Time evolution and observables in constrained systems

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Abstract

The discussion is limited to first-class parametrized systems, where the definition of time evolution and observables is not trivial, and to finite dimensional systems in order that technicalities do not obscure the conceptual framework. The existence of reasonable true, or physical, degrees of freedom is rigorously defined and called local reducibility. A proof is given that any locally reducible system admits a complete set of perennials. For locally reducible systems, the most general construction of time evolution in the Schroedinger and Heisenberg form that uses only geometry of the phase space is described. The time shifts are not required to be symmetries. A relation between perennials and observables of the Schroedinger or Heisenberg type results: such observables can be identified with certain classes of perennials and the structure of the classes depends on the time evolution. The time evolution between two non-global transversal surfaces is studied. The problem is posed and solved within the framework of the ordinary quantum mechanics. The resulting non-unitarity is different from that known in the field theory (Hawking effect): state norms need not be preserved so that the system can be lost during the evolution of this kind.
1 Introduction

A truly intriguing feature of the general relativity is the lack of any fixed background spacetime that would serve as a stage for its dynamics. There are many different spacetimes that solve Einstein’s equations; but the time evolution of the given gravitating system in the strict sense that we are used to from the study of other systems does not take place in any of them. The discussion of these problems is somewhat confined to the group of people who are trying to quantize the theory and the issue is called “the problem of time in quantum gravity” (see, e.g. [1] and [2]). However, even within the classical version of Einstein’s theory, the concept of time evolution and the related one of observable which would be sufficiently closely analogous to that of other models of theoretical physics are either not suitable for Einstein’s theory or not yet completely developed.

An impressive work in this direction has been done by Kuchař. His method is to reconstruct the naked spacetime manifold (that is, without metric) from the phase space by separating the kinematical variables from the dynamical ones; the kinematical variables describe the position at the naked manifold and the dynamical variables become observables which evolve along it. The approach of the present paper—the so-called perennial formalism—owes much to Kuchař’ ideas. However, it attempts to construct the dynamics directly within the phase space so that no form of spacetime is needed at any stage. The construction is based on some well-known (even very old) ideas. First of them is Dirac’s theory of the so-called “three forms of relativistic dynamics” [3] for a system of massive particles in Minkowski spacetime. This is based on one hand on the Poincaré group or algebra and on the other hand on three kinds of surfaces defining the three forms. Although Dirac considered these surfaces as lying in the spacetime, each of them defines a unique surface in the phase space and the properties of these surfaces that are essential for the method to work can even more easily be understood within the phase space: they are “transversal surfaces” (see [4]); any reasonable system possesses such surfaces. Similarly the Poincaré group or algebra is a structure which can be found in the phase space of reasonable systems: it is a group of symmetries or an algebra of perennials. These are the basic notions used by some “modern” methods of quantizing the parametrized systems, in particular the group and algebraic quantization ([5], [6] and [7]). The corresponding generalization of Dirac’s idea to any finite-dimensional parametrized system has been given in [8]; an infinite-dimensional system (the massive scalar field in curved background spacetime) was studied in [9], where the geometric theory of infinite-dimensional Hamiltonian systems by Marsden and his collaborators ([10] and [11]) helped to solve the problems [12].

Roughly, the present paper contains three new ideas. First, a distinction between integrals of motion and perennials is recognized; this yields several useful insights.
Second, a general construction of time evolution for parametrized systems is given and the importance of the time evolution for the notion of observable is clarified. Third, the time evolution between two non-global transversal surfaces is considered as an exercise in, and is solved using just the tools of, ordinary quantum mechanics. The plan of the paper is as follows.

Section 2 contains an extension of the notion of perennial and symmetry to the so-called singular perennial and symmetry, which is necessary for the method to work for non-global transversal surfaces of certain kind that are often met (for example inextensible transversal surfaces which are not global). The property of local reducibility, that is the existence of reasonable true degrees of freedom, is rigorously defined and shown to imply the existence of a complete system of perennials. A relation between integrals of motion of (unconstrained) Hamiltonian systems and perennials of the corresponding parametrized systems is clarified. In particular, “chaotic” Hamiltonian systems that do not admit any integral of motion except for the Hamiltonian, do admit a complete system of perennials if parametrized. Section 3 recalls briefly the quantization theory as given in [8], and it brings some improvements, especially the use of universal enveloping algebras. Section 4 contains a construction of time evolution using the so-called time shifts, which is, in a sense explained in subsection 4.1, the most general time evolution possible. In particular, no symmetry is now necessary for such a construction. This generality should not be understood in the sense, however, that each possible time evolution which can be constructed for a given system according to our prescription is sensible—a choice has to be done. A differentiable one-dimensional case of time evolution is studied in subsection 4.2, where the general form of the Heisenberg and Schroedinger equations of motion is derived. An example shows that our new construction contains time evolution that can also be obtained by the method of reduction (see e. g. [13] or [14]). In the final subsection 4.3 of the section 4 we investigate the evolution between two non-global transversal surfaces. In fact, an example of such an evolution for a field system was studied in (11): the Hawking effect. It was shown that a careful consideration of domains and ranges of time shifts can explain the well-known non-unitarity of time evolution in this case. However, in the field system case, the evolution just looses information; the normalization of states is preserved, because (roughly speaking) even the state of no excitation is a normalized state (vacuum) of the system. Surprisingly, the situation is worse for finite dimensional systems, where the non-arrival of the system at a final transversal surface must be interpreted as a loss of the system during the evolution—the non-unitarity is then of a different type (not preserving norms). However, the conclusion about the non-unitarity of the evolution follows necessarily once the choice of the two transversal surfaces is met. In section 5, we discuss the notion of observable and its relation to that of perennial.
It turns out that this notion is related to, but not completely identical with, that of “evolving constant of motion” by Carlo Rovelli [15]. Thus, the observables are not identical with perennials (in general); formally, they are classes of perennials. These observables are of the ordinary quantum mechanical type; they are measured each by a distinguished measurement process or apparatus that is well-defined independently of time (“the same measurement at different times”). Finally, in section 3 we illustrate the construction of the time evolution between two non-global transversal surfaces using a simple model of a completely solvable system that does not admit global transversal surfaces.

2 Singular perennials and symmetries

In this section we generalize the notion of perennial and of symmetry in a way that will lead to simplifications in our subsequent work on non-global transversal surfaces.

Let us first recall the few basic facts about the first-class parametrized systems (for details see [8]). We restrict ourselves to finite-dimensional models so the phase space will be a $2N$-dimensional manifold $\Gamma$ with a symplectic form $\Omega$. The dynamics is determined by the constraint surface $\Gamma$ of a special kind (for a first-class system): $\Gamma$ is a $(2n - \nu)$-dimensional submanifold of $\Gamma$ such that the pull-back $\Omega$ of $\Omega$ to $\Gamma$ is a pre-symplectic form whose singular subspace $L_p$ at the point $p \in \Gamma$ has the dimension $\nu$ for all $p$. Then $L_p$ is an integrable distribution on $\Gamma$; the maximal integral manifolds $\gamma$ of $L_p$ are called c-orbits. Each c-orbit represents a unique maximal classical solution in all possible gauges and foliations. Each point $p \in \Gamma$ lies at exactly one c-orbit, which will be denoted by $\gamma_p$.

A perennial is defined as a differentiable function $o : \Gamma \mapsto \mathbb{R}$ that is constant along each c-orbit. Our generalization will allow perennials to be $C^\infty$ only on a subset, $\mathcal{D}(o)$ of $\Gamma$, the so-called domain of $o$. The set $\mathcal{D}(o)$ must have the following properties:

1. $\mathcal{D}(o)$ is open in $\Gamma$,
2. $\Gamma \subset \overline{\mathcal{D}(o)}$.

Such perennials will be called singular. Let $o_1$ and $o_2$ be two perennials with domains $\mathcal{D}(o_1)$ and $\mathcal{D}(o_2)$. Then the linear combination, functional multiplication and Poisson brackets of $o_1$ and $o_2$ are all well-defined on $\mathcal{D}(o_1) \cap \mathcal{D}(o_2)$. As this set has again the properties of a domain, the three operations will result in singular perennials. All singular perennials form a Poisson algebra which we denote by $\mathcal{P}$.

A useful objects will be the projectors associated with some open subsets of $\Gamma$; we define them as maps in $\mathcal{P}$. Let $\mathcal{D}$ be an open subset of $\Gamma$ with the property: if
p ∈ (D ∩ Γ), then γ_p ∈ D. Let χ_D : ŠΓ → R be the characteristic function of D, that is

χ_D(p) = 1 ∀p ∈ D,
χ_D(p) = 0 ∀p ∈ ŠΓ \ D.

Then χ_D is a (singular) perennial. χ_D defines a map Π_D : P ↦→ P by Π_D(o) = χ_D o for all o ∈ P. It follows easily that D(Π_D(o)) = (D ∪ (ŠΓ \ D)) ∩ D(o), that Π_D(P) = P_D is a Poisson algebra, and that Π_D is a Poisson algebra homomorphism. Π_D has all properties of a projection operator.

The next notion that plays an important role in the perennial formalism is that of transversal surface. Recall that such a surface is a submanifold Γ_1 of the constraint manifold Γ which has no common tangent vectors with the c-orbits (except for zero vector) and which intersects each c-orbit in at most one point. The set D(Γ_1) := {p ∈ Γ | γ_p ∩ Γ_1 ≠ ∅} is called domain of Γ_1 and Γ_1 is called a global trasversal surface, if D(Γ_1) = Γ. The pull-back Ω_1 of the symplectic form Ω to Γ_1 is non-degenerate so that the pair (Γ_1, Ω_1) is a symplectic manifold; we denote the corresponding Poisson brackets by {·,·}_1. This symplectic manifold can be considered as the phase space of the corresponding reduced system; in particular, the number of true degrees of freedom is half the dimension of Γ_1. Symmetries and perennials can be projected to transversal surfaces: Let i_1 be the embedding of Γ_1 in ŠΓ and π_1 : Γ ↦→ Γ_1 be defined by π_1(p) = γ_p ∩ Γ_1; π_1 is called projector associated with Γ_1. Then each symmetry φ which preserves the domain of Γ_1 defines a map a_1(φ) : Γ_1 ↦→ Γ_1 by a_1(φ)(p) = π_1(φ(p)). The map a_1 preserves the composition of the symmetries; thus it defines an action of groups of symmetries provided that all elements of the group preserve the domain of Γ_1. If o is a perennial, then o_1 = i_1* o is a function on Γ_1; i_1* preserves the linear combination, product of functions and the Poisson bracket, i.e. i_1*{o,o′} = {i_1* o,i_1* o′}_1. Thus, i_1* is a homomorphism of Poisson algebras. For details see [8].

The definition of the first-class parametrized systems as given above and in [8] is too general for physicist’s purposes. Generically, such a system cannot be reduced even locally, that is, there will be no transversal surfaces in any neighbourhood of any point of Γ. To exclude this pathology, we restrict ourselves to the locally reducible systems, which can be defined as follows.

**Definition 1** A first-class parametrized system (ŠΓ, ŠΩ, Γ) is called locally reducible, if a dense open subset of Γ/γ is a quotient manifold (not necessarily Hausdorff).

