On semiclassical equivalence of Green–Schwarz and pure spinor strings in AdS(5) × S(5)

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Abstract
We present a method to study the equivalence at the semiclassical level of the Green–Schwarz and pure spinor formulations of string theory in AdS(5) × S(5). This method provides a clear separation of the physical and unphysical sectors of the pure spinor formulation and allows us to prove that the two models have not only equal spectra of the fermionic fluctuations but also equal conformal weights (the bosonic ones being equal by construction).

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1. Introduction
String theory in AdS(5) × S(5) is of great relevance, being the first and best-studied example of AdS/CFT correspondence. The Ramond–Neveu–Schwarz formulation is not suitable for describing string theory in this background due to the presence of a non-vanishing Ramond–Ramond (RR) flux. Two approaches are available in this case: the Green–Schwarz (GS) formulation [1] and the pure spinor (PS) one [2]. The GS formulation for type II superstrings in a general curved background has been known for a long time [3]. The PS formulation in a generic superbackground has been constructed in [4] by considering the more general conformal invariant action, involving also the momenta $d_{L,R}$, $d_{g,R,a}$ of the Grassmann-odd superspace coordinates $\theta_{L,R}^a$ and the ghosts $\lambda_{L,R}^a$, $\omega_{L,R,a}$. The BRS invariance of the PS action implies the on-shell supergravity constraints and holomorphicity of BRS currents. For further discussions see [5–7]. Alternatively, starting from an extended free differential algebra of the superspace geometry [8–10], one can add to the GS action conformal invariant terms involving $d_{L,R}$ and ghosts to promote the $\kappa$-symmetry of the GS approach to BRS symmetry [9, 10], following a method proposed in [11] for the heterotic case.

The GS and PS actions in AdS(5) × S(5) are obtained from the corresponding actions in a generic curved superbackground by setting the background superfields to their AdS(5) × S(5) values. An incomplete list of papers that discuss the AdS(5) × S(5) string theory are [12–17] in the GS formulation and [18–22] in the PS formulation. In the PS case there is a term,
proportional to \((dLMdR)\) where \(M\) is a superfield related to the RR flux. In \(\text{AdS}(5) \times S(5)\) \(M\) is a constant invertible matrix and, since the other terms are almost linear in \(dL/R\), one can integrate over \(dL\) and \(dR\) to get an action that depends on the same superspace variables as in the GS model (in addition to the ghosts). Since \(\text{AdS}(5) \times S(5)\) is the coset \(\text{PSU}(2,2|4)/\text{SO}(4,4) \times \text{SO}(5)\), it is convenient to express the GS and PS actions in terms of the currents valued in this coset [12]. The bosonic sectors of the GS and PS actions are equal but the fermionic sectors are different. In particular, in the PS formulation the fermionic modes have a second-order kinetic term. Then, a natural question to ask is whether the two formulations are equivalent.

It does not seem easy to answer this question in general. In a flat background, the quantum equivalence of the GS superstring (in the semi-light-cone gauge) and the PS superstring was proven long ago in [23, 24]. This was essentially because in flat background string theories are free field theories on the worldsheet. There have also been attempts to relate these two approaches in a curved background, but only at the classical level. For instance, in [25], the classical PS formulation (for the heterotic string) was obtained from a twistor-like (or superembedding) formulation which is equivalent to the GS one. To prove this equivalence at the full quantum level in a curved background (where the strings are not free) seems an almost hopeless problem.

However it is possible to study this problem in a simplified setup, that is, at the semiclassical level: one expands the GS and PS actions around a classical solution of the bosonic field equations of the string up to terms quadratic in the fluctuations, and then compares their (fermionic) spectra (equivalently their one-loop (fermionic) partition functions). In the \(\text{AdS}(5) \times S(5)\) background, this problem has been faced in two recent papers [26, 27]. In [26] the equality of the fermionic spectra has been proven for a simple family of string motions and in [27] this result has been extended to a generic motion of the string in \(\text{AdS}(5) \times S(5)\).

In this paper we propose a different method to deal with this problem at the semiclassical level. The method has the advantage that it provides a clear separation of the physical and unphysical sectors of the PS approach and allows us to prove not only the equivalence of the spectra of the physical fermionic fluctuations of the GS and PS formulations but also their conformal weights. Indeed it is shown that the PS model in the considered approximation contains \(8L + 8R\) fermions with conformal weight \((0,0)\) and the same ‘mass’ (in general worldsheet dependent) as in the GS approach, and \(2L + 2R\) massless fermions with conformal weights \((2,0), (0,2), (0,−1)\) which match the masses and the conformal weights of the b–c ghosts, which are present in the GS approach to fix the worldsheet diffeomorphisms in the conformal gauge.

The unphysical (bosonic and fermionic) fluctuations give rise to eleven left-handed and eleven right-handed massless BRS quartets so that their contributions to the one-loop partition function cancel.

2. GS action in \(\text{AdS}(5) \times S(5)\)

In the conformal gauge, in a generic curved background the GS action is

\[
I_{\text{GS}} = \frac{R^2}{\pi \alpha'} \int dz^+ dz^- [E^a_+(Z) E_{-a} (Z) + B_{+-} (Z)]
\]

where \(E^a_+(Z) \equiv (E^a_+, E^a_L, E^a_R)\) are the vector-like \((E^a, a = 0, \ldots, 9)\) and spinor-like \((E^a_L/R, \alpha = 1, \ldots, 16)\) supervielbeins and \(B\) is the NS–NS 2-superform. \(E^a_+\) and \(B_{+-}\) are the pull-backs of these forms onto the worldsheet parametrized by the coordinates \(z^\pm\), and \(Z^M \equiv (x^m, \theta_L^\alpha, \theta_R^\beta)\)
are the target superspace coordinates. If the superspace geometry satisfies the (on-shell) supergravity constraints, the action (1) is invariant under $\kappa$-symmetry

\[ \delta Z^a E^a_{\alpha \beta} = (E^a_{\alpha \beta} \Gamma_a k_{L})^a \]

\[ \delta Z^a E^a_{\alpha \beta} = (E^a_{\alpha \beta} \Gamma_a k_{R})^a \]

($k_{L,R}$ being local parameters) which halves the number of fermionic degrees of freedom. In AdS($5 \times S(5)$) $E^a$ decomposes as $E^a \equiv (E^a, E^i)$ where $\alpha = 0, 1, \ldots, 4$ refers to AdS($5$) and $i = 5, \ldots, 9$ refers to S($5$). A similar decomposition holds for the Lorentz connection $\omega^{[ab]} \equiv (\omega^{[ab]}_1, \omega^{[ab]}_2)$ with curvature

\[ R^{abcd} = (-\delta^{[ab]} \eta^{c]d}, \delta^{[ab]} \eta^{c]d}) \].

(2)

We will assume $R = 1$ in the following. Moreover in this background the only non-vanishing component of $B$ is $[13]$ $(EL B^{[ab]})_{\alpha \beta}$ where, with $R = 1$,

\[ B^{(10)} = (\gamma^{01234})_{\alpha \beta} \equiv (\gamma_*)_{\alpha \beta} \]

(3)

and

\[ (\gamma_*)^2 = -1. \]

The matrix $\gamma_*$ is also equal to the constant matrix $M^{(0)}$ that describes the RR flux in this background:

\[ M^{(0)} = \frac{1}{2 \cdot 5!} F^{abcd} \gamma_{abcd} = \gamma_* \]

(4)

Then the GS action in AdS($5 \times S(5)$) is

\[ I_{GS} = \int L_{GS} = \int \left[ E^a_{\alpha} (Z) E_{-\alpha} (Z) + \frac{1}{2} (E_{L+} (Z) \gamma_* E_{R-} (Z)) - \frac{1}{2} (E_{L-} (Z) \gamma_* E_{R+} (Z)) \right] \]

(5)

Here and in the following $\int \cdots$ stands for $\int \frac{dx^- d^4 \phi}{\pi a}$.

