Transport of and Radiation Production by Trans-relativistic/Non-relativistic Particles Moving Through Sub-Larmor-Scale Electromagnetic Turbulence

Brett D. Keenan‡, Alexander L. Ford† and Mikhail V. Medvedev‡
Department of Physics and Astronomy, University of Kansas, Lawrence, KS 66045

Plasmas with electromagnetic fields turbulent at sub-Larmor-scales are a feature of a wide variety of high-energy-density environments, and are essential to the description of many astrophysical/laboratory plasma phenomena. Radiation from particles, whether they be relativistic or non-relativistic, moving through small-scale magnetic turbulence has spectral characteristics distinct from both synchrotron and cyclotron radiation. The radiation, carrying information on the statistical properties of the magnetic turbulence, is also intimately related to the particle diffusive transport. We have investigated, both theoretically and numerically, the transport of non-relativistic and trans-relativistic particles in plasmas with high-amplitude isotropic sub-Larmor-scale magnetic turbulence, and its relation to the spectra of radiation simultaneously produced by these particles. Consequently, the diffusive and radiative properties of plasmas turbulent on sub-Larmor scales may serve as a powerful tool to diagnosis laboratory and astrophysical plasmas.

I. INTRODUCTION

High-amplitude sub-Larmor-scale electromagnetic turbulence is a phenomenon largely associated with high-energy density environments. Such turbulence is a common feature of astrophysical and space plasmas, e.g., at high-Mach-number collisionless shocks in weakly magnetized plasmas \[ \frac{v}{c} \lesssim 0.1 \] and others. Additionally, these sub-Larmor-scale, or “small-scale”, fields play a critical role in laboratory plasmas; especially in high-intensity laser plasmas – as observed in facilities such as the National Ignition Facility (NIF), OmegaEP, Hercules, Trident, and others \[ [8–11] \]. Experimental and numerical studies of non-relativistic collisionless shocks also show that they are mediated by small-scale electromagnetic turbulence \[ [12, 13] \]. Thus, studies of plasmas and turbulence in these environments are important for the fusion energy sciences and the inertial confinement concept \[ [8,11] \].

Small-scale electromagnetic turbulence can be of various origin and thus have rather different properties, from being purely magnetic (Weibel) turbulence \[ [14–16] \], to various types of electromagnetic turbulence (for example, whistler wave turbulence or turbulence produced by filamentation/mixed mode instability \[ [17, 18] \]), to purely electrostatic Langmuir turbulence \[ [19,20] \].

Despite substantial differences, these small-scale fields share one thing in common: they vary on scales much smaller than the characteristic curvature scale of the particles traversing the field, i.e. the plasma inertial length (skin depth) which are on the order of the particle Larmor radius. The particle trajectory through these turbulent fields will, consequently, never form a well-defined Larmor circle.

If the electromagnetic fields are random, which is usually the case of turbulence because of the random phases of fluctuations, the paths of the particles diffusively diverge due to pitch-angle diffusion. Radiation simultaneously produced by these particles is neither cyclotron nor synchrotron (for non-relativistic or relativistic particles, respectively) but, instead, carries information about the spectrum of turbulent fluctuations.

Previously, see Ref. \[ [21] \], we found the relation between the transport of relativistic particles in isotropic three-dimensional magnetic turbulence and the radiation spectra simultaneously produced by these particles. In particular, we found that the radiation spectrum agrees with the small-angle jitter radiation prediction, in the small deflection angle regime \[ [13,22–25] \]. Furthermore, we demonstrated that the pitch-angle diffusion coefficient is directly related to, and can readily be deduced from, the spectra of the emitted radiation. This inter-relation between radiative and transport properties provides a unique way to remotely diagnose high-energy-density plasmas, both in laboratory experiments and in astrophysical systems.

We extend our previous work to now consider non-relativistic \( (v \lesssim 0.1c) \) and trans-relativistic \( (0.1c \lesssim v \lesssim 0.5c) \) particles moving through three-dimensional sub-Larmor-scale magnetic turbulence. We demonstrate, once more via numerical and theoretical analysis, that an analogous inter-relation holds in these regimes as well, which naturally generalizes the relativistic small-angle jitter radiation regime and the pitch-angle diffusion coefficient.

This trans-relativistic regime is applicable to laboratory plasmas, particularly high-intensity laser plasmas – where bulk plasma motion is below \( v \lesssim 0.5c \). Multi-dimensional relativistic Particle-In-Cell (PIC) simulations have revealed that non-relativistic collisionless shocks, mediated by Weibel-like instabilities, can occur in an overcritical plasma via interaction with an ultraintense laser pulse \[ [12] \]. In the laboratory setting, laser-produced supersonic counter-streaming plasmas have been observed to give rise to self-organized electromagnetic fields \[ [26] \]. Recently, the formation of Weibel-like magnetic fields, perfectly consistent with the shock model offered by 3D PIC simulations and theoretical instability analysis, has been directly observed in a scaled laboratory experiment \[ [27] \].
Consequently, given the role of trans-relativistic particle motion in these environments, the study of the small-scale electromagnetic turbulence may be aided by the diagnostic tool offered via this inter-relation between the transport and radiative properties.

The rest of the paper is organized as follows. Section II presents the analytic theory. Sections III and IV describe the numerical techniques employed and the obtained simulation results. Section V is the conclusions. All equations appear in cgs units.

II. ANALYTIC THEORY

A. Pitch-angle diffusion

Consider a trans-relativistic electron moving (with velocity, \(v\)) by a non-uniform, random, small-scale magnetic field (and assume that this magnetic "micro-turbulence" is statistically homogeneous and isotropic). Because the Lorentz force on the electron is random, it’s velocity and acceleration vectors vary stochastically, leading to a random (diffusive) trajectory. We define the field turbulence to be "small-scale" when the electron’s Larmor radius, \(r_L = \gamma \beta m_e c^2 / \langle B^2 \rangle^{1/2}\) (where \(\beta = v/c\) is the dimensionless particle velocity, \(m_e\) is the electron mass, \(c\) is the speed of light, \(\epsilon\) is the electric charge, \(\langle B^2 \rangle^{1/2}\) is the rms component of the magnetic field perpendicular to the electron's velocity vector, and \(\gamma\) is the electron's Lorentz factor) is greater than, or comparable to, the characteristic correlation scale of the magnetic field, \(\lambda_B\), i.e., \(r_L \gtrsim \lambda_B\).

For small deflections, the defection angle of the velocity (with respect to the particle’s initial direction of motion) is approximately the ratio of the change in the electron’s transverse momentum to its initial transverse momentum. The former is \(\sim F_L \tau_\lambda\), where \(F_L = (e/c) v \times B\) is the transverse Lorentz force, and \(\tau_\lambda\) is the transit time, which is the time required to traverse the scale of the field’s inhomogeneity, i.e., the field correlation length, \(\lambda_B\). This is, \(\tau_\lambda \sim \lambda_B / v_\perp\) where \(v_\perp\) is the component of the velocity perpendicular to the magnetic field. The change in the transverse momentum is thus, \(\Delta p_\perp = F_L \tau_\lambda = e(B/c) \lambda_B\). Given that the particle’s total transverse momentum is \(p_\perp = \gamma m_e v_\perp\), the deflection angle over the field correlation length will be \(\alpha_L \approx \Delta p_\perp / p_\perp \sim e(B/c) \lambda_B / \gamma m_e v_\perp\). The subsequent deflection will be in a random direction, because the field is uncorrelated over scales greater than \(\lambda_B\), hence the particle motion is diffusive. For any diffusive process, the ensemble-averaged squared deviation grows linearly with time. Hence, for the pitch-angle deviation, we have

\[
\langle \alpha^2 \rangle = D_{\alpha \alpha} t.
\]

The pitch-angle diffusion coefficient is, by definition, the ratio of the square of the deflection angle in a coherent path to the transit time over this path, that is

\[
D_{\alpha \alpha} \sim \frac{\alpha^2}{\tau_\lambda} \sim \left( \frac{e^2}{m_e^2 c^2} \right) \frac{1}{\langle B^2 \rangle^{1/2}} \frac{\lambda_B}{\gamma^2} \langle B^2 \rangle,
\]

where a volume-averaged square magnetic field, \(\langle B^2 \rangle\) and perpendicular rms velocity, \(\langle B^2 \rangle^{1/2}\) have been substituted for \(B^2\) and \(\beta \equiv v_\perp / c\). Note that the diffusion coefficient depends on both statistical properties of the magnetic field, namely its strength and the typical correlation scale.

