A Radon-type transform arising in Photoacoustic Tomography with circular detectors

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Abstract

PAT is the best-known example of a hybrid imaging method. In this article, we define a Radon-type transform arising in a version of PAT that uses integrating circle detectors and describe how the Radon transform integrating over all circles with a fixed radius is determined from this Radon-type transform. Also, the inversion of the Radon transform integrating over all circles is provided. Various configuration of detectors have their own geometry problems. Here we consider three situations: when the centers of the detectors are located on a cylinder, on a plan, and on a sphere.

This transform is similar to a toroidal transform, which maps a given function to its integrals over a set of tori. We also study this mathematically similar object.

1 Introduction

Photoacoustic Tomography (PAT) is a noninvasive medical imaging technique based on the reconstruction of an internal photoacoustic source. Its principle is based on the excitation of high bandwidth acoustic waves with pulsed non-ionizing electromagnetic energy [22, 25, 27]. Ultrasound imaging often has high resolution but displays low contrast. Optical or radio-frequency EM illumination, on the other hand, gives an enormous contrast between unhealthy and healthy tissues, although it has low resolution. PAT combines these advantages of pure optical imaging and ultrasound imaging. The photo-acoustic effect which is discovered by A.G. Bell in 1880 made it possible [2]. It can provide information about the chemical composition as well as the optical structure of an object. PAT, proposed for medical diagnostic applications, shows great potential in important medical applications.

In PAT, one induces an acoustic wave inside an object of interest by delivering optical energy [14, 22], and then one measures the restriction of the acoustic wave to a surface outside of the object of interest [5, 22, 25]. The initial data of the three dimensional wave equation contain diagnostic information. The one mathematical problem of PAT boils down to recovering this initial acoustic pressure field.

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The type of detector most often studied is a point transducer, which approximately measures the pressure at a given point. It, however, is difficult to manufacture small detectors with high bandwidth and sensitivity. Hence, various other types of detectors to measure the acoustic data are introduced, such as line detectors, planar detectors, cylindrical detectors and circular detectors. Measurements are modeled by the integrals of pressure over the shape of the detectors. Some works [25, 26, 27] have dealt with PAT with the circular detectors. They showed that the data from PAT with circular detectors is the solution of an initial value problem and used this to reduce this problem to the problem of inverting a circular Radon transform. Also, they assume that the circle detectors are centered on a cylinder. In our approach, we define a new Radon-type transform arising in this version of PAT, and consider the situation when the set of the centers of detector circles is a cylinder (only this situation is discussed in previous works [25, 27]) and two more situations when the set of the centers of detector circles is a plane or a sphere. This transform is similar to a toroidal transform, which maps a given function to its integrals over a set of tori. We also study a mathematically similar object.

This paper is organized as follows. Section 2 is devoted to a Radon-type transform arising in PAT with circular integrating detectors. We reduce this Radon-type transform to the Radon transform on circles with a fixed radius. In section 3, we define a toroidal Radon transform and reduce this transform to the circular Radon transform.

2 PAT with circular integrating detectors

In PAT, the acoustic pressure $p(x, t)$ satisfies the following initial value problem:

$$
\begin{align*}
\partial_t^2 p(x, t) &= \Delta_x p(x, t) \quad (x, t) = (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty), \\
p(x, 0) &= f(x) \quad x \in \mathbb{R}^3, \\
\partial_t p(x, 0) &= 0 \quad x \in \mathbb{R}^3.
\end{align*}
$$

(We assume that the sound speed in the interior of the object is equal to one.) The goal of PAT is to recover the initial pressure $f$ from measurements of $p$ outside the support of $f$.

Throughout this section, it is assumed that an initial pressure field $f$ is smooth and circular detectors are parallel to $x_1x_2$-plane. As mentioned before, we will consider three geometries: the centers of detector circles are located on a cylinder, a plane or a sphere. In other words, it is assumed that the acoustic signals are measured by a stack of parallel circle detectors where these circles are centered on a cylinder $\partial B^2_R(0) \times \mathbb{R}$, on the $x_1 = 0$ plane, or on a sphere $\partial B^3_R(0)$ and their radii are a constant $r_{det}$. Here $B^k (x) = B^k(x, t)$ is a ball in $\mathbb{R}^k$ centered at $x \in \mathbb{R}^k$ with radius $t$.

Let $A \subset \mathbb{R}^3$ be the set of the centers of the detector circles. In deed, $A$ is a cylinder $\partial B^2_R(0) \times \mathbb{R}$, the $x_1 = 0$ plane, or a sphere $\partial B^3_R(0)$ (See figure 1). The measured data $P(a, t)$ for $(a, t) \in A \times (0, \infty)$ can be written as

$$
P(a, t) = \frac{1}{2\pi} \int_0^{2\pi} p(a + (r_{det}\alpha, 0), t) d\alpha,
$$

2
where $\vec{\alpha} = (\cos \alpha, \sin \alpha)$. Also, it is a well-known fact that

$$p(x, t) = \partial_t \left( \frac{1}{4\pi t} \int_{\partial B^3(x, t)} f dS \right),$$

is a solution of PDE system (1). Hence $P(a, t)$ becomes

$$P(a, t) = \frac{1}{2\pi} \int_0^{2\pi} \partial_t \left( \frac{1}{4\pi t} \int_{\partial B^3(a + (r_{\text{det}} \vec{\alpha}, 0), t)} f dS \right) d\alpha$$

$$= \frac{1}{8\pi^2} \partial_t \left( \frac{1}{t} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(a + (r_{\text{det}} \vec{\alpha}, 0) + t\vec{\beta}) \sin \beta_2 d\beta_1 d\beta_2 d\alpha \right),$$

where $\vec{\beta} = (\cos \beta_1 \sin \beta_2, \sin \beta_1 \sin \beta_2, \cos \beta_2) \in S^2$. Let us define a transform $R_P$ by

$$R_P f(a, t) := \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(a + (r_{\text{det}} \vec{\alpha}, 0) + t\vec{\beta}) \sin \beta_2 d\beta_1 d\beta_2 d\alpha.$$

We will demonstrate a relation between the Radon transform on circles with a fixed radius-a well studied problem-and $R_P f$. This will allow us to recover $f$ from $P$.

