Optimisation in some Banach Algebras related to the Fourier Algebra.

Edmond E. Granirer

ABSTRACT. Let $A_p(G)$ denote the Figa-Talamanca-Herz Banach Algebra of the locally compact group $G$, thus $A_2(G)$ is the Fourier Algebra of $G$. If $G$ is commutative then $A_2(G) = L^1(\hat{G})$. Let $A^r_p(G) = A_p \cap L^r(G)$ with norm $\|u\|_{A^r_p} = \|u\|_{A_p} + \|u\|_{L^r}$. We investigate a property which insures not only existence of solutions to optimization problems but moreover, facility in testing that an algorithm converges to such solutions, namely the RNP. Theorem (a): If $G$ is weakly amenable then $A^r_p$ is a dual Banach space with the RNP if $1 \leq r \leq p'$. This does not hold if $G = SL(2,R)$, $p = 2$ and $r > 2$. Theorem (b): If $G$ is weakly amenable and second countable and $A^r_p$ has the RNP for $t = s$, then it has the RNP for all $1 \leq t \leq s$, where $s = \infty$ is allowed. In particular second countable noncompact groups $G$, for which $A_p(G)$ has the RNP, namely Fell groups, have to satisfy that $A^r_p(G)$ has the RNP for all $1 \leq r < \infty$. The results are new, even if $G = Z$, the additive integers.

1. INTRODUCTION

Let $G$ be a locally compact group and let $A_p(G)$ denote the Figa-Talamanca-Herz Banach Algebra of $G$, as defined in [Hz1], thus generated by $L^{p'} \ast L^{yp}(G)$, where $1 < p < \infty$, and $1/p + 1/p' = 1$, see sequel. Hence $A_2(G)$ is the Fourier algebra of $G$ as defined and studied by Eymard in [Ey1]. If $G$ is abelian then $A^r_p(G) = L^1(\hat{G})$. Denote $A^r_p(G) = A_p \cap L^r(G)$, for $1 \leq r \leq \infty$, equipped with the norm $\|u\|_{A^r_p} = \|u\|_{A_p} + \|u\|_{L^r}$. If $r = \infty$ let $A^r_p(G) = A_p(G)$.

If $G$ is abelian then, $A^r_2(G) = \{ u \in L^1(\hat{G}) \wedge \hat{f} \in L^r(G) \}$, with the norm $\|u\| = \|f\|_{L^1(\hat{G})} + \|\hat{f}\|_{L^r(G)}$, if $u = \hat{f}$.

2010 Mathematics Subject classification. Primary 43A15, 46J10, 43A25, 46B22. Secondary 46J20, 43A30, 43A80, 22A30. Key words and phrases: Fourier Algebra, Radon-Nikodym property, weakly amenable, locally compact groups.
The study of these Banach Algebras started in a beautiful paper of Larsen Liu and Wang [LLW] in the abelian case, and continued in [La1], [La2], [LCh], etc.

Let $X$ be a Banach space. Then

$X$ has the *Krein-Milman Property* (KMP), if each norm closed convex bounded subset $B$ of $X$ is the norm closed convex hull of its extreme points, $\text{ext}(B)$, (namely, $B = \overline{\text{co}\text{ext}(B)}$)

Hence: “Optimisation problems in such sets $B$ have solutions”.

The closed unit ball $B$ of $L^1(\mu)$, for a nonatomic measure $\mu$, has no extreme points, hence it does not have the KMP. And yet, if $\mu$ is atomic (for example in case of $\ell^1$) then $B$ has many extreme points. In fact this space does even have the KMP and is in addition a dual space. Known results (see [DU1]) then imply that this space has a stronger property denoted RNP, namely:

$X$ has the *Radon-Nikodym Property* (RNP) if each $B$ as above is the norm closed convex hull of its strongly exposed points ($\text{strexp}(B)$), see sequel or [DU1], p. 190 and p. 218.

Points in $\text{strexp}(B)$ are points of $\text{ext}(B)$ that have beautiful smoothness properties. In particular they are weak to norm continuity points of $B$ and are peak points of $B$. Hence: “Optimisation problems in such sets $B$ have solutions, but moreover, it is easy to test if an algorithm converges to a solution”.

