NEW ACCUMULATIVE SCORE FUNCTION BASED BOUND FOR SPARSITY LEVEL OF $L_1$ MINIMIZATION

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ABSTRACT

This paper discusses a fundamental problem in compressed sensing: the sparse recoverability of $l_1$ minimization with an arbitrary sensing matrix. We develop a new accumulative score function (ASF) to provide a lower bound for the recoverable sparsity level ($SL$) of a sensing matrix while preserving a low computational complexity. We first define a score function for each row of a matrix, and then ASF sums up large scores until the total score reaches 0.5. Interestingly, the number of involved rows in the summation is a reliable lower bound of $SL$. It is further proved that ASF provides a sharper bound for $SL$ than coherence. We also investigate the underlying relationship between the new ASF and the classical RIC and achieve a RIC-based bound for $SL$.

Index Terms— accumulative score function, compressive sensing, $l_1$ minimization, sparsity level, sparse recovery.

1. INTRODUCTION

Recently, compressed sensing (CS) has become a powerful technique for exploring sparse representation of a signal given a redundant dictionary. In CS, it is a fundamental problem to study the sparsity level ($SL$) of $l_1$ minimization. Specifically speaking, $s \leq SL$ means for any $s$-sparse vector $x$ (i.e., $\|x\|_0 \leq s$) can be correctly recovered by solving the following $l_1$ minimization problem.

$$(L_1) \quad \hat{x} = \arg \min_{x \in \mathbb{R}^m} \|x\|_1, \text{ s.t. } y = Ax,$$

where $A \in \mathbb{R}^{n \times m}$ ($n < m$) is a sensing matrix, $y \in \mathbb{R}^n$ is the sensed data and $\hat{x} \in \mathbb{R}^m$ is the $l_1$ minimizer. Generally, the larger the $l_0$ norm of the vector $x$, the less possible to precisely recover this vector by solving a corresponding $l_1$ minimization problem. Therefore, we are more interested in providing a precise lower bound of $SL$. Researchers have explored many tools to study the sparse recovery property of the $l_1$ minimization problem. Some useful and powerful tools include null space property (NSP) [1, 2], coherence [1, 3], restricted isometry property (RIP) [4, 5], which will be briefly introduced below.

NSP states that $l_1$-minimization holds a sparsity level of $s$ if and only if the following condition holds

$$(NSP) \quad \sum_{v \in S} |v_i| < \frac{1}{2} \|v\|_1, \forall v \neq 0 \in ker A,$$

where $|S| = s$ and $ker A$ means kernel of matrix $A$. NSP itself is NP-hard but it reveals the fact that the sparse recovery ability of the $l_1$ minimization problem is actually determined by its sensing matrix. NSP has also been widely applied to derive other useful sparsity-related properties on a sensing matrix, such as coherence and RIP.

Coherence is an important measure to get a lower bound of $SL$ as shown below [1, 3]:

$$SL \geq \frac{1}{2} \left(1 + \mu^{-1}\right) - 1,$$

where $\mu$ denotes the coherence of $A$. Usually the smaller the coherence, the higher the $SL$. However, coherence may not be sufficient to estimate a lower bound of $SL$ because, as pointed by [6], rescaling the columns of a sensing matrix may enhance the sparsity recovery ability of $l_1$ minimization in some cases but does not change the value of the coherence.

Restricted isometry property (RIC) is a more sophisticated sparsity-relevant constant than coherence [7]. The RIP condition of order $s$ states that, there exists a RIC, $\sigma_s \in (0, 1)$, that makes the following inequality holds.

$$(RIP) \quad (1 - \sigma_s) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \sigma_s) \|x\|_2^2,$$

for any $s$-sparse vector $x \in \mathbb{R}^m$. $\sigma_s$ increases with the sparsity number $s$. In order to guarantee the correct recovery of a sparse vector via $l_1$ minimization, RIC is required to be less than a certain bound. Tremendous effort has been made to sharpen this bound [8]. But it is still combinatorial complexity to compute RIC of a certain sensing matrix.