2.1 Reconstruction

Let a transform $M_{r_{\text{det}}} f$ be defined by

$$M_{r_{\text{det}}} f(x) := \int_0^{2\pi} f(r_{\text{det}} \vec{\alpha} + (x_1, x_2), x_3) d\alpha,$$
where \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \).

We will show that \( M_{r_{\text{det}}} f \) can be obtained from \( R_P f \) when \( A \) is a cylinder, a plane or a sphere.

**Remark 1.** When we have two circular detectors with different radii, say \( r_1, r_2 \), we have two different values \( M_{r_1} f, M_{r_2} f \) for each \( x \), i.e., two Radon transforms on circles with different fixed radii. Some works \([4, 21, 23]\) show how \( f \) can be reconstructed from \( M_{r_1} f, M_{r_2} f \) under a certain assumption.

The following theorem describes an inversion for the Radon transform \( M_{r_{\text{det}}} \) over all circles with a fixed radius.

**Theorem 2.** Let \( f \in C_\infty^\infty(\mathbb{R}^3) \). Then we have

\[
\hat{f}(\rho, \theta, x_3) = \frac{1}{2\pi} M_{r_{\text{det}}} \hat{f}(\rho_0, \theta, x_3) \left( 1 + h_1 \left( \frac{r_{\text{det}} \rho}{2} \right)^2 + \cdots + h_n \left( \frac{r_{\text{det}} \rho}{2} \right)^{2n} + \cdots \right),
\]

where \( h_1 = 1 \) and \( h_n = h_{n-1} - h_{n-2} 2n!^{-2} + h_{n-3} 3n!^{-2} - \cdots + (-1)^{n-2} h_1 (n-1)!^{-2} + (-1)^{n-1} n!^{-2} \).

**Proof.** To obtain an inversion of \( M_{r_{\text{det}}} f \), let us take the Fourier transform of \( M_{r_{\text{det}}} f \) with respect to \((x_1, x_2)\). Then we have \( M_{r_{\text{det}}} \hat{f}(\xi, x_3) = 2\pi \hat{f}(\xi, x_3) J_0(r_{\text{det}} |\xi|) \) or \( M_{r_{\text{det}}} \hat{f}(\rho, \theta, x_3) = 2\pi \hat{f}(\rho, \theta, x_3) J_0(r_{\text{det}} \rho) \)

where \( \rho = |\xi|, \theta = \xi/|\xi| \), and \( M_{r_{\text{det}}} \hat{f} \) and \( \hat{f} \) are the Fourier transforms of \( M_{r_{\text{det}}} f \) and \( f \) with respect to \((x_1, x_2)\). It is well-known in \([11]\) that

\[
J_0(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} \left( \frac{s}{2} \right)^{2k}.
\]

Hence, we have

\[
1 = J_0(s) \left( 1 + h_1 \left( \frac{s}{2} \right)^2 + h_2 \left( \frac{s}{2} \right)^4 + \cdots + h_n \left( \frac{s}{2} \right)^{2n} + \cdots \right),
\]

which implies our assertion. \( \square \)

### 2.1.1 Cylindrical geometry

Let \( B_R^2(0) \times \mathbb{R} \) be the solid cylinder \( \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq R \} \). Let the centers of the detector circles be located on the cylinder \( A = \partial B^2(0) \times \mathbb{R} \). We can represent \( a \in A \) by \((R\theta, z)\) for \((\theta, z) \in S^1 \times \mathbb{R} \). Then \( R_P f(R\theta, z, t) \) is equal to

\[
\int_0^{2\pi} \int_0^{2\pi} \int_0^\pi f((R\theta, z) + (r_{\text{det}} \alpha, 0) + t\beta) \sin \beta_2 d\beta_2 d\beta_1 d\alpha.
\]

Consider the definition of \( R_P f \). The inner integral with respect to \( \beta_1 \) in the definition formula of \( R_P f \) can be thought of as the circular Radon transform with weight \( \sin \beta_2 \) and centers at the
$(\vec{\beta}, z)$ with radius $t$. We will first remove this integral by applying a similar method which is used in obtaining an inversion formula for the circular Radon transform.

Let us define the operator $\mathcal{R}^\#_P$ for an integrable function $g$ on $\partial B^2_R(0) \times \mathbb{R} \times [0, \infty)$ by

$$\mathcal{R}^\#_P g(R\theta, x_3, \rho) = \int_{\mathbb{R}} g(R\theta, z, \sqrt{(z - x_3)^2 + \rho^2}) |(z - x_3, \rho)| \, dz.$$  

Let us define the linear operator $I^{1/2}_2$ by $\hat{I}^{-1/2}_2 h(R\theta, \xi) = |\xi|^2 \hat{h}(R\theta, \xi)$, where $h$ is a function on $\partial B^2_R(0) \times \mathbb{R}^2$ with its Fourier transform $\hat{h}$ in the last two-dimensional variable.

**Lemma 3.** Let $f \in C^\infty_c(B^2_R(0) \times \mathbb{R})$. Then we have

$$\int_0^{2\pi} \int_0^{2\pi} f(R\theta + r_{det} \vec{\alpha} + t \vec{\beta}_1, x_3) \, d\beta_1 \, d\alpha = -\frac{1}{\pi^2 t} I^{1/2}_2 \mathcal{R}^\#_P \mathcal{R}_P f(R\theta, x_3, t).$$  

(2)

To prove this theorem, we follow the similar method in [1, 8, 17, 18].

