Selanjutnya, \( \langle \cdot, \cdot \rangle \) merupakan produk skalar standar dalam ruang Hilbert \( L^2([-a,a]) \):

\[
\langle u, v \rangle = \int_{[-a,a]} u(t)\overline{v(t)} \, dt, \quad \forall u, v \in L^2([-a,a]).
\]

1. Introduction.

Pencarian spektrum teori Fourier operator terbatas pada interval simetris tak terhingga \([-a,a] \):

\[
(F_E x)(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{it\xi} x(\xi) \, d\xi, \quad t \in E, \quad E = [-a,a],
\]

\( F_E : L^2(E) \rightarrow L^2(E), \quad (1.1) \)
is closely related to study of the differential operator generated by the differential expression (or formal differential operator) $L$:

$$\left( Lx \right)(t) = -\frac{d}{dt} \left( \left( 1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} \right) + t^2 x(t).$$  \quad (1.2)

The relationship between the spectral theory of the integral operator $\mathcal{F}_E^* \mathcal{F}_E$, $E = [-a, a]$, and a differential operator generated by the differential expression $L$ was discovered in the series of remarkable papers [SlPo], [LaP1], [LaP2], where this relationship has been ingeniously used for developing the spectral theory of the operator $\mathcal{F}_E^* \mathcal{F}_E$. (See also [Sl2], [Sl3].) Actually the reasoning of [SlPo], [LaP1], [LaP2] can be easily applied to the spectral theory of the operator $\mathcal{F}_E$ itself.

It is a certain system of eigenfunctions related to the differential expression $L$ which was used in [SlPo], [LaP1], [LaP2]. These eigenfunctions are known as prolate spheroidal wave functions. The prolate spheroidal wave functions themselves were used much before the series of the papers [SlPo], [LaP1], [LaP2] was published. These functions naturally appear by separation of variables for the Laplace equation in spheroidal coordinates. However, it was the works [SlPo], [LaP1], [LaP2] where the prolate functions were first used for solving the spectral problem related to the Fourier analysis on a finite symmetric interval. Until now, there is no clear understanding why the approach used in [SlPo], [LaP1], [LaP2] works. This is a lucky accident which still waits for its explanation. (See [Sl3].)

Actually eigenfunctions are related not to the differential expression itself but to a certain differential operator. This differential operator is generated not only by the differential expression but also by certain boundary conditions. In the case $E = (-\infty, \infty)$, the differential operator generated by the differential expression $-\frac{d^2}{dt^2} + t^2$ on the class smooth finite functions (or the class of smooth fast decaying functions) is essentially self-adjoint: the closure of this operator is a self-adjoint operator. Thus in the case $E = (-\infty, \infty)$ there is no need to discuss the boundary condition.

In contrast to the case $E = (-\infty, \infty)$, in the case $E = [-a, a]$, $0 < a < \infty$, the minimal differential operator related to the formal differential operator $-\frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \frac{d}{dt} \right) + t^2$ is symmetric but is not self-adjoint. This minimal operator admits the family of self-adjoint extensions. Each of this self-adjoint extensions is described by a certain boundary conditions at the end points of the interval $[-a, a]$. The set of all such extensions can be parameterized by the set of all $2 \times 2$ unitary matrices.

It turns out that only one of these extensions commutes with the truncated Fourier operator $\mathcal{F}_E$, $E = [-a, a]$. To our best knowledge, until now no attention was paid to this aspect. In the present paper, we investigate the question which extensions of the minimal differential operator generated by $L$, (1.2), commute with $\mathcal{F}_E$, $E = [-a, a]$. 

The formal operator $L$ is of the form

$$L = M + Q,$$

(1.3a)

where

$$(Mx)(t) = -\frac{d}{dt} \left( \left( 1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} \right),$$

(1.3b)

$$(Qx)(t) = t^2 x(t).$$

(1.3c)

The operator $L$ is said to be the prolate spheroid differential operator. The operator $M$ is said to be the Legendre differential operator. The operator $Q$ is a bounded self-adjoint operator in $L^2([-a,a])$. Therefore the operators $L$ and $M$ are "equivalent" from the viewpoint of the extension theory: if one of these operators is self-adjoint on some domain of definition $D$, then the other is self-adjoint on $D$ as well.

2. Analysis of solutions of the equation $Mx = \lambda x$ near singular points.

For the differential equation

$$-\frac{d}{dt} \left( \left( 1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} \right) = \lambda x(t), \quad t \in \mathbb{C},$$

(2.1)

considered in complex plane, the points $-a$ and $a$ are the regular singular point. Let us investigate the asymptotic behavior of solutions of the equation (2.1) near these points. (Actually we need to know this behavior only for real $t \in (-a,a)$ only, but it is much easier to investigate this question using some knowledge from the analytic theory of differential equation.) Concerning the analytic theory of differential equation see [Sm, Chapter 5].

Let us outline an analysis of solution of the equation near the point $t = -a$. Change of variable

$$t = -a + s, \quad x(-a + s) = y(s)$$

reduces the equation (2.1) to the form

$$s \frac{d^2 y(s)}{ds^2} + f(s) \frac{dy(s)}{ds} + g(s,\lambda)y(s) = 0,$$

(2.2)

where $f(s)$ and $g(s)$ are functions holomorphic within the disc $|s| < 2a$, moreover $f(0) = 1$:

$$f(s) = 1 + \sum_{k=1}^{\infty} f_k s^k, \quad g(s,\lambda) = \sum_{k=0}^{\infty} g_k(\lambda) s^k.$$  

(2.3)

An explicit calculation with power series gives:

$$f_1 = -\frac{1}{2a}; \quad g_0 = \frac{\lambda a}{2}, \quad g_1 = \frac{\lambda}{4}.$$  

(2.4)
Now we turn to the analytic theory of differential equations. The results of this theory which we need are presented for example in [Sm, Chapter 5], see especially section 98 there. We seek the solution of the equation (2.2)-(2.3) in the form
\[ y(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k. \]
Substituting this expression to the left-hand side of the equation (2.2)-(2.3) and equating the coefficients, we obtain the equations for the determination of \( \rho \) and \( c_k \). In particular, the equation corresponding to the power \( s^\rho - 1 \) is of the form:
\[ c_0 \rho^2 = 0. \]
The coefficient \( c_0 \) plays the role of a normalizing constant, and we may take
\[ c_0 = 1. \] (Equation 2.5)
Equation for \( \rho \), the so called characteristic equation, is of the form
\[ \rho^2 = 0. \] (Equation 2.6)
This equation has the root \( \rho = 0 \) and this root is of multiplicity two. According to general theory, the equation (2.2)-(2.3) has two solutions \( y_1(s) \) and \( y_2(s) \) possessing the properties:

1. There exist two solutions \( x_1^-(t, \lambda) \) and \( x_2^-(t, \lambda) \) of the equation \( Mx(t) = \lambda x(t) \) possessing the properties:
   (a) The function \( x_1^-(t, \lambda) \) is holomorphic in the disc \( |t + a| < 2a \), and satisfies the normalizing condition \( x_1^-(0, \lambda) = 1 \);
   (b) The function \( x_2^-(t, \lambda) \) is of the form
      \[ x_2^-(t, \lambda) = x_1^-(t, \lambda) \ln (t + a) + w^-(t, \lambda), \]
      where the function \( w^-(t, \lambda) \) is holomorphic in the disc \( |t + a| < 2a \) and satisfies the condition \( w^-(0, -a) = 0 \).
2. There exist two solutions \( x_1^+(t, \lambda) \) and \( x_2^+(t, \lambda) \) of the equation \( Mx(t) = \lambda x(t) \) possessing the properties:
   (a) The function \( x_1^+(t, \lambda) \) is holomorphic in the disc \( |t - a| < 2a \), and satisfies the normalizing condition \( x_1^+(0, a) = 1 \);
   (b) The function \( x_2^+(t, \lambda) \) is of the form
      \[ x_2^+(t, \lambda) = x_1^+(t, \lambda) \ln (a - t) + w^+(t, \lambda), \]
      where the function \( w^+(t, \lambda) \) is holomorphic in the disc \( |t + a| < 2a \) and satisfies the condition \( w^+(0, a) = 0 \).
For a fixed $\lambda$, the solutions $x_1^-(t, \lambda)$, $x_2^-(t, \lambda)$ are linearly independent. Therefore arbitrary solution $x(t, \lambda)$ of the equation $(2.1)$ can be expanded into a linear combination

$$x(t, \lambda) = c_1^- x_1^-(t, \lambda) + c_2^- x_2^-(t, \lambda).$$

The solutions $x_1^+(t, \lambda)$, $x_2^+(t, \lambda)$ also are linearly independent, and the solution $x(t, \lambda)$ can be also expanded into the other linear combination

$$x(t, \lambda) = c_1^+ x_1^+(t, \lambda) + c_2^+ x_2^+(t, \lambda).$$

Here $c_1^\pm$, $c_2^\pm$ are constants (with respect to $t$). The solution $x_1^-(t, \lambda)$ is bounded and the solution $x_2^-(t, \lambda)$ grows logarithmically as $t \to -a$. Therefore the solution $x(t, \lambda)$ is square integrable near the point $t = -a$. For the same reason, the solution $x(t, \lambda)$ is square integrable near the point $t = a$. Thus we prove the following result.

**Lemma 2.2.** Given $\lambda \in \mathbb{C}$, then every solution $x(t, \lambda)$ of the equation $(2.1)$ is square integrable:

$$\int_{-a}^{a} |x(t, \lambda)|^2 \, dt < \infty.$$  

(2.8)

3. Maximal and minimal differential operators generated by the differential expression $M$.

Various differential operators may be related to the differential expression

$$M = -\frac{d}{dt} \left(1 - \frac{t^2}{a^2}\right) \frac{d}{dt}.$$ 

(3.1)

Such operators are determined by boundary conditions which are posed on functions from their domains of definition.

**Definition 3.1.** The set $\mathcal{A}$ is the set of complex-valued functions $x(t)$ defined on the open interval $(-a, a)$ and satisfying the following conditions:

1. The derivative $\frac{dx(t)}{dt}$ of the function $x(t)$ exists at every point $t$ of the interval $(-a, a)$;
2. The function $\frac{dx(t)}{dt}$ is absolutely continuous on every compact subinterval of the interval $(-a, a)$;

**Definition 3.2.** The differential operator $\mathcal{M}_{\text{max}}$ is defined as follows:

1. The domain of definition $\mathcal{D}_{\mathcal{M}_{\text{max}}}$ of the operator $\mathcal{M}_{\text{max}}$ is:

$$\mathcal{D}_{\mathcal{M}_{\text{max}}} = \{ x : x(t) \in L^2((-a, a)) \cap \mathcal{A} \text{ and } (Mx)(t) \in L^2((a, a)) \},$$  

(3.2a)

where $(Mx)(t)$ is defined by (1.3b).
2. The action of the operator $\mathcal{M}_{\text{max}}$ is:

$$\mathcal{M}_{\text{max}} x = Mx, \quad \forall x \in \mathcal{D}_{\mathcal{M}_{\text{max}}}.$$ 

(3.2b)
The operator \( \mathcal{M}_{\text{max}} \) is said to be the \textit{maximal differential operator generated by the differential expression} \( M \).

