Non linear travelling waves with diffusion

Nota di M.De Angelis, E.Mazziotti

Presentata dal Socio Pasquale Renno
(Adunanza del 3 Marzo 2006)

Key words: Viscoelastic models, Superconductivity, Boundary-layer, Partial differential equations

Abstract- A third order parabolic operator $L_\varepsilon$, typical of a non linear wave operator $L_0$ perturbed by viscous terms, is analyzed. Some particular solutions related to $L_0$ are explicitly determined and the initial value problem for $L_\varepsilon$ is considered. The parabolic-hyperbolic behaviour is analyzed and rigorous approximations of the solution are achieved.

Riassunto- Si prende in esame un operatore parabolico del terzo ordine $L_\varepsilon$ tipico di perturbazioni introdotte su equazioni iperboliche non lineari. Si ricavano anzitutto classi di soluzioni dell’equazione non perturbata e si considera per $L_\varepsilon$ il problema di valori iniziali in tutto lo spazio. Mediante una opportuna equazione integrale si determinano stime utili alla valutazione rigorosa di approssimazioni di strato limite.

---

1Facoltà di Ingegneria, Dip. Mat. Appl. "R. Caccioppoli", via Claudio 21, 80125, Napoli. E-mail: modeange@unina.it
1 Introduction

In a vast number of realistic mathematical models (ecology, superconductivity, spread of epidemics, neurobiology, viscoelasticity), the evolution is characterized by deep interactions between wave propagation and diffusion. These models are generally given by non linear hyperbolic equations perturbed by viscous terms described by higher - order derivatives with small diffusion coefficients $\varepsilon$. Such small parameters imply very long times for the transmission of signals, at least when the diffusion is the principle process involved.

In order to have an enough knowledge of the dynamic response, the order of diffusion effects and the time - intervals where the wave behavior prevails must be estimated. The valuation of these aspects leads to the analysis of non linear parabolic - hyperbolic boundary layers. A typical example is related to equations such as:

\[
\mathcal{L}_\varepsilon u \equiv \left[ \varepsilon \partial_{xx} + c^2 \partial_{xx} - \partial_{tt} - a \partial_t \right] u = f(x, t, u, u_x, u_t) \tag{1.1}
\]

where $u = u(x, t)$ and $\varepsilon, a, c$ are positive constants.

The parabolic equation (1.1) includes models of the viscoelasticity [1]-[3] and biology [4]. In particular, when $f = \gamma + \text{sen} u$, one has the perturbed sine - Gordon equation (PSGE)which is basic in superconductivity [5],[6].

In this paper, the initial value problem $\mathcal{P}_\varepsilon$ for the PSGE in all of the space is considered and the fundamental solution $K_\varepsilon(x, t)$ of the linear operator $\mathcal{L}_\varepsilon$ is explicitly determined. So, when $f$ is linear ($f = f(x, t)$), the problem $\mathcal{P}_\varepsilon$ is quite solved. More, $K_\varepsilon$ is a $C^\infty$ function endowed with properties like those of the fundamental solution of the heat operator [7]. For this, when $f$ is not linear, it is convenient to reduce the problem $\mathcal{P}_\varepsilon$ to an integral equation whose kernel is just $K_\varepsilon$.

The qualitative analysis of this equation allows not only existence and uniqueness results, but rigorous estimates when $\varepsilon \to 0$ too. Let $\mathcal{L}_0$ the operator to which $\mathcal{L}_\varepsilon$ is reduced for $\varepsilon \equiv 0$ and let $w$ the solution of the following initial value problem $\mathcal{P}_0$:
\( L_0 w = \hat{f}(x,t,u) \) with \( w(x,0) = f_0(x), \ w_t(x,0) = f_1(x) \).

