A variational principle for discrete integrable systems

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Abstract

For multidimensionally consistent systems we can consider the Lagrangian as a form, closed on the multidimensional equations of motion. For 2-dimensional systems this allows us to define an action on a 2-dimensional surface embedded in a higher dimensional space of independent variables. It is then natural to propose that the system should be derived from a variational principle which includes not only variations with respect to the dependent variables, but also variations of the surface in the space of independent variables. Here we give the resulting set of Euler-Lagrange equations firstly in 2 dimensions, and show how they can specify equations on a single quad in the lattice. We give the defining set of Euler-Lagrange equations also for 3-dimensional systems, and in general for n-dimensional systems. In this way the variational principle can be considered as supplying Lagrangians as solutions of a system of equations, as much as the equations of motion themselves.

1 Introduction

It is a notion going back to the 1600s that a dynamical system should minimize some quantity, i.e., that the equations of motion should arise as critical points of some action functional. This action is a quantity depending in principle on both the dependent and independent variables, and finding a minimum, or more generally a critical point, specifies a path followed by the dependent variable. The condition that specifies this path is the Euler-Lagrange equation.

A discrete calculus of variations was first developed outside the scope of integrable systems in the 1970s by Cadzow [6], Logan [11] and later Maeda [12, 13, 14]. Cadzow’s original motivation was the use of the digital computer in modern systems and the solution of control problems, and it became clear that the formulation of a discrete calculus of variations was important for numerical methods, in optimization and engineering problems. In the discrete realm instead of the action being an integral of a Lagrangian, it is a sum over the independent variable(s).

In the case of multidimensionally consistent systems, we are able to embed the system in a higher-dimensional space, with compatible systems living in each subspace. Indeed, we may have an infinite number of compatible systems in an infinite number of dimensions, and we do not need to restrict to any particular subspace; we could have a system following a path through an arbitrary number of dimensions. So we now have to consider not only the path taken by the dependent variable(s) with respect to the independent variable(s), but also the path through this space of independent variables. Then it is natural to ask that the action be critical with respect to a change in the path of independent variables.

This postulate was first put forward by the authors in [8], initially for 2-dimensional systems, both discrete and continuous. Requiring the action functional to be invariant under small changes in the path (which for 2-dimensional systems is a surface) through the space of independent variables leads to a condition on the Lagrangian, a closure relation, which was shown to be satisfied for many examples of multidimensionally consistent systems [8, 15, 2, 10, 17, 5]. This serves as an answer to the question of how to encode an entire multidimensionally consistent system in a variational principle.

An issue with the usual variational principle is that it is often impossible to obtain the desired system of equations as Euler-Lagrange equations, but only an integrated or derived form of those equations. This can be seen in the continuous realm in the case of the potential Korteweg-de Vries...
(pKdV) equation, where the Euler-Lagrange equation gives a derivative of the pKdV; and it can be seen in the discrete realm in the case of quad equations, such as those in the Adler-Bohnenk-Suris (ABS) classification \[1\], where the Euler-Lagrange equations give only consequences of the quad equations, which can be considered as discrete derivatives. One does not obtain a quad equation directly as an Euler-Lagrange equation on a fixed surface.

We show in this paper that the variational principle of \[8\], which considers variations of the surface, provides a set of Euler-Lagrange equations, specifying conditions on both the Lagrangian and Euler-Lagrange equations. In the 2-dimensional discrete case this set is enough to specify an equation on a single quad. The key point we wish to make is that the variational principle should be considered as supplying Lagrangians as solutions of a system of equations, as much as the equations of motion themselves. It is the latter perspective, invited by the phenomenon of multidimensional consistency as the defining aspect of integrability, that forms the main departure of our new variational principle from any of the conventional variational theories.

The case of 1-dimensional systems was examined in \[19\] \[18\] and subsequently from the Hamiltonian perspective in \[16\] \[3\]. Further work has also appeared recently on 2-dimensional systems in \[4\].

This paper is concerned with discrete systems. In Section 2 we examine the variational principle for 2-dimensional discrete systems: defining the action, listing the Euler-Lagrange equations for the basic configurations in the surface, and deriving quad equations as consequences of these Euler-Lagrange equations. We give examples of H1 and H3 to serve as illustrations. In Section 3 we give the defining set of Euler-Lagrange equations for 3-dimensional discrete systems, and show that these are compatible with the bilinear discrete Kadomsev-Petviashvili (KP) equation. Section 4 provides some further discussion and perspectives.

2 Discrete 2-dimensional systems

2.1 Defining the action

For a large class of equations defined on a quadrilateral, namely those in the ABS list \[1\], we have Lagrangians involving 3 points of the quadrilateral. Since the equations are multidimensionally consistent, we can think of them as being defined on a surface embedded in arbitrary dimensions, instead of the regular 2-dimensional lattice. And we can consider the action to be defined as the sum of Lagrangian contributions from all elementary plaquettes in this surface.

To this end, consider the surface \(\sigma\) to be a connected configuration of elementary plaquettes \(\sigma_{ij}(n)\), where \(\sigma_{ij}(n)\) is specified by the position \(n = (n, n + e_i, n + e_j)\) of one of its vertices in the lattice and the lattice directions given by the base vectors \(e_i, e_j\), as in Figure 1. The surface can be closed, or have a fixed boundary.

