Harmonics on the quantum Euclidean space related to the quantum orthogonal group

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Abstract

The aim of this paper is to study harmonic polynomials on the quantum Euclidean space $E_q^N$ generated by elements $x_i$, $i = 1, 2, \cdots, N$, on which the quantum group $SO_q(N)$ acts. The harmonic polynomials are defined as solutions of the equation $\Delta_q p = 0$, where $p$ is a polynomial in $x_i$, $i = 1, 2, \cdots, N$, and the $q$-Laplace operator $\Delta_q$ is determined in terms of the differential operators on $E_q^N$. The projector $H_m: A_m \rightarrow H_m$ is constructed, where $A_m$ and $H_m$ are the spaces of homogeneous of degree $m$ polynomials and homogeneous harmonic polynomials, respectively. By using these projectors, a $q$-analogue of the classical zonal polynomials and associated spherical polynomials with respect to the quantum subgroup $SO_q(N-2)$ are constructed. The associated spherical polynomials constitute an orthogonal basis of $H_m$. These polynomials are represented as products of polynomials depending on $q$-radii and $x_j$, $x_{j}'$, $j' = N-j+1$. This representation is in fact a $q$-analogue of the classical separation of variables. The dual pair $(U_q(\mathfrak{sl}_2), U_q(\mathfrak{so}_n))$ is related to the action of $SO_q(N)$ on $E_q^N$. Decomposition into irreducible constituents of the representation of the algebra $U_q(\mathfrak{sl}_2) \times U_q(\mathfrak{so}_n)$ defined by the action of this algebra on the space of all polynomials on $E_q^N$ is given.

I. INTRODUCTION

The Laplace operator, harmonic polynomials and related separations of variables are of a great importance in classical analysis. They are closely related to the rotation group $SO(N)$ and its subgroups (see, for example, [1], chapter 10).

Harmonic polynomials are defined by the equation $\Delta p = 0$, where $\Delta$ is the Laplace operator and $p$ belongs to the space $R$ of polynomials on the Euclidean space $E_N \sim \mathbb{R}^N$. The space $\mathcal{H}$ of all harmonic polynomials on $E_N$ decomposes as a direct sum of the subspaces $\mathcal{H}_m$ of homogeneous harmonic polynomials of degree $m$: $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$. The Laplace operator $\Delta$ on the Euclidean space $E_N$ commutes with the natural action of the rotation group $SO(N)$ on this space. This means that the subspaces $\mathcal{H}_m$ are invariant with respect to $SO(N)$. The irreducible representation $T_m$ of the group $SO(N)$ with highest weight $(m, 0, \cdots, 0)$ is realized on $\mathcal{H}_m$.

The Laplace operator $\Delta$ permits separation of variables on the space $\mathcal{H}_m$. In other words, there exist different coordinate systems (spherical, polyspherical) on $E_N$ and for each of them it is possible to find the corresponding basis of the space of solutions of the equation $\Delta p = 0$ consisting of products of functions depending on separated variables. To different coordinate systems there correspond different separations of variables. From the other side, to different coordinate systems there correspond different chains of subgroups of the group $SO(N)$ (see [1], chapter 10, for detail of this correspondence). The basis of the space $\mathcal{H}_m$ in separated variables (for a fixed coordinate system) consists of products of Jacobi polynomials multiplied by $r^{2m}$, where $r$ is the radius. These polynomials (considered only on the sphere $S^{N-1}$) are matrix elements of the class 1 (with respect to the subgroup $SO(N-1)$) irreducible representations $T_m$ of $SO(N)$ belonging to the zero column.

Many new directions of contemporary mathematical physics are related to quantum groups and noncommutative geometry. It is natural to generalize the theory described above to noncommutative spaces. Such generalizations can be of a great importance for further development of some branches of mathematical and theoretical physics related to noncommutative geometry.
The aim of this paper is to construct a $q$-deformation of some aspects of the classical theory described above. In the $q$-case, instead of the Euclidean space we have the quantum Euclidean space. It is defined in terms of the associative algebra $\mathcal{A}$ generated by the elements $x_1, x_2, \cdots, x_N$ satisfying the certain defining relations. These elements play a role of Cartesian coordinates of $E_N$.

The $q$-Laplace operator $\Delta_q$ on $\mathcal{A}$ is defined in terms of $q$-derivatives (see formula (17) below). Instead of the group $SO(N)$ we have the quantum group $SO_q(N)$ or the corresponding quantum algebra $U_q(so_N)$. In our exposition it is more convenient to deal with the algebra $U_q(so_N)$. The $q$-harmonic polynomials on the quantum Euclidean space are defined as elements $p$ of $\mathcal{A}$ (that is, polynomials in quantum coordinates $x_1, x_2, \cdots, x_N$) for which $\Delta_q p = 0$. By using the algebra $U_q(so_N)$ or the quantum group $SO_q(N)$ it is possible to construct for $q$-harmonic polynomials the theory similar to the theory for classical harmonic polynomials described above. Namely, we construct projectors $H_m : \mathcal{A}_m \rightarrow \mathcal{H}_m$, where $\mathcal{A}_m$ and $\mathcal{H}_m$ are the subspaces of homogeneous (of degree $m$) polynomials in $\mathcal{A}$ and in the space $\mathcal{H}$ of all $q$-harmonic polynomials from $\mathcal{A}$, respectively. Using these projectors we construct in $\mathcal{H}_m$ a $q$-analogue of associated spherical harmonics with respect to the quantum subgroup $SO_q(N - 2)$. They constitute an orthogonal basis of the space $\mathcal{H}_m$ corresponding to the chain of the quantum subgroups

$$SO_q(N) \supset SO_q(N - 2) \supset SO_q(N - 4) \supset \cdots \supset SO_q(3) \ (\text{or } SO_q(2)).$$

Here we obtain a $q$-analogue of the corresponding spherical separated coordinates. Our construction is similar to one used by us in [2] for the case of quantum complex vector space with the quantum unitary group $U_q(N)$ as a quantum motion group.

The operator $\Delta_q$ and the operator $\hat{Q}$ of multiplication by the squared $q$-radius together with the certain operator equivalent to operator of homogeneity degree on the sets of homogeneous polynomials generate the quantum algebra $U_q(sl_2)$. Thus, the algebra $U_q(sl_2) \times U_q(so_N)$ acts on the space $\mathcal{A}$. We decompose this representation into irreducible ones. The pair $(U_q(sl_2), U_q(so_N))$ constitutes a dual pair of quantum algebras which is a $q$-analogue of the corresponding classical dual pair.

Our constructions use essentially the results of paper [3], where the operator $\Delta_q$ and the spaces $\mathcal{H}_m$ were defined.

Everywhere below we suppose that $q$ is not a root of unity. We shall use two different definitions of $q$-numbers:

$$[a] = \frac{1 - q^a}{1 - q}, \quad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}.$$ 

It is necessary to pay attention which of these definitions is used in each concrete case.

**II. THE QUANTUM ALGEBRAS $U_q(sl_N)$ AND THE QUANTUM EUCLIDEAN SPACE**

It is well known that the rotation group $SO(N)$ naturally acts on the $N$-dimensional Euclidean space $E_N$. All this machinery (the group $SO(N)$, the Euclidean space $E_N$, the action of $SO(N)$ on $E_N$, etc) can be quantized. As a result, we have a "quantum" action of the quantum rotation group $SO_q(N)$ on the quantum Euclidean space (see [4]). The quantum Euclidean space $E_q^N$ is defined by means of the algebra of polynomials $\mathcal{A} \equiv \mathbb{C}_q[x_1, x_2, \cdots, x_N]$ in noncommutative elements $x_1, x_2, \cdots, x_N$ which are called quantum Cartesian coordinates (see [4] and [5]). The number $N$ can be even or odd and we represent it as $N = 2n$ or $N = 2n + 1$, respectively. Moreover, for $j = 1, 2, \cdots, N$ we shall use the notation $j' = N - j + 1$.

The algebra $\mathcal{A}$ is the associative algebra generated by elements $x_1, x_2, \cdots, x_N$ satisfying the defining relations

$$x_i x_j = qx_j x_i, \quad i < j \quad \text{and} \quad i \neq j',$$

(1)
The set of monomials $x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_N^{\nu_N}$, $\nu_i = 0, 1, 2, \cdots$, form a basis of the algebra $A$ (see [3]).

