Energy Density Fluctuations in Inflationary Cosmology

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Abstract
We analyze the energy density fluctuations contributed by scalar fields Φ with vanishing expectation values, ⟨Φ⟩ = 0, which are present in addition to the inflaton field. For simplicity we take Φ to be non–interacting and minimally coupled to gravity. We use normal ordering to define the renormalized energy density operator ρ, and we show that any normal ordering gives the same result for correlation functions of ρ. We first consider massless fields and derive the energy fluctuations in a single mode $\vec{k}$, the two–point correlation function of the energy density, the power spectrum, and the variance of the smeared energy density, $\langle \delta^2_R \rangle$. Mass effects are investigated for energy fluctuations in single modes. All quantities considered are scale invariant at the second horizon crossing (Harrison–Zel’dovich type) for massless and for unstable massive fields. The magnitude of the relative fluctuations $\delta \rho / \rho_{\text{tot}}$ is of order $(H_1 / M_{\text{Pl}})^2$ in the massless case, where $H_1$ is the Hubble constant during inflation. For an unstable field of mass $m_\Phi \ll H_1$ with a decay rate $\Gamma_\Phi$ the magnitude is enhanced by a factor $\sqrt{m_\Phi / \Gamma_\Phi}$. Finally, the prediction for the cosmic variance of the average energy density in a sample is given in the massless case.

PACS numbers: 04.62.+v, 98.80.Cq, 98.65.Dx
1. Introduction and Conclusions

In the early universe gravity acting on a quantum matter field [1] amplifies energy density fluctuations above the unavoidable vacuum fluctuations [2,3]. These primordial fluctuations may serve as seeds for the subsequent formation of the observable large-scale structure [4].

In the standard literature [5] the fluctuations in the energy density of the inflaton field, which drives inflation through the energy density of its vacuum expectation value, were thoroughly analyzed.

In a recent paper [6] we have started to investigate the energy density fluctuations contributed by scalar fields Φ with vanishing expectation values, $\langle \Phi \rangle = 0$, which are present during inflation in addition to the inflaton field. For simplicity we take Φ to be non–interacting and minimally coupled to gravity. The emphasis of the first paper has been on the quantum field theoretical concepts and methods. (They are briefly reviewed in the next section.) The present paper contains the application to cosmology. We evolve the energy density fluctuations through the radiation era up to the time of matter and radiation equality. A related investigation for the case of classical cosmological perturbations can be found in ref. [7].

The energy density $\rho$ of the (non–inflaton) field Φ is bilinear in the fluctuating field, and $\rho$ is therefore not a Gaussian variable. When Φ is non–interacting and in external gravity, Φ is a Gaussian variable and $\rho$ is $\chi^2$–distributed. This is in contrast to the Gaussian energy density fluctuations for the inflaton field, which arise from the interference between the background field and the fluctuating part of the inflaton. Various observable measures of non–Gaussianity in our model are discussed in ref. [8].

We use normal ordering to define the renormalized energy density operator, and we show that any normal ordering prescription gives the same results for correlation functions of $\rho$. This is so because the difference of two normal orderings is a c–number, which drops out in the correlations.

The cosmological model is given as an inflationary universe, represented by de Sitter space with Hubble constant $H_I$, followed by a radiation dominated universe. The total energy density present is always critical, $\rho_{\text{tot}} = \rho_{\text{crit}}$. The transition between the two eras can be approximated as instantaneous, because the physical transition time is much shorter than the characteristic time for the evolution of the cosmologically relevant modes, which have $\lambda_{\text{phys}} > e^{70} H_I^{-1}$. The scalar field Φ is assumed to be non–dominant, therefore it evolves in the background curved space–time. Back reaction of Φ on the geometry is neglected, and gauge ambiguities are eliminated. We consider massless and massive fields, the latter either stable or decaying into radiation. In this paper we have not put in a decoherence rate, i.e. the fields Φ stay coherent until they eventually decay. With respect to decoherence properties the situation is similar to axion models (axions, however, have a non–zero expectation value) [9]. Since we have not yet included the transfer of the fluctuations from the resulting massless fields to matter which is dominating today (cold dark matter and baryons), we give our predictions at
the end of the radiation era.

The state of the quantum field $\Phi$ is the Bunch–Davies state initially, during inflation: Every observationally relevant mode, which has today $(\lambda_{\text{phys}})_{0} < H_{0}^{-1}$, had at early times in inflation $R/k_{\text{phys}}^{2} \to 0$ and is taken to be initially in the Minkowski vacuum state. A physical discussion of this state can be found in ref. [6].

In section 3 we analyze the contributions of massless fields. We present several quantities which characterize the energy density fluctuations: (1) First we derive the energy fluctuations $\delta E_{\vec{k}}$ in a single mode $\vec{k}$. As long as the mode is super–horizon, i.e. between the first and second horizon crossing, $\delta E_{\vec{k}}$ grows linearly with the scale factor $a$. This law for massless fields is independent of the cosmological era. After the second horizon crossing $\delta E_{\vec{k}} \sim a^{-1}$. (2) Next the equal time two–point correlation function $\xi(\ell) = \langle \delta(\vec{x})\delta(\vec{x'}) \rangle$ of the energy density contrast $\delta(\vec{x})$ is investigated. At the second horizon crossing $\xi(\ell)$ is independent of the point separation $\ell$, i.e. it is of Harrison–Zel’dovich type. On super–horizon scales, for $\ell \gg H^{-1}$ where $H$ is the Hubble parameter, the correlation function falls off as $\ell^{-4}$. For $\ell \ll H^{-1}$ we find a logarithmic increase as $(\log H\ell)^{2}$ towards smaller scales. $\xi(\ell)$ has a cusp at $\ell = 2H^{-1}$, which gives extra strength at large scales. (3) The variance $\langle \delta_{R}^{2} \rangle$ of the smeared energy density contrast $\delta_{R}$ is also scale independent at the second horizon crossing. It grows as $(\log H R)^{2}$ towards smaller scales for $R \ll H$, and it decreases as $R^{-3}$ on scales $R \gg H$. (4) The power spectrum $P(q)$, i.e. the Fourier transform of $\xi(\ell)$, is such that $q^{3}P(q)$ is scale independent at the second horizon crossing, which is again the characteristics of Harrison–Zel’dovich type fluctuations. For $q \gg H$ the quantity $q^{3}P(q)$ is $q$–independent, while on super–horizon scales ($q \ll H$) the power spectrum $P(q)$ itself tends to a constant. (5) The correlations between the pressure $P$ and energy density $\rho$ are computed during the inflationary era. On super–horizon scales (where they are independent of the cosmological era) they are related as $\delta P = -\frac{1}{3}\delta\rho$. Using the covariant conservation of the energy–momentum tensor this again gives the $a^{2}$–growth of $\sqrt{\xi(\ell)}$ for fixed comoving length.

The magnitude at the second horizon crossing for the relative energy density fluctuations $\delta\rho/\rho$ is of order $(H_{1}/M_{P1})^{2}$. Let us assume that during inflation there were $N$ massless scalar fields fields present and take for simplicity $N = 100$ (in the minimal supersymmetric standard model we have $N = 44$). In order to obtain an amplitude of order $10^{-5}$, we need $H_{1}/M_{P1} \approx 10^{-3}$, i.e. $H_{1} \approx 10^{16} \text{ GeV}$ is required. For the vacuum energy density during inflation, $\rho_{1} := 3H^{2}_{1}M^{2}_{P1}/8\pi$, this means $\rho_{1}^{1/4} \approx 2 \cdot 10^{17} \text{ GeV}$. This is in conflict with the upper bound of $\rho_{1}^{1/4} \lesssim 5.2 \cdot 10^{16} \text{ GeV}$ which was derived in [10] from an estimate of the distortions in the cosmic microwave background radiation caused by inflationary gravitational waves.

