A construction of dualizing categories by tensor products of categories

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Abstract

It is shown that the idempotent completion of the additive hull of the tensor product of the residue category of the category of paths of a locally finite quiver modulo an admissible ideal and a dualizing category is dualizing. Furthermore, the category of finitely presented functors over such tensor product category is dualizing and has almost split sequences. As applications, the categories of all kinds of complexes are proved to have almost split sequences.

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1 Introduction

Throughout this paper we assume that $k$ is a commutative artin ring unless stated otherwise. Denote by $\text{Mod}_k$ the category of $k$-modules and $\text{mod}_k$ the full subcategory of $\text{Mod}_k$ consisting of all finitely generated $k$-modules. Let $E$ be an injective envelope of the factor module $k/\text{rad}k$ of $k$ modulo its radical $\text{rad}k$ in $\text{Mod}_k$, and $D := \text{Hom}_k(-, E)$. A dualizing $k$-category or dualizing $k$-variety $\mathcal{A}$ is a Hom-finite Krull-Schmidt $k$-category such that the duality $D : \text{Fun}_k(\mathcal{A}^{\text{op}}, \text{mod}_k) \to \text{Fun}_k(\mathcal{A}, \text{mod}_k)$ induces a duality $D : \text{mod}\mathcal{A} \to \text{mod}\mathcal{A}^{\text{op}}$. Dualizing $k$-categories were introduced by Auslander and Reiten as a generalization of artin $k$-algebras [2]. It is well-known that the
existence of almost split sequences is quite useful in the representation theory of algebras (Ref. [30, Chapter 2]). A $k$-category $\mathcal{A}$ being dualizing ensures that the category $\text{mod}\mathcal{A}$ of finitely presented functors in $\text{Mod}\mathcal{A}$ has almost split sequences (Ref. [30, Theorem 7.1.3]). From a given dualizing $k$-category $\mathcal{A}$, there are some known constructions of dualizing $k$-categories such as $\text{mod}\mathcal{A}$ (Ref. [2, Proposition 2.6]), the functorially finite Krull-Schmidt full $k$-subcategory $\mathcal{A}$ (Ref. [4, Theorem 2.3] and [20, §9.7 Example 5]), the residue categories $\mathcal{A}/(1_A)$ of $\mathcal{A}$ modulo the ideal $(1_A)$ of $\mathcal{A}$ generated by the identity morphism $1_A$ of an object $A$ in $\mathcal{A}$ (Ref. [20, §9.7, Example 8]), and the category $\mathcal{C}^b(\text{mod}\mathcal{A})$ of bounded complexes over $\text{mod}\mathcal{A}$ (Ref. [6, Theorem 4.3]).

In this paper, we will give another construction of dualizing $k$-categories by tensor products of $k$-categories which can be applied to construct a large number of new dualizing $k$-categories from a given dualizing $k$-category. Let $Q$ be a locally finite quiver, $kQ$ the $k$-category of paths of $Q$, $I$ an admissible ideal of $kQ$ generated by a set of paths in $Q$, $\mathcal{B} := kQ/I$ the residue category of $kQ$ modulo $I$, and $\mathcal{A}$ a dualizing $k$-category. We will prove that the idempotent completion $\oplus (\mathcal{B} \otimes_k \mathcal{A})$ of the additive hull $\oplus (\mathcal{B} \otimes_k \mathcal{A})$ of the tensor product $\mathcal{B} \otimes_k \mathcal{A}$ of $k$-categories $\mathcal{B}$ and $\mathcal{A}$, is a dualizing $k$-category. Furthermore, we will show that $\text{mod}(\mathcal{B} \otimes_k \mathcal{A})$ is a dualizing $k$-category and has almost split sequences. This is a natural generalization of [2, Proposition 2.6], [6, Theorem 4.3], and so on.

The $n$-complexes was introduced by Mayer in 1942 for setting up a new homology theory [27, 32]. This generalized homology theory was studied in [25, 17, 16, 26, 14]. The projectives and injectives in the category of $n$-complexes were described in [34, 19]. The homotopy category and derived category of the category of $n$-complexes were studied in [36, 21, 5, 24]. The $n$-complexes were also applied to study generalized Koszul algebras [9, 11, 23]. Moreover, the categories of complexes with amplitude in an interval play important roles in the theory of derived representation types [6, 7, 8, 37]. As one application of our results, we will prove that the category $\mathcal{C}^b_n(\text{mod}\mathcal{A})$ (resp. $\mathcal{C}^b_n(\mathcal{A})$) of bounded $n$-complexes over $\text{mod}\mathcal{A}$ (resp. $\mathcal{A}$ with $\text{gl.dim}(\text{mod}\mathcal{A}) < \infty$) and the category $\mathcal{C}^m_n(\text{mod}\mathcal{A})$ (resp. $\mathcal{C}^m_n(\mathcal{A})$) of $n$-complexes over $\text{mod}\mathcal{A}$ (resp. $\mathcal{A}$) with amplitude in the interval $[1, m]$ have almost split sequences. These generalize some results in [6]. The $n$-cyclic complexes or $n$-cycle complexes were introduced by Peng and Xiao (Ref. [29, §7, Appendix]) which were used to realize simple Lie algebras and their quantum enveloping algebras (Ref. [29, 12, 13]). As the other
application of our results, we will show that the category $C_{Z_n}(mod\mathcal{A})$ (resp. $C_{Z_n}(\mathcal{A})$, $C_{Z_n}(\mathcal{A})$) of $n$-cyclic complexes (resp. bounded $n$-cyclic complexes, $n$-cyclic complexes) over $mod\mathcal{A}$ (resp. $\mathcal{A}$ with $\text{gl.dim}(mod\mathcal{A}) < \infty$, $\mathcal{A}$ with $\text{gl.dim}(mod\mathcal{A}) \leq 1$) has almost split sequences. Note that the almost split sequences in the category $C_{Z_n}(\text{proj}\mathcal{A})$ of $n$-cyclic complexes over the category $\text{proj}\mathcal{A}$ of finitely generated projective modules over a finite dimensional hereditary algebra $\mathcal{A}$, are described in [31, 13].

2 Preliminaries

In this section, we will fix some notations and terminologies on all kinds of categories and functors, and almost split sequences.

2.1 Categories and functors

A category $\mathcal{C}$ is said to be skeletally small if all isomorphism classes of objects in $\mathcal{C}$ form a set. Note that a skeletally small category is called a svelte category in [20, 2.1]. For a skeletally small category $\mathcal{C}$ and a category $\mathcal{D}$, we denote by $\text{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ whose objects are all functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are all natural transformations between these functors.

A $k$-category is a category $\mathcal{A}$ whose morphism sets are endowed with $k$-module structures such that the composition maps are $k$-bilinear (Ref. [20, §2.1]). A $k$-functor from a $k$-category $\mathcal{A}$ to a $k$-category $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that the defining maps $F(A, A') : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$ are $k$-linear for all $A, A' \in \mathcal{A}$. For a skeletally small $k$-category $\mathcal{A}$ and a $k$-category $\mathcal{B}$, we denote by $\text{Fun}_k(\mathcal{A}, \mathcal{B})$ the category of $k$-functors from $\mathcal{A}$ to $\mathcal{B}$, i.e., the full subcategory of the functor category $\text{Fun}(\mathcal{A}, \mathcal{B})$ consisting of all $k$-functors, which is also a $k$-category.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $k$-categories. The tensor product of $\mathcal{A}$ and $\mathcal{B}$ is the $k$-category $\mathcal{A} \otimes_k \mathcal{B}$ whose objects are pairs $(A, B)$ and whose Hom sets $(\mathcal{A} \otimes_k \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \otimes_k \mathcal{B}(B, B')$ for all $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ (Ref. [28, Page 13]). It is well-known that if $\mathcal{A}, \mathcal{B}$ are skeletally small $k$-categories and $\mathcal{C}$ is a $k$-category then $\text{Fun}_k(\mathcal{A}, \text{Fun}_k(\mathcal{B}, \mathcal{C})) \cong \text{Fun}_k(\mathcal{A} \otimes_k \mathcal{B}, \mathcal{C})$ (Ref. [28, Page 13]).

For a skeletally small $k$-category $\mathcal{A}$, we denote by $\text{Mod}\mathcal{A}$ the category of right $\mathcal{A}$-modules, i.e., the category $\text{Fun}_k(\mathcal{A}^{\text{op}}, \text{Mod}k)$ of $k$-functors from the opposite category $\mathcal{A}^{\text{op}}$ of $\mathcal{A}$ to $\text{Mod}k$. Clearly, $\text{Mod}\mathcal{A}$ is an abelian
category. A functor $M \in \text{Mod}\mathcal{A}$ is representable if $M \cong \mathcal{A}(-, A)$ for some $A \in \mathcal{A}$. A functor $M \in \text{Mod}\mathcal{A}$ is finitely generated if there is an epimorphism $\bigoplus_{i \in I} \mathcal{A}(-, A_i) \twoheadrightarrow M$ for a finite index set $I$ and $A_i \in \mathcal{A}$. A functor $M \in \text{Mod}\mathcal{A}$ is finitely presented if there is an exact sequence $\bigoplus_{j \in J} \mathcal{A}(-, A'_j) \to \bigoplus_{i \in I} \mathcal{A}(-, A_i) \to M$ for two finite index sets $I, J$ and $A_i, A'_j \in \mathcal{A}$. Note that once $\mathcal{A}$ is a skeletally small additive $k$-category then a functor $M \in \text{Mod}\mathcal{A}$ is finitely generated if and only if there is an epimorphism $\mathcal{A}(-, A) \twoheadrightarrow M$ for some $A \in \mathcal{A}$, and a functor $M \in \text{Mod}\mathcal{A}$ is finitely presented if and only if there is an exact sequence $\mathcal{A}(-, A') \to \mathcal{A}(-, A) \to M$ for some $A, A' \in \mathcal{A}$. Denote by $\text{mod}\mathcal{A}$ the full subcategory of $\text{Mod}\mathcal{A}$ consisting of all finitely presented functors (Ref. [20, Page 22]). It is well-known that $\text{mod}\mathcal{A}$ is abelian if and only if $\mathcal{A}$ has pseudo-kernels, i.e., for any morphism $f \in \mathcal{A}(A', A)$ there is a morphism $f' \in \mathcal{A}(A'', A')$ such that

$\mathcal{A}(-, A'') \mathcal{A}(-, f') \mathcal{A}(-, f) \mathcal{A}(-, A)$

is exact (Ref. [1] Page 102, Proposition]).

