ORTHOGONAL POLYNOMIALS FOR THE WEIGHT $x^\nu \exp(-x-t/x)$

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ABSTRACT. Orthogonal polynomials for the weight $x^\nu \exp(-x-t/x), x,t \geq 0, \nu \in \mathbb{R}$ are investigated. Differential-difference equations, recurrence relations, explicit representations, generating functions and Rodrigues-type formula are obtained.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let $\nu \in \mathbb{R}, t > 0$ be parameters and consider the sequence of orthogonal polynomials $\{P_n^\nu(x,t)\}_{n \geq 0}$ of degree $n$, satisfying the orthogonality condition

$$\int_0^\infty P_n^\nu(x,t)P_m^\nu(x,t) e^{-x-t/x} dx = \delta_{n,m}, \quad (1.1)$$

where $\delta_{n,m}, n,m \in \mathbb{N}_0$ is the Kronecker symbol. Admitting the limit case $t = 0$ and $\nu > -1$, it gives normalized Laguerre polynomials

$$P_n^\nu(x,0) = L_n^\nu(x), \quad (1.2)$$

where $\{L_n^\nu(x)\}_{n \geq 0}$ are classical Laguerre polynomials [2], Vol. II and $\Gamma(z)$ is the Euler gamma function [2], Vol. I. It satisfies the three term recurrence relation

$$xL_n^\nu(x) = -(n + \nu + 1)L_n^\nu(x) + (2n + \nu + 1)L_{n-1}^\nu(x) - (n + \nu)L_{n-1}^\nu(x), \quad n \in \mathbb{N}. \quad (1.3)$$

As it follows from the theory of orthogonal polynomials, the three term recurrence relation for the sequence $\{P_n^\nu(x,t)\}_{n \geq 0}$ can be written in the form

$$xP_n^\nu(x,t) = A_{n+1}(t)P_{n+1}^\nu(x,t) + B_n(t)P_n^\nu(x,t) + A_n(t)P_{n-1}^\nu(x,t), \quad (1.4)$$

where $P_{n-1}^\nu(x,t) \equiv 0$, $P_n^\nu(x,t) = a_n(t)x^n + b_n(t)x^{n-1} + \ldots$, $a_n(t) \neq 0$ and

$$A_n(t) = \frac{a_{n-1}(t)}{a_n(t)}, \quad B_n(t) = \frac{b_n(t)}{a_n(t)} \frac{b_{n+1}(t)}{a_{n+1}(t)}. \quad (1.5)$$

As a consequence of (1.4) the Christoffel-Darboux formula takes place

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\[ \sum_{k=0}^{n} P_k^n(x,t)P_k^n(y,t) = A_{n+1}(t) \frac{P_{n+1}^n(x,t)P_{n+1}^n(y,t) - P_n^n(x,t)P_n^n(y,t)}{x - y}. \]  

Using the value of the integral [2, Vol. II]

\[ \int_0^\infty e^{-t/s}x^{\nu-1}dx = 2\nu/(\nu^2/K_\nu(2\sqrt{t}) \equiv \rho_\nu(t), \quad t > 0, \]  

where \( K_\nu(z) \) is the modified Bessel function or Macdonald function [6], we can find the moments of the weight \( x^\nu \exp(-t/x) \). Namely, we find

\[ \int_0^\infty e^{-t/s}x^{\nu+n}dx = \rho_{\nu+n+1}(t), \quad n \in \mathbb{N}_0. \]  

The asymptotic behavior of the modified Bessel function at infinity and near the origin [2, Vol. II gives the corresponding values for the function \( \rho_\nu, \nu \in \mathbb{R} \). Precisely, we have

\[ \rho_\nu(t) = O\left(t^{(\nu-|\nu|)/2}\right), \quad t \to 0, \quad \nu \neq 0, \quad \rho_0(t) = O(\log t), \quad t \to 0, \]  

\[ \rho_\nu(t) = O\left(t^{\nu/2-1/4}e^{-2\sqrt{t}}\right), \quad t \to +\infty. \]  

Moreover, it can be represented in terms of Laguerre polynomials (cf. [4, Vol. II, Entry 2.19.4.13)

\[ \frac{(-1)^n n!}{n!} \rho_\nu(t) = \int_0^\infty x^{\nu+n-1}e^{-x}L_n^\nu(x)dx, \quad n \in \mathbb{N}_0. \]  

Further, it has a relationship with the Riemann-Liouville fractional integral [6]

\[ (\mathbb{I}^\nu f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (x-t)^{\alpha-1}f(x)dx, \quad \text{Re}\alpha > 0, \]  

namely, we get the formula

\[ \rho_\nu(t) = (\mathbb{I}^{\nu}\rho_0)(t), \quad \nu > 0. \]  

Hence the index law for fractional integrals immediately implies

\[ \rho_{\nu+\mu}(t) = (\mathbb{I}^{\nu}\rho_\mu)(t) = (\mathbb{I}^{\nu}\rho_\nu)(t). \]  

The corresponding definition of the fractional derivative presumes the relation \( D_{\nu}^\mu = -\mathbb{D}_\nu^{1-\mu} \). Hence for the ordinary \( n \)-th derivative of \( \rho_\nu \) we find

\[ D^n\rho_\nu(t) = (-1)^n \rho_{\nu-n}(t), \quad n \in \mathbb{N}_0. \]  

Recalling (1.7) and integrating by parts, it is not difficult to establish the following recurrence relation for \( \rho_\nu \)

\[ \rho_{\nu+1}(t) = \nu \rho_\nu(t) + t \rho_{\nu-1}(t), \quad \nu \in \mathbb{R}. \]  

In the operator form it can be written as follows

\[ \rho_{\nu+1}(t) = (\nu - tD) \rho_\nu(t). \]  

Finally in this section we observe that up to a normalization factor the orthogonality (1.1) is equivalent to the following \( n \) equalities
Besides, taking into account (1.4), (1.5), we get the identities

\[
\int_0^\infty p_n^\nu (x,t) e^{-x-t/x} x^{\nu+1} dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{1.18}
\]

and with (1.8) it yields

\[
P_0^\nu (x,t) = [\rho_{\nu+1}(t)]^{-1/2}, \quad P_1^\nu (x,t) = [\rho_{\nu+1}(t)(\rho_{\nu+3}(t)\rho_{\nu+1}(t) - \rho_{\nu+2}^2(t))]^{-1/2}
\]

\[
\times [-\rho_{\nu+1}(t)x + \rho_{\nu+2}(t)], \tag{1.23}
\]

where \( \rho_{\nu+3}(t)\rho_{\nu+1}(t) - \rho_{\nu+2}^2(t) > 0 \) for all \( t > 0 \). The latter fact follows immediately from the Ismail integral representation of the quotient of functions \( \rho_\nu, \rho_{\nu+1} [3] \)

\[
\frac{\rho_\nu(x)}{\rho_{\nu+1}(x)} = \frac{1}{\pi^2} \int_0^\infty \frac{y^{-1} dy}{(x+y) [J_{\nu+1}^2(2\sqrt{y}) + J_{\nu+1}^2(2\sqrt{y})]} \tag{1.24}
\]

and the Nicholson’s integral [2], Vol. II

\[
J_\nu^0(2\sqrt{x}) + Y_\nu^0(2\sqrt{x}) = \frac{8}{\pi^2} \int_0^\infty K_0(4\sqrt{x} \sinh(t)) \cosh(2\nu t) dt. \tag{1.25}
\]

2. DIFFERENTIAL AND DIFFERENTIAL-DIFFERENCE EQUATIONS

We begin this section with the following auxiliary lemma.

**Lemma 1.** Let \( \nu \in \mathbb{R}, \ t > 0, \ n \in \mathbb{N} \). Then it has the identities

\[
\int_0^\infty [p_n^\nu (x,t)]^2 e^{-x-t/x} x^{\nu-1} dx = \frac{1}{t} [B_n(t) - \nu - 1 - 2n], \tag{2.1}
\]

\[
\int_0^\infty [p_n^\nu (x,t)]^2 e^{-x-t/x} x^{\nu-2} dx = \frac{1}{t} \left[ 1 - \frac{\nu}{t} [B_n(t) - \nu - 1 - 2n] \right], \tag{2.2}
\]

\[
\int_0^\infty p_n^\nu (x,t) p_{n-1}^\nu (x,t) e^{-x-t/x} x^{\nu-1} dx = \frac{1}{t} \left[ A_n(t) + \frac{b_n(t)}{a_n(t) A_n(t)} \right], \tag{2.3}
\]

\[
\int_0^\infty p_n^\nu (x,t) p_{n-1}^\nu (x,t) e^{-x-t/x} x^{\nu-2} dx = - \frac{1}{t} \left[ \frac{\nu}{t} A_n(t) + \frac{b_n(t)}{a_n(t) A_n(t)} + \frac{n}{A_n(t)} \right]. \tag{2.4}
\]
In fact, employing (1.18), (1.19), (1.20) and integration by parts we derive

\[ \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y-1}dx = \frac{1}{t} \left[ \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y+1}dx \right] \]

\[ -(v+1) \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y+1}dx - 2 \int_0^\infty P_n^v(x,t) \frac{\partial}{\partial x} [P_n^v(x,t)] e^{-x/t}x^{y+1}dx \]

\[ = \frac{1}{t} [B_n(t) - v - 1 - 2n], \]

which proves (2.1). Then, analogously, recalling (1.1) and using (2.1), we get

\[ \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y-2}dx = \frac{1}{t} \left[ \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y}dx \right] \]

\[ -v \int_0^\infty [P_n^v(x,t)]^2 e^{-x/t}x^{y}dx - 2 \int_0^\infty P_n^v(x,t) \frac{\partial}{\partial x} [P_n^v(x,t)] e^{-x/t}x^{y}dx \]

\[ = \frac{1}{t} \left[ 1 - \frac{v}{t} [B_n(t) - v - 1 - 2n] \right]. \]

Concerning (2.3), we deduce, invoking (1.1), (1.5), (1.20), (1.21), (1.22),

\[ \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y-1}dx = \frac{1}{t} \left[ \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y+1}dx \right] \]

\[ -(v+1) \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y+1}dx - \int_0^\infty P_n^v(x,t) \frac{\partial}{\partial x} [P_{n-1}^v(x,t)] e^{-x/t}x^{y+1}dx \]

\[ - \int_0^\infty P_{n-1}^v(x,t) \frac{\partial}{\partial x} [P_{n}^v(x,t)] e^{-x/t}x^{y+1}dx \]

\[ = \frac{1}{t} \left[ A_n(t) + \frac{b_n(t)}{a_n(t)A_n(t)} \right]. \]

Finally, a similar reasoning gives via (2.3)

\[ \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y-2}dx = \frac{1}{t} \left[ \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y}dx \right] \]

\[ -v \int_0^\infty P_n^v(x,t)P_{n-1}^v(x,t) e^{-x/t}x^{y}dx - \int_0^\infty P_n^v(x,t) \frac{\partial}{\partial x} [P_{n-1}^v(x,t)] e^{-x/t}x^{y}dx \]

\[ - \int_0^\infty P_{n-1}^v(x,t) \frac{\partial}{\partial x} [P_{n}^v(x,t)] e^{-x/t}x^{y}dx \]

\[ = \frac{1}{t} \left[ v \left[ A_n(t) + \frac{b_n(t)}{a_n(t)A_n(t)} \right] + \frac{n}{A_n(t)} \right], \]

and we establish (2.4), completing the proof of Lemma 1.

\[ \square \]

Now we are ready to prove the following theorem.

