Crust-core interactions and the magnetic dipole orientation in neutron stars.

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We develop an effective model for a neutron star with a magnetosphere. It takes into account the electromagnetic torques acting on the magnetic dipole, the friction forces between the crust and the core, and the gravitational corrections. Anomalous electromagnetic torques, usually neglected in a rigid star model, play here a crucial role for the alignment of the magnetic dipole. The crust-core coupling time implied by the model is consistent with the observational data and other theoretical estimations. This model describes the main features of the behavior of the magnetic dipole during the life of the star, and in particular gives a natural explanation for the $n < 3$ value of the breaking index in a young neutron star.

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1. Introduction

Pulsars are identified with rapidly rotating, highly magnetized neutron stars. The basic information we receive from this kind of stars is a sequence of electromagnetic pulses with a very stable frequency, which is interpreted as directly related to the star rotation. The angular velocity $\Omega$ generally decreases gradually as a result of the torque exerted on the star by the radiation reaction. In the vacuum dipole model the star magnetic field is assumed to be a magnetic dipole $M$ that forms an angle $\theta$ with the rotation axis, so that the star loses energy by electromagnetic radiation due to its rotation (Pacini 1967, 1968; Gunn & Ostriker 1969). This turns out to be the main source for the star energy loss. In this simple model the evolution of the angular velocity for a neutron star with a momentum of inertia $I$ is

$$\dot{\Omega} = -\frac{2}{3} \frac{M^2 \Omega^3}{I c^3} \sin^2 \theta,$$

or, more generally, $\dot{\Omega} = -\mu \Omega^3$. Several characteristics of the dynamics of the $\mu$ parameter have been observed. After glitches a sudden increment of $\mu$ has been noticed, which does not completely relax back (Link, Epstein & Baym 1992; Link & Epstein 1997). There is also evidence that for old pulsars, with an age of $\frac{\Omega}{2\Omega} \sim 10^7 \text{yr}$, $\mu$ is smaller than for younger pulsars (Ruderman, Zhu & Chen 1997). The first behavior can be interpreted as an increase of the external torque after de glitches, whereas the second one suggests a slow decrease of the torque with the age of the star.

Besides this, the breaking index $n = \frac{\Omega}{I^2}$ has been measured in four young pulsars, and it takes values between 1.4 and 2.8. If $\mu$ were constant the breaking index would be exactly equal to 3, but the measured values are smaller than this canonical value implying an increasing $\mu$. A number of factors might affect the breaking index. One of them is the presence of mechanisms of loss of energy different from the dipolar electromagnetic radiation that could change the $\Omega$ exponent in eq. (1). For example, pure multipolar electromagnetic
radiation gives $n \geq 5$, gravitational quadrupole radiation $n = 5$ (Manchester & Taylor 1977), and early neutrino emission $n < 0$ (Alpar & Ögelman 1990). These effects are expected to be relatively weak for the pulsars with a measured breaking index, and in particular the first two would increase $n$ and thus are unable to explain the observed values smaller than 3. Another possible factor that could be considered is the effect of the pulsar magnetosphere (Goldreich & Julian 1961), but although it would change appreciably the angular dependence and the numerical coefficients in eq. (1), it leaves the $\Omega$ exponent equal to 3 (Good & Ng 1983), and thus does not affect the value of the breaking index provided that $\mu$ remains constant (see references (Michel 1983, 1985) for an alternative). According to this, the most natural explanation seems to require variations of the parameters $I, M$ and $\theta$, which change the breaking index in the following way for a star without magnetosphere:

$$n = 3 + \frac{\Omega \dot{\mu}}{\Omega \mu} = 3 + \frac{\Omega}{\dot{\Omega}} \left( 2 \frac{\dot{\theta}}{\tan \theta} + 2 \frac{\dot{M}}{M} - \frac{\dot{I}}{I} \right).$$ (2)

Given that $(n - 3)$ is of the order of the unity, the parameters responsible for the variation should have characteristic evolution times of the order of the pulsar age $\tau_\Omega = \frac{\Omega}{2 \dot{\Omega}}$, which is of around $1000 \text{yr}$ in the case of the stars of known breaking index.

With respect to the momentum of inertia, sensible variations during such times due to changes in the structure, sphericity, or in the coupling of the internal superfluids to the crust, seem to be unable to explain the observed deviations (Link & Epstein 1997).

Significant changes in the magnetic moment also seem to be inhibited by the high crust conductivity, which can keep the magnetic fields unchanged and thus fixed to the crust along times of order $10^7 \text{yr}$ or more (Lamb 1991; Chanmugan 1992; Phynney & Kulkarni 1994; Goldreich & Reisenegger 1992) and hence the breaking index $n$ is unaffected in young pulsars. Furthermore, the decay of the magnetic field reduces the radiation rate and hence would lead to $n > 3$. However, there is no general consensus about the behavior of the intensity of the effective magnetic moment and other hypotheses have been proposed.
Muslinov and Page (1995) consider magnetic fields trapped under the surface during the birth of the star, which could rapidly be freed and would thus increase the total magnetic moment (see also (Blandford, Applegate & Hernquist 1983; Chanmugam & Sang 1989)). Ruderman (1991a, 1991b, 1991c) considers the possibility of crust cracking, reordering due to vortex pinning and stresses. If flux tubes in the core proton superconductor are pinned to neutron superfluid vortices, the latter could drag the magnetic structure to the equator as the star spins down (Ruderman, Zhu & Chen 1997; Srinivasan et al. 1990). In this work we adopt the first hypothesis and simply assume a stable external magnetic field rigidly fixed to the crust, except perhaps in very short periods at the glitches.

