Diameter, edge-connectivity, and $C_4$-freeness

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Abstract

Improving a recent result of Fundikwa, Mazorodze, and Mukwembi, we show that $d \leq (2n - 3)/5$ for every connected $C_4$-free graph of order $n$, diameter $d$, and edge-connectivity at least 3, which is best possible up to a small additive constant. For edge-connectivity at least 4, we improve this to $d \leq (n - 3)/3$. Furthermore, adapting a construction due to Erdős, Pach, Pollack, and Tuza, for an odd prime power $q$ at least 7, and every positive integer $k$, we show the existence of a connected $C_4$-free graph of order $n = (q^2 + q - 1)k + 1$, diameter $d = 4k$, and edge-connectivity $\lambda$ at least $q - 6$, in particular, $d \geq 4(n - 1)/(\lambda^2 + O(\lambda))$.

Keywords: Diameter; edge-connectivity; $C_4$-freeness; Brown graph

1 Introduction

All graphs considered here are finite, simple, and undirected. The fact that a connected graph of minimum degree $\delta$ and diameter $d$ has order at least $\frac{d(\delta + 1)}{\delta} + O(1)$ has been discovered several times [2,9,14,15]. Lower bounds on the order of graphs realizing a certain diameter subject to conditions such as triangle-freeness [9,12], $C_4$-freeness [9,11], or conditions involving the chromatic number [5,6] or the edge-connectivity have been studied [1,11,12]. Similar problems for notions related to the diameter also received attention [7,8,13].

In the present paper we relate the order and the diameter for $C_4$-free graphs subject to edge-connectivity conditions. Let $G$ be a connected graph of order $n$, minimum degree $\delta \geq 2$, diameter $d$, and edge-connectivity $\lambda$ that is $C_4$-free, that is, the graph $G$ contains no cycle of length four as a (not necessarily induced) subgraph. Exploiting the simple fact that there are at least $\delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1$ vertices within distance at most 2 from every vertex of $G$, Erdős et al. [9] showed

$$d \leq \frac{5n}{\delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1}. \quad (1)$$

Furthermore, they showed that, provided that $q = \delta + 1$ is a prime power, there is a graph $G$, derived from the so-called Brown graph $B(q)$ [3,4,10], with the above properties that satisfies

$$d \geq \frac{5n}{\delta^2 + 3\delta + 2} - 1, \quad (2)$$

that is, asymptotically, there is only little room for improvements of (1). If $\lambda \geq 3$, then $\delta \geq 3$, and (1) implies $d \leq \frac{5n}{8}$, which was recently improved by Fundikwa et al. [11] to

$$d \leq \frac{3n - 3}{7}. \quad (3)$$
They also constructed graphs with \( d \geq \frac{2n-5}{8} \) for infinitely many \( n \), that is, the factor 3/7 in (3) cannot be improved beyond 3/8.

Our first result is the following improvement of (3).

**Theorem 1.** If \( G \) is a connected \( C_4 \)-free graph of order \( n \), diameter \( d \), and edge-connectivity at least 3, then

\[
d \leq \frac{2n-3}{5}.
\]  

(4)

The graphs illustrated in Figure 1 show that the factor 2/5 in (4) is best possible, that is, our result is tight up to a small additive constant.

Figure 1: A family of \( C_4 \)-free graphs \( G \) with \((n_0, n_1, \ldots, n_d) = (1, 3, 4, 2, 3, 2, 3, \ldots, 2, 3, 2, 4, 3, 1)\), where \( n_i \) is the number of vertices at distance \( i \) from \( u \). The graph \( G \), depending on the integer parameter \( k \), is 3-edge-connected, and satisfies \( d = 2k + 6 \) and \( n = 5k + 18 \), that is, \( d = \frac{2n-6}{5} \).

For \( \delta \geq \lambda \geq 4 \), the inequality (4) implies \( d \leq \frac{5n}{13} \), which we improve as follows.

