ON THE EQUIVARIANT ALGEBRAIC JACOBIAN FOR CURVES OF GENUS TWO

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Abstract. We present a treatment of the algebraic description of the Jacobian of a generic genus two plane curve which exploits an $SL_2(k)$ equivariance and clarifies the structure of E.V.Flynn’s 72 defining quadratic relations. The treatment is also applied to the Kummer variety.

1. Introduction

The work described in this paper is a reflection on some material in Chapters 2 and 3 of [4]. We intend to present a simplification of the explicit description of the algebraic Jacobian, $\mathcal{J}(C)$, for a genus 2 curve given there and at [7].

The Jacobian of a non-singular, compact Riemann surface, $X$, is the group $\text{Pic}^0$ of divisors of degree zero factored out by principle divisors. This can be constructed analytically using the Abel map [6]. As such, $g$ being the genus of $C$, the Jacobian is $\mathbb{C}^g/\Lambda$, $\Lambda$ being the $g$-dimensional lattice of periods.

The Riemann-Roch theorem describes the dimensions of linear spaces of functions with prescribed poles on $X$. These functions are coordinates on $X$ and relations between them provide us with (generally singular) models of the surface as an algebraic curve in some projective space. An economical description is obtained by taking $P \in X$ to be a Weierstraß point. In the case of the genus 2 (hyperelliptic) surface, there are coordinates $x : X \mapsto \mathbb{P}^1$ and $y : X \mapsto \mathbb{P}^1$ with poles of orders 2 and 5 respectively which satisfy a relation of the form

$$y^2 = 4x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0,$$

the $\lambda_i$ being constants in the ground field. As a curve in $\mathbb{P}^2$ this is singular at infinity. Functions associated with more general special divisors provide us with other models. Such models are related by birational transformations. Thus we will be concerned, as is [4], with (singular) models of the genus 2 curve in the form

$$y^2 = g_6 x^6 + 6g_5 x^5 + 15g_4 x^4 + 20g_3 x^3 + 15g_2 x^2 + 6g_1 x + g_0,$$

which are related amongst themselves and to the quintic by simple Möbius maps:

$$x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta},$$

$$y \mapsto \frac{y}{(\gamma x + \delta)^3}.$$

A similar philosophy allows the algebraic construction of $\mathcal{J}(C)$ and such constructions are of use over fields other than $\mathbb{C}$ which is what partly motivates [4].

Another construction goes back to Jacobi and is described in [10]. In [3] $\text{Pic}^0$ is identified with $\text{Pic}^2$ and $\mathcal{J}(C)$ is constructed as a quadric variety in $\mathbb{P}^{15}$, the locus of seventy two linearly independent quadratic identities. Sixteen
homogeneous coordinates on $\mathbb{P}^{15}$ are chosen to be symmetric functions in two points on the curve: in our notation, $(x_1, y_1)$ and $(x_2, y_2)$. These coordinates are allowed to have poles of order up to 4 on the special divisor $D = (x, y) + (x, -y)$; that is, when $x_1 = x_2$ and $y_1 = -y_2$ and up to order 2 at the (singular) point at infinity. By looking at quadratic expressions in these coordinates and by balancing poles [4] construct the seventy two identities to be found explicitly at [7].

The purpose of the current paper is to use a little representation theory to oil the wheels of this machinery and to uncover some structure intrinsic to the collection of quadratic identities. Such an approach has already proven valuable in the analytic context [11, 2] following upon the work of [3].

The idea is that the coordinates on $\mathcal{J}(C)$ can be chosen to belong to irreducible $G$-modules where $G$ is a group of birational transformations. Quadratic functions arise by tensoring up these modules and decomposing into irreducibles. It suffices to work only with highest weight elements and it turns out that the dimensions of the components of the decomposition are graded by degrees of poles on the divisor in a “helpful” way. Because of this we need do less work and the identities are arranged for us by the representation theory into patterns.

We use the group $PSL_2(k)$ defined by the Möbius transformations above and take $k$ to be an algebraically closed field of zero characteristic so that we can stay close to classical representation theory as presented in, say, [9].