![Figure 1: Elementary oriented plaquette.](image)

Since the 3-point Lagrangians depend on two directions in the lattice, and when embedded in a multidimensional lattice at each point can be associated with an oriented plaquette \(\sigma_{ij}(n)\), we can think of these Lagrangians as defining a discrete 2-form \(L_{ij}(n)\) whose evaluation on that plaquette is given by the Lagrangian function as follows

\[
L_{ij}(n) = L(u(n), u(n + e_i), u(n + e_j); \alpha_i, \alpha_j).
\]  (2.1)
The Lagrangians given in [8] are all antisymmetric with respect to the interchange of lattice
directions $i, j$, and so this is well-defined. Then the action $S$ is also well-defined by

\[
S[u(n); \sigma] = \sum_{\sigma_{ij}(n) \in \sigma} L_{ij}(n).
\]  

(2.2)

Note that in performing this sum we must be careful to take into account the orientation of the
plaquettes.

### 2.2 The Euler-Lagrange equations

To derive the set of Euler-Lagrange equations stemming from the action (2.2), we look at what
happens at a particular point $n$ in the lattice. For ease of notation we will suppress the depe ndence
on $n$, writing $u = u(n)$, and make use of shift operators $T_i$, writing $T_i u = u(n + e_i), T_j u =
\begin{align*}
u(n + e_j), T_{ij}^{-1} u &= u(n - e_i), \text{ etc.}
\end{align*}

The postulate is that the system lies at a critical point of the action, and our point of view
is that it lies at a critical point with respect to not only the dependent variable $u$, but also the
independent variables, i.e., the surface $\sigma$. Since we are considering discrete surfaces here, the
notion of infinitesimal variations of the independent variables does not make sense, and we can
make only finite variations. Thus our postulate is that the action is independent of $\sigma$ (keeping any
boundary fixed) on solutions to the system.

It suffices to consider a collection of fixed surfaces embedded in 3 dimensions, and compute
variations with respect to $u$ on that surface. For an action which is the sum of 3-point Lagrangians
$L(u, T_i u, T_j u; \alpha_i, \alpha_j)$, there are various possible configurations involving the arbitrary point $u$. The
first is the usual flat 2-dimensional configuration shown in Figure 2:

Figure 2: Usual configuration in 2 dimensions.

This corresponds to the Euler-Lagrange equation

\[
\frac{\partial}{\partial u}\left( L(T_i^{-1} u, T_i^{-1} T_j u; \alpha_i, \alpha_j) + L(u, T_i u, T_j u; \alpha_i, \alpha_j) + L(T_j^{-1} u, T_j T_i^{-1} u, u; \alpha_i, \alpha_j) \right) = 0.
\]  

(2.3)

The elementary configurations in 3 dimensions are shown in Figure 3; all other configurations
can be obtained as combinations of these. A statement to this effect appears in [4].

Note that in the final picture in Figure 3, only two plaquettes contribute, because of the 3-point
nature of the Lagrangians we are considering here.

Each of these pictures corresponds to a different Euler-Lagrange equation. Since all surfaces
in the lattice can be obtained by combining these elementary configurations, the Euler-Lagrange
equation for any surface can be obtained by combining the Euler-Lagrange equations corresponding
to the respective elementary configurations.

**Theorem 1** The following form a complete set of Euler-Lagrange equations for the quadrilateral
lattice system with action defined by (2.1) and (2.2).

\[
\frac{\partial}{\partial u} \left( L(u, T_i u, T_j u; \alpha_i, \alpha_j) + L(u, T_j u, T_k u; \alpha_j, \alpha_k) + L(u, T_k u, T_i u; \alpha_k, \alpha_i) \right) = 0,
\]
(2.4a)

\[
\frac{\partial}{\partial u} \left( L(T_i^{-1} u, u, T_j u; \alpha_i, \alpha_j) - L(u, T_j u, T_k u; \alpha_j, \alpha_k) + L(T_i^{-1} u, T_i^{-1} T_k u, u; \alpha_k, \alpha_i) \right) = 0,
\]
(2.4b)

\[
\frac{\partial}{\partial u} \left( L(T_j^{-1} u, T_j^{-1} T_k u; \alpha_j, \alpha_k) + L(T_i^{-1} u, T_i^{-1} T_k u, u; \alpha_k, \alpha_i) \right) = 0,
\]
(2.4c)

for all \( i, j, k \in I \), where \( I \) is the index set labelling the lattice directions.