In the superconductive case, it results [8]

\[
(1.3) \quad f(u) - \hat{f}(w) = \sin w - \sin u
\]

and the error \( v = u - w \) related to the approximation of \( u \) by means of the wave solution \( w \), is the solution of a problem \( P_\varepsilon \) with initial data vanishing and a source term given by

\[
(1.4) \quad F_w(v) = \sin w - \sin u - \varepsilon w_{xt}.
\]

Then, according to several classes of special initial data \((f_0, f_1, \ldots)\), the solution \( w \) of the reduced problem \( P_0 \) is explicitly determined and, in correspondence, the remainder term \( v = u - w \) is rigorously estimated, together with the time - interval where the diffusion effects are negligible for \( \varepsilon \rightarrow 0 \). As meaningful result one can see that whatever a positive constant \( k < 1 \) is prefixed, the evolution is characterized by the travelling wave \( w \) and by diffusion effects of the order of \( \varepsilon^k \), in each interval- time \([0, T_\varepsilon]\), with \( T_\varepsilon \) defined in (6.12).

\section{Statement of the problem}

If \( T \) is a positive constant and

\[
\Omega_T = \{(x,t) : x \in R, \ 0 < t \leq T\},
\]

let consider the following initial value problem \( P_\varepsilon \):
\[
\begin{align*}
&\partial_{xx}(\varepsilon u_t + c^2u) - \partial_t(u_t + au) = f(x,t,u), \quad (x,t) \in \Omega_T, \\
u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), \quad x \in R,
\end{align*}
\]
(2.1)

where \( f, f_0, f_1 \) are arbitrary specified functions, and \( \varepsilon, a, c, \) are positive constants.

When \( \varepsilon \equiv 0 \), the parabolic equation (2.1) turns into the hyperbolic telegraph equation

\[
(c^2 \partial_{xx} - \partial_{tt} - a \partial_t)w = \bar{f}(x,t,w)
\]
(2.2)

and the problem \( P_\varepsilon \) changes into a problem \( P_0 \) for \( w(x,t) \) which has the same initial conditions of \( P_\varepsilon \).

When \( \varepsilon \) is negligible, a possible boundary-layer region could appear for \( T = \infty \).

In order to estimate the influence of the dissipative term \( \varepsilon u_{xxt} \) on the wave behavior of \( w \), the difference

\[
u(x,t,\varepsilon) - w(x,t) = v(x,t,\varepsilon)
\]
(2.3)

is to be evaluated. For this, one has the following problem \( P_v \) related to the remainder term \( v \) :

\[
\begin{align*}
&\varepsilon v_{xxt} + c^2 v_{xx} - v_{tt} - av_t = F_w(x,t,v) \quad (x,t) \in \Omega_T, \\
v(x,0) = 0, \quad v_t(x,0) = 0, \quad x \in R,
\end{align*}
\]
(2.4)

where the source term \( F_w \) is given by:

\[
F_w(x,t,v) = f(x,t,w + v) - \bar{f}(x,t,w) - \varepsilon w_{xxt}.
\]
(2.5)
3 Linear case: fundamental solution and properties

When \( F_w = F(x,t) \) is linear and the Laplace transform is applied to the problem (2.4), the transform \( \hat{v}(x,s) \) of the solution \( v(x,t) \) is:

\[
\hat{v}(x,s) = \int_{\mathbb{R}} \hat{K}_\varepsilon(x - \xi,s) \hat{F}(\xi,s) \, d\xi,
\]

(3.1)

where \( \hat{F} \) is the transform of \( F \) and

\[
\hat{K}_\varepsilon(x,s) = \frac{e^{-|x|\sqrt{s(s+a)/(\varepsilon s+c^2)}}}{2\sqrt{s(s+a)(\varepsilon s+c^2)}}.
\]

(3.2)

To obtain the inverse Laplace transform \( \mathcal{L}_t^{-1} \) of the function (3.2), let \( r = |x|/\varepsilon, \ b = c^2/\varepsilon \) and

\[
\hat{G}(r,s) = \frac{e^{-r\sqrt{s(s+a)/(s+b)}}}{2\sqrt{\varepsilon}\sqrt{(s+a)(s+b)}},
\]

(3.3)

so that:

\[
\hat{K}_\varepsilon(x,s) = \frac{1}{\sqrt{s}} \hat{G}(r,s) = \mathcal{L}_t \left[ \frac{1}{\sqrt{\pi t}} * G(r,t) \right].
\]

(3.4)

If \( I_n(z) \) denotes the modified Bessel function, let

\[
G(r,t) = \frac{r}{4\sqrt{\varepsilon}} \int_0^t \frac{e^{-\frac{r^2}{v}}}{v\sqrt{v}} e^{-b(t-v)} I_0(r\sqrt{(b-a)(t-v)/v}) \, dv.
\]

(3.5)
Then, the following theorem holds:

**Theorem 3.1** - For all \( r > 0 \), the Laplace integral \( \mathcal{L}_t G(r, t) \) converges absolutely in the half-plane \( \Re s > \max(-a, -b) \), and one has:

\[
\mathcal{L}_t G(r, t) = \hat{G}(r, s),
\]

with \( \hat{G} \) and \( G \) defined by (3.3)-(3.5).

