 Recent observational data suggests a high compacticity (the quotient $M/R$) of some "neutron" stars. Motivated by these works we revisit models based on quark-diquark degrees of freedom and address the question of whether that matter is stable against diquark disassembling and hadronization within the different models. We find that equations of state modeled as effective $\lambda\phi^4$ theories do not generally produce stable self-bound matter and are not suitable for constructing very compact star models, that is the matter would decay into neutron matter. We also discuss some insights obtained by including hard sphere terms in the equation of state to model repulsive interactions. We finally compare the resulting equations of state with previous models and emphasize the role of the boundary conditions at the surface of compact self-bound stars, features of a possible normal crust of the latter and related topics.

1. Introduction

Recent works on compact stars have attracted the attention because not only the masses may be determined, but also indications of the radii are available for the first time. In some cases the compactness of the sources has been claimed, although these results have yet to be confirmed carefully. Nonetheless, it is worthwhile to entertain the possibility that at least some compact stars are extremely compact, or in other words, that their radii are $\sim 30-40\%$ less than the "canonical" $10\ km$ favored by neutron matter models for $M \sim 1\ M_\odot$. 

\*Email: foton@astro.iag.usp.br
\*Email: glugones@astro.iag.usp.br
\*Email: pacheco@obs-nice.fr
Claims of high compactness from the analysis of the binary Her X-1 \(^1\)(\(M = 0.98 \pm 0.12 M_\odot\) and \(R = 6.7 \pm 1.2 \text{ km}\)) and of the isolated nearby RX J185635-3754 \(^3\) (\(M \approx 0.9 M_\odot\) and \(R \approx 6 \text{ km}\)) have been made. In both cases, the results have been revisited and challenged by other groups \(^4\) which found radii in the ballpark of conventional neutron star models. This stresses the cautionary remarks made by several researchers about the high-compactness objects and guarantees further studies, already undertaken in most cases. We should add that Li et al. \(^6\) have also added an argument about the source 4U 1728-34, showing that conventional accretion models would indicate a very compact source, as quantified by their limit in the mass-radius plane. Again the actual distance to the source is a matter of concern. If really present in these sources compactness would be extremely difficult (and perhaps impossible) to model using underlying equations of state based on ordinary hadrons alone, and a natural alternative would be to consider deconfined matter (or other component, like kaon matter or hyperons). Other stars that have been claimed to be made up of deconfined matter are the compact objects associated with the X-ray bursters GRO J1744-28 \(^7\), and SAX J1808.4-3658 \(^8\).

It is widely believed that at high temperatures and/or high baryon number densities deconfinement of color is achieved. It is not very clear how high the densities/temperatures must be because of the extensively discussed failure of perturbative expansions around the transition point. The physical realization of deconfined matter immediately above the deconfinement point is also doubtful. The phase structure seems to be very rich and lattice simulations suggest that "asymptotia" is not sharply reached. There is an expectation that diquarks (e. g. a spin-0, color-antitriplet bound state of two quarks) might occur as a component in the QCD plasma in these conditions (see Ref. 9 and references therein for a review). Such diquarks would be expected to be favored by Bose statistics. If bound, the diquark binding energy would also contribute in making this phase favorable to ordinary uncorrelated quark matter. At very high densities diquarks are expected to lose their identity and must eventually dissolve into quarks, even if (as stated) there is no consensus about the onset of the asymptotic regime.

As a working hypothesis, we shall treat diquarks as single entities, and explore the possibility of a quark-diquark phase being absolutely stable, quite analogously to the well known strange quark matter hypothesis. The difference here is that we shall not invoke a new quantum number to bind the mixture, this role is partially played by the expected bosonic character of diquarks.

We stress that absolute stability of the mixture does not require the diquark itself to be bound (see Ref. 10, where a value of \(m_D \sim 700 \text{ MeV}\) has been claimed), but rather that the energy per baryon of the quark-diquark mixture to be less than the nucleon mass. The functional form of the free energy may or may not allow this to happen, and therefore the model-dependence of the description is the actual motive of concern, rather than the exact value of the diquark mass at a given density. As we shall see, this is crucial within \(\lambda \phi^4\) theories. We shall return to this point later. We devote the next two sections to discuss the effective models of the quark-diquark
phase and present a critical appraisal in the last section.

2. Bag-inspired quark-diquark Model

2.1. The equation of state

Some time ago it was the interesting controversy about the compactness of Her X-1 that prompted two of us to consider models in which diquarks are considered fundamental degrees of freedom and treated as effective bosons. The results were derived using an effective $\lambda\phi^4$ model and the equation of state given in $^{12}$, shown to be valid in this case. This approach goes back to Refs. $^{13,14}$ in which similar bosonic diquark dynamics were employed.

