Right order Turán-type converse Markov inequalities for convex domains on the plane

Szilárd Gy. Révész

November 13, 2018

Abstract

For a convex domain $K \subset \mathbb{C}$ the well-known general Bernstein-Markov inequality holds asserting that a polynomial $p$ of degree $n$ must have $\|p'\| \leq c(K)n^2\|p\|$. On the other hand for polynomials in general, $\|p'\|$ can be arbitrarily small as compared to $\|p\|$.

The situation changes when we assume that the polynomials in question have all their zeroes in the convex body $K$. This was first investigated by Turán, who showed the lower bounds $\|p'\| \geq (n/2)\|p\|$ for the unit disk $D$ and $\|p'\| \geq c\sqrt{n}\|p\|$ for the unit interval $I := [-1, 1]$. Although partial results provided general lower estimates of lower order, as well as certain classes of domains with lower bounds of order $n$, it was not clear what order of magnitude the general convex domains may admit here.

Here we show that for all compact and convex domains $K$ with nonempty interior and polynomials $p$ with all their zeroes in $K$ $\|p'\| \geq c(K)n\|p\|$ holds true, while $\|p'\| \leq C(K)n\|p\|$ occurs for any $K$. Actually, we determine $c(K)$ and $C(K)$ within a factor of absolute numerical constant.

MSC 2000 Subject Classification. Primary 41A17. Secondary 30E10, 52A10.

Keywords and phrases. Bernstein-Markov Inequalities, Turán’s lower estimate of derivative norm, logarithmic derivative, Chebyshev constant, convex domains, width of a convex domain.

Supported in part in the framework of the Hungarian-French Scientific and Technological Governmental Cooperation, Project # F-10/04 and the Hungarian-Spanish Scientific and Technological Governmental Cooperation, Project # E-38/04.

This work was not supported by Hungarian National Foundation for Scientific Research.

1
§0. Introduction

On the complex plane polynomials of degree $n$ admit a Bernstein-Markov inequality $\|p'\|_K \leq c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on $K$.

Sixty-five years ago Paul Turán studied converse inequalities of the form $\|p'\|_K \geq c_K n^4 \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials $p$. Turán assumed that all zeroes of the polynomials must belong to $K$. So denote the set of complex (algebraic) polynomials of degree (exactly) $n$ as $\mathcal{P}_n$, and the subset with all the $n$ (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n(K)} M(p) \quad \text{with} \quad M := M(p) := \frac{\|p'\|}{\|p\|}.$$  \hfill (1)

**Theorem A [Turán].** If $p \in \mathcal{P}_n(D)$, where $D$ is the unit disk, then we have

$$\|p'\|_D \geq \frac{n}{2} \|p\|_D.$$  \hfill (2)

**Theorem B [Turán].** If $p \in \mathcal{P}_n(I)$, where $I := [-1, 1]$, then we have

$$\|p'\|_I \geq \frac{\sqrt{n}}{6} \|p\|_I.$$  \hfill (3)

Theorem A is best possible. Regarding Theorem B, Turán pointed out by example of $(1 - x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [5], and the value of the constant is computed for all fixed $n$ precisely in [4].

The key to Theorem A was the following observation, which had already been present implicitly in [10] and [11] and was later formulated explicitly in [5].

**Lemma C [Turán].** Assume that $z \in \partial K$ and that there exists a disc $D_R$ of radius $R$ so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n(K)$ we have

$$|p'(z)| \geq \frac{n}{2R} |p(z)|.$$  \hfill (4)

Drawing from the work of Turán, Erőd [4] already addressed the question: "For what kind of domains does the method of Turán provide $cn$ order of oscillation for the derivative?" In particular, he showed

\[ \text{Namely, to each point } z \text{ of } K \text{ there exists another } w \in K \text{ with } |w - z| \geq \text{diam}(K)/2, \text{ and applying Markov’s inequality on the segment } [z, w] \subset K \text{ yields } |p'(z)| \leq (1/\text{diam}(K))n^2\|p\|_K. \]
**Theorem D [Erőd].** Let \(0 < b < 1\) and let \(E_b\) denote the ellipse domain with major axes \([-1, 1]\) and minor axes \([-ib, ib]\). Then for all \(p \in \mathcal{P}_n(E_b)\) we have

\[
\|p'\| \geq \frac{b}{2n}\|p\|. \quad (5)
\]

Moreover, he elaborated on the inverse Markov factors belonging to domains with some favorable geometric properties, such as having positive curvature exceeding a given fixed positive bound at all boundary points, or at all boundary points with the exception of a given (finite) set of vertices, etc. For a detailed account of results of Erőd in this direction, as well as even further results applying basically Turán’s Lemma see the recent works [5], [2] and [9].

A lower estimate of the inverse Markov factor for any convex set and of at least the same order as for the interval was obtained in full generality only in about three years ago.

