Segre varieties and Lie symmetries

Alexandre SUKHOV

Univesité des Sciences et Technologies de Lille, Laboratoire d’Arithmétique - Géométrie - Analyse - Topologie, Unité Mixte de Recherche 8524, U.F.R. de Mathématique, 59655 Villeneuve d’Ascq, Cedex, France.
e-mail: Alexandre.Sukhov@agat.univ-lille1.fr

Abstract. We show that biholomorphic automorphisms of a real analytic hypersurface in $\mathbb{C}^{n+1}$ can be considered as (pointwise) Lie symmetries of a holomorphic completely overdetermined involutive second order PDE system defining its Segre family. Using the classical S.Lie method we obtain a complete description of infinitesimal symmetries of such a system and give a new proof of some well known results of CR geometry.

1 Introduction and result

In the present paper we apply the Lie method of studying PDE symmetries to a special but geometrically important class of holomorphic completely overdetermined second order PDE systems, i.e. systems of the form

$$(S): u^k_{x_i x_j} = F^k_{ij}(x, u, u^{(1)})$$

where $x = (x_1, ..., x_n)$ are independent variables, $u(x) = (u^1(x), ..., u^m(x))$ are unknown holomorphic functions (dependent variables), $u_x = (u^i_{x_j})$ and $F^k_{ij} = F^k_{ji}$ are holomorphic functions.

Denote by $J^r_{n,m}$ the $r$-jet space of $r$-jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$. Set $u^{(1)} = (u^1_1, ..., u^1_n, ..., u^m_1, ..., u^m_n)$, $u^{(s)} = (u^j_\tau)$ with $j = 1, ..., m$, $\tau = (\tau_1, ..., \tau_s)$, $\tau_1 \leq \tau_2 \leq ... \leq \tau_s$ and use it as the natural coordinates $(x, u, u^{(1)}, ..., u^{(r)})$ on the jet space: for every holomorphic (near a point $p$) function $u = f(x) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ the natural coordinates of the corresponding $r$-jet $j^r_p(f) \in J^r_{n,m}$ defining by $f$ at $p$ are $x_j = p_j$, $u^k = f^k(p)$, $u^k_{\tau_1, ..., \tau_s} = \frac{\partial^s f^k(p)}{\partial x_{\tau_1} ... \partial x_{\tau_s}}$.

We say that a system $[S]$ is involutive if the differential forms $du^k_i - \sum_j F^k_{ij}(x, u, u^{(1)})dx_j$, $du^k - \sum_i u^k_i dx_i$ define a completely integrable distribution on the tangent bundle $T(J^1_{n,m})$, i.e. satisfy the Frobenius involutivity condition.

Solutions of such a system are holomorphic vector valued functions $u = u(x)$; denote by $\Gamma_u$ the graph of a solution $u$. A symmetry group $\text{Sym}(S)$ (see for instance [1], [8]) of a system is a local complex transformation group $G$ acting on a domain in the space $\mathbb{C}^n_x \times \mathbb{C}^m_u$ of independent and dependent variables with the following property: for every solution $u(x)$ of $(S)$ and every $g \in G$ such that the image $g(\Gamma_u)$ is defined, it is a graph of a solution of $(S)$. Sometimes the largest symmetry group $\text{Sym}(S)$ is of main interest (and so
we write the symmetry group); for us this is not very essential since our methods give a description of any symmetry group for given system. A holomorphic vector field

\[ X = \sum_j \theta_j(x, u) \frac{\partial}{\partial x_j} + \sum \eta^\mu \frac{\partial}{\partial u^\mu}, \]  

(2)
generating a complex one-parameter group of symmetries of a system of PDE (\( \mathcal{S} \)) is called an infinitesimal symmetry of this system. All these fields form a complex Lie algebra with respect to the Lie bracket which is denoted by \( \text{Lie}(\mathcal{S}) \).

