AN $L_p$-LIPSCHITZ THEORY FOR PARABOLIC EQUATIONS WITH TIME MEASURABLE PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. In this article we prove the existence and uniqueness of a (weak) solution $u$ in $L_p((0,T); \Lambda_{\gamma+m})$ to the Cauchy problem

$$
\frac{\partial u}{\partial t}(t,x) = \psi(t, i\nabla)u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^d
$$

$$
u(0,x) = 0,
$$

(0.1)

where $d \in \mathbb{N}$, $p \in (1, \infty)$, $\Lambda_{\gamma+m}$ is the Lipschitz space on $\mathbb{R}^d$ whose order is $\gamma+m$, $f \in L_p((0,T); \Lambda_{\gamma})$, and $\psi(t,i\nabla)$ is a time measurable pseudo-differential operator whose symbol is $\psi(t,\xi)$, i.e.

$$
\psi(t,i\nabla)u(t,x) = \mathcal{F}^{-1}[\psi(t,\xi)\mathcal{F}[u(t,\cdot)](\xi)](x),
$$

with the assumptions

$$
\mathbb{R}[\psi(t,\xi)] \leq -\nu|\xi|^\gamma,
$$

and

$$
|D_\xi^\alpha \psi(t,\xi)| \leq \nu^{-1}|\xi|^{\gamma-|\alpha|}.
$$

Furthermore, we show

$$
\int_0^T \|u(t,\cdot)\|_{\Lambda_{\gamma+m}}^p dt \leq N \int_0^T \|f(t,\cdot)\|_{\Lambda_{\gamma}}^p dt,
$$

(0.2)

where $N$ is a positive constant depending only on $d$, $p$, $\gamma$, $\nu$, $m$, and $T$.

The unique solvability of equation (0.1) in $L_p$-Hölder space is also considered. More precisely, for any $f \in L_p((0,T); C^{n+\alpha})$, there exists a unique solution $u \in L_p((0,T); C^{n+\alpha}(\mathbb{R}^d))$ to equation (0.1) and for this solution $u$,

$$
\int_0^T \|u(t,\cdot)\|_{C^{n+\alpha}}^p dt \leq N \int_0^T \|f(t,\cdot)\|_{C^{n+\alpha}}^p dt,
$$

(0.3)

where $n \in \mathbb{Z}_+$, $\alpha \in (0,1)$, and $\gamma + \alpha \notin \mathbb{Z}_+$.

1. Introduction

The class of pseudo-differential operators is a very large class of differential operators including second-order, $2m$-order, and generators of Markov processes. Therefore, theories for pseudo-differential operators have been applied in many areas of science and contain lots of mathematical interesting properties. For classical and modern theories to pseudo-differential operators, we refer books [6, 16, 11, 7, 1].

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Pseudo-differential operators are treated mostly in elliptic setting and commonly independent of $t$ or regular with respect to $t$ even though there are a few results in parabolic setting.

Recently, the author with collaborators studied pseudo-differential operators which have no regularity with respect to $t$. In [8, 9] we obtained BMO estimates and $L^q((0, \infty); L^p)$ estimates for the singular integral operator

$$Tf(t,x) := \int_0^t \int_{\mathbb{R}^d} K(t,s,x-y)f(s,y)dyds,$$

where

$$K(s,t,x) := \mathcal{F}^{-1}\left( |\xi|^\gamma \exp \left( \int_s^t \psi(r,\xi)dr \right) \right)$$

and the symbol $\psi(t,\xi)$ satisfies (2.1) and (2.2).

In this article, we study the well-posedness of Cauchy problem (0.1) in the $L^p$-Lipschitz space $L^p((0,T); \Lambda^{\gamma+m})$ (see Definition 2 for the Lipschitz space $\Lambda^{\gamma+m}$). Especially, we obtain the optimal regularity estimate (0.2) when the given datum $f$ is in $L^p((0,T); \Lambda^m)$. To the best of our knowledge, this article is the first result which handles the unique solvability of equations with pseudo-differential operators in the Lipschitz space, although there are a few results related to the boundedness of certain elliptic pseudo-differential operators in the Lipschitz space (see [16, Chapter VI] and [12]).

There is a close relation between the Lipschitz space and the classical Hölder space. If $\gamma \in (0,1)$, then $\Lambda^\gamma = C^\gamma$. In general, Lipschitz space $\Lambda^\gamma$ is a bigger class than the classical Hölder space $C^\gamma$. Hence it is needed to remark related results in $L^p$-Hölder space. If the operator is second-order, then there exists an $L^p$-Hölder theory. In [10], Krylov obtained the unique solvability to the parabolic second-order equation

$$u_t(t,x) = a^{ij}(t)u_{x^i x^j}(t,x) - \lambda u(t,x) \quad (t,x) \in \mathbb{R}^{d+1}$$

in $L^p(\mathbb{R}; C^{2+\alpha})$-space, where $p \in (1,\infty]$ and $\alpha \in (0,1)$. Here $a^{ij}(t)$ are merely measurable and satisfy an ellipticity condition, i.e. there exists a constant $\delta > 0$ so that

$$\delta \leq a^{ij}(t)\xi^i \xi^j \leq \delta^{-1} \quad \forall (t,\xi) \in \mathbb{R}^{d+1}.$$ 

Except Krylov’s work, we could not find any other result studying parabolic equations in $L^p$-Hölder space with all $p \in (1,\infty]$. However, if we restrict $p = \infty$, many results can be found. We refer the reader to [13, 2, 3] (second-order equations) and [14, 15] (integro-differential equations).

The novelty of our result is that we handle parabolic equations with arbitrary positive order operator and the unique solvability is considered in the space $L^p((0,T); \Lambda^{\gamma+m})$ which is rougher than $L^p((0,T); C^{\gamma+m})$ with all $\gamma, m > 0$. We emphasize that our estimates hold for all $p \in (1,\infty]$. In this sense, even the $L^p$-Hölder estimate given by an application of (0.2) is new since most previous results are proved only when $\gamma \in (0,2]$, $p = \infty$, and $m \in (0,1)$.

Another innovation of this paper is the method we use. Maximum principles play an important role in the proofs of [13, 2, 3] and the proof of [10] highly depends on explicit form of the heat kernel and upper bounds of its derivatives. For integro-differential operators, the methods obtaining estimates are connected with a probability theory. For instance, probability tools such as Itô’s formula and the integral representation of generators of Lévy processes are used in the proofs.
of \cite{14, 15}. However we do not have such rich information for pseudo-differential operators and thus the method we use in this article is different from previous one. We adopt Littlewood-Paley operators which are recognized as one of most powerful tools in modern Fourier analysis. Since most computations are related to the Fourier transforms of kernels instead of kernels themselves, calculation becomes much simpler even though estimates in this paper are stronger than previous results.

This article is organized as follows. We introduce our main results in Section 2. In section 3, we prove required kernel estimates related to pseudo-differential operators. In section 4, an \( L_{\infty}((0,T); \Lambda_{\gamma+n}) \)-estimate is obtained. Finally, proofs of main theorems are given in Section 5.

We finish the introduction with notation used in the article.

- \( \mathbb{N} \) and \( \mathbb{Z} \) denote the natural number system and the integer number system, respectively. \( \mathbb{Z}_+ := \{ k \in \mathbb{Z}; k \geq 0 \} \). As usual \( \mathbb{R}^d \) stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \). For \( i = 1, \ldots, d \), multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i \in \{0,1,2,\ldots\} \), and a function \( u(x) \) we set

\[
  u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad \nabla u = (u_{x^1}, u_{x^2}, \ldots, u_{x^d}).
\]

Sometimes we use \( D^\alpha x \) to denote the variable to which differentiation is taken. \( C(\mathbb{R}^d) \) denotes the space of bounded continuous functions on \( \mathbb{R}^d \). For \( n \in \mathbb{N} \), we write \( u \in C^n(\mathbb{R}^d) \) if \( u \) is \( n \)-times continuously differentiable in \( \mathbb{R}^d \) and the supremum of all derivatives up to \( n \) is bounded, i.e. \( \sup_{x \in \mathbb{R}^d, |\alpha| \leq n} |D^\alpha u(x)| < \infty \). Simply we put \( C^n := C^n(\mathbb{R}^d) \).

- For \( p \in [1, \infty) \), a normed space \( F \), and a measure space \((X, \mathcal{M}, \mu)\),

\[
  L_p(X, \mathcal{M}, \mu; F)
\]

denotes the space of all \( F \)-valued \( \mathcal{M}^\mu \)-measurable functions \( u \) so that

\[
  \|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|_F^p \mu(dx) \right)^{1/p} < \infty,
\]

where \( \mathcal{M}^\mu \) denotes the completion of \( \mathcal{M} \) with respect to the measure \( \mu \).

For \( p = \infty \), we write \( u \in L_\infty(X, \mathcal{M}, \mu; F) \) iff

\[
  \sup_x |u(x)| := \|u\|_{L_\infty(X, \mathcal{M}, \mu; F)} := \inf \left\{ \nu \geq 0 : \mu(\{x : \|u(x)\|_F > \nu\}) = 0 \right\} < \infty.
\]

If there is no confusion for the given measure and \( \sigma \)-algebra, we usually omit the measure and \( \sigma \)-algebra. In particular, we denote \( L_p = L_p(\mathbb{R}^d, \mathcal{L}, \ell; \mathbb{R}) \), where \( \mathcal{L} \) is the Lebesgue measurable sets, and \( \ell \) is the Lebesgue measure.