**Theorem 1.** Let \( n \in \mathbb{N} \). Orthogonal polynomials \( P_n^v(x,t) \) satisfy the first order linear differential-difference equation

\[ x^2 \frac{\partial}{\partial x} [P_n^v(x,t)] = \left[ nx - A_n^2(t) - \frac{b_n(t)}{a_n(t)} \right] P_n^v(x,t) + A_n(t) [x + B_n(t) - v - 1 - 2n] P_{n-1}^v(x,t). \]
Proof. Since \( \frac{\partial}{\partial x}[P_n^\nu(x,t)] \) is a polynomial of degree \( n-1 \), we write it in the form

\[
\frac{\partial}{\partial x}[P_n^\nu(x,t)] = \sum_{k=0}^{n-1} c_{n,k}(t) P_k^\nu(x,t),
\]

where, owing to the orthogonality,

\[
c_{n,k}(t) = \int_0^\infty \frac{\partial}{\partial x}[P_n^\nu(x,t)] P_k^\nu(x,t) e^{-x^2/2} dx.
\]

Then, integrating by parts and using the orthogonality relation, we obtain

\[
c_{n,k}(t) = -t \int_0^\infty P_n^\nu(x,t) P_k^\nu(x,t) e^{-x^2/2} dx - \nu \int_0^\infty P_n^\nu(x,t) P_k^\nu(x,t) e^{-x^2/2} dx.
\]

Moreover, we observe that

\[
\int_0^\infty P_n^\nu(y,t) \sum_{k=0}^{n-1} P_k^\nu(y,t) P_k^\nu(x,t) e^{-y^2/2} y^{-1} dy = 0,
\]

\[
\int_0^\infty P_n^\nu(y,t) \sum_{k=0}^{n-1} P_k^\nu(y,t) P_k^\nu(x,t) e^{-y^2/2} y^{-2} dy = 0.
\]

Therefore from (2.6), (2.8) and Christoffel-Darboux formula (1.6) we derive

\[
\frac{\partial}{\partial x}[P_n^\nu(x,t)] = -t \sum_{k=0}^{n-1} P_k^\nu(x,t) \int_0^\infty P_n^\nu(y,t) P_k^\nu(y,t) e^{-y^2/2} y^{-1} \left[ \frac{1}{y^2} - \frac{1}{x^2} \right] dy
\]

\[
-\nu \sum_{k=0}^{n-1} P_k^\nu(x,t) \int_0^\infty P_n^\nu(y,t) P_k^\nu(y,t) e^{-y^2/2} y^{-1} \left[ \frac{1}{y^2} + \frac{1}{x^2} \right] dy
\]

\[
= -t A_n(t) P_n^\nu(x,t) \int_0^\infty P_n^\nu(y,t) P_n^\nu(y,t) e^{-y^2/2} y^{-1} \left[ \frac{1}{y^2} + \frac{1}{x^2} \right] dy
\]

\[
+ t A_n(t) P_n^\nu(x,t) \int_0^\infty [P_n^\nu(y,t)]^2 e^{-y^2/2} y^{-1} \left[ \frac{1}{y^2} + \frac{1}{x^2} \right] dy
\]

\[
- \frac{\nu}{x} A_n(t) P_n^\nu(x,t) \int_0^\infty P_n^\nu(y,t) P_n^\nu(y,t) e^{-y^2/2} y^{-1} dy
\]

\[
+ \frac{\nu}{x} A_n(t) P_n^\nu(x,t) \int_0^\infty [P_n^\nu(y,t)]^2 e^{-y^2/2} y^{-1} dy.
\]

Hence by virtue of Lemma 1 equalities (2.9) become

\[
\frac{1}{A_n(t)} \frac{\partial}{\partial x}[P_n^\nu(x,t)] = \frac{1}{x} P_n^\nu(x,t) \left[ \left( \frac{\nu}{t} - \frac{1}{x} \right) A_n(t) + \frac{b_n(t)}{a_n(t)} \right] + \frac{n}{A_n(t)}
\]

\[
+ \frac{1}{x} P_{n-1}^\nu(x,t) \left[ 1 - \left( \frac{\nu}{t} - \frac{1}{x} \right) [B_n(t) - \nu - 1 - 2n] \right]
\]

\[
- \frac{\nu}{xt} P_n^\nu(x,t) \left[ A_n(t) + \frac{b_n(t)}{a_n(t)} \right] + \frac{\nu}{xt} P_{n-1}^\nu(x,t) [B_n(t) - \nu - 1 - 2n].
\]

Hence after simplification we arrive at the differential-difference equation (2.5). Theorem 1 is proved. \( \square \)
Corollary 1. Denoting by $a_{n,0}(t)$ the free term of the polynomial $P_n^y(x,t)$, it has the value

$$a_{n,0}(t) = \frac{1}{a_n(t)\rho_{\nu+1}(t)} \prod_{k=1}^{n} \frac{B_k(t) - \nu - 1 - 2k}{A_k^2(t) + \frac{b_n(t)}{a_n(t)}}.$$

(2.10)

Proof. In fact, letting $x = 0$ in (2.5), we find the recurrence relation

$$\left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] a_{n,0}(t) = [B_n(t) - \nu - 1 - 2n]A_n(t)a_{n-1,0}(t).$$

(2.11)

Hence formula (2.10) comes immediately, solving recurrence (2.11) with the use of (1.5) and (1.23).

Remark 1. In the limit case $t = 0$ we use (1.2), (1.5), (1.8) to have $\rho_{\nu+1}(0) = \Gamma(\nu+1)$,

$$a_n(0) = \frac{(-1)^n}{n!\Gamma(n+\nu+1)^{1/2}}, \quad b_n(0) = (-1)^n+1 \left( \frac{n(n+v)}{(n-1)!\Gamma(n+v)} \right)^{1/2},$$

(2.12)

$$B_n(0) = 2n + \nu + 1, \quad A_n(0) = -(n(n+\nu))^{1/2}, \quad a_{n,0}(0) = \frac{1}{\Gamma(\nu+1)} \left( \frac{\Gamma(n+\nu+1)}{n!} \right)^{1/2},$$

$$\lim_{t \to 0} \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] B_n(t) - \nu - 1 - 2n \right]^{-1} = -n.$$  (2.13)

Moreover, taking the limit of the indeterminate form under the product sign in (2.10) when $t \to 0$, it implies the identity

$$\prod_{k=1}^{n} \frac{B_k'(0)}{(-1)^k(k!\Gamma(k+\nu+1))^{1/2}[P_k'(0) + k(k+\nu)d_k'(0)] - 2(k(k+\nu))^{1/2}A_k'(0)} = \frac{(-1)^n}{n!}.$$  (2.14)

Now, assuming that polynomial coefficients in (1.4), (1.5) $a_{n,k}(t) \in C^1(\mathbb{R}_+)$, $a_{n,n}(t) \equiv a_n(t)$, $a_{n,n-1}(t) = b_n(t)$ as functions of $t$, we differentiate equality (1.1) by $t$ to obtain

$$0 = \frac{\partial}{\partial t} \int_{0}^{\infty} P_n^y(x,t)^2 e^{-x^{-t}/x^y} dx = 2 \int_{0}^{\infty} P_n^y(x,t) \frac{\partial}{\partial t} \left[ P_n^y(x,t) \right] e^{-x^{-t}/x^y} dx$$

$$- \int_{0}^{\infty} \left[ P_n^y(x,t) \right]^2 e^{-x^{-t}/x^y-1} dx,$$

where the differentiation under the integral sign can be easily motivated by virtue of the absolute and uniform convergence. Then, appealing to (1.20) and (2.1), it implies the equality

$$\frac{d_n(t)}{\rho_n(t)} = \frac{B_n(t) - \nu - 1 - 2n}{2t}.$$  (2.15)

This first order differential equation with respect to the leading term $a_n(t)$ can be uniquely solved under initial condition (2.12), and we get

$$a_n(t) = \frac{(-1)^n}{(n!\Gamma(n+\nu+1))^{1/2}} \exp \left( \frac{1}{2} \int_{0}^{t} \frac{B_n(y) - \nu - 1 - 2n}{y} dy \right),$$  (2.16)

where the integral under the exponential function exists since, evidently, (see (2.13), (2.15))

□
Furthermore, employing (1.5), we find

\[
A_{n+1}(t) = -\left((n+1)(n+1+\nu)\right)^{1/2} \exp\left(\frac{1}{2} \int_0^t \frac{B_n(y) - B_{n+1}(y) + 2}{y} dy\right). \tag{2.18}
\]

Hence after differentiation we obtain

\[
A'_{n+1}(0) = \frac{1}{2\nu}(n+1)(n+1+\nu)^{1/2} \left[B'_{n+1}(0) - B'_n(0)\right]. \tag{2.19}
\]

On the other hand, via (1.15), (1.23) we have

\[
B_0(t) = \frac{\rho_{v+2}(t)}{\rho_{v+1}(t)}, \quad B'_0(t) = \frac{\rho_{v+2}(t)\rho_v(t)}{\rho_{v+1}^2(t)} - 1, \quad B'_0(0) = \frac{1}{\nu}, \quad \nu > 0. \tag{2.20}
\]

Hence it gives the equality

\[
B'_{n+1}(0) = \frac{1}{\nu} + 2 \sum_{k=0}^{n} \frac{A'_k(0)}{(k+1)(k+1+\nu)^{1/2}}. \tag{2.21}
\]

The following theorem shows the first order partial differential-difference equation whose solutions are polynomials \(P^\nu_n(x,t)\).

