Abstract

In this paper it is numerically shown that the dynamics of a heterogeneous Cournot oligopoly model depending on two bifurcation parameters can exhibit hidden and self-excited attractors. The system has a single equilibrium and a line of equilibria. The bifurcation diagrams show that the system admits several attractor coexistence windows, where the hidden attractors can be found. Depending on the parameters ranges, the coexistence windows present combinations of periodic, quasiperiodic and chaotic attractors.

Keywords: Hidden attractor; Self-excited attractor; Cournot oligopoly model

1. Introduction

Since 1838, when A. Cournot [1] proposed the first treatment of oligopoly (a duopoly case), the theory of a market form in which a market has a dominant influence on a small number of sellers (oligopolists) has been deeply researched. The first (and crucial) additions to the theory were made by H. von Stackelberg [2], and later on significant advances of the theory were made from a different point of view.

As the Cournot-Nash equilibria of the corresponding game reflects given oligopoly behavior, its stability has to be investigated depending on the number of players, and also how the game is modeled.

For the second case, it was shown in [3] (see also [4], p. 237) that an oligopoly model constructed under constant marginal costs with a linear demand function is neutrally stable for three competitors, and unstable for more than three competitors (for more details see

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It is noted in [5] that linear demand functions are very easy to use, but they do not avoid negative supplies and prices, so they can only be used to study local behavior.

Hence, nonlinear demand functions such as piecewise linear functions or other more complex functions were applied; for duopoly [6], and later [7], for a triopoly using isoelastic demand functions. These types of demand function were later studied by [8] and [9] for a nonlinear (iso-elastic) demand function and constant marginal costs, and it was concluded that this Cournot model for \( n \) competitors is neutrally stable if \( n = 4 \), and is unstable if the number of competitors is greater than five (see also [5]). Finally, a complete characterization of the Cournot-Nash stability was done in [10] which depended on the number of competitors.

All the above approaches were done for a homogeneous approach. In the heterogeneous decision mechanism, introduced in [11], two different types of quantity setting players characterized by different decision mechanisms that coexist and operate simultaneously are considered. In this case, competitors adaptively use their choices to increase their profits. This model’s Cournot-Nash equilibria stabilities were described showing its periodic and also chaotic regimes. Moreover in [12], an addition to the foregoing approach was made where the role of the intensity of scenario choice was taken into consideration.

On the other hand, hidden attractors represent an important recently introduced notion in applications because they might allow unexpected and potentially disastrous systems responses to some perturbations in a structure like a bridge or aircraft wing. However, except for some examples of theoretical models (see e.g. [13, 14, 15]), there are no important investigations of hidden attractors in real and applied examples of chaotic maps.

In this paper by attractor one understands the numerical attractor obtained after transients are discarded.

The generally accepted classification of the attractors we are considering is described by

**Definition 1.** [16, 17] An attractor is called a self-excited attractor if its basin of attraction intersects an open neighborhood of an equilibrium, otherwise it is called a hidden attractor.

The sudden appearance of some hidden chaotic attractor, could represent a major disadvantage for the underlying system. Thus, the consequences could be dramatic, such as in the case of pilot-induced oscillations that caused the YF-22 crash in April 1992 and Gripen crash in August 1993 [18]. It is understandable that identifying unwanted hidden chaotic behavior is desirable. There exists the risk of a sudden jump from a desirable attractor to possible undesired behavior of some hidden attractor. Recently, it has been shown that multistability is connected with the occurrence of hidden attractors. If there are unstable fixed points, the basins of attraction of the hidden attractors do not touch them. Note that if the system exhibits chaotic or regular behavior and the systems equilibria are stable, then the chaotic or regular underlying attractors are implicitly hidden. Therefore, the stability of equilibria is important
For a hidden attractor, its attraction basin is not connected with unstable equilibria. For example, hidden attractors can be found in e.g. systems without equilibria or with stable equilibria [18].

Also, as in the case of the studied discrete-time system in this paper, systems with an infinite number of equilibria (also called line of equilibria), can admit hidden attractors. Systems with a line of equilibria are very few (see e.g. [19, 20]). Hidden attractors in an impulsive discrete dynamical system have been found in [21], where the case of a supply and demand economical system is studied.

The present paper on the oligopoly model follows the usual path in research development: revealing at the first step the existence of hidden chaos in the model. Our second step to explain the essence of the new finding of hidden chaos will be carried out, hopefully in the near future, since there is no precise rigorous theory about hidden attractors yet in the current literature where researchers are still intensively working on the topic. Besides, this important but difficult issue is beyond the scope of the present paper of first attempt.