For the definition of quotient manifolds see, e.g. [16]. In particular, the natural projection π : Γ ↦→ Γ/γ is a submersion. Then, as shown in [16], there is a differentiable section of π through any point of Γ. A section of π is a map ψ : Γ/γ ↦→ Γ
such that \( \pi \circ \psi \) is the identity on the domain of \( \psi \) (which is necessarily Hausdorff). This implies easily that the image of \( \psi \) is a transversal surface. Inversely, suppose that every point \( p \in \Gamma \) lies at some transversal surface and that the associated projectors are differentiable. Then the quotient set can be given a quotient manifold structure by pasting all these transversal surfaces by their associated projectors in the overlapping domains \( D(\Gamma_i) \cap D(\Gamma_j) \). This justifies the definition. The general relativity may be locally reducible (see [11]).

The locally reducible systems have the nice property that they admit complete systems of (singular) perennials. We will say that a system of perennials is complete, if it separates separable c-orbits; the c-orbits \( \gamma_1 \) and \( \gamma_2 \) are separable if there is a continuous perennial \( o \) such that \( o(\gamma_1) \neq o(\gamma_2) \). Indeed, in the special case that \( \Gamma/\gamma \) is Hausdorff, we can construct such a system as follows. According to the classical theorem by Whitney, \( \Gamma/\gamma \) can be globally embedded in \( \mathbb{R}^\kappa \), where \( \kappa = 4N - 4\nu + 1 \) (because the dimension of \( \Gamma/\gamma \) is \( 2N - 2\nu \) [17]). Let \( X^k, k = 1, \ldots, \kappa \) be the natural coordinates on \( \mathbb{R}^\kappa \) and let \( \Phi : \Gamma/\gamma \mapsto \mathbb{R}^\kappa \) be the embedding. Then \( X^k \circ \Phi \circ \pi, k = 1, \ldots, \kappa \) is a complete system of perennials on \( \Gamma \). Moreover, the gradients of all elements of the system span the subspace of \( T^*_p\Gamma \) that is transversal to \( T_p\gamma \) at each point \( p \in \Gamma \). In the general case, when \( \Gamma/\gamma \) need not be Hausdorff, one can find a complete system of singular perennials as follows. Let us recall that any non-Hausdorff manifold \( M \) can be decomposed in its maximal Hausdorff submanifolds \( M_i \); any point of \( M \) lies at some Hausdorff submanifold of \( M \) (namely, the corresponding chart), and all Hausdorff submanifolds of \( M \) form a partially ordered set with the right properties so that one easily obtains the desired existence. Let, then, \( \Gamma/\gamma = \bigcup_i M_i \) be this decomposition of \( \Gamma/\gamma \). For each \( M_i \), a complete set of perennials can be constructed according to the procedure described above. The functions we find in this way, however, need not possess differentiable extensions to the boundaries \( \partial M_i \) of \( M_i \) in \( \Gamma/\gamma \) (for examples, see section 3 and [18]). This motivates our introduction of singular perennials: we can define such perennials everywhere on \( \Gamma/\gamma \) by setting them equal to zero in \( \Gamma/\gamma \setminus \pi^{-1}M_i \). Working this out for each \( i \), one obtains a (hopefully finite) complete set of singular perennials.

The construction of perennials in \( D(\Gamma_1) \), where \( \Gamma_1 \) is a transversal surface can start from \( \Gamma_1 \) instead of \( \Gamma/\gamma \). Indeed, \( \pi|_{\Gamma_1} \) is a diffeomorphism between \( \Gamma_1 \) and \( \pi\Gamma_1 \). Thus, a differentiable function \( o_1 \) on \( \Gamma_1 \) can be pulled back by \( (\pi|_{\Gamma_1})^{-1} \) to \( \pi\Gamma_1 \) and the resulting perennial \( o \) is given by \( o = o_1 \circ (\pi|_{\Gamma_1})^{-1} \circ \pi = o_1 \circ \pi_1 \), because \( \pi_1 = (\pi|_{\Gamma_1})^{-1} \circ \pi \). \( o \) will be referred to as defined by the initial datum \( o_1 \) at \( \Gamma_1 \).

To prevent misunderstanding, some comment is in order. On one hand, perennials can be considered as “integrals of motion” of the system. On the other, many completely regular and physically reasonable Hamiltonian systems do not admit any integrals of motion. This seems to be a paradox. In order to remove the paradox,
we must become a little more precise. A Hamiltonian system \((V, \Omega, H)\) consists of a symplectic manifold \((V, \Omega)\) and a differentiable function \(H\) whose Hamiltonian vector field on \(V\) is complete. An integral of motion is a function on \(V\) which is constant along the orbits of \(H\). It has been shown in \([19]\) that such systems generically do not admit any integral of motion independent from the Hamiltonian. For example, the movement of a material point on a frictionless surface \(\Sigma\) without external forces is such a system, if \(\Sigma\) is a compact Riemannian manifold with constant negative curvature \((V = T^*\Sigma, \text{ see } [20])\); there is nothing pathological with this system. \((V, \Omega, H)\) is no constrained system, however. To obtain a first-class parametrized system from it that will describe the same motion, one must parametrize it. This is the following procedure. Let \(\tilde{V} := V \times \mathbb{R}^2\) and let the natural coordinates on \(\mathbb{R}^2\) be \(t\) and \(p_t\). Define the symplectic form \(\tilde{\Omega}\) on \(\tilde{V}\) by \(\tilde{\Omega} := \Omega + dp_t \wedge dt\) and the constraint surface \(\Gamma\) by the equation \(p_t + H = 0\). One easily verifies that the corresponding \(c\)-orbits, if projected down to \(V\) by the natural projection in the cartesian product \(V \times \mathbb{R}^2\), coincide with the dynamical trajectories of \((V, \Omega, H)\). However, this correspondence is many-to-one; \(c\)-orbits that are mapped on the same trajectory are obtained from different time parametrizations of the trajectory. Thus, perennials of \((\tilde{V}, \tilde{\Omega}, \Gamma)\) need not coincide with the integrals of motion of \((V, \Omega, H)\): an integral defines a perennial, but a perennial need not determine any integral. Let us show that the system \((\tilde{V}, \tilde{\Omega}, \Gamma)\) is locally reducible. For this aim, we define the map \(\Psi : (V \times \mathbb{R}) \mapsto \Gamma\) by \(\Psi(p, t) = (\Phi_t(p), t, H(p))\), where \(\Phi_t\) is the flow of the Hamiltonian vector field of \(H\) on \(V\). \(\Psi\) is a diffeomorphism, because \(\Phi_t\) is a diffeomorphism for each \(t \in \mathbb{R}\) and \(\Phi_t\) is a differentiable curve at each \(t \in \mathbb{R}\) and for each \(p \in V\). Moreover, \(\Psi(p, R)\) is the \(c\)-orbit through the point \((p, 0, H(p))\) of the surface \(t = 0\) in \(\Gamma\) for any \(p \in V\). Consider the map \(\pi_V \circ \Psi^{-1}\), where \(\pi_V : (V \times \mathbb{R}) \mapsto V\) is the natural projection of a Cartesian product of manifolds. \(\pi_V \circ \Psi^{-1}\) maps all points of any \(c\)-orbit to just one point of \(V\). Thus, \(\pi_V \circ \Psi^{-1}\) can be considered as mapping \(\Gamma/\gamma\) to \(V\); as such it is a bijection. We may use \(\pi_V \circ \Psi^{-1}\) to define a manifold structure on \(\Gamma/\gamma\); with this structure, \(\Gamma/\gamma\) is a quotient manifold. Indeed, \(\pi : \Gamma \mapsto \Gamma/\gamma\) can be identified with \(\pi_V \circ \Psi^{-1}\), and this is a submersion. As a byproduct, we have that \(\Gamma_0 := \{(p, 0, H(p))|p \in V\}\) is a global transversal surface.

To summarize: this example shows that parametrizing a Hamiltonian system always results in a constrained system with a complete set of perennials independently of how many integrals of motion the Hamiltonian system possesses. Clearly, a parametrized system without a complete system of perennials has a different status than a Hamiltonian system without integrals of motion: the former is pathological, the latter is not. The locally reducible systems are, however, rather rare among all first-class parametrized systems. To understand that, the following observation is useful. Formally, another parametrized system can be constructed from the Hamil-
tonian system \((V, \Omega, H)\): this is \((V, \Omega, \Gamma')\), where \(\Gamma'\) is defined by the equation \(H = E\) and \(E \in H(V)\). Such a parametrized system is not locally reducible, if \((V, \Omega, H)\) does not admit a complete system of integrals of motion (that is, separating dynamical trajectories).

3 Quantization

In this section, we wish to combine the algebraic Ashtekar method of quantization with the group method by Isham and simultaneously allow for the singular perennials.

Let \(\tilde{G}_0\) be a Lie group of symmetries; that is, each element of \(\tilde{G}_0\) is a symmetry, and there is a common invariant domain, \(\mathcal{D}(\tilde{G}_0)\) of all elements of \(\tilde{G}_0\) such that \(\mathcal{D}(\tilde{G}_0) \cap \Gamma = \Gamma\). Recall that a group \(G\) is called almost transitive if there is a c-orbit \(\gamma\) such that \(\tilde{G}(\gamma) = \Gamma\). All elements of \(\tilde{G}_0\) that leave the c-orbits invariant form a normal subgroup \(N\). Let \(\tilde{S}_0\) be the Lie algebra of \(\tilde{G}_0\). The action of \(\tilde{G}_0\) on \(\tilde{\Gamma}\) enables us to realize \(\tilde{S}_0\) as a Lie algebra of vector fields on \(\mathcal{D}(\tilde{G}_0)\). Let us call the group \(\tilde{G}_0\) Hamiltonian, if all these vector fields are globally Hamiltonian. Then each element of \(\tilde{S}_0\) determines a unique class \(\{o\}\) of perennials (each two elements of the class differ by a constant function). These perennials will in general be singular, but they will have a common domain containing \(\mathcal{D}(\tilde{G}_0)\). One can either choose representatives of the classes \(\{o\}\) in such a way that they form a Lie algebra \(\tilde{S}\) with respect to the Poisson bracket—and which is then isomorph to the algebra \(\tilde{S}_0\)— or, if this is not possible, that they generate the Lie algebra \(\tilde{S}\) which is isomorph to a central extension of \(\tilde{S}_0\). Let \(\tilde{G}\) be the Lie group which is obtained from \(\tilde{G}_0\) by the corresponding central extension; then \(\tilde{G}\) has a well-defined action on \(\Gamma\), given by that of \(\tilde{G}_0\) and by the requirement that the central elements act trivially. One can show \((\S)\) that \(N\) is still a normal subgroup of \(\tilde{G}\). Thus if we assume that

(a) \(\tilde{G}_0\) is almost transitive,

(b) \(\tilde{G}_0\) is Hamiltonian,

(c) \(N\) is a closed subgroup,

then \(G := \tilde{G}/N\) is a Lie group; we call \(G\) first-class canonical group (FCC group). FCC subgroup of \(G\) is a subgroup which itself satisfies the conditions a, b, and c above. The quantum theory is to be constructed via some representations of the FCC group.