In this paper, we introduce a new method, accumulative score function (ASF), to analyze the sparsity recovery ability of the $l_1$ minimization. With the help of NSP, we derive a
lower bound of $SL$ based on ASF. ASF considers not only the angle information between columns but also the scale information of each column in a sensing matrix. Importantly, we prove that ASF tends to provide a sharper bound for $SL$ than coherence, and, thus, generally RIP is a more sophisticated tool to study sparse recovery of the $l_1$ minimization. Furthermore, we investigate the underlying relationship between the new ASF and the classical RIC, and derive a new RIC-based formulation for the lower bound of $SL$.

The rest of this paper is organized as follows. Section 2 introduces the ASF and analyze its performance in sparsity estimation. Section 3 is devoted to prove that ASF-based bound is relatively sharper than the coherence-based bound for $SL$, and to derive an alternative RIC-based bound for $SL$. We finally conclude this paper in Section 4.

2. ACCUMULATIVE SCORE FUNCTION

**Notations** Let $[m]$ denote the set $\{1, ..., m\}$ and let $T$ be a subset of $[m]$. Use $|T|$ to denote the cardinality of a set $T$. For a matrix $B \in \mathbb{R}^{n \times m}$, $B_T$ denotes a submatrix constructed by columns of matrix $B$ indexed by elements in $T$. $\Lambda_{max}(B)$ and $\Lambda_{min}(B)$ respectively denote the maximal and minimal eigenvalue of $B$.

Let $\{\alpha_1, ..., \alpha_m\}$ denote columns of the measurement matrix $A \in \mathbb{R}^{n \times m}$. Let $C = A^T A \in \mathbb{R}^{m \times m}$. Obviously, the diagonal entries of matrix $C$, denoted as $c_{ii}$, equal to $\|\alpha_i\|_2^2$ for $i \in [m]$. The non-diagonal entries, $c_{ij}$ ($i \neq j$), are equal to inner products between $\alpha_i$ and $\alpha_j$ (i.e. $c_{ij} = <\alpha_i, \alpha_j>$).

We define the score function $\rho(i)$ for each row $i$ of matrix $C$ as:

$$\rho(i) = \frac{\nu(i)}{\nu(i) + 1},$$

where

$$\nu(i) = \max_{j \neq i} \left| \frac{c_{ij}}{c_{ii}} \right|, \forall j \neq i.$$  \hspace{1cm} (1)

By summing up the first $s$ largest scores, we define the ASF as

$$\rho(S^*) = \max_{|S| = s} \sum_{i \in S} \rho(i),$$

where $S^*$ denotes optimal index set that makes $\sum_{i \in S} \rho(i)$ largest. Without loss of generality, we can assume sequence $\{\rho(1), ..., \rho(m)\}$ is in a non-increasing order, that is

$$\rho(i) \geq \rho(j), \forall i \leq j.$$  \hspace{1cm} (2)

Then $\rho(S^*)$ with $|S^*| = s$ can be re-written as

$$\rho(S^*) = \sum_{i = 1}^{s} \rho(i)$$  \hspace{1cm} (3)

### 2.1. Sparsity Analysis

We now aim to show how to apply ASF to obtain a lower bound of sparsity level of the $l_1$ minimization problem with an arbitrary sensing matrix. First we show a lemma which can be taken as an alternative interpretation of NSP.

**Lemma 1** Let $S$ denote the support set of a sparse vector $x$, i.e. the index set of nonzero-entry of $x$, while $\hat{x}$ denotes an $l_1$ minimizer, then we have

$$\|x - \hat{x}\|_1 \leq 2 \|(x - \hat{x})_S\|_1.$$  \hspace{1cm} (4)

**Proof** Firstly we have $\|x\|_1 \geq \|\hat{x}\|_1$ which means

$$\|(x - \hat{x})_S\|_1 \leq \|x\|_1 - \|\hat{x}_S\|_1 \geq \|\hat{x}\|_1 = \|\hat{x}_S\|_1.$$  \hspace{1cm} (5)

Let $e = x - \hat{x}$, then we have

$$\|e\|_1 = \|e_S\|_1 + \|\hat{x}_S\|_1 \leq \|e_S\|_1 + \|(x - \hat{x})_S\|_1 = 2\|e_S\|_1.$$  \hspace{1cm} (6)

Thus Lemma 1 is proven. \hspace{1cm} \blacksquare

Based on the above lemma, we can then show how to apply ASF to assess the sparsity level of the $l_1$ minimization problem in the following theorem.

