Network model for higher-order topological insulators

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We introduce a two-dimensional network model that realizes a higher-order topological insulator (HOTI) phase. We find that in the HOTI phase a total of 16 corner states are protected by the combination of a four-fold rotation, a phase-rotation, and a particle-hole symmetry. In addition, the model exhibits a strong topological phase at a point of maximal coupling. This behavior is in opposition to conventional network models, which are gapless at this point. By introducing the appropriate topological invariants, we show how a point group symmetry can protect a topological phase in a network. Our work provides the basis for the realization of HOTI systems in alternative experimental platforms implementing the network model.

I. INTRODUCTION

Topological phases of matter are known to support boundary modes which are protected by bulk topological invariants through the so-called bulk-boundary correspondence. The topological protection is typically associated with fundamental symmetries (time-reversal, particle-hole, and chiral symmetry) which have led to the ten-fold classification of (gapped) topological phases [1–3]. By further including discrete spatial symmetries, new phases such as topological crystalline insulators [4–8] and, very recently, higher-order topological insulators (HOTIs) have been identified [9–29].

A prototypical HOTI model is the Benalcazar-Bernevig-Hughes (BBH) model [9], where fundamental and point group symmetries protect zero-energy corner states in an otherwise gapped, two-dimensional (2D) system [30]. HOTIs have already been realized in various platforms ranging from photonic, phononic, and electronic systems to microwave and topoelectric circuits [31–41].

Ever since the discovery of the quantum Hall effect, a main focus of research has been to study the robustness of topological states against disorder, and the associated localization-delocalization transitions that destroy a topological phase [42]. Among other approaches, a powerful tool in this study has been the network model description of topological phases [42–44], first introduced by Chalker and Coddington (CC) in the context of the quantum Hall effect [43]. The network model idea has subsequently been adapted to a variety of different topological systems, including the quantum spin-Hall effect, weak topological insulators, Floquet systems, as well as different types of topological superconductors [45–51]. However, despite their versatility, no network models for point group symmetry-protected topological phases have been introduced to date. This includes HOTIs, which are very recent additions to the list of topological systems, but also conventional topological crystalline insulators, which are by now a decade old [4, 5].

II. NETWORK MODEL

Here, we construct a network model for a HOTI protected by particle-hole and by four-fold rotation ($C_4$) symmetry, which hosts mid gap corner modes. Unlike the $C_4$-symmetric HOTIs in static systems, which have 4 corner states [9, 10], or their periodically driven counterparts, where this number can go up to 8 [52–54], our network model shows up to 16 corner modes, due to an additional, phase-rotation symmetry [55]. Upon varying system parameters, we find that trivial and HOTI phases are separated by an intermediate, strong topological phase (STP), which surprisingly is present even at the point of maximal mode mixing, i.e., when the reflection and transmission probabilities are equal. This behavior is in sharp contrast to the conventional Chalker-Coddington model and its generalizations [45–51], all of which are gapless at this point.

The rest of this work is organized as follows. In Sec. II, we describe the construction of the network model and examine its symmetries. In Sec. III we analyze several simple limits of the network model, which realize HOTI, trivial, and STP. A detailed study of the phase diagram is performed in Sec. IV, and the topological invariants of the system are calculated in Sec. V. We study the effect of disorder in Sec. VI, and conclude in Sec. VII.
the scattering matrices are real, the network correspond to chiral Majorana modes and model belongs to the AZ class D. The directed links of Fig. 1. We impose particle-hole symmetry, such that the lattice structure of the network model should be modified. meaning that both the form of \([42, 45–51, 56, 57]\).

The above equation is reminiscent of the Floquet treatment of periodically-driven systems, where now the role of the Floquet operator is taken by the unitary matrix \(S\), and its eigenphases \(\varepsilon\) play the role of the quasienergies. By solving Eq. (4), we gain access to the spectrum of the network model, enabling us to identify gaps in the spectrum of eigenphases and potential corner modes. Furthermore, by considering an infinite, translationally invariant network model, Eq. (4) provides access to the ‘bandstructure’ \(\varepsilon(k)\) of the system with \(k = (k_x, k_y)\) the two wavenumbers.

In our network model, the momentum space Ho-Chalker operator in the translationally invariant setting is a \(16 \times 16\) matrix, due to the fact that there are 16 links per unit cell. Using the labeling convention of Fig. 1, it reads

\[
S(k) = \begin{pmatrix}
S_1(k) & 0 & 0 & S_4(k) \\
0 & S_2(k) & 0 & 0 \\
0 & 0 & S_3(k) & 0
\end{pmatrix},
\]

where the blocks are given by

\[
S_1(k) = \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 & 0 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 & 0 \\
0 & 0 & \cos \theta_1 & \sin \theta_1 \\
0 & 0 & \sin \theta_1 & -\cos \theta_1
\end{pmatrix},
\]

\[
S_2(k) = \begin{pmatrix}
e^{-ik_y} \sin \theta_4 & 0 & 0 & \cos \theta_4 \\
\cos \theta_4 & 0 & \sin \theta_4 & 0 \\
0 & \sin \theta_2 & -\cos \theta_2 & 0 \\
\cos \theta_2 & 0 & 0 & -e^{-ik_x} \sin \theta_4
\end{pmatrix},
\]
\[ S_3(k) = S_1(k) \{ k_y \rightarrow -k_y, \theta_3 \rightarrow -\theta_3, \theta_1 \rightarrow \pi - \theta_1 \}, \]

\[ S_4(k) = S_2(k) \{ k_x \rightarrow -k_x \}. \]

In the translationally invariant system, particle-hole symmetry is expressed as \( S(k) = S^*(-k) \). This is reminiscent of Floquet systems and indicates that for any eigenstate of the network model at eigenphase \( \varepsilon \) and momentum \( k \), there exist another eigenstate at \( -\varepsilon \) and \(-k\). Unlike in periodically-driven systems, however, the Ho-Chalker operator \( S \) also possesses a so-called phase-rotation symmetry (PRS) \( ZS(k)Z^\dagger = e^{i\pi/2}S(k) \), with \( Z = \text{diag}(i, -1, -i, 1) \otimes \mathbb{1}_4 \) the phase-rotation operator \([55]\). The latter symmetry implies that for any eigenstate with eigenphase \( \varepsilon \) there also exists an eigenstate at \( \varepsilon + \pi/2 \). As such, the spectrum of the network model repeats four times in \( \varepsilon \in [-\pi, \pi] \), allowing to focus on the ‘fundamental phase domain’ with \( \varepsilon \in [-\pi/4, \pi/4] \). Finally, along the line \( \theta_1 = \theta_2 = \theta_3 = \theta_4 \), the Ho-Chalker operator has \( C_4 \) symmetry \( \mathcal{R} \) with \( \mathcal{R}S(k_x, k_y)\mathcal{R}^\dagger = S(k_y, -k_x) \), where

\[ \mathcal{R} = \begin{pmatrix}
0 & 0 & 0 & \mathcal{R}_4 \\
0 & 0 & 0 & 0 \\
0 & \mathcal{R}_2 & 0 & 0 \\
0 & 0 & \mathcal{R}_1 & 0
\end{pmatrix}, \quad \mathcal{R}_i = K_i = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}; \]

\( K_i \) is a \( 4 \times 4 \) diagonal matrix whose \( i \)-th diagonal element is \(-1 \) while the others are \(1 \).

### III. DECOUPLED LIMITS

We study the network model by first focusing on the \( C_4 \)-symmetric case where the network properties are controlled by the two angles \( \theta_1(= \theta_2) \) and \( \theta_3(= \theta_4) \), corresponding to the mixing angle at the inner and outer nodes of the unit cell, see Fig. 1. To determine the properties of the edge and corner states, we construct finite-sized network models by imposing open boundary conditions (OBC) in both directions. As shown in Fig. 2, the OBC are obtained by demanding that the Majorana wavefunctions impinging on the boundary are reflected with unit probability, which implies a vanishing of the probability current across the boundary.

First, we investigate the four ‘decoupled limits’ of the network model, obtained when \( \theta_i \) are either 0 or \( \pi/2 \). For these values, the incoming Majorana modes do not get mixed by node scattering but instead turn either clockwise or counter-clockwise with unit probability, see Fig. 2. The system then consists of Majorana modes propagating along closed loops that are fully decoupled from each other.

In this limit, we can determine the spectrum of the network model entirely from the structure of these closed loops. For a closed path consisting of \( \ell \) links, a Majorana wavefunction comes back to itself after \( \ell \) discrete time steps, i.e., after \( \ell \) applications of the Ho-Chalker operator of Eq. (3). Note that there are only two options after a full loop. Either the link amplitude picks up a total phase of 0 (periodic) or \( \pi \) (anti-periodic), since the Majorana wavefunction is real. The spectrum of the system can thus be determined from the lengths of these loops. All periodic loops of length \( \ell \) contribute to the spectrum of the Ho-Chalker operator with the eigenphases \( \varepsilon = 2\pi k/\ell \), where \( k = 0, 1, \ldots, \ell - 1 \). As such, they admit a gapless solution with an eigenphase \( \varepsilon = 0 \). In contrast, the spectrum of the anti-periodic loops is shifted by \( \frac{\pi}{2} \) with \( \varepsilon = 2\pi(k + \frac{1}{2})/\ell \) and no gapless solution is possible in this case. In the following, we discuss the four different decoupled limits shown in Fig. 2, in order of appearance.

Setting \( \theta_1 = \theta_2 = \pi/2 \), the network model is in a HOTI phase, see Fig. 2(a). We find that the bulk and edges are gapped, consisting only of anti-periodic Majorana loops. At the four corners, however, periodic loops of length 4 lead to mid-gap modes in the spectrum. There are a total of four modes localized at each one of the corners, with eigenphases given by the fourth roots of unity, i.e., \( \varepsilon = 0, \pm \pi/2, \) and \( \pi \). The 0 and \( \pi \) modes cannot be shifted away from their value without breaking particle-hole symmetry. On the other hand, the \( \varepsilon = \pm \pi/2 \) modes cannot be moved away from their value due to the combination of particle-hole and phase-rotation symmetry. As such, the only mechanism through which a corner mode may be removed is by coupling it with another corner mode to form a dimerized pair. Since the system obeys a \( C_4 \) symmetry, the hybridization of the corner mode cannot happen along an edge of the system, but requires all four mid-gap states to be simultaneously moved to the center (or bulk) of the network. Therefore, the \( C_4 \) symmetry protects the mid-gap corner states of the bulk HOTI phase through a mechanism that is fully analogous to that of the BBH model (see Appendix A).

For \( \theta_1 = \theta_3 = 0 \), the network model is topologically trivial, see Fig. 2(b). All Majorana loops are anti-periodic, so that no mid-gap states exist in the spectrum. Notice, however, that the pattern of closed loops is the same as that of the HOTI network in Fig. 2(a). The main difference between the two limits is that for the trivial system the loops are fully contained inside each unit cell, whereas in the HOTI phase, they extend across the boundaries of the unit cells. This feature is analogous to the BBH model, where the HOTI phase of the tight-binding Hamiltonian is obtained when sites are dimerized across the unit cell boundary and the trivial phase contains sites which are dimerized within the unit cell. As an additional common feature, this means that HOTI and trivial phases can be mapped onto each other by a re-definition of the unit cell (see Appendix A).

For \( \theta_1 = 0 \) and \( \theta_2 = \pi/2 \), the network model is in a STP, see Fig. 2(c), equivalent to that realized in the Cho-Fisher model [60]. The bulk is gapped, and a single chiral Majorana mode propagates along the edge, akin to topological \( p \)-wave superconductors [61]. The large Majorana loop extending along the perimeter of the net-
Figure 2. The top panels show the network model, consisting of $3 \times 3$ unit cells, in the four decoupled limits: HOTI ($\theta_1 = \theta_3 = \pi/2$) in panel (a), trivial ($\theta_1 = \theta_3 = 0$) in panel (b), STP ($\theta_1 = 0, \theta_3 = \pi/2$) in panel (c), and Majorana flat band ($\theta_1 = \pi/2, \theta_3 = 0$) in panel (d). The bottom panels show the spectra of the Ho-Chalker operators in the same limits. The inset in panel (a) is a closeup of the mid-gap corner modes. Gapped (gapless) modes and the Majorana loops producing them are shown in light (dark) blue. The horizontal dotted lines mark the boundaries of the fundamental phase domain, which repeats four times due to phase-rotation symmetry. Notice that all gapped Majorana loops contain an odd number of minus signs (denoted by ⊖) in a closed loop. See Fig. 1 for the minus sign conventions within a unit cell.

work is anti-periodic, such that it does not admit exact zero-eigenphase states. Instead, like for the chiral Majorana mode on the boundary of $p$-wave superconductors, there is a finite-size a gap in the spectrum of the edge mode [62]. The mini-gap $\pi/U$ is inversely proportional to the perimeter $U$ and vanishes in the thermodynamic limit. Note that the spectrum of the Ho-Chalker operator is periodic. As a result, every bulk band has an edge mode in the gap above and an edge mode in the gap below. Due to this fact, which is reminiscent of so-called anomalous Floquet topological phases [63–66], the bulk bands all have a vanishing Chern number, despite the presence of the chiral edge mode, which winds around the torus formed by the momentum (along the edge) and eigenphase.

Finally, setting $\theta_1 = \pi/2$ and $\theta_3 = 0$, the network model realizes a gapless, Majorana flat band, as shown in Fig. 2(d). All closed loops are periodic, such that the number of zero-eigenphase states is extensive and scales with the system size. In the other three decoupled limits, the presence of a bulk gap protects the resulting phase against small, symmetry preserving parameter changes. Here, in contrast, the Majorana flat bands may be gapped out by small changes of the parameters. In fact, as we will show in the following, the decoupled limit of Fig. 2(d) represents a tri-critical point in the phase diagram of the network model at which the other three phases meet.

IV. PHASE DIAGRAM

We determine the phase diagram of the network model by computing its two-terminal dimensionless conductance $G$, given by the total transmission probability. To this end, we attach semi-infinite leads composed of Majorana modes to its left and right boundaries and calculate the two-lead scattering matrix associated to the whole network (see Appendix B for details). We consider systems with either open boundary conditions (OBC) or with periodic boundary conditions (PBC) in the transverse direction. This enables us to distinguish between bulk and edge contributions to the two-terminal transmission probability.

We begin by preserving the $C_4$ symmetry and determining the phase diagram in the $\theta_1–\theta_3$ plane, shown in Figs. 3(a) and (b). We observe that the trivial and the HOTI phases have a vanishing two-terminal transmission probability $G$ for both OBC and PBC. The reason is that both the bulk and the edges are gapped. There are four trivial insulating regions in the $\theta_1–\theta_3$ plane, appearing as horizontally elongated regions of $G = 0$ in Fig. 3(b), centered around $\theta_3 = 0, \pi$. The four HOTI regions appear in the same panel as vertically elongated $G = 0$ regions
The closing condition is given by related to it by the phase-rotation symmetry. The gap which show a quantized value $\varepsilon$ symmetries of the Ho-Chalker operator, as we explain in each other in parameter space is a consequence of several phases meet. The position of these phases relative to of Fig. 2(d) forming a critical point at which the three trivial, and STP regions, with the Majorana flat band shown in Fig. 2 correspond to the centers of the HOTI, chiral edge mode. Notice how the decoupled limits $G = \pi/\varepsilon = 0$ closes, as do all of the other gaps (at $\varepsilon = \pm \pi/2, \pi$) related to it by the phase-rotation symmetry. The gap closing condition is given by

$$\det \left( \prod_{j=1}^4 S_j - I_{4\times 4} \right) = 0, \quad (11)$$

with solutions of the form

$$| \cos \theta_1 \mp \sin \theta_3 | = \sqrt{2} | \sin \theta_4 \cos \theta_1 |, \quad (12)$$

at $k = (0, 0)$ and $k = (\pi, \pi)$, respectively. Note that the two solutions, shown as solid black and dashed green lines in Fig. 3(b), nicely match the numerics.

In Figs. 3(a) and (b), there is no path in the phase diagram which connects the HOTI to the trivial phase without a bulk gap closing. As discussed in the previous section, this is a consequence of $C_4$ symmetry, which forbids the dimerization of corner modes along the edge of the system. Conversely, if $C_4$ symmetry is broken, it becomes possible to connect the HOTI and trivial phase while preserving the bulk gap. We explore this possibility in Figs. 3(c) and (d), where the transmission probability is plotted as a function of $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$. These two axes correspond to different dimerizations between Majorana loops of the network in the horizontal and vertical directions, analogous to how dimerization can be varied independently for the horizontal and vertical hoppings of the BBH model (see Appendix A).

Along the diagonals of Fig. 3(c), $\theta_1 = \theta_2$ and $\theta_1 = -\theta_2$, a $C_4$ symmetry is preserved (albeit with a different symmetry operator, see Appendix D). When all $\theta$ are equal, there is an intermediate STP centered around $\theta = \pi/4$ which separates the trivial ($\theta = 0$) and HOTI ($\theta = \pi/2$) phases. In this case, the analytical expression for the gap closing is given by $| \sin(\pi/4 \pm \theta_1) \sin(\pi/4 \pm \theta_2) | = \sqrt{3}/2$ at $k = (0, 0), (\pi, \pi)$. The corresponding contours are shown as black solid and green dashed lines in Fig. 3(d). Note that along the anti-diagonal with $\theta_1 = -\theta_2$, the system remains trivial. However, we observe bulk gap closings at the points $(\theta_1, \theta_2) = (\pi/4, -\pi/4)$ and $(-\pi/4, \pi/4)$ corresponding to $k = (0, \pi)$ and $(\pi, 0)$. These are visible as small spots of increased transmission probability in the numerical data of Fig. 3(c).

Away from the diagonal, there exists a continuous path, shown as a black arrow in Fig. 3(c), which connects the HOTI to the trivial phase without closing the bulk gap. Note that along this path the topology is changed by an edge gap closing. Indeed, for any path connecting the HOTI and trivial regions either the bulk or the edge gap must close, since the Majorana corner modes can only be gapped out by pairwise coupling. When adjacent corner modes hybridize through the top and bottom edges, they form counter-propagating edge modes which are visible in the transmission map of Fig. 3(d); for instance at $\theta_1 = \pi/2, \theta_2 = \pi/4$ (blue cross). In fact, the closing of the edge gap is the reason for the thin horizontal lines of Fig. 3(d) at which we observe $G = 2$, consistent with the presence of counter-propagating edge modes on both the top and bottom edges. In contrast, corner modes overlapping via the left and right edges of the network (such as at the point $\theta_1 = \pi/4, \theta_2 = \pi/2$) do not contribute to transmission, since the leads are attached to the left and right sides of the network and thus we only probe transport along the top and bottom edges.

Note that at the point $(\theta_1, \theta_2) = (\pi/4, \pi/4)$, the links
of the network are maximally coupled, in the sense that each incoming Majorana mode has an equal probability of turning clockwise or counter-clockwise at each node of the network. In the CC model and, as far as we know, in all of its generalizations \cite{42,45–51,57}, the maximally coupled limit is a gapless critical point separating topologically distinct phases. Interestingly, here it corresponds to a gapped, STP. For the parametrization shown in Figs. 3(c) and (d), in which \( \theta_1 = \theta_2 \) and \( \theta_2 = \theta_3 \), the point \( (\theta_1, \theta_2) = (\pi/4, \pi/4) \) is in the middle of the strong topological region of the phase diagram, at which the bulk gap is even largest.

To gain insight into the behavior of the network in the vicinity of \( (\theta_1, \theta_2) = (\pi/4, \pi/4) \), we study a half-plane geometry (on \( y < 0 \)). We use an ansatz wavefunction \( \Phi_{\text{tot}} = (\Phi, \lambda \Phi, \lambda^2 \Phi, \ldots)^T \) for the edge state, with a decay factor \( \lambda \) between successive unit cells along the \( y \) direction. The eigenvalue equation yielding stationary states then becomes

\[
[S_0(k_x) + \lambda S'] \Phi = e^{-i\varepsilon(k_x)} \Phi, \tag{13}
\]

\[
[\lambda^{-2} S'' + S(k_x)] \Phi = e^{-i\varepsilon(k_x)} \Phi, \tag{14}
\]

where \( S_0(k_x) \) and \( S(k_x) \) represent scattering terms within each unit cell, located at the boundary of the network (\( y = 0 \)) and away from it, respectively. The two other Ho-Chalker operators \( S' \) and \( S'' \) represent coupling to the adjacent unit cells, located respectively below and above. As discussed before, the open boundary condition is implemented by having the scattering nodes on the edge of the system completely reflect the incoming waves with unit probability amplitude (this corresponds to setting \( \theta_3 = 0 \) for those particular nodes). Considering \( \varepsilon(k_x) = 0 \) and fixing one angle as \( \theta_2 = \pi/2 \), we find two possible values for the decay constant, \( \lambda_{\pm} = (\sqrt{2} \pm 1)^2 \tan(\pi/4 - \theta_1/2)/\tan(\theta_1/2) \) which are shown in Fig. 4(a). Since normalizable edge modes require \( |\lambda| < 1 \), this criterion leads to one edge state for \( \theta_1 \in (\pi/12, \pi/3) \cup (-11\pi/12, -\pi/3) \), and two counter-propagating edge states for \( \theta_1 \in (7\pi/12, 11\pi/12) \). The ranges for \( \theta_1 \) correspond to the regions of the STP and the horizontal line of \( G = 2 \) emanating from it, respectively, as shown in Fig. 3(d). Furthermore, by performing perturbation theory in \( k_x \), we obtain the linear dispersion \( \varepsilon(k_x) = v_E k_x \), with the velocity \( |v_E| = \sqrt{2}/8 \approx 0.18 \) of the edge state, see Fig. 4(b).

V. TOPOLOGICAL INVARIANTS

In this section, we classify the different topological phases, realized by the network model, by topological invariants based on the scattering matrix of the system. In the HOTI phase, this approach is augmented with a symmetry indicator analysis whenever possible.