The vector space of the algebra $A$ can be represented as a direct sum of the vector subspaces $A_m$ consisting of homogeneous polynomials of homogeneity degree $m$, $m = 0, 1, 2, \cdots$:

$$A = \bigoplus_{m=0}^{\infty} A_m.$$ 

A $*$-operation (that is, an involutive algebra anti-automorphism) can be defined on the algebra $A$ turning it into a $*$-algebra. This $*$-operation is uniquely determined by the relations $x_i^* = q^{\rho_i} x_i$, $i = 1, 2, \cdots, N$.

The quantum rotation group $SO_q(N)$ and the corresponding quantized universal enveloping algebra $U_q(so_N)$ act on the algebra $A$. These actions are determined by each other. It will be convenient for us to use the action of the algebra $U_q(so_N)$. The last algebra is the Hopf algebra generated by the elements $K_i, K_i^{-1}, E_i, F_i$, $i = 1, 2, \cdots, n$, satisfying the certain defining relations (see, for example, section 6.1.3 in [5]), where $n$ is an integral part of $N/2$. The algebra $U_q(so_N)$ is supplied by the Hopf algebra operations. We adopt these operations determined in [3]. The action of $X \in U_q(so_N)$ on an element $a \in A$ will be denoted as $X \triangleright a$.

A $*$-operation can also be introduced on $U_q(so_N)$ (see, for example, [5]) which determines the compact form of $U_q(so_N)$. We denote this compact form by $U_q(so(N))$. The action of $U_q(so(N))$ on the $*$-algebra $A$ is compatible with the $*$-action, that is, $(X \triangleright a)^* = S(X)^* \triangleright a^*$ for $X \in U_q(so(N))$ and $a \in A$, where $S$ is the antipode on $U_q(so_N)$ (see [3]).

The action of $U_q(so_N)$ on $A$ is explicitly given in [3], Lemma 2.5. For $U_q(so_{2n+1})$ and $U_q(so_{2n})$, the action of elements $E_k$ and $F_k$, $k = 1, 2, \cdots, n-1$, are determined as

$$E_k \triangleright x^\nu = [\nu_{k+1}] q^{-\nu_{k+1}+1} x^{\nu_{k+1}} - [\nu_k] q^{\nu_k - \nu_{k+1} + 1} x^{\nu_k - \nu_{k+1} + 1},$$

$$F_k \triangleright x^\nu = [\nu_k] q^{-\nu_k + 1} x^{\nu_k} - [\nu_{k+1}] q^{\nu_{k+1} + 1} x^{\nu_{k+1}}.$$ 

The action of elements $E_n$ and $F_n$ are given by the formulas

$$E_n \triangleright x^\nu = [\nu_{n+1}] q^{-\nu_{n+1} + 3/2} x^{\nu_{n+1} + 3/2} - [\nu_{n+2}] q^{-\nu_{n+2} + 1} x^{\nu_{n+2} + 1},$$

$$F_n \triangleright x^\nu = [\nu_{n+1}] q^{-\nu_{n+1} + 1} x^{\nu_{n+1}} - [\nu_{n+2}] q^{-\nu_{n+2} + 1} x^{\nu_{n+2} + 1}.$$ 

if $N = 2n + 1$ and by the formulas

$$E_n \triangleright x^\nu = [\nu_{n+1}] q^{-\nu_{n+1} + 1} x^{\nu_{n+1} - 1/2} - [\nu_{n+2}] q^{-\nu_{n+2} + 1} x^{\nu_{n+2} + 1},$$

$$F_n \triangleright x^\nu = [\nu_{n+1}] q^{-\nu_{n+1} - 1/2} x^{\nu_{n+1} - 1/2} - [\nu_{n+2}] q^{-\nu_{n+2} + 1} x^{\nu_{n+2} + 1}.$$ 

if $N = 2n$. In these formulas $\epsilon_i$ is the vector with $i$th coordinate equal to 1 and all others equal to 0.
The monomials $x^\nu$ are weight vectors with respect to the action of $U_q(\mathfrak{so}_N)$ on $A$. We represent weights $\lambda$ in the well known orthogonal coordinate system, that is, as $\lambda = \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \cdots + \mu_n \varepsilon_n$ (in this system highest weights are given by the coordinates $\mu_1, \mu_2, \cdots, \mu_n$ such that $\mu_1 \geq \mu_2 \geq \cdots$). The weight of the monomial $x^\nu$ is

$$\lambda = (\nu_1 - \nu_{1'}) \varepsilon_1 + (\nu_2 - \nu_{2'}) \varepsilon_2 + \cdots + (\nu_n - \nu_{n'}) \varepsilon_n.$$ 

The action of the element $K_i, \ i < n$, on the monomial $x^\nu$ is given by the formula

$$K_i \triangleright x^\nu = q^{(\nu_i - \nu_{i'}) - (\nu_{i+1} - \nu_{(i+1)'})} x^\nu.$$ 

Moreover, in the algebra $U_q(\mathfrak{so}_N)$ there exist elements $\hat{K}_i, \ i = 1, 2, \cdots, n$, such that

$$\hat{K}_i \triangleright x^\nu = q^{(\nu_i - \nu_{i'})} x^\nu.$$ (5)

A differential calculus is developed on the quantum Euclidean space which is determined by the $R$-matrix of the quantum algebra $U_q(\mathfrak{so}_N)$. There exist different formulations for this differential calculus. We adopt the definition of the differential operators $\partial_i, \ i = 1, 2, \cdots, N,$ used in [3]. These operators act on the monomials $x^\nu$ as

$$\partial_k \triangleright x^\nu = [\nu_k] q^k x^\nu, \ k \leq n,$$ (6)

$$\partial_{n+1} \triangleright x^\nu = [\nu_{n+1}] q^{n+1} x^\nu, \ \text{if} \ N = 2n + 1,$$

$$\partial_{k'} \triangleright x^\nu = [\nu_{k'}] q^{k'} x^\nu, \ \text{if} \ N = 2n + 1,$$

$$\sum_{j=k+1}^n [\nu_j] [\nu_j'] q(q - q^{-1}) q^{\rho_k - \rho_j} q^{d_{kj}} x^\nu_{\varepsilon_j \varepsilon_j'}, \ k \leq n,$$ (7)

where $d_{kj} = \nu_k + \cdots + \nu_{j-1} + \nu_{j-1}' + \cdots + \nu_{j'}$ and $e_k = (\nu_k + \cdots + \nu_{1'}) - 2\nu_{n+1}$. The last summand in (7) must be omitted for $N = 2n$. The operators $\partial_i, \ i = 1, 2, \cdots, N,$ satisfy the relations

$$\partial_i \partial_j = q^{-1} \partial_j \partial_i, \ \ i < j, \ i \neq j',$$ (8)

$$\partial_i \partial_i' - \partial_i' \partial_i = -q^{-1} \frac{q - q^{-1}}{q^{n+1} + q^{-n+1}} \sum_{k=1}^{i+1} \partial_k \partial_{k'} q^{\rho_k}, \ i < n,$$ (9)

$$\partial_{N+1} \partial_{n+1} - \partial_{n+1} \partial_n = -(q^{1/2} - q^{-1/2})^2 q_{n+1}^2, \ \text{if} \ N = 2n + 1,$$ (10)

$$\partial_{n'} \partial_{n'} = \partial_n \partial_n', \ \text{if} \ N = 2n.$$ 

The operators $\partial_k$ and the operators $\hat{x}_i$ of left multiplication by $x_i$ satisfy certain relations which can be represented by means of the quantum $R$-matrix of the algebra $U_q(\mathfrak{so}_N)$. These relations are given in [3]. We need the following ones:

$$\partial_k \hat{x}_k = \hat{x}_k \partial_k q^{\delta_{kk'} - 1} - (q - q^{-1}) \sum_{j<k} \hat{x}_j \partial_j + (q - q^{-1}) \sigma_k q^{\rho_k} \hat{x}_k \partial_{k'} + c, \ k = 1, 2, \cdots N,$$ (11)

$$\partial_k \hat{x}_j = \hat{x}_j \partial_k + (q - q^{-1}) \sigma_{kj} q^{\rho_j' - \rho_k} \hat{x}_k \partial_{j'}, \ k \neq j, j',$$ (12)

$$\partial_k \hat{x}_{k'} = q \hat{x}_{k'} \partial_k, \ k \neq k', \ c \partial_k = q^{-1} \partial_k c,$$ (13)
where $\sigma_k = 1$ if $k > k'$ and $\sigma_k = 0$ otherwise, $\sigma_{kj} = 1$ if $k > j'$ and $\sigma_{kj} = 0$ otherwise, $c$ is the linear operator which acts on the monomials $x^\nu$ as $c \triangleright x^\nu = q^{\nu_1 + \cdots + \nu_s} x^\nu$.

