Non-relativistic Nambu-Goldstone modes
propagating along a skyrmion line

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Abstract

We study Nambu-Goldstone (NG) modes or gapless modes propagating along a skyrmion (lump) line in relativistic and non-relativistic $O(3)$ sigma model, the latter of which describes isotropic Heisenberg ferromagnets. We show for the non-relativistic case that there appear two coupled gapless modes with a quadratic dispersion; In addition to the well-known Kelvin mode consisting of two translational zero modes transverse to the skyrmion line, we show that the other consists of a magnon and dilaton, that is, a NG mode for a spontaneously broken internal $U(1)$ symmetry and a quasi-NG mode for a spontaneously broken scale symmetry of the equation of motion. We find that the commutation relations of Noether charges admit a central extension between the dilatation and phase rotation, in addition to the one between two translations found recently. The counting rule is consistent with the Nielsen-Chadha and Watanabe-Brauner relations only when we take into account quasi-NG modes.
I. INTRODUCTION

When a symmetry of Hamiltonian or Lagrangian is broken in the ground state, it is said that the symmetry is spontaneously broken. When a continuous symmetry is spontaneously broken, there appear Nambu-Goldstone (NG) modes as gapless excitations which are dominant at low-energy. It is enough to incorporate these degrees of freedom to construct low-energy effective theories. In relativistic theories, there is one-to-one correspondence between broken symmetry generators and NG modes, at least for internal symmetries. In non-relativistic cases, this is not the case; In addition to type-I (relativistic) NG modes with a linear dispersion relation each of which corresponds to one broken generator, there are type-II (non-relativistic) NG modes with a quadratic dispersion relation, each of which corresponds to two broken generators \[1–5\]. The criteria was summarized as the Watanabe-Brauner relation \[3\] stating that when two broken generators \(X_i\) and \(X_j\) do not commute in the ground state, \(\langle [X_i, X_j] \rangle \neq 0\), they give rise to one type-II NG mode, such as magnons in ferromagnets. This has been recently proved for internal symmetries \[4, 5\].

However, there are no general arguments for spontaneously broken space-time symmetry, see, e.g., Refs. \[6–8\] for recent studies. In the presence of topological defects, space-time symmetries are spontaneously broken. For instance, it has been long known that there appears only one type-II NG mode known as a Kelvin mode, or Kelvon if quantized, corresponding to two translational symmetries spontaneously broken in the presence of a quantized vortex in superfluids or a skyrmion (lump) \[13\] in ferromagnets, while there are two type-I NG modes in the case of a relativistic string, see, e.g., Ref. \[9, 10\] for a vortex and Ref. \[11\] for a skyrmion. Recently, Watanabe and Murayama \[12\] have found

\[
[P_x, P_y] = B \neq 0
\]

in the background of a quantized vortex or a skyrmion line, where \(P_x\) and \(P_y\) are the Noether charges of translations perpendicular to the skyrmion or vortex line, and \(B\) is a topological charge for the skyrmion or vortex line. The two translational generators give one type-II NG mode, a Kelvon, to be consistent with the Watanabe-Brauner relation \[3\]. In our previous work \[14\], we have further found

\[
[P_x, \Theta] = W \neq 0
\]
in the background of a domain wall \[15\] (a magnetic domain wall in ferromagnets \[16\]), where \(P_x\) is the Noether charge of the translation perpendicular to the wall, \(\Theta\) is the Noether charge of an internal \(U(1)\) symmetry and \(W\) is a topological charge of the domain wall. A similar result has been obtained in Ref. \[17\] for a domain wall in two-component Bose-Einstein condensates \[18\]. In the relativistic case, the two operators in Eq. (2) commute and there are two type-I NG modes. The latter non-commutative relation \(2\) resembles supersymmetry algebras in the presence of Bogomol’nyi-Prasad-Sommerfield (BPS) solitons in supersymmetric field theories \[19, 20\] and \(p\)-branes in supergravity and string theory \[21\].

In this paper, in addition to Eq. (1) for the translational modes in the presence of a skyrmion line, we show

\[[D, \Theta + M_{12}] \neq 0 (3)\]

where \(D\) is the Noether charge of a dilatation, \(M_{12}\) is the Noether charge of a rotation around the \(z\)-axis along which the skyrmion line is placed. The skyrmion solution has four moduli \(X, Y, \theta\) and \(R\). While \(X, Y\) and \(\theta\) are NG modes of two translations and the internal \(U(1)\) symmetry, respectively, we point out that a dilaton \(R\) is a quasi-NG mode \[22\], which appears when a symmetry of the equation of motion, but not that of the Lagrangian or action, is spontaneously broken. By constructing the low-energy effective field theory on a 1+1 dimensional skyrmion world-sheet via the moduli approximation \[23\], we find that the dilaton \(R\) and the \(U(1)\) NG mode (magnon) \(\theta\) are coupled to give rise to one type-II gapless mode, which is consistent with the Watanabe-Brauner relation only when we count quasi-NG modes, while, in the relativistic case, the dilaton and the \(U(1)\) NG mode appear independently as type-I (quasi-)NG modes. We further study fluctuations around the solution in the Bogoliubov analysis and find the same result.