Mass effects for $m \ll H_{1}$ are analyzed in section 4 for the energy fluctuations $\delta E_{\vec{k}}$ in a single mode $\vec{k}$. After the super–horizon mode has become non–relativistic, $k_{\text{phys}} \ll m$, we find a growth of $\delta E_{\vec{k}} \sim a^{3}$, i.e. it is enhanced over the $\delta E_{\vec{k}} \sim a$ law for massless modes. The growth of $\delta E_{\vec{k}}$ stops when $H$ drops below $m$, i.e. when the Compton wavelength $\lambda_{C} = 2\pi/m$ becomes sub–horizon. If the massive field is stable on the scale
of the Hubble time $H_{EQ}$, $\delta E_k$ is enhanced over the contributions from massless fields by a $k$–dependent factor at the second horizon crossing. On cosmologically relevant scales the enhancement factor is $10^{12}$ or more. To compensate one must take a lower $H_I$. For an unstable massive field which decays with a rate $\Gamma$ into radiation the magnitude of $\delta E_k$ is only moderately enhanced by a factor $\sqrt{m/\Gamma}$, independent of the wave number $k$.

The topic of section 5 is the cosmic variance of $\bar{\rho}$. In a finite sample $S$, i.e. a finite patch of the universe, the spatial average of the density $\bar{\rho}_S = E_S/V_S$ is one measurable number. In a quantum field theoretic model one can predict the quantum fluctuations of the operator $E_S$, i.e. the variance $(\delta E_S)^2 = \langle E_S^2 \rangle - \langle E_S \rangle^2$, where $\langle \ldots \rangle$ indicates the (quantum theoretical) ensemble average. This variance within the ensemble is identified with the variance which would be observed when measuring $E_S$ in many different patches (of size $S$) within one universe. For a patch size $\ell_S \approx 30 \, h^{-1} \, \text{Mpc}$ there are many patches today, and such a variance is accessible to observations and can be compared to models. For $\ell_S \approx 200 \, h^{-1} \, \text{Mpc}$ $\bar{\rho}_S$ is (in principle) one measurable number today, but one might be able to measure its variance in a decade or so when observations reach much deeper than $200 \, h^{-1} \, \text{Mpc}$. For $\ell_S \approx H_0^{-1}$ observations can give one single number for $\bar{\rho}_S$, but its variance can be inferred from COBE measurements under the assumption of adiabatic fluctuations together with a modest extrapolation from $\ell = 2$ multipoles to $\ell = 0$. This gives the cosmic variance of $\delta \bar{\rho}_S/\rho \simeq 10^{-5}$ for $\ell_S \approx H_0^{-1}$. Our model gives a contribution to $\delta \bar{\rho}_S/\rho_{tot}$ which drops off as $1/\sqrt{V_S}$ for large sample volumes $V_S$. Finally we relate the experimentally accessible correlation function for $(\rho(x) - \bar{\rho}_S)$, which by definition must have at least one zero, to the quantum field theoretic correlation function for $(\rho(x) - \langle \rho \rangle)$, which in our model turns out to be strictly positive. On small separation scales, $\ell \ll \ell_S$, the difference between the two correlation functions is $\ell$–independent and equal to the cosmic variance.

2. Model and Methods

The background is given by a Friedmann–Robertson–Walker (FRW) space–time with spatially flat sections, i.e. we take the mean energy density in the universe to be critical, $\rho_{tot} = \rho_{crit}$. Using conformal time $\eta$, we have $ds^2 = a(\eta)^2 (d\eta^2 - \vec{dx}^2)$. The Hubble constant during inflation is denoted by $H_I$. We fix the coordinates in such a way that at the time $\eta_1$ of the transition from the inflationary to the radiation dominated stage we have $\eta_1 = H_I^{-1}$ and $a(\eta_1) = 1$. This fixes the scale factor $a(\eta)$ for our cosmological model as

$$a(\eta) = \begin{cases} \frac{1}{(2 - H_I \eta)} & (\eta \leq H_I^{-1}) \\ H_I \eta & (\eta \geq H_I^{-1}) \end{cases}$$

(2.1)

The Hubble parameter $H = a^{-2} \partial_\eta a$ at time $\eta$ during the radiation era reads $H = a^{-2} H_I$. The cosmological model (2.1) is determined by one single model parameter, the Hubble
constant $H_I$ during inflation.

We investigate the energy fluctuations contributed by a neutral scalar quantum field $\Phi$ with minimal coupling to gravity. $\Phi$ has no self–interactions and zero expectation value, $\langle \Phi \rangle = 0$. Its action reads

$$S_\Phi = \frac{1}{2} \int d^4x \sqrt{-g} \left( \partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 \right). \quad (2.2)$$

Variation of the action $S_\Phi$ with respect to the field $\Phi$ gives the equation of motion (Klein–Gordon equation)

$$\left[ \frac{d^2}{d\eta^2} + \frac{2}{a} \frac{d}{d\eta} - \frac{d^2}{d\tilde{x}^2} + a^2 m^2 \right] \Phi = 0. \quad (2.3)$$

The energy density $\rho$ is always defined to be the one measured by a comoving observer, who has the 4–velocity $u^\mu = (a^{-1}, 0, 0, 0)$ in conformal coordinates. Thus we obtain

$$\rho = \frac{1}{2a^2} \left[ (\partial_\eta \Phi)^2 + (\partial_{\tilde{x}} \Phi)^2 + a^2 m^2 \Phi^2 \right]. \quad (2.4)$$

The energy–energy correlation at equal times is defined by

$$C(\tilde{x}, \tilde{x}') := \langle \Omega | \rho(\tilde{x}) \rho(\tilde{x}') | \Omega \rangle - \langle \Omega | \rho(\tilde{x}) | \Omega \rangle \langle \Omega | \rho(\tilde{x}') | \Omega \rangle. \quad (2.5)$$

The quantum state $|\Omega\rangle$ is initially (during inflation) the Bunch–Davies state: Every observationally relevant mode, which has today ($\lambda_{\text{phys}} \ll H_0^{-1}$, had at early times in inflation $R/k_{\text{phys}}^2 \to 0$ and is taken to be initially in the Minkowski vacuum state. Since $|\Omega\rangle$ is translation invariant, $C$ depends only on the physical distance $\ell = a |\tilde{x} - \tilde{x}'|$ of the two points.

We use normal ordering to define the renormalized energy density operator $\rho(x) := N[\rho(x)]$. The key observation [6] is that different choices for the normal ordering prescription give identical results for the energy autocorrelation function $C(\tilde{x}, \tilde{x}')$. This is so because two normal orderings $N[\rho]$ differ by a c–number, which drops out in the connected two–point function $C(\tilde{x}, \tilde{x}')$.

The mode functions in a spatially flat FRW space–time are eigenfunctions of the comoving wave vector $\vec{k}$. Because of the translational invariance of the gravitational field in 3–space different $\vec{k}$'s decouple, and $\vec{k}$ is a conserved quantity. We make the ansatz for the modes of fixed $\vec{k}$

$$\varphi_k(\eta, \tilde{x}) =: u_k(\eta) e^{i\vec{k}\tilde{x}}. \quad (2.6)$$

The modes $u_k(\eta)$ are normalized solutions of the dynamics throughout time, i.e. they are solutions of the Klein–Gordon equation (2.3) in both eras of our cosmological model, and $u_k$ and $\partial_\eta u_k$ are continuous at the transition. We call $u_{k,\text{in}}$ those modes which have
the time dependence $e^{-ik\eta}$ at early times, as $k\eta \to -\infty$ in inflation. The $u_{k,\text{in}}(\eta)$ are those evolving modes which approached Minkowski single particle waves at early times (when their physical wavelengths were much smaller than the Hubble radius).