**Proof** - By Fubini-Tonelli theorem it results

\[
(3.7) \mathcal{L}_t G = \frac{r}{4\sqrt{\pi \varepsilon}} \int_0^\infty e^{-(sv^2 + \frac{2}{v})} \frac{dv}{v} \int_0^\infty e^{-(s+b)z} I_0(r \sqrt{(b-a)z/v}) dz
\]

and further [9] it is:

\[
(3.8) \int_0^\infty e^{-(s+b)z} I_0(r \sqrt{(b-a)z/v}) dz = \frac{e^{\frac{r^2}{4}} b^{-a}}{s + b}.
\]

By means [9] of known formulae, (3.7) and (3.8) imply (3.6). The absolute convergence holds whatever the constants \( a, b \) may be. In fact, when \( a > b \), for all real \( z \geq 0 \), one has:

\[
(3.9) | J_0(r \sqrt{(a-b)z}) | \leq 1 \leq I_0(r \sqrt{(b-a)z}).
\]

**Remark 3.1** - The function \( G(r, t) \) given by (3.5) is defined also when \( r = \frac{|x|}{\sqrt{\varepsilon}} = 0 \). In fact, if in the integral of (3.5) one puts : \( r^2 (t-v)/v = z \), one has

\[
(3.10) G(r, t) = \frac{e^{-\frac{r^2}{4\sqrt{\varepsilon}}}}{4\sqrt{\pi \varepsilon} t} \int_0^\infty e^{-\frac{z^2}{4}} \frac{z^{b} t}{\sqrt{z^2 + r^2}} I_0(\frac{\sqrt{(b-a)z}}{\sqrt{z^2 + r^2}}) dz
\]

and for \( r = 0 \), it results:
(3.11) \[ G(0,t) = \frac{e^{-bt}}{4\sqrt{\pi \varepsilon t}} \int_0^\infty e^{-\frac{z^2}{4t}} I_0(\sqrt{(b-a)z}) \frac{dz}{\sqrt{z}} = \]
\[ = \frac{1}{2\varepsilon} e^{-\frac{a+b}{2}t} I_0\left(\frac{b-a}{2}t\right). \]

As consequence, by (3.4) it follows

(3.12) \[ K_\varepsilon(x,t) = \int_0^t G(r,\tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}} \]

and, for \( x = 0 \), it is:

(3.13) \[ K_\varepsilon(0,t) = \frac{1}{2} \int_0^t \frac{e^{-\frac{a+b}{2} \tau}}{\sqrt{\pi \varepsilon (t-\tau)}} I_0\left(\frac{b-a}{2} \tau\right) d\tau. \]

Moreover, it’s easy to prove the following theorem:

**Theorem 3.2-** The function \( K_\varepsilon \) defined by (3.10)- (3.12) is a \( C^\infty(\Omega_T) \) solution of the equation \( L_\varepsilon u = 0 \). More, when \( a < b = c^2/\varepsilon \), \( K_\varepsilon \) is never negative in \( \Omega_T \). \( \blacksquare \)

As (3.10)- (3.12) show, the fundamental solution \( K_\varepsilon(x,\cdot) \), qua function of \( x \), is an even function which results positive when \( a < b \) and so

(3.14) \[ 0 \leq \int_0^\infty K_\varepsilon(x-\xi,t) d\xi = 2 \int_0^\infty K_\varepsilon(y,t) dy. \]

More by (3.2) one has

(3.15) \[ 0 \leq 2 \int_0^\infty \hat{K}_\varepsilon(y,s) dy = \frac{1}{s^2 + as} = L_t\left[\frac{1}{a} (1 - e^{-at})\right] \]

and the absolute convergence of the integral (3.6) implies the following estimate:
Theorem 3.3- When \( a \leq b \) it results

\[
0 \leq \int_{0}^{\infty} K_{\varepsilon} (x - \xi, t) \, d\xi = \left( \frac{1}{a} \right)(1 - e^{-at}) \leq 1/a.
\]  

This property of \( K_{\varepsilon} \) is useful to various estimates for the non linear case.

