$W_\infty$ AND ANOMALIES OF SELF-DUAL EINSTEIN THEORIES

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ABSTRACT

Recently it has been demonstrated that self-dual Einstein Euclidean instantons possess an infinite dimensional group of symmetries which contain the standard $w_\infty$. Since $w_\infty$ has a central extension only in its $w_2$ subalgebra one may claim that there is an anomaly in the symmetry group of these instantons which is described by a three dimensional $w_2$ effective action. Thus one may have a bosonization of the 3/2 spin anomaly of the Eguchi-Hanson instantons. In analogy to $w_\infty$ we construct the three dimensional effective action for the Lone Star $W_\infty$ algebra. This suggests that there is a quantum deformation of the instantons that contributes to the effective action of four dimensional self-dual gravity. We comment on this possibility.
Two dimensional conformal field theory \[1\] has firmly rooted itself into the theoretical understanding of many different physical phenomena. For some time now there has been evidence that suggests that self-dual gauge theories in four dimensions can be thought of as self-dual theories in two dimensions \[2\]. In Ref. \[3\], Park is able to show that 4 dimensional self-dual gravitational systems have the symmetries of 2D conformal theories. In particular, four dimensional self-dual Einstein theories which admit a single rotational Killing vector have the symmetry of \(w_\infty\). Since the standard \(w_\infty\) of Bakas\[4\] has a central extension only in its \(w_2\) subalgebra it appears that the three dimensional \(w_2\) effective action of \[5\] is the anomalous contribution for Eguchi-Hanson \[6\] effective actions. Furthermore different coadjoint orbits of the \(w_2\) algebra correspond to different isotropy groups of the self-dual Einstein equations. The isotropy groups of the various covectors are used to construct different solutions to the self-dual field equations.

Just as in WZW and Polyakov gravity one may think of these effective actions as arising from functionally integrating chiral fermions coupled to gauge fields. Atiyah has shown, using index theorems, that the Dirac equation in the presence of an Eguchi-Hanson instanton has index zero. This implies that the chiral Dirac operator with a self-dual spin connection, has no anomaly. However, as shown by Hanson and Römer, the Rarita-Schwinger operator does have a non-zero index, revealing a \(\frac{3}{2}\) spin anomaly in the presence of these instantons \[7\]. The fact that these instantons should contribute to the \(\frac{3}{2}\) spin axial anomaly as opposed to the \(\frac{1}{2}\) spin axial anomaly has been attributed to supersymmetry \[8\]. All this strongly suggest that the three dimensional \(w_2\) effective action is the response of the \(\frac{3}{2}\) spin fermionic measure to the underlying 2D conformal symmetry in self-dual \(D = 4\) theories. These actions suggest a bosonization for the spin \(\frac{3}{2}\) fields coupled to self-dual connections.

We would like to extend our analysis of \(w_2\) to the symplectic geometry of the “Lone Star” \(W_\infty\) \[9\] case. Throughout we will make a distinction between the \(w_\infty\), due to Bakas \[4\] and Bilal \[10\], and \(W_\infty\), the “Lone Star” algebra. The \(W_\infty\) algebra corresponds to a linear deformation of the standard \(w_\infty\) algebra in
such a way that all higher conformal spin fields have central extensions but the generators themselves do not have definite conformal spin.

Recall that the standard $w_\infty$ algebra [4,10] can be written as

$$\{f, g\} = \partial_x f \partial_y g - \partial_y f \partial_x g$$

where $f = f_{m,s} x^{m+s+1} y^{s+1}$, $g = g_{m,t} x^{m+t+1} y^{t+1}$ (implied sum) where $\{*, *\}$ is the Poisson bracket of the coordinates $x$ and $y$ and $s, t \geq 0$. By restricting the values of $s$ and $t$ to 0 one recovers the $w_2$ subalgebra of $w_\infty$. This subalgebra admits a central extension and the algebra may be written as

$$[w_m, w_n] = (m - n)w_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n}.$$ 

In [5] we showed that one may extract an effective action from almost any infinite dimensional Lie algebra by knowing the algebra’s two cocycle. That action is

$$S = \int d\lambda d\tau \left( (k(g) \tilde{f}[\xi_\tau, \xi_\lambda]) + \tilde{t}c(\xi_\tau g, \xi_\lambda g) \right), \quad (1)$$

where the vector $\xi_\tau$ is used to denote a Hamiltonian vector field generating changes along the time direction. $\xi_\lambda$ is the Hamiltonian vector field generating changes in a direction $\lambda$ which corresponds to a one parameter family of group transformations from, say, the identity to some group element $g(x, y)$ [11-13]. $k(g) \tilde{f}[\ast]$ is just the coadjoint action of the group $G$ on the coadjoint vector $f$. The pure coadjoint orbit in the $w_2$ case corresponds to an $SL(2, R)$ Chern-Simons theory, since $SL(2, R)$ is the isotropy group of this coadjoint vector.