**Theorem E [Levenberg-Poletsky].** If \(K \subset \mathbb{C}\) is a compact, convex set, \(d := \text{diam } K\) and \(p \in \mathcal{P}_n(K)\), then we have

\[
\|p'\| \geq \frac{\sqrt{n}}{20 \text{diam } (K)}\|p\|. \quad (6)
\]

Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order, while for convex compact domains \(K\) and for all \(p \in \mathcal{P}_n(K)\) we have at least \(M(p) \geq C_1(K)n^{2/3}\), see [5]. Recall that here the term convex domain stands for a compact, convex subset of \(\mathbb{C}\) having nonempty interior. Clearly, assuming boundedness is natural, since all polynomials of positive degree have \(\|p\|_K = \infty\) when the set \(K\) is unbounded. Also, all convex sets with nonempty interior are fat, meaning that \(\text{cl}(K) = \text{cl}((\text{int}K))\). Hence taking the closure does not change the sup norm of polynomials under study. The only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his lower estimate is of the right order.

The case of the unit disk and the example of \(p(z) = 1 + z^n\) shows that in general the order of the inverse Markov factor can not be higher than \(n\). On the other hand, some general classes of domains were found to have order \(n\) inverse Markov factors. Let us list a few examples of such domains.

1. All convex domains with \(C^2\)-smooth boundary and curvature above a given fixed parameter \(\kappa > 0\) (Erőd [4] and Révész [9]).

2. Convex domains bounded by finitely many \(C^2\)-smooth Jordan arcs and a finite number of vertices, with the curvature of any relative interior points of the arcs bounded away from 0 (Erőd [4] and Révész [9]).

---

[5] For further details on the constants and more precise details of the slightly incomplete proofs in [4] for items 1-3 see [9].
3. Convex domains of smooth boundary and curvature bounded away from 0, with the exception of one straight line segment on the boundary having length $< \text{diam} (K)/4$, (Erőd [4]).

4. A square (Erdélyi, [2], [3]).

5. Convex domains with finitely many vertices having vertices of only acute supplementary angles and finitely many smooth Jordan arcs connecting the vertices [8, 9].

6. Smooth convex domains [8, 9].

7. Convex domains of fixed positive depth [8, 9].

8. Convex domains with their almost everywhere (with respect to arc length measure) existing curvature exceeding almost everywhere a given positive lower bound [9].

For further details and a discussion of the results of Erőd [4], see the references, in particular [9]. On the other hand, it was not known whether the inverse Markov factor can be $o(n)$ or not.

To study (1) some geometric parameters of the convex domain $K$ are involved naturally. We write $d := d(K) := \text{diam} (K)$ for the diameter of $K$, and $w := w(K) := \text{width} (K)$ for the minimal width of $K$. Note that a convex domain is a closed, bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leq d(K) < \infty$. Our main result is the following.

**Theorem 1.** Let $K \subset \mathbb{C}$ be any convex domain. Then for all $p \in \mathcal{P}_n(K)$ we have

$$\frac{\|p'\|}{\|p\|} \geq C(K)n \quad \text{with} \quad C(K) = 0.0006 \frac{w(K)}{d^2(K)}.$$  

(7)

Clearly this result contains all the above results apart from the precise value of the absolute constant factor. Moreover, the result is essentially sharp for all convex domains $K$: see [2] below.

§1. Proof of Theorem 1

Our proof will follow the argument of [8], with one key alteration, suggested to us by Gábor Halász. Namely, we start with picking up a boundary point $\zeta \in \partial K$ of maximality of $|p|$, and consider a supporting line at $\zeta$ to $K$ as in [8]. However, then we do not use the normal direction to compare values of $p$ at $\zeta$ and on the intersection of $K$ and this normal line, but instead here

---

3Erdélyi also proves similar results on rhombuses, under the further condition of some symmetry of the polynomials in consideration – e.g. if the polynomials are real, or odd. Note also that his work [2] preceded [8, 9] and apparently was accomplished without being aware of details of [4].
we compare the values of \( p \) at \( \zeta \) and on a line slightly slanted off from the normal. Comparing the calculations here and in [8] the reader will detect how this change led to an essential improvement of the result through improving the contribution of the factors belonging to zeroes close to the supporting line. In [8] we could get a square term (in \( h \) there) only, due to orthogonality and the consequent use of the Pithagorean Theorem in calculating the distances. However, here we obtain linear dependence in \( \delta \) via the general cosine theorem for the slanted segment \( J \). (That insightful observation was provided by G. Halász.) One of the major geometric features still at our help is the fact, that when the intersection of a normal or close-to-normal line with \( K \) is small, then one part of the convex domain \( K \), cut into two by the line, will also be small in the same order. That was explicitly formulated in [8], and is used implicitly even here through various calculations with the angles: this is the key feature which allows us to bend the direction of the normal a bit towards the smaller portion of \( K \). As a result of the improved estimates squeezed out this way, we do not need to employ the second technique, also going back to Turán, i.e. integration of \( (p'/p') \) over a suitably chosen interval. As pointed out already in [8], this part of the proof yields weaker estimates than \( cn \), so avoiding it is not only a matter of convenience, but is an essential necessity.

