SCHRÖDINGER OPERATOR WITH NON-ZERO ACCUMULATION POINTS OF COMPLEX EIGENVALUES

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Abstract. We study Schrödinger operators

\[ H = -\Delta + V \]

in \( L^2(\Omega) \) where \( \Omega \) is \( \mathbb{R}^d \) or the half-space \( \mathbb{R}^d_+ \), subject to (real) Robin boundary conditions in the latter case. For \( p > d \) we construct a non-real potential \( V \in L^p(\Omega) \cap L^\infty(\Omega) \) that decays at infinity so that \( H \) has infinitely many non-real eigenvalues accumulating at every point of the essential spectrum \( \sigma_{ess}(H) = [0, \infty) \). This demonstrates that the Lieb-Thirring inequalities for self-adjoint Schrödinger operators are no longer true in the non-selfadjoint case.

1. Introduction

In three seminal papers [15, 16, 17] from the 1960s, Pavlov studied Schrödinger operators \( H = -\Delta + V \) in \( L^2(0, \infty) \) with real-valued rapidly decaying potentials \( V \), subject to a non-selfadjoint Robin boundary condition \( f'(0) = hf(0) \) for some \( h \in \mathbb{C} \). In contrast to the selfadjoint case, for non-real \( h \) the discrete eigenvalues are complex and can, in principle, accumulate at a non-zero point of the essential spectrum \( \sigma_{ess}(H) = [0, \infty) \). Using inverse spectral theory, Pavlov proved the existence of a potential \( V \) and a boundary condition so that \( H \) has infinitely many non-real eigenvalues that accumulate at a prescribed point \( \lambda \) of the essential spectrum \( \sigma_{ess}(H) = [0, \infty) \). He further studied the structure of the set of accumulation points.

Since then, it has been an open question whether these results can be modified so that the non-selfadjointness is not coming from the boundary conditions but from a non-real potential \( V \).

The aim of the present paper is to fill this gap by proving the following two results. In the first theorem we address non-selfadjoint Schrödinger operators in \( L^2(\mathbb{R}^d) \) for any dimension \( d \in \mathbb{N} \).

Theorem 1. Let \( p > d \) and \( \mathcal{E} > 0 \). There exists \( V \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) with \( \max\{\|V\|\infty, \|V\|_p\} \leq \mathcal{E} \) that decays at infinity so that the Schrödinger operator

\[ H := -\Delta + V, \quad \mathcal{D}(H) := W^{2,2}(\mathbb{R}^d), \]

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in \([0, \infty)\).

In the second main result we replace the whole Euclidean space \( \mathbb{R}^d \) by the half-space \( \mathbb{R}^d_+ := \{x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d : x_d > 0\} \) and impose (real) Robin boundary conditions.

Theorem 2. Let \( p > d \) and \( \mathcal{E} > 0 \), and let \( \phi \in [0, \pi) \). There exists \( V \in L^\infty(\mathbb{R}^d_+) \cap L^p(\mathbb{R}^d_+) \) with \( \max\{\|V\|\infty, \|V\|_p\} \leq \mathcal{E} \) that decays at infinity so that the Schrödinger operator

\[ H := -\Delta + V, \quad \mathcal{D}(H) := \{f \in W^{2,2}(\mathbb{R}^d_+) : \cos(\phi)\partial_{x_d}f + \sin(\phi)f = 0 \text{ on } \partial\mathbb{R}^d_+\}, \]

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in \([0, \infty)\).

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Theorem [1] is also relevant in the context of Lieb-Thirring inequalities (after Lieb and Thirring [14], see also [12] for an overview) and their (possible) generalisation to complex potentials [8, 13, 5]. In the selfadjoint case the Lieb-Thirring inequalities state that, if

\[ p \geq \frac{d}{2} \quad \text{for } d \geq 3; \quad p > 1 \quad \text{for } d = 2; \quad p \geq 1 \quad \text{for } d = 1, \tag{1} \]

then there exists \( C_{d,p} > 0 \) so that for every real \( V \in L^p(\mathbb{R}^d) \) the negative eigenvalues of the Schrödinger operator \( H = -\Delta + V \) satisfy

\[ \sum_{\lambda \in \sigma(H) \setminus (0, \infty)} |\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_p^p \tag{2} \]

where in the sum each eigenvalue is repeated according to its algebraic multiplicity. In fact, the inequality remains true if \( V \) on the right hand side is replaced by the negative part \( V_- := \max\{0, -V\} \). Now Theorem [1] demonstrates that, if \( p > d \), an inequality like (2) cannot hold in the non-selfadjoint case since, for the constructed \( V \) in Theorem [1], the left hand side is infinite whereas the right hand side is finite (and, in fact, arbitrarily small). The sharpness of \( p > d \) (in relation to \( p \) in (1)) is discussed in Remark [1] below. For possible modifications of Lieb-Thirring inequalities see [6] and the references therein.

