On q-analogues of Riemann’s zeta

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The aim of the paper is to define $q$-deformations of the Riemann zeta function and to extend them to the whole complex plane. The construction is directly related to the recent difference generalization of the Harish-Chandra theory of zonal spherical functions [C1,C2,C3]. The Macdonald truncated theta function [M1] replaces $x^s$. The analytic continuation is based on the shift operator technique and a more traditional approach close to Riemann’s first proof of the functional equation (see [E,R]).

We introduce $q$-deformations using the real and imaginary integration. The construction depends on the particular choice of the path and the integrand much more than in the classical theory. We come to several different functions with different properties. All have meromorphic continuations to the whole complex plane. The basic ones approach $\zeta$ (up to a $\Gamma$-factor) as $q \to 1$ for all $s$ thanks to the Stirling-Moak formula for $\Gamma_q$ [Mo]. There are also equally interesting transitional $q$-zeta functions converging to $\zeta$ for $\Re s > 1$ and to proper combinations of gamma functions for $\Re s < 1$. They approximate $\zeta$ very well in the critical strip when $q$ is close enough but not too close to 1. Then the tendency $\Gamma_q \to \Gamma$ suppresses the Stirling formula and eventually they go to their $\Gamma$-limits.

Our $q$-integrals do not satisfy the celebrated relation between $\zeta(s)$ and $\zeta(1 - s)$ in the $q$-setting as well as the Euler product formula. However certain analytic properties of the $q$-zeta functions are better than the classical ones. For instance, the imaginary one is periodic in the imaginary direction and stable as $\Re s \to \pm \infty$. The number of its zeros can be calculated exactly upon the symmetrization $s \leftrightarrow 1 - s$. Its leading term coincides with that from the Riemann estimate of the number of zeros of $\zeta(s)$ in the critical strip. The $q$-zeros do not belong to the critical line anymore but their distribution is far from random.

Let us try to summarize the key results of the paper in more detail.

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1) We construct the meromorphic continuation of the imaginary $q$-zeta functions (defined in terms of the imaginary $q$-integration) to all $s$ by means of the shift operator method, mainly Theorem 4.2. The existence of such a continuation is not too surprising because we can also apply Cauchy’s theorem almost following Riemann’s first proof of the functional equation. The functions with poles dense everywhere appear in this approach.

2) What is surprising is that the convergence of the meromorphic continuation to the classical zeta (up to a factor) holds for the $q$-plus-zeta corresponding to the integration with the kernel $(q^{x^2} + 1)^{-1}$, but fails for the one defined for $(q^{x^2} - 1)^{-1}$. The latter tends to $\Gamma(s)\zeta(s)$ for $\Re s > 1$ only. In the plus case, the limit for negative $\Re s$ is calculated via the shift operator. We were not able to employ other methods.

3) Numerical experiments show that the $q$-plus-zeta does not have zeros for $\Re s > 1/2$ when $0 < q < 0.96$. Actually it is very surprising because the imaginary $q$-zeta functions are periodic in the imaginary direction and the convergence to the classical zeta (multiplied by a proper factor) is for relatively small $\Im s$. Presumably all zeros of the $q$-plus-zeta come from the classical zeros for $\Re s > 0$ and $a > 1$.

4) The imaginary $q$-integral can be exactly calculated for the kernel $q^{-x^2}$. It is an important part of the paper (Section 2) related to Ramanujan’s $q$-integrals and closely connected with [C1]. The resulting function has no zeros for $\Re s > 1/2$. However it explains the above numerical phenomenon for small $q$ only. Analogous exact formulas are obtained for the Jackson summation instead of the imaginary integration.

5) The most promising result of the paper may be the definition of the sharp $q$-zeta functions, which have no classical counterparts. They are Jackson sums in terms of the plus-minus kernels discussed above, convergent for all $s$ except for the poles. Their limits $q \to 1$ are similar to those for the imaginary integration. The analysis of their zeros leads to some interesting properties of Riemann’s zeta.

6) Still the main applications are expected upon the (anti)symmetrization of the integral $q$-zeta functions with respect to $s \leftrightarrow 1 - s$ (Section 6). In the imaginary case, the number of the corresponding zeros matches well the famous Riemann estimate. The antisymmetrization of the sharp $q$-zeta may satisfy the Riemann hypothesis. Conjecturally, all its zeros in a natural horizontal strip are imaginary.

Actually our methods are of more general nature. They can be applied to study difference Fourier transforms of functions of $q^{x^2}$-decay for a quadratic form $x^2$ associated to a root system in $\mathbb{R}^n$ (or in the imaginary variant). Without going into detail, we mention that such class of functions is Fourier-invariant in the $q$-theory in contrast to the Harish-Chandra transform. The
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q-Fourier transforms “at zero” of \((q^{-x^2} \pm 1)^{-1}\), simplest deformations of the \(q^{x^2}\), are natural multi-dimensional counterparts of the \(q\)-integrals studied below.

Note that \(q\)-zeta functions are mainly considered in the paper as a good starting point for the general analytic theory of the \(q\)-Fourier transform. We use them to demonstrate the shift operator method and to show some typical problems. Thanks to such a concrete choice (and the one-dimensional setup) we are able to illustrate all claims numerically. There are of course more specific results and applications to the classical zeta, which hopefully will be continued.

The first section contains a certain motivation and the main constructions (for the standard minus-kernel). Section 2 is devoted to the exact formulas for the integrals of the truncated theta with and without the Gaussian. In Section 3, we introduce the \(q\)-zeta functions systematically using the real and imaginary integrations and the analytic continuation. Section 4 contains the approach via the shift operator. The next one is a discussion of the \(q\)-zeros. The last section is an attempt to establish connections with the Riemann hypothesis.

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1. Main constructions

Switching from integers to their \(q\)-analogues is a common way of defining \(q\)-counterparts. For instance, one can deform \(\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\) as follows [UN]:

\[
\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{sn}}{[n]_q^s} \text{ for } [n]_q \overset{\text{def}}{=} 1 - q^n, \frac{1}{1-q}.
\]

Here we assume that \(0 < q < 1\) and \(\Re s > 0\). The numerator ensures the convergence. It becomes a constant when \(\zeta_q\) is rewritten in terms of \([n]_q^{-1}\). This function approaches \(\zeta\) as \(q \to 1\) for \(\Re s > 1\). We note that similar \(q\)-deformations were considered before in connection with Carlitz’s Bernoulli numbers and \(p\)-adic \(L\)-functions (see [Sa]). Ueno and Nishizawa constructed a meromorphic continuation of \(\zeta_q\) to all \(s\) with simple poles in \(\{-Z_+ + 2\pi aiZ\}\) for \(a = -(\log(q))^{-1}\), \(Z_+ = \{0 \leq n \in Z\}\). The procedure is a certain substitute of the functional equation which cannot be saved. As a rule relations between special functions which involve the multiplicative property of \(x^n\) have no direct \(q\)-counterparts.

A modification of \(\zeta_q\) tends to \(\zeta\) for all \(s\). The latter property is important but not sufficient. Once the functional equation is sacrificed, something else should be gained. At least, certain analytic advantages can be expected. Generally speaking, \(q\)-functions (and their representations as sums, products
or integrals) are single-valued and have better analytic properties than their classical counterparts. It does not seem to happen with \( \zeta_q(s) \). Actually its definition is a transition from the differential theory to the difference one because the complex powers are still involved. The latter are redefined via \( \Gamma_q \) in a systematic \( q \)-theory.

Indeed, substituting \( q^{sn} \to q^{sn/2} \) in (1.1), the resulting zeta (up to a simple factor) is the classical \( \zeta(s) \) where \( x^s \) is replaced by \( \sinh^s(x/(2a)) \). By the way, such numerators are more relevant to associate \( \zeta_q(s) \) with the quantum \( SL_2 \).

This passage is well-known in the theory of special functions. Instead of the Bessel functions, which are pairwise orthogonal with respect to the measure \( x^s \, dx \), one gets the hypergeometric and spherical functions. The multi-dimensional theory of \( \sinh(x)^s \) is the cornerstone of the harmonic analysis on symmetric spaces (see e.g. [He,HIO]). It is of obvious importance to incorporate the hypergeometric function into the theory of zeta. The paper [UN] is a step in this direction. However the basic (difference) hypergeometric function seems more appropriate.

A straightforward definition. Let us switch from \([x]^{2k}_q\) to the following combination of \( q \)-shifted factorials (see e.g. [A,M1]):

\[
\delta_k(x; q) = \prod_{j=0}^{\infty} \frac{(1 - q^{j+2x})(1 - q^{j-2x})}{(1 - q^{j+k+2x})(1 - q^{j+k-2x})}. 
\]

Trying to improve the \( z\zeta \) due to Ueno - Nishizawa, we put

\[
(1.3) \quad \tilde{\zeta}_q^u(s)_c = \sum_{n=1}^{c} \delta_{-s}(\sqrt{au}; q), \quad \tilde{\zeta}_q^u(s) = \tilde{\zeta}_q^u(s)_{\infty} \quad \text{for} \\
q = \exp(-1/a), \ R \ni a > 0, \ u \in \mathbb{C}, \ c = 1, 2, \ldots.
\]

From now on, \( q \) and \( a \) will be always connected as in (1.3). Since \( \delta(x) \) is even the sign of \( \sqrt{ } \) can be arbitrary. As we will see, \( \sqrt{a} \) and the “direction” \( u \) are necessary to ensure the convergence and go back to the classical theory via the following Stirling-Moak formula [Mo,UN].

**Lemma 1.1.**  

a) Setting \( z^k = \exp(k \log z) \) for complex (fixed) \( z,k \) and the usual branch of log,

\[
(1.4) \quad \lim_{a \to \infty} (a/4)^k \delta_k(\sqrt{az}; q) = (-z)^k \\
\text{if either } z \notin \mathbb{R}_+ \text{ or } k \in \mathbb{Z}.
\]

b) Replacing \( az \) by \( v \) for any (fixed) \( v \in \mathbb{C} \):

\[
(1.5) \quad \lim_{a \to \infty} (a)^{2k} \delta_k(\sqrt{v}; q) = \frac{\Gamma(k + 2\sqrt{v})\Gamma(k - 2\sqrt{v})}{\Gamma(2\sqrt{v})\Gamma(-2\sqrt{v})}.
\]
The singularities of the denominator in (1.4) go to infinity together with \( a \) for \( z \not\in \mathbb{R}_+ \) and disappear in the limit. When \( k \in \mathbb{Z} \) the claim is obvious since \( \delta_k \) is a rational function in this case. The Moak theorem also gives a certain estimate of the convergence. The second formula is well-known.

It is clear why the first formula fails for \( z \in \mathbb{R}_+ \). The numerator of (1.2) becomes 0 for infinitely many \( z \) provided that \( k \not\in \mathbb{Z} \). However the limit cannot be zero. Another reason is that we switch to (1.5) when \( z = v/a \) for fixed \( v \). The \( a \)-factor in this formula is \( a^{2k} = o(a^k) \) for \( \Re k < 0 \). Hence (1.4) diverges as \( a \to \infty \) for \( z \) approaching zero and such \( k \). The latter argument will be applied to analyze the limiting behavior of the sharp \( q \)-zeta functions.

Thus the simplest choice \( u = 1 \) in (1.3) is impossible, since it destroys the termwise convergence to the classical sum for \( \zeta(s) \). The lemma gives that

\[
\lim_{a \to \infty} (-4u/a)^s \tilde{\zeta}_q^u(s)_c = \sum_{n=1}^c n^{-s}
\]

for any \( s \in \mathbb{C} \) (and fixed \( c \)) if \( u \not\in \mathbb{R}_+ \). However we must avoid \( u \in -\mathbb{R}_+ \) because \( \lim_{c \to \infty} \tilde{\zeta}_q^u(s)_c \) does not exist for such \( u \). Let us examine the imaginary unit \( u = i \).

**Theorem 1.2.** Let \( \tilde{\zeta}_q(s) = \tilde{\zeta}_q^i(s) \), \( \Re s > 0 \). Then \( \tilde{\zeta}_q(s) \) exists apart from the set \( S(a) \) of the poles of all terms in (1.3). If \( \Re s > 1 \), given a sequence \( a_m \to \infty \) and \( k \not\in \bigcup_m \infty S(a_m) \),

\[
\lim_{m \to \infty} (-4i/a_m)^s \tilde{\zeta}_q(s) = \zeta(s) = \sum_{n=1}^\infty n^{-s}.
\]

We see the first special feature of the \( q \)-zeta functions. The convergence is for \( \Re s > 0 \) in contrast to the usual inequality \( \Re s > 1 \). However (1.6) holds for \( \Re s > 1 \) only.

The function \( \tilde{\zeta}_q(s) \) is not real for real \( s \). The following function is:

\[
\sum_{n=1}^\infty (\delta_-(\sqrt[n]{a}; q) + \delta_-(\sqrt[n]{a}^*; q))/2.
\]

Here and further \( z^* \) is the complex conjugation. The limit of this function as \( a \to \infty \) remains the same.

The analytic behavior of \( \tilde{\zeta}_q(s) \) is totally singular. The set \( S(a) \) is dense everywhere in \( \mathbb{C} \) unless \( 2\pi a \in \mathbb{Q} \). If \( 2\pi a = n/d \) for coprime integers \( n, d \), then \( S(a) \) belongs to the union \( \mathcal{C}_d \) of the translations of the diagonal cross \( \mathcal{C} = \{ x^2 \in i\mathbb{R} \} \) by \( \{(l/d)i, l \in \mathbb{Z} \} \). Choosing \( a_m = m^2a \), we have the sum for \( \tilde{\zeta}_q(s) \) is pointwise convergent in spite of infinitely many poles in any
It seems that “totally singular” functions are inevitable in the difference calculus. However we prefer to avoid them in the basic definitions. The next step is to remove as many poles as possible turning from sums to integrals. We note that $\tilde{\zeta}_q(s)$ will be directly related to $3_q^\lambda(k|d)$ from (3.25) playing an important role in the analytic continuation of the imaginary $q$-zeta functions.

**Real integration.** Actually the $\delta$-function and its multi-dimensional generalizations due to Macdonald are important because they make selfadjoint remarkable difference operators which have many applications (see [M2,C3]). Hence it is more logical to use them as measures, which is exactly what we are going to do.

From now on we will switch from $s$ to the “root multiplicity” $k = s - 1/2$, standard in the harmonic analysis on symmetric spaces and in the corresponding theories of orthogonal polynomials. This choice is convenient in the theory of $\zeta$ as well: the critical line $\Re s = 1/2$ coincides with the $k$-imaginary axis.

First, we will express the $\Gamma$-function and $\zeta \Gamma$ in terms of the Gaussian $\exp(-x^2)$ modifying a little the classical representations:

$$\Gamma(k + 1/2) = 2 \int_0^\infty x^{2k} \exp(-x^2) dx, \ \Re k > 1/2,$$

(1.8)  $3(k) \overset{def}{=} \zeta(k + 1/2) \Gamma(k + 1/2) = 2 \int_0^\infty x^{2k} (\exp(x^2) - 1)^{-1} dx.$

Since the first formula has a nice $q$-generalization with $\delta_k(x; q)$ instead of $x^{2k}$ (see [C1] and the next section), we can try to use the second to deform $3(k)$.

