Incorporating Heisenberg’s Uncertainty Principle into Quantum Multiparameter Estimation

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The quantum multiparameter estimation is very different with the classical multiparameter estimation due to Heisenberg’s uncertainty principle in quantum mechanics. When the optimal measurements for different parameters are incompatible, they cannot be jointly performed. We find a correspondence relationship between the inefficiency of a measurement for estimating the unknown parameter with the measurement error in the context of measurement uncertainty relations. Taking this correspondence relationship as a bridge, we incorporate Heisenberg’s uncertainty principle into quantum multiparameter estimation by giving a tradeoff relation between the measurement inefficiencies for estimating different parameters. For pure quantum states, this tradeoff relation is tight, so it can reveal the true quantum limits on individual estimation errors in such cases. We apply our approach to derive the tradeoff between attainable errors of estimating the real and imaginary parts of a complex signal encoded in coherent states and obtain the joint measurements attaining the tradeoff relation. We also show that our approach can be readily used to derive the tradeoff between the errors of jointly estimating the phase shift and phase diffusion without explicitly parameterizing quantum measurements.

The random nature of quantum measurement imposes ultimate limits on the precision of estimating unknown parameters with quantum systems. Quantum parameter estimation theory has been developing for more than half a century to reveal and pursue the quantum-limited measurement [1–9]. In classical parameter estimation theory, the Cramér-Rao bound (CRB) together with the asymptotic normality of the maximum likelihood estimator give a satisfactory approach to derive the asymptotically attainable accuracy of estimation, where the Fisher information matrix (FIM) plays a pivotal role [10–17]. The CRB and the FIM have been extended to quantum regime [1–6], where not only estimators—data processing—but also quantum measurements are taken into consideration in optimization.

For single parameter estimation, Helstrom’s version of quantum CRB can be attained at large samples due to the asymptotic efficiency of adaptive measurements [8, 18–20]. However, unlike the classical parameter estimation, the quantum CRB does not possess the asymptotic attainability in general for multiparameter estimation. This can be understood as a consequence of the fact that the optimal measurements for different parameters may be incompatible in quantum mechanics so that they cannot be jointly performed according to Heisenberg’s uncertainty principle (HUP) [21, 22]. Many application scenarios, e.g., superresolution imaging [23, 24], quantum enhanced estimation of a magnetic field [25, 26], and joint estimation of phase shift and phase diffusion [27], essentially belong to quantum multiparameter estimation problems. Therefore, the characterization of the quantum-limited bound on the estimation errors is of great importance to many practical applications of quantum estimation. Nevertheless, it is still challenging to derive, characterize, and understand the quantum limit on accuracies for the multiparameter estimation [9, 28–45].

To approach the attainable bounds on the estimation errors, various versions of quantum CRBs have been proposed [1–3, 6, 9, 34, 46–48]. These quantum CRBs are formulated by taking a scalar figure of merit, e.g., the weighted mean errors, and formulating the inequalities for the scalar mean errors by utilizing the mathematical structures of the operators on the Hilbert space (e.g., the Cauchy-Schwarz inequality). It is still unclear how the HUP affects the multiparameter estimation when the optimal measurement for individual parameters are incompatible. In other words, the HUP has not been incorporated at the first place in quantum multiparameter estimation.

In this work, we tackle the problem of the fundamental limit on the errors of estimating multiple parameters by incorporating the HUP into quantum multiparameter estimation. We define the regret of Fisher information for a quantum measurement that is used to estimate an unknown parameter and shall derive the following correspondence relation:

Information Regret ↔ Measurement Error.

Taking this relationship as a bridge, we obtain tradeoff relations between the information regrets for different parameters through Branciard’s and Ozawa’s versions of measurement uncertainty relations in terms of the state-dependent measurement error defined by Ozawa [49–54]. This tradeoff relation is tight for pure quantum states, so it can faithfully reveal the quantum limits on multiparameter estimation errors with pure quantum states. We shall apply the regret tradeoff relation to the coherent state estimation and the joint estimation of phase shift and phase diffusion.
Let us start with a brief introduction on quantum multiparameter estimation. Let \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \) be an unknown vector parameter, which can be estimated via observing a quantum system. The state of the quantum system depends on the true value of \( \theta \) and is described by a parametric density operator \( \rho_\theta \). The quantum measurement can be characterized by a positive-operator-valued measure (POVM) \( M = \{M_x | M_x \geq 0, \sum_x M_x = \mathbb{1} \} \), where \( x \) denotes the outcome and \( \mathbb{1} \) is the identity operator. Denote the estimator for \( \theta \) by \( \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n) \), which is a map from the observation data to the estimates. The estimation error can be characterized by the error-covariance matrix defined by its entries \( E_{jk} = \mathbb{E}_\hat{\theta}[(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)] \), where the expectation \( \mathbb{E}_\hat{\theta}[\cdot] \) is taken with respect to the observation data with the joint probability mass function \( p_\theta(x_1, x_2, \ldots, x_\nu) = \prod_{j=1}^\nu \text{tr}(M_{x_j} \rho_\theta) \) with \( \nu \) being the number of experimental runs. The error-covariance matrix of any unbiased estimator \( \hat{\theta} \) obeys the CRB \( \mathcal{E} \geq \nu^{-1}F^{-1} \) in the sense that the matrix \( \mathcal{E} - \nu^{-1}F^{-1} \) is positive semi-definite [14-17], where \( F \) is the (classical) FIM for a single experimental run and defined by

\[
F_{jk} = \mathbb{E}_\theta \left[ \frac{\partial \ln p_\theta(x)}{\partial \theta_j} \frac{\partial \ln p_\theta(x)}{\partial \theta_k} \right]
\]

with \( p_\theta(x) = \text{tr}(M_x \rho_\theta) \). The CRB is asymptotically attainable by the maximum likelihood estimator [10, 11], whose distribution at a large \( \nu \) is approximate to a multivariate normal distribution with the mean being the true value of \( \theta \) and the covariance matrix being \( \nu^{-1}F^{-1} \), according to the central limit theorem [15, Theorem 9.27].

