Inconsistencies of the Adiabatic Theorem and the Berry Phase

Arun K. Pati(1) and A. K. Rajagopal(2)

(1) Institute of Physics, Sainik School Post, Bhubaneswar-751005, Orissa, India
(2) Naval Research Laboratory, Washington D.C. 20375-5320, USA

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The adiabatic theorem states that if we prepare a quantum system in one of the instantaneous eigenstates then the quantum number is an adiabatic invariant and the state at a later time is equivalent to the instantaneous eigenstate at that time apart from phase factors. Recently, Marzlin and Sanders have pointed out that this could lead to apparent violation of unitarity. We resolve the Marzlin-Sanders inconsistency within the quantum adiabatic theorem. Yet, our resolution points to another inconsistency, namely, that the cyclic as well as non-cyclic adiabatic Berry phases may vanish under strict adiabatic condition. We resolve this inconsistency and develop an unitary operator decomposition method to argue for the validity of the adiabatic approximation.

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I. INTRODUCTION

Adiabatic theorem is one of the most important and widely studied theorems in quantum mechanics [1,2]. It states that if we have a slowly changing Hamiltonian that depends on time, and the system is prepared in one of the instantaneous eigenstates of the Hamiltonian then the state of the system at any time is given by an instantaneous eigenfunction of the Hamiltonian up to multiplicative phase factors [3–5]. It has potential application in diverse areas of physics such as in molecular physics, nuclear physics, chemical physics, quantum field theory and so on. It is the revisit of adiabatic theorem that enabled Berry to discover his now famous geometric phase [6]. (Some authors mention that δ is usually neglected in the text book but can give dynamical value equation H(t)|n(t)⟩ = E_n(t)|n(t)⟩. If the state of the system was initially in one of the eigenstate (say n), then under adiabatic approximation (AA) the state at a later time t is given by

|ψ(t)⟩ ≈ e^i δ_n(t) e^i γ_n(t) |n(t)⟩ (1)

where δ_n(t) = −∫ E_n(t′)dt′ (h=1) is the dynamical phase and γ_n(t) = ∫ i⟨n(t′)|i(t′)⟩dt′ is the extra phase that is usually neglected in the text book but can give rise to Berry phase during a cyclic change of the Hamiltonian [6]. (Some authors mention that γ_n(t) is the Berry phase which is incorrect. Unless we have a cyclic evolution γ_n(t) is not gauge invariant, hence not observable.)

Since the MS inconsistency is a very curious one, (as well as for the sake of completeness), we spell out their argument. Let us define a state |ψ(t)⟩ = U^†(t)|ψ(0)⟩, where U(t) = T exp(−i ∫ H(t) dt). This state satisfies a Schrödinger equation with a Hamiltonian ̃H(t), i.e.,

i|ψ̇(t)⟩ = ̃H(t)|ψ(t)⟩, (2)

where ̃H = −U^†(t)H(t)U(t). Note that This holds irrespective of adiabatic approximation. Now, what is the
solution to the above equation in the adiabatic approximation? First, note that if $|n(t)\rangle$ is an eigenstate of $H(t)$ with eigenvalue $E_n(t)$, then $|\tilde{n}(t)\rangle = U(t)^\dagger |n(t)\rangle$ is an eigenstate of $H(t)$ with eigenvalue $-E_n(t)$. Since $|\tilde{n}(t)\rangle$'s are orthonormal and satisfies completeness criterion, this forms an instantaneous basis. We can expand

$$|\tilde{\psi}(t)\rangle = \sum_n d_n(t) |\tilde{n}(t)\rangle,$$

(3)

and using AA the amplitudes $d_n(t)$'s are given by

$$id_n(t) + E_n(t)d_n(t) + i\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle d_n(t) + \sum_{m\neq n} i\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle d_m(t) = 0.$$  

(4)

Under adiabatic approximation we can drop the terms like $\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle$ because

$$\frac{|\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle}{|E_n - E_m|} < 1, \quad n \neq m.$$  

(5)