Let

$$3_q^\pm(k) \overset{def}{=} \int_0^{\infty \pm i\epsilon} (\exp(x^2/a) - 1)^{-1} \delta_k(x; q) dx.$$

(1.9)  The contour of integration $C^\epsilon_\pm$ for $3^\pm$ goes from 0 to $\pm \epsilon i$ and then to $\infty \pm i\epsilon$ for fixed $\epsilon > 0$, which is assumed to be sufficiently small. Note that we cannot integrate over $\mathbb{R}_+$ for real $k$ because $\delta_k$ has infinitely many poles there. In contrast to (1.8), there is no problem with the point $x = 0$. Indeed, $\delta_k$ has a zero of the second order at 0 which annihilates the pole of the same order coming from $(\exp(x^2/a) - 1)^{-1}$. To be more exact, we need to exclude $k = 0$ in this argument. The functions $3_q^\pm$ are regular apart from

(1.10)  $K^\epsilon_\pm(a) = \{2C^\epsilon_\pm - Z_+ + (2\pi ai)\mathbb{Z}\} \cup \{-2C^\epsilon_\pm - Z_+ + (2\pi ai)\mathbb{Z}\}.$

They are obviously $2\pi ai$-periodic. However it is of no use, since the connected components of $C \setminus K^\epsilon_\pm(a)$ have the same periodicity. In this definition, we loose the reality of $\zeta$ on the real axis. The following functions have this important
property:

\begin{align}
3_q^{re}(k) & \overset{\text{def}}{=} \frac{3_q^{-}(k) + 3_q^{+}(k)}{2} = \frac{1}{2} \int_{-\infty - \epsilon i}^{+\infty - \epsilon i} \frac{\delta_k(x; q)}{\exp(x^2/a) - 1} \, dx, \\
3_q^\sharp(k) & \overset{\text{def}}{=} \frac{3_q^{-}(k) - 3_q^{+}(k)}{2i} = \frac{1}{2i} \int_{\infty + \epsilon i}^{\infty - \epsilon i} \frac{\delta_k(x; q)}{\exp(x^2/a) - 1} \, dx.
\end{align}

The second integration is with respect to the path \( \{ C^- - C^+ \} \). The corresponding function is well-defined in the horizontal strip \( \{-2\epsilon < \Im k < +2\epsilon\} \) and has no singularities for \( \Re k > 0 \). The natural domain of the first function is \( K^{re} \overset{\text{def}}{=} C \setminus \{ K^- \cup K^+ \} \).

**Theorem 1.3.**  
a) The functions \( 3_q^{\pm} \) coincide when

\[ \Im k + 2\pi a \mathbf{Z} \notin [-2\epsilon, +2\epsilon]. \]

The functions \( 3_q^{re}, 3_q^\sharp \) have meromorphic continuations from \( \mathbf{R}_+ \) to \( \mathbf{C} \). The first has no poles in the strip

\[ K^\sharp(a) \overset{\text{def}}{=} \{ k, -2\epsilon a < \Im k < 2\epsilon a \} \text{ for } \epsilon_a \overset{\text{def}}{=} \min\{\sqrt{\pi a}, \pi a/2\}. \]

The set of poles of the continuation of \( 3_q^\sharp \) in this strip is \( \{-\mathbf{Z}_+/2\} \). It also has simple zeros at \( k = 1, 2, \ldots \).

b) Let \( 3(k) = \zeta(k + 1/2)\Gamma(k + 1/2) \) (as above) and \( \Re k > 1/2 \). Then

\begin{align}
\lim_{a \to \infty} (a/4)^{k-1/2} \exp(\pm\pi ik)3_q^{\pm}(k) & = 3(k), \\
\lim_{a \to \infty} (a/4)^{k-1/2}3_q^\sharp(k) & = \sin(\pi k)3(k).
\end{align}

For any \( k \in \mathbf{C} \),

\[ \lim_{a \to \infty} (a/4)^{k-1/2}3_q^{re}(k) = \cos(\pi k)3(k). \]

The calculation of the limits of \( 3_q^{\pm} \) is based directly on (1.8) and Lemma 1.1; (1.16) is a formal corollary of the (1.15).

The limit of \( 3_q^{re}(k) \) is calculated using the following well-known integral representation for the zeta (see e.g. [E]):

\[ (1/2) \int_{\infty - \epsilon i}^{\infty + \epsilon i} \frac{(-z)^k}{i\sqrt{-z}(e^z - 1)} \, dz = -\sin(\pi s)3(k) = \cos(\pi k)3(k). \]

The path of integration begins at \( z = -\epsilon i + \infty \), moves to the left down the positive real axis till \(-\epsilon i\), then circles the origin and returns up the positive real axis to \( \epsilon i + \infty \) (for small \( \epsilon \)). Note the difference from the integral \( \int_{\infty + \epsilon i}^{\infty - \epsilon i} \) above, where the orientation is different and the path goes through the origin. The left-hand side multiplied by \( (a/4)^{k-1/2} \) is exactly the integral (1.11) after the
substitution: $z = x^2/a$. The factor $i\sqrt{-z}$ is $(a/2)dz/dx$, $\sqrt{x}$ is the standard branch positive on $R_+$. 

In the paper, we prefer to replace $(\exp(x^2/a) - 1)^{-1}$ by $(\exp(x^2/a) + 1)^{-1}$, which improves the analytic properties of the q-zeta functions. For instance, the left-hand side of (1.16) diverges for $\Re k < 1/2$, whereas its “plus-counterpart” approaches the zeta (up to a proper factor) for all $k$. However the asymptotic behavior of $3_q^\nu(k)$ for $\Re k < 1/2$ is very interesting:

\begin{equation}
\lim_{a\to\infty} a^{2k-1} 3_q^\nu(k) = \tan(\pi k)\Gamma(k)^2 \quad \text{for} \quad \Re k < 1/2,
\end{equation}

\begin{equation}
(a/4)^{k-1/2} 3_q^\nu(k) = (4a)^{1/2-k} \tan(\pi k)\Gamma(k)^2 + \sin(\pi k)3(k) + o(1)
\end{equation}

for $\Re k = 1/2$, where the error $o(1)$ tends to 0 as $a \to \infty$. In fact the latter formula holds for all $k$. Indeed, the term $\sin(\pi k)3(k)$ is suppressed in (1.20) for $\Re k > 1/2$ and vice versa for $\Re k < 1/2$ because of the difference in the $a$-factors. Moreover, the error is of order $\tan(\pi k)\Gamma(k)^2$ in the strip $-1/2 + \epsilon < \Re k < 1/2 - \epsilon$ for any small $\epsilon > 0$, when $a$ are big but not too big. Since $\Gamma(k)^2$ quickly approaches 0 as $\Im k$ increases, the function $(a/4)^{k-1/2} 3_q^\nu(k)$ first approximates $\sin(\pi k)3(k)$ with high accuracy and then slowly switches to the leading term $(4a)^{1/2-k} \tan(\pi k)\Gamma(k)^2$ in the critical strip. Numerically, the latter (dominant) regime is very difficult to reach for big $\Im k$, even for $k \sim 10i$.

The main advantage of $3_q^\nu(k)$ is the following representation (Cauchy’s theorem), which has no counterpart in the classical theory:

\begin{eqnarray}
3_q^\nu(k) &=& -\frac{a\pi}{2} \prod_{j=0}^\infty \frac{(1 - e^{-(j+k)/a})(1 - e^{-(j-k)/a})}{(1 - e^{-(j+2k)/a})(1 - e^{-j/1/a})} \\
& & \sum_{j=0}^\infty \prod_{l=1}^j \frac{(1 - e^{-(l+2k-1)/a})(1 - e^{l+k}/a)}{(1 - e^{-(l+k-1)/a})(1 - e^{l/2/a})} \frac{1}{\exp((k+j)^2/4a)} - 1.
\end{eqnarray}

Thanks to the factor $(\exp((k+l)^2/(4a)) - 1)^{-1} \sim \exp(-l^2/(4a))$, it converges for all $k$ ($0 < q < 1$) except for the singularities, and very quickly. For instance, we can calculate $3_q^\nu(k)$ numerically for $a \sim 10000$ (i.e. when $q \sim 0.9999$), and $\Im k \sim 100$. For integral $k$, the range is almost unlimited. The numerical simulation of the $3_q^\nu\nu(k)$ is approximately $10 \cdots 100$ times worse.

In a sense, the series (1.21) is complimentary to $\zeta_q(s)$ (Theorem 1.2). They appear together in the following construction.

**Imaginary integration.** The integration in the real and imaginary directions are practically interchangable in the classical theory of zeta function (the substitution $x \to ix$ multiplies $x^s$ by a constant). In the harmonic analysis, the difference is fundamental. The imaginary case (integrating with $\sin(x)^{2k}$) is much simpler than the real one (for $\sinh(x)^{2k}$ as the kernel). The real and
imaginary integrations are closer to each other for $\delta_k(x; q)$. However the latter has certain analytic advantages.

For $\Re k > 0$, we set

$$Z_{\text{im}}^q(k) \overset{\text{def}}{=} (-i) \int_0^\infty \exp(-x^2/a) - 1)^{-1} \delta_k(x; q) dx.$$  \hspace{1cm} (1.22)

The complex conjugation is denoted by $\ast$.

**Theorem 1.4.** a) The function $Z_{\text{im}}^q(k)$ is analytic for $\Re k > 0$, and has a $2\pi ai$-periodic meromorphic continuation $Z_{\text{an}}^q(k)$ to all $k \in \mathbb{C}$ with the following set of poles (simple for generic $a$):

$$\mathcal{P} = \{-Z_+/2 + \pi ai \mathbb{Z}\} \cup \Lambda,$$  \hspace{1cm} (1.23)

where

$$\Lambda = \{-2\sqrt{2\pi ai N} - Z_+ + 2\pi ai \mathbb{Z}\} \cup \{-2\sqrt{2\pi ai N} - Z_+ + 2\pi ai \mathbb{Z}\}.$$  \hspace{1cm} (1.25)

The poles of the $2\pi ai$-periodic meromorphic continuation $\mathring{Z}_{\text{an}}^q(k)$ of the function

$$Z_{\text{im}}^q(k) = Z_{\text{im}}^q(k) \prod_{j=0}^\infty \frac{(1-q^{j+2k})(1-q^{j+1})}{(1-q^{j+k})(1-q^{j+k+1})},$$  \hspace{1cm} (1.24)

belong to the set $\mathcal{P} = \Lambda \cup \{-Z_+/2 + 2\pi ai \mathbb{Z}\}$ (they are double for $\Re(z) \in -\mathbb{N} = -1 - Z_+$).

b) If $\Re k > 1/2$, then

$$\lim_{a \to \infty} (a/4)^{k-1/2} Z_{\text{an}}^q(k) = Z(k) = \Gamma(k+1/2)\zeta(k+1/2),$$  \hspace{1cm} (1.25)

$$\lim_{a \to \infty} a^{k-1/2} \mathring{Z}_{\text{an}}^q(k) = \sqrt{\pi} \Gamma(k+1)\zeta(k+1/2).$$  \hspace{1cm} (1.26)

For all $k$,

$$\lim_{a \to \infty} (a/4)^{k-1/2} \mathring{Z}_{\text{an}}^q(k) \equiv (4a)^{1/2-k} \tan(\pi k)\Gamma(k)^2 + Z(k),$$  \hspace{1cm} (1.27)

as $a \to \infty$.

c) The function $Z_{\text{im}}^q(k)$ tends to a nonzero limit as $\Re k \to \infty$. Given $\kappa \in -\mathbb{R}_+$ such that $\{\kappa + i\mathbb{R} + Z\} \cap \mathcal{P} = \emptyset$, the limit

$$\psi(\nu; \kappa) = \lim_{Z \to m+i\nu} \mathring{Z}_{\text{an}}^q(k - m + i\nu)$$  \hspace{1cm} (1.28)

exists and is a continuous $2\pi a$-periodic function of $\nu \in \mathbb{R}$.

The term $Z$ in (1.27) can be skipped for $\Re k$ strictly less than 1/2 due to the $a$-factor. However, the formula with this term gives more than was stated in the theorem. In the critical strip $\{-1/2 < \Re k < 1/2\}$, the right-hand side approximates well the left-hand side even when $a$ is not too big. For instance, $(a/4)^{k-1/2} \mathring{Z}_{\text{an}}^q$ is getting very close to $Z$ when $\Im k \sim 10$ or more to make $\Gamma(k)^2$ small enough. It is the same phenomenon (with the same $\Gamma$-limit) as in (1.20).
The set of admissible $\kappa$ in c) is dense everywhere in $-\mathbb{R}^+$ because so is $\tilde{P} + i\mathbb{R} + \mathbb{Z} \subset \mathbb{C}$ (for generic $a$). However the function $\psi(\nu; \kappa)$ is well-defined. It is related to the existence of the totally singular zeta $\tilde{\zeta}_q$ considered above. In fact, $\psi$ is a limit of a certain sum similar to (1.3).

Thanks to c), the number of zeros of $\tilde{3}_q^{an}(k)$ in the vertical strip
\[\{\kappa - m - 1 < \Re k < \kappa - m\}\]mod $2\pi ai\mathbb{Z}$
equals the number of poles of $\tilde{3}_q^{im}(k)$ (counted with multiplicities) in this strip for sufficiently big $m$. It is likely that even the positions of individual zeros (modulo $\mathbb{Z} + 2\pi ai\mathbb{Z}$) approach certain limits when $m \to \infty$. This can be seen numerically.

The number of zeros of $\tilde{3}_q^{an}$ in the half-planes $\Re k > C$ for any $C \in \mathbb{R}$ is finite, which readily results from a). It is not difficult to calculate $\psi(\nu; \kappa)$ numerically for $a \sim 10$ or less. The change of $\text{arg}(\psi)/(2\pi)$ from $\nu = 0$ through $\nu = 2\pi a$ combined with the periodicity of $\tilde{3}_q^{an}$ can be used to calculate the exact number of its zeros modulo the period $2\pi ai$ in the half-planes $\Re k > C$.

We can ensure the convergence to Riemann’s zeta for all $k$ switching to the “plus-zeta functions” defined for $(\exp(-x^2/a) + 1)^{-1}$. In the classical theory, this substitution adds a simple factor to $\zeta$, terminating the pole at $s = k+1/2 = 1$ and improving the analytic properties. The zeros for $\Re k \neq 1/2$ remain the same.

Setting
\begin{equation}
(1.29) \quad \tilde{3}_q^{im}(k) \overset{\text{def}}{=} (-i) \int_{0}^{\infty i} (\exp(-x^2/a) + 1)^{-1} \delta_k(x; q) dx \quad \text{for} \quad \Re k > 0,
\end{equation}
\begin{equation}
(a/4)^{k-1/2} \tilde{3}_q^{im}(k) \to \tilde{3}_q^{+}(k) \overset{\text{def}}{=} (1 - 2^{1/2-k}) \tilde{3}(k).
\end{equation}
The limit holds for all $k \in \mathbb{C}$ except for the poles if $\tilde{3}_q^{im}$ is replaced by the meromorphic continuation $\tilde{3}_q^{an}(k)$.

**Locating the zeros.** The numerical experiments demonstrate that $\tilde{3}_q^{im}$ has only two zeros (mod $2\pi ai$) in the right half-plane $\Re k > \epsilon$ at least for $\epsilon = 0.05$, $a \leq 25(q < 0.96)$. These zeros are connected to each other:
\[z_0 = \xi + \eta i, \quad z_1 = \xi + (2\pi a - \eta)i, \quad \eta \leq \pi a.\]

In a sense, these zeros appear because of the poles of $\tilde{3}_q^{im}$ at $k = 0, \pi ai$. They vanish for $\tilde{3}_q^{im}(k)$.

The $z_0$ slowly approaches $k = 1/2$, the pole of $\zeta(k + 1/2)$:
\begin{equation}
z_{0}(a = 3) = 0.0215 + 1.2746i, \quad z_{0}(100) = 0.4419 + 0.7216i
\end{equation}
(1.30) $z_{0}(1000) = 0.4741 + 0.5560i$, $z_{0}(2000) = 0.4788 + 0.5208i$.

