CROSS VARIETIES OF APERIODIC MONOIDS WITH COMMUTING IDEMPOTENTS

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Abstract. A variety of algebras is called Cross if it is finitely based, finitely generated, and has finitely many subvarieties. In present article, we classify all Cross varieties of aperiodic monoids with commuting idempotents.

1. Introduction

A variety of algebras is called finitely based if it has a finite basis of its identities. A variety is called finitely generated if it is generated by a finite algebra. A variety with finitely many subvarieties is said to be small. A finitely generated, finitely based, small variety of algebras is called a Cross variety. Finite members from several classical classes of algebras such as groups and associative rings generate Cross varieties (see [13] and [7, 12], respectively). But this result does not hold in general. For example, there are a lot of finite semigroups or monoids that generate non-Cross varieties.

For any class of algebras, one method of describing Cross varieties in this class is to find its minimal non-Cross varieties, or almost Cross varieties. Any non-Cross variety contains some almost Cross subvariety by Zorn’s lemma. It follows that a variety is Cross if and only if it does not contain any almost Cross variety. So, if one manages to classify all almost Cross varieties within some class of varieties, then this classification implies a description of all Cross varieties in this class.

The present article is concerned with the class of aperiodic monoids, i.e., monoids that have trivial subgroups only. For a long time, it was known only two explicit examples of almost Cross varieties of monoids: the variety of all commutative monoids [4] and the variety of all idempotent monoids [16]. The second of these varieties is the first example of an almost Cross variety of aperiodic monoids. More recently, Jackson [5] found two new examples of almost Cross monoid varieties $M$ and $N$. It turned out that $M$ and $N$ are finitely generated subvarieties of the class $A_{cen}$ of aperiodic monoids with central idempotents. In [8], Lee showed that $M$ and $N$ are the unique finitely generated almost Cross subvarieties of $A_{cen}$. Further, Lee [10] found one more (non-finitely generated) almost Cross subvariety $L$ of $A_{cen}$ and established that only $L$, $M$ and $N$ are almost Cross subvarieties of $A_{cen}$.

It is of fundamental interest to generalize this result by Lee from $A_{cen}$ to some larger class of monoids. The class $A_{com}$ of aperiodic monoids with commuting...
idempotents is a natural candidate. The first step in describing almost Cross subvarieties of $A_{\text{com}}$ was taken by Lee [11]. Namely, he proved that a subvariety of $A_{\text{com}}$ that satisfies the identity $xyx^2 \approx x^2yx$ is Cross if and only if it excludes the almost Cross varieties $L$, $M$ and $N$. However, there exist almost Cross subvarieties of $A_{\text{com}}$ that are different from $L$, $M$ and $N$. The two of such varieties $K$ and $\overleftarrow{K}$ were exhibited in [3]. Finally, just recently, two more almost Cross subvarieties $J$ and $\overleftarrow{J}$ of $A_{\text{com}}$ were found in [1]. In present article, we provide two new examples of almost Cross subvarieties $P$ and $\overleftarrow{P}$ of $A_{\text{com}}$ and then completely classify all Cross varieties within the class $A_{\text{com}}$.

The article consists of five sections. Some background results are first given in Section 2. In Section 3, we provide two new examples of almost Cross subvarieties of $A_{\text{com}}$. Let $O$ denote the monoid variety given by the identities
\begin{align*}
(1.1) & \quad xtyzx \approx xtyzyx, \\
(1.2) & \quad xtyzx \approx xtyzyx.
\end{align*}

The subvarieties of $O$ have been actively studied. In particular, any subvariety of $O$ is finitely based and $O$ is a maximal variety with such a property [9]. Section 4 is devoted to the description of aperiodic almost Cross subvarieties of $O$. Finally, in Section 5, the main result of the article is formulated and is proved. Namely, it is established that a subvariety of $A_{\text{com}}$ is Cross if and only if it excludes the nine almost Cross subvarieties of $A_{\text{com}}$.

\section{Preliminaries}

\subsection{Locally finite varieties.}
Recall that a variety of universal algebras is called \textit{locally finite} if all of its finitely generated members are finite.

\begin{lemma}[	ext{[6, Lemma 2.1]}] A locally finite variety $V$ of algebras is finitely generated if and only if there exists no strictly increasing infinite chain $V_1 \subset V_2 \subset V_3 \subset \cdots$ of varieties such that $V = \bigvee_{i \geq 1} V_i$. Consequently, any locally finite, small variety is finitely generated. \hfill $\square$
\end{lemma}

The following assertion directly follows from [14, Proposition 3.1].

\begin{lemma} Each monoid variety that satisfies the identity
\begin{equation}
xyx \approx xyx^2
\end{equation}
is locally finite. \hfill $\square$
\end{lemma}

\subsection{Subvarieties of $A_{\text{com}}$.}
For each $n \geq 1$, let $A_n$ denote the variety defined by the identities
\begin{align*}
(2.1) & \quad x^2y^n \approx y^n x^n, \\
(2.2) & \quad x^n \approx x^{n+1}, \\
(2.3) & \quad x^ny^n \approx y^n x^n.
\end{align*}

It is easily shown that the inclusions $A_1 \subset A_2 \subset \cdots \subset A_{\text{com}}$ hold and are proper. The following claim is evident.

\begin{observation}
The class $A_{\text{com}}$ is not a variety, but each of its subvarieties is contained in $A_n$ for all sufficiently large $n$. \hfill $\square$
\end{observation}

A variety of monoids is called \textit{completely regular} if it consists of \textit{completely regular monoids} (i.e., unions of groups). Let $T$ and $SL$ denote the trivial variety of monoids and the variety of all semilattice monoids, respectively. We note that $A_1 = SL$. 

\begin{lemma}[	ext{2.2}]
Each monoid variety that satisfies the identity
\begin{equation}
xyx \approx xyx^2
\end{equation}
is locally finite. \hfill $\square$
\end{lemma}

\begin{lemma}[	ext{2.2}]
Each monoid variety that satisfies the identity
\begin{equation}
xyx \approx xyx^2
\end{equation}
is locally finite. \hfill $\square$
\end{lemma}

\begin{lemma}[	ext{2.2}]
Each monoid variety that satisfies the identity
\begin{equation}
xyx \approx xyx^2
\end{equation}
is locally finite. \hfill $\square$
\end{lemma}

Observation 2.4. If $V$ is a completely regular subvariety of $A_{\text{com}}$, then $V \subseteq \{T, SL\}$.

Proof. Observation 2.3 implies that $V$ satisfies the identities $(2.2)$ and $(2.3)$ for some $n \geq 1$. If $n$ is the least number with such a property, then $n = 1$ because $V$ is completely regular. Hence $V \subseteq SL$. It is well known that $SL$ covers $T$, whence $V \in \{T, SL\}$. □

Observation 2.5. Each commutative or completely regular subvariety of $A_{\text{com}}$ is Cross.

Proof. A completely regular subvariety of $A_{\text{com}}$ is either $T$ or $SL$ by Observation 2.4 and so Cross. A commutative aperiodic variety is Cross by the results of [4]. □

2.3. Words, identities, Rees quotients of free monoids. Let $\mathcal{X}^*$ denote the free monoid over a countably infinite alphabet $\mathcal{X}$. Elements of $\mathcal{X}$ are called letters and elements of $\mathcal{X}^*$ are called words. We treat the identity element of $\mathcal{X}$ as the empty word. The content of a word $w$, i.e., the set of all letters occurring in $w$, is denoted by $\text{con}(w)$. An identity is an expression $u \approx v$ where $u, v \in \mathcal{X}^*$.

Observation 2.6. Let $a, a', b, b' \in \mathcal{X}^*$, $x, t_1, t_2, \ldots, t_k \in \mathcal{X}$ and $V$ be a monoid variety that satisfies the identity $(2.2)$. Suppose that $t_1, t_2, \ldots, t_k \notin \text{con}(aa'bb')$. Then the identities

$$\sigma : a t_1 x^n b \approx a' t_1 x^n b'$$

and

$$\tau : a \prod_{i=1}^{k} (t_i x^n) b \approx a' \prod_{i=1}^{k} (t_i x^n) b'$$

are equivalent in $V$.

Proof. We substitute $x^n \prod_{i=2}^{k} (t_i x^n)$ for $t_1$ in the identity $\sigma$ and obtain the identity $\tau$. So, $\tau$ follows from $\sigma$.