2.2 Dualizing categories and almost split sequences

A $k$-category $\mathcal{A}$ is said to be Hom-finite if all Hom sets $\mathcal{A}(A, A')$ are finitely generated $k$-modules for all $A, A'$ in $\mathcal{A}$. A skeletally small additive $k$-category $\mathcal{A}$ is said to be Krull-Schmidt if each object in $\mathcal{A}$ is a finite direct sum of indecomposables with local endomorphism algebras. Note that a Krull-Schmidt category is called a multilocular category in [20, 3.1]. A skeletally small Hom-finite additive $k$-category $\mathcal{A}$ is Krull-Schmidt if and only if all idempotents in $\mathcal{A}$ split (Ref. [20, Theorem 3.3]), i.e., for each idempotent $e \in \mathcal{A}(A, A)$ there are $A' \in \mathcal{A}$, $f \in \mathcal{A}(A, A')$ and $g \in \mathcal{A}(A', A)$ such that $e = gf$ and $fg = 1_{A'}$.

A dualizing $k$-category or dualizing $k$-variety $\mathcal{A}$ is a Hom-finite Krull-Schmidt $k$-category such that the natural duality $D : \text{Fun}_k(\mathcal{A}^\text{op}, \text{mod}k) \to \text{Fun}_k(\mathcal{A}, \text{mod}k), F \mapsto DF$, where $(DF)(A) := D(F(A))$ and $(DF)(f) := D(F(f))$ for all $A \in \mathcal{A}$ and $f \in \mathcal{A}(A, A')$, induces a duality $D : \text{mod}\mathcal{A} \to \text{mod}\mathcal{A}^\text{op}$. Dualizing $k$-categories were introduced by Auslander and Reiten as a generalization of artin $k$-algebras (Ref. [2]). For an artin $k$-algebra $\Lambda$, the category proj$\Lambda$ of finitely generated projective $\Lambda$-modules and the category mod$\Lambda$ of finitely generated $\Lambda$-modules are dualizing $k$-categories (Ref. [2] Proposition 2.5 and Proposition 2.6). A locally bounded $k$-category is a Hom-finite Krull-Schmidt $k$-category $\mathcal{A}$ satisfying that for each $A$ in $\mathcal{A}$ there are only finitely many isomorphism classes of indecomposable objects $A'$ in $\mathcal{A}$ with $\mathcal{A}(A', A) \neq 0$ or $\mathcal{A}(A, A') \neq 0$. This is the additivization of the

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locally bounded $k$-category in \[11\ §2.1\]. Every locally bounded $k$-category is a dualizing $k$-category (Ref. \[30\ Proposition 7.1.5\]).

Let $\mathcal{A}$ be a Krull-Schmidt $k$-category. A \emph{right almost split morphism} in $\mathcal{A}$ is a morphism $g : Y \to Z$ in $\mathcal{A}$ which is not a retraction and for any non-retraction $g' : Y' \to Z$ there is a morphism $g'' : Y' \to Y$ such that $g' = gg''$. We say that $\mathcal{A}$ has right almost split morphisms if for any indecomposable $Z$ in $\mathcal{A}$ there is a right almost split morphism ending in $Z$. A \emph{left almost split morphism} in $\mathcal{A}$ is a morphism $f : X \to Y$ in $\mathcal{A}$ which is not a section and for any non-section $f' : X \to Y'$ there is a morphism $f'' : Y \to Y'$ such that $f' = f''f$. We say that $\mathcal{A}$ has left almost split morphisms if for any indecomposable $X$ in $\mathcal{A}$ there is a left almost split morphism starting in $X$. We say that $\mathcal{A}$ has almost split morphisms if $\mathcal{A}$ has both right and left almost split morphisms.

Let $\mathcal{A}$ be an additive $k$-category. A pair $(i, d)$ of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ of $\mathcal{A}$ is said to be \emph{exact} if $i$ is a kernel of $d$ and $d$ is a cokernel of $i$. Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{i} Y \xrightarrow{d} Z$ which is closed under isomorphisms. An exact pair in $\mathcal{E}$ is called a \emph{conflation}. The morphisms $i$ and $d$ appearing in a conflation $(i, d)$ are called an \emph{inflation} and a \emph{deflation} respectively. The class $\mathcal{E}$ is called an \emph{exact structure} on $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ is called an \emph{exact $k$-category} (Ref. \[20\ §9.1\] and \[15\ §1.1\ and Appendix]) if the following axioms are satisfied:

- (E1) The composition of two deflations is a deflation.
- (E2) For each $f \in \mathcal{A}(Z', Z)$ and each deflation $d \in \mathcal{A}(Y, Z)$, there is $Y' \in \mathcal{A}$, $f' \in \mathcal{A}(Y', Y)$ and a deflation $d' \in \mathcal{A}(Y', Z')$ such that $df' = fd'$. 
- (E3) Identities are deflations. If $de$ is a deflation, then so is $d$.
- (E3\op) Identities are inflations. If $ji$ is an inflation, then so is $i$.

It is well-known that if $(\mathcal{A}, \mathcal{E})$ is a skeletally small exact $k$-category then $(\mathcal{A}^\op, \mathcal{E}^\op)$ with $\mathcal{E}^\op := \{(d^\op, i^\op) \mid (i, d) \in \mathcal{E}\}$ is an exact $k$-category as well (Ref. \[20\ §9.1\ Example 2\]). Note also that an abelian category admits a natural exact structure whose conflations are all short exact sequences.

Let $(\mathcal{A}, \mathcal{E})$ be a Krull-Schmidt exact $k$-category. An \emph{almost split sequence} in $(\mathcal{A}, \mathcal{E})$ is a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ with $f$ a left almost split morphism and $g$ a right almost split morphism (Ref. \[15\ §2.2\]). We say that $(\mathcal{A}, \mathcal{E})$ has almost split sequence if $\mathcal{A}$ has almost split morphisms, for any indecomposable non-$\mathcal{E}$-projective object $Z$ there is an almost split sequence ending in $Z$, and for any indecomposable non-$\mathcal{E}$-injective object $A$ there is an almost split sequence starting in $X$. It is well-known that if $\mathcal{A}$ is a dualizing $k$-category then $\text{mod}\mathcal{A}$ has almost split sequences (Ref. \[30\ Theorem 7.1.3\]).
Let $\mathcal{A}$ be an additive $k$-category. A full additive $k$-subcategory $C$ of $\mathcal{A}$ is said to be contravariantly finite in $\mathcal{A}$ if for each $A \in \mathcal{A}$ the restriction $\mathcal{A}(-, A)|C$ of $\mathcal{A}(-, A)$ to $C$, i.e., there is an epimorphism $C(-, C) \to \mathcal{A}(-, A)|C$ for some $C \in \mathcal{C}$. Equivalently, $C$ is contravariantly finite in $\mathcal{A}$ if for each $A \in \mathcal{A}$, there is an epimorphism $C \to A$ (called the right $C$-approximation of $A$ in $C$) with $C \in \mathcal{C}$ such that $C(C', C) \to \mathcal{A}(C', A) \to 0$ is exact for all $C' \in \mathcal{C}$. Dually, a full additive $k$-subcategory $C$ of $\mathcal{A}$ is said to be covariantly finite in $\mathcal{A}$ if for each $A \in \mathcal{A}$, there is a morphism $C \to A$ (called the left $C$-approximation of $A$ in $C$) with $C \in \mathcal{C}$ such that $C(C, C') \to \mathcal{A}(A, C') \to 0$ is exact for all $C' \in \mathcal{C}$. Furthermore, a $k$-subcategory $C$ of $\mathcal{A}$ is said to be functorially finite in $\mathcal{A}$ if it is both contravariantly and covariantly finite in $\mathcal{A}$. These definitions were introduced by Auslander and Smalø in [3, §3, Page 81]. If a Krull-Schmidt exact $k$-category $(\mathcal{A}, \mathcal{E})$ has almost split sequences, and $C$ is a functorially finite full $k$-subcategory of $\mathcal{A}$ closed under conflations and direct summands, then $(C, \mathcal{E}|C)$ has almost split sequences (Ref. [4, Theorem 2.4]).

3 A construction of dualizing categories

In this section, we will provide a new construction of dualizing categories from a given dualizing category by idempotent completions, additive hulls, and tensor products of $k$-categories. The representations of quivers in $k$-linear categories will play a key role in the construction as well.

3.1 Idempotent completions and additive hulls

The idempotent completion of a $k$-category $\mathcal{A}$ is the $k$-category $|\mathcal{A}|$ whose objects are all pairs $(A, e)$ with $A \in \mathcal{A}$ and $e \in \mathcal{A}(A, A)$ being idempotent, and whose Hom sets are $|\mathcal{A}|((A, e), (A', e')) := e' \mathcal{A}(A, A')e$ (Ref. [20 §2.1, Example 7]). Clearly, all idempotents in $|\mathcal{A}|$ split, and if $\mathcal{A}$ is skeletally small (resp. Hom-finite, additive) then so is $|\mathcal{A}|$. Moreover, $(|\mathcal{A}|)^{op} \cong (|\mathcal{A}|)^{op}$.

The additive hull of a $k$-category $\mathcal{A}$ is the $k$-category $\oplus \mathcal{A}$ whose objects are all $m$-tuples $(A_1, \cdots, A_m)$ with $m \in \mathbb{N}$ and $A_i \in \mathcal{A}$, and whose Hom sets are $(\oplus \mathcal{A})((A_1, \cdots, A_m), (A'_1, \cdots, A'_m)) := (\mathcal{A}(A_i, A'_i))_{j,i}$ (Ref. [20 §2.1, Example 8]). Clearly, $\oplus \mathcal{A}$ is additive, and if $\mathcal{A}$ is skeletally small (resp.
Hom-finite) then so is $\oplus A$. Moreover, $(\oplus A)^{\text{op}} \cong (A^{\text{op}})^{\text{op}}$.

**Lemma 1.** Let $\mathcal{A}$ be a skeletally small Hom-finite $k$-category. Then the idempotent completion $| \oplus A$ of the additive hull $\oplus A$ of $\mathcal{A}$ is a Krull-Schmidt $k$-category.

**Proof.** Obviously, $| \oplus A$ is a skeletally small Hom-finite additive $k$-category. Moreover, all idempotents in $| \oplus A$ split. It follows from [20, Theorem 3.3] that $| \oplus A$ is Krull-Schmidt. \qed

Let $\mathcal{A}$ be a skeletally small $k$-category. Then the functor $F : \mathcal{A} \to \mathcal{A} \to (A, 1_A)$, is fully faithful, which induces an equivalence $F_* : \text{Mod}(| \mathcal{A}) \to \text{Mod}A, M \mapsto MF$ (Ref. [20, §2.2, Example 6]). Moreover, the functor $G : \mathcal{A} \to \oplus A, A \mapsto (A)$, is also fully faithful, which induces an equivalence $G_* : \text{Mod}(\oplus A) \to \text{Mod}A, N \mapsto NG$ (Ref. [20, §2.2, Example 7]). Restricted to the categories of finitely presented functors, we have equivalences $\text{mod}(\mathcal{A}) \cong \text{mod}(A)$, and further $\text{mod}(| \oplus A) \cong \text{modA}$.

