Abrikosov Vortex Lattices at Weak Magnetic Fields

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Abstract

We prove existence of Abrikosov vortex lattice solutions of the Ginzburg-Landau equations in two dimensions, for magnetic fields larger than but close to the first critical magnetic field.

1 Introduction

1.1 Ginzburg-Landau equations

In this paper we prove existence of Abrikosov lattice solutions of Ginzburg-Landau equations of superconductivity at weak magnetic fields. In the Ginzburg-Landau theory the equilibrium configurations are described by the Ginzburg-Landau equations:

\[-\Delta_A \Psi - \kappa^2 (1 - |\Psi|^2) \Psi = 0,\]
\[\text{curl}^2 A - \text{Im}(\bar{\Psi} \nabla_A \Psi) = 0,\]

(1)

where \(\Psi : \mathbb{R}^2 \to \mathbb{C}\) is the order parameter, \(A : \mathbb{R}^2 \to \mathbb{R}^2\) is the vector potential of the magnetic field \(B(x) := \text{curl} A(x)\), and \(\nabla_A = \nabla - iA\), and \(\Delta_A = -\nabla_A^* \nabla_A\), the covariant gradient and covariant Laplacian, respectively. \(|\Psi(x)|^2\) gives the local density of (Cooper pairs of) superconducting electrons and the vector-function \(J(x) := \text{Im}(\bar{\Psi}(\nabla - iA)\Psi)\), on the r.h.s. of the second equation, is the superconducting current.

The parameter \(\kappa\) is a material constant depending, among other things, on the temperature. It is called the Ginzburg-Landau parameter and it is the ratio of the length scale for \(A\) (penetration depth) to the length scale for \(\Psi\) (coherence length). The value \(\kappa = 1/\sqrt{2}\) divides all superconductors into two groups, type I superconductors \((\kappa < 1/\sqrt{2})\) and type II superconductors \((\kappa > 1/\sqrt{2})\).

The Ginzburg-Landau equations (1) have the trivial solutions corresponding to physically homogeneous states:

1. the perfect superconductor solution: \((\Psi_s \equiv 1, A_s \equiv 0)\) (so the magnetic field \(B_s = \text{curl} A_s \equiv 0\)),

2. the normal metal solution: \((\Psi_n \equiv 0, A_n)\), the magnetic field \(B_n = \text{curl} A_n\) is constant.

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We see that the perfect superconductor is a solution only when the magnetic field $B = \text{curl } A$ is zero. On the other hand, there is a normal solution for any constant $B$.

Though the equations (1) depend explicitly on only one parameter, $\kappa$, there is another - hidden - parameter determining solutions. It can be alternatively expressed as the average magnetic field, $b$, in the sample, or as an applied magnetic field, $h$. As it increases in type II superconductors from 0, the pure superconducting state turns into a mixed state, which after further increase becomes the normal state. (For type I superconductors, the behaviour is quite different: the transitions from superconducting to normal state and back are abrupt and occur at different values of magnetic field - hysteresis behaviour.)

One of the greatest achievements of the Ginzburg-Landau theory of superconductivity is the discovery by A.A. Abrikosov (Abr) of solutions with symmetry of square and triangular lattices (Abrikosov vortex lattice solutions) and one unit of magnetic flux per lattice cell, for type II superconductors in the regime just before the mixed state becomes the normal one (the regime (71) below). The rigorous proof of existence of such solutions was provided in Odeh Lash BGT Dut Al TS. Moreover, important and fairly detailed results on asymptotic behaviour of solutions, for $\kappa \to \infty$ and the applied magnetic fields, $h$, satisfying $h \leq \frac{1}{4} \log \kappa + \text{const}$ (the London limit), were obtained in AS (see this paper and the book SS for references to earlier works). Further extensions to the Ginzburg-Landau equations for anisotropic and high temperature superconductors can be found in ABS1 ABS2.

In this paper we prove existence of Abrikosov lattice solutions in the regime just after the superconducting state became the mixed one (the regime (72) in Appendix A) for all values of the Ginzburg-Landau parameter $\kappa$’s, all lattice shapes and all (quantized) values of magnetic flux per lattice cell. We also show that in each lattice cell, the solution looks like and $n$-vortex place at the center of the cell.

1.2 Ginzburg-Landau free energy

The Ginzburg-Landau equations are Euler-Lagrange equations for the Ginzburg-Landau (Helmholtz) free energy

$$E_Q(\Psi, A) := \frac{1}{2} \int_Q \left( |\nabla_A \Psi|^2 + \frac{\kappa^2}{2} (1 - |\Psi|^2)^2 + |\text{curl } A|^2 \right) d^2 x,$$  \hspace{1cm} (2)

where $Q$ is the domain occupied by the superconducting sample. This energy depends on the temperature (through $\kappa$) and the average magnetic field, $b = \lim_{Q' \to Q} \frac{1}{|Q'|} \int_{Q'} \text{curl } A$, in the sample, as thermodynamic parameters. Alternatively, one can consider the free energy depending on the temperature and an applied magnetic field, $h$. This leads (through the Legendre transform) to the Ginzburg-Landau Gibbs free energy $G_Q(\Psi, A) := E_Q(\Psi, A) - \Phi_Q h$, where $\Phi_Q = b|Q| = \int_Q \text{curl } A$ is the total magnetic flux through the sample. $b$ or $h$ do not enter the equations (1) explicitly, but they determine the density of vortices, which we describe below.
1.3 Symmetries and equivariant solutions

The Ginzburg-Landau equations (1) admit several symmetries, that is, transformations which map solutions to solutions:

**Gauge symmetry:** for any sufficiently regular function \( \eta : \mathbb{R}^2 \to \mathbb{R} \),

\[
\Gamma_\gamma : (\Psi(x), A(x)) \mapsto (e^{i\eta(x)}\Psi(x), A(x) + \nabla \eta(x)); \tag{3}
\]

**Translation symmetry:** for any \( h \in \mathbb{R}^2 \),

\[
T_h : (\Psi(x), A(x)) \mapsto (\Psi(x + h), A(x + h)); \tag{4}
\]

**Rotation and reflection symmetry:** for any \( R \in O(2) \) (including the reflections \( f(x) \mapsto f(-x) \))

\[
T_R : (\Psi(x), A(x)) \mapsto (\Psi(Rx), R^{-1}A(Rx)). \tag{5}
\]

The symmetries allow us to introduce special classes of solutions, called equivariant solutions. They are defined as solutions having the property that they are gauge equivalent under the action, \( T_\gamma \), of a subgroup, \( G \), of the group of rigid motions which is a semi-direct product of the groups of translations and rotations, i.e., for any \( g \in G \), there is \( \gamma = \gamma(g) \) s.t.

\[
T_g(\Psi, A) = \Gamma_\gamma(\Psi, A),
\]

where \( T_g \) for the groups of translations, and rotations, is given (4) and (5), respectively, and \( \Gamma_\gamma \) is the action of for the gauge group, given in (3).

For \( G \) the group of rotations, \( O(2) \), we arrive at the notion of the (magnetic) vortex, which is labeled by the equivalence classes of the homomorphisms of \( S^1 \) into \( U(1) \), i.e. by integers \( n \),

\[
\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta), \tag{6}
\]

where \((r, \theta)\) are the polar coordinates of \( x \in \mathbb{R}^2 \). Such vortices exist and are unique, up to symmetry transformation, for every \( n \in \mathbb{Z} \) and their profiles have the following properties (see [GST] and references therein):

\[
|\partial^\alpha (1 - f^{(n)}(r))| \leq ce^{-m_\alpha r}, \quad |\partial^\alpha (1 - a^{(n)}(r))| \leq ce^{-r},
\]

\[
f^{(n)}(r) = r^n + O(r^{n+2}) \quad \text{and} \quad a^{(n)}(r) = r^2 + O(r^4), \text{ as } r \to 0. \tag{7}
\]

Here \( m_\alpha := \min(\sqrt{2}k, 1) \). The exponential decay rates at infinity for \( f^{(n)}(r) \) and \( a^{(n)}(r) \) are called the coherence length and penetration depth, respectively.

For \( G \) a finite subgroup of the group of rotations, \( O(2) \), say \( C_k \) (see [DFN]), a possible solution would be a polygon of vortices, similar to the one described in [OS].

If \( G \) is the subgroup of the group of lattice translations for a lattice \( L \), then we call the corresponding solution a lattice, or \( L \)-gauge-periodic state. Explicitly,

\[
T_s(\Psi(x), A(x)) = \Gamma_{g_s(x)}(\Psi(x), A(x)), \quad \forall s \in L, \tag{9}
\]
where \( g_s : \mathbb{R}^2 \to \mathbb{R} \) is, in general, a multi-valued differentiable function, with differences of values at the same point \( \in 2\pi \mathbb{Z} \), and satisfying

\[
g_{s+t}(x) - g_s(x + t) - g_t(x) \in 2\pi \mathbb{Z}.
\]  

(10)

The latter condition on \( g_s \) can be derived by computing \( \Psi(x + s + t) \) in two different ways. In the special case described above this is the Abrikosov (vortex) lattice.

The characteristic property of \( L \)-gauge-periodic states is their physical characteristics \(|\Psi|^2, B(x)\) and \( J(x) \), where, recall \( B(x) := \text{curl} \, A(x) \) and \( J(x) := \text{Im} \, (\overline{\Psi} \nabla A \Psi) \), are doubly periodic with respect to the lattice \( L \). The converse is also true: a state whose physical characteristics are doubly periodic with respect to some lattice \( L \) is a \( L \)-gauge-periodic state.

An important property of lattice states is flux quantization: The flux, \( \int_{\Omega} \text{curl} \, A \), through the fundamental lattice cell \( \Omega \) (and therefore through any lattice cell) is

\[
\int_{\partial \Omega} A = 2\pi n,
\]

(11) for some integer \( n \). (Indeed, if \(|\Psi| > 0 \) on the boundary of the cell, we can write \( \Psi = |\Psi|e^{ix} \), for \( 0 \leq x < 2\pi \). The periodicity of \(|\Psi|^2\) and \( J(x) := \text{Im} \, (\overline{\Psi} \nabla A \Psi) \) ensure the periodicity of \( \nabla \chi - A \) and therefore by Green’s theorem, \( \int_{\Omega} \text{curl} \, A = \int_{\partial \Omega} A = \int_{\partial \Omega} \nabla \chi \) and this function is equal to \( 2\pi n \) since \( \Psi \) is single-valued.) Now, due to (9), the equation \( \int_{\partial \Omega} A = 2\pi n \) is equivalent to the condition

\[
- \int_{\partial_1 \Omega} \nabla g_{\omega_1}(x) + \int_{\partial_2 \Omega} \nabla g_{\omega_2}(x) = 2\pi n,
\]

(12)

where \( \{\omega_1, \omega_2\} \) is the basis of \( \Omega \) and

\( \partial_1 \Omega/\partial_2 \Omega \) is the part of the boundary of \( \Omega \) parallel to \( \omega_2/\omega_1 \). Finally, note that the flux quantization can be written as \( b = \frac{2\pi n}{|\Omega|} \), where \( b \) is the average magnetic flux per cell, \( b = \frac{1}{|\Omega|} \int_{\Omega} \text{curl} \, A \). Using the reflection symmetry, we can assume that \( b \), and therefore \( n \), is positive.