The Lie group \(N\) determines the Lie algebra \(I_S\) of perennials; \(I_S\) is a Lie ideal of \(\tilde{S}\) and it consists of all elements of \(\tilde{S}\) which vanish at \(\Gamma\). Then \(S := \tilde{S}/I_S\) is a Lie algebra. If we replace the point (a) of the definition of FCC group by
(a’) $S$ is a complete system of perennials,

then $S$ is called the **algebra of elementary perennials**. This algebra will satisfy (cf. §) the following requirements

(c) $S$ is a Lie algebra with respect to the operations of linear combination and Poisson bracket (these operations are well-defined for the classes of perennials in $S$);

(d) $S$ is a complete system of perennials;

(e) let $D(S) = \bigcap_{o \in S} D(o)$ and let $\xi_o$ be the Hamiltonian vector field of the function $o$; then $\xi_o$ is complete in $D(S)$ for all $o \in S$. $D(S)$ is called the **common invariant domain** of $S$.

Clearly, $D(S)$ coincides with $D(\tilde{G}_0)$. An important observation is that each element of $S$—which is a class of perennials—defines exactly one function on $\Gamma$ (which is constant along c-orbits). Another observation is that a complete system of perennials (whose existence has been shown in section 2) does not necessarily form an algebra of elementary perennials: the Hamiltonian vector fields need not be complete, and the algebra need not close. There are symplectic manifolds that do not admit any finite system of functions that separate points, whose elements possess complete Hamiltonian vector fields, and whose Poisson-bracket algebra closes. An example is an orientable two-dimensional Riemannian manifold of genus two (sphere with two handles), the symplectic form being the volume form. Still, there is a finite set of functions that separates points of this manifold.

The last key object of the classical part of the theory is the universal enveloping algebra $A$ of $S$. This algebra $A$ is a counterpart of the ‘abstract associative algebra’ introduced by Ashtekar ([4]). $A$ is needed for a formulation of some important conditions on the representations of the FCC group. These conditions—the so-called **relations**—come about because the elements of $S$ considered as functions on $\Gamma$ often are functionally dependent; it holds e. g. that $F(o_1, \ldots, o_k) = 0$ for $o_1 \in S, \ldots, o_k \in S$. We would like to transfer these relations into the quantum theory. The popular way to do that is to identify $F$ with an element of the algebra $A$. This will be possible if $F$ is a polynomial. Even if $F$ is a real analytic function, one can define $F$ by a series; one can extend the algebra $A$ by formal series’ to an associative algebra $\bar{A}$ (cf. [21]) and then try to place the series for $F$ in $\bar{A}$. However, each such identification is a particular choice of factor ordering, so one has to solve the ‘factor ordering problem’ in each case (there are always some reasonable requirements on the physical factor ordering, see e. g. [22], but the factor ordering is still not uniquely determined in many cases, must be chosen and represents another ambiguity in the way from a classical to the quantum theory). Suppose that this
problem is solved. Then we have some elements of the algebra $\bar{A}$—which will again be called relations—that should be represented by zero operators. It can happen that some of the relations lies in the center of $\bar{A}$; this was observed by Pohlmayer in the cases of a massive relativistic particle on Minkowski spacetime and of the string theory $^2$. In this form, some constraints may reappear in the quantum theory.

The last step in the quantization is to find a unitary representation $R$ of the Lie group $G$ on a Hilbert space $K$ that satisfies the conditions

1. the representation $R$ of all FCC subgroups of $G$ is irreducible;
2. all relations are represented by zero operators.

The second conditions is sensible, because any unitary representation of a Lie group will induce a representation of its Lie algebra by operators which have a common linear invariant domain in the representation space; this domain is the well-known Gårding subspace. Thus, the representation of the Lie algebra can be extended to that of the universal enveloping algebra. In addition, the operators representing the elements of the Lie algebra are essentially self-adjoint on the Gårding domain (representations of topological groups are automatically assumed to be continuous, cf. $^2$). The algebraic quantization method ($^1$) proceeds in a different (but more or less equivalent) way: the relations generate an ideal $I_R$ in the algebra $A$; then, one is to look for the representations of the algebra $A/I_R$. We must use a different procedure, because we are looking for a representation of a group (and the group structure does not contain information about relations); our procedure can be quite practical, however: the relations that lie in the center of the algebra can give the Casimir operators of the group some definite values. Then, the physical representation is determined or limited strongly (for examples, see $^2$).

4 Time evolution

In this section, we will generalize the construction of the time evolution as described in $^3$. The key idea in $^3$ is to introduce an auxiliary rest frame in the phase space and to describe the movement of the system with respect to this frame. The rest frame is constructed in such a way that the resulting time evolution reproduces the usual results for parametrized systems with well-known time evolution.

4.1 General theory

Let $\{\Gamma_t\}$ be a family of transversal surfaces and $t \in \mathcal{T}$, where $\mathcal{T}$ is an index set (it can contain just two elements, it can coincide with the real axis, etc.). There is a symplectic form $\Omega_t$ associated with each $t$ as described in section $^3$. Thus, we have
the symplectic manifolds \((\Gamma_t, \Omega_t)\), which will be called \textit{time levels}. Let \(\vartheta_{tt'} : \Gamma_t \rightarrow \Gamma_{t'}\) be a symplectic diffeomorphism for each pair \((t, t')\); this maps will be called \textit{time shifts}. Finally, the system \(\{\Gamma_t, \vartheta_{tt'}\}\) is called \textit{auxiliary rest frame}. A dynamical trajectory of the system with respect of the auxiliary rest frame can be defined as follows. Let \(\gamma\) be a c-orbit (a maximal classical solution in all possible gauges and foliations). Suppose that \(\gamma \cap \Gamma_t \neq \emptyset\) for all \(t \in \mathcal{T}\). Then \(\gamma\) determines a map \(\eta_\gamma : \mathcal{T} \rightarrow \Gamma\) by
\[
\eta_\gamma(t) = \gamma \cap \Gamma_t \quad \forall t \in \mathcal{T},
\]
and this map will be called \textit{dynamical trajectory}.

The time shifts define what might be intuitively described as “the same measurements at different times”. Let \(o_t\) be a perennial whose value is measurable at the time level \(\Gamma_t\). Thus, \(o\) is associated with a particular measurement at this time level (an apparatus in a particular position, etc.). We define the same measurement at the time level \(\Gamma_{t'}\) by the perennial \(\theta_{tt'}o\) that is given by the relation
\[
(\theta_{tt'}o)|_{\Gamma_{t'}} = o|_{\Gamma_t} \circ \vartheta_{tt'}^{-1}.
\]
(The initial datum for \(\theta_{tt'}o\) at \(\Gamma_{t'}\) is obtained by mapping that of \(o\) at \(\Gamma_t\) by \(\vartheta_{tt'}\) to \(\Gamma_{t'}\).) The map \(\theta_{tt'} : C^\infty(\Gamma_t) \rightarrow C^\infty(\Gamma_{t'})\) is a Poisson algebra homomorphism (\(\vartheta_{tt'}\) is a symplectic diffeomorphism). We will denote the results of the time shifting described above by \(o_{t'}\).

Finally, the time evolution of the system is the \(t\)-dependence of the results of the same measurements made along the dynamical trajectory of the system. Thus, it is given by the \(t\)-functions \(o_t(\eta_\gamma(t))\). All this is analogous to the ideas in [8], but the time shifts used in [8] were much more special: they were defined by a one-dimensional symmetry group.

The above way of defining the time shifts seems to be the most general one in the following sense. If we assume that each measurement at a given time level is represented by a perennial and that two systems of the same measurements at different time levels are to be represented by permannials with the same Poisson bracket algebra, then the time shift must be a symplectic diffeomorphism between the two time levels. This follows from the following proposition:

**Proposition 1** Let \((\Gamma_1, \Omega_1)\) and \((\Gamma_2, \Omega_2)\) be two symplectic manifolds, \(\vartheta : \Gamma_1 \rightarrow \Gamma_2\) a diffeomorphism, \(S_1\) a set of functions which separates points at \(\Gamma_1\), and let \(\vartheta\) preserve the Poisson brackets,
\[
\{f, g\}_1 = \{f \circ \vartheta^{-1}, g \circ \vartheta^{-1}\}_2
\]
for any two elements \(f\) and \(g\) of \(S_1\). Then,
\[
\Omega_1 = \vartheta^*\Omega_2.
\]
vector field \( X \) in \( U \) such that \( \langle X, df \rangle = 0 \) for all \( f \in S \) and each \( p \in U \), because all differentials are smooth forms. As a consequence, \( f \) is constant along any integral curve of \( X \) in \( U \) for any \( f \in S \). However, then \( S \) does not separate points of the curve. Next, let \( \Omega \) and \( \Omega' \) be two symplectic forms on \( \Gamma \); if \( S \) is a set of functions whose differentials span \( T_p^* \Gamma \) at \( p \in \Gamma \); and if \( \{ f, g \}_p = \{ f, g \}'_p \) for all functions \( f \) and \( g \) from \( S \) at \( p \), then \( \Omega(p) = \Omega'(p) \). Indeed, let \( J : T_p^* \Gamma \mapsto T_p^* \Gamma \) be defined by \( \langle \xi, X \rangle = \Omega(X, J(\xi)) \); \( J \) is a linear isomorphism. Define \( \Omega^{-1} : T_p^* \Gamma \times T_p^* \Gamma \mapsto \mathbb{R} \) by \( \Omega^{-1}(X, Y) = \Omega(J^{-1}X, J^{-1}Y) \). \( \Omega^{-1} \) is a non-degenerated skew-symmetric two-form on \( T_p \Gamma \times T_p \Gamma \), uniquely determined by \( \Omega \) and satisfying the relation \( \{ f, g \}' = \Omega^{-1}(df, dg) \). Similarly, \( \{ f, g \}' = \Omega'^{-1}(df, dg) \). It follows that \( \Omega^{-1} = \Omega'^{-1} \) and this implies that \( \Omega = \Omega' \). Finally, let \( \Omega_2' = \vartheta^{-1*} \Omega_1' ; \Omega_2' \) and \( \Omega_2 \) are two symplectic forms on \( \Gamma_2 \). Define \( S_2 \) by \( S_2 = \{ f \in C^\infty(\Gamma_2, \mathbb{R}) | f = f_1 \circ \vartheta^{-1}, \ f_1 \in S_1 \} \). As \( \vartheta \) is a diffeomorphism, \( S_2 \) separates points on \( \Gamma_2 \). Hence, \( \Omega_2' = \Omega_2 \) on a dense subset of \( \Gamma_2 \). However, \( \Omega_2' \) and \( \Omega_2 \) are smooth. Thus, they are equal everywhere on \( \Gamma_2 \), QED.

The next task is to calculate the numbers \( o_t(\eta_t(t)) \). For this purpose, the information represented by the two \( t \)-functions \( o_t \) and \( \eta_t(t) \) is somewhat superfluous and we are lead to the Schroedinger and Heisenberg pictures of dynamics (within the classical theory). In general, the Heisenberg phase space \((\Gamma_H, \Omega_H)\) will not be the same as the Schroedinger one \((\Gamma_S, \Omega_S)\).

