Classification of rational 1-forms on the Riemann sphere up to $\text{PSL}(2, \mathbb{C})$

Julio C. Magaña-Cáceres$^{1,2}$

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Abstract
We study the family $\Omega^1(-s)$ of rational 1-forms on the Riemann sphere, having exactly $-s \leq -2$ simple poles. Three equivalent $(2s - 1)$-dimensional complex atlases on $\Omega^1(-s)$, using coefficients, zeros–poles and residues–poles of the 1-forms, are recognized. A rational 1-form is called isochronous when all their residues are purely imaginary. We prove that the subfamily $\mathcal{RI}\Omega^1(-s)$ of isochronous 1-forms is a $(3s - 1)$-dimensional real analytic submanifold in the complex manifold $\Omega^1(-s)$. The complex Lie group $\text{PSL}(2, \mathbb{C})$ acts holomorphically on $\Omega^1(-s)$. For $s \geq 3$, the $\text{PSL}(2, \mathbb{C})$-action is proper on $\Omega^1(-s)$ and $\mathcal{RI}\Omega^1(-s)$. Therefore, the quotients $\Omega^1(-s)/\text{PSL}(2, \mathbb{C})$ and $\mathcal{RI}\Omega^1(-s)/\text{PSL}(2, \mathbb{C})$ admit a stratification by orbit types. Realizations for the quotients $\Omega^1(-s)/\text{PSL}(2, \mathbb{C})$ and $\mathcal{RI}\Omega^1(-s)/\text{PSL}(2, \mathbb{C})$ are given, using an explicit set of $\text{PSL}(2, \mathbb{C})$-invariant functions.

Keywords Rational 1-forms · Isochronous centers · Proper $\text{PSL}(2, \mathbb{C})$ 1-action · Principal $\text{PSL}(2, \mathbb{C})$-bundle · Stratified space by orbit types

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1 Introduction

A compact Riemann surface $M_g$ of genus $g \geq 0$ is determined by its space of holomorphic 1-forms. For $g \geq 1$, the Jacobian variety $J(M_g)$ is defined using the vector space of holomorphic 1-forms. Very roughly speaking, Torelli’s theorem says that we can recover $M_g$ from $J(M_g)$; see [12, p. 359]. Moreover, for $g \geq 2$, a classical result of Hurwitz asserts that the automorphism group $\text{Aut}(M_g)$ is finite; see [7, Ch.
Looking at the moduli space of compact Riemann surfaces \( \mathcal{M}_{g,0} \), generically \( \text{Aut}(\mathcal{M}_g) \) is trivial. For \( g = 0 \), three special features appear. Obviously there exists only one complex structure, set the Riemann sphere \( \hat{\mathbb{C}} \). Secondly, any holomorphic 1-form over \( \hat{\mathbb{C}} \) is identically zero. Finally, the automorphism group \( \text{PSL}(2, \mathbb{C}) \) of \( \hat{\mathbb{C}} \) is the biggest between the automorphism groups of any Riemann surface.

A natural problem is the classification of rational 1-forms on \( \hat{\mathbb{C}} \) with \(-s \leq -2\) poles up to the automorphism group \( \text{PSL}(2, \mathbb{C}) \).

The family of rational 1-forms on \( \hat{\mathbb{C}} \) is an infinite dimensional vector space. A natural idea is to consider the stratification by the multiplicities of zeros and poles. Let \( \Omega^1(-1^s) \) be the family of rational 1-forms having exactly \(-s \leq -2\) simple poles. Our techniques naturally allows us to study rational 1-forms with zeros of any multiplicity and simple poles. We recognized three equivalent \((2s - 1)\)-dimensional complex atlases on \( \Omega^1(-1^s) \), looking at different expressions of the rational 1-forms coefficients \( \Omega^1_{\text{cof}}(-1^s) \), zeros–poles \( \Omega^1_{zp}(-1^s) \) and residues–poles \( \Omega^1_{rp}(-1^s) \). Our main result is the following theorem:

**Theorem 1** The complex manifolds \( \Omega^1_{\text{cof}}(-1^s) \), \( \Omega^1_{zp}(-1^s) \) and \( \Omega^1_{rp}(-1^s) \) are biholomorphic.

For the proof, we construct explicitly the biholomorphisms \( \Omega^1_{\text{cof}}(1^s) \rightarrow \Omega^1_{\text{cof}}(1^s) \) and \( \Omega^1_{zp}(1^s) \rightarrow \Omega^1_{cof}(1^s) \) using the Viète map defined in (2).

The group \( \text{PSL}(2, \mathbb{C}) \) acts naturally on \( \Omega^1(-1^s) \) by coordinate changes. As a result, this \( \text{PSL}(2, \mathbb{C}) \)-action is proper for \( s \geq 3 \); see Lemma 1. Using the classical theory of proper Lie group actions as in [6, Ch. 2], we recognize a principal \( \text{PSL}(2, \mathbb{C}) \)-bundle \( \pi_s : \mathcal{G}(-1^s) \rightarrow \mathcal{G}(-1^s)/\text{PSL}(2, \mathbb{C}) \), where \( \mathcal{G}(-1^s) \subset \Omega^1(-1^s) \) is the open and dense subset of generic rational 1-forms with trivial isotropy group in \( \text{PSL}(2, \mathbb{C}) \), and \( \pi_s \) denotes the natural projection to the orbit space. For \( s = 3 \) (resp. \( s = 4 \)), we prove that the principal \( \text{PSL}(2, \mathbb{C}) \)-bundle is trivial (resp. nontrivial). On the other hand, for all \( \omega \in \Omega^1(-1^s) \setminus \mathcal{G}(-1^s) \), the isotropy group \( \text{PSL}(2, \mathbb{C})_\omega \) is a nontrivial finite subgroup of \( \text{PSL}(2, \mathbb{C}) \). A classical result of F. Klein classifies the finite subgroups of \( \text{PSL}(2, \mathbb{C}) \); see [17, p. 126]. In our framework, the realization problem is which finite subgroups of \( \text{PSL}(2, \mathbb{C}) \) are realizable as the isotropy groups of suitable \( \omega \in \Omega^1(-1^s) \)? A positive answer is as follows.

**Proposition 1** Every finite subgroup \( G < \text{PSL}(2, \mathbb{C}) \) appears as the isotropy group of suitable \( \omega \in \Omega^1(-1^s) \).

Obviously, the degree of the divisor of poles \( s \) depends on the order of \( G \). Recalling the rotation groups of a pyramid, bipyramid and platonic solids, the proof is done.

As a second goal, recall that \( \omega \in \Omega^1(-1^s) \) is isochronous when all their residues are purely imaginary. We study the subfamily \( \mathcal{RI} \Omega^1(-1^s) \subset \Omega^1(-1^s) \) of isochronous 1-forms. Our result is below.

**Corollary 1** The subfamily \( \mathcal{RI} \Omega^1(-1^s) \) is a \((3s - 1)\)-dimensional real analytic submanifold of \( \Omega^1(-1^s) \).

For the proof, we use the complex atlas by residues–poles. Proposition 1 is fulfilled for suitable \( \omega \in \mathcal{RI} \Omega^1(-1^s) \). The residues are naturally a set of \( \text{PSL}(2, \mathbb{C}) \)-invariant functions. They can be used in order to describe a realization of 1-forms;
see [9]. The $\text{PSL}(2, \mathbb{C})$-action on $RI\Omega^1(-1^s)$ is well defined. Hence, the quotients $\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ and $RI\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ admit a complex and a real stratification by orbit types, respectively. For $s \geq 4$, in order to get a complete set of $\text{PSL}(2, \mathbb{C})$-invariant functions, we enlarge the set of residues by adding the cross-ratio of poles and explicitly recognize realizations for the quotients $\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ and $RI\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$; see, respectively, Proposition 3 and Corollary 2.

In Sect. 5, for $\omega \in \Omega^1(-1^s)$, resp. $\omega \in RI\Omega^1(-1^s)$, we obtain a realization for the quotient of the associated flat surfaces $S_\omega$ up to isometries $M(-s)$, resp. $RI\Omega M(-s)$, extending naturally the $\text{PSL}(2, \mathbb{C})$-action.

Our results have applications in dynamical systems and the geometry of flat surfaces since there is a one-to-one correspondence between rational 1-forms $\omega = (Q(z)/P(z))dz$, oriented rational quadratic differentials $\omega \otimes \omega = (Q^2(z)/P^2(z))dz^2$, rational complex vector fields $X_\omega = P(z)/Q(z) \partial / \partial z$, pairs of singular real analytic vector fields $(\Re (X_\omega), \Im (X_\omega))$ on $\hat{\mathbb{C}} \setminus \{Q = 0\}$, and singular flat surfaces $S_\omega = (\hat{\mathbb{C}}, g_\omega)$ provided with two real singular geodesic foliations. The metrics $g_\omega$ are associated with the quadratic differentials $\omega \otimes \omega$, and the foliations come from the horizontal and vertical trajectories. This correspondence is used in many works, e.g. [2, 16, 20]. For $\omega \in \Omega^1(-1^s)$, its associated quadratic differential $\omega \otimes \omega$ has poles of multiplicities 2 and they were studied by Strebel [24, 25].