**Theorem 2.** Orthogonal polynomials \(P^\nu_n(x,t)\) satisfy the first order partial differential-difference equation

\[
\left(\frac{\partial}{\partial t} + x \frac{\partial}{\partial x}\right) P^\nu_n(x,t) = \left(\frac{d_n'(t)}{d_n(t)} + n\right) P^\nu_n(x,t) + A_n(t)P^\nu_{n-1}(x,t). \tag{2.22}
\]

**Proof.** Indeed, differentiating both sides of (1.18) with respect to \(t\), we have

\[
\int_0^\infty \frac{\partial}{\partial t} [P^\nu_n(x,t)] e^{-x-t/x}\nu^{m} dx - \int_0^\infty P^\nu_n(x,t) e^{-x-t/x}\nu^{m-1} dx = 0, \quad m = 0, 1, \ldots, n-1. \tag{2.23}
\]

Meanwhile, the second integral on the left-hand side of (2.23) can be treated via integration by parts which gives

\[
\int_0^\infty P^\nu_n(x,t) e^{-x-t/x}\nu^{m+1} dx = \frac{1}{t} \int_0^\infty P^\nu_n(x,t) e^{-x-t/x}\nu^{m+1} dx
\]

\[ - \frac{1}{t} \int_0^\infty \frac{\partial}{\partial x} [P^\nu_n(x,t)] e^{-x-t/x}\nu^{m+1} dx = \frac{\nu + m + 1}{t} \int_0^\infty P^\nu_n(x,t) e^{-x-t/x}\nu^{m} dx,
\]

and, combining with (2.23), (1.4) and (1.18), we obtain

\[
\int_0^\infty \left[\left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}\right) P^\nu_n(x,t) - A_n(t)P^\nu_{n-1}(x,t)\right] e^{-x-t/x}\nu^{m} dx = 0, \quad m = 0, 1, \ldots, n-1. \tag{2.24}
\]

Hence by unicity we therefore have

\[
\left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}\right) P^\nu_n(x,t) - A_n(t)P^\nu_{n-1}(x,t) = c_n(t)P^\nu_n(x,t). \tag{2.25}
\]
The function $c_n(t)$ is defined, equating coefficients of $x^n$ on both sides in (2.25) to find

$$c_n(t) = \frac{a'_n(t)}{a_n(t)} + n.$$  

This completes the proof of Theorem 2. \hfill \Box

**Corollary 2.** Equation (2.22) can be written in the form

$$
\left( t \frac{d}{dt} + x \frac{d}{dx} \right) P_n^\nu(x,t) = \frac{1}{2} [B_n(t) - \nu - 1] P_n^\nu(x,t) + A_n(t) P_{n-1}^\nu(x,t).
$$

**Proof.** The proof is immediate with the use of (2.15). \hfill \Box

**Corollary 3.** Let $t > 0$. The following equalities take place

\begin{align*}
\frac{d}{dt} \left[ \frac{b_n(t)}{a_n(t)} \right] &= \frac{1}{t} \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right], \quad (2.26) \\
B'_n(t) &= \frac{1}{t} \left[ A_n^2(t) - A_{n+1}^2(t) + B_n(t) \right], \quad (2.27) \\
a'_{n,0}(t) &= \frac{B_n(t) - \nu - 1}{2t} a_{n,0}(t) + \frac{A_n(t)}{t} a_{n-1,0}(t). \quad (2.28)
\end{align*}

**Proof.** The proof is immediate, equating coefficients of $x^{n-1}$ and free terms on both sides of (2.22), and the use of (1.5), (2.15). \hfill \Box

Further, differentiating (1.19) by $t$ and employing (1.1), (1.21), we deduce

$$
B'_n(t) = 2 \int_0^\infty P_n^\nu(x,t) \frac{d}{dt} \left[ P_n^\nu(x,t) \right] e^{-x/t}x^{\nu+1}dx - 1
$$

$$
= -2 \frac{a_n'(t) b_{n+1}(t)}{a_{n+1}(t) a_n(t)} + 2 \frac{b_n'(t)}{a_n(t)} - 1,
$$

i.e. (see (1.5))

$$
B'_n(t) = 2B_n(t) \frac{a'_n(t)}{a_n(t)} + 2 \frac{d}{dt} \left[ \frac{b_n(t)}{a_n(t)} \right] - 1. \quad (2.29)
$$

**Corollary 4.** Let $\nu > 0$, $n \in \mathbb{N}_0$. It has the values

\begin{align*}
A'_n(0) &= 0, \quad B'_n(0) = \frac{1}{\nu}, \quad a'_n(0) = \frac{(-1)^n}{2\nu n! \Gamma(n + \nu + 1)}^{1/2}, \quad (2.30) \\
b'_n(0) &= (-1)^{n+1} \frac{n+\nu+2}{2\nu(n+\nu+1)!^{1/2}}, \quad a'_{n,0}(0) = \frac{1}{2\nu \Gamma(\nu + 1)} \left( \frac{\Gamma(n+\nu+1)}{n!} \right)^{1/2}. \quad (2.31)
\end{align*}
Proof. In fact, taking a limit in (2.26) when $t \to 0$, we find
\[
\lim_{t \to 0} \frac{d}{dt} \left[ A_n^2(t) \right] = 2A_n(0)A'_n(0) = 0.
\]
Hence via (2.13) $A'_n(0) = 0$ and the sum in (2.21) is zero. Thus $B'_n(0) = \frac{1}{v}$. The value for $A'_n(0)$ comes from (2.12), (2.17) and $b'_n(0)$ is obtained from (2.29). Finally, $a'_{n,0}(0)$ is a consequence of (2.28) when $t \to 0$.

Now, assuming that polynomial coefficients are twice continuously differentiable functions of $t$, we solve a simple Cauchy problem for the first order differential equation (2.26) to find
\[
b_n(t) = 2t \int_0^t \frac{A_n(y)A'_n(y)}{y} dy - A_n^2(t) - \frac{nt}{v}.
\] (2.32)
Moreover, via (1.5), (2.27) and integration by parts we derive
\[
B_n(t) = \frac{1}{v} t \int_0^t \left( A_n^2(y) - A_{n+1}^2(y) \right) dy + A_n^2(t) - A_{n+1}^2(t) + \frac{t}{v}.
\] (2.33)
\[
B'_n(t) = \frac{1}{v} \int_0^t \left( A_n^2(y) - A_{n+1}^2(y) \right) \frac{y}{y^2 + \frac{1}{v}} dy.
\] (2.34)

**Corollary 5.** Coefficients $A_n$ satisfy the following second kind nonlinear differential-difference equation
\[
A''_n(t)A_n(t) - \left( A'_n(t) \right)^2 - \frac{A^2_n(t)}{2t^2} \left[ A_{n-1}^2(t) - 2A_n^2(t) + A_{n+1}^2(t) - 2 \right] = 0.
\] (2.35)

**Proof.** Indeed, from (1.5) and (2.18) we get
\[
\frac{A'_n(t)}{A_n(t)} = \frac{B_{n-1}(t) - B_n(t) + 2}{2t}, \quad t > 0.
\] (2.36)
Then by virtue of (2.27)
\[
2t \frac{d}{dt} \left[ \frac{A'_n(t)}{A_n(t)} \right] = t \frac{d}{dt} \left[ B_{n-1}(t) - B_n(t) \right] = B_{n-1}(t) - B_n(t) + A_{n-1}^2(t) - 2A_n^2(t) + A_{n+1}^2(t)
\]
and the result follows via (2.36) and simple differentiation.

**Corollary 6.** The free term $a_{n,0}(t)$ can be determined by the formula
\[
a_{n,0}(t) = (-1)^n \left( 1 + \nu \right) n a_n(t) \exp \left( \int_0^t \left[ n + \frac{A_n^2(y) + b_n(y)}{B_n(y) - \nu - 1 - 2n} \right] \frac{dy}{y} \right),
\] (2.37)
where $(z)_n$ is the Pochhammer symbol.
Theorem 3. \( P_n^\nu(x,t) \) obey the second order differential equation

\[
\begin{align*}
X^4 \left[ x + B_n(t) - \nu - 1 - 2n \right] \frac{\partial^2}{\partial x^2} [P_n^\nu(x,t)] - x^2 \left[ x + B_n(t) - \nu - 1 - 2n \right] \\
\times \left[ (2n-3)x - A_n^2(t) - A_{n-1}^2(t) - \frac{b_n(t)}{a_n(t)} - \frac{b_{n-1}(t)}{a_{n-1}(t)} \right] \\
+ [x + B_{n-1}(t) - \nu + 1 - 2n] [x - B_{n-1}(t)] \frac{\partial}{\partial x} [P_n^\nu(x,t)] \\
+ \left[ nx - A_n^2(t) - \frac{b_n(t)}{a_n(t)} \right] \left[ (n-1)x - A_{n-1}^2(t) - \frac{b_{n-1}(t)}{a_{n-1}(t)} [x + B_{n-1}(t) - \nu + 1 - 2n] [x - B_{n-1}(t)] \right] \\
- x^2 \left[ n [B_n(t) - \nu - 2n - 1] + A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] P_n^\nu(x,t) = 0. \tag{2.38}
\end{align*}
\]

Proof. Differentiating both sides of (2.5) with respect to \( x \), we have

\[
X^4 \frac{\partial^2}{\partial x^2} [P_n^\nu(x,t)] = n P_n^\nu(x,t) + \left[ (n-2)x - A_n^2(t) - \frac{b_n(t)}{a_n(t)} \right] \frac{\partial}{\partial x} [P_n^\nu(x,t)] \\
+ A_n(t) P_{n-1}^\nu(x,t) + A_n(t) [x + B_n(t) - \nu - 1 - 2n] \frac{\partial}{\partial x} [P_n^\nu(x,t)].
\]

