) will break supersymmetry\(^{[10]} , [11]\). So the mentioned comparison is possible only because of the absence (4.12) of proper susy vacua, i.e. one can have a non-trivial $G$-flux just\(^{32}\) for all $u \neq 0$. One tries to choose\(^{49}\) $\int_B G := \log \beta u$ (mimicking the quantum vev). Recall that the notion of superpotential was somewhat improper because of the non-compactness of $X_7$; similarly one does not have a proper Kähler potential\(^{45}\) in the infinite volume case, or a flux with support on a closed cycle (here $B$ has effectively the boundary\(^{46}\) $D$). In the end all of this should be embedded in compact $G_2$ holonomy manifolds. But at least it is suggestive to see how the form of the membrane instanton superpotential may reappear here. Let us recall first the flux superpotentials \([5]\), \([9]\), \([10]\).

One has, schematically, the flux-generated superpotential in type IIB on a Calabi-Yau manifold. But at least it is suggestive to see how the form of the membrane instanton superpotential may reappear here. Let us recall first the flux superpotentials \([5]\), \([9]\), \([10]\).

\(W_H = \int_{\text{CY}} H \wedge \Omega \), \(V_H = \int_{\text{CY}} H \wedge *H \) ( + n ) (4.56)

(with holomorphic three-form $\Omega$ and $H$ and gains\(^{48}\) on a potential the boundary $H$.) One has, schematically, the flux-generated superpotential in type IIB on a Calabi-Yau superpotential may reappear here. Let us recall first the flux superpotentials \([5]\), \([9]\), \([10]\).

\(W_G = \int_X G \wedge (C + i\mathcal{T}) \), \(V_G = \int_X G \wedge *G + (\int_X G \wedge C)^2 \) (4.57)

and gains\(^{47}\) in the scalar potential the Kaluza-Klein reduction of the kinetic term $H^2$ (including a topological integer $n \sim \int_X H^{NS} \wedge H^R$). Similarly one has, schematically, on a $G_2$ holonomy manifold $X$ with covariant constant three-form $\mathcal{T}$ \([10]\), \([6]\)

\(\int_X G \wedge *G = \int_{B_i} G \int_{Q_i} *G \leftrightarrow^{49} \int_{D_i} C \int_{Q_i} *G \) (4.58)

\(^{44}\)strictly speaking this may concern in general a 'dual' $G_2$ holonomy manifold; the difference may concern in our case of the $M$-theory conifold just a transition to a phase with the role of $S^3$'s exchanged

\(^{45}\)Actually $W$ is a section of a line bundle $L$ of $c_1(L) = \frac{1}{2\pi} \partial\bar{\partial}K$ over the moduli space $\mathcal{M}$.

\(^{46}\)Compactification of $X$ gives the closed cone of boundary $Y = S^3_{3} \times S^3_{3}$: for $M$-theory on manifolds with boundary \([28]\) open membrane instanton contributions to the superpotential become important \([12]\).

\(^{47}\)Precise normalizations \([6]\) give $\text{vol}(X) = \frac{1}{4} \int \mathcal{T} \wedge *\mathcal{T}$, $\theta = \frac{1}{4\pi} \int G \wedge C \in \mathbb{R}/2\pi\mathbb{Z}$, $e^K = \frac{(2\pi)^3}{\text{vol}(X)}$

\(^{48}\)The two effective supersymmetry transformations of the double fermionic integration in $V = \int d^2\theta W$ lead for $u$ \([2]\) from the volume (metric) in $\int_Q \mathcal{T} \sim \int_Q d^3x \sqrt{-g}$ ($Q$ is supersymmetric) first to the gravitino $\psi$ and then again to the bosonic field $C$, more precisely to $\int_{Q_i} *G$; so symbolically one gets from the $\int G \wedge \mathcal{T}$ in $W_G$ the $\int G \wedge *G$ in $V_G$ via $\int d^2\theta \int_Q \mathcal{T} \sim \int_{Q_i} *G$ just as with $\Omega$ and $*H$ for a Calabi-Yau.
(at least in a schematic product ansatz where also \( \theta \sim \int_{B_i} G \int_{Q_i} C \leftrightarrow \int_{D_i} C \int_{Q_i} C \) so one has still \( V_G \sim \int_{D_i} C \)). Now we want to argue for the \( P_i \) as representing supersymmetric vacua. For this note that the relations \( u_i = 0 \) and \( \int_{D_i} C = 0 \) (which hold at the semiclassical end \( P_i \) where \( r_0 \approx \infty \)) give \( W|_{P_i} = 0 \) and \( \partial W|_{P_i} = 0 \): the first from \( \int_B G = \log \beta u \to 0 \), the latter by \( V \sim |\partial W|^2 \) (at \( P_i \) where \( W|_{P_i} = 0 \)) together with \( V_G \sim \int_{D_i} C \) by (4.58) (conversely (4.7), (4.58) would give \( P \neq P_i \implies \int_D C \neq 0 \implies \partial W \neq 0 \)

\[
P = P_i \implies W|_P = 0 = \partial W|_P
\]

(4.59)

Note further that (specialising to our non-compact \( X_7 \))

\[
\frac{\partial W_G}{\partial \Phi_i} = \int_X G \wedge \delta_{B_i} = \int_{B_i} G
\]

suggesting\(^{49} \) with (3.7) again the differential equation (4.10) (and (4.59) by (4.12))

\[
\frac{dW_G}{d \log u_i} = \log \beta u_i
\]

(4.61)

Remark: There is a formal similarity between two invariance-adjustments of the superpotential \( W_G = -i L_i \). The adjusted imaginary part \( \mathcal{L} \), depending on \( u = e^{i\Phi} \), is invariant under monodromy from the \( \pi_1(\mathcal{P}_u^1 \setminus \{0, 1, \infty\}) \) action (with \textit{additive} shifts \( \Delta_0 \Phi = 2\pi \), \( \Delta_1 W_G = 2\pi i \Phi \)); a term like \( DW \) (now dependent on \( \Phi \)) is invariant under Kähler transformations (with \textit{multiplicative} shift \( W \to We^{-F} \) and \( K \to K + F + \bar{F} \); so the section \( W \) is adjusted to a well-defined function in \( e^G = e^K|W|^2 \)). From the general relation

\[
\partial_i K = \frac{i}{2 \text{vol}(X)} \int_X \chi_i \wedge * \Upsilon \quad \text{(where } \Upsilon = \text{Im}\Phi^i \chi_i \text{ is the harmonic decomposition)} \]  

[6] one gets in our local situation of \( X_7 \) with one 3-cycle \( Q, \chi = \delta_B \) and \( \text{vol}(X) = \text{Im}\Phi \int_B * \Upsilon \) the finite expression \( \partial_i K = \frac{i}{2 \text{vol}(X)} \int_B * \Upsilon - \frac{1}{2 \text{Im}\Phi_i} \), which gives for the covariant derivative

\[
DW = \partial W + \frac{i}{2 \text{Im} \Phi_i} W \quad \text{(so that the difference to the ordinary derivative } \partial W \text{ vanishes when approaching via } \text{vol}(Q) \to \infty \text{ the end)}. \]

So one can compare (with \( Li(u = e^{i\Phi}) \))

\[
\text{Im}\Phi \cdot \text{Im}DW_G = \frac{1}{2} \text{Im}(iW_G) + \text{Im}\Phi \text{Im} \frac{\partial(iW_G)}{\partial(i\Phi)}
\]

(4.62)

\[
\mathcal{L} = \text{Im} Li + \text{Im} \Phi \text{Im} \frac{\partial Li}{\partial(i\Phi)}
\]

(4.63)

which leads, up to the factor 1/2, with \( iW_G = Li \) to a certain formal parallelism.

\(^{49} \) In the compact case note that \( G \)-flux, on a compact \( K3 \) fibre, say, of a \( K3 \) fibered \( X_7 \), is quantised (in units of \( 2\pi \) over tension), so constant over the moduli space (and the duality \([29]\) with the heterotic string might then be obstructed as for type IIA \([30]\)). Here \( \int_B G = \int_D C \), being zero classically, becomes in the quantum domain the varying expression \( \text{Im} \log \eta \), which may now be mimicked (!) by prescribing for each \( u \) a corresponding classical flux background (which entails the ensuing formal similarities).
5 codimension 4 singularities

When $X = \mathbb{R}^4 \times S^3$ is divided through by a finite subgroup $\Gamma$ of $SU(2)$ one obtains $X_{1,\Gamma} \cong \mathbb{R}^4/\Gamma \times S^3$ and $X_{2,\Gamma} \cong X_{3,\Gamma} \cong S^3/\Gamma \times \mathbb{R}^4$ leading as effective four-dimensional field theories to an $ADE$ gauge theory and to a theory without massless fields, respectively (the latter explaining the conjectured mass gap of the former) ($\Gamma$ operates always on the first factor of $SU(2)^3/SU(2)_D$, the $i$ in $X_i$ denotes which factor is filled in).

Relations between the observables given by the $\eta_i$-variables

In more detail [2] let $N = |\Gamma|$ and $Y_\Gamma = \Gamma \setminus S^3 \times S^3 = \Gamma \setminus SU(2)^3/SU(2)$ where $\Gamma$ acts on the first factor. Clearly the triality symmetry $\Sigma_3$ is broken down to $\mathbb{Z}_2$. The three-cycles $D_i$ project to the $D'_i$ with $D_1 \simeq ND'_1$ and $D_i \simeq D'_i$ for $i > 1$ and one has $M = \frac{1}{2} \sum_{n \in \mathbb{Z}} \delta_{n,1}$, fulfilling $N = h + 2h'$. The latter 'exotic' case leads to the new semiclassical limit point $\eta_1 = -1$ (beyond $0, 1, \infty$) of the quantum modulio space $\mathcal{N}_1$, the relations are

$$A - \text{series} \quad \eta_2 = (\beta^2 \eta_1)^N, \quad \eta_3 = (\beta \eta_1)^N \quad (5.1)$$

This is actually only the case if $\Gamma$ is just a cyclic group, corresponding to the $A$-series (in the type IIA reinterpretation this means that one has wrapped $N$ $D$-6-branes on the $S^3$ respectively has $N$ units of Ramond flux on the $S^2$). For the two different types of $D_n$ singularity in $M$-theory ($n \geq 4$), with gauge group $SO(2n)$ and $Sp(n-4)$ (of dual Coxeter numbers $h = 2n - 2$ and $h' = n - 3$, fulfilling $N = h + 2h'$), respectively, where the latter 'exotic' case leads to the new semiclassical limit point $\eta_1 = -1$ (beyond $0, 1, \infty$) of the quantum modulio space $\mathcal{N}_1$, the relations are

$$D - \text{series} \quad \eta_2 = (\beta^2 \eta_1)^h (\beta^2 (-\eta_1))^{2h'}, \quad \eta_3 = (\beta \eta_1)^h (\beta (-\eta_1))^{2h'} \quad (5.2)$$

Finally for the $E$-series one has (with $\omega_t = e^{2\pi i/t}$) new classical limits at $\eta_1 = \omega_t^\mu$: the different $E$-singularities in $M$-theory are indexed by an integer $t$ dividing some of the Dynkin indices $k_i$ of the $E$-group and an integer $\mu$ with $1 \leq \mu < t$ and $(\mu, t) = 1$ (for $t \geq 2$; for $t = 1$ is $\mu = 0$). With $h_t = \frac{1}{t} \sum_{t|k_i} k_i$ the dual Coxeter number of the gauge group $K_t$ in $M$-theory at a $G$-singularity of index $t$ one has (in general)

$$E - \text{series} \quad \eta_2 = \prod_{t,\mu} (\beta^2 \omega_t^\mu \eta_1)^{ht}, \quad \eta_3 = \prod_{t,\mu} (\beta \omega_t^\mu \eta_1)^{ht} \quad (5.3)$$
Relation to the instanton expansion parameters $u_i$