In general if $H(t)$ varies slowly a unitarily related Hamiltonian need not vary slowly. But in this case it happens to be so. Under the above condition the solution is found to be

$$d_n(t) \approx \exp(-i\delta_n(t)) \exp(i \int i\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle dt) d_n(0).$$  

(6)

Therefore, the state $|\tilde{\psi}(t)\rangle$ is given by

$$|\tilde{\psi}(t)\rangle \approx \exp(-i\delta_n(t)) \exp(i \int i\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle dt) |\tilde{n}(t)\rangle.$$  

(7)

Using $|\tilde{n}(t)\rangle = \exp(-i\delta_n(t)) \exp(-i\gamma_n(t)) |n(0)\rangle$ and $i\langle \tilde{n}(t)|\dot{\tilde{n}}(t)\rangle = i\langle n(t)|\dot{n}(t)\rangle - E_n(t)$ one will have

$$|\tilde{\psi}(t)\rangle \approx e^{i \int E_n(t')dt'} |n(0)\rangle.$$  

(8)

This was the result of MS. Now we give their contradiction. Since $|\tilde{\psi}(t)\rangle$ is unitarily related to the initial state, so it must be normalized to unity. However, the above solution together with the standard adiabatic ansatz (1) gives a non-unit norm! Explicitly, one can see that

$$\langle \tilde{\psi}(t)|\tilde{\psi}(t)\rangle = \langle \psi(0)|U(t)|\tilde{\psi}(t)\rangle \approx e^{i\gamma_n(t)} \langle n(0)|n(t)\rangle \neq 1.$$  

(9)

Hence a contradiction. So the standard adiabatic theorem apparently violates unitarity! Does it really do so?

III. RESOLUTION OF THE MS INCONSISTENCY

The resolution to the above contradiction is now given within the adiabatic theorem. Now let us take a close look at the transition amplitude $A_n(t) = \langle n(0)|n(t)\rangle$ between the initial and the instantaneous eigenstates, and ask how does that change with time as we slowly change the Hamiltonian. Consider the transition amplitude defined by

$$A_n(t) = \langle n(0)|n(t)\rangle.$$  

(10)

Its time rate of change is given by $\dot{A}_n(t)$

$$\dot{A}_n(t) = \sum_m \langle n(0)|m(t)\rangle \langle m(t)|\dot{n}(t)\rangle = \langle n(0)|n(t)\rangle \langle n(t)|\dot{n}(t)\rangle$$  

(11)

$$+ \sum_{m \neq n} \langle n(0)|m(t)\rangle \langle m(t)|\dot{n}(t)\rangle.$$  

Now under standard adiabatic approximation one can drop the terms $\langle m(t)|\dot{n}(t)\rangle$, for $m \neq n$. Then we have the transition amplitude $A_n(t)$ as follows:

$$i\dot{A}_n(t) \approx \gamma_n(t) A_n(t),$$  

(12)

where $\gamma_n(t) = i\langle n(t)|\dot{n}(t)\rangle$ is the Berry frequency. This leads to

$$A_n(t) \approx e^{-i\gamma_n(t)} A_n(0).$$  

(13)

Since $A_n(0) = 1$, this implies that $\langle n(0)|n(t)\rangle \approx e^{-i\gamma_n(t)}$. Using this solution one can easily see that the MS contradiction is resolved, i.e., the unitarity is preserved. This also tells us that the transition probability between initial eigenstate and the later instantaneous eigenstate is unity for all time. In terms of the ‘minimum-normed distance’ [12] we have $D^2(|n(0)\rangle, |n(t)\rangle) = 2(1 - |\langle n(0)|n(t)\rangle|) \approx 0$ which is almost zero. So under strict adiabatic condition the instantaneous eigenstate apparently stays almost close to the original one and hence there is no violation of norm preservation.

IV. THE BERRY PHASE-YET ANOTHER INCONSISTENCY

The above resolution could have been a satisfying situation if the following is not true. However, as we will show below under strict adiabatic evolution, i.e., under the condition (5) the cyclic as well as the non-cyclic Berry phases almost vanish!