To construct \((\Gamma_S, \Omega_S)\), we consider the set \( \Gamma := \cup_{t \in T} \{ \Gamma_t \} \) and the equivalence relation \( \sim_S \) on \( \Gamma \) defined as follows: \( p \sim_S q \) if there is \( (t, t') \in T \times T \) such that \( q = \vartheta_{tt'} p \). Then, \( \Gamma_S := \Gamma / \sim_S \). As \( \vartheta_{tt'} \) is a diffeomorphism between \( \Gamma_t \) and \( \Gamma_{t'} \), \( \Gamma_S \) is diffeomorphic to any of \( \Gamma_t \)'s. As \( \vartheta_{tt'} \) is symplectic map, \( \Omega_S \) is well-defined on \( \Gamma_S \). The class \( \{ o_t \} := \{ o_t | t \in T \} \) of perennials defines a unique function on \( \Gamma_S \), as \( o_t \) and \( o_{t'} \) are related by the pasting \( \vartheta_{tt'} \); let us denote this function by \( o_S \) and call it Schroedinger observable. Any dynamical trajectory \( \eta_t \) defines the map \( \eta_t^S : T \mapsto \Gamma_S \); this will be called Schroedinger trajectory of the system. We obtain easily that

\[
o_t(\eta_t(t)) = o_S(\eta_t^S(t)).\]

For the construction of \((\Gamma_H, \Omega_H)\), the procedure is analogous, but the relation \( \sim_H \) is defined by the maps \( \rho_{tt'} : \Gamma_t \mapsto \Gamma_{t'} \) where \( \rho_{tt'} = \pi_{t'}|_{\Gamma_t} \) and \( \pi_{t'} \) is the projector associated with transversal surface \( \Gamma_{t'} \). The resulting manifold \( \Gamma_H \) is not necessarily Hausdorff. If the domains of all \( \Gamma_t \)'s cover \( \Gamma \), then \( \Gamma_H \) coincides with the quotient space \( \Gamma / \gamma \). Again, there is a well-defined symplectic form \( \Omega_H \), because the maps \( \rho_{tt'} \) are symplectic (see [3]). A dynamical trajectory \( \eta_t \) defines a unique point \( \eta^H_t \) on \( \Gamma_H \).
as $\rho_{tt'}$ acts along the $c$-orbits. Any perennial $o'$ defines a function, $o_H$, on $\Gamma_H$, as it is invariant with respect to $\rho_{tt'}$. Thus, the perennials $o_t$ define the set of functions $o_t^H := o_t[H]$, which is called Heisenberg observable. We easily find that

$$o_t(\eta_\tau(t)) = o_t^H(\eta_\tau^H).$$

We observe that the whole class $\{o_t\}$ of perennials collapses into one observable of Schroedinger or Heisenberg type. This gives us the motivation to call such classes observables. In fact, a proposal to distinguish between observables and perennials is not new. It has been made by Kuchař [25]. His proposal is, however, not equivalent to ours, because Kuchař defines observables in a different way. Some discussion of these and related questions is contained in the section 5.

The construction of quantum evolution follows closely the classical one. Let us assume in this subsection that all transversal surfaces are global; for the modifications due to non-global surfaces, see subsection 4.3. Then, we obtain that $(\Gamma_S, \Omega_S) \cong (\Gamma_H, \Omega_H) \cong (\Gamma_0, \Omega_0)$, where 0 symbolizes a fixed element of $\mathcal{T}$ and $\cong$ is the isomorphism of symplectic manifolds. From the definition of $\theta_{ts}$ it follows immediately that $o^H_t = \theta_{0t} o^S = \theta^{st} o^H_s$. The maps $\theta_{st}$ have the physical meaning of time evolution maps for the classical Heisenberg picture; they have to be taken over into the quantum theory. Recall that we have the representation $R: S \rightarrow L(K)$ already at our disposal. If $\theta_{st} S \subset S$ (this happens in linear theories, like quantum field theory on curved background, cf. [4]), then it is straightforward to define $\hat{\theta}_{st}$ by the commuting diagram:

$$\begin{array}{ccc}
S & \xrightarrow{\theta_{st}} & S \\
\downarrow R & & \downarrow R \\
L(K) & \xrightarrow{\hat{\theta}_{st}} & L(K)
\end{array}$$

In the opposite case, one has to choose one element of the algebra $\tilde{A}$ for each $o \in S$, $s \in \mathcal{T}$ and $t \in \mathcal{T}$ to play the role of $\theta_{st}(o)$ (which is an element of $\mathcal{P}$; this is another factor ordering problem). The result would be a map $\theta^a_{st}: S \mapsto \tilde{A}$. There are some reasonable restriction on this map $\theta^a_{st}$ (or else the choice is practically unlimited!): we require the following two conditions:

1. $\theta^a_{st}$ is a Lie-algebra isomorphism of $S$ and $\theta^a_{st}(S)$,

2. $\theta^a_{st} = \theta^a_{ts} \circ \theta^a_{su}$ for all $s, t$ and $u$ for which the equation $\theta_{st} = \theta_{ut} \circ \theta_{su}$ is satisfied.

Then the corresponding quantum map $\hat{\theta}_{st}$ is defined by the following diagram

$$\begin{array}{ccc}
S & \xrightarrow{\theta^a_{st}} & \tilde{A} \\
\downarrow R & & \downarrow R \\
L(K) & \xrightarrow{\hat{\theta}_{st}} & L(K)
\end{array}$$
because the representation $R$ can be extended to the algebra $\bar{A}$.

Having the quantum map $\hat{\theta}_{st}$, we can attempt to implement it by a unitary map $U_{st} : K \mapsto K$ so that $\hat{\theta}_{st}(\hat{o}) = U_{st}\hat{o}U_{st}^{-1}$ for $\hat{o} \in L(K)$. \{\{U_{st}\}\} is the system of unitary evolution operators for the system, and the construction of the (quantum) Schroedinger and Heisenberg picture can be completed in a straightforward way.

We will clarify and develop the general concepts as introduced in this section by studying some particular cases.

4.2 Continuous, one dimensional case

For the sake of simplicity, we will assume in this subsection that all transversal surfaces are global. This assumption can easily be removed by working within the domain of a non-global surface.

Let \{\{\Gamma_t\}\} be a one-dimensional differentiable family of global transversal surfaces; then $\mathcal{T} = \mathbb{R}$. Let $\vartheta_t : \Gamma_0 \mapsto \Gamma_t$ be a symplectic diffeomorphism for each $t$ such that $\vartheta_t(p)$ is a smooth curve for each $p \in \Gamma_0$; these curves define the “rest trajectories”.

Each ordered pair of time levels defines the time shift by $\vartheta_{tt'} = \vartheta_{t'} \circ \vartheta_t^{-1}$. Any dynamical trajectory is a curve $\eta_\gamma : \mathbb{R} \mapsto \Gamma$. This curve is a classical solution in a particular gauge and foliation; this is why it is a one-dimensional object. A perennial $o$ measurable at $\Gamma_0$ defines an observable \{\{o_t\}\} as described in section 4.1.

The Schroedinger phase space is isomorphic to $(\Gamma_0, \Omega_0)$, the Schroedinger trajectory $\eta^S_\gamma(t)$ is obtained by $\eta^S_\gamma(t) = \theta_t^{-1}\eta_\gamma(t)$ and the Schroedinger observable is given by $o_S = \theta_t^{-1}(o_t) = o_0$. The Heisenberg phase space is also isomorphic to $(\Gamma_0, \Omega_0)$ as the maps $\rho_{tt'}$ are symplectic diffeomorphisms. Each c-orbit $\gamma$ defines a Heisenberg trajectory, the point $\eta^H_\gamma = \gamma \cap \Gamma_0$. Each observable \{\{o_t\}\} defines the Heisenberg observable $o^H_t$ on $\Gamma_0$ by projecting each perennial $o_t$ to $\Gamma_0$: $o^H_t = o_t|_{\Gamma_0}$. The $t$-functions $\eta^S_\gamma(t)$ and $o^H_t$ satisfy ordinary differential equations, which we are going to derive.

The Schroedinger trajectories define a set of maps $\chi_{tt'} : \Gamma_0 \mapsto \Gamma_0$ on the Schroedinger phase space as follows. Let $p \in \Gamma_0$ and $(t, t') \in \mathbb{R}^2$; then

$$\chi_{tt'} := \theta_{t'}^{-1} \circ \rho_{0t'} \circ \rho_{0t}^{-1} \circ \theta_t. \quad (1)$$

From this definition, it follows directly:

1. the relation $\eta^S_\gamma(t') = \chi_{tt'}(\eta^S_\gamma(t))$,  
2. that $\chi_{tt'}$ is a symplectic diffeomorphism for each $(t, t') \in \mathbb{R}^2$,  
3. the composition law $\chi_{tt'} = \chi_{st'} \circ \chi_{ts}$ for all $(t, t', s) \in \mathbb{R}^3$. 

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In particular, $\chi_{ts}^{-1} = \chi_{st}$. However, $\chi_{st} \neq \chi_{0(t-s)}$, in general, i. e., the set of maps $\chi_0$ does not form a group (it is no flow!).

It is easy to prove that $\dot{\eta}_{S\gamma}(t) = X^S_t(\dot{\eta}_{S\gamma}(t))$, where $X^S_t$ is a locally Hamiltonian vector field, as $\chi_{st}$ is symplectic and because of Eq. (1). Suppose that $X^S_t$ is globally Hamiltonian; let us call the auxiliary rest frame Hamiltonian in this case. Then there is a function $H^S_t : \Gamma_0 \mapsto \mathbb{R}$ such that $X^S_t$ is its Hamiltonian vector field, and we have

$$\dot{\eta}^S_t(t) = X^S_t.$$  \hfill (2)

The set of functions $\{ H^S_t | t \in \mathbb{R} \}$ is called Schrodinger Hamiltonian and Eq. (2) is the Schrodinger equation of motion.

For the Heisenberg observable $o^H_t$, we have that $o^H_t(p) = o_t(\rho_{0t}(p))$ and $o_t(\rho_{0t}(p)) = o_0((\theta^{-1}_t \circ \rho_{0t})(p))$. Thus,

$$o^H_t = o_S \circ \chi_{0t},$$  \hfill (3)

because $o_0 = o_S$. Let us calculate the derivative of $o^H_t$ with respect to $t$; using Eq. (3), we obtain:

$$\dot{\eta}^H_t(p) = (X^S_t \cdot o_S)\chi_{0t}(p)) = (\chi_{0t}^{-1}X^S_t)(o_s \circ \chi_{0t})|_p = X^H_t \cdot o^H_t,$$  \hfill (4)

where $X \cdot f$ is the action of the vector field $X$ (as a differential operator) on the function $f$ and $X^H_t = \chi_{0t}^{-1}X^S_t$; as $X^S_t$ is a Hamiltonian vector field of $H^S_t$ and $\chi_{0t}$ is a symplectic diffeomorphism, $X^H_t$ is a Hamiltonian vector field of the function

$$H^H_t = H^S_t \circ \chi_{0t}.$$  \hfill (5)

The set of functions $\{ H^H_t | t \in \mathbb{R} \}$ is called Heisenberg Hamiltonian and Eq. (5) implies the Heisenberg equation of motion

$$\dot{o}^H_t = \{ o^H_t, H^H_t \}.$$  \hfill (6)

Now, we can return to the discussion of the relation between integrals of motion and perennials. Clearly, each perennial $o$ defines a function on the reduced phase space $\Gamma_0$ by $i^*_0 o$. Eq. (3) shows that $i^*_0 o$ is an integral of motion, if the time shifts are chosen so as to preserve $o$. Thus, any given perennial can become an integral for some time evolution.