**Proof:** Consider the action of a closed surface. The smallest non-trivial closed surface is a cube, for which the action is

\[
S[u; \text{cube}] = L(u, T_i u, T_j u; \alpha_i, \alpha_j) + L(u, T_j u, T_k u; \alpha_j, \alpha_k) + L(u, T_k u, T_i u; \alpha_k, \alpha_i)
- L(T_k u, T_i u, T_j u; \alpha_i, \alpha_j) - L(T_i u, T_j u, T_k u; \alpha_j, \alpha_k)
- L(T_j u, T_k u, T_i u; \alpha_k, \alpha_i).
\]
(2.5)

We require variations of the action with respect to the dependent variables to be zero. That is,

\[
\frac{\partial}{\partial u} \left( L(u, T_i u, T_j u; \alpha_i, \alpha_j) + L(u, T_j u, T_k u; \alpha_j, \alpha_k) + L(u, T_k u, T_i u; \alpha_k, \alpha_i) \right) = 0,
\]
(2.6a)

\[
\frac{\partial}{\partial T_i u} \left( L(u, T_i u, T_j u; \alpha_i, \alpha_j) - L(T_i u, T_j u, T_k u; \alpha_i, \alpha_j) + L(u, T_i u, T_k u; \alpha_k, \alpha_i) \right) = 0,
\]
(2.6b)

\[
\frac{\partial}{\partial T_j T_k u} \left( L(T_k u, T_i u, T_j u; \alpha_k, \alpha_i) + L(T_j u, T_i u, T_k u; \alpha_i, \alpha_k) \right) = 0,
\]
(2.6c)

and cyclic permutations. Shifting these in the lattice we see that they are equivalent to (2.4a)-(2.4c). Any closed surface can be constructed from cubes, so at least away from any boundary all possible Euler-Lagrange equations are consequences of (2.6a)-(2.6c).

\[\square\]

2.3 Consequences of the Euler-Lagrange equations

As in the previous subsection, we consider actions which are the sum of 3-point Lagrangians \( \mathcal{L}(u, T_i u, T_j u; \alpha_i, \alpha_j) \), where the Lagrangians are anti-symmetric with respect to the interchange
of the lattice directions, so that the equations (2.4a)-(2.4c) hold. The one further assumption we will make is that we may choose initial conditions \( u, T_j u, T_k u \) independently and arbitrarily.

If we impose that the action remains invariant under perturbations of the surface, then it is independent of the surface \( \mathcal{S} \), and all of these equations must hold simultaneously. Note that (2.5) is a consequence of (2.4a)-(2.4c) and their cyclic permutations.

**Theorem 2** Suppose \( u, T_i u, T_j u, T_k u \) are independent and can be chosen arbitrarily. The Euler-Lagrange equation (2.4a) implies that the anti-symmetric Lagrangian \( L(u, T_i u, T_j u; \alpha_i, \alpha_j) \) has the form

\[
L(u, T_i u, T_j u; \alpha_i, \alpha_j) = A(u, T_i u; \alpha_i) - A(u, T_j u; \alpha_j) + C(T_i u, T_j u; \alpha_i, \alpha_j),
\]

for some functions \( A, C \).

**Proof:** Consider equation (2.4a). If all of \( u, T_i u, T_j u, T_k u \) are independent and can be chosen arbitrarily, then writing

\[
l(u, T_i u, T_j u; \alpha_i, \alpha_j) = \frac{\partial}{\partial u} L(u, T_i u, T_j u; \alpha_i, \alpha_j),
\]

we must have

\[
\frac{\partial}{\partial T_i u} \left(l(u, T_i u, T_j u; \alpha_i, \alpha_j) + l(u, T_k u, T_i u; \alpha_k, \alpha_i)\right) = 0,
\]

\[
\Rightarrow l(u, T_i u, T_j u; \alpha_i, \alpha_j) = a(u, T_i u; \alpha_i) + b(u, T_j u; \alpha_i, \alpha_j),
\]

for some functions \( a, b \). This, plus the various cyclic permutations of the lattice directions, gives in fact

\[
l(u, T_i u, T_j u; \alpha_i, \alpha_j) = a(u, T_i u; \alpha_i) - a(u, T_j u; \alpha_j).
\]

Thus if \( \partial A(u, T_i u; \alpha_i)/\partial u = a(u, T_i u; \alpha_i) \), we should have

\[
L(u, T_i u, T_j u; \alpha_i, \alpha_j) = A(u, T_i u; \alpha_i) - A(u, T_j u; \alpha_j) + C(T_i u, T_j u; \alpha_i, \alpha_j),
\]

for some function \( C \), which is (2.7). Note that since \( L(u, T_i u, T_j u; \alpha_i, \alpha_j) \) is antisymmetric under the interchange of lattice directions \( i, j \), then the same must be true of \( C(T_i u, T_j u; \alpha_i, \alpha_j) \). □

**Theorem 3** The Euler-Lagrange equations (2.4a)-(2.4c) determine the following relation on each single quad:

\[
\frac{\partial}{\partial u} \left(L(T_i^{-1} u, u; \alpha_i, \alpha_j)\right) = \frac{\partial}{\partial u} \left(A(u, T_i u; \alpha_j)\right) - h(u),
\]

where \( h(u) \) is an arbitrary function, which can be absorbed into \( A \).