After the mutual $\eta_i$ relations let us also give the analogues of the relation (3.7) between the $\eta_i$ and the instanton parameters $u_i$. Let us restrict us to the $A$-series (in general a membrane wrapped on $S^3_Q \subset X_{i,\Gamma}$ corresponds to $t$ instantons in $K_{i,\Gamma}$). The different $X_{i,\Gamma}$ are defined by the 'filling in' condition $D'_i \simeq 0$. Consider the two cases $i = 1$ resp. $i > 1$ separately. At the center of $X_{1,\Gamma} = R^4/\Gamma \times S^3$ lies $S^3 = Q'_1 \simeq \pm D'_{i>1}$ (as the membrane instanton corresponds in the four-dimensional supersymmetric $SU(N)$ gauge theory to a point-like Yang-Mills instanton, note that, because of chiral symmetry breaking as detected by the gluino condensate, the local parameter at $P_1$ is not $u_1 = \exp\{i(Q'_1 C + i\Upsilon)\}$ but rather $u_1^{1/N}$). As earlier one gets $u_1 = \eta_2$ or with (5.1)

$$\beta(u_1^{1/N}) = \eta_1 \quad (5.4)$$

(again footn.'s 11, 13 apply). For $i > 1$ one has at the center of $X_{i,\Gamma} = R^4 \times S^3/\Gamma$ lying $S^3/\Gamma = Q'_i = \pm D'_i$ (this time $u_i$ is a good local parameter at $P_i$). $u_3 = \eta_1$ leads now to

$$\beta u_3 = \eta_3^{1/N} \quad (5.5)$$

Superpotential

Let us consider the ensuing superpotential evaluations (4.10) (again for the $A$-series). On $X_{i,\Gamma}$, where $i = 1$ or 3, one gets for $dW/d\log u_i$ now $\int_{S^3/\Gamma} C = \Im \log \eta_1$ and $\int_{S^3} C = \Im \log \eta_3$, respectively, and finds (after holomorphic completion) with (5.4), (5.5) ($k \in Z_N$)

$$W(u_1,k) = \int_0^{u_1} \log \beta(t^{1/N}) d\log t = NLi(\omega_N^k u_1^{1/N}) \quad (5.6)$$

$$W(u_3) = \int_0^{u_3} \log(\beta t)^N d\log t = NLi(u_3) \quad (5.7)$$

($u_i^{1/N}$ principal value, $\omega_N = e^{2\pi i/N}$). $u_1^{1/N} = \beta^2 u_3$ by $u_1 = \eta_2$, $u_3 = \eta_1$ and (5.1). By (4.29)

$$\sum_{k \in Z_N} W(u_{1,k}) = Li(u_1) \quad (5.8)$$

$$W(u_3) = NLi(u_3) \quad (5.9)$$

Now consider the four-dimensional interaction $\Im \int_{R^4} d^4y \int d^2\theta S \Phi$ with $S = tr W^\alpha W_\alpha$ the 'glueball' chiral superfield of lowest component the gaugino bilinear $tr \lambda^2$ and highest component $\int d^2\theta tr W^2 = F^2 + i F \wedge F$ ($W^\alpha$ the field-strength superfield of highest component $F + i*4F$) and $\Phi$ the superfield of lowest component $\Phi = \int_{Q} C + i\Upsilon$ and highest component $\int_{Q} *(X_7) G$ by (4.6). Just as the coupling constant in front of the kinetic term of the seven-dimensional gauge fields on $R^4 \times S^3$ gets rescaled by $\operatorname{vol}(S^3) = \Im \Phi$
in four dimensions, one has an interaction \( \int_{\mathbb{R}^4 \times S^3} tr F \wedge F \wedge C \) (gauge theory instantons carry membrane charge) so \( \text{Re} \Phi \) leads to the four-dimensional theta-angle \( \theta \). On the other hand one has the interaction \( \int_{\mathbb{R}^4 \times S^3} d^4 y tr \lambda^2 \wedge \ast_e (\nabla \gamma) G \) and as \( tr \lambda^2 \) gets a vev \( \sim \lambda^3 e^{i \Phi} \omega_N^k \) \( (k \in \mathbb{Z}_N \), the \( N \) different vacua from chiral symmetry breaking) one finds \( \) again (cf. (4.6)) the interaction proportional to \( \int_{S^3} *G \).

Taken together (with relative weight factor\(^{50} c \sim e^{-1/g^2_{YM} \Phi^0} \) with the membrane instanton contribution \( u \) one finds as superpotential \( W_{YM,\text{mem}}(\Phi) = S\Phi + N c u_{1/N}^1 \) of critical point \( S = -i N c u_{1/N}^1 \). This gives \( \Phi = -i \log(\frac{1}{Nc} S)^N \) and \( W_{\text{eff}}(S) = -i S \log(\frac{1}{Nc} S)^N + i N S \) of critical point \( (\frac{1}{Nc} S)^N = 1 \) or \( tr \ W^2 = -i N c \omega_N^k \). We will consider elsewhere the critical point and the effective superpotential for the full

\[
W_{YM,\text{mem}} = S\Phi + N c W(u_{1,k})
\]

Remark: There is another \( \mathbb{Z}_N \) relation (5.11) besides (5.8) (i.e. (4.29)) which would be interesting to relate with the \( \mathbb{Z}_N \) of \( SU(N) \) or to provide a gauge-theoretic meaning.

The expression \( R(z) = \frac{1}{2} \left( Li(z) - Li(1 - z) \right) + \frac{\pi^2}{12} \), cf. (B.15), is more suitable for expressing some \( Li \) relations. Actually one has a relation (cf. appendix, sect. B)

\[
\sum_{i=1}^{N-1} R \left( \frac{\sin^2 \frac{\pi}{N}}{\sin^2 \frac{i \pi}{N}} \right) = R(1) + \sum_{i=1}^{N-2} R \left( \frac{\sin^2 \frac{\pi i}{N}}{\sin^2 \frac{\pi (i+1)}{N}} \right) = \frac{\pi^2}{6} \left( 1 + \frac{3(N-2)}{N} \right)
\]

(5.11)

Here the argument is \( 1/Q_{0}^2 \) (where \( i + 1 = Sym^i 2 \) with action \( \text{diag}(z^i, z^{i-2}, \ldots, z^{-i}) \))

\[
Q_{0} = \frac{\sin(i+1) \pi}{\pi} = \frac{1 - \omega_{N}^{i+1}}{1 - \omega_{N}} = tr_{i+1} \omega_N
\]

(5.12)

with \( \mathbb{Z}_N \leftrightarrow Sl(2, \mathbb{C}) \) via \( \text{diag}(z, z^{-1}) \) for \( z = \omega_N \).

(5.11) is interpretable \(^{43} \) as the evaluation (cf. (6.27)) of a Cheeger Chern-Simons class on a generator of \( H_B(\mathbb{Z}_N, \mathbb{Z}) \): consider the embedding \( \mathbb{Z}_N \hookrightarrow P\text{Sl}(2, \mathbb{R}) \) given by

\[
\left( \begin{array}{cc}
\cos \frac{\pi}{N} & -\sin \frac{\pi}{N} \\
\sin \frac{\pi}{N} & \cos \frac{\pi}{N}
\end{array} \right)
\]

(5.13)

Furthermore one has a geometrical interpretation \(^{44} \) that \( \sum_{j=1}^{k} R(t_j) = \frac{\pi^2}{6} n \) for a certain integer \( n \) when a 3-manifold \( M \) is triangulated by \( k \) oriented tetrahedra \( T_j \). Here for each vertex \( i \) \( (i = 1, \ldots, N) \) a real number \( x_i \) is given and one has associated to the tetrahedron \( T_j \) the corresponding cross ratio \( t_j = cr\{x_a, x_b, x_c, x_d\} \). As the set of tetrahedra forms a triangulation the boundary \( \partial \sum_{j=1}^{k} T_j = 0 \) of the associated 3-chain is zero and this implies the relation \( \sum_{j=1}^{k} t_j \wedge (1 - t_j) = 0 \) which then implies (5.11). (Cf. app., sect. D)

\(^{50} \)by a shift \( \Phi \to \Phi + \Phi_0 \) the \( c \) can be identified with a shift in the bare coupling constant, so \( c \sim e^{\pm \Phi_0} \); further there is an order \( N^2 \) factor \(^{33} \), \(^{34} \)}
6 Interpretation and Outlook

The preceding investigations cause three sorts of questions. First, one may dwell on some of the points touched already: the identification of the relevant coordinates \((u_i, \eta_i)\) (cf. discussion around (3.7)), especially the globalization question with possibly a direct connection to the type IIA approach [14] which uses special flat coordinates of \(N = 2\); the question of holomorphic completion of the superpotential in \(\frac{\partial W}{\partial \log u} = \text{Im} \log \beta u \rightarrow \log \beta u\), respectively the holomorphy violation (by the boundary; cf. the \(E_2\) anomaly, even in superpotential contexts); the interpretation of the transformation rules of \(W\) (section of Heisenberg bundle, or even a balancing argument as in (4.17); \(W_G\) monodromy from the \(C\) field shifts); also to see directly, before evaluation, the connection \(X_7 \rightarrow \Delta(z)\) (and that a \(G\) flux (?) evaluates \(W_G\) on \(X_7\) to the (complexified, cf. below) invariant \(\text{vol}(\Delta)\)).

The second type of questions concerns an interpretation of the results obtained so far (sect. 6.1). Finally their possible extension to more generic (compact) cases and placement in a greater conceptual context (sketched in the more speculative sect. 6.2, described in more detail elsewhere [41]). (For relations to type IIA string theory cf. [35].)

6.1 Local interpretation

The universal object over the quantum moduli space

We now want to compare the structures found with corresponding constructions in the description of pure \(N = 2\) \(SU(2)\) gauge theory as given in the Seiberg/Witten (S/W) set-up [25]. There one was interested in the section \(\left(\frac{\partial_a a_D}{\partial_a a}\right) = \left(\int_\beta \partial_a \lambda \quad \int_\alpha \partial_a \lambda\right)\) of the flat bundle given by the first cohomology of the universal elliptic curve \(E \rightarrow \mathbb{P}^1_u\) over the quantum moduli space \(\mathbb{P}^1_u = \Gamma(2) \backslash \mathbb{H}_2\); the fibre over \(u\) is \(H^1(E_u, \mathbb{C})\) with the elliptic curve \(E_u\) given as two-fold covering of \(\mathbb{P}^1\) branched at \(\infty, 0, 1, u\) (degenerating for \(u\) being one of the three points \(\infty, 0, 1\); encircling the corresponding singularities gives monodromy elements generating \(\Gamma(2)\)).

Recall the relation\(^{51}\) of the (meromorphic) periods of the S/W curve \(C\) (describing a gauge theory engineered on a \(K3\)-fibered Calabi-Yau \(X\) in type IIA) and the (holomor-\(^{51}\) where \(C\) is built up as a covering over \(\mathbb{P}^1\) 'the same way' (replacing the intersection lattice \(H^2(K3, \mathbb{Z})\) by a zero-dimensional spectral set) that \(W\) is built up as \(K3\) fibration over \(\mathbb{P}^1\)

\(^{51}\)
phic) periods of the mirror Calabi-Yau $W$; one has with cycles $C_3 \subset W, C_1 \subset C$

\[ \int_{C_3} \Omega \sim \int_{C_1} \lambda \]  

(6.14)

Now replace this universal family of elliptic curves by a family of three-manifolds, or rather (in our local case) three-simplices: we associate to $z \in \mathbb{P}^1_z$ the hyperbolic geodesic three-simplex given by the ideal tetrahedron $\Delta(z)$ in $\mathbb{H}_3$ with vertices $\infty, 0, 1, z$; again the construction degenerates for $z$ being one of the three points $\infty, 0, 1$.

So the quantum regime of the universal local structure provided by the non-compact $M$-theory conifold $X_7$ (of quantum moduli space $\mathbb{P}^1_C$) corresponds to the variation of $\Delta(z)$ over $z \in \mathbb{P}^1_C = \partial \mathbb{H}_3$ (cf. vol $\Delta(z) = \mathcal{L}(z)$ in (4.45)).

We will compare to corresponding expressions in our set-up the pair $a, a_D$ and the Kähler potential (in the S/W set-up) as relations of dual torus periods ($\mathcal{F}$ prepotential)

\[ a_D = \partial \mathcal{F} / \partial a, \quad K = - \text{Im} a a_D \]  

(6.15)

\[ a = \oint_\alpha \lambda, \quad a_D = \oint_\beta \lambda \]  

(6.16)

The quantum coordinate is not $a^2$ but rather the corrected (cf. remark 3 below) quantity

\[ \frac{1}{2\pi i} u = \frac{1}{8} (\mathcal{F} - \frac{1}{2} a \partial_a \mathcal{F}) \]  

(6.17)

and in the stringy realization the quantum coordinate $u$ becomes purely geometrical\(^{58,53}\)

\[ u = \Xi^2 = \int_{C_3} \Omega \]  

(6.18)

which is made possible by going to the mirror description in type IIB.

With the modification $R$ of $Li$ (cf. (B.15)), whose relevance will emerge below repeatedly, we can compare to (6.17) as one has (so $a, \mathcal{F}, u$ are related to\(^{54}\) $i \Phi = \log z, Li, R$)

\[ R = Li - \frac{1}{2} \log z \partial_{\log z} Li \]  

(6.19)

---

\(^{52}\)think of a different copy of $\mathbb{H}_3$ over each point $z \in \mathbb{P}^1_z$ as ambient space for $\Delta(z)$ just as one thinks of a different copy of the Weierstrass embedding plane $\mathbb{P}^2_{x,y,z}$ over each $u \in \mathbb{P}^1_u$ as ambient space for $E_u$ so that in both cases one really ends up with a fibration (where the fibres are disjoint)

\(^{53}\)The prepotential $F(X^0, X^1, \ldots, X^n)$ of the periods $X^i$ is related to the prepotential $\mathcal{F}(t^1, \ldots, t^n)$ of the Kähler coordinates $t^A = X^A / X^0$ via $F = (X^0)^2 \mathcal{F}$, giving the relation $\frac{1}{4} (X^0)^{-1} \partial_{X^0} F = \mathcal{F} - \frac{1}{6} t^n \partial_{t^n} \mathcal{F}$. The period $a_D$ is related to the conifold via $\Xi_6 \sim a_D \sim x_+ \sim \bar{u} - 1$ with $\Xi_6$ the period related to the 6-cycle in type IIA, respectively the vanishing $S^3$ of the conifold in type IIB. The type IIA perspective on the conifold, the period $a_D$ and its relation to the dilogarithm are discussed further in [35].

\(^{54}\)we call here the $u$ modulus of (2.15) $z$ to avoid mix-up with the S/W $u$
So the distinctive feature of the S/W solution, that a quantity in the bulk of the quantum moduli space has a purely geometric expression like the mentioned periods (typical for string dualities), resembles the way how in our \( N = 1 \) set-up the quantum corrected quantity \( W(u) \) becomes an integral of classical geometry on a 'dual object' (lying over a respective point in the quantum moduli space), i.e. \( \mathcal{L} = \int_{\Delta} \text{vol} \) (respectively its complexified extensions described below which suggest the whole point of view).

