Random hopping fermions on bipartite lattices: Density of states, inverse participation ratios, and their correlations in a strong disorder regime

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We study Anderson localization of non-interacting random hopping fermions on bipartite lattices in two dimensions, focusing our attention to strong disorder features of the model. We concentrate ourselves on specific models with a linear dispersion in the vicinity of the band center, which can be described by a Dirac fermion in the continuum limit. Based on the recent renormalization group method developed by Carpentier and Le Doussal for the XY gauge glass model, we calculate the density of states, inverse participation ratios, and their spatial correlations. It turns out that their behavior is quite different from those expected within naive weak disorder approaches.

I. INTRODUCTION

During the last decade, we have acquired much wisdom of Anderson localization. Multifractal scaling dimensions, for example, have been calculated exactly for a random Dirac fermion model, and a lot of novel universality classes for disordered systems have been discovered. Among the universality classes found so far, chiral (or sublattice) classes have been attracting much interest, since they are exactly on random critical points. Gade has predicted for these classes that the density of states (DOS) diverges at the critical points.

Recently, Motrunich et.al. have claimed that such DOS as was predicted within a naive weak disorder approach is incorrect if one considers strong disorder features of the model. They have predicted an alternative expression of the DOS, which shows a bit weaker divergence.

On the other hand, Carpentier and Le Doussal have proposed a renormalization group (RG) method to analyze strong disorder features at low temperatures of the XY gauge glass model. They have derived generalized RG equations for the Coulomb gas model, which is remarkably intimate relationship with a nonlinear diffusion equation called KPP equation. Their idea has been applied to a Dirac fermion model with a random vector field, and properties of a strong disorder regime as well as a weak disorder regime are successfully described in a unified way.

Interestingly, Mudry et al. have shown that the same RG method can apply to the random hopping problem: It has turned out that the disorder strength flows to a strong disorder regime, which provides the same DOS as Motrunich et al. have predicted.

Recent numerical calculations for a random hopping fermion model on a square lattice with flux suggest the divergent DOS, and we can now believe that the model actually belongs to the chiral orthogonal class. Therefore, we would like to confirm as the next step whether strong disorder features of the chiral orthogonal class reveal themselves. Unfortunately, the DOS may not be a good quantity to grasp them, since the divergent behavior itself is technically hard to observe in numerical computations.

In order to clarify the strong disorder features of the model furthermore, we calculate, in this paper, some alternative quantities, the inverse participation ratios (IPR) and their spatial correlations, where such features also appear manifestly.

This paper is organized as follows: In the next section, we introduce lattice models and their field theory in the scaling limit. In Sec. II, we derive the scaling equations taking into account higher powers of energy terms. In Sec. III, we convert the scaling equations into KPP-type equation and discuss its asymptotic solution. By the use of them, we then calculate the DOS, IPR, and their spatial correlations in Sec. IV. We give concluding remarks in Sec. V.

II. MODEL

In this section, we first introduce the following two kinds of lattice models with sublattice symmetry as well as time-reversal symmetry: One is the model studied by Hatsugai et al. defined on a square lattice with flux, and the other is the one on a honeycomb lattice. Next we derive a Dirac fermion model in the continuum limit of the lattice models at the band center.

A. Lattice models

We consider the following non-interacting random hopping fermions on bipartite lattices in two dimensions:

$$H = \sum_{\langle i,j \rangle} t_{ij} c_i^\dagger c_j,$$  \hspace{1cm} (2.1)

where $\langle i,j \rangle$ stands for summation over nearest neighbor pairs and $t_{ij} = t_{0,ij} + \delta t_{ij}$ is pure and random hopping amplitude, both of which can be chosen as real numbers due to time-reversal symmetry. The model defined on a square lattice with flux is here described with a gauge...
\[ t_{0,ij} = (-)^j t_0. \] In the case of the honeycomb lattice, \( t_{0,ij} = t_0. \)

We are interested in the ensemble-average of the correlation functions
\[
P^{(q_1,q_2)}(i-j) = \langle \Psi_n | C_{i}^{(q_1)} \Psi_n \rangle \langle \Psi_n | C_{j}^{(q_2)} \Psi_n \rangle
\]
\[
= \frac{1}{\Psi_n(i)} \frac{\Psi_n(j)}{\sum_n | \Psi_n(i) \rangle \langle \Psi_n(j) |}^{2q_1} \delta(E - E_n),
\]
where \( \Psi_n(i)\rangle \langle i | \) with \( |i\rangle = c_{i}^{\dagger} |0\rangle \) is a normalized wave function with an energy eigenvalue \( E_n \). Especially \( P^{(q,0)} = P(q) \) defines the moments of the wavefunction, the IPR. To calculate the above correlation functions, the following Green functions may be useful:
\[
\Gamma(i) = \text{Im} \langle \Psi_n | \langle \Psi_n | i \rangle, \Gamma^{(q_1,q_2)}(i,j) = \Gamma^{q_1}(i) \Gamma^{q_2}(j).
\]

