A generalization of c-Supplementation *

Shiheng Li, Dengfeng Liang, Wujie Shi
School of Mathematic Science, Suzhou University, Suzhou 215006, China
E-mail: lishiheng01@163.com; dengfengliang@163.com; wjshi@suda.edu.cn

Abstract

A subgroup $H$ is said to be nc-supplemented in a group $G$ if there is a subgroup $K \leq G$ such that $HK \trianglelefteq G$ and $H \cap K$ is contained in $H_G$, the core of $H$ in $G$. We characterize the solvability of finite groups $G$ with some subgroups of Sylow subgroups nc-supplemented in $G$. We also give a result on c-supplemented subgroups.

Keywords solvable, nc-supplemented, $p$-nilpotent.

Wang[6] introduces the notation of c-supplemented subgroups and determines the structure of finite groups $G$ with some subgroups of Sylow subgroups c-supplemented in $G$. Here we give a new concept called nc-supplementation that is a weak version of c-supplementation and characterize the solvability of groups $G$ with some maximal or 2-maximal subgroups nc-supplemented in $G$, respectively.

In this paper, $\pi$ denotes a set of primes. We say $G \in E_\pi$ if $G$ has a Hall $\pi$-subgroup; $G \in C_\pi$ if $G \in E_\pi$ and any two Hall $\pi$-subgroups of $G$ are conjugate in $G$; $G \in D_\pi$ if $G \in C_\pi$ and every $\pi$-subgroup of $G$ is contained in a Hall $\pi$-subgroups of $G$. We say that a number $n$ is a $\pi$-number if every of its prime divisor is in $\pi$. $|G|_\pi$ denotes the largest $\pi$-number that divides $|G|$. $H < \cdot G$ denotes that $H$ is a maximal subgroup of $G$. $L$ is called a 2-maximal subgroup of $G$, if there exists a maximal subgroup $M$ of $G$ such that $L < \cdot M$.

Definition 1. Let $G$ be a group and $H$ be a subgroup of $G$.

(1) $H$ is said to be c-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $HK = G$ and $H \cap K \leq H_G$, where $H_G = \bigcap_{g \in G} H^g$ is the core of $H$ in $G$. We say that $K$ is a c-supplement of $H$ in $G$.

(2) $H$ is said to be nc-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $H \cap K \leq H_G$. We say that $K$ is a nc-supplement of $H$ in $G$. $\square$

Remark 1. Let $H$ be nc-supplemented in $G$. Evidently, $H$ is c-supplemented in $G$ if $H$ is simple; $H$ is c-supplemented in $G$ if $H$ is a maximal subgroup of $G$. $\square$

In general, nc-supplementation does not implies c-supplementation.

Example. Let $G = A_4$ and $B = \{(1), (12)(34), (13)(24), (14)(23)\}$. Set $C = \{(1), (12)(34)\}$ and $D = \{(1), (14)(23)\}$. Then $C \times D = B \trianglelefteq G$ and $C$ is nc-supplemented in $G$. However, $C$ is not c-supplemented since $C_G = 1$ and $A_4$ has no subgroup of order 6. $\square$

Lemma 1. If $H$ is nc-supplemented in $G$, then there exists a subgroup $C$ of $G$ such that $H \cap C = H_G$ and $HC \trianglelefteq G$.

*Supported by the National Natural Science Foundation of China (Grant No. 10171074).
Suppose \( H \) is \( nc \)-supplemented in \( G \). Then there is a subgroup \( C_1 \leq G \) such that \( H \cap C_1 \leq H_G \) and \( H C_1 \leq G \). Let \( C = C_1 H_G \). Then \( HC = (H C_1) H_G \leq G \) and \( H \cap C = H \cap C_1 H_G = H_G (H \cap C_1) = H_G \).

**Lemma 2.** Let \( H \) be \( nc \)-supplemented in \( G \).

1. If \( H \leq H \leq G \), then \( H \) is \( nc \)-supplemented in \( M \).
2. If \( N \leq H \) and \( N \leq H \), then \( H/N \) is \( nc \)-supplemented in \( G/N \).
3. If \( N \leq G \) and \( (|N|, |H|) = 1 \), then \( H/N \) is \( nc \)-supplemented in \( G/N \).