Although the assumption that \(\alpha \ll 1\) is valid in the ultra-relativistic limit: \(\beta \to 1\) (see Ref. [22]), it is not evident that it holds for trans-relativistic and non-relativistic velocities. As we will demonstrate via numerical simulation, pitch-angle diffusion will occur in accordance with Eq. (2), so long as the magnetic turbulence is sub-Larmor-scale, i.e. \(r_L \gtrsim \lambda_B\).

The average square magnetic field, \(\langle B^2 \rangle\) is related to \(\langle B^2 \rangle\) by a multiplicative factor. For isotropic magnetic turbulence, \(\langle B^2 \rangle = \langle B^2 \rangle = \langle B^2 \rangle\). Thus, \(\frac{1}{3} \langle B^2 \rangle = \langle B^2 \rangle\). Alternatively, \(\mathbf{B}\) may be expressed as a linear combination of parallel and perpendicular components. Given isotropy, \(\langle B^2 \rangle = \langle B^2 \rangle + \langle B^2 \rangle\), so

\[
\langle B^2 \rangle = \frac{2}{3} \langle B^2 \rangle.
\]

Recognizing that \(v_\perp B = eB_\perp\) allows the expression of the rms perpendicular velocity as

\[
\langle B^2 \rangle^{1/2} = \sqrt{\frac{2}{3}} \beta,
\]

Next, the correlation length, \(\lambda_B\) lacks a formal definition. It is, nonetheless, commonplace in the literature – e.g. Ref. [28], to define the two-point autocorrelation tensor,

\[
R^{ij}(r, t) = \langle B^i(x, \tau) B^j(x + r, \tau + t) \rangle_{x, \tau},
\]

with the formally path and time dependent correlation length tensor defined as

\[
\lambda_B^{ij}(r, t) = \int_0^\infty \frac{R^{ij}(r, t)}{R^{ij}(0, 0)} \, dt.
\]

Note that we make no distinction between co-variant and contra-variant components; the usage of upper and lower indices is only for convenience.

Since the component of the magnetic field perpendicular to the particle trajectory alters the motion, we choose an integration path along \(v_\perp\) and only consider a transverse magnetic field component. In accord with standard practice (see, for example, Ref. [29]), we choose \(r = z\hat{z}\) and \(i = j = x\). Thus, we define the magnetic field correlation length as

\[
\lambda_B \equiv \lambda_B^{xx}(z, t) = \int_0^\infty \frac{R^{xx}(z\hat{z}, t)}{R^{xx}(0, 0)} \, dz.
\]

The correlation length has a convenient representation in Fourier “k-space” and “ξ-space”. Let \(\mathbf{B}_{k, \Omega}\) be the spatial and temporal Fourier transform of the magnetic field, i.e.

\[
\mathbf{B}_{k, \Omega} = \int \mathbf{B}(x, t) e^{-i(k_x x - \Omega t)} \, dx dt,
\]

where \(k\) and \(\Omega\) are the corresponding wave vector and frequency, respectively. We may define a complementary spectral correlation tensor \(\Phi_{ij}(k, \Omega)\), such that

\[
R_{ij}(r, t) = (2\pi)^{-4} \int \Phi_{ij}(k, \Omega) e^{i(k \cdot r - \Omega t)} \, dk d\Omega,
\]
Isotropy, homogeneity, time-independence, and $\nabla \cdot B = 0$ require that the spectral correlation tensor take the simple form: 

$$
\Phi_{ij}(k, \Omega) = \frac{1}{2V} |B_k|^2 \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) 2\pi \delta(\Omega),
$$

where $V$ is the volume of the space considered, $\hat{k}$ is the unit vector in the direction of the wave vector, and $\delta_{ij}$ is the Kronecker delta. The normalization has been chosen such that $\sum R_{ij}(0, 0) = \langle B^2 \rangle$, $\tau = \langle B^2 \rangle$. Given Eq. (9) and Eq. (10), the correlation length may be reformulated as

$$
\lambda_B = \int_0^\infty \frac{|B_k|^2 k^{-2} (k^2 - k_z^2) \phi_{\Omega}(|k|) \text{d}k}{\int |B_k|^2 k^{-2} (k^2 - k_z^2) \text{d}k}.
$$

By assuming isotropic turbulence, the magnetic field has azimuthal and polar symmetry in $k$-space, hence $B_k$ is only a function of $|k| \equiv k$. After the integration over $z$ and all solid angles in Fourier space, Eq. (11) becomes

$$
\lambda_B = \frac{3\pi}{8} \int_0^\infty \frac{|B_k|^2 k^{-2} \text{d}k}{\int |B_k|^2 \text{d}k}.
$$

It may be noted that $\lambda_B \approx k_B^{-1}$, where $k_B$ is the characteristic (dominant) wave number of turbulence.

Thus, with Eqs. (2), (4), and (12), the pitch-angle diffusion coefficient is

$$
D_{\alpha\alpha} = \frac{3\pi}{8} \sqrt{\frac{\tau}{2}} \left( \frac{e^2}{m_e c^2} \right) \int_0^\infty \frac{|B_k|^2 \text{d}k}{\int |B_k|^2 \text{d}k} \langle B^2 \rangle.
$$

To continue, we must specify a magnetic spectral distribution, $|B_k|^2$. As in our previous work (Ref. [21]), we assume the isotropic three-dimensional magnetic turbulence has a static, i.e. time-independent, power law turbulent spectrum:

$$
|B_k|^2 = \begin{cases} 
C k^{-\mu}, & k_{min} \leq k \leq k_{max} \\
0, & \text{otherwise}
\end{cases}
$$

Here the magnetic spectral index, $\mu$ is a real number, and

$$
C \equiv \frac{2\pi^2 V \langle B^2 \rangle}{\int_{k_{min}}^{k_{max}} k^{-\mu+2} \text{d}k}.
$$

is a normalization, such that

$$
V^{-1} \int |B^2(x)| \text{d}x = (2\pi)^{-3} \int |B_k|^2 \text{d}k.
$$

It should be noted that our principal results strictly apply only to static turbulence. One should, in principle, consider time-dependent fields as well. However, if the transit time of a particle over a correlation length is shorter than the field variability time-scale, then the static field approximation is valid. Magnetic turbulence of this kind is a natural outcome of the non-linear Weibel/filamentation instability, which occurs at relativistic collisionless shocks and in laser-produced plasmas [13, 22, 23].

### B. The relativistic jitter theory

Now we consider the radiative properties of these sub-Larmor-scale plasmas. First, the ultra-relativistic radiation regime in sub-Larmor-scale magnetic turbulence is well understood. This regime is characterized by a single parameter, the ratio of the deflection angle, $\alpha_\lambda$ to the relativistic beaming angle, $\Delta \theta \sim 1/\gamma$. The ratio

$$
\frac{\alpha_\lambda}{\Delta \theta} \sim \frac{eB_\perp \lambda_B}{m_e c^2} \sim 2\pi \frac{e^2 \langle B^2 \rangle^{1/2}}{m_e c^2 k_B} \equiv \delta_j
$$

is known as the jitter parameter. From this, we recover four distinct radiation regimes. Firstly, if $\delta_j \rightarrow \infty$, the regime is the classical synchrotron radiation regime; the particle orbits are circular in the plane orthogonal to a perfectly homogeneous magnetic field. With $\delta_j > \gamma$, the regime is very similar to synchrotron, but the particle’s guiding center is slowly drifting, due to slight inhomogeneity in the magnetic field. The produced spectrum is well represented by the synchrotron spectrum, and it evolves slowly in time due to the particle diffusion through regions of differing field strength. This regime may be referred to as the diffusive synchrotron regime.

