We examine the extent of gerrymandering for the 2010 General Assembly district map of Wisconsin. We find that there is substantial variability in the election outcome depending on what maps are used. We also found robust evidence that the district maps are highly gerrymandered and that this gerrymandering likely altered the partisan make up of the Wisconsin General Assembly in some elections. Compared to the distribution of possible redistricting plans for the General Assembly, Wisconsin’s chosen plan is an outlier in that it yields results that are highly skewed to the Republicans when the statewide proportion of Democratic votes comprises more than 50-52% of the overall vote (with the precise threshold depending on the election considered). Wisconsin’s plan acts to preserve the Republican majority by providing extra Republican seats even when the Democratic vote increases into the range when the balance of power would shift for the vast majority of redistricting plans.

We generate an ensemble of 19,184 redistricting plans drawn from a distribution placed on redistricting plans of the state of Wisconsin. The probability distribution used is concentrated on redistricting plans that satisfy design criteria laid out in the Wisconsin constitution, statutes, and relevant court cases: compactness and contiguity of districts, equal partition of votes, resistance to splitting counties across districts, and compliance with the Voting Rights Act (VRA). With the possible exception of satisfying the VRA, none of these design criteria have any partisan tilt.

We explore three basic questions: the variability of elections results across redistricting plans; the degree to which the Wisconsin Act 43 is typical or an outlier with respect to its partisan bias; and lastly, the structural source of any bias.

Our approach has a number of inherent advantages. We do not presume any notion of proportional representation based on statewide vote counts. By sampling, we are able to factor in the inherent geopolitical structure of Wisconsin as most of the redistricting maps were generated on the restricting maps used in Wisconsin and give graphical aids for understanding and detecting gerrymandering. In Section 4, we explore a number of summary statistics which quantify the understanding and detecting gerrymandering. In Section 3, we explore the structural source of any bias.

In Section 1, we discuss how the election results may vary depending on the restricting maps used. In Sections 2 and 3, we explore the geopolitical structure of Wisconsin and give graphical aids for understanding and detecting gerrymandering. In Section 4, we explore a number of summary statistics which quantify the understanding and insights developed in Sections 2 and 3.

We find that the Wisconsin redistricting plan is highly gerrymandered and less representative than at least 99% of all plans in our ensemble and shows more Republican bias than over 99% of the plans. The gerrymandering results are stable over a number of different sets of votes and years. These results are summarized in Tables 1 and 2 in Section 4. These results further suggest that the election outcomes produced by the Wisconsin maps systematically become less representative of our ensemble as the overall percentage of Republican votes decreases to 50% and below. This is further supported by graphical analysis in Figures 3–7 in Section 3. The Act 43 map acts as a kind of firewall, keeping a Republican Assembly majority in place even as voter preferences becomes increasingly Democratic. In Figure 9, we show that there is a fundamental asymmetry in the geopolitical structure of Wisconsin as most of the redistricting maps in our ensemble require less than 50% of the votes to be Republican to make equal the chance of either party being in the legislative majority. Nonetheless, the Act 43 map is again a clear outlier in favor of the Republicans as it requires a much lower percentage of Republican votes to produce an equal chance of having a legislative majority.

In Section 5, we describe how the ensemble is generated by Markov Chain Monte Carlo. In Section 7, we give evidence that our results are robust and that the algorithm is sufficiently converged. In particular we show that the our all of our results remain unchanged when a larger ensemble of 84,500 redistricting plans is used. In Sections 6 and 8, we make some technical comments about data curation.

This report continues our work started in (1–3). It is related to other works on sampling and computation in the redistricting context (4–11). In particular, the recent papers (12, 13) apply sampling to the Wisconsin redistricting setting that we consider here.

1. The Inherent Variability of Election Results

For each redistricting plan in the ensemble, the outcome of the election is computed using votes from either the Wisconsin General Assembly elections from 2012 (denoted WSA12), from 2014 (denoted WSA16) and from 2016 (denoted WSA16). In all cases, the actual votes were used at the ward level. However, the existence of unopposed races necessitated interpolating the data using votes from other elections in a number of wards: 27% in WSA12, 46% in WSA14, and 49% in WSA16. The details of this interpolation are given in Section 6, but the vote counts are based on actual Wisconsin election data in the years given.