In fact [4] does mention that their coordinates transform nicely under translations and inversion of $x$ and goes so far as to write down a more complicated basis which would presumably be similar to that we present below. But the authors do not pursue the observation.

In the next section we define our notation, presenting the Lie algebraic action of the coordinate transformations on the variables and the coefficients of the curve and we define the construction of a highest weight element that we use for a component of the decomposition. We define our inhomogeneous coordinates on $\mathbb{P}^{15}$ and classify them according to dimension and degree of pole divisor. The inhomogeneous coordinates seem easier to use with the $PSL_2(k)$ action.

We give the decompositions of tensor products of coordinates according to dimension and divisor degree and indicate how the strategy of balancing dimensions and poles works to create quadratic identities. The quadratic identities themselves we summarise in the next section. We also need to construct various invariants and covariants out of the coefficients of the curve which themselves are a basis for a 7 dimensional $SL_2(k)$ module.

Some of the algebraic manipulation is done by hand. Some of it is best done using a computer algebra package such as MAPLE, used in this instance. Since one knows exactly where to look for cancelations the only issue is calculating the coefficients, a matter of linear algebra.

After that we give a construction of the Kummer variety associated to the genus 2 curve which is perhaps a little different to the [3] construction, though the result is entirely, equivariantly, equivalent.

Finally we make some closing remarks.
2. Notation

2.1. Modules and Tensor products. We use the following normalisation for a basis \{v_0, v_1, \ldots, v_{n-1}\} of a standard irreducible \(\mathfrak{sl}_n\)-module, \(V_n\) of dimension \(n\):

\[
\begin{align*}
e(v_i) &= (n-i)v_{i-1} \\
e(v_0) &= 0 \\
f(v_i) &= (i+1)v_{i+1} \\
f(v_{n-1}) &= 0 \\
h(v_i) &= (n-2i-1)v_i
\end{align*}
\]

for \(i = 0, \ldots, n-1\).

We call \(v_0 \in \ker e\) a highest weight element.

The coefficients \(g_i\) of the curve are a basis for a seven dimensional dual module so, for the sake of uniformity, we introduce a new set of coefficients

\[g_i^* = (-1)^i \binom{6}{i} g_{6-i} \quad i = 0, \ldots, 6\]

carrying the standard representation.

The tensoring of modules of dimensions \(n\) and \(m\), \(n \geq m\), leads to a decomposition of the form

\[V_n \otimes V_m \simeq \bigoplus_{i=n-m+1}^{n+m-1} V_i\]

and we construct the highest weight elements of the components in this plethysm according to the following rule:

\[(U_n \otimes V_m)_{n+m-p,0} = \sum_{i=0}^p (-1)^i \frac{(n-i-1)!}{(n-1)!} \frac{(m-p+1-1)!}{(m-1)!} u_i v_{p-i}\]

The basis elements of the representations we use are to be inhomogeneous coordinate functions defined on the Jacobian of the genus two curve in \(\mathbb{P}^{15}\). Each has a singularity on the diagonal \(x_1 = x_2, y_1 = y_2\) and on the divisor. All the elements of a given irreducible have, in fact, the same singularities as can be verified by inspection of all the terms to be defined. We denote by \(\text{div}_{\text{diag}}\) the class of \(n\) dimensional modules with poles of order \text{diag} (respectively \text{div}) on the diagonal (respectively divisor) of the product \(C \times C\).

2.2. Fundamental Irreducibles. Let \(\Delta = x_1 - x_2\). There is an invariant, related to the polar form of the curve, namely:

\[
\mathcal{I} = \frac{F(x_1, x_2) - y_1 y_2}{\Delta^3} \in \mathfrak{g}_3^1
\]

where \(F(x_1, x_2)\) is the (equivariant) polar form:

\[
F(x_1, x_2) = g_0 + 3(x_1 + x_2)g_1 + 3(x_1^2 + 3x_1x_2 + x_2^2)g_2 \\
+ (x_1^3 + 9x_1x_2^2 + 3x_1^2x_2 + x_2^3)g_3 + 3x_1x_2(x_1^2 + 3x_1x_2 + x_2^2)g_4 \\
+ 3x_1^2x_2(x_1 + x_2)g_5 + x_1^3 x_2^3 g_6
\]