**Theorem 1** Suppose $S$ is the support set of a sparse vector $x$. If $\rho(S) < \frac{1}{2}$, $x$ can be correctly recovered by solving the $l_1$ minimization problem.

**Proof** Assume $x \neq \hat{x}$ and let $e = x - \hat{x}$, then we have $A^T A e = 0$ since $A x = A \hat{x}$. For each row $i$, we have

$$\sum_{j \neq i} c_{ij} e_j + c_{ii} e_i = 0,$$

from which we can derive the following inequality:

$$|c_{ii} e_i| \leq \sum_{j \neq i} |c_{ij} e_j| \leq |c_{ik}| \sum_{j \neq i} |e_j|,$$

where $|c_{ik}| = \max_{j \neq i} |\{c_{ij}\}|$. Then we get

$$|c_{ii} + |c_{ik}| |e_i| \leq |c_{ik}| \|e\|_1.$$  \hspace{1cm} (7)

Furthermore, we have

$$\|e\|_1 \leq \frac{|c_{ik}|}{c_{ii} + |c_{ik}|} \|e\|_1 = \rho(i) \|e\|_1$$  \hspace{1cm} (8)

Combining (3) in lemma 1 with the inequality in (5), we get

$$\frac{1}{2} \|e\|_1 \leq \|e_S\|_1 \leq \rho(S) \|e\|_1 \leq \frac{1}{2} \|e\|_1,$$

which is contradictory. Then we have $x = \hat{x}$. Thus we prove the theorem. \hspace{1cm} \blacksquare

It should be noted that $\rho(S) < \frac{1}{2}$ is a sufficient condition to guarantee that the $l_1$ minimization problem can recover the correct sparse vector. In general, information about the support set of a sparse vector is unknown. But we can still use ASF to provide some useful information about the sparsity
level of a sensing matrix. The following corollary presents a simple way to get a lower bound of SL via ASF.

**Corollary 1** Suppose sequence \( \{\rho(1), \ldots, \rho(m)\} \) is in a non-increasing order, then we have \( SL \geq l^* \), where \( l^* \) is determined below:

\[
  l^* = \arg \min_l \left\{ \sum_{i=1}^{l} \rho(i) \geq \frac{1}{2} \right\} - 1. 
\]  

(6)

In fact, \( l^* \) denotes the largest integer that makes

\[
  \sum_{i \in S} \rho(i) < \frac{1}{2},
\]

for an arbitrary index set \( S \subset [m] \) with \( |S| \leq l^* \). If sequence \( \{\rho(1), \ldots, \rho(m)\} \) is not originally in a non-increasing order, one can sort it in a non-increasing order. And such a sort operation does not affect the conclusion in corollary 1.

According to corollary 1, we give a corresponding algorithm in table I to compute the exact value of \( l^* \) when given a sensing matrix \( A \). The computational complexity of the presented algorithm is \( O(m^2) \), which is the same as that of computing the coherence of \( A \).

**3. DISCUSSIONS**

3.1. Relation to Coherence

In this subsection, we show that ASF improves on the bound of \( SL \) derived by coherence. To make the comparison between ASF and coherence straightforward, we first assume \( A \) consists of \( l_2 \) normalized columns. Therefore we have

\[
  \nu(i) \leq \mu, \forall i \in [m],
\]

which means

\[
  \rho(i) = \frac{\nu(i)}{1 + \nu(i)} \leq \frac{\mu}{1 + \mu}.
\]

Then we get

\[
  \max_{|S|=k} \rho(S) \leq \frac{k}{1 + \mu}.
\]

According to (1) and (3), to ensure correct \( l_1 \) recovery of a sparse vector, the sparsity \( k \) should be less than \( \frac{1}{2}(1 + \mu^{-1}) \), which guarantees that

\[
  \max_{|S|=k} \rho(S) < \frac{1}{2}.
\]

Above inequality implies that \( l^* \geq k \). Therefore, we can say ASF is a finer measure for sparsity estimation than coherence. It is important to note that, different from coherence which only considers the angle information between columns in a sensing matrix, ASF takes into account angle information as well as scale information of each column, which can improve ASF’s performance in sparsity estimation.