**Proof.** By definition, $\mathcal{R}_P f(R\theta, z, t)$ can be written as

$$-\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 f(R\theta + r_{det} \vec{\alpha} + t \vec{\beta}_1, x_3) \, dsd\beta_1 \, d\alpha$$

Taking the Fourier transform of $\mathcal{R}_P f$ with respect to $z$ yields

$$\hat{\mathcal{R}_P f}(R\theta, \xi_1, t) = -\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(R\theta + r_{det} \vec{\alpha} + t \vec{\beta}_1, \xi_1) e^{its\xi_1} \, dsd\beta_1 \, d\alpha,$$

where $\hat{f}$ and $\hat{\mathcal{R}_P f}$ are the 1-dimensional Fourier transforms of $f$ and $\mathcal{R}_P f$ with respect to $x_3$ and $z$, respectively. Taking the Hankel transform of order zero of $t\hat{\mathcal{R}_P f}$ with respect to $t$, we have

$$H_0(t\hat{\mathcal{R}_P f})(R\theta, \xi_1, \eta)$$

$$= -\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(R\theta + r_{det} \vec{\alpha} + t \sqrt{1 - s^2 \vec{\beta}_1}, \xi_1) e^{its\xi_1} \, dsd\beta_1 \, d\alpha \ t^2 J_0(t\eta) \, dt$$

$$= -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} \hat{f}(R\theta + r_{det} \vec{\alpha} + t \sqrt{1 - s^2 \vec{\beta}_1}, \xi_1) t^2 J_0(t\eta) \cos(ts\xi_1) \, dsd\beta_1 \, d\alpha dt$$

$$= -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} \hat{f}(R\theta + r_{det} \vec{\alpha} + b\vec{\beta}_1, \xi_1) b \cos(\rho\xi_1) \sqrt{\rho^2 + b^2} \, dpdbd\beta_1 \, d\alpha$$

5
where in the last line, we made a change of variables \((t, s) \to (\rho, b)\) where \(t = \sqrt{\rho^2 + b^2}\) and 
\(s = \rho / \sqrt{\rho^2 + b^2}\). We use the following identity: for \(0 < \xi_1 < a\),
\[
\int_0^\infty J_0(a \sqrt{\rho^2 + b^2}) \cos(\rho \xi_1) d\rho = \begin{cases} 
\cos(b \sqrt{a^2 - \xi_1^2}) & \text{if } 0 < \xi_1 < a, \\
0 & \text{otherwise.}
\end{cases} \tag{3}
\]
\[\text{p.55 (35) vol.1}]\]. Using this identity, \(H_0(t \mathcal{R} P \hat{f})(R\theta, \xi_1, \eta)\) is equal to
\[
-2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(R\theta + r_{det} \vec{\alpha} + b \vec{\beta}_1, \xi_1) b \frac{\cos(b \sqrt{\eta^2 - \xi_1^2})}{\sqrt{\eta^2 - \xi_1^2}} db d\beta_1 d\alpha \quad \text{if } 0 < \xi_1 < \eta,
\]
\[
\text{otherwise.}
\]
Substituting \(\eta = \sqrt{\xi_1^2 + \xi_2^2}\) yields
\[
H_0(t \mathcal{R} P \hat{f})(R\theta, \xi_1, |\xi|) = -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(R\theta + r_{det} \vec{\alpha} + b \vec{\beta}_1, \xi_1) b \frac{\cos(b \xi_2) db d\beta_1 d\alpha}{\xi_2} \quad \text{if } 0 < \xi_1 < \eta. \tag{4}
\]
The inner integral in the right hand side of the last equation is the Fourier cosine transform with
respect to \(b\), so taking the Fourier cosine transform of formula \((4)\), we get
\[
\int_0^{2\pi} \int_0^{2\pi} \hat{f}(R\theta + r_{det} \vec{\alpha} + s \vec{\beta}_1, \xi_1) s db d\beta_1 d\alpha 
= -\pi^{-1} \int_0^{\infty} H_0(t \mathcal{R} P \hat{f})(R\theta, \xi_1, |\xi|) \cos(s \xi_2) s \xi_2 ds. \tag{5}
\]
where \(\hat{f}\) is the Fourier transform of \(f\) with respect to the last variable \(x_3\).

We change the right hand side of \((5)\) into a term containing the operator \(\mathcal{R}_P^\#\). Taking the Fourier transform of \(\mathcal{R}_P^\# g\) on \(A \times \mathbb{R}^2\) with respect to \((z, \rho)\) yields
\[
\mathcal{R}_P^\# g(R\theta, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot z} \mathcal{R}_P^\# g(R\theta, x_3, \rho) dx_3 d\rho 
= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot z} \int_{\mathbb{R}} |g(\theta, z, \sqrt{(z - x_3)^2 + \rho^2}) dz dx_3 d\rho 
= \int_{\mathbb{R}} e^{-i x_3 \cdot z} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot x_3} |g(\theta, z, \sqrt{(z - x_3)^2 + \rho^2}) dz dx_3 d\rho 
= \int_{\mathbb{R}} e^{-i x_3 \cdot z} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot \xi} |(x_3, \rho) |g(\theta, z, |(x_3, \rho)|) dx_3 d\rho 
= 2\pi \int_{\mathbb{R}} e^{-i x_3 \cdot z} H_0(t \hat{g})(R\theta, z, |\xi|) dz 
= 2\pi H_0(t \hat{g})(R\theta, \xi_1, |\xi|), \tag{6}
\]
where \( \hat{R}_{p}^#g \) is the Fourier transform with respect to the last variable \((x_3, \rho)\). Combining this with formula (5), we have for \( g = R_{p}f \),

\[
\int_0^{2\pi} \int_0^{2\pi} \hat{f}(R\theta + r_{det} \vec{\alpha} + s\vec{\beta}_1, \xi_1) d\beta_1 d\alpha = -\frac{1}{2\pi^2 s} \int_0^{\infty} \hat{R}_{p}^#g(R\theta, \xi) \cos(s\xi_2) \xi_2 d\xi_2
\]

\[
= -\frac{1}{\pi^2 s} \int_{\mathbb{R}} \hat{R}_{p}^#g(R\theta, \xi) e^{is\xi_2} |\xi_2| d\xi_2.
\]