### 1.4 Parity

It is convenient to restrict the class of solutions we are looking for as follows. We place the co-ordinate origin at the center of the fundamental cell \( \Omega \) so that \( \Omega \) (as well as \( L \)) is **invariant under the reflection** \( x \to -x \). We define reflection (parity) operation

\[
Rf(x) = f(-x).
\]

(13)

We say that a function \( f \) on \( \mathbb{R}^2 \), or on the fundamental cell \( \Omega \) is even or odd, if it is even or odd under reflection in any cite of the lattice. A pair \( w = (\xi, \alpha) \) of functions on \( \mathbb{R}^2 \), or on \( \Omega \) is said to be even/odd if its \( \psi \)- and \( \alpha \)-component are even/odd and odd, respectively. Note that, since \( \theta(-x) = \theta(x) + \pi \), the \( n \)-vortex solutions, \( U^{(n)} := (\Psi^{(n)}, A^{(n)}) \), are odd, if \( n \) are odd, and are even, if \( n \) are even.
Since the Ginzburg-Landau equations (11), the fundamental cell $\Omega$ and the lattice $\mathcal{L}$ are invariant under the reflection $x \rightarrow -x$, we can restrict ourself to either odd or even lattice state solutions. For convenience, we consider in what follows only odd solutions and odd vortices:

$$(\Psi(x), A(x)) \quad \text{and} \quad n \text{ are odd.} \quad (14)$$

Even solutions and $n$ are treated in exactly the same way.

1.5 First result: Existence of vortex lattice states

We describe here our main result. First we identify $\Omega$ with $\mathcal{L} \subseteq \mathbb{C}$ and note that any lattice $\mathcal{L} \subseteq \mathbb{C}$ can be given a basis $r, r'$ such that the ratio $\tau = \frac{r'}{r}$ satisfies the inequalities $|\tau| \geq 1$, Im $\tau > 0$, $-\frac{1}{2} < \text{Re} \tau \leq \frac{1}{2}$, and $\text{Re} \tau \geq 0$ if $|\tau| = 1$ (see [Ahlf], where the term discrete module, rather than lattice, is used). Although the basis is not unique, the value of $\tau$ is, and we will use that as a measure of the shape of the lattice. Let $\mathcal{L} \equiv \mathcal{L}_R$ be a family of lattices of a fixed shape, with the minimal distance $R > 1$ between the nearest neighbour sites. Then the area of the fundamental cells, $\Omega$, of $\mathcal{L}$ is $\geq R^2$ and the average magnetic field $b = O(1)$.

We have

**Theorem 1.** Let $\kappa \neq \frac{1}{1/\sqrt{2}}$ and $n \neq 0$. For any $n \in \mathbb{Z}$ there is $R_0 = R_0(\kappa) \sim (\kappa - 1/\sqrt{2})^{-1} > 0$ such that for $R \geq R_0$, there exists a $\mathcal{L}$-periodic, odd solution $U^\mathcal{L} \equiv (\Psi^\mathcal{L}, A^\mathcal{L})$ of (11) on the space $\mathbb{R}^2$, s.t. for any $\alpha \in \mathcal{L}$ we have on $\Omega + \alpha$

$$U^\mathcal{L}(x) = U^{(n)}(x - \alpha) + O(e^{-cR}), \quad (15)$$

where, recall, $U^{(n)} := (\Psi^{(n)}, A^{(n)})$ is the $n$-vortex and $c > 0$, in the sense of the local Sobolev norm of any index.

**Discussion of the result.**

1) Theorem 1 shows that, for every $\kappa \neq 1/\sqrt{2}$ and every lattice shape $\tau$, there is a unique, up to symmetries, Abrikosov lattice solution, $(\Psi^\mathcal{L}, A^\mathcal{L})$, of the Ginzburg-Landau equations (11), satisfying (14) and (13), as long as $R$ sufficiently large. (Existence for $\kappa = 1/\sqrt{2}$ is actually trivial.)

2) One can modify our proof to make $R_0$ uniform in $\kappa - 1/\sqrt{2}$, see Remark II.

3) Let $U_s := (\Psi_s = 1, A_s = 0)$, the pure superconducting state and $h_{c1} := \frac{E^{(1)}}{\Phi^{(1)}}$, where $E^{(n)} := E(U^{(n)})$ and $\Phi^{(n)} := \int B^{(n)}$, the energy and flux of individual $n$-vortex, respectively, the first critical magnetic field (see Appendix A). For $R$ sufficiently large and for the applied magnetic field $h > h_{c1}$, we have, for the fundamental cell $\Omega$, that the Gibbs energy satisfies

$$G_\Omega(U^\mathcal{L}) < G_\Omega(U_s).$$

Indeed, due to (11a), $G_\Omega(U^\mathcal{L}) = G_\Omega(U^{(n)}) + O(e^{-cR})$. Hence, since $h_{c1} := \frac{E^{(1)}}{\Phi^{(1)}}$ and $G_\Omega(U_s) = 0$, the result follows.

4) One expects (based on results of [GS2] on the Ginzburg-Landau energy, that for $\kappa > 1/\sqrt{2}$, $n = 1$ and for $R$ sufficiently large, the average energy, $E_\Omega(\mathcal{L}) := \frac{1}{|\Omega|} G_\Omega(\Psi^\mathcal{L}, A^\mathcal{L})$, of the fundamental cell $\Omega$ of the lattices $\mathcal{L}$ is minimized by the triangular lattice.
5) One might be able to prove existence of solutions of the Ginzburg-Landau equation \( \mathbb{I} \) in a large domain \( Q \), which are close to the Abrikosov lattice solution \( U^\mathbb{C} := (\Psi^\mathbb{C}, A^\mathbb{C}) \). To do this we first construct an almost solution \( \tilde{U}^\mathbb{C} := (\tilde{\Psi}^\mathbb{C}, \tilde{A}^\mathbb{C}) \) by gluing together \( U^\mathbb{C} \) in \( Q' \subset Q \) with an appropriate function in \( Q/Q' \). This would give us the solution \( \tilde{U}_Q^\mathbb{C} := (\tilde{\Psi}_Q^\mathbb{C}, \tilde{A}_Q^\mathbb{C}) \) in \( Q \), close to \( U^\mathbb{C} := (\Psi^\mathbb{C}, A^\mathbb{C}) \).

Our approach to proving Theorem \( \mathbb{I} \) is as follows. First we show that the existence problem on \( \mathbb{R}^2 \) can be reduced to \( \Omega \) with the boundary conditions on \( \Omega \) induced by the periodicity condition \( \mathbb{J} \) (Subsection 2.1). Then we solve the Ginzburg-Landau equations \( \mathbb{I} \) on \( \Omega \) with the obtained boundary conditions. To this end we construct an approximate solution, \( v \) (Subsection 2.3) and use the Lyapunov-Schmidt reduction to obtain an exact solution (Subsection 2.7). (Then we glue together copies of the translated and gauged solution on \( \Omega \) (according to the prescription of Subsection 2.4) to obtain a solution on \( \mathbb{R}^2 \).)

1.6 Second result: Spectrum of fluctuations

To formulate our second result which concerns the spectrum of fluctuations around the solution \( U^\mathbb{C} \equiv (\Psi^\mathbb{C}, A^\mathbb{C}) \) found above, we have to introduce the linearized operators and their zero modes. Denote by \( F(U) \), \( U = (\Psi, A) \), the map defined by the.l.h.s. of \( \mathbb{I} \). Let \( L_{U^\mathbb{C}} := F'(U^\mathbb{C}) \) be the linearization of \( F(U) \) around a solution \( U^* := (\Psi^*, A^*) \) of \( \mathbb{I} \) \((F'(U) = \text{the } L^2\text{-gradient of } F \text{ at } U)\). Note that \( L_{U^\mathbb{C}} \) is a real-linear operator, symmetric, \( \langle v, L_{U^\mathbb{C}}v \rangle = \langle L_{U^\mathbb{C}}v, v \rangle \), with respect to the inner product

\[
\langle w, w' \rangle = \int_{\mathbb{R}^2} (\text{Re } \tilde{\xi}' + \alpha \cdot \alpha'),
\]

where \( w = (\xi, \alpha), \) etc.. Unless \( U^* \) is trivial, it breaks the translational and gauge symmetry and as a result the linearized operator \( L_{U^\mathbb{C}} \) has translation and gauge symmetry zero modes: \( LT_k^* = 0, LG_k^* = 0 \), where \( T_k^*(x) := ((\nabla A)_k \Psi^*(x), B^*(x) \gamma_k) \) and \( G_k^*(x) := (i\gamma \Psi^*(x), \nabla \gamma(x)) \), with \( B^*(x) := \text{curl } A^*(x) \) and \( J \), the symplectic matrix

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In particular, this applies to \( U^* = U^\mathbb{C}, U^{(n)} \), with the corresponding zero modes denoted by \( T_k, G_\gamma \) and \( T_k^{(n)}, G_\gamma^{(n)} \), respectively, so that e.g.

\[
T_k^{(n)}(x) := ((\nabla A^{(n)})_k \Psi^{(n)}(x), B^{(n)}(x) \gamma_k),
\]

where \( B^{(n)}(x) := \text{curl } A^{(n)}(x) \), and

\[
G_\gamma^{(n)}(x) := (i\gamma(x) \Psi^{(n)}(x), \nabla \gamma(x)),
\]

are translation and gauge zero modes, respectively, zero modes for the \( n \)-vortex \( U^{(n)} := (\Psi^{(n)}, A^{(n)}) \): \( L^{(n)}T_k^{(n)} = 0, L^{(n)}G_\gamma^{(n)} = 0 \), with \( L^{(n)} := F'(U^{(n)}). \) (See [GS1] for a discussion of \( T_k^{(n)} \) and \( G_\gamma^{(n)} \).)
Define the shifted translational zero modes $T_{jk}(x) = T^{(n)}_k(x - j)$, associated with the $n$-vortices located at the sites $j$ and let $L := L_{U_L} := F(U_L)$. We emphasize that while $T_k(x)$ are zero modes of $L$, $T_{jk}(x)$ are not. We have

**Theorem 2.** Suppose either $\kappa > 1/\sqrt{2}$ and $n = 1$ or $\kappa < 1/\sqrt{2}$ and $n \neq 0$. There is $R_0 = R_0(\kappa) \sim (\kappa - 1/\sqrt{2})^{-1} > 0$ such that for $R \geq R_0$, we have

1) [approximate zero-modes of $L$] $\|LT_{jk}\|_{H^r} \lesssim e^{-cR}$, for any $r$;
2) [Coercivity away from the translation and gauge modes] $\langle \eta, L\eta \rangle \geq c'\|\eta\|^2_{H^1}$, for any $\eta \perp \text{Span}\{T_{jk}, G_\gamma : \forall j \in \mathcal{L}, k = 1, 2, \gamma \in H^2(\mathbb{R}^2, \mathbb{R})\}$,

and $c' > 0$ independent of $R$.

Above and in sequel, the norms and inner products without subindices stand for those in $L^2$, while the Sobolev norms on $\Omega$ are distinguished by the symbol $H^r$ in the subindex.

We prove this theorem in Section 5. In exactly the same way one proves a similar, but stronger, result about a complex-linear extension, $K$, of the operator $L$ (the latter result implies the former one). The spectrum of fluctuations around $U_L$ is the spectrum of $K$.

This paper is self-contained. In what follows we write $e^{-R}$ for $e^{-cR}$.

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## 2 Proof of Theorem 1

In this section we prove Theorem 1 modulo technical statements proved in the next section.