Since AdS($5 \times S(5)$) is the coset

\[ \frac{G}{H} = \text{PSU}(2, 2/4) \]

\[ \frac{G}{H} = \text{SO}(4, 1) \times \text{SO}(5) \]

it is convenient to express this action in terms of the currents

\[ g^{-1} dg = J \equiv J^0 + J^1 + J^2 + J^3 \]

(6)

where $g$ is an element of $G$ that transforms under $G$ from the left and under the structure group $H$ on the right, and $J = J^a T_a$ is valued in the Lie algebra of $G$, $T_a$ being the generators of $\text{psu}(2, 2/4)$, that is $J^a = J^a T_a, J^i = J^i T_{\alpha a}, J^3 = J^3 T_{\alpha a}, J^0 = J^0 [ab] T^{[ab]}$ As shown in [13], in terms of these currents the action (5) can be written as

\[ I_{GS} = \frac{1}{2} \int \text{str} \left[ J^+ J^- - \frac{1}{2} (J^+ J^- - J^3 J^1) \right] \]

(7)

where str denotes the supertrace. This form of the action is useful since it reveals the hidden $Z_2$ automorphism of the AdS($5 \times S(5)$) action, the index $r$ in $J^r$ being its grading under $Z_2$. The relation with the superspace notations of (1)–(5) is: $J^{2,0} = E^a, J^{1,0} = E^a_L, J^{3,0} = E^a_R$ and $J^{0,0} = \omega^{[ab]}$. We are interested in studying the motion of a string in AdS($5 \times S(5)$) at the semiclassical level, i.e., to compute the spectra of the quantum fluctuations around the classical solution. For that one considers the AdS($5 \times S(5)$) GS action expanded around a classical solution of the bosonic field equations. A generic classical motion of the bosonic string is described by the pull-back of the classical vielbeins $e^a_+, e^a_-$ which in the conformal gauge satisfy the field equations

\[ \nabla_+ e^a_- = 0 = \nabla_- e^a_+ \]

(8)
and the Virasoro constraints
\[ e^a_{\mu} e^{-\mu} = 0 = e^a_{\mu} e^{+\mu} \]
(9)
\[ \nabla_{\pm} \] are the pull-back of the covariant derivative \( \nabla = d + \omega \) where \( \omega^{[ab]} \) is the classical Lorentz connection considered previously. It will be convenient to define
\[ \kappa_{\pm} = e^a_{\mu} \Gamma_{a} \equiv e^a_{\mu} \Gamma_{a} + e^a_{\mu} \Gamma_{i}. \]
(10)
Expanding equation (5) (or (7)) around this classical solution up to terms quadratic in fluctuations, the action decomposes into a bosonic part and a fermionic part. Since, as already noted and as will be seen below, the bosonic parts are the same for the GS and PS formulations, for the sake of comparison of the two approaches, one can forget the bosonic parts. The fermionic part of the GS Lagrangian, expanded up to quadratic terms, is
\[ L_{GS} = \frac{1}{2} \theta \kappa_{-} \nabla_{+} \theta_{L} + \frac{1}{2} \theta \kappa_{+} \nabla_{-} \theta_{R} - \frac{1}{2} \theta \kappa_{-} \gamma \ast \kappa_{+} \theta_{L} \]
(11)
which is invariant under the simplified \( \kappa \)-symmetry
\[ \delta \theta_{L/R} = (\kappa \pm \kappa_{L/R}). \]
(12)
If one defines the fermions
\[ \theta^{(\pm)}_{L/R} = P_{\pm} \theta_{L/R}, \]
where \( P_{\pm} \) are the projectors
\[ P_{\pm} = \frac{1}{2(e^{+} e^{-})} e^{\pm} \kappa_{\pm} \]
(12)
that project on an 8-dimensional subspace of the 16-dimensional spinorial space, the fermions \( \theta^{(\pm)}_{L/R} \) have eight components each. The \( \kappa \)-symmetry implies that (36) does not depend on \( \theta^{(-)}_{L} \) and \( \theta^{(+)}_{R} \) so that the Lagrangian
\[ L_{GS} = \frac{1}{2} (\theta^{(+)}_{L} \kappa_{-} \nabla_{+} \theta^{(+)}_{L} + \frac{1}{2} (\theta^{(-)}_{R} \kappa_{+} \nabla_{-} \theta^{(-)}_{R} - \frac{1}{2} (\theta^{(+)}_{L} \kappa_{-} \gamma \ast \kappa_{+} \theta^{(-)}_{R} \]
(13)
describes eight left-handed and eight right-handed massive fermions with conformal weights \((0,0)\), which are the fermionic partners of the eight transverse bosons \( x^{(a)} \). The values of the fermion masses are determined by the last term of (13). Notice that since this term involves the classical vielbeins \( e_{\pm}(x^{(z)}) \), in general the ‘masses’ depend on the worldsheet coordinates \( z^{\pm} \).

Let us also recall that fixing the worldsheet diffeomorphisms by imposing the conformal gauge requires the GS action to be supplemented with the b–c ghost action
\[ \int L_{ghost} = \int [b_{L,--} e_{-L}^{+} c_{L} + b_{R,++} e_{+R}^{+} c_{R}^{+}] \]
(14)
where the ghosts \( b_{L,--} \) and \( b_{R,++} \) have conformal weights \((2,0)\), \((-1,0)\) and \((0,2)\), \((0,-1)\) respectively.

3. PS action in AdS(5) × S(5)

As said before, the PS action in a generic curved background is obtained by adding to the GS action conformal invariant terms involving the fermionic momenta \( d_{L/R} \) and the PS ghosts, so that the PS action becomes invariant under BRS transformations generated by the BRS charge
\[ Q_{BRS} = \int dz^{-}(d_{-L} \lambda_{L}) + \int dz^{+}(d_{+R} \lambda_{R}) \]
(15)
if the supergravity background is on-shell. In AdS(5) × S(5) the PS action takes the form

$$I = \int L_{GS} + \int [(E_{L+} d\lambda_L^-) + (E_{R-} d\lambda_R^-) - (d\gamma^a d\lambda_R^-)] + I_{\omega\lambda}$$  \hspace{1cm} (16)

where

$$I_{\omega\lambda} = \int [(\omega_{L+} \nabla \lambda_L^+) + (\omega_{R-} \nabla \lambda_R^-) + R_{abcd} (\omega_{L-} \Gamma^a \lambda_L^-) (\omega_{R+} \Gamma^d \lambda_R^+)]$$  \hspace{1cm} (17)

with $L_{GS}$ defined in (5) and $R_{abcd}$ in (2). $\lambda_{L/R}$ are PSs satisfying the constraints

$$(\lambda_L \Gamma^a \lambda_L) = 0 = (\lambda_R \Gamma^a \lambda_R)$$

and $I_{\omega\lambda}$ is invariant under the $\omega$-symmetry

$$\delta \omega_{L/R} = \Lambda_{L/R} \Gamma^a \lambda_{L/R}$$

where $\Lambda^{a}_{L/R}$ are local parameters. Equation (16) can be integrated over $d\lambda_{L/R}$ and expressed in terms of the currents (6), as in the GS case. The result is [18]

$$I_{PS} = \frac{1}{2} \int \text{tr} \left[ J^+ J_0^- + \frac{1}{2} J^+ J_0^0 + \frac{3}{2} J^0 J_0^- \right] + I_{\omega\lambda}.$$  \hspace{1cm} (18)