Quoting Jerry Uhl: “A Banach space has the RNP if its unit ball wants to be weakly compact, but just cannot make it”.

**Definition:** Let $B$ be a bounded subset of the Banach space $X$ and $b \in B$. $b$ is a strongly exposed point of $B$ (and $\text{strexp}(B)$ denotes the set of all such), if

$\exists b^* \in X^*$ such that

$$\Re b^*(x) < \Re b^*(b), \forall x \in B \text{ and } x \neq b, \text{ and}$$

$$\Re b^*(x_n) \rightarrow \Re b^*(b) \text{ for } x_n \in B \text{ implies } \|x_n - b\| \rightarrow 0. \text{ (see [DU1] p.138)}$$

Hence in order to test an algorithm for $b \in \text{str exp} B$, it is enough to test it on one element of $X^*$.

Any $X$ which is norm isomorphic to $\ell^1$ has the RNP. If $X$ a dual Banach space, and $B$ is $w^*$ compact convex, then the functional $b^*$ can be chosen in the predual of $X$, see [DU1] p.213.

It follows from above that if $G$ is abelian then $A_2(G)$ has the RNP if $G$ is compact and does not have the RNP if $G$ is not compact.

And yet, for any abelian $G$ and any compact subset $K$,

$A_2^2(G) = \{u \in A_2(G) : \text{spt } u \subset K\}$, does have the RNP, where spt denotes support.
In fact we have proved in [Gr1] that for any $G$ and any compact subset $K$ and any $1 < p < \infty$, $A^p_K = \{ u \in A_p(G) ; \text{spt } u \subseteq K \}$ has the RNP. Tools for abelian $G$ are not available in this case.

It has been proved by W. Braun, in an unpublished preprint [Br], that if $G$ is amenable, then $A^1_p(G)$ is a dual Banach space with the RNP. The result in [Br] uses the method in [Gr1] and the involved machinery of [BrF], which is avoided below.

Denote as in [Hz1]

\[ A_p(G) = \{ u = \sum u_n^* v_n^\vee ; u_n \in L^p, v_n \in L^{p'}, \sum ||u_n||_L^{p'} ||v_n||_L^p < \infty \}, \]

where the norm of $u \in A_p$ is the infimum of the last sum over all the representations of $u$ as above.

We will omit at times $G$ and write $L^p, A_p, \text{ etc.}$ instead of $L^p(G), A_p(G), \text{ etc.}$

Denote by $PM_p(G) = A_p(G)^*$, and by $PF_p(G)$, the norm closure in $PM_p(G)$ of $L^1(G)$, (as a space of convolutors on $L^p(G)$).

Let $W_p(G) = PF_p(G)^*$. Then $W_p(G)$ is a Banach algebra of bounded continuous functions on $G$ containing the ideal $A_p(G)$, studied by Cowling in [Co1]. Let

\[ W^r_p(G) = W_p \cap L^r(G) \]

If $G$ is abelian and $p=2$ then $W_2(G) = M(\hat{G})^\wedge$, where $M(G)$ is the space of bounded Borel measures on $G$. Let $C_0(G) [C_c(G)]$ denote the continuous functions which tend to 0 at $\infty$, [with compact support], with norm $||u||_{C_0} = \sup \{ |u(x)| ; x \in G \}$.

The group $G$ is weakly amenable if $A^2_2(G)$ has an approximate identity $\{ v_\alpha \}$ bounded in the norm of $B^2_2(G)$, the space of Herz-Schur multipliers, see [Hz1]-[Hz2], [Ey2] (or [DCH], [Gr5]). As known the free group on $n > 1$ generators is weakly amenable but non amenable. For much more see [DCH].

Our first result hereby is the

**Theorem 1:** Let $p=2$ or $G$ be weakly amenable and $1 < p < \infty$.

Then

\[ (*) \ W_p \cap L^r(G) = A^r_p \cap L^r(G), \forall 1 \leq r \leq p'. \]

Hence (by [Gr5] Thm. 2.2) $A^r_p(G)$ is a dual Banach space for such $r$.

If $G$ is in addition unimodular then the above holds for $1 \leq r \leq \max(p, p')$.