Thirdly, when $1 < \delta_j < \gamma$, the particle does not complete its Larmor orbit because the $B$-field varies on a shorter scale. In this case, an onlooking observer would see radiation from only short intervals of the particle’s trajectory (i.e., whenever the trajectory is near the line-of-sight), as in synchrotron, but these intervals are randomly distributed. This is the case of the large-angle jitter regime. The radiation is similar to synchrotron radiation near the spectral peak and above, but differ significantly from it at lower frequencies, see Ref. [23] for details.

Finally, If $\delta_j \ll 1$, a distant observer on the line-of-sight will see the radiation along, virtually, the entire trajectory of the particle (which will be approximately straight with small, random, transverse deviations). This is known as small-angle jitter radiation [13, 22, 23]. The resulting radiation markedly differs from synchrotron radiation, although the total radiated power of radiation, $P_{tot} \equiv dW/dt$, produced by a particle in all these regimes, e.g., jitter and synchrotron, is identical:

$$
P_{tot} = \frac{2}{3} \frac{e^2}{m_e c^2} \gamma^2 \langle B^2 \rangle,
$$

where $r_e = e^2/m_e c^2$ is the classical electron radius.

For ultra-relativistic electrons, the radiation spectra are wholly determined by $\delta_j$ and the magnetic spectral distribution. It has been shown [13, 22, 23] that monoenergetic relativistic electrons in the sub-Larmor-scale magnetic turbulence given by Eq. (14) produce a flat angle-averaged spectrum below the spectral break and a power-law spectrum above the break, that is

$$
P(\omega) \propto \begin{cases} 
\omega^0, & \text{if } \omega < \omega_j \\
\omega^{-\mu+2}, & \text{if } \omega_j < \omega < \omega_b \\
0, & \text{if } \omega_b \leq \omega,
\end{cases}
$$

where the spectral break is

$$
\omega_j \approx \gamma^2 k_{min} c,
$$

$$
\omega_b \approx \gamma^2 k_{max} c.
$$
which is called the jitter frequency. Similarly, the high-frequency break is
\[ \omega_b = \gamma^2 k_{\text{max}} c. \] (21)

C. Non-relativistic jitter radiation

In contrast, radiation from non-relativistic particles is not beamed along a narrow cone of opening angle, \( \Delta \theta \). The jitter parameter is, consequently, without meaning in the non-relativistic radiation regime. Instead, the “dimensionless scale” (or “gyro-number”), i.e. \( r_L \lambda_B^{-1} \), is the only meaningful parameter:
\[ r_L \lambda_B^{-1} \sim k_B r_L = k_B \frac{\gamma m_e v c}{e (B^2)^{1/2}} \equiv \rho, \] (22)

Given the magnetic spectral distribution exhibited by Eq. (14), \( k_B \sim k_{\text{min}} \), so
\[ \rho = k_{\text{min}} r_L. \] (23)

As we shall see below, the radiation spectrum in this regime markedly differs from the single-harmonic cyclotron spectrum. We call this radiation “pseudo-cyclotron” radiation or “non-relativistic jitter” radiation.

Regardless of the regime, the radiation spectrum (which is the radiative spectral energy, \( dW \) per unit frequency, \( d\omega \), and per unit solid-angle, \( d\Omega \)) seen by a distant observer is obtained from the equation [80, 81]
\[ \frac{d^2 W}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} A_k(t) e^{i\omega t} dt \right|^2, \] (24)
where
\[ A_k(t) = \hat{n} \times \frac{[\hat{n} - \beta] \times \dot{\beta}}{(1 - \hat{n} \cdot \beta)^2} e^{-ik \cdot r(t)}. \] (25)

In this equation, \( r(t) \) is the particle’s position at the retarded time \( t, k \equiv \omega/c \) is the wave vector which points along \( \hat{n} \) from \( r(t) \) to the observer and \( \beta \equiv d\delta/dt \). Since the observer is distant, \( \hat{n} \) is approximated as fixed in time to the origin of the coordinate system. This fully relativistic equation is obtained from the Liénard-Wiechart potentials. If \( v \ll c \), Eq. (24) simplifies to
\[ \frac{d^2 W}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \beta) e^{i\omega t} dt \right|^2, \] (26)

Next, integrating Eq. (26) over all solid-angles gives the radiated energy per frequency
\[ \frac{dW}{d\omega} = \frac{2e^2}{3\pi c^3} |w_\omega|^2, \] (27)
where \( w_\omega \) is the Fourier component of the electron’s acceleration with frequency, \( \omega \). Eq. (27), valid for \( v \ll c \), is known as the dipole approximation [30]. This expression may also be obtained from the Larmor formula, i.e.
\[ P_{\text{tot}} = \frac{2e^2}{3\pi c^3} |w|^2, \] (28)
using the identity [30]:
\[ \frac{1}{2} \int_{-\infty}^{\infty} |w(t)|^2 dt = (2\pi)^{-1} \int_{0}^{\infty} |w_\omega|^2 d\omega. \] (29)

To proceed further, we use our previous assumption that the particle deflection angle over a field correlation length is small (i.e. \( \alpha \lambda \ll 1 \)). This condition implies that the particle trajectory is approximately straight. For a particle moving in a magnetic field, \( |w_\omega|^2 \) is given by the Lorentz force. In this limiting case of small deflections, we may write
\[ |w_\omega|^2 = \left( \frac{e\beta^2}{m_e} \right)^2 (\delta\bar{n} \cdot \hat{v}_\epsilon) B^i_\omega B^j_\omega, \] (30)
where \( B_\omega \) is the temporal variation of the magnetic field along the trajectory of the electron, i.e.
\[ B_\omega = (2\pi)^{-4} \int e^{i\omega t} dt \int B_{k,\Omega} e^{ik \cdot (r(t) - r_0)} dkd\Omega. \] (31)

Since the trajectory is approximately straight, \( r(t) \approx r_0 + vt \), consequently
\[ B_\omega = (2\pi)^{-4} \int e^{ik \cdot r_0} B_{k,\Omega} dkd\Omega \int e^{i(\omega + k \cdot v - \Omega)t} dt, \] (32)
After the time integration, this becomes
\[ B_\omega = (2\pi)^{-3} \int \delta(\omega + k \cdot v - \Omega) e^{ik \cdot r_0} B_{k,\Omega} dkd\Omega. \] (33)

Now, since the magnetic turbulence is assumed to be homogeneous (at least over a time scale greater than the particle transit time) the product of \( B^i_\omega B^j_\omega \) along a particular trajectory starting at \( r_0 \) is representative of the magnetic field as a whole [23]. Thus, we may consider only the volume-average of \( B^i_\omega B^j_\omega \). Performing the integration leads to
\[ \langle B^i_\omega B^j_\omega \rangle_{r_0} = (2\pi)^{-3} V^{-1} \int \delta(\omega + k \cdot v - \Omega) B^i_{k,\Omega} B^j_{k,\Omega} dkd\Omega. \] (34)

