FETI-DP FOR THE THREE-DIMENSIONAL VIRTUAL ELEMENT METHOD

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ABSTRACT. We deal with the Finite Element Tearing and Interconnecting Dual Primal (FETI-DP) preconditioner for elliptic problems discretized by the virtual element method (VEM). We extend the result of [16] to the three dimensional case. We prove polylogarithmic condition number bounds, independent of the number of subdomains, the mesh size, and jumps in the diffusion coefficients. Numerical experiments validate the theory.

1. INTRODUCTION

Methods for the solution of PDEs based on polytopal meshes have recently attracted an increasing attention, mainly due to the necessity of tackling what is nowadays a bottleneck in the overall process of simulating real life phenomena, namely the task of mesh generation. Several methods have been recently introduced which allow for quite general polygonal or polyhedral elements, such as Mimetic Finite Differences [11, 19], Discontinuous Galerkin-Finite Element Method (DG-FEM) [1, 22], Hybridizable and Hybrid High-Order Methods [24, 26], Weak Galerkin Method [47], BEM-based FEM [42] and Polygonal FEM [44] to name a few.

Here we deal with the Virtual Element Method (VEM) [5], a discretization technique which can be considered as an extension of the Finite Element Method to polytopal tessellations. In such a method, local approximation spaces containing polynomial functions are defined and assembled in a global conforming approximation space, but the explicit construction and integration of the associated shape functions is avoided, whence the name virtual [5]. The evaluation of the operators and matrices needed in the implementation of the method are carried out by only relying on an implicit knowledge of the local shape functions, as described in [7] (see also [4, 10, 37], where the $p$ and $hp$ versions of the method are discussed and analyzed). Though introduced fairly recently, such a method has already been applied and extended to a wide variety of different model problems; we recall applications to: parabolic problems [46], Cahn-Hilliard, Stokes, Navier-Stokes and Helmholtz equations [2, 3, 12, 13, 40], linear and nonlinear elasticity problems [25, 6, 29], general elliptic problems in mixed form [8], fracture networks [14], Laplace-Beltrami equation [28].

In this paper we focus on the linear system of equations associated with the VEM discretization. As it happens in the case of finite element, the efficient solution of such a linear system is of paramount importance to fully exploit the potential of the method. Little work has been done on this issue up to now, all limited to the spatial dimension two. First works in the literature tackled the increase of the condition number appearing already at the level of the elementary stiffness matrix, due either to a degradation of the quality of the tessellation and/or to the increase in the polynomial order of the method [10, 36, 15]. If we rather consider the increase of the condition number resulting from refining the discretization, to the best of our knowledge the approaches considered up to now are domain decomposition ([21, 20, 16, 41]) and multigrid ([4], for $p$ refinement). In the present paper we extend to the three dimensional case the results obtained in [16]. More precisely, we focus on one of the most efficient preconditioning techniques: the Dual-Primal Finite Element Tearing
and Interconnecting (FETI-DP) [27, 45], a non overlapping domain decomposition method where
the problem is reformulated as a constrained optimization problem and solved by iterating on
the set of Lagrange multipliers representing the fluxes across the interface between the non overlapping
subdomains. The FETI-DP method has been already extensively studied in the context of many
different discretization methods – spectral elements [38, 31], mortar discretizations [30], NURBS
discretizations in isogeometric analysis [39].

Following the approach presented in [16] for two dimensional domains, we prove that the prop-
erties of scalability, quasi-optimality and independence on the discontinuities of the elliptic operator
coefficients across subdomain interfaces, that are known for the finite element case, still hold when
dealing with VEM. More specifically, we show that the condition number of the preconditioned
matrix is bounded by a constant times the factor \((1 + \log\left(H/h\right))^2\), where \(H\) and \(h\) are, respectively,
mesh-size of the subdomain decomposition and of the tessellation, see Theorem 4.7. In order to do
so, we need to prove several inequalities related to the VEM approximation space, by only relying
on the implicit definition of the discrete functions, which, we recall, are not explicitly known.

We observe that, since we are in the framework of [35], the equivalence of the BDDC (Balancing
Domain Decomposition by Constraint) and the FETI-DP preconditioners holds. Therefore the
bound for the condition number here obtained also yields an estimate on the BDDC preconditioner
for VEM.

The paper is organized as follows. The basic notation, functional setting and the description of
the Virtual Element Method are given in Section 2. The dual-primal preconditioner is introduced
and analyzed in Section 4, whereas some relevant properties of the VE discretization space mainly
used for the proof are presented in Section 3. The analysis of the preconditioner, with the proof of
the estimate for the condition number (Theorem 4.7), is carried out in section 5 where we also give
some detail specific to its implementation in the VEM framework. Numerical experiments that
validate the theory are presented in Section 6.

2. THE VIRTUAL ELEMENT METHOD (VEM)

Notations. As we are interested here in explicitly studying the dependence of the estimates that
we are going to prove on the number and size of the subdomains and the number and size of
the elements of the tessellations, throughout the paper we will employ the notation \(A \lesssim B\) (resp.
\(A \gtrsim B\)) to say that the quantity \(A\) is bounded from above (resp. from below) by \(cB\), with a
constant \(c\) independent of the diffusion coefficient \(\rho\) in the PDE (and in particular, independent of
\(M, \alpha\) and of its jump across the interface of the decomposition), and depending on the tessellation
and on the decomposition only via the (possibly implicit) constants in Assumptions 2.2 and 4.1.
The expression \(A \simeq B\) will stand for \(A \lesssim B \lesssim A\).

The Virtual Element Discretization. We start by recalling the definition and the main prop-
erties of the Virtual Element Method [5] and, to fix the ideas, we focus on the following elliptic
model problem:

\[-\nabla \cdot (\rho \nabla u) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

with \(g \in L^2(\Omega)\), where \(\Omega \subset \mathbb{R}^3\) is (for simplicity) a convex polyhedron. We assume that the
coefficient \(\rho\) is a scalar such that for almost all \(x \in \Omega\), \(\alpha \leq \rho(x) \leq M\) for two constants \(M \geq \alpha > 0\).
The variational formulation of such an equation reads

\[
\begin{cases}
\text{find } u \in V := H^1_0(\Omega) \text{ such that } \\
\quad a(u, v) = (f, v) \quad \forall v \in V
\end{cases}
\]

with

\[
a(u, v) = \int_{\Omega} \rho(x) \nabla u(x) \cdot \nabla v(x) \, dx, \quad (g, v) = \int_{\Omega} g(x) v(x) \, dx.
\]
We consider a family \( \{T_h\}_h \) of tessellations of \( \Omega \) into a finite number of polyhedra \( K \).

**Definition 2.1.** We say that a polyhedron \( K \) is shape regular of diameter \( h \) if there exist \( \gamma_K > 0 \) such that \( K \) satisfies the following assumptions (2.3):

1. for every element \( K \), every face \( f \) and every edge \( e \) we have 
   \[
   \gamma_K h^3 \leq |K|, \quad \gamma_K h^2 \leq |f|, \quad \gamma_K h \leq |e|.
   \]
2. for each face \( f \), there exists a point \( x_f \in f \) such that \( f \) is star-shaped with respect to every point in the disk of radius \( \gamma_K h \) centered at \( x_f \);
3. for each element \( K \), there exists a point \( x_K \) such that \( K \) is star-shaped with respect to every point in the sphere of radius \( \gamma_K h \) centered at \( x_K \).
4. for every element \( K \), and for every face \( f \) of \( K \), there exists a pyramid contained in \( K \) such that its base equals to \( f \), its height equals to \( \gamma_K h \) and the projection of its vertex onto \( f \) is \( x_f \).

**Assumption 2.2.** We assume that there exist two constants \( N_* \) and \( \gamma_* \) such that the tessellation \( T_h \) verifies the following assumptions

1. \( T_h \) is geometrically conforming, that is for each \( K, K' \) in \( T_h \), \( \bar{K} \cap \bar{K}' \) is either the empty set, a vertex, an edge or a face of both \( K \) and \( K' \);
2. every element \( K \) has at most \( N_* \) faces and each face has at most \( N_* \) edges.
3. all \( K \in T_h \) are shape regular of diameter \( h_K \) with \( \gamma_K \geq \gamma_* \);
4. the tessellation is quasi uniform, that is there exists an \( h \) such that for all \( K \in T_h \) \( h_K \simeq h \).

The lowest order Virtual Element discretization space is defined element by element starting from the faces of the tessellation, where the discrete functions are defined as linears. On the boundary of each face \( f \) we then introduce the space:

\[
\mathbb{B}_1(\partial f) = \{ g \in C^0(\partial f) : g|_e \in \mathbb{P}_1(e) \text{ for all edge } e \subseteq f \}
\]

where, for any one, two or three dimensional domain \( D \), \( \mathbb{P}_1(D) \) denotes the set of order 1 polynomials on \( D \). On each face \( f \) we then define the space \( V^{f,1} \) of local discrete functions as follows

\[
(2.2) \quad V^{f,1} = \{ v \in C^0(f) : v|_{\partial f} \in \mathbb{B}_1(\partial f), \Delta v \in \mathbb{P}_1(f) \text{ and } \int_f qv = \int_f q\Pi_f^v, \forall q \in \mathbb{P}_1(f) \},
\]

where \( \Pi_f^v : C^0(f) \rightarrow \mathbb{P}_1(f) \) denotes the projection onto the space of order one polynomials, orthogonal with respect to the scalar product

\[
(2.3) \quad (v, w)_{1,f} = \int_f \nabla v(x) \cdot \nabla w(x) + \left( \sum_{\text{vertex of } f} v(V) \right) \left( \sum_{\text{vertex of } f} w(V) \right).
\]

Observe that the values at the vertices of \( f \) uniquely determine the (piecewise linear) trace of any function \( v \) in \( V^{f,1} \) on \( \partial f \), which, in turn, uniquely determines (and allows to compute) \( \Pi_f^v \). Indeed, for \( q \in \mathbb{P}_1(K) \), Green’s formula yields

\[
(2.4) \quad \int_f \nabla v(x) \cdot \nabla q(x) \, dx = \int_{\partial f} v(s)\nabla q(s) \cdot n_f(s) \, ds.
\]

As a function in \( H^1(f) \) with linear Laplacian is uniquely defined by its trace on \( \partial f \) and by its moments up to order one (which, by definition, for \( v \in V^{f,1} \) coincide with those of \( \Pi_f^v \)), the values of any function \( v \in V^{f,1} \) at the vertices of \( f \) uniquely determine \( v \).