### 2.1 Reduction to the basic cell

Assume we are given a multi-valued differentiable function $g_s : \mathbb{R}^2 \rightarrow \mathbb{R}$, with differences of values at the same point $x \in 2\pi \mathbb{Z}$ and satisfying (10). An example of such a function is $g_s := \frac{1}{2}s \wedge x = -\frac{1}{2}s \cdot Jx$ used in [15]. Another example will be given below. Given a continuous function $U \equiv (\Psi, A)$ on the space $\mathbb{R}^2$, satisfying the gauge-periodicity conditions (9) (a $\mathcal{L}$—gauge-periodic function), its restriction, $u \equiv (\psi, a)$, to the fundamental cell $\Omega$ satisfies the boundary conditions induced by (10):

\[
\begin{aligned}
\psi(x + s) &= e^{ig_s(x)}\psi(x), \\
\frac{\partial}{\partial_1 \Omega} a + \frac{\partial}{\partial_2 \Omega} a &= \frac{\partial}{\partial_1 \Omega} (a(x) + \nabla g_s(x)), \\
x \in \partial_1 \Omega, s = \omega_1/\omega_2.
\end{aligned}
\]

Here $\partial_1 \Omega/\partial_2 \Omega$ is the left/bottom boundary of $\Omega$ and $\{\omega_1, \omega_2\}$ is a basis in $\mathcal{L}$. 
In the opposite direction, given a continuous function $u \equiv (\psi, a)$ on the fundamental cell $\Omega$, satisfying the boundary conditions (19), we lift it to a $L-$periodic function $U \equiv (\Psi, A)$ on the space $\mathbb{R}^2$, satisfying the gauge-periodicity conditions (9), by setting, for any $\alpha \in L$,

$$
\Psi(x) = \psi(x - \alpha)e^{i\Phi_\alpha(x)}, \quad A(x) = a(x - \alpha) + \nabla\Phi_\alpha(x), \quad x \in \Omega + \alpha,
$$

where $\Phi_\alpha(x)$ is a real, possibly multi-valued, function to be determined. (Of course, we can add to it any $L-$periodic function.) We define

$$
\Phi_\alpha(x) := g_\alpha(x - \alpha), \quad \text{for } x \in \Omega + \alpha.
$$

The periodicity condition (9), applied to the cells $\Omega + \alpha - \omega_i$ and $\Omega + \alpha$ and the continuity condition on the common boundary of the cells $\Omega + \alpha - \omega_i$ and $\Omega + \alpha$ imply that $\Phi_\alpha(x)$ should satisfy the following two conditions:

$$
\Phi_\alpha(x) = \Phi_{\alpha - \omega_i}(x - \omega_i) + g_{\omega_i}(x - \omega_i), \quad \text{mod } 2\pi, \quad x \in \Omega + \alpha,
$$

$$
\Phi_\alpha(x) = \Phi_{\alpha - \omega_i}(x) + g_{\omega_i}(x - \alpha), \quad \text{mod } 2\pi, \quad x \in \partial\Omega + \alpha,
$$

where $i = 1, 2$, and, recall, $\{\omega_1, \omega_2\}$ is a basis in $L$ and $\partial_1\Omega/\partial_2\Omega$ is the left/bottom boundary of $\Omega$.

To show that (21) satisfies the conditions (22) and (23), we note that, due to (10), we have $g_\alpha(x - \alpha) = g_{\alpha - \omega_i}(x - \alpha) + g_{\omega_i}(x - \omega_i)$, mod $2\pi$, $x \in \Omega + \alpha$, and $g_\alpha(x - \alpha) = g_{\alpha - \omega_i}(x - \alpha + \omega_i) + g_{\omega_i}(x - \alpha)$, mod $2\pi$, $x \in \partial\Omega + \alpha$, which are equivalent to (22) and (23), with (24).

Finally, note that

(a) Since $\Psi, A$ satisfy the gauge-periodicity conditions (9) in the entire space $\mathbb{R}^2$ and are smooth in $\mathbb{R}^2/(\cup_{x \in L}\partial\Omega)$, $\nabla A\Psi$, $\Delta A\Psi$ and $\mathrm{curl}^2 A$ are continuous and satisfy the gauge-periodicity condition (9);

(b) Since $u \equiv (\psi, a)$ satisfies the Ginzburg-Landau equations (11) in $\Omega$, then $U \equiv (\Psi, A)$ satisfies (11) in $\mathbb{R}^2/(\cup_{x \in L}S_t\partial\Omega)$, where $S_t: x \rightarrow x + t$;

(c) Since $\Psi, A$ satisfy the gauge-periodicity conditions (9) in the entire space $\mathbb{R}^2$, we conclude by the first equation in (11) that $\Delta A\Psi$ is continuous and satisfies the periodicity conditions (in the first equation of (9)) in $\mathbb{R}^2$ and therefore, by the Sobolev embedding, theorem so is $\nabla A\Psi$. Hence, by the second equation in (11), $\mathrm{curl}^2 A$ is continuous and satisfies the periodicity conditions (9) in $\mathbb{R}^2$. Therefore, by iteration of the above argument (i.e. elliptic regularity), $\Psi, A$ are smooth functions obeying (11) and (14).

We summarize the conclusions above as

**Lemma 1.** Assume twice differentiable functions $(\psi, a)$ on $\Omega$ obey the boundary conditions (19) and the Ginzburg-Landau equations (11). Then the functions $(\Psi, A)$ constructed in (20) and (21) are smooth in $\mathbb{R}^2$ and satisfy the periodicity conditions (9) and the Ginzburg-Landau equations (11).
2.2 Existence of solutions in the basic cell

In what follows we look for odd solutions, \((\psi, a_0)\), of the Ginzburg-Landau equations \((1)\) in \(\Omega\). Our goal now is prove the following

**Theorem 3.** For any \(n \in \mathbb{Z}\) there is \(R_0 > 0\) such that for \(R \geq R_0\), there exists a smooth, odd solution \(u^L \equiv (\psi^L, a^L)\) of \((1)\) on the fundamental lattice cell \(\Omega\), satisfying the boundary conditions \((19)\) and the estimate, in a Sobolev norm of arbitrary index,

\[
\int L(x) = U(n)(x) + O(e^{-R}).
\] (24)

To prove this theorem we construct an approximate solution of \((1)\) on \(\Omega\) and then use a perturbation theory (Lyapunov-Schmidt decomposition), starting with this approximate solution. This is done in Subsections 2.3 - 2.7, modulo technical estimates proven in Section 4.

Using this result and gluing together copies of the translated and gauged solution on \(\Omega\), (see Subsection 2.1 and especially \((20)\) and \((21)\)), we derive Theorem 3.

2.3 Construction of an approximate solution

In this subsection we construct test functions, \((\psi_0, a_0)\), describing a vortex of the degree \(n\), centered at the center of the fundamental cell \(\Omega\).

Let \(\eta\) and \(\bar{\eta}\) be smooth, nonnegative, spherically symmetric (hence even), cut-off functions on \(\Omega\), such that \(\eta = 1\) on \(|x| \leq \frac{1}{3}R\) and \(\eta = 0\) on \(\Omega/\{|x| \leq \frac{2}{3}R\}\) and

\[|\partial^\alpha \eta(x)| \lesssim R^{-|\alpha|}\]

inside \(\Omega\) and \(\eta + \bar{\eta} = 1\) on \(\Omega\). Fix an odd integer \(n\). We define on \(\Omega\)

\[
\psi_0(x) := [f^{(n)} + \eta]/|x - n\theta|, \quad a_0(x) := [A^{(n)} + n\nabla \eta]/|x - n\theta|.
\] (25)

These functions belong to Sobolev spaces \(H^r_{odd}(\Omega) := H^r_{odd}(\Omega, \mathbb{C}) \times H^r_{odd}(\Omega, \mathbb{R})\) of odd functions, for any \(r \geq 0\), and satisfy the boundary conditions \((19)\) with

\[g_s(x) := n\theta(x + s) - n\theta(x) \quad \text{and} \quad x \in \mathbb{R}^2.\] (26)

Note that, though the function \(g_s(x)\) is multi-valued on \(\mathbb{R}^2\), it is well-defined for \(x \in \partial_1 \Omega\) and \(s = \omega_i\), \(i = 1, 2\). Indeed, \(g_s(x)\) can be written as

\[
g_s(x) = n \int_0^1 dr \frac{J_{x \cdot s}}{|x + rs|^2} = n \frac{J_{x \cdot \hat{s}}}{|x|^2 - (x \cdot \hat{s})^2} \lambda_1 \int_0^{\lambda_2} \frac{dt}{t^2 + 1},
\]

where \(\lambda_1 = \frac{x \cdot \hat{s}}{|x|^2 - (x \cdot \hat{s})^2}\), \(\lambda_2 = \frac{|x| + x \cdot \hat{s}}{|x|^2 - (x \cdot \hat{s})^2}\), and \(\hat{x} := x/|x|\), etc. (Note that, taking for simplicity lattices with equal sides, by our choice, \(\Omega = \{r_1 \omega_1 + r_2 \omega_2 \mid -\frac{R}{2} \leq r_i \leq \frac{R}{2} \forall i\}\), \(\partial_1 \Omega := \{-\frac{1}{2} \omega_1 + r \omega_2 \mid -\frac{R}{2} \leq r \leq \frac{R}{2}\}\) and \(\partial_2 \Omega := \{r \omega_1 - \frac{R}{2} \omega_2 \mid -\frac{R}{2} \leq r \leq \frac{R}{2}\}\),

\[
\partial_1 \Omega := \{-\frac{1}{2} \omega_1 + r \omega_2 \mid -\frac{R}{2} \leq r \leq \frac{R}{2}\}\) and \(\partial_2 \Omega := \{r \omega_1 - \frac{R}{2} \omega_2 \mid -\frac{R}{2} \leq r \leq \frac{R}{2}\},
\]
so that $|x|^2 - (x \cdot \hat{s})^2$ never vanishes for $s = \omega_1$, $x \in \partial_1 \Omega$.) It can be also verified directly that (26) satisfies the conditions (10) and (12):
\[
g_s(x) - g_s(x + t) - g_t(x) \in 2\pi \mathbb{Z}
\]
and
\[
-\int_{\partial_1 \Omega} \nabla g_{\ell_1}(x) + \int_{\partial_2 \Omega} \nabla g_{\ell_2}(x) = \int_{\partial_\Omega} \nabla n \theta(x) = 2\pi n.
\]
Finally, by the construction we have $\psi_0 = \Psi^{(n)} + (1 - f^{(n)}) e^{in \theta} \tilde{\eta}$, $a_0 = A^{(n)} + n \nabla \theta(1 - a^{(n)}) \tilde{\eta}$. This, the definition of $\tilde{\eta}$ and the estimates (7) imply, for $v := (\psi_0, a_0)$, $U^{(n)} = (\Psi^{(n)}, A^{(n)})$, that
\[
\|v - U^{(n)}\|_{H^r} \lesssim e^{-R} \quad \forall r \geq 0.
\]

2.4 Spaces

We consider the spaces $L^2_{\text{odd}}(\Omega) := L^2_{\text{odd}}(\Omega, \mathbb{C}) \times L^2_{\text{odd}}(\Omega, \mathbb{R}^2)$ of odd square integrable functions on $\Omega$, with the real inner product (16). Fixing an odd integer $n$, we define $H^r(\Omega)$ to be the Sobolev space of order $r \geq 0$ of odd functions $w = (\xi, \alpha)$ : $\Omega \rightarrow \mathbb{C} \times \mathbb{R}^2$, satisfying the gauge periodic boundary conditions
\[
\begin{align*}
\xi(x + s) &= e^{isg_s(x)} \xi(x), \\
\alpha(x + s) &= \alpha(x),
\end{align*}
\]
for $x \in \partial_1 \Omega / \partial_2 \Omega$ (= the left/bottom boundary of $\Omega$), $s = \omega_1/\omega_2$ ($\{\omega_1, \omega_2\}$, a basis in $\mathcal{L}$), and $g_s$ given in (26). (Note that $\nabla a_0 \xi$ satisfies the boundary conditions on the first line on (30).) For $r > \frac{1}{2}$, the restrictions of functions in $H^r(\Omega)$ to the boundary exist as $H^{r-\frac{1}{2}}(\Omega)$ functions and therefore (30) is well defined; for $0 \leq r \leq \frac{1}{2}$, one can define the corresponding spaces by observing that if $\xi \in H^r(\Omega)$, then $e^{-in \theta} \xi$ is periodic w.r. to the lattice $\mathcal{L}$ and the corresponding norms can be defined in terms of its 'Fourier' coefficients. (We need $H^r(\Omega)$ for $r = 2$.)