We first discuss the HOTI phase with its limiting case shown in Fig. 2(a). The presence of corner modes can be detected in a four-terminal geometry, in which one lead is attached to each of the corners of the system \cite{15,67}. We provide more details on this transport geometry in Appendix B.

Crucially, this setup only allows us to characterize the HOTI as an extrinsic topological phase \cite{15}, as corner leads cannot distinguish between topology due to a nontrivial bulk and due to nontrivial edges.

In the HOTI phase, both bulk and edges are gapped, as exemplified in Fig. 2(a). As such, in each of the four corner leads, incoming modes are fully back-reflected and the corresponding corner reflection matrices \( r_j \) (\( j = 1, 2, 3, 4 \)) are unitary. Furthermore, the reflection blocks \( r_j \) are also real due to particle-hole symmetry, leading to the \( \mathbb{Z}_2 \) invariants \cite{68,69}

\[
\nu_j = \text{sgn} \det r_j. \tag{15}
\]

The values \( \nu_j = \pm 1 \) are topologically distinct since the only way to interpolate between them is by going through a point for which \( \det r_j = 0 \), indicating that there is at least one fully transmitting mode connecting the leads at the different corners, either through the bulk or through the edges. Since in the HOTI phase the corner modes simultaneously appear at all four corners, we will only investigate \( r_1 \) in the following.

Conventionally, a value \( \nu_1 = 1 \) indicates the trivial phase and \( \nu_1 = -1 \) the topological phase, due to the \( \pi \) phase shift caused by the resonant reflection of waves on a zero-energy state \cite{67–69}. However, the relation is exactly inverted in our model, cf. Fig. 2(a). In particular, we see that the loop containing the corner mode is periodic and thus does not lead to a \( \pi \) phase shift. As a result, we find \( \nu_1 = 1 \) in the topological phase. Analogously, the invariant is \( \nu_1 = -1 \) in the trivial phase, with decoupled limit shown in Fig. 2(b) due to the fact that the loop at the corner is anti-periodic. Therefore, \( \nu_1 = -1 \) describes the trivial phase of the network model \cite{70}.
In Fig. 5, we show how det \( \tau_1 \) depends on the angles \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \). In Fig. 5(a), we consider the \( C_4 \) symmetric network model. We observe a good agreement with a phase diagram of Fig. 2(b), which has been obtained from the two-terminal transmission probability. The invariant correctly identifies the HOTI and trivial regions, while the STP boundaries between the HOTI (or trivial) phase and the STP are not as easily seen as in the two-terminal calculations. For this reason, we plot \( 1 - |\text{det} \tau_1| \) in Fig. 5(b), and look at small deviations from 0. The resulting phase diagram is now in excellent agreement with Fig. 2(b).

Once \( \theta_1 = \theta_3 \) and \( \theta_2 = \theta_4 \), such that \( C_4 \) is broken, the system produces the phase diagram shown in Fig. 5(c). Comparing to Fig. 2(c), it differs significantly. The difference occurs because the two-terminal transmission probability is only sensitive to gap closings along one set of edges, while the four-terminal geometry detects gap closings along all edges. For this reason, in Fig. 2(c), we only observe the gap closings on the top and bottom edges, while Fig. 5(c) contains information on gap closings also on the left and right edges. Again the features are better visible in Fig. 5(d), where we plot \( 1 - |\text{det} \tau_1| \). We see that the HOTI phase is separated from the trivial phase by a gap closing either in the bulk or along the edges.

In the presence of a \( C_4 \) symmetry, the HOTI phase is a bulk topological phase, such that a transition to a different phase requires the closing and reopening of the bulk (and not just the edge) gap [10]. To show this fact mathematically, we define a bulk \( Z_2 \) topological invariant \( Q = \pm 1 \) based on the \( C_4 \) symmetry eigenvalues of the Ho-Chalker eigenstates for a translationally invariant system at the \( C_4 \) invariant points of Brillouin zone \( \Gamma = (0, 0) \) and \( M = (\pi, \pi) \), similar to how this is done for static Hamiltonian systems [10, 71–74]. It reads

\[
Q = d_4^+ (M) d_4^- (\Gamma)^* = d_4^- (M) d_4^+ (\Gamma)^*,
\]

where \( d_4^\pm \) are the \( C_4 \) eigenvalues that square to \( \pm i \) in the ‘occupied bands’ of the Ho-Chalker operator. Even though the latter has a periodic spectrum, the phase-rotation and particle-hole symmetries enable us to unambiguously define the occupied bands as those in the interval \(-\pi/4 < \epsilon < 0 \) (which is half of the fundamental phase domain).

Since the \( C_4 \) symmetry operator of Eq. (10) obeys \( R^4 = -1 \), its eigenvalues are \( e^{\pm i\pi/4} \) and \( e^{\pm 3i\pi/4} \). In the HOTI phase, we find that the two occupied bands in the fundamental domain have eigenvalues \( e^{-i\pi/4} \) and \( e^{-3i\pi/4} \) at the \( \Gamma \) point. At the \( M \) point, these bands change their symmetry eigenvalues to \( e^{i3\pi/4} \) and \( e^{i\pi/4} \), respectively. We thus obtain \( Q = -1 \) which corresponds to the HOTI phase. In the trivial phase, we find the \( C_4 \) symmetry indicators of occupied bands do not change between \( \Gamma \) and \( M \), which leads to a value of \( Q = 1 \).