As anticipated and as is now clear from (7) and (18), the physical bosonic sectors in the GS and PS approaches are identical. Therefore, in order to compare the two approaches at the semiclassical level, it is sufficient to consider the fermionic sector of (16) or (18), expanded around the bosonic solution considered in (8), (9), up to terms quadratic in the fluctuations. For the purposes of the present paper it is convenient to start from the non-integrated action (16). In this approximation the Lagrangian for the fermionic sector is

$$L = \frac{1}{2} \left[ \theta_L (\nabla \lambda_L - \frac{1}{2} \gamma_a \lambda_L \gamma^a) + \theta_R (\nabla \lambda_R + \frac{1}{2} \gamma_a \lambda_R \gamma^a) \right]$$

$$- \frac{1}{2} \left[ d_L (\nabla \lambda_L - \frac{1}{2} \gamma_a \lambda_L \gamma^a) + d_R (\nabla \lambda_R + \frac{1}{2} \gamma_a \lambda_R \gamma^a) \right]$$

$$+ \Gamma^a \lambda_L \gamma^a \lambda_L + \Gamma^a \lambda_R \gamma^a \lambda_R \right] .$$  \hspace{1cm} (19)

Upon integrating (19) over $dL$ and $dR$ one gets

$$L = \frac{1}{2} \left[ \theta_L (\nabla \lambda_L - \frac{1}{2} \gamma_a \lambda_L \gamma^a) + \theta_R (\nabla \lambda_R + \frac{1}{2} \gamma_a \lambda_R \gamma^a) \right]$$

$$- \frac{1}{2} \left[ \theta_L \gamma_a \lambda_L \gamma^a \lambda_L + \theta_R \gamma_a \lambda_R \gamma^a \lambda_R \right]$$

$$+ \Gamma^a \lambda_L \gamma^a \lambda_L + \Gamma^a \lambda_R \gamma^a \lambda_R \right] .$$  \hspace{1cm} (20)

which is the quadratic string action considered in [26, 27].

Let us come back to equation (19). Let us define the projectors\(^1\)

$$K_L = \frac{1}{2} (\Gamma^a \gamma^a \lambda_L) \frac{1}{(\lambda_R \gamma^a \lambda_L)} \Gamma^a$$  \hspace{1cm} (21)

$$K_R = \frac{1}{2} (\Gamma^a \gamma^a \lambda_L) \frac{1}{(\lambda_R \gamma^a \lambda_L)} \Gamma^a$$  \hspace{1cm} (22)

and $\tilde{K}_L$ and $\tilde{K}_R$ are the transposed forms of $K_L$ and $K_R$. Since $\text{Tr} K_{L/R} = 5$, $K_{L/R}$ and $1 - K_{L/R}$ decompose the 16-dimensional spinorial space into 5-dimensional and 11-dimensional subspaces, respectively. Notice that

$$K_L \lambda_L = 0 = K_R \lambda_R$$  \hspace{1cm} (23)

so that the PSs $\lambda_{L/R}$ have 11 components. Moreover

$$\lambda_L \Gamma^a (1 - K_L) = 0 = \lambda_R \Gamma^a (1 - K_R).$$  \hspace{1cm} (24)

\(^1\) In AdS(5) × S(5), $\lambda_{R/L}\gamma^a\lambda_{L/R}$ belong to the BRS cohomology and can be assumed to be non-vanishing [20].
Other useful identities are
\[ \hat{K}_L + \gamma s K_R \gamma s = 0 = \hat{K}_R + \gamma s K_L \gamma s \]
\[ (\hat{K}_L \Gamma^a K_L) = 0 = (\hat{K}_R \Gamma^a K_R) \]
and, from \((K_L)^2 = K_L\),
\[ K_L (\nabla K_L) K_L = 0 = (1 - K_L)(\nabla K_L)(1 - K_L) \]
and the same for \(K_R\). Moreover, by gauge fixing the \(\omega\)-symmetry one can impose the conditions
\[ \omega L K_L = 0 = \omega R K_R. \]

In order to avoid problems with the semiclassical approximation it is convenient to assume that \(\lambda_{L/R} \) decompose as \(\lambda_{L/R} = \lambda_{0,L/R} + \hat{\lambda}_{L/R}\) where \(\lambda_{0,L/R}\) are classical fields subject to the field equations \(\nabla_L \lambda_{0,L/R} = 0\) and \(\lambda_{0,L/R}, \hat{\lambda}_{L/R}\) are PS. Then \(K_{L/R}\) become
\[ K_L = \frac{1}{2} (\Gamma^a \gamma s \lambda_{0,R}) \frac{1}{(\lambda_{0,R} \gamma s \lambda_{0,L})} (\lambda_{0,L} \Gamma^a) + O(\hat{\lambda}). \]
Notice that now in (19) (and in (20)) one must add the term
\[ -(\omega L - [R_{abc} \Gamma^a \lambda_{0,L}(\lambda_{0,R} \Gamma^b)] \omega_{R,+}) \]
coming from the last term of (17). This term does not affect the masslessness of the ghosts \(\lambda_{L/R}\) and \(\omega_{L/R}\). The action based on the Lagrangian (19) is invariant under the BRS transformations
\[ s \nabla_+ \theta_L = \nabla_+ \lambda_L, \quad s \nabla_- \theta_R = \nabla_- \lambda_R. \]
\[ s d \xi_- = \lambda_L \xi_- \quad s d \xi_+ = \lambda_R \xi_+ \]
\[ s d \xi_- = -(d_- (1 - K_L)) \quad s d \xi_+ = -(d_+ (1 - K_R)) \]
\[ s d K_-. = (\lambda_L \xi_-) K_L \quad s d K_+ = (\lambda_R \xi_+) K_R. \]

Notice that we abstain from defining the BRS transformations of \(\theta_{L/R}\) themselves, i.e. \(s \theta_{L/R} = \lambda_{L/R}\), since in a curved background such as AdS(5) x S(5), \(\lambda_{L/R}\) transform as spinors under the target-space structure group and \(\theta_{L/R}\) transform as odd superspace coordinates. Only the space–time superfields or forms like \(E_{L/R}, B\) etc have definite transformation properties under the BRS symmetry. In addition notice that, allowing the BRS transformations \(s \theta = \lambda\), we would obtain that \(s \theta_{L/R} = \lambda_{L/R} \theta_{L/R}\), which trivializes the cohomology. Therefore, we never use them to study the cohomology of our model. However, since in our approximation the model is free, for all other instances the use of \(s \theta_{L/R} = \lambda_{L/R}\) is safe.

From (29), (30) and (31) it follows that \(11_L + 11_R\) components \((1 - K_L)\theta_L\) and \((1 - K_R)\theta_R\), as well as \(\omega_L\) and \(\omega_R\), are not BRS invariant and therefore the states involving these fields are not present in the physical Fock space, which is defined as
\[ |\psi\rangle \in \mathcal{F}_{ph} \subset \mathcal{F} \quad \text{iff} \quad Q_{BRS} |\psi\rangle = 0 \]
where \(\mathcal{F}\) is the Fock space of the system. The physical space is defined as
\[ \mathcal{H}_{ph} = \frac{\text{Ker}(Q_{BRS}) \mathcal{F}}{\text{Im}(Q_{BRS}) \mathcal{F}}. \]
At the same time
\[ \theta_{L}^{ph} = K_L \theta_L, \quad \theta_{R}^{ph} = K_R \theta_R \]
are BRS invariant and are physical fields. Notice that, since \( e'_{\ell} \) are classical fields, it follows from (32) that \( d_{L_{-}}K_L \) and \( d_{R_{+}}K_R \) are not invariant under BRS. However, if one defines the combinations

\[
d'_{L_{-}} = d_{L_{-}} - \theta_{L}\hat{\theta}_{-}, \quad d'_{L_{+}} = d_{L_{+}} - \theta_{R}\hat{\theta}_{+}
\]

then

\[
d'^{ph}_{L_{-}} = \hat{d}_{L_{-}}K_L, \quad d'^{ph}_{R_{+}} = \hat{d}_{R_{+}}K_R
\]

are physical, being BRS invariant but not BRS exact. They can be considered as the conjugate momenta of \( \theta_{L}^{ph} \) and \( \theta_{R}^{ph} \).

