ON LEGENDRIAN CURVES IN $\mathbb{P}^3$

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Abstract. We show that if a smooth projective curve $C \subset \mathbb{P}^3$ (over an algebraically closed field of characteristic zero) is Legendrian with respect to a contact structure (it is well known that a contact structure on $\mathbb{P}^3$ is unique up to a linear automorphism) and $C$ is linearly normal (i.e., not an isomorphic linear projection of a smooth curve $C' \subset \mathbb{P}^n$, $n > 3$, where $C'$ does not lie in a hyperplane) then $C$ is a twisted cubic or a line.

Earlier, the author showed that any smooth projective curve can be embedded in $\mathbb{P}^3$ as a Legendrian curve.

1. Introduction

As far as 1926, Beniamino Segre [Seg26] proved, according to the Zentralblatt review, that the algebraic curve “of arbitrary genus and moduli” can be embedded in $\mathbb{P}^3$ as a Legendrian curve (Segre used a slightly different language; see below).

In [Bry82, Theorem G] R. Bryant gave a modern proof of a result close to Segre’s: in $\mathbb{P}^3$ it is shown that any smooth projective curve over $\mathbb{C}$ can be embedded in $\mathbb{P}^3$ as a Legendrian curve with respect to a contact structure on $\mathbb{P}^3$. It is well known that any contact structure on $\mathbb{P}^3 = \mathbb{P}(E)$ is defined by some non-degenerate skew-symmetric form on the four-dimensional linear space $E$, so such a structure is unique up to a linear automorphism of $\mathbb{P}^3$ (see details below). Cf. the paper [AFL17], where it is proved that for any $k \geq 2$ any open Riemann surface admits a proper embedding in $\mathbb{C}^{2k-1}$ that is tangent to the contact structure defined by a non-degenerate bilinear form.

When discussing this result, Fyodor Zak observed that for the Legendrian curves obtained by the construction from [Bry82] the linear system of hyperplane sections is as a rule dramatically incomplete (i.e., these curves are far from being linearly normal) and suggested to find out which Legendrian curves in $\mathbb{P}^3$ are linearly normal. The answer to this question turns out to be very short.
Theorem 1.1. Suppose that $C \subset \mathbb{P}^3$ is a smooth and connected projective curve that is Legendrian with respect to some contact structure on $\mathbb{P}^3$. If $C$ is linearly normal, then $C$ is a twisted cubic or a line.

It is well known that both twisted cubics and lines are Legendrian indeed (with respect to appropriate contact structures). To wit, if a contact structure is defined by a skew-symmetric form $B$, then projectivisations of isotropic planes in $E$ are contact lines, and see for example Remark 1.4 for twisted cubics.

A problem closely related to that of description of linearly normal Legendrian curves was considered as far as 1926 by Beniamino Segre [Seg26]. In the modern language, Segre investigated curves $C \subset \mathbb{P}^r$, $r \geq 3$, for which the Gaussian image $\gamma(C) \subset \text{Gr}(1, \mathbb{P}^r)$ (the Grassmannian of lines) has the smallest possible dimension of linear span if we regard the Grassmannian as Plücker embedded in $\mathbb{P}^\left(\binom{r+1}{2}-1\right)$. If $r = 3$, a non-degenerate curve satisfies this condition if and only if it is Legendrian.

For $r \geq 4$, Segre proves that the only curves satisfying this condition are normal rational curves in $\mathbb{P}^r$; a modern proof of Segre’s result was given by Ciliberto and Miranda in [CM92]. However, to the best of my knowledge, Segre did not treat the case of linearly normal curves in $\mathbb{P}^3$.

As a by-product of the proof, we establish the following result as well.

Proposition 1.2. Suppose that $C \subset \mathbb{P}^3$ is a smooth and connected projective curve of genus greater than zero. If $C$ is Legendrian with respect to a contact structure on $\mathbb{P}^3$, then $C$ is not contained in a quadric.

Proposition 1.2 fails for curves of genus 0 (see Remark 4.4).

In the concluding Section 6 we show that Theorem 1.1 cannot be readily extended to singular curves: we give an example of a connected reducible Legendrian curve in $\mathbb{P}^3$ that is linearly normal (and even a complete intersection). Needless to say, this curve is not a twisted cubic nor a line.