**Theorem 2.** If \( G \) is a connected, \( C_4 \)-free graph of order \( n \), diameter \( d \), and edge-connectivity \( \lambda \) at least 4, then

\[
d \leq \frac{n - 3}{3}.
\]  

(5)

We believe that the factor 1/3 in (5) can be improved to 2/7, which would be best possible in view of the graphs illustrated in Figure 2.

Figure 2: A family of \( C_4 \)-free 4-edge-connected graphs \( G \) of order \( n \) and diameter \( d \) with \( d = \frac{2}{7}n + O(1) \). The graph \( G \) contains two disjoint copies of the graph \( G_0 \) explained after Theorem 3.
The proofs of (1), (3), and (5) all rely on rather local arguments counting vertices within some bounded distance. While this leads to essentially best possible results in some settings, for the proof of (4), involving the best possible factor \(2/5\), a non-local argument was required. We believe that also the proof of a best possible version of (5) would require a non-local argument. The two presented improvements of the consequences of (1) for small values of \(\lambda\) both require a detailed case analysis, that is, their proofs do not seem to generalize to larger values of \(\lambda\).

The graphs constructed by Erdős et al. [9] for (2) contain numerous bridges, that is, their edge-connectivity is small. Analyzing the edge-connectivity of the building blocks \(B(q)\) for that construction, cf. Lemma 4 below, allows to show the following.

**Theorem 3.** For an odd prime power \(q\) at least 7, and every positive integer \(k\), there is a connected \(C_4\)-free graph of order \(n = (q^2 + q - 1)k + 1\), diameter \(d = 4k\), and edge-connectivity \(\lambda\) at least \(q - 6\), in particular,

\[
d \geq \frac{4(n - 1)}{\lambda^2 - 11\lambda + 29}
\]

Note that Theorem 3 implies the existence of a specific \(C_4\)-free, 4-edge-connected graph \(G_0\) of bounded order for which \(G_0^3\) contains an independent set \(I\) of order 6, which is used in Figure 2.

All proofs are given in the following section.

In a final concluding section, we discuss directions for further research.

## 2 Proofs

We begin with the proof of Theorem 1; the non-local argument mentioned above relies on Claim 1(c)-(g) as well as (6).

**Proof of Theorem 1.** Let \(G\) be a connected \(C_4\)-free graph of order \(n\), diameter \(d\), and edge-connectivity \(\lambda\) at least 3. Let \(u\) be a vertex in \(G\) of maximum eccentricity. For \(i \in \{0, 1, \ldots, d\}\), let \(V_i\) be the set of vertices of \(G\) at distance \(i\) from \(u\). By the choice of \(u\), the number \(n_i\) of vertices in \(V_i\) is positive for every \(i \in \{0, 1, \ldots, d\}\).

**Claim 1.**

(a) If \(n_i = 1\) for some \(i \leq d - 1\), then \(n_{i+1} \geq 3\).

(b) If \(n_i = 2\) for some \(i \leq d - 1\), then \(n_{i+1} \geq 2\).

(c) If \((n_i, n_{i+1}) = (1, 3)\) for some \(i \leq d - 1\), then \(i \leq d - 2\) and \(n_{i+2} \geq 4\).

(d) If \((n_i, n_{i+1}) = (2, 2)\) for some \(i \leq d - 1\), then \(i \leq d - 2\) and \(n_{i+2} \geq 3\).

(e) If \((n_i, n_{i+1}) = (2, 3)\) for some \(i \leq d - 1\), then \(i \leq d - 2\) and \(n_{i+2} \geq 2\).

(f) If \((n_i, n_{i+1}, n_{i+2}) = (2, 3, 2)\) for some \(i \leq d - 2\), then \(V_{i+1}\) contains exactly one or two edges, and \(i \leq d - 3\).

(g) If \((n_i, n_{i+1}, n_{i+2}, n_{i+3}) = (2, 3, 2, 2)\) for some \(i \leq d - 3\), then \(i \leq d - 4\) and \(n_{i+4} \geq 4\).