In constructing the EOS for quark-diquark matter there are some quantities that had to be treated parametrically: the diquark mass $m_D$, a vacuum energy density $B$ and the coupling constant $\lambda$. In fact, the effective character of this description allows for other possibilities than the previously selected ($m_D = 575 MeV$, $B = 57 MeV fm^{-3}$, $\lambda = 27.8$) to be considered.

Following the approach of Ref. 11 we have modeled the quark-diquark plasma at zero temperature as a mixture of a free Fermi gas of $u$ and $d$ quarks, and a $ud$ diquark gas treated in the framework of a $\lambda\phi^4$ theory. Confinement is introduced by hand by means of a phenomenological bag constant $B$.

As shown by Donoghue et al. $^{13}$ a $\lambda\phi^4$ theory leads to a quadratic expression for the pressure of a pure diquark gas in the low-density regime (valid for densities up to $\sim 10 \times \rho_0$, being $\rho_0$ the nuclear saturation density $^{14}$):

$$P_D = \frac{\lambda}{2m_D^2} n_D^2$$  \hspace{1cm} (1)

whereas the energy density is given simply by a rest mass term $m_D n_D$. Therefore, the full EOS is quite simple and given by

$$P = \frac{\lambda}{2m_D^2} n_D^2 + \frac{1}{5} \frac{\hbar}{m_q} (n_u^{5/3} + n_d^{5/3}) - B,$$

$$\epsilon = m_u n_u + m_d n_d + m_D n_D + \frac{3}{10} \frac{\hbar}{m_q} (n_u^{5/3} + n_d^{5/3}) + B,$$  \hspace{1cm} (3)

and must be subject to the following conditions:

1) chemical equilibrium between diquarks and quarks, $D \leftrightarrow u+d$, which requires $\mu_D = \mu_u + \mu_d$

$$\frac{\epsilon_D + P_D}{n_D} = \mu_u + \mu_d,$$  \hspace{1cm} (4)

2) bulk electrical charge neutrality $(1/3)n_D + (2/3)n_u - (1/3)n_d = 0$,

3) baryon number conservation $n_B = (2/3)n_D + (1/3)(n_u + n_d)$. 


Quark-diquark equations of state models . . .

Once the parameters $B$, $\lambda$ and $m_D$ are given, the equation of state can be calculated for a given value of, for example, $n_D$. Note that we did not include strange quarks in the admixture, thus ignoring CFL phases. Rather than extrapolating down in density (as done for CFL models), we work “bottom up” here by postulating a gradual loss of correlations (diquarks) starting from the deconfinement point.

2.2. Effective hard-sphere model: interactions in the quark-diquark mixture

The effective models of a quark-diquark mixture based on a $\lambda \phi^4$ theory do not say much about some fundamental questions. One of these is the strength of the repulsion between particles, which is closely related to their size and must be guessed by resorting to experimental data. Alternatively, the interactions may be modeled with a potential at relatively low densities, the latter being much more amenable to well-known analysis. It would be desirable to have a better insight to this and related questions since, in principle, the diquark size and scattering behavior are measurable (see Ref. 9 and references therein). We can account for this by introducing a scattering length in both the pressure and energy density of the Bose gas. Thus, we can consider the following expressions for the pressure and the energy density of diquarks

\begin{align}
P_D &= \frac{4\pi a^2}{m_D} n_D^2 \\
\epsilon_D &= \frac{4\pi a}{m_D} n_D^2 \left( 1 + \frac{128}{15\pi^{1/2}} (a n_D)^{1/2} \frac{a^3 n_D}{1/2} \right)
\end{align}

As shown below quantitatively (and expected qualitatively), the repulsive interactions makes the quark-diquark mixture less favourable energetically than the ”optimal” Fermi-Bose mixture (see also Ref. 17).

As the diquark pressure depends on $n_D^2$ in both the $\lambda \phi^4$-based EOS and the approximation of the present section, we can establish immediately a relation between the coupling constant $\lambda$ and the scattering length $a$ by equating the pressure in both approaches. This gives

\begin{align}
\lambda &= 30.6 \times \left( \frac{a}{0.4 \text{ fm}} \right) \left( \frac{m_D}{600 \text{ MeV}} \right).
\end{align}

It is interesting to note that although the comparison is very crude, it gives a kind of self-consistent result: the scattering length reflects the diquark size, which is expected to be of the order of the typical instanton radius, i.e. 1/3 fm. For a ”reasonable” diquark mass of 600 MeV the above formula gives a value of the coupling constant which is very close to the claimed value of $\lambda = 27.8$.

3. Stability of quark-diquark matter
Different compositions for diquark matter have been analyzed in the literature. Although it is not clear at all whether any of these compositions can actually happen in neutron stars, a mixture of quarks $u$ and $d$ and $ud$ diquarks is perhaps the simplest possibility since it may be thought to arise from pure neutron matter deconfinement, and therefore it seems a good starting point for our analysis. Weak interaction decays must alter this simplest composition and remains to be studied in detail.