Here a comment is in order. The appropriate frame to work in the PS approach is the Wick rotated Euclidean version of the theory with \( SO(10) \) as the structure group. The PSs (as well as any spinor projected with \( K_L, K_R, (1 - K_L), (1 - K_R) \) and any tensor of \( SO(10) \)) belong to representations of the subgroup \( U(5) \subset SO(10) \). Our distinction between physical and un-physical sectors refers to the Euclidean framework. In this case the states involving ten bosonic fields \( x^\alpha \) and the physical fermionic fields \( \theta_{L}^{ph} \) and \( \theta_{R}^{ph} \) have positive norm and \( \mathcal{H}_{ph} \) is the physical Hilbert space. When the Wick rotation is reversed to get the Lorentz version with \( SO(1,9) \) as the structure group, \( U(5) \) becomes one of its non-compact versions and the physical space further reduces, to be formed of eight bosonic and eight fermionic fields, as will be discussed in the last section of the paper.

The Lagrangian (19) can be obtained as follows. One starts from the GS Lagrangian

\[
L_{GS} = \frac{1}{2} (\theta_{L}\hat{\theta}_{-})_{\gamma_{+}\gamma_{-}\theta_{L}} + \frac{1}{2} (\theta_{R}\hat{\theta}_{+})_{\gamma_{+}\gamma_{-}\theta_{R}} - \frac{1}{2} (\theta_{L}\hat{\theta}_{-})_{\gamma_{+}\gamma_{-}\theta_{L}}.
\]

Notice that its variation is

\[
\delta L_{GS} = \left( \delta \hat{\theta}_{-} \left[ (\gamma_{+}\theta_{L}) - \frac{1}{2} (\gamma_{+}\theta_{R}) \right] \right) + \left( \delta \theta_{R}\hat{\theta}_{+} \left[ (\gamma_{-}\theta_{R}) - \frac{1}{2} (\gamma_{-}\theta_{L}) \right] \right)
\]

so that \( L_{GS} \) is invariant under the \( \kappa \)-symmetry \( \delta \hat{\theta}_{-} = k_{\ell}\hat{\theta}_{-}, \delta \theta_{R} = k_{R}\hat{\theta}_{+} \) as a consequence of the Virasoro constraints \( (e'_{\ell} e_{-\alpha}) = 0 = (e'_{\alpha} e_{\alpha}) \). Then one adds the new term

\[
L_{new} = - (d_{L_{-}}K_{L} (\gamma_{+}\theta_{L}) - \frac{1}{2} (\gamma_{+}\theta_{R})) - (d_{R_{+}}K_{R} (\gamma_{-}\theta_{R}) + \frac{1}{2} (\gamma_{-}\theta_{L}))
\]

\[
- \frac{1}{2} (d_{L_{-}}K_{L}\gamma_{+}K_{R}d_{R_{+}}).
\]

The addition of \( L_{new} \) promotes the \( \kappa \)-symmetry to a BRS symmetry involving the PS \( \lambda_{L}, \lambda_{R} \) and \( L_{GS} + L_{new} \) is invariant under the BRS transformations (29), (30), (31) and (32).

The action \( \int L \), with \( L \) defined in (19), is obtained by adding to \( \int (L_{GS} + L_{new}) \) a suitable BRS-exact gauge fixing term

\[
\int L_{gf} = s \int F_{gf}
\]

where \( F_{gf} \) is the so-called gauge fermion of ghost number \( n_{gh} = -1 \). Choosing

\[
F_{gf} = - (\omega_{L_{-}}\gamma_{+}\theta_{L}) - (\omega_{R_{+}}\gamma_{-}\theta_{R}) + \frac{1}{2} (\omega_{L_{-}}\gamma_{+}\gamma_{-}\theta_{L}) - \frac{1}{2} (\omega_{R_{+}}\gamma_{+}\gamma_{-}\theta_{R})
\]

\[
- \frac{1}{2} (\omega_{L_{-}}\gamma_{+}(1 - K_{R})d_{R_{+}} - \omega_{R_{+}}\gamma_{+}(1 - K_{L})d_{L_{-}})
\]

one gets

\[
L_{gf} = sF_{gf} = - (\omega_{L_{-}}\gamma_{+}\lambda_{L}) - (\omega_{R_{+}}\gamma_{-}\lambda_{R})
\]

\[
- (d_{L_{-}}(1 - K_{L})\gamma_{+}\theta_{L}) - (d_{R_{+}}(1 - K_{R})\gamma_{-}\theta_{R})
\]

\[
+ \frac{1}{2} (d_{L_{-}}(1 - K_{L})\gamma_{+}\theta_{L}) - \frac{1}{2} (d_{R_{+}}(1 - K_{R})\gamma_{+}\theta_{L})
\]

\[
- \frac{1}{2} (d_{L_{-}}(1 - K_{L})\gamma_{+}(1 - K_{R})d_{R_{+}}).
\]
With this choice any dependence on $K_L$ and $K_R$ disappears from the total action and $L_{\text{GS}} + L_{\text{new}} + L_{\text{gf}}$ coincides with $L$ in (19). Projected with $K_{L/R}$, the first two terms of $L_{\text{GS}}$, neglecting a total derivative, give
\[
\frac{1}{2}(\theta_{L/R} \gamma_+ \gamma_1 \theta_{L/R} + \theta_{L/R} \gamma_+ \gamma_1 \theta_{L/R}) 
\]
\[
+ \frac{1}{2}(\theta_{L/R} - \tilde{K}_{L/R} - K_{L/R} \gamma_+ \gamma_1 \theta_{L/R} + (1 - K_{L/R} - \theta_{L/R})) + L_{L/R}^{(1)}
\] (41)
and the last term of $L_{\text{GS}}$ yields
\[
- \frac{1}{2}(\theta_{L/R} \gamma_+ \gamma_1 \theta_{L/R}) = - \frac{1}{2}(\theta_1 \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R}) - \frac{1}{2}(\theta_1 \tilde{K}_{L/R} \gamma_+ \gamma_1 \theta_{L/R})
\]
\[
- \frac{1}{2}(\theta_1 (1 - \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R} + (1 - \tilde{K}_{L/R} - \theta_{L/R}))) + L_{L/R}^{(2)}
\] (42)
where
\[
L_{L/R}^{(1)} = \left(\theta_{L/R} (1 - \frac{1}{2} \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R}) \right) - \frac{1}{2}(\theta_1 (1 - \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R}))
\] (43)
and
\[
L_{L/R}^{(2)} = - \frac{1}{2}(\theta_1 \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R} - \frac{1}{2}(\theta_1 (1 - \tilde{K}_{L/R} - \gamma_+ \gamma_1 \theta_{L/R})).
\] (44)