The polar form plays a fundamental role in all approaches to the theory. In the analytic description of the Jacobian the equivariant \(\mathfrak{g}\)-functions \([1]\).
We introduce the set of inhomogeneous projective coordinates on $\mathbb{P}^{15}$ designated by lists of standard basis elements, $P(n)_2^6 \in n_6^6$:

\[
P(5)_2^2 = \left( \frac{1}{\Delta^2}, \frac{2(x_1 + x_2)}{\Delta^2}, \frac{x_1^2 + 4x_1x_2 + x_2^2}{\Delta^2}, \frac{2x_1x_2(x_1 + x_2)}{\Delta^2}, \frac{x_1^2x_2}{\Delta^2} \right)
\]

\[
P(4)_2^3 = \left( \frac{y_1 - y_2}{\Delta^3}, \frac{3(x_2y_1 - x_1y_2)}{\Delta^3}, \frac{3(x_2^2y_1 - x_1^2y_2)}{\Delta^3}, \frac{x_2^3y_1 - x_1^3y_2}{\Delta^3} \right)
\]

\[
P(3)_2^4 = \left( \frac{\mathcal{I}}{\Delta}, \frac{(x_1 + x_2)\mathcal{I}}{\Delta}, \frac{2x_1x_2\mathcal{I}}{\Delta} \right)
\]

\[
P(2)_2^5 = \left( \frac{y_1\mathcal{I}, x_1 + y_2\mathcal{I}, x_2y_1\mathcal{I}, x_1y_2\mathcal{I}}{\Delta} \right)
\]

\[
P(1)_2^6 = \mathcal{I}^2
\]

These coordinates are inspired by the sixteen homogeneous coordinate functions of Cassels and Flynn [4], but have one crucial difference. The fifteen inhomogeneous coordinates, created by dividing through by the coordinate $(x_1 - x_2)^2$ in [4], above have been adapted to a decomposition of $\mathbb{P}^{15}$ into irreducible $sl_2$-modules:

\[
15 \simeq 5_2^2 \oplus 4_2^3 \oplus 3_2^4 \oplus 2_2^5 \oplus 1_2^6
\]

It is important to notice that there is a pole grading on the divisor related to the module dimension. As inhomogeneous coordinates they are well-behaved at infinity and have poles of orders up to $4 + 2 = 6$ on the divisor.

3. Tensor products and pole gradings

Quadratic functions on the Jacobian arise by tensoring up the coordinates. In the (symmetric) table below are summarised the reducible decompositions of the symmetric tensor products (denoted $\otimes$) of all the coordinate modules.

| $\otimes$ | $P(5)_2^2$ | $P(4)_2^3$ | $P(3)_2^4$ | $P(2)_2^5$ | $P(1)_2^6$ |
|-----------|------------|------------|------------|------------|------------|
| $P(5)_2^2$ | $9_1^4 \oplus 5_2^3 \oplus 1_0^6$ |
| $P(4)_2^3$ | $8_5^3 \oplus 6_3^3 \oplus 4_2^3 \oplus 7_0^1 \oplus 3_2^4$ |
| $P(3)_2^4$ | $7_0^1 \oplus 3_2^4 \oplus 6_7^1 \oplus 4_2^5 \oplus 5_3^4 \oplus 1_2^6$ |
| $P(2)_2^5$ | $6_7^1 \oplus 4_2^5 \oplus 5_8^5 \oplus 3_2^6 \oplus 4_2^7 \oplus 2_2^7 \oplus 3_7^4 \oplus 1_2^10$ |
| $P(1)_2^6$ | $5_8^5 \oplus 4_2^9 \oplus 3_7^10 \oplus 2_2^{11} \oplus 1_2^{13}$ |

We make some remarks about these decompositions.

Firstly the highest dimensional component in each decomposition has predictable singularity structure: the highest weight element is simply the product of the highest weight elements in each factor. More generally the pole and dimension grading are connected in an interesting but currently obscure manner.