Finally, the topological invariant of the STP can be expressed as the winding number [69, 75, 76]

\[
W = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \frac{d}{d\varphi} \log \text{det} \tau(\varphi),
\]

where \( \tau(\varphi) \) is the reflection block of the two terminal scattering matrix. Here, we impose twisted boundary conditions in the vertical direction, with twist angle \( \varphi \), such that \( \varphi = 0 \) corresponds to PBC. We have checked that all phases of Fig. 3 in which \( G = 0 \) with PBC and \( G = 1 \) with OBC have \( W = -1 \), confirming that they are indeed STPs.

VI. DISORDER

We study the effect of disorder on the network model by adding a random term \( \delta \theta_i \) to each \( \theta_i \), drawn uniformly for each angle from the uniform distribution \([-w, w]\), with the disorder strength \( w \), \( 0 \leq w \leq \pi \). The disorder breaks the translation symmetry but preserves particle-hole symmetry since the disordered scattering matrices remain orthogonal. As the system is still in the class D, we expect that, with sufficiently strong disorder, the phase boundaries separating HOTI, STP, and trivial networks to evolve into a delocalized, thermal metal phase [42, 45, 46, 60, 77]. In contrast, for weak disorder, the system should be insensitive to disorder, with only minor modifications of the phase diagram with respect to the clean case.
We verify this behavior by computing the average two-terminal transmission of the disordered network model. In Fig. 6, without disorder ($w = 0$), we observe all three phases: trivial, STP, and HOTI. All of them have a localized bulk and thus are robust against weak disorder ($w \ll 1$). However, as disorder strength is increased beyond $w \approx 1$, the bulk begins to delocalize, such that each phase undergoes a metal-insulator transition.

Next, we fix $w = 1.1$ and in Fig. 7 plot the average transmission probability as a function of different $\theta$s, to show how the phase diagram of Fig. 3 changes in the presence of moderate disorder. Among the four decoupled limits of the clean $C_4$ symmetric case, the Majorana flat band is first to be delocalized due to its gapless bandstructure. In addition, the network model near the phase boundaries (or any other gap closing points) can also be easily delocalized, as seen in Figs. 7(a) and (b). As such, we observe that in the disordered phase diagram the phase transition lines separating HOTI, STP, and trivial phases evolve into finite-width delocalized regions, as do the gapless points at $(\theta_1, \theta_2) = (\pi/4, -\pi/4)$ and $(-\pi/4, \pi/4)$. Nevertheless, the STP and HOTI phases retain their nontrivial nature, despite the presence of disorder. The transmission probability still changes from $G = 1$ with PBC to $G = 0$ with OBC in the STP, signaling that the chiral Majorana edge mode is robust against disorder. We have confirmed that the scattering matrix invariants characterizing the HOTI and STP retain their nontrivial values. Finally, we observe that the counter-propagating edge modes separating HOTI and trivial phases in the $C_4$ broken case [Fig. 7(d)] remain conducting, albeit with a conductance $G < 2$. In fact, they are protected by an average translation symmetry of the ensemble of disorder realizations [78–82].

VII. CONCLUSION

We have introduced a network model realization of a higher-order topological phase. The system shows many similarities to the HOTI phases present in static or periodically-driven Hamiltonian systems, namely it hosts zero-modes localized at the corners, which are protected by the bulk and edge gaps due to a combination of a particle-hole symmetry and a four-fold rotation symmetry. However, the network model additionally features a phase-rotation symmetry. As a result, there are a total of 16 topologically protected corner states, more than for a four-fold symmetric static or Floquet system. The HOTI phase is robust against disorder, being characterized by nontrivial scattering matrix invariants as well as by a more conventional invariant based on symmetry indicators.

We hope that our work will motivate further theoretical and experimental research into network models. The latter (as well as HOTIs) have been successfully realized using optical fibers and coupled ring resonators [83–86]. In some of these platforms it is also possible to directly measure the scattering matrix invariants [87, 88]. Therefore, the HOTI network proposed here, and its invariants could be experimentally probed in these platforms. Finally, our work provides an example of a network model for a point group symmetry protected topological phase. These are by now the most studied types of topology in condensed-matter, Hamiltonian systems, but have not yet been explored on the level of network models. We believe that different types of topological crystalline insulators protected by different lattice symmetries may be
realized using arrays of scattering matrices, and we plan to further explore this direction in the future.

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Appendix A: BBH Model

In this section, we briefly review the main features of the BBH model and explain its similarities to our network model.

The BBH model consists of non-interacting spinless electrons hopping on a square lattice. Only the nearest-neighbor hoppings are nonzero, and they are staggered in both directions, consisting of alternating weak and strong hopping strengths, as shown in Fig. 8(a). A magnetic π-flux pierces each plaquette, ensuring that the bulk and edges are gapped.

The momentum-space Hamiltonian reads

\[ h(k) = (\gamma_x + \lambda_x \cos k_x) \tau_x \sigma_0 - \lambda_x \sin k_x \tau_y \sigma_z - (\gamma_y + \lambda_y \cos k_y) \tau_y \sigma_y - \lambda_y \sin k_y \tau_y \sigma_z. \] (A1)

Here, \( k = (k_x, k_y) \) are the two momenta and Pauli matrices \( \tau \) and \( \sigma \) represent the sublattice degree of freedom and the site degree of freedom, respectively. Intracell hoppings are \( \gamma_x \) and \( \gamma_y \), while \( \lambda_x \) and \( \lambda_y \) denote intercell couplings.