Switching to $\tilde{3}_q^{im}$, we hope that its zeros in the strips $\Re k > -\epsilon, \epsilon > 0$ are always deformations of the zeros of $\zeta(k + 1/2)$ for sufficiently big $a > a(\epsilon)$.
This might be true for $3_q^{an}$ as well with a reservation about the “low” zeros-going towards $k = 1/2$ (similar to (1.30)) and their reflections.

Let us mention without going into detail, that the following “theta”-deformation

$$3_q^\theta(k) \overset{\text{def}}{=} (-i) \int_0^\infty \left( \sum_{n=1}^\infty \exp(-nx^2/a) \right) \delta_k(x; q) \, dx \quad \text{for } \Re k > 0$$

also has only low zeros in the same range of $a$. Here the functional equation for the theta is used to integrate near 0. The analytic continuation of $3_q^\theta$ to the left half-plane is not known.

We note that the roles of the left and right half-planes are not symmetric in the $q$-theory. For instance, the zeros of $3_q^{an} + q$ mainly appear in the left one. There is an explanation for this.

Given a zero $k = z \in i\mathbb{R}$ of $\zeta(k + 1/2)$, the linear approximations $\tilde{z}_+(a)$ for the zero $z_+(a)$ of $3_q^{an}$ associated to a given zero $k = z$ of $\zeta(k + 1/2)$ can be calculated as follows:

$$\tilde{z}_+(a) = z(1 + \frac{4(z + 1/2)\zeta_+(z + 3/2) - (z - 1)\zeta_+(z - 1/2)}{12a\zeta'(z + 1/2)(1 - 2^{1/2-s})})$$

(1.32) for $\zeta'(s) = \partial\zeta(s)/\partial s$, $\zeta_+(s) = (1 - 2^{1-s})\zeta(s)$.

The formula results from (1.29). However we did not check the necessary estimates, so it is not completely justified at the moment. The first $\tilde{z}_+(a)$ which we found in the right half-plane corresponds to $z = 1977.2714i$, well beyond the range where we can calculate $3_q^{an}$ numerically. After this point, the number of linear deformations in the right half-plane grows but not quickly. Generally speaking, they appear when the distances between the corresponding consecutive zeros of $\zeta$ are getting too close, i.e. exactly when the linear approximation could not be reliable (numerically).

The same strong tendency holds for the “real” zeta $3_q^{re}$. However now the deformations of the zeros prefer the left half-plane. The formula for the linear approximation $\tilde{z}(a)$ of the zero $z(a)$ of $3_q^{re}(k)$ corresponding to a given root $z$ of $\zeta(k + 1/2)$ reads:

$$\tilde{z}(a) = z(1 - \frac{4(z + 1/2)\zeta(z + 3/2) - (z - 1)\zeta(z - 1/2)}{12a\zeta'(z + 1/2)}).$$

(1.33)

Note the opposite sign of the term after 1 and the missing factor $(1 - 2^{1-s})$. The formula for the plus-counterpart $3_q^{re+}(k)$ of $3_q^{re}(k)$ is (1.32) with the minus after 1. The tendency to move to the right is practically the same for $3_q^{st+}(k)$. The factor $(1 - 2^{1-s})$ does not contribute too much.

Formula (1.33) gives a very good approximation for the zeros $z^\sharp(a)$ of $3_q^\sharp(k)$ as well. Of course $a$ must be big but not enormously big, because eventually
$\mathcal{Z}^z_q(k)$ will go to its $\Gamma$-limit, which has no zeros in the critical strip. For instance, $\bar{z}(100) = 0.7091 + 14.1326i$ for $z = 14.1347i$, the first zero of $\zeta(k + 1/2)$. The exact zero $z^z(100)$ equals $0.7041 + 14.0779i$. Enlarging $a$:

$$\bar{z}(1000) = 0.0709 + 14.1345i, \quad z^z(1000) = 0.0709 + 14.1340i.$$  

The first zero $z$ of $\zeta(k + 1/2)$ with the “real” linear approximation $\bar{z}$ in the left half-plane (which we found) is $z = 1267.5706i$. The next one is $1379.6833i$. The exact zero $z^z(100)$ equals $0.7041 + 14.0779i$. Enlarging $a$:

$$\bar{z}(1000) = 0.0709 + 14.1345i, \quad z^z(1000) = 0.0709 + 14.1340i.$$  

The $a$-evolution of the $z$-zeros is very nontrivial. It is not too difficult to trace the evolution of the first ones back in $a$ (at least, through $a \sim 1$). For instance, if $z = 14.1347i$

$$z^z(35) = 1.92 + 13.71i, \quad z^z(11) = 4.25 + 11.36i,$$
$$z^z(5) = 0 + 8.72i, \quad z^z(2) = -2.98 + 5.01i.$$  

Concerning big $a$, we were not able to reach the moment when the zero under consideration leaves the vicinity of $z = 14.1347i$. The linear approximation works reasonably well at least until $a = 50000$. One may expect $z^z$ to go to the vertical line $\Re k = 1/2$. Employing the formula (1.20), $z^z \simeq 0.0015 + 14.1320i$ when $\log(a) = 122.25$, so it is still close to the classical zero for such $a$. When $\log(a) = 1020.34$, approximately $z^z \simeq 0.4361 + 14.1334i$. This calculation is of course qualitative (as well as (1.20)), but presumably demonstrates the tendency.

There are several (more than 1) examples of “low” zeros of $\mathcal{Z}^z_q$ traveling from the left half-plane towards the natural “attractor” $k = 1/2$. The evolution of the first of them (a counterpart of the zero from (1.30)) is as follows:

$$a = 5 : z^z = 0.037 + 0.735i, \quad a = 2733 : z^z = 0.409 + 0.477i,$$
$$a = 4500 : z^z = 0.418 + 0.460i, \quad a = 15187 : z^z = 0.436 + 0.424i,$$
$$a = 76882 : z^z = 0.452 + 0.383i, \quad a = 172984 : z^z = 0.459 + 0.365i.$$  

The convergence is very slow. It explains to a certain extent why the deformations of the zeros of $\zeta$ stay put. The speed of decay at big $a$ is expected to be much smaller for the “high” zeros.

Summarizing, the distribution of the zeros of the imaginary $q$-zeta functions looks surprisingly regular. As to the $\mathcal{Z}^z_q$ and its plus-counterpart, the main objects of our computer simulation, the $q$-zeros are also far from random in the horizontal strip $\{-2\epsilon_a < 3k < 2\epsilon_a\}$ for $\epsilon_a = \sqrt{\pi a}$ (see (1.14)).

There must be a reason for this. Numerically, a little decrease of the accuracy in the formula for $\mathcal{Z}^z_q$ (say, $\sim (5 \cdots 10)\sqrt{a}$ terms in the range $k = 14i \cdots 50i$ instead of the necessary $\sim (15 \cdots 20)\sqrt{a}$ for $a \sim 500 \cdots 1000$) and one can see how irregular the zeros could be.
2. Exact formulas

First we will drop the factor \( \exp\left((k + l)^2/(4a)\right) - 1 \) in the formula (1.21) for \( 3_q^{\ast}(k) \) (see (1.12)) and calculate the resulting sum. It is a variant of the celebrated constant term conjecture \([AI,M1,M2,C3]\). The latter corresponds to the imaginary integration over the period. The following theorem readily results from the \( q \)-binomial theorem. Let \( \epsilon > 0 \),

\[
\mathcal{C}_q^{\ast}(k) \overset{\text{def}}{=} \frac{1}{2i} \int_{\infty+i\epsilon}^{\infty-i\epsilon} \delta_k(x; q) \, dx = -\frac{a\pi}{2} \prod_{j=0}^{\infty} \frac{1 - e^{-(j+k)/a}}{(1 - e^{-(j+2k)/a})(1 - e^{-(j+1)/a})} \times \\
\sum_{j=0}^{\infty} \prod_{l=1}^{j} \frac{1 - e^{-(l+2k-1)/a}}{(1 - e^{-(l+k-1)/a})(1 - e^{l/a})}.
\]

The last series is convergent for all \( k \) except for \( k \in -1/2 - Z_+ + 2\pi ai\mathbb{Z} \). The coincidence of the integral and the series holds provided that \( k/2 \) is inside the contour of integration. We will always use the series if \( k \) is outside.

**Theorem 2.1.** For all \( k \) such that \( \Re k < 0 \),

\[
-\frac{2}{a\pi} \mathcal{C}_q^{\ast}(k) = 2 \prod_{j=0}^{\infty} \frac{(1 - e^{-(j+k)/a})(1 - e^{-(j-k)/a})}{(1 - e^{-(j+2k)/a})(1 - e^{-(j+1)/a})} \sum_{j=0}^{\infty} e^{kj/a} \prod_{l=1}^{j} \frac{1 - e^{-(l+2k)/a}}{1 - e^{l/a}} = 2 \prod_{j=0}^{\infty} \frac{(1 - e^{-(j+k)/a})(1 - e^{-(j+k+1)/a})}{(1 - e^{-(j+2k)/a})(1 - e^{-(j+1)/a})}.
\]

The limit of the latter expression as \( a \to \infty \) is \( \frac{2\Gamma(2k)}{\Gamma(k)\Gamma(k+1)} \).

**Integrating the Gaussians.** Now let us integrate the Gaussians \( q^{\pm x^2} \) with respect to \( \delta_k \). See \([M1,O,J]\) about the classical (i.e. differential) multidimensional generalizations. The formula for the imaginary integral can be deduced from \([C1]\) (in the case of \( A_1 \)). Let us discuss it first.

**Theorem 2.2.** Provided that \( \Re k > 0 \),

\[
\mathcal{C}_q^{im}(k) \overset{\text{def}}{=} \frac{1}{2i} \int_{-i\infty}^{i\infty} e^{x^2/a} \delta_k(x; q) \, dx = \sqrt{a\pi} \prod_{j=0}^{\infty} \frac{1 - e^{-(j+k)/a}}{1 - e^{-(j+2k)/a}},
\]

\[
a^{k-1/2} \mathcal{C}_q^{im}(k) \to \sqrt{\pi} \frac{\Gamma(2k)}{\Gamma(k)} \text{ as } a \to \infty.
\]
Actually the theorem can be considered as a variant of the Ramanujan integrals, namely (3.12) and (3.13) from Askey’s paper in [A]. They correspond to the two special cases $k = \pm \infty$, when $\delta_k$ has either no denominator or numerator, but involve more parameters. Richard Askey writes, “try to imagine what someone with Ramanujan’s ability could do now with the computer algebra systems ...”. Of course the computers are useful for algebraic $q$-calculations, but still supporting the $q$-analytic theory (meromorphic continuations, zeros etc.) seems a more promising application. The theory is not mature yet and we need numerical experiments. However, first, computers must become at least 100 times faster.

We note that the integral from (2.4) does not give the analytic continuation of $G_{q \mu}^m(k)$ for $\Re k < 0$. There is a simple way to extend the theorem to $\Re k > -1/2$ replacing $\delta_k$ by its asymmetric variant

\[
\mu_k(x; q) = \prod_{j=0}^{\infty} \frac{(1 - q^{1+2x})(1 - q^{j+1-2x})}{(1 - q^{1+k+2x})(1 - q^{j+k+1-2x})}.
\]

The $\mu$-measure has many applications and appears naturally in the proof of the Macdonald constant term conjecture (formulated in terms of $\delta$ only). Actually it is not surprising because the symmetry $x \to 1/2 - x$ of $\mu$ is quite common for many $q$-functions (including the theta functions). The connection with the $\delta$-measure is through the symmetrization:

\[
\int_{-\infty}^{+\infty} f(x^2) \mu_k \, dx = \frac{1 + e^{-k/a}}{2} \int_{-\infty}^{+\infty} f(x^2) \delta_k \, dx \text{ for any } f.
\]

The function

\[
\mathfrak{g}_{q \mu}^{\sharp}(k) \equiv -\frac{i}{1 + e^{-k/a}} \int_{1/4-\infty i}^{1/4+\infty i} e^{x^2/a} \mu_k(x; q) \, dx
\]

coincides with $G_{q \delta}^m(k)$ for $\Re k > 0$ and gives its meromorphic continuation to $\Re k > -1/2$ with the poles \( \{ \pi ai + 2\pi ai \mathbb{Z} \} \) coming from the factor before the integral. Respectively, (2.4) holds for such $k$ if $G_{q \delta}^m(k)$ is replaced by $G_{q \mu}^m(k)$. The passage $\delta \to \mu$ will be also used later for the zeta.

**Real integration.** A more interesting exact formula is for

\[
\mathfrak{g}_{q \delta}^{\sharp}(k) \equiv \frac{1}{2i} \int_{\infty+\epsilon i}^{\infty-\epsilon i} e^{-x^2/a} \delta_k(x; q) \, dx =
\]

\[
- \frac{a\pi}{2} \prod_{j=0}^{\infty} \frac{(1 - e^{-(j+k)/a})(1 - e^{-(j-k)/a})}{(1 - e^{-(j+2k)/a})(1 - e^{-(j+1)/a})} \times
\]

\[
\sum_{j=0}^{\infty} \frac{(k+j)^2}{4a} \prod_{l=1}^{j} \frac{(1 - e^{-(l+2k-1)/a})(1 - e^{(l+k)/a})}{(1 - e^{-(l+k-1)/a})(1 - e^{l/a})}.
\]
Here the convention is the same as for $\mathbf{c}_q^2(k)$: we use the integral if and only if it coincides with series, i.e., when $k/2$ is inside the contour of integration. For such $k$, (2.10) is nothing else but an application of the Cauchy theorem.

**Theorem 2.3.** Let $Q = \exp(-8a\pi^2)$. Then for all $k$ except for the poles $\mathbf{c}_q^2(k) = \phi(k)\sqrt{\pi a} \exp\left(\frac{k(k+1)}{2a}\right) \prod_{j=0}^{\infty} \frac{1 - e^{-(j+k)/a}}{1 - e^{-(j+2k)/a}}$ for $\mathbf{c}_q^2(k) = \phi(k)\sqrt{\pi a} \prod_{j=1}^{\infty} \frac{1 + Q^{j/2}}{1 - Q^{j/2}}$.

(2.11) $$(1 - e^{2\pi ik}Q^j)(1 - e^{-2\pi ik}Q^j)(1 - e^{4\pi ik}Q^{2j-1})(1 - e^{-4\pi ik}Q^{2j-1})$$

(2.12) $$\prod_{j=1}^{\infty} (1 - q^{j-k})(1 - q^{j+k})(1 + q^{j/2-1/4+k/2})(1 + q^{j/2-1/4-k/2}).$$

As $a \to \infty$,

(2.13) $\phi(k) \to \sin(\pi k), \quad (a/4)^{k-1/2} \mathbf{c}_q^2(k) \to \sin(\pi k)\Gamma(k + 1/2)$.