Actually we have
\[
\omega^{q_1 + q_2 - 1} \Gamma^{(q_1,q_2)}(i,j) \rightarrow \frac{\omega^{q_1 + q_2 - 1} \Gamma^{(q_1,q_2)}(i,j) }{\rho(E)}.
\]

In the following, we will take the scaling limit of the lattice model near the zero energy and calculate the above correlations by the use of the field theory. In that case, it is difficult to define normalized wave functions which is crucial to the IPR. To avoid this, we alternatively define
\[
\bar{P}^{(q_1,q_2)}(i-j) = \frac{\omega^{q_1 + q_2 - 1} \Gamma^{(q_1,q_2)}(i,j) }{\rho(E)}.
\]

With the normalization condition relaxed. Using these, we can define generalized IPR as \( \bar{P}(q) \equiv \bar{P}(q,0) \). Note that \( \rho \) in the denominator in Eq. (2.5) is needed for the “normalization” of the generalized IPR, i.e., \( \bar{P}^{(1)} = 1 \). In what follows, we calculate Eq. (2.5) using continuum theory and the RG method.

### B. Continuum limit

In the continuum limit around the zero energy of the above lattice models, via \( ai \rightarrow x \) and \( c_{i}/\sqrt{a} \rightarrow \sum_{x} e^{i k_{x} / \sqrt{a}} \psi_{x} \) with a lattice constant \( a \), we have the following action \[ \int \mathcal{L}_{\text{QG}} \] after the choice of a suitable basis
\[
S = \int d^{2} x \bar{\psi}_{i} \left( i \gamma_{5}(\partial_{\mu} + A_{\mu}) + i M_{1} + M_{2} \gamma_{5} \right) \psi_{i},
\]
where \( \gamma_{1} = \sigma_{1}, \gamma_{2} = \sigma_{2}, \) and \( \gamma_{5} = \sigma_{3} \) are usual Pauli matrices, and \( i = 1,2 \) denotes a “flavor” due to the species doubling. The probability distributions of the randomness are
\[
P[A_{\mu}] \propto \exp \left( -\frac{1}{2 g_{A}} \int d^{2} x A_{\mu}^{2} \right),
\]
\[
P[M_{\mu}] \propto \exp \left( -\frac{1}{2 g_{M}} \int d^{2} x M_{\mu}^{2} \right).
\]

The difference between two lattice models introduced above is that the initial strength \( g_{A} \) and \( g_{M} \) is given by \( g_{A} = g_{M} \) for the model on a square lattice while \( g_{A} \neq g_{M} \) for the one on a honeycomb lattice. However, it has nothing to do with the long distance behavior, since \( g_{A} \) only is renormalized, as we shall see soon.

The energy term in Eq. (2.8) are
\[
S_{y} = y \int d^{2} x \mathcal{Y},
\]
where \( y = \omega - i E \) and \( \mathcal{Y} = \mathcal{O}_{+} + \mathcal{O}_{-} \). Then, averaging DOS is given by
\[
\rho(E) = \int \mathcal{Y}(x) d^{2} x.
\]

Here, chiral fermions have been defined by
\[
\bar{\psi}_{i} = -i \left( \bar{\psi}_{L_{i}}, \bar{\psi}_{R_{i}} \right), \quad \psi_{i} = \left( \psi_{R_{i}}, \psi_{L_{i}} \right).
\]

In the following section, we add powers of this energy term to the action by the use of the replica method. They play a crucial role in the RG analysis as we shall see.

### III. RENORMALIZATION GROUP ANALYSIS

In the previous work within usual weak disorder approaches, the replica method \[ \left[ \right. \] and supersymmetry method \[ \right. \] have been used to take quenched average. The latter method is quite interesting, since it enables us to derive exact beta functions for the present model. The recent work by Mudry et. al. has also used the supersymmetry method, but with higher energy (or “fugacity” in the context of the Coulomb gas model by Carpenter and Le Doussal) terms being added, anomalous dimensions of them may be hard to obtain beyond one-loop order.