Then, adding to $L_{\text{GS}}$ the terms $L_{\text{new}}$ given in (37) and $L_{\text{gf}}$ given in (40), and using the definitions (33)–(35), as well as
\[
\theta_{L}^{\text{unph}} = (1 - K_{L}) \theta_{L}, \quad \theta_{L}^{\text{unph}} = (1 - K_{L}) \theta_{L},
\]
\[
d_{L,R}^{\text{unph}} = \left[ d_{L,R} - \frac{1}{2} \theta_{L}^{\text{unph}} \phi_{L,R} \right] (1 - K_{L}),
\] (45)

one obtains
\[
L_{\text{GS}} + L_{\text{new}} + L_{\text{gf}} = L_{L,R}^{\text{ph}} + L_{L,R}^{\text{unph}} + L_{L,R}^{(1)} + L_{L,R}^{(2)} +
\]
\[
- \frac{1}{2} \left[ (d_{L,R} - (1 - K_{L}) \gamma_+ \gamma_1 \theta_{L,R}) - (d_{L,R} + (1 - K_{L}) \gamma_+ \gamma_1 \theta_{L,R}) \right]
\]
\[
+ \frac{1}{2} (d_{L,R} - (1 - K_{L}) \gamma_+ \gamma_1 \theta_{L,R} + (1 - \tilde{K}_{L,R}) d_{L,R})
\] (47)
where
\[
L_{L,R}^{\text{ph}} = \left[ (d_{L,R}^{\text{ph}} \gamma_+ \gamma_1^{\text{ph}}) + (d_{R,L}^{\text{ph}} \gamma_+ \gamma_1^{\text{ph}}) + \frac{1}{2} (\theta_{L,R}^{\text{ph}} \gamma_+ \gamma_1^{\text{ph}}) + \frac{1}{2} (\theta_{L,R}^{\text{ph}} \gamma_+ \gamma_1^{\text{ph}}) \right]
\] (48)
\[
L_{L,R}^{\text{unph}} = - \left[ (d_{L,R}^{\text{unph}} \gamma_+ \gamma_1^{\text{unph}}) - (d_{R,L}^{\text{unph}} \gamma_+ \gamma_1^{\text{unph}})
\]
\[
+ 2 (d_{L,R}^{\text{unph}} \gamma_+ \gamma_1^{\text{unph}}) - (\omega_{L,R} \gamma_+ \omega_{L,R} \gamma_+ \omega_{L,R}) \right]
\] (49)
and
\[
L_{L,R}^{(1)} = L_{L,R}^{(1)} + L_{R,L}^{(1)}
\] (50)
where $L_{L,R}^{(1)}$ are defined in (43).

For the purpose of this paper it is not relevant to have an action which would be independent of $K_{L/R}$ and therefore there is great freedom in the choice of $L_{\text{gf}}$. In other words, adding BRS-exact terms to $L$ gives rise to Lagrangians $L'$ which are equivalent to $L$. Of course the terms in the last two rows of (40) (as well as the last two terms in (47)) are, by construction, BRS exact and can be subtracted. (Those of the first two rows of (40) must be retained if one insists on having propagating ghosts $\lambda_{L/R}$ and $\omega_{L/R}$.)
Moreover a BRS invariant term which contains $\lambda_L \phi_-$ or $\lambda_R \phi_+$ is BRS exact, being the BRS variation of the same term with $\lambda_L / R \phi_-$ replaced by $d_L / R K_L / R$. For instance, the first term $L^{(2)}$ in (44) is BRS exact due to the identity
\[
(\theta_L \gamma_a \Gamma_a \lambda_L)(\lambda_L \phi_+ \Gamma^a (1 - K_R) \theta_R) = \frac{2(\lambda_L \gamma_s \lambda_R)}{2(\lambda_L \gamma_s \lambda_R)}
\]
where $\theta = \nu^a \Gamma_a$ and $\nu^a$ is the 1-form with components
\[
u_a^\pm \equiv (e_a^\pm, -e^\pm_a)
\]
so that
\[
\gamma_s \phi_\pm = \nu_a^\pm \gamma_s.
\]
A similar identity holds for the second term of (44), and therefore $L^{(2)}$ can be subtracted from the action. The same happens for BRS invariant terms that contain a factor $\nabla_\pm \lambda_L$ or $\nabla_\pm \lambda_R$ which are the BRS variation of the same terms with $\nabla_\pm \lambda_L$ replaced by $\nabla_\pm (1 - K_L / R) \theta_L / R$. In particular, terms like $L^{(1)}_L / R$ in (43) (or $L^{(1)}$ in (50)) that contain the factors $\nabla_\pm K_L / R$ are BRS exact.

However this claim requires clarification. Indeed it stays on the implicit assumption that $\nabla_\pm K_L / R$ produces only terms proportional to $\nabla_\pm \lambda_L$, i.e. that $\nabla_\pm \gamma_s = 0$. Since $\gamma_s$ is related to the value of the RR flux in $\text{AdS}_5 \times S_5$ which connects left-handed and right-handed spinors, and the PS formulation allows for different left-handed and right-handed Lorentz (and Weyl) connections, one might assume that the BRS variation of $\nabla_\pm \gamma_s$ vanishes only if the left-handed and right-handed Lorentz connections are equal. However $L^{(1)}$ can also be BRS trivial if they are different but satisfy a certain condition. If the connections are different
\[
\nabla_\pm \gamma_s = \Omega_{\pm, \pm} \gamma_s + \gamma_s \Omega_{\pm, \pm} = (\Omega_{\pm, \pm} - \Omega_{\pm, \pm}) \gamma_s = \Omega_{\pm, \pm} \gamma_s
\]
where $\tilde{\Omega}_{L / R} = \gamma_s \Omega_{L / R} \gamma_s$, $\Omega_{L / R} = \frac{1}{4} \Omega_{L / R}^{ab} \Gamma_{ab}$ are (classical) left-handed and right-handed Lorentz connections and $\Omega = (\Omega_{L / R} - \Omega_{R / L})$. If $\Omega \neq 0$, the terms in the second line of $L^{(1)}_L / R$ in (43) are still BRS exact and the first terms are exact provided that
\[
\tilde{\Omega}_{\pm, \pm} = 0.
\]
As we will see, this condition is satisfied by the connection $\Omega$ that we will need for consistency.

Therefore, neglecting BRS-exact terms, the action $\int L$ is equivalent to the action
\[
\int L' = \int L^{ph} + \int L^{unph}.
\]
$L^{ph}$ describes (in the Euclidean signature) the physical fermionic sector and $L^{unph}$ describes the unphysical sector.

The Lagrangian of the physical sector describes a set of $5 + 5$ left-handed and $5 + 5$ right-handed physical fields with field equations that are linear in derivatives. This is in agreement with the number of the ten physical bosonic fields $\lambda^a$ (five for the Euclidean version of $\text{AdS}_5$ and five for $S_3$) with the field equations which are quadratic in derivatives.

As will be seen in the next section, in the case of the Lorentzian signature this result also agrees with that of the GS formulation, where there are $8 + 8$ fermionic physical fields and $2 + 2$ Grassmann-odd degrees of freedom provided by the $b - c$ ghosts.

As for the unphysical sector described by the action $\int L^{unph}$, it follows from equations (29) and (31) that $\omega L_{\pm}, \omega R_{\pm}$ and $\theta L^{unph}, \theta R^{unph}$ do not belong to the physical Fock space since they are not BRS invariant and $\lambda_L, \lambda_R$ and $d_L = (1 - K_L), d_R = (1 - K_R)$, being BRS exact, are swept away by the quotient that defines the physical space $\mathcal{H}_{\text{ph}}$. Moreover it follows from (49) by computing the functional determinant that $d_L^{unph}, \theta L^{unph}, \lambda_L / R$ and $\omega L / R$ are massless and form 11 left-handed and 11 right-handed BRS quartets. Therefore, they give a vanishing contribution to the one-loop partition function, since the contributions of the members of each quartet cancel each other.
4. The physical fermionic sector of the PS action in AdS(5) × S(5)

Now let us discuss the action \( \int L^{ph} \) for the physical sector, in order to clarify the relation of this free action and its spectrum to the corresponding action and spectrum of the GS theory. To this end we shall work in the Lorentzian frame.