**Proof.** Similar to the argument in the proof of [6, Lemma 2.1 (1) and (2)], we get (1) and (2), respectively.

Similar to the argument in the proof of [6, Theorem 2.2], we get the following result by computing \(|G|_2\):

**Lemma 3** [1, Proposition 2.1]. If \( K \) is a normal subgroup of the group \( G \) such that \( K \leq C_\pi \) and \( G/K \leq C_\pi \), then \( G \in E_\pi \).

From [7, p.485, Theorem] we get the following result by computing \(|G|_2\):

**Lemma 4.** Let \( G \) be a simple group having a Sylow 2-subgroup isomorphic to \( C_2 \times C_2 \). Then \( G \cong L_2(q) \), where \( q \equiv 3 \pmod{8} \) or \( q \equiv 5 \pmod{8} \).

**Theorem 1.** Let \( G \) be a finite group and let \( P \) be a Sylow \( p \)-subgroup of \( G \), where \( p \) is a prime divisor of \( |G| \). Suppose that there is a maximal subgroup \( P_1 \) of \( P \) such that \( P_1 \) is \( nc \)-supplemented in \( G \).

1. If \( P_1 \neq 1 \), then \( G \) is not a non-Abelian simple group.
2. If \( P_1 \neq 1 \), then \( G \in E_2' \) and every composition factor of \( G \) is either a cyclic group of prime order or isomorphic to \( L_2(r) \), where \( r = 2^p - 1 \) is a Mersenne prime.

**Proof.** (a) Suppose \( P_1 \) is \( nc \)-supplemented in \( G \). If \( G \) is simple then \( P_1 \) is \( c \)-supplemented in \( G \). By [6, Theorem 2.2], it follows that \( G \) is not simple, a contradiction.

(b) Suppose \( p = 2 \). If \( (P_1)_G \neq 1 \), then \( G/(P_1)_G \) satisfies the hypothesis by Lemma 2. Hence \( G/(P_1)_G \in E_2' \) by induction on \( |G| \) and thus \( G \in E_2' \) by Lemma 3. So we may assume that \( (P_1)_G = 1 \). Since \( P_1 \) is \( nc \)-supplemented in \( G \), there is a subgroup \( K \) of \( G \) such that \( P_1 K \leq G \) and \( P_1 \cap K \leq (P_1)_G = 1 \). Let \( N = P_1 K \). Then \( G/N \) is solvable since \( |G/N|_2 \leq 2 \).

**Remark 2.** In (a) of Theorem 1, the hypothesis \( P_1 \neq 1 \) is necessary. For example, \( G = L_2(7) \) and \( p = 7 \). The example also shows that the hypothesis \( P_1 \neq 1 \) should be in the first conclusion in [6, Theorem 2.2]. Otherwise, from the example it is certain that the conclusion that \( G : K = p^r, r \geq 1 \), in the proof of [6, Theorem 2.2], is impossible.

**Theorem 2.** Let \( G \) be a finite group. Then \( G \) is solvable if and only if every Sylow subgroup of \( G \) is \( nc \)-supplemented in \( G \).

**Proof.** If \( G \) is solvable, then every Sylow subgroup of \( G \) has a complement in \( G \) and thus \( nc \)-supplemented in \( G \).
Conversely, suppose that $G$ is a counterexample of smallest order.

(1) If $N \leq G$, then $G/N$ is solvable.

Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$. Then there is a subgroup $C$ of $G$ such that $PC \leq G$ and $P \cap C \leq G$. Since $|PC : C| = |P : P \cap C|$ and $P \in Syl_pG$, $(|PC : C|, |PC : P|) = 1$. Hence $PC \cap N = (N \cap P)(N \cap C)$ by [2, A.1.2 Lemma]. Thus $NP \cap NC = N(P \cap C)$ by [2, A.1.6 Lemma (c)] and $PN/N \cap NC/N = N(P \cap C)/N \leq PGN/N \leq (PN/N)_{G/N}$. On the other hand, $(PN/N)/C(N/N) = (PC/N)/G/N$. And for every Sylow $p$-subgroup $S/N$ of $G/N$, we set $P \in Syl_pS$. Then $P \in Syl_pG$ and $PN/N = S/N \in Syl_pG/N$. Therefore, $G/N$ satisfies the hypothesis of the theorem. Then $G/N$ is solvable since $G$ is a counterexample of smallest order.