**Proof of Claim 1.** (a) and (b) follow because \(\lambda \geq 3\) implies \(n_in_{i+1} \geq 3\).

(c) Since \(\lambda \geq 3\), the unique vertex in \(V_i\) is adjacent to all three vertices in \(V_{i+1}\). If \(i = d - 1\), then \(V_{i+1}\) must be complete, which implies the contradiction that \(G\) contains some \(C_4\). Hence, we have
$i \leq d - 2$. Suppose, for a contradiction, that $(n_i, n_{i+1}, n_{i+2}) = (1, 3, p)$ for some $p \leq 3$. If $p \leq 2$, then some vertex in $V_{i+2}$ has two neighbors in $V_{i+1}$, which implies the contradiction that $G$ contains some $C_4$. It follows that $p = 3$, and that every vertex in $V_{i+2}$ has exactly one neighbor in $V_{i+1}$, which implies that the edges between $V_{i+1}$ and $V_{i+2}$ form a matching. Since every vertex in $V_{i+1}$ has degree at least 3, the set $V_{i+1}$ contains at least two edges, which implies the contradiction that $G$ contains some $C_4$. This completes the proof of (c).

(d) If $i = d - 1$, then $\lambda \geq 3$ implies that there are all four possible edges between $V_i$ and $V_{i+1}$, which implies the contradiction that $G$ contains some $C_4$. Hence, we have $i \leq d - 2$. Suppose, for a contradiction, that $(n_i, n_{i+1}, n_{i+2}) = (2, 2, p)$ for some $p \leq 2$. Since $\lambda \geq 3$, both $V_i$ as well as $V_{i+2}$ contain a vertex that is adjacent to both vertices in $V_{i+1}$, which implies the contradiction that $G$ contains some $C_4$. This completes the proof of (d).

(e) If $i = d - 1$, then the $C_4$-freeness implies that two vertices in $V_{i+1}$ have only one neighbor in $V_i$. Since $\lambda \geq 3$, this implies that $V_{i+1}$ is complete. Since two vertices in $V_{i+1}$ have a common neighbor in $V_i$, we obtain the contradiction that $G$ contains some $C_4$. Hence, we have $i \leq d - 2$. Suppose, for a contradiction, that $(n_i, n_{i+1}, n_{i+2}) = (2, 3, 1)$. Since $\lambda \geq 3$, the unique vertex in $V_{i+2}$ is adjacent to all three vertices in $V_{i+1}$, and some vertex in $V_i$ has two neighbors in $V_{i+1}$, which implies the contradiction that $G$ contains some $C_4$. This completes the proof of (e).

(f) First, suppose, for a contradiction, that $V_{i+1}$ is independent. Since $\lambda \geq 3$, the set $V_{i+1}$ contains two vertices that are adjacent to both vertices in $V_i$ or $V_{i+1}$ contains two vertices that are adjacent to both vertices in $V_{i+2}$. In both cases $G$ contains some $C_4$, which implies that $V_{i+1}$ is not independent. Now, suppose, for a contradiction, that $V_{i+1}$ is complete, then some vertex in $V_{i+2}$ has two neighbors in $V_{i+1}$, which implies the contradiction that $G$ contains some $C_4$. Hence, the set $V_{i+1}$ contains exactly one or two edges. If $i = d - 2$, then $\lambda \geq 3$ and the $C_4$-freeness imply that both vertices in $V_{i+2}$ are adjacent and have two neighbors in $V_{i+1}$. Together with a suitable edge within $V_{i+1}$, we obtain the contradiction that $G$ contains some $C_4$. Hence, we have $i \leq d - 3$. This completes the proof of (f).

(g) By (d), we obtain $i \leq d - 4$ and $n_{i+4} \geq 3$. Suppose, for a contradiction, that $n_{i+4} = 3$. Let $V_{i+1} = \{a_1, a_2, a_3\}$, $V_{i+2} = \{b_1, b_2\}$, $V_{i+3} = \{c_1, c_2\}$, and $V_{i+4} = \{d_1, d_2, d_3\}$.