The FIM depends on the quantum measurement via \( p_\theta(x) = \text{tr}(M_x \rho_\theta) \), so does the CRB. We use \( F(M) \) to explicitly indicate the dependence of \( F \) on a POVM \( M \). Quantum parameter estimation takes into consideration the optimization over quantum measurements. For any quantum measurement, the FIM is bounded by the following matrix inequality: [18, 55]

\[
F(M) \leq \mathcal{F},
\]

where \( \mathcal{F} \) is the so-called quantum FIM, also known as the Helstrom information matrix [1, 2]. The quantum FIM is the real part of a Hermitian matrix \( Q \) (i.e., \( \mathcal{F} = \text{Re} \mathcal{Q} \)) defined by

\[
Q_{jk} = \text{tr}(L_j L_k \rho_\theta),
\]

where \( L_j \) is the symmetric logarithmic derivative (SLD) operator for \( \theta_j \), is a Hermitian operator satisfying \( (L_j \rho_\theta + \rho_\theta L_j)/2 = \partial \rho_\theta / \partial \theta_j \). Combining Eq. (2) with the CRB yields the quantum CRB \( \mathcal{E} \geq \nu^{-1}F^{-1} \) for any quantum measurement and any unbiased estimator. This quantum CRB was first obtained by Helstrom with a different method [1, 2].

To characterize the efficiency of a quantum measurement for multiparameter estimation, we here define the regret of Fisher information by

\[
R(M) = \mathcal{F} - F(M).
\]

This matrix \( R(M) \) is positive semi-definite due to Eq. (2) and real symmetric as both the quantum and classical FIMs are real symmetric according to their definitions. For single-parameter estimation, Braunstein and Caves proved that the classical Fisher information can equal the quantum Fisher information with an optimal quantum measurement [18] and thus the regret \( R(M) \) thereof vanishes. In the multiparameter setting, for any column vector \( v \in \mathbb{R}^n \), there exist a quantum measurement \( M \) such that \( v^\top R(M)v = 0 \), where \( \top \) denotes matrix transpose. This is because \( v^\top F(M)v \) can be interpreted as the classical and quantum Fisher information, respectively, about a parameter \( \varphi \) satisfying \( \partial \varphi / \partial \theta_j = \sum_k v_j \partial \varphi / \partial \theta_k \). The POVM \( M \) making \( v^\top R(M)v \) vanish can be considered as an optimal measurement for estimating \( \varphi \) and in general depends on \( v \). For different parameters, the optimal measurement may be different and even incompatible. Consequently, the entries of \( R(M) \) in general cannot simultaneously vanish, which is a manifestation of HUP. In what follows, we shall give a quantitative characterization of the mechanism in which the HUP affects the regret matrix of Fisher information.

Define by \( \Delta_j = \sqrt{R_{jj}/F_{jj}} \) the normalized-square-root regret of Fisher information with respect to \( \theta_j \). Note that \( \Delta_j \) takes value in the interval \([0,1]\). Our main result is the following tradeoff relation:

\[
\Delta_j^2 + \Delta_k^2 + 2 \sqrt{1 - c_{jk}^2} |\Delta_j - \Delta_k| \geq c_{jk}^2,
\]

where \( c_{jk} \) is a real number given by

\[
c_{jk} = \frac{\text{Im} Q_{jk}}{\sqrt{\text{Re} Q_{jj} \text{Re} Q_{kk} - |\text{Im} Q_{jj}|^2}} = \frac{|\text{Im} Q_{jk}|}{\sqrt{F_{jj} F_{kk}}},
\]

with \( Q_{jk} \) being given by Eq. (3). For nonzero \( c_{jk} \), Eq. (5) describes the tradeoff between the regrets of Fisher information with respect to different parameters. For a family \( \rho_\theta \) of pure states, the inequality Eq. (5) is tight, in the sense that there exists a quantum measurement \( M \) such that the equality in Eq. (5) holds; In such a case, our result fully reflects the tradeoff between different regrets of Fisher information. For mixed states \( \rho_\theta \), the inequality Eq. (5) can be tightened by replacing \( c_{jk} \) thereof by its variant

\[
c_{jk} = \frac{\text{tr} |\rho_\theta [L_j, L_k] \rho_\theta|}{2 \sqrt{F_{jj} F_{kk}}},
\]

where \( |X| = \sqrt{X^\dagger X} \) for an operator \( X \). Note that the coefficient \( c_{jk} \) is not less than \( c_{jk} \) for all quantum states and equal to \( c_{jk} \) for all pure states. We also give the second form of the tradeoff relation in terms of the estimation errors:

\[
\gamma_j + \gamma_k - 2 \sqrt{1 - c_{jk}^2} \sqrt{(1 - \gamma_j)(1 - \gamma_k)} \leq 2 - c_{jk}^2,
\]

where \( \gamma_j = 1 - \frac{\text{tr} \rho_\theta [L_j, L_j] \rho_\theta}{\text{tr} \rho_\theta [L_j, L_j] \rho_\theta} \) for pure states; the upper bound is not tight in general.
where we have defined $\gamma_j = 1/(\nu E_{jj} F_{jj})$ for simplicity. The above inequality is a result of combining Eq. (5) with the classical CRB $\mathcal{E}_{jj} \geq 1/(F_{jj} \nu)$.

