A simplified climate model and maximum entropy production

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Abstract A simplified climate model based on maximum entropy production, described by a variational principle, is revisited and an analytical solution to its Euler–Lagrange equation is found. Mindful of controversy about maximum or minimum entropy production in open thermodynamical systems, we show that the solution extremizing the action integral corresponds to a maximum.

1 Introduction

The principle that open thermodynamical systems in nature tend to maximize entropy production (MEP) has been debated extensively in the earth sciences, and in atmospheric science in particular. The climate is an open thermodynamical system in which energy flows from tropical to polar regions. A simple model based on MEP was proposed by Paltridge in 1975 [1] to calculate the temperature $T(\vartheta)$ and the cloud cover as functions of the latitude $\vartheta$.

MEP has been the subject of much discussion and controversy [2–4]. Studies aimed at testing this model use simple one-dimensional energy balance models [1,5,5–15], with a few exceptions studying general circulation models [16–20]. Although, in numerical studies of these models, one can check easily whether entropy production is maximized, minimized, or has a saddle point, in these studies one is always committed to particular choices of parameters and initial conditions, and numerical confirmation does not constitute mathematical proof. Some authors in the earth sciences contend that that there is no need for a general mathematical proof and that numerical checks are sufficient, but we argue that this attitude is contrary to the spirit of science. Anyway, it is not difficult to obtain a proof for simple systems, as shown below. From the mathematical point of view, most statements about MEP are not backed by rigorous mathematical proofs [21,22] and this probably reflects the fact that MEP is not understood. It is quite possible that MEP is not a fundamental law, but just an approximation in which different scales are separated, but the parameter, or ratio of variables, characterizing the MEP regime has not been identified. In this case, MEP could be a quasi-static approximation in which effects occurring on a longer timescale are neglected in favour of processes occurring on a shorter timescale [23].
If one takes a slightly broader view of open non-equilibrium thermodynamical systems beyond atmospheric models, the spectrum of possible situations appears rich and complicated. In one-dimensional diffusion problems for systems in steady state, a maximum of entropy production is usually associated with closed boundary conditions and a minimum with open ones [24] (diffusion is quite distinct from convection and turbulence, but this example shows that opposite outcomes can sometimes occur in the same kind of physical processes). For other systems, the situation is not clear. Therefore, a priori statements should be backed by testing. A similar controversy on entropy production rate can be found in the determination of equilibrium beach profiles in oceanography. In the zone seaward from the wave breaking point, wave friction against the sea bed and transport of sediments dissipate energy and a one-dimensional model of this open thermodynamical system expressed by a variational principle can be constructed [25,26], with different authors disputing whether the entropy production rate is maximized or minimized (recent work determines that it is a minimum [27–29]). A similar problem occurs in the erosion of glacial valleys, where friction1 is maximized instead [30], which has also been the subject of a minimum/maximum controversy [31–35]. The uncertainty in the theoretical foundations of MEP reflects our incomplete knowledge of non-equilibrium thermodynamics. A practical lesson gained from the literature is that intuition often fails in MEP-based models and the nature of the extremum attained by the system should be assessed—which is not hard to do numerically—and proved rigorously in general whenever possible.

Here, we revisit a simple one-dimensional model with energy transport in the meridional direction, based on MEP, which was proposed in [36]. Given the latitudinal distribution of energy absorbed at short wavelengths, the model calculates the latitudinal distribution of energy emitted at long wavelengths and the meridional heat transport by means of a variational principle that extremizes the entropy production rate. The Euler–Lagrange equation produced in this way was solved numerically in [36].

First, we solve analytically the central equation of the model. Second, we show explicitly that this solution of the Euler–Lagrange equation indeed corresponds to a maximum of entropy production. This is necessary because Ref. [36] does not prove mathematically that the extremum of the entropy production rate is a maximum (a similar, but long and rather involved proof for Paltridge’s 1975 model [1] was sketched only twenty years later [37]). While the derivation of the model’s main equation requires only that the solution be an extremum of the action integral, to understand the physics and validate MEP it is crucial to determine whether this extremum is a maximum or a minimum. A recurrent puzzle in MEP-based system is that sometimes there is a maximum and sometimes a minimum of entropy production [1,5–20,27–29]: here we prove (as opposed to checking numerically for special configurations) that a maximum always occurs in this model.

2 The model

The Murakami and Kitoh one-dimensional climate model in the meridional direction [36] is based on a very idealized radiative formulation. It assumes that:

- The system is in steady state;
- The maximum entropy production hypothesis (MEP);
- The distribution of the absorbed solar radiation is a given (even) function $I(\theta)$.