Conversely, substitute 1 for the letters $t_2, t_3, \ldots, t_k$ in the identity $\tau$. We get the identity $a t_1 x^{kn} b \approx a' t_1 x^{kn} b'$. Evidently, this identity together with $(2.2)$ implies $\sigma$. Therefore, $\sigma$ and $\tau$ are equivalent in $V$. □

For any set of words $W \subseteq \mathcal{X}^*$, let $S(W)$ denote the Rees quotient over the ideal of $\mathcal{X}^*$ consisting of all words that are not subwords of words in $W$. A word $w$ is an isoterms for a variety $V$ if it violates any non-trivial identity of the form $w \approx w'$. The following statement shows it is relatively easy to check if $S(W)$ is contained in some given variety.

Lemma 2.7 ([5, Lemma 3.3]). For any variety $V$ and any set of words $W$, the inclusion $S(W) \in V$ holds if and only if all words from $W$ are isoterms for $V$. □

2.4. Decompositions of words. A letter is called simple [multiple] in a word $w$ if it occurs in $w$ once [at least twice]. The set of all simple [multiple] letters in a word $w$ is denoted by $\text{sim}(w)$ [respectively, $\text{mul}(w)$]. The number of occurrences of the letter $x$ in $w$ is denoted by $\text{occ}_x(w)$. For a word $w$ and letters $x_1, x_2, \ldots, x_k \in \text{con}(w)$, let $w(x_1, x_2, \ldots, x_k)$ denote the word obtained from $w$ by deleting all letters except $x_1, x_2, \ldots, x_k$.

Let $w$ be a word and $\text{sim}(w) = \{t_1, t_2, \ldots, t_m\}$. We may assume without loss of generality that $w(t_1, t_2, \ldots, t_m) = t_1 t_2 \cdots t_m$. Then $w = t_0 w_0 t_1 w_1 \cdots t_m w_m$ where $w_0, w_1, \ldots, w_m$ are possibly empty words and $t_0$ is the empty word. The words $w_0, w_1, \ldots, w_m$ are called blocks of $w$, while $t_0, t_1, \ldots, t_m$ are said to be dividers of $w$. 

...
The representation of the word $w$ as a product of alternating dividers and blocks starting with the divider $t_0$ and ending with the block $w_m$ is called a decomposition of the word $w$. For a given word $w$, a letter $x \in \text{con}(w)$ and a natural number $i \leq \text{occ}_x(w)$, we denote by $h_i(w, x)$ the right-most divider of $w$ that precedes the $i$th occurrence of $x$ in $w$.

**Lemma 2.8** ([3, Proposition 2.13 and Lemma 2.14]). Let $V$ be a non-completely regular non-commutative monoid variety and $u \approx v$ be an identity that holds in $V$. Suppose that

\begin{equation}
(2.4)
\end{equation}

\[ t_0 u_0 t_1 u_1 \cdots t_m u_m \]

is the decomposition of $u$. Then the decomposition of $v$ has the form

\begin{equation}
(2.5)
\end{equation}

\[ t_0 v_0 t_1 v_1 \cdots t_m v_m \]

for some words $v_0, v_1, \ldots, v_m$ and $\text{con}(u_0 u_1 \cdots u_m) = \text{con}(v_0 v_1 \cdots v_m)$. \qed

**2.5. The variety $F$.** Let $F$ denote the monoid variety given by the identities $(2.1)$ and

\begin{align}
(2.6) & \quad x^2 y \approx x^2 yx, \\
(2.7) & \quad x^2 y^2 \approx y^2 x^2, \\
(2.8) & \quad xyzxy \approx yxzxy.
\end{align}

**Lemma 2.9** ([3, Proposition 6.9(i)]). A non-trivial identity $u \approx v$ holds in the variety $F$ if and only if $\text{sim}(u) = \text{sim}(v)$, $\text{mul}(u) = \text{mul}(v)$ and $h_1(u, x) = h_1(v, x)$, $h_2(u, x) = h_2(v, x)$ for all $x \in \text{con}(u)$.

If $u, v \in \mathcal{R}^*$ and $\Sigma$ is an identity or an identity system, then we will write $u \Sigma v$ in the case when the identity $u \approx v$ follows from $\Sigma$.

**Lemma 2.10.** Let $V$ be a non-completely regular monoid variety that satisfies the identity $(2.2)$ for some $n \geq 2$. Then $F \not\subseteq V$ if and only if $V$ satisfies the identity

\begin{equation}
(2.9)
\end{equation}

\[ xyx^n \approx x^n yx^n. \]

**Proof.** The variety $F$ violates the identity $(2.9)$ for any $n \geq 2$ by Lemma 2.9. Therefore, if $V$ satisfies $(2.9)$, then $F \not\subseteq V$.

Conversely, suppose that $F \not\subseteq V$. If $V$ is commutative, then it satisfies the identity $(2.9)$ because

\[ x^n yx^n \approx x^n yx^n. \]

Hence it suffices to assume that the variety $V$ is non-commutative. By assumption, there exists an identity $u \approx v$ that is satisfied by the variety $V$ but not by the variety $F$. Let $(2.4)$ be the decomposition of $u$. In view of Lemma 2.8, the decomposition of $v$ has the form $(2.5)$ and $\text{mul}(u) = \text{mul}(v)$. Lemma 2.9 implies that there are a letter $x \in \text{mul}(u)$ and $k \in \{1, 2\}$ such that $h_k(u, x) \neq h_k(v, x)$. If $k = 1$ then we multiply the identity $u \approx v$ by $xt$ on the left where $t \notin \text{con}(u)$. So, we may assume that $k = 2$. Suppose that $h_2(u, x) = t_i$ and $h_2(v, x) = t_j$ where $i \neq j$. We may assume without any loss that $i > j$. Then $V$ satisfies the identity

\[ u(x, t_i) = xt_i x^p \approx x^q t_i x^r = v(x, t_i). \]
where \( p \geq 1, q \geq 2 \) and \( r \geq 0 \). Now we multiply this identity by \( x^n \) on the right. Taking into account the identity (2.2), we get that \( \mathbf{V} \) satisfies the identity (2.10) \[
.xy x^n \approx x^q y x^n.
\]
Since \( xy x^n \overset{(2.10)}{\approx} x^{1+q(n-1)} y x^n \overset{(2.2)}{\approx} x^n y x^n \), the variety \( \mathbf{V} \) satisfies the identity (2.9). \( \square \)

2.6. The variety \( \mathbf{Q} \). For an identity system \( \Sigma \), we denote by \( \text{var} \Sigma \) the variety of monoids given by \( \Sigma \). Put \[ \mathbf{Q} = \text{var}\{(2.1), (2.7), xy x^2 \approx x^2 y x^2\}. \]

Lemma 2.11 ([11, Lemma 5.1]). An identity of the form (2.11) \[
\sum u_0 \prod_{i=1}^m (t_i u_i) \approx \sum v_0 \prod_{i=1}^m (t_i v_i)
\]
where \( u_0, v_0, u_1, v_1, \ldots, u_m, v_m \) are words not containing any of the letters \( t_1, t_2, \ldots, t_m \) holds in \( \mathbf{Q} \) if and only if \( \text{con}(u_i) = \text{con}(v_i) \) for any \( i = 0, 1, \ldots, m \). \( \square \)

In fact, the proof of the following assertion is similar to the proof of Lemma 5.3 in [11]. We provide its proof for the sake of completeness.

Lemma 2.12. Let \( \mathbf{V} \) be a non-completely regular monoid variety that satisfies the identity (2.2) for some \( n \geq 2 \). Then \( \mathbf{Q} \not\subseteq \mathbf{V} \) if and only if \( \mathbf{V} \) satisfies the identity (2.12) \[
x^n y x z x^n \approx x^n y x z x^n.
\]

Proof. The variety \( \mathbf{Q} \) violates the identity (2.12) for any \( n \geq 1 \) by Lemma 2.11. Therefore, if \( \mathbf{V} \) satisfies (2.12), then \( \mathbf{Q} \not\subseteq \mathbf{V} \).

Conversely, suppose that \( \mathbf{Q} \not\subseteq \mathbf{V} \). If \( \mathbf{V} \) is commutative, then it satisfies the identity (2.12) because \( x^n y x z x^n \text{ comm.} \approx x^{n+1} y x z x^n \overset{(2.2)}{\approx} x^n y x z x^n \).

