Asymptotic structure of general metric spaces at infinity

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Abstract

Let \((X, d)\) be an unbounded metric space and \(\tilde{r} = (r_n)_{n \in \mathbb{N}}\) be a scaling sequence of positive real numbers tending to infinity. We define the pretangent space \(\Omega^X_{\infty, \tilde{r}}\) to \((X, d)\) at infinity as a metric space whose points are equivalence classes of sequences \((x_n)_{n \in \mathbb{N}} \subset X\) which tend to infinity with the speed of \(\tilde{r}\). It is proved that the pretangent spaces \(\Omega^X_{\infty, \tilde{r}}\) are complete for every unbounded metric space \((X, d)\) and every scaling sequence \(\tilde{r}\). The finiteness conditions of \(\Omega^X_{\infty, \tilde{r}}\) are found.

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1 Introduction

Under the asymptotic structure of an unbounded metric space \((X, d)\) at infinity we mean the set of metric spaces which are the limits of rescaling metric spaces \(\left(X, \frac{1}{r_n}d\right)\) for \(r_n\) tending to infinity. The Gromov–Hausdorff convergence and the asymptotic cones are most often used for construction of such limits. Both of these approaches are based on higher-order abstractions (see, for example, [17] for details), which makes them very powerful, but it does away the constructiveness. In this paper we propose a more elementary, sequential approach for describing the asymptotic structure of unbounded metric spaces at infinity.

Let \((X, d)\) be a metric space and let \(\tilde{r} = (r_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers with \(\lim_{n \to \infty} r_n = \infty\). In what follows \(\tilde{r}\) will be called a scaling sequence and the formula \((x_n)_{n \in \mathbb{N}} \subset A\) will be mean that all elements of the sequence \((x_n)_{n \in \mathbb{N}}\) belong to the set \(A\).
Definition 1.1. Two sequences $\tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X$ and $\tilde{y} = (y_n)_{n \in \mathbb{N}} \subset X$ are mutually stable with respect to the scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ if there is a finite limit
\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_\tilde{r}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}
\]

Let $p \in X$. Denote by $\text{Seq}(X, \tilde{r})$ the set of all sequences $\tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X$ for which there is a finite limit
\[
\lim_{n \to \infty} \frac{d(x_n, p)}{r_n} := \tilde{d}_p(\tilde{x}) \tag{1.2}
\]
and such that $\lim_{n \to \infty} d(x_n, p) = \infty$.

Definition 1.2. A set $F \subseteq \text{Seq}(X, \tilde{r})$ is self-stable if any two $\tilde{x}, \tilde{y} \in F$ are mutually stable. $F$ is maximal self-stable if it is self-stable and, for arbitrary $\tilde{y} \in \text{Seq}(X, \tilde{r})$, we have either $\tilde{y} \in F$ or there is $\tilde{x} \in F$ such that $\tilde{x}$ and $\tilde{y}$ are not mutually stable.

The maximal self-stable subsets of $\text{Seq}(X, \tilde{r})$ will be denoted as $\tilde{X}_\infty, \tilde{r}$.

Remark 1.3. If $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \text{Seq}(X, \tilde{r})$ and $p, b \in X$, then the triangle inequality implies
\[
\lim_{n \to \infty} \frac{d(x_n, p)}{r_n} = \lim_{n \to \infty} \frac{d(x_n, b)}{r_n}. \tag{1.3}
\]

In particular, $\text{Seq}(X, \tilde{r})$, the self-stable subsets and the maximal self-stable subsets of $\text{Seq}(X, \tilde{r})$ are invariant w.r.t. the choosing a point $p \in X$ in $\text{Seq}(X, \tilde{r})$.

Consider a function $\tilde{d} : \tilde{X}_\infty, \tilde{r} \times \tilde{X}_\infty, \tilde{r} \to \mathbb{R}$ satisfying (1.1) for all $\tilde{x}, \tilde{y} \in \tilde{X}_\infty, \tilde{r}$. Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for $\tilde{d}$ gives us the triangle inequality for $\tilde{d}$,
\[
\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y}).
\]

Hence $(\tilde{X}_\infty, \tilde{r}, \tilde{d})$ is a pseudometric space.

Now we are ready to define the main object of our research.

Definition 1.4. Let $(X, d)$ be an unbounded metric space, let $\tilde{r}$ be a scaling sequence and let $\tilde{X}_\infty, \tilde{r}$ be a maximal self-stable subset of $\text{Seq}(X, \tilde{r})$. The pretangent space to $(X, d)$ (at infinity, with respect to $\tilde{r}$) is the metric identification of the pseudometric space $(\tilde{X}_\infty, \tilde{r}, \tilde{d})$. 

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Since the notion of pretangent space is basic for the paper, we recall the metric identification construction. Define a relation \( \equiv \) on \( \text{Seq}(X, \tilde{r}) \) as
\[
(\tilde{x} \equiv \tilde{y}) \iff \left( \tilde{d}_\tilde{r}(\tilde{x}, \tilde{y}) = 0 \right).
\] (1.4)
The reflexivity and the symmetry of \( \equiv \) are evident. Let \( \tilde{x}, \tilde{y}, \tilde{z} \in \text{Seq}(X, \tilde{r}) \) and \( \tilde{x} \equiv \tilde{y} \) and \( \tilde{y} \equiv \tilde{z} \). Then the inequality
\[
\limsup_{n \to \infty} \frac{d(x_n, z_n)}{r_n} \leq \lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} + \lim_{n \to \infty} \frac{d(y_n, z_n)}{r_n}
\]
implies \( \tilde{x} \equiv \tilde{z} \). Thus \( \equiv \) is an equivalence relation.

Write \( \Omega^X_{\infty, \tilde{r}} \) for the set of equivalence classes generated by the restriction of \( \equiv \) on the set \( \tilde{X}_{\infty, \tilde{r}} \). Using general properties of pseudometric spaces we can prove (see, for example, [12]) that the function \( \rho: \Omega^X_{\infty, \tilde{r}} \times \Omega^X_{\infty, \tilde{r}} \to \mathbb{R} \) with
\[
\rho(\alpha, \beta) := \tilde{d}_\tilde{r}(\tilde{x}, \tilde{y}), \quad \tilde{x} \in \alpha \in \Omega^X_{\infty, \tilde{r}}, \quad \tilde{y} \in \beta \in \Omega^X_{\infty, \tilde{r}},
\] (1.5)
is a well-defined metric on \( \Omega^X_{\infty, \tilde{r}} \). The metric identification of \( (\tilde{X}_{\infty, \tilde{r}}, \tilde{d}) \) is the metric space \( (\Omega^X_{\infty, \tilde{r}}, \rho) \).

Let \( (n_k)_{k \in \mathbb{N}} \subset \mathbb{N} \) be a strictly increasing sequence. Denote by \( \tilde{r}' \) the subsequence \( (r_{n_k})_{k \in \mathbb{N}} \) of the scaling sequence \( \tilde{r} = (r_n)_{n \in \mathbb{N}} \) and, for every \( \tilde{x} = (x_n)_{n \in \mathbb{N}} \in \text{Seq}(X, \tilde{r}) \), write \( \tilde{x}' := (x_{n_k})_{k \in \mathbb{N}} \). It is clear that
\[
\{ \tilde{x}' \in \text{Seq}(X, \tilde{r}') \} \subseteq \text{Seq}(X, \tilde{r}')
\]
and \( \tilde{d}_{\tilde{r}'}(\tilde{x}') = \tilde{d}_\tilde{r}(\tilde{x}) \) holds for every \( \tilde{x} \in \text{Seq}(X, \tilde{r}) \). Moreover, if sequences \( \tilde{x}, \tilde{y} \in \text{Seq}(X, \tilde{r}) \) are mutually stable w.r.t. \( \tilde{r} \), then \( \tilde{x}' \) and \( \tilde{y}' \) are mutually stable w.r.t. \( \tilde{r}' \) and
\[
\tilde{d}_\tilde{r}(\tilde{x}, \tilde{y}) = \tilde{d}_\tilde{r} (\tilde{x}', \tilde{y}').
\] (1.6)
By Zorn’s lemma, for every \( \tilde{X}_{\infty, \tilde{r}} \subseteq \text{Seq}(X, \tilde{r}) \), there is \( \tilde{X}_{\infty, \tilde{r}'} \subseteq \text{Seq}(X, \tilde{r}') \) such that
\[
\{ \tilde{x}' : \tilde{x} \in \tilde{X}_{\infty, \tilde{r}} \} \subseteq \tilde{X}_{\infty, \tilde{r}'}.
\] (1.7)
Denote by \( \varphi_{\tilde{r}'} \) the mapping from \( \tilde{X}_{\infty, \tilde{r}} \) to \( \tilde{X}_{\infty, \tilde{r}'} \) with \( \varphi_{\tilde{r}'}(\tilde{x}) = \tilde{x}' \) for \( \tilde{x} \in \tilde{X}_{\infty, \tilde{r}} \). It follows from (1.6) that, after metric identifications, the mapping \( \varphi_{\tilde{r}'}: \tilde{X}_{\infty, \tilde{r}} \to \tilde{X}_{\infty, \tilde{r}'} \) passes to an isometric embedding \( \text{em}' : \Omega^X_{\infty, \tilde{r}} \to \Omega^X_{\infty, \tilde{r}'} \) such that the diagram
\[
\begin{array}{ccc}
\tilde{X}_{\infty, \tilde{r}} & \xrightarrow{\varphi_{\tilde{r}'}} & \tilde{X}_{\infty, \tilde{r}'} \\
\pi & \downarrow & \pi' \\
\Omega^X_{\infty, \tilde{r}} & \xrightarrow{\text{em}'} & \Omega^X_{\infty, \tilde{r}'}
\end{array}
\] (1.8)
is commutative. Here $\pi$ and $\pi'$ are the corresponding natural projections,

$$
\pi(\tilde{x}) := \{ \tilde{y} \in \tilde{X}_\infty, \tilde{r} : d_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0 \}, \quad \pi'(\tilde{t}) := \{ \tilde{y} \in \tilde{X}_\infty, \tilde{r}' : d_{\tilde{r}'}(\tilde{t}, \tilde{y}) = 0 \}
$$

for all $\tilde{x} \in \tilde{X}_\infty, \tilde{r}$ and $\tilde{t} \in \tilde{X}_\infty, \tilde{r}'$.

**Definition 1.5.** Let $(X, d)$ be an unbounded metric space and let $\tilde{r}$ be a scaling sequence. A pretangent $\Omega^X_{\infty, \tilde{r}}$ is tangent if $em': \Omega^X_{\infty, \tilde{r}} \rightarrow \Omega^X_{\infty, \tilde{r}'}$ is surjective for every $\tilde{X}_\infty, \tilde{r}'$.

It is can be proved that the following statements are equivalent.

- The metric space $\Omega^X_{\infty, \tilde{r}}$ is tangent.
- The mapping $em' : \Omega^X_{\infty, \tilde{r}} \rightarrow \Omega^X_{\infty, \tilde{r}'}$ is an isometry for every $\tilde{r}'$.
- The set $\{ \tilde{x}' : \tilde{x} \in \tilde{X}_\infty, \tilde{r} \}$ is a maximal self-stable subset of $Seq(X, \tilde{r}')$ for every $\tilde{r}'$.
- The mapping $\varphi_{\tilde{r}'} : \tilde{X}_\infty, \tilde{r} \rightarrow \tilde{X}_\infty, \tilde{r}'$ is onto for every $\tilde{r}'$.

In conclusion of this brief introduction we note that there exist other techniques which allow to investigate the asymptotic properties of metric spaces at infinity. As examples, we mention only the Gromov product which can be used to define a metric structure on the boundaries of hyperbolic spaces [9], [18], the balleans theory [16] and the Wijsman convergence [14], [21], [22].

### 2 Basic properties of pretangent spaces

Let us denote by $\tilde{X}_\infty$ the set of all sequences $(x_n)_{n \in \mathbb{N}} \subset X$ satisfying the limit relation $\lim_{n \to \infty} d(x_n, p) = \infty$ with $p \in X$. It is clear that $Seq(X, \tilde{r}) \subseteq \tilde{X}_\infty$ holds for every scaling sequence $\tilde{r}$.

**Proposition 2.1.** Let $(X, d)$ be an unbounded metric space. Then the following statements hold.

(i) The set $Seq(X, \tilde{r})$ is nonempty for every scaling sequence $\tilde{r}$.

(ii) For every $\tilde{x} \in \tilde{X}_\infty$, there exists a scaling sequence $\tilde{r}$ such that $\tilde{x} \in Seq(X, \tilde{r})$.

**Proof.** (i) Let $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ be a scaling sequence and let $p \in X$. Let us denote by $B(p, r_{\frac{1}{n}})$ the closed ball

$$
\{ x \in X : d(x, p) \leq r_{\frac{1}{n}} \}.
$$


Write
\[ k_n := \sup \{ d(x, p) : x \in B(p, r_n^{\frac{1}{2}}) \} \] \hfill (2.2)

\( n = 1, 2, \ldots \). We can find \( \tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X \) such that
\[ \lim_{n \to \infty} \frac{k_n}{d(x_n, p)} = 1. \] \hfill (2.3)

Since \( X \) is unbounded, the limit relation \( \lim_{n \to \infty} k_n = \infty \) holds. Consequently \( \lim_{n \to \infty} d(x_n, p) = \infty \), i.e., \( \tilde{x} \in \tilde{X}_\infty \). It follows from (2.1) and (2.2) that the inequality \( k_n \leq r_n^{\frac{1}{2}} \) holds for every \( n \in \mathbb{N} \). The last inequality and (2.3) imply \( \lim_{n \to \infty} \frac{d(x_n, p)}{r_n} = 0 \). Thus, \( \tilde{x} \in \text{Seq}(X, \tilde{\rho}) \).