The action \( \int L^{ph} \) in (48) is similar to the gauge fixed GS action (13) but with a relevant difference. The GS action describes 8 left-handed and 8 right-handed fermionic fields whereas the physical action \( \int L^{ph} \) describes 5 + 5 left-handed and 5 + 5 right-handed fermions. Therefore, in order to compare them one needs projectors \( \Pi_{L/R}^{(4)} \) and \( \Pi_{L/R}^{(1)} \) which commute with \( K_{L/R} \) and among themselves and which project on a four-dimensional and a one-dimensional subspace respectively. The projectors satisfy the conditions

\[
\Pi_{L/R}^{(4)} + \Pi_{L/R}^{(1)} = K_{L/R}, \quad (56)
\]

\[
\Pi_{L/R}^{(1)} \Pi_{L/R}^{(1)} = \Pi_{L/R}^{(1)}, \quad \Pi_{L/R}^{(4)} \Pi_{L/R}^{(4)} = \Pi_{L/R}^{(4)},
\]

\[
\Pi_{L/R}^{(4)} \Pi_{L/R}^{(1)} = 0 = \Pi_{L/R}^{(1)} \Pi_{L/R}^{(4)}. \quad (57)
\]

Notice, however, that it is sufficient that these properties hold at the cohomological level i.e. modulo BRS trivial terms. In fact, since the physical Lagrangian \( L^{ph} \) depends only on BRS invariant fields (i.e. \( \theta^{ph}_{L/R} \) and \( d^{ph}_{L/R} \)), then when \( \Pi_{L/R}^{(4)} \) and \( \Pi_{L/R}^{(1)} \) are used in \( L^{ph} \), these BRS trivial contributions give rise to terms in the action that are BRS exact and can therefore be subtracted, as discussed previously. Therefore, in the following we shall imply that the identities (including (57) and (58)) hold modulo BRS trivial terms. We shall use the notation \( \equiv \) to mean ‘equal modulo BRS-exact terms’ (then also in (57) and (58) the signs of the equality should be replaced with \( \doteq \)).

Let us first discuss the case when the string motion is restricted to AdS(5). In this case it is not difficult to guess the form of these projectors:

\[
\Pi_{L/R}^{(4)} = \frac{1}{2(e_+ e_-)} K_{L/R} \hat{e}_+ \hat{e}_-, \quad \Pi_{L/R}^{(1)} = \frac{1}{2(e_+ e_-)} K_{L/R} \hat{e}_+ \hat{e}_-.
\]

\[
\Pi_{L/R}^{(1)} = \frac{1}{2(e_+ e_-)} K_{L/R} \hat{e}_+ \hat{e}_- \equiv \frac{(\hat{e}_+ \gamma^{RL} \hat{e}_-)}{2(e_+ e_-) (\lambda_L \gamma \lambda_R)} K_{L/R},
\]

\[
\Pi_{L/R}^{(4)} \Pi_{L/R}^{(1)} = 0 = \Pi_{L/R}^{(1)} \Pi_{L/R}^{(4)}, \quad (58)
\]

which indeed satisfy, at the cohomological level, the conditions (56)–(58). Moreover

\[
\text{Tr}(\pi_{L/R}^{(4)}) \equiv 4 \quad \text{Tr}(\pi_{L/R}^{(1)}) \equiv 1
\]

so that, at the cohomological level, they project onto four-dimensional and one-dimensional subspaces.

Now let us come back to the physical action (48) which we write in the form

\[
\int L^{ph} = \int \left[ d^{ph}_{L} K_{L} \gamma_{+} (K_{L} \theta_{L}) + d^{ph}_{R} K_{R} \gamma_{-} (K_{R} \theta_{R}) - \frac{1}{2} (d^{ph}_{L} K_{L} \gamma_{+} K_{R} d^{ph}_{R}) - \frac{1}{2} (\theta_{L} K_{L} \theta_{-} K_{R} \theta_{R}) \right], \quad (61)
\]

and use the identity (56) to replace \( K_{L/R} \) with the projectors \( \Pi_{L/R}^{(4)} \) and \( \Pi_{L/R}^{(1)} \) defined in (59) and (60). Then one gets
Using the identity (56) the action (61) splits into four parts:

\[ d_{L_4}^\text{ph} \Pi_L^{(4)} = \frac{1}{2(e_+ e_-)} (d_{L_4}^\text{ph} \Phi_L) \Phi_L - K_L = \hat{\theta}_L^{(p)} \Phi_L - K_L \]

\[ d_{L_1}^\text{ph} \Pi_L^{(1)} = \frac{1}{2(e_+ e_-)(\lambda_L \gamma_L \lambda_R)} (d_{L_1}^\text{ph} \Phi_L) \lambda_L \Phi_L = \hat{\theta}_{L_1}^{(p)} \Phi_L \]

\[ d_{R_4}^\text{ph} \Pi_R^{(4)} = \frac{1}{2(e_+ e_-)} (d_{R_4}^\text{ph} \Phi_R) \Phi_R = \hat{\theta}_R^{(p)} \Phi_R \]

\[ d_{R_1}^\text{ph} \Pi_R^{(1)} = \frac{1}{2(e_+ e_-)} (d_{R_1}^\text{ph} \Phi_R) \lambda_R \Phi_R = \hat{\theta}_{R_1}^{(p)} \Phi_R \]

where we have defined

\[ \hat{\theta}_L^{(p)} = \frac{1}{2(e_+ e_-)} (d_{L_4}^\text{ph} \Phi_L) \]

\[ \hat{\theta}_R^{(p)} = \frac{1}{2(e_+ e_-)} (d_{R_4}^\text{ph} \Phi_R) \]

and

\[ \hat{\theta}_{L_1}^{(p)} = \frac{1}{2(e_+ e_-)} (d_{L_1}^\text{ph} \Phi_L) \]

\[ \hat{\theta}_{R_1}^{(p)} = \frac{1}{2(e_+ e_-)} (d_{R_1}^\text{ph} \Phi_R) \]

The definitions (64) look natural since \( \hat{\theta}_L^{(p)}, \hat{\theta}_R^{(p)} \) (like \( \theta_L, \theta_R \)) have conformal weights (0,0). Moreover we define

\[ \Pi_L^{(4)} \theta_L = \hat{\theta}_L^{(p)}, \quad \Pi_R^{(4)} \theta_R = \hat{\theta}_R^{(p)} \]

and

\[ \Pi_L^{(1)} \theta_L = \frac{1}{2(e_+ e_-)} \frac{1}{(\lambda_L \gamma_L \lambda_R)} (d_{L_1}^\text{ph} \Phi_L) \lambda_L \Phi_L = \frac{1}{2(e_+ e_-)} \frac{1}{(\lambda_L \gamma_L \lambda_R)} (d_{L_1}^\text{ph} \Phi_L) \lambda_L \Phi_L \]

\[ \Pi_R^{(1)} \theta_R = \frac{1}{2(e_+ e_-)} \frac{1}{(\lambda_L \gamma_L \lambda_R)} (d_{R_1}^\text{ph} \Phi_R) \lambda_R \Phi_R = \frac{1}{2(e_+ e_-)} \frac{1}{(\lambda_L \gamma_L \lambda_R)} (d_{R_1}^\text{ph} \Phi_R) \lambda_R \Phi_R \]

where

\[ \hat{c}_L^+ = \frac{1}{2(e_+ e_-)} (d_{L_4}^\text{ph} \Phi_L) \lambda_R \Phi_R \]

\[ \hat{c}_R^+ = \frac{1}{2(e_+ e_-)} (d_{R_4}^\text{ph} \Phi_R) \lambda_R \Phi_R \]