Theorem [1] is proved in Section 2, and Theorem [2] in Section 3. In contrast to Pavlov’s inverse spectral theory approach using an elaborate analysis of Weyl m-functions, our proofs are constructive. For both \( \Omega = \mathbb{R}^d \) and \( \Omega = \mathbb{R}^d_+ \) the proof relies on the following two main ingredients (see Lemmas [1, 2] and [3, 4] for the precise formulation):

(I) For an arbitrary \( \lambda \in (0, \infty) \) we construct \( V_0 \in L^\infty(\Omega) \cap L^p(\Omega) \) with arbitrarily small \( \|V_0\|_\infty, \|V_0\|_p \) and that decays at infinity so that \( -\Delta + V_0 \) in \( L^2(\Omega) \) has an eigenvalue \( \mu \) close to \( \lambda \).

(II) For two potentials \( V_1 \in L^\infty(\Omega), V_2 \in L^\infty(\mathbb{R}^d) \) decaying at infinity, consider the corresponding Schrödinger operators

\[ H_1 := -\Delta + V_1 \quad \text{in } L^2(\Omega), \quad H_2 := -\Delta + V_2 \quad \text{in } L^2(\mathbb{R}^d), \]

and assume that there exists \( \mu \in \sigma(H_2) \setminus \sigma(H_1) \). If we shift \( V_2 \) in direction of the \( d \)-th coordinate vector \( e_d \) to \( V_2(-te_d) \) for a sufficiently large \( t > 0 \), then \( H_1 + \chi_{\Omega} V_2(-te_d) \) in \( L^2(\Omega) \) has an eigenvalue \( \mu_t \) close to \( \mu \).

The potential \( V \) in Theorems [1, 2] is then an infinite sum of functions \( V_j, j \in \mathbb{N} \), that we construct inductively using (I) and (II) above.

Since we do not know the exact value of the “sufficiently large” shift \( t \) in (II), we cannot control the exact decay rate of \( V \) at infinity. For \( \Omega = \mathbb{R}^3 \) or \( \Omega = (0, \infty) \), subject to the boundary condition \( f(0) = 0 \) or \( f'(0) = h f(0) \), \( h \in \mathbb{C} \), in the half-line case, Pavlov [15] proved that if

\[ \exists \varepsilon > 0 : \sup_{x \in \Omega} |V(x) e^{\varepsilon \sqrt{|x|}}| < \infty, \tag{3} \]

then \( -\Delta + V \) in \( L^2(\Omega) \) has only finitely many eigenvalues. Therefore, the potential \( V \) in Theorem [1] (for \( d = 3 \)) and Theorem [2] (for \( d = 1 \)) has to decay so slow to violate (3). The condition (3) for \( \Omega = (0, \infty) \) is sharp; Pavlov [15] proved that it cannot be relaxed to \( \sup_{x \in (0, \infty)} |V(x) e^{\beta x^d}| < \infty \) for any \( \beta \in (0, \frac{1}{2}) \). For an arbitrary odd dimension \( d \), see [9] and the references therein for conditions guaranteeing a finite number of eigenvalues.

We employ the following notation and conventions. The open ball in \( \mathbb{R}^d \) with radius \( r > 0 \) around \( v \in \mathbb{R}^d \) is \( B(v,r) := \{ x \in \mathbb{R}^d : |x-v| < r \} \), and analogously \( B(z,r) \subset \mathbb{C} \) denotes the open disk of radius \( r > 0 \) around \( z \in \mathbb{C} \). For a subset
\( \Lambda \subset \mathbb{C} \) the complex conjugated set is \( \Lambda^* := \{ \overline{\lambda} : \lambda \in \Lambda \} \), and for \( z \in \mathbb{C} \) its distance to \( \Lambda \) is \( \text{dist}(z, \Lambda) := \inf_{\lambda \in \Lambda} |z - \lambda| \). Take a domain \( \Omega \subset \mathbb{R}^d \) and \( p \in [1, \infty) \). A function \( f \in L^p(\Omega) \) is viewed as an element of \( L^p(\mathbb{R}^d) \) by extending it by zero outside \( \Omega \), with \( L^p \) norm \( \| f \|_p \); conversely, if we multiply a function \( g \in L^p(\mathbb{R}^d) \) with the characteristic function \( \chi_{\Omega} \) of \( \Omega \), then \( \chi_{\Omega} g \in L^p(\Omega) \). If not specified by an index, the norm \( \| \cdot \| \) always refers to the one of the Hilbert space \( L^2(\mathbb{R}^d) \).

The operator domain, spectrum and resolvent set of an operator \( H \) are denoted by \( \mathcal{D}(H) \), \( \sigma(H) \) and \( \varrho(H) \), and the Hilbert space adjoint operator is \( H^* \). An identity operator is denoted by \( I \), and scalar multiples \( \lambda I \) for \( \lambda \in \mathbb{C} \) are written as \( \lambda \).