2.5 Generators of translations and gauge transformations

An important role in the analysis of vortices is played by the generators of translations and gauge transformations, $T_k$, $k = 1, 2$, and $G_\gamma$, $\gamma : \Omega \rightarrow \mathbb{R}$, defined as
\[
T_k(x) := ((\nabla a_0)_k \psi_0(x), b_0(x) J e_k),
\]
where $b_0(x) := \text{curl} a_0$, and
\[
G_\gamma := (i \gamma \psi_0, \nabla \gamma), \quad \gamma : \Omega \rightarrow \mathbb{R}.
\]
These generators are almost zero modes of the operator $L_0 := F'(v)$ (= the $L^2$-gradient of $F$ at $v$), where, recall, $F$ is the map defined by the l.h.s. of (1).
Since \((T_k)_\psi\) and \((T_k)_a\) are even, by our definition in Subsection 1.4, so are \(T_k, \ k = 1, 2\), and therefore \(T_k, \ k = 1, 2\), do not belong to our spaces. On the other hand, \(G_\gamma\) belongs to our space \(\mathcal{H}^r(\Omega), \forall r\), iff \(\gamma\) is periodic and even, with appropriate smoothness conditions.

2.6 Orthogonal decomposition

Let \(v = (\psi_0, a_0)\) with \(\psi_0\) and \(a_0\) defined in (25). Consider odd functions, \(u = (\psi, a) \in L^2_{\text{odd}}(\Omega)\), satisfying the boundary conditions (19) with (26) and s.t.

\[ u = v + w, \text{ with } w \perp G_\gamma, \forall \gamma \in H^{2+r}_{\text{per}}(\Omega, \mathbb{R}), \]  

(33)

where \(H^{2+r}_{\text{per}}(\Omega, \mathbb{R})\) is the Sobolev space of real, periodic, even functions on \(\Omega\) of order \(2 + r\).

The function \(w\), defined by (33), has the following properties

- Since \(v = (\psi_0, a_0)\) is odd and since scalar products of even functions with odd ones vanish, \(w \perp T_k, \ k = 1, 2\).
- Since \(v\) and \(u\) satisfy the boundary conditions (19) with (26), we conclude that \(w\) satisfies the boundary conditions (30) with (26).
- Since \(v \in H^r(\Omega)\), for any \(r \geq 0\), we have that, if \(u \in H^r(\Omega)\), then \(w \in H^r(\Omega)\).

Note that by integration by parts, \(w \perp G_\gamma, \forall \gamma \in H^{2+r}_{\text{per}}(\Omega, \mathbb{R})\), is equivalent to

\[ \text{Im}(\bar{\psi}_0 \xi) + \text{div} \alpha = 0. \]  

(34)

2.7 Lyapunov-Schmidt decomposition.

Recall that \(F\) is the map defined by the l.h.s. of (1) and denote \(u = (\psi, a) : \Omega \to \mathbb{C} \times \mathbb{R}^2\). The Ginzburg-Landau equations (1) on \(\Omega\) can be written as

\[ F(u) = 0. \]  

(35)

Clearly, \(F\) maps \(v + \mathcal{H}^{r+2}(\Omega)\) to \(\mathcal{H}^r(\Omega)\). Let \(L_0 := F'(v)\). It is a real-linear operator on \(L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^2)\) mapping \(\mathcal{H}^{r+2}(\Omega)\) to \(\mathcal{H}^r(\Omega)\). Now, we assume \(u \in \mathcal{H}^r(\Omega), \ r \geq 0\), and substitute the decomposition (33) into (35) to obtain

\[ F(v) + L_0w + N_v(w) = 0, \]  

(36)

where \(N_v(w)\) is the nonlinearity \(N_v(w) := F(u) - F(v) - F'(v)w\).

Let \(P\) denote the orthogonal projection from \(L^2_{\text{odd}}(\Omega, \mathbb{C}) \times L^2_{\text{odd}}(\Omega, \mathbb{R}^2)\) onto the subspace \(\{G_\gamma, | \gamma \in H^{2+r}_{\text{per}}(\Omega, \mathbb{R})\}\), and let \(\bar{P} := 1 - P\). We split (35) into two equations:

\[ P[F(v) + L_0w + N_v(w)] = 0, \]  

(37)

and

\[ \bar{P}[F(v) + L_0w + N_v(w)] = 0. \]  

(38)
Recall the notation \( \| \cdot \|_{H^r} \) for the norm in the Sobolev space \( H^r(\Omega) \). The following estimates are proven in Section 4: \( \forall r > r' + 1, \ r' \geq 0, \)

\[
\| F(v) \|_{H^{r'}} \lesssim e^{-R}, \tag{39}
\]

\[
\bar{L} := PL_0 P|_{\text{ran } \bar{P}} \text{ is invertible and } \| \bar{L}^{-1} \|_{H^{r'} \to H^{r' + 2}} \lesssim 1, \tag{40}
\]

\[
\| N_v(w) \|_{H^{r'}} \leq c_r (\| w \|_{H^{r'}}^2 + \| w \|_{H^r}^3), \tag{41}
\]

\[
\| N_v(w') - N_v(w) \|_{H^{r'}} \leq c_r (\| w \|_{H^{r'}} + \| w \|_{H^r}^2 + \| w' \|_{H^{r'}} + \| w' \|_{H^r}) \| w' - w \|_{H^r}. \tag{42}
\]

\((r = 2, \ r' = 0 \text{ suffices for us.})\)

**Proposition 1.** Let \( n \) be odd and assume (39) - (41) hold. Then, for \( R \) sufficiently large, Eqn (38) has a solution, \( w = v(w) \), unique in a ball in \( H^r \) of the radius \( \ll 1 \), which is odd and satisfies the estimate

\[
\| w \|_{H^r} \lesssim e^{-R}, \ r \geq 1. \tag{43}
\]

**Proof.** Since the operator \( \bar{L} := PL_0 P|_{\text{ran } \bar{P}} \) is invertible by (40), the equation (38) can be rewritten as

\[
w = -\bar{L}^{-1} P[F(v) + N_v(w)]. \tag{44}
\]

Using the estimates on \( F(v) \), \( \bar{L}^{-1} \) and \( N_v(w) \), given in (39) - (42), one can easily see that the map on the r.h.s. of (44) maps a ball in \( H^r \) of the radius \( \ll 1 \) into itself and is a contraction, provided \( R \) is sufficiently large. Hence the Banach fixed point theorem yields the existence of a unique \( w = v(w) \) and the estimate

\[
\| w \|_{H^{r'} + 2} \lesssim \| F(v) \|_{H^{r'}}. \]

This equation together with (39) implies (43). Since \( v \) is odd and since \( \bar{L}^{-1} \) and \( N_v(\cdot) \) are invariant under the reflections, \( w = v(w) \) is odd, by the construction. \( \square \)

Now we turn to the equation (37). With \( u := v + w(v) \), this equation can be rewritten as

\[
\langle G_\gamma, F(u) \rangle = 0, \ \forall \gamma \in H^{2+r}_{per}(\Omega, \mathbb{R}). \tag{45}
\]

(Note that Eqn (30), the symmetry of \( L_0 \) and the fact that \( G_\gamma \) is a zero mode of \( L_0 \) imply \( \langle G_\gamma, F(u) \rangle = \langle G_\gamma, N_v(w) \rangle \). To show that (45) is satisfied by \( u := v + w(v) \) we differentiate the equation \( \mathcal{E}_\lambda(e^{i\gamma} \psi, a + s \nabla \gamma) = \mathcal{E}_\lambda(\psi, a) \), w. r. t \( s \) at \( s = 0 \), to obtain

\[
\partial_s \mathcal{E}_\lambda(\psi, a) i \gamma \psi + \partial_s \mathcal{E}_\lambda(\psi, a) \nabla \gamma = 0,
\]

or \( \langle F(\psi, a), G_\gamma \rangle = 0 \). By either varying the Sobolev index \( r \) or invoking elliptic regularity one shows smoothness of solutions. This proves, Theorem \( 3 \) modulo the statements (39) - (41). Combining the latter with the lifting procedure, (20) and (21), gives Theorem \( 1 \) \( \square \)
\section{Complex-linear extension $K$ of $L$}

In order to be able to use spectral theory, we construct a complex-linear extension $K$ of the operator $L$ defined on $\mathcal{H}(\mathbb{R}^{2}) := L^{2}_{\text{odd}}(\mathbb{R}^{2}; \mathbb{C}) \oplus L^{2}_{\text{odd}}(\mathbb{R}^{2}; \mathbb{R})$, or on $\mathcal{H}(\Omega) := L^{2}_{\text{odd}}(\Omega; \mathbb{C}) \oplus L^{2}_{\text{odd}}(\Omega; \mathbb{R})$, with the boundary conditions (30). The (complex-) linear operator $K$ is defined on $\mathcal{H}^{c}(\mathbb{R}^{2}) := [L^{2}_{\text{odd}}(\mathbb{R}^{2}; \mathbb{C})]^{4} \equiv [L^{2}_{\text{odd}}(\mathbb{R}^{2}; \mathbb{C})]^{2} \oplus [L^{2}_{\text{odd}}(\mathbb{R}^{2}; \mathbb{C})]^{2}$, or on $\mathcal{H}^{c}(\Omega) := [L^{2}_{\text{odd}}(\Omega; \mathbb{C})]^{4}$, as follows. We first identify $\mathcal{H}^{c}(\mathbb{R}^{2})$ with the function $\alpha : \mathbb{R}^{2}/\Omega \to \mathbb{R}^{2}$ with the function $\alpha^{c} = \alpha_{1} + i\alpha_{2} : \mathbb{R}^{2}/\Omega \to \mathbb{C}$.

The space $\mathcal{H}(\mathbb{R}^{2})/\mathcal{H}(\Omega)$ is embedded in $\mathcal{H}^{c}(\mathbb{R}^{2})/\mathcal{H}^{c}(\Omega)$ via the isometric injection

$$
\sigma : w = \left( \begin{array}{c} \xi \\
\alpha \end{array} \right) \to w^{c} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \xi \\
\xi \alpha^{c} \\
\alpha^{c} \end{array} \right),
$$

in the sense of the inner product (10). (Below we drop the superscript $c$ from $\alpha^{c}$.) This embedding transfers the operator $L$ to the operator $L^{c}\sigma w := \sigma Lw$ on the real-linear subspace, $\sigma \mathcal{H}$, of $\mathcal{H}^{c}$. (Here $\mathcal{H}$ ($\mathcal{H}^{c}$) stands for either $\mathcal{H}(\mathbb{R}^{2})$ ($\mathcal{H}^{c}(\mathbb{R}^{2})$) or $\mathcal{H}(\Omega)$ ($\mathcal{H}^{c}(\Omega)$).) Next, we define the projection $\pi$ from $\mathcal{H}^{c}$ to $\sigma \mathcal{H} \subset \mathcal{H}^{c}$, by

$$
\pi \left( \begin{array}{c} \xi \\
\chi \\
\alpha \\
\beta \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} \xi + \bar{\chi} \\
\xi + \chi \\
\alpha + \beta \\
\bar{\alpha} + \beta \end{array} \right),
$$

and observe that $\pi + i\pi i^{-1} = 1$. We extend the operator $L^{c}$ from the subspace $\sigma \mathcal{H}$ to the complex-linear operator $K$ on the entire $\mathcal{H}^{c}$ as

$$
K \hat{w} := \sigma L \pi \hat{w} + i\sigma L \pi i^{-1} \hat{w}.
$$

Similarly we proceed on the fundamental cell $\Omega$. The explicit form of $K$ is the same on $\mathcal{H}^{c}(\mathbb{R}^{2})$ and on $\mathcal{H}^{c}(\Omega)$ and is given in Appendix [3].