**Proof:** Substituting (2.7) into equations (2.4b) and (2.4c) gives

\[
\frac{\partial}{\partial u} \left(-A(T_i^{-1} u, T_j^{-1} T_i u; \alpha_j) - A(u, T_j u; \alpha_j) + A(u, T_k u; \alpha_k) + A(T_i^{-1} u, T_j^{-1} T_k u; \alpha_k)\right)
\]

\[
\quad + C(u, T_i^{-1} T_j u; \alpha_i, \alpha_j) - C(u, T_j^{-1} T_i u; \alpha_i, \alpha_j)\right) = 0,
\]

\[
\frac{\partial}{\partial u} \left(A(T_j^{-1} u, u; \alpha_j) - A(T_j^{-1} u, T_j^{-1} T_k u; \alpha_k) + A(T_i^{-1} u, T_j^{-1} T_k u; \alpha_k) - A(T_i^{-1} u, u; \alpha_i)\right)
\]

\[
\quad + C(u, T_j^{-1} T_k u; \alpha_j, \alpha_k) - C(u, T_i^{-1} T_k u; \alpha_i, \alpha_k)\right) = 0,
\]

where we have already cancelled some of the terms (provided we assume that \( T_j u \) and \( T_k u \) can be independently chosen, so that they don’t depend on \( u \)). We see that we can rewrite these in a suggestive way, isolating dependence on particular lattice directions:

\[
\frac{\partial}{\partial u} \left(A(u, T_i u; \alpha_j) + A(T_i^{-1} u, T_i^{-1} T_j u; \alpha_j) - C(u, T_i^{-1} T_j u; \alpha_i, \alpha_j)\right)
\]

\[
\quad = \frac{\partial}{\partial u} \left(A(u, T_k u; \alpha_k) + A(T_i^{-1} u, T_i^{-1} T_k u; \alpha_k) - C(u, T_i^{-1} T_k u; \alpha_i, \alpha_k)\right),
\]

\[
\frac{\partial}{\partial u} \left(A(T_j^{-1} u, u; \alpha_j) - A(T_j^{-1} u, T_j^{-1} T_k u; \alpha_k) + C(u, T_j^{-1} T_k u; \alpha_j, \alpha_k)\right)
\]

\[
\quad = \frac{\partial}{\partial u} \left(A(T_i^{-1} u, u; \alpha_i) - A(T_i^{-1} u, T_i^{-1} T_k u; \alpha_k) + C(u, T_i^{-1} T_k u; \alpha_i, \alpha_k)\right),
\]
and of course this must be true for all $i, j, k$. Thus we must have
\[
\frac{\partial}{\partial u} \left( A(u, T_i u; \alpha_i) + A(T_i^{-1} u, T_i^{-1} T_i u; \alpha_j) - C(u, T_i^{-1} T_i u; \alpha_i, \alpha_j) \right) = f(\ldots, T_i^{-1} u, u, T_i u, \ldots; \alpha_i), \tag{2.15a}
\]
\[
\frac{\partial}{\partial u} \left( A(T_i^{-1} u, u; \alpha_i) - A(T_i^{-1} u, T_i^{-1} T_i u; \alpha_j) + C(u, T_i^{-1} T_i u; \alpha_i, \alpha_j) \right) = g(\ldots, T_i^{-1} u, u, T_i u, \ldots; \alpha_j). \tag{2.15b}
\]
for some $f, g$ depending on $u$ and its shifts in only one lattice direction. Adding these expressions together, we deduce that
\[
f(\ldots, T_i^{-1} u, u, T_i u, \ldots; \alpha_i) = \frac{\partial}{\partial u} \left( A(T_i^{-1} u, u; \alpha_i) \right) + h(u), \tag{2.16a}
\]
\[
g(\ldots, T_j^{-1} u, u, T_j u, \ldots; \alpha_j) = \frac{\partial}{\partial u} \left( A(u, T_j u; \alpha_j) \right) - h(u), \tag{2.16b}
\]
for some function $h$. Thus we obtain the relation on a single quad
\[
\frac{\partial}{\partial u} \left( A(T_i^{-1} u, u; \alpha_i) - A(T_i^{-1} u, T_i^{-1} T_i u; \alpha_j) + C(u, T_i^{-1} T_i u; \alpha_i, \alpha_j) \right) = \frac{\partial}{\partial u} \left( A(u, T_j u; \alpha_j) \right) - h(u), \tag{2.17}
\]
which is in fact exactly (2.12). It is easy to check that (2.7) and (2.12) are enough to satisfy all Euler-Lagrange equations (2.4a)-(2.4c). Of course, so far $A$ is only defined up to an arbitrary function of $u$, so $h(u)$ can w.l.o.g. be chosen to be zero.

Therefore, in fact, we can rewrite the Euler-Lagrange equations (or rather, the consequences thereof) in the following two equivalent ways:
\[
\frac{\partial}{\partial T_i u} \left( A(u, T_i u; \alpha_i) - A(T_i u, T_j T_i u; \alpha_j) + C(T_i u, T_j T_i u; \alpha_i, \alpha_j) \right) = 0, \tag{2.18a}
\]
\[
\frac{\partial}{\partial T_j u} \left( A(T_j u, T_i T_j u; \alpha_i) - A(u, T_j u; \alpha_j) + C(T_i u, T_j T_i u; \alpha_i, \alpha_j) \right) = 0. \tag{2.18b}
\]
By construction, on solutions to the equations, the Lagrangians satisfy a closure relation
\[
\Delta_i L_{ij} + \Delta_j L_{ij} + \Delta_k L_{ij} = 0, \tag{2.19}
\]
where $\Delta_i$ is a difference operator defined by $\Delta_i = T_i - id$.