Analogously, in \( L^2(\mathbb{R}^d) \) the operator of multiplication with an \( L^\infty(\mathbb{R}^d) \) function \( V \) is simply \( V \); its adjoint operator is the multiplication operator with the complex conjugated function \( \overline{V} \). Weak convergence in \( L^2(\mathbb{R}^d) \) is denoted by \( f_n \rightharpoonup f \), and strong operator convergence is \( H_n \rightarrow H \).

### 2. Schrödinger Operator in \( L^2(\mathbb{R}^d) \)

Throughout this section, all operator domains are \( W^{2,2}(\mathbb{R}^d) \). The functions \( V_j \), \( j \in \mathbb{N} \), mentioned in the introduction will be of the form

\[
U_{c,t,a}(x) := \begin{cases} 
c, & x \in B(te_d, a), \\
\frac{(d - 3)(d - 1)}{4|x - te_d|^2}, & x \in \mathbb{R}^d \setminus B(te_d, a),
\end{cases}
\]

where \( c \in \mathbb{C}, t \in \mathbb{R} \) and \( a > 0 \). Note that in dimension \( d = 1 \) and \( d = 3 \) the function \( U_{c,t,a} \) vanishes outside the ball \( B(te_d, a) \).

Before we study finite or infinite sums, we reduce our attention to a potential of the form \( U_{c,t,a} \).

**Lemma 1.** Let \( \lambda \in (0, \infty) \) and \( p > d \). For any \( \varepsilon, \delta, r > 0 \) there exist \( a > 0, c \in \mathbb{C} \) and \( \mu \in \mathbb{C} \) with \( \text{Im} \mu < 0 \) such that, for every \( t \in \mathbb{R} \),

\[
\|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_{\infty} < \delta, \quad |\mu - \lambda| < r,
\]

and \( \mu \) is an eigenvalue of \( -\Delta + U_{c,t,a} \).

**Proof.** Define \( \nu := \sqrt{\lambda} > 0 \) and

\[
a_m := \frac{\frac{d\nu}{d} + \pi m}{\nu} > 0, \quad m \in \mathbb{N}_0.
\]

For \( m \in \mathbb{N}_0 \) let \( \eta_m > 0 \) be the unique solution of

\[
\eta_m e^{2\eta_m a_m} = \nu.
\]

Note that \( a_m \rightarrow \infty \) and \( \eta_m \rightarrow 0 \) as \( m \rightarrow \infty \). We set

\[
\tau_m := \nu + i\eta_m, \quad m \in \mathbb{N}_0,
\]

and

\[
k_m := J_{\frac{d-2}{2}}(\tau_m a_m) J_{\frac{d-1}{2}}(\tau_m a_m) \tau_m + \frac{i(d - 3)}{2a_m}, \quad m \in \mathbb{N}_0,
\]

where \( J_n \) is the Bessel function of the first kind of order \( n \) (see [2, Chapter 9]). It satisfies

\[
J'_n(z) = J_{n-1}(z) - \frac{nJ_n(z)}{z}, \quad z^2 J''_n(z) + z J'_n(z) = (n^2 - z^2) J_n(z),
\]
Using (5) implies e

(see [2, Equation 9.1.2]), then for large \( \eta \)

\[ g_m(r) := \begin{cases} 
\frac{e^{ik_m a_m}}{\sqrt{a_m} d_{\frac{d}{2}-1}(\tau_m a_m)} \frac{\tau_m^{\frac{d}{2}-1}}{r^{\frac{d}{2}-1}}, & \text{if } r = 0, \\
\frac{e^{ik_m a_m}}{\sqrt{a_m} d_{\frac{d}{2}-1}(\tau_m a_m)} \frac{J_{\frac{d}{2}-1}(\tau_m r)}{r^{\frac{d}{2}-1}}, & \text{if } 0 < r < a_m, \\
\frac{e^{ik_m r}}{r^{\frac{d}{2}-1}}, & \text{if } r > a_m.
\end{cases} \]

Using (6) and [2, Equation 9.1.10], one may check that both \( g_m \) and \( g_m' \) are continuous; for small \( r > 0 \) we expand \( g_m(r) = g_m(0) + O(r^2) \), hence \( \lim_{r \to 0} g_m'(r) = 0 \).