Again, the inner integral with respect to \( \beta_1 \) in the left hand side of equation (2) is the circular Radon transform with centers on \( \partial B_2^{(0)} \) and radius \( t \). Hence, if applying an inversion formula of the circular Radon transform, we get \( M_{r_{det}}f(x) \).

**Theorem 4.** Let \( f \) be a smooth function supported in \( B_2^2 (0) \times \mathbb{R} \). Then for any \( x \in \mathbb{R}^3 \), \( M_{r_{det}}f(x) \) can be determined through

\[
-\frac{1}{2\pi^2 R} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} f(R\vec{\theta} + r_{det} \vec{\alpha} + t\vec{\beta}_1, x_3) q(t) \log |t^2 - |(x_1, x_2) - \vec{R}\vec{\theta}|^2| dt d\theta.
\]

To prove this theorem, we follow the method discussed in [9].

**Proof.** It is computed in [9] that

\[
\int_{0}^{2\pi} \log \left| |(x_1, x_2) - \vec{R}\vec{\theta}|^2 - |(y_1, y_2) - \vec{R}\vec{\theta}|^2 \right| d\theta = 2\pi R \log |(x_1, x_2) - (y_1, y_2)| + 2\pi R \log R.
\]

For any measurable function \( g \) on \( \mathbb{R} \), it is easily shown that

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\mathbb{R}^2} f(R\vec{\theta} + r_{det} \vec{\alpha} + t\vec{\beta}_1, x_3) q(t) dt d\beta_1 d\alpha = \int_{\mathbb{R}^2} f(R\vec{\theta} + r_{det} \vec{\alpha} + w, x_3) q(|w|) dwd\alpha.
\]

Applying this with \( q(t) = \log \left| t^2 - |(x_1, x_2) - \vec{R}\vec{\theta}|^2 \right| \) and making the change of variables \( (y_1, y_2) =\)
\[ R\tilde{\theta} + t\tilde{\beta}_1 \in \mathbb{R}^2 \text{ give} \]
\[
\begin{align*}
2\pi & \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} tf(R\tilde{\theta} + r_{\text{det}}\tilde{\alpha} + t\tilde{\beta}_1, x_3) \log \left| t^2 - \|(x_1, x_2) - R\tilde{\theta}\|^2 \right| d\beta_1 d\alpha dt d\theta \\
&= \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{\text{det}}\tilde{\alpha} + (y_1, y_2), x_3) \log \left| (x_1, x_2) - R\tilde{\theta}\|^2 - \|(y_1, y_2) - R\tilde{\theta}\|^2 \right| dy d\alpha d\theta \\
&= 2\pi R \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{\text{det}}\tilde{\alpha} + (y_1, y_2), x_3)(\log \left| (x_1, x_2) - (y_1, y_2) \right| + \log R)dy d\alpha,
\end{align*}
\]

where in the last line, we used the Fubini-Tonelli theorem. Since \((2\pi)^{-1} \log \|(x_1, x_2) - (y_1, y_2)\| + \log R\) is a fundamental solution of the Laplacian in \(\mathbb{R}^2\), we have
\[
\begin{align*}
\triangle & \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} tf(R\tilde{\theta} + r_{\text{det}}\tilde{\alpha} + t\tilde{\beta}_1, x_3) \log \left| t^2 - \|(x_1, x_2) - R\tilde{\theta}\|^2 \right| d\beta_1 d\alpha dt d\theta \\
&= R \int_0^{2\pi} \int_{\mathbb{R}^2} f((x_1, x_2) + r_{\text{det}}\tilde{\alpha}, x_3) d\alpha,
\end{align*}
\]

where \(\triangle\) is the Laplacian on \((x_1, x_2)\). Lemma 3 completes this proof. \(\square\)

**Remark 5.** When \(A\) is a cylinder, we can reconstruct \(f\) from \(\mathcal{R}_P f\) by applying Theorems 2 and 4.

### 2.1.2 Planar geometry

Let the centers of detector circles be located on the \(x_1 = 0\) plane. Then we denote \(a \in A\) by \((y, z) \in \mathbb{R}^2\). Also, \(\mathcal{R}_P f\) is equal to zero if \(f\) is an odd function in \(x_1\). We thus assume the function is even in \(x_1\). Then \(\mathcal{R}_P f\) can be written by
\[
\mathcal{R}_P f(y, z, t) = \int_0^{2\pi} \int_{S^2} f((0, y, z) + r_{\text{det}}(\alpha, 0) + t\tilde{\beta}) d\tilde{\beta} d\alpha.
\]

Also, when the spherical Radon transform \(M_P\) maps a locally integrable function \(f\) into
\[
M_P(y, z, t) = \int_{S^2} f((0, y, z) + t\tilde{\beta}) d\tilde{\beta},
\]
we have
\[
\mathcal{R}_P f(y, z, t) = \int_{S^2} M_{r_{\text{det}}} f((0, y, z) + t\tilde{\beta}) d\tilde{\beta} = M_P(M_{r_{\text{det}}} f)(y, z, t).
\]
It is well-known in [1, 8, 17, 18] that
\[ \hat{f}(\xi) = \frac{|\xi_1|}{16\pi^3} \mathcal{F}(M_P^*M_Pf)(\xi), \]
where \( \mathcal{F} = \hat{f} \) is the \( n \)-dimensional Fourier transform of \( f \) and for an integrable function \( g \) on \( \mathbb{R}^2 \times [0, \infty) \)
\[ M_P^*g(x) = \int_{\mathbb{R}^2} g(y, z, \sqrt{x_1^2 + (y - x_2)^2 + (z - x_3)^2}) dy dz. \]

**Theorem 6.** Let \( f \in C_c^{\infty}(\mathbb{R}^3) \) be even in \( x_1 \). Then we have
\[ M_{r_{\det}}f(x) = \frac{1}{2\pi^6} \int_{\mathbb{R}^3} |\xi| |\xi_1| \mathcal{F}(M_P^*R_Pf)(\xi) e^{i\xi \cdot x} d\xi. \]

Now \( f \) can be determined applying Theorem 2.

**Remark 7.** Redding and Newsam derived another inversion formula in [20]. Using this inversion formula, we can reconstruct \( M_{r_{\det}}f \) from \( R_Pf \).

**Remark 8.** Let us consider the centers of detector circles be located on the \( x_3 = 0 \) plane. Then the circular detectors are located on the \( x_1x_2 \)-plane, but our analysis can still be applied. In this case, \( f \) should be assumed even in \( x_3 \).

### 2.1.3 Spherical geometry

Let the centers of detector circles be located on the sphere \( \partial B^3_{R}(0) \). Then we denote \( a \in A \) by \( R\omega \) for \( \omega \in S^2 \). Then \( R_Pf \) can be written by
\[ R_Pf(R\omega, t) = \int_{S^2} \int f(R\omega + r_{\det}(\bar{\alpha}, 0) + t\bar{\beta}) d\bar{\beta} d\alpha. \]

Also, when the spherical Radon transform \( M_S \) maps a locally integrable function \( f \) into
\[ M_S(\omega, t) = \int_{S^2} f(R\omega + t\bar{\beta}) d\bar{\beta}, \]
we have
\[ R_Pf(R\omega, t) = \int_{S^2} M_{r_{\det}}f(R\omega + t\bar{\beta}) d\bar{\beta} = M_S(M_{r_{\det}}f)(\omega, t). \]

It is well-known in [10] that
\[ f(x) = -\frac{1}{2\pi^2 r_{\det}^2} \int_{S^2} \frac{\partial^2 M_Sf(\alpha, |r_{\det}\alpha - x|)}{|r_{\det}\alpha - x|} d\alpha. \]
Theorem 9. Let \( f \in C^\infty_c(B^3_R(0)) \). Then we have

\[
M_{r\det} f(x) = -\frac{1}{2\pi^2 r_{\det}} \int_{S^2} \frac{\partial^2 \mathcal{R}_P f(R\omega, |r_{\det}\alpha - x|)}{|r_{\det}\alpha - x|} d\omega.
\]

Again \( f \) can be determined applying Theorem 2.

Remark 10. Kunyansky derived two other inversion formulas in \([15, 16]\). Using these inversion formulas, we can reconstruct \( M_{r\det} f \) from \( \mathcal{R}_P f \).

3 A toroidal Radon transform

The cylindrical Radon transform was introduced by Haltmeier in \([12]\). This transform arose in PAT with line detectors. The detector line corresponds to the central axis of a cylinder and the propagation distance of the acoustic wave gives the radius of a cylinder.

As mentioned before, circular detectors as well as line detectors have been used to measure the acoustic data in PAT. If we think that the detector circle corresponds to the central circle of a torus and the propagation distance of the acoustic wave gives the radius of the tube of this torus, then it looks reasonable that PAT with circular detectors brings about the toroidal Radon transform, which assigns a given locally integrable function to its integrals over a set of tori. Unfortunately, we have not been able to establish the direct link between PAT with circular detectors and the toroidal Radon transform. Nevertheless, studying the toroidal Radon transform is an interesting geometric problem in its own right.

3.1 Definition of the Toroidal Radon Transform

We assume that all tori are parallel to \( xy \)-plane and consider two geometries: the centers of tori are located on a cylinder or a plane.

Definition 11. Let \( u > 0 \) be a radius of the central circles of tori. Let \( A \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} \) be the set of the centers of tori. The toroidal Radon transform \( \mathcal{R}_T \) maps \( f \in C^\infty_c(\mathbb{R}^3) \) into

\[
\mathcal{R}_T f(\mu, p, r) = \frac{1}{2\pi} \int_{S^1} \int_0^{2\pi} f(\mu + (u - r \cos \beta)\alpha, p + r \sin \beta) d\beta d\alpha,
\]

for \((\mu, p, r) \in A \times \mathbb{R} \times (0, \infty)\). Here \( \alpha \) is the angular parameter along the central circle, \((\mu, p)\) is the center of the torus, and \( \beta \) and \( r \) are the polar angle and radius of the tube of the torus, respectively.

We consider the two situations when \( A \) is a circle or a line and thus the set of the centers of tori is a cylinder or a plane. We then present the relation between the circular Radon transform and
the toroidal Radon transform. This relation leads naturally to an inversion formula, if one uses an inversion formula for the circular Radon transform (already discussed in [9, 15] or [1, 8, 17, 18, 20]).

**Definition 12.** Let $f$ be a compactly supported function in $\mathbb{R}^3$. The circular Radon transform $M$ maps a function $f$ into

$$Mf(\mu, x_3, r) = \int_{S^1} f(\mu + r\vec{\alpha}, x_3) d\vec{\alpha} \quad \text{for} \ (\mu, x_3, r) \in A \times \mathbb{R} \times (0, \infty).$$

### 3.2 Inversion of the toroidal Radon Transform

The inner integral with respect to $\beta$ in (7) can be thought of as the circular Radon transform with centers at $(\mu + u\vec{\alpha}, p)$ and radius $r$. As in subsection 2.1, we will first invert this transform.