Hence it suffices to assume that the variety \( \mathbf{V} \) is non-commutative. By assumption, there exists an identity \( u \approx v \) that is satisfied by the variety \( \mathbf{V} \) but not by the variety \( \mathbf{Q} \). Let (2.4) be the decomposition of \( u \). In view of Lemma 2.8, the decomposition of \( v \) has the form (2.5). According to Lemma 2.11, \( \text{con}(u_i) \neq \text{con}(v_i) \) for some \( i \in \{0, 1, \ldots, m\} \), say \( x \in \text{con}(u_i) \setminus \text{con}(v_i) \).

Evidently, the identity \( u \approx v \) implies the identity \( x h_1 u h_2 x \approx x h_1 v h_2 x \) where \( h_1, h_2 \notin \text{con}(u) = \text{con}(v) \). The last identity implies the identity (2.13) \[
x^p t_i x^q t_i x^r \approx x^s t_i x^t \]
for some \( p, q, r, s, t \geq 1 \). Then the identity (2.14) \[
x^n y x^n z x^n \approx x^n y z x^n.
\]
holds in \( \mathbf{V} \) because \[
x^n y x^n z x^n \overset{(2.2)}{\approx} x^n y x^n \overset{(2.13)}{\approx} x^n y z x^n \overset{(2.2)}{\approx} x^n y z x^n.
\]
Finally, since \[
x^n y z x^n \overset{(2.14)}{\approx} x^n y x^n z x^n \overset{(2.2)}{\approx} x^n y x^n z x^n \overset{(2.14)}{\approx} x^n y x^n z x^n,
\]
the variety \( \mathbf{V} \) satisfies the identity (2.12). \( \square \)
A word $w$ is said to be $n$-limited if $\text{occ}_{x}(w) \leq n$ for any $x \in \text{con}(w)$.

**Corollary 2.13.** Let $w$ be a word and $V$ a monoid variety that satisfies the identities (2.2) and

$$\kappa_{n,j} : \prod_{i=0}^{n}(t_{i,x}) \approx (\prod_{i=0}^{j-1}(t_{i,x})) (t_{j,x^{2}}) (\prod_{i=j+1}^{n}(t_{i,x}))$$

for some $n \geq 1$ and $0 \leq j \leq n$. If $Q \not\subseteq V$, then there is a $(2n + 2)$-limited word $w'$ such that $w \approx w'$ is satisfied by $V$.

**Proof.** In view of Lemma 2.12, $V$ satisfies (2.12). It follows that the identities

$$x \left( \prod_{i=1}^{n}(t_{i,x}) \right) yz x \left( \prod_{i=1}^{n}(h_{i,x}) \right)$$

$$\kappa_{n,j} x \left( \prod_{i=1}^{j-1}(t_{i,x}) \right) (t_{j,x^{n}}) \left( \prod_{i=j+1}^{n}(t_{i,x}) \right) yz x \left( \prod_{i=1}^{j-1}(h_{i,x}) \right) (h_{j,x^{n}}) \left( \prod_{i=j+1}^{n}(h_{i,x}) \right)$$

$$(2.12) \approx x \left( \prod_{i=1}^{j-1}(t_{i,x}) \right) (t_{j,x^{n}}) \left( \prod_{i=j+1}^{n}(t_{i,x}) \right) yxz x \left( \prod_{i=1}^{j-1}(h_{i,x}) \right) (h_{j,x^{n}}) \left( \prod_{i=j+1}^{n}(h_{i,x}) \right)$$

$$\kappa_{n,j} x \left( \prod_{i=1}^{n}(t_{i,x}) \right) yxz x \left( \prod_{i=1}^{n}(h_{i,x}) \right)$$

hold in $V$. We see that $V$ satisfies the identity

$$x \left( \prod_{i=1}^{n}(t_{i,x}) \right) yz x \left( \prod_{i=1}^{n}(h_{i,x}) \right) \approx x \left( \prod_{i=1}^{n}(t_{i,x}) \right) yxz x \left( \prod_{i=1}^{n}(h_{i,x}) \right).$$

We note that the identity (2.15) allows us to delete the $(n + 2)$th occurrence of some letter $x$ in a word $w$ whenever $\text{occ}_{x}(w) > 2n + 2$. This implies that, for any $w \in \mathcal{X}^{*}$, there is a $(2n + 2)$-limited word $w'$ such that $w \approx w'$ is satisfied by $V$. □

2.7. **The variety $R$ and its subvarieties.** Let $R$ denote the monoid variety given by the identities (2.7) and

$$xzxxyt \approx xzyxt.$$  

Put $E = \var{x^{2} \approx x^{3}, x^{2}y \approx xy, x}$. If $V$ is a monoid variety, then we denote by $\overline{V}$ the variety dual to $V$, i.e., the variety consisting of monoids antiisomorphic to monoids from $V$. For any $n \geq 1$, we put

$$e_{n} = \begin{cases} x & \text{if } n \text{ is odd,} \\ y & \text{if } n \text{ is even.} \end{cases}$$

The following statement establishes restrictions on the type of identities that can be used to define some subvarieties of $R$.

**Lemma 2.14** ([2, Lemmas 3.4 and 3.5]). Each subvariety of $R$ containing $F \vee \overline{E}$ is defined by the identities (2.7) and (2.16) together with some of the following
identities: (2.8)

\[
\begin{align*}
(2.17) & \quad yx^2txy \approx xytxy, \\
(2.18) & \quad x^2ytxy \approx xytxy, \\
(2.19) & \quad xyxztx \approx xyxztx,
\end{align*}
\]

\[
\alpha_n : \quad xy \prod_{i=1}^{n+1} (t_i e_i) \approx yx \prod_{i=1}^{n+1} (t_i e_i),
\]
\[
\beta_n : \quad yx^2 \prod_{i=2}^{n+1} (t_i e_i) \approx xy \prod_{i=2}^{n+1} (t_i e_i),
\]
\[
\gamma_n : \quad x^2y \prod_{i=1}^{n+1} (t_i e_i) \approx xy \prod_{i=1}^{n+1} (t_i e_i),
\]
\[
\gamma'_n : \quad x^2y \prod_{i=2}^{n+1} (t_i e_i) \approx xy \prod_{i=2}^{n+1} (t_i e_i),
\]

where \( n \geq 1 \).

\[\square\]

3. The Variety \( P \)

Put \( P = \text{var}\{ (2.1), (2.7), (2.8), \beta_1, \gamma_1' \} \). The main goal of this section is to verify that \( P \) and \( P \preceq P \) are almost Cross varieties.

**Proposition 3.1.** The variety \( P \) is a non-finitely generated almost Cross subvariety of \( A_{\text{com}} \). The lattice \( L(P) \) has the form shown in Fig. 1, where

\[
\begin{align*}
\mathcal{C} &= \var{ x^2 \approx x^3, xy \approx yx }, \\
\mathcal{D} &= \var{ x^2 \approx x^3, x^2y \approx xy \approx yx^2 }, \\
\mathcal{H} &= P\{ (2.19) \}, \\
\mathcal{P}_n &= P\{ \alpha_n \}, \quad n \geq 1.
\end{align*}
\]

**Proof.** Let \( V \) be a proper subvariety of \( P \). If \( V \) is completely regular, then \( V \in \{ T, SL \} \) by Observation 2.4. So, we may assume that \( V \) is not completely regular. If \( Q \not\subseteq V \), then \( V \) satisfies the identity

\[
x^2yzx^2 \approx x^2yxzx^2
\]

by Lemma 2.12. Then the identities

\[
xyxztx \approx xy^2zx^2 \approx xy^2zx^2 \approx xyxztx
\]

hold in \( V \), whence \( V \subseteq \mathcal{H} \). The lattice \( L(H) \) has the form shown in Fig. 1 [1, Proposition 3.1]. So, we may assume that \( Q \not\subseteq V \). If \( F \not\subseteq V \), then \( V \subseteq Q \) by Lemma 2.10. The results of [11, Section 5] imply that the lattice \( L(Q) \) is as shown in Fig. 1. Thus, we may assume that \( F \cup Q \subseteq V \). Clearly, \( P \) satisfies the identities (2.8), (2.17), (2.18), \( \beta_n, \gamma_n \) and \( \gamma'_n \) for any \( n \geq 1 \). Then Lemma 2.14 and the fact that \( Q \) violates (2.19) imply that \( V = P_n \) for some \( n \geq 1 \). It follows that \( F \cup Q = P_1 \). The variety \( H \) violates the identity \( \alpha_1 \) by [1, Lemma 3.6(i)] and satisfies the identities

\[
xyhsyt \approx xyhsyt \approx xyhsyt \approx yxhsyt \approx yxhsyt,
\]
whence $H \lor Q = P_2$. Thus, the lattice $L(P_2)$ has the form shown in Fig. 1.