### III. Squared q-Radius and q-Laplace Operator

The element

$$Q = \sum_{i=1}^{N} q^{\rho_i} x_i x_{i'}$$

(14)

of the algebra $\mathcal{A}$ is called the **squared q-radius** on the quantum Euclidean space. It is an important element in $\mathcal{A}$. It is shown in [3] that the center of $\mathcal{A}$ is generated by $Q$.

Using relations between the elements $x_j$ it is shown that

$$Q = (1 + q^{N-2}) \left( \sum_{i=1}^{n} q^{\rho_i} x_i x_{i'} + \frac{q}{q+1} x_{n+1}^2 \right) \text{ if } N = 2n + 1,$$

$$Q = (1 + q^{N-2}) \sum_{i=1}^{n} q^{\rho_i} x_i x_{i'} \text{ if } N = 2n.$$

We shall also use the elements

$$Q_j = \sum_{i=j}^{j'} q^{\rho_i} x_i x_{i'}, \quad 1 < j \leq n,$$

(15)

which are squared q-radii for the subalgebras $\mathbb{C}_q[x_j, \cdots, x_{j'}]$. They satisfy the relations

$$Q_j Q_k = Q_k Q_j, \quad x_i x_{i'} = q^{\rho_i} \left( \frac{Q_i}{1 + q^{N-2i}} - \frac{Q_{i+1}}{1 + q^{N-2i-2}} \right), \quad 1 \leq i \leq n,$$

$$x_i Q_j = q^2 Q_j x_i, \quad x_{i'} Q_j = q^{-2} Q_j x_{i'} \text{ for } i < j, \quad x_i Q_j = Q_j x_i \text{ for } j \leq i \leq j'.$$

It can be checked by direct computation that

$$x_i x_j = Q'_{k}(Q_2/Q_1; q^2)_k,$$

(16)

where $Q'_1 = q^{-\rho_j} Q_1/(1 + q^{N-2})$, $Q'_2 = q^{-\rho_j} Q_2/(1 + q^{N-4})$ and

$$(a; q)_s = (1 - a)(1 - aq) \cdots (1 - a q^{s-1}).$$

To the element (15) there corresponds the operator $\hat{Q}$ on $\mathcal{A}$ defined as

$$\hat{Q} = \sum_{i=1}^{N} q^{\rho_i} \hat{x}_i \hat{x}_{i'},$$

where $\hat{x}_i$ is the operator of left multiplication by $x_i$. It is clear that $\hat{Q} : \mathcal{A}_m \to \mathcal{A}_{m+2}$.

We also consider on $\mathcal{A}$ the operator

$$\Delta_q = \sum_{i=1}^{N} q^{\rho_i} \partial_i \partial_{i'}$$

(17)

which is called the **q-Laplace operator** on the quantum Euclidean space. We have $\Delta_q : \mathcal{A}_m \to \mathcal{A}_{m-2}$. 


The important property of the operators $\hat{Q}$ and $\Delta_q$ is that they commute with the action of the algebra $U_q(\mathfrak{so}_N)$ on $\mathcal{A}$ (see [3]).

The operators $\hat{Q}$ and $\Delta_q$ satisfy the relations

$$\Delta_q \hat{Q}^k - q^{2k} \hat{Q}^k \Delta_q = \hat{Q}^{k-1} q^{-N+3}[2k][N + 2k + 2\gamma - 2] \frac{(1 + q^{N-2})^2}{(1 + q)^2}, \tag{18}$$

$$\Delta_q(\hat{Q}^k) = Q^{k-1} q^{-N+3}[2k][N + 2k - 2] \frac{(1 + q^{N-2})^2}{(1 + q)^2}, \tag{19}$$

where $\gamma$ is the operator acting on the monomials $x^\nu$ as $\gamma x^\nu = (\nu_1 + \cdots + \nu_N) x^\nu$ (see [3]). We shall also use the following formula from [3]:

$$\Delta_q(x^\nu) = (1 + q^{N-2}) q^{\nu_1 + \cdots + \nu_{i-1} - 1} \times \left( \sum_{j=1}^{n} [\nu_j]_q x^{\nu_j - \epsilon_j} \right) \frac{d}{e} x^{\nu - 2\epsilon_{n+1}}, \tag{20}$$

where $d = \nu_1 + \cdots + \nu_{i-1} + \nu_{i+1} + \cdots + \nu_N$, $e = \nu_1 + \cdots + \nu_{i-1} - 2\nu_{n+1} + 2$, and the last summand must be omitted for $N = 2n$.

IV. $q$-HARMONIC POLYNOMIALS

A polynomial $p \in \mathcal{A}$ is called $q$-harmonic if $\Delta_q p = 0$. The linear subspace of $\mathcal{A}$ consisting of all $q$-harmonic polynomials is denoted by $\mathcal{H}$. If $\mathcal{H}_m = \mathcal{A}_m \cap \mathcal{H}$, then $\mathcal{H}_m$ is the subspace of $\mathcal{H}$ consisting of all homogeneous of degree $m$ harmonic polynomials.

**Proposition 1** [3]. If $m \geq 2$, then the space $\mathcal{A}_m$ can be represented as the direct sum

$$\mathcal{A}_m = \mathcal{H}_m \oplus Q \mathcal{A}_{m-2}. \tag{21}$$

We shall need the following consequences of the decomposition (21):

**Corollary 1.** If $p \in \mathcal{H}_m$, then $p$ cannot be represented as $p = Q^k p'$, $k \neq 0$, with some polynomial $p' \in \mathcal{A}$.

**Corollary 2.** The space $\mathcal{A}_m$ decomposes into the direct sum $\mathcal{A}_m = \bigoplus_{j=0}^{[m/2]} Q^j \mathcal{H}_{m-2j}$, where $[m/2]$ is the integral part of the number $m/2$.

**Corollary 3.** For dimension of the space of $q$-harmonic polynomials $\mathcal{H}_m$ we have the formula

$$\dim \mathcal{H}_m = \frac{(m + N - 3)!(2m + N - 2)}{(N - 2)! m!}.$$

**Corollary 4.** The linear space $\mathcal{H}$ can be represented in the form of a direct sum

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

Corollary 1 is a direct consequence of formula (21). Corollary 2 easily follows from repeated application of (21). Corollary 3 is proved in the same way as in the classical case (see, for example, [1], Chap. 10). For this we note that

$$\dim \mathcal{A}_m = \frac{(N + m - 1)!}{(N - 1)! m!}.$$
Hence, for \( \dim H_m = \dim \mathcal{A}_m - \dim \mathcal{A}_{m-2} \) we obtain the expression stated in the corollary. In order to prove Corollary 4 we note that any \( p \in \mathcal{H} \) can be represented as \( p = \sum_m p_m, \) \( p_m \in \mathcal{A}_m. \) We have \( \Delta_q p = \sum_m \Delta_q p_m = 0. \) Since \( \Delta_q p_m, m = 0, 1, 2, \cdots, \) have different homogeneity degrees, it follows from the last equality that \( \Delta_q p_m = 0 \) for all values of \( m. \) Thus, \( \mathcal{H} = \bigoplus_{m=0}^\infty H_m. \)

Remark: If \( n = 2, \) then \( \mathcal{A} \) consists of all polynomials in commuting elements \( x_1 \) and \( x_2. \) In this case, the space \( \mathcal{H} \) of \( q \)-harmonic polynomials has a basis consisting of the polynomials

\[
1, \ x_1^k, \ x_2^k, \ k = 1, 2, \cdots.
\]  

(22)

**Proposition 2.** The linear space isomorphism \( \mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H} \) is true, where \( \mathbb{C}[Q] \) is the space of all polynomials in \( Q. \)

This proposition follows from Corollary 2.