Under the above assumptions, the only responsible for the anomalous breaking index could be the variation of the angle between the magnetic dipole moment $\mathbf{M}$ and the rotation axis $\hat{\Omega}$. In this case eq. (2) becomes:

$$n - 3 = 2 \frac{\Omega}{\dot{\Omega}} \tan \theta .$$

The angular momentum of the star decreases during its life by radiation, and thus $\dot{\Omega}$ is negative. Therefore a value $n < 3$ requires an increase in time of $\theta$, e.g. $\dot{\theta} > 0$. In brief, for a rigid star in vacuum the observed dynamics is related to the behavior of the $\theta$ angle. The observational data suggest that its evolution is controlled by three characteristic times. There is a fast dynamics, associated to the reorientation of the magnetic dipole during the glitches, with a characteristic time less than or of the order of 50 days, an intermediate dynamics that dominates during the early stages of the stars, where the magnetic moment slides towards the equator and gives the breaking index values smaller than 3, and finally a slow dynamics, where the magnetic moment slides towards the rotation axis of the star, which involves characteristic times of the order of $\tau_\Omega \sim 10^7 \text{yr}$ (Lyne & Manchester 1988). The intermediate dynamics competes with the slow one, and the observational evidence shows that for young pulsars it dominates, leading to the anomalous values of the breaking
index, whereas for old pulsars ($\tau \Omega \sim 10^7$ years) it is the slow one which dominates, giving place to the alignment of the magnetic dipole with the rotation axis. However, if we only took into account the radiation reaction of the magnetic dipole on a rigid star, the evolution of $\theta$ would perform a slow alignment during the life of the star ([Pandey & Prasad 1996] with a breaking index $n > 3$, in contradiction with the features mentioned before.

In this work we consider a more realistic description given by a simple effective model, where the star is considered as constituted by two rigid interacting components in presence of a magnetosphere. One of them is the core which contains the bulk of the mass, with a momentum of inertia $I_o$ and an angular velocity $\Omega_o$. Its angular momentum is mainly given by the vortices of superfluid neutrons. The other component is the crust, with a momentum of inertia $I_c \ll I_o$ and an angular velocity $\Omega_c$, and with a dipole magnetic moment $M$. The evolution of the system is governed by three kinds of torques:

a.- torques acting on the crust due to electromagnetic interactions,

b.- friction torques between the core and the crust due to the interaction of the neutron vortices, the proton flux tubes and the electrons in the star,

c.-torques due to gravitational effects.

Although this model could seem a rather simplified approach, it is sufficient for obtaining a good qualitative comprehension and can be considered as a first step to a more sophisticated description.

In the following section we present a review of the different effective torques we take into account in our model. In Section III we write the system of equations that describes the dynamics of the model and we analyze the resulting behavior. Finally, the last two sections are devoted to the discussion of the results and their phenomenological implications.
2. The effective torques

In this section we review the torques that act on both components of the star. In principle the coordinates of our model are represented by three vectors, \( \mathbf{M} \), \( \Omega_0 \) and \( \Omega_c \), which are the magnetic moment, the core angular velocity and the crust angular velocity respectively. It should be remarked that the two components considered in the present work are identified with the crust and core, and they are not the ones used to describe the long relaxation time after glitches, which is currently identified as an effect of the crustal superfluid (Anderson & Itoh 1973; Alpar, Cheng & Pines 1989; Link & Epstein 1996). Once we have chosen a direction of reference there remains five degrees of freedom, because we have assumed the magnetic dipole moment constant in modulus and fixed to the crust. So the equation of motion for the magnetic field is simply

\[
\dot{\mathbf{M}} = \Omega_c \times \mathbf{M}.
\]

In the following subsections we describe the torques that arise from the different interactions.

2.1. Electromagnetic torques on the magnetic dipole

We have a magnetic dipole \( \mathbf{M} \) which forms an angle \( \theta \) with the angular velocity \( \Omega_c \) of the crust. There are several dipole and quadrupole torque terms that act on the magnetic dipole, and as we have assumed that it is bound to the crust, these torques are directly applied to the latter. They have been carefully discussed by Good and Ng (1985). The effect of the average torques that involve quadrupole terms are strongly suppressed, and hence it is enough to consider only the dipolar components. There are two types of dipolar torques that act on the crust, the non-anomalous torques of order \( \frac{M^2 \Omega^3}{c^3} \), which are of the same order as the classical spin down torque (Davis & Goldstein 1970; Michel & Goldwire 1970), and the anomalous torques that are \( \frac{c}{\Omega_c R} \) times larger than the former ones, where \( R \)
is the radius of the star. In the following we will use an orthogonal system of reference with \( \mathbf{\hat{z}} \) in the \( \Omega_c \) direction, \( \mathbf{\hat{y}} \) in the plane \((\Omega_c, \mathbf{M})\), and \( \mathbf{\hat{x}} \) orthogonal to such a plane, according to Fig. 1. The \( \mathbf{\hat{y}} \), and \( \mathbf{\hat{z}} \) torques, which cause the loss of energy and the alignment of the magnetic dipole in a rigid star, are non anomalous. In this plane the torque is