**Proof.** We list the zeroes of a polynomial \( p \in \mathcal{P}_n(K) \) according to multiplicities as \( z_1, \ldots, z_n \), and the set of these zero points is denoted as \( \mathcal{Z} := \mathcal{Z}(p) := \{ z_j : j = 1, \ldots, n \} \subset K \). (It suffices to assume that all \( z_j \) are distinct, so we do not bother with repeatedly explaining multiplicities, etc.) Assume, as we may, \( p(z) = \prod_{j=1}^{n}(z - z_j) \).

We start with picking up a point \( \zeta \) of \( K \), where \( p \) attains its norm. By the maximum principle, \( \zeta \in \partial K \), and by convexity there exists a supporting line to \( K \) at \( \zeta \) with inward normal vector \( \nu \), say. Without loss of generality we can take \( \zeta = 0 \) and \( \nu = i \). Now by definition of the minimal width \( w = w(K) \), there exists a point \( A \in K \) with \( \Im A \geq w \); by symmetry, we may assume \( \Re A \leq 0 \), say.

Sometimes we write the zeroes in their polar form

\[
z_j = r_j e^{i \varphi_j} \quad (r_j := |z_j|, \ \varphi_j := \arg z_j \quad (j = 1, \ldots, n)). \tag{8}
\]

Throughout the proofs with \( [\varphi, \psi] \) being any open, closed, halfopen-half-closed or halfclosed-halfopen interval we use the notations

\[
S[\varphi, \psi] := \{ z \in \mathbb{C} : \arg(z) \in [\varphi, \psi] \} \tag{9}
\]

and

\[
\mathcal{Z}[\varphi, \psi] := \mathcal{Z} \cap S[\varphi, \psi] \quad \text{and} \quad n[\varphi, \psi] := \# \mathcal{Z}[\varphi, \psi]. \tag{10}
\]

for the sectors, the zeroes in the sectors, and the number of zeroes in the sectors determined by the angles \( \varphi \) and \( \psi \).

Let us formulate a well-known but useful fact in advance.
Lemma 1. Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$ and let $J \subset R \subset \mathbb{C}$ be any set containing $J$. Then for all $k \in \mathbb{N}$ we have

$$
\min_{w_1, \ldots, w_k \in R} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \left( \frac{|J|}{4} \right)^k. \tag{11}
$$

Proof. This is essentially the classical result of Chebyshev for a real interval, cf. [1, 6], and it holds for much more general situations (perhaps with the loss of the factor 2) from the notion of Chebyshev constants and capacity, cf. Theorem 5.5.4. (a) in [7]. \qed

In all our proof we fix the angles

$$
\psi := \arctan \left( \frac{w}{d} \right) \in (0, \pi/4] \quad \text{and} \quad \theta := \psi/20 \in (0, \pi/80]. \tag{12}
$$

Since $|p(0)| = \|p\|, M \geq |p'(0)/p(0)|$. Observe that for any subset $\mathcal{W} \subset \mathcal{Z}$ we then have

$$
M \geq \left| \frac{p'}{p}(0) \right| \geq -\Im \frac{p'}{p}(0) = \sum_{j=1}^{n} \Im \frac{-1}{z_j} \geq \sum_{z_j \in \mathcal{W}} \Im \frac{-1}{z_j} = \sum_{z_j \in \mathcal{W}} \sin \frac{\varphi_j}{r_j}, \tag{13}
$$

since all terms in the full sum are nonnegative.

Let us consider now the ray (straight half-line) emanating from $\zeta = 0$ in the direction of $e^{i(\pi/2 - 2\theta)}$. This ray intersects $K$ in a line segment $[\zeta, D]$, and if $D = \zeta$, then $K \subset S[\pi/2 - 2\theta, \pi]$ and a standard argument using e.g. Turán’s Lemma yields $M \geq n/(2d)$. Hence we may assume $D \neq 0$.

Consider now any point $B \in K$ with maximal real part, and take $B' := \Re B = \max \{ \Re z : z \in K \}$. Since $D \neq 0$, $B' > 0$, and as $\Re A \leq 0$ and $\Re B$ is maximal, $[A, B']$ intersects $[0, D]$ in a point $D' \in [0, D]$, i.e. $[0, D'] \subset [0, D] \subset K$. Moreover, the angle at $B'$ between the real line and $AB'$ is $-\arg(B' - A) = -\arg(B' - D') \in [\psi, \pi/2]$. Indeed, $\Im (A - B') \geq w$ and $\Re (B' - A) = \Re (B - A) \leq d$ (resulting from $A, B \in K$) imply $-\arg(B' - A) \geq \arctan(w/d) = \psi$.

In the following let us write $\delta := |D'| > 0$; it can not vanish, as $B' \neq 0$ and the line segment $[B', A]$ intersects the real line only in $B'$. Consider the point $B'' \in \mathbb{R}$ with $B'' \geq B' > 0$ and $-\arg(B'' - D') = \psi$. We can say now that $K$ lies both in the upper half of the disk with radius $d$ around 0 (which we denote by $U$), and the halfplane $\Re z \leq B''$ (which we denote by $H$); moreover, $[0, D'] \subset K \subset (U \cap H)$.

