Cold dark matter subhaloes at arbitrarily low masses

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ABSTRACT

A defining prediction of the cold dark matter (CDM) cosmological model is the existence of a very large population of low mass haloes, down to planet-size masses. However, their fate as they are accreted onto haloes many orders of magnitude more massive remains fundamentally uncertain. A number of numerical explorations have found subhaloes to be very resilient to tides, but resolution limits make it difficult to explore tidal evolution at arbitrarily low masses. What are the structural properties of heavily stripped subhaloes? Do tidal effects destroy low-mass CDM subhaloes? Here we focus on cosmologically motivated subhaloes orbiting CDM hosts, and show that subhaloes of any initial mass can be stripped to arbitrarily small mass fractions. We show that previous numerical results can be reproduced by a simple model that describes tidal evolution as a progressive ‘peeling’ in energy space and subsequent re-virialization. Tidal heating can effectively be neglected at all masses, because its importance i) does not increase for subhaloes of decreasing mass at accretion, and ii) it decreases as stripping proceeds. This allows us to predict analytically the structural properties of subhaloes with any initial mass and arbitrary degrees of mass loss. Under the hypotheses that CDM haloes have centrally divergent density profiles and approximately isotropic phase space distributions, our results prove that subhaloes of very low masses are a robust prediction of the CDM model: as for haloes, CDM subhalo populations extend and are abundant down to very low masses.

Key words: dark matter — galaxies: kinematics and dynamics

1 INTRODUCTION

Cosmological numerical simulations of the process of structure formation show that cold dark matter (CDM) haloes are populated by an abundance of gravitationally bound substructures, or subhaloes (Springel et al. 2005; Frenk & White 2012, and references therein). Since these substructures are resolved with individual particles, numerical simulations can only address the existence and properties of subhaloes above a minimum bound mass – usually taken to correspond to the mass of about 100 particles. It is undisputed understanding that CDM simulations find the abundance of bound subhalos to grow with a distinctive power law behaviour towards low masses, \( dN/dm \propto m^{-1.9} \), all the way down to the mass resolution limit (e.g., Ghigna et al. 1998; Jenkins et al. 2001; Giocoli, Tormen, & van den Bosch 2008; Diemand et al. 2008; Springel et al. 2008). Taken at face value, this would suggest very abundant populations of subhaloes at even lower masses. But how low?

For a 100 GeV WIMP dark matter (DM) particle, very large populations of isolated CDM haloes form with masses as low as the mass of the Earth (~ \( 10^{-6} M_\odot \)) (Green, Hoffmann, & Schwarz 2005; Wang et al. 2020). This would suggest that subhalo mass functions may in fact extend down to similarly small masses, and in fact to even lower values as a result of tidal evolution. Needless to say, this is an extremely bold extrapolation, and skepticism on its legitimacy is not without motivation. Haloes that get accreted by more massive hosts experience strong tidal fields, which strip their mass and may cause complete disruption (e.g., Taylor & Babul 2001; Hayashi et al. 2003; Sanders, Evans, & Dehnen 2018; Bahé et al. 2019).

This fundamental uncertainty on the properties and the very existence of the low-mass subhalo populations of CDM haloes undermines a number of research lines that aim at using small-scale substructures to constrain the properties of DM itself. Subhaloes at the mass scale targeted by gravitational lensing (~ \( 10^{-5} M_\odot \)) (e.g., Koopmans 2005; Vegetti et al. 2012; Hezaveh et al. 2016; Gilman et al. 2020) are resolved or close to being resolved in current high-resolution zoom ins (Richings et al. 2021). The subhaloes responsible for disturbing the thin stellar streams of Globular Clusters (GCs, ~ \( 10^{-5} M_\odot \)) (e.g., Ibata et al. 2002; Johnston, Spergel, & Haydn 2002; Erkal et al. 2016; Bonaca et al. 2019) are likely to be resolved in the near future, despite the difficulties in-
roduced by baryonic processes. Unfortunately, however, the extrapolation above is unlikely to be bridged fully by cosmological simulations in the near future. For a galaxy-sized host with $M_{\text{host}} = 10^{12} M_\odot$, a simulation with $N = 10^{10}$ particles has a mass resolution limit of about $10^5 M_\odot$, 10 orders of magnitude higher than one Earth mass. This frustrates those studies that hinge on the existence, abundance and structural properties of subhaloes with even lower masses. These include the survival of heavily stripped ‘microgalaxies’ in the Milky Way halo (e.g., Errani & Peñarrubia 2020; Newton et al. 2020); the possibility of DM annihilation, and the associated signal ‘boost’ caused by substructures (e.g., Diemand, Kuhlen, & Madau 2007; Springel et al. 2008; Delos 2019; Wang et al. 2020); possible correlations in the signals from pulsar timing arrays (e.g., Kashiwaya & Oguri 2018; Ramani, Trickle, & Zurek 2020) the properties and survival of weakly bound wide binaries (Peñarrubia 2019a,b; Kervick et al. 2021).

As a consequence, a self-contained and predictive description of the tidal evolution of gravitationally bound collisionless structures would be extremely valuable. As highlighted by countless works before (e.g., Aguilar & White 1985; Taylor & Babul 2001; van den Bosch et al. 2018, and references therein), tidal evolution is a complex non-linear phenomenon, in which a number of physical mechanisms operate at the same time. For the case of CDM subhaloes, at first order, one may ignore i) subhalo-subhalo interactions, which are present in the cosmological case but seem to have a secondary role in their evolution (van den Bosch et al. 2018); ii) the contribution of dynamical friction, which is essentially negligible for subhaloes of sufficiently low masses (e.g., White 1983; Taylor & Babul 2001; Amorisco 2017). However, there are still tidal heating, tidal stripping, and re-virialization, each of which are not easily tackled with analytical methods. This makes a self-contained description an elusive objective, especially when looking for a model that can be extended to arbitrary initial masses and degrees of mass loss.

Controlled numerical experiments have therefore become increasingly prevalent in trying to characterize the process of tidal evolution. With regard to CMD subhaloes, these studies have highlighted two fundamental results.

First, that systems with a centrally divergent density cusp, $ρ ∝ r^{-2}$, and an isotropic phase space distribution, as displayed in CDM cosmological simulations (Navarro, Frenk, & White 1997; Wang et al. 2020), are extremely resilient to tides (Kazantzidis et al. 2004; Peñarrubia, Navarro, & McConnachie 2008; van den Bosch et al. 2018; Errani & Peñarrubia 2020; Errani & Navarro 2021, EN21 in the following). It should be noted that numerical effects – insufficient force resolution in primis – cause spurious alterations to the density profile and kinematic properties in the innermost regions of the system, and are therefore responsible for artificial disruption (van den Bosch et al. 2018; van den Bosch & Ogiya 2018). However, when the necessary numerical criteria are satisfied (Power et al. 2003; van den Bosch & Ogiya 2018; Ogiya et al. 2019; Green & van den Bosch 2019), subhaloes with the above properties are not observed to fully disrupt above the mass resolution limit of the simulation. If the mass resolution allows it and the tidal field is strong enough, subhaloes are seen shedding the largest majority of their mass and still surviving as a self-bound remnant. Complete disruption is only observed as a consequence of limited mass resolution (Kazantzidis et al. 2004; Peñarrubia, Navarro, & McConnachie 2008; van den Bosch et al. 2018; Errani & Peñarrubia 2020, EN21).

Second, that tidally stripped CDM subhaloes populate an essentially uni-dimensional sequence, which has been dubbed ‘tidal track’ (Hayashi et al. 2003; Peñarrubia, Navarro, & McConnachie 2008). If the undisturbed progenitor (stripped bound remnant) have the maximum values of their circular velocity profiles $v_{\text{circ}}(r_{\text{max}})$ at the radii $r_{\text{max}}$, both dimensionless ratios $v_{\text{circ}}/v_{\text{max}}$ and $r_{\text{max}}/r_{\text{max}}$ are found to only depend on the bound mass fraction of the remnant itself, $M_\ast/M_\odot$. It is possible this sequence displays some ‘thickness’, with some numerical results showing it correlates with the subhaloes’ initial concentration (Green & van den Bosch 2019), but the bound mass fraction essentially sets the structural properties of the remnant univocally.

The observed resilience of simulated CDM subhaloes has prompted the suggestion that these are in fact ‘indestructible’, and can not be fully disrupted by smooth tidal fields (van den Bosch et al. 2018; Errani & Peñarrubia 2020, EN21). However, without an understanding of the physical reasons underlying this resilience, it is difficult to justify the extrapolation of numerical results into possibly different regimes, to arbitrary degrees of mass loss and arbitrary satellite-to-host mass ratios. On the other hand, the observation that subhaloes evolve on a simple uni-dimensional tidal track seems to suggest that such an understanding may in fact be attainable.

