An infinite family of generalized Kalnajs discs

Guillermo A. González* and Jerson I. Reina*
Escuela de Física, Universidad Industrial de Santander, AA 678, Bucaramanga, Colombia

Accepted 2006 July 17. Received 2006 June 29; in original form 2006 January 11

ABSTRACT
An infinite family of axially symmetric thin discs of finite radius is presented. The family of discs is obtained by means of a method developed by Hunter and contains, as its first member, the Kalnajs disc. The surface densities of the discs present a maximum at the centre of the disc and then decrease smoothly to zero at the edge, in such a way that the mass distribution of the higher members of the family is more concentrated at the centre. The first member of the family has a circular velocity proportional to the radius, thus representing a uniformly rotating disc. On the other hand, the circular velocities of the other members of the family increase from a value of zero at the centre of the discs to a maximum and then decrease smoothly to a finite value at the edge of the discs, in such a way that, for the higher members of the family, the maximum value of the circular velocity is attained nearest the centre of the discs.

Key words: stellar dynamics – galaxies: kinematics and dynamics.

1 INTRODUCTION
A fact usually assumed in astrophysics is that the main part of the mass of a typical spiral galaxy is concentrated in a thin disc (Binney & Tremaine 1987). Accordingly, the obtention of the gravitational potential generated by an idealized thin disc is a problem of great astrophysical relevance and so, through the years, different approaches have been used to obtain such kinds of thin disc models. Wyse & Mayall (1942) studied thin discs by superposing an infinite family of elementary discs of different radii. Brandt (1960) and Brandt & Belton (1962) constructed flat galaxy discs by the flattening of a distribution of matter whose surfaces of equal density were similar spheroids. A simple potential–density pair for a thin disc model was introduced by Kuzmin (1956) and then rederived by Toomre (1963, 1964) as the first member of a generalized family of models. The Toomre models are obtained by solving the Laplace equation in cylindrical coordinates subject to appropriated boundary conditions on the disc and at infinity. The Kuzmin and Toomre models of thin discs, although they have surface densities and rotation curves with remarkable properties, represent discs of infinite extension and thus they are rather poor flat galaxy models. Accordingly, in order to obtain more realistic models of flat galaxies, it is better to consider methods that permit the obtention of finite thin disc models.

A simple method to obtain the surface density, the gravitational potential and the rotation curve of thin discs of finite radius was developed by Hunter (1963). The Hunter method is based in the obtention of solutions of the Laplace equation in terms of oblate spheroidal coordinates, which are ideally suited to the study of flat discs of finite extension. By superposition of solutions of the Laplace equation, expressions for the surface density of the discs, the gravitational potential and its rotational velocity can be obtained as series of elementary functions.

The simplest example of a thin disc obtained by means of the Hunter method is the well-known Kalnajs disc (Kalnajs 1972), which can also be obtained by flattening a uniformly rotating spheroid (Wyse & Mayall 1942; Brandt 1960; Brandt & Belton 1962). The Kalnajs disc has a well-behaved surface density and represents a uniformly rotating disc, so that its circular velocity is proportional to the radius, and its stability properties have been extensively studied (see, for instance, Hunter 1963, 1965; Kalnajs 1972; Kalnajs & Athanassoula-Georgala 1974).

In this paper, we use the Hunter method in order to obtain an infinite family of thin discs of finite radius. We particularize the Hunter general model by considering a family of thin discs with a well-behaved surface mass density. We will require that the surface density be a monotonically decreasing function of the radius, with a maximum at the centre of the disc and vanishing at the edge, in such a way that the mass distribution of the higher members of the family be more concentrated at the centre.

The paper is organized as follows. In Section 2, we present a summary of the Hunter method used to obtain the thin disc models of finite radius and also we obtain the general expressions for the gravitational potential, the surface density and the circular velocity. In the next section, Section 3, we present the particular family of models obtained by imposing the required behaviour of the surface densities and then, in Section 4, we analyse its physical behaviour. Finally, in Section 5, we summarize our main results.

2 GENERAL FINITE THIN DISC MODELS
In order to obtain finite axially symmetric thin disc models, we need to find solutions of the Laplace equation that represent the outer potential of a thin disc-like source. According to this, we need...
to solve the Laplace equation for an axially symmetric potential,
\[ \Phi,_{RR} + \frac{\Phi,_{R}}{R} + \Phi,_{zz} = 0, \]
(1)
where \( R, \phi, z \) are the usual cylindrical coordinates. We will suppose that, besides the axial symmetry, the gravitational potential has a symmetry of reflection with respect to the plane \( z = 0 \),
\[ \Phi(R, z) = \Phi(R, -z), \]
(2)
so that the normal derivative of the potential, \( \partial \Phi / \partial z \), satisfies the relation
\[ \frac{\partial \Phi}{\partial z}(R, -z) = -\frac{\partial \Phi}{\partial z}(R, z), \]
(3)
in agreement with the attractive character of the gravitational field. We also assume that \( \partial \Phi / \partial z \) does not vanish on the plane \( z = 0 \), in order to have a thin distribution of matter that represents the disc.

Given a potential \( \Phi(R, z) \) with the above properties, the density \( \Sigma(R) \) of the surface distribution of matter can be obtained using the Gauss law (Binney & Tremaine 1987). So, using the equation (3), we obtain
\[ \Sigma(R) = \frac{1}{2 \pi a G} \left( \frac{\partial \Phi}{\partial z} \right)_{z=0^+}. \]
(4)

Now, in order to have a surface density corresponding to a finite disc-like distribution of matter, we impose the boundary conditions
\[ \frac{\partial \Phi}{\partial z}(R, 0^+) \neq 0; \ R \leq a, \]
(5a)
\[ \frac{\partial \Phi}{\partial z}(R, 0^+) = 0; \ R > a, \]
(5b)
so that the matter distribution is restricted to the disc \( z = 0, 0 \leq R < a \).

We introduce now the oblate spheroidal coordinates, whose symmetry adapts in a natural way to the geometry of the model. These coordinates are related to the usual cylindrical coordinates by the relation (Morse & Feshbach 1953)
\[ R = a \sqrt{(1 + \xi^2)(1 - \eta^2)}, \]
(6a)
\[ z = a \xi \eta, \]
(6b)
where \( 0 \leq \xi < \infty \) and \(-1 \leq \eta < 1\). The disc has the coordinates \( \xi = 0, 0 \leq \eta < 1 \). On crossing the disc, \( \eta \) changes sign but does not change in absolute value. This singular behaviour of the coordinate \( \eta \) implies that an even function of \( \eta \) is a continuous function everywhere but has a discontinuous \( \eta \) derivative at the disc.

In terms of the oblate spheroidal coordinates, the Laplace equation can be written as
\[ [(1 + \xi^2) \Phi,_{\xi}]_{\xi} + [(1 - \eta^2) \Phi,_{\eta}]_{\eta} = 0, \]
(7)
and we need to find solutions that are even functions of \( \eta \) and with the boundary conditions
\[ \Phi,_{\xi}(0, \eta) = \Phi(q), \]
(8a)
\[ \Phi,_{\eta}(\xi, 0) = 0, \]
(8b)
where \( F(q) \) is an even function that can be expanded in a series of Legendre polynomials in the interval \(-1 \leq \eta \leq 1\) (Bateman 1944).

According to this, the Newtonian gravitational potential for the exterior of a finite thin disc with axially symmetric matter density can be written as (Bateman 1944)
\[ \Phi(\xi, \eta) = -\sum_{n=0}^{\infty} C_{2n} Q_{2n}(\xi) P_{2n}(\eta), \]
(9)
where \( C_{2n} \) are arbitrary constants, and \( P_{2n}(\eta) \) and \( Q_{2n}(\xi) \) are the Legendre polynomials and the Legendre functions of the second kind, respectively. With this general solution for the gravitational potential, the surface matter density is given by
\[ \Sigma(R) = \frac{1}{2 \pi a G} \sum_{n=0}^{\infty} C_{2n}(2n + 1) q_{2n+1}(0) P_{2n}(\eta) \]
(10)
and, as we will show later, the arbitrary constants \( C_{2n} \) must be chosen properly so that the surface density presents a physically reasonable behaviour.

Besides the matter density, another quantity commonly used to characterize galactic matter distributions is the circular velocity \( V(R) \), also called the rotation curve, defined as the tangential velocity of the stars in circular orbits around the centre. Now, given \( \Phi(R, z) \), we can easily evaluate \( V \) through the relation
\[ V^2 = R \left( \frac{\partial \Phi}{\partial R} \right)_{z=0^+}, \]
(11)
in such a way that, by using equation (9), we obtain
\[ V^2(R) = \frac{R^2}{(a^2 - R^2)\xi} \sum_{n=0}^{\infty} C_{2n} q_{2n}(0) P_{2n}(\eta), \]
(12)
for the circular velocity.

### 3 THE GENERALIZED KALNAJS DISCS

We will now particularize the above general model by considering a family of finite thin discs with a well-behaved surface mass density. We will require that the surface density will be a monotonically decreasing function of the radius, with a maximum at the centre of the disc and vanishing at the edge. In order to do this, we impose the conditions
\[ \Sigma(0) = \Sigma_{\text{max}}, \]
(13)
\[ \Sigma(\eta) = 0, \]
(14)
and we also require that
\[ M = 2\pi \int_0^\infty \Sigma(R) R dR, \]
(15)
where \( M \) is the total mass of the disc.