We can then assemble the local face spaces to build local spaces on the boundary of \( K \):

\[
\mathbb{B}_1(\partial K) := \{ g \in C^0(\partial K) : g|_f \in V^{f,1} \text{ for all face } f \subseteq \partial K \}
\]
and on \( K \)
\[
V^{K,1} := \{ v \in H^1(K) : v_{|\partial K} \in H^1(\partial K) \text{ and } v \text{ harmonic in } K \}.
\]

Finally we define:
\[
V_h := \{ V \in H^1_0(\Omega) : v_{|K} \in V^{K,1} \text{ for all } K \in \mathcal{T}_h \}.
\]

It is not difficult to check that a function in \( V^{K,1} \) is uniquely determined by its values at the vertices of \( K \), and consequently, that all functions in \( V_h \) are uniquely determined by their values at the vertices of the tessellation. However, they are not known in closed form, so that it is not possible to directly evaluate the bilinear form \( a \) on two of such functions (this would imply solving a Poisson equation in each element). The Virtual Element Method is constructed by replacing the bilinear form \( a \) with a suitable approximation. This can be achieved starting from the observation that given any \( v \in V^{K,1} \) and any \( w \in P_1(K) \) we can easily compute \( a_K(u, v) \) by using Green’s formula (we are assuming, for the sake of simplicity, that \( \rho \) is piecewise constant on the elements of the tessellation, that is, \( \rho_{|K} = \rho_K \))
\[
a_K(u, w) = -\rho_K \int_K v \Delta w + \rho_K \int_{\partial K} v \frac{\partial w}{\partial n} = \rho_K \int_{\partial K} v \frac{\partial w}{\partial n}.
\]
The right hand side can in fact be computed exactly since on each edge of \( K \) \( \partial w/\partial n \) is a known constant, and, in view of the the definition of \( V^{f,1} \), it is possible to compute the integral of \( v \) against such a constant in terms of the degrees of freedom. It is then possible to evaluate, for each \( v \in V^{K,1} \), its projection \( \Pi_{\nabla}K : V^{K,1} \to P_1(K) \), onto the space of linears, orthogonal with respect to the scalar product \( (2.3) \). Clearly we have
\[
a_K(u, v) = a_K^h(\Pi_{\nabla}K u, \Pi_{\nabla}K v) + a_K(u - \Pi_{\nabla}K u, v - \Pi_{\nabla}K v).
\]
The virtual element method stems from replacing the second term on the right hand side, which cannot be computed exactly, with an “equivalent” operator \( S^K a \). We then define
\[
a_h^K(u, v) = a_K(\Pi_{\nabla}K u, \Pi_{\nabla}K v) + S^K a(u - \Pi_{\nabla}K u, v - \Pi_{\nabla}K v),
\]
where \( S^K a \) is any continuous symmetric bilinear form satisfying
\[
a_K^2(v, v) \geq S^K a(v, v) \quad \forall v \in V^{K,1} \text{ with } \Pi_{\nabla}K v = 0.
\]
Equations \( (2.6) \) and \( (2.7) \) immediately yield
\[
a_K^2(v, v) \geq a_h^K(v, v) \quad \forall v \in V^{K,1} \quad \text{and} \quad a_K^2(v, w) = a_h^K(v, w) \quad \forall v \text{ or } w \in P_1(K).
\]

Finally, we let \( a_h : V_h \times V_h \to \mathbb{R} \) be defined by
\[
a_h(u_h, v_h) = \sum_K a_K^h(u_h, v_h),
\]
and we consider the following discrete problem:

**Problem 2.3.** Find \( u_h \in V_h \) such that
\[
a_h(u_h, v_h) = \int_{\Omega} gv_h \quad \forall v_h \in V_h.
\]

For the study of the convergence, stability and robustness properties of the method we refer to [5, 9].
3. Some relevant properties of the VE discretization space

In this section we present some bounds that will play a role in the forthcoming analysis. More precisely, letting $K \subseteq \mathbb{R}^3$ be a shape regular polyhedron of diameter $h$, and letting $f$ be any face of $K$, we have the following bounds.

**Agmon inequality.** For all functions in $H^1(f)$, it holds [33]

\[
\int_{\partial f} |u|^2 \lesssim h^{-1} \int_f |u|^2 + h \int_f |\nabla u|^2.
\]

**Inverse estimates.** The following two inverse inequality hold for all $v_h \in VF^1_1$ [23]:

\[
\int_f |\Delta v_h|^2 \lesssim h^{-2} \int_f |\nabla v_h|^2,
\]

\[
\int_f |\nabla v_h|^2 \lesssim h^{-2} \int_f |v_h|^2.
\]

**Riesz basis property.** We have the following Lemma.

**Lemma 3.1.** Let $K$ be a shape regular polyhedron in three dimensions, let $f$ be a face of $K$, with vertices $V_1, \ldots, V_N$, and let $v_h \in VF^1_1$. Then

\[
\int_f |v_h|^2 \simeq h^2 \sum_{i=1}^N |v_h(V_i)|^2.
\]

**Proof.** Let us at first prove that the left hand side of (3.4) is less or equal than a constant times the right hand side. We start by observing that

\[
\int_f |\hat{v}_h|^2 \lesssim h^2 \sum_i |\hat{v}_h(V_i)|^2,
\]

where $\hat{v}_h$ is the solution to

\[-\Delta \hat{v}_h = 0 \text{ in } f, \quad \hat{v}_h = v_h \text{ on } \partial f.\]

In fact, using the maximum principle, since $\hat{v}_h$ is harmonic on the faces, $|f| \lesssim h^2$, and $f$ has at most $N_*$ vertices, we have

\[
\int_f |\hat{v}_h|^2 \lesssim h^2 \max_x |\hat{v}_h(x)|^2 \lesssim h^2 \max_{\partial f} |v_h|^2 \lesssim h^2 \max_i |v_h(V_i)|^2 \lesssim h^2 \sum_i |v_h(V_i)|^2.
\]

Now we can write

\[
\int_f |v_h|^2 \lesssim \int_f |\hat{v}_h|^2 + \int_f |v_h - \hat{v}_h|^2.
\]

It is not difficult to check that, as $v_h = \hat{v}_h$ on $\partial f$, it holds that $\Pi_f^\Sigma v_h = \Pi_f^\Sigma \hat{v}_h$. Thus, integrating by part and using the definition of $VF^1_1$, we can write (as $\Delta(v_h - \hat{v}_h) \in P_1(f)$)

\[
\int_f |\nabla (v_h - \hat{v}_h)|^2 = - \int_f (v_h - \hat{v}_h) \Delta(v_h - \hat{v}_h) = - \int_f (\Pi_f^\Sigma \hat{v}_h - \hat{v}_h) \Delta(v_h - \hat{v}_h),
\]

since, from [2.2], for all $v_h \in VF^1_1$ and all $q \in P_1(f)$ it holds

\[
\int_f v_h q = \int_f \Pi_f^\Sigma v_h q.
\]
Then

\[(3.7) \quad \int_f |\nabla (v_h - \tilde{v}_h)|^2 \leq \left( \int_f |\Pi_f \tilde{v}_h - \tilde{v}_h|^2 \right)^{1/2} \left( \int_f |\Delta (v_h - \tilde{v}_h)|^2 \right)^{1/2} \leq \left( \int_f |\nabla \tilde{v}_h|^2 \right)^{1/2} \left( \int_f |\nabla (v_h - \tilde{v}_h)|^2 \right)^{1/2}, \]

where we used a direct inequality for the first term (\(\Pi_f\) is a bounded operator that preserves linears), and an inverse inequality for the second. Dividing both sides by the second factor in the product on the right hand side we obtain

\[\left( \int_f |\nabla (v_h - \tilde{v}_h)|^2 \right)^{1/2} \lesssim \left( \int_f |\nabla \tilde{v}_h|^2 \right)^{1/2}.\]

Then, using Poincaré inequality, we can write

\[(3.8) \quad \int_f |v_h - \tilde{v}_h|^2 \lesssim h^2 \int_f |\nabla (v_h - \tilde{v}_h)|^2 \lesssim h^2 \int_f |\nabla \tilde{v}_h|^2 \lesssim \int_f |\tilde{v}_h|^2,\]

where the last bound is obtained by using once again an inverse inequality. Hence collecting (3.6), (3.5) and (3.8), we get

\[\int_f |v_h|^2 \lesssim h^2 \sum_i |v_h(V_i)|^2.\]

Let us prove the converse inequality. We have

\[h^2 \sum_i |v_h(V_i)|^2 \lesssim h\|v_h\|_{0,\partial f} \lesssim \int_f |v_h|^2 + h^2 |v_h|^2_{1,f} \lesssim \int_f |v_h|^2,\]

where we used (3.1) and (3.3).

\[\square\]

4. Domain decomposition for the Virtual Element Method

4.1. The subdomain decomposition. We assume that \(T_h\) can be split as \(T_h = \cup \ell T_h^\ell\), inducing a decomposition of \(\Omega\) as the union of \(L\) disjoint polyhedral subdomains \(\Omega^\ell\)

\[(4.1) \quad \tilde{\Omega} = \cup \ell \tilde{\Omega}^\ell \quad \text{with} \quad \tilde{\Omega}^\ell = \cup K \in T_h K.\]

We will refer to the edges and faces of the \(\Omega^\ell\)'s as macro edges and macro faces. We let \(\Gamma = \cup \partial \Omega^\ell \setminus \partial \Omega\) denote the skeleton of the decomposition, \(E_H\) and \(F_H\) denote respectively the set of macro-edges \(E\) and of macro-faces \(F\) of the subdomain decomposition interior to \(\Omega\), and \(E_H^\ell\) and \(F_H^\ell\) denote the set of, respectively, macro faces and macro edges of the subdomain \(\Omega^\ell\).

Assumption 4.1. We make the following assumptions:

1. the subdomain decomposition is geometrically conforming, that is, for each \(\ell, m\), \(\partial \Omega^\ell \cap \partial \Omega^m\) is either a vertex or a whole edge or a whole face of both \(\Omega^\ell\) and \(\Omega^m\);
2. the subdomains \(\Omega^\ell\) are shape regular (in the sense of Definition 2.1) of diameter \(H_\ell\);
3. for all \(\ell\), there exists a scalar \(\rho_\ell > 0\) such that \(\rho_\ell |\Omega^\ell| \simeq \rho_\ell\);
4. the decomposition is quasi uniform: there exists an \(H\) such that for all \(\ell\) we have \(H_\ell \simeq H\).

Remark 4.2. We would like to point out that Assumption 4.1 is actually also an assumption on the tessellation \(T_h\), satisfied, for instance, if \(T_h\) is built by first introducing the \(\Omega^\ell\)'s and then refining them.
those subdomains whose boundary \( y \)

For each edge \( \hat{\Omega} \) and norms for the Sobolev spaces defined on faces and edges of the subdomains. More precisely, letting \( \mathcal{Y}_i \in E \)

Remark that for all \( i \in Y \)

In the following we will make use of suitably scaled norms and seminorms.

Notation: global vs local degrees of freedom. In the following we will need to single out subsets of nodes and edges. To this end, letting \( \Upsilon \) denote the set of vertices of \( T_h \),

\[
\mathcal{Y} = \{ y_i : y_i \in \Gamma \} = \text{set of (pointers to the) nodes on } \Gamma.
\]

For each subdomain \( \Omega^\ell \) we let \( \mathcal{Y}^\ell \subset \mathcal{Y} \)

denote the set of indices pointing to nodes on the boundary of \( \Omega^\ell \). We let \( \mathcal{X} \) and \( \mathcal{X}^\ell \) denote, respectively, the set of indices of cross-points (vertices of the subdomain decomposition) and of the vertices of the subdomain \( \Omega^\ell \)

\[
\mathcal{X} = \{ i : y_i \text{ is a cross point} \}, \quad \mathcal{X}^\ell = \mathcal{X} \cap \mathcal{Y}^\ell.
\]

Finally we let \( \mathcal{W} \) be the set of indices of nodes on the wire basket, i.e. on the union of edges of the decomposition.

For each macroedge \( E \) and for each macroface \( F \) we let

\[
\mathcal{Y}_E = \{ i : y_i \in E \}, \quad \mathcal{Y}_F = \{ i : y_i \in F \}
\]

denote the set of indices of nodes belonging to \( E \) and \( F \) respectively and

\[
\mathcal{W}^\ell = \mathcal{W} \cap \mathcal{Y}^\ell, \quad \mathcal{W}_F = \mathcal{W} \cap \mathcal{Y}_F = \{ i : y_i \in \partial F \}.
\]

For each node \( y_i \) on the skeleton of the decomposition we let \( \mathcal{N}_i \) denote the set of the indices of those subdomains whose boundary \( y_i \) belongs to:

\[
\mathcal{N}_i = \{ \ell : y_i \in \partial \Omega^\ell \}, \quad n_i = \#(\mathcal{N}_i).
\]

For each edge \( E \in \mathcal{E}_H \) and face \( f \in \mathcal{F}_H \) we can also define the set \( \mathcal{N}_E \) and \( \mathcal{N}_F \) of the indices of those subdomains that share, respectively, \( E \) and \( F \) as an edge or face:

\[
\mathcal{N}_E = \{ \ell : E \subset \partial \Omega^\ell \}, \quad \mathcal{N}_F = \{ \ell : F \subset \partial \Omega^\ell \}.
\]

Remark that for all \( i \in \mathcal{Y}_E \setminus \mathcal{X} \) we have \( \mathcal{N}_i = \mathcal{N}_E \).