The quantity, \( B^i_{k,\Omega} B^j_{k,\Omega} \), is proportional to the Fourier image of the two-point auto-correlation tensor – i.e. Eq. (10). Thus, with Eqs. (27), (30), (33), and (10), the angle-averaged radiation spectrum of a non-relativistic electron moving in static, statistically homogeneous and isotropic sub-Larmor-scale magnetic turbulence is
\[ dW/d\omega = \left( \frac{T r^2 \beta^2 c}{12\pi^3 V} \right) \int \delta(\omega + k \cdot v) \left[ 1 + (\hat{k} \cdot \hat{v})^2 \right] |B_k|^2 d(k), \] (35)
where \( T \) is the duration of the observation, and where we have used
\[ \delta(\omega + k \cdot v) = \int \delta(\omega + k \cdot v - \Omega) \delta(\Omega) d\Omega. \] (36)
We see that the radiation spectrum is fully determined by the magnetic spectral distribution, $|B_k|^2$. It is instructive to consider one of the simplest such distributions – the isotropic spectrum of a magnetic field at a single scale, $k_B$, i.e.

$$|B_k|^2 = (2\pi)^3 V(B^2) \frac{\delta(k - k_B)}{4\pi k_B^3}. \quad (37)$$

Substitution of Eq. (37) into Eq. (35) produces the radiation spectrum

$$\frac{dW}{d\omega} = \begin{cases} \frac{T^2}{3k_B} r_c^2 \beta \delta(B^2) \left(1 + \frac{\omega^2}{\omega_{jn}^2}\right), & \text{if } \omega \leq \omega_{jn} \\ 0, & \text{if } \omega_{jn} < \omega \end{cases} \quad (38)$$

where $\omega_{jn} \equiv k_B v$.

The non-relativistic spectrum may be generalized to trans-relativistic velocities by considering the virtual photon approximation; i.e. we treat the magnetic field inhomogeneity as a virtual photon that is Compton scattered off the electron to produce the observed radiation. Consider a frame where the electron is approximately at rest. In this frame, the correlation scale $\lambda_B'$ is related to $\lambda_B$ (i.e. the correlation length in a frame moving at a relativistic velocity) by a Lorentz transformation (i.e. $\lambda_B' \approx \lambda_B/\gamma$). Recall that the characteristic time scale of the electron’s transverse acceleration is the transit time over a field correlation length; In the approximate rest frame, this is $\tau'_{\bot} \sim \lambda_B'/v$. Thus, the characteristic frequency of the virtual photon, in the electron’s rest frame, is $\omega' \sim 2\pi/\tau'_{\bot} \sim 2\pi v/\lambda_B \sim 2\pi v\gamma/\lambda_B$. Transforming back to the relativistic frame picks up another factor of $\gamma$ from the Lorentz transformation, giving the characteristic frequency of $\omega \sim 2\pi v\gamma^2/\lambda_B \sim \gamma^2 k_B v$.

Although the virtual photon approximation is only formally valid in the ultra-relativistic limit, our numerical simulations demonstrate that it holds well for trans-relativistic particles moving through magnetic turbulence, at least if the turbulence is sub-Larmor-scale. However, as relativistic beaming becomes more prominent, the spectrum approaches ultra-relativistic small-angle jitter radiation.

Next, the total radiated power may be obtained by integrating Eq. (35) over all frequencies and dividing by the total observation time, yielding

$$P_{tot} = \frac{2}{3} r_c^2 \beta^2 c (B^2_\perp), \quad (39)$$

where we have used Eq. (3). Compare this to the total power radiated by a non-relativistic electron moving through a uniform magnetic field,

$$P_{tot} = \frac{2}{3} r_c^2 \beta^2 c B_L^2, \quad (40)$$

which follows directly from Eq. (28). Evidently, the total power of non-relativistic jitter radiation is identical to the total power of cyclotron radiation – with $B^2 \to \langle B^2 \rangle$; this is exactly analogous to the relation between synchrotron and relativistic jitter radiation.

Finally, for the magnetic spectral distribution of Eq. (14), the corresponding trans-relativistic and non-relativistic jitter spectrum, is given by

$$\frac{dW}{d\omega} = \begin{cases} A + D\omega^2, & \text{if } \omega \leq \omega_{jn} \\ F\omega^{-\mu+2} + G\omega^2 + K, & \text{if } \omega_{jn} \leq \omega \leq \omega_{bn} \\ 0, & \text{if } \omega_{bn} < \omega \end{cases} \quad (41)$$

where $\mu \neq 2$ and

$$A \equiv \frac{v}{2 - \mu} \left(k_{\max}^{-\mu+2} - k_{\min}^{-\mu+2}\right), \quad (42)$$

$$D \equiv -\frac{1}{v\mu\gamma^4} \left(k_{\max}^{-\mu} - k_{\min}^{-\mu}\right), \quad (43)$$

$$F \equiv \frac{v^\mu}{v} \left(\frac{1}{\mu - 2} + \frac{1}{\mu}\right) \left(\frac{1}{\gamma^2}\right)^{-\mu+2}, \quad (44)$$

$$G \equiv -\frac{v}{\mu\gamma^2} k_{\min}^{-\mu}, \quad (45)$$

$$K \equiv \frac{v}{2 - \mu} k_{\max}^{-\mu+2}, \quad (46)$$

with the (trans-relativistic) jitter frequency defined as

$$\omega_{jn} \equiv \gamma^2 k_{\min} v, \quad (47)$$

and

$$\omega_{bn} \equiv \gamma^2 k_{\max} v, \quad (48)$$

which is the (trans-relativistic) break frequency. In the non-relativistic regime, $\gamma \simeq 1$, of course. With this in mind, and for the sake of convenience, we retain the n subscript for both the trans-relativistic and non-relativistic expressions. Notice the structural similarity between the spectrum at frequencies less than $\omega_{jn}$ and the delta function spectrum in Eq. (38).

From Eqs. (41) and (13), we see that an inter-relation between the diffuse and radiative properties of trans-relativistic/non-relativistic plasmas with sub-Larmor-scale magnetic turbulence exists. Furthermore, this inter-relation owes its existence to the statistical properties of the magnetic turbulence (e.g. $\langle B^2 \rangle$ and $\lambda_B$). We note, however, that our results in both cases depend on the presumption of a small deflection angle over a field correlation scale. By way of numerical simulation, we will demonstrate that, indeed, this condition holds as long as $\rho > 1$ (i.e. the turbulence is sub-Larmor in scale).

### III. NUMERICAL MODEL

Using the method from our previous work (see Ref. [21]), here we explore the inter-relation between the diffusive and radiative properties of these plasmas, and thereby test our theoretical predictions. As before, this was done via simulations of particle dynamics in sub-Larmor-scale magnetic turbulence. In our simulations, only first-principles were used.
Non-relativistic and trans-relativistic electrons are test particles moving in preset magnetic fields defined over a 3D simulation box, with periodic boundary conditions in all directions. The particles do not interact with each other, as in collisionless plasmas, nor do they induce any fields. Additionally, any radiative energy losses are neglected. An individual electron’s motion is, consequently, determined only by the Lorentz force equation given by:

\[
\frac{d\beta}{dt} = -\frac{1}{\gamma} (\beta \times \Omega_B),
\]

where \(\Omega_B = eB/m_e c\). For simplicity, we define our simulation magnetic field as \(B \equiv \Omega_B\). In this manner, our arbitrary simulation units are always related to a physical magnetic field via the definition of \(\Omega_B\). Notice that the purely magnetic Lorentz force conserves particle energy; hence, the velocity vector varies in direction but has a constant magnitude.

The simulation can be divided into two principle stages (see Ref. [32] for a detailed description of the numerical implementation). First, the turbulent magnetic field is created using a predetermined spectral distribution in Fourier space. This field is generated on a cubic lattice that is then interpolated, so as to represent a “continuous” field. The interpolation implements divergenceless matrix-valued radial basis functions (see Ref. [33], for a discussion). This interpolation method starts with a radial function – in our case, one of the simplest, \(\phi(r) = e^{-\epsilon r^2}\) (where \(\epsilon\) is a scaling factor, and \(r^2 = x^2 + y^2 + z^2\)). Then, a set of divergence-free matrix-valued radial basis functions is obtained from the transformation [33]:

\[
\Phi(r) = (\nabla^T - \mathbb{I}_{3\times3} \nabla) \phi(r),
\]

where \(\nabla^T\) is the second-order, \(3 \times 3\)-matrix differential operator and \(\mathbb{I}_{3\times3}\) is the \(3 \times 3\) identity matrix.