- We use the notation \( N \) to denote a generic constant which may change from line to line. If we write \( N = N(a, b, \cdots) \), this means that the constant \( N \) depends only on \( a, b, \cdots \).

- We use “:=” or “\(:=\)” to denote a definition. For \( a, b \in \mathbb{R} \), \( a \wedge b := \min\{a, b\} \), \( a \vee b := \max\{a, b\} \), and \( \lfloor a \rfloor \) is the biggest integer which is less than or equal to \( a \). For a set \( A \), we use \( 1_A(x) \) to denote the indicator of \( A \), i.e. \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \). For a Lebesgue measurable set \( B \), \(|B|\) is the Lebesgue measure of \( B \). For a complex number \( z \), \( \Re[z] \) is the real part of \( z \) and \( \bar{z} \) is the complex conjugate of \( z \).
\begin{itemize}
\item By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the $d$-dimensional Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$ and $\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi$.
\end{itemize}

\section{Main results}

Recall the assumptions on the symbol $\psi(t, \xi)$. Let $\gamma \in (0, \infty)$ and $\psi(t, \xi)$ be a measurable function on $[0, T] \times \mathbb{R}^d$ satisfying
\begin{equation}
\mathbb{R}[\psi(t, \xi)] \leq -\nu|\xi|^{\gamma}, \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}^d
\end{equation}
and
\begin{equation}
|D^\alpha_x \psi(t, \xi)| \leq \nu^{-1}|\xi|^{\gamma-|\alpha|}, \quad \forall (t, \xi) \in [0, T] \times (\mathbb{R}^d \setminus \{0\}), \quad |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1,
\end{equation}
where $\nu$ is a positive constant. The $\left\lfloor \frac{d}{2} \right\rfloor + 1$ differentiability on the symbol has been known as an optimal differentiability (cf. Mihlin’s condition and Hörmander’s condition \cite[Theorem 5.2.7]{4}). For notational convenience, we put
\begin{equation}
d_0 := \left\lfloor \frac{d}{2} \right\rfloor + 1.
\end{equation}

Define a pseudo-differential operator $\psi(t, i\nabla)$ on $C_c^\infty(\mathbb{R}^d)$ as
\begin{equation}
\psi(t, i\nabla)\phi(x) := \mathcal{F}^{-1}[\psi(t, \xi)\mathcal{F}(\phi)(\xi)](x)
\end{equation}
and its adjoint operator $\psi^*(t, i\nabla)$ as
\begin{equation}
\psi^*(t, i\nabla)\phi(x) := \mathcal{F}^{-1}[\psi(t, -\xi)\mathcal{F}(\phi)(\xi)](x) = \mathcal{F}^{-1}[\psi(t, \xi)\mathcal{F}(\phi^*)(\xi)](x),
\end{equation}
where $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\psi(t, \xi)$ denotes the complex conjugate of $\psi(t, \xi)$. Observe that
\begin{equation}
\mathcal{F}[\psi^*(t, i\nabla)\phi(x)](\xi) = \psi(t, -\xi)\mathcal{F}(\phi)(\xi).
\end{equation}

\begin{remark}
If $\psi(t, \xi)$ is defined on a interval $[0, T] \times \mathbb{R}^d$, then there exists a trivial extension to $\mathbb{R}^{d+1}$ by putting
\begin{equation}
\psi(t, \xi) := \psi(0, \xi) \quad t \in (-\infty, 0)
\end{equation}
and
\begin{equation}
\psi(t, \xi) := \psi(T, \xi) \quad t \in (T, \infty).
\end{equation}
Therefore we may assume that the symbol $\psi(t, \xi)$ is defined on $\mathbb{R}^{d+1}$.
\end{remark}

We adopt the definition of a solution to equation \eqref{0.1} in the weak sense as usual.

\begin{definition}[Definition of a solution $u$]
Let $T \in (0, \infty)$. We say that a locally integrable function $u$ on $(0, T) \times \mathbb{R}^d$ is a (weak) solution to equation \eqref{0.1} iff for any $\phi \in C_c^\infty((0, T) \times \mathbb{R}^d)$
\begin{equation}
\int_0^T \int_{\mathbb{R}^d} u(t, x) (-\phi_t(t, x) - \psi^*(t, i\nabla)\phi(t, x)) \, dt \, dx = \int_0^T \int_{\mathbb{R}^d} f(t, x)\phi(t, x) \, dt \, dx
\end{equation}
\end{definition}

We introduce function spaces needed to handle solvability of equation \eqref{0.1} in $L_{p'}$-Lipschitz space.
Definition 2.3. For $f \in C(\mathbb{R}^d)$ and $x, h \in \mathbb{R}^d$, define the difference operator as 
$$D_h(f)(x) := f(x + h) - f(x).$$
Inductively, for any $n \in \mathbb{N} \setminus \{1\}$, we define 
$$D_h^n(f)(x) = D_h(D_h^{n-1}f)(x).$$

Definition 2.4 (Lipschitz space). For $m \in (0, \infty)$ and $f \in C(\mathbb{R}^d)$, we define 
$$\|f\|_{\Lambda_m} := \sup_{x \in \mathbb{R}^d} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{|D_h^{\lfloor m \rfloor + 1}(f)(x)|}{|h|^m}$$
and
$$\|f\|_{\Lambda_m} := \|f\|_{L_\infty} + \sup_{x \in \mathbb{R}^d} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{|D_h^{\lfloor m \rfloor + 1}(f)(x)|}{|h|^m}.$$  \tag{2.4}

The space of continuous functions $f$ with $\|f\|_{\Lambda_m} < \infty$ is called homogeneous Lipschitz space whose order is $m$, which is denoted by $\Lambda_m$. Similarly, $\Lambda_m$ denotes the spaces of continuous functions $f$ with $\|f\|_{\Lambda_m} < \infty$ and is called (inhomogeneous) Lipschitz space.

Definition 2.5 ($L_p$-Lipschitz space). For $T \in (0, \infty)$, $p \in (1, \infty)$, $m \in (0, \infty)$, and a measurable function $f(t, x)$ on $(0, T) \times \mathbb{R}^d$, we denote
$$\|f\|_{L_p((0, T); \Lambda_m)} := \left( \int_0^T \|f(t, \cdot)\|_{\Lambda_m}^p dt \right)^{1/p}.$$ \tag{2.6}

We say that $f \in L_p((0, T); \Lambda_m)$ iff $\|f\|_{L_p((0, T); \Lambda_m)} < \infty$.

Remark 2.6. (i) Since $\|\cdot\|_{\Lambda_m}$ does not recognize polynomials of degree up to order $\lfloor m \rfloor$ and thus it is not a norm. By identifying two continuous functions whose difference is a polynomial of degree up to order $\lfloor m \rfloor$, we can regard $\|\cdot\|_{\Lambda_m}$ as a norm.
(ii) $\Lambda_m$ and $L_p((0, T); \Lambda_m)$ are Banach spaces.
(iii) From the definition of $\Lambda_m$, one can easily check that for any $f \in L_p((0, T); \Lambda_m)$ and $0 < t \leq T$,
$$\left\| \int_0^t f(s, \cdot) ds \right\|_{\Lambda_m} \leq \int_0^t \|f(s, \cdot)\|_{\Lambda_m} ds.$$ \tag{2.7}
Moreover, if we consider the integral $\int_0^t f(s, \cdot) ds$ as Bochner’s integral, then 2.7 is one of simple properties of the Bochner integral.

Here is the main result of this paper.

Theorem 2.7. Let $m, T \in (0, \infty)$ and $p \in (1, \infty]$. Then for any $f \in L_p((0, T); \Lambda_m)$, there exists a unique solution $u \in L_p((0, T); \Lambda_{\gamma+m})$ to equation 0.1. Furthermore, for this solution $u$,
$$\int_0^T \|u(t, \cdot)\|_{\Lambda_{\gamma+m}}^p dt \leq N \int_0^T \|f(t, \cdot)\|_{\Lambda_m}^p dt,$$ \tag{2.8}
where $N$ depends only on $d$, $p$, $\gamma$, $\nu$, $m$, and $T$. 


The proof of this theorem will be given in Section 5.

There is a close relation between the Lipschitz space \( \Lambda_{n+\alpha} \) and the classical Hölder space the \( C^{n+\alpha} \). We recall the definition of the classical Hölder spaces and introduce a comparison between the Lipschitz space and the Hölder space briefly.

**Definition 2.8** (Hölder space). For \( n \in \mathbb{Z}_+ \), \( \alpha \in (0,1) \), and \( f \in C^n \), we define

\[
\|f\|_{C^{n+\alpha}} := \|f\|_{L_\infty} + \sum_{|\beta|=n} \sup_{x\in \mathbb{R}^d} \sup_{h\in \mathbb{R}^d \setminus \{0\}} \frac{|D_h(D^\beta(f))(x)|}{|h|^\alpha}.
\] (2.9)

The space of continuous functions \( f \) such that \( \|f\|_{C^{n+\alpha}} < \infty \) is called Hölder space whose order is \( n + \alpha \).