Let us take a moment to examine the $w_\infty$ algebra more closely. As we just mentioned the $w_\infty$ algebra is usually written as the Poisson bracket algebra of functions on a two-dimensional phase space. The $w_2$ central extension is easily added to this algebra to yield the full centrally extended $w_\infty$ algebra. Indeed for centrally extended generators $\{w^i_m, \alpha_i\}$, we write

$$\left[ \{w^i_m, \alpha_i\}, \{w^j_n, \alpha_j\}\right]$$

$$= \{(j + 1)m - (i + 1)n\}w^{i+j}_{m+n} + \frac{1}{12} c(m^3 - m) \delta_{m+n,0} \delta^{i,0} \delta^{j,0}.$$
The $\alpha_i$’s denote the generators for the central extensions of the algebra and here only $\alpha_2$ is non zero. From here we attempt to define the coadjoint representation of $w_\infty$. We seek the generators $(\tilde{w}_m^l, \tilde{\alpha}_l)$ such that

$$< \{ \tilde{w}_k^l, \tilde{\alpha}_l \} | \{ w_n^j, \alpha_j \} > = \delta^{l,j} \delta_{n,k} + \tilde{\alpha}_l(\alpha_j),$$

where $\tilde{\alpha}_l(\alpha_j) = \delta^{l,2} \delta^{j,2}$. The action of the Lie algebra element $\{ w_m^{(i)}, \alpha_i \}$ on $\{ \tilde{w}_k^l, \tilde{\alpha}_l \}$ is given by

$$\{ w_m^{i}, \alpha_i \} \{ \tilde{w}_k^l, \tilde{\alpha}_l \} = -\{ ((l + 2)m - (i + 1)k)\tilde{w}_{k-m}^{l-i} + \frac{1}{12}c(m^3 - m)\delta_{m,k-n}\delta^{0,l}\delta^{0,i}, 0 \}.$$

Observe that if we demand that $i, l \geq 0$ the dual space or coadjoint representation for this subalgebra is not invariant under the action of the adjoint representation. Therefore one cannot enforce this condition and also define a smooth dual for this algebra. One must allow all conformal spin field generators in order to describe the symplectic geometry associated with the $w_\infty$ algebra. Hence by using the natural bilinear map which takes the unrestricted algebra to complex numbers we can define the coadjoint representation. In this sense the $w_\infty$ algebra is more analogous to the WZW type theories than two dimensional Polyakov gravity. In other words, recall that the Kac-Moody algebras also enjoy the presence of a bilinear symmetric form, where as Diff $S^1$ does not. As we will see shortly this problem persist for the $W_\infty$ algebra. One representation of the symmetric form on $w_\infty$ (from here on we assume all conformal spins are allowed) is given by ( promoting $x$ and $y$ to complex variables )

$$< w_m^i | w_n^j > = \oint \oint w_m^i w_n^j dx \, dy.$$  

The contour is a unit circle about the origin. From here it is easy to see that

$$w_m^i = w_{-m}^{1-i}.$$

The action for $w_\infty$ was essentially constructed in Ref.[5] with the exception that now coadjoint vectors can come from full dual of the area preserving diffeomorphism algebra. Observe that the pure covector, $B = (0, \tilde{t})$, is invariant with
respect to all the generators except the \( w_2 \) generators (modulo \( \text{SL}(2,\mathbb{R}) \)) which belong to \( \text{Diff}^+_0 \mathbb{R}^2 \). This orbit could correspond to the self-dual Einstein vacuum. We will now focus on the \( W_{\infty} \) algebra.