Therefore, we will use the replica method for simplicity and derive the RG equations up to one-loop order for consistency. We follow a similar formulation developed by Mudry et. al. \[ \left. \right. \].

#### A. Scaling equations for fugacities

Let us introduce the replica, \( \psi_{i} \rightarrow \psi_{i 0} \) with a replica number \( m; a = 1,2, \ldots, m. \) Then ensemble-averaged correlation functions can be calculated as
\[
\overline{\Gamma^{(q_1,q_2)}(x - y)} = \langle \mathcal{Y}_{a_1} \mathcal{Y}_{a_2} \cdots \mathcal{Y}_{a_m} (x) \mathcal{Y}_{b_1} \mathcal{Y}_{b_2} \cdots \mathcal{Y}_{b_m} (y) \rangle,
\]
where
\[
\mathcal{Y}_{a} = \psi_{a} \psi_{a}^{\dagger}, \quad \psi_{a} = \psi_{R_{a}} \psi_{L_{a}}.
\]
in the replica limit \( m \rightarrow 0 \).

We utilize the Hodge decomposition to take ensemble-average over the vector field,

\[
A_\mu = \epsilon_{\mu\nu}\partial_\nu \varphi + \partial_\mu \phi, \tag{3.2}
\]

which converts the probability distribution into

\[
P[A] \propto \exp \left\{ -\frac{1}{2g_A} \int d^2x \left[ (\partial_\mu \varphi)^2 + (\partial_\mu \phi)^2 \right] \right\}. \tag{3.3}
\]

The gauge transformation

\[
\psi_{ia} \rightarrow \psi_{ia} e^{-i\varphi \gamma_5 + \phi}, \quad \psi_{ia} \rightarrow e^{-i\varphi \gamma_5 - \phi} \psi_{ia}, \tag{3.4}
\]

together with \( M_1 \pm iM_2 \rightarrow e^{\pm 2i\varphi}(M_1 \pm iM_2) \), which gives rise to no change to the probability distribution \( P[M_\mu] \), yields the replicated action

\[
S^{(m)} = \int d^2x \left( \mathcal{L}_0 + \mathcal{L}_1 + 2g_M \mathcal{O}_M + \sum_{n=1}^{m} n \gamma_n \right), \tag{3.5}
\]

with

\[
\mathcal{L}_0 = \frac{1}{2g_A} (\partial_\mu \phi)^2, \quad \mathcal{L}_1 = \bar{\psi}_{R\alpha} \partial_\alpha \psi_{R\alpha} + \bar{\psi}_{L\alpha} \partial_\alpha \psi_{L\alpha}, \quad \mathcal{O}_M = J_{R\alpha \beta} J_{L\beta \alpha}, \tag{3.6}
\]

where we have denoted \( \alpha = ia \) and \( \beta = jb \) for simplicity, and defined currents as

\[
J_{R\alpha \beta} = \bar{\psi}_{R\alpha} \psi_{R\beta}, \quad J_{L\alpha \beta} = \bar{\psi}_{L\alpha} \psi_{L\beta}. \tag{3.7}
\]

Note that the field \( \varphi \) has been decoupled and hence omitted in Eq. \( \text{[3.6]} \).

The last term in Eq. \( \text{[3.6]} \) is the \( n \)th power of the energy term, \( \gamma_n = \sum_{a} \gamma_{a_1} \cdots \gamma_{a_n} \). Expanding these with respect to \( \mathcal{O}_M \) defined in Eq. \( \text{[2.9]} \), it turns out that the most relevant operators are \( \sum_{a} (\mathcal{O}_{a_1} \cdots \mathcal{O}_{a_n} + \mathcal{O}_{a_1} \cdots \mathcal{O}_{a_n}) \equiv \mathcal{O}_+^a + \mathcal{O}_-^a \equiv \gamma_{mn} \), which is actually converted via the gauge transformation into

\[
\gamma_{mn} = e^{2n\phi} \mathcal{O}_+^n + e^{-2n\phi} \mathcal{O}_-^n. \tag{3.8}
\]

As we shall see momentarily, it is important to include all these terms even if we started initially with \( \gamma_1 \) only, since these terms are generated in the process of the renormalization. To see this, note that the dimension of the vertex operator \( e^{n\phi} \) is \(-n^2g_A/\pi \) and therefore, the following OPE holds;

\[
e^{n\phi(\bar{z})} e^{n'\phi(0)} \sim \frac{1}{|z|^{2n+n'}g_A/\pi} e^{(n+n')\phi(0)}. \tag{3.9}
\]

This equation is actually involved with a “fusion” of the energy terms, generating the higher powers of operators \( \gamma_n \).