But, owing to (1.4) and (2.5), the latter equality becomes

\[
X^4 \frac{\partial^2}{\partial x^2} [P_n^\nu(x,t)] = x^2 \left[ (n-2)x - A_n^2(t) - \frac{b_n(t)}{a_n(t)} \right] \frac{\partial}{\partial x} [P_n^\nu(x,t)] \\
+ \left[ nx^2 - A_n^2(t) [x + B_n(t) - \nu - 1 - 2n] [x + B_{n-1}(t) - \nu + 1 - 2n] \right] P_n^\nu(x,t) \\
+ \left[ nx - A_n^2(t) - \frac{b_n(t)}{a_n(t)} \right] \left[ (n-1)x - A_{n-1}^2(t) - \frac{b_{n-1}(t)}{a_{n-1}(t)} \right] \\
+ [x + B_{n-1}(t) - \nu + 1 - 2n] [x - B_{n-1}(t)] A_n(t) [x + B_n(t) - \nu - 1 - 2n] P_{n-1}^\nu(x,t).
\]

Finally, recalling (2.5) to express \( P_n^\nu(x,t) \), we end up with the equation (2.37), completing the proof of Theorem 3.
3. Explicit representations. Recurrence relations for coefficients

In this section we will deduce recurrence relations for coefficients of orthogonal polynomials $P_n^\nu(x,t)$ and their explicit representations. In fact, we have

**Theorem 4.** Let $n \in \mathbb{N}_0$, $t > 0$. Then the following identities hold

\begin{equation}
A_{n+1}^2(t) + B_n^2(t) + A_n^2(t) - (2n + \nu + 2)B_n(t) + 2 \frac{b_n(t)}{a_n(t)} - t = 0, \tag{3.1}
\end{equation}

\begin{equation}
A_n^2(t) [B_n(t) - \nu - 1 - 2n] [B_{n-1}(t) - \nu + 1 - 2n] - \left[A_n^2(t) + \frac{b_n(t)}{a_n(t)}\right]^2 + t \left[A_n^2(t) + \frac{b_n(t)}{a_n(t)}\right] = 0, \tag{3.2}
\end{equation}

\begin{equation}
t \left[B_{n-1}(t) + B_n(t)\right] + \left[B_{n-1}(t) + B_n(t)\right] [B_{n-1}(t) - B_n(t) + 1] - (2n + \nu) [B_{n-1}(t) - B_n(t)] = 0. \tag{3.3}
\end{equation}

**Proof.** In order to prove (3.1), we appeal to the three term recurrence relation (1.4) and the orthogonality condition (1.1). Hence, integrating by parts, we derive

\begin{align*}
A_{n+1}^2(t) + B_n^2(t) + A_n^2(t) &= \int_0^\infty \left|xP_n^\nu(x,t)\right|^2 e^{-x-t/x^\nu} dx = (v + 2) \int_0^\infty \left[P_n^\nu(x,t)\right]^2 e^{-x-t/x^\nu+1} dx \\
&+ t \int_0^\infty \left[P_n^\nu(x,t)\right]^2 e^{-x-t/x^\nu} dx + 2 \int_0^\infty \frac{d}{dx} \left[P_n^\nu(x,t)\right] e^{-x-t/x^\nu+2} dx \\
&= (v + 2)B_n(t) + t + 2nB_n(t) + 2A_n(t) \int_0^\infty xP_{n-1}^\nu(x,t) \frac{d}{dx} \left[P_n^\nu(x,t)\right] e^{-x-t/x^\nu} dx \\
&+ (2n + \nu + 2)B_n(t) - 2nA_n(t) \frac{b_n(t)}{a_{n-1}(t)} + 2(n - 1)A_n(t) \frac{b_n(t)}{a_{n-1}(t)} \\
&= t + (2n + \nu + 2)B_n(t) - 2 \frac{b_n(t)}{a_{n}(t)}.
\end{align*}

This gives (3.1). On the other hand, writing (3.1) for the index $n - 1$ in the form

\begin{equation}
A_n^2(t) + \frac{b_n(t)}{a_n(t)} - t = B_{n-1}(t) [2n + \nu - 1 - B_{n-1}(t)] - A_{n-1}^2(t) - \frac{b_{n-1}(t)}{a_{n-1}(t)}, \tag{3.4}
\end{equation}

we let $x = 0$ in (2.38) to find the equality

\begin{align*}
A_n^2(t) [B_n(t) - \nu - 1 - 2n] [B_{n-1}(t) - \nu + 1 - 2n] \\
+ \left[A_n^2(t) + \frac{b_n(t)}{a_n(t)}\right] [A_{n-1}^2(t) + \frac{b_{n-1}(t)}{a_{n-1}(t)}] + [B_{n-1}(t) - 2n - \nu + 1] B_{n-1}(t) = 0. \tag{3.5}
\end{align*}

Hence a simple comparison leads to (3.2). Finally, writing (3.1) for $n - 1$ and subtracting one equality from another, we use (2.27) to establish (3.3). \(\square\)
Corollary 7. Differential equation (2.38) reduces to the equality
\[ x^2 (x + B_n(t) - \nu - 1 - 2n) \frac{d^2}{dx^2} \left[ P_n^\nu (x, t) \right] \]
\[ - \left[ x^3 + B_n(t) - 2(\nu + n + 1) \right] x^2 - \left[ t + (B_n(t) - \nu - 1 - 2n)(\nu + 2) \right] x - t (B_n(t) - \nu - 1 - 2n) \frac{d}{dx} \left[ P_n^\nu (x, t) \right] \]
\[ + \left[ n x^2 - \left( \frac{b_n(t)}{a_n(t)} - n (B_n(t) - 2\nu - 3n - 1) \right) x + (B_n(t) - \nu - 2n - 1) \left[ A_n^2(t) - n(n + \nu + 1) \right] \right] \]
\[ + \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] (2n + \nu - B_n(t)) \left[ P_n^\nu (x, t) \right] = 0. \quad (3.6) \]

Proof. In fact, appealing to (3.4), (3.5), we get from (2.38)
\[ x^3 \left[ x + B_n(t) - \nu - 1 - 2n \right] \frac{d^2}{dx^2} \left[ P_n^\nu (x, t) \right] - x \left[ x^2 + [x + B_n(t) - \nu - 1 - 2n] \right] \]
\[ \times \left[ (2n - 3)x - A_n^2(t) - A_n^2(t) - \frac{b_n(t)}{a_n(t)} - \frac{b_n(t)}{a_n(t)} \right] \]
\[ + \left[ x + B_{n-1}(t) - \nu - 1 - 2n \right] \frac{d}{dx} \left[ P_n^\nu (x, t) \right] \]
\[ \times \left[ A_n^2(t) \left[ B_n(t) + B_{n-1}(t) + x - 2\nu - 4n \right] + n(x^2 - (n + \nu)x - t) \right] \]
\[ + \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] \left[ B_n(t) - \nu - 1 - 2n \right] (2n + \nu - x) + x(2n + \nu - 1 - x) \]
\[ - nx [B_n(t) - \nu - 2n - 1] \left[ P_n^\nu (x, t) \right] = 0. \quad (3.7) \]

Hence, letting \( x = 0 \) in (3.7), we find the identity
\[ A_n^2(t) \left[ B_n(t) + B_{n-1}(t) - 2\nu - 4n \right] + (2n + \nu) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right] - nt = 0. \quad (3.8) \]

Consequently, (3.7) becomes
\[ x^3 \left[ x + B_n(t) - \nu - 1 - 2n \right] \frac{d^2}{dx^2} \left[ P_n^\nu (x, t) \right] - x \left[ x^2 + [x + B_n(t) - \nu - 1 - 2n] \right] \]
\[ \times \left[ (2n - 3)x - A_n^2(t) - A_n^2(t) - \frac{b_n(t)}{a_n(t)} - \frac{b_n(t)}{a_n(t)} \right] \]
\[ + \left[ x + B_{n-1}(t) - \nu - 1 - 2n \right] \frac{d}{dx} \left[ P_n^\nu (x, t) \right] \]
\[ + x \left[ n(x - n - \nu) - \frac{b_n(t)}{a_n(t)} \right] + (B_n(t) - \nu - 2n - 1) \left( A_n^2(t) + n(x - n - \nu - 1) \right) \]
orthogonal polynomials for exponential weights

\[ d^2_n(t) + \frac{b_n(t)}{a_n(t)} \left( 2n + \nu - B_n(t) \right) P_n^\nu(x,t) = 0. \]

Dividing by \( x \) and appealing to (3.4), we arrive at (3.6).

\[ \square \]

**Remark 2.** For the limit case \( t = 0 \) we employ (1.5), (2.12), (2.13) to reduce (3.6) to the classical differential equation for Laguerre polynomials.

The following theorem gives the integro-differential-difference equation for orthogonal polynomials \( P_n^\nu(x,t) \).

Precisely, it has

**Theorem 5.** Orthogonal polynomials \( P_n^\nu(x,t) \) satisfy the integral-difference equation of the form

\[
P_n^\nu(x,t) = P_{n-1}^\nu(x,t) - a_n(t) \exp \left( \frac{b_n(t)}{x a_n(t)} \right) \int_0^t \exp \left( - \frac{b_n(y)}{x a_n(y)} \right) \frac{d}{dy} \left[ \frac{P_n^\nu(x,y)}{a_{n-1,0}(y) a_n(y)} \right] dy
\]

\[ + (-1)^n(n-1)! a_n(t) \exp \left( \frac{1}{x} \left[ \frac{b_n(t)}{a_n(t)} + n(n+\nu) \right] \right) \left[ n L_n^\nu(x) - \Gamma(1+\nu) \left( \frac{n!(n+\nu)}{\Gamma(n+\nu)} \right)^{1/2} L_{n-1}^\nu(x) \right]. \quad (3.9) \]

**Proof.** Recalling the differential-difference equations (2.5), (2.28), the recurrence relation (2.11) and Corollary 2, we rewrite equation (2.22) in the form

\[
\frac{\partial}{\partial t} \left[ \frac{P_n^\nu(x,t)}{a_n(t)} \right] + \frac{1}{x} A_n^2(t) + \frac{B_n(t)}{a_n(t)} - \frac{1}{x} \left( A_n^2(t) + \frac{b_n(t)}{a_n(t)} \right) P_n^\nu(x,t) = \frac{1}{x} \left( \frac{b_n(t)}{a_n(t)} \right) P_{n-1}^\nu(x,t).
\]

Then by virtue of (2.26), (2.37) it implies

\[
\frac{\partial}{\partial t} \left[ \frac{P_n^\nu(x,t)}{a_n(t)} \right] + \left[ \frac{a_n'(t)}{a_n(t)} - \frac{a_n(t)}{a_n'(t)} \right] \frac{d}{dt} \left[ \frac{b_n(t)}{a_n(t)} \right] P_n^\nu(x,t) = \frac{1}{x} \left( \frac{b_n(t)}{a_n(t)} \right) P_{n-1}^\nu(x,t).
\]

Solving this first order differential equation in terms of \( P_n^\nu \) under initial condition (1.2) and taking into account (2.12), we find

\[
P_n^\nu(x,t) = - \frac{a_n(t)}{x} \exp \left( \frac{b_n(t)}{x a_n(t)} \right) \int_0^t \exp \left( - \frac{b_n(y)}{x a_n(y)} \right) \frac{d}{dy} \left[ \frac{b_n(y)}{a_n(y)} \right] \frac{P_{n-1}^\nu(x,y)}{a_{n-1,0}(y) a_n(y)} dy
\]

\[ + (-1)^n(n-1)! a_n(t) \exp \left( \frac{1}{x} \left[ \frac{b_n(t)}{a_n(t)} + n(n+\nu) \right] \right) L_n^\nu(x). \quad (3.11) \]

Finally, integrating by parts in (3.11), we end up with (3.9). Theorem 5 is proved.

\[ \square \]

Concerning recurrence relations for the coefficients of the polynomials \( P_n^\nu \), we have the following result.