Notation: Scaled norms and seminorms. In the following we will make use of suitably scaled norms for the Sobolev spaces defined on faces and edges of the subdomains. More precisely, letting \( \hat{\Omega} \) and \( \hat{\Gamma} \) denote any three-dimensional and two-dimensional domain with diameter \( \bar{H} \) we set:

\[
\| w \|_{L^2(\hat{\Omega})}^2 = \bar{H}^{-2} \int_{\hat{\Omega}} |w(x)|^2 \, dx, \quad \| w \|_{L^2(\hat{\Gamma})}^2 = \bar{H}^{-1} \int_{\hat{\Gamma}} |w(\tau)|^2 \, d\tau,
\]

\[
|w|_{H^1(\hat{\Omega})}^2 = \int_{\hat{\Omega}} |\nabla w(x)|^2 \, dx, \quad |w(\tau)|_{H^1(\hat{\Gamma})}^2 = \bar{H} \int_{\hat{\Gamma}} |w'(\tau)|^2 \, d\tau,
\]

\[
|w|_{H^{s}(\hat{\Gamma})}^2 = \bar{H}^{2s-1} \int_{\hat{\Gamma}} d\sigma \int_{\hat{\Gamma}} d\tau \frac{|w(\sigma) - w(\tau)|^2}{|\sigma - \tau|^{2s+1}}, \quad 0 < s < 1,
\]

\[
\| w \|_{H^1(\hat{\Omega})}^2 = \| w \|_{L^2(\hat{\Omega})}^2 + |w|_{H^1(\hat{\Omega})}^2, \quad \| w \|_{H^s(\hat{\Gamma})}^2 = \| w \|_{L^2(\hat{\Gamma})}^2 + |w|_{H^s(\hat{\Gamma})}^2, \quad 0 < s \leq 1.
\]
Domain decomposition and FETI-DP preconditioner. The subdomain spaces $V^\ell_h$ and bilinear forms $a^\ell_h : V^\ell_h \times V^\ell_h \to \mathbb{R}$ are defined, as usual, as

$$V^\ell_h = V^\ell_h|_{\Omega^\ell}, \quad a^\ell_h(u_h, v_h) = \sum_{K \in T^\ell_h} a^K_h(u_h, v_h).$$

In view of (2.8) we immediately obtain that for all $u_h, v_h \in V^\ell_h$

$$a^\ell_h(u_h, v_h) \lesssim |u_h|_{1, \Omega^\ell} |v_h|_{1, \Omega^\ell}, \quad a^\ell_h(u_h, u_h) \simeq |u_h|^2_{1, \Omega^\ell}.$$

Solving Problem 2.3 is then reduced to finding $u_h = (u^\ell_h)_\ell \in \prod V^\ell_h$ minimizing

$$J(u_h) = \frac{1}{2} \sum_{\ell} a^\ell_h(u^\ell_h, u^\ell_h) - \int_{\Omega} g u_h$$

subject to a continuity constraint across the interface.

The FETI-DP preconditioner is then constructed according to the same strategy used in the finite element case, which we recall, mainly to fix some notation. We let

$$\tilde{V}_h = \prod V^\ell_h, \quad \tilde{V}_h = V^\ell_h \cap H^1_0(\Omega^\ell),$$

and

$$W_h = \prod W^\ell_h, \quad W_h = V^\ell_h|_{\partial \Omega^\ell}.$$ (4.6)

Moreover, for each macro face $F$ and macro edge $E$ we let

$$W^F_h = V^\ell_h|_{F} \quad \text{and} \quad W^E_h = V^\ell_h|_{E}$$

denote, respectively, the trace on $F$ and $E$ of $V_h$. On $W_h$ we define a norm and a seminorm:

$$\|w_h\|^2_{1/2, s} = \sum_{\ell} \rho^\ell \|w^\ell_h\|^2_{H^{1/2}(\partial \Omega^\ell)}, \quad |w_h|^2_{1/2, s} = \sum_{\ell} \rho^\ell |w^\ell_h|^2_{H^{1/2}(\partial \Omega^\ell)}.$$

As usual, we define local discrete harmonic lifting operators $L^\ell_h : W^\ell_h \to V^\ell_h$ as

$$a^\ell_h(L^\ell_h w_h, v_h) = 0 \quad \forall v_h \in \tilde{V}^\ell_h, \quad L^\ell_h w_h = w_h \quad \text{on} \ \partial \Omega^\ell.$$ (4.7)

The following proposition holds

**Proposition 4.3.** $L^\ell_h$ is well defined, and it verifies

$$|L^\ell_h w_h|_{H^1(\Omega^\ell)} \simeq |w_h|_{H^{1/2}(\partial \Omega^\ell)}.$$

**Proof.** We start by recalling that there exists a linear operator $\Pi_{SZ} : H^1(\Omega^\ell) \to V^\ell_h$ such that, for $v \in H^{1+s}$, $0 \leq s \leq 1$ one has (23)

$$\|v - \Pi_{SZ} v\|_{L^2(\Omega^\ell)} + \frac{h}{H} |v - \Pi_{SZ} v|_{H^s(\Omega^\ell)} \lesssim \left( \frac{h}{H} \right)^{1+s} |v|_{H^{1+s}(\Omega^\ell)},$$ (4.8)

(note that we are using scaled norms, see (2.2.4.5)). Moreover $\Pi_{SZ}$ is constructed in such a way that if $v|_{\partial \Omega^\ell} \in W^\ell_h$ one has $\Pi_{SZ} v = v$ on $\partial \Omega^\ell$. In view of this result it is not difficult to construct an operator $L^\ell_h : W^\ell_h \to V^\ell_h$ satisfying $L^\ell_h w_h|_{\partial \Omega^\ell} = w_h$ and

$$|L^\ell_h w_h|_{H^1(\Omega^\ell)} \leq |w_h|_{H^{1/2}(\partial \Omega^\ell)}.$$ (4.9)
$L^\ell_h$ can be for instance defined as $\Pi_{SZ}$ applied to the harmonic lifting of $w_h$; (4.9) follows then from the stability of the harmonic lifting and of $\Pi_{SZ}$, exactly as in the two dimensional case [16]. We can then write

$$|\mathcal{L}_h^\ell w_h - L_h^\ell w_h|_{H^1(\partial\Omega)}^2 \lesssim a_h^\ell(\mathcal{L}_h^\ell w_h - L_h^\ell w_h, \mathcal{L}_h^\ell w_h - L_h^\ell w_h)$$

$$= -a_h^\ell(L_h^\ell w_h, \mathcal{L}_h^\ell w_h - L_h^\ell w_h) \lesssim |L_h^\ell w_h|_{H^1(\partial\Omega)} |\mathcal{L}_h^\ell w_h - L_h^\ell w_h|_{H^1(\partial\Omega)}.$$  

In view of (4.9) it is easy to conclude by applying a triangular inequality.

Let now $\hat{W}_h \subset W_h$ denote the subset of functions which are single valued across $\Gamma$:

(4.10) $\hat{W}_h = \{ w_h \in W_h : \forall x \in \tilde{\Omega}^\ell \cap \tilde{\Omega}^k, w_h^\ell(x) = w_h^k(x) \}$

$$= \{ w_h \in W_h : \forall i \in \mathcal{V}, \ell, m \in \mathcal{N}_i \Rightarrow w_h^\ell(y_i) = w_h^m(y_i) \}.$$  

For $w_h = (w_h^\ell)_\ell \in \hat{W}_h$ we let $\mathcal{L}_h(w_h) = (\mathcal{L}_h^\ell w_h^\ell)_\ell$, so that $V_h$ is split as

(4.11) $V_h = \hat{V}_h \oplus \mathcal{L}_h \hat{W}_h.$

We next define the bilinear for $s : W_h \times \hat{W}_h \rightarrow \mathbb{R}$ as

$$s(w_h, v_h) := \sum_{\ell} a_h^\ell(\mathcal{L}_h^\ell w_h, \mathcal{L}_h^\ell v_h).$$

The proof of the following proposition is trivial.

**Proposition 4.4.** For all $v_h, w_h \in \hat{W}_h$ we have

(4.12) $s(v_h, w_h) \lesssim |w_h|_{1/2,*} v_h|_{1/2,*}$, $s(w_h, v_h) \gtrsim |w_h|_{1/2,*}^2.$

**Problem 2.3** is then split as the combination of two independent problems

**Problem 4.5.** Find $\tilde{u}_h \in \hat{V}_h$ such that for all $v_h \in \hat{V}_h$

$$a_h(\tilde{u}_h, v_h) = \int_\Omega g v_h.$$  

**Problem 4.6.** Find $w_h \in \hat{W}_h$ such that for all $v_h \in \hat{W}_h$

$$s(w_h, v_h) = \int_\Omega g \mathcal{L}_h v_h.$$  

Exactly as in the finite element case, the design of different versions of the FETI-DP method will rely on the choice of a subspace $\hat{W}_h$ of $W_h$ whose elements have some degree of continuity on $\Gamma$, ensuring that the restriction to $\hat{W}_h$ of the bilinear form $a_h$ is coercive (which is equivalent to asking that the seminorm $|\cdot|_{1/2,*}$ is a norm on $\hat{W}_h$). We recall that, while in two dimensions the space $\hat{W}_h$ can be defined as the subspace of functions continuous at the vertices of the subdomains, it is known that this is not sufficient to get a quasi-optimal result in the three dimensional case, so that we also need to impose continuity of either edge or face averages (or both). While later in the paper we will analyze the different possible choices, for now we will only assume that $a_h$ is coercive on $\hat{W}_h$.

Following the approach of [18], we introduce the operators $\hat{\mathcal{S}} : \hat{W}_h \rightarrow \hat{W}_h'$ and $\mathcal{S} : \hat{W}_h \rightarrow \hat{W}_h'$ defined respectively as

(4.13) $(\hat{\mathcal{S}} w_h, v_h) = s(w_h, v_h) \forall v_h \in \hat{W}_h$, $(\mathcal{S} w_h, v_h) = s(w_h, v_h) \forall v_h \in \hat{W}_h.$
and we let $\mathcal{R} : \tilde{W}_h \to \tilde{W}_h$ denote the natural injection operator. We observe that 
$$\hat{S} = \mathcal{R}^T \hat{S} \mathcal{R}. $$
Problem 4.6 becomes
(4.14) 
$$\hat{S} u = \hat{g}, $$
with $\hat{f}$ suitable right-hand side.