Figure 1 shows the frequency of different election outcomes in our ensemble using the votes from WSA12, WSA14, and WSA16. Across the redistricting plans for the 99-seat Wisconsin General Assembly, the expected number of seats won by Republicans was typically concentrated within a range of 3-5 seats. However, a small proportion of redistricting plans are outliers, which extend the range to as many as 10 seats. This wide range of possible outcomes shows that the state’s choice of redistricting plan can have an effect on the same order as the typical changes in the popular vote across elections (e.g. a swing of 60 to 64 elected Republicans from 2012 to 2016 in the Wisconsin General Assembly). The fact that the different redistricting plans in the ensemble give such different results speaks to the need for a concept of acceptable redistricting, lest the state’s redistricting exercise become as, or more important than, the democratic expression of voters.

While the precise definition of a typical result may be debatable,
it is clear that some extreme ranges clearly represent anomalous behavior: the results should be labeled as outliers. The view that some points would clearly be labeled as outliers is the starting point for our analysis.

2. Situating the Wisconsin Act 43 Redistricting in the Ensemble

We now turn to situating the actual redistricting plan established by Act 43 of the 2011 Wisconsin General Assembly within our ensemble of 19,184 redistricting plans. This was the redistricting plan actually used in the WSA12, WSA14, and WSA16 elections. The annotation “WI” on each plot in Figure 2 indicates the number of seats produced by this redistricting. We note that the use of our modified election data in 2012 and 2014, which interpolates the missing data caused by unopposed races, does not change the balance of power. However, in 2016 the results of the actual election differed from those our interpolated vote data produces. The actual results had three fewer Republican seats than the interpolated results would have had, due to unopposed races in which Democrats ran unopposed in districts that tended to vote Republican. The number of unopposed races was least in 2012 with 27%, growing to 46% in 2014, and then to 49% in 2016.

Any reasonable sense of outlier would label the Wisconsin result in 2012 as anomalous. Yet, the actual result produced by the same map is well within the distribution for 2014 and 2016, in which the Republican share of the vote was considerably higher. As we see below, this behavior turns out to reflect an unusual property of Wisconsin’s redistricting plan: it gives an anomalously high number of seats to Republicans in elections in which Democrats perform well, but a typical number of seats in elections in which Republicans perform well.

To better understand this situation, we consider the outcome that would occur if votes from a number of other elections were used as if they had been cast for the Wisconsin General Assembly. Specifically we compare the effect of using results from U.S. House, U.S. Senate, and Presidential elections from 2012, 2014, and 2016 in addition to our interpolated results for WSA12, WSA14, and WSA16. Figure 3 presents an interesting trend: whenever the election would have typically produced around 55 or fewer Republican seats, the Wisconsin plan behaves very anomalously in the sense that it is far to the right in the histogram. In fact, even though the expected number of Republican seats falls below 50 in one election and the statewide percentage of Republican votes falls well below 50% in three elections, the number of seats elected stays pinned in the high 50s; it is almost constant despite the fact that the Republican vote continues to fall as one moves down the plot.

The plot shows that to determine whether the outcome of a given map will be anomalous for a particular election, it is not enough to consider only the total vote count or expected number of seats. The USH12 and WSA12 are similar by those two metrics, yet the outcome in the first is typical but the second is anomalous. This detail shows the importance of the geopolitical structure of the votes in determining the outcome and the pitfalls of coarse, global measures.

Based on these insights, we propose a method to evaluate the extent to which a state redistricting plan is an outlier with respect to its ability to protect a party from losing seats: we examine the impact of shifting the proportion of votes up or down within each election examined. We shift the proportions in WSA14 and WSA16 uniformly up and down in all districts so that the statewide Republican vote fraction varies from 45% to 55%. We then plot the histograms and the election results for each shifted vote count in Figure 4.

Unlike the plots in Figure 3, the geopolitical structure of all of these shifted votes is identical. Both plots in Figure 4 exhibit the trend we already observed in Figure 3. As the percentage of Republican votes decreases, the election results for both WSA14 and WSA16 (shown with red dots in Figure 4) move from being representative
(located in the center of the histograms) to being outliers (located in the extrema of the histograms).