Recall that the \( W_{\infty} \) algebra is defined by the commutation relations,

\[
[W_m^i, W_n^j] = \sum_{p=0}^{s=[j+k]/2} q^{2p} g_{2p}^{jk}(m, n) W_{m+n}^{j+k-2p} + q^{2j} c_j m^{2j+3} \delta_j, k \delta_{m+n,0}.
\]

where \( s \) is the maximal integer and the structure function have the form,

\[
g_{2p}^{jk}(m, n) = \phi_{2p}^{jk} N_{2p}^{jk}(m, n)
\]

with

\[
\phi_{2p}^{jk} = \sum_{r=0}^{p} \prod_{l=1}^{r} \frac{(2l-3)(2l+1)(2p-2l+3)(p-l+1)}{l!(2j-2l+3)(2k-2l+3)(2j+2k-4r+2l+3)},
\]

and

\[
c_j = \frac{2^{2j} j!(j+2)!}{(2j+1)!!(2j+3)!!} c.
\]

The \( N_{2p}^{jk}(m, n) \) can be computed from the prescription described below. The structure functions, \( g_{2r}^{ij} \), vanish when \( i + j - 2r < 0 \) [9]. This guarantees that the algebra will terminate to generators with conformal spin two and higher. However this is to no advantage in the construction of the coadjoint representation. Again we must allow the generators with conformal spin less than two, since only the \( W_2 \) subalgebras have an invariant dual space. Let us construct the dual vectors, \( (\tilde{W}_m^l, \tilde{\alpha}_l) \), to this algebra such that

\[
< \{ \tilde{W}_k^l, \tilde{\alpha}_l \} | \{ W_n^j, \alpha_j \} > = \delta^{l,j} \delta_{n,k} + \tilde{\alpha}_l(\alpha_j).
\]

Again the \( \alpha_l \)'s and \( \tilde{\alpha}_l \)'s are the corresponding generators for the \( l \)'th central extension in the adjoint and coadjoint vector spaces. From the invariance of this expression one finds that

\[
\{ W_m^i, \alpha_1 \} [\{ \tilde{W}_k^l, \tilde{\alpha}_l \}] = -\sum_{r=0}^{s=\frac{1}{2}} \{ g(m, k - m)_{2r}^{i,l-i+2r} \tilde{W}_{k-m}^{l-i+2r} + c_l(k) \delta^{i,l} \delta_{m,-k}, 0 \}.
\]
and we see that in order to define an invariant dual space, we need to include all the conformal spin generators. This will technically facilitate matters, as we will see shortly.

In Ref. [4], Bakas has emphasized the importance of using a “\( q \)” bracket or Moyal bracket [14] in the construction of the symplectic geometry for generators with conformal spin 2 and higher. The Moyal bracket provides a suitable deformation of the Poisson bracket that will permit Dirac quantization. Ambiguities due to operator ordering are avoided. In the limit as \( \hbar \to 0 \), this bracket reduces to the Poisson bracket. The “Lone Star” \( W_\infty \) [9] has (at least formally) a realization in terms of the Moyal bracket for all conformal spin fields [15]. Since we cannot restrict to positive conformal spins, we will use the notation of Fairlie and Nuyts, Ref. [15], to describe this algebra. We define the Moyal bracket [14,15] between functions \( f(x, y) \) and \( g(x, y) \) by,

\[
\{f, g\} = \sum_{p=0}^{\infty} \frac{(-1)^p \hbar^{2p+1}}{(2p + 1)!} \sum_{k=0}^{2p+1} \frac{(-1)^k}{k!} \left( \partial_x^k \partial_y^{2p+1-k} f \right) \left( \partial_x^{2p+1-k} \partial_y^k g \right)
\]

Then, using this bracket, one may obtain a renormalized \( W_\infty \) algebra by using the generators

\[
W^j_m = i\lambda \exp(\frac{imx}{y}) y^{2(j+1)},
\]

with \( i, j \geq 0 \). This algebra admits central extensions for all positive conformal spins. We write the centrally extended \( W_\infty \) algebra as (here the \( i = 0 \) term will correspond to the spin two components)

\[
\left[ W^i_m, W^j_n \right] = \sum_{p=0}^{\infty} q^{2p} N^{jk}_{2p}(m, n) W^{j+k-2p}_{m+n} + q^{2j} c_j m^{2j+3} \delta j, k \delta_{m+n,0}.
\]