We here outline the proof of Eq. (5) and leave the details in the Supplemental Material [56]. Denote by $\mathcal{H}_s$ the Hilbert space associated with the underlying quantum system. For a given POVM $M$ on $\mathcal{H}_s$, we define a measurement channel $\Phi(\rho) = \sum_x \text{tr}(M_x \rho) |x\rangle\langle x|$, where $\{|x\rangle\}$ is an orthonormal basis associated with the measurement outcomes $x$’s and span another Hilbert space $\mathcal{H}_s$. Note that the density operators $\Phi(\rho_{\theta})$ are always diagonal with the basis $\{|x\rangle\}$. As a result, the SLD operators $\Phi(\rho_{\theta})$ are also diagonal with the basis $\{|x\rangle\}$ and can be represented as

$$\bar{L}_j = \sum_x \frac{\partial \ln \text{tr}(M_x \rho_{\theta})}{\partial \theta_j} |x\rangle\langle x|.$$  

The measurement channel $\Phi$ can be implemented by a unitary operation $U$ acting on $\mathcal{H}_s \otimes \mathcal{H}_s \otimes \mathcal{H}_s$ such that

$$\text{tr}_{1,3} \left[U (\rho_{\theta} \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|) U^\dagger \right]$$

for all density operators $\rho$ on $\mathcal{H}_s$, where $\text{tr}_{1,3}$ denotes the partial trace over the first and third tensor factors of the Hilbert space and $|0\rangle$ can be an arbitrary initial state [57, Chapter 2] (see Fig. 1 for a schematic illustration). Using the techniques developed in Ref. [58], we show that [56]

$$R_{jj} = \text{tr} \left[ (L_j - L_j \otimes \mathbb{1}_s \otimes \mathbb{1}_s)^2 \rho_{\text{total}} \right],$$

where $L_j = U \mathbb{1}_s \otimes L_j \otimes \mathbb{1}_s U$ with $\mathbb{1}_s$ and $\mathbb{1}_s$ being the identity operators on $\mathcal{H}_s$ and $\mathcal{H}_s$, respectively, and $\rho_{\text{total}} = \rho_{\theta} \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$. We observe that $R_{jj}$ expressed in Eq. (11) is of the same form as the square of Ozawa’s definition of measurement error [49, 50, 59], when taking $L_j$ as the ideal observable we intend to measure and $L_j$ as the observable actually measured. We list in Table I the correspondence relation between the parameter estimation scenario and the measurement error scenario. Notice that the Hermitian operators $L_j$ and $L_k$ always commutes, as both $\bar{L}_j$ and $\bar{L}_k$ are diagonal with the basis $\{|x\rangle\}$. Therefore, the observables $L_j$ and $L_k$ can be jointly measured in quantum mechanics. When two ideal observables $L_j$ and $L_k$ do not commute, it may be impossible to make their measurement errors, which equals the regrets $R_{jj}$ and $R_{kk}$ in our context, simultaneously vanish. By invoking the measurement uncertainty relations [49–51, 53, 54, 60] in terms of Ozawa’s definition of measurement error, we can derive the tradeoff relation between the regrets of Fisher information with respect to different parameters. Concretely, the inequality Eq. (5) follows from Branciard’s version of measurement uncertainty relation, which is tight for pure states [53]. Using Ozawa’s work on strengthening Branciard’s inequality for mixed states [60], the inequality Eq. (5) can be tightened through replacing $c_{jk}$ by $\tilde{c}_{jk}$.

It is worthy to point out that we do not designate the SLD operator as the ideal observable in reality to optimally estimate an individual parameter. Although the eigenstates of the SLD operator, which possibly depend on the true value of the parameter, in principle constitute a measurement basis extracting the maximum Fisher information at a parameter point [18], it is possible for some models to find a global optimal measurement that is independent of the parameter [6, 19]: A global optimal measurement is often more ideal than a local one for estimating the unknown parameter.

We can give an operational significance to the coefficients $\tilde{c}_{jk}$ through the tradeoff relation Eq. (5) as follows. If the QFI about a parameter $\theta_j$ is exhaustively extracted by a quantum measurement $M$, i.e., $\Delta_j = 0$, then it follows from Eq. (5) that the regret for any other parameter $\theta_k$ obeys $\Delta_k \geq \tilde{c}_{jk}$. That is, $\tilde{c}_{jk}$ is the lower bound on the residual regret for $\theta_k$ when there is no regret for $\theta_j$. For pure states, this lower bounds $c_{jk}$ can be attained as Eq. (5) is tight in such cases.

Let us now consider as an example the estimation of a complex number $\alpha$ encoded in a coherent state [61] $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} (\alpha^n/\sqrt{n!}) |n\rangle$, where $|n\rangle$’s are the number states. The parameters of interest here are the real and imaginary parts of $\alpha$, i.e., $\theta_1 = \text{Re } \alpha$ and $\theta_2 = \text{Im } \alpha$. After some algebra, we get $Q = 4 \left( \frac{1}{2} \right)$ and thus $c_{12} = 1$. The regret tradeoff Eq. (5) then becomes $\Delta_1^2 + \Delta_2^2 \geq 1$, which is equivalent to $F_{11} + F_{22} \leq 4$ in terms of Fisher

| Estimation-regret scenario | Measurement-error scenario |
|---------------------------|---------------------------|
| regret of Fisher information | measurement error |
| $L_j$ of $\rho_{\theta}$ | ideal observable $L_1$ on $\rho_{\theta}$ |
| $L_j$ of $\Phi(\rho_{\theta})$ | approximate observable on $\Phi(\rho_{\theta})$ |
| $L_j = U^\dagger (\mathbb{1}_s \otimes L_j \otimes \mathbb{1}_s) U$ | approximate observable on $\rho_{\theta}$ |