II. NONLINEAR SIGMA MODELS AND SKYRMIONS

We consider the following relativistic and non-relativistic \(\mathbb{C}P^1\) Lagrangian densities \(L_{\text{rel}}\) and \(L_{\text{nrel}}\):

\[
L_{\text{rel}} = \frac{|\dot{u}|^2 - |\nabla u|^2}{(1 + |u|^2)^2}, \quad L_{\text{nrel}} = \frac{i(u^*\dot{u} - \dot{u}^*u)}{2(1 + |u|^2)} - \frac{|\nabla u|^2}{(1 + |u|^2)^2},
\]

where, \(u \in \mathbb{C}\) is the complex projective coordinate of \(\mathbb{C}P^1\), defined as \(\phi^T = (1, u)^T/\sqrt{1 + |u|^2}\) with normalized two complex scalar fields \(\phi = (\phi_1, \phi_2)^T\). The non-relativistic Lagrangian
\( \mathcal{L}_{\text{nrel}} \) is obtained by taking the non-relativistic limit of \( \mathcal{L}_{\text{rel}} \) (see Appendix A of Ref. \cite{14}). \( \mathcal{L}_{\text{rel}} \) and \( \mathcal{L}_{\text{nrel}} \) can be rewritten as \( O(3) \) nonlinear sigma models,

\[
\mathcal{L}_{\text{rel}} = \frac{1}{4} \{ |\dot{n}|^2 - |\nabla n|^2 \}, \quad \mathcal{L}_{\text{nrel}} = \frac{\dot{n}_1 n_2 - n_1 \dot{n}_2}{2(1 + n_3)} - \frac{1}{4} |\nabla n|^2,
\]

(5)

under the Hopf map for a three-vector of real scalar fields \( n \equiv \phi^\dagger \sigma \phi \) with the Pauli matrices \( \sigma \). These models describe isotropic Heisenberg ferromagnets. In this paper, we use the \( \mathbb{C}P^1 \) model notation.

The Lagrangians \( \mathcal{L}_{\text{rel}} = \int d^3 x \mathcal{L}_{\text{rel}} \) and \( \mathcal{L}_{\text{nrel}} = \int d^3 x \mathcal{L}_{\text{nrel}} \) are invariant under a global \( SU(2) \) rotation of \( \phi \), the Poincaré (for \( \mathcal{L}_{\text{rel}} \)) or Galilean (for \( \mathcal{L}_{\text{rel}} \)) transformation. In the vacuum of the system, i.e., the arbitrary uniform \( u \), the internal \( SU(2) \) symmetry is spontaneously broken down to a \( U(1) \) symmetry with the identification of the global phase of \( \phi \) (phase of \( \phi_1 \)). The vacuum manifold is, therefore, isomorphic to \( \mathcal{M}_1 \equiv SU(2)/U(1) \equiv \mathbb{C}P^1 \equiv S^2 \).

The dynamics of \( u \) can be described by the Euler-Lagrange equation for \( \mathcal{L}_{\text{rel}} \) and \( \mathcal{L}_{\text{nrel}} \):

\[
(1 + |u|^2)\ddot{u} - 2u^* \dot{u}^2 \equiv (1 + |u|^2) \nabla^2 u - 2u^*(\nabla u)^2,
\]

\[
-i(1 + |u|^2)\dot{u} \equiv (1 + |u|^2) \nabla^2 u - 2u^*(\nabla u)^2,
\]

(6)

where, \( \equiv \) and \( \equiv \) correspond to dynamics derived from \( \mathcal{L}_{\text{rel}} \) and \( \mathcal{L}_{\text{nrel}} \) respectively. The equations of motion (6) enjoy an additional symmetry of a scaling transformation \((t, x, y, z) \rightarrow (st, sx, sy, sz)\) for the relativistic case and \((t, x, y, z) \rightarrow (s^2 t, sx, sy, sz)\) for the non-relativistic case with \( s \in \mathbb{R}^+ - \{0\} \approx \mathbb{R} \). These are not symmetry of the Lagrangians or the actions.