To complete the description of $L(P)$ it remains to verify that $P_n \subseteq P_{n+1}$ for any $n \geq 1$. Clearly, $P_n \subseteq P_{n+1}$ for any $n \geq 1$ because $\alpha_{n+1}$ follows from $\alpha_n$. So, it suffices to establish that if $P_{n+1}$ satisfies an identity $u \approx v$, then $v = xy \prod_{i=1}^{n+1} t_i e_i$ for some $\ell_1, \ell_2, \ldots, \ell_{n+1} \geq 1$. Put 

$$\Psi = \{(2.1), (2.7), (2.8), \alpha_{n+1}, \beta_1, \gamma_1 \}.$$ 

By induction, we can reduce our considerations to the case when $\{u, v\} = \{a \xi(s)b, a \xi(t)b\}$ for some words $a, b \in X^*$, an endomorphism $\xi$ of $X^*$ and $s \approx t \in \Psi$. We may assume without loss of generality that the words $u$ and $v$ are different.

**Observation 3.2.** Let $a$ and $b$ be different letters. If $ab$ is a subword of $u$, then this subword has exactly one occurrence in $u$ and $u$ does not contain the subword $ba$. □

If $\xi(x)$ is the empty word, then $\xi(s) = \xi(t)$, but this is impossible because $u \neq v$. Thus, $\xi(x)$ is non-empty. Then, since $\text{con}(\xi(x)) \subseteq \text{mul}(\xi(s)) \subseteq \text{mul}(u)$, Observation 3.2 implies that $\xi(x) = c^k$ for some $k \geq 1$ and $c \in \{x, y\}$.
The identity (2.1) allows us to add and delete the occurrences of the letter \( x \) next to the non-first occurrence of this letter. This implies the required conclusion whenever \( s \approx t \) coincides with (2.1). Suppose now that \( s \approx t \in \Psi \setminus \{ (2.1) \} \). Note that \( \xi(y) \) is non-empty because the identity \( u \approx v \) is non-trivial. Then \( \xi(y) = d' \) for some letter \( d \) and some \( r \geq 1 \) by Observation 3.2. Since the words \( u \) and \( v \) are different, \( c \neq d \). So, \( \{ c, d \} = \{ x, y \} \).

For any \( u \approx v \), Lemmas 2.9 and 2.11 imply that \( u \) contains at least two occurrences of the form \( c^{2k}d' \), \( d'c^{2k} \) and \( c^k d'c^k \). So, it remains to consider the case when the identity \( s \approx t \) coincides with the identity \( \alpha_{n+1} \). Evidently, there is \( j \in \{ 1, 2, \ldots, n+2 \} \) such that \( t_i \not\in \text{con}(\xi(t_j)) \) for any \( i \in \{ 1, 2, \ldots, n+1 \} \). It follows that \( \text{con}(\xi(t_j)) \subseteq \{ x, y \} \). Then \( u \) contains either at least two different occurrences of some subwords of the form \( xy \) or \( yx \). A contradiction with Observation 3.2.

So, we have proved that the lattice \( L(P) \) has the form shown in Fig. 1. This fact and Lemmas 2.1 and 2.2 imply that any proper subvariety of \( P \) is Cross. Then \( P \) is almost Cross. Finally, \( P \) is non-finitely generated by Lemmas 2.1 and 2.2. \( \square \)

If \( w \in \mathcal{B}^* \) and \( X \subseteq \mathcal{B} \), then \( w_X \) denote the word obtained from \( w \) by deleting all letters from \( X \).

**Lemma 3.3.** Let \( k \geq 1 \) and \( V \) be a monoid variety that satisfies the identity (2.2) for some \( n \geq 1 \). Suppose that \( F \vee Q \subseteq V \). Then \( V \) satisfies the identity

\[
\delta_{k,n} : xy \prod_{i=1}^{k+1} (t_i e_i^n) \approx yx \prod_{i=1}^{k+1} (t_i e_i^n)
\]

if and only if \( P_{k+1} \not\in V \).

**Proof.** For any \( n \geq 1 \), \( P_{k+1} \{ \alpha_k \} = P_{k+1} \{ \beta_k \} = P_k \) because

\[
xy \prod_{i=1}^{k+1} (t_i e_i) \overset{(2.1)}{=} xy \prod_{i=1}^{k+1} (t_i e_i^n) \approx yx \prod_{i=1}^{k+1} (t_i e_i^n) \overset{(2.1)}{=} yx \prod_{i=1}^{k+1} (t_i e_i).
\]

This fact and Proposition 3.1 imply that if \( V \) satisfies \( \delta_{k,n} \), then \( P_{k+1} \not\in V \).

Conversely, suppose that \( P_{k+1} \not\in V \). Then there exists an identity \( u \approx v \) that is satisfied by the variety \( V \) but not by the variety \( P_{k+1} \). The variety \( V \) is non-completely regular and non-commutative because \( F \vee Q \subseteq V \). Let (2.4) be the decomposition of \( u \). Lemma 2.8 implies that the decomposition of \( v \) has the form (2.5).

As in [3], a letter \( x \) is said to be a 1-divisor of a word \( w \) if the first and the second occurrences of this letter in \( w \) lie in different blocks. Equivalently, \( x \) is a 1-divisor of a word \( w \) if \( h_1(w, x) \neq h_2(w, x) \). For any \( 0 \leq i \leq m \), let \( X_i = \text{con}(u_i) \setminus X_i \). The inclusion \( F \vee Q \subseteq V \) and Lemmas 2.9 and 2.11 imply that \( X_i \) coincides with the set of all 1-dividers of \( v \) in the block \( v_i \) and \( Y_i = \text{con}(v_i) \setminus X_i \). Let \( Y_i = \{ y_{i1}, y_{i2}, \ldots, y_{ir_i} \} \). It is routinely checked that \( P_{k+1} \) satisfies \( u \approx \prod_{i=0}^{m} (t_i u'_i) \) and \( v \approx \prod_{i=0}^{m} (t_i v'_i) \) where \( u'_i = (u_i)_{Y_i}, h_i, v'_i = (v_i)_{Y_i}, h_i \) and \( h_i = y_{i1} y_{i2}^2 \cdots y_{ir_i}^2 \).

For any \( s = 0, 1, \ldots, m+1 \), we put \( w_s = \prod_{i=s}^{m-1} (t_i v'_i) \prod_{i=s}^{m} (t_i u'_i) \). Since \( u \approx v \) does not hold in \( P_{s+1} \), there is \( j \in \{ 0, 1, \ldots, m \} \) such that \( P_{s+1} \) violates \( w_j \approx w_{j+1} \). Clearly, all the letters of both \( (u_j)_{Y_j} \) and \( (v_j)_{Y_j} \) are simple in these words. By induction, we may assume that there are \( x, y \) such that the first occurrence of \( x \)
said to be \( a \) if \( \ell > j \), then we obtain a contradiction with the fact that (3.2) is not satisfied by \( \mathbf{P}_{k+1} \) because

\[
\begin{align*}
\mathbf{w}_{j+1} &\equiv \left( \prod_{i=0}^{j-1} (t_i v_i') \right) (t_j a x y b h) \left( \prod_{i=j+1}^{m} (t_i u_i') \right) .
\end{align*}
\]

If \( x, y \in \text{con}(u_i') \) for some \( \ell > j \), then we may assume that \( r > k + 1 \) because this identity follows from \( (2.1), (2.7) \) for some \( \{ x, y \} = \text{con}(e_{a}) \).

Clearly, \( r > 2 \) because \( x, y \in \text{con}(\prod_{i=j+1}^{m} u_i') = \text{con}(\prod_{i=j+1}^{m} v_i) \). If \( r > k + 1 \), then we get a contradiction with the fact that (3.2) is not satisfied by \( \mathbf{P}_{k+1} \) because this identity follows from \( \alpha_k \).