Table I. Correspondence relation.
as follows. Denote by \( a \) the annihilation operators for the mode for which the coherent state is defined. The measurements of the quadrature components \( Q = (a + a^\dagger)/2 \) and \( P = (a - a^\dagger)/(2i) \) are natural for estimating the coherent signal, as \( \langle \alpha | Q | \alpha \rangle = \text{Re} \alpha \) and \( \langle \alpha | P | \alpha \rangle = \text{Im} \alpha \). Indeed, the maximum Fisher information about \( \theta_1 \) and \( \theta_2 \) can be obtained by measuring \( Q \) and \( P \), respectively, corresponding to either \( F_{11} = 4 \) or \( F_{22} = 4 \). However, \( Q \) and \( P \) are not commuting so that they cannot be jointly measured. It is known that we can jointly measure the commuting operators \( Q - Q' \) and \( P + P' \), where \( Q' \) and \( P' \) are the quadrature components of an ancillary mode (whose annihilation operator is denoted by \( a' \)) in the vacuum state, to estimate the real and imaginary parts of \( \alpha \), see Refs. [3, 5, 6]. This measurement strategy attains the minimum unweighted arithmetic mean error of estimation with \( F_{11} = F_{22} = 2 \), see the blue circle in

\[
\frac{1}{\nu E_{11}} + \frac{1}{\nu E_{22}} \leq 4 \tag{12}
\]

in terms of estimation errors. As shown in Fig. 2, Eq. (12) gives the most informative lower bound on the estimation error, compared with the error bounds that was previously investigated [3, 5, 6, 34].

The regret tradeoff relation in the above example can be attained by an optimal measurement, and thus the most informative lower bound Eq. (12) can be asymptotically attained. We construct the optimal measurement. The other three curves stand for the tradeoff relations that have been systematically analyzed in Ref. [34]; They are obtained by generalized-mean Cramér-Rao bounds based on the SLD and the right logarithmic derivative (RLD).

In our second example, we consider the joint estimation of phase shift and phase diffusion [27]. For a two-mode probe state, the parametric density operator can be effectively simplified as \( \rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + e^{-i\theta_1 - i\theta_2} |0\rangle\langle 1| + e^{i\theta_1 - i\theta_2} |1\rangle\langle 0|) \), where \( \theta_1 \) stands for the phase shift and \( \theta_2 \) the phase diffusion. In Ref. [27], Vidrighin et al. obtained the tradeoff relation \( F_{11}/F_{11} + F_{22}/F_{22} \leq 1 \) by explicitly parameterizing the rank-1 POVMs and then taking optimization. We here show that Vidrighin et al.’s tradeoff relations follows from our regret tradeoff relation Eq. (5) in a very easy way. We only need to show \( \Delta_2 = 1 \) by a straightforward calculation according to its definition (see the Supplemental Material [56] for the details). As a result, we get \( \Delta_1^2 + \Delta_2^2 \geq 1 \), which is equivalent to Vidrighin et al.’s tradeoff relation by recognizing \( \Delta_1^2 = 1 - F_{jj}/F_{jj} \).

In conclusion, we have incorporated the HUP into quantum multiparameter estimation by deriving a tradeoff relation between the regrets of Fisher information about different parameters. Unlike the quantum CRBs on scalar mean errors, the regrets tradeoff quantitatively characterizes how the HUP affects the combinations of estimation errors for multiple parameters. The correspondence relationship we found between information regret and measurement error also, as a bonus, supplies an operational meaning to Ozawa’s definition of the state-dependent measurement error, on which there exists a controversy for a long time [59, 62, 63].

Our approach also opens a new perspective on quantum geometry. The matrix \( Q \) defined by Eq. (3) is known as the quantum geometric tensor on the manifold of physical quantum state, up to an insignificant constant factor [64, 65]. The real part of \( Q \)—the quantum FIM—gives a Riemannian metric on the manifolds of quantum states. The imaginary part of \( Q \) gives a curvature form of Berry’s connection [65], which has relations to the quantum FIM [28, 66] and the density of quantum states [67]. It is known that a zero curvature is necessary for the simultaneous vanishing of the regrets of Fisher information about different parameters [8, 33, 36, 48]. Note that in our tradeoff relation, \( c_{jk} \) is the curvature divided by a scalar related to the entries of the quantum FIM. So our tradeoff relation quantitatively characterize the intricate mechanism in which the simultaneous reduction of the regrets of Fisher information about different parameters
is restricted by a nonzero quantum curvature, which is indicated as

Information Regret $\leftrightarrow$ Quantum Curvature.

Carollo et al. has proposed an incompatibility index, which is similar to $c_{jk}$, based on the ratio between the curvature and the quantum FIM as a figure of merit for the quantumness of a quantum multiparameter estimation model [28]. In addition, since $c_{jk}$ is better than $c_{jk}$ to manifest the regrets tradeoff for mixed states, it may be possible to take the quantity $\text{tr} [\sqrt{\rho} L_j L_k \sqrt{\rho}]$ as an alternative form of quantum curvature.