### 2.4 Example: H1

If we consider the example of H1, the Lagrangian (which was first given in [7]) evaluated on a plaquette in the $(i, j)$-direction has the form
\[
L(u, T_i u, T_j u; \alpha_i, \alpha_j) = (T_i u - T_j u) u - (\alpha_i - \alpha_j) \ln(T_i u - T_j u). \tag{2.20}
\]
Then the usual Euler-Lagrange equation (2.3) coming from an action on a flat 2-d surface is
\[
\frac{\partial}{\partial u} \left( (T_i u - T_j u + T_i^{-1} u - T_j^{-1} u) u - (\alpha_i - \alpha_j) \ln(u - T_i^{-1} T_j u) - (\alpha_i - \alpha_j) \ln(T_i^{-1} T_j u - u) \right) = 0,
\]
\[
\Rightarrow \quad T_i u - T_j u + T_i^{-1} u - T_j^{-1} u - \frac{\alpha_i - \alpha_j}{u - T_i^{-1} T_j u} + \frac{\alpha_i - \alpha_j}{T_i^{-1} T_j u - u} = 0, \tag{2.21a}
\]
which consists of 2 shifted copies of H1 lying on a 7-point configuration, i.e., a consequence of H1. The Euler-Lagrange equations on non-flat surfaces (2.4a)-(2.4c) are respectively
\[
\frac{\partial}{\partial u} \left( 0 \right) = 0, \tag{2.21b}
\]
\[
\frac{\partial}{\partial u} \left( u(-T_j u + T_k u) - (\alpha_i - \alpha_j) \ln(u - T_i^{-1} T_j u) - (\alpha_k - \alpha_i) \ln(T_i^{-1} T_k u - u) \right) = 0,
\]
\[
\Rightarrow -T_j u + T_k u - \frac{\alpha_i - \alpha_j}{u - T_i^{-1} T_j u} + \frac{\alpha_k - \alpha_i}{T_i^{-1} T_k u - u} = 0.
\]
\[
(2.21c)
\]
\[
\frac{\partial}{\partial u} \left( u(T_j^{-1} u - T_i^{-1} u) - (\alpha_j - \alpha_k) \ln(u - T_j^{-1} T_k u) - (\alpha_k - \alpha_i) \ln(T_i^{-1} T_k u - u) \right) = 0,
\]
\[
\Rightarrow T_j^{-1} u - T_i^{-1} u - \frac{\alpha_j - \alpha_k}{u - T_j^{-1} T_k u} + \frac{\alpha_k - \alpha_i}{T_i^{-1} T_k u - u} = 0.
\]
\[
(2.21d)
\]
In fact, \((2.21d)\) is a consequence of \((2.21c)\) and its copies under permutation of lattice directions. Also equation \((2.12)\) with \(a\) taken to be zero is
\[
T_i^{-1} u - \frac{\alpha_i - \alpha_j}{u - T_i^{-1} T_j u} = \frac{\partial}{\partial u} \left( A(u, T_j u; \alpha_j) \right)
\]
\[
\Rightarrow u - \frac{\alpha_i - \alpha_j}{T_i u - T_j u} = \frac{\partial}{\partial T_i u} \left( A(T_i u, T_j u; \alpha_j) \right).
\]
\[
(2.22)
\]
Of course by swapping the lattice directions we also get
\[
u - \frac{\alpha_i - \alpha_j}{T_i u - T_j u} = \frac{\partial}{\partial T_j u} \left( A(T_j u, T_i u; \alpha_i) \right).
\]
\[
(2.23)
\]
Combining \((2.22)\) and \((2.23)\) we get
\[
\frac{\partial}{\partial T_i u} \left( A(T_i u, T_j u; \alpha_j) \right) = \frac{\partial}{\partial T_j u} \left( A(T_j u, T_i u; \alpha_i) \right)
\]
\[
(2.24)
\]
\[
f(T_i T_j u),
\]
\[
(2.25)
\]
for some function \(f\). This implies
\[
A(u, T_j u; \alpha_j) = u f(T_j u) + g(T_j u).
\]
\[
(2.26)
\]
Here, we know that
\[
A(u, T_i u; \alpha_i) - A(u, T_j u; \alpha_j) = (T_i u - T_j u) u,
\]
\[
(2.27)
\]
and so we must have
\[
A(u, T_i u; \alpha_i) = u(T_i u + \lambda) + \mu,
\]
\[
(2.28)
\]
where \(\lambda, \mu\) are arbitrary constants. Then the Euler-Lagrange equation is
\[
u + \lambda - T_i(T_j u)(T_i u - T_j u) - \alpha_i + \alpha_j = 0,
\]
\[
(2.29)
\]
which is consistent around a cube for arbitrary \(\lambda\).