Now, by using the boundary condition (8a), the surface density can be written in the form
\[ \Sigma(R) = \frac{F(\eta)}{2\pi a G \eta}, \]
(16)
where \( F(\eta) \) is an even function of \( \eta \), monotonically increasing at the interval \( 0 \leq \eta \leq 1 \), and is such that
\[ \lim_{\eta \to 0} \frac{F(\eta)}{\eta} = 0. \]
(17)
Furthermore, we must impose the condition
\[ \int_0^1 F(\eta) d\eta = \frac{MG}{a}, \]
(18)
in agreement with equation (15).

A simple function \( F(\eta) \) that agrees with all the above requirements was given by Letelier & Oliveira (1987) and can be written as
\[ F(\eta) = (2m + 1) \frac{MG}{a} \eta^{2m}. \]
(19)
As we can easily see, the disc with \(m = 1\) corresponds to the well-known Kalnajs disc (Kalnajs 1972). Accordingly, this family of finite thin discs can then be considered as a generalization of the Kalnajs disc. 

Now, from equation (10), the function \(F(\eta)\) can be written as

\[
F(\eta) = \sum_{n=0}^{+\infty} K_{2n} P_{2n}(\eta),
\]

with

\[
K_{2n} = (2n + 1)q_{2n+1}(0)C_{2n}.
\]

The coefficients \(K_{2n}\) are founded, by using the orthogonality property of the Legendre polynomials, through the expression

\[
K_{2n} = \frac{4n+1}{2} \int_{-1}^{1} F(\eta) P_{2n}(\eta) d\eta.
\]

The above equation can be expressed as (Bateman 1953)

\[
K_{2n} = \frac{MG}{2a} \left[ \frac{\pi \Gamma(2n+1)(2m+1}\Gamma(2m+1)}{2^{2n+1}\Gamma(n+1/2)\Gamma(n+3/2)} \right],
\]

so that, using the gamma function properties, we obtain

\[
C_{2n} = \frac{MG}{2a} \left[ \frac{\pi \Gamma(2n+1)(2m+1)!}{2^{2n}(2n+1)(m-n)!\Gamma(m+n+3/2)} q_{2n+1}(0) \right],
\]

for \(n \leq m\) and \(C_{2n} = 0\) for \(n > m\).

## 4 BEHAVIOUR OF THE MODELS

With the above values of \(C_{2n}\), we can compute the different physical quantities that characterize the behaviour of the models. So, for instance, the gravitational potential of the first three members of the family are given by

\[
\Phi_1(\xi) = -\frac{MG}{a} \left[ \cot^{-1} \xi + A(3\eta^2 - 1) \right],
\]

\[
\Phi_2(\xi) = -\frac{MG}{a} \left[ \cot^{-1} \xi + \frac{10A}{3}(3\eta^2 - 1) \right. \\
\left. + B(35\eta^4 - 30\eta^2 + 3) \right],
\]

\[
\Phi_3(\xi) = -\frac{MG}{a} \left[ \cot^{-1} \xi + \frac{10A}{6}(3\eta^2 - 1) \right. \\
\left. + \frac{21B}{11}(35\eta^4 - 30\eta^2 + 3) \right. \\
\left. + C(231\eta^6 - 315\eta^4 + 105\eta^2 - 5) \right],
\]

where

\[
A = \frac{1}{4} [(3\xi^2 + 1) \cot^{-1} \xi - 3\xi],
\]

\[
B = \frac{3}{448} \left[ 35\xi^4 + 30\xi^2 + 3 \right] \cot^{-1} \xi - 35\xi^3 - \frac{55}{3} \xi^5.
\]

The above equation can be expressed as (Bateman 1953)

\[
C = \frac{3}{448} \left[ 35\xi^4 + 30\xi^2 + 3 \right] \cot^{-1} \xi - 35\xi^3 - \frac{55}{3} \xi^5,
\]

\[
-231\xi^6 - 238\xi^4 - \frac{231}{5} \xi^5,
\]

\[
\Phi_1(R, 0) = \frac{3\pi MG}{8a} R^2,
\]

for \(R \leq a\), and this expression is completely equivalent to the corresponding expression in Kalnajs (1972).

