Explicit elements of norm one for cyclic groups

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Let $R$ be a ring (with unit element 1) and $G$ a finite group acting on $R$ by ring automorphisms. For any subgroup $U$ of $G$ define the norm map (sometimes called the trace map) $N_U : R \to R^U$ by

$$N_U(x) = \sum_{g \in U} g(x).$$

Ginosar and the first author reduced the surjectivity of the norm map $N_G$ for a group $G$ to the surjectivity of the norm maps for its elementary abelian subgroups; more precisely, they proved that $N_G : R \to R^G$ is surjective if and only if $N_U : R \to R^U$ is surjective for every elementary abelian subgroup $U$ of $G$ (see [2, Theorem 1]).

The $R^U$-linearity of $N_U$ implies that it is surjective if and only there exists an element $x_U \in R$ such that $N_U(x_U) = 1$. Suppose we have such an element $x_U$ for every elementary abelian subgroup $U$ of $G$. Then by the result mentioned above there is a “global” element $x_G \in R$ such that $N_G(x_G) = 1$. Using this last statement, Shelah observed (see [2, Proposition 6]) that there exists a formula in which $x_G$ is a finite sum of the form

$$x_G = \sum a \sigma_{i_1}(x_{U_{j_1}}) \sigma_{i_2}(x_{U_{j_2}}) \cdots \sigma_{i_r}(x_{U_{j_r}}),$$

where $a \in \mathbb{Z}$ and $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_r} \in G$.

Using a tensor induction argument, the first author found such a formula in case the ring $R$ is commutative (see [1, Theorem 2.1]). When $R$ is not commutative, the only formulas known so far hold in the following two cases:

(a) $G$ is an abelian 2-group and $R$ is an algebra over the field $\mathbb{F}_2$ (cf. [2, Section 2]),
(b) $G = \mathbb{Z}/4$ and $R$ is any ring (see Formula (2) below).

The aim of this article is to show how to find an explicit formula of the above form in the case of a cyclic $p$-group $G$ acting on an arbitrary (non-necessarily commutative) ring $R$. More precisely, in the theorem below, we express a norm one element for a cyclic group $G$ of order $p^n$ in terms of a norm one element for a subgroup of order $\geq p^{n/2}$. This allows to find by induction a norm one element for a cyclic $p$-group in terms of a norm one element for its unique elementary subgroup ($\cong \mathbb{Z}/p$).

The theorem appears as the main result of Section 1. Its proof is given in Section 2. In Section 3 we give some cohomological explanations for the proof.

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1. Statement of results

We fix a prime number $p$, and integers $n$ and $k$ verifying $n \geq 2$ and $1 \leq k \leq n/2$. Consider the cyclic group $G = \mathbb{Z}/p^n$ of order $p^n$ with a generator $\sigma$, and the cyclic subgroup $U$ of order $p^{n-k}$, generated by $\sigma^p$. Our main result is the following.

Theorem.— Let $x \in R$ satisfy $N_U(x) = 1$. Define $z$ and $a \in R$ by

$$z = p^{n-2k} (1 + \sigma + \sigma^2 + \cdots + \sigma^{p^k-1})(x) - 1$$

and

$$a = p^{n-2k} x + (1 - \sigma) \left( \sum_{i=1}^{p^{n-k}-1} \left( 1 + \sigma^{p^k} + \sigma^{2p^k} + \cdots + \sigma^{(i-1)p^k} \right)(x\sigma^{-ip^k}(z)) \right).$$

(1)

Then $N_G(ax) = 1$.

When $G = \mathbb{Z}/p^2$, the theorem can be reformulated as follows.

Corollary.— Let $\sigma$ be an automorphism of order $p^2$ of a ring $R$, and $x \in R$ satisfying $(1 + \sigma + \sigma^2 + \cdots + \sigma^{(p-1)p})(x) = 1$. Define $z = (1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1})(x) - 1$ and

$$a = x + (1 - \sigma) \left( \sum_{i=1}^{p-1} \left( 1 + \sigma^p + \sigma^{2p} + \cdots + \sigma^{(i-1)p} \right)(x\sigma^{-ip}(z)) \right).$$