The paper is organized as follows: Section 2 presents the considered oligopoly model, underlining equilibria stability, necessary in the study of hidden attractors; Section 3 deals with hidden and self-exited attractors. Conclusion ends the paper.

2. The heterogeneous Cournot oligopoly model

Consider the heterogeneous Cournot oligopoly model (HCOM) introduced in [12] defined for identical quantity setting agents \( N = \{1, 2, \ldots, N\} \) that compete in the same market for a homogeneous good, whose demand is summarized by a linear inverse-demand function, or price function \( P(Q) = \max\{a - bQ, 0\} \) \( (P \) treats price as a function of quantity demanded). Denote by \( q_i^n \) the quantity of goods that is a generic \( i \)-th agent, with \( i \in N \), selling in the market at time-period \( n \). All the agents bear the same constant marginal production cost \( c \), so that the generic \( i \)-th agent earns the profit

\[ \pi_i = P(Q)q_i - cq_i. \]

The oligopoly in this case is characterized by introducing heterogeneous decision mechanisms, used to decide what quantity of goods to produce by considering a population structured into two groups of agents of different kinds. The first group, whose representative is denoted by \( q_1 \), includes boundedly rational players that use the gradient rule, and are hence called gradient player. The second group, denoted by \( q_2 \), includes agents that adopt an imitation-based decision mechanism, and are called imitator players.

The collective behavior of the whole heterogeneous population of \( N \) players is described by the following 2-dimensional non-linear autonomous discrete dynamical system [12]:

\[
\text{HCOM} : \begin{cases} 
q_1^{n+1} = q_1^n + \gamma q_1^n (a - b ((N(1 - \omega) + 1)q_1^n + \omega N q_2^n) - c), \\
q_2^{n+1} = \frac{\pi_2^n}{\pi_2^n + \pi_1^n} q_2^n + \frac{\pi_1^n}{\pi_2^n + \pi_1^n} q_1^n, 
\end{cases}
\]
where

\[ \begin{align*}
\pi_1^n &= (a - c - bN((1 - \omega)q_1^n + \omega q_2^n))q_1^n, \\
\pi_2^n &= (a - c - bN((1 - \omega)q_1^n + \omega q_2^n))q_2^n,
\end{align*} \]

In theoretical analysis, \( \omega \), which for the standard Cournot model is assumed to be \( \omega \equiv 1 \), is considered within the range \([0, 1]\). \( \omega \) is determined as a fraction of imitators, i.e. \( \omega = N^I / N \), where \( N^I \) includes the imitators agents that adopt an imitation-based decision mechanism. The other group of \( N^G \) players, are the gradient players and determines the expression of \( N \): \( N = N^I + N^G \). \( \gamma \), which is a measure of the reactiveness of gradient players, determines interesting dynamics within \( \gamma \in [0, 3] \) (see [11, 12]). To note that \( \omega \) can take unbounded values for sufficiently small values of \( \gamma \) for whatever values of the other parameters. In this paper the parameters selection is considered only to make the model capable to give rise to a wide range of the complex dynamics.

The system admits a line of equilibria

\[ X_0^*(0, q_2), \quad q_2 \geq 0, \]

along the positive axis \((Oq_2)\), and also the single equilibrium

\[ X_1^*(\frac{a - c}{b(N + 1)}, \frac{a - c}{b(N + 1)}). \]

The stability of equilibria is stated by the following result

**Theorem 1.** [12]

i) Equilibrium \( X_1^1 \) is asymptotically stable if

\[ \omega \in (\Omega_1, \Omega_2), \]

with

\[ \begin{align*}
\Omega_1 &= \frac{3}{2} \frac{N + 1}{N} \left( \frac{1}{2} - \frac{1}{\gamma(a - c)} \right), \\
\Omega_2 &= \frac{1}{2} \frac{N + 1}{N} \left( \frac{1}{\gamma(a - c)} + 1 \right). 
\end{align*} \]

ii) Equilibria \( X_0^*(0, q_2) \) are stable for \( q_2 > q_2^* = \frac{a - c}{\lambda N} \).

Note that the stability, established by Theorem 1 is locally\(^1\).

**Graphical interpretation**

Consider in the parameter space \((\gamma, \omega)\), the lattice domain \( D = [0.35, 0.525] \times [0.329, 0.721] \) with corners \( M_1(0.35, 0.329), M_2'(0.525, 0.329), M_3'(0.525, 0.791), M_4(0.35, 0.791) \) (Fig. 1).