The (complex-) linear operator $K$ has the following properties:

1) $K = K^{*}$ on $\mathcal{H}^{c}$;
2) $\langle \sigma w', K \sigma w \rangle = \langle w', Lw \rangle$;
3) $K \sigma w = \sigma Lw$;
4) $0 \in \sigma_{\text{ess}}(K)$;
5) $[K, R] = 0$ (recall, $R$ is the parity transformation).

Note that the third statement and the property that $G_{\gamma}, \gamma \in H^{2}(\mathbb{R}^{2}, \mathbb{R})$, and $T_{k}, k = 1, 2$, given above, are a zero mode of $L$ implies that their complexifications, $G_{\gamma}^{c}, \gamma \in H^{2}(\mathbb{R}^{2}, \mathbb{C})$ and $T_{k}^{c}, k = 1, 2$, are zero modes of $K : KG_{\gamma}^{c} = 0$, $LT_{k}^{c} = 0$. Since $T_{k}^{c} \notin [L^{2}(\mathbb{R}^{2}; \mathbb{C})]^{4}$, $k = 1, 2$, but are bounded (or since $\gamma \in H^{2}(\mathbb{R}^{2}, \mathbb{R})$), we have that the operator $K$ defined in the entire $L^{2}$ space has 0 in its essential spectrum. The same statements, but with 4 replaced by 0 $\in \sigma(K)$, hold if we replace $\mathbb{R}^{2}$ by $\Omega$.

Due to the properties above, Theorem [2] follows from the following result.

\textbf{Theorem 4.} Suppose $\kappa \neq 1/\sqrt{2}$ and $n = 1$ if $\kappa > 1/2$. There is $R_{0} > 0$ such that for $R \geq R_{0}$, we have

1) [approximate zero-modes] $\|KT_{j_{0}}^{c}\|_{H^{*}} \leq e^{-R}$;
2) [Coercivity away from the translation and gauge modes] \[ \langle \eta, K\eta \rangle \geq c \|\eta\|_H^2, \] for any \( \eta \perp T_{jk}, \) \( k = 1, 2, \) \( j \in \mathbb{L}, G_\gamma, \gamma \in H_2(\mathbb{R}^2, \mathbb{R}), \) and \( c > 0 \) independent of \( R. \)

A proof of Theorem 3 is identical to the proof of Theorem 2 and is given in Section 5.

Next, we introduce, for an odd integer \( n \), the Sobolev space \( H^c_r(\Omega) \) of order \( r \) of odd functions \( w = (\xi, \chi, \alpha, \beta) : \Omega \to \mathbb{C}^4 \), satisfying the gauge periodic boundary conditions

\[
\begin{align*}
\xi(x + s) &= e^{ig_s(x)}\xi(x), \\
\chi(x + s) &= e^{-ig_s(x)}\chi(x), \\
\alpha(x + s) &= \alpha(x), \\
\beta(x + s) &= \beta(x),
\end{align*}
\]

for \( x \in \partial_1\Omega/\partial_2\Omega, s = \omega_1/\omega_2, \) and for \( g_s \) given in (26). These conditions extend (30).

Finally, let \( K_0 \) be the complex-linear extension of \( L_0 \), defined as above, i.e. \( K_0 \) is the restriction of \( K \) to \( \Omega \). We remark that \( K_0 \) maps \( H^c_r + 2(\Omega) \) into \( H^c_r(\Omega) \), for \( s \geq 0 \). Moreover, it is shown in Appendix B that \( K_0 \), defined on \( L^2_{odd}(\Omega, \mathbb{C}^4) \) with the domain \( H^c_2(\Omega) \), is self-adjoint.

4 Key properties

In this section we prove the inequalities, (39) - (42), used in the proof of Theorem 1.

4.1 Approximate static solution property

Lemma 2. For \( R \geq 1 \) and for any \( r > 0 \), we have

\[ \| F(v) \|_{H^r} \lesssim e^{-\min(\sqrt{\kappa}, 1)R}. \] (50)

Proof. The proof is a computation using the facts that \( U^{(n)} = (\Psi^{(n)}, A^{(n)}) \) satisfies the Ginzburg-Landau equations, together with the exponential decay (7). We write

\[ \psi_0 = \Psi^{(n)} + \xi, \quad a_0 = A^{(n)} + \alpha, \] (51)

where \( \xi \) and \( \alpha \) are defined by this expressions. Using the first Ginzburg-Landau equation, we find

\[ [F(v)]_\psi = \Delta_{A^{(n)}} \xi + (2i\alpha \cdot \nabla_{A^{(n)}} + i \text{div} \alpha + |\alpha|^2)(\Psi^{(n)} + \xi) \]
\[ -\kappa^2(2 \text{Re}(\overline{\Psi^{(n)}}\xi) + |\xi|^2)(\Psi^{(n)} + \xi) - (1 - |\Psi^{(n)}|^2)|\xi|. \] (52)

Furthermore, using the second Ginzburg-Landau equation, \( \text{curl}^2 A^{(n)} - \text{Im}(\overline{\Psi^{(n)}}\nabla_{A^{(n)}}\Psi^{(n)}) = 0 \), we arrive at

\[ [F(v)]_a = \text{curl}^2 \alpha + |\Psi^{(n)} + \xi|^2\alpha \]
\[ -\text{Im}(\overline{\Psi^{(n)}}\nabla_{A^{(n)}}\xi + \xi\nabla_{A^{(n)}}\Psi^{(n)} + \xi\nabla_{A^{(n)}}\xi). \] (53)

Since by (29), \( \alpha = O(e^{-R}) \) in any Sobolev norm, the estimates (4.1) and (4.1) imply (50). \( \square \)
4.2 Approximate zero-mode property

Recall the translational and gauge zero-modes $T_k$, $k = 1, 2$, and $G_\gamma$ are given in (31) and (33).

**Lemma 3** (approximate zero-modes). For any $k = 1, 2$, $\gamma$ twice differentiable and bounded together with its derivatives, and $r > 0$, we have

\[
\|L_0 T_k\|_{H^r} \lesssim e^{-R}, \quad \|L_0 G_\gamma\|_{H^r} \lesssim e^{-R}.
\]

**Proof.** Let $L^{(n)} := E_{GL}(U^{(n)})$. We may write

\[L_0 = L^{(n)} + V^{(n)},\]

where $V^{(n)}$ is a multiplication operator defined by this relation. Using the explicit form (73) of $L$, given in Appendix B, we see that $V^{(n)}$ satisfies

\[
\|V^{(n)}(x)\| \lesssim e^{-R}.
\]

Since $T_k - T_k^{(n)}$ is expressed in terms of $v - U^{(n)}$, the definitions (17), (18), (31) and (32) and the estimates (29), imply

\[
\|T_k - T_k^{(n)}\|_{H^r} \lesssim e^{-R}, \quad \|G_\gamma - G_\gamma^{(n)}\|_{H^r} \lesssim e^{-R},
\]

Using Eqn (55) and (56), the definitions of $T_k^{(n)}$ and $G_\gamma^{(n)}$, in (17) and (18) and the facts

\[L^{(n)} T_k^{(n)} = 0, \quad L^{(n)} G_\gamma^{(n)} = 0,
\]

we obtain the estimates in (54).  

Recall from Section 3 that $K_0$ is a complex-linear extension of $L_0$ and the vectors $T_k^c$, $k = 1, 2$, $G_\gamma^c$, $\gamma \in H^{2+r}_{per}(\Omega)$, are complexifications of the vectors $T_k$, $k = 1, 2$, $G_\gamma$, $\gamma \in H^{2}_{per}(\Omega)$, defined in (31) and (33) (see (19)). The properties $\sigma L_0^{-1} = K_0^{-1} \sigma$ and $\|\sigma w\| = ||w||$ (see Section 3) imply

**Corollary 1** (approximate zero-modes). We have

\[
\|K_0 T_k^c\|_{H^r} \lesssim e^{-R}, \quad \|K_0 G_\gamma^c\|_{H^r} \lesssim e^{-R}.
\]

4.3 Coercivity of the Hessian

In this subsection we prove (40). With the notation as at the end of the last subsection, let $P^c$ be the projection on the span of the vector $G_\gamma^c$, $\gamma \in H^{2+r}_{per}(\Omega)$. We begin with a lower bound on the complexification $K_0$ of $L_0$.

**Lemma 4** (coercivity). For $R$ sufficiently large and for any $w \in \text{Ran}(1 - P^c)$ and $r \geq 0$, we have

\[
\|K_0 w\|_{H^r} \geq c \|w\|_{H^{2+r}}.
\]

(For $n = 1$ if $\kappa > \frac{1}{\sqrt{2}}$ and for any $n$ if $\kappa < \frac{1}{\sqrt{2}}$ we have $c \|w\|_{H^1}^2 \leq \langle w, K_0 w \rangle \leq \frac{1}{c} \|w\|_{H^1}^2$, which could be also extended to a larger class of Sobolev spaces.)
Proof. We omit the subindex 0 in $K_0$ and superindex $c$ in $P^c$ and to simplify the exposition we conduct the proof only for $r = 0$. The proof for general $r \geq 0$ requires an extra technical step (commuting $(-\Delta + 1)\gamma$ through $K_0$). Let $\{\chi_0, \chi_1\}$ be a partition of unity associated to the ball of the radius $R/2$ and its exterior, i.e. $\sum_{j=0}^{\infty} \chi_j^2 = 1$, $\chi_0$ is supported in the ball of the radius $3R/5$ and $\chi_1$ is supported outside the ball of the radius $R/2$. We also assume $|\partial^\alpha \chi_j| \lesssim R^{-|\alpha|}$. Using these properties and commuting $\chi_j$ through $K$, with the help of

$$[\chi_j, \Delta] = -2(\nabla \chi_j) \cdot \nabla - (\Delta \chi_j),$$

we obtain

$$\|Kw\|^2 = \sum_0^1 \|\chi_j Kw\|^2 \geq \sum_0^1 \|K\chi_j w\|^2 - C R^{-2}\|w\|^2_{H^1}.$$ 

We extend the function $w$ to an $L^2-$function on $\mathbb{R}^2$ for which we keep the same notation. Since $\chi_1$ is supported outside the ball of the radius $R/2$, it follows from Lemma 6 of Appendix B that

$$\|K\chi_1 w\| \geq c_1 \|\chi_1 w\|_{H^2},$$

for some $c_1 > 0$.

Now, since $w \in \text{Ran}(1 - P)$, we have that $w \perp G_{r^*}, \gamma \in H^2_{\text{per}}(\Omega)$, and, since $w$ is odd and $T^{(n)}_k, k = 1, 2$, are even, we have that $w \perp T_k^{(n)}, k = 1, 2$. Therefore, due to (58), we have, for the vortex translational and gauge zero-eigenfunctions, $T_k^{(n)}, k = 1, 2$, $G_{r^*}^\gamma, \gamma \in H^2_{\text{per}}(\Omega)$, of $L^{(n)}$,

$$\|T_k^{(n)}, \chi_0 w\| \lesssim e^{-R}, \quad \|G_{r^*}^\gamma, \chi_0 w\| \lesssim e^{-R}, \quad \gamma \in H^2_{\text{per}}(\Omega).$$

Let $P^{(n)}$ be the orthogonal projection on the span of $G_{r^*}^{\gamma}, \gamma \in H^2_{\text{per}}(\Omega)$, and $T_k^{(n)}, k = 1, 2$. Writing $K^{(n)}\chi_0 w = K^{(n)}(1 - P^{(n)})\chi_0 w + K^{(n)}P^{(n)}\chi_0 w$ and using the estimate above and the $n$-vortex stability result of [15] (see Theorem 6 of Appendix B), we obtain

$$\|K^{(n)}\chi_0 w\| \geq c_2 \|\chi_0 w\|_{H^2} - C e^{-R}\|w\|.$$ 

Since on the other hand we have trivially that $\|K^{(n)}\chi_0 w\| \geq c_4 \|\chi_0 w\|_{H^2} - c_4\|w\|$, for some $c_3, c_4 > 0$, the above estimate can be lifted to

$$\|K^{(n)}\chi_0 w\| \geq c_0 \|\chi_0 w\|_{H^2} - C e^{-R}\|w\|,$$

where $K^{(n)}$ is the complex linear extension of $L^{(n)}$. Now, as with $L_0$ in Subsection 1.2, we write $K = K^{(n)} + V^{(n)}$, where recall $V^{(n)}$ satisfies the estimate

$$\|V^{(n)}(x)\| \lesssim e^{-R}.$$ 

Then the last two estimates imply

$$\|K\chi w\| \geq c_0 \|\chi w\|_{H^2} - C e^{-R}\|w\|.$$ 

Collecting the estimates above and using the fact that $\sum \|\chi_j w\|^2_{H^2} \geq \|w\|^2_{H^2} - C R^{-2}\|w\|_{H^1}^2$, we find

$$\|Kw\|^2 \geq (\min c_j)\|w\|^2_{H^2} - C(e^{-R} + R^{-2})\|w\|_{H^1}^2,$$

which for $R$ sufficiently large gives (58) for $r = 0$. As was mentioned above, an extension to arbitrary $r$ is standard. \qed
Let $\bar{P}^c := 1 - P^c$. Lemma 4 and the self-adjointness of $K_0$ imply

**Corollary 2** (invertibility of $K_0$). For $R$ sufficiently large and and $r \geq 0$, the operator $K_0 := P^c K_0 \bar{P}^c : \mathcal{H}^{r+2}_c(\Omega) \to \mathcal{H}^r_c(\Omega)$ is invertible and its inverse, $K_0^{-1}$, satisfies the estimate

$$\|K_0^{-1} w\|_{H^{r+2}} \leq c\|w\|_{H^r}.$$  

(60)

This estimate, the definition of $G^c_\gamma$, $\gamma \in H^{2+r}_{\text{per}}(\Omega)$, and the relations $\sigma L_0 = K_0 \sigma$, $\sigma P = P^c \sigma$ and $\|\sigma w\| = \|w\|$ (see Section 3) imply that the operator $\bar{L}_0 := \bar{P}L_0 \bar{P}$ is invertible and the inverse satisfies $\sigma \bar{L}_0^{-1} = K_0^{-1} \sigma$ and $\|\sigma\|$. 

### 4.4 Nonlinearity estimate

**Lemma 5.** For any $r > r' + 1$, $r' \geq 0$ and $w \in H^r$,

$$\|N_\psi(w)\|_{H^{r'}} \leq c_r(\|w\|_{H^r}^2 + \|w\|_{H^r}^2),$$

$$\|N_\psi(w') - N_\psi(w)\|_{H^{r'}} \leq c_r(\|w\|_{H^r} + \|w\|_{H^r} + \|w'\|_{H^r} + \|w'\|_{H^r})\|w' - w\|_{H^r}.$$  

(61)

**Proof.** We prove only the first estimate. The second one is proved similarly. Explicitly, $N_\psi(w)$ is given by

$$\begin{cases}
N_\psi(w)_\phi = (2i \alpha \cdot \nabla A^{\alpha}) + i \text{div} \alpha \xi + |\alpha|^2(\Psi^{(n)} + \xi) - \kappa^2[2 \Re(\bar{\Psi}^{(n)}\xi) + \xi^2(\Psi^{(n)} + \xi)], \\
N_\psi(w)_a = (2 \text{Re}(\bar{\Psi}^{(n)}\xi) + |\xi|^2)\alpha - \text{Im}(\xi \nabla A^{\alpha})\xi).
\end{cases}$$  

(62)

The most problematic term in $N_\psi(w)$ is of the form $\xi \nabla \xi$, so we will just bound this one (the rest are straightforward). Using Sobolev embedding theorems of the type $\|\xi\|_\infty \lesssim \|\xi\|_{H^r}$, for any $s > 1$, etc, and using the Leibnitz-type property of fractional derivatives (see Stein [SW]), we obtain, for $r > r' + s > 1$,

$$\|\xi \nabla \xi\|_{H^{r'}} \lesssim \|\xi\|_{H^r} \|\nabla \xi\|_{H^{r'}} + \|\xi\|_{H^{r'+s}} \|\nabla \xi\|_{H^s} \lesssim \|\xi\|_{H^r} \|\nabla \xi\|_{H^{r'+1}} + \|\xi\|_{H^{r'+s}} \|\xi\|_{H^1},$$

which gives $\|\xi \nabla \xi\|_{H^{r'}} \lesssim \|\xi\|_{H^r}^2$.  

5 Proof of Theorems 2 and 4

Let $U^L \equiv (\Psi, A)$ be the $L$–periodic solution of (11) found in Theorem 1 (In this section we omit the superindex $L$ in $\Psi^L, A^L$.) The proofs of Theorems 2 and 4 are identical and we give the proof of Theorem 2. It follows from the two propositions given below. Define the shifted gauge zero modes, $G_\gamma^{(n)}(x) = G_\gamma^{(n)}(x - j)$, where $G_\gamma^{(n)}(x)$ are the gauge zero modes of the linearized operator, $L^{(n)} := F'(U^{(n)})$, given in (18).
5.1 Zero and almost zero modes

Proposition 2 (approximate zero-modes of \(L\)). With definitions given in Subsections 1.5 and 1.6 and under the additional condition that \(\gamma \in H^2_{\text{per}}(\mathbb{R}^2)\) is exponentially localized, \(|\gamma(x)| \lesssim e^{-cR}\) for some \(c > 0\), we have

\[
\|LT_{jk}\| \lesssim e^{-R}, \quad \|LG_{j\gamma}\| \lesssim e^{-R}.
\] (63)

Proof. For each \(j \in \mathcal{L}\), we write \(L = L_j + V_j\), where \(L_j\) is the shifted vortex linearized operator

\[
L_j := F'(U^{(n)}(\cdot - j)) \equiv L^{(n)}|_{x \to x-j},
\]

and \(V_j\) is a multiplication operator defined by this relation. Due to the explicit form (73) of \(L\), given in Appendix B, and the estimates

\[
U^L(x) = U^{(n)}(x - \alpha) + O_{H^1}(e^{-R}), \text{ on } \Omega + \alpha, \forall \alpha \in \mathcal{L}.
\] (64)
on the \(\mathcal{L}\)-periodic solution \(U^L \equiv (\Psi, A)\) of (1), given in Theorem 1, \(V_j\) satisfies

\[
|V_j(x)| \lesssim e^{-\delta R}, \quad \text{if} \quad |x - j| \leq \delta R.
\] (65)

By the definition, \(L_j\) has the zero modes, which are shifted translation and gauge zero modes,

\[
T_{jk}(x) = T_k^{(n)}(x - j), \quad k = 1, 2, \quad \text{and} \quad G_{j\gamma}(x) = G_{\gamma}^{(n)}(x - j),
\]

\[
L_jT_{jk} = 0, \quad L_jG_{j\gamma} = 0.
\] (66)

This, the estimates (7) and the condition that \(\gamma\) is exponentially localized, yield that

\[
|T_{jk}|, \quad |G_{j\gamma}| \lesssim e^{-\delta R}, \quad \text{if} \quad |x - j| \geq \delta R.
\] (67)

Using these estimates and using (65), (66) and the relations \(L = L_j + V_j\), we obtain the estimates (63) of Proposition 2. \(\square\)

5.2 Coercivity away from the translation and gauge modes

Proposition 3. Under conditions of Theorem 2, there is \(c > 0\) s.t.

\[
\langle \eta, L\eta \rangle \geq c\|\eta\|_{H^1}^2,
\] (68)

for any \(\eta \perp \text{Span}\{T_{jk}, \quad k = 1, 2, \quad \forall j \in \mathcal{L}, \quad G_{j\gamma}, \quad \gamma \in H^2(\mathbb{R}^2, \mathbb{R})\}\).

Proof. Recall that the lattice \(\mathcal{L}\) is defined in such a way that vortices are located at the centers of its cells. Let \(\mathcal{L}'\) be a shifted lattice having vortices at its verices and let \(\mathcal{L}'' := \mathcal{L}' \cup \{\infty\}\). Let \(\{\chi_j, \quad j \in \mathcal{L}''\}\) be a partition of unity associated to the balls of radius \(R/3\), centered at the points of the lattice \(\mathcal{L}'\), i.e. \(\chi_j, \quad j \in \mathcal{L}'\), are supported in the balls, \(B(j, R/3)\),
of the radius $R/3$ about $j \in \mathcal{L}'$, $\chi_\infty$ is supported in $\mathbb{R}^2 / \bigcup_{j \in \mathcal{L}'} B(j, R/4)$, i.e. away from all the vortices, and $\sum_{j \in \mathcal{L}''} \chi_j^2 = 1$. We can choose $\{\chi_j\}$ such that $|\nabla \chi_j| \lesssim R^{-1}$. By the IMS formula ([CPKS]),

$$L = \sum \chi_j L \chi_j - 2 \sum |\nabla \chi_j|^2.$$  

(69)

As in the previous subsection, we write $L = L_j + V_j$, for each $j \in \mathcal{L}'$. By our choice of $\{\chi_j, j \in \mathcal{L}'\}$, we have that $\|V_j|\text{Supp} \chi_j\|_\infty \lesssim e^{-R}$ (see (65)), and so, for $j \in \mathcal{L}'$,  

$$\langle \chi_j \eta, L \chi_j \eta \rangle \geq \langle \chi_j \eta, L_j \chi_j \eta \rangle - C e^{-R} \|\chi_j \eta\|^2.$$  

Let $\gamma_j(x) = \gamma(x-j)$. Since $\eta \perp G_\gamma$, $\gamma \in H^2(\mathbb{R}^2, \mathbb{R})$, we have $\langle G_{j\gamma}, \chi_j \eta \rangle = \langle G_{j\gamma} - G_{j\gamma}, \chi_j \eta \rangle + \langle G_{j\gamma}, (\chi_j - 1) \eta \rangle$. By (15) and the exponential localization of $\gamma \in H^2_{\text{per}}(\mathbb{R}^2, \mathbb{R})$, the first term on the r.h.s. is $\lesssim e^{-R}$. By exponential localization of $G_{j\gamma}$ the same is true for the second term as well. Hence we obtain $|\langle G_{j\gamma}, \chi_j \eta \rangle| \lesssim e^{-R}$. Next, since $\eta \perp T_{j_k}$, $k = 1, 2, \forall j \in \mathcal{L}'$, and $\|T_{j_k} \chi_j\|_2 \lesssim e^{-R}$, we have $|\langle T_{j_k}, \chi_j \eta \rangle| \lesssim e^{-R}$. To sum up, for $j \in \mathcal{L}'$, and for all $\gamma \in H^2(\mathbb{R}^2, \mathbb{R})$, exponentially localized, we have that  

$$|\langle T_{j_k}, \chi_j \eta \rangle| \lesssim e^{-R} \|\eta\|, \ |\langle G_{j\gamma}, \chi_j \eta \rangle| \lesssim e^{-R} \|\eta\|.$$  

So by the $n$-vortex stability result of [GS1] (for all $n$ if $\kappa < \frac{1}{\sqrt{2}}$ and for $n = 1$ if $\kappa > \frac{1}{\sqrt{2}}$), we have, for $R$ sufficiently large and $\forall j \in \mathcal{L}'$,  

$$\langle \chi_j \eta, L_j \chi_j \eta \rangle \geq c_1 \|\chi_j \eta\|^2_{L^2}.$$  

Also, since $\chi_\infty$ is supported away from all the lattice sites, where the vortices are centered, we have that  

$$\langle \chi_\infty \eta, L \chi_\infty \eta \rangle \geq c_2 \|\chi_\infty \eta\|^2_{L^2},$$  

for some $c_1 > 0$. The above estimates together with (69) and the fact that $\text{Supp} \nabla \chi_j$ for different $j$’s do not overlap and therefore $\sum_j |\nabla \chi_j| \lesssim R^{-2}$, give, for $R$ sufficiently large,

$$\langle \eta, L \eta \rangle \geq \left[c_3 - CR^{-2}\right] \|\eta\|_{L^2}^2 \geq c \|\eta\|_{L^2}^2.$$  

(70)

Hence we have shown (68). \hfill \Box

Propositions 2 and 3 imply Theorem 4. \hfill \Box

Theorem 4 is obtained by replacing, in the proof above, $L$ with $K$.