We next consider a static skyrmion line solution. A straight skyrmion line solution parallel to the \( z \)-axis is \[13\]

\[
u_0(x, y, z) = \exp \left\{ i \left( \tan^{-1} \frac{y-Y}{x-X} + \theta \right) \right\} \frac{\sqrt{(x-X)^2 + (y-Y)^2}}{R_0 + R},
\]

(7)

where \( R_0 \in \mathbb{R} \) is the characteristic radius of the skyrmion line, and \( X, Y \in \mathbb{R}, \theta (0 \leq \theta < 2\pi) \), and \( R \in \mathbb{R} \) are the translational, phase, and dilatation moduli of the skyrmion line respectively. The tension of the skyrmion line (the energy per unit area) is

\[
T = \int dx dy \frac{|\nabla u_0|^2}{(1 + |u_0|^2)^2} = 2\pi,
\]

(8)

independent of \( X, Y, \theta, \) and \( R \). The \( U(1) \) phase rotation of \( u \), the translation \( \mathbb{R}^2_{x,y} \) inside the \( xy \)-plane, and the dilatation \( \mathbb{R}_{dxy} \) inside the \( xy \)-plane are spontaneously broken in the
vicinity of the skyrmion line. The four moduli \( X, Y, \theta, \) and \( R \) in Eq. (7) are regarded as NG modes corresponding to \( \mathbb{R}_{txy}^2, U(1) \) and \( \mathbb{R}_{dxy} \), respectively, localized in the vicinity of the skyrmion line \[24\]. The NG modes \( X \) and \( Y \) are the translational modes, which are called as Kelvin waves of the skyrmion line in the non-relativistic case. The NG mode \( \theta \) may be called as the localized magnon. We call \( R \) as the dilaton, associated with the spontaneously broken \( \mathbb{R}_{dxy} \). This \( \mathbb{R}_{dxy} \) is merely a symmetry of the equation of motion but not that of the full theory. Consequently \( R \) is a so-called quasi-NG mode \[22\] but not a genuine NG mode, which is gapless at least classically, see, e.g., Ref. \[25\] for an example of a quasi-NG mode in condensed matter physics. The dilaton \( R \) is similar to the localized varicose mode excited along a superfluid vortex in terms of the radius wave of the string \[26\], but it has a gap in the absence of the dilatational symmetry.

III. LOW-ENERGY EFFECTIVE THEORY OF A SKYRMION LINE

We next discuss the dynamics of the localized NG modes in the vicinity of the skyrmion line, by constructing the effective theory on a skyrmion line using the moduli approximation \[23\]. Let us introduce the \( z \) and \( t \) dependences of the moduli in Eq. (7) as \( X(z,t), Y(z,t), \theta(z,t), \) and \( R(z,t) \):

\[
\mathbf{u}(x, y, z) = \exp \left[ i \left\{ \tan^{-1} \left( \frac{y - Y(z,t)}{x - X(z,t)} \right) + \theta(z,t) \right\} \right] \frac{\sqrt{\{x - X(z,t)\}^2 + \{y - Y(z,t)\}^2}}{R_0 + R(z,t)}.
\]

By inserting Eq. (9) back into Eq. (4), the two effective Lagrangians \( \mathcal{L}_{\text{eff}, \text{rel}} \) and \( \mathcal{L}_{\text{eff}, \text{nrel}} \) defined as \( \mathcal{L}_{\text{eff}, \text{rel}} = \int_{-L}^{L} dx \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} dy \mathcal{L}_{\text{rel}} \) and \( \mathcal{L}_{\text{eff}, \text{nrel}} = \int_{-L}^{L} dx \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} dy \mathcal{L}_{\text{nrel}} \) can be calculated, to yield

\[
\mathcal{L}_{\text{eff}, \text{rel}} = \pi (\dot{X}^2 + \dot{Y}^2 - X_z^2 - Y_z^2) + 2\pi \log \left( \frac{L}{R_0} \right) (R_0^2 \dot{\theta}^2 + \dot{R}^2 - R_0^2 \dot{\theta}_z^2 - R_z^2) - 2\pi + O(\partial_z^3),
\]

\[
\mathcal{L}_{\text{eff}, \text{nrel}} = -\pi L^2 \dot{\theta} + \pi (\dot{X}Y - \dot{Y}X) - \pi (X_z^2 + Y_z^2)
\]

\[
+ 2\pi \log \left( \frac{L}{R_0} \right) \{2R_0 \dot{R} - \pi (R_0^2 \dot{\theta}_z^2 + R_z^2)\} - 2\pi + O(\partial_z^3),
\]

up to the quadratic order in derivatives.

The low-energy dynamics of \( X, Y, \theta \) and \( R \) derived from the Euler-Lagrange equation...
becomes
\[
\begin{align*}
\dot{X}^\text{rel} &= X_{zz}, & \dot{Y}^\text{rel} &= Y_{zz}, & \dot{\theta}^\text{rel} &= \theta_{zz}, & \dot{R}^\text{rel} &= R_{zz}, \\
\dot{X}^\text{nrel} &= -Y_{zz}, & \dot{Y}^\text{nrel} &= X_{zz}, & \dot{\theta}^\text{nrel} &= -R_{zz}, & \dot{R}^\text{nrel} &= R_0\theta_{zz}.
\end{align*}
\]
(11a)
(11b)

For the relativistic case, all dynamics of \(X, Y, \theta\), and \(R\) are independent of each other, giving linear dispersions:
\[
\omega_{\text{rel}} = \pm |k|,
\]
with the frequencies \(\omega_{\text{rel}}\), and the wave-number \(k\). Oscillations \(X\) and \(Y\) of the skyrmion line into the \(x\) and \(y\)-directions, a localized magnon \(\theta\) and a dilaton \(R\) independently propagate along the \(z\)-axis.

With being different from the relativistic case, there are two different coupled modes, \(X\) and \(Y\), and \(\theta\) and \(R\) in the non-relativistic case. Typical solutions of Eq (11b) are
\[
\begin{align*}
X &= A_{(XY)}\pm \sin(kz \mp \omega_{nrel}t + \delta_{(XY)}), & Y &= \mp A_{(XY)}\pm \cos(kz \mp \omega_{nrel}t + \delta_{(XY)}), \\
\theta &= A_{(\theta R)}\pm \sin(kz \mp \omega_{nrel}t + \delta_{(\theta R)}), & R &= \mp R_0A_{(\theta R)}\pm \cos(kz \mp \omega_{nrel}t + \delta_{(\theta R)}),
\end{align*}
\]
(13a)
(13b)

where \(A_{(XY)},(\theta R)\in \mathbb{R}\) and \(\delta_{(XY)},(\theta R)\in \mathbb{R}\) are arbitrary constants. Waves of \(X\) and \(Y\) couple and propagate as a spiral Kelvin wave, and \(\theta\) and \(R\) couple to each other and propagate as a coupled magnon-dilaton, both with a quadratic dispersion
\[
\omega_{nrel} = k^2.
\]
(14)

For the upper and lower signs in Eqs. (13a) and (13b), each coupled NG mode propagates in the direction parallel and anti-parallel to \(z\)-axis, respectively. In contrast to the Kelvin wave in Eq. (13a) which are combinations of two translational modes in real space, the localized coupled magnon-dilaton mode in Eq. (13b) is a combination of the phase mode of the internal degrees of freedom and the dilatation in real space. Figures 1 shows the schematic picture of coupled localized magnon-dilaton mode for Eq. (13b).

IV. LINEAR RESPONSE THEORY

Linear response theory is another technique to study the dynamics of the gapless modes \(X, Y, \theta, R\). Let us consider the ansatz of the straight skyrmion line solution and its fluctuation: \(u = u_0 + \delta u = u_0 + a_+ e^{i(kz - \omega t)} + a_-^* e^{-i(kz - \omega t)}\). By inserting this ansatz into the
The arrows show the direction of \( \mathbf{n} = \phi^\dagger \sigma \phi \) with \( \phi^T = (1, u)/\sqrt{1 + |u|^2} \) and their colors show the value of \( \theta \). The transparent surface shows the isosurface for \( |u| = 0 \) \( (n_3 = 0) \). For left (right) figures, the coupled localized magnon-dilaton propagate in the upper (lower) direction.

By expanding \( a_{\pm} \) as \( a_{\pm} = \sum_l a_{\pm,l} e^{il\phi} \), we obtain

\[
\omega_{\text{rel}, \pm}^2 a_{\pm,l} \equiv \left\{ (k^2 - \nabla_r^2 - \partial_r/r + l^2/r^2) + \frac{4(r \partial_r \pm i \partial_\theta)}{r^2 + R_0^2} \right\} a_{\pm,l} + O(a_{\pm}^2),
\]
\[
\omega_{\text{nrel},\pm} a_{\pm,l} \equiv \pm\left\{ (k^2 - \nabla_r^2) + \frac{4(r \partial_r \pm i \partial_\theta)}{r^2 + R_0^2} \right\} a_{\pm,l} + O(a_{\pm}^2),
\]
up to the linear order of \( a_{\pm} \). Here, \( \nabla_r = (\partial_x, \partial_y) \) denotes the derivative in the \( xy \)-plane. By expanding \( a_{\pm} \) as \( a_{\pm} = \sum_l a_{\pm,l} e^{il\phi} \), we obtain

