THE ROCHE PROBLEM: SOME ANALYTICS

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Received 2003 October 12; accepted 2003 November 12

ABSTRACT

Exact analytical formulae are derived for the potential and mass ratio as a function of Lagrangian point positions, in the classical Roche model of close binary stars.

Subject headings: binaries: close — stars: rotation

1. INTRODUCTION

The Roche model is widely used in the interpretation of close binary star observations. Several authors have derived various approximations for solving the Roche problem and have presented numerical tables (see, e.g., Plavec & Kratochvil 1964; Kippenhahn & Thomas 1970; Paczyński 1971; Eggleton 1983; Mochnacki 1984; Morris 1985, 1994). Not intending to solve the whole problem analytically, I rather show here a way in which some analytical relations can be found. The idea is to reverse the problem: instead of finding, e.g., the first Lagrangian point $x_1$ as function of binary mass ratio $q$, we seek a solution for $q$ as function of $x_1$.

2. BASIC EQUATION

The basic equation of the classic Roche problem is the formula describing the surface of the (primary) star as an equipotential:

$$
\Psi(x, y, z) = \left(x - \frac{q}{1 + q}\right)^2 + y^2 + \frac{2q}{1 + q}\sqrt{(-1 + x)^2 + y^2 + z^2} + \frac{2}{1 + q}\sqrt{x^2 + y^2 + z^2}.
$$

I use in equation (1) the notation of Mochnacki (1984): the $x$-axis is aligned along the stars’ centers, the $z$-axis is parallel to the rotation axis, $\Psi$ is the “normalized” potential in units of $G(M_1 + M_2)/(2A)$, $q = M_2/M_1 < 1$, and $x, y$, and $z$ are in units of $A$, the distance between the centers of the binary components with masses $M_1$ and $M_2$. Since I consider here only the first (inner) and second (outer) Lagrangian points problem, it is sufficient to consider equation (1) at the $x$-axis.

3. THE FIRST LAGRANGIAN POINT

We write equation (1) with $y = z = 0$ and $0 < x < 1$ and look for the minimum of the function

$$
\Psi_1(x) = \frac{2q}{(1 + q)(1 - x)} + \frac{2}{1 + q} + \left(x - \frac{q}{1 + q}\right)^2
$$

at some $x = x_1$, with $0 < x_1 < 1$. The important observation from equation (2) is that $\Psi_1(q, x) = \Psi_1(1/q, 1 - x)$ if $0 < x < 1$ (not in the general case!). From equation (2) we find the condition $d\Psi_1(x)/dx = 0$, which we rewrite after some algebra as a function $q(x_1)$:

$$
q(x_1) = \frac{(1 - x_1)^3(1 + x_1 + x_1^2)}{x_1^2(3 - 3x_1 + x_1^2)}.
$$

We notice, from equation (3), the elegant relation (also with a clear physical meaning) $q(x_1)q(1 - x_1) = 1$. We emphasize that equation (3) gives the fully analytical and exact relation between values of $q$ and the first Lagrangian point $x_1$. See, e.g., Table 1 in Mochnacki (1984), in which $Q$ and $X_1$ correspond to our $q$ and $x_1$, respectively, and compare the difference in difficulty of calculations between the method of Mochnacki (1984) and formulae (3), (4), (6), and (9) of this paper.

Now, in order to find the value of the potential corresponding to the first Lagrangian point, we can either use equations (2) and (3) together or, in the spirit of this paper, exclude $q$ from equations (2) and (3) and find the explicit function $\Psi_1(x_1)$:

$$
\Psi_1(x_1) = \frac{3 - 12r + 15r^2 - 10r^3 - 4r^4}{(1 + 2r + r^2)^2},
$$

$$
t = x_1(1 - x_1).
$$

4. THE SECOND LAGRANGIAN POINT

Now we look for the minimum of the function (note the difference from eq. [2])

$$
\Psi_2(x) = \frac{2q}{(1 + q)(x - 1)} + \frac{2}{1 + q} + \left(x - \frac{q}{1 + q}\right)^2
$$

at some $x = x_2$, with $1 < x_2 < 2$. Repeating the procedure of § 3, we get the solution for $q$ as function of $x_2$:

$$
q(x_2) = \frac{(x_2 - 1)^3(1 + x_2 + x_2^2)}{x_2^2(2 - x_2)(1 - x_2 + x_2^2)}.
$$
We note that in contrast to equation (3), it is not evident from equation (6) that we have the relation \( q(x_2)q(1-x_2) = 1 \), because here \( x_2 > 1 \), but \((1-x_2) < 0\), while equation (6) is derived under the condition \( 1 < x_2 < 2 \). In the case of equation (3), both \( x_1 \) and \((1-x_1)\) are in the (open) interval \((0, 1)\).

To prove the validity of the relation \( q(x_2)q(1-x_2) = 1 \), we should return to the basic equation (1), set \( y = z = 0 \) and \( x < 0 \), and look for the minimum of the function

\[
\Psi_3(x) = \frac{2q}{(1+q)(1-x)} - \frac{2}{(1+q)x} + \left(x - \frac{q}{1+q}\right)^2
\]  

(7)
at some \( x = x_1 \), with \( x_1 < 0 \). Note the differences between the functions \( \Psi_1(x) \), \( \Psi_2(x) \), and \( \Psi_3(x) \). Repeating the above procedure for \( \Psi_3(x) \), we get

\[
q(x_3) = \frac{(2-x_3)x_3^2(1-x_3 + x_3^2)}{(x_3 - 1)^2(1 + x_3 + x_3^2)}.
\]  

(8)

Now, from equations (6) and (8), we have \( q(x_2)q(x_3) = 1 \) if \( x_3 = 1 - x_2 \), and if \( x_2 > 1 \), \( x_3 = 1 - x_2 < 0 \), Q.E.D. Equation (6) (together with eq. [8]) gives the fully analytical and exact relation between \( q \) and the value of the second Lagrangian point \( x_2 \). See, e.g., Table 1 in Mochnacki (1984), in which \( Q \) and \( X' \) correspond to our \( q \) and \( x_2 \), respectively.

Now, in order to find the value of the potential corresponding to the second Lagrangian point, we can either use equations (5) and (6) and find the explicit function \( \Psi_2(x_2) \):

\[
\Psi_2(x_2) = \frac{-1 - 4x_2 + 27x_2^2 - 36x_2^3 + 9x_2^4 + 18x_2^5 - 14x_2^6 + 4x_2^7}{(-1 + 2x_2 + x_2^2 - 2x_2^3 + x_2^4)^2}.
\]  

(9)

Equation (9) gives the fully analytical and exact relation between the values of the second Lagrangian point \( x_2 \) and the corresponding potential \( \Psi_2(x_2) \). See, e.g., Table 1 in Mochnacki (1984), in which \( C_2 \) and \( X' \) correspond to our \( \Psi_2(x_2) \) and \( x_2 \), respectively.

5. SUMMARY

In this short paper we present the exact analytical relations for the first and second Lagrangian points of the classical Roche problem. In practice, the “exact” formulae are not necessarily the most convenient ones; hence the different approximate expressions in the literature (see, e.g., Kopal 1959; Plavec & Kratochvil 1964; Kippenhahn & Thomas 1970; Paczyński 1971; Eggleton 1983; Mochnacki 1984; Morris 1985, 1994), and some relevant approximate formulae will be given elsewhere. Still, the exact formulae have their own beauty and are more relevant as solutions to classical problems such as the problem by Roche (1847).

My thanks are due to the anonymous referee for encouraging criticism.

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