Naked Singularities in Spherically Symmetric, Self-Similar Spacetimes

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Abstract

We show that all known naked singularities in spherically symmetric self-similar spacetimes arise as a result of singular initial matter distribution. This is a result of the peculiarity of the coordinate transformation that takes these spacetimes into a separable form. Therefore, these examples of naked singularities are of no apparent consequence to astrophysical observations or theories.

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In recent years, many workers appear to consider naked or visible spacetime singularities to be of serious astrophysical significance [1]. This is in contrast to the general research atmosphere of most of the second half of the twentieth century where the emphasis was on proving that naked singularities are “unphysical” in the sense and spirit of the Cosmic Censorship Hypothesis (CCH). This “paradigm shift” can be attributed to different, specific, known examples of naked singularities.

It is noteworthy that such solutions have been found to the highly nonlinear coupled partial differential equations as are the field equations of General Relativity for some realistic equations of state for the matter in the spacetime. There is however a general consensus that these examples, being mostly spherically symmetrical in nature, do not counter the spirit of the Cosmic Censorship Hypothesis (CCH). It is then hoped that these examples of naked singularities, apart from their special symmetry, may help sharpen the statement of the CCH that is as yet unproven.

Many of the known examples occur in the highly specialized spherically symmetric, self-similar spacetimes (SSSSS). Such spacetimes admit a homothetic Killing vector. All points along the integral curves of the homothetic Killing vector field are equivalent. Consequently, for a self-similar spacetime the field equations of General Relativity reduce to ordinary differential equations.

It is the purpose of this letter, to re-examine the applicability of the CCH as regards SSSSS notwithstanding the results in, for example, [2, 3].

The requirement that the general spherically symmetric line element

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + Y^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (1)$$

admits a homothetic Killing vector of the form

$$X^a = (0, f(r,t), 0, 0)$$  \hspace{1cm} (2)$$

reduces [4] this metric uniquely to:

$$ds^2 = -y^2(r) dt^2 + \gamma^2 B^2(t) (y')^2 dr^2 + y^2(r) Y^2(t) \left[d\theta^2 + \sin^2 \theta d\phi^2 \right]$$  \hspace{1cm} (3)$$
with an overhead prime denoting differentiation with respect to \( r \), \( f(r,t) \) in (2) now becoming only a function of \( r \), viz.

\[
f(r, t) = \frac{y}{\gamma y'}
\]

(4)

where \( \gamma \) is a constant and we have absorbed the temporal coefficient of \( dt^2 \) by a redefinition of \( t \). This spacetime admits an energy density of the form

\[
\rho = \frac{1}{y^2} \left( \frac{\dot{Y}^2}{Y^2} + \frac{2\dot{B} \dot{Y}}{\dot{B} Y} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} \right)
\]

(5)

Clearly, the density is non-singular/regular for \( y(r) \neq 0 \) for all \( r \). We note that the density is a decreasing function of \( r \) when \( y' > 0 \) and an increasing function of \( r \) when \( y' < 0 \). (We also note that for constant \( y \), the spacetime is homogeneous and for \( y = \infty \), the spacetime is a vacuum solution.) The singularity develops in the collapse of this regular density distribution of matter as a result of its temporal evolution, that is, when the temporal part in (3) becomes infinite at some \( \tau = t_0 \). By symmetry considerations, this singularity must, of necessity, lie at \( r = 0 \) when \( \tau = t_0 \).

As can be shown [6], (3) does not admit any naked singularities, provided the initial data is regular. We recall here some of the relevant results from [6] for the sake of completeness. From (3) it follows that the radial null geodesic satisfies

\[
\frac{dt}{dr} = \pm \gamma \frac{y'}{y} B
\]

(6)

A non-singular density profile requires \( y(r) \neq 0 \) for all \( r \). In this case, we shall have

\[
\lim_{r \to 0} \frac{y'}{y} = \ell_o
\]

say, where \( \ell_o \) is the corresponding finite limiting value. (Note that \( \ell_o \) can be positive or negative depending on the sign of \( y' \).)

Now, for the geodesic tangent to be uniquely defined and to exist at the singular point, \( \tau = 0 \) and \( r = 0 \), the following must hold so that an outgoing, future-directed photon trajectory exists at the singularity:

\[
\lim_{\tau \to 0} \frac{\tau}{r} = \lim_{\tau \to 0} \frac{dt}{dr} = X_o
\]

(7)
where $X_o$ is required to be real and positive. As we approach the singular point, we have

$$X_o = \pm \gamma \lim_{r \to 0} \frac{y'}{y} B(t) = \pm \gamma \lim_{l \to 0} \ell_o B(t) \tag{8}$$

Clearly, the limit in (8) implies that $X_o = 0$. Therefore, in the limit of the singularity of (3), that is, $t \to 0, r \to 0$ there does not exist a radially outgoing tangent to the radial null geodesic. Hence, the spacetime singularity is not naked or visible if the initial density profile corresponds to a density distribution that is non-singular.

The obvious question is the reconciliation of this result with the proliferation of naked singularities in SSSSSS widely reported in the literature.

In this connection, we note that the usual form of the homothetic Killing vector is taken to be [5]:

$$\bar{X}^a = (T, R, 0, 0) \tag{9}$$

Indeed most authors impose (3) on (1) before performing any further analysis. As a result, the metric they focus on is

$$ds^2 = -P \left(\frac{T}{R}\right)^2 dT^2 + Q \left(\frac{T}{R}\right)^2 dR^2 + R^2 S \left(\frac{T}{R}\right)^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right] \tag{10}$$

where $P, Q, S$ are the metric functions of the self-similarity variable $T/R$. In the discussion of self-similar collapse as considered by most of these authors, one is therefore led to consider the singularity of (11) at $R = 0, T = 0$.

We emphasize that any vector of the form (2) can be transformed into (3) via the coordinate transformation

$$R = l(t) \exp \left(\int f^{-1} dr\right) \quad T = k(t) \exp \left(\int f^{-1} dr\right) \tag{11}$$

which, in view of (4), reduces to

$$R = l(t) y^\gamma(r) \quad T = k(t) y^\gamma(r) \tag{12}$$

(Note: If we invoke (11), the resulting metric will not be diagonal. The imposition of diagonality of the metric will require a relationship between $l(t)$ and $k(t)$. Such a relation can always be imposed.)
As we noted above, a relationship exists between $l(t)$ and $k(t)$. Thus $l(t) = 0$ for (3) corresponds to both $R = 0$ and $T = 0$ for (4). There is thus no constraint on the radial function $y(r)$ in (3). This will correspond to those sectors in SSSSS in which no naked singularities arise.

However, we also note that $R = 0$ and $T = 0$ also corresponds to $y(r) = 0$. We have seen [6] that the only case of obtaining a naked singularity for (3) is that of assuming $y(r) = 0$ for $r = 0$ which makes (3) initially singular. This corresponds to those sectors of SSSSS in which naked singularities do arise. It is therefore not surprising that a naked singularity arises in (4) as one assumes a naked singularity at the outset. This is also evident from the density distribution for (4)

$$\rho = \frac{1}{R^2} \eta \left( \frac{T}{R} \right)$$

(13)

also being initially singular for $R = 0$. We emphasize that this is true for any equation of state for matter in the spacetime.

In conclusion, we note that the metric (3) contains all spacetimes admitting a homothetic Killing vector, that have vanishing or non-vanishing shear and/or energy flux provided the transformation (12) is non-singular, ie when $y' \neq 0$ (inhomogeneous) and $y' \neq \infty$ (non-vacuum) for all $r$. We therefore conclude that naked singularities do not arise in SSSSS for regular initial data. The known examples of naked singularities in SSSSS must correspond to singular initial data for matter fields. Consequently, these examples of naked singularities neither form counter-examples to CCH nor provide any implications vis-a-vis astrophysical observations. In fact, the metric (3) not possessing any naked singularities for regular initial data for matter fields is in complete agreement with the strong CCH [7].
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[1] See, as an example, Joshi, P. S., Dadhich, N. K. and Maartens, R. (2000) *Mod. Phys. Lett.* A15 991. gr-qc/0005080

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[2] Hod, S. and Piran, T. (2000) *Gen. Rel. Grav.* 32 2333-2338. gr-qc/0011003

[3] Nolan, B. C. (2001) *Class. Quantum Grav.* 18 1651-1675.

[4] Wagh, S. M. and Govinder, K. S. (2001) *Class. Quantum Grav.* submitted. *Spherically Symmetric, Self-Similar Spacetimes*. gr-qc/0112035 Pre-print No. - CIRI/01-swkg01

[5] See Joshi, P. S. (1993) *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford) and references therein.

[6] Wagh, S. M. and Govinder, K. S. (2001) *Spherically Symmetric, Self-Similar Gravitational Collapse* in preparation. Pre-print No. - CIRI/01-swkg03

[7] Penrose, R. (1998) in *Black Holes and Singularities: S. Chandrasekhar Symposium* (Ed. R. M. Wald, Yale: Yale University Press) and references therein.