Based on the above OPE, let us now compute the OPE between \( \gamma_{mn} \) and \( \gamma_{mn'} \). One problem is the cross term such as \( e^{n\phi(\bar{z})} \mathcal{O}_+^n(z) e^{-n'\phi(0)} \mathcal{O}_-^{n'}(0) \). Its most singular term in the OPE yields the exponent \(-2mn'g_A/\pi + |n-n'| \), and therefore at least when \( g_A \geq \pi/2 \), such OPE gives rise to no singularity. Therefore, we will neglect these terms, and then we reach the following OPE for \( \gamma_{mn} \);

\[
\gamma_{mn}(z) \gamma_{mn'}(0) \sim \frac{1}{|z|^{2n+n'}g_A/\pi} \gamma_{mn+n'}(0). \tag{3.10}
\]

This leads to the scaling equations

\[
\frac{dy_n}{dt} = \beta_n = (2 - x_n)y_n - \pi \sum_{n'=1}^{m} y_{n'} y_{n'-n'}, \tag{3.11}
\]

where \( x_n \) is the scaling dimension of the operator \( \gamma_{mn} \) given by \( x_n = n - \frac{4\pi}{g_M} n^2 \) for the present. However, the \( \mathcal{O}_M \) yields an anomalous dimension, which is determined in the next subsection. The correlation function \( \Gamma^{(q_1,q_2)}(x-y) \) is dominated by

\[
\Gamma^{(q_1,q_2)}(x-y) = \langle \gamma_{mn}(x) \gamma_{mn'}(y) \rangle. \tag{3.12}
\]

In Sec. \( \text{V} \) we will determine its long distance behavior using the RG equation for these correlation functions.

\[\text{B. Scaling equations for random mass and vector-field couplings}\]

In this subsection, we derive the scaling equation for \( g_M \). The basic OPE for the currents are

\[
J_{R\alpha \beta}(z) J_{R\alpha \beta}(w) \sim \frac{\delta_{\alpha \lambda} \delta_{\beta \kappa}}{4\pi^2(z-w)^2} + \frac{1}{2\pi^2(z-w)} \left[ \delta_{\beta \kappa} J_{R\alpha \lambda}(w) - \delta_{\alpha \lambda} J_{R\beta \kappa}(w) \right],
J_{R\alpha \beta}(z) \bar{\psi}_{R\alpha} \sim \frac{\delta_{\beta \kappa}(z-w)}{2\pi(z-w)} \bar{\psi}_{R\beta}, \quad J_{R\alpha \beta}(z) \psi_{R\alpha} \sim \frac{-\delta_{\beta \kappa}(z-w)}{2\pi(z-w)} \psi_{R\beta}, \tag{3.13}
\]
and similar for \(L\)-movers. Therefore, we have the following OPE for \(O_M\) \cite{17}

\[
O_M(z)O_M(0) \sim \frac{1}{4\pi^2|z|^2} [2O_A(0) - 4mO_M(0)],
\]

(3.14) and similar for \(O_A\). We also have to neglect them in order to close the OPE algebra. The RG equations for \(g_A\) and \(g_M\) in the replica limit \((m \to 0)\) are then given by

\[
\frac{dg_A}{dl} = \beta_A = \frac{g_A^2}{\pi},
\]

\[
\frac{dg_M}{dl} = \beta_M \equiv 0,
\]

(3.17) and the anomalous dimension of \(Y_{mn}\) is calculated as

\[
x_n = (1 - \frac{g_M}{\pi}) n - \frac{g_A}{\pi} n^2.
\]

(3.18) If we neglected the fusion of the energy terms, this would serve as the scaling dimension of the operator \(Y_{mn}\) and govern the long distance behavior of the correlation function \(\langle Y^n \rangle\).