\(^1\)In [12] this relation seems to be wrong.
All numerical tests in this paper have been done for the constants \( a = 10, \ b = 1, \ c = 1, \ N = 5 \) and the parameters \( \gamma, \ \omega \) as bifurcation parameters. For all these values, the equilibrium \( X_1^* = (1.5, 1.5) \).

As functions of \( \gamma, \ \Omega_1 \), with graph \( \Gamma_1 \) and \( \Omega_2 \), with graph \( \Gamma_2 \), are bifurcation curves representing the flip bifurcation and Neimark-Sacker bifurcation, respectively, of \( X_1^* \). For all \((q_1, q_2^*)\), \( X_0^* \) suffers a Neimark-Sacker bifurcation [12].

The stability of point \( X_{0_1}^* \) reads as follows: the dark green area \( S \subset D \) limited by the curves \( \Gamma_{1,2} \) and lines \( \gamma = 0.35 \) and \( \gamma = 0.525 \) and defined by the corners \( M_1(0.35, 0.329) \), \( M_2(0.525, 0.519) \), \( M_3(0.525, 0.727) \), and \( M_4(0.35, 0.791) \) (Fig. 1), contains the parameter sets \((\gamma, \omega)\) which generate the stability of the equilibrium \( X_{1}^* \), while the light green areas (outside the curves \( \Gamma_{1,2} \)) represent the instability sets of the equilibrium \( X_{1}^* \).

The vertical dotted line through \( \gamma = 0.48 \) and the horizontal dotted line through \( \omega = 0.4 \) represent the bifurcation diagrams with respect \( \omega \) and \( \gamma \), respectively.

3. Attractors coexistence and hidden attractors

As is known, for a fixed parameter value, to each initial condition corresponds uniquely an attractor.

Therefore, in a bifurcation diagram (BD) generated with a fixed initial condition, to every bifurcation parameter value corresponds a unique attractor represented in the BD as a vertical line, composed from a set of isolated points (periodic attractors), or a band of an infinity of points (quasiperiodic or chaotic attractors)\(^2\).

Because in this paper every BD is generated with two different initial conditions (IC), namely \( IC_1 = (1, 1.5) \) and \( IC_2 = (1.75, 1.5) \), one obtains two sets of attractors represented by vertical lines of points in the BD denoted with fraktur letters with the indices 1 and 2: \((A_1, A_2), (B_1, B_2)\) and so on. Therefore, at the considered resolution of 800 points on the bifurcation parameter axis correspond two sets of 800 attractors denoted by the calligraphic letters indexed 1 or 2 depending on the initial conditions \( IC_{1,2}, A_1 \in A_1, B_1 \in B_1, A_2 \in A_2, B_2 \in B_2 \) and so on. All these attractors are a function of the considered bifurcation parameter, \( A_1 = A_1(p), B_1 = B_1(p) \) and so on, the parameter \( p \) being either \( \gamma \) or \( \omega \), and are plotted red and blue corresponding to \( IC_1 \), or to \( IC_2 \), respectively.

For simplicity, hereafter we drop the parameter \( p \) in the attractors notation and all attractors in this paper are generated by starting from one of the initial conditions \( IC_1 \), or \( IC_2 \). Note that every attractor can also be generated from the indicated initial conditions from underlying attraction basins, denoted \( q_0 \).

For some parameter ranges, the two sets of attractors could be different (inside the coexistence windows), when the existence of hidden attractors is possible, or identical (outside

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\(^2\)Note, in this paper every chaotic orbit is as usual understood as an orbit approaching a chaotic attractor, even if the exiting (interior and exterior) crises might imply chaotic but non-attracting sets (see e.g. the non-attracting chaotic set after the saddle-node bifurcation of the logistic map for \( r \approx 4.83 \)).
Within the studied coexistence windows, the system (1) presents three different kinds of attractors: periodic attractors or limit cycles, quasiperiodic attractors, and chaotic attractors.

The tools utilized in this paper to identify attractors are: BDs, time series, planar phase representations, the maximal local finite-time Lyapunov exponent \( \lambda \), the output \( K \) of the 0-1 test for chaos (see e.g. [22]) and Power Spectrum Density (PSD). Because the PSD is two-sided symmetric, only the left-side is considered. The numerical integration of the system (1) has been effectuated for \( n = 3000 \) iterations.

Following Definition 1, the algorithm used to detect numerically hidden and self-excited attractors of the considered system (1) is presented in Fig 2. In systems in spaces with higher dimensions with unstable equilibria, the attraction basins are chosen usually as planar sections containing unstable equilibria. The case of three-dimensional neighborhoods of the Fabrikant-Rabinovich system is treated in [23]. As the diagram shows, the main steps in finding hidden attractors are based on testing if the analyzed attractor has initial conditions within neighborhoods of all unstable equilibria \((X_0^*, X_1^*)\), however small they are. If there exists a neighborhood of at least one of the unstable equilibria containing the initial conditions of the considered attractor, the attractor is self-excited. Otherwise, if the attraction basin does not intersect any of unstable equilibria, the attractor is hidden.