Historically, C. Huygens [27, p. 72] gave formulas for the period of isochronous centers in the model of a simple pendulum as differential form $\omega$ extending naturally the PSL-s classification of rational 1-forms on the Riemann sphere up to PSL($\mathbb{C}$). For $\omega \in \Omega^1(-1^s)$, its associated quadratic differential $\omega \otimes \omega$ has poles of multiplicities 2 and they were studied by Strebel [24, 25].

Another interesting facet of isochronous 1-forms $\omega \in RI\Omega^1(-1^s)$, comes from dynamical systems. The phase portrait of the associated vector field $\Re (X_\omega)$ is a union of isochronous centers or annulus. In other words, any pair of trajectories of $\Re (X_\omega)$ that share a center basin has the same period. Looking at isochronous centers on $\mathbb{R}^2$; Mardešić et al. [19] studied the linearization problem and Gavrilov [11] considered the appearance of isochronous centers in polynomial of Hamiltonian systems on $\mathbb{C}^2$ and its relation to the famous Jacobian conjecture. A constructive result for isochronous vector fields on $\mathbb{C}$, using the residues, is provided by Muciño-Raymundo in [20, § 8]. A topological and analytic classification of complex polynomials vector fields on $\mathbb{C}$ with only isochronous centers was performed by Frías-Armenta and Muciño-Raymundo [9].

\section{Rational 1-forms with simple poles}

\subsection{Stratification}

We define a stratification on the set $\Omega^1(-1^s)$ of rational 1-forms on the Riemann sphere $\hat{\mathbb{C}}$, having exactly $-s \geq -2$ simple poles. First, recall that the infinite dimensional vector space of rational 1-forms admits a stratification fixing the multiplicities of the poles.
zeros \{k_1, \ldots, k_m\} and poles \{-s_1, \ldots, -s_n\}, where \(k_j, s_i \in \mathbb{N}\). The stratum of rational 1-forms with these multiplicities is denoted by \(\Omega^1[k_1, \ldots, k_m; -s_1, \ldots, -s_n]\); they are connected \((m + n + 1)\)-dimensional complex manifolds in the vector space of rational 1-forms; our notation is similar to \([8,20]\). On the other hand, for \(g \geq 2\) the stratum are not necessarily connected. Kontsevich and Zorich describe the connected components of holomorphic 1-forms in each stratum on \(M_g\); see \([18]\).

Our framework allows us to study

\[
\Omega^1(-1^s) = \bigsqcup_{s} \Omega^1[k_1, \ldots, k_m; -1, \ldots, -1],
\]

where the union takes all the multiplicities \(\{k_1, \ldots, k_m; -1, \ldots, -1\}\) such that \(\{k_1, \ldots, k_m\}\) is an integer partition of \(s - 2\), i.e. the sum \(k_1 + \cdots + k_m = s - 2\). The expression \((-1^s)\) in \((1)\) is motivated by the “exponential” notation for multiple poles of the same degree in the stratification by multiplicities of rational 1-forms; see \([4,18]\).

2.2 Polynomials

Let us recall the existence of two natural complex atlases for polynomials \(\mathbb{C}[z] = s\) with degree \(s\). For complex manifolds, we use notation as in \([10, \text{Ch. IV}]\). First, given the natural homeomorphism

\[
f_1 : \mathbb{C}[z] = s \longrightarrow \mathbb{C}^{s+1}\setminus\{b_s = 0\}
\]

\[
b_s z^s + \cdots + b_0 \longmapsto (b_s, \ldots, b_0),
\]

we obtain that \((\mathbb{C}[z] = s, f_1)\) is an \((s + 1)\)-dimensional complex coordinate system in \(\mathbb{C}[z] = s\). Clearly, the subset of polynomials with degree \(s\) and simple roots \(\mathbb{C}[z] = s \setminus \mathcal{D}(P, P')\) is open, where \(\mathcal{D}(P, P')\) denotes the discriminant of \(P\).

Second, the action of the symmetric group \(S(s)\) of \(s\) elements on \(\hat{\mathbb{C}}^s \setminus \Delta := \{(p_1, \ldots, p_s) \in \hat{\mathbb{C}}^s | p_i \neq p_{\kappa}, \text{ for all } i \neq \kappa\}\) is properly discontinuous. In fact, the quotient \((\hat{\mathbb{C}}^s \setminus \Delta) / S(s)\) is an \(s\)-dimensional complex manifold. Moreover, \((\mathbb{C}[z] = s \setminus \mathcal{D}(P, P'), v_s \circ f_2)\) is an \((s + 1)\)-dimensional complex coordinate system in \(\mathbb{C}[z] = s \setminus \mathcal{D}(P, P')\), where the map

\[
f_2 : \mathbb{C}[z] = s \setminus \mathcal{D}(P, P') \longrightarrow \mathbb{C}^s \times \left(\frac{\hat{\mathbb{C}}^s \setminus \Delta}{S(s)}\right)
\]

\[
b_s \prod_{i=1}^s (z - p_i) \longmapsto (b_s, \{p_1, \ldots, p_s\})
\]

is a natural bijection and
\[ v_s : \mathbb{C}^s \times \left( \hat{\mathbb{C}}^s \setminus \Delta \right) \rightarrow \mathbb{C}^{s+1} \setminus \{ b_s = 0 \} \]

\[
(b_s, \{ p_1, \ldots, p_s \}) \mapsto \left( b_s, -b_s \left( \sum_{i=1}^s p_i \right), \ldots, (-1)^s b_s \prod_{i=1}^s p_i \right)
\]

(2) is called the Viète map.

Finally, it is easy to prove that the coordinate systems \((\mathbb{C}[z]_{=s} \setminus \mathcal{D}(P, P'), f_1)\) and \((\mathbb{C}[z]_{=s} \setminus \mathcal{D}(P, P'), v_s \circ f_2)\) are holomorphically compatible; see [15]. A remarkable fact is that some properties of polynomials are easy to see in one coordinate system but others are kept hidden.

### 2.3 Complex atlases on \(\Omega^1(-1^s)\)

Similarly as in Sect. 2.2, three equivalents \((2s - 1)\)-dimensional complex atlases on \(\Omega^1(-1^s)\) will be constructed.

1. **Coefficients** We consider the Zariski open subset of \(\mathbb{CP}^{2s-1}\),

\[
\Omega^1_{\text{coef}}(-1^s) := \left\{ [a_{s-2} : \ldots : a_0 : b_s : \ldots : b_0] \in \mathbb{CP}^{2s-1} \bigg| \mathcal{D}(P, Q) \neq 0, \mathcal{D}(P, P') \neq 0 \right\},
\]

where \(Q(z) = a_{s-2}z^{s-2} + \ldots + a_0, P(z) = b_sz^{s} + \ldots + b_0\) and \(\mathcal{D}(P, Q)\) denotes the resultant of the polynomials \(P\) and \(Q\). If \(\{ (U_j^{\text{coef}}, \varphi_j^{\text{coef}}) \}\) denotes the complex atlas on \(\Omega^1_{\text{coef}}(-1^s)\), then \(\mathcal{U}_{\text{coef}} := \{ (f^{-1}_{\text{coef}}(U_j^{\text{coef}}), \varphi_j^{\text{coef}} \circ f_{\text{coef}}) \}\) is a complex atlas on \(\Omega^1(-1^s)\), where

\[
f_{\text{coef}} : \Omega^1(-1^s) \rightarrow \Omega^1_{\text{coef}}(-1^s)
\]

\[
\omega = \frac{a_{s-2}z^{s-2} + \ldots + a_0}{b_sz^{s} + \ldots + b_0} \mathrm{d}z \mapsto [a_{s-2} : \ldots : a_0 : b_s : \ldots : b_0]
\]

is a natural bijection map.