(ii) Let \( \tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}_\infty \) and let \( p \in X \). Then \( \lim_{n \to \infty} d(x_n, p) = \infty \) holds. Define a sequence \( \tilde{\rho} = (r_n)_{n \in \mathbb{N}} \) as
\[ r_n := \begin{cases} d(x_n, p), & \text{if } x_n \neq p \\ 1, & \text{if } x_n = p \end{cases} \]
for \( n \in \mathbb{N} \). From \( \tilde{x} \in \tilde{X}_\infty \) it follows that \( \lim_{n \to \infty} r_n = \infty \). Hence \( \tilde{\rho} \) is a scaling sequence. It is clear that \( d_{\tilde{\rho}}(\tilde{x}) = 1 \). Thus, \( \text{Seq}(X, \tilde{\rho}) \ni \tilde{x} \). \( \square \)

For every unbounded metric space \( (X, d) \) and every scaling sequence \( \tilde{\rho} \) define the subset \( \tilde{X}^0_{\infty, \tilde{\rho}} \) of the set \( \text{Seq}(X, \tilde{\rho}) \) by the rule:
\[ (z_n)_{n \in \mathbb{N}} \in \tilde{X}^0_{\infty, \tilde{\rho}} \iff (z_n)_{n \in \mathbb{N}} \in \tilde{X}_\infty \quad \text{and} \quad \lim_{n \to \infty} \frac{d(z_n, p)}{r_n} = 0 \], \hfill (2.4)

where \( p \) is a point of \( X \).

Below we collect together some basic properties of the set \( \tilde{X}^0_{\infty, \tilde{\rho}} \).

**Proposition 2.2.** Let \( (X, d) \) be an unbounded metric space and let \( \tilde{\rho} \) be a scaling sequence. Then the following statements hold.

(i) The set \( \tilde{X}^0_{\infty, \tilde{\rho}} \) is nonempty.

(ii) If \( \tilde{z} \in \tilde{X}^0_{\infty, \tilde{\rho}}, \tilde{y} \in \tilde{X}_\infty \) and \( d_{\tilde{\rho}}(\tilde{z}, \tilde{y}) = 0 \), then \( \tilde{y} \in \tilde{X}^0_{\infty, \tilde{\rho}} \) holds.

(iii) If \( F \subseteq \text{Seq}(X, \tilde{\rho}) \) is self-stable, then \( \tilde{X}^0_{\infty, \tilde{\rho}} \cup F \) is also a self-stable subset of \( \text{Seq}(X, \tilde{\rho}) \).

(iv) The set \( \tilde{X}^0_{\infty, \tilde{\rho}} \) is self-stable.
(v) The inclusion $\tilde{X}_0^0,\tilde{r} \subseteq \tilde{X}_\infty,\tilde{r}$ holds for every maximal self-stable subset $\tilde{X}_\infty,\tilde{r}$ of $\text{Seq}(X, \tilde{r})$.

(vi) Let $\tilde{z} \in \tilde{X}_\infty,\tilde{r}$ and $\tilde{x} \in \tilde{X}_\infty$. Then $\tilde{x} \in \text{Seq}(X, \tilde{r})$ if and only if $\tilde{x}$ and $\tilde{z}$ are mutually stable. For $\tilde{x} \in \text{Seq}(X, \tilde{r})$ we have

$$\tilde{d}_r(\tilde{x}, \tilde{z}) = \tilde{d}_r(\tilde{x}) + \tilde{d}_r(\tilde{z}).$$

(vii) Denote by $\Omega_\infty^X,\tilde{r}$ the set of all pretangent to $X$ at infinity (with respect to $\tilde{r}$) spaces. Then the membership relation

$$\tilde{X}_\infty,\tilde{r} \in \bigcap_{\Omega_\infty^X,\tilde{r}} \Omega_\infty^X,\tilde{r}$$

holds.

Proof. (i) It follows from the proof of statement (i) in Proposition 2.1.

(ii) To prove $\tilde{y} \in \tilde{X}_\infty,\tilde{r}$ note that

$$0 \leq \limsup_{n \to \infty} \frac{d(y_n, p)}{r_n} \leq \tilde{d}_r(\tilde{z}, \tilde{y}) + \tilde{d}_r(\tilde{z}) = 0.$$ 

(iii) Let $p \in X$ and let $F \subseteq \text{Seq}(X, \tilde{r})$ be self-stable. It is clear that $\tilde{d}_r(\tilde{y})$ exists for every $\tilde{y} \in F \cup \tilde{X}_\infty,\tilde{r}$. Hence $F \cup \tilde{X}_\infty,\tilde{r}$ is self-stable if and only if $\tilde{z}$ and $\tilde{x}$ are mutually stable for all $\tilde{x}, \tilde{z} \in F \cup \tilde{X}_\infty,\tilde{r}$. If $\tilde{z}, \tilde{x} \in F$, then $\tilde{z}$ and $\tilde{x}$ are mutually stable by the condition. Suppose $\tilde{x} \in F$ and $\tilde{z} \in \tilde{X}_\infty,\tilde{r}$. The inequalities

$$d(x_n, p) - d(z_n, p) \leq d(x_n, z_n) \leq d(x_n, p) + d(z_n, p)$$

and the equality

$$\lim_{n \to \infty} \frac{d(z_n, p)}{r_n} = 0$$

imply the existence of $\tilde{d}_r(\tilde{x}, \tilde{z})$. The case $\tilde{x}, \tilde{z} \in \tilde{X}_\infty,\tilde{r}$ is similar. Thus the set $F \cup \tilde{X}_\infty,\tilde{r}$ is self-stable.

(iv) This follows from (iii) with $F = \emptyset$.

(v) Using statement (iii) with $F = \tilde{X}_\infty,\tilde{r}$ we see that $\tilde{X}_\infty,\tilde{r} \cup \tilde{X}_\infty,\tilde{r}$ is self-stable. Since $\tilde{X}_\infty,\tilde{r}$ is maximal self-stable, the equality $\tilde{X}_\infty,\tilde{r} \cup \tilde{X}_\infty,\tilde{r} = \tilde{X}_\infty,\tilde{r}$ holds. Thus, $\tilde{X}_\infty,\tilde{r} \subseteq \tilde{X}_\infty,\tilde{r}$.

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(vi) Suppose \( \tilde{x} \) and \( \tilde{z} \) are mutually stable. Then using (2.5) we obtain
\[
\limsup_{n \to \infty} \frac{d(x_n, p)}{r_n} \leq \tilde{d}_r(\tilde{x}, \tilde{z}) \leq \liminf_{n \to \infty} \frac{d(x_n, p)}{r_n}.
\]
Hence \( \tilde{x} \in Seq(X, \hat{r}) \). Similarly, if \( \tilde{x} \in Seq(X, \hat{r}) \), then we have
\[
\limsup_{n \to \infty} \frac{d(x_n, z_n)}{r_n} \leq \tilde{d}_r(\tilde{x}) \leq \liminf_{n \to \infty} \frac{d(x_n, z_n)}{r_n}.
\]
Consequently \( \tilde{x} \) and \( \tilde{z} \) are mutually stable and \( \tilde{d}_r(\tilde{x}) = \tilde{d}_r(\tilde{x}, \tilde{z}) \) holds.

(vii) It follows from (ii), (v) and the definition of pretangent spaces.

\[
\text{Remark 2.3. The set } \hat{X}^0_{\infty, \hat{r}} \text{ is invariant under replacing of } p \in X \text{ by an arbitrary point } b \in X \text{ in (2.4).}
\]

\[
\text{Lemma 2.4. Let } (X, d) \text{ be an unbounded metric space, } p \in X \text{ and } \hat{y} \in \hat{X}_\infty, \text{ let } \hat{r} \text{ be a scaling sequence and let } \hat{X}_{\infty, \hat{r}} \text{ be a maximal self-stable set. If } \hat{y} \text{ and } \hat{x} \text{ are mutually stable for every } \hat{x} \in \hat{X}_{\infty, \hat{r}}, \text{ then } \hat{y} \in \hat{X}_{\infty, \hat{r}}.
\]

\[
\text{Proof. Suppose } \hat{y} \text{ and } \hat{x} \text{ are mutually stable for every } \hat{x} \in \hat{X}_{\infty, \hat{r}}. \text{ To prove } \hat{y} \in \hat{X}_{\infty, \hat{r}}, \text{ it suffices to show that there is a finite limit } \lim_{n \to \infty} \frac{d(y_n, p)}{r_n} \text{ that follows from statement (vi) of Proposition 2.2.}
\]

\[
\text{Lemma 2.5. Let } (X, d) \text{ be an unbounded metric space and let } \hat{r} \text{ be a scaling sequence. If } \hat{x}, \hat{y}, \hat{t} \in \hat{X}_\infty \text{ such that } \hat{x} \text{ and } \hat{y} \text{ are mutually stable with respect to } \hat{r} \text{ and } \hat{d}_r(\hat{x}, \hat{t}) = 0, \text{ then } \hat{y} \text{ and } \hat{t} \text{ are mutually stable with respect to } \hat{r}.
\]

\[
\text{Proof. The statement follows from the equality } \hat{d}_r(\hat{x}, \hat{y}) = 0 \text{ and the inequalities}
\]
\[
\hat{d}_r(\hat{x}, \hat{y}) - \hat{d}_r(\hat{x}, \hat{t}) \leq \liminf_{n \to \infty} \frac{d(y_n, t_n)}{r_n} \leq \limsup_{n \to \infty} \frac{d(y_n, t_n)}{r_n} \leq \hat{d}_r(\hat{x}, \hat{y}) + \hat{d}_r(\hat{x}, \hat{t}).
\]

\[
\text{The set } \hat{X}^0_{\infty, \hat{r}} \text{ is a common distinguished point of all pretangent spaces } \Omega_{\infty, \hat{r}}^X \text{ (with given scaling sequence } \hat{r}). \text{ We will consider the pretangent spaces to } (X, d) \text{ at infinity as the triples } (\Omega_{\infty, \hat{r}}^X, \rho, \nu_0), \text{ where } \rho \text{ is defined by (1.5) and } \nu_0 := \hat{X}^0_{\infty, \hat{r}}. \text{ The point } \nu_0 \text{ can be informally described as follows. The points of pretangent space } \Omega_{\infty, \hat{r}}^X, \text{ are infinitely removed from the initial space}
\]
(X, d), but \( \Omega_{\infty, \tilde{r}}^X \) contains a unique point \( \nu_0 \) which is close to \( (X, d) \) as much as possible.

Let \( (X, d) \) be an unbounded metric space and let \( \tilde{r} \) be a scaling sequence. Write \( \Omega_{\tilde{r}, \infty}^X \) for the set of the equivalence classes generated by the relation \( \equiv \) on the set \( Seq(X, \tilde{r}) \) (see (1.2)). Let us consider the simple graph \( G_{X, \tilde{r}} \) consisting of the vertex set \( V(G_{X, \tilde{r}}) := \Omega_{\tilde{r}, \infty}^X \) and the edge set \( E(G_{X, \tilde{r}}) \) defined by the rule:

\[
\text{u and v are adjacent if and only if } u \neq v \text{ and } \lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} \text{ exists for } \hat{x} \in u \text{ and } \hat{y} \in v.
\]

Recall that a clique in a graph \( G = (V, E) \) is a set \( C \subseteq V \) such that every two distinct vertices of \( C \) are adjacent. A maximal clique is a clique \( C_1 \) such that the inclusion

\[
V(C_1) \subseteq V(C)
\]

implies the equality \( V(C_1) = V(C) \) for every clique \( C \) in \( G \).

**Theorem 2.6.** Let \( (X, d) \) be an unbounded metric space and let \( \tilde{r} \) be a scaling sequence. A set \( C \subseteq \Omega_{\tilde{r}, \infty}^X \) is a maximal clique in \( G_{X, \tilde{r}} \) if and only if there is a pretangent spaces \( \Omega_{\infty, \tilde{r}}^X \) such that \( C = \Omega_{\infty, \tilde{r}}^X \).

**Proof.** Lemma 2.4 and Lemma 2.5 imply the equality

\[
\{ \hat{x} \in \hat{X}_{\infty, \tilde{r}} : \hat{d}_{\tilde{r}}(\hat{x}, \hat{y}) = 0 \} = \{ \hat{x} \in Seq(X, \tilde{r}) : \hat{d}_{\tilde{r}}(\hat{x}, \hat{y}) = 0 \} \quad (2.6)
\]

for every \( \hat{y} \in \hat{X}_{\infty, \tilde{r}} \) and every \( \hat{X}_{\infty, \tilde{r}} \). Since, for every \( \hat{y} \in Seq(X, \tilde{r}) \), there is \( \hat{X}_{\infty, \tilde{r}} \) such that \( \hat{X}_{\infty, \tilde{r}} \ni \hat{y} \), equality (2.6) implies

\[
\Omega_{\infty, \tilde{r}}^X = \bigcup_{\Omega_{\infty, \tilde{r}}^X \in \Omega_{\infty, \tilde{r}}^X} \Omega_{\infty, \tilde{r}}^X, \quad (2.7)
\]

where \( \Omega_{\infty, \tilde{r}}^X \) is the set of all spaces which are pretangent to \( X \) at infinity with respect to \( \tilde{r} \). Now the theorem follows from the definitions of the pretangent spaces and the maximal cliques.