Concerning the limits, the convergence of $\phi(k)$ to $\sin(\pi k)$ is very fast. They practically coincide even for $a \sim 1$ because $Q$ is getting very close to zero. The $Q^{1/2}$ is dual to $q^{1/4} = \exp(-1/(4a))$ in the sense of the functional equation for the theta function:

(2.14) $$\sum_{j=-\infty}^{\infty} Q^{j^2/2} = (\phi(k)/\sin(\pi k))(k = 0) = \frac{1}{2\sqrt{\pi a}} \sum_{j=-\infty}^{\infty} q^{j^2/4}.$$

Here we first check (2.11) and then use the functional equation to go to (2.12). The latter is straightforward. Combining (2.10) with (2.12) we can eliminate $Q$. Indeed, dividing by the common factors and distributing $\exp\left(\frac{k(k+1)}{2a}\right)$ properly one arrives at the $q$-identity:

(2.15) $$\sum_{j=0}^{\infty} q^{j^2-2bk} \prod_{l=1}^{j} \frac{1 - q^{j+k-1}}{1 - q^k} \prod_{l=1}^{j} \frac{1 - q^{l+2k-1}}{1 - q^l} =$$

$$\prod_{j=1}^{\infty} (1 - q^{j/2})(1 - q^{j+k})(1 + q^{j/2-1/4+k/2})(1 + q^{j/2-1/4-k/2})(1 - q^j).$$

It is nothing else but the formula (1.11) from [C1] for the root system $A_1$ and the polynomial spherical representation. The simplest particular cases
\( k = 0,1,2,1/2 \) readily result from the product formula for the classical \( \varphi = \sum_{n=-\infty}^{\infty} q_n^m \) (see (2.14)). It is possible that this identity for \( k \in \mathbb{Z}_+/2 \) follows from known formulas. Then one can extend it analytically to arbitrary \( k \). However here we establish (2.15) via (2.11).

The key observation is that

\[
(2.16) \quad \phi(k+1) = -\phi(k), \quad \phi(k+2p) = q^{3kp+3p^2} \phi(k) \quad \text{for} \quad p \equiv 2\pi ai.
\]

The second formula results directly from the definition, the first one requires this definition, \( k \neq -1,-2,-3,\ldots,-m+1 \) modulo \( 2\pi ai\mathbb{Z} \). For instance,

\[
(2.17) \quad p_m^{(k)} = q^{mx} + q^{-mx} + \text{lower powers, except for } p_0^{(k)} = 1.
\]

The multi-dimensional generalization is due to Macdonald (see e.g. [M2]). In this definition, \( k \neq -1,-2,-3,\ldots,-m+1 \) modulo \( 2\pi ai\mathbb{Z} \). For instance,

\[
(2.18) \quad p_1^{(k)} = q^x + q^{-x}, \quad p_2^{(k)} = q^{2x} + q^{-2x} + \frac{(1-q^k)(1+q)}{1-q^{k+1}}.
\]

Setting

\[
\mathfrak{G}_q^z(k;m,n) \overset{def}{=} \frac{1}{2} \int_{\infty+\epsilon i}^{\infty-\epsilon i} p_m^{(k)}(x)p_n^{(k)}(x)e^{-x^2/a} \delta_k(x;q) \, dx =
\]

\[
- \frac{a\pi}{2} \prod_{j=0}^{\infty} \frac{(1-e^{-(j+k)/a})(1-e^{-(j-k)/a})}{(1-e^{-(j+2k)/a})(1-e^{-(j+1)/a})} \times
\]

\[
\sum_{j=0}^{\infty} p_m^{(k)}(\frac{k+j}{2})p_n^{(k)}(\frac{k+j}{2})e^{-\frac{(k+j)^2}{4a}} \prod_{l=1}^{j} \frac{(1-e^{-(l+k-1)/a})(1-e^{-(l+k)/a})}{(1-e^{-(l+k-1)/a})(1-e^{l/a})},
\]

we deduce from [C1] the identity:

\[
(2.20) \quad \mathfrak{G}_q^z(k;m,n) = q^{-\frac{m^2+n^2+2kn+mn}{4}} p_m^{(k)}(\frac{n+k}{2})p_n^{(k)}(\frac{k}{2}) \mathfrak{G}_q^z(k).
\]

The formula also holds for the \( q \)-spherical functions, generalizations of \( p_m^{(k)}, p_n^{(k)} \) to arbitrary complex \( m,n \), and in the multi-dimensional setting (see ibid. and [AI]). Moreover, the Gaussian \( \gamma = q^{x^2} = e^{-x^2/a} \) can be replaced by \( (h_t)^* \gamma \) for the \( q \)-Hermite polynomials \( \{h_t(x)\} \) (the Rogers polynomials at
\( k = \infty \) and the formal conjugation \((q)^* = q^{-1}\). This is possible (in one dimension only) because \((h_l)^* \gamma\) are eigenfunctions of the shift operator (see [BI]). We can make \( l \) complex too switching to the \(q\)-Hermite spherical functions. So a natural setting for Theorem 2.3 is with three complex parameters \( \{k, l, m\}\). Hopefully one can add more replacing \( \delta_k \) by an arbitrary truncation of the theta-function and involving \( BC_1 \). Also (2.20) can be extended to nonsymmetric Macdonald polynomials.

We note that both (2.4) and (2.15) can be deduced from known identities (Macdonald, Andrews). The general one involving Macdonald polynomials (or spherical functions) seems new.

3. Analytic continuation

In this section we construct meromorphic continuations of the “plus-counteparts” of \( Z^q \) and \( Z^q_{im} \) and give the formulas for the \( a\)-limits. We mainly use the Cauchy theorem. There is another approach based on the shift operator technique, which will be applied in the next section.

Let us start with the “sharp” function:

\[
Z^q_{\sharp}(k|d) \overset{\text{def}}{=} \frac{1}{2i} \int_{-\infty}^{+\infty} \delta_k(x;q) \frac{dx}{\exp(dx^2/a) + 1} \quad \text{for}
\]

\[
\varepsilon = \epsilon_+(a, d) = \min\{\sqrt{\frac{\pi a}{2d}}, \frac{2a}{d}\}, \quad d > 0, \quad \Re k > -1/2.
\]

The integration is with respect to the path \(\{C^\varepsilon_--C^\varepsilon_+\}\) from (1.10). Recall that it goes from \(\infty + \varepsilon i\) to \(\infty - \varepsilon i\) through the origin. The function is well-defined and has no singularities in the horizontal strip \(K^\sharp_+ = \{-2\varepsilon < \Im k < +2\varepsilon\}\) for \(\Re k > -1/2\).

**Theorem 3.1.** The meromorphic continuation of \( Z^q_{\sharp}(k|d) \) to all \( k \) is given by the formula

\[
Z^q_{\sharp}(k|d) = -\frac{a\pi}{2} \prod_{j=0}^{\infty} \left(1 - e^{-\frac{(j+k)}{a}}\right) \left(1 - e^{\frac{(j-k)}{a}}\right) \times
\]

\[
\sum_{j=0}^{\infty} \frac{1 - e^{\frac{-(j+k)}{a}}}{1 - e^{-\frac{k}{a}}} \prod_{l=1}^{\frac{1}{4}} \frac{1 - e^{\frac{-(l+2k-1)}{a}}}{1 - e^{-\frac{l}{a}}} \frac{\exp(kj/a)}{\exp\left(\frac{d(k+j)^2}{4a}\right) + 1}.
\]

The set of its poles in the strip \(K^\sharp_+\) is \(\{-1/2 - Z_+\}\). It also has zeros at all integral \( k \). For any \( k \in \mathbb{C} \) except for the poles,

\[
\lim_{a \to \infty} a^{1/4} Z^q_{\sharp}(k|d) = \sin(\pi k) Z_{\sharp}(k|d),
\]

\[
Z_+(k|d) \overset{\text{def}}{=} d^{-1/2-k} Z(k)(1 - 2^{1/2-k}).
\]
Formula (3.2) results from Cauchy’s theorem. Thanks to the \( \exp(dx^2/a) \), the convergence is for all \( k \) (except for the poles). The zeros of \( 3^♯_q(k) \) at \( k \in \mathbb{Z}_+ \) come from the product before the summation. It is less obvious for negative integers (use Theorem 4.2 below). Actually it is a general property of the difference Fourier transform (cf. (2.20)). If \( \Re k > -1/2 \), the limit (3.3) is due to the integral representation. For the other \( k \), one can use the shift operators (the next section). We note that the calculation of the limit for \( \Re k < -1/2 \) is equivalent to that for the imaginary \( q \)-zeta.

**Imaginary zeta.** We define it for \( \Re k > 0 \) as follows (cf. (1.29)):

\[
(3.4) \quad 3^{im}_q(k|d) = 3_q(k|d) \overset{def}{=} (-i) \int_0^{i\infty} (\exp(-dx^2/a) + 1)^{-1} \delta_k(x;q)dx.
\]

We will drop \( \{im\} \) till the end of the section. Its limit as \( a \to \infty \) can be readily calculated:

\[
(3.5) \quad (a/4)^{k-1/2} 3_q(k) \to 3_q(k|d).
\]

The first step is to extend this function to small negative \( \Re k \). Let

\[
(3.6) \quad \epsilon = \min\{ \sqrt{\pi a/2d}, 1/4 \}, -2\epsilon < \Re k < 2\epsilon.
\]

The following function is analytic for such \( k \) except for the poles at \( \{\pi ai + 2\pi ai\mathbb{Z}\} \) and coincides with \( 3_q(k|d) \) when \( \Re k > 0 \):

\[
(3.7) \quad 3^♭_q(k|d) \overset{def}{=} \frac{1}{2i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} (\exp(-dx^2/a) + 1)^{-1} \delta_k(x;q)dx + 3^♭_0(k),
\]

\[
3^♭_n(k|d) \overset{def}{=} \frac{a\pi}{2} \prod_{j=0}^{\infty} \frac{(1 - e^{-(j+k)/a})(1 - e^{-(j-k)/a})}{(1 - e^{-(j+2k)/a})(1 - e^{-(j+1)/a})} \times 
\]

\[
\prod_{l=1}^{n} \frac{(1 - e^{-(l+2k-1)/a})(1 - e^{(l+k)/a})}{(1 - e^{-(l+k-1)/a})(1 - e^{l/a})} \sum_{m=-\infty}^{+\infty} \frac{1}{\exp\left(-d(k+n+2\pi aim)^2\right) + 1}.
\]

To see this, we replace \( (-i) \int_0^{i\infty} \) in (3.4) by \( (-i/2) \int_{-\infty}^{i\infty} \) and then by \( (-i/2) \int_{-\infty}^{\epsilon+i\infty} + 3^♭_0 \) provided that \( 0 < \Re k < 2\epsilon \). Here we applied Cauchy’s theorem.
THEOREM 3.2. Let $\xi_l = ((2l + 1)\pi ai/d)^{1/2}$, where $l = 0, 1, \ldots$,
\begin{align*}
\mathcal{S} &\stackrel{\text{def}}{=} \Pi \cup \Lambda \cup \Lambda_+, \quad \Pi = \{-Z_+/2 + \pi aiZ\} \setminus \{-Z_+ + 2\pi aiZ\}, \\
\Lambda &\stackrel{\text{def}}{=} \{-2\xi_l - Z_+ + 2\pi aiZ\} \cup \{-2\xi_l^* - Z_+ + 2\pi aiZ\}, \\
(3.8) \quad \Lambda_+ &\stackrel{\text{def}}{=} \{2\xi_l - Z_+ + 2\pi aiZ\} \cup \{2\xi_l^* - Z_+ + 2\pi aiZ\}.
\end{align*}
By * we mean the complex conjugation. The series
\begin{align*}
3_{-q}^\pi(k|d) &= \sum_{n=0}^\infty 3_n^\pi(k|d), \quad 3_{+q}^\lambda(k|d) = \sum_{l=0}^\infty 3_l^\lambda(k|d), \\
3_l^\lambda(k|d) &= \frac{a\pi}{2d} \left( \xi_l^{-1} \prod_{j=0}^\infty \left( 1 - e^{-(j+2\xi_l)/a} \right) \left( 1 - e^{-(j+2\xi_l^*)/a} \right) \right)^{1/2} \\
(3.9) \quad \left( \xi_l^{-1} \right)^{1/2} \prod_{j=0}^\infty \left( 1 - e^{-(j+2\xi_l)/a} \right) \left( 1 - e^{-(j+2\xi_l^*)/a} \right)
\end{align*}
are absolutely convergent for any fixed $k \notin \mathcal{S}$ with $Rek < 0$. The difference
\begin{align*}
(3.10) \quad 3_{+q}^{neq}(k|d) &\stackrel{\text{def}}{=} 3_{+q}^\pi(k|d) - 3_{+q}^\lambda(k|d)
\end{align*}
represents the meromorphic continuation of $3_{+q}(k|d)$ to all $Rek < 0$ except for the set of the poles $\mathcal{P} = \Pi \cup \Lambda$.

It is not difficult to check that the poles of $3_{+q}^\pi$ and $3_{+q}^\lambda$ apart from $\mathcal{P}$ have the same residues and cancel each other in (3.10). The latter set is discrete in contrast to $\mathcal{S}$ which is dense everywhere (for generic $a$). We will skip the convergence estimates because the theorem can be reproved using the shift operators without termwise consideration.

Actually the self-termination of the poles readily follows from the integral representation (3.7). The latter coincides with $3_{+q}^{neq}$ and has no poles for $0 > Rek > -2\epsilon$. If the differences of residues of the poles from $\Lambda_+$ are nonzero they can be seen in this strip (for generic $a$). This argument and the theorem can be somewhat generalized as follows.

PROPOSITION 3.3. Given $r \in \mathbb{Z}_+$, the meromorphic continuation of the integral $3_{-q}^{im}$ (see (3.4)) from the strip $K_r = \{k, \ -r < Rek < -r + 1\}$ to all negative $Rek$ is given by the series
\begin{align*}
(3.11) \quad 3_{+q}^{-r}(k|d) &\stackrel{\text{def}}{=} \sum_{j=0}^\infty \text{sign}(j - r + 0.1) 3_j^\pi(k|d) - 3_j^\lambda(k|d).
\end{align*}
Its poles ($Rek < 0, r > 0$) belong to $\mathcal{P}_r = \{-r + \Lambda\} \cup \Pi \setminus \{1/2 - r + \pi ai\mathbb{Z}\}$. 
This statement will be involved to analyze $3_{+q}^{neg}$ as $\Re k \to -\infty$ upon the following renormalization:

\begin{equation}
(3.12) \hspace{1cm} \tilde{3}_{+q}(k|d) = \beta(k)\tilde{3}_{+q}(k|d), \hspace{0.5cm} \bar{3}_{+q}^{-r}(k|d) = \beta(k)\bar{3}_{+q}^{-r}(k|d) \quad \text{for}
\end{equation}

\begin{equation}
(3.13) \hspace{1cm} \beta(k) \defeq \prod_{j=0}^{\infty} \frac{(1 - q^{j+2k})(1 - q^{j+1})}{(1 - q^{j+k})(1 - q^{j+k+1})}.
\end{equation}

These functions also have meromorphic continuations to $k \in \mathbb{C}$. The additional factor simplifies the singularities. The poles of the continuation of $\tilde{3}_{+q}$ (all $k$) form the set $\tilde{\mathcal{P}} = \Lambda \cup \{-1 - \mathbb{Z}_+ + 2\pi ai\mathbb{Z}\}$. The poles of the second function (negative $\Re k$, $r > 0$) constitute $\tilde{\mathcal{P}}_r = \{-r + \Lambda\} \cup \{-1 - \mathbb{Z}_+ + 2\pi ai\mathbb{Z}\}$. It also has “two” zeros $\{1/2 - r + \pi ai\mathbb{Z}\}$ modulo $2\pi ai\mathbb{Z}$.

Similar methods can be applied to the $3_{+q}^{-r}$ from Section 1 corresponding to the standard kernel $(\exp(-x^2/a) - 1)^{-1}$. One has to replace $3_{+q}^{-r}$ switching to the factors $(\exp(-(k + n + 2\pi aim)^2/(4a)) - 1)$ and use $\xi_l = (l\pi ai/d)^{1/2}$ starting the summation from $l = 1$. Other ingredients remain the same.

**k-limits.** The behavior of $q$-zeta functions as $k \to \infty$ is very non-classical. Let us first tend $\Re k$ to $+\infty$. We may use the integral representation:

\begin{equation}
(3.14) \hspace{1cm} 3_{+q}(k|d) \to \psi_+ = (-i) \int_0^\infty \frac{\prod_{k=0}^{\infty} (1 - q^{j+2x})(1 - q^{j-2x}) \, dx}{\exp(-dx^2/a) + 1}
\end{equation}

as $\Re k \to +\infty$. So the limit is a positive constant.

The calculation is more involved when $\Re k$ goes to $-\infty$. Namely, the integral representation cannot be used anymore and the renormalization (3.12) is necessary. We will apply Proposition 3.3 as $r \to \infty$ and $k = \kappa - r + \nu i$ for $0 < \kappa < 1$. First of all,

\begin{equation}
(3.15) \hspace{1cm} \phi(\nu; \kappa) \leftarrow \tilde{3}_{+q}^{-r}(k|d) \to -\tilde{3}_{+q}^{-r}(k|d) - \tilde{3}_{+q}^\lambda(k|d)
\end{equation}

for a continuous function $\phi(\nu; \kappa)$ which is $2\pi a$-periodic with respect to $\nu \in \mathbb{R}$.

Here we use the integral representation for $\tilde{3}_{+q}^{-r}(k|d)$ and evaluate the change of the integrand as $k \mapsto k - 1$ (the multiplicator) for big $-\Re k$. Thanks to the renormalization, the integral tends to a $\mathbb{Z}$-periodic function of $k$. Of course it is necessary to avoid the singularities $k \in \mathcal{S}$.

Thus we established a relation between $3_{+q}^\pi$ and $3_{+q}^\lambda$ in the considered limit. Let us use it:

\begin{equation}
(3.16) \hspace{1cm} \lim 3_{+q}^{neg} = \lim (\tilde{3}_{+q}^\pi - \tilde{3}_{+q}^\lambda) = -\phi(\nu; \kappa) - 2\lim \tilde{3}_{+q}^\lambda(k|d).
\end{equation}

If $k$ is fixed (and is in a general position),

\begin{equation}
(3.17) \hspace{1cm} 3_{+q}^\lambda(k|d) = \frac{1}{2i} \int \frac{\beta(k)\delta_k(x; q) \, dx}{\exp(-dx^2/a) + 1},
\end{equation}
where the integration contour \( \ll \) is the boundary of a neighborhood of the union
\[
\{(\epsilon + \mathbf{R}_+)(1 + i) \cup (\epsilon + \mathbf{R}_+)(1 - i)\}.
\]
The orientation is standard. The neighborhood must contain no points from 
\( \pm \{(k/2 + \mathbf{Z}_+/2) + \pi ai\mathbf{Z}\} \). So it depends on \( \kappa, \nu \) and approaches the diagonals for big \( \Re x \). Now we can go from \( k \) to \( k - 1 \) evaluating
\[
\beta(k - 1)\delta_{k-1} - \beta(k)\delta_k
\]
as \( r \to \infty \).

**Theorem 3.4.** Given \( 0 < \kappa < 1 \) such that \( \{\kappa + i\mathbf{R} + \mathbf{Z}\} \cap \bar{P} = \emptyset \), the limit
\[
\psi(\nu; \kappa) = \lim_{\mathbf{Z} \ni r \to \infty} \bar{3}_{+q}^{neg}(\kappa - r + i\nu)
\]
exists and is a continuous \( 2\pi a \)-periodic function of \( \nu \in \mathbf{R} \).

The theorem is the “plus-variant” of Theorem 1.4, where \( \bar{P} \) was somewhat different. The set of admissible \( \kappa \) is dense everywhere in \([0, 1]\) because so is \( \bar{P} + i\mathbf{R} + \mathbf{Z} \subset \mathbb{C} \) (for generic \( a \)). However the function \( \psi(\nu; \kappa) \) is well-defined. It is directly related to the existence of the totally singular zeta \( \bar{\zeta}_q \) from Section 1, because \( \bar{3}_{+q}^{1} \) is nothing else but a variant of the function from (1.7).

Thanks to the theorem, the asymptotic number of zeros \( N_r \) of \( \bar{3}_{+q}^{neg}(k|d) \) for big (generic) \( r \) in the box
\[
B_r \overset{\text{def}}{=} \{\kappa - r - 1 < \Re k < \kappa - r\} \text{ modulo } 2\pi ai\mathbf{Z}
\]
coincides with that for \( \bar{3}_{+q}^{neg}(k|d) \) and equals the number of the poles, i.e. the points from \( \bar{P} \cap B_r \). Indeed, it suffices to calculate the difference of the changes of arg(\( \psi(\nu) \)) over the “periods” at \( \kappa = 0 \) and \( \kappa = 1 \). Note that \( N_r \) is quadratic in \( r \):
\[
N_r = 2\left[\frac{1 + d(r + 1 - \kappa)^2}{4a\pi}\right] + 1, \ 0 < \kappa < 1, \ r \gg 1.
\]

It is likely that the positions of individual zeros (modulo \( \mathbf{Z} + 2\pi ai\mathbf{Z} \)) approach certain limits when \( r \to \infty \). To ensure this, the convergence to \( \psi(\nu; \kappa) \) must be fast enough. This can be seen numerically. For instance, if \( a = 1 = d \) the function \( \bar{3}_{+q}(k|d) \) has the following (unique?) stable real zero:
\[
z_6 = -6.668819069126, \ z_7 = -7.669834033011, 
\]
\[
\text{stable zero } z_r = -r - 0.67042591089415.
\]
The stabilization is rapid (for small \( a \)). At least one real zero must exist in \( B_r \) for sufficiently big \( r \) because \( N_r \) is odd. Recall that \( \bar{3}_{+q}(k|d) \) is real on \( \mathbf{R} \), so the complex zeros appear in pairs.
**a-limits.** Let us establish the relations to the classical counterparts of the considered zeta functions:

\[
3_+^\lambda(k|d) = d^{-1/2-k}(1 - 2^{1/2-k}) \zeta(k),
\]

(3.21) \[3_+^\lambda(k|d) = d^{-1/2-k}(1 - 2^{1/2-k})\sqrt{\pi} \Gamma(k + 1) \zeta(k + 1/2).
\]

See (3.3) and (1.26).

**Theorem 3.5.** Let \(3_{\pm}^{an}, \tilde{3}_{\pm}^{an}\) be the meromorphic continuations of \(3_{\pm}^{an}\) and \(\tilde{3}_{\pm}^{an}\) to all \(k\) constructed above. Then

\[
\lim_{a \to \infty} (a/4)^{k-1/2} 3_{\pm}^{an}(k|d) = 3_{\pm}(k|d) \quad \text{for} \quad k \neq -1/2, -3/2, \ldots ,
\]

(3.22) \[\lim_{a \to \infty} a^{k-1/2} \tilde{3}_{\pm}^{an}(k|d) = \tilde{3}_{\pm}(k|d) \quad \text{for} \quad k \neq -1, -2, \ldots .
\]

The proof is based on the integral representation (3.4) for \(\Re k > 0\). As to \(\tilde{3}_{\pm}^{an}\), we also use (1.5) from Lemma 1.1. When \(\Re k > -1/2\), we can go from \(\delta_k\) to \(\mu_k\) (see Section 2 and below) or involve \(3_{\pm}^\lambda\) from (3.7). Let us deduce the formula for the limit of \(3_{\pm}^{an}\) for \(\Re k < -1/2\) from (3.3), Theorem 3.1. Generally speaking, the real and imaginary \(q\)-zeta functions are not connected, because the contours of integration are different and the signs of the Gaussians are opposite. However, there is a relation in the limit.

First, let us consider \(3_{\pm}^{an}(k|d)\) (see (3.10) and (3.25)). Since \(a \to \infty\), the terms in the sums (3.7) for \(3_\pm^\sigma(k|d)\) with \(m > 0\) are very small and can be ignored. Therefore

\[
3_{\pm}^{\sigma} + 3_{\pm}^\lambda \approx \sum_{j=0}^{\infty} \frac{2\pi}{\sigma} \prod_{j=0}^{\infty} \frac{1 - e^{-(j+k)/\sigma})(1 - e^{-(j-k)/\sigma})}{(1 - e^{-(j+2k)/\sigma})(1 - e^{-(j+1)/\sigma})}
\]

\[
\prod_{l=1}^{j} \frac{1 - e^{-(l+2k-1)/\sigma})(1 - e^{-(l+k)/\sigma})}{(1 - e^{-(l+k-1)/\sigma})(1 - e^{l/\sigma})} \frac{1}{\exp\left(-\frac{d(k+j)^2}{4a}\right) + 1} = \]

(3.24) \[-3_\pm(k) + 3_{\pm}^\lambda.
\]

Here we used the formula for the “sharp constant term” from (2.2) and of course the identity \((\exp(x) + 1)^{-1} + (\exp(-x) + 1)^{-1} = 1\).

Since \(3_\pm^\sigma(k) = O(a)\) (Theorem 2.1) and \(3_{\pm}^\lambda = O(a^{1/2-k})\), the latter will dominate for big \(a\) because \(\Re k < -1/2\). Thus \(3_{\pm}^{\sigma neg} \approx 3_{\pm}^{\sigma neg} - 3_{\pm}^\lambda\), and thanks to (3.3) it suffices to calculate the limit of \(3_{\pm}^{\lambda neg}\). This can be done via the Moak theorem (Lemma 1.1) exactly as we did in (1.6) with \(\tilde{\zeta}\):

\[
3_{\pm}^\lambda(k|d) \approx \sum_{l=0}^{\infty} \frac{2\pi}{\sigma} (a/4)^{-k} ((ax)^{-1/2}(-x)^k + (a_x)^{-1/2}(x)^k)
\]

(3.25) \[3_{\pm}^\lambda(k|d) \approx \sum_{l=0}^{\infty} \frac{2\pi}{\sigma} (a/4)^{-k} ((ax)^{-1/2}(-x)^k + (a_x)^{-1/2}(x)^k)
\]
for $x_l = ((2l + 1)\pi i/d)$. Then we use the classical functional equation.

Summarizing, our technique of analytic continuation of the imaginary $q$-zeta is a $q$-variant of the so-called Riemann’s first proof of the functional equation (see [E], 1.6 and the appendix, the translation of Riemann’s “On the number of primes less than a given magnitude”). In contrast to the classical theory, the resulting functions are periodic in the imaginary direction and have limits as $\Re k \to \pm \infty$. The “sharp” $q$-zeta functions $\zeta_q^\sharp$ and $\zeta_{q+q}^\sharp$ can be represented as convergent series for all $k$, which has no counterpart for Riemann’s zeta.

## 4. Shift operator

The shift operator in the setup of this paper is the so-called Askey-Wilson operator, the following $q$-deformation of the differentiation:

$$Sf(x) \overset{def}{=} (q^x - q^{-x})^{-1} (\tau - \tau^{-1}) f(x),$$

$$\tau f(x) = f(x - \frac{1}{2}), \quad q = \exp(-1/a).$$

(4.1)

For instance,

$$S\gamma = -q^{1/4}\gamma, \quad S\gamma^{-1} = q^{-1/4}\gamma^{-1} \quad \text{for} \quad \gamma = q^{x^2}. $$

(4.2)

The name is “shift operator” because its action on the hypergeometric functions, both classical and basic (difference), results in the shift of the parameters [AW]. For example,

$$Sp_m^{(k)}(x; k) = (q^{-m/2} - q^{m/2})p_{m-1}^{(k)}(x; k + 1), \quad m = 1, 2, \cdots $$

(4.3)

for the Rogers polynomials from (2.17).

The differential shift operators for arbitrary root systems are due to Opdam (see [O] and also [H] for the interpretation via the Dunkl operators). The general difference ones are considered in [C3]. They depend on $k$ in contrast to the simplest case considered here ($A_1$). The celebrated constant term conjecture and the Macdonald-Mehta conjecture were verified using these operators (see [M1, O, C3, C1]). They were also used in [O] for analytic continuations.

In this section, the shift operators will be applied to go from positive $\Re k$ to the left without involving Cauchy’s theorem and convergence estimates of the pole decomposition. Some properties of the difference shift operators have no counterparts in the classical theory. For instance, they give a connection between the values of $q$-zeta functions at $k$ and $k + 1$ which collapses in the limit. We do not discuss the technique of the shift operators in detail. It is similar to the algebraic theory [C1,C3]. See also [C2].
Shift-formula. Let \( \pi a > \epsilon > 0 \) and \( \int \) be one of the following integrations:

\[
\int_{ct} = \frac{1}{\pi ai} \int_{-\pi ai/2}^{\pi ai/2} dx, \quad \int_{im} = \frac{1}{2i} \int_{c-\epsilon i}^{c+\epsilon i} dx, \\
\int_{re} = \frac{1}{2} \int_{-\infty - \epsilon i}^{\infty + \epsilon i} dx, \quad \int_{z} = \frac{1}{2i} \int_{\infty - \epsilon i}^{\infty + \epsilon i} dx.
\]

(4.4)

Given \( u > 0 \), a function \( E(x) \) is called \( u \)-regular, if it is even, analytic in \( D_{ct} = \{ x | -u \leq \Re x \leq u \} \) for \( CT \), \( D_{im} = \{ x | -u - \epsilon \leq \Re x \leq u + \epsilon \} \) for \( \int_{im} \), \( D_{re} = \{ x \in \pm \epsilon i + R \} \) for \( \int_{re} \) (no \( u \)), \( D_{z} = D_{re} \cup \{ x | -\epsilon \leq \Im x \leq \epsilon, -u \leq \Re x \leq u \} \) for \( \int_{z} \), and continuous on the closure of \( D \). It is also assumed to be \( \pi ai \)-periodic for \( CT \) and have coinciding continuous limits

\[
e^{\pm}(\kappa + \nu i) = \lim_{\nu \to \pm \infty} E(x)\delta_{\kappa + \nu i}(x)
\]

in the case of the imaginary integration \( \int_{im} \).

The following proposition results from (4.3). The proof is similar to [C2] (in the one-dimensional setup).

**Proposition 4.1.** Let \( K_{\epsilon} \) be the set of \( k \) such that \( \Re k > 2 \epsilon \) for the imaginary integration, \( \Re k > 0 \) for \( CT \), \( \{ \Re k > 0, |\Im k| < 2 \epsilon \} \) for the sharp one \( \int_{z} \), and \( K_{\epsilon} = \{ 3k + 2 \pi aZ \neq \pm 2 \epsilon \} \) for the real integration. Given a \( 1/2 \)-regular \( E \) and a Rogers polynomial \( p^{(k)}_{m}(m > 1) \),

\[
\int S(E)p^{(k+1)}_{m-1}\delta_{k+1} = q^{k}(q^{-k-m/2} - q^{k+m/2})\int E p^{(k)}_{m}\delta_{k}
\]

(4.6)

for \( k \in K_{\epsilon} \), provided the existence of the integrals.

Note that we do not take \( iR \) as the path for the imaginary integration because \( S(E) \) may have singularities at \( 2\pi aZ \). Concerning the convergence of the integrals from (4.6), the function \( E \) can be arbitrary (say, continuous) for \( CT \). For the imaginary integration, we can use the Stirling formula from Lemma 1.1. For instance, the integrability of \( |E(x)||x|^{-2\Re k} \) in the imaginary directions inside \( D_{im} \) is sufficient. The estimates in the real direction are more involved but not difficult either.

Next, we put

\[
\delta_{k+1} = (1 - q^{2x+k})(1 - q^{-2x+k})\delta_{k},
\]

(4.7)
express \((1 - q^{2r+k})(1 - q^{-2r+k})\) in terms of \(p_2^{(k)}\) (see (2.18)), and then apply the proposition twice to turn \(p_2^{(k)}\) into a constant.

**Theorem 4.2.** Given \(\epsilon > 0\), let \(E(x)\) be 1-regular. Assuming (again) that \(k \in K_\epsilon\) and the integrals are well-defined,

\[
\frac{(1 - q^{2k+1})(1 - q^{2k})}{(1 - q^{k+1})(1 - q^k)} \int E\delta_k = \int E\delta_{k+1} + \frac{q^{3/2+k}}{(1 - q^{2k+2})(1 - q^{2k+3})} \int S^2(E)\delta_{k+2}.
\]

(4.8)

The generalization of the theorem to arbitrary root systems is straightforward. However the explicit formulas for \(\delta_{k+1}/\delta_k\) in terms of “small” Macdonald’s polynomials are getting complicated.

An immediate application of the theorem is the constant term conjecture in the \(A_1\)-case (see [AI,M1] and the papers of Milne and Stanton from [A]). Let \(\int = \int_{ct}\), \(E = 1\). Then

\[
\int_{ct} \delta_k = \frac{(1 - q^k)(1 - q^{k+1})}{(1 - q^{2k})(1 - q^{2k+1})} \int_{ct} \delta_{k+1} = \ldots = 2 \prod_{j=0}^{\infty} \frac{(1 - q^{k+j})(1 - q^{k+j+1})}{(1 - q^{2k+j})(1 - q^{j+1})}.
\]

(4.9)

The resulting product formula holds for \(\Re k > 0\). The reasoning is the same for other integrations with a reservation about the convergence conditions. For instance, we get (2.3) for the sharp integration. However now the convergence is for \(\Re k < 0\), so we need to employ a variant of the theorem for the negative contours of integration and use the chain

\[
\int \delta_k = \frac{(1 - q^{2k-1})(1 - q^{2k-2})}{(1 - q^k)(1 - q^{k-1})} \int \delta_{k-1} = \ldots .
\]

(4.10)

Actually this chain can be applied to establish (4.9) as well. First, we assume that \(k \in \mathbb{Z}_+\), second, get the exact formula and, third, use the analytic continuation.

It is not surprising that the products in (4.9) and (2.3) coincide. Formally,

\[-\pi ai \int_{ct} E\delta_k = 2i \int E\delta_k\]

if \(E\) is regular everywhere.

Another application is to the Gaussians. Using (4.2) we come to (2.4) for \(E = \gamma^{-1}\). When the integration is sharp and \(E = \gamma\), we establish the symmetry (2.16), which played the key role in Theorem 2.3.
Let us apply $S^2$ to our real and imaginary plus-kernels $E^{(\pm)} = (1 + q^{\pm x^2})^{-1}$ when $d = 1$:

\begin{equation}
S^2(E^{(\pm)}) = \pm \frac{q^{x^2+1/2}(q^{x^2+1} - 1)}{(1 + q^{x^2})(1 + q^{(x+1)^2})(1 + q^{(x-1)^2})}. \tag{4.11}
\end{equation}

So it is the same for the imaginary and real integration up to a sign. The complete factorization is a special feature of $S(E^{(\pm)})$ and $S^2(E^{(\pm)})$. The formulas are not that nice for $d \neq 1$ and for higher $S$-powers of $E^{(\pm)}$. We mention that big $S$-powers have remarkable stabilization properties which we hope to discuss somewhere else.

**Analytic continuation.** The theorem can be applied infinitely many times if the function $E$ is $\infty$-regular, for instance, in the case of the real integration $\int_{re}$. Of course it is necessary to ensure the convergence of the involved integrals and the resulting series. It may lead to interesting transformations and identities. However it seems that the most promising applications are analytic. Namely, using (4.8), we can extend the function $\int E \delta_k$ to negative $\Re k$ when $E$ is analytic in the entire complex plane or proper vertical strips. If $E$ is meromorphic the method can still be used. Of course the poles will contribute to (4.8), but the extra terms are meromorphic for reasonable $E$.

**Corollary 4.3.** In the setup of Theorem 4.2, let $\mathfrak{E}(k) \overset{\text{def}}{=} \int E(x) \delta_k$ for $k \in K_\epsilon$. The function

\begin{equation}
\mathfrak{E}^{(1)}(k) = \frac{(1 - q^{k+1})(1 - q^k)}{(1 - q^{2k+1})(1 - q^{2k})} \mathfrak{E}(k + 1) + \frac{q^{3/2 + k}(1 - q^{k+2})(1 - q^{k+1})}{(1 - q^{2k+1})(1 - q^{2k+3})(1 - q^{2k+2})(1 - q^{2k+1})} \int S^2(E) \delta_{k+2} \tag{4.12}
\end{equation}

is a meromorphic continuation of $\mathfrak{E}(k)$ to $K_\epsilon - 1$, with $\int S^2(E) \delta_{k+2}$ being analytic.

Assuming that $E$ is $r$-regular for $r \in 1 + \mathbb{Z}_+$ and all the integrals are convergent, the iterations of (4.12) provide a meromorphic continuation of $\mathfrak{E}(k)$ to $K_\epsilon - r$ with simple poles belonging to the set

\[ \{-\mathbb{Z}_+/2 + \pi ai \mathbb{Z}\} \setminus \{-\mathbb{Z}_+ + 2\pi ai \mathbb{Z}\}. \]

The described procedure becomes somewhat sharper for the imaginary integration when $\delta_k$ is replaced by $\mu_k$ from (2.6). Namely,

\begin{equation}
\mathfrak{E}(k) = 2(1 + q^k)^{-1} \int_{\text{im}} E(x) \mu_k \tag{4.13}
\end{equation}
for \( k \in K_\varepsilon \), where the latter integral gives a meromorphic continuation of \( \mathcal{E}(k) \) to \( K_\varepsilon - 1/2 \) with the poles in \( \pi ai + 2\pi ai\mathbb{Z} \). Respectively, substituting \( \mu \) for \( \delta \), the corollary provides the continuation to \( K_\varepsilon - r - 1/2 \) for \( r \)-regular \( E \).

Note that the above set of poles coincides with \( \Pi \) from (3.8). Thus the results of the previous section can be reproved by means of the shift operator technique for \( E(x) = (\exp(-dx^2/a) + 1)^{-1} \) with a reservation about meromorphic \( E \).

Using (4.12) only, we have to stop once the first pole of \( E \) appears, i.e. when \( \Re k = -2\Re(\xi_0) \) (Theorem 3.2).

**Limits.** An important application of the shift operator is to the \( k \)-limits and \( a \)-limits. We will discuss here only the latter. Formula (3.3) is still not justified for \( k \leq -1/2 \). Now we can proceed as follows. Let us take \( E = (\exp(dx^2/a) + 1)^{-1} \).

The diagonal poles are departing from the origin as \( a \to \infty \), so we can define the analytic continuation using \( \mathcal{E}, \mathcal{E}^{(1)}, \mathcal{E}^{(2)} \) etc. (any number of them). It suffices to consider the first step, the passage from \( \Re k > -1/2 \) to \( \Re k > -3/2 \). Then we can repeat the procedure. For big \( a \),

\[
\left( \frac{a}{4} \right)^{k-1/2} \mathcal{E}^{(1)}(k) \approx \frac{(k + 1)}{a(k + 1/2)} \left( \frac{a}{4} \right)^{k+1/2} \mathcal{E}(k + 1) +
\]

\[
\frac{1}{(k + 3/2)(k + 1/2)} \left( \frac{a}{4} \right)^{k+3/2} \int S^2(E) \delta_{k+2}.
\]

We already know (thanks to the initial integral representation) that

\[
\left( \frac{a}{4} \right)^{k+1/2} \mathcal{E}(k + 1) \to \sin(\pi k) \mathfrak{3}_+(k + 1|d) \text{ for } k > -3/2.
\]

Actually the exact formula does not matter since this term comes with the factor \( (1/a) \) and will vanish. So we need to evaluate the second term only. For the sake of simplicity, let \( d = 1 \). Using (4.11),

\[
\lim_{a \to \infty} \left( \frac{a}{4} \right)^{k-1/2} \mathcal{E}^{(1)}(k) =
\]

\[
\frac{1}{2i(k + 3/2)(k + 1/2)} \int_{\infty+\varepsilon i}^{\infty-\varepsilon i} z^{-1/2}(-z)^{k+2} \mathcal{E}^{(2)}dz,
\]

(4.15)

\[
\mathcal{E}^{(2)} = -\frac{z(1 - e^z)}{(1 + e^z)^3}.
\]

Here we substituted \( z = x^2/a \). The integration path must be of course recalculated. However it can be deformed to the same “sharp” shape for a small \( \varepsilon > 0 \) because \( k + 2 > 1/2 \). In fact, \( k + 2 > -1/2 \) is sufficient. Then we may integrate by parts: \( \mathcal{E}^{(2)} \) is the second derivative of \( \mathcal{E} = (1 + e^z)^{-1} \). The latter is not surprising because \( S^2 \) is a deformation of the second derivative.
Finally,
\[
\lim_{a \to \infty} \left( \frac{a}{4} \right)^{k-1/2} e^{(1)}(k) = \frac{1}{2i} \int_{\infty+\epsilon i}^{\infty-\epsilon i} z^{-1/2} (-z)^k e^{(1)} d\zeta = (1 - 2^{1/2-k}) \sin(\pi k) \zeta(k + 1/2) \Gamma(k + 1/2) = \sin(\pi k) \zeta(k + 1/2) \Gamma(k + 1/2).
\]  

The last equality holds for \( \Re k > -1/2 \). The integral from (4.15) represents \( \mathfrak{Z}_+^{(k)}(1) \) for \( \Re k > -5/2 \) (we need it only when \( \Re k > -3/2 \)). So, indeed, we moved by \(-1\) to the left. Similarly, we can use \( e^{(2)} \) and so on. This proves (3.3). One can (try to) apply this method for finding other terms of the \((1/a)\)-expansion of \( \mathfrak{Z}_+^{(k)}(1) \) and other \( q \)-zeta functions. Moreover, it may work as \( a \to 0 \) and \( q \) tends to roots of unity, however this is beyond the framework of the paper.

Summarizing, the technique of analytic continuation from this section has the following classical origin. Each integration by parts shifts by \(-1\) the \( k \)-domain of the integral representation (4.16) for the “plus-zeta”. There are of course other similar examples in the classical theory of zeta. However the difference integrating by parts, namely (4.12), seems more promising. It has better analytic properties and, what is important, establishes a certain connection between the values of \( q \)-zeta functions (or more general integrals) at \( k \) and \( k + 1 \), collapsing in the limit. The stability of our zeta functions modulo \( \mathbb{Z} \), which has no classical counterpart, is also directly related to (4.12).

5. Locating the zeros

In this section we discuss (mainly numerically) the zeros of \( q \)-zeta functions and demonstrate that their behavior is quite regular at least for \( \Re k > -1/2 \). Almost certainly the evolution of the \( q \)-zeros is related to the distribution of the classical ones. We will give some evidence.

Imaginary zeta. Let us begin with the “plus-imaginary” \( \mathfrak{Z}_+^{im}(k|d) \) from (3.4). Recall that
\[
\mathfrak{Z}_+^{im}(k|d) = (-i) \int_0^{\infty} \frac{1}{e^{-dx^2/a} + 1} \delta_k(x; q) dx \quad \text{for} \quad \Re k > 0,
\]
\[
\lim_{a \to \infty} (a/4)^{k-1/2} \mathfrak{Z}_+^{im}(k|d) = \mathfrak{Z}_+^{im}(k|d) = (1 - 2^{1/2-k}) d^{-1/2-k} \mathfrak{Z}_+(k),
\]
where \( \mathfrak{Z}_+(k) = \zeta(k + 1/2) \Gamma(k + 1/2) \).

Its analytic continuation has the same limit for all \( k \) except for the poles. Since \( \mathfrak{Z}_+^{im} \) is \( 2\pi a i \)-periodic and has the limit (a positive constant) as \( \Re k \to \infty \) the
number of its zeros $N^\epsilon_+(a; d)$ in the strip $K^\epsilon_+ = \{ k \mid \Re k > \epsilon > 0 \}$ modulo $2\pi ai$ is as follows:

\begin{equation}
N^\epsilon_+(a; d) = \frac{1}{\pi} \arg(3_{i-q}^{im})_{k=\epsilon} \approx \pi a + \pi i.
\end{equation}

Here we used that $3_{i-q}^{im}$ is real on $\mathbb{R}_+$ and $\pi ai + \mathbb{R}_+$. Numerically, it has no zeros in the right half-plane at least for $a = 25$, $\epsilon = 0.05$, $d = 2$. Here is the table of the arguments $\alpha(n) = \arg(3_{i-q}^{im})(\pi ai/21 + \epsilon)$ for $n = 1, \ldots , 20$ ($\alpha(0) = 0 = \alpha(21)$):

\begin{align*}
\alpha(1 - 5) &= -7.45845 - 11.6586 - 12.1104 - 12.4365 - 11.5873 \\
\alpha(6 - 10) &= -10.1478 - 8.26116 - 6.03690 - 3.56672 - 1.01060 \\
\alpha(11 - 15) &= 1.61364 + 4.19644 + 6.66558 + 8.91015 + 10.8286 \\
\alpha(16 - 20) &= 12.2974 + 13.1573 + 13.1854 + 12.0256 + 8.96469.
\end{align*}

The total number of points on the half-period is $\sim 320$ in this calculation. The behavior of the argument becomes more complicated approximately after $a = 30$. Our computer program (it adds points automatically to ensure the required accuracy) was not able to reach a reliable answer for such $a$. Eventually the zeros will appear in the right half-plane. Indeed, the $q$-deformations of the classical zeros with $\Re k = 1/2$ coming from the factor $(1 - 2^{1/2 - k})$ must arrive.

Without going into detail, let us mention that similar results hold when $(\exp +1)^{-1}$ is replaced by regular modular functions (cf. (1.31)). The “imaginary” $q$-deformations of the corresponding $L$-functions (with proper gamma-factors) are well defined for $\Re k > 0$. Sometimes elementary functions can be used instead of modular ones in the integral representations. Say, for the $L$-function associated with the “standard” Dirichlet character modulo 3, we can proceed as follows:

\begin{equation}
3_{\Delta q}^{im}(k|d) = (-i) \int_0^{\infty i} \frac{1}{e^{-dx^2/a} + 1 + e^{dx^2/a}} \delta_k(x; q) dx \quad \text{for } \Re k > 0.
\end{equation}

Its analytic continuation approaches the classical $L$-function up to a gamma-factor for all $k \in \mathbb{C}$ as $a \to \infty$. It has no $q$-zeros in the right half-plane at least when $a \sim 25, d = 2$.

Recall that in the critical strip $1/2 > \Re k > -1/2$, we use the analytic continuation, namely, either

\begin{equation}
3_{i-q}^{b} = 3_{i-q}^{neg} \text{ from (3.7), or}
\end{equation}

\begin{equation}
3_{i-q}^{\mu} = -i(1 + q^k)^{-1} \int_{1/4 - \infty i}^{1/4 + \infty i} (\exp(-dx^2/a) + 1)^{-1} \mu_k(x; q) dx.
\end{equation}
They coincide (see (3.10) and (4.13)). The passage to negative $\Re k$ is necessary because the most interesting zeros, the $q$-deformations of the classical ones, prefer the left half-plane. The theoretical explanation is as follows.

Given a zero $k = z$ of the classical $\zeta(k + 1/2)$ (only $z \in i\mathbb{R}_+$ will be considered) we can calculate the linear approximation $\tilde{z}_+(a)$ for the corresponding exact zero $z_+(a)$ of $\mathcal{Z}^\sharp_q$:

$$
\tilde{z}_+(a) = z(1 + \frac{4d^{-1}(z + 1/2)\zeta_+(z + 3/2) - d(z - 1)\zeta_+(z - 1/2)}{12a\zeta_+(z + 1/2)})
$$

(5.6) for $\zeta_+'(s) = \partial\zeta_+(s)/\partial s$, $\zeta_+(s) = (1 - 2^{1-s})\zeta(s)$.