It should be stressed that the existence of a uni-dimensional tidal track is an extremely surprising and non-trivial result. For instance, and against all expectations, the properties of the remnant appear to be independent of its orbit. This fact suggests that tidal heating may be a secondary factor in the evolution of the remnant, as, at fixed host and satellite, the amount of energy injected in the remnant itself is a strong function of its trajectory (e.g., Spitzer 1987; Gnedin & Ostriker 1999). The key reason for this independence is likely connected to the internal dynamical times of CDM subhaloes, a property which has been recently highlighted by Errani & Peñarrubia (2020). As the dynamical time decreases like $t_{\text{dyn}} \propto r^{1/2}$ towards the center of the system, the central parts are effectively shielded from tidal shocks, to which they respond adiabatically (e.g., Spitzer 1987; Weinberg 1994a,b; Gnedin & Ostriker 1999).

Notice however, that the tidal track is also independent of the satellite-to-host mass ratio. Ultimately, the existence of a tidal track suggests that the properties of the remnant are in fact only dependent on the remnant itself. Remnants with the same bound mass fractions, $M_\ast/M_\odot$, but different initial masses, different hosts or different orbits have identical properties once these are scaled to their initial values. This calls for a description of tidal evolution of CDM subhaloes that only involves the subhalo itself.

The simplest approach, however, fails. A simple description of the process of tidal stripping of collisionless systems as an effective truncation at the tidal radius, $r_{\text{tid}}$, does not reproduce the observed phenomenology, independently of the specific details. Most evidently, it mistakenly predicts tidal disruption for those haloes that are stripped beyond the threshold tidal radius of $r_{\text{tid}} = 0.77 r_s$ (Hayashi et al. 2003), where $r_s$ is the characteristic radius of the initial
NFW density profile (Navarro, Frenk, & White 1997)

\[ \rho(r) = \frac{\rho_s}{\left(1 + \frac{r}{r_s}\right)^2}, \tag{1} \]

Clearly, numerical experiments show otherwise (Kazantzidis et al. 2004; Peã¡arrubia, Navarro, & McConnachie 2008; van den Bosch et al. 2018; Errani & Peã¡arrubia 2020, EN21).

An alternative approach is to describe stripping as an effective truncation in energy space, rather than in radius. Historically, this is certainly not a new proposal. Most notably, it motivated the introduction of lowered distribution functions as a model for tidally truncated GCs (e.g., Woolley 1954; Michie 1963; King 1966; Varri & Bertin 2012). More recently, Choi, Weinberg, & Katz (2009) and Drakos, Taylor, & Benson (2020) have shown that, while a sharp truncation remains an approximation, the stripping of subhaloes with centrally divergent density profiles indeed proceeds as an outside-in process (see also Read et al. 2006; Stã¼cker, Angulo, & Busch 2021). Starting from the least bound material, the energy distribution of the system is ‘peeled’ in a systematic manner, proceeding towards more bound regions. These results make it attractive to try and describe tidal stripping as a process of progressive energy truncation. This, however, does not complete the model. It remains to be determined how to approach the re-virialization that follows the energy peeling.

Focussing on NFW haloes, Drakos, Taylor, & Benson (2017) and Drakos, Taylor, & Benson (2020) have proposed an approach based on the hypothesis that tides cause the remnant to evolve on a sequence of lowered versions of the initial, unperturbed distribution function. Their calculations show tantalising agreement with the results of numerical simulations. However, their model appears to systematically overestimate the density in the innermost regions of the remnants, which is worrying when looking for predictions that can be extended towards even lower initial masses and bound mass fractions. By construction, the model proposed by Drakos, Taylor, & Benson (2017) addresses re-virialization by making explicit choices on both the form of the energy distribution of the system and on the quantities that link the latter to the actual density distribution, through the Poisson equation. The mismatch seems to suggest that some of these choices do not entirely capture the physics of the tidal evolution process.

Here, we follow a perhaps simpler approach. We model tidal stripping as a sharp truncation in energy space and allow the ‘peeled’ remnants to re-virialize freely in controlled, isolated N-body simulations. This remains entirely agnostic in the dynamics of the re-virialization process. We show that this automatically reproduces the tidal track. This then allows us to investigate the regime of arbitrarily small bound mass fractions. In Section 3 we connect energy truncations to the external tidal field, which results in a predictive model of tidal evolution for CDM subhaloes with arbitrary initial mass. In Section 4 we briefly address the role of the central power-law slope of the satellite’s density profile. In Section 5 we apply our model to cosmologically motivated CDM halos. Section 6 sets out the Conclusions.

2 THE TIDAL TRACK AS A SEQUENCE IN ENERGY TRUNCATION

In this Section, we focus on satellites with initial density profiles, \( \rho_{\text{sat}}, \) that diverge like \( \rho_{\text{sat}} \propto r^{-1} \) at the center. In particular, we consider spherically symmetric NFW density profiles, as in equation (1), which we exponentially truncate at some virial radius \( r_{\text{vir}} = c r_s \)

\[ \rho_{\text{sat}}(r) = \frac{\rho_s e^{-r/c r_s}}{\left(1 + \frac{r}{c r_s}\right)^2}, \tag{2} \]

where \( c \) is the satellites’ concentration. It is useful to also introduce the Dehnen family of density profiles (Dehnen 1993), which have varying central power-law cusps

\[ \rho_{\text{sat}}(r) = \frac{\rho_s}{\left(1 + \frac{r}{c r_s}\right)^{3-\gamma}}, \tag{3} \]

The Hernquist model (Hernquist 1990) corresponds to the case \( \gamma = 1, \) and therefore shares the same density cusp as the NFW profiles of equation (2). Unless otherwise specified, in the rest of this Section we refer to NFW density profiles. Across this study, we indicate with \( M_0 \) the total mass of the unperturbed satellite, and with \( v_0^2 \) its circular velocity profile.

We start by considering the phase space distribution of the undisturbed satellite, \( f, \) which we assume is a function of the mechanical Energy, \( E, \) alone. This is equivalent to assuming that the satellite’s phase space distribution is isotropic. In this case, we can recover the distribution function \( f \) by Abel transform1 (Eddington 1916; Widrow 2000):

\[ f(\varepsilon) \equiv \frac{\rho_s}{\Phi_0^2} F(\varepsilon) = \frac{1}{\Phi_0^2} \frac{1}{\sqrt{8 \pi^2}} \int_0^{\infty} \varepsilon^{1/2} \rho_{\text{sat}} \frac{d\Psi}{d\varepsilon} \frac{d\Psi}{\sqrt{\Psi - \varepsilon}}, \tag{4} \]

where \( \Phi_0 \) is the value at the centre of the gravitational potential of the satellite, \( \Phi_0 = \Phi_{\text{sat}}(r = 0); \) \( \Psi \) is the normalised gravitational potential of the satellite, \( \Psi = \Phi_{\text{sat}}/\Phi_0; \) \( \varepsilon \) is the normalised energy, \( \varepsilon \equiv E/\Phi_0. \) By definition, the normalised energy \( \varepsilon \) ranges in the \((0, 1)\) interval, with \( \varepsilon \rightarrow 1 \) corresponding to deeply bound material. It is well known that, for \( 0 < \gamma < 2, \) the distribution function \( F(\varepsilon) \) diverges as a power-law in the limit \( \varepsilon \rightarrow 1, \) with an exponent that is uniquely fixed by the central density cusp (see e.g., Dehnen 1993; Widrow 2000). We have

\[ F(\varepsilon) \sim (1 - \varepsilon)^{-6(1+\gamma)}, \tag{5} \]

The upper panel2 of Figure 1 shows the normalised distribution function \( F(\varepsilon) \) for an NFW density profile with concentration \( c \in \{5, 10, 20\} \). In agreement with equation (5), the three distribution functions share the same asymptotic behaviour as \( \varepsilon \rightarrow 1, \) and the different properties of the truncation at large radii only affect the relative abundance of material towards lower energies.

1 We neglect the term in \((dp_{\text{sat}}/d\Psi)_{\Psi=0}\) as this is zero for all of the density profiles considered in this work.

2 All across this study, we use the symbol \( \log \) to refer to the base 10 logarithm.
Such a truncation corresponds to the density profile \( F \) entirely due to the divergence of the distribution function \( \rho \). This is the case for any value of the concentration, and is useful for the un-truncated original NFW profile: for any truncation \( c \), the density profiles of equation (7) are asymptotically identical.\(^3\) The density is identically zero at larger radii as no material is physically removed from the innermost regions because of tides. This is not the case for a cored density profile. In fact, the central regions of the satellite are ‘asymptotically decoupled’. As all kinematic properties obey asymptotic relations analogous to equation (9), the innermost regions of the system remain in dynamical equilibrium while the outskirts are stripped. Therefore, the re-virialization process that follows the energy ‘peeling’ also proceeds through equilibrium states of the innermost regions.

It is interesting to note that these facts are not unique to the \( \gamma = 1 \) case. Equations (7) and (5) show that asymptotic statements like the one of equation (9) remain true as long as \( F(\ell) \) diverges as \( \ell \to 1 \): any density cusp \( \gamma > 0 \) is sufficient for this. In turn, energy peeling has fundamentally different consequences for density profiles characterized by a finite value of the distribution function \( F \) as \( \ell \to 1 \). This is the case of a cored density profile. If \( F \) does not diverge at the center, energy truncation affects the density profile at all radii. Stripping causes depletion of the central regions, which find themselves out of both virial and dynamical equilibrium as a consequence of the energy peeling, which in turn facilitates further stripping.