Following [45], we let $\delta_\ell$ denote the weighted counting function associated with $\partial \Omega^\ell$ and defined for $\gamma \in [1/2, \infty)$ by a sum of contribution from $\Omega^\ell$ and its neighbors. More precisely, for $i \in \mathcal{Y}^\ell$, we set
(4.15) 
$$\delta_\ell(y_i) = \frac{\sum_{j \in \mathcal{N}_i} \rho_\gamma^j}{\rho_\gamma^i}. $$
As in [34, 32], we define the scalar product $d : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$ as
(4.16) 
$$d(w_h, v_h) = \sum_\ell \sum_{i \in \mathcal{Y}^\ell} d^{\ell,i} v^\ell(y_i) w^\ell(y_i), $$
where, for $i \in \mathcal{Y}^\ell$, the scaling coefficient $d^{\ell,i}$ is defined as
(4.17) 
$$d^{\ell,i} = (\delta_\ell(y_i))^{-1} = \frac{\rho_\gamma^i}{\sum_{j \in \mathcal{N}_i} \rho_\gamma^j}. $$
Next, we introduce the projection operator $\mathcal{E}_D : \mathcal{W}_h \to \tilde{W}_h$, orthogonal with respect to the scalar product $d$:
(4.18) 
$$d(\mathcal{E}_D w_h, v_h) = d(w_h, v_h), \quad \forall v_h \in \tilde{W}_h. $$

Following [18], we introduce the quotient space $\Sigma_h = \tilde{\mathcal{W}}_h / \hat{\mathcal{W}}_h$. We let $\Lambda_h = \Sigma'_h$ denote its dual and we let $\mathcal{B} : \tilde{W}_h \to \Sigma_h = \Lambda'_h$ denote the quotient mapping, defined as
$$\mathcal{B} w_h = w_h + \tilde{W}_h. $$
Observe that two elements $w_h$ and $v_h$ are representative of the same equivalence class in $\Sigma_h$ if and only if they have the same jump across the interface: for $F \in \mathcal{F}_H$ with $F = \Gamma^m \cap \Gamma^\ell$, $w_h^m - w_h^\ell = v_h^m - v_h^\ell$. We can then identify $\Sigma_h$ with the set of jumps of elements of $\mathcal{W}_h$. The quotient map $\mathcal{B}$ can then be interpreted as the operator that maps an element of $\mathcal{W}_h$ to its jump on the interface. Clearly
$$\tilde{W}_h = \ker(\mathcal{B}) = \{w_h \in \tilde{W}_h : b(w_h, \lambda) = 0, \quad \forall \lambda \in \Lambda_h\}, $$
where $b : \tilde{W}_h \times \Lambda_h \to \mathbb{R}$ is defined as $b(w_h, \lambda_h) = \langle Bw_h, \lambda_h \rangle$. Problem 2.3 is then equivalent to the following saddle point problem: find $w_h \in \tilde{W}_h$, $\lambda_h \in \Lambda_h$ solution to
(4.19) 
$$\hat{S} w_h - \mathcal{B}^T \lambda_h = \hat{g}, \quad \mathcal{B} w_h = 0, $$
where $\hat{g} \in \mathcal{W}'_h$ is defined by
$$\langle \hat{g}, w_h \rangle = \int_\Omega g \mathcal{L}_h w_h = \sum_\ell \int_{\Omega^\ell} g \mathcal{L}_h^\ell w_h^\ell. $$
Using the first equation in (4.19) to express $w_h$ as a function of $\lambda_h$, we eliminate the former unknown and finally reduce the solution of Problem 4.6 to the solution of a problem in the unknown $\lambda_h \in \Lambda_h$ of the form
(4.20) 
$$\mathcal{B} \hat{S}^{-1} \mathcal{B}^T \lambda_h = -\mathcal{B} \hat{S}^{-1} \hat{f}. $$
Letting now \( B^+ : \Sigma_h \rightarrow \tilde{W}_h \) be any right inverse of \( B \) (for \( \eta \in \Sigma_h \), \( B^+ \eta \) is any element in \( \tilde{W}_h \) such that \( BB^+ \eta = w_h \)), we set
\[
B_D^T = (1_{\tilde{W}_h} - \mathcal{E}_D)B^+,
\]
where \( 1_{\tilde{W}_h} \) denotes the identity operator in the space \( \tilde{W}_h \). Recall that, as in the two dimensional case, the definition of \( B_D^T \) is independent of the actual choice of the operator \( B^+ \). Indeed \( B(B_1^+ \eta - B_2^+ \eta) = \eta - \eta = 0 \) implies \( B_1^+ \eta - B_2^+ \eta \in \tilde{W}_h \) and hence \((1_{\tilde{W}_h} - \mathcal{E}_D)B_1^+ \eta - (1_{\tilde{W}_h} - \mathcal{E}_D)B_2^+ \eta = 0 \).

We finally let the FETI preconditioner \( M : \Sigma_h \rightarrow \Lambda_h \) be defined by
\[
M = B_D S B_D^T.
\]

Let us now come to the choice of the space \( \tilde{W}_h \), or, equivalently, to the choice of the so called primal degrees of freedom, which are taken as single valued, thus directly imposing the corresponding degree of continuity across the interface, whereas continuity for the remaining dual degrees of freedom is imposed via Lagrange multipliers in \( \Lambda_h \). Exactly as in the finite element case there are several possibilities. Letting
\[
\tilde{W}_h^V = \{ w_h \in W_h : w_h^\ell(y_i) = w_h^m(y_i), \forall i \in \mathcal{X}, \ell, m \in \mathcal{N}_i \},
\]
\[
\tilde{W}_h^E = \{ w_h \in W_h : \int_E w_h^\ell = \int_E w_h^k, \forall E \in \mathcal{E}_H, \ell, k \in \mathcal{N}_E \},
\]
\[
\tilde{W}_h^F = \{ w_h \in W_h : \int_F w_h^\ell = \int_F w_h^k, \forall F \in \mathcal{F}_H, \ell, k \in \mathcal{N}_F \},
\]
denote the subset of \( W_h \) of traces of functions which, respectively, are continuous at cross-points, have same average at all edges, and have same average at all faces, different choices for \( \tilde{W}_h \) can be considered, see Section 6, resulting in different versions of the FETI algorithms. More precisely we have

- **E**: \( \tilde{W}_h = \tilde{W}_h^E \) (primal d.o.f.’s are the edge averages);
- **F**: \( \tilde{W}_h = \tilde{W}_h^F \) (primal d.o.f.’s are the face averages);
- **VE**: \( \tilde{W}_h = \tilde{W}_h^V \cap \tilde{W}_h^E \) (primal d.o.f.’s are the values at cross points and the edge averages);
- **VF**: \( \tilde{W}_h = \tilde{W}_h^V \cap \tilde{W}_h^F \) (primal d.o.f.’s are the values at cross points and the face averages);
- **EF**: \( \tilde{W}_h = \tilde{W}_h^E \cap \tilde{W}_h^F \) (primal d.o.f.’s are edge and face averages);
- **VEF**: \( \tilde{W}_h = \tilde{W}_h^V \cap \tilde{W}_h^E \cap \tilde{W}_h^F \) (primal d.o.f.’s are the values at cross points and the face and edge averages).

As it happens for the finite element case, all the above choices lead to a quasi-optimal (in terms of dependence on \( h \) and \( H \)) preconditioning, whereas robustness with respect to the jumps in the coefficient \( \rho \) is achieved for algorithms VE and VEF, as stated by the following theorem.

**Theorem 4.7.** Letting \( \kappa \) denote the condition number of the matrix corresponding to the operator \( M(BS^{-1}B^T) \), depending on the choice of the space \( \tilde{W}_h \) we have the following bounds:

**Algorithms E / EF:** If \( \tilde{W}_h \subseteq \tilde{W}_h^E \) then
\[
\kappa \lesssim \left(1 + \log \left(\frac{H}{h}\right) + \tau_E\right) \left(1 + \log \left(\frac{H}{h}\right)\right);
\]
Algorithm F: If $\tilde{W}_h \subseteq \tilde{W}_h^F$ then
$$\kappa \lesssim \left( 1 + \log \left( \frac{H}{h} \right) + \tau_F \right) \left( 1 + \log \left( \frac{H}{h} \right) \right);$$

Algorithms VE / VEF: If $\tilde{W}_h \subseteq \tilde{W}_h^V \cap \tilde{W}_h^E$ then
$$\kappa \lesssim \left( 1 + \log \left( \frac{H}{h} \right) \right)^2;$$

Algorithm VF: If $\tilde{W}_h \subseteq \tilde{W}_h^V \cap \tilde{W}_h^F$ then
$$\kappa \lesssim \left( 1 + \log \left( \frac{H}{h} \right) + \tau_{EF} \right) \left( 1 + \log \left( \frac{H}{h} \right) \right),$$

where $\tau_E$, $\tau_F$ and $\tau_{EF}$ are constants depending on the diffusion coefficient $\rho$ that satisfy
$$0 < \tau_E, \tau_F, \tau_{EF} \leq \max \rho / \min \rho, \quad \tau_E \leq \tau_F, \quad \tau_{EF} \leq \tau_F.$$

Remark 4.8. The precise definition of the constant $\tau_E$, $\tau_F$ and $\tau_{EF}$ (which will be detailed in Section 5) is the same as in the finite element case [45], and it is quite technical. We would like to remark that the bounds (4.23) are often quite pessimistic, as we will see in Section 6.

4.2. Implementation in the Virtual Element context. As in the Finite Element case, the implementation of the FETI-DP method entails the need for numerically imposing the continuity constraints on the primal degrees of freedom, either directly or via additional Lagrange multipliers (using which, the computation of $S^{-1}$ will imply solving an algebraic saddle point problem) [32]. In order to do this, it is necessary to be able to evaluate the primal degrees of freedom for any given discrete functions, and, if imposing the continuity directly, to explicitly construct a basis function for each primal degree of freedom, being it a vertex value, an edge average, or a face average. While for vertex and edge degrees of freedom this is not difficult, the same is not true for face averages. Indeed, contrary to what happens for finite elements, discrete functions are explicitly known only on the edges of the tessellation, where they are linear. On faces and within the elements VEM basis functions are not explicitly known and all quantities needed for the implementation (of the preconditioner, but also of the stiffness matrix and load vector) have to be retrieved in terms of the values at the nodes and on the edges, by exploiting the definitions of the spaces $V^{1,f}$ and $V^{1,K}$ (in the terminology of VEM, they must be computable). Extended details on the computability of the stiffness matrix and the right hand side can be found in [7]. Here we concentrate on the quantities needed for implementing the FETI-DP method, and, more specifically, on the face averages of discrete functions. Let $w_h^\ell \in W_h^\ell$ and let $F$ denote a macro face of $\Omega^\ell$. We observe that $F$ can be written as the union of a certain number of faces $f$ of polyhedra of the tessellation $T_h$. We have then
$$|F|^{-1} \int_F w_h = |F|^{-1} \sum_{f \subset F} \int_f w_h = |F|^{-1} \sum_{f \subset F} \int_f \Pi_f^\ell w_h,$$

where the last identity stems from the definition (2.2) of the space $V^{f,1}$ to which $w_h|_f$ belongs. We recall that, thanks to (2.4), $\Pi_f^\ell w_h$ is computable in terms of the (known) value of $w_h$ on $\partial f$.

Implementation of the FETI-DP preconditioner can then be carried out following different approaches [32]. As already stated, continuity of the primal variables can be imposed directly by explicitly constructing a basis allowing to decoupling the primal variables, for which continuity is strongly imposed, from the dual variables, or by introducing additional Lagrange multipliers. For the numerical experiments that we present in Section 6, we chose the first option, which leads to smaller and computationally more efficient coarse problems.
5. Proof of Theorem 4.7

We start by remarking that, by construction, we are in the framework of \([35]\). In particular we have the identity
\[
B_D^T B + \mathcal{E}_D = 1_{\tilde{W}_h}.
\]
In fact, it is not difficult to see that for all \(w_h \in \tilde{W}_h\) we have that \((1_{\tilde{W}_h} - B_D^T B)w_h \in \tilde{W}_h\) and that we have
\[
d((1_{\tilde{W}_h} - B_D^T B)w_h, v_h) = d(w_h, v_h) \quad \text{for all } v_h \in \tilde{W}_h.
\]
Then we have \((35)\)
\[
(5.1) \quad \kappa \lesssim \max_{w_h \in \tilde{W}_h} \frac{s(B_D^T Bw_h, B_D^T Bw_h)}{s(w_h, w_h)} \simeq \max_{w_h \in \tilde{W}_h} \frac{s(\mathcal{E}_D w_h, \mathcal{E}_D w_h)}{s(w_h, w_h)}.
\]
In order to have a bound on the condition number, we then only need to bound \(s(\mathcal{E}_D w_h, \mathcal{E}_D w_h)\) in terms of \(s(w_h, w_h)\).