**Remark 1.** One can modify the proof of proposition 7 to make $R_0$ uniform in $\kappa - 1/\sqrt{2}$. To this end one would have to ‘project out’ also the $(\kappa = 1/\sqrt{2})$—zero modes (see [GS1]).

### A Critical magnetic fields

In superconductivity there are several critical magnetic fields, two of which (the first and the second critical magnetic fields) are of special importance:

- $h_{c1}$ is the field at which the first vortex enters the superconducting sample.
- $h_{c2}$ is the field at which a mixed state bifurcates from the normal one.
The critical field $h_{c1}$ is defined as $h$ for which $G_Q(\Psi_s, A_s) = G_Q(\Psi^{(1)}, A^{(1)})$, for $Q = \mathbb{R}^2$.

For type I superconductors $h_{c1} > h_{c2}$ and for type II superconductors $h_{c1} < h_{c2}$. In the former case, the vortex states have relatively large energies, i.e. are metastable, and therefore are of little importance.

For type II superconductors, there are two important regimes to consider: 1) average magnetic fields per unit area, $b$, are less than but sufficiently close to $h_{c2}$,

$$0 < h_{c2} - b \ll h_{c2}$$

and 2) the external (applied) constant magnetic fields, $h$, are greater than but sufficiently close to $h_{c1}$,

$$0 < h - h_{c1} \ll h_{c1}.$$  \hspace{1cm} (72)

The reason the first condition involves $b$, while the second $h$ is that the first condition comes from the Ginzburg-Landau equations (which do not involve $h$), while the second from the Ginzburg-Landau Gibbs free energy.

One of the differences between the regimes (71) and (72) is that $|\Psi|^2$ is small in the first regime (the bifurcation problem) and large in the second one. If a superconductor fills in the entire $\mathbb{R}^2$, then in the second regime, the average magnetic field per unit area, $b \to 0$, as $h \to h_{c1}$.

B The operators $L$ and $K$

This appendix combines the construction of the complex $K$ extension of $L$ and statement of its fiber decomposition and its properties, due to [GS1], which is essential to our analysis, with the proof of self-adjointness of $K$ and the group theoretical elucidation of the fiber decomposition of the operator $K$.

B.1 Explicit form of $L$ and $K$

First, we write out explicitly the operators $L$ and $K$ introduced in Subsection 1.6 and Section 3 and discuss a different way to treat the operator $L$. In this section we write operators $L$ and $K$ for any solution $U = (\Psi, A)$ of (1). The arguments below are presented on $\mathbb{R}^2$ but are also applicable on $\Omega$.

Let $\mathcal{R}$ be the operation of taking the real part. The operator $L$ is given explicitly as (GSII)

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

with

$$L_{11} = -\Delta_A + \kappa^2 (2|\Psi|^2 - 1) + \kappa^2 \sqrt{2} \mathcal{C},$$

$$L_{12} = i[\nabla_A \Psi + \nabla_A \Psi] = i[2(\nabla_A \Psi) + \Psi \nabla \Psi],$$

$$L_{21} = -\mathcal{R} i[\nabla_A \Psi - \Psi \nabla \Psi],$$

$$L_{22} = -\Delta + |\Psi|^2.$$  \hspace{1cm} (74)
(Here \((\nabla_A \Psi)\) stands for the function resulting in application \(\nabla_A\) to \(\Psi\), while \(\nabla_A \Psi\) stands for the product of operators \(\nabla_A\) and multiplication by \(\Psi\).) To prove symmetry of \(L\), we have

\[
\text{Re} \int \bar{\xi} (2i\omega \cdot \nabla_A \Psi + i\Psi \div \omega) = \int -2\omega \cdot \text{Im} (\bar{\xi} \nabla_A \Psi) - \text{Im}(\bar{\xi} \Psi) \div \omega
\]

\[
= \int -2\omega \cdot \text{Im}(\bar{\xi} \nabla_A \Psi) + \text{Im}(\bar{\xi} \Psi - \bar{\Psi} \nabla \xi) \cdot \omega
\]

\[
= \int -2\omega \cdot \text{Im}(\bar{\xi} \nabla_A \Psi) + \text{Im}(\bar{\xi} \Psi - i\Psi A + i\bar{\Psi} \nabla \xi) \cdot \omega
\]

\[
= \int -\omega \cdot \text{Im}(\bar{\xi} \nabla_A \Psi) + \bar{\Psi} \nabla \xi \cdot \omega
\]

To extend the operator \(L\) to a complex-linear operator \(K\) we recall \(\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \leftrightarrow \alpha^c = \alpha_1 - i\alpha_2\), use the complex notation \(\partial = \partial_{x_1} - i\partial_{x_2}\), \(\partial A^c = \partial - iA^c\), \((75)\), and introduce the complex conjugate, \(\bar{A}\), of an operator \(A\) as the operator \(\bar{A} := C A C\), where \(C\) denotes complex conjugation. Straightforward calculations show that

\[
\text{div} \alpha = \frac{1}{2} \partial \bar{\alpha}^c + \frac{1}{2} \partial \alpha^c,
\]

\[
2i\alpha \cdot \nabla \Psi = -i(\partial A^c \Psi) \alpha^c + i(\partial A^c \Psi) \bar{\alpha}^c,
\]

and

\[
-\text{Im}(\bar{\xi} \nabla_A \Psi)^c = \frac{i}{2}(\partial A^c \Psi) \xi + \frac{i}{2}(\partial A^c \Psi) \bar{\xi}.
\]

In what follows we drop the superscript \(c\) in \(A^c\). Using the above relations one shows that the complex-linear extension, \(K\), of the operator \(L\), is given explicitly as

\[
K = \begin{pmatrix}
-\Delta_A + \kappa^2(2|\Psi|^2 - 1) & \kappa^2 \Psi^2 & -i(\partial A^c \Psi) + \frac{i}{2} \Psi \bar{\partial} & \frac{i}{2}(\partial A^c \Psi) + \frac{i}{2} \Psi \bar{\partial} \\
\kappa^2 \Psi^2 & -\Delta_A + \kappa^2(2|\Psi|^2 - 1) & -i(\partial A^c \Psi) - \frac{i}{2} \Psi \bar{\partial} & \frac{i}{2}(\partial A^c \Psi) - \frac{i}{2} \Psi \bar{\partial} \\
\frac{i}{2}(\partial A^c \Psi) + \frac{i}{2} \Psi \bar{\partial} & \frac{i}{2}(\partial A^c \Psi) + \frac{i}{2} \Psi \bar{\partial} & -\Delta + |\Psi|^2 & 0 \\
-\frac{i}{2}(\partial A^c \Psi) - \frac{i}{2} \Psi \bar{\partial} & -\frac{i}{2}(\partial A^c \Psi) - \frac{i}{2} \Psi \bar{\partial} & 0 & -\Delta + |\Psi|^2
\end{pmatrix}
\]

(76)

It is not hard to check that \(K\) restricted to vectors on the r.h.s. of (46) gives \(L^c\).

We consider the linearized operator \(L\), on a space of pairs \((\Psi, A)\), satisfying the gauge condition

\[
\text{Im}(\bar{\Psi} \xi) - \nabla \cdot \alpha = 0.
\]

(77)

We mention a convenient way to treat the condition (77) by passing to a modified real-linear operator \(L_\#\), defined by the quadratic form \((\text{CSI})\)

\[
\langle w, L_\# w \rangle = \langle w, L_0 w \rangle + \int_{\mathbb{R}^2} (\text{Im}(\bar{\Psi} \xi) - \nabla \cdot \alpha)^2,
\]
where $w = (\xi, \alpha) \in L^2(\mathbb{R}^2, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^2)$. Clearly, $L_\#$ agrees with $L$ on the subspace of $L^2(\mathbb{R}^2, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{R}^2)$ specified by the gauge condition (77). This modification has the important effect of shifting the essential spectrum away from zero. A straightforward computation gives the following expression for $L_\#$:

$$L_\# \begin{pmatrix} \xi \\ \alpha \end{pmatrix} = \begin{pmatrix} -\Delta_A + \frac{\kappa^2}{2} |\Psi|^2 - 1 + \frac{1}{2} |\Psi| \xi + \frac{1}{2} (\kappa^2 - 1) \Psi^2 \bar{\xi} + 2i \nabla_A \Psi \cdot \alpha \\ 2 \text{Im}[\nabla_A \Psi \xi] + [\Delta + |\Psi|^2] \alpha \end{pmatrix}.$$  

The complex-linear extension, $K_\#$, of $L_\#$, defined on $[L^2(\mathbb{R}^2; \mathbb{C})]^4$, is given by

$$K_\# = \text{diag} \{-\Delta_A, -\Delta_A, -\Delta, -\Delta\} + V,$$  

where $V$ is the matrix-multiplication operator given, using the notation (75), by

$$V = \begin{pmatrix} \frac{\kappa^2}{2} |\Psi|^2 - 1 + \frac{1}{2} |\Psi|^2 & \frac{1}{2} (\kappa^2 - 1) |\Psi|^2 \\ 0 & 0 & -i(\partial_A \Psi) & i(\partial_A \Psi) \\ i(\partial_A \Psi) & i(\partial_A \Psi) & |\Psi|^2 & 0 \\ -i(\partial_A \Psi) & -i(\partial_A \Psi) & 0 & |\Psi|^2 \end{pmatrix}.$$  

The components of $V$ are bounded, and it follows from standard results that $K_\#$ is a self-adjoint operator on $[L^2(\mathbb{R}^2; \mathbb{C})]^4$, with domain $\mathcal{D}(K_\#) = [H^2(\mathbb{R}^2; \mathbb{C})]^4$.

### B.2 Self-adjointness of $K_0$

Next, we sketch a proof of

**Theorem 5.** The operator $K_0$, defined by the expression (76) on $L^2(\Omega, \mathbb{C}^4)$ with the domain $\mathcal{H}^\infty_2(\Omega)$, is self-adjoint.