Because the case of stability of both equilibria is trivial (for example the case of chaotic attractors which, in this case, are all hidden by Definition 1), we consider the complicated case of unstable equilibria, when each equilibrium must be analyzed.

The main step in verifying if the attractors are hidden or self-excited, following the algorithm in Fig. 2, is to check neighborhoods of equilibria \( X_{0,1}^* \). Precisely, one has to verify the connection of the attraction basins of the considered attractor with both equilibria. For this purpose we examine the neighborhoods of both equilibria \( X_0^* \) and \( X_1^* \), considered separately for clarity. Fig. 1 shows the lattice of points \((q_1, q_2)\), considered as initial conditions for numerical integration of the IVP (1), containing equilibria \( X_0^* \) and \( X_1^* \), respectively. For \( X_0^* \), which is a line of equilibria, the region containing the equilibrium, is a rectangular neighborhood with the width \( 1e - 5 \) and height taken so that it includes \( q_2^* \). Thus, the neighborhood contains a part of the line equilibria \( X_0^* \) including the critical point \( q_2^* \) (see Theorem 1 ii). Conforming to Theorem 1 ii), the yellow points with \( q_2 > q_2^* \) represent initial conditions leading to the vertical axis, which is attractive, while for \( q_2 < q_2^* \), \( X_0^* \) it is repulsive. For \( X_1^* \), the examined region is a square lattice with side 1 centered on \( X_1^* \). On both neighborhoods, red plots represent the initial conditions leading to attractors of the first set, corresponding to \( IC_1 \), while blue plots are the points leading to attractors belonging to the second coexisting set of attractors corresponding to \( IC_2 \). Black points represent the divergence points, for which the system is unbounded.
3.1. Hidden $\gamma$-attractors

Let the BD of $q_1$ and $q_2$ vs $\gamma$ in Fig. 3 for $\omega = 0.4$, denoted with $BD_\gamma$ (dotted line through $\omega = 0.4$ in Fig. 1). Note that at $\gamma = 0.4$, the diagram crosses the curve $\Gamma_1$ (point $F$ in Fig. 1), marking the first flip bifurcation of $X_1^*$, which at $\gamma = 0.4$ loses its stability.

Due to the symmetry of the $\text{BD}$s with respect to the $\gamma$ and $\omega$ axes, for simplicity, hereafter only the component $q_1$ is considered.

As specified below, attractors belonging to $A_1$ are denoted with $A_1$, while the attractors of $A_2$ are denoted with $A_2$ (red and blue plots, respectively, in Fig. 4 (a) and (b)).

Because the system dynamics related to hidden attractors regard mainly unstable equilibria, we consider for $BD$ the values $\gamma > 0.4$, where the equilibrium $X_1^*$ is unstable (see Fig. 1), and where the coexistence window is denoted $A$ and the magnified window $B$, delimited by $\gamma \in (0.5032, 0.5084)$ and $\gamma \in (0.507292, 0.50741)$, respectively (Fig. 4 (a) and (b)). The window $A$ starts with an exterior crisis at $\gamma = 0.5032$, after which a cascade of flip bifurcations leads to the chaotic behavior of the attractor $A_1$, and ends with an interior crisis at $\gamma = 0.5084$.

Window $B$ starts with an exterior crisis at $\gamma = 0.5068$, which begins a cascade of flip bifurcations for both attractors $A_1, A_2$, the bifurcations of $A_2$ being “delayed” with respect to $\gamma$, compared to the bifurcations of the attractor $A_1$. At $\gamma = 0.508$ the window ends with an interior crisis of the attractor $A_2$.

Consider first $\gamma = \gamma_1 = 0.506$ (Fig. 4 (a)) to which correspond two coexisting attractors, $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. The type of the attractor $A_1$ is, is revealed by the three red points in $BD_\gamma$ (Fig. 4 (a)), $\lambda$, which is negative (the point on the light green curve in Fig. 4 (c)) and $K$ which is 0 (the point on the light magenta curve in Fig. 4 (c)). Moreover, the time series (Fig. 5 (a)) and the phase plot (Fig. 5 (e)) indicate that the attractor $A_1$ is period-3 (the three numbered red points in Fig. 5 (a) and (c)). The attractor $A_2$ presents a so called chaotic band (light blue band in Fig. 4 (a)). $\lambda$ is positive (the point on dark green curve in Fig. 4 (c)) and $K \approx 1$ (the point on the dark magenta curve in Fig. 4 (c)). The time series (Fig. 5 (b)) and the chaotic band appearing in the phase plot as projection of the attractor on $q_1$ axis (Fig. 5 (e)) indicate the chaotic characteristic of $A_2$.