**Definition 3.3.** The set \( \hat{\mathcal{A}} \) is the set of complex-valued functions \( x(t) \) defined on the open interval \((-a, a)\) and satisfied the following conditions:

1. The function \( (x)(t) \) belongs to the set \( \mathcal{A} \) defined above;
2. The support \( \text{supp} x \) of the function \( x(t) \) is a compact subset of the open interval \((-a, a)\): \( (\text{supp} x) \in (-a, a) \).

The minimal differential operator \( \mathcal{M}_{\text{min}} \) is a restriction of the maximal differential operator \( \mathcal{M}_{\text{max}} \) on the set of functions which is some sense vanish at the endpoint of the interval \((-a, a)\). The precise definition is presented below.

**Definition 3.4.** The operator the \( \hat{\mathcal{M}} \) is the restriction of the operator \( \mathcal{M}_{\text{max}} \) on the set \( \hat{\mathcal{A}} \) compactly supported in \((a, a)\) functions from \( \mathcal{A} \):

\[
\mathcal{D}_{{\hat{\mathcal{M}}}} = \mathcal{D}_{{\mathcal{M}_{\text{max}}}} \cap \hat{\mathcal{A}}; \quad \hat{\mathcal{M}} \subset \mathcal{M}_{\text{max}}.
\]

(3.3a)

The operator \( \mathcal{M}_{\text{min}} \) is the closure\(^2\) of the operator \( \hat{\mathcal{M}} \):

\[
\mathcal{M}_{\text{min}} = \text{clos}(\hat{\mathcal{M}}).
\]

(3.3b)

The operator \( \mathcal{M}_{\text{min}} \) is said to be the \textit{minimal differential operator generated by the differential expression} \( M \).

**Theorem 3.5.**

1. The operator \( \mathcal{M}_{\text{min}} \) is symmetric:

\[
\langle \mathcal{M}_{\text{min}} x, y \rangle = \langle x, \mathcal{M}_{\text{min}} y \rangle; \quad \forall x, y \in \mathcal{D}_{{\mathcal{M}_{\text{min}}}};
\]

In other words, the operator \( \mathcal{M}_{\text{min}} \) is contained in its adjoint:

\[
\mathcal{M}_{\text{min}} \subseteq (\mathcal{M}_{\text{min}})^*;
\]

(3.4)

2. The operators \( \mathcal{M}_{\text{min}} \) and \( \mathcal{M}_{\text{max}} \) are mutually adjoint:

\[
(\mathcal{M}_{\text{min}})^* = \mathcal{M}_{\text{max}}, \quad (\mathcal{M}_{\text{max}})^* = \mathcal{M}_{\text{min}}.
\]

(3.5)

Proof. The proof of this theorem can be found in [HuPy, 10.4.7-10.4.11]. \( \square \)

4. The boundary linear forms related to the Legendre operator \( M \).

We use the notations

\[
p(t) = 1 - \frac{t^2}{a^2}; \quad -a < t < a.
\]

In this notation, the formal differential operator \( M \) introduced in (3.1) is:

\[
(Mx)(t) = -\frac{d}{dt} \left( p(t) \frac{dx(t)}{dt} \right), \quad -a < t < a.
\]

---

\(^1\)Since \( x \in \mathcal{A} \), the expression \( (Mx)(t) \) is well defined.

\(^2\)Since the operator \( \mathcal{M} \) is symmetric and densely defined, it is closable.
For every \( x, y \in A \),
\[
(Mx)(t) \overline{y(t)} - x(t) (My)(t) = \frac{d}{dt} [x, y](t), \quad -a < t < a,
\]
where
\[
[x, y](t) = -p(t) \left( \frac{dx(t)}{dt} \overline{y(t)} - x(t) \frac{dy(t)}{dt} \right), \quad -a < t < a \tag{4.1}
\]
Therefore, for every \( x, y \in A \) and for every \( \alpha, \beta : -a < \alpha < \beta < a \),
\[
\int_{\alpha}^{\beta} \left( (Mx)(t) \overline{y(t)} - x(t) (My)(t) \right) dt = [x, y](\beta) - [x, y](\alpha). \tag{4.2}
\]

**Lemma 4.1.** For each \( x, y \in D_{M_{\text{max}}} \), there exist the limits
\[
[x, y]_{-a} \overset{\text{def}}{=} \lim_{\alpha \to -a+0} [x, y](\alpha), \quad [x, y]^a \overset{\text{def}}{=} \lim_{\beta \to a-0} [x, y](\beta), \tag{4.3}
\]
where the expression \([x, y](t)\) is defined in \((4.1)\).

**Proof.** Since the functions \( x(t), y(t), (Mx)(t), (My)(t) \) belong to \( L^2((-a, a)) \), then
\[
\int_{-a}^{a} \left| (Mx)(t) \overline{y(t)} - x(t) (My)(t) \right| dt < \infty.
\]
Therefore the limit
\[
\lim_{\alpha \to -a+0} \int_{\alpha}^{\beta} \left( (Mx)(t) \overline{y(t)} - x(t) (My)(t) \right) dt = \int_{a}^{b} \left( (Mx)(t) \overline{y(t)} - x(t) (My)(t) \right) dt \tag{4.4}
\]
nexists. Comparing \((4.4)\) with \((4.2)\), we conclude that the limits in \((4.3)\) exist. \(\square\)

Concerning Lemma 4.1 and related results see [HuPy 10.4.12-10.4.13].

**Lemma 4.2.** The expressions \([x, y]_{-a}\) and \([x, y]^a\), which were introduced by \((4.1)\) and \((4.3)\), are well defined for \( x \in D_{M_{\text{max}}}, y \in D_{M_{\text{max}}} \). Considered as functions of \( x, y \in D_{M_{\text{max}}} \), they are sesquilinear forms. The forms \([x, y]_{-a}\) and \([x, y]^a\) are skew-hermitian:
\[
[x, y]_{-a} = -[y, x]_{-a}, \quad [x, y]^a = -[y, x]^a, \quad \forall x, y \in D_{M_{\text{max}}}. \tag{4.5}
\]

**Definition 4.3.** The forms \([x, y]_{-a}\) and \([x, y]^a\) are said to be the end point sesquilinear forms related to the differential operator \( M \).

**Theorem 4.4.** For every \( x \in D_{M_{\text{max}}}, y \in D_{M_{\text{max}}} \), the equality
\[
\langle M_{\text{max}} x, y \rangle - \langle x, M_{\text{max}} y \rangle = [x, y]^a - [x, y]_{-a} \tag{4.6}
\]
holds, where \([x, y]_{-a}\), \([x, y]^a\) are the end point forms related to the differential operator \( M \).

**Proof.** The equality \((4.6)\) is a consequence of the equalities \((4.4)\), \((4.2)\) and \((4.3)\). \(\square\)
5. The deficiency indices of the operator $\mathcal{M}_{\text{min}}$.

In 1930 John von Neumann, [Neu], has found a criterion for the existence of a self-adjoint extension of a symmetric operator $A$ and has described all such extensions. This criterion is formulated in terms of deficiency indices of the symmetric operator.

**Definition 5.1.** Let $A$ be an operator in a Hilbert space $\mathcal{H}$. We assume that the domain of definition $\mathcal{D}_A$ is dense in $\mathcal{H}$ and that the operator $A$ is symmetric, that is

$$\langle Ax, y \rangle = \langle x, Ay \rangle, \ \forall x, y \in \mathcal{D}_A.$$  \hfill (5.1)

For $\lambda \in \mathbb{C}$, consider the orthogonal complement

$$\mathcal{N}_\lambda = \mathcal{H} \ominus (A - \lambda I)\mathcal{D}_A,$$  \hfill (5.2)

of the subspace $(A - \lambda I)\mathcal{D}_A$, or, what is equivalent,

$$\mathcal{N}_\lambda = \{x \in \mathcal{D}_{A^*} : A^*x = \lambda x\},$$  \hfill (5.3)

where $A^*$ is the operator adjoint to the operator $A$, $\mathcal{D}_{A^*}$ is the domain of definition of $A^*$.

The subspace $\mathcal{N}_\lambda$ is said to be the deficiency subspace of the operator $A$ corresponding to the value $\lambda$.

**Remark 5.2.** The equality (5.1) implies that $\mathcal{D}_A \subseteq \mathcal{D}_{A^*}$. So the factor space $\mathcal{D}_{A^*}/\mathcal{D}_A$ is defined.

**Theorem (von Neumann).** Let $A$ be an operator in the Hilbert space $\mathcal{H}$. We assume that the domain of definition $\mathcal{D}_A$ is dense in $\mathcal{H}$ and that the operator $A$ is symmetric. Then

1. The dimension $\dim \mathcal{N}_\lambda$ is constant in the upper half-plane and in the lower half-plane:

$$\dim \mathcal{N}_\lambda = n_+, \ \forall \lambda : \text{Im} \lambda > 0,$$  \hfill (5.4a)

$$\dim \mathcal{N}_\lambda = n_-, \ \forall \lambda : \text{Im} \lambda < 0,$$  \hfill (5.4b)

each of $n_+, n_-$ may be either non-negative integer or $+\infty$. The numbers $n^+$ and $n^-$ are said to be the deficiency indices of the operator $A$.

2. For the dimension of the factor space $\mathcal{D}_{A^*}/\mathcal{D}_A$ the equality

$$\dim (\mathcal{D}_{A^*}/\mathcal{D}_A) = n_+ + n_-$$  \hfill (5.5)

holds.

**Theorem (von Neumann).** Let $A$ be a densely defined symmetric operator and $n_+, n_-$ are its deficiency indices. Then

1. The operator $A$ is self-adjoint if and only if $n_+ = n_- = 0$.

2. The operator $A$ admits self-adjoint extensions if and only if its deficiency indices are equal:

$$n_+ = n_-.$$  \hfill (5.6)
3. Assume that the deficiency indices of the operator $A$ are equal and non-zero: $0 < n_+ = n_- \leq \infty$. Choose a pair of non-real conjugated complex numbers, for example $\lambda = i$, $\overline{\lambda} = -i$. The set of all self-adjoint extensions of the operator $A$ is in one-to-one correspondence with the set of all unitary operators acting from the deficiency subspace $\mathcal{N}_i$ into the deficiency subspace $\mathcal{N}_{-i}$.

We apply the von Neumann Theorem to the situation where the operator $\mathcal{M}_{\text{min}}$ is taken as the operator $A$. Then the equation

$$A^* x = \lambda x$$

takes the form

$$\mathcal{M}_{\text{max}} x = \overline{\lambda} x.$$  

This is the differential equation

$$-\frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} = \overline{\lambda} x(t), \quad t \in (-a,a),$$  

(5.7)

under the extra condition $x(t) \in L^2(-a,a)$. In particular, the dimension of the deficiency space $\mathcal{N}_\lambda$ coincides with the dimension of the linear space of the set of solutions of the equation (5.7) belonging to $L^2(-a,a)$. According to Lemma 2.2, every solution of the equation (5.7) belongs to $L^2(-a,a)$. Thus we prove the following

Lemma 5.3. For the operator $\mathcal{M}_{\text{min}}$, the deficiency indices are:

$$n_+(\mathcal{M}_{\text{min}}) = 2, \quad n_-(\mathcal{M}_{\text{min}}) = 2.$$  

(5.8)

Thus, the operator $\mathcal{M}_{\text{min}}$ is symmetric, but not self-adjoint, and the set of all its self-adjoint extensions can be parameterized by the set of all unitary operators acting from the two-dimensional deficiency subspace $\mathcal{N}_i$ into the two-dimensional deficiency subspace $\mathcal{N}_{-i}$. However we use another parametrization.