IV. KPP EQUATION

So far we have derived the scaling equations for \(y_n\), \(g_A\), and \(g_M\). Those for \(g_A\) and \(g_M\) are actually scaling equations in the replica limit. Basically, we also have to solve the equations for \(y_n\) and take the replica limit. To this end, we utilize the method developed by Carpentier and Le Doussal \cite{20}. Define the following distribution function \(P(l, u)\) of \(y_n\)

\[
y_n(l) = \frac{2}{\pi n!} \int e^{nu} P(u,l) = \frac{1}{\pi n!} \langle e^{nu} \rangle_P.
\]

(4.1) Then it turns out that the resultant equation for \(P(u,l)\) is free from \(m\), and we need not to worry about the replica limit. Moreover, define

\[
G(x,l) = 1 - \left\langle e^{u - x(1 - g_M/\pi)} \right\rangle_P.
\]

(4.2) It is readily seen that this function \(G\) obeys the following KPP-type equation,

\[
\frac{1}{2} \partial_l G = D(l) \partial^2_x G + G(1 - G),
\]

(4.3) where \(D(l) = \frac{g_A(l)}{2\pi^2}\) is a diffusion constant. The boundary condition is \(G(-\infty, 0) = 1\) and \(G(\infty, 0) = 0\). The difference from the normal KPP equation is that the diffusion constant \(D\) depends on \(l\). In analyzing the above equation, we take an “adiabatic” approximation as Mudry et. al. applied to the same problem. Namely, we first neglect the l-dependence of the diffusion constant \(D\) for a while, and after solving the KPP equation, we take the \(l\)-dependence into account. This scheme may hold at least in the leading approximation. It should be noted that the function \(G\) is a generating functional of \(y_n\). Actually, expanding with respect to \(e^x\) leads to

\[
G(x,l) = \frac{\pi}{2} \sum_{n=1}^{\infty} (-1)^n y_n(l) e^{-(x - (1 - g_M/\pi)) l}.
\]

(4.4) In the limit \(l \to \infty\), the KPP equation asymptotically allows the following traveling wave solution \(G(x,l) \sim G(x - m(l))\), where \(m(l)\) is a velocity (times \(l\)) of the traveling wave. In order to determine the dimension of the operator \(Y_{mn}\), let us perturb the action with it, setting the initial coupling \(y_n = y_n \delta_{n,q}\) in Eq. (3.4). Then it turns out that

\[
m(l) = \begin{cases} 
2(Dq + 1/q) l + O(1), & \text{for } q < 1/\sqrt{D}, \\
4\sqrt{D}(l - \frac{1}{3} \ln l) + O(1), & \text{for } q = 1/\sqrt{D}, \\
4\sqrt{D}(l - \frac{3}{2} \ln l) + O(1), & \text{for } q > 1/\sqrt{D},
\end{cases}
\]

(4.5) The average value of \(u\) can be evaluated as \cite{20, 32}

\[
\langle u \rangle_p \sim m(l) - \left(1 - \frac{g_M}{\pi}\right) l.
\]

(4.6) Using this, we define the typical value of \(y_n\) as

\[
y_{\text{typ},n}(l) = -\frac{2}{\pi n!} e^{n(u_p)}.
\]

(4.7) It is readily seen that \(y_{\text{typ},n}\) satisfies the following “scaling equation”

\[
\frac{dy_{\text{typ},n}}{dl} = n \left[ \frac{dm(l)}{dl} - \left(1 - \frac{g_M}{\pi}\right) y_{\text{typ},n} \right]
\]

(4.8) To be explicit, Eq. (4.8) yields
\begin{align}
z_n &= \begin{cases} 
2 - (1 - \frac{2 \alpha_n}{\pi}) n + 2 D n^2, & \text{for } n < 1/\sqrt{D}, \\
4\sqrt{D} \left(1 - \frac{1}{\sqrt{D}}\right) n - (1 - \frac{2 \alpha_n}{\pi}) n, & \text{for } n = 1/\sqrt{D}, \\
4\sqrt{D} \left(1 - \frac{1}{\sqrt{D}}\right) n - (1 - \frac{2 \alpha_n}{\pi}) n, & \text{for } n > 1/\sqrt{D}. 
\end{cases}
\end{align}

Notice that the first line of the equation reproduces the naive dynamical exponent of \( y_n \), i.e., \( z_n = 2 - x_n \), where \( x_n \) is given by Eq. (4.13). We denote the exponent thus defined as \( z_w, n \), which we shall discuss in Sec. \( \text{VCl} \). This result implies that effective scaling dimensions of \( y_n \) can be modified due to the fusion of the energy terms.