\[
\omega_{\text{rel}, \pm}^2 a_{\pm,l} \equiv \left\{ (k^2 - \partial_r^2 - \partial_r/r + l^2/r^2) + \frac{4(r \partial_r \pm l)}{r^2 + R_0^2} \right\} a_{\pm,l} + O(a_{\pm}^2),
\]
\[
\omega_{\text{nrel},\pm} a_{\pm,l} \equiv \pm\left\{ (k^2 - \partial_r^2 - \partial_r/r + l^2/r^2) + \frac{4(r \partial_r \pm l)}{r^2 + R_0^2} \right\} a_{\pm,l} + O(a_{\pm}^2).
\]

There are two characteristic solutions related to the Kelvin wave, localized magnon, and dilaton: \( a_{\pm,0} = 1 \) and \( a_{\pm,1} = r/R_0 \), and eigenvalues are \( \omega_{\text{rel}}^2 = k^2 \) and \( \omega_{\text{nrel},\pm} = \pm k^2 \). For the
relativistic case, NG modes for $X$, $Y$, $\theta$, and $R$ are obtained as

\begin{align}
X &: \delta u = \Delta X (a_{+0} e^{i(kz + \delta X)} + a_{-0} e^{-i(kz + \delta X)}), \\
Y &: \delta u = \Delta Y (a_{+0} e^{i(kz + \delta Y)} - a_{-0} e^{-i(kz + \delta Y)}), \\
\theta &: \delta u = \Delta \theta (a_{+1} e^{i\phi} e^{i(kz + \delta \theta)} - a_{-1} e^{i\phi} e^{-i(kz + \delta \theta)}), \\
R &: \delta u = \Delta R (a_{+1} e^{i\phi} e^{i(kz + \delta R)} + a_{-1} e^{i\phi} e^{-i(kz + \delta R)}),
\end{align}

(17)

with arbitrary constant $\Delta X, Y, \theta, R, \delta X, Y, \theta, R \in \mathbb{R}$. The upper (lower) sign in Eq. (17) shows the NG modes propagating in the direction parallel (anti-parallel) to $z$-axis. For non-relativistic case, the coupled NG modes for, $(X, Y)$, and $(\theta, R)$ are obtained as

\begin{align}
(X, Y) &: \delta u = \Delta (XY) (a_{+0} e^{i(kz + \delta XY)}), \\
(\theta, R) &: \delta u = \Delta (\theta R) (a_{+1} e^{i\phi} e^{i(kz + \delta \theta R)}).
\end{align}

(18)

We shortly note that there are countably infinite number of gapless solutions to the Bogoliubov-de Gennes equation (15) : $a_{+,n} = r^n / R_0^n (n \in \mathbb{Z}^+_0)$ and corresponding zero modes $\omega^2_{rel} = k^2$ and $\omega_{nrel,\pm} = \pm k^2$ besides the present solutions $a_{+,0}$ and $a_{-,1}$. Solutions for $n = 0, 1, 2$ correspond to the (quasi-)NG modes, i.e., $a_{+,0}$ corresponds to the Kelvin waves $a_{+,1}$ corresponds to the localized magnon and dilaton, and $a_{+,2}$ corresponds to the bulk magnon far from the skyrmion line, for which we have not discussed in this paper. The other solutions $a_{+,n} \neq 0, 1, 2$ do not originate from any symmetry of the Lagrangians and cannot be regarded as NG modes. We will soon discuss this in detail elsewhere.