I am glad to thank Fyodor Zak for the discussion that gave rise to this paper.

2. Notation, conventions, and preliminaries

We adopt the notation and conventions of our paper [Lvo].

The base field is an arbitrary algebraically closed field of characteristic zero.

All varieties are supposed to be reduced (not necessarily irreducible), the word “point” means “closed point”.

If $E$ is a finite-dimensional linear space over the base field, then points of $\mathbb{P}(E)$ are lines in $E$.

If $X \subset \mathbb{P}^n$ is a connected projective subvariety, we will say that $X$ is linearly normal if the linear system of its hyperplane sections is complete (i.e., if the natural homomorphism $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{O}_X(1))$ is surjective).
We recall the folklore result according to which the contact structure on an (odd-dimensional) projective space is essentially unique: if $\mathbb{P}^{2k-1} = \mathbb{P}(E)$, where $E$ is a $2k$-dimensional linear space, then there exists a non-degenerate skew-symmetric bilinear form $B$ on $E$ such that if $x \in \mathbb{P}(E)$, then the contact hyperplane $S_x \subset \mathbb{P}_x(E)$, where $\mathbb{T}_x$ stands for the tangent space at $x$, is $\mathbb{T}_x \mathbb{P}((\hat{x})^\perp)$, where $(\hat{x})^\perp \subset E$ is the skew-orthogonal complement to $\hat{x}$ with respect to $B$. See [Lvo, Proposition 3.1].

If $X \subset \mathbb{P}^n$ is a projective subvariety, then $\mathcal{N}_{\mathbb{P}^n|X}^* = \mathcal{I}_X/\mathcal{I}_X^2$, where $\mathcal{I}_X$ is the ideal sheaf, is the conormal sheaf of $X$; if $X \subset \mathbb{P}^n$ is a smooth projective subvariety, then $\mathcal{N}_{\mathbb{P}^n|X}^*$ is locally free and its dual $\mathcal{N}_{\mathbb{P}^n|X}$ is the (locally free) normal sheaf aka normal bundle of $X$.

The terms “locally free sheaf” and “vector bundle” will be used interchangeably. If $\mathcal{E}$ is a vector bundle of rank $m$ over a variety $X$ and $x$ is a point of $X$, then $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at $x$ in the naïve sense (an $m$-dimensional linear space over the base field). If $s$ is a section of $\mathcal{E}$, then $s_x \in \mathcal{E}_x$ is the image of $s$ in $\mathcal{E}_x$. If $\varphi : \mathcal{E} \to \mathcal{F}$ is a homomorphism of vector bundles, then $\varphi_x : \mathcal{E}_x \to \mathcal{F}_x$ is the corresponding homomorphism of fibers.

If $E$ is a finite-dimensional linear space over the base field and $C \subset \mathbb{P}(E)$ is a smooth projective curve, then $\Gamma_C \subset E \otimes \mathcal{O}_C$ is the vector bundle of rank $2$ such that, for each $x \in C$, projectivisation of its fiber over $x$ is $\mathbb{T}_x C$ (the embedded tangent space to $C$ at $x$). The bundle $\Gamma_C$ can be included in the exact sequences

\[(2.1) \quad 0 \to \mathcal{O}_C(-1) \to \Gamma_C \to \omega_C^{-1}(-1) \to 0,\]
\[(2.2) \quad 0 \to \Gamma_C \to E \otimes \mathcal{O}_C \to \mathcal{N}_{\mathbb{P}(E)|C}(-1) \to 0.\]

The exact sequence (2.1) never splits. Actually, the element of $\text{Ext}^1_{\mathcal{O}_C}(\omega_C^{-1}(-1), \mathcal{O}_C(-1)) \cong H^1(C, \omega_C)$ defined by (2.1), coincides with $c_1(\mathcal{O}_C(1))$.

Some details on the sequences (2.1) and (2.2) can be found in [Lvo92].

3. Sections of twisted conormal bundle

**Proposition 3.1.** If $C \subset \mathbb{P}^3$ is a smooth Legendrian curve with respect to a contact structure on $\mathbb{P}^3$, then $\mathcal{N}_{\mathbb{P}^3|C}^*(1) \cong \Gamma_C$.