Using the notation \( w = (x, u) \), we fix a point \((x, u)\) and set

\[
\begin{align*}
\alpha_j(x, u) &= (\theta_{j_{\omega_1}}(x, u), ... , \theta_{j_{\omega_{n+m}}}(x, u)), \\
\alpha(x, u) &= (\alpha_1, ..., \alpha_n), \\
\beta^k(x, u) &= (\eta^k_{\omega_1}(x, u), ... , \eta^k_{\omega_{n+m}}(x, u)), \\
\beta(x, u) &= (\beta^1, ..., \beta^m), \\
\gamma(x, u) &= (\theta_{1_{\omega_1}}(x, u), ... , \theta_{1_{\omega_{n+m}}}(x, u)), \\
\delta(x, u) &= (\eta^1(x, u), ... , \eta^m(x, u)), \\
\varepsilon(x, u) &= (\theta_1(x, u), ... , \theta_n(x, u))
\end{align*}
\]

We call the vector \( \omega(x, u) = (\alpha(x, u), \beta(x, u), \gamma(x, u), \delta(x, u), \varepsilon(x, u)) \)
of \( \mathbb{C}^{(n+m+2)(n+m)} \) the initial date of an infinitesimal symmetry \( X \) of the form (2) at the point \((x, u)\).

Our main result is the following

**Theorem 1.1** Let \( \mathcal{S} \) be a holomorphic completely overdetermined second order involutive system with \( n \) independent and \( m \) dependent variables. Then the Taylor expansions of coefficients of any infinitesimal symmetry \( X \in \text{Lie}(\mathcal{S}) \) at a fixed point \((x, u)\) are uniquely determined by the initial date \( \omega(x, u) \) that is the linear map \( \text{Lie}(\mathcal{S}) \rightarrow \mathbb{C}^{(n+m+2)(n+m)} \) defined by \( X \mapsto \omega(x, u) \) is injective.

In the special case \( n = 1, m = 1 \) i.e. in the case of ordinary second order differential equation this result was obtained by A.Tresse [11] (a student of S.Lie) in 1896: he proved that the symmetry group of a second order differential equation is a Lie group of dimension \( \leq 8 \) (as it was observed by B.Segre, this implies that the automorphism group of a real analytic Levi nondegenerate hypersurface in \( \mathbb{C}^2 \) is a finite dimensional real Lie group). A very clear proof of the same fact is contained in the known paper of L.E. Dickson [5] (another former student of S.Lie) inspired by the lectures of S.Lie at the end of the XIX century. So it is quite possible that the basic idea goes back to S.Lie himself. Our proof is based on the direct generalization of the Lie - Tresse - Dickson method.

We point out that this method is quite elementary and constructive and gives an efficient recursive algorithm for determination of the Taylor expansions of the coefficients.
of an infinitesimal symmetry at a given point. The symmetry group of \((S)\) then can be parametrized by the exponential map (in a suitable neighborhood of the identity). If a point \((x, u)\) is fixed, the components of the initial date \(\omega(x, u)\) are local parameters for the symmetry group. The number of these parameters is equal to \((n + m + 2)(n + m)\); however, in general they are not independent. In our proof of theorem we consider only those equations on the coefficients of an infinitesimal symmetry which are necessary in order to conclude. In general, additional equations can occur. So in the general case the parameters may satisfy some additional relations and the actual dimension of a symmetry group may be smaller than \((n + m + 2)(n + m)\).

**Corollary 1.2** A symmetry group of a holomorphic completely overdetermined involutive second order system with \(n\) independent and \(m\) dependent variables is a local complex Lie transformation group of dimension \(\leq (n + m + 2)(n + m)\).

We will show that this estimate is precise. In the special case \(m = 1\) this last result can also be deduced from Chern’s solution of the equivalence problem for completely overdetermined involutive second order systems with one dependent variable \([4]\).

In the next section we consider applications of theorem 1.1 to CR geometry.