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Supplemental material

DETAILED DERIVATION OF THE REGRET TRADEOFF RELATION

Regret of Fisher information

Firstly, let $\mathcal{H}_s$ be the Hilbert space associated with the underlying quantum system, and $\mathcal{S}(\mathcal{H})$ the set of all density operators on a Hilbert space $\mathcal{H}$. For a given POVM $M$ acting on $\mathcal{H}_s$, define the measurement channel $\Phi : \mathcal{S}(\mathcal{H}_s) \to \mathcal{S}(\mathcal{H}_t)$ by

$$\Phi : \rho \mapsto \sum_x \text{tr}(M_x \rho) |x\rangle\langle x|,$$

where $\mathcal{H}_t$ is the Hilbert space for a register that associates the outcomes $x$ with an orthonormal basis $\{ |x\rangle \}$. The output density operators $\Phi(\rho_0)$ are diagonal with the basis $\{ |x\rangle \}$ and its SLD operator with respect to $\theta_j$ can be expressed as

$$\bar{L}_j = \frac{\partial \ln \text{tr}(M_x \rho_0)}{\partial \theta_j} |x\rangle\langle x|.$$  \hfill (S2)

Denote by $\bar{F}$ the quantum Fisher information matrix of $\Phi(\rho_0)$. Note that $\bar{F}$ equals the classical Fisher information matrix under the measurement $M$, i.e., $\bar{F} = F(M)$.

Secondly, it can be shown that

$$\bar{F}_{jk} = \text{Re} \text{tr} \left( \bar{L}_j \bar{L}_k \Phi(\rho_0) \right) = \text{tr} \left[ \bar{L}_j \left( \frac{\bar{L}_k \Phi(\rho_0) + \Phi(\rho_0) \bar{L}_k}{2} \right) \right] = \text{tr} \left[ \bar{L}_j \left( \frac{\partial \Phi(\rho_0)}{\partial \theta_k} \right) \right],$$  \hfill (S3)

where we have used the SLD equation

$$\frac{\partial \Phi(\rho_0)}{\partial \theta_j} = \frac{1}{2} \left[ \bar{L}_j \Phi(\rho_0) + \Phi(\rho_0) \bar{L}_j \right]$$  \hfill (S4)

in the third equality and $\partial \Phi(\rho_0)/\partial \theta_j = \Phi(\partial \rho_0/\partial \theta_j)$ in the fourth equality. Now, let us introduce the dual map $\Phi^\dagger$ that satisfies

$$\text{tr}[\Phi^\dagger(X)\rho] = \text{tr}[X\Phi(\rho)]$$  \hfill (S5)

for any density operator $\rho$ on $\mathcal{H}_s$ and any bounded operator $X$ on $\mathcal{H}_t$. It then follows that

$$\bar{F}_{jk} = \text{tr} \left[ \Phi^\dagger(\bar{L}_j) \frac{\partial \rho_0}{\partial \theta_k} \right] = \text{Re} \text{tr} \left( \Phi^\dagger(\bar{L}_j)L_k \rho_0 \right),$$  \hfill (S6)

where we have used the SLD equation $\partial \rho_0/\partial \theta_k = (L_k \rho_0 + \rho_0 L_k)/2$. Since the quantum Fisher information matrix is symmetric, by interchanging the subscripts $j$ and $k$ in the right hand side of Eq. (S6), we can also get

$$\bar{F}_{jk} = \text{Re} \text{tr} \left[ \Phi^\dagger(\bar{L}_k)L_j \rho_0 \right].$$  \hfill (S7)

For the simplicity of notation, let us define $(\bullet) := \text{tr}(\bullet \rho_0)$. By noting that

$$\bar{F}_{jk} = \text{Re} \langle L_j \Phi^\dagger(\bar{L}_k) \rangle = \text{Re} \langle \Phi^\dagger(\bar{L}_j)L_k \rangle = \text{Re} \langle \Phi^\dagger(\bar{L}_j \bar{L}_k) \rangle,$$

we show that the entries of the regret matrix can be expressed as

$$R_{jk} := F_{jk} - \bar{F}_{jk} = \text{Re} \langle L_j L_k - L_j \Phi^\dagger(\bar{L}_k) - \Phi^\dagger(\bar{L}_j)L_k + \Phi^\dagger(\bar{L}_j \bar{L}_k) \rangle.$$  \hfill (S8)

Thirdly, we shall show that the expression Eq. (S8) of the regret matrix entries can be rewritten in a more elegant way through the open-system representation of quantum channels. There always exist a unitary operator $U$ on $\mathcal{H}_s \otimes \mathcal{H}_t \otimes \mathcal{H}_r$ such that

$$\Phi(\rho) = \text{tr}_{1,3}[U(\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)U^\dagger]$$  \hfill (S9)
for all density operators $\rho$ on $\mathcal{H}$, where $|0\rangle$ can be any state in $\mathcal{H}$, and $\text{tr}_{1,3}$ is the partial trace over the first and third tensor factors of $\mathcal{H}_s \otimes \mathcal{H}_r \otimes \mathcal{H}_t$ (see Ref. [57, Chapter 2]). With this open-system representation of $\Phi$, it can be shown that

$$\text{tr}[X \Phi(\rho)] = \text{tr}[(\mathds{1}_s \otimes X \otimes \mathds{1}_r)U(\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)U^\dagger],$$  \hspace{1cm} (S10)

implying that

$$\Phi^\dagger(X) = \text{tr}_{2,3} \left[ U^\dagger(\mathds{1}_s \otimes X \otimes \mathds{1}_r)U(\mathds{1}_s \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|) \right].$$  \hspace{1cm} (S11)

With this open-system representation of the dual map of $\Phi$, Eq. (S8) can be expressed as

$$R_{jk} = \text{Re} \text{tr}[N_j N_k (\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)],$$  \hspace{1cm} (S12)

where $N_j := \mathcal{L}_j - L_j \otimes \mathds{1}_r \otimes \mathds{1}_t$ with $\mathcal{L}_j := U^\dagger(\mathds{1}_s \otimes L_j \otimes \mathds{1}_r)U$.