To start, let us recall some functional inequalities that will be useful in the following. Let \(F\) be a shape regular polygon (in the following \(F\) will be a face of one of the \(\Omega^\ell\)'s). Then, for all \(\eta \in H^s(F), \ 1/2 < F \leq 1\), we have, uniformly in \(s\), the following trace inequalities
\[
\|\eta\|_{H^{s-1/2}(\partial F)} \lesssim \frac{1}{\sqrt{2s-1}} \|\eta\|_{H^s(F)}, \quad |\eta|_{H^{s-1/2}(\partial F)} \lesssim \frac{1}{\sqrt{2s-1}} \|\eta\|_{H^s(F)}.
\]
On the other hand, for all \(\eta \in H^s(F), \ 0 \leq s < 1/2\), and for all \(\alpha \in \mathbb{R}\) it holds that
\[
\|u\|_{H^s_0(F)} \lesssim \frac{1}{1/2-s} \|u - \alpha\|_{H^s(F)} + \frac{1}{\sqrt{1/2-s}} |\alpha|,
\]
once again uniformly in \(s\) (recall that we are using scaled norms, as defined in Section 2 so that the bounds are uniform in \(H\)).

We also observe that, thanks to the inverse inequality \((3.3)\) and to the scaling of the norms, by using a standards space interpolation technique it is not difficult to prove that for all \(r, s \in [0, 1]\) with \(r < s\), and for all \(w_h \in W_{h|F}\)
\[
(5.4) \quad \|w_h\|_{H^s(F)} \lesssim \left(\frac{H}{h}\right)^{r-s} \|w_h\|_{H^r(F)}.
\]
An analogous bound holds for the norms in the spaces \(H^s_0(F)\) and \(H^s_0(F)\) (with the usual care when either \(s\) or \(r\) are equals to 1/2), provided \(w_h \in W_{h|F} \cap H^s_0(F)\). In particular, in such case we have
\[
(5.5) \quad \|w_h\|_{H^{1/2}_0(F)} \lesssim \left(\frac{H}{h}\right)^{r-s} \|w_h\|_{H^{1/2}_0(F)}.
\]

The following proposition holds.

**Proposition 5.1.** Let \(\Omega^\ell\) be a shape regular subdomain and let \(F\) be a face of \(\Omega^\ell\). Then for \(w_h \in W_{h|F}\) we have
\[
\|w_h\|_{L^2(\partial F)} \lesssim \sqrt{1 + \log(h/H)} \|w_h\|_{H^{1/2}(F)}.
\]
Proof. Using inequalities (5.4) and (5.2) we can write, for $0 < \varepsilon \leq 1/2$ arbitrary,
\[
\|w_h\|_{L^2(\partial F)} \leq \|w_h\|_{H^{s}(\partial F)} \lesssim \frac{1}{\sqrt{\varepsilon}}\|w_h\|_{H^{1/2+\varepsilon}(F)} \lesssim \left(\frac{h}{H}\right)^{-\varepsilon} \frac{1}{\sqrt{\varepsilon}}\|w_h\|_{H^{1/2}(F)} \lesssim \sqrt{1+\log(h/H)}\|w_h\|_{H^{1/2}(F)},
\]
where the last bound is obtained by choosing $\varepsilon = (1 + \log(H/h))^{-1}$. $\square$

We now prove the following lemma, which is the equivalent, for the Virtual Element Method, of Lemma 5.6 of [13] and Lemma 4.3 of [17].

Lemma 5.2. Let $w_h \in W_h^F$ and let $\tilde{w}_h \in W_h^F$ be defined by $\tilde{w}_h(y_i) = 0$, for all $i \in \mathcal{W}_h$, $\tilde{w}_h(y_i) = w_h(y_i)$, for all $i \in \mathcal{Y}_h \setminus \mathcal{W}_h$. Then, for all face $F$ of $\Omega_h$ it holds that
\[
\|\tilde{w}_h\|_{H^{1/2}(F)}^2 \lesssim (1 + \log(H/h))^2\|w_h\|_{H^{1/2}(F)}^2.
\]
Moreover, if $w_h$ is constant on $F$ then
\[
\|\tilde{w}_h\|_{H^{1/2}(F)}^2 \lesssim (1 + \log(H/h))\|w_h\|_{H^{1/2}(F)}^2
\]

Proof. We let $\tilde{W}_h^F$ be defined as
\[
\tilde{W}_h^F = \{ w_h \in W_h^F : w_h|_F = 0 \text{ for all } i \in \mathcal{W}_F \}.
\]
Let $\pi_h : L^2(F) \to W_h^F$ and $\pi_h^0 : L^2(F) \to \tilde{W}_h^F$ denote the $L^2$-projection onto $W_h^F$ and onto $\tilde{W}_h^F$, respectively. Recall that for all $u \in H^1(F)$, using (4.8) we have
\[
(u - \pi_h u)_{L^2(\Omega)} \leq \|u - \Pi_{SZ} u\|_{L^2(\Omega)} \lesssim \frac{h}{H} |u|_{H^1(\Omega)}
\]
and, since $u \in H^1_0(F)$ implies that $\Pi_{SZ} u \in \tilde{W}_h^F$, we also have for all $u \in H^1_0(F)$
\[
(u - \pi_h^0 u)_{L^2(\Omega)} \leq \|u - \Pi_{SZ} u\|_{L^2(\Omega)} \lesssim \frac{h}{H} |u|_{H^1(\Omega)}.
\]

Let now $i_h^0 : W_h^F \to \tilde{W}_h^F$ be defined by $i_h^0 w_h(y_i) = w_h(y_i)$ for all $i \in \mathcal{Y}_h \setminus \mathcal{W}_h$, so that, on $F$, $\tilde{w}_h^F = i_h^0 w_h$. Remark that, thanks to Lemma 3.1, we have
\[
\|i_h^0 w_h\|_{L^2(F)}^2 \lesssim \left(\frac{h}{H}\right)^2 \sum_{i \in \mathcal{Y}_h \setminus \mathcal{W}_h} |w_h(y_i)|^2 \lesssim \left(\frac{h}{H}\right)^2 \sum_{i \in \mathcal{Y}_h} |w_h(y_i)|^2 \lesssim \|w_h\|_{L^2(F)}^2.
\]
Consider now the operator $\pi_h^1 = i_h^0 \circ \pi_h : L^2(F) \to \tilde{W}_h^F$ obtained by first projecting onto $W_h^F$ and then setting the values at nodes on $\partial F$ to zero. We will prove that the restriction of $\pi_h^1$ to $H^s_0(F)$ is uniformly bounded for all $s < 1/2$, that is, that for all $w \in H^s_0(F)$ we have
\[
\|\pi_h^1 w\|_{H^s_0(F)} \lesssim \|w\|_{H^s_0(F)}
\]
with a constant independent of $s$. Then, for $\varepsilon \in [0,1/2]$ arbitrary, using (5.5) and (5.5), we can write
\[
\|\pi_h^1 w_h\|_{H^{1/2}(F)} \lesssim \left(\frac{h}{H}\right)^{-\varepsilon} \|\pi_h^1 w_h\|_{H^{1/2-\varepsilon}(F)} \lesssim \left(\frac{h}{H}\right)^{-\varepsilon} \|w_h\|_{H^{1/2-\varepsilon}(F)} \lesssim \frac{1}{\varepsilon} \left(\frac{h}{H}\right)^{-\varepsilon} \|w_h\|_{H^{1/2-\varepsilon}(F)} \lesssim (1 + \log(H/h))\|w_h\|_{H^{1/2}(F)}.
\]
which by choosing \( \varepsilon = 1/|\log(H/h)| \), yields
\[
\|\pi_h^1 w_h\|_{H^{1/2} (F)} \lesssim (1 + \log(H/h)) \|w_h\|_{H^{1/2}(F)}.
\]
Observing that for \( w_h \in W^F_h \) we have
\[
\tilde{w}_h^\ell|_E = i^0_h w_h|_E = \pi_h^1 w_h|_E,
\]
we immediately get the thesis (the result for \( w_h^\ell \) constant on \( F \) is obtained by setting \( \alpha = w_h^\ell \) in (5.3)).

Let us then prove (5.9). We easily see that \( \pi_h^1 \) is \( L^2 \) bounded: for all \( w \in L^2(F) \),
\[
\|\pi_h^1 w\|_{L^2(F)} \lesssim \|\pi_h w\|_{L^2(F)} \lesssim \|w\|_{L^2(F)}.
\]
On the other hand, observing that \( i^0_h \circ \pi_h^0 = \pi_h^0 \), using (5.8) we see that, for \( w \in H^1_0(F) \)
\[
\|w - \pi_h^1 w\|_{L^2(F)} = \|w - \pi_h^0 w + i^0_h \pi_h^0 w - i^0_h \pi_h^1 w\|_{L^2(F)} \lesssim \|w - \pi_h^0 w\|_{L^2(F)} + \|\pi_h^0 w - \pi_h^1 w\|_{L^2(F)}.
\]
By adding and subtracting \( w \) in the second term on the right hand side and using (5.6) and (5.7), we obtain, for \( w \in H^1_0(F) \),
\[
\|w - \pi_h^1 w\|_{L^2(F)} \lesssim \|w - \pi_h^0 w\|_{L^2(F)} + \|w - \pi_h^1 w\|_{L^2(F)} \lesssim \frac{h}{H} |w|_{H^1(F)}.
\]
This allows us to prove, by a standard argument, that \( \pi_h^1 \) is \( H^1_0 \)-bounded. In fact, letting \( \Pi_h^1 : H^1_0(F) \to \tilde{W}^F_h \) denote the \( H^1_0 \) projection, for \( w \in H^1_0(F) \), using an inverse inequality, adding and subtracting \( w \) and then using an approximation bound, we have
\[
|\pi_h^1 w|_{H^1(F)} = |\Pi_h^1 w|_{H^1(F)} + |\pi_h^1 w - \Pi_h^1 w|_{H^1(F)} \lesssim |w|_{H^1(F)} + \left( \frac{h}{H} \right)^{-1} \|\pi_h^1 w - \Pi_h^1 w\|_{L^2(F)} \lesssim |w|_{H^1(F)} + \left( \frac{h}{H} \right)^{-1} \|\pi_h^1 w - \Pi_h^1 w\|_{L^2(F)} \lesssim \frac{h}{H}\|w|_{H^1(F)} \lesssim |w|_{H^1(F)}.
\]
The bound (5.9) follows by a standard space interpolation argument.

Let us now consider the projector \( \mathcal{E}_D : W_h \to \tilde{W}_h \). We have the following Lemma.

**Lemma 5.3.** For all \( w_h \in W_h \) it holds that
\[
(5.10) \quad |\mathcal{E}_D w_h|_{1/2,*}^2 \lesssim (1 + \log(H/h))^2 |w_h|_{1/2,*}^2 + \left( \frac{h}{H} \right) \Delta^{X} + \Delta^{E} + (1 + \log(H/h)) \Delta^{F},
\]
with
\[
(5.11) \quad \Delta^{X} = \sum_{i \in X} \sum_{\ell,k \in \mathcal{N}_i} \min\{\rho^i, \rho^k\} |w_h^\ell(y_i) - w_h^k(y_i)|^2,
\]
\[
(5.12) \quad \Delta^{E} = \sum_{E \in \mathcal{E}_h} \sum_{\ell,k \in \mathcal{N}_E} \min\{\rho^E, \rho^k\} |\alpha^E_\ell - \alpha^E_k|^2,
\]
\[
(5.13) \quad \Delta^{F} = \sum_{F \in \mathcal{F}_h} \sum_{\ell,k \in \mathcal{N}_F} \min\{\rho^F, \rho^k\} |\alpha^F_\ell - \alpha^F_k|^2.
\]
\[ \alpha^F = |F|^{-1} \int_F w^F_h, \quad \alpha^E = |E|^{-1} \int_E w^E_h. \]

Proof. It is not difficult to check that, for \( i \in \mathcal{Y} \) we have
\[ \mathcal{E}_D w_h(y_i) = \theta_i^{-1} \sum_{k \in \mathcal{N}_i} \rho_k^\gamma w^k_h(y_i), \quad \text{where } \theta_i = \sum_{k \in \mathcal{N}_i} \rho_k^\gamma, \]
and that these relations completely define \( \mathcal{E}_D \).