The Wisconsin redistricting seems to create a firewall which resists Republicans falling below 50 seats. The effect is striking around the mark of 60 seats where the number of Republican seats remains constant, despite the fraction of votes dropping from 51% to 48%.

Figure 5 gives a more stylized version of Figure 4. Rather than the entire histogram, we plot the mean, variance, a region containing 90% of all sampled redistricting plans, and the extrema for a larger number of finer shifts that swing the election 10 percentage points in both directions of the observed result. The fine black line gives the number of seats produced by the Wisconsin Act 43 redistricting. Though the local geography of the votes in the three elections is different, each produces a clear deviation from the typical results starting around 50%. This deviation continues as the fraction of Republican votes decreases. All three elections (especially WSA14 and WSA16) show a significant range of Republican votes where the partisan outcome of the election (expressed in the number of Republican seats) does not change even though the percentage of the Republican votes decreases substantially.

Finally, we seek to summarize the extent to which Wisconsin’s redistricting plan is an outlier (compared to the ensemble of redistricting plans). Toward this end, we defined a statistic as follows: for each of various shifts around an equally split election, we (i) calculate the extent to which a state’s redistricting plan produces results different from the results produced by the distribution of possible redistricting plans, and (ii) take the average across the shifts. For any election, we then measure the extent to which a state’s plan is an extreme outlier with respect to the same statistic. (See Section 4 for more details.) As show in Figure 6, we find that the Wisconsin plan is an extreme outlier. In each of the three elections (2012, 2014, and 2016), it is more extreme than 99% of all possible redistricting plans in our ensemble (See the values of $H$ in Table 2 in Section 4). This statistic is essentially the log-likelihood of seeing the election outcome produced in the Wisconsin plan averaged across the shifted elections.

The above statistic is symmetric in that it measures if anomalous results favors one political party over the other, not which party. Using a second set of statistics, we measure if one party is favored over the other by the Wisconsin plan. For each party, we measure (i) what fraction of redistricting plans from our ensemble produce fewer legislative seats than the state’s plan, and (ii) take the minimum of these fractions across all the shifts considered above. As before, for any election, we then measure the extent to which a state’s plan is an outlier with respect to these statistics measuring party bias. In each of the three elections (2012, 2014, and 2016), the Wisconsin plan is more favorable to the Republicans with respect to this statistic than 99% of all possible redistricting plans in our ensemble. In contrast, the percentage of plans that favor the Democrats less at some shift is much smaller (only 3%, 73%, and 8% in the 2012, 2014, and 2016 elections respectively). See the values of $L_{rep}$ and $L_{dem}$ in Table 2 in Section 4). We will further see that, in the contexts in which the Act 43 map aids Democrats, its effect is to make a large GOP majority somewhat smaller. In Section 7, we also consider a complementary approach in which we assess shifts of up to ±7.5% centered around the outcomes of each election. Wisconsin’s plan is again seen to be an extreme outlier.

In the next section, we explain why the Wisconsin plan is such an outlier by exploring the structure of the vote in more detail. The graphical understanding of the structure of the vote developed in the next section, and Figure 7 in particular, is encapsulated in the Gerrymandering Index defined in (3). In Section 4, we explore the Gerrymandering Index of the Wisconsin plan over a number of historical Wisconsin elections (WSA12, WSA14, WSA16, Governor 2012, US House 2012 and 2014, US Senate 2012 and 2016, Wisconsin Secretary of State 2014, and Presidential 2012 and 2016). Again situating the result in our ensemble, we fine that at worse 98% of our ensemble had a better Gerrymandering Index. For the majority of the elections considered, none of the redistrictings in our ensemble had a worst Gerrymandering score (See Table 1 in Section 1).
3. Exposing the Geopolitical Structure of Wisconsin

To understand the structure which leads to the results of the previous section, we repeat the marginal analysis developed in (2, 3). Fixing a set of votes, for each redistricting we calculate the percentage of Republican votes and then place this vector of 99 numbers in increasing order. To gain insight into the distribution of this 99-dimensional vector when varied over our ensemble, we plot a box-plot for each marginal direction. As standard in box-plots, the box contains 50% of the values, the outer whiskers bracket whichever is smaller – 1.5 times the interquartile range from each quartile or the furthest outlier – and the central line through the box marks the median value.