Since the results are independent of the normal ordering, we can make the most convenient choice, the one adapted to the quantum state $|\Omega\rangle$. We use the mode expansion of the field operator $\Phi(\eta, \vec{x})$ as

$$\Phi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[ a_{k,\text{in}} u_{k,\text{in}}(\eta) e^{i\vec{k}\cdot\vec{x}} + a_{k,\text{in}}^\dagger u_{k,\text{in}} (\eta)^* e^{-i\vec{k}\cdot\vec{x}} \right]. \quad (2.7)$$

The state $|\Omega\rangle$ of the quantum field is annihilated by all the operators $a_{k,\text{in}}$,

$$a_{k,\text{in}} |\Omega\rangle = 0, \quad \text{for all } k. \quad (2.8)$$

A key tool of ref. [6] was that the computation of the correlation function $C(\vec{x}, \vec{x}')$ in a given quantum state $|\Omega\rangle$ can be reduced to the computation of the Wightman function of this state,

$$W(x, x') := \langle \Omega | \Phi(x) \Phi(x') | \Omega \rangle = \int \frac{d^3k}{(2\pi)^3} u_{k,\text{in}}(\eta) u_{k,\text{in}}(\eta')^* e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}.$$

In the massless case, the two–point correlation function $C(\vec{x}, \vec{x}')$ is given as

$$C(\ell) = \frac{1}{2^{d+4}} \sum_{\alpha, \beta = 0}^3 \left[ W(x, x')_{,\alpha\beta} \right]^2 \bigg|_{\eta = \eta'}, \quad (2.10)$$

where primed derivatives act exclusively on the primed coordinates.

The inflationary era was treated in ref. [6]. In particular, for $m = 0$ the modes $u_{k,\text{in}}$ have the functional form

$$u_{k,\text{in}}(\eta) = \frac{H_I}{\sqrt{2k^3}} (i - k\tilde{\eta}) e^{-ik(\tilde{\eta} + \eta)} \quad (m = 0, \eta \leq H_I^{-1}), \quad (2.11)$$

where $\tilde{\eta} = \eta - 2H_I^{-1}$, i.e. they have the time dependence of a Hankel function, $u_{k,\text{in}} \sim \tilde{\eta}^{3/2} H_{3/2}^{(2)}(k\tilde{\eta})$. The two–point correlation function $C$ was found as

$$C(\ell) = \frac{1}{(2\pi)^4} \left[ \frac{24}{\ell^8} + \frac{14H_I^2}{\ell^6} + \frac{3H_I^4}{2\ell^4} \right] \quad (m = 0, \eta \leq H_I^{-1}). \quad (2.12)$$

The energy density contrast contributed by the scalar field under consideration is

$$\delta(x) = \frac{1}{\rho_{\text{tot}}} \left( \rho(x) - \langle \rho \rangle \right), \quad (2.13)$$
where \( \rho_{\text{tot}} = 3H^2 M_{\text{Pl}}^2 /8\pi \). The two–point function of \( \delta(x) \),

\[
\xi(\ell) = \langle \Omega | \delta(\vec{x})\delta(\vec{x}') | \Omega \rangle,
\]

is a measure of the relative energy density fluctuations. The two–point functions (2.5) and (2.14) are related as \( \xi(\ell) = C(\ell)/\rho_{\text{tot}}^2 \).

3. The Energy Fluctuations for Massless Fields

3.1. The evolved modes

After the transition to the radiation dominated era, at times \( \eta \geq \eta_1 = H_i^{-1} \), the Klein–Gordon equation (2.3) reads

\[
\left[ \frac{d^2}{d\eta^2} + \frac{2}{\eta} \frac{d}{d\eta} + k^2 + m^2 H_i^2 \eta^2 \right] u_k(\eta) = 0. \tag{3.1}
\]

We choose as fundamental solutions for \( m = 0 \)

\[
v_k(\eta) = \frac{1}{\sqrt{2k H_i \eta}} e^{-ik(\eta-\eta_1)} \tag{3.2}
\]

and its complex conjugate \( v_k^* \). They have the time dependence \( \eta^{-1/2} H_i^{1/2}(k\eta) \). The mode functions \( u_{k,\text{in}} \), which correspond to the initial Bunch–Davies state, are linear combinations of the two fundamental solutions \( v_k \) and \( v_k^* \),

\[
u_{k,\text{in}}(\eta) = \alpha_k v_k(\eta) + \beta_k v_k(\eta)^* \tag{3.3}\]

The \( \alpha_k \) and \( \beta_k \) in eq. (3.3) are called Bogolubov coefficients. They are fixed by requiring a continuous transition of the mode functions \( u_{k,\text{in}}(\eta) \), given by eqs. (2.11) and (3.3), and their first derivative at time \( \eta = \eta_1 = H_i^{-1} \). We obtain

\[
\alpha_k = 1 + \frac{iH_i}{k} - \frac{H_i^2}{2k^2} \quad \text{and} \quad \beta_k = \frac{H_i^2}{2k^2}. \tag{3.4}
\]

This satisfies the general condition for Bogolubov coefficients \( |\alpha_k|^2 - |\beta_k|^2 = 1 \). We get the functional form of the modes \( u_{k,\text{in}} \) during the radiation era as

\[
u_{k,\text{in}}(\eta) = \frac{1}{\sqrt{2k H_i \eta}} \left[ \left( 1 + \frac{iH_i}{k} \right) e^{-ik(\eta-\eta_1)} + i\frac{H_i^2}{k^2} \sin k(\eta-\eta_1) \right]. \tag{3.5}\]

There are two types of modes, those which never went beyond the Hubble radius, \( i.e. \lambda_{\text{phys}} < H^{-1} \) at all times, and those modes which spent some time with \( \lambda_{\text{phys}} > H^{-1} \). The separation point is \( k = H_i \). From now on we look exclusively at the cosmologically
relevant modes, which spent a long time outside the Hubble radius, i.e. at modes with $k \ll H_1$,

$$u_{k,\text{in}}(\eta) = \frac{i}{\sqrt{2k}} \frac{H_1 \sin k\eta}{k\eta} \quad \text{(for } k \ll H_1\text{).} \tag{3.6}$$

Since $k\eta = k_{\text{phys}}/H$ we see that well before the second horizon crossing, $k_{\text{phys}} \ll H$, $u_{k,\text{in}}(\eta)$ is time independent. After the second horizon crossing it oscillates in time.

### 3.2. Energy fluctuations in one mode

We start our investigation of the energy fluctuations by looking at one single mode $\vec{k}$. The mode decomposition (Fourier transform) will be defined in a comoving cube $L^3$ with periodic boundary conditions. We compute the fluctuations of the total measured energy $E_{\vec{k}}$ within the periodicity cube, contributed by the given mode $\vec{k}$,

$$\langle \delta E_{\vec{k}} \rangle^2 := \langle \Omega | E_{\vec{k}}^2 | \Omega \rangle - \langle \Omega | E_{\vec{k}} | \Omega \rangle^2. \tag{3.7}$$

The energy of a single–particle quantum measured at a given time is

$$e_{\vec{k}} = \sqrt{k_{\text{phys}}^2 + m^2}, \tag{3.8}$$

and the fluctuations of the number of $\vec{k}$–quanta at a given time are $\delta n_{\vec{k}} = \delta E_{\vec{k}} / e_{\vec{k}}$.

The operator $E_V$ of the total energy in the physical volume $V = (aL)^3$ relates the energies $E_{\vec{k}}$ to the energy density $\rho$, eq. (2.4),

$$E_V = a^3 \int L^3 d^3x \rho(x) = \sum_{\vec{k}} E_{\vec{k}}. \tag{3.9}$$

Using the discrete version of the mode decomposition (2.7) of the field operator $\Phi(x)$ we find the operator $E_{\vec{k}}$ in terms of bilinears of the annihilation and creation operators $a_{k,\text{in}}$ and $a_{k,\text{in}}^\dagger$, accompanied by $u_{k,\text{in}}$ and $u_{k,\text{in}}^*$ with their derivatives. The fluctuations of $E_{\vec{k}}$ are given by $|\langle \Omega | E_{\vec{k}}^* | \vec{k}, -\vec{k} \rangle|^2$, and we obtain

$$\langle \delta E_{\vec{k}} \rangle^2 = \frac{a^6}{4} \left[ \frac{(W_{k,\eta^\prime})^2}{a^4} + 2 \frac{\text{Re}(W_{k,\eta})^2}{a^2} + (W_k)^2 e_k^2 + (W_k)^2 e_k^4 \right] \bigg|_{\eta = \eta^\prime}, \tag{3.10a}$$

where

$$W_k := u_{k,\text{in}}(\eta) u_{k,\text{in}}(\eta^\prime)^* \tag{3.10b}$$

is Fourier transform of the Wightman function (2.9).