(2) $G$ has a unique minimal normal subgroup $N$, $\Phi(G) = 1$ and $O_p(G) = 1$ for any $p||G|$. Since the class of all solvable groups is a saturated formation, $G$ has only one minimal subgroup $N$ and $\Phi(G) = 1$ by (1). If $O_p(G) \neq 1$, then $G/O_p(G)$ is solvable by (1) and $G$ is solvable, which contradicts that $G$ is a counterexample.

For any $p||G|$, and $P \in Syl_pG$, there exists a subgroup $C$ of $G$ such that $PC \leq G$ and $P \cap C \leq O_p(G) = 1$ by our hypothesis and (2). Then $PC \leq N$ by (2) and $C$ is a $p$-complement of $PC$. Thus $C \cap N$ is a $p$-complement of $N$ for any $p||G|$ and $N$ is solvable by [2, I.3.5 Theorem]. Hence $G$ is solvable, which contradicts that $G$ is a counterexample. The final contradiction completes the proof. 

Theorem 3. Let $G$ be a finite group and let $P$ be a Sylow 2-subgroup of $G$. Suppose that every maximal subgroup of $P$ is no-supplemented in $G$. Then $G$ is solvable.

Proof Assume that $G$ is a counterexample of smallest order. In particular, $G$ is non-
solvable.

(1) $O_2(G) = 1$ and $O_2'(G) = 1$.

Assume that $O_2(G) \neq 1$. Then $G/O_2(G)$ either is of odd order or satisfies the hypothesis of the theorem by Lemma 2. In the first case $G/O_2(G)$ is solvable by the odd order theorem. In the second case $G/O_2(G)$ is also solvable since $G$ is a counterexample of smallest order. Thus in both cases $G/O_2(G)$ is solvable, $G$ is also solvable, a contradiction.

Assume that $O_2'(G) \neq 1$. Then $G/O_2'(G)$ satisfies the hypothesis of the theorem by Lemma 2 and thus $G/O_2'(G)$ is solvable since $G$ is a counterexample of smallest order. In addition $O_2'(G)$ is solvable by the odd order theorem again. Hence $G$ is solvable, a contradiction.

(2) $G$ has a unique minimal normal subgroup $N$ and $N$ is a direct product of some simple
groups, which are isomorphic to each other. Moreover, $G = PN$.

Let $N$ be a minimal normal subgroup of $G$. We consider $PN$.

We assume that $PN < G$. By Lemma 2 $PN$ satisfies the hypothesis of the theorem, then $PN$ is solvable since $G$ is a counterexample of smallest order. In particular, $N$ is solvable. Then either $O_2(N) \neq 1$ or $O_2'(N) \neq 1$ and thus either $O_2(G) \geq O_2(N) > 1$ or $O_2'(G) \geq O_2'(N) > 1$, which contradicts (1). Now we in the case $PN = G$. Then $G/N \cong G/P \cap N$ is solvable. Since the class of all solvable groups is a saturated formation, $G$ has a unique minimal normal subgroup $N$. Evidently $N$ is not solvable and $N$ is a direct product of some simple groups, which are isomorphic with each other.

(3) The final contradiction.

Let $P_1$ be a maximal subgroup of $P$. Then there is a $K \leq G$ such that $P_1K \leq G$ and $P_1 \cap K \leq (P_1)_G \leq O_2(G) = 1$ by (1). Thus $|K|_2 \leq 2$, $K$ has a normal 2-complement $K_2$. Evidently, $K_2$ is also a Hall $2'$-subgroup of $P_1K$. In addition $N \leq P_1K$ by (2) since $P_1 \neq 1$. Hence $K_2 \cap N$ is a 2-complement of $N$. On the other hand $G = PN$. So $K_2'$
is a 2-complement of $N$ and $G$. Set $N_2 = P \cap N$, $H = N_G(K_2)$ and $P' = P \cap H$. Then $N_2 \in \text{Syl}_2 N$, $G = P \cap N = PK_2$ and $G = NH = N_2 H$ by Frattini argument and [3, Theorem A]. So $P = P \cap (N_2 H) = N_2 (P \cap H) = N_2 P'$. If $G$ is simple, then $G$ is solvable by [6, Corollary 3.2] and Remark 1. Now we assume that $G$ is not simple. Then $P > N_2$ and $P' \neq 1$. Since $G = N_2 H = PH$, $P' = P \cap H \in \text{Syl}_2 H$. Since $H < G$ by (1), $P > P'$. Then there is a maximal subgroup $P'_1$ of $P$ such that $P' \leq P'_1$. Then there is a $K' \leq G$ such that $P'_1 K' \unlhd G$ and $P'_1 \cap K' \leq (P'_1 K') \leq O_2 (G) = 1$ by (1). By the same argument as above, with $(P'_1, K')$ in place of $(P, K)$, we get: the normal 2-complement $K''_2$ of $K'$ is also a 2-complement of $N$ and $G$, and $P'_1 K''_2 \geq N$. Then $P'_1 K''_2 = (P'_1 N_2 K''_2) = PK''_2 = G$ since $P'' \leq P'_1$. Since $G = PK''_2$, we may assume $K''_2 = K'_2$ by [5, VI,4.5] and [3, Theorem A]. Then $K' \leq N_G(K'_2) = H$ and $G = P'_1 K''_2 = P'_1 H = P'_1 P'' K''_2 = P'_1 K'_2$. Hence $|G| = |P'_1|/|K'_2| < |P|/|K_2| = |G|$, a contradiction. The final contradiction completes the proof. □

