A HITCHHIKER’S GUIDE TO KHOVANOV HOMOLOGY

by

Paul Turner

I dedicate these notes to the memory of Ruty Ben-Zion

Abstract. — These notes from the 2014 summer school Quantum Topology at the CIRM in Luminy attempt to provide a rough guide to a selection of developments in Khovanov homology over the last fifteen years.

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Foreword

There are already too many introductory articles on Khovanov homology and another is not really needed. On the other hand by now - 15 years after the invention of subject - it is quite easy to get lost after having taken those first few steps. What could be useful is a rough guide to some of the developments over that time and the summer school Quantum Topology at the CIRM in Luminy has provided the ideal opportunity for thinking about what such a guide should look like. It is quite a risky undertaking because it is all too easy to offend by omission, misrepresentation or other. I have not attempted a complete literature survey and inevitably these notes reflects my personal view, jaundiced as it may often be. My apologies in advance for any offense caused.
In this preprint version I have added arXiv reference numbers as marginal annotations. If you are reading this electronically these are live links taking you directly to the appropriate arXiv page of the article being referred to.

I would like to express my warm thanks to David Cimasoni, Lukas Lewark, Alex Shumakovitch, Liam Watson and Ben Webster.

1. A beginning

There are a number of introductions to Khovanov homology. A good place to start is Dror Bar-Natan’s exposition of Khovanov’s work

– On Khovanov’s categorification of the Jones polynomial (Bar-Natan, [BN02]),

followed by Alex Shumakovitch’s introduction

– Khovanov homology theories and their applications (Shumakovitch, [Shu11a]),

not forgetting the original paper by Mikhail Khovanov

– A categorification of the Jones polynomial (Khovanov, [Kho00]).

Another possible starting point is

– Five lectures on Khovanov homology (Turner, [Tur06a]).

1.1. There is a link homology theory called Khovanov homology. — What are the minimal requirements of something deserving of the name link homology theory? We should expect a functor

\[ H : \text{Links} \rightarrow A \]

where \text{Links} is some category of links in which isotopies are morphisms and \( A \) another category, probably abelian, where we have in mind the category of finite dimensional vector spaces, \( \text{Vect}_R \), or of modules, \( \text{Mod}_R \), over a fixed ring \( R \). This functor should satisfy a number of properties.

– Invariance. If \( L_1 \rightarrow L_2 \) is an isotopy then the induced map \( H(L_1) \rightarrow H(L_2) \) should be an isomorphism.

– Disjoint unions. Given two disjoint links \( L_1 \) and \( L_2 \) we want the union expressed in terms of the parts

\[ H(L_1 \sqcup L_2) \cong H(L_1) \square H(L_2) \]

where \( \square \) is some monoidal operation in \( A \) such as \( \oplus \) or \( \otimes \).

– Normalisation. The value of \( H(\text{unknot}) \) should be specified. (Possibly also the value of the empty knot).

– Computational tool. We want something like a long exact sequence which relates homology of a given link with associated “simpler” ones - something like the Meyer-Vietoris sequence in ordinary homology.

If these are our expectations then Khovanov homology is bound to please. Let us take \text{Links} to be the category whose objects are oriented links in \( S^3 \) and whose morphisms are link cobordisms, that is to say compact oriented surfaces-with-boundary in \( S^3 \times I \) defined up to isotopy. All manifolds are assumed to be smooth.
Theorem 1.1 (Existence of Khovanov homology). — There exists a (covariant) functor
\[ \text{Kh} : \text{Links} \to \text{Vect}_{\mathbb{F}_2} \]
satisfying
1. If \( \Sigma : L_1 \to L_2 \) is an isotopy then \( \text{Kh}(\Sigma) : \text{Kh}(L_1) \to \text{Kh}(L_2) \) is an isomorphism,
2. \( \text{Kh}(L_1 \sqcup L_2) \cong \text{Kh}(L_1) \otimes \text{Kh}(L_2) \),
3. \( \text{Kh} (\text{unknot}) = \mathbb{F}_2 \oplus \mathbb{F}_2 \) and \( \text{Kh}(\emptyset) = \mathbb{F}_2 \),
4. If \( L \) is presented by a link diagram a small piece of which is \( \text{ } \) then there is an
exact triangle
\[
\text{Kh}(\text{ }) \longrightarrow \text{Kh}(\text{ } \text{ }) \longrightarrow \text{Kh}(\text{ } \text{ } )
\]

In fact a little more is needed to guarantee something non-trivial and in addition to the
above we demand that \( \text{Kh} \) carries a bigrading
\[ \text{Kh}^*,* (L) = \bigoplus_{i,j \in \mathbb{Z}} \text{Kh}^{i,j}(L) \]
and with respect to this
- a link cobordism \( \Sigma : L_1 \to L_2 \) induces a map \( \text{Kh}(\Sigma) \) of bidegree \((0, \chi(\Sigma))\),
- the generators of the unknot have bidegree \((0, 1)\) and \((0, -1)\) (and for the empty knot
bidegree \((0, 0)\)),
- the exact triangle unravels as follows:
  Case I: \( \text{ } \) For each \( j \) there is a long exact sequence
\[
\delta \longrightarrow \text{Kh}^{i,j+1}(\text{ }) \longrightarrow \text{Kh}^{i,j}(\text{ } \text{ }) \longrightarrow \text{Kh}^{i-\omega,j-1-3\omega}(\text{ } \text{ }) \longrightarrow \delta \longrightarrow \text{Kh}^{i+1,j+1}(\text{ } \text{ }) \longrightarrow
\]
where \( \omega \) is the number of negative crossings in the chosen orientation of \( \text{ } \) minus
the number of negative crossings in \( \text{ } \).
  Case II: \( \text{ } \) For each \( j \) there is a long exact sequence
\[
\longrightarrow \text{Kh}^{i+1,j-1}(\text{ } \text{ }) \delta \longrightarrow \text{Kh}^{i-1-c,j-2-3c}(\text{ } \text{ }) \longrightarrow \text{Kh}^{i,j}(\text{ } \text{ }) \longrightarrow \text{Kh}^{i,j-1}(\text{ } \text{ }) \delta \longrightarrow
\]
where \( c \) is the number of negative crossings in the chosen orientation of \( \text{ } \) minus
the number of negative crossings in \( \text{ } \).

To prove the theorem one must construct such a functor, but first let’s see a few conse-
quences relying only on existence and standard results.

Proposition 1.2. — If a link \( L \) has an odd number of components then \( \text{Kh}^{*,\text{even}}(L) \) is triv-
ial. If it has an even number of components then \( \text{Kh}^{*,\text{odd}}(L) \) is trivial.

Proof. — The proof is by induction on the number of crossing and uses the following
elementary result.
Lemma 1.3. — In the discussion of the long exact sequences above (i) if the strands featured at the crossing are from the same component then $\omega$ is odd and $c$ is even, and (ii) if they are from different components then $\omega$ is even and $c$ odd.

For the inductive step we use this and, depending on the case, one of the long exact sequence shown above, observing that in each case two of the three groups shown are trivial.

Proposition 1.4. — If $L^!$ denotes the mirror image of the link $L$ then $Kh^{i,j}(L^!) \cong Kh^{i,-j}(L)$.

Proof. — There is a link cobordism $\Sigma : L^! \sqcup L \to \emptyset$ with $\chi(\Sigma) = 0$ obtained by bending the identity cobordism (a cylinder) $L \to L$. Since $Kh$ is a functor there is an induced map of bidegree $(0,\chi(\Sigma)) = (0,0)$

$$\Sigma_* : Kh^{*,*}(L^!) \otimes Kh^{*,*}(L) \to Kh^{*,*}(\emptyset) = F_2.$$

By a standard “cylinder straightening isotopy” argument

$$\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{cylinder.png}
\end{array}
\end{array}$$

the bilinear form is non-degenerate, and the result follows recalling that we are in a bi-graded setting so

$$(Kh^{*,*}(L^!) \otimes Kh^{*,*}(L))^{(0,0)} = \bigoplus_{i,j} Kh^{i,j}(L^!) \otimes Kh^{i,-j}(L).$$

Exercise 1. — Theorem 1.1 includes the statement that $Kh(\text{unknot}) = F_2 \oplus F_2$. In fact we could assume the weaker statement: the homology of the unknot is concentrated in degree zero. Use this along with the diagram $\includegraphics[scale=0.5]{unknot.png}$, the long exact sequence, the property on disjoint unions and the invariance of Khovanov homology to show that $\dim(Kh(\text{unknot})) = 2$.

Proposition 1.5. — For any oriented link $L$,

$$\frac{1}{t^{\frac{i}{2}} + t^{-\frac{i}{2}}} \sum_{i,j} (-1)^{i+j+1} t^{\frac{i}{2}} \dim(Kh^{i,j}(L))$$

is the Jones polynomial of $L$.

Proof. — Let $P(L) = \Sigma_{i,j} (-1)^j q^i \dim(Kh^{i/j}(L))$ and suppose $L$ is represented by a diagram $D$. The alternating sum of dimensions in a long exact sequence of vector spaces is always zero, so from the long exact sequence for a negative crossing we have that for each $j \in \mathbb{Z}$ the sum

$$\sum_i (-1)^i \dim(Kh^{i,j+1}(\includegraphics[scale=0.5]{negative_crossing.png})) - \sum_i (-1)^i \dim(Kh^{i,j}(\includegraphics[scale=0.5]{positive_crossing.png})) + \sum_i (-1)^i \dim(Kh^{i,-j-3}(\includegraphics[scale=0.5]{negative_crossing.png})).$$
is zero. Written in terms of the polynomial $P$ this becomes

$$q^{-1}P\left(\begin{array}{c}
q
\end{array}\right) - P\left(\begin{array}{c}
q
\end{array}\right) + (-1)^{\omega}q^{1+3\omega}P\left(\begin{array}{c}
q
\end{array}\right) = 0.$$  

Similarly, using the long exact sequence for a positive crossing (noting that $c = \omega + 1$) we get

$$(-1)^{\omega}q^{5+3\omega}P\left(\begin{array}{c}
q
\end{array}\right) - P\left(\begin{array}{c}
q
\end{array}\right) + qP\left(\begin{array}{c}
q
\end{array}\right) = 0.$$  

Combining these gives

$$q^{-2}P\left(\begin{array}{c}
q
\end{array}\right) - q^2P\left(\begin{array}{c}
q
\end{array}\right) + (q-q^{-1})P\left(\begin{array}{c}
q
\end{array}\right) = 0$$  

which becomes the skein relation of the Jones polynomial when $q = -t^\frac{1}{2}$. Since $P(\text{unknot}) = q + q^{-1} = -(t^\frac{1}{2} + t^{-\frac{1}{2}})$, the uniqueness of the Jones polynomial gives

$$P(D)\big|_{q=-t^{\frac{1}{2}}} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})J(D)$$  

whence the result. □

**Remark 1.6.** — Rasmussen has given tentative definition of what a knot homology theory should encompass (somewhat different from the expectations given above) discussing both Khovanov homology and Heegaard-Floer knot homology (Rasmussen, [Ras05]).