4 Reduced equation and travelling waves

As for the model of Superconductivity, the approximation \( w \) is characterized by the source \( \bar{f} = \sin w + \gamma \), so that one has

\[
w_{xx} - w_{tt} - aw_{t} = \sin w + \gamma
\]

where it’s assumed \( c^2 = 1 \) without loss of generality.

Let \( \psi \) an arbitrary function and let \( \Pi(\psi) \) the Lobachevsky’s angle of parallelism [10], defined by

\[
\begin{align*}
\Pi(\psi) &= 2 \arctg e^\psi = 2 \arctg e^{-\psi} \quad [\psi \geq 0] \\
\Pi(\psi) &= \pi - \Pi(-\psi) \quad [\psi < 0],
\end{align*}
\]  

and such that

\[
\sin \Pi(\psi) = \frac{1}{\cosh(\psi)}.
\]

A class of travelling wave solutions of (4.1) can be explicitly evaluated whatever \( \gamma \) may be; it suffices to consider functions \( \psi = \psi(\xi) \) such as:
\[ w = \Pi[\psi(\xi)] = \varphi(x,t), \quad \text{with} \quad \xi = \frac{1}{a} (x-t). \]

Then (4.1) becomes

\[ -a\varphi_t = \sin \varphi + \gamma \]

and the function \( \psi \) is given by

\[ \frac{d\psi}{1 + \gamma \cosh(\psi)} = d\xi. \]

According to all the values of \( \gamma \), indicating by \( k \) an arbitrary constant of integration, one has the following special solutions of the non linear equation (4.1).

- **Case \( \gamma = 0 \)**

\[ w(x,t) = \Pi(\xi + k), \]

- **Case \( \gamma = 1 \)**

If \( \xi - k = \bar{\xi} \), one has:

\[ w(x,t) = 2 \arctg(-\frac{\bar{\xi} + 2}{\xi}), \]

- **Case \( \gamma^2 < 1 \)**

If \( \alpha = \sqrt{1-\gamma^2}, \quad (\alpha / k) \xi = \bar{\xi} \), one has:

\[ w(x,t) = 2 \arctg\left[ \frac{\alpha}{\gamma} \left( \frac{1 + k e^{\bar{\xi}}}{1 - k e^{\bar{\xi}}} - \frac{1}{\alpha} \right) \right]. \]

- **Case \( \gamma^2 > 1 \)**

If \( \sigma = \sqrt{\gamma^2 - 1}, \quad (\sigma / 2) \xi = \bar{\xi} \), one has

\[ w(x,t) = 2 \arctg\left\{ \frac{1}{\gamma} \left[ 1 + \sigma \tan(\bar{\xi} + k) \right] \right\}. \]
5 An integral equation

Consider now the non linear problem defined in (2.4) and assume that the prefixed approximation \( w(x,t) \) is compatible with the following assumptions on the function \( F_w \) given by (2.5):

**Conditions A**

1) \( F_w(x,t,v) \) is defined and continuous on the set

\[
\Upsilon = \{ (x,t,v) \in \Omega_T \times (-\infty, +\infty) \}
\]

2) For each \( k > 0 \) and for \( |v| < k \), the function \( F_w(x,t,v) \) is uniformly Holder continuous in \( x \) and \( t \) for each compact subset of \( \Omega_T \).

3) There exists a constant \( C_F \) such that

\[
|F_w(x,t,v_1) - F_w(x,t,v_2)| \leq C_F |v_1 - v_2|
\]

holds for all \( v_1, v_2 \) and \( w \).

4) \( F_w \) is bounded in \( \Omega_T \) for bounded \( v \) and \( w \).