During inflation the mode functions $u_{k,\text{in}}$ are given by eq. (2.11), and we obtain

$$\delta E_{\vec{k}} = \frac{H_I^2}{4k_{\text{phys}}} \sqrt{1 + 4 \frac{k_{\text{phys}}^2}{H_I^2}}. \tag{3.11}$$
In the radiation era the mode functions \( u_{k,\text{in}} \) are given by eq. (3.6), and we obtain the energy fluctuations

\[
\delta E_k = \frac{H_I^2 \mathcal{H}^2}{4k_{\text{phys}}^3} \left( 1 - \frac{\sin 2x}{x} + \frac{\sin^2 x}{x^2} \right), \quad x = \frac{k_{\text{phys}}}{\mathcal{H}} = k\eta. \tag{3.12}
\]

For very small and very large wave numbers compared to the Hubble parameter this simplifies to

\[
\delta E_k \simeq \begin{cases} 
\frac{H_I^2}{4k_{\text{phys}}} & \text{(for } k_{\text{phys}} \ll \mathcal{H} \text{)} \\
\frac{H_I^2 \mathcal{H}^2}{4k_{\text{phys}}^3} & \text{(for } \mathcal{H} \ll k_{\text{phys}} \ll \frac{H_I}{a} \text{)}.
\end{cases} \tag{3.13}
\]

Let us summarize the time evolution of \( \delta E_k \) for a fixed comoving wave number \( k \ll H_1 \), see fig. 3.1: (1) Before the first horizon crossing early in the inflationary era, \( \delta E_k \) is constant, and \( \delta n_k \to 0 \) for \( k_{\text{phys}} \gg H_1 \). (2) At the first horizon crossing \( \delta E_k = \mathcal{O}(H_1) \) and \( \delta n_k = \mathcal{O}(1) \). (3) Between the first and the second horizon crossing, when \( k_{\text{phys}} \ll H_1 \) and \( k_{\text{phys}} \ll \mathcal{H} \), the fluctuations \( \delta E_k \) grow in time linearly with the scale factor \( a \), and \( \delta n_k \) grows as \( a^2 \). This law remains valid as long as the mode is super–horizon sized, independent of the cosmological era. (4) After the second horizon crossing, for \( k_{\text{phys}} \gg \mathcal{H} \) during the radiation dominated era, the fluctuations \( \delta E_k \) decrease in time \( \sim a^{-1} \). The fluctuations \( \delta n_k \) do not evolve in time any more. The total growth factor for \( \delta n_k \) between the first and second horizon crossing is \( H^4_1/k^4 \), a factor \( H^2_1/k^2 \) between the first horizon crossing and the end of inflation, another factor \( H^2_1/k^2 \) during the radiation era up to the second horizon crossing.

At fixed time in the radiation era we find \( \delta E_k \sim k^{-3} \) for sub–horizon modes, \( k_{\text{phys}} \gg \mathcal{H} \). For super–horizon modes, with \( k_{\text{phys}} \ll \mathcal{H} \), we have \( \delta E_k \sim k^{-1} \). The two asymptotes meet at \( k_{\text{phys}} = \mathcal{H} \).

### 3.3. Two–point correlation function

We now sum up the contributions of all the \( \vec{k} \)–modes to the energy density fluctuations. To do this we approximate the mode functions \( u_{k,\text{in}}(\eta) \) again by eq. (3.6) alone. Thus we will obtain an approximation for \( \xi(\ell) \) which is valid only on length scales \( \ell \gg aH_1^{-1} \). But it will fail for the short–distance behaviour of \( \xi(\ell) \), on scales \( \ell \) which never went beyond the Hubble radius. However, the short–distance behaviour of \( \xi(\ell) \) is universal, independent of quantum state and external curved space–time. This term was discussed in ref. [6], and it can be read off eq. (2.12).

The integral (2.9) for \( W(x, x') \) is logarithmically divergent in the infrared, but the divergent term is independent of the coordinates of the two points \( x \) and \( x' \) and therefore drops out when computing \( \xi(\ell) \). We insert (3.6) into (2.9) and obtain for the infrared finite terms of the Wightman function

\[
W(x, x') = \frac{H_I^2}{(2\pi)^2} \frac{1}{24\eta \sqrt{\eta'}} \left[ \Delta^3_+ \log |H_1 \Delta_{+-}| + \Delta^3_- \log |H_1 \Delta_{-+}| \\
- \Delta^3_- \log |H_1 \Delta_{-+}| - \Delta^3_+ \log |H_1 \Delta_{++}| \right]. \tag{3.14}
\]
We have used the notation $\Delta x := |\vec{x} - \vec{x}'|$ and $\Delta_{\pm \pm} := \Delta x \pm \eta \pm \eta'$.

First we investigate the correlation function $\xi(\ell)$ on sub–horizon scales, $aH^{-1}_I \ll \ell \ll H^{-1}$. We obtain

$$\xi(\ell) \simeq \frac{1}{6} \left( \frac{2H^2}{3\pi M_{Pl}^2} \right)^2 \left( \log \frac{\mathcal{H}\ell}{2} \right)^2 \left( aH^{-1}_I \ll \ell \ll H^{-1} \right), \quad (3.15)$$

a plateau–like structure on scales well inside the Hubble radius. Here $\xi(\ell)$ increases logarithmically towards the smaller scales. On super–horizon scales, $\mathcal{H}\ell \gg 1$, we find

$$\xi(\ell) \simeq \frac{3}{2} \left( \frac{2H^2}{3\pi M_{Pl}^2} \right)^2 \frac{1}{(\mathcal{H}\ell)^4} \left( \ell \gg H^{-1} \right), \quad (3.16)$$

a power law decay. In the range of intermediate length scales $\ell \approx H^{-1}$ we need the exact expression for the correlation function $\xi(\ell)$,

$$\xi(\ell) = \left( \frac{2H^2}{3\pi M_{Pl}^2} \right)^2 \left[ \left( \frac{1}{48y^4} + \frac{1}{144y^2} + \frac{1}{72} \right) + \mathcal{L}_1 \left( -\frac{1}{48y^6} + \frac{1}{27y^3} - \frac{5}{144y} \right) \right.
\left. + \mathcal{L}_2 \left( \frac{5}{72} - \frac{y^2}{36} \right) + \mathcal{L}_1^2 \left( \frac{1}{192y^6} - \frac{5}{144y^4} + \frac{17}{576y^2} + \frac{1}{64} \right) \right.
\left. + \mathcal{L}_1\mathcal{L}_2 \left( -\frac{1}{18y} - \frac{y}{36} \right) + \mathcal{L}_2^2 \left( \frac{1}{24} - \frac{y^2}{72} + \frac{y^4}{72} \right) \right]. \quad (3.17)$$

We have used the dimensionless variable $y = \frac{1}{2}\mathcal{H}\ell$, further $\mathcal{L}_1 := \log |(y + 1)/(y - 1)|$ and $\mathcal{L}_2 := \log |y^2/(y^2 - 1)|$. The function $\xi(\ell)$ is strictly positive for all $\ell$, see fig. 3.2.

The correlation function $\xi(\ell)$ is finite for all $\ell > 0$, but it is not differentiable at the point $\ell = 2\mathcal{H}^{-1}$. This can already be seen in the Wightman function $W(x, x')$, eq. (3.14). The term involving $\Delta_{-\pm}$ has a pole in its third derivatives. This cusp singularity has its origin in the instantaneous transition from inflation to the radiation era. We can mimic a finite duration by introducing an exponential high frequency cut–off $e^{-kT}$ into the Bogolubov coefficient $\beta_k$ of eq. (3.4), where $T$ is the fastest comoving time scale involved in the phase transition (see the discussion in ref. [11]). However, since in any case the time scale $T$ is at most one or two orders of magnitude away from the Hubble time during inflation $H^{-1}_I$, the smoothing of the cusp occurs on an extremely small scale at the time of matter and radiation equality. The effect is invisible on the plot.