### 3.2 Representations of a quiver in a category

Let $Q$ be a quiver, i.e., a directed graph. Denote by $Q_0$ the set of vertices of $Q$ and by $Q_1$ the set of arrows of $Q$. For an arrow $a \in Q_1$, denote by $s(a)$ and $t(a)$ the source and target of $a$ respectively. A path $p$ of length $l \geq 1$ with source $s(p) = v$ and target $t(p) = w$ is a sequence $a_1 \cdots a_{l-1}$ of arrows $a_i$ such that $s(a_1) = v, s(a_{i+1}) = t(a_i)$ for all $1 \leq i \leq l-1$, and $t(a_l) = w$. Besides the paths of length $\geq 1$, there are also trivial paths $1_v$ of length 0 with both source and target $v$ for all $v \in Q_0$. A quiver $Q$ gives rise to the category of paths of $Q$, denoted by $Q$ as well, whose objects are vertices of $Q$, whose Hom sets $Q(v, w)$ consists of the paths with source $v$ and target $w$, and the composition is the juxtaposition of paths. Usually, we view a quiver as a category — its category of paths. A quiver $Q$ also gives rise to the $k$-category of paths of $Q$, denoted by $kQ$, whose objects are vertices of $Q$ and whose Hom sets $(kQ)(v, w)$ are the $k$-vector spaces with a basis $Q(v, w)$ (Ref. [20, §2.1 Example 6]). The opposite quiver of a quiver $Q$ is the quiver $Q^{\text{op}}$ with $(Q^{\text{op}})_0 := Q_0$ and $(Q^{\text{op}})_1 := \{ a^{\text{op}} \mid a \in Q_1, s(a^{\text{op}}) := t(a), t(a^{\text{op}}) := s(a) \}$. Obviously, the $(k)$-category of paths of $Q^{\text{op}}$ is isomorphic to the opposite category of the $(k)$-category of paths of $Q$.

To a quiver $Q$, we associate a quiver $P(Q)^l$ called the left path space of $Q$, whose vertices are the paths $p$ of $Q$ and whose arrows are pairs $(p, ap)$:
Let $I$ be an **ideal** of $Q$ consisting of some paths of length at least 2, i.e., a set of paths of length at least 2 closed under left or right concatenation with any concatenatable path of $Q$, or equivalently, a set of paths of length at least 2 such that all the paths of $Q$ containing a path in $I$ as a subpath are in $I$. We define $Q_I$, called **quiver $Q$ with monomial relations $I$**, to be the category whose objects are all vertices of $Q$ and whose morphisms are all the paths of $Q$ not in $I$. Clearly, the category $\text{Rep}(Q_I, A)$ of representations of $Q_I$ in a pre-additive category $A$ is isomorphic to the full subcategory of $\text{Rep}(Q, A)$ consisting of all representations $R$ in $\text{Rep}(Q, A)$ such that $R(p) = 0$ for all $p \in I$. By abuse of terminology, we still denote by $I$ the ideal of $k Q$ generated by all paths in $I$, which will not cause any confusion. Then the category $\text{Rep}(Q_I, A)$ of representations of $Q_I$ in a $k$-category $A$ is isomorphic to $\text{Fun}_k(k Q / I, A)$.

For a vertex $v \in Q_0$, we define $P(Q)_v^I = \{ p \in P(Q)_v | p \notin I \}$. Denote by $A_1$ the quiver having just one vertex $\bullet$ and no arrows. We define functors $f_v : A_1 \to Q_I, \bullet \mapsto v, g_v : A_1 \to P(Q)_v^I, \bullet \mapsto 1_v$, and $t_v : P(Q)_v^I \to Q_I$ as follows: for a vertex $v$ of $P(Q)_v^I, t_v(p) := t(p)$; for an arrow $(p, ap)$ of $P(Q)_v^I, t_v(p, ap) := a$. Thus $f_v = t_v \circ g_v$. Note that each functor $h : \text{Fun}(\mathcal{C}, A) \to \text{Fun}(\mathcal{B}, A), F \mapsto F \circ h$. So we have induced functors $f_{v*} : \text{Rep}(Q_I, A) \to \text{Rep}(A_1, A), g_{v*} : \text{Rep}(P(Q)_v^I, A) \to \text{Rep}(A_1, A)$ and $t_{v*} : \text{Rep}(Q_I, A) \to \text{Rep}(P(Q)_v^I, A)$. Moreover, $f_{v*} = (t_v \circ g_v)_* = g_{v*} \circ t_{v*}$. 

$p \to ap$ with $a \in Q_1$ satisfying $t(p) = s(a)$. For a vertex $v$ of $Q$, denote by $P(Q)_v^I$ the connected component of $P(Q)_v^I$ that $1_v$ lies in.

A **representation** of a quiver $Q$ in a category $\mathcal{A}$ is a functor $R \in \text{Fun}(Q, \mathcal{A})$. Obviously, a representation $R$ of $Q$ in $\mathcal{A}$ is determined by assigning to each object $R(v) \in \mathcal{A}$ to each vertex $v \in Q_0$ and a morphism $R(a) \in \mathcal{A}(R(s(a)), R(t(a)))$ to each arrow $a \in Q_1$. A **morphism** $\phi$ between two representations $R$ and $R'$ is a natural transformation, i.e., a family of morphisms $\phi_v \in \mathcal{A}(R(v), R'(v))$ with $v \in Q_0$ such that $R'(a) \circ \phi_{s(a)} = \phi_{t(a)} \circ R(a)$, i.e., the following diagram is commutative:

$$
\begin{array}{ccc}
R(s(a)) & \xrightarrow{\phi_{s(a)}} & R'(s(a)) \\
\downarrow R(a) & & \downarrow R'(a) \\
R(t(a)) & \xrightarrow{\phi_{t(a)}} & R'(t(a))
\end{array}
$$

for all arrows $a \in Q_1$. Obviously, the category $\text{Rep}(Q, \mathcal{A})$ of representations of a quiver $Q$ in a $k$-category $\mathcal{A}$, i.e., the functor category $\text{Fun}(Q, \mathcal{A})$, is isomorphic to the $k$-functor category $\text{Fun}_k(k Q, \mathcal{A})$. 

% # Diagrams

% # Proof

% # Examples

% # Further discussion

% # Conclusion

% # References
Let $\mathcal{A}$ be a cocomplete category, i.e., small coproducts exist in $\mathcal{A}$. Completely analogous to [18, Proposition 3.1 and 3.2], the functors $g_{v*}$ and $t_{v*}$ have left adjoints $g^*_v : \text{Rep}(\mathbb{A}_1, \mathcal{A}) \to \text{Rep}(P(Q_I)^l_v, \mathcal{A})$ and $t^*_v : \text{Rep}(P(Q_I)^l_v, \mathcal{A}) \to \text{Rep}(Q_I, \mathcal{A})$ respectively. Thus the functor $f_{v*}$ has a left adjoint $f^*_v := t^*_v \circ g^*_v$. The functor $g^*_v : \text{Rep}(\mathbb{A}_1, \mathcal{A}) \to \text{Rep}(P(Q_I)^l_v, \mathcal{A})$ is defined as follows: for a representation $A$ in $\text{Rep}(\mathbb{A}_1, \mathcal{A})$, or equivalently, an object $A \in \mathcal{A}$, we define $g^*_v(A)$ to be the representation of $P(Q_I)^l_v$ which sends each vertex to $A$ and each arrow to $1_A$; for a morphism in $\text{Rep}(\mathbb{A}_1, \mathcal{A})$, or equivalently, a morphism $f : A \to A'$ in $\mathcal{A}$, we define $g^*_v(f)_p = f$ for all vertex $p$ of $P(Q_I)^l_v$. The left adjoint $t^*_v$ of $t_{v*}$ is defined as follows: for a given representation $M$ of $P(Q_I)^l_v$, we define $t^*_v(M)$ as follows: for a vertex $w$ of $Q$, $t^*_v(M)(w) := \oplus_{p \in Q_I(w,v)} M(p)$ where $M(p)$ is the object in $\mathcal{A}$ corresponding to the vertex $p$ of $P(Q_I)^l_v$ under the representation $M$ of $P(Q_I)^l_v$; for an arrow $a \in Q_1$, the morphism $t^*_v(M)(a) : t^*_v(M)(s(a)) \to t^*_v(M)(t(a))$ is defined as

$$\alpha_{qp} := M(p, ap) : M(p) \to M(ap) \quad \text{is the morphism in } \mathcal{A} \text{ corresponding to the arrow } (p, ap) : p \to ap \text{ of } P(Q_I)^l_v \text{ under the representation } M \text{ of } P(Q_I)^l_v \text{ if } q = ap, \text{ and } \alpha_{qp} := 0 \text{ otherwise. Moreover, we need to define } t^*_v \text{ for morphisms. If } f : M \to M' \text{ is a morphism in } \text{Rep}(P(Q_I)^l_v, \mathcal{A}) \text{ then, for each vertex } p \text{ of } P(Q_I)^l_v, \text{ we have a morphism } f_p \in \mathcal{A}(M(p), M'(p)). \text{ For each vertex } w \text{ of } Q, \text{ we define the morphism } t^*_v(f)_w := \oplus_{p \in Q_I(w,v)} f_p \in \mathcal{A}(\oplus_{p \in Q_I(w,v)} M(p), \oplus_{p \in Q_I(v,w)} M'(p)).$$

**Lemma 2.** (See [18, Theorem 3.3] and [19, Page 3217]) Let $Q$ be a quiver, $I$ a closed set of monomial relations of $Q$, and $\mathcal{A}$ a cocomplete abelian category having enough projective objects. Then $\text{Rep}(Q_I, \mathcal{A})$ is a cocomplete abelian category having enough projective objects and $\{ f^*_v(P) \mid v \in Q_0, \ P \in \mathcal{A} \text{ projective} \}$ is a family of projective generators for $\text{Rep}(Q_I, \mathcal{A})$.

**Proof.** Clearly, $\text{Rep}(Q_I, \mathcal{A})$ is a cocomplete abelian category. Now we show that it has enough projective objects. We have known that there are adjoint isomorphisms

$$(\text{Rep}(Q_I, \mathcal{A}))(f^*_v(A), R) \cong (\text{Rep}(\mathbb{A}_1, \mathcal{A}))(A, f_{v*}(R))$$

for all $R \in \text{Rep}(Q_I, \mathcal{A})$ and $A \in \text{Rep}(\mathbb{A}_1, \mathcal{A}) = \mathcal{A}$. Note that we always identify the category $\text{Rep}(\mathbb{A}_1, \mathcal{A})$ with $\mathcal{A}$. For any $R \in \text{Rep}(Q_I, \mathcal{A})$, we have $f_{v*}(R) \in \mathcal{A} = \text{Rep}(\mathbb{A}_1, \mathcal{A})$. Since $\mathcal{A}$ has enough projective objects, there
is a projective object \( P_v \in \mathcal{A} \) and an epimorphism \( d_v : P_v \to f_{v*}(R) \). By the adjoint isomorphism, there is a unique morphism \( \phi^v : f^*_v(P_v) \to R \) such that \( (\phi^v)_v : (f^*_v(P_v))(v) \to R(v) \) is just \( d_v : P_v \to f_{v*}(R) \). Repeat this procedure for every vertex \( v \) of \( Q_I \), we get morphisms of representations \( \phi^v : f^*_v(P_v) \to R \) for all \( v \) of \( Q_I \). By definition, \( f_{v*} \) preserves exactness. Thus \( f^*_v \) preserves projectiveness. Hence \( f^*_v(P_v) \) is a projective representation of \( Q_I \). So is \( \oplus_{v \in Q_0} f^*_v(P_v) \). Furthermore, the morphism of representations \( \phi = (\phi^v)_{v \in Q_0} : \oplus_{v \in Q_0} f^*_v(P_v) \to R \) is an epimorphism, since the restriction \( (\phi^v)_v = d_v \) of \( \phi_v \) on the \( v \)-component \( f^*_v(P_v)(v) \) of \( (\oplus_{v \in Q_0} f^*_v(P_v))(v) \) is. It follows that \( \{ f^*_v(P) \mid v \in Q_0, P \in \mathcal{A} \text{ projective} \} \) is a family of projective generators for \( \text{Rep}(Q_I, \mathcal{A}) \).