The interval $[1, p']$ cannot be improved even if $p=2$ and $G = \mathbb{Z}$, the additive integers, see sequel.
Use of Theorem 1 is made in proving the main result of this section, namely:

**Theorem 2:** Let $G$ be a weakly amenable locally compact group. Then
\[ \forall 1 \leq r \leq p', A_p^r(G) = W_p^r(G) \text{ and } A_p^r(G) \text{ is a dual Banach Algebra with the RNP.} \]

If $G$ is in addition unimodular this is the case $\forall 1 \leq r \leq \max(p,p')$.

**Remark:** If $G = \text{SL}(2,\mathbb{R})$, $p = p' = 2$ and $r > 2$, then $A_2^r(G) = A_2(G)$ (see [KuS], [Co2]) and $A_2^r(G)$ does not have the RNP and is not a dual space, see [Gr4] p. 4382.

Note that $G$ is a weakly amenable, but non amenable group.

The unimodular case, for both the above results, was proved in our paper [Gr5].

**Remarks:**
1. A group $G$ with completely reducible regular representation is called in [T] an [AR] group. $G$ is such iff $A_2(G)$ has the RNP, as proved by Keith Taylor [T]. (see[T] for much more). A noncompact [AR] group is called a Fell group, see [B] section IV.

   Larry Baggett and Keith Taylor construct in [BT] p.596 (iii) an example of a connected nonunimodular Lie group $G = G_3$ such that $A_2(G) \neq W_2 \cap C_0(G)$ and such that $G$ is a Fell group. Our next result will imply that for this group $G$, $A_2^r(G)$ has the RNP, for all $r$.

   If [BT] could be improved to show that for some finite $s > 2$, $A_2^s(G) \neq W_2^s(G)$, it would follow that $A_2^s(G)$ having the RNP does not imply that $A_2^s(G) = W_2^s(G)$.

   (Necessarily $s > 2$, if $G = G_4$ in [BT] p.597, (which is amenable) since then $A_2^r(G) = W_2^r(G), \forall 1 \leq r \leq 2$, by Theorem 2).

2. Assume that for arbitrary $G$, $A_2^s(G)$ having the RNP for $s > 2$ would imply the equality $W_2^s(G) = A_2^s(G).$ It would then follow for $G = Z$, that $A_2^s(Z)$ does not have the RNP for all $s > 2$. This is implied by the fact that $A_2^s(Z) = W_2^s(Z), \forall s > 2$ by the Hewitt-Zuckerman [HZ] result, noted in [LiR] and used in [Gr4] p. 4379. Hence there would be no need to take $G = \text{SL}(2,\mathbb{R})$ in the remark above, and $Z$ would suffice.

   In the next section we are interested in the following question:

   Given $p$ and the group $G$, determine the set $O(p,G)$ of those $r$ for which $A_p^r(G)$ has the RNP, (O for optimization!)

   We will show, using results on semi embedings due to H.P.Rosenthal, (see [R], [LPP]) the following
Theorem 3: Let $G$ be second countable and $A_p(G)$ have a multiplier bounded approximate identity: If $A^l_p(G)$ has the RNP for $t = s$ then $A^r_p(G)$ it has the RNP for all $1 \leq t \leq s$, where $s = \infty$ is allowed.

The above results show that:

(I) $[1, p'] \subset O(p, G)$, if $G$ is weakly amenable, and $[1, \max(p, p')] \subset O(p, G)$ if $G$ is in addition unimodular. Note that weak amenability depends only on $2$, yet the result holds for all $p$.

(2) $[1, 2] = O(2, G)$, if $G = SL(2, \mathbb{R})$. This shows that (I) is the best one can do.

(3) $[1, \infty] = O(2, G)$, if $G$ is a Fell group.

2. MAIN RESULTS

(I) NO UNIMODULARITY. We improve hereby results in [Gr4], [Gr5], by removing the unimodularity of the group $G$.

Theorem 1: Let $G$ be a locally compact group. If $p = 2$, or if $G$ is weakly amenable and $1 < p < \infty$, then

\[ (*) \quad W_p \cap L^r(G) = A_p \cap L^r(G), \forall 1 \leq r \leq p'. \]

If $G$ is in addition unimodular then (*) holds for $\forall 1 \leq r \leq \max(p, p')$.

Hence, for the above values of $r$, $A^r_p(G)$ is a dual Banach space.