V. COMMUTATION RELATION

By the two techniques of the effective theory with the moduli approximation and the linear-response theory, we have shown the independence of the four gapless modes in the presence of the skyrmion line; the two translational modes, the localized magnon and the dilaton. They are independent of each other with the linear dispersion relations (12) for the relativistic theory with $L_{rel}$, while the coupled spiral Kelvin wave and the coupled localized magnon-dilaton are formed showing the quadratic dispersion relations (14) for the non-relativistic theory with $L_{nrel}$. These modes are (quasi-)NG modes appearing as a consequence of the spontaneous breaking continuous symmetries; the $\mathbb{R}^2_{txy}$ translational symmetry for Kelvin waves, the $U(1)$ symmetry for the localized magnon, and the two-dimensional $\mathbb{R}_{dxy}$
scaling symmetry for the dilaton. The Lorentz invariance in the relativistic model supports that the number of (quasi-)NG modes is equivalent to that of symmetry generators $N_{BG}$ corresponding to spontaneously broken symmetries, and all (quasi-)NG modes are type-I for the linear dispersion \[12\]. Without the Lorentz invariance, there appear not only type I (quasi-)NG modes but also type II (quasi-)NG modes with the quadratic dispersion and the relation between the number of (quasi-)NG modes and $N_{BG}$ becomes more complicated. In both cases, the numbers of (quasi-)NG modes saturates the equality of the Nielsen-Chadha inequality \[1\]: $N_I + 2N_{II} \geq N_{BG}$, where $N_I$ and $N_{II}$ are the total numbers of the type-I NG modes and the type-II NG modes. In the case of internal symmetries, it has been shown in Refs. \[4, 5\] that the equality of the Nielsen-Chadha inequality is saturated as the Watanabe-Brauner’s relation \[3\]:

$$N_{BG} - N_{NG} = \frac{1}{2} \text{rank} \rho, \quad \rho_{i,j} = \lim_{V \to \infty} \frac{1}{V} \int d^3x [\Omega_i, \Omega_j] \bigg|_{u=u_0}. \quad (19)$$

Here, $N_{NG} = N_I + N_{II}$ is the total number of NG modes, $V$ is the volume of the system, $\Omega_i$ is the Noether charge or a generator of broken symmetries, and $[\cdot, \cdot]$ is a commutator or the Poisson bracket in classical level. We see that a mismatching $N_{BG} \neq N_{NG}$ takes place when commutators of broken generators are non-vanishing in the ground states in non-relativistic theories. This relation has been proven for cases of the bulk magnons (two internal symmetries) and the Kelvin wave (two space-time symmetries) in the massless $\mathbb{C}P^1$ model for the isotropic Heisenberg ferromagnet \[12\]. It has been also proved for the translational and internal $U(1)$ zero modes in the background of a domain wall in the massive $\mathbb{C}P^1$ model for the Heisenberg ferromagnet with one easy axis \[14\]. As the case of a domain wall in Ref. \[14\], the two broken generators corresponding to the coupled localized magnon-dilaton, that is, the internal $U(1)$ symmetry and translational symmetry, intuitively commute, because underlying symmetries are the direct product and are independent of each other, \textit{i.e.}, $U(1) \times \mathbb{R}_{dxy}$. In order to check whether the relation \[19\] also holds in our case or not, let us directly calculate the commutation relation between Noether charges of symmetry generators of the localized magnon and the dilaton.

Before calculating the commutator for the localized magnon and the dilatation mode, we briefly overview the commutator of the two translations for the spiral Kelvin wave \[12\] which also intuitively commute with each other. Let us define the momenta $v$ conjugate to
Then, the Noether’s charges for the translations for $x$ and $y$-directions are obtained as

$$P_x = \int d^2x \ J_0^0, \quad P_y = \int d^2x \ J_1^0, \quad J_0^0 = u_x v, \quad J_1^0 = u_y v.$$ (21)

The commutator between $P_x$ and $P_y$ can be calculated from $[u(x_1, y_1), v(x_2, y_2)] = \delta(x_1 - x_2)\delta(y_1 - y_2) \equiv \delta^2(\mathbf{x}_1 - \mathbf{x}_2)$, to yield

$$[P_x, P_y] = \int d^2x \ J_0^0 \ J_1^0 = \int d^2x_1 \int d^2x_2 \ [u_x(x_1, y_1)v(x_1, y_1), u_y(x_2, y_2)v(x_2, y_2)]$$

$$= \int d^2x_1 \int d^2x_2 \ \{u_x(x_1, y_1)[v(x_1, y_1), u_y(x_2, y_2)]v(x_2, y_2)$$

$$\quad + u_y(x_2, y_2)[u_x(x_1, y_1), v(x_2, y_2)]v(x_1, y_1)\}$$

$$= \int d^2x_1 \int d^2x_2 \ \{u_x(x_1, y_1)v_{y_2}(x_2, y_2) - u_{y_2}(x_2, y_2)v(x_1, y_1)\} \delta^2(\mathbf{x}_1 - \mathbf{x}_2)$$

$$= \int d^2x \ \{u_x(x, y)v_{y}(x, y) - u_y(x, y)v(x, y)\}$$

$$= \int d^2x \ \frac{u_r v_\phi - u_\phi v_r}{r},$$

with the cylindrical coordinate $(r, \phi)$.