Let us now take account of the dependence of \( D \) on \( l \) through \( g_A \). It should be stressed that even if one takes into account the \( l \)-dependence of \( D \) in deriving Eq. (5.1), from Eq. (1.8), the following calculations hold. For sufficiently large \( l \), \( z_n \) is given by the last of the selection rule above, since \( 1/\sqrt{D} \sim 1/\sqrt{l} \ll 1 \). Namely, we have

\begin{align}
z_n &\sim 4 n \sqrt{D} \\
&= n z(l), \\
z(l) &\sim \text{const.} \left( \frac{g_M}{\pi} \right) l^{rac{2}{\beta}},
\end{align}

and using this, we define the typical scaling exponent \( x_{t, n} \) of the operator \( Y_{n, n}^\prime \) as

\begin{align}
x_{t, n}(l) &= 2 - z_n \sim 2 - n z(l), \quad (l \to \infty).
\end{align}

There are two differences between \( z_n \) and \( z_w, n \): One is the \( l \)-dependence, which is essential to the behavior of the DOS, and the other is the \( n \)-dependence, which plays a role in the spatial correlation functions of the IPR.

V. THE RG EQUATIONS

In order to calculate the DOS, IPR, and their spatial correlations, we utilize the RG equations for the correlation functions \( \Gamma^{\psi, \phi^2}(x - y) \). Unfortunately, the equations are too hard to solve, so that we calculate their “typical” values based on the scaling equations for typical exponents derived in the previous section.

A. One-point functions

In this subsection, we first derive the RG equation for \( \Gamma^{\psi, \psi^0} \) to calculate the DOS and the IPR. Recall that \( Y_{n, n}^\prime \) includes two fermion correlation functions such as \( \langle \psi_{R_1, a}(x) \psi_{L_2, a}(x) \rangle \), which diverges for a finite \( \omega \). In order to get finite renormalized correlation functions, point-splitting the fields is necessary, \( \langle \psi_{R_1, a}(x) \psi_{L_2, a}(x + a) \rangle \). Then \( \langle Y_{n, n}^0(x) \rangle \) does not depend on \( x \) but does on the cut-off \( a \). Therefore under the scale transformation \( a \to (1 + dl)a \), it turns out that \( \langle Y_{n, n}^\prime \rangle \) obeys the RG equations

\begin{align}
\left( \frac{\partial}{\partial l} - \beta_A \frac{\partial}{\partial g_A} - \beta_A \frac{\partial}{\partial g_A} + \gamma_q \right) \langle Y_{n, n}^\prime \rangle = 0, \quad (5.1)
\end{align}

where the beta functions are defined in Eqs. (4.10) and (5.11), and the \( (\text{matrix of}) \) anomalous dimension \( \gamma_q \) is given by

\begin{align}
\gamma_q Y_{n, n}^\prime = x_q Y_{n, n}^\prime + 2 \pi \sum_{n=1}^{m-q} Y_{n, n}^\prime, \quad (5.2)
\end{align}

with \( x_q \) defined in Eq. (3.15). Since these equations are coupled together, it may be difficult to solve and take the replica limit.

One of possible approximations is to neglect the fusion of the operators \( Y_{n, n}^\prime \) in Eqs. (4.10) and (5.11). This approximation corresponds to the conventional weak disorder approaches. The anomalous dimension \( \gamma_q \) becomes in this case trivially diagonal and the RG equation is readily integrated. The results thus obtained are summarized in Sec. \( \text{VCl} \). However, this scheme is not able to grasp essential strong disorder features the present model should have.

An alternative way is to compute the “typical” values of the correlation function, since we have been able to calculate \( y_{t, n} \) in the last section. This includes strong disorder effects, as it should be. Assuming that these typical values \( y_{t, n} \) correspond to the typical values of the correlation functions, which we denote as \( Y_{n, n}^\prime \), one can simplify the RG equations above for these typical values, since the scaling equations for \( y_{t, n} \) in Eq. (4.13) are “diagonalized”, and so is the matrix of anomalous dimension \( \gamma_n \). The RG equations for the typical values should be

\begin{align}
\left( \frac{\partial}{\partial l} - \beta_A \frac{\partial}{\partial g_A} + x_{t, q} \right) \langle Y_{n, n}^\prime \rangle = 0.
\end{align}

We are then led to

\begin{align}
\langle Y_{n, n}^\prime \rangle_{t, n} \sim \exp \left[ - \int \, dl' x_{t, q}(l') \right].
\end{align}

On the other hand, recall that the energy \( E \) is related to \( y_1 \) as \( y_1 = \omega - iE \). Corresponding to \( y_{t, n} \), let us define the typical energy \( y_{t, n} = \omega_{t, n} - iE_{t, n} \). From Eq. (4.13), it follows that

\begin{align}
\frac{A_{t, n}}{E_{t, n}} \sim \exp \left[ \int \, dl' z(l') \right] = \exp \left[ \text{const.} \left( \frac{g_M}{\pi} \right) l^\frac{2}{\beta} \right],
\end{align}