2. **Zeros–poles** We consider

\[
M_s := \left\{ \{ c_1, \ldots, c_{s-2}, p_1, \ldots, p_s \} \in \frac{\hat{\mathbb{C}}^{s-2}}{\mathcal{S}(s-2)} \times \frac{\hat{\mathbb{C}}^s \setminus \Delta}{\mathcal{S}(s)} \bigg| c_j \neq p_i \right\}.
\]

Recalling that there exists a biholomorphism between \(\hat{\mathbb{C}}^s / \mathcal{S}(s)\) and \(\mathbb{CP}^{s}\), \(M_s\) is a \((2s - 2)\)-dimensional open and dense complex submanifold of \(\mathbb{CP}^{s-2} \times \mathbb{CP}^{s}\). Consider the transition functions for a nontrivial principal \(\mathbb{C}^*\)-bundle over \(M_s\) as follows. Let \(u = \{ c_1, \ldots, c_{s-2}, p_1, \ldots, p_s \} \in M_s\) and set
\[ Q_u(z) := \begin{cases} (z - c_1) \cdots (z - c_{s-2}) & \text{where } c_j \neq \infty \text{ for all } j, \\ (z - c_1) \cdots (z - c_{j-1})(z - c_{j+1}) \cdots (z - c_{s-2}) & \text{if } c_j = \infty, \end{cases} \]

\[ P_u(z) := \begin{cases} (z - p_1) \cdots (z - p_s) & \text{where } p_i \neq \infty \text{ for all } i, \\ (z - p_1) \cdots (z - p_{i-1})(z - p_{i+1}) \cdots (z - p_s) & \text{if } p_i = \infty. \end{cases} \]

For \( \alpha \in I := \{1, 2, \ldots, 2s - 1\} \), we define \( \Omega^1_{zp}(-1^s) \) as the total space of the principal \( \mathbb{C}^* \)-bundle over \( M_s \) with the transition functions

\[ \phi_{\alpha \beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathbb{C}^* \]

\[ u \longmapsto \frac{Q_u(\alpha)}{P_u(\alpha)} \left( \frac{P_u(\beta)}{Q_u(\beta)} \right), \tag{3} \]

where \( \mathcal{U}_\alpha := \{ u \in M_s \mid Q_u(\alpha) \neq 0 \text{ and } P_u(\alpha) \neq 0 \} \). If \( \{(U_j^{zp}, \varphi_j^{zp})\} \) denotes the complex atlas on \( \Omega^1_{zp}(-1^s) \), then \( \mathfrak{A}_{zp} = \{(f_{zp}^{-1}(U_j^{zp}), \varphi_j^{zp} \circ f_{zp})\} \) is a complex atlas on \( \Omega^1(-1^s) \) where

\[ f_{zp} : \Omega^1(-1^s) \longrightarrow \Omega^1_{zp}(-1^s) \]

\[ \omega = \lambda \frac{(z - c_1) \cdots (z - c_{s-2})}{(z - p_1) \cdots (z - p_s)} dz \longmapsto \left[ (c_1, \ldots, c_{s-2}, p_1, \ldots, p_s), \lambda \right] \]

is a bijection map.

(3) Residues–poles Let \( H_s := \{(r_1, \ldots, r_s) \in (\mathbb{C}^*)^s \mid r_1 + \cdots + r_s = 0\} \) and recall that \( \widehat{\mathbb{C}^s} \setminus \Delta = \{(p_1, \ldots, p_s) \in \widehat{\mathbb{C}^s} \mid p_i \neq p_\kappa, \text{ for all } i \neq \kappa\} \). We consider a diagonal action of the symmetric group \( S(s) \) of \( s \) elements

\[ S(s) \times (H_s \times (\widehat{\mathbb{C}^s} \setminus \Delta)) \longrightarrow H_s \times (\widehat{\mathbb{C}^s} \setminus \Delta) \]

\[ (\sigma, (r_1, \ldots, r_s, p_1, \ldots, p_s)) \longmapsto (r_\sigma(1), \ldots, r_\sigma(s), p_\sigma(1), \ldots, p_\sigma(s)). \tag{4} \]

Clearly, the action above is properly discontinuous and the quotient

\[ \Omega^1_{zp}(-1^s) \cong \frac{H_s \times (\widehat{\mathbb{C}^s} \setminus \Delta)}{S(s)} \]

is a \((2s - 1)\)-dimensional open complex manifold. We denote the equivalence class under the action (4) as \( \langle r_1, \ldots, r_s ; p_1, \ldots, p_s \rangle \). Geometrically, an element in \( \Omega^1_{zp}(-1^s) \) is a configuration\(^1\) of \( s \) points \( \{p_i\} \) in the Riemann sphere with weights \( \{r_i\} \subset \mathbb{C}^* \) which satisfy the residue theorem. If \( \{(U_j^{rp}, \varphi_j^{rp})\} \) denotes the complex atlas on \( \Omega^1_{zp}(-1^s) \), then \( \mathfrak{A}_{rp} := \{(f_{zp}^{-1}(U_j^{rp}), \varphi_j^{zp} \circ f_{zp})\} \) is a complex atlas on \( \Omega^1(-1^s) \)

\(^1\) We convene that a configuration is an unordered set of points different between them.
where

\[ f_{\text{rp}} : \Omega^1(-1^s) \to \Omega^1_{\text{rp}}(-1^s) \]

\[ \omega = \sum_{i=1}^s \frac{r_i}{z - p_i} \, dz \mapsto \langle r_1, \ldots, r_s; p_1, \ldots, p_s \rangle \]

is a bijection map. For a 1-form \( \omega \) with the pole \( p_\kappa = \infty \), the term \( r_\kappa/(z - p_\kappa) \) is omitted in the sum above. Obviously, the residue theorem is the unique obstruction to realize \( \omega \in \Omega^1(-1^s) \). Moreover,

\[ 2s - 1 = \dim_\mathbb{C}(\Omega^1(-1^s)) \geq \dim_\mathbb{C}(\Omega^1_{\text{coef}}[1, \ldots, m; -1, \ldots, -1]) = m + s + 1. \]

Recall that the complex atlases above are valid only for rational 1-forms with simple poles. The study of rational 1-forms with poles of multiplicities greater or equal than 2 will be consider in a future work. Our main result is as follows.

Theorem 1 The complex manifolds \( \Omega^1_{\text{coef}}(-1^s) \), \( \Omega^1_{zp}(-1^s) \) and \( \Omega^1_{\text{rp}}(-1^s) \) are biholomorphic.

Proof We construct explicitly two biholomorphisms from \( \Omega^1_{\text{coef}}(-1^s) \) to \( \Omega^1_{\text{rp}}(-1^s) \) and \( \Omega^1_{zp}(-1^s) \) to \( \Omega^1_{\text{coef}}(-1^s) \).

First, we show that \( \Omega^1_{\text{rp}}(-1^s) \) and \( \Omega^1_{\text{coef}}(-1^s) \) are biholomorphic. For \( \langle P \rangle = \langle r_1, \ldots, r_s; p_1, \ldots, p_s \rangle \in \Omega^1_{\text{rp}}(-1^s) \), consider the map \( C_s : \Omega^1_{\text{rp}}(-1^s) \to \Omega^1_{\text{coef}}(-1^s) \) such that

\[
C_s(P) := \begin{cases} 
- \sum_{j=1}^s r_j \sum_{i \neq j} p_i : \sum_{j=1}^s \sum_{i, \kappa \neq j} p_i p_\kappa : \ldots : \\
(-1)^s - \sum_{j=1}^s r_j \prod_{i \neq j} p_i : \nu_s(1, \{p_1, \ldots, p_s\}) \end{cases}
\]

for \( p_i \in \mathbb{C} \),

\[
\begin{cases} 
\sum_{j \neq \kappa} r_j : - \sum_{j \neq \kappa} r_j \sum_{i \neq j} p_i : \ldots : \\
(-1)^{s-2} \sum_{j \neq \kappa} r_j \prod_{i \neq j} p_i : 0 : \nu_{s-1}(1, \{p_1, \ldots, \hat{\kappa}, \ldots, p_s\}) \end{cases}
\]

for \( p_\kappa = \infty \),

where \( \nu_s \) is the Viète map in (2). The hat over the pole \( p_\kappa \) indicates that it is omitted. A direct computation prove that the map \( C_s \) is a bijective map. If \( p_j \neq \infty \) for all \( j = 1, \ldots, s \), then the Jacobian matrix is

\[
DC_s(r_1, \ldots, r_s, p_1, \ldots, p_s) = \begin{pmatrix} A \ast & 0 \\ 0 & D_{\nu_s^s}\{p_1, \ldots, p_s\} \end{pmatrix}.
\]
where

\[
A = \begin{pmatrix}
\sum_{j \neq 1} p_j & \sum_{j \neq 2} p_j & \ldots & \sum_{j \neq s} p_j \\
- \sum_{j,i \neq 1} p_j p_i & - \sum_{j,i \neq 2} p_j p_i & \ldots & - \sum_{j,i \neq s} p_j p_i \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^s \prod_{j \neq 1} p_j & (-1)^s \prod_{j \neq 2} p_j & \ldots & (-1)^s \prod_{j \neq s} p_j
\end{pmatrix}, \tag{5}
\]

and \(v_\nu^\sigma\) denotes the Viète map \(v_\nu\) by removing the first coordinate. The rows of \(DC_s\) are linear independent and the map \(C_s\) is a biholomorphism. The case \(p_j = \infty\) is analogous. We are done, \(\Omega_{\text{coef}}^1(-1^s)\) is biholomorphic to \(\Omega_{\text{rp}}^1(-1^s)\).