3.2. Relation to RIC

RIP condition of order \( s \) is commonly understood as a measure of “overall conditioning” of the set of \( n \times s \) submatrices of \( A \). For simplicity, we first make several definitions: Let \( T \) be an arbitrary set that has elements in \([m]\) with its cardinality equal to or less than \( s \). Let \( A_T \) denote a \( n \times s \) submatrix of \( A \). Then we can get two critical sets as below:

\[
  T_1 = \arg \max_{|T|=s} \Lambda_{\max}(A_T^T A_T),
\]

\[
  T_2 = \arg \min_{|T|=s} \Lambda_{\min}(A_T^T A_T).
\]

Then we set \( k_{\text{max}}, k_{\text{min}} \) as

\[
  k_{\text{max}} = \Lambda_{\max}(A_{T_1}^T A_{T_1}),
\]

\[
  k_{\text{min}} = \Lambda_{\min}(A_{T_2}^T A_{T_2}).
\]

(7)

Since the RIP condition does not hold the homogeneity property \( [9] \), to avoid this problem, we use the following condition instead:

\[
  k_{\text{min}} \|x\|_2^2 \leq \|Ax\|_2^2 \leq k_{\text{max}} \|x\|_2^2, \forall \|x\|_0 \leq s
\]

where \( k_{\text{max}} > k_{\text{min}} > 0 \) and one can check parameter \( \frac{k_{\text{max}}}{k_{\text{min}}} \) holds the homogeneity property. Obviously RIP condition is a special case of the above formulation by setting \( k_{\text{max}} = 1 + \sigma_s \) and \( k_{\text{min}} = 1 - \sigma_s \) where \( \sigma_s \in (0, 1) \).

**Lemma 2** Set \( C = A^T A \), \( k_{\text{max}} \) and \( k_{\text{min}} \) are defined in (7). There exist indices \( h \) and \( l \) such that

\[
  c_{hh} - k_{\text{min}} \leq \sum_{j \neq h, j \in T_1} |c_{hj}| \leq (s - 1)|c_{hh}|
\]

\[
  k_{\text{max}} - c_{ll} \leq \sum_{j \neq l, j \in T_2} |c_{lj}| \leq (s - 1)|c_{lf}|,
\]
where $|c_{hh}| = \max_{j \neq h} \{ |c_{hj}| \}$ and $|c_{lf}| = \max_{j \neq l} \{ |c_{lj}| \}$. This Lemma can be easily derived using the Gersgorin Disc Theorem.

Based on Lemma 2, we get the following theorem:

**Theorem 2** With $k_{\text{max}}$ and $k_{\text{min}}$ defined in (7), we have

$$\frac{k_{\text{min}}}{k_{\text{max}}} \geq \min_{j \neq l} \frac{|c_{hh}|}{|c_{lf}|} - 2(s-1) \max_i \nu(i)$$

**Proof** According to lemma 2, we have

$$c_{hh} - k_{\text{min}} \leq (s-1)|c_{hh}|$$

$$\frac{c_{hh}}{c_{lf}} k_{\text{max}} - c_{hh} \leq (s-1)\frac{c_{hh}}{c_{lf}} |c_{lf}|,$$

then we get

$$\frac{c_{hh}}{c_{lf}} - \frac{k_{\text{min}}}{k_{\text{max}}} \leq (s-1)\frac{|c_{hh}|}{k_{\text{max}}} + (s-1)\frac{|c_{lf}|}{c_{lf}} \frac{c_{hh}}{k_{\text{max}}}$$

$$\leq 2(s-1) \max_i \nu(i)$$

with $\nu(i)$ is defined in [2]. Furthermore, we get

$$\frac{k_{\text{min}}}{k_{\text{max}}} \geq \min_{j \neq l} \frac{c_{ij}}{c_{jj}} - 2(s-1) \max_i \nu(i).$$