For $\gamma = \gamma_2 = 0.5081$ (Fig. 4 (a)), both attractors, $A_1$ and $A_2$, are chaotic, as shown by the $BD_\gamma$, $\lambda$, $K$ (Figs. 4 (c)), the time series in Figs 5 (d) and (e), and the phase plots in Fig. 5 (f). While the chaotic attractor $A_1$ contains three chaotic bands (purple lines in Fig. 4 (a) and Fig. 5 (a) and (f)), the chaotic attractor $A_2$ is composed by a single chaotic band (light blue in Fig. 4 (a), Fig. 5 (e), and Fig. 5 (f)). Note that the three chaotic bands born from the previous stable three red points in Fig. 5 (a) and (c) which lose stability.

At $\gamma = \gamma_3 = 0.507292$, a value which can be viewed in Fig. 4 (b), the two corresponding attractors $A_1$, and $A_2$, are two periodic attractors. The stable cycle $A_1$ is a period-6 attractor (the apparent cycle point at about $q_1 = 2$, marked with * in the time series (Fig. 5 (g)), is actually a superposition of two points of the cycle: points 2 and 5 in Fig. 5 (i)).
The other stable cycle, $A_2$, is a period-7 attractor. The periods are revealed by the red and blue points, respectively, in the $BD_\gamma$, negative $\lambda$ and zero $K$ (Fig. 4 (d)).

The last considered value, $\gamma = \gamma_4 = 0.50741$ generates the period-6 attractor $A_1$ (the six red points in Fig. 5 (j)) and the six pieces chaotic attractor $A_2$, which presents six light blue chaotic bands (see Fig. 5 (k) and the phase plot in Fig. 5 (l)). The periodic characteristic of $A_1$ is underlined by the negative $\lambda$ (light green, Fig. 4 (d)) and the zero value of $K$ (light magenta, Fig. 4 (d)). Again, at $q_1 = 2$ there are two overplotted points (3 and 6 in Fig. 5 (j) and (l)).

The analysis made in Fig. 6 for $\gamma$-attractors shows that for every considered cases of $\gamma$, the attraction basins of attractors $A_1$ for $\gamma \in \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ (red plot) have no connection with $X_1^1$ or $X_0^*$ and, therefore, they are hidden, while attractors $A_2$, for $\gamma \in \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, have initial conditions which intersect any neighborhood of $X_1^*$ (blue plot), and, therefore, are self-excited.

Note that, for the considered $\gamma$ values, attractors $A_1$ and $A_2$ can be generated, beside $IC_1$ and $IC_2$, from the indicated initial conditions $q_0$ too. Also, the critical value of $q_2^*$ is $q_2^* \approx 4.5$.

To summarize:

1. For $\gamma = \gamma_1$, the period-3 stable cycle $A_1$ is hidden, while the chaotic attractor $A_2$ is self-excited (Fig. 6 (a));
2. For $\gamma = \gamma_2$, the chaotic attractor $A_1$ is hidden, while the chaotic attractor $A_2$ is self-excited (Fig. 6 (b));
3. For $\gamma = \gamma_3$, the period-6 stable cycle $A_1$ is hidden, while the stable period-7 stable cycle $A_2$ is self-excited (Fig. 6 (c));
4. For $\gamma = \gamma_4$, the period-6 stable cycle $A_1$ is hidden, while the chaotic attractor $A_2$ is self-excited (Fig. 6 (d)).

3.2. Hidden $\omega$-attractors

Consider the $BD$ versus $\omega$, $BD_\omega$, for $\omega \in [0.329, 0.791]$ and $\gamma = 0.48$ generated with the same initial conditions, $IC_1$ and $IC_2$ (Fig. 7). Like the $BD_\gamma$, the $BD_\omega$ crosses the stability and instability domains $S$ and $I$ (Fig. 1). Compared to the $BD_\gamma$ which intersects only the flip bifurcation curve, $\Gamma_1$, $BD_\omega$ intersects the NS bifurcation curve, $\Gamma_2$, at points $Q'$ and $Q''$, as well, which is a prerequisite for quasiperiodic oscillations. At $\omega = 0.48325$, $X_1^1$ becomes stable (Fig. 1).