Now we put $D'' := 3D'/4$ and take

$$
J := [D'', D'] \subset K \quad \text{i.e.} \quad J := \{ \tau := te^{i(\pi/2 - 2\theta)} \delta : 3/4 \leq t \leq 1 \}. \tag{14}
$$
Denoting \( D_r(0) := \{ z : |z| \leq r \} \) we split the set \( \mathcal{Z} \) into the following parts.

\[
\mathcal{Z}_1 := \mathcal{Z}[0, \theta], \quad \mu := \# \mathcal{Z}_1 = n[0, \theta]
\]

\[
\mathcal{Z}_2 := \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \mathfrak{R}(e^{i2\theta}z) < \frac{3}{8} \delta \right\}, \quad \nu := \# \mathcal{Z}_2
\]

\[
\mathcal{Z}_3 := \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \mathfrak{R}(e^{i2\theta}z) \geq \frac{3}{8} \delta \right\} \cap D_{2\delta}(0), \quad \kappa := \# \mathcal{Z}_3
\]

\[
\mathcal{Z}_4 := \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \mathfrak{R}(e^{i2\theta}z) \geq \frac{3}{8} \delta \right\} \setminus D_{2\delta}(0) = \mathcal{Z}(\theta, \pi - \theta) \setminus \mathcal{Z}_2 \setminus \mathcal{Z}_3, \quad k := \# \mathcal{Z}_4
\]

\[
\mathcal{Z}_5 := \mathcal{Z}[\pi - \theta, \pi], \quad m := \# \mathcal{Z}_5 = n[\pi - \theta, \pi].
\]

In the following we establish an inequality from condition of maximality of \( |p(0)| \). First we estimate the distance of any \( z_j \in \mathcal{Z}_1 \) from \( J \). In fact, taking any point \( z = re^{i\varphi} \in H \cap S[0, \theta] \) the sine theorem yields \( r \cos \varphi = \Re z \leq |B^\ast| = \delta (\sin(\pi/2 + 2\theta - \psi)/\sin \psi) = \delta \cos(\psi - 2\theta)/\sin(\psi) < \delta \cot(18\theta) \), and so

\[
|z - \tau| \leq \frac{\sin \theta}{\cos \varphi \tan(18\theta)} \leq \frac{\delta \tan \theta}{\tan(18\theta)} < \frac{\delta}{18}.
\]

Applying also \( \boxed{16} \) to estimate \( \delta/r \) in the last but one step. Now \( \delta/d \leq 1 \) and \( 5 \sin \theta < 0.2 \), hence we can apply \( \log(1 + x) \geq x - x^2/2 \geq 0.9x \) for \( 0 < x < 0.2 \) to get

\[
\frac{|z - \tau|}{|z|^2} \geq \exp \left( 0.9 \frac{5 \sin \theta \delta}{d} \right) > \exp \left( \frac{4 \sin \theta \delta}{d} \right) \quad (\tau \in J).
\]

Applying this estimate for all the \( \mu \) zeros \( z_j \in \mathcal{Z}_1 \) we finally find

\[
\prod_{z_j \in \mathcal{Z}_1} \left| \frac{z_j - \tau}{z_j} \right| \geq \exp \left( \frac{2 \sin \theta \delta \mu}{d} \right) \quad (\tau = t\delta e^{i(\pi/2 - 2\theta)} \in J). \quad (17)
\]

The estimate of the contribution of zeroes from \( \mathcal{Z}_5 \) is somewhat easier, as now the angle between \( z_j \) and \( \tau \) exceeds \( \pi/2 \). By the cosine theorem again, we obtain for any \( z = re^{i\varphi} \in S[\pi - \theta, \pi] \cap U \) the estimate

\[
|z - \tau|^2 \geq r^2 + t^2 \delta^2 - 2 \cos(\varphi - (\pi/2 - 2\theta)) \, r t \delta \geq r^2 + t^2 \delta^2 + 2 \sin \theta \, r t \delta > r^2 \left( 1 + \frac{3 \sin \theta \delta}{2d} \right) \quad (\tau \in J), \quad (18)
\]
as \( t \geq 3/4 \) and \( r \leq d \). Hence using again \( \delta/d \leq 1 \) and \( 1.5 \sin \theta < 0.06 \) we can apply \( \log(1+x) \geq x - x^2/2 \geq 0.97x \) for \( 0 < x < 0.06 \) to get

\[
\frac{|z - \tau|}{|z|} \geq \exp \left( \frac{1}{2} 0.97 \frac{3 \sin \theta \ \delta}{2d} \right) \geq \exp \left( \frac{18 \sin \theta \ \delta}{25d} \right) \quad (\tau \in J),
\]

whence

\[
\prod_{z_j \in \mathcal{Z}_3} \left| \frac{z_j - \tau}{z_j} \right| \geq \exp \left( \frac{18 \sin \theta \ \delta \ m}{25d} \right) \quad (\tau = t \delta e^{i(\pi/2 - 2\theta)} \in J). \quad (19)
\]