Now, for any \( i = 1, 2, \ldots, r - 1 \), we substitute \( t_i e_i^n \) for \( t_k \) and 1 for all the other letters occurring in the identity \( u \approx v \) except \( x \) and \( y \). We obtain the identity

\[
xy \prod_{i=1}^{r-1} (t_i e_i^n) \approx yx \prod_{i=1}^{r-1} (t_i e_i^n)
\]

for some \( p_1, q_1, p_2, q_2, \ldots, p_{r-1}, q_{r-1} > n \). Then, since (2.2) holds in \( \mathbf{V} \), we obtain that \( \mathbf{V} \) satisfies \( \delta_{r-2,n} \). Thus, we may assume that \( r \leq k + 1 \). It remains to note that \( \delta_{r-2,n} \) implies \( \delta_{k,n} \). \( \square \)

4. The variety \( \mathbf{O} \)

Recall that \( \mathbf{O} = \text{var}\{ (1,1), (1,2) \} \). Consider an identity of the form (2.11) where \( \text{sim}(u) = \text{sim}(v) = \{ t_1, t_2, \ldots, t_m \} \) and \( u_0, v_0, u_1, v_1, \ldots, u_m, v_m \) are words not containing any of the letters \( t_1, t_2, \ldots, t_m \). If \( (u_0, v_0), (u_1, v_1), \ldots, (u_m, v_m) \neq (\emptyset, \emptyset) \), then this identity is said to be \textit{efficient}. For any \( i \geq 2 \), the word \( w \) is said to be \textit{i-free} if, for any \( a_1, a_2, a_3 \in \mathcal{F}^{*} \), the equality \( w = a_1(a_2)^{a_3} \) implies that \( a_2 \) is the empty word. For any variety \( \mathbf{V} \) and any identity system \( \Sigma \), we put \( \mathbf{V}\{\Sigma\} = \mathbf{V} \land \text{var} \Sigma \). Let

\[
\mathbf{L} = \mathbf{O}\{ x^2 \approx x^3, x^2 y \approx y x^2, a_1 \}.
\]

The main result of this section is the following description of aperiodic almost Cross subvarieties of \( \mathbf{O} \).

**Proposition 4.1.** The varieties \( \mathbf{L}, \mathbf{P} \) and only they are aperiodic almost Cross subvarieties of \( \mathbf{O} \).
Proof. Let $V$ be an aperiodic subvariety of $O$ that does not contain $L$ and $P$. We need to verify that $V$ is a Cross variety. In view of Observation 2.5, we may assume that $V$ is non-commutative and non-completely regular. Since $V$ is aperiodic, it satisfies the identity $x^s \approx x^{s+1}$ for some $s \geq 1$. According to [11, Lemma 4.4], $V$ satisfies $\kappa_{s'}$ for some $0 \leq j \leq s'$. If we multiply the identity $\kappa_{s'}$ by $t_{s'+1}x$ on the left, then we obtain the identity $\kappa_{s'+1,j}$. Evidently, the identity $x^s \approx x^{s+1}$ implies the identity $x^{s'+1} \approx x^{s'+2}$. It follows that we may assume without any loss that $V$ satisfies the identities (2.2) and $\kappa_{n,j}$ for some $n \geq 1$ and $0 \leq j \leq n$.

It is proved in [10, Lemma 7] that any small subvariety of $O$ is Cross. In view of this fact, it suffices to verify that $V$ is small. Let $u \approx v$ be an identity that does not hold in $V$. We verify that there is an identity $u' \approx v'$ such that $V\{u \approx v\} = V\{u' \approx v'\}$ and $\ell(u'), \ell(v') \leq 50n^2$. Then any subvariety of $V$ can be given by identities from some fixed finite system of identities. The proof of the fact that $V$ is small and so the proof of Proposition 4.1 are thus complete.

The identity (2.2) allows us to assume that $u$ and $v$ are $(n+1)$-free. In view of [10, Lemma 8 and Remark 11], we may assume without any loss that one of the following claims holds:

(a) $u \approx v$ coincides with the efficient identity

\[(4.1) \quad x^{e_0} \prod_{i=1}^{m} (t_i x^{e_i}) \approx x^{f_0} \prod_{i=1}^{m} (t_i x^{f_i})\]

for some $m \geq 0$, $0 \leq e_0, f_0, e_1, f_1, \ldots, e_m, f_m \leq n$;

(b) $u \approx v$ coincides with an efficient identity

\[(4.2) \quad x^{e_0} y^{f_0} \prod_{i=1}^{m} (t_i x^{e_i} y^{f_i}) \approx y^{f_0} x^{e_0} \prod_{i=1}^{m} (t_i x^{e_i} y^{f_i})\]

for some $m \geq 0$, $e_0, f_0 \geq 1$, $0 \leq e_1, f_1, e_2, f_2, \ldots, e_m, f_m \leq n$ and $\sum_{i=0}^{m} e_i, \sum_{i=0}^{m} f_i \geq 2$.

Further considerations are divided into two cases corresponding to the claims (a) and (b).

The claim (a) holds. Suppose that $Q \not\subseteq V$. Lemma 2.8 and Corollary 2.13 imply that there are $(2n+2)$-limited words $u_1 = x^{e_0} \prod_{i=1}^{m} (t_i x^{e_i})$ and $v_1 = x^{f_0} \prod_{i=1}^{m} (t_i x^{f_i})$ such that $u \approx u_1$ and $v \approx v_1$ hold in $V$. Let $X = \{t_i \mid 1 \leq i \leq m \text{ and } e_i = f_i = 0\}$. Put $u' = (u_1)_x$ and $v' = (v_1)_x$. Clearly, the identity $u' \approx v'$ is equivalent to $u \approx v$ in $V$. Since $u' \approx v'$ is efficient and $u'$, $v'$ are $(2n+2)$-limited, both $u'$ and $v'$ have at most $(4n+3)$ simple letters. Hence $\ell(u'), \ell(v') \leq 6n + 5 < 50n^2$. So, we may assume below that $Q \subseteq V$.

Since the identity $u \approx v$ is efficient, Lemma 2.11 implies that $e_i, f_i > 0$ for any $i = 0, 1, \ldots, m$. If $m \leq 2n$, then $\ell(u), \ell(v) \leq n + 2n(n+1) < 50n^2$ because $u$ and $v$ are $(n+1)$-free. Therefore, we may assume without any loss that $m > 2n$. Then $V$ satisfies the identities $u \approx_{\kappa_{n,j}} u_1$ and $v \approx_{\kappa_{n,j}} v_1$ where

\[
u_1 = x^{e_0} \prod_{i=1}^{m} (t_i x^{e_i}) \prod_{i=m-n+1}^{m} (t_i x^{e_i})
\]

\[
u_1 = x^{f_0} \prod_{i=1}^{m} (t_i x^{f_i}) \prod_{i=m-n+1}^{m} (t_i x^{f_i})
\]
Now Observation 2.6 applies with the conclusion that $u_1 \approx v_1$ is equivalent to $u' \approx v'$ in $V$ where

$$u' = x^{e_0} \left( \prod_{i=1}^{n-1} (t_i x^{e_i}) \right) \left( \prod_{i=m-n+1}^{m} (t_i x^{e_i}) \right),$$

$$v' = x^{f_0} \left( \prod_{i=1}^{n-1} (t_i x^{f_i}) \right) \left( \prod_{i=m-n+1}^{m} (t_i x^{f_i}) \right).$$

Then $V\{u \approx v\} = V\{u_1 \approx v_1\} = V\{u' \approx v'\}$. Since $u'$ and $v'$ are $(n+1)$-free, $\ell(u') \leq n + (n^2 - 1) + (n + 1) + n(n + 1) < 50n^2$.

and we are done.