By combining $k_{\text{max}}, k_{\text{min}}$ with the definition of RIP, we have:

$$\frac{k_{\text{min}}}{k_{\text{max}}} = \frac{1 - \sigma_s}{1 + \sigma_s}.$$  

According to [3], we further have

$$\frac{1 - \sigma_s}{1 + \sigma_s} \geq \min_{i \neq j} \frac{c_{ii}}{c_{jj}} - 2(s-1) \max_i \nu(i)$$

Given $\max_i \nu(i) < \frac{1}{2(s-1)} \min_{i \neq j} \min_{j \neq j} \frac{c_{ii}}{c_{jj}}$, we have

$$\sigma_s \leq 1 - \left( \min_{i \neq j} \frac{c_{ii}}{c_{jj}} - 2(s-1) \max_i \nu(i) \right)$$

$$\frac{1}{1 + \min_{i \neq j} \frac{c_{ii}}{c_{jj}} - 2(s-1) \max_i \nu(i)}.$$  

When given a specific number $t \in (0, 1)$, if we have

$$s < \frac{1}{2 \max_i \nu(i)} \left( \min_{i \neq j} \frac{c_{ii}}{c_{jj}} - \frac{1 - t}{1 + t} \right) + 1,$$

then it is guaranteed that $\sigma_s < t$. On the other hand, $t = \frac{1}{8}$ is a sharp bound to guarantee correct recovery of any $s$-sparse vector via $l_1$ minimization [8]. Therefore we can get another lower bound for $SL$ as follows:

$$SL \geq \frac{1}{2 \max_i \nu(i)} \left( \min_{i \neq j} \frac{c_{ii}}{c_{jj}} - \frac{1}{2} \right).$$

**4. CONCLUSION**

In this paper, we successfully developed a new ASF for analyzing the sparsity recovery behavior of $l_1$ minimization. We prove that ASF provides a sharper bound for sparsity level of a sensing matrix than coherence. Also we further analyze the underlying relationship between ASF and RIC and derive an alternative RIC-based bound for sparsity level. ASF may find applications in areas like sparse coding and dictionary learning, coherent sampling and so on. ASF may provide hints on how to re-scale the columns of a sensing matrix in order to enhance sparsity. In future work, we are interested in applying ASF to analyze the stable recovery ability of the $l_1$ minimization problem in noisy environment.

**5. REFERENCES**

[1] Rémi Gribonval and Morten Nielsen, “Sparse representations in unions of bases,” *Information Theory, IEEE Transactions on*, vol. 49, no. 12, pp. 3320–3325, 2003.

[2] Yin Zhang, “Theory of compressive sensing via $l_1$-minimization: a non-rip analysis and extensions,” *Journal of the Operations Research Society of China*, vol. 1, no. 1, pp. 79–105, 2013.

[3] David L Donoho and Michael Elad, “Optimally sparse representation in general (nonorthogonal) dictionaries via $l_1$ minimization,” *Proceedings of the National Academy of Sciences*, vol. 100, no. 5, pp. 2197–2202, 2003.

[4] Emmanuel J Candès and Terence Tao, “Decoding by linear programming,” *Information Theory, IEEE Transactions on*, vol. 51, no. 12, pp. 4203–4215, 2005.

[5] Emmanuel J Candès, “The restricted isometry property and its implications for compressed sensing,” *Comptes Rendus Mathematique*, vol. 346, no. 9, pp. 589–592, 2008.

[6] Emmanuel J Candès, Michael B Wakin, and Stephen P Boyd, “Enhancing sparsity by reweighted $l_1$ minimization,” *Journal of Fourier analysis and applications*, vol. 14, no. 5-6, pp. 877–905, 2008.

[7] Simon Foucart and Holger Rauhut, *A mathematical introduction to compressive sensing*, Springer, 2013.

[8] T Tony Cai and Anru Zhang, “Sharp rip bound for sparse signal and low-rank matrix recovery,” *Applied and Computational Harmonic Analysis*, vol. 35, no. 1, pp. 74–93, 2013.

[9] Simon Foucart and Ming-Jun Lai, “Sparsest solutions of underdetermined linear systems via $l_q$-minimization for $0 < q \leq 1$,” *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 395–407, 2009.