6. Self-adjoint extensions of operators and self-orthogonal subspaces.

J. von Neumann, [Neu], reduced the construction of a self-adjoint extension for a symmetric operator $A_0$ to an equivalent problem of construction of an unitary extension of an appropriate isometric operator - the Caley transform of this symmetric operator. This approach was also developed by M. Stone, [St], and then used by many others.

In some situations, it is much more convenient to use the construction of extensions based on the so called boundary forms. The usage of such construction is especially convenient for differential operators. The first version of the extension theory based on abstract symmetric boundary conditions was developed by J.W. Calkin, [Cal]. Afterwards, various versions of the extension...
theory of symmetric operators were developed in terms of abstract boundary conditions. The problem of the descriptions of extensions of symmetric relations was also considered. See [RoB], [Koch], [Br].

Let $A$ be a symmetric operator acting in a Hilbert space $H$. We assume that the domain of definition $D_A$ of the operator $A$ is dense in $H$ and that the operator $A$ is closed. Since $A$ is symmetric and densely defined, the adjoint operator $A^*$ exists, and $A \subseteq A^*$, that is $D_A \subseteq D_{A^*}$, $Ax = A^*x$, $\forall x \in D_A$. Since $A$ is closed, the equality $(A^*)^* = A$ holds.

We relate the form $\Omega$ to the operator $A$:

$$\Omega(x, y) = \frac{\langle A^*x, y \rangle - \langle x, A^*y \rangle}{i}, \quad \Omega : D_{A^*} \times D_{A^*} \to \mathbb{C}. \quad (6.1a)$$

The form $\Omega$ is hermitian:

$$\Omega(x, y) = \overline{\Omega(y, x)}, \quad \forall x, y \in D_{A^*}, \quad (6.1b)$$

and possesses the property

$$\Omega(x, y) = 0, \quad \forall x \in D_{A^*}, y \in D_A. \quad (6.1c)$$

This property allows to consider the form $\Omega$ as a form on the factor-space $\mathcal{E}$:

$$\mathcal{E} = D_{A^*}/D_A. \quad (6.2)$$

We use the same notation for the form induced on the factor space $\mathcal{E}$:

$$\Omega(x, y) = \frac{\langle A^*x, y \rangle - \langle x, A^*y \rangle}{i}, \quad \Omega : \mathcal{E} \times \mathcal{E} \to \mathbb{C}. \quad (6.3)$$

**Definition 6.1.** The form $\Omega$, (6.1), is said to be the boundary form. The factor space $\mathcal{E}$ is said to be the boundary space.

According to von Neumann Theorem,

$$\dim \mathcal{E} = n_+ + n_-, \quad (6.4)$$

where $n_+$ and $n_-$ are deficiency indices of the operator $A$.

**Lemma 6.2.** The form $\Omega$ is not degenerate on $\mathcal{E}$. In other words, for each non-zero $x \in \mathcal{E}$, there exists $y \in \mathcal{E}$ such that $\Omega(x, y) \neq 0$.

**Proof.** Let $x \in D_{A^*}$ be given. We assume that $\Omega(x, y) = 0$, $\forall y \in D_{A^*}$. This means that $\langle x, A^*y \rangle = \langle A^*x, y \rangle$, $\forall y \in D_{A^*}$. The last equality means that $x \in D_{(A^*)^*}$ and $A^*x = (A^*)^*x$. Since $(A^*)^* = A$, we conclude that $x \in D_A$. $\square$

The definitions of the boundary form and the boundary space can be found in [Str] §1.

Let $\mathcal{S}$ be a subspace of the factor space $\mathcal{E}$:

$$\mathcal{S} \subseteq \mathcal{E}. \quad (6.5a)$$
We identify $S$ with its preimage with respect to the factor-mapping $\mathcal{D}_{A^*} \to \mathcal{D}_{A^*}/\mathcal{D}_A (= \mathcal{E})$ and use the same notation $S$ for a subspace in $\mathcal{E}$ and for its preimage in $\mathcal{D}_A$:

$$\mathcal{D}_A \subseteq S \subseteq \mathcal{D}_{A^*}. \quad (6.5b)$$

To every $S$ satisfying $(6.5b)$, an extension of the operator $A$ is related. We denote this extension by $A_S$:

$$\mathcal{D}_{A_S} = S, \quad A_S x = A^* x, \quad \forall x \in S.$$  

The operator $(A_S)^*$, which is the operator adjoint to the the operator $A_S$, is related to the subspace $S^{\perp_\Omega}$:

$$(A_S)^* = A_{S^{\perp_\Omega}}, \quad (6.6)$$

where $S^{\perp_\Omega}$ is the orthogonal complement of the subspace $S$ with respect to the hermitian form $\Omega$:

$$S^{\perp_\Omega} = \{x \in \mathcal{E} : \Omega(x,y) = 0 \forall y \in S\}. \quad (6.7)$$

In particular the following result holds:

**Lemma 6.3.** The extension $A_S$ of the symmetric operator $A$ is a self-adjoint operator: $A_S = (A_S)^*$, if and only if the subspace $S$ which appears in $(6.5b)$ possesses the property:

$$S = S^{\perp_\Omega}. \quad (6.8)$$

**Definition 6.4.** The subspace $S$ of the boundary space $\mathcal{E}$ is said to be $\Omega$-self-orthogonal if it possess the property $(6.8)$.

Thus, the problem of description of all self-adjoint extension of a symmetric operator $A_0$ can be reformulate as the problem of description of subspaces of the space $\mathcal{E}$, $(6.2)$, which are self-orthogonal with respect to the (non-degenerated) form $\Omega$, $(6.3)$.

It turns out that self-orthogonal subspaces exist if and only if the form $\Omega$, $(6.3)$, has equal numbers of positive and negative squares. (Which conditions is equivalent to the condition $n_+ = n_-$.)

### 7. Self-adjoint extensions of symmetric differential operators.

The description of self-adjoint extensions of a symmetric operator $A$ becomes especially transparent in the case when this symmetric operator is a formally self-adjoint ordinary differential operator, regular or singular. In this case the boundary form $\Omega$, $(6.1a)$, can be expressed in term of the endpoint forms $[x,y]_{-a}$ and $[x,y]_a$, which were introduced in section 4. See Definition 4.3.

This justifies the terminology introduced in Definition 6.1.

We illustrate the situation as applied to the case where the symmetric operator $A$ is the minimal differential operator $\mathcal{M}_{\text{min}}$ generated by the formal Legendre differential operator $M$. Then the adjoint operator $A^*$ is the maximal differential operator $\mathcal{M}_{\text{max}}$ (See Definitions 3.4 and 3.2.)

The problem of description of self-adjoint differential operators generated by a given formal differential operator has the long history. See, for
example, [Kr], [Nai Chapter 5]. The book of [DuSch] is the storage of wisdom in various aspects of the operator theory, in particular is self-adjoint ordinary differential operators. See especially Chapter XIII of [DuSch].

We could incorporate this issue to one or another existing abstract scheme. However to adopt our question to such a scheme one need to agree the notation, the terminology, etc. This auxiliary work may obscure the presentation. To make the presentation more transparent, we prefer to act independently on the existing general considerations.

Let us consider the boundary form $\Omega_M$, constructed from the operator $A = \mathcal{M}_{\text{min}}$ according to (6.1a). Using Theorem 3.5 we conclude that

$$\Omega_M(x, y) = \langle \mathcal{M}_{\text{max}} x, y \rangle - \langle x, \mathcal{M}_{\text{max}} y \rangle, \quad \forall x, y \in \mathcal{D}_{M_{\text{max}}}. \quad (7.1)$$

The appropriate boundary space $\mathcal{E}_M$ is:

$$\mathcal{E}_M = \mathcal{D}_{M_{\text{max}}}/\mathcal{D}_{M_{\text{min}}}. \quad (7.2)$$

According to (6.4) and Lemma 5.3

$$\dim \mathcal{E}_M = 4. \quad (7.3)$$

By Theorem 4.1 the boundary form $\Omega_M$ can be expressed in the term of the end point forms $[x, y]_{-a}, [x, y]^a$:

$$\Omega_M(x, y) = \frac{[x, y]^a - [x, y]_{-a}}{i}, \quad \forall x, y \in \mathcal{D}_{M_{\text{max}}}. \quad (7.4)$$

To make calculation explicit, we choose a special basis in the space $\mathcal{E}_M$. The asymptotic behavior of solutions of the equation $Lx = 0$ near the endpoints of the interval $(-a, a)$, described in Lemma 2.1 prompts us the choice of such a basis.

Let us choose and fix smooth real valued functions $\varphi_-(t), \psi_-(t), \varphi_+(t), \psi_+(t)$ defined on the interval $(-a, a)$ such that

$$\varphi_-(t) = 1, \quad -a < t < -a/2, \quad \varphi_-(t) = 0, \quad a/2 < t < a, \quad (7.5a)$$

$$\psi_-(t) = \ln(a + t), \quad -a < t < -a/2, \quad \psi_-(t) = 0, \quad a/2 < t < a, \quad (7.5b)$$

$$\varphi_+(t) = 0, \quad -a < t < -a/2, \quad \varphi_+(t) = 1, \quad a/2 < t < a, \quad (7.5c)$$

$$\psi_+(t) = 0, \quad -a < t < -a/2, \quad \psi_+(t) = \ln(a - t), \quad a/2 < t < a. \quad (7.5d)$$

It is clear that

$$\varphi_\in \mathcal{D}_{M_{\text{max}}}, \quad \psi_\in \mathcal{D}_{M_{\text{max}}}, \quad \varphi_+ \in \mathcal{D}_{M_{\text{max}}}, \quad \psi_+ \in \mathcal{D}_{M_{\text{max}}}. \quad (7.6)$$

The next calculations are based on the representation (7.4). Since the end point forms $[x, y]_{-a}, [x, y]^a$ are skew-hermitian, then $\Omega_M(\chi, \chi) = 0$ for each real valued function $\chi \in \mathcal{D}_{M_{\text{max}}}$. In particular,

$$\Omega_M(\chi, \chi) = 0, \quad \text{if } \chi \text{ is one of the functions } \varphi_-, \psi_-, \varphi_+, \psi_+. \quad (7.7a)$$
It is clear that
\[ \Omega_M(\chi_-, \chi_+) = 0, \text{ if } \chi_\pm \text{ is one of the functions } \varphi_\pm, \psi_\pm. \] (7.7b)

Direct calculation shows that
\[ \Omega_M(\varphi_-, \psi_-) = \frac{2i}{a}, \quad \Omega_M(\varphi_+, \psi_+) = -\frac{2i}{a}. \] (7.7c)

Thus, the Gram matrix (with respect to the hermitian form \( \Omega_M \)) of the vectors \( \varphi_- \), \( \psi_- \), \( \varphi_+ \), \( \psi_+ \) is:
\[
\frac{a}{2} \begin{bmatrix}
\Omega_M(\varphi_-, \varphi_-) & \Omega_M(\varphi_-, \psi_-) & \Omega_M(\varphi_-, \varphi_+) & \Omega_M(\varphi_-, \psi_+)
\end{bmatrix} = J, \quad (7.8)
\]
where
\[
J = \begin{bmatrix}
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{bmatrix}. \quad (7.9)
\]

The rank of the Gram matrix is is equal to the dimension of the space \( \mathcal{E}_M \):
\[ \text{rank } J = \dim \mathcal{E}_M (= 4). \] (7.10)

**Lemma 7.1.** The functions \( \varphi_- \), \( \psi_- \), \( \varphi_+ \), \( \psi_+ \) generate the boundary space \( \mathcal{E}_M \).