Note that for $\omega \in (0.48325, 0.73855)$, the range of $\omega$ which starts with the last reverse flip bifurcation (point $Q'$) and ends at the first NS bifurcation (Point $Q''$), the system dynamics do not depend on $\omega$, a potentially useful system characteristic. We denote the two coexisting sets of attractors with $B_1$ and $B_2$ with the corresponding elements, attractors $B_1$ and $B_2$, respectively.

The considered coexistence windows starts with successive reversed flip bifurcations (period halving bifurcation) of the attractors $B_1$, while the attractors $B_2$ remain chaotic.
for a large parameter range $\omega \in (0.3608, 0.3681)$. The window ends with an interior crisis of $B_1$.

The windows of interest generated by the parameter $\omega$ are denoted by $C$, and the successive magnified areas $D$ and $E$ (Fig. 7 (a), (b) and (c)).

Within area $C$ and its magnified area $D$, one must consider three representative cases: $\omega_1 = 0.3612$, $\omega_2 = 0.3631$, and $\omega_3 = 0.3669$.

For $\omega = \omega_1$, the values of $\lambda$, and $K$ suggest chaotic dynamics for both underlying attractors, $B_1$ and $B_2$ (Fig. 7 (c)). The time series in Fig. 8 (a) and the phase plot (Fig. 8 (b)) shows that the attractor $B_1$ produces three chaotic bands $V_1$ (magenta plot). The attractor $B_2$ presents a single large chaotic band (see Fig. 8 (b) and the phase plot in Fig. 8 (c), the light blue plot).

For $\omega = \omega_2$, by a reverse flip bifurcation, the former attractor $B_1$, transforms into a stable period-6 cycle (see Fig. 8 (d) and (f) where elements 1 and 4 have the same $q_1$ value), while the attractor $B_2$ remains chaotic but with a reduced size of the underlying chaotic band (Fig. 8 (e) and (f)).

At the last considered value $\omega = \omega_3$, by reverse flip bifurcation both attractors $B_1$ and $B_2$ transform into stable cycles of period-3 and period-14, respectively (Fig. 8 (g), (h), and (i)).

To see which $\omega$-attractors are hidden, one applies the algorithm presented in Fig. 2 for each considered case of $\omega$.

Again, for the considered $\omega$ values, attractors $B_1$ and $B_2$ can be generated, beside $IC_1$ and $IC_2$, from indicated initial conditions $q_0$ in the attraction basins too (Figs. 9). The critical value of $q_2^*$ is $q_2^* \approx 4.9$.

Attractors of $B_1$ are hidden (the red plot corresponding to the ICs of the attractor $B_1$, indicates that the attraction basins do not touch equilibria), while those of $B_2$ are self-excited.

1. For $\omega = \omega_1$, the chaotic attractor $B_1$ is hidden, while the chaotic attractor $B_2$ is self-excited;
2. For $\omega = \omega_2$, the period-6 cycle $B_1$ is hidden, while the chaotic attractor $B_2$ is self-excited;
3. For $\omega = \omega_3$, the period-3 cycle $B_1$ is hidden, while the period-14 cycle $B_2$ is self-excited;

Quasiperiodic attractors

One interesting case of $\omega$-attractors unlike the previous cases involves the coexisting window $E$, defined by $\omega \in [0.7423, 0.75216]$ (Fig. 10 (a)).

Denote the two sets of attractors with $C_1$ and $C_2$ with elements $C_1$ and $C_2$, generated as for all cases from $q_0 = (1.6, 1.4)$.

Consider in this window $\omega = 0.745$ (Fig. 10 (a)). In this case $\lambda$ is negative for $C_1$ and zero for $C_2$, and $K$ is zero for both attractors (Fig. 10 (b)), which indicates that $C_2$ is
quasiperiodic, while $C_1$ has a period-4 cycle. This conclusion is sustained by the four red points in the time series in Fig. 11 (a), the grey quasiperiodic band in Fig. 11 (b), and the phase plot in Fig. 11 (c). Note that the points of the quasiperiodic curve in the phase plot tending to fill the entire closed quasiperiodic orbit (invariant circle) indicates that the orbit neither closes nor repeats itself. The quasiperiodicity is also revealed by the PSD which clearly shows that the periodic orbit ($C_1$) presents the first fundamental frequency $f_0$ and the harmonic $f_1$ situated at distance $\delta_0$ (Fig. 11 (d)). Regarding $C_2$, because of the NS bifurcation, which generally generates quasiperiodicity, a new set of subharmonics are created, like $f_{01}$, close to first frequencies $f_0$ and $f_1$, at smaller distances $\delta_{01}$.