\[
T_{yz} = I_c \omega_{yz} \mathbf{\hat{M}} \times (\mathbf{\hat{M}} \times \Omega_c) - I_c \omega_{yz} \Omega_c, \tag{5}
\]

where \( \mathbf{\hat{M}} = M^{-1} \mathbf{M} \), \( \omega_{yz} = \frac{2}{3} \frac{M^2 \Omega_c^2}{I_c c} \nu_{yz} \) and \( \omega_{yz} = \frac{2}{3} \frac{M^2 \Omega_c^2}{I_c c} \nu_{yz} \). The \( \nu_{yz} \) and \( \tilde{\nu}_{yz} \) coefficients contain information on the magnetosphere contribution to the torque. The first term changes the magnitude and direction of \( \Omega_c \), whereas the last one alters only its magnitude. For a neutron star without magnetosphere we have \( \nu_{yz} = 1 \) and \( \tilde{\nu}_{yz} = 0 \), but the presence of the magnetosphere may greatly change the value of these coefficients. Its effects are in general of comparable size to the ones in the vacuum case, and could depend on the angle between \( \mathbf{\hat{M}} \) and \( \Omega_c \). Their values are related not only to the currents flowing in the near magnetosphere, but also to the ones in distant regions, and so cannot be calculated at present. For this reason we will maintain these coefficients of the order of the unity as unknown adimensional functions of the angle between the dipole magnetic moment and the angular velocity, to be phenomenologically estimated.

The \( \mathbf{\hat{x}} \) component has anomalous and normal terms, but the first ones dominate because the normal terms are much smaller than the anomalous ones. The anomalous contribution is given by

\[
T_x = I_c \omega_x \cos \theta \Omega_c \times \mathbf{\hat{M}}, \tag{6}
\]

where \( \omega_x = \frac{4}{5} \frac{M^2 \Omega_c^2}{I_c c} \left( \frac{c}{R c R} \right) \nu_x \). Here the \( \nu_x \) coefficient contains information on the magnetosphere, and reduces to \( \nu_x = 1 \) when it is absent. In the case of the anomalous torque the magnetospheric effects depend mainly on the near region and can be evaluated under reasonable assumptions. For example, if we assume that there is a dominant
contribution from the Goldreich-Julian charge density on the closed lines of forces, it leads to \( \nu_x = -1/4 \).

As long as \( \frac{\omega_x}{\Omega_c R} \sim 10^2 - 10^4 \) for most pulsars, \( \omega_x \) is much greater than \( \omega_{yz} \) and \( \omega_{yz} \).

Despite this, in the rigid dipole model the anomalous term does not contribute to the alignment of the magnetic dipole with respect to \( \Omega_c \) or to the spin down of the star. However, in the two-component model considered here the situation changes and the anomalous term acquires a significant role.

### 2.2. Dissipative effects in the superfluid hydrodynamics

In the outer-core region of a neutron star there is a mixture of superfluid neutrons, superconducting protons and normal electrons. The effects of viscosity and friction due to the scattering of the electrons by the neutron and proton vortices give place to effective torques between the core and the crust. The resulting crust-core friction can be characterized by a time parameter \( \tau_f \). There are several phenomenological estimations on the basis of the glitch dynamics. For example, upper bounds are given for the Vela pulsar of \( \tau_f < 10s \) for a crust initiated glitch and of \( \tau_f < 440s \) for a core initiated glitch (Abney, Epstein & Olinto 1996). These values are deduced from the December 24, 1980 Christmas glitch data (Mc Cullock et al. 1990).

This point has also awakened a great deal of theoretical interest. The interior plasma couples the neutron superfluid due to mixing superfluid effects (Alpar, Langer & Sauls 1984, Alpar & Sauls 1988), and it is locked to the crust due to Alfven waves or cyclotron vortex waves. An extensive analysis of these phenomena has been performed by Mendell (1991a, 1991b, 1997) on the basis of the Newtonian superfluid hydrodynamics, generalized to include dissipation. The characteristic times related to all these couplings are of the
order of the second. The mutual friction torque simplifies to

\[ T_f = f (\Omega_c - \Omega_o), \]

when the angular velocities differences are small. When the system is taken out of equilibrium such as in a glitch, the time response frequency is given by

\[ \frac{1}{\tau_f} = \omega_f = f \frac{I_t}{I_c I_o}, \]

where \( I_c \) and \( I_o \) are the moments of inertia of the components, and \( I_t = I_c + I_n \) is the total star moment of inertia. Therefore this torque is relevant for the rapid nucleus spin up during glitches. The effective friction could depend on the angular velocity. Non dissipative effects give torques of the same kind as the gravitational dragging, which is to be discussed in the next subsection. If they are also characterized by a time of the order of the second they should be smaller than the gravitational torque for a rapidly rotating star.