In the same way, for the surface mass densities of the first three members of the family, we obtain the expressions

\[
\Sigma_1 = \frac{3M}{\pi a^2} (1 - R^2)^{1/2},
\]

\[
\Sigma_2 = \frac{5M}{\pi a^2} (1 - R^2)^{3/2},
\]

\[
\Sigma_3 = \frac{7M}{\pi a^2} (1 - R^2)^{5/2},
\]

and, for the corresponding circular velocities, the expressions

\[
V_1^2 = \frac{3\pi MG}{4a} R^2,
\]

\[
V_2^2 = \frac{15\pi MG}{32a} \tilde{R}^2(4 - 3\tilde{R}^2),
\]

\[
V_3^2 = \frac{105\pi MG}{256a} \tilde{R}^2(5\tilde{R}^4 - 12\tilde{R}^2 + 8),
\]

where the dimensionless radial variable \(\tilde{R} = R/a\) has been introduced.

In order to graphically illustrate the behaviour of the different particular models, we first introduce the dimensionless surface density of the discs, defined as

\[
\Sigma_n(R) = \frac{M}{\pi a^2} \tilde{\Sigma}_n(\tilde{R}),
\]

for \(0 \leq \tilde{R} \leq 1\). So, in Fig. 1 are depicted the dimensionless surface mass densities \(\tilde{\Sigma}_n\) for the models corresponding to \(m = 1, \ldots, 8\). As we can see, the discs with higher values of \(m\) present a mass distribution that is more concentrated at the centre and less at the edge. Accordingly, these discs can then be considered as appropriated models of galaxies with a central bulge.

Now, in order to graphically illustrate the behaviour of the circular velocities or rotation curves, we introduce the dimensionless quantity

\[
V_n(R) = \sqrt{\frac{MG}{a} \tilde{V}_n(\tilde{R})}.
\]

We plot, in Fig. 2, the dimensionless rotation curves for the models corresponding to \(m = 1, \ldots, 10\). The circular velocity corresponding to \(m = 1\) is proportional to the radius, thus representing a uniformly rotating disc. On the other hand, for \(m > 1\), the circular velocity increases from a value of zero at the centre of the discs until it attains a maximum at a critical radius and then decreases to a finite value at the edge of the disc. Also we can see that the value of the critical radius decreases as the value of \(m\) increases.
5 CONCLUDING REMARKS

We presented an infinite family of axially symmetric thin discs of finite radius obtained by means of a particularization of the Hunter method. The disc models so obtained are generalizations of the well-known Kalnajs disc, which corresponds to the first member of the family. The particularization of the Hunter model was obtained by requiring that the surface density was a monotonically decreasing function of the radius, with a maximum at the centre of the disc and vanishing at the edge, in such a way that the mass distribution of the higher members of the family were more concentrated at the centre.

We also analysed the rotation curves of the models and we find: for the first member of the family, the Kalnajs disc, a circular velocity proportional to the radius, thus representing a uniformly rotating disc; whereas, for the other members of the family, the circular velocity increases from a value of zero at the centre of the discs until it reaches a maximum at a critical radius and then decreases to a finite value at the edge of the disc. Also, we find that the value of the critical radius decreases as the value of \( m \) increases.

We believe that the obtained thin disc models have some remarkable properties and so they can be considered as appropriated realistic flat galaxy models, in particular if the superposition of these thin discs with appropriated halo distributions (Binney & Tremaine 1987) is considered. We are now considering some research in this direction. We are now also working on the non-axially symmetric generalization of the discs models presented here and also on the obtention of the relativistic generalization of them for the axially symmetric case.

ACKNOWLEDGMENTS

The authors want to give thanks for the financial support from COLCIENCIAS, Colombia.

REFERENCES

Bateman H., 1944, Partial Differential Equations. Dover, New York
Bateman H., 1953, Higher Transcendental Functions Vol. 1. McGraw Hill, New York
Binney J., Tremaine S., 1987, Galactic Dynamics. Princeton Univ. Press, Princeton
Brandt J. C., 1960, ApJ, 131, 211
Brandt J. C., Belton M. J. S., 1962, ApJ, 136, 352
Hunter C., 1963, MNRAS, 126, 299
Hunter C., 1965, MNRAS, 129, 321
Kalnajs A. J., 1972, ApJ, 175, 63
Kalnajs A. J., Athanassoula-Georgala E., 1974, MNRAS, 168, 287
Kuzmin G., 1956, Astron. Zh., 33, 27
Letelier P. S., Oliveira S. R., 1987, J. Math. Phys., 28, 165
Morse P. M., Fesbach H., 1953, Methods of Theoretical Physics. McGraw Hill, New York
Toomre A., 1963, ApJ, 138, 385
Toomre A., 1964, ApJ, 139, 1217
Wyse A. B., Mayall N. U., 1942, ApJ, 95, 24

This paper has been typeset from a \TeX/\LaTeX file prepared by the author.