### 1.2. Reduced Khovanov homology

There is a further piece of structure induced on Khovanov homology defined in the following way. The Khovanov homology of the unknot is a ring with unit courtesy of the cobordisms $\bigcirc$ and $\otimes$ which induce multiplication and unit respectively. The Khovanov homology of a link $L$ together with a chosen point $p$ is a module over this ring, using the link cobordism indicated. A priori this module structure depends on the point $p$ and in particular on the component of $L$ to which $p$ belongs. Although it does not follow from the existence theorem directly, for the version of Khovanov homology presented above (namely over $\mathbb{F}_2$), this structure does not depend on these choices. In fact more is true (again not immediate from the existence theorem) and the structure of $Kn^{\ast\ast}(L)$ over $U^{\ast\ast} = \overline{Kh}^{\ast\ast}(\text{unknot})$ can be described as follows: there exists a bigraded vector space $\overline{Kh}^{\ast\ast}(L)$ with the property that

$$Kn^{\ast\ast}(L) \cong \overline{Kh}^{\ast\ast}(L) \otimes U^{\ast\ast}.$$  

**Theorem 1.7 (Existence of reduced Khovanov homology).** — There exists a (covariant) functor

$$\overline{Kh}^{\ast\ast}: \text{Links} \rightarrow \text{Vect}_{\mathbb{F}_2}$$  

satisfying

1. If $\Sigma: L_1 \rightarrow L_2$ is an isotopy then $\overline{Kh}(\Sigma)$ is an isomorphism,
2. $\overline{Kh}^{\ast\ast}(L_1 \sqcup L_2) \cong \overline{Kh}^{\ast\ast}(L_1) \otimes \overline{Kh}^{\ast\ast}(L_2) \otimes U^{\ast\ast},$
3. $\overline{Kh}^{\ast\ast}(\text{unknot}) = \mathbb{F}_2$ in bidegree $(0, 0),$
4. $\overline{Kh}^{\ast\ast}$ satisfies the same long exact sequence (with the same bigradings) written down previously for the unreduced case.
There is a question about which category of links we should be using here. A natural one would be links with a marked point and cobordisms with a marked line. If $L$ has an odd number of components then $\widetilde{Kh}^{\text{odd}}$ is trivial and if it has an even number of components then $\widetilde{Kh}^{\text{even}}$ is trivial. By a similar argument to what we saw previously we have

$$\sum_{i,j} (-1)^{i+j} t^{i} \dim(\widetilde{Kh}^{i,j}(L)) = J(L).$$

In order to compute Khovanov homology we should use our tools for that purpose which are the two long exact sequences.

**Exercise 2 (beginners).** — Using the long exact sequences calculate the reduced Khovanov homology of the Hopf link, left and right trefoils, and the figure eight knot.

**Exercise 3 (experts).** — Find the first knot in the tables for which the reduced Khovanov homology can not be calculated using only the long exact sequences and calculations of the reduced Khovanov homology of knots and links occurring previously in the tables.

Alternating links have particularly simple Khovanov homology (Lee, [Lee05]).

**Proposition 1.8.** — For a non-split alternating link $L$ the vector space $\widetilde{Kh}^{i,j}(L)$ is trivial unless $j - 2i$ is the signature of $L$.

As a corollary we note that for an alternating link $\widetilde{Kh}^{*,*}(L)$ is completely determined by the Jones polynomial and signature. Lee’s result can also be proved using an approach to Khovanov homology using spanning trees (Wehrli, [Weh08]).

**Remark 1.9.** — The Khovanov homology of the unknot has more structure than that of a ring. There are also cobordisms $\emptyset$ and $\odot$ which induce maps at the algebraic level. These maps and the ones above are subject to relations determined by the topology of surfaces. The upshot is that $U^{**}$ is a Frobenius algebra.

### 1.3. Integral Khovanov homology.

One can also define an integral version of the theory which has long exact sequences as above, but some changes are necessary.

1. Functoriality is much trickier (see section 2.3 below for references)
   - up to sign $\pm 1$ everything works okay,
   - strict functoriality requires work.

2. There is a reduced version but
   - it is dependent on the component of the marked point,
   - the relationship to the unreduced theory is more complicated and is expressed via a long exact sequence,

   $$\delta : \widetilde{Kh}^{i,j+1}_Z(L, L_\alpha) \rightarrow \widetilde{Kh}^{i,j}_Z(L) \rightarrow \widetilde{Kh}^{i,j-1}_Z(L, L_\alpha) \rightarrow \widetilde{Kh}^{i,j+1}_Z(L, L_\alpha) \delta$$

   where $L_\alpha$ is a chosen component of $L$,
   - in this exact sequence the coboundary map $\delta$ is zero modulo 2.
The integral theory is related to the $\mathbb{F}_2$-version by a universal coefficient theorem. There is a short exact sequence:

$$0 \rightarrow Kh_i^j(L) \otimes \mathbb{F}_2 \rightarrow Kh_i^j(L) \rightarrow \text{Tor}(Kh_i^{i+1,j}(L), \mathbb{F}_2) \rightarrow 0$$

There is a similar universal coefficient theorem relating the two reduced theories.

Any theory defined over the integers has a chance of revealing interesting torsion phenomena. Unreduced integral Khovanov homology has a lot of 2-torsion and much of this arises in the passage from reduced to unreduced coming from that fact that in the long exact sequence relating the two theories the coboundary map is zero mod 2. Correspondingly the reduced theory has much less 2-torsion.

**Proposition 1.10.** — The reduced integral Khovanov homology of alternating links has no 2-torsion.

**Proof.** — Suppose that $L$ is non-split. Any 2-torsion in $\widetilde{Kh}_i^j(L)$ would contribute non-trivial homology in $\widetilde{Kh}_i^j(L)$ via the leftmost group in the universal coefficient theorem for reduced theory and also in $\widetilde{Kh}_i^{i-1,j}(L)$ via the Tor group. This contradicts the conclusion of Proposition 1.8, namely that there is only non-trivial homology when $j-2i = \text{signature}(L)$.

In general torsion is not very well understood. Calculations (by Alex Shumakovitch) show

– the simplest knot having 2-torsion in reduced homology has 13 crossings, for example 13n3663,
– the simplest knot having odd torsion in unreduced homology is $T(5, 6)$ which has a copy of $\mathbb{Z}/3$ and a copy of $\mathbb{Z}/5$,
– the simplest knot having odd torsion in reduced homology is also $T(5, 6)$ which has a copy of $\mathbb{Z}/3$,
– some knots, e.g. $T(5, 6)$, have odd torsion in unreduced homology which is not seen in the reduced theory, but the other way around is also possible: $T(7, 8)$ has an odd torsion group in reduced that is not seen in unreduced.

Torus knots are a very interesting source of odd torsion and in fact almost all odd torsion observed so far has been for torus knots. There is, however, an example of a non-torus knot 5-braid which has 5-torsion (Przytycki and Sazdanović, [PS14]). In general torsion remains quite a mystery.

### 2. Constructing Khovanov homology

The central combinatorial input in the construction of Khovanov homology is a hypercube decorated by modules known variously as “the cube”, “the cube of resolutions” and “the Khovanov cube” and it is constructed from a diagram representing the link in question. This cube of resolutions is actually an example of something more general: it is a Boolean lattice equipped with a local coefficient system and this is the point of view we take in these notes.
2.1. Posets and local coefficient systems. — Let \( P \) be a partially ordered set and \( R \) a ring. For Khovanov homology the posets of most importance are Boolean lattices: for a set \( S \), the Boolean lattice on \( S \) is the poset \( \mathbb{B}(S) \) consisting of the set of subsets of \( S \) ordered by inclusion. The Hasse diagram of the Boolean lattice \( \mathbb{B}(\{a, b, c\}) \) is shown here. The Hasse diagram of a poset \( P \) is the graph having vertices the elements of \( P \) and an edge \( A \rightarrow B \) if and only if \( A < B \) and there is no \( C \) such that \( A < C < B \) (in this case say \( B \) covers \( A \)). We will adopt the pictorial convention that if \( A < B \) then the Hasse diagram features \( A \) to the left of \( B \).

A poset \( P \) can be regarded as a category with a unique morphism \( A \rightarrow B \) whenever \( A \leq B \) and a system of local coefficients for \( P \) consists of a (covariant) functor \( F : P \rightarrow \text{Mod}_R \).

Example 2.1. — Let \( D \) be an oriented link diagram and let \( X_D \) be the set of crossings. In this example we will construct a local coefficient system on the boolean lattice \( \mathbb{B}(X_D) \). We will define a functor \( F_D : \mathbb{B}(X_D) \rightarrow \text{Mod}_R \) first on objects and then on morphisms.

Objects: Let \( A \subset X_D \). Near each crossing \( c \in X_D \) do “surgery” as follows:

\[
\begin{align*}
\text{F}_D(A) &= \mathbb{P}[x_\gamma \mid \gamma \text{ a component of the resolution associated to } A]/(x_\gamma^2 = 0) \\
\end{align*}
\]

Now define \( F_D(A) \) to be the truncated polynomial algebra with one generator for each component of the resolution associated to \( A \).

Morphisms: Suppose that \( B \) covers \( A \). We must define a map \( F_D(A < B) : F_D(A) \rightarrow F_D(B) \) which to simplify the notation we will denote by \( d_{A,B} \). By assumption \( B \) has exactly one more element than \( A \) and in a neighbourhood the additional crossing we see the following local change:
There are now two cases.

- if $\alpha \neq \alpha'$ (in which case we also have $\beta = \beta'$) we define $d_{A,B}$ to be the algebra map defined to be the identity on all generators apart from $x_\alpha$ and $x_{\alpha'}$ where
  \[ x_\alpha, x_{\alpha'} \mapsto x_\beta, \]
  - if $\alpha = \alpha'$ (in which case we also have $\beta \neq \beta'$) we define $d_{A,B}$ to be the module map given by
    \[ x \mapsto (x_\beta + x_{\beta'})x \]
    where $x$ is a monomial in which $x_\alpha$ does not appear, and
    \[ xx_\alpha \mapsto xx_\beta x_{\beta'}. \]

Thus, in particular, $1 \mapsto x_\beta + x_{\beta'}$ and $x_\alpha \mapsto x_\beta x_{\beta'}$.

Finally, if $A < B$ is an arbitrary morphism, decompose it as $A = A_0 < A_1 < \cdots < A_k = B$ where $A_{i+1}$ covers $A_i$ and define $F_D(A < B)$ to be the composition of the maps $d_{A_i, A_{i+1}}$ defined above.

**Exercise 4.** — Check that the definition for an arbitrary morphism is independent of the decomposition into one-step morphisms.

The example above is in fact central to the construction of Khovanov homology but there is one embellishment needed, namely that the local coefficient system takes values in graded modules. Let $A \subset X_D$ and let $\Vert A \Vert$ denote the number of components in the resolution associated to $A$. The monomial $x_{y_1} \cdots x_{y_m}$ is defined to have grading $\vert A \vert + \Vert A \Vert - 2m$. If, for example, $A$ has 5 elements and the associated resolution has two components then there are four generators 1, $x_1$, $x_2$ and $x_1x_2$ of degrees 7, 5, 5 and 3 respectively. With these gradings the functor $F_D: \mathbb{B}(X_D) \to \text{GrMod}_R$ is graded in the sense that morphisms induce maps of graded modules of degree zero.

### 2.2. Extracting information from local coefficient systems on Boolean lattices.

Let $S$ be a set and $F: \mathbb{B}(S) \to \text{GrMod}_R$ a graded local coefficient system (the assumption is that morphisms induce maps of degree zero). We can now define a bigraded cochain complex $(\mathcal{C}^*, d)$ with cochain groups

\[ \mathcal{C}^i,j = \bigoplus_{A \subset S, \vert A \vert = i} F^i(A). \]

In order to define a differential $d: \mathcal{C}^i,j \to \mathcal{C}^{i+1,j}$ we use the “matrix elements” $d_{A,B}: F(A) \to F(B)$ where $A$ and $B$ range over subsets of size $i$ and $i+1$ respectively and there is such a matrix element whenever $B$ covers $A$. Explicitly, for $v \in F^i(A) \subset \mathcal{C}^i,j$

\[ d(v) = \sum_{B \text{ covers } A} d_{A,B}(v). \]

As it stands $d^2$ is zero only mod 2 but this can be rectified by introducing a *signage* function $\epsilon: \{\text{edges}\} \to \mathbb{Z}/2$ which satisfies $\epsilon(e_1) + \epsilon(e_2) + \epsilon(e_3) + \epsilon(e_4) = 1 \mod 2$ whenever
$e_1, \cdots, e_4$ are the four edges of a square in the Hasse diagram. The definition above is modified to read

$$d(v) = \sum_{B \text{ covers } A} (-1)^{e(A,B)} d_{A,B}(v).$$

This will give $d^2 = 0$ over any ring and a different choice of signage will give an isomorphic complex.

**Definition 2.2.** — Let $L$ be an oriented link and let $D$ be a diagram representing $L$ having $n$ crossings of which $n_-$ are negative and $n_+$ positive. Let $F_D : \mathbb{B}(X_D) \to \text{GrMod}_k$ be as defined in Example 2.1. Applying the above construction gives a bigraded cochain complex $C^{\star, \star}(D)$. The Khovanov homology of $L$ with coefficients in the ring $R$ is the (shifted) homology of this complex:

$$Kh^{i,j}(D) = H^{i+n_-, j+2n_--n_+}((C^{\star, \star}(D), d))$$

**Remark 2.3.** — The shifts by $n_-$ and $2n_--n_+$ are global shifts needed in order to obtain an invariant (see Theorem 2.5 below) in much the same way as the Kauffman bracket formulation of Jones polynomial requires an additional factor depending on the writhe of the diagram used.

**Remark 2.4.** — For $A \subset X_D$, the monomial $x_{y_1} \cdots x_{y_m} \in F(A)$ defines a cochain in bidegree

$$(|A| - n_, |A| + ||A|| - 2m - 2n_+ + n_+)$$

**Exercise 5.** — Read sections 3.1 and 3.2 of *On Khovanov’s categorification of the Jones polynomial* (Bar-Natan, [BN02]) and/or sections 2.1 and 2.2 of *Khovanov homology theories and their applications* (Shumakovitch, [Shu11a]) and marry the description of the construction of Khovanov homology with what is written above.

This construction appears to depend on the diagram, but Khovanov’s first main result is that it doesn’t (Khovanov, [Kho00], see also Bar-Natan, [BN02]).

**Theorem 2.5.** — Up to isomorphism the definition above does not depend on the choice of diagram representing the link.

**Remark 2.6.** — In fact one need not go all the way to taking homology: the cochain complex itself is an invariant up to homotopy equivalence of complexes.

**Exercise 6.** — Using the construction above show that $Kh(L_1 \sqcup L_2) \cong Kh(L_1) \otimes Kh(L_2)$.

**Exercise 7.** — Show that if $L$ is presented by a diagram part of which is $\bigtimes$ then there is a short exact sequence of complexes

$$0 \longrightarrow C^{\star-1}(\bigtimes) \longrightarrow C^\star(\bigtimes) \longrightarrow C^\star(\bigtimes) \longrightarrow 0$$

Verify that the induced long exact sequence has gradings as presented in Theorem 1.1.
2.3. Functoriality. — The existence theorem asserts that there is a functor from a category of links and link cobordisms to modules. Courtesy of the construction in the previous section we know how to define this functor on links, but what is still needed is how to define module maps associated to link cobordisms.

Exercise 8. — Find out how link cobordisms are represented by movies and how to associate maps in Khovanov homology to such things (by looking in the papers cited below for example).

Because of the dependence on diagrams there are things to check. One can show that up to an over all factor of ±1 there is no dependence of the maps on the diagrams chosen. This is enough to give a functor over $\mathbb{F}_2$. The papers showing functoriality up to ±1 are

− An invariant of link cobordisms from Khovanov homology (Jacobsson, [Jac04]) 0206303
− An invariant of tangle cobordisms (Khovanov, [Kho06a]) 0207264

and

− Khovanov’s homology for tangles and cobordisms (Bar-Natan, [BN05]) 0410495

It is hard work to remove the innocent looking “up to ±1” and something additional is needed to make it work. One approach is to using Bar-Natan’s local geometric point of view (see section 4.2 below)

− Fixing the functoriality of Khovanov homology (Clark, Morrison and Walker, [CMW09]) 0701339

which requires working over $\mathbb{Z}[i]$. A somewhat similar point of view is developed in

− An sl(2) tangle homology and seamed cobordisms (Caprau, [Cap08]) 0707.3051

A different construction working over $\mathbb{Z}$ is in

− An oriented model for Khovanov homology (Blanchet, [Bla10]) 1405.7246

2.4. Aside for algebraic topologists: another extraction technique. — There is another, more abstract, way of extracting information from the cube. To motivate this kind of approach think about the definitions of group cohomology where one can either define an explicit cochain complex using the bar resolution or use derived functors. Each approach has its uses and if the definition is taken to be the explicit complex then the derived functors approach becomes an “interpretation”, but if the definition is in terms of derived functors then the explicit complex becomes a “calculation”.

There is a way of defining cohomology of posets equipped with coefficient systems by using the right derived functors of the inverse limit. With a small modification to the underlying Boolean lattice, this gives an alternative way of getting Khovanov homology. Let $Q$ be the poset formed from $\mathcal{B}(X_D)$ by the addition of a second minimal element. Extend the functor $F_D$ to $Q$ by sending this new element to the trivial group. Khovanov homology can be interpreted as the right derived functors of the inverse limit functor (Everitt and Turner, [ET12]):

\[ Kh(D) \cong R^* \lim_{\leftarrow Q} F_D \]
3. Odd Khovanov homology

The construction of Khovanov homology makes no demands on the order of the circles appearing in a resolution. At the algebraic level this is reflected in the polynomial variables commute among themselves. If one could impose a local ordering of strands near crossings then one might hope that this commutativity requirement could be removed. The subject of odd Khovanov homology is one approach to achieving this. The defining paper is

– Odd Khovanov homology (Ozsváth, Rasmussen and Szabó, [ORS13]) 0710.4300

and there is also a nice expository article with many calculations

– Patterns in odd Khovanov homology (Shumakovitch, [Shu11b]) 1101.5607

The construction of odd Khovanov homology is a refinement of the construction of (ordinary) Khovanov homology given in the last section. Let \( D \) be an oriented link diagram and let \( X_D \) be the set of crossings. We will construct a local coefficient system \( F_{odd}^D : \mathbb{B}(X_D) \to \text{Mod}_R \) on the boolean lattice \( \mathbb{B}(X_D) \).

– Objects: Let \( A \subset X_D \). Near each crossing \( c \in X_D \) replace the crossing according to the following two rules

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\uparrow \quad c \notin A \\
\quad \rightarrow
\end{array} & \quad \begin{array}{c}
\downarrow \quad c \notin A
\end{array}
\end{array}
\end{align*}
\]

The result is a collection of closed circles in the plane with a number of additional dotted arrows and we refer to this as the odd resolution associated to \( A \). The right-handed trefoil with crossing set \{1, 2, 3\} has typical resolution as shown here (for the subset \( A = \{1, 3\} \)).

Note that if presented with a local piece around a crossing for which the two strands are from different components we may order these by the decree: tail before head. This does not give a global ordering on the circles.

Now define \( F_{odd}^D(A) \) to be the exterior algebra with one generator for each component of the odd resolution associated to \( A \).

\[
F_{odd}^D(A) = \Lambda_A = \Lambda[x_\gamma \mid \gamma \text{ a component of the odd resolution associated to } A]
\]

– Morphisms: Suppose that \( B \) covers \( A \). We must define a map \( d_{odd}^{A,B} : F_{odd}^D(A) \to F_{odd}^D(B) \). By assumption \( B \) has exactly one more element than \( A \) and in a neighbourhood the additional crossing we see the following local change:
There are now two cases.

- if $\alpha \neq \alpha'$ (in which case we also have $\beta = \beta'$) we define $d_{A,B}$ to be the algebra map given by

$$1 \mapsto 1, \quad x_\alpha, x_{\alpha'} \mapsto x_\beta, \quad x_\gamma \mapsto x_\gamma \text{ for } \gamma \neq \alpha, \alpha'$$

- if $\alpha = \alpha'$ (in which case we also have $\beta \neq \beta'$) we define $d_{A,B}$ to be the module map given by

$$x_\alpha \wedge v \mapsto x_\beta \wedge x_{\beta'}, \quad v \mapsto (x_\beta - x_{\beta'}) \wedge v$$

where $x_\alpha$ is assumed not to appear in $v$. Thus $1 \mapsto x_\beta - x_{\beta'}$ and $x_\alpha \mapsto x_\beta \wedge x_{\beta'}$.

The first of these makes no use of the local ordering, but in the second the asymmetry is very clear.

Exercise 9. — Check that if the underlying ring is the field $F_2$ then the exterior algebra is isomorphic to the truncated polynomial algebra and the matrix element maps $d_{A,B}^{\text{odd}}$ agree with the ones used in construction of (ordinary) Khovanov homology.

For ordinary Khovanov homology the construction gives a functor $B(X_D) \to \text{Mod}_R$ without further trouble. Or, put differently, the square faces of the cube commute. (Immediately afterwards a sign assignment is made, but that is to turn commuting squares into anti-commuting ones which is only necessary because of the particular extraction technique used to obtain a complex out of the functor.) Here, for odd Khovanov homology things are not so simple and there is not obviously functor $B(X_D) \to \text{Mod}_R$; some squares commute, others anti-commute and others still produce maps which are zero. After a fair bit of digging into the possible cases Ozsváth-Rasmussen-Szabó prove:

**Proposition 3.1.** — There exists a signage making all squares commute.

This gives a functor $F_{D}^{\text{odd}}: B(X_D) \to \text{Mod}_R$ and by one of the extraction techniques discussed previously this yields a complex whose homology defines odd Khovanov homology, denoted $Kh_A^{*,*}(D; R)$.

**Remark 3.2.** — The bigrading is as follows: for $A \subset X$, the $m$-form $x_{\gamma_1} \wedge \cdots \wedge x_{\gamma_m} \in \Lambda^m \subset \Lambda_A$ defines a cochain with bigrading

$$||A|| - n_- |A| - ||A|| - 2m - 2n_- + n_+$$

where as before $||A||$ is the number of circles in the odd resolution defined by $A$.

Odd Khovanov homology shares many properties of ordinary Khovanov homology, but there are some crucial differences. Here is a summary of some of its properties:

- there are skein long exact sequences precisely as for ordinary Khovanov homology (with the same indices),
the Jones polynomial is obtained as
\[ \sum_{i,j} (-1)^i q^i \dim(Kh_{od}(L))|_{q = -t^\frac{1}{2}} = -(t^\frac{1}{2} + t^{-\frac{1}{2}}) J(D), \]

there is a reduced version, \( \tilde{Kh}_{od} \), satisfying
\[ Kh_{i,j} \approx \tilde{Kh}_{i,j+1} \oplus \tilde{Kh}_{i,j-1} \]

and which does not depend on the component of the base-point. So \( Kh_{od} \) stands in the same relationship to \( \tilde{Kh}_{od} \) as \( Kh_{F_2} \) to \( \tilde{Kh}_{F_2} \) which is very different to the relationship between \( Kh_{\mathbb{Z}} \) and \( \tilde{Kh}_{\mathbb{Z}} \).

over \( \mathbb{F}_2 \) odd and ordinary Khovanov homology coincide (reduced and unreduced); this is courtesy of Exercise 9,

\( \tilde{Kh}_{od}(\text{alternating}) \approx Kh^{**}(\text{alternating}) \) but in general \( \tilde{Kh} \) neither determines or is determined by \( Kh_{od} \).

**Remark 3.3.** — The is a spectral sequence with \( E_2 \)-page \( Kh_{F_2}(L') \) converging to the Heegaard-Floer homology of the double branched cover branched along \( L \) (Ozsváth and Szabó, [OS05]). To lift this integrally the correct theory to put at \( E_2 \) is (conjecturally) odd integral Khovanov homology. Indeed this was one of the motivations for the invention of odd Khovanov homology.

**Remark 3.4.** — There are other interesting spectral sequences featuring odd Khovanov homology at the \( E_2 \)-page. There is one starting with odd Khovanov homology and converging to an integral version of a theory made by Szabo (Beier, [Bei12]). Another starts with odd Khovanov homology and converging to the framed instanton homology of the double cover (Scaduto, [Sca14]).

**Remark 3.5.** — On seeing a typical odd resolution it is tempting to re-draw it as a graph whose vertices are the circles and whose directed edges are the dotted arrows. There is a description of odd Khovanov homology in terms of arrow graphs (Bloom, [Blo10]).

### 4. Tangles

We now return to (ordinary) Khovanov homology. The topology of the resolutions of a link diagram requires knowledge of the whole diagram and this is used in the construction of Khovanov homology (circles fuse or split depending on global information). None the less diagrams are made up of more basic pieces, namely tangles, and so it is natural to ask if Khovanov homology may be defined more locally. The difficulty is that while piecing together geometric data is easy, doing the same with algebraic data is never so simple.

**4.1. Khovanov’s approach.** — The first approach is due to Khovanov who studies \((m,n)\)-tangles such as the one shown here (Khovanov, [Kho02]).
For such a diagram $T$ there is a cube of resolutions (in our language: a local coefficient system on the Boolean lattice on crossings) as before. To $A \subset X_T$ one associates

$$M_A = \bigoplus P[x_\gamma \mid \gamma \text{ a circle }] / (x_\gamma^2 = 0)$$

where the direct sum is over all tangle closures. Each $M_A$ is an $(H^m, H^n)$-bimodule where $\{H^i\}$ is a certain family of rings (with elements the top-bottom closures of a set of $2i$ points; these rings can also be related to parabolic category $\mathcal{O}$ (Stroppel, [Str09])). By the usual extraction of a complex from a cube this yields a complex of bi-modules $C(T)$. When $m = n = 0$ one recovers the usual Khovanov complex. Isotopic tangles produce complexes that are homotopy equivalent and the construction is functorial (up to $\pm 1$) with respect to tangle cobordisms (Khovanov, [Kho06a]).

The key new property is that by using the bi-module structure tangle composition can be captured algebraically.

**Proposition 4.1.** — Let $T_1$ be a $(m, n)$-tangle and $T_2$ a $(k, m)$-tangle. Then,

$$C(\begin{array}{c} 2n \\ T_1 \\ \end{array}) \cong C(\begin{array}{c} 2m \\ T_2 \end{array}) \otimes_{H^m} C(\begin{array}{c} 2k \\ T_1 \end{array})$$

**4.2. Bar-Natan’s approach.** — A different approach to locality is due to Bar-Natan as his viewpoint has turned out to be very influential (Bar-Natan, [BN05]). The functor $F_D: \mathbb{B}(X_D) \to \text{GrMod}_R$ comes from two step process: firstly make resolutions (which are geometric objects) and secondly associate to these some algebraic data. The first step can be re-cast as a functor from $\mathbb{B}(X_D)$ to a cobordism category and the second step consists of applying a $1+1$-dimensional TQFT to the first step. Bar-Natan’s central idea is to work with the “geometric” functor (or cube) as long as possible delaying the application of the TQFT.

$$F_D: \mathbb{B}(X_D) \xrightarrow{TQFT} \text{Cob}_{1+1} \xrightarrow{\text{TQFT}} \text{GrMod}_R$$

Distilling the essential operations used to construct a cochain complex from the functor $F_D$ one sees that we needed to 1) take direct sums of vector spaces (in the step often referred to as “flattening the cube”), and 2) assemble a linear map out of the matrix elements which involved taking linear combinations of maps between vector spaces. In order to delay the passage to the algebra and to build some notion of “complex” in the setting of a cobordism category we need some equivalent of these two operations. What is done is to
replace direct sum by the operation of taking formal combinations of objects (closed 1-manifolds) and allowing linear combinations of cobordisms. A typical morphism will be a matrix of formal linear combinations of cobordisms. In this way it is possible to define a “formal” complex $[D]$ associated to $D$. It is no longer possible to take homology of such formal complexes because we are working in a non-abelian category (the kernel of a linear combination of cobordisms makes no sense, for example) but one still has the notion of homotopy equivalence of formal complexes and indeed if $D \sim D'$ then $[D] \cong [D']$.

This approach works perfectly well for tangles too. Given a tangle $T$ of the type shown below a resolution will typically involved 1-manifolds with- and without-boundary and the cobordism category must be adapted appropriately but a formal complex $[[T]]$ may be constructed as above. Things are as they should be because given isotopic tangles $T_1$ and $T_2$ then there is a equivalence of formal complexes $[[T_1]] \cong [T_2]]$. Moreover this construction is functorial (up to $\pm 1$) with respect to tangle cobordisms.

But now comes the beauty of this approach: by insisting on staying on the geometric side of the street for so long, the composition of tangles is accurately reflected at the level of formal complexes as well. The combinatorics of tangle composition is captured by the notion of a planar algebra: to each $d$-input arc diagram $D$ like the one shown here

there is an operation

$$D : \text{Tang} \times \cdots \times \text{Tang} \rightarrow \text{Tang}$$

defined by plugging the holes. For example

These operations are subject to various composition criteria that make up the structure of a planar algebra. The category $\text{Tang}$ is very naturally a planar algebra, but other categories may admit the structure of a planar algebra too - all that is needed is operations of the type above. The category of formal complexes above (the one in which $[[T]]$ lives) is an example - for the details of the construction you should read Bar-Natan’s paper.
Proposition 4.2. — The construction sending a tangle $T$ to the associated formal complex $\llbracket T \rrbracket$ respects the planar algebra structures defined on tangles and formal complexes. (In other words $\llbracket - \rrbracket$ is a morphism of planar algebras)

Using the planar algebra structure all tangles can be built out of single crossings. What the proposition is telling us is that the same is true of the formal complex: it is enough to specify $\llbracket - \rrbracket$ on single crossings and the rest comes from the planar algebra structure.

The next question to ask about this approach is: how does the planar algebra structure interact with link cobordisms? What is needed is an extension of the notion of planar algebra to the situation where there are morphisms between the planar algebra constituents. The name given to the appropriate structure is a canopolis. Again read Bar-Natan’s paper for details. Working with this local approach makes far more digestible the proofs of invariance and functoriality.

Remark 4.3. — If one wishes to apply a TQFT to get something algebraic out of Bar-Natan’s geometric complex one needs something slightly different capable of handling manifolds with boundary. The appropriate thing is an open-closed TQFT (Lauda and Pfeiffer, [LP09]). The question of algebraic gluing of tangle components has also been studied (Roberts, [Rob13]) where inspiration is drawn from bordered Heegaard Floer homology and the skein module of tangles in the context of Khovanov homology (Asaeda, Przytycki and Sikora, [APS04]).

Remark 4.4. — The complex constructed above can be simplified at an early stage by a technique called de-looping (Bar-Natan, [BN07]). Though very different in approach the de-looped complex is closely related to the one used by Viro in his description of Khovanov homology (Viro, [Vir04]).

Remark 4.5. — In order to place odd Khovanov homology into a Bar-Natan-like geometric framework it is necessary to enrich the theory by working with 2-categories (Putyra, [Put13]; Beliakova and Wagner [BW10]).

5. Variants

In the definition of Khovanov homology we assigned a truncated polynomial algebra to a given resolution with one variable for each component of the resolution. It is possible to take the quotient by other ideals and still obtain a link homology theory. In fact for $h, t ∈ R$ a functor $F^{h,t}_{D} : \mathbb{B}(X_D) → \text{Mod}_R$ may be defined by

$$F^{h,t}_{D}(A) = P[x_γ | γ \text{ a component of the resolution associated to } A]/(x_γ^2 = t + hx_γ)$$

with maps defined in a similar way to previously:

- if $α ≠ α′$ we define $d_{A,B}$ to be the algebra map defined to be the identity on all generators apart from $x_α$ and $x_α′$ where

  $$x_α, x_α′ \mapsto x_β,$$
– if $\alpha = \alpha'$ we define $d_{A,B}$ to be the module map given by

$$x \mapsto (x_\beta + x_{\beta'} - h)x$$

where $x$ is a monomial in which $x_\alpha$ does not appear, and

$$x x_\alpha \mapsto x x_\beta x_{\beta'} + t.$$ 

Thus, in particular, $1 \mapsto x_\beta + x_{\beta'} - h$ and $x_\alpha \mapsto x_\beta x_{\beta'} + t$.

In each case this gives rise to a link homology theory (Khovanov, [Kho06b]; Naot [Nao06]). To ensure a bigraded theory the ground ring must also be graded and contain $h$ and $t$ of degree $-2$ and $-4$ respectively. For Khovanov homology we take $t = h = 0$ and the bigrading of the ground ring can be concentrated in degree zero unproblematically.

5.1. Lee Theory. — The first variant of Khovanov homology to appear was the case $h = 0$ and $t = 1$ working over $\mathbb{Q}$ (Lee, [Lee05]). The ring $\mathbb{Q}$ is ungraded which means that Lee’s theory is a singly graded theory. The two most important facts about this theory are:

– it can be completely calculated for all links in terms of linking numbers
– there is a filtration on the chain complex

For calculation, Lee proves the following. (There is another proof this using Bar-Natan’s local theory (Bar-Natan and Morisson, [BNM06]).)

**Theorem 5.1.** — Let $K$ be a knot. Then

$$\text{Lee}^i(K) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0 \\ 0 & \text{else.} \end{cases}$$

Let $L$ be a two component link. Then

$$\text{Lee}^i(K) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0, \text{ or lk(the two components)} \\ 0 & \text{else.} \end{cases}$$

In general, for a $k$ component link $\sum \dim(\text{Lee}^i(L)) = 2^k$ and there is a formula for the degrees of the generators in terms of linking numbers.

The filtration leads to a spectral sequence (implicit in Lee’s paper, made explicit in (Rasmussen, [Ras10]).

**Theorem 5.2.** — Let $L$ be a link and $\gamma$ its number of components modulo two. There exists a spectral sequence, the Lee-Rasmussen spectral sequence, which has the form

$$E^{p,q}_2 = \text{Kh}_{\mathbb{Q}}^{p+q,2p+\gamma} \Rightarrow \text{Lee}^*(L).$$

The differentials have the form $d_r : E^{p,q}_r \to E^{p+q,q-r+1}_r$. Moreover, each page of the spectral sequence is a link invariant.

**Remark 5.3.** — If one re-grades so that the differentials are expressed in terms of the gradings of Khovanov homology (rather than the pages of the spectral sequence) the differential $d_r$ is zero for $r$ odd and has bigrading $(1, 2r)$ when $r$ is even.
Remark 5.4. — In all known examples this spectral sequence (over $\mathbb{Q}$) collapses at the $E_2$-page. It is still an open question as to whether this is always the case or not.

The utility of this spectral sequence is that it puts considerable restrictions on the allowable shape of Khovanov homology. As an example consider an attempted calculation of the rational (unreduced) Khovanov homology of the right-handed trefoil only using the skein long exact sequences. At some point you will find that you need additional information (some boundary map may or may not be zero and you have no way of telling without some further input). You can conclude that the Khovanov homology must be one of the two possibilities shown here. The existence of the Lee-Rasmussen spectral sequence tells you that the correct answer is on the right: the two generators that survive to the $E_\infty$-page of the spectral sequence are the two in homological degree zero and all the others must be killed by differentials; if the Khovanov homology were as given on the left, then a quick look at the degrees of the differentials shows that the generator in bi-degree (2,7) could never be killed, giving a contradiction.

Remark 5.5. — Over other rings Lee theory behaves as follows:

1. over $\mathbb{F}_p$ for $p$ odd it behaves as Lee theory over $\mathbb{Q}$ (i.e. it is “degenerate”) and the proof of Lee’s theorem works verbatim.
2. over $\mathbb{F}_2$ a change of variables shows that it is isomorphic (as an ungraded theory) to $\mathbb{F}_2$ Khovanov homology
3. over $\mathbb{Z}$ it has a free part of rank $2^{\text{no. of components}}$, no odd torsion, but a considerable amount of 2-torsion.

Remark 5.6. — Over $\mathbb{Q}$ the above family (parametrized by $h$ and $t$) produces only two isomorphism classes of theories: when $h^2 + 4t = 0$ the theory is isomorphic to rational Khovanov homology and otherwise it is isomorphic to rational Lee theory (Mackaay, Turner and Vaz, [MTV07]).

5.2. Bar-Natan Theory. — Another interesting case is to take $t = 0$ and $h = 1$ (Bar-Natan, [BN05]). This theory is quite similar to Lee theory in the sense that it is a “degenerate” theory requiring only linking numbers for a full calculation and it is filtered with an attendant Lee-Rasmussen type spectral sequence (Turner, [Tur06b]). There are, however, some differences (which possibly make it a better theory than Lee theory): the integral version also degenerates and there is a reduced version with a reduced Lee-Rasmussen type spectral sequence.
6. Generalisations: $sl(N)$-homology and HOMFLYPT-homology

There is a general procedure, due to Witten and Reshetikhin-Turaev, for the construction of (quantum) link invariants using the representation theory of quantum groups as input. Starting with a simple Lie algebra $g$, link components are labelled with irreducible representations of the quantum group $U_q(g)$ to produce a link invariant. From this point of view the Jones polynomial arises from the two dimensional representation when $g = sl(2)$.

An important and natural question is: are there link homology theories associated to other Lie algebras which generalise Khovanov homology in some appropriate sense?

6.1. Khovanov-Rozansky $sl(N)$-homology. — An obvious place to start is $g = sl(N)$; the case $N = 2$ is already done and the analogue of the Jones polynomial, the $sl(N)$-polynomial has been extensively studied.

A very nice summary is given in

- Khovanov-Rozansky homology of two-bridge knots and links (Rasmussen, [Ras07]) 0508510

and the details are contained in the original paper:

- Matrix factorizations and link homology (Khovanov and Rozansky, [KR08a]). 0401268

**Theorem 6.1 (Existence of $sl(N)$-homology).** — There exists a (covariant) projective functor

$$KR_N^{*,*}: \text{Links} \to \text{Vect}_\mathbb{Q}$$

satisfying

1. If $\Sigma: L_1 \to L_2$ is an isotopy then $KR_N(\Sigma)$ is an isomorphism.
2. $KR_N^{*,*}(L_1 \sqcup L_2) \cong KR_N^{*,*}(L_1) \otimes KR_N^{*,*}(L_2)$.
3. $KR_N^{i,j}(\text{unknot}) = \begin{cases} \mathbb{Q} & i = 0 \text{ and } j = 2k - N - 1 (k = 1, \ldots, N) \\ 0 & \text{else} \end{cases}$
4. There are long exact sequences:

$$\delta \rightarrow KR_N^{i-1,j+N} \rightarrow KR_N^{i,j} \rightarrow KR_N^{i,j+N-1} \rightarrow \delta$$

$$\delta \rightarrow KR_N^{i-j-N+1} \rightarrow KR_N^{i-1,j-N} \rightarrow KR_N^{i,j} \rightarrow \delta$$

Immediately we see there is something fishy with this: the long exact sequences feature an as yet undefined object. In fact $KR_N^{*,*}$ assigns a bigraded vector space to each singular link diagram (where crossing of the form $\includegraphics[width=0.05\textwidth]{image1}$, $\includegraphics[width=0.05\textwidth]{image2}$ and $\includegraphics[width=0.05\textwidth]{image3}$ are allowed). Up to isomorphism this assignment is invariant on deforming the diagram by Reidemeister moves away from singularities. The situation is not quite as good as for Khovanov homology because even with perfect information about the long exact sequences, the basic normalising set of object consists not of one single simple object (the unknot) but an infinity of 4-valent planar graphs such as the one shown here.

**Remark 6.2.** — If $\Sigma$ is a cobordism then $KR_N(\Sigma)$ has bi-degree $(0, (1 - N)\chi(\Sigma))$. 

Exercise 10. — Attempt a computation of \( KR_N^*(\text{Hopf link}) \) from the existence theorem, carefully observing why this is much harder than when attempting the same computation for Khovanov homology.

Proposition 6.3. — Let \( P_N(D) = \sum_{i,j} (-1)^i q^i \dim(KR_N^{i,j}(L)) \). We have
\[
q^{-N} P_N(\begin{array}{c}
\text{
} \\
\text{
} \end{array}) - q^N P_N(\begin{array}{c}
\text{
} \\
\text{
} \end{array}) + (q - q^{-1}) P_N(\begin{array}{c}
\text{
} \\
\text{
} \end{array}) = 0
\]
and
\[
P_N(\text{unknot}) = \frac{q^N - q^{-N}}{q - q^{-1}} = \sum_{k=1}^{N} q^{2kN-1}
\]

We recognise these two properties as the ones characterising the \( sl(N) \)-polynomial showing that \( P_N \) is the \( sl(N) \)-polynomial.

Exercise 11. — Prove this proposition using the long exact sequences and the fact that the alternating sum of dimensions in a long exact sequence is always zero.

The construction of \( sl(N) \)-homology (and the proof of the existence theorem above) proceeds once again by defining a local coefficient system on a the Boolean lattice of crossings of a link diagram (a decorated “cube” if you prefer that language). As before this begins by constructing a resolution for each subset \( A \) of the set of crossings \( X_D \). Each resolution is a planar singular graph and the rules for its construction are:

\[
\begin{align*}
\text{c} & \in A \\
\text{c} & \notin A
\end{align*}
\]

If \( B \) covers \( A \) (it contains exactly one more crossing) then the corresponding resolutions are identical except in a small neighbourhood of the additional crossing where one of the following two local changes is seen: \( \begin{array}{c}
\text{
} \\
\text{
} \end{array} \rightarrow \begin{array}{c}
\text{
} \\
\text{
} \end{array} \) or \( \begin{array}{c}
\text{
} \\
\text{
} \end{array} \rightarrow \begin{array}{c}
\text{
} \\
\text{
} \end{array} \). What is now needed is a way of associating a module to each resolution and maps corresponding to cover relations (to cube edges) giving a functor \( F_{D,N} : \mathbb{B}(X_D) \rightarrow \text{GrVect}_Q \) from which a complex and its homology can be extracted as before. For this to be worth anything it must result in a link invariant and therein, of course, lies the difficulty.

Khovanov and Rozansky employ matrix factorizations in order to carry this out. There are some guiding principles coming from the description of the \( sl(N) \)-polynomial given by H. Murakami, Ohtsuki and Yamada who describe it in terms of certain graphs which suitably interpreted are the ones considered here. One may think of their construction as associating a Laurent polynomial \( \text{MOY}(\gamma) \) to each planar singular graph \( \gamma \) and the \( sl(N) \)-polynomial is then expressed as a sum (over resolutions) of such polynomials. Matrix factorizations can be used to make an assignment
\[
A_N^*(\cdot) : \text{Planar singular graphs} \rightarrow \text{GrVect}_Q
\]
such that $\sum_i q^{\dim A_i^*(\Gamma)} = \text{MOY}(\Gamma)$. The local relations satisfied by $\text{MOY}(\cdot)$ are lifted to $A_N^*(\cdot)$; for example

$$\text{MOY}(\begin{array}{c}\includegraphics{fig1.png} \\
\end{array}) = (q + q^{-1}) \text{MOY}(\begin{array}{c}\includegraphics{fig2.png} \\
\end{array})$$

becomes

$$A_N^*(\begin{array}{c}\includegraphics{fig1.png} \\
\end{array}) \cong A_N^{i-1}(\begin{array}{c}\includegraphics{fig2.png} \\
\end{array}) \oplus A_N^{i+1}(\begin{array}{c}\includegraphics{fig2.png} \\
\end{array})$$

Moreover, if two planar singular graphs are identical except in a small neighbourhood where they are as shown above then there are maps

$$A_N^*(\begin{array}{c}\includegraphics{fig1.png} \\
\end{array}) \rightarrow A_N^*(\begin{array}{c}\includegraphics{fig2.png} \\
\end{array}) \quad A_N^*(\begin{array}{c}\includegraphics{fig2.png} \\
\end{array}) \rightarrow A_N^*(\begin{array}{c}\includegraphics{fig1.png} \\
\end{array})$$

A functor is constructed by taking

$$F_{D,N}(A) = A_N^*(\text{resolution associated to } A)$$

and applying the maps above. Making the complex and taking homology defines $sl(N)$-homology.

**Remark 6.4.** — (on gradings for which we follow the conventions given in Rasmussen’s paper cited above). Let $A \subset X$ and let $\Gamma$ be the associated resolution. If $x \in A_N^k(\Gamma)$ then the corresponding element of the (bi-graded) complex has bi-degree

$$(|A| - n_+, k - i + (N - 1)(n_+ - n_-))$$

**Remark 6.5.** — $KR_{**}^2$ should be isomorphic to $Kh^{**}$ and indeed it is (Hughes, [Hug13]).

**Remark 6.6.** — There are both reduced and un-reduced versions of the theory and they are related by a spectral sequence (Lewark, [Lew14]).

Calculations with $sl(N)$-homology are much harder than for Khovanov homology. An understanding of why this is so can be obtained by attempting to compute the $sl(N)$-homology of Hopf link and comparing it to the Khovanov homology calculation. Here are some calculational results.

- The $sl(N)$-homology of two bridge knots has been completely determined and the result can be expressed in terms of the HOMFLYPT polynomial and signature (Rasmussen, [Ras07]). The torus knots $T(2, n)$ are a special case and this computation confirms previous conjectures (Dunfield-Gukov-Rasmussen, [DGR06]).

- There is an explicit conjecture about the $sl(N)$-homology of 3-stranded torus knots (Gorsky and Lewark, [GL14]).

- As $n \rightarrow \infty$ the homology $KR_N^{**}(T(k, n))$ stabilises in bounded degree (Stošić, [Sto07]) so it makes sense to consider the $sl(N)$-homology of $T(k, \infty)$ and there are conjectures to what this should be (Gorsky, Oblomov and Rasmussen, [GOR13]).
Remark 6.7. — One might hope that taken collectively the family of $sl(N)$-homologies are a complete invariant but this is not so and there are families of distinct knots undistinguishable by the $KR_N^{*,*}$ (Watson, [Wat07]; Lobb, [Lob14]).

Remark 6.8. — There was a different construction for $N = 3$ available before matrix factorizations entered the picture (Khovanov, [Kho04]) using Kuperberg’s theory of webs.

The construction of the local coefficient system used in the construction is once again a two step process: a geometric step is followed by an algebraic step and it is natural to ask if Bar-Natan’s approach can be carried over to $sl(N)$-homology. For simplicity we have been using the language of singular link diagrams in which we allowed crossings looking like $\bigotimes$. In fact the notation used by Khovanov and Rozansky is to elongate the vertex into a thick edge (or double edge) $\bigotimes$. This depiction more accurately reflects the viewpoint of Murakami, Ohtsuki and Yamada and with this in mind Bar-Natan’s approach can be made to work for $sl(N)$-homology, but cobordisms must be generalised to foams which take into account the existence of thick edges (Mackaay and Vaz, [MV08a]; Mackaay, Stošić and Vaz, [MSV09]).

Remark 6.9. — The main new algebraic ingredient in the construction of $sl(N)$-homology is the notion of a matrix factorization and these are used locally: a matrix factorization is associated to a small neighbourhood of a resolved diagram. Locality of diagrams was best patched together (when incorporating link cobordisms too) using the formalism of a canopolis. There is a certain category of matrix factorizations that can be given the structure of a canopolis and the construction of $sl(N)$-homology can be presented in these terms (Webster, [Web07]).

Remark 6.10. — The construction of Khovanov homology can be modified to produced Lee theory and in a similar way there are “degenerate” variants of $sl(N)$ homology (Gornik, [Gor04]). Like Lee theory these are filtered theories and can be completely computed in terms of linking numbers. There is also an analogue of the Lee-Rasmussen spectral sequence for these theories (Wu, [Wu09]).

Remark 6.11. — Throughout this section we have been assuming that we are working over $\mathbb{Q}$ (or $\mathbb{C}$), but integral theories have also been studied (Krasner, [Kra10]).

6.2. Khovanov-Rozansky HOMFLYPT-homology. — For each $N$ the (normalised) $sl(N)$-polynomial $\hat{P}_N$ is a specialisation of the following version of the HOMFLYPT polynomial $\hat{P}$

$$a\hat{P}(\bigotimes) - a\hat{P}(\bigotimes) + (q - q^{-1})\hat{P}(\bigotimes) = 0$$

and

$$\hat{P}(\text{unknot}) = 1.$$
There is such a theory constructed once again by Khovanov and Rozansky.

\[ \widetilde{KR}^{*,*}_N \] (In asking this question it is not immediately clear what “specialise” should mean.)

A good place to start might be the expository sections of

- Matrix factorizations and link homology II, (Khovanov and Rozansky, [KR08b])

Khovanov and Rozansky’s HOMFLYPT homology (reduced version) assigns to each braid closure diagram \( D \) a triply graded \( \mathbb{Q} \)-vector space \( \widetilde{H}^{*,*,*}(D) \) such that

1. \( D_1 \sim D_2 \implies \widetilde{H}(D_1) \cong \widetilde{H}(D_2) \),
2. The HOMFLYPT polynomial is recovered
   \[ \sum_{r,s,t} (-1)^{rs} a^r q^s \dim(\widetilde{H}^{r,s,t}(D)) = \widetilde{P}(D), \]
3. \( \widetilde{H}(\text{unknot}) \cong \mathbb{Q} \) in grading \((0,0,0)\),
4. \( \widetilde{H}(L_1 \sqcup L_2) \cong \widetilde{H}(L_1) \otimes \widetilde{H}(L_2) \otimes \mathbb{Q}[x] \),
5. There are skein long exact sequences.

Remark 6.12. — The notation and grading conventions we are following are Rasmussen’s. In fact Khovanov and Rozansky work with the unreduced theory \( H \) which is related to reduced by \( H \cong \widetilde{H} \otimes \mathbb{Q}[x] \). The reduced theory has the nice property that for any connected sum we have \( \widetilde{H}(L_1 \# L_2) \cong \widetilde{H}(L_1) \otimes \widetilde{H}(L_2) \).

Remark 6.13. — The construction of HOMFLYPT homology uses matrix factorizations (though see section 7.1 below). They are graded matrix factorizations which is the grading which pushes through to ultimately give three gradings.

Remark 6.14. — The theory is not as well behaved as previous theories. For one thing it is restricted to braid closures (a priori this could be removed but is required for the proof of Reidemeister invariance). Another drawback is that there is no functoriality.

HOMFLYPT-homology reproduces the polynomial \( \widetilde{P} \) but what about its relationship to \( sl(N) \)-homology? Since \( \widetilde{P} \) specialises to the \( sl(N) \)-polynomial one would hope there is a relationship. This question has been thoroughly investigated in a wonderful paper which analyses a family of spectral sequences relating these theories (Rasmussen, [Ras06]). The main result is:

**Theorem 6.15.** — 1. For each \( N > 0 \) there exists a spectral sequence starting with \( \widetilde{H} \) and converging to \( \widetilde{KR}^{*,*}_N \). Moreover each page is a knot invariant.
2. There is a spectral sequence starting with \( \widetilde{H} \) and converging to \( \mathbb{Q} \).

As a consequence one has:

**Proposition 6.16.** — For large \( N \)

\[ \widetilde{KR}^{i,j}_N(L) \cong \bigoplus_{j=r+N \delta \atop 2m-z-s} \widetilde{H}^{i,j}(L). \]
Remark 6.17. — The existence of these spectral sequences gives information about the form of HOMFLYPT-homology in much the same way as the Lee-Rasmussen spectral sequence gives information about the form of Khovanov homology. As there is now a whole family of spectral sequences to be considered this gives a large amount of information. Rasmussen carries out low crossing number computations, and a nice worked example of how to glean information from the existence of the spectral sequences is the calculation showing that $\tilde{H}(\text{Conway knot})$ and $\tilde{H}(\text{Kinoshita-Terasaka knot})$ are isomorphic (Mackaay and Vaz, [MV08b]). Some of this structure was predicted before the construction of the spectral sequences in particular for torus knots (Dunfield, Gukov, and Rasmussen, [DGR06]).

Remark 6.18. — One can construct other theories similar to $\tilde{KR}^*$ and there are Rasmussen-type spectral sequences starting with $\tilde{H}$ and converging to these theories (Wu, [Wu09]).

Remark 6.19. — There is a spectral sequence converging to knot Floer homology which has $E_1$-page the HOMFLYPT-homology (Manolescu, [Man14]).

7. Generalisations: further developments

7.1. Other constructions of Khovanov-Rozansky HOMFLYPT-homology. — There is an alternative construction of triply-graded HOMFLYPT-homology which uses Hochschild homology (Khovanov, [Kho07]). One can associate a cochain complex $F^*(\sigma)$ to a word $\sigma$ representing a braid group element (Rouquier, [Rou06]). This assignment is such that if two words represent the same group element then the associated complexes are isomorphic. Khovanov uses this construction in the following way. Suppose we have a link presented as the closure of an $m$-braid diagram $D$ and let $\sigma$ be the corresponding braid word and $F^*(\sigma)$ its Rouquier complex. Now apply Hochschild homology $HH(R, -)$ to this to get a complex

$$\cdots \rightarrow HH(R, F^i(\sigma)) \rightarrow HH(R, F^{i+1}(\sigma)) \rightarrow HH(R, F^{i+2}(\sigma)) \rightarrow \cdots$$

where $R$ is a certain ring ($R = \mathbb{Q}[x_1 - x_2, \ldots, x_{m-1} - x_m] \subset \mathbb{Q}[x_1, \ldots, x_m]$). There are internal gradings and each term in the sequence is in fact bigraded.

Theorem 7.1. — The homology of this complex, denoted $H^*,*,*$, is independent (up to isomorphism) of the choices made and (module juggling grading conventions) is isomorphic to Khovanov and Rozansky’s HOMFLYPT-homology.

There is another more geometric construction of HOMFLYPT homology which uses the cohomology of sheaves on certain algebraic groups (Webster and Williamson, [WW09]).
7.2. Coloured link homologies. — Coloured Jones polynomials arise from $sl(2)$ using higher dimensional representations rather than the fundamental two dimensional one. The natural question here is to ask if there are link homology theories that stand in the same relationship to these polynomials as Khovanov homology does to the Jones polynomial. There is a formula for each coloured Jones polynomial as a sum of Jones polynomials of cables and this can also be used to define link homology theories.

- Categorifications of the colored Jones polynomial (Khovanov, [Kho05])
  For this to work coefficients in $\mathbb{Z}/2$ are needed, but can be extended to $\mathbb{Z}[\frac{1}{2}]$.

- Categorification of the colored Jones polynomial and Rasmussen invariant of links (Beliakova-Wehrli, [BW08])

The $sl(N)$-polynomial is associated to the fundamental representation of $sl(N)$ and by allowing other representations one obtains coloured versions. Khovanov and Rozansky’s $sl(N)$-homology can be extended in a similar way.

- Generic deformations of the colored $sl(N)$-homology (Wu, [Wu11])

Using the definition of HOMFLYPT homology in terms of Hochschild homology it is possible to consider coloured versions of this theory too.

- The 1,2-colored HOMFLY-PT link homology (Mackaay, Stošić and Vaz, [MSV11])

The algebro-geometric construction of HOMFLYPT homology can be extended to a coloured version which agrees with the above when restricted.

- A geometric construction of colored HOMFLYPT homology (Webster and Williamson, [WW09])

7.3. Higher representation theory. — This is now a vast and important subject providing the most comprehensive answers to the question of how one should generalise Khovanov homology to other Lie algebras. To get an idea of the state of play you could read the introductory sections of:

- Khovanov homology is a skew Howe 2-representation of categorified quantum $sl_m$ (Lauda, Queffelec and Rose, [LQR12])

- An introduction to diagrammatic algebra and categorified quantum $sl_2$ (Lauda, [Lau12])

- Knot invariants and higher representation theory (Webster, [Web13a])

For the latter, the theory is explained separately in detail for the $sl(2)$ case (Webster, [Web13b]).

The work of Rouquier has been of fundamental importance in this area and you can get an idea of his vision from:

- Quiver Hecke algebras and 2-Lie algebras (Rouquier, [Rou12])

8. Applications of Khovanov homology

8.1. Concordance invariants. — The first paper to read on this subject is

- Khovanov homology and the slice genus (Rasmussen, [Ras10])
Recall that for a knot the Lee-Rasmussen spectral sequence leaves only two generators on the $E_\infty$-page. If we use the grading conventions which impose the differentials on the usual picture for Khovanov homology, then denoting the $E_\infty$ page by $K_{i,j}^\infty$, the statement that the spectral sequence converges to Lee theory means that

$$K_{i,j}^\infty = \frac{F^j \text{Lee}^j}{F^{j+1} \text{Lee}^j}$$

where $F^j \text{Lee}^j$ is the induced filtration on Lee theory. Since we are working over $\mathbb{Q}$ (so the spectral sequence has no extension problems) this means that $\text{Lee}^j \cong \bigoplus_j K_{i,j}^\infty$.

A priori the filtration grading (here the grading denoted by $j$) of the $E_\infty$-page of a spectral sequence is not particularly meaningful, but in this case the entire spectral sequence from the second page onwards is a knot invariant and thus the filtration gradings of the generators (two of them) surviving to the $E_\infty$-page are too. In fact these two generators lie in filtration gradings that differ by two.

**Proposition 8.1.** — For a knot $K$ there exists an even integer $s(K)$ such that the two surviving generators in the Lee-Rasmussen spectral sequence have filtration degrees $s(K) \pm 1$.

**Definition 8.2.** — The integer $s(K)$ is called the Rasmussen $s$-invariant of the knot $K$.

**Remark 8.3.** — For an alternating knot, Rasmussen’s invariant agrees with the signature.

By digging down a bit into the filtration, Rasmussen shows that his invariant has the following properties:

1. $s(\text{unknot}) = 0$,
2. $s(K_1 \# K_2) = s(K_1) + s(K_2)$,
3. $s(K^c) = -s(K)$.

A cobordism $\Sigma: K_1 \to K_2$ induces a filtered map $\text{Lee}(\Sigma): \text{Lee}^*(K_1) \to \text{Lee}^*(K_2)$ of filtered degree $\chi(\Sigma)$ meaning that $\text{im}(F^j \text{Lee}^*(K_1) \subset F^{j+\chi(\Sigma)} \text{Lee}^*(K_2)$. Denoting the filtration grading by $gr$ this means that for $\alpha \in \text{Lee}^*(K)$ we have

$$gr(\text{Lee}(\Sigma)(\alpha) \geq gr(\alpha) + \chi(\Sigma)$$

Also, Rasmussen shows:

**Proposition 8.4.** — If $\Sigma$ is connected then $\text{Lee}(\Sigma)$ is an isomorphism.

Using these properties one can show that Rasmussen’s invariant provides an obstruction to a knot being smoothly slice. (Recall that all manifolds and link cobordims are assumed to be smooth).

**Proposition 8.5.** — If $K$ is a smoothly slice knot then $s(K) = 0$

**Proof.** — Let $\Sigma$ be a slice disc with another small disc removed. This can be viewed as (connected) link cobordism $\Sigma: K \to U$ (the unknot). Since the Euler characteristic of $\Sigma$ is zero, this cobordism induces a filtered isomorphism of filtered degree zero

$$\text{Lee}(\Sigma): \text{Lee}^0(K) \to \text{Lee}^0(U).$$
Thus for any $\alpha \in \text{Lee}^0(K)$ we have

$$\gr(\text{Lee}(\Sigma)(\alpha)) \geq \gr(\alpha).$$

Now $\text{Lee}^0(U)$ has two generators in filtration degrees $\pm 1$ and $\text{Lee}(\Sigma)$ is an isomorphism, from which we have $-1 \leq \gr(\text{Lee}(\Sigma)(\alpha)) \leq 1$ giving

$$\gr(\alpha) \leq \gr(\text{Lee}(\Sigma)(\alpha)) \leq 1.$$ 

Now $s(K)$ is equal to $\gr(\alpha) - 1$ for some $\alpha$ so $s(K) \leq 0$. Finally, a similar argument applies to $K'$ giving $s(K') \leq 0$ and so $s(K) = -s(K') \geq 0$. □

Remark 8.6. — This proof uses the fact that Lee theory is a functor.

In fact more is true and $s$ gives a lower bound for the slice genus.

Theorem 8.7. — Rasmussen’s invariant is a concordance invariant and for a knot $K$

$$|s(K)| \leq 2g_s(K),$$

where $g_s(K)$ denotes the smooth slice genus of $K$.

Exercise 12. — Prove this theorem by modifying the proof of Proposition 8.5 above.

Remark 8.8. — By studying the $s$-invariant for positive knots and using the theorem above, Rasmussen gives a simple proof of the Milnor conjecture (Rasmussen, [Ras10]):

the slice genus of the torus knot $T(p, q)$ is $\frac{1}{2}(p - 1)(q - 1)$.