**Proof.** Suppose that $\mathbb{P}^3 = \mathbb{P}(E)$, $\dim E = 4$, and that the contact structure on $\mathbb{P}(E)$ is defined by a skew-symmetric bilinear form $B$ on $E$. Since $C$ is Legendrian with respect to this contact structure, for any point $x \in C$ the fiber $\Gamma_{C,x} \subset E$ (i.e., the deprojectivisation of the embedded tangent space $\mathbb{T}_x C \subset \mathbb{P}(E)$) is isotropic with respect to $B$ (see [Lvo] Proposition 3.2]). Hence, the non-degenerate bilinear form $B$ induces an isomorphism $\Gamma_{C,x} \to (E/\Gamma_{C,x})^*$. Globalizing and taking into account that, in view of (2.2), $(E \otimes \mathcal{O}_C)/\Gamma_C \cong \mathcal{N}_{\mathbb{P}(E)|C}(-1)$, one obtains the result. $\square$
Proposition 3.2. If \( C \subset \mathbb{P}^3 = \mathbb{P}(E) \) is a smooth Legendrian curve with respect to the contact structure defined by a skew-symmetric bilinear form \( B \) on \( E \), then the form \( B \) induces a section \( \sigma \) of the vector bundle \( N_{\mathbb{P}^3|C}(2) \) such that \( \sigma_x \neq 0 \) for any \( x \in C \).

Proof. For any \( x \in C \subset \mathbb{P}(E) \), let \( \hat{x} \subset E \) be its deprojectivisation. Since \( C \) is Legendrian, the homomorphism \( \hat{x} \to (E/\hat{x})^* \) induced by the bilinear form \( B \), factors through \( (E/\Gamma_{C,x})^* \). The resulting homomorphism \( \hat{x} \to (E/\Gamma_{C,x})^* \) is never zero since \( B \) is non-degenerate. Globalizing, one obtains a nowhere vanishing homomorphism

\[
O_C(-1) \to (E \otimes O_C/\Gamma_C)^* \cong N_{\mathbb{P}(E)|C}(2),
\]

which is the same as a section of \( N_{\mathbb{P}(E)|C}(2) \).

\( \square \)

Proposition 3.3. If \( C \subset \mathbb{P}^3 \) is a smooth Legendrian curve of genus greater than zero, then \( \dim H^0(C, N_{\mathbb{P}^3|C}(2)) = 1 \).

Proof. Proposition 3.1 implies that \( N_{\mathbb{P}^3|C}(2) \cong \Gamma_C(1) \). Twisting (2.1) one obtains the exact sequence

\[
(3.1) \quad 0 \to O_C \to \Gamma_C(1) \to \omega_C \to 0.
\]

If genus of \( C \) is greater than 1, (3.1) implies the result immediately. If genus is equal to 1, then \( \omega_C \cong O_C \), so (3.1) takes the form

\[
(3.2) \quad 0 \to O_C \to \Gamma_C(1) \to O_C \to 0.
\]

Since the exact sequence (2.1) does not split, (3.2) does not split either, so the homomorphism \( H^0(\Gamma_C(1)) \to H^0(O_C) \) induced by this sequence must be zero. This implies the result. \( \square \)

We conclude this section with the following construction. Suppose that \( X \subset \mathbb{P}^n = \mathbb{P}(E) \) is a smooth projective subvariety and that \( \mathcal{L} \) is an invertible sheaf on \( X \). If \( \sigma \in H^0(X, N^*_{\mathbb{P}(E)|X} \otimes \mathcal{L}) \) and \( x \in X \) is such that \( \sigma_x \neq 0 \), then the section \( \sigma \) defines a hyperplane in \( \mathbb{P}(E) \) containing \( x \). To wit, \( \sigma_x \) may be regarded as a non-zero linear functional on the fiber \( (N_{\mathbb{P}(E)|X} \otimes \mathcal{L}^{-1})_x \); since \( N_{\mathbb{P}(E)|X} \otimes \mathcal{L}^{-1} \) is, by virtue of (2.2), a quotient of \( E \otimes \mathcal{L}^{-1}(1) \), this linear functional may be regarded as a linear functional on

\[
(E \otimes \mathcal{L}^{-1}(1))_x \cong E \otimes (\mathcal{L}^{-1}(1))_x.
\]

The projectivisation of the kernel of this non-zero functional is a well-defined hyperplane in \( \mathbb{P}(E) = \mathbb{P}^n \).