Consider the cyclic variation of the Hamiltonian, i.e., $H(R(T)) = H(R(0))$ over a period of time $T$. Then we know that the state of the system at time $t = T$ is given by

$$|\psi(T)\rangle \approx e^{-i \int_0^T E_n(t) dt} e^{i\gamma_n(C)} |\psi(0)\rangle$$  

(14)

where $\gamma_n(C)$ is given by

$$\gamma_n(C) = i \int_0^T \langle n(t)|\dot{n}(t)\rangle dt = \oint \langle n(R)|\nabla n(R)\rangle dR$$  

(15)
is the gauge-invariant Berry phase that depends only on the geometry of the path in the parameter space and is also measurable [6]. Therefore, we have \( \langle \psi(0)|\psi(T) \rangle = \exp(i\delta_n(T) + \gamma_n(C)) \). However, our solution to MS inconsistency suggest that \( \langle \psi(0)|\psi(T) \rangle = \exp(i\delta_n(T) + i\gamma_n(C)|n(0)\rangle\langle n|n(T) \rangle = \exp(i\delta_n(T)) \). This implies that the initial and the final state differ only by the dynamical phase and there is no observable Berry phase. Hence, a contradiction!

Next, consider the general definition of the Berry phase when a pure state vector undergoes a time evolution \( \langle \psi(0)| \rightarrow |\psi(t) \rangle \). Invoking Pancharatnam’s idea of relative phase shift one can define the geometric phase for non-cyclic evolution of quantum systems [13]. The non-cyclic geometric phase is then given by a gauge invariant functional of \( \psi(t) \) along an open path \( \Gamma \) [14]

\[
\Phi_G[\Gamma] = \text{Arg}\langle \psi(0)|\psi(t) \rangle + i \int dt \langle \psi(t)|\dot{\psi}(t) \rangle \tag{16}
\]

Using the reference-section state vector \( |\chi(t) \rangle \) we can express the general geometric phase during an arbitrary evolution as

\[
\Phi_G[\Gamma] = i \int dt \langle \chi(t)|\dot{\chi}(t) \rangle = \int G, \tag{17}
\]

where the reference-section \( |\chi(t) \rangle = \frac{\langle \psi(t)|\psi(0) \rangle}{|\langle \psi(t)|\psi(0) \rangle|} \psi(t) \rangle \) and \( G = i\langle \chi|d\chi \rangle \) is the generalized gauge potential or connection form [15,16]. Thus, the generalized geometric phase can be written as a line integral of a vector potential in the projective Hilbert space of a quantum system. For differential geometric formulation of the general Berry phase see [15].

Note that the above definition holds irrespective of adiabatic, cyclic, and Schrödinger time evolution. So this is the generalized geometric phase during a time evolution of a quantum system described by a pure state vector. Under adiabatic approximation there is an open-path Berry phase which was introduced in [17] and is given by

\[
\Phi^{(n)}_G[\Gamma] = \text{Arg}\langle n(0)|n(t) \rangle + i \int dt \langle n(t)|\dot{n}(t) \rangle = \int G_n(R).dR, \tag{18}
\]

where \( G_n(R) \) is the generalized gauge potential that gives rise to the adiabatic open-path Berry phase. Under strict adiabatic approximation using our previous solution we find that \( \Phi^{(n)}_G[\Gamma] \approx 0 \). Thus, it almost vanishes for all time! Therefore, it appears that under strict adiabatic evolution a quantum system cannot acquire any Berry phase (non-cyclic as well as cyclic). However, there are many physical systems that show the existence of non-cyclic and cyclic Berry phases under adiabatic approximation. Thus our solution, though resolves Marzlin-Sanders inconsistency, yet points to another important inconsistency.

Hence, a possible way out is not to drop off-diagonal terms that are usually done. One has to be careful when to drop and when not to. The statement that matrix elements causing transition to other eigenstates are dropped under adiabatic approximation could lead to internal inconsistencies either MS type or our type (i.e. the vanishing Berry phase). Thus the source of our inconsistency is dropping of the terms like \( \langle m(t)|\dot{n}(t) \rangle \), for \( m \neq n \).