The claim (b) holds. Suppose that $Q \notin V$. Lemma 2.8, Corollary 2.13 and its proof imply that there are $(2n+2)$-limited words $u_1 = x^{e_0} y^{f_0} \prod_{i=1}^{n} (t_i x^{e_i} y^{f_i})$ and $v_1 = y^{f_0} x^{e_0} \prod_{i=1}^{n} (t_i x^{e_i} y^{f_i})$ such that $u \approx u_1$ and $v \approx v_1$ hold in $V$. Let $X = \{ t_i \mid 1 \leq i \leq m \text{ and } e_i' = f_i' = 0 \}$. Put $u'(1) = (u_1)_X$ and $v'(1) = (v_1)_X$. Clearly, the identity $u' \approx v'$ is equivalent to $u \approx v$ in $V$. Since $u' \approx v'$ is efficient and $u'$, $v'$ are $(2n+2)$-limited, both $u'$ and $v'$ have at most $(4n+2)$ simple letters. Hence $\ell(u'), \ell(v') \leq 8n + 6 < 50n^2$. So, we may assume below that $Q \subseteq V$. Then $(e_i, f_i) \neq (0, 0)$ for any $i = 0, 1, \ldots, m$ because the identity $u \approx v$ is efficient. If $m \leq 8n$, then $\ell(u), \ell(v) \leq 2n + 8n(2n + 1) < 50n^2$ because $u$ and $v$ are $(n+1)$-free. Therefore, we may assume without any loss that $m > 8n$. Then either $e = \sum_{i=0}^{m} e_i > 2n$ or $f = \sum_{i=0}^{m} f_i > 2n$ because $(e_i, f_i) \neq (0, 0)$ for any $i = 0, 1, \ldots, m$. By symmetry, we may assume that $e > 2n$. Further considerations are divided into two cases: $f \leq 2n$ or $f > 2n$.

Case 1: $f \leq 2n$. There are $\ell \leq 2n$ and a maximal subsequence $k_1, k_2, \ldots, k_\ell$ of $4n, 2n + 1, \ldots, m - 4n + 1$ such that $f_{k_1}, f_{k_2}, \ldots, f_{k_\ell} > 0$. For convenience, we put $k_{\ell+1} = m - 4n + 1$. Then $u = x^{e_0} y^{f_0} w$ and $v = y^{f_0} x^{e_0} w$ where

$$w = \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \prod_{i=4n}^{k_1-1} (t_i x^{e_i} y^{f_i}) \prod_{s=1}^{\ell} \left( \prod_{i=k_s}^{k_{s+1}-1} (t_i x^{e_i} y^{f_i}) \right) \prod_{i=k_{\ell+1}}^{m} (t_i x^{e_i} y^{f_i}).$$

Then $\prod_{i=0}^{4n-1} e_i, \prod_{i=m-4n+1}^{m} e_i > n$ because the identity $u \approx v$ is efficient and $f \leq 2n$. It follows that $V$ satisfies the identities $u \approx u_1$ and $v \approx v_1$ where $u_1 = x^{e_0} y^{f_0} w_1$, $v_1 = y^{f_0} x^{e_0} w_1$ and

$$w_1 = \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \prod_{i=4n}^{k_1-1} (t_i x^{e_i} y^{f_i}) \prod_{s=1}^{\ell} \left( \prod_{i=k_s}^{k_{s+1}-1} (t_i x^{e_i} y^{f_i}) \right) \prod_{i=k_{\ell+1}}^{m} (t_i x^{e_i} y^{f_i}).$$

Now Observation 2.6 applies with the conclusion that $u_1 \approx v_1$ is equivalent to $u' \approx v'$ in $V$ where $u' = x^{e_0} y^{f_0} w'$, $v' = y^{f_0} x^{e_0} w'$ and

$$w' = \left( \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \right) \left( \prod_{i=4n}^{k_1-1} (t_i x^{e_i} y^{f_i}) \right) \prod_{s=1}^{\ell} \left( \prod_{i=k_s}^{k_{s+1}-1} (t_i x^{e_i} y^{f_i}) \right) \prod_{i=m-4n+1}^{m} (t_i x^{e_i} y^{f_i}).$$

Then $V\{u \approx v\} = V\{u_1 \approx v_1\} = V\{u' \approx v'\}$. Since $u'$ and $v'$ are $(n+1)$-free and $\ell \leq 2n$,

$\ell(u'), \ell(v') \leq 2n + (4n - 1)(2n + 1) + (n + 1) + 2n(3n + 2) + 4n(2n + 1) \leq 50n^2$.
and we are done.

Case 2: $f > 2n$. Then $V(u \approx v)$ satisfies the identity

\[(4.3) \quad x^{c_o} y^{f_o} tx^n y^n \approx y^{f_o} x^{c_o} tx^n y^n\]

because

\[
x^{c_o} y^{f_o} tx^n y^n \approx x^{c_o} y^{f_o} tx^n y^n + f_n \quad (4.1) \quad x^{c_o} y^{f_o} tx^n y^n \prod_{i=1}^{m} (x^{c_i} y^{f_i}) \quad (4.2)
\]

\[
y^{f_o} x^{c_o} tx^n y^n \prod_{i=1}^{m} (x^{c_i} y^{f_i}) \quad (1.1) \quad y^{f_o} x^{c_o} tx^n y^n + f_n \quad (2.2) \approx y^{f_o} x^{c_o} tx^n y^n.
\]

Suppose that $F \not\subseteq V$. Then $V$ satisfies (2.9) by Lemma 2.10. Since $e, f > 2n$, there are $c$ and $d$ such that $e, f_d > 0$ and $n \leq \sum_{i=1}^{e} e_i, \sum_{i=1}^{d} f_i, \sum_{i=1}^{m} f_i$. By symmetry, we may assume that $c \leq d$. Then $V(u \approx v) = V((2.9), (4.3))$ because

\[
u \approx^{s, s'} x^{c_o} y^{f_o} w^{(2.9)} \approx x^{n-1+e_0} y^{n-1+f_0} w^{(2.2)} \approx x^{n+e_0} y^{n+f_0} w^{(1.2)} \approx x^{c_o} y^{f_o} x^n y^n w^{(4.3)}
\]

\[
y^{f_o} x^{c_o} x^n y^n w^{(1.2)} \approx y^{n+f_0} x^n + e_0 w^{(2.2)} \approx y^{n-1+f_0} x^{n-1+e_0} w^{(2.9)} \approx y^{f_o} x^{c_o} w_{n, j} \approx v,
\]

where

\[
w = \left( \prod_{i=1}^{c-1} (t_i x^{c_i} y^{f_i}) \left( t_c x^n y^{f_c} \right) \left( \prod_{i=c+1}^{d-1} (t_i x^{c_i} y^{f_i}) \right) \left( t_d x^n y^n \right) \left( \prod_{i=d+1}^{m} (t_i x^{c_i} y^{f_i}) \right) \right).
\]

and we are done. So, we may assume below that $F \subseteq V$. Then $V$ satisfies $\delta_{s, s'}$ for some $s, s' \geq 1$ by Proposition 3.1, Lemma 3.3 and the fact that $P \not\subseteq V$. Clearly, $\delta_{s, s'}$ implies $\delta_{s+1, s}$ and $\delta_{s, s'+1}$. Then the arguments quite analogous to ones from the first paragraph of the proof of this proposition allow us to assume that $V$ satisfies $\delta_{n, n}$.

Since $u \approx v$ is efficient, either $\sum_{i=0}^{4n-1} e_i \geq n$ or $\sum_{i=0}^{4n-1} f_i \geq n$ and either $\sum_{i=m-4n+1}^{4n-1} e_i \geq n$ or $\sum_{i=m-4n+1}^{4n-1} f_i \geq n$. By symmetry, we may assume that $\sum_{i=0}^{4n-1} e_i \geq n$. Further considerations are divided into two cases: $\sum_{i=m-4n+1}^{4n-1} e_i \geq n$ or $\sum_{i=m-4n+1}^{4n-1} f_i \geq n$.