Theorem 2.6 gives some grounds for calling the graph \( G_{X, \tilde{r}} \) a net of pretangent spaces.

In the next proposition we follow terminology used in [7]. Recall only that a vertex \( v \) of a graph \( G = (V, E) \) is a dominating vertex if \( \{u, v\} \in E \) for all \( u \in V \setminus \{v\} \).
Proposition 2.7. Let \((X,d)\) be an unbounded metric space and let \(\tilde{r}\) be a scaling sequence. Then the following statements hold.

(i) The vertex \(\nu_0 = \tilde{X}_\infty^0,\tilde{r}\) is a dominating vertex of the graph \(G_{X,\tilde{r}}\).

(ii) \(G_{X,\tilde{r}}\) is complete if and only if there is a unique pretangent space \(\Omega_{\infty,\tilde{r}}^X\).

(iii) \(G_{X,\tilde{r}}\) is a star if and only if

\[
\sup\{|\Omega_{\infty,\tilde{r}}^X| : \Omega_{\infty,\tilde{r}}^X \in \Omega_{\infty,\tilde{r}}^X\} = 2,
\]

where \(|\Omega_{\infty,\tilde{r}}^X|\) is the cardinal number of \(\Omega_{\infty,\tilde{r}}^X\).

The proof is simple. Note only that (i) follows from statement (vii) of Proposition 2.2.

3 Completeness of pretangent spaces

It is well known that the Gromov–Hausdorff limits and the asymptotic cones of metric spaces are always complete. The quasi-metrics on the boundaries of hyperbolic spaces are also complete (see, for example, Proposition 6.1 in [18]). The goal of this section is to show that every pretangent space is complete. For the proof of this fact we shall use the following lemmas.

Lemma 3.1. Let \((X,d)\) be an unbounded metric space, \(\tilde{r}\) be a scaling sequence, \(\tilde{X}_\infty,\tilde{r}\) be maximal self-stable, \(\tilde{x} \in \tilde{X}_\infty\) and let \((\tilde{\gamma}_m)_{m \in \mathbb{N}} \subset \tilde{X}_\infty,\tilde{r}\) such that \(\tilde{\gamma}_m\) and \(\tilde{x}\) are mutually stable for every \(m \in \mathbb{N}\) and let

\[
\lim_{m \to \infty} \tilde{d}((\tilde{\gamma}_m)) = 0. \tag{3.1}
\]

Then \(\tilde{x}\) belongs to \(\tilde{X}_\infty,\tilde{r}\).

Proof. By Lemma 2.4 \(x \in \tilde{X}_\infty,\tilde{r}\) if and only if for every \(\tilde{y} \in \tilde{X}_\infty,\tilde{r}\) there is a finite limit

\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n}. \tag{3.2}
\]

Let \(\tilde{y} \in \tilde{X}_\infty,\tilde{r}\). It follows from the triangle inequality for \(\tilde{d}\) that

\[
|\tilde{d}(\tilde{y}, \tilde{\gamma}_m) - \tilde{d}(\tilde{y}, \tilde{\gamma}_{m'})| \leq \tilde{d}(\tilde{\gamma}_m, \tilde{\gamma}_{m'}) \leq \tilde{d}(\tilde{x}, \tilde{\gamma}_m) + \tilde{d}(\tilde{x}, \tilde{\gamma}_{m'}) \tag{3.3}
\]

for all \(m_1, m_2 \in \mathbb{N}\). Now (3.1) and (3.3) imply that \((\tilde{d}(\tilde{y}, \tilde{\gamma}_m))_{m \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{R}\). Consequently, there is a finite limit \(\lim_{m \to \infty} \tilde{d}(\tilde{y}, \tilde{\gamma}_m)\). We claim that limit (3.2) exists and

\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m \to \infty} \tilde{d}(\tilde{y}, \tilde{\gamma}_m). \tag{3.4}
\]
This statement holds if and only if
\[
\limsup_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m \to \infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m) \tag{3.5}
\]
and
\[
\liminf_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m \to \infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m). \tag{3.6}
\]
Equality (3.5) holds if and only if
\[
\lim_{m \to \infty} \left| \tilde{d}(\tilde{y}, \tilde{\gamma}^m) - \limsup_{n \to \infty} \frac{d(x_n, y_n)}{r_n} \right| = 0.
\]
It is clear that
\[
\lim_{m \to \infty} \left| \tilde{d}(\tilde{y}, \tilde{\gamma}^m) - \limsup_{n \to \infty} \frac{d(x_n, y_n)}{r_n} \right| \leq \lim_{m \to \infty} \limsup_{n \to \infty} \frac{d(\gamma^m_n, x_n)}{r_n} = \lim_{m \to \infty} \tilde{d}(\tilde{\gamma}^m, \tilde{x}) = 0,
\]
where \((\gamma^m_n)_{n \in \mathbb{N}} = \tilde{\gamma}^m\). Equality (3.5) follows. Equality (3.6) can be proved similarly.

**Lemma 3.2.** Let \((X, d)\) be an unbounded metric space and let \(\tilde{r} = (r_n)_{n \in \mathbb{N}}\) be a scaling sequence. Then for every \(\tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X\) there is \(\tilde{y} = (y_n)_{n \in \mathbb{N}} \in \hat{X}_\infty\) such that
\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = 0.
\]

**Proof.** Let \(\tilde{z} = (z_n)_{n \in \mathbb{N}} \in \hat{X}^0_{\infty, \tilde{r}}\) and let \(p \in X\). For every \(\tilde{x} \subset X\) define a sequence \(\tilde{y} = (y_n)_{n \in \mathbb{N}} \subset X\) by the rule
\[
y_n := \begin{cases} x_n, & \text{if } d(x_n, p) \geq d(z_n, p) \\ z_n, & \text{if } d(x_n, p) < d(z_n, p). \end{cases} \tag{3.7}
\]
It follows from (3.7) that the inequality
\[
d(y_n, p) \geq d(z_n, p) \tag{3.8}
\]
holds for every \(n \in \mathbb{N}\). Since we have \(\tilde{z} \in \hat{X}^0_{\infty, \tilde{r}} \subset \hat{X}_\infty\), inequality (3.8) implies \(\tilde{y} = (y_n)_{n \in \mathbb{N}} \in \hat{X}_\infty\). Moreover, from (3.8) we also have
\[
0 \leq d(x_n, y_n) \leq 2d(z_n, p). \tag{3.9}
\]
The equality
\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = 0
\]
follows from (3.9) and \(\lim_{n \to \infty} \frac{d(z_n, p)}{r_n} = 0\). \qed
Theorem 3.3. Let \((X, d)\) be an unbounded metric space. Then all pretangent spaces to \((X, d)\) at infinity are complete.

Proof. Let \(p \in X\), let \(\tilde{r} = (r_n)_{n \in \mathbb{N}}\) be a scaling sequence and let \(\bar{X}_{\infty, \tilde{r}}\) be a maximal self-stable set with metric identification \((\Omega_{\infty, \tilde{r}}, \rho)\). The metric space \((\Omega_{\infty, \tilde{r}}, \rho)\) is complete if and only if the pseudometric space \((\bar{X}_{\infty, \tilde{r}}, \bar{d})\) is complete, i.e., for every Cauchy sequence \((\tilde{x}^m)_{m \in \mathbb{N}} \subset \bar{X}_{\infty, \tilde{r}}\) there is \(\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \bar{X}_{\infty, \tilde{r}}\) such that

\[
\lim_{m \to \infty} \bar{d}(\tilde{x}, \tilde{x}^m) = 0. \tag{3.10}
\]

By Lemma 3.1 if (3.10) holds with some \(\tilde{x} \in \bar{X}_{\infty}\), then \(\tilde{x} \in \bar{X}_{\infty, \tilde{r}}\). Let \((\tilde{x}^m)_{m \in \mathbb{N}}\) be a Cauchy sequence in \((\bar{X}_{\infty, \tilde{r}}, \bar{d})\). We first find \(\tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X\) for which (3.10) holds. Then, using Lemma 3.2 we obtain \(\tilde{x} \in \bar{X}_{\infty}\) satisfying (3.10). Let \((\varepsilon)_{k \in \mathbb{N}} \subset (0, \infty)\) be an decreasing sequence such that

\[
\sum_{k=1}^{\infty} \varepsilon_k < \infty. \tag{3.11}
\]

There is \((m_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) such that, for all \(k \in \mathbb{N}\), \(m_{k+1} > m_k\) and \(\bar{d}(\tilde{x}^m, \tilde{x}^{m_k}) \leq \varepsilon_k\) holds whenever \(m \geq m_k\). Now we construct \(\tilde{x} = (x_n)_{n \in \mathbb{N}} \subset X\) such that \(\tilde{x}\) and \(\tilde{x}^{m_k}\) are mutually stable for every \(k \in \mathbb{N}\) and

\[
\lim_{k \to \infty} \bar{d}(\tilde{x}^{m_k}, \tilde{x}) = 0. \tag{3.12}
\]

For every \(m \in \mathbb{N}\) we set \(\tilde{x}^m = (\gamma_n^m)_{n \in \mathbb{N}}\). Let \((N_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) and \(\tilde{\beta}^k = (\beta^k_n)_{n \in \mathbb{N}} \in \bar{X}_{\infty}\) be inductively defined by the rule: if \(k = 1\), then \(N_1 = 1\) and \((\beta^1_n)_{n \in \mathbb{N}} = (\gamma_n^m)_{n \in \mathbb{N}}\); if \(k \geq 2\), then \(N_k\) is the smallest \(l \in \mathbb{N}\) which satisfies the inequalities \(l > N_{k-1}\) and

\[
\beta^k_n := \begin{cases} 
\beta^1_n & \text{if } N_1 \leq n < N_2, \\
\beta^2_n & \text{if } N_2 \leq n < N_1, \\
\ldots & \ldots \ldots \ldots \\
\beta^{k-1}_n & \text{if } N_{k-1} \leq n < N_k, \\
\gamma^m_n & \text{if } n \geq N_k.
\end{cases} \tag{3.13}
\]

Define \(\tilde{x} = (x_n)_{n \in \mathbb{N}}\) as

\[
x_n := \beta^k_n \text{ for } n \in [N_k, N_{k+1}), k = 1, 2, 3, \ldots \tag{3.14}
\]

It follows from (3.12) and (3.13) that

\[
\lim_{k \to \infty} \frac{\bar{d}(\beta^k_n, \gamma^m_n)}{r_n} = 0 \tag{3.15}
\]
for every \( k \in \mathbb{N} \), and
\[
\frac{d(\beta_n^k, \beta_n^{k-1})}{r_n} < 2\varepsilon_k
\]  
(3.16)
for all \( n, k \in \mathbb{N} \). Limit relation (3.15) implies that (3.12) holds if and only if
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left( \frac{d(x_n, \beta_n^{k})}{r_n} \right) = 0.
\]  
(3.17)
Using (3.14), we see that for every \( n \in \mathbb{N} \) there is \( K(n) \in \mathbb{N} \) such that
\[
\frac{d(x_n, \beta_n^{k})}{r_n} = \frac{d(\beta_{K(n)}^k, \beta_n^{k})}{r_n}.
\]
If \( k \) is given, then, for sufficiently large \( n \), the inequality \( K(n) > k \) holds. Consequently by (3.16) we have
\[
\frac{d(\beta_{K(n)}^k, \beta_n^{k})}{r_n} \leq \sum_{i=0}^{K(n)-k} \frac{d(\beta_n^{k+i+1}, \beta_n^{k+i})}{r_n} \leq 2 \sum_{i=k}^{\infty} \varepsilon_i.
\]  
(3.18)
Inequalities (3.11) and (3.18) imply (3.17).

If \( Y \) is an unbounded subspace of metric space \( X \) and for a scaling sequence \( \tilde{r} \) and maximal self-stable \( \tilde{X}_{\infty, \tilde{r}} \) and \( \tilde{Y}_{\infty, \tilde{r}} \) we have \( \tilde{Y}_{\infty, \tilde{r}} \subseteq \tilde{X}_{\infty, \tilde{r}} \), then there is a unique isometric embedding \( \text{in}_Y : \Omega_{\infty, \tilde{r}}^Y \to \Omega_{\infty, \tilde{r}}^X \) such that the diagram

\[
\begin{array}{ccc}
\tilde{Y}_{\infty, \tilde{r}} & \xrightarrow{\text{in}_Y} & \tilde{X}_{\infty, \tilde{r}} \\
\pi_Y \downarrow & & \downarrow \pi_X \\
\Omega_{\infty, \tilde{r}}^Y & \xrightarrow{\text{in}_Y} & \Omega_{\infty, \tilde{r}}^X
\end{array}
\]

is commutative, where \( \pi_Y \) and \( \pi_X \) are the natural projections and \( \text{in}_Y(\tilde{y}) = \tilde{y} \) for every \( \tilde{y} \in \tilde{Y}_{\infty, \tilde{r}} \).