The attractor $C_1$ is hidden, while the quasiperiodic orbit $C_2$ is self-excited (see the attraction basins in Fig. 12 (a)). Compared to previous cases, the attraction basins in this case have a more complicated shape; remember the riddled attraction basins (see e.g. [24]), where any arbitrary neighborhoods of every point of the attraction basin seems to contain points from some other basin (see e.g. circled regions of $X_0^{*}$ neighborhoods).

As can be seen, there exist thin yellow strips of points ($q_1, q_2$) with $q_2 < q_2^*$ for which $X_0^{*}$ is attractive, in contradiction with Theorem 1 ii). For example for the initial condition $q_0 = (0.0005347, 0.4380547)$, with $q_2 = 0.4380547 < 2.42 = q_2^*$, the orbit is attracted by the vertical axis of equilibria $X_0^{*}$ (Fig. 12 (b)). The second component of the orbit tends to a value situated beyond $q_2^*$, where $X_0^{*}$ is stable (see $q_2 \approx 3.5$ in Fig. 12 (b)), but the initial condition belongs to an instability domain established by Theorem 1 i). Also, all considered neighborhoods of $X_0^{*}$ reveal the fact that $q_2^*$ is not constant with respect to $q_1$, but is a function of $q_1$ too, and therefore the graph of $q_2^*$ (separatrix between the yellow and blue domain) is not a constant horizontal line. These apparent contradictions could be related to the local character results of the stability given by Theorem 1.

All results are presented in Table 1.