### 2.5 Example: H3

The Lagrangian evaluated on a plaquette in the \((i, j)\)-direction has the form
\[
L(u, T_i u, T_j u; \alpha_i, \alpha_j) = \ln(\alpha_i^2) \ln u - \text{Li}_2 \left( -\frac{u T_i u}{\alpha_i \delta} \right) - \ln(\alpha_j^2) \ln u - \text{Li}_2 \left( -\frac{u T_j u}{\alpha_j \delta} \right)
\]
\[
+ \text{Li}_2 \left( \frac{\alpha_j T_i u}{\alpha_i T_j u} \right) - \text{Li}_2 \left( \frac{\alpha_i T_i u}{\alpha_j T_j u} \right) + \ln(\alpha_i^2) \ln \left( \frac{T_i u}{T_j u} \right),
\]
\[
(2.30)
\]
where \(\delta\) is an arbitrary constant parameter, so for an as yet unspecified function \(f\) we see
\[
A(u, T_i u; \alpha_i) = \ln(\alpha_i^2) \ln u - \text{Li}_2 \left( -\frac{u T_i u}{\alpha_i \delta} \right) + f(u),
\]
\[
(2.31)
\]
\[
C(T_i u, T_j u; \alpha_i, \alpha_j) = \text{Li}_2 \left( \frac{\alpha_j T_i u}{\alpha_i T_j u} \right) - \text{Li}_2 \left( \frac{\alpha_i T_i u}{\alpha_j T_j u} \right) + \ln(\alpha_j^2) \ln \left( \frac{T_i u}{T_j u} \right).
\]
\[
(2.32)
\]
The equation (2.18a) is then

\[
\frac{\partial}{\partial T_i u} \left\{ A(u, T; u; t) - A(T_i u, T_j u; t) + C(T_i u, T_j u; t) + \frac{\alpha_i T_j u}{\alpha_j T_i u} - f(T_i u) \right\} = 0
\]

where \( \frac{\partial}{\partial T_i u} \) is defined as

\[
\frac{1}{T_i u} \ln \left( 1 + \frac{u T_i u}{\alpha_i \delta} \right) - \frac{1}{T_i u} \ln \left( 1 + \frac{u T_j u}{\alpha_j \delta} \right) - \frac{1}{T_i u} \ln \left( 1 + \frac{T_i u T_j u}{\alpha_j T_i u} \right) - f'(T_i u) = 0.
\]

(2.33)

where \( f'(z) = df/dz \). If we define

\[
t_i = \exp \left\{ T_i u f'(T_i u) \right\},
\]

then

\[
\alpha_i (u T_i u + t_i T_i u T_j u) - \alpha_j (u T_j u + t_i T_i u T_j u) + \delta (\alpha_i^2 - t_i \alpha_j^2) + \alpha_i \alpha_j \frac{T_j u}{T_i u} (t_i - 1) = 0.
\]

(2.35)

Note from (2.18b) that we also have the equation

\[
\alpha_i (u T_i u + t_j T_j u T_i u) - \alpha_j (u T_j u + t_i T_i u T_j u) + \delta (\alpha_i^2 - t_j \alpha_j^2) - \alpha_i \alpha_j \frac{T_i u}{T_j u} (t_j - 1) = 0,
\]

(2.36)

and so we must have

\[
(\alpha_i T_i u - \alpha_j T_j u) T_i u (t_i - t_j) + \delta \left[ \alpha_i (t_i - 1) \left( \frac{T_j u}{T_i u} - \alpha_j \right) + \alpha_j (t_j - 1) \left( \frac{T_i u}{T_j u} - \alpha_i \right) \right] = 0.
\]

(2.37)

Therefore \( t_i = t_j \), and if \( \delta \neq 0 \) then \( t_i = t_j = 1 \), and we have the usual equation H3. If on the other hand \( \delta = 0 \) we have a little more freedom, and we can let \( t_i = t_j = t \) for some arbitrary constant \( t \). In that case, the equation is

\[
\alpha_i (u T_i u + t T_j u T_i u) - \alpha_j (u T_j u + t T_i u T_j u) = 0,
\]

(2.38)

and this equation is also consistent around the cube.

3 Discrete 3-dimensional systems

3.1 Defining the action

A Lagrangian for a 3-dimensional system can be defined on an elementary cube \( \nu_{ijk}(n) \), where \( \nu_{ijk}(n) \) is specified by the position \( n = (n, \mathbf{n}, \mathbf{n}, \mathbf{e}_i, \mathbf{n} + \mathbf{e}_j, \mathbf{n} + \mathbf{e}_k) \) of one of its vertices in the lattice and the lattice directions given by the base vectors \( \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \), as in Figure 4.

The Lagrangian can depend in principle on the fields at all 8 vertices of the elementary cube:

\[
L_{ijk}(n) = L(u(n), u(n+e_i), u(n+e_j), u(n+e_k), u(n+e_i+e_j), u(n+e_i+e_k), u(n+e_j+e_k)),
\]

(3.1)

The action can then be defined as a connected configuration \( \nu \) of these elementary cubes,

\[
S[u(n); \nu] = \sum_{\nu_{ijk}(n) \in \nu} L_{ijk}(n).
\]

(3.2)

This action is of course still perfectly valid if the Lagrangian doesn’t depend on the fields at all vertices of the cube. For example, in the case of the bilinear discrete KP equation, one can write a Lagrangian depending on fields at 6 vertices.
3.2 The Euler-Lagrange equations