Remark The interval $[1, p']$ is the best one can do even for $G = \mathbb{Z}$ and $p = 2$ as proved by Hewitt and Zuckerman in [HZ], and as noted in [LiR]. (see also [Gr4] p.4379.)

If $G = SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$ and $p = 2$ then (*) does not hold for any $r > 2$, and in addition $A^r_p$ is not a dual space for $r > 2$.

Proof: By weak amenability, for all $1 < p < \infty$, the $W_p$ norm restricted to $A_p$ is equivalent to the $A_p$ norm, ([Gr5] Corollary 3.7.). If $p = 2$ then Kaplansky’s density theorem will yield the same result.

Now with the notations of [Gr4] Thm. 2.1. p. 4379, if $e_{\alpha} \in C_c(G)$ is an approximate identity for $L_1(G)$, such that each $e_{\alpha}$ is the square of a special operator, a la Fendler [Fe] p.129, we have, loc. cit. $\|e_{\alpha} \ast w - w\| \to 0 \forall w \in W_p$, 
afortiori \( \forall w \in W_p \cap L^{p'} \).

But, since \( e_\alpha \in C_c (G) \), we have for such \( w \), that

\[ e_\alpha * w \in L^p \cap L^{p'} \subset A_p \], thus \( e_\alpha * w \) is a Cauchy sequence in \( A_p \).

Hence \( w \in A_p \). It follows that \( W_p \cap L^{p'} = A_p \cap L^{p'} \).

However by [Co] p.91, \( W_p \cap L^{p'} = A_p \cap L^{p'} \). Hence

\[ W_p \cap L^{p'} = A_p \cap L^{p'}, \forall 1 < p < \infty . \]

But \( W_p \) contains only bounded functions, hence

\[ \forall r \leq p', \ W_p \cap L^r = W_p \cap L^{p'} \cap L^r = A_p \cap L^{p'} \cap L^r = A_p \cap L^r . \] Thus

\[ (i) \quad W_p \cap L^r = A_p \cap L^r, \forall r \leq p'. \]

If \( G \) is unimodular then, since \( (W_p \cap L^r)^\vee = (A_p \cap L^r)^\vee \),

it follows that

\[ (ii) W_p' \cap L^r = A_p \cap L^r, \forall r \leq p', \] which holds for all \( 1 < p' < \infty . \)

Replace now \( p' \) by \( p \) in (ii) and then

\[ (iii) W_p' \cap L^r = A_p \cap L^r, \forall r \leq p. \]

Thus (i) and (iii) imply the unimodular case.

By Theorem 2.2 of [Gr5] \( W_p (G) \cap L^r (G) \) is a dual Banach space for all

\( 1 < p < \infty \) and \( 1 \leq r \leq \infty \), and all locally compact groups \( G \). \( \square \)

**Lemma 1:** Let \( G \) be a locally compact group. Assume that \( A_p (G) \) has an approximate identity \( u_\alpha \) such that \( \sup \| u_\alpha \|_\infty \leq B < \infty \). Then

(a) \( A_p \cap C_c \) is norm dense in \( A_p \) and

(b) If \( G \) is second countable then \( A_p \) is norm separable.

**Proof:** (a) Let \( e_\alpha \in A_p \cap C_c \) satisfy \( \| e_\alpha - u_\alpha \|_{A_p} \to 0 \) and \( \| e_\alpha - u_\alpha \|_{A_p} \leq 1, \forall \alpha \).
Then \( \| \epsilon_{\alpha} \|_{\infty} \leq \| \epsilon_{\alpha} - u_{\alpha} \|_{\infty} + \| u_{\alpha} \|_{\infty} \leq 1 + B \). Hence
\[
\| \epsilon_{\alpha} v - v \|_{A_p} \leq \| \epsilon_{\alpha} - u_{\alpha} \|_{A_p} + \| u_{\alpha} v - v \|_{A_p} \to 0 , \forall v \in A_p .
\]
But if \( w \in A_p^r \) and \( K \subset G \) is compact such that \( \int_{G-K} |w|^r dx < \in \) then
\[
\int_{G-K} |(\epsilon_{\alpha} - 1)w|^r dx \leq \int_{G-K} (2 + B) |w|^r = (2 + B) \in .
\]
But \( \int_{G-K} |(\epsilon_{\alpha} - 1)w|^r \to 0 \). It thus follows that \( \| \epsilon_{\alpha} w - w \|_{A_p^r} \to 0 \). But \( \epsilon_{\alpha} w \in A_p \cap C_c \).