**Definition 2.9** (\( L_p \)-Hölder space). For \( T \in (0,\infty) \), \( p \in (1,\infty) \), \( n \in \mathbb{Z}_+ \), \( \alpha \in (0,1) \), and a measurable function \( f(t,x) \) on \((0,T)\times \mathbb{R}^d\), we denote

\[
\|f\|_{L_p((0,T);C^{n+\alpha})} := \left( \int_0^T \|f(t,\cdot)\|^p_{C^{n+\alpha}} dt \right)^{1/p}.
\] (2.10)

We say that \( f \in L_p((0,T);C^{n+\alpha}) \) iff \( \|f\|_{L_p((0,T);C^{n+\alpha})} < \infty \).

**Remark 2.10.** From the definitions of the Lipschitz space and the Hölder space, one can easily check the following two properties:

(i) For any \( \alpha \in (0,\infty) \), \( C^\alpha \subset \Lambda_\alpha \) and the inclusion is continuous i.e. there exists a constant \( N \) so that

\[
\|f\|_{\Lambda_\alpha} \leq N\|f\|_{C^\alpha} \quad \forall f \in C^\alpha.
\]

(ii) \( C^\alpha = \Lambda_\alpha \) if \( \alpha \in (0,1) \).

**Theorem 2.11.** Let \( T \in (0,\infty) \), \( p \in (1,\infty) \), \( n \in \mathbb{Z}_+ \), and \( \alpha \in (0,1) \) so that \( \gamma + \alpha \notin \mathbb{Z}_+ \). Then for any \( f \in L_p((0,T);C^{n+\alpha}) \), there exists a unique solution \( u \in L_p((0,T);C^{\gamma+n+\alpha}) \) to equation (2.7). Furthermore, for this solution \( u \),

\[
\int_0^T \|u(t,\cdot)\|^p_{C^{\gamma+n+\alpha}} dt \leq N \int_0^T \|f(t,\cdot)\|^p_{C^{n+\alpha}} dt,
\] (2.11)

where \( N \) depends only on \( d \), \( p \), \( \gamma \), \( \nu \), \( \alpha \), and \( T \).

The proof of this theorem will be given in Section 5.

3. Preliminaries

We introduce kernels related to the symbol \( \psi(t,\xi) \) satisfying (2.1) and (2.2). For \( s < t \), \( a, b \in \mathbb{R} \), and a multi-index \( \alpha \), denote

\[
p_{\alpha,a,b}(s,t,x) := \mathcal{F}^{-1} \left[ D^\alpha_x \left( (\psi(t,\xi))^a |\xi|^b \exp \left( \int_s^t \psi(r,\xi) dr \right) \right) \right](x),
\]

\[
q_{\alpha,a,b}(s,t,\xi) := D^\alpha_x \left[ (t-s)^{\alpha} (t-s)^{-1/\gamma} \xi)^a |\xi|^b \exp \left( \int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right],
\]

and

\[
q_{\alpha,a,b}(s,t,x) := \mathcal{F}^{-1} \left[ q_{\alpha,a,b}(s,t,\cdot) \right](x).
\]
In particular, we set
\[ p_a(s, t, x) := p_{0, a, b}(s, t, x), \quad q_a(s, t, \xi) := q_{0, a, b}(s, t, \xi), \quad q_a(s, t, x) := q_{0, a, b}(s, t, x), \]
and
\[ p(s, t, x) := p_{0, 0, 0}(s, t, x). \]

**Remark 3.1.** If \( a \geq 0 \) and \( b \geq 0 \), then obviously for each \( s < t \)
\[ (\psi(t, \cdot))^a \cdot | \psi(t, \cdot) | exp \left( \int_s^t | \psi(r, \cdot) | dr \right) \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \]
due to (2.1) and (2.2). Therefore
\[ p_{a,b}(s, t, \cdot) \in L_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \]
and \( p_{a,b}(s, t, \cdot) \in L_p(\mathbb{R}^d) \) for all \( p \geq 2 \). However, we do not know \( p_{a,b}(s, t, \cdot) \in L_p(\mathbb{R}^d) \) for \( p \in [1, 2] \) yet.

**Lemma 3.2.** Let \( a \in \mathbb{Z}_+ \), \( b \in \mathbb{R} \), and \( \alpha \) be a multi-index so that \( |\alpha| \leq d_0 \). Then there exists a constant \( N(d, a, b, \gamma, \nu) \) such that for all \( s < t \) and \( \xi \neq 0 \),
\[ |q_{a,b}(s, t, \xi)| \leq N1_{a=b=0, \alpha \neq 0} \cdot |\xi|^{-|\alpha|} e^{-\nu|\xi|} \]
\[ + N (1_{a \neq 0} + 1_{b \neq 0} + 1_{\alpha = 0}) \cdot |\xi|^{\gamma a + b - |\alpha|} e^{-\nu|\xi|}. \]
(3.1)

In particular, if either (i) \( a = b = 0 \), \( \alpha \neq 0 \) or (ii) \( \gamma a + b - |\alpha| > -d \) holds then
\[ \sup_{s < t} \int_{\mathbb{R}^d} |q_{a,b}(s, t, \xi)| d\xi \leq N(d, a, b, \gamma, \nu). \]
(3.2)

**Proof.** This is an easy consequence of (2.1) and (2.2). □

**Corollary 3.3.** Let \( a \in \mathbb{Z}_+ \), \( b \in \mathbb{R} \), and \( \alpha \) be a multi-index so that \( |\alpha| \leq d_0 \). Assume that either (i) \( a = b = 0 \) and \( \alpha \neq 0 \) or (ii) \( \gamma a + b - |\alpha| > -d \) holds. Then there exists a constant \( N(d, a, b, \gamma, \nu) \) such that
\[ \sup_{s < t, x \in \mathbb{R}^d} |q_{a,b}(s, t, x)| \leq N. \]
(3.3)

**Proof.** This is an easy consequence of a property of the Fourier inverse transform and (3.2). Indeed,
\[ \sup_{s < t, x \in \mathbb{R}^d} |q_{a,b}(s, t, x)| \leq \sup_{s < t} \|q_{a,b}(s, t, \cdot)\|_{L_1(\mathbb{R}^d)} \leq N. \]
The corollary is proved. □

**Lemma 3.4.** Let \( a \in \mathbb{Z}_+ \), \( b \in \mathbb{R} \), and \( \alpha \) be a multi-index so that \( |\alpha| \leq d_0 \). Then there exists a constant \( N \) such that for all \( \varepsilon \in (0, 1) \), \( s < t \), and \( \xi \neq 0 \),
\[ \int_{|\xi| \geq \varepsilon} |\hat{q}_{a,b}(s, t, \xi)|^2 d\xi \leq N1_{a=b=0, \alpha \neq 0} \cdot \left( 1 + \varepsilon^{2(\gamma - |\alpha|) + d} + 1_{2(\gamma - |\alpha|) = -d} \cdot \ln \varepsilon^{-1} \right) \]
\[ + N (1_{a \neq 0} + 1_{b \neq 0} + 1_{\alpha = 0}) \left( 1 + \varepsilon^{2(\gamma a + b - |\alpha|) + d} + 1_{2(\gamma a + b - |\alpha|) = -d} \cdot \ln \varepsilon^{-1} \right), \]
(3.4)
where \( N = N(d, a, b, \gamma, \nu) \). In particular, if either (i) \( a = b = 0 \), \( \alpha \neq 0 \), \( 2(\gamma - |\alpha|) > -d \) or (ii) \( 2(\gamma a + b - |\alpha|) > -d \) holds, then

\[
\sup_{s < t} \int_{\mathbb{R}^d} |\hat{q}_{a, b}(s, t, \xi)|^2 d\xi \leq N(d, a, b, \gamma, \nu). \tag{3.5}
\]

**Proof.** By (3.1), Lemma 3.5.

\[
\int_{|\xi|\geq r} |\hat{q}_{a, b}(s, t, \xi)|^2 d\xi = 0 \quad \text{and recall the definition of the fractional Laplacian operator}
\]

Then by properties of the Fourier inverse transform and Plancherel’s theorem,

\[
\int_{|\xi|\geq r} |\hat{q}_{a, b}(s, t, \xi)|^2 d\xi \leq N \left( 1 + 1_{a=b=0, \alpha \neq 0} \cdot |\xi|^2(\gamma|\alpha|) + 1_{a \neq 0, 1_{\gamma a + b > |\alpha|} = 0} \cdot |\xi|^2(\gamma a + b - |\alpha|) \right) d\xi
\]

\[
\leq N \left( 1 + 1_{a=b=0, \alpha \neq 0} \cdot |\xi|^2(\gamma|\alpha|) + 1_{d(\gamma - |\alpha|) = -d \cdot \ln \epsilon^{-1}} + 1_{\gamma a + b > |\alpha|} \cdot \left( 1 + \epsilon^{2(\gamma a + b - |\alpha|) + d} \right) \right).
\]

The lemma is proved. \( \square \)

**Lemma 3.5.** Let \( a \in \mathbb{Z}_+ \) and \( b \geq 0 \) be constants such that either (i) \( a = b = 0 \) or (ii) \( \gamma a + b > 0 \) holds. Assume

\[
0 < \delta < \frac{1}{2} \wedge (\gamma a + b) \quad \text{if} \quad \gamma a + b > 0
\]

\[
0 < \delta < \frac{1}{2} \wedge \gamma \quad \text{if} \quad a = b = 0.
\]

Then there exists a constant \( N(d, a, b, \gamma, \nu, \delta) \) such that

\[
\sup_{s < t} \int_{\mathbb{R}^d} \left| |x|^{\frac{d}{2} + \delta} q_{a, b}(s, t, x) \right|^2 dx \leq N. \tag{3.7}
\]

**Proof.** Set

\[
\delta_0 := \frac{d}{2} + 1 - d_0 + \delta = \frac{d}{2} - \left\lfloor \frac{d}{2} \right\rfloor + \delta \in (0, 1)
\]

and recall the definition of the fractional Laplacian operator

\[
(-\Delta)^{\delta_0/2} f(x) = \mathcal{F}^{-1} \left[ |\xi|^{\delta_0} \mathcal{F}[f](\xi) \right](x).
\]

Then by properties of the Fourier inverse transform and Plancherel’s theorem,

\[
\sup_{s < t} \int_{\mathbb{R}^d} \left| |x|^{\frac{d}{2} + \delta} q_{a, b}(s, t, x) \right|^2 dx
\]

\[
\leq N \sum_{j=1}^{d} \sup_{s < t} \int_{\mathbb{R}^d} \left| |x|^{\frac{d}{2} - \left\lfloor \frac{d}{2} \right\rfloor + \delta} (ix^j)^{\frac{\delta}{2}} q_{a, b}(s, t, x) \right|^2 dx
\]

\[
= N \sum_{j=1}^{d} \sup_{s < t} \int_{\mathbb{R}^d} \left| |x|^{\frac{d}{2} - \left\lfloor \frac{d}{2} \right\rfloor + \delta} \mathcal{F}^{-1} \left[ \hat{q}(\frac{\xi}{2}) \right] e_j \eta_s(t, \cdot) \right|^2 dx
\]

\[
= N \sum_{j=1}^{d} \sup_{s < t} \int_{\mathbb{R}^d} \left| (-\Delta)^{\delta_0/2} \hat{q}(\frac{\xi}{2}) e_j \eta_s(t, \cdot) \right|^2 dx,
\]
where $e_j$ ($j = 1, \ldots, d$) is the standard orthonormal basis on $\mathbb{R}^d$, i.e., $e_j$ is the vector in $\mathbb{R}^d$ whose $j$-th coordinate is 1 and the others are zero. It is well-known that the fractional Laplacian operator has the integral representation

$$(-\Delta)^{\delta_0/2} \left[ \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, \cdot \right) \right] (x)$$

where

$$N \int_{\mathbb{R}^d} \frac{\hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x + y \right) - \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x \right)}{|y|^{d+\delta_0}} dy = N \left( \mathcal{I}_1(s, t, x) + \mathcal{I}_2(s, t, x) + \mathcal{I}_3(s, t, x) \right),$$

where

$$\mathcal{I}_1(s, t, x) = \int_{|y| \geq 1} \frac{\hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x + y \right) - \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x \right)}{|y|^{d+\delta_0}} dy,$$

$$\mathcal{I}_2(s, t, x) = \int_{|x|/2 \leq |y| \leq 1} \frac{\hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x + y \right) - \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x \right)}{|y|^{d+\delta_0}} dy,$$

and

$$\mathcal{I}_3(s, t, x) = \int_{|y| < 1, |y| < |x|/2} \frac{\hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x + y \right) - \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, x \right)}{|y|^{d+\delta_0}} dy.$$

First we estimate $\mathcal{I}_1$. By Minkowski’s inequality and (3.5),

$$\sup_{s < t} \| \mathcal{I}_1(s, t, \cdot) \|_{L_2}^2 \leq 2 \sup_{s < t} \| \hat{q} \left( \frac{\xi}{4} \right) e_j, a, b \left( s, t, \cdot \right) \|_{L_2}^2 \left( \int_{|y| \geq 1} |y|^{-d-\delta_0} dy \right)^2 \leq N.$$

Next we estimate $\mathcal{I}_2$. By the the fundamental theorem of calculus and (3.1),

$$\mathcal{I}_2(s, t, x)$$

$$\leq \sum_{|\alpha| = d_0} \int_{|x|/2 \leq |y| \leq 1} |y|^{1-d-\delta_0} \int_0^1 |\hat{q}_{\alpha, a, b}(s, t, x + \theta y)| d\theta dy$$

$$\leq N \int_{|y| < 1} 1_{|x| \leq 2|y|} |y|^{1-d-\delta_0} \times$$

$$\int_0^1 (1_{a=b=0} \cdot |x + \theta y|^{\gamma-d_0} + (1_{a\neq 0} + 1_{b\neq 0}) \cdot |x + \theta y|^{\gamma + b - d_0}) d\theta dy.$$

Noe that $\mathcal{I}_2 = 0$ if $|x| > 2$. Therefore by Minkowski’s inequality,

$$\| \mathcal{I}_2(s, t, \cdot) \|_{L_2(\mathbb{R}^d)}$$

$$\leq N \int_{|y| < 1} |y|^{\frac{\gamma}{2}} (1_{a=b=0} \cdot |y|^{\gamma-d_0-d-\delta_0+1} + (1_{a\neq 0} + 1_{b\neq 0}) \cdot |y|^{\gamma + b - d_0 - d - \delta_0 + 1}) dy$$

$$\leq N,$$
where (3.6) is used in the last inequality. It only remains to estimate $I_3$. By the fundamental theorem of calculus, Minkowski’s inequality, (3.4), and (3.6),

$$
\|I_3(s, t, \cdot)\|_{L^2_d(R^d)}^2 \\
\leq \left\| \sum_{|\alpha| = \delta_0} \int_{|y| < 1, |y| < |\cdot|/2} |y| \int_0^1 \frac{|q_{\alpha, a, b}(s, t, \cdot + \theta y)|}{|y|^{d+\delta_0}} \, d\theta dy \right\|_{L^2_d(R^d)}^2,
$$

$$
\leq N \int_{|y| < 1} |y|^{1-d-\delta_0} \left( \int_{|x| \geq |y|} |\hat{q}_{d, e, a, b}(s, t, x)|^2 \, dx \right)^{1/2} dy
$$

$$
\leq N \int_{|y| < 1} |y|^{1-d-\delta_0} \cdot \left( 1_{a = b = 0} \cdot |y|^{-d_0 + \frac{d}{2} + (1_{a \neq 0} + 1_{b \neq 0})} \cdot |y|^{\gamma a + b - d_0 + \frac{d}{2}} \right) dy
$$

$$
\leq N.
$$

Therefore, the lemma is proved.

\[ \square \]

**Corollary 3.6.** Let $a \in \mathbb{Z}_+$ and $b \in \mathbb{R}$ such that $a = b = 0$ or $\gamma a + b > 0$.

Then there exists a constant $N(d, a, b, \gamma, \nu)$ such that for all $s < t$,

$$
\int_{R^d} |p_{a, b}(s, t, x)| \, dx \leq N(t-s)^{-a-b/\gamma}.
$$

In particular,

$$
\sup_{s < t} \int_{R^d} |p(s, t, x)| \, dx \leq N,
$$

$$
\sup_{s < t} \int_{R^d} |p_{0, \gamma}(s, t, x)| \, dx \leq N(t-s)^{-1},
$$

and

$$
\sup_{s < t} \int_{R^d} |p_{1, \gamma}(s, t, x)| \, dx \leq N(t-s)^{-2}.
$$

**Proof.** Set

$$
\delta = \left\{ \begin{array}{ll}
\frac{1}{4} \wedge \frac{\gamma}{2} & \text{if } a = b = 0 \\
\frac{1}{4} \wedge \frac{\gamma a + b}{2} & \text{if } \gamma a + b > 0.
\end{array} \right. \quad (3.8)
$$

By Hölder’s inequality, (3.3), and (3.1),

$$
\sup_{s < t} \int_{R^d} |q_{a, b}(s, t, x)| \, dx \leq N \left( 1 + \sup_{s < t} \int_{|x| \geq 1} |q_{a, b}(s, t, x)| \, dx \right)
$$

$$
\leq N \left[ 1 + \left( \sup_{s < t} \int_{|x| \geq 1} |x|^\frac{d}{2} q_{a, b}(s, t, x)^2 \, dx \right)^{1/2} \right] \leq N.
$$

Therefore

$$
\int_{R^d} |p_{a, b}(s, t, x)| \, dx = (t-s)^{-a-b/\gamma} \int_{R^d} |q_{a, b}(s, t, x)| \, dx \leq N(t-s)^{-a-b/\gamma}.
$$
The corollary is proved.

4. $L_\infty(\Lambda_{\gamma+m})$-ESTIMATE

In this section, we introduce Littlewood-Paley operators, which play a crucial role in modern analysis and are very helpful to characterize diverse function spaces such as Sobolev space, Besov space, Lipschitz space, Hardy space, and Triebel-Lizorkin space. Applying this powerful characterization, we obtain an $L_\infty(\Lambda_{\gamma+m})$-estimate.

Choose a nonnegative function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta(\xi) = 1$ for all $|\xi| \leq 1$ and $\eta(\xi) = 0$ for all $|\xi| \geq 2$. For $n \in \mathbb{Z}$, define $\delta_n(\xi) = \eta(2^{-n}\xi) - \eta(2^{-n+1}\xi)$. Then obviously $\delta_n$ has a support in $(2^n - 1, 2^n + 1)$ and

$$\sum_{n=-\infty}^{\infty} \delta_n(\xi) = \eta(\xi) + \sum_{n=1}^{\infty} \delta_n(\xi) = 1.$$  \hspace{1cm} (4.1)

Denote $\Phi(x) := F^{-1}[\eta(\xi)](x)$ and $\Psi_n(x) := F^{-1}[\delta_n(\xi)](x) = F^{-1}[\eta(2^{-n}\xi)](x) - F^{-1}[\eta(2^{-n+1}\xi)](x)$. For a function $f \in C(\mathbb{R}^d)$, we define the Littlewood-Paley operators as

$$S_0(f)(x) := \Phi * f(x)$$

and

$$\Delta_n f(x) := \Psi_n * f(x),$$

where $*$ denotes the convolution on $\mathbb{R}^d$, i.e.

$$f * g(x) := f(\cdot) * g(\cdot)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$  

Here is the Littlewood-Paley characterization for the Lipschitz space.

**Theorem 4.1.** Let $m > 0$ and $f \in C(\mathbb{R}^d)$. Then

(i) \hspace{1cm} $\|f\|_{\dot{\Lambda}_m} < \infty$

if and only if

$$\sup_{n \in \mathbb{Z}} 2^{nm}\|\Delta_n(f)\|_{L_\infty} < \infty.$$  \hspace{1cm} (4.2)

(ii) \hspace{1cm} $\|f\|_{\Lambda_m} < \infty$

if and only if

$$\|S_0(f)\|_{L_\infty} + \sup_{n \geq 1} 2^{nm}\|\Delta_n(f)\|_{L_\infty} < \infty.$$  \hspace{1cm} (4.3)

Moreover, there exists a constant $N(d, m)$ so that

$$N^{-1}\|f\|_{\dot{\Lambda}_m} \leq \sup_{n \in \mathbb{Z}} 2^{nm}\|\Delta_n(f)\|_{L_\infty} \leq N\|f\|_{\dot{\Lambda}_m}$$

and

$$N^{-1}\|f\|_{\Lambda_m} \leq \|S_0(f)\|_{L_\infty} + \sup_{n \geq 1} 2^{nm}\|\Delta_n(f)\|_{L_\infty} \leq N\|f\|_{\Lambda_m}.$$
Proof. For the proof of this theorem, see [5 Theorems 6.3.6 - 6.3.7]. □

Recall that
\[ p(s, t, x) := F^{-1} \left[ \exp \left( \int_s^t \psi(r, \xi)dr \right) \right](x) \]
and by Corollary 3.6
\[ \sup_{s < t} \|p(s, t, \cdot)\|_{L_1(\mathbb{R}^d)} < \infty. \]
Thus for a bounded measurable function \( f \) on \([0, T] \times \mathbb{R}^d\), one can define
\[ \mathcal{G}f(t, x) := \int_0^t p(s, t, \cdot) * f(s, \cdot)(x)ds \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \]
and obtain
\[ \sup_{t \in [0, T]} \|\mathcal{G}f(t, \cdot)\|_{L_\infty(\mathbb{R}^d)} \leq \int_0^T \|p(s, t, \cdot)\|_{L_1(\mathbb{R}^d)} \|f(s, \cdot)\|_{L_\infty(\mathbb{R}^d)}ds \]
\[ \leq N(d, \gamma, \nu, T) \sup_{s \in [0, T]} \|f(s, \cdot)\|_{L_\infty(\mathbb{R}^d)}. \quad (4.4) \]

Remark 4.2. Let \( f \in L_p \left( (0, T); \Lambda_m \right), f \in L_p ((0, T); \Lambda_m), \) or \( f \in L_p ((0, T); C^{n+\alpha}) \),
where \( p \in (1, \infty), m > 0, n \in \mathbb{Z}_+, \) and \( \alpha \in (0, 1). \) Then for any \((t, x) \in (0, T) \times \mathbb{R}^d, \)
\( \mathcal{G}f(t, x) \) is well-defined and for each \( t \in (0, T), \) \( \mathcal{G}(t, x) \) is continuous with respect to \( x. \) Moreover, we have
\[ \|\mathcal{G}f(t, \cdot)\|_{\Lambda_m} \leq Nt^{1/q} \|f\|_{L_p((0, T); \Lambda_m)}, \]
\[ \|\mathcal{G}f(t, \cdot)\|_{\Lambda_m} \leq N^{1/q} \|f\|_{L_p((0, T); \Lambda_m)}, \]
and
\[ \|\mathcal{G}f(t, \cdot)\|_{C^{n+\alpha}} \leq N^{1/q} \|f\|_{L_p((0, T); C^{n+\alpha})}, \quad (4.5) \]
where \( N = N(d, \gamma, \nu) \) and \( q \) is the Hölder conjugate of \( p, \) i.e. \( 1/p + 1/q = 1. \)

Lemma 4.3. Let \( m > 0 \) and \( f \in C(\mathbb{R}^d). \) Then there exists a positive constant \( N(d, \gamma, \nu, m) \) so that
(i) for all \( s < t, \)
\[ \|p(s, t, \cdot) * f(\cdot)\|_{\Lambda_{\gamma+m}} \leq N(t - s)^{-1/2} \|f\|_{\Lambda_m} \]
and
\[ \|p(s, t, \cdot) * f(\cdot)\|_{\Lambda_{\gamma+m}} \leq N \left( 1 + (t - s)^{-1/2} \right) \|f\|_{\Lambda_m} \quad (4.6) \]
(ii) for all \( s \lor t_0 < t, \)
\[ \|p(s, t, \cdot) - p(t_0, t, \cdot)\|_{\Lambda_{\gamma+m}} \leq N|s - t_0| \left( \frac{1}{t - (s \lor t_0)} \right) \|f\|_{\Lambda_m}, \]
and
\[ \|p(s, t, \cdot) - p(t_0, t, \cdot)\|_{\Lambda_{\gamma+m}} \leq N|s - t_0| \left( 1 + (t - (s \lor t_0))^{-2} \right) \|f\|_{\Lambda_m}, \quad (4.7) \]
where \( N = N(d, \gamma, \nu, m). \)
Proof. Due to the similarity of the proof, we only prove inhomogeneous type estimates \([4.4]\) and \([4.7]\). Moreover, we may assume that \(|f|_{\Lambda_m} < \infty\) without loss of generality.

(i) Choose a \(\zeta \in C_c^\infty (\{ \xi \in \mathbb{R}^d : 2^{-2} \leq |\xi| \leq 2^2 \})\) so that \(\zeta (\xi) = 1\) if \(2^{-1} \leq |\xi| \leq 2\). Recall \(\delta_n (\xi) := \eta (2^{-n} \xi) - \eta (2^{-n+1} \xi), \eta (\xi) = 1\) for all \(|\xi| \leq 1\) and \(\eta (\xi) = 0\) for all \(|\xi| \geq 2\). Thus \(\delta_n (\xi)\) has a support in \(\{ \xi \in \mathbb{R}^d : 2^{n-1} \leq |\xi| \leq 2^{n+1}\}\) and \(\delta_n (\xi) = \delta_n (\xi) \zeta (2^{-n} \xi)\). Observe that

\[
S_0 \left( p(s,t,\cdot) \ast f(\cdot) \right) (x) = p(s,t,\cdot) \ast S_0 (f(\cdot))(x) \quad (4.8)
\]

and

\[
\Delta_n \left( \left( p(s,t,\cdot) \ast f(\cdot) \right) \right) (x) = F^{-1} \left[ \zeta (2^{-n} \xi) \delta_n (\xi) \exp \left( \int_s^t p(r,\xi) dr \right) \right] \ast f(\cdot) (x) = F^{-1} \left[ \zeta (2^{-n} \xi) |\xi|^{-\gamma} \right] \ast p_{0,\gamma} (s,t,\cdot) \ast \Psi_n (\cdot) \ast f(\cdot) (x) = F^{-1} \left[ \zeta (2^{-n} \xi) |\xi|^{-\gamma} \right] \ast p_{0,\gamma} (s,t,\cdot) \ast \Delta_n (f(\cdot))(x) ds, \quad (4.9)
\]

Therefore by Theorem \(4.3\) Young’s inequality, and Corollary \(3.6\)

\[
\|p(s,t,\cdot) \ast f(\cdot)\|_{\Lambda_{\gamma+m}} \leq N \left( 1 + (t-s)^{-1} \sup_{n \in \mathbb{N}} 2^{n\gamma} \| F^{-1} \left[ \zeta (2^{-n} \xi) |\xi|^{-\gamma} \right] \|_{L_1 (\mathbb{R}^d)} \right) \|f\|_{\Lambda_m}.
\]

It only remains to observe that

\[
\| F^{-1} \left[ \zeta (2^{-n} \xi) |\xi|^{-\gamma} \right] \|_{L_1 (\mathbb{R}^d)} \leq N 2^{-n \gamma}.
\]

The proofs of (ii) is similar to (i). We only remark that by the mean-value theorem

\[
p(s,t,\cdot) - p(t_0,t,\cdot) = \frac{\partial}{\partial s} p(\theta s + (1 - \theta) t_0, t, \cdot) = \cdot \theta \in [0,1]\]

and by Corollary \(3.6\)

\[
\left\| \frac{\partial}{\partial s} p_{0,\gamma} (\theta s + (1 - \theta) t_0, t, \cdot) \right\|_{L_1 (\mathbb{R})} = \| p_{1,\gamma} (\theta s + (1 - \theta) t_0, t, \cdot) \|_{L_1 (\mathbb{R})} \leq N |t - (\theta s + (1 - \theta) t_0)|^{-2}.
\]

The lemma is proved. \(\square\)

Since \(\int_0^1 (t-s)^{-1} ds = \infty\), Lemma \(4.3\) is not enough to show that \(G\) is a bounded operator from \(L_\infty((0,T); \Lambda_m)\) into \(L_\infty((0,T); \Lambda_{\gamma+m})\). We need more delicate estimates.

**Lemma 4.4.** Let \(\zeta (\xi) \in C_c^\infty (\{ \xi \in \mathbb{R}^d : 2^{-2} \leq |\xi| \leq 2^2 \})\). Then there exists a constant \(c > 0\) and \(N > 0\) so that for all \(s < t\) and \(n \in \mathbb{N}\),

\[
\left\| \frac{\partial}{\partial s} p_{0,\gamma} (\theta s + (1 - \theta) t_0, t, \cdot) \right\|_{L_1 (\mathbb{R})} \leq N e^{-c(t-s)2^{\gamma}},
\]

where \(c\) and \(N\) depend only on \(d, \gamma, \nu, \) and \(\zeta\).
Proof. It suffices to show that for all $j = 1, \ldots, d$,
\[
\int_{\mathbb{R}^d} \left( F^{-1} \left[ \zeta(\xi) \exp \left( \int_s^t \psi(r, 2^n \xi) dr \right) \right] (x) \right)^2 dx \\
+ \int_{\mathbb{R}^d} \left( F^{-1} \left[ D^{d, e_j} \zeta(\xi) \exp \left( \int_s^t \psi(r, \xi) dr \right) \right] (x) \right)^2 dx \leq Ne^{-2c(t-s)2^{n\gamma}}.
\]
Indeed, if (4.10) holds, then by the change of variable, Hölder’s inequality, and the property of the Fourier inverse transform that
for a function $f(t, x)$, we use the notation that $S_0(f(t, x)) := S_0 f(t, \cdot)(x)$ and $\Delta_n(f(t, x)) := \Delta(f(t, \cdot))(x)$. Here is the main result of this section.
Theorem 4.5. Let \( m > 0 \) and \( T \in (0, \infty) \). Then \( \mathcal{G} \) is a bounded operator from \( L_{\infty}((0,T); \dot{A}_m) \) into \( L_{\infty}((0,T); \dot{A}_{\gamma + m}) \) and from \( L_{\infty}((0,T); \Lambda_m) \) into \( L_{\infty}((0,T); \Lambda_{\gamma + m}) \). More precisely, there exist positive constants \( N_1 \) and \( N_2 \) so that

\[
\| \mathcal{G} f \|_{L_{\infty}((0,T); \dot{A}_{\gamma + m})} \leq N_1 \| f \|_{L_{\infty}((0,T); \Lambda_m)} \quad \forall f \in L_{\infty}((0,T); \dot{A}_m),
\]

and

\[
\| \mathcal{G} f \|_{L_{\infty}((0,T); \Lambda_{\gamma + m})} \leq N_2 \| f \|_{L_{\infty}((0,T); \Lambda_m)} \quad \forall f \in L_{\infty}((0,T); \Lambda_m),
\]

where \( N_1 = N_1(d, \gamma, \nu, m) \) and \( N_2 = N_2(d, \gamma, \nu, m, T) \).

Proof. Choose a \( \zeta \in C_0^\infty(\{ \xi \in \mathbb{R}^d : 2^{-2} \leq |\xi| \leq 2^2 \}) \) so that \( \zeta(\xi) = 1 \) if \( 2^{-1} \leq |\xi| \leq 2 \) as in the proof of Lemma 4.3. Similarly to (4.8) and (4.9), we have

\[
S_0 (\mathcal{G} f(t, \cdot))(x) = \int_0^t F^{-1} \left[ \eta(\xi) \exp \left( \int_s^t \psi(r, \xi)dr \right) \right] (\cdot) \ast f(s, \cdot)(x)ds
\]

\[
= \int_0^t p(s, t, \cdot) \ast \Phi(\cdot) \ast f(s, \cdot)(x)ds
\]

\[
= \int_0^t p(s, t, \cdot) \ast S_0 f(s, \cdot)(x)ds
\]

(4.11)

and

\[
\Delta_n (\mathcal{G} f(t, \cdot))(x) = \int_0^t F^{-1} \left[ \zeta(2^{-n}\xi) \delta_n(\xi) \exp \left( \int_s^t \psi(r, \xi)dr \right) \right] (\cdot) \ast f(s, \cdot)(x)ds
\]

\[
= \int_0^t p^\kappa_n(s, t, \cdot) \ast \Psi_n(\cdot) \ast f(s, \cdot)(x)ds
\]

\[
= \int_0^t p^\kappa_n(s, t, \cdot) \ast \Delta_n(f)(s, \cdot)(x)ds,
\]

(4.12)

where

\[
p^\kappa_n(s, t, x) := F^{-1} \left[ \zeta(2^{-n}\xi) \exp \left( \int_s^t \psi(r, \xi)dr \right) \right](x).
\]

Thus by Theorem 4.1 (4.11), and (4.12),

\[
\| \mathcal{G} f(t, \cdot) \|_{\dot{A}_{\gamma + m}} \leq N \sup_{n \in \mathbb{Z}} \int_0^t 2^{n(m+\gamma)} \sup_{x \in \mathbb{R}^d} |p^\kappa_n(t, s, \cdot) \ast \Delta_n(f)(s, \cdot)(x)| ds
\]

\[
\leq N \sup_{n \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^d} 2^{n\gamma} |p^\kappa_n(t, s, x)| dx ds \| f \|_{L_{\infty}((0,T); \dot{A}_m)}
\]

and

\[
\| \mathcal{G} f(t, \cdot) \|_{\Lambda_{\gamma + m}} \leq N \int_0^t \sup_{x \in \mathbb{R}^d} |p(s, t, \cdot) \ast S_0 f(s, x)| ds
\]

\[
+ N \sup_{n \geq 1} \int_0^t 2^{n(m+\gamma)} \sup_{x \in \mathbb{R}^d} |p^\kappa_n(t, s, \cdot) \ast \Delta_n(f)(s, \cdot)(x)| ds
\]

\[
\leq N \sup_{n \geq 1} \int_0^t \int_{\mathbb{R}^d} (|p(s, t, x)| + 2^{n\gamma} |p^\kappa_n(t, s, x)|) dx ds \| f \|_{L_{\infty}((0,T); \Lambda_m)}.
\]

Therefore it suffices to find positive constants \( N_1(d, \gamma, \nu, m) \) and \( N_2(d, \gamma, \nu, m, T) \) so that

\[
\sup_{t \in [0,T]} \int_0^t \int_{\mathbb{R}^d} |p(s, t, x)| dx ds \leq N_2
\]
and
\[ \sup_{t \in [0, T]} \sup_{n \in \mathbb{Z}} 2^{n\gamma} \int_0^T \int_{\mathbb{R}^d} |p^\delta_n(t, s, x)| dx ds \leq N_1 \]

But this is an easy consequence of Corollary 3.6 and Lemma 4.4. The theorem is proved. \(\square\)

5. Proof of main theorems

To prove that \(G\) is a bounded operator from \(L_p((0, T); \hat{A}_m)\) into \(L_p((0, T); \hat{A}_{\gamma+m})\) with \(p \in (1, \infty)\), we use the Marcinkiewicz interpolation theorem. Since it is already shown that \(G\) is a bounded operator from \(L_\infty((0, T); \hat{A}_m)\) into \(L_\infty((0, T); \hat{A}_{\gamma+m})\), we only need to check that \(G\) satisfies the weak type \((1, 1)\)-estimate.

**Lemma 5.1.** Let \(T > 0\) and \(f \in L_1\left( (0, T); \hat{A}_m \right) \cap L_\infty\left( (0, T); \hat{A}_m \right)\). Then there exists a positive constant \(N\) so that for all \(\lambda > 0\),
\[ \lambda|\{ t \in (0, T) : \|Gf(t, \cdot)\|_{\hat{A}_{m+\gamma}} > \lambda \}| \leq N \int_0^T \|f(t, \cdot)\|_{\hat{A}_m} dt, \quad (5.1) \]
where \(N = N(d, \gamma, \nu, m)\).

**Proof.** It suffices to find positive constants \(N_1, N_2\) which only depend on \(d, \gamma, \nu, \) and \(m\) so that for all \(\lambda > 0\),
\[ \lambda|\{ t \in (0, T) : \|Gf(t, \cdot)\|_{\hat{A}_{m+\gamma}} > N_1 \lambda \}| \leq N_2 \int_0^T \|f(t, \cdot)\|_{\hat{A}_m} dt, \quad (5.2) \]
If we consider \(N_1 f(t, x)\) instead of \(f\) in (5.2), then (5.1) is obtained.