### 2 Segre varieties, holomorphic maps and PDE symmetries

Let \(\Gamma\) be a real analytic Levi nondegenerate hypersurface in \(\mathbb{C}^{n+1}\) and let \(\text{Aut}(\Gamma)\) denote its biholomorphism group (all our considerations are local). Denote by \(Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}\) the standard coordinates in \(\mathbb{C}^{n+1}\). For a fixed point \(\zeta \in \mathbb{C}^{n+1}\) close enough to \(\Gamma\) consider the complex hypersurface \(Q(\zeta) = \{Z : r(Z, \zeta) = 0\}\). It is called the Segre variety \([9]\).

The basic property of the Segre varieties \([8, 12]\) is their biholomorphic invariance: for every automorphism \(f \in \text{Aut}(\Gamma)\) and any \(\zeta\) one has \(f(Q(\zeta)) = Q(f(\zeta))\). B.Segre observed that for \(n = 1\) i.e. in \(\mathbb{C}^2\) the set of Segre varieties of \(\Gamma\) (which is called the Segre family of \(\Gamma\)) is a regular two parameter family of holomorphic curves and so represents the trajectories of solutions of a holomorphic second order ordinary differential equation (see also \([8, 11, 12]\)). This important observation can be generalized as follows.

After a biholomorphic change of coordinates in a neighborhood of the origin \(\Gamma\) is given by the equation \(\{w + i\bar{\tau} + \sum_{j=1}^n \epsilon_j z_j \bar{z}_j + R(Z, \bar{Z}) = 0\}\) where \(\epsilon_j = 1\) or \(-1\) and \(R = o(|Z|^2)\).

For every point \(\zeta\) the corresponding Segre variety is given by \(w + \zeta_{n+1} + \sum_{j=1}^n \epsilon_j z_j \zeta_j + R(Z, \zeta) = 0\). If we consider the variables \(x_j = z_j\) as independent ones and the variable \(w = u(x)\) as dependent one, then this equation can be rewritten in the form

\[
u + \zeta_{n+1} + \sum_{j=1}^n \epsilon_j x_j \zeta_j + R((x, u), \zeta) = 0 \tag{3}\]

Taking the derivatives in \(x_k\) we obtain the equations

\[
u_{x_k} + \epsilon_k \zeta_k + R_{x_k}(x, u, \zeta) + R_u(x, u, \zeta)u_{x_k} = 0, k = 1, \ldots, n \tag{4}\]
The equations (3), (4) and the implicit function theorem imply that $\phi$ is an analytic function $\phi = \phi(x, u, u_1, ..., u_n)$; taking again the partial derivatives in $x_j$ in (4) and using the obtained expression for $\phi$ in order to eliminate it from the equations, we obtain that every $u$ given by (3) is a solution of a completely overdetermined second order holomorphic PDE system $(S_T)$ of the form (I). This system is necessarily involutive since the family of its solutions (3) define a completely integrable distribution on the tangent space $T(J^1_{n,1})$. The property of biholomorphic invariance of the Segre varieties means that any biholomorphism of $\Gamma$ transforms the graph of a solution of $(S_T)$ to the graph of another solution, i.e. is a Lie symmetry of $(S_T)$. So the study of $Aut(\Gamma)$ can be reduced as a very special case to the general problem of study of Lie symmetries of holomorphic completely overdetermined second order involutive systems. We emphasize that the systems describing the Segre families form a very special subclass in the class of holomorphic completely overdetermined second order involutive systems with one dependent variable. So theorem (I) generalize known results in several directions.

Indeed, since $Sym(S_T)$ is a complex Lie transformation group in view of our theorem and $Aut(\Gamma)$ is its closed subgroup, we conclude that it is a local real Lie transformation subgroup of $Sym(S_T)$. Hence theorem (I) (which we apply in the special case of one dependent variable) gives an upper estimate for its dimension: the real dimension of $Aut(\Gamma)$ is majorated by $2dim Sym(S_T)$, in particular, by $2(n^2 + 4n + 3)$. In order to improve this estimate, we recall the following useful observation due to E.Cartan [2]. Let a holomorphic vector field $X$ generate a local real one-dimensional subgroup of $Aut(\Gamma)$. This is equivalent to the fact that $Re X$ is a tangent vector field to $\Gamma$. Since this subgroup is a real one-parameter subgroup of $Sym(S_T)$, we have necessarily $X \in Lie(\Gamma)$. $\Gamma$ is Levi nondegenerate, so the field $Re(iX)$ cannot be tangent to $\Gamma$ simultaneously with $Re X$ i.e. $Lie(\Gamma)$ is a totally real subspace of $Lie(S_T)$. Therefore, the real dimension of $Aut(\Gamma)$ is majorated by the complex dimension of $Lie(S_T)$. In particular, it is smaller that $n^2 + 4n + 3$. The example of the sphere shows that this estimate is precise. We obtain the following

**Corollary 2.1** $Aut(\Gamma)$ is a local real Lie transformation subgroup of $Sym(S_T)$ and the dimension of $Aut(\Gamma)$ is majorated by the complex dimension of $Sym(S_T)$. In particular, it is always majorated by $n^2 + 4n + 3$. Moreover, every infinitesimal automorphism of $\Gamma$ is uniquely determined by its second order Taylor development at a fixed point.

Thus, we find here some classical results of N.Tanaka [I], S.S.Chern - J.Moser [3] in the infinitesimal form.

# 2. Lie method and proof of the main theorem

Let $G$ be a local group of biholomorphic transformations acting on a domain in $\mathcal{C}^n \times \mathcal{C}^m$. Every biholomorphism $g \in G$, $g : (x, u) \mapsto (x^*, u^*)$ close enough to the identity lifts canonically to a fiber preserving biholomorphism $g^{(r)} : J^r_{n,m} \longrightarrow J^r_{n,m}$ as follows: if $u = f(x)$ is a holomorphic function near $p$, $q = f(p)$ and $u^* = f^*(x^*)$ is its image under $g$ (i.e. the graph of $f^*$ is the image of the graph of $f$ under $g$ near the point $(p^*, q^*) = g(p, q)$), then the jet $j_{p^*}^r(f^*)$ is by the definition the image of $j_p^r(f)$ under $g^{(r)}$. In particular, a
one-parameter local Lie group of transformations $G$ canonically lifts to $J^{(r)}_{n,m}$ as a one-parameter Lie group of transformations $G^{(r)}$ which is called the $r$-prolongation of $G$. The infinitesimal generator $X^{(r)}$ of $G$ is called the $r$-prolongation of the infinitesimal generator $X$ of $G$.

Let $(S)$ be a holomorphic PDE system of $r$th order with $n$ independent and $m$ dependent variables. Then it defines naturally a complex subvariety $(S)^{(r)}$ in the jet space $J^{(r)}_{n,m}$ obtained by replacing of derivatives of dependent variables by the corresponding coordinates in the jet space. So $u$ is a holomorphic solution of the system $(S)$ if and only if the section $(p, u(p), j^r_u(u))$ of the holomorphic fibre bundle $\pi_n : J^{(r)}_{n,m} \rightarrow \mathbb{C}^n$ (with the natural projection $\pi_n$) is contained in the variety $(S)^{(r)}$. A key proposition of the Lie theory states that if the r-prolongation $X^{(r)}$ of a vector field $X$ is a tangent field to $(S)^{(r)}$ then $X$ is an infinitesimal symmetry of $(S)$ ([8], Theorem 2.31).

For the system (1) one has

$$(S)^{(2)} : u^k_{ij} = \hat{F}^k_{ij}(x, u, u^{(1)})$$

Here $\hat{F}^k_{ij}$ denote the natural lifting of $F^k_{ij}$ to the jet space obtained via the replacing of the derivatives $u^\mu_{x_i}$ by the jet coordinates $u^\mu_i$ in power expansions of $F^k_{ij}$, i.e. $\hat{F}^k_{ij} = F^k_{ij}(x, u, u^{(1)})$.