(5.5)
where $\Lambda_{\text{typ}}$ is a renormalized energy and const. denotes a nonuniversal positive constant. Hence, as a function of $E_{\text{typ}}$, we can rewrite the typical DOS $\rho_{\text{typ}} = (\gamma_{\text{mr}}^1)_{\text{typ}}$ into

$$\rho_{\text{typ}}(E_{\text{typ}}) \sim \frac{\Lambda_{\text{typ}}}{E_{\text{typ}}} \exp \left[ -c \left( \ln \frac{\Lambda_{\text{typ}}}{E_{\text{typ}}} \right)^{\kappa} \right], \quad (5.6)$$

with

$$c \sim \text{const.} \left( \frac{\pi}{2M} \right), \quad \kappa = \frac{2}{3}. \quad (5.7)$$

This has been derived for the first time by Motrunich et al. [10] with the help of a strong coupling expansion, and rederived by Mudry et al. [32] by using the supersymmetry method. What is responsible for this value of $\kappa$ is the velocity selection rule (4.9) and resultant dynamical exponents (4.10). Recent numerical calculations [33] support, to be sure, the divergent behavior of the DOS, but the nonuniversal constant in Eq. (5.7) may be too small for us to observe the exponent $\kappa$ in the numerical calculations, unfortunately.

To overcome the difficulty, we calculate the IPR for general $q$, which could make the strong disorder features manifest. For general $q$, we have

$$\omega_{\text{typ}}^{q-1}(\gamma_{\text{mr}}^q)_{\text{typ}} \sim \rho_{\text{typ}}(l), \quad (5.8)$$

which follows from

$$(q-1)z + x_{\text{typ},q} = (q-1)z + (2 - qz)$$

$$= 2 - z$$

$$= x_{\text{typ},1}. \quad (5.9)$$

Therefore, we reach the following IPR:

$$\bar{P}_{\text{typ}}^{(q)}(E) \sim \text{const.} \quad (5.10)$$

This implies that the multifractal scaling exponent $\tau(q)$, defined by $P(q) \sim L^{-\tau(q)}$, is given by

$$\tau(q) = 0, \quad (5.11)$$

for the typical values. This result is also attributed to the velocity selection rule (4.9) and therefore the strong disorder features of the present model. For references, we summarize in Sec. B the results of a naive weak disorder approach.

### B. Two-point functions

Constant IPR obtained in the last subsection usually implies a localized wavefunction. However, it is believed in general that the zero-energy wavefunction of chiral models is extended. To clarify this point, we calculate the spatial correlations of the IPR in this subsection. The correlation function $\Gamma(q_1,q_2)(x-y) = \langle \gamma_{\text{mr}}^{q_1}(x)\gamma_{\text{mr}}^{q_2}(y) \rangle$ obeys the following OPE

$$\langle \gamma_{\text{mr}}^{q_1}(x)\gamma_{\text{mr}}^{q_2}(y) \rangle \sim C_{q_1q_2}(x-y)(\gamma_{\text{mr}}^{q_1+q_2}(y)). \quad (5.12)$$

This may be valid for $a \ll |x-y| \ll L$, where $L$ is a system size, and we will determine the $r = |x-y|$-dependence of the coefficient $C_{q_1q_2}$. It is easily verified that the coefficient $C_{q_1q_2}$ obeys the RG equations,

$$\left( r \frac{\partial}{\partial r} - \beta_B \frac{\partial}{\partial q_B} + X_{\text{typ},q_1+q_2} \right) C_{q_1q_2}(r) - C_{q_1,q_2}(r) \gamma_{q_1+q_2} = 0, \quad (5.13)$$

where $\gamma$ is defined in Eq. 4.22. These coupled set of equations become tractable for the typical values:

$$\left( r \frac{\partial}{\partial r} - \beta_B \frac{\partial}{\partial q_B} + X_{\text{typ},q_1+q_2} \right) C_{q_1,q_2}(r) = 0, \quad (5.14)$$

where $X_{\text{typ},q_1+q_2} = x_{\text{typ},q_1} + x_{\text{typ},q_2} - x_{\text{typ},q_1+q_2}$. For sufficiently large $l$, it follows from Eqs. (4.10) and 4.11 that $X_{\text{typ},q_1+q_2} \sim 2$. Thus we obtain

$$\bar{P}_{\text{typ}}^{(q_1,q_2)}(x-y) \sim \text{const.} \frac{1}{(|x-y|^2)^{\kappa}}, \quad (5.15)$$

where we have used that previous result 5.10. Remarkably, the result shows that the exponent is a constant, 2, independent of $q_1$ and $q_2$ as well as the disorder strength. Since the correlation functions show power-law behavior, the wavefunction should not be localized.