Let \( X \) and \( Y \) be a metric spaces. Recall that a map \( f : X \to Y \) is called closed if the image of each closed set is closed.

**Corollary 3.4.** Let \( (X, d) \) be an unbounded metric space, \( Y \) be an unbounded subspace of \( X \), \( \tilde{r} \) be a scaling sequence and let \( \Omega_{\infty, \tilde{r}}^X \) and \( \Omega_{\infty, \tilde{r}}^Y \) be pretangent spaces such that for corresponding \( \tilde{X}_{\infty, \tilde{r}} \) and \( \tilde{Y}_{\infty, \tilde{r}} \) we have \( \tilde{X}_{\infty, \tilde{r}} \supseteq \tilde{Y}_{\infty, \tilde{r}} \). Then the isometric embedding \( \text{in}_Y : \Omega_{\infty, \tilde{r}}^Y \to \Omega_{\infty, \tilde{r}}^X \) is closed.

**Proof.** The map \( \text{in}_Y \) is an isometric embedding. Hence, \( \text{in}_Y \) is closed if and only if the set \( \text{in}_Y(\Omega_{\infty, \tilde{r}}^X) \) is a closed subset of \( \Omega_{\infty, \tilde{r}}^X \). The space \( \Omega_{\infty, \tilde{r}}^X \) is complete by Theorem 3.3. Since a metric space is complete if and only if this space is closed in every its superspace (see, for example, [19, Theorem 10.2.2]), \( \text{in}_Y(\Omega_{\infty, \tilde{r}}^X) \) is closed in \( \Omega_{\infty, \tilde{r}}^X \).
4 When are the pretangent spaces finite?

The main goal of this section is to find conditions under which the inequality $|\Omega_{\infty, \tilde{r}}^X| \leq n$ holds, with given $n \in \mathbb{N}$, for all pretangent at infinity spaces $\Omega_{\infty, \tilde{r}}^X$.

Lemma 4.1. Let $(X, d)$ be an unbounded metric space. Then there exists a pretangent space $\Omega_{\infty, \tilde{r}}^X$ such that $|\Omega_{\infty, \tilde{r}}^X| \geq 2$.

The proof of the lemma is similar to the proof of statement (ii) from Proposition 2.1.

The following lemma is an analog of Lemma 5 from [1].

Lemma 4.2. Let $(X, d)$ be an unbounded metric space, $p \in X$, let $B$ be a countable subset of $\tilde{X}_\infty$ and let $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ be a scaling sequence. Suppose that
\[
\limsup_{n \to \infty} \frac{d(b_n, p)}{r_n} < \infty
\tag{4.1}
\]
holds for every $\tilde{b} = (b_n)_{n \in \mathbb{N}} \in B$. Then there is a strictly increasing sequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that the family
\[
B' := \{ \tilde{b}' = (b_{n_k})_{k \in \mathbb{N}} : \tilde{b} \in B \}
\]
is self-stable at infinity with respect to $\tilde{r}' = (r_{n_k})_{k \in \mathbb{N}}$.

Proof. It is sufficient to consider the case when $B$ is countably infinite. Then the set of all ordered pairs $(\tilde{b}, \tilde{x}) \in B^2$ can be enumerated as $(\tilde{b}^1, \tilde{x}^1), (\tilde{b}^2, \tilde{x}^2), \ldots$. The triangle inequality and (4.1) imply
\[
\sup_{n \in \mathbb{N}} \frac{d(b_n^1, x_n^1)}{r_n} < \infty
\]
for each pair $(\tilde{b}^1, \tilde{x}^1) \in B^2$, $\tilde{b}^1 = (b_n^1)_{n \in \mathbb{N}}$ and $\tilde{x}^1 = (x_n^1)_{n \in \mathbb{N}}$. In particular, we have
\[
\sup_{n \in \mathbb{N}} \frac{d(b_n^1, x_n^1)}{r_n} < \infty.
\]
Since every bounded, infinite sequence contains a convergent subsequence, there is a strictly increasing sequence $n^1_k = (n^1_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that
\[
\lim_{k \to \infty} \frac{d(b_{n_k^1}, x_{n_k^1})}{r_{n_k^1}}, \quad \lim_{k \to \infty} \frac{d(x_{n_k^1}, p)}{r_{n_k^1}} \quad \text{and} \quad \lim_{k \to \infty} \frac{d(b_{n_k^1}, p)}{r_{n_k^1}}
\]
are finite. Hence, the sequences $(b_{n_k^1})_{k \in \mathbb{N}}$ and $(x_{n_k^1})_{k \in \mathbb{N}}$ are mutually stable with respect to $(r_{n_k^1})_{k \in \mathbb{N}}$. Analogously, by induction, we can prove that for
every integer \( i \geq 2 \) there is a subsequence \( \tilde{n}_i = (n^i_k)_{k \in \mathbb{N}} \) of sequence \( \tilde{n}_{i-1} \) such that \((b^i_k)_{k \in \mathbb{N}}\) and \((x^i_k)_{k \in \mathbb{N}}\) are mutually stable with respect to \((r^i_k)_{k \in \mathbb{N}}\).

Using Cantor’s diagonal construction, write \( \tilde{r}' := (r^i_k)_{k \in \mathbb{N}} \) and, for every \( \tilde{b} = (b_n)_{n \in \mathbb{N}} \in \mathcal{B} \), define \( \tilde{b}' := (b^i_k)_{k \in \mathbb{N}} \). Then the family \( \mathcal{B}':= \{ \tilde{b}' : \tilde{b} \in \mathcal{B} \} \) is self-stable at infinity with respect to \( \tilde{r}' \).

Let \((X, d)\) be an unbounded metric space and let \( p \) be a point of \( X \). Denote by \( X^n \) the set of all \( n \)-tuples \( x = (x_1, \ldots, x_n) \) with \( x_k \in X \) for \( k = 1, \ldots, n \), \( n \geq 2 \) and define the function \( F_n : X^n \to \mathbb{R} \) as

\[
F_n(x_1, \ldots, x_n) := \begin{cases} 
0 & \text{if } (x_1, \ldots, x_n) = (p, \ldots, p) \\
\min_{1 \leq k \leq n} d(x_k, p) \prod_{1 \leq k < l \leq n} d(x_k, x_l) & \text{otherwise.}
\end{cases} \tag{4.2}
\]

**Theorem 4.3.** Let \((X, d)\) be an unbounded metric space and let \( n \geq 2 \) be an integer number. Then the inequality

\[
|\Omega^X_{\infty, \tilde{r}}| \leq n \tag{4.3}
\]

holds for every \( \Omega^X_{\infty, \tilde{r}} \) if and only if

\[
\lim_{x_1, \ldots, x_n \to \infty} F_n(x_1, \ldots, x_n) = 0. \tag{4.4}
\]

**Proof.** Let (4.3) hold for all pretangent spaces \( \Omega^X_{\infty, \tilde{r}} \). Suppose \( \tilde{x}^i = (x^i_m)_{m \in \mathbb{N}} \in \tilde{X}_\infty \), \( i = 1, \ldots, n \) such that

\[
\lim_{m \to \infty} F_n(x^1_m, \ldots, x^n_m) = \limsup_{x_1, \ldots, x_n \to \infty} F_n(x_1, \ldots, x_n) > 0. \tag{4.5}
\]

If \( \tilde{r} = (r_m)_{m \in \mathbb{N}} \) is a scaling sequence with

\[
r_m = \max\{1, d(x^1_m, p), \ldots, d(x^n_m, p)\}
\]

for every \( m \in \mathbb{N} \), where \( p \) is a point of \( X \) in definition (4.2), then the inequality

\[
\limsup_{m \to \infty} \frac{d(x^k_m, p)}{r_m} \leq 1
\]

holds for every \( k \in \{1, \ldots, n\} \). Using Lemma 4.2 we may suppose that the family \( \{\tilde{x}^1, \ldots, \tilde{x}^n\} \) is self-stable with respect to \( \tilde{r} \). Now (4.5) and (4.2) imply

\[
\tilde{d}_{\tilde{r}}(\tilde{x}^k) > 0 \quad \text{and} \quad \tilde{d}_{\tilde{r}}(\tilde{x}^k, \tilde{x}^j) > 0.
\]
for all distinct \( k, j \in \{1, \ldots, n\} \). Adding \( \tilde{z} \in \tilde{X}_\infty^{0, \tilde{r}} \) to the family \( \{\tilde{z}, \tilde{x}^1, \ldots, \tilde{x}^n\} \) we see that the family \( \{z, \tilde{x}^1, \ldots, \tilde{x}^n\} \) is self-stable by statement (iii) of Proposition 2.2. Consequently there is \( \Omega^X_{\infty, \tilde{r}} \) with \( |\Omega^X_{\infty, \tilde{r}}| \geq n + 1 \), contrary to (4.3). Equality (4.4) follows.

To prove the converse statement it suffices to consider some different \( n + 1 \) points \( \nu_0, \nu_1, \ldots, \nu_n \in \Omega^X_{\infty, \tilde{r}} \) such that \( \nu_0 = X^0_{\infty, \tilde{r}} \) (see (1.8)). Then, for the sequences \( \tilde{x}^1, \ldots, \tilde{x}^n \) with 
\[
\pi(\tilde{x}^k) = \nu_k, \quad \tilde{x}^k = (x^k_m)_{m \in \mathbb{N}}, \quad k \in \{1, \ldots, n\},
\]
we obtain
\[
\lim_{m \to \infty} F_n(x^1_m, \ldots, x^n_m) = \frac{\min_{1 \leq k \leq n} \rho(\nu_0, \nu_k) \prod_{1 \leq k < l \leq n} \rho(\nu_k, \nu_l)}{\left( \max_{1 \leq k \leq n} \rho(\nu_k, \nu_0) \right)^{\frac{n(n-1)}{2} + 1}} \neq 0. \tag{4.6}
\]

\[\square\]

**Corollary 4.4.** Let \((X, d)\) be an unbounded metric space and let \( n \geq 2 \) be an integer number. Suppose \( \lim_{x_1, \ldots, x_n \to \infty} F_n(x_1, \ldots, x_n) = 0 \) holds. Then every pretangent space \( \Omega^X_{\infty, \tilde{r}} \) with \( |\Omega^X_{\infty, \tilde{r}}| = n \) is tangent.

**Theorem 4.5.** Let \((X, d)\) be an unbounded metric space and let \( n \geq 2 \) be an integer number such that the inequality
\[
|\Omega^X_{\infty, \tilde{r}}| \leq n \tag{4.7}
\]
holds for every \( \Omega^X_{\infty, \tilde{r}} \). Then the following statements are equivalent.

(i) For every scaling sequence \( \tilde{r} \), the function \( \rho^0 : V(G_X, \tilde{r}) \to \mathbb{R} \) defined as
\[
\rho^0(\nu) = d_{\tilde{r}}(\tilde{x}), \quad \tilde{x} \in \nu \in V(G_X, \tilde{r})
\]
is injective.

(ii) For every scaling sequence \( \tilde{r} \), the inequality
\[
|V(G_X, \tilde{r})| \leq n \tag{4.8}
\]
holds.
If, in addition, \( \tilde{r} \) is a scaling sequence such that

\[
\rho^0(\nu_1) = \rho^0(\nu_2) \tag{4.9}
\]

for some distinct \( \nu_1, \nu_2 \in V(G_{X, \tilde{r}}) \), then we have

\[
|V(G_{X, \tilde{r}})| \geq c, \tag{4.10}
\]

where \( c \) is the continuum.

**Remark 4.6.** For every vertex \( \nu \in V(G_{X, \tilde{r}}) \) there is a pretangent space \( (\Omega^X_{\infty, \tilde{r}}, \rho) \) such that \( \nu \in \Omega^X_{\infty, \tilde{r}} \). Then \( \rho^0(\nu) \) is the distance from \( \nu \) to the distinguished point \( \nu_0 = \hat{X}^0_{\infty, \tilde{r}} \) in the metric space \( (\Omega^X_{\infty, \tilde{r}}, \rho), \rho^0(\nu) = \rho(\nu_0, \nu) \).

**Proof of Theorem 4.5.**

(i) Suppose (ii) does not hold. Then there is a scaling sequence \( \tilde{r}_1 \) such that

\[
|V(G_{X, \tilde{r}_1})| \geq n + 1.
\]

Consequently, we can find \( \bar{x}^0 \in \hat{X}^0_{\infty, \tilde{r}_1} \) and \( \tilde{x}_1, \ldots, \tilde{x}_n \in \text{Seq}(\hat{X}, \tilde{r}_1) \) such that

\[
0 = \tilde{d}_{\tilde{r}_1}(\bar{x}^0) < \tilde{d}_{\tilde{r}_1}(\bar{x}_k) < \tilde{d}_{\tilde{r}_1}(\bar{x}_{k+1}) < \infty \tag{4.11}
\]

for every \( k \in \{1, \ldots, n - 1\} \). By Lemma 4.2 there is an infinite subsequence \( \tilde{r}'_1 \) of the sequence \( \tilde{r}_1 \) such that the set \( \{\tilde{x}^0_0, \tilde{x}^1_1, \ldots, \tilde{x}^n_n\} \) is self-stable. Let \( \hat{X}_{\infty, \tilde{r}} \supseteq \{\tilde{x}^0_0, \tilde{x}^1_1, \ldots, \tilde{x}^n_n\} \) and let \( (\Omega^X_{\infty, \tilde{r}'_1}, \rho) \) be the metric identification of \( \hat{X}_{\infty, \tilde{r}} \). Write \( \nu_i = \pi(\tilde{x}^i_i), i = 0, \ldots, n, \) where \( \pi : \hat{X}_{\infty, \tilde{r}_1} \to \Omega^X_{\infty, \tilde{r}'_1} \) is the natural projection. Now (4.11) implies that

\[
0 = \rho(\nu_0, \nu_0) < \rho(\nu_0, \nu_1) < \ldots < \rho(\nu_0, \nu_n).
\]

Consequently \( \nu_0, \ldots, \nu_n \) are distinct points of the space \( \Omega^X_{\infty, \tilde{r}} \). Thus

\[
|\Omega^X_{\infty, \tilde{r}_1}| \geq n + 1,
\]

which contradicts (4.7) with \( \tilde{r} = \tilde{r}'_1 \).