2.1 Energy-truncated halos

We truncate the distribution function \( F(\ell) \) at some truncation energy \( \ell_{\text{tr}} \) by simply setting

\[
F_{\text{tr}}(\ell, \ell_{\text{tr}}) = \begin{cases} F(\ell) & \text{if } \ell > \ell_{\text{tr}} \\ 0 & \text{if } \ell \leq \ell_{\text{tr}}. \end{cases}
\]

Such a truncation corresponds to the density profile

\[
\rho_{\text{sat}}^{\text{tr}}(r) = 4\sqrt 2 \pi r \int_{\ell_{\text{tr}}}^{\infty} F(\ell) \sqrt{\psi - \frac{\ell}{\ell_{\text{tr}}}} d\ell.
\]

In the following, we use the symbol \( M^{\ell_{\text{tr}}} \) to indicate the total mass implied by such truncation.

The lower panel of Fig. 1 displays the density profiles associated with different values of the truncation energy \( \ell_{\text{tr}} \). The parent un-truncated density profile is an NFW profile with concentration \( c = 20 \). These density profiles are truncated at the radius \( r_{\text{tr}} \), where

\[
\Psi(r_{\text{tr}}) = \ell_{\text{tr}}.
\]

The density is identically zero at larger radii as no material is present at higher energies. Instead, at small radii, the density profiles of equation (7) are asymptotically identical to the un-truncated original NFW profile: for any truncation energy \( \ell_{\text{tr}} \), it is true that

\[
\rho_{\text{sat}}^{\text{tr}}(r) \to \rho_{\text{sat}}(r) = \rho_0 \frac{r_{\text{tr}}}{r}.
\]

This is the case for any value of the concentration, and is entirely due to the divergence of the distribution function \( F(\ell) \) reported in equation (5). While trivial, this asymptotic condition is the first key ingredient of the resilience to tides of satellites with an NFW density profile.

Under the hypothesis that tidal stripping proceeds as an outside-in process in energy, the asymptotic identity of equation (9) ensures that the central regions are preserved while the outskirts get stripped. The stripping process will cause the system to re-virialize, which also affects the central regions. However, no material is physically removed from the innermost regions because of tides. This is not the case for a cored density profile. In fact, the central regions of the satellite are ‘asymptotically decoupled’. As all kinematic properties obey asymptotic relations analogous to equation (9), the innermost regions of the system remain in dynamical equilibrium while the outskirts are stripped. Therefore, the re-virialization process that follows the energy ‘peeling’ also proceeds through equilibrium states of the innermost regions.

2.2 The tidal track from re-virialization

We simulate the re-virialization of a set of energy-truncated satellites with initial NFW density profiles. We consider the three cases \( c \in \{5, 10, 20\} \), and for each of them, we focus on the set of truncation energies \( \ell_{\text{tr}} \in \{0.36, 0.48, 0.58, 0.68, 0.76, 0.83, 0.89, 0.93, 0.97, 0.99\} \). For the same set of energies, we also examine the case of an Hernquist density profile. We generate initial conditions by sampling the associated distribution functions with \( N = 10^5 \) particles, independently of the truncation energy. This means that particle masses decrease considerably for higher values of \( \ell_{\text{tr}} \), as these correspond to smaller values of the total mass \( M^{\ell_{\text{tr}}} \). This is shown in the lower panel of Figure 2, which shows the ratio between the mass of the energy truncated satellites, \( M^{\ell_{\text{tr}}} \), and the total mass of the unperturbed density profile\(^3\), \( M_0 \).

We evolve the truncated satellites in controlled simulations as isolated systems, for a total of 10\( t_{\text{tr}} \), where \( t_{\text{tr}} \) is the period of a circular orbit at the truncation radius in the un-truncated density profile. Simulations are run with Gadget2 \(^6\) (Springel 2005). We put special care to making sure that the adopted softening length is such to preserve the central density cusp \( \text{(van den Bosch & Ogiya 2018; van den Bosch et al.)} \).

\(^3\) The masses \( M^{\ell_{\text{tr}}} \), displayed in Figure 2, include all material in the truncated satellite, both bound and unbound, as defined by equations (6) and (7).
In our case, this means that we scale the softening length with the truncation radius of the satellites.

Figure 2 shows the location of the re-virialised truncated satellites in the plane \((r_{\text{max}}/r_{\text{max}}^0, v_{\text{max}}/v_{\text{max}}^0)\), where \(r_{\text{max}}\) and \(v_{\text{max}}\) are the location and value of the maximum rotational velocity of the re-virialized satellite, and \(r_{\text{max}}^0\) and \(v_{\text{max}}^0\) are the corresponding values for the unperturbed, un-truncated density profile. We measure \(r_{\text{max}}\) and \(v_{\text{max}}\) at the end of our simulations, which provides the systems with enough time to settle in new equilibrium states. In fact, the re-virialization process is rather quick in the central regions, which stop evolving in an appreciable manner within a handful of dynamical times \(t_d\).

Dashed lines in the same Figure display the tidal tracks observed in previous work\(^4\) (Peñarrubia, Navarro, & McConnachie 2008, EN21). Our results appear to delineate a very similar locus. It is important to stress the fundamental difference between our results and the tidal track. The displayed tidal tracks have been observed by letting NFW profiles evolve within the tidal field of a host. Our results involve no host or tidal field, and simulate the re-virialization of an energy-truncated density profile – NFW or Hernquist in this specific case. Despite this difference, we find that the locus described by our results reproduces extremely well the tidal tracks reported in the literature. It is worth noticing that published tidal tracks differ from each other for \(\log(r_{\text{max}}/r_{\text{max}}^0) \lesssim -0.6\). Smaller values of the ratio \(r_{\text{max}}/r_{\text{max}}^0\) correspond to bound mass fractions \(M_\text{b}/M_\text{b0} \lesssim 10^{-2}\). This strongly suggests that these mismatches originate from numerical effects, which, in turn, may operate differently in different numerical implementations (see also EN21).

Since our approach implies that the different satellites are resolved with the same number of particles, our results extend to significantly smaller values of the ratio \(r_{\text{max}}/r_{\text{max}}^0\), corresponding to higher degrees of mass loss. In particular, we find that our locus scales like \(v_{\text{max}}/v_{\text{max}}^0 \sim (r_{\text{max}}/r_{\text{max}}^0)^{1/2}\) for small values of \(r_{\text{max}}/r_{\text{max}}^0\), a result on which we return in the Section 2.5. The gray-dashed line in Figure 2 is a parametric curve which closely follows previously proposed tidal tracks in the regime of ‘low’ mass loss, \(\log(r_{\text{max}}/r_{\text{max}}) \lesssim -0.6\), and reproduces our results in the regime of high mass loss. A description is available in Appendix A.

### 2.3 The relevance of tidal heating I

The identity between the locus of our re-virialized truncated satellites and the tidal track is an extremely surprising result, and appears to highlight an unexpected simplicity of the tidal evolution process, at least for systems with centrally divergent density profiles. It is astonishing that our calculations entirely ignore tidal heating, and still reproduce the results of the full tidal evolution process! Despite so, our results do show that the inner bound regions of the systems simulated by Peñarrubia, Navarro, & McConnachie (2008) and EN21 do not absorb appreciable amounts of energy as a consequence of tidal heating; their structural properties are reproduced by truncated satellites that experience no energy injection.

In fact, this allows us to extend such a surprising result to satellites with any initial mass. For cosmologically motivated CDM haloes, the characteristic radius \(r_s\) scales like \(r_s \sim M_\text{b}^{1/3}\) down to planetary masses, or to the truncation due to free streaming (Wang et al. 2020). As a consequence,
the ratio between the energy injected by tidal heating, $E_{\text{th}}$, and the binding energy of the undisturbed satellite at accretion, $U_0$, is independent of the initial satellite mass:

$$\frac{E_{\text{th}}}{U_0} \sim \frac{M_0 r_s^2}{M_0^2/r_s} \sim (M_0)^0. \quad (10)$$

This allows us to conclude that the importance of tidal heating does not increase for decreasing initial masses $M_0$ for cosmologically motivated systems. As the satellites simulated in the mentioned works, CDM subhalos of any mass are affected by tidal heating in similarly negligible amounts. This is the second key ingredient of the resilience of CDM subhaloes. It should be highlighted, however, that equation (10) does not prove that heating remains negligible at any degree of tidal stripping. We address this in Section 3.2.

The results above validate an approach to the process of tidal evolution based on first principles – namely energy truncation and re-virialization – in which tidal heating is entirely neglected. We build such a model in Section 3. Before that, we use our numerical results to characterize the dimensionless structural properties of the re-virialized truncated satellites.

### 2.4 The internal structure of the re-virialized satellites

Figure 3 shows examples of the density profiles (top panels), circular velocity profiles (middle panels) and energy distributions (bottom panels) of our energy truncated satellites. The different columns display different values of the truncation energy $\mathcal{E}_t$, all plots pertain to an initial NFW profile with $c = 20$, results are analogous for the other values of the concentration. Black curves refer to the system at $t = 0$, red curves show the re-virialized systems at $t = 10 t_t$. In the upper panels, lighter red curves include material that is not self-bound as a consequence of the re-virialization. Radii are shown in units of the truncation radius $r_t$, circular velocities are displayed in units of the circular velocity at the truncation radius in the unperturbed, un-truncated satellite, $v_t(r_t)$.