Then

$$(1 + \sigma + \sigma^2 + \cdots + \sigma^{p^2-1})(ax) = 1.$$  

For $p = 2, 3$, the corollary yields the following explicit formulas. When $p = 2$, starting from $x \in R$ such that $x + \sigma^2(x) = 1$, we obtain the norm one element

$$ax = \sigma(x)x - \sigma(x)x^2 + x\sigma^2(x)x + x\sigma^3(x)x - \sigma(x)x^3(x)x$$

$$= 2x^2 - x^3 - x\sigma(x)x - \sigma(x)x^2 + \sigma(x)x^2.$$  

In this case, Péter P. Pálfy had shown the first author the following simpler formula:

$$x\sigma(x)x + x\sigma(x) - x^2\sigma(x).$$  

(2)

When $p = 3$, starting now from $x \in R$ such that $x + \sigma^3(x) + \sigma^6(x) = 1$, we obtain the norm one element

$$ax = -x^2 + x\sigma(x)x - \sigma^3(x)x + \sigma^4(x)x$$

$$+ x\sigma^3(x)x + x\sigma^4(x)x + x\sigma^5(x)x + x\sigma^6(x)x + x\sigma^7(x)x + x\sigma^8(x)x$$

$$- \sigma(x)\sigma^4(x)x - \sigma(x)\sigma^5(x)x - \sigma(x)\sigma^6(x)x - \sigma(x)\sigma^7(x)x - \sigma(x)\sigma^8(x)x - \sigma(x)x^2$$

$$+ \sigma^3(x)\sigma^6(x)x + \sigma^3(x)\sigma^7(x)x + \sigma^3(x)\sigma^8(x)x$$

$$- \sigma^4(x)\sigma^7(x)x + \sigma^4(x)\sigma^8(x)x - \sigma^4(x)x^2.$$  

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Using the theorem repeatedly, we can find an explicit expression for an element \( x_G \in R \) with \( N_G(x_G) = 1 \) as a noncommutative polynomial in the variables \( \sigma^i(x_E) \) \((0 \leq i < p^n)\), where \( x_E \in R \) satisfies \( N_E(x_E) = 1 \) for the unique subgroup \( E \cong \mathbb{Z}/p \) of \( G \). Formula (1) also gives an upper bound for the number of monomials appearing in this polynomial. Indeed, the number of monomials for \( a \) in (1) is
\[
\leq p^{n-k}(p^{n-k} - 1)(p^k + 1) + 1.
\]
We obtain a crude upper bound by setting \( k = 1 \) and by summing from 2 to \( n \). The upper bound we get in this way is
\[
\frac{p(p + 1)(p^{n-1} - 1)(p^n - 1)}{p^2 - 1} + n - 1 \sim \frac{p + 1}{p^2 - 1} p^{2n}
\]
when \( n \) becomes large. By taking \( k \) as the largest integer \( \leq n/2 \), we get a smaller upper bound, which is equivalent to \( p^{1/2} p^{3n/2} \) \((n \gg 0)\).

2. Proof of the theorem

We shall need a special case of the following lemma, directly inspired from Proposition 1.3 of [3, Chap. XII].

**Lemma 1.**— Let \( U \) be a finite group acting by ring automorphisms on a ring \( R \). If there exists an element \( x \in R \) such that \( N_U(x) = 1 \), then every element \( z \in R \) such that \( N_U(z) = 0 \) can be written as
\[
z = \sum_{g \in U} (g - 1)(xg^{-1}(z)).
\]

**Proof.**— We have
\[
\sum_{g \in U} (g - 1)(xg^{-1}(z)) = \sum_{g \in U} g(x) g(g^{-1}(z)) - \sum_{g \in U} xg^{-1}(z)
\]
\[
= N_U(x) z - x N_U(z) = z.
\]

Let us apply Lemma 1 to the case when \( U \) is a cyclic group of order \( r \). If we denote a generator of \( U \) by \( t \), then every element \( z \in R \) such that \( N_U(z) = 0 \) is of the form \( z = (t - 1)(w) \), where
\[
w = \sum_{i=1}^{r-1} (1 + t + t^2 + \cdots + t^{i-1})(xt^{-i}(z)).
\]

We now start the proof of the theorem. Recall that \( G \) is a cyclic group of order \( p^n \), generated by \( \sigma \), and \( U \) is the subgroup generated by \( \sigma^k \), where \( 1 \leq k \leq n/2 \).

Consider the group \( B = \text{Hom}(\mathbb{Z}[G], R) \) of \( \mathbb{Z} \)-linear maps from the group ring \( \mathbb{Z}[G] \) to \( R \), and equip it with the \( G \)-action given by \( (g \varphi)(s) = \varphi(sg) \) for all \( g, s \in G \) and \( \varphi \in B \).
We embed $R$ into $B$ by considering an element $x \in R$ as the element $\varphi_x \in B$ determined by $\varphi_x(g) = g(x)$ for all $g \in G$. The embedding $R \subset B$ preserves the $G$-action.

Define $\varphi \in B$ by

$$\varphi(g) = \begin{cases} 1 & \text{if } g = \sigma^{ip^k} \ (0 \leq i < p^{n-k}), \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that $\varphi$ is invariant under the action of the subgroup $U$. Since the action of $G$ on the group $B^U$ of $U$-invariant elements of $B$ factors through an action of the quotient cyclic group $G/U$, we may consider the action of

$$N_{G/U} = 1 + \sigma + \sigma^2 + \cdots + \sigma^{p^k-1}$$
on $B^U$. We clearly have

$$N_{G/U}(\varphi) = \varphi_1 \quad \text{and} \quad N_G(\varphi) = p^{n-k} \varphi_1. \quad (4)$$

On the other hand, $N_U(x) = 1$ implies $N_G(x) = p^k$. Therefore

$$N_G(\varphi - p^{n-2k} \varphi_x) = 0. \quad (5)$$

Define $\psi \in B$ inductively by $\psi(\sigma^0) = 0$ and for $1 \leq i < p^n$ by

$$\psi(\sigma^i) = \psi(\sigma^{i-1}) - \varphi(\sigma^{i-1}) + p^{n-2k} \sigma^{i-1}(x).$$

It follows from the definition of $\psi$ and from (5) that

$$\varphi = (1 - \sigma)(\psi) + p^{n-2k} \varphi_x. \quad (6)$$

**Lemma 2.**— The element $\psi \in B$ is $U$-invariant modulo $R$.

**Proof.**— By (4) and (6) we have

$$\varphi_1 = N_{G/U}(\varphi) = N_{G/U}((1 - \sigma)(\psi))$$

$$\equiv ((1 + \sigma + \sigma^2 + \cdots + \sigma^{p^k-1})(1 - \sigma))(\psi)$$

$$\equiv (1 - \sigma^{p^k})(\psi) \quad \text{modulo } R.$$

We conclude by observing that $\varphi_1 \equiv \varphi_0 \equiv 0$ modulo $R$. \hfill \Box

It follows from Lemma 2 that there exists $z \in R$ such that

$$(\sigma^{p^k} - 1)(\psi) = \varphi_z. \quad (7)$$

In order to find $z$, it suffices to apply both sides of (7) to the unit element of $G$; we thus obtain $z = \psi(\sigma^{p^k}) - \psi(\sigma^0)$, which, by definition of $\psi$, equals

$$z = p^{n-2k} (1 + \sigma + \sigma^2 + \cdots + \sigma^{p^k-1})(x) - 1. \quad (8)$$
We claim that the element \( z \in R \) is killed by \( N_U \). Indeed, by (7),

\[
N_U(z) = N_U \left( (\sigma^{p^k} - 1)(\psi) \right) = (\sigma^{p^n} - 1)(\psi) = 0.
\]

Applying Formula (3), we have \( z = (\sigma^{p^k} - 1)(w) \), where

\[
w = \sum_{i=1}^{p^{n-k} - 1} \left( 1 + \sigma^{p^k} + \sigma^{2p^k} + \cdots + \sigma^{(i-1)p^k} \right) (x^{\sigma^{ip^k}}(z)).
\]

Formulas (7) and (9) imply

\[
(\sigma^{p^k} - 1)(\psi - \varphi_w) = 0.
\]

In other words, \( \psi - \varphi_w \) is \( U \)-invariant.

**Lemma 3.**— The element \( a = p^{n-2k} x + (1 - \sigma)(w) \in R \) is \( U \)-invariant and \( N_{G/U}(a) = 1 \).

**Proof.**— By definition of \( a \) and \( w \), and by (6), we have

\[
\varphi_a = \varphi - (1 - \sigma)(\psi - \varphi_w),
\]

which is \( U \)-invariant in view of (10). By (4) and (10), we obtain

\[
N_{G/U}(a) = N_{G/U}(\varphi) - N_{G/U}((1 - \sigma)(\psi - \varphi_w)) = \varphi_1 - (1 - \sigma^{p^k})(\psi - \varphi_w) = 1.
\]