**Theorem 6.** For the orthogonal polynomial \( P_n^\nu(x,t) = \sum_{k=0}^n a_{n,k}(t)x^k \), \( a_{n,n}(t) \equiv a_n(t) \) and its coefficients fulfil the differential-recurrence relations

\[
a_{n,k}(t) = a_{n,0}(t) \sum_{m=k}^n a_m(t) \left[ a_m(t) a_{m,k-1}(t) - a_{m,k-1}(t) a_m(t) \right] = a_{n,0}(t) \frac{a_m(t)a_{m,k-1}(t) - a_{m,k-1}(t)a_m(t)}{a_m(t)b_m(t) - b_m(t)a_m(t)} a_{m,0}(t), \quad k = 1, \ldots, n. \quad (3.12) \]
Proof. By virtue of (2.26) and taking into account that \( A_n^2(t) + \frac{b_n(t)}{a_n(t)} \neq 0 \), \( t > 0 \) (see (2.1), (3.2)), we write equality (3.10) in the form

\[
\frac{d^2}{dt^2} \left( \frac{A_n(t)}{b_n(t)^2} \right) - \frac{a_n(t)}{b_n(t)^3} \frac{d}{dt} \left( \frac{A_n(t)}{b_n(t)^2} \right) \frac{a_n(t)}{a_n(t)} = \frac{1}{x} \left[ \frac{P_n^x(x,t)}{a_{n,0}(t)} - \frac{P_{n-1}^x(x,t)}{a_{n-1,0}(t)} \right].
\]

Hence it yields

\[
\sum_{m=1}^{n} \frac{a_n(t)}{a_m(t)b_m(t) - b_m(t)a_m(t)} \frac{a_{n,0}(t)}{a_m,0(t)} \sum_{k=1}^{m} \left[ a_m(t)a_{m,k-1}(t) - a_{m,k-1}(t)a_m(t) \right] a_k(t) = \frac{P_n^x(x,t)}{a_{n,0}(t)} - 1.
\]

Therefore changing the order of summation on the left-hand side of the latter equality, we arrive at the representation

\[
P_n^x(x,t) = a_{n,0}(t) + \sum_{k=1}^{n} \frac{a_n(t)}{a_m(t)b_m(t) - b_m(t)a_m(t)} \frac{a_{n,0}(t)}{a_m,0(t)} \sum_{m=k}^{n} \left[ a_m(t)a_{m,k-1}(t) - a_{m,k-1}(t)a_m(t) \right] a_k(t),
\]

which leads to (3.12).

The three term recurrence relation (1.4) can be written in another form. Indeed, it has

**Theorem 7.** Orthogonal polynomials \( P_n^x \) satisfy the following recurrence relation

\[
\begin{aligned}
A_{n+1}^2(t) + \frac{b_{n+1}(t)}{a_{n+1}(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] A_{n+1}(t) P_n^x(x,t) + A_n(t) P_{n-1}^x(x,t) \\
+ [xt B_n(t) + A_n(t) [B_{n-1}(t) - \nu + 1 - 2n] - A_{n+1}^2(t) [B_n(t) - \nu - 1 - 2n] \\
- (x - B_n(t)) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] P_n^x(x,t)] = 0. 
\end{aligned}
\]

**Proof.** Differentiating the three term recurrence relation (1.4), we have

\[
\begin{aligned}
xt A_{n+1}^2(t) + A_{n+1}(t) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] P_{n+1}^x(x,t) \\
+ [xt B_n(t) + A_n(t) [B_{n-1}(t) - \nu + 1 - 2n] - A_{n+1}^2(t) [B_n(t) - \nu - 1 - 2n] \\
- (x - B_n(t)) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] P_n^x(x,t)] = 0.
\end{aligned}
\]

Then, using (1.4), (2.5) and Corollary 2, after straightforward simplifications the latter equality becomes

\[
\begin{aligned}
xt A_{n+1}^2(t) + A_{n+1}(t) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] P_{n+1}^x(x,t) \\
+ [xt B_n(t) + A_n(t) [B_{n-1}(t) - \nu + 1 - 2n] - A_{n+1}^2(t) [B_n(t) - \nu - 1 - 2n] \\
- (x - B_n(t)) \left[ A_n^2(t) + \frac{b_n(t)}{a_n(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] P_n^x(x,t)] = 0.
\end{aligned}
\]
Proof.
The proof is immediate, letting $Hence, taking into account (2.27), (3.4), equality (3.14) leads to the final form (3.15).

It can be rewritten, appealing to (2.36), and we obtain

$$
\left[A_{n+1}^2(t) + \frac{b_{n+1}(t)}{a_{n+1}(t)} + \frac{x}{2} [B_n(t) - \nu - 1 - 2n] \right] A_{n+1}(t) P_n^e(x, t)
$$

$$
+ \left[ xB_n(t) + A_n^2(t) [x + B_{n-1}(t) - \nu + 1 - 2n] - A_{n+1}^2(t) [x + B_{n+1}(t) - \nu - 3 - 2n] \right]
$$

Hence, taking into account (2.27), (3.4), equality (3.14) leads to the final form (3.15).

**Corollary 8.** The following identity takes place

$$
tB_n(t)B'_n(t) + A_n^2(t) [B_{n-1}(t) - \nu + 1 - 2n] - A_{n+1}^2(t) [B_{n+1}(t) - \nu - 3 - 2n] = 0. \tag{3.16}
$$

*Proof.* The proof is immediate, letting $x = 0$ in (3.14) and recalling (2.27), (3.4).

**Corollary 9.** Coefficients $B_n$ obey the following integral-recurrence relation

$$
B_n(t) + B_{n-1}(t) - 2n - \nu + 1 = \exp \left( \int_0^t \frac{B_{n-1}(y) - B_n(y) + 2}{y} dy \right)
$$

$$
\times \left( 2 \int_0^t \exp \left( - \int_0^y \frac{B_{n-1}(u) - B_n(u) + 2}{u} du \right) [B_{n-1}(y) - 2n - \nu + 1] \frac{dy}{y} + 2n + \nu + 1 \right). \tag{3.17}
$$

*Proof.* In fact, writing (3.3) in the form

$$
\frac{d}{dt} \left[ B_n(t) + B_{n-1}(t) - 2n - \nu + 1 \right] + \frac{1}{t} [B_{n-1}(t) - B_n(t) + 2] [B_{n-1}(t) + B_n(t) - 2n - \nu + 1]
$$

$$
- \frac{2}{t} [B_{n-1}(t) - 2n - \nu + 1] = 0,
$$

we solve a simple Cauchy problem for the first order differential equation to obtain (3.17).
In the sequel, let us consider polynomial coefficients \((3.12)\) \(a_{n,k}(t)\) as functions of \(t\) as well, i.e. \(a_{n,k} \equiv a_{n,k}^\nu\). Then, returning to the formula \((1.8)\) for the moments, we represent orthogonal polynomials \(P_n^\nu\) in terms of the Hankel determinant

\[
P_n^\nu(x,t) = \frac{1}{\left[G_{n-1}^\nu(t)G_n^\nu(t)\right]^{1/2}} | \begin{array}{cccc}
\rho_{v+1}(t) & \rho_{v+2}(t) & \ldots & \rho_{v+n+1}(t) \\
\rho_{v+2}(t) & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\rho_{v+n}(t) & \ldots & \ldots & \rho_{v+2n}(t) \\
1 & x & \ldots & x^n 
\end{array} |, \quad (3.18)
\]

where

\[
G_n^\nu(t) = \left| \begin{array}{cccc}
\rho_{v+1}(t) & \rho_{v+2}(t) & \ldots & \rho_{v+n+1}(t) \\
\rho_{v+2}(t) & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\rho_{v+n+1}(t) & \ldots & \ldots & \rho_{v+2n+1}(t) 
\end{array} \right|. \quad (3.19)
\]

Hence, employing the Laplace theorem to the last row, we find from \((3.18), (3.19)\)

\[
G_n^\nu(x,t) = (-1)^n \rho_{v+n+1}(t) G_{n-1}^{\nu+1}(t) + \rho_{v+2n+1}(t) G_{n-1}^\nu(t) + (-1)^{n+1} \sum_{k=2}^{n} (-1)^k \rho_{v+n+k}(t) G_{n-k}^\nu(t)
\]

\[
= [G_{n-1}^\nu(t)G_n^\nu(t)]^{1/2} \sum_{k=0}^{n} \rho_{v+n+k+1}(t) a_{n,k}^\nu(t), \quad (3.20)
\]

where

\[
G_{n,k}^\nu(t) = \left| \begin{array}{cccc}
\rho_{v+1}(t) & \rho_{v+2}(t) & \ldots & \rho_{v+k}(t) & \rho_{v+k+1}(t) & \ldots & \rho_{v+n+1}(t) \\
\rho_{v+2}(t) & \rho_{v+3}(t) & \ldots & \rho_{v+k}(t) & \rho_{v+k+1}(t) & \ldots & \rho_{v+n+2}(t) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\rho_{v+n}(t) & \ldots & \rho_{v+n+k}(t) & \rho_{v+n+k+1}(t) & \ldots & \rho_{v+2n}(t) 
\end{array} \right|. \quad (3.21)
\]