The system has a particle-hole symmetry \( P = \tau_x \mathcal{K} \), where \( \mathcal{K} \) denotes complex-conjugation. Once \( \gamma_x = \gamma_y \equiv \gamma \) and \( \lambda_x = \lambda_y \equiv \lambda \), the system is \( C_4 \) symmetric. It obeys \( \mathcal{R} h(k_x, k_y) \mathcal{R}^{-1} = h(k_y, -k_x) \), where the \( C_4 \) symmetry operator reads \[ \mathcal{R} = \begin{pmatrix} 0 & \sigma_0 \\ -i \sigma_y & 0 \end{pmatrix}. \] (A2)

We now discuss the similarities between BBH model and network model, and focus first on the \( C_4 \) symmetric limit. If the intracell coupling strength is much weaker than intercell coupling strength, both systems with open boundaries would support four topologically-protected zero-energy corner states. Then, in the opposite limit, both systems would also yield a trivially gapped bulk.

Figure 8. (a) Sketch of the BBH model. The black dots represent sites. The intracell/intercell hoppings are shown in red/blue, respectively, and dashed lines denote negative hopping strengths (used to implement the \( \pi \) fluxes). The gray rectangle depicts a unit cell. (b) Phase diagram of the BBH model Eq. (A1). The gray area indicates the HOTI phase, while the red circles denote the points of the bulk gap closings. Moreover, the gap closings along the \( x \)- and \( y \)-edges are shown in blue and green, respectively. The arrows indicate possible paths in parameter space that lead to bulk gap (red), \( x \)-edge (blue) and \( y \)-edge (green) gap closings.

Furthermore, for the BBH model, the bulk gap closing point is at \( k = (0, 0) \) \( [k = (\pi, \pi)] \) and occurs when \( \gamma = -\lambda \) \( [\gamma = \lambda] \), as shown on Fig. 8(b). Similarly, the network model also possesses two types of gap closing at \( k = (0, 0) \) and \( k = (\pi, \pi) \), shown in Fig. 3.

Once the \( C_4 \) symmetry is broken, the BBH model Eq. (A1) is in the HOTI phase as long as \( \gamma_x < \lambda_x \) and \( \gamma_y < \lambda_y \). As denoted in Fig. 8(b), the phase without corner states can be reached by the \( x \)-edge gap closing \( (\gamma_x = \lambda_x) \) or the \( y \)-edge gap closing \( (\gamma_y = \lambda_y) \). The consequence of the edge gap closing is the appearance of two counter-propagating modes per edge. Then in the network model, once the \( C_4 \) symmetry is broken, the system is also in a HOTI phase as long as this phase doesn’t reach edge gap closing lines, which corresponds to a counter-propagating modes with \( G = 2 \), see Figs. 3(d) and 5(c-d).

Appendix B: Transport geometries

In this section, we detail two transport geometries used for obtaining the phase diagrams in Figs. 3 and 5, respectively. We attach the code used to perform these calculations as an ancillary file.

For the conductance calculations in Fig. 2, we use two semi-infinite leads attached to opposite vertical edges of the 2D system, as shown in Fig. 9(a). Each lead is composed of an array of Majorana modes, thus preserving particle-hole symmetry. As such, in the Majorana basis the scattering matrix is real, and it has a size of \( 4N_y \times 4N_y \) for a \( N_x \times N_y \) unit cell system. To compute this scattering matrix, we consider that the system consists of a sequence of 1D slices, as shown for a \( 2 \times 2 \) system.
If we label the scattering matrices for two adjacent slices from left to left and from right to right, respectively. The scattering matrix blocks [89] in a HOTI phase. The lead modes are depicted in red. The other modes are represented with $k$ ($k > 4$). The vertical green lines mark the different 1D slices which are combined to yield the full scattering matrix (see text). In panel (b), we show the four-terminal transport geometry with the leads only connected to the corners of the system.

in Fig. 9(a). Each slice contains an array of scattering nodes arranged vertically. Every slice has $4N_y$ incoming/outgoing modes, related by the scattering matrix

$$S_{\text{slice}} = \begin{pmatrix} t_{\text{slice}} & t'_{\text{slice}} \\ t'_{\text{slice}} & t_{\text{slice}} \end{pmatrix}, \quad (B1)$$

where $t_{\text{slice}}$ and $t'_{\text{slice}}$ are reflection matrices of the slice from left to left and from right to right, respectively. The transmission matrix from left to right, and vice versa are denoted by $t_{\text{slice}}$ and $t'_{\text{slice}}$.

The final scattering matrix of the full network is obtained by combining scattering matrices of all $2N_x$ slices. If we label the scattering matrices for two adjacent slices as $S_L$ (left) and $S_R$ (right), respectively, then the scattering matrix $S_C$ of the new slice after combination has matrix blocks [89]

$$t_C = t_L + t'_L t_R (1 - t'_R t_R^{-1}) t_L,$$

$$t'_C = t'_R (1 - t_R t'_R^{-1}) t'_L,$$

$$t'_C = t'_L + t_R t'_L (1 - t_R t'_R^{-1}) t'_L. \quad (B2)$$

Therefore, after multiple combination processes, one can gradually include all slices and obtain the combined scattering matrix for the whole system. With this, we can calculate the conductance as $G = \text{tr}(t t'^\dagger)$, where $t$ is the transmission block of the full, two-terminal scattering matrix: $S_{2\text{-term}}$.