Let us define the operator $R^*_T$ for $g \in C^\infty_c(A \times \mathbb{R} \times [0, \infty))$ by

$$R^*_Tg(\mu, z, \rho) = \int_{\mathbb{R}} g(\mu, p, \sqrt{(z - p)^2 + \rho^2}) dp,$$

where $(\mu, z, \rho) \in A \times \mathbb{R}^2$. The following two lemmas show the relation between the circular and the toroidal Radon transforms.

**Lemma 13.** Let $f \in C^\infty_c(\mathbb{R}^3)$. Then we have

$$\frac{1}{2} I^{-1}_2 R^*_T R_T f(\mu, x_3, r) = \begin{cases} Mf(\mu, x_3, u - r) + Mf(\mu, x_3, u + r) & \text{if } u > r, \\ Mf(\mu, x_3, r - u) + Mf(\mu, x_3, u + r) & \text{otherwise}. \end{cases} \quad (8)$$

To prove this lemma, we follow the method discussed in [11, 17, 18].

**Proof.** By definition, we have

$$R_T f(\mu, p, r) = \frac{1}{2\pi} \sum_{j=1}^{2} \int_{S^1} \int_{-1}^{1} f(\mu + (u + (-1)^j r \sqrt{1 - s^2}) \vec{\alpha}, p + rs) \frac{ds}{\sqrt{1 - s^2}} d\vec{\alpha}.$$

We take the Fourier transform of $R_T f$ with respect to $p$ and the Hankel transform of order zero of $\tilde{R}_T f$ with respect to $r$. Then $H_0 \tilde{R}_T f(\mu, \xi_1, \eta)$ can be written as

$$\frac{1}{2\pi} \sum_{j=1}^{2} \int_{0}^{\infty} \int_{0}^{\infty} \hat{f}(\mu + (u + (-1)^j b) \vec{\alpha}, \xi_1) \cos(\rho \xi_1) J_0(\eta \sqrt{\rho^2 + b^2}) dp db d\vec{\alpha}, \quad (9)$$
where \( \hat{f} \) and \( \hat{R}_T f \) are the 1-dimensional Fourier transforms of \( f \) and \( R_T f \) with respect to \( z \) and \( p \), respectively. Lastly, we change variables \((r, s) \rightarrow (\rho, b)\), where \( r = \sqrt{\rho^2 + b^2} \) and \( s = \rho/\sqrt{\rho^2 + b^2} \).

Applying equation (3) to equation (9), we get

\[
H_0 \hat{R}_T f(\mu, \xi_1, |\xi|) = \frac{1}{2\pi} \sum_{j=1}^{2} \int_{S^1} \hat{f}(\mu + (u + (-1)^j b)\tilde{\alpha}, \xi_1) \cos(\xi_2) \frac{d\tilde{\alpha}d\theta}{\xi_2}.
\]

The inner integral in the right hand side of the last equation is the Fourier cosine transform with respect to \( b \), so taking the inverse Fourier cosine transform of the above formula, we get

\[
\sum_{j=1}^{2} \int_{S^1} \hat{f}(\mu + (u + (-1)^j s)\tilde{\alpha}, \xi_1) d\tilde{\alpha} = 4 \int_{0}^{\infty} H_0 \hat{R}_T f(\mu, \xi_1, |\xi|) \cos(s\xi_2) d\xi d\xi_2.
\]

For a fixed \( \xi_1 \), one recognizes the sum of two circular Radon transforms on the left.

Similarly to equation (6), we can change the right hand side of equation (10) into a term containing operator \( R^*_T \), i.e.,

\[
\hat{R}^*_T g(\mu, \xi) = 2\pi H_0 \hat{g}(\mu, \xi_1, |\xi|).
\]

Here \( \hat{R}^*_T g \) is the Fourier transform with respect to the variable \((z, \rho)\). Combining this equation (11) with equation (10), we have for \( g = R_T f \),

\[
\sum_{j=1}^{2} \int_{S^1} \hat{f}(\mu + (u + (-1)^j s)\tilde{\alpha}, \xi_1) d\tilde{\alpha} = 2\pi \int_{0}^{\infty} \hat{R}^*_T g(\mu, \xi) \cos(s\xi_2) d\xi d\xi_2
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} \hat{R}^*_T g(\mu, \xi) e^{is\xi_2} |\xi_2| d\xi_2,
\]

since \( \hat{R}^*_T g \) is even in \( \xi_2 \).

**Lemma 14.** Let \( f \in C_c^\infty(\mathbb{R}^3) \). Then we have

\[
\frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{\infty} rs R_T f(\mu, -\eta, s) e^{-i(s^2 + 2x_3 \eta + x_3^2 - \eta^2 + r^2)\xi} ds d\eta d\xi
\]

\[
= \begin{cases} 
Mf(\mu, x_3, u - r) + Mf(\mu, x_3, u + r) & \text{if } u > r, \\
Mf(\mu, x_3, r - u) + Mf(\mu, x_3, u + r) & \text{otherwise}.
\end{cases}
\]

To prove this lemma, we follow the method discussed in [20].