The decomposition \( \mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H} \) is a \( q \)-analogue of the theorem on separation of variables for Lie groups in an abstract form (see [6]). It follows from this decomposition that

\[
\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H} \simeq \mathbb{C}[Q] \otimes \bigoplus_{m \geq 0} \mathcal{H}_m = \bigoplus_{m \geq 0} (\mathbb{C}[Q] \otimes \mathcal{H}_m).
\]  

(23)

Since the operator \( \Delta_q \) commutes with the action of the algebra \( U_q(\mathfrak{so}_n) \) the subspaces \( \mathcal{H}_m \) are invariant with respect to the action of this algebra. It is proved in [3] that the irreducible representation \( T_m \) of \( U_q(\mathfrak{so}_N) \) with highest weight \( (m, 0, \cdots, 0) \) is realized on \( \mathcal{H}_m. \)

We denote by \( \mathcal{A}^{U_q(\mathfrak{so}_n)} \) the space of elements of \( \mathcal{A} \) consisting of invariant elements with respect to the action of \( U_q(\mathfrak{so}_n). \) We have \( \mathcal{A}^{U_q(\mathfrak{so}_n)} = \mathbb{C}[Q] \) (see [3]). In what follows we shall consider the subalgebra \( U_q(\mathfrak{so}_{N-2}) \) generated by the elements \( H_i, E_i, F_i, \) \( i = 2, 3, \cdots, n. \)

**Proposition 3.** We have

\[
\mathcal{A}^{U_q(\mathfrak{so}_{N-2})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_2] x_1^k x_1^l, \quad \bigoplus_{k,l} \mathbb{C}[Q] x_1^k x_1^l.
\]

**Proof.** In order to prove this proposition we note that for \( U_q(\mathfrak{so}_{N-2}) \)-module \( \mathcal{A} \) we have

\[
\mathcal{A} = \mathbb{C}_q[x_1, x_2, \cdots, x_N] = \bigoplus_{k,l} \mathbb{C}_q[x_2, x_3, \cdots, x_{N-1}] x_1^k x_1^l.
\]

The action of \( U_q(\mathfrak{so}_{N-2}) \) on monomials \( x_1^k x_1^l \) is trivial. Moreover, \( \mathbb{C}_q[x_2, x_3, \cdots, x_{N-1}] U_q(\mathfrak{so}_{N-2}) = \mathbb{C}[Q_2]. \) Since \( Q = c_1 Q_2 + c_2 x_1 x_N, \) where \( c_1 \) and \( c_2 \) are constants, we have \( \mathcal{A}^{U_q(\mathfrak{so}_{N-2})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_2] x_1^k x_1^l \simeq \bigoplus_{k,l} \mathbb{C}[Q] x_1^k x_1^l. \) Proposition is proved.

**V. THE DUAL PAIR** \((U_q(\mathfrak{sl}_2), U_q(\mathfrak{so}_N))\)

The formulas

\[
ke = q^2 ek, \quad kf = q^{-2} f k, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}
\]  

(24)

determine the quantum algebra \( U_q(\mathfrak{sl}_2) \) generated by the elements \( k, k^{-1}, e, f. \) Let \( L(A) \) be the space of linear operators on the algebra \( \mathcal{A}. \) It was shown in [3] that the operators

\[
\omega(k) = q^{N/2} q^\gamma, \quad \omega(e) = \hat{Q}, \quad \omega(f) = -\Delta_q q^{-\gamma} \frac{q^{N/2}}{(1 + q^{N-2})^2}
\]  

(25)

satisfy relations (24). This means that the algebra homomorphism \( \omega : U_q(\mathfrak{sl}_2) \rightarrow L(A), \) uniquely determined by formulas (25), is a representation of \( U_q(\mathfrak{sl}_2). \)
Since the operators $\omega(k), \omega(e), \omega(f)$ commute with the operators $L(X), X \in U_q(so_N)$, we can introduce the representation $\omega \times L$ of the algebra $U_q(sl_2) \times U_q(so_N)$ on $A$, where $L$ is the above defined natural action of $U_q(so_N)$ on $A$. This representation is reducible. Let us decompose it into irreducible constituents.

By (23), we have $A = \bigoplus_{m \geq 0} (\mathbb{C}[Q] \otimes \mathcal{H}_m)$. The subspaces $\mathbb{C}[Q] \otimes \mathcal{H}_m$ are invariant under $U_q(sl_2) \times U_q(so_N)$, since the space $\mathbb{C}[Q]$ is elementwise invariant under $U_q(so_N)$, and for $f \in \mathbb{C}[Q]$ and $h_m \in \mathcal{H}_m$ we have

$$\omega(e)(f(Q) \otimes h_m) = Qf(Q) \otimes h_m, \quad (26)$$

$$\omega(f)(Q^r \otimes h_m) = -[r]_q [r + m + (N/2) - 1]_q Q^{r-1} \otimes h_m, \quad (27)$$

$$\omega(k)(Q^r \otimes h_m) = q^{2r+m+N/2}(Q^r \otimes h_m) \quad (28)$$

(we used formula (18) for obtaining (27)). These formulas show that $U_q(sl_2)$ acts on $\mathbb{C}[Q]$ and $U_q(so_N)$ acts on $\mathcal{H}_m$. However, this action of $U_q(sl_2)$ depends on the component $\mathcal{H}_m$. Taking the basis

$$|r\rangle := [r + m + (N/2) - 1]_q!^{-1} Q^r, \quad r = 0, 1, 2, \ldots,$$

in the space $\mathbb{C}[Q]$, we find from (26)–(28) that

$$\omega(k)|r\rangle = q^{2r+m+N/2}|r\rangle \quad \omega(f)|r\rangle = -[r]_q |r-1\rangle,$$

$$\omega(e)|r\rangle = [r + m + N/2]_q |r+1\rangle.$$ 

Comparing this representation with the known irreducible representations of $U_q(sl_2)$ (see, for example, [7]) we derive that the irreducible representation of $U_q(sl_2)$ of the discrete series with lowest weight $m + N/2$ is realized on the component $\mathbb{C}[Q] \otimes \mathcal{H}_m$ of the space $\mathbb{C}[Q] \otimes \mathcal{H}_m$. We denote this representation of $U_q(sl_2)$ by $D_{m+N/2}$.

Thus, we have derived that on the subspace $\mathbb{C}[Q] \otimes \mathcal{H}_m \subset A$ the irreducible representation $D_{m+N/2} \times T_m$ of the algebra $U_q(sl_2) \times U_q(so_N)$ acts. This means that for the reducible representation $\omega \otimes L$ we have the following decomposition into irreducible components:

$$\omega \times L = \bigoplus_{m=0}^{\infty} D_{m+N/2} \times T_m,$$

that is, each irreducible representation of $U_q(so_N)$ in this decomposition determines uniquely the corresponding irreducible representation of $U_q(sl_2)$ and vice versa. This means that $U_q(sl_2)$ and $U_q(so_N)$ constitute a dual pair under the action on $A$. It is a $q$-analogue of the well known classical dual pair $(sl_2, so_N)$ (see, for example, [1], Chapter 12).

VI. RESTRICTION OF $q$-HARMONIC POLYNOMIALS ONTO THE QUANTUM SPHERE

The associative algebra $\mathcal{F}(S_q^{N-1})$ generated by the elements $x_1, \ldots, x_N$ satisfying the relations (1)–(3) and the relation $Q = 1$ is called the algebra of functions on the quantum sphere $S_q^{N-1}$ (see [4] and [5], Chap. 11). It is clear that the following canonical algebra isomorphism has place:

$$\mathcal{F}(S_q^{N-1}) \simeq A/I,$$

where $I$ is the two-sided ideal of $A$ generated by the element $Q - 1$. We denote by $\tau$ the canonical algebra homomorphism

$$\tau : A \to A/I \simeq \mathcal{F}(S_q^{N-1}).$$

This homomorphism is called the restriction of polynomials of $A$ onto the quantum sphere $S_q^{N-1}$. 