*Proof.* Lemma (7.1) is a consequence of (7.6) and of the equality (7.10). \( \square \)

**Lemma 7.2.** The domain of definition \( \mathcal{D}_{\mathcal{M}_{\text{min}}} \) of the minimal differential operator \( \mathcal{M}_{\text{min}} \) can be characterized by means of the conditions:
\[
\mathcal{D}_{\mathcal{M}_{\text{min}}} = \{ x(t) \in \mathcal{D}_{\mathcal{M}_{\text{max}}}: \Omega_M(x, \varphi_-) = 0, \quad \Omega_M(x, \psi_-) = 0, \quad \Omega_M(x, \varphi_+) = 0, \quad \Omega_M(x, \psi_+) = 0 \}. \quad (7.11)
\]

*Proof.* According to Lemma (7.1) from (7.11) it follows that \( \Omega_M(x, y) = 0 \), \( \forall y \in \mathcal{M}_{\text{max}} \). Now we refer to Lemma 6.2 and to Theorem 3.5 taking the operator \( \mathcal{M}_{\text{min}} \) as the operator \( A \). \( \square \)

**Lemma 7.3.** Let \( \Omega_M \) be a bilinear form in the boundary space \( \mathcal{E} \) defined by (7.1), and \( J \) be the matrix (7.9).

The vector \( x^1 = \alpha^1 \varphi_+ + \beta^1 \psi_+ + \alpha^2 \varphi_- + \beta^2 \psi_- \in \mathcal{E}_L \) is \( \Omega_M \)-orthogonal to the vector \( x^2 = \alpha^2 \varphi_+ + \beta^2 \psi_+ + \alpha^2 \varphi_- + \beta^2 \psi_- \in \mathcal{E}_L \), that is
\[ \Omega_M(x^1, x^2) = 0, \] (7.12a)
if and only if the vector-row \( v_{x^1} = [\alpha^1, \beta^1, \alpha^2, \beta^2] \in \mathcal{V} \) is \( J \)-orthogonal to the vector-row \( v_{x^2} = [\alpha^2, \beta^2, \alpha^2, \beta^2] \in \mathcal{V} \), that is
\[ v_{x^1}^* J v_{x^2}^* = 0, \] (7.12b)
where \( \mathcal{V} \) is the space \( \mathbb{C}^4 \) of vector-rows equipped by the standard hermitian metric, and the star * is the Hermitian conjugation.
Thus, the problem of description of self-adjoint extensions of the operator $M_{\text{min}}$ is equivalent to the problem of description of $\Omega_M$-self-orthogonal subspaces in $\mathcal{E}$, which in its turn is equivalent to the problem of description of $J$-self-orthogonal subspaces in $\mathbb{C}^4$. The last problem is a problem of the indefinite linear algebra and admits an explicit solutions. We set

$$P_+ = \frac{1}{2}(I + J), \quad P_- = \frac{1}{2}(I - J), \quad (7.13a)$$

More explicitly,

$$P_+ = \frac{1}{2} \begin{bmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{bmatrix}, \quad P_- = \frac{1}{2} \begin{bmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{bmatrix}. \quad (7.13b)$$

The matrix $J$, (7.9), possesses the properties

$$J = J^*, \quad J^2 = I.$$  

Therefore the matrices $P_+$, $P_-$, (7.13a), possess the properties

$$P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+ = P_+^*, \quad P_- = P_-^*, \quad (7.14)$$

$$P_+ P_- = 0, \quad P_+ + P_- = I. \quad (7.15)$$

In other words, the matrices $P_+$, $P_-$ are orthogonal projector matrices. These matrices project the space $\mathcal{V}$ onto subspaces $\mathcal{V}_+$ and $\mathcal{V}_-$:

$$\mathcal{V}_+ = \mathcal{V}P_+, \quad \mathcal{V}_- = \mathcal{V}P_-.$$ \quad (7.16)

These subspaces are orthogonally complementary:

$$\mathcal{V}_+ \oplus \mathcal{V}_- = \mathcal{V}. \quad (7.17)$$

The vector rows

$$e_+^1 = [1, \ i, 0, 0], \quad e_+^2 = [0, 0, 1, \ i] \quad (7.18a)$$

and

$$e_-^1 = [1, -i, 0, 0], \quad e_-^2 = [0, 0, 1, -i] \quad (7.18b)$$

form orthogonal bases in $\mathcal{V}_+$ and $\mathcal{V}_-$ respectively.

It turns out that $J$-self-orthogonal subspaces of the space $\mathcal{V}$ are in one-to-one correspondence with unitary operators acting from $\mathcal{V}_+$ onto $\mathcal{V}_-$. 

**Definition 7.4.** Let $U$ be an unitary operator acting from $\mathcal{V}_+$ onto $\mathcal{V}_-$. As the vector-row $v$ runs over the whole subspace $\mathcal{V}_+$, the vector $v + vU$ runs over a subspace of the space $\mathcal{V}$. This subspace is denoted by $\mathcal{S}_U$:

$$\mathcal{S}_U = \{v + vU\}, \text{ where } v \text{ runs over the whole } \mathcal{V}_+. \quad (7.19)$$

---

3 As soon as the notion of $J$-orthogonality of two vectors is introduced, (7.12b), the notions of $J$-orthogonal complement and $J$-self-orthogonal subspaces can be introduced as well.

4 In the standard scalar product on $\mathcal{V} = \mathbb{C}^4$. 
Lemma 7.5.

1. Let $U$ be an unitary operator acting from $\mathcal{V}_+$ onto $\mathcal{V}_-$. Then the subspace $\mathcal{S}_U$ is $J$-self-orthogonal, that is

$$\mathcal{S}_U = \mathcal{S}_U^\perp.$$  

2. Every $J$-self-orthogonal subspace $\mathcal{S}$ of the space $\mathcal{V}$ is of the form $\mathcal{S}_U$:

$$\mathcal{S} = \mathcal{S}_U$$

for some unitary operator $U : \mathcal{V}_+ \rightarrow \mathcal{V}_-$.  

3. The correspondence between $J$-self-orthogonal subspaces and unitary operators acting from $\mathcal{V}_+$ onto $\mathcal{V}_-$ is one-to-one;

$$(U_1 = U_2) \Leftrightarrow (\mathcal{S}_{U_1} = \mathcal{S}_{U_2}).$$

Proof.  

1. The mapping $v \rightarrow v + Uv$ is one-to-one mapping from $\mathcal{V}_+$ onto $\mathcal{S}_U$. Indeed, this mapping is surjective by definition of the subspace $\mathcal{S}_U$. This mapping is also injective. The equality $v + Uv = 0$ implies that $v = Uv = 0$ since $v \perp Uv$. In particular, $\dim \mathcal{S}_U = \dim \mathcal{V}_+$ ($= 2$).

If $v_1$ and $v_2$ are two arbitrary vectors from $\mathcal{V}_+$, then the vectors $w_1 = v_1 + v_1U$ and $w_2 = v_2 + v_2U$ are $J$-orthogonal: $w_1Jw_2^* = 0$. Indeed, since $J = P_+ - P_-$ and $v_k = v_kP_+, v_kU = v_kUP_-, k = 1, 2$, then, using the properties (7.14) of $P_+$ and $P_-$, we obtain

$$w_1Jw_2^* = (v_1P_+ + v_1^*UP_-)(P_+ - P_-)(P_+^*v_2^* + P_-^*v_2) =$$

$$= v_1v_2^* - v_1UU^*v_2^*.$$  

Since the unitary operator $U$ preserves the scalar product, then $v_1v_2^* = v_1UU^*v_2^*$, hence $w_1Jw_2^* = 0$. Thus, $\mathcal{S}_U \subseteq (\mathcal{S}_U^\perp)^\perp$. (The symbol $\perp$ means $J$-orthogonal complement.) Since the Hermitian form $(v_1, v_2) \rightarrow v_1Jv_2^*$ is non-degenerate on $\mathcal{V}$, then $\dim(\mathcal{S}_U^\perp) = \dim \mathcal{V} - \dim \mathcal{S}_U$. Because $\dim \mathcal{V} - \dim \mathcal{S}_U = \dim \mathcal{S}_U$, we have $\dim \mathcal{S}_U = \dim(\mathcal{S}_U^\perp)$. Hence, $\mathcal{S}_U = (\mathcal{S}_U^\perp)^\perp$, i.e. the subspace $\mathcal{S}_U$ is $J$-self-orthogonal.

2. Let $\mathcal{S}$ be a $J$-self-orthogonal subspace. If

$$v \in \mathcal{S}, v = v_1 + v_2, v_1 \in \mathcal{V}_+, v_2 \in \mathcal{V}_-,$$

then the condition $v \perp Jv = 0$, that is the condition $vJv^* = 0$ means that $v_1v_1^* = v_2v_2^*$. Therefore, if $v_1 = 0$, then also $v = 0$. This means that the projection mapping $v \rightarrow vP_+$, considered as a mapping from $\mathcal{S} \rightarrow \mathcal{V}_+$, is injective. For $J$-self-orthogonal subspace $\mathcal{S}$ of the space $\mathcal{V}$, the equality $\dim \mathcal{S} = \dim \mathcal{V} - \dim \mathcal{S}$ holds. Hence $\dim \mathcal{S} = \dim \mathcal{V}_+$. Therefore, the injective linear mapping $v \rightarrow P_+$ is surjective. The inverse mapping is defined on the whole subspace $\mathcal{V}_+$ and can by presented in the form $v = v_1 + v_1U$, where $U$ is a linear operator acting from $\mathcal{V}_+$ into $\mathcal{V}_-$. This mapping $v_1 \rightarrow v_1 + v_1U$ maps the subspace $\mathcal{V}_+$ onto the subspace $\mathcal{S}$.

\[\text{Recall that } v \in \mathcal{V}_+, Uv \in \mathcal{V}_-, \text{ and } \mathcal{V}_+ \perp \mathcal{V}_-.\]
Lemma 7.7. Let \( nJn^* = 0 \), then \( v_1v_1^* = v_2v_2^* \), where \( v_2 = v_1U \). Since \( v_1 \in \mathcal{V}_+ \) is arbitrary, this means that the operator \( U \) is isometric. Since \( \dim \mathcal{V}_+ = \dim \mathcal{V}_- \), the operator \( U \) is unitary. Thus, the originally given \( J \)-self-orthogonal subspace \( S \) is of the form \( S_U \), where \( U \) is an unitary operator acting from \( \mathcal{V}_+ \) to \( \mathcal{V}_- \).

3. The coincidence \( S_{U_1} = S_{U_2} \) means that every vector of the form \( v_1 + v_1U_1 \), where \( v_1 \in \mathcal{V}_+ \) can also be presented in the form \( v_2 + v_2U_2 \) with some \( v_2 \in \mathcal{V}_+ \):

\[
v_1 + v_1U_1 = v_2 + v_2U_2.
\]

Since \( v_1, v_2 \in \mathcal{V}_+, v_1U_1, v_1U_2 \in \mathcal{V}_- \), then \( v_1 = v_2 \), and \( v_1U_1 = v_1U_2 \). The equality \( v_1U_1 = v_1U_2 \) for every \( v_1 \in \mathcal{V}_+ \) means that \( U_1 = U_2 \). Thus, \( (S_{U_1} = S_{U_2}) \Rightarrow (U_1 = U_2) \).

Choosing the orthogonal bases (7.18) in the subspaces \( \mathcal{V}_+ \) and \( \mathcal{V}_+ \), we represent an unitary operator \( U \) by the appropriate unitary matrix:

\[
e_+^1U = e_+^1u_{11} + e_+^2u_{21},
\]

\[
e_+^2U = e_+^1u_{12} + e_+^2u_{22}.
\]

The following result is a reformulation of Lemma 7.5.

**Lemma 7.6.** Let \( \mathcal{V} \) be the space \( \mathbb{C}^4 \) of four vector-rows, \( J \) be a matrix of the form (7.9). With every \( 2 \times 2 \) matrix \( U = \| u_{pq} \|_{1 \leq p,q \leq 2} \), we associate the pair of vectors \( v^1(U), v^2(U) \):

\[
v^1(U) = e_+^1 + e_-^1u_{11} + e_-^2u_{21},
\]

\[
v^2(U) = e_+^2 + e_-^1u_{12} + e_-^2u_{22},
\]

where \( e_k^k, k = 1,2 \), are the vector-rows of the form (7.18), and the subspace \( S_U \) of \( \mathcal{V} \) is the linear hull of the vectors \( v^1(U), v^2(U) \),

\[
S_U = \text{hull}(v^1(U), v^2(U)).
\]

1. If the matrix \( U \) is unitary, then the vectors \( v^1(U), v^2(U) \) are linearly independent, and the subspace \( S_U \) is \( J \)-self-orthogonal.

2. Let \( S \) be a \( J \)-self-orthogonal subspace of the space \( \mathcal{V} \). Then \( S = S_U \) for some an unitary matrix \( U \).

3. For unitary matrices \( U_1, U_2 \),

\[
(S_{U_1} = S_{U_2}) \Leftrightarrow (U_1 = U_2).
\]

The "coordinate" form of the vectors \( v^1(U), v^2(U) \) is:

\[
v^1(U) = [1 + u_{11}, i(1 - u_{11}), u_{21}, -iu_{21}],
\]

\[
v^2(U) = [u_{12}, -iu_{12}, 1 + u_{22}, i(1 - u_{22})].
\]

Taking in account Lemma 7.3 we formulate the following result

**Lemma 7.7.** Let us associate the pair of vectors \( d^1(U), d^2(U) \in \mathcal{E}_M \) with every \( 2 \times 2 \) matrix \( U = \| u_{pq} \|_{1 \leq p,q \leq 2} \):

\[
d^1(U) = (1 + u_{11})\varphi_- + i(1 - u_{11})\psi_- + u_{21}\varphi_+ - iu_{21}\psi_+,
\]

\[
d^2(U) = u_{12}\varphi_- - iu_{12}\psi_- + (1 + u_{22})\varphi_+ + i(1 - u_{22})\psi_+.
\]
where the functions $\varphi_{\pm}, \psi_{\pm}$ are defined in (7.5). The subspace $\mathcal{G}_U$ of the space $\mathcal{E}_M$ is defined as the linear hull of the vectors $d^1(U), d^2(U)$:

$$\mathcal{G}_U = \text{hull}(d^1(U), d^2(U)).$$  \hfill (7.23)

1. If the matrix $U$ is unitary, then the subspace $\mathcal{S} = \mathcal{G}_U$ is $\Omega_M$-self-orthogonal.
2. Let $\mathcal{S}$ be a $\Omega_M$-self-orthogonal subspace of the space $\mathcal{E}_M$. Then $\mathcal{S} = \mathcal{G}_U$ for some an unitary matrix $U$.
3. For unitary matrices $U_1, U_2,$

$$\mathcal{G}_{U_1} = \mathcal{G}_{U_2} \iff (U_1 = U_2).$$

It is clear that a subspace $\mathcal{S} \subseteq \mathcal{E}_M$ is an $\Omega_M$-self-orthogonal subspace if and only if its $\Omega_M$-orthogonal complement $\mathcal{S}^\perp_{\Omega_M}$ is an $\Omega_M$-self-orthogonal subspace. The subspace $(\mathcal{S}_U)^{\perp_{\Omega_M}}$ can be described as:

$$(\mathcal{S}_U)^{\perp_{\Omega_M}} = \left\{ x \in \mathcal{E}_M : \Omega_M(x, d^1(U)) = 0, \Omega_M(x, d^2(U)) = 0 \right\},$$

where $d^1, d^2$ are defined in (7.22), (7.5). Thus Lemma 7.7 can be reformulated in the following way:

**Lemma 7.8.** Let us associate the pair of vectors $d^1(U), d^2(U)$ with every $2 \times 2$ matrix $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$ by (7.22), (7.5). The subspace $\mathcal{O}_U$ is defined as

$$\mathcal{O}_U = \left\{ x \in \mathcal{E}_L : \Omega_M(x, d^1(U)) = 0, \Omega_M(x, d^2(U)) = 0 \right\}. \hfill (7.24)$$

1. If the matrix $U$ is unitary, then the subspace $\mathcal{S} = \mathcal{O}_U$ is $\Omega_M$-self-orthogonal.
2. Let $\mathcal{S}$ be a $\Omega_M$-self-orthogonal subspace of the space $\mathcal{E}_M$. Then $\mathcal{S} = \mathcal{O}_U$ for some an unitary matrix $U$.
3. For unitary matrices $U_1, U_2,$

$$\mathcal{O}_{U_1} = \mathcal{O}_{U_2} \iff (U_1 = U_2).$$

Thus there is one-to-one correspondence between the set of all $2 \times 2$ unitary matrices $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$ and the set of all $\Omega_M$-self-orthogonal subspaces $\mathcal{S}$ of the space $\mathcal{E}_M = \mathcal{D}_{\mathcal{M}_{\text{max}}}/\mathcal{D}_{\mathcal{M}_{\text{min}}}$. This correspondence is described as

$$\mathcal{S} = \mathcal{O}_U, \hfill (7.25)$$

where $\mathcal{O}_U$ is defined in (7.24), (7.22), (7.5).

On the other hand, the subspaces of the space $\mathcal{E}_M = \mathcal{D}_{\mathcal{M}_{\text{max}}}/\mathcal{D}_{\mathcal{M}_{\text{min}}}$ which are self-orthogonal with respect to the Hermitian form $\Omega_M$, (7.1), are in one-to-one correspondence to self-adjoint differential operators generated by the formal differential operator $\mathcal{M}$, (3.1). Every self-adjoint differential operators $\mathcal{M}$ generated by the formal differential operator $\mathcal{M}$ is the restriction of the maximal differential operator $\mathcal{M}_{\text{max}}$, (3.2), on the appropriate domain of definition. According to Lemma 6.3, as applied to the operators $A = \mathcal{M}_{\text{min}}, A^* = \mathcal{M}_{\text{max}}$, the domains of definition of a self-adjoint extension $\mathcal{S}$ of the operator $\mathcal{M}_{\text{min}}$ are those subspaces $\mathcal{S}$:

$$\mathcal{D}_{\mathcal{M}_{\text{min}}} \subseteq \mathcal{S} \subseteq \mathcal{D}_{\mathcal{M}_{\text{max}}} \hfill (7.26)$$
which are self-orthogonal with respect to the Hermitian form $\Omega_M$, (4.3). According to Lemma 7.8, $\Omega_M$-self-orthogonal subspaces $S$ can be described by means of the conditions

$$S = \{ x(t) \in D_{M_{\text{max}}} : \Omega_M(x, d^1(U)) = 0, \Omega_M(x, d^2(U)) = 0 \}, \quad (7.27)$$

where $d^1(U), d^2(U)$ are the same that in (7.22), (7.5), $U$ is an unitary $2 \times 2$ matrix.

8. Description of the selfadjoint extensions $M_U$ in terms of the end point linear forms.

The conditions $\Omega_M(x, d^1(U)) = 0, \Omega_M(x, d^2(U)) = 0$ may be interpreted as a boundary conditions posed on functions $x \in D_{M_{\text{max}}}$. Let us present these conditions in more traditional form.

**Definition 8.1.** For each fixed $y \in D_{M_{\text{max}}}$, the expressions $[x, y]_a$, $[x, y]^a$, considered as function of $x$, are linear forms defined on $D_{M_{\text{max}}}$. These forms are said to be the end point linear forms related to the differential operator $M$.

In view of (7.6), all four endpoint linear forms $[x, \varphi_+ - a]$, $[x, \psi_+ - a]$, $[x, \varphi_+]^a$, $[x, \psi_+]^a$ (8.1) are well defined for $x \in D_{M_{\text{max}}}$.

**Lemma 8.2.**

1. For every $x \in D_{M_{\text{max}}}$, the end point linear forms $[x, \varphi_+]_a$, $[x, \psi_+]_a$, $[x, \varphi_+]^a$, $[x, \psi_+]^a$ can be expressed as:

$$[x, \varphi_+]_a = -\frac{2}{a} b_{-a}(x), \quad (8.2a)$$

$$[x, \varphi_+]^a = \frac{2}{a} b_a(x), \quad (8.2b)$$

$$[x, \psi_+_a] = -\frac{2}{a} c_{-a}(x), \quad (8.2c)$$

$$[x, \psi_+]^a = \frac{2}{a} c_a(x), \quad (8.2d)$$

where

$$b_{-a}(x) = \lim_{t \to -a+0} (t + a) \frac{dx(t)}{dt}, \quad (8.3a)$$

$$b_a(x) = \lim_{t \to a-0} (t - a) \frac{dx(t)}{dt}, \quad (8.3b)$$

$$c_{-a}(x) = \lim_{t \to -a+0} \left( (t + a) \ln(a + t) \frac{dx(t)}{dt} - x(t) \right), \quad (8.3c)$$

$$c_a(x) = \lim_{t \to a-0} \left( (t - a) \ln(a - t) \frac{dx(t)}{dt} - x(t) \right). \quad (8.3d)$$

In particular, the limits exist in (8.3).
2. The end points linear forms \([x, \varphi_-]^a, [x, \psi_-]^a, [x, \varphi_+]^a, [x, \psi_+]^a\) vanish identically on \(D_{\mathcal{M}_{\text{max}}}\).

**Proof.** Let us introduce
\[
\begin{align*}
b_{-a}(x) &= \frac{ia}{2} \Omega_M(x, \varphi_-), \quad c_{-a}(x) = \frac{ia}{2} \Omega_M(x, \psi_-), \\
b^a(x) &= \frac{ia}{2} \Omega_M(x, \varphi_+), \quad c^a(x) = \frac{ia}{2} \Omega_M(x, \psi_+),
\end{align*}
\] (8.4a)
\[
\begin{align*}
b_{-a}(x) &= \frac{ia}{2} \Omega_M(x, \varphi_-), \quad c_{-a}(x) = \frac{ia}{2} \Omega_M(x, \psi_-), \\
b^a(x) &= \frac{ia}{2} \Omega_M(x, \varphi_+), \quad c^a(x) = \frac{ia}{2} \Omega_M(x, \psi_+),
\end{align*}
\] (8.4b)

From (7.21) it follows that the equalities (8.2) hold. The existence of the limits in (8.3) follows from Lemma 4.1 applied to the functions \(x(t)\) and \(y(t) = \varphi_\pm(t)\) or \(y(t) = \psi_\pm(t)\). The equalities (8.4) can be obtained by the direct computation using the explicit expressions (7.5) for the functions \(\varphi_\pm(t), \psi_\pm(t)\). \(\square\)

**Remark 8.3.** The values \(b_{-a}(x), c_{-a}(x), b_a(x), c_a(x)\) may be considered as generalized boundary values related to the function \(x(t) \in D_{\mathcal{M}_{\text{max}}}\) at the end points \(-a\) and \(a\) of the interval \((-a, a)\).

In view of (7.11) and (8.4), Lemma 7.2 can be reformulated as follow.

**Theorem 8.4.** The domain of definition \(D_{\mathcal{M}_{\text{min}}}\) of the minimal differential operator \(\mathcal{M}_{\text{min}}\) can be characterized by means of the boundary conditions:
\[
D_{\mathcal{M}_{\text{min}}} = \{ x(t) \in D_{\mathcal{M}_{\text{max}}} : \quad b_{-a}(x) = 0, \quad b^a(x) = 0, \quad c_{-a}(x) = 0, \quad c^a(x) = 0 \}. \tag{8.5}
\]

Due to (8.4), the equality (7.8) can be rewritten as
\[
\begin{bmatrix}
b_{-a}(\varphi_-) & c_{-a}(\varphi_-) & b_{-a}(\varphi_-) & c_{a}(\varphi_-) \\
\phantom{b_{-a}(\varphi_-)} & \phantom{c_{-a}(\varphi_-)} & \phantom{b_{-a}(\varphi_-)} & \phantom{c_{a}(\varphi_-)} \\
b_{-a}(\psi_-) & c_{-a}(\psi_-) & b_{-a}(\psi_-) & c_{a}(\psi_-) \\
\phantom{b_{-a}(\psi_-)} & \phantom{c_{-a}(\psi_-)} & \phantom{b_{-a}(\psi_-)} & \phantom{c_{a}(\psi_-)} \\
b_{-a}(\varphi_+) & c_{-a}(\varphi_+) & b_{-a}(\varphi_+) & c_{a}(\varphi_+) \\
\phantom{b_{-a}(\varphi_+)} & \phantom{c_{-a}(\varphi_+)} & \phantom{b_{-a}(\varphi_+)} & \phantom{c_{a}(\varphi_+)} \\
b_{-a}(\psi_+) & c_{-a}(\psi_+) & b_{-a}(\psi_+) & c_{a}(\psi_+) \\
\phantom{b_{-a}(\psi_+)} & \phantom{c_{-a}(\psi_+)} & \phantom{b_{-a}(\psi_+)} & \phantom{c_{a}(\psi_+)}
\end{bmatrix}
= \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}. \tag{8.6}
\]

According to (8.4), the equalities \(\Omega_M(x, d^1(U)) = 0, \ \Omega_M(x, d^2(U)) = 0\) take the form
\[
(1 + u_{11}) b_{-a}(x) - i(1 - u_{11}) c_{-a}(x) + u_{12} b_a(x) + iu_{12} c_a(x) = 0, \tag{8.7a}
\]
\[
u_{21} b_{-a}(x) + iu_{21} c_{-a}(x) + (1 + u_{22}) b_a(x) - i(1 - u_{22}) c_a(x) = 0 \tag{8.7b}
\]

**Remark 8.5.** Since the form \(\Omega_M(x, y)\) is antilinear with respect to the argument \(y\); \(\Omega_M(x, \mu y) = \overline{\mu} \Omega_M(x, y)\) for \(\mu \in \mathbb{C}\), the numbers \(i, -i\) which occurs in (7.22) must be replaced with the numbers \(-i, i\) in appropriate positions in the equality (8.7). For the same reason, the numbers \(u_{pq}\) which occurs in (7.22) must be replaced with the numbers \(\overline{u_{pq}}\) in (8.7). However to simplify the notation, we replace the number \(u_{pq}\) with the number \(u_{qp}\) rather with the numbers \(\overline{u_{pq}}\). In other words, we use the matrix \(U^*\) as a matrix which parameterizes the set of all \(\Omega_M\)-self-orthogonal subspaces. The matrix \(U^*\) is an arbitrary unitary matrix if \(U\) is an arbitrary unitary matrix.
Definition 8.6. Let $U$ be an arbitrary $2 \times 2$ matrix. The operator $\mathcal{M}_U$ is defined in the following way:

1. The domain of definition $\mathcal{D}_{\mathcal{M}_U}$ of the operator $\mathcal{M}_U$ is the set of all $x(t) \in \mathcal{D}_{\mathcal{M}_{\max}}$ which satisfy the conditions (8.7a)-(8.7b), (8.3).
2. For $x \in \mathcal{D}_{\mathcal{M}_U}$, the action of the operator $\mathcal{M}_U$ is 
   $$\mathcal{M}_U x = \mathcal{M}_{\max} x.$$ 

Remark 8.7. In view of (8.5) and (8.7), for any matrix $U$, 

$$\mathcal{D}_{\mathcal{M}_{\min}} \subseteq \mathcal{D}_{\mathcal{M}_U}.$$ 

Thus for any matrix $U$, the operator $\mathcal{M}_U$ is an extension of the operator $\mathcal{U}_{\min}$: 

$$\mathcal{M}_{\min} \subseteq \mathcal{M}_U \subseteq \mathcal{M}_{\max}. \quad (8.8)$$ 

The equalities (8.7) which determine the domain of definition of the extension $\mathcal{M}_U$ can be considered as boundary conditions posed on functions $x \in \mathcal{D}_{\mathcal{M}_{\max}}$. (See Remark 8.3.)

The following Theorem is a reformulation of Lemma 7.8 in the language of extensions of operators.

Theorem 8.8.

1. If $U$ is an unitary matrix, then the operator $\mathcal{M}_U$ is a self-adjoint differential operator, and $\mathcal{M}_{\min} \subseteq \mathcal{M}_U \subseteq \mathcal{M}_{\max}$. 
2. Every differential operator $\mathcal{M}$ which is self-adjoint extension of the minimal differential operator $\mathcal{M}_{\min}$, $\mathcal{M}_{\min} \subseteq \mathcal{M} \subseteq \mathcal{M}_{\max}$, is of the form $\mathcal{M} = \mathcal{M}_U$ for some unitary matrix $U$. 
3. For unitary matrices $U_1$, $U_2$, 
   $$(U_1 = U_2) \Leftrightarrow (\mathcal{M}_{U_1} = \mathcal{M}_{U_2}).$$ 

The equalities (1.3), which relate the formal Legendre operator $L$ and formal prolate spheroid operator $M$, lead to the equalities 

$$L_{\max} = M_{\max} + Q, \quad (8.9a)$$ 
$$L_{\min} = M_{\min} + Q, \quad (8.9b)$$ 

where $Q$ is the multiplication operator: 

$$D_Q = L^2([-a,a]), \quad (Qx)(t) = t^2 x(t). \quad (8.9c)$$ 

The operator $Q$ is a bounded self-adjoint operator: 

$$Q = Q^*. \quad (8.10)$$ 

So there are no problems with the equalities (8.9). We may consider the operators in the right hand sides of the equalities (8.9) as definitions for the operators in the left hand sides. In particular, the domains of definition coincide: 

$$D_{L_{\max}} = D_{M_{\max}}, \quad D_{L_{\min}} = D_{M_{\min}}. \quad (8.11)$$ 

The relations 

$$L_{\min} \subseteq (L_{\min})^*; \quad (L_{\min})^* = L_{\max}, \quad (L_{\max})^* = L_{\min}.$$
are consequences of the relations (3.4), (3.5), of the definitions (8.9) and of the equality (8.10). In view of (8.10), the boundary forms \( \Omega_M \) and \( \Omega_L \) coincide. The boundary linear forms related to the operators \( L \) and \( M \) are the same and are expressed by (8.3). Finally the self-adjoint extensions of the symmetric operator \( L_{\text{min}} \) are in one-to-one correspondence with \( 2 \times 2 \) unitary matrices \( U \). This correspondence is of the form \( U \mapsto L_U \), where the domain of definitions \( D_{L_U} = D_{M_U} \) is described by linear boundary conditions (8.7). Moreover the equality

\[
L_U = M_U + Q \quad (8.12)
\]
holds.

9. Spectral analysis of the operators \( L_U \).

The matrix \( I \) is \( 2 \times 2 \) identity matrix: \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). The operators \( L_I \) and \( M_I \) are the operators \( L_U \) and \( M_U \) corresponding to the choice \( U = I \). In particular, for \( U = I \) the boundary conditions (8.7) take the form

\[
\lim_{|\xi| \to a-0} \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} = 0, \quad \forall x \in D_{M_I} = D_{L_I}, \quad (9.1)
\]

Lemma 9.1. Let \( x \in D_{M_I} \), and

\[
\int_{-a}^{a} \left| \frac{d}{d\xi} \left( \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} \right) \right|^2 d\xi = C^2 < \infty, \quad C = C(x) > 0. \quad (9.2)
\]

Then

\[
\left| \frac{dx(t)}{dt} \right| \leq \sqrt{2} C a^{3/2}, \quad \forall t \in (-a, a). \quad (9.3)
\]

Proof. From (9.2) and the Schwarz inequality we obtain

\[
\int_{-a}^{a} \left| \frac{d}{d\xi} \left( \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} \right) \right| d\xi \leq \sqrt{2} a C. \quad (9.4)
\]

From (9.1) and (9.4) we derive the inequality

\[
\left| \left( 1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} \right| \leq \sqrt{2} a C \min(a+t, a-t), \quad \forall t \in (-a, a).
\]

Since \( \min(a+t, a-t) \leq a \left( 1 - \frac{t^2}{a^2} \right) \), we obtain the inequality (9.3). \( \square \)

Lemma 9.2. Let \( x \in D_{L_I} \). Then the limits

\[
x(-a + 0) = \lim_{t \to -a+0} x(t), \quad x(a - 0) = \lim_{t \to a-0} x(t) \quad (9.5a)
\]

exist and are finite:

\[
|x(-a + 0)| < \infty, \quad |x(a - 0)| < \infty. \quad (9.5b)
\]
Theorem 9.3.

1. The self-adjoint operator $M_I$ is non-negative:
   \[ \langle M_I x, x \rangle \geq 0, \quad \forall x \in D_{M_I}, \quad x \neq 0. \]  
   \[ (9.6) \]

2. The self-adjoint operator $L_I$ is positive:
   \[ \langle L_I x, x \rangle > 0, \quad \forall x \in D_{L_I}, \quad x \neq 0. \]  
   \[ (9.7) \]

Proof.

1. Integrating by parts, we obtain
   \[
   \int_{-a}^{a} -\frac{d}{dt} \left( \left(1 - \frac{t^2}{a^2} \right) \frac{dx}{dt} \right) \cdot \overline{x(t)} dt = \int_{-a}^{a} \left(1 - \frac{t^2}{a^2} \right) \left| \frac{dx}{dt} \right|^2 dt.
   \]

In view of (9.1) and (9.5), the summands corresponding to the endpoints $-a$ and $a$ disappear. The last equality can be interpreted as
   \[ \langle M_I x, x \rangle = \int_{-a}^{a} \left(1 - \frac{t^2}{a^2} \right) \left| \frac{dx}{dt} \right|^2 dt, \quad \forall x \in D_{M_I}. \]

So the inequality (9.6) holds.

