Discrete complex image method for periodic Green’s function

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Abstract. The electromagnetic scattering of periodic structure may be expressed by integral equation with periodic Green function as the core. The periodic Green's function can be expressed as infinite sum in spectral domain. In order to avoid the numerical calculation of infinite sum, the discrete complex image method is proposed. By using the Generalized pencil-of-function method, the closed form can be obtained without any numerical integration. It can be expressed as one group of real sources and two groups of complex sources with complex amplitude and complex position. The accuracy of the method is demonstrated through numerical examples.

1. Introduction
It is necessary to calculate the periodic green’s function quickly and accurately when using the integral equation method to calculate the electromagnetic scattering of periodic structures [1-11]. Unfortunately, periodic green’s functions are slow to converge and are very time-consuming and complex [1]. In order to speed up the calculation of these sequences, many acceleration techniques have been proposed in the literature, which can be divided into analytical method and numerical method [2-11]. The spatial or spectral Kummer’s transformation [2], the spectral Kummer-Poisson’s method [3], the Veisoglu’s transformation [4], the perfectly matched layers method [5] and the Ewald’s method [6] are the analytical methods while the Shanks’s transformation [7], the T algorithm [8], the Levin’s T transform [9], the Chebyschev-Toeplitz algorithm [10] belong to the numerical methods. As shown in [11], The Bessel function's spectrum kummer-poisson method is more efficient than the Ewald’s method, which was previously thought to be the most efficient.

In this paper, the discrete complex image method is developed, which avoids the numerical calculation of infinite sum. By using the Generalized pencil-of-function method, the remaining quotients in the spectral domain can be approximated by exponential series. The periodic green's function can be expressed by a closed periodic green's function, which is composed of a finite series real source and two finite series complex images.

2. Formulation of the Problem
The two-dimensional Green's function of one-dimensional periodic structure in homogeneous medium is the field potential generated by an array of infinite periodic line sources with the same amplitude and linear progressive phase. We introduce a rectangular Cartesian coordinate system Oxyz in free space and suppose the line source is on the z=0 plane, parallel to the y axis, and has the same spacing p along the x axis. Ignoring the harmonic time dependence $e^{jwt}$, the periodic Green’s function can be written as [1]...
\[ G_p(x, z; x', z') = \sum_{n=-\infty}^{\infty} G(x, z; x' + np, z') e^{-jnpk} = \sum_{n=-N}^{N} G(x, z; x' + np, z') e^{-jnpk}, \]
\[ + \sum_{n=0}^{N-1} G(x, z; x' + np, z') e^{-jnpk} + \sum_{n=N+1}^{\infty} G(x, z; x' + np, z') e^{-jnpk}, \]  
(1)

Figure 1. The geometry of the problem

where \( N \) is a positive integer. In order to deduce the complex image representation of the two infinite summation terms on the right side of the equation (1), the following identities can be used

\[ G(x, z; x' + np, z') = \frac{H_0^{(2)}(k\sqrt{(x-x' - np)^2 + (z-z')^2})}{4j}, \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\beta_x|x-x'|} e^{-j\beta_z(z-z')} dk_z, \]  
(2)

Where \( \beta_x = \sqrt{k^2 - k_z^2} \). By substituting equation (2) in equation (1) and changing the order of summation and integration operators, the last two infinite series can be easily written as

\[ \sum_{n=-N}^{N} G(x, z; x' + np, z') e^{-jnpk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\beta_x|x-x'|} e^{-j\beta_z(z-z')} \frac{e^{j(k_x-\beta_x)p}}{1-e^{j(k_x-\beta_x)p}} dk_z, \]  
(3)

\[ \sum_{n=N+1}^{\infty} G(x, z; x' + np, z') e^{-jnpk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=N+1}^{\infty} e^{-j\beta_x|x-x'|} e^{-j\beta_z(z-z')} \frac{e^{j(k_x-\beta_x)p}}{1-e^{j(k_x-\beta_x)p}} dk_z, \]  
(4)

The quotients in equation (3) and equation (4) can be estimated by the generalized pencil-of-function method (GPOF) with finite exponential series

\[ \frac{e^{j(k_x-\beta_x)p}}{1-e^{j(k_x-\beta_x)p}} = \sum_{n=1}^{N_1} a_n e^{b_n\beta_x} \]  
(5)

And
2.1 One-level complex image method for approximation of the spectral-domain Green’s function

As shown in figure 2, it requires the variable \( t \) to vary evenly between 0 and \( T_0 \), that is, to sample uniformly along the path

\[
\beta_x = k[-jt + (1-t/T_0)], \quad 0 \leq t \leq T_0
\]  

(7)