The commutator vanishes for the relativistic case because of $v = 0$ ($\dot{u} = 0$), implying two type I NG modes.

For the non-relativistic case, the commutator becomes

$$[P_x, P_y] = \int d^2x \ \frac{u_r v_\phi - u_\phi v_r}{r} = \int d^2x \ b = B \neq 0.$$ (23)

where $b$ and $B$ are the topological charge density and the total topological charge of the skyrmion line:

$$b = u_x v_y - u_y v_x = \frac{u_r v_\phi - u_\phi v_r}{r} = \frac{R_0^2}{(r^2 + R_0^2)^2}, \quad B = \int_0^\infty dr \int_0^{2\pi} d\theta r b = \pi.$$ (24)

In the last equalities in (24), we have used the background of a single skyrmion line $u = u_0 = re^{i\phi}/R_0$. 
We next calculate the commutator for the localized magnon and dilaton. The Noether’s charges for the phase shift and the dilatation are obtained as

\[
\Theta = \int d^2 x \, J_\theta^0, \quad J_\theta^0 = iuv,
\]

\[
D = \int d^2 x \, J_R^0, \quad J_R^0 = (xu_x + yu_y)v,
\]

respectively. The commutator between them reads

\[
[D, \Theta] = \int d^2 x_1 \int d^2 x_2 \left[ J_D^0(x_1, y_1), J_\theta^0(x_2, y_2) \right]
\]

\[
= i \int d^2 x_1 \int d^2 x_2 \left[ \{ x_1 u_x, (x_1, y_1) + y_1 u_y, (x_1, y_1) \} v(x_1, y_1), u(x_2, y_2)v(x_2, y_2) \right]
\]

\[
= i \int d^2 x_1 \int d^2 x_2 \left[ x_2 \{ u(x_1, y_1)[u_{x_2}(x_2, y_2), v(x_1, y_1)]v(x_2, y_2)
\]

\[
+ u_{x_2}(x_2, y_2)[v(x_2, y_2), u(x_1, y_1)]v(x_1, y_1) \}
\]

\[
+ y_2 \{ u(x_1, y_1)[u_{y_2}(x_2, y_2), v(x_1, y_1)]v(x_2, y_2)
\]

\[
+ u_{y_2}(x_2, y_2)[v(x_2, y_2), u(x_1, y_1)]v(x_1, y_1) \} \right].
\]

\[
= -i \int d^2 x_1 \int d^2 x_2 \left[ x_2 \{ u(x_1, y_1)v_{x_2}(x_2, y_2) + u_{x_2}(x_2, y_2)v(x_1, y_1)\}
\]

\[
+ y_2 \{ u(x_1, y_1)v_{y_2}(x_2, y_2) + u_{y_2}(x_2, y_2)v(x_1, y_1)\}
\]

\[
+ 2u(x_1, y_1)v(x_2, y_2)\delta^2(x_1 - x_2)
\]

\[
= -i \int d^2 x \left[ r\{ u_r(r, \phi)v(r, \phi) + u(r, \phi)v_r(r, \phi)\} + 2u(r, \phi)v(r, \phi) \right].
\]

For the non-relativistic case, the commutator becomes

\[
[D, \Theta] = \int d^2 x \frac{r^2(r^2 + 2R_0^2)}{(r^2 + R_0^2)^2} = \int d^2 x \left(\frac{1}{r^2 + R_0^2} \right) \neq 0,
\]

while it vanishes for the relativistic case.

The ansatz in Eq. (9) with \( X = Y = 0 \)

\[
u(r, \phi, z) = \frac{r \exp\{i(\phi + \theta)\}}{R_0 + R}
\]

implies that the localized magnon \( \theta \) can be induced not only by the phase shift of \( u, u \to ue^{i\theta} \), but also by a spatial rotation along \( z \)-axis, \( \phi \to \phi + \theta \). Therefore, we further calculate the commutator between the spatial rotation and the dilatation. The Noether’s charge for the
rotation is
\[ M_{12} = \int d^2 x \, J^0_\phi, \quad J^0_\phi = (x u_y - y u_x)v. \]  
(30)

As well as Eq. (27), the commutator becomes

\[
[D, M_{12}] = \int d^2 x_1 \int d^2 x_2 \left[ J^0_D(x_1, y_1), J^0_\phi(x_2, y_2) \right] \\
= \int d^2 x \left[ (x^2 + y^2)\{u_x(x_2, y_2)v_y(x_1, y_1) - u_y(x_1, y_1)v_x(x_2, y_2)\} \right. \\
\left. + 2\{y u_x(x_1, y_1) - x u_y(x_2, y_2)\}\}v(x, y) \right] \\
= \int d^2 x [r\{u_r(r, \phi)v_\phi(r, \phi) - u_\phi(r, \phi)v_r(r, \phi)\} - 2u_\phi(r, \phi)v(r, \phi)] \\
= \int d^2 x r^2 \left( b + \frac{1}{r^2 + R_0^2} \right) \\
= [D, \Theta].
\]  
(31)