These interpolants are then applied to the interior of each lattice “cell” (the significance of the interpolant’s divergence is explored in Appendix [3]). The second stage in our model involves the numerical solution of the equation of motion for each particle, i.e. Eq. (49). From the solution, \(\langle \alpha^2 \rangle\) and the radiation spectra are obtained. We first turn our attention to the generation of the magnetic field.

As discussed previously (see Ref. [21]), generation of the magnetic field distribution is more convenient in Fourier space. There are two principal reasons for this. Firstly, it is an easier task to specify a particular spectral distribution in Fourier space directly, rather than attempting to approximate the corresponding field in real space. Secondly, any physically realizable field should satisfy Maxwell’s equations, thus its divergence must be zero. This divergence-less condition is more readily met in Fourier space, because Gauss’ law, \(\nabla \cdot \mathbf{B} = 0\), is an algebraic equation there; \(\mathbf{k} \cdot \mathbf{B}_k = 0\), for each Fourier component. Although our code can handle a wide variety of magnetic spectral distributions, we limit our study to isotropic magnetic turbulence, described in Eq. (14) – leaving more sophisticated models for the future.

After the magnetic field is generated, the next step is the numerical solution of the equation of motion, Eq. (49). This was done via a fixed step 4th-order Runge-Kutta-Nyström method. With all the particle positions, velocities, and accelerations calculated, the radiation spectrum is obtained from Eq. (24).

Next, the total radiation spectrum is obtained by “summing” over the spectra of the individual particles. There are two, usually equivalent, methods for doing the summation. First, one can add the spectra coherently by summing over each particle’s \(A_k\), and then performing a single integration via Eq. (24). This is a more physical method. In the second method we add the spectra incoherently (i.e., by integrating each particle’s \(A_k\) separately, and then summing the results of each integration). As discussed in Ref. [34], both methods will result in the same spectra, since the wave phases are uncorrelated. However, an incoherent sum will produce spectra that are less noisy, for a given number of simulation particles, than the coherently summed spectra. Hence we use the incoherent approach in our study.

IV. NUMERICAL RESULTS

In Section II we made a number of theoretical predictions concerning the transport and radiation properties of plasmas with small-scale turbulent magnetic fields. Additionally, we anticipated that an inter-connection between the transport and radiative properties of non-relativistic/trans-relativistic particles moving through sub-Larmor-scale magnetic turbulence exists, as it does for ultra-relativistic particles [21]. Here we check our predictions, and further explore the radiation spectra.

First of all, we explore how the pitch-angle diffusion coefficient depends on various parameters, cf. Eq. (13), namely the particle’s velocity, \(\beta\), the magnetic field strength, \(\langle B^2 \rangle\), the field correlation scale, \(\lambda_B\), and the “gyro-number”, \(\rho\).

To start, we tested our fundamental assumption that the particle velocity vector only varies slightly over a correlation length, \(\lambda_B\). This is the key assumption that underlies our theoretical predictions for pitch-angle diffusion and the radiation spectra. If this assumption were to not hold (i.e. if \(\alpha_\infty \ll 1\) then pitch-angle diffusion would break down, and the plot of \(\langle \alpha^2 \rangle\) vs time will deviate from linear behavior. In Figure 1, \(\langle \alpha^2 \rangle\) is plotted as a function of time for seven different cases. In each run, \(\langle B^2 \rangle\), \(k_{\text{min}}\), and \(N_p\) (number of simulation particles) are fixed to the values of 0.01, \(4\pi/5\) (both in arbitrary simulation units), and 2000, respectively. The particles are monoenergetic, and are isotropically distributed in their initial velocities. Each case differs in particle velocities; which range from \(\frac{1}{\Omega_B}c\) to \(\frac{1}{\gamma}c\). As can be seen, the curves begin as straight lines that increase with slope as \(\beta\) decreases. Eventually, the linear behavior breaks down as \(\beta\) decreases. A decrease in \(\rho\) occurs concurrently, in accordance with Eq. (22).

As expected, the breakdown in linear behavior, and hence diffusion, occurs when \(\rho \sim 1\). Later, we did the same experiment, only this time we varied \(\langle B^2 \rangle\) in such a way as to keep \(\rho\) constant (\(\rho = 24.5\)). In this way, each case is securely in the small-scale regime. In Figure 2, we see that the linear behavior of \(\langle \alpha^2 \rangle\) vs time is preserved for all velocities, as anticipated. Consequently, our assumption of a small \(\alpha_\infty\) is valid, as
appear with increasing average slope as $\beta$ decreases. Clearly, the linear form of the curves is retained in all seven cases.

Figure 2. (Color online) Average square pitch-angle vs. time (in simulation units). Relevant parameters are $N_p = 2000$, $k_{\text{min}} = 4\pi/5$, $k_{\text{max}} = 8\pi$, $(B^2)^{1/2} = 0.01$, and $\mu = 3$. The particle velocities range (in the opposite order) from $\frac{1}{2}c$ to $\frac{1}{4}c$ (by multiples of two). The curves appear with increasing average slope as $\beta$ decreases. As $\beta$ decreases, eventually $\rho \sim 1$ (at $\beta = \frac{1}{24}$, i.e. the fifth most sloped, “green” line), after which the deflection angle becomes large, and pitch-angle diffusion breaks down.

Next, we tested the correlation length dependence, i.e. whether or not the numerical simulations agree with Eq. (11). In Figure 3, the numerically obtained diffusion coefficients are compared to the analytical result of (13). In Figure 4, the numerical diffusion coefficient is plotted against the magnetic spectral index, $\mu$. The (blue) empty “squares” indicate the $D_{\alpha\alpha}$ obtained directly from simulation, while the (red) filled “triangles” are the analytical, given by Eq. (13), pitch-angle diffusion coefficients. Simulation parameters are identical to those used in Figure 2.

With $k_{\text{min}} = \pi$ and $k_{\text{max}} = 8\pi$, we varied the magnetic spectral index, $\mu$ from 2 to 5. This is plotted in Figure 4, where the numerical diffusion coefficient closely matches the analytical result.

In Figure 5 the numerical diffusion coefficient is plotted against the analytical coefficient for the same range of $\mu$ values, but now the $k_{\text{min}}$, $k_{\text{max}}$, and $\beta$ values differ among the three (with $\rho$ fixed to 24.5). Included are the results of Figure 5.
All three cases give a nearly linear relationship between the numerical and analytical coefficients, with slopes approximately equal to unity. Another concern worth addressing is the dependence of the numerical diffusion coefficient on the total number of simulation particles. In Figure 6, a test case was repeated with an increasing number of simulation particles. The number of particles was increased from 500 to 64000, by factors of 2. There is little variation to be seen in the numerical result, as the number of particles is increased. Next, we explored the trans-relativistic jitter radiation regime by calculating the radiation spectra, using Eq. (24), with variable simulation parameters. We aimed to test the radiation spectra’s dependence upon the key turbulent parameters: $k_{\text{min}}$, $k_{\text{max}}$, $\langle B^2 \rangle$, and $\mu$, as well as the particle velocity, $v$. To start, we considered the $k_{\text{min}}$ dependence. In Figure 7, we have plotted spectra for an initially isotropically distributed, monoenergetic, ensemble of trans-relativistic electrons ($v = 0.5c$) moving through sub-Larmor-scale magnetic turbulence with three different values of $k_{\text{min}}$. The key parameters are: $\rho = 18.1$, 36.3, and 72.6, with $k_{\text{min}} = \pi/5$, $2\pi/5$, and $4\pi/5$, respectively (see Table 3 for a complete listing of simulation parameters used in every figure). The spectra of Figure 7 at least superficially, resemble our theoretical prediction; cf. Eq. (41). We have normalized the $dW/d\omega$ and $\omega$ axes by $\lambda_B$ and $k_{\text{min}}$, respectively. As expected, the frequency of the spectral peak scales by $k_{\text{min}}$. The precise scaling of the peak frequency is revealed in Figure 8. In this figure, we have varied the particle velocities, keeping all other parameters fixed. Three velocities appear: $v = 0.125c$, $0.25c$, and $0.5c$. Clearly, the overall shape of the spectra is not strongly dependent upon the particle velocities. We have identified the proper scaling on the horizontal axis. With this result, and Figure 7 we may conclude that the frequency of the peak of the radiation spectrum is $\omega \sim \gamma^2 k_{\text{min}} v = \omega_j \gamma n$. This is jitter frequency given in Eq. (41).