The correlation function $\xi(\ell)$ depends on $\ell$ only via $\mathcal{H}\ell$, i.e. it is time independent if evaluated at fixed $\ell/\ell_\mathcal{H}$, where $\ell_\mathcal{H}$ is the Hubble radius. At the second horizon crossing, when $\mathcal{H}\ell = \text{fixed} = \mathcal{O}(1)$, the correlation is $\ell$–independent and therefore of Harrison–Zel’dovich type. The magnitude is of order $(H_1/M_{Pl})^4$. For instance, we obtain $\sqrt{\xi(\ell = \mathcal{H}^{-1})} = 0.098(H_1/M_{Pl})^2$. 

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3.4. Variance of the smeared energy density contrast

We introduce the spatially smeared energy density contrast

\[ \delta_R := \int d^3x \, \delta(\vec{x}) W_R(\vec{x}). \]  

(3.18)

For convenience, we take a Gaussian window function \( W_R \) of scale \( R \),

\[ W_R(\vec{x}) = \frac{1}{(\sqrt{2\pi R})^3} e^{-\frac{\vec{x}^2}{2R^2}}. \]  

(3.19)

The variance is given by \( \langle \delta_R^2 \rangle = \int d^3x \int d^3y \, W_R(\vec{x}) W_R(\vec{y}) \xi(\vec{x}, \vec{y}) \). Since the equal time two–point function \( \xi \) depends only on the physical separation \( \ell \) of the two points, one integration is trivial. We obtain

\[ \langle \delta_R^2 \rangle = \frac{1}{2\sqrt{\pi R^3}} \int_0^\infty d\ell \, \frac{\ell^3}{\ell} e^{-\frac{\ell^2}{4R^2}} \xi(\ell). \]  

(3.20)

We first evaluate \( \langle \delta_R^2 \rangle \) for sub–horizon smearing scales, \( \mathcal{H} R \ll 1 \). For sub–horizon separations, \( \mathcal{H} \ell \ll 1 \), the two–point function \( \xi(\ell) \) varies only logarithmically, see eq. (3.15). Therefore the integral (3.20) is strongly peaked at \( \ell = \sqrt{6} R \), and we obtain

\[ \langle \delta_R^2 \rangle \simeq \frac{\sqrt{3}}{2\pi e^3} \left( \frac{2}{3\pi} \right)^2 \left( \frac{H_1}{M_{\text{Pl}}} \right)^4 \left( \log \mathcal{H} R \right)^2 \left( \mathcal{H} R \ll 1 \right). \]  

(3.21)

For super–horizon smearing scales, \( \mathcal{H} R \gg 1 \), the exponential in eq. (3.20) can be dropped and

\[ \langle \delta_R^2 \rangle \simeq \frac{88\sqrt{\pi}}{1890} \left( \frac{H_1}{M_{\text{Pl}}} \right)^4 \left( \frac{1}{\mathcal{H} R^3} \right) \left( \mathcal{H} R \gg 1 \right). \]  

(3.22)

The variance \( \langle \delta_R^2 \rangle \) is plotted in fig. 3.3. For \( \mathcal{H} R \gg 1 \) we have an \( R^{-3} \) fall–off, in contrast to the \( \ell^{-4} \) super horizon fall–off for \( \xi(\ell) \). The reason for this is simple: Because \( \xi(\ell) \) falls off rapidly for \( \ell \gg \mathcal{H}^{-1} \), the integral (3.20) is strongly peaked at \( \ell \approx \sqrt{6} R \), independent of the scale \( R \), if \( R \gg \mathcal{H}^{-1} \). In this limit the \( R \)–dependence is entirely due to the prefactor in eq. (3.20), which reflects the fact that \( \langle \delta_R^2 \rangle \) involves three non–trivial integrations.

3.5. Power spectrum

Up to now we have concentrated on the two–point correlation function in \( x \)–space. For many purposes it is useful to have the Fourier transform of \( C(\ell) \),

\[ C(q) = \frac{1}{a^3} \int d^3\ell \, C(\ell) e^{-i\vec{q}\vec{\ell}/a} \]  

(3.23)

(or correspondingly the power spectrum \( \xi(q) = C(q)/\rho_{\text{tot}}^2 \)), where \( a^{-1} \ell \) is the comoving separation. By inserting eq. (2.10) we obtain in terms of \( u_k \equiv u_{k,\text{in}}(\eta) \)

\[ C(q) = \frac{1}{2a^4} \int \frac{d^3k}{(2\pi)^3} \left\{ \left| \partial_\eta u_k \right|^2 \left| \partial_\eta u_{|\vec{q}-\vec{k}|} \right|^2 + (\vec{q}\vec{k} - \vec{k}^2)^2 \left| u_k \right|^2 \left| u_{|\vec{q}-\vec{k}|} \right|^2 \right. 

-2(\vec{q}\vec{k} - \vec{k}^2) \text{Re} \left[ u_k (\partial_\eta u_k)^* \langle \partial_\eta u_{|\vec{q}-\vec{k}|} (\partial_\eta u_{|\vec{q}-\vec{k}|})^* \rangle \right] \left. \right\}. \]  

(3.24)
Note that the Fourier transform \( C(q) \) of the two-point correlation function \( C(\ell) \) involves convolutions of the Wightman function in Fourier space, \( W_{\vec{k}} := \langle \Omega | \Phi(\vec{k}) \Phi(-\vec{k}) | \Omega \rangle = |u_{k,\text{in}}(\eta)|^2 \) corresponding to the squares occurring in eq. (2.10). The mode functions \( u_{k,\text{in}} \) are given by eq. (3.6). The resulting power spectrum \( C(q) \) is plotted in Fig. 3.4.

In the limit \( q \to 0 \) the correlation function \( C(q) \) tends to the integral of \( C(\ell) \) over all space, see eq. (3.23). Since \( C(\ell) \) is positive everywhere, \( C(q) \) must tend to a positive constant for \( q \to 0 \). We obtain

\[
C(q) \approx \frac{1}{a^3} \int d^3 \ell \ C(\ell) = \frac{11}{420 \pi} \frac{H_1^4 \mathcal{H}}{a^3} \quad \text{for} \quad q_{\text{phys}} \ll \mathcal{H}.
\] (3.25)

We write \( C(q = 0) \) in terms of single mode contributions. From eqs. (3.24) and (3.10) we obtain in the continuum limit \( L \to \infty \)

\[
C(q = 0) = \int \frac{d^3 k}{(2\pi)^3} (\delta \rho_{\vec{k}})^2,
\] (3.26)

where \( \delta \rho_{\vec{k}} := V^{-1} \delta E_\vec{k} \) are the fluctuations in the mean energy density \( \rho_{\vec{k}} = V^{-1} E_\vec{k} \) in the physical volume \( V = (aL)^3 \) contributed by the mode \( \vec{k} \). \( C(q = 0) \) is a sum of squares, and it cannot vanish unless \( \delta E_{\vec{k}} = 0 \) for all modes \( \vec{k} \). In the standard treatment of inflaton fluctuations only the interference terms between the fluctuations \( \delta \Phi_{\vec{k}} \) and the homogeneous background are kept. The interference terms cannot contribute to the sum of squares on the right hand side of eq. (3.26), and therefore \( C(q)_{\text{inflaton}} \to 0 \) for \( q \to 0 \).

In the subhorizon wavelength region, \( q_{\text{phys}} \gg \mathcal{H} \), we find numerically

\[
C(q) \approx \frac{5}{32 \pi^2} \frac{H_1^4 \mathcal{H}}{a^3} \left( \frac{\mathcal{H}}{q_{\text{phys}}} \right)^3 \quad \text{for} \quad \mathcal{H} \ll q_{\text{phys}} \ll \frac{H_1}{a}.
\] (3.27)

When evaluated at the second horizon crossing, where \( q_{\text{phys}}/\mathcal{H} = \text{fixed} = \mathcal{O}(1) \), the quantity \( q^3 \xi(q) \) is scale independent. The energy density fluctuations are of Harrison–Zel’dovich type. For instance, we find numerically \( q^3 \xi(q_{\text{phys}} = \mathcal{H}) \approx \frac{1}{6} (H_1/M_{\text{Pl}})^4 \).