### C. Weak disorder approximation

So far we have obtained several formulas taking into account strong disorder effects. As has been discussed by Motrunich et. al. as well as Mudry et. al., the exponent $\kappa$ in the DOS is $\kappa = 2/3$, which is different from $\kappa = 1/2$ obtained previously by Gade with the help of a naive weak disorder approach [7]. In order to clarify the strong disorder features of the IPR and their spatial correlations obtained in the previous subsection, we calculate them within the conventional weak disorder approach, for reference.

Without considering the fusion process of higher energy terms, the dynamical exponents of $\gamma^n$ is given just
by the first line of Eq. \ref{eq:10}. Namely,
\[ z_{w,n} \sim 2Dn^2 = n^2z_w(l), \]
\[ z_w(l) \sim \text{const.} \left( \frac{g_M}{\pi} \right) l. \tag{5.16} \]
Here, the label \( w \) means exponents within a weak disorder approximation. Note the difference of the \( l \)-dependence between Eqs. \ref{eq:10} and \ref{eq:16}: This reflects the exponent \( \kappa \) in the DOS. Neglecting the fusion of the operators \( \langle \gamma_{nn}^q \rangle \), we find that the RG equation for \( \langle \gamma_{nn}^q \rangle \) is diagonal, given by Eq. \ref{eq:3} but with the naive scaling dimension \( x_q \sim 2 - q^2z_w(l) \) in Eq. \ref{eq:15}. Therefore, together with
\[ \frac{\Lambda}{E} \sim \exp \left[ \int dl' z_w(l') \right] \]
\[ = \exp \left[ \text{const.} \left( \frac{g_M}{\pi} \right) l^2 \right]. \tag{5.17} \]
we have the same DOS \ref{eq:6} but with
\[ \kappa_w = \frac{1}{2}. \tag{5.18} \]

Next, let us consider the correlation functions for general \( q \). The solution of the RG equations yields
\[ \omega^{q-1} \langle \gamma_{nn}^q \rangle / \rho \]
\[ \sim \exp \left\{ - \int dl' \left[ (q - 1)z_w - x_q + x_1 \right] \right\} \]
\[ = \exp \left[ q(q - 1) \int dl' z_w(l') \right], \tag{5.19} \]
from which, together with Eq. \ref{eq:17}, we reach
\[ \bar{F}_w^{(q)}(E) \sim \left( \frac{E}{\Lambda} \right)^{-q(q-1)}. \tag{5.20} \]
Negative exponent in the above equation suggests that the present weak disorder approach is incorrect and the result in the last subsection \ref{eq:10} is more reasonable. Since we are dealing with the same model in the same formalism, this result also implies that the exponent \( \kappa \) in the DOS should be \( \kappa = 2/3 \) rather than \( 1/2 \).

In a similar way, we reach,
\[ \bar{F}_w^{(q_1,q_2)}(x - y) \sim \exp \left[ -c_1q_2 (\ln |x - y|)^2 \right] \times \bar{F}_w^{(q_1+q_2)}(E), \tag{5.21} \]
where \( c \) is given by the same disorder-dependent constant as in Eq. \ref{eq:7}.

VI. CONCLUDING REMARKS

In this paper we have reconsidered random hopping fermion models, based on the recent developments of the RG method for a strong disorder regime. We have used the replica method and calculated the DOS, IPR, and their spatial correlations. We have been able to reproduce the DOS predicted by Motrunich et. al. as well as Mudry et. al. Moreover, we have calculated the IPR and their spatial correlations, whose behavior near the zero energy is also different from the ones predicted by using conventional weak disorder approach.

In the present calculations, including higher powers of the energy perturbation (or higher fugacity terms in the context of Coulomb gas model) plays a crucial role, which has been invented for the first time by Carpentier and Le Doussal and applied to the present model by Mudry et. al.

Although recent numerical calculations have supported the divergent DOS toward the zero energy, it may be difficult to distinguish the exponent \( \kappa \). We hope that IPR and their correlations calculated in the present paper may be useful to establish the strong disorder features of the random hopping models.

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