First, we show that $V_{i+1}$ contains exactly one edge. Therefore, by (f), we suppose, for a contradiction, that $V_{i+1}$ contains exactly two edges; say $a_1a_2, a_2a_3 \in E(G)$. Since $\lambda \geq 3$ and $G$ contains no $C_4$, we may assume, by symmetry, that $c_1$ is adjacent to $b_1$ and $b_2$, that $b_1$ is adjacent to $a_1$ and $a_2$, that $b_2$ is adjacent to $a_3$, and that $c_2$ has exactly one neighbor in $V_{i+2}$. If $c_2$ is adjacent to $b_1$, then, since $G$ contains no $C_4$, the vertex $b_2$ has degree 2, which contradicts $\lambda \geq 3$. Conversely, if $c_2$ is adjacent to $b_2$, then, since $G$ contains no $C_4$, the two edges $a_3b_2$ and $b_1c_1$ form a 2-edge cut, which contradicts $\lambda \geq 3$. Hence, the set $V_{i+1}$ contains exactly one edge; say $a_1a_2 \in E(G)$.

Next, we show that $V_{i+2}$ is complete. Therefore, suppose, for a contradiction, that $b_1$ and $b_2$ are not adjacent. Since $\lambda \geq 3$ and $G$ contains no $C_4$, we may assume, by symmetry, that $c_1$ is adjacent to $b_1$ and $b_2$, that $a_3$ is adjacent to $b_2$, and that $b_1$ is adjacent to $a_1$. Since the two edges $a_3b_2$ and $b_1c_1$ form no 2-edge cut, it follows that $b_2$ is adjacent to $a_2$, and that $b_1$ is adjacent to $c_2$. Now, the two edges $a_1b_1$ and $b_2c_1$ form a 2-edge cut, which contradicts $\lambda \geq 3$. Hence, the set $V_{i+2}$ is complete.

Since $\lambda \geq 3$ and $G$ contains no $C_4$, we may assume, by symmetry, that $c_1$ is adjacent to $b_1$ and $b_2$, that $c_2$ is adjacent to $b_2$, that $c_2$ is adjacent to $d_2$ and $d_3$, and that $c_1$ is adjacent to $d_1$. Now, the two edges $b_2c_2$ and $c_1d_1$ form a 2-edge cut, which contradicts $\lambda \geq 3$. This completes the proof of (g).

Since $n_0 = 1$, Claim 1(a,c) implies that either $n_1 \geq 4$ or $n_1 = 3$ and $n_2 \geq 4$. Furthermore, if
\[ n_1 = 4, \text{ then } n_2 \geq 3. \] In all cases we obtain \( \frac{n_0 + n_1 - \frac{1}{2}}{i + 1} \geq \frac{5}{2} \) or \( \frac{n_0 + n_1 + n_2 - \frac{1}{2}}{2i + 1} \geq \frac{5}{2} \). Let \( i \in \{0, 1, \ldots, d\} \) be maximum such that
\[
\frac{n_0 + n_1 + \cdots + n_i - \frac{1}{2}}{i + 1} \geq \frac{5}{2}. \tag{6}
\]
The above observations imply that \( i \) is well-defined and that \( i \geq 1 \).