Let \( t \in \mathbb{R} \) be arbitrary. Then \( f_m(x) := g_m(|x - te_d|) \), \( x \in \mathbb{R}^d \), belongs to \( W_{1,2,loc}^{2,2}(\mathbb{R}^d) \) and

\[ -\Delta f_m(x) = -g_m''(|x - te_d|) - \frac{d-1}{|x - te_d|} g_m'(|x - te_d|) \]

\[ = \frac{\tau_m^2 f_m(x)}{x^2 f_m(x)} + \frac{(d-3)(d-1)}{4|x - te_d|^2} f_m(x), \quad |x - te_d| > a_m. \]

Hence

\[ -\Delta f_m + U_{cm,t,a_m} f_m = \mu_m f_m \quad \text{with} \quad \mu_m := k_m^2, \quad \nu_m := \frac{k_m^2 - \tau_m^2}{m}. \]

In order to ensure \( f_m \in W_{1,2,loc}^{2,2}(\mathbb{R}^d) \), \( D(-\Delta + U_{cm,t,a_m}) \) we need \( \Im k_m > 0 \). We use the asymptotics of the Bessel function for \( z \in \mathbb{C} \) with \( \arg z < \pi \) and large \( |z| \) (see [2, Equation 9.2.1]),

\[ J_n(z) = \sqrt{\frac{2}{\pi z}} \left( \cos \left( z - \frac{(n + 1)\pi}{4} \right) + e^{\Im z} O(|z|^{-1}) \right). \]

A straightforward calculation reveals that, if

\[ \Re z \in \left( \frac{n + 1}{2} \pi \right) + \pi \mathbb{Z}, \quad \Im z > 0, \quad (7) \]

then for large \( |z| \) we have

\[ \frac{J_{n-1}(z)}{J_n(z)} = \frac{e^{-\Im z} + i e^{\Im z} + e^{\Im z} O(|z|^{-1})}{i e^{-\Im z} + e^{\Im z} + e^{\Im z} O(|z|^{-1})} = -2 e^{-2\Im z} + i (e^{-4\Im z} - 1) + O(|z|^{-1}). \]

The point \( z = \tau_m a_m \) satisfies (7) for \( n = \frac{d}{2} - 1 \), and hence, for large \( m \), (6) yields

\[ k_m = -i \tau_m \left( -2 e^{-2\Im \tau_m a_m} + i (e^{-4\Im \tau_m a_m} - 1) + O(|\tau_m a_m|^{-1}) \right) + O(a_m^{-1}) \]

\[ = -\nu (1 - e^{-2\Im \tau_m a_m}) - 2 \eta_m e^{-2\Im \tau_m a_m} + i (2 \nu e^{-2\Im \tau_m a_m} - \eta_m (1 - e^{-4\Im \tau_m a_m})) + O(a_m^{-1}). \]

Using that (5) implies \( e^{-2\Im \tau_m a_m} = \frac{\eta_m}{r} \) and \( a_m = \frac{\ln(\nu/\eta_m)}{2 \eta_m} \), we arrive at

\[ k_m = -\nu + i \eta_m \left( 1 + O\left( \ln \left( \frac{\nu}{\eta_m} \right)^{-1} \right) \right). \]

Since \( \eta_m > 0 \) and \( \ln(\nu/\eta_m)^{-1} \to 0 \) as \( m \to \infty \), we conclude that \( \Im k_m > 0 \) for all sufficiently large \( m \in \mathbb{N}_0 \). In addition, for large \( m \in \mathbb{N}_0 \) the eigenvalue \( \mu_m = k_m^2 \) satisfies

\[ \mu_m = \lambda - i 2 \nu \eta_m \left( 1 - O\left( \ln \left( \frac{\nu}{\eta_m} \right)^{-1} \right) \right), \]
and hence \( \text{Im } \mu_m < 0 \) for all sufficiently large \( m \in \mathbb{N}_0 \). One may check that

\[
\mu_m - \lambda = \mathcal{O}(\eta_m), \quad c_m = k_m^2 - \tau_m^2 = \mathcal{O}(\eta_m)
\]

close to 0 as \( m \to \infty \). Further note that

\[
\|U_{m,t,a_m}\|_p^p = \text{Vol}(B(0,1)) \left( \frac{|c_m|^d a_m^d}{d} + \frac{|d-3|^p|d-1|^p}{4p(2p-d)a_m^{2p-d}} \right) = \mathcal{O} \left( \eta_m^{p-d} \ln \left( \frac{\mu}{\eta_m} \right)^d \right),
\]

\[
\|U_{m,t,a_m}\|_\infty = \max \left\{ |c_m|, \frac{|d-3||d-1|}{4a_m^2} \right\} = \mathcal{O}(\eta_m).
\]

Since \( p > d \) by the assumptions, both norms converge to 0 as \( m \to \infty \). Altogether, we see that the claim is satisfied if we set \( a := a_m, c := c_m, \mu := \mu_m \) for a sufficiently large \( m \in \mathbb{N}_0 \).

\[ \square \]

**Remark 1.** In dimension \( d = 1 \) the assumption \( p > d = 1 \) of Lemma 1 is sharp. In fact, due to Abramov et al. [1], for every \( V \in L^1(\mathbb{R}) \) every eigenvalue \( \mu \in \sigma(-d^2/dx^2 + V) \setminus [0, \infty) \) satisfies

\[
|\mu|^{\frac{1}{2}} \leq \frac{1}{2} \|V\|_1;
\]

hence \( \delta > 0 \) cannot be chosen arbitrarily small as in Lemma 1. In addition, in Theorem 1 for \( d = 1 \) it is impossible to construct \( V \in L^1(\mathbb{R}) \) since then [3] forces the non-real eigenvalues to lie in the disk \( B(0, \mathcal{E}/4) \), so they cannot accumulate at every point in \( (0, \infty) \).