(b) \( A_p^r(G) \) is norm separable, hence so is \( A_p[K] = \{ u \in A_p(G) ; spt u \subset K \} \), where \( K \subset G \). Let \( A_p^r[K] = \{ u \in A_p^r(G) ; spt u \subset K \} \). If \( K \) is compact then the identity
\[
I : A_p^r[K] \to A_p[K]
\]
is \( 1-1 \), onto and continuous, hence it is bi continuous. Hence
\( A_p^r[K] \) is separable. Let now \( K_n \subset \text{int} \, K_{n+1} \subset G \), be compact (int denotes interior), such that \( \bigcup K_n = G \). It is hence enough to show that \( \bigcup A_p^r[K_n] \) is norm dense in \( A_p^r(G) \).

By (a) we know that \( A_p \cap C_c \) is norm dense in \( A_p^r(G) \). But if \( v \in A_p^r(G) \) has compact support \( S \) then \( S \subset K_j \) for some \( j \), hence \( v \in A_p^r[K_j] \). Thus \( \bigcup A_p^r[K_n] \) is norm dense in \( A_p^r(G) \).

**Remark:** We do not know if \( A_p \cap C_c(G) \) is norm dense in \( A_p^r(G) \) even for \( G = SL(2, R) \lt R^2 \), if \( p = 2 \) and all \( r \), (see [Do]).

**Corollary1:** Let \( G \) be a second countable locally compact group. If \( G \) is weakly amenable then \( \forall 1 \leq r \leq p' \), \( A_p^r(G) \) is a separable dual Banach algebra and thus has the RNP.

If \( G \) is in addition unimodular, this is the case \( \forall 1 \leq r \leq \max(p, p') \).

**Remark:** Weak amenability, namely the existence in \( A_2 \) of an approximate identity norm bounded in \( B_2 \) depends only on \( p = 2 \), yet the result holds for all \( p \). Since by Furuta’s Thm.2.4 in [Fu], \( B_2 \subset B_p \) contractively, see also [Gr5] p.23.

**Proof:** By Thm. 2.2 on of [Gr3] and the above Proposition \( A_p^r(G) \) is a dual Banach space \( \forall 1 \leq r \leq p' \). But since \( G \) is weakly amenable \( A_p(G) \), has a multiplier
bounded approximate identity $\forall 1 < p < \infty$, by [Gr2013] p.24. It thus follows by the Lemma above, that $A_p^r(G)$ is norm separable. But separable dual Banach spaces have the RNP by [DU] p.218. □

The main result of this section is the

**Theorem 2:** Let $G$ be a weakly amenable locally compact group. Then

$$\forall 1 \leq r \leq p', A_p^r(G) = W_p^r(G)$$

and $A_p^r(G)$ is a dual Banach algebra with the RNP.

If $G$ is unimodular, this is the case $\forall 1 \leq r \leq \max(p, p')$.

**Proof:** Based on the above Corollary follow the proof of Theorem 3.1 on p.p.22-24 of [Gr2013] and [Gr2011P] p.4381. □

**Remark:** If $G=SL(2,R), p=p'=2$ and $r > 2$ then $A_2^r(G)=A_2(G)$ and $A_2^r(G)$ does not have the RNP and is not a dual space, see [Gr4] p.4382.

(II) INTERVALS WITH THE RNP. We will show that if $G$ is second countable and $A_p(G)$ has a multiplier bounded approximate identity then $A_p^t(G)$ having the RNP for $t = \infty$ implies that it has it for all $1 \leq t \leq s$, where $s = \infty$ is allowed.

**Definition:** Let $X, Y$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. $T$ is a semi-embedding if it is one to one and it maps the closed unit ball in $X$ into a closed set in $Y$. If such $T$ exists we say that $X$ semi embeds in $Y$.

**Theorem [H.P.Rosenthal]:** A separable Banach space has the RNP if it semi-embeds in a Banach space with the RNP. See [DU2] p.160 or [Ro], [LPP].