(ii) \( \Rightarrow \) (i) Suppose now that (i) does not hold. Then there exist \( \tilde{r} \) and \( \nu_1, \nu_2 \in V(G_{X, \tilde{r}}) \) such that \( \nu_1 \neq \nu_2 \) and (4.9) holds. It suffices to show that inequality (4.10) holds for an infinite subsequence \( \tilde{r}' \) of \( \tilde{r} \). Let \( \tilde{x}^1 = (x^1_n)_{n \in \mathbb{N}} \in \nu_1 \) and \( \tilde{x}^2 = (x^2_n)_{n \in \mathbb{N}} \in \nu_2 \) and let \( p \in X \). Since \( \nu_1 \neq \nu_2 \) and \( \rho^0(\nu_1) = \rho^0(\nu_2) \), we have

\[
\lim_{n \to \infty} \frac{d(x^1_n, p)}{r_n} = \lim_{n \to \infty} \frac{d(x^2_n, p)}{r_n} > 0 \tag{4.12}
\]

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and
\[ \infty > \lim_{n \to \infty} \sup \frac{d(x^1_n, x^2_n)}{r_n} > 0. \]

Let \( N_e \) be an infinite subset of \( \mathbb{N} \) such that \( \mathbb{N} \setminus N_e \) is also infinite and
\[ \lim_{n \to \infty} \sup \frac{d(x^1_n, x^2_n)}{r_n} = \lim_{n \in N_e} \frac{d(x^1_n, x^2_n)}{r_n}. \]  \hspace{1cm} (4.13)

To prove (4.10) we consider a relation \( \asymp \) on the set \( 2^{N_e} \) of all subsets of \( N_e \) defined by the rule:
\[ A \asymp B, \text{ if and only if } \left| A \triangle B \right| < \infty. \]

It is clear that \( \asymp \) is reflexive and symmetric. Since for all \( A, B, C \subseteq N_e \) we have
\[ A \triangle C \subseteq (A \triangle B) \cup (B \triangle C), \]
the relation \( \asymp \) is transitive. Thus \( \asymp \) is an equivalence on \( 2^{N_e} \). If \( A \subseteq N_e \), then for every \( B \subseteq N_e \) we have
\[ B = (B \setminus A) \cup (A \setminus (A \setminus B)). \] \hspace{1cm} (4.14)

For every \( A \subseteq N_e \) write
\[ [A] := \{ B \subseteq N_e : B \asymp A \}. \]

The set of all finite subsets of \( N_e \) is countable. Consequently equality (4.14) implies \( |[A]| = \aleph_0 \) for every \( A \subseteq N_e \). Hence we have
\[ |\{ [A] : A \subseteq N_e \}| = |2^{N_e}| = \mathfrak{c}. \] \hspace{1cm} (4.15)

Let \( \mathcal{N} \subseteq 2^{N_e} \) be a set such that:
- For every \( A \subseteq N_e \) there is \( N \in \mathcal{N} \) with \( A \asymp N \) holds;
- The implication
\[ (N_1 \asymp N_2) \Rightarrow (N_1 = N_2) \] \hspace{1cm} (4.16)
holds for all \( N_1, N_2 \in \mathcal{N} \).

It follows from (4.15) that \( |\mathcal{N}| = \mathfrak{c} \). For every \( N \in \mathcal{N} \) define the sequence
\[ \hat{x}(N) = (x_n(N))_{n \in \mathbb{N}} \] as
\[ x_n(N) := \begin{cases} x^1_n & \text{if } n \in N \\ x^2_n & \text{if } n \in \mathbb{N} \setminus N, \end{cases} \] \hspace{1cm} (4.17)
where \((x_1^n)_{n \in \mathbb{N}}, (x_2^n)_{n \in \mathbb{N}} \in \text{Seq}(X, \tilde{r})\) satisfy (4.12) and (4.13). It follows from (4.12) and (4.13) that
\[
\lim_{n \to \infty} \frac{d(x_n(N), p)}{r_n} = \tilde{d}_r(x^1) = \tilde{d}_r(x^2)
\]
for every \(N \in \mathcal{N}\). Thus \(\tilde{x}(N) \in \text{Seq}(\tilde{X}, \tilde{r})\). If \(N_1\) and \(N_2\) are distinct elements of \(\mathcal{N}\), then the equality
\[
d(x_n(N_1), x_n(N_2)) = d(x_n^1, x_n^2)
\]
holds for every \(n \in N_1 \triangle N_2\). Using (4.13) and the definition of \(\simeq\) we see that
\[
\text{the set } \nu_N = \{\tilde{x} \in \text{Seq}(X, \tilde{r}) : \tilde{d}_r(\tilde{x}, \tilde{x}(N)) = 0\}.
\]
(4.19) The first inequality in (4.18) implies that \(\nu_{N_1} \neq \nu_{N_2}\) if \(N_1 \neq N_2\). From \(|\mathcal{N}| = \mathfrak{c}\), we obtain
\[
|\{\nu_N : N \in \mathcal{N}\}| = \mathfrak{c}.
\]
Inequality (4.10) follows.

**Remark 4.7.** The existence of continuum many sets \(A_\gamma \subseteq \mathbb{N}\) satisfying, for all distinct \(\gamma_1\) and \(\gamma_2\), the equalities
\[
|A_{\gamma_1} \setminus A_{\gamma_2}| = |A_{\gamma_2} \setminus A_{\gamma_1}| = |A_{\gamma_1} \cap A_{\gamma_2}| = \aleph_0
\]
are well know. See, for example, Problem 41 of Chapter 4 in [13].

**Corollary 4.8.** Let \((X, d)\) be an unbounded metric space, \(n \geq 2\) be an integer number such that \(|\Omega_{\infty, \tilde{r}}^X| \leq n\) holds for every pretangent space \(\Omega_{\infty, \tilde{r}}\). Then, for every \(\tilde{r}\), we have
\[
\text{either } |\Omega_{\infty, \tilde{r}}^X| \leq 2^{n-1} \text{ or } |\Omega_{\infty, \tilde{r}}^X| \geq \mathfrak{c},
\]
where \(|\Omega_{\infty, \tilde{r}}^X|\) is the cardinal number of distinct pretangent spaces to \((X, d)\) at infinity with respect to \(\tilde{r}\).

**Proof.** Since every pretangent space \(\Omega_{\infty, \tilde{r}}^X\) is a subset of \(V(G_{X, \tilde{r}})\), inequality (4.18) implies that the number of pretangent is less than or equal to \(2^{n-1}\) if \(\rho^0 : V(G_{X, \tilde{r}}) \to \mathbb{R}\) is injective. Otherwise (4.18) implies that distinct \(\nu_{N_1}\) and \(\nu_{N_2}\) defined by (4.19) belong to the distinct pretangent spaces \(\Omega_{\infty, \tilde{r}}^X\) and \(2\Omega_{\infty, \tilde{r}}^X\).
Recall that a graph $G = (V, E)$ is trivial if $|V| = 1$. Moreover, if $|V| = 2$ and $G$ is connected, then $G$ is called a 1-path (see Figure 1).

Theorem 4.5 and Theorem 4.3 imply the next result for $n = 2$.

**Corollary 4.9.** Let $(X, d)$ be an unbounded metric space. Then the following statements are equivalent.

(i) The inequality
\[ \left| \Omega^X_{\infty, \hat{r}} \right| \leq 2 \] (4.20)
holds for every $\Omega^X_{\infty, \hat{r}}$.

(ii) The limit relation
\[ \lim_{x,y \to \infty} F_2(x,y) = 0 \] (4.21)
holds.

(iii) For every scaling sequence $\hat{r}$, the net $G_{X, \hat{r}}$ of pretangent spaces is either trivial or this is a 1-path.

**Proof.** It suffices to note that the function $\rho^0: V(G_{X, \hat{r}}) \to \mathbb{R}$ is injective for every $\hat{r}$ if (i) holds. Indeed, if $\hat{r}$ is a scaling sequence such that $\rho^0(\nu_1) = \rho^0(\nu_2)$ and $\nu_1, \nu_2$ are distinct vertices of $G_{X, \hat{r}}$, then $\rho^0(\nu_1) = \rho^0(\nu_2) > 0$. Consequently we can find a subsequence $\hat{r}'$ of $\hat{r}$ and $\Omega^X_{\infty, \hat{r}'}$ such that $\nu_0, \nu_1, \nu_2 \in \Omega^X_{\infty, \hat{r}'}$. Hence the inequality $|\Omega^X_{\infty, \hat{r}'}| \geq 3$ holds which contradicts (i).

**Corollary 4.10.** Let $(X, d)$ be an unbounded metric space such that every pretangent space to $(X, d)$ at infinity contains at most two points. Then, for every scaling sequence $\hat{r}$, there is a unique pretangent spaces $\Omega^X_{\infty, \hat{r}}$.

**Proof.** Suppose contrary that $^1\Omega^X_{\infty, \hat{r}}$ and $^2\Omega^X_{\infty, \hat{r}}$ are distinct pretangent spaces to $X$ with the same scaling sequence $\hat{r}$. Let $^1\tilde{X}_{\infty, \hat{r}}$ and $^2\tilde{X}_{\infty, \hat{r}}$ be maximal self-stable subsets of $Seq(X, \hat{r})$ such that $^1\Omega^X_{\infty, \hat{r}}$ is the metric identification of $^1\tilde{X}_{\infty, \hat{r}}$, $i = 1, 2, \ldots$. Since $^1\Omega^X_{\infty, \hat{r}} \neq ^2\Omega^X_{\infty, \hat{r}}$ we have also $^1\tilde{X}_{\infty, \hat{r}} \neq ^2\tilde{X}_{\infty, \hat{r}}$. It implies
\[ ^1\tilde{X}_{\infty, \hat{r}} \setminus ^2\tilde{X}_{\infty, \hat{r}} \neq \emptyset \neq ^2\tilde{X}_{\infty, \hat{r}} \setminus ^1\tilde{X}_{\infty, \hat{r}} \] (4.22)
because $^1\tilde{X}_{\infty, \hat{r}}$ and $^2\tilde{X}_{\infty, \hat{r}}$ are maximal self-stable. Using (4.22) we obtain
\[ ^1\Omega^X_{\infty, \hat{r}} \setminus ^2\Omega^X_{\infty, \hat{r}} \neq ^2\Omega^X_{\infty, \hat{r}} \setminus ^1\Omega^X_{\infty, \hat{r}}. \]
Moreover, $\tilde{X}_{\infty, \tilde{r}} \in \Omega_{\infty, \tilde{r}} \cap \Omega_{\infty, \tilde{r}}$. Consequently, we have

$$|\tilde{\Omega}_{\infty, \tilde{r}}| \geq |\Omega_{\infty, \tilde{r}} \cap \Omega_{\infty, \tilde{r}}| + |\Omega_{\infty, \tilde{r}} \setminus \Omega_{\infty, \tilde{r}}| + |\Omega_{\infty, \tilde{r}} \setminus \Omega_{\infty, \tilde{r}}| \geq 3,$$

contrary to statement (iii) of Theorem 4.9.

The following example shows that, for any $n \geq 3$, the equality

$$\lim_{x_1, \ldots, x_n \to \infty} F_n(x_1, \ldots, x_n) = 0$$

is not sufficient for the finiteness of the net $G_{X, \tilde{r}}$. In what follows $\mathbb{C}$ is the complex plane.

**Example 4.11.** Let $\tilde{r} = (r_m)_{m \in \mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that

$$\lim_{m \to \infty} \frac{r_{m+1}}{r_m} = \infty,$$

let $n \geq 2$ be an integer number, let $R_i = \{z \in \mathbb{C}: \arg z = \theta_i\}$ be the rays starting at origin with the angles of $\theta_i = \frac{\pi}{n} i$ with the positive real axis, $i = 0, \ldots, n - 1$ and let

$$C_m = \{z \in \mathbb{C}: |z| = r_m\}$$

be the circles in $\mathbb{C}$ with radius $r_m$, $m \in \mathbb{N}$, and the center 0. Write

$$X_n := \left( \bigcup_{i=0}^{n-1} R_i \right) \cap \left( \bigcup_{m=1}^{\infty} C_m \right)$$

and define the distance function $d$ on $X_n$ as

$$d(z, w) = |z - w|$$

(see Figure 2 for $X_n$ with $n = 3$).

