The Large Time Behaviour in Quantum Field Theory and Quantum Chaos

I.Ya. Aref’eva and I.V. Volovich

Steklov Mathematical Institute, Russian Academy of Sciences
Gubkin St.8, GSP-1, 117966, Moscow, Russia
arefeva@mi.ras.ru, volovich@mi.ras.ru

Abstract

An exact general formula for the matrix elements of the evolution operator in quantum theory is established. The formula (“ABC-formula”) has the form $\langle U(t) \rangle = \exp(At + B + C(t))$. The constants $A$ and $B$ and the decreasing function $C(t)$ are computed in perturbation theory. The ABC-formula is valid for a general class of Hamiltonians used in statistical physics and quantum field theory. The formula includes the higher order corrections to the well known Weisskopf-Wigner approximation and to the stochastic (van Hove) limit which are widely used in considerations of problems of radiation, decay, quantum decoherence, derivation of master and kinetic equations etc. The function $C(t)$ admits an interpretation as an analogue of the autocorrelation function describing quantum chaos for the quantum baker’s map.
1 Introduction

Scattering theory is one of the very old and developed topics in quantum theory. The study of the large time behaviour of the evolution operator in statistical physics and quantum field theory is the subject of numerous investigations. The basic object to study in quantum field theory is the scattering matrix, see in particular [1]–[6] and references therein.

However, there are many important problems in quantum field theory where we are interested in the large but not infinite time and where the standard $S$-matrix description is not very convenient or even not applicable. These include processes with unstable particles [7, 8], atom-photon interactions [9], elementary particles in ”semidressed states” with non-equilibrium proper fields [10], electroweak baryogenesis and phase transitions in the early Universe and in high-energy collisions [11], quantum optics [12], quantum decoherence (see for example [12, 13]) etc. In the consideration of such processes we are interested in the time regime smaller than the ”infinite” time when the $S$-matrix description becomes applicable. The consideration of such processes belongs to non-equilibrium quantum field theory, see [14] for more discussions.

Various approximate methods of consideration of time evolution for classical and quantum systems have been developed by Bogoliubov [15], Weisskopf and Wigner (see [5]), van Hove [16], Prigine [17] and many others, see [18, 19]. The purpose of this paper is to obtain an exact general result about the large time behaviour of certain matrix elements of the evolution operator

If $H$ is a generic self-adjoint operator in a Hilbert space and we are interested in the study of the corresponding evolution operator $e^{-itH}$ then we can say nothing interesting about the behaviour of its matrix elements

$$(\psi, e^{-itH} \psi) = \int e^{-it\sigma} d\rho(\sigma)$$

We have to restrict ourself to the Hamiltonians of some special forms if we want to obtain useful results. In this paper we consider a general class of Hamiltonians used in solid state physics and quantum field theory

$$H = H_0 + \lambda V,$$

where $H_0$ is a free Hamiltonian, $V$ describes an interaction and $\lambda$ is the coupling constant. We shall study the evolution operator

$$U(t) = e^{itH_0} e^{-itH}. \quad (1.2)$$

The main result of this paper is the following exact formula (we call it the $ABC$-formula) which is valid for arbitrary time $t$

$$\langle U(t) \rangle = e^{A t + B + C(t)}. \quad (1.3)$$

Here $A$ and $B$ are constants for which a representation in perturbation theory will be given and $C(t)$ is a function which under rather general assumptions can be represented for large time $t$ as

$$C(t) = \frac{f(t)}{t^\alpha}. \quad (1.4)$$
Here $f(t)$ is a bounded function and the exponent $\alpha$ depends on the model and on the dimension of space ($\alpha = 3/2$ for the physical 3-dimensional space). The function $C(t)$ admits a representation which has the form similar to the autocorrelation function describing quantum chaos, see below.

The well known Weisskopf-Wigner approximation is given by

$$\langle U(t) \rangle \simeq e^{at},$$

(1.5)

where $a$ is a constant. It corresponds to the particular case of the $ABC$ - formula (1.3).

Expectation value in (1.3) is taken over the vacuum vector. For the case of one-particle states we obtain

$$\langle p | U(t) | p' \rangle = e^{iA(p)t + B(p) + C(t,p)} \delta(p - p').$$

(1.6)

The formulae (1.3) and (1.6) have a very general character. We prove (1.3) in Section 3 for a wide class of Hamiltonians. The class of considered Hamiltonians includes the Bose and Fermi gases, phonon self-interaction and electron-phonon interaction, quantum electrodynamics in external fields etc.

We derive the main formula (1.3) by using the theory of perturbation of spectra and renormalized wave operators [5, 6]. This method can be used only under some restrictions to the form of the Hamiltonian when one has not decay. Another method based on the direct examination of perturbation theory which can be used also in the case of decay will be considered in another publication, see [20].

2 Notations and auxiliary results

2.1 Hamiltonians

We consider Hamiltonians of the form (1.1) where $H_0$ is a free Hamiltonian

$$H_0 = \sum_i \int \omega_i(k) a_i^\dagger(k) a_i(k) d^dk$$

(2.7)

and $V$ is the sum of Wick monomials. Creation and annihilation operators $a_i^\dagger(k)$, $a_i(k)$ describe particles or quasiparticles and they satisfy the commutation or anticommutation relations

$$[a_i(k), a_j^\dagger(k')]_{\pm} = \delta_{ij} \delta(k - k')$$

(2.8)

Here $k, k' \in R^d$ and $i, j = 1, ..., N$ label the finite number of different types of (quasi)particles. Creation and annihilation operators have the standard realization in the Fock space with the vacuum vector which will be denoted $|0\rangle$ or $\Phi_0$. Examples of one-particle energy $\omega_i(k)$ include the relativistic ($\omega(k) = (k^2 + m^2)^{1/2}$) and non-relativistic ($\omega(k) = k^2/2 - \omega_0$) laws, the Bogoliubov spectrum ($\omega(k) = (bk^4 + k^2v(k))^{1/2}$), the Fermi quasiparticle spectrum ($\omega(k) = |k^2/2m - \mu|$) etc.

We consider two different types of Wick polynomials. The first type describes an interaction in the case when there is not the translation invariance.
\[
V = \sum_{I,J,i,j} \int v(p_1, i_1 \ldots p_I, i_I | q_1, j_1 \ldots q_J, j_J) \prod_{l=1}^{I} a_{i_l}^*(p_l) dp_l \prod_{r=1}^{J} a_{j_r}(q_r) dq_r \tag{2.9}
\]

were \(v(p_1, i_1 \ldots p_I, i_I | q_1, j_1 \ldots q_J, j_J)\) are some test functions.

The second type is described by the translation invariant Hamiltonian

\[
V = \sum_{I,J} V_{I,J} = \sum_{I,J,i,j} \int \hat{v}(p_1, i_1 \ldots p_I, i_I | q_1, j_1 \ldots q_J, j_J) \tag{2.10}
\]

\[
\delta \left( \sum_l p_l - \sum_r q_r \right) \prod_{l=1}^{I} a_{i_l}^*(p_l) dp_l \prod_{r=1}^{J} a_{j_r}(q_r) dq_r
\]

Clearly the delta function causes the trouble and there are singular terms in (2.10). Namely, \(V_{I,0}\) does not belong to the Fock space unless the vector \(\phi = 0\). This singularity is called the volume singularity. To give a meaning to the Hamiltonian with interaction (2.10) one has to introduce a volume cut-off, then perform the vacuum renormalization and vacuum dressing and only after that remove the cut-off. This procedure defines the Hamiltonian in a new space (see [5, 21] for details). To avoid this difficulty in this paper we will assume that for translation invariant interaction there are no pure creation and annihilation terms.

### 2.2 Friedrichs diagrams

We will study the evolution operator

\[
U(t) = e^{i t H_0} e^{-i t (H_0 + \lambda V)}.	ag{2.11}
\]

In perturbation theory the evolution operator (2.11) has the representation

\[
U(t) = 1 - i \lambda \int_0^t V(t_1) dt_1 + (-i \lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 V(t_1)V(t_2) + \ldots \tag{2.12}
\]

where

\[
V(t) = e^{i t H_0} V e^{-i t H_0}
\]

We will use for \(V_{I,J}\) the Friedrichs diagram representation [3]. The corresponding diagram has one vertex and \(I\) lines going from the vertex to the left and \(J\) lines going to the right. The first \(I\) lines represent creation operators and the last \(J\) lines represent annihilation operators. In what follows we will use the Wick theorem and also the following notions. The line of the graph is called the internal if it connects two vertices of the graph. A graph is the connected graph if all its vertices are connected by a set of internal lines otherwise it is called the disconnected one. A connected graph is called the one-particle reducible (1PR) if after the removing a line it becomes disconnected. A connected graph is called the one-particle irreducible (1PI) if after the removing any line it is still connected.

The following "linked cluster theorem" [5] will be used:

\[
U(t) =: e^{U_c(t)} : \tag{2.13}
\]
where
\[ U_c(t) = \sum_{n=1}^{\infty} (-i\lambda)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n (V(t_1) \ldots V(t_n))_c \]

Here :: means the Wick normal ordering and the index \( c \) in \( U_c \) indicates that one takes only the connected diagrams.

Below for simplicity of notations we consider interactions with the only one type of particles, but the main results are valid for arbitrary number of types of particles.

3 Non-translation invariant Hamiltonians

3.1 Second order

To get an insight to the problem it is very instructive to start with the consideration of matrix elements of the evolution operator in the second order of perturbation theory. For the vacuum matrix element of the evolution operator we obtain from (2.13) the representation:

\[ \langle 0 | U(t) | 0 \rangle = e^{E(t)} \tag{3.14} \]

where
\[ E(t) = \langle 0 | (-i\lambda \int_0^t dt_1 V(t) + (-i\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 V(t_1) V(t_2) + \ldots )_c | 0 \rangle. \tag{3.15} \]

Here the symbol \((\ldots )_c\) means that we keep only connected diagrams.

Representation (3.14), (3.15) permits us to calculate the leading terms of the asymptotic behaviour of the matrix elements of the evolution operator for large time \( t \) as well the corrections to the leading terms. In fact we will show that \( E(t) \) has the following form

\[ E(t) = At + B + C(t) \tag{3.16} \]

where one has the perturbative expansions

\[ A = \lambda^2 A_2 + \lambda^3 A_3 + \ldots, \quad B = \lambda^2 B_2 + \lambda^3 B_3 + \ldots, \quad C(t) = \lambda^2 C_2(t) + \lambda^3 C_3(t) + \ldots \tag{3.17} \]

and \( C_n(t) \) vanishes for large \( t \).

Let us find explicitly these terms in the second order of perturbation theory for the Hamiltonian

\[ H = H_0 + \lambda V, \tag{3.18} \]

where
\[ H_0 = \int \omega(p) a^*(p) a(p) dp \tag{3.19} \]

and the interaction has the form
\[ V = \int (v(p_1, \ldots, p_n) a^*(p_1) \ldots a^*(p_n) + h.c.) dp_1 \ldots dp_n \tag{3.20} \]
Here \( \omega(p) \) is a positive smooth function, for example \( \omega(p) = \sqrt{p^2 + m^2}, m > 0 \) and \( v(p_1, ..., p_n) \) is a test function. For this interaction the first term in (3.15) is identically zero. The second term in (3.15) equals to

\[
E^{(2)}(t) = (-i\lambda)^2 \int dp_1 ... dp_n |v(p_1, ..., p_n)|^2 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{it_1 E_1 + it_2 E_2}
\]

(3.21)

where

\[
E_2 = -E_1 = E(p_1, ..., p_n) = \sum_{i=1}^{n} \omega(p_i)
\]

(3.22)

By using the equality

\[
\int_0^t dt_1 \int_0^{t_1} dt_2 e^{-it_1 E + it_2 E} = -\frac{i}{E} t + \frac{1}{E^2} - \frac{1}{E^2} e^{-it E}
\]

(3.23)

we get

\[
E^{(2)}(t) = (-i\lambda)^2 \int dp_1 ... dp_n |v(p_1, ..., p_n)|^2 \left(-\frac{i}{E} t + \frac{1}{E^2} - \frac{1}{E^2} e^{-it E}\right)
\]

(3.24)

Therefore we obtain the expression of the form (3.16)

\[
E(t) = \lambda^2 A_2 t + \lambda^2 B_2 + \lambda^2 C_2(t) + ...
\]

(3.25)

where

\[
A_2 = i \int \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n)} dp_1 ... dp_n,
\]

(3.26)

\[
B_2 = - \int \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n)^2} dp_1 ... dp_n,
\]

(3.27)

\[
C_2(t) = \int \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n)^2} e^{-it E(p_1, ..., p_n)} dp_1 ... dp_n
\]

(3.28)

We have obtained the following

**Theorem 1.** The vacuum expectation value of the evolution operator for the Hamiltonian (3.18) in the second order of perturbation theory has the form

\[
< U(t) > = e^{\lambda^2 A_2 t + \lambda^2 B_2 + \lambda^2 C_2(t)}
\]

(3.29)

where \( A_2, B_2 \) and \( C_2(t) \) are given by (3.26), (3.27) and (3.28).

**Remark.** By using the stationary phase method one can prove that the function \( C_2(t) \) vanishes as \( t \to \infty \) (see below).

### 3.2 Decay

We have proved theorem 1 under the assumption \( \omega(p) > 0 \). However the obtained formula (3.29) is valid in the more general case when one has the decay. In this case formula (3.29) still is true but in the expressions (3.26)-(3.28) one has to substitute \( E \to E - i0 \). Let us consider the important case when

\[
\omega(p) = \frac{p^2}{2} - \omega_0, \quad \omega_0 > 0
\]

(3.30)
Instead of (3.23) we will use now the identity
\[ \int_0^t dt_1 \int_0^{t_1} dt_2 e^{i(t_2-t_1)E} = t \int_0^t (1 - \frac{\sigma}{t})e^{-i\sigma E} d\sigma. \] (3.31)

We have
\[ E^{(2)}(t) = (-i\lambda)^2 \int dp_1...dp_n |v(p_1, ..., p_n)|^2 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{i(t_2-t_1)E(p_1, ..., p_n)} \] (3.32)
\[ = -\lambda^2 t \int_0^t d\sigma (1 - \frac{\sigma}{t}) \int dp_1...dp_n |v(p_1, ..., p_n)|^2 e^{-i\sigma E(p_1, ..., p_n)} \]
\[ = \lambda^2 t A_2(t) + \lambda^2 B_2(t) \]
where
\[ A_2(t) = -\int_0^t d\sigma F(\sigma), \quad B_2(t) = \int_0^t d\sigma \sigma F(\sigma) \] (3.33)

and
\[ F(\sigma) = \int dp_1...dp_n |v(p_1, ..., p_n)|^2 e^{-i\sigma E(p_1, ..., p_n)}. \] (3.34)

By using the stationary phase method we obtain the following asymptotic behaviour of the function \( F(\sigma) \) as \( \sigma \to \infty \):
\[ F(\sigma) = \left( \frac{2\pi i}{\sigma} \right)^{dn/2}e^{in\omega_0} |v(0)|^2 [1 + o\left( \frac{1}{\sigma} \right)]. \]

Therefore for \( dn \geq 3 \) there exist the limits
\[ \lim_{t \to \infty} A_2(t) = A_2 = -\int_0^{\infty} d\sigma F(\sigma) \] (3.35)
\[ \lim_{t \to \infty} B_2(t) = B_2 = \int_0^{\infty} d\sigma \sigma F(\sigma) \]
because there exists the limit
\[ \lim_{t \to \infty} \int_1^t e^{i\sigma \omega_0} \frac{d\sigma}{\sigma^{1/2}}. \]

Moreover one has
\[ A_2(t) = -\int_0^{\infty} d\sigma F(\sigma) + o\left( \frac{1}{t^2} \right), \quad B_2(t) = \int_0^{\infty} d\sigma \sigma F(\sigma) + o\left( \frac{1}{t} \right). \]

If \( dn \geq 5 \) one gets also
\[ A_2 = i \int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n) - i0} \] (3.36)
\[ B_2 = -\int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{(E(p_1, ..., p_n) - i0)^2}. \] (3.37)

Indeed one has
\[ A_2 = -\int_0^{\infty} d\sigma F(\sigma) = -\lim_{\epsilon \to 0} \int_0^{\infty} d\sigma F(\sigma) e^{-\sigma \epsilon}. \]
This is true due to the Lebesgue theorem since \( |F(\sigma)e^{-\alpha t}| \leq |F(\sigma)| \) and \( F(\sigma) \in L_1(R_+) \) \((L_1(\R^+))\) is the space of absolute integrable functions if \( nd \geq 3 \). Substituting in the above formula the representation (3.34) and changing the order of integrations (we can do this due to the Fubini theorem since for positive \( \epsilon \) the function \( |v(p_1, ..., p_n)|^2 e^{-i \sigma (E(p_1, ..., p_n) - \epsilon)} \) belongs to the space \( L_1(R_+ \times \R^{nd}) \) of absolute integrable functions), we can perform the integration over \( \sigma \) explicitly

\[
A_2 = \lim_{\epsilon \to 0} (i) \int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n) - i\epsilon} = i \int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{E(p_1, ..., p_n) - i\epsilon}.
\]

The same calculation is true for \( B_2 \) with the more strong assumption : \( dn \geq 5 \),

\[
B_2 = \int_0^\infty d\sigma F(\sigma) = \lim_{\epsilon \to 0} \int_0^\infty d\sigma \int dp_1...dp_n |v(p_1, ..., p_n)|^2 e^{-i \sigma (E(p_1, ..., p_n) - \epsilon)}
\]

\[
= -\lim_{\epsilon \to 0} \int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{(E(p_1, ..., p_n) - i\epsilon)^2} = - \int dp_1...dp_n \frac{|v(p_1, ..., p_n)|^2}{(E(p_1, ..., p_n) - i\epsilon)^2}.
\]

We have proved the following theorem.

**Theorem 2.** The asymptotic behaviour as \( t \to \infty \) of the vacuum expectation value of the evolution operator for the Hamiltonian (3.18) with the dispersion low (3.30) in the second order of perturbation theory is

\[
< U(t) > = e^{A_2 t + \lambda^2 B_2 + \lambda^2 o(1/t)}
\]

(3.38)

where \( A_2 \) and \( B_2 \) are given by (3.33) (or (3.36) and (3.37)). After the rescaling \( t \to t/\lambda^2 \) one gets the \( \lambda^2 \) corrections to the stochastic limit

\[
< U(t/\lambda^2) > = e^{A_2 t + \lambda^2 B_2 + \lambda^2 o(\lambda^2/t)}.
\]

(3.39)

### 3.3 Example

We discuss here the evolution operator for the simple explicitly solvable model described by the Hamiltonian

\[
H = \int \omega(k) a^+(k)a(k) dk + \lambda \int (a(k)\bar{\pi}(k) + a^*(k)v(k)) dk.
\]

(3.40)

We will see that the vacuum expectation value of the evolution operator has the form obtained in theorems and 2. Under assumptions

\[
A = \lambda^2 A_2 = i\lambda^2 \int \frac{|v(k)|^2}{\omega(k)} dk < \infty, \quad B = \lambda^2 B_2 = -\lambda^2 \int \frac{|v(k)|^2}{\omega^2(k)} dk < \infty,
\]

(3.41)

one has the following

**Proposition 1.** The vacuum expectation value of the evolution operator \( U(t) = e^{i t H_0} e^{-i t H} \) for the model (3.40) is

\[
\langle U(t) \rangle = \exp [At + B + C(t)]
\]

(3.42)
where $A$ and $B$ are given by (3.41) and $C(t)$ is

$$C(t) = \lambda^2 \int dk \frac{|v(k)|^2}{\omega^2(k)} e^{-i\omega(k)t}$$

**Proof.** It follows from the known explicit solution of the model. From Proposition 1 we obtain Proposition 2. The asymptotic behaviour of the expectation value (3.42) for $t \to \infty$ has the form

$$\langle U(t) \rangle = \exp \left[ A t + B + \lambda^2 \left( \frac{1}{t} \right)^\frac{d}{2} (2i\pi)^\frac{d}{2} \int dk \frac{|v(k)|^2}{\omega^2(k)} e^{-i\omega(k)t} + \ldots \right].$$

(3.43)

where $k_0$ is a critical point, $\nabla \omega(k_0) = 0$ and we assume there is only one nondegenerate critical point.

**Proof.** It follows immediately from (3.42) by using the stationary phase method.

**Remark.** If $\omega(k) = k^2 - \omega_0$, $\omega_0 > 0$ then one has the decay. We can not use in this case the diagonalization of the Hamiltonian (3.40) but the formula (3.14) still is true. We have

$$\langle U(t) \rangle = \exp[-\lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \int d^d k |v(k)|^2 e^{i(t_2-t_1)\omega(k)}]$$

$$= \exp[\lambda^2 t A_2(t) + \lambda^2 B_2(t)]$$

where

$$A_2(t) = -\int_0^t d\sigma F(\sigma), \quad B_2(t) = \int_0^t d\sigma F(\sigma)$$

and

$$F(\sigma) = \int d^d k |v(k)|^2 e^{-i\sigma \omega(k)}$$

Again we obtain the ABC-formula.

### 3.4 Wave operators and the main formula

In this section we show how the spectral theory and renormalized wave operators can be used for the derivation of the main formula. In particular explicit expressions for the parameters $A$, $B$ and $C(t)$ will be obtained.

We consider the Hamiltonian (2.9) for one type of particles with $\omega(p) = \sqrt{p^2 + m^2}$, $m > 0$ in the space $R^d$, $d > 2$. We will work with the formal perturbation series for the evolution operator. In fact if the interaction (2.9) includes only fermionic operators or it is linear in bosonic operators then one can prove that the series are absolutely convergent. The following operator plays the crucial role in the scattering theory

$$T =: \exp \sum_{n=1}^\infty (-\lambda)^n \Gamma(V \ldots \Gamma(V) \ldots)_L :$$

(3.44)

Here $\Gamma$ means the Friedrichs $\Gamma$ - operation

$$\Gamma(V) = \lim_{\epsilon \to +0} \langle -i \rangle \int_0^\infty e^{-\epsilon t} e^{itH_0} V e^{-itH_0} dt$$

(3.45)
and \((\cdot)_L\) means that only connected non-vacuum diagrams are included. The operator \(T\) is equal in fact to the non-vacuum part of the conjugate wave operator:

\[
T = \lim_{\epsilon \to 0} \lim_{t \to \infty} U^*_\epsilon(t) / \langle U^*_\epsilon(t) \rangle
\]

One has the following relations [5]:

\[
HT = T(H_0 + E_0),
\]

\[
E_0 = \sum_{n=1}^{\infty} (-\lambda)^{n+1} \langle V\Gamma(V)...\Gamma(V)... \rangle c >, \]

\[
T^*T = TT^* = Z^{-1}, \quad Z^{-1} = ||T\Phi_0||^2
\]

We will use these relations to derive the main formula for matrix elements of the evolution operator and in particular to compute corrections to the stochastic limit.

From (3.46) it follows

\[
H = T(H_0 + E_0)T^*e^B
\]

where

\[
B = \ln Z
\] (3.47)

Therefore one has

\[
U(t) = e^{itH_0}e^{-itH} = e^{itH_0}Te^{-it(H_0+E_0)}T^*e^B = e^{At+B}e^{itH_0}Te^{-itH_0}T^*
\] (3.48)

where

\[
A = -iE_0
\] (3.49)

By taking the expectation value of the equality (3.48) we obtain

\[
\langle \psi, U(t)\psi \rangle = e^{At+B+C(t)}
\] (3.50)

where

\[
e^{C(t)} = \langle \psi, e^{itH_0}Te^{-itH_0}T^*\psi \rangle
\]

If \(\psi\) is the vacuum vector then one can prove that \(C(t) \to 0\) as \(t \to \infty\) and we obtain the main formula (1.3). If \(\psi\) is a non-vacuum vector then the asymptotic behaviour of \(C(t)\) is more complicated. We have proved the following theorem.

**Theorem 3.** If the Hamiltonian satisfies the indicated above assumptions then there exists the following representation for the vacuum expectation value of the evolution operator

\[
\langle \Phi_0, U(t)\Phi_0 \rangle = e^{At+B+C(t)}
\]

where constants \(A\) and \(B\) are given by (3.49) and (3.47) and \(C(t)\) is defined by

\[
e^{C(t)} = \langle \Phi_0, T(t)T^*\Phi_0 \rangle, \quad T(t) = e^{itH_0}Te^{-itH_0}
\] (3.51)

Here the weak limit of \(T(t)\) as \(t \to \infty\) is equal to 1 and \(\lim_{t \to \infty} C(t) = 0\).

This theorem also shows the physical meaning of constants \(A, B\) and the function \(C(t)\).
4 One particle matrix elements of the evolution operator for translation invariant Hamiltonian

In this section we study the asymptotic behaviour of one particle matrix elements of evolution operator

\[ <p|U(t, \lambda)|p'> = \delta(p - p')U_{1,1}(t, p, \lambda), \]  

(4.52)

for translation invariant Hamiltonian (2.10) without vacuum polarization, i.e. when \( V_{I0} = V_{0,I} = 0 \), and we assume also \( V_{1,1} = 0 \). We will prove that the following presentation is true

\[ U_{1,1}(t, p, \lambda) = \exp\{itA(p, \lambda) + B(p, \lambda)\} (1 + C(t, p, \lambda)) \]  

(4.53)

where \( A(p, \lambda) \), \( B(p, \lambda) \) and \( C(t, p, \lambda) \) are formal series in \( \lambda \)

\[ A(p, \lambda) = \sum_{n=1}^{\infty} \lambda^n A_n(p), \quad B(p, \lambda) = \sum_{n=1}^{\infty} \lambda^n B_n(p), \quad C(t, p, \lambda) = \sum_{n=1}^{\infty} \lambda^n C_n(t, p) \]  

(4.54)

and functions \( C_n(t, p) \) vanish as \( t \to \infty \).

We show how the spectral theory and renormalized wave operators can be used for the derivation of the main formula. In particular, explicit recursive relations for the parameters \( A_n, B_n \) and \( C_n(t) \) will be obtained. Note that for this derivation we have to assume that there is no decay.

The intertwining operator \( T \) is defined as a solution of the following equation

\[ HT = T(H_0 + M), \]  

(4.55)

Here \( M \) has the form

\[ M = \int m(p)a^*(p)a(p)dp \]  

(4.56)

The operator \( T \) plays the crucial role in the scattering theory. Its singular part defines the renormalized wave operators. The renormalized wave operators also give a solution of intertwining condition. Taking \( T \) in the following form

\[ T =: \exp W :, \quad W = \Gamma(Q) \]  

(4.57)

one gets \( \Box \) equations to define \( Q \) and \( M \)

\[ Q + V \bot T - (V \bot T)_{1,1} - W - \circ - M = 0 \]  

(4.58)

\[ M = (V \bot T)_{1,1} \]  

(4.59)

Here the symbol \( \bot \) means that for connected \( A \) in the operators \( A \bot B \) all connected parts of \( B \) are paired with \( A \). If \( B \) is connected then \( A \bot B = A - \circ - B = (AB)_c \). For a special form of the interaction, when \( V_{0,I} = V_{I0} = 0 \) we can write \( M = (V \bot T)_{1,1} = (V\Gamma(Q))_{1,1} \).
Expanding $M$ and $Q$ in the power series in $\lambda$

\[ M = \sum_{n=1}^{\infty} \lambda^{2n} M_{2n}, \quad Q = \sum_{n=1}^{\infty} \lambda^{2n+1} Q_{2n+1}, \quad (4.60) \]

we get recursive relations to define $M_{2n}$ and $Q_{2n+1}$. Let us compute explicitly the first terms solving these equations. We obtain

\[ Q_1 = -V \quad (4.61) \]
\[ M_2 = -(VT\Gamma V)_{1,1} \quad (4.62) \]
\[ Q_2 = (VT\Gamma)_{c} - (VT\Gamma)_{1,1} \quad (4.63) \]
\[ Q_3 = -(VT\Gamma r(V)(VT(V)))_{c} - \frac{1}{2} V \perp \perp : \Gamma(V)^2 : + \Gamma V - \circ - M_2 \quad (4.64) \]
\[ M_4 = -(VT(VT r(V)(VT(V))))_{1,1,1,1} + (VT^2(V)M_2)_{1,1} \quad (4.65) \]
or
\[ M_4 = -(VT(VT r(V)(VT(V))))_{1,1,1,1} - (VT^2(V)(VT(V)))_{1,1,1,1} \quad (4.66) \]

Here we use the notation

\[ \Gamma_r(Q) = \Gamma(Q - Q_{1,1}) \quad (4.67) \]

One can construct in perturbation theory the operator $Z$ such that

\[ TT^* Z = 1 \quad (4.68) \]

In particular, at the second order of perturbation theory in $\lambda$ one has

\[ <p|Z|p'>(1 + \lambda^2 B_2(p))\delta(p - p') \quad (4.69) \]

By using (4.55) and (4.68) one gets

\[ U(t) = e^{itH_0}e^{-itH} = e^{-itM}e^{it(H_0 + M)T}e^{-it(H_0 + M)T^*}Z \quad (4.70) \]

By taking the one particle expectation value of the equality (4.70) we obtain

\[ <p|U(t)|p'> = e^{-itM(p,\lambda)}Z(p,\lambda)(1 + C(p, t, \lambda)) \quad (4.71) \]

where

\[ (1 + C(t, p, \lambda))\delta(p - p') = <p|e^{it(H_0 + M)T}e^{-it(H_0 + M)T^*}|p'> \quad (4.72) \]

and $M$ and $T$ should be computed from the recursive relations.

We have proved the following

**Theorem 4.** For translation invariant Hamiltonians without vacuum polarization the one particle matrix elements of evolution operator are given by the formula (4.71) where functions $M(p, \lambda)$ and $Z(p, \lambda)$ are solutions of equations (4.57)-(4.59) and (4.68). Function $C(t, p, \lambda)$ is defined by (4.72) and it vanishes as $t \to \infty$.  

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5 Quantum Chaos

The representations (3.51) and (4.72) for the function $C(t)$ are similar in fact to the representation for the autocorrelation function describing quantum chaos for the quantum baker’s map. The quantum baker’s map is a simple model invented to study quantum chaos, see for example [22, 23, 24, 25, 26] and references therein. The evolution operator for the quantum baker’s map has the form $U(t) = B^t$ where $B$ is a unitary matrix and $t$ is an integer number. The autocorrelation function $F(t) = \langle \psi | U(t) | \psi \rangle$ where $\psi$ is a coherent vector, admits a representation of the form 

$$F(t) = a(t) \sum e^{W(\beta,p,q,t)}$$

where $a(t)$ falls off exponentially for large $t$ and $W$ is quadratic polynomial in $q, p$ and $\beta$. Here $p$ and $q$ are parameters in the coherent vectors, $\beta$ is a string in the symbolic dynamics representation and the sum runs over all the strings. This form is similar to the representations (3.51) and (4.72) for the function $C(t)$, compare also with the explicit formula (3.28) for $C_2(t)$. It would be very interesting to study further this analogy between the functions $F(t)$ and $C(t)$. In particular one has to investigate the sensitivity of the function $C(t)$ to the initial data, i.e. in the one particle case the dependence of the function $C(t, p, \lambda)$ on $p$. Also one has to investigate the analogous function for coherent states. To this end one can use the perturbation theory as well the semiclassical expansion. Actually it is possible that a quantum system exhibits chaotic properties, although its classical counterpart is non-chaotic. Such phenomena are called wave chaos [27]. It would be very interesting to study a relation of the properties of the function $C(t)$ with the wave chaos and also with the spectral and holographic properties of quantum chaos, see [28].

6 Conclusion

We have obtained in this paper the explicit representations (1.3) for the vacuum and one-particle matrix elements of the evolution operator. By using these representations we have computed the corrections to the known results for the large time exponential behaviour of these matrix elements. This opens the way for further investigations of the large time behaviour in quantum theory. In particular the problems of quantum decoherence and decay and also the relation with quantum chaos require the further study.

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