Observe that zeroes belonging to \( \mathcal{Z}_2 \) have the property that they fall to the opposite side of the line \( \Im(e^{2i\theta}z) = 3\delta/8 \) than \( J \), hence they are closer to 0 than to any point of \( J \). It follows that

\[
\prod_{z_j \in \mathcal{Z}_2} \left| \frac{z_j - \tau}{z_j} \right| \geq 1 \quad (\tau = t \delta e^{i(\pi/2 - 2\theta)} \in J). \quad (20)
\]

Next we use Lemma \( \text{II} \) to estimate the contribution of zero factors belonging to \( \mathcal{Z}_3 \). We find

\[
\max_{\tau \in J} \prod_{z_j \in \mathcal{Z}_3} \left| \frac{z_j - \tau}{z_j} \right| \geq 2 \left( \frac{|J|}{4} \right)^{\kappa} \prod_{z_j \in \mathcal{Z}_3} \frac{1}{r_j} > \left( \frac{1}{32} \right)^{\kappa} > \exp(-3.5\kappa), \quad (21)
\]

in view of \( |J| = \delta/4 \) and \( r_j \leq 2\delta \).

Note that for any point \( z = re^{i\varphi} \in D_{2\delta}(0) \cap \{ \Im(e^{2i\theta}z) \geq 3\delta/8 \} \) we must have

\[
\frac{3\delta}{8} \leq \Im(e^{2i\theta}re^{i\varphi}) = r \sin(\varphi + 2\theta),
\]

hence by \( r \leq 2\delta \) also

\[
\sin(\varphi + 2\theta) \geq \frac{3\delta}{8r} \geq \frac{3}{16}
\]

and \( \sin \varphi \geq \sin(\varphi + 2\theta) - 2\theta \geq 3/16 - \pi/40 > 1/10 \). Applying this for all the zeroes \( z_j \in \mathcal{Z}_3 \) we are led to

\[
1 \leq \frac{2\delta}{r_j} \leq 20\delta \frac{\sin \varphi_j}{r_j} \quad (z_j \in \mathcal{Z}_3). \quad (22)
\]

On combining \( (21) \) with \( (22) \) we are led to

\[
\max_{\tau \in J} \prod_{z_j \in \mathcal{Z}_3} \left| \frac{z_j - \tau}{z_j} \right| \geq \exp \left( -70\delta \sum_{z_j \in \mathcal{Z}_3} \frac{\sin \varphi_j}{r_j} \right). \quad (23)
\]

Finally we consider the contribution of the zeroes from \( \mathcal{Z}_4 \), i.e. the ”far” zeroes for which we have \( \Im(z_je^{2i\theta}) \geq 3\delta/8, \varphi_j \in (\theta, \pi - \theta) \) and \( |r_j| \geq 2\delta \). Put
\[ w := z e^{2i\theta} = u + iv = re^{i\alpha}, \text{ and } s := |\tau| = t\delta, \text{ say.} \]

We then have
\[ \left| \frac{z_j - \tau}{z_j} \right|^2 = \left| \frac{w - t\delta i}{r^2} \right|^2 = \frac{u^2 + (v - s)^2}{r^2} = 1 - \frac{2\nu s}{r^2} + \frac{s^2}{r^2}. \]

Recall that \( \log(1 - x) > -x - \frac{x^2}{2} - \frac{1}{3x} \geq -x(1 + 1/2) \) whenever \( 0 \leq x \leq 1/2 \). We can apply this for \( x := \delta |\sin \alpha|/r_j \leq \delta/r_j \leq 1/2 \) using \( r = r_j = |z_j| = |w| \geq 2\delta \). As a result, \((24)\) leads to
\[ \left| \frac{z_j - \tau}{z_j} \right| \geq \exp \left( -\frac{3\delta}{2} \left| \sin(\varphi_j + 2\theta) \right| \right), \tag{25} \]

and using \( |\sin(\varphi_j + 2\theta)| \leq |\sin(\varphi_j) + \sin(2\theta)| \leq 3|\varphi_j| \) (in view of \( \varphi_j \in (\theta, \pi - \theta) \)), finally we get
\[ \prod_{z_j \in Z} \left| \frac{z_j - \tau}{z_j} \right| \geq \exp \left( -\frac{9\delta}{2} \sum_{z_j \in Z} \frac{\sin \varphi_j}{r_j} \right) \quad (\tau = t\delta e^{i(\theta/2 - \varphi)} \in J) \tag{26} \]

Collecting the estimates \((17), (19), (20), (23)\) and \((26)\) gives for a certain point of maxima \( \tau_0 \in J \) in \((23)\) the inequality

\[ 1 \geq \frac{|p(\tau_0)|}{|p(0)|} = \prod_{z_j \in Z} \left| \frac{z_j - \tau_0}{z_j} \right| > \exp \left\{ \frac{18}{25} \sin \theta \delta \frac{\mu + m}{d} - 70\delta \sum_{z_j \in Z_2 \cup Z_3 \cup Z_4} \frac{\sin \varphi_j}{r_j} \right\}, \]

or, after taking logarithms and cancelling by \(18\delta/25\)
\[ \sin \theta \frac{\mu + m}{d} < \frac{875}{9} \sum_{z_j \in Z_2 \cup Z_3 \cup Z_4} \frac{\sin \varphi_j}{r_j}. \tag{28} \]