We have seen that under strict adiabatic approximation the transition amplitude between the initial and the instantaneous eigenstate obeys a linear homogeneous equation that results in a vanishing Berry phase. However, if we investigate the full solution and one can save the adiabatic Berry phase. Note that Eq(4) can be written as

\[
\frac{d}{dt}A_n(t) = \gamma_n(t)A_n(t) + S_n(t), \tag{19}
\]

where \( S_n(t) = \sum_{m\neq n} \langle n(0)|m(t)\rangle \langle m(t)|\dot{n}(t) \rangle \). The solution to the above equation can be written as

\[
A_n(t) = e^{-\gamma_n(t)}[1 - i \int_0^t dt' S_n(t')e^{i\gamma_n(t')}]. \tag{20}
\]

The second term clearly represents the correction to the adiabatic approximation and its presence can only make a non-zero Berry phase in cyclic as well as non-cyclic case. In particular, the generalized non-cyclic adiabatic Berry phase given by Eq(8) can be expressed as

\[
\Phi^{(n)}_G[\Gamma] = \tan^{-1}\left[\frac{-\text{Re}Q_n(t)}{1 + \text{Im}Q_n(t)}\right], \tag{21}
\]

where \( Q_n(t) = \int dt' S_n(t')e^{i\gamma_n(t')} \). This can be shown to be related to the response function of a many body quantum system that explains damping of collective excitations [8].

V. UNITARY OPERATOR FOR ADIABATIC EVOLUTION

We can develop a general solution to the unitary evolution operator and show that it has two pieces; a diagonal and a non-diagonal piece. It is the diagonal piece that gives what we want - adiabatic theorem but there is no inconsistency if we keep the order of approximation in our development.

The time evolution operator of a quantum system obeys

\[
i\dot{U}(t) = H(t)U(t) \tag{22}
\]

with \( U(0) = I \), and the unitarity condition for all \( t \) holds, i.e., \( U(t)U(t)^\dagger = U(t)^\dagger U(t) = I \), where \( I \) is the identity
operator. Since the Hamiltonian is Hermitian for any $t$, it admits an eigen-expansion (for simplicity, we consider here discrete and non-degenerate case) $H(t)|n(t)⟩ = E_n(t)|n(t)⟩$, $⟨n(t)|m(t)⟩ = δ_{nm}$ and $\sum_n |n(t)⟩⟨n(t)| = I$, for all $t$. We can express the unitary operator in terms of these instantaneous eigenstates as

$$U(t) = \sum_{nm} |n(t)⟩U_{nm}(t)⟨m(0)|,$$

where the matrix elements satisfy

$$i\dot{U}_{nm}(t) + E_n(t)U_{nm}(t) + \sum_p i⟨n(t)|\dot{p}(t)|U_{pn}(t) = 0. \quad (24)$$

The condition at $t = 0$ implies that $U_{nm}(0) = δ_{nm} = U_{n0}^*(0)$ and the unitarity relation implies that $\sum_p U_{pn}(t)U_{np}^*(t) = δ_{nm}$ for all $t$. Now let us introduce a “smallness” parameter, $\epsilon$, such that

$$U_{nm}(t) = U_{nn}(t)δ_{nm} + \epsilon \delta U_{nm}(t)(1 - δ_{nm}). \quad (25)$$

Then the unitarity condition to leading order in $\epsilon$, implies $|U_{nn}(t)|^2 = 1$ for all $t$. Hence, $U_{nn}(t) = e^{i\phi_n(t)}$, with $U_{nn}(0) = 1$ or $\phi_n(0) = 0$. The explicit form of the phase to this order is given by the real number

$$\phi_n(t) = -\int_0^t E_n(t')dt' + \int_0^t i⟨n(t')|\dot{\psi}(t')⟩ dt'. \quad (26)$$

The first term is the dynamical phase and the second one gives us the familiar Berry term. The corresponding $U(t)$ is then of the form

$$U(t) = \sum_n e^{i\phi_n(t)}|n(t)⟩⟨n(0)|$$

$$+ \epsilon \sum_{n\neq m} |n(t)⟩δU_{nm}(t)⟨m(0)|. \quad (27)$$

Thus the adiabatic theorem is presented with a consistent form.