**Remarks**

1) The membrane anomaly becomes manifest in the described global quantum model

\[
\sum_{\mathbb{Z}_3} \int_{S^3_{b_i}} C = \pi \leftrightarrow \sum_{\mathbb{Z}_3} \text{tors}(\gamma_i) = \pi \tag{6.20}
\]

(using \( S^3_Q \) for the \( S^3_D \) (phases)\(^{13} \)). This points to a connection\(^{55} \) between the direct manifestation (C.10) of the anomaly in the dual hyperbolic model and the proof (2.19).

2) The different *three-dimensional* structures we encounter (membrane instantons and the modulus \( \Phi = \int_Q C + i \Upsilon; \) the Heisenberg bundle \( \mathcal{H} \) as built up from \( Li, \) log and 1; the (solid) tetrahedron \( \Delta(z) \) and its volume\(^{56} \)) have corresponding two-dimensional structures in the S/W set-up (world-sheet instantons; \( H^1(E_u) \) as built up by \( (a_D, a); E_u \)).

3) In the S/W set-up there is also the relation with a flux superpotential which we contemplated for our case in subsect. 4.5. For this recall the stringy realization of the \( N = 2 \rightarrow N = 1 \) mass breaking. According to [5] the quantum corrected version \( W = mu = m < tr\Phi^2 > \) of the classical \( (u \approx a^2/2) \) mass deformation in the field theory\(^{57} \) is realised as a flux induced superpotential \( W = \int_W \Omega \wedge H_3 \) in the type IIB string, essentially because \( u \) occurs among the Calabi-Yau periods (cf. below). The stringy realization proposed in [5] of this scenario started from the type IIA superpotential

\[
W_{\text{flux}} \sim \int_X H_2 \wedge t \wedge t \sim (\int_{P^1_b} H_2) \cdot \text{vol}(K3) = n_{\text{flux}} \partial_s \mathcal{F} \tag{6.21}
\]

where\(^{58} \) the entries \( S, \partial_s \mathcal{F} \) of the type IIA period vector correspond to \( \text{vol}(P^1_b), \text{vol}(K3). \)

---

\(^{55}\)One may look at an anologue of \( P^2_H \rightarrow \Delta, \) a \( T^2 \cong U(1)^3/U(1)_D \) fibration \( P^2_I \rightarrow \Delta \) (cf. [36]).

\(^{56}\)or some hyperbolic 3-manifold \( M_3 \) with its volume and Chern-Simons form \( C^{CS}, \) cf. below

\(^{57}\)giving mass to the chiral multiplet \( \Phi \) of the vector multiplet, and so the breaking \( N = 2 \rightarrow N = 1. \)

As near \( \tilde{u} = \pm 1 \) a monopole and a dyon become massless one gets by including the light states \( W = mu + (a_D - a_0)\phi \phi \) which leads to monopole condensation and locking on \( \tilde{u} = \pm 1 \leftrightarrow a_D = a_0. \)

\(^{58}\)Actually analytic continuation shows that this expectation has to be refined [27]: \( W = mu \) is then given by \( W = mu \sim 2i\Xi_\infty^2 + \Xi_\infty^4 = 2it + \partial_s \mathcal{F}. \)
4) Having emphasized analogies between \( X_7 \) and the S/W set-up let us point also to a difference. In the S/W set-up at the three special points \( u = \infty, +1, -1 \) BPS states become massless, the \( W \)-boson, a monopole and a dyon, respectively. Being BPS states, in the string theory embedding the relation between mass and volume is saturated, the respective cycles of homology classes \( N, N^+, N^- \) shrink at the special points and fulfill

\[
N = N^+ + N^-
\]  

(6.22)

Note that not all of the three special points are on the same footing but some of them \( (u = \pm 1) \) are more equal (in the stringy representation in type II these correspond to \( S^3 \)'s in the mirror Calabi-Yau whereas \( u = \infty \) corresponds to \( S^2 \times S^1 \), indicating hypermultiplets and a vector multiplet, respectively. In the type II conifold transition the two small resolutions are also more equal, i.e. the type II reduction breaks \( \Sigma_3 \) to \( Z_2 \). These two points \( u = \pm 1 \) lie on the curve of marginal stability. The potential decay of BPS states when crossing such a curve were considered in investigations about singularities of special Lagrangian three-cycles [37] from the perspective of transitions that occur for corresponding supersymmetric three-cycles in the Calabi-Yau manifold. In [37] two different types of such singularities are considered, modelled (in \( C^3 \)) respectively on a \( T^2 \)-cone and two real 3-planes. The latter case was exemplified above in (6.22) and considered also in [38], [39] (and [2] when considering the cone over \( P^3 \)) whereas the former is related to the case considered in [14], [2] (the case of the cone over \( S^3 \times S^3 \)) and the present paper. Here the corresponding relation between the homology classes of the three respective cycles which become nullhomologous at the three special points is

\[
D_1 + D_2 + D_3 = 0
\]  

(6.23)

6.2 Global interpretation

Compact \( G_2 \) holonomy manifolds

Now consider compact \( G_2 \) holonomy manifolds \( X_7, K3 \) fibered over \( S^3 \) (replacing the previous local fibre \( K3^{decomp} = R^4 \), with singularities not just of codimension 7 (and potentially 4) but also codimension 6. The latter case where the discriminant in the base \( S^3 \) of \( X_7 \) is of codimension two, the 'discriminant link' \( l = \cup_j \gamma_j \) (a union of \( h \) circles), will be especially relevant to make the connection to the hyperbolic 3-manifold \( M_3 \).

More precisely, we will be concerned on the one hand with the case of a codimension 7 singularity of the classical geometry locally modelled after the cone over \( Y = S^3 \times S^3 \),
or even a situation with many, say $h_X$, local ‘ends’ modelled that way (having one $S^3$ as base of the $K3$ fibration brings a certain asymmetry into the description). On the other hand codimension 7 singularities arise as the cone over $P^3_C$, the $S^2$-twistor space over $S^4$. Consider here a component in the discriminant link given by just an unknot $\gamma$ and let $S \subset S^3$ be a spanning Seifert surface, not touching, say, the other link components. The total cycle traced out by following the cycle $S^2_{x,y,z}$ in the $K3$ fibre, which vanishes over $\gamma = \partial S$, through the whole $S = D^2_{v,w}$ leads to an $S^4 = \{x^2 + y^2 + z^2 + v^2 + w^2 = 1\}$ contributing to $b_4(X) = b_3(X)$.

Hyperbolic 3-manifolds

Above we considered the 3-dimensional structure given by the ideal tetrahedron $\Delta$ in $H_3$ and studied its volume. Actually one will consider a two-fold generalisation. One refines the volume invariant and generalises the tetrahedron to smooth manifolds. We recall an universal cohomological interpretation of the dilogarithm superpotential starting from the hyperbolic simplex volume computation related to its single-valued cousin $L$ which points to the consideration of the complexified Chern-Simons invariant of hyperbolic 3-manifolds (the volume combined with the Chern-Simons invariant).

Concerning the first issue one pairs the volume with the Chern-Simons invariant as suggested by the cohomological interpretation of the occurrence of $Li$ (or $R$) in the hyperbolic volume computations (cf. app. D and below) together with the Chern-Simons reformulation of three-dimensional gravity [40]. Concerning the second issue one will consider general hyperbolic 3-manifolds $M$ and the way the refined (complexified) Chern-Simons invariant varies over the hyperbolic deformation moduli space of $M$; this shows [41] (from a triangulation by simplices) how the dilogarithm occurs in this variation.

Concerning the generalisation to smooth manifolds note that just as in the case of the upper half-plane one can study now discrete torsion-free subgroups $\Gamma$ of the full group $PSl(2, \mathbb{C})$ of orientation-preserving isometries of $H_3$ and look for the corresponding (orientable) hyperbolic three-manifold $M$ (complete Riemannian manifold of constant curvature $-1$ of finite volume) given by the non-compact quotient $\Gamma \backslash H_3$ (any such $M$ arises this way)\(^60\). The geodesic simplices we studied occur, just as in the well known

\(^{59}\)Fibre singularities relate to the cohomology of a total space: for an elliptic $K3$ for example one gets $S^2$’s building up $H^2(K3)$ (besides base and fibre) from paths $P$ connecting points $p, p'$ (of codimension 2 in the base as is our link $l \subset S^3$) in the base $P^1$ (so $\partial P = p' - p$) over which an $S^1$ in the fibre shrinks.

\(^{60}\)Cf. that a closed surface of genus $g > 1$ admits a metric of constant curvature $-1$ and is isometric to $\Gamma \backslash H_2$. Note that by the Mostow/Prasad rigidity theorem two hyperbolic threefolds of finite volume
upper half-plane case, as (parts of) fundamental domains for suitable group actions and the sum of their volumes gives the volume of the quotient manifold.

Cohomological interpretation
For $E \to M$ a differentiable $\text{Gl}(n, \mathbb{C})$ bundle with flat connection $\theta$ one finds from the Bockstein exact sequence that $c_2(E)$ lies in the image of the Bockstein homomorphism $\beta$

$$H^3(M, \mathbb{C}/\mathbb{Z}) \xrightarrow{\beta} H^4(M, \mathbb{Z}) \rightarrow H^4(M, \mathbb{C}) \quad (6.24)$$

The second Cheeger Chern-Simons class (app. D) gives a canonical choice of a preimage

$$\hat{C}_2(\theta) \in H^3(M, \mathbb{C}/\mathbb{Z}) \quad (6.25)$$

With $\omega$ a $\mathbb{C}$-valued $\text{Sl}(2, \mathbb{C})$ invariant three-form on $\text{Sl}(2, \mathbb{C})/SU(2) = H_3$ one finds a $\mathbb{C}/\mathbb{Z}$ valued Eilenberg-MacLane cochain $I(\omega)$ with

$$\hat{C}_2 = I(\omega)(g_1, g_2, g_3) = \int_{\Delta(z)} \omega \quad (6.26)$$

One finds then\(^{61}\) (with $L(z) = R(z) - \frac{\pi^2}{6}$, cf. (B.15))

$$2\text{Re } \hat{C}_2 = \frac{1}{2\pi^2} L(z) \quad (\text{mod } 1/24), \quad 2\text{Im } \hat{C}_2 = \frac{1}{2\pi^2} \mathcal{L}(z) \quad (6.27)$$

Transition to hyperbolic 3-manifolds
Let $M$ now be a closed 3-manifold of hyperbolic structure given by $M \cong \Gamma_{\text{hol}} \backslash H_3$ or equivalently by the holonomy representation $h : \pi_1(M) \to (P)\text{Sl}(2, \mathbb{C})$, respectively by a flat $(P)\text{Sl}(2, \mathbb{C})$ bundle over $M$; this is pulled back from the universal bundle $U$ over the classifying space $BS\text{Sl}(2, \mathbb{C})^\delta$ by a base map $m : M \to B\text{Sl}(2, \mathbb{C})^\delta$ so one can evaluate\(^{62}\)

$$\hat{C}_2 \in H^3(B\text{Sl}(2, \mathbb{C})^\delta) \quad (\text{cf. app. D})$$

on the class in $H_3(B\text{Sl}(2, \mathbb{C})^\delta)$ given by $M$

$$\text{Re } \hat{C}_2(M) \sim CS(M), \quad \text{Im } \hat{C}_2(M) \sim \text{vol}(M) \quad (6.28)$$

So the proper cohomological interpretation of $\text{vol} \Delta(z) = \mathcal{L}(z)$ leads to the consideration of hyperbolic 3-manifolds $M$ for which the second Cheeger Chern-Simons class is given by (6.28) with universal evaluation (6.27). For a hyperbolic 3-manifold $M$ the invariant $\int_{M_3} C^{CS} + i \text{vol}$ is studied (in this complex Chern-Simons theory (as in 3D gravity) one has naturally the complex pairing of volume and the $CS$ 3-form field, cf. [41]).

---

\(^{61}\) with $\text{Re } \hat{C}_2$ evaluated on $H_3(\text{Sl}(2, \mathbb{R})^\delta)$ for $z \in \mathbb{R}$. One can give a similar interpretation for $\text{Re } \log$ and its relation to $\hat{C}_1 \in H^1(\text{Gl}(\mathbb{C}), \mathbb{R})$ just as $\mathcal{L}$ represents part of $\hat{C}_2 \in H^3(\text{Gl}(\mathbb{C}), \mathbb{R})$.