Then we obtain

$$\lim_{x_1, \ldots, x_{n+1} \to \infty, x_1, \ldots, x_{n+1} \in X_n} F_{n+1}(x_1, \ldots, x_{n+1}) = 0 < \limsup_{x_1, \ldots, x_n \to \infty, x_1, \ldots, x_n \in X_n} F_n(x_1, \ldots, x_n)$$

and $|\Omega_{X, \tilde{r}}| = c$. In particular, for $n = 3$, the equality

$$\limsup_{x,y,z \to \infty, x,y,z \in X_3} F_3(x, y, z) = 2\sqrt{2} - 2$$

holds. (See Figure 3 for all possible pretangent spaces to $X_3$ at infinity with respect to $\tilde{r}$.)
Figure 2: The graphical representation of $X_3$. The points of $X_3$ are depicted here as small circles.

It can be shown that for every finite metric space $Y$ there is an unbounded metric space $X$ and a scaling sequence $\tilde{r}$ such that $Y$ is isometric to a pretangent space $\Omega_{\tilde{r}}^X$. We will consider the case when $Y$ is strongly rigid.

**Definition 4.12.** A metric space $(Y, \delta)$ is said to be strongly rigid if for all $x, y, z, w \in Y$ the conditions

$$\delta(x, y) = \delta(w, z) \quad \text{and} \quad x \neq y$$

imply that $\{x, y\} = \{z, w\}$.

**Remark 4.13.** See [11] and [15] for some interesting properties of strongly rigid metric spaces.

**Example 4.14.** Let $(Y, \delta)$ be a finite, nonempty and strongly rigid metric space, $Y = \{y_1, ..., y_k\}$ and let $\tilde{r} = (r_j)_{j \in \mathbb{N}}$ be a scaling sequence such that

$$\lim_{j \to \infty} \frac{r_{j+1}}{r_j} = \infty. \quad (4.23)$$

We will define a metric space $(X, d)$ as a subset of the finite dimensional normed space $l_k^\infty$ of $k$-tuples $x = (x_1, ..., x_k)$ of real numbers with the sup norm

$$\|x\|_\infty = \sup_{1 \leq i \leq k} |x_i|.$$
Figure 3: Every pretangent space $\Omega_{\infty, \tilde{r}}^{X_3}$ (with respect to $\tilde{r}$ given above) is isometric either $1\Omega_{\infty, \tilde{r}}^{X_3}$ or $2\Omega_{\infty, \tilde{r}}^{X_3}$.

Let $y^*$ be a fixed point of $Y$. The Kuratowski embedding $s : Y \rightarrow l^\infty_k$ can be defined as

$$s(y) = \begin{pmatrix} \delta(y, y_1) - \delta(y_1, y^*) \\ \vdots \\ \delta(y, y_k) - \delta(y_k, y^*) \end{pmatrix}.$$  (4.24)

Write

$$X := \bigcup_{n \in \mathbb{N}} r_n s(Y),$$  (4.25)

where $r_n s(Y) := \{r_n s(y) : y \in Y\}$ and define $d(x, y) := \| x - y \|_\infty$ for all $x, y \in X$. It follows directly from (4.24) that

$$s(y^*) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$  

For convenience we can suppose that the distinguished point $p$ of $X$ is $s(y^*)$,

$$p = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$  

Hence, for every $x \in X$, we have

$$d(x, p) = \| x \|_\infty.$$

(4.26)

Suppose now

$$\tilde{x} = (x_j)_{j \in \mathbb{N}} \in Seq(X, \tilde{r})$$
such that
\[
\tilde{d}_r(\tilde{x}) = \lim_{j \to \infty} \frac{d(x_j, p)}{r_j} = \lim_{j \to \infty} \frac{\|x_j\|_\infty}{r_j} > 0. \tag{4.27}
\]

By (4.25), for \(j \in \mathbb{N}\), there are \(n \in \mathbb{N}\) and \(y \in Y\) satisfying the equality \(x_j = r_ns(y)\). It is well known that the Kuratowski embeddings are isometric (see, for example, [8, the proof of Theorem III.8.1]). Consequently
\[
\frac{1}{r_j} \|x_j\|_\infty = \frac{r_n}{r_j} \|s(y)\|_\infty = \frac{r_n}{r_j} \delta(y, y^*). \tag{4.28}
\]

Now (4.27) implies \(y \neq y^*\) for all sufficiently large \(j\). Moreover, using (4.23) and (4.28) we obtain \(r_n = r_j\) for all sufficiently large \(j\) which is an equivalent for \(n = j\) for all sufficiently large \(j\). Hence, if \(\tilde{x} = (x_j)_{j \in \mathbb{N}}\) belongs \(Seq(X, \tilde{r})\) and \(\tilde{d}_r(x) > 0\), then for every sufficiently large \(j\) there is \(y(j) \in \mathbb{N}\) such that
\[
\frac{1}{r_j} \|x_j\|_\infty = \delta(y(j), y^*).
\]
Since \((Y, \delta)\) is strongly rigid, the limit \(\lim_{j \to \infty} \delta(y(j), y^*)\) exists if and only if there is \(y' \in Y\) such that
\[
y(j) = y'
\]
holds for all sufficiently large \(j\). This result and statement (vi) of Proposition 2.2 imply that every two \(\tilde{x}, \tilde{y} \in Seq(X, \tilde{r})\) are mutually stable and
\[
\lim_{j \to \infty} \frac{d(x_j, y_j)}{r_j} = \|s(x') - s(y')\|_\infty = \delta(x', y'),
\]
where \(x'\) and \(y'\) are the points of \(Y\) for which
\[
y(j) = y' \quad \text{and} \quad x(j) = x'
\]
hold for all sufficiently large \(j\). It is clear that, for every \(y \in Y\), we have
\[
(r_j s(y))_{j \in \mathbb{N}} \in Seq(X, \tilde{r}).
\]
Thus there is the unique pretangent space \(\Omega_{\infty, \tilde{r}}^X\) and this space is isometric to \((Y, \delta)\).

Analyzing the construction of Example 4.14 we obtain the following proposition.

**Proposition 4.15.** Let \((Y, \delta)\) be a finite nonempty metric space containing a point \(y^*\) for which
\[
\delta(y^*, x) \neq \delta(y^*, z)
\]
whenever \(x\) and \(z\) are distinct points of \(Y\). Then there are an unbounded metric space \((X, d)\) and a scaling sequence \(\tilde{r}\) such that:
There is a unique pretangent space $\Omega^X_{\infty, \tilde{r}}$.

The pretangent space $\Omega^X_{\infty, \tilde{r}}$ is tangent.

There is an isometry $f : \Omega^X_{\infty, \tilde{r}} \to Y$ such that $f(\nu_0) = y^*$, where $\nu_0 = X^0_{\infty, \tilde{r}}$ is the distinguished point of $\Omega^X_{\infty, \tilde{r}}$.

For the proof note that statement (ii) can be obtained by application of Corollary 4.4 and Theorem 4.3 to $(X, d)$ constructed in Example 4.14.

A simple modification of Example 4.14 gives us the following result.

**Theorem 4.16.** For every finite nonempty metric space $(Y, \delta)$ and every $y^* \in Y$ there are an unbounded metric space $(X, d)$, and a scaling sequence $\tilde{r}$ and an isometry $f : \Omega^X_{\infty, \tilde{r}} \to Y$ such that $f(\nu_0) = y^*$ holds and $\Omega^X_{\infty, \tilde{r}}$ is tangent.

The last theorem does not have direct generalization to the case of infinite $(Y, \delta)$ even if $(Y, \delta)$ is complete, separable and strongly rigid. (See Example 6.7 in the last section of the paper.)

## 5 Finite tangent spaces and strong porosity at a point

Theorem 4.3 gives, in particular, a condition guaranteeing the finiteness of all pretangent spaces. The goal of present section is to obtain the existence conditions for finite tangent spaces.

Let $(X, d)$ be an unbounded metric space and let $p \in X$. The finiteness of $\Omega^X_{\infty, \tilde{r}}$ is closely connected with a porosity of the set

$$Sp(X) := \{d(x, p) : x \in X\}$$

at infinity.

**Definition 5.1.** Let $E \subseteq \mathbb{R}^+$. The porosity of $E$ at infinity is the quantity

$$p^+(E, \infty) := \lim_{h \to \infty} \sup \frac{l(\infty, h, E)}{h},$$

where $l(\infty, h, E)$ is the length of the longest interval in the set $[0, h] \setminus E$. The set $E$ is strongly porous at infinity if $p^+(E, \infty) = 1$ and, respectively, $E$ is nonporous at infinity if $p^+(E, \infty) = 0$. 

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The standard definition of porosity at a finite point can be found in [20]. See [2, 4–6] for some applications of porosity to studies of pretangent spaces at finite points of metric spaces.

**Lemma 5.2.** Let $E \subseteq \mathbb{R}^+$ and let $p_1 \in (0, 1)$. If the double inequality
\begin{equation}
\frac{p^+(E, \infty)}{p_1} < 1
\end{equation}
holds, then for every infinite, strictly increasing sequence of real numbers $r_n$ with $\lim_{n \to \infty} r_n = \infty$ there is a subsequence $(r_{n_k})_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$ there are points $x_1^{(k)}, \ldots, x_k^{(k)} \in E$ which satisfy the inequalities
\begin{equation}
\frac{r_{n_k}}{1 - p_1} \leq x_1^{(k)} \leq \frac{r_{n_k}}{(1 - p_1)^2} < x_2^{(k)} \leq \frac{r_{n_k}}{(1 - p_1)^3} < \cdots < x_k^{(k)} \leq \frac{r_{n_k}}{(1 - p_1)^{2k}} < r_{n_{k+1}}.
\end{equation}

**Proof.** Suppose that (5.2) holds. Let $n_1$ be the first natural number such that $l(\infty, h, E) < p_1 h$ for all $h \in (r_{n_1}, \infty)$. If $r_{n_1}, \ldots, r_{n_k}$ are defined, then write $n_{k+1}$ for the first $m$ with
\begin{equation}
r_m > (1 - p_1)^{-2k} r_{n_k}.
\end{equation}

It is easy to show that the equality
\begin{equation}
\frac{r_{n_k}}{(1 - p_1)^m} - \frac{r_{n_k}}{(1 - p_1)^{m-1}} = p_1 \cdot \frac{r_{n_k}}{(1 - p_1)^m}
\end{equation}
holds for all $m \in \mathbb{N}$. Using the last equality, Definition 5.1 and inequality (5.2) we obtain
\begin{equation}
E \cap \left[ \frac{r_{n_k}}{(1 - p_1)^m}, \frac{r_{n_k}}{(1 - p_1)^{m+1}} \right] \neq \emptyset
\end{equation}
for all $m \in \mathbb{N}$. Hence, there are points $x_1^{(k)}, \ldots, x_k^{(k)} \in E$ which satisfy (5.3).

**Theorem 5.3.** Let $(X, d)$ be an unbounded metric space, $p \in X$. The following statements are equivalent:

(i) The set $Sp(X)$ is strongly porous at infinity;

(ii) There is a single-point, tangent space $\Omega^X_{\infty, p}$;
(iii) There is a finite tangent space $\Omega^X_{\infty, \tilde{r}}$;
(iv) There is a compact tangent space $\Omega^X_{\infty, \tilde{r}}$;
(v) There is a bounded, separable tangent space $\Omega^X_{\infty, \tilde{r}}$.

Proof. (i)$\Rightarrow$(ii) Suppose the equality

$$p^+(Sp(X), \infty) = 1$$

holds. Let $\tilde{h} = (h_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive numbers such that

$$\lim_{n \to \infty} h_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{l(\infty, h_n, Sp(X))}{h_n} = 1.$$

Let us consider a sequence of intervals $(c_n, d_n) \subseteq [0, h_n] \setminus Sp(X)$ for which

$$\lim_{n \to \infty} \frac{|d_n - c_n|}{h_n} = 1. \quad (5.4)$$

Passing, if it is necessary, to a subsequence we suppose that

$$0 < c_n < d_n \leq h_n \quad (5.5)$$

holds for every $n \in \mathbb{N}$. A pretangent $\Omega^X_{\infty, \tilde{r}}$ is single-point if and only if $X^\infty_{\infty, \tilde{r}} = \tilde{X}^0_{\infty, \tilde{r}}$ holds for the corresponding $\tilde{X}^\infty_{\infty, \tilde{r}}$. Hence, it suffices to prove that

$$\lim_{n \to \infty} \frac{d(x_n, p)}{r_n} = 0 \quad (5.6)$$

holds for every $x \in \tilde{X}^\infty_{\infty, \tilde{r}}$. Write

$$r_n := \sqrt{d_n c_n} \quad (5.7)$$

for every $n \in \mathbb{N}$ and define $\tilde{r} := (r_n)_{n \in \mathbb{N}}$.

Let us prove equality (5.6). It is evident that (5.4) and (5.5) imply the limit relations

$$\lim_{n \to \infty} \frac{c_n}{h_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{d_n}{h_n} = 1. \quad (5.8)$$

Consequently, we obtain

$$\lim_{n \to \infty} \frac{c_n}{d_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{d_n}{c_n} = \infty. \quad (5.9)$$

Since $(c_n, d_n) \subseteq [0, h_n] \setminus Sp(X)$, we have either $d(x_n, p) \leq c_n$ or $d(x_n p) \geq d_n$ for all $n \in \mathbb{N}$. Thus, either

$$\frac{d(x_n, p)}{r_n} \leq \frac{c_n}{d_n} \quad (5.10)$$
or
\[
\frac{d(x_n, p)}{r_n} \geq \sqrt{\frac{d_n}{c_n}} \tag{5.11}
\]
holds for every \( n \in \mathbb{N} \). The second relation in (5.11) implies that (5.11) cannot be valid for sufficient large \( n \) because \( \tilde{d}_{\tilde{x}}(\tilde{x}) \) is finite. Now, (5.11) follows from (5.10).

It is proved that if \( \tilde{r} \) is defined by (5.7), then there is a unique pretangent space \( \Omega_{x,\tilde{r}} \) and this space is single-point. Note also that \( \Omega_{x,\tilde{r}} \) is tangent. To prove it we can consider the subsequences \( \tilde{x}' = (x_{n_k})_{k \in \mathbb{N}}, \tilde{z}' = (z_{n_k})_{k \in \mathbb{N}} \), and \( \tilde{r}' = (r_{n_k})_{k \in \mathbb{N}} \) of \( \tilde{x}, \tilde{z} \) and \( \tilde{r} \), and repeat the proof of equality (5.6) substituting \( d_n, c_n, h_n \) and \( r_n \), respectively. The implication (i) \(\Rightarrow\) (ii) follows.

(ii) \(\Rightarrow\) (iii), (iii) \(\Rightarrow\) (iv), (iv) \(\Rightarrow\) (v) The implications are evident.

(v) \(\Rightarrow\) (i) Suppose statement (v) holds but there is \( p_1 \in (0, 1) \) such that \( p^+(Sp(X), \infty) < p_1 \). Let \( \tilde{r} = (r_n)_{n \in \mathbb{N}} \) be a scaling sequence and let \( \Omega_{x,\tilde{r}} \) be bounded, separable and tangent. Applying Lemma 5.2 with \( E = Sp(X) \) we can find a subsequence \( \tilde{r}' = (r_{n_k})_{k \in \mathbb{N}} \) of the sequence \( \tilde{r} \) such that for every \( k \in \mathbb{N} \) there are points \( x_{1}^{(k)}, \ldots, x_{k}^{(k)} \in X \) for which

\[
\frac{1}{1 - p_1} \leq \frac{d(x_{1}^{(k)}, p)}{r_{n_k}} \leq \frac{1}{(1 - p_1)^2},
\]
\[
\frac{1}{(1 - p_1)^3} \leq \frac{d(x_{2}^{(k)}, p)}{r_{n_k}} \leq \frac{1}{(1 - p_1)^4},
\]
\[\vdots\]
\[
\frac{1}{(1 - p_1)^{2k-1}} \leq \frac{d(x_{k}^{(k)}, p)}{r_{n_k}} \leq \frac{1}{(1 - p_1)^{2k}}. \tag{5.12}
\]

Let \( \tilde{z} = (z_k)_{k \in \mathbb{N}} \in \tilde{X}_{x,\tilde{r}} \) and \( \tilde{q} = (q_k)_{k \in \mathbb{N}} \). Write \( \tilde{x}_j \) for the \( j \)-th column of the following infinite matrix

\[
\begin{pmatrix}
 x_{1}^{(1)} & z_1 & z_1 & z_1 & \ldots \\
 x_{1}^{(2)} & x_{2}^{(2)} & z_2 & z_2 & \ldots \\
 x_{1}^{(3)} & x_{2}^{(3)} & x_{3}^{(3)} & z_3 & \ldots \\
 x_{1}^{(4)} & x_{2}^{(4)} & x_{3}^{(4)} & x_{4}^{(4)} & z_4 & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \ldots
\end{pmatrix}
\]

It follows from (5.12) that the inequalities

\[
\frac{1}{(1 - p_1)^{2j-1}} \leq \liminf_{k \to \infty} \frac{d(x_{j}^{(k)}, p)}{r_{n_k}} \leq \limsup_{k \to \infty} \frac{d(x_{j}^{(k)}, p)}{r_{n_k}} \leq \frac{1}{(1 - p_1)^{2j}} \tag{5.13}
\]

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holds for all \( j \in \mathbb{N} \). Let \( \tilde{X}_{\infty, \tilde{r}} \) be the maximal self-stable family with the metric identification \( \Omega_{\infty, \tilde{r}}^X \) and let \( \tilde{X}'_{\infty, \tilde{r}} = \{ (x_n)_{n \in \mathbb{N}} : (x_n)_{n \in \mathbb{N}} \in \tilde{X}_{\infty, \tilde{r}} \} \). The family \( \mathfrak{B} := \{ \tilde{x}_1, \tilde{x}_2, \ldots \} \) satisfies the conditions of Lemma 4.2. Hence there is a subsequence \( \tilde{r}'' \) of \( \tilde{r}' \) such that \( \tilde{X}'_{\infty, \tilde{r}''} \supseteq \mathfrak{B}' \). The first inequality in (5.12) implies that \( \Omega_{\infty, \tilde{r}''} \) is unbounded, contrary to (v). \qed

6 Kuratowski limits of subsets of \( \mathbb{R} \) and their applications to pretangent spaces

Let \( (Y, \delta) \) be a metric space. For any sequence \( (A_n)_{n \in \mathbb{N}} \) of nonempty sets \( A_n \subseteq Y \), the Kuratowski limit inferior of \( (A_n)_{n \in \mathbb{N}} \) is the subset \( \text{Li}_{n \to \infty} A_n \) of \( Y \) defined by the rule:

\[
( y \in \text{Li}_{n \to \infty} A_n ) \iff ( \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : B(y, \varepsilon) \cap A_n \neq \emptyset ),
\]

where \( B(y, \varepsilon) \) is the open ball of radius \( \varepsilon > 0 \) centered at the point \( y \in Y \),

\[
B(y, \varepsilon) = \{ x \in Y : \delta(x, y) < \varepsilon \}.
\]

Similarly, the Kuratowski limit superior of \( (A_n)_{n \in \mathbb{N}} \) can be defined as the subset \( \text{Ls}_{n \to \infty} A_n \) of \( Y \) for which

\[
( y \in \text{Ls}_{n \to \infty} A_n ) \iff ( \forall \varepsilon > 0 \forall n \in \mathbb{N} \exists n_0 \geq n : B(y, \varepsilon) \cap A_{n_0} \neq \emptyset ).
\]

Remark 6.1. The Kuratowski limit inferior and limit superior are basic concepts of set-valued analysis in metric spaces and have numerous applications (see, for example, [3]).

We denote \( tA := \{ tx : x \in A \} \) for any nonempty set \( A \subseteq \mathbb{R} \) and \( t \in \mathbb{R} \), and, \( \nu_0 := \tilde{X}_0^0 \in \Omega_{\infty, \tilde{r}}^X \) for any pretangent space \( \Omega_{\infty, \tilde{r}}^X \) of an unbounded metric space \( (X, d) \). Moreover, for every scaling sequence \( \tilde{r} \), we denote by \( \Omega_{\infty, \tilde{r}}^X \) the set of all pretangent at infinity spaces to \( (X, d) \) with respect to \( \tilde{r} \). Write

\[
Sp(\Omega_{\infty, \tilde{r}}^X) := \{ \rho(\nu_0, \nu) : \nu \in \Omega_{\infty, \tilde{r}}^X \} \quad \text{and} \quad Sp(X) := \{ d(p, x) : x \in X \}.
\]

Proposition 6.2. Let \( (X, d) \) be an unbounded metric space, \( p \in X \), \( \tilde{r} = (r_n)_{n \in \mathbb{N}} \) be a scaling sequence and let \( R \) be the set of all infinite subsequences of \( \tilde{r} \). Then the equalities

\[
\bigcup_{\Omega_{\infty, \tilde{r}}^X \in \Omega_{\infty, \tilde{r}}^X} Sp(\Omega_{\infty, \tilde{r}}^X) = \text{Li}_{n \to \infty} \left( \frac{1}{r_n} Sp(X) \right),
\]

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\[
\bigcup_{\Omega^X_{\infty,\tilde{r}} \in \Omega^X_{\infty,\tilde{r}'}, \tilde{r}' \in \mathbb{R}} Sp(\Omega^X_{\infty,\tilde{r}}) = Ls_{n \to \infty} \left( \frac{1}{r_n} Sp(X) \right) \quad (6.5)
\]

hold.

**Proof.** Let us prove the inclusion

\[
\bigcup_{\Omega^X_{\infty,\tilde{r}} \in \Omega^X_{\infty,\tilde{r}'}, \tilde{r}' \in \mathbb{R}} Sp(\Omega^X_{\infty,\tilde{r}}) \subseteq Ls_{n \to \infty} \left( \frac{1}{r_n} Sp(X) \right). \quad (6.6)
\]

Let \( \Omega^X_{\infty,\tilde{r}} \in \Omega^X_{\infty,\tilde{r}'} \) and \( \nu \in \Omega^X_{\infty,\tilde{r}'} \) be arbitrary. Let \( \tilde{X}_{\infty,\tilde{r}'} \) be a maximal self-stable family with the metric identification \( \Omega^X_{\infty,\tilde{r}} \), and let \( \tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}'} \), \( \tilde{z} = (z_n)_{n \in \mathbb{N}} \in \tilde{X}^0_{\infty,\tilde{r}'} \) such that

\[
\pi(\tilde{x}) = \nu \quad \text{and} \quad \pi(\tilde{z}) = \nu_0.
\]

Then, by the definition of pretangent spaces, we have

\[
\lim_{n \to \infty} \frac{d(x_n, z_n)}{r_n} = \rho(\nu_0, \nu).
\]

Consequently, for every \( \varepsilon > 0 \) the inequality

\[
\left| \frac{1}{r_n} d(x_n, p) - \rho(\nu_0, \nu) \right| < \varepsilon
\]

holds for all sufficiently large \( n \). Since \( \Omega^X_{\infty,\tilde{r}} \) is an arbitrary element of \( \Omega^X_{\infty,\tilde{r}'} \) and \( \nu \) is an arbitrary point of \( \Omega^X_{\infty,\tilde{r}'} \) and \( \frac{1}{r_n} d(x_n, p) \in \frac{1}{r_n} Sp(X) \), inclusion \( (6.6) \) follows.

To obtain the converse inclusion, we consider an arbitrary

\[
t \in Ls_{n \to \infty} \left( \frac{1}{r_n} Sp(X) \right). \quad (6.7)
\]

It is evident that \( 0 \in Sp\left( \Omega^X_{\infty,\tilde{r}'} \right) \) holds for every \( \Omega^X_{\infty,\tilde{r}'} \). Suppose \( t > 0 \) and write

\[
\text{dist}\left( t, \frac{1}{r_n} Sp(X) \right) := \inf \left\{ |t - s| : s \in \frac{1}{r_n} Sp(X) \right\}.
\]

Using \( (6.1) \), we see that \( (6.7) \) holds if and only if

\[
\lim_{n \to \infty} \text{dist}\left( t, \frac{1}{r_n} Sp(X) \right) = 0. \quad (6.8)
\]

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Consequently, there is a sequence \((\tau_n)_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to \infty} |\tau_n - t| = 0
\]
and \(\tau_n \in \frac{1}{r_n} Sp(X)\) for every \(n \in \mathbb{N}\). Using the definition of \(\frac{1}{r_n} Sp(X)\), we may rewrite the last statement as: “There is a sequence \((x_n)_{n \in \mathbb{N}} \subset X\) such that
\[
\lim_{n \to \infty} \left| \frac{d(x, p)}{r_n} - t \right| = 0
\]
holds”. Thus, we have
\[
\lim_{n \to \infty} \frac{d(x, p)}{r_n} = t.
\]

The inequality \(t > 0\) implies that \((x_n)_{n \in \mathbb{N}} \in \tilde{X}_{\infty, \tilde{r}}\). Let \(\tilde{X}_{\infty, \tilde{r}}\) be a maximal self-stable family for which \((x_n)_{n \in \mathbb{N}} \in \tilde{X}_{\infty, \tilde{r}}\) and let \(\Omega^X_{\infty, \tilde{r}}\) be the metric identification of \(\tilde{X}_{\infty, \tilde{r}}\). Limit relation (6.10) implies \(t \in Sp(\Omega^X_{\infty, \tilde{r}})\). Since \(t\) is an arbitrary positive number from \(Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right)\), we obtain
\[
\bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}}} Sp(\Omega^X_{\infty, \tilde{r}}) \supseteq Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right).
\]

Equality (6.10) follows.

Equality (6.5) follows from (6.2) because, for every \(t \geq 0\), we have \(t \in Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right)\) if and only if \(t \in Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right)\) holds for some \((r_n)_{n \in \mathbb{N}} \in \tilde{R}\).