The ‘smallness’ parameter actually decides when to keep the off-diagonal terms and when the approximation with diagonal ones is satisfactory. But how small can it be? One can obtain a non-trivial lower bound on the $\epsilon$ as follows. Let the initial state of the system is $|\psi(0)⟩ = |n(0)⟩$. The state at a later time $t$ is $|\psi(t)⟩ = U(t)|n(0)⟩$. Now the transition amplitude between the initial and the final state is

$$⟨\psi(0)|\psi(t)⟩ = ⟨n(0)|U(t)|n(0)⟩ = U_{nn}(t)⟨n(0)|n(t)⟩$$

$$+ \epsilon \sum_{m\neq n} δU_{mn}(t)⟨n(0)|m(t)⟩. \quad (28)$$

This leads to

$$|⟨\psi(0)|\psi(t)⟩|^2 ≤ |⟨n(0)|n(t)⟩|^2 + \epsilon \sum_{m\neq n} |δU_{mn}(t)|.$$

The above inequality can be written as

$$\epsilon \geq \frac{D(|n(0)⟩, |n(t)⟩) - D(|\psi(0)⟩, |\psi(t)⟩)}{\sum_{m\neq n} |δU_{mn}(t)|}, \quad (30)$$

where $D(|n(0)⟩, |n(t)⟩)$ is the ‘minimum-normed’ distance functions between the initial eigenstate and final eigenstate, and $D(|\psi(0)⟩, |\psi(t)⟩)$ is the similar one between the initial state and the final state of the system $[12]$. When we are within adiabatic regime $D(|n(0)⟩, |n(t)⟩)$ and $D(|\psi(0)⟩, |\psi(t)⟩)$ are same, hence we have $\epsilon = 0$. There is no lower bound on it. However, if we move away from adiabatic regime $D(|n(0)⟩, |n(t)⟩)$ and $D(|\psi(0)⟩, |\psi(t)⟩)$ differ. Then we will have a lower bound on the ‘smallness’ parameter.

One can also resolve the MS inconsistency using the unitary operator method. Note that we can obtain $|\tilde{n}(t)⟩ = \sum_p |p(t)⟩U_{pn}^*(t) = U_{n0}^*(t)|n(0)⟩$+o-diagonal terms, and thus as before we have $|\tilde{n}(t)⟩ \approx \exp(i \int E_n(t')dt') \exp(-i\gamma_n(t)|n(0)⟩$. Then using $⟨n(0)|n(t)⟩ \approx \exp(-i\gamma_n(t))$ one can resolve the MS inconsistency. Below we give two examples to illustrate the power of unitary operator method developed in this letter.

VI. TWO EXAMPLES

Here we consider two examples that will illustrate the main points.

A. The MS Example in the instantaneous representation

Let us consider the precession of a spin-half particle in a magnetic field of strength proportional to $\omega_0$ and in addition it rotates in the $x - y$ plane with a frequency $\Omega = 2\pi/τ$. This is also considered by MS [11]. We work here in the instantaneous representation. The Hamiltonian is given by $H(t) = R(t)\sigma$, where

$$H(t) = \left( \begin{array}{cc} \frac{Ω}{2}(1 - \cos 2\omega_0t) & -i\frac{Ω}{2}(\omega_0 - i\frac{Ω}{2}\sin 2\omega_0t) \\ e^{i\Omega t}(\omega_0 + i\frac{Ω}{2}\sin 2\omega_0t) & -\frac{Ω}{2}(1 - \cos 2\omega_0t) \end{array} \right). \quad (32)$$