Higher values of the truncation energy $\mathcal{E}_t$ correspond to more significant departures from virial equilibrium. This causes larger evolution as a result of the re-virialization itself. In fact, as shown by the energy distributions in the bottom row, part of the material in the satellite is unbound as a result of the energy truncation, even before the re-virialization process itself. This is the case for $\mathcal{E}_t \approx 0.76$ and higher values of the truncation energy. The red histograms show that more material becomes unbound as the satellites relax and expand\(^5\). As a consequence, the value $\mathcal{E}_t \approx 0.76$ sets a divide in the properties of the density distribution of the re-virialized satellites. Lower truncation energies satellites have sharply truncated density profiles, as their energy distributions do not extend to $\mathcal{E} \to 0$. At higher truncation energies, the presence of a population of particles at $\mathcal{E} \to 0$ results in density profiles with outer

\(^5\) With a slight abuse of notation, in the bottom row of Fig. 3 we define the normalized energy, $\mathcal{E}$, as the total energy of the satellites' particles in their own potential, both bound and unbound, normalised by the value of the central potential in the un-truncated density profile, $\Phi_0$. 

Figure 3. Top (middle) panels: the density (circular velocity) profiles of the truncated satellites. Black lines indicate all material at $t = 0$. Red lines represent self-bound material at $t = 10 t_t$. Light-red lines include both bound and unbound material. The yellow dashed lines are guiding lines for the power-law slope $d \ln \rho / d \ln r = -4$. The vertical dot-dashed lines display the truncation radius $r_t$. Bottom panels: black (red) histograms show the normalised energy distributions at $t = 0$ ($t = 10 t_t$). All distributions include both bound and unbound material.
The tidal stripping of CDM subhaloes

power-law slopes of $d \log \rho / d \log r \approx -4$ (e.g., Jaffe 1987), as shown by the yellow guiding lines in the top row panels. In the presence of such diffuse ‘haloes’, dynamical timescales become very long for the loosely bound material at large radii, which causes the outer wings of the density profiles to remain partially not phase mixed at the end of our simulations. This does not appreciably affect the re-virialization in the central regions.

In all cases, the density profile at the center appears to retain a power-law slope of $\gamma = -1$, as seen during tidal evolution in many numerical works before. The density normalization decreases, and so does the circular velocity profile in the center. As a result of the re-virialization process, the satellite expands, increasingly so for higher truncation energies. In particular, this results in a gradual increase of the ratio $v_{\text{max}}/r_{\text{tr}}$ for increasing values of $\varepsilon_{\text{tr}}$.

Figure 4 collects all dimensionless structural properties of our re-virialized satellites, as a function of the truncation energy. The displayed quantities represent the set of dimensionless figures that we could not have determined analytically without making some combination of assumptions on the re-virialization process itself, and that therefore we determine numerically. In detail, these are the ratios

$$\mu \equiv M_b/M_0(<r_{\text{tr}})$$

$$\varpi \equiv r_{\text{max}}/r_{\text{tr}}$$

$$\nu \equiv v_{\text{max}}/v_c(\varepsilon_{\text{tr}}).$$

Here, $M_b$ is the bound mass after re-virialization and $M_0(<r_{\text{tr}})$ is the mass within the truncation radius in the unperturbed, un-truncated density profile. The dimensionless quantity $\mu$ is displayed in the left panel of Fig. 4. The middle panel shows the quantity $\varpi$, the location of the maximum circular velocity in terms of the truncation radius. The right panel shows $\nu$, the maximum circular velocity itself in terms of the value of the circular velocity at the truncation radius in the un-truncated density profile, $v_c^0(\varepsilon_{\text{tr}})$.

It appears that the values of $\mu$, $\varpi$, and $\nu$ are not strongly affected by the precise value of the concentration $c$. In fact, values of the same quantities pertaining the Hernquist profile are seen to approximately follow the relations valid for the NFW density profile. In the case of the quantity $\varpi$, we observe differences of the order of 10% between the different models. We interpret these as scatter, most likely associated to the numerical precision of our determination of the location of the maximum circular velocity. In Section 2.5, we show that the similarity of $\mu$, $\varpi$, and $\nu$ for different initial density profiles is justified, and that it is associated with the identical asymptotic divergence of the systems’ distribution functions, reported in equation (5).

It should be stressed that, for a given initial density profile $\rho_0(r)$, the combination of equation (8) and the dimensionless quantities in Figure 4 are all that is needed to calculate the structural properties ($r_{\text{max}}, v_{\text{max}}$) of a satellite that re-virialises after an arbitrary energy truncation $\varepsilon_{\text{tr}}$. We also report the relation between the truncation radius $r_{\text{tr}}/r_c$ and the bound mass fraction $M_b/M_0$ in Figure 5. For convenience, we collect some simple parametric functions that reproduce the behaviours of $\mu(\varepsilon)$, $\varpi(\varepsilon)$ and $\nu(\varepsilon)$ in Appendix A. These are shown as grey dash-dotted lines in Fig. 4. The asymptotic behaviour of the bound mass fraction is also shown as as grey dash-dotted lines in Fig. 5 and reported in Appendix A.

2.5 Truncated satellites with arbitrarily small bound mass fractions

In the previous Sections we have shown that the tidal track is essentially a uni-dimensional sequence in the truncation energy $\varepsilon_{\text{tr}}$. In the common interpretation, the tidal track is parameterized by the remnant’s bound mass fraction, which in our approach is determined by the truncation energy itself. Here, we consider the limit of ‘asymptotic truncation’, $\varepsilon_{\text{tr}} \rightarrow 1$, in which the truncation energy approaches the bottom of the potential, causing the bound mass fraction to vanish.

In this regime, the distribution function is fully described by its asymptotic behaviour, as in equation (5). Similarly, the gravitational potential of the unperturbed satellite is $\Psi = -1 \sim (r/r_c)^{2-\gamma}$. As a consequence, equation (7) simplifies to

$$\rho_{\text{sat}}^{\text{asy}}(x) \propto \int_{x^{-2+\gamma}}^{x^{-2}} t^{(1-\gamma)/2+\gamma} t_1^{1-2+\gamma} \, dt,$$

Figure 4. The dimensionless structural properties of energy-truncated satellites as a function of the truncation energy. The left panel shows the ratio between the bound mass $M_b$ and the initial total mass. The middle panel shows the ratio between $r_{\text{max}}$ and the truncation radius $r_{\text{tr}}$. The right panel shows the ratio between $v_{\text{max}}$ and the circular velocity at the truncation radius in the unperturbed, un-truncated satellite, $v_c^0(\varepsilon_{\text{tr}})$. (\(\text{MNFRAS} 000, 1–16 (2021)\))
where we have used \( x \equiv r/r_{\text{tr}} \). The necessary dimensional multiplicative constant is set by the asymptotic condition at \( x \to 0 \), equation (9). Analogous expressions enable the calculation of any other moment of the distribution function, and therefore of any kinematic property of the ‘asymptotically truncated’ satellites.

Note that the central power-law slope \( \gamma \) entirely determines the truncated satellites’ properties in this regime. In particular, these are independent of the full details of the initial un-truncated density profile. This justifies the similarity of the numerical results recorded in Sections 2.2 and 2.4 for different values of the NFW concentration, as well as between NFW and Hernquist density profiles. Furthermore, equation (14) shows that all satellites in the regime of asymptotic truncation \( \delta_{\text{tr}} \to 1 \) have identical dimensionless structural properties:

\[
\rho_{\text{sat}}^\text{asy}(r) = \rho_{\text{sat}}(r = r_{\text{tr}}) \tilde{\rho}(x),
\]

where the function \( \tilde{\rho} \) does not depend on the specific value of the truncation energy \( \delta_{\text{tr}} \). The function \( \tilde{\rho}(x) \) is unique for all sufficiently small values of the truncation radius \( r_{\text{tr}}/r_s \).

In other words, at infinitely low bound mass fractions, the dimensionless structural properties of the energy truncated satellites are the same. For instance, in the case \( \gamma = 1 \), equation (9) simplifies to

\[
\tilde{\rho}(x) = \frac{(1-x)^{-3/2}}{x}.
\]

The identity of the dimensionless structural properties before re-virialization implies their identity after the virialization process is complete. This means that the limits of the quantities \( \mu \), \( \omega \) and \( \nu \) as \( \delta_{\text{tr}} \to 1 \) are well-defined. These values describe the re-virialization of all energy truncated satellites with sufficiently small values of the truncation radius, despite their different bound mass fractions. In practice, this means that it is sufficient to run a single numerical simulation to capture the re-virialization and subsequent structural properties at arbitrarily low bound mass fractions. We do so for the case \( \gamma = 1 \) using a controlled simulation analogous to those described in Section 2.2. We find

\[
\{\mu(1), \Omega(1), \nu(1)\} \approx \{10^{-0.75}, 10^{-0.17}, 10^{-0.50}\}.
\]

In particular, we find that, even in the ultimate case of an asymptotic truncation, \( \delta_{\text{tr}} \to 1 \), the fraction of mass that remains bound to the satellite as a consequence of the re-virialization process is

\[
\frac{M_b}{M_{\text{vir}}}(1) \approx 0.75.
\]

This proves that it is possible to truncate an isotropic and centrally divergent density profile with \( \gamma = 1 \) arbitrarily high truncation energies. The re-virialized structure will be self-bound, independently of its small mass. Certainly, steeper cusps will behave similarly. In Section 4 we address the case of shallower cusps \( 0 < \gamma < 1 \).