If \( i = d \), then \( \frac{n_i - \frac{1}{2}}{i + 1} \geq \frac{5}{2} \), which implies (11). Similarly, if \( i = d - 1 \), then \( \frac{n_i - \frac{1}{2}}{d} \geq \frac{n_{d-1} - \frac{1}{2}}{d} \geq \frac{5}{2} \), which implies (11). Hence, we may assume that \( i \leq d - 2 \). The choice of \( i \) implies \( n_{i+1} \in \{1, 2\} \). If \( n_{i+1} = 1 \), then, by Claim (11)(a,c), we obtain \( \frac{n_{i+1} + n_{i+3}}{2} \geq \frac{5}{2} \) or \( i \leq d - 3 \) and \( \frac{n_{i+1} + n_{i+2} + n_{i+3}}{3} \geq \frac{5}{2} \), which implies the contradiction that \( i + 2 \) or \( i + 3 \) satisfies (6) (replacing \( i \) with the larger value). Hence, we obtain \( n_{i+1} = 2 \). By the choice of \( i \), Claim (11)(b) implies \( n_{i+2} = 2 \). By Claim (11)(d), we obtain \( i \leq d - 3 \) and \( n_{i+3} \geq 3 \). By the choice of \( i \), we have \( n_{i+3} = 3 \). Let the positive integer \( k \) be maximum such that \( i + 2k + 1 \leq d \) and
\[
(n_{i+1}, n_{i+2}, n_{i+3}, \ldots, n_{i+2k}, n_{i+2k+1}) = \underbrace{(2, 2, 3, \ldots, 2, 3)}_{k \times (2, 3)}.\]

By Claim (11)(d) and the choice of \( i \), we obtain \( i + 2k + 2 \leq d \) and \( n_{i+2k+2} = 2 \). By Claim (11)(b)(f) and the choices of \( i \) and \( k \), we obtain \( i + 2k + 3 \leq d \) and \( n_{i+2k+3} = 2 \). By Claim (11)(g), we obtain \( i + 2k + 4 \leq d \) and \( n_{i+2k+4} \geq 4 \). Now, we have \( \frac{n_{i+1} + \cdots + n_{i+2k+4}}{2k+4} \geq \frac{5}{2} \), which implies the contradiction that \( i + 2k + 4 \) satisfies (6) (replacing \( i \) with this larger value). This final contradiction completes the proof. \( \square \)

The following proof of Theorem 2 is similar to the general approach from (11).

**Proof of Theorem 2.** Let \( G \) be as in the statement. Let \( u, V_i, \) and \( n_i \) be as in the proof of Theorem 1.

**Claim 2.**
\[
n_{i-1} + n_i + n_{i+1} \geq 9 \text{ for every } i \in \{1, \ldots, d - 1\}. \tag{7}
\]

**Proof of Claim 2.** Let \( i \in \{1, \ldots, d - 1\} \). We consider different cases according to the value of \( n_i \).

First, we assume that \( n_i = 1 \). Since \( \lambda \geq 4 \), we have \( n_{i-1}, n_{i+1} \geq 4 \), which implies (7).

Next, we assume that \( n_i = 2 \). This implies \( n_{i-1}, n_{i+1} \geq 3 \), since, otherwise, the at least 4 edges between \( V_i \) and either \( V_{i-1} \) or \( V_{i+1} \) already yield a \( C_4 \). If \( n_{i-1} = n_{i+1} = 3 \), then some vertex in \( V_{i-1} \) as well as some vertex in \( V_{i+1} \) is adjacent to both vertices in \( V_i \), which yields a \( C_4 \). Hence, \( \min\{n_{i-1}, n_{i+1}\} \geq 3 \) and \( \max\{n_{i-1}, n_{i+1}\} \geq 4 \), which implies (7).

Next, we assume that \( n_i = 3 \). Let \( V_i = \{b_1, b_2, b_3\} \). Since \( \lambda \geq 4 \), we obtain \( n_{i-1}, n_{i+1} \geq 2 \). If \( (n_{i-1}, n_{i+1}) = (2, 3) \), then some vertex \( c_1 \) in \( V_{i+1} \) has two neighbors, say \( b_1 \) and \( b_2 \), in \( V_i \). Since \( \lambda \geq 4 \) and \( G \) is \( C_4 \)-free, one vertex in \( V_{i-1} \) is adjacent to \( b_1 \) and \( b_2 \), and the other vertex in \( V_{i-1} \) is adjacent to \( b_2 \) and \( b_3 \). Since \( G \) is \( C_4 \)-free, there is at most one edge with both endpoints in \( V_i \). Since \( \delta \geq \lambda \geq 4 \), this implies the existence of at least four edges between \( V_i \) and \( V_{i+1} \) that are distinct from the two edges \( b_1 c_1 \) and \( b_3 c_1 \), which easily implies the contradiction that \( G \) contains a \( C_4 \) using one vertex in \( V_{i-1} \), two vertices in \( V_i \), and one vertex in \( V_{i+1} \). Similarly, the assumption \( (n_{i-1}, n_{i+1}) \in \{(2, 2), (3, 2)\} \) yields a contradiction. Hence, we obtain \( n_{i-1} + n_{i+1} \geq 6 \), which implies (7).