The resulting 99 box-plots arranged on one graph for WSA12, WSA14, and WSA16 give insight into the inherent geopolitical geometry of Wisconsin due to the interaction of the state’s geometry with population density and partisan distribution (Figure 7). We see that typically there are at least 25 districts with less than 40% Democratic vote.

These plots give a method to determine the typical partisan makeup of each district. This inherently reflects the geopolitical structure of the state. For a given redistricting plan, if a given district’s percentage falls below the horizontal 50% line then the district elects a Republican. If it falls above the line, it elects a Democrat. This plot has proven useful in detecting redistricting plans with packed or fractured districts. See (3, 14). In some sense, they give quantitative definitions to these concepts.

If a given district’s Democratic vote percentage is at the bottom or below the box plot, the district has fewer Democrats than expected. If the percentage is above or at the top of the box plot, the district has more Democrats than expected. It is clear that the Wisconsin Act 43 redistricting plan produces election results with Democratic votes depleted from the center of the plot and places those votes in the districts which already have a large number of Democratic voters.

We also understand why the actual results of the WSA12 elections were not representative while the actual results of the WSA14 and WSA16 elections were representative. It is simply a result of where the 50% line hits the box plot graph in Figure 7. If the 50% line crosses the graph in the region in which the location of the current Wisconsin redistricting plan (the red dots) falls outside of the boxplot (which encodes typical behavior) then the results will be anomalous. This is the case in WSA12 but not in WSA 14 and WSA16. On the other hand, in all three years the district corresponding to the 50th seat, which dictates who is in the majority, is always an outlier, requiring less Republican votes than expected.

A. Inherent Lack of Proportionality. Notice that there is a structural tilt to the Republicans – in all of our analyses, a 50% vote fraction for both parties leads to a majority of Republicans. We see that one only needs the Republican vote to be around 47% to 49% to obtain 50 seats with the structure of the WSA12 votes over the
majority of redistricting plans. Similarly, WSA14 and WSA16 require between 46% and 47% Republican vote fractions and between 45% and 46% Republican vote fractions, respectively, to obtain 50 seats. This shows clearly that it is not reasonable to expect that 50% of the vote leads to 50% of the seats. This does not explain all of the shift in the Republican favor produced by the Wisconsin Act 43 redistricting plan. Our analysis allows us to separate out the effect of the geopolitical landscape, and to show that the Act 43 map generates extreme partisan asymmetry above and beyond this effect.

### B. Exploring Parity

To further explore the impact of the structure in Figures 5 and 7, we explore two ideas around parity. We begin by shifting the votes in WSA12, WSA14, and WSA16 so that there are an equal number of redistricting plans in the ensemble in which the Republicans and Democrats are in the majority. When shifting the votes in this way, the Wisconsin Act 43 redistricting produces significantly more Republican seats – 56 with WSA12, 57 with WSA14, and 54 with WSA16. In the first two cases, this is a result seen in very few redistricting plans of the ensemble redistricting plans while in WSA16 it has a very low probability (that is to say a relatively small number redistricting plans compared to the just under 20,000 plans considered in the ensemble).

Of course, one could perform a similar analysis around another point than the 50% mark. One can see whether if the Wisconsin redistricting would still be an outlier when the votes are centered around a different line by drawing a vertical line in Figure 5 at a different value and noting where the thin black line corresponding to the Wisconsin Act 43 redistricting crosses this vertical line. For instance for WSA12, any vertical line up to at least 52% and above 41% results in a result with Wisconsin Act 43 which is well outside the results of 90% of the ensemble – very few redistricting plans which give this result exist in the ensemble. Similar in WSA14 from about 43% to 50.5% the results produced by Wisconsin Act 43 are outliers as they lie outside the region containing 5% to 95% of the ensemble. Lastly for WSA16, from 42% to 50% the Wisconsin redistricting produces results which are outside the region containing 5% to 95% of the ensemble. In all cases the results are skewed to the Republicans precisely in the region where the Democrats threaten to move into the majority.