Finally, it is worth noticing that the identity of the structural properties of the re-virialized satellites forces their locus in the plane \((r_{\text{max}}/v_{\text{max}}^0, v_{\text{max}}/v_{\text{max}}^0)\) to scale like \( v_{\text{max}}/v_{\text{max}}^0 \sim (r_{\text{max}}/v_{\text{max}}^0)^{1/2} \) at low values of \( r_{\text{max}}/v_{\text{max}} \). This is a direct consequence of the value of \( \gamma \): since \( M \sim r_{\text{tr}}^{-3-\gamma} \) at the center, we have that

\[
v_{\text{max}} \sim v_{\text{tr}}^0(1) \left( \frac{M_b(< r_{\text{tr}})}{r_{\text{tr}}^3} \right)^{\frac{1}{2}} \sim r_{\text{tr}}^{\gamma} \sim r_{\text{max}}^{\gamma/2}.
\]

### 3 A SIMPLE MODEL OF TIDAL EVOLUTION

Having described the properties of satellites truncated at arbitrary values of \( \delta_{\text{tr}} \) and subsequently re-virialized, we look at connecting the truncation energy with the properties of the tidal field. We consider a subhalo with an NFW density profile at accretion, with initial properties \( r_s, \rho_s \), and total mass \( M_0 \). For the moment, we assume that the subhalo evolves towards some long-lived state characterized by the structural properties \((r_{\text{max}}, v_{\text{max}})\), which allow it to orbit within the tidal field of its host without experiencing significant additional evolution. For example, the numerical results of EN21 suggest that the structural evolution induced by the tidal process appears to converge on a remnant whose properties are uniquely determined by the host’s tidal field at pericenter.

The classical definition of the tidal radius stems from the analysis of the forces in the vicinity of the satellite’s centre. At the tidal radius\(^6\) \( r_{\text{tid}} \) the gravitational attraction of the satellite itself and the forces from the main host are balanced:

\[
\left[ -\frac{d}{dr} \Phi_{\text{sat}}(r) + \lambda_1^2 r \right]_{r_{\text{tid}}} = -\frac{GM_{\text{sat}}(r_{\text{tid}})}{r_{\text{tid}}^2} + \lambda_1^2 r_{\text{tid}} = 0.
\]

Here, \( \lambda_1 \) is the largest eigenvalue of the effective tidal tensor (e.g., Renaud, Gieles, & Boily 2011), and \( \lambda_1^2 \) is its value at the pericentric radius, \( r_p \).

Our results of Section 2 motivate using equation (20) as an equation for the value of the truncation energy itself, assuming that the properties of the tidally evolved satellites

\[\text{(17)}\]

\[\text{(18)}\]

\[\text{(19)}\]

\[\text{(20)}\]
are well described by the uni-dimensional sequence of truncation energies. We can expect such an approach to return the correct dimensional scalings of the structural properties of tidally evolved remnants. In turn, it is unclear to which accuracy this approach can return dimensionless coefficients. First, as shown by EN21 and previous works, tidally evolved satellites appear to have exponentially truncated density profiles for the bound material. This is not the case for the density profiles that result from re-viralization, which are characterized by $\rho \sim r^{-3}$ at large radii in the case of significant mass loss. Second, our model adopts a sharp energy truncation, while numerical results suggest a more gentle transition and secondary dependences (e.g., Read et al. 2006; Dracos, Taylor, & Benson 2020). This may suggest that the values of the structural parameters collected in Section 2.5 may not fully capture the dimensionless structural properties of stripped satellites. We stress, however, that this is an issue of coefficients rather than dimensional scalings. As long as the energy distribution is unaffected below some energy value our dimensional scalings are still valid. Therefore, we proceed using the same set of symbols defined in Section 2.

### 3.1 Subhaloes in a tidal field

We focus on the regime of important mass loss $\xi_{tr} \to 1$, or equivalently $r_{tr}/r_h \to 0$. We re-write $r_{tr} = \xi r_h$, where $\xi$ is a number, and, for brevity, we are assuming $\xi = \xi_{tr} = 1$ – we drop the explicit dependence on the truncation energy from now on. Analogously, we highlight the bound mass of the re-viralized satellite by rewriting $M_{tot}(< \xi r_h) = \zeta M_h$. With these substitutions, and using that, for $\gamma = 1$, $M_h = 2\pi \mu \rho_h r_h^2 r_{tr}$, equation (20) gives us

$$r_{tr} = r_s \frac{G \rho_h}{\zeta^3} 2\pi \mu^2 \zeta \frac{\xi}{\zeta}. \quad (21)$$

This equation determines the energy truncation – i.e. the truncation radius $r_{tr}$ – which corresponds to the re-viralized satellite having a tidal radius $r_{tid}$ of $r_{tid} = \xi r_h$. Under the assumption that the coefficients of equation (17) are approximatively valid here, the dimensionless quantity $\zeta/\xi^3$ is the only unknown in this equation. The initial properties of the satellite $\rho_h$ and $r_h$ are fixed, and so is the host’s tidal field at pericenter. In terms of physical mechanisms, it seems plausible to imagine that the satellite experiences a progressive energy truncation in the form of tidal stripping. This essentially stops when the truncation radius $r_{tr}$ is such that the majority of the re-viralized structure sits within the remnant’s tidal radius $r_{tid}$. In other words, once $\xi^3/\zeta$ is large enough, the tidal stripping and the subsequent evolution slow down considerably, and a long lived state is achieved. This has the structural properties:

$$r_{max} = r_{tr} \equiv \frac{2\pi \mu \rho_h}{\zeta^3} 2\pi \mu \zeta \frac{\xi}{\zeta}.$$

$$v_{max} = \nu \frac{\rho_h}{\nu} v_{tr} = \left(\frac{r_{tr}}{r_h}\right)^{\frac{1}{2}},$$

where we have defined $v_{tr} \equiv \sqrt{\frac{4\pi G \rho_h r_{tr}^2}{\zeta}}$. Note that the left-hand sides of these equations describe the tidal track in the limit of small characteristic radii and velocities, parametrized by the value of the truncation radius $r_{tr}$. This model then suggest a process of tidal evolution which starts with an initial fast phase of inequilibrium, in which material is quickly lost because the initial density profile is too extended with respect to its own tidal radius, and proceeds in a progressively slower manner, as the system approaches a state in which it is almost fully contained within its own tidal radius.

It is well known that a tidal truncation as in equation (20) is equivalent to requiring that the mean density of the remnant and the mean density of the host within the pericenter are multiples of each other, with a coefficient that depends of the host’s density profile and any assumptions on the remnant’s orbit (e.g., Taylor & Babul 2001; Dracos, Taylor, & Benson 2020). This is also equivalent to requiring that the characteristic dynamical timescale of the remnant, $T_{max} \equiv r_{max}/v_{max}$, is a fixed fraction of the dynamical timescale of the host at the pericentric radius, $T_{host} \equiv r_p/v_c(r_p)$, where $V_c(r_p)$ is the circular velocity of the host at the pericentric radius. This fact has recently been highlighted in numerical simulations by EN21. In the context of our simple model, we find that

$$T_{max} = r_{max}/v_{max} = \frac{\nu}{v_{tr}} \frac{r_{tr}}{v_{tr}} = \frac{\nu (r_{tr}/v_{tr})}{2\pi \mu \rho_h r_{tr}^2 \zeta} \left(\frac{\xi}{\zeta}\right)^{\frac{1}{2}},$$

which, using equation (22) can be written as

$$T_{max} = \frac{\nu (r_{tr}/v_{tr})}{2\pi \mu \rho_h r_{tr}^2 \zeta} \left(\frac{\xi}{\zeta}\right)^{\frac{1}{2}}.$$

In the case of a self-similar density profile like used by EN21 in their simulations, or in the more general case in which the pericentric radius lies considerably within the host’s scale radius, $(\lambda_0^p)^{-1/2} \propto T_{host}$. It is useful to consider explicitly the case of an NFW host halo, with characteristic radius $r_h$ and characteristic density $\rho_h$. For orbits that result in strong tidal fields, $r_p \ll r_h$, the largest eigenvalue of the tidal tensor is

$$\lambda_0^p = 2\pi G \rho_h r_h \frac{r_p}{r_h}.$$

This gives us

$$r_{max} = r_s = \frac{r_h \lambda_0^p}{\nu \rho_h} \frac{\xi}{\zeta},$$

$$v_{max} = v_{tr} = \nu \frac{\rho_h}{\nu} \left(\frac{r_{tr}}{r_h}\right)^{\frac{1}{2}} \left(\frac{\xi}{\zeta}\right)^{\frac{1}{2}}.$$

These provide us with the scalings for the structural properties of CDM subhaloes orbiting the innermost regions of an NFW host in the case of significant mass loss. Equations (28) and (29) are valid for any initial subhalo mass $M_0$ and arbitrarily low bound mass fractions. The basic hypothesis behind these equations is that tidal heating is negligible.