The relative energy density fluctuations for a fixed comoving \( \vec{q} \) grow in time as \( \xi(q) \approx a^3 \) before the second horizon crossing. This is in contrast to the energy density fluctuations in the inflaton scenario which are also of Harrison–Zel’dovich type (i.e. scale independent at the second horizon crossing), but which show a superhorizon growth as \( \xi(q)_{\text{inflaton}} \approx a^4 \). Therefore \( C(q)_{\text{inflaton}} \sim q \) for \( q_{\text{phys}} \ll \mathcal{H} \).

### 3.6. Two-point correlations of the energy–momentum–stress tensor

We investigate the equal-time connected two-point correlation functions of the energy–momentum–stress tensor,

\[
C_{\hat{a}\hat{b}\hat{c}\hat{d}}(\vec{\ell}) = \langle \Omega | T_{\hat{a}\hat{b}}(\vec{x}) T_{\hat{c}\hat{d}}(\vec{x}') | \Omega \rangle - \langle \Omega | T_{\hat{a}\hat{b}} | \Omega \rangle \langle \Omega | T_{\hat{c}\hat{d}} | \Omega \rangle,
\] (3.28)

where the hatted indices refer to the local orthonormal frame of a comoving observer. For simplicity, we restrict ourselves to the inflationary de Sitter era. These correlation
functions depend on the separation vector \( \vec{\ell} = (\vec{x} - \vec{x}')_{\text{phys}} \). The computation goes along the same line as outlined in section 2, and the correlation functions can be expressed in terms of the Wightman function \( W(x, x') \) as

\[
C_{\hat{a}\hat{b}\hat{c}\hat{d}}(\vec{\ell}) = W_{\hat{a}\hat{c}'} W_{\hat{b}\hat{d}'} + W_{\hat{a}\hat{d}'} W_{\hat{b}\hat{c}'} - \eta_{\hat{a}\hat{b}} W_{\hat{a}\hat{z}'} W_{\hat{b}\hat{z}'} - \eta_{\hat{c}\hat{d}} W_{\hat{a}\hat{z}'} W_{\hat{b}\hat{z}'} + \frac{1}{2} \eta_{\hat{a}\hat{b}} \eta_{\hat{c}\hat{d}} W_{\hat{a}\hat{z}'} W_{\hat{b}\hat{z}'} .
\]

(3.29)

For \( a = b = c = d = 0 \) we recover eq. (2.10). The tensor structure of \( C_{\hat{a}\hat{b}\hat{c}\hat{d}} \) can be built using \( \eta_{\hat{a}\hat{b}} \) and \( \epsilon_{\hat{a}} = \ell_{\hat{a}}/\ell \). To simplify we average over the directions \( \vec{\ell} \), and therefore the indices of \( C_{\hat{a}\hat{b}\hat{c}\hat{d}} \) must be pairwise equal. The possibilities are \( \rho\rho \), \( \rho P \), \( P P \), and \( \vec{S} \cdot \vec{S} \)-correlations, where \( P = \frac{1}{3}(T_{11} + T_{22} + T_{33}) \) is the isotropic pressure, and \( \vec{S} \) is the energy flow density (or momentum density), \( S^\hat{a} = T^0_{\hat{a}} \).

The computation goes along the same lines as outlined in section 2. For super–horizon separations, \( \ell \gg H^{-1} \), the energy density and pressure correlations are all of the order \( H^4 \ell^{-4} \) with relative magnitudes

\[
C_P P(\ell) = \frac{1}{9} C_\rho\rho(\ell), \quad C_\rho P(\ell) = -\frac{1}{3} C_\rho\rho(\ell) \quad \text{for } H_1 \ell \gg 1, \quad (3.30)
\]

with \( C_\rho\rho \) given in eq. (2.12). The ratios correspond to an effective equation of state

\[
\delta P = -\frac{1}{3} \delta \rho . \quad (3.31)
\]

Using this effective equation of state and the conservation law \( T^\nu_{\mu\nu} = 0 \) we can derive the growth of the super–horizon perturbations. In a small comoving volume \( V \) of the external FRW space the conservation law is equivalent to \( dU = -PdV \) and to \( d\rho = -3(\rho + P)(da/a) \). Writing \( \rho - \langle \rho \rangle = \delta \rho \) we obtain

\[
d(\delta \rho) = -3(\delta \rho + \delta P) \frac{da}{a} . \quad (3.32)
\]

The equation of state (3.31) yields \( \delta \rho \sim a^{-2} \), the super–horizon growth factor which we have found earlier using a different method, see eq. (3.16).

For the autocorrelation function of the energy flow density \( \vec{S} \) we find on super–horizon scales

\[
C_{SS}(\ell) \sim -\frac{1}{(2\pi)^4} \frac{6H^2_1}{\ell^6} \quad \text{for } H_1 \ell \gg 1. \quad (3.33)
\]

The correlation function of the flow density decreases much faster with \( \ell \) than the correlations of the densities \( \rho \) and \( P \).

In ref. [12] correlations of the energy–momentum–stress tensor were computed which are not averaged over directions. In this case one must distinguish pressures and energy flows longitudinal and transversal to \( \vec{\ell} \).
4. Mass Effects

4.1. Stable massive fields

In this section we investigate the time evolution of \( \delta E_k \) for a mode \( k \) of a "light" massive scalar with \( m \ll H_1 \). As in the massless case of section 3, we consider only modes that spend a considerable amount of time outside the horizon, i.e. with \( k_{\text{phys}} \ll H_1 \) at the end of inflation. As long as \( m \ll k_{\text{phys}} \) the mode is ultra–relativistic and behaves as if it was massless. It becomes non–relativistic \( (k_{\text{phys}} \approx m) \) only after the first horizon crossing.

The mass \( m \) introduces a second length scale, the Compton wavelength \( m^{-1} \), in addition to the Hubble length \( H^{-1} \) of the external background. Recall that in the massless case everything depends on the relation of the two characteristic scales \( e_k = k_{\text{phys}} \) and \( H \). The horizon crossings occur when \( k_{\text{phys}} \approx H \). If \( m \neq 0 \) the ratio \( e_k / H \), where \( e_k^2 = k_{\text{phys}}^2 + m^2 \), is dynamically relevant for the horizon crossing and not \( k_{\text{phys}} / H \). In addition, the ratio \( k_{\text{phys}} / m \) determines the transition from the relativistic to the non–relativistic regime. A typical scenario is plotted in fig. 4.1.

During inflation the massive modes are given in terms of Hankel functions [6], \( u_{k,\text{in}}(\eta) = \frac{1}{2} \sqrt{\pi \eta^3} H_1^{(2)}(k\tilde{\eta}) \) of order \( \nu^2 = \frac{9}{4} - m^2 H_1^{-2} \), where \( \tilde{\eta} = \eta - 2H_1^{-1} \). Using eq. (3.10) we obtain for super–horizon modes

\[
\delta E_k \simeq \frac{H_1^2}{4} \frac{e_k^2}{k_{\text{phys}}^3} \quad \text{(for } e_k \ll H_1 \text{ and } m \ll H_1). \tag{4.1}
\]

Here we consider only masses \( (m/H_1)^2 \ll \frac{1}{70} \). Since cosmologically relevant modes have \( k_{\text{phys}}/H_1 = O(e^{-70}) \) at the end of inflation, factors \( (k_{\text{phys}}/H_1)^{m^2/H_1^2} \) are irrelevant in eq. (4.1). As long as the mode is relativistic, i.e. when \( e_k \simeq k_{\text{phys}} \), the fluctuations \( \delta E_k \) grow in time linearly with the scale factor \( a \). In the non–relativistic regime the growth is accelerated to \( \delta E_k \sim a^3 \), since the physical energy \( e_k \simeq m \) is no longer redshifted.