**Remark 6.3.** Let \(\rho^0 : V(G, X) \to \mathbb{R}^+\) be the function defined in Theorem 4.5. Then using equality (2.7) we see that
\[
\rho^0(V(G, X)) = \{d_{\tilde{r}}(\tilde{x}) : \tilde{x} \in Seq(X, \tilde{r})\} = \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}}} Sp(\Omega^X_{\infty, \tilde{r}}).
\]

**Corollary 6.4.** Let \((X, d)\) be an unbounded metric space, \(\tilde{r}\) be a scaling sequence and let \(1\Omega^X_{\infty, \tilde{r}}\) be tangent and separable. Then we have
\[
Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right) = Lin_{n \to \infty} \left(\frac{1}{r_n} Sp(X)\right) = Sp(\Omega^X_{\infty, \tilde{r}}).
\]

**Proof.** Using Lemma 4.2 for every \(\tilde{r}' \in \tilde{R}\) and every
\[
s \in \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}}^*} \Omega^X_{\infty, \tilde{r}'}.
\]
we can find $\nu$ belonging to the tangent space $\Omega^X_{\infty, \tilde{r}}$ such that $\rho(\nu_0, \nu) = s$. Consequently

$$Sp \left( \Omega^X_{\infty, \tilde{r}} \right) \supseteq \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}, \tilde{r}^\prime \in \mathbb{R}} Sp(\Omega^X_{\infty, \tilde{r}})$$

holds. It is evident that

$$\bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}, \tilde{r}^\prime \in \mathbb{R}} Sp(\Omega^X_{\infty, \tilde{r}}) \supseteq \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}} Sp(\Omega^X_{\infty, \tilde{r}^\prime}) \supseteq Sp(\Omega^X_{\infty, \tilde{r}}).$$

Hence we have the equalities

$$\bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}, \tilde{r}^\prime \in \mathbb{R}} Sp(\Omega^X_{\infty, \tilde{r}^\prime}) = \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}} Sp(\Omega^X_{\infty, \tilde{r}^\prime}) = Sp(\Omega^X_{\infty, \tilde{r}}).$$

The last statement, (6.2) and (6.5) imply (6.11).

Since the Kuratowski limit inferior and limit superior are closed (see, for example, [3, p. 18]), we obtain the following corollary of Proposition 6.2.

**Corollary 6.5.** Let $(X, d)$ be an unbounded metric space, $\tilde{r}$ be a scaling sequence. Then the sets

$$\bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}, \tilde{r}^\prime \in \mathbb{R}} Sp(\Omega^X_{\infty, \tilde{r}^\prime}) \text{ and } \bigcup_{\Omega^X_{\infty, \tilde{r}} \in \Omega^X_{\infty, \tilde{r}^\prime}, \tilde{r}^\prime \in \mathbb{R}} Sp(\Omega^X_{\infty, \tilde{r}^\prime})$$

are closed subsets of $[0, \infty)$.

Proposition 4.15 claims that every finite, nonempty and strongly rigid metric space $Y$ is isometric to a tangent space $\Omega^X_{\infty, \tilde{r}}$. Using Corollary 6.5, we will show that this is, generally speaking, not so to infinite strongly rigid metric spaces.

Let us consider a strongly rigid metric space $(Y, \delta)$ such that:

(i) $\delta(x, y) < 2$ for all points $x, y \in Y$;
(ii) $\sup\{\delta(x, y) : x, y \in Y\} = 2$;
(iii) The cardinality of the open ball

$$B(y^*, r) = \{y \in Y : \delta(y, y^*) < r\}$$

is finite for every $r \in (0, 2)$ and every $y^* \in Y$.

**Corollary 6.6.** Let $(X, d)$ be an unbounded metric space, $\tilde{r}$ be a scaling sequence, $\Omega^X_{\infty, \tilde{r}}$ be tangent and let $(Y, \delta)$ be a strongly rigid metric space satisfying conditions (i)-(iii). If $Y_1 \subseteq Y$ and $f : \Omega^X_{\infty, \tilde{r}} \rightarrow Y_1$ is an isometry, then $\Omega^X_{\infty, \tilde{r}}$ is finite.
Proof. Let $Y_1$ and $f$ satisfy the above conditions and let $y^* = f^{-1}(ν_0)$, $ν_0 = X_0^{∞, r}$. Conditions $(i_2)$ and $(i_3)$ imply that $Y$ is countable. Consequently $Ω^{X, r}_{∞, r}$ is separable. Using Corollary 6.3, Corollary 6.4 and $(i_2)$ we obtain that $Sp (Ω^{X, r}_{∞, r})$ is a closed subset of $[0, 2]$. Since

$$ Sp (Ω^{X, r}_{∞, r}) = \{ δ(y, y^*) : y ∈ Y_1 \} $$

holds, the set $\{ δ(y, y^*) : y ∈ Y_1 \}$ is also closed. If $Ω^{X, r}_{∞, r}$ is infinite, then $Y_1$ is infinite and, for every sequence $(y_n)_{n∈N}$ of distinct points $y_n ∈ Y_1$, we have

$$ \lim_{n→∞} δ(y^*, y_n) = 2. $$

Hence

$$ 2 ∈ \{ d(y, y^*) : y ∈ Y_1 \} $$

holds, contrary to $(i_1)$.

Example 6.7. Let $(Y, δ)$ be a metric space with $Y = N$ and the metric $δ$ defined such that:

$$ δ(1, 2) = 1 + \frac{1}{2}; \quad δ(1, 3) = 1 + \frac{2}{3}, \quad δ(2, 3) = 1 + \frac{3}{4}; $$

$$ δ(1, 4) = 1 + \frac{4}{5}, \quad δ(2, 4) = 1 + \frac{5}{6}, \quad δ(3, 4) = 1 + \frac{6}{7}; $$

$$ δ(1, 5) = 1 + \frac{7}{8}, \quad δ(2, 5) = 1 + \frac{8}{9}, \quad δ(3, 5) = 1 + \frac{9}{10}, \quad δ(4, 5) = 1 + \frac{10}{11}; $$

.................................................................................................................. .

Then $(Y, δ)$ is a countable, complete and strongly rigid metric space satisfying conditions $(i_1)$-$(i_3)$. By Corollary 6.6 no tangent space $Ω^{X, r}_{∞, r}$ is isometric to $(Y, δ)$.

Corollary 6.8. Let $(X, d)$ be an unbounded metric space and let $r$ be a scaling sequence. Then the following statements are equivalent:

(i) There is a single-point pretangent space $Ω^{X, r}_{∞, r}$;

(ii) All $Ω^{X, r}_{∞, r}$ are single-point;

(iii) The equality

$$ \lim_{n→∞} \left( \frac{1}{r_n} Sp(X) \right) = \{ 0 \} $$

holds;
The net $G_{X,\tilde{r}}$ of pretangent spaces to $(X, d)$ at infinity is trivial,

$$|V(G_{X,\tilde{r}})| = 1.$$  

Proof. It suffices to show that the implication (i) $\Rightarrow$ (ii) is valid. Suppose contrary that there exist pretangent spaces $^1\Omega^{X,\tilde{r}}_\infty$ and $^2\Omega^{X,\tilde{r}}_\infty$ such that

$$|^1\Omega^{X,\tilde{r}}_\infty| = 1 \quad \text{and} \quad |^2\Omega^{X,\tilde{r}}_\infty| \geq 2.$$  

Write $^1\tilde{X}^{X,\tilde{r}}_\infty$ and $^2\tilde{X}^{X,\tilde{r}}_\infty$ for the maximal self-stable sets corresponding $^1\Omega^{X,\tilde{r}}_\infty$ and $^2\Omega^{X,\tilde{r}}_\infty$ respectively. Then the equality $|^1\Omega^{X,\tilde{r}}_\infty| = 1$ implies the equality

$$^1\tilde{X}^{X,\tilde{r}}_\infty = \tilde{X}^{0,\tilde{r}}_\infty.$$  

By statement (v) of Proposition 2.2 we have $\tilde{X}^{0,\tilde{r}}_\infty \in ^2\Omega^{X,\tilde{r}}_\infty$. It follows from the inequality $|^2\Omega^{X,\tilde{r}}_\infty| \geq 2$ that

$$^2\tilde{X}^{X,\tilde{r}}_\infty \setminus \tilde{X}^{0,\tilde{r}}_\infty \neq \emptyset.$$  

Consequently we have

$$^1\tilde{X}^{X,\tilde{r}}_\infty \subseteq ^2\tilde{X}^{X,\tilde{r}}_\infty \quad \text{and} \quad ^2\tilde{X}^{X,\tilde{r}}_\infty \setminus ^1\tilde{X}^{X,\tilde{r}}_\infty \neq \emptyset.$$  

Since $^2\tilde{X}^{X,\tilde{r}}_\infty$ is self-stable, the set $^1\tilde{X}^{X,\tilde{r}}_\infty$ is not maximal self-stable, contrary to the definition.

Using Corollary 6.8 we can construct an unbounded metric space $(X, d)$ such that there exist single-point pretangent spaces but these spaces are never the tangent spaces to $(X, d)$ at infinity.

Example 6.9. Let $\mathbb{Z}$ be the set of all integer numbers, $t \in (1, \infty)$ and let $X$ be a subset of the real line $\mathbb{R}$ (with the standard metric $d(x, y) = |x - y|$) such that $x \in X$ if and only if $x = 0$ or $x = t^i$ for some $i \in \mathbb{Z}$. Let us define a scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ as

$$r_n := t^{n/2}, \quad n \in \mathbb{N} \quad (6.12)$$  

and put $p = 0$. Then we have

$$Sp(X) = \{|x - 0| : x \in X\} = X$$  

and

$$\frac{1}{r_n}Sp(X) = \begin{cases} X & \text{if } n \text{ is even} \\ \sqrt{t}X & \text{if } n \text{ is odd}. \end{cases} \quad (6.13)$$
It is easy to see that the inclusion
\[ \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right) \subseteq \lim_{k \to \infty} \left( \frac{1}{r_{n_k}} \text{Sp}(X) \right) \tag{6.14} \]
holds for every infinite subsequence \((r_{n_k})_{k \in \mathbb{N}}\) of \(\tilde{r}'\). Using (6.13) and (6.14) with \(n_k = 2k, k \in \mathbb{N}\) and with \(n_k = 2k + 1\) we obtain
\[ \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right) \subseteq X \]
and, respectively,
\[ \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right) \subseteq \sqrt{t}X. \]
Consequently, we have
\[ \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right) \subseteq (\sqrt{t}X) \cap X = \{0\}. \]
It is clear that
\[ 0 \in \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right). \]
Thus we obtain the equality
\[ \lim_{n \to \infty} \left( \frac{1}{r_n} \text{Sp}(X) \right) = \{0\}. \]
Now Corollary 6.8 implies that, for \(\tilde{r} = (r_n)_{n \in \mathbb{N}}\) defined by (6.12), there is a unique pretangent space \(\Omega^X_{\infty, \tilde{r}}\) and this space is single-point. A simple calculation shows that the equality
\[ p^+(\text{Sp}(X), \infty) = \frac{t - 1}{t} \tag{6.15} \]
holds. Consequently, by Theorem 5.3 the metric space \((X, d)\) does not have any single-point tangent spaces at infinity.

Letting, at equality (6.15), \(t\) to 1 we obtain the following proposition.

**Proposition 6.10.** For every \(\varepsilon > 0\) there are an unbounded metric space \((X, d)\) and a scaling sequence \(\tilde{r}\) such that \(G_{X, \tilde{r}}\) is trivial and
\[ p^+(\text{Sp}(X), \infty) < \varepsilon \]
holds.
In the previous proposition, we considered the metric space $s$ having an arbitrary small positive porosity at infinity. What happens if this porosity becomes zero?

**Proposition 6.11.** Let $(X, d)$ be an unbounded metric space, $p \in X$. If $Sp(X)$ is a nonporous set, then the inequality

$$|\Omega^X_{\infty, \tilde{r}}| \geq 2$$

holds for every pretangent space $\Omega^X_{\infty, \tilde{r}}$.

**Proof.** Suppose $Sp(X)$ is nonporous at infinity, i.e.,

$$p^+(Sp(X), \infty) = 0$$

holds. Let $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ be a scaling sequence. By Corollary 6.8 it suffices to show that there is a pretangent space $\Omega^X_{\infty, \tilde{r}}$ satisfying (6.16). From Definition 5.1 and (6.17) it follows that

$$\lim_{n \to \infty} \frac{l(\infty, r_n, Sp(X))}{r_n} = 0,$$

where $l(\infty, r_n, Sp(X))$ is the length of the longest interval in $[0, r_n) \setminus Sp(X)$. Write

$$\tau_n := \sup([0, r_n) \cap Sp(X)), \quad n \in \mathbb{N}.$$  

Then (6.18) implies the equality

$$\lim_{n \to \infty} \frac{r_n - \tau_n}{r_n} = 0.$$

Thus

$$\lim_{n \to \infty} \frac{\tau_n}{r_n} = 1$$

hold. It is easy to see that, for every $n \in \mathbb{N}$, we have

$$\tau_n = \sup\{d(p, x) : x \in B(p, r_n)\},$$

where $B(p, r_n)$ is the open ball $\{x \in X : d(x, p) < r_n\}$. It follows from (6.19), (6.20) and the definition of $Seq(X, \tilde{r})$ that there is $\tilde{x} \in Seq(X, \tilde{r})$ such that

$$\tilde{d}_p(\tilde{x}) = \lim_{n \to \infty} \frac{d(x_n, p)}{r_n} = 1.$$

Consequently if $\tilde{X}_{\infty, \tilde{r}}$ is maximal self-stable subset of $Seq(X, \tilde{r})$ such that $\tilde{x} \in \tilde{X}_{\infty, \tilde{r}}$, then the inequality

$$|\Omega^X_{\infty, \tilde{r}}| \geq 2$$

holds for the metric identification $\Omega^X_{\infty, \tilde{r}}$ of $\tilde{X}_{\infty, \tilde{r}}$. 

\[ \square \]
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