With Lemma 4, by the same argument as in the proof of theorem 3, we get the following:

**Theorem 4.** Let $G$ be a finite group and $P$ a Sylow 2-subgroup of $G$. If every 2-maximal subgroup of $P$ is $nc$-supplemented in $G$ and $L_2(q)$-free, where $q \equiv 3(\text{mod } 8)$ or $q \equiv 5(\text{mod } 8)$, then $G$ is solvable.  

**Remark 3.** With the condition $nc$-supplemented in place of the condition $c$-supplemented in [6, Theorem 3.1 and Theorem 4.2], we can get that $G$ is solvable by Theorem 3 and Theorem 4 respectively, but cannot conclude that $G/O_p(G)$ is $p$-nilpotent. For example:

Let $G = H \wr \langle a \rangle$ is a wreath product of $H$ and $\langle a \rangle$, where $H = Z_7 \times Z_3$ is a semi-direct product of $Z_7$ by $Z_3$ but $H \not\cong Z_7 \times Z_3$, and $a = (1234567)$. Let $p = 3$ and $P \in \text{Syl}_p G$. Then $P$ is an elementary Abelian $p$-subgroup. For every subgroup $P_1$ of $P$ there exists a subgroup $C$ of $P$ such that $P_1 \times C = P$. Let $F = O_7 (G)$. Then $F = Z_7 \times \cdots \times Z_7$ (7 copies of $Z_7$). Hence $FC \leq G$, $(FC)P_1 = FP = H \triangleleft G$ and $FC \cap P_1 = 1$. Thus $P_1$ is $nc$-supplemented in $G$. However, $O_p(G) = 1$ and $G$ is not $p$-nilpotent. □

The following is related to $c$-supplemented subgroups.

**Lemma A**[6, Lemma 4.1]. Let $G$ be a finite group and let $p$ be a prime divisor of $|G|$ such that $(|G|, p - 1) = 1$. Assume that the order of $G$ is not divisible by $p^3$ and $G$ is $A_4$-free. Then $G$ is $p$-nilpotent. In particular, if there exists odd prime $p$ with $(|G|, p - 1) = 1$ and the order of $G$ is not divisible by $p^3$. Then $G$ is $p$-nilpotent. □

**Remark 4.** From the hypotheses of Lemma A, we cannot get that $G$ is $p$-nilpotent. For example:

Let $G = (Z_{19} \times Z_{19}) \times \langle a \rangle$ and $p = 19$, where $o(a) = 5$ and $a \in \text{Aut}(Z_{19} \times Z_{19})$. Then $(G, p)$ satisfies the hypotheses of Lemma A. However, $G$ is not $p$-nilpotent. From the example, it is certain that the conclusion that $p = 2$ and $q = 3$, in the proof of [6, Lemma 4.1], is impossible. But the mistake cannot affect the results and arguments after [6, Lemma 4.1] in [6], since Lemma A holds by [4, Lemma 3.12] if $p$ is the smallest prime divisor of $|G|$. □

From the hypotheses of Lemma A, we get the following:

**Theorem 5.** Let $G$ be a finite group and let $p$ be a prime divisor of $|G|$ such that $(|G|, p - 1) = 1$. Assume that the order of $G$ is not divisible by $p^3$ and is $A_4$-free. Then $G$ is $p$-nilpotent or $G/O_p'(G) \cong (Z_p \times Z_p) \rtimes H$, where $H \leq \text{Aut}(Z_p \times Z_p)$, and $H$ is a cyclic group whose order is odd and divides $\frac{p^4 - 1}{2}$. 

