Optimal $\ell_1$ Column Subset Selection and a Fast PTAS for Low Rank Approximation

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Abstract

We study the problem of entrywise $\ell_1$ low rank approximation. We give the first polynomial time column subset selection-based $\ell_1$ low rank approximation algorithm sampling $\tilde{O}(k)$ columns and achieving an $\tilde{O}(k^{1/2})$-approximation for any $k$, improving upon the previous best $\tilde{O}(k)$-approximation and matching a prior lower bound for column subset selection-based $\ell_1$-low rank approximation which holds for any poly($k$) number of columns. We extend our results to obtain tight upper and lower bounds for column subset selection-based $\ell_p$ low rank approximation for any $1 < p < 2$, closing a long line of work on this problem.

We next give a $(1 + \varepsilon)$-approximation algorithm for entrywise $\ell_p$ low rank approximation of an $n \times d$ matrix, for $1 \leq p < 2$, that is not a column subset selection algorithm. First, we obtain an algorithm which, given a matrix $A \in \mathbb{R}^{n \times d}$, returns a rank-$k$ matrix $\hat{A}$ in $2^{\text{poly}(k/\varepsilon)} + \text{poly}(nd)$ running time that achieves the following guarantee:

$$\|A - \hat{A}\|_p \leq (1 + \varepsilon) \cdot OPT + \frac{\varepsilon}{\text{poly}(k)} \|A\|_p$$

where $OPT = \min_{A_k \text{ rank } k} \|A - A_k\|_p$. Using this algorithm, in the same running time we give an algorithm which obtains error at most $(1 + \varepsilon) \cdot OPT$ and outputs a matrix of rank at most $3k$ — these algorithms significantly improve upon all previous $(1 + \varepsilon)$- and $O(1)$-approximation algorithms for the $\ell_p$ low rank approximation problem, which required at least $n^{\text{poly}(k/\varepsilon)}$ or $n^{\text{poly}(k)}$ running time, and either required strong bit complexity assumptions (our algorithms do not) or had bicriteria rank $3k$. Finally, we show hardness results which nearly match our $2^{\text{poly}(k)} + \text{poly}(nd)$ running time and the above additive error guarantee.
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1 Introduction

Low rank approximation is one of the most fundamental problems in data science and randomized numerical linear algebra. In this problem, one is given an $n \times d$ matrix $A$ and a rank parameter $k$, and one would like to approximately decompose $A$ as $U \cdot V$, where $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{k \times d}$ are the low rank factors. This gives a form of compression, since rather than storing the $nd$ parameters needed to represent $A$, we can store only $(n + d)k$ parameters to store the low rank factors. One can also multiply $U \cdot V$ by a vector in $O((n + d)k)$ time, rather than the $O(nd)$ time needed for multiplication by $A$. The singular value decomposition (SVD) can be used to find the best low rank approximation of $A$ with respect to the sum of squares of differences, i.e., the Frobenius norm error measure, but this measure is often not robust enough in applications, since the need to fit single large outliers in $A$ is exacerbated by the squared error measure, causing $U$ and $V$ to overfit such outliers and not capture enough of the remaining entries of $A$.

To overcome this, a large body of work has studied other, more robust error measures, with a notable one being the entrywise $\ell_1$-low rank approximation problem: given $A \in \mathbb{R}^{n \times d}$, a rank parameter $k$, and an approximation parameter $\alpha \geq 1$, find $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{k \times d}$ so that:

$$\|U \cdot V - A\|_1 \leq \alpha \cdot \min_{U' \in \mathbb{R}^{n \times k}, V' \in \mathbb{R}^{k \times d}} \|U' \cdot V' - A\|_1,$$

where for a matrix $B \in \mathbb{R}^{n \times d}$, $\|B\|_1 = \sum_{i=1}^{n} \sum_{j=1}^{d} |B_{i,j}|$ is its entrywise 1-norm. Since we take the sum of absolute values of differences of entries of $A$ and corresponding entries of $U \cdot V$, this is often considered more robust than the Frobenius norm error measure, which takes the squared differences. There is a large body of work on applications of $\ell_1$ matrix factorization, as well as practical improvements obtained by optimizing the $\ell_1$ error measure rather than the more well-understood Frobenius norm error measure, in areas such as computer vision and machine learning. For instance, in [27] and later in [47] it was shown that optimizing the $\ell_1$-based objective above, or a regularized version of it, yields much better performance than an IRLS-based approach or other $\ell_2$-based methods on the structure-from-motion problem. Various other works mentioned below, also motivated by problems in machine learning and signal processing, developed heuristics for $\ell_1$ low rank approximation and similar problems.

The $\ell_1$-low rank approximation problem, for $\alpha = 1$, was shown to be NP-hard in [21], and assuming the Exponential Time Hypothesis, it requires $2^{\Omega(1/\epsilon)}$ time for $\alpha = 1 + \epsilon$. While these results rule out extremely accurate solutions to the $\ell_1$-low rank approximation problem in polynomial time, they leave open the possibility of larger approximation factors. Such approximation factors are of considerable interest, and we note that a low rank approximation $U \cdot V$ corresponding to an approximation factor $\alpha \ll \sqrt{nd}$ for the $\ell_1$-low rank approximation problem may be much better in applications than a low rank approximation corresponding to an exact solution to the Frobenius norm error measure, since the error measures are incomparable$^1$. A number of heuristics were proposed for the $\ell_1$-low rank approximation problem in [26, 27, 28, 29, 47, 7, 8, 33, 31, 32, 30, 35]. The first rigorous approximation factors were proven in [41], where it was shown that $\alpha = \text{poly}(k) \log n$-approximation is achievable in $\text{poly}(ndk)$ time$^2$. Later, in [14] an alternative $\text{poly}(ndk)$ time algorithm with $\text{poly}(k \log (nd))$-approximation factor was given which held for every entrywise $\ell_p$ norm, for any constant $p \geq 1$.

In fact, each of the algorithms above is, or can be used to obtain a $\text{poly}(dk)$ time column subset selection algorithm for the $\ell_1$-low rank approximation problem with a $\text{poly}(k \log (nd))$ approximation factor. Column subset selection is a special type of low rank approximation in which the left factor $U$ corresponds to a subset of columns of $A$ itself. Column subset selection is a widely studied and extremely important special case of low rank approximation (see, e.g., [3, 16, 6] and the references therein); it has a number of advantages - for example, if the columns of $A$ are sparse, then the columns in the left factor $U$ are also sparse. One might argue that the column subset selection problem, also known as feature selection (since the columns of $U$ can be thought of as the important features), is sometimes more important than the low rank approximation problem itself. Due to these advantages, it has been of interest to determine how well column subset selection algorithms can do compared to the optimal low-rank approximation error, in the Frobenius norm [19], spectral norm [16, 6] and in the $\ell_1$ norm [10, 32] and even for more general loss functions [44].

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$^1$We note that the exact solution to the Frobenius norm error measure gives a $\sqrt{nd}$-approximation to $\ell_1$-low rank approximation by relating the entrywise 2-norm to the entrywise 1-norm.

$^2$Note that the problem is symmetric in $n$ and $d$, so if $d < n$, we can instead write this as $\text{poly}(k) \log d$. 
Often for column subset selection, we allow for a bicriteria approximation, namely, for outputting a low rank matrix of rank r with r a bit larger than k. Bicriteria approximations are common in the low rank approximation literature [13, 20, 11], and correspond to the case when k is not known or is not a hard constraint; note that bicriteria approximations still capture the original compression and fast multiplication motivations of low rank approximation discussed above. Moreover, bicriteria approximations are necessary in the context of column subset selection for some norms; indeed, for Frobenius norm it is known [19] that with exactly k columns the best approximation factor possible is Θ(k), while with O(k) columns an O(1) approximation is possible.

A natural question is what the limits of column subset selection algorithms for $\ell_1$-low rank approximation are. It was shown in [44] that the $O(k \log(nd))$-approximation can be improved to $O(k \log k)$ with bicriteria rank $r = O(k \log n)$, via a polynomial time algorithm. Moreover, if one is willing to spend $n^{\Omega(k)}$ time, it was shown in [11] how to obtain an approximation factor of $O(\sqrt{k \log k})$. From a purely combinatorial perspective, this is close to best possible as [11] shows that there exist matrices for which any subset of at most poly(k) columns provides at best a $k^{1/2-\gamma}$-approximation, for an arbitrarily small constant $\gamma > 0$. We note that for every $p \geq 2$, there are tight bounds on the size of the best column subset selection algorithms for entrywise $\ell_p$-low rank approximation known, and there are polynomial time algorithms achieving these (see Theorem 4.1 of [13]). For $1 \leq p \leq 2$, however, the best known upper bounds have an approximation factor of roughly $k^{1/p}$, while the known lower bounds are only $k^{1-1/p}$ for $1 < p < 2$ [13]. In addition, the lower bounds of [13] only hold when $k$ columns are selected, and do not rule out smaller approximation factors through slightly larger bicriteria ranks. While these results impose a limit on the approximation error achievable through column subset selection, the current gap of an $O(k \log k)$ approximation factor versus a $k^{1/2-\gamma}$ lower bound for column subset selection algorithms for $p = 1$ has remained elusive:

**Question 1:** What is the best approximation factor for column subset selection for $\ell_1$-low rank approximation achievable by a polynomial time algorithm? What about for $1 < p < 2$?

While the above approximation factors are highly non-trivial, the lower bound of [11] rules out better than a $k^{1/2-\gamma}$-approximation with column subset selection methods, for arbitrarily small constant $\gamma > 0$. A natural question is if one can improve these approximation factors to $O(1)$ or even $(1 + \varepsilon)$ in polynomial time. This was the main question underlying [2], which, motivated by the fact that $k$ is often small, took a parameterized complexity approach and showed how to obtain a $(1 + \varepsilon)$-approximation in $n^{\text{poly}(k/\varepsilon)}$ time under the assumption that the entries of the input matrix $A$ are integers in the range $\{-\text{poly}(n), \ldots, \text{poly}(n)\}$ (or alternatively, outputting a matrix of rank $3k$ instead of $k$ with the same approximation factor guarantee, while removing this assumption on the entries of $A$). This algorithm has several drawbacks though: (1) the $n^{\Omega(k)}$ running time even for constant $\varepsilon$ is prohibitive and makes the algorithm super-polynomial time for any $k = \omega(1)$, i.e., if $k$ is larger than any constant, and (2) the $O(\log n)$ bit complexity assumption on the entries of $A$ may not be realistic; ideally one should allow $\text{poly}(nd)$ bit complexity. This motivates the following central question:

**Question 2:** Is it possible to obtain a polynomial time algorithm for $\ell_1$-low rank approximation for $k = \omega(1)$ with a small bicriteria rank $r$ and with a $(1 + \varepsilon)$ approximation factor?

1.1 Our Results

In this work we obtain results for both column subset selection and general low rank approximation.

1.1.1 $\ell_p$ Column Subset Selection for $1 \leq p < 2$

We resolve Question 1 above up to small factors by giving a $\text{poly}(ndk)$ time algorithm for finding a subset of $O(k \log k \log d)$ columns providing an $O(\sqrt{k \log^{3/2} k})$-approximation for $\ell_1$-low rank approximation. This nearly matches the lower bound of [44] which shows that there exist matrices for which the span of any subset of $\text{poly}(k)$ columns has approximation factor at least $k^{1/2-\gamma}$ for an arbitrarily small constant $\gamma > 0$. See

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3See Theorem C.1, part III, of [44]. Note that part I of that theorem also obtains an $O(\sqrt{k \log k})$-approximation, but it is not a column subset selection algorithm.

4For entrywise $\ell_p$-low rank approximation, we seek to find rank-$k$ factors $U$ and $V$ so as to minimize $\sum_{i,j} |(U \cdot V - A)_{i,j}|^p$. **
approximation algorithm with bi-criteria rank, and a poly($k$)- approximation factor or rank), and a poly($k$)-approximation factor independent of $\gamma$. We are not aware of any matching lower bounds for column subset selection for $1 < p < 2$ (the tight lower bounds in [13] are specifically for $p > 2$ and the lower bound in [11] is only for $p = 1$), so we also prove the first lower bounds for $1 < p < 2$, showing that there exist matrices for which the span of any column subset of size $k \cdot \text{poly}(k)$ has approximation factor at least $k^{1/p-1/2-\gamma}$ for an arbitrarily small constant $\gamma > 0$. This shows that our algorithm is nearly optimal for $1 < p < 2$ as well.

Our results complement the results of [13] for the case of $p > 2$; together with our results we obtain optimal bounds on column subset selection, up to small factors, for every $\ell_p$-norm, $p \geq 1$, closing a line of work on this problem [11, 14, 42, 10, 13]. We note that all the column subset selection algorithms in this line of work are bicriteria algorithms, as motivated and defined above, meaning that the actual number of columns returned is $k \cdot \text{poly}(\log(nk))$, which is optimal up to polylogarithmic factors.

We use the above algorithm as a subroutine to achieve two additional results, a polynomial time $O(k)$-approximation algorithm with bi-criteria rank $O(k^2)$ (and no dependence on $n$ and $d$, even logarithmic, in either the approximation factor or rank), and a poly($k$)-approximation with rank $k$ (and no dependence on $n$ and $d$, even logarithmic, in the approximation factor) with running time $2^{O(k \log k)} + \text{poly}(nd)$ — the first of these algorithms is a column subset selection algorithm, while the second is not. These algorithms may be of independent interest — note that all previously known algorithms for getting a poly($k$)-approximation factor independent of $n$ or $d$ require either $d^8$ time or a log$d$ term in the bi-criteria rank.

We note that there are also works studying $\ell_1$ column subset selection, which obtain guarantees in terms of the error from the optimal column subset — two such works are [11, 23]. Our column subset selection results are not comparable to these. [11] studies the problem of fitting the columns of a matrix $B$ using a subset of the columns of a (potentially different) matrix $A$, and considers the case where both $A$ and $B$ are nonnegative. The following guarantee is obtained by [11]: if there is a column subset of size $k$ obtaining error at most $\varepsilon\|B\|_1$, then the algorithm of [11] (given a $\delta > 0$) finds a subset of columns of $A$ of size $O(k \log(1/\delta)/\delta^2)$ that fits $B$ with error at most $\sqrt{\delta + \varepsilon\|B\|_1}$. This error could potentially be larger than that of our algorithm which obtains relative error guarantees, if $\|B\|_1$ is much larger than $\text{OPT}$ — in that case, even if there is a column subset obtaining error at most $\text{OPT}$, the algorithm of [11] could obtain error up to $\sqrt{\text{OPT} \cdot \|B\|_1}$, according to this guarantee. Hence, our algorithm could potentially be significantly better in the case where $\|B\|_1 \gg \text{poly}(k)\text{OPT}$. [23] gives a protocol for distributed column subset selection in the $\ell_p$-norm, which obtains an $O(\sqrt{k \log(d)})$-approximation relative to the error of the best column subset when $p = 1$. This could be larger than the error from our algorithm in cases where the error due to the best column subset is significantly larger than $\text{OPT}$ — note that by the lower bound of [11], the error of the best column subset could be $\Omega(k^{1/2-\alpha} \cdot \text{OPT})$ for an arbitrarily small constant $\alpha$.
1.1.2 General $\ell_p$ Low Rank Approximation for $1 \leq p < 2$

We next consider Question 2 and give a new bicriteria algorithm for entrywise $\ell_1$ low rank approximation which goes beyond column subset selection. We improve upon the previous best bicriteria algorithms achieving $O(1)$ and $(1 + \varepsilon)$-approximation, which require at least $n^{\text{poly}(k/\varepsilon)}$ time and are not polynomial time when $k = \omega(1)$. In contrast, our algorithm is polynomial time even for slow-growing functions $k$ of $n$, such as $k = \Theta((\log^c n)$, for an absolute constant $c > 0$.

As mentioned above, bicriteria algorithms are natural and well-studied in this context. We note though that the main algorithm of \cite{2} outputs a rank-$k$ matrix while ours is bicriteria; however, no $(1 + \varepsilon)$ or $O(1)$ approximations in less than $n^{\text{poly}(k)}$ time were known even for bicriteria algorithms. In particular, even the bicriteria algorithm of \cite{2} requires $n^{\text{poly}(k/\varepsilon)}$ time.

We give a $(1 + \varepsilon)$-approximation algorithm with $2^{\text{poly}(k/\varepsilon)} + \text{poly}(nd)$ running time and output rank at most $3k$. See Table 2 for a table containing our result and a comparison to prior work for $\ell_1$-low rank approximation, where we list all previous $O(1)$ and $(1 + \varepsilon)$ approximation algorithms. Our algorithm also works for $\ell_p$ low rank approximation for $1 \leq p < 2$.

An interesting aspect of our result is that it shows entrywise $\ell_p$-low rank approximation for $p \in [1, 2)$, with the above approximation factor and bicriteria rank, is not $W[1]$-hard, in the language of parameterized complexity, which a priori may have been the case as all previous algorithms required $n^{\text{poly}(k)}$ time.

| Paper | Bicriteria Rank | Approximation Factor | Time Complexity | Notes |
|-------|-----------------|----------------------|-----------------|-------|
| \cite{1}, Lemma C.10 | $k$ | $O(1)$ | $n^{\text{poly}(k)}$ | BCA |
| \cite{1}, Theorem C.9 | $3k$ | $1 + \varepsilon$ | $n^{\text{poly}(k/\varepsilon)}$ | |
| \cite{2} | $k$ | $O(1)$ | $n^{\text{poly}(k)}$ | BCA |
| \cite{2} | $3k$ | $1 + \varepsilon$ | $n^{\text{poly}(k/\varepsilon)}$ | |
| This paper | $3k$ | $1 + \varepsilon$ | $2^{\text{poly}(k/\varepsilon)} + \text{poly}(nd)$ | |

Table 2: Summary of our results for general $\ell_1$-Low Rank Approximation. We write BCA in the Notes next to each algorithm if it makes bit-complexity assumptions on the entries of the input matrix. We note that the non-bicriteria algorithm of \cite{2} actually has an $(Mn)^{\text{poly}(k)}$ running time, where $M$ is the maximum value of an entry of the input matrix, and thus is not even polynomial time for constant $k$ if the entries of the input matrix are expressed with more than $\log n$ bits. In contrast, our algorithm is polynomial in the input description length, and thus for example, can handle entries as large as $M = 2^{\text{poly}(n)}$. The prior algorithms from \cite{1} and \cite{2} require time $n^{\text{poly}(k)}$, and thus are not polynomial time for any $k = \omega(1)$. In contrast, our algorithms are all polynomial time even if $k$ is a slow-growing function of $n$, such as $\Theta((\log^c n)$ for an absolute constant $c > 0$.

1.1.3 Hardness for $\ell_p$ Low Rank Approximation with Additive Error - Appendix A

As an intermediate step for $\ell_p$ low rank approximation, we obtain an algorithm (Algorithm 2 for $p = 1$ and Algorithm 5 for general $p$) which achieves the following guarantee: given $A \in \mathbb{R}^{n \times d}$ and $k \in \mathbb{N}$, it obtains a matrix $\hat{A}$ of rank at most $k$ such that

$$\|\hat{A} - A\|_p \leq (1 + \varepsilon) \min_{A_k \text{ rank } k} \|A_k - A\|_p + \frac{\varepsilon}{\beta} \|A\|_p$$

(1)

in $f^{\text{poly}(k/\varepsilon)} + \text{poly}(nd)$ time, where $f$ can be any desired number greater than 1. To our knowledge, there do not exist hardness results for such a guarantee (or for bi-criteria approximations). It is known, due to \cite{2}, that achieving an $O(1)$-approximation for $\ell_p$ low rank approximation ($p \in (1, 2]$) requires at least $2^{\text{poly}(1)}$ running time assuming the Small Set Expansion Hypothesis (SSEH) \cite{30} and Exponential-Time Hypothesis (ETH) \cite{22}. We extend the techniques of \cite{2} to show that even achieving the guarantee in Equation 1 requires $2^{\text{poly}(1)}$ time assuming SSEH and ETH, for $p \in (1, 2]$, if $f \geq 2^{\text{poly}(k)}$. Even when $\varepsilon = \Theta(1)$. Hence, our algorithm is close to optimal in a sense, since it can achieve that guarantee in $2^{\text{poly}(k)} + \text{poly}(nd)$ time.

\footnote{There is a slight typo in \cite{1} in Theorem C.9 and its proof, where the bicriteria rank is said to be $2k$ — the bicriteria rank of that algorithm is actually $3k$, since once the poly(n)-approximation $B$ is subtracted off from the target matrix $A$, a good rank-2k approximation $M$ for $B - A$ is needed to recover the original optimum for $A$, from $B - A$.}
We also show that it is optimal in the following related sense: it achieves a similar guarantee in $2^{\text{poly}(k)} + \text{poly}(nd)$ time for the related problem of constrained $\ell_1$ low rank approximation.

**Problem 1** (Constrained $\ell_1$ Low Rank Approximation). Given a matrix $A \in \mathbb{R}^{n \times d}$ and a subspace $V \subset \mathbb{R}^n$, find a matrix $\hat{A}$ of rank at most $k$ minimizing $\|\hat{A} - A\|_1$, such that the columns of $\hat{A}$ are in $V$.

Our algorithm for $p = 1$ (see Algorithm 2) can be modified very slightly so that it achieves the same guarantee for this problem as well — the modified algorithm can compute, in $2^{\text{poly}(k)} + \text{poly}(nd)$ time, a matrix $\hat{A}$ such that

$$
\|\hat{A} - A\|_1 \leq O(1) \min_{A_k} \|A_k - A\|_1 + \frac{1}{2^{\text{poly}(k)}} \|A\|_1
$$

such that $\hat{A}$ has rank $k$ and the columns of $A$ are contained in $V$ — here, the minimum on the right-hand side is also taken over $A_k$ with rank at most $k$, such that the columns of $A_k$ are in $V$. Assuming the SSEH and randomized ETH (see [15], page 5), we show that achieving this guarantee also requires at least $2^\Omega(k)$ time.

### 1.2 Our Techniques

We give an overview of our arguments for $\ell_1$-column subset selection, then for general $\ell_1$-low rank approximation. The arguments for $\ell_p$-column subset selection and $\ell_p$-low rank approximation, for $p \in (1, 2)$, are similar and are included in Section 2 and Section 3 respectively.

#### 1.2.1 $\ell_1$ Column Subset Selection

**Algorithm and Initial $O(\sqrt{k}) \log(d)$ Approximation Factor.** In our algorithm, we uniformly sample a column subset $S^{(0)}$ of size $t = O(k \log k)$ of our input matrix $A$ and argue that we can approximately span a constant fraction of remaining columns of $A$ using $S^{(0)}$, where to approximately span the $i$-th column means to obtain a column vector $v^i$ for which $\|v^i - A_i\|_1 = O(\sqrt{k \log k})OPT/d$, where $OPT$ is the cost of the optimal rank-$k$ approximation to $A$. Thus, our total cost to cover a constant fraction of columns will be $O(\sqrt{k \log k})OPT$. We then recurse on the remaining fraction of columns. If there are $d_1$ columns in the next recursive call, we argue we approximately span a constant fraction of the remaining columns, where now to approximately span the $i$-th column means to obtain a column vector $v^i$ for which $\|v^i - A_i\|_1 = O(\sqrt{k \log k})OPT/d_1$. Again, the total cost to cover a constant fraction of remaining columns is $O(\sqrt{k \log k})OPT$. We then recurse on the columns still remaining. Since we approximately span a constant fraction of columns in each recursive step, after $O(\log d)$ recursive steps we will have spanned all $d$ columns. The total number of columns we have will have chosen is $O(k \log k \log d)$ and the overall approximation factor will be $O(\sqrt{k \log k} \log d)$.

The algorithm described above is simple, and reminiscent of column sampling algorithms [10 13 44] in prior work. However, our analysis is completely new and does not involve going through maximum determinant subsets, as in each of these previous column sampling algorithms. Those works argued that if one uniformly samples a set $S$ of $2k$ columns of $V^* \in \mathbb{R}^{k \times n}$, where $A_k = U^* V^*$ is the best rank-$k$ approximation to $A$, and considers a random additional column $c$, then with probability at least $1/2$, the maximum determinant subset (which is of size $k$) of columns of $V_{S \cup \{c\}}^* \in \mathbb{R}^{k \times (2k+1)}$ does not contain $c$, and consequently $c$ can be expressed as a linear combination of columns in our sample set $S$ with coefficients of absolute value at most $1$, and thus by the triangle inequality one pays a cost at most what the subset $S$ pays to approximate the $c$-th column of $A$, which since $S$ was chosen uniformly at random, is at most $O(k)OPT/d$ with constant probability.

We do not know how to reduce the approximation factor in the analysis of all of these previous algorithms; intuitively, the difficulty stems from the fact that the maximum determinant subset may not be the best subset to look at for $\ell_1$; indeed, it could be that for a random $c$, one needs coefficients of absolute value 1 to span it using the columns in the set $S$, and it is unclear how the error propagates other than through the triangle inequality. We thus give the first analysis of the above sampling framework that *does not go through maximum determinant subsets*.

Instead, we argue that if one had $V^*$, then one could sample columns using its so-called *Lewis weights* [12], creating a sampling and rescaling matrix $R$ with $t/2 = O(k \log k)$ columns, so that the solution $U =$
arg\min_c \| A_{S \cup c} R - U(V^*)_{S \cup c} R \|_1 \) would be an \( O(1) \)-approximate rank-\( k \) left factor for the submatrix of \( A \) indexed by \( S \cup c \). By relating \( \ell_1 \) and \( \ell_2 \)-norms of the rows of \( A_{S \cup c} R - U(V^*)_{S \cup c} R \) and using the normal equations for least squares regression, we get that \( U' = A_{S \cup c} R((V^*)_{S \cup c} R)^+ \) provides an \( O(\sqrt{k \log k}) \)-approximate rank-\( k \) left factor for the submatrix of \( A \) indexed by \( S \cup c \). The advantage of \( U' \) is that it is in the column span of \( A_{S \cup c} R \). Note that this subroutine should be reminiscent of the algorithm of [41], which argued one could obtain an \( O(\sqrt{k \log k}) \)-approximate column subset selection algorithm by enumerating over all subsets of \( t/2 \) columns of \( A \); however [41] actually does this and suffers \( d^3(k \log k) \) time.

Here, we would instead just like to argue that \( A_{S \cup c} R \) is unlikely to contain the column \( A_c \). If that were true, then that means we approximately span the column \( A_c \) with our sample \( A_S \), and this property is all that is needed for the above recursive algorithm! Note we never have to guess or enumerate over subsets of columns of \( V^* \) as in previous work. However, why should it be the case that \( A_{S \cup c} R \) is unlikely to contain the column \( A_c \), when the sampling matrix \( R \) is allowed to depend on \( A_c \)? Here we use the fact [12] that for a rank-\( k \) space \( V^* \), one only needs to sample \( O(k \log k) \) columns so that with large constant probability, one has that the solution \( U = \arg\min_U \| A_{S \cup c} R - U(V^*)_{S \cup c} R \|_1 \) is an \( O(1) \)-approximate rank-\( k \) left factor for the submatrix of \( A \) indexed by \( S \cup c \). In the above discussion, we are denoting this number \( O(k \log k) \) by \( t/2 \). Now, since we are considering a set \( S \cup \{c\} \) of \( t + 1 \) uniformly random columns, it follows that if we were to compute the Lewis weights of \( V^* \) of \( S \cup \{c\} \) and then sample our matrix \( R \) using them, then with constant probability the column \( c \) would not be in our sample of \( t/2 \) columns chosen by \( R \). But here we are conditioning on column \( c \) not being in our sample set \( S \), so we still need to show that conditioned on having not chosen the \( c \)-th column, the cost of approximately spanning column \( c \) by the Lewis weight sample is still small. This follows from the facts that (1) \( c \) is not chosen by the Lewis weight sampling matrix with good probability and therefore the cost of the Lewis weight sampling remains small conditioned on this occurring (2) the overall cost of using Lewis weight sampling to approximately span all remaining columns is \( O(\sqrt{k \log k})OPT_{S \cup \{c\}}(c) \) where \( OPT_{S \cup \{c\}}(c) \) is the total cost on \( A_{S \cup \{c\}} \), (3) since \( S \cup \{c\} \) was a uniformly random subset, the value \( OPT_{S \cup \{c\}}(c) \) is \( O(t/d)OPT \) in expectation, and (4) \( c \) is among \( t/2 \) random columns all of which are not in the Lewis weight sample, and thus is likely to have a \( 2/t \) fraction of the total conditional expected cost. Chaining these statements together gives us that with large constant probability, the cost of approximately spanning the \( c \)-th column from our sample set is \( O(\sqrt{k \log k})OPT/d \), completing the argument.

We generalize this approach to obtain optimal column subset selection algorithms for entrywise \( \ell_p \) low rank approximation, for every \( 1 < p < 2 \), replacing the \( \ell_1 \) Lewis weights with the \( \ell_p \) Lewis weights in the above analysis. Note we cannot use earlier sampling distributions, such as \( \ell_p \) leverage scores or total sensitivities, as it is also important in the argument above that one only needs to sample \( O(k \log k) \) columns for a rank-\( k \) space and these latter sampling distributions would require a larger \( k^{1+c} \) samples for a constant \( c > 0 \) (see, e.g., [40] for a survey): this is important not only for the overall number of sampled columns but also for the approximation factor, since we also relate the \( p \)-norm to the 2-norm through this number.

**Removing \( \log d \) from the Approximation Factor.** An improved version of the above argument, inspired by [41], gives an \( O(\sqrt{k \log k}) \) approximation rather than \( O(\sqrt{k \log k \log d}) \) — to achieve this, we note that in each round, we can condition on the event that the columns being chosen are not among the \( \frac{1}{t} \)-fraction of columns which have the highest cost, under the optimal \( \ell_1 \) rank-\( k \) approximation. This event occurs with constant probability. Moreover, for each of the other columns which are not in this top \( \frac{1}{t} \)-fraction of columns (which are indexed by a subset \( F \subset [d] \)), we can bound the cost by \( O(\sqrt{k \log k})OPT_{F} \) where \( OPT_{F} \) denotes the cost under the optimal rank-\( k \) approximation, excluding the errors from the top \( \frac{1}{t} \)-fraction of columns \( F \). Finally, we show that over the course of the \( O(\log d) \) recursive rounds, a particular column of \( A \) can contribute to \( OPT_{F} \) in at most \( O(\log k) \) rounds — that is, it cannot be outside of \( F \) for more than \( O(\log k) \) rounds without being approximately covered and discarded. A similar technique was used in [41] to obtain an \( O(k \log k \) approximation factor independent of \( \log d \).

**Lower Bound for \( \ell_p \) Column Subset Selection.** \( 1 \leq p < 2 \). Our nearly matching lower bound for entrywise \( \ell_p \) low rank approximation is a technical generalization of that for \( \ell_1 \)-low rank approximation given in [41] and we defer the details to Appendix B. For \( p \in (1, 2) \), we show that a proof strategy similar to that of [41] can be used to show that any subset selection algorithm that selects at most \( O(k \log k)^{\varepsilon_1} \)
columns (where $c_1$ can be any constant) achieves at least an $\Omega(\frac{k^\frac{3}{2}}{p} \cdot \frac{1}{2})$ approximation factor in the worst case, where $\alpha$ can be an arbitrary number in $(0, \frac{1}{p} - \frac{1}{2})$.

**Additional Column Subset Selection Results (Appendix C): Decreasing the Bicriteria Rank.**

Now, our polynomial time $O(k (\log k)^2)$-approximation algorithm with $O(k^2 (\log k)^2)$ bicriteria rank makes use of this improved approximation factor that is independent of $\log d$, and relies on the following simple observation. Let $U \in \mathbb{R}^{n \times O(k \log k \log d)}$ be the left factor ultimately returned by our main column subset selection algorithm. In each recursive round, if $S$ is the set of $t = O(k \log k)$ columns which are sampled, then for each column $A_i$ which is discarded during that round, $A_i$ can be approximately covered using only the $t = O(k \log k)$ columns belonging to $S$. The implication of this is that there exists $M$ having rank $O(k \log k \log d)$, which provides an $O(\sqrt{k \log k)^{\frac{2}{3}}}$-approximation for $A$, such that each column of $M$ can be written exactly as a linear combination of $O(k \log k)$ columns of $U$ (more specifically, $O(k \log k)$ columns of $U$ that were obtained in a single round of sampling from $A$).

Now, as mentioned above, it was shown in [41] that any matrix has a column subset of size $O(k \log k)$ spanning a $\sqrt{k \log k}$-approximation — hence, $M$ has a column subset of size $O(k \log k)$ which spans a $\sqrt{k \log k}$-approximation to $M$. By the triangle inequality, one can show that since $M$ is an $O(\sqrt{k \log k)^{\frac{2}{3}}}$-approximation for $A$, this $O(k \log k)$-sized column subset of $M$ spans an $O(k \log k^{\frac{2}{3}})$-approximation for $A$.

To form our left factor, for each of these columns $A_i$, we could collect the $O(k \log k)$ columns that were sampled in the round when $A_i$ was covered. Since we do not actually know this poly$(k)$-approximate subset of columns of $M$, we could naively try all of them — however, rather than checking all column subsets of $M$ of size $O(k \log k)$, which takes time $d^{O(k \log k)}$, it suffices to check all $O(k \log k)$-subsets of the $O(d \log d)$ rounds of sampling done by our main algorithm, and take the best subset.

There are $\binom{O(k \log k)}{d \log d} \leq d^{O(1)}$ such subsets, so this algorithm is polynomial time. Finally, to obtain a poly$(k)$-approximate matrix with rank at most $k$, we combine this algorithm with one of the algorithms of [10], which converts an arbitrary bicriteria solution into a rank-$k$ solution, at the cost of a slight increase in the approximation factor, and a $2^{O(k \log k)}$ term in the running time. The algorithm of [10] cannot be directly applied to previously studied bicriteria algorithms, which had a log $d$ dependence in either the rank or the approximation factor, since the algorithm of [10] relies on well-conditioned bases, which lead to an additional distortion proportional to the rank of the bi-criteria approximation.

**1.2.2 General $\ell_1$ Low Rank Approximation**

We next turn to general low rank approximation, where it is possible to obtain much smaller approximation factors than with column subset selection. A crucial novelty in our algorithm is the use of a randomized rounding technique for solving an integer linear program (ILP) — to our knowledge, such an approach was not previously considered in the context of $\ell_1$ or $\ell_p$ low rank approximation. Randomized rounding of relaxations has been previously used in other subspace optimization problems, such as by [17] in the related problem of subspace approximation (in the $\ell_{p,2}$ norm). However, [17] uses it to select random linear combinations of singular vectors of a matrix obtained by solving a convex relaxation, while we use randomized rounding of an LP to choose columns satisfying multiple linear constraints. One appealing aspect of our algorithm is that it does not use polynomial system solvers, which are somewhat impractical — these have been used for several other NP-hard matrix factorization problems, such as in [38, 43].

**Background: The Algorithm of [2] and its Bicriteria Variant.**

As a starting point, we recall the bicriteria variant of the main algorithm of [2] (shown in Algorithm 1), which uses a median-based sketch to obtain a $(1 + \varepsilon)$-approximation for $\ell_1$ low rank approximation. This sketch was previously considered in [1], where it was shown that for a $k$-dimensional subspace $V$ of $\mathbb{R}^n$, if $S \in \mathbb{R}^{\text{poly}(k/\varepsilon) \times n}$ is a random matrix with i.i.d. standard Cauchy entries, then with constant probability, for all $v \in V$, $\text{med}(Sv)$ is within a $(1 + \varepsilon)$ factor of $\|v\|_1$, where $\text{med}(v)$ is the median of the absolute values of the entries of the vector $v$. This sketch was then considered in the context of $\ell_1$ low rank approximation by [2], where the following “one-sided embedding” property of this sketch is shown: for a given matrix $U \in \mathbb{R}^{n \times k}$, and a fixed matrix $A \in \mathbb{R}^{n \times d}$, if $S \in \mathbb{R}^{\text{poly}(k/\varepsilon) \times n}$, then with constant probability $\text{med}(SUV - SA) \geq (1 - \varepsilon)\|UV - A\|_1$ for all matrices.
$V \in \mathbb{R}^{k \times d}$, where if $M$ is a matrix with $d$ columns, then med($M$) := $\sum_{i=1}^{d}$ med($M_i$). These properties make this sketch useful in $\ell_1$ low rank approximation, as we now see.

**Algorithm 1** \((1 + \varepsilon)\)-approximation algorithm from \cite{2} that gives bi-criteria rank \(3k\). This is adapted from Algorithm 1 of \cite{2} and Theorems 10 and 23 of \cite{2}.

**Require:** \(A \in \mathbb{R}^{n \times d}, k \in \mathbb{N}, \varepsilon > 0\) with \(n \geq d\)

**Ensure:** \(\hat{A} \in \mathbb{R}^{n \times d}\)

**procedure** \textsc{PreviousOnePlusEpsApproximation} \((A, k, \varepsilon)\)

\[
B \leftarrow \text{The rank-}k \text{ SVD of } A
\]

\[
C \leftarrow A - B
\]

\[
S \leftarrow \text{A poly}(k/\varepsilon) \times n \text{ matrix of i.i.d. standard Cauchy random variables}
\]

\[
U, V \leftarrow \text{The } n \times 2k \text{ and } 2k \times d \text{ zero matrices}
\]

\[
\hat{S} \leftarrow S \text{ with each entry rounded to the nearest integer multiple of } \frac{\varepsilon}{\text{poly}(n)}
\]

Guess all possible values of $\hat{S}U^*$ with each entry rounded to the nearest integer multiple of $\frac{\varepsilon}{\text{poly}(n)}\|C\|_1$.

**for** each guessed value \(M\) of $\hat{S}U^*$ **do**

\[
V_{\text{guess}} \leftarrow \arg\min_{V} \text{med}(MV' - SC) \text{ such that } \|V'\|_\infty \leq \text{poly}(k)
\]

\[
U_{\text{guess}} \leftarrow \arg\min_{U} \|UV_{\text{guess}} - C\|_1
\]

**if** $\|U_{\text{guess}}V_{\text{guess}} - C\|_1 \leq \|UV - C\|_1$ **then**

\[
U \leftarrow U_{\text{guess}}, \ V \leftarrow V_{\text{guess}}
\]

**end if**

**end for**

**return** $B + UV$

**end procedure**

Algorithm 1 is the bicriteria variant of the main algorithm of \cite{2} (we cannot directly modify the main algorithm since it requires bit complexity assumptions). The main algorithm is given in Algorithm 1 of \cite{2}, and Theorem 10 of \cite{2} gives the analysis of that algorithm, while Theorem 23 of \cite{2} describes how the algorithm should be modified to remove the need for bit complexity assumptions at the cost of a bicriteria rank of \(3k\). Hence, our summary of the analysis largely follows Theorem 10 of \cite{2}, with minor modifications as given by Theorem 23 of \cite{2}.

Briefly, its analysis proceeds as follows. Define $U^* \in \mathbb{R}^{n \times 2k}$ and $V^* \in \mathbb{R}^{2k \times d}$ so that $U^*V^*$ is the optimal rank-\(2k\) approximation for $C$. First, $V^*$ is assumed without loss of generality to be a poly\((k)\)-well-conditioned basis, meaning for all row vectors $x \in \mathbb{R}^{2k}$, $\frac{1}{\text{poly}(k)}\|x\|_1 \leq \|x^T V^*\|_1 \leq \text{poly}(k)\|x\|_1$. Then, $U^*$ is assumed to have all of its entries rounded to the nearest integer multiple of $\frac{\varepsilon\|C\|_1}{\text{poly}(n)}$. This is not an issue, because if $\tilde{U}$ is the rounded version of $U^*$, then

$$
\|(\tilde{U} - U^*)V^*\|_1 \leq \text{poly}(k) \cdot \|\tilde{U} - U^*\|_1 \leq \text{poly}(k) \cdot O(nk) \cdot \frac{\varepsilon\|C\|_1}{n \cdot \text{poly}(n)} = \frac{\varepsilon\|C\|_1}{\text{poly}(n)} \leq \varepsilon \cdot \text{OPT}
$$

where the second inequality is because $\tilde{U}, U^*$ have $2nk$ entries, and the last inequality is because the rank-\(k\) SVD of $A$ gives an $O(n)$-approximation, assuming $n \geq d$.

In addition, $S$ is also assumed to be discretized, and the discretized version is written as $\hat{S}$. This is done as follows. First, because $V^*$ is a poly\((k)\)-well-conditioned basis, each of its entries is at most poly\((k)\) (note that in Theorem 10 of \cite{2}, it is mentioned that we can assume each entry of $V^*$ is at most poly\((nk/\varepsilon)\), but this can be decreased further to poly\((k)\)). Hence, we can restrict ourselves to right factors $V'$ for which each of its entries is at most poly\((k)\). Now, if $V \in \mathbb{R}^{2k \times d}$ has each entry at most poly\((k)\), then

$$
\|U^*V - C\|_1 \leq \|U^*V\|_1 + \|C\|_1 = \sum_{i=1}^{d} \|U^*_i V_i\|_1 + \|C\|_1 \leq \text{poly}(k)d\|U^*\|_1 + \|C\|_1 = \text{poly}(k) \cdot d \cdot \|C\|_1
$$

where the first inequality is by the triangle inequality, the second inequality is because $\|V_i\|_\infty \leq \text{poly}(k)$, and the last equality is because $\|U^*\|_1 \leq \text{poly}(k)\|U^*V^*\|_1 = \text{poly}(k)\|C\|_1$ since $V^*$ is a well-conditioned basis.
Now, the reason for rounding the entries of $S$ to the nearest integer multiple of $\frac{\varepsilon}{\text{poly}(n)}$ (and why this does not significantly increase the error) is that, for any $V$ with $\|V\|_\infty \leq \text{poly}(k)$,

$$|\text{med}(SU^*V - \tilde{S}C) - \text{med}(SU^*V - SC)| \leq \sum_{i=1}^{d} \|\tilde{S}U^*V_i - \tilde{S}C_i - (SU^*V_i - SC_i)\|_\infty$$

$$= \sum_{i=1}^{d} \|\tilde{S} - S\| \cdot \|U^*V_i - C_i\|_1$$

$$\leq \sum_{i=1}^{d} \|\tilde{S} - S\|_\infty \cdot \|U^*V_i - C_i\|_1$$

$$= \|S - \tilde{S}\|_\infty \cdot \|U^*V - C\|_1$$

where the first inequality is due to the fact that, if $v_1, v_2 \in \mathbb{R}^n$, then $|\text{med}(v_1 + v_2) - \text{med}(v_1)| \leq \|v_2\|_\infty$, and the second is because, for a matrix $B$ and a vector $v$, $\|Bv\|_\infty \leq \|B\|_\infty \cdot \|v\|_1$.

Since the entries of $S$ are rounded to the nearest multiple of $\frac{\varepsilon}{\text{poly}(n)}$, $\|S - \tilde{S}\|_\infty \leq \frac{\varepsilon}{\text{poly}(n)}$ and $\|S - \tilde{S}\|_\infty \cdot \|U^*V - C\|_1$ is at most $\frac{\varepsilon}{\text{poly}(n)} \cdot \|C\|_1 = \varepsilon \cdot \text{OPT}$. Therefore, not much additional error is incurred when minimizing $\text{med}(SU^*V - \tilde{SC})$, subject to the constraint that $\|V\|_\infty \leq \text{poly}(k)$, as opposed to minimizing $\text{med}(SU^*V - SC)$.

In summary, the algorithm works by guessing all possible values of $\tilde{S}U^*$. By well-known properties of Cauchy matrices, the entries of $S$ are bounded above by $\text{poly}(n)$, and those of $U^*$ can also be bounded above by $\text{poly}(n)/\varepsilon$, meaning there are $\text{poly}(n/\varepsilon)$ choices for each entry of $SU^*$, and $SU^*$ has $\text{poly}(k/\varepsilon)$ entries, meaning the overall running time is $n^{\text{poly}(k/\varepsilon)}$.

**Our Approach: Achieving FPT Time by Reducing the Number of Guesses Per Entry.** Our approach, like those of [2] and Theorem C.9 of [41], follows the general strategy of first taking a good initialization $B$, subtracting it from $A$, and finding a good rank $2k$ approximation for the residual $C := B - A$. Now, the running time of Algorithm [41] is dominated by the time it takes to guess $\tilde{S}U^*$, and there are $\text{poly}(n/\varepsilon)$ guesses per entry of $\tilde{S}U^*$. We now describe how with our approach, we reduce the number of possibilities per entry to $\text{poly}(k/\varepsilon)$, while still obtaining a $(1 + \varepsilon)$-approximation.

Perhaps the most obvious optimization to make is to use a better initialization — rather than letting $B$ be the rank-$k$ SVD of $A$, we could run a $\text{poly}(k)$-approximation algorithm on $A$ to obtain $B$, such as our Algorithm [8] which gives a $\text{poly}(k)$-approximation with rank at most $k$ in $2^{O(k \log k)} + \text{poly}(nd)$ time. A $\text{poly}(k) \log(d)$-approximation algorithm, such as that of [41], would also suffice.

Using an initialization algorithm with a better approximation factor reduces the number of guesses per entry of $SU^*$, but the number of guesses remains $\text{poly}(n/\varepsilon)$, rather than $\text{poly}(k/\varepsilon)$, mainly for the following reasons:

- **When $U^*$ is being discretized** (recall that this is not explicitly done in the algorithm, but the analysis assumes $U^*$ is discretized in order to have a finite number of entries to guess) the entries still need to be rounded to the nearest integer multiple of $\frac{\varepsilon}{n \cdot \text{poly}(k)}$. This is because $U^*$ has $n$ rows, and therefore, if $\tilde{U}$ denotes the rounded version of $U^*$, then the additional error from using $\tilde{U}$ instead of $U^*$ can still only be upper bounded by

$$\|(\tilde{U} - U^*)V^*\|_1 \leq \text{poly}(k) \cdot \|\tilde{U} - U^*\|_1 \leq \text{poly}(k) \cdot \text{OPT} \cdot \|\tilde{U} - U^*\|_\infty$$

in the worst case — meaning a rounding granularity of at least $\frac{1}{n}$ is needed.

- **When $S$ is being discretized** to obtain $\tilde{S}$, then as mentioned above in Equation [2] the additive error is at most $\|S - \tilde{S}\|_\infty \cdot \|U^*V - C\|_1$, where $V$ is the right factor that Algorithm [41] obtains. Recall that the upper bound for $\|U^*V - C\|_1$ is $\text{poly}(k) \cdot d \cdot \|C\|_1$ for all $V$ with no entry larger than $\text{poly}(k)$ (in our exposition of the algorithm of [2], we showed that we just need to consider $V$ with no entry larger than $\text{poly}(k)$, while the original algorithm in [2] in fact allowed $V$ to have entries at most $\text{poly}(n/\varepsilon)$ —
with granularities of \( \frac{2}{n} \) or \( \frac{2}{n^2} \), we round \( SU^* \) itself in our analysis. Specifically, we show the following. If \( M \) is \( SU^* \), but with each entry rounded to the nearest power of \( 1 + \frac{\epsilon}{\text{poly}(k)} \), or set to 0 if it is below poly(\( \epsilon/k \)) \cdot OPT, then we obtain a small additive error by solving the following problem instead:

\[
\min_{V'} \text{med}(MV' - SC) \text{ subject to } \|V'\|_1 \leq \text{poly}(k)
\]

and then again finding a good left factor \( U' \) for \( V' \) through linear programming. The number of choices for each entry of \( M \) is then \( \text{poly}(k/\epsilon) \), and since \( M \) is a \( \text{poly}(k/\epsilon) \times k \) matrix (in fact, a \( k \)-\( \text{poly}(1/\epsilon) \times k \) matrix) the number of guesses for \( M \) is \( 2^{O(k^2 \cdot \text{poly}(1/\epsilon) \cdot \text{polylog}(k/\epsilon))} \). Note that the constraint on \( V' \) is now different — instead of having the constraint that \( \|V'\|_\infty \leq \frac{\text{poly}(n)}{\epsilon} \), as in the main algorithm of [2], or \( \|V'\|_\infty \leq \text{poly}(k) \), as in our presentation of that paper’s algorithm, we instead enforce a constraint on the \( \ell_1 \)-norm of \( V' \). This has the benefit that the additive error obtained by minimizing \( \text{med}(MV' - SC) \) instead of \( \text{med}(SU^* V' - SC) \) is small — this is necessary because we are now rounding \( SU^* \) to a coarser granularity, \( \frac{1}{\text{poly}(k/\epsilon)} \) instead of \( \frac{\epsilon^2}{\text{poly}(n)} \). However, enforcing the constraint that \( \|V'\|_1 \leq \text{poly}(k) \) is nontrivial, as we now see.

**Remark 1.1.** We round each entry of \( SU^* \) to the nearest power of \( 1 + \frac{\epsilon}{\text{poly}(k/\epsilon)} \) — a similar alternative approach that seems to work is rounding the entries to the nearest multiple of \( \frac{1}{\text{poly}(k/\epsilon)} \cdot \|C\|_1 \), which is \( \frac{1}{\text{poly}(k/\epsilon)} \cdot \text{OPT} \) if \( C \) is the residual from a \( \text{poly}(k) \)-approximation algorithm.

### Ensuring that the Candidate Right Factor Has Norm At Most poly(\( k \)) Through Randomized Rounding of an ILP

How do we enforce the \( \ell_1 \)-norm constraint on \( V' \)? First, let us discuss how the original median-based problem is solved in [2]. For each column index \( i \in [d] \), Algorithm \( \square \) finds \( V_i \in \mathbb{R}^k \) such that \( V_i \) minimizes \( \text{med}(\tilde{S}U^* V_i - \tilde{SC}_i) \) subject to the constraint that \( \|V_i\|_\infty \) is small. Note that there are \( r! \) orderings of the coordinates of \( \tilde{S}U^* V_i - \tilde{SC}_i \), where \( r \) is the number of rows in \( S \), meaning that all of those orderings can be tried — for a fixed ordering of the coordinates, the ordering turns into a linear constraint, and the \( \ell_\infty \) norm constraint can also be written as a linear constraint, meaning this can be solved with linear programming, and the overall running time is \( r! \text{poly}(nd) = 2^{O(r \log r)} \text{poly}(nd) \), and this is less than the \( n^{\text{poly}(k/\epsilon)} \) running time of Algorithm \( \square \).

Enforcing the constraint that \( \|V'\|_1 \leq \text{poly}(k) \) is more subtle. If we solve a similar problem on each \( i \in [d] \) — for instance, minimizing \( \text{med}(MV'_i - SC_i) \) such that \( \|V'_i\|_1 \leq \text{poly}(k) \) — then the overall norm of \( \|V'_i\|_1 \) could still depend on \( d \) in the worst case, if each minimizer \( V'_i \) has norm roughly equal to \( \text{poly}(k) \). It is also not easy to directly include this constraint inside a median-based optimization problem that includes information from all the columns. For instance, one naive way of minimizing \( \text{med}(MV' - SC) \) such that \( \|V'\|_1 \leq \text{poly}(k) \) is to do the following: simultaneously try all possible orderings, for each \( i \in [d] \), of the coordinates of \( MV'_i - SC_i \). The number of such orderings is \( (r!)^d \), and this does not lead to an FPT running time.

Instead, we still solve separate median-based optimization problems for each \( i \in [d] \), and combine the results for different \( i \) using a relaxation of a suitable ILP. Instead of finding \( V_i \) minimizing \( \text{med}(MV'_i - SC_i) \), we instead seek to minimize the \( \ell_1 \) norm of \( V_i \). At the same time, we would like the cost \( \text{med}(MV'_i - SC_i) \) to be small enough. Hence, for each column index \( i \in [d] \), we find a column \( V_{i,c} \) minimizing \( \|V_{i,c}\|_1 \), subject to the constraint that \( \text{med}(MV_{i,c} - SC_i) \leq c \) for a well-chosen \( c \). For any \( c \), the running time of this step is \( r! \text{poly}(nd) \) (by trying all orderings of the coordinates of \( MV_{i,c} - SC_i \) and including the orderings as linear constraints in the LP), and this fits in our desired FPT running time.

Here, \( c \) should be chosen so that it is not much higher than the cost \( \text{med}(MV_{i,c}^* - SC_i) \) of \( V_{i,c}^* \) — precisely, it should be within a \( (1 + O(\epsilon)) \) factor of \( \text{med}(MV_{i,c}^* - SC_i) \). Although we do not know \( \text{med}(MV_{i,c}^* - SC_i) \), we can guess all powers of \( (1 + \epsilon) \) less than \( O(1)\|C\|_1 \) and greater than \( O\left(\frac{\epsilon^2}{\text{poly}(k/\epsilon)}\|C\|_1\right) \) in the place of \( c \).

(The lower bound is chosen so that, even if the cost on some of the columns is \( O\left(\frac{\epsilon^2}{\text{poly}(k/\epsilon)}\|C\|_1\right) \), the overall additive error of these columns is at most \( O\left(\frac{\epsilon^2}{\text{poly}(k/\epsilon)}\|C\|_1\right) \) which is acceptable.) The number of such cost bounds \( c \) is thus polynomial in \( d, k \) and \( 1/\epsilon \).
For each \( i \in [d] \), we now have several minimizers \( V_{i,c} \) for each possible cost bound \( c \). Now, the question is, which cost bound \( c \) should we pick for each \( i \in [d] \)? We can decide this through the following \( 0-1 \) integer linear program. For each \( i \in [d] \) and each possible cost bound \( c \), we create a variable \( x_{i,c} \) which can be 0 or 1 (1 representing the minimizer \( V_{i,c} \) being chosen as the \( i^{th} \) column of \( V' \), and 0 representing \( V_{i,c} \) not being chosen). It is then natural to add the constraint that \( \sum x_{i,c} = 1 \) for each \( i \in [d] \), since only one \( V_{i,c} \) can be chosen as the \( i^{th} \) column of \( V' \).

In addition, we wish to have \( \|V'\|_1 \leq \text{poly}(k) \) and \( \text{med}(MV' - SC) \leq (1 + O(\varepsilon))OPT_{C,2k} + O(\varepsilon^2/f)\|C\|_1 \) (where \( OPT_{C,2k} \) is the optimal rank-2k approximation error for \( C \)). Note that these can be made to hold if \( V' \) is taken to be \( V^* \), since for each \( i \in [d] \), there is at least one cost bound \( c \) for which \( V_i^* \) is feasible. These can be represented as constraints that are linear in the \( x_{i,c} \), since

\[
\|V'\|_1 = \sum_{i=1}^{d} \sum_{c} x_{i,c}\|V_{i,c}\|_1
\]

and

\[
\text{med}(MV' - SC) = \sum_{i=1}^{d} \sum_{c} x_{i,c}\text{med}(MV_{i,c} - SC_i)
\]

Now, solving this ILP will again take at least \( 2^{O(kd)} \) time — instead, we can relax the \( 0-1 \) constraint on the \( x_{i,c} \), so that we now have the constraints \( x_{i,c} \in [0,1] \) for all \( i,c \). Since, for each \( i \in [d] \), we also have the constraint \( \sum c x_{i,c} = 1 \), this means that the \( x_{i,c} \) give us a probability distribution on each of the columns of \( V' \). By the constraints of the new LP, if we sample for each \( i \in [d] \) a single \( V_{i,c} \) to be the \( i^{th} \) column of \( V' \), according to the distribution given by the \( x_{i,c} \) (i.e., for each \( i \in [d] \), \( V_{i,c} \) is chosen with probability \( x_{i,c} \)) then the expectation of \( \|V'\|_1 \) is \( \text{poly}(k) \), while the expectation of \( \text{med}(MV' - SC) \) is at most \( (1 + O(\varepsilon))OPT + \frac{\varepsilon}{\text{poly}(k)}\|C\|_1 \).

This gives the desired result, but a few subtleties arise when sampling according to the \( x_{i,c} \) and analyzing this using Markov’s inequality. To obtain a \( (1 + O(\varepsilon))\)-approximation, we would need to have \( \text{med}(MV' - SC) \) be at most \( (1 + O(\varepsilon)) \) times its expectation. By Markov’s inequality, we find that \( \text{med}(MV' - SC) \) is at most \((1 + 2\varepsilon)\)-times its expectation with probability at least \( \varepsilon \) (meaning this fails to occur with probability at most \( 1 - \varepsilon \)). To apply a union bound to control \( \|V'\|_1 \) as well, we note that \( \|V'\|_1 \) is at most \( \frac{\varepsilon}{2} \) times its expectation with failure probability at most \( \frac{\varepsilon}{2} \) — having \( \|V'\|_1 \leq \frac{\text{poly}(k)}{\varepsilon} \) is enough for our purposes, since we round \( SU^* \) with a granularity of \( \frac{1}{\text{poly}(k)\varepsilon} \). By a union bound, we find that \( \|V'\|_1 \) and \( \text{med}(MV' - SC) \) are both small enough with probability \( \frac{\varepsilon}{2} \) (i.e., failure probability \( 1 - \Omega(\varepsilon) \)), and we can simply sample \( V' \) a total of \( O(1/\varepsilon) \) times, choosing the best solution found, in order to reduce this failure probability to a small constant independent of \( \varepsilon \).

Finally, since \( V' \) has norm at most \( \frac{\text{poly}(k)}{\varepsilon} \), the difference between \( \text{med}(MV' - SC) \) and \( \text{med}(SU^*V' - SC) \) is at most \( \frac{\text{poly}(1/\varepsilon)}{\text{poly}(k)\varepsilon} \|C\|_1 \), meaning \( \text{med}(SU^*V' - SC) \) is also small enough, and so is \( \|U^*V' - C\|_1 \). At this point, we can find an appropriate left factor \( U' \) for \( V' \) through linear programming.

As a summary of this discussion, we show our algorithm in the \( \ell_1 \)-case in Algorithms 4 and 5. Algorithm 3 simply shows the process of obtaining an initial crude approximation \( B \) and subtracting it from \( A \) to obtain \( C \), and Algorithm 2 shows how we obtain a matrix \( UV \) such that

\[
\|UV - C\|_1 \leq (1 + O(\varepsilon))\min_{\text{rank } 2k} \|C^* - C\|_1 + O\left(\frac{\varepsilon}{\text{poly}(k)}\right)\|C\|_1
\]

meaning that \( UV + B \) is a \( (1 + \varepsilon) \)-approximation to the optimal rank-\( k \) approximation error for \( A \).

**Remark 1.2.** Note that we need to know \( OPT \) in order to enforce the linear constraint that \( \text{med}(MV' - SC) \) is at most \( (1 + O(\varepsilon))OPT + \frac{\varepsilon}{\text{poly}(k)}\|C\|_1 \). Observe that it suffices to have an estimate \( \widehat{OPT} \) of \( OPT \) that is accurate within a \( (1 + \varepsilon) \)-factor. We can obtain such a \( \widehat{OPT} \) as follows — if \( E \) is the error achieved by the rank-\( k \) SVD of \( C \), then \( E \) is within a \( \sqrt{n}d \) factor of \( OPT \), meaning it suffices to guess all powers of \( (1 + \varepsilon) \) that are between \( OPT \) and \( \frac{\varepsilon}{\sqrt{n}d}OPT \), and one of these will give a \( (1 + O(\varepsilon)) \)-approximate factorization with additive \( \frac{\varepsilon}{\text{poly}(k)}\|C\|_1 \) error.
Remark 1.3. For each column index \( i \) and cost bound \( c \), we minimize the norm of \( V_{i,c} \) such that \( \text{med}(MV_{i,c} - SC_i) \leq c \). The argument would also proceed similarly if we minimized \( \text{med}(MV_{i,c} - SC_i) \) while having a constraint on the norm of \( V_{i,c} \). In particular, we can try all powers of \((1 + \varepsilon)\) between \( \frac{\text{poly}(k)}{n} \) and \( \text{poly}(k) \), and the linear program will still be feasible because \( \|V^*\|_1 \leq \text{poly}(k) \).

**Hardness for Additive Error - Appendix A.** Our techniques for our hardness results are based on the proof by [2] that, assuming the Small-Set Expansion Hypothesis and the Exponential Time Hypothesis, finding the optimal rank-\( k \) approximation takes at least \( 2^{k^c} \) time for some constant \( c > 0 \). To obtain our first result, that computing a matrix \( A \) and rank at most \( k \) such that

\[
\|\hat{A} - A\|_p \leq O(1) \min_{A_k \text{ rank } k} \|A_k - A\|_p + \frac{1}{2^{\text{poly}(k)}} \|A\|_p
\]

requires \( 2^{k^c} \) time for \( p \in (1, 2) \) with the same hardness assumptions, we show that the reduction of [2] from the Small Set Expansion problem to \( \ell_p \)-Low Rank Approximation can be performed in such a way that each entry of the input matrix \( A \) ultimately has \( \text{poly}(k) \) bits in both its numerator and denominator. If this holds, then we can assume without loss of generality that the entries of \( A \) are in fact integers with at most \( \text{poly}(k) \) bits, meaning \( \|A\|_p \leq 2^{\text{poly}(k)} \min_{A_k \text{ rank } k} \|A_k - A\|_p \), and the above guarantee is in fact equivalent to obtaining an \( O(1) \)-approximation.

To obtain a similar guarantee for constrained \( \ell_1 \) low rank approximation, we reduce from \( \ell_p \)-low rank approximation to constrained \( \ell_1 \)-low rank approximation through the use of the following theorem:

**Theorem 1.4 (Embedding \( \ell_p \) into \( \ell_1 \) - Restatement of Theorem 1 of [24]).** Let \( \tau > 0 \), and let \( p \in (1, 2) \). Moreover, let \( n, m \in \mathbb{N} \). Then, there exists a family of random matrices \( R \in \mathbb{R}^{m \times n} \) such that, if \( m \geq \beta_{p, \tau} n \), then for all \( x \in \mathbb{R}^n \),

\[
(1 - \tau)\|x\|_p \leq \|Rx\|_1 \leq (1 + \tau)\|x\|_p
\]

Here, \( \beta_{p, \tau} \) is a constant depending only on \( p \) and \( \tau \).

Combining this theorem with our hardness result for \( \ell_p \)-low rank approximation with \( \frac{1}{2^{\text{poly}(k)}} \|A\|_p \) additive error, and additionally assuming a randomized version of the Exponential-Time Hypothesis (which was used, for instance, in [13] — this is necessary since the embedding \( R \) is randomized, meaning there is a constant probability of error on the original 3-SAT instance), we find that \( 2^{\Omega(k^c)} \) time is needed (for some constant \( c > 0 \)) to achieve the following guarantee, with constant probability of error: given a matrix \( R \in \mathbb{R}^{m \times n}, \ A \in \mathbb{R}^{m \times d} \) and \( k \in \mathbb{N} \), find \( \hat{A} \in \mathbb{R}^{m \times d} \) with rank \( k \), such that the columns of \( \hat{A} \) are contained in the column span of \( R \) and

\[
\|\hat{A} - A\|_1 \leq O(1) \min_{A_k} \|A_k - A\|_1 + \frac{1}{2^{\text{poly}(k)}} \|A\|_1
\]

where the minimum on the right-hand side is taken over all matrices \( A_k \) with rank at most \( k \), whose columns are contained in the span of \( R \). To see this, we can multiply the family of hard instances for \( \ell_p \)-low rank approximation by the randomized embedding matrix \( R \) — note that we can let \( R \) have \( \Theta(n) \) rows since we can let \( \tau \) be a constant, and fix \( p = \frac{3}{2} \), so that \( \beta_{p, \tau} \) is also constant.

1.3 Paper Outline

1.3.1 Main Results - Algorithms

In Section [4] we describe our \( \widetilde{O}(k^{1/p-1/2}) \)-approximate algorithm for \( \ell_p \) column subset selection for \( p \in [1, 2) \), and its analysis. In Section [3] we analyze our \((1 + \varepsilon)\)-approximation algorithm for \( \ell_p \) low rank approximation, for \( p \in [1, 2) \), which returns a matrix of rank at most \( 3k \) in \( 2^{\text{poly}(k/\varepsilon)} + \text{poly}(nd) \) time.

1.3.2 Appendices - Additional Results

In Appendix [A] we show how to extend the hardness results of [2] to show that even obtaining an \( O(1) \)-approximation for \( \ell_p \) low rank approximation with \( \frac{1}{2^{\text{poly}(k)}} \|A\|_p \) additive error is hard, when \( p \in (1, 2) \). Using this, we show that obtaining an \( O(1) \)-approximation for constrained \( \ell_1 \) low rank approximation with \( \frac{1}{2^{\text{poly}(k)}} \|A\|_1 \) additive error is hard.
**Algorithm 2** Obtaining a matrix $\hat{A}$ such that $\|\hat{A} - A\|_1 \leq (1 + \varepsilon)OPT_{A,k} + \frac{\varepsilon}{2}\|A\|_1$, where $OPT_{A,k} = \min_k \text{rank}_k (A - A_k)_1$. This is done by guessing a sketched left factor $SU$, and finding an appropriate right factor $V$ with norm at most $\text{poly}(k)$. The argument $\varepsilon$ is assumed to be at most $c$ for some absolute constant $c$.

**Require:** $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, $\varepsilon \in (0, c)$, $f > 1$, $\hat{OPT} \geq 0$

**Ensure:** $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{k \times d}$

**procedure** GuessingAdditiveEpsApproximation($A, k, \varepsilon, f, \hat{OPT}$)

- If $A$ has rank $k$, return $A$.
- $r \leftarrow O(\max(k/\varepsilon^0 \log(k/\varepsilon), 1/\varepsilon^0))$
- $q \leftarrow \text{poly}(k)$
- $S \leftarrow$ An $r \times n$ matrix of i.i.d. standard Cauchy random variables

\[ I \leftarrow \{0\} \cup \left\{ \sigma \cdot (1 + \frac{1}{\text{poly}(k/\varepsilon)})^t \mid t \in \mathbb{Z}, \frac{1}{\text{poly}(k/\varepsilon)}\|A\|_1 \leq (1 + \frac{1}{\text{poly}(k/\varepsilon)})^t \leq \text{poly}(k/\varepsilon)\|A\|_1, \sigma = \pm 1 \right\} \]

\[ C \leftarrow \left\{ M \in \mathbb{R}^{r \times k} \mid M_{i,j} \in I \forall i \in [r], j \in [k] \right\} \quad \text{This is the set of (sketched) left factors we will guess.} \]

// Guess possible rounded (sketched) left factors and find a good right factor $V$, with $\|V\|_1 \leq \text{poly}(k)$.

$U_{\text{best}} \leftarrow 0$, $V_{\text{best}} \leftarrow 0$

for $M \in C$ do

\[ \text{CostBounds} \leftarrow \left\{ \frac{\varepsilon^2\|A\|_1}{d} \leq c \leq O(\|A\|_1) \text{ and } c \text{ is an integer power of } (1 + \varepsilon) \right\} \]

for $i \in [d]$, $c \in \text{CostBounds}$ do

\[ V_{i,c} \leftarrow \text{argmin}_{V_{i,c}} \|V_{i,c}\|_1 \text{ subject to the constraint that } \text{med}(MV_{i,c} - SA_i) \leq c \]

\[ C_{i,c} \leftarrow \text{med}(MV_{i,c} - SA_i) \]

end for

// Create LP to find a good distribution over $c \in \text{CostBounds}$ for each $i \in [d]$.

$\text{Variables} \leftarrow \{ x_{i,c} \forall i \in [d], c \in \text{CostBounds} \}$

$\text{Constraints} \leftarrow \left\{ 0 \leq x_{i,c} \forall i \in [d], c \in \text{CostBounds} \text{ and } \sum_{c \in \text{CostBounds}} x_{i,c} = 1 \forall i \in [d] \right\}$

$\text{Constraints} \leftarrow \text{Constraints} \cup \left\{ \sum_{i \in [d], c \in \text{CostBounds}} x_{i,c}\|V_{i,c}\|_1 \leq kq = \text{poly}(k) \right\}$

$\Delta \leftarrow (1 + O(\varepsilon))\hat{OPT} + O\left(\frac{\varepsilon^2}{\hat{OPT}}\right)\|A\|_1$

$\text{Constraints} \leftarrow \text{Constraints} \cup \left\{ \sum_{i \in [d], c \in \text{CostBounds}} x_{i,c}C_{i,c} \leq \Delta \right\}$

$x_{i,c} \leftarrow \text{Solution to the LP given by } \text{Variables and Constraints}, \text{ for all } i \in [d], c \in \text{CostBounds}$

If LP is infeasible, then continue to next $M \in C$.

// For each column, sample an appropriate cost bound. Do this $O(1/\varepsilon)$ times, then $V'$ meets both
// the cost and norm constraints with constant probability.

for $t = 1 \to 10/\varepsilon$ do

\[ c_t \leftarrow \text{An element } c \in \text{CostBounds} \text{ sampled according to the distribution on CostBounds} \]

\[ \text{given by } \{ x_{i,c} \mid c \in \text{CostBounds} \} \text{ (note that for a fixed } i \in [d], \text{ the } x_{i,c} \text{ are nonnegative and sum to 1) } \]

\[ V'_{i,c} \leftarrow V_{i,c}, \text{ for all } i \in [d] \]

Break if $\|V'\|_1 \leq \frac{2kq}{\varepsilon}$ and $\text{med}(MV' - SA) \leq (1 + 2\varepsilon)\Delta$

end for

$U' \leftarrow \text{argmin}_U \|UV' - A\|_1$

If $\|U'V' - A\|_1 \leq \|U_{\text{best}}V_{\text{best}} - A\|_1$ then $U_{\text{best}} \leftarrow U'$ and $V_{\text{best}} \leftarrow V'$.

end for

return $U'$, $V'$
Lemma 2.1. Let $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, $\varepsilon \in (0, c)$

 Require: $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, $\varepsilon \in (0, c)$

 Ensure: $\hat{A} \in \mathbb{R}^{n \times d}$ having rank $3k$

 procedure RoundingGuessingEpsilonApproximation($A, k, \varepsilon$)
     $W, Z \leftarrow$ PolyKErrorNotBiCriteriaApproximation($A, k$)
     $B \leftarrow WZ$
     $C \leftarrow A - B$
     $f \leftarrow$ poly($k$), the approximation factor of Algorithm 8
     $\hat{A} \leftarrow 0 \in \mathbb{R}^{n \times d}$
     for $t = 0 \rightarrow O(\frac{\log nd}{\varepsilon})$ do
         $\hat{\text{OPT}} \leftarrow$ SvdError/$(1 + \varepsilon)^t$
         $U, V \leftarrow$ GuessingAdditiveEpsilonApproximation($C, 2k, \varepsilon, f, \hat{\text{OPT}}$)
         If $(B + UV) - A \leq \|\hat{A} - A\|_1$ then $\hat{A} \leftarrow B + UV$
     end for
     return $\hat{A}$
 end procedure

Next, in Appendix B we show how the lower bound for $\ell_1$ column subset selection due to [11] and its analysis can be extended to obtain a lower bound for $\ell_p$ column subset selection, for $p \in (1, 2)$. Finally, in Appendix C we show how to obtain a poly($k$)-approximation algorithm with running time $2^{O(k \log k)} + \text{poly}(nd)$, based on Algorithm 4 and techniques from [10].

2 Optimal $\ell_1$ Column Subset Selection via Random Sampling

2.1 Preliminaries: Notation and Lewis Weight Sampling

Suppose $A \in \mathbb{R}^{n \times d}$. We use the following notation for the rows, columns, and submatrices of $A$. For $i \in [n]$, we let $A^i$ be the $i^{th}$ row of $A$, and for $j \in [d]$, we let $A_j$ be the $j^{th}$ column of $A$. Moreover, for $S \subset [d]$, we let $A_S$ be the submatrix of $A$, such that its columns are those of $A$ whose indices are in $S$, and for $R \subset [n]$, we let $A^R$ be the submatrix of $A$, such that its rows are those of $A$ whose indices are in $R$.

We now recall basic facts about row sampling with Lewis weights. Since we use Lewis weight sampling as a black-box, we do not go into the details of the construction of the sampling matrix — for these details we refer the reader to [12].

Lemma 2.1. (Sampling and Rescaling Matrix Based on Lewis weights - Adapted from Theorem 7.1 of [12])

Let $A \in \mathbb{R}^{n \times d}$, $1 \leq p < 2$, and let $r = O(d \log d)$ if $p = 1$ and $r = O(d \log d \log(d) \log(d))$ otherwise. There exists a distribution $(p_1, p_2, \ldots, p_n)$ on the rows of $A$ such that if we generate a matrix $S$ with $r$ rows, each chosen independently as the $i^{th}$ standard basis vector, times $\frac{1}{(rp_i)^{p}}$, with probability $p_i$, then with probability $1 - O(1)$ we have

$$\Omega(1)|Ax|_p \leq \|SAx\|_p \leq O(1)|Ax|_p$$

for all $x \in \mathbb{R}^d$. The distribution $(p_1, p_2, \ldots, p_n)$ can be computed in $mnz(A) + \text{poly}(d)$ time.

Lemma 2.2 (O(1) Dilation and Contraction for Lewis Weights - Lemma D.11 (p = 1) and E.11 (p ∈ (1, 2)) of [11])

Let $M \in \mathbb{R}^{n \times d}$ and $U \in \mathbb{R}^{n \times t}$. Let $r = O(t \log t)$ if $p = 1$ and $O(t \log t \log \log t)$ if $p \in (1, 2)$, and
suppose $S \in \mathbb{R}^{r \times n}$ is a sampling and rescaling matrix whose entries are generated according to the $\ell_p$ Lewis weights of $U$. Then, with probability $1 - O(1)$, $\|SM\|_p^p \leq O(1)\|M\|_p^p$, and with probability $1 - O(1)$, for all $x \in \mathbb{R}^d$, $\|SUx\|_p^p \geq \Omega(1)\|Ux\|_p^p$.

**Lemma 2.3 (O(1) Contraction on Affine Subspace for Lewis Weights - From Lemmas D.11 and D.7 of [44] for $p = 1$, and Lemmas E.11 and E.7 of [41] for $p \in (1, 2)$).** Let $A \in \mathbb{R}^{n \times d}$, $U \in \mathbb{R}^{n \times k}$, and let $V^* = \arg\min_{V \in \mathbb{R}^{k \times d}} \|UV - A\|_p$. Let $r = O(k \log k)$ if $p = 1$ and $r = O(k \log k \log \log k)$ if $p \in (1, 2)$. Let $S \in \mathbb{R}^{r \times n}$ be a sampling and rescaling matrix whose entries are generated according to the $\ell_p$ Lewis weights of $U$. Then, with probability $1 - O(1)$, simultaneously for all $V \in \mathbb{R}^{k \times d}$,

$$\|SU - SA\|_p^p \geq \Omega(1)\|UV - A\|_p^p - O(1)\|UV^* - A\|_p^p$$

### 2.2 \(\ell_p\) Column Subset Selection Algorithm and Analysis

In this section, we analyze Algorithm 4 with the difference being that we sample columns per iteration, instead of $2k$ — however, its analysis is significantly different, and does not use maximum-determinant column subsets.

**Algorithm 4** Randomly sample columns of $A$ repeatedly, to obtain $O(k \cdot \text{polylog}(k) \log(d))$ columns of $A$ spanning a good approximation. This is a variant of Algorithm 1 in [41] with the difference being that we sample $O(k \cdot \text{polylog}(k))$ columns in each round instead of $O(k)$ columns. Here, $\text{MULTIPLEREGRESSION-SOLVER}(n, d, m, U, B)$ (where $U \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{n \times m}$) is a subroutine which computes $\min_x \|Ux - B\|_2$ for each $j \in [m]$. The call $\text{BOTTOMK}($$\text{SORT}$$($$\text{cost}$$,$$\Omega(m))$$)$ serves to find the $\Omega(m)$ column indices in $[m]$ having the smallest regression cost.

**Require:** $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, $p \in [1, 2)$

**Ensure:** $S \subseteq [d]$, $|S| = O(k \log k \log d)$ if $p = 1$ and $O(k \log k \log \log d)$ otherwise

**procedure** $\text{RANDOMCOLUMNSUBSETSELECTION}(A, k, p)$

Samples $\leftarrow O(\log d)$

$T_0 \leftarrow [d]$

$r \leftarrow O(k \log k)$ if $p = 1$ and $O(k \log k \log \log k \log d)$ otherwise

for $i = 1$ to samples do

$m \leftarrow |T_{i-1}|$

for $j = 1$ to $O(\log d)$ do

Sample $S^{(j)}$ from $\{T_{1-1/2} \} \text{ uniformly at random}$

$m \leftarrow |T_{i-1} \setminus S^{(j)}|$

$\{\text{cost}_t\}_{t \in T_{i-1} \setminus S_j} \leftarrow \text{MULTIPLEREGRESSION-SOLVER}(n, 2r, m, A_{S^{(j)}}, A_{T_{i-1} \setminus S^{(j)}})$

$R^{(j)} \leftarrow \text{BOTTOMK}(\text{SORT}(\text{cost}), \Omega(m))$

$c_j \leftarrow \sum_{t \in R^{(j)}} \text{cost}_t$

end for

$j^* \leftarrow \min_{j \in [O(\log d)]} c_j$

$S_i \leftarrow S^{(j^*)} \cup R^{(j^*)}$

$T_i \leftarrow T_{i-1} \setminus S_i$

end for

$S \leftarrow \cup_i S_i$

return $S, S_1, S_2, \ldots, S_{O(\log d)}$

end procedure

The following result was shown in Theorem C.1 of [41]. That is the key tool in the analysis of Algorithm 4.

**Theorem 2.4. (Existence of a Good Column Subset) Let $A \in \mathbb{R}^{n \times d}$, $p \in [1, 2)$, $k \in \mathbb{N}$, and $m = O(k \log k)$ if $p = 1$ and $m = O(k \log k \log \log k)$ if $1 < p < 2$. Then, there exist matrices $U \in \mathbb{R}^{n \times m}$ and $V \in \mathbb{R}^{m \times d}$, (1)

[6]Our presentation is based on Algorithm 2 of [https://arxiv.org/pdf/1811.01442v1.pdf](https://arxiv.org/pdf/1811.01442v1.pdf) which is version 1 of [41] on arXiv.
such that the columns of $U$ are columns of $A$, and

$$\|UV - A\|_p \leq O\left(m^{\frac{1}{p} - \frac{1}{2}}\right) \min_{A_k \text{ rank } k} \|A - A_k\|_p$$

Moreover, this occurs with probability $\frac{2999}{3000}$ if the columns of $U$ are sampled from the columns of $A$ according to the Lewis weights of $V^*$, where $U^*V^*$ is an optimal rank-$k$ approximation to $A$.

Proof. This result was essentially shown in Theorem C.1 of [11] for the case $p = 1$ — we give a brief sketch here. The key point is that $U = AR$, where $R$ is given by (column) Lewis weights of $V^*$ (here we first generate a sampling matrix based on the Lewis weights of $(V^*)^T$, then transpose the sampling matrix).

Indeed, if $R$ is chosen according to the column Lewis weights of $V^*$, then letting $U'$ be the minimizer of $\|U'V^*R - A'R\|_2$ instead of $\|U^2V^*R - A'R\|_1$ for all $i \in [n]$ gives an $O(m^{\frac{1}{2}})$-approximation. This minimizer $U'$ is given by $U' = A'R(V^*R)^+$, where $(V^*R)^+$ is the pseudo-inverse of $V^*R$. Hence we can choose $U' = ARV^*R^+$, and hence there exists an $O(m^{\frac{1}{2}})$-approximation for $A$ with left factor $AR$. In Theorem C.1 of [11], it is mentioned that this occurs with probability $\frac{2999}{3000}$, which is shown by an application of Markov’s inequality — we can increase this probability to $\frac{2999}{3000}$ by increasing the constant used in Markov’s inequality, and increasing $m$ by a constant factor.

For $1 < p < 2$, the proof is nearly identical [11]. Instead of Lemma D.11 of [11] we can use part (III) of Lemma E.11 of the same work to extend the result to $p \neq 1$. Similarly, instead of Lemma D.8, we can use Lemma E.8 of [11]. Finally, we apply Lemma B.10 of [11] to convert between the $\ell_p$-norm and the $\ell_2$-norm, rather than between the $\ell_1$-norm and the $\ell_2$-norm as was done in [11] — now we obtain a distortion of $m^{\frac{1}{p} - \frac{1}{2}}$.

Now, we analyze Algorithm 4.

**Theorem 2.5** (Column Sampling Approximation Factor). Let $A \in \mathbb{R}^{n \times d}$, $p \in [1, 2]$ and $k \in \mathbb{N}$. In addition, let $r = O(k \log k)$ if $p = 1$ and $r = O(k \log k \log \log k)$ otherwise. Let $S \subset [d]$ be the set of columns of $A$ that is returned by Algorithm 4. Let $U = AS \in \mathbb{R}^{n \times O(r \log d)}$, and let $V = \arg \min_{V' \in \mathbb{R}^{O(r \log d) \times d}} \|UV' - A\|_p$, i.e., $V$ is obtained by performing multiple-response $\ell_p$-regression on the columns of $A$ using the columns of $AS$. Then,

$$\|UV - A\|_p \leq O(r^{\frac{1}{p} - \frac{1}{2}}(\log d)^{\frac{1}{p}}) \min_{A_k \text{ rank } k} \|A - A_k\|_p$$

with probability $1 - o(1)$.

Proof. We first show the following claim (whose role is somewhat analogous to Lemma 6 of [10]), which is that after sampling $2r$ columns of $A$, at least a constant fraction of the remaining columns are covered, up to our desired approximation factor, with constant probability. The main theorem is then a consequence of this claim, together with a Markov bound (to show that a constant fraction of the columns of $A$ are covered with constant probability) and a union bound over the samples in each of the iterations (note that we perform $O(\log d)$ repetitions in each iteration, and use the result of the best repetition, to boost the probability of a constant fraction of columns being covered).

**Claim 2.6.** Let $B \in \mathbb{R}^{n \times 2r}$ be a submatrix of $A$, whose columns are a uniformly random subset of those of $A$, of size $2r$. Furthermore, let $A_i$ be an additional, uniformly random, column of $A$ not among those of $B$. Then, with probability $1 - O(1)$ (where the probability is taken over $B$ and $A_i$),

$$\min_{x \in \mathbb{R}^{2r}} \|Bx - A_i\|_p \leq O\left(r^{1 - \frac{1}{p}} \frac{OPT^p}{d}\right)$$

where $OPT = \min_{A_k \text{ rank } k} \|A - A_k\|_p$.

Proof. We prove this claim by first showing that with at least constant probability, the column Lewis weight of $B'$ corresponding to $A_i$ is small (i.e. $O(\frac{1}{d})$), where $B' = [B, A_i]$. Then, if we sample $r$ columns of $B'$ using the column Lewis weights of the right factor in the best rank-$k$ approximation to $B'$, then with constant probability, we have that $A_i$ is both covered well by the Lewis weight sample, and is not in the Lewis weight sample (meaning it is covered well by $B$).
Let $B' \in \mathbb{R}^{n \times (2r + 1)}$ be $B$ with $A_i$ adjoined. Then, the columns of $B'$ also form a uniformly random subset of the columns of $A$. Define $\Delta \in \mathbb{R}^{n \times d}$ so that if $U^*V^*$ is the optimal rank-$k$ approximation to $A$, then $\Delta = A - U^*V^*$. For a subset $S \subset [d]$, let $\Delta_S$ denote the submatrix of $\Delta$ containing those columns whose indices are in $S$.

Let $T \subset [d]$ such that $B' = A_T$, and let $B'_k$ be the best rank-$k$ approximation to $B'$. Then, by the definition of $B'_k$, $\|B' - B'_k\|_p^p \leq \|\Delta_T\|_p^p$. Hence, taking expectations gives

$$E[\|B' - B'_k\|_p^p] \leq E[\|\Delta_T\|_p^p] = \frac{|T|}{d} \text{OPT}^p = O\left(\frac{r \text{OPT}^p}{d}\right)$$

where the last equality is because $T$ is a uniformly random subset of $[d]$ of size $2r + 1$.

Now, write $B'_k = U^*V^*$, for some $U^* \in \mathbb{R}^{n \times k}$ and $V^* \in \mathbb{R}^{k \times |T|}$. As described in Theorem 1.1, consider the probability distribution on $T$ given by the (column) Lewis weights of $V^*$. Denote this distribution by $\pi$, and denote the probability on $j \in T$ by $\pi_j$. Note that $\sum_{j \in T} \pi_j = 1$ and $|T| = 2r + 1$. Therefore, if $j$ is an element of $T$ chosen uniformly at random, then by Markov’s inequality,

$$P\left[\pi_j \geq \frac{5}{r}\right] \leq E[\pi_j] = \frac{1}{18r} \leq \frac{1}{10000}$$

(where the last inequality holds if $r$ is sufficiently large, i.e. we multiply it by a sufficiently large constant). Denote the event that $\pi_i \leq \frac{5}{r}$ by $\mathcal{E}_1$ (recall that $i$ is the additional column of $A$ that was adjoined to the column subset $B$ of $A$ that we sample).

In addition, suppose we sample columns of $B'$ according to $\pi$. Let $S$ denote a subset of $T$ of size $r$, where the elements of $S$ are sampled independently, with replacement, according to $\pi$. Let $B'_k$ denote the corresponding submatrix of $B'$. As before, let $V = \text{argmin}_{V \in \mathbb{R}^{k \times |T|}} \|B'_k V - B'_k\|_p$. Then, by Theorem 1.1, with probability $\frac{999}{1000}$ (over both the sampling matrix and the uniformly random submatrix $B'$), $\|B'_k V - B'_k\|_p \leq O\left(\frac{r^{\frac{1-s}{d}}}{d}\right)\text{OPT}^p$ — denote the event that this holds by $\mathcal{E}_2$ (Here, the last inequality is by Markov’s inequality and $\mathbb{E}$ — note that this is included in the calculation of the probability of $\mathcal{E}_2$). Observe that, by our definition of $\mathcal{E}_2$,

$$E\left[\|B'_k V - B'_k\|_p \mid \mathcal{E}_2\right] \leq O\left(\frac{r^{\frac{1-s}{d}}}{d}\right)\text{OPT}^p$$

where the expectation is taken over uniformly random $i \in T$.

Finally, let $\mathcal{E}_3$ be the event that the above $S \subset T$ (sampled according to $\pi$) does not contain $i$. Then,

$$P(\mathcal{E}_3) \geq P(\mathcal{E}_1)P(\mathcal{E}_3 \mid \mathcal{E}_1) \geq \frac{999}{1000} \left(1 - \frac{5}{r}\right)^r \geq \frac{6}{1000}$$

(where the last inequality holds for $r$ sufficiently large, since the left-hand side converges to $\frac{1}{e}$ as $r \to \infty$).

Therefore,

$$O\left(\frac{r^{\frac{1-s}{d}}}{d}\right)\text{OPT}^p \geq E\left[\|B'_k V - B'_k\|_p \mid \mathcal{E}_3\right]$$

$$= P(\mathcal{E}_3)E\left[\|B'_k V - B'_k\|_p \mid \mathcal{E}_2 \cap \mathcal{E}_3\right] + P(\neg \mathcal{E}_3)E\left[\|B'_k V - B'_k\|_p \mid \mathcal{E}_2 \cap \neg \mathcal{E}_3\right]$$

$$\geq \frac{6}{1000} \cdot E\left[\|B'_k V - B'_k\|_p \mid \mathcal{E}_2 \cap \mathcal{E}_3\right]$$

Therefore, by Markov’s inequality, if $\mathcal{E}_2$ and $\mathcal{E}_3$ hold simultaneously, then with constant probability,

$$\|B'_k V - B'_k\|_p \leq O\left(\frac{r^{\frac{1-s}{d}}}{d}\right)\text{OPT}^p$$

where $S \subset T$ does not contain $i$. By a union bound, $\mathcal{E}_2$ and $\mathcal{E}_3$ both hold with positive constant probability (the failure probability of $\mathcal{E}_2$ is at most $\frac{1}{1000}$ and that of $\mathcal{E}_3$ is at most $\frac{999}{1000}$, so by a union bound $\mathcal{E}_2 \cap \mathcal{E}_3$ occurs with probability at least $\frac{5}{1000}$), meaning $\mathcal{E}_2$, $\mathcal{E}_3$ and the above inequality holds with constant probability.
The above statements together imply that with constant probability, there exists \( x \in \mathbb{R}^{2r} \) such that
\[
||Bx - A_i||_p \leq O\left(\frac{r^{1-\frac{p}{2}}}{d}\right)OPT_p
\]
where the probability is taken over \( B \) and \( A_i \), whose columns form a uniformly random subset of those of \( A \) of size \( 2r + 1 \). This proves the claim. \(\Box\)

We can combine the above claim with a Markov bound, as follows. Let \( B \in \mathbb{R}^{n \times 2r} \) be a submatrix of \( A \), such that the column indices of \( B \) form a uniformly random subset of \( T_{i-1} \) (here, we are using the notation of Algorithm 4 — \( T_i \) is the set of indices of columns of \( A \) which have not been discarded after \( i \) iterations). For \( i \in [m] \), let \( Z_i \) be equal to 1 if \( A_i \) is approximately covered by \( B \), i.e.,
\[
\min_{x \in \mathbb{R}^r} ||Bx - A_i||_p \leq O\left(\frac{r^{1-\frac{p}{2}}OPT_p}{d}\right)
\]
and 0 otherwise. Then, \( E_{B,i}[Z_i] = c \), where \( 0 < c < 1 \), and the expectation is taken over uniformly random \( B \) and \( i \). Hence, if we let \( Z := \sum_i Z_i \) be the number of approximately covered columns, then \( E[Z] = cm \), and \( E[m - Z] = (1-c)m \). Therefore, by Markov’s inequality, with constant probability \( F > 0 \), \( (m - Z) \leq (1-\frac{c}{2})m \), meaning \( Z \geq \frac{2}{5}m \). In other words, with probability \( F \), there exists \( R \subset T_{i-1} \) with \( |R| \geq \frac{2}{5} |T_{i-1}| \) such that for each \( j \in R \),
\[
\min_{V \in \mathbb{R}^{k \times |R|}} ||BV_j - A_j||_p \leq O\left(\frac{r^{1-\frac{p}{2}}OPT_p |T_{i-1}|}{|T_{i-1}|}\right)
\]
meaning
\[
\min_{V \in \mathbb{R}^{k \times |R|}} ||BV_j - A_j||_p \leq O\left(\frac{r^{1-\frac{p}{2}}OPT_p |T_{i-1}|}{|T_{i-1}|}\right) = O\left(\frac{r^{1-\frac{p}{2}}OPT_p}{|T_{i-1}|}\right)
\]
Denote this event by \( \mathcal{E} \), meaning \( \mathcal{E} \) occurs with probability \( F \). Then, as done in Algorithm 4, it is sufficient to sample the submatrix \( B \) in \( O(\log d) \) independent iterations, and take the sampled submatrix which minimizes the sum of the lowest \( \frac{2}{5} |T_{i-1}| \) residuals. By choosing \( B \) in this way, we ensure that \( \mathcal{E} \) has failure probability at most \( (1-F)^{O(\log d)} = \frac{1}{2^{\Omega(r)}} \). Since we perform \( O(\log d) \) iterations, we only have to perform a union bound over \( O(\log d) \) such events, meaning the overall failure probability of Algorithm 4 is \( O(\log d) \cdot \frac{1}{2^{\Omega(r)}} = o(1) \). \(\Box\)

Remark 2.7. Note that our application of linearity of expectation in the above proof is valid. For a fixed \( i \in [m] \), it may not seem that we can take the expectation \( E_{B,i}[Z_i] \) over a uniformly random column index \( i \), since \( i \) is determined by \( Z_i \). However, we could for instance shuffle the indices \( i \in [m] \), and sum the \( Z_i \) in the shuffled order — then, each index \( i \in [m] \) is a uniformly random column index, and we can use linearity of expectation.

We can also remove the \( O(\log d) \) term in the approximation factor from Theorem 2.5 with the following refined analysis, reminiscent of one performed in [44] — we examine the number of times a column of \( A \) can remain “uncovered” before it is removed:

Theorem 2.8 (Column Sampling - Better Approximation Factor). Let \( A \in \mathbb{R}^{n \times d}, p \in \{1, 2\} \) and \( k \in \mathbb{N} \). In addition, let \( r = O(k \log k) \) if \( p = 1 \) and \( r = O(k \log k \log \log k) \) otherwise. Let \( S \subset [d] \) be the set of columns of \( A \) that is returned by Algorithm 4. Let \( U = A_S \in \mathbb{R}^{n \times O(r \log d)} \) and let \( V = \arg\min_{V' \in \mathbb{R}^{n \times O(r \log d)}} ||UV' - A||_p \), i.e., \( V \) is obtained by performing multiple-response \( \ell_p \)-regression on the columns of \( A \) using the columns of \( A_S \). Then,
\[
||UV - A||_p \leq O\left(r^{\frac{1}{p} - \frac{1}{2}}(\log k)^{\frac{1}{p}}\right) \min_{A_k \text{ rank } k} ||A - A_k||_p
\]
with constant probability.

Proof. Define \( \Delta \) as in the proof of Theorem 2.5. Consider the \( i^{th} \) iteration of Algorithm 4 and recall that \( T_{i-1} \) is the set of remaining column indices. Let \( m = |T_{i-1}| \). Finally, let \( T_{i-1, \text{big}} \subset T_{i-1} \) consist of the \( \frac{2}{5} \) indices of \( T_{i-1} \) with greatest cost (i.e. \( j \in T_{i-1, \text{big}} \) if \( ||\Delta_j||_p \) is among the top \( \frac{2}{5} \) column norms of \( \Delta \)).

Note that with constant probability, \( S^{(j)} \) will be disjoint from \( T_{i-1, \text{big}} \) (provided \( r \) is multiplied by a sufficiently large constant and \( m \geq \Omega(r) \)), since the probability of not selecting an element of \( T_{i-1, \text{big}} \) in a uniformly random subset of size \( 2r \) is at least \( 1 - O(\frac{1}{r})^{2r} \geq \Omega(1) \) for \( r \) sufficiently large.
Condition on this event (which we can call $\mathcal{E}_1$) occurring — then, for uniformly random $j \in T_{i-1} \setminus T_{i-1, \text{big}}$, with constant probability, if $B = A S^{(j)}$,

$$
\min_{x \in \mathbb{R}^2} \| B x - A_j \|^p_p \leq O\left( r^{1-\frac{2}{p}} \frac{OPT^p_{T_{i-1} \setminus T_{i-1, \text{big}}}}{m} \right)
$$

This is by Claim 2.6, since $S^{(j)}$ is a uniformly random subset of $T_{i-1} \setminus T_{i-1, \text{big}}$. Therefore, conditioning on $\mathcal{E}_1$, with constant probability, the smallest $\Omega(m)$ columns have a cost of $O(r^{1-\frac{2}{p}} OPT^p_{T_{i-1} \setminus T_{i-1, \text{big}}})$. Finally, $\mathcal{E}_1$ occurs in the $i^{th}$ iteration with probability at least $1 - \frac{1}{\text{poly}(d)}$ since we repeat the sampling process $O(\log d)$ times and take the sample which gives the smallest cost on the lowest $\Omega(m)$ columns — hence, with probability $1 - \frac{1}{\text{poly}(d)}$, the smallest $\Omega(m)$ columns have a cost of $O(r^{1-\frac{2}{p}} OPT^p_{T_{i-1} \setminus T_{i-1, \text{big}}})$, and this occurs on every iteration with probability $1 - o(1)$ by a union bound (since there are $O(\log d)$ iterations).

It just remains to bound $\sum_i OPT^p_{T_{i-1} \setminus T_{i-1, \text{big}}}$ where $i$ ranges across all of the iterations. We show that for $j \in [d]$, $j$ can be a member of $T_{i-1} \setminus T_{i-1, \text{big}}$ for at most $O(\log k)$ different $i$, which will imply that $\sum_i OPT^p_{T_{i-1} \setminus T_{i-1, \text{big}}} = O((\log k)OPT^p)$.

Assume without loss of generality that on each iteration $i$, the column indices $j$ which are “covered” (and hence discarded) have the smallest $\| \Delta_j \|_p$ (otherwise, the upper bound on the cost incurred in future iterations will only decrease, as we discuss briefly at the end of the proof). Then, in every iteration, the indices $j$ in $T_{i-1}$ with the $\Omega(m)$ lowest values of $\| \Delta_j \|_p$ will be removed. The size of $T_{i-1}$ decreases by a constant factor in each round, meaning that for $i' \geq i + O(\log r)$ (i.e. after $O(\log r)$ more iterations) $j$ will not be in $T_{i' - 1}$, since $T_{i' - 1}$ will be equal to $T_{i-1, \text{big}}$.

In the general case (where the discarded columns $j \in [d]$ are not always the ones with smallest $\| \Delta_j \|_p$), suppose $j \in T_{i-1} \setminus T_{i-1, \text{big}}$, and at the end of iteration $i' \geq i + O(\log r)$, column index $j$ is still remaining (otherwise, we are done). In this case, there must exist $j' \in T_{i-1, \text{big}}$, such that $j'$ was removed at some point before iteration $i'$ (but would not have been removed if the smallest $\Omega(m)$ indices were removed every iteration). We can “replace” $\| \Delta_j \|_p$ with $\| \Delta_{j'} \|_p$ in the cost calculations, to reduce to the special case discussed above. \qed
3 \ (1 + \varepsilon)-Approximation in FPT Time with Bicriteria Rank \ 3k

In this section, we give an algorithm for $\ell_p$-low rank approximation, for $p \in [1, 2]$, which runs in $2^{\text{poly}(k/\varepsilon)} + \text{poly}(nd)$ time and outputs a matrix of rank $3k$. Our algorithm discussed in Subsubsection 2.2.2 is a special case of this algorithm, in the case $p = 1$. The analysis here is similar to the analysis given in the introduction, but uses sketching matrices whose entries are $p$-stable random variables, rather than Cauchy random variables.

3.1 Preliminaries: Median-Based Estimator for $\ell_p$-norm Dimension Reduction from [2]

We first recall some concepts from [2] related to sketches based on medians and dense $p$-stable random matrices (Section 2 of [2] for the $p = 1$ case and Subsection 3.2 of [2] for the $p \in (1, 2)$ case). In the following, we let $B \in \mathbb{R}^{n \times d}$.

**Definition 3.1** ($p$-Stable Random Variables - As Defined in Section 3.2 of [2]). Suppose $Z, Z_1, Z_2, \ldots, Z_n$ are i.i.d. random variables, and $p \in [1, 2]$. We say $Z$ and $Z_i$ are $p$-stable if, for any $x \in \mathbb{R}^n$, $\|x\|_p Z$ and $\sum_{i=1}^n x_i Z_i$ have the same distribution.

Note that $p$-stable random variables only exist for $p \in (0, 2]$ — we consider $p \in [1, 2]$. 1-stable random variables are also called Cauchy random variables, as in our description of our algorithm in the introduction. For $p \in [1, 2)$, we use med$_p$ to denote the median of a half-$p$-stable random variable — that is, if $Z$ is a $p$-stable random variable, then $|Z|$ is a half-$p$-stable random variable. Note that the median of a half-Cauchy random variable is just 1, but for $p \in (1, 2)$, there is no simple closed form for med$_p$. med$_p$ can be computed up to a $(1 \pm \varepsilon)$-factor, as described in [20] — this is enough for our purposes. This definition is relevant for the median-based sketch of [2], as we see below.

**Definition 3.2** (Medians and Quantiles of Vectors (Definition 4 in [2])). For a vector $v \in \mathbb{R}^n$, we let med$(v)$ be the median of $|v_i|$ for $i \in [n]$. In addition, for $\alpha \in [0, 1]$, we let $q_\alpha(v)$ denote the minimum value greater than $\lceil \alpha n \rceil$ of the values $|v_1|, |v_2|, \ldots, |v_n|$.

The following lemmas from [2] allow us to obtain very accurate estimates of the $\ell_p$-norms of matrices after first multiplying by a dense $p$-stable matrix to reduce the dimension.

**Lemma 3.3** ($p$-stable Matrix + Median Preserves Norms (from [2])). Let $S$ be an $m \times n$ matrix with i.i.d. standard $p$-stable entries and let $M$ be an $n \times d$ matrix. For $\varepsilon > 0$, with probability $1 - \frac{1}{\Omega(1)}$,

$$(1 - \varepsilon)\|M\|_p \leq \left(\sum_i \text{med}(SM_i)^p\right)^{\frac{1}{p}} / \text{med}_p \leq (1 + \varepsilon)\|M\|_p$$

as long as $m = \text{poly}(1/\varepsilon)$.

**Proof.** This is Lemma 6 from [2] in the case $p = 1$, and Lemma 12 of [2] in the case $p \in (1, 2)$.

**Remark 3.4.** From inspecting the proof of the above lemma in the $p = 1$ case, it can be applied as long as $m \geq \frac{1}{\varepsilon}$. This is because $\Pr[\text{med}(SM_i) = (1 \pm \varepsilon)\|M_i\|_1] \leq e^{-\Theta(\varepsilon^2 m)}$ (by Lemma 6 of [2]) and as long as $m \geq \frac{1}{\varepsilon}$, this probability is at most $e^{-\frac{1}{\varepsilon}} \leq \varepsilon$, which is sufficient for this lemma.

**Lemma 3.5** ($p$-stable Matrix + Top Quantile Does Not Cause Dilation (from [2])). When $S$ is an $m \times n$ matrix with i.i.d. standard $p$-stable entries, $m = \text{poly}(1/\varepsilon)$, and $M$ is an $n \times d$ matrix, then with probability $1 - \frac{1}{\Omega(1)}$,

$$\left(\sum_i q_{1 - \frac{1}{\varepsilon}}(SM_i)^p\right)^{\frac{1}{p}} / \text{med}_p \leq O\left(\frac{1}{\varepsilon}\right)\|M\|_p$$

**Proof.** This is Lemma 7 from [2] in the $p = 1$ case, and Lemma 13 of [2] in the $p \in (1, 2)$ case.
Remark 3.6. From inspecting the proof of the above lemma in the \( p = 1 \) case, we see that it can be applied as long as \( m \geq \frac{1}{\epsilon} \). This is because \( \Pr(\|X\|_{1} \geq \frac{n}{\epsilon} \|M\|_{1}) \leq e^{-\Theta(\epsilon^{-3})} \leq e^{-O(1/\epsilon)} \leq \epsilon \) (by Lemma 4 of [2]) as long as \( m \geq \frac{1}{\epsilon} \).

Remark 3.7. In addition, the failure probabilities in the above two lemmas can be as small as desired, since they are obtained from Markov bounds.

Lemma 3.8 (Quasi-Subspace Embedding with Median Estimator (from [2])). Let \( X \subset \mathbb{R}^{n/k} \) be a \( k \)-dimensional subspace and \( \epsilon, \delta > 0 \). Let \( S \) be an \( m \times n \) matrix whose entries are i.i.d. standard \( p \)-stable random variables, where \( m = O(1/\epsilon^{2} \cdot k \log(k/\epsilon\delta)) \). Then, with probability at least \( 1 - \Theta(\delta) \), for all \( x \in X \),
\[
(1 - \Theta(\epsilon))\|x\|_{p} \leq q_{\frac{1}{2} - \epsilon}(Sx/\text{med}_{p}) \leq q_{\frac{1}{2} + \epsilon}(Sx/\text{med}_{p}) \leq (1 + O(\epsilon))\|x\|_{p}
\]

Proof. This is Lemma 5 of [2] in the \( p = 1 \) case, and Lemma 11 of [2] in the \( p \in (1, 2) \) case.

Remark 3.9. Note that we can select any number of rows that is larger than some \( O(\frac{1}{\epsilon^{2}} k \log(\frac{1}{\epsilon\delta})) \), by inspecting the proof of Lemma 5 of [2] — any number of rows larger than the specified \( O(\frac{1}{\epsilon^{2}} k \log(\frac{1}{\epsilon\delta})) \) allows the net argument in that proof to work.

The following lemma shown in [2] allows this sketch to serve as a “quasi-affine embedding”:

Lemma 3.10. Let \( U \in \mathbb{R}^{n \times k} \) and \( A \in \mathbb{R}^{n \times d} \). Let \( V^{*} \) be chosen to minimize \( \|UV^{*} - A\|_{p} \). Suppose \( S \) is an \( m \times n \) random matrix such that:

1. \( q_{\frac{1}{2} - \epsilon}(SUx/\text{med}_{p}) \geq (1 - \Theta(\epsilon))\|Ux\|_{p} \) for all \( x \in \mathbb{R}^{k} \)
2. For each \( i \in [d] \), with probability at least \( 1 - \epsilon^{3} \), \( \text{med}(S[U, A_{i}]x/\text{med}_{p}) \geq (1 - \epsilon^{3})\|U, A_{i}\|_{p} \) for all \( x \in \mathbb{R}^{k+1} \)
3. \( (\sum_{i} \text{med}(SU_{i}^{*} - SA_{i})^{p})^{\frac{1}{p}}/\text{med}_{p} \leq (1 + \epsilon^{3})\|UV^{*} - A\|_{p} \)
4. \( (\sum_{i} q_{1 - \epsilon/2}(SU_{i}^{*} - SA_{i})^{p})^{\frac{1}{p}}/\text{med}_{p} \leq O(\frac{1}{\epsilon^{2}})\|UV^{*} - A\|_{p} \)

If statements 1, 3 and 4 each hold with probability \( 1 - \frac{1}{\text{poly}(n,m)} \), then with probability \( 1 - \frac{1}{\text{poly}(n,m)} \), for all \( V \in \mathbb{R}^{k \times d} \),
\[
(\sum_{i} \text{med}(SU_{i}^{*} - SA_{i})^{p})^{\frac{1}{p}}/\text{med}_{p} \geq (1 - O(\epsilon))\|UV - A\|_{p}
\]

Proof. This is Theorem 11 of [2] in the \( p = 1 \) case and Theorem 12 of [2] in the \( p \in (1, 2) \) case. Note that in the statements of these Theorems in [2], it is not explicitly stated that statements 1, 3 and 4 only need to hold with constant probability. However, this is true because, by inspecting the proof of Theorem 11 of [2], we see that statement 2 is only used to perform a Markov bound, which needs to hold with constant probability — once that Markov bound is obtained, a union bound can be performed over statements 1, 2, 3 and 4.

Specializing this lemma to \( p \)-stable matrices gives:

Lemma 3.11 (Lower Bound for One-Sided Embedding with Median Estimator (from [2])). Let \( U \in \mathbb{R}^{n \times k} \) and \( A \in \mathbb{R}^{n \times d} \). If \( S \) is an \( m \times n \) random matrix with i.i.d. standard \( p \)-stable entries, where \( m = O(\max(k/\epsilon^{6} \log(k/\epsilon), 1/\epsilon^{9})) \), then with probability \( 1 - \frac{1}{\text{poly}(n,m)} \),
\[
(\sum_{i} \text{med}(SU_{i}^{*} - SA_{i})^{p})^{\frac{1}{p}}/\text{med}_{p} \geq (1 - O(\epsilon))\|UV - A\|_{p}
\]
for all \( V \in \mathbb{R}^{k \times d} \).

Proof. This is a corollary of all the above lemmas. For statement 1 of the previous lemma to hold, it is sufficient for \( S \) to have at least \( O(k/\epsilon^{2} \log(k/\epsilon)) \) rows by Lemma 3.8. For statement 2 of the previous lemma to hold, it is sufficient for \( S \) to have at least \( O(k/\epsilon^{6} \log(k/\epsilon)) \) rows, again by Lemma 3.8. For statement 3 of the previous lemma to hold, it is sufficient for \( S \) to have \( O(1/\epsilon^{3}) \) rows, by Lemma 3.11. Finally, for statement 4 of the previous lemma to hold, it is sufficient for \( S \) to have \( O(1/\epsilon^{2}) \) rows, by Lemma 3.9.
We recall another useful lemma on $p$-stable matrices, which bounds the $\ell_p$-norm of $SM$ if $S$ is a $p$-stable matrix and $M$ is a fixed matrix:

Lemma 3.12 (Distortion in $\ell_p$-norm with $p$-stable Matrices - Lemma E.11 of [41]). Let $M \in \mathbb{R}^{n \times d}$, and let $S \in \mathbb{R}^{n \times n}$ be a random matrix whose entries are i.i.d. standard $p$-stable random variables. Then, with probability $1 - \frac{1}{n^{1/2}}$,  

$$\|SM\|_p \leq O(r \log d)\|M\|_p$$

Note that in the original statement of this lemma in [41], the $p$-stable matrices are rescaled by $\Theta(1/r^{1/p})$ — since our sketch uses $p$-stable matrices that are not rescaled, we include this factor in the distortion.

In the course of our analysis, it will also be useful to note that $med_p$ is bounded away from 0 — that is, there exists a constant $K > 0$ such that $med_p \geq K$. To our knowledge, this fact was not explicitly shown elsewhere, and we prove it below.

Lemma 3.13 ($med_p$ is $\Omega(1)$). There exists an absolute constant $K > 0$ such that $med_p \geq K$ for all $p \in [1, 2]$.

Proof. First, we recall the following formula from [34] for the c.d.f. of a standard $p$-stable random variable for $p \in (1, 2]$. For $x > 0$, if $X$ is a standard $p$-stable random variable, then

$$\Pr[X > x] = 1 - \frac{1}{\pi} \int_0^\pi e^{-x \sqrt{\frac{x}{\sin p \theta}}} V(\theta; p) d\theta$$

where

$$V(\theta; p) = \left(\frac{\cos \theta}{\sin p \theta}\right)^{\frac{1}{p-1}} \frac{\cos((p-1)\theta)}{\cos \theta}$$

(This is a corollary of Theorem 1 of [34].) Hence, because $X$ is symmetric, $x > 0$ is less than $med_p$ if and only if

$$\frac{3}{4} > \Pr[X > x] = 1 - \frac{1}{\pi} \int_0^\pi e^{-x \sqrt{\frac{x}{\sin p \theta}}} V(\theta; p) d\theta$$

or equivalently

$$I := \int_0^\pi e^{-x \sqrt{\frac{x}{\sin p \theta}}} V(\theta; p) d\theta > \frac{\pi}{4}$$

(recall that $med_p$ is the median of $|X|$ rather than $X$). Now, our goal is to bound the integral $I$ on the left-hand side from below. As a first step, we show the following claim.

Claim 3.14. Let $c > 0$ be a sufficiently small absolute constant (to be chosen outside of this claim). There exists a constant $D > 0$ (which may depend on $c$) such that for all $p \in [1, 2]$ and $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2} - c\right]$,

$$\frac{\cos \theta}{\sin(p \theta)} \cdot \left(\frac{\cos((p-1)\theta)}{\cos \theta}\right)^{\frac{1}{p-1}} \leq D$$

Proof. Observe that for $p \in [1, 2]$ and $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2} - c\right]$, $\frac{\pi}{6} \leq p \theta \leq \pi - 2c$, meaning $\sin(p \theta) > 0$ and is in fact bounded away from 0 because $[1, 2] \times \left[\frac{\pi}{6}, \frac{\pi}{2} - c\right]$ is compact. Let $C_0 > 0$ be such that $\sin(p \theta) \geq C_0$ for all $(p, \theta) \in [1, 2] \times \left[\frac{\pi}{6}, \frac{\pi}{2} - c\right]$. Then,

$$\frac{\cos \theta}{\sin(p \theta)} \cdot \left(\frac{\cos((p-1)\theta)}{\cos \theta}\right)^{\frac{1}{p-1}} \leq \cos \theta \cdot \left(\cos((p-1)\theta)\right)^{\frac{1}{p-1}} \cdot \frac{1}{C_0} \leq \frac{1}{C_0}$$

where the second inequality holds because $\cos \theta$ and $\cos((p-1)\theta)$ are at most 1. (Note that $\cos \theta \cdot \left(\cos((p-1)\theta)\right)^{\frac{1}{p-1}}$ are well-defined because $\cos \theta$ and $\cos((p-1)\theta)$ are nonnegative for this choice of $\theta$.)

Hence, for $p \in [1, 2]$ and $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2} - c\right]$,

$$V(\theta; p) \leq \left(\frac{\cos \theta}{\sin(p \theta)}\right)^{\frac{1}{p-1}} \cdot \frac{\cos((p-1)\theta)}{\cos \theta} \leq \left(\frac{\cos \theta}{\sin(p \theta)}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\cos((p-1)\theta)}{\cos \theta}\right)^{\frac{1}{p-1}} \leq D^{\frac{1}{p-1}}$$
Hence, we can bound $I$ from below:

$$
\int_{0}^{\frac{3\pi}{4} - c} e^{-\frac{x}{\pi^{2}}} V(\theta, p) d\theta \geq \int_{0}^{\frac{3\pi}{4} - c} e^{-\frac{x}{\pi^{2}}} D^{\frac{2}{\pi^{2}}} d\theta = \left(\frac{3\pi}{4} - c\right) e^{-\frac{(xD)}{\pi^{2}}}.
$$

(6)

We can choose $c = \frac{\ln(2/5)}{5}$, meaning $I > \frac{\ln(2/5)}{5}$ as long as $\frac{5\pi}{8} e^{-\frac{(xD)}{\pi^{2}}} > \frac{\ln(2/5)}{5}$ or $e^{-\frac{(xD)}{\pi^{2}}} > \frac{\ln(2/5)}{5}$. This holds as long as $x < \frac{1}{D}(\log(2/5)) \frac{5\pi}{8}$. Letting $C = -\ln(2/5)$, we observe that $0 < C < 1$, meaning $\sqrt{C} \leq C^{\frac{5}{1-4C}} \leq 1$.

In summary, $x \leq \text{med}_{p}$ as long as $x < \frac{\sqrt{C}}{2D}$, and hence we can take $\frac{\sqrt{C}}{2D}$ to be the desired $K$.

\[\Box\]

### 3.2 $(1 + \varepsilon)$-Approximation Algorithm and Analysis

We now present and analyze our $(1 + \varepsilon)$-approximation algorithm with bicriteria rank $3k$ for $\ell_p$-low rank approximation, where $p \in [1, 2]$.

**Theorem 3.15** (Correctness and Running Time of Algorithm 5). Let $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, $\varepsilon \in (0, C)$ (where $C$ is a sufficiently small absolute constant), $f > 1$ and $p \in [1, 2]$. Furthermore, if $\text{OPT} = \min_{A_k \in \text{rank} k} \|A_k - A\|_p$, then suppose $(1 - O(\varepsilon))\text{OPT} \leq \hat{\text{OPT}} \leq (1 + O(\varepsilon))\text{OPT}$. Finally, let $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{k \times d}$ be the output of GUESSINGEpsilonAPPROXIMATION($A, k, \varepsilon, f, \hat{\text{OPT}}, p$) (shown in Algorithm 5). Then,

$$
\|UV - A\|_p \leq (1 + O(\varepsilon))\text{OPT} + O\left(\frac{\varepsilon}{f}\right)\|A\|_p
$$

The running time of Algorithm 5 is at most $f^{O(rk)} + 2^{O((rk \log(k/\varepsilon)) + \text{poly}(\text{nd}/\varepsilon))}$, where $r$, the number of rows in the $p$-stable sketching matrix, is $O(\max(k^2/\varepsilon^6 \log(k/\varepsilon), 1/\varepsilon^9))$.

**Remark 3.16**. We can obtain $\hat{\text{OPT}}$ efficiently as follows. If $A_{SV,D,k}$ is the rank-$k$ SVD of $A$ (which can be computed in polynomial time) then $\text{OPT} \leq \|A_{SV,D,k} - A\|_p \leq (nd)^{1/p - 1/2}\text{OPT}$, meaning we can guess all integer powers of $1 + \varepsilon$ between $\|A_{SV,D,k} - A\|_p$ and $(nd)^{1/p - 1/2}\|A_{SV,D,k} - A\|_p$ — the number of guesses is $O(\log(nd))$. We can input all of those guesses to Algorithm 5 and one of them will produce the right answer. This is done when applying Algorithm 5 within Algorithm 6.

**Remark 3.17**. Note that $f$ represents the approximation factor of the initialization algorithm used to obtain $A$. In case $f$ will equal $\text{poly}(k)$, but we will analyze this algorithm for a general $f$.

**Proof**. Let $U^* \in \mathbb{R}^{n \times k}$, $V^* \in \mathbb{R}^{k \times d}$ such that $\|U^*V^* - A\|_p = \text{OPT}$. Without loss of generality, assume $V^*$ is a $q$-well-conditioned basis (where $q = \text{poly}(k)$), meaning that for all $x \in \mathbb{R}^k$,

$$
\frac{\|x\|_q}{q} \leq \|x^T V^*\|_p \leq q \|x\|_p
$$

(Note that well-conditioned bases exist for all $p$, for instance see Lemma 10 of [10.]) In particular, this implies that $\|V^*\|_p^q \leq k q^p$ by letting $x$ be each of the standard basis vectors. Now, let $r = O(\max(k/\varepsilon^6 \log(k/\varepsilon), 1/\varepsilon^9))$ as in Algorithm 5 and let $S \in \mathbb{R}^{r \times n}$ be a random matrix where each entry is an i.i.d. $p$-stable random variable.

We first analyze the effect of rounding $SU^*$, multiplicatively, so that the absolute value of each entry is rounded to the nearest power of $1 + \frac{1}{\text{poly}(k/\varepsilon)}$, or set to 0 if it is too small (here, $\text{poly}(k/\varepsilon)$ in the denominator is $(k/\varepsilon)^c$ for a sufficiently large constant $c$). Note that $\|U^*V^*\|_p \leq O(1)\|A\|_p$ by the triangle inequality. Hence, because $V^*$ is a well-conditioned basis, $O(1)\|A\|_p \geq \frac{1}{2}\|U^*\|_p$, and $\|U^*\|_p \leq O(q)\|A\|_p$. Therefore, $\|SU^*\|_\infty \leq \|SU^*\|_p \leq \text{poly}(k/\varepsilon)\|U^*\|_p \leq \text{poly}(k/\varepsilon)\|A\|_p$, where the second inequality is due to Lemma 3.12 since $U^*$ has $k$ columns and $S$ has $\text{poly}(k/\varepsilon)$ rows.

Now, let $M_1$ be $SU^*$, but with the absolute value of each entry rounded to the nearest power of $1 + \frac{1}{\text{poly}(k/\varepsilon)}$. In addition, let $M_2$ be $M_1$, but with each entry having absolute value less than $\frac{1}{\text{poly}(k/\varepsilon)}\|A\|_p$ being replaced by 0. Then,

$$
\|M_2 - M_1\|_p \leq \left(\frac{1}{\text{poly}(k/\varepsilon)}\right)^p \|A\|_p \cdot O\left(\frac{k^2}{\varepsilon^9} \log(k/\varepsilon)\right) \leq \left(\frac{1}{\text{poly}(k/\varepsilon)}\right)^p \|A\|_p
$$
Algorithm 5 Guessing a sketched left factor \( SU \), and finding an appropriate right factor \( V \) with norm at most \( \poly(k) \), to obtain a \((1+\varepsilon)\)-approximation with additive \( \varepsilon/(\poly(k))\|A\|_p^2 \) error.

Require: \( A \in \mathbb{R}^{n \times d}, k \in \mathbb{N}, \varepsilon \in (0, c), f > 1, \hat{\OPT} \geq 0, p \in [1, 2) \)
Ensure: \( U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d} \)

procedure GUESSINGADDITIVEEPSAPPROXIMATION\((A, k, \varepsilon, f, \hat{\OPT}, p)\)
\hspace{1em} If \( A \) has rank \( k \), return \( A \).
\hspace{1em} \( r \leftarrow \Theta(m(k/\varepsilon^6 \log(k/\varepsilon), 1/\varepsilon^n)) \)
\hspace{1em} \( q \leftarrow \poly(k) \)
\hspace{1em} \( S \leftarrow \{0\} \cup \{\sigma \cdot (1 + \frac{1}{\poly(k/\varepsilon)})^t \mid t \in \mathbb{Z}, \frac{1}{\poly(k/\varepsilon)} \|A\|_p \leq (1 + \frac{1}{\poly(k/\varepsilon)})^t \leq \poly(k/\varepsilon) \|A\|_p, \sigma = \pm 1\} \)
\hspace{1em} \( C \leftarrow \{M \in \mathbb{R}^{r \times k} \mid M_{i,j} \in \mathcal{I} \forall i \in [r], j \in [k]\} \) — This is the set of (sketched) left factors we will guess.
\hspace{1em} // Guess possible rounded (sketched) left factors and find a good right factor \( V \), with \( \|V\|_p \leq \poly(k) \).
\hspace{1em} \( U_{\text{best}} \leftarrow 0, V_{\text{best}} \leftarrow 0 \)
\hspace{1em} for \( M \in C \) do
\hspace{1em} \hspace{1em} COSTBOUNDS \leftarrow \left\{ \frac{c^2}{2} \|A\|_p/d_k^2 \leq c \leq \Theta(\|A\|_p) \text{ and } c \text{ is an integer power of } (1 + \varepsilon) \right\} \)
\hspace{1em} \hspace{1em} for \( i \in [d], c \in \text{COSTBOUNDS} \) do
\hspace{1em} \hspace{1em} \hspace{1em} \( V_{i,c} \leftarrow \underset{V_{i,c}}{\text{argmin}}\|V_{i,c}\|_p \text{ subject to the constraint that } \text{med}(MV_i - SA_i)/\text{med}_p \leq c \)
\hspace{1em} \hspace{1em} \hspace{1em} \( C_{i,c} \leftarrow \text{med}(MV_i - SA_i)/\text{med}_p \)
\hspace{1em} \hspace{1em} end for
\hspace{1em} \hspace{1em} end for
\hspace{1em} // Create LP to find a good distribution over \( c \in \text{COSTBOUNDS} \) for each \( i \in [d] \).
\hspace{1em} \( \text{VARIABLES} \leftarrow \{x_{i,c} \mid i \in [d], c \in \text{COSTBOUNDS} \} \)
\hspace{1em} \( \text{CONSTRAINTS} \leftarrow \left\{ 0 \leq x_{i,c} \forall i \in [d], c \in \text{COSTBOUNDS} \text{ and } \sum_{c \in \text{COSTBOUNDS}} x_{i,c} = 1 \forall i \in [d] \right\} \)
\hspace{1em} \( \text{CONSTRAINTS} \leftarrow \text{CONSTRAINTS} \cup \left\{ \sum_{i \in [d], c \in \text{COSTBOUNDS}} x_{i,c} \|V_{i,c}\|_p^p \leq kq^p \right\} \)
\hspace{1em} \( \Delta \leftarrow (1 + O(\varepsilon))^p (\hat{\OPT} + \frac{1}{\poly(k/\varepsilon)} \|A\|_p^p) + \left(\frac{q}{f}\right)^p \|A\|_p^p \)
\hspace{1em} \( \text{CONSTRAINTS} \leftarrow \text{CONSTRAINTS} \cup \left\{ \sum_{i \in [d], c \in \text{COSTBOUNDS}} x_{i,c} \sigma_{i,c}^p \leq \Delta \right\} \)
\hspace{1em} \( x_{i,c} \leftarrow \text{Solution to the LP given by VARIABLES and CONSTRAINTS, for all } i \in [d], c \in \text{COSTBOUNDS} \)
\hspace{1em} if LP is infeasible, then continue to next \( M \in C \).
\hspace{1em} // For each column, sample an appropriate cost bound. Do this \( O(1/\varepsilon) \) times, then \( V' \) meets both // the cost and norm constraints with constant probability.
\hspace{1em} for \( t = 1 \to 10/\varepsilon \) do
\hspace{1em} \hspace{1em} \( c_t \leftarrow \text{An element } c \in \text{COSTBOUNDS} \text{ selected according to the distribution on COSTBOUNDS} \)
\hspace{1em} \hspace{1em} given by \( \{x_{i,c} \mid c \in \text{COSTBOUNDS}\} \) (note that for \( i \in [d] \), the \( x_{i,c} \) are nonnegative and sum to 1)
\hspace{1em} \hspace{1em} \( V_{i}' \leftarrow V_{i,c_t} \text{ for all } i \in [d] \)
\hspace{1em} \hspace{1em} Break if \( \|V'\|_p^p \leq \frac{2kq^p}{\varepsilon} \text{ and } \sum_{i=1}^d \text{med}(MV_i - SA_i)^p/\text{med}_p^p \leq (1 + 2\varepsilon)\Delta \)
\hspace{1em} end for
\hspace{1em} \( U' \leftarrow \underset{U}{\text{argmin}}\|UV' - A\|_p \)
\hspace{1em} if \( \|U'V' - A\|_p \leq \|U_{\text{best}}V_{\text{best}} - A\|_p \) then \( U_{\text{best}} \leftarrow U' \text{ and } V_{\text{best}} \leftarrow V' \).
\hspace{1em} end for
\hspace{1em} \hspace{1em} return \( U', V' \)
\hspace{1em} end procedure
Algorithm 6 First apply \textsc{PolyKErrorNotBiCriteriaApproximation} from Algorithm 8 to obtain a poly\((k)\)-approximation \(B\) — then, apply Algorithm 5 to the residual to obtain an approximation \(UV\) with additive error \(1/poly(k/\varepsilon)\|A - B\|_p \leq \varepsilon \cdot \text{OPT}\). Finally, \(B + UV\) gives a \((1 + \varepsilon)\)-approximation with rank \(3k\).

Require: \(A \in \mathbb{R}^{n \times d}, k \in \mathbb{N}, \varepsilon \in (0, c), p \in [1, 2)\)

Ensure: \(\hat{A} \in \mathbb{R}^{n \times d}\) having rank \(3k\)

procedure \textsc{RoundingGuessingEpsApproximation}\((A, k, \varepsilon, p)\)

\(W, Z \leftarrow \textsc{PolyKErrorNotBiCriteriaApproximation}(A, k, p)\)

\(B \leftarrow WZ\)

\(C \leftarrow A - B\)

\(f \leftarrow \text{poly}(k)\), the approximation factor of Algorithm 8

\[
\begin{align*}
// & \text{ Guess all } O((\log nd)/\varepsilon) \text{ possible values for } \hat{OPT} \text{ and try them for Algorithm 5} \\
// & \text{ as described in Remark 3.10} \\
C_{SVD, 2k} & \leftarrow \text{The optimal rank-2k approximation for } C \text{ under the } \ell_2 \text{ norm.} \\
\text{SvdError} & \leftarrow \|C - C_{SVD, 2k}\|_p \\
\hat{A} & \leftarrow 0 \in \mathbb{R}^{n \times d}\end{align*}
\]

for \(t = 0 \rightarrow O(\log nd/\varepsilon)\) do

\[
\begin{align*}
\hat{OPT} & \leftarrow \text{SvdError}/(1 + \varepsilon)^t \\
U, V & \leftarrow \text{GuessingAdditiveEpsApproximation}(C, 2k, \varepsilon, f, \hat{OPT}, p) \\
\text{If } \|(B + UV) - A\|_p & \leq \|\hat{A} - A\|_p \text{ then } \hat{A} \leftarrow B + UV
\end{align*}
\]

end for

end procedure

where the first inequality holds because \(M_1\) and \(M_2\) have \(rk = O(\max(k^2/\varepsilon^6 \log(k/\varepsilon), 1/\varepsilon^6))\) entries. Moreover,

\[
\|SU^* - M_1\|_p \leq \left(\frac{1}{f\text{poly}(k/\varepsilon)}\right)^p \|SU^*\|_p \leq \left(\frac{1}{f\text{poly}(k/\varepsilon)}\right)^p \cdot \text{poly}(k/\varepsilon)\|A\|_p \leq \left(\frac{1}{f\text{poly}(k/\varepsilon)}\right)^p \|A\|_p
\]

where the first inequality is because for any \(a \in \mathbb{R}\), and \(t \in (0, 1)\), if \(\tilde{a}\) is \(a\) with its absolute value rounded to the nearest power of \((1 + t)\), then \(|a - \tilde{a}|^p \leq |a|^p|\tilde{a}|^p\). The second inequality is simply because \(\|SU^*\|_p \leq \text{poly}(k/\varepsilon)\|A\|_p\) as mentioned above. Hence,

\[
\begin{align*}
\|SU^* - M_2\|_p & \leq \|SU^* - M_1\|_p + \|M_1 - M_2\|_p \\
& \leq \left(\frac{1}{f\text{poly}(k/\varepsilon)}\right)^p \|A\|_p + \left(\frac{1}{f\text{poly}(k/\varepsilon)}\right)^p \|A\|_p \\
& \leq \frac{1}{f\text{poly}(k/\varepsilon)}\|A\|_p \\
\end{align*}
\]

(7)

where the first inequality is due to the triangle inequality.

Now, observe that Algorithm 8 will guess \(M_2\) at some point — that is, since \(M_2 \in C\), \(M\) will be equal to \(M_2\) at some point. Suppose \(M = M_2\). Let us condition on the following events involving \(S\). Let \(\mathcal{E}_1\) denote the event that for all \(V \in \mathbb{R}^{k \times d}\),

\[
\left(\sum_i \text{med}(SU_i^*V_i - SA_i)^p\right)^{1/p} / \text{med}_{p} \geq (1 - O(\varepsilon))\|U^*V - A\|_p
\]

\(\mathcal{E}_2\) the event that

\[
\left(\sum_i \text{med}(SU_i^*V_i - SA_i)^p\right)^{1/p} / \text{med}_{p} \leq (1 + \varepsilon)\|U^*V - A\|_p = (1 + \varepsilon)\text{OPT}
\]
and $\mathcal{E}_3$ the event that

$$\|SU^*\|_p \leq \text{poly}(k/\varepsilon)\|U^*\|_p$$

By Lemma 3.11, 3.3 and 3.12 respectively, these each occur with probability $1 - O(1)$ (where the constant probability can be made as small as desired by increasing $r$ by a constant factor), and by a union bound they occur simultaneously with probability $1 - O(1)$. First, we use these events to examine the effect of rounding $SU^*$ to $M$, when the right factor is $V^*$, and more generally, when the $\ell_p$ norm of the right factor $V$ is at most $\text{poly}(k/\varepsilon)$.

**Claim 3.18.** Suppose $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ hold, and suppose $V \in \mathbb{R}^{k \times d}$ such that $\|V\|_p \leq \text{poly}(k/\varepsilon)$. Then,

$$\left| \left( \sum_{i=1}^d \text{med}(MV_i - SA_i)^p \right)^{\frac{1}{p}} / \text{med}_p - \left( \sum_{i=1}^d \text{med}(SU^*V_i - SA_i)^p \right)^{\frac{1}{p}} / \text{med}_p \right| \leq \frac{1}{f\text{poly}(k/\varepsilon)\text{med}_p}\|A\|_p$$

**Proof.** Let $x \in \mathbb{R}^d$ be the vector whose $i^{th}$ coordinate is $\text{med}(MV_i - SA_i)$, and let $y \in \mathbb{R}^d$ be the vector whose $i^{th}$ coordinate is $\text{med}(SU^*V_i - SA_i)$. Then, observe that

$$\left| \left( \sum_{i=1}^d \text{med}(MV_i - SA_i)^p \right)^{\frac{1}{p}} - \left( \sum_{i=1}^d \text{med}(SU^*V_i - SA_i)^p \right)^{\frac{1}{p}} \right| = \|x\|_p - \|y\|_p$$

Now, by the triangle inequality, this is at most $\|x - y\|_p$, which we can bound from above:

$$\|x - y\|_p = \sum_{i=1}^d \text{med}(MV_i - SA_i) - \text{med}(SU^*V_i - SA_i)^p \leq \sum_{i=1}^d \|M - SU^*\|V_i\|_\infty^p \leq \sum_{i=1}^d \|M - SU^*\|V_i\|_p \leq \|M - SU^*\|V\|_p$$

(8)

Here, the first inequality is because $|\text{med}(v_1 + v_2) - \text{med}(v_1)| \leq \|v_2\|_\infty$ for any two vectors $v_1, v_2 \in \mathbb{R}^n$, and the second is because $\|v\|_\infty \leq \|v\|_p$ for any vector $v$.

Hence,

$$\|x - y\|_p \leq \|M - SU^*\|V\|_p \leq \text{poly}(k/\varepsilon)\|M - SU^*\|_p \leq \frac{1}{f\text{poly}(k/\varepsilon)}\|A\|_p$$

Here, the second inequality holds because, even though $V$ is not necessarily a well-conditioned basis, for any vector $x \in \mathbb{R}^k$,

$$\|x^TV\|_p = \left\| \sum_{i=1}^k x_iV_i \right\|_p \leq \sum_{i=1}^k |x_i|\|V_i\|_p \leq k\|x\|_p\|V\|_p = \text{poly}(k/\varepsilon)\|x\|_p$$

where the first inequality is due to the triangle inequality, and the last equality is because $\|V\|_p \leq \text{poly}(k/\varepsilon)$. Hence, for any matrix $D \in \mathbb{R}^{n \times k}$,

$$\|DV\|_p = \sum_{i=1}^n \|D^iV\|_p \leq \sum_{i=1}^n \text{poly}(k/\varepsilon)\|D^i\|_p \leq \text{poly}(k/\varepsilon)\|D\|_p$$

and taking $p^{th}$ roots gives $\|DV\|_p \leq \text{poly}(k/\varepsilon)\|D\|_p$.

**Claim 3.19.** Suppose $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ occur. Then, the linear program constructed in Algorithm 3 with variables in the set VARIABLES, and constraints in the set CONSTRAINTS, is feasible when $M = M_2$ is guessed.
Proof. Consider a particular \( i \in [d] \). Note that

\[
\frac{\text{med}(MV_i^* - SA_i)}{\text{med}_p} \leq \frac{\text{med}(SU_i^* V_i^* - SA_i)}{\text{med}_p} + \frac{\| (M - SU^*) V_i^* \|_{\infty}}{\text{med}_p}
\]

\[
\leq \frac{\text{med}(SU_i^* V_i^* - SA_i)}{\text{med}_p} + \frac{\| (M - SU^*) V_i^* \|_p}{\text{med}_p}
\]

\[
\leq \frac{\text{med}(SU_i^* V_i^* - SA_i)}{\text{med}_p} + \text{poly}(k) \frac{\| (M - SU^*) \|_p}{\text{med}_p}
\]

\[
\leq \frac{\text{med}(SU_i^* V_i^* - SA_i)}{\text{med}_p} + \frac{1}{f \text{poly}(k/\varepsilon)} \| A \|_p
\]

\[
\leq \left( \sum_{j \in [d]} \text{med}(SU_j^* - SA_j) \right)^{\frac{1}{p}} + \frac{1}{f \text{poly}(k/\varepsilon)} \| A \|_p
\]

\[
\leq (1 + \varepsilon) \text{OPT} + \frac{1}{f \text{poly}(k/\varepsilon)} \| A \|_p
\]

\[
\leq (1 + O(1)) \| A \|_p
\]

The first inequality is true because for \( v_1, v_2 \in \mathbb{R}^n \), \( |\text{med}(v_1 + v_2) - \text{med}(v_1)| \leq \| v_2 \|_{\infty} \), meaning \( |\text{med}(MV_i^* - SA_i) - \text{med}(SU_i^* V_i^* - SA_i)| \leq \| (M - SU^*) V_i^* \|_{\infty} \). Here, the second inequality is by \( \| x \|_p \geq \| x \|_\infty \), and the third is because \( V_i^* \) is a poly\((k)\) well-conditioned basis. The fourth inequality is because \( \text{med}_p \) is nonnegative and bounded away from 0 by Lemma \ref{lem:bounded} and because \( \| M - SU^* \|_p \leq \frac{1}{\text{poly}(k/\varepsilon)} \| A \|_p \) by Equation \ref{eq:poly_bound}.

Finally, the sixth inequality is because \( E_2 \) holds and the seventh is because \( \text{OPT} \leq \| A \|_p \).

As a summary, we have shown that for each \( i \in [d], \text{med}(MV_i^* - SA_i)/\text{med}_p \leq O(1) \| A \|_p \). Hence, there are two cases — either there exists \( c_0 \in \text{CostBounds} \) such that \( c_0 \geq \text{med}(MV_i^* - SA_i)/\text{med}_p \geq \frac{c_0}{1 + \varepsilon} \), or \( \text{med}(MV_i^* - SA_i)/\text{med}_p \leq \left( \frac{2}{f} \right) \| A \|_p / d^{\frac{1}{p}} \).

- In the first case, we can assign \( x_{i,c} = 1 \) and \( x_{i,c'} = 0 \) for \( c' \neq c_0 \). Define \( V_{i,c_0} \) as in Algorithm \ref{alg:cost_bounds} meaning it is equal to \( \text{argmin}_V \| V_i \|_p \) subject to the constraint that \( \text{med}(MV_i^* - SA_i)/\text{med}_p \leq c_0 \). Then, \( \| V_{i,c_0} \|_p \leq \| V_i \|_p \) and moreover, since \( \text{med}(MV_i^* - SA_i)/\text{med}_p \geq \frac{c_0}{1 + \varepsilon} \), \( \text{med}(MV_{i,c_0} - SA_i)/\text{med}_p \leq c_0 \leq (1 + O(\varepsilon))\text{med}(MV_i^* - SA_i)/\text{med}_p \).

- In the second case, we can assign \( x_{i,c} = 1 \) and \( x_{i,c'} = 0 \) for \( c' \neq c_0 \), where \( c_0 \) is now \( \left( \frac{2}{f} \right) \| A \|_p / d^{\frac{1}{p}} \).

Again, \( V_{i,c_0} \) is equal to \( \text{argmin}_V \| V_i \|_p \) subject to the constraint that \( \text{med}(MV_i^* - SA_i)/\text{med}_p \leq c_0 \). Hence, since \( \text{med}(MV_i^* - SA_i)/\text{med}_p \leq c_0, \| V_{i,c_0} \|_p \leq \| V_i \|_p \). Moreover, \( \text{med}(MV_{i,c_0} - SA_i)/\text{med}_p \leq \left( \frac{2}{f} \right) \| A \|_p / d^{\frac{1}{p}} \), which is sufficient for our purposes, as we see now.

We now conclude that this assignment to the \( x_{i,c} \) satisfies all the constraints of the linear program. Clearly the constraints \( x_{i,c} \geq 0 \) (for all \( i \in [d], c \in \text{CostBounds} \)) and \( \sum_{c \in \text{CostBounds}} x_{i,c} = 1 \) (for all \( i \in [d] \)) are satisfied (since for each \( i \in [d] \), exactly one \( x_{i,c} \) is set to 1 and the rest are 0). For each \( i \in [d] \), let \( c_i \) be the unique element of \( \text{CostBounds} \) such that \( x_{i,c_i} = 1 \) according to our assignment. Then, for all \( i \in [d], \| V_{i,c_i} \|_p \leq \| V_i \|_p \) as mentioned above (where we wrote \( \| V_{i,c_0} \|_p \leq \| V_i \|_p \)), meaning

\[
\sum_{i=1}^{d} \sum_{c \in \text{CostBounds}} x_{i,c} \| V_{i,c} \|_p = \sum_{i=1}^{d} \| V_{i,c_i} \|_p \leq \sum_{i=1}^{d} \| V_i \|_p = \| V_i \|_p \leq k \| V_i \|_p \leq k q^p = \text{poly}(k)
\]

where the last inequality is because each row of \( V_i \) has norm at most \( q \). In addition,

\[
\sum_{i=1}^{d} \sum_{c \in \text{CostBounds}} x_{i,c} C_{i,c} \leq \sum_{i=1}^{d} \text{med}(MV_{i,c_i} - SA_i)/\text{med}_p
\]

\[
\leq \sum_{i=1}^{d} \left( (1 + O(\varepsilon))^p \text{med}(MV_i^* - SA_i)/\text{med}_p + (\varepsilon^2 / f)^p \| A \|_p / d \right)
\]

\[
= (1 + O(\varepsilon))^p \sum_{i=1}^{d} \frac{\text{med}(MV_i^* - SA_i)}{\text{med}_p} + \left( \frac{\varepsilon^2}{f} \right)^p \| A \|_p
\]
By Claim 3.18, this is at most
\[ (1 + O(\varepsilon))p\left(\left(\sum_{i=1}^{\ell} \text{med}(SU^*V^*_{i} - SA_i)^p\right)^{1/p} \text{med}_p + \frac{1}{f^\text{poly}(k/\varepsilon)}\text{med}_p\|A\|_p\right)^p + \left(\frac{\varepsilon^2}{f}\right)^p\|A\|_p^p \]
and this is in turn at most
\[ (1 + O(\varepsilon))p\left(\text{OPT} + \frac{1}{f^\text{poly}(k/\varepsilon)}\|A\|_p\right)^p + \left(\frac{\varepsilon^2}{f}\right)^p\|A\|_p^p \]
Here, we used the fact that \( \mathcal{E}_2 \) occurs, meaning \( \left(\sum_{i=1}^{d} \text{med}(SU^*V^*_{i} - SA_i)^p\right)^{1/p} \text{med}_p \leq (1 + \varepsilon)\text{OPT} \), \( \text{OPT} \leq \frac{1}{1 - O(\varepsilon)}\text{OPT} \leq (1+ O(\varepsilon))\text{OPT} \), and moreover, \( \text{med}_p = \Omega(1) \) (regardless of \( p \)), meaning \( \frac{1}{\text{med}_p} \) is bounded above.

Hence, we can conclude that
\[
\sum_{i=1}^{d} \sum_{c \in \text{CostBounds}} x_{i,c}C_{i,c}^p \leq (1 + O(\varepsilon))p\left(\text{OPT} + \frac{1}{f^\text{poly}(k/\varepsilon)}\|A\|_p\right)^p + \left(\frac{\varepsilon^2}{f}\right)^p\|A\|_p^p
\]
and we have shown that this assignment to the \( x_{i,c} \) satisfies all the constraints of the LP.

\[ \square \]

**Claim 3.20.** Suppose \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) occur. Then, when \( M = M_2 \) is guessed, with probability \( 1 - O(1) \), Algorithm 5 finds \( V' \) such that \( \|V'\|_p \leq \frac{2kq^p}{\varepsilon} \) and \( \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p/\text{med}_p \leq (1 + 2\varepsilon)\Delta \), where \( \Delta \) is defined in Algorithm 5.

**Proof.** The proof is by Markov’s inequality. First, by the previous claim, if \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) hold, then solving the LP within Algorithm 5 gives \( x_{i,c} \) for \( i \in [d] \) and \( c \in \text{CostBounds} \) such that
\[
\sum_{i \in [d], c \in \text{CostBounds}} x_{i,c}\|V_{i,c}\|_p \leq kq^p \text{ and } \sum_{i \in [d], c \in \text{CostBounds}} x_{i,c}C_{i,c}^p \leq \Delta
\]
Now, for each column \( i \in [d] \), suppose we sample a single \( c_i \in \text{CostBounds} \) according to the distribution on \( \text{CostBounds} \) given by the \( x_{i,c} \), and we let \( V'_i = V_{i,c_i} \). Denote this distribution by \( \pi_i \). Then, conditioning on a fixed value of the \( p \)-stable matrix \( S \) such that \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) hold,
\[
E_{\pi_i \sim \pi, \forall i \in [d]} \left[ \|V'\|_p | S \right] = \sum_{i=1}^{d} \sum_{c \in \text{CostBounds}} x_{i,c}\|V_{i,c}\|_p \leq kq^p
\]
The first equality is by the definition of expectation. Note that prior to sampling \( c_i \) for each \( i \in [d] \), the only source of randomness is \( S \), and the appropriate rounded matrix \( M \), as well as the minimizers \( V_{i,c} \) and their costs \( C_{i,c} \) are all determined by \( S \). The second inequality holds if \( S \) is such that \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) are satisfied. By the same argument
\[
E_{\pi_i \sim \pi, \forall i \in [d]} \left[ \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p/\text{med}_p | S \right] = \sum_{i \in [d], c \in \text{CostBounds}} x_{i,c}C_{i,c}^p \leq \Delta
\]
Hence, letting \( \mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \),
\[
E_{\pi_i \sim \pi, \forall i \in [d]} \left[ \|V'\|_p | \mathcal{E} \right] = \frac{\int_{\mathcal{E}} E_{\pi_i \sim \pi, \forall i \in [d]} \left[ \|V'\|_p | S \right] p(S)dS}{P[\mathcal{E}]} \leq \frac{\int_{\mathcal{E}} kq^p : p(S)dS}{P[\mathcal{E}]} = kq^p
\]
where the first equality is by the definition of conditional expectation (here, \( p(S) \) denotes the p.d.f. of the Cauchy matrix \( S \)), and the second is because the integral is over \( S \) for which \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) hold. Similarly,

\[
E_{c_i \sim \pi_i, \forall i \in [d]} \left[ \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \mid \mathcal{E} \right] = \frac{\int_{\mathcal{E}} E_{c_i \sim \pi_i, \forall i \in [d]} \left[ \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \mid S \right] p(S) dS}{P[\mathcal{E}]} \\
\leq \frac{\int_{\mathcal{E}} \Delta \cdot p(S) dS}{P[\mathcal{E}]} = \Delta
\]

(11)

Now, by Markov’s inequality, if we sample \( c_i \) according to \( \pi_i \) for \( i \in [d] \), then

\[
P \left[ \|V'\|_p^p \geq \frac{2kq^p}{\varepsilon} \mid \mathcal{E} \right] \leq \frac{\varepsilon}{2}
\]

and

\[
P \left[ \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \geq (1 + 2\varepsilon)\Delta \mid \mathcal{E} \right] \leq \frac{1}{1 + 2\varepsilon} \leq 1 - \varepsilon
\]

where the second inequality holds because for \( \varepsilon \in (0, \frac{1}{2}) \), \((1 + 2x)(1 - x) = 1 + x - 2x^2 = 1 + x(1 - 2x) \geq 1 \), meaning \( 1 - x \geq \frac{1}{1 + 2x} \). Hence, by a union bound,

\[
P \left[ \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \geq (1 + 2\varepsilon)\Delta \text{ or } \|V'\|_p^p \geq \frac{2kq^p}{\varepsilon} \mid \mathcal{E} \right] \leq 1 - \varepsilon
\]

Finally, if we repeatedly sample \( c_i \) for \( i \in [d] \) over the course of \( \frac{10}{\varepsilon} \) trials, then the probability that none of the \( V' \) satisfy the desired properties is at most

\[
\left( 1 - \frac{\varepsilon}{2} \right)^{10/\varepsilon} \leq \frac{1}{100}
\]

This completes the proof of the claim — if we condition on \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \), then with probability \( \frac{99}{100} \), we obtain \( V' \) such that \( \|V'\|_p \leq \frac{2kq^p}{\varepsilon} \) and

\[
\sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \leq (1 + 2\varepsilon)\Delta
\]

\( \square \)

Now, suppose \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \) hold, and that we have obtained \( V' \) with \( \|V'\|_p \leq \frac{O(kq^p)}{\varepsilon} \) and \( \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p / \text{med}_p \leq (1 + 2\varepsilon)\Delta \) — by the previous claim, this occurs with constant probability. By Claim 3.18 since \( \|V'\|_p \leq \text{poly}(k/\varepsilon) \), we know that

\[
\left( \sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p \right)^{\frac{1}{p}} / \text{med}_p \geq \left( \sum_{i=1}^{d} \text{med}(SU^*V'_i - SA_i)^p \right)^{\frac{1}{p}} / \text{med}_p - \frac{1}{f \text{poly}(k/\varepsilon) \text{med}_p} \|A\|_p \geq (1 - O(\varepsilon))\|U^*V' - A\|_p - \frac{1}{f \text{poly}(k/\varepsilon) \text{med}_p} \|A\|_p \geq (1 - O(\varepsilon))\|U'V' - A\|_p - \frac{1}{f \text{poly}(k/\varepsilon) \text{med}_p} \|A\|_p
\]

(12)

Here, the first inequality is by Claim 3.18, the second is because we are conditioning on \( \mathcal{E}_1 \) (meaning the median-based sketch and \( S \) provide a one-sided embedding for \( U^* \) and \( A \)), and the third inequality is because
\[ U' = \text{argmin}_U \| UV' - A \|_p. \]

Therefore,

\[
\| U'V' - A \|_p \leq (1 + O(\varepsilon))(\sum_{i=1}^{d} \text{med}(MV'_i - SA_i)^p)^{\frac{1}{p}} + \frac{1}{f_{\text{poly}(k/\varepsilon)} \text{med}_p} \| A \|_p
\]

\[
\leq (1 + O(\varepsilon))(1 + 2\varepsilon)^{1/p} \Delta^{1/p} + \frac{1}{f_{\text{poly}(k/\varepsilon)} \text{med}_p} \| A \|_p
\]

\[
\leq (1 + O(\varepsilon))\Delta^{1/p} + \frac{1}{f_{\text{poly}(k/\varepsilon)} \text{med}_p} \| A \|_p
\]

where the last inequality is because \( \text{med}_p = O(1) \) as \( p \) ranges through \([1, 2]\). Finally, we can bound \( \Delta^{1/p} \) from above by observing that the function \( f : x \rightarrow |x|^{1/p} \) is subadditive:

\[
\Delta^{1/p} = \left( (1 + O(\varepsilon))^p \left( \hat{O}^{\text{OPT}} + \frac{1}{f_{\text{poly}(k/\varepsilon)}} \| A \|_p \right)^p + \left( \frac{\varepsilon^2}{f} \| A \|_p \right)^{1/p} \right)^{1/p}
\]

\[
\leq \left( (1 + O(\varepsilon))^p \left( \hat{O}^{\text{OPT}} + \frac{1}{f_{\text{poly}(k/\varepsilon)}} \| A \|_p \right)^p + \frac{\varepsilon^2}{f} \| A \|_p \right)^{1/p}
\]

\[
\leq (1 + O(\varepsilon))(\hat{O}^{\text{OPT}} + \frac{1}{f_{\text{poly}(k/\varepsilon)}} \| A \|_p) + \frac{\varepsilon^2}{f} \| A \|_p
\]

where the first inequality holds because \( (|x| + |y|)^{1/p} \leq |x|^{1/p} + |y|^{1/p} \), and the last is because \( \hat{O}^{\text{OPT}} \leq (1 + O(\varepsilon))\text{OPT} \). In summary,

\[
\| U'V' - A \|_p \leq (1 + O(\varepsilon))\Delta^{1/p} + \frac{1}{f_{\text{poly}(k/\varepsilon)} \text{med}_p} \| A \|_p
\]

\[
= (1 + O(\varepsilon))\text{OPT} + O\left( \frac{\varepsilon^2}{f} \| A \|_p \right)
\]

Finally, we analyze the running time of Algorithm 5.

**Claim 3.21.** The running time of Algorithm 5 is at most \( f^{O(rk)} + 2^{O(rk \log(k/\varepsilon))} + \text{poly}(fnud/\varepsilon) \), where \( r \), the number of rows in the \( p \)-stable matrix \( S \), is at most \( O(\max(k/\varepsilon^6 \log(k/\varepsilon), 1/\varepsilon^9)) \).

**Proof.** First, let us find the number of matrices in \(|C|\). Note that each \( M \in C \) has \( rk \) entries (recall that \( r \) is the number of rows of \( S \)). For each entry, there are \(|I|\) choices. The magnitude of the largest possible guess is \( \text{poly}(k/\varepsilon) \| A \|_p \), while that of the smallest possible guess is \( \frac{1}{f_{\text{poly}(k/\varepsilon)} \text{med}_p} \| A \|_p \). Therefore, the number of guesses for each entry is \( \frac{O(\log(fk/\varepsilon))}{\log(1 + \frac{1}{\varepsilon})} \). Since for \( x < 1 \), \( \log(1 + x) \geq \frac{x}{2} \), the number of guesses per each entry of \( M \) is in fact \( O(f_{\text{poly}(k/\varepsilon)} \log(fk/\varepsilon)) \). In summary,

\[
|C| = (f_{\text{poly}(k/\varepsilon)} \log(fk/\varepsilon))^{rk} = f^{O(rk)}2^{O(rk \log(k/\varepsilon))}
\]

Let us now calculate the running time needed for each guess. The size of \( \text{CostBounds} \) is

\[
\log \left( \frac{fd}{\varepsilon} \right) \cdot \frac{1}{\log(1 + \varepsilon)} \leq O\left( \frac{1}{\varepsilon} \log \left( \frac{fd}{\varepsilon} \right) \right)
\]

For each \( i \in [d] \) and \( c \in \text{CostBounds} \), we solve the problem of minimizing \( \| V_i \|_p \) subject to the constraint that \( \text{med}(MV'_i - SA_i)/\text{med}_p \leq c \). Computing \( SA_i \) takes \( O(nr) \) time. If \( V_{i,c,best} \) is the solution to this problem, then there are at most \( r! \) possible orderings for the coordinates of \( MV_{i,c,best} - SA_i \). Therefore, we can make this optimization problem into a convex program by trying all orderings of the coordinates of \( MV_i - SA_i \) (meaning that the orderings are added in the form of \( \text{poly}(k/\varepsilon) \) additional constraints). Once we
do this, \( \text{med}(MV_i - SA_i)/\text{med}_p \leq c \) becomes a linear constraint. Hence, we solve \( r! \) distinct convex programs. Since \( M \) is an \( r \times k \) matrix, and \( SA_\i \) is an \( r \)-dimensional vector, finding \( V_{i,c,best} \) takes at most \( r! \cdot \text{poly}(r) \) running time. In summary, the total time taken to find \( V_{i,c} \) and \( C_{i,c} \) for all \( i \in [d] \) and \( c \in \text{CostBounds} \) is

\[
d \cdot |\text{CostBounds}| \cdot r! \cdot \text{poly}(r) + O(nr) \leq O \left( \frac{d}{\varepsilon} \log \left( \frac{fd}{\varepsilon} \right) \right) \cdot r^{O(r)} + O(nr) \\
\leq \text{poly}(fnd/\varepsilon) 2^{O(r \log r)}
\]

Next comes the running time for the convex program used to find the \( c \). As calculated above, the number of variables is \( d \cdot |\text{CostBounds}| = O \left( \frac{d}{\varepsilon} \log \left( \frac{fd}{\varepsilon} \right) \right) \) and the number of constraints is \( d |\text{CostBounds}| + d + 2 = O \left( \frac{d}{\varepsilon} \log \left( \frac{fd}{\varepsilon} \right) \right) \). Therefore, the convex program can be solved in \( \text{poly}(fnd/\varepsilon) \) time.