Using the identity (56) the action (61) splits into four parts: \( L_{4,4}^\text{ph} \) when both \( K_{L,R} \) are replaced by \( \Pi_L^{(4)} \), \( L_{1,1} \) when both \( K_{L,R} \) are replaced by \( \Pi_L^{(1)} \), \( L_{1,4} \) when the first \( K_{L,R} \) is replaced by \( \Pi_L^{(4)} \) and the second one by \( \Pi_R^{(1)} \), \( L_{1,4} \) when the first \( K_{L,R} \) is replaced by \( \Pi_L^{(1)} \) and the second one by \( \Pi_R^{(4)} \). Then, modulo BRST-exact terms, one obtains

\[ L_{4,4}^\text{ph} = (\hat{\theta}_L^{(p)} \Phi_L + \hat{\theta}_R^{(p)} \Phi_R) - \frac{1}{2} (\hat{\theta}_L^{(p)} \Phi_L + \hat{\theta}_R^{(p)} \Phi_R) - \frac{1}{2} (\hat{\theta}_L^{(p)} \Phi_L + \hat{\theta}_R^{(p)} \Phi_R) \]

\[ L_{1,4} = [\hat{b}_{L_1} - \nabla_\Phi \hat{c}_L^+] + [\hat{b}_{R_1} + \nabla_\Phi \hat{c}_R^+] \]

where

\[ \hat{c}_L^+ = \frac{1}{2(e_+ e_-)^2} \]

and

\[ L_{4,1} + L_{1,4} \sim L^a + L^b + L^c \]

where

\[ L^a = \frac{1}{2(e_+ e_-)^2} \left[ \hat{b}_{L_1} + \lambda_L \phi_+ \phi_+ \phi_+ + \hat{b}_{R_1} + \lambda_R \phi_+ \phi_+ \phi_+ \right] \]
Our procedure holds for a generic motion of the string with the exception of those singular motions where their functional determinants are equal. As for the GS action, the action does not involve \( G \) being proportional to \( \Omega_1 \). However, these two actions have the same spectrum, as can be seen by showing that their functional determinants are equal. As for \( \Omega_1 \), defined in (80), indeed satisfies the condition (54), as anticipated.

If the string moves only in AdS, \( \Gamma_\pm \) commute with \( \gamma_\pm \) so that \( G \) reduces to

\[
G = \frac{(\lambda_L \gamma_\pm \lambda_R)}{(e_\pm e_+)}
\]

and \( L^{(a)} \) in (74) vanishes. As for \( L^{(b)} \) and \( L^{(c)} \), equation (75) can be rewritten as

\[
L^{(b)} = \frac{1}{(e_\pm e_+)} \left[ \hat{b}_{L,-} \left( \lambda_L \hat{\Gamma}_+ \left( \nabla_+ \left[ e_+^{[a} e_-^{b]} \right] \frac{1}{2} \Gamma_{ab} \right) \right) \theta_+^{(p)} \right] \\
+ \hat{b}_{R,++} \left( \lambda_R \hat{\Gamma}_- \left( \nabla_- \left[ e_-^{[a} e_+^{b]} \right] \frac{1}{2} \Gamma_{ab} \right) \right) \theta_-^{(p)} \right] \tag{78}
\]

and, according to the discussion before equation (53), \( L^{(c)} \) can be rewritten as

\[
L^{(c)} = \frac{1}{(e_\pm e_+)} \left[ \hat{b}_{L,-} \left( \lambda_L \hat{\Omega}_\pm \theta_+^{(p)} \right) + \hat{b}_{R,++} \left( \lambda_R \hat{\Omega}_\pm \theta_-^{(p)} \right) \right] \tag{79}
\]

where the Lorentz-valued 1-form \( \Omega \) is the difference between the left-handed and right-handed Lorentz connections. Then, if for consistency we take for \( \Omega_\pm \)

\[
\Omega_\pm = -\nabla_\pm \left[ e_\pm^{[a} e_\mp^{b]} \right] \frac{1}{2} \Gamma_{ab} \tag{80}
\]

\( L^{(c)} \) and \( L^{(b)} \) in (78) and (79) cancel and \( L^{(b)}_{4,4} + L^{(b)}_{1,1} \) vanishes at cohomological level. Therefore the action \( \int L^{(b)} \) is equivalent to the action

\[
\int L' = \int L^{(b)}_{4,4} + \int L_{1,1}
\]

where \( L^{(b)}_{4,4} \) and \( L_{1,1} \) are defined in (70) and (71).

Notice that \( \Omega_\pm \) defined in (80), indeed satisfies the condition (54), as anticipated.

As the GS action, the action \( \int L^{(b)}_{4,4} \) describes a set of eight left-handed and eight right-handed 'massive' fermions with conformal weight \((0,0)\). This action is similar but not identical to the GS action. However, these two actions have the same spectrum, as can be seen by showing that their functional determinants are equal. As for \( \int L_{1,1} \), the functional determinant of this action does not involve \( G \) being proportional to \( \nabla_+ \nabla_- \nabla_+ \nabla_- \) and therefore \( \int L_{1,1} \) describes a massless b–c system.

It is not difficult to slightly modify the procedure to cover the general case in which the string moves in the whole AdS\(5 \times S(5)\).\(^2\) For that purpose let us consider the 1-form \( v^\mu \) defined in (52). Let us recall that

\[
\gamma_\pm \hat{\Gamma}_\pm = \hat{\Gamma}_\pm \gamma_\pm
\]

and also notice that

\[
(v_\pm v_\pm) = (e_\pm v_\mp) = (v_+ v_+) = (v_- v_-)
\]

\(^2\) Our procedure holds for a generic motion of the string with the exception of those singular motions where \((v_\pm e_\pm) = 0\).
then define the modified projectors
\[
\Pi_{L/R}^{(4)} = \frac{1}{2(v_\pm e_\pm)} K_{L/R} \hat{p}_\pm \hat{e}_+ K_{L/R},
\]
\[
\Pi_{L/R}^{(1)} = \frac{1}{2(v_\pm e_\pm)} (\hat{e}_+ \gamma_5 \lambda_{L/R}) (\lambda_{L/R} \hat{p}_\pm). \tag{83}
\]

They satisfy, at the cohomological level, the same conditions (56)–(58) satisfied by \( \Pi_{L/R}^{(4)} \) and \( \Pi_{L/R}^{(1)} \). Projecting with these modified projectors, the analogues of (62), (63), (67), (68) are obtained by replacing in (62), (67) \( e^\mu_\pm \) with \( v^\mu_\pm \) and in (63), (68) \( e^\mu_\pm \) with \( v^\mu_\pm \). The left-handed fields \( \tilde{\theta}_L^{(p)}, \theta_L^{(p)}, \hat{b}_{L,--}, \hat{c}_L^- \), and the right-handed fields \( \tilde{\theta}_R^{(p)}, \theta_R^{(p)}, \hat{b}_{R,++}, \hat{c}_R^+ \) are changed accordingly.