### 3.3 A construction of dualizing categories

A quiver \( Q \) is said to be \textit{locally finite} if for each vertex \( v \) of \( Q \), there are only finitely many arrows of \( Q \) with source or target \( v \) (Ref. [20, §8.3, Example 2]).

Let \( Q \) be a quiver, \( kQ \) the \( k \)-category of paths of \( Q \), \( k^1Q \) the ideal of \( kQ \) generated by all arrows of \( Q \), and \( k^rQ := (k^1Q)^r \) for all \( r \in \mathbb{N} \). An ideal \( I \) of \( kQ \) is said to be \textit{admissible} if \( I \subseteq k^2Q \) and for each vertex \( v \) of \( Q \) there is an \( l_v \in \mathbb{N} \) such that \( I \) contains all paths of length \( \geq l_v \) with source or target \( v \) (Ref. [20, §8.3, Example 2]).

From now on, we always assume that \( I \) is an admissible ideal of \( kQ \) generated by a set of paths in \( Q \). So all paths of \( Q \) in \( I \) form an ideal \( J \) of \( Q \) consisting of some paths of length at least 2. Once we denote by \( Q_I \) the category whose objects are vertices of \( Q \) and whose morphisms are all paths of \( Q \) that are not in \( I \), then \( Q_I = Q/I \). Moreover, the opposite category \( (kQ/I)\) is a residue category of \( (kQ)\) modulo an admissible ideal \( I \) of \( kQ \) generated by a set of paths in \( Q \). Indeed, all paths of \( Q \) in \( I \) form an ideal \( J = \{ a_1^{op}a_2^{op} \cdots a_l^{op} \mid a_1 \cdots a_l \in J \} \) of \( Q \). Let \( \mathcal{A} \) be an additive \( k \)-category. We denote by \( \text{rep}(kQ/I, \mathcal{A}) \) (resp. \( \text{rep}(Q_I, \mathcal{A}) \)) the full subcategory of \( \text{Rep}(kQ/I, \mathcal{A}) \) (resp. \( \text{Rep}(Q_I, \mathcal{A}) \)) consisting of all \textit{support-finite} representations, i.e., the representations \( R \) satisfying \( R(v) \neq 0 \) for only finitely many vertices \( v \) of \( Q \). Then \( \text{rep}(kQ/I, \mathcal{A}) \simeq \text{rep}(Q_I, \mathcal{A}) \).

**Proposition 1.** Let \( Q \) be a locally finite quiver, \( I \) an admissible ideal of \( kQ \) generated by a set of paths of \( Q \), \( \mathcal{B} := kQ/I \) and \( \mathcal{A} \) a skeletonially small additive \( k \)-category with pseudo-kernels. Then the equivalence \( \text{Rep}(\mathcal{B}, \text{Mod}\mathcal{A}) \simeq \text{Mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A}) \) restricts to an equivalence \( \text{rep}(\mathcal{B}, \text{mod}\mathcal{A}) \simeq \text{mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A}) \).
Proof. First of all, we have natural equivalences

\[
\begin{array}{ccc}
\text{Rep}(\mathcal{B}, \text{Mod}\mathcal{A}) & \xrightarrow{\sim} & \text{Mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A}) \\
\text{rep}(\mathcal{B}, \text{mod}\mathcal{A}) & \xrightarrow{\sim} & \text{mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A})
\end{array}
\]

The equivalence functor \( \Phi : \text{Rep}(\mathcal{B}, \text{Mod} \mathcal{A}) \to \text{Mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A}) \) is given by \( \Phi(R)((v, A)) := R(v)(A) \) and \( \Phi(R)(a \otimes f) := R(a)_A \circ R(s(a))(f) = R(t(a))(f) \circ R(a)_{A'} : R(s(a))(A') \to R(t(a))(A) \)

\[
\begin{array}{ccc}
R(s(a))(A) & \xrightarrow{R(a)_A} & R(t(a))(A) \\
\downarrow R(s(a))(f) & & \downarrow R(t(a))(f) \\
R(s(a))(A') & \xrightarrow{R(a)_{A'}} & R(t(a))(A')
\end{array}
\]

for all arrows \( a \in Q_1 \) and morphisms \( f : A \to A' \) in \( \mathcal{A} \). The quasi-inverse \( \Psi : \text{Mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A}) \to \text{Rep}(\mathcal{B}, \text{Mod} \mathcal{A}) \) of \( \Phi \) is given by \( \Psi(M)(v)(A) := M((v, A)) \) and \( \Psi(M)(v)(f) := M(1_v \otimes f) \) for all \( v \in Q_0, A \in \mathcal{A}, f \in \mathcal{A}(A, A') \), \( \Psi(M)(a)_A := M(a \otimes 1_A) \) for all \( a \in Q_1, A \in \mathcal{A} \), and \( (\Psi(F)_v)_A := F(v)_A : \Psi(M)(v)(A) = M((v, A)) \to \Psi(M')(v)(A) = M'((v, A)) \) for all \( v \in Q_0, A \in \mathcal{A}, F \in (\text{Mod}(\mathcal{B}^{\text{op}} \otimes_k \mathcal{A})) \).

Since \( \text{Mod} \mathcal{A} \) has enough projective objects \( \mathcal{A}(\_ , A) \) with \( A \in \mathcal{A} \), by Lemma 2 \( \text{Rep}(\mathcal{B}, \text{Mod} \mathcal{A}) \simeq \text{Rep}(Q_I, \text{Mod} \mathcal{A}) \) has enough projective objects \( \{ P_{v, A} := f^*_v(\mathcal{A}(\_ , A)) \mid v \in Q_0, A \in \mathcal{A} \} \). Note that \( Q \) is locally finite and \( I \) is admissible, by the definition of \( f^*_v \), we have \( P_{v, A} \in \text{rep}(Q_I, \text{mod} \mathcal{A}) \). Since \( \mathcal{A} \) has pseudo-kernels, mod\( \mathcal{A} \) is abelian [1, Page 102, Proposition]. So is rep\( (\mathcal{B}, \text{mod} \mathcal{A}) \simeq \text{rep}(Q_I, \text{mod} \mathcal{A}) \). For each \( R \in \text{rep}(\mathcal{B}, \text{mod} \mathcal{A}) \), by the proof of Lemma 2 there is an exact sequence

\[
\bigoplus_{v \in Q_0, A \in |\mathcal{A}|} P^m_{v, A} \to \bigoplus_{v \in Q_0, A \in |\mathcal{A}|} P^n_{v, A} \to R
\]

where \( |\mathcal{A}| \) is a complete set of representatives of isomorphism classes of objects in \( \mathcal{A} \), and the nonnegative integers \( m_{v, A} \) and \( n_{v, A} \) are nonzero for only finitely many pairs \( (v, A) \in Q_0 \times |\mathcal{A}| \). It is not difficult to check
that $\Phi(P_{v,A}) = B(v,-) \otimes_k A(-,A) = B^{\text{op}}(-,v) \otimes_k A(-,A) = (B^{\text{op}} \otimes_k A)(-(v,A))$. Applying the equivalence functor $\Phi$ to the above exact sequence, we obtain an exact sequence

$$
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}} \otimes_k A)(-(v,A))^{m_{v,A}} \rightarrow \\
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}} \otimes_k A)(-(v,A))^{n_{v,A}} \rightarrow \Phi(R).
$$

Namely, $\Phi(R) \in \text{mod}(B^{\text{op}} \otimes_k A)$.

Conversely, for any $M \in \text{mod}(B^{\text{op}} \otimes_k A)$, there is an exact sequence

$$
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}} \otimes_k A)((v',-),(v,A))^{m_{v,A}} \rightarrow \\
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}} \otimes_k A)((v',-),(v,A))^{n_{v,A}} \rightarrow M(v',-) = \Psi(M)(v'),
$$

where the nonnegative integers $m_{v,A}$ and $n_{v,A}$ are nonzero for only finitely many pairs $(v,A)$. Thus for each $v' \in Q_0$, there is an exact sequence

$$
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}}(v',v) \otimes_k A)(-,A)^{m_{v,A}} \rightarrow \\
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}}(v',v) \otimes_k A)(-,A)^{n_{v,A}} \rightarrow M(v',-) = \Psi(M)(v'),
$$

i.e.,

$$
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}}(v',v) \otimes_k A)(-,A)^{m_{v,A}} \rightarrow \\
\oplus_{v \in Q_0, A \in |A|} (B^{\text{op}}(v',v) \otimes_k A)(-,A)^{n_{v,A}} \rightarrow M(v',-) = \Psi(M)(v').
$$

Since $Q$ is locally finite and $I$ is admissible and generated by a set of paths, all $B(v,v') = B^{\text{op}}(v',v)$ are finitely generated free $k$-modules and there are only finitely many $v' \in Q_0$ such that $B(v,v') = B^{\text{op}}(v',v) \neq 0$. Hence $\Psi(M) \in \text{rep}(B,\text{mod},A)$. \hfill $\square$

**Remark 1.** A sequence $0 \xrightarrow{R} f R' \xrightarrow{f'} R'' \xrightarrow{f''} 0$ in the functor category $\text{Rep}(kQ/I,\text{Mod}A)$ is exact if $0 \xrightarrow{R(v)} f_0 R'(v) \xrightarrow{f_0} R''(v) \xrightarrow{f_0'} 0$ is exact in $\text{Mod}A$ for all $v \in Q_0$. It gives a natural exact structure $\mathcal{F}$ on the abelian category $\text{Rep}(kQ/I,\text{Mod}A)$ such that the natural equivalence $\text{Rep}(kQ/I,\text{Mod}A) \simeq \text{Mod}((kQ/I)^{\text{op}} \otimes_k A)$ preserves exactness.