Next, we tested the $\mu$ dependence. In Figure 2, $\mu = 4.5$. For each spectrum, $v = 0.125c$, and the total simulation time was $T_g$, where $T_g = c\langle B^2 \rangle^{1/2}/\gamma m_e c$ is the gyroperiod. The

Figure 5. (Color online) Numerical pitch-angle diffusion coefficient vs the analytical pitch-angle diffusion coefficient, for three different cases. In each case, the magnetic spectral index has been varied from 2 to 5, by intervals of unity. Relevant parameters are $k_{\text{min}} = \pi/2$ (red) “circles” and (blue) “triangles”, $\pi$ (green) “diamonds”, $k_{\text{max}} = 5.12\pi$ (red) “circles”; $k_{\text{max}} = 8\pi$ (green) “diamonds” and (blue) “triangles”; $B_{\text{min}}/B_{\text{max}} = 0.064$ (green) “diamonds”; $B_{\text{min}}/B_{\text{max}} = 0.25$ (red) “circles”, 0.5 (blue) “triangles” and (green) “diamonds”. In each case, a line of best fit is applied. The slopes are 0.979 (circles), 0.972 (diamonds), and 1.06 (triangles).

Figure 6. (Color online) Pitch-angle diffusion coefficient, $D_{\alpha\alpha}$ vs the total number of simulation particles, $N_p$. The “blue squares” indicate the $D_{\alpha\alpha}$ obtained directly from simulation, while the red dotted line is the analytical result, given by Eq. (13). Relevant parameters are $k_{\text{min}} = \pi/2$, $k_{\text{max}} = 8\pi$, $\langle B^2 \rangle^{1/2} = 0.032$, $\beta = 0.5$, and $\rho = 24.5$. There appears to be no strong dependence of the numerical pitch-angle diffusion coefficient upon the total number of simulation particles; nevertheless, there appears to be some convergence to the analytical result.

Figure 7. (Color online) Radiation spectra given variable $k_{\text{min}}$, with all other parameters fixed. The number of simulation particles, $N_p$, is 2000, and $v = 0.5c$ in each case. In each trial, the particles moved for a total simulation time of $T = T_g$, where $T_g = 2\pi\gamma m_e c/e\langle B^2 \rangle^{1/2}$ is the “gyroperiod”. Here, the axes are in arbitrary, simulation units. We see that the frequency scales as $k_{\text{min}}$ and $dW/d\omega$ scales as $\lambda_B$. 


Table I. Table of parameters used for the radiation spectra figures. Here, and elsewhere, $\Delta t$ is the simulation time step, the simulation time is denoted in multiples of the “gyroperiod” (i.e. $T_g = 2\pi\gamma m_c e/\langle B^2 \rangle^{1/2}$), and $N_p$ is the total number of simulation particles.

| # | $\rho$ | $\Delta t$ | $\beta$ | $\mu$ | $k_{\text{min}}$ | $k_{\text{max}}$ | $\sqrt{\langle B^2 \rangle}$ | $N_p$ | $T_g$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 18.1 | 0.005 | 0.5 | 3 | $\pi/5$ | 10.24$\pi$ | 0.02 | 2000 | 1 |
| 2 | 36.3 | 0.005 | 0.5 | 3 | $2\pi/5$ | 10.24$\pi$ | 0.02 | 2000 | 1 |
| 3 | 72.6 | 0.005 | 0.5 | 3 | $4\pi/5$ | 10.24$\pi$ | 0.02 | 2000 | 1 |
| 4 | 15.8 | 0.005 | 0.125 | 3 | $4\pi/5$ | 10.24$\pi$ | 0.02 | 1000 | 10 |
| 5 | 32.4 | 0.005 | 0.25 | 3 | $4\pi/5$ | 10.24$\pi$ | 0.02 | 1000 | 10 |
| 6 | 72.6 | 0.005 | 0.5 | 3 | $4\pi/5$ | 10.24$\pi$ | 0.02 | 1000 | 10 |
| 7 | 0.5 | 0.125 | 4 | $8\pi$ | 0.064 | 8000 | 1 |
| 8 | 6.18 | 0.005 | 0.125 | 5 | $\pi$ | 8$\pi$ | 0.064 | 8000 | 1 |
| 9 | 6.18 | 0.005 | 0.125 | 5 | $\pi/2$ | 4$\pi$ | 0.064 | 2000 | 1 |
| 10 | 6.18 | 0.065 | 0.25 | 5 | $\pi/2$ | 4$\pi$ | 0.064 | 2000 | 1 |
| 11 | 6.34 | 0.005 | 0.25 | 5 | $8\pi$ | 0.064 | 2000 | 1 |
| 12 | 7.9 | 0.005 | 0.125 | 5 | 0.125 | 0.064 | 2000 | 1 |
| 13 | 0.125 | 100 | $\pi$ | 8$\pi$ | 0.032 | 8000 | 10 |
| 14 | 12.4 | 0.05 | 0.125 | 4 | $2\pi/5$ | 8$\pi$ | 0.02 | 4000 | 10 |
| 15 | 6.2 | 0.005 | 0.125 | 5 | $\pi$ | 8$\pi$ | 0.064 | 5000 | 1 |

Figure 8. (Color online) Radiation spectra given variable $v$. In each trial, 1000 particles move for a total simulation time of $T = 10T_g$, where $T_g \equiv 2\pi\gamma m_c e/\langle B^2 \rangle^{1/2}$ is the “gyroperiod”. We see that the overall shape of the spectra is not appreciably altered with decreasing $v$. The spectra are normalized by $T^2 \gamma^3 v$, vertically. Given Figure 7, we may conclude that the peak frequency of these spectra is $\omega \sim \gamma^2 k_{\text{min}} v$ —cf. Eq. 47.

Figure 9. (Color online) Radiation spectra given two different values of the magnetic spectral index: $\mu = 5$ (red) “thick” line and $\mu = 4$ (blue) “thin” line. Included are the analytical solutions given by Eq. 41. Note that the $\mu = 5$ solution has been multiplied by an overall factor of two for easier visualization. For frequencies near $\omega \sim \gamma^2 k_{\text{min}} v$, the numerical spectra agree decently with the analytical results. However, for frequencies near the break, $\omega \sim \gamma^2 k_{\text{max}} v$, there is considerable deviation between the predicted and numerical spectra — for both values of the magnetic spectral index. The origin of this discrepancy is explored in Appendix A.

The numerical and analytical spectra show close agreement for frequencies less than the break frequency, $\omega \sim \gamma^2 k_{\text{max}} v$. In Figure 10 we have plotted two spectra that differ in their $k_{\text{max}}$ values (all other parameters kept fixed). The $k_{\text{max}}$ values employed differ by a factor of 2. We see that, roughly, the spectra approach zero near $\omega \sim \gamma^2 k_{\text{min}} v$. The proceeding power law “tail” feature is a numerical artifact that arises from a steep drop to zero power (this fact is more readily apparent in a linear plot — see Appendix A). Next, we examined the apparent structure in the radiation spectra for $\omega < \omega_{jn}$. This is most clearly seen in Figure 8 where it appears as a distinctive "bump". According to Eq. 41, this bump-like feature has a functional form of $A + D \omega^2$. To assure that this form is correctly identified, we considered a large magnetic spectral index of $\mu = 100$ with $\beta = 0.125c$. Such a large $\mu$ makes the feature more prominent, helping to magnify it. As can be seen, the curve that best fits the bump-like feature at $\omega < \omega_{jn}$ is given by a function of the form $A + D \omega^2$.

One may consider the magnetic correlation tensor and its relation to the shape of the radiation spectra. Anisotropic
Figure 11. (Color online) Radiation spectrum with $\mu = 100$ ($\beta = 0.125c$). Evidently, the spectral feature presented directly prior to $\omega_{jn}$ has a functional form given by $A + D\omega^2$ (dashed line). This is consistent with Eq. (41).

The altered correlation tensor will affect the particle diffusion coefficient as well. In fact, as can be seen in Figure (13), the pitch-angle diffusion coefficient of particles moving in the monopolar field is twice as large as the divergenceless field equivalent. This follows from the fact that

$$\lambda_B^{\text{monopole}} = 2\lambda_B^{\text{div. free}},$$

which results from substitution of Eq. (51) into Eq. (7).

It is a noteworthy observation that the preceding results are identical, up to overall multiplicative factors, to the radiation spectra and pitch-angle diffusion coefficient for the more physically plausible situation of a trans-relativistic monopole moving through “small-scale” electrostatic turbulence, such as Langmuir turbulence.

V. CONCLUSIONS

In this paper we explored non-relativistic and trans-relativistic particle transport (diffusion) and radiation production in small-scale electromagnetic turbulence. Principally, we demonstrated that in the regime of small deflections, i.e. when the particle’s deflection angle over a correlation length is small $\alpha \ll 1$, the pitch-angle diffusion coefficient and the simultaneously produced radiation spectrum are wholly determined by the particle velocity and the statistical/spectral properties of the magnetic turbulence; which is a result most...
transparently offered by Eqs. (12) and (35). Additionally, we showed that the condition of a small deflection angle is satisfied if \( \rho > 1 \), i.e. if the magnetic turbulence is small-scale.

These results generalize the ultra-relativistic regime first discussed in Ref. [21]. In fact, the pitch-angle diffusion coefficient remains unchanged, in both the non-relativistic and relativistic regimes. Significantly, just as small-angle jitter radiation strongly differs from synchrotron radiation, so too does the analogous non-relativistic jitter radiation distinguish itself from cyclotron radiation. Given the isotropic 3D power law magnetic spectral distribution from Eq. (14), the resulting trans- and non-relativistic radiation spectrum is a piece-wise function of a quadratic equation in frequency, \( \omega \) up to the characteristic (jitter) frequency, \( \omega_{jn} = \gamma^2 k_{\text{min}} v \), after which it is the sum of a power law and a quadratic term up to the break frequency, \( \omega_{bn} = \gamma^2 k_{\text{max}} v \), where it then goes to zero – see Eq. (52). We have, further, confirmed our theoretical results via first-principle numerical simulations.

Lastly, we have considered the change in the radiative and transport properties of trans-relativistic particles moving through magnetic turbulence due to a topological change in the field. Namely, we supposed the generation of sub-Larmor-scale magnetic turbulence from a distribution of magnetic monopoles. We showed that the radiation spectra and pitch-angle diffusion coefficient are modified; i.e. the pitch-angle diffusion coefficient doubles in magnitude, à la Eq. (52), and the shape of the radiation spectrum is dramatically altered for frequencies less than the jitter frequency, \( \omega_{jn} \). These results, furthermore, generalize to the case of a magnetic monopole moving through “small-scale” electrostatic turbulence.

To conclude, the obtained results, coupled with our previous work, reveal strong inter-relation of transport and radiative properties of plasmas turbulent at sub-Larmor scales – whether they be relativistic or non-relativistic. We have demonstrated how spectral information can be a powerful tool to diagnose magnetic micro-turbulence in laboratory and astrophysical plasmas.

ACKNOWLEDGMENTS

This work has been supported by the DOE grant DE-FG02-07ER54940 and the NSF grant AST-1209665.

Appendix A: the Spectral Tail

As can be seen in Figure 7 and Figure 10, there is additional structure to the radiation spectra beyond the break frequency, \( \sim \gamma^2 k_{\text{max}} v \). This feature is, in fact, a numerical artifact that is magnified by the use of a log-log plot. Here we have plotted Figure 7 on a linear scale, and have normalized the frequency axis by the spectral break frequency \( \omega_{bn} = \gamma^2 k_{\text{max}} v \).

Appendix B: Interpolation of the Magnetic Field

One might consider the importance of using a divergenceless set of interpolants for the magnetic field. In Figure 15 we show a spectrum obtained via the divergenceless radial-basis interpolants of Eq. (50) with a spectrum obtained using a simple, non-divergenceless, trilinear interpolation of the magnetic field. For small frequencies, there is little disagreement between the two spectra. However, as the curves approach the break frequency \( \omega_{bn} = \gamma^2 k_{\text{max}} v \), considerable deviation between the trilinear and radial basis interpolants occurs. In our previous work on the relativistic small-angle jitter regime (see Ref. [21]), little deviation in these spectra was observed in our test runs. One possible explanation is that, since the particle velocities were \( \sim c \), the total distance traveled by a particle in one time step was \( \Delta x \sim c \Delta t \). The spacing between lattice points is, typically, within an order of \( c \Delta t \). In this case, the interpolant should not play an important role in determining the particle trajectories. If, however, \( v \) is much less than \( c \), then the difference may be significant. In Figure 15 \( v = 0.125c \), thus \( \Delta x \sim 0.125 c \Delta t \) (an order of magnitude smaller). In this case, frequencies in the radiation spectrum at scales comparable to the grid resolution (i.e. large \( k \)'s) will suffer the most from this deviation.
Figure 15. Radiation spectra given two different interpolations of the magnetic field and a “continuous” field. Relevant parameters are $v = 0.125c$, $\rho = 24.7$, $N_p = 2000$ (for a complete listing, see Table I). The number of wave modes employed to produce the “continuous” magnetic field was $N_m = 10000$. For small frequencies, there is little deviation between the spectra. It is only near the “break” frequency (i.e., $\omega_{bn} = \gamma^2 k_{max} v$) that the three differ considerably. Both of the interpolation derived spectra largely deviate from the analytical solution at the high frequency end; however, the “continuous” field derived spectrum differs noticeably only at the outermost frequencies. Whether or not this deviation is solely to blame on the quality of the interpolant or the discrete nature of lattice derived field, has yet to be determined. At any rate, both interpolants fail to preserve the slope of the spectra up to $\omega_{bn}$, and there is considerable difference between the divergence-free and trilinear cases.

Another question worth addressing is the influence of the discrete implementation of the magnetic field on the spectral shape. Recall that the random magnetic field is initially generated on a lattice in $k$-space, after which it is subsequently transformed by FFT to real space. The interpolation is then applied on the lattice of points. Due to memory limitations, the lattice dimensions are limited to $\sim 500^3$; this can be a very severe limitation on the spatial resolution of the magnetic field.

An alternative generation of the magnetic field – which is grid-less and, therefore, not requiring interpolation – employs a large sum of sinusoidal wave modes which are evaluated at each time step (as needed). Thus, the magnetic field is effectively “continuous” in this representation. Each wave mode is constructed with a random phase and random polarization vector (which is constrained to the plane perpendicular to $k$; thus satisfying Gauss’s law). The polarization vector may be generated by a variety of methods, but we have chosen the implementation described by Ref. [36]. This representation of the polarization vectors is designed specifically to simultaneously satisfy the required properties of isotropic, homogeneous, and divergence-free magnetic turbulence. Additionally, the wave numbers, ranging from $k_{min}$ to $k_{max}$, are logarithmically spaced.