To complete the model, we require a determination of the value $\zeta/\xi^3$. Other numerical coefficients in equations (28) and (29) are estimated in Section 2.5. Since $r_{max}$ and $r_{tr}$ have similar magnitudes, it seems reasonable to argue that $\zeta$ has to be of the order of a few if most of the remnant itself is to be contained within the tidal radius. However, the only way to provide a reliable determination of the value $\zeta/\xi^3$ is through a numerical simulation. For this purpose, we consider the results of EN21, who find that $T_{max} \approx T_{host}/4$ once tidal evolution has slowed down substantially. Using
equation (26), this gives us $\zeta/\xi^2 \approx \nu^2/8\pi^2\mu$. In conclusion, for an NFW host, we have

$$r_{\text{max}} \approx r_s \frac{\rho_s \nu^2}{\rho_h \rho_h} \approx 1.8 \times 10^{-2} r_s \frac{\rho_s \rho_s}{\rho_h \rho_h},$$

(30)

$$v_{\text{max}} \approx v_{\text{th}} \left( \frac{\rho_s \rho_s}{\rho_h \rho_h} \right)^{1/2} \approx 3.7 \times 10^{-2} v_{\text{th}} \left( \frac{\rho_s \rho_s}{\rho_h \rho_h} \right)^{1/2},$$

(31)

$$T_{\text{max}} \approx \frac{T_{\text{host}}}{2\sqrt{2}}.$$  

(32)

We test these dimensional scalings and coefficients in Section 3.3. First, however, we return on the issue of tidal heating.

### 3.2 The relevance of tidal heating II

Equation (10) shows that the importance of tidal heating does not increase for subhalos with lower mass at accretion. Therefore, the combination of equation (10) and Figure 2 have allowed us to conclude that tidal heating is essentially negligible in the initial phases of the tidal evolution of subhalos of any mass at accretion. Equation (10), however, does not guarantee that tidal heating remains negligible once tidal evolution has substantially affected the structural properties of the subhalo itself, and made them depart from the properties of CDM haloes. The evolution may, in principle, proceed towards structural configurations of the remnant in which tidal heating is important.

Numerical results seem to suggest that that is not the case: numerically determined tidal tracks coincide with the locus of re-virialised energy-truncated satellites down to $r_{\text{max}}/r_0 \approx 0.1$. This shows that tidal heating remains of little importance at least while $\delta_s \lesssim 0.9$, or equivalently, for $M_b/M_\bullet \gtrsim 10^{-3}$. In fact, we can settle this for all degrees of mass loss using our model.

Our model shows that tidal heating remains unimportant at all values of the bound mass fraction: evolution along the sequence of re-virialized energy truncated remnants does not cause the importance of tidal heating to increase with decreasing bound fractions. That is because, for $\delta_t \rightarrow 1$, we have that $M_b \approx r_{\text{max}}^2$. Therefore, we find that

$$\frac{E_{\text{th}}}{U} \sim \frac{M_b r_{\text{max}}^2}{M_\bullet^2 T_{\text{max}}} \sim \frac{M_b^2}{M_\bullet}.$$  

(33)

This indicates that heavily stripped satellites are in fact less susceptible to tidal heating than their progenitors, and that the stripping process itself causes tidal heating to become progressively less important. Since the long lived states described by equations (30) and (31) are obtained under the hypothesis that tidal heating is always negligible, equation (33) guarantees that our model is consistent. Furthermore, it ensures that, once the remnant is described by equations (30) and (31), it does not evolve further due to heating. Equation (33) is the third and last ingredient in the tidal resilience of CDM subhalos.

We should clarify that equations (10) and (33) do not guarantee that tidal heating is uniformly negligible for all CDM subhalos, in all tidal fields. Our anchor points are the numerical results of Peñarrubia, Navarro, & McConnachie (2008); Ogiya et al. (2019) and EN21, which have analysed cosmologically motivated haloes orbiting NFW or $\rho \sim r^{-2}$ hosts. First, deviations from the mean mass-concentration relation in the subhalos at accretion certainly affect the value of the ratio in equations (10) and (33) by some dimensionless factor. It is therefore possible that tidal heating may become important for significantly under-concentrated CDM subhaloes (see e.g., Amorisco 2019). The combination of equations (10) and (33) suggests that this may be the case during the initial phases of the tidal evolution process. Second, the presence of a massive disk may also result in a substantial increase of the energy injected in the remnants by tidal shocks (Peñarrubia et al. 2010; D’Onghia et al. 2010; Kelley et al. 2019). This is not included in our model and further numerical study is required to ascertain whether this invalidates our basic hypothesis.

### 3.3 A numerical test

We test the scalings and coefficients of equations (30) and (31) by simulating the tidal evolution of three highly stripped subhaloes orbiting the inner regions of NFW hosts, with different combinations of properties. We choose a fiducial model in which:

$$\left( \frac{r_s}{r_h}, \frac{\rho_s}{\rho_h}, \frac{\nu}{\nu_{\text{th}}} \right) = \left( \frac{1}{20}, 10, \frac{1}{10} \right).$$  

(34)

Two additional models are chosen so that the same set of parameters has values ($1/10, 10, 1/10$) and ($1/40, 20, 1/10$). Such a set allows us to test the dimensional scalings derived in Section 3.1.

For this set of properties, our model predicts bound mass fractions of the order of $M_b/M_\bullet \approx 10^{-4.5}$. This makes it challenging to follow the long term evolution of these systems numerically. On the one hand, preserving the cusp requires high force resolution, i.e. small softening lengths. On the other, this implies that the remnant has to be resolved with large numbers of particles in order to suppress relaxation processes, which, in the long term, also tend to erase the cusp. For example, in order to resolve the remnant with $\approx 10^5$ particles, the system should be composed of $N \gtrsim 10^9$ particles at accretion, which is unrealistic. Therefore, we choose to initialize these simulations using a set of ‘partially truncated’ satellites, i.e. with systems with a truncation energy that is lower than imposed by the tidal field. This spares us the computational cost of simulating the largest majority of particles that are quickly lost from the system. We stress that the satellites in these simulations experience significant tidal stripping during the simulations themselves: the final state of the remnants is not determined by the energy truncation chosen as initial conditions, rather, it is set by the tidal field of the host. For definiteness, we use an $\alpha_t = 0.97$ energy truncation, and a parent concentration of $c = 5$. We resolve the truncated satellites with $N = 5 \times 10^5$ particles, which guarantees $\approx 10^5$ particles in the remnants. The three subhaloes are evolved on circular orbits for more than 60 orbital periods, which, through equation (32), is equivalent to about 170 dynamical times of the remnants themselves.

Figure 6 shows the results of these simulations, in terms of the circular velocity profile of the remnant at equally spaced time intervals, corresponding to $\approx 1.7$ dynamical
times of the remnant, as quantified by equation (32). Both radius and circular velocity are normalised by the characteristic scales predicted by our model, equations (30) and (31). The fact that the three panels appear essentially identical when normalised in this way shows that the dimensional scalings of equations (30) and (31) are correct.

The dashed lines show the circular velocity profiles of the energy-truncated satellites at the beginning of the simulation. As the simulation starts, the system re-virializes and starts to get stripped. This results in an initial phase of rapid evolution: within the first two dynamical times, all three satellites lose a large fraction of their mass. After that, we see continuous, but increasingly slower evolution. This appears to validate the picture laid down in Section 3.1.

The red crosses highlight the location and values of the maximum circular velocity $v_{\text{max}}$. These points are seen tracking the relationship set by equations (23) and (24). This is highlighted by a green dashed line, which has the characteristic slope captured by equation (19). This confirms that our simple model captures the evolution of highly stripped subhaloes. Furthermore, the coefficients collected in Section 2.5, and determined exclusively by re-virialization, appear to provide a good approximation here. There appears to be a small shift between the model curve and the locus of the numerically determined points of maximum circular velocity, of the order of $\approx -0.1$ dex in $r_{\text{max}}$. This may be caused by numerical effects or, as mentioned earlier, by the details of the actual energy truncation associated with stripping. Similarly, the characteristic values $r_{\text{max}}$ and $v_{\text{max}}$ are not seen to reproduce the values predicted in equations (30) and (31) exactly. Rather, within the simulated time-span, we see the remnants approach such values. Differences however, are small, of the order of 0.05 dex in $v_{\text{max}}$. This also appears to confirm that, once the remnant is approximately entirely contained its tidal radius, further evolution slows down considerably.

We should notice that, despite the care put to adjust the numerical resolution in our experiments, our remnants appear to become affected by spurious numerical effects towards the end of our simulations. In particular, the central regions of the circular velocity profile shows signs of progressive steepening as time increases, a clear sign of artificial core formation. In turn, this may play some role in the fact that, while we observe a progressively slower evolution, this is not seen to entirely stall. At the same time, the orbit of our remnants appears to decay, by about 5% by the end of the simulations. This is the result of dynamical friction on the already stripped material (e.g., van den Bosch et al. 2018). Although this is a small effect, it can clearly also contribute to the continued evolution at late times. In conclusion, we are unable to disentangle these small effects from the ideal, unperturbed, tidal evolution of the remnants. This means that it is not possible to ascertain whether the remnants would actually stop evolving when the properties predicted by equations (30) and (31) are reached, or whether they would continue evolving ever more slowly following the tidal track.