For dimension \( d \geq 2 \) the sharpness of the assumption \( p > d \) is directly related to the following conjecture of Laptev and Safronov [13]: For \( p \in \left( \frac{d}{2}, d \right] \) there exists \( C_{d,p} > 0 \) such that

\[
|\mu|^{p-d} \leq C_{d,p}\|V\|_p^p
\]

for every \( V \in L^p(\mathbb{R}^d) \) and every \( \mu \in \sigma(-\Delta + V) \setminus [0, \infty) \). In [10] the conjecture was proved for radial potentials. Note that the potential in Lemma 1 is radial, so \( p > d \) is sharp. In general (for non-radial potentials) the conjecture has been confirmed for \( p \in \left( \frac{d}{2}, \frac{d}{2} \right] \) (see [7]) and is still open for \( p \in \left( \frac{d}{2}, d \right] \). If the conjecture is false, then it may also be possible to modify Lemma 1 for a non-radial potential and hence prove Theorems 1-2 for a \( p \leq d \).

**Lemma 2.** Let \( V_1, V_2 \in L^\infty(\mathbb{R}^d) \) be decaying at infinity and such that there exists \( \mu \in \sigma(-\Delta + V_2) \setminus \sigma(-\Delta + V_1) \). Then there are

\[
\mu_t \in \sigma(-\Delta + V_1 + V_2(\cdot - te_d)), \quad t > 0,
\]

with \( \mu_t \to \mu \) as \( t \to \infty \).

**Proof.** First note that

\[
\sigma(-\Delta + V_1 + V_2(\cdot - te_d)) = \sigma(-\Delta + V_1(\cdot + te_d) + V_2), \quad t > 0.
\]

Next we prove that, for every \( z \in \mathbb{C} \) with \( \text{dist}(z,[0,\infty)) > \|V_1\|_\infty + \|V_2\|_\infty \), we have strong resolvent convergence

\[
(-\Delta + V_1(\cdot + te_d) + V_2 - z)^{-1} \xrightarrow{s} (-\Delta + V_2 - z)^{-1}, \quad t \to \infty,
\]

\[ \square \]
and the same holds for the adjoint operators. To this end, first note that a Neumann series argument yields
\[
z \in \bigcap_{t > 0} \sigma(-\Delta + V_1(\cdot + te_d) + V_2) \cap \sigma(-\Delta + V_2),
\]
\[
\sup_{t > 0} \left\| \left( -\Delta + V_1(\cdot + te_d) + V_2 - z \right)^{-1} \right\|
\leq \left\| (-\Delta - z)^{-1} \right\| \sup_{t > 0} \left\| (I + (V_1(\cdot + te_d) + V_2)(-\Delta - z)^{-1})^{-1} \right\|
\leq \frac{1}{\text{dist}(z, [0, \infty))} \frac{1}{\text{dist}(z, [0, \infty)) - (\|V_1\|_\infty + \|V_2\|_\infty)}.
\]
The space $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$ and hence a core of $-\Delta + V_2$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $f \in W^{2,2}(\mathbb{R}^d)$, and the assumption $V_1(x) \to 0$ as $|x| \to \infty$ yields
\[
\left\| (-\Delta + V_1(\cdot + te_d) + V_2)f - (-\Delta + V_2)f \right\| \leq \sup_{x \in (\text{supp} f + te_d)} |V_1(x)||f| \to 0, \quad t \to \infty.
\]
Now the strong resolvent convergence in (11) follows from [3, Theorem 3.1, Proposition 2.16 ii]), and the strong resolvent convergence of the adjoint operators
\[
(-\Delta + V_1(\cdot + te_d) + V_2)^* = -\Delta + V_1(\cdot + te_d) + V_2, \quad t > 0,
\]
to $(-\Delta + V_2)^*$ is proved analogously.

By [4, Theorem 2.3 ii]), in the limit $t \to \infty$ the isolated eigenvalue $\mu \in \sigma(-\Delta + V_2) \setminus \sigma(-\Delta + V_1)$ is approximated by points $\mu_t \in \sigma(-\Delta + V_1(\cdot + te_d) + V_2)$, $t > 0$, provided that the so-called \textit{limiting essential spectrum} satisfies
\[
\mu \notin \sigma_{\text{ess}}((-\Delta + V_1(\cdot + te_d) + V_2)_{t>0}) \cup \sigma_{\text{ess}}((-\Delta + V_1(\cdot + te_d) + V_2)^*)_{t>0}.
\]
This, together with (10), then proves the claim. So it is left to prove (12).