### 4.2.1 An example

Let $\tilde{\Gamma}$ be $\mathbb{R}^{2n+2}$ with canonical coordinates $T, P, q^1, \ldots, q^n, p_1, \ldots, p_n$ and let $\tilde{\Omega} = dP \wedge dT + dp_k \wedge dq^k$. The constraint surface is given by the equation $C = 0$, where $C$ is a differentiable function on $\tilde{\Gamma}$; let the equation $C = 0$ be equivalent to

$$P = -\mathcal{H}(T, q^1, \ldots, q^n, p_1, \ldots, p_n),$$  \hfill (7)
where $\mathcal{H}$ is a smooth function on $\mathbb{R}^{2n+1}$. This defines our first-class parametrized system.

We choose an auxiliary rest frame as follows. Let $\Gamma_t$ be the image of embedding $\mathbb{R}^{2n}$ with canonical coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$ by the embedding maps $i_t$ into $\tilde{\Gamma}$ that are given by

$$i_t(x^1, \ldots, x^n, y_1, \ldots, y_n) = (t, -\mathcal{H}(t, x^1, \ldots, x^n, y_1, \ldots, y_n), x^1, \ldots, x^n, y_1, \ldots, y_n).$$

Observe that $\mathcal{H}(x^1, \ldots, x^n, y_1, \ldots, y_n) = -P|_{\Gamma_t}$. Clearly, $\Omega_t = dy_k \wedge dx^k$. Let the maps $\vartheta_t$ be given by $\vartheta_t = i_t \circ i_t^{-1}$.

A tangent vector field $L$ to the c-orbits is easily calculated from the constraint in the form (7):

$$L = \frac{\partial}{\partial T} - \frac{\partial \mathcal{H}}{\partial T} \frac{\partial}{\partial P} + \frac{\partial \mathcal{H}}{\partial p_k} \frac{\partial}{\partial q^k} - \frac{\partial \mathcal{H}}{\partial q^k} \frac{\partial}{\partial p_k}.$$

For the Schroedinger dynamical trajectory, we obtain simply that $\eta^S_t = \vartheta_t \circ \vartheta_t^{-1}$; thus, the tangent vector $X^S_t$ to this trajectory is given by $X^S_t(p) = \vartheta_t^{-1} L(\vartheta_t(p))$, which results in

$$X^S_t = \left( \frac{\partial \mathcal{H}}{\partial y_k} \frac{\partial}{\partial x^k} - \frac{\partial \mathcal{H}}{\partial x^k} \frac{\partial}{\partial y_k} \right)_{T=t}.$$

It follows that the Schroedinger Hamiltonian is

$$H^S(t, x^1, \ldots, x^n, y_1, \ldots, y_n) = \vartheta_t^* \mathcal{H}|_{T=t}.$$

Observe that the family of rest trajectories is not generated by $P$ in general.

Next, we define the family of perennials $Q^k_t$ and $P_{tk}$ by

$$Q^k_t|_{\Gamma_t} = x^k, \quad P_{tk}|_{\Gamma_t} = y_k.$$ 

Clearly, $Q^k_s = \theta_{ts} Q^k_t$ and $P_{sk} = \theta_{ts} P_{tk}$ for any pair $(t, s)$ of real numbers. For each value of $t$, we obtain a complete system of perennials with a well-known algebra.

The Heisenberg Hamiltonian $H^H(p) = H^S(\chi_0(p))$ is not available in explicit form, as $\chi_0$ can only be obtained by integrating the differential equation $d\chi_0/dt = X^S_t$.

The procedure described in this subsection is, on one hand, equivalent to that in [8], if the time shifts $\vartheta_t$ are generated by a perennial $\vartheta$; then $H^S = H^H = -h|_{\Gamma_0}$.

On the other hand, the example shows that it is related to the so-called reduction procedure, which is the reversal of the parametrization procedure that was described in section 2, see also [13] or [14].

### 4.3 Non-global transversal surfaces

Consider the following situation. Let $\Gamma_1$ and $\Gamma_2$ be two transversal surfaces; let $\Gamma'_1 := \Gamma_1 \cap \mathcal{D}(\Gamma_2)$ and $\Gamma'_2 := \Gamma_2 \cap \mathcal{D}(\Gamma_1)$; let $\rho : \Gamma'_1 \rightarrow \Gamma'_2$ be given by $\rho := \pi_2|_{\Gamma'_1}$,
where \( \pi'_{2} \) is the projector associated with \( \Gamma'_2 \); and finally let \( \vartheta : \Gamma_1 \mapsto \Gamma_2 \) be a time shift (\( \rho \) and \( \vartheta \) are symplectic diffeomorphisms). Our aim is to construct a quantum time evolution from the time level \( \Gamma_1 \) to \( \Gamma_2 \) in the general case when \( \Gamma''_1 \) and \( \Gamma'_2 \) are proper submanifolds of \( \Gamma_1 \) and \( \Gamma_2 \).

In order to get some hint of how one can proceed and how the problem is to be posed, let us stay within the classical theory and consider an evolution of an ensemble of classical systems on \( \Gamma \); let this ensemble be described by a measure \( \mu \) on \( \Gamma/\gamma \); let \( \mu_1 \) and \( \mu_2 \) be the measures induced by \( \mu \) on \( \Gamma_1 \) and \( \Gamma_2 \), respectively.

The problem can now be posed as follows. Suppose that we can control the input only at \( \Gamma_1 \), and that we can measure the output only at \( \Gamma_2 \). In particular, we can prepare the \( \Gamma_1 \)-part of the ensemble arbitrarily so that \( \mu_1 \) can be normalized, \( \int_{\Gamma_1} d\mu_1 = 1 \). Which perennials have then a mean value at \( \Gamma_2 \) that is calculable from the knowledge of \( \mu_1 \)? Let us first study a simpler question: Suppose that a transversal surface \( \Gamma_0 \), not necessarily global, and the measure \( \mu \) are given. Which perennial has a mean value calculable from what is known at \( \Gamma_0 \)? The problem is that the data at \( \Gamma_0 \) do not determine the measure outside of \( D(\Gamma_0)/\gamma \) so that the mean value of a perennial that does not vanish there is not determined. This leads to the following definition.

**Definition 2** The perennial \( o \) is called pertinent to the transversal surface \( \Gamma_0 \), if

1. \( o(\Gamma \setminus D(\Gamma_0)) = 0 \),
2. the Hamiltonian vector field \( \xi_o \) of \( o \) is complete on \( D(\Gamma_0) \).

Then clearly,

\[
\text{mean}(o) = \int_{\Gamma_0} d\mu_1 \cdot o.
\]  

(8)

For example, if the perennial \( o \) generates a symmetry group which leaves \( D(\Gamma_0) \) invariant, then \( \Pi_{\Gamma_0} o \) is pertinent to \( \Gamma_0 \). The condition 1 is sufficient for Eq. (8) to hold, but the condition 2 will turn out to be vital for the quantum theory.

If \( o_1 \) is pertinent to \( \Gamma_1 \) then \( o_2 \) := \( \vartheta o_1 \) is pertinent to \( \Gamma_2 \). The pair \( (o_1, o_2) \) is an observable associated with the auxiliary rest frame \( (\Gamma_1, \Gamma_2, \vartheta) \).

Let us study the evolution of the mean values of observables. Let \( (o_1, o_2) \) be an observable; we want to calculate \( \text{mean}(o_2) \) using only \( \mu_1 \), \( o_1 \), \( \rho \) and \( \vartheta \). \( o_2 \) is determined everywhere at \( \Gamma_2 \) by these data, but \( \mu_2 \) is determined only at \( \Gamma'_2 \):

\[
\mu_2|_{\Gamma'_2} = \rho^{-1*}(\mu_1);
\]

the rest \( \mu_2|_{\Gamma_2 \setminus \Gamma'_2} \) of \( \mu_2 \), which is not controlled from \( \Gamma_1 \), can be considered as noise; we assume that it is completely independent of \( \mu_1 \). Thus, \( o_2 \) must be pertinent to \( \Gamma'_2 \) and \( o_1 \) to \( \Gamma'_1 := \vartheta^{-1}(\Gamma'_2) \).
Let $\mathcal{P}_{\Gamma'_2}$ and $\mathcal{P}_{\Gamma'_1}$ be the algebras of all perennials that are pertinent to $\Gamma'_2$ and $\Gamma'_1$, respectively. We have:

$$\theta \mathcal{P}_{\Gamma'_1} = \mathcal{P}_{\Gamma'_2}.$$ 

Moreover,

$$\mathcal{P}_{\Gamma'_2} = \mathcal{P}_{\Gamma''_1} \subset \mathcal{P}_{\Gamma_1}.$$ 

Thus, $\mathcal{P}_{\Gamma'_2}$ is determined by generators of $\mathcal{P}_{\Gamma_1}$ (even by those of the subalgebra $\mathcal{P}_{\Gamma''_1}$ of $\mathcal{P}_{\Gamma_1}$).

In the (classical) Schroedinger picture, the two transversal surfaces $\Gamma_1$ and $\Gamma_2$ are identified by $\vartheta$ to, say, $\Gamma_1$. $\Gamma''_1$ remains as it is and $\Gamma'_2$ becomes to $\Gamma'_1$. The map $\chi : \Gamma''_1 \mapsto \Gamma'_1$ was defined in the subsection [1] by $\chi = \vartheta^{-1} \circ \rho$. Let $(o_1, o_2)$ be an observable such that $o_2$ pertains to $\Gamma'_2$. Then it holds that

$$\int_{\Gamma'_2} d\mu_2 o_2 = \int_{\Gamma'_1} d(\chi^{-1} \mu_1) o_1.$$ 

Thus, the evolution is given by the map $\chi^{-1}$ of $\mu_1$.

In the (classical) Heisenberg picture, we identify $\Gamma_1$ and $\Gamma_2$ by $\rho$ along $\Gamma''_1$ and $\Gamma'_2$. There is only one measure, $\mu$. Only the time shifts of the observables from $\mathcal{P}_{\Gamma'_1}$ screen the noise automatically; their images by the time shift $\theta$ lie in $\mathcal{P}_{\Gamma'_2}$. The mean values calculated in the Schroedinger picture coincide with the corresponding Heisenberg picture ones.

The analysis above suggests that the following groups and algebras will play an important role in the quantization. Let $G_1$ and $G_2$ be two groups of symplectic diffeomorphisms acting on $\Gamma_1$ and $\Gamma_2$, respectively, and let $S_1$ and $S_2$ be the Lie algebras of functions on $\Gamma_1$ and $\Gamma_2$ that generate these groups via Poisson brackets. The groups $G_1$ and $G_2$ may result as projections to $\Gamma_1$ and $\Gamma_2$ of some groups of symmetries in $\tilde{\Gamma}$. The functions from $S_1$ and $S_2$ define perennials with the same algebras and we will denote these algebras of perennials by the same symbols. Let the groups $G_1$ and $G_2$ satisfy the following requirements:

1. $G_2 = \{ \vartheta \circ g \circ \vartheta^{-1} \mid g \in G_1 \}$,

2. $S_2 = \{ o \circ \vartheta^{-1} \mid o \in S_1 \}$.