### 2.3. Gravitational effects

The neutron star is essentially constituted by superfluid matter, and hence the observed rotation can be achieved only by the presence of vortices. The gravitational fields are strong enough to be relevant. They induce a change in the shape of the vortex lines, and also affect the density of vortices (Casini & Montemayor 1997). The main contribution of these effects is a correction on the coefficients of the dissipative torques of the order of 15% with respect to the flat space-time values, but they do not introduce new terms.

An additional term arises from the gravitational dragging, which gives place to a new torque if the components of the star have different angular velocities. If we neglect the gravitational radiation this interaction is conservative and the corresponding torque is

\[ T_g = z \, \Omega_c \times \Omega_o, \]

where \( z \simeq I_c I_o \frac{2G}{c^2 R^3} \simeq I_c \frac{R_g}{R} \left( \frac{R_s}{R} \right)^2 \simeq 0.1 \, I_c \) for small velocities, \( \Omega R \ll c \), where \( R \) is the star radius and \( R_g \) and \( R_s \) are the gyration and Schwarzschild radius of the star. This
torque has a significant modulus, and at a first glance it could have sensible effects on the dynamics of the magnetic dipole.

The torques $T_f$ and $T_g$ could be seen as the first terms of the mutual torque in an expansion around the point of equal angular velocities, and as such the form of the interaction is largely independent of the model. We are only considering terms up to the first order in the angle between $\Omega_c$ and $\Omega_o$. As we will discuss later this is enough for our analysis. In the following section we will use all these torques to construct a set of equations of motion that defines the dynamics of the star components.

3. The equations of motion

The equations of motion for the system described in the preceding section are:

\[
\dot{\mathbf{M}} = \Omega_c \times \mathbf{M}, \tag{9}
\]

\[
\dot{\Omega}_c = \omega_z \cos \theta \Omega_c \times \dot{\mathbf{M}} + \omega_{yz} \dot{\mathbf{M}} \times (\dot{\Omega}_c \times \Omega_c) - \frac{f}{I_c}(\Omega_c - \Omega_o) - \frac{z}{I_c} \Omega_c \times \Omega_o, \tag{10}
\]

\[
\dot{\Omega}_o' = \frac{f}{I_o} (\Omega_c - \Omega_o) + \frac{z}{I_o} \Omega_c \times \Omega_o. \tag{11}
\]

If $f \to 0$ the crust decouples from the core and if $f \to \infty$ the two components act as a rigid body. In both cases we recover the dynamics of the model of a rigid star with a magnetic dipole, with momentum of inertia $I_c$ and $I_t = I_c + I_o$ respectively, which has been already discussed in the Introduction.

The eq. (9) implies that the magnitude of $\mathbf{M}$ is constant. Therefore we have only five variables in the system, three angles, $\alpha$, $\beta$, and $\theta$, defined in Fig. 1, and the two moduli of the angular velocities, $\Omega_c$ and $\Omega_o$. For these variables the equations of motion are:

\[
\dot{\Omega}_c = - \frac{f}{I_c} (\Omega_c - \Omega_o \cos \alpha) - \omega_{yz} \Omega_c \sin^2 \theta - \bar{\omega}_{yz} \Omega_c, \tag{12}
\]

\[
\dot{\Omega}_o = \frac{f}{I_o} (\Omega_c \cos \alpha - \Omega_o), \tag{13}
\]
\[
\begin{align*}
\frac{d(\cos \theta)}{dt} &= \frac{f}{I_c \Omega_c} (\cos \beta - \cos \alpha \cos \theta) + \omega_{yz} \sin^2 \theta \cos \theta + \frac{z}{I_c} \Omega_o \left( \hat{\Omega}_o \cdot (\hat{\Omega}_c \times \hat{M}) \right), \\
\frac{d(\cos \beta)}{dt} &= \frac{f}{I_c \Omega_o} (\cos \theta - \cos \beta \cos \alpha) + \Omega_c \left( 1 - \frac{z}{I_o} \right) \left( \hat{\Omega}_o \cdot (\hat{\Omega}_c \times \hat{M}) \right), \\
\dot{\alpha} &= -\frac{\omega_x \cos \theta}{\sin \alpha} \left( \hat{\Omega}_o \cdot (\hat{\Omega}_c \times \hat{M}) \right) - \sin \alpha \left( \frac{f}{I_c \Omega_c} + \frac{f}{I_o \Omega_o} \right) \\
&- \frac{\omega_{yz} \cos \theta}{\sin \alpha} (\cos \beta - \cos \alpha \cos \theta).
\end{align*}
\]

In principle this system of equations would be very difficult to solve, but in fact it contains several dynamics with very different time scales, given by

\[\omega_{yz} \sim \tilde{\omega}_{yz} \ll \omega_x \ll \min(\omega_f \simeq \frac{I_c}{I}, \Omega_c),\]

which greatly simplify its treatment as we will see now. In the first place, from eqs. \((\ref{eq:12})\) and \((\ref{eq:13})\), if \(\Omega_c\) and \(\Omega_o\) are very different at a given instant, they will attain equilibrium in a relatively short time of the order of \(\tau_f\). The last equation tells us that the transient of \(\alpha\) is also characterized by \(\tau_f\), and thus, after a time of this order, this variable will acquire a value of the order \(\frac{\omega_x}{\omega_f} \ll 1\). Hence, after this transient we will have \(\Omega_c \sim \Omega_o\), \(\alpha \ll 1\) and therefore \(\theta \sim \beta\), all of them satisfying a slow dynamics with characteristic frequencies of the order of \(\omega_x\) or \(\omega_{yz}\).