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It was shown in [3] that $\tau: \mathcal{H} \to \mathcal{F}(S^N_q)$ is a one-to-one mapping, that is, the restriction of a $q$-harmonic polynomial to the sphere $S^N_q$ determines this polynomial uniquely. This statement allows us to determine a scalar product on $\mathcal{H}$. For this, we use the invariant functional $h$ on the quantum sphere $S^N_q$ defined in [3]. In order to give this functional we introduce the linear subspace $(\tau\mathcal{A})^0$ of $\mathcal{F}(S^N_q)$ spanned by the elements $\tau x^\nu$ such that

$$\nu_1 = \nu_1', \ldots, \nu_n = \nu_n', \quad \nu_{n+1} = 2m, \quad m = 0, 1, 2, \ldots,$$

(for $N = 2n$ the last condition must be omitted). The functional $h$ vanishes on the elements $\tau x^\nu \notin (\tau\mathcal{A})^0$ and on the monomials $\tau x^\nu \in (\tau\mathcal{A})^0$ it is given by the formula

$$h(\tau x^\nu) = \frac{(q^{-2}; q^{-2})_{\nu_1} \cdots (q^{-2}; q^{-2})_{\nu_n} (q^{-1}; q^{-2})_{m} (1 + q)^m}{(q^{1+\nu_1+\cdots+\nu_n} + m (1 + q^{N-2})^{\nu_1 + \cdots + \nu_n + m} (q^{-N}; q^{-2})_{\nu_1 + \cdots + \nu_n + m}},$$

where, as before, $m = 0$ for $N = 2n$. The following assertions (similar to ones proved in [8] for the case of the quantum group $U_q(N)$) are true:

(a) The subalgebra $(\tau\mathcal{A})^0$ is a commutative algebra generated by $Q_{n-1}, Q_{n-2}, \ldots, Q_1$ and also by $x^2_{n+1}$ if $N = 2n + 1$.

(b) The algebra $(\tau\mathcal{A})^0$ is isomorphic to the polynomial algebra in $n - 1$ commuting indeterminates if $N = 2n$ and in $n$ commuting indeterminates if $N = 2n + 1$.

A scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is introduced by the formula:

$$\langle p_1, p_2 \rangle = h((\tau p_1)^*(\tau p_2)),$$  \hspace{1cm} (29)

where $a^*$ determines an element conjugate to $a \in \mathcal{A}$ under action of the $*$-operation introduced in section 2.

**Proposition 4.** We have $\mathcal{H}_m \perp \mathcal{H}_r$ if $m \neq r$.

*Proof* follows from the fact that $(\tau p_1)^*(\tau p_2) \notin (\tau\mathcal{A})^0$ if $p_1 \in \mathcal{H}_m, p_2 \in \mathcal{H}_r$, and $m \neq r$.

**VII. THE PROJECTION $\mathcal{A}_m \to \mathcal{H}_m$**

Let us go back to the decomposition (21) and construct the projector

$$\mathcal{H}_m : \mathcal{A}_m = \mathcal{H}_m \oplus Q \mathcal{A}_{m-2} \to \mathcal{H}_m.$$

We present this projector in the form

$$\mathcal{H}_m p = \sum_{k=0}^{[m/2]} \alpha_k \hat{Q}^k \Delta^k p, \quad \alpha_k \in \mathbb{C}, \quad p \in \mathcal{A}_m, \hspace{1cm} (30)$$

where $[m/2]$ is the integral part of the number $m/2$. Let us show that the summands on the right hand side are linearly independent at least for one nontrivial $p \in \mathcal{A}_m$ (in this case the coefficients $\alpha_k$ are determined uniquely up to a common constant). Let $p = x^m_{n+1}$ if $N = 2n + 1$. Using formula (20) we derive that

$$\Delta^k (x^m_{n+1}) = q^k \left(1 + \frac{q^{N-2}}{1 + q}\right)^{k} \frac{[m]!}{[m - 2k]!} x^{m-2k}_{n+1}, \quad 2k \leq m,$$  \hspace{1cm} (31)

where $[m]! = [1][2] \cdots [m]$. Then the right hand side of (30) is a linear combination of the elements $Q^k x^m_{n+1}, k = 1, 2, \ldots, [m/2]$. It is easy to see that these elements are linearly independent. If
\( N = 2n \), then instead of \( p = x_{n+1}^m \) we take \( p = x_1^m x_{1'}^m \) and make the same reasoning (see these calculation in the next section).

We have to calculate values of the coefficients \( \alpha_k \) in (30). In order to do this, we act by the operator \( \Delta_q \) upon both sides of (30) and use the relation (18). Under this action, the left hand side vanishes. Equating the right hand side to 0 and taking into account that the elements \( \hat{Q}^k \Delta_q^{k+1} p, k = 1, 2, \ldots, [m/2] \), are linearly independent for generic elements \( p \in \mathcal{A}_m \), we derive the recurrence relation

\[
q^{-N-2k+5} \frac{(1 + q^{N-2})^2}{(1 + q)^2} [2k] [N + 2m - 2k - 2] \alpha_k + \alpha_{k-1} = 0
\]

for \( \alpha_k \) which gives

\[
\alpha_k = (-1)^k \frac{q^{(N-4)k + k^2}}{(1 - q)^k} \frac{(1 + q)^{2k} [N + 2m - 2k - 4]!!}{(1 + q^{N-2})^{2k} [2k]!! [N + 2m - 4]!!},
\]

where \([s]!! = [s][s - 2][s - 4] \cdots [2] \) (or \([1]\)) for \( s \neq 0 \) and \([0]!! = 1\). Using the relations

\[
[2k]!! = \frac{q^2; q^2)_k}{(1 - q^2)_k}, \quad \frac{[N + 2m - 2k - 4]!!}{[N + 2m - 4]!!} = \frac{(1 - q)^k}{(q^{N-2m-2k+2}; q^2)_k},
\]

\[
(q^{N+2m-2k-2}; q^2)_k = (q^{-N-2m+4}; q^2)_k q^{-2k-k(k-1)}(-q^{N+2m-2k})^k,
\]

we derive that

\[
\alpha_k = \frac{q^{2k^2 - 2mk - k}(1 - q^2)^{2k}}{(1 + q^{N-2})^{2k}(q^{-N-2m+4}; q^2)_k (q^2; q^2)_k}. \tag{32}
\]

Note that the coefficients \( \alpha_k \) are determined by the recurrence relation uniquely up to a common constant. In (30) we have chosen this constant in such a way that \( H_m p = p \) for \( p \in \mathcal{H}_m \). This means that \( H_m^2 = H_m \).

**Proposition 5.** The operator \( H_m \) commutes with the action of \( U_q(\mathfrak{so}_N) \).

**Proof.** This assertion follows from the fact that the action of \( X \in U_q(\mathfrak{so}_N) \) commutes with \( \hat{Q} \) and \( \Delta_q \). Proposition is proved.

The operator \( H_m \) can be used for obtaining explicit forms of \( q \)-harmonic polynomials. As an example, we derive here formulas for harmonic projection of the polynomial \( x_{n+1}^m \in \mathcal{A}_m \) when \( N = 2n + 1 \). For this we use formula (31) for \( \Delta_q^k(x_{n+1}^m) \). Since

\[
\frac{[m]!}{[m-2k]!} = \frac{(q^{m-2k+2}; q^2)_k (q^{m-2k+1}; q^2)_k}{(1 - q)^{-2k}},
\]

\[
(q^{m-2k+2}; q^2)_k = (-1)^k q^{mk - k(k-1)}(q^{-m}; q^2)_k,
\]

\[
(q^{m-2k+1}; q^2)_k = (-1)^k q^{mk - k^2}(q^{-m+1}; q^2)_k
\]

we derive that

\[
\Delta_q^k(x_{n+1}^m) = \left( 1 + \frac{q^{N-2}}{1 + q} \right)^k \frac{q^{2mk - 2k^2 + 2k}}{(1 - q)^{2k}}(q^{-m}; q^2)_k (q^{-m+1}; q^2)_k x_{n+1}^{m-2k}.
\]

Using the expression (30) for \( H_m x_{n+1}^m \) and formula (32) for coefficients \( \alpha_k \) we obtain

\[
H_m x_{n+1}^m = x_{n+1}^m \sum_{k=0}^{[m/2]} \frac{(q^{-m}; q^2)_k (q^{-m+1}; q^2)_k}{(q^2; q^2)_k (q^{-N-2m+4}; q^2)_k} (aQ x_{n+1}^{-2})^k, \tag{33}
\]
where

\[ a = \frac{q(1 + q)}{1 + q^{N-2}}. \]

Note that we used \( x_{n+1}^{-2} \) in (33). However, since there exists the multiplier \( x_{n+1}^m \) before the sign of sum, negative powers of \( x_{n+1} \) in fact are absent.

The expression (33) for \( H_n x_{n+1}^m \) can be represented in terms of the basic hypergeometric function \( \phi_1 \) (see [9] for the definition of this function):

\[ H_n x_{n+1}^m = x_{n+1}^m \phi_1(q^{-m}, q^{-m+1}; q^{-N-2m+4} : q^2, aQx_{n+1}^{-2}). \]

Using the definition

\[ P_k^{(\alpha, \beta)}(x; q) = 2 \phi_1(q^{-k}, q^{\alpha + k + 1}; q^{\alpha + 1}; q, qx) \]

of the little \( q \)-Jacobi polynomials we can represent \( H_n x_{n+1}^m \) in the form

\[ H_n x_{n+1}^m = x_{n+1}^m P^{(\frac{N}{2} + m + 1, N-2)}_{m/2} \left( \frac{1 + q}{q(1 + q^{N-2})} Q x_{n+1}^{-2} \right) \]

if \( m \) is even and in the form

\[ H_n x_{n+1}^m = x_{n+1}^m P^{(\frac{N}{2} + m + 1, N-2)}_{(m-1)/2} \left( \frac{1 + q}{q(1 + q^{N-2})} Q x_{n+1}^{-2} \right) \]

if \( m \) is odd.

Up to a constant the restriction of the expression for \( H_n x_{n+1}^m \) to \( S_q^{N-1} \) was found (by other method: as a zonal spherical function with respect to a certain ideal) in [3]. It was expressed in term of the big \( q \)-Jacobi polynomials which in fact are the basic hypergeometric functions \( \psi_2 \) of the argument \( q \). Thus, we found another expression for these zonal spherical functions.

VIII. ZONAL POLYNOMIALS WITH RESPECT TO \( SO_q(N-2) \)

A polynomial \( \varphi \) of the space \( H_m \) is called zonal with respect to the quantum subgroup \( SO_q(N-2) \) (or with respect to the subalgebra \( U_q(so_{N-2}) \)) if it is invariant with respect to action of elements \( X \in U_q(so_{N-2}) \). In order to find zonal polynomials \( \varphi \in H_m \) we have to take polynomials \( p \in A_m \) invariant with respect to the subalgebra \( U_q(so_{N-2}) \) and to act on them by the projection \( H_m \).

It follows from Proposition 3 that in the space \( A_m \) there exist \( m + 1 \) elements which are \( U_q(so_{N-2}) \)-invariant and linearly independent over \( \mathbb{C}[Q] \). They coincide with \( x_{1}^{m_1} x_{1'}^{m_1'} \), \( m_1 + m_1' = m \). Therefore, \( H_m(x_{1}^{m_1} x_{1'}^{m_1'}) \), \( m_1 + m_1' = m \), are zonal polynomials with respect to \( U_q(so_{N-2}) \). Let us find explicit form of these polynomials.

Using formula (20) we find that

\[ \Delta^k_q(x_{1}^{m_1} x_{1'}^{m_1'}) = (1 + q^{2N-2})^k q^{(m-k)k} q^{(-n-\epsilon)k} \frac{[m_1]_q [m_1']_q!}{[m_1 - k]_q [m_1' - k]_q!} x_{1}^{m_1-k} x_{1'}^{m_1'-k}, \] (34)

where \( \epsilon = 1 \) for \( N = 2n \) and \( \epsilon = \frac{1}{2} \) for \( N = 2n + 1 \). Since

\[ \frac{[m_1]_q!}{[m_1 - k]_q!} = q^{(2m_1 - k + 1)k/2} \frac{(q^{-2m_1}; q^2)_k}{(q^{-1})^k}, \]

we have

\[ \Delta^k_q(x_{1}^{m_1} x_{1'}^{m_1'}) = (1 + q^{2N-2})^k q^{2(m-k)k} q^{(-n+3+\epsilon)k} \frac{(q^{-2m_1}; q^2)_k (q^{-2m_1'}; q^2)_k}{(1 - q^2)^{2k}} x_{1}^{m_1-k} x_{1'}^{m_1'-k}. \]
Now using formulas (30) and (32) we derive that

$$\varphi_{m_1m_1'}^m \equiv H_m(x_1^{m_1} x_1'^{m_1'}) = \sum_{k=0}^{\min(m_1, m_1')} C_{m_1m_1'}^k Q_k x_1^{m_1-k} x_1'^{m_1'-k}, \quad (35)$$

where

$$C_{m_1m_1'}^k = \frac{(q^{-2m_1}; q^2)_k (q^{-2m_1'}; q^2)_k}{(q^2; q^2)_k (q^{-N-2m+2}; q^2)_k (1 + q^{N-2})^k}. \quad (36)$$

Using formula (16) the polynomials $\varphi_{m_1m_1'}^m$ can be represented as

$$\varphi_{m_1m_1'}^m = x_1^{m_1-m_1'} \sum_{k=0}^{m_1'} C_{m_1m_1'}^k Q^k (Q'_2/Q'_1; q^2)^{m_1'-k} \quad (35')$$

if $m_1 \geq m_1'$ and as

$$\varphi_{m_1m_1'}^m = \left( \sum_{k=0}^{m_1} C_{m_1m_1'}^k Q^k (Q'_2/Q'_1; q^2)^{m_1'-k} \right) x_1^{m_1'-m_1} \quad (35'')$$

if $m_1' \geq m_1$, where $Q' \equiv Q'_1$ and $Q'_2$ are such as in (16).

**Theorem 1.** The zonal polynomials $\varphi_{m_1m_1'}^m$, $m_1 + m_1' = m$, of $H_m$ are orthogonal with respect to the scalar product introduced in section VI. These polynomials constitute a full set of zonal polynomials in the space $H_m$.

**Proof.** We have $\hat{K}_1 \triangleright (x_1^{m_1} x_1'^{m_1'}) = q^{m_1-m_1'} (x_1^{m_1} x_1'^{m_1'})$, that is, the monomials $x_1^{m_1} x_1'^{m_1'}$, $m_1 + m_1' = m$, are eigenfunctions of the operator defined by the action of $\hat{K}_1$ on $A$ which belong to different eigenvalues. Since the projection $H_m : A_m \rightarrow H_m$ commutes with the action of $U_q(so_N)$, then $\hat{K}_1 \triangleright \varphi_{m_1m_1'}^m = q^{m_1-m_1'} \varphi_{m_1m_1'}^m$. The scalar product of section VI is defined in terms of the invariant functional, that is, this scalar product is invariant with respect to the action of $\hat{K}_i$, $i = 1, 2, \ldots, n$. Since the zonal polynomials $\varphi_{m_1m_1'}^m$, $m_1 + m_1' = m$, belong to different eigenvalues of $\hat{K}_1$, they are orthogonal. Theorem is proved.

It is possible to define zonal polynomials of the space $H_m$ with respect to the subalgebra $A := U_q(so_2) \times U_q(so_{N-2})$, where $U_q(so_2)$ is the subalgebra of $U_q(so_N)$ generated by the element $\hat{K}_1$. Then the following assertions are true.

**Theorem 2.** The subspace of zonal polynomials of the space $H_m$ with respect to the subalgebra $A$ is not more than one-dimensional. The space $H_m$ contains a zonal polynomial if and only if $m$ is even. This zonal polynomial coincides with the polynomial $\varphi_{m/2m/2}^m$ given by formula (35).

Proof easily follows from the above results.

**IX. ASSOCIATED SPHERICAL POLYNOMIALS WITH RESPECT TO $SO_q(N-2)$**

The aim of this section is to construct an orthogonal basis of the space $H_m$ of homogeneous $q$-harmonic polynomials which corresponds to the chain

$$U_q(so_N) \supset U_q(so_{N-2}) \supset \cdots \supset U_q(so_3) \quad (or \quad U_q(so_2)).$$

This basis is a $q$-analogue of the set of associated spherical harmonics on the classical Euclidean space which are products of Jacobi polynomials and correspond to the chain of the subgroups.
SO(N) ⊃ SO(2) × SO(N − 2) ⊃ ... (see [1], Chap. 10). The basis elements give solutions of the equation \( \Delta_q p = 0 \) in “separated coordinates”. So, we obtain a \( q \)-analogue of the classical separation of variables.

Let us note that

\[
\Delta_q \equiv \Delta^{(N)}_q = \sum_{j=1}^{N} q^{\rho_j} \partial_j \partial_j' = (q^{\rho_1} \partial_1 \partial_1' + q^{\rho_2} \partial_2 \partial_2') + \Delta^{(N-2)}_q,
\]

(37)

where \( \Delta^{(N-2)}_q \) is the \( q \)-Laplace operator on the subspace \( A^{(N-2)} \equiv \mathbb{C}[x_2, \ldots, x_{N}] \). We also have from (9) that

\[
\partial_1 \partial_1' - \partial_1' \partial_1 = -\frac{q - q^{-1}}{q^{\rho_1-1} + q^{-\rho_1+1}} \Delta^{(N-2)}_q.
\]

(38)

Let \( p(x_2, \ldots, x_{N}) \) be a polynomial of \( A \) which does not depend on \( x_1 \) and \( x_1' \equiv x_N \). Then it is easy to see from (6) that \( \partial_1 p(x_2, \ldots, x_{N}) = 0 \).

Lemma 1. Let \( p(x_2, \ldots, x_{N}) \in A \) and \( \Delta^{(N-2)}_q p = 0 \). Then \( \partial_1 p = 0 \) and \( \Delta_q p = 0 \).

Proof. Let \( p(x_2, \ldots, x_{N}) \) be a polynomial of \( A \) which does not depend on \( x_1 \) and \( x_1' \equiv x_N \). Then it is easy to see from (6) that \( \partial_1 p(x_2, \ldots, x_{N}) = 0 \).

(6)

Then due to (38) we have \( \partial_1 \partial_1' p = 0 \), and \( \Delta_q p = 0 \) by (37). From formula (7) for \( \partial_1' \), it follows that \( \partial_1 p = x_1 p'(x_2, \ldots, x_{N}) \), where \( p'(x_2, \ldots, x_{N}) \) is a polynomial in \( x_2, x_3, \ldots, x_{N} \). Let us show that from \( \partial_1 \partial_1' p = 0 \) the equality \( \partial_1 p = 0 \) follows. Indeed, due to (6) we have \( \partial_1 \partial_1' p = \partial_1 x_1 p' = \tilde{p}(x_2, \ldots, x_{N}) \), where \( \tilde{p}(x_2, \ldots, x_{N}) \) is some polynomial in \( x_2, x_3, \ldots, x_{N} \), which is a linear combination (with nonvanishing coefficients) of the same monomials as \( p' \) does. Moreover, if \( p' \neq 0 \) then \( \tilde{p} \neq 0 \). Since \( \tilde{p} = 0 \) then \( p' = 0 \) and we have \( \partial_1 p = x_1 p'(x_2, \ldots, x_{N}) = 0 \). It proves the lemma.

Lemma 2. If \( p(x_2, \ldots, x_{N}) \in A \), then

\[
\Delta^{(N-2)}_q (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} x_1'^{m_1'} \Delta^{(N-2)}_q p(x_2, \ldots, x_{N}).
\]

(39)

If \( p(x_2, \ldots, x_{N}) \) is \( \Delta^{(N-2)}_q \)-harmonic, then

\[
\Delta_q (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = (q^{m_1} + q^{-m_1}) \partial_1 \partial_1 x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(40)

Proof. Since \( \partial_1 x_1' = q x_1, \partial_1 = q^{-1} \), it follows that

\[
\Delta_q (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = (q^{m_1} + q^{-m_1}) \partial_1 \partial_1 x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(40)

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]

(see formulas (11)-(13)) we obtain

\[
\partial_2 (x_1^{m_1} x_1'^{m_1'} p(x_2, \ldots, x_{N})) = x_1^{m_1} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q x_1 + (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1') x_1'^{m_1'} - q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N})
\]

\[
= x_1^{m_1} (q - q^{-1}) q^{\rho_1 - \rho_2} \partial_2 x_1'^{m_1'} p(x_2, \ldots, x_{N}).
\]
\[ \Delta q(x_1^m x_2^{m'} p(x_2, \ldots, x_2') = \hat{x}_1^m x_2^{m'} \partial_2 p(x_2, \ldots, x_2'). \]

We have the same results when \( \partial_2 \) and \( \partial_2' \) are replaced by \( \partial_i \) and \( \partial_i' \), \( i = 3, 4, \ldots \). This leads to the relation (39). If \( p \) is \( \Delta_q^{(N-2)} \)-harmonic, then it follows from (38) that \( (\partial_l \partial_1 - \partial_1 \partial_l) x_1^m x_2^{m'} p(x_2, \ldots, x_2') = 0. \) From here and from (37) we derive that

\[ \Delta_q(x_1^m x_2^{m'} p) = (q^{\rho_1} \partial_1 \partial_1' + q^{\rho_2} \partial_1 \partial_1')(x_1^m x_2^{m'} p) = (q^{\rho_1} + q^{\rho_2}) \partial_1 \partial_1' (x_1^m x_2^{m'} p) \]

and the relation (40) is proved. Lemma is proved.

**Proposition 6.** Let \( h \equiv h(x_2, \ldots, x_2') \) be a \( \Delta_q^{(N-2)} \)-harmonic polynomial of degree \( l \). Then

\[ \Delta_q(x_1^m x_2^{m'} h(x_2, \ldots, x_2')) = (q^{\rho_1} + q^{-\rho_1})[m_1 q]\Delta_q^{m_1 + m'_1 - 1} x_1^{m_1 - 1} x_2^{m'_1 - 1} h. \]  

(41)

**Proof.** Using (40) and then (1), (6), (13), we derive that

\[ \Delta_q(x_1^m x_2^{m'} h(x_2, \ldots, x_2')) = (q^{\rho_1} + q^{-\rho_1})[m_1 q]\Delta_q^{m_1 + m'_1 - 1} x_1^{m_1 - 1} \partial_1 \partial_1' x_2^{m'_1} h(x_2, \ldots, x_2'). \]  

(42)

By (11) we have for \( \partial_1 x_2^{m'_1} h \) the expression

\[ \partial_1 x_2^{m'_1} h(x_2, \ldots, x_2') = \left( q^{-1} x_1 \partial_1 - (q - q^{-1}) \sum_{j < N} \hat{x}_j \partial_j \right) \]

\[ + (q - q^{-1}) \hat{x}_1 \partial_1 q^{\rho_1} + c \partial_1 x_2^{m'_1 - 1} h(x_2, \ldots, x_2') \]

\[ = (q \hat{x}_1 \partial_1 - (q - q^{-1}) E + q^{m'_1 - 1} \partial_1 x_2^{m'_1} h(x_2, \ldots, x_2'), \]  

(43)

where \( E = \sum_{k=1}^{N} \hat{x}_k \partial_k \). It is proved by using the relation between \( \hat{x}_i \) and \( \partial_j \) that

\[ E \hat{x}_k = q^{-1} \hat{x}_k E + \frac{q - q^{-1}}{1 + q^{N-2} Q} Q \partial_1 x_2^{m'_1} h(x_2, \ldots, x_2'). \]

(see also [3], Proposition 2.9). Then

\[ E(x_2^{m'_1} h(x_2, \ldots, x_2')) = (q^{-1} \hat{x}_1 E + x_1 q^{l + m'_1 - 1}) x_2^{m'_1 - 1} h(x_2, \ldots, x_2') \]

\[ = (q^{l + m'_1 - 1} + q^{l + m'_1 - 3} + \ldots + q^{l - m'_1 + 1}) x_2^{m'_1} h(x_2, \ldots, x_2') + q^{-m'_1} \hat{x}_1 E h(x_2, \ldots, x_2'). \]

By direct calculation it is proved that

\[ E = \frac{c - c^{-1}}{q - q^{-1}} + \frac{q - q^{-1}}{1 + q^{N-2} Q} Q \Delta_q c^{-1} \]

(see also [3]). Since \( h \in \mathcal{H}_l \), then \( Eh = [l] q h \). Now we have for \( E x_2^{m'_1} h \) the expression

\[ E(x_2^{m'_1} h(x_2, \ldots, x_2')) = (q^{l} m'_1 q + q^{-m'_1} [l] q) x_2^{m'_1} h(x_2, \ldots, x_2') = [m'_1 + l] q x_2^{m'_1} h(x_2, \ldots, x_2'). \]