We will make use of the above Theorem of Rosenthal, to prove the main Theorem 3.

We will need the following

**Lemma 2:** If $r < s$ then the identity $I : A_p^r(G) \to A_p^s(G)$ is a semi-embedding, for any $s \leq \infty$. (If $s = \infty$, $A_p^s(G) = A_p(G)$).

**Proof:** Denote by $B_r$ the closed unit ball of $A_p^r(G)$. Let $v_n \in B_r$ satisfy that

$\|v_n - w\|_{A_p^r} = \|v_n - w\|_{A_p} + \|v_n - w\|_{L^1}\to 0$, for some $w \in A_p^r(G)$.
(If $s = \infty$ only $\|v_n - w\|_{L^p}$ appears). Clearly $|v_n(x)| \to |w(x)|$, $\forall x \in G$. And by Fatou's Lemma we have $\int |w|^r \, dx \leq \liminf \int |v_n|^r \, dx \leq 1$. Thus $w \in A_p^r$. But $1 \geq \limsup(\|v_n\|_{A_p} + \|v_n\|_{L^r}) \geq \lim\|v_n\|_{A_p} + \liminf\|v_n\|_{L^r} \geq \|w\|_{A_p} + \|w\|_{L^r}$. Thus $w \in B_r$. □

**Theorem 3:** Assume that $G$ is second countable and $A_p(G)$ has a multiplier bounded approximate identity.

If for some $t \leq \infty$, $A_p^t(G)$ has the RNP, then so does $A_p^r(G)$, $\forall 1 \leq r \leq t$.

In particular, if $A_p(G)$ has the RNP then $A_p^r(G)$ has the RNP for all $1 \leq r < \infty$.

**Proof:** Apply Rosenthal’s Theorem and the above Lemma 2, and note that, by Lemma 1 $A_p^r(G)$ is necessarily norm separable, due to the existence of the multiplier bounded approximate identity (thus bounded in the uniform norm). □

**Remark:** Note that by [Fu] Thm. 2.4 and p.581, weak amenability implies existence of a multiplier bounded approximate identity.

Note that the Fell group in [B] p.142 is unimodular and CCR.

Mauceri and Picardello have constructed in [MP], amenable and nonamenable, totally disconnected unimodular Fell groups, generalizing the original Fell group. Many of these are $p$-adic matrix groups.

Baggett and Taylor present in [BT] examples of Fell groups which are connected Lie groups and which are (i) solvable, (ii) amenable nonsolvable, (iii) nonamenable, (iv) non-TypeI. All of which are not unimodular.

For all the above ones, $A_p^r(G)$, $\forall 1 \leq r \leq \infty$ has the RNP, by Rosenthal’s theorem and the above lemma.

**Question:** If $G$ is noncompact abelian then $A_p^r(G)$ does not have the RNP, since its closed unit ball has no extreme points see [DU] p.219. Yet, $A_p^r(G)$, $\forall 1 \leq r \leq 2$ does have the RNP, by Theorem 1. For such $G$ nothing is known if $r > 2$.

**REFERENCES**

[B] L. Baggett, A separable group having discrete dual is compact. J. Functional Anal. 10 (1972), 131-148.

[BT] Larry Baggett and Keith Taylor, Groups with Completely Reducible Regular Representation. Proc. Amer. Math. Soc. 72 (1978), 593-600.

[Br] W. Braun: Einige Bemerkungen Zu S_0(G) und A^p(G)intL^1(G). Preprint.

[BrF] W. Braun and Hans G. Feichtinger. Banach Spaces of Distributions Having Two Module Structures. J.Funct. Analysis. 51 (1983) 174-212.

[Co1] Michael Cowling, An Application of Littlewood-Paley Theory in Harmonic Analysis. Math. Ann. 241 (1979), 83-96.
[Co2] Michael Cowling, The Kunze-Stein phenomenon. Ann. Of Math. 106 (1978), 209-234.

[DCH] J. deCanniere and U. Haagerup: multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), 455–500.

[DU1] J. Diestel and J.J. Uhl, Jr: Vector Measures. Math. Surveys. Amer. Math. Soc. 1977.