Thus, the groups $G_1$ and $G_2$ are isomorphic, and their actions are related by $\vartheta$. Let $G'_2 \subset G_2$ be the subgroup which preserves $\Gamma'_2$. $G'_2$ acts on $\Gamma'_2 \in \Gamma_2$, but it has also an action $a_1$ on $\Gamma'_1$, because it preserves the common domain of $\Gamma'_2$ and $\Gamma''_1$; $a_1$ is defined by $a_1(g) := \rho^{-1} \circ g \circ \rho$ for all $g \in G'_2$. Thus, $a_1(G'_2)$ acts on $\Gamma_1$, but it is no subgroup of $G_1$, in general. The algebra of perennials that generate $G'_2$ will be denoted by $S'_2$. The projections of the perennials from $S'_2$ generate the action of $G'_2$ on both $\Gamma'_2$ and $\Gamma''_1$. Finally, $G'_1$ is the subgroup of $G_1$ which is related by $\vartheta$ to $G'_2$, that is
$G'_1 := \{ \vartheta^{-1} \circ g \circ \vartheta | g \in G'_2 \}$. Then $G'_1$ preserves $\Gamma'_1$. It follows that each element $g_2 \in G'_2$ defines an element $g_1 \in G'_1$ such that $a_1(g_2) = \rho^{-1} \circ \vartheta \circ g_1 \circ \vartheta^{-1} \circ \rho = \chi^{-1} \circ g_1 \circ \chi$.

Let $S'_1$ be the algebra of perennials that generates $G'_1$; then $S'_2 = \theta S'_1$. For the projections of the algebras, we obtain easily $S'_2|_{\Gamma_1} = \{ o_1 \circ \chi^{-1} | o_1 \in S'_1|_{\Gamma_1} \}$.

As the elements of $\theta S'_1$ do not lie in $S''_1$, we have to look for them in the universal enveloping algebra $\bar{A}_1$ of $S_1$. Suppose that we have solved this “factor-ordering problem”. Let us denote by $\theta_o(o)$ the element of $\bar{A}_1$ which is associated in this way with $o \in S'_1$.

The construction of the corresponding quantum mechanical evolution is based on an analogous problem setting: if we can prepare a state at the time level $\Gamma_1$, what can be said about measurements at the time level $\Gamma_2$? The answer can be worked out with the tools of the ordinary quantum mechanics and it consists of the following steps.

1. With the two phase spaces $\Gamma_1$ and $\Gamma_2$, we associate the Hilbert spaces $K_1$ and $K_2$ and the representations $R_1$ and $R_2$ of the groups and algebras, $R_1 : G_1 \mapsto L(K_1)$ and $R_2 : G_2 \mapsto L(K_2)$. These are unitarily equivalent, irreducible unitary representations, and let the unitary equivalence be realized by the map $U(\vartheta) : K_1 \mapsto K_2$. In most cases, one just takes two copies of the same representation, so the search for $U(\vartheta)$ is trivial.

2. We try to find the Hilbert subspaces that correspond to the symplectic manifolds $\Gamma'_1$, $\Gamma''_1$ and $\Gamma'_2$ using the method described in [18]. Consider $\Gamma'_1$. The representation $R_1 : G'_1 \mapsto L(K_1)$ is not (in general) irreducible. Thus, $K_1$ decomposes into irreducible representations subspaces. Each such subspace is usually characterized by values of invariants (in particular, the Casimirs elements) of $G'_1$. A comparison with the classical values of these invariants on $\Gamma'_1$ helps to identify the subspace $K'_1$ that corresponds to the classical submanifold $\Gamma'_1$ in the quantum mechanics. Similarly for $\Gamma''_1$ and $\Gamma'_2$ we find the subspaces $K''_1 \subset K_1$ and $K'_2 \subset K_2$. Clearly, $K'_2 = U(\vartheta)K'_1$.

3. The construction of the Schrödinger picture proceeds by identifying the Hilbert spaces $K_1$ and $K_2$ using the map $U(\vartheta)$. Then $\chi : \Gamma''_1 \mapsto \Gamma'_1$ is to be implemented by a unitary map $U(\chi) : K''_1 \mapsto K'_1$. $\chi$ is a symplectic diffeomorphism with domain $\Gamma''_1$ that may be singular at the boundary $\partial \Gamma''_1$. One possible method is to look for a function $h$ on $\Gamma_1$ (it may be singular at the boundary) that generates a flow such that the map $\chi$ is the element of the flow at the value 1 of the flow parameter. Then the factor order problem has to be solved: the function $h$ is to be identified with an element $h_o$ of $\bar{A}_1$. Finally, we set $U(\chi) = \exp(R_1h_o)|_{K''_1}$. An example in which this method works is given in section [3]. The dynamics in the Schrödinger picture is given by
$U(\chi)$ in the Schroedinger Hilbert space $K_1$. As such, it is not a unitary map in general: formally, neither its domain nor its range coincide with $K_1$; less formally, the evolution of the state $\psi \in K_1$ is given by $U(\chi)P_1'\psi$, where $P_1'$ is the projection operator on the subspace $K_1'$ of $K_1$. Thus, if $\psi$ has norm 1, its time evolution will have norm $\leq 1$.

4. To construct the Heisenberg picture, we have to identify the Hilbert spaces $K_1$ and $K_2$ along the subspaces $K_1''$ and $K_2'$ using an implementation of the symplectic diffeomorphism $\rho : \Gamma_1'' \mapsto \Gamma_2'$. As $\rho = \vartheta \circ \chi$, we can set $U(\rho) = U(\vartheta) \circ U(\chi)$ utilizing our knowledge of the map $U(\chi)$. An alternative way is to define the map $\hat{\theta}$ by the commuting diagram

$$
\begin{array}{ccc}
S_1' & \xrightarrow{\theta_a} & \tilde{A}_1 \\
\downarrow R_1 & & \downarrow R_1 \\
L(K_1') & \xrightarrow{\hat{\theta}} & L(K_1)
\end{array}
$$

Then we attempt to implement $\hat{\theta}$ by a unitary map $U(\chi)$ so that $\hat{\theta}(\hat{\phi}) = U(\chi) \circ \hat{\phi} \circ U^{-1}(\chi)$ for all $\hat{\phi} \in R_1(S_1')$ such that $\text{Dom}(U(\chi)) = K_1''$ and $\text{Ran}(U(\chi)) = K_1'$. This may be a problem, because $\theta_a$ is defined on a proper subalgebra of $S_1$ only. An example in which it works is described in section 3. Using $U(\chi)$, one can paste the Hilbert spaces as above finishing the construction. In the Heisenberg picture, the measurement of the observables from $\hat{\theta}(R_1S_1')$ is predictable, because they leave the subspace $K_2' = K_1''$ invariant. The expansion of any state $\psi$ of $K_2$ into the eigenvectors of these observables is well-defined even if we know only the $K_2'$-projection of $\psi$. This is the main reason behind the point 2 of the definition 1. Quantum mechanically, one can equivalently require that the elements of $S_1'$ or $S_1''$ commute with the projectors $P_1'$ or $P_1''$; then, one can always multiply the elements of $S_1'$ or $S_1''$ by the projector $P_1$ or $P_1''$ obtaining again self-adjoint operators. Thus, one of the main features of the Heisenberg picture—the time-independence of the states—can be preserved if we limit ourselves to the measurement of just the observables that are pertinent to $K_2'$. If $G_1'$ or $G_1''$ do not act transitively on $\Gamma_1'$ or $\Gamma_1''$ or if $S_1'$ or $S_1''$ do not separate points in $\Gamma_1'$ or $\Gamma_1''$, then the system of measurements defined by the observables from $S_1'$ or $S_1''$ is not complete in $K_1'$ or $K_1''$ and the genuine Heisenberg picture of a complete quantum evolution cannot be constructed. However, one can pass to a kind of a mixed picture instead. One can obtain a complete information by performing measurements corresponding to the elements of the algebra $S_2$ that is pertinent to the whole space $K_2$, if one can screen away the noise from the states by the projection operator $P_2'$ before these measurements are done (for an example of such a case, see section 3).
Thus, the time evolution of the states is given by the projection and that of
the observables by the map \( U(\vartheta) \) (which coincides with \( U(\chi) \) on \( K'' \), because
\( U(\rho) \) is an identity).

One may be able to find pathological classical systems for which this construction
cannot be performed, but we hope that it will work in physically interesting cases.

5 Meaning of perennials

The perennial formalism is based on two ideas:

- Study the systems whose time evolution is well-understood like Newtonian
  systems \([8]\), the massive particle in Minkowski spacetime \([4]\) or the scalar field
  in curved spacetime \([4]\). These systems all posses a background spacetime and
  some structure of this spacetime plays a crucial role in the construction of
  quantum evolution.

- Replace this spacetime structure by or transform it into some phase space
  structure so that the quantum time evolution of the systems can be recon-
  structed *solely from some phase space objects*. Try to use similar phase space
  objects to construct a quantum time evolution for systems without any back-
  ground spacetime.

The approach seems a little formal in comparison with attempts to reconstruct time
by using some physical system playing the role of a clock \([24]\) or in which time is to
emerge in the semiclassical approximation \([27]\). The hope is that we can reconcile
our approach with these attempts (this is a project for future research).

A key mathematical notion that keeps everything together and allows elegant
proofs and formulations is that of a perennial. The perennial formalism is a kind of
language that is adequate to describe the relevant structure of parametrized systems.
However, there has been some discussion in the literature about perennials (or about
equivalent notions), cf. \([28]\), \([24]\), \([29]\), \([30]\), or \([15]\). What is the relation of our
perennial formalism to the ideas that come out of this discussion?

Two problems were already discussed: that of existence of perennials (section \([4]\),
and that of having explicit expressions for perennials \([8]\).