**Proof.** Let \( G \) be defined by

\[
G(\mu, p, \xi) := \int_{0}^{\infty} r R_T f(\mu, p, r) e^{-ir^2 \xi} dr.
\]
Then we have

\[
G(\mu, p, \xi) = \frac{1}{2\pi} \int_0^\infty \int_0^{\pi} \int_0^{\pi} r f(\mu + (u - r \cos \beta)\vec{a}, p + r \sin \beta)e^{-ir^2\xi}d\beta d\vec{a} dr
\]

\[
= \frac{1}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{a}, p + z)e^{-i(y^2 + z^2)\xi}dy dz d\vec{a}
\]

\[
= \frac{1}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{a}, z)e^{-i(y^2 + (z - p)^2)\xi}dy dz d\vec{a}
\]

\[
= \frac{e^{-ip^2\xi}}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{a}, z)e^{-i(y^2 + z^2)\xi}e^{2ipz\xi}dy dz d\vec{a},
\]

where in the second line, we switched from the polar coordinates \((r, \beta)\) to the Cartesian coordinates \((y, z)\) \(\in \mathbb{R}^2\). Making the change of variables \(r = y^2 + z^2\) yields

\[
G(\mu, p, \xi) = \frac{e^{-ip^2\xi}}{2\pi} \sum_{j=1}^2 \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u + (-1)^j \sqrt{r - z^2})\vec{a}, z)e^{-ir\xi}e^{2ipz\xi}2\sqrt{r - z^2} dr dz d\vec{a}.
\]

Let us define the function

\[
k_\mu(\alpha, z, r) := \begin{cases} 
\sum_{j=1}^2 f(\mu + (u + (-1)^j \sqrt{r - z^2})\vec{a}, z)/\sqrt{r - z^2} & \text{if } 0 < z^2 < r, \\
0 & \text{otherwise.}
\end{cases}
\]

Then we have

\[
G(\mu, p, \xi) = \frac{e^{-ip^2\xi}}{4\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} k_\mu(\alpha, z, r)e^{-ir\xi}e^{2ipz\xi}dr dz d\vec{a},
\]

where \(\hat{k}_\mu\) is the 2-dimensional Fourier transform of \(k_\mu\) with respect to the variables \((z, r)\). Also, we have

\[
\sum_{j=1}^2 \int_{S^1} f(\mu + (u + (-1)^j s)\alpha, x_3)d\alpha = \int_{S^1} sk_\mu(\alpha, x_3, x_3^2 + s^2)d\alpha
\]

\[
= \frac{1}{4\pi^2} \int_{S^1} \int_{\mathbb{R}^2} sk_\mu(\alpha, \eta, \xi)e^{-i(x_3\eta + (x_3^2 + s^2)\xi)}d\alpha d\eta d\xi
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} \int_{S^1} \int_{\mathbb{R}^2} G(\mu, -\frac{\eta}{2\xi}, \xi)e^{-i(x_3\eta + (x_3^2 + s^2)\xi)}d\eta d\xi
\]

\[
= \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} sG(\mu, -\eta, \xi)e^{-i((2x_3\eta + (x_3^2 + s^2) - \eta^2)\xi)}d\eta d\xi,
\]

where in the last line, we changed the variable \(\eta\) to \(2\xi\eta\). \(\square\)
3.2.1 Cylindrical geometry

Let the centers of the central circles be located on the a cylinder $\partial B^2_R(0) \times \mathbb{R} = A \times \mathbb{R}$. That is, $A$ is the circle centered at the origin with radius $R$. The next two results show that the circular Radon transform can be recovered from the toroidal Radon transform. Both theorems are easily obtained using Lemma \ref{lem:toroidal_transformation}.

**Theorem 15.** If $R/2 < u < R$ and $f \in C^\infty_c(B^2_R(0) \times \mathbb{R})$, then

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1}I_2^{-1}R_2^*R_Tf(\mu, x_3, r - u) & \text{if } r > u, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 16.** Let $f \in C^\infty_c(B^2_R(0) \times \mathbb{R})$. Then

$$Mf(\mu, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{[R/2u]} (-1)^jI_2^{-1}R_2^*R_Tf(\mu, x_3, (2j+1)u - r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{[R/u]} (-1)^jI_2^{-1}R_2^*R_Tf(\mu, x_3, (2j+1)u + r) & \text{otherwise.} \end{cases}$$

**Remark 17.** One can obtain other relations similar to Theorems 15 and 16, by using Lemma \ref{lem:inversion_formula} instead of Lemma \ref{lem:toroidal_transformation}.

**Remark 18.** When the set of centers of the detector circles lie on a cylinder, (i.e., $A$ is a circle,) one can recover $f$ from its toroidal transform $R_Tf$ by applying inversion formulas (see e.g. \cite{9, 15}) to the left hand sides of equations in Theorems 15 and 16.

**Remark 19.** If $u > 2R$ (i.e., the radius of central circles is bigger than the diameter of domain cylinder), then

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1}I_2^{-1}R_2^*R_Tf(\mu, x_3, u - r) & \text{if } r \leq u, \\ 2^{-1}I_2^{-1}R_2^*R_Tf(\mu, x_3, u + r) & \text{otherwise.} \end{cases}$$

3.2.2 Planar geometry

Let $A \subset \mathbb{R}^2$ be the $x_1 = 0$ line. (i.e., the centers of tori are located on the $x_1 = 0$ plane in $\mathbb{R}^3$.) Then $R_Tf(\mu, x_3, r)$ is equal to zero if $f$ is an odd function in $x_1$. We thus assume the function $f$ to be even in $x_1$.

**Theorem 20.** Let $f \in C^\infty_c(B^2_R(0))$ be even in $x_1$. Then we have

$$Mf(\mu, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{[R/2u]} (-1)^jI_2^{-1}R_2^*R_Tf(\mu, x_3, (2j+1)u - r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{[R/2u]} (-1)^jI_2^{-1}R_2^*R_Tf(\mu, x_3, (2j+1)u + r) & \text{otherwise.} \end{cases}$$

(12)
Remark 21. When $A$ is a line, we can determine $f$ from $R_T f$ by applying inversion formulas of [1, 8, 17, 18, 20] to the left hand side of equation (12).