Note that from the action of \( W^0_m \) on \( W^i_n \) that the generators do not have definite conformal spin. The structure functions, \( N(m, n)^{jk}_{2p} \), can be computed directly using the Moyal bracket and are the same as those used by Pope, Romans, and Shen in Ref. [9]. By enforcing the Jacobi identity one finds that

\[
c_{j+1} = c_j \frac{(2)^{j+1}}{(j+1)!}.
\]
The duality condition for the representation given in Eq.(5) is

\[
< \{ \tilde{W}_k^l, \tilde{\alpha}_l \} | \{ W_n^j, \alpha_j \} >= \\
\frac{1}{4\pi^2} \oint dy \int_0^{2\pi} (-i\lambda \exp(-ilx/y)y^{-2k-3})(i\lambda \exp(\frac{inx}{y})y^{2(j+1)}) dx + \tilde{\alpha}_l(\alpha_j),
\]

where care is taken to do the \( x \) integration first. Following Eq.(1) we may write down the action for \( W_\infty \). For the generators given in Eq.(5), the two cocycle for the renormalized \( W_\infty \) is given by

\[
c(f, g) = \sum_{i=0}^{\infty} \frac{c_i(-1)^{i+1}}{4\pi^2(2i+1)!^2} \oint \oint D_y^{2(i+1)} \partial_x^{i+1} (y^{i+1}f) \frac{1}{y} D_y^{2(i+1)} \partial_x^{i+2} (y^{i+2}g) \ dx \ dy,
\]

where the derivative operator \( D_y = \partial_y + \frac{\tilde{x}}{y} \partial_x \). Instead of using the Hamiltonian vector fields we may use the Hamiltonians corresponding to generators in the direction of time, \( \tau \), and the group direction, \( \lambda \). Using the Moyal bracket and its correspondence principle [16] we can identify the canonical variables as \( x \) and \( y \).

Then given an area preserving diffeomorphism

\[
x \to \varphi_1(x, y), \quad y \to \varphi_2(x, y),
\]

For the generators of time translations and group transformations we may write

\[
\partial_b H_\tau = \frac{\partial_\tau \varphi^\alpha(x, y)}{\partial_a \varphi^\alpha} \epsilon_{ab}, \quad \partial_b H_\lambda = \frac{\partial_\lambda \varphi^\alpha(x, y)}{\partial_a \varphi^\alpha} \epsilon_{ab}
\]

respectively. Here \( a, b \in \{ 1, 2 \} \). The \( W_\infty \) effective action for the orbit corresponding to the coadjoint vector \( B = (\tilde{B}, t^l \tilde{\alpha}_l) \) is

\[
S(\tilde{B}) = \\
\sum_{i=0}^{\infty} t^i c_i \oint \oint D_y^{2(i+1)} \partial_x^{i+1} (y^{i+1}H_\tau) D_y^{2i+2} \partial_x^{i+2} (y^{i+2}H_\lambda) \ dx \ dy \ d\lambda \ d\tau \\
+ \oint \oint \tilde{B}(x, y)\{H_\tau, H_\lambda\}_n \ dx \ dy \ d\lambda \ d\tau.
\]

First notice that the coordinates in Eq.(4) represent the algebra on \( S^1 \otimes R \).

As long as one is interested in only the algebraic structure this compactification
offers no significant changes as compared to $R^1 \otimes R^1$. However in our case we are actually interested in group transformation on coadjoint orbits. One may recall [12,13] that the isotropy groups for the orbits in Polyakov gravity depend on whether we specify two dimensional gravity on $R^2$ or $S^1 \otimes R$. In the first case, the pure covector corresponds to the field space (orbit) $\text{Diff} S^1 / \text{SL}(2, R)$ and in the second case to $\text{Diff} S^1 / S^1$. The action is a $2 + 1$ space-time action with one more dimension for the group direction. We have left it in the four dimensional form since WZ terms are present. The existence of such terms will require that $t^i c_i$ be integers. Since this action arises from four dimensional theories with a Killing symmetry, the number of dimensions is correct. Recently a two dimensional realization of $W_\infty$ has been found for a free complex bosonic field [17].

One may characterize the orbits by the subalgebras that leave $B$ invariant. By knowing the available subalgebras one can work backwards to find representative covectors for each orbit. In this case the pure covector, $B = (\tilde{B} = 0, \sum_{i=0} t^i c_i)$, will certainly be invariant under $\text{Diff}_0 R^2$ and the Abelian subgroup generated by $W_k^{-1}$. Thus $\frac{\text{Diff}}{\text{Diff}_0 R^2 \otimes H}$ is the field space and also corresponds to vacuum solutions. Non-pure orbits could be realized as cosmological terms coupled to gravity and higher spin tensor fields.