A very important point is the relation between perennials and observables. A
thorough discussion of this relation is given in \([28]\). The conclusion was that “One
can observe dynamical variables which are not perennial, and...Perennials are often
difficult to observe.” The results of the present paper support Kuchař’ opinion in
that the perennials and observables turn out to be two different notions in general.
More precisely, if we are looking for the classical counterparts of *quantum mechanical*
observables—which possess the Schroedinger and Heisenberg forms—then these are
definitely not perennials, because some “time information” is contained in them
(it is an interesting question to be studied whether or not there are observable
quantities of different kind). We have identified such observables with classes of
perennials, each two elements of which are time shifted with respect to each other.
It seems to follow that perennials are in principle measurable, but only in relation to
a particular instant of time (in general, to a transversal surface; for systems equipped
with a unique time, to a particular instant of that time): the value of a perennial at
a given time coincides with the value of an observable that contains the perennial
at the time as an element of the corresponding class. This seems to be a natural
consequence of our approach. However, this touches another controversy. For those
who would consider perennials as exactly analogous to “gauge-invariant quantites”
of gauge theories, the way of their measurement must also be “gauge-invariant”; that
is, it is either not associated with any time instant (which is, in fact, a particular
location at a “gauge orbit”) at all, or it can be performed at any time instant with
the same result. This would also apply to the quantum version of the theory, and
for this version, a very interesting counterexample has been constructed by Kuchař
[31]. Suppose that perennials turn to be observables in quantum theory that are
measurable at any time instant and that the results of such measurements of one
and the same perennial at different time instants are time independent. Consider a
set of non-commuting perennials. Let us perform two measurements of all perennials
in the set, each in a different time order. From the assumptions, it follows that the
two measurements must give the same result. This, however, contradicts the basic
postulates of the quantum theory of measurement. The counterexample seems to
speak in favour of the distinction between observables and perennials as it results
from our theory. The next remark concerns the nature of observables. The form that
the observables obtain in this paper (namely, classes of perennials) is not the only
form possible. They may be equivalently described in a way that makes no reference
to perennials. An example is provided by the system studied in the section [1.2.1].
There, e. g. the classes \{Q^k_t\} of the perennials Q^k_t are observables; each such class is
determined by the coordinate function x^k (assuming the time foliation as known); the
coordinate x^k would provide such an equivalent (but non-geometrical) description
of the observable. How may such an object be measurable at all being no “gauge
invariant?” The old discussion of this problem is nicely summarized in [30]. Briefly,
a quantity \(x\) that is not gauge invariant within a given model A can be associated
with another quantity \(y\) of a model B such that \(y\) is gauge invariant within B and
acquires the same (or approximately the same) values as \(x\) in the same physical
situation. The system B contains the system A as a subsystem together with some
auxiliary matter system (“material reference frames”). For the measurement of \(x\),
the coupling of A to the auxiliary matter system is in any case necessary. Thus, what is measurable in a given model A is determined by all possible couplings to other models, not just by A itself.

Finally, there has been some discussion about perennials of a particular form, namely “evolving constants of motion”: roughly, such a perennial is the value of a quantity taken at the hypersurface in the phase space that is defined as a level of some other quantity (reference quantity), see [15]. One problem with these perennials is that they are too complicated functions (for general reference quantities) to be easily representable by quantum operators; they will be (continuous) functions with diverging derivatives; their Hamiltonian vector fields will practically never be complete, etc. (It seems also that the perennial defined e. g. as the coordinate the system had at 5 o’clock is measurable only at five o’clock, cf. previous paragraph.) A deeper critics of such quantities is contained in [29]: a general reference quantity will often lead to a perennial that describes a dynamically very involved information so that its time ordering is not well defined. One has to restrict the reference quantities to the so-called “good time functions”, etc. We have to deal with these objections, because the perennial formalism also uses quantities analogous to the evolving constants—in fact, the “observables” are a kind of such evolving constants, and the perennials that are defined by their “initial values” at some transversal surface are similar to them, too. However, the reference quantity in all these cases is chosen such that its levels are transversal surfaces. It seems then that it must be “a good time function”, but this is still to be studied in more detail.

6 A system without global transversal surfaces

6.1 The model

An example of a system that did not admit global transversal surfaces was studied in [18]. This system possessed, however, connected transversal surfaces that were almost global: their domains were dense in the constraint surface. The quantum theory of this system did not exhibit, however, much consequence of the complicated topology of the classical model; this could be shown in [18]. In the present paper, we will give a more interesting example: there will be inextensible connected transversal surfaces whose domains will be “small” parts of the constraint surface.

The phase space \( \tilde{\Gamma} \) is \( \mathbb{R}^4 \) with the natural coordinates \( q_1, q_2, p_1, p_2 \) and the symplectic form is given by \( \tilde{\Omega} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \). The constraint surface is the hyperboloid given by the equation \( C = 0, \) where

\[
C = p_1^2 - p_2^2 - q_1^2 - q_2^2 + 1.
\]
The system is completely integrable as there are two integrals of motion that are in involution; let us denote these integrals as follows: \( A := (1/2)(p_1^2 - q_1^2) \) and \( B := (1/2)(p_2^2 + q_2^2) \). Thus, the c-orbits will lie at the cylinders \( A = \text{const}, B = \text{const} \); their projections to the \((p_1, q_1)\)-plane are hyperbolas \( A = \text{const} \) and those to the \((p_2, q_2)\)-plane are circles \( B = \text{const} \). The general solution of the equations of motion is easily found: the c-orbit through the point \((\bar{q}_1, \bar{p}_1, \bar{q}_2, \bar{p}_2)\) is given by the equations:

\[
\begin{align*}
q_1 &= \bar{q}_1 \cosh t + \bar{p}_1 \sinh t, \\
p_1 &= \bar{q}_1 \sinh t + \bar{p}_1 \cosh t, \\
q_2 &= \bar{q}_2 \cos t - \bar{p}_2 \sin t, \\
p_2 &= \bar{q}_2 \sin t + \bar{p}_2 \cos t.
\end{align*}
\]

The set \( E := \{ X \in \Gamma | A = 0 \} \) plays a very special role (at \( E, B = 2 \)); there is one critical c-orbit \( E_0 \in E \) with \( p_1 = q_1 = 0 \) and \( 4 \times S^1 \) exceptional (imprisoned) orbits on the four separating manifolds \( E_1 \subset E, E_2 \subset E, E_3 \subset E \) and \( E_4 \subset E \) defined as follows:

\[
\begin{align*}
E_1 : \quad &p_1 - q_1 = 0, \quad p_1 + q_1 > 0, \\
E_2 : \quad &p_1 - q_1 = 0, \quad p_1 + q_1 < 0, \\
E_3 : \quad &p_1 + q_1 = 0, \quad p_1 - q_1 > 0, \\
E_4 : \quad &p_1 + q_1 = 0, \quad p_1 - q_1 < 0.
\end{align*}
\]

They separate the constraint surface into four quadrants \( T_{13}, T_{14}, T_{23} \) and \( T_{24} \), each \( T_{ab} \) lying between the two separating manifolds \( E_a \) and \( E_b \). \((\Gamma \setminus E_0)\) is a (non-Hausdorff) manifold and the sets \((T_{13} \cup E_1 \cup T_{14})/\gamma, (T_{23} \cup E_2 \cup T_{24})/\gamma, (T_{23} \cup E_3 \cup T_{13})/\gamma \) and \((T_{24} \cup E_4 \cup T_{14})/\gamma\) form its maximal Hausdorff submanifolds.

The equations \((9), (10), (11) \) and \((12)\) imply the following statement: Let \( \{ \gamma_n \} \) be a sequence of c-orbits within the quadrant \( T_{ab} \) that converges pointwise to a c-orbit at \( E_a \), and let \( p \) be any point of \( E_0 \cup E_b \), a set at the boundary of \( T_{ab} \). Then there is a sequence \( p_n \) such that \( p_n \in \gamma_n \forall n \) and \( \lim_{n \to \infty} p_n = p \). It follows that the space \( \Gamma/\gamma \) is non-Hausdorff, each two c-orbits at \( E \) being non-separable (that is: each neighbourhood of the first c-orbit intersects each neighbourhood of the second one, cf. section \[3\]). Moreover, it follows that each analytical perennial \( o \) must have the form \( o = f(A, B) \). Indeed, any continuous perennial must be constant along the set \( E \). Consider a point \( X \in E \setminus E_0 \). In a neighbourhood \( U \) of \( X \), \( A \) and \( B \) are two independent analytical functions that are constant along \( E \cap U \); any two other functions \( x^1 \) and \( x^2 \) that form an analytical chart together with \( A \) and \( B \) in \( U \) must not be constant along \( E \cap U \). Any analytical function \( F \) can be written in \( U \) as \( f(A, B, x^1, x^2) \), where \( f \) is analytical. However, \( F \) will be constant along \( E \cap U \) only if \( f \) does not depend on \( x^1 \) and \( x^2 \), which proves the claim. The next
consequence is that there is no complete system of perennials (i.e. that separates separable c-orbits) that will all be analytical: indeed, $A$ and $B$ are not independent on $\Gamma$, and we need at least two perennials to form a complete system. We will use singular perennials and symmetries that will be associated with transversal surfaces.

6.2 Transversal surfaces

From the fact that the set $\Gamma/\gamma$ is non-Hausdorff, it follows that there is no \textit{global} transversal surface (see [32]). The next interesting kind of transversal surface is the inextensible connected one: such surfaces play, for example, a key role in the Hawking effect [9]. In our model, the following four surfaces, $\Gamma_i$, $i = 1, 2, 3, 4$, are of this kind; their domains together cover $\Gamma \setminus E_0$, they are all of the topology $\mathbb{R}^2$ and they can be defined by equations: $\Gamma_1$ by $p_1 - q_1 = 1$, $\Gamma_2$ by $p_1 - q_1 = -1$, $\Gamma_3$ by $p_1 + q_1 = 1$, and $\Gamma_4$ by $p_1 + q_1 = -1$, together with the constraint equation, $C = 0$. Observe that the sets $\mathcal{D}(\Gamma_i)$, $i = 1, 2, 3, 4$ coincide with the maximal Hausdorff submanifolds. The disconnected transversal surfaces $\Gamma_1 \cup \Gamma_2$ and $\Gamma_3 \cup \Gamma_4$ are almost global.

We will construct a time evolution between the surfaces $\Gamma_1$ and $\Gamma_4$ and so illustrate the procedure described in section 4.3. As for the choice of the two surfaces, let us just remark that $\Gamma_4$ lies in the future of $\Gamma_1$, if one takes seriously the time-orientation of the c-orbits that is defined by the Hamiltonian vector field of the constraint function $C$. Let the natural coordinates be $(x_1, y_1)$ on $\Gamma_1$ and $(x_4, y_4)$ on $\Gamma_4$, and let the injection maps be given for $\Gamma_1$ by

$$q_1 = B_1 - 1, \quad p_1 = B_1, \quad q_2 = x_1, \quad p_2 = y_1,$$

and for $\Gamma_4$ by

$$q_1 = B_4 - 1, \quad p_1 = -B_4, \quad q_2 = x_4, \quad p_2 = y_4,$$

where $B_i := \frac{1}{2}(y_i^2 + x_i^2)$, $i = 1, 4$. For the pull-back $\Omega_i$ of the symplectic form $\tilde{\Omega}$ we obtain simply $\Omega_i = dy_i \wedge dx_i$.

The time shift $\vartheta : \Gamma_1 \mapsto \Gamma_4$ can be defined by

$$x_4(\vartheta(x_1, y_1)) = x_1, \quad y_4(\vartheta(x_1, y_1)) = y_1.$$  \hspace{1cm} (13)

This is a “natural” choice, because $\vartheta$ is to define the same measurements at the different time levels $\Gamma_4$ and $\Gamma_1$, and the coordinates $(x_i, y_i)$ coincide at each of these surfaces with the values of the phase functions $(q_2, p_2)$; one usually assumes that the same symbol is used in the canonical formalism to denote a quantity which is always measured in the same way. The dynamical map $\rho : \Gamma_1 \mapsto \Gamma_4$ has the domain $\Gamma_1'' := \{(x_1, y_1) \in \Gamma_1 | B_1 < \frac{1}{2}\}$ and the range $\Gamma_4' := \{(x_4, y_4) \in \Gamma_4 | B_4 < \frac{1}{2}\}$.