We expand the integral representation in terms of $(1/a)$ using a refined variant of Lemma 1.1. Another approach is based on the shift operator technique. However we did not check all the estimates in (5.6) so the exact range (both $a, k$) where it can be used is not determined at the moment.

The first $\tilde{z}_+(a)$ which we found in the right half-plane corresponds to $z = 1977.2714i$. The formula predicts that this $z$ could have the $q$-deformation $z_+(a)$ to the right, but we did not even try to find it numerically. It is well beyond the capacity of our computer program and the speed of existing (available) computers. The applicability of the linear formula for so big $z$ is also unclear. A more systematic analysis of the Moak-Stirling formula is necessary.

The negativity of $\Re(\tilde{z}_+(a))$ is a certain property of Riemann’s zeta and its derivative. As we will see, it is connected with known facts. However there is no explanation why the tendency is that strong.

Sharp zeta. Let us switch entirely to the main object of our computer simulation, the plus-sharp $q$-zeta function $\mathcal{Z}_+^\sharp_q(k|d)$ from (3.2). It tends to $(a/4)^{1/2-k}\sin(\pi k)\mathcal{Z}_+(k|d)$ for all $k$ except for the poles (Theorem 3.1). We will analyze it in the strip

$$
K_+^\sharp = \{-2\varepsilon < \Im k < +2\varepsilon\} \text{ for } \varepsilon = \sqrt{\frac{\pi a}{2d}},
$$

(5.7) where it has poles at $\{-1/2 - Z_+\}$ and vanishes at all integral $k$. We assume that $a > 2/(\pi d^2)$ which provides that the poles with $\Im k = \pi a$ will not appear in this strip.

The formula for the linear approximation $\tilde{z}_+^{\sharp}(a)$ for the exact zero $z_+^{\sharp}(a)$ of $\mathcal{Z}_+^\sharp_q$ corresponding to a given zero $z$ of $\zeta(k + 1/2)$ reads

$$
\tilde{z}_+^{\sharp}(a) = z(1 - \frac{4d^{-1}(z + 1/2)\zeta_+(z + 3/2) - d(z - 1)\zeta_+(z - 1/2)}{12a\zeta_+(z + 1/2)}).
$$

(5.8) The reservation about the applicability is the same as above.

We will give the numerical values for the zeros $z_+^{\sharp}$ deforming the classical ones $z$ in the strip $K_+^\sharp$ and the values of the corresponding linear approxima-
We did not try to reach high accuracy. However the existence of the zeros was carefully justified by means of sequences of integrations over diminishing rectangles around the consecutive approximations. The same integrations were used to calculate the \( q \)-zeros themselves.

Note that the standard Newton method and its modifications work for the first 5 zeros but diverge for the other 4 even when applied to the zeros we found via the contour integrating \((z^\#_+ + \varepsilon)\).

For \( a \) till 10000, there is no problem with finding the first zeros, but calculating the last ones is difficult.

The zeros exist for small \( a \) as well. If \( d = 2 \) the least \( a \) with the zero in the strip \( K^\#_+ \) is \( a = 61 \) (anyway \( a = 60 \) is smaller than necessary). The zero equals \( z^\#_+ = 1.1721 + 13.8088i \) whereas \( 2\varepsilon = 13.8433 \). The corresponding linear approximation \( \tilde{z}^\#_+ = 1.60159 + 14.2789i \) is beyond the \( K \)-strip.

This does not mean that we cannot consider the \( q \)-deformations of the classical zeros for small \( a \). For instance, tracing \( 1.1721 + 13.8088i \) back in \( a \):

\[
a = 30 : z^\#_+ = -3.6951 + 10.5620i, \quad \tilde{z}^\#_+ = 3.2566 + 14.4279i,
\]

\[(5.10) a = 1.98952 : z^\#_+ = -11.5000 + 2.5004i.\]

The first is very far from its linear approximation \( \tilde{z}^\#_+ \), the second seems to have nothing to do with the classical \( z = 14.1347i \) at all. However both are indeed the \( q \)-deformations of the latter.

We note that outside \( K^\#_+ \), the appearance of diagonal poles (from \( \Lambda \)) changes the picture of zeros (and their number) dramatically. For instance, \( 2\varepsilon = 9.70813 \) for \( a = 30, d = 2 \) and there is a zero \( 0.3862 + 9.9175i \). It is also necessary to take into consideration the classical zeros due to the factor \((1 - 2^{1/2-k})\). For instance, \( 0.8223 + 9.0518i, 1.9601 + 17.6542i \) are exact deformations of \( 0.5 + 9.0647i, 0.5 + 18.1294i \) for \( d = 2, a = 100 \).
According to our calculations, it is possible that all zeros $z^♯_+$ inside $K^♯_+$ (or a bit smaller strip) and with $\Re k > -1/2$ ($a > 2/(\pi d^2)$) are $q$-deformations of the classical ones ($\Re k = 0, 1/2$). However this must be checked more systematically. Vice versa, all classical zeros presumably have the deformations for all $q$, but maybe far from the critical strip (see (5.9), (5.10)).

**Why do the zeros move to the right?** The last topic we are going to discuss is a qualitative analysis of the positivity of $\Re(\tilde{z}^♯)$. A similar question is about ups and downs of the linear deformations. They look random in the table (5.9), but there are interesting tendencies, especially for the usual “minus-zeta”. We will not touch it upon here.

Since $z \in iR_+$, we need to examine the positivity of

\[(5.11) \quad \eta_+ (z|d) \overset{def}{=} \Im \left( \frac{4d^{-1}(z + \frac{1}{2})(1 - 2^{-1/2 - z})\zeta(z + \frac{3}{2}) - d(z - 1)(1 - 2^{3/2 - z})\zeta(z - \frac{1}{2})}{12a\zeta'(z + 1/2)(1 - 2^{1/2 - z})} \right).\]

If $d = 2$ the smallest zero $z$ when (5.11) becomes negative is $z = 1977.27i$. The next one is $2254.56i$. Actually this property has almost nothing to do with the zeros of $\zeta$. Indeed, the first interval where $\eta_+(z|2)$ takes negative values ($\Im z > 6$) is $[1977.24i, 1977.35i]$. Since there are many zeros of Riemann’s zeta, there is nothing unusual that one of them appeared in this interval. The tendency is practically the same from $d = 0.2$ till $d = \infty$. Only around $d = 0.1$, a (much) smaller “negative” zero appears: $163.03i$. The next one is $353.49i$.

Thus the positivity phenomenon is mainly because of the coefficient of $d$ in (5.11):

\[(5.12) \quad \Im \left(\frac{-(z - 1)(1 - 2^{3/2 - z})\zeta(z - 1/2)}{12a\zeta'(z + 1/2)(1 - 2^{1/2 - z})}\right).\]

Here the factor $(1 - 2^{3/2 - z})/(1 - 2^{1/2 - z})$ is not too significant. Numerically, it is because its real part, always positive, provides the major contribution. Let us drop it (together with $12a$). So we need to interpret the tendency

\[(5.13) \quad \eta(z) \overset{def}{=} \Im ((1 - z)\zeta(z - 1/2)/\zeta'(z + 1/2)) > 0\]

for the zeros $z \in iR_+$ of $\zeta$.

Note that $\eta_+(z|d)$ without the factors $(1 - 2^{-\cdot})$ describes the linear deformations in the case of the usual zeta (see (1.33)). Therefore a little influence of the skipped factors could be expected a priori. Actually we can simplify $\eta$ even more replacing $(z - 1)$ by $i$, since it almost belongs to $R_+i$. However this factor is necessary to fix the positivity for small zeros, so we will not touch it.

The phenomenon is again not about the zeros. The first interval where $\eta(z)$ takes negative values is $[1267.47i, 1267.70i]$ whereas the first zero with
negative $\eta$ is 1267.57$i$. However we will use that $z$ is a zero in the following calculation.

Applying the functional equation $\cos \left( \frac{s\pi}{2} \right) \Gamma(s) \zeta(s) = \zeta(1-s)\pi^{s-1}$ and the reality of $\zeta$ on $\mathbb{R}$,

$$
\frac{(1-z)\zeta(z-1/2)}{\zeta'(z+1/2)} = \frac{(1-z)(z-1/2)\Gamma(z-1/2)\zeta(z-1/2)}{\Gamma(z+1/2)\zeta'(z+1/2)} = \frac{(1-z)(1/2-z)}{2\pi} \frac{1-s}{\sin(\pi z)} \frac{1}{\zeta(1/2-z)} \\
\approx \frac{y^2 + 1/2}{2\pi} (-i) \left( \frac{\zeta(3/2 + z)}{-\zeta'(1/2 + z)} \right),
$$

(5.14)

where $z = iy$, $y \gg 0$. Thus we need to check that

$$
\rho(y) \overset{\text{def}}{=} \Re(\zeta(\frac{3}{2} + iy)(\zeta'(\frac{1}{2} + iy))^*) = \Re(i\zeta(\frac{3}{2} + iy)(\zeta_y(\frac{1}{2} + iy))^*) = \\
\Re(\zeta(\frac{3}{2} + iy)\Im(\zeta_y(\frac{1}{2} + iy)) - \Im(\zeta(\frac{3}{2} + iy)\Re(\zeta_y(\frac{1}{2} + iy)))
$$

(5.15)

is mainly positive for $\zeta_y(1/2 + iy) = \partial\zeta(1/2 + iy)/\partial y$. The first term of the last difference dominates (we will skip the explanation). Let us show that it has a very good reason to be positive on the zeros $z = iy$.

In the first place, $\Re(\zeta(3/2 + iy))$ is always positive ([E], 6.6, pg.129). Then, following [E], 6.5,

$$
\zeta\left(\frac{1}{2} + iy\right) = e^{-i\theta(y)} Z(y), \ \theta(y) = \Im \log(\Gamma\left(\frac{1}{4} + i\frac{y}{2}\right)) - \frac{y}{2} \log(\pi)
$$

(5.16)

for the classical real-valued function $Z(y)$ of $y \in \mathbb{R}_+$. For instance, the zeros of $\Im \zeta(1/2 + iy) = -Z(y) \sin(\theta(y))$ are either the zeros of $\zeta(k + 1/2)$ (coming from $Z$) or the zeros of $\sin(\theta(y))$, the so-called Gram points. For small $y$, they appear alternately (pg.125, Gram’s law), i.e. indeed $\Im(\zeta_y(1/2 + iy)) > 0$ on the imaginary (+0.5) zeros of the zeta. Even when Gram’s law fails the rule due to Rosser et al. states that excluding the zeros from the so-called Gram blocks the sign is plus (8.4, pg.180). It holds for at least 13.4 millions of zeros of $\zeta$, many more than we need here.

However the above reduction and the last argument are not sufficient to explain why the first “negative” zero for the initial $\eta_+(z|2)$ (see (5.11)) is $z = 1977.2714i$. The sign of $\Im(\zeta_y(1/2 + iy))$ becomes minus for the first time at $y = 282.4651$. The next one is $y = 295.5733$. Not very impressive.

There is something remarkable about $z = 1977.2714i$. Exactly at this point, the computer program looking for the zeros of the zeta has to become more sophisticated. The reason is that this zero is very close to the previous one $z = 1977.1739i$. If there is a correlation between the zeros with the linear $q$-approximations to the left and such pairs of zeros then it could be of a certain importance. The existence of nearly coincident zeros with low extrema of $Z$.
between them “almost” contradicts the Riemann hypothesis ([E], 8.4, pg.178) and, indeed, “must give pause to even the most convinced believer”.

6. Symmetrization

In conclusion, let us try to establish connections with Riemann’s estimate of the number of zeros in the critical strip and the Riemann hypothesis. We will examine the (anti)symmetrizations of the $q$-zeta functions considered above with respect to $k \leftrightarrow -k$. It is the simplest way to ensure the functional equation. Let us begin with $3_{+q}^{im}(k|d)$ from (3.4) and its meromorphic continuation $3_{+q}^{neg}(k|d)$ to negative $\Re k$ from (3.10). Since they behave differently for positive and negative $\Re k$, one may hope to avoid unwanted zeros upon the symmetrization.

Let us renormalize $3_{+q}^{im}$ (cf. (3.13)) as follows:

$$
\gamma(k) \overset{df}{=} \prod_{j=0}^{\infty} \frac{(1-q^{j+2k})(1-q^{j+1})}{(1-q^{j+k+1/2})(1-q^{j+k+1})},
$$

(6.1) $$
\tilde{3}_{+q}^{im}(k|d) \overset{df}{=} \gamma(k)3_{+q}^{im}(k|d), \quad \tilde{3}_{+q}^{neg}(k|d) \overset{df}{=} \gamma(k)3_{+q}^{neg}(k|d),
$$

lim_{a \to \infty} a^k \tilde{3}_{+q}^{im}(k|d) = \tilde{3}_{+}^{im}(k|d), \quad \Re k > 0,

lim_{a \to \infty} a^k \tilde{3}_{+q}^{neg}(k|d) = \tilde{3}_{+}^{neg}(k|d), \quad \Re k < 0, \quad \text{for}

$$
3_{+}(k|d) = d^{-1/2-k}(1 - 2^{1/2-k})\Gamma(k + 1/2)\zeta(k + 1/2).
$$

Cf. (3.12), (3.13), and (3.23). For $\Re k \geq 0$, we set

(6.2) $$
\tilde{3}_{+q}^{sym}(k|d) \overset{df}{=} a^k \tilde{3}_{+q}^{im}(k|d) + a^{-k} \tilde{3}_{+q}^{neg}(-k|d),
$$

lim_{a \to \infty} \tilde{3}_{+q}^{sym}(k|d) = \tilde{3}_{+}^{sym}(k|d) \overset{df}{=} k\sqrt{\pi}(3_{+}(k|d) - 3_{+}(-k|d))
$$

The function $\tilde{3}_{+q}^{sym}$ can be naturally extended to a symmetric (even) meromorphic function defined for all $k \in \mathbb{C}$: $\tilde{3}_{+q}^{sym}(-k|d) = \tilde{3}_{+q}^{sym}(k|d)$. It has zeros at $k \in 2\pi i a \mathbb{Z}$. Its poles modulo $2\pi i a \mathbb{Z}$ form (see (3.8)) the set

(6.3) $$
\tilde{\mathcal{P}} = (-\Lambda) \cup \Lambda \cup \{1/2 + \mathbb{Z}\}, \quad \Lambda = \{-2\xi_l - \mathbb{Z}_+\} \cup \{-2\xi^*_l - \mathbb{Z}_+\}
$$

They are all simple for generic $a$.

Given $0 < \kappa < 1$ such that $\{\kappa + i\mathbb{R} + \mathbb{Z}\} \cap \tilde{\mathcal{P}} = \emptyset$,

$$
\lim_{r \to \infty} a^{-r-n} \tilde{3}_{+q}^{sym}(k - r + iv) = a^{iv}\psi_+,
$$

(6.4) where $\psi_+ = \lim_{\Re k \to \infty} 3_{+q}^{im}(k) > 0$, $r \in \mathbb{N}$.