\(^{62}\) for $M$ closed $CS(M)$ is essentially the $\eta$ invariant, this is suitably extended for $M$ non-compact
The hyperbolic deformation moduli space is defined via periods of the generator loops \( m_i, l_i \) (for \( h \) a suitable one-form) for the (assumed) toroidal ends

\[
v_i(u) = \partial G / \partial u_i \quad , \quad K(u) := 2\pi \sum_i \text{length}(\gamma_i) = -\sum_i \text{Im}u_i \bar{v}_i \tag{6.29}
\]

\[
u_i = \pm \int_{m_i} 2h^* \quad , \quad v_i = \pm \int_{l_i} 2h^* \tag{6.30}
\]

In other words there exists again a prepotential \( G \) and again the expression

\[
f = \frac{1}{4}(2 - u \partial_u) G \tag{6.31}
\]

has a purely geometrical description (Dehn filling the ends via solid tori will be involved)

\[
f = \text{vol}(M) + iC^{CS}(M) = \int_{M} \omega \tag{6.32}
\]

(6.29)-(6.32) compare to (6.15)-(6.18) \((a, F, u \text{ relate to } u, G, f)\). One defines invariants

\[
I(M) = \exp\{\int_{M} \frac{2}{\pi} \text{vol} + iC^{CS}\} = \exp\{\int_{M} \omega\}
\]

\[
\lambda(\gamma) = \exp\{\text{length}(\gamma) + i \text{tors}(\gamma)\} = \exp\{\int_\gamma 2h^*\} \tag{6.33}
\]

generalising the occurrence of \( Li \) and \( \log \), or their real cousins \( \mathcal{L} \) and \( \text{Re} \log \) as three- resp. one-dimensional volumes (4.45), (C.6) in \( \Delta(z) \). So one has corresponding triples

\[
\begin{pmatrix}
Li(y) \\
\log y \\
1
\end{pmatrix},
\begin{pmatrix}
\mathcal{C}_2 \\
\mathcal{C}_1 \\
1
\end{pmatrix},
\begin{pmatrix}
I(M_3) \\
\lambda(\gamma_1) \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mathcal{L} \\
\text{Re} \log \\
1
\end{pmatrix},
\begin{pmatrix}
\text{vol}(\Delta) \\
\text{length}(\gamma) \\
1
\end{pmatrix} \tag{6.34}
\]

**Interpretation**

Now having generalised our local \( G_2 \) holonomy manifold \( X_7 \) to a global manifold (compact and \( K3 \) fibered over \( S^3 \)) and furthermore having generalised the hyperbolic geometry of the simplex \( \Delta(z) \) to smooth hyperbolic 3-manifolds let us indicate a potential connection. The quantum expression comprising all the corrections may have again a purely geometric description (when going to the dual description provided by the hyperbolic 3-manifold \( M \), a 'thinned out' (spectral) version of the dual 7-fold just as the S/W curve is the \( K3 \)-integrated-out version of the mirror CY). In the dual evaluation the membrane instanton superpotential \( W = Li(z) \), generalising \( \mathcal{L} \), occurs as a complexified volume of a simplex in hyperbolic 3-space (respectively of a 3-manifold). So just as the periods of the S/W curve were periods of the type IIB Calabi-Yau (mirror to the original theory in type IIA) now the dual 3-manifold \( M_3^u \) (analogue of the S/W curve \( E_u \)) and its
'period' \( f(u) = \int_{M_3^u} \text{vol} + iC^{CS} \) (evaluated in the local case by \( R(e^u) \), i.e. essentially by \( Li(e^u) \)) reflects a \((W_G ?) \) 'period' (evaluated locally by \( W \sim Li(e^\phi) \)).

Supersymmetric (associative) \( S^3 \)'s, which sit in \( X \) locally like in \( S^3 \times \mathbb{R}^4 \), contribute to \( H^3(X) \); in the dual 3-manifold \( M_3 \) the hyperbolic moduli space has dimension \( h \), the number of ends, i.e. the number of link components in the description of the discriminant of a \( K3 \) fibration \( X \to S^3 \) (responsible for the codimension 6 singularities); first these numbers and then the moduli of \( X_7 \) and \( M_3 \) have to be brought into relation for the dual description. Relating the moduli spaces \( \text{cf. (6.14)} \) might include relations

\[
\log u = \int_{S^3_{\text{contractible}}} -\Upsilon + iC \quad \longleftrightarrow \quad \log \lambda = \int_{\gamma} \text{length} + i \text{tors}
\]

(6.35)

\( \frac{\partial W}{\partial \log u_k} = \log v_k \quad \longleftrightarrow \quad \frac{\partial G}{\partial u_i} = v_i \)

(the coordinate \( u = \exp \{ i \int_Q C + i \Upsilon \} \) for the non-contractible \( S^3 \) (and \( \Phi_k = \log u \)) are replaced with \( \lambda = \exp \{ \text{length}(\gamma) + i \text{tors}(\gamma) \} \) for the non-contractible \( S^1 \) (and \( u = \log \lambda \)).

The base part \( S^3 \) \( - l \), over which the fibre is non-degenerate, is a 3-manifold \( M_3 \). Concerning the moduli spaces we want to compare note that for the superpotential we are interested in the number \( h_X \) of these \( S^3 \) (related to codimension 7 singularities \( C(S^3 \times S^3) \)) whereas in the description of the hyperbolic 3-manifold and the complexified Chern-Simons invariant we are interested in the number \( h = h_M \) of ends of \( M_3 \) (or components of the discriminant link describing codimension 6 singularities, related to \( S^4 \) or the deformed \( C(P^3_\mathbb{C}) \)). The 3-manifold \( M_3 \) having three representations (the quotient \( \Gamma \backslash H_3 \), a triangulation \( M_3 = \bigcup_k \Delta(z_k) \) and the link complement \( S^3 \) \( - l \) of \( l = \bigcup_j \gamma_j \)) one now connects its second and third representation: the question of translating the different dimensions of moduli spaces is then captured by the reshuffling of the different summation boundaries in \( K = \sum_i^h (\text{length} + i \text{tors})(\gamma_i) \) and \( f = \sum_k R(z_k) \) (where \( R \) is essentially \( Li \) and for \( X_7 \), very naively, \( W \approx \sum_j^{h_X} Li(u_j) \)) inherent in (6.29)-(6.32) [41].

The comparison (considered in more detail elsewhere [41]) will describe the actual form which the analogy between a local description of a singularity of a \( G_2 \) manifold by \( X_7 = \mathbb{R}^4 \times S^3 \) and the Dehn filling of an end of a hyperbolic 3-manifold with the solid torus \( \mathcal{T} = \mathbb{D}^2 \times S^1 \) takes, i.e. the mapping between the moduli spaces of \( X_7 \) and \( M_3 \) (including the prepotential of hyperbolic deformation space) and the connection between the membrane instanton superpotential and (possibly \( G \)-flux superpotentials resp.) the complex \( CS \) theory on \( M_3 \).

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Appendix

A  The triality symmetry group on the moduli space

$\Sigma_3$, the permutation group of three elements, is built up from $\mathbb{Z}_2$ and an invariant subgroup $\mathbb{Z}_3 = \{e, \beta, \beta^2\}$; one has the relation $\alpha \beta^i \alpha = \beta^{-i}$ (read $i$ mod 3).

$$1 \rightarrow \mathbb{Z}_3 \rightarrow \Sigma_3 \rightarrow \mathbb{Z}_2 \rightarrow 1 \quad \text{(A.1)}$$

The non-trivial coset consists of the three order two elements $\alpha, \alpha \beta, \alpha \beta^2$

$$
\begin{array}{ccc}
\alpha & \beta & \beta^2 \\
\alpha & \alpha \beta & \alpha \beta^2
\end{array}
\quad \text{(A.2)}
$$

There are three conjugacy classes (CC) given by the elements of order one, two and three, respectively. We denote a conjugacy class by $c$, and the number of its elements by $n_c$. We index the classes by the common order of its elements, so $n_{c_1} = 1, n_{c_2} = 3, n_{c_3} = 2$.

Some representations are: the trivial representation $1 = \text{triv}$; the sign character

$$\text{sign} : \Sigma_3 \rightarrow \Sigma_3/\mathbb{Z}_3 = \mathbb{Z}_2 = \{\pm 1\} \quad \text{(A.3)}$$

and the fundamental representation $3 = \text{fund}$ induced by permutation of $\{1, 2, 3\}$. In general, for a representation $R$, one has the following projection operators: first the 'invariant projector' $P^+(v) = \sum_{\gamma \in \Sigma_3} \gamma v$ ($v$ in the representation space of $R$) which gives an $\Sigma_3$ invariant element; and analogously the 'anti-invariant projection'

$$P^-(v) = \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma) \gamma v \quad \text{(A.4)}$$

which transforms with the sign character (both may be normalized by $1/6$).

Note that the representation $3$ is not irreducible. Think of it in real three-space to see the invariant (Euler) axis $\sum_i e_i$ and the $2\pi/3$ rotation in the orthogonal ('barycentric') plane. So it decomposes into a sum of the trivial representation and a two-dimensional irreducible representation, called $2$ (we will also denote $-1 = \text{sign}$ and $-k = -1 \otimes k$)

$$3 = 2 \oplus 1 \quad \text{(A.5)}$$

Let us denote the degree and character of a representation $d$ by $\deg d$ and $\chi d$, respectively. The representations $1, -1, 2$ exhaust the irreducible representations as $3 = \sharp \text{ CC or}$

$$|\Sigma_3| = \deg_1^2 + \deg_{-1}^2 + \deg_2^2 \quad \text{(A.6)}$$

At this point it might be appropriate to give the character table
Let us furthermore point to the following facts which we will use later

\[
\begin{array}{ccc}
2 \otimes 2 & \cong & 2 \oplus 1 \oplus -1 \\
\Lambda^2 3 & \cong & 2 \oplus -1 \\
\text{Sym}^2 3 & \cong & 2 \oplus 2 \oplus 1 \oplus 1
\end{array}
\] (A.7) (A.8) (A.9)

For (A.7) note\(^{63}\) that the multiplicities \(m_d = 1\) of each of our three building blocks \(1, -1, 2\) occurring as isotypic components \(d\) in \(2 \otimes 2\) follow from the character relations

\[
m_d = \frac{1}{|\Sigma_3|} \sum_{\gamma \in \Sigma_3} \chi_{2 \otimes 2}(\gamma) \chi_d(\gamma) = \frac{1}{|\Sigma_3|} \sum_{c \in C} n_c \chi_{2 \otimes 2}(c) \chi_d(c)
\] (A.10)

(A.8) follows by inspection\(^{64}\) and (A.9) from\(^{65}\) \(\text{Sym}^2 3 \cong (3 \otimes 3)/\Lambda^2 3\).

Note also that if a system \((z_i)_{i \in \mathbb{Z}_3}\) spans a \(-3\), i.e. \(\alpha z_i = -z_{\alpha i}\), then the system of \(w_i := z_{i+1} - z_{i-1} = \beta^2 z_i - \beta z_i\) spans a \(2\) (by \(\sum_i w_i = 0\) and \(\alpha w_i = w_{\alpha i}\) from \(\alpha \beta^2 = \beta \alpha\))

\[\bigoplus_i z_i C \cong -3 \implies \sum_i w_i C \cong 2\] (A.11)

The \(f_i\) and \(\alpha_i\) transform essentially (shifted by \(\alpha \rightarrow \alpha \beta^2\)) under \(3\) and \(-3\); then (A.11) leads to the introduction of the (log) \(y_i\) (similar to the relation of the (log) \(\eta_i\) to the (log) \(y_i\)).

**Some representation theory for \(\Sigma_3\) acting on \(\mathbb{P}^1\)**

For a \(\Sigma_3\) action on \(\mathbb{P}^1_C\) consider the induced operation on functions\(^{66}\) \(\Lambda^0 \mathbb{P}^1\) on \(\mathbb{P}^1\). Consider now a \(\mathbb{Z}_3\)-orbit of an \(\mathbb{Z}_2\)-anti-invariant function \(f\), i.e. of a function with \(f(\alpha z) = -f(z)\), like the logarithm (here \(\alpha z = 1/z\) like for the \(Sl_2\) action). One has

\[\bigoplus_{i \in \mathbb{Z}_3} f(\beta_i \cdot) C \subset \Lambda^0 \mathbb{P}^1(-3), \text{ so}
\]

\[\bigoplus_i \log \beta^i z C \cong -3\] (A.12)

\(^{63}\)or: \(2 \otimes 2\) is represented by the span of \(e_i \otimes e_j\) with \(i = 1, 2; j = 1, 2\); clearly the diagonal provides a \(2\); the \(+1\) and \(-1\) are spanned by \(e_1 \otimes e_1 + e_2 \otimes e_2 + e_1 \otimes e_2 - e_2 \otimes e_1\), respectively.