Case 2.1: $\sum_{i=m-4n+1}^{4n-1} e_i \geq n$. Put

\[Y_1 = \{s \mid 4n \leq s \leq m-4n, n \leq \sum_{i=0}^{s} f_i \} \quad \text{and} \quad Y_2 = \{s \mid 4n \leq s \leq m-4n, n \leq \sum_{i=s}^{m} f_i \}.
\]

Let

\[p = \begin{cases} \min Y_1 & \text{if } Y_1 \neq \emptyset, \\ m-4n & \text{otherwise} \end{cases} \quad \text{and} \quad q = \begin{cases} \max Y_2 & \text{if } Y_2 \neq \emptyset, \\ 4n & \text{otherwise}. \end{cases}
\]

Clearly, $p \leq q$ because $f > 2n$. There exist $t, r \leq n$ and a maximal subsequence $k_1, k_2, \ldots, k_t$ of $4n, 4n+1, \ldots, p-1$ such that $f_{k_1}, f_{k_2}, \ldots, f_{k_t} > 0$, and a maximal subsequence $h_1, h_2, \ldots, h_r$ of $q+1, q+2, \ldots, m-4n$ such that $f_{h_1}, f_{h_2}, \ldots, f_{h_r} > 0$. 
For convenience, we put \( k_{\ell+1} = p \) and \( k_{r+1} = m - 4n + 1 \). Then

\[
\prod_{i=4n}^{p-1} (t_i x^{e_{i}} y^{f_{i}}) = \prod_{i=4n}^{k_{1}-1} (t_i x^{e_{i}}) \prod_{s=1}^{\ell} ((t_{k_s} x^{e_{k_s}} y^{f_{k_s}}) \prod_{i=k_s+1}^{k_{s+1}-1} (t_i x^{e_{i}})) \prod_{i=q+1}^{m-4n} (t_i x^{e_{i}} y^{f_{i}}).
\]

For any \( i = p, p + 1, \ldots, q \), put

\[
e_i' = \begin{cases} n & \text{if } e_i > 0, \\ 0 & \text{otherwise; } \end{cases} \quad \text{and} \quad f_i' = \begin{cases} n & \text{if } f_i > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Then \( V \) satisfies the identities \( u \overset{\kappa_{n,j}}{\approx} u_1 \) and \( v \overset{\kappa_{n,j}}{\approx} v_1 \) where \( u_1 = x^{e_0} y^{f_0} w_1 w_2 w_3, \ v_1 = y^{f_0} x^{e_0} w_1 w_2 w_3, \ w_2 = \prod_{i=p}^{q} (t_i x^{e_i'} y^{f_i'}) \) and

\[
w_1 = \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \prod_{i=4n}^{k_{1}-1} (t_i x^{n}) \prod_{s=1}^{\ell} ((t_{k_s} x^{e_{k_s}} y^{f_{k_s}}) \prod_{i=k_s+1}^{k_{s+1}-1} (t_i x^{n})) = \prod_{i=q+1}^{m-4n} (t_i x^{e_{i}} y^{f_{i}}).
\]

If \( e'_i = f'_i \) for some \( p \leq s \leq q \), then \( e'_i = f'_i = n \) because \( u \approx v \) is efficient. Then \( V\{u \approx v\} = V\{(4.3)\} \) because \( u \overset{\kappa_{n,j}}{\approx} u_1 \overset{(4.3)}{\approx} v_1 \overset{\kappa_{n,j}}{\approx} v, \) and we are done. So, we may assume that \((e'_i, f'_i) \in \{(n, 0), (0, n)\}\) for any \( i = p, p + 1, \ldots, q \). By symmetry, we may assume that there exist \( b \geq 0 \) and a subsequence \( g_1 = p, g_2, \ldots, g_{b+1} = q + 1 \) of \( p, p + 1, \ldots, q + 1 \) such that \( w_2 = \prod_{i=1}^{b} (\prod_{j=g_i}^{g_{i+1}-1} (t_s e_i')) \). If \( b > n \), then \( V\{u \approx v\} = V \) because \( u \overset{\kappa_{n,j}}{\approx} u_1 \overset{\delta_{n,n}}{\approx} \overset{\kappa_{n,j}}{v_1} \overset{\kappa_{n,j}}{\approx} v, \) and we are done. Thus, we may assume that \( b \leq n \).

Now Observation 2.6 applies with the conclusion that \( u_1 \approx v_1 \) is equivalent to \( u' \approx v' \) where \( u' = x^{e_0} y^{f_0} w_1' w_2' w_3', v' = y^{f_0} x^{e_0} w_1' w_2' w_3', \ w_2' = \prod_{i=1}^{b} (t_s e_i') \) and

\[
w_1' = \left( \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \right) (t_2n x^{n}) \prod_{s=1}^{\ell} ((t_{k_s} x^{e_{k_s}} y^{f_{k_s}}) t_{k_{s+1}} x^n) = \left( \prod_{i=q+1}^{m-4n} (t_i x^{e_{i}} y^{f_{i}}) \right).
\]

Then \( V\{u \approx v\} = V\{u_1 \approx v_1\} = V\{u' \approx v'\} \). Since \( u', v' \) are \((n + 1)-\)free and \( \ell, r, b \leq n \), we have that

\[
\ell(w_1') \leq (4n - 1)(2n + 1) + (n + 1) + n(3n + 2) < 20n^2,
\]

\[
\ell(w_2') \leq n(n + 1) \leq 2n^2,
\]

\[
\ell(w_3') \leq (n + 1) + n(3n + 2) + 4n(2n + 1) < 20n^2.
\]

It follows that \( \ell(u'), \ell(v') < 50n^2 \), and we are done.
Case 2.2: $\sum_{i=m-4n+1}^m f_i \geq n$. Put

$$Y_1 = \{ s \mid 4n \leq s \leq m-4n, n \leq \sum_{i=s}^m f_i \} \text{ and } Y_2 = \{ s \mid 4n \leq s \leq m-4n, n \leq \sum_{i=s}^m c_i \}.$$  

Let

$$p = \begin{cases} \min Y_1 & \text{if } Y_1 \neq \emptyset; \\ m-4n & \text{otherwise} \end{cases} \quad \text{and} \quad q = \begin{cases} \max Y_2 & \text{if } Y_2 \neq \emptyset; \\ 4n & \text{otherwise}. \end{cases}$$

Case 2.2.1: $q < p$. Then there exist $\ell, r \leq n$ and a maximal subsequence $k_1, k_2, \ldots, k_\ell$ of $4n, 4n+1, \ldots, q$ such that $f_{k_1}, f_{k_2}, \ldots, f_{k_\ell} > 0$, and a maximal subsequence $h_1, h_2, \ldots, h_r$ of $p, p+1, \ldots, m-4n$ such that $c_{h_1}, c_{h_2}, \ldots, c_{h_r} > 0$. For convenience, we put $k_{\ell+1} = q + 1$ and $k_{r+1} = m - 4n + 1$. Then

$$\prod_{i=4n}^{k_{\ell+1}} (t_i x^{e_i} y^{f_i}) \prod_{i=k_{\ell+1}+1}^{k_{r+1}} (t_i x^{e_i} y^{f_i}) \prod_{i=h_{r+1}+1}^{h_{r+1}+1} (t_i y^{g_i}).$$

Then $V$ satisfies the identities $u \approx v_1$ and $v \approx v_1$ where $u_1 = x^{e_0} y^{f_0} w_1 w_2 w_3$, $v_1 = y^{f_0} x^{e_0} w_1 w_2 w_3$, $w_2 = \prod_{i=q+1}^{p-1} (t_i x^{e_i} y^{f_i})$ and

$$w_1 = \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \prod_{i=4n}^{k_{\ell+1}} (t_i x^{e_i}) \prod_{i=k_{\ell+1}+1}^{k_{r+1}} (t_i x^{e_i}),$$

$$w_3 = \prod_{i=p+1}^{h_{r+1}+1} (t_i y^{g_i}) \prod_{i=1}^{r} (t_i x^{e_i} y^{f_i}) \prod_{i=h_{r+1}+1}^{h_{r+1}+1} (t_i y^{g_i}).$$

Now Observation 2.6 applies with the conclusion that $u_1 \approx v_1$ is equivalent to $u' \approx v'$ where $u' = x^{e_0} y^{f_0} w'_1 w'_2 w'_3$, $v' = y^{f_0} x^{e_0} w'_1 w'_2 w'_3$ and

$$w'_1 = \left( \prod_{i=1}^{4n-1} (t_i x^{e_i} y^{f_i}) \left( t_4 n x^n \right) \prod_{i=1}^{\ell} (t_i x^{e_i} y^{f_i} t_{k_{\ell+1}} x^n) \right),$$

$$w'_3 = \left( t_{p+1} y^{g_i} \right) \left( \prod_{i=1}^{r} (t_i x^{e_i} y^{f_i} t_{h_{r+1}} y^n) \right) \left( \prod_{i=m-4n+1}^{m} (t_i x^{e_i} y^{f_i}) \right).$$

Then $V \{ u \approx v \} = V \{ u_1 \approx v_1 \} = V \{ u' \approx v' \}$. Since $u'$, $v'$ are $(n+1)$-free and $\ell, r \leq n$, we have that

$$\ell(w'_1) \leq (4n-1)(2n+1) + (n+1) + n(3n+2) < 20n^2,$$

$$\ell(w'_3) \leq (n+1) + n(3n+2) + 4n(2n+1) < 20n^2.$$

The definitions of $p$ and $q$ imply that $\ell(w_2) \leq 4n$. It follows that $\ell(u'), \ell(v') < 50n^2$, and we are done.