The fact \([D, \Theta - M_{12}] = 0\) implies that \(u_0\) is invariant under a simultaneous action of the phase shift \(u \rightarrow u e^{i\theta}\) and the spatial rotation \(\phi \rightarrow \phi - \theta\). Therefore, we find an independent non-vanishing commutation relation

\[
[D, \Theta + M_{12}] = 2 \int d^2 x \, r^2 \left( b + \frac{1}{r^2 + R_0^2} \right) \neq 0,
\]  
(32)

being consistent with our result for the coupled localized magnon-dilaton.

VI. CONCLUSION AND DISCUSSION

In conclusion, we have considered (quasi-)NG modes excited along one straight skyrmion line in the relativistic and non-relativistic \(\mathbb{C}P^1\) or \(O(3)\) sigma models. The non-relativistic model describes isotropic Heisenberg ferromagnets. The (quasi-)NG modes in the relativistic model consist of the two translational (Kelvin) modes, the localized magnon, and the dilatation mode which are independent of each other and have linear dispersions. In the non-relativistic model, on the other hand, there are the coupled spiral Kelvin wave and localized magnon-dilaton mode with quadratic dispersions. Only when we count quasi-NG modes, the numbers of gapless modes saturate the equality of the Nielsen-Chadha inequality and satisfies the Watanabe-Brauner’s relation, in which the commutator between two generators of the internal phase mode and the dilatation mode is related to the topological charge of skyrmions.
Several comments and discussions are addressed here. The coupled magnon-dilaton found in this paper is non-normalizable; The effective Lagrangian for that is divergent for infinite volume limit \( L \to \infty \). When there are multiple skyrmion strings, one coupled dilaton-magnon is localized on each of them. While the “overall” mode, which is a NG mode of the global symmetry, is non-normalizable, “relative” modes, which can be regarded as locally NG modes for approximate local transformations, are normalizable, as was shown in Refs. \[27, 28\].

The \( \mathbb{C}P^1 \) manifold has the Kähler form \( \omega = i du \wedge du^*/(1 + |u|^2)^2 \) and the topological charge, \( \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z} \), is the pulback of this form into a two-dimensional space perpendicular to the skyrmion string. Skyrmion strings are admitted in any nonlinear sigma model with Kähler target spaces \( M \) with \( \pi_2(M) \neq 0 \), such as the projective space \( \mathbb{C}P^N \) and the Grassmann sigma model \[28\]. With a locally defined one-form \( \alpha \) satisfying \( \omega = d\alpha \), the first-order time derivative term can be constructed in the Lagrangian, and so our results can be extended to general Kähler manifolds.

Quantum effects on localized type-II modes remain as an important problem, which was studied before for a vortex with localized type-II non-Abelian NG modes \[29\]; localized type-II NG modes remain gapless, unlike the case of relativistic theories in which all NG modes in 1+1 dimensions are gapped through quantum corrections in consistent with the Coleman-Mermin-Wagner theorem. Quasi-NG modes are in general gapped with taking into account quantum corrections even in the bulk 3+1 dimensions, because they are not associated with an exact symmetry of Lagrangians. In our case, the magnon-dilaton is a half genuine NG mode, therefore the fate in quantum corrections is a non-trivial question.

In our previous paper \[14\], we have studied the NG modes of a domain wall in the \( O(3) \) sigma model with a potential term admitting two discrete vacua \[15\], that describes ferromagnets with one easy axis. A skyrmion studied in this paper and a domain wall in the massive \( O(3) \) sigma model can be related by a dimensional reduction \[30\], as in the case between Yang-Mills instantons and BPS magnetic monopoles. How type-II NG modes and corresponding commutation relations for a skyrmion and a domain wall are related to each other remains as a future problem.

In \( d = 3 + 1 \) dimensions, the massive \( O(3) \) sigma model admits a composite soliton of skyrmion strings ending on a domain wall, known as a D-brane soliton \[31, 32\]. D-brane solitons exist also in two-component Bose-Einstein condensates \[33\], for which NG modes
have been studied in the presence of a domain wall \[17, 18\]. Investigating NG modes for such a composite soliton will be one interesting direction.

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