\(^{64}\)or: \(\Lambda^2 3 = \bigoplus_{i \in \mathbb{Z}_3} e_i \wedge e_{i+1} C\) the split (A.5) leads now to the anti-invariant line \(\sum_{i \in \mathbb{Z}_3} e_i \wedge e_{i+1} C\)

\(^{65}\)or: among the \(e_i \cdot e_j\) \((i \leq j)\) of \(\text{Sym}^2 3\) the diagonal and the \(i < j\) part span each a \(3\)

\(^{66}\)functions are considered for now just formally, regardless of poles or the question of single-valuedness.
Now (3.2) implies for some $z := \eta_i$

$$P^+ \log z = \sum_{i \in \mathbb{Z}_3} \log \beta^i z = \pm \pi i$$  \hfill (A.13)

$$P^- \log z = \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma) \log \gamma z = \pm 2\pi i$$  \hfill (A.14)

$$\sum_{i \in \mathbb{Z}_3} \log (-1)^{\delta_{ij}} \beta^i z = 0 \quad (j \in \mathbb{Z}_3)$$  \hfill (A.15)

$$\sum_{i \in \mathbb{Z}_3} d \log \beta^i z = 0$$  \hfill (A.16)

$$\sum_{i \in \mathbb{Z}_3} \text{Re} \log \beta^i z = 0$$  \hfill (A.17)

One has the exact sequence (by (A.13) the left term are the constants (= ker $d$) in the middle term)

$$0 \rightarrow (\sum_i \log \beta^i z) C \rightarrow \oplus_i \log \beta^i z C \xrightarrow{d} \sum_i (d \log \beta^i z C) \rightarrow 0$$  \hfill (A.18)

$$0 \rightarrow -1 \rightarrow -3 \rightarrow -2 \rightarrow 0$$

Similarly one has an interpretation of the real vector spaces with $\Sigma_3$ action (note (A.17))

$$\oplus_i \text{Im} \log \beta^i z \mathbb{R} \cong -3$$  \hfill (A.19)

$$\sum_i (\text{Re} \log \beta^i z \mathbb{R}) \cong -2$$  \hfill (A.20)

### A.1 Anti-invariance of $\mathcal{L}$: first argument

We show how $dLi = \log \beta z d \log z$ and $\text{Im} \log \beta z \text{Re} \log z$ behave under $e - \text{sign}(\gamma)\gamma$. The complete parallelism shows (cf. footn. 68) that $(\text{Im} \int dLi) - \text{Im} \log \beta z \text{Re} \log z$ vanishes for all $e - \text{sign}(\gamma)\gamma$ transformations, i.e. the anti-invariant transformation behaviour.

$$\beta \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ \log \beta z & \log \beta^2 z & \log \beta z \\ \log z & \log \beta z & \log \beta^2 z \end{pmatrix}: \quad \text{(the last equalities from (A.13))}$$

$$(e - \beta) dLi = \log \beta z d \log z - \log \beta^2 z d \log \beta z$$

$$= d (\log \beta z \cdot \log z) - \log z d \log \beta z - \log \beta^2 z d \log \beta z$$

$$= d (\log \beta z \cdot \log z) + \frac{1}{2} d \log^2 \beta z \mp \pi i d \log \beta z$$

$$(e - \beta) \text{Im} \log \beta z \text{Re} \log z = \text{Im} \log \beta z \text{Re} \log z - \text{Im} \log \beta^2 z \text{Re} \log \beta z$$

$$= \text{Im}(\log \beta z \cdot \log z) - \text{Im} \log z \text{Re} \log \beta z - \text{Im} \log \beta^2 z \text{Re} \log \beta z$$

$$= \text{Im}(\log \beta z \cdot \log z) + \frac{1}{2} \text{Im} \log^2 \beta z \mp \pi \text{Im} \log \beta z$$
\[ \beta^2 \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ \log \beta^2 z & \log z & \log \beta z \end{pmatrix} : \text{(the last equalities from (A.16))} \]

\[ (e - \beta^2) dL_i = \log \beta z \, d \log z - \log z \, d \log \beta^2 z \]

\[ = d (\log \beta z \cdot \log z) - \log z \, d \log \beta z - \log z \, d \log \beta^2 z \]

\[ = d (\log \beta z \cdot \log z) + \frac{1}{2} d \log^2 z \]

\[ (e - \beta^2) \text{Im} \log \beta z \, \text{Re} \log z = \text{Im} \log \beta z \, \text{Re} \log z - \text{Im} \log z \, \text{Re} \log \beta^2 z \]

\[ = \text{Im}(\log \beta z \cdot \log z) - \text{Im} \log z \, \text{Re} \log \beta z - \text{Im} \log z \, \text{Re} \log \beta^2 z \]

\[ = \text{Im}(\log \beta z \cdot \log z) + \frac{1}{2} \text{Im} \log^2 z \]

\[ \alpha \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log z & -\log \beta^2 z & -\log \beta z \end{pmatrix} : \text{(the last equalities from (A.13))} \]

\[ (e + \alpha) dL_i = \log \beta z \, d \log z + \log \beta^2 z \, d \log z \]

\[ = -\frac{1}{2} d \log^2 z \pm \pi id \log z \]

\[ (e + \alpha) \text{Im} \log \beta z \, \text{Re} \log z = \text{Im} \log \beta z \, \text{Re} \log z + \text{Im} \log \beta^2 z \, \text{Re} \log z \]

\[ = -\frac{1}{2} \text{Im} \log^2 z \pm \pi \text{Re} \log z \]

\[ \alpha \beta \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log \beta^2 z & -\log \beta z & -\log z \end{pmatrix} : \]

\[ (e + \alpha \beta) dL_i = \log \beta z \, d \log z + \log z \, d \log \beta z \]

\[ = d (\log \beta z \cdot \log z) \]

\[ (e + \alpha \beta) \text{Im} \log \beta z \, \text{Re} \log z = \text{Im} \log \beta z \, \text{Re} \log z + \text{Im} \log z \, \text{Re} \log \beta z \]

\[ = \text{Im}(\log \beta z \cdot \log z) \]

\[ \alpha \beta^2 \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log \beta^2 z & -\log \beta z & -\log z \end{pmatrix} : \text{(the last equalities from (A.16) and (A.17))} \]

\[ (e + \alpha \beta^2) dL_i = \log \beta z \, d \log z + \log \beta z \, d \log \beta^2 z \]

\[ = -\frac{1}{2} d (\log^2 \beta z) \]

\[ (e + \alpha \beta^2) \text{Im} \log \beta z \, \text{Re} \log z = \text{Im} \log \beta z \, \text{Re} \log z + \text{Im} \log \beta z \, \text{Re} \log \beta^2 z \]

\[ = -\frac{1}{2} \text{Im} (\log^2 \beta z) \]
A.2 Anti-invariance of $\mathcal{L}$: second argument

To investigate the potential anti-invariant transformation behaviour of $\mathcal{L}(z)$ let us take up now our representation-theoretic considerations from section 3. The proper reason for the anti-invariance of $\mathcal{L} = \text{Im} Li(z) - \text{Im} \log \beta z \Re \log z$ is the following fact: when one operates on bilinear product expressions like $\log \beta^i z \log \beta^j z$ with either $d$ or Im one finds as image elements in their respective target spaces (of expressions $\log \beta^i z d \log \beta^j z$ and $\text{Im} \log \beta^i z \Re \log \beta^j z$) just the symmetric combinations by reason of the (pseudo-)derivative nature of these operations

\[
d(f \cdot g) = df \cdot g + f \cdot dg
\]
\[
\text{Im}(fg) = \text{Im}f \Re g + \Re f \text{Im} g
\] (A.21)

This is for $Li = \int \log \beta z d \log z$ an indication that the integral can not be done elementary (the integrand is not symmetric, thereby not naturally a derivative of the presumptive candidate functions, cf. footn. 71). Now both terms, whose imaginary parts add up to $\mathcal{L}(z)$, i.e. $\int \log \beta z d \log z$ and $\psi = \log \beta z \Re \log z$ (or equally well $i \text{Im} \log \beta z \log z$), constitute the one missing piece which, when linearly combined with the elementary expressions $\log \beta^i z \log \beta^j z$, gives after application of $d$ and Im respectively not just the symmetric elements (im $d$ and im Im in (A.23) and (A.26), respectively) of their natural target space but all elements; furthermore both of these missing ‘non-symmetric’ elements are built from the same underlying element\(^{67}\) $\log \beta z \otimes \log z$

\[
d Li = \log \beta z d \log z
\]
\[
\text{Im} \psi = \text{Im} \log \beta z \Re \log z
\] (A.22)

We will see in a moment that the non-vanishing elements in the respective one-dimensional quotient space (target space modulo image) they generate transform with the sign character (A.24), (A.27). The common origin of the non-trivial terms and the complete parallelism of the (pseudo-)derivative operations mentioned above shows then that the classes, when lifted back to the proper elements, acquire exactly the same correction terms which gives finally the anti-invariance of their difference.\(^{68}\)

\(^{67}\) of the tensor product which one has to take, instead of the symmetric product used above in the elementary expressions, to be able to apply $d$ or Im to individual factors

\(^{68}\) when going back and forth in $\text{Im} \circ d^{-1}$ the interrelations are kept, i.e. the integration constants are real (actually rational multiples of $(\pi i)^2$) (note that $\ker d = \mathbb{C}$, $\ker \text{Im} = \mathbb{R}$ on holomorphic functions)
Now consider the following exact diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Sym}^2\text{fund} & \rightarrow & \text{Sym}^2\text{fund} \oplus Li C & \rightarrow & Li C & \rightarrow & 0 \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \rightarrow & \text{im } d & \rightarrow & \oplus_{i,j} \log \beta^i z \log \beta^j z C & \rightarrow & [d Li]C & \rightarrow & 0
\end{array}
\]

(A.23)

\[\text{im } d \cong 2 \oplus 1 \oplus 2 \quad \text{(A.25)}\]

¿From consideration of the lower horizontal exact sequence one finds

\[ [d Li]C \cong -1 \quad \text{(A.24)} \]

For, by (A.18), the space of elementary bilinear expressions \( \log \beta^i z \log \beta^j z \) gives a \( \text{Sym}^2(-3) = \text{Sym}^2 3 \); concerning \( \text{im } d \) note that the kernel of constants in the vertical short exact sequence is, by (A.13), \( (\sum_i \log \beta^i z)^2 C \cong \text{Sym}^2(-1) = 1 \); so by (A.9)

\[ \text{im } d \cong 2 \oplus 1 \oplus 2 \quad \text{(A.25)} \]

The middle term in the lower sequence, is given by\(^{69}\) \(-3 \otimes -2 = 3 \otimes 2 = 2 \otimes 2 \oplus 2\). Thereby \([d Li]C \cong (-3 \otimes -2) / \text{im } d = (2 \otimes 2 \oplus 2) / (2 \oplus 1 \oplus 2)\); (A.7) now implies (A.24).

Now consider the following companion exact diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Sym}^2\text{fund} & \rightarrow & \text{Sym}^2\text{fund} \oplus \psi C & \rightarrow & \psi C & \rightarrow & 0 \\
\downarrow \text{Im} & & \downarrow \text{Im} & & \downarrow \text{Im} & & \downarrow \text{Im} & & \\
0 & \rightarrow & \text{im Im} & \rightarrow & \oplus_{i,j} \text{Im } \log \beta^i z \text{Re } \log \beta^j z R & \rightarrow & [\text{Im } \psi] R & \rightarrow & 0
\end{array}
\]

(A.26)

By completely parallel arguments, where now the second embodiment of \(-2\) in (A.20) replaces the first one (A.18) used before, here too one finds (as representations over \( R \))

\[ [\text{Im } \psi] R \cong -1 \quad \text{(A.27)} \]

\(^{69}\text{the tensor (instead of the symmetric) product applies as the symmetry between the factors is broken}\)
A.3 Formal anti-invariance of a modified superpotential

To understand the anti-invariance property of $\tilde{W}$ in (4.19) one would like to see the corresponding symmetry becoming manifest. This can be done on the derivative level: $dW/dz$ becomes an elementary logarithmic function just as the correction terms $dC/dz$, with the only difference that $C$ (in contrast to $W$) is already itself an elementary function. To avoid an additional transformation factor $d(\gamma z)/dz$ obscuring the transformation properties, we actually consider the one-form $d\tilde{W}$ which again transforms with the sign character: i.e. we are using the $\Sigma_3$ equivariant map $d: \Lambda^0 C_z \rightarrow \Lambda^1 C_z$.

One finds (cf. (A.32)) for $d\tilde{W}$ the manifestly anti-invariant expression

$$6d\tilde{W} = \sum_{i\in Z_3} \log \frac{\beta^{i+1} z}{\beta^{i-1} z} d \log \beta^i z \quad (= \sum_{i\in Z_3} d \log \frac{\beta^{i-1} z}{\beta^{i+1} z} \log \beta^i z)$$  \hspace{1cm} (A.28)

The first of the six terms is just the original term we started with

$$dW = \log \beta u \ d \log u$$ \hspace{1cm} (A.29)

\(\text{From (A.29) one can read off directly that } W \text{ is not anti-invariant (compare to (A.28)).}\)

Note that our solution (A.28) is actually indeed of the form (4.20) (by integration), i.e.

$$6d\tilde{W}(\cdot) = \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)(dW)(\gamma \cdot) = \sum_{\gamma \in \Sigma_3} \text{sign}(\gamma)d(W(\gamma \cdot))$$  \hspace{1cm} (A.30)

Matching to (A.28) is obvious for $\gamma \in Z_3$ by (A.29) which also gives $(dW)(\alpha z) = \log \beta^2 z \ d \log z$. (A non-zero integration constant would violate $\tilde{W}(\gamma \cdot) = \text{sign}(\gamma)\tilde{W}(\cdot)$.)

Now, to prove (A.28), one finds from (4.21)

$$6dC = -2 \log \beta z \ d \log z + 2 \log \beta^2 z \ d \log \beta z$$

\[-2 \log \beta z \ d \log \beta z + 3 \log \beta z \ d \log \beta z + 2 \log \beta^2 z \ d \log \beta z \]

\[+ 2 \log \beta z \ d \log \beta^2 z + \log \beta^2 z \ d \log \beta^2 z \quad (A.31)\]

Combining with (A.29) one gets\(^71\) (A.28) after a regrouping in the ‘verticals’ via (A.16)

$$6d\tilde{W} = + \log \beta z \ d \log z - \log \beta^2 z \ d \log z$$

\[- \log \beta z \ d \log \beta z + \log \beta^2 z \ d \log \beta z \]

\[+ \log \beta z \ d \log \beta^2 z - \log \beta z \ d \log \beta^2 z \quad (A.32)\]

\(^70\text{meaning } (df)(\gamma \cdot) = (\frac{df}{dz})(\gamma \cdot) = \frac{df}{dz}(\gamma \cdot) d(\gamma \cdot) = d(f(\gamma \cdot))\)

\(^71\text{Note the anti-symmetry of the coefficient matrix which guarantees the non-triviality of the expression (of course, being non-symmetric is enough; if one starts form an elementary expression } F = \sum_{i,j} a_{i,j} \log \beta^i z \log \beta^j z \text{ one ends up with a symmetric quantity } dF = \sum_{i,j} (a_{i,j} + a_{j,i}) \log \beta^i z d \log \beta^j z.\)

The missing symmetry did account already for the non-triviality of the original term (A.29); cf. (A.23)).
B  The monodromy representation

The polylogarithms are (with $Li(x) := Li_2(x), Li_1(x) = \log \beta x, Li_0(x) = x \cdot \beta x$)

$$Li_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}, \quad \frac{d}{dx} Li_{k+1}(e^x) = Li_k(e^x) \tag{B.1}$$

To express the multi-valuedness of $W = Li_2$ define the matrix differential form [18]

$$\Omega = \begin{pmatrix} 0 & d \log \beta z & 0 \\ 0 & 0 & d \log z \\ 0 & 0 & 0 \end{pmatrix} \tag{B.2}$$

The one-forms $\omega_i = d \log \beta^i z$ are related with the loops $l_i$ by $\frac{1}{2\pi i} \int_{l_i} \omega_k = \delta_{jk}$. Now consider for a (multi-valued) function $F : P^1 \setminus \{0,1,\infty\} \to gl(3, \mathbb{C})$ the matrix differential equation

$$dF = F \cdot \Omega \tag{B.3}$$

A fundamental solution is provided by the principal branch (on $|z - 1/2| < 1/2$) of

$$L(z) = \begin{pmatrix} 1 & \log \beta z & Li(z) \\ 0 & 1 & \log z \\ 0 & 0 & 1 \end{pmatrix} \tag{B.4}$$

Analytic continuation of the principal branch of $L(z)$ about a loop $l$ in $P^1 \setminus \{0,1,\infty\}$ (based at 1/2, say) leads to another fundamental solution $M(l)L(z)$ where

$$M : \pi_1(P^1 \setminus \{0,1,\infty\}) \to Gl(3, \mathbb{C}) \tag{B.5}$$

defines the monodromy representation. One finds for the images of the generator loops $l_i(t) (i = 0,1)$ the representing matrices\textsuperscript{72} $M(l_i)$ in (4.33).

In a column vector picture the three columns $c_k, k = 1,2,3,$ of $L$ fulfill $d c_k = c_k \cdot \Omega$ and one gets from (4.33) the monodromies (4.31) for $c_3$. Similarly in the row picture the rows $r^{(j)} (j = 1,2,3)$ in (B.4) are flat sections\textsuperscript{72} of a meromorphic connection $\nabla$ (on the trivial $\mathbb{C}^3$ bundle over $P^1$) given for a section $s = (s_1, s_2, s_3) : P^1 \setminus \{0,1,\infty\} \to \mathbb{C}^3$ by

$$\nabla s = ds - s\Omega = (ds_1, ds_2 - s_1 d \log \beta z, ds_3 - s_2 d \log z) \tag{B.6}$$

The Heisenberg picture involves the complexified Heisenberg group. Consider first the situation over the reals with the following central extension of the group $(\mathbb{R}^2, +)$ by $(\mathbb{S}^1, \cdot)$

$$1 \to \mathbb{S}^1 \to \mathcal{H} \to \mathbb{R}^2 \to 0 \tag{B.7}$$

\textsuperscript{72}Multiplying the rows $r^{(j)}$ by $(2\pi i)^{-1}$ the factor $2\pi i$ can be put in (B.4). The $r^{(j)}$ are multi-valued but the $\mathbb{Q}$-linear span of the $(2\pi i)^{-1}r^{(j)}$ is well-defined (the monodromy representation is then rational.)
So the normal subgroup $S^1$ of $\mathcal{H}$ constitutes the centre and one has the group law

$$(X, \lambda) \cdot (Y, \mu) = (X + Y, e(X,Y)\lambda\mu) \quad (B.8)$$

with a skew-multiplicative\(^{73}\) pairing\(^{74}\) $e : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow S^1$ given by $e(X,Y) = e^{2\pi i A(X,Y)}$ for $A : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ a non-degenerate, bilinear, skew-symmetric pairing. With the parameterisation $\lambda = e^{2\pi i c}, \mu = e^{2\pi i d}$ one finds as multiplication law on $\mathbb{R} \times \mathbb{R}^2$

$$(x_1, x_2 \mid c) \cdot (y_1, y_2 \mid d) = (x_1 + y_1, x_2 + y_2 \mid A(X,Y) + c + d) \quad (B.9)$$

Choosing for $A$ the pairing $A(X,Y) = x_1y_2 - x_2y_1$ (for $X = (x_1, x_2), Y = (y_1, y_2)$) one sees that the group law (B.9) on triples $(x_1, x_2 \mid c) \in \mathbb{R}^2 \times \mathbb{R} := \mathcal{H'}$ is induced from matrix multiplication under the following association of $\mathcal{H'}$ with the upper triangular matrices

$$(a, b \mid c) \cong \begin{pmatrix} 1 & a & \frac{c+ab}{2} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad (a, b \mid c) \cdot (u, v \mid w) = (a + u, b + v \mid av - bu + c + w) \quad (B.10)$$

Note that one has a slightly different induced group law by the following association

$$(a, b \mid c) \cong \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad (a, b \mid c) \cdot (u, v \mid w) = (a + u, b + v \mid av + c + w) \quad (B.11)$$

Define the complexified Heisenberg group $\mathcal{H}_C$ (with the $S^1$ from $e^{2\pi i \cdot}$ replaced by $\mathbb{C}^*$) where $\mathcal{H}_C$ is $\mathbb{C}^3$ with this composition (and so with inverse $(a, b \mid c)^{-1} = (-a, -b \mid ab - c)$) which makes $\mathcal{H}_C$ a non-commutative group with normal subgroups $(\ast, 0 \mid \ast')$ and $(0, \ast \mid \ast')$, both isomorphic to $(\mathbb{C}^2, +)$, whose intersection $(0, 0 \mid \ast)$ is the centre of $\mathcal{H}_C$.

The adjusted imaginary part $\mathcal{L}$ of $W$

Consider the Heisenberg bundle (4.39) with section $s$ (where $e(\mathcal{H}_Z(a, b \mid c)) = (e^a, e^b)$)

$$\xymatrix{ \mathcal{H} \ar[r] \ar[d] & \mathcal{H}_Z \ar[d]^e \\ \mathbb{P}^1 \setminus \{0, 1, \infty\} \ar[r]^{(1-z/z)} & \mathbb{C}^* \times \mathbb{C}^* } \quad (B.12)$$

\(^{73}\)So one has $e(X + X', Y) = e(X, Y)e(X', Y)$, similarly in $Y$ and $e(Y, X) = e(X, Y)^{-1}, e(X, X) = 1$.

\(^{74}\) $X \rightarrow e(X, \cdot)$ will then provide an isomorphism of $\mathbb{R}^2$ with its character group.
The $\mathcal{H}_Z$ coset expresses the fact that $z \to c$ is not given as a function (the monodromy increments of $L_i$). This comes as the right vertical sequence in (B.11) does not split, i.e. there is no map $\alpha$ with $(0,0|c) \to \mathcal{H}_Z(0,0|c) \xrightarrow{\alpha} (0,0|c)$. For the imaginary part of the fibre $(2\pi i)^2 \mathbb{Z} \setminus \mathbb{C}_c = ((2\pi i)^2 \mathbb{Z} \setminus \mathbb{R}_{Re_c}) \oplus i \mathbb{R}_{Im_c}$ there is such a map. The function $f$ on $\mathcal{H}_C$

$$f(u,v,w) = \text{Im } w - \text{Re } u \text{ Im } v \quad \text{(B.13)}$$

is invariant under action of $\mathcal{H}_R$ from the left, so a fortiori under $\mathcal{H}_Z$ which according to the remark after (4.35) represents the monodromy increments. So [19] the combination

$$\mathcal{L} : \mathbb{P}^1 \setminus \{0,1,\infty\} \ni z \xrightarrow{\lambda} \mathcal{H}_Z \left(-\frac{\log \beta z}{2\pi i}, \frac{\log z}{2\pi i}, -\frac{\text{Li}(z)}{(2\pi i)^2}\right) \in \mathcal{H}_Z \mathcal{H}_C \xrightarrow{(2\pi i)^2f} \mathbb{R} \quad \text{(B.14)}$$

(making factors $2\pi i$ manifest) gives a $\pi_1$-invariant, i.e. single-valued function (4.40).

**Some expressions related to $L_i$ and the Rogers CFT relation**

The $\mathbb{Z}_2$ anti-projector $P_{\mathbb{Z}_2}f = \frac{1}{2} \sum_{i \in \mathbb{Z}_2} \text{sign}(\gamma^i) \gamma^i f = (f - \gamma f)/2$ does not reproduce $L_i$ (not anti-invariant; $\gamma \in \Sigma_3 \setminus \mathbb{Z}_3$), so we introduce the Rogers function $R$ (for $\gamma = \alpha \beta$)

$$R(z) = \frac{1}{2} \left(\text{Li}(z) - \text{Li}(1 - z)\right) + \frac{\pi^2}{12} = \text{Li}(z) - \frac{1}{2} \log \beta z \log z \quad \text{(B.15)}$$

$$\text{Li}(z) = \int_0^z \log \beta w \, d \log w \quad , \quad d \text{Li}(z) = \log \beta z \, d \log z$$

$$\mathcal{L}(z) = \text{Im} \text{Li}(z) - \text{Im} \log \beta z \Re \log z \quad , \quad d \mathcal{L}(z) = \frac{1}{2} \left(\Re \log \beta z \, d \log z - \Re \log z \, d \log \beta z\right)$$

$$R(z) = \text{Li}(z) - \frac{1}{2} \log \beta z \log z \quad , \quad d R(z) = \frac{1}{2} \left(\log \beta z \, d \log z - \log z \, d \log \beta z\right) \quad \text{(B.16)}$$

For background on (5.11) recall that Calabi-Yau hypersurfaces in weighted projective space have a Gepner point in their moduli space with the underlying exactly solvable RCFT a tensor product of $N = 2$ superconformal minimal models of central charge $c = \frac{3k}{k+2}$ (the central charge of an integrable level $k$ representation of the affine Kac-Moody Lie algebra of $Sl(2)$). Recall the character $\chi_n(\theta) = \sin(n \pi \theta)/\sin(\pi \theta)$ of the $n$-dimensional representation of $SU(2)$. The characters $\chi_i(\tau, z) = tr_{\mathcal{H}_i q^{L_0 - \frac{c}{24}}} q^{2\pi i u_0}$ transform like $\chi_i(\frac{1}{\tau}, \frac{z}{\tau}) = e^{\pi i k z^2/2} \sum_j S_{ij} \chi_j(\tau, z)$ ($Q_{ij}$ generalized quantum dimensions)

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin(i+1)(j+1) \frac{\pi}{k+2} \quad , \quad Q_{ij} = \frac{S_{ij}}{S_{0j}} = \frac{\sin(i+1)(j+1)(\frac{\pi}{k+2})}{\sin(j+1)(\frac{\pi}{k+2})}$$

giving ($j$ fixed) $\sum_{i=1}^k R(i) \left(\frac{1}{\mathcal{Q}_{ij}}\right) = \frac{\pi^2}{6} \left(\frac{3k}{k+2} - 24 \Delta_j^{(k)} + 6j\right)$. $j = 0, N = k+2$ give (5.11) [42].

\footnote{For $\sum_{i=0}^N a_i^\sigma = 0$ in $\mathbb{P}^4(w_1, w_2, w_3, w_4, w_5)$ the CFT is a suitably interpreted tensor product of five $SU(2)$ theories of level $a_i - 2$ and chiral primary operators with integral anomalous dimensions come from operators in the $SU(2)^{k=a_i-2}$ factors with anomalous dimensions $\Delta_j^{(k)} = \frac{j(j+2)}{4(k+2)}$ ($j = 0, \ldots, k$).}

\footnote{Concerning $Q_{i0}$ recall that $Z(S^2 \times S^1) = 1, Z(S^3) = S_{00}$ give for the vev $<C> = \frac{\sin N \pi \theta}{\sin \pi \theta}$ of the unknot as Wilson line in $S^3$ (for $G = SU(N)$) that $<C> = \frac{Z(S^3, R_S)}{Z(S^3)} = \frac{S_{01}}{S_{00}}$ for $G = SU(2)$.}

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C Volume of a hyperbolic ideal tetrahedron

Hyperbolic three-space

As model for the hyperbolic space $\mathbb{H}_3$ we take the half-space model constructed in analogy to the upper half-plane $\mathbb{H} = \{ x = x_1 + x_2i | x_1, x_2 \in \mathbb{R}, x_2 > 0 \}$; we consider $\mathbb{C}$ embedded at $x_3 = 0$ so that $\mathbb{H}_3$ is $\{ (w := x_1 + ix_2, t := x_3) \in \mathbb{C} \times \mathbb{R}^{\geq 0} \}$. Now, $\mathbb{H}$ is also a homogeneous space $PSl(2, \mathbb{R})/SO(2)$ from the operation of $Sl(2, \mathbb{R})$ on $i$ by fractional linear transformations. Consider here the following subspace of the quaternions

$$\mathbb{H}_3 = \{ x = x_1 + x_2i + x_3j | x_1, x_2, x_3 \in \mathbb{R}, x_3 > 0 \}$$

(C.1)

$g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in Sl(2, \mathbb{C})$ operates on $\mathbb{H}_3$ by

$$g \cdot x = (ax + b)(cx + d)^{-1}$$

(C.2)

With the norm $||c(w + tj) + d||^2 = |cw + d|^2 + |c|^2t^2$ in the quaternions this is given by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (w, t) = \frac{1}{||c(w + tj) + d||^2} ((aw + b)(\bar{c}w + \bar{d}) + |c|^2t^2, t)$$

(C.3)

So for $z \in \mathbb{C}^\times \hookrightarrow Sl(2, \mathbb{C})$ (via $z = a^2, b = c = 0$) one has $z \cdot (w, t) = (zw, |z|t)$.

Now just as for $\mathbb{H} = \mathbb{H}_2$ one has here that the map $q : Sl(2, \mathbb{C}) \to \mathbb{H}_3$ given by $g \cong g \cdot j$ induces an equivariant diffeomorphism $q : Sl(2, \mathbb{C})/SU(2) \xrightarrow{\cong} \mathbb{H}_3$

$$1 \longrightarrow SU(2) \longrightarrow Sl(2, \mathbb{C}) \xrightarrow{q} \mathbb{H}_3 \longrightarrow 1$$

(C.4)

This may be equally well expressed by considering the quotient $PSl(2, \mathbb{C})/SO(3)$.

Prolonging the analogy to the real case note that the boundary of the upper half plane is identified with $\mathbb{P}^1_\mathbb{R} = \mathbb{R} \cup \{ \infty \}$ with $\mathbb{R}$ the locus $x_2 = 0$ whereas here the boundary is $\mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{ \infty \}$ with $\mathbb{C}$ the locus $x_3 = 0$. The group of (orientation preserving) isometries of $\mathbb{H}$ is isomorphic to $PSl(2, \mathbb{R})$ and for $\mathbb{H}_3$ to$^{77} PSl(2, \mathbb{C})$.

The standard hyperbolic metric $ds^2$ and the volume form $vol$ are given by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}, \quad vol = \frac{dx_1dx_2dx_3}{x_3^3}$$

(C.5)

For example, using the mentioned embedding $z \in \mathbb{C}^\times \hookrightarrow Sl(2, \mathbb{C})$, one finds for the length of the geodesic line-segment $\gamma_z$ from $q(e)$ to $q(z)$ (with $e = 1_2 \in Sl(2, \mathbb{C})$)

$$\text{length}(\gamma_z) = \text{distance}(j, |z|j) = \int_{\frac{1}{|z|}}^{\frac{1}{|z|}} \frac{dx_3}{x_3} = \text{Re} \log z$$

(C.6)

$^{77}$the latter acts on the boundary $\mathbb{P}^1_\mathbb{C}$ fractionally linear, so acts three-fold transitively and maps (uniquely) two quadruples of points onto another exactly if they have the same cross ratio
Volume of a hyperbolic ideal tetrahedron

Now an ideal tetrahedron is determined (up to congruence) by the dihedral angles \( \gamma_1, \gamma_2, \gamma_3 \) of the edges incident to any vertex. Then chosen any vertex\(^78\) one has

\[
\sum_i \gamma_i = \pi \tag{C.7}
\]

We will choose the vertex at \( \infty \) so that the angles become angles of an euclidean triangle in\(^79\) \( \mathbb{C} \) given by the remaining three vertices \( u, v, w \).\(^80\)

Now concerning the parametrization of an euclidean triangle \( \Delta(u, v, w) \subset \mathbb{C} \) (the vertices labeled in the mathematical positive sense) note that if one associates to each vertex the ratio of the adjacent sides

\[
z(u) = \frac{w-u}{v-u}, \quad z(v) = \frac{u-v}{w-v} = \beta z(u), \quad z(w) = \frac{v-w}{u-w} = \beta^2 z(u) \tag{C.8}
\]

then these vertex invariants depend only on the orientation preserving similarity class of \( \Delta(u, v, w) \) which in turn is determined by \( z(u) \) (\( \arg z(u) \) is the angle of \( \Delta(u, v, w) \) at \( u \); \( \text{Im} z(u) > 0 \)). So after the usual normalization in our tetrahedron set-up we are considering the angles of the euclidean triangle with vertices 0, 1, \( z \) in \( \mathbb{C} \), the angle \( \alpha_0 \) at 0 is \( \arg z \) and the angles \( \alpha_1 \) at 1 and \( \alpha_z \) at \( z \) are given by (cf. footn. 37)

\[
\alpha_0 = \arg z, \quad \alpha_1 = \arg \beta z, \quad \alpha_z = \arg \beta^2 z \tag{C.9}
\]

In other words this gives a geometric manifestation of the membrane anomaly (3.2)

\[
\sum_{i \in \mathbb{Z}_3} \text{Im} \log \beta^i z = \pi \tag{C.10}
\]

(cf. (1.9), (2.20)). As \( z, \beta z, \beta^2 z \) give the same tetrahedron one must pick an edge of \( \Delta \) (the dihedral angle of the faces adjacent at this edge is then \( \arg z \)) to specify \( z \) uniquely.

Furthermore one has with (4.25) that (where \( \gamma_{1,2,3} = \alpha_{0,1,z} \))

\[
\text{vol} \Delta(z) = \sum_i \Pi(\gamma_i) \tag{C.11}
\]

For convenience let us choose a slightly different 'circle gauge' of the points \( z_i \): the one actual complex degree of freedom (which is left after the \( SL(2, \mathbb{C}) \) operation) will not

\(^78\)as opposite dihedral angles are equal the \( \gamma_i \) are independent of the vertex chosen

\(^79\)the corresponding face of \( \Delta \) is a hemisphere over \( \mathbb{C} \) through \( u, v, w \) bounded by semi-circles over \( \mathbb{C} \)

\(^80\)The link \( L \) (parametrizing the rays in \( \Delta \) through \( v \)) of a vertex \( v \) of an ideal tetrahedron \( \Delta \) is an euclidean triangle (well-defined up to orientation preserving similarity) given by the intersection of the boundary of \( \Delta \) with a horizontal euclidean plane (a "horosphere"); \( L \) determines \( \Delta \) up to congruence.
be encoded in the complex number \( z \) with the other three points fixed (leaving two real degrees of freedom); rather we gauge to a situation where one of the points again becomes \( \infty \) and the other three points \( a, b, c \) lie on the unit circle \( |z| = 1 \) (these are three real degrees of freedom) with the one further condition that \( \text{Re } b = \text{Re } c \) (leaving two real degrees of freedom). In this situation, where we assume that the face opposite to \( \infty \) lies in the hemisphere \( x_1^2 + x_2^2 + x_3^2 = 1 \) (\( x_3 \geq 0 \)) and has vertices \( a, b, c \) (of \( x_3 = 0 \)), project \( \Delta \) orthogonally down to the unit disk \( D_{x_1,x_2} \) where you get the picture of an euclidean triangle (of vertices \( a, b, c \)) whose angles sum up to \( \pi \). Subdividing this triangle by drawing the heights from the origin on the sides one gets six smaller right triangles. Then one computes with (C.5) for the volume \( \text{vol}_1 \) of the region lying over one of these triangles (with angle \( \gamma \), \( A^2 := 1 - x_1^2 \), \( \cos \theta = x_1 \) and (4.26)) ([21])

\[
\text{vol}_1 = \int_0^{\cos \gamma} dx_1 \int_0^{x_1 \tan \gamma} dx_2 \int_0^{\infty} \frac{dx_3}{\sqrt{1-x_1^2-x_2^2}} = \frac{1}{2} \int_0^{\cos \gamma} dx_1 \int_0^{x_1 \tan \gamma} \frac{dx_2}{1-x_1^2-x_2^2}
\]

\[
= \frac{1}{4} \int_0^{\cos \gamma} dx_1 \log \frac{A+x_1 \tan \gamma}{A-x_1 \tan \gamma} = -\frac{1}{4} \int_{\pi/2}^{\gamma} d\theta \log \frac{2 \sin(\theta + \gamma)}{2 \sin(\theta - \gamma)}
\]

\[
= \frac{1}{4} \left( \Pi(2\gamma) - \Pi(\pi/2 - \gamma) - \Pi(\pi/2 - \gamma) - \Pi(0) \right) = \frac{1}{2} \Pi(\gamma)
\]

(C.12)

By summing over the six partial triangles one gets thereby (C.11). This gives the connection of (4.43) with (4.45) in view of the remark following (C.7).

cross ratios

The \( \Sigma_3 \) transformation properties of a cross ratio can be understood as follows. For four points \( z_1, z_2, z_3, z_4 \) of \( \mathbb{P}^1(\mathbb{C}) \) one defines their cross ratio

\[
\text{cr}\{z_1, z_2, z_3, z_4\} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_3}{z_2 - z_4}
\]

(C.13)

For example \( \text{cr}\{0, 1, \infty, z\} = z \). Clearly a \( \Sigma_4 \) is operating. One has the equalities

\[
\text{cr}\{z_1, z_2, z_3, z_4\} = \text{cr}\{z_2, z_1, z_3, z_4\} = \text{cr}\{z_3, z_4, z_1, z_2\} = \text{cr}\{z_4, z_3, z_2, z_1\}
\]

(C.14)

but the index four subgroup \( \Sigma_3 \) operates effectively which gives the following realisation

\[
x = \text{cr}\{z_1, z_2, z_3, z_4\} \quad \frac{1}{1-x} = \text{cr}\{z_1, z_3, z_4, z_2\} \quad \frac{x-1}{x} = \text{cr}\{z_1, z_4, z_2, z_3\}
\]

(C.15)

of the isomorphism \( \Sigma_3 \cong Sl(2, \mathbb{Z})/\Gamma(2) \) in

\[
1 \rightarrow V \rightarrow \Sigma_4 \rightarrow \Sigma_3 \cong Sl(2, \mathbb{Z})/\Gamma(2) \rightarrow 1
\]

(C.16)
\[ (6.25) \text{ gives for the classifying space } BGl(n,\mathbb{C})^\delta \text{ of flat bundles an universal class } \]
\[ \hat{C}_2 \in H^3(BGl(n,\mathbb{C})^\delta,\mathbb{C}/\mathbb{Z}) \cong H^3_{EM}(Gl(n,\mathbb{C})^{\text{disc}},\mathbb{C}/\mathbb{Z}) \] 
\[ (D.1) \]
where we also indicated the isomorphism of the topological homology with the Eilenberg-MacLane group cohomology\(^{81}\) of the underlying discrete group of \( Gl(n,\mathbb{C}) \).

One defines a geodesic simplex for three elements \( g_i \) of \( G \) by \( \Delta(z) \) with (cf. (C.13))
\[ \sigma((g_1,g_2,g_3)) = z = cr\{\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty \} \]
\[ (D.2) \]
With \( \omega \) a \( \mathbb{C} \)-valued \( Sl(2,\mathbb{C}) \) invariant three-form on \( Sl(2,\mathbb{C})/SU(2) = H_3 \) one finds a \( \mathbb{C}/\mathbb{Z} \) valued Eilenberg-MacLane cochain \( T(\omega) \) with \( \hat{C}_2 = T(\omega)(g_1,g_2,g_3) = f_\Delta(z) \omega \). This is evaluated [22] as \( 2\hat{C}_2 = c \) via the exterior square version (D.5) of the Heisenberg bundle

\[ \begin{array}{ccc}
\mathbb{Q}\backslash\mathbb{C} & \rightarrow & \mathbb{Q}\backslash\mathbb{C} \\
\alpha \uparrow \downarrow & & \downarrow 1 \wedge id \\
\rho \uparrow \downarrow & & \\
\mathcal{C} \wedge_{\mathbb{Z}} \mathcal{C} & \rightarrow & \mathcal{C} \wedge_{\mathbb{Z}} \mathcal{C} \\
\sigma & & \rho \\
H^3(Sl(2,\mathbb{C}))/\mathcal{Q}/\mathcal{Z} & \rightarrow & \mathbb{P}^1 \backslash \{0,1,\infty\} \\
\downarrow c & & \\
\mathcal{C}/\mathcal{Q} & \leftarrow & \Lambda^2_{\mathbb{Z}}(\mathcal{C}^*) \\
\end{array} \]
\[ (D.3) \]
(for the proper target of \( \sigma \) cf. (D.4)) The arrows in the lower row compose to zero.\(^{82}\)

This is the commutative diagram with exact rows\(^{83}\) [22], [23]

\[ \begin{array}{ccc}
\mathbb{H}^3(Sl(2,\mathbb{C}),\mathbb{Z})/\mathcal{Q}/\mathcal{Z} & \rightarrow & \mathbb{P}_C \\
\downarrow c & & \downarrow \rho \\
\mathcal{C}/\mathcal{Q} & \leftarrow & \Lambda^2_{\mathbb{Z}}(\mathcal{C}) \\
\end{array} \]
\[ (D.4) \]
with\(^{84}\) \( \mathbb{P}_C = F(\mathbb{P}^1_C)/\Sigma^3_3 \), i.e. free generators from \( \mathbb{P}^1_C \) modulo the equivalence relation given by the non-linear \( \Sigma_3 \) action with order two elements operating together with a minus sign (\( \lambda \) is then still welldefined)\(^{85}\). Therefore \( \rho \circ \sigma \) comes from an element in \( \mathbb{Q}\backslash\mathbb{C} \),

\(^{81}\)homology is of chain complex of elements of \( G^n \) with boundary \( \partial(g_1,\ldots,g_n) = (g_2,\ldots,g_n) + \sum_{i=1}^{n-1}(-1)^i(g_1,\ldots,g_{i+1},\ldots,g_n) + (-1)^n(g_1,\ldots,g_{n-1}) \); so \( H^0(G) = \mathbb{Z}, H^1(G) = G^{ab} = G/G^{comm} \)

\(^{82}\)The target space of \( e \) has to be ('log'-)interpreted so that \( (ab) \wedge c = a \wedge c + b \wedge c, \frac{1}{b} \wedge c = -(b \wedge c), 0 = \pm 1 \wedge c \) hold; in particular \( \lambda(z) = 0 \) for \( z \in \mu_C \) (the complex roots of unity) as \( z \wedge (1-z) = (1/n)(z^n \wedge (1-z)) \).