In the natural coordinates one has

$$X^{(r)} = X + \sum_{j,\mu} \eta^\mu_i \frac{\partial}{\partial u^\mu_j} + \ldots + \sum_{i_1,\ldots,i_r,\mu} \eta^\mu_{i_1\ldots i_r} \frac{\partial}{\partial u^\mu_{i_1\ldots i_r}}$$

where $\eta^\mu_i = D_i \eta^\mu - \sum_j (D_j \theta_j) u^\mu_{i j}$, $\eta^\mu_{i_1\ldots i_r} = D_{i_1} \eta^\mu_{i_2\ldots i_r} - \sum_j (D_j \theta_j) u^\mu_{i_1\ldots i_r}$ and

$$D_i = \frac{\partial}{\partial x_i} + \sum_k u^k_i \frac{\partial}{\partial u^k} + \sum_{\mu, j} u^\mu_{ij} \frac{\partial}{\partial u^\mu_j} + \ldots$$

is the operator of total derivative ([3], [8]).

In our case a direct computation gives an explicit expression for the coefficients of $X^{(2)}$:

$$\eta^\mu_{i_1} = \frac{\partial \eta^\mu}{\partial x_{i_1}} + \sum_k u^k_{i_1} \frac{\partial \eta^\mu}{\partial u^k} - \sum_j \left( \frac{\partial \theta_j}{\partial x_{i_1}} + \sum_k u^k_{i_1} \frac{\partial \theta_j}{\partial u^k} \right) u^\mu_j,$$

$$\eta^\mu_{i_1i_2} = \frac{\partial^2 \eta^\mu}{\partial x_{i_2} \partial x_{i_1}} + u^\mu_{i_1} \left[ \frac{\partial^2 \eta^\mu}{\partial x_{i_2} \partial u^k} - \frac{\partial^2 \theta_{i_1}}{\partial x_{i_2} \partial x_{i_1}} \right] + u^\mu_{i_2} \left[ \frac{\partial^2 \eta^\mu}{\partial x_{i_1} \partial u^k} - \frac{\partial^2 \theta_{i_2}}{\partial x_{i_2} \partial x_{i_1}} \right]$$

$$+ \sum_{k \neq \mu} u^k_{i_1} \frac{\partial^2 \eta^\mu}{\partial x_{i_2} \partial u^k} + \sum_{k \neq \mu} u^k_{i_2} \frac{\partial^2 \eta^\mu}{\partial x_{i_1} \partial u^k} - \sum_{k \neq i_1, k \neq i_2} u^\mu_{i_1} \frac{\partial^2 \theta_k}{\partial x_{i_2} \partial x_{i_1}} - \sum_{k : j \neq i_2} u^k_{i_1} \frac{\partial^2 \theta_j}{\partial x_{i_2} \partial x_{i_1}}$$
\[- \sum_{i,s \neq i_1} u^i_{2s} \frac{\partial^2 \theta_s}{\partial x_i \partial u^s} + \sum_{r \neq \mu, p \neq \mu} u^p_{2r} \frac{\partial^2 \eta^p}{\partial x_i \partial u^r} + \sum_{t \neq \mu} u^t_{12} \left[ \frac{\partial^2 \theta_t}{\partial x_i \partial u^t} + \frac{\partial^2 \eta^t}{\partial x_i \partial u^t} \right] + \sum_{q \neq \mu} u^q_{12} \left[ \frac{\partial^2 \theta_q}{\partial x_i \partial u^q} + \frac{\partial^2 \eta^q}{\partial x_i \partial u^q} \right] \]

\[- \sum_{a,b,s} u^a_{2b} u^s_{12} \frac{\partial^2 \theta_s}{\partial x_i \partial u^b} + \Lambda^\mu_{1i2} \]

for \( i_1 \neq i_2 \) and

\[ \eta^\mu_{ii} = \frac{\partial^2 \eta^\mu}{\partial (x_i)^2} + u^i_1 \left[ 2 \frac{\partial^2 \eta^\mu}{\partial (x_i)^2} - \frac{\partial^2 \theta_i}{\partial (x_i)^2} \right] + 2 \sum_{k \neq \mu} u^k_{2i} \frac{\partial^2 \eta^k}{\partial (x_i) \partial u^k} - \sum_{k \neq \mu} u^k_{2i} \frac{\partial^2 \theta_k}{\partial (x_i)^2} \]

\[-2 \sum_{k,j \neq \mu} u^k_{ij} \frac{\partial^2 \theta_j}{\partial (x_i) \partial u^k} + \sum_{r \neq \mu, t \neq \mu} u^t_{12} \frac{\partial^2 \eta^t}{\partial (x_i) \partial u^t} + \sum_{t \neq \mu} u^t_{12} \left[ \frac{\partial^2 \theta_t}{\partial (x_i) \partial u^t} + \frac{\partial^2 \eta^t}{\partial (x_i) \partial u^t} \right] + \sum_{q \neq \mu} u^q_{12} \left[ \frac{\partial^2 \theta_q}{\partial (x_i) \partial u^q} + \frac{\partial^2 \eta^q}{\partial (x_i) \partial u^q} \right] \]

\[- \sum_{a,b,s} u^a_{2b} u^s_{12} \frac{\partial^2 \theta_s}{\partial (x_i) \partial u^b} + \Lambda^\mu_{ii} \]

with

\[ \Lambda^\mu_{1i2} = \sum_s u^s_{12} \frac{\partial \eta^\mu}{\partial u^s} - \sum_p u^p_{12} \frac{\partial \theta_p}{\partial x_i} - \sum_j u^a_{12} \frac{\partial \theta_j}{\partial x_i} - \sum_{q,j} u^q_{12} u^s_{2j} \frac{\partial \theta_j}{\partial u^q} \]

Since the system (1) is involutive, for every point \( P \in (S)^{(2)} \) with the natural projection \( \pi_{n,m}(P) = (p,q) \in \mathbb{C}^n \times \mathbb{C}^m \) there exists a solution \( u(x) \) of (S) holomorphic near \( P \) such that \( (p,q,j^p_p(u)) = P \). So the Lie criterion (8), Theorem 2.72) implies that \( X \in \text{Lie}(S) \) if and only if the second prolongation satisfies the following system of equations in \( J^2_{n,m} \):

\[ X^{(2)}(u^\mu_{ij} - \dot{F}^\mu_{ij}) = 0, u^\mu_{ij} = \dot{F}^\mu_{ij} \]

This system implies that \( \eta^\mu_{ij} = X^{(2)}(\dot{F}^\mu_{ij}) = X^{(1)}(\dot{F}^\mu_{ij}) \). Replace now in the expressions of \( \Lambda^\mu_{1i2} \) the jet coordinates \( u^\mu_{ij} \) by \( \dot{F}^\mu_{ij} \) and denote obtained functions by \( \dot{\Lambda}^\mu_{ij} \). Transfer them to the right side; we get the equations of the form

\[ \dot{\eta}^\mu_{ij} = X^{(1)}(\dot{F}^\mu_{ij}) - \dot{\Lambda}^\mu_{ij} \]  

(5)
Without loss of generality assume that every $\hat{F}_{ij}^\mu$ is represented by a power series with respect to $u_i^k$. Then we can develop the right sides $X^{(1)}(\hat{F}_{ij}^\mu) - \hat{A}_{ij}^\mu$ of our equations in power series with respect to $u_i^k$. Clearly, the coefficients of these expansions are completely determined by the coefficients of the expansions of $\hat{F}_{ij}^\mu$ and one can effectively compute them in a concrete case.