During the radiation dominated era the Klein–Gordon equation is given by eq. (3.1). For a highly non–relativistic mode, with \( k_{\text{phys}} \ll m \), we can drop the \( k \)–dependence. We choose as fundamental solutions

\[
v_k(\eta) \simeq \sqrt{\frac{\pi}{32H_1^2\eta}} H_1^{(2)} \left( \frac{H_1 m}{2} \right)^{\frac{1}{4}} \eta^2 \quad \text{\( (k_{\text{phys}} \ll m) \) } \tag{4.2}
\]

and its complex conjugate, \( v_k(\eta)^* \). The mode functions \( u_{k,\text{in}} \) are linear combinations of \( v_k \) and \( v_k^* \), constructed in such a way that \( u_{k,\text{in}} \) and its first derivative are continuous at the time \( \eta = H_1^{-1} \) of the transition from inflation to the radiation era. We obtain

\[
u_{k,\text{in}}(\eta) \simeq \frac{i\Gamma(\frac{5}{4})}{\sqrt{8H_1^2 \eta}} \left( \frac{H_1}{m} \right)^{\frac{1}{4}} \left( \frac{k}{2H_1} \right)^{-\frac{3}{2}} J_{\frac{1}{4}} \left( \frac{H_1 m}{2} \eta^2 \right) \quad \text{\( (k_{\text{phys}} \ll m) \).} \tag{4.3}
\]
In the modes $u_{k,\text{in}}$ and in the following expressions we neglect powers of $(k/H_I)^2/H_I^2$. The energy fluctuations (3.10) are

$$\delta E_k = \Gamma \frac{\delta^2 H^2 m^2}{2 k_{\text{phys}}^3} \sqrt{\frac{H}{m}} \left( J_\frac{5}{4} \left( \frac{m}{2H} \right)^2 + J_\frac{1}{4} \left( \frac{m}{2H} \right)^2 \right) \quad (k_{\text{phys}} \ll m). \quad (4.4)$$

We distinguish two asymptotic limits

$$\delta E_k \simeq \begin{cases} 
\frac{H^2 m^2}{4 k_{\text{phys}}^3} & (e_k \simeq m \ll H) \\
\Gamma \frac{(\frac{1}{4})^2 \sqrt{H_I m}}{8\pi} \frac{k^3}{k^3} & (e_k \simeq m \gg H). 
\end{cases} \quad (4.5)$$

As long as the mode is super–horizon, $\delta E_k$ goes on growing in time as $a^3$ independent of the cosmological era. After the second horizon crossing, when $e_k \simeq m \gg H$, the fluctuations $\delta E_k$ do not evolve in time any more, $\delta E_k$ is constant in time. The time evolution of $\delta E_k$ is plotted in fig. 4.2.

The most interesting effect of the mass $m$ is that the energy density fluctuations $\delta E_k$ grow faster in time as soon as the super–horizon mode becomes non–relativistic, when $k_{\text{phys}} \approx m$ and $e_k \ll H$. As a consequence, the fluctuations are much bigger for a massive than for a massless scalar field. It can easily be read off figs. 3.1 and 4.2 that the ratio at the time when $k_{\text{phys}} = H$ (i.e. at the second horizon crossing for $k_{\text{phys}}$) is

$$\frac{(\delta E_k)_{m \neq 0}}{(\delta E_k)_{m = 0}} = \sqrt{\frac{H_I m}{k}} \quad (\text{for } k_{\text{phys}} = H). \quad (4.6)$$

The ratio depends on the comoving wave number $k$, which equals $k_{\text{phys}}$ at the end of inflation. The magnitude of this ratio for modes with $k_{\text{phys,eq}} = H_{\text{eq}}$, i.e. with $\lambda_{\text{phys}} \simeq 160 h^{-1} \text{Mpc}$ today, is

$$\frac{(\delta E_k)_{m \neq 0}}{(\delta E_k)_{m = 0}} = \sqrt{\frac{m}{H_{\text{eq}}}} \approx \sqrt{\frac{m \cdot M_{\text{Pl}}}{1 \text{ eV}}} \approx \sqrt{\frac{m}{10^{-28} \text{ eV}}} \quad (\text{for } k_{\text{phys,eq}} = H_{\text{eq}}). \quad (4.7)$$

Note that the dependence on $H_1$ has dropped out from this ratio since for fixed $k_{\text{phys,eq}}$ the physical wave number at the end of inflation varies as $\sqrt{H_1}$. The massive fluctuations are enhanced over the massless fluctuations by an enormous factor for physically reasonable non–zero masses. For instance with $m \gtrsim 10^{-3} \text{ eV}$ the enhancement factor is larger than $10^{12}$. Since massless modes with horizon crossing at equality have $(\delta E_k)_{m = 0} \approx H_1^2 / H_{\text{eq}}$ at the time of horizon crossing, see eq. (3.13), massive modes give an order of magnitude of fluctuations compatible with observations if $(H_1/10^{16} \text{ GeV})^2 \cdot (m/10^{-28} \text{ eV})^{1/2} = \mathcal{O}(1)$. For stable massive fields with $m \geq 10^{-3} \text{ eV}$ this requires $H_1 \leq 10^{10} \text{ GeV}$, i.e. the energy density during inflation has to fulfil $\rho_1 \leq (3 \cdot 10^{14} \text{ GeV})^4$. 


4.2. Unstable massive fields

In particle physics all massive bosons are unstable. They will decay into lighter or massless particles. We consider the following simple scenario: The massive scalar field decays with a rate \( \Gamma \) into ultra–relativistic particles, \( \text{i.e.} \) the scalar field has a mean lifetime \( \Gamma^{-1} \). In addition to the two horizon crossings and the time when the mode becomes non–relativistic, there is now a fourth important point in the history of the mode, namely the time of decay, where \( \Gamma \approx \mathcal{H} \). We have included this point in fig. 4.1. The decay does not alter the spatial distribution of the energy density on large scales. The decay products immediately decohere and thermalize, and they will behave like classical radiation. Therefore after the decay the fluctuations \( \delta \rho \) in the energy density of the massless, thermalized decay products decrease in time as \( a^{-4} \) in external FRW space–time.

Instead of evolving in time a true measure of the energy fluctuations in the energy density of the decay products, like for instance the two–point correlation function, we will give a quick estimate for their magnitude. We continue the energy fluctuations \( \delta E \) from the time of decay by redshifting with \( a^{-1} \), as if the massive mode had transformed into a massless (sub–horizon) mode at \( \Gamma = \mathcal{H} \). This scenario is plotted in fig. 4.3. In this case we obtain the ratio

\[
\frac{(\delta E_k)_{m \neq 0, \Gamma \neq 0}}{(\delta E_k)_{m = 0}} = \sqrt{\frac{m}{\Gamma}} \quad \text{(for } k_{\text{phys}} = \mathcal{H}).
\]  

(4.8)

The ratio is scale independent at the reference point \( k_{\text{phys}} = \mathcal{H} \), and does not change afterwards. Therefore we can conclude that the energy fluctuations \( \delta \rho \) in the massless, thermalized decay products have the same scale dependence as in the case of a massless, fully coherent scalar field. But their magnitude is enhanced over the massless case by a factor \( \sqrt{m \Gamma^{-1}} \) in the decay scenario. This simple estimate indicates that with an unstable massive scalar field the scale invariant (Harrison–Zel’dovich) fluctuations are preserved, and the small magnitude \( (H_1/M_{\text{Pl}})^2 \) of the massless case can easily be enhanced by a factor of ten or more.

5. The Cosmic Variance

5.1. The variance of \( \bar{\rho}_S \)

In a finite sample \( S \) with volume \( V_S \) and mean linear extension \( \ell_S := \frac{3}{2} \sqrt[3]{V_S} \), \( \text{i.e.} \) a finite patch of the universe, the quantum fluctuations of the operator \( E_S \) of the total energy, \( E_S := \int_S d^3x \rho(x) \), play a crucial role. (In this section we choose \( a = 1 \) at the time for which the analysis is performed.) For an infinite sample \( S \) the fluctuations of \( E_S \) are negligible compared to the expectation value \( \langle E_S \rangle \), but in a finite \( S \) the variance \( (\delta E_S/E_S)^2 \) is relevant.