Consider a class of dyadic cubes in \(\mathbb{R}\) such that
\[ Q_{k,l} := [2^k l, 2^k(l+1)) \quad k,l \in \mathbb{Z} \]
and denote
\[ Q^*_k,l := [2^k(l-1), 2^k(l+2)) \]
For a function \(\bar{f}(t) := 1_{0 < t < T} \|f(t, \cdot)\|_{\hat{A}_m}\), we apply the Calderón-Zygmund decomposition (for instance, see [4, Theorem 4.3.1]). Then for any \(\lambda > 0\) we have
\[ \bar{f}(t) = g_\lambda(t) + b_\lambda(t), \]
where
\[ g_\lambda(t) = \begin{cases} \bar{f}(t) & \text{if } t \in (\cup_j Q_j)^c, \\ \frac{1}{|Q_j|} \int_{Q_j} \bar{f}(r)dr & \text{if } t \in Q_j, \end{cases} \]
\[ b_\lambda(t) := \bar{f}(t) - g_\lambda(t) = \sum_j 1_{Q_j}(t) \left( \bar{f}(t) - \int_{Q_j} \bar{f}(r)dr \right) =: \sum_j b_{\lambda,j}(t), \]
and \(Q_j = Q_{k_j,l_j}\) for some \(k_j, l_j \in \mathbb{Z}\). Here \(g_\lambda\) and \(b_\lambda\) satisfy the followings:
(i) \(\|g_\lambda\|_{L_1(\mathbb{R})} \leq \|\bar{f}\|_{L_1(\mathbb{R})}\) and \(\|g_\lambda\|_{L_\infty} \leq 2\lambda\),
(ii) Each \(b_j\) is supported in a dyadic cube \(Q_j\). Furthermore, \(Q_j\) and \(Q_\ell\) are disjoint if \(j \neq \ell\).
(iii) \[ \int_{Q_j} b_{\lambda,j}(x) dx = 0. \]

(iv) \[ \| b_{\lambda,j} \|_{L^2(\mathbb{R})} \leq 2^2 \lambda |Q_j|. \]

(v) \[ \sum_j |Q_j| \leq \lambda^{-1} \| \bar{f} \|_{L^2(\mathbb{R})}. \]

Denote \( f_T(t,x) = 1_{0 < t < T} f(t,x) \) and consider the decomposition
\[ f_T(t,x) = f_{\lambda,1}(t,x) + f_{\lambda,2}(t,x), \]
where
\[ f_{\lambda,1}(t,x) = \left\{ \begin{array}{ll}
  f_T(t,x) & \text{if } t \in (\cup_j Q_j)^c \\
  \frac{1}{|Q_j|} \int_{Q_j} f_T(r,x) dr & \text{if } t \in Q_j,
\end{array} \right. \]
and
\[ f_{\lambda,2}(t,x) = \sum_j 1_{Q_j}(t) \left( f_T(t,x) - \frac{1}{|Q_j|} \int_{Q_j} f_T(r,x) dr \right). \]

Note that \( \| f_T(\cdot) \|_{\hat{A}_m} = \bar{f}(t) \) and thus
\[ \| f_{\lambda,1}(t,\cdot) \|_{\hat{A}_m} \leq |g_\lambda(t)|. \]
Hence by Theorem 4.5 and (i), there exists a positive constant \( N_0(d,\gamma,\nu,m) \) so that
\[ \| \mathcal{G}(f_{\lambda,1}) \|_{L_\infty((0,T) ; \hat{A}_{m+\gamma})} \leq \frac{N_0}{2} \sup_t |g_\lambda(t)| \leq N_0 \lambda. \]

Therefore,
\[ |\{ t \in (0,T) : \| \mathcal{G} f(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 4N_0 \lambda \}| \]
\[ \leq |\{ t \in (0,T) : \| \mathcal{G} f_{\lambda,1}(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 2N_0 \lambda \}| + |\{ t \in (0,T) : \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 2N_0 \lambda \}| \]
\[ = |\{ t \in (0,T) : \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 2N_0 \lambda \}|. \]

We split the last term above into two parts. By (v) and Chebyshev's inequality,
\[ |\{ t \in (0,T) : \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 2N_0 \lambda \}| \]
\[ \leq \sum_j |Q_j^*| + |\{ t \in (0,T) \cap (\cup_j Q_j)^c : \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} > 2N_0 \lambda \}| \]
\[ \leq 3 \lambda^{-1} \int_0^T \| f(t,\cdot) \|_{A_m} dt + N \lambda^{-1} \int_{(0,T) \cap (\cup_j Q_j)^c} \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} dt, \]
where \( (\cup_j Q_j)^c = \mathbb{R} \setminus \cup_j Q_j^* \). It only remains to show that
\[ \int_{(0,T) \cap (\cup_j Q_j)^c} \| \mathcal{G} f_{\lambda,2}(t,\cdot) \|_{\hat{A}_{m+\gamma}} dt \leq N \int_0^T \| f(t,\cdot) \|_{A_m} dt. \]
We put \( Q_j = [t_j, t_j + \delta_j] \). Then \( Q_j^* = [t_j - \delta_j, t_j + 2\delta_j] \). Obviously, if \( t \in (-\infty, t_j - \delta_j) \) and \( s \in Q_j \), then \( 1_{s < t} p(s, t, x) = 0 \). Moreover, for any \( t \in (0, T) \),

\[
\mathcal{G} f_{\lambda, 2}(t, x) = \int_0^t p(s, t, \cdot) \ast f_{\lambda, 2}(s, \cdot)(x) \, ds
\]

\[
= \sum_j \int_{\mathbb{R}} \left( 1_{0 < s < t < T} p(s, t, \cdot) - 1_{t_j < s < t_j + \delta_j} p(t_j, t, \cdot) \right) * 1_{Q_j}(s) \left( f_T(s, \cdot) - \frac{1}{|Q_j|} \int_{Q_j} f_T(r, \cdot) \, dr \right)(x) \, ds
\]

since

\[
\int_{\mathbb{R}} 1_{Q_j}(s) \left( f_T(s, y) - \frac{1}{|Q_j|} \int_{Q_j} f_T(r, y) \, dr \right) \, ds = 0 \quad \forall y \in \mathbb{R}^d.
\]

Therefore by Lemma 4.3,

\[
\int_{(0, T) \cap (\cup_j Q_j^* \cup (\cup_j Q_j^*)))} \| \mathcal{G} f_{\lambda, 2}(t, \cdot) \|_{\tilde{\Lambda}_{m+\gamma}} \, dt
\]

\[
\leq \sum_j \int_{(t_j + 2\delta_j, t_j + T)} \| \mathcal{G} f_{\lambda, 2}(t, \cdot) \|_{\tilde{\Lambda}_{m+\gamma}} \, dt
\]

\[
\leq \sum_j \int_{(t_j + 2\delta_j, t_j + T)} \int_{Q_j} \left\| (p(s, t, \cdot) - p(t_j, t, \cdot)) * \left( f_T(s, \cdot) - \frac{1}{|Q_j|} \int_{Q_j} f_T(r, \cdot) \, dr \right) \right\|_{\tilde{\Lambda}_{m+\gamma}} \, ds \, dt
\]

\[
\leq N \sum_j \int_{(t_j + 2\delta_j, t_j + T)} \left( \int_{Q_j} \delta_j(t - s)^{-2} \| f_T(s, \cdot) \|_{\tilde{\Lambda}_{m}} \, ds + \int_{Q_j} \delta_j(t - s)^{-2} ds \frac{1}{|Q_j|} \int_{Q_j} \| f_T(r, \cdot) \| \, dr \right) \, dt
\]

\[
\leq N \sum_j \int_0^T \| f(s, \cdot) \|_{\tilde{\Lambda}_{m}} \, ds.
\]

The lemma is proved. \( \square \)

**Corollary 5.2.** Let \( T > 0 \) and \( p \in (1, \infty) \). Then there exist positive constant \( N_1 \) and \( N_2 \) so that

\[
\| \mathcal{G} f \|_{L_p((0, T); \tilde{\Lambda}_{\gamma+m})} \leq N_1 \| f \|_{L_p((0, T); \tilde{\Lambda}_m)} \quad \forall f \in L_1 ((0, T); \tilde{\Lambda}_m) \cap L_\infty ((0, T); \tilde{\Lambda}_m),
\]

(5.3)

and

\[
\| \mathcal{G} f \|_{L_p((0, T); \tilde{\Lambda}_{\gamma+m})} \leq N_2 \| f \|_{L_p((0, T); \tilde{\Lambda}_m)} \quad \forall f \in L_1 ((0, T); \tilde{\Lambda}_m) \cap L_\infty ((0, T); \tilde{\Lambda}_m),
\]

(5.4)

where \( N_1 = N_1(d, p, \gamma, \nu, m) \) and \( N_2 = N_2(d, p, \gamma, \nu, m, T) \).
Proof. (5.3) is an easy application of the Marcinkiewicz interpolation theorem with Theorem 4.5 and Lemma 5.1. (5.4) comes from (4.4) and (5.3) □

The following lemma will be used to show the existence of a solution to equation (0.1).