In particular, we easily find from \((3.18), (3.20)\) the expression for the free coefficient \(a_{n,0}^\nu(t)\) in terms of the determinant by the formula

\[
a_{n,0}^\nu(t) = \frac{(-1)^n G_{n,1}^\nu(t)}{[G_{n-1}^\nu(t)G_n^\nu(t)]^{1/2}}, \quad (3.22)
\]

Moreover, since the leading term \(a_n^\nu\) (cf. [2], Vol. II) has the representation

\[
|a_n^\nu(t)| = \left[\frac{G_{n-1}^\nu(t)}{G_n^\nu(t)}\right]^{1/2}, \quad (3.23)
\]

we find from \((3.22), (3.23)\) the equality

\[
\frac{G_{n-1}^\nu(t)}{G_n^\nu(t)} = (-1)^n a_{n,0}^\nu(t) a_{n,0}^\nu(t). \quad (3.24)
\]
Proof. Indeed, appealing to (2.16) and (3.23), we have

\[ (-1)^n a_n^\nu(t) a_{n,0}^\nu(t) = \frac{G_{n-1}^\nu(t) G_{n+1}^\nu(t)}{G_n^\nu(t) G_n^\nu(t)} \]

Therefore, it yields

\[ (-1)^n+1 \left[ a_n^\nu(t) \right]^2 a_{n+1,0}^\nu(t) a_{n+1}^\nu(t) \frac{G_{n+1}^\nu(t)}{G_n^\nu(t)} \]

Hence, recalling (2.11), we derive the identities

\[ \left( \frac{a_n^\nu(t)}{a_{n+1}^\nu(t)} \right)^2 = -A_n^\nu(t) \frac{a_{n,0}^\nu(t)}{a_{n+1,0}^\nu(t)} = \frac{[A_n^\nu(t)]^2 + b_{n+1}^\nu(t)}{2n + \nu + 3 - B_{n+1}^\nu(t)}, \]

\[ \left( a_n^\nu(t) \right)^2 = \frac{[a_n^\nu(t)]^2 + a_{n+1}^\nu(t)b_{n+1}^\nu(t)}{2n + \nu + 3 - B_{n+1}^\nu(t)}, \quad (3.25) \]

\[ \prod_{k=0}^n [a_k^\nu(t)]^2 = \frac{(-1)^{n+1} a_{n,0}^\nu(t)}{a_{n+1,0}^\nu(t)} \prod_{k=0}^n [a_k^\nu(t)]^2. \quad (3.26) \]

Moreover, (3.20), (3.23) imply

\[ \sum_{k=0}^n B_{n+k}^\nu(t) a_{n,k}^\nu(t) = \frac{1}{a_n^\nu(t)}, \quad (3.27) \]

and this agrees with (1.20).

**Corollary 10.** Coefficients \( B_n^\nu(t), A_{n+1}^\nu(t) \) (1.5) of the three term recurrence relation (1.4) are equal, correspondingly,

\[ B_n^\nu(t) = 2n + \nu + 1 + \frac{G_n^\nu(t) - G_{n-1}^\nu(t)}{G_n^\nu(t) G_{n-1}^\nu(t)}, \quad (3.28) \]

\[ A_{n+1}^\nu(t) = \frac{[G_{n-1}^\nu(t) G_n^\nu(t)]^{1/2}}{G_n^\nu(t)}. \quad (3.29) \]

**Proof.** Indeed, appealing to (2.16) and (3.23), we have

\[ \frac{G_n^\nu(t)}{G_{n-1}^\nu(t)} = \frac{1}{n! \Gamma(n + \nu + 1)} \exp \left( \int_0^t B_n(y) - y - 1 - 2n dy \right). \quad (3.30) \]

Therefore,

\[ G_n^\nu(t) = n! \Gamma(n + \nu + 1) G_{n-1}^\nu(t) \exp \left( \int_0^t \frac{y + 1 + 2n - B_n(y)}{y} dy \right), \]

and, solving this recurrence relation owing to (1.5) and (3.19), we easily derive

\[ G_n^\nu(t) = \prod_{k=0}^n \frac{k! \Gamma(k + \nu + 1) \exp \left( \int_0^t \frac{b_n^\nu(y)}{a_{n+1}^\nu(y)} + (n + 1)(n + \nu + 1) }{y} \right) dy . \]

Consequently,
\[ G'_n(t) = \frac{dG'_n(t)}{dt} = \frac{G'_n(t)}{t} \left[ \frac{b'_n(t)}{a'_{n+1}(t)} + (n+1)(n+1) \right]. \] (3.31)

Hence equality (3.28) follows immediately from (1.5). Formula (3.29) is a direct consequence of (1.5), (3.23) or (2.18), (3.30).

**Corollary 11.** The following equality holds

\[ t^2 \frac{d}{dt} \left[ \frac{G'_n(t)}{G'_n(t)} \right] = A_{n+1} - (n+1)(n+1+\nu). \] (3.32)

**Proof.** The proof is immediate, involving differentiation in (3.31) and employing (2.26).

On the other hand, the key identity (1.16) for the moments and properties of the determinants allow to treat (3.19) when \( n = 2, 3, \ldots \), as follows

\[
tG'_n(t) = \begin{vmatrix}
  t\rho_{\nu+1}(t) & \rho_{\nu+2}(t) & \cdots & \rho_{\nu+n+1}(t) \\
  t\rho_{\nu+2}(t) & \cdots & \cdots & \rho_{\nu+n+2}(t) \\
  \cdots & \cdots & \cdots & \cdots \\
  t\rho_{\nu+n+1}(t) & \cdots & \cdots & \rho_{\nu+2n+1}(t)
\end{vmatrix}
= -\begin{vmatrix}
  (\nu+2)\rho_{\nu+2}(t) & \rho_{\nu+3}(t) & \cdots & \rho_{\nu+n+2}(t) \\
  (\nu+3)\rho_{\nu+3}(t) & \rho_{\nu+4}(t) & \cdots & \rho_{\nu+n+3}(t) \\
  \cdots & \cdots & \cdots & \cdots \\
  (\nu+n+2)\rho_{\nu+n+2}(t) & \rho_{\nu+n+3}(t) & \cdots & \rho_{\nu+2n+1}(t)
\end{vmatrix}
= -\begin{vmatrix}
  0 & \rho_{\nu+2}(t) & \cdots & \rho_{\nu+n+1}(t) \\
  \rho_{\nu+3}(t) & \rho_{\nu+3}(t) & \cdots & \rho_{\nu+n+2}(t) \\
  2\rho_{\nu+4}(t) & \rho_{\nu+4}(t) & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \ddots \\
  n\rho_{\nu+n+2}(t) & \rho_{\nu+n+2}(t) & \cdots & \rho_{\nu+2n+1}(t)
\end{vmatrix}
\]

Continuing this process, we find via Laplace’s theorem and (1.16)

\[
(-1)^{n+1}t^{n-1}G'_n(t) = \begin{vmatrix}
  0 & 0 & \cdots & 0 & \rho_{\nu+n}(t) & \rho_{\nu+n+1}(t) \\
  \rho_{\nu+3}(t) & \rho_{\nu+4}(t) & \cdots & \rho_{\nu+n+1}(t) & \rho_{\nu+n+2}(t) \\
  2\rho_{\nu+4}(t) & 2\rho_{\nu+5}(t) & \cdots & 2\rho_{\nu+n+2}(t) & \rho_{\nu+n+3}(t) \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  n\rho_{\nu+n+2}(t) & n\rho_{\nu+n+3}(t) & \cdots & n\rho_{\nu+2n}(t) & \rho_{\nu+2n+1}(t)
\end{vmatrix}. \] (3.33)

On the other hand, using continuously (1.16) from the last column in (3.19), we arrive in the same manner at the equality
Generally, we find via (1.16)

\[
G_n^t = \begin{vmatrix}
\rho_{v+1}(t) & \rho_{v+2}(t) & 0 & 0 & \ldots & 0 \\
\rho_{v+2}(t) & \rho_{v+3}(t) & \rho_{v+4}(t) & \rho_{v+4}(t) & \ldots & \rho_{v+n+1}(t) \\
\rho_{v+3}(t) & \rho_{v+4}(t) & 2\rho_{v+4}(t) & 2\rho_{v+5}(t) & \ldots & 2\rho_{v+n+2}(t) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{v+n+1}(t) & \rho_{v+n+2}(t) & np_{v+n+2}(t) & np_{v+n+3}(t) & \ldots & np_{v+2n}(t)
\end{vmatrix}.
\]  

(3.34)

Hence, it is not difficult to observe from (3.33), (3.34), (3.35) that

\[
H_{j,j}^{v,n}(t) = 0, \quad j \in \mathbb{Z}.
\]

(3.36)

Hence, let us consider the following double sequence of determinants

\[
H_{j,j}^{v,n}(t) = \begin{vmatrix}
\rho_{v+i}(t) & \rho_{v+j}(t) & 0 & 0 & \ldots & 0 \\
\rho_{v+i+1}(t) & \rho_{v+j+1}(t) & \rho_{v+j+1}(t) & \rho_{v+j+1}(t) & \ldots & \rho_{v+v+1}(t) \\
\rho_{v+i+2}(t) & \rho_{v+j+2}(t) & 2\rho_{v+j+2}(t) & 2\rho_{v+j+2}(t) & \ldots & 2\rho_{v+v+2}(t) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{v+n+i}(t) & \rho_{v+n+j}(t) & np_{v+n+2}(t) & np_{v+n+3}(t) & \ldots & np_{v+2n}(t)
\end{vmatrix}.
\]

(3.35)

where \((i, j) \in \mathbb{Z}^2\). Generally, we find via (1.16)

\[
H_{j,j}^{v,n}(t) = (v + i - 1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

\[
+ \begin{vmatrix}
0 & \rho_{v+j}(t) & 0 & 0 & \ldots & 0 \\
\rho_{v+i}(t) & \rho_{v+i+1}(t) & \rho_{v+i+1}(t) & \rho_{v+i+1}(t) & \ldots & \rho_{v+v+1}(t) \\
2\rho_{v+i+1}(t) & \rho_{v+i+2}(t) & 2\rho_{v+i+2}(t) & 2\rho_{v+i+2}(t) & \ldots & 2\rho_{v+v+2}(t) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
np_{v+n+i-1}(t) & \rho_{v+n+j}(t) & np_{v+n+2}(t) & np_{v+n+3}(t) & \ldots & np_{v+2n}(t)
\end{vmatrix}
\]

\[
= (v + i - 1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

\[
- n!\rho_{v+j}(t)
\]

\[
H_{j,j}^{v,n}(t) = (v + i - 1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

\[
- n!\rho_{v+j}(t)
\]

\[
H_{j,j}^{v,n}(t) = (v + j - 1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

(3.36)

Analogously, it has

\[
H_{j,j}^{v,n}(t) = (v + j - 1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]
above to the derivative, we deduce the equalities

\[
\begin{bmatrix}
\rho_{v+j(t)} & \rho_{v+3(t)} & \rho_{v+4(t)} & \cdots & \rho_{v+n+1(t)} \\
\rho_{v+j+1(t)} & \rho_{v+4(t)} & \rho_{v+5(t)} & \cdots & \rho_{v+n+2(t)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{v+n+j-1(t)} & \rho_{v+n+2(t)} & \rho_{v+n+3(t)} & \cdots & \rho_{v+2n(t)}
\end{bmatrix},
\]

(3.38)

Thus we find

\[
(v+i-1)H_{i-1,j}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

\[
- n! \rho_{v+j(t)}
\]

\[
(v+j-1)H_{i,j-1}^{v,n}(t) + tH_{i-2,j}^{v,n}(t)
\]

\[
+ n! \rho_{v+i(t)}
\]

\[
\begin{bmatrix}
\rho_{v+j(t)} & \rho_{v+3(t)} & \rho_{v+4(t)} & \cdots & \rho_{v+n+1(t)} \\
\rho_{v+j+1(t)} & \rho_{v+4(t)} & \rho_{v+5(t)} & \cdots & \rho_{v+n+2(t)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{v+n+j-1(t)} & \rho_{v+n+2(t)} & \rho_{v+n+3(t)} & \cdots & \rho_{v+2n(t)}
\end{bmatrix},
\]

(3.39)

As an immediate consequence, when \((i,j) \in [3, \ldots, n+1] \times [3, \ldots, n+1]\) the following recurrence relation holds

\[
(v+i-1)H_{i-1,j}^{v,n}(t) + (v+j-1)H_{j-1,i}^{v,n}(t) + t \left[ H_{i-2,j}^{v,n}(t) + H_{j-2,i}^{v,n}(t) \right] = 0.
\]

(3.40)

Recalling (3.36), (3.37), it yields

\[
H_{n+2,n+1}^{v,0}(t) = (-1)^{n+1} t^{n-1} G_{n}^{v}(t) - n! \rho_{v+n+1(t)} G_{n-1}^{v+2}(t).
\]

(3.41)

On the other hand, differentiating determinant (3.19) by virtue of (1.15) and applying the same process as above to the derivative, we deduce the equalities

\[
d\frac{dG_{n}^{v}}{dt} =
\begin{bmatrix}
\rho_{v+2(t)} & \cdots & \rho_{v+n+1(t)} \\
\rho_{v+3(t)} & \cdots & \rho_{v+n+2(t)} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\rho_{v+n+1(t)} & \cdots & \rho_{v+2n+1(t)}
\end{bmatrix},
\]

\[
\begin{bmatrix}
0 & \rho_{v+2(t)} & \cdots & \rho_{v+n+1(t)} \\
\rho_{v+3(t)} & \cdots & \rho_{v+n+2(t)} \\
2\rho_{v+3(t)} & \rho_{v+4(t)} & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
np_{v+n+1(t)} & \rho_{v+n+2(t)} & \cdots & \rho_{v+2n+1(t)}
\end{bmatrix}
\]

\[
= (v+1)G_{n}^{v}(t)
\]
But from (3.35) we observe

Consequently, it gives by virtue of (3.19), (1.16)

\[ \frac{dG_n^\nu}{dt} = (v + 1)G_n^\nu(t) - n! \rho_{v+2}(t)G_{n-1}^{\nu+1}(t) \]

But from (3.35) we observe

\[ \frac{d}{dt} \left| \begin{array}{ccc} 0 & \rho_{v+2}(t) & \rho_{v+3}(t) & 0 & \cdots & 0 & 0 \\ \rho_{v+2}(t) & \rho_{v+3}(t) & \rho_{v+4}(t) & \rho_{v+4}(t) & \cdots & \rho_{v+n}(t) & \rho_{v+n+1}(t) \\ \rho_{v+3}(t) & \rho_{v+4}(t) & \rho_{v+5}(t) & 2\rho_{v+5}(t) & \cdots & 2\rho_{v+n}(t) & 2\rho_{v+n+1}(t) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n\rho_{v+n+1}(t) & \rho_{v+n+2}(t) & \rho_{v+n+3}(t) & \cdots & \cdots & \cdots & \cdots \\ \end{array} \right| = 0 

\[ \frac{d}{dt} \left| \begin{array}{ccc} 0 & \rho_{v+2}(t) & \rho_{v+1}(t) & 0 & \cdots & 0 & 0 \\ \rho_{v+1}(t) & \rho_{v+2}(t) & \rho_{v+2}(t) & \rho_{v+4}(t) & \cdots & \rho_{v+n}(t) & \rho_{v+n+1}(t) \\ \rho_{v+2}(t) & \rho_{v+3}(t) & \rho_{v+3}(t) & 2\rho_{v+3}(t) & \cdots & 2\rho_{v+n}(t) & 2\rho_{v+n+1}(t) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{v+n}(t) & \rho_{v+n+1}(t) & \rho_{v+n+1}(t) & \rho_{v+n+1}(t) & \cdots & \cdots & \cdots \\ \end{array} \right| + t \]

Hence we derive from (3.42) the identity
\[
\frac{dG^y}{dt} - (v + 1)G^y(t) + n!\rho_{v+2}(t)G^{y+1}(t) + t^2 \left[ \frac{d}{dt} \left[ H^y_{0,1}(t) + H^{y,m}_{-1,1}(t) \right] \right] = 0. \tag{3.43}
\]

**Remark 3.** By using these features an interesting open question is to obtain a differential-difference recurrence relation for determinants (3.19).

4. **Rodrigues-Type Formula. Generating Function**

Let us expand the function \( e^{-t/x}P^v_n(x,t) \) in terms of the Laguerre polynomials \( L^y_n(x) \). It gives

\[
e^{-t/x}P^v_n(x,t) = \sum_{k=0}^{\infty} d^v_{n,k}(t)L^y_k(x), \tag{4.1}
\]

where

\[
d^v_{n,k}(t) = \frac{k!}{\Gamma(k + v + 1)} \int_{0}^{\infty} e^{-x/t}P^v_n(x,t)L^y_k(x)x^vdx. \tag{4.2}
\]

But from (1.18) we have \( d^v_{n,k}(t) = 0, \ k = 0, \ldots, n - 1 \). The uniform estimate with respect to \( k \) for coefficients \( d^v_{n,k}(t) \) is given by the following lemma.

**Lemma 2.** Let \( t > 0 \) \( v > -1 \). Coefficients \( d^v_{n,k}(t) \), \( n, k \in \mathbb{N}_0 \) satisfy the upper bound of the form

\[
|d^v_{n,k}(t)| \leq \frac{k!h^v_n(t)}{\Gamma(k + v + 1)}, \tag{4.3}
\]

where

\[
h^v_n(t) = 2^{v-1/2} \int_{0}^{t} Q^v_n(t-y)\rho_{2v+1}^{1/2}(2y) \frac{dy}{\sqrt{y}} \tag{4.4}
\]

and

\[
Q^v_n(x) = \sum_{m=0}^{n} |a^v_{n,m}(t)| \frac{x^m}{m!}. \tag{4.5}
\]

**Proof.** Writing polynomial \( P^v_n \) in the explicit form, equality (4.2) becomes

\[
d^v_{n,k}(t) = \frac{k!}{\Gamma(k + v + 1)} \sum_{m=0}^{\infty} a^v_{n,m}(t) \int_{0}^{\infty} e^{-x/t}L^y_k(x)x^v+mdx. \tag{4.6}
\]

Recalling (1.11), the latter integral can be represented as follows

\[
\int_{0}^{\infty} e^{-x/t}L^y_k(x)x^v+mdx = (-1)^{k+m+1} \frac{d^{k-m-1}}{dt^{k-m-1}} \int_{0}^{\infty} x^{v+k-1}e^{-x/t}L^y_k(x)dx
\]

\[
= \frac{(-1)^{m+1}}{k!} \frac{d^{k-m-1}}{dt^{k-m-1}} \left[ t^k\rho_{v}(t) \right] = \frac{(-1)^{m+1}}{k!m!} \int_{0}^{t} (t-y)^m \frac{d^k}{dy^k} y^k\rho_{v}(y)dy, \tag{4.7}
\]

where we mean (cf. (1.12))

\[
\frac{d^{-q}f}{dt^{-q}} \equiv (t^q \cdot f)(t) = \frac{1}{(q-1)!} \int_{0}^{t} (t-y)^{q-1}f(y)dy, \quad q \in \mathbb{N}_0.
\]

Taking into account (4.6), coefficients \( d^v_{n,k}(t) \) can be written in the operator form.
Meanwhile, via (1.11) and the Rodrigues formula for Laguerre polynomials it has
\[ d^k_y \rho_\nu(y) = k! \int_0^\infty x^{\nu-1} e^{-x} L_k^{(\nu)}(x) dx = \int_0^\infty x^{\nu-1} e^{-x} \frac{d^k}{dx^k} \left[ e^{-x} x^{\nu+k} \right] dx. \]

Integrating by parts with the use of Entry 1.1.3.2 on p. 4 in [1], we get
\[ \frac{d^k}{dy^k} \left[ y^k \rho_\nu(y) \right] = (-1)^k \int_0^\infty \frac{d^k}{dx^k} \left[ x^{\nu-1} e^{-x} \right] e^{-x} x^{\nu+k} dx \]
\[ = k! \int_0^\infty e^{-x} x^{\nu-1} L_k \left( \frac{x}{y} \right) dx, \]
where \( L_k \) are Laguerre polynomials of the index zero. Then owing to (1.7), Schwarz’s inequality and orthogonality of Laguerre polynomials, we find the estimate
\[ \left| \frac{d^k}{dy^k} \left[ y^k \rho_\nu(y) \right] \right| \leq k! \left( \int_0^\infty e^{-x} x^{\nu-2} x^{2(\nu+1)} dx \right)^{1/2} \left( \int_0^\infty e^{-x} [L_k(x)]^2 dx \right)^{1/2} \]
\[ = \frac{2^{\nu-1/2} k!}{\sqrt{\nu}} \rho_{2\nu+1}(2y). \]

Therefore, returning to (4.6), (4.7), we derive
\[ |d^\nu_{n,k}(t)| \leq \frac{2^{\nu-1/2} k!}{\Gamma(k+\nu+1)} \int_0^t \sum_{m=0}^n \left| a^\nu_{n,m}(t) \right| \frac{(t-y)^m}{m!} \rho_{2\nu+1}(2y) \frac{dy}{\sqrt{\nu}}, \]
which implies (4.3) and completes the proof of Lemma 2.

\[ \square \]

**Corollary 12.** Under the condition \( \nu > 3/2 \) the Laguerre series (4.1) converges absolutely and uniformly on closed intervals of \( \mathbb{R}_+ \).