Secondly there are “holes” in the table. We expect to find, for example, a 2 inside $5 \otimes 4$. Its absence is due to its vanishing. Such cancelations are the simplest
of the quadratic identities which will describe the Jacobian as a locus in \( \mathbb{P}^{15} \). The identities arising in this way are nine in number and shown in section 4.2.

We will use the notation \( [\mathbf{\cdot}]_n \) to denote projection onto the \( n \)-dimensional irreducible of the decomposition.

In the next section we will summarise the seventy one quadratic relations in such a concise form but before doing so we explain how they are derived by considering the most complicated of them.

Consider \( \mathbf{P}(2)^5 \otimes \mathbf{P}(2)^5 \in \mathbb{P}^{10} \). The only possible cancelation is with \( \mathbf{P}(3)^2 \otimes \mathbf{P}(1)^6 \). We take an arbitrary linear combination of the highest weight elements, look at the worst singularity on the divisor and choose the (one) free parameter to kill it. This leaves us with a highest weight element in \( \mathbf{3}^4 \). Such singularites do not occur in the table except as \( \mathbf{5}^4 \). We can form elements of \( \mathbf{3}^4 \) by tensoring up the the three occurances of \( \mathbf{5}^4 \) with the seven dimensional module of degree one in the coefficients of the curve, \( g_i^* \), denoted \( g \). Choosing the free parameters in a linear combination appropriately we can cancel down to a highest weight in \( \mathbf{3}^4 \).

We continue this process systematically until we reach a vanishing element and we are done. In the current instance it becomes necessary to use modules arising from tensor products of degrees two and three in the curve coefficients. We summarise the structure of such representations in the next section, giving their highest weight elements before presenting the full list of quadratic identities.

4. Quadratic relations

4.1. Irreducibles in the curve coefficients. Standard partition counting arguments [8] yield plethysms for symmetric tensor products of the \( V_i \). From the decomposition \( V_7 \otimes V_7 \simeq V_{13} \oplus V_9 \oplus V_5 \oplus V_1 \) we obtain the following highest weight elements for quadratic representations.

\[
[g \otimes g]_{13,0} = g_6^2 \\
[g \otimes g]_{9,0} = g_6g_4 - g_5^2 \\
[g \otimes g]_{5,0} = \frac{1}{2^2.3^2}(g_6g_2 - 4g_5g_3 + 3g_4^2) \\
[g \otimes g]_{1,0} = \frac{1}{2^3.3^2.5}(g_6g_0 - 6g_5g_1 + 15g_4g_2 - 10g_3^2)
\]

The cubic irreducibles, \( V_7 \otimes V_7 \otimes V_7 \simeq V_{19} \oplus V_{15} \oplus V_{13} \oplus V_{11} \oplus V_9 \oplus V_7 \oplus V_5 \oplus V_4 \), have the following highest weight elements. Note there are two seven dimensional
irreducibles.

\[
\begin{align*}
[g \odot g \odot g]_{19.0} & = g_6^3 \\
[g \odot g \odot g]_{15.0} & = \frac{8}{11} g_6 (g_6 g_4 - g_5^2) \\
[g \odot g \odot g]_{11.0} & = \frac{3}{2.11} (g_6^2 g_3 + 3 g_6 g_5 g_4 - 2 g_5^3) \\
[g \odot g \odot g]_{9.0} & = \frac{13}{24.3^2.11} (g_5 g_6^2 - 5 g_4 g_1 g_0 + 2 g_3 g_2 g_0 + 8 g_1^2 g_3 - 6 g_4 g_2^2) \\
[g \odot g \odot g]_{7.0} & = \frac{1}{25.3^3.5.7.11} (-2778 g_5 g_1 g_0 + 3795 g_4 g_2 g_0 + 3150 g_1^2 g_4 \\
& - 1480 g_3^2 g_0 - 6300 g_1 g_2 g_3 + 3150 g_2^3 + 463 g_6^2 g_0) \\
[g \odot g \odot g]_{7.0}' & = \frac{1}{23.3^2.7} (-3 g_5 g_3 g_0 + 3 g_5 g_1 g_2 + 2 g_4^2 g_0 - g_4 g_3 g_1 - 3 g_4 g_2^2 \\
& + 2 g_2 g_3^2 + g_2 g_0 g_6 - g_1 g_6) \\
[g \odot g \odot g]_{7.0} & = 1
\end{align*}
\]

Cancelations of poles occur between linear combinations of quadratic expressions
at each pole order. In the following paragraphs we list the identities by pole order
and to simplify notation use \( n \) for \( P(n) \).