Observe that for the zeroes in \( Z_2 \cup Z_3 \cup Z_4 \) we have \( \sin \varphi_j > \sin \theta \), whence also
\[ (\nu + \kappa + k) \frac{\sin \theta}{d} \leq \sum_{z_j \in Z_2 \cup Z_3 \cup Z_4} \frac{\sin \varphi_j}{r_j}. \tag{29} \]

Adding \((28)\) and \((29)\) and taking into account \( \#Z = \sum_{j=1}^5 \#Z_j \), we obtain
\[ \sin \theta \frac{n}{d} = \sin \theta \frac{\mu + m + \nu + \kappa + k}{d} < \frac{884}{9} \sum_{z_j \in Z_2 \cup Z_3 \cup Z_4} \frac{\sin \varphi_j}{r_j}. \tag{30} \]

Making use of \((13)\) with the choice of \( W := Z_2 \cup Z_3 \cup Z_4 \) we arrive at
\[ \sin \theta \frac{n}{d} < \frac{884}{9} M, \]

9
that is,
\[
M > \frac{9 \sin \theta}{884d} n.
\] (31)

It remains to recall (12) and to estimate
\[
\sin \theta = \sin \left( \frac{\arctan(w/d)}{20} \right).
\]

As \( \theta \in (0, \pi/80] \), \( \sin \theta > \theta (1 - \theta^2/6) \geq \theta (1 - \pi/240) > 0.98\theta \) and as \( 0 < w/d \leq 1 \), \( \arctan(w/d) \geq (w/d)/(\pi/4) \), whence
\[
\sin \theta \geq 0.98 \frac{\arctan(w/d)}{20} \geq \frac{0.98 w}{5\pi d}.
\]

Substituting this last estimate into (31) yields
\[
M > \frac{9}{884} \cdot \frac{0.98}{5\pi} \cdot \frac{w}{d^2} \cdot n > 0.0006 \frac{w}{d^2} n,
\]
concluding the proof.

\[\square\]

§2. On sharpness of the main result

**Theorem 2.** Let \( K \subset \mathbb{C} \) be any compact, connected set with diameter \( d \) and minimal width \( w \). Then for all \( n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w) \) there exists a polynomial \( p \in P_n(K) \) of degree exactly \( n \) satisfying
\[
\|p'\| \leq C'(K) n \|p\| \quad \text{with} \quad C'(K) := 600 \frac{w(K)}{d^2(K)}.
\] (32)

**Remark 1.** Note that here we do not assume that \( K \) be convex, but only that it is a connected, closed (compact) subset of \( \mathbb{C} \). (Clearly the condition of boundedness is not restrictive, \( \|p\| \) being infinite otherwise.)

**Proof.** Take \( a, b \in K \) with \( |a - b| = d \) and \( m \in \mathbb{N} \) with \( m > m_0 \) to be determined later. Consider the polynomials \( q(z) := (z - a)(z - b) \), \( p(z) = (z - a)^m(z - b)^m = q^m(z) \) and \( P(z) = (z - a)^m(z - b)^{m+1} = (z - b)q^m(z) \). Clearly, \( p, P \in P_n(K) \) and \( \deg p = 2m, \deg P = 2m + 1 \). We claim that these polynomials satisfy inequality (32) for appropriate choice of \( m_0 \).

First we make a few general observations. One obvious fact is that if the unit vector \( e := (b - a)/d \), then the line \( \ell := \{(a + tb)/2 + i(e : t \in \mathbb{R}) \} \) separates \( a \) and \( b \). Since \( K \) is connected, also \( \ell \) contains some point \( c \) of \( K \). Therefore, \( \|q\| \geq |q(c)| = (d/2)^2 + t^2 \geq (d/2)^2 \). Also, it is clear that \( q'(z) = 2z - a - b \) and hence \( \|q'\| \leq |z - a| + |z - b| \leq 2d \), by definition of the diameter.

As for \( p \), we have \( p' = mq'q^{m-1} \), hence
\[
\|p'\| \leq m\|q'||q||^{m-1} \leq m2d\frac{\|p\|}{\|q\|} \leq \frac{2md\|p\|}{(d/2)^2} = \frac{8m}{d} \|p\|.
\] (33)
Concerning $P$ we can write using also (33) above
\[ \|P'\| \leq \|p\| + \|p'\| \|z - b\| \leq \|p\| \left(1 + \frac{8m}{d} \right) = (8m + 1)\|p\|. \] (34)