Secondly, we prove that \(\Omega_{zp}^1(-1^s)\) and \(\Omega_{\text{coef}}^1(-1^s)\) are biholomorphic. For \(u = \{c_1, \ldots, c_{s-2}, p_1, \ldots, p_s\} \in U_\alpha \subset M_s\), we consider the map

\[
F_s : \Omega_{zp}^1(-1^s) \longrightarrow \Omega_{\text{coef}}^1(-1^s)
\]

\[
[u, \lambda] \longmapsto \left[ \frac{P_u(\alpha)}{Q_u(\alpha)} v_{s-2}(1, \{c_1, \ldots, c_{s-2}\}) : v_\nu(1, \{p_1, \ldots, p_s\}) \right].
\]

For all \(\alpha, \beta \in I\), the transition functions \(\phi_{\beta\alpha}\) make that the diagram below commutes.

\[
\begin{array}{ccc}
(U_\alpha \cap U_\beta) \times \mathbb{C}^* & \xrightarrow{\phi_{\beta\alpha}} & (U_\alpha \cap U_\beta) \times \mathbb{C}^* \\
\downarrow{\mathcal{F}_s} & & \downarrow{\mathcal{F}_s} \\
\Omega_{\text{coef}}^1(-s),
\end{array}
\]

In fact, the map \(\mathcal{F}_s\) is a biholomorphism. \(\square\)

From now, we only use the complex atlas by residues–poles \(\mathcal{A}_{\text{rp}}\) on \(\Omega^1(-1^s)\). However, by Theorem 1, the results in this paper are valid independently of the complex atlas.

**Definition 1** A rational 1-form \(\omega \in \Omega^1(-1^s)\) is *isochronous* when all their residues are purely imaginary. The family of rational isochronous 1-forms is denoted by

\[
\mathcal{RI} \Omega^1(-1^s) := \{\omega \in \Omega^1(-1^s) \mid \omega \text{ is isochronous}\}.
\]

**Corollary 1** The subfamily \(\mathcal{RI} \Omega^1(-1^s)\) is a \((3s - 1)\)-dimensional real analytic submanifold of \(\Omega^1(-1^s)\).

**Proof** The result follows using the complex atlas by residues–poles \(\mathcal{A}_{\text{rp}}\) and the \((s - 1)\)-dimensional real analytic submanifold \(\mathcal{Im}(H_s) := \{(ir_1, \ldots, ir_s) \in H_s \mid r_i \in \mathbb{R}^s\}\) of \(H_s\). \(\square\)
3 Classification of isotropy groups

3.1 The PSL(2, ℂ)-action

In this section, we prove that the natural holomorphic PSL(2, ℂ)-action on \( \Omega^1(-1^s) \), defined as:

\[
A_s : \text{PSL}(2, \mathbb{C}) \times \Omega^1(-1^s) \rightarrow \Omega^1(-1^s)
\]

\[
(T, \omega) \mapsto T_s \omega
\]

(6)

is proper for \( s \geq 3 \).

**Remark 1** Using the complex atlas by residues–poles \( \mathfrak{A}_{rp} \), the expression for the action is

\[
A_s(T, \langle r_1, \ldots, r_s; p_1, \ldots, p_s \rangle) = \langle r_1, \ldots, r_s; T(p_1), \ldots, T(p_s) \rangle.
\]

The residues are a set of \( \text{PSL}(2, \mathbb{C}) \)-invariant functions under the above action.

The class of an \( \omega \) is denoted by \( \langle \langle \omega \rangle \rangle \in \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C}) \). Recalling the definition of proper action as in [6, p. 53], we have the next result.

**Lemma 1** For \( s \geq 3 \), the holomorphic (resp. real analytic) \( \text{PSL}(2, \mathbb{C}) \)-action \( A_s \) on \( \Omega^1(-1^s) \) (resp. on \( \mathcal{R}\mathcal{I}\Omega^1(-1^s) \)) is proper.

**Proof** We will show that the map \( \tilde{A}_s : \text{PSL}(2, \mathbb{C}) \times \Omega^1(-1^s) \rightarrow \Omega^1(-1^s) \times \Omega^1(-1^s) \), defined as \( \tilde{A}_s(T, \omega) := (T_s \omega, \omega) \), is closed and the preimage for all points is a compact set. Applying Thm. 1 in [5, Sec. § 10.2 p. 101], the action \( A_s \) is proper.

First, we want to prove that the map \( \tilde{A}_s \) is closed. Consider a closed subset \( C \subset \text{PSL}(2, \mathbb{C}) \times \Omega^1(-1^s) \) and a convergent sequence \( \{ (\eta_m, \omega_m) \} \subset \tilde{A}_s(C) \) with a limit point \( (\eta, \omega) \in \tilde{A}_s(C) \) with a limit point \( (\eta, \omega) \in \tilde{A}_s(C) \), the sets of residues for \( \omega_m \) and \( \eta_m \) coincide. Explicitly, for each \( m \) we choose \( \eta_m = \{ r'_m, \ldots, r'_m; q_m, \ldots, q_m \} \) and \( \omega_m = \langle r_m, \ldots, r_m; p_m, \ldots, p_m \rangle \), without loss of generality \( r'_m = r_m \); here, our assertions will work for all \( i = 1, \ldots, s \).

If \( s \geq 3 \), then there exists a unique \( T_m \in \text{PSL}(2, \mathbb{C}) \) such that \( T(p_m) = q_m \). Since \( C \) is closed and \( (\eta, \omega) \) is the limit point of the sequence \( \{ (\eta_m, \omega_m) \} \), say \( \eta = \{ r'_1, \ldots, r'_s; q_1, \ldots, q_s \} \) and \( \omega = \langle r_1, \ldots, r_s; p_1, \ldots, p_s \rangle \); it follows that there is a unique limit transformation \( T \in \text{PSL}(2, \mathbb{C}) \) with \( T(p_i) = q_i \) and \( r'_i = r_i \), thus, the sequence \( \{ T_m \} \) converges to \( T \). Therefore, \( (\eta, \omega) \in \tilde{A}_s(C) \), and the map \( \tilde{A}_s \) is closed.

Secondly, we prove that \( A_{s-1}(\eta, \omega) \) is a compact set. Since there are at most \( s! \)! permutations of the configuration of poles \( \{ p_i \} \) to \( \{ q_i \} \), a configuration of poles with residues \( \{ r_1, \ldots, r_s; p_1, \ldots, p_s \} \) has at least two residues satisfying \( r_i \neq r_j \); hence there are at most \( (s-1)! \)! admissible permutations of \( \{ r_1, \ldots, r_s; p_1, \ldots, p_s \} \) to \( \{ r'_1, \ldots, r'_s; q_1, \ldots, q_s \} \). In fact, \( \tilde{A}_{s-1}(\eta, \omega) \) is a finite set, hence compact in \( \text{PSL}(2, \mathbb{C}) \times \Omega^1(-1^s) \). \( \square \)
3.2 Nontrivial isotropy groups

For \( \omega \in \Omega^1(-1^s) \), we denote by

\[
\text{PSL}(2, \mathbb{C})_\omega := \{ T \in \text{PSL}(2, \mathbb{C}) \mid T_* \omega = \omega \}
\]

its isotropy group.

A direct computation prove that \( \tilde{\mathcal{A}}^{-1}(\omega, \omega) = \text{PSL}(2, \mathbb{C})_\omega \times \{ \omega \} \) is a finite set when \( s \geq 3 \); see prove of Lemma 1. In fact, every \( \omega \in \Omega^1(-1^s) \) has finite isotropy group. A well-known result of Klein [17, p. 126] classifies the finite subgroups of \( \text{PSL}(2, \mathbb{C}) \); for a modern reference see [14, Sec. 2.13]. These finite subgroups are cyclic \( \mathbb{Z}_n \), dihedral \( D_n \) and the rotation groups \( G(S) \) of platonic solids \( S \); \( A_4 \) for tetrahedron, \( S(4) \) for octahedron and cube, and \( A_5 \) for dodecahedron and icosahedron. A natural question is which finite subgroups of \( \text{PSL}(2, \mathbb{C}) \) are realizable as isotropy groups of \( \omega \in \mathcal{RI} \Omega^1(-1^s) \)? The answer is as follows.

**Proposition 1** Every finite subgroup \( G < \text{PSL}(2, \mathbb{C}) \) appears as the isotropy group of suitable \( \omega \in \mathcal{RI} \Omega^1(-1^s) \).

**Proof** Fixing \( G < \text{PSL}(2, \mathbb{C}) \) finite subgroup, we construct explicitly a rational 1-form \( \omega \in \mathcal{RI} \Omega^1(-1^s) \) such that \( \text{PSL}(2, \mathbb{C})_\omega \cong G \). Consider \( \zeta_1, \ldots, \zeta_n \) the \( n \)th roots of unity; \( n \geq 2 \).

**Case** \( G = \mathbb{Z}_n \). Recall that the rotation group of a pyramid with polygonal base and triangular faces is \( \mathbb{Z}_n \), where \( n \) is the number of sides on the base. In particular, the set \( \{ \zeta_1, \ldots, \zeta_n, 0 \} \) in the Riemann sphere are the vertices of a pyramid as above. In fact, if

\[
\omega = \left( i, \ldots, i, -n i; \zeta_1, \ldots, \zeta_n, 0 \right) \in \mathcal{RI} \Omega^1(-1^{(n+1)}),
\]

then its isotropy group is \( \text{PSL}(2, \mathbb{C})_\omega \cong \mathbb{Z}_n \).