Remark 8.9. — Gompf gives a way of constructing non-standard smooth structures on $\mathbb{R}^4$ from the data of a topologically slice but not smoothly slice knot. By work of Freedman if the Alexander polynomial $\Delta_K$ is 1 then $K$ is topologically slice. Thus a non-standard smooth structure on $\mathbb{R}^4$ can be inferred from a knot $K$ satisfying $\Delta_K = 1$ and $s(K) \neq 0$. Examples of such knots are readily found, for example, the pretzel knot $P(-3, 5, 7)$.

Remark 8.10. — There is a similar invariant to Rasmussen’s, called the $\tau$-invariant, coming from Heegaard-Floer knot homology. While in many cases $2\tau = s$ in general this is not the case (Hedden and Ording, [HO08]). There is even an example of a topologically slice knot for which $s \neq 2\tau$ (Livingston, [Liv08]).

Remark 8.11. — For a short time it looked like Rasmussen’s invariant might help to find a counter-example to the smooth 4-dimensional Poincaré conjecture (Freedman, Gompf, Morrison and Walker, [FGMW10]), but this hope was short lived and the potential counter-examples are all standard spheres (Akbulut, [Akb10]). In fact Rasmussen’s invariant can be related to a similar invariant from instanton homology which leads to the conclusion that Rasmussen’s invariant will never detect counter-examples to the 4-dimensional Poincaré conjecture of this nature (Kronheimer and Mrowka, [KM13]).
By replacing Khovanov homology by $sl(N)$-homology and allowing Gornik’s theory $G_N^*$ to play the role of Lee theory, there is once again a spectral sequence starting with the former and converging to the latter. Since Gornik’s theory has dimension $N$ concentrated in (homological) degree 0 one can get a Rasmussen-like invariant (Wu, [Wu09]; Lobb, [Lob09]; Lobb, [Lob12]).

**Theorem 8.12.** — (1) Let $K$ be a knot. There exists and integer $s_N(K)$ such that

$$
\sum q^{|s_N(K)|} G_N^{q^{-1}}(K) = q^{s_N(K)} - q^{-N} \overline{q}^{-1}
$$

where the second grading on Gornik theory is the filtration grading.

(2) This provides a lower bound for the slice genus:

$$|s_N(K)| \leq 2(n - 1)g_s(K).$$

**Remark 8.13.** — It is interesting to ask if these invariants are related or not for various $N$. Lewark conjectures that the invariants $\{s_N(K)\}_{N \geq 2}$ are linearly independent with evidence from the result that $s_2(K)$ is not a linear combination of $\{s_N(K)\}_{N \geq 3}$ and a similar statement for $s_3(K)$ (Lewark, [Lew14]).

Another way of obtaining a Rasmussen-type invariant is by using the spectral sequence to Bar-Natan theory. This gives invariants $s^{BN}_R(K)$ for a variety of rings $R$. It was thought (incorrectly) that over $\Z$, $\Q$ and finite fields that these invariants always coincide with Rasmussen’s original invariant (Mackaay, Turner and Vaz, [MTV07]) but this is wrong and Cotton Seed has done some calculations which show that the knot $K = K_{14n19265}$ has $s(K) \neq s^{BN}_F(K)$. These invariants have been further refined (Lipshitz and Sarkar, [LS14b]) and a discussion of the $K_{14n19265}$ example can be found there.

**Remark 8.14.** — For links (rather than knots) there is also a way to obtain a Rasmussen-type invariant (Beliakova and Wehrli, [BW08]).

### 8.2. Unknot detection.

Khovanov homology is a nice functorial invariant which is known not to be complete and it is not hard to find distinct knots with the same Khovanov homology. However, the weaker question of whether or not Khovanov homology detects the unknot remained open until recently.

There are a number of partial results applying somewhat the same approach: make something else out of the knot and use a spectral sequence to Heegaard-Floer homology to make a conclusion about the minium size of the $E_2$-page (Hedden and Watson, [HW10]; Hedden, [Hed09]; Grigsby and Wehrli, [GW10]). For example, the following is a result of Hedden and Watson:

**Theorem 8.15.** — The dimension of the reduced Khovanov homology of the $(2,1)$-cable of a knot $K$ is exactly 1 if and only if $K$ is the unknot.

It is now known that Khovanov homology itself detects the unknot (Kronheimer and Mrowka, [KM11a]).

**Theorem 8.16.** — Khovanov homology detects the unknot.
This result also uses a spectral sequence but this time using another theory defined using instantons (Kronheimer and Mrowka, [KM11b]). This is a deep result requiring mastery of huge amount of low dimensional topology.

8.3. Other applications. — After the slice genus one of the first places Khovanov homology found application was to bounding the Thurston-Bennequin number (Shumakovitch, [Shu07]; Plamenevskaya, [Pla06]; Ng, [Ng05]). Let $L$ be a link and let $\overline{tb}(L)$ be the maximum Thurston-Bennequin number over all Legendrian representatives of $L$. The result of Ng is:

**Theorem 8.17.** — *There is a bound for maximum Thurston-Bennequin number give by*

\[
\overline{tb}(L) \leq \min\{k \mid \oplus_{j-i=k} Kh^{i,j}(L) \neq 0\}
\]

*and this bound is sharp for alternating links.*

The reduced Khovanov homology of an alternating knot lies exclusively on the line $j - 2i = \text{signature}(K)$ and as a general rule the Khovanov homology of a link clusters around this line. The homological width of a link the width of the diagonal band in which the non-trivial Khovanov homology lies: if

\[
\begin{align*}
\text{width}(L) &= \max\{j - 2i \mid Kh^{i,j}(L) \neq 0\} - \min\{j - 2i \mid Kh^{i,j}(L) \neq 0\} + 1.
\end{align*}
\]

then $\text{width}(L) = w_{\text{max}} - w_{\text{min}} + 1$. Alternating links, for example, have width 1. Since width involves both gradings available to Khovanov homology it is revealing something new not available to, say, the Jones polynomial. It can be used to provide certain obstructions to Dehn fillings (Watson, [Wat12]; Watson, [Wat13b]).

In a somewhat different direction, there is also (vector space-valued) invariant of tangles using an natural inverse system of Khovanov homology groups which can be applied to strongly invertible knots to show that a strongly invertible knot is the trivial knot if and only if the invariant is trivial (Watson, [Wat13a]).

It is possible to use a variant of Khovanov homology to distinguish between braids and other tangles (Grigsby and Ni, [GN13]).

9. Geometrical interpretations and related theories

What is Khovanov homology *really*? What geometrical features of knots and links does it measure? Is there an intrinsic definition starting with an actual knot in $S^3$ rather than a diagrammatic representation of it? The Jones polynomial has a good “physical” interpretation - how about Khovanov homology? Many of the ingredients of Khovanov homology are familiar to other areas of mathematics - what bridges can be built? In this brief final section we gather together a few places where these questions have been addressed.
9.1. Symplectic geometry. — There is a way to define a (singly) graded vector space invariant of links by using Lagrangian Floer cohomology. This approach has the very nice property that it starts with an actual link rather than a diagram. It is conjectured to be isomorphic to Khovanov homology after collapsing the grading of the later.

- A link invariant from the symplectic geometry of nilpotent slices (Seidel and Smith, [SS06])

This approach has been generalised to Khovanov-Rozansky $sl(N)$-homologies:

- Link homology theories from symplectic geometry (Manolescu, [Man07])

9.2. Knot groups and representation varieties. — For low crossing number examples the (singly graded) Khovanov homology of a link is isomorphic to a graded group constructed from the cohomology of the space of $SU(2)$ representations of the fundamental group of the link complement. These latter spaces can be understood in terms of intersections of Lagrangian submanifolds of a certain symplectic manifold.

- Symplectic topology of $SU(2)$-representation varieties and link homology, I: symplectic braid action and the first Chern class (Jacobsson and Rubinsztein, [Jac08])

9.3. Instanton knot homology. — The observation connecting Khovanov homology to $SU(2)$-representation varieties is also the starting point for the construction of a functorial link homology theory defined as an instanton Floer homology theory. This is the theory used to show that Khovanov homology detects the unknot.

- Knot homology groups from instantons (Kronheimer and Mrowka, [KM11b])
- Filtrations on instanton homology (Kronheimer and Mrowka, [KM14])
- Gauge theory and Rasmussen’s invariant (Kronheimer and Mrowka, [KM13])

9.4. Derived categories of coherent sheaves. — There is an algebro-geometric construction of a link homology theory isomorphic to Khovanov homology.

- Knot homology via derived categories of coherent sheaves I, $sl(2)$ case (Cautis and Kamnitzer, [CK08])

9.5. Physics. — The Jones polynomial (famously) has a description as a path integral of a 3-dimensional gauge theory using the Chern-Simons action. Mathematical physics and string theory also have things to say about Khovanov homology.

- Khovanov-Rozansky homology and topological strings (Gukov, Schwarz and Vafa, [GSV05])
- Link homologies and the refined topological vertex (Gukov, Iqbal, Kozcaz and Vafa, [GIKV10])
- Fivebranes and knots (Witten, [Wit12a])
- Khovanov homology and gauge theory (Witten, [Wit12b])
- Two lectures on the Jones polynomial and Khovanov homology (Witten, [Wit14])
9.6. Homotopy theory. — Homotopy theory is a subject rich in computational and theoretical tools and it would be nice to have a way of applying these to Khovanov homology. One basic question to ask is if the Khovanov homology of a link can be obtained from the application of some classical invariant from algebraic topology (cohomology for example) to a space defined from the link. This is the notion of Khovanov homotopy type.

– A Khovanov homotopy type (Lipshitz and Sarkar, [LS14a])

One can also carry out the entire construction of Khovanov homology in a homotopy theoretic setting using Eilenberg-Mac Lane spaces, homotopy limits and the description of Khovanov homology in terms of right derived functors of the inverse limit.

– The homotopy theory of Khovanov homology (Everitt and Turner, [ET12])

9.7. Factorization homology. — A vast machine from algebraic topology (factorization homology of singular manifolds) may - on specialising - provide new link homology theories related to Khovanov homology.

– Structured singular manifolds and factorization homology (Ayala, Francis and Tanaka, [AFT12])

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**Paul Turner**, Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, CH-1211, Geneva, Switzerland. • E-mail: prt.maths@gmail.com