Consider any point $z \in K$ where $\|q\|$, and thus also $\|p\|$ is attained. We clearly have $\|P\| \geq |P(z)| = |z - b|\|p\|$. But here $|z - b| \geq d/5$: for in case $|z - b| \leq d/5$ we also have $|z - a| \leq 6d/5$ by the triangle inequality, thus $|q(z)| \leq 6d^2/25 < (d/2)^2 \leq \|q\|$, as shown above. Therefore, we conclude $\|P\| \geq (d/5)\|p\|$ and (34) leads to
\[ \|P'\| \leq \frac{5(8m + 1)}{d} \|P\| < \frac{20n}{d} \|P\| \quad (n := 2m + 1 = \deg P). \] (35)

Now consider first the case $w > d/25$. Using $(25w/d) \geq 1$ we obtain both for $p$ and for $P$ the estimate
\[ M(p), M(P) \leq \frac{20n}{d} \leq 500 \frac{w}{d^2} n \quad (n := \deg p \text{ or } \deg P, \text{ respectively}). \] (36)

Note that here we have these estimates for any $n \in \mathbb{N}$, without bounds on $n$.

Let now $w < d/25$. Note that if $S$ is the strip $S := \{\omega = \alpha a + (1 - \alpha)b + ite \in \mathbb{C} : 0 \leq \alpha \leq 1, \ t \in \mathbb{R}\}$, then $K \subset S$, since points outside of this strip are further than $d$ either from $a$ or from $b$. In the following we even introduce $w^+ := \sup_K \Im(\omega/e)$ and $w^- := \inf_K \Im(\omega/e)$. In the current second case of $w < d/25$, we can estimate $w^\pm$ by $1.02w$. That is, we claim that for a point $\omega = \alpha a + (1 - \alpha)b + i\beta e \in K$ with $(\alpha \in [0, 1] \text{ and } \beta \geq 0$, say, we necessarily have $\beta \leq 1.02w$. By symmetry, we may assume that $\alpha \geq 1/2$. Put $z := (\alpha - 1/2)a + (3/2 - \alpha)b \in [a, b]$. We then find
\[
\begin{align*}
w(K) & \geq w(\{a, b, \omega\}) = w(\operatorname{con}\{a, b, \omega\}) \\
& \geq w(\operatorname{con}\{z, \alpha a + (1 - \alpha)b, \omega\}) \\
& = \operatorname{dist}(\alpha a + (1 - \alpha)b, [z, \omega]) = \frac{(d/2)\beta}{\sqrt{(d/2)^2 + \beta^2}} = \frac{\beta}{\sqrt{1 + 4(\frac{d}{2})^2}}.
\end{align*}
\] (37)

Since $\beta \leq d$ is obvious, we conclude
\[ \frac{d}{25} \geq w \geq \frac{\beta}{\sqrt{5}}, \] (38)

hence from (37) we even have
\[ \beta \leq w\sqrt{1 + 4(\frac{\beta}{d})^2} \leq w\sqrt{1 + \frac{4}{125}} < 1.02w. \] (39)

It follows that $w^\pm \leq w' := 1.02w$, as stated. Therefore, the domain $K$ lies not only in the strip $S$, but also within the rectangle $R := \operatorname{con}\{a - iw'e, b - iw'e, b + iw'e, a + iw'e\}$. For the central part $Q := \{\omega \in S : |\alpha - 1/2| \leq 10w/d\}$ of $R$ we have
\[ \|q'\|_{K \cap Q} = \|2z - a - b\|_{K \cap Q} \leq 2\sqrt{10w^2} + w^2 < 21w, \] (40)
while for the remaining part
\[ \|q\|_{K \setminus Q} \leq 2d \]  \hspace{1cm} (41)
remains valid as above.

Next we estimate \(|q|\) in \(K \setminus Q\). It is easy to see that here
\[ \|q\|_{K \setminus Q} \leq \|q\|_{R \setminus Q} = \|q(1/2 + 10w/d)a + (1/2 - 10w/d)b + iw'e)\|, \]
hence
\[
\|q\|_{K \setminus Q}^2 \leq \left( \frac{d}{2} \right)^4 - \left( \frac{d}{2} \right)^2 \left[ 200w^2 - 2w^2 \right] + 10^4w^4 + 200w^2w^2 + w^4
\]

applying also (39), i.e. \(w' \leq 1.02w\) in the last step. Now \(w \leq d/25\) yields
\[
\|q\|_{K \setminus Q}^2 \leq \left( \frac{d}{2} \right)^4 \left[ 1 - 788 \left( \frac{w}{d} \right)^2 + 16 \cdot 10212 \left( \frac{w}{d} \right)^4 \right]
\]
\[
\leq \left( \frac{d}{2} \right)^4 \left[ 1 - 522 \left( \frac{w}{d} \right)^2 \right] \leq \left( \frac{d}{2} \right)^4 \left[ 1 - \left( 16w \right)^2 \right]^2,
\]
that is, using also \(\|q\|_K \geq (d/2)^2\), we find
\[
\frac{\|q\|_{K \setminus Q}}{\|q\|_K} \leq 1 - \left( \frac{16w}{d} \right)^2. \hspace{1cm} (42)
\]