For these modified fields we shall maintain the same notations as for the unmodified ones. In particular, \( \hat{b}_{L,--} \) and \( \hat{b}_{R,++} \) in (65) remain unchanged while \( \tilde{\theta}_L^{(p)} \) in (64) become
\[
\tilde{\theta}_L^{(p)} = \frac{1}{2(e_+, v_-)} \theta_L^{(p)} + \frac{1}{2(e_+, v_-)} \theta_L^{(p)} \tag{84}
\]
and \( \hat{c}_L^- \) in (69) become
\[
\hat{c}_L^- = \frac{1}{2(e_-, v_+)} (\lambda_L \gamma_5 \lambda_R) \tag{85}
\]
As for the action \( \int L \), \( L_{4,4} \) and \( L_{1,1} \) remain unchanged but now \( G \) becomes
\[
G = \frac{(\lambda_L \gamma_5 \lambda_R)}{2(e_+, v_-)} \text{ or, using (81) and modulo a BRS trivial term,}
\]
\[
G = \frac{(\lambda_L \gamma_5 \lambda_R)}{e_-, v_+},
\]
which agrees with (77) if the string moves only in AdS(5).

Now for \( L_{4,1} + L_{1,4} \) one has
\[
L^{(a)} = \frac{1}{2(e_-, v_+)} \left[ \hat{b}_{L,--} \left( \lambda_L \gamma_5 \lambda_R \Theta_L^{(p)} \right) + \hat{b}_{R,++} \left( \lambda_R \gamma_5 \lambda_L \Theta_L^{(p)} \right) \right], \tag{86}
\]
\[
L^{(b)} = \frac{1}{2(e_-, v_+)} \left[ \hat{b}_{L,--} \left( \lambda_L \gamma_5 \lambda_R \Theta_L^{(p)} \right) + \hat{b}_{R,++} \left( \lambda_R \gamma_5 \lambda_L \Theta_L^{(p)} \right) \right], \tag{87}
\]
\[
L^{(c)} = \frac{1}{2(e_-, v_+)} \left[ \hat{b}_{L,--} \left( \lambda_L \gamma_5 \lambda_R \Theta_L^{(p)} \right) + \hat{b}_{R,++} \left( \lambda_R \gamma_5 \lambda_L \Theta_L^{(p)} \right) \right]. \tag{88}
\]
As before, \( L^{(a)} \) vanishes as a consequence of (81) and (82), and \( L^{(b)} \) and \( L^{(c)} \) cancel each other by choosing
\[
\hat{Q}_\pm = -\nabla_\pm \left[ \frac{v^{(a)b}_\pm}{(v_+ e_+)} \right] \frac{1}{2} \Gamma_{ab}, \tag{89}
\]
which also satisfies the condition (54). Therefore, as before, \( L_{4,1} + L_{1,4} \) vanishes at the cohomological level.

The action \( \int L \) is then equivalent to the action
\[
\int L = \int L_{4,4}^{(b)} + \int L_{1,1},
\]
where \( L_{4,4}^{(b)} \) and \( L_{1,1} \) are defined in (70) and (71).
The action \( \int L_{4,4} \) describes a set of eight left-handed and eight right-handed massive fermions with conformal weight \((0,0)\) and the same ‘mass’ of the fermions in the GS action. Indeed (70) reduces to equation (13) with the redefinitions

\[
\theta_L^{(+)n} = e^{\frac{i\pi}{2}} \left( \theta_L^{(1)} \right), \quad \theta_R^{(-)n} = e^{\frac{i\pi}{2}} \left( \theta_R^{(1)} \right)
\]

where

\[
\theta_L^{(1)} = e^{\frac{i\pi}{2} \left( \theta_L^{(p)} - \theta_L^{(p)} \right)}, \quad \theta_R^{(2)} = e^{\frac{i\pi}{2} \left( \theta_L^{(p)} + \theta_L^{(p)} \right)}.
\]

As for the action \( \int L_{1,1} \), it describes a left-handed and a right-handed pair of massless anticommuting scalars with conformal weights \((2,0)\), \((-1,0)\), \((0,2)\) and \((0,-1)\) which are equivalent to the b–c ghost system of the GS approach.

As the last remark, it is interesting to notice the relation between our fields \( \hat{b}_{l/R,\mp} \) and the \( \hat{b} \)-fields [20–22] for the PS string theory in AdS(5) \( \times \) S(5). Indeed, in our notation the AdS(5) \( \times \) S(5) \( b \)-fields are

\[
b = \frac{1}{(\lambda_L \gamma_4 \lambda_R)} \left[ \frac{1}{2} (E_{R,-} - \gamma_4 E^{a}_{R} \Gamma_{a} \gamma_4 \lambda_R) + \frac{1}{4} (E_{R,-} N_{L/R}^{ab} \Gamma_{ab} \lambda_R) + \frac{1}{4} (E_{R,-} j_{L/R} \lambda_R) \right] \tag{90}
\]

and

\[
\tilde{b} = \frac{1}{(\lambda_L \gamma_4 \lambda_R)} \left[ \frac{1}{2} (E_{L,+} E^{a}_{R} \Gamma_{a} \gamma_4 \lambda_R) + \frac{1}{4} (E_{L,+} N_{R/L}^{ab} \Gamma_{ab} \lambda_R) + \frac{1}{4} (E_{L,+} j_{R,L} \lambda_R) \right] \tag{91}
\]

where \( N_{L/R,\mp} = (\omega_{L/R} \mp \Gamma_{ab} \lambda_L \lambda_R) \) and \( j_{L/R,\mp} = (\omega_{L/R} \mp \lambda_L \lambda_R) \) and one recovers our expressions for \( b_{L/R,\mp} \) by neglecting the last two terms in (90) and (91) (which are of higher order), replacing \( E^{a}_{R} \) with the classical solutions \( \epsilon^{a}_{R} \) and using the field equations of \( d_{L/R} \). Also let us notice that in equation (3.35) of [20] the expression for the zero-modes \( \epsilon_0 \) and \( \tilde{\epsilon}_0 \) of the \( c \)-fields in an RR plane-wave background seems to be related to our fields \( \tilde{c}_{L} \), \( \tilde{c}_{R}^{+} \) in (69) (or (85)).

5. Conclusion

In contrast to the case of string theories in a flat background [23, 24], the equivalence of the Green–Schwarz (GS) and pure spinor (PS) formulations in curved backgrounds, at the worldsheet quantum level, is still an open problem. As noted in the introduction, the reason for this is that string theories in curved backgrounds are not free field theories. This problem can be addressed perturbatively by using, for instance, the background field method. The semiclassical approximation amounts to computing and comparing the one-loop partition functions of the quantum fluctuations around the given background in the GS and PS formulation, as has been done in [26, 27]. Going beyond the semiclassical approximation is possible in principle by computing higher-loop contributions but the calculations become quite hard. It would be interesting to perform these calculations, even only at two loops, since they involve the interactions among quantum fluctuations which are clearly different in the two approaches (they also affect the bosonic sector).

In this paper, we have proved at the level of the semiclassical approximation the equivalence of the GS and PS formulations of a string moving in AdS(5) \( \times \) S(5) with a method that allows for a clear separation of the physical and unphysical fermionic sectors of the PS formulation. It has been show that the unphysical fluctuations amount to eleven left-handed and eleven right-handed massless (bosonic and fermionic) BRS quartets whose contributions to the one-loop partition function cancel and that the quadratic physical fluctuations in the two approaches have not only the same spectrum but also the same conformal weights. This result

\[3\] I am grateful to Luca Mazzuccato for this remark.
has also revealed an interesting connection between the fields $\hat{b}_{L,--}$ and $\hat{b}_{R,++}$ defined in (65) and the $b$-fields of the PS approach in $\text{AdS}(5) \times S(5)$ [20–22].

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