Case 2.2.2: $p \leq q$. This case is considered similarly to Case 2.1.

So, we have proved that each subvariety of $V$ can be given by identities from some fixed finite system of identities both hand-sides of which are of length $\leq 50n^2$. It follows that the variety $V$ is small and so Cross by aforementioned Lemma 7 of [10].
5. Main result

To formulate the main result of the article we put

\[ \mathbf{K} = \text{var}\{(2.1), (2.6), (2.7)\}, \]

\[ \mathbf{J} = \text{var}\left\{(2.1), (2.7), (2.8), (2.19), \right. \]

\[ \left. xz_1z_2 \cdots z_nx = \left( \prod_{i=1}^{n} t_i z_i \right) \approx x^2 z_1 z_2 \cdots z_n \left( \prod_{i=1}^{n} t_i z_i \right) \right\} \quad \text{for} \quad n \geq 1, \quad \pi \in S_n \]

where \( S_n \) denote the full symmetric group on the set \( \{1, 2, \ldots, n\} \). Let \( \mathbf{M} \) and \( \mathbf{N} \) denote the monoid varieties generated by the monoids \( S(\pi xyzty) \) and \( S(\pi xzytxy) \), respectively.

The main result of the article is the following

**Theorem 5.1.** A subvariety of \( \mathbf{A}_{com} \) is Cross if and only if it excludes the varieties \( \mathbf{J}, \mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{P} \) and \( \mathbf{\tilde{P}} \). Consequently, the class \( \mathbf{A}_{com} \) contains precisely nine almost Cross subvarieties.

**Proof.** The varieties listed in Theorem 5.1 are almost Cross:

- \( \mathbf{J} \) and \( \mathbf{\tilde{J}} \) by [1, Corollary 1.2 and the dual to it];
- \( \mathbf{K} \) and \( \mathbf{\tilde{K}} \) by [3, Proposition 6.1 and the dual to it];
- \( \mathbf{L} \) by [10, Theorem 2];
- \( \mathbf{M} \) and \( \mathbf{N} \) by [5, Proposition 5.1];
- \( \mathbf{P} \) and \( \mathbf{\tilde{P}} \) by Proposition 3.1 and the dual to it.

Therefore, every Cross subvariety of \( \mathbf{A}_{com} \) does not contain these varieties. So, to complete the proof it remains to verify that if \( \mathbf{V} \) is a subvariety of \( \mathbf{A}_{com} \) and \( \mathbf{V} \) excludes the varieties listed in Theorem 5.1, then \( \mathbf{V} \) is Cross.

Suppose that \( S(xyxy) \in \mathbf{V} \). Since \( \mathbf{M} \nsubseteq \mathbf{V} \) and \( \mathbf{N} \nsubseteq \mathbf{V} \), either \( \mathbf{V} \subseteq \mathbf{O} \) or \( \mathbf{V} \subseteq \mathbf{\tilde{O}} \) by Lemma 2.7 and [15, Fact 3.1]. Then Proposition 4.1 and the dual to it imply that \( \mathbf{V} \) is a Cross variety because \( \mathbf{V} \) does not contain the varieties \( \mathbf{L}, \mathbf{P} \) and \( \mathbf{\tilde{P}} \).

Suppose now that \( S(xyxy) \notin \mathbf{V} \). In view of Observation 2.5, we assume that \( \mathbf{V} \) is non-commutative and non-completely regular. According to Lemma 2.7, \( \mathbf{V} \) satisfies a non-trivial identity of the form \( xyxy \approx w \). Then \( w = x^p y x^q \) for some \( p \) and \( q \) such that \( p \geq 2 \) or \( q \geq 2 \) by Lemma 2.8. By symmetry, we may assume that \( q \geq 2 \).

Suppose at first that \( p = 0 \). Then \( \mathbf{V} \) satisfies the identity \( x^2 \approx x^q \) and, therefore, the identity \( xyxy \approx yx^2 \). It follows from [11, the dual to Corollary 3.6] that every periodic variety that satisfies the latter identity is Cross. Thus, \( \mathbf{V} \) is a Cross variety.

Suppose now that \( p \geq 1 \). Then \( \mathbf{V} \) satisfies the identity \( x^2 \approx x^{p+q} \) and, therefore, the identity \( x^2 \approx x^3 \) because \( \mathbf{V} \) is aperiodic. Then the identities

\[ xyxy \approx x^p y x^q \approx x^p y x^{q+1} \approx xyx^2 \]

hold in \( \mathbf{V} \). Thus, (2.1) is satisfied by \( \mathbf{V} \). In view of Observation 2.3, \( \mathbf{V} \) satisfies the identity (2.3) for some \( n \geq 1 \). This identity together with (2.1) implies (2.7).

Consider an arbitrary subvariety \( \mathbf{X} \) of \( \mathbf{V} \). If \( \mathbf{E} \nsubseteq \mathbf{X} \), then \( \mathbf{X} \subseteq \mathbf{K} \) by [2, Lemma 2.3(ii)]. Since \( \mathbf{V} \neq \mathbf{K} \) and \( \mathbf{K} \) is an almost Cross variety, we obtain that \( \mathbf{X} \) is a Cross subvariety of \( \mathbf{K} \). If \( \mathbf{F} \nsubseteq \mathbf{X} \), then \( \mathbf{X} \subseteq \mathbf{Q} \) by Lemma 2.10. In view of [11, Proposition 5.4], \( \mathbf{X} \) is a Cross variety again.

In particular, it remains to consider the case when \( \mathbf{F} \nsubseteq \mathbf{E} \). Since \( \mathbf{V} \neq \mathbf{J} \), [2, Lemma 4.1] implies that \( \mathbf{V} \) satisfies the identity (2.16). Hence \( \mathbf{V} \subseteq \mathbf{R} \). Further,
suppose that $Q \not\subseteq V$. Then $V$ satisfies the identity

$$x^2yzx \approx x^2yxx$$

(5.1)

by Lemma 2.12. Clearly, this identity together with (2.1) imply the identity (2.19). It is easy to see that every identity of the form $p_1e_1 \cdots t_ke_k \approx q_1e_1 \cdots t_ke_k$ is equivalent modulo \{2.1, (5.1)\} to the identity

$$p \cdot t_1e_1t_2e_2t_3e_3t_4e_4 \cdot t_5 \cdots t_{k-1}e_{k-1}t_ke_k \approx q \cdot t_1e_1t_2e_2t_3e_3t_4e_4 \cdot t_5 \cdots t_{k-1}e_{k-1}t_ke_k.$$ 

Evidently, the latter identity is equivalent modulo (2.1) to

$$p \cdot t_1e_1t_2e_2t_3e_3t_4e_4 \cdot t_{k-1}e_{k-1}t_ke_k \approx q \cdot t_1e_1t_2e_2t_3e_3t_4e_4 \cdot t_{k-1}e_{k-1}t_ke_k.$$ 

This fact and Lemma 2.14 imply that each subvariety of $V$ containing $F \lor \overline{E}$ may be given by some identities from the identity system

$$\{(2.7), (2.8), (2.16), (2.17), (2.18), (2.19), \alpha_r, \beta_r, \gamma_r, \gamma'_r \mid r \leq 6\}.$$ 

The evident fact that this identity system is finite and the previous paragraph imply that the variety $V$ is small and finitely based. Now Lemmas 2.1 and 2.2 apply with the conclusion that $V$ is finitely generated. So, we have proved that $V$ is a Cross variety. Let now $Q \subseteq V$. Then, since $V \neq P$, Proposition 3.1 implies that $P_{k+1} \not\subseteq V$ for some $k \geq 0$. Lemma 3.3 applies with the conclusion that $V$ satisfies $\delta_{k,2}$. Since (2.1) holds in $V$, the identity $\delta_{k,2}$ is equivalent to the identity $\alpha_k$ in $V$. Evidently, for any $r > k$, the identities $\alpha_r, \beta_r, \gamma_r$ and $\gamma'_r$ follow from $\alpha_k$. This fact and Lemma 2.14 imply that each subvariety of $V$ containing $F \lor \overline{E}$ may be given by some identities from the identity system

$$\{(2.7), (2.8), (2.16), (2.17), (2.18), (2.19), \alpha_r, \beta_r, \gamma_r, \gamma'_r \mid r \leq k\}.$$ 

Since this identity system is finite, the same arguments as above imply that $V$ is a Cross variety again. \qed

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