**Definition 5.1** - For each prefixed \( w(x,t) \), by a solution of the problem \( \mathcal{P}_w \) we mean a function \( v \) continuous and bounded in \( \Omega_T \) such that, whatever \( \varepsilon > 0 \) may be, the derivatives \( v_t, v_{tt}, v_{xx}, v_{xxt} \) are continuous in \( \Omega_T \) and verify (2.4).

According to the formal representation (3.1) it results:

\[
v(x,t) = \int_0^t d\tau \int_R K_\varepsilon(|x - \xi|, t - \tau) F_w [\xi, \tau, v(\xi, \tau)] d\xi
\]
and the conditions A imply that, for a solution \( v(x,t) \) of the problem \( P_v \), the function \( \lambda(\xi,\tau) = F_w[\xi, \tau, v(\xi,\tau)] \) is bounded in \( \Omega_T \) and is uniformly Holder continuous in \( \xi \) and continuous in \( \tau \) on each compact of \( \Omega_T \).

If

\[
(5.2) \quad ||F_w||_t = \sup_{0 \leq \tau \leq t} \sup_{x \in R} |\lambda(\xi,\tau)|,
\]

by (5.1) and theorem 3.3 one deduces

\[
(5.3) \quad |v(x,t)| \leq (t/a) \ ||F_w||_t \quad x \in R, \ t \geq 0
\]

and so, for each compact subset of \(-\infty < x < \infty, \ 0 < t \leq T\), the function \( v \) is uniformly Holder continuous. Consequently, by Conditions A, the function \( F_w[x,t,v(x,t)] \) is uniformly Holder continuous too. More, by means of the properties of the kernel \( K_\varepsilon \), it’s possible the following statement.

**Theorem 5.1** - For each prefixed \( w \), the initial-value problem (2.4) admits a unique solution if and only if the integral equation (5.1) possesses a unique solution \( v \) continuous and bounded for \( (x,t) \in (-\infty, +\infty) \times (0,T) \).

Consider now a positive \( \eta < T \) and let

\[
(5.4) \quad ||v||_\eta = \sup_{0 \leq t \leq \eta} \sup_{x \in R} |v(x,t)|
\]

\[
(5.5) \quad B_\eta \equiv \{ v(x,t) : v \in C(-\infty, \infty) \times [0,\eta] \ \text{and} \ ||v||_\eta < \infty \}
\]

The set \( B_\eta \) is a Banach space and the mapping

\[
(5.6) \quad Fv(x,t) = \int_0^t d\tau \int_R K_\varepsilon(x-\xi,t-\tau) F_w[\xi, \tau, v(\xi,\tau)] d\xi,
\]
owing to theorems 3.2 - 3.3, maps $B_{\eta}$ into $B_{\eta}$. Then, by means of the well-known techniques of the fixed point, the following theorem can be proved.

**Theorem 5.2** - When $F_w(x,t,v)$ satisfies the conditions A, then the initial-value problem $\mathcal{P}_v$ given by (2.1) admits a unique solution. $lacksquare$

In conclusion, according to the prefixed initial data $f_0(x)$, $f_1(x)$ of the problem $\mathcal{P}_\varepsilon$, one can solve explicitly the reduced problem $\mathcal{P}_0$ by means of the research of travelling waves outlined in sect.4 and by numerous class of solutions of sine-Gordon equations available in literature.

When the solution $w$ of $\mathcal{P}_0$ is determined, the source term $F_w$ is known and its properties can be analyzed in order to show that theorems 5.1 and 5.2 can be applied. Then, in this case, the integral equation (5.1) allows to estimate the remainder term $v$.

6 Estimate of the diffusion - An example

For example, consider the approximation $w$ induced by the initial conditions

\[
\begin{aligned}
\begin{cases}
w(x,0) &= 2\arctg(e^{x/a}) \\
w_t(x,0) &= -\frac{2}{a} \frac{e^{x/a}}{1+e^{2x/a}}
\end{cases}
\end{aligned}
\tag{6.1}
\]

and let $\xi = \frac{1}{a}(x-t)$. Then, the problem $\mathcal{P}_0$ given by (4.1) - (6.1) admits the solution

\[
w(x,t) = 2 \arctg(e^\xi)
\tag{6.2}
\]

which represents a travelling wave with speed $c = 1$. It results:

\[
w_{xxt} = -\frac{2}{a^3} e^\xi \frac{e^{4\xi} - 6e^{2\xi} + 1}{(e^{2\xi} + 1)^3} \quad (\xi \in \mathbb{R})
\tag{6.3}
\]