Its eigenvalues and eigenfunctions are $E_1(t) = \sqrt{\omega_0^2 + \Omega^2 \sin^2 \omega_0t}$, $E_2(t) = \omega_0$; $|n_1(t)⟩ = \cos(a_1(t), b_1(t))$, $|n_1(0)⟩ = 1/\sqrt{2}\cos(1, 1)$, with

$$a_1(t) = \sqrt{\frac{E_0(t) + \Omega \sin^2 \omega_0t}{2E_0(t)}} e^{-i(Ωt+θ(t))/2}$$

$$b_1(t) = \sqrt{\frac{E_0(t) - \Omega \sin^2 \omega_0t}{2E_0(t)}} e^{i(Ωt+θ(t))/2}. \quad (31)$$
And similarly, we have $E_2(t) = -\varepsilon_0(t)$, $E_2(0) = -\omega_0$; 
$|n_2(t)\rangle = \text{col}(a_2(t), b_2(t))$, $|n_2(0)\rangle = 1/\sqrt{2}\text{col}(-1, 1)$, with 
\[
a_2(t) = -\sqrt{\frac{\varepsilon_0(t) - \Omega \sin^2 \omega_0 t}{2\varepsilon_0(t)}} e^{-i(\Omega t + \theta(t))/2},
\]
\[
b_2(t) = \sqrt{\frac{\varepsilon_0(t) + \Omega \sin^2 \omega_0 t}{2\varepsilon_0(t)}} e^{i(\Omega t + \theta(t))/2}.
\]
(32)
In the above equations, $\theta(t) = \tan^{-1}(\Omega/2\omega_0)\sin 2\omega_0 t$.
The unitary time evolution associated with the Hamiltonian $H(t)$ is given by
\[
U(t) = \begin{pmatrix} \cos \omega_0 t & -ie^{-i\Omega t} \sin \omega_0 t \\ -ie^{i\Omega t} \sin \omega_0 t & \cos \omega_0 t \end{pmatrix}.
\]
We need to express this in terms of the instantaneous eigenstates to isolate the adiabatic term from the non-adiabatic pieces. We present here the diagonal terms, being the adiabatic contributions to the evolutions: 
\[
\langle n_1(t)|U(t)|n_1(0)\rangle = \frac{1}{\sqrt{2}}[a_1(t)(\cos \omega_0 t - ie^{-i\Omega t} \sin \omega_0 t) + b_1(t)(\cos \omega_0 t - ie^{i\Omega t} \sin \omega_0 t)] 
\]
and 
\[
\langle n_2(t)|U(t)|n_2(0)\rangle = \frac{1}{\sqrt{2}}[a_2(t)(\cos \omega_0 t + ie^{-i\Omega t} \sin \omega_0 t) - b_2(t)(\cos \omega_0 t + ie^{i\Omega t} \sin \omega_0 t)].
\]
These exact solutions can be evaluated explicitly and compared with the solutions based on adiabatic approximation.

B. The Schwinger example

This example is an elegant one originally due to Schwinger [5] for describing the precession of a spin in a transverse time-dependent field. Here the Hamiltonian of the system is given by $H(t) = -g\mu_B (\sigma_x \sin \theta \cos \phi + \sigma_y \sin \theta \sin \phi + \sigma_z \cos \theta) = -g\mu_B \left( \begin{array}{cc} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{array} \right)$, where the magnetic field $H$ and $\theta$ are independent of time, and $\phi = \omega t$. The MS model is a version of the Schwinger’s NMR precession problem described above. The instantaneous eigenstates are $E_1 = g\mu_B H$ and $E_2 = -g\mu_B H$. The instantaneous eigenstates are $|n_1(t)\rangle = \text{col}(e^{-i\phi/2} \sin \theta/2, -e^{i\phi/2} \cos \theta/2)$ and $|n_2(t)\rangle = \text{col}(e^{-i\phi/2} \cos \theta/2, e^{i\phi/2} \sin \theta/2)$. From these we have $\langle n_1(t)|\dot{n}_1(0)\rangle = i(\omega \sin \theta)/2 = \langle n_2(t)|\dot{n}_2(0)\rangle$. Similarly, we have $\langle n_1(t)|\dot{n}_2(0)\rangle = -i(\omega \sin \theta)/2 = \langle n_2(t)|\dot{n}_1(0)\rangle$.
The solution to the time evolution operator $U(t)$ in terms of the instantaneous eigenstates can be obtained as $U(t) = \sum_{i,j=1}^{2} |n_i(t)\rangle U_{ij}(t)|n_j(0)\rangle$ with $U_{ij}(0) = \delta_{ij}$. The equations to be solved are the matrix elements of the time evolution operator obey Eq.(13).
\[
iU_{11} + i\langle n_1|\dot{n}_2\rangle U_{21} = (E_1 - i\langle n_1|\dot{n}_1\rangle) U_{11}
\]
\[
iU_{21} + i\langle n_2|\dot{n}_1\rangle U_{11} = (E_2 - i\langle n_2|\dot{n}_2\rangle) U_{21}.
\]
(33)
A similar pair of equations for the other two. We can solve these equations by Laplace transforms and obtain the solutions: The solutions are given by $U_{11}(t) = \frac{[\tilde{E}_1 \cos \tilde{E}_1 t - i(g\mu_B H + \frac{\omega}{2} \cos \theta) \sin \tilde{E}_1 t]/\tilde{E}_1 = U_{22}(t)$
\[
U_{21}(t) = i[\omega \sin \theta \sin \tilde{E}_1 t/2\tilde{E}_1 = U_{12}(t),
\]
(34)
where $\tilde{E}_1 = [(g\mu_B H)^2 + g\mu_B H \omega \cos \theta + \omega^2/4]^{1/2}$. The unitarity of the this $U$-matrix is obeyed, as it should be. These are the exact solutions of the Schwinger problem.