4.2. \((\cdot)^0\) 10 identities.

\[
\begin{align*}
[5 \odot 5]_1 & = \frac{1}{24.3^2} \\
[5 \odot 4]_2 & = 0 \\
[5 \odot 3]_5 & = 0 \\
[4 \odot 3]_2 & = 0
\end{align*}
\]

4.3. \((\cdot)^2\) 5 identities.

\[
\begin{align*}
[5 \odot 5]_5 & + \frac{1}{2^2.3} 5 = 0 \\
[5 \odot 4]_4 & - \frac{1}{2^2.3} 4 = 0 \\
[5 \odot 3]_3 & + \frac{1}{2.3} 3 = 0
\end{align*}
\]

4.4. \((\cdot)^3\) 4 identities.

\[
\begin{align*}
[5 \odot 4]_4 & - \frac{1}{2^2.3} 4 = 0
\end{align*}
\]

4.5. \((\cdot)^4\) 3 identities.

\[
\begin{align*}
[5 \odot 3]_3 & + \frac{1}{2.3} 3 = 0
\end{align*}
\]

4.6. \((\cdot)^5\) 2 identities.

\[
\begin{align*}
2 - \left[ \frac{2^5.3^3.5 g \odot [5 \odot 4]_8 + 24.3^3.5}{7} g \odot [5 \odot 4]_6 \right]_2 = 0
\end{align*}
\]
4.7. \((\cdot)^5_2\) 2 identities.

\[
[3 \odot 3]_1 + \frac{1}{2} = 0
\]

\[
1 - \left[ 2^4.3^2.5 g \odot [4 \odot 4]_7 - 2^7.3^4.5.7 [g \odot g]_9 \odot [5 \odot 5]_9 - \frac{2^8.3^3}{7} [g \odot g]_5 \odot 5 - 108 g \odot g \right]_1 = 0
\]

4.8. \((\cdot)^3_1\) 0 identities.

4.9. \((\cdot)^4_1\) 3 identities.

\[
\left[ 4 \odot 4 + 2^4.3^2.5 (g \odot [5 \odot 5]_9) - \frac{2^2.3^2}{7} (g \odot 5) \right]_3 = 0
\]

4.10. \((\cdot)^3_1\) 8 identities.

\[
\left[ 4 \odot 3 + 2^3.3^3.5 g \odot [5 \odot 4]_8 - \frac{2^4.3^3}{7} g \odot [5 \odot 4]_6 + \frac{2.3^3}{5} g \odot 4 \right]_4 = 0
\]

\[
\left[ 5 \odot 2 - 2^4.3^2.5 g \odot [5 \odot 4]_8 + \frac{2^5.3^3}{7} g \odot [5 \odot 4]_6 + \frac{2^3.3^3}{5} g \odot 4 \right]_4 = 0
\]

4.11. \((\cdot)^5_0\) 10 identities.

\[
\left[ 4 \odot 2 + 2^3.3^2.5 g \odot [4 \odot 4]_7 \right]_3 = 0
\]

\[
\left[ 4 \odot 4 - 25 \odot 3 - 2^3.3^3 g \odot [5 \odot 5]_9 - \frac{2^4.3}{7} g \odot 5 - \frac{3}{2.5} g \right]_7 = 0
\]

4.12. \((\cdot)^7_7\) 8 identities.

\[
\left[ 3 \odot 2 - 2^2.3^2.5 g \odot [4 \odot 3]_6 \right]_2 = 0
\]

\[
\left[ 5 \odot 2 - 3.4 \odot 3 + 2^4.3^2 g \odot [5 \odot 4]_8 - \frac{2^3.3^4}{7} g \odot [5 \odot 4]_6 + \frac{2.3^3}{5} g \odot 4 \right]_6 = 0
\]

4.13. \((\cdot)^5_1\) 10 identities.