A complimentary perspective is instead to ask to what percentage of Republican vote does one have to shift the election to produce a 50/50 split of the seats with a given redistricting. A histogram of the quantity tabulated over the ensemble is shown in Figure 9 along with the percentage needed for the Wisconsin Act 43 redistricting plan. Again we see there is a systematic tilt towards the Republicans built into the geopolitical structure of the state. However, in all cases the percentage needed to produce parity in the Wisconsin Act 43 plan is abnormally low.

### 4. Summary Statistics

We now develop a number of summary statistics that highlight and make quantitative the graphical analysis developed in the last section.
**Representative Index.** The Gerrymandering Index directly measures how anomalous the partisan composition of a redistricting is. It is possible for a redistricting to be gerrymandered, yet still be representative of the vote count, as we have seen in Figure 1 for WSA14 and WSA16. Therefore, we also measure how representative a redistricting is in the context of different vote counts.

In (3), we also define a Representative Index which quantifies how representative the result obtained by using a particular redistricting and vote combination is. It is essentially the distance from the mean representative the result obtained by using a particular redistricting has districts whose vote margins for each election deviate from what is expected in Figure 7. For a given election, the square of the Gerrymandering index is the sum of the square deviations of each of the sorted Democratic percentages from the means of the marginals in the ninety-nine box-plots in Figure 7. To contextualize the Gerrymandering Index, we situate a given score within the distribution of scores from our ensemble of redistricting plans. Redistricting plans which have unusually large Gerrymandering index should be view as Gerrymandered. The percentage of the ensemble with Gerrymandered redistricting plans is highly gerrymandered index worst than the Wisconsin Act 43 redistricting is shown in Figure 7. There is a strong correlation between Republican vote fraction and Representativeness. (PRE = President, WAG=Attorney General, GOV= Governor, USS= US Senate, USH=US House)

| Voting data | % more gerrymandered | % less representative | Rep. Vote Fraction |
|-------------|----------------------|----------------------|-------------------|
| WSA16       | 0.01                 | 87.98                | 52.91             |
| GOV12       | 0                    | 48.61                | 52.59             |
| USH14       | 0.09                 | 46.78                | 51.89             |
| WSA14       | 0                    | 82.34                | 51.28             |
| USS16       | 0                    | 40.93                | 51.09             |
| WSA12       | 0                    | 0.44                 | 50.05             |
| PRE16       | 1.52                 | 60.00                | 49.91             |
| USH12       | 0.47                 | 25.02                | 48.68             |
| USS12       | 0.47                 | 0                    | 46.65             |
| PRE12       | 0                    | 0                    | 45.98             |

Table 1. We show the percentage of redistricting plans within the ensemble that are (i) more gerrymandered and (ii) less representative than Wisconsin Act 43 redistricting; we also display the republican vote fraction. There is a strong correlation between Republican vote fraction and Representativeness. (PRE = President, WAG=Attorney General, GOV= Governor, USS= US Senate, USH=US House)

We compliment this nonpartisan statistic with one designed to measure deviations in the Republicans’s advantage, denoted by $L_{\text{rep}}$, and one to measure deviations in the Democrats’s advantage, denoted by $L_{\text{dem}}$. In Table 1, we see based on the $H$ statistics that the Wisconsin Act 43 districts are outliers being much less representative than most of the redistricting plans in the ensemble. The $L$ statistic shows that the Wisconsin redistricting is tilted to favor the Republicans. In one year the $L_{\text{dem}}$ raises to the almost 75%; however the box-plots in Figure 7 show that the benefit to Democrats comes in elections where the Republicans hold a strong majority in any event, so that the benefit does not affect majority control.

To capture the representativeness over a range of election outcomes, we consider shifted election votes over a range of outcomes. We consider a measure which registers both the worst-case deviation from the typical and one which measures the average deviation.

Fixing a set of votes to evaluate election outcomes, we define the index $\ell_{\text{rep}}$ to be the minimum, overall shifts of the percentage Republican vote between 45% and 55%, of the probability that the number of Republican seats for a given map is greater than one drawn.
from our distribution. We estimate this probability using the ensemble we generated. We then define $L_{\text{rep}}$ to be the fraction of maps in our ensemble for which $\ell_{\text{rep}}$ is greater than it is for the map in question. We define $\ell_{\text{dem}}$ and $L_{\text{dem}}$ in the same fashion but with Republicans replaced by Democrats.