Comparing now the coefficients near the powers of $u_i^k$ of degree $\leq 3$ in the equations (5) and using explicit expressions for the coefficients of $X^{(2)}$, we obtain the following PDE system for the coefficients of the infinitesimal symmetry $X$:

\[
(A) : \frac{\partial^2 \eta^\mu}{\partial x_i \partial x_i} = A_{ij}^\mu, \quad \frac{\partial^2 \eta^\mu}{\partial x_i \partial u_k} = B_{ik}^\mu, k \neq \mu, \quad \frac{\partial^2 \eta^\mu}{\partial u^r \partial u^p} = C_{rp}^\mu, r \neq \mu, p \neq \mu,
\]

\[
(B) : \frac{\partial^2 \theta^k}{\partial x_i \partial x_i} = D_{ik}^\mu, k \neq i, k \neq i_2, \quad \frac{\partial \theta^k}{\partial x_i} = E_{i}^j, j \neq i, (C) : \frac{\partial^2 \theta^k}{\partial u^r \partial u^p} = G_{ab}^r
\]

\[
(D_1) : \frac{\partial^2 \eta^\mu}{\partial x_i \partial u^\mu} - \frac{\partial^2 \theta^k}{\partial x_i \partial x_{i_1}} = H_{i_1}^\mu, i_1 \neq i_2, (D_2) : \frac{\partial^2 \eta^\mu}{\partial u^r \partial u^p} - \frac{\partial^2 \theta^k}{\partial x_{i_1} \partial u^t} = I_{i_1}^\mu, t \neq \mu
\]

\[
(D_3) : 2\frac{\partial^2 \eta^\mu}{\partial x_i \partial u^\mu} - \frac{\partial^2 \theta^k}{(\partial u^\mu)^2} = J_i^\mu, (D_4) : \frac{\partial^2 \eta^\mu}{(\partial u^\mu)^2} - 2\frac{\partial \theta^k}{\partial x_i} = K_i^\mu
\]

where the right sides are analytic functions in $\lambda(x, u) = (x, u, \alpha(x, u), \beta(x, u), \delta(x, u), \varepsilon(x, u))$. We point out that we do not write here all obtained equations; we consider only those who will be enough for the proof of our results.

Denote by $\Omega$ a holomorphic vectorvalued function whose components coincide with the right sides of our system: $\Omega = (A_{i_1 i_2}^\mu, B_{i k}^\mu, ..., K_i^\mu)$.

An important property of obtained PDE system $(A) - (D_4)$ is its linearity with respect to the second order derivatives of dependent variables. Denote by $v$ the vector $\mathbb{C}^L$ (for a suitable $L$) whose components are the second order partial derivatives of $\theta_j$ and $\eta^\mu$; then our system can be written in the form $Mv = \Omega$, where $M$ is an integer matrix. An elementary linear algebra argument shows that this system can be represented in the form $M'v' = P\gamma + \Omega$, where $v'$ is a vector formed by components of $v$ which are not components of $\gamma$. $P$ is an integer matrix and $M'$ is an invertible square integer matrix. Therefore $v' = M^{-1}P\gamma + M^{-1}\Omega$, so every second order partial derivative of $\theta_j$, $\eta^k$ is a linear combination of components of $\gamma$, $\Omega$. In particular, the second order partial derivatives of $\theta_j$, $\eta^k$ at $(x, u)$ are determined by $\omega(x, u)$. Denote by $V$ the vector whose components are the third order partial derivatives of $\theta_j$, $\eta^k$ and write the system obtained by taking the partial derivatives in $(A) - (D_4)$ in the form $NV = \Omega'$ where $\Omega'$ denote the vector with components $\frac{\partial \omega^k}{\partial x_i}, \frac{\partial \omega^k}{\partial u^\mu}$ and $N$ is an integer matrix. A direct computation shows that one can choose a subsystem of this system with an invertible square matrix $N'$, so we obtain that for every multi-indice $\tau$, $|\tau| = 3$, there are polynomials $R_\tau^k$, $S^k_j$ with rational coefficients, such that the following holds:

\[
\frac{\partial^3 \omega^k}{\partial x_1^{\tau_1} \partial x_n^{\tau_n} \partial (u_1)^{\tau_{n+1}} \partial (u_m)^{\tau_{n+m}}}(x, u) = R^k_j((\partial \omega^k)(\lambda(x, u)), (\partial^2 \theta)(x, u), (\partial^2 \eta)(x, u)) \quad (6)
\]
\[ \frac{\partial^3 \eta^k}{\partial x_1^{\tau_1} \cdots \partial x_n^{\tau_n} \partial (u^m)^{\tau_{n+m}}} (x, u) = S^k_k(\lambda(x, u)), (\partial^2 \theta)(x, u), (\partial^2 \eta)(x, u) \] (7)

where \((\partial \Omega)(\lambda(x, u))\) denote the vector function whose components are the first order partial derivatives of \(\Omega_j\) evaluated at \(\lambda(x, u)\), \((\partial^2 \theta)(x, u)\) (resp. \((\partial^2 \eta)(x, u)\)) denote the vector function whose components are the partial derivatives of all \(\theta_j\) (resp. \(\eta^k\)) of order \(\leq 2\). This means that the third order partial derivatives of \(\theta_j, \eta^k\) at \((x, u)\) are determined by \(\omega(x, u)\). If \(\omega(x, u)\) is given now, (6), (7) and the chain rule show that all coefficients of the Taylor expansions of \(\theta_j, \eta^k\) are determined by recursion. This completes the proof of the theorem.

It is also clear from the construction that the corresponding homogeneous system \(Mv = 0\) describes infinitesimal symmetries of the system \((S_0)\) of the form (8) with \(F^k_i \equiv 0\). We obtain that the set \(\text{Lie}(S_0)\) of infinitesimal symmetries of this system is a complex Lie algebra of dimension \((n + m + 2)(n + m)\) generated by the following holomorphic vector fields:

- \(U_k = \frac{\partial}{\partial x_k}\), \(V_{\mu} = \frac{\partial}{\partial u^\mu}\), \(W_{jk} = x_j \frac{\partial}{\partial x_k}\), \(A_{jk} = u_j \frac{\partial}{\partial x_k}\), \(B_{k\mu} = x_k \frac{\partial}{\partial u^\mu}\), \(C_{k\mu} = u_k \frac{\partial}{\partial u^\mu}\), \(X_j = \sum_k x_j x_k \frac{\partial}{\partial x_k} + \sum_{j\mu} x_j u^\mu \frac{\partial}{\partial u^\mu}\), \(Y_{\nu} = \sum_k x_k u^\nu \frac{\partial}{\partial x_k} + \sum_{j\mu} u^\nu u^\mu \frac{\partial}{\partial u^\mu}\), \(k, j = 1, \ldots, n\), \(\mu, \nu = 1, \ldots, m\).

In particular, \(\text{dim} \text{Lie}(S_0) = (n + m + 2)(n + m)\) so the dimension estimate given by our theorem is precise.

In the present paper we restrict an application of the Lie method only by the classical case of a Levi nondegenerate hypersurface. But this method can also be applied in other situations which are of interest for the CR geometry and form an area of research activity of several authors. For instance, the Levi degenerate case leads to considerations of holomorphic second order completely overdetermined involutive PDE systems which are not solved with respect to the second order derivatives. A study of their symmetries requires a combination of the Lie method with some tools of the local complex analytic geometry (compare with \([3, 4]\)). On the other hand, the Segre families of Cauchy - Riemann manifolds of higher codimension are described by holomorphic second order completely overdetermined involutive PDE systems with additional first order relations, i.e. with additional holomorphic equations including first order derivatives of dependent variables. Clearly, the Lie method allows to study this class of systems and just requires more involved computations.

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