\[
[3 \odot 3 - 5 \odot 1]_5 = 0
\]

\[
[4 \odot 2 - 65 \odot 1 - 2^4.3^2.5 g \odot [5 \odot 3]_7
- \frac{2^5.3^3.5 [g \odot g]_9 \odot [5 \odot 5]_9 + \frac{2^5.3^6}{7} [g \odot g]_5 \odot [5 \odot 5]_9}{7}
+ \frac{2^4.3^3}{7} [g \odot g]_9 \odot 5 - \frac{2^4.3^5.5}{7^2} [g \odot g]_5 \odot 5 + 2^3.3^4 [g \odot g]_1 \odot 5 + \frac{2^2.3^2}{5} g \odot g \right]_5 = 0
\]

4.14. \((\cdot)^4_4\) 4 identities.

\[
\left[ 3 \odot 2 - 3.4 \odot 1 + 2^4.3^2 g \odot [4 \odot 3]_6 + \frac{2^2.3^3}{5} g \odot [4 \odot 3]_4 \right]_4 = 0
\]
4.15. \((\cdot)^4\) 3 identities.

\[
\begin{align*}
2 \circ 2 - 2.3^2 \circ 1 - & \frac{2^5.3^5}{5} g \circ [5 \circ 1]_5 + \frac{2^2.3^2.3^1}{5} g \circ [4 \circ 2]_5 \\
- & + \frac{2^4.3^4.5^2.7}{5} [g \circ g]_9 \circ [5 \circ 3]_7 - \frac{2^4.3^4}{5} [g \circ g]_5 \circ [4 \circ 4]_3 - 2^3.3^5 [g \circ g]_1 \circ 3 \\
- & - \frac{2^{11}.3^7.11}{5} [g \circ g \circ g]_{11} \circ [5 \circ 5]_9 + \frac{2^9.3^8.7.11}{5.13} [g \circ g \circ g]_9 \circ [5 \circ 5]_9 \\
+ & + \frac{2^7.3^6.11.61}{5} [g \circ g \circ g]_7 \circ [5 \circ 5]_9 - \frac{2^6.3^6.23.211}{5.7} [g \circ g \circ g]'_7 \circ [5 \circ 5]_9 \\
+ & + \frac{2^9.3^7.11.157}{5^2.7} [g \circ g \circ g]_7 \circ [5 \circ 5]_5 - \frac{2^{16}.3^6.11.13}{5^2.7^2} [g \circ g \circ g]'_7 \circ [5 \circ 5]_5 \\
- & - \frac{2^9.3^6.11}{5^2} [g \circ g \circ g]_3 \circ [5 \circ 5]_5 + \frac{2^5.3^6.7}{5^2} [g \circ g \circ g]_3 \circ [5 \circ 5]_5 \\
& = 0
\end{align*}
\]

These relations are all linearly independent. This is guaranteed by the pole grading and by checking within each graded component. Thus at \((\cdot)^6\), for example, we have possible cancelations between poles in \([4 \circ 4]_7, [5 \circ 3]_7\) and \([4 \circ 2]_3\). Seven relations obtain from canceling \([4 \circ 4]_7\) against \([5 \circ 3]_7\). These cannot then involve \([4 \circ 2]_3\). Three relations come from expressing \([4 \circ 2]_3\) in terms of a tensor product of \(g\) with either \([4 \circ 4]_7\) or \([5 \circ 3]_7\). But the difference of these two possibilities arises exactly from tensoring the seven former identities with \(g\). Hence there are exactly 7+3=10 linearly independent relations.

5. THE KUMMER VARIETY

The Kummer is a simple, quartic relation in \(P^3\) which contains important information about the Jacobian. It is a degree four homogeneous relation between four of the above coordinates on \(P^{15}\). In \(1\) the Kummer relation is expressed in terms of the variable set \((1, x_1 + x_2, x_1 x_2, (x_1 - x_2)^2)\) where \(\mathcal{I}\) is a non-equivariant version of \(\mathcal{I}\). From the equivariant point of view the appropriate variables are \(3\) and \(1\).

So we seek the corresponding quartic relation between the one- and three-dimensional irreducible parts of \(P^{15}\). Either by eliminating \(1\) and \([4 \circ 4]_7\) between relations belonging to \((\cdot)^6\) and \((\cdot)^4\) above or by employing the same methodology we used to obtain them, we find:

\[
[3 \circ 3 + 2^3.3^2.5 g \circ [5 \circ 3]_7 + 2^5.3^4.5.7 [g \circ g]_9 \circ [5 \circ 5]_9 \\
+ \frac{2^6.3^3}{7} [g \circ g]_5 \circ 5 + 3^3 g \circ g ]_1 = 0
\]

to which we may add the relations already found above, under \((\cdot)^4\),

\[
[3 \circ 3 - 5 \circ 1]_5 = 0.
\]

Since tensoring by the invariant \(1\) is simply ordinary multiplication we can eliminate all occurrences of \(5\) by multiplying the first relation by \(1^2\) to obtain an invariant, quartic, homogeneous expression in \(3\) and \(1\):

\[
[3^2]_1 1^2 + 2^3.3^2.5 g \circ [3^3]_7 1 \\
+ 2^5.3^4.5.7 [g \circ g]_9 \circ [3^4]_9 - \frac{2^8.3^3}{7} ([g \circ g]_5 \circ 3^2)[3^7]_1 \\
+ 2^4.3^3 [g \circ g]_1 [3^2]_1^2 = 0.
\]
For economy of notation we have written $3^n$ for the symmetric $n$-fold tensor product of $3$ and taken for granted the projection onto the one dimensional (invariant) component throughout. We have used the identity $1 = -4 \cdot 3^2$ in deriving the above. This could be used again to write the Kummer as an inhomogeneous sextic in the three variables $3$ alone.

This expression for the Kummer should be compared to that in [4]. It has the same structure, it being understood that the variables are differently defined.

6. Conclusions

Moving up to higher genus in any approach to the algebraic Jacobian would appear to be an insane endeavour although an equivariant description of the Kummer for higher genus might be informative.

However a few specific points deserve to be pursued.

Is there a reason for the dimensional breakdown of the quadratic identities? The seventy two identities decompose as

$$1 \oplus 3 \oplus 2 \oplus 4 \oplus 3 \oplus 4 \oplus 5 \oplus 4 \oplus 6 \oplus 7.$$ 

What is the reason for the sequence of multiplicities $(3, 4, 4, 4, 1, 1, 0, \ldots)$?

A related question is why the tensor products of the coordinate modules have the pole structures they do, related as they are to the dimensions of the irreducible components in a not quite linear way. This kind of structure is also apparent in analytic treatments of the Jacobian via generalised $\wp$-functions [5] where the Hirota derivative plays an important rôle.

This in turn leads to the question of the relation of the algebraic approach to the equivariant Kleinian approach of [1, 2]. It is implicit in the definition of the equivariant $\wp$-functions. Putting $\wp = (\wp_{22}, \wp_{12}, \wp_{11})$ this is, in the current notation,

$$[3 \circ \wp]_1 = 1.$$ 

References

1. Athorne, C., J. Phys. A 41 (2008) 415202–21
2. Athorne, C., Phys. Lett. A, 375 (2011) 2689-2693.
3. Buchstaber, V.M., Enolskii V.Z. and Leykin, D.V., Reviews in Mathematics and Mathematical Physics, (London), Eds. Novikov, S.P. and Krichever, I.M., (Gordon and Breach) 10 (1997) 1–125.
4. Cassels, J.W.S. and Flynn, E.V., Prolegomena to a Middlebow Arithmetic of Curves of Genus 2, LMS Lecture Note Series 230, CUP (1996)
5. England, M. and Athorne, C., Generalised $\wp$-functions, ArXiv?
6. Farkas, H.M. and Kra, I., Riemann Surfaces, Graduate Texts in Mathematics 71, Springer (1980)
7. http://people.maths.ox.ac.uk/flynn/genus2/
8. Fulton, W. and Harris, J., Representation Theory: A First Course, Graduate Texts in Mathematics 129, Springer (1991)
9. Humphreys, J.E, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9, Springer (1972)
10. Mumford, D., Tata Lectures on Theta II, Progress in Mathematics 43, Birkhäuser (1984)

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