As was already commented, the first two equations imply that the moduli of the angular velocities of the crust and the core will rapidly reach an equilibrium regime where \(\dot{\Omega}_c \simeq \dot{\Omega}_o\) and \(\Omega_c - \Omega_o \simeq - \left( \tilde{\omega}_{yz} + \omega_{yz} \sin^2 \theta \right) \frac{\Omega}{\omega_f}\), i.e., the difference between the angular velocities is quickly suppressed. The common decrease of both components is given by

\[\dot{\Omega}_o \simeq \dot{\Omega}_c \simeq - \frac{I_c}{I_o} \left( \tilde{\omega}_{yz} + \omega_{yz} \sin^2 \theta \right) \Omega_c,\]

which coincides with the one obtained in the dipole model for a rigid star when there is a magnetosphere. This means that the torque component along \(x\) will not affect the decrease of the angular velocity. This is analogous to the case of a rigid star, where this torque does not cause any spin-down. It means that the dissipative effects of the core-crust interactions do not change the star energy significantly, and therefore the main mechanism of energy loss
is still due to the electromagnetic radiation. In fact, the ratio between the power lost by friction and radiation is \( \frac{W_f}{W_r} \approx \frac{\omega_x}{\omega_f} \ll 1 \). However, as we will see, \( T_x \) produces a significant effect, because it makes an important contribution to the orientation of the magnetic axis.

Returning now to the angular variables, we can use a first order approximation in \( \alpha \) and consider the quasi-stationary regime. To simplify the expressions, instead of dealing with the very similar variables \( \theta \) and \( \beta \), we will replace the last angle by a new angle \( \gamma \) defined as in Fig. 1. It satisfies \( \cos \beta = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos \gamma \), and we have \( \hat{\Omega}_o \hat{\Omega}_c \times \hat{M} = \sin \gamma \sin \alpha \sin \theta \). Thus, with this substitution and at first order in \( \alpha \), assuming \( \frac{\omega_x}{\omega_f} \approx 1 \), \( \Omega_o \approx \Omega_c = \Omega \), we have:

\[
\begin{align*}
\dot{\theta} &= -\omega_f \alpha \cos \gamma - \omega_{yz} \sin \theta \cos \theta \frac{z}{I_c} \Omega \alpha \sin \gamma, \\
\dot{\alpha} &= -\omega_x \cos \theta \sin \theta \sin \gamma - \omega_f \alpha - \omega_{yz} \cos \gamma \sin \theta \cos \theta, \\
\dot{\gamma} &= -\Omega \left(1 - \frac{z}{I_c}\right) - \frac{\omega_x}{\alpha} \cos \gamma \sin \theta \cos \theta + \frac{\omega_{yz}}{\alpha} \sin \theta \cos \theta \sin \gamma. \tag{18, 19, 20}
\end{align*}
\]

The solutions for \( \alpha \) and \( \gamma \) can be decomposed in a transient dynamics, with a characteristic time \( \tau_f \), plus a slow varying time-function. This implies that \( \alpha \) and \( \gamma \) will reach a quasi-stationary regime in a few seconds. From here on they will be driven by the slow time-dependence of \( \theta \). This assertion can be verified by evaluating the Liapunov exponents at the equilibrium point whose real parts are \(-\omega_f\). The equation for \( \dot{\theta} \) contains only small frequencies. The explicit solutions at the fixed point for \( \alpha \) and \( \gamma \) are:

\[
\begin{align*}
\sin \gamma &= \pm \sqrt{\frac{\omega_x}{\omega_f} \left(1 - \frac{z}{I_c}\right) \Omega \omega_{yz}} \frac{\omega_x \omega_y + (1 - \frac{z}{I_c})^2 \Omega^2}{\sqrt{(\omega_x^2 + \omega_{yz}^2) (\omega_f^2 + (1 - \frac{z}{I_c})^2 \Omega^2)}}, \tag{21}
\\
\alpha &= \pm \sqrt{\frac{\omega_y^2 + (1 - \frac{z}{I_c})^2 \Omega^2}{\omega_x^2 + (1 - \frac{z}{I_c})^2 \Omega^2}} \sin \theta \cos \theta. \tag{22}
\end{align*}
\]

These results are consistent with our previous discussion. In particular \( \alpha \) becomes of order \( \frac{\omega_x}{\max(\omega_f, \Omega)} \), and thus the approximation \( \alpha \ll 1 \) is totally justified. We can also see here
that the gravitational dragging torque, despite its noticeable magnitude according to eq. (8), has the only effect of renormalizing the angular velocity \( \Omega \) by a correction of the order of 10\%. In what follows we use \( \kappa = 1 - \frac{z}{I_c} \sim 0.9 \).