Our main result in this paper is the following theorem:

**Theorem 1.** Let $Q$ be a locally finite quiver, $I$ an admissible ideal of $kQ$ generated by a set of paths in $Q$, $B := kQ/I$ and $A$ a dualizing $k$-category. Then the following statements hold:

1. The $k$-category $| \oplus (B \otimes_k A)$ is dualizing.
2. The abelian $k$-category $\text{mod}(B \otimes_k A)$ is dualizing.
3. The abelian $k$-category $\text{mod}(B \otimes_k A)$ has almost split sequences.
**Proof.** (1) By assumption, $\mathcal{B}$ is a skeletally small Hom-finite $k$-category. So is $\mathcal{B} \otimes_k \mathcal{A}$. It follows from Lemma 1 that $\bigoplus (\mathcal{B} \otimes_k \mathcal{A})$ is Hom-finite and Krull-Schmidt. So we only need to show that the duality $D : \text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) \rightarrow \text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A}))$. 

Since $(\bigoplus (\mathcal{B} \otimes_k \mathcal{A}))^\text{op} \cong \bigoplus (\mathcal{B} \otimes_k \mathcal{A})^\text{op} \cong (\bigoplus (\mathcal{B}^\text{op} \otimes_k \mathcal{A}^\text{op}))$, we have $\text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A}))^{\text{op}} \cong \text{mod}(\mathcal{B}^\text{op} \otimes_k \mathcal{A}^\text{op})$. On the other hand, we have $\text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) \cong \text{mod}(\mathcal{B} \otimes_k \mathcal{A})$ as well. Thus it is enough to prove that the duality $D$ above induces a duality $D : \text{mod}(\mathcal{B} \otimes_k \mathcal{A}) \rightarrow \text{mod}(\mathcal{B} \otimes_k \mathcal{A})$. 

By Proposition 1, we have equivalences $\text{mod}(\mathcal{B} \otimes_k \mathcal{A}) \cong \text{rep}(kQ/I, \mathcal{A})$ and $\text{mod}(\mathcal{B}^\text{op} \otimes_k \mathcal{A}^\text{op}) \cong \text{rep}(kQ/I, \mathcal{A}^\text{op})$. Thus it suffices to prove that the duality $D$ above induces a duality $D : \text{rep}(kQ/I, \mathcal{A}) \rightarrow \text{rep}(kQ/I, \mathcal{A}^\text{op})$, or equivalently, a duality $D : \text{rep}(Q_{I^\text{op}}, \mathcal{A}) \rightarrow \text{rep}(Q_I, \mathcal{A}^\text{op})$. 

$$
\begin{array}{ccc}
\text{Fun}_k(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) & \rightarrow & \text{Fun}_k(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) \\
\text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) & \rightarrow & \text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A}))^{\text{op}} \\
\text{mod}(\mathcal{B} \otimes_k \mathcal{A}) & \rightarrow & \text{mod}(\mathcal{B} \otimes_k \mathcal{A})^{\text{op}} \\
\text{rep}(kQ/I, \mathcal{A}) & \rightarrow & \text{rep}(kQ/I, \mathcal{A}^\text{op}) \\
\text{rep}(Q_{I^\text{op}}, \mathcal{A}) & \rightarrow & \text{rep}(Q_I, \mathcal{A}^\text{op})
\end{array}
$$

Since $\mathcal{A}$ is dualizing, the duality $D : \text{Fun}_k(\mathcal{A}^\text{op}, \text{mod}k) \rightarrow \text{Fun}_k(\mathcal{A}, \text{mod}k)$ induces a duality $D : \text{mod}A \rightarrow \text{mod}A^\text{op}$. In the last row of the above diagram, the functor $D : \text{rep}(Q_{I^\text{op}}, \text{mod}A) \rightarrow \text{rep}(Q_I, \text{mod}A^\text{op})$ is defined as follows: For any $R \in \text{rep}(Q_{I^\text{op}}, \text{mod}A)$, $D(R)(v) = D(R(v)) \in \text{mod}A^\text{op}$ for all $v \in Q_0$ and $D(R)(a) = D(R(a))$ for all $a \in Q_1$. For any morphism $\phi : R \rightarrow R'$ in $\text{rep}(Q_{I^\text{op}}, \text{mod}A)$, $D(\phi) : D(R') \rightarrow D(R)$ is the morphism in $\text{rep}(Q_I, \text{mod}A^\text{op})$ given by $D(\phi)_v = D(\phi_v)$ for all $v \in Q_0$. The duality $D : \text{mod}A \rightarrow \text{mod}A^\text{op}$ implies that $D : \text{rep}(Q_{I^\text{op}}, \text{mod}A) \rightarrow \text{rep}(Q_I, \text{mod}A^\text{op})$ is a duality.

(2) It follows from (1) and [2] Proposition 2.6 that $\text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A}))$ is...
an abelian dualizing \( k \)-category. Thus \( \text{mod}(\mathcal{B} \otimes_k \mathcal{A}) \simeq \text{mod}(| \oplus (\mathcal{B} \otimes_k \mathcal{A})) \) is also an abelian dualizing \( k \)-category.

(3) It follows from (1) and \([30, \text{Theorem 7.1.3}]\) that \( \text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) \) has almost split sequences. Thus \( \text{mod}(\mathcal{B} \otimes_k \mathcal{A}) \simeq \text{mod}(\bigoplus (\mathcal{B} \otimes_k \mathcal{A})) \) has almost split sequences.

\[ \square \]

**Remark 2.** By Theorem \([1, (1)]\), we can construct a large number of new dualizing \( k \)-categories from a given dualizing \( k \)-category. In practice, it is more convenient to apply the conclusion that \( \text{rep}(kQ/I, \text{mod} \mathcal{A}) \) is a dualizing \( k \)-category and has almost split sequences which can be obtained from Proposition \([1]\) and Theorem \([1]\) (2) and (3). This is a generalization of \([6, \text{Theorem 4.3 and Corollary 4.4}]\), which will be clear in the next section. In the case that \( Q \) is just one vertex and has no any arrows, it is nothing but \([2, \text{Proposition 2.6}]\).

### 4 Applications

In this section, we will apply our main theorem to show that the categories of all kinds of complexes have almost split sequences.

#### 4.1 Categories of \( n \)-complexes

Let \( \mathcal{A} \) be an additive category and \( n \geq 2 \). An \( n \)-complex \( X \) on \( \mathcal{A} \) is a collection \((X^i, d^i_X)_{i \in \mathbb{Z}}\) with \( X^i \in \mathcal{A} \) and \( d_X^i \in \mathcal{A}(X^i, X^{i+1}) \) such that \( d_X^{i+n-1} \cdots d_X^i = 0 \) for all \( i \in \mathbb{Z} \). An \( n \)-complex \( X = (X^i, d^i_X)_{i \in \mathbb{Z}} \) on \( \mathcal{A} \) can be visualized as the following diagram:

\[
\cdots \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots
\]

A morphism \( f \) from an \( n \)-complex \( X = (X^i, d^i_X)_{i \in \mathbb{Z}} \) on \( \mathcal{A} \) to an \( n \)-complex \( Y = (Y^i, d^i_Y)_{i \in \mathbb{Z}} \) on \( \mathcal{A} \) is a collection \((f^i)_{i \in \mathbb{Z}}\) with \( f^i \in \mathcal{A}(X^i, Y^i) \) such that \( f^{i+1}d_X^i = d_Y^if^i \) for all \( i \in \mathbb{Z} \), i.e., the following diagram is commutative:

\[
\cdots \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots
\]

\[
\cdots \xrightarrow{d_Y^{i-2}} Y^{i-1} \xrightarrow{d_Y^{i-1}} Y^i \xrightarrow{d_Y^i} Y^{i+1} \xrightarrow{d_Y^{i+1}} \cdots
\]

The composition of morphisms \( f = (f^i)_{i \in \mathbb{Z}} : X \to Y \) and \( g = (g^i)_{i \in \mathbb{Z}} : Y \to Z \) is \( gf := (g^if^i)_{i \in \mathbb{Z}} : X \to Z \). All \( n \)-complexes on \( \mathcal{A} \) and all morphisms between them form the category of \( n \)-complexes on \( \mathcal{A} \), denoted by \( C_n(\mathcal{A}) \).
Let \((\mathcal{A}, \mathcal{E})\) be an exact category and \(\mathcal{E}_n\) the class of composable morphisms \(X \xrightarrow{f} Y \xrightarrow{g} Z\) in \(\mathcal{C}_n(\mathcal{A})\) such that \(X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i\) is a conflation in \(\mathcal{E}\) for all \(i \in \mathbb{Z}\). Then \((\mathcal{C}_n(\mathcal{A}), \mathcal{E}_n)\) is an exact category. An \(n\)-complex \(X = (X^i, d^i_X)_{i \in \mathbb{Z}}\) on \(\mathcal{A}\) is said to be bounded if \(X^i = 0\) for all but finitely many \(i \in \mathbb{Z}\). Denoted by \(\mathcal{C}^b_n(\mathcal{A})\) the full subcategory of \(\mathcal{C}_n(\mathcal{A})\) consisting of all bounded \(n\)-complexes on \(\mathcal{A}\). Let \(\mathcal{E}^b_n\) be the class of the composable morphisms in both \(\mathcal{C}^b_n(\mathcal{A})\) and \(\mathcal{E}_n\). Then \((\mathcal{C}^b_n(\mathcal{A}), \mathcal{E}^b_n)\) is a full exact subcategory of \((\mathcal{C}_n(\mathcal{A}), \mathcal{E}_n)\).

The following result is a generalization of [6, Theorem 4.3]:

**Corollary 1.** Let \(\mathcal{A}\) be a dualizing \(k\)-category, \(\mathcal{F}\) the natural exact structure on the abelian category \(\text{mod}\, \mathcal{A}\), and \(n \geq 2\). Then \((\mathcal{C}^b_n(\text{mod}\, \mathcal{A}), \mathcal{F}^b_n)\) has almost split sequences.

**Proof.** Let \(Q\) be the quiver with vertices \(i\) and arrows \(a_i : i \to i + 1\) for all \(i \in \mathbb{Z}\), i.e.,

\[
\cdots \xrightarrow{a_{-2}} \cdot \xrightarrow{a_{-1}} \cdot \xrightarrow{a_0} \cdot \xrightarrow{a_1} \cdots
\]

\(I\) the ideal of \(kQ\) generated by all paths of length \(n\), and \(B := kQ/I\). By Proposition [1] and Theorem [1] we know \(\mathcal{C}^b_n(\text{mod}\, \mathcal{A}) \simeq \text{rep}(Q, \text{mod}\, \mathcal{A}) \simeq \text{rep}(B, \text{mod}\, \mathcal{A}) \simeq \text{mod}(B^{op} \otimes_k \mathcal{A})\) has almost split sequences.