\(^{83}\)the upper row is well-defined as the part \( \mathcal{Q}/\mathcal{Z} \) modded out comes from \( H^3(\mu_C,\mathcal{Z}) \) embedded diagonally and this goes to zero under \( \sigma \); note also that \( \lambda(z) = 0 \) for \( z \in \mu_C \) by footn. 82

\(^{84}\)in [22], [23] actually a group \( \mathbb{P}_C = \mathbb{P}_C/\sim \) for a certain 5-term equivalence relation \( \sim \) is considered

\(^{85}\)Note that the mapping \( \lambda : \mathbb{C} \ni z \rightarrow z \wedge (1-z) \in \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^* \) transforms anti-invariantly (as \(-\lambda(z) = z \wedge \beta z \) and \( z \cdot \beta z \cdot \beta^2 z = -1 \)), i.e. \( z \wedge (1-z) \in -1 \) (cf. sect. A.2).
i.e. one defines \( c = \alpha \circ \rho \circ \sigma \) (a natural continuous option for the splitting \( \alpha \) is given only for the imaginary part); so \( c \) is essentially given by \( \rho \), i.e. the Rogers 'function' (in the end the dilogarithm). One finds then that \( 2\hat{C}_2 = c \), more precisely\(^{86}\) (6.27).

**Representation via exterior squares**

We are interested in the diagram analogous to (B.11) (again \( e(z, w) = (e^{2\pi iz}, e^{2\pi iw}) \); the tilde indicates a pullback of the \( e \) projection map along the indicated base map)

\[
\begin{array}{ccc}
Q \setminus C & \xrightarrow{\downarrow 1 \wedge \text{id}} & C \wedge_Z C \\
\downarrow p & & \downarrow e \\
\tilde{\mathbb{P}^1 \setminus \{0, 1, \infty\}} & \xrightarrow{z \wedge (1-z)} & C^* \wedge_Z C^*
\end{array}
\]

(D.5)

Note that one has the expression which is *not* welldefined as a function (cf. (B.15))

\[
\frac{1}{2\pi^2} Li(z) = \frac{1}{2\pi i} \log z \frac{1}{2\pi i} \log(1-z) + \frac{-2}{(2\pi i)^2} R(z)
\]

(D.6)

and can define a map \( \rho : C \ni z \rightarrow \rho(z) \in \Lambda_Z^2(C) \) which *is* indeed welldefined [20]

\[
\rho(z) = \frac{1}{2\pi i} \log z \frac{1}{2\pi i} \log(1-z) + 1 \wedge \frac{-2}{(2\pi i)^2} R(z)
\]

(D.7)

This is a section of \( p \) in (D.5) just as (4.35) was a section of \( pr \) in (B.11). Note that if one wants to go back from a value in \( \Lambda_Z^2(C) \) to a complex number (to have a function instead of a section of a non-trivial projection), i.e. if one wants to define a splitting \( \alpha : C \wedge_Z C \rightarrow Q \setminus C \) to \( Q \setminus C \xrightarrow{1\wedge \text{id}} C \wedge_Z C \) one has *natural* option just for the imaginary part (this replaces (B.13); cf. also the alternative identification before (B.10)) [22], [23]

\[
\text{Im } \alpha(z \wedge w) = \text{Re } z \text{Im } w - \text{Re } w \text{Im } z
\]

(D.8)

The relative minus sign escapes the symmetry in (A.21); for by (A.26) \( \psi \), and so \( \mathcal{L} \), is not an imaginary part of ordinary (rather than wedge) products. One has\(^{87}\) (cf. (B.14))

\[
\text{Im } \alpha \rho = \frac{1}{2\pi^2} \mathcal{L}
\]

(D.9)

---

\(^{86}\)It suffices to evaluate \( 2\text{Re} \hat{C}_2 \) on the cohomology \( H_3(\text{Sl}(2, \mathbb{R})^\delta) \) of the real subgroup (actually the universal cover \( \tilde{\text{PSl}}(2, \mathbb{R})^\delta \) is concerned) where one finds that in \( H_3(\text{Sl}(2, \mathbb{R}), \mathbb{R}/\mathbb{Z}) \) it is congruent to \( \frac{1}{2\pi^2} R(z) \) \( \mod \frac{1}{2\pi} \left( = \frac{1}{2\pi} \cdot \frac{\pi^2}{6} \right) \); \( \text{Im } \hat{C}_2 \) is a *continuous* cochain and so uniquely determined (up to a factor) as \( H^3_{\text{cont}}(\text{Sl}(2, \mathbb{C}), \mathbb{R}) \cong \mathbb{R} \) [22].

\(^{87}\)Such a relation without taking the imaginary part would be inappropriate (\( \mathcal{L} \neq \text{Im} R \) as \( \alpha \rho \neq \frac{1}{2\pi^2} R \)).
References

1. M. Atiyah, J. Maldacena and C. Vafa, "An M-theory Flop as a Large N Duality", J.Math.Phys. 42 (2001) 3209, hep-th/0011256.

2. M. Atiyah and E. Witten, "M-Theory Dynamics On A Manifold Of $G_2$ Holonomy", hep-th/0107177.

3. B. S. Acharya, "On Realising N=1 Super Yang-Mills in M theory", hep-th/0011089.

4. A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, "Gauge Theory at Large N and New $G_2$ Holonomy Metrics", Nucl.Phys. B611 (2001) 179, hep-th/0106034.

5. T.R. Taylor and C. Vafa, "RR Flux on Calabi-Yau and Partial Supersymmetry Breaking", Phys. Lett. B474 (2000) 130, hep-th/9912152.

6. C. Beasley and E. Witten, "A Note on Fluxes and Superpotentials in M-theory Compactifications on Manifolds of $G_2$ Holonomy", JHEP 0207 (2002) 046, hep-th/0203061.

7. E. Witten, "Phases of $N=2$ Theories In Two Dimensions", Nucl. Phys. B403 (1993) 159, hep-th/9301042.

8. F. Cachazo, K. Intriligator and C. Vafa, "A Large N Duality via a Geometric Transition", Nucl. Phys. B 603 (2001) 3, hep-th/0103067.

9. S. Gukov, "Solitons, Superpotentials and Calibrations", Nucl. Phys. B574 (2000) 169, hep-th/9911011. S. Gukov, C. Vafa and E. Witten, "CFT’s From Calabi-Yau Four-folds", Nucl. Phys. B584 (2000) 69; Erratum-ibid. B608 (2001) 477, hep-th/9906070.

10. B.S. Acharya and B. Spence, "Flux, Supersymmetry and M theory on 7-manifolds", hep-th/0007213.

11. G. Curio and A. Krause, Four-Flux and Warped Heterotic M-Theory Compactifications, Nucl. Phys. B602 (2001) 172, hep-th/0012152.

12. G. Curio and A. Krause, G-Fluxes and Non-Perturbative Stabilisation of Heterotic M-Theory, Nucl. Phys. B643 (2002) 131, hep-th/0108220.

13. M. Aganagic and C. Vafa, Mirror Symmetry, D-Branes and Counting Holomorphic Discs, hep-th/0012041.
14. M. Aganagic and C. Vafa, "Mirror Symmetry and a $G_2$ Flop", hep-th/0105225.

15. M. Aganagic, A. Klemm and C. Vafa, "Disk Instantons, Mirror Symmetry and the Duality Web", Z. Naturforsch. A57 (2002) 1, hep-th/0105045.

16. A. Brandhuber, "$G_2$ Holonomy Spaces from Invariant Three-Forms", hep-th/0112113, Nucl.Phys. B629 (2002) 393.

17. E. Witten, "Five-Brane Effective Action In M-Theory", hep-th/9610234, J.Geom.Phys. 22 (1997) 103.

18. R. Hain, "Classical Polylogarithms", alg-geom/9202022. D. Ramakrishnan, "A regulator for curves via the Heisenberg group", Bull. Amer. Math. Soc. 52 (1981) 191.

19. R. Hain and R. MacPherson, "Higher logarithms", Ill. Journ. Math. 342 (1990) 392.

20. S. Bloch, "Applications of the dilogarithm function", Intl. Symp. on Algebraic Geometry, Kyoto (1981) 103.

21. J. Milnor, "Hyperbolic Geometry: The first 150 Years", Bull. Amer. Math. Soc. 6 (1982) 9. J. G. Ratcliffe, "Foundations of Hyperbolic Manifolds", Graduate Texts in Mathematics 149, Springer 1994.

22. J.L. Dupont, "The dilogarithm as a characteristic class for flat bundles", Journ. of Pure a. Appl. Alg. 44 (1987) 137.

23. J.L. Dupont and C.-H. Sah, "Scissors congruences II", Journ. of Pure and Appl. Alg. 25 (1982) 159.

24. J.A. Harvey and G. Moore, "Superpotentials and Membrane Instantons", hep-th/9907026.

25. N. Seiberg and E. Witten, "Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory", Nucl. Phys. B426 (1994) 19, Erratum-ibid. B430 (1994) 485, hep-th/9407087.

26. M. Matone, "Instantons and recursion relations in N=2 Susy gauge theory", Phys. Lett. B357 (1995) 342, hep-th/9506102.
27. G. Curio, A. Klemm, D. Lüst and S. Theisen, "On the Vacuum Structure of Type II String Compactifications on Calabi-Yau Spaces with H-Fluxes", Nucl. Phys. B609 (2001) 3, hep-th/0012213.

28. P. Horava and E. Witten, "Heterotic and Type I String Dynamics from Eleven Dimensions", Nucl. Phys. B460 (1996) 506, hep-th/9510209; "Eleven-Dimensional Supergravity on a Manifold with Boundary", Nucl. Phys. B475 (1996) 94, hep-th/9603142. E. Witten, "Strong Coupling Expansion Of Calabi-Yau Compactification", Nucl. Phys. B471 (1996) 135, hep-th/9602070.

29. B. Acharya and E. Witten, "Chiral Fermions from Manifolds of $G_2$ Holonomy", hep-th/0109152.

30. G. Curio, A. Klemm, B. Körs and D. Lüst, "Fluxes in Heterotic and Type II String Compactifications", Nucl. Phys. B620 (2002) 237, hep-th/0106155.

31. B.R. Greene, K. Schalm and G. Shiu, "Dynamical Topology Change in M Theory", J. Math. Phys. 42 (2001) 3171, hep-th/0010207.

32. G. Tian and S.T. Yau, "Three dimensional algebraic manifolds with $c_1 = 0$ and $\chi = -6$", in Mathematical Aspects of String theory, Proc. San Diego (1986) 543.

33. C. Vafa, "Superstrings and Topological Strings at Large N", J. Math. Phys. 42 (2001) 2798, hep-th/0008142.

34. E. Witten, "Branes And The Dynamics Of QCD", Nucl. Phys. B507 (1997) 658, hep-th/9706109.

35. G. Curio, "Superpotential of the M-theory conifold and type IIA string theory", hep-th/0212233.

36. G. Curio, B. Kors and D. Lüst, "Fluxes and Branes in Type II Vacua and M-theory Geometry with $G(2)$ and Spin(7) Holonomy", hep-th/0111165.

37. D. Joyce, "On counting special Lagrangian homology 3-spheres", hep-th/9907013.

38. S. Kachru and J. McGreevy, "Supersymmetric Three-cycles and (Super)symmetry Breaking", Phys. Rev. D61 (2000) 026001, hep-th/9908135.

39. F. Denef, "Supergravity flows and D-brane stability" JHEP 0008 (2000) 050, hep-th/0005049. "(Dis)assembling Special Lagrangians", hep-th/0107152.
40. E. Witten, "2+1 dimensional gravity as an exactly soluble system", Nucl. Phys. B 311 (1988) 46.

41. in preparation.

42. B. Richmond and G. Szekeres, "Some formulas related to dilogarithms, the zeta function and the Andrews-Gordon identities", J. Austral. Math. Soc. 31 (1981) 362. A.N. Kirillov and Yu.N. Reshetikhin, "Exact solutions of the integrable XXZ Heisenberg model with arbitrary spin: I, II", J. Phys. A: Math. Gen. 20 (1987) 1565, 1587.

43. J.L.Dupont and C.H. Sah, "Dilogarithm Identities in Conformal Field Theory and Group Homology", Commun. Math. Phys. 161 (1994) 265, hep-th/9303111.

44. R. Caracciolo, F. Gliozzi and R.Tateo, "A topological invariant of RG flows in 2D integrable quantum field theories", Int. J. Mod. Phys. B13 (1999) 2927, hep-th/9902094.