**Case** \( G = D_n \). A bipyramid is a polyhedron defined by two pyramids glued together by their basis. If all their faces are isosceles triangles, then its rotation group is \( D_n \) where \( n \) are the number of sides in the base for both pyramids. In particular, the set \( \{ \zeta_1, \ldots, \zeta_n, 0, \infty \} \) in the Riemann sphere are the vertices of a bipyramid as above. In fact, if

\[
\omega = \left( i, \ldots, i, -\frac{n}{2} i, -\frac{n}{2} i; \zeta_1, \ldots, \zeta_n, 0, \infty \right) \in \mathcal{RI} \Omega^1(-1^{(n+2)}),
\]

then its isotropy group is \( \text{PSL}(2, \mathbb{C})_\omega \cong D_n \).

**Case** \( G = G(S) \). We consider the union of the vertices of a platonic solid \( S \) and its dual \( S^* \) in the Riemann sphere. The suitable 1-form \( \omega \) has poles in both sets of vertices. The choice of the residues is as follows, residue \( i \) at the vertices of \( S \) and \( -ki \) at the
vertices of $S^*$, where

$$k := \begin{cases} 1 & \text{for } S \text{ the tetrahedron}, \\ 4/3 & \text{for } S \text{ the cube}, \\ 3/4 & \text{for } S \text{ the octahedron}, \\ 3/5 & \text{for } S \text{ the dodecahedron}, \\ 5/3 & \text{for } S \text{ the icosahedron}, \end{cases}$$

whence the isotropy group $\text{PSL}(2, \mathbb{C})_\omega \cong G(S)$. Concrete examples are provided below.

**Example 1**

1. For $\epsilon_1, \epsilon_2, \epsilon_3$ roots of $z^3 + 1 = 0$, the isotropy group of

$$\omega = \left\langle \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, 1, 0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \infty \right\rangle \in \mathcal{RI} \Omega^1(-1^8)$$

is isomorphic to the rotation group $A_4$ of a tetrahedron.

2. For $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ roots of $z^4 + 1 = 0$ and $\lambda = (\sqrt{6} - \sqrt{2})/2$, the isotropy group of

$$\omega = \left\langle \frac{i}{\sqrt{6}}, \frac{-i}{\sqrt{6}}, \frac{i}{\sqrt{6}}, \frac{-i}{\sqrt{6}}, 1, 0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, 0, \infty \right\rangle \in \mathcal{RI} \Omega^1(-1^{14})$$

is isomorphic to the rotation group $S_4(4)$ of a cube (octahedron).

Obviously, the degree $s$ depends on the order of $G$ and the 1-forms in the above proposition are isochronous. Solynin [23] constructs quadratic differentials on compact Riemann surface $\mathcal{R}$ associated with a weight graph embedded in $\mathcal{R}$. He explicitly gives quadratic differentials, with zeros in the vertices of a platonic solid and poles in the center of each face. They are different from our 1-forms. Alvarez-Parrilla et al. [1] classify the isotropy groups of rational 1-forms on the Riemann sphere using the complex atlas by zeros–poles.

For $2 \leq s \leq 11$, we classify the isotropy groups $\text{PSL}(2, \mathbb{C})_\omega$. These results will help in Sect. 4.

**Example 2**

1. For all $\omega = \langle r, -r; p_1, p_2 \rangle \in \Omega^1(-1^2)$, the isotropy group is $\text{PSL}(2, \mathbb{C})_\omega \cong \mathbb{C}^* \cong \{ T(z) = az \}$.

2. For $\omega \in \Omega^1(-1^3)$, the isotropy group is $\text{PSL}(2, \mathbb{C})_\omega \cong \mathbb{Z}_2$ if and only if $\omega = \langle r_1, r_2, r_3; p_1, p_2, p_3 \rangle$, i.e. two residues are equal.

**Lemma 2** Consider $\omega = \langle r_1, r_2, r_3, r_4; p_1, p_2, p_3, p_4 \rangle \in \Omega^1(-1^4)$.

1. If $\omega$ has exactly two equal residues and the cross-ratio

$$\text{cross-ratio} = \frac{(p_4 - p_1)(p_3 - p_2)}{(p_4 - p_2)(p_3 - p_1)},$$

The cross-ratio is defined as $(p_1, p_2, p_3, p_4) := (p_4 - p_1)(p_3 - p_2)$.
2. If $\omega$ has two pairs of equal residues and

$$(p_1, p_2, p_3, p_4) \notin \{-1, \frac{1}{2}, 2\}, \text{ then } PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_2.$$ 

$$ (p_1, p_2, p_3, p_4) \in \{-1, \frac{1}{2}, 2\}, \text{ then } PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_2.$$ 

3. If $\omega$ has three equal residues and

$$(p_1, p_2, p_3, p_4) \in \left\{ (1 \pm i\sqrt{3})/2 \right\}, \text{ then } PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_3.$$ 

4. For any other case, the isotropy group $PSL(2, \mathbb{C})_\omega$ is trivial.

**Proof** Case 1. Consider $\omega = \langle r_1, r_2, r_3, r_4; p_1, p_2, p_3, p_4 \rangle \in \Omega^1(-1^4)$ with $r_1 = r_2, r_3 \neq r_4$ and $(p_1, p_2, p_3, p_4) = -1$. We can verify that $(p_1, p_2, p_3, p_4) = (p_2, p_1, p_3, p_4)$, and there is a nontrivial $T \in PSL(2, \mathbb{C})$ such that $T\omega = \langle r_1, r_2, r_3, r_4; p_2, p_1, p_3, p_4 \rangle = \omega$. In fact, $T \in PSL(2, \mathbb{C})_\omega$. Since $r_3 \neq r_4$, there are no more elements in the isotropy; therefore $PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_2$. The result is analogous for

$$ r_1 \neq r_2, r_3 = r_4, \lambda = -1, \quad r_1 = r_4, r_2 \neq r_3, \lambda = 2, $$

$$ r_1 = r_3, r_2 \neq r_4, \lambda = 1/2, \quad r_1 \neq r_4, r_2 = r_3, \lambda = 2, $$

$$ r_1 \neq r_3, r_2 = r_4, \lambda = 1/2. $$

We leave the reader to perform the other cases. \hfill \Box

Obviously, for $s \geq 5$ the specific conditions to determine the isotropy groups are more complicated.

**Example 3** For $\omega \in \Omega^1(-1^5)$, the nontrivial isotropy groups $PSL(2, \mathbb{C})_\omega$ are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ or $D_3$.

Let us explicitly describe it. If $\omega = \langle r_1, \ldots, r_5; p_1, \ldots, p_5 \rangle$ has nontrivial isotropy group, then there are at least two pairs of equal residues or three equal residues. Case $r_1 = r_2$ and $r_3 = r_4$. Since $r_5$ is different from the other residues, the pole $p_5$ is a fixed point in the action of $PSL(2, \mathbb{C})_\omega$ on $\hat{\mathbb{C}}$. In fact, the isotropy group is cyclic. If $r_1 \neq r_3$, then $PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_2$. If $r_1 = r_3$, then $PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_2, \mathbb{Z}_3$, or $\mathbb{Z}_4$.

Case $r_1 = r_2 = r_3$. We suppose that $PSL(2, \mathbb{C})_\omega$ is not isomorphic to a cyclic group. Since $PSL(2, \mathbb{C})_\omega$ is nontrivial, $r_4 = r_5$, and $\{p_4, p_5\}$ is an orbit of order 2 in the action of $PSL(2, \mathbb{C})_\omega$ on $\hat{\mathbb{C}}$. In fact, the isotropy group is dihedral. Since $r_1 = r_2 = r_3$, the isotropy group $PSL(2, \mathbb{C})_\omega \cong D_3$.

Numerical conditions on $s \geq 3$, to realize $G < PSL(2, \mathbb{C})$ as an isotropy group for some $\omega \in \Omega^1(-1^s)$, are as follows.

**Proposition 2** Consider $n \geq 2$ and $n_1, n_2 \in \mathbb{N} \cup \{0\}$ such that $n_1 + n_2 \geq 2$. There exists $\omega \in \Omega^1(-1^s)$ such that

1. $PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_n$ if and only if $s \equiv 0, 1 \text{ or } 2 \text{ (mod } n\text{)}$, where $s > n$.
2. $PSL(2, \mathbb{C})_\omega \cong D_n$ if and only if $s \equiv 0 \text{ or } 2 \text{ (mod } n\text{)}$, where $s > n$.
3. $PSL(2, \mathbb{C})_\omega \cong A_4$ if and only if $s = 12n_1 + n_2$, where $n_2 \in \{0, 8, 10, 14, 16, 18\}$. 

Using Proposition 2, we complete the Table 1.

4. $PSL(2, \mathbb{C})_\omega \cong S(4)$ if and only if $s = 24n_1 + n_2$, where $n_2 \in \{0, 14, 18, 20, 26, 30, 32, 36\}$.

5. $PSL(2, \mathbb{C})_\omega \cong A_5$ if and only if $s = 60n_1 + n_2$, where $n_2 \in \{0, 32, 42, 50, 62, 72, 80, 90\}$.