If we make the adiabatic approximation, we would drop the $U_{21}$ term in Eq.(22) and obtain the following result:
\[
U_{11}(t) = \exp[-it(g\mu_B H + \frac{\omega}{2} \cos \theta)] = U_{22}(t)
\]
\[
U_{21}(t) = i\frac{\omega}{2} \sin \theta \frac{\sin t((\omega/2) \cos \theta + g\mu_B H)}{(i(\omega/2) \cos \theta + g\mu_B H)} = U_{12}(t).
\]
(35)
This means that we drop terms of the order of $(\omega/2) \sin \theta$ in the exact expressions, so that $\tilde{E}_1 \approx [g\mu_B H + (\omega/2) \cos \theta]$. We thus see that the “smallness” parameter in the “adiabatic” treatment is $(n_1(t)|\dot{n}_2(0)\rangle$. Note that the unitarity of this $U(t)$ is verified to be obeyed consistent with the adiabatic approximation stated above. Also, a way to see the departure of the result in the adiabatic approximation from the exact result is to compute the overlap $A\langle n_1(t)|n_1(0)\rangle$ and hence the fidelity, $F(t) = |A\langle n_1(t)|n_1(0)\rangle|^2$. The overlap is given by $A\langle n_1(t)|n_1(0)\rangle = \exp(it\Omega)(\cos E_1 t - i(\Omega/\tilde{E}_1) \sin E_1 t) + (\omega \sin \theta/2)(\sin \tilde{E}_1 t/\tilde{E}_1)(\sin \tilde{\Omega}/\tilde{\Omega})$, where $\tilde{\Omega} = (g\mu_B H + (\omega/2) \cos \theta)$. One sees at once that to the leading order in $(\omega \sin \theta/2)$, the fidelity is unity.

VII. CONCLUSIONS

Adiabatic theorem though widely studied, can reveal surprising phenomena, and sometime even apparent inconsistencies, namely, MS type or vanishing Berry phase type. Thus, it is of utmost importance to know where such inconsistencies may arise. Besides showing how to resolve these inconsistencies associated with the adiabatic theorem (AT), we have presented here several novel results. We have established an equation of motion for the transition amplitude leading to the Berry phase, developed the unitary time evolution operator expressed in instantaneous basis as a powerful tool in AT as illustrated with two examples, estimated the “smallness” parameter in the unitary operator, and quantified the departure of AT from the exact by means of “Fidelity” expressed in terms of the respective unitary evolution operators. Given the current interest in the geometric phases for mixed states [18], this work opens up the possibility of studying the Berry phase and adiabatic theorem for density operators and entangled quantum systems in a new perspective which one may like to take up in future.
Note Added: After completion of our work we noticed Ref. [19] which addresses the MS inconsistency. Our present work goes beyond this aspect of the adiabatic theorem.

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