Substituting the expressions (21) and (22) into eq. (18), we finally obtain

\[
\dot{\theta} = \frac{2 M^2 \Omega^2}{3 I_c c^3} \frac{\Omega^2}{\omega_f^2 + \kappa^2 \Omega^2} \left( \frac{6 \nu_y \omega_f c}{5 \Omega^2 R} - \kappa \nu_y z \right) \sin \theta \cos \theta,
\]

(23)

and we have for the angular velocity of the star

\[
\dot{\Omega} = -\frac{2 M^2}{3 I_c c^3} \left( \nu_y \sin^2 \theta + \tilde{\nu}_y z \right) \Omega^3.
\]

(24)

With these results, the expression for the breaking index becomes:

\[
n = 3 \left[ 1 + \frac{1}{3} \frac{I_t}{I_c} \frac{\Omega^2}{\omega_f^2 + \kappa^2 \Omega^2} \left( \frac{6 \nu_y \omega_f c}{5 \Omega^2 R} - \kappa \nu_y z \right) \sin \theta \cos \theta \frac{d}{d\theta} \left( \nu_y \sin^2 \theta + \tilde{\nu}_y z \right)^{-1} \right].
\]

(25)

The equations of motion (23), (24), and the expression (25) for the breaking index show that the effects of the magnetosphere are indeed relevant to explain the evolution of the magnetic dipole and the angular velocity of a neutron star.

4. Discussion

The model depends on several parameters. The mechanical characterization of the system is given by \( I_t/I_c \) and \( R \). The lower magnetosphere is described by \( \nu_y \), whereas the upper one is represented by \( \nu_y z \) and \( \tilde{\nu}_y z \). The effective friction between the core and the crust is given by \( \omega_f \). We have observational information on \( \Omega, \dot{\Omega} \) and \( n \). Furthermore, we expect to have a star radius of the order of \( R \sim 10 \text{ km} \), a ratio of the total momentum of inertia and the crust momentum of inertia \( I_t/I_c \sim 10^2 \) and \( \kappa \sim 1 \). The angular velocities of neutron stars are in the range \( 1 \text{ s}^{-1} > \Omega > 10^3 \text{s}^{-1} \), whereas a reasonable value for \( \omega_f \) is of the order of \( 1 \text{s}^{-1} \), and thus we can assume that \( \frac{\Omega^2}{\omega_f^2 + \kappa^2 \Omega^2} \sim 1 \).
The simplest situation we can consider corresponds to a star without a magnetosphere. In this case $\nu_x = \nu_{yz} = 1$ and $\bar{\nu}_{yz} = 0$, and hence we have:

\[
\dot{\Omega} = -\frac{2}{3} \frac{M^2 \Omega^3}{I_c c^3} \sin^2 \theta ,
\]

\[
\dot{\theta} \simeq \frac{2}{3} \frac{M^2 \Omega^2}{I_c c^3} \left( \frac{6 \omega_f c}{5 \Omega^2 R} - 1 \right) \sin \theta \cos \theta ,
\]

\[
n \simeq 3 \left[ 1 - \frac{2}{3} \frac{I_t}{I_c} \left( \frac{6 \omega_f c}{5 \Omega^2 R} - 1 \right) \cot^2 \theta \right] .
\]

The first equation states that $\dot{\Omega}$ is always negative, which is consistent with the fact that the star is losing energy by electromagnetic radiation, and thus the angular velocity is constantly decreasing. The second equation implies that the magnetic dipole slides toward the direction of the axis of rotation if $\frac{6 \omega_f c}{5 \Omega^2 R} < 1$ or to the equator if $\frac{6 \omega_f c}{5 \Omega^2 R} > 1$, and hence the breaking index, as is shown by the last equation, becomes greater or smaller than 3 respectively. For example, if we consider the Crab pulsar, for which $\Omega = 190 \, s^{-1}$, $\dot{\Omega} = -2.4 \times 10^{-9} \, s^{-2}$ and $n = 2.5$, they lead to:

\[
\frac{I_t c^3}{M^2 \Omega^2} \simeq 2 \times 10^3 \sin^2 \theta \, yr ,
\]

\[
\omega_f = 1.2 \left( 1 + 1.7 \times 10^{-3} \tan^2 \theta \right) \, s^{-1} ,
\]

\[
\dot{\theta} \simeq 10^{-4} \tan \theta \, yr^{-1} .
\]

The resulting value for $\omega_f$, of the order of seconds, is in good agreement with the theoretical expectations and the bounds derived from the Vela and the Crab pulsars. But this value for $\omega_f$ in fact corresponds to a fine tuning to have $\frac{6 \omega_f c}{5 \Omega^2 R} - 1 \lesssim 10^{-3}$, which is rapidly spoiled as $\Omega$ decreases. The effects of the electromagnetic aligning torques are much higher than the corresponding ones for a rigid star. In particular for a young star they give a characteristic time of the order of 10 yr. For this reason the magnetic moment of a neutron star in the vacuum will reach the rotation axis or the equator, and stabilize in a rather short time compared with the age of the star. If it tends to the rotation axis the
breaking index grows much bigger than 3, and if it falls in the equatorial plane the breaking index becomes exactly 3. Clearly, this is not the situation observed in the known pulsars.