**Proof** Case 1. Consider $\eta = (-n, i, \ldots, i; \infty, \zeta_1, \ldots, \zeta_n)$, where $\zeta_i$ are the $n$th roots of unity. Obviously, its isotropy group $PSL(2, \mathbb{C})_\eta = \{e^{2k\pi i / n_2}\} \cong \mathbb{Z}_n$. For $\omega \in \Omega^1(-1^s)$, if $PSL(2, \mathbb{C})_\omega \cong \mathbb{Z}_n$ then there is $T \in PSL(2, \mathbb{C})$ such that $PSL(2, \mathbb{C})_{T,\omega} = T \cdot PSL(2, \mathbb{C})_{\omega} \cdot T^{-1} = PSL(2, \mathbb{C})_\eta$; see [6, p. 107] and [14, p. 44]. It is easy to see that for $p_i \in \mathbb{C}$ pole of $T_\omega$, its orbit $PSL(2, \mathbb{C})_{T_e,\omega} \cdot p_i$, under the action of $PSL(2, \mathbb{C})_{T_e,\omega}$ on $\mathbb{C}$, is a set of poles for $T_\omega$. In other words, if $\ell$ is the number of poles $p_i$ with different orbits, then

$$s = \# \{\text{poles of } \omega\} = \# \{\text{poles of } T_\omega\} = \sum_{i=1}^{\ell} \#(PSL(2, \mathbb{C})_{T_\omega} \cdot p_i).$$

Since $\#(PSL(2, \mathbb{C})_{T_e,\omega} \cdot p_i) = n$, for $p_i \neq 0$ or $\infty$ and $\#(PSL(2, \mathbb{C})_{T_e,\omega} \cdot 0) = \#(PSL(2, \mathbb{C})_{T_e,\omega} \cdot \infty) = 1$, the result is proved. The other cases are analogous.

Using Proposition 2, we complete the Table 1.

### 4 Quotients

#### 4.1 Stratification by orbit types

For $s \geq 3$, the $PSL(2, \mathbb{C})$-action is proper and the classical theory of Lie groups can be applied. Mainly, we follow the theory and notation of Duistermaat and Kolk [6]. In order to describe the quotients $\Omega^1(-1^s)/PSL(2, \mathbb{C})$ and $R\mathcal{L}\Omega^1(-1^s)/PSL(2, \mathbb{C})$, recall that if a Lie group $G$ acts properly on a manifold $M$, then every closed subgroup $H$ of $G$ acts in a proper and free way on $G$; see [6, p. 93]. Moreover, the right coset $G/G_x$ is a manifold of dimension $\dim(G) - \dim(G_x)$ diffeomorphic to the orbit $G \cdot x$.  

---

**Table 1** Finite subgroups of $PSL(2, \mathbb{C})$ that appear as isotropy for $\omega \in \Omega^1(-1^s)$

| $s$ | Nontrivial isotropy groups for $\omega \in \Omega^1(-1^s)$ |
|-----|---------------------------------------------------------|
| 3   | $\mathbb{Z}_2$                                        |
| 4   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| 5   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, D_3$        |
| 6   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2, D_3, D_4$ |
| 7   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, D_3, D_4$    |
| 8   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_2, D_3, D_4, D_6, A_4$ |
| 9   | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_8, D_3, D_7$ |
| 10  | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2, D_4, D_5, D_8$ |
| 11  | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_{10}, D_3, D_9$ |
In our case, since the $\text{PSL}(2, \mathbb{C})$-action is proper and all isotropy groups are finite, the orbits $\text{PSL}(2, \mathbb{C}) \cdot \omega$ under $\mathcal{A}_s$ are 3-dimensional complex submanifolds of $\Omega^1(-1^s)$, biholomorphic to the right coset $\text{PSL}(2, \mathbb{C})/\text{PSL}(2, \mathbb{C})_\omega$. Similarly, for $\omega \in \mathcal{RI} \Omega^1(-1^s)$ its orbit $\text{PSL}(2, \mathbb{C}) \cdot \omega$ is a 6-dimensional real analytic submanifold of $\mathcal{RI} \Omega^1(-1^s)$, and $\text{PSL}(2, \mathbb{C}) \cdot \omega$ is diffeomorphic to the right coset $\text{PSL}(2, \mathbb{C})/\text{PSL}(2, \mathbb{C})_\omega$.

Furthermore, by applying the Theorem 2.7.4 in [6] the quotients $\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ and $\mathcal{RI} \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ admit a stratification by orbit types. The action $\mathcal{A}_s$ is proper and free in the generic open and dense subset

$$G(-1^s) := \{ \omega \in \Omega^1(-1^s) \mid \text{PSL}(2, \mathbb{C})_\omega \cong \{ \text{Id} \} \}.$$  

(7)

The quotient $\mathcal{E}(-1^s) := G(-1^s)/\text{PSL}(2, \mathbb{C})$ is a $(2s - 4)$-dimensional complex manifold. By following [6, p. 107], for each $\omega \in \Omega^1(-1^s)$ its orbit type is

$$\Omega^1(-1^s)_\omega := \left\{ \eta \in \Omega^1(-1^s) \mid \text{PSL}(2, \mathbb{C})_\eta \cong \text{PSL}(2, \mathbb{C})_\omega \right\}.$$  

Note that in our case the isotropy groups are isomorphic in $\Omega^1(-1^s)_\omega$ instead of conjugates since for finite subgroups of $\text{PSL}(2, \mathbb{C})$, they are equivalents; see [14, p. 50]. Similarly, its orbit type on the quotient is $\Omega^1(-1^s)_\omega/\text{PSL}(2, \mathbb{C})$. Looking at $\Omega^1(-1^s)_\omega$, its connected components $\{E_j\}$ are the stratum and they are complex submanifolds of $\Omega^1(-1^s)$ with dimension $\dim(E_j) \leq 2s - 1$. The higher dimensional stratum is $G(-1^s)$. For the quotient, the connected components of $\Omega^1(-1^s)_\omega/\text{PSL}(2, \mathbb{C})$ are the stratum and they are complex manifolds with dimension less or equal to $2s - 4$. The higher dimensional stratum is $\mathcal{E}(-1^s)$.

**Remark 2** There exists a holomorphic principal $\text{PSL}(2, \mathbb{C})$-bundle

$$\text{PSL}(2, \mathbb{C}) \longrightarrow G(-1^s) \quad \xrightarrow{\pi_s} \quad \mathcal{E}(-1^s),$$

where $\pi_s$ denotes the natural projection to the $\text{PSL}(2, \mathbb{C})$-orbits.

For $\mathcal{RI} \Omega^1(-1^s)$, the generic open and dense real analytic submanifold is

$$\mathcal{RI} G(-1^s) := \{ \omega \in \mathcal{RI} \Omega^1(-1^s) \mid \text{PSL}(2, \mathbb{C})_\omega \cong \{ \text{Id} \} \}.$$  

The quotient $\mathcal{RI} \mathcal{E}(-1^s) := \mathcal{RI} G(-1^s)/\text{PSL}(2, \mathbb{C})$ is a $(3s - 7)$-dimensional real analytic manifold. For $\mathcal{RI} \Omega^1(-1^s)$ and $\mathcal{RI} \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$, their stratification by orbit types are analogous as for $\Omega^1(-1^s)$ and $\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$, respectively.
4.2 Realizations

Let us define a realization\(^3\) for the quotient \(\Omega^1(-1^S)/\text{PSL}(2, \mathbb{C})\) using a complete set of \(\text{PSL}(2, \mathbb{C})\)-invariant functions. First, we consider the ordered set of residues as the complement of an arrangement of \(s\) hyperplanes

\[
A_s = \mathbb{C}^{s-1}_{(r_1, \ldots, r_{s-1})} \setminus \{ r_1 + \cdots + r_{s-1} = 0, \; r_i = 0 \; i = 1, \ldots, s-1 \}.
\]

For \(s = 2, 3\), the residues are a complete set of \(\text{PSL}(2, \mathbb{C})\)-invariant functions.

**Example 4** Case \(s = 2\), the natural projection

\[
\pi_2: \Omega^1(-1^2) \longrightarrow \Omega^1(-1^2)/\text{PSL}(2, \mathbb{C}): \quad (r_1, r_2; p_1, p_2) \longmapsto r_1
\]
determines a fiber bundle. Obviously, the base space is biholomorphic to \(\mathbb{C}^*/\mathbb{Z}_2\). Similarly, the quotient \(\mathcal{RI} \Omega^1(-1^2)/\text{PSL}(2, \mathbb{C})\) is diffeomorphic to \(\mathbb{R}^+ = \{ r_1 | r_1 > 0 \}\). For both cases, the fibers are \(\mathbb{C}^2\setminus \Delta = \{ (p_1, p_2) | p_1 \neq p_2 \}\).