□

We can now complete the proof of the theorem. First observe that the elements \( a \in R^U \) and \( z \in R \) of the theorem are exactly the ones introduced in this section. So it is enough to check that \( N_G(ax) = 1 \). Indeed, using the \( R^U \)-linearity of \( N_U \), Lemma 3, and \( N_U(x) = 1 \), we have

\[
N_G(ax) = \sum_{g \in G} g(ax) = \sum_{t \in G/U} t \left( \sum_{s \in U} s(ax) \right)
= \sum_{t \in G/U} t \left( N_U(ax) \right) = \sum_{t \in G/U} t \left( aN_U(x) \right)
= \sum_{t \in G/U} t(a) = N_{G/U}(a) = 1.
\]

□
3. Some cohomological considerations

In the proof of the theorem as given in Section 2, the computations take place in the co-induced module \( B = \text{Hom}(\mathbb{Z}[G], R) \), and the elements \( \varphi, \psi \in B \) play a central rôle. This can be explained through the following cohomological considerations.

We first claim that the existence of \( x \in R \) such that \( N_U(x) = 1 \) implies the vanishing of the cohomology of the group \( U \) with coefficients in \( R \) in positive degrees: \( H^i(U, R) = 0 \) for all \( i > 0 \). Indeed, since \( U \) is cyclic with generator \( \sigma^p \), we have

\[
H^1(U, R) = \text{Ker}(N_U : R \to R)/(\sigma^p - 1)(R),
\]

which is zero by Lemma 1. The surjectivity of \( N_U : R \to R^U \) implies the vanishing of

\[
H^2(U, R) = R^U/N_U(R).
\]

Then the claim follows from the periodicity of the cohomology of cyclic groups.

In view of [4, Section VII.6] (or of Hochschild-Serre’s spectral sequence), the vanishing of \( H^i(U, R) = 0 \) for \( i > 0 \) implies that the inflation maps

\[
\text{Inf} : H^*(G/U, R^U) \to H^*(G, R)
\]

are isomorphisms.

Now consider the short exact sequence of \( \mathbb{Z}[G] \)-modules

\[
0 \to R \to B \to C \to 0,
\]

where \( C = B/R \). Applying \( H^*(U, -) \) to (11), we obtain a sequence of \( \mathbb{Z}[G/U] \)-modules

\[
0 \to R^U \to B^U \to C^U \to 0,
\]

which is exact because of the vanishing of \( H^1(U, R) \). Observe that \( B^U \cong \text{Hom}(\mathbb{Z}[G/U], R) \) is a co-induced module for \( G/U \).

The naturality of the inflation maps gives rise to the commutative square

\[
\begin{array}{ccc}
H^1(G/U, C^U) & \xrightarrow{\text{Inf}} & H^1(G, C) \\
\delta \downarrow & & \delta \downarrow \\
H^2(G/U, R^U) & \xrightarrow{\text{Inf}} & H^2(G, R)
\end{array}
\]

(13)

where the vertical maps \( \delta \) are connecting maps for the short exact sequences (11) and (12). We claim that all maps in the square (13) are isomorphisms. We have already proved this for the lower inflation map. The connecting maps \( \delta \) are isomorphisms because co-induced modules are cohomologically trivial. It follows that the upper inflation map is an isomorphism as well. Moreover, by [2, Theorem 1], the surjectivity of \( N_U : R \to R^U \) implies the surjectivity of \( N_G : R \to R^G \). Therefore, \( H^2(G, R) = R^G/N_G(R) = 0 \), which implies the vanishing of all cohomology groups in (13).
The central rôle played by the element $\varphi \in B^U$ in the proof of the theorem follows from the following two facts:

(i) If $\bar{\varphi}$ denotes the class of $\varphi$ in $C$, then by (4) it induces an element $[\bar{\varphi}] \in H^1(G/U, C^U) = \text{Ker}(N_{G/U} : C^U \to C^U )/(\sigma - 1)(C^U)$.

(ii) The image $\delta([\bar{\varphi}])$ of $[\bar{\varphi}]$ under the connecting map for the short exact sequence (12) is computed as follows: lift $\bar{\varphi}$ to $\varphi \in B^U$ and apply $N_{G/U}$. By (4) again, we obtain

$$\delta([\bar{\varphi}]) = 1 \in H^2(G/U, R^U) = R^G/N_{G/U}(R^U).$$

The existence of $\psi$ satisfying (6) follows from the vanishing of

$$H^1(G, C) = \text{Ker}(N_G : C \to C)/(\sigma - 1)(C)$$

and Lemma 2 follows from the injectivity of $\text{Inf} : H^1(G/U, C^U) \to H^1(G, C)$.

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