2. The operator $Q$ is positive:
   \[ \langle Qx, x \rangle > 0, \quad \forall x \in L^2([-a, a]), \quad x \neq 0. \]  
   \[ (9.8) \]

The inequality (9.7) is a consequence of the inequalities (9.6), (9.8) and of the equality (8.12) with $U = I$. \[ \square \]

Let $I$ be the identity operator in $L^2([-a, a])$.

Lemma 9.4. Given $\lambda \in \mathbb{C} \setminus [0, \infty)$, the operators $(M_I - \lambda I)^{-1}$ and $(L_I - \lambda I)^{-1}$ are compact operators.

Proof. Since both operators $M_I$ and $L_I$ are self-adjoint and non-negative, both resolvents $(M_I - \lambda I)^{-1}$ and $(L_I - \lambda I)^{-1}$ exist and are bounded operators.

The spectral analysis of the operator $M_I$ can be done explicitly. Let $P_k(t)$ be the Legendre polynomials:
   \[ P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k, \quad k = 0, 1, 2, \ldots, \]

and
   \[ v_k(t) = P_k(t/a), \quad t \in [-a, a], \quad k = 0, 1, 2, \ldots. \]  
   \[ (9.9) \]

The system \( \{v_k(t)\}_{k=0,1,2,\ldots} \) is a complete orthogonal system in $L^2([-a, a])$. The functions $v_k(t)$ are eigenfunctions of the operator $M_I$:
   \[ (M_I v_k)(t) = \mu_k v_k(t), \]  
   \[ (9.10a) \]

where
   \[ \mu_k = \frac{k(k + 1)}{a^2}, \quad k = 0, 1, 2, \ldots. \]  
   \[ (9.10b) \]
Thus the operator $\mathcal{M}_I$ is an operator with discrete spectrum and the resolvent $(\mathcal{M}_I - \lambda \mathcal{I})^{-1}$ is a compact operator. Since 

$$(\mathcal{L}_I - \lambda \mathcal{I})^{-1} = (\mathcal{M}_I - \lambda \mathcal{I})^{-1} - (\mathcal{M}_I - \lambda \mathcal{I})^{-1} Q(\mathcal{L}_I - \lambda \mathcal{I})^{-1},$$

the operator $(\mathcal{L}_I - \lambda \mathcal{I})^{-1}$ is a compact operator as well. □

**Lemma 9.5.** Given $\lambda \in \mathbb{C} \setminus (-\infty, \infty)$ and an unitary matrix $U$, the operator $(\mathcal{L}_U - \lambda \mathcal{I})^{-1}$ is a compact operator.

**Proof.** Since $\lambda \notin \mathbb{R}$, both resolvents $(\mathcal{L}_U - \lambda \mathcal{I})^{-1}$, $(\mathcal{L}_I - \lambda \mathcal{I})^{-1}$ exist. Since both operators $\mathcal{L}_U$ and $\mathcal{L}_I$ and extensions of the same operator $\mathcal{L}_{\min}$ with deficiency indices

$$n_+(\mathcal{L}_{\min}) = n_-(\mathcal{L}_{\min}) = 2,$$

(9.11) the difference of the resolvents $(\mathcal{L}_U - \lambda \mathcal{I})^{-1} - (\mathcal{L}_I - \lambda \mathcal{I})^{-1}$ is an operator which rank does not exceed two. According to Lemma 9.4, the operator $(\mathcal{L}_I - \lambda \mathcal{I})^{-1}$ is compact. Hence the operator $(\mathcal{L}_U - \lambda \mathcal{I})^{-1}$ is compact. □

**Theorem 9.6.**

1. For any unitary matrix $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, the spectrum of the operator $\mathcal{L}_U$ is discrete. This spectrum is formed by the sequence $\{\lambda_k(\mathcal{L}_U)\}_{1 \leq k < \infty}$ of the eigenvalues of $\mathcal{L}_U$:

$$\lambda_1(\mathcal{L}_U) \leq \lambda_2(\mathcal{L}_U) \leq \ldots \leq \lambda_k(\mathcal{L}_U) \leq \ldots,$$

$$\lambda_k(\mathcal{L}_U) \to \infty \text{ as } k \to \infty. \quad (9.12)$$

2. Not more than two of these eigenvalues can be negative:

$$\lambda_k(\mathcal{L}_U) \geq 0, \quad 3 \leq k < \infty. \quad (9.13)$$

3. The multiplicity of the eigenvalue $\lambda_k(\mathcal{L}_U)$ does not exceed two:

$$\text{mult}(\lambda_k(\mathcal{L}_U)) \leq 2, \quad 1 \leq k < \infty. \quad (9.14)$$

4. If at least one of the entries $u_{11}, u_{22}$ of the matrix $U$ is equal to one; i.e if $(1-u_{11})(1-u_{22}) = 1$, then all eigenvalues $\lambda_k(\mathcal{L}_U)$ are of multiplicity one:

$$\text{mult}(\lambda_k(\mathcal{L}_U)) = 1, \quad 1 \leq k < \infty. \quad (9.15)$$

**Proof.** According to Lemma 9.5, the spectrum of the self-adjoint operator $\mathcal{L}_U$ consists of isolated points which are eigenvalues. The operator $\mathcal{L}_U$ is an extension of the symmetric operator $\mathcal{L}_{\min}$ which is non-negative. (The inequality 9.7 for $x \in \mathcal{D}_{\mathcal{L}_{\min}}$.) Since the deficiency indices of the operator $\mathcal{L}_{\min}$ are finite, (9.11), the spectrum of the operator $\mathcal{L}_U$ is bounded from below. Hence the sequence $\{\lambda_k(\mathcal{L}_U)\}_{1 \leq k < \infty}$ of the eigenvalues of $\mathcal{L}_U$ can be enumerated such that the conditions (9.12) holds. The condition (9.13) is a consequence of [K] Theorem 18. The inequality (9.14) holds because the equation $\mathcal{L}_U x - \lambda x = 0$ is a differential equation of order two. If $u_{11} = 1$ then the boundary condition (8.7a) is of the form $b_{-a}(x) = 0$. According to a version of Lemma 9.2 formulated for the operator $L$, any solution $x(t, \lambda)$ of the eigenvalue problem $\mathcal{L}_U x = \lambda x$ is bounded as $t \to -a + 0$. According to Lemma 2.1 any solution $x(t, \lambda)$ of the differential equation $(Lx)(t, \lambda) = \ldots$

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**Self-adjoint boundary conditions**
\( \lambda x(t, \lambda) \) must be of the form \((2.7a)\). Since the function \( x_1^-(t, \lambda) \) is bounded
and the function \( x_2^-(t, \lambda) \) is unbounded as \( t \to -a + 0 \), the coefficient \( c_2^- \) in
\((2.7a)\) must vanish. \( \Box \)

Among all self-adjoint extensions \( \mathcal{L}_U \) of the minimal symmetric non-negative operator \( \mathcal{L}_{\text{min}} \), we distinguish the extension \( \mathcal{L}_I \) which corresponds
to the choice of the identity matrix \( I \) as the matrix \( U \). The operator \( \mathcal{L}_I \) plays
a special role. We shall see in the next section that among all extensions \( \mathcal{L}_U \) of the operator \( \mathcal{L}_{\text{min}}, \) only the operator \( \mathcal{L}_I \) commutes with the truncated
Fourier operator \( \mathcal{F}_{[-a,a]} \).

**Theorem 9.7.**

1. The spectrum of the operator \( \mathcal{L}_I \) is formed by the sequence \( \{ \lambda_k \}_{1 \leq k < \infty} \) of positive eigenvalues of multiplicity one:

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots, \quad \lambda_k \to \infty \text{ as } k \to \infty. \quad (9.16)
\]

2. The system of the eigenfunctions \( \{ \chi_k \}_{1 \leq k < \infty} \):

\[
(L\chi_k)(t) = \lambda_k \chi_k(t), \quad t \in (-a, a), \\
b_{-a}(\chi_k) = 0, \quad b_a(\chi_k) = 0, \quad (9.17a, b)
\]

is a complete orthogonal system in \( L^2([-a,a]) \).

**Definition 9.8.** The functions \( \chi_k(t) \), which are the eigenfunction of the boundary value problem \((9.17)\), are said to be the prolate spheroidal wave functions.

**Remark 9.9.** Traditionally the prolate spheroidal wave functions \( \chi_k \) are defined as those solutions of the equation \((9.17a)\) which are bounded on \( (-a, a) \):

\[
\sup_{t \in (-a,a)} |\chi_k(t)| < \infty \quad (9.18)
\]

The traditional definitions is equivalent to the definition \( \chi_k \) by means of the eigenvalue problem \((9.17a), (9.17b)\).

**10. Commutator of the operators \( \mathcal{F}_E \) and \( \mathcal{L}_U \).**

For \( x \in D_{\mathcal{L}_{\text{max}}} \), let us calculate the difference \( \mathcal{F}_E \mathcal{L}_{\text{max}} x - \mathcal{L}_{\text{max}} \mathcal{F}_E x \). Since \( \mathcal{L}_{\text{max}} x \in L^2([-a,a]), \) the expression \( \mathcal{F}_E(\mathcal{L}_{\text{max}} x) \) is defined. The functions \( (\mathcal{F}_E x)(t) \) and \( (\mathcal{F}_E \mathcal{L}_{\text{max}} x)(t) \) are smooth on the closed interval \( [-a,a] \). (In fact these function are analytic in the whole real axis.) All the more, \( \mathcal{F}_E x \in D_{\mathcal{L}_{\text{max}}} \),
Thus for \( x \in D_{\mathcal{L}_{\text{max}}} \), the difference \( \mathcal{F}_E \mathcal{L}_{\text{max}} x - \mathcal{L}_{\text{max}} \mathcal{F}_E x \) is well defined.
Assuming that \( x \in \mathcal{D}_{L_{\text{max}}} \) and that \(-a < \alpha < \beta < a\), we integrate by parts twice
\[
\int_{\alpha}^{\beta} \left( -\frac{d}{d\xi} \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} \right) e^{it\xi} d\xi =
\]
\[
= - \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} e^{it\xi} \bigg|_{\xi=\beta}^{\xi=\alpha} + it \left( 1 - \frac{\xi^2}{a^2} \right) x(\xi) e^{it\xi} \bigg|_{\xi=\beta}^{\xi=\alpha} -
\]
\[
- it \int_{\alpha}^{\beta} x(\xi) \frac{d}{d\xi} \left( \left( 1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi. \tag{10.1}
\]

For \( x \in \mathcal{D}_{L_{\text{max}}} \), both limits \( \lim_{t \to \pm a} \frac{dx(t)}{dt} \) exist, are finite, and
\[
\lim_{t \to -a} \frac{dx(t)}{dt} = \frac{2}{a^2} b_{-a}(x), \tag{10.2a}
\]
\[
\lim_{t \to +a} \frac{dx(t)}{dt} = -\frac{2}{a^2} b_{a}(x). \tag{10.2b}
\]
where \( b_{-a}(x), b_{a}(x) \) are defined in (8.3) and also appear in the boundary conditions (8.7). Since the limits in (10.2) are finite, we conclude that \( |x(t)| = O(\ln(a^2 - t^2)) \) as \( t \to \pm a, |t| < a \). All the more, for \( x \in \mathcal{D}_{L_{\text{max}}} \)
\[
\lim_{t \to -a+0} \left( 1 - \frac{t^2}{a^2} \right) x(t) = 0. \tag{10.3}
\]