Next, we assume that \( n_i = 4 \). If \( n_{i-1} = 1 \), then no vertex in \( V_{i+1} \) has two neighbors in \( V_i \), which implies \( n_{i+1} \geq 4 \), and, hence, (7). Similarly, if \( n_{i+1} = 1 \), then (7) follows. Suppose, for a contradiction, that \( (n_{i-1}, n_{i+1}) = (2, 2) \). Let \( V_{i-1} = \{a_1, a_2\} \), \( V_i = \{b_1, b_2, b_3, b_4\} \), and \( V_{i+1} = \{c_1, c_2\} \).
Since \( \lambda \geq 4 \) and \( G \) is \( C_4 \)-free, we may assume that \( c_1 \) is adjacent to \( b_1 \) and \( b_2 \), \( b_1 \) is adjacent to \( a_1 \), and \( b_2 \) is adjacent to \( a_2 \). Since every vertex in \( V_i \) has a neighbor in \( V_{i-1} \), this implies that \( c_1 \) has no further neighbor in \( V_i \). Hence, the vertex \( c_2 \) has exactly two neighbors in \( V_i \). First, suppose that \( c_2 \) is adjacent to \( b_3 \) and \( b_4 \). By symmetry, we may assume that \( b_3 \) is adjacent to \( a_1 \), and \( b_4 \) is adjacent to \( a_2 \). Now, \( b_2 \) has two neighbors in \( V_i \). Since \( G \) is \( C_4 \)-free, this implies that \( b_2 \) is adjacent to \( b_1 \) and \( b_4 \). Symmetrically, it follows that \( b_3 \) is adjacent to \( b_4 \) and \( b_1 \), which implies the contradiction that \( G \) contains a \( C_4 \) completely within \( V_i \). Next, suppose that \( c_2 \) is adjacent to \( b_2 \) and \( b_3 \). Since \( \lambda \geq 4 \) and \( G \) is \( C_4 \)-free, the vertex \( b_4 \) has at most one neighbor in \( V_i \), and at most one neighbor in \( V_{i+1} \). Hence, \( b_4 \) is adjacent to \( a_1 \) and \( a_2 \). Since \( G \) is \( C_4 \)-free, \( b_4 \) has no neighbor in \( V_{i+1} \), which implies the contradiction that \( b_4 \) has degree at most 3. Altogether, we obtain \( n_{i-1} + n_{i+1} \geq 5 \), which implies (7).

Next, we assume that \( n_i \in \{5,6\} \). If \((n_{i-1}, n_{i+1}) = (1,1), (1,2), \) or \((2,1)\), then some vertex in \( V_{i+1} \) has two neighbors in \( V_i \) that have a common neighbor in \( V_{i-1} \), which is a contradiction. This implies (7) in this case.

Finally, since \( n_{i-1} + n_{i+1} \geq 2 \), the inequality (7) is trivial for \( n_i \geq 7 \), which completes the proof of the claim.

Since \( \lambda \geq 4 \) and \( G \) is \( C_4 \)-free, we have \( n_{d-1} + n_d \geq 5 \). Now, together with Claim 2 this implies that

- if \( d \equiv 2 \mod 3 \), then \( n = (n_0 + n_1 + n_2) + (n_3 + