Existence Theorems for $\frac{\pi}{n}$ Vortex Scattering

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Abstract The analysis of $90^\circ$ vortex-vortex scattering is extended to $\frac{\pi}{n}$ scattering in all head-on collisions of $n$ vortices in the Abelian Higgs model. A Cauchy problem with initial data that describe the scattering of $n$ vortices is formulated. It is shown that this Cauchy problem has a unique global finite-energy solution. The symmetry of the solution and the form of the local analytic solution then show that $\frac{\pi}{n}$ scattering is realised.

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In recent years, analytic and numerical studies have uncovered quite unexpected scattering processes of soliton-like objects in (2 + 1) and (3 + 1) dimensions. The analytic studies were mainly based on the geodesic approximation [1], whose validity has recently been proved for vortices [2] and monopoles [3], and were mainly concerned with the scattering of 2 objects. There are, however, some interesting analytic results for the scattering of more than 2 extended objects [4] [5] [6]. Among the most interesting processes is $90^\circ$ scattering of 2 vortices in a head-on collision. In Ref [7], existence theorems for this process were given, which act as an underpinning to numerical methods and approximation techniques. In this Letter this analysis is extended to $\frac{\pi}{n}$ scattering in all head-on collisions of $n$ vortices.

The model we study is the Abelian Higgs model given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(D_\mu \Phi)(D^\mu \Phi)^* - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{\lambda}{8}(|\Phi|^2 - 1)^2. \quad (1)$$

$\Phi$ is the complex Higgs field, $D_\mu \Phi = \partial_\mu \Phi - i A_\mu \Phi$, $\mu = 0, 1, 2$, is the covariant derivative and the gauge fields $F_{\mu \nu}$ are defined in terms of the real gauge potentials $A_\mu$ as $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 0, 1, 2$. The indices are raised and lowered with the metric tensor $g = \text{diag}(+1, -1, -1)$. The Euler-Lagrange equations are

$$D_\mu D^\mu \Phi + \frac{\lambda}{2} \Phi(|\Phi|^2 - 1) = 0, \quad \partial_\mu F^{\mu \nu} + \frac{i}{2}(\Phi^*(D^\nu \Phi) - \Phi(D^\nu \Phi)^*) = 0. \quad (2)$$

For all $\lambda$ the Euler-Lagrange equations have static, finite energy $n$-vortex solutions of the form [3],

$$A_i(r, \theta) = -\epsilon_{ij} r^j n \frac{a(r)}{r^2}, \quad A_0 = 0, \quad \Phi(r, \theta) = f(r) \exp[im\theta] \quad (3)$$

for $i, j = 1, 2$, where

$$r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial a}{\partial r} \right) - f^2(r)[a(r) - 1] = 0, \quad (4)$$

$$2r \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) - 2n^2 f(r)[a(r) - 1]^2 - \lambda r^2 f(r)[f^2(r) - 1] = 0, \quad (5)$$

with

$$f(0) = 0, \quad a(0) = 0, \quad \lim_{r \to \infty} f(r) = \lim_{r \to \infty} a(r) = 1. \quad (6)$$
In the special case $\lambda = 1$, it can be shown \[9\] that the solutions actually satisfy the first order Bogomolnyi equations, so $f$ and $a$ satisfy,

$$rf' - n(1 - a)f = \frac{2n}{r}a' + f^2 - 1 = 0. \quad (7)$$

In this case, there also exists a $2n$ parameter family of static $n$-vortex solutions describing vortices located at arbitrary positions. The reason for its existence is the fact that for $\lambda = 1$, the net force between static vortices is zero.

We now study the scattering of vortices during the time from shortly before to shortly after the collision. The fields are taken to be of the form,

$$\Phi(t, \vec{x}) = \hat{\Phi}(\vec{x}) + t\zeta(\vec{x}), \quad A_0(t, \vec{x}) = 0, \quad A_i(t, \vec{x}) = \hat{A}_i(\vec{x}) + tB_i(\vec{x}). \quad (8)$$

Here $(\hat{\Phi}, \hat{A}_i)$ is taken to be the solution (3). It is assumed that equations (2) can be linearized in $(t\zeta, tB_i)$ for $t \in (-\epsilon, \epsilon)$, $\epsilon \ll 1$. These linearized equations are solved for

$$\zeta = nf(r)k(r)$$

$$(B_1, B_2) = \left(-n \sin[(n-1)\theta]\frac{rk'(r) + nk(r)}{r}, -n \cos[(n-1)\theta]\frac{rk'(r) + nk(r)}{r}\right) \quad (9)$$

where $k$ is the solution of the equation,

$$r^2k''(r) + rk'(r) - k(r)[n^2 + r^2f^2(r)] = 0, \quad (10)$$

with $k(r)e^r \to 1$ as $r \to \infty$. (The functions (9) have been used by Weinberg [10] in his study of the zero modes of the static solutions.)

We use the zeroes of $|\Phi|^2$ to define the positions of the vortices. For the solution (3) one obtains

$$|\Phi|^2 = f^2(1 + 2ntk\cos(n\theta) + n^2t^2k^2)$$

$$\geq f^2(1 - n |t| k)^2$$

For $t \neq 0$, $|\Phi|^2$ has exactly $n$ zeroes, namely at $r = \rho$ and $\theta = 0, \frac{2\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}$ for $t < 0$, and at $r = \rho$ and $\theta = \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{(2(n-1)+1)\pi}{n}$ for $t > 0$. This shows that the solution (3) describes $\frac{2\pi}{n}$ scattering. Here $\rho$ is the point where $k(\rho) = \frac{1}{n|t|}$. That exactly one such $\rho$ exists follows from the fact that $k(r)$ is monotonic decreasing, and behaves like $k = n r^{-n}$ for small $r$. 

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The problem with this approach is that the linearization, though plausible for times shortly before and shortly after the collision, has not been justified in an entirely rigorous fashion. Guided in our choice of initial data by the results just discussed we will now bring our discussion to mathematically rigorous conclusions.

The first rigorous result concerns the global existence of the solution. We can now show that for certain initial data a unique global finite-energy time-dependent solution of equations \( \frac{\partial}{\partial \mu} A^\mu = 0 \).

To do this, we first subtract a background field \( (\hat{\Phi}, \hat{A}^\mu) \) and write,

\[
\Phi(t, \vec{x}) = \hat{\Phi}(\vec{x}) + \phi(t, \vec{x}), \quad A^\mu(t, \vec{x}) = \hat{A}^\mu(\vec{x}) + a^\mu(t, \vec{x}).
\] (11)

For the background field we choose the static solution \( (3) \). As initial data we choose:

\[
\phi(0, \vec{x}) = 0, \quad a_0(0, \vec{x}) = 0, \quad a_i(0, \vec{x}) = 0, \quad \text{for } i = 1, 2;
\]

\[
\partial^t \phi(0, \vec{x}) = nf(r)k(r), \quad \partial^t a_0(0, \vec{x}) = 0,
\]

\[
\partial^t a_1(0, \vec{x}) = -n \sin[(n - 1)\theta] \frac{[rk'(r) + nk(r)]}{r},
\]

\[
\partial^t a_2(0, \vec{x}) = -n \cos[(n - 1)\theta] \frac{[rk'(r) + nk(r)]}{r}.
\]

(12)

For this choice of background field and initial data, a unique global finite-energy solution exists, since all the conditions from Ref \([11]\) are satisfied.

An essential element of the proof which is based on Segal’s existence and uniqueness theorem \([12]\), is an iteration method. The method starts with writing the Cauchy problem in the form,

\[
\partial^t \Psi = -i\tilde{A}\Psi + J
\]

(13)

where \( \Psi^t = (a_0, p_0, a_1, p_1, a_2, p_2, \phi, \pi^*) \), \( p_\mu = \partial_\mu a_\mu \), \( \pi^* = \partial_0 \phi - ia_0 \phi \), and where the operator \( \tilde{A} \) is defined by,

\[
\tilde{A} = \begin{pmatrix}
\Gamma & 0 & 0 & 0 \\
0 & \Gamma & 0 & 0 \\
0 & 0 & \Gamma & 0 \\
0 & 0 & 0 & \Gamma
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
0 & 1 \\
\Delta - m^2 & 0
\end{pmatrix}.
\]

(14)
The vector $J$ can be calculated from equations (2) (and is given in [7]).

The solution of the Cauchy problem (12,13) can be obtained as the solution of the integro-differential equation,

$$
\Psi(t, \vec{x}) = e^{-i\tilde{A}t}\Psi(0, \vec{x}) + \int_0^t ds \exp[-i\tilde{A}(t-s)]J(\Psi(s, \vec{x})).
$$

(15)

In turn this integro-differential equation is solved by the limit of a sequence of successive approximations $\Psi_n$ defined by the formula:

$$
\Psi_{n+1}(t, \vec{x}) = e^{-i\tilde{A}t}\Psi(0, \vec{x}) + \int_0^t ds \exp[-i\tilde{A}(t-s)]J(\Psi_n(s, \vec{x})),
$$

(16)

where $\Psi_0 = \Psi(0, \vec{x})$, with the initial data (12). We now establish certain symmetries of the initial data $\Psi_0$ and use (16) to establish these symmetries for the successive approximations $\Psi_n$, and finally for the solution of (13).

The first transformation we study is $\vec{x} \rightarrow \vec{x}' = S\vec{x}$ where $S$ is the orthogonal matrix

$$
S = \begin{pmatrix}
\cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\
\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n}
\end{pmatrix}
$$

(17)

Under this transformation the initial data change as follows:

$$
\Psi(0, \vec{x}') = M_1\Psi(0, \vec{x}),
$$

with

$$
M_1 = \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & A & -B & 0 \\
0 & B & A & 0 \\
0 & 0 & 0 & I
\end{pmatrix},
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
A = \cos \frac{2\pi}{n}I, B = \sin \frac{2\pi}{n}I.
$$

(18)

We can see that $J(M_1\Psi(s, \vec{x})) = M_1J(\Psi(s, \vec{x}))$, $[M_1, \tilde{A}] = 0$ and

$$
\exp(-i\tilde{A}t)M_1\Psi_n(s, \vec{x}) = M_1 \exp(-i\tilde{A}t)\Psi_n(s, \vec{x}),
$$

which implies that $\Psi_n(t, \vec{x}') = M_1\Psi_n(t, \vec{x})$ for all $n \in \mathbb{N}$. From this follows $\Psi(t, \vec{x}') = M_1\Psi(t, \vec{x})$ for the solution $\Psi$.

Next we study the reflection $(x_1, x_2) \rightarrow (x_1, -x_2)$. Under this transformation the initial data change as follows: $\psi(t, x_1, -x_2) = M_2\psi(t, x_1, x_2)$,
where
\[
M_2 = \begin{pmatrix}
-I & 0 & 0 & 0 \\
0 & -I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & C
\end{pmatrix}, \quad CV = V^*
\tag{19}
\]
Furthermore, \( J(M_2 \Psi(s, \vec{x})) = M_2 J(\Psi(s, \vec{x})) \). Again we have \([M_2, \tilde{A}] = 0\), and \( \Psi_n(t, x_1, -x_2) = M_2 \Psi_n(t, x_1, x_2) \). From this follows \( \Psi(t, x_1, -x_2) = M_2 \Psi(t, x_1, x_2) \), for the solution \( \Psi \).

Under the transformations considered, all terms in the energy density:
\[
E = \frac{1}{2} |D_0 \Phi|^2 + \frac{1}{2} |D_i \Phi|^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{2} F_{0i}^2 + \frac{\lambda}{8}(|\Phi|^2 - 1)^2,
\tag{20}
\]
are invariant. This leads to the following conclusion: If by using functions like \( |\Phi|^2, F_{ij}^2 \) or \( E \), there is a way of defining the positions \((x_1^a(t), x_2^a(t)), a = 1, 2, \) of exactly \( n \) separate vortices, these \( n \) positions must lie on \( n \) radial lines separated by an angle \( \frac{2\pi}{n} \) with equal distance from the origin. (Below we will use the the minima of \( |\Phi|^2 \) to define these positions.) Furthermore, one of these radial lines must be the positive \( x_1 \)-axis, or make an angle \( \frac{\pi}{n} \) with the positive \( x_1 \)-axis. Any vortex that does not satisfy these conditions immediately leads to \( 2n - 1 \) other vortices, because of the symmetries of our solution. Since our solution is continuous, these positions will change continuously such that at \( t = 0 \) the \( n \) positions coincide, and after the collision the vortices move again on the radial lines just described. Therefore, they can either go back on the radial lines they came in on, or go back on radial lines shifted by an angle \( \frac{\pi}{n} \). We will study a further symmetry and use the Cauchy-Kowalewskiy theorem \([13]\) to show that the second case is realised.

The last transformation we study is \( \vec{x} \rightarrow M\vec{x} \) where \( M \) is the orthogonal matrix
\[
M = \begin{pmatrix}
\cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\
\sin \frac{\pi}{n} & \cos \frac{\pi}{n}
\end{pmatrix}.
\tag{21}
\]
Under this transformation the initial data change as follows:
\[
\Psi(0, M\vec{x}) = M_3 \Psi(0, \vec{x}),
\]
with

\[
M_3 = \begin{pmatrix}
-\sigma & 0 & 0 & 0 \\
0 & C & -D & 0 \\
0 & D & C & 0 \\
0 & 0 & 0 & -\sigma
\end{pmatrix}, \quad \sigma = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad C = \cos \frac{\pi}{n} \sigma, \quad D = \sin \frac{\pi}{n} \sigma.
\]

We can see that \( J(M_3 \Psi(s, \vec{x})) = -M_3 J(\Psi(s, \vec{x})) \), \( \{M_3, \vec{A}\} = 0 \) and

\[
\exp(\im \vec{A} t) M_3 = M_3 \exp(-\im \vec{A} t).
\]

From this follows \( \Psi(-t, M\vec{x}) = M_3 \Psi(t, \vec{x}) \), and we see that all terms in the energy density are invariant under the transformation \((t, \vec{x}) \rightarrow (-t, M\vec{x})\). This establishes \( \pi \) scattering for \( n \) vortices. To show that, for small time \( t \), \( |\Phi| \) has exactly \( n \) minima (so that we can identify \( n \) vortices), we use the Cauchy-Kowalewskyi theorem \[13\].

From Ref \[8\], we know that \( f \) starts with an \( r^n \) term and \( a \) starts with an \( r^2 \) term. Equation (10) shows that asymptotically near \( r = 0 \),

\[
k \sim k_{-n} r^{-n} + k_n r^n.
\]

Using the same techniques as in Ref \[7\], we can now establish the analyticity of the initial data, and verify that all the conditions of the Cauchy-Kowalewskyi theorem \[13\] are satisfied. We therefore have an analytic solution near the origin. This solution leads to the following asymptotic expression for \( |\Phi|^2 \):

\[
|\Phi|^2(t, \vec{x}) \sim f_n^2(x_1^2 + x_2^2)^n + 2ntf_n^2k_{-n}\sum_{p=0}^{[n/2]}(-1)^{n+p}\left(\frac{n}{2p}\right)x_1^{n-2p}x_2^{2p} + Kt^2,
\]

where we have assumed that \( x_1 \) and \( x_2 \) are of order \( \epsilon \), and \( t \) is of order \( \epsilon^n \), so that higher order terms can be ignored. \( K \) is a constant that can be expressed in terms of the coefficients of the expansions of \( a, f, \) and \( k \). \([n/2]\) is the highest integer that does not exceed \( n/2 \). For \( t \neq 0 \), the approximation (24) of \( |\Phi|^2 \) has exactly \( n \) minima which lie at \( r = (n|t|k_{-n})^{1/n} \) and \( \theta = 0, \frac{2\pi}{n}, ..., \frac{2(n-1)\pi}{n} \) for \( t < 0 \), and at \( r = (n|t|k_{-n})^{1/n} \) and \( \theta = \frac{\pi}{n}, \frac{3\pi}{n}, ..., \frac{(2(n-1)+1)\pi}{n} \) for \( t > 0 \). This establishes the \( \frac{\pi}{n} \) symmetry of the process. Our rigorous method is, however, not capable of following the minima for large time \( t \).

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