In practical terms, however, this is of little importance. Our numerical experiments evolve for a total of $\approx 170$ dynamical times, or 60 orbital periods. Scaling the host to a MW-sized halo would give us a total of 12 Gyr. This is considerably longer than the number of orbital periods experienced by most cosmological subhaloes. Our simulated haloes do not appear to reach the values predicted by equations (30) and (31), but these still provide acceptable estimates during a very long time interval. For more precise estimates, one should accompany our model of the tidal track with a suitable description of the the rate at which subhaloes move along it. This is deferred to future study.

Finally, it is interesting to notice such a long phase of slow evolution is characterized by a perhaps unexpected scaling law, $M_h \sim t^{-1/3}$. This is illustrated in Figure 7,

![Figure 6. Numerical tests of the model. The three panels show the circular velocity profiles of three different tidal evolution experiments, corresponding to combinations of subhalos/host properties in the regime of extreme mass loss, $M_h/M_0 \approx 10^{-4.5}$. The radius and circular velocity values are scaled by the predictions of our model. The different curves span a time interval of $\approx 170$ dynamical timescales of the remnant. The green dashed line shows the locus of re-virialized truncated satellites in the regime of extreme mass loss, as predicted by equations (23) and (24).](image_url)
which displays the product \( r_{\text{max}} v_{\text{max}}^2 \) – in units of the value predicted by equations (30) and (31) – as a function of time for our fiducial numerical simulation. The black line shows the scaling \( M_b \sim t^{-1/3} \).

Figure 7. The time dependence of the product \( r_{\text{max}} v_{\text{max}}^2 \), proportional to the bound mass \( M_b \), as a function of time for our fiducial numerical experiment. The black line shows the scaling \( M_b \sim t^{-1/3} \).

Figure 8. The ratio between the bound mass after the re-virialization process, \( M_b \), and the total mass of the energy truncated systems, \( M_{\text{tr}}^* \), for asymptotically truncated systems, \( \delta_{\text{tr}} \rightarrow 1 \), and for different values of the central density slope \( \gamma \).

4 THE ROLE OF THE CUSP INDEX \( \gamma \)

We have seen that the asymptotic condition (9) is satisfied for any value of the central power-law index \( \gamma > 0 \). This does not ensure, however, that density profiles with cusps shallower than \( \gamma = 1 \) can also be stripped down to arbitrarily low bound mass fractions. First of all, that is because equation (9) does not guarantee that the re-virialization process results in bound remnants for arbitrary values of the truncation energy \( \delta_{\text{tr}} \).

For instance, shallower power law slopes \( \gamma \) result in stronger departures from virial equilibrium at constant values of the truncation energy \( \delta_{\text{tr}} \). Even before re-virialization, we find that values \( \gamma \leq 0.5 \) result in positive values of the total energy of the truncated satellite when \( \delta_{\text{tr}} \rightarrow 1 \). While this does not exclude the possibility of a bound remnant following re-virialization (see also van den Bosch et al. 2018), this suggests that re-virialization will cause the truncated satellite to lose a larger fraction of its mass.

We perform a set of simulations addressing the re-virialization of asymptotically truncated satellites, \( \delta_{\text{tr}} \rightarrow 1 \), with different values of the central power-law slope \( \gamma \). These simulations are analogous to those described in Section 2: the truncated satellites are evolved as isolated systems for a total of \( 10 t_{\text{tr}} \). Figure 8 shows our results for the ratio between the mass of the bound remnant after re-virialization, \( M_b \), and the total mass of the truncated satellite \( M_{\text{tr}}^* \). This ratio appears to decrease very sharply for shallower and shallower cusps. However, we detect bound re-virialized remnants in most of our simulations, and certainly down to our mass resolution limit. Our last bound remnant is for the case \( \gamma = 0.35 \), and comprises only \( N \approx 290 \) particles. Therefore, we cannot exclude that re-virialization does result in a bound remnant for even lower values of \( \gamma \).

Clearly, it still remains a question to what degree density profiles with \( \gamma < 1 \) are affected by tidal heating. A blind use of our model would still result in consistent predictions: using that \( M_b \sim r_{\text{tr}}^{3-\gamma} \), we have

\[
\frac{E_{\text{th}}}{U} \sim \frac{M_b r_{\text{max}}^2}{M_{\text{tr}}^*/r_{\text{max}}} \sim r_{\text{tr}}^\gamma,
\]

with a similar decreasing behaviour as in equation (33). However, this is clearly unjustified for arbitrary values of \( \gamma \). Despite that, Figure 8 and equation (35) do suggest that the infinite resilience to tides of centrally divergent density profiles is not singularly tied a value of power-law index of \( \gamma = 1 \). All indications are that this is a ‘continuous property’, which likely extends to density profiles with somewhat shallower slopes. Nonetheless, the importance of tidal heating does increase as \( \gamma \) decreases, which suggests that the fate of systems with shallower slopes may depend on their specific properties. This, however, can not be detailed without further numerical study.

5 COSMOLOGICAL SUBHALOES

Here, we concentrate on the long lived state of subhalos described by equations (30) and (31), in the context of cosmological CDM subhalo populations. First, note that equations (30) and (31) suggest a population of highly stripped subhaloes. To get a quick estimate, consider a typical cosmological orbit with \( r_p/r_h \approx 1 \) (e.g., Wetzel 2011; Jiang et al. 2015). For haloes on the mass concentration relation, the ratio \( \rho_s/\rho_h \) at accretion is independent of the accretion redshift itself, and is a function of the concentrations of satellite and host, \( c_{\text{sat}} \) and \( c_{\text{host}} \) respectively. Assuming \( \rho_s/\rho_h \approx 2 \) we find \( r_{\text{tr}}/r_h \approx 5 \times 10^{-4} \). From Fig. 5 we see that this implies a considerable degree of mass loss, of the order of \( M_h/M_0 \approx 10^{-4} \).
The tidal stripping of CDM subhaloes

\[ \log(\frac{r_{\text{max}}}{r_s}) \]

\[ \log(\frac{M_b}{M_0}) \]

Figure 9. The values of the ratios \( \frac{r_{\text{max}}}{r_s} \) and \( \frac{M_b}{M_0} \) predicted by our model for subhaloes on orbits with \( r_p = r_h \), as a function of the concentrations of satellite and host.

Figure 9 shows a more comprehensive quantification. The left panel displays the ratio \( \frac{r_{\text{max}}}{r_s} \) as a function of the concentration of satellite and host, assuming \( r_p = r_h \). Through the relationships in Fig. 5, these imply the bound mass fractions \( \frac{M_b}{M_0} \) displayed in the right panel. Unless the subhalo is significantly more concentrated than the host at accretion, it will experience a considerable level of mass loss. Note in particular that lower satellite-to-host mass ratios at infall imply higher final bound mass fractions. This may appear counter intuitive, but is due to the higher concentrations of low mass CDM haloes. A lower subhalo mass at accretion means higher concentration, which therefore causes low-mass subhaloes to retain a larger fraction of their mass with respect to more massive subhaloes on the same orbit. It should be highlighted that the quantities displayed in Fig. 9 do not take into account either the growth of the host after accretion or the evolution of the subhalo’s orbit. It is well known that massive subhaloes experience significant dynamical friction and experience orbital radialization (Amorisco 2017). These will result in increased stripping. Here, however, we are mainly interested in subhaloes with low satellite-to-host mass ratios, for which dynamical friction is negligible. Similarly, we neglect the effects of the host’s evolution.

5.1 Dimensional scalings

A full calculation of the abundance of CDM subhaloes based on our model requires making a number of assumptions on the population of haloes at accretion, as well as on the accretion process itself. We defer such a detailed analysis to future work. Here, we investigate how the characteristic scales \( r_{\text{max}} \) and \( v_{\text{max}} \) vary with the mass of the subhalo \( M_b \), based on a simplified model.

In order to do so, we require a prescription for the distribution of pericentric radii of accretion orbits in CDM. We adopt the distribution displayed in the Appendix of Wetzel (2011). In terms of the virial radius \( r_{\text{vir}} \), this has a mean pericentric radius of \( r_p/r_{\text{vir}} \approx 0.32 \) and a standard deviation of 0.23. We assume that the same distribution is valid for all subhalo masses. First, we explore what is the effect of this distribution on the final bound mass fractions. This is shown in Figure 10. As for Figure 9, we consider subhaloes with concentration \( c_{\text{sat}} \), and hosts with concentration \( c_{\text{host}} \). The distributions shown in Fig. 10 reflect the distribution in pericentric radii, together with the scatter in the mass concentration relation for the subhaloes (we use a lognormal scatter of 0.15 dex; the properties of the host are kept fixed). As mentioned earlier, higher subhalo concentrations allow it to experience comparatively smaller levels of mass loss. All distributions, however span at least 3 orders of magnitudes in \( \frac{M_b}{M_0} \) between their 10% and 90% quantiles. Accretions at high redshifts, i.e. the case \( c_{\text{sat}} = c_{\text{host}} = 3 \), display the most extreme levels of mass-loss. The case of subhaloes of extremely low mass on MW-sized hosts is represented by the choice \( (c_{\text{sat}}, c_{\text{host}}) = (30, 10) \) and displays comparatively much higher mass fractions. We now fix a concentration of \( c = 8 \) for the host, approximately representative of a MW-sized halo at around MNRAS 000, 1–16 (2021)
For the subhaloes, we assume the mass-concentration relation determined by Wang et al. (2020), (we do not include the cutoff due to thermal effects). We use such concentrations to calculate the characteristic radii $r_z$ and densities $\rho_z$ of the subhaloes at accretion, assuming an accretion redshift $z = 1$, independent of mass. This provides us with all necessary ingredients to use our model to predict the long term properties of the same subhaloes: bound masses, $M_b$, characteristic radius $r_{\text{max}}$ and velocity $v_{\text{max}}$.