Some interesting question emerge from this action. In the case of $w_\infty$, Park has shown that the equations of motion of the action

$$S(u) = \int \left( \frac{1}{2} u \tilde{\partial} u + \partial^2 u \exp u \right) dx \ dy \ dz,$$

corresponding to self-dual Einstein solutions with one Killing symmetry, have $w_\infty$ symmetry. Indeed we have tried to argue that coupling the Eguchi-Hanson instanton to spectating Rarita-Schwinger fields yields the $W_2$ effective action. This further suggests that the pure central orbit of $W_2$ may correspond to a bosonization prescription of the Rarita-Schwinger field on a self-dual manifold with one Killing vector field. By working backwards from the $W_\infty$ algebra we may be able to reconstruct instantons with hidden $W_\infty$ symmetry. We expect the $W_\infty$ action to be a consistent coupling of higher spin fields to a self-dual(?) four dimensional, linearly deformed, gravitational instanton. This is reminiscent of string theories,
since a consistent string theory also requires an infinite tower of coupled fields. This may be a realistic expectation of quantum gravity in four dimensions.

Also, recall that one may define the conformal spin of a field through the action of the Virasoro subalgebra of these \( W \) algebras [18]. Indeed in a one dimensional realization of \( w_2 \) we know that,

\[
w_k^0(z^{m+1})dz^\Delta = -((m+1)+\Delta(k+1))z^{m+k+1}dz^\Delta.
\]

In the case of quantum two dimensional Polyakov gravity [18,10-13], the adjoint vectors correspond to conformal spin -1 operators and the covectors dual to them are the quadratic differentials. The importance of the quadratic differentials in the quantization of two dimensional gravity is that they are the pseudo-metrics corresponding to conformally gauge fixed metrics, \( ds^2 = g_{ab}dx^a dx^b \). Here there is a clear distinction between the contravariant and covariant vectors in terms of the conformal spin. Because we required all conformal spin generators for the construction of the symplectic structure of the \( w_\infty \) and \( W_\infty \) algebras, the distinction between derivative operators and the fields can no longer be determined just by checking the conformal spin. In other words the higher order differentials can be related to “conformally” gauge fixed fields, i.e. \( ds^3 = A_{abc}dx^a dx^b dx^c \), and be assigned positive conformal spin. However, in the \( w_\infty \) and \( W_\infty \) algebras, the familiar “in” and “out” states as mentioned in the original work of Belavin, Polyakov, and Zamolodchikov [18] are dual to each other, but there are two spaces of all the conformal spins that are respectively dual.

Let us remark on one more observation. We have seen that there are two separate algebras that contain either one or an infinite number of central extensions. It has been argued that both are related to the area preserving diffeomorphism in 2D. As far as we know \( w_\infty \) and \( W_\infty \) are not unitarily related. Furthermore it is well known that there may be many different ways to take the limit of the Zamolodchikov \( W_N \) algebras as \( N \rightarrow \infty \) and recover a Lie algebra.

Then how do these ambiguities translate into the four dimensional theories? We suspect that the four dimensional self-dual manifolds support instantons, which arise from different limits, are not diffeomorphic to each other. Recall that for
compact four manifolds, the diffeomorphism class is specified by the cup product of $H^2(M^4, \mathbb{Z})$ with itself [20]. In other words, the intersection of the two manifolds associated with the homology classes determines what functions, if any, will be smooth on the four manifold. Even in the case of $R^4$, there can be many (at least two-fold uncountably infinite) inequivalent notions of a ring of smooth functions. Roughly speaking, one surgers a hole about the topology of a compact manifold that does not admit a ring of smooth functions to produce an open four manifold which is homeomorphic but not diffeomorphic to $R^4$, yet still admits a smooth ring of functions. To add further relationships between 2D and 4D, recall that on compactified Euclidean space-times that all (stable, finite) solutions of the pure Yang-Mills equations are self-dual [21]. This together with the results of [2] and [3] tends to support the fact that 2D conformal theories may indeed drive the differential structure of theories in four-dimensions. We are presently investigating these issues.

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