Notation 3.4. In the above setting, the hyperplane corresponding to the section \( \sigma \in H^0(X, N^*_{\mathbb{P}^n|X} \otimes \mathcal{L}) \) at the point \( x \in X \) will be denoted \( H_{\sigma,x} \subset \mathbb{P}^n \).

It is clear that \( H_{\sigma,x} \) does not change if one replaces the section \( \sigma \) by its multiple.
4. Quadrics containing Legendrian curves

Construction 4.1. Suppose that \( X \subset \mathbb{P}^n \) is a projective subvariety and \( f \in H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \) is a nonzero homogeneous polynomial of degree \( d \). If \( X \subset V(f) \), then \( f \) defines a section \( \sigma_f \in H^0(\mathcal{N}_{\mathbb{P}^n|X}^*(d)) \), as follows.

Observe that constructing such a section amounts to constructing a homomorphism \( \lambda_f : \mathcal{O}_X(-d) \to \mathcal{N}_{\mathbb{P}^n|X}^* \). To construct it, suppose that \( s \) is a local section of \( \mathcal{I}_X \). Then \( f s \) is a well-defined local section if \( f \) is a well-defined local section of \( \mathcal{I}_X/\mathcal{I}_X^2 = f(s) \) is a well-defined local section if \( \mathcal{I}_X/\mathcal{I}_X^2 = \mathcal{N}_{\mathbb{P}^n|X}^* \). It is clear that \( d(f s) = 0 \) whenever \( s \) is a local section of \( \mathcal{I}_X(-d) \subset \mathcal{O}_X(-d) \), so the mapping \( s \mapsto d(f s) \) factors through \( \mathcal{I}_X(-d) \) and induces the required homomorphism \( \lambda_f \), hence the section \( \sigma_f \).

Proposition 4.2. If, in the above setting, \( x \in X \) is a smooth point of \( V(f) \), then \( (\sigma_f)_x \neq 0 \).

If we put \( \mathcal{L} = \mathcal{O}_X(d) \), then, in terms of Notation 3.4, \( H_{\sigma_f,x} = T_x V(f) \subset \mathbb{P}^n \).

The straightforward proof of this proposition is left to the reader.

Proof of Proposition 1.2. Recall the statement: \( C \subset \mathbb{P}^3 = \mathbb{P}(E) \) is a smooth connected Legendrian curve of genus greater than zero, not lying in a hyperplane. We are to show that \( C \) cannot be contained in a quadric.

Arguing by contradiction, suppose that \( C \subset Q \), where \( Q = \mathcal{O}_X(q) \) is a quadric and \( q \in H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \) is a quadratic form. Let \( \sigma_q \in H^0(\mathcal{N}_{\mathbb{P}^3|C}^*(2)) \) be the section corresponding to \( q \) via Construction 4.1. Since \( Q \) is a quadric and \( C \), being of genus \( > 0 \), is not a line, \( C \not\subset \text{sing}(Q) \), so Proposition 4.2 implies that, for all points \( x \in X \) (except possibly a finite number), \( (\sigma_q)_x \neq 0 \), \( H_{\sigma,x} \) is well-defined and \( H_{\sigma,x} = T_x Q \subset \mathbb{P}^3 \).

If \( C \) is Legendrian with respect to the contact structure corresponding to a skew-symmetric bilinear form \( B \) on \( E \), and if \( \sigma \in H^0(\mathcal{N}_{\mathbb{P}^3|C}^*(2)) \) is the section corresponding to \( B \) by the construction of Proposition 3.2, then it is clear that, for any \( x \in C \), \( H_{\sigma,x} = \mathbb{P}(\hat{x}^\perp) \). Proposition 4.1 from [Lvo] implies that \( \mathbb{P}(\hat{x}^\perp) = \text{Osc}_x^2 C \).

Since genus of \( C \) is greater than \( 0 \), Proposition 3.3 implies that
\[
\dim H^0(C, N_{\mathbb{P}^3|C}^*(2)) = 1,
\]
so the sections \( \sigma_q \) and \( \sigma \) are proportional. Hence, for all points \( x \in X \) (except possibly a finite number) one has \( H_{\sigma_q,x} = H_{\sigma,x} \), whence \( T_x Q = \text{Osc}_x^2 C \). Now we state the following result, which is an analogue of Meusnier’s theorem from elementary differential geometry.