The quotients in equation (5) and equation (6) sampled on the path \( C_{ap} \) in the \( \beta_x \)-plane and estimated in the exponential form of \( t \) using the GPOF, and the exponential form of \( t \) can be easily converted into the exponential form of \( \beta_x \). This kind of solution is called the one-level complex image method [12] because the complex functions being approximated sample between zero and \( T_0 \) and are assumed to be negligible beyond \( T_0 \).

\[
e^{-j(k_s + \beta_x)p} \over 1 - e^{-j(k_s + \beta_x)p} = \sum_{n=1}^{N_2} c_n e^{d_n \beta_x} \]

(6)

2.2 Two-level complex image method for approximation of the spectral-domain Green’s function

As shown in figure 3, the parametric equations that describe the paths for \( C_{ap1} \) and \( C_{ap2} \) are as follows:

\[
C_{ap1} : \beta_x = k[-jt + (1-t/T_{02})] \]

(8)

\[
C_{ap2} : \beta_x = -jk[T_{02} + t] \]

(9)

Figure 2. The paths \( C_{ap} \) used in one-level approximation

Figure 3. The paths \( C_{ap1} \) and \( C_{ap2} \) used in two-level approximation
The two-level complex image method first approximated the quotients in equation (3) and equation (4) along the path $C_{ap1}$, and then estimate the compensable exponential series for the contribution neglected beyond $T_0$ along the path $C_{ap2}$.

The spatial Green’s functions can be obtained by inverse Hankel transformation. By substituting equation (5) in equation (3) and equation (6) in equation (4), respectively, and by using identity equation (2), the following results are obtained:

\[
G_p(x,z;x',z') = \frac{1}{4j} \sum_{n=-N}^{N} e^{-j \cdot n \cdot p} H_0^{(2)}(k \sqrt{(x-x')^2+(z-z')^2})
\]
\[
+ \sum_{n=1}^{N1} a_n \frac{e^{j \cdot N_k \cdot p}}{4j} H_0^{(2)}(k \sqrt{|x-x'|+Np+jb_n)^2+(z-z')^2})
\]
\[
+ \sum_{n=1}^{N2} c_n \frac{e^{-j \cdot N_k \cdot p}}{4j} H_0^{(2)}(k \sqrt{|x-x'|-Np+jd_n)^2+(z-z')^2})
\]  

(10)

3. Numerical Results and Conclusion

In this paragraph, we examine the accuracy of the technique presented in the previous section by considering two typical examples. The first example is the case with $\lambda=p$, $k_x= k \sin \theta = \frac{\sqrt{2}}{2} k$, in which the period $p$ is the same order of magnitude as $\lambda$, and the one-level discrete complex image method is feasible under the circumstances. The second example is the case with $\lambda=0.1p$, $k_x= k \sin \theta = \frac{\sqrt{2}}{2} k$, in which the frequency is 10 times of the first example, in this case we should choose a small $T_0$ and the contribution of the spectral function beyond $T_0$ should be considered. So we have to calculate this example with the two-level discrete complex image method. Modulus value and the phase of scattered field on the $z=0.2 \lambda$ planes related to first and second examples are drawn in figure 4.5. As can be seen from the diagram, the result of the proposed method and spectral Kummer-Poisson’s method match very accurately.

![Figure 4](image-url)  
Figure 4. Scattered field calculated on 0.2$\lambda$ planes with one-level discrete complex image method

(a) modulus value (b) phase($\lambda=p$, $k_x= k \sin \theta = \frac{\sqrt{2}}{2} k$, N=50)
Figure 5. Scattered field calculated on 0.2\(\lambda\) planes two-level discrete complex image method

(a) modulus value (b) phase (\(\lambda=0.1p\), \(k_\parallel = k \sin \theta = \frac{\sqrt{2}}{2}k N=4\))

In a word, this paper presents a discrete complex image method, which accelerates the slow convergence of the periodic green’s function by approximating the two infinite series with two finite series of complex sources with complex positions and amplitudes, and finite series of real sources in the middle 2N+1 period are reserved. The complex sources are obtained by approximating the remaining quotients as a finite series of complex exponential using the GPOF. The two finite series of complex sources are used to represent the scattering of real sources with infinite periods in \(N + 1 \rightarrow \infty\) and \(-\infty \rightarrow -N \rightarrow 1\) periods separately. At low frequency, the one-level discrete complex image method is feasible to calculate the periodic Green’s function. As the frequency increases, the change of the quotients on the path \(C_{ap}\) are accelerated, in order to obtain as many sampling points as possible, it is reasonable to select a small \(T_0\), and the contribution of spectral functions beyond \(T_0\) should be considered. In these cases, the one-level discrete complex image method has difficulty in finding appropriate parameters, and the two-level discrete complex image method is proposed to solve this problem.

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