The $L$ statistics described above compare the worst case between two redistricting plans and are inherently one-sided, hence the Democratic and Republican versions. It is also useful to consider a statistic which is an average over a range of shifts. Again fixing a set of votes and a redistricting to be investigated, we define $h$ to be the sum over a set of shifts of the logarithm of the probability that two of the redistricting plans in question produce the same number of Republicans as a random redistricting drawn from our distribution. As with the preceding statistic, we determine a sense of scale for $h$ by defining $H$ to be the probability that the $h$ of a given redistricting is greater than a randomly drawn redistricting from our ensemble.

$$H(\xi, \pi) = P\left(h(\xi, \pi) \geq h(\Xi, \pi)\right).$$

In calculating $H$, we extrapolate the observed histogram using a Gaussian tail approximation whenever a values is needed outside the range observed in the histogram.

We report the summary statistics in Table 2. We find that the Wisconsin Act 43 redistricting is an extreme outlier in terms of how probable it is to observe its value of $H$. We also find that in the worst case, it can benefit the Republicans by more than 99% of all redistricting plans in our ensemble. Conversely, when shifting between 45%-55% of the vote fraction, the Democrats are significantly impeded in WSA12 and WSA16, and are aided to a much lesser degree in other elections and vote shifts. We remark that when we re-examine Figure 5, the Democrats are only ‘aided,’ once the Republicans have obtained a super majority, as can be seen by the thin continuous line falling below the 90% region.

### 5. Generating the Ensemble of Redistricting Plans

Our method begins by first placing a probability distribution on all the reasonable redistricting plans. The probability distribution will be concentrated on redistricting plans which better satisfy the design criteria. The Wisconsin State Assembly (WSA) districting plans will correspond to better compliance with the associated design criteria.

#### A. The Distribution on Redistricting Plans

Following the prescription from (3) (see also (1, 14)), we consider distributions with a density proportional to

$$e^{-\beta J(\xi)}$$

where $\xi$ is the function which assigns to each ward a district which we label with the numbers 1 to 99 for convenience. The score function $J$ will be the sum of a number of different score functions

$$J(\xi) = w_{\text{comp}} J_{\text{comp}}(\xi) + w_{\text{pop}} J_{\text{pop}}(\xi) + w_{\text{county}} J_{\text{county}}(\xi) + w_{\text{vra}} J_{\text{vra}}(\xi)$$

where $J_{\text{comp}}(\xi)$ measures compactness, $J_{\text{pop}}(\xi)$ measure population deviation from the ideal, $J_{\text{county}}(\xi)$ the number of counties split across counties, and $J_{\text{vra}}(\xi)$ measures the compliance with the Voting Rights Act (VRA); the $w_i$’s are positive weights. In all cases, low scores will correspond to better compliance with the associated design criteria.

We will use the population and compactness score functions from (3). The county and VRA score functions will versions of those from (3) with modifications to adapt to the details of the Wisconsin redistricting context. The Wisconsin State Assembly (WSA) districting required that many counties and towns be split into more than two districts (in contrast to the work in (3)). Hence minor alterations were required to our previous score functions.

We determine the weight parameters with a nearly identical process to that described in (3). We have determined $w_{\text{pop}} = 2200$, $w_{\text{comp}} = 0.8$, $w_{\text{county}} = 0.6$, $w_{\text{vra}} = 100$.
B. Markov Chain Monte Carlo Sampling. We sample redistricting plans according to the algorithm presented (3). For simulated annealing parameters, we take 20,000 accepted steps at $\beta = 0$, 80,000 accepted steps as $\beta$ linearly increasing to one, and 20,000 steps for $\beta = 1$ (see (3) for more details about the meaning of these parameters). In our reported ensemble we take 19,184 samples. In Section 7, we show evidence that this is sufficient to correctly sample the distribution on redistricting plans.