Lemma 4.3. Suppose that \( F \subset \mathbb{P}^3 \) is a quasiprojective surface and \( C \subset F \) is a smooth curve not lying in a plane. If, for a point \( x \in C \), one has \( \text{Osc}_x^2 C = T_x F \), then the tangent direction of \( C \) at \( x \) is isotropic with respect to the second fundamental form of \( F \) at \( x \).
Proof of Lemma 4.3. Put $H = \text{Osc}_x^2 C = T_x F$. Since $H$ is the osculating plane to $C$ at $x$, one has
\begin{equation}
(H \cdot C)_x \geq 3,
\end{equation}
where the quantity in the left-hand side is the local intersection index.

Recall the definition of the second fundamental form (for surfaces in $\mathbb{P}^3$). Suppose that $x \in F$, $H = T_x X$, and $s \in H^0(\mathcal{O}_F(1))$ is a section defining the (hyper)plane $H$. Fixing an isomorphism between the stalk of $\mathcal{O}_F(1)$ at $x$ and the local ring $\mathcal{O}_{F,x}$, we may and will regard $s$ as an element of the local ring. Now if $v, w \in T_x F$ are two tangent vectors, then let $V$ and $W$ be germs of vector fields at $x$ such that $V(x) = v$, $W(x) = w$; if we put $\Phi(v, w) = V(W(s))(x)$ (i.e., the value of the function $V(W(s))$ at $x$), then $\Phi$ becomes a symmetric bilinear form on $T_x F$ (defined up to proportionality); this form is called second fundamental form.

Suppose now that $v \in T_x C \subset T_x F$; we are to show that $\Phi(v, v) = 0$. In this case the germ $V$ can be chosen so that $V(\mathcal{I}_{C,x}) \subset \mathcal{I}_{C,x}$, where, $\mathcal{I}_{C,x} \subset \mathcal{O}_{F,x}$ is the ideal of $C$ in the local ring; the germ $V \in T_F,x$ induces the germ $V'$ of a vector field on $C$ at $x$, and if $r: \mathcal{O}_{F,x} \to \mathcal{O}_{C,x}$ is the natural surjection, then $V' \circ r = r \circ V$. Thus, if we put $s' = r(s)$, it suffices to show that $V'(V'(s'))(x) = 0$. Now it follows from (4.1) that $s' \in m_{C,x}^3$, where $m_{C,x}$ is the maximal ideal of $\mathcal{O}_{x,C}$. Observe that if $V: A \to A$ is a derivation of a ring $A$ and if $I \subset A$ is an ideal, then $V(I^k) \subset I^{k-1}$ for any natural $k$. Hence, $V'(V'(s')) \in m_{C,x}$, and we are done. \hfill \Box

Resuming the proof of Proposition 1.2, observe that, since $T_x Q = \text{Osc}_x^2 C$ for the general $x \in C$, Lemma 4.3 implies that the curve $C \subset Q$ must be, at the general point $x \in C$, tangent to a line lying on $Q$. Hence, $C$ is a line, which contradicts the hypothesis. This completes the proof. \hfill \Box

Remark 4.4. Proposition 1.2 fails for curves of genus 0. To wit, if $m \geq 3$ and $C$ is the image of the mapping $\mathbb{P}^1 \to \mathbb{P}^3$ defined by the formula
\[(s : t) \mapsto (s^m : s^{m-1}t : st^{m-1} : t^m),\]
then $C$ is Legendrian with respect to the contact structure corresponding to the skew-symmetric form with the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 2 - m \\
0 & 0 & m - 1 & 0 \\
0 & 1 - m & 0 & 0 \\
m - 2 & 0 & 0 & 0
\end{pmatrix}
\]
and $C$ is contained in the quadric with the equation $x_0x_3 = x_1x_2$.

5. Proof of Theorem 1.1

Suppose that $C \subset \mathbb{P}^3 = \mathbb{P}(E)$ is a smooth projective linearly normal Legendrian curve; we are to show that $C$ is a line or a twisted cubic.
If \( C \) lies in a plane, then \( C \) is a line by virtue of Proposition 17 from [LM07]; so, from now on we assume that \( C \) is not contained in a plane.

If genus of \( C \) is zero, then \( C \) is linearly normal if and only if it is a twisted cubic. From now on we assume that genus of \( C \) is not zero; let us show that this case is impossible.