**Proof** If $O_p'(G) \neq 1$, then $G/O_p'(G)$ satisfies the hypotheses and $G/O_p'(G)$ is $p$-nilpotent or $G/O_p'(G) \cong (Z_p \times Z_p) \rtimes H$ by induction on $|G|$. So we assume $O_p'(G) = 1$.

If $|G|_p = p$, then $G$ is $p$-nilpotent by [5, VI,2.6], since $|N_G(P)/C_G(P)||\text{|Aut}(P)| = (p - 1, |G|) = (p - 1, |G|) = 1,$
where $P \in Syl_p G$. Now we assume $|G|_p = p^2$. If $p$ is the smallest prime divisor of $|G|$, then $G$ is $p$-nilpotent by [4, Lemma 3.12]. So we may assume $p$ is odd. Then $|G|$ is odd since $(|G|, p - 1) = 1$ and thus solvable by the odd order theorem. This gives $O_p(G) \neq 1$ since $O_p(G) = 1$.

In case $|O_p(G)| = p$. Then $|G/O_p(G)|_p = p$ and $G/O_p(G)$ is $p$-nilpotent as above. Let $T/O_p(G)$ be the normal $p$-complement of $G/O_p(G)$. Then $|T|_p = p$ and thus $T$ is $p$-nilpotent as above again. So $T$ has normal $p$-complement $T_p'$ and $T_p'$ is a character subgroup of $T$. Then $T_p' \leq G$ and $T_p' \leq O_p(G) = 1$. Thus $G$ is a $p$-group and thus $p$-nilpotent.

In case $|O_p(G)| = p^2$. Then $O_p(G) \in Syl_p G$. Since $O_p'(G) = 1$, $F(G) = O_p(G)$ and then $C_G(O_p(G)) = O_p(G)$ by [2, A,10.6 Theorem]. Thus $G/O_p(G) \leq Aut(O_p(G))$. If $O_p(G)$ is cyclic then $|Aut(O_p(G))| = p(p - 1)$. Thus, $|G/O_p(G)||G/O_p(G)||G/O_p(G)||G/O_p(G)||G/O_p(G)| = 1$ and $G = O_p(G)$ is a $p$-group. Now we assume $O_p(G)$ is an elementary Abelian subgroup of $(p, p)$-type. Then $|Aut(O_p(G))| = (p + 1)p(p - 1)^2$ and $|G/O_p(G)||G/O_p(G)|p^2$ since $(|G|, p - 1) = 1$. In addition, $G$ has a subgroup $H$ such that $G = HO_p(G)$ and $H \cap O_p(G) = 1$ by Shur-Zassenhaus theorem. Therefore, $G \simeq (Z_p \times Z_p) \times H$, where $H \leq Aut(Z_p \times Z_p)$, $|H||p^2$ and $|H|$ is odd. In addition, $|L_2(p)| = (p+1)p(p-1)/2$, $|Aut(Z_p \times Z_p)| = |GL_2(p)| = (p + 1)p(p - 1)^2$, and $L_2(p)$ is a section of $GL_2(p)$. Hence $H$ is isomorphic to a subgroup of $L_2(p)$ and thus $H$ is cyclic by [5, II,8.27].

References

[1] Z. Arad, E. Fisman, On finite factorizable groups, J. Algebra, 86(1984), 522-548.

[2] K. Doerk, T. Hawkes, Finite Soluble Groups, Walter de Gruyter, New York, 1992.

[3] F. Gross, Conjugacy of odd order Hall subgroups, Bull. London. Math. Soc., 19(1987), 311-319.

[4] X. Guo, K. P. Shum, Cover-avoidance properties and the structure of finite groups, J. Pure Appl. Algebra, 181(2003), 297-308.

[5] B. Huppert, Endliche Gruppen I, Spring-Verlag, New York, 1979.

[6] Y. Wang, Finite groups with some subgroups of Sylow subgroups $c$-supplemented, J. Algebra, 224(2000), 467-478.

[7] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, 1980.