[DU2] J. Diestel and J.J. Uhl, Jr: Progress in Vector measures – 1977-83. Measure Theory and Applications. LNM. 1033. Springer. 1983

[Do] B. Dorofaeff: The Fourier Algebra of $SL(2,R)/\mathbb{R}$, n>1, has no multiplier bounded approximate unit. Math. Ann. 297 (1993), 707-724.

[Ey1] P. Eymard: L’algebre de Fourier d’un groupe localement compacte. Bull. Soc. Math. France. 92 (1964), 181–236.

[Ey2] P. Eymard: Algebre $A_p$ et convoluteurs de $L_p$. Lecture Notes in Math. No. 180 (Springer 1971), 364–381.

[Fe] Gero Fendler: An Lp version of a theorem of D. A. Raikov. Ann. Inst. Fourier, Grenoble. 35 (1985), 125-135.

[Fu] Koji Furuta: Algebras $A_p$ and $B_p$ and amenability of locally compact groups. Hokkaido Math. J. 20 (1991) 579-591.

[Gr1] Edmond E. Granirer: An Application of the Radon Nikodym Property in Harmonic Analysis. Bull. U.M.I. (5) 18-B (1981), 663-671.

[Gr2]: amenability and semisimplicity for second duals of quotients of the Fourier Algebra $A(G)$. J. Austral. Math. Soc. (Series A) 63 (1997), 289-296.

[Gr3] -------------- The Figa-Talamanca-Herz-Lebesgue Banach Algebras $A^p_r(G)$= $A^p_1 \cap L^r_1(G)$. Math. Proc. Camb. Phil. Soc. 140 (2006), 401-416.

[Gr4] -------------- The Radon-Nikodym Property for some Banach Algebras related to the Fourier Algebra. Proc. Amer. Math. Soc. 139 (2011), 4377-4384.

[Gr5] -------------- Weakly Amenable Groups and the RNP for some Banach Algebras related to the Fourier Algebra. Coll. Math. 130 (2013), 19-26.

[HZ] Edwin Hewitt and Herbert Zuckerman: Singular measures with absolutely continuous convolution squares. Proc. Camb. Phil. Soc. 62 (1966), 399-420.

[HZ1] C. Herz: Harmonic Synthesis for Subgroups. Ann. Inst. Fourier, Grenoble. 23 (1973) 91-123.

[HZ2] C. Herz: The theory of p spaces with an application to convolution operators. Trans. Amer. Math. Soc. 154 (1971) 69–82.

[LLW] Ron Larsen, Ten-sun Liu, Ju-kwei Wang: On functions with Fourier Transforms in $L_p$. Mich. Math. J. 11 (1964), 369-378.

[La1] Hang-Chin Lai: On some properties of $A^p(G)$-algebras. Proc. Japan Acad. 45 (1969), 572-576.

[La2] Hang-Chin Lai: A remark on $A^p(G)$-algebras. Proc. Japan Acad. 46 (1970) 58- 63.

[LPP] Lotz.H.P, Peck.N.T., Porta. H. Semi-embeddings of Banach Spaces. Proc.Edinburgh Math. Soc. 22 (1979), 233-240

[LiR] Teng-sun Liu and Arnoud van Rooij: Sums and Intersections of Normed Linear Spaces. Math. Nachrichten. 42 (1969), 29-42.
[MP] G. Mauceri and M.A. Picardello : Noncompact unimodular groups with purely atomic Plancherel measures . Proc. Amer. Math. Soc.78 (1980) , 77-84.

[Ri] N.W.Rickert : Convolutions of Lp functions. Proc. Amer. Math. Soc. 18 (1967), 762-763. MR0216301 (35:7136)

[R] Rosenthal H.P. Convolution by a Biased Coin. The Altgelt Book 1975/76. University of Illinois Functional Analysis Seminar.

[Sa] Sadahiro Saeki : The Lp conjecture and Young’s Inequality. Ill. J. Math. 34 (1990), 614-627.

[T] Keith Taylor, Geometry of the Fourier Algebras and Locally Compact Groups with Atomic Unitary Representations. Math. Ann. 262 (1983), 183-190.

Dept. of Math. Univ. of B.C. Vancouver B.C. V6T IZ4, Canada.
E-mail address: granirer@math.ubc.ca