Here we employ Theorem 3.4 and (3.14) and use that $(\gamma \beta^{-1})(\kappa - r + iv)$ (see (3.13)) tends to zero as $\mathbb{N} \ni r \to -\infty$.
Because of the appearance of the $a$-factors the function $\hat{Z}_{+q}^{\text{sym}}(k)$ is not $2\pi ai$-periodic. However we may assume that $a 2\pi ai = 1$, i.e.

$$ai \log(a) = M \text{ for } M \in \mathbb{Z}.$$ 

Then it becomes $2T i$-periodic for $T \overset{\text{def}}{=} \pi a$ and we can readily calculate the number of its zeros minus the number of poles modulo $2T i$ in the limit of big $r$. Moreover, because of the reality of $\hat{Z}_{+q}^{\text{sym}}(k)$ on the real axis, if $z$ is a zero (a pole) so is $2Ti - z$. Hence we can find this number for the half-period too.

**Corollary 6.1.** Let $\kappa$ be as above, $\hat{O}_r$ the number of zeros of $\hat{Z}_{+q}^{\text{sym}}(k)$ in the strip

$$\mathcal{S}_r \overset{\text{def}}{=} \{ \kappa - r \leq \Re k \leq r - \kappa, \ 0 \leq \Im k \leq T \},$$

$\hat{P}_r$ the number of poles. We calculate zeros(poles) with multiplicities and with the coefficient $1/2$ if they belong to the boundary. Then

$$\hat{N}_\infty \overset{\text{def}}{=} \hat{O}_r - \hat{P}_r = \frac{T}{\pi} (\log \frac{T}{\pi}) \text{ for } N \ni r \gg 0. \tag{6.5}$$

Formula (6.5) can be considered as a certain counterpart of the celebrated formula (see [E, 6.7])

$$N(T) = \frac{T}{2\pi} (\log \frac{T}{2\pi} - 1) + O(\log T) \tag{6.6}$$

for the number of the zeros $z$ of the zeta with $0 < \Im z < T$ (in the critical strip). The total number of imaginary zeros of $\hat{Z}_{+q}^{\text{sym}}(k|d)$ such that $0 < \Im z < T$ results from this formula modulo the Riemann hypothesis:

$$\hat{N}(T|d) = \frac{T}{\pi} (\log T - 1 - \log(2d)) + O(\log T). \tag{6.7}$$

All zeros of $\zeta(k + 1/2)$ are those of $\hat{Z}_{+q}^{\text{sym}}$. However the latter may have extra zeros. According to our analysis their number is finite for $d \leq 1$ and is not bigger than $(T/\pi) \log 2$ for $d > 1$ For big $\Im k$ they may appear only near the zeros of $(1 - 2^{1/2 \pm k})$. Anyway the non-imaginary zeros do not change (6.7) too much.

Concerning $\hat{Z}_{+q}^{\text{sym}}(k|d)$, first, it must have exactly one zero approaching each pole from the set $\{ \pm k = 1/2 + 2Z_+ \}$ as $q \to 1$. Indeed, the limit $\hat{Z}_{+q}^{\text{sym}}(k|d)$ has neither poles nor zeros there. It is likely that the same holds for the remaining poles at $\{ \pm k = -1/2 + 2Z_+ \}$. At least it is true for sufficiently big $|\Re k|$ because of the stable $\mathbb{Z}$-periodicity of $\hat{Z}_{+q}^{\text{neg}}(k|d)$. Numerical experiments confirm this: the “symmetric” $q$-zeros practically coincide with those from (3.20) as $a = 1 = d$. 


Second, there is a tendency for the poles from \( \Lambda \) to be moving accompanied by the zeros. We cannot prove that this always happens. Anyway the number of \( q \)-poles equals the number of \( q \)-zeros inside a closed curve in the form \( C_a = \sqrt{\alpha C_1} \) for a given \( C_1 \) if

\[
|a^k \hat{3}^{im}_{+q}(k|d)| > |a^{-k} \hat{3}^{neg}_{+q}(k|d)| \quad \text{for} \quad k \in C_a,
\]

assuming that \( \Re C_1 > 0 \) and \( \hat{3}^{im}_{+q}(k|d) \) has no zeros inside \( C_a \). Generally speaking, it holds. However it is not clear at the moment whether this can be used for quantitative estimates.

Anyway if it is true (or almost true) that the poles of \( \hat{3}^{sym}_{+q}(k|d) \) from \( \Lambda \) are compensated by the zeros in the limit \( a \to \infty \), then we come to the following conjecture.

Conjecture 6.2. Assuming that \( d \geq \pi/(2e) \), the \( q \)-deformation \( z_q \) of any imaginary zero \( z = iy(y > 0) \) of \( \hat{3}^{sym}_{+q}(k|d) \), namely a zero of \( \hat{3}^{sym}_{+q}(k|d) \) approaching \( z \) in the limit \( q \to 1 \), exists and remains in the horizontal strip \( 0 < \Im z_q < y + C \) for all \( 0 < q < 1 \) and a proper constant \( C > 0 \). The zeros of \( \hat{3}^{sym}_{+q}(k|d) \) in the box

\[
K_a(c) = \{-2\sqrt{\frac{\pi a}{2d}} + c < \Re k < 2\sqrt{\frac{\pi a}{2d}} - c, \ 0 < \Im k < \pi a\}
\]

are \( q \)-deformations of the classical ones for a constant \( c > 0 \).

The first part of the conjecture for \( d = \pi/(2e) \) results in the estimate

\[
N(T) < \frac{T}{2\pi}(\log \frac{T}{2\pi} - 1) + O(\log T),
\]

speculating that the number of poles of \( \hat{3}^{sym}_{+q}(k|d) \) for big \( a \) with \( 0 < |\Re k| < t \) which have more than one neighboring zero is \( O(\log t) \) at most. As \( a \to 0 \) (i.e. \( q = \exp(-1/a) \to 0 \)), the \( q \)-deformations must approach the real line. So eventually they go down for small \( a \). The box \( K_a(0) \) in the second part does not contain the poles of \( \hat{3}^{sym}_{+q} \). Thus we expect that once a \( q \)-zero gets distant enough from the \( q \)-poles it always approaches a classical zero in the limit.

Variants. Now let us modify the imaginary \( q \)-zeta to avoid the gamma-factors. For \( \Re k \geq 0 \), we set

\[
\hat{\zeta}_{+q}(k|d) \overset{\text{def}}{=} 4\sigma(k)\hat{3}^{im}_{+q}(k|d) - 4\sigma(-k)\hat{3}^{neg}_{+q}(-k|d),
\]

where \( \sigma(k) \overset{\text{def}}{=} \prod_{j=0}^{\infty} (1 - q^{k+1/2+j})/(1 - q^{j+1}), \)

\[
\lim_{a \to \infty} \hat{\zeta}_{+q}(k|d) = \hat{\zeta}_{+}(k|d) \overset{\text{def}}{=} \hat{\zeta}_{+}(k|d) - \hat{\zeta}_{+}(-k|d),
\]

\[
\zeta_{+}(k|d) = (d/4)^{-1/2-k}(1 - 2^{1/2-k})\zeta(k + 1/2).
\]
The function \( \tilde{\zeta}_{+q} \) can be extended to an odd meromorphic \( 2\pi ai \)-periodic function with the poles at \( \{ \pi ai + Z/2 \} \). Since \((\sigma \beta^{-1})(k - r + vi) \to 0 \) as \( r \to 0 \) (see (6.4)),
\[
(6.10) \quad \lim_{r \to \infty} \tilde{\zeta}_{+q}(k|d)(\kappa - r + iv) = 4\psi_{+}, \ r \in N.
\]
Hence the (stable) numbers of zeros of \( \tilde{\zeta}_{+q}(k|d) \) and poles on the half-period \( 0 \leq \Re k < T = \pi a \) coincide. So there is no relation to the Riemann estimate in the considered case, which can be explained as follows. First, the classical function \( \tilde{\zeta}_{+}(k|d) \) has infinitely many real zeros approaching the points from \( \pm \{1/2 + 2Z_+\} \) for big \( |\Re k| \). It can also have zeros which are neither real nor imaginary (but finitely many). Second, the poles of \( \tilde{\zeta}_{+}(k|d) \) on the line \( \Re k = \pi a \) may “loose” the neighboring zeros as \( a \to \infty \). In line with Conjecture 6.2, such lost \( q \)-zeros could be a significant source of the classical ones. Hopefully it may be seen numerically.

It is instructional to get a \( q \)-deformation of the modified zeta
\[
\tilde{\zeta}(k) = \pi^{-(k/2+1/4)}\Gamma(\frac{k}{2} + \frac{1}{4})\zeta(k + \frac{1}{2})
\]
satisfying the plane functional equation \( \tilde{\zeta}(k) = \tilde{\zeta}(-k) \). As always, we need to switch to its plus-variant to ensure the convergence for \( \Re k \leq 1/2 \):
\[
(6.11) \quad \tilde{\zeta}_{+}(k|d) = p(k|d)\tilde{\zeta}(k), \ p(k|d) = d^{-1/2-k}(1 - 2^{1/2-k}) - d^{-1/2+k}(1 - 2^{1/2+k}).
\]
All zeros of the function \( p(k|d) \) are imaginary for \( d \geq 1/2 \). The number of imaginary zeros in the interval \( 0 < \Re k < T \) equals \( |(T/\pi)\log(2d)| \) up to \(-1 \leq \epsilon \leq 1 \) for any \( d > 0 \). Let us assume that \( d > 1/2 \).

We set
\[
(6.12) \quad \alpha(k) \defeq \gamma(k) \prod_{j=0}^{\infty} \frac{1 - q^{2j+1}}{1 - q^{k/2+1/4+j}}, \quad (4\pi)\tilde{\zeta}_{+q}(k|d) \defeq \left(\frac{a}{16\pi}\right)^{k/2-3/4}\alpha(k)\mathcal{B}_{-q}^{im}(k|d) - \left(\frac{a}{16\pi}\right)^{-k/2-3/4}\alpha(-k)\mathcal{B}_{+q}^{neg}(-k|d).
\]
Then \( \lim_{a \to \infty} \tilde{\zeta}_{+q}(k|d) = \tilde{\zeta}_{+}(k|d) \).

The function \((a\beta^{-1})(\kappa - r + vi)\) tends to zero as \( r \to \infty \) too, so we can calculate the stable (see above) difference \( \tilde{N}_{\infty} \) between the total number of zeros and poles on the “half-period” for \( \tilde{\zeta}_{+q}(k|d) \). We may assume that \( (a/(16\pi))^{\pi ai/2} = 1 \), i.e.
\[
(\pi ai)\log(a/(16\pi)) = M \quad \text{for} \quad M \in Z,
\]
to provide the \( 2\pi ai \)-periodicity of \( \tilde{\zeta}_{+q} \). Then \( \tilde{N}_{\infty} = M \). Let us express it in terms of \( T = \pi a \):
\[
(6.13) \quad \tilde{N}_{\infty}(T) = M = \frac{T}{2\pi}(\log \frac{T}{16\pi} - \log \pi).
\]
Here the leading term coincides with that from (6.6) again. More exactly, the number of zeros of $\zeta_+(k|d)$ such that $0 < \Im z < T$ equals

$$\tilde{N}(T|d) = \frac{T}{2\pi} (\log \frac{T}{2\pi} - 1 + 2\log(2d)) + O(\log T).$$

The zeros of $\tilde{\zeta}_{q+q}$ either a) approach its real poles at $\{\pm k = 1/2 + 2\mathbb{Z}_+\}$, or b) go together with the corresponding poles from $\Lambda$ (see (6.3)) upwards, or c) tend to the zeros of zeta in the critical strip. We may conjecture that a $q$-deformation $z_q$ of a zero $z = iy(y > 0)$ of $\zeta(k+1/2)$ remains in the horizontal strip $0 < \Im z_q < y + C_d y / \log y$ for all $0 < q < 1$ and a constant $C_d > 0$ depending on $d$. One can also speculate that $C_d \sim \log(32\pi d^2/e)$ comparing the formulas for $\tilde{N}(T|d)$ and $\tilde{N}_\infty(T)$ in more detail.

The above estimates can be extended to the case of the standard zeta. The factors $(1 - 2^{1/2-k})$ will disappear from the formulas. The calculation of the stable number of zeros minus poles remains the same. However the minus-counterparts $\tilde{\mathcal{Z}}_{q+q}^\text{sym}$, $\tilde{\zeta}_q$, and $\tilde{\zeta}$ of the functions considered above diverge in the critical strip. So the relations to the classical zeros have to be discussed under the constraint $\log(a) = o(|\Gamma(k)^2 \tan(\pi k)|^{-1/2})$. The conjecture and other considerations can be also extended to the $q$-deformations of the Dirichlet $L$-function (see (5.4)), where the convergence holds for all $k$ except for the poles.

**Towards Riemann $q$-hypothesis.** The conjecture states nothing about the pure imaginary zeros of $\tilde{\mathcal{Z}}_{q+q}^\text{sym}(k|d)$. It is more convenient to discuss this problem for the sharp zeta-function. For the sake of concreteness, we assume that $d = 1$ and will $q$-deform the $\zeta$ from (6.9) only. A similar behavior is expected for the sharp-variants of the remaining two symmetric $q$-zeta functions considered above, their minus-counterparts (for $a$ which are not too big), and the sharp Dirichlet $L$-functions.

We set

$$\tilde{\zeta}_{q+q}(k) \overset{\text{def}}{=} \chi(k)^{-1} \tilde{\mathcal{Z}}_{q+q}(k) - \chi(-k)^{-1} \tilde{\mathcal{Z}}_{q+q}(-k),$$

$$\chi(k) \overset{\text{def}}{=} \mathcal{O}_q^\ast(k) = -\frac{\pi a}{2} q^{k^2} (1 - q^{-k}) \prod_{j=0}^{\infty} \frac{(1 - q^{i+k})(1 - q^{j/2+1/2})}{(1 - q^{j+2k})(1 - q^{j+1/2})} \times$$

$$\prod_{j=1}^{\infty} (1 - q^{i-k})(1 - q^{i+k})(1 + q^{j/2-1/4+k/2})(1 + q^{j/2-1/4-k/2}),$$

$$\lim_{a \to \infty} \tilde{\zeta}_{q+q}(k) = \tilde{\zeta}_+(k) = \overset{\text{def}}{=} \zeta_+(k) - \zeta_+(-k).$$

Thus we just divided the sharp-integral for the kernel $(1 + q^{-x^2})$ by that for the Gaussian $q^{x^2}$. This very choice of the normalization is not too important. However in this form the definition can be naturally extended to the multi-dimensional case. Thanks to Theorem 2.3 and Theorem 3.1, the function
\( \zeta_{+q}(k) \) is regular in the horizontal strip

\[
K_{a}^{\pm}(c) \overset{def}{=} \{- \sqrt{2\pi a} + c < \Im k < \sqrt{2\pi a} - c \}
\]

for \( c \geq 0 \).

**Conjecture 6.3.** Let us fix \( q = \exp(-1/a) \) for \( a > 0 \). Then all zeros of \( \zeta_{+}(k) \) inside \( K_{a}^{\pm}(0) \) have unique \( q \)-deformations, i.e. the zeros of \( \zeta_{+}(k) \) convergent to the corresponding classical ones in the limit, which belong to \( K_{a}^{\pm}(-c) \) for a constant \( c > 0 \). Vice versa, all zeros of \( \zeta_{+}(k) \) inside \( K_{a}^{\pm}(c) \) are such deformations for proper \( c > 0 \). The \( q \)-deformations of the imaginary zeros of \( \zeta_{+}(k) \) in \( K_{a}^{\pm}(0) \) are imaginary.

The sharp-counterpart of the Riemann hypothesis is the claim that all zeros \( z_{q} \) of \( \zeta_{+q}(k) \) in the critical strip \( \{ |\Re k| < 1/2 \} \) such that \( 0 < \Im z_{q} < \sqrt{2\pi a} - c \) for a certain constant \( c > 0 \) are imaginary. Confirmations of the conjecture are entirely numerical. No theoretical approach is known at the moment.

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