Let \(\mathcal{A}\) be a dualizing \(k\)-category and \(\mathcal{E}\) the trivial exact structure on \(\mathcal{A}\), i.e., \(\mathcal{E}\) consists of all split short exact sequences \(X \xrightarrow{f} Y \xrightarrow{g} Z\) in \(\mathcal{A}\). Denote by \(\text{proj}\, \mathcal{A}\) the full subcategory of \(\text{mod}\, \mathcal{A}\) consisting of all projective \(\mathcal{A}\)-modules, i.e., all representable functors in \(\text{Mod}\, \mathcal{A}\). Note that the exact category \((\mathcal{A}, \mathcal{E})\) is equivalent to the exact category \((\text{proj}\, \mathcal{A}, \mathcal{F}|\text{proj}\, \mathcal{A})\). Indeed, \(\mathcal{A} \to \text{proj}\, \mathcal{A}\) is an equivalence functor.

For any \(M \in \text{mod}\, \mathcal{A}\) and \(j \in \mathbb{Z}\), denote by \(J_j(M)\) the \(n\)-complex \(X = (X^i, d^i_X)_{i \in \mathbb{Z}}\) where \(X^i := M\) for all \(i \in [j, j + n - 1]\) and \(X^i := 0\) otherwise, and \(d^i_X := 1_M\) for all \(i \in [j, j + n - 2]\) and \(d^i_X := 0\) otherwise.

**Corollary 2.** Let \(\mathcal{A}\) be a dualizing \(k\)-category with trivial exact structure \(\mathcal{E}\) and \(\text{gl.dim}(\text{mod}\, \mathcal{A}) < \infty\), and \(n \geq 2\). Then the exact category \((\mathcal{C}^b_n(\mathcal{A}), \mathcal{E}^b_n)\) has almost split sequences.

**Proof.** First of all, we show that \((\mathcal{C}^b_n(\mathcal{A}), \mathcal{E}^b_n)\) has right almost split morphisms. We have known \((\mathcal{A}, \mathcal{E}) \simeq (\text{proj}\, \mathcal{A}, \mathcal{F}|\text{proj}\, \mathcal{A})\). Thus \((\mathcal{C}^b_n(\mathcal{A}), \mathcal{E}^b_n) \simeq \text{proj}\, \mathcal{A}\).
\( (\mathcal{C}_n^b(\text{proj}\mathcal{A}), (\mathcal{F}|\text{proj}\mathcal{A})_n^b) \). So it is enough to prove that the exact category \( (\mathcal{C}_n^b(\text{proj}\mathcal{A}), (\mathcal{F}|\text{proj}\mathcal{A})_n^b) = (\mathcal{C}_n^b(\text{proj}\mathcal{A}), \mathcal{F}_n^b|\mathcal{C}_n^b(\text{proj}\mathcal{A})) \) has right almost split morphisms. For this, it suffices to show that \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \) is a contravariantly finite subcategory of \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \) closed under \( \mathcal{F}_n^b \)-extensions and direct summands. Obviously, we need only to prove that \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \) is contravariantly finite over \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \), i.e., for any \( Z \in \mathcal{C}_n^b(\text{mod}\mathcal{A}) \), there exists a right \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \)-approximation \( g : Y \to Z \).

By \cite{24} Page 6, Formula (17), there is an epimorphism \( p : \oplus_{j \in \mathbb{Z}} J_j(Z^j) \to Z \) in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \). Since \( \text{mod}\mathcal{A} \) has enough projective objects, for any \( j \in \mathbb{Z} \), we have an epimorphism \( P_j \to Z^j \) in \( \text{mod}\mathcal{A} \) with \( P_j \in \text{proj}\mathcal{A} \) where we take \( P_j = 0 \) in the case of \( Z^j = 0 \). Thus there is an epimorphism \( p' : \oplus_{j \in \mathbb{Z}} J_j(P_j) \to \oplus_{j \in \mathbb{Z}} J_j(Z^j) \) in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \), and further a morphism

\[
\begin{align*}
r := pp' : \oplus_{j \in \mathbb{Z}} J_j(P_j) & \longrightarrow \oplus_{j \in \mathbb{Z}} J_j(Z^j) \\
& \longrightarrow Z
\end{align*}
\]

in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \).

By the assumption \( \text{gl.dim}(\text{mod}\mathcal{A}) < \infty \) and \cite{24} Proposition 41, \( Z \) admits a homotopically projective resolution \( q : Q \to Z \) with \( Q \in \mathcal{C}_n^b(\text{proj}\mathcal{A}) \). Now we check \( g := (q, r) : Y := Q \oplus (\oplus_{j \in \mathbb{Z}} J_j(P_j)) \to Z \) is a right \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \)-approximation of \( Z \). In another words, for any morphism \( g' : Z' \to Z \) in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \) with \( Z' \in \mathcal{C}_n^b(\text{proj}\mathcal{A}) \), we need to show that there is a morphism \( g'' : Z' \to Y \) in \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \) such that \( g' = g g'' \).

Since \( q \) is a quasi-isomorphism, \( q^{-1} g' : Z' \to Q \) is a morphism in the bounded derived category \( \mathcal{D}_n^b(\text{mod}\mathcal{A}) \). Since \( Z' \in \mathcal{C}_n^b(\text{proj}\mathcal{A}) \) is homotopically projective, there is a morphism \( h : Z' \to Q \) in \( \mathcal{C}_n^b(\text{proj}\mathcal{A}) \) such that \( h = q^{-1} g' \) in \( \mathcal{D}_n^b(\text{mod}\mathcal{A}) \).

Thus \( g' = q h \) in \( \mathcal{D}_n^b(\text{mod}\mathcal{A}) \), and further in the bounded homotopy category \( \mathcal{K}_n^b(\text{mod}\mathcal{A}) \) due to \( Z' \in \mathcal{C}_n^b(\text{proj}\mathcal{A}) \). Hence \( g' = q h + l \) in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \) for some null-homotopy \( l : Z' \to Z \) in \( \mathcal{C}_n^b(\text{mod}\mathcal{A}) \). Since \( l \) is a null-homotopy, \( l \) is factored through \( p \) (cf. \cite{24} Proof of Theorem 19), say \( l = p l' \). Since \( Z' \in \mathcal{C}_n^b(\text{proj}\mathcal{A}) \), each component of \( Z' \) is projective. Thus \( l' \) is factored
through \( p' \), say \( l' = p'l'' \). Hence \( l \) is factored through \( r \).

\[ \begin{array}{ccc}
\oplus_{j \in \mathbb{Z}} J_j(P_j) & \xrightarrow{i''} & Z' \\
\downarrow{p''} & & \downarrow{i} \\
\oplus_{j \in \mathbb{Z}} J_j(Z) & \xrightarrow{\nu} & Z
\end{array} \]

So we get

\[ g' = qh + l = qh + rl'' = (q, r) \left( \begin{array}{c} h \\ \nu \end{array} \right) = gg'' \]

where \( g'' := \left( \begin{array}{c} h \\ \nu \end{array} \right) : Z' \to Y = Q \oplus (\oplus_{j \in \mathbb{Z}} J_j(P_j)) \). Thus \( g : Y \to Z \) is a right \( C^b_n(\text{projA}) \)-approximation of \( Z \). Therefore, \( C^b_n(\text{projA}) \) is contravariantly finite over \( C^b_n(\text{modA}) \).

Since \( C^b_n(A) \) is Krull-Schmidt and \( (C^b_n(A), \mathcal{E}^b_n) \) has right almost split morphisms, \( (C^b_n(A), \mathcal{E}^b_n) \) has minimal right almost split morphisms. Furthermore, one can prove that if \( Z \in C^b_n(A) \) is indecomposable and non-\( \mathcal{E}^b_n \)-projective then there is an almost split sequence in \( (C^b_n(A), \mathcal{E}^b_n) \) ending in \( Z \).

Since \( A \) is dualizing, \( \text{modA}^{\text{op}} \simeq (\text{modA})^{\text{op}} \). Thus \( \text{gl.dim} (\text{modA}^{\text{op}}) = \text{gl.dim} (\text{modA}) < \infty \) (Ref. [23, Page 42]). Applying the obtained result to dualizing \( k \)-variety \( A^{\text{op}} \), we know that \( (C^b_n(A^{\text{op}}), (\mathcal{E}^{\text{op}})^b_n) \) has right almost split morphisms and if \( Z \in C^b_n(A^{\text{op}}) \) is indecomposable and non-\( (\mathcal{E}^{\text{op}})^b_n \)-projective then there is an almost split sequence in \( (C^b_n(A^{\text{op}}), (\mathcal{E}^{\text{op}})^b_n) \) ending in \( Z \). Since \( (C^b_n(A), \mathcal{E}^b_n) \simeq (C^b_n(A^{\text{op}}), (\mathcal{E}^{\text{op}})^b_n) \), \( (C^b_n(A), \mathcal{E}^b_n) \) has left almost split morphisms and if \( X \in C^b_n(A) \) is indecomposable and non-\( \mathcal{E}^b_n \)-injective then there is an almost split sequence in \( (C^b_n(A), \mathcal{E}^b_n) \) starting in \( X \). Therefore, \( (C^b_n(A), \mathcal{E}^b_n) \) has almost split sequences. \( \square \)

For \( m \in \mathbb{N} \), denote by \( C^m_n(A) \) the full subcategory of \( C_n(A) \) consisting of all \( n \)-complexes \( X = (X^i, d_X^i) \in Z \) on \( A \) with amplitude in the interval \([1, m]\), i.e., \( X^i = 0 \) for all \( i \notin \{1, 2, \cdots, m\} \). Let \( \mathcal{E}^m_n \) be the class of the composable morphisms in both \( C^m_n(A) \) and \( \mathcal{E}_n \). Then \( (C^m_n(A), \mathcal{E}^m_n) \) is a full exact subcategory of \( (C^b_n(A), \mathcal{E}^b_n) \).

The following result is a generalization of [4, Corollary 4.4]:

**Corollary 3.** Let \( A \) be a dualizing \( k \)-category, \( \mathcal{F} \) the natural exact structure on the abelian category \( \text{modA}, m \geq 1 \), and \( n \geq 2 \). Then \( (C^m_n(\text{modA}), \mathcal{F}^m_n) \) has almost split sequences.
Proof. Let $Q$ be the quiver with vertices $i$ and arrows $a_i : i \to i + 1$ for all $1 \leq i \leq m - 1$, i.e.,

$$
\begin{array}{ccccccc}
& & a_1 & & a_2 & & \cdots & & a_m & & \\
1 & & 2 & & 3 & & \cdots & & m - 1 & & m
\end{array}
$$

$I$ the ideal of $kQ$ generated by all paths of length $n$, and $B := kQ/I$. By Proposition 11 and Theorem 11, we know $C^n_m(\text{mod}A) \cong \text{rep}(Q_1, \text{mod}A) \cong \text{rep}(B, \text{mod}A) \cong \text{mod}(B^{op} \otimes_k A)$ has almost split sequences.

The following result is a generalization of [6, Theorem 4.5]:

Corollary 4. Let $A$ be a dualizing $k$-category with trivial exact structure $\mathcal{E}$, $m \geq 1$, and $n \geq 2$. Then the exact category $(C^n_m(A), \mathcal{E}_n^m)$ has almost split sequences.