**Example 5** Case \(s = 3\), using Example 2.2 the quotient \(\Omega^1(-1^3)/\text{PSL}(2, \mathbb{C})\) admits a stratification with two orbit types. It is homeomorphic to \(A_3/S(3)\), where the symmetric group \(S(3)\) acts linearly on \(A_3\) using the isomorphism

\[
S(3) \cong \{(0 \ 1 \ 0), (-1 \ 0 \ -1)\}.
\]

Similarly, the quotient \(\mathcal{RI} \Omega^1(-1^3)/\text{PSL}(2, \mathbb{C})\) has two connected components. A fundamental domain is

\[
\{(r_1, r_2) | r_1r_2 > 0 \text{ and } r_1 \leq r_2 \} \subset A_3.
\]

Here, \((r_1, r_2)\) determines the 1-form \((ir_1, ir_2, -i(r_1 + r_2); 0, \infty, 1)\). Their connected components come from \(\{ r_1 > 0 \}\) and \(\{ r_1 < 0 \}\). The orbit types are \(\{ r_1 = r_2 \}\) and \(\{ r_1 < r_2 \}\). Since the connected components are contractibles, the corresponding principal \(\text{PSL}(2, \mathbb{C})\)-bundle \(\pi_3 : \mathcal{RI}G(-1^3) \longrightarrow \mathcal{RI}E(-1^3)\) is trivial.

For \(s \geq 4\), the residues are not a complete set of \(\text{PSL}(2, \mathbb{C})\)-invariant functions. To enlarge our set, we fix three poles in \([0, \infty, 1]\) and consider the ordered set of poles with

\[
[C^*\setminus\{1\}]^{s-3} \setminus \Delta := \{ (p_4, \ldots, p_s) \in [C^*\setminus\{1\}]^{s-3} | \; p_i \neq p_\kappa \text{ for } i \neq \kappa \}.
\]

Given a configuration \(\{q_1, \ldots, q_s\} \subset \widehat{\mathbb{C}}\), there exist \((s^s)3!\) Möbius transformations \(T \in \text{PSL}(2, \mathbb{C})\) such that \(\{T(q_1), \ldots, T(q_s)\} = [0, \infty, 1, p_4, \ldots, p_s]\).

\(^3\) We use definition of realization as in [28, p. 6].
Remark 3 For an ordered collection

\[(r_1, \ldots, r_{s-1}, p_4, \ldots, p_s) \in A_s \times \left[ \mathbb{C}^* \setminus \{1\} \right]^{s-3} \setminus \Delta := \mathcal{M}(-s)\]

and each permutation \(\sigma \in S(s)\), there exists a unique \(T_\sigma \in \text{PSL}(2, \mathbb{C})\) that

\[(r_1, \ldots, r_s, p_4, \ldots, p_s) \mapsto \{ (r_{\sigma(1)}, 0), (r_{\sigma(2)}, \infty), (r_{\sigma(3)}, 1), (r_{\sigma(4)}, T_\sigma(p_4)), \ldots, (r_{\sigma(s)}, T_\sigma(p_s)) \} \]

:= \langle r_1, \ldots, r_s; 0, \infty, 1, p_4, \ldots, p_s \rangle \in \Omega^1(-s)/\text{PSL}(2, \mathbb{C})\).

Note the appearance of \(r_s = -(r_1 + \cdots + r_{s-1})\) and 0, \(\infty\), 1 on the right side. There is a natural \(S(s)\)-action on \(\mathcal{M}(-s)\). To recognize it, we define a group representation in the Coexeter generators of \(S(s)\), see [3, Sec. 1.2], as

\[\rho_s : S(s) \longrightarrow GL_{s-1}(\mathbb{Z}) \times \text{Bir}(\tilde{\mathbb{C}}^{s-3})\]

\[\sigma_j = (j, j+1) \mapsto (A_j, f_j) = \begin{cases} (A_1, \left( \frac{1}{z_4}, \ldots, \frac{1}{z_s} \right)) \\ (A_2, \left( \frac{z_4}{z_4-1}, \frac{z_5}{z_4}, \ldots, \frac{z_s}{z_4-1} \right)) \\ (A_3, \left( \frac{1}{z_4}, \frac{z_5}{z_4}, \frac{z_6}{z_4}, \ldots, \frac{z_s}{z_4} \right)) \\ (A_j, \left( z_\sigma(4), \ldots, z_\sigma(s) \right)) \end{cases}, \]

where \(j = 4, \ldots, s - 1\).

Here, \(\text{Bir}(\tilde{\mathbb{C}}^{s-3})\) denotes the group of complex birational maps on \(\tilde{\mathbb{C}}^{s-3}\); the birational map \(f_1\) from \(\sigma_1\) must be understood as \(f_1 : (z_4, \ldots, z_s) \mapsto (1/z_4, \ldots, 1/z_s)\). Since there is a biholomorphism between the Torelli space of the \(s\)-punctured sphere and \([\mathbb{C}^* \setminus \{1\}]^{s-3} \setminus \Delta\), the subgroup of birational maps \{\(f_\sigma\)\} is the corresponding Torelli modular group; see [22].

For \(j = 1, \ldots, s - 2\), the matrices \(A_j\) come from the identity matrix by exchanging the \(j\)th-row with the \((j+1)\)th-row; for \(j = s - 1\), \(A_{s-1}\) results from replacing, in the identity matrix, the \((s-1)\)th-row with \((-1, \ldots, -1)\).

It is a straightforward computation that \(\{\rho_s(\sigma_j)\}\) satisfy the relations in Coxeter’s presentation. Using \(\rho_s\), we define a \(S(s)\)-action on \(\mathcal{M}(-s)\) as:

\[S(s) \times \mathcal{M}(-s) \longrightarrow \mathcal{M}(-s)\]

\[(\sigma, (r_1, \ldots, r_{s-1}, p_4, \ldots, p_s)) \mapsto \left( A_\sigma \left( \begin{array}{c} r_1 \\ \vdots \\ r_{s-1} \end{array} \right), f_\sigma (p_4, \ldots, p_s) \right). \tag{8} \]

To recognize the quotient \(\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})\), the map

\[\mu_s : \mathcal{M}(-s) \longrightarrow \Omega^1(-1^s)\]

\[(r_1, \ldots, r_{s-1}, p_4, \ldots, p_s) \mapsto \langle r_1, \ldots, r_s; 0, \infty, 1, p_4, \ldots, p_s \rangle = \omega\]
will be useful. The number of preimages $\mu_s^{-1}(\omega)$ is $(s - 3)!$ Furthermore, the number of preimages $(\pi_s \circ \mu_s)^{-1}(\langle \omega \rangle)$ is less than or equal to $s!$ and the equality is fulfilled when $\omega \in G(-1^s)$; recall (7).

**Proposition 3** For $s \geq 4$, the realization of the quotient $\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ is $\mathcal{M}(-s)/\mathcal{S}(s)$.

**Proof** Let us prove that $\pi_s \circ \mu_s$ is a $\mathcal{S}(s)$-equivariant map, i.e.

$$(\pi_s \circ \mu_s)(\sigma \cdot (r_1, \ldots, r_{s-1}, p_4, \ldots, p_5)) = (\pi_s \circ \mu_s)(r_1, \ldots, r_{s-1}, p_4, \ldots, p_5)$$

for all $\sigma \in \mathcal{S}(s)$. For example, consider $\sigma_1 = (1\ 2) \in \mathcal{S}(s)$, the explicit calculation is

$$(\pi_s \circ \mu_s)(\sigma_1 \cdot (r_1, \ldots, r_{s-1}, p_4, \ldots, p_5))$$

$$= (\pi_s \circ \mu_s)(r_2, r_1, r_3, \ldots, r_{s-1}, 1/p_4, \ldots, 1/p_5)$$

$$= \langle (r_2, r_1, r_3, \ldots, r_{s}; 0, \infty, 1, 1/p_4, \ldots, 1/p_5) \rangle$$

$$= \langle (1/z) \langle r_2, r_1, r_3, \ldots, r_{s}; \infty, 0, 1, p_4, \ldots, p_5 \rangle \rangle$$

$$= \langle (r_1, \ldots, r_{s}; 0, \infty, 1, p_4, \ldots, p_5) \rangle$$

$$= (\pi_s \circ \mu_s)(r_1, \ldots, r_{s-1}, p_4, \ldots, p_5).$$

On the other hand, $(\pi_s \circ \mu_s)$ is surjective. Therefore, there exists a homeomorphism $\tilde{\mu}_s : \mathcal{M}(-s)/\mathcal{S}(s) \rightarrow \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ such that the diagram below commutes.

$$\begin{array}{ccc}
\mathcal{M}(-s) & \xrightarrow{\pi_s \circ \mu_s} & \Omega^1(-1^s) \\
\mathcal{S}(s) \downarrow & & \downarrow \mu_s \\
\mathcal{M}(-s)/\mathcal{S}(s) & \xrightarrow{\tilde{\mu}_s} & \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})
\end{array}$$

$\square$

Similarly, we define $\Imm(\mathcal{A}_s) := \{(ir_1, \ldots, ir_{s-1}) \in \mathcal{A}_s \mid r_i \in \mathbb{R}^s, \ i = 1, \ldots, s-1\}$. Since the $\mathcal{S}(s)$-action (8) is well defined on $\Imm(\mathcal{M}(-s)) := \Imm(\mathcal{A}_s) \times [\mathbb{C}^s \backslash \{1\}]^{s-3} \backslash \Delta$, the result below was proved.