Finally, we calculate the running time of the stage where we sample \( c_i \) for every \( i \in [d] \) in order to find a right factor \( V' \). The time needed to sample \( c_i \) is \( d \cdot |\text{CostBounds}| = O \left( \frac{d}{\varepsilon} \log \left( \frac{fd}{\varepsilon} \right) \right) \), and for each sample of the \( c_i \), computing the norm of the new \( V' \) takes \( O(nk) \) time, and computing \( \sum_{i \in [d]} \text{med}(MV'_i - SA_i)^p/\text{med}_p \) takes \( n \cdot \text{poly}(k) \) time. The number of samples is \( O(1/\varepsilon) \), meaning the total running time of this stage is \( \text{poly}(fnd/\varepsilon) \). After \( V' \) has been found, the running time needed to find \( U' \) by solving \( \min \|UV' - A\|_p \) using convex programming is \( \text{poly}(nd) \). In summary, the total running time of Algorithm 6 is

\[
f^{O(\text{rk})} 2^{O(rk \log (k/\varepsilon))} \left( \text{poly}(fnd/\varepsilon) 2^{O(r \log r)} + \text{poly}(fnd/\varepsilon) \right)
\]

and this is at most \( f^{O(\text{rk})} 2^{O(rk \log (k/\varepsilon))} \text{poly}(fnd/\varepsilon) \). The runtime in the theorem statement is due to the inequality \( abc \leq \frac{a^3 + b^3 + c^3}{3} \), for \( a, b, c \geq 0 \).

We use this to obtain a \((1 + \varepsilon)\)-approximation algorithm with bicriteria rank 3k.

**Theorem 3.22** (Correctness and Running Time of Algorithm 6). Let \( A \in \mathbb{R}^{n \times d}, \ k \in \mathbb{N}, \ \text{and} \ \varepsilon \in (0, c) \) where \( c \) is a sufficiently small absolute constant. Then, Algorithm 6, with these inputs, returns \( \hat{A} \in \mathbb{R}^{n \times d} \) such that \( \hat{A} \) has rank 3k and

\[
\|\hat{A} - A\|_p \leq (1 + O(\varepsilon)) \min_{\text{rank } k} \|A - A_k\|_p
\]

The running time of Algorithm 6 is \( 2^{O(rk \log (k/\varepsilon))} + \text{poly}(nd/\varepsilon) \), where \( r \) is the same as in the statement of Theorem 3.14.

**Proof.** Let \( A_k \) be the optimal rank-\( k \)-approximation for \( A \), and suppose

\[
(1 - O(\varepsilon))\text{OPT}_{C,2k} \leq \hat{O}^P \leq (1 + O(\varepsilon))\text{OPT}_{C,2k}
\]

where \( \text{OPT}_{C,2k} \) is the error from the optimal rank-2k approximation for \( C \) (as the algorithm will indeed guess such an \( \hat{O}^P \) at some point). If this is the case, then when we call \( \text{GUESSINGADDITIVEEPSAPPROXIMATION}(C,2k,\varepsilon,f,\hat{O}^P,p) \) we obtain \( U, V \) such that

\[
\|UV - C\|_p \leq (1 + O(\varepsilon)) \min_{\text{rank } 2k} \|C_{2k} - C\|_p + O\left( \frac{\varepsilon}{f} \right) \|C\|_p \\
\leq (1 + O(\varepsilon))\|A_k - B\|_p + O\left( \frac{\varepsilon}{\text{poly}(k)} \right) \cdot \text{poly}(k) \|A - A_k\|_p \\
\leq (1 + O(\varepsilon))\|A_k - A\|_p + O(\varepsilon)\|A - A_k\|_p \\
\leq (1 + O(\varepsilon))\|A_k - A\|_p
\]

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where the second inequality is because $A_k - B$ has rank at most $2k$, and because $B$ is the result of a poly($k$)-approximation for $A$. Since $\|UV - C\|_p = \|UV - (A - B)\|_p = \|(UV + B) - A\|_p$, this completes the proof of correctness for Algorithm 6. Note that $UV + B$ has rank at most $3k$ because $UV$ has rank at most $2k$ and $B$ has rank at most $k$.

Now, we analyze the running time of Algorithm 6. First, the running time of Algorithm 8 (used to find $B$) is $2^{O(k \log k)} + \text{poly}(nd)$. The running time of GuessingAdditiveEpsApproximation is at most $f^{O(k)} + 2^{O(rk \log(k/\varepsilon))} + \text{poly}(f nd/\varepsilon)$, and since $f = \text{poly}(k)$, this is

$$2^{O(rk \log k)} + 2^{O(rk \log(k/\varepsilon))} + \text{poly}(nd/\varepsilon) = 2^{O(rk \log(k/\varepsilon))} + \text{poly}(nd/\varepsilon)$$

The number of times Algorithm 6 is called is $O((\log nd)/\varepsilon)$, meaning the overall running time of Algorithm 6 is also the above. All other steps in Algorithm 6 can be done in polynomial time. 

\[\square\]
A Hardness for $\ell_p$ Low Rank Approximation with Additive Error, based on [2]

A.1 Background: Small Set Expansion Hypothesis

The hardness proof in [2] proceeds by a reduction from the Small Set Expansion problem — our presentation of this problem follows that of [2].

Problem 2 (Small Set Expansion Problem - As Presented in [2]). Let $G = (V, E)$ be a regular graph. For any subset $S \subset V$, the measure of $S$ is defined to be $\mu(S) := |S|/|V|$. The distribution $G(S)$ over the vertices of $G$ is generated as follows: first a uniformly random vertex $x \in S$ is selected, then a uniformly random neighbor $y$ of $x$ is selected (as a sample of $G(S)$). For $S \subset V$, the expansion of $S$ is defined to be $\Phi_G(S) := \Pr_{y \sim G(S)}[y \notin S]$. Finally, for $\delta \in (0, 1)$, $\Phi_G(\delta) := \min_{S \subset V: \mu(S) \leq \delta} \Phi_G(S)$.

The Small Set Expansion Problem is as follows: given a graph $G = (V, E)$ and $\varepsilon, \delta \in (0, 1)$, the goal is to decide whether $\Phi_G(\delta) \leq \varepsilon$ or $\Phi_G(\delta) \geq 1 - \varepsilon$.

In other words, $\Phi_G(S)$ is the proportion of neighbors that $S$ has, which do not belong to $S$ — this can be considered as the “expansion” of $S$ — and $\Phi_G(\delta)$ is the smallest expansion among all subsets of $V$ which have at most $\delta|V|$ vertices. The Small Set Expansion Hypothesis is as follows:

Conjecture 1 (Small Set Expansion Hypothesis - Conjecture 1.3 of [36]). For any fixed $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that it is NP-hard to decide whether $\Phi_G(\delta) \leq \varepsilon$ or $\Phi_G(\delta) \geq 1 - \varepsilon$.

A.2 Background: Hardness Proof from [2] — $\ell_p$ Low Rank Approximation for $p \in (1, 2)$

Now, we summarize the reduction due to [2] from the Small Set Expansion Problem to $\ell_p$ low rank approximation for $p \in (1, 2)$. The reduction in Section 5 of [2] shows that given $k \in \mathbb{N}$ and $p \in (1, 2)$, and given a matrix $A \in \mathbb{R}^{n \times d}$, it is NP-hard to find an $O(1)$-approximation for the error from the best rank-$(k-1)$ approximation for $A$ in the $\ell_p$-norm. Ultimately, [2] reduces from Problem 2 to finding the best rank-$(k-1)$ approximation. The first step is to reduce from Problem 2 to computing the $2 \to q$ norm (in this subsection, $q$ denotes the Holder conjugate of $p$):

Theorem A.1 (Theorem 2.4 of [3] for $q \in 2\mathbb{Z} \setminus \{2\}$, and Theorem 21 of [2] for general $q \in (2, \infty)$). Let $G$ be a regular graph, $\lambda \in (0, 1)$ and $q \in (2, \infty)$. Let $M$ be the normalized adjacency matrix of $G$, and let $V_{\geq \lambda}(G)$ be the subspace spanned by the eigenvectors of $M$ with eigenvalue at least $\lambda$. Finally, let $P_{\geq \lambda}(G)$ be the orthogonal projection matrix onto $V_{\geq \lambda}(G)$. Then,

- For all $\delta > 0$, $\varepsilon > 0$, $\|P_{\geq \lambda}(G)\|_{2 \to q} \leq \varepsilon/\delta^{(q-2)/2q}$ implies that $\Phi_G(\delta) \geq 1 - \lambda - \varepsilon^2$.

- There is a constant $a = a(q)$ such that for all $\delta > 0$, $\Phi_G(\delta) > 1 - a\lambda^2$ implies $\|P_{\geq \lambda}(G)\|_{2 \to q} \leq 2/\sqrt{\delta}$.

This implies that the $2 \to q$ norm is hard to approximate — for completeness, we include the proof of this from [2]. In the next subsection, we show that this proof can be modified to obtain hardness for a multiplicative $O(1)$-approximation with additive $\frac{1}{2^{\sqrt{\log\log\log n}}}\|A\|_p$ error.

Theorem A.2 (Theorem 7 of [2]). Assuming the Small Set Expansion Hypothesis, for any $q \in (2, \infty)$, and $r > 1$, it is NP-hard to approximate the $\|\cdot\|_{2 \to q}$ norm within a factor $r$.

Proof. (From [2], Page 46). Using [37], the Small Set Expansion Hypothesis implies that for any sufficiently small numbers $0 < \delta \leq \delta'$, there is no polynomial time algorithm that can distinguish between the following cases for a given graph $G$:

- Yes case: $\Phi_G(\delta) < 0.1$

- No case: $\Phi_G(\delta') > 1 - 2^{-a'\log(1/\delta')}$
In particular, for all $\eta > 0$, if we let $\delta' = \delta^{(q-2)/8q}$ and make $\delta$ small enough, then in the No case $\Phi_G(\delta^{(q-2)/8q}) > 1 - \eta$. (Since $q > 2$, $\delta' \to 0$ as $\delta \to 0$.)

Using Theorem A.7 in the Yes case we know $\|P_{\geq 1/2}(G)\|_{2\rightarrow q} \geq 1/(10\delta^{(q-2)/2q})$, while in the No case, if we choose $\delta$ sufficiently small so that $\eta$ is smaller than $a(1/2)^{2q}$, then we know that $\|P_{\geq 1/2}(G)\|_{2\rightarrow q} \leq 2/\sqrt{\delta} = 2/(\delta^{(q-2)/4q})$. The gap between the Yes case and the No case is at least $\delta^{-(q-2)/4q}/20$, which goes to $\infty$ as $\delta$ decreases.

This proof shows that computing $\|P_{\geq 1/2}(G)\|_{2\rightarrow q}$ within a constant factor is sufficient to decide the Small Set Expansion Problem. Now (if $G$ is assumed to be a graph with $k$ vertices) the following steps are used to reduce the $2 \rightarrow q$ norm of the $k \times k$ matrix to the problem of rank-$(k-1)$ $\ell_p$ low rank approximation. For convenience, let $P := P_{\geq 1/2}(G)$.

First, computing the $2 \rightarrow q$ norm of $P$ is equivalent to computing the $2 \rightarrow q$ norm of some invertible matrix $P_1$ which can be constructed in poly($k$) time:

**Lemma A.3** (Claim 14 of [2]). Let $A$ be a nonzero $n \times d$ matrix. For any $p, q \in (1, \infty)$ and any $\varepsilon > 0$, there is an invertible and polynomial-time computable $\max(n, d) \times \max(n, d)$ matrix $B$ such that $(1-\varepsilon)\|A\|_{p\rightarrow q} \leq \|B\|_{p\rightarrow q} \leq (1+\varepsilon)\|A\|_{p\rightarrow q}$.

The proof of the above lemma proceeds as follows: first, $A$ is made into a square matrix by adding rows/columns of zeros — then, $\frac{\varepsilon}{\|M\|_{0\rightarrow q}}$ is added to every diagonal entry to make it invertible, where $M$ is the absolute value of the largest entry of $A$. The next step, after obtaining the invertible matrix $P_1$, is to compute a matrix $P_2$ such that finding $\|P_2\|_{p\rightarrow q}$ allows us to find $\|P_1\|_{2\rightarrow q}$.

**Lemma A.4** (Claim 13 of [2]). For any $A \in \mathbb{R}^{n \times d}$ and $p \in (1, \infty)$, if $p^*$ is its Holder conjugate, then $\|AA^T\|_{p\rightarrow p^*} = \|A\|_{2\rightarrow p^*}^2$.

Hence, by this claim, the appropriate $P_2$ will simply be $P_1P_1^T$. Finally, it is useful to reduce this problem to computing $\min_{p\rightarrow q}(P_3)$ of some well-chosen matrix $P_3$, since $\min_{p\rightarrow q}$ was shown by [2] to in fact be equivalent to rank-$(k-1)$ $\ell_p$-low rank approximation:

**Lemma A.5** (Fact 4 of [2]). For $p, q \in (1, \infty)$, if $A$ is an invertible matrix, then $\min_{p\rightarrow q}(A^{-1}) = (\|A\|_{p\rightarrow q})^{-1}$.

**Lemma A.6** (Lemma 1/Lemma 27 of [2] - Equivalent of $\min_{p\rightarrow p^*}$ and Rank-$(k-1)$ $\ell_p$ Low Rank Approximation). Let $p \in (1, \infty)$ and $p^*$ be the Holder conjugate of $p$. Let $A \in \mathbb{R}^{n \times d}$ with $n \geq d$ and $k = d - 1$. Then,$$
\min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \|UV - A\|_p = \min_{x \in \mathbb{R}^d \mid \|x\|_{p^*} = 1} \|Ax\|_p
$$

Hence, $P_3$ is in fact $P_3^{-1}$, and the $2 \rightarrow q$ norm of $P_{\geq 1/2}(G)$ can be approximated up to a constant factor if $\min_{p\rightarrow p^*}(P_3) = \min_M \text{rank}_k \|M - P_3\|_p$ can be approximated up to a constant factor. In [2], it is also mentioned that assuming the Exponential-Time Hypothesis [22] together with SSEH implies that the running time required to obtain an $O(1)$-approximation is $2^{k^{O(1)}}$ — this is also true for our hardness result, as we mention in the proof of Theorem A.7.

### A.3 Hardness for $\ell_p$ Low Rank Approximation with Additive Error, $p \in (1, 2)$

Now, we show that the same reduction as [2] can in fact be used to show the following somewhat stronger hardness result:

**Theorem A.7** (Hardness for $\ell_p$ Low Rank Approximation with Additive Error). Suppose the SSEH holds. Then, for $p \in (1, 2)$, it is NP-hard to achieve the following guarantee: given a matrix $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$, find a matrix $\hat{A} \in \mathbb{R}^{n \times d}$ of rank at most $k$ such that

$$\|\hat{A} - A\|_p \leq O(1) \min_{A_k \text{ rank } k} \|A - A_k\|_p + \frac{1}{2^{\text{poly}(k)}} \|A\|_p$$

Hence, as mentioned in [2], if we assume the Exponential Time Hypothesis (due to [22]) along with the SSEH, then achieving this guarantee takes at least $2^{ck}$ time for some $c > 0$. 

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It is worth noting that Algorithm 5 can in fact be used to obtain this guarantee in $2^{\text{poly}(k)} + \text{poly}(nd)$ time, by letting $f = 2^{\text{poly}(k)}$ (instead of letting $f = \text{poly}(k)$) as is done when Algorithm 4 is called by Algorithm 0.

Proof. The main idea is that while performing the reduction due to [2], we can ensure that the matrix $A$ has entries with at most $\text{poly}(k)$ bits. As a result of this, we show that the above guarantee is in fact equivalent to achieving an $O(1)$-approximation. Throughout this proof, we let $q$ denote the H"older conjugate of $p$.

As in the reduction of [2], let $G$ be a regular graph that has $k$ vertices, so that $P := P_{\leq 1/2}(G)$ is a $k \times k$ matrix. Note that we can write $P = U U^T$, where $U \in \mathbb{R}^{k \times t}$ (for some $t \in \mathbb{N}$) has orthonormal columns. Note that the entries of $U$ have absolute value at most 1, meaning the entries of $P$ have absolute value at most $k$ by the triangle inequality. In addition, the entries of $P$ can be rounded to the nearest integer multiple of $\frac{1}{D}$, where $D = \text{poly}(k)$ is a sufficiently large power of 2, without significantly changing $\|P\|_{2 \to q}$. We can see this as follows. Let $\hat{P}$ be $P$ with its entries rounded to the nearest integer multiple of $\frac{1}{D}$. Then,

$$\|P\|_{2 \to q} - \|\hat{P}\|_{2 \to q} \leq \|P - \hat{P}\|_{2 \to q} \leq \|P - \hat{P}\|_{2} \leq \|P - \hat{P}\|_{F} \leq \frac{k}{D}$$

(18)

Here, the first inequality is by the triangle inequality. The second is because $\|x\|_q \leq \|x\|_2$ for any $x \in \mathbb{R}^n$, and the third is because the spectral norm is at most the Frobenius norm. Finally, the last inequality is because $\|P - \hat{P}\|_{\infty} \leq \frac{1}{D}$, and $P$ and $\hat{P}$ are $k \times k$ matrices.

Now, we must simply choose a large enough $D$ so that we can distinguish between the Yes and No cases of the Small Set Expansion Problem using $\|\hat{P}\|_{2 \to q}$, where the Yes and No cases are as specified in the proof of Theorem A.2. As mentioned in the proof of Theorem A.2 in the Yes case,

$$\|P\|_{2 \to q} \geq 1/(10 \delta^{(q-2)/2q}) := C_1$$

while in the No case

$$\|P\|_{2 \to q} \leq 2/\delta^{(q-2)/4q} := C_2$$

(where $\delta \in (0,1)$ can be any sufficiently small number). Hence, it suffices to choose $D \geq 50k \delta^{(q-2)/4q}$. To see why, note that in the No case,

$$\|\hat{P}\|_{2 \to q} \leq \|P\|_{2 \to q} + \frac{k}{D} \leq \frac{2}{\delta^{(q-2)/4q}} + \frac{1}{50 \delta^{(q-2)/4q}} \leq \frac{3}{\delta^{(q-2)/4q}}$$

while in the Yes case,

$$\|\hat{P}\|_{2 \to q} \geq \|P\|_{2 \to q} - \frac{k}{D} \geq \frac{1}{10 \delta^{(q-2)/2q}} - \frac{1}{50 \delta^{(q-2)/4q}} \geq \frac{1}{20 \delta^{(q-2)/2q}}$$

where the last inequality holds because $\frac{1}{2} > 1$ and $(q - 2)/2q > (q - 2)/4q$, meaning $\frac{1}{20 \delta^{(q-2)/2q}} \geq \frac{1}{50 \delta^{(q-2)/4q}}$. Note that the gap between the Yes and No case is still

$$\frac{1}{(20 \delta^{(q-2)/2q})} = \frac{1}{60 \delta^{(q-2)/4q}}$$

and this can be made arbitrarily large as $\delta \to 0$, meaning computing $\|\hat{P}\|_{2 \to q}$ within an arbitrary constant factor is NP-hard. We now reduce the problem of computing $\|\hat{P}\|_{2 \to q}$ to that of computing the optimal rank-$(k-1)$ approximation of a well-chosen matrix where each entry has at most $\text{poly}(k)$ bits.

First, observe that each entry of $\hat{P}$ in fact has at most $O(\log k)$ bits in its numerator and denominator, since each entry is at most $k$ in absolute value, and each entry has denominator $D$, which has $O(\log k)$ bits. Now, we show that each of the steps in the reduction of [2] preserves the bit complexity of $\hat{P}$,
that none of the $P_i$ mentioned in the previous subsection will have more than poly$(k)$ bits. First, we can construct $P_1$ so that $(1 - \varepsilon)\|\hat{P}\|_{2\rightarrow q} \leq \|P_1\|_{2\rightarrow q} \leq (1 + \varepsilon)\|\hat{P}\|_{2\rightarrow q}$ by Lemma A.3 — recall that this can be done by adding $\varepsilon M$ to each diagonal entry of $\hat{P}$, where $I$ is the identity matrix. Since we are only interested in a constant-factor approximation to $\|\hat{P}\|_{2\rightarrow q}$, we can let $\varepsilon = \Theta(1)$. In addition, recall that $M$ has at most $O(\log k)$ bits in both its numerator and denominator (where $M$ is the largest entry of $\hat{P}$). Finally, $\|I\|_{2\rightarrow q} = 1$ since $q > 2$, meaning $\|x\|_q \leq \|x\|_2$ for all $x$. Hence, all entries of $P_1$ have at most $O(\log k)$ bits in the numerator and denominator.

Furthermore, recall that all entries of $\hat{P}$ are integer multiples of $\frac{1}{q}$, meaning all entries of $P_1$ are as well. Hence, we can compute $P_2 = P_1 P_1^T$, and all of the entries of $P_2$ are at most poly$(k)$ and are integer multiples of $\frac{1}{q}$. meaning all entries of $P_2$ have at most $O(\log k)$ bits in their numerators and denominators. Note that by Lemma A.3 $\|P_2\|_{p\rightarrow q} = \|P_1\|_{2\rightarrow q}^p$, meaning in order to get a constant-factor approximation to $\|P_1\|_{2\rightarrow q}$ it suffices to get a constant-factor approximation to $\|P_2\|_{p\rightarrow q}$.

Finally, we compute $P_3 = P_2^{-1}$ as described in the previous subsection. Each entry of $P_3$ is a cofactor of $P_2$, divided by the determinant of $P_2$. Each cofactor is the sum of at most $k!$ terms, each of which is a product of $(k - 1)$ entries of $P_2$. Similarly, the determinant of $P_2$ is the sum of $k!$ products of $k$ entries of $P_2$. Since each entry of $P_2$ is an integer multiple of $\frac{1}{q}$, each of these products has a denominator of $D^k$, which is at most $2^{\text{poly}(k)}$ (meaning it has poly$(k)$ bits). The numerator of each of these products has absolute value at most $k! \cdot \text{poly}(k)$, meaning it has at most poly$(k)$ bits. In summary, for each entry of $P_3$, its numerator and denominator have at most poly$(k)$ bits.

So far, we have found a matrix $A := P_3 \in \mathbb{R}^{k \times k}$ such that all of its entries have at most poly$(k)$ bits in their numerators and denominators, such that computing a constant-factor approximation for $\|\hat{P}\|_{2\rightarrow q}$ reduces to computing a constant-factor approximation to $\min_{M_{k-1}} \text{rank} (k-1) \|M_{k-1} - A\|_p$. To show the desired result, we then simply need to show the following claim:

**Claim A.8.** $\frac{1}{2^{\text{poly}(k)}} \|A\|_p \leq \text{OPT} := \min_{M_{k-1}} \text{rank} (k-1) \|M_{k-1} - A\|_p$

**Proof.** First, note that we can assume without loss of generality that the entries of $A$ are integers that are at most $2^{\text{poly}(k)}$. To see this, let $L$ denote the least common multiple of the denominators of all of the entries of $A$. Then, since $A$ is a $k \times k$ matrix, $L$ is at most $2^{\text{poly}(k)}$, since the denominators of the entries of $A$ are also at most $2^{\text{poly}(k)}$. Hence, we can multiply through by $L$ — note that the desired claim is scale-invariant.

We now use an argument similar to Claim 1 on page 15 of [2]. First note that $A$ has full rank since it is invertible. Now, note that if $\sigma_k$ is the $k^{\text{th}}$ singular value of $M$, then $\|M_k - A\|_F \geq \sigma_k$ since $\sigma_k$ is the optimal Frobenius norm rank-$(k-1)$ error for $A$. Hence, $\|M_k - A\|_p \geq \|M_k - A\|_F \geq \sigma_k$.

Consider the invertible $k \times k$ matrix $A^T A$ — it has integer entries, meaning its characteristic polynomial has integer coefficients. The last term of the characteristic polynomial of $A^T A$ is $\prod_{i=1}^{k} \sigma_i^2$, where $\sigma_i$ is the $i^{\text{th}}$ singular value of $A$. Since $A$ has full rank and the characteristic polynomial of $A^T A$ has integer coefficients, this means $\prod_{i=1}^{k} \sigma_i^2 \geq 1$. On the other hand, for each $i \in [k]$, $\sigma_i \leq \|A\|_F \leq \text{poly}(k)$ (since $A$ is a $k \times k$ matrix). Hence, $\sigma_k^2 \geq \frac{1}{2^{\text{poly}(k)}}$, meaning $\sigma_k \geq \frac{1}{2^{\text{poly}(k)}}$.

In summary, $\text{OPT} \geq \sigma_k \geq \frac{1}{2^{\text{poly}(k)}}$, while $\|A\|_p \leq 2^{\text{poly}(k)}$ since the entries of $A$ are at most $2^{\text{poly}(k)}$ — hence, $\|A\|_p \leq 2^{\text{poly}(k)} \text{OPT}$, and this completes the proof of the claim.

Hence, if we find $\hat{A}$ of rank at most $k - 1$ such that

$$\|\hat{A} - A\|_p \leq O(1) \min_{M_{k-1}} \text{rank} (k-1) \|M_{k-1} - A\|_p + \frac{1}{2^{\text{poly}(k)}} \|A\|_p$$

then the claim above in fact shows that $\|\hat{A} - A\|_p \leq O(1) \min_{M_{k-1}} \text{rank} (k-1) \|M_{k-1} - A\|_p$ — this is sufficient for obtaining a constant-factor approximation to $\|\hat{P}\|_{2\rightarrow q}$, and hence deciding the Small Set Expansion instance.

Now, suppose we assume the Exponential-Time Hypothesis (ETH) in addition to the Small Set Expansion Hypothesis — ETH is the assumption that any algorithm for solving 3-SAT takes at least $2^{\Omega(n)}$ time, where $n$ is the number of variables in the 3-SAT instance. Since SSEH is the assumption that the Small Set Expansion Problem is NP-hard, suppose that there is a reduction from 3-SAT to Small Set Expansion that takes an instance with $n$ variables to a graph with at most $m = n^\frac{1}{2}$ vertices for some $c > 0$ (since the reduction takes
polynomial time, it creates a graph with size at most $\text{poly}(n)$. Hence, assuming SSEH and ETH, there is no algorithm which can decide any Small Set Expansion instance in $2^{o(m^c)}$ time, since this would imply that any 3-SAT instance could be decided in $2^{o(n)}$ time.

A.4 Hardness for Constrained $\ell_1$ Low Rank Approximation with Additive Error

The hardness results of [2] and our above hardness result do not apply to $\ell_1$ low rank approximation. Intuitively, this is because the reduction of [2] ultimately shows that finding the best rank-$(d-1)$ approximation to an $n \times d$ matrix is NP-hard, assuming the Small Set Expansion Hypothesis. However, finding the best rank-$(d-1)$ subspace can actually be done in polynomial time when $p = 1$ [7, 40].

On the other hand, it is also possible to show a similar hardness result for a somewhat more general version of $\ell_1$ low rank approximation:

Problem 3 (Constrained $\ell_1$ Low Rank Approximation). Given a matrix $A \in \mathbb{R}^{n \times d}$ and a subspace $V \subset \mathbb{R}^n$, find a matrix $\hat{A}$ of rank $k$ minimizing $\|\hat{A} - A\|_1$, such that the columns of $\hat{A}$ are in $V$.

Notice that $\ell_1$ low rank approximation is the special case of this problem where $V = \mathbb{R}^n$. We show the hardness of constrained $\ell_1$ low rank approximation by reducing from $\ell_p$ low rank approximation. We again assume the Small Set Expansion Hypothesis — since the reduction from $\ell_p$ low rank approximation to constrained $\ell_1$ low rank approximation is randomized, it is also useful to assume the following randomized version of ETH used in [15], which can be considered a stronger version of $\text{BPP} \neq \text{NP}$:

Conjecture 2 (Randomized ETH [15]). There is a constant $c > 0$ such that no randomized algorithm can decide 3-SAT in time $2^{cn}$ with error probability at most $\frac{1}{3}$.

Finally, the following is the main tool in our reduction:

Theorem A.9 (Embedding $\ell_p$ into $\ell_1$ - Restatement of Theorem 1 of [24]). Let $\tau > 0$, and let $p \in (1, 2)$. Moreover, let $n, m \in \mathbb{N}$. Then, there exists a family of random matrices $R \in \mathbb{R}^{m \times n}$ such that, if $m \geq \beta_p, \tau n$, then for all $x \in \mathbb{R}^n$,

$$ (1 - \tau)\|x\|_p \leq \|Rx\|_1 \leq (1 + \tau)\|x\|_p $$

with probability greater than $\frac{1}{2}$. Here, $\beta_p, \tau$ is a constant depending only on $p$ and $\tau$.

We can take $\tau = \Theta(1)$ since we only need to preserve the error by a constant factor. Hence, we arrive at the following result:

Theorem A.10 (Hardness for Constrained $\ell_1$ Low Rank Approximation with Additive Error). Suppose the SSEH holds, and assume $\text{NP} \not\subset \text{BPP}$. Then, for $k \in \mathbb{N}$, the following guarantee cannot be achieved in polynomial time, with constant probability: given a matrix $A \in \mathbb{R}^{n \times d}$, $k \in \mathbb{N}$ and $Y \in \mathbb{R}^{n \times t}$ for some $t \leq n$, find a matrix $\hat{A} \in \mathbb{R}^{n \times d}$ of rank at most $k$ such that

$$ \|\hat{A} - A\|_1 \leq O(1) \min_{A_k} \|A_k - A\|_1 + \frac{1}{2^{\text{poly}(k)}} \|A\|_1 $$

and the columns of $\hat{A}$ are contained in the column span of $Y$ — here, the minimum is taken over $A_k$ having rank $k$, such that the columns of $A_k$ are contained in the column span of $Y$. Moreover, if we assume Randomized ETH, along with SSEH, then achieving this guarantee takes at least $2^{k^c}$ time for some $c > 0$.

Proof. For convenience, let $p = \frac{3}{2}$. Let $A \in \mathbb{R}^{n \times d}$ and $k \in \mathbb{N}$. By Theorem A.9 there exists a random matrix $R \in \mathbb{R}^{m \times n}$, for some $m = \Theta(n)$, such that for all $x \in \mathbb{R}^n$,

$$ \Omega(1)\|x\|_p \leq \|Rx\|_1 \leq O(1)\|x\|_p $$

with constant probability. (Note that the number of rows of $R$ is $\Theta(n)$ since we are taking $\tau = \Theta(1)$ and $p = \frac{3}{2}$, meaning $\beta_p, \tau = \Theta(1)$). Now, suppose we could find $M \in \mathbb{R}^{n \times d}$, with rank at most $k$, such that

$$ \|RM - RA\|_1 \leq O(1) \min_{A_k} \|RA_k - RA\|_1 + \frac{1}{2^{\text{poly}(k)}} \|RA\|_1 \tag{19} $$
where the minimum is taken over all \( A_k \) of rank at most \( k \). Note that this is a solution to the constrained \( \ell_1 \) low rank approximation problem, that achieves the guarantee described in the statement of this problem (replacing \( A \) with \( RA \), and the subspace \( Y \in \mathbb{R}^{n \times t} \) with \( R \)).

However, Equation \( \text{[19]} \) implies, with constant probability, that

\[
\| M - A \|_p \leq O(1) \min_{A_k \text{ rank } k} \| A_k - A \|_p + \frac{1}{2 \text{poly}(k)} \| A \|_p
\]

Recall that achieving this guarantee is NP-hard, meaning \( NP \subset BPP \) if an \( M \) satisfying Equation \( \text{[19]} \) can be obtained in polynomial time, with constant probability.

Finally, we show that \( 2^{\Omega(k^c)} \) running time is required to achieve this guarantee with constant probability, for some constant \( c > 0 \). To do this, first observe that, assuming randomized ETH, at least \( 2^{\Omega(k^c)} \) running time is required to decide the Small Set Expansion Problem with constant probability, since there is a polynomial-time reduction from 3-SAT to the Small Set Expansion Problem assuming SETH holds. Since achieving the guarantee in Equation \( \text{[19]} \) is sufficient for deciding the Small Set Expansion Problem, this means achieving this guarantee with constant probability takes at least \( 2^{\Omega(k^c)} \) time.

\( \square \)

Note that Algorithm \( \text{[2]} \) can be modified to work for constrained \( \ell_1 \) low rank approximation, with the same running time guarantee. The only modification to the algorithm itself would be that, once the right factor \( V' \) is obtained, then \( U \) is set to be \( \arg\min_y \| YUV' - YA \|_1 \) instead of \( \arg\min\| U V' - A \|_1 \), where \( Y \) is the subspace to which the low-rank solution is constrained. The only change that will be made to the analysis in the introduction and in Theorem \( \text{[3.15]} \) is that \( U^* \) will be the optimal rank-\( k \) factor whose columns are contained in the span of \( Y \), rather than simply being the optimal rank-\( k \) factor.

### B \( \ell_p \) Column Subset Selection Lower Bound

We use a construction similar to \( \text{[41]} \), to show that column subset selection algorithms cannot give an approximation factor better than \( O(k^{\frac{p}{p+1}}) \) for \( \ell_p \)-low rank approximation, for \( 1 < p < 2 \), if \( k \cdot \text{polylog}(k) \) columns are chosen. The hard distribution for \( \ell_1 \) column subset selection from \( \text{[41]} \), which we use here for \( \ell_p \) column subset selection, is as follows: \( A \in \mathbb{R}^{(k+n) \times n} \) is a random matrix where each of the first \( k \) rows has i.i.d. \( N(0,1) \) entries, and the remaining \( n \times n \) submatrix is the identity matrix. We show that if \( n = k \cdot \text{poly}(\log k) \), then any subset of \( r \leq k \cdot \text{poly}(\log k) \) columns cannot give an approximation factor better than \( O(k^{\frac{p}{p+1}}) \), for sufficiently large \( k \), unless \( n - r = o(n) \). The proofs in this section follow those of the analogous lemmas in Section G.3 of \( \text{[41]} \) with slight modifications — for each of our lemmas, we note the corresponding lemma in \( \text{[41]} \). First, we state some definitions.

**Definition B.1.** Suppose \( \beta, \gamma \in \mathbb{R} \) and \( p \in (1,2) \). Then, we define

\[
Y_{\beta, \gamma, p} = \left\{ y \in \mathbb{R}^n \mid \| y \|_p \leq O(\gamma), |y_i| \leq \frac{1}{k^\beta} \leq 1 \forall i \in [n] \right\}
\]

**Definition B.2.** (Similar to Definition G.19 of \( \text{[41]} \)) Let \( V \in \mathbb{R}^{n \times r} \) have orthonormal columns, and \( A \in \mathbb{R}^{k \times r} \) be a random matrix, for which each entry is i.i.d. \( N(0,1) \). Then, we define the event \( \tilde{E}(A, V, \beta, \gamma, p) \) as follows: \( \forall y \in Y_{\beta, \gamma, p}, AV^T y \) has at most \( O(\frac{1}{\log k}) \) coordinates with absolute value at least \( \Omega(\frac{1}{\log k}) \), and \( \| A \|_2 = O(\sqrt{r}) \).

**Lemma B.3** (Gaussian Matrices Have Small Operator Norm - Due to \( \text{[39]} \), as stated in \( \text{[41]} \)). Let \( A \in \mathbb{R}^{r \times k} \) be a random matrix for which each entry is i.i.d. \( N(0,1) \). With probability at least \( 1 - e^{-\Theta(r)} \), the maximum singular value of \( A \) is at most \( O(\sqrt{r}) \).

**Lemma B.4** (Flat Vectors Stay Flat - Similar to Lemma G.20 of \( \text{[41]} \)). Suppose \( k \geq 1 \), \( c_2 \geq c_1 \geq 1 \), and \( k \leq r = O(k(\log k)^{c_2}) \), \( r \leq n = O(k(\log k)^{c_1}) \). Let \( V \in \mathbb{R}^{n \times r} \) have orthonormal columns, and let \( A \in \mathbb{R}^{k \times r} \) be a random matrix with each entry being i.i.d. \( N(0,1) \). Finally, suppose \( \beta, \gamma \in \mathbb{R} \) such that \( 0 < \gamma \leq p \gamma < \beta(2 - p) \leq \beta \). Then,

\[
Pr[\tilde{E}(A, V, \beta, \gamma, p)] \geq 1 - 2^{-\Theta(k)}
\]
Proof. The proof is the same as that of Lemma 20 in [41] — it uses a net argument and a union bound. Rather than constructing a net for all of \( Y_{\beta,\gamma,p} \), the coordinates of a point \( y \in Y_{\beta,\gamma,p} \) are first divided between points \( y^j \) with disjoint supports, such that \( y^j \) has coordinates between \( \frac{1}{2^j} \) and \( \frac{1}{2^j} \). Then, for each \( j \), an \( \varepsilon \)-net \( \mathcal{N}_j \) (for a suitable \( \varepsilon \)) is constructed for all points of the same form as \( y^j \), with coordinates between \( \frac{1}{2^j} \) and \( \frac{1}{2^j} \). It is shown that with high probability, all points in \( \mathcal{N}_j \) have at most \( O(k/\log^2 k) \) coordinates which are at least \( \Omega\left(\frac{1}{\log k}\right) \) — if this occurs, then this implies that \( \hat{E}(A,V,\beta,\gamma,p) \) occurs. Moreover, it is only necessary to consider \( O(\log k) \) distinct values of \( j \) — for sufficiently large values of \( j \geq j^* \), \( AV^Ty^j \) makes a very small contribution to the coordinates of \( AV^Ty \), and in fact, \( \sum_{j \geq j^*} AV^Ty^j \) makes a very small contribution to the coordinates of \( AV^Ty \) (the proof of this last statement is where Lemma 3.3 is used).

We now begin the proof. Let \( Y_{\beta,\gamma,p} \) be as in Definition 1.1 and take \( y \in Y_{\beta,\gamma,p} \). As mentioned before, write \( y = \sum_{j=j_0}^{\infty} y^j \), for \( j \geq j_0 \), \( y^j \) has coordinates in the interval \([\frac{1}{2^j}, \frac{1}{2^j}]\). Note that the \( y^j \) have disjoint supports, and we can take \( j_0 \) so that \( \frac{1}{2^j_0} = \frac{1}{k^2} \) by the definition of \( Y_{\beta,\gamma} \). Let \( s_j \) be the support size of \( y^j \). Then,

\[
\left(\frac{s_j}{2^{j(j+1)}}\right) \frac{1}{\log k} \leq |y^j|_p \leq |y|_p \leq O(k^\gamma)
\]

meaning

\[
s_j \leq O(2^{p(j+1)}k^{p\gamma})
\]

In addition, it will later be useful to bound from above \(|y^j|_2^2\):

\[
|y^j|_2^2 \leq \frac{1}{2^{2j}} \cdot s_j \leq O(k^{p\gamma} \cdot 2^{p(j+1)-2})
\]

We use this to construct an \( \varepsilon \)-net \( \mathcal{N}_j \) for points of the same form as \( y^j \), and bound its size:

\[
\mathcal{N}_j := \{ x \in \mathbb{R}^n : \exists x' \in \mathbb{Z}^n, x = \varepsilon x', |x|_p \leq O(k^\gamma), \forall i \in [n], \text{ either } \frac{1}{2^j} \leq |x_i| < \frac{1}{2^j} \text{ or } x_i = 0 \}
\]

In other words, it is the integer grid, but contracted by a factor of \( \varepsilon \), and with coordinates and \( \ell_p \)-norm in the same range as \( x \). This is an \( \varepsilon \)-net in the \( \ell_\infty \) norm, meaning it is an \( \varepsilon \sqrt{n} \)-net in the \( \ell_2 \) norm. We will choose \( \varepsilon = O\left(\frac{1}{\log k}\right) = O\left(\frac{1}{k^{1+\varepsilon/2}}\right), \) meaning \( \mathcal{N}_j \) is an \( O\left(\frac{1}{k^{\varepsilon/2}}\right) \)-net for the \( \ell_2 \) norm. Now, we bound the size of \( \mathcal{N}_j \) — for each coordinate \( x_i \), since it is between \( \frac{1}{2^j} \) and \( \frac{1}{2^j} \), the net has a granularity of \( \varepsilon \), the number of choices for \( x_i \) is \( 1 + \frac{1}{2^j} \leq 1 + \frac{\varepsilon}{k^2} \) since \( \frac{1}{2^j} \leq \frac{\varepsilon}{k^2} \leq 2 \). The number of coordinates of \( y^j \) is at most \( n = O(k(\log k)^{\varepsilon/2}) \), meaning

\[
|\mathcal{N}_j| \leq \left(1 + \frac{2}{\varepsilon}\right)^{O(k(\log k)^{\varepsilon/2})} \leq 2^{O(k(\log k)^{\varepsilon/2} \log \frac{1}{\varepsilon})} \leq 2^{O(k(\log k)^{\varepsilon/2+1})}
\]

(20)

In preparation for the union bound over points in \( \mathcal{N}_j \), we consider the event \( E(y^j) \) that for a single \( y^j \), \( AV^Ty^j \) has at least \( O\left(\frac{k}{\log^2 k}\right) \) coordinates which are at least \( O\left(\frac{1}{\log^2 k}\right) \) in absolute value. Notice that a single coordinate of \( AV^Ty^j \) is \( N(0, |V^T y^j|_2^2) \) since the rows of \( A \) are i.i.d. \( N(0,1) \). Hence, the probability \( q \) that a particular coordinate of \( AV^Ty^j \) is greater than \( O\left(\frac{1}{\log^2 k}\right) \) is, by properties of the Gaussian distribution, at most

\[
\exp\left(-\frac{1}{2 |V^T y^j|_2^2} \cdot \frac{1}{\log^2 k}\right)
\]

since the probability that a Gaussian random variable \( N(0, \sigma^2) \) has absolute value greater than \( t \) is at most \( e^{O(-\frac{t^2}{\sigma^2})} \). Hence, letting \( i_0 = O\left(\frac{1}{\log^2 k}\right) \), the probability that \( AV^Ty^j \) has at least \( O\left(\frac{1}{\log^2 k}\right) \) coordinates greater
\[
\sum_{i=1}^{k} q^i (1 - q)^i \binom{k}{i} \leq k^2 q^k \\
\leq k^2 \exp \left( -\frac{i}{\|V^T y\|^2} \cdot \frac{1}{\log^4 k} \right) \\
\leq k^2 \exp \left( -\Theta \left( \frac{k}{\|V^T y\|^2 \log^6 k} \right) \right) \\
\leq k^2 \exp \left( -\Theta \left( \frac{2p^{-(2-p)} j \cdot \log^6 k}{k^1-p^\gamma} \right) \right) \\
\leq k^2 \exp \left( -\Theta \left( \frac{k^1+p^\gamma(2-p)-p^\gamma}{\log^6 k} \right) \right) \exp \left( -\Theta \left( \frac{k^1-p^\gamma}{\log^6 k} \right) \right) \\
\leq \exp \left( -\Theta \left( \frac{k^1-p^\gamma}{2p^{-(2-p)} j \cdot \log^6 k} \right) \right)
\]

The first inequality is because \((1-q)^i \leq 1, \binom{k}{i} \leq k^i\), and there are at most \(k\) summands. The third inequality is because \(i \geq i_0 = \Theta \left( \frac{k}{\log^4 k} \right)\). The fourth inequality is because \(\|V^T y\|^2_2 \leq \|V^T\|^2_2 \|y\|^2_2 \leq O(k^{p\gamma} \cdot 2^{(p+1) - 2j})\) because \(V\) has orthonormal columns and because of the upper bound on \(\|y\|^2_2\) mentioned above. The fifth inequality is because \(2^{(2-p)j} \geq 2^{(2-p)j_0} = \frac{1}{k^{p\gamma}}\). Finally, the sixth inequality is because \(\beta(2-p) > p\gamma\), meaning \(k^{\beta(2-p)-p^\gamma}/(\log^6 k) = o(1)\), and \(k^2\) is absorbed in the remaining exponential — this is where we use the hypothesis that \(p\gamma < (2-p)\beta\).

Now we show that we can ignore the effect of \(y^j\) for \(j \geq \Theta(\log k)\) — we then perform a union bound over the remaining \(O(\log k)\) nets \(\mathcal{N}_j\) to show that \(E(y^j)\) holds for all \(y^j\) in those \(\mathcal{N}_j\). In particular, as in [41], let \(j_1 = \lceil 100(c_1 + c_2 + 1) \log k \rceil\). Then, for \(j \geq j_1\), \(\|\sum_{j=j_1}^{\infty} y^j\|_2 \leq \Theta(2^{-j_1} \sqrt{m}) \leq \frac{1}{k^{100(1+c_1)}}\), since all coordinates of \(\sum_{j=j_1}^{\infty} y^j\) are at most \(\frac{1}{2m}\) and the \(y^j\) have disjoint supports. Hence,

\[
\|AV^T y^j\|_\infty \leq \|AV^T y^j\|_2 \\
\leq \|A\|_2 \|V^T\|_2 \|y^j\|_2 \\
\leq O(\sqrt{r}) \cdot \frac{1}{k^{100(1+c_2)}} \\
\leq O(\frac{\sqrt{k}(\log k)^{\epsilon_1/2}}{k^{100(1+c_1)}}) \\
\leq \frac{1}{k^{100}}
\]

where the third inequality is because \(A\) has top singular value at most \(O(\sqrt{r})\) and \(V\) has orthonormal columns. Hence, if \(y^{j_1-j_1} := \sum_{j=j_0}^{j_1} y^j\), then \(y^j\) has at least \(O(\frac{k}{\log k})\) coordinates with absolute value at least \(O(\frac{1}{\log k})\), if and only if this is the case for \(y^{j_1-j_1}\).

Finally, by a union bound over \(O(\log k)\) nets \(\mathcal{N}_{j_0, \ldots, j_1}\), we can show that for all \(j\) between \(j_0\) and \(j_1\), with high probability, \(AV^T y^j\) has at most \(O(k/\log^2 k)\) coordinates with absolute value greater than or equal to \(O(1/\log^2 k)\) — therefore, outside of a set of \(O(\log k) \cdot O(k/\log^2 k) = O(k/\log k)\) coordinates, each coordinate of \(AV^T y^j\) is at most \(O(\log k) \cdot O(1/\log^2 k) = O(1/\log k)\).

First, the probability that there exists \(y^j \in \mathcal{N}_j\) for some \(j\) between \(j_0\) and \(j_1\), such that \(E(y^j)\) occurs, is
\[ P\left[ \exists y^j \in \bigcup_{j=\hat{j}}^{j_1} \mathcal{N}_j, \mathcal{E}(y^j) \text{ happens} \right] \leq \sum_{j=\hat{j}}^{j_1} |\mathcal{N}_j| \cdot \exp \left( -\Theta \left( \frac{k^1 \cdot p^\gamma}{2^p - (2-p)j \log^b k} \right) \right) \]
\[ \leq \sum_{j=\hat{j}}^{j_1} 2^{O(k(\log k)^{2+1})} \cdot \exp \left( -\Theta \left( \frac{k^1 \cdot p^\gamma \cdot 2^{(2-p)j}}{\log^b k} \right) \right) \]
\[ \leq O(\log k) \cdot 2^{O(k(\log k)^{2+1})} \cdot \exp \left( -\Theta \left( \frac{k^1 \cdot p^\gamma \cdot k^{\beta(2-p)}}{\log^b k} \right) \right) \]
\[ \leq O(\log k) \cdot 2^{-\Theta(k)} \leq 2^{-\Theta(k)} \]

The first inequality above is by a union bound and our upper bound on the probability of \( \mathcal{E}(\hat{y}^j) \) for an individual \( \hat{y}^j \) in \( \mathcal{N}_j \). The third inequality is because \( 2^j \geq k^\beta \). The fourth inequality is because \( \beta(2-p) - p\gamma > 0 \), meaning that \( k\text{poly}(\log k) = o(k^{1-p\gamma + \beta(2-p)}/\log^b k) \).

Finally, given a vector \( y^j \) with each entry having absolute value in \( \left[ \frac{1}{\sqrt{T}}, \frac{1}{T} \right] \) such that \( \|y^j\|_p \leq k^\gamma \) and each entry is at most \( \frac{1}{k^r} \), there exists \( \hat{y}^j \in \mathcal{N}_j \) such that \( \|y^j - \hat{y}^j\|_2 \leq O(\frac{1}{\sqrt{kr}}) \). Hence, assuming that \( \mathcal{E}(\hat{y}^j) \) does not occur for any \( \hat{y}^j \) in \( \mathcal{N}_j \) for \( j \) between \( \hat{j} \) and \( j_1 \),

\[
\|AV^Ty^j - AV^T\hat{y}^j\|_\infty \leq \|AV^Ty^j - AV^T\hat{y}^j\|_2 \\
\leq O(\sqrt{T})\|y^j - \hat{y}^j\|_2 \\
\leq O\left(\frac{\sqrt{T}}{k^{c_1+c_2/2+3}}\right) \\
\leq O\left(\frac{1}{k^c}\right)
\]

since \( A \) has operator norm \( O(\sqrt{T}) \) and \( V^T \) has operator norm \( \leq 1 \). Hence, with probability \( 1 - 2^{-\Theta(k)} \), for all \( y \in Y_{\beta,\gamma,p} \), and all \( j \) between \( \hat{j} \) and \( j_1 \), \( AV^Ty^j \) has at most \( O(k/\log k) \) coordinates with absolute value more than \( O(1/\log k) \).

In summary, as discussed above, this implies that with probability \( 1 - 2^{-\Theta(k)} \), for all \( y \in Y_{\beta,\gamma,p} \), \( AV^Ty \) has at most \( O(k/\log k) \) coordinates with absolute value more than \( O(1/\log k) \). This completes the proof.

**Lemma B.5** (Paying with Either Regression Cost or Norm - Similar to Lemma G.22 of [41]). For any \( t, k \geq 1 \), and any constants \( c_2 \geq c_1 \geq 1 \), let \( k \leq r = O(k(\log k)^{c_1}) \), and \( r \leq n = O(k(\log k)^{c_2}) \). Let \( V \in \mathbb{R}^{n \times r} \) be a matrix with orthonormal columns. For an arbitrary constant \( \alpha \in (0, 0.5) \), if \( A \in \mathbb{R}^{k \times r} \) such that \( \hat{E}(A, V, \frac{1+\alpha}{2}, \frac{1}{p} - \frac{1}{2} - \alpha, p) \) holds, then with probability \( 1 - 2^{-\Theta(k)} \), there are at least \( \left\lceil \frac{t}{10} \right\rceil \) such \( j \in [t] \) that \( \forall x \in \mathbb{R}^r \), either \( \|Ax - v_j\|_p \geq \Omega(k^{-\frac{p}{2} - \frac{1}{2} - \alpha}) \) or \( \|Vx\|_p \geq \Omega(k^{-\frac{p}{2} - \frac{1}{2} - \alpha}) \).

**Remark B.6.** The meaning of this lemma is that when a subset \( A_S \) of columns of the hard instance \( A \in \mathbb{R}^{(k+n)\times n} \) is chosen, then there exists a large enough subset of the remaining columns such that for each of those columns \( v_j \), the regression coefficient vector \( y_j \) used to fit those \( v_j \) (using \( A_S \) as the left factor) either leads to a large regression error on the top \( k \) rows, or has a large norm, in which case it leads to a large regression cost on the bottom \( n \) rows, which are simply the identity matrix.

**Proof.** The proof is similar to that of the analogous Lemma G.22 of [41]. Here we use a simplified version of that argument, that was given to us by Peilin Zhong.

We show the desired statement using a net argument. For convenience, let \( \gamma = \frac{1}{p} - \frac{1}{2} - \alpha \), and let \( \beta = \frac{1+\alpha}{2} \). Then, note that \( p\gamma = 1 - \frac{p}{2} - p\alpha \), which is less than \( \beta(2-p) = 1 - \frac{p}{2} + \alpha \cdot \frac{2-p}{2} \). Hence, by Lemma B.4 \( \hat{E}(A, V, \frac{1+\alpha}{2}, \frac{1}{p} - \frac{1}{2} - \alpha, p) \) holds with probability \( 1 - 2^{-\Theta(k)} \). For a particular \( x \in \mathbb{R}^r \), if we let \( y := Vx \), meaning \( x = V^Ty \), then the statement becomes equivalent to showing that there are at least \( \left\lceil \frac{t}{10} \right\rceil \) indices \( j \in [t] \) such that \( \forall y \in \mathbb{R}^n \), either \( \|y\|_p \geq \Omega(k^\gamma) \), or \( \|AV^Ty - v_j\|_p \geq \Omega(k^\gamma) \). Throughout this
Let $B_{\gamma,p}$ denote the $\ell_p$-ball of radius $O(k^{\gamma})$ with center 0. Then, we must show that for all $y \in B_{\gamma,p}$, $\|AV^Ty - v\|_p \geq \Omega(k^{\gamma})$ with high probability. For each such $y$, we can write it as $y_0 + y_1$, where all the coordinates of $y_0$ are either 0 or greater than $\frac{1}{k^{\gamma}}$ in absolute value, all the coordinates of $y_1$ are at most $\frac{1}{k^{\gamma}}$, and the supports of $y_0$ and $y_1$ are disjoint. Recall that this means $y_1 \in Y_{\beta,\gamma,p}$ as defined above. Since $\hat{E}(A,V,\beta,\gamma,p)$ holds, for each $y \in B_{\gamma,p}$, its corresponding $y_1$ is such that $AV^Ty_0$ has at most $O(k/\log k)$ coordinates that are at least $O(1/\log k)$ in absolute value.

We first build a net for all possible $y_0$. For convenience, let $T_{\beta,\gamma,p}$ be the set of all possible $y_0$, i.e. $T_{\beta,\gamma,p} := \{x \in \mathbb{R}^n \mid \forall i \in [n], \text{ either } |x_i| \geq \frac{1}{k^{\gamma}} \text{ or } x_i = 0\}$. Then, for any $\varepsilon > 0$, the following is an $\varepsilon$-net in the $\ell_\infty$ norm for $T_{\beta,\gamma,p}$ (and hence an $\varepsilon\sqrt{n}$-net in the $\ell_2$ norm):

$$N := \{x \in \mathbb{R}^n \mid \exists x' \in \mathbb{Z}^n, x = \varepsilon x', \|x\|_p \leq O(k^{\gamma}), \forall i \in [n], \text{ either } |x_i| \geq \frac{1}{k^{\gamma}} \text{ or } x_i = 0\}$$

Let us calculate the size of $|N|$. Note that if $s$ is the support size of $y_0$, then $\frac{1}{k^{\gamma}}s \leq \|y_0\|_p \leq \|y\|_p \leq k^{\gamma}$. Moreover, for each coordinate of $x \in N$, the number of choices is at most $O(\frac{k^n}{s})$ (since each coordinate is at most $k^{\gamma}$ and has a granularity of $\varepsilon$). Hence, if we let $\varepsilon = O(1/rnk^3)$ as in [41], then

$$|N| \leq \left(\frac{n}{s}\right) \cdot (k^{\gamma}/\varepsilon)^{O(s)} = 2^{O(s)\log k} = 2^{k^{\beta+\gamma}p \log k}$$

since $n = O(k(\log k)^{c_2})$.

We will perform a union bound over all $y_0 \in N$, to show that the failure event $E_1(y_0)$ does not occur, where $E_1(y_0)$ is the event that for some $y_1 \in Y_{\beta,\gamma,p}$ with disjoint support from $y_0$, if $y := y_0 + y_1$, then $\|AV^Ty - v\|_p \leq O(k^{\gamma})$. First, let $E_2(y_0)$ be the event that $AV^Ty_0 - v$ has at most $O(k/\log k)$ coordinates that are at least $O(1/\log k)$ in absolute value.

Claim B.7 (Similar to Claim G.23 of [41]). Assume $\hat{E}(A,V,\beta,\gamma,p)$ holds. Then, for all $y_0 \in T_{\beta,\gamma,p}$, $E_1(y_0)$ implies $E_2(y_0)$.

Proof. The proof is similar to that of Claim G.23 of [41]. Suppose $E_1(y_0)$ occurs, meaning there is $y_1 \in Y_{\beta,\gamma,p}$ such that if $y := y_0 + y_1$, then $\|AV^Ty - v\|_p \leq O(k^{\gamma})$. Let $s$ be the number of coordinates of $AV^Ty - v$ that are greater than $\frac{1}{k^{\gamma}}$ in absolute value — then, $\frac{s}{\log k} \leq \|AV^Ty - v\|_p \leq k^{\gamma}$, meaning $s \leq k^{\gamma} \log^p k$, and this is $o(k/\log k)$, since $p \gamma < p \cdot (\frac{1}{p} - \frac{1}{2}) < 1$.

In addition, observe that

$$AV^Ty_0 - v = (AV^Ty - v) - AV^Ty_1$$

since $y = y_0 + y_1$. Since $\hat{E}(A,V,\beta,\gamma,p)$ occurs, $AV^Ty_1$ has at most $O(k/\log k)$ coordinates that are greater than $\frac{1}{k^{\gamma}}$ in absolute value, by the previous lemma. Hence, since $AV^Ty - v$ has at most $o(k/\log k)$ coordinates that are greater than $\Omega(1/\log k)$ in absolute value, $AV^Ty_0 - v$ also has at most $O(k/\log k)$ coordinates that are greater than $\Omega(1/\log k)$ in absolute value, meaning $E_2(y_0)$ holds.

Now, we show that the probability of $E_2(y_0)$ is small, where the randomness is over the entries of a vector $v \in \mathbb{R}^k$ with i.i.d. $N(0,1)$ entries.

Claim B.8. Let $z \in \mathbb{R}^k$, and let $v$ be a $k$-dimensional vector with i.i.d. $N(0,1)$ entries. Then, with probability $1 - 2^{-\Theta(k)}$, there exist at most $O(k/\log k)$ coordinates $i \in [k]$ such that $|z_i - v_i| \leq O(1/\log k)$.

Proof. Let $i \in [k]$. Then,

$$\Pr[|z_i - v_i| \leq O(1/\log k)] = \int_{v_i - O(1/\log k)}^{v_i + O(1/\log k)} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\leq O(1/\log k)$$

(25)

Hence, if $Z_i$ is 1 if $|z_i - v_i| \leq O(1/\log k)$ and 0 otherwise, and $Z = \sum Z_i$ is the number of coordinates on which the difference between $z$ and $v$ is at most $O(1/\log k)$, then $E[Z] \leq O(k/\log k)$. Since the $Z_i$ are independent, by a Chernoff bound, with probability $1 - 2^{-\Theta(k)}$, $Z \leq O(k/\log k)$. This proves the claim. □
By applying the claim above with $z = AV^Ty_0$ for any particular $y_0 \in \mathbb{R}^n$, the probability that $AV^Ty_0 - v$ has at least $\frac{1}{10}$ coordinates on which the difference is at most $O(1/\log k)$ (meaning $AV^Ty_0 - v$ has less than $\frac{9k}{10}$ coordinates on which the difference is greater than $O(1/\log k)$) is at most $2^{-\Theta(k)}$ — since this implies $\mathcal{E}_2(y_0)$, the probability of $\mathcal{E}_2(y_0)$ is also at most $2^{-\Theta(k)}$. Now, we union bound over all such $y_0 \in \mathcal{N}$ with high probability:

$$\Pr[\exists y_0 \in \mathcal{N}, \mathcal{E}_2(y_0)] \leq |\mathcal{N}| \cdot 2^{-\Theta(k)} \leq 2^{k(\beta + \gamma)p \log k} 2^{-\Theta(k)} \leq 2^{-\Theta(k)}$$

(26)

Here we use the hypothesis that $\beta + \gamma < \frac{1}{p}$, meaning $k(\beta + \gamma)p = o(k)$.

The above argument shows that for all $y_0 \in \mathcal{N}$, with probability at least $1 - 2^{-\Theta(k)}$, the event $\mathcal{F}$ holds that there does not exist $y_1 \in Y_{\beta, \gamma, p}$ such that $\|AV^Ty_1 - v\|_p \leq O(k')$ where $y_1 = y_0 + y_1$. Assume that $\mathcal{F}$ holds. Now, suppose $y_0 \in T_{\beta, \gamma, p}$, and there exists $y_1 \in Y_{\beta, \gamma, p}$ such that, if $y = y_0 + y_1$, then $\|AV^Ty - v\|_p \leq O(k')$. Let $\hat{y}_0 \in \mathcal{N}$ such that $\|\hat{y}_0 - y_0\|_2 \leq \varepsilon \sqrt{n} = O(1/r \sqrt{n}k^3)$. Then, letting $\hat{y} := \hat{y}_0 + y_1$, we obtain

$$\|AV^T\hat{y} - v\|_p \leq \|AV^Ty - v\|_p + \|AV^T(\hat{y} - y)\|_p \leq O(k') + \|AV^T\hat{y} - \hat{y}\|_2 \leq O(k') + \|AV^T\hat{y} - \hat{y}\|_2 \leq O(k')$$

Here, the first inequality is by the triangle inequality, and the second is because $\hat{y}_0 \in \mathcal{N}$, meaning $\|AV^T\hat{y} - v\|_p \leq O(k')$. Finally, because of our choice of $\hat{y}_0$ as the net vector closest to $y_0$, and because $k\frac{\sqrt{\delta}}{\sqrt{n}k^3} \leq \sqrt{K}$, we obtain $\|AV^T\hat{y} - v\|_p \leq O(k') + \sqrt{K} \sqrt{\frac{\sqrt{\delta}}{\sqrt{n}k^3}} = O(k')$ since $\|A\|_2 \leq O(\sqrt{r})$, and $\|V\|_2 \leq 1$.

Therefore, if for all $y_0 \in \mathcal{N}$, there does not exist a corresponding $y_1$ for which $y := y_0 + y_1$ satisfies $\|AV^Ty - v\|_p$, then the same holds for all $y_0 \in T_{\beta, \gamma, p}$. This means that, with probability $1 - 2^{-\Theta(k)}$ over the entries of $v$, for all $y_0 \in T_{\beta, \gamma, p}$, there does not exist $y_1 \in Y_{\beta, \gamma, p}$ with disjoint support from $y_0$ such that $\|AV^Ty - v\|_p \leq O(k')$, meaning for all $y \in B_{\gamma, p}$, $\|AV^Ty - v\|_p \geq \Omega(k')$. We have shown that for any $j \in [t]$, the event $\mathcal{E}_3(v_j)$ holds with probability $1 - 2^{-\Theta(k)}$, where $\mathcal{E}_3(v_j)$ is the event that with probability $1 - 2^{-\Theta(k)}$, for all $y \in \mathbb{R}^n$, either $\|y\|_p \geq \Omega(k')$ or $\|AV^Ty - v_j\|_p \geq \Omega(k')$.

Note that the $v_j$ are independent for different $j$. Hence, we can show that $\mathcal{E}_3(v_j)$ holds simultaneously for $\Omega(t)$ indices $j \in [t]$ with the desired probability, as follows. Let $Z_j = 1$ if $\mathcal{E}_3(v_j)$ does not hold. Then, the probability that at least $\frac{9t}{10}$ of the $Z_j$ are equal to 1 is at most

$$\sum_{d=\frac{9t}{10}}^{t} \binom{t}{d} (2^{-\Theta(k)})^d \leq O(t) \cdot 2^{-\Theta(t)} = 2^{-\Theta(tk)}$$

meaning that with probability $1 - 2^{-\Theta(tk)}$, $\mathcal{E}_3(v_j)$ holds for at least $\frac{t}{10}$ indices $j \in [t]$.

Now we prove the main result of this appendix:

**Theorem B.9 (Lower Bound for $\ell_p$ Low Rank Approximation through Column Subset Selection - Similar to Theorem G.28 of [11]).** For a sufficiently large $k \in \mathbb{N}$, and any constant $c \geq 1$, let $n = O(k(\log k)^{c})$ and let $M \in \mathbb{R}^{(k+n) \times n}$ be a random matrix such that the top $k \times n$ submatrix has i.i.d. $N(0, 1)$ entries, and the bottom $n \times n$ submatrix is the identity matrix. Then, with probability $1 - 2^{-\Theta(k)}$, for any subset $S \subset [n]$ with $|S| \leq \frac{n}{2} =: r$,

$$\min_{X \in \mathbb{R}^{r \times k}} \|M_S X - M\|_p \geq \Omega(k^{\frac{3}{2}} - \frac{\alpha}{\sqrt{n}}) \min_{M_{k, \text{rank } k}} \|M_k - M\|_p$$

where $\alpha \in (0, \frac{1}{p} - \frac{1}{2})$ can be arbitrary.

**Proof.** Throughout the proof, fix $\alpha \in (0, \frac{1}{p} - \frac{1}{2})$, $\gamma = \frac{1}{p} - \frac{1}{2} - \alpha$, and $\beta = \frac{1}{2} - \alpha$. In addition, let $A \in \mathbb{R}^{k \times n}$ be the submatrix of $M$ consisting of its top $k$ rows. The proof of this theorem follows that of Theorem G.28 of [11]: we apply Lemma B.3.1 with $A_S$ in the place of $A$, to show that each subset $S$ of size at most $r$ incurs a large cost, then perform a union bound over all such $S$. For convenience, we consider the $p^{th}$ power of the $\ell_p$ norm throughout the proof.
First, notice that \( \min_{M} \text{rank}_k M_k - M \|_p \leq n \), since we could take \( M_k \) to be the \((k + n) \times n\) matrix whose first \( k \) rows are the same as those of \( M \), and whose last \( n \) rows are 0. Hence, it suffices to show that for any subset \( S \subset [n] \) of size at most \( r \), \( \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p \geq k^n p^n n \). First, fix a subset \( S \subset [n] \) with \( |S| \leq r \). Observe that the cost of fitting a single column \( M_l \) of \( M \) using \( M_S \) is

\[
\text{cost}(S, l) := \min_{x_l \in \mathbb{R}^r} \left( \| A_S x_l - A_l \|_p + \| x_l \|_p - 1 \right)
\]

(where the notation \( \text{cost}(S, l) \) is from the proof of Theorem G.28 in [41]). This is because the first \( k \) entries of \( M_l \) are given by \( A_l \), and one of the last \( n \) entries of \( M_l \) is 1 and the others are 0. We can apply Lemma B.10 now, as follows. Suppose \( \hat{\mathcal{E}}(A_S, I_r, \beta, \gamma, p) \) occurs. Then, since the columns of \( A \) are independent, and \( A_I \) is independent from \( A_S \) for \( l \not\in S \), by Lemma B.15 with probability \( 1 - 2^{-\Theta(nk)} \), for at least \( \Omega(n) \) of the indices \( l \in [n] \setminus S \), either \( \| A_S x_l - A_l \|_p \geq \Omega(k^n p^n) \) or \( \| x_l \|_p \geq \Omega(k^n p^n) \) (since while applying that lemma, we can take \( n = r \) and the matrix \( V \in \mathbb{R}^{n \times r} \) to be \( I_r \), which has orthonormal columns — note that we are taking \( t = n \) in that lemma).

\[
\text{cost}(S, l) \geq \Omega(n \cdot k^n p^n) \geq \Omega(k^n p^n \cdot \text{OPT})
\]

Now, instead of conditioning on \( \hat{\mathcal{E}}(A_S, I_r, \beta, \gamma, p) \) simultaneously for all \( S \), we can simply condition on \( \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \):

**Claim B.10** (Similar to Claim G.29 of [41]), \( \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \) implies \( \hat{\mathcal{E}}(A_S, I_r, \beta, \gamma, p) \).

**Proof.** In the proof of this claim, we use \( Y_{\beta, \gamma, p, t} \) to denote the instance of \( Y_{\beta, \gamma, p} \) in \( \mathbb{R}^t \). Suppose \( \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \) occurs. Then, for all \( y \in Y_{\beta, \gamma, p, n} \subset \mathbb{R}^n \), \( \hat{A} y \) has at most \( O(k/\log k) \) coordinates with absolute value at least \( \Omega(1/\log k) \). In particular, for any subset \( S \subset [n] \), let \( y_S \in \mathbb{R}^n \) be a point whose support is contained in \( S \), and let \( \tilde{y}_S = y_S \), with all coordinates not indexed by \( S \) removed. Observe that \( A_S I_r \tilde{y}_S = A_I y_S \), and since \( \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \) holds, \( A_I y_S \) has at most \( O(k/\log k) \) coordinates with absolute value at least \( \Omega(1/\log k) \).

This implies that for any \( y \in Y_{\beta, \gamma, p, r} \), \( A_S I_r y \) has at most \( O(k/\log k) \) coordinates with absolute value at least \( \Omega(1/\log k) \). Moreover, since \( A_S \) is a subset of columns of \( A \), the operator norm of \( A_S \) is at most \( \|A\|_2 \leq \sqrt{n} = O(\sqrt{n}) \), since \( r = \frac{n}{2} \).

Hence, we can perform a union bound over all subsets \( S \subset [n] \) with \( |S| \leq r \), while conditioning on \( \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \). For convenience, let \( \tilde{\mathcal{E}} = \hat{\mathcal{E}}(A, I_n, \beta, \gamma, p) \). Then,

\[
\begin{align*}
\Pr\left[ \exists S \subset [n], |S| \leq r \text{ s.t. } \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p < O(k^n p^n \cdot \text{OPT}) \right] \\
= \Pr\left[ \exists S \subset [n], |S| \leq r \text{ s.t. } \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p < O(k^n p^n \cdot \text{OPT}) \right] \cdot \Pr[\tilde{\mathcal{E}}] \\
+ \Pr\left[ \exists S \subset [n], |S| \leq r \text{ s.t. } \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p < O(k^n p^n \cdot \text{OPT}) \right] \cdot \Pr[-\tilde{\mathcal{E}}] \\
\leq \Pr\left[ \exists S \subset [n], |S| \leq r \text{ s.t. } \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p < O(k^n p^n \cdot \text{OPT}) \right] \cdot \Pr[\tilde{\mathcal{E}}] \\
+ \Pr[-\tilde{\mathcal{E}}] \\
\leq \sum_{S \subset [n], |S| \leq r} \Pr\left[ \min_{X \in \mathbb{R}^{n \times n}} \| M_S X - M \|_p < O(k^n p^n \cdot \text{OPT}) \right] \cdot \Pr[\tilde{\mathcal{E}}] + 2^{-\Theta(k)} \\
\leq \sum_{S \subset [n], |S| \leq r} 2^{-\Theta(nk)} + 2^{-\Theta(k)} \\
= \left( \frac{n+1}{r+1} \right) 2^{-\Theta(nk)} + 2^{-\Theta(k)} \\
\leq 2^{O(r \log n)} 2^{-\Theta(nk)} + 2^{-\Theta(k)} \\
= 2^{O(n \log k)} 2^{-\Theta(nk)} + 2^{-\Theta(k)} \\
= 2^{-\Theta(k)}
\end{align*}
\]
Here, the first inequality is because probabilities are at most 1. The second is because \( \tilde{E} \) occurs with probability at least \( 1 - 2^{-\Theta(k)} \), and by a union bound over \( S \cap [n] \) with \(|S| \leq r\). The third is because of our observation above, that \( \tilde{E} \) implies \( \tilde{E}(A_S, I_r, \beta, \gamma, p) \), and if \( A_S \) is such that \( \tilde{E}(A_S, I_r, \beta, \gamma, p) \) holds, then the probability that \( \min_{X \in \mathbb{R}^{r \times n}} \| M_S X - M \|_F^2 < O(k^p \cdot OPT) \) is at most \( 2^{-\Theta(nk)} \). After that, the second equality is because the number of subsets of \([n]\) of size at most \( r \) is \( \binom{n+1}{r+1} \).

In summary, the probability that there is a subset achieving error less than \( O(k^p \cdot OPT) \) is at most \( 2^{-\Theta(k)} \).

\[ \square \]

\[ \text{C} \quad \text{poly}(k)\text{-Approximation Algorithms with Bicriteria Rank Independent of } n \text{ and } d \]

\[ \text{C.1 A poly}(k)\text{-Approximation with poly}(k) \text{ Bicriteria Rank} \]

We can remove the \( O(\log d) \)-factor in the bicriteria rank of Algorithm 4 at the cost of an increase in the approximation factor to \( O(k^{\frac{1}{3}} \text{poly}(\log k)) \). This works as follows: note that Algorithm 4 selects columns in \( O(\log d) \) blocks, each having size \( r \) (which is \( O(k \log k) \) if \( p = 1 \) and \( O(k \log k \log \log k) \) if \( p \in (1, 2) \)) — out of these \( O(\log d) \) blocks, we show that there exists a subset of blocks of size \( r \) (hence giving a column subset of size \( r^2 \)) which spans an \( O(k^{\frac{1}{3}} \text{poly}(\log k)) \)-approximation to \( A \). An advantage of this algorithm is that both the rank and the approximation factor are polynomial in \( k \), and do not depend on \( n \) and \( d \).

\begin{algorithm}
\caption{Obtaining a \( \text{poly}(k) \)-approximation of rank \( O(r^2) \), where \( r = O(k \log k) \) if \( p = 1 \) and \( O(k \log k \log \log k) \) if \( p \in (1, 2) \). This algorithm simply finds the left factor, and the right factor can be obtained using linear programming.}
\begin{algorithmic}
\REQUIRE \( A \in \mathbb{R}^{n \times d}, k \in \mathbb{N}, p \in [1, 2) \)
\ENSURE \( U \in \mathbb{R}^{n \times O(r^2)} \)
\PROCEDURE {PolyKErrorAndRankApproximation} \((A, k, p)\)
\STATE \( S_1, S_2, \ldots, S_b \leftarrow \text{RandomColumnSubsetSelection}(A, k, p) \) (Note that the output \( S \) is being discarded, since all we need are the blocks. Also note that the number of blocks \( b \) is \( O(\log d) \).)
\STATE \( U \leftarrow \text{The } n \times O(r^2) \text{ zero matrix} \)
\STATE \( \text{MINERROR} \leftarrow 0 \)
\IF \( b \leq \frac{C}{r} \)
\STATE // Here \( C \) is a sufficiently large absolute constant. In this case \text{RandomColumnSubsetSelection} \( \text{already gives } O(k^2 \log^2 k/\varepsilon) \text{ columns.} \)
\STATE \( S \leftarrow \bigcup_{i=1}^{p} S_i \)
\STATE \( U \leftarrow A_S \)
\ELSE
\FOR \( I \subseteq [b], |I| = r \)
\STATE \( T \leftarrow \bigcup_{i \in I} S_i \)
\STATE \( U_{\text{temp}} \leftarrow AT \)
\STATE \( V_{\text{temp}} \leftarrow \arg\min_{V \in \mathbb{R}^{r \times d}} \| U_{\text{temp}} V - A \|_p \)
\IF \( \| U_{\text{temp}} V_{\text{temp}} - A \|_p \leq \text{MINERROR} \)
\STATE \( \text{MINERROR} \leftarrow \| U_{\text{temp}} V_{\text{temp}} - A \|_p \)
\STATE \( U \leftarrow U_{\text{temp}} \)
\ENDIF
\ENDFOR
\ENDIF
\ENDPROCEDURE
\end{algorithmic}
\end{algorithm}

\textbf{Theorem C.1} (Column Subset Selection - Removing Dependence on \( \log d \)). Let \( A \in \mathbb{R}^{n \times d}, p \in [1, 2), \) and \( k \in \mathbb{N} \). In addition, let \( r = O(k \log k) \) if \( p = 1 \) and \( r = O(k \log k \log \log k) \) otherwise. Then, with constant
If the number of blocks $b < \frac{C}{\epsilon}$ (where $C$ is a sufficiently large absolute constant mentioned in Algorithm 7 and will be chosen appropriately later), then \textsc{RandomColumnSubsetSelection} already returns a subset of $O(r^2/\varepsilon)$ columns.

Otherwise, $r \leq \frac{cb}{\varepsilon}$, and Algorithm 7 checks all subsets of $[b]$ of size $r$. In this case, the number of subsets checked is

$$\binom{b}{r} \leq \left(\frac{b}{b/\varepsilon}\right) \leq 2^H_2(\varepsilon)b = d^{O(1)H_2(\varepsilon)} \leq d^\varepsilon.$$
where the first inequality holds as long as $r \leq \frac{\ell}{\Omega} \leq \frac{b}{2}$, and in the second inequality $H_2$ is the binary entropy function. The first equality is because $b = O(\log d)$, meaning $2^b = d^{O(1)}$. Finally, the fourth inequality is because we can choose $C$ to be sufficiently large.

Hence, the number of subsets checked is at most $d^c$ if $C$ is chosen to be sufficiently large. For each subset, we perform multiple-response $\ell_p$-regression. We can speed up this $\ell_p$-regression as follows. Let $S \in \mathbb{R}^{O(k \log d) \times n}$ be a sampling and rescaling matrix generated according to the Lewis weights of $A_T$ (recall that $T$ is the subset of columns returned by Algorithm 4). Recall that by Lemma 2.2, with probability $1 - O(1)$, for all $V \in \mathbb{R}^{|T|^d}$,

$$\|SA_T V - SA\|_p \geq (\Omega(1))\|A_T V - A\|_p - O(1)\|A_T V^* - A\|_p$$

where $V^* = \text{argmin}_V\|A_T V' - A\|_p$.

Now, out of all the subsets tried by Algorithm 7, let $U_{temp}^*$ be the optimal subset of blocks, and let $U_{temp}$ be an arbitrary subset of blocks that is tried. We make the following definitions: $V_{temp}^* = \text{argmin}_V\|U_{temp} V - A\|_p$, $V_{temp,S}^* = \text{argmin}_V\|SU_{temp} V - SA\|_p$, and $V_{temp,S} = \text{argmin}_V\|SU_{temp} V - SA\|_p$. Then, since $U_{temp}$ and $U_{temp}^*$ are both given by subsets of columns of $A$, we can apply Equation 29. Applying it to $U_{temp}$ gives us

$$\|SU_{temp} V_{temp,S} - SA\|_p \geq (\Omega(1))\|U_{temp} V_{temp,S} - A\|_p - O(1)\|A_T V^* - A\|_p$$

Now, let $U_{temp}^*$ be the subset of blocks tried by Algorithm 7, which minimizes the sketched error, i.e. it is the subset $U_{temp}$ minimizing $\|SU_{temp} V_{temp,S} - SA\|_p$. Define $V_{temp,S}^* = \text{argmin}_V\|SU_{temp}^* V - SA\|_p$ Then,

$$\|SU_{temp} V_{temp,S} - SA\|_p \geq \|SU_{temp} V_{temp,S} - SA\|_p$$

$$\geq \|SU_{temp}^* V_{temp,S} - SA\|_p$$

$$\geq (\Omega(1))\|U_{temp}^* V_{temp,S} - A\|_p - O(1)\|A_T V^* - A\|_p$$

where the first inequality is because $V_{temp,S}^*$ is the minimizer of the sketched error for the left factor $U_{temp}$, the second is because $U_{temp}^*$ is the subset of blocks minimizing the sketched error, and the third is by Equation 30.

Finally, by Lemma 2.2 with probability $1 - O(1)$,

$$\|SU_{temp} V_{temp}^* - SA\|_p \leq (\Omega(1))\|U_{temp} V_{temp}^* - A\|_p$$

with probability $1 - O(1)$.

Putting this together, if $U_{temp}^*$ is the subset of blocks minimizing the sketched $\ell_p$ regression error, and $V_{temp,S}^*$ is its corresponding right factor (that minimizes the sketched $\ell_p$ regression error), then with probability $1 - O(1)$,

$$\|U_{temp}^* V_{temp,S}^* - A\|_p \leq (\Omega(1))\|U_{temp} V_{temp}^* - A\|_p + O(1)\|A_T V^* - A\|_p$$

$$\leq (O(r^{\frac{1}{\beta^2}}(\log k)^{\frac{1}{\beta}}))^{\frac{1}{p}} \min_{k} \|A - A_k\|_p$$

where the second inequality is because we showed above that $U_{temp}$ provides an $O(r^{\frac{1}{\beta^2}}(\log k)^{\frac{1}{\beta}})$-approximation, while $A_T$ provides an $O(r^{\frac{1}{\beta^2}}(\log k)^{\frac{1}{\beta}})$-approximation by our analysis of Algorithm 4.

Hence, by taking $p^{th}$ powers, we find that performing multiple-response $\ell_p$ regression for each of the subsets of blocks, while reusing a single sampling and rescaling matrix $S$ generated using the $\ell_p$ Lewis weights of $A_T$, only worsens the approximation guarantees by an $O(1)$ factor.

Now we finish analyzing the running time when Algorithm 7 is implemented using $\ell_p$ Lewis weights. Computing the $\ell_p$ Lewis weights of $A_T$, and generating the sampling matrix $S$, takes at most $\text{nnz}(A) + \text{poly}(k)$ time, by Lemma 2.1 and performing the multiplication $SA$ takes time at most $\text{nnz}(A)$. Now, for each of the $d^c$ subsets we try, we perform multiple-response $\ell_p$ regression on the $\tilde{O}(k \log d) \times d$ matrix $SA$, fitting the columns of $SA$ using a $\tilde{O}(k \log d) \times r^2$ matrix $SU_{temp}$. Each of the $d$ $\ell_p$-regression steps takes time $\text{poly}(k \log d)$. Therefore, each multiple-response $\ell_p$ regression step takes $d \text{poly}(k \log d)$ time, and overall, trying all subsets takes $d^{3+\epsilon} \text{poly}(k \log d)$ time. This completes the proof. \qed
Remark C.2. Note that in [10], it was shown that every \( A \in \mathbb{R}^{n \times d} \) has a subset of \( k \) columns spanning an \( O(k) \)-approximation to the optimum. Using this in our analysis, instead of Theorem 2.4 allows us to get an \( O(r^{\frac{p}{2}} \cdot \log k) \)-approximation instead (in particular, for \( p = 1 \), this is an \( O(k^{\frac{3}{2}}) \)-approximation) while giving us a rank \( O(rd) \)-solution, in particular removing one \( \log(k) \) factor from the rank in the \( p = 1 \) case.

Remark C.3. In the analysis of our algorithm with rank at most \( k \), we will take \( \varepsilon \) from Theorem C.4 to just be a sufficiently small constant to optimize the running time, but to also keep the rank at most \( O(k^2 \log^2 k) \).

Remark C.4. Note that this kind of analysis is not applicable to any bi-criteria column subset selection algorithm, and it uses special properties of Algorithm 4 — given an arbitrary bi-criteria column subset selection algorithm, we cannot use this analysis to reduce the rank. The key property is that for each column that is discarded by Algorithm 4, it can be fit using at most \( r \) columns belonging to the left factor that Algorithm 4 returns. Note that Algorithm 1 of [44] also has this property, and hence it can also be used to obtain a poly(\( k \))-approximation algorithm with poly(\( k \)) bicriteria rank, albeit with a somewhat larger approximation factor.

C.2 Reducing the Rank to At Most \( k \)

We can combine our algorithm from the previous subsection with Algorithm 4 of [10] to give a poly(\( k \))-approximation algorithm that returns a matrix of rank at most \( k \), with running time \( 2^{O(k \log k)} + \text{poly}(nd) \).

Algorithm 8 Obtaining a poly(\( k \))-approximation of rank \( k \). As before, \( r = O(k \log k) \) if \( p = 1 \) and \( O(k \log k \log log k) \) if \( p \in (1, 2) \). This algorithm is essentially Algorithm 4 of [10], but rather than using our own Algorithm 4 or one of its previously studied variants [44] as the initialization (i.e. instead of using it to provide the initial \( U \) and \( V \) in the first two steps), we use Algorithm 7 for which the bi-criteria rank is poly(\( k \)) and has no dependence on \( \log d \).

Require: \( A \in \mathbb{R}^{n \times d} \), \( k \in \mathbb{N} \), \( p \in [1, 2] \)
Ensure: \( W \in \mathbb{R}^{n \times k} \), \( Z \in \mathbb{R}^{k \times d} \)

procedure POLYKERRORNOTBICRITERIAAPPROXIMATION(\( A, k, p \))

\( U \leftarrow \text{POLYKERRORANDRANKAPPROXIMATION} (A, k, p) \)
\( V \leftarrow \arg \min_{V'} \| UV' - A \|_p \)
\( W_0 \leftarrow A \) well-conditioned basis for the span of \( U \), with \( O(r^2) \) distortion
\( Z_0 \leftarrow \arg \min_{V'} \| W_0 V' - A \|_p \)
\( Y \leftarrow A \) \( k \times d \) matrix consisting of the size \( k \) subset of rows of \( Z_0 \), which minimizes the error incurred when fitting all rows of \( Z_0 \)
\( X \leftarrow \arg \min_{X \in \mathbb{R}^{r \times k}} \| U'Y - Z_0 \|_p \) (the best left factor for \( Y \))
\( W \leftarrow W_0 X \in \mathbb{R}^{n \times k} \)
\( Z \leftarrow Y \in \mathbb{R}^{k \times d} \)
end procedure

The algorithm is shown in Algorithm 8. First, we give Algorithm 7 as the initialization. The rest of the algorithm is Algorithm 4 from [10], which is designed to take any bi-criteria approximation and reduce its rank to exactly \( k \) without significantly increasing the approximation factor. The reason that Algorithm 4 from [10] does not provide a poly(\( k \))-approximation, and instead provides a poly(\( k \log d \))-approximation, is that it takes a previously studied variant of our Algorithm 4 as the initialization — however, the bicriteria rank of that initialization is \( O(k \log d) \), and hence, the distortion from the well-conditioned basis (see Lemma C.6 below) is \( O(k \log d) \), which leads to the presence of \( \log d \) in the approximation factor. This is clarified in Appendix E of [10].

Theorem C.5 (Analysis of Algorithm 8). Let \( A \in \mathbb{R}^{n \times d} \), and \( k \in \mathbb{N} \). Then, with constant probability, Algorithm 8 returns \( W \in \mathbb{R}^{n \times k} \) and \( Z \in \mathbb{R}^{k \times d} \) such that

\[ \| WZ - A \|_p \leq O(k^\frac{3}{2} \cdot \| \log k \|_2) \cdot \min_{A_k \text{ rank } k} \| A_k - A \|_p \]

where \( r = O(k \log k) \) if \( p = 1 \) and \( r = O(k \log k \log k) \) if \( p \in (1, 2) \). The running time of Algorithm 8 is \( 2^{O(k \log k)} + \text{poly}(nd) \).
Proof. The analysis is largely the same as in [10] and Appendix E of [13]. We recall the following two useful lemmas. First, the existence of a well-conditioned basis, which is used in step 3 of Algorithm 8.

**Lemma C.6** (Existence of A Well-Conditioned Basis - Lemma 10 of [10], based on [14]). Given a matrix $A \in \mathbb{R}^{n \times m}$ having full column rank, there exists $B \in \mathbb{R}^{n \times m}$ such that the span of $B$ is the same as that of $A$, and for all $x \in \mathbb{R}^n$,

$$\frac{\|x\|_p}{O(m)} \leq \|Bx\|_p \leq \|x\|_p$$

Moreover, $B$ can be computed from $A$ in poly$(mn)$ time.

Next, the fact that every matrix has a subset of $k$ columns (and equivalently rows) which span an $O(k)$-approximation:

**Lemma C.7** (Subset of $k$ Columns Spanning an $O(k)$-Approximation - Theorem 3 of [10]). Let $A \in \mathbb{R}^{n \times d}$ and $k \in \mathbb{N}$. Then, there exists $S \subset [d]$, with $|S| = k$, such that

$$\min_{V \in \mathbb{R}^{k \times d}} \|AV - A\|_p \leq O(k) \min_{A_k \in \text{rank } k} \|A_k - A\|_p$$

The desired subset $S$ from the above lemma can be found in time $d^k$ through brute-force search. Note that we cannot rely on Theorem [24] since we wish to obtain a factorization with rank exactly $k$, rather than $O(k \log k)$. Using these lemmas, we analyze Algorithm 8, following the analysis of [10]. Throughout this proof, let $OPT := \min_{A_k \in \text{rank } k} \|A_k - A\|_p$.

First, note that since the span of $W_0$ is equal to that of $Z_0$, $W_0Z_0 = UV$. Now, let $OPT_{Z_0}$ denote the optimal rank-$k$ approximation cost for $Z_0$ — that is, $OPT_{Z_0} := \min_{M \text{ rank } k} \|M - Z_0\|_p$. Then, by Lemma C.7 there exists a subset of rows $Y$ of $Z_0$ such that if $X$ is the corresponding optimal left factor for $Y$, then $\|XY - Z_0\|_p \leq O(k)OPT_{Z_0}$. Moreover, because $W_0$ is a well-conditioned basis for the column span of $U$, by Lemma C.6

$$\|W_0XY - W_0Z_0\|_p \leq \|XY - Z_0\|_p \leq O(k)OPT_{Z_0}$$

and therefore, by the triangle inequality,

$$\|W_0XY - A\|_p \leq \|W_0XY - UV\|_p + \|UV - A\|_p$$

$$= \|W_0XY - W_0Z_0\|_p + \|UV - A\|_p$$

$$\leq O(k)OPT_{Z_0} + O(r \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT$$

where the first inequality is by the triangle inequality, the first equality is because $UV = W_0Z_0$, and the second inequality is by the approximation guarantee from Theorem C.1, where $r$ has the same meaning here. The only thing remaining is to bound $OPT_{Z_0}$ from above.

First, if we let $A_k$ be the optimal rank-$k$ approximation for $A$, then

$$\|W_0Z_0 - A_k\|_p \leq \|W_0Z_0 - A\|_p + \|A_k - A\|_p$$

$$= \|UV - A\|_p + OPT$$

$$= O(r \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT + OPT$$

$$= O(r \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT$$

meaning $OPT_{W_0Z_0} := \min_{M \text{ rank } k} \|M - W_0Z_0\|_p$ is at most $O(r \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT$. Moreover, by Lemma C.7 $W_0Z_0$ has a subset of $k$ columns spanning an $O(k)$-approximation to $W_0Z_0$. We can write this subset of columns as $W_0Z_0C$, where $C$ is a matrix with $k$ columns having a single 1 in each column, with the rest of its entries being 0. Then, there exists a right factor $D$ for $W_0Z_0C$ such that

$$\|W_0Z_0CD - W_0Z_0\|_p \leq O(k)OPT_{W_0Z_0} \leq O(kr \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT$$

Finally, since $W_0$ is a well-conditioned basis,

$$\|Z_0CD - Z_0\|_p \leq O(r^2)\|W_0Z_0CD - W_0Z_0\|_p \leq O(kr \hat{\varepsilon}^{-1}(\log k)^{\frac{1}{5}})OPT$$
since $W_0$ leads to at most $O(r^2)$ distortion by Lemma C.6 because the approximation from Algorithm 4 has bicriteria rank at most $O(r^2)$.

Since $C$ has $k$ columns, meaning $Z_0CD$ has rank at most $k$, this implies that
\[ \text{OPT}_{Z_0} \leq O(kr^{\frac{1}{2}+1} (\log k)^{\frac{1}{2}}) \text{OPT} \]
and in summary,
\[ \|W_0XY - A\|_p \leq O(k)\text{OPT}_{Z_0} \leq O(k^2r^{\frac{1}{2}}+1 (\log k)^{\frac{1}{2}}) \text{OPT} \]

Finally, to calculate the running time of Algorithm 8, note that all steps other than the fifth step can be done in $\text{poly}(nd)$ time. For the fifth step, note that $Z_0$ has $O(r^2)$ rows, and we check all $k$-subsets of those, meaning there are $(r^2)^O(k) = 2^{O(k\log k)}$ subsets of rows to check — computing the error for each subset takes $\text{poly}(nd)$ time (by performing $\ell_p$ regression). Hence, the overall running time is $2^{O(k\log k)}\text{poly}(nd)$, which is $2^{O(k\log k)} + \text{poly}(nd)$ due to the inequality $ab \leq a^2 + b^2$. \qed

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