Let both \( w_h \) and \( v_h = \mathcal{E}_D w_h \) be split as the sum of the contributions of nodes on the wirebasket, which we will denote by \( w^\sharp_h \) and \( v^\sharp_h \), respectively, and the contribution of nodes interior to the faces, which we will denote by \( \hat{w}_h \) and \( \hat{v}_h \), respectively. More precisely, we let \( w^\sharp_h \in \mathcal{W}_h \) and \( v^\sharp_h \in \hat{\mathcal{W}}_h \) be defined by
\[ w^\sharp_h = (w^\sharp)_{\ell} = \begin{cases} w^\ell_h(y_i), & i \in \mathcal{W}^\ell, \\ 0, & i \in \mathcal{Y} \setminus \mathcal{W}^\ell, \end{cases} \quad \text{and} \quad v^\sharp_h = \begin{cases} v_h(y_i), & i \in \mathcal{W}, \\ 0, & i \in \mathcal{Y} \setminus \mathcal{W}, \end{cases} \]
and we set \( \hat{w}_h = w_h - w^\sharp_h \) and \( \hat{v}_h = v_h - v^\sharp_h \). Remark that
\[ v^\sharp_h = \mathcal{E}_D w^\sharp_h, \quad \hat{v}_h = \mathcal{E}_D \hat{w}_h. \]

To start, let us consider the contribution of the faces. We have
\[ \rho_e |\hat{v}_h|^2_{H^{1/2}(\partial \Omega_e)} \lesssim \rho_e \sum_{F \in \mathcal{F}_h^F} \|\hat{v}_h\|^2_{H^{1/2}_0(F)}, \]

We recall that for \( a, b > 0 \) and \( \gamma \geq 1/2 \) we have \( ab^{2\gamma}/(a^\gamma + b^\gamma)^2 \lesssim \min\{a, b\} \). Let \( F \) be the common face of the subdomains \( \Omega^\ell \) and \( \Omega^k \). On \( F \) we have \( \mathcal{E}_D \hat{w}_h = \theta^\ell F^{-1}(\rho^\gamma_{\ell} \hat{w}^\ell_h + \rho^\gamma_{k} \hat{w}^k_h) \) where \( \theta_F = \rho^\gamma_{\ell} + \rho^\gamma_{k} \). Letting \( \Theta^F \in \hat{\mathcal{W}}^F \) denote the function defined by
\[ \Theta^F(y_i) = 1 \quad \forall i \in \mathcal{Y} \setminus \mathcal{W}^F, \quad \Theta^F(y_i) = 0 \quad \forall i \in \mathcal{W}^F, \]
we can write
\[ \rho_e \|\hat{w}^\ell_h - \hat{v}_h\|^2_{H^{1/2}_0(F)} = \rho_e (\theta^\ell F \rho^\gamma_{\ell})^2 \|\hat{w}^\ell_h - \hat{w}^\ell_h\|^2_{H^{1/2}_0(F)} \]
\[ \lesssim \min\{\rho_e, \rho_k\} \|\hat{w}^\ell_h - \alpha^F_{\ell} \Theta^F \Theta^F - \hat{w}^\ell_h\|^2_{H^{1/2}_0(F)} + \min\{\rho_e, \rho_k\} \|\alpha^F_{\ell} - \alpha^F_{k}\|^2_{H^{1/2}_0(F)} \]
\[ \lesssim \rho_e \|\hat{w}^\ell_h - \Theta^F \Theta^F\|^2_{H^{1/2}_0(F)} + \rho_k \|\hat{w}^k_h - \alpha^F_{k}\|^2_{H^{1/2}_0(F)} + \min\{\rho_e, \rho_k\} \|\alpha^F_{\ell} - \alpha^F_{k}\|^{2}_{H^{1/2}_0(F)} \].

We now apply Lemma 5.2 which, thanks to Poincaré inequality, gives us
\[ \rho_e \|\hat{w}^\ell_h - \hat{v}_h\|^2_{H^{1/2}_0(F)} \lesssim (1 + \log(H/h))^2 \left( \rho_e \|\hat{w}^\ell_h\|^2_{H^{1/2}_0(F)} + \rho_k \|\hat{w}^k_h\|^2_{H^{1/2}_0(F)} \right) + \min\{\rho_e, \rho_k\} (1 + \log(H/h))\alpha^F_{\ell} - \alpha^F_{k}\|^2. \]

Adding up over all subdomains and over all faces (each face is counted twice) we obtain
\[ |\hat{w}_h - \hat{v}_h|_{1/2,*}^2 \lesssim (1 + \log(H/h))^2 |w_h|_{1/2,*}^2 + (1 + \log(H/h))\Delta F. \]
We now consider the contribution of the wirebasket. Using an inverse inequality analogous to (5.4), and Lemma 3.1 we can write

\[
(5.15) \quad \rho_{\ell}|w_{\ell}^{\rho} - w_{\ell}^{\rho,2}|^2 \leq \rho_{\ell}\left(\frac{h}{H}\right)^{-1}\|w_{\ell}^{\rho} - v_{\ell}^{\rho,2}\|_{L^2(\partial\Omega)}^2 \leq \rho_{\ell}\left(\frac{h}{H}\right)\sum_{i\in\mathcal{W}^e}|w_{\ell}^{\rho}(y_i) - v_{\ell}(y_i)|^2.
\]

We can write

\[
(5.16) \quad \rho_{\ell}|w_{\ell}^{\rho}(y_i) - v_{\ell}(y_i)|^2 = \rho_{\ell}|\theta_i^{-1}\sum_{k\in\mathcal{N}_i}\rho_k^\gamma(w_{\ell}^k(y_i) - w_{\ell}^k(y_i))|^2
\]

\[
\lesssim \sum_{k\in\mathcal{N}_i}\rho_{\ell}|\theta_i^{-1}\rho_k^\gamma|^2|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2 \lesssim \sum_{k\in\mathcal{N}_i}\min\{\rho_{\ell},\rho_k\}|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2.
\]

Plugging (5.16) in (5.15) and adding up over all \(\ell\) we obtain

\[
(5.17) \quad |w_{\ell}^{\rho} - v_{\ell}^{\rho,2}|_{1/2,*} \leq \left(\frac{h}{H}\right)\sum_{\ell}\sum_{k\in\mathcal{N}_i\cap\mathcal{V}^e}\min\{\rho_{\ell},\rho_k\}|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2
\]

\[
\lesssim \left(\frac{h}{H}\right)\Delta^\mathcal{X} + \sum_{E\in\mathcal{E}_H}\sum_{k\in\mathcal{N}_E}\left(\frac{h}{H}\right)\sum_{i\in\mathcal{Y}_E}\min\{\rho_{\ell},\rho_k\}|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2.
\]

Now, given \(E\in\mathcal{E}_H\), for \(\ell, k \in \mathcal{N}_E\) and \(i\in\mathcal{Y}_E\) we can write

\[
\min\{\rho_{\ell},\rho_k\}|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2 \lesssim \rho_{\ell}|w_{\ell}^k(y_i) - \alpha_{\ell}^E|^2 + \rho_k|w_{\ell}^k(y_i) - \alpha_k^E|^2 + \min\{\rho_{\ell},\rho_k\}|\alpha_{\ell}^E - \alpha_k^E|^2,
\]

yielding

\[
(5.18) \quad \left(\frac{h}{H}\right)\sum_{i\in\mathcal{Y}_E}\min\{\rho_{\ell},\rho_k\}|w_{\ell}^k(y_i) - w_{\ell}^k(y_i)|^2 \leq \left(\frac{h}{H}\right)\sum_{i\in\mathcal{Y}_E}\rho_{\ell}|w_{\ell}^k(y_i) - \alpha_{\ell}^E|^2
\]

\[
+ \left(\frac{h}{H}\right)\sum_{i\in\mathcal{Y}_E}\rho_k|w_{\ell}^k(y_i) - \alpha_k^E|^2 + \left(\frac{h}{H}\right)\min\{\rho_{\ell},\rho_k\}|\mathcal{Y}_E|\alpha_{\ell}^E - \alpha_k^E|^2.
\]

\[
\lesssim \rho_{\ell}|w_{\ell}^k - \alpha_{\ell}^E|^2_{L^2(E)} + \rho_k|w_{\ell}^k - \alpha_k^E|^2_{L^2(E)} + \min\{\rho_{\ell},\rho_k\}|\alpha_{\ell}^E - \alpha_k^E|^2,
\]

where we used Lemma 3.1 and the fact that, under the assumptions made on the tessellation, we have that \(\#(\mathcal{Y}_E) \lesssim H/h\). Then

\[
|w_{\ell}^{\rho} - v_{\ell}^{\rho,2}|_{1/2,*} \lesssim \left(\frac{h}{H}\right)\Delta^\mathcal{X} + \Delta^\mathcal{E} + \sum_{E\in\mathcal{E}_H}\sum_{\ell\in\mathcal{N}_E}\rho_{\ell}|w_{\ell}^k - \alpha_{\ell}^E|^2_{L^2(E)}.
\]

We conclude by observing that

\[
\sum_{E\in\mathcal{E}_H}\sum_{\ell\in\mathcal{N}_E}\rho_{\ell}|w_{\ell}^k - \alpha_{\ell}^E|^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}_H}\sum_{\ell\in\mathcal{N}_H}\rho_{\ell}|w_h^\ell - \alpha_{\ell}^E|^2_{L^2(\partial \Omega_F)}
\]

(we used that \(\alpha_{\ell}^E\) minimizes \(|w_{\ell}^k - \alpha_{\ell}^E|_{L^2(E)}\)). Applying Proposition 5.1 we then obtain

\[
\sum_{E\in\mathcal{E}_H}\sum_{\ell\in\mathcal{N}_E}\rho_{\ell}|w_h^\ell - \alpha_{\ell}^E|^2_{L^2(E)} \lesssim (1 + \log(h/H))|w_h - \alpha|^2_{1/2,*} \lesssim (1 + \log(h/H))|w_h^2|_{1/2,*}.
\]

\[
\square
\]

Observe that \(\Delta^\mathcal{X}, \Delta^\mathcal{E}\) and \(\Delta^\mathcal{F}\) vanish provided that \(w_h\) belongs to \(\tilde{W}^V, \tilde{W}^E\) and \(\tilde{W}^F\), respectively, so that, depending on the choice of \(\tilde{W}_h\), some of the terms at the right hand side of (5.10) disappear. In order to get a bound for \(\mathcal{E}_D\) for the different choices of \(\tilde{W}_h\), we then need to bound the remaining
we fall back in the previous case. To this aim we start by introducing the following definitions:

**Lemma 5.4.** For $E$ edge of $F \subset \partial \Omega^\ell$ it holds that

\[ |\alpha^E_\ell - \alpha^F_\ell|^2 \lesssim (1 + \log(h/H))|w^E_h|_{H^{1/2}(F)}^2. \]

**Proof.** Let $\pi_E$ denote the $L^2(E)$ projection onto constant functions, which is defined by

\[ \pi_E w = |E|^{-1} \int_E w. \]

Thanks to the trace inequality (5.2) it holds that

\[ \|\pi_E w\|_{L^2(E)} \leq \|w\|_{L^2(E)} \leq \|w\|_{L^2(\partial F)} \lesssim \|w\|_{H^s(\partial F)} \lesssim \frac{1}{\sqrt{2s-1}}\|w\|_{H^s(F)}. \]

Trivially, such an operator preserves the constants. Then, by using Proposition 5.1 and a Poincaré type inequality, we can write

\[ |\alpha^E_\ell - \alpha^F_\ell|^2 \lesssim \|\pi_E (w^E_h - \alpha^F_\ell)\|_{L^2(E)}^2 \lesssim \|w^E_h - \alpha^F_\ell\|_{L^2(E)}^2 \lesssim (1 + \log(h/H))|w^E_h|_{H^{1/2}(F)}^2. \]

We can now bound $\Delta^X$ in terms of either $\Delta^e$ or $\Delta^F$.