The left panel of Figure 11 shows the distribution of tidally stripped subhaloes in the plane of bound mass $M_b$ and characteristic radius $r_{\text{max}}$. We have considered subhaloes with masses at accretion $\log(M_b/M_\odot) \in \{9, 6, 3, 0, -3, -6\}$. Each of these values correspond to a distribution of pairs $(M_b, r_{\text{max}})$. As in Figure 10, this is the result of the distribution of pericentric radii and of the spread in the mass-concentration relation. Note that highly stripped subhaloes with identical initial mass are distributed along a $r_{\text{max}} \sim M_b^{1/3}$ locus, reflecting equation (27). However, the global relation defined by the mix of different initial subhalo masses still follows the same dimensional scaling associated with isolated CDM haloes, $r_{\text{max}} \sim M_b^{1/3}$. This relation is shown by the red dashed line, which displays the characteristic radius $r_{\text{max}} = 2.16r_s$ for haloes on the same mass concentration relation, at redshift $z = 0$. Subhaloes appear to have somewhat lower characteristic radii, but a detailed quantification of this systematic shift would require a more accurate model for the mix of accreted haloes.

The left panel of Fig. 11 shows the same tidally stripped subhaloes in the plane of characteristic velocities $v_{\text{max}}$ and characteristic radii $r_{\text{max}}$. The red dashed line shows the location in this plane of haloes on the mass concentration relation at $z = 0$: $(r_{\text{max}}, v_{\text{max}}) = (2.16r_s, 0.46v_s)$. As for the relation between bound mass and characteristic radius, subhaloes appear to follow the same dimensional scaling of isolated CDM haloes: $r_{\text{max}} \sim v_{\text{max}}$. Subhaloes appear on average denser than isolated haloes, which is a well known effect (e.g., Springel et al. 2008), but we defer the quantification of the shift between the two populations to future study.

6 DISCUSSION AND CONCLUSIONS

In this work, we have shown that the following two hypotheses are sufficient to ensure that CDM subhaloes orbiting CDM hosts can never be disrupted. This is valid for any mass of the subhalo at accretion. Therefore, this result implies that, like the halo mass function, the subhalo mass function of CDM models extends down to very low masses. For a 100 GeV WIMP model, very low masses mean planet-sized masses (Green, Hofmann, & Schwarz 2005; Wang et al. 2020). Thanks to mass loss, subhalo mass functions extend even below that.

This follows from the realization that the tidal track of tidally evolving CDM subhaloes is very well described by the locus of subhaloes that re-virialize after a truncation in energy space. This is illustrated in Figure 2, and proves that tidal heating is essentially negligible for systems that satisfy the two conditions above: the bound regions of subhaloes evolved in previous numerical work have experienced negligible energy injection. In fact, this is the primary reason for the existence of a simple, uni-dimensional tidal track. The properties of cosmologically motivated haloes are such that the regime of very low mass subhaloes is not different: the relative importance of tidal heating does not increase for subhaloes with smaller masses at accretion – see equation (10). Therefore, subhaloes with any arbitrarily low initial mass are not more susceptible to heating, and evolve on the same tidal track.
This motivates building a simple model of tidal evolution which entirely neglects the effects of heating. In such a model, subhaloes are peeled in energy space and subsequently re-virialize. While a sharp truncation in energy may not be a fully accurate model of tidal stripping, it captures the fact that material is removed in an outside-in fashion. If material at high binding energies remains untouched while the outskirts are stripped, this model is bound to return the correct dimensional scalings of the structural properties of the evolved subhaloes.

While the tidal track is commonly assumed to be parametrized by the bound mass fraction, within this model it is parametrized by the value of the energy truncation \( \varepsilon_{\text{tr}} \), or equivalently, by the truncation radius \( r_{\text{tr}} \). This better highlights the fact that, if a subhalo satisfies the two conditions above, it is possible to strip it down to arbitrarily small masses. A bound remnant will always ensue. This limit of small bound mass fractions is the limit \( \varepsilon_{\text{tr}} \rightarrow 1 \). We find that such truncations always result in bound remnants, proving that arbitrarily small values of the bound mass fraction are indeed accessible, for any initial mass of the subhalo at accretion.

In particular, the central power law index of the density profile, \( \gamma = 1 \), fixes the slope of the tidal track in the regime of extreme mass loss: for low values of \( r_{\text{max}}/r_{\text{max}}^0 \) and \( v_{\text{max}}/v_{\text{max}}^0 \) we find \( v_{\text{max}} \sim r_{\text{max}}^{1/2} \). Furthermore, satellites on such a tidal track grow progressively more resilient to tidal heating. This ensures that our model is consistent.

Using the classical, local description of the tidal field of the host, we connect the latter to the truncation energy. This provides us with an estimate for the characteristic values \( r_{\text{max}} \) and \( v_{\text{max}} \) in the final phase of tidal evolution. These are the values along the tidal track that first allow the stripped subhalo to be almost entirely contained within its tidal radius. As the remnant approaches these structural properties, its evolution slows down substantially. We confirm this picture and test these estimates with a set of numerical experiments.

A full quantification of the subhalo mass function based on our model is beyond the scope of this work. We have shown, however, that cosmological subhaloes experience significant degrees of mass loss, and that the spread in orbital properties and the scatter in the mass concentration relation result in wide distributions for the final bound mass fraction. In particular, subhaloes with higher satellite-to-host mass ratios are due to lose a larger fraction of their mass to stripping, as a consequence of their lower concentrations, even before dynamical friction is taken into account. Similarly, accretion events at higher redshifts, when the concentrations of satellite and host are more similar to each other, result in lower bound mass fractions, with a median of \( M_b/M_h \approx 10^{-4} \). In comparison, subhaloes of very low masses, and therefore high concentrations, are significantly less affected, with a median of \( M_b/M_h \approx 10^{-1} \) for the case \((v_{\text{sat}},v_{\text{host}}) = (30, 10)\).

While subhaloes with identical masses at accretion can lose very different fractions of their mass, the wide dynamic range covered by CDM subhaloes is such that the dimensional scaling between their characteristic properties remains tied to the scaling that describe isolated CDM haloes. More explicitly, \( r_{\text{max}} \sim M_h^{1/3} \) and \( v_{\text{max}} \sim r_{\text{max}}^{1/2} \). Subhaloes appear more dense, on average, than isolated haloes. However, a more detailed model of the accretion process would be required to quantify such a systematic shift in more detail.

As mentioned in Section 3.2, it is possible that significantly under-concentrated subhaloes may be susceptible to tidal heating, and therefore escape from being described by our simple model. If this is the case, they may evolve following different trajectories in the \((r_{\text{max}}/r_{\text{max}}^0, v_{\text{max}}/v_{\text{max}}^0)\) plane, which will depend on the subhalo’s orbit, effectively negating the intrinsic simplicity of a uni-dimensional tidal track. Similarly, the case of a host featuring a stellar disk should be considered separately. Our results, however, suggest that these factors may in fact not be crucial to the survival itself of the subhaloes, as this is directly connected to the survival of its density cusp. This, in turn, is effectively shielded from energy injection. The framework we have laid down will certainly facilitate further study on these outstanding questions.

Our results settle the longstanding issue of the properties and survival of CDM subhaloes. If the halo mass function of CDM haloes extends to masses of the order of \( 10^{-8}M_\odot \), so does the subhalo mass function. In fact, our analysis directly shows why, despite the complexities of the tidal evolution process, the mass dependence of subhalo abundances is found to effectively track the one of halo abundances in cosmological simulations. This results from the combination of the structural properties of CDM haloes (equation (33)) and the characteristic central divergence of their density profile. Even before further detailed analysis, our results imply that, as halo populations, CDM subhalo populations are indeed extremely abundant at low masses. In fact, it is not unlikely subhalo mass functions may be well approximated by simple extrapolation of current numerical results. This provides renewed confidence on the hope of using such small scales properties to finally test the CMD model.

SOFTWARE CITATIONS

This work used the following software packages:

- Gadget2 (Springel 2005)
- matplotlib (Hunter et al. 2007)
- NumPy (van der Walt et al. 2011)
- Python (Van Rossum et al. 2009)
- Scipy (Virtanen et al. 2020)

DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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