It is usually assumed as a criterion for stability of a given deconfined EOS that the energy per baryon of the deconfined phase (at $P = 0$ and $T = 0$) must be lower than $939 \text{MeV}$ (the neutron mass). The effect of considering a scattering length amplitude $a$ (or, equivalently, a positive value of $\lambda$) makes more difficult for the quark-diquark matter to fulfill this condition and therefore to be absolutely stable.

With the above condition we can determine the set of parameters $B$, $m_D$ and $\lambda$ (or $a$) that gives a stable EOS (see Figures 1 and 2). Except for a marginal choice of the parameters (e.g. $B \sim 10 \text{MeV fm}^{-3}$, $\lambda \sim 10$ and very small diquark masses $m_D \sim 400 \text{MeV}$, the EOS posses an energy per baryon higher than the neutron mass. That is, it seems difficult that in the frame of a $\lambda\phi^4$ theory quark-diquark matter could be the ground state of strong interactions.

Another condition that has to be considered is the empirically known stability of normal nuclear matter against deconfinement at zero pressure. In other words the energy per baryon of deconfined matter (a pure gas of quarks $u$ and $d$) at zero pressure and temperature must be higher than the neutron mass. In the framework of a MIT-based EOS it has been shown that this condition imposes that the MIT bag constant must be greater than $57 \text{MeV fm}^{-3}$ (see Ref. 18). This is marked in Figs.1 and 2 with vertical lines.

4. Discussion

We have found in this work that for a wide range of the bag constant $B$, the diquark mass $m_D$ and the coupling constant $\lambda$, the quark-diquark mixture has an energy per baryon higher than the neutron mass. Absolute stability is obtained only for very low values of $B$, and $m_D$, if at all.

Within this description diquarks are favoured over a pure mixture of quarks $u$ and $d$ due to their Bose character and eventually its hypothetical binding energy. However, we must note, as stressed in Ref. 19, that the effect of Bose condensation seems to be much more important for quark-diquark matter self-binding than the binding energy of the diquark itself. In other words, even an unbound diquark may have left room for stable quark-diquark matter, although this is finally not the case, at least within this model. Recently, detailed calculations of the equation of state have been performed in a model in which confinement and masses are related, as the quark mass-density-dependent model, finding a wide set of parameters for which quark-diquark matter is stable $^{19}$. Therefore the absolute stability of quark-diquark matter seems to be quite model dependent.

A corollary of these findings has a big potential impact on compact star models.
made up of quark-diquark matter. Without repeating the analysis here, we just point out that (as expected) in the frame of the $\lambda \phi^4$ model a self-bound quark-diquark mixture indeed produces very compact models. The mass-radius relation of quark-diquark models presents the same qualitative shape as the curves found for strange stars. This is a direct consequence of the existence of a zero pressure point at finite density (of the order of the nuclear saturation density). However, although they give a very compact star structure, models based in the $\lambda \phi^4$ model are not found to be stable against hadronization, although those based on quark mass-density dependent are. Therefore we do not claim a statement about the binding energy of quark-diquark matter but rather a feature of the particular model description of the equation of state.

The work of Ref. 20 has discussed how to check the viability of a quark-diquark picture of the source Her X-1 (claimed to be a strange star in Refs. 1 and 2 by using an approach pioneered by Vaidya and Tikekar 21, in which a fluid is found to support a given geometry. The proposed quark-diquark mixture is one such a fluid and the authors calculations seem to agree with previous works. This is quite expected, although the proof may have helped to enlighten the issue. The authors have criticized the boundary conditions used in former works and also the (claimed) inconsistency between the models of Ref. 11 and a deconfinement density higher than the outer densities found. However, we believe that this is a misleading argument since the feature happens in all self-bound stars made out of (hypothetic) self-bound states like SQM. In the latter, the pressure has always a zero for a non-zero value of the energy density, and this energy density is always lower than the deconfinement density. Therefore the boundary condition at the surface is adequate because the radius is found by imposing $P = 0$ there, and the total energy density is non-zero in a perfectly consistent way. These "pure" quark or quark-diquark stars without any crust are certainly more compact than any counterpart having a normal crust. It is also well known that there can not be in these models a coexistence of both types of matter because the lower (exotic) energy state would swallow the normal matter liberating energy. In quark-diquark models a possible normal crust must be supported by electrostatic forces as well, and therefore it is the charge density at the surface which determines the mass of the former. No electrostatically supported crust is possible without electrons, not present here because of the assumptions of the model. Should a more complete model been developed (including strange quarks and detailed beta equilibrium), a tiny crust may have been added, analogously to strange star models and with a similar thickness $\sim 100$ m. Given that the uncertainties in the microscopic description of the quark-diquark phase are likely to have a much larger effect on the radius than the presence or absence of a normal crust, we think that this is not really an issue unless very detailed models have to be constructed. However, the message here is that there is no inconsistency whatsoever in the surface boundary conditions and, of course, also no "large" crusts unless the electrostatic forces happen to be orders of magnitude more important than for strange stars. Given the state-of-the-art of quark-diquark
Quark-diquark equations of state models

models, more studies are guaranteed to see whether they may play a role in compact star structure.

Acknowledgements

G. Lugones acknowledges the Instituto de Astronomia Geofísica e Ciências Atmosféricas da Universidade de São Paulo for its hospitality and the financial support received from the Fundação de Amparo à Pesquisa do Estado de São Paulo. J.E. Horvath wishes to acknowledge the CNPq Agency (Brazil) for partial financial support.

References

1. X. -D. Li, Z. -G. Lai and Z. -R. Wang, *Astron. Astrophys.* 303, (1995) L1
2. M. Dey, I. Bombaci, J. Dey, S. Ray and B.C. Samanta, *Phys. Lett.* B438, (1998) 123; Addendum (1999), B447, 352; Erratum B467, (1999) 303.
3. J. Pons, F. M. Walter, J. M. Lattimer, M. Prakash, R. Neuhäuser, and Penghui An., *Astrophys. J.*, 564, (2002) 981.
4. A.P. Reynolds; H. Quaintrell; M.D. Still; P. Roche; D. Chakrabarty and S.E. Levine, *MNRA*, 288 (1997), 43.
5. D. L. Kaplan, M. H. van Kerkwijk, J. Anderson, astro-ph/0111174 (2001).
6. X. Li, S. Ray, J. Dey, M. Dey & I. Bombaci, *Astrophys. J.* 527, (1999) L51.
7. K. S. Cheng, Z. G. Dai, D. M. Wai and T. Lu, Science 280, 407 (1998).
8. X. Li, I. Bombaci, M. Dey, J. Dey & E. P. J. van den Heuvel, *Phys. Rev. Lett.* 83, (1999) 3776.
9. M. Anselmino, E. Pedrazzi, S. Ekelin, S. Fredriksson and D.B. Lichtenberg, *Rev. Mod. Phys.*65 (1993) 1199.
10. M. Hess, F. Karsch, E.Learmann and I. Wetzorke, *Phys.Rev.* D58 (1998) 111502
11. J. E. Horvath and J. A. de Freitas Pacheco, *Int. J. of Mod. Phys.* D 7, (1998) 19.
12. M. Colpi, F. L. Shapiro and I. Wasserman, *Phys. Rev. Lett.* 57 (1986) 2485.
13. J. F. Donoghue and K. S. Sateesh, *Phys.Rev.* D38 (1988) 360.
14. D. Kastor and J. Traschen, *Phys. Rev.* D 44, (1991) 3791
15. see M. Alford, invited talk to the *Proceedings of the Workshop on Compact Stars and the QCD Phase Diagram*, Copenhagen 2001, astro-ph/0110150 for a recent review.
16. T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957).
17. R. Rapp, T. Schäfer, E. V. Shuryak and M. Velkovsky, Phys. Rev. Lett. 81, 53 (1998).
18. E. Farhi and R. L. Jaffe, Phys. Rev. D 30, 2379 (1984).
19. G. Lugones and J. E. Horvath, submitted to LIMPD.
20. R. Sharma and S. Mukherjee, *Mod. Phys. Lett.* A 16 (2001) 1049.
21. P.C. Vaidya and R. Tikekar, *Jour. Astron.Astrophys.*, 3 (1982), 325
Fig. 1. The energy per baryon for quark-diquark matter within the $\lambda \phi^4$-based EOS as a function of the bag constant $B$, for different values of the coupling constant $\lambda$. The mass of quarks $u$ and $d$ is set to 360 $\text{MeV}$ and the diquark mass to $m_D = 400 \text{MeV}$. In order to be absolutely stable the energy per baryon must be lower than 939 $\text{MeV}$ (horizontal line) and the bag constant must be greater than 57 $\text{MeV}/\text{fm}^{-3}$ (vertical line). This condition is never fulfilled. If we relax the condition imposed on $B$ there is a small region in the parameter space where stability is allowed.

Fig. 2. The same as the previous figure but for $m_D = 600 \text{MeV}$. Absolute stability is never fulfilled even if we relax the condition imposed on $B$. 
$m_D = 400 \text{ MeV}$

- $\lambda = 60$
- $\lambda = 30$
- $\lambda = 0$
$m_D = 600\,\text{MeV}$

$E / n_B$ [MeV] at $P=0$

$B$ [MeV fm$^{-3}$]

$\lambda = 60$

$\lambda = 30$

$\lambda = 0$