### Measurement uncertainty relation

We here briefly introduce the measurement uncertainty relations, which will be invoked to derive the regret tradeoff relation. Let $A$ and $B$ be two Hermitian operators standing for the ideal observables we intend to measure. In quantum mechanics, when $[A,B] \neq 0$, these two observables cannot be jointly measured. To approximate the joint measurement of $A$ and $B$ when $[A,B] \neq 0$, we can measure another pair of commuting observables $\mathcal{A}$ and $\mathcal{B}$ acting on the system possibly dilated by adding an ancilla whose state is denoted by a density operator $\eta$ [68]. Ozawa proposed to quantify the (state-dependent) measurement errors for the ideal observables $A$ and $B$ in the quantum state $\rho$ by

$$\epsilon_A = \sqrt{\text{tr}[(A - A \otimes \mathds{1})^2(\rho \otimes \eta)]} \quad \text{and} \quad \epsilon_B = \sqrt{\text{tr}[(B - B \otimes \mathds{1})^2(\rho \otimes \eta)]},$$  \hspace{1cm} (S13)

respectively, and derived the following measurement uncertainty relation [49, 50]:

$$\epsilon_A \epsilon_B + \epsilon_A \sigma_B + \epsilon_B \sigma_A \geq C_{AB} := \frac{1}{2} |\text{tr}([A,B]\rho)|,$$  \hspace{1cm} (S14)

where $\sigma_A := \sqrt{\text{tr}(A^2\rho) - \text{tr}(A\rho)^2}$ and $\sigma_B := \sqrt{\text{tr}(B^2\rho) - \text{tr}(B\rho)^2}$. Branciard obtained a stronger inequality [53]:

$$\epsilon_A^2 \sigma_B^2 + \epsilon_B^2 \sigma_A^2 + 2 \sqrt{\epsilon_A^2 \sigma_B^2 \epsilon_B^2 \sigma_A^2} - C_{AB}^2 \epsilon_A \epsilon_B \geq C_{AB}^2,$$  \hspace{1cm} (S15)

which implies Ozawa’s inequality and is tight when $\rho$ is a pure state. For mixed states, Ozawa showed that Branciard’s inequality can be strengthened by replacing $C_{AB}$ by

$$D_{AB} := \frac{1}{2} \text{tr} |\sqrt{\rho}[A,B]\sqrt{\rho}|$$  \hspace{1cm} (S16)

with $|X| := \sqrt{X^\dagger X}$ for an operator $X$.

### Derivation of the regret tradeoff relation

Now, we derive our regret tradeoff relation. It follows from Eq. (S12) that

$$R_{jj} = \text{tr} \left[ (\mathcal{L}_j - L_j \otimes \mathds{1}_r \otimes \mathds{1}_t)^2(\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|) \right],$$  \hspace{1cm} (S17)

which is in the form of Ozawa’s definition of measurement error by taking $L_j \rightarrow A$, $\mathcal{L}_j \rightarrow \mathcal{A}$, and $|0\rangle\langle 0| \otimes |0\rangle\langle 0| \rightarrow \eta$. Correspondingly, we have

$$\sigma_A = \sqrt{\text{tr}(L_j^2 \rho) - \text{tr}(L_j \rho)^2} = \sqrt{F_{jj}}.$$  \hspace{1cm} (S18)

Let us consider another parameter $\theta_k$ and take $L_k \rightarrow B$ and $\mathcal{L}_k \rightarrow \mathcal{B}$. It is easy to see that $[L_j, L_k] = 0$ for $[L_j, \mathcal{L}_k] = 0$. Consequently, the square roots of the regret of Fisher information for different parameters $\theta_j$ and $\theta_k$, i.e., $\sqrt{R_{jj}}$ and
The coherent state in the coordinate representation is given by the wave function $|\alpha\rangle$, where $\Delta Q, \Delta P$ it can be shown that:

$$R_{jj}F_{kk} + R_{kk}F_{jj} + 2\sqrt{F_{jj}F_{kk} - C_{jk}^2} \geq C_{jk}^2,$$

where $C_{jk} := \frac{1}{2} |\text{tr}([L_j, L_k]\rho)|$. Dividing both sides of the above inequality by $F_{jj}F_{kk}$, we get our regret tradeoff relation in the main text.

**ESTIMATING COHERENT STATE**

Here, we give the detailed calculations for the first example in the main text, i.e., the joint estimation of parameters $\theta_1 = \text{Re}\alpha$ and $\theta_2 = \text{Im}\alpha$ in coherent states $|\alpha\rangle$. Due to:

$$\frac{\partial |\alpha\rangle}{\partial \theta_1} = (-\theta_1 + a\dagger)|\psi\rangle, \quad \frac{\partial |\alpha\rangle}{\partial \theta_2} = (-\theta_2 + ia\dagger)|\psi\rangle$$  \hspace{1cm} (S20)

and

$$Q_{jk} = 4\left(\frac{\partial \langle \alpha |}{\partial \theta_j} (1 - |\alpha\rangle \langle \alpha|) \frac{\partial |\alpha\rangle}{\partial \theta_k}\right),$$  \hspace{1cm} (S21)

it can be shown that:

$$Q = 4\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$  \hspace{1cm} (S22)

We then get the regret tradeoff relation $R_{11} + R_{22} \geq 4$, which is equivalent to $F_{11} + F_{22} \leq 4$.