Now for \(z \in K \cap Q\) we have in view of (40)
\[
|p'(z)| = m \cdot |q'(z)| \cdot |q^{m-1}(z)| \leq 21wm\|q\|^{m-1} = 21wm\frac{\|p\|}{\|q\|}
\]
\[
\leq \frac{21wm\|p\|}{(d/2)^2} = 42\frac{w}{d^2}m\|p\|, \hspace{1cm} (43)
\]
and for \(z \in K \setminus Q\) using \(\|q\|_K \geq (d/2)^2\), (41) and (42) we get
\[
|p'(z)| \leq m \cdot 2d \cdot \|q\|_{K \setminus Q} \leq 2md\frac{\|p\|}{\|q\|} \left[ 1 - \left( \frac{16w}{d} \right)^2 \right]^{m-1}
\]
\[
\leq \frac{8m}{d} \|p\| \left[ 1 - \left( \frac{16w}{d} \right)^2 \right]^{m-1}. \hspace{1cm} (44)
\]

Now in view of \(w < d/25\), a standard calculation shows that
\[
\left[ 1 - \left( \frac{16w}{d} \right)^2 \right]^{m-1} \leq \frac{25w}{d} \quad \text{if} \quad m \geq m_0 := \left( \frac{d}{16w} \right)^2 \log \left( \frac{d}{16w} \right). \hspace{1cm} (45)
\]
Indeed, as \( \log(1 - x) < -x \) for all \( 0 < x < 1 \), using \( w < d/25 \) we find
\[
(m - 1) \log \left[ 1 - \left( \frac{16w}{d} \right)^2 \right] < -(m - 1) \left( \frac{16w}{d} \right)^2 < -m \left( \frac{16w}{d} \right)^2 + 0.41,
\]
which entails for \( m \geq m_0 \) that
\[
\left[ 1 - \left( \frac{16w}{d} \right)^2 \right]^{m-1} < e^{-m_0 \left( \frac{16w}{d} \right)^2 + 0.41} = e^{-\log \left( \frac{d}{16w} \right) + 0.41} < \frac{25w}{d}.
\]

It follows from (44) and (45) that
\[
\|p'\|_{K \setminus Q} \leq 100 \frac{w}{d^2} n \|p\|.
\]

Collecting (43) and (46) we get also in this case of \( w < d/25 \) the estimate
\[
\|p'\| \leq 100 \frac{w}{d^2} n \|p\| \quad (n = 2m = \deg p, \; m \geq m_0).
\] (47)

It remains to consider the odd degree case of \( n = 2m + 1 \), i.e. \( P \). Now write
\[
|p'(z)| \leq |p(z)| + |p'(z)| \cdot |z - b| \leq |p(z)| + d \|p'\| \leq (1 + 100 \frac{w}{d} m) \|p\| \quad (m \geq m_0),
\] (48)
in view of (47). As shown above, we have \( \|P\| \geq \|p\|/(d/5) \), while \( m \geq m_0 \) entails \( 1 \leq m/m_0 < m(w/d)(16^2/25((1/\log(25/16)) < 30mw/d) \), hence (48) yields
\[
\|P'\| \leq \frac{230mw}{d} \|p\| \leq \frac{1150mw}{d^2} \|P\|.
\]

Since now \( n = 2m + 1 > 2m \), we finally find
\[
\|P'\| < 600 \frac{w}{d^2} n \|P\| \quad (n = 2m + 1 = \deg P, \; m > m_0).
\] (49)

\[\blacksquare\]

§3. Acknowledgements and comments

Because [8] will not be published in a journal, a full, self-contained proof was presented here. At the same time, this was meant to provide also a clear explanation and documentation of the origin and development of the various ideas that have led to the result.

The author is indebted to Gábor Halász for his generous contribution of an essential idea, explained at the beginning of §1 above. Although he did not accept being included as coauthor, the paper contains, in fact, a joint result with him.
Bibliography

[1] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics 161, Springer Verlag, New York, 1995.

[2] T. Erdélyi, Turán type inequalities on diamonds, *manuscript*, 2004.

[3] T. Erdélyi, Inequalities for exponential sums via interpolation and Turán type reverse Markov inequalities, *manuscript*, 2005.

[4] J. Erőd, Bizonyos polinomok maximumának alsó korlátjáról, *Mat. Fiz. Lapok* 46 (1939), 58-82 (in Hungarian).

[5] N. Levenberg, E. Poletsky, Reverse Markov inequalities, *Ann. Acad. Fenn. 27* (2002), 173-182.

[6] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[7] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts 28 Cambridge University Press, 1994.

[8] Sz. Gy. Révész, Turán-Markov inequalities for convex domains on the plane, *Preprint of the Alfréd Rényi Institute of Mathematics, #3/2004*.

[9] Sz. Gy. Révész, On a paper of Erőd, *manuscript*, 2005.

[10] P. Turán, Über die Ableitung von Polynomen, *Comp. Math. 7* (1939), 89-95.

Alfréd Rényi Institute of Mathematics, 
Hungarian Academy of Sciences, 
Budapest, POB 127, 
1364 Hungary 
E-mail: revesz@renyi.hu