Proof. First of all, we show that $(\mathcal{C}^n_m(A), \mathcal{E}_n^m)$ has right almost split morphisms. We have known $(\mathcal{A}, \mathcal{E}) \cong (\text{proj}A, \mathcal{F}|\text{proj}A)$. Thus $(\mathcal{C}^n_m(A), \mathcal{E}_n^m) \cong (\mathcal{C}^n_m(\text{proj}A), (\mathcal{F}|\text{proj}A)^m)$. So it is enough to prove that the exact category $(\mathcal{C}^n_m(\text{proj}A), (\mathcal{F}|\text{proj}A)^m)$ has right almost split morphisms. For this, it suffices to show that $\mathcal{C}^n_m(\text{proj}A)$ is a contravariantly finite subcategory of $\mathcal{C}^n_m(\text{mod}A)$ closed under $\mathcal{F}_n^m$-extensions and direct summands. Obviously, we need only to prove that $\mathcal{C}^n_m(\text{proj}A)$ is contravariantly finite over $\mathcal{C}^m_m(\text{mod}A)$, i.e., for any $Z \in \mathcal{C}^m_m(\text{mod}A)$, there exists a right $\mathcal{C}^m_m(\text{proj}A)$-approximation $g : Y \to Z$.

By [24, Page 6, Formula (17)], there is an epimorphism $p : \oplus_{j=1}^m J_j(Z^j) \to Z$ in $\mathcal{C}^n_m(\text{mod}A)$. Since $\text{mod}A$ has enough projective objects, for any $j \in \{1, m\}$, we have an epimorphism $P_j \to Z$ in $\text{mod}A$ with $P_j \in \text{proj}A$. Thus there is an epimorphism $p' : \oplus_{j=1}^m J_j(P_j) \to \oplus_{j=1}^m J_j(Z^j)$ in $\mathcal{C}^n_m(\text{mod}A)$. Compose these morphisms with the natural injection $p'' : \oplus_{j=1}^{m-n+1} J_j(P_j) \hookrightarrow \oplus_{j=1}^m J_j(P_j)$, we obtain a morphism

$$
\begin{array}{ccccccc}
r := pp'' & : \oplus_{j=1}^{m-n+1} J_j(P_j) & \oplus_{j=1}^m J_j(P_j) & \oplus_{j=1}^m J_j(Z^j) & \to & Z
\end{array}
$$

in $\mathcal{C}^m_m(\text{mod}A)$.

By [24, Proposition 41], $Z$ admits a homotopically projective resolution $q : Q \to Z$ with $Q \in \mathcal{C}^m_n(\text{proj}A)$. Now we check $g := (\tau_{\geq 1}(q), r) : Y := \tau_{\geq 1}(Q) \oplus (\oplus_{j=1}^{m-n+1} J_j(P_j)) \to Z$ is a right $\mathcal{C}^m_n(\text{proj}A)$-approximation of $Z$, where $\tau_{\geq 1} : C_n(\text{Mod}A) \to C^1_n(\text{Mod}A)$ is the hard truncation functor, i.e., for any $n$-complex $X$ and any morphism of $n$-complexes $f$, $(\tau_{\geq 1}(X))^i := X^i$.
for all $i \geq 1$ and $(\tau_{\geq 1}(X))^i := 0$ otherwise and $(\tau_{\geq 1}(f))^i := f^i$ for all $i \geq 1$ and $(\tau_{\geq 1}(f))^i := 0$ otherwise (Ref. [24, Definition 21]). In another words, for any morphism $g' : Z' \to Z$ in $C_n^m(\text{mod}\mathcal{A})$ with $Z' \in C_n^m(\text{proj}\mathcal{A})$, we need to show that there is a morphism $g'' : Z'' \to Y$ in $C_n^m(\text{proj}\mathcal{A})$ such that $g' = gg''$.

Since $q$ is a quasi-isomorphism, $q^{-1}g' : Z' \to Q$ is a morphism in the upper bounded derived category $D_n^-(\text{mod}\mathcal{A})$. Since $Z' \in C_n^m(\text{proj}\mathcal{A})$ is homotopically projective, there is a morphism $h : Z' \to Q$ in $C_n^-(\text{proj}\mathcal{A})$ such that $h = q^{-1}g'$ in $D_n^-(\text{mod}\mathcal{A})$.

Thus $g' = qh$ in $D_n^-(\text{mod}\mathcal{A})$, and further in the upper bounded homotopy category $K_n^-(\text{mod}\mathcal{A})$ due to $Z' \in C_n^m(\text{proj}\mathcal{A})$. Hence $g' = qh + l$ in $C_n^m(\text{mod}\mathcal{A})$ for some null-homotopy $l : Z' \to Z$ in $C_n^m(\text{mod}\mathcal{A})$. Since $l$ is a null-homotopy, $l$ is factored through $p$ (Ref. [24, Proof of Theorem 19]), say $l = pl'$. Since $Z' \in C_n^m(\text{proj}\mathcal{A})$, each component of $Z'$ is projective. Thus $l'$ is factored through $p''$, say $l' = pl''$. It is easy to see that a morphism from $Z' \in C_n^m(\text{proj}\mathcal{A})$ to $J_j(P)$ must be zero for all $m - n + 2 \leq j \leq m$. Thus $l''$ is factored through $p''$, say $l'' = pl'''$. Hence $l$ is factored through $r$.

Acting the hard truncation functor $\tau_{\geq 1}$ on $g' = qh + l$, we get

$$g' = \tau_{\geq 1}(q)\tau_{\geq 1}(h) + l = \tau_{\geq 1}(q)\tau_{\geq 1}(h) + rl''' = (\tau_{\geq 1}(q), r) \left( \begin{array}{c} \tau_{\geq 1}(h) \\ l'' \end{array} \right) = gg''$$

where $g'' := \left( \begin{array}{c} \tau_{\geq 1}(h) \\ l'' \end{array} \right) : Z' \to Y = \tau_{\geq 1}(Q) \oplus (\oplus_{j=1}^{m-n+1} J_j(P_j))$. So $g : Y \to Z$ is a right $C_n^m(\text{proj}\mathcal{A})$-approximation of $Z$. Therefore, $C_n^m(\text{proj}\mathcal{A})$ is contravariantly finite over $C_n^m(\text{mod}\mathcal{A})$.

Since $C_n^m(\mathcal{A})$ is Krull-Schmidt and $(C_n^m(\mathcal{A}), E_n^m)$ has right almost split morphisms, $(C_n^m(\mathcal{A}), E_n^m)$ has minimal right almost split morphisms. Furthermore, one can prove that if $Z \in C_n^m(\mathcal{A})$ is indecomposable and non-$E_n^m$-projective then there is an almost split sequence in $(C_n^m(\mathcal{A}), E_n^m)$ ending in $Z$.  

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Applying the obtained result to dualizing $k$-variety $A^{\text{op}}$, we know that $(C^m_n(A^{\text{op}}), (E^{\text{op}})^m_n)$ has right almost split morphisms and if $Z \in C^m_n(A^{\text{op}})$ is indecomposable and non-$(E^{\text{op}})^m_n$-projective then there is an almost split sequence in $(C^m_n(A^{\text{op}}), (E^{\text{op}})^m_n)$ ending in $Z$. Since $(C^m_n(A), E^m_n) \simeq (C^m_n(A^{\text{op}}), (E^{\text{op}})^m_n)$, $(C^m_n(A), E^m_n)$ has left almost split morphisms and if $X \in C^m_n(A)$ is indecomposable and non-$E^m_n$-injective then there is an almost split sequence in $(C^m_n(A), E^m_n)$ starting in $X$. Therefore, $(C^m_n(A), E^m_n)$ has almost split sequences.

4.2 Categories of $n$-cyclic complexes

Let $A$ be an additive category, $n \in \mathbb{N}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ the additive cyclic group of order $n$. An $n$-cyclic complex $X$ on $A$ is a collection $(X^i, d^i_X)_{i \in \mathbb{Z}_n}$ with $X^i \in A$ and $d^i_X \in A(X^i, X^{i+1})$ such that $d^{i+1}_X d^i_X = 0$ for all $i \in \mathbb{Z}_n$ (Ref. [29], §7, Appendix). An $n$-cyclic complex $X = (X^i, d^i_X)_{i \in \mathbb{Z}_n}$ on $A$ can be visualized as the following diagram:

$$\begin{array}{ccc}
X^0 & \xrightarrow{d^0_X} & X^1 \\
\downarrow{d^{n-1}_X} & & \downarrow{d^n_X} \\
X^{n-1} & \xrightarrow{d^{n-2}_X} & X^{n-2}
\end{array}$$

A morphism $f$ from an $n$-cyclic complex $X = (X^i, d^i_X)_{i \in \mathbb{Z}_n}$ on $A$ to an $n$-cyclic complex $Y = (Y^i, d^i_Y)_{i \in \mathbb{Z}_n}$ on $A$ is a collection $(f^i)_{i \in \mathbb{Z}_n}$ with $f^i \in A(X^i, Y^i)$ such that $f^{i+1} d^i_X = d^i_Y f^i$ for all $i \in \mathbb{Z}_n$, i.e., we have the following
commutative diagram:

The composition of morphisms \( f = (f_i)_{i \in \mathbb{Z}_n} : X \to Y \) and \( g = (g_i)_{i \in \mathbb{Z}_n} : Y \to Z \) is \( gf := (g_i f_i)_{i \in \mathbb{Z}_n} : X \to Z \). All \( n \)-cyclic complexes on \( A \) and all morphisms between them form the category of \( n \)-cyclic complexes on \( A \), denoted by \( C_{\mathbb{Z}_n}(A) \).

Let \( (A, \mathcal{E}) \) be an exact category and \( \mathcal{E}_{\mathbb{Z}_n} \) the class of composable morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \) such that \( X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \) is a conflation in \( \mathcal{E} \) for all \( i \in \mathbb{Z}_n \). Then \( (C_{\mathbb{Z}_n}(A), \mathcal{E}_{\mathbb{Z}_n}) \) is an exact category.

**Corollary 5.** Let \( A \) be a dualizing \( k \)-category, \( \mathcal{F} \) the natural exact structure on the abelian category \( \text{mod} A \), and \( n \in \mathbb{N} \). Then \( (C_{\mathbb{Z}_n}(\text{mod} A), \mathcal{F}_{\mathbb{Z}_n}) \) has almost split sequences.