**Lemma 5.5.** The following inequalities hold

\[ \left(\frac{h}{H}\right)\Delta^X \lesssim \tau_E \left( (1 + \log(H/h))|w^E_h|_{1/2,s}^2 + \Delta^e \right), \]

\[ \left(\frac{h}{H}\right)\Delta^X \lesssim \tau_F (1 + \log(H/h)) \left( |w^F_h|_{1/2,s}^2 + \Delta^F \right), \]

with $\tau_E, \tau_F$ constants depending on the diffusion coefficient $\rho$, satisfying $0 < \tau_E \leq \tau_F \leq \max \rho/\min \rho$.

**Proof.** We start by proving (5.20). Let $i \in X$ and let $\ell, k \in N_i$. Assume at first that $\Omega^\ell$ and $\Omega^k$ share an edge $E$ having $y_i$ as one of the vertices. Adding and subtracting $(\alpha^F_\ell - \alpha^F_k)$, using Proposition 5.1 as well as a Poincaré inequality for function with vanishing average in a portion of the boundary (allowing to bound the $H^{1/2}$ norm with the $H^{1/2}$ seminorm), we can write

\[ \left(\frac{h}{H}\right) \min\{\rho_\ell, \rho_k\} |w^F_\ell(y_i) - w^F_k(y_i)|^2 \lesssim \min\{\rho_\ell, \rho_k\} \|w^F_\ell - w^F_k\|_{L^2(E)}^2 \lesssim (1 + \log(h/H)) (\rho_\ell |w^F_\ell|_{H^{1/2}(\partial \Omega^\ell)} + \rho_k |w^F_k|_{H^{1/2}(\partial \Omega^k)}) + \min\{\rho_\ell, \rho_k\} |\alpha^F_\ell - \alpha^F_k|^2. \]

Let now $\ell, k \in N_i$ be two subdomains sharing a vertex $y_i$ but not an edge. In this case we bound $|w^F_\ell(y_i) - w^F_k(y_i)|$ by adding and subtracting a suitable sequence of values $w^F_n(y_i)$ in such a way that we fall back in the previous case. To this aim we start by introducing the following definitions:

- a path $\mathcal{P}$ of length $N$ is any sequence of subdomains $\Omega^{n_0}, \ldots, \Omega^{n_N}$ such that for all $i$, $\Omega^{n_i}$ and $\Omega^{n_{i+1}}$ share at least a vertex.
- for a given path $\mathcal{P} = (\Omega^{n_0}, \ldots, \Omega^{n_N})$ we set $\tau_\mathcal{P} = (\min \rho_{n_i})^{-1}, i \in [0, \ldots, N]$,
- a path $\mathcal{P} = (\Omega^{n_0}, \ldots, \Omega^{n_N})$ connects $\Omega^\ell$ and $\Omega^k$ via edges (resp. via faces) if $n_0 = \ell, n_N = k$ and for all $i = 1, \ldots, N$ the subdomains $\Omega^{n_i}$ and $\Omega^{n_{i+1}}$ share an edge (resp. a face).

Letting $K^*$ be the maximum number of subdomains sharing a vertex, we denote by $\mathbf{P}^\ell_k$ (resp. $\mathbf{P}^\ell_k$) the set of paths of length $\leq K^*$ connecting $\Omega^\ell$ and $\Omega^k$ via edges (resp. via faces). For all
path $\mathcal{P} = (\Omega^{n_0},\ldots,\Omega^{N}) \in \mathbf{F}_E^{\ell,k}$ we can bound
\[
\min \{\rho_\ell, \rho_k\}|w_h^\ell(y_i) - w_h^k(y_i)|^2 \lesssim \sum_{j=1}^N \frac{\min \{\rho_\ell, \rho_k\}}{\min \{\rho_n, \rho_{n-1}\}} \min \{\rho_{n_j}, \rho_{n_{j-1}}\}|w_h^{n_j}(y_i) - w_h^{n_{j-1}}(y_i)|^2
\]
and, using the bound for subdomains sharing an edge, we obtain \([5.20]\), with $\tau_E$
\[
\tau_E = \max_{(\ell,k)} \left( \min \{\rho_\ell, \rho_k\} \tau_E^{\ell,k} \right)
\]
where
\[
\tau_E^{\ell,k} = \min_{\mathcal{P} \in \mathbf{F}_E^{\ell,k}} \tau_P.
\]
Bound \([5.21]\) with $\tau_F^{\ell,k} = \min_{\mathcal{P} \in \mathbf{F}_F^{\ell,k}} \tau_P$ and
\[
\tau_F = \max_{(\ell,k)} \left( \min \{\rho_\ell, \rho_k\} \tau_F^{\ell,k} \right)
\]
is obtained by a similar argument. As $\mathbf{F}_F^{\ell,k} \subseteq \mathbf{F}_E^{\ell,k}$ (if two subdomains share a face they also share a vertex) we easily get that $\tau_E \leq \tau_F$.

Finally, we bound $\Delta^\ell$ and $\Delta^F$ in terms of each other.

**Lemma 5.6.** The following bound hold:
\[
\Delta^F \lesssim (1 + \log(H/h))|w_h|_{1/2,*}^2 + \Delta^\ell,
\]
\[
\Delta^\ell \lesssim \tau_{EF} \left( (1 + \log(H/h))|w_h|_{1/2,*}^2 + \Delta^F \right),
\]
with $\tau_{EF}$ a constant that satisfies $\tau_{EF} \leq \tau_F$.

**Proof.** Let us consider the first inequality. Let $F$ be the face common to the subdomains $\Omega^\ell$ and $\Omega^k$, and let $E$ be any of the edges of $F$. By Lemma 5.4, we have
\[
\min \{\rho_\ell, \rho_k\}|\alpha^F_\ell - \alpha^F_k|^2 \lesssim \rho_\ell|\alpha^F_\ell - \alpha^E_\ell|^2 + \rho_k|\alpha^F_k - \alpha^E_k|^2 + \min \{\rho_\ell, \rho_k\}|\alpha^E_\ell - \alpha^E_k|^2 \lesssim \sum_{j=1}^N \frac{\min \{\rho_\ell, \rho_k\}}{\min \{\rho_n, \rho_{n-1}\}} \min \{\rho_{n_j}, \rho_{n_{j-1}}\}|w_h^{n_j}(y_i) - w_h^{n_{j-1}}(y_i)|^2.
\]
In view of the definition of $\Delta^F$ and $\Delta^\ell$, the bound \([5.22]\) follows up adding the contribution of all faces.

Let us now consider the second bound. Let $E$ be an edge, and let $\ell, k \in \mathcal{N}_E$. Assume at first that $\Omega^\ell$ and $\Omega^k$ share a face $F$. Then, as we did for the previous bound, it is not difficult to prove that
\[
\min \{\rho_\ell, \rho_k\}|\alpha^E_\ell - \alpha^E_k|^2 \lesssim (1 + \log(H/h)) \left( \rho_\ell|w_h^{2}_{H^{1/2}(F)} + \rho_k|w_h^{2}_{H^{1/2}(F)} \right) + \min \{\rho_\ell, \rho_k\}|\alpha^E_\ell - \alpha^E_k|^2.
\]
Let us now assume that $\Omega^\ell$ and $\Omega^k$ do not share a face. Proceeding as in the proof of the previous lemma it is easy to see that bound \([5.23]\) holds with
\[
\tau_{EF} = \max_{(\ell,k)} \left( \min \{\rho_\ell, \rho_k\} \tau_F^{\ell,k} \right).
\]
As two subdomains that share an edge also share a vertex, we have that $\tau_{EF} \leq \tau_F$. \qed
Now we have all the ingredients to prove Theorem 4.7. From 5.1 we get that to bound the condition number, we only need to bound \( s(E_D w_h, E_D w_h) \) in terms of \( s(w_h, w_h) \). By using Lemma 5.20, 5.21, 5.22 and 5.23 we can easily obtain that:

\[
\begin{align*}
(5.24) & \quad \text{if } \widetilde{W}_h \subseteq \widetilde{W}_E^h, \quad |E_D w_h|^2 \lesssim ((1 + \log(H/h)) + \tau_E)(1 + \log(H/h))|w_h|^2; \\
(5.25) & \quad \text{if } \widetilde{W}_h \subseteq \widetilde{W}_F^h, \quad |E_D w_h|^2 \lesssim ((1 + \log(H/h)) + \tau_F)(1 + \log(H/h))|w_h|^2; \\
(5.26) & \quad \text{if } \widetilde{W}_h \subseteq \widetilde{W}_V^h \cap \widetilde{W}_E^h, \quad |E_D w_h|^2 \lesssim (1 + \log(H/h))^2|w_h|^2; \\
(5.27) & \quad \text{if } \widetilde{W}_h \subseteq \widetilde{W}_V^h \cap \widetilde{W}_F^h, \quad |E_D w_h|^2 \lesssim ((1 + \log(H/h)) + \tau_{EF})(1 + \log(H/h))|w_h|^2.
\end{align*}
\]

Remark 5.7. Observe that, since we are in the framework of [35], we have the equivalence of the BDDC preconditioner with the FETI-DP preconditioner. Therefore the analysis presented also yields an estimate on the BDDC preconditioner for the Virtual Element Method.

6. Numerical tests

We consider the model problem

\[
\begin{align*}
(6.1a) & \quad -\nabla \cdot (\rho \nabla u) = f \quad \text{in } \Omega = (0, 1)^3, \\
(6.1b) & \quad u = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

We deal with meshes made of either truncated octahedra or Voronoi cells (see Figure 1); Table 1 lists the values of the following geometrical parameters for the reference meshes used in the experiments:

- \( h = \max_{K \in \mathcal{T}_h} h_K \), where \( h_K \) is the diameter of element \( K \in \Omega_h \).
- \( h_{\min} = \min_{K \in \mathcal{T}_h} h_{\min,K} \), where \( h_{\min,K} \) is the minimum distance between any two vertices of \( K \).
- \( \gamma_* = \min_{K \in \mathcal{T}_h} \gamma_K \), where \( \gamma_K \) is the parameter given in Definition 2.1.

From Table 1 we see that Voronoi meshes satisfy Assumption 2.2 but with worse constants than the octahedra ones.