Passing to the limit in (10.1) and taking into account (10.3) and (10.2), we obtain
\[
\int_{-a}^{a} \left( -\frac{d}{d\xi} \left( 1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} \right) e^{it\xi} d\xi = \frac{2}{a} \left( b_{a}(x)e^{iat} + b_{-a}(x)e^{-iat} \right) -
\]
\[
- it \int_{-a}^{a} x(\xi) \frac{d}{d\xi} \left( \left( 1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi. \tag{10.4}
\]

Transforming the last summand of the right hand side of (10.4), we obtain
\[
- it \int_{-a}^{a} x(\xi) \frac{d}{d\xi} \left( \left( 1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi =
\]
\[
= t^2 \int_{-a}^{a} x(\xi) e^{it\xi} d\xi + \frac{it}{a^2} \int_{-a}^{a} x(\xi) \frac{d}{d\xi} \left( \xi^2 e^{it\xi} \right) d\xi =
\]
\[
\left( \text{since } \frac{d}{d\xi} (\xi^2 e^{it\xi}) = \frac{d}{d\xi} \left( -\frac{d^2}{dt^2} e^{it\xi} \right) = -\frac{d^2}{dt^2} (ite^{it\xi}) \right)
\]
\[ = t^2 \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi + \frac{t}{a^2} \frac{d^2}{dt^2} \left( t \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi \right) = \]

\[ = t^2 \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi + \frac{d}{dt} \left( \frac{t^2}{a^2} \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi \right) = \]

\[ = t^2 \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi - \frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \right) \frac{d}{dt} \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi \]

\[ = \frac{2}{a} \left( b_a(x) e^{iat} + b_{-a}(x) e^{-iat} \right) + \left( -\frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2 \right) \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi. \]

(10.5)

Unifying (10.4) and (10.5), we obtain the equality

\[ \int_{-a}^{a} \left( \left( -\frac{d}{d\xi} \left( 1 - \frac{\xi^2}{a^2} \right) \frac{d}{d\xi} + \xi^2 \right) x(\xi) \right) e^{it\xi} \, d\xi = \]

\[ = \frac{2}{a} \left( b_a(x) e^{iat} + b_{-a}(x) e^{-iat} \right) + \left( -\frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2 \right) \int_{-a}^{a} x(\xi) e^{it\xi} \, d\xi. \]

(10.6)

We summarize the above calculation as

**Lemma 10.1.** Let \( \mathcal{F}_E \) be the Fourier operator truncated on the finite symmetric interval \( E = [-a,a] \). Let \( \mathcal{L}_{\text{max}} \) be the maximal differential operator with domain of definition \( \mathcal{D}_{\mathcal{L}_{\text{max}}} \) generated by the formal differential operator

\[ L = -\frac{d}{dt} \left( 1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2. \]  

(See Definition 3.2.)

If \( x \in \mathcal{D}_{\mathcal{L}_{\text{max}}} \), then \( \mathcal{F}_E x \in \mathcal{D}_{\mathcal{L}_{\text{max}}} \), and the equality holds

\[ (\mathcal{F}_E \mathcal{L}_{\text{max}} x) (t) - (\mathcal{L}_{\text{max}} \mathcal{F}_E x) (t) = \frac{2}{a} \left( b_a(x) e^{iat} + b_{-a}(x) e^{-iat} \right). \]  

(10.7)

Every self-adjoint differential operator generated by the formal differential operator \( L \) is a restriction of the maximal differential operator \( \mathcal{L}_{\text{max}} \) on the appropriate domain of definition. According to Theorem 8.8, the set of such self-adjoint operators coincides with the set of operators \( \mathcal{L}_U \), where \( U \) is an arbitrary \( 2 \times 2 \) unitary matrix. The domain of definition \( \mathcal{D}_{\mathcal{L}_U} \) of the operator \( \mathcal{L}_U \) is distinguished from the domain \( \mathcal{D}_{\mathcal{L}_{\text{max}}} \) by the boundary conditions (8.7) constructed from \( U \). The next theorem answers the question which operators \( \mathcal{L}_U \) commute with the truncated Fourier operator \( \mathcal{F}_E \), \( E = [-a,a] \).

**Theorem 10.2.**

1. If \( U = I \), where \( I \) is \( 2 \times 2 \) identity matrix, then the differential operator \( \mathcal{L}_I \) commutes with the truncated Fourier operator \( \mathcal{F}_{[-a,a]} \):

\[ \mathcal{F}_{[-a,a]} \mathcal{L}_I x = \mathcal{L}_I \mathcal{F}_{[-a,a]} x \quad \forall x \in \mathcal{D}_{\mathcal{L}_I}. \]  

(10.8)
2. If $U \neq I$, then the operator $\mathcal{L}_U$ do not commute with the operator $\mathcal{F}[-a,a]$: 
(a) There exist vectors $x \in \mathcal{D}_{\mathcal{L}_U}$ such that $\mathcal{F}[-a,a] \in \mathcal{D}_{\mathcal{L}_U}$, so both operators $\mathcal{F}[-a,a] \mathcal{L}_U$ and $\mathcal{L}_U \mathcal{F}[-a,a]$ are applicable to $x$, but
\[ \mathcal{F}[-a,a] \mathcal{L}_U x \neq \mathcal{L}_U \mathcal{F}[-a,a] x; \]  
(10.9)

(b) There exist vectors $x \in \mathcal{D}_{\mathcal{L}_U}$ such that $\mathcal{F}[-a,a] x \notin \mathcal{D}_{\mathcal{L}_U}$, so the operator $\mathcal{L}_U \mathcal{F}[-a,a]$ even can not be applied to such $x$.

Proof.
1. For $U = I$, the boundary conditions (8.7) take the form
\[ b_{-a}(x) = 0, \quad b_a(x) = 0. \]  
(10.10)
Thus, the domain of definition $\mathcal{D}_{\mathcal{L}_I}$ of the operator $\mathcal{L}_I$ is:
\[ \mathcal{D}_{\mathcal{L}_I} = \{ x : x \in \mathcal{D}_{\mathcal{L}_{\max}}, \ b_{-a}(x) = 0, \ b_a(x) = 0 \}. \]  
(10.11)
Every function $x(t)$ on $(-a, a)$ which derivative is bounded: $\sup_{t \in (-a, a)} |x'(t)| < \infty$, belongs to $\mathcal{D}_{\mathcal{L}_{\max}}$. Moreover, according to (8.3), every such a function satisfies the boundary condition (10.11), i.e. $b_{-a}(x) = 0, b_a(x) = 0$. Hence every smooth function on $(-a, a)$ which derivative is bounded on $(-a, a)$, belongs to domain of definition $\mathcal{D}_{\mathcal{L}_I}$ of the operator $\mathcal{L}_I$. In particular, if $x \in L^2((-a, a)$ and $y = \mathcal{F}[-a,a] x$, then $y \in \mathcal{D}_{\mathcal{L}_I}$. Thus for $x \in \mathcal{D}_{\mathcal{L}_I}$ both summands in the expression $\mathcal{F}[-a,a] \mathcal{L}_I x - \mathcal{L}_I \mathcal{F}[-a,a] x$ are well defined. Since the operator $\mathcal{L}_I$ is a restriction of the operator $\mathcal{L}_{\max}$, then
\[ \mathcal{F}[-a,a] \mathcal{L}_I x - \mathcal{L}_I \mathcal{F}[-a,a] x = \mathcal{F}[-a,a] \mathcal{L}_{\max} x - \mathcal{L}_{\max} \mathcal{F}[-a,a] x \text{ for } x \in \mathcal{D}_{\mathcal{L}_I}. \]
In view of (10.7) and (10.10), the equality (10.8) holds.

2. Let $U \neq I$. Then at least of one value $u_{11} - 1$ or $u_{22} - 1$ differs from zero. For definiteness, let $u_{11} - 1 \neq 0$. Set
\[ \gamma = \frac{1 + u_{11}}{t(1 - u_{11})}, \quad x(t) = \psi_-(t) + \gamma \varphi_-(t) + x_0(t), \]  
(10.12)
where $x_0(t)$ is a smooth function which support is a compact subset of the open interval $(-a, a)$:
\[ \text{supp } x_0 \subset (-a, a). \]  
(10.13)
The function $x_0$ will be chosen later. According to (8.6), (10.13) and the choice of $\gamma$, for any choice of $x_0(t)$, the function $x(t)$ from (10.12) satisfy the boundary conditions (8.7). Thus,
\[ x(t) \in \mathcal{D}_{\mathcal{L}_U}. \]  
(10.14)
for any choice of $x_0$. Moreover
\[ b_{-a}(x) = 1, \quad b_a(x) = 0. \]  
(10.15)
For the function $y(t) = (\mathcal{F}[-a,a] x)(t)$, the boundary conditions (8.7) either hold, or does not hold. This depends on the choice of the function $x_0$. If $L_I = L_U$ for $U = I$.\footnote{\textcolor{red}{Note: This footnote needs to be referenced elsewhere in the text.}}
hold for this $y$, then $\mathcal{F}_{(-a,a)}x \in \mathcal{D}_{LU}$ and the equality (10.7) can be interpreted as the equality

$$(\mathcal{F}_{(-a,a)}\mathcal{L}_U x)(t) - (\mathcal{L}_U \mathcal{F}_{(-a,a)}x)(t) = \frac{2}{a} \left( b_a(x)e^{iat} + b_{-a}(x)e^{-iat} \right).$$  (10.16)

In view of (10.15), $(\mathcal{F}_{(-a,a)}\mathcal{L}_U x)(t) - (\mathcal{L}_U \mathcal{F}_{(-a,a)}x)(t) \neq 0$.

Let us show that both of the possibilities $\mathcal{F}_{(-a,a)}x \in \mathcal{D}_{LU}$ and $\mathcal{F}_{(-a,a)}x \not\in \mathcal{D}_{LU}$ are realizable. Since the function $y(t) = (\mathcal{F}_{(-a,a)}x)(t)$ is smooth on $[-a,a]$,

$$b_{-a}(y) = 0, b_a(y) = 0, c_{-a}(y) = -y(-a), c_{-a}(y) = -y(a).$$

Thus the boundary conditions (8.7) take the form

$$(1 - u_{11})y(-a) - u_{12}y(a) = 0, \quad (10.17a)$$

$$u_{21}y(-a) - (1 - u_{22})y(a) = 0. \quad (10.17b)$$

If, using the freedom of choice of the function $x_0(t)$ in (10.12), we can arbitrary prescribe the values $y(-a)$ and $y(a)$, then we can either satisfy the boundary conditions (10.17) (prescribing $y(-a) = 0, y(a) = 0$), or violate them (if $u_{11} \neq 1$, we prescribe $y(-a) = 1, y(a) = 0$, if $u_{22} \neq 1$, we prescribe $y(-a) = 0, y(a) = 1$.) The reference to Lemma below finishes the proof.

Lemma 10.3. Given the complex numbers $y_1$ and $y_2$, there exists a smooth function $x_0(t)$ on $[-a,a]$ which possesses the properties:

1. $\text{supp } x_0 \subseteq (-a,a)$.
2. $y_0(-a) = y_1, y_0(a) = y_2$, where $y_0 = \mathcal{F}_{[-a,a]}(x_0)$.

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Victor Katsnelson
Department of Mathematics, Weizmann Institute, Rehovot, 7610001, Israel
e-mail: victor.katsnelson@weizmann.ac.il; victorkatsnelson@gmail.com