The Euler-Lagrange equation in the usual 3-dimensional space is

\[ 0 = \frac{\partial}{\partial u} \left( L_{ijk} + T_i^{-1} L_{ijk} + T_j^{-1} L_{ijk} + T_k^{-1} L_{ijk} \right) \]

\[ + T_i^{-1} T_j^{-1} L_{ijk} + T_j^{-1} T_k^{-1} L_{ijk} + T_k^{-1} T_i^{-1} L_{ijk} + T_i^{-1} T_j^{-1} T_k^{-1} L_{ijk} \right), \quad (3.3) \]

where we take into account all Lagrangian contributions that involve the field \( u \). We have suppressed the dependence on the variables, writing

\[ L_{ijk} = L_{ijk}(u, T_i u, T_j u, T_k u). \]

Note that any point in \( \mathbb{Z}^3 \) belongs to 8 cubes, so we have in principle 8 terms in the above equation. This is the analogue of the “flat” equation (2.3) we had in 2 dimensions.

Embed the system in 4 dimensions. In 3 dimensions, the smallest closed 2-dimensional space is a cube, consisting of 6 faces; in 4 dimensions, the smallest closed 3-dimensional space is a hypercube, consisting of 8 cubes. The action on the elementary hypercube will have the form

\[ S(u; \text{hypercube}) = \Delta_i L_{ijk} + \Delta_j L_{jkl} + \Delta_k L_{kli} \]

\[ - \Delta_i L_{ijkl} - \Delta_j L_{ijkl} - \Delta_k L_{ijkl} - \Delta_l L_{ijkl}. \quad (3.4) \]

Because of the symmetry, we need only take derivatives with respect to \( u, T_i u, T_j u, T_k u, T_l u \) and \( T_i T_j T_k u \), and the other equations will follow by cyclic permutation of the lattice directions. Then we have the set of equations

\[ 0 = \frac{\partial}{\partial u} \left( -L_{ijk} + L_{jkl} - L_{kli} + L_{lij} \right), \quad (3.5a) \]

\[ 0 = \frac{\partial}{\partial T_i u} \left( -L_{ijk} - T_i L_{jkl} - L_{kli} + L_{lij} \right), \quad (3.5b) \]

\[ 0 = \frac{\partial}{\partial T_j u} \left( -L_{ijk} - T_j L_{jkl} + T_i L_{kli} + L_{lij} \right), \quad (3.5c) \]

\[ 0 = \frac{\partial}{\partial T_i T_j u} \left( -L_{ijk} - T_i L_{jkl} + T_j L_{kli} + T_k L_{lij} \right), \quad (3.5d) \]

\[ 0 = \frac{\partial}{\partial T_i T_j T_k u} \left( T_i L_{ijk} - T_i L_{jkl} + T_j L_{kli} + T_k L_{lij} \right). \quad (3.5e) \]
along with the equivalent shifted versions

\[
0 = \frac{\partial}{\partial u} \left( -\mathcal{L}_{ijk} + \mathcal{L}_{jkl} - \mathcal{L}_{kli} + \mathcal{L}_{lij} \right),
\]  
\[(3.6a)\]

\[
0 = \frac{\partial}{\partial u} \left( -T_i^{-1} \mathcal{L}_{ijk} - \mathcal{L}_{jkl} - T_i^{-1} \mathcal{L}_{kli} + T_i^{-1} \mathcal{L}_{lij} \right),
\]  
\[(3.6b)\]

\[
0 = \frac{\partial}{\partial u} \left( -T_i^{-1} T_j^{-1} \mathcal{L}_{ijk} - T_j^{-1} \mathcal{L}_{jkl} + T_j^{-1} \mathcal{L}_{kli} + T_j^{-1} T_i^{-1} \mathcal{L}_{lij} \right),
\]  
\[(3.6c)\]

\[
0 = \frac{\partial}{\partial u} \left( -T_i^{-1} T_j^{-1} T_k^{-1} \mathcal{L}_{ijk} - T_j^{-1} T_k^{-1} \mathcal{L}_{jkl} + T_j^{-1} T_k^{-1} \mathcal{L}_{kli} - T_i^{-1} T_j^{-1} T_k^{-1} \mathcal{L}_{lij} \right),
\]  
\[(3.6d)\]

\[
0 = \frac{\partial}{\partial u} \left( T_i^{-1} T_j^{-1} T_k^{-1} \mathcal{L}_{ijk} - T_j^{-1} T_k^{-1} T_i^{-1} \mathcal{L}_{jkl} + T_j^{-1} T_k^{-1} T_i^{-1} \mathcal{L}_{kli} - T_i^{-1} T_j^{-1} T_k^{-1} \mathcal{L}_{lij} \right).
\]  
\[(3.6e)\]