24
$\Gamma''_1$ coincides with $\Gamma'_1 := \vartheta^{-1}\Gamma'_4$ in this case. Let us observe that the symplectic manifold $(\Gamma'_k, \Omega_k)$—a disc of a finite symplectic volume—does not admit any Lie (that is, finite-dimensional) group of symplectic diffeomorphisms; this can be shown by studying the candidate Lie algebras. The coordinate expression for the map $\rho$ can easily be obtained from the Eqs. (9–12):

\begin{align*}
x_4(\rho(x_1, y_1)) &= x_1 \cos T_1 + y_1 \sin T_1, \quad (15) \\
y_4(\rho(x_1, y_1)) &= -x_1 \sin T_1 + y_1, \cos T_1 \quad (16)
\end{align*}

where

$$T_1 := \log(1 - 2B_1). \quad (17)$$

Thus, $T_1$ diverges at the boundary of the domain $\Gamma'_1$ of $\rho$. Finally, the map $\chi : \Gamma'_1 \mapsto \Gamma'_1$, defined by $\chi = \vartheta^{-1} \circ \rho$ has the following expression in the coordinates

\begin{align*}
x_1(\chi(x_1, y_1)) &= x_1 \cos T_1 + y_1 \sin T_1, \quad (18) \\
y_1(\chi(x_1, y_1)) &= -x_1 \sin T_1 + y_1 \cos T_1. \quad (19)
\end{align*}

Next, we prove a property of the map $\chi$ that will be important for the quantum implementation of this map by one of the methods described in section 4.3. Let $f$ be a function with a complete Hamiltonian vector field $\xi_f$ and let the flow of $\xi_f$ be denoted by $\Phi[f]_t$. Then the map $\chi$ satisfies the equation

$$\chi = \Phi[h]_1, \quad (20)$$

where

$$h = \frac{1 - 2B_1}{2} \log \frac{1 - 2B_1}{e}, \quad (21)$$

and $e$ is the basis of natural logarithms. To show this property, we consider the family of curves defined by

\begin{align*}
x'_1 &= \bar{x}_1 \cos \bar{T}_1 t + \bar{y}_1 \sin \bar{T}_1 t, \quad (22) \\
y'_1 &= -\bar{x}_1 \sin \bar{T}_1 t + \bar{y}_1 \cos \bar{T}_1 t, \quad (23)
\end{align*}

t $\in \mathbb{R}$, each starting for $t = 0$ at the point $\bar{x}_1, \bar{y}_1$; $\bar{T}_1$ is the function defined by Eq. (17) with the arguments $\bar{x}_1$ and $\bar{y}_1$. The tangent vector $(\hat{x}_1, \hat{y}_1)$ to the curve at the point $(x_1, y_1)$ is

\begin{align*}
\hat{x}_1 &= -\bar{x}_1 \bar{T}_1 \sin \bar{T}_1 t + \bar{y}_1 \bar{T}_1 \cos \bar{T}_1 t, \\
\hat{y}_1 &= -\bar{x}_1 \bar{T}_1 \cos \bar{T}_1 t + \bar{y}_1 \bar{T}_1 \sin \bar{T}_1 t.
\end{align*}

The Eqs. (22) and (23) imply that $B_1(x'_1, y'_1) = B_1(\bar{x}_1, \bar{y}_1)$, so $T_1(x'_1, y'_1) = T_1(\bar{x}_1, \bar{y}_1)$, and so we obtain that

\begin{align*}
\hat{x}_1 &= T_1 y_1, \quad (24) \\
\hat{y}_1 &= -T_1 x_1. \quad (25)
\end{align*}
It follows that the curves (22) and (23) are identical with the flow of the vector field (24) and (25); moreover, \( \chi \) is an element of this flow. Thus, we have to find a function \( h \) such that

\[
\dot{x}_1 = \{x_1, h\}_1 = \frac{\partial h}{\partial y_1},
\]

\[
\dot{y}_1 = \{y_1, h\}_1 = -\frac{\partial h}{\partial x_1},
\]

where \( \{\cdot, \cdot\}_1 \) denotes the Poisson bracket of the symplectic manifold \( (\Gamma_1, \Omega_1) \) (cf section 2). An obvious ansatz \( h = h(B_1) \) leads to the desired result, Eq. (21).

We have all classical maps that we need for the construction of the time evolution between the two surfaces \( \Gamma_1 \) and \( \Gamma_4 \). What is still missing are algebras of elementary perennials and/or first-class canonical groups. We will construct some such algebras first, and then look which groups they generate. The simplest procedure is to define the singular perennials \( X_i \) and \( Y_i \) by their initial data along the transversal surfaces \( \Gamma_i \) as follows:

\[
X_i|_{\Gamma_i} = x_i,
\]

\[
Y_i|_{\Gamma_i} = y_i.
\]

An easy calculation using the Eqs. (9–12) gives the following results

\[
X_1 = q_2 \cos T_+ - p_2 \sin T_+,
\]

\[
Y_1 = q_2 \sin T_+ + p_2 \cos T_+,
\]

for \( p_1 - q_1 > 0 \) and \( X_1 = Y_1 = 0 \) (30) for \( p_1 - q_1 < 0 \);

\[
X_4 = q_2 \cos T_+ + p_2 \sin T_+,
\]

\[
Y_4 = -q_2 \sin T_+ + p_2 \cos T_+,
\]

for \( p_1 + q_1 < 0 \) and \( X_4 = Y_4 = 0 \) (33) for \( p_1 + q_1 > 0 \); here,

\[
T_\pm = \log |p_1 \pm q_1|.
\]

The perennials \( X_i \) and \( Y_i \) are pertinent (see section 4.3) to the surface \( \Gamma_i, i = 1, 4 \), and they are singular at \( p_1 - q_1 = 0 \) for \( i = 1 \) and at \( p_1 + q_1 = 0 \) for \( i = 4 \). Indeed, the Hamiltonian vector fields of these perennials are complete (this is the only property of pertinent perennials which is non-trivial to prove); we can show
this as follows. The Eqs. (28) and (29) imply immediately that \( \{ X_1, p_1 - q_1 \} = \{ Y_1, p_1 - q_1 \} = 0 \). Hence, the Hamiltonian vector fields of these functions are tangent to the planes \( p_1 - q_1 = \text{const} \) and their integral curves can never meet the singularity at \( p_1 - q_1 = 0 \). Inside these planes, the vector fields can easily be integrated and found to be complete. The common domain of the perennials \( X_1 \) and \( Y_1 \) is \( \tilde{\Gamma} \setminus (E_3 \cup E_0 \cup E_4) \); together with the perennial \( B \), they generate the four-dimensional “harmonic oscillator Lie algebra”, which we will denote by \( S_1 \). From Eqs. (28) and (29), a relation follows, namely \( B = \frac{1}{2}(X_1^2 + Y_1^2) \). \( S_1 \) generates, in turn, the four-dimensional harmonic oscillator group with the same common invariant domain. We will call this group \( G_1 \). Similarly for the other two perennials \( X_4 \) and \( Y_4 \): they define another copy of the harmonic oscillator algebra \( S_4 \) with the common domain \( \tilde{\Gamma} \setminus (E_1 \cup E_0 \cup E_2) \) and another copy of the harmonic oscillator group \( G_4 \). Observe that the groups must be kept segregated, because the elements of one move the domain of the other so that all transformations that result from composition of the elements of the two groups would have no common domain at all. The groups \( G_1 \) and \( G_4 \) have a common subgroup that is generated by \( B \); in accordance with the rules of section 4.3, this subgroup can be denoted by \( G_1' \) or \( G_4' \), because it is the subgroup that leaves the submanifolds \( \Gamma_1' \) and \( \Gamma_4' \) invariant (without acting transitively on them).

The definitions above imply that \( \theta S_1 = S_4 \) and \( \varphi G_1 \varphi^{-1} = G_4 \). It is easy to construct the (classical) Schroedinger and the Heisenberg phase spaces and the time evolution according to the prescription given in the section 4.3. We pass directly to the quantum mechanics.

### 6.3 Quantum mechanics

As quantum mechanical counterparts of the phase spaces \( (\Gamma_k, \Omega_k) \), let us consider two Hilbert spaces \( K_k \) together with harmonic oscillator annihilation operators \( a_k \), \( k = 1, 4 \), acting in the well-known way. In particular, there is a basis \( \{ \psi_k^n \} \) in \( K_k \) \( n = 0, 1, \ldots \), \( k = 1, 4 \) such that

\[
\begin{align*}
    a_k \psi_k^n &= \sqrt{n\hbar} \psi_k^{n-1}, \\
    a_k^\dagger \psi_k^n &= \sqrt{(n+1)\hbar} \psi_k^{n+1},
\end{align*}
\]

and the algebra \( S_k \) is represented on \( K_k \) by

\[
\begin{align*}
    \hat{X}_k &= \frac{i}{\sqrt{2}}(a_k - a_k^\dagger), \\
    \hat{Y}_k &= \frac{1}{\sqrt{2}}(a_k + a_k^\dagger), \\
    \hat{B} &= a_k^\dagger a_k + \frac{1}{2} \hbar.
\end{align*}
\]
The two representations of the corresponding groups are irreducible and equivalent; the map $U(\vartheta)$ which realizes the equivalence and implements the time shift $\vartheta$ is given by

$$U(\vartheta)\psi_n^1 = \psi_n^4.$$ 

The next step is to find the subspaces $K'_k$ which are the quantum counterparts of the submanifolds $\Gamma'_k$. This is straightforward: the states $\psi_k^h$ are the eigenstates of the operator $\hat{B}$ with eigenvalues $\hbar(n + \frac{1}{2})$, and they also define the invariant subspaces of the group $G'_k$. This is analogous to the classical observable $B$ generating the group $G'_k$ that leaves the submanifolds $\Gamma'_k$ invariant so that the orbits of the group are defined by $B = \text{const}$ with $B < \frac{1}{2}$. Thus we can identify $K'_k$ with the subspace spanned by the states $\psi_k^h$ with $\hbar(2n + 1) < 1$; let us denote the projection operator onto these subspaces by $P'_k$. Then, we can directly implement the map $\chi$ because of the relations (20) and (21): let us set $U(\chi) = \exp(i\hat{h})$ on $K'_1$, that is:

$$U(\chi)\psi_n^1 = \left(\frac{\hbar(2n + 1) - 1}{e}\right)^{-\frac{1}{\hbar(2n+1)-1}}\psi_n^1.$$  

(35)

for all $n < (\frac{1}{2\hbar} - \frac{1}{2})$. The Schroedinger dynamics is then defined by the evolution operator $U(\chi)P'_1$ on $K'_1$. The perennials that are pertinent to $\Gamma'_1$ form just a one-dimensional space spanned by $B$. The time evolution of the operator $\hat{B}$ by $U(\chi)$ is trivial: $U(\chi)\hat{B}U^{-1}(\chi) = \hat{B}$. This is in fact all to be said about the Heisenberg picture. However, the change of phases defined by Eq. (35) is measurable: one has to screen the “chaos” in the states by the operator $P'_2$ and then just perform measurements corresponding to the observables from the algebra $S_2$.

Thus, our model nicely illustrates sections 2 and 4.3; it is intriguing, how the necessarily bizarre properties resulted from the extremely simple definition equations of the system.

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