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4. Conclusions

In this paper hidden and self-excited attractors of a discrete heterogeneous Cournot oligopoly model were numerically found. The system proved to have extremely rich dynamics including attractors’ coexistence. All studied coexistence windows had embedded hidden attractors and self-excited attractors. To identify hidden attractors, the neighborhoods of unstable equilibria were analyzed in order to see if they had connections with the considered attractors. The flowchart of the numerical algorithm utilized to find the hidden attractors of this system can be used for other alike systems. The stability of the equilibria was determined numerically so that it can be used in the study of hidden attractors. Compared with [12], where this system was introduced, our results show supplementarily that the system presents hidden attractors, a desirable or an avoidable characteristic for an economic system.
Figure 1: The explored rectangular area $M_1M_2M_3M_4$ on the parameters plane $(\gamma, \omega)$. The dark green area $M_1, M_2, M_3, M_4$ represents the area where $X_1^*$ is stable. Parameters points $(\gamma, \omega)$ within this area generate counterclockwise orbits spiralling toward the stable equilibrium $X_1^*$. Light green represents the instability domain of $X_1^*$. Curves $\Gamma_1, \Gamma_2$ represent the limits stability. $\Gamma_1$ is the flip bifurcation curve and $\Gamma_2$ the NS bifurcation curve. Horizontal dotted line through $\omega = 0.4$ represents the considered bifurcation line versus $\gamma$, $BD_\omega$. The near red points represent the studied values of $\gamma$ in this bifurcation diagram. The vertical dotted line through $\gamma = 0.48$ represents the bifurcation diagram versus $\omega$, $BD_\gamma$. Blue near points on this line represent the values of $\omega$ studied in this bifurcation diagram. Points $F, Q', Q''$, and $Q$ are points of flip and NS bifurcations, respectively.
Figure 2: Algorithm utilized in this paper to identify hidden attractors.
Figure 3: Bifurcation diagram of the system (1) versus $\gamma, BD_{\gamma}$, for both components $q_{1,2}$, for $\omega = 0.4$. 
Figure 4: Magnified areas of the bifurcation diagram $BD_\gamma$. (a) Magnified area for $\gamma \in [0.5032, 0.5084]$; (b) Magnified area for $\gamma \in [0.5068, 0.508]$; (c) $K$ and $\lambda$ for the magnified area $A$; (d) $K$ and $\lambda$ for the magnified area $B$. 
Figure 5: (a) Time series of the periodic attractor $A_1$ for $\gamma_1 = 0.506$ with initial condition $q_0 = (1.37, 1.32)$; (b) Time series of the chaotic attractor $A_2$ for $\gamma_1 = 0.506$ with initial condition $q_0 = (1.6, 1.4)$; (c) Phase plot of attractors $A_1$ and $A_2$ for $\gamma = \gamma_1$; (d) Time series of the chaotic attractor $A_1$ for $\gamma_2 = 0.5081$ with initial condition $q_0 = (1.865, 1.59)$; (e) Time series for the chaotic attractor $A_2$ for $\gamma_2 = 0.5081$ with initial condition $q_0 = (1.6, 1.4)$; (f) Phase plot of attractors $A_1$ and $A_2$ for $\gamma = \gamma_2$. 
Figure 5: Continuation: (g) Time series of the periodic attractor $A_1$ for $\gamma_3 = 0.507292$ with initial condition $q_0 = (1.98125, 1.3625)$; (h) Time series of the periodic attractor $A_2$ for $\gamma_4 = 0.507292$; (i) Phase plot of attractors $A_1$ and $A_2$ for $\gamma = \gamma_3$; (j) Time series of the periodic attractor $A_1$ for $\gamma_4 = 0.50741$ with initial condition $q_0 = (1.02875, 1.53125)$; (k) Time series of the chaotic attractor $A_2$ for $\gamma_4 = 0.50741$; (l) Phase plot of attractors $A_1$ and $A_2$ for $\gamma = \gamma_4$. 
Figure 6: Attraction basins of attractors $A_1$ and $A_2$ for the considered four $\gamma$ values, considered around both equilibria $X^*_0, 1$. The red area represents initial conditions of the attractor $A_1$, while blue parts represent the attraction basin of $A_2$. Yellow points are attracted by $X^*_0$ which, for $q_2 > q^*_2$ is attractive. Points in the black area tend to infinity. (a) $\gamma = \gamma_1$; (b) $\gamma = \gamma_2$; (c) $\gamma = \gamma_3$; (d) $\gamma = \gamma_4$. 
Figure 7: (a) Bifurcation diagram of $q_1$ versus $\omega$, $BD_\omega$ for $\gamma = 0.48$; (b) Magnified area $C$ of the $BD_\omega$ for $\omega \in [0.3608, 0.3681]$; (c) Magnified area of $C$; (d) $K$ and $\lambda$ of the area $C$. 
Figure 8: (a) Time series for the chaotic attractor $B_1$ for $\omega_1 = 0.3612$ with initial condition $q_0 = (1.2, 1.44)$; (b) Time series of the chaotic attractor $B_2$ for $\omega_1 = 0.3612$; (c) Phase plot of attractors $B_1$ and $B_2$ for $\omega = \omega_1$; (d) Time series of the periodic attractor $B_1$ for $\omega_2 = 0.3631$ with initial condition $q_0 = (1.9, 1.51)$; (e) Time series of the chaotic attractor $B_2$ for $\omega_2 = 0.3631$; (f) Phase plot of attractors $B_1$ and $B_2$ for $\omega = \omega_2$; (g) Time series of the periodic attractor $B_1$ for $\omega_3 = 0.3669$ with initial condition $q_0 = (1.3, 1.35)$; (h) Time series of the chaotic attractor $B_2$ for $\omega = \omega_3$ with initial condition $q_0 = (1.6, 1.4)$; (i) Phase plot of attractors $B_1$ and $B_2$ for $\omega = \omega_3$. 
Figure 9: Attraction basins of attractors $B_1$ and $B_2$ for the considered three $\omega$ values, considered around both equilibria $X_{0,1}$. The red area represents parts of the attraction basin of the attractor $B_1$, while blue parts represent the attraction basin of $B_2$. Yellow points are attracted by $X_0^*$ which, for $\omega_2 > \omega_2^*$ is attractive. Points in the black area tend to infinity. (a) $\omega_1 = \omega_1^*$; (b) $\omega = \omega_2^*$; (c) $\omega = \omega_3$. 
Figure 10: (a) Magnified area $E$ of the $BD_\omega$ for $\omega \in [0.7423, 0.75216]$; (b) $K$ and $\lambda$. 

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Figure 11: (a) Time series of the periodic attractor $C_1$ for $\omega_4 = 0.745$; (b) Time series of the quasiperiodic attractor $C_2$; (c) Phase plot of the attractors $C_1$ and $C_2$; (d) PSD of the component $q_1$ of the attractor $C_1$; (e) PSD of the component $q_1$ of the attractor $C_2$. 
Figure 12: (a) Attraction basin of attractors $C_1$ and $C_2$ for $\omega_4 = 0.745$, considered around both equilibria $X_{0,1}$. The red area represents parts of the attraction basin of the attractor $C_1$, while blue parts of the attraction basin of $C_2$. Yellow points are attracted by $X_0^*$ which, for $q_2 > q_2^*$ is attractive. Points in the black area tend to infinity. Circled regions reveal the complicated shape of attraction basins; (b) The orbit from $q_0 = (0.0005347, 0.4380547)$ is attracted by the equilibria line $X_0^*$, even $q_0$ is within unstable area, as defined by Theorem 1 (for clarity only the first 100 iterations are shown).
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