It is crucial to distinguish between spatial averages and ensemble averages (\text{i.e.} quantum mechanical expectation values). The spatial averages of an operator \( A \) will be
denoted by a bar, $\overline{A}$, while for the ensemble average brackets will be used, $\langle A \rangle$. We note that the spatial average of the operator $\rho(x)$ over the sample $S$, i.e. $\overline{\rho}_S$, is proportional to the operator $E_S$,

$$\overline{\rho}_S := \int_S \frac{d^3x}{V_S} \rho(x) = \frac{E_S}{V_S}. \quad (5.1)$$

The ensemble average $\langle \rho(x) \rangle$ of the operator $\rho(x)$ must be $x$–independent in a cosmological context, therefore $\langle \rho \rangle = \langle \overline{\rho}_S \rangle$. The variance of the total energy in the sample is $(\delta E_S)^2 = \langle E_S^2 \rangle - \langle E_S \rangle^2$, and the variance of the average energy density in $V_S$ is

$$(\delta \overline{\rho}_S)^2 = \langle \overline{\rho}_S^2 \rangle - \langle \overline{\rho}_S \rangle^2. \quad (5.2)$$

For a spherical sample of radius $R$ the variance $(\delta \overline{\rho}_S)^2$ is the same as $\rho_{\text{tot}}^2 (\delta \overline{\rho}_S)$ for a top hat window function, $W_R(\vec{x}) \equiv 1$ for $|\vec{x}| \leq R$ and $W_R(\vec{x}) \equiv 0$ otherwise.

Consider now the (equal time) autocorrelation function $C(\ell)$, eq. (2.5). The spatial (double) integral of $C(x, x') \equiv C(\ell)$ over the sample yields the variance of $E_S$,

$$\int_S d^3x \int_S d^3x' C(\ell) = \langle E_S^2 \rangle - \langle E_S \rangle^2 \equiv V_S^2 \cdot (\delta \overline{\rho}_S)^2. \quad (5.3)$$

In the limit $V_S \rightarrow \infty$ the boundary effects of $S$ become negligible, and we obtain $\int_S d^3x \int_S d^3x' C(\ell) \approx V_S \cdot \int_S d^3\ell C(\ell)$. Thus we can rewrite eq. (5.3) as

$$\begin{cases} (\delta E_S)^2 \approx V_S \int_S d^3\ell C(\ell) \\
(\delta \overline{\rho}_S)^2 \approx \frac{1}{V_S} \int_S d^3\ell C(\ell) \end{cases} \quad \text{if } V_S \rightarrow \infty. \quad (5.4)$$

This is equivalent to $(\delta \overline{\rho}_S)^2 \approx V_S^{-1} C(q = 0)$ for $V_S \rightarrow \infty$. Since $C(q = 0)$ in our class of models is a positive constant, the absolute fluctuations $\delta E_S$ increase with $\sqrt{V_S}$, while the relative fluctuations $\delta \overline{\rho}_S$ decrease with $1/\sqrt{V_S}$. In the standard treatment of the inflaton fluctuations only the contributions of the interference terms between background and fluctuating part are kept, and $\delta \overline{\rho}_S$ decreases as $\ell_S^{-2}$ for large samples. Because of these different asymptotic scaling laws the fluctuations in our model inevitably dominate over the fluctuations in the inflaton scenario on large enough scales. Our model predicts $\int d^3\ell C(\ell) = 11H_4^4H/420\pi$. Dividing by the total energy density $\rho_{\text{tot}}$ we obtain

$$\frac{\delta \overline{\rho}_S}{\rho_{\text{tot}}} \approx \sqrt{\frac{11\pi}{105}} \frac{4H_4^4}{3M_{\text{Pl}}^2} \left(\frac{\ell_H}{\ell_S}\right)^{\frac{3}{2}} \text{ for } \ell_S \gg \ell_H := H^{-1}. \quad (5.5)$$

### 5.2. The finite sample correlation function

We define the operator

$$\Delta \rho_S(x) := \rho(x) - \overline{\rho}_S, \quad (5.6)$$
the difference between $\rho(x)$ and its spatial average in the sample $S$. We consider the (equal time) autocorrelation function $C_S(\ell)$ of the operator $\Delta \rho_S$,

$$C_S(\ell) := \langle \Delta \rho_S(\vec{x}) \Delta \rho_S(\vec{x}') \rangle. \quad (5.7)$$

In a cosmological context $C_S(\ell)$ is only a function of the physical separation $\ell$ of the two points. The (double) integral of $C_S(\ell)$ over the sample vanishes by definition, $\int_S d^3x \int_S d^3x' C_S(\ell) \equiv 0$, therefore the correlation function $C_S(\ell)$ must change its sign at some value $\ell$ in the sample (or at several values). The correlation function $C_S(\ell)$, which by definition has at least one zero, is the experimentally accessible one, see ref. [13]. But $C(\ell)$, which in our model has turned out to be strictly positive, contains the ensemble average $\langle \rho \rangle$, which is unmeasurable (since only one sample $S$ is available).

For small separations, $\ell \ll \ell_S$, the two correlation functions are related as

$$C(\ell) \simeq C_S(\ell) + (\delta \bar{\rho}_S)^2 = C_S(\ell) + \frac{1}{V_S} \int d^3\ell \, C(\ell) \quad \text{for } \ell \ll \ell_S, \quad (5.9)$$

their difference is $\ell$-independent and equal to the cosmic variance. For any fixed $\ell$ the two correlation functions are identical in an infinite sample. In our model the zero of $C_S(\ell)$ moves to infinity as $V_S \to \infty$.

We would like to thank Christophe Massacand and Slava Mukhanov for many valuable discussions and comments.
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Figure Captions

**Fig. 3.1:** Time dependence of the energy fluctuations $\delta E_{\vec{k}}$ for $m = 0$

**Fig. 3.2:** The energy–energy correlation function $C(\ell)$ at equality time. Two domains: On scales $\ell \ll \lambda_{\text{EQ}} = H_{\text{EQ}}^{-1} = 13(\Omega h^2)^{-1} \text{Mpc}$ a logarithmic increase towards smaller $\ell$. For $\ell \gg \lambda_{\text{EQ}}$ a fall–off as $\ell^{-4}$. The cusp in the curve is at $\ell_{\text{cusp}} = 2\lambda_{\text{EQ}}$.

**Fig. 3.3:** The variance $\langle \delta \rho_R^2 \rangle := \delta \rho_{\text{tot}}^2 \langle \delta R^2 \rangle$ as a function of the smearing width $R$ at equality time. Two domains: For $R \ll \lambda_{\text{EQ}} = H_{\text{EQ}}^{-1} = 13(\Omega h^2)^{-1} \text{Mpc}$ a logarithmic increase towards smaller $R$. On scales $R \gg \lambda_{\text{EQ}}$ a fall–off as $R^{-3}$.

**Fig. 3.4:** The energy–energy correlation function $C(q)$ at equality time. Two domains: On scales $q \gg H_{\text{EQ}} = (13(\Omega h^2)^{-1} \text{Mpc})^{-1}$ a fall–off as $q^{-3}$. For super–horizon wavelengths, $q \ll H_{\text{EQ}}$, $C(q)$ is constant.

**Fig. 4.1:** Time development of the physical length scales. Three relevant scales for a stable massive mode: the Hubble parameter $\mathcal{H}$, the mass $m$, and the energy $e_{\text{phys}}$. A fourth scale for an unstable field, the decay rate $\Gamma$. Three important points: the first horizon crossing when $e_{\text{phys}} \simeq k_{\text{phys}} = H_1$, the time when the mode becomes non–relativistic, $k_{\text{phys}} = m$, and the second horizon crossing when $e_{\text{phys}} \simeq m = \mathcal{H}$. For an unstable particle a fourth important point where $\Gamma = \mathcal{H}$. The point $k_{\text{phys}} = \mathcal{H}$ is dynamically irrelevant for massive modes.

**Fig. 4.2:** Time evolution of the energy fluctuations $\delta E_{\vec{k}}$ for a stable massive field with $0 < m \ll H_1$.

**Fig. 4.3:** Time evolution of the energy fluctuations $\delta E_{\vec{k}}$ for an unstable massive field with $0 < m \ll H_1$ and lifetime $\Gamma^{-1}$. Comparison between the fluctuations $\delta E_{\vec{k}}$ for unstable massive (solid) and massless scalars (dashed). At $k_{\text{phys}} = \mathcal{H}$ we see the enhancement.
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