Remark 22. If $u > R$ (i.e., the radius of detector is bigger than the radius of the ball containing $\text{supp } f$), then

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1} I_2^{-1} R_T^* R_T f(\mu, x_3, u - r) & \text{if } r < u, \\ 2^{-1} I_2^{-1} R_T^* R_T f(\mu, x_3, u + r) & \text{otherwise.} \end{cases}$$

4 Conclusion

Here we studied a Radon-type transform arising in PAT with circular detectors and the toroidal Radon transform which has a chance to have a direct link to PAT. We proved that these transforms reduce to a well-studied transforms, the Radon transform over circles with a fixed radius or the circular Radon transform.

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References

[1] L. Andersson. On the determination of a function from spherical averages. SIAM Journal on Mathematical Analysis, 19(1):214–232, 1988.

[2] A.G. Bell. On the production and reproduction of sound by light. American Journal of Science, 20:305–324, October 1880.

[3] C. Berenstein and L. Zalcman. Pompeiu’s problem on spaces of constant curvature. Journal d’Analyse Mathématique, 30(1):113–130, 1976.

[4] C.A. Berenstein, R. Gay, and A. Yger. Inversion of the local Pompeiu transform. Journal d’Analyse Mathématique, 54(1):259–287, 1990.

[5] P. Burgholzer, J. Bauer-Marschallinger, H. Grün, M. Haltmeier, and G. Paltauf. Temporal back-projection algorithms for photoacoustic tomography with integrating line detectors. Inverse Problems, 23(6):S62–S80, 2007.

[6] L. Ehrenpreis. The Universality of the Radon transforms. Oxford Mathematical Monographs. Clarendon Press, 2003.
[7] A. Erdelyi. *Tables of Integral Transforms, Vols. I and II*. Batemann Manuscript Project, McGrawHill, New York, 1954.

[8] J. Fawcett. Inversion of n-dimensional spherical averages. *SIAM Journal on Applied Mathematics*, 45(2):336–341, 1985.

[9] D. Finch, M. Haltmeier, and Rakesh. Inversion of spherical means and the wave equation in even dimensions. *SIAM Journal on Applied Mathematics*, 68:392–412, 2007.

[10] D. Finch, S. Patch, and Rakesh. Determining a function from its mean values over a family of spheres. *SIAM Journal on Mathematical Analysis*, 35(5):1213–1240, 2004.

[11] G.B. Folland. *Fourier analysis and its applications*. Pure and Applied Undergraduate Texts. American Mathematical Society, 1992.

[12] M. Haltmeier. Inversion formulas for a cylindrical Radon transform. *SIAM Journal on Imaging Sciences*, 4(3):789–806, 2011.

[13] P. Kuchment. Mathematics of hybrid imaging: A brief review. In Irene Sabadini and Daniele C Struppa, editors, *The Mathematical Legacy of Leon Ehrenpreis*, volume 16 of *Springer Proceedings in Mathematics*, pages 183–208. Springer Milan, 2012.

[14] P. Kuchment and L. Kunyansky. Mathematics of thermoacoustic tomography. *European Journal of Applied Mathematics*, 19:191–224, 2008.

[15] L.A. Kunyansky. Explicit inversion formulæ for the spherical mean Radon transform. *Inverse Problems*, 23(1):373, 2007.

[16] L.A. Kunyansky. A series solution and a fast algorithm for the inversion of the spherical mean radon transform. *Inverse Problems*, 23(6):S11, 2007.

[17] F. Natterer and F. Wübbeling. *Mathematical methods in image reconstruction*. SIAM Monographs on mathematical modeling and computation. SIAM, Society of industrial and applied mathematics, Philadelphia (Pa.), 2001.

[18] S. Nilsson. *Application of fast backprojection techniques for some inverse problems of integral geometry*. Linköping studies in science and technology: Dissertations. Department of Mathematics, Linköping University, 1997.

[19] G. Paltauf and R. Nuster. Iterative reconstruction method for photoacoustic section imaging with integrating cylindrical detectors. *Proceedings of SPIE*, 8581:85814N–85814N–9, 2013.

[20] N.T. Redding and G.N. Newsam. Inverting the circular Radon transform. *DTSO Research Report DTSO-Ru-0211*, August 2001.

[21] S. Thangavelu. Spherical means and CR functions on the heisenberg group. *Journal d’Analyse Mathématique*, 63(1):255–286, 1994.

[22] Y. Xu and L.V. Wang. Photoacoustic imaging in biomedicine. *Review of Scientific Instruments*, 77(4):041101–041122, April 2006.
[23] L. Zalcman. Offbeat integral geometry. *The American Mathematical Monthly*, 87(3):pp. 161–175, 1980.

[24] L. Zalcman. A bibliographic survey of the Pompeiu problem. In B. Fuglede, M. Goldstein, W. Haussmann, W.K. Hayman, and L. Rogge, editors, *Approximation by Solutions of Partial Differential Equations*, volume 365 of *NATO ASI Series*, pages 185–194. Springer Netherlands, 1992.

[25] G. Zangerl and O. Scherzer. Exact series reconstruction in photoacoustic tomography with circular integrating detectors. *Communications in Mathematical Sciences*, 7(3):665–678, 2009.

[26] G. Zangerl and O. Scherzer. Exact reconstruction in photoacoustic tomography with circular integrating detectors II: Spherical geometry. *Mathematical Methods in the Applied Sciences*, 33(15):1771–1782, 2010.

[27] G. Zangerl, O. Scherzer, and M. Haltmeier. Circular integrating detectors in photo and thermoacoustic tomography. *Inverse Problems in Science and Engineering*, 17(1):133–142, 2009.