Lemma 5.3. For \( \phi \in C_\infty_c((0,T) \times \mathbb{R}^d) \), denote

\[
f_\phi(t,x) = -\phi_t(t,x) - \psi^*(t,i\Delta)\phi(t,x).
\]

Then for all \((s,y) \in (0,T) \times \mathbb{R}^d\),

\[
\int_0^T \int_{\mathbb{R}^d} 1_{s<t} p(s, t, x-y)(f_\phi(s, x))dxdt = \phi(s, y).
\]

Proof. It suffices to show that

\[
\mathcal{F}\left[ \int_s^T \int_{\mathbb{R}^d} p(s, t, x)(f_{\phi^*}(s, x + \xi))dt dx \right](\xi) = \mathcal{F}[\phi(s, \cdot)](\xi).
\]  

(5.5)

Denote \( \hat{\phi}(t, \xi) := \mathcal{F}[\phi(t, \cdot)](\xi) \) and recall the property of the Fourier transform that

\[
\mathcal{F}\left[ \int_{\mathbb{R}^d} f(x)g(x + \xi)dx \right](\xi) = \mathcal{F}[f](t \xi)g(\xi).
\]

Thus by (2.3) and the integration by parts, the left hand side of (5.5) is equal to

\[
\int_s^T \exp\left( \int_s^t \psi(r, -\xi)dr \right) \left(-\frac{\partial \hat{\phi}}{\partial t}(t, \xi) - \psi(t, -\xi)\hat{\phi}(t, \xi) \right) dt
\]

\[
= \hat{\phi}(s, \xi) + \int_s^T \psi(t, -\xi) \exp\left( \int_s^t \psi(r, -\xi)dr \right) \hat{\phi}(t, \xi) dt
\]

\[- \int_t^s \exp\left( \int_t^r \psi(r, -\xi)dr \right) \psi(t, -\xi)\hat{\phi}(t, \xi) dt = \hat{\phi}(s, \xi).
\]

The lemma is proved. □

The proof of Theorem 2.7

Part I. (Uniqueness)

Let \( u \in L_p((0,T); \Lambda_{m+\gamma}) \) be a solution to the equation

\[
\frac{\partial u}{\partial t}(t, x) = \psi(t,i\nabla)u(t, x), \quad (t, x) \in (0,T) \times \mathbb{R}^d
\]

\[u(0, x) = 0,\]

Then by the definition of the solution, for any \( \phi \in C_\infty_c((0,T) \times \mathbb{R}^d) \)

\[
\int_0^T \int_{\mathbb{R}^d} u(t, x) (-\phi_t(t, x) - \psi^*(t,i\nabla)\phi(t, x)) dt dx = 0.
\]

Since \( u \) is a locally integrable function on \((0,T) \times \mathbb{R}^d\), it suffices to show that

\[
\{(\phi(t, x) - \psi^*(t,i\nabla)\phi(t, x)) : \phi(t, x) \in C_c((0,T) \times \mathbb{R}^d)\}
\]

(5.6)
is dense in $L_p((0, T) \times \mathbb{R}^d)$. However, this is an application of an $L_p$-theory, which is proved in author's previous paper [9]. Note that in [9], we considered the equation:

$$\frac{\partial u}{\partial t}(t, x) = \psi^*(t, i\nabla)u(t, x) - \lambda u(t, x) + g(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad \lambda > 0.$$ 

However following [11, Section 2.5], we can obtain the unique solvability of the Cauchy problem [11] and considering the change of variable $t \to T - t$, we can also have the unique solvability of the terminal value problem. Thus for any $g \in L_p((0, T) \times \mathbb{R}^d)$, there exists a unique solution $v \in L_p((0, T); H^\gamma_p(\mathbb{R}^d)) \cap H^1_p((0, T); L_p(\mathbb{R}^d))$ to the equation

$$-\frac{\partial v}{\partial t}(t, x) = \psi^*(t, i\nabla)v(t, x) + g(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d$$

and

$$\|v\|_{L_p((0, T); H^\gamma_p(\mathbb{R}^d))} + \|v\|_{L_p((0, T); H^1_p(\mathbb{R}^d))} \leq N\|g\|_{L_p((0, T) \times \mathbb{R}^d)},$$

where $N$ is independent of $g$, $H^\gamma_p(\mathbb{R}^d)$ is the fractional Sobolev space with the order $\gamma$ and the exponent $p$, and $H^1_p((0, T); L_p(\mathbb{R}^d))$ is the $L_p(\mathbb{R}^d)$-valued Sobolev space with the order 1 and the exponent $p$. Since the terminal value is zero and the operator $\phi \to -\phi_t - \psi^*(t, i\nabla)\phi$ is a continuous operator from $L_p((0, T); H^\gamma_p(\mathbb{R}^d)) \cap H^1_p((0, T); L_p(\mathbb{R}^d))$ to $L_p((0, T) \times \mathbb{R}^d)$, we can find a sequence of $v_n(t, x) \in C^\infty_c((0, T) \times \mathbb{R}^d)$ such that

$$\|v - v_n\|_{H^\gamma_p((0, T); H^\gamma_p(\mathbb{R}^d))} \to 0$$

and

$$\|g - g_n\|_{L_p((0, T) \times \mathbb{R}^d)} \to 0$$

as $n \to \infty$, where $g_n = -\frac{\partial v}{\partial t}(t, x) - \psi^*(t, i\nabla)v_n(t, x)$. Therefore the set in [5,0] is dense in $L_p((0, T) \times \mathbb{R}^d)$ and the uniqueness is proved.

**Part II. (Existence)**

For a $f \in L_p((0, T); \Lambda_m)$, we claim that

$$u(t, x) := \mathcal{G}f(t, x) := \int_0^t p(s, t, \cdot) * f(s, \cdot)(x)ds$$

is a solution to equation [0.1]. By Definition 2.2, it is sufficient to show that for any $\phi \in C^\infty_c((0, T) \times \mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} u(t, x)(-\phi_t(t, x) - \psi^*(t, i\nabla)\phi(t, x)) dt dx = \int_0^T \int_{\mathbb{R}^d} f(t, x)\phi(t, x) dt dx.$$ 

Recall the notation

$$f_\phi(t, x) := -\phi_t(t, x) - \psi^*(t, i\Delta)\phi(t, x).$$
By Fubini’s theorem and Lemma [5.3]

\[
\int_0^T \int_{\mathbb{R}^d} u(t,x) \left( -\phi_t(t,x) - \psi^*(t,i\nabla)\phi(t,x) \right) \, dt \, dx
= \int_0^T \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{R}^d} 1_{s<t} \phi(s,t,x-y) f_\phi(t,x) \, dt \, dx \, f(s,y) \, dy \, ds
= \int_0^T \int_{\mathbb{R}^d} \phi(s,y) f(s,y) \, dy \, ds.
\]

Part III. \((L^p((0,T);\Lambda_m+\gamma))-estimate\)

Due to Part I and II, \(Gf(t,x)\) is the unique solution to (0.1). Therefore (2.8) holds due to Corollary 5.2 if \(f\in L^1((0,T);\Lambda_m)\cap L^\infty((0,T);\Lambda_m)\). For general \(f\in L^p((0,T);\Lambda_m)\), it suffices to find an approximation \(f_n\in L^1((0,T);\Lambda_m)\cap L^\infty((0,T);\Lambda_m)\) so that

\[
\|f_n - f\|_{L^p((0,T);\Lambda_m)} \to 0
\]
as \(n \to \infty\). However, one can easily find this approximation by mollifying and cutting off \(f\) with respect to \(t\). The theorem is proved. \(\square\)

The proof of Theorem 2.11

Since the existence and uniqueness of a solution \(u\) to equation (0.1) is already proved, it suffices to show (2.11). Recall that \(u(t,x) := Gf(t,x)\) is the solution to equation (0.1). Due to (4.5),

\[
\|u\|_{L^p((0,T);C^{n+\gamma})} \leq N\|f\|_{L^p((0,T);C^{n+\gamma})}. \tag{5.7}
\]

Note that \(\Lambda_\alpha = C^{\alpha}\). Thus due to (5.3), for any multi-index \(|\beta| = n, \gamma + \alpha / \in \mathbb{Z}^+\),

\[
\|D^\beta u\|_{L^p((0,T);\Lambda_{\gamma+\alpha})} \leq N\|D^\beta f\|_{L^p((0,T);C^{\alpha})}. \tag{5.8}
\]

Moreover, due to [5, Corollary 6.3.10], for any multi-index \(\alpha_1 = [\gamma + \alpha]\),

\[
\|D^{\beta + \alpha_1} u\|_{L^p((0,T);C_{\gamma+\alpha-\gamma+\alpha})} = \|D^{\beta + \alpha_1} u\|_{L^p((0,T);\Lambda_{\gamma+\alpha-\gamma+\alpha})} \leq N\|D^\beta u\|_{L^p((0,T);\Lambda_{\gamma+\alpha})}. \tag{5.9}
\]
since \(\gamma + \alpha / \in \mathbb{Z}^+\). Finally, combining (5.7)-(5.9), we obtain (2.11). The theorem is proved. \(\square\).

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