For the calculation of the topological invariant $\nu_1$ to capture the presence of corner modes, we use a four-terminal geometry depicted in Fig. 9(b). The associated scattering matrix, $S_{4\text{-term}}$, can be calculated from $S_{2\text{-term}}$ in the following manner. We reorder the incoming and outgoing modes of $S_{2\text{-term}}$ such that it has the structure

$$S_{2\text{-term}} = \begin{pmatrix} S_A & S_B \\ S_C & S_D \end{pmatrix}. \quad (B3)$$

Here, $S_A$ is a $4 \times 4$ matrix that related four corner incoming and outgoing modes, denoted in Fig. 9(b). Therefore, $(\Psi^+_1, \Psi^+_2, \Psi^+_3, \Psi^+_4)^T = S_A (\Psi^-_1, \Psi^-_2, \Psi^-_3, \Psi^-_4)^T$. In general, $S_A$ is not necessarily a unitary matrix as the corner incoming modes can contribute to the outgoing modes not pinned at corners. Matrix $S_B$ has $4 \times (4N_y - 4)$ entries, and relates four outgoing corner modes with all remaining incoming amplitudes, while $S_C$ does the opposite. Finally, the matrix $S_D$ has $(4N_y - 4) \times (4N_y - 4)$ entries, and relates all non-corner incoming to non-corner outgoing modes.

In order to remove leads from the $x$-edges of the network model, except at the points of corner terminals, we assume $\Psi^+_k = \Psi^-_k$, for $4 < k < 4N_y$. Then, solving the set of these $4N_y - 4$ number of equations, we can eliminate all but four incoming/outgoing modes. The resulting scattering matrix reads

$$S_{4\text{-term}} = S_A + S_B (1 - S_D)^{-1} S_C. \quad (B4)$$

**Appendix C: Symmetries of the network model**

We discuss here how different symmetries of the Ho-Chalker operator can be used to fully characterize the phase diagram discussed in the main text.

In the Ho-Chalker operator, each non-zero element is proportional to either $\sin \theta_i$ or $\cos \theta_i$ ($i = 1, 2, 3, 4$). This leads to

$$S(\theta_1, \theta_2, \theta_3, \theta_4) = -S(\theta_1 + \pi, \theta_2 + \pi, \theta_3 + \pi, \theta_4 + \pi). \quad (C1)$$

The minus sign corresponds to a $\pi$ shift of all eigenphases, which, together with the $\pi/2$ shift caused by phase-rotation symmetry, means that both $S(\theta_1, \theta_2, \theta_3, \theta_4)$ and $S(\theta_1 + \pi, \theta_2 + \pi, \theta_3 + \pi, \theta_4 + \pi)$ have identical spectra and therefore share the same topological properties.

Another symmetry constraining the phase diagram is

$$U_1 S(\theta_1, \theta_2, \theta_3, \theta_4) U_1^\dagger = S(-\theta_1, -\theta_2, \theta_3, \theta_4) \quad (C2)$$

with $U_1 = 1_{4 \times 4} \otimes \text{diag}(1, 1, -1, 1)$. Because $S(\theta_1, \theta_2, \theta_3, \theta_4)$ and $S(-\theta_1, -\theta_2, \theta_3, \theta_4)$ have identical spectra, they are in the same topological phase.

Next, one can find that

$$S(\theta_1, \theta_2, \theta_3, \theta_4, k_x, k_y) = S(\theta_1, \theta_2, -\theta_3, -\theta_4, k_x + \pi, k_y + \pi). \quad (C3)$$

Therefore, these two systems are also in the same topological class: STP, HOTI, or trivial.

Finally, we introduce a fourth symmetry,

$$U_2 S(\theta_1, \theta_2, \theta_3, \theta_4) U_2^\dagger = S(\theta_1 + \pi, \theta_2 + \pi, -\theta_3, -\theta_4). \quad (C4)$$

Where

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (C5)$$
Taken together, the symmetries listed above help to explain how the different topological phases of Fig. 3 are related to each other.

Appendix D: $C_4$ symmetry when $\theta_1 = \theta_3 = -\theta_2 = -\theta_4$

Along the diagonal line of $\theta_1 = -\theta_3$ in Figs. 3(c) and (d), the system still preserves $C_4$ symmetry, but the $C_4$ symmetry condition becomes,

$$\mathcal{R}'S(k_x, k_y)\mathcal{R}'S = S(-k_y + \pi, k_x + \pi),$$  \hspace{1cm} (D1)

with

$$\mathcal{R}' = \begin{pmatrix} 0 & 0 & 0 & \mathcal{R}_4' \\ \mathcal{R}_3' & 0 & 0 & 0 \\ 0 & \mathcal{R}_2' & 0 & 0 \\ 0 & 0 & \mathcal{R}_1' & 0 \end{pmatrix}, \quad \mathcal{R}_i' = K_{i},$$  \hspace{1cm} (D2)

with the $K_i$ matrices defined as in Eq. (10). If momentum space is discretized (for instance in a torus geometry), this symmetry constraint needs an even number of discretized momenta $k_x$ and $k_y$ in order to be preserved. Correspondingly, in real space the rotation axis is positioned at the corner of the unit cell.

At the two high symmetry points $X = (\pi, 0)$ and $Y = (0, \pi)$ we have $[\mathcal{R}', S] = 0$ in momentum space. Like Section V, we can again determine the $C_4$ eigenvalues of the occupied bands. Since $(\mathcal{R}')^4 = -1$, these eigenvalues are $e^{\pm i\pi/4}$ and $e^{\pm 3i\pi/4}$. When changing $\theta_1 = -\theta_3$, so as to move on the diagonal of Fig 3(c), we observe that there is a bulk gap closing and reopening which occurs at the point $\pi/4$. This gap closing is parabolic as a function of the $\theta$s, and so does not involve a band inversion in which states with different $C_4$ eigenvalues cross $\varepsilon = 0$ and are interchanged. As such, no topological phase transition takes place at $\theta_1 = -\theta_2 = \pi/4$.

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Even though the BBH model is probably well known by now, we briefly summarize its main features and compare it with our network model in Appendix A.

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