**Figure 1.** (Left): clipped view of a mesh made of truncated octahedra; (right) example cells of a Voronoi mesh.

Problem (6.1) is discretized with virtual elements of degree 1. In all experiments \( \Omega \) is divided into \( L = N \times N \times N \) cubic subdomains with side length \( H = 1/N \). For simplicity, subdomain meshes
Table 1. Data for the meshes used in the experiments: (left) meshes made of truncated octahedra; (right) meshes made of Voronoi cells.

| Mesh  | $h$      | $h_{\text{min}}$ | $\gamma_*$ | Mesh  | $h$      | $h_{\text{min}}$ | $\gamma_*$ |
|-------|----------|------------------|------------|-------|----------|------------------|------------|
| oct$_1$ | $4.33 \cdot 10^{-1}$ | $6.25 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_1$ | $7.16 \cdot 10^{-1}$ | $2.40 \cdot 10^{-4}$ | $9.08 \cdot 10^{-7}$ |
| oct$_2$ | $2.89 \cdot 10^{-1}$ | $4.17 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_2$ | $5.66 \cdot 10^{-1}$ | $2.48 \cdot 10^{-4}$ | $4.83 \cdot 10^{-7}$ |
| oct$_3$ | $2.17 \cdot 10^{-1}$ | $3.13 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_3$ | $5.37 \cdot 10^{-1}$ | $1.26 \cdot 10^{-4}$ | $3.48 \cdot 10^{-8}$ |
| oct$_4$ | $1.73 \cdot 10^{-1}$ | $2.50 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_4$ | $3.62 \cdot 10^{-1}$ | $2.60 \cdot 10^{-5}$ | $6.77 \cdot 10^{-9}$ |
| oct$_5$ | $1.44 \cdot 10^{-1}$ | $2.08 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_5$ | $2.81 \cdot 10^{-1}$ | $5.69 \cdot 10^{-6}$ | $1.92 \cdot 10^{-8}$ |
| oct$_6$ | $1.24 \cdot 10^{-1}$ | $1.79 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_6$ | $2.46 \cdot 10^{-1}$ | $3.12 \cdot 10^{-6}$ | $3.38 \cdot 10^{-9}$ |
| oct$_7$ | $1.08 \cdot 10^{-1}$ | $1.56 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_7$ | $1.90 \cdot 10^{-1}$ | $1.31 \cdot 10^{-6}$ | $7.24 \cdot 10^{-11}$ |
| oct$_8$ | $9.62 \cdot 10^{-2}$ | $1.39 \cdot 10^{-2}$ | $6.06 \cdot 10^{-2}$ | voro$_8$ | $1.48 \cdot 10^{-1}$ | $5.93 \cdot 10^{-8}$ | $5.32 \cdot 10^{-12}$ |

Concerning the space $\widehat{W}_h$, the following choices from Section 5 are tested:

- E: $\widehat{W}_h = \widehat{W}_h^E$;
- F: $\widehat{W}_h = \widehat{W}_h^F$;
- VE: $\widehat{W}_h = \widehat{W}_h^V \cap \widehat{W}_h^E$;
- VF: $\widehat{W}_h = \widehat{W}_h^V \cap \widehat{W}_h^F$;

For the sake of completeness, we also test

- V: $\widehat{W}_h = \widehat{W}_h^V$,

for which we expect a worse behaviour, as for the finite element case, where the condition number increases as $(1 + \log(H/h))^2(H/h)$ (see [15], Remark 6.39).

Edge and face constraints are imposed using an explicit change of basis. To simplify the implementation, we work with a fully redundant set of Lagrange multipliers for the dual part of the solution.

In order to analyze the performance of these algorithms, we carry out two series of experiments:

**Test 1:** FETI-DP scalability. We fix the subdomain problem size by choosing a reference mesh (either oct$_3$ or voro$_5$ in Table 1) and increase the number of subdomains and thus the overall problem size, but keeping $H/h$ fixed. Table 2 shows the dimension of the primal spaces $\widehat{W}_h$ for the different algorithms. According to Theorem 4.7, we expect the condition number for the FETI-DP preconditioner to remain constant asymptotically.

**Test 2:** FETI-DP quasi-optimality. We fix the number of subdomains $L$ to 216 ($6 \times 6 \times 6$), so that $H$ is kept constant, and increase the size of the local problems by choosing finer and finer reference meshes (from oct$_1$ to oct$_8$, or from voro$_1$ to voro$_8$ in Table 1), thereby incrementing the overall problem size. This results in a smaller $h$ and a bigger $H/h$. According to Theorem 4.7, we now expect the condition number for the FETI-DP preconditioner to exhibit a polylogarithmic behavior asymptotically.

To test the robustness of FETI-DP, each test is run with two types of data

- constant coefficients $\rho_{\ell} = 1$ \ \forall \ell, f = \sin(2\pi x) \sin(2\pi y) \sin(2\pi z)$;
coefficient jumps of $10^{10}$ in a 3D checkerboard distribution ($\rho_1 = 10^5, \rho_2 = 10^{-5}$). The right hand side $f$ is implicitly chosen by choosing a right hand side vector with values uniformly randomly distributed in $[-1, 1]$.

We use the conjugate gradient with the FETI-DP Dirichlet preconditioner, with zero initial guess and, as a stopping criterion, the relative reduction of the dual residual by either $10^{-6}$ or $10^{-12}$ when using the first or the second type of data, respectively. MATLAB® R2016b is used as the subdomain and coarse sparse direct solver. All the experiments are run on a machine equipped with processor Intel® Core™ i7-7820HQ, operating system Ubuntu Linux 16.04 LTS, memory 64GB, 2400MHz DDR4 Non-ECC SDRAM.

| $L$ | Octahedra | Voronoi | Coarse |
|-----|------------|---------|--------|
|     | $h$ D.o.f. | $h$ D.o.f. | V E F VE VF |
| 8   | 1.08$\cdot 10^{-1}$ | 1.40$\cdot 10^{-1}$ | 1 | 6 | 12 | 7 | 13 |
| 64  | 5.41$\cdot 10^{-2}$ | 7.02$\cdot 10^{-2}$ | 388 280 | 27 | 108 | 144 | 135 | 171 |
| 216 | 3.61$\cdot 10^{-2}$ | 4.68$\cdot 10^{-2}$ | 1 330 240 | 125 | 450 | 540 | 575 | 665 |
| 512 | 2.71$\cdot 10^{-2}$ | 3.51$\cdot 10^{-2}$ | 3 176 952 | 343 | 1176 | 1344 | 1519 | 1687 |
| 1000| 2.17$\cdot 10^{-2}$ | 2.81$\cdot 10^{-2}$ | 6 233 072 | 729 | 2430 | 2700 | 3159 | 3429 |
| 1728| 1.80$\cdot 10^{-2}$ | 2.34$\cdot 10^{-2}$ | 10 803 256 | 1331 | 4356 | 4752 | 5687 | 6083 |

Table 2. Number of global and primal d.o.f. for the different algorithms considered and the meshes of Test 1: constant $H/h$, reference subdomain mesh oct$_3$ (octahedra) or vor$_5$ (Voronoi).

| $L$ | V | E | F | VE | VF |
|-----|---|---|---|----|----|
|     | $\kappa$ | it | $\kappa$ | it | $\kappa$ | it | $\kappa$ | it | $\kappa$ | it |
| 8   | 1.00 | 1 | 1.00 | 1 | 1.00 | 1 | 1.00 | 1 |
| 64  | 3.23 | 9 | 2.39 | 8 | 2.33 | 8 | 1.58 | 7 | 2.15 | 8 |
| 216 | 12.85 | 12 | 2.67 | 10 | 2.95 | 10 | 1.58 | 7 | 2.64 | 10 |
| 512 | 13.75 | 18 | 2.59 | 10 | 3.05 | 11 | 1.61 | 7 | 2.57 | 10 |
| 1000| 13.92 | 18 | 2.53 | 10 | 3.27 | 11 | 1.62 | 7 | 2.70 | 10 |
| 1728| 13.98 | 18 | 2.51 | 10 | 3.38 | 11 | 1.63 | 7 | 2.75 | 10 |

Table 3. Test 1 - truncated octahedra meshes, $H/h = 4.6188$ and constant coefficients $\rho = 1$. Condition number estimates and iteration numbers for the different choices of $\tilde{W}_h$.

6.1. **FETI-DP scalability.** Results for the first series of experiments (Test 1) with constant coefficients $\rho$ are reported in Table 3 for meshes of truncated octahedra, and in Table 4 for Voronoi meshes. The results are in accordance with the theoretical bounds for both set of meshes, confirming the robusteness of the preconditioner. Tables 3 and 4 show that, without any jumps in the coefficients, the results for $E$, $F$, $VE$ and $VE$ are similar. In switching from the octahedra to the Voronoi mesh, which satisfies Assumption 2.2 but with worse constants, we observe an increase in the number of iterations and in the condition number which, however, still display the expected behavior. As one expects, the choice $V$ is instead not competitive.
The numerical tests with varying coefficients \( \rho \) are displayed in Table 5 and Table 6 for octahedra and Voronoi meshes respectively. With highly oscillating coefficients, the choice \( \tilde{W}_h \) of only continuity of face averages across subdomains is imposed, performs very poorly on both types of meshes, despite their different degree of regularity. We note that, with jumping coefficients, the bound given in Theorem 4.7 for \( E, F \) or \( VF \) would become meaningless if \( \tau_E, \tau_F \) or \( \tau_{EF} \) are high enough.

Table 4. Test 1 - Voronoi meshes, \( H/h = 6.171687 \) and constant coefficients \( \rho = 1 \).
Condition number estimates and iteration numbers for the different choices of \( \tilde{W}_h \).

| \( L \) | \( V \) | \( E \) | \( F \) | \( VE \) | \( VF \) |
|-------|-------|-------|-------|-------|-------|
| \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it |
| 8 | 1.00 | 1 | 1.00 | 1 | 1.00 | 1 | 1.00 | 1 | 1.00 | 1 |
| 64 | 27.88 | 19 | 4.72 | 15 | 6.36 | 15 | 3.92 | 13 | 5.51 | 14 |
| 216 | 26.53 | 25 | 6.03 | 17 | 9.45 | 19 | 4.16 | 14 | 7.79 | 17 |
| 512 | 29.05 | 31 | 6.09 | 17 | 10.54 | 21 | 4.16 | 14 | 8.38 | 18 |
| 1000 | 29.55 | 32 | 6.14 | 17 | 11.03 | 21 | 4.16 | 14 | 8.73 | 19 |
| 1728 | 30.47 | 32 | 6.17 | 17 | 11.43 | 22 | 4.16 | 14 | 8.89 | 19 |

Table 5. Test 1 - truncated octahedra meshes, \( H/h \) constant, checkerboard \( (\rho_1 = 10^5, \rho_2 = 10^{-5}) \), \( f \) uniform random in \([-1, 1]\). Condition number estimates and iteration numbers for the different choices of \( \tilde{W}_h \).

| \( L \) | \( V \) | \( E \) | \( F \) | \( VE \) | \( VF \) |
|-------|-------|-------|-------|-------|-------|
| \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it | \( \kappa \) | it |
| 8 | 1.87 | 6 | 1.75 | 5 | 2.73 | 6 | 1.25 | 5 | 2.17 | 6 |
| 64 | 10.33 | 13 | 2.57 | 8 | 4.48 \times 10^9 | 31 | 1.28 | 5 | 11.69 | 11 |
| 216 | 10.36 | 15 | 2.53 | 8 | 5.59 \times 10^9 | 88 | 1.28 | 5 | 12.62 | 15 |
| 512 | 10.35 | 15 | 2.54 | 8 | 6.06 \times 10^9 | 118 | 1.29 | 5 | 13.37 | 16 |
| 1000 | 10.36 | 15 | 2.54 | 8 | 6.29 \times 10^9 | 132 | 1.29 | 5 | 13.28 | 16 |
| 1728 | 10.37 | 15 | 2.54 | 8 | 6.44 \times 10^9 | 138 | 1.29 | 5 | 13.17 | 16 |

Table 6. Test 1 - Voronoi meshes, \( H/h \) constant, checkerboard \( (\rho_1 = 10^5, \rho_2 = 10^{-5}) \), \( f \) uniform random in \([-1, 1]\). Condition number estimates and iteration numbers for the different choices of \( \tilde{W}_h \).
Indeed, with coefficient jumps of $10^{10}$ in a checkerboard distribution, we have $\tau_E = 1$, $\tau_F = 10^{10}$, and $\tau_{EF} = 10^{10}$, in agreement with the better performance of E with respect to F and VF, as shown in Tables 5-6. The independence from the jumps of the coefficients is shown by VE as predicted by the bound of Theorem 4.7.

6.2. FETI-DP quasi-optimality. Results for our second set of runs (Test 2) with both smooth and random data are shown in Figures 2 and 3 for meshes of truncated octahedra, and in Figures 4 and 5 for Voronoi meshes. Both sets of runs are in agreement with the condition number estimates of Theorem 4.7. In particular, the experiments show that FETI-DP achieves good scalability for our model problem if VE is used, i.e. $\tilde{W}_h \subset W^V_h \cap W^E_h$.

![Figure 2](image-url)

**Figure 2.** Test 2 - constant coefficients $\rho = 1$, truncated octahedra meshes. Plots of $\kappa^{1/2}$ as a function of the global degrees of freedom on meshes of Voronoi meshes for the different choices of $W^h$.

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