**Proof.** Let \( Q \) be the quiver with vertices \( i \) and arrows \( a_i : i \to i + 1 \) for all \( i \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} \), i.e.,

\[
\begin{array}{ccc}
0 & \xrightarrow{a_0} & 1 \\
& \xrightarrow{a_{n-1}} & \\
(n-1) & \xrightarrow{a_{n-2}} & n-2
\end{array}
\]

\( I \) the ideal of \( kQ \) generated by all paths of length 2, and \( B := kQ/I \). By Proposition \[\] and Theorem \[\] we know \( C_{\mathbb{Z}_n}(\text{mod} A) \simeq \text{rep}(B, \text{mod} A) \simeq \text{mod}(B^{\text{op}} \otimes_k A) \) has almost split sequences. \( \square \)
For any $M \in \text{mod} \mathcal{A}$ and $j \in \mathbb{Z}_n$, denote by $J_j(M)$ the $n$-cyclic complex $X = (X^i, d_X^i)_{i \in \mathbb{Z}_n}$: if $n \geq 2$ then $X^i := M$ for $i = j, j + 1$ and $X^i := 0$ otherwise, and $d_X^i := 1_M$ for $i = j$ and $d_X^i := 0$ otherwise; if $n = 1$ then $X^0 := M \oplus M$ and $d_X^i := \begin{pmatrix} 0 & 0 \\ 1_M & 0 \end{pmatrix}$. We say an $n$-cyclic complex $X = (X^i, d_X^i)_{i \in \mathbb{Z}_n}$ is stalk if $X^i \neq 0$ for at most one $i \in \mathbb{Z}_n$ and $d_X^i = 0$ for all $i \in \mathbb{Z}_n$. Denote by $C_{Z_n}^b(\mathcal{A})$ the smallest full subcategory of $C_{Z_n}(\mathcal{A})$ containing all stalk $n$-cyclic complexes and closed under finite extensions, and $\mathcal{E}_{Z_n}^b := \mathcal{E}_{Z_n}|C_{Z_n}^b(\mathcal{A})$. Then $(C_{Z_n}^b(\mathcal{A}), \mathcal{E}_{Z_n}^b)$ is a full exact subcategory of $(C_{Z_n}(\mathcal{A}), \mathcal{E}_{Z_n})$.

**Corollary 6.** Let $\mathcal{A}$ be a dualizing $k$-category with trivial exact structure $\mathcal{E}$ and $\text{gl.dim}(\text{mod} \mathcal{A}) < \infty$, and $n \in \mathbb{N}$. Then the exact category $(C_{Z_n}^b(\mathcal{A}), \mathcal{E}_{Z_n}^b)$ has almost split sequences.

**Proof.** First of all, we show that $(C_{Z_n}^b(\mathcal{A}), \mathcal{E}_{Z_n}^b)$ has right almost split morphisms. We have known $(\mathcal{A}, \mathcal{E}) \simeq (\text{proj} \mathcal{A}, \mathcal{F}|\text{proj} \mathcal{A})$. Thus $(C_{Z_n}^b(\mathcal{A}), \mathcal{E}_{Z_n}^b) \simeq (C_{Z_n}^b(\text{proj} \mathcal{A}), (\mathcal{F}|\text{proj} \mathcal{A})_{Z_n}^b)$. So it is enough to prove that the exact category $(C_{Z_n}^b(\text{proj} \mathcal{A}), (\mathcal{F}|\text{proj} \mathcal{A})_{Z_n}^b) = (C_{Z_n}^b(\text{proj} \mathcal{A}), \mathcal{F}|\mathcal{Z}_n|C_{Z_n}^b(\text{proj} \mathcal{A}))$ has right almost split morphisms. For this, it suffices to show that $C_{Z_n}^b(\text{proj} \mathcal{A})$ is a contravariantly finite subcategory of $C_{Z_n}(\text{mod} \mathcal{A})$ closed under $\mathcal{F}|\mathcal{Z}_n$-extensions and direct summands. Clearly, we need only to prove that $C_{Z_n}^b(\text{proj} \mathcal{A})$ is contravariantly finite over $C_{Z_n}(\text{mod} \mathcal{A})$, i.e., for any $Z \in C_{Z_n}(\text{mod} \mathcal{A})$, there exists a right $C_{Z_n}^b(\text{proj} \mathcal{A})$-approximation $g : Y \rightarrow Z$.

By [29] Page 53, Proof of Proposition 7.1], there is an epimorphism $p : \oplus_{j=0}^{n-1} J_j(Z^i) \rightarrow Z$ in $C_{Z_n}(\text{mod} \mathcal{A})$. Since $\text{mod} \mathcal{A}$ has enough projective objects, for any $j \in \mathbb{Z}_n$, we have an epimorphism $P_j \rightarrow Z^i$ in $\text{mod} \mathcal{A}$ with $P^j \in \text{proj} \mathcal{A}$. Thus there is an epimorphism $p' : \oplus_{j=0}^{n-1} P_j \rightarrow \oplus_{j=0}^{n-1} J_j(Z^i)$ in $C_{Z_n}(\text{mod} \mathcal{A})$. So we obtain a morphism

$$r = pp' : \oplus_{j=0}^{n-1} J_j(P_j) \longrightarrow \oplus_{j=0}^{n-1} J_j(Z^i) \longrightarrow Z$$

in $C_{Z_n}(\text{mod} \mathcal{A})$.

From $n$-cyclic complex $Z \in C_{Z_n}(\text{mod} \mathcal{A})$, we can construct a complex $\tilde{Z} = (\tilde{Z}^i, d_{\tilde{Z}}^i)_{i \in \mathbb{Z}}$ where $\tilde{Z}^i := Z^i$ and $d_{\tilde{Z}}^i := d_Z^i$ for all $i \in \mathbb{Z}$. By [35] Lemma 5.7.2], $\tilde{Z}$ admits a Cartan-Eilenberg resolution $P_{**}$ whose total complex $\text{Tot}^\oplus(P_{**})$, denoted by $\tilde{Q}$, is quasi-isomorphic to $\tilde{Z}$ in $C_2(\text{Mod} \mathcal{A})$. Let $\tilde{q} : \tilde{Q} \rightarrow \tilde{Z}$ be such a quasi-isomorphism. Due to $\text{gl.dim}(\text{mod} \mathcal{A}) < \infty$, we can choose projective resolutions of all cohomologies and coboundaries, and thus cocycles and components, of $\tilde{Z}$ during the construction of $P_{**}$ to
be of finite length. Namely, we can assume that each component of $\tilde{Q}$ is finitely generated projective, i.e., $\tilde{Q} \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$. From $\tilde{Q}$, we can construct an $n$-cyclic complex $Q = (Q^i, d_Q^i)_{i \in \mathbb{Z}_n}$ where $Q^i := \tilde{Q}^i$ and $d_Q^i := d_{\tilde{Q}}^i$ for all $i \in \mathbb{Z}_n$. Then $Q \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$ and there is a quasi-isomorphism $q : Q \to Z$ (cf. \cite{33} Proposition 2.5 and \cite{33} Lemma 3.5). Now we check $g := (q, r) : Y := Q \oplus (\oplus_{j=0}^{n-1}J_j(P_j)) \to Z$ is a right $C^b_{\mathbf{Z}_n}(\mathbf{projA})$-approximation of $Z$. In another words, for any morphism $g' : Z' \to Z$ in $C_{\mathbf{Z}_n}(\mathbf{modA})$ with $Z' \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$, we need to show that there is a morphism $g'' : Z' \to Y$ in $C^b_{\mathbf{Z}_n}(\mathbf{projA})$ such that $g' = gg''$.

Since $q$ is a quasi-isomorphism, we have a morphism $q^{-1}g' : Z' \to Q$ in $D_{\mathbf{Z}_n}(\mathbf{modA})$. Since $Z' \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$ is homotopically projective (Ref. \cite{33} Proposition 2.4)), there is a morphism $h : Z' \to Q$ in $C^b_{\mathbf{Z}_n}(\mathbf{projA})$ such that $h = q^{-1}g'$ in the derived category $D_{\mathbf{Z}_n}(\mathbf{modA})$.

\[
\begin{array}{c}
Z' \\
\downarrow h \\
Q \\
\downarrow q \\
Z
\end{array}
\]

So $g' = qh$ in $D_{\mathbf{Z}_n}(\mathbf{modA})$, and further in the homotopy category $K_{\mathbf{Z}_n}(\mathbf{modA})$ due to $Z' \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$. Hence $g' = qh + l$ in $C_{\mathbf{Z}_n}(\mathbf{modA})$ for some null-homotopy $l : Z' \to Z$ in $C_{\mathbf{Z}_n}(\mathbf{modA})$. Since $l$ is a null-homotopy, $l$ is factored through $p$ (Ref. \cite{29} Page 53, Proof of Proposition 7.1]), say $l = pl'$. Since $Z' \in C^b_{\mathbf{Z}_n}(\mathbf{projA})$, each component of $Z'$ is projective. Thus $l'$ is factored through $p'$, say $l' = pl''$. Hence $l$ is factored through $r$.

\[
\begin{array}{c}
Z' \\
\downarrow l' \\
\oplus_{j=0}^{n-1}J_j(Z') \\
\downarrow p \\
Z
\end{array}
\]

Furthermore,

\[
g' = qh + l = qh + rl'' = (q, r) \left( \begin{pmatrix} h \\ l'' \end{pmatrix} \right) = gg''
\]

where $g'' := \left( \begin{pmatrix} h \\ l'' \end{pmatrix} \right) : Z' \to Y = Q \oplus (\oplus_{j=0}^{n-1}J_j(P_j))$. So $g : Y \to Z$ is a right $C^b_{\mathbf{Z}_n}(\mathbf{projA})$-approximation of $Z$. Therefore, $C^b_{\mathbf{Z}_n}(\mathbf{projA})$ is contravariantly finite over $C_{\mathbf{Z}_n}(\mathbf{modA})$. 

23
Since \(C_{bn}^b(A)\) is Krull-Schmidt and \((C_{bn}^b(A), E_{bn}^b)\) has right almost split morphisms, \((C_{bn}^b(A), E_{bn}^b)\) has minimal right almost split morphisms. Furthermore, one can prove that if \(Z \in C_{bn}^b(A)\) is indecomposable and non-\(E_{bn}^b\)-projective then there is an almost split sequence in \((C_{bn}^b(A), E_{bn}^b)\) ending in \(Z\).

Applying the obtained result to dualizing \(k\)-variety \(A^{\text{op}}\), we know that \((C_{bn}^b(A^{\text{op}}), E_{bn}^b)\) has right almost split morphisms and if \(Z \in C_{bn}^b(A^{\text{op}})\) is indecomposable and non-\(E_{bn}^{\text{op}}\)-projective then there is an almost split sequence in \((C_{bn}^b(A^{\text{op}}), E_{bn}^{\text{op}})\) ending in \(Z\). Therefore, \((C_{bn}^b(A), E_{bn}^b)\) has almost split sequences.

**Corollary 7.** Let \(A\) be a dualizing \(k\)-category with trivial exact structure \(E\) and \(\text{gl.dim}(\text{mod}A) \leq 1\), and \(n \in \mathbb{N}\). Then the exact category \((C_{bn}^b(A), E_{bn}^b)\) has almost split sequences.

**Proof.** Analogous to [12, Lemma 4.2] (cf. [22, Proposition 9.7]), we can prove \(C_{bn}^b(A) = C_{bn}(A)\) due to \(\text{gl.dim}(\text{mod}A) \leq 1\). Then this corollary follows from Corollary 6.

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