By definition (see [4]), the point $\mu$ belongs to set on the right hand side of (12) only if there exist an infinite subset $I \subset (0, \infty)$ and $f_t \in W^{2,2}(\mathbb{R}^d)$, $t \in I$, with $\|f_t\| = 1$, $f_t \rightharpoonup 0$ and, in the limit $t \to \infty$,
\[
\left\| \left( -\Delta + V_1(\cdot + te_d) + V_2 - \mu \right)f_t \right\| \to 0
\quad \text{or}
\left\| \left( -\Delta + V_1(\cdot + te_d) + V_2 - \overline{\mu} \right)f_t \right\| \to 0.
\]
It is easy to see that the latter implies that $\|f_t\|_{W^{1,2}(\mathbb{R}^d)}$, $t \in I$, are uniformly bounded. Since, for any $r > 0$, the space $W^{1,2}(B(0, r))$ is compactly embedded in $L^2(B(0, r))$ by the Rellich-Kondrachov theorem, the weak convergence $f_t \rightharpoonup 0$ implies $\|\chi_{B(0, r)}f_t\| \to 0$ and hence $\|\chi_{B(0, r)}V_2f_t\| \to 0$ as $t \to \infty$. Moreover, the assumption $V_2(x) \to 0$ as $|x| \to \infty$ yields
\[
\sup_{t > 0} \|\chi_{\mathbb{R}^d \setminus B(0, r)}V_2f_t\| \leq \sup_{|x| > r} |V_2(x)| \to 0, \quad r \to \infty.
\]
Altogether, in the limit $t \to \infty$ we obtain $\|V_2f_t\| \to 0$ and hence, by (13),
\[
\left\| \left( -\Delta + V_1 - \mu \right)f_t(\cdot - te_d) \right\| = \left\| \left( -\Delta + V_1(\cdot + te_d) - \mu \right)f_t \right\| \to 0
\quad \text{or}
\left\| \left( -\Delta + V_1 + \overline{\mu} \right)f_t(\cdot - te_d) \right\| = \left\| \left( -\Delta + V_1(\cdot + te_d) + \overline{\mu} \right)f_t \right\| \to 0.
\]
Therefore, in either case $\mu$ needs to belong to $\sigma(-\Delta + V_1) = \sigma(-\Delta + V_1)^*$, which is excluded by the assumptions. This proves the claim (12).

Now we are ready to prove the main result.

\textit{Proof of Theorem 1} Consider an enumeration of $(\mathbb{Q} \cap (0, \infty)) \times \mathbb{N}$, i.e. a bijective map
\[
\mathbb{N} \ni n \mapsto (q_n, m_n) \in (\mathbb{Q} \cap (0, \infty)) \times \mathbb{N}.
\]
Set $\gamma_0 := \infty$. By induction over $n \in \mathbb{N}$ we construct $c_n, t_n, a_n$ and $\gamma_n$ such that

$$H_n := -\Delta + \sum_{j=1}^n U_{c_j, t_j, a_j}$$

satisfies the following:

i) The norms of the functions are bounded by

$$\|U_{c_n, t_n, a_n}\|_p < \varepsilon_n := \frac{6\varepsilon}{\pi^2 n^2},$$

$$\|U_{c_n, t_n, a_n}\|_\infty < \delta_n := \frac{6 \min \{\gamma_{n-1}, \varepsilon\}}{\pi^2 n^2},$$

and

$$\exists \mu_n \in \sigma(H_n) : \text{Im} \mu_n < 0, \ |\mu_n - q_n| < \frac{1}{2m_n}. \tag{15}$$

ii) We have $0 < \gamma_n \leq \gamma_{n-1}$ and for any $U_n \in L^\infty(\mathbb{R}^d)$ with $\|U_n\|_\infty < \gamma_n$ there is $\lambda_n \in \sigma(H_n + U_n)$ such that

$$|\lambda_n - \mu_n| < \frac{\text{dist}(\mu_n, [0, \infty))}{2}.$$ 

We start with $n = 1$. By Lemma 1 applied to

$$\lambda = q_1, \ \varepsilon = \varepsilon_1, \ \delta = \delta_1, \ r = \frac{1}{2m_1}$$

and an arbitrary $t_1 \in \mathbb{R}$, there exist $c_1 \in \mathbb{C}, a_1 > 0$ and an eigenvalue satisfying (15) for $n = 1$. By [11] Theorems IV.2.14, 3.16, there exists $\gamma_1$ satisfying claim ii) for $n = 1$.