Dualizing and twisting exact sequences (2.1) and (2.2), one obtains the exact sequences

\[
\begin{align*}
0 & \to \omega_C(2) \to \Gamma_C^*(1) \xrightarrow{f} O_C(2) \to 0, \\
0 & \to N^*_P|C(2) \to E^* \otimes O_C(1) \xrightarrow{g} \Gamma_C^*(1) \to 0.
\end{align*}
\]

The natural mapping of sheaves \( E^* \otimes O_C \to O_C(1) \) induces a homomorphism \( \mu : E^* \otimes O_C(1) \to O_C(2) \), and it is clear that the diagram

\[
\begin{array}{ccc}
E^* \otimes O_C(1) & \xrightarrow{\mu} & O_C(2) \\
\downarrow{g} & & \downarrow{f} \\
\Gamma_C^*(1) & \xrightarrow{\mu} & \Gamma_C^*(1)
\end{array}
\]

is commutative.

Putting

\[ m = H^0(\mu) : H^0(E^* \otimes O_C(1)) \to H^0(O_C(2)) \]

and passing to global sections, one obtains the commutative diagram

\[
\begin{array}{ccc}
H^0(E^* \otimes O_C(1)) & \xrightarrow{H^0(g)} & E^* \otimes H^0(O_C(1)) \xrightarrow{H^0(m)} H^0(O_C(2)) \\
H^0(\Gamma_C^*(1)) & \xrightarrow{H^0(f)} & H^0(\Gamma_C^*(1)).
\end{array}
\]

Since \( C \) is linearly normal and does not lie in a plane, the natural homomorphism \( E^* \to H^0(O_C(1)) \) is bijective, so \( m \) coincides with the natural multiplication mapping

\[ m' : E^* \otimes E^* \to H^0(O_C(2)). \]

Proposition 3.2 and exact sequence (5.2) imply that \( \ker H^0(g) \neq 0 \), so it follows from diagram (5.3) that \( \ker m' = \ker m \neq 0 \). This means that there exists a quadric containing \( C \); since genus of \( C \) is not zero, this contradicts Proposition 1.2 and this contradiction completes the proof.

6. A CONCLUDING REMARK

The following example shows that Theorem 1.1 cannot be readily extended to curves with singularities. To wit, let \( X \subset \mathbb{P}^3 \) be a twisted cubic; choose a contact structure on \( \mathbb{P}^3 \) with respect to which \( X \) is Legendrian. Now pick a point \( x \in X \) and let \( L \) be the tangent line to \( C \) at \( x \); put \( C = X \cup L \). It is clear that the reducible curve \( C \) is Legendrian: one has only to check that \( L \) is tangent to the contact structure at any point, but
this follows immediately from the fact that $X$ is Legendrian and Proposition 3.2 from [Lvo]. On the other hand, the curve $C$ is linearly normal. It is even a complete intersection of two quadrics.

I do not know whether Theorem 1.1 can be extended to singular and irreducible curves.

References

[AFL17] Antonio Alarcón, Franc Forstnerič, and Francisco J. López, Holomorphic Legendrian curves, Compos. Math. 153 (2017), no. 9, 1945–1986. MR 3705282

[Bry82] Robert L. Bryant, Conformal and minimal immersions of compact surfaces into the 4-sphere, J. Differential Geometry 17 (1982), no. 3, 455–473. MR 679067

[CM92] Ciro Ciliberto and Rick Miranda, Gaussian maps for certain families of canonical curves, Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser., vol. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 106–127. MR 1201378

[LM07] J. M. Landsberg and L. Manivel, Legendrian varieties, Asian J. Math. 11 (2007), no. 3, 341–359. MR 2372722

[Lvo] Serge Lvovski, Some remarks on osculating self-dual varieties, preprint arXiv:1602.07450 [math.AG].

[Lvo92] ———, Extensions of projective varieties and deformations. I, Michigan Math. J. 39 (1992), no. 1, 41–51. MR 1137887

[Seg26] B. Segre, Sulle curve algebriche le cui tangenti appartengono al massimo numero di complessi lineari indipendenti, Mem. Accad. naz. Lincei, Cl. Sci. fis. mat. nat. (6) 2 (1926), 577–592 (Italian).