C. Redistricting Plans in the Ensemble. We ensure that the districts are contiguous, all redistricting plans are more compact than the Wisconsin Act 43 plan. We only kept samples such that the maximum population deviation is below 5%. To account for the VRA, we only retain redistricting plans containing six districts that have at least 40% African Americans and one district that has at least a 40% Hispanic population. All sampled redistricting plans are described at the ward level, and no ward is split.

With the above criteria, we have account for all districting criteria present in the Wisconsin constitution, with the exception of splitting townships. We also gather a smaller number of samples (2043) from a distribution that concentrates on redistricting plans that also preserve townships. The township consideration requires an additional term in the score function, and is similar in form to the county splitting energy. We compare the effect of preserving townships below in Section 7.

6. Interpolating Election Data

We now explain how we interpolate the election data which is missing due to unopposed races. Let $V_{tot}(i), V_{dem}(i),$ and $V_{rep}(i)$ be respectively the total vote, the Democratic vote, and the Republican vote in ward $i$. We split the ward indices into the good $G$ and $B$ wards. Typically the wards in $B$ are the wards where the race is unopposed. We also make use of a second set of voting data $(U_{tot}(i), U_{dem}(i), U_{rep}(i))$ for which no data is missing. We begin by considering the pairs $(U_{tot}(i), V_{tot}(i)) : i \in \{1, \ldots, 99\}$ which we assume to be sorted by the first value. To interpolate $V_{tot}(i)$ for some $i \in B$, we select the pairs

$$\{(U_{tot}(i_1), V_{tot}(i_2)), (U_{tot}(i_2), V_{tot}(i_1))\}$$

where $i_1$ and $i_2$ are the next two elements in the increasing ordered sequence of $V_{tot}$ values after $U_{tot}(i)$ so that $i_1, i_2 \in G$. Similarly $U_{tot}(i_1)$ and $U_{tot}(i_2)$ are the previous two elements in the ordered sequence again so that both indices are in $G$. If no such point exists, we proceed with the points we have. We then perform a linear least-squares fit to this collection of points. Observe that there are always at least two points in the collection. We then evaluate this linear fit at the point $U_{tot}(i)$ to obtain our estimate of $V_{tot}(i)$ which we will denote by $\hat{V}_{tot}(i)$. Then, in the same fashion, we estimate $\rho_{rep}(i) = U_{rep}(i)/U_{tot}(i)$ and $\rho_{dem}(i) = U_{dem}(i)/U_{tot}(i)$ with $r_{rep}(i) = V_{rep}(i)/V_{tot}(i)$ and $r_{dem}(i) = V_{dem}(i)/V_{tot}(i)$ to obtain $\hat{\rho}_{rep}(i)$ and $\hat{\rho}_{dem}(i)$. We then set $\hat{V}_{rep}(i)$ to be the average of floor($\hat{\rho}_{rep}(i)\hat{V}_{tot}(i)$) and $\hat{V}_{tot}(i) - \hat{\rho}_{dem}(i)\hat{V}_{tot}(i)$ and similarly $\hat{V}_{dem}(i)$ to be the average of $\hat{\rho}_{dem}(i)\hat{V}_{tot}(i)$ and $\hat{V}_{tot}(i) - \hat{\rho}_{rep}(i)\hat{V}_{tot}(i)$ for each choice of reference vote $(U_{tot}(i), U_{dem}(i), U_{rep}(i))$, we obtain such an estimate. In some cases, we obtain multiple such estimates associated with different reference votes. We then average all of the estimates to obtain a final estimate which we then round to the nearest integer.

To decide which of the many possible combinations of reference votes $(U_{tot}(i), U_{dem}(i), U_{rep}(i))$ produce the best results, we also predict the values for the $i \in G$ and select the collection of reference votes which produces the smallest total squared error. This leads to the choices of reference votes presented in Table 3 in the following elections with unopposed elections.

In 2012 and 2014, the interpolated votes yield the same number of seats with the Wisconsin Act 43 maps as the original vote counts with the unopposed races not interpolated. In 2016, the number of seats changed from 64 to 67. To understand why this occurred, observe that districts 54, 73 and 74 were uncontested by Republicans and thus went to the Democratic candidate; however the votes in these districts leaned Republican for the President and the Senate, explaining why the interpolated result disagrees with the actual result.