Therefore, returning to (43) we obtain

\[ \partial_1 x_2^{m'_1} h(x_2, \ldots, x_2') = (q \hat{x}_1 \partial_1 + q^{-m'_1 - 1} l q) x_2^{m'_1 - 1} h(x_2, \ldots, x_2') \]

\[ = (q \hat{x}_1 \partial_1 + q^{-m'_1 - l - 1}) x_2^{m'_1 - 1} h(x_2, \ldots, x_2'). \]
Applying this relation for $x_{1'}^{m_i'-1}h, x_{1'}^{m_i'-2}h, \ldots, x_{1'}h$ and Lemma 1 we receive

$$
\partial_1 x_{1'}^{m_i'} h(x_2, \ldots, x_2) = (q^{1-m_i'-l} + q^{1-m_i'-l+2} + \cdots) x_{1'}^{m_i'-1} h(x_2, \ldots, x_2)
$$

$$
+ q^{m_i'} x_{1'}^{m_i'} \partial_1 h(x_2, \ldots, x_2) = q^{-1} [m'_1]_q x_{1'}^{m_i'-1} h(x_2, \ldots, x_2).
$$

Now using (42) we derive (41). Proposition is proved.

Theorem 1 it follows from here that the sum (47) is orthogonal. Proposition is proved.

operators $\hat{t}$ from different subspaces are linearly independent. Therefore, on the right hand side of (47) we show that dimensions of spaces on both sides of (47) are equal to each other. Now in order to prove our proposition we have to show that the sum on the right hand side of (47) do not pairwise intersect and elements from different subspaces are linearly independent. Therefore, on the right hand side of (47) we have a direct sum. Besides, we have $\mathcal{H}_m \supseteq \bigoplus_{m_1, m'_1} t_{m_1 m'_1}^{N, m} \mathcal{H}_{m=m-m'_1}^{(N-2)}$. By direct computation (by using Corollary 3) we show that dimensions of spaces on both sides of (47) are equal to each other. Now in order to prove our proposition we have to show that the sum on the right hand side is orthogonal.

By means of formula (16) the polynomials $t_{m_1 m'_1}^{N, m}$ can be represented in the form similar to (35') and (35'').

**Proposition 7.** The space $\mathcal{H}_m$ can be represented as the orthogonal sum

$$
\mathcal{H}_m = \bigoplus_{m_1, m'_1} t_{m_1 m'_1}^{N, m} \mathcal{H}_{m-m_1-m'_1}^{(N-2)},
$$

where $\mathcal{H}_{m-m_1-m'_1}^{(N-2)}$ is the space of $\Delta_q^{(N-2)}$-harmonic polynomials in $x_2, x_3, \ldots, x_{2'}$ and summation is over all nonnegative values of $m_1$ and $m'_1$ such that $m - m_1 - m'_1 \geq 0$.

**Proof.** The subspaces $t_{m_1 m'_1}^{N, m} \mathcal{H}_{m-m_1-m'_1}^{(N-2)}$ from (47) do not pairwise intersect and elements from different subspaces are linearly independent. Therefore, on the right hand side of (47) we have a direct sum. Besides, we have $\mathcal{H}_m \supseteq \bigoplus_{m_1, m'_1} t_{m_1 m'_1}^{N, m} \mathcal{H}_{m-m_1-m'_1}^{(N-2)}$. By direct computation (by using Corollary 3) we show that dimensions of spaces on both sides of (47) are equal to each other. Now in order to prove our proposition we have to show that the sum on the right hand side is orthogonal.

It is easy to prove that the subspaces on the right hand side of (47) are eigenspaces of operators $\bar{K}_i, i \leq n$, from formula (5) belonging to different eigenvalues. As in the proof of Theorem 1 it follows from here that the sum (47) is orthogonal. Proposition is proved.

Now we apply the decomposition (47) to the subspaces $\mathcal{H}_{m-m_1-m'_1}^{(N-2)}$ and obtain

$$
\mathcal{H}_m = \bigoplus_{m_1, m'_1, m_2, m'_2} t_{m_1 m'_1}^{N, m} t_{m_2 m'_2}^{N-2, m} \mathcal{H}_{l=m_1-m'_1}^{(N-4)}, \quad l = m - m_1 - m'_1,
$$

15
where $\mathcal{H}^{(N-4)}_{l-m_2-m_2'}$ are the subspaces of homogeneous $q$-harmonic polynomials in $x_3, x_4, \ldots, x_{3'}$. Continuing such decompositions we obtain the decomposition

$$\mathcal{H}_m = \bigoplus_{\mathbf{m}, \mathbf{m}', k} \mathbb{C}\Xi_{\mathbf{m}, \mathbf{m}', k}(x_1, \ldots, x_{1'}) \quad (48)$$

if $N = 2n$ and the decomposition

$$\mathcal{H}_m = \bigoplus_{\mathbf{m}, \mathbf{m}', \sigma} \mathbb{C}\Xi_{\mathbf{m}, \mathbf{m}', \sigma}(x_1, \ldots, x_{1'}) \quad (49)$$

if $N = 2n + 1$, where $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, $\mathbf{m}' = (m_1', m_2', \ldots, m'_{n-1})$ in the first case, $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $\mathbf{m}' = (m_1', m_2', \ldots, m'_{n})$ in the second case and $m_j$ are nonnegative integers, $k$ take integral values, and $\sigma = 0$ or $1$. The basis $q$-harmonic polynomials $\Xi_{\mathbf{m}, \mathbf{m}', k}(x_1, \ldots, x_{1'})$ of $\mathcal{H}_m$ are given by the formula

$$\Xi_{\mathbf{m}, \mathbf{m}', k}(x_1, \ldots, x_{1'}) = t^{N, m}_{m_1 m_1'} t^{N-2, m-m_1-m_1'}_{m_2 m_2'} \cdots t^{4, m-\sum_{i=1}^{n-2} m_i-\sum_{i=1}^{n-2} m_i'}_{m_{n-1} m_{n-1}'} t^{2, k}_{2} \quad (50)$$

if $N = 2n$ and by the formula

$$\Xi_{\mathbf{m}, \mathbf{m}', \sigma}(x_1, \ldots, x_{1'}) = t^{N, m}_{m_1 m_1'} t^{N-2, m-m_1-m_1'}_{m_2 m_2'} \cdots t^{3, m-\sum_{i=1}^{n-1} m_i-\sum_{i=1}^{n-1} m_i'}_{m_{n} m_{n}'} t^{1, \sigma}_{1} x_{n+1} \quad (51)$$

if $N = 2n + 1$. In (50), $t^{2, k} = x^n_k$ if $k > 0$, $t^{2, k} = 1$ if $k = 0$, and $t^{2, k} = x^{-k}_{n'}$ if $k < 0$. Note that the integers $k$, $\sigma$, $\mathbf{m} = (m_1, m_2, \ldots)$ and $\mathbf{m}' = (m_1', m_2', \ldots)$ take such values that

$$m_1 + m_1' + m_2 + m_2' + \cdots + m_{n-1} + m_{n-1}' + k = m \quad (52)$$

for $N = 2n$ and

$$m_1 + m_1' + m_2 + m_2' + \cdots + m_n + m_n' + \sigma = m \quad (53)$$

for $N = 2n + 1$. Besides, conditions like the condition $m - m_1 - m_1' \geq 0$ of Proposition 7 must be fulfilled on each step.

**Theorem 3.** If $N = 2n$ then the polynomials (50) for which the equality (52) is satisfied constitute an orthogonal basis of the space $\mathcal{H}_m$. If $N = 2n + 1$ then the polynomials (51) for which the equality (53) is satisfied constitute an orthogonal basis of the space $\mathcal{H}_m$.

**Proof.** The fact that the polynomials (50) for $N = 2n$ and the polynomials (51) for $N = 2n + 1$ constitute a basis of $\mathcal{H}_m$ was proved above. Orthogonality of basis elements is proved in the same method as in Theorem 1. Theorem is proved.

It is interesting to have explicit formulas how the generators $K_i, E_i, F_i$ of $U_q(\mathfrak{so}_n)$ act on the basis elements of Theorem 3. However, derivation of these formulas are very awkward and the formulas are not simple. We shall consider them in a separate paper.

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