Now assume that for $j = 1, \ldots, n-1$ the constants $c_j, t_j, a_j$ and $\gamma_j$ have been constructed. We construct $c_n, t_n, a_n$ and $\gamma_n$ so that $H_n$ satisfies i) and ii). We apply Lemma 2 to

$$\lambda = q_n, \ \varepsilon = \varepsilon_n, \ \delta = \delta_n, \ r = \min \left\{\text{dist}(\lambda, \sigma(H_{n-1})), \frac{1}{4m_n}\right\}.$$ 

In this way we obtain $c_n \in \mathbb{C}$ and $a_n > 0$ such that, for any $t \in \mathbb{R}$, the Schrödinger operator $-\Delta + U_{c_n, t, a_n}$ has an eigenvalue $\mu \in \sigma(-\Delta + U_{c_n, t, a_n}) \setminus \sigma(H_{n-1})$ with $\text{Im} \mu < 0$ and

$$\|U_{c_n, t, a_n}\|_p < \varepsilon_n, \ \|U_{c_n, t, a_n}\|_\infty < \delta_n, \ |\mu - q_n| < \frac{1}{4m_n}.$$ 

Lemma 2 implies that, for $t_n := t$ sufficiently large, the operator $H_n = H_{n-1} + U_{c_n, t, a_n}$ has an eigenvalue $\mu_n$ with $\text{Im} \mu_n < 0, |\mu_n - \mu| < 1/(4m_n)$ and hence $|\mu_n - q_n| < 1/(2m_n)$. This proves claim i), and claim ii) follows again from [11] Theorems IV.2.14, 3.16.

Finally we prove that the potential

$$V := \sum_{j=1}^\infty U_{c_j, t_j, a_j}$$

satisfies the claims of the theorem. By Minkowski’s inequality and (14),

$$\max\{\|V\|_p, \|V\|_\infty\} < \sum_{j=1}^\infty \max\{\varepsilon_j, \delta_j\} \leq \frac{6\varepsilon}{\pi^2} \sum_{j=1}^\infty \frac{1}{j^2} = \mathcal{E}.$$ 

Moreover, for $n \in \mathbb{N}$ the $L^\infty(\mathbb{R}^d)$ norm of $U_n := \sum_{j=n+1}^\infty U_{c_j, t_j, a_j}$ is estimated as

$$\|U_n\|_\infty < \sum_{j=n+1}^\infty \delta_j \leq \frac{6\gamma_n}{\pi^2} \sum_{j=n+1}^\infty \frac{1}{j^2} < \gamma_n.$$
So the above claim ii) implies for $H_n + U_n = H$ that
\[ \exists \lambda_n \in \sigma(H) : |\lambda_n - \mu_n| < \frac{\text{dist}(\mu_n, [0, \infty))}{2}. \]
Hence $\text{Im} \lambda_n < 0$ and
\[ |\lambda_n - q_n| \leq |\lambda_n - \mu_n| + |\mu_n - q_n| < \frac{\text{dist}(\mu_n, [0, \infty))}{2} + |\mu_n - q_n| < \frac{1}{m_n}, \]
i.e. $\lambda_n \in B(q_n, \frac{1}{m_n})$, $n \in \mathbb{N}$. Now it is easy to see that every point in $[0, \infty)$, which is the closure of $\mathbb{Q} \cap (0, \infty)$, is an accumulation point of $\{\lambda_n : n \in \mathbb{N}\}$. □

\section{3. Schrödinger operator in $L^2(\mathbb{R}^d_+)$}

In this section we study Schrödinger operators on the half-space $\mathbb{R}^d_+$, and for the proof of Lemma \[\text{Lemma 4}\] below also on the shifted half-space $\mathbb{R}^d_+ + tc_d$ for some $t \in \mathbb{R}$. We fix an angle $\phi \in [0, \pi)$ which determines the Robin boundary condition. Throughout this section, every operator in $L^2(\mathbb{R}^d_+ + tc_d)$ for some $t \in \mathbb{R}$ is assumed to have the operator domain
\[ \{ f \in W^{2,2}(\mathbb{R}^d_+ + tc_d) : \cos(\phi)\partial_{\alpha} f + \sin(\phi) f = 0 \text{ on } \partial(\mathbb{R}^d_+ + tc_d) \}, \]
and operators in $L^2(\mathbb{R})$ have domains $W^{2,2}(\mathbb{R})$.