Following Helstrom [69] and Holevo [6], we consider the measurement for jointly estimating the two parameters $\theta_1$ and $\theta_2$ by using an ancillary mode whose annihilation operator is denoted by $a'$ and satisfies $[a', a\dagger] = 0$. Define the dimensionless coordinate and momentum operators for these two modes by:

$$Q = \frac{a + a\dagger}{2}, \quad P = \frac{a - a\dagger}{2i},$$

$$Q' = \frac{a' + a'\dagger}{2}, \quad P' = \frac{a' - a'\dagger}{2i}.$$  \hspace{1cm} (S23)

Note that $[Q, P] = i/2$ due to $[a, a\dagger] = 1$. The two observables $A = Q - Q'$ and $B = P + P'$ can be jointly measured, as they are commuting. It can be shown that $\langle A \rangle = \theta_1$, $\langle B \rangle = \theta_2$, $\Delta A^2 = \Delta Q^2 + \Delta Q'^2 = 1/2$, and $\Delta B^2 = \Delta P^2 + \Delta P'^2 = 1/2$.

To calculate the Fisher information matrix under the joint measurement of $Q - Q'$ and $P + P'$, we need obtain the joint probability density function with respect to the corresponding outcomes, which are denoted by $\xi$ and $\eta$. Denote by $|\xi, \eta\rangle$ the simultaneous eigenstates of the commuting observables $Q - Q'$ and $P + P'$. It is known that [69]

$$|\xi, \eta\rangle = \pi^{-1/2} \int e^{2i\eta q} |q\rangle_{Q'} \otimes |q - \xi\rangle_{Q'} dq,$$  \hspace{1cm} (S24)

where $|q\rangle_{Q}$ and $|q - x\rangle_{Q'}$ are the eigenstates of $Q$ and $Q'$ with the eigenvalues $q$ and $q - \xi$, respectively. Note that $|\xi, \eta\rangle$ are normalized so that $\langle \xi', \eta'| \xi, \eta\rangle = \delta(\xi - \xi')\delta(\eta - \eta')$. According to Born’s rule in quantum mechanics, the joint probability density function of the outcomes of measuring $Q - Q'$ and $P + P'$ is given by $p(\xi, \eta) = |\langle \xi, \eta | |\alpha\rangle \otimes |0\rangle|^2$. The coherent state in the coordinate representation is given by the wave function [70, Section 5.1.1]

$$\psi_\alpha(q) := \langle q | \alpha\rangle = \left(\frac{2}{\pi}\right)^{1/4} \exp[-(q - \theta_1)^2] \exp(2i\eta\theta_2).$$  \hspace{1cm} (S25)

Therefore,

$$p(\xi, \eta) = \frac{1}{\pi} \left| \int e^{-2i\eta q} \psi_\alpha(q) \psi_0(q - \xi) dq \right|^2$$  \hspace{1cm} (S26)

$$= \frac{2}{\pi^2} \left| \int e^{-2i\eta q} e^{-(q - \theta_1)^2} e^{2i\eta\theta_2} e^{-(q - \xi)^2} dq \right|^2$$  \hspace{1cm} (S27)

$$= \frac{1}{\pi} \exp[-(\eta - \theta_2)^2 - (\xi - \theta_1)^2].$$  \hspace{1cm} (S28)
The classical Fisher information matrix of \( p(\xi, \eta) \) is

\[
F = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},
\]

which saturates the tradeoff \( F_{11} + F_{22} \leq 4 \).

Now, we consider the joint measurement of \( Q - Q' \) and \( P + P' \) with the ancillary mode being in the squeezed state \( S(\xi) |0\rangle \), where \( S(\xi) = \exp[[\xi^* a^2 - \xi a'^* 2]]/2 \) with \( \xi = re^{i\varphi} \) being an arbitrary complex number is the squeeze operator. Here, we set \( \varphi = 0 \). It then can be shown that

\[
S(\xi)^\dagger a^* S(\xi) = a' \cosh r - a'' \sinh r,
\]

implying that

\[
S(\xi)^\dagger Q^* S(\xi) = Q' e^{-r} \quad \text{and} \quad S(\xi)^\dagger P^* S(\xi) = P' e^r.
\]

Therefore, jointly measuring \( Q - Q' \) and \( P + P' \) with the ancillary mode being in the squeezed state \( S(\xi) |0\rangle \) is equivalent to jointly measuring \( A_r := Q - e^{-r} Q' \) and \( B_r := P + e^r P' \) with the ancillary mode being in the vacuum state.

The normalized simultaneous eigenstates of \( A_r \) and \( B_r \) are given by

\[
|\xi, \eta\rangle_r = e^{r/2} \int \int \int \exp[2i\eta q - 2ip_1 q - 2ip_2 e^{r} (q - \xi)] |p_1\rangle_P \otimes |p_2\rangle_P \, dq \, dp_1 \, dp_2
\]

\[
= e^{r/2} \frac{1}{\sqrt{\pi}} \int \delta(\eta - p_1 - p_2 e^{r}) \exp(2ip_2 \xi e^{r}) |p_1\rangle_P \otimes |p_2\rangle_P \, dp_1 \, dp_2
\]

\[
= e^{-r/2} \frac{1}{\sqrt{\pi}} \int \exp[2i(\xi - p)] |p\rangle_P \otimes |e^{-r}(\eta - p)\rangle_P \, dp.
\]

With Eq. (S35), it is easy to see that \( A_r |\xi, \eta\rangle_r = \xi |\xi, \eta\rangle_r \) and \( B_r |\xi', \eta'\rangle_r = \delta(\xi - \xi') \delta(\eta - \eta') \). To show that \( |\xi, \eta\rangle_r \) is the eigenstate of \( B_r \) with the eigenvalue \( \eta \), we need to write \( |\xi, \eta\rangle_r \) with the momentum representation. Using \( p_r \langle q | p \rangle_Q = \frac{1}{\sqrt{\pi}} e^{-2ipq} \) with \( |p\rangle_P \) denoting the eigenstate of \( P \), we get

\[
p(\xi, \eta) = \frac{e^r}{\pi} \left| \int e^{-2i\eta q} \psi_\alpha(q) \psi_0(e^{r} (q - \xi)) \, dq \right|^2
\]

With the wave function of coherent state, namely, Eq. (S25), we get

\[
p(\xi, \eta) = \frac{2e^r}{\pi^2} \left| \int \exp[-2i\eta q - (q - \theta_1)^2 + 2iq\theta_2 - e^{2r}(q - \xi)^2] \, dq \right|^2
\]

\[
= \frac{1}{\pi} \frac{2e^r}{e^{2r} + 1} \left| \exp \left[ - \frac{2(\eta - \theta_2)^2}{e^{2r} + 1} - \frac{2e^{2r}(\xi - \theta_1)^2}{e^{2r} + 1} \right] \right|
\]

This probability density function is Gaussian with the covariance matrix as follows:

\[
\Sigma = \begin{pmatrix} e^{2r} + 1 & 0 \\ 0 & e^{2r} + 1 \end{pmatrix}.
\]

The classical Fisher information matrix with respect to \( \theta_1 \) and \( \theta_2 \) is then given by

\[
F = \Sigma^{-1} = \begin{pmatrix} 4 & 0 \\ 0 & \frac{4e^{2r}}{e^{2r} + 1} \end{pmatrix},
\]

which saturates the tradeoff relation \( F_{11} + F_{22} \leq 4 \).
JOINT ESTIMATION OF PHASE AND PHASE DIFFUSION

Here, we give the details of the calculation for the second example in the main text, i.e., the joint estimation of phase shift and phase diffusion. The density operators of a two-level quantum system can always be represented by \( \rho = \frac{1}{2}(I + \mathbf{r} \cdot \mathbf{\sigma}) \), where \( I \) is the \( 2 \times 2 \) identity matrix, \( \mathbf{r} = (r_1, r_2, r_3) \in \mathbb{R}^3 \), and \( \mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) is the vector of Pauli matrices. For the joint estimation of phase and phase diffusion [27], the Bloch vector \( \mathbf{r} \) is given by \( \mathbf{r} = e^{-\theta^2} \mathbf{n} \), where \( \mathbf{n} = (\sin \chi \cos \theta_1, \sin \chi \sin \theta_1, \cos \chi) \) is a unit vector. Here, \( \theta_1 \) and \( \theta_2 \) are the parameters of interest and \( \chi \) is a parameter determined by the initialization of the quantum system.

To obtain the SLD operators, we use the eigenvalue decomposition of \( \rho \). The eigen-projections of \( \rho_\theta \) are \( \Pi_{\pm} = \frac{1}{2}(I \pm \mathbf{n} \cdot \mathbf{\sigma}) \) with the eigenvalues \( \lambda_{\pm} = [1 \pm \exp(-\theta^2)]/2 \). The SLD operators about \( \theta_j \) can be expressed as

\[
L_j = \sum_{u, v = \pm} \frac{2}{\lambda_u + \lambda_v} \Pi_u (\partial_j \rho_\theta) \Pi_v = \sum_{u, v = \pm} \frac{1}{\lambda_u + \lambda_v} \Pi_u (\partial_j \mathbf{r} \cdot \mathbf{\sigma}) \Pi_v.
\]  

(S41)

Substituting the expressions of \( \lambda_{\pm}, \Pi_{\pm}, \) and \( \mathbf{r} \) into the above formula, we get

\[
L_1 = \begin{pmatrix}
0 & -ie^{-\theta^2-i\theta_1} \\
-ie^{\theta^2+i\theta_1} & 0
\end{pmatrix}, \quad L_2 = \frac{1}{e^{2\theta^2} - 1} \begin{pmatrix}
2\theta_2 & -2\theta_2 e^{\theta^2-i\theta_1} \\
-2\theta_2 e^{\theta^2+i\theta_1} & 2\theta_2
\end{pmatrix}.
\]  

(S42)

Therefore, we get the quantum geometric tensor:

\[
Q = \begin{pmatrix}
e^{-2\theta^2} & 0 \\
0 & \frac{4\theta^2}{e^{2\theta^2} - 1}
\end{pmatrix},
\]  

(S43)

from which we can see \( c_{12} = 0 \). To calculate \( \tilde{c}_{12} \), we note that

\[
[L_1, L_2] = \frac{-i}{e^{2\theta^2} - 1} \begin{pmatrix}
-4\theta_2 & 0 \\
0 & 4\theta_2
\end{pmatrix}.
\]  

(S44)

After some algebras, we get

\[
\text{tr} |\sqrt{\rho}[L_1, L_2]|\sqrt{\rho}| = \frac{4e^{-\theta^2}\theta_2}{\sqrt{e^{2\theta^2} - 1}}
\]  

(S45)

and thus

\[
\tilde{c}_{12} = \frac{\text{tr} |\sqrt{\rho}[L_1, L_2]|\sqrt{\rho}|}{2\sqrt{\text{Re}Q_{11}\text{Re}Q_{22}}} = 1.
\]  

(S46)