7. Robustness of Results

To check the robustness of our results, we (i) take longer runs, and (ii) generate a second ensemble in which we additionally account for town splitting. In considering more runs, we extend our sampling algorithms to examine $84500$ redistricting plans. This extended sampling tests whether we have appropriately sampled the space of redistricting plans. The box plots are the most detailed of our results and all other results may be derived from the data contained within them; to show even more detail we present marginal histogram plots. We plot the marginal histograms of the extended samples compared with the reported samples in Figure 10. We find the histogram structures are visually identical for WSA12, WSA14, and WSA16 voting data which provides evidence that we have appropriately sampled the space of redistricting plans.

To consider the effect of keeping townships contiguous, we add a fifth term to the score function that is similar to the county splitting score reported in (3). We consider townships to be all wards with the same name within the shapefile provided by the Legislative Technology Services Bureau (15). For example, using this criteria the city of Wausau is comprised of 41 wards. The new score function is weighted with a value of 0.005, which we have found only marginally affects the overall districting compactness and keeps townships together in a similar way to that of the current plan in Wisconsin. We sample 2043 redistricting plans that preserve townships.

We compare the marginal histogram plots when considering town-ship splitting and for the ensemble we have reported above in Figure 11. We find the histogram structures are visually identical for WSA12, WSA14, and WSA16 voting data. Because this new ensemble predicts identical district level results, we have evidence that (1) the ensemble used throughout the paper is robust and (2) reflects all of Wisconsin’s stated redistricting criteria according to the state constitution.

Lastly, we considered alternative definitions of the summary statistics $H, L_{rep}$, and $L_{dem}$. Instead of shifting the election data so the resulting global elections margins varied between 45% and 55% on might want to take a symmetric interval around the actual global elections margins. Taking a range of $\pm7.5\%$ for the shift, we produced a second set statistics: $\bar{H}, \bar{L}_{rep}$, and $\bar{L}_{dem}$. Again we see that the Wisconsin’s plan is still an extreme outlier. The only change is that the $\bar{L}_{rep}$ statistic is much higher. As we discuss bellow, this is because

| Election to interpolate | Reference elections |
|------------------------|---------------------|
| WSA12                  | PRE12, USS12, USH12 |
| WSA14                  | GOV14, WAG14        |
| WSA16                  | PRE16, USS16        |

Table 3. Data used to interpolate Wisconsin State Assembly date. See Table 1 for abbreviations.
Extended number of samples
Reported Ensemble
Fraction of Democratic vote

District from most to least Republican (WSA12)

Fig. 10. Testing the effect of using an ensemble with more samples.

Accounting for township splitting

District from most to least Republican (WSA14)

Fig. 11. Testing the effect of favoring townships not being split by district boundaries on the districting results.
the range now includes a range of percentages where the Wisconsin plan causes the Democrats to perform better than expected in the typical plan. However the results in this range have little effect on the balance of power as the Republicans are already solidly in the majority in those elections.

We prefer $H$, $L_{\text{rep}}$, and $L_{\text{dem}}$ to $\tilde{H}$, $\tilde{L}_{\text{rep}}$, and $\tilde{L}_{\text{dem}}$ because the range is limited to 45% to 55%. While the others are more symmetric, they often pull information from the low 60% or high 30% in global vote. These ranges seem less relevant. The effect of this difference is seen in the values of $L_{\text{dem}}$ which is much higher than $L_{\text{dem}}$ because it includes elections with a large global percentage of Republican votes. From Figure 7, we see that the Democratic votes depleted from districts with partisan make up around 50% often is packed into districts with more that 60%. This causes a tilt in favor of the Democrats from what is expected should the global vote get that high. Of course if the vote is above 60% Republican, a few seats shifted to the Democrats will have little effect operationally.

8. Adjustments to Wisconsin General Assembly Redistricting

Data provided in (15) is incomplete in terms of the current redistricting plan for Wisconsin. We provide the script that we used to assign districts to unreported wards in our repository. The number of wards affected is relatively small.

9. Supplementary Materials

Database with redistricting plans and other data:

git@git.math.duke.edu:gjh/WIRedistrictingData.git

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