1 Friction, however, does not coincide with entropy production rate.
The model calculates the long-wave radiation emitted \( O(\vartheta) \). No assumptions are made about the relationship between heat transport and temperature gradient. This model neglects the vertical structure of the atmosphere,\(^2\) cloud radiative processes, the oceans, and the atmosphere–ocean coupling. In reality, the incoming radiation \( I(\vartheta) \) is not an even function on Earth, because of the asymmetry of the albedo, and it does not vanish at the poles because of the Earth obliquity.

More in detail: assume that the climate (an open thermodynamical system) is in steady state with energy flowing from the equator to polar regions, maximum entropy production, and that the radiative flux density absorbed by the Earth at short wavelengths (the insolation), \( I(\vartheta) \), is a given function of the latitude \( \vartheta \in [-\pi/2, \pi/2] \). By symmetry,\(^3\) it must be an even function, \( I(-\vartheta) = I(\vartheta) \) and it must vanish at the poles, \( I(\pm \pi/2) = 0 \) and decrease going from the equator to a pole, \( dI/d\vartheta > 0 \) if \(-\pi/2 < \vartheta < 0\) and \( dI/d\vartheta < 0 \) if \( 0 < \vartheta < \pi/2 \). Let \( O(\vartheta) = \sigma T^4(\vartheta) \) be the flux density radiated by the Earth into space at the latitude \( \vartheta \), where \( \sigma \) is the Stefan–Boltzmann constant and \( T(\vartheta) \) is the absolute temperature at the same latitude. At low latitudes absorption dominates and \( O(\vartheta) < I(\vartheta) \), while at high latitudes it is \( O(\vartheta) > I(\vartheta) \).

The net radiative flux density is \( I(\vartheta) - O(\vartheta) \) and the heat flux transported to higher latitudes is

\[
2\pi R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \cos \vartheta \left[ I(\vartheta) - O(\vartheta) \right],
\]

where \( R \) is the Earth’s radius. The elementary entropy production rate is related to the elementary heat production rate \( \mathrm{d}q \) by \( \mathrm{d}s = \mathrm{d}q/T \) and the finite entropy production rate is

\[
A = 2\pi R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \cos \vartheta \left[ I(\vartheta) - O(\vartheta) \right] / T(\vartheta),
\]

a functional of the function \( O(\vartheta) \). The variational principle consists of extremizing this entropy production rate subject to the constraint

\[
\int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \cos \vartheta \left[ I(\vartheta) - O(\vartheta) \right] = 0
\]

expressing the fact that the climate system is in steady state \([36,38]\). This constrained variational principle is simplified as follows in Ref. \([36]\): define \( x \equiv \sin \vartheta \) and

\[
y(x) \equiv \int_{-1}^{x} \mathrm{d}\bar{x} \left[ I(\bar{x}) - O(\bar{x}) \right];
\]

then

\[
y'(x) = I(x) - O(x)
\]

(where a prime denotes differentiation with respect to \( x \)) and \( y(\pm 1) = 0 \) \([36]\). The action integral \((2)\) (divided by the irrelevant constant \(2\pi R^2 \sigma^{1/4})\) is converted into \([36]\)

\[
J[y(x)] = - \int_{-1}^{+1} \mathrm{d}x \ y'(x) \left[ I(x) - O(x) \right]^{-1/4} \equiv \int_{-1}^{+1} \mathrm{d}L \left( y'(x), x \right)
\]

where \( I \geq y' \) (equivalent to \( O \geq 0 \)) is always satisfied. Now we have an unconstrained variational principle \( \delta J = 0 \) with fixed boundaries. Since the Lagrangian \( L \) does not depend

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\(^2\) The radiative budget is usually non-local, i.e., a functional of the vertical temperature profile, even in simple models when several layers of atmosphere are considered.

\(^3\) In the end, Ref. \([36]\) assumes \( I(\vartheta) = \beta - \alpha \sin^2 \vartheta \), with \( \alpha \) and \( \beta \) constants.
explicitly on \( y \), the Euler–Lagrange equation
\[
\frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 0
\] (7)
gives conservation of the momentum \( \Pi_y = \frac{\partial L}{\partial y'} \) canonically conjugated to \( y \), or
\[
(I - y')^{5/4} + \frac{3y'}{C} = \frac{4I}{C},
\] (8)
where \( C \) is an integration constant (which is fixed by the boundary conditions, as described below). Equation (5) gives
\[
C = \frac{I}{O^{5/4}} > 0.
\] (9)

The first integral (8) of the Euler–Lagrange equation apparently was missed in [36], where the authors report the second-order Euler–Lagrange equation, which they integrate numerically. A second integration is unnecessary. In fact, the derivative \( y' \) was introduced in [36] to simplify the original variational problem, which is solved by determining \( O \) or \( y' \). Equation (5) gives immediately the analytical solution \( O(x) \) of the problem through
\[
I(O) = C O^{5/4} - 3O
\] (10)
or, equivalently,
\[
I(\vartheta) = C \sigma^{5/4} T^{5}(\vartheta) - 3\sigma T^{4}(\vartheta).
\] (11)
Equation (10) cannot be inverted to obtain \( O(I) \) explicitly, but this is not crucial.