In Figure 15 we also included a radiation spectrum obtained by electrons moving in the “continuous” magnetic field (with, otherwise, identical properties). Evidently, the “continuous” field derived spectrum closely matches the analytical solution, Eq. (41) – preserving the high-frequency end better than the interpolation derived spectra.

### Appendix C: Comment on pitch-angle diffusion in the ultra-relativistic regime

We wish to address an error in our previous paper on relativistic pitch-angle diffusion in sub-Larmor-scale magnetic turbulence, Ref. [21]. The paper contains a table for a plot (Figure 7) of the diffusion coefficient vs the corresponding radiation spectral peak, for relativistic particles moving through a small-scale magnetic field. The magnetic field has identical properties to those employed in this paper. The table contains some errors, which we address here by providing a corrected table (see Table II).

Table II. Corrected table of parameters used in Figure 7 of ref. [21], and Figure 16. The correction is as follows: #2 $\rightarrow$ #1, #3 $\rightarrow$ #2, and #1 $\rightarrow$ #3; in what was previously #1, $k_{min}$ has been changed from 1.3 to 0.6 and $\langle B^2 \rangle^{1/2}$ has been changed from 0.024 to 0.047.

| #   | $\delta_j$ | $\Delta t$ | $\gamma$ | $\mu$ | $k_{min}$ | $k_{max}$ | $\langle B^2 \rangle^{1/2}$ | $N_p$ |
|-----|------------|------------|---------|-------|-----------|-----------|----------------|-------|
| 1   | 0.63       | 0.0100     | 8       | 3     | 1.0       | 16.1      | 0.100          | 2000  |
| 2   | 0.47       | 0.0100     | 7       | 3     | 0.6       | 16.1      | 0.047          | 500   |
| 3   | 0.12       | 0.0025     | 5       | 3     | 0.6       | 32.2      | 0.047          | 4000  |
| 4   | 0.47       | 0.0100     | 3       | 3     | 0.6       | 16.1      | 0.047          | 500   |
| 5   | 0.94       | 0.0100     | 5       | 3     | 0.3       | 16.1      | 0.047          | 500   |

Additionally, we have opted to reproduce Figure 7 from Ref. [21], to recalculate the analytical pitch-angle diffusion coefficient. In our previous paper, we used Eq. (2), as we have here, but with cruder approximations for $\lambda_B$ and $\langle \beta_{L1}^2 \rangle^{1/2}$ – namely, $\langle \beta_{L1}^2 \rangle^{1/2} \approx 1$ and $\lambda_B \approx k_{min}$.
Figure 16. (Color online) Modified figure of $D_{\alpha \alpha}$ vs the frequency of $\omega_j = \gamma^2 k_{min} c$, from Ref. [21]. Once more, the (blue) empty “squares” indicate the $D_{\alpha \alpha}$ obtained directly from simulation while the (red) filled “triangles” are the analytical $D_{\alpha \alpha}$, given by Eq. (13). The analytical solution from Ref. [21] appears as (green) filled “circles”. Notice that the redefined analytical $D_{\alpha \alpha}$’s (red) empty “triangles” eliminate the wider variance seen in the cruder approximation (green) filled “circles”.

Now, the refined definition for $D_{\alpha \alpha}$, Eq. (13), eliminates the wider variance between the theoretical and numerical results (see Figure 16). There continues to exist a small difference between the analytical and numerical pitch-angle diffusion coefficients, but this variation is relatively small in each case; despite the variability in the simulation parameters employed.

[1] Medvedev, M. V. 2009, American Institute of Physics Conference Series, 1183, 189
[2] Frederiksen, J. T., Hededal, C. B., Haugbølle, T., & Nordlund, Å. 2004, ApJL, 608, L13
[3] Nishikawa, K.-I., Hardee, P., Richardson, G., et al. 2003, ApJ, 595, 555
[4] Sironi, L., & Spitkovsky, A. 2009, ApJ, 698, 1523
[5] Plotnikov, I., Pelletier, G., & Lemoine, M. 2013, MNRAS, 430, 1280
[6] Swisdak, M., Liu, Y.-H., & Drake, J. F. 2008, ApJ, 680, 999
[7] Liu, Y.-H., Swisdak, M., & Drake, J. F. 2009, Physics of Plasmas, 16, 042101
[8] Ren, C., Tzoufras, M., Tsung, F. S., et al. 2004, Physical Review Letters, 93, 185004
[9] Huntington, C. M., 2012, Ph.D. Thesis.
[10] Mondal, S., Narayanan, V., Ding, W. J., et al. 2012, Proceedings of the National Academy of Science, 109, 8011
[11] Tatarakis, M., Beg, F. N., Clark, E. L., et al. 2003, Physical Review Letters, 90, 175001
[12] Fiuza, F., Fonseca, R. A., Tonge, J., Mori, W. B., & Silva, L. O. 2012, Physical Review Letters, 108, 235004
[13] Medvedev, M.V. 2006, ApJ, 637, 869
[14] Weibel, E.S. 1959, PRL, 2, 83
[15] Fried, B. D. 1959, Physics of Fluids, 2, 337
[16] Medvedev, M. V. 2009, Astrophysics and Space Science, 322, 147
[17] Lemoine, M., & Pelletier, G. 2010, MNRAS, 402, 321
[18] Bret, A., & Deutsch, C. 2005, Physics of Plasmas, 12, 082109
[19] Treumann, R. A., & Baumjohann, W., Advanced Space Plasma Physics (Imperial College Press, London, 1997).
[20] Bret, A., Firpo, M.-C., & Deutsch, C. 2005, PRE, 72, 016403
[21] Keenan, B. D., & Medvedev, M. V. 2013, PRE, 88, 013103
[22] Medvedev, M.V., 2000, ApJ, 40, 704
[23] Medvedev, M.V., Frederiksen, J.T., Haugbølle, T., Nordlund, Å. 2011, ApJ, 737, 55
[24] Reville, B., & Kirk, J. G. 2010, ApJ, 724, 1283
[25] Teraki, Y. & Takahara, F. 2011, ApJ, 735, L44
[26] Kugland, N. L., Ryutov, D. D., Chang, P.-Y., et al. 2012, Nature Physics, 8, 809
[27] Huntington, C. M., Fiuza, F., Ross, J. S., et al. 2013, 1310.3337
[28] Biswas, G. & Eswaran, V. 2002, Turbulent Flows: Fundamentals, Experiments and Modeling, IIT Kanpur series of advanced texts. (CRC Press)
[29] Batchelor, G. K. 1982, The Theory of Homogeneous Turbulence, by G. K. Batchelor, pp. 197. ISBN 0521041171. Cambridge, UK: Cambridge University Press, June 1982.
[30] Landau, L.D. & Lifshitz, E.M. 1975, The Classical Theory of
Fields, Course of Theoretical Physics, Vol. 2 (New York: Pergamon)

[31] Jackson, J.D. 1999, Classical Electrodynamics, 3rd. ed. (New York: Wiley)

[32] Keenan, Brett D. (2012). Transport Properties and Radiation Production in Plasmas with Sub-Larmor-Scale Magnetic Turbulence. (Master’s thesis). Retrieved from ProQuest Dissertations and Theses. (1533116)

[33] McNally, C. P. 2011, MNRAS, 413, L76

[34] Hededal, C. 2005, Ph.D. Thesis

[35] Rädler, K. H., Brandenburg, A., Del Sordo, F., & Rheinhardt, M. 2011, PRE, 84, 046321

[36] Tautz, R. C., & Dosch, A. 2013, Physics of Plasmas, 20, 022302