**Corollary 2** For $s \geq 4$, the realization of the quotient $\mathcal{RI}\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C})$ is $\Imm(\mathcal{M}(-s))/\mathcal{S}(s)$.

For $s = 4$, the number of connected components of $\Imm(\mathcal{M}(-4))$ depends only on the number of connected components of $\Imm(\mathcal{A}_4)$. In fact, $\Imm(\mathcal{M}(-4))$ has 14
Remark 4
1. The complex manifold \( \omega \)

Recall that for \( 5.1 \) Isometries

S

The associated singular flat surfaces

\( X \) connected components,

\[
\begin{align*}
X^+_j :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_4 > 0, r_j > 0, r_t < 0 \ i \neq j \}, \\
X^-_j :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_4 < 0, r_j > 0, r_t < 0 \ i \neq j \}, \\
X^+_{j_1j_2} :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_4 > 0, r_{j_1} > 0, r_{j_2} > 0, r_{j_3} < 0 \}, \\
X^-_{j_1j_2} :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_4 < 0, r_{j_1} > 0, r_{j_2} > 0, r_{j_3} < 0 \}, \\
X_+ :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_j > 0, \ j = 1, 2, 3 \}, \\
X_- :&= \{(ir_1, ir_2, ir_3, p_4) \in \mathbb{H}(\mathcal{M}(-4)) \mid r_j < 0, \ j = 1, 2, 3 \}.
\end{align*}
\]

By applying the \( \mathcal{S} \) (4)-action (8), these components are identified as:

\[
\begin{align*}
X^+_+ \sim X^+_{23} \sim X^+_{13} \sim X^+_{12}, \\
X^-_- \sim X^-_1 \sim X^-_2 \sim X^-_3, \\
X^+_1 \sim X^+_2 \sim X^+_3 \sim X^-_{23} \sim X^-_{13} \sim X^-_{12}.
\end{align*}
\]

Using Proposition 3, Corollary 2 and Table 1, the result below was proved.

Lemma 3
1. The quotient \( \Omega^1(-1^4)/\text{PSL}(2, \mathbb{C}) \) is connected and it admits a stratification with 4 orbit types and 5 stratum.
2. The quotient \( \mathcal{RI}\Omega^1(-1^4)/\text{PSL}(2, \mathbb{C}) \) admits a stratification with 4 orbit types, 3 connected components and 10 stratum. The corresponding principal \( \text{PSL}(2, \mathbb{C}) \)-bundle \( \pi_4 : \mathcal{RI}\mathcal{G}(-1^4) \longrightarrow \mathcal{RI}\mathcal{E}(-1^4) \) is nontrivial.

Remark 4
1. The complex manifold \( \Omega^1(-1^s) \) and the quotient \( \Omega^1(-1^s)/\text{PSL}(2, \mathbb{C}) \) are connected.
2. The real analytic manifold \( \mathcal{RI}\Omega^1(-1^s) \) and the quotient \( \mathcal{RI}\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C}) \) have \( s - 1 \) connected components.

For odd numbers \( 3 \leq s \leq 11 \), Table 2 shows the number of orbit types and stratum on the quotient \( \mathcal{RI}\Omega^1(-1^s)/\text{PSL}(2, \mathbb{C}) \).

5 The associated singular flat surfaces \( S_\omega \)

5.1 Isometries

Recall that for \( \omega \in \Omega^1(-1^s) \), there is a complex atlas \( \{(V_j, \psi_j)\} \) on \( X_\omega = \mathbb{C}\backslash \{\text{zeros and poles of } \omega\} \), where \( \{V_j\} \) is an open cover by simply connected sets and
the functions

\[ \Psi_j(z) = \int_{z_0}^{z} \omega : V_j \to \mathbb{C} \]

are well defined for all \( j \). Moreover, \( \Psi_{jk}(z) = z + a_{jk} \), for \( a_{jk} \in \mathbb{C} \). If \( \omega \in \Omega^1\{k_1, \ldots, k_m; -1, \ldots, -1\} \subset \Omega^1(-1^s) \), then the zero of multiplicity \( k_j \) is a singularity of cone angle \((2k_j + 2)\pi\) and the pole \( p_i \) is a cylindrical end of diameter \( T_i = 2\pi|r_i| \), where \( r_i = \text{Res}(\omega, p_i) \), \( j = 1, \ldots, m \) and \( i = 1, \ldots, s \); see [20,21].

For \( \omega = (Q(z)/P(z))dz \in \Omega^1(-1^s) \), its associated singular flat surface \( S_\omega = (\hat{\mathbb{C}}, g_\omega) \) has the Riemannian metric

\[
g_\omega(z) := \begin{pmatrix}
|q(z)|^2 & 0 \\
0 & |p(z)|^2
\end{pmatrix}.
\]

We denote by \( S^1 \) the unit circle on \( \mathbb{C} \). The result below is well known; see [21].

**Proposition 4** For \( \omega, \eta \in \Omega^1(-1^s) \), their associated singular flat surfaces \( S_\omega \) and \( S_\eta \) are isometric if and only if there exist \( \lambda \in S^1 \) and \( T \in \text{PSL}(2, \mathbb{C}) \) such that \( \eta = \lambda T \omega \).

### 5.2 The \((S^1 \times \text{PSL}(2, \mathbb{C}))\)-action

By applying Proposition 4, we can extend naturally the \( \text{PSL}(2, \mathbb{C}) \)-action (6) as follows.

\[
\hat{A}_s : (S^1 \times \text{PSL}(2, \mathbb{C})) \times \Omega^1(-1^s) \to \Omega^1(-1^s)
\]

\[
((\lambda, T), \omega) \mapsto \lambda T \omega.
\]

Similarly, we can extend the \( S \) \((s)\)-action (8) as:

\[
(S^1 \times S(s)) \times \mathcal{M}(-s) \to \mathcal{M}(-s)
\]

\[
((\lambda, \sigma), (r_1, \ldots, r_{s-1}, p_4, \ldots, p_s)) \mapsto \left( \lambda A_{r_1} \begin{pmatrix} r_1 \\ \vdots \\ r_{s-1} \end{pmatrix}, \ f_{\sigma}(p_4, \ldots, p_s) \right).
\]

The expression for the \((S^1 \times \text{PSL}(2, \mathbb{C}))\)-action, using the complex atlas by residues–poles on \( \Omega^1(-1^s) \), is

\[
\hat{A}_s(\lambda, T, (r_1, \ldots, r_s; p_1, \ldots, p_s)) = (\lambda r_1, \ldots, \lambda r_s; T(p_1), \ldots, T(p_s)).
\]

We use the techniques developed in Sects. 3 and 4 to prove the results below.

**Remark 5** Since the \((S^1 \times \text{PSL}(2, \mathbb{C}))\)-action is proper, the quotient

\[
\frac{\Omega^1(-1^s)}{S^1 \times \text{PSL}(2, \mathbb{C})} = \frac{\{S_\omega \mid \omega \in \Omega^1(-1^s)\}}{\text{Isometries}} := \mathcal{M}(-1^s)
\]

admits a stratification by orbit types. Furthermore, the realization for the quotient \( \mathcal{M}(-1^s) \) is \( \mathcal{M}(-1^s)/S^1 \times S(s) \).
For the subgroups $\mathbb{Z}_2 \times \text{PSL}(2, \mathbb{C}) < S^1 \times \text{PSL}(2, \mathbb{C})$ and $\mathbb{Z}_2 \times S(s) < S^1 \times S(s)$, the actions (9) and (10) are well defined on $\mathcal{RI}\Omega^1(-1^s)$ and $\mathfrak{Im}(\mathcal{M}(-s))$, respectively.

**Remark 6** The $(\mathbb{Z}_2 \times \text{PSL}(2, \mathbb{C}))$-action on $\mathcal{RI}\Omega^1(-1^s)$ is proper; therefore, the quotient

$$\frac{\mathcal{RI}\Omega^1(-1^s)}{\mathbb{Z}_2 \times \text{PSL}(2, \mathbb{C})} = \left\{ [\omega] \mid \omega \in \mathcal{RI}\Omega^1(-1^s) \right\} / \text{Isometries} := \mathcal{RI}\mathfrak{M}(-1^s)$$

admits a stratification by orbit types. Furthermore, the realization for the quotient $\mathcal{RI}\mathfrak{M}(-1^s)$ is homeomorphic to $\mathfrak{Im}(\mathcal{M}(-s)) / \mathbb{Z}_2 \times S(s)$.

**Example 6** The quotient $\mathfrak{M}(-3)$ and $\mathcal{RI}\mathfrak{M}(-3)$ are connected and both admit a stratification with two orbit types. For $\mathcal{RI}\mathfrak{M}(-3)$, a fundamental domain is

$$\left\{ (r_1, r_2) \mid 0 < r_1 \leq r_2 \right\},$$

and the orbit types are $\{r_1 = r_2\}$ and $\{r_1 < r_2\}$.

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