**Proof.** Due to representation of the (78) type for $K$ and standard arguments, the question of self-adjointness for $K$ reduces to the same question for $\Delta_{a_0}$. To prove the latter we proceed as in [RSH], Theorem X.28. Namely, we use that, by construction and properties of $a_n$, $a_0$ is $C^1$ and the fact that since $-\Delta_{a_0} \geq 0$, it suffices to show that $(-\Delta_{a_0} + 1)^* \xi = 0$ implies $\xi = 0$, which is equivalent to showing that $(-\Delta_{a_0} + 1) \xi = 0$, $\xi \in L^2$, (in the weak sense) implies $\xi = 0$. Now, we use Kato’s inequality $\Delta |\xi| \geq \text{Re}[(\text{sign} \xi) \Delta_{a_0} \xi]$, where $(\text{sign} \xi)(x) = \xi(x)/|\xi(x)|$ if $\xi(x) \neq 0$ and $(\text{sign} \xi)(x) = 0$ if $\xi(x) = 0$, (see e.g. RSH, Theorem X.33). By this inequality, $\Delta |\xi| \geq \text{Re}[(\text{sign} \xi) \Delta_{a_0} \xi] = |\xi| \geq 0$. Let now $\omega_\delta \geq 0$ be an approximation of identity and $f_\delta := \omega_\delta * |\xi|$. Then by the above $\Delta f_\delta := \omega_\delta * \Delta |\xi| \geq 0$ and therefore $\langle f_\delta, \Delta f_\delta \rangle \geq 0$. On the other hand, by integration by parts, $\langle f_\delta, \Delta f_\delta \rangle \leq 0$. Therefore we have $\langle f_\delta, \Delta f_\delta \rangle = 0$, which implies $f_\delta = 0$. Since $f_\delta \rightarrow |\xi|$, as $\delta \rightarrow 0$, we conclude that $|\xi| = 0$. This completes the argument. (For more more general results on self-adjointness of Schrödinger type operators on Hermitian vector bundles see [BMS].) □
B.3 Lower bound on $K_0$ away from vortices

Lemma 6. For $R$ sufficiently large, there is $c_1 > 0$ such that for any $w$ satisfying (77) and supported outside the ball of the radius $R/2$, we have that
\[
\|K_0 w\| \geq c_1 \|w\|_{H^2}, \tag{79}
\]

Proof. In this proof we omit the subindex 0 in $K_0$. First we prove that $\|K w\| \geq c_1 \|w\|$. By the Schwarz inequality it suffices to show that $\langle w, K w \rangle \geq c_1 \|w\|_2^2$. To prove the latter inequality we use that for any $w$ satisfying (77), $K$ and $K_\#$ induce the same quadratic form, $\langle w, K w \rangle = \langle w, K_\# w \rangle$. Observe that $\|K w\| \geq \langle w, K_\# w \rangle$ and estimate the r.h.s. of the latter expression. To this end we use the explicit construction of $\psi_0$ and $a_0$ or the estimates (29) which imply that outside the ball of the radius $R/2$
\[
\|\psi_0\|^2 - 1\|_\infty, \|\partial_{a_0} \psi_0\|_\infty, \|\partial_{a_0} \psi_0\|_\infty \leq Ce^{-R},
\]
and the explicit expression for $K_\#$ which is given by (78), with $\Psi$ and $A$ replaced by $\psi_0$ and $a_0$, to obtain that outside the ball of the radius $R/2$,
\[
K_\# = \begin{pmatrix}
-\Delta_0 + \frac{1}{2} (\kappa^2 + 1) & \frac{1}{2} (\kappa^2 - 1) \psi_0^2 & 0 & 0 \\
\frac{1}{2} (\kappa^2 - 1) \psi_0^2 & -\Delta_0 + \frac{1}{2} (\kappa^2 + 1) & 0 & 0 \\
0 & 0 & -\Delta + 1 & 0 \\
0 & 0 & 0 & -\Delta + 1
\end{pmatrix} + O(e^{-R}), \tag{80}
\]
and therefore $\langle w, K_\# w \rangle \geq c \|w\|^2$. As was argued above this gives, by the Schwarz inequality,
\[
\|K_\# w\| \geq c \|w\|.
\]
Next, (78) implies that for some $C > 0$, $\|K_\# w\| \geq \frac{1}{2} \|\Delta w\| - C \|w\|$. Writing $\|K_\# w\| = \delta \|K_\# w\| + (1 - \delta) \|K_\# w\|$ and applying the second inequality to the first term and the first inequality to the second one and choosing $\delta$ appropriately (say $\delta = \frac{2C}{1 + 2C + C}$), we arrive at (78).

B.4 Fibre decomposition of $K_\#$

Now we consider the operator $K_\#$ for the vortex solution $U^{(n)} = (\Psi^{(n)}, A^{(n)})$. We denote the resulting operator by $K^{(n)}_\#$ and present the important decomposition of $K^{(n)}_\#$, which is due to the fact that vortices are gauge equivalent under the action of rotation, i.e.,
\[
\Psi(\alpha x) = e^{i n \alpha} \Psi(x), \quad R_{-\alpha} A(\alpha x) = A(x),
\]
where $R_{\alpha}$ is counterclockwise rotation in $\mathbb{R}^2$ through the angle $\alpha$. This property induces the following symmetry property of $K^{(n)}_\#$. Let $\rho_n : U(1) \rightarrow \text{Aut}([L^2(\mathbb{R}^2; \mathbb{C})]^4)$ be the representation whose action is given by
\[
\rho_n(e^{i \theta})(\xi, \chi, \alpha, \beta)(x) = (e^{i n \theta} \xi, e^{-i n \theta} \chi, e^{-i \theta} \alpha, e^{i \theta} \beta)(R_{-\theta} x).
\]
It is easily checked that the linearized operator $K^{(n)}_\#$ commutes with $\rho_n(g)$ for any $g \in U(1)$. It follows that $K^{(n)}_\#$ leaves invariant the eigenspaces of $d\rho_n(s)$ for any $s \in i \mathbb{R} = \text{Lie}(U(1))$. 
(The representation of $U(1)$ on each of these subspaces is multiple to an irreducible one.) This results in (fiber) block decomposition of $K_\#^{(n)}$, which is described below. In particular, the translational zero-modes each lie within a single subspace of this decomposition. In what follows we write functions on $\mathbb{R}^2$ in polar coordinates, so that

$$H^c(\mathbb{R}^2) := [L^2(\mathbb{R}^2; \mathbb{C})]^4 = [L^2_{\text{rad}} \otimes L^2(S^1; \mathbb{C})]^4$$

where $L^2_{\text{rad}} \equiv L^2(\mathbb{R}^+, rdr)$. Let $C$ be the operation of taking the complex conjugate.

**Theorem 6.** (a) Let $H_m := [L^2_{\text{rad}}]^4$ and define $U : H^c(\mathbb{R}^2) \to H$, where $H = \bigoplus_{m \in \mathbb{Z}} H_m$, so that on smooth compactly supported $v$ it acts by the formula

$$(Uv)_m(r) = J^{-1}_m \int_0^{2\pi} \chi_m^{-1}(\theta) \rho_n(e^{i\theta}) v(x) d\theta.$$ 

where $\chi_m(\theta)$ are characters of $U(1)$, i.e., all homomorphisms $U(1) \to U(1)$ (explicitly we have $\chi_m(\theta) = e^{im\theta}$) and

$$J_m : H_m \to e^{i(m+n)\theta} L^2_{\text{rad}} \oplus e^{i(m-n)\theta} L^2_{\text{rad}} \oplus -ie^{i(m-1)\theta} L^2_{\text{rad}} \oplus ie^{i(m+1)\theta} L^2_{\text{rad}}$$

acting in the obvious way. Then $U$ extends uniquely to a unitary operator.

(b) Under $U$ the linearized operator around the vortex, $K_\#^{(n)}$, decomposes as

$$UK_\#^{(n)}U^{-1} = \bigoplus_{m \in \mathbb{Z}} K_m^{(n)},$$

where the operators $K_m^{(n)}$ act on $H_m$ as $J_m^{-1} K_\#^{(n)} J_m$.

(c) The operators $K_m^{(n)}$ have the following properties:

$$K_m^{(n)} = RK_m^{(n)} R^T,$$

where $R = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$, $Q = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$

$$\sigma_{\text{ess}}(K_m^{(n)}) = [\min(1, \lambda), \infty),$$

for $|n| = 1$ and $m \geq 2$, $K_m^{(n)} - K_1^{(n)} \geq 0$ with no zero-eigenvalue,

$$K_0^{(n)} \geq c > 0 \quad \text{for all } \kappa,$$

$$K_1^{(\pm1)} \geq 0 \quad \text{with non-degenerate zero-mode given by}$$

$$T := (f' - \frac{n(1-a)}{r} f, f' + \frac{n(1-a)}{r} f, 2n a', 0).$$

**Proof.** We prove (a) and (b). The properties (83) - (87) in (c) were proven in [GS1] (the latter paper did not articulate the construction in (a) and (b) explicitly).
A straightforward calculation shows that for \( \hat{v}_m = \int_0^{2\pi} \chi_m^{-1}(\theta)\rho_n(e^{i\theta})v(x)\frac{d\theta}{2\pi}, \) \( \rho_n(e^{i\theta})\hat{v}_m = \chi_m(\theta)\hat{v}_m, \) from which it follows that \( \hat{v}_m \) lies in the range of \( J_m. \) Therefore \( U \) is well-defined.

We now calculate that for smooth compactly supported \( v, \)

\[
\sum_{m \in \mathbb{Z}} \int_0^\infty \left\| \int_0^{2\pi} \chi_m^{-1}(\theta)\rho_n(e^{i\theta})v(x)\frac{d\theta}{2\pi} \right\|^2 r dr
\]

\[
= \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m \in \mathbb{Z}} e^{im(\theta-\phi)} \right) \rho_n(e^{i\theta})v(x)\rho_n(e^{i\phi})v(x)r dr d\theta d\phi
\]

\[
= \int_0^\infty \int_0^{2\pi} |\rho_n(e^{i\theta})v(x)|^2 r dr d\theta = \|v\|^2
\]

It then follows that \( U \) extends to all of \( L^2(\mathbb{R}^2)^4 \) with norm \( \|U\| = 1. \) To show that \( U \) is in fact a unitary map, we consider the map \( U^* : \mathcal{H} \to L^2(\mathbb{R}^2)^4 \) given by

\[
U^* g = \sum_{m \in \mathbb{Z}} J_m g_m.
\]

Similar calculations as above show that \( U^* \) is indeed the adjoint of \( U \) and also has norm 1. This proves (a).

To prove (b), the essential fact is that \( K^{(n)}_\# \) commutes with the \( \rho_n. \) We have for any \( g = Uv \in \mathcal{H} \)

\[
(UK^{(n)}_\#U^{-1} g)_m = J_m^{-1} \int_0^{2\pi} \chi_m^{-1}(\theta)\rho_n(e^{i\theta})K^{(n)}_\# v(x)d\theta
\]

\[
= J_m^{-1} K^{(n)}_\# \int_0^{2\pi} \chi_m^{-1}(\theta)\rho_n(e^{i\theta})v(x)d\theta
\]

\[
= (J_m^{-1} K^{(n)}_\# J_m) g_m.
\]

This then completes the proof of (b).

Since, by (84) and (87), \( K^{(\pm 1)}_\# \geq \tilde{c} > 0 \) and, by (86) and (87), \( K^{(\pm 1)}_m \geq c' > 0 \) for \( |m| \geq 2, \) this theorem implies that \( K^{(\pm 1)}_\# \geq c > 0 \) on the subspace of \( \mathcal{H}^c(\mathbb{R}^2) \) orthogonal to the translational zero-modes.

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