The effects of the magnetosphere introduce a qualitative change in the dipole alignment behavior. The lower magnetosphere can be modeled by the Goldreich-Julian currents, in which case we can take $\nu_x = -1/4$. At the moment there is no suitable calculation for the non anomalous torques, generated by the upper magnetosphere, and thus we will maintain the corresponding coefficients as phenomenological parameters. Then we have:

$$\dot{\theta} = -\frac{2}{3} \frac{M^2 \Omega^2}{I_c c^3} \left( \frac{3 \om_f c}{10 \Omega^2 R} + \nu_{yz} \right) \sin \theta \cos \theta,$$

$$\dot{\Omega} = -\frac{2}{3} \frac{M^2}{I_c c^3} \left( \nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz} \right) \Omega^3,$$

$$n = 3 \left[ 1 - \frac{1}{3} \frac{I_t}{I_c} \left( \frac{3 \om_f c}{10 \Omega^2 R} + \nu_{yz} \right) \sin \theta \cos \theta \frac{d}{d\theta} \left( \nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz} \right)^{-1} \right].$$

There is a very important remark to be made about the $\theta$ behavior. Taking into account that we have $-\frac{M^2 \Omega^2}{I_c c^3} \simeq \frac{\dot{\Omega}}{\Omega}$, the evolution of $\theta$ when $\frac{3 \om_f c}{10 \Omega^2 R} + \nu_{yz} \simeq 1$ is dominated by $\tan \theta \propto e^{-\frac{\dot{\Omega} t}{\Omega I_c}}$. For a typical neutron star $\frac{\Omega t}{\Omega I_c} \simeq 10^{-2} \tau_{\Omega}$, and thus in a very short time the magnetic dipole lines up with the rotation axis, and from there on $\dot{\theta} = 0$. This behavior is a consequence of the relative freedom of the crust respect to the core, which increases the velocity of alignment by a factor $\frac{I_t}{I_c}$. But this is not the only possible behavior. Another one can be realized if there is an equilibrium point for the dynamics of $\theta$ at

$$\nu_{yz}(\theta) = -\frac{3 \om_f c}{10 \Omega^2 R},$$

satifying $\nu_{yz}(\theta) \cos \theta > 0$ to be stable, where the prime indicates a derivative with respect to $\theta$. In this case $\theta$ will rapidly adjust to the equilibrium value and its dynamics will be tied to the angular velocity dynamics

$$\dot{\theta} = -\frac{3c}{10 \nu_{yz}'(\theta) R} \left( \frac{\omega_f}{\Omega^2} \right)' \dot{\Omega},$$
which implies that $\theta$ slides towards the rotation axis. Besides this, from eq. (35) we have
\[
\frac{\dot{\Omega}}{\Omega} \simeq \frac{2}{3} \frac{M^2}{I c^3} \left( \frac{3 \nu_f c}{10 R} \sin^2 \theta - \tilde{\nu}_{yz} \Omega^2 \right) .
\] (37)

If $\tilde{\nu}_{yz} \gtrsim \frac{3 \nu_f c}{100 R} \sin^2 \theta \simeq -\nu_{yz} \sin^2 \theta$, the angular velocity $\Omega$ decreases throughout the life of the star; otherwise, if $\tilde{\nu}_{yz} < \frac{3 \nu_f c}{100 R} \sin^2 \theta$, the surroundings accelerates the star. The first situation seems to apply to the known pulsars.

There are theoretical arguments suggesting that $\omega_f$ depends on $\Omega^k$, with $k < 2$ (Alpar, Langer & Sauls 1984; Mendell 1997). The angular function $\nu_{yz}(\theta)$ is of order one, but during the life of the star the angular velocity constantly decreases. Thus, at a given moment the equilibrium point condition (35) cannot be maintained any more. When $\frac{\omega_f c}{\Omega R}$ becomes significantly greater than $\nu_{yz}(\theta)$ there is a change of regime and the dipolar moment rapidly aligns with the rotation axis. For a young star evolving at the equilibrium point, where $\nu_{yz}$ is of order one, we can estimate a lower bound for $\tau_f$ because it must be close to or greater than $\frac{3 c}{100 R}$. For example, for the Crab pulsar it is $\tau_f \gtrsim 0.25 s$ and for the Vela pulsar $\tau_f \gtrsim 1.8 s$, consistent with the known upper bound. In this regime the breaking index is given by
\[
n - 3 = (k - 2) \nu_{yz} \frac{\nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz}}{\nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz}} ,
\] (38)
which has the correct order of magnitude. To have $n < 3$ it must be
\[
\left( \nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz} \right) \cos \theta < 0 \text{ at the equilibrium point.}
\]