The following result is almost the same as Lemma \[\text{Lemma 1}\] note that here $t$ is not arbitrary but needs to be sufficiently large, and the eigenvalue $\mu_t$ depends on $t$.

\begin{lemma}
Let $\lambda \in (0, \infty)$ and $p > d$. For any $\varepsilon, \delta, r > 0$ there exist $a > 0$ and $c \in \mathbb{C}$ with
\[ \|U_{c,t,a}\|_p < \varepsilon, \quad |\|U_{c,t,a}\|_\infty| < \delta, \quad (16) \]
and such that, for every sufficiently large $t > 0$, the operator
\[ -\Delta + \chi_{\mathbb{R}^d_+} U_{c,t,a} \text{ in } L^2(\mathbb{R}^d_+) \]
has an eigenvalue $\mu_t$ with $\text{Im} \mu_t < 0$ and $|\mu_t - \lambda| < r$.
\end{lemma}

For the proof we use the following result, which is the analogue of Lemma \[\text{Lemma 2}\]

\begin{lemma}
Let $V_1 \in L^\infty(\mathbb{R}^d_+), V_2 \in L^\infty(\mathbb{R}^d)$ be decaying at infinity, and define the operators
\[ H_1 := -\Delta + V_1 \text{ in } L^2(\mathbb{R}^d_+), \quad H_2 := -\Delta + V_2 \text{ in } L^2(\mathbb{R}^d). \]
Assume that there exists $\mu \in \sigma(H_2) \setminus \sigma(H_1)$. Then, for any $t > 0$, the operator
\[ H_1 + \chi_{\mathbb{R}^d_+} V_2 \cdot (-tc_d) \text{ in } L^2(\mathbb{R}^d_+) \]
has an eigenvalue $\mu_t$ with $\mu_t \to \mu$ as $t \to \infty$.
\end{lemma}

\begin{proof}
Define operators
\[ H_{2,t} := -\Delta + \chi_{(\mathbb{R}^d_+-tc_d)} V_2 \text{ in } L^2(\mathbb{R}^d_+ - tc_d), \quad t > 0. \]
Note that
\[ \sigma(H_1 + \chi_{\mathbb{R}^d_+} V_2 \cdot (-tc_d)) = \sigma(H_{2,t} + V_1 \cdot (+tc_d)), \quad t > 0. \quad (17) \]
Analogously as in the proof of Lemma \[\text{Lemma 2}\] one can show that for every $z \in \mathbb{C}$ with $\text{dist}(z, [0, \infty))$ sufficiently large, we have strong resolvent convergence
\[ (H_{2,t} + V_1 \cdot (+tc_d) - z)^{-1} \xrightarrow{t \to \infty} (H_2 - z)^{-1}, \quad t \to \infty, \]
and the same holds for the adjoint operators; note that here we use that every $f \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{D}(H_{2,t})$ for all $t > 0$ so large that $\text{supp } f \subset (\mathbb{R}^d_+-tc_d)$. Therefore, by [\[\text{[4 Theorem 2.3 i)]\]}, in the limit $t \to \infty$ the isolated eigenvalue
\( \mu \in \sigma(H_2) \setminus \sigma(H_1) \) is approximated by points \( \mu_t \in \sigma(H_{2,t} + V_1(\cdot + t \varepsilon d)), \ t > 0, \) provided that
\[
\mu \notin \sigma_{\text{ess}}\left(H_{2,t} + V_1(\cdot + t \varepsilon d)\right)_{t>0} \cup \sigma_{\text{ess}}\left((H_{2,t} + V_1(\cdot + t \varepsilon d)) \right)_{t>0}^*.
\]

Similarly as in the proof of Lemma 2, one may check that the set on the right is contained in \( \sigma(H_1) = \sigma(H_1^*) \), and \( \mu \notin \sigma(H_1) \) by the assumptions. This, together with (17), proves the claim. \( \square \)

Proof of Lemma 3. First we return to the problem on the whole \( \mathbb{R}^d \). By Lemma 4 applied to \( t, \varepsilon, \delta \) and \( r/2 \), there exist \( a > 0 \) and \( c \in \mathbb{C} \) such that \( U_{c,t,a} \) satisfies (10), and so the operator \(-\Delta + U_{c,t,a} \) in \( L^2(\mathbb{R}^d) \) has an eigenvalue \( \mu \) (independent of \( t \)) with \( \text{Im} \mu < 0 \) and \( |\mu - \lambda| < r/2 \). By Lemma 4 applied to \( V_1 \equiv 0, V_2 = U_{c,t,a^*} \), for every \( t > 0 \) sufficiently large, the operator \(-\Delta + c_{R,\varepsilon} U_{c,t,a^*} \) in \( L^2(\mathbb{R}^d_+) \) has an eigenvalue \( \mu_t \) with \( \text{Im} \mu_t < 0 \) and \( |\mu_t - \mu| < r/2 \), hence \( |\mu_t - \lambda| < r \). \( \square \)

Now the proof of the main result is straightforward.

Proof of Theorem 2. We proceed analogously as in the proof of Theorem 1 but use Lemmas 3, 4 instead of Lemmas 1, 2. Note that here \( t_1 \) is not arbitrary but given (sufficiently large) by Lemma 3. \( \square \)

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