12
and so

(6.4) \[ |w_{xxt}| \leq \beta \quad \forall (x,t) \in \Omega_T \]

where the constant $\beta$ depends only on $a$. As consequence, the source $F_w(x,t,v)$ related to the problem $\mathcal{P}_v$ for the superconductive model is

(6.5) \[ F_w(x,t,v) = \sin (v + w) - \sin w - \varepsilon w_{xxt} \]

and satisfies the conditions A. More, by (6.4) one has:

(6.6) \[ |F_w(x,t,v)| \leq |v(x,t,\varepsilon)| + \varepsilon \beta. \]

Then, by theorems 5.1 and 5.2, one deduces that the error $v$ related to the approximation $u_\varepsilon \sim w$ is such that:

(6.7) \[ |v(x,t,\varepsilon)| \leq \int_0^t d\tau \int_R \left[ |v(\xi,\tau,\varepsilon)| + \varepsilon \beta \right] |K_\varepsilon(x - \xi, t - \tau)| d\xi. \]

When $\varepsilon \to 0$, it results $a < b = \frac{c^2}{\varepsilon}$ and $K_\varepsilon \geq 0$ according to theorem 3.2. If one puts

(6.8) \[ r_\varepsilon(t) = \sup_R |v(x,t,\varepsilon)|, \]

by (6.7) and theorem 3.3 it follows

(6.9) \[ 0 \leq r_\varepsilon(t) \leq \frac{1}{a} \int_0^t r_\varepsilon(\tau) d\tau + \frac{\beta}{a} \varepsilon t \]

and so the Gronwall Lemma implies

(6.10) \[ 0 \leq r_\varepsilon(t) \leq \left[ \beta \left( \frac{T}{a} \right) e^{T/a} \right] \varepsilon \quad \forall t \in [0,T]. \]
This estimate allows to specify the infinite time - intervals where the
effects of diffusion are of the order $\varepsilon^k$ with $k < 1$. In fact, whatever
$0 < k < 1$ may be, let

\begin{equation}
T_\varepsilon = \frac{a}{2} \ln \left[ \frac{1}{\beta} \frac{1}{\varepsilon^{1-k}} \right].
\end{equation}

Then, for $T \leq T_\varepsilon$, by (6.10) it results

\begin{equation}
0 \leq r_\varepsilon(t) \leq \beta \, e^{2T/a} \, \varepsilon \leq \varepsilon^k \ \forall t \in [0, T_\varepsilon].
\end{equation}

So, when $\varepsilon$ is vanishing, the evolution of the superconductive model is
characterized by the travelling wave $w$ given in (6.2) and by diffusion effects
which are of the order of $\varepsilon^k$ in each interval-time $[0, T_\varepsilon]$, with $T_\varepsilon$ given
by (6.11) and $k < 1$.

References

[1] M. Renardy, *On localized Kelvin - Voigt damping*, ZAMM Z. Aangew Math Mech 84 no 4, 280-283 (2004)

[2] V.P. Maslov, P. P. Mosolov, *Non linear wave equations perturbed by viscous terms* Walter deGruyher Berlin N. Y. 329 (2000).

[3] P. Haupt, Continuous Mechanics and theory of Materials, (2000).

[4] J.D. Murray, *J Mathematical Biology* Vol . I, II N.Y. Springer (2002-2003).

[5] A.Barone, G. Paterno’, *Physics and Application of the Josephson Effect* Wiles and Sons N. Y. 530 (1982).

[6] Lonngren and Scott editors *Solitons in action* edited by Academic press N.Y. (1978).
[7] J.R. Cannon, *The One - Dimensional Heat Equation* Addison-Wesley Publishing Company Menio park, california (1984).

[8] A. Scott, *Active and nonlinear wave propagation in electronics* Wiley-Interscience (1989).

[9] Erdelyi, Magnus, Oberhettinger, Tricomi, *Tables of integral transforms* vol I MacGraw Hill N.Y. (1954).

[10] J.S. Gradshteyn, I.M.Ryzhik, *Table of integrals, series and products*, Academic Press (1980).