3.3 Example: bilinear discrete KP

The Lagrangian for the bilinear discrete KP equation was first given in \[\text{[2]},\] and in 3-dimensional space gives as Euler-Lagrange equations 12 copies of the bilinear discrete KP equation itself, on 6 elementary cubes. The Lagrangian \(\mathcal{L}_{ijk}\) depends on the six fields \(T_i u, T_j u, T_k u, T_i T_j u, T_i T_k u,\) and \(T_i T_j T_k u,\) and has the following explicit form:

\[
\mathcal{L}_{ijk} = \ln \left( \frac{T_k u T_j u}{T_j u T_k u} \right) \ln \left( \frac{A_{ik} T_j u}{A_{jk} T_i u} \right) - L_{12} \left( \frac{A_{ij} T_k u T_j u}{A_{ik} T_j u T_k u} \right)
\]  
\[(3.6a)\]

\[
+ \ln \left( \frac{T_j u T_k u}{T_k u T_j u} \right) \ln \left( \frac{A_{ij} T_k u}{A_{ki} T_u} \right) - L_{12} \left( \frac{A_{ij} T_k u T_j u}{A_{ki} T_u T_j u} \right)
\]  
\[(3.6b)\]

\[
+ \ln \left( \frac{T_j u T_k u}{T_k u T_j u} \right) \ln \left( \frac{A_{ij} T_k u}{A_{ki} T_u} \right) - L_{12} \left( \frac{A_{ij} T_k u T_j u}{A_{ki} T_u T_j u} \right)
\]  
\[(3.6c)\]

\[
- \frac{1}{2} \left( (\ln(T_i T_j u) \right)^2 + (\ln(T_j T_k u) \right)^2 + (\ln(T_k T_i u) \right)^2
\]  
\[(3.6d)\]

Here the \(A_{ij}\) are constants which are antisymmetric with respect to swapping the indices.

If we introduce the quantity \(C_{ijk}\), defined by

\[
C_{ijk} = \frac{A_{ij} T_k u T_i T_j u + A_{jk} T_i u T_j T_k u}{A_{ik} T_j u T_k u},
\]  
\[(3.7)\]

then the bilinear discrete KP equation itself can be written \(C_{ijk} = 1\). The usual Euler-Lagrange equation is

\[
0 = \frac{1}{u} \ln \left\{ \begin{array}{l}
T_i^{-1} C_{jk} T_j^{-1} T_i^{-1} \mathcal{C}_{ijk} \\
T_j^{-1} C_{ik} T_k^{-1} T_j^{-1} \mathcal{C}_{ijk} \\
T_k^{-1} C_{ij} T_i^{-1} T_k^{-1} \mathcal{C}_{ijk}
\end{array} \right\}.
\]  
\[(3.9)\]

The Euler-Lagrange equations (3.5a) and (3.5b) are trivial in this case, while (3.5b) and (3.5d) are

\[
0 = \frac{1}{T_i T_j T_k u} \ln \left\{ \begin{array}{l}
T_i C_{ijk} T_j T_k u \\
T_j C_{ijk} T_k T_i u \\
T_k C_{ijk} T_i T_j u
\end{array} \right\},
\]  
\[(3.10a)\]

\[
0 = \frac{1}{T_i T_j} \ln \left\{ \begin{array}{l}
C_{ijk} T_j C_{ik} T_k C_{ij} \\
C_{ijk} C_{ij} T_k C_{ik} \\
C_{ijk} C_{ij} T_k C_{ik}
\end{array} \right\},
\]  
\[(3.10b)\]

\[
0 = \frac{1}{T_i T_j T_k} \ln \left\{ \begin{array}{l}
T_i C_{ijk} T_k C_{ij} T_j C_{ik} \\
T_j C_{ijk} T_k C_{ij} T_i C_{ik} \\
T_k C_{ijk} T_i C_{ij} T_j C_{ik}
\end{array} \right\}.
\]  
\[(3.10c)\]
4 Summary and conclusions

Multidimensionally consistent systems can be considered as critical points of an action: critical with respect to the dependent variable, and also with respect to the curve or surface in the space of independent variables. In the case of discrete systems, this means the action is required to be independent of the curve or surface on which it is defined, whilst keeping any boundary it may have fixed. This leads to a set of Euler-Lagrange equations, corresponding to basic configurations of points in a surface, which should be satisfied simultaneously.

In the case of 2-dimensional discrete systems, we have shown that the set of Euler-Lagrange equations arising from this variational principle specify firstly a particular form of the Lagrangian, and furthermore quad equations themselves, whereas previously only a weaker form of the equations could be derived. Starting from known examples of Lagrangians, we can show that the resulting quad equations are compatible with previous results.

It would be interesting to see if the results of this paper can be extended to higher than 3 dimensions where we will have a Lagrangian function evaluated on an n-dimensional object, in particular on an n-dimensional cube. Embedding this in higher dimensions, we consider an action on the smallest closed n-dimensional surface in (n+1) dimensions, a hypercube. Then the minimal set of Euler-Lagrange equations are obtained by demanding that the derivative of this action with respect to each variable is zero.

As we pointed out earlier, the set of Euler-Lagrange equations could, and maybe should, be viewed as a system of equations for the Lagrangian itself. This constitutes a significant departure from the conventional point of view where the Lagrangian is a given object (usually obtained from considerations of physics) and the main issue is to derive the equations of the motion of the system from a variational approach. In the integrable case of Lagrangian multiforms, the Lagrangians themselves are part of the solution of the extended system of equations obtained from varying not only the field variables on a given space-time of independent variables, but by also varying the geometry of space-time itself. It would be of interest to see whether Lagrangians associated with descriptions of known physical processes could be obtained from such a novel variational theory.

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