The integration constant \( C \) is fixed by the boundary condition \( y(1) = 0 \) (the other boundary condition \( y(-1) = 0 \) is satisfied by construction and does not provide new information):
\[
y(1) = \int_{-1}^{+1} dx \left[ I(x) - O(x) \right] = 0
\] (12)
becomes, using Eq. (10),
\[
C = \frac{4 \int_{-1}^{+1} dx \ O(x)}{\int_{-1}^{+1} dx \ O^{5/4}(x)} = \frac{4 \int_{-1}^{+1} dx \ I(x)}{\int_{-1}^{+1} dx \ O^{5/4}(x)}.
\] (13)
Alternatively, using the information that \( I \) vanishes at the poles, Eq. (10) yields
\[
C = \frac{3}{O^{1/4}(\pm \pi/2)}
\] (14)
(this value can be obtained also by setting \( I(\pm \pi/2) = 0 \) in Eq. (9)).

3 Maximum or minimum?

Consider varied paths \( y(x, a) \), parametrized by the parameter \( a \), around the actual solution \( y(x, 0) \) that extremizes the functional \( J[y(x)] \),
\[
y(x, a) = y(x, 0) + a\eta(x).
\] (15)
We have
\[
\frac{\partial y}{\partial a} = \eta, \quad \frac{\partial y'}{\partial a} = \frac{d\eta}{dx}, \quad \frac{\partial^2 y}{\partial a^2} = \frac{\partial^2 y'}{\partial a^2} = 0.
\] (16)
The second variation of the functional $J$ gives (eg., [39])

$$\frac{\partial^2 J}{\partial a^2} = \int_{-1}^{+1} dx \left[ \frac{\partial^2 L}{\partial y'^2} \left( \frac{d\eta}{dx} \right)^2 + 2 \frac{\partial^2 L}{\partial y \partial y'} \eta \frac{d\eta}{dx} + \frac{\partial^2 L}{\partial y^2} \eta^2 \right].$$

(17)

Since the Lagrangian $L$ does not depend explicitly on $y$, this integral reduces to

$$\frac{\partial^2 J}{\partial a^2} = \int_{-1}^{+1} dx \frac{\partial^2 L}{\partial y'^2} \left( \frac{d\eta}{dx} \right)^2 = -\frac{1}{4} \int_{-1}^{+1} dx \frac{(I - 3y'/8)}{(I - y')^{9/4}} \left( \frac{d\eta}{dx} \right)^2,$$

(18)

where $I - y' = O = \sigma T^4 > 0$ so that $I > y' > 3y'/8$, hence the numerator of the fraction in the integral is positive; the denominator is positive, and $(d\eta/dx)^2 \geq 0$ cannot vanish everywhere. As a result, the integral (18) is positive-definite, not only on-shell (i.e., when evaluated on the extremizing trajectories) but always. Then $\partial^2 J/\partial a^2 < 0$, and the extremum is a maximum [39] of the entropy production rate described by the action integral (6).

4 Conclusions

Neglecting the vertical structure of the atmosphere, the obliquity of the earth, the albedo asymmetry between the two hemispheres, and many other factors no doubt oversimplifies the real description of the Earth’s climate, but conceptual box models still have value. They provide insight into the essentials of the energy balance without the burden of a myriad of complicated details; they can be used for quick tests of numerical codes; and they are valuable pedagogical tools.

We have revisited the simplified climate model of [36] and have simplified and solved its main equation, which is derived from an action integral corresponding to the entropy production rate and involves radiative absorbed and radiated fluxes and poleward transport of energy from tropical regions. Per se, the analytical equation (10) does not require numerical integration (although some numerics are still needed to plot the physical quantities, which are given in [36] following full numerical integration and are not reproduced here). Although MEP is routinely verified numerically in this kind of model, it is not understood. A first step consists of proving rigorously that entropy production is maximum for all values of the parameters and initial conditions in their physical ranges. Moreover, there have often been surprises in natural processes associated with open thermodynamical systems, in which entropy production is sometimes maximized and other times minimized, and it is useful to set MEP on a firm footing with rigorous statements before proceeding with numerical studies. Indeed, MEP is taken as an assumption or principle, while it is likely to be just an approximation valid in a certain regime that is not even beginning to being characterized in terms of known variables and parameters. Here we have shown that, in the one-dimensional climate model of [36], the entropy production rate is indeed maximized. Our proof could perhaps stand as an example for more realistic models in which the atmosphere is stratified and the temperature and the radiative budget are non-local. These would be necessary steps in order to make the model more realistic.

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