**Proof.** In fact, Stirling’s asymptotic formula for gamma-function [2], Vol. I yields
\[ \frac{k!}{\Gamma(k+\nu+1)} = O(k^{-\nu}), \quad k \to \infty. \]
Since Laguerre polynomials \( L_k^{(\nu)}(x) \) behave as \( O(k^{\nu/2-1/4}), k \to \infty \) uniformly on closed intervals \( x \in [\alpha, \beta] \) of \( \mathbb{R}_+ \), the absolute and uniform convergence of the series (4.1) is guaranteed under the assumption \( \nu > 3/2 \).

\[ \square \]

Further, writing (4.1) as follows
\[ e^{-t/x} P_\nu^\nu(t, x) = \sum_{k=n}^\infty d^\nu_{n,k}(t)L_k^{(\nu)}(x) = x^{-\nu} e^t \sum_{k=n}^\infty \frac{d^\nu_{n,k}(t)}{k!} \frac{d^k}{dx^k} \left[ x^{k+\nu} e^{-x} \right] \]
\[ = x^{-\nu} e^t \sum_{k=0}^\infty \frac{d^\nu_{n,k+n}(t)}{(k+n)!} \frac{d^{k+n}}{dx^{k+n}} \left[ x^{k+n+\nu} e^{-x} \right] \]
the problem arises to interchange the order of differentiation and summand. It can be done by virtue of the absolute and uniform convergence of the series

\[ \sum_{k=0}^{\infty} \frac{d_{n,k+n}^n(t) k!}{(k+n)!} \frac{d^n}{dx^n} \left[ e^{-x} x^{y+n} L_k^{n+y}(x) \right], \]

\[ = x^{-v} e^x \sum_{k=0}^{n} \frac{d_{n,k+n}^n(t) k!}{(k+n)!} \frac{d^n}{dx^n} \left[ e^{-x} x^{y+n} L_k^{n+y}(x) \right], \]

Recurrence relations for coefficients \( d_{n,k}^n(t) \) are given by

**Theorem 8.** Coefficients \( d_{n,k}^n(t) \) obey recurrence relations of the form

\[ (2k + v + 1 - (k + 1)B_v^n(t)) d_{n,k}^n(t) - (k + 1)A_{n+1}^v(t) d_{n+1,k}^n(t) - (k + v + 1) d_{n,k+1}^n(t) \]

\[ - (k + 1) A_v^v(t) d_{n-1,k}^n(t) - k d_{n,k-1}^n(t) = 0, \]

\[ (2k + n + 1 + v) A_{n+1}^v(t) d_{n+1,k}^n(t) - (k + v + 1) A_{n+1}^v(t) d_{n+1,k+1}^n(t) \]

\[ + \left[ (1 + n) B_v^n(t) - (k + v + 2)(2k + v + 1) - t + \left[ A_v^v(t)^2 - A_{n+1}^v(t) B_v^n(t) - v - 1 - 2n \right] \right] d_{n,k}^n(t) \]

\[ + (k + v + 1)(k + v + 2 - B_v^n(t)) d_{n,k+1}^n(t) - (k + v + 1) A_v^v(t) d_{n-1,k+1}^n(t) \]

\[ + \left[ (2k + n + v + 1 + B_v^n(t)) A_v^v(t) - \left[ A_v^v(t)^2 + \frac{b_v^n(t)}{a_v^v(t)} \right] \right] d_{n-1,k}^n(t) \]

\[ + k(k + v + 2) d_{n-1,k-1}^n(t) + A_v^v(t) A_{n-1}^v(t) d_{n-2,k}^n(t) = 0. \]

**Proof.** Indeed, relation (4.11) follows immediately from the three recurrence relations (1.3), (1.4) for Laguerre polynomials and polynomials \( P_n^v \), respectively. In order to prove (4.12), we integrate by parts in (4.2) to obtain

\[ d_{n,k}^n(t) = \frac{k!}{i^k(k + v + 1)} \left[ \int_0^\infty e^{-x_i/s_P^n(x,t)} L_k^n(x) x^{v+1} dx \right] \]

\[ - (v + 2) \int_0^\infty e^{-x_i/s_P^n(x,t)} L_k^n(x) x^{v+1} dx + \int_0^\infty e^{-x_i/s_P^n(x,t)} L_k^{n+1}(x) x^{v+2} dx \]

\[ - \int_0^\infty e^{-x_i/s_P^n(x,t)} \frac{d}{dx} \left[ P_n^v(x,t) L_k^v(x) x^{v+2} dx \right]. \]
But using the known relation for Laguerre polynomials [5]

\[ xL_{k+1}^\nu(x) = (k + \nu)L_k^\nu(x) - kL_{k-1}^\nu(x), \]
equality (4.13) becomes

\[ d_{n,k}^\nu(t) = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} p_n^\nu(x,t) L_k^\nu(x) x^{\nu + 2} dx \]

\[ - (v + 2 + k) \int_0^\infty e^{-x-t/\nu} p_n^\nu(x,t) L_k^\nu(x) x^{\nu + 1} dx + (k + v) \int_0^\infty e^{-x-t/\nu} p_n^\nu(x,t) L_{k-1}^\nu(x) x^{\nu + 1} dx \]

\[ - \int_0^\infty e^{-x-t/\nu} \frac{\partial}{\partial x} [p_n^\nu(x,t) L_k^\nu(x) x^{\nu + 2}] dx. \] (4.14)

Then, employing again recurrence relations (1.3), (1.4) and differential-difference equation (2.5), we derive

\[ \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} p_n^\nu(x,t) L_k^\nu(x) x^{\nu + 2} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left[ \left( 2k + 1 + v \right) L_k^\nu(x) \right] x^\nu dx \]

\[ = \frac{1}{t} \left[ \left( 2k + 1 + v \right) L_k^\nu(x) \right] x^\nu dx \]

\[ = \frac{1}{t} \int_0^\infty e^{-x-t/\nu} \left[ A_{n+1}^\nu(t) d_{n+1,k}^\nu(t) + B_n^\nu(t) d_{n,k}^\nu(t) + A_n^\nu(t) d_{n-1,k}^\nu(t) \right] \]

\[ - \left( k + 1 + v \right) \left[ A_{n+1}^\nu(t) d_{n+1,k+1}^\nu(t) + B_n^\nu(t) d_{n,k+1}^\nu(t) + A_n^\nu(t) d_{n-1,k+1}^\nu(t) \right] \]

\[ - \left( k + v \right) \left[ A_{n+1}^\nu(t) d_{n+1,k-1}^\nu(t) + B_n^\nu(t) d_{n,k-1}^\nu(t) + A_n^\nu(t) d_{n-1,k-1}^\nu(t) \right] \]

\[ \times \left( (2k + 1 + v) L_k^\nu(x) - (k + 1) L_{k+1}^\nu(x) - (k + v) L_{k-1}^\nu(x) \right) x^\nu dx \]

\[ = \frac{k + v + 2}{t} \int_0^\infty e^{-x-t/\nu} \left[ (2k + 1 + v) d_{n,k}^\nu(t) - (k + 1 + v) d_{n,k+1}^\nu(t) - k d_{n,k-1}^\nu(t) \right] \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]

\[ = \frac{k!}{t\Gamma(k + v + 1)} \int_0^\infty e^{-x-t/\nu} \left( \left( 2k + 1 + v \right) L_k^\nu(x) \right) x^{\nu + 1} dx \]
Hence, substituting these values into (4.13), we get after simplification the desired relation (4.12).

Finally, defining as usual the generating function $G(x, w, t)$ in terms of the series

$$G(x, w, t) = \sum_{n=0}^{\infty} P_n^\nu(x, t) \frac{w^n}{n!}, \quad x > 0, \ w \in \mathbb{C},$$

it can be written due to (4.10) in the form

$$G(x, w, t) = x^{-\nu} e^{x/t} \sum_{n=0}^{\infty} \frac{w^n}{n!} d^n x \left[ e^{-x} x^\nu \sum_{k=0}^{\infty} \frac{d^{k+n} t}{(k+n)!} L_k^{n+\nu}(x) \right].$$

The convergence of the series (4.15) is guaranteed at least in $L_2(\mathbb{R}_+: e^{-x-t} x^\nu dx)$. Indeed, by virtue of the Minkowski inequality it has

$$\left( \int_0^\infty e^{-x-t} x^\nu \left[ \sum_{n=0}^{\infty} P_n^\nu(x, t) \frac{w^n}{n!} x^\nu dx \right] \right)^{1/2} \leq \sum_{n=0}^{\infty} \frac{|w|^n}{n!} \left( \int_0^\infty e^{-x-t} \left[ P_n^\nu(x, t) x^\nu dx \right] \right)^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{|w|^n}{n!} \rightarrow 0, \quad N \rightarrow \infty.$$

REFERENCES

1. Y. A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas, Chapman and Hall/CRC, 2008.
2. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher Transcendental Functions, Vols. I and II, McGraw-Hill, New York, London, Toronto, 1953.
3. M.E.H. Ismail, Bessel functions and the infinite divisibility of the student r-distribution, Ann. Prob. 5 (1977), 582-585.
4. A. P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, Vol. I: Elementary Functions, Gordon and Breach, New York, London, 1986; Vol. II: Special Functions, Gordon and Breach, New York, London, 1986; Vol. III: More Special Functions, Gordon and Breach, New York, London, 1990.
5. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq, Publ. XXIII, 1939.
6. S. Yakubovich and Yu. Luchko, The Hypergeometric Approach to Integral Transforms and Convolutions, Kluwer Academic Publishers, Mathematics and Applications. Vol.287, 1994.