Although the aim of this model is to describe the overall evolution of the magnetic dipole, a question that naturally arises is what it can say about the glitches, which up to this point have not been considered. If we suppose that when a glitch happens there is a change in $\theta$, from eq. (33) we have
\[
\frac{\Delta \dot{\Omega}}{\Omega} \simeq \frac{\left( \nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz} \right)'}{\nu_{yz} \sin^2 \theta + \tilde{\nu}_{yz}} \Delta \theta .
\] (39)
The magnetosphere-dependent factor can be assumed of the order of the unity, whereas the change in the angle $\theta$ is at most of the same order as $\frac{\Delta \Omega_c}{\Omega} \simeq \frac{I_t}{I_c} \frac{\Delta \Omega}{\Omega}$, where $\Delta \Omega_c$ is the crust angular velocity jump just after the glitch. Thus this relation implies that the relative change of $\dot{\Omega}$ is at most $\frac{\Delta \dot{\Omega}}{\dot{\Omega}} \simeq \frac{I_t}{I_c} \frac{\Delta \Omega}{\Omega}$. For example, in the case of the Crab pulsar this relation tells us that $\frac{\Delta \dot{\Omega}}{\dot{\Omega}} \simeq 10^2 \frac{\Delta \Omega}{\Omega} \simeq 10^{-6}$, whereas for the observed glitches it is $10^{-6} \lesssim \frac{\Delta \dot{\Omega}}{\dot{\Omega}} \lesssim 10^{-3}$. This shows that these phenomena involve some non systematic and probably very complicated factors, such as deformations, earthquakes and breakings of the crust, which are not considered in the present model. For this reason it gives only a lower boundary for the ratio between the increment of the angular acceleration and the acceleration itself. The change of the $\theta$ and $\omega_f$ parameters during the glitches could produce a departure from the equilibrium point that would be reached again after a transient time of the order of ten years.

5. Concluding remarks

In this paper we have developed a two-component model for the evolution of a neutron star in presence of a magnetosphere. It considers the core and the crust with a fixed magnetic dipole moment, taking into account the normal and anomalous torques that act on the dipole and the friction between the core and the crust. We also analyze the effects of gravitational corrections and show that they only introduce a renormalization in some parameters, but do not affect the qualitative behavior of the system. We have solved the equations of motion taken advantage of the very different characteristic time scales that emerge from the complete dynamics.

The anomalous torques are usually supposed to be irrelevant for the alignment of the magnetic dipole, but here we show that in fact they are very relevant. This is because the crust-core interaction allows the angular momentum of the crust to evolve independently.
of the magnetic moment by interchanging angular momentum with the core. Despite this qualitative change in the alignment, the energy loss due to dissipation is very small and the equation governing the angular velocity is the same as for a rigid star.

Another effect of the crust-core interaction is an amplification of the classical alignment velocities by a factor \( \frac{I_0}{I_c} \) that greatly reduces the alignment times. This result and the observations lead us to the conclusion that these torques are not directly governing the dynamics, but there is an equilibrium point which effectively drives the angle \( \theta \). This equilibrium is reached by the interplay of the aligning effect of the \( \hat{x} \) axis torque, which depends on the friction between core and crust, and the \( \hat{y} - \hat{z} \) plane aligning torques, which depends in a crucial way on the magnetosphere. The equilibrium point naturally moves with the characteristic times of the dynamics of the angular velocity and leads to a breaking index near 3, as is actually observed. This equilibrium point regime works for rapidly rotating young stars. When the angular velocity becomes small it cannot be established and the magnetic moment will rapidly reach its final state. The equilibrium point regime requires a crust-core friction in reasonable agreement with the theoretical expectations and the observational boundaries.

One can be tempted to extend this model to the study of the glitches, but the results do not agree well with the observational data from the Crab pulsar. We obtain only a lower boundary for \( \frac{\Delta \dot{\Omega}}{\dot{\Omega}} \). This is to be expected, because in these phenomena it is very likely that more complicated processes, such as crust deformation, fractures and earthquakes, not taken into account here, have a very significant role. This suggests that other effects could be included in this framework. For the electromagnetic interaction, the actual effect of anomalous torques due to quadrupolar magnetic moments should be investigated, although their average effect is expected to be strongly suppressed in a rigid star model (Good & Ng 1985). The effects due to crust elasticity and oblateness could also be important as analized
in references (Goldreich 1970; Macy 1973), and they might be relevant to extend this model for describing the glitches dynamics.

In summary, this model gives a consistent description of the overall evolution of the main parameters of a neutron star. As long as the magnetic moment can be considered constant it is a suitable approach for understanding the magnetic field and angular velocity behavior and, as has already been said, it can be considered a first step for constructing a reliable description of the global dynamics of a neutron star. In particular it reconciles the fact that the angular velocity can be permanently decreasing by radiation, and that simultaneously for a young and rapidly rotating star we have a breaking index $n \lesssim 3$. For an older one the magnetic dipole aligns with the rotation axis and stabilize.

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Fig. 1.— The crust angular velocity $\Omega_c$ defines the z-axis. The magnetic moment $\mathbf{M}$ is in y-z plane, with an angle $\theta$ with respect to the z-direction. The core angular velocity $\Omega_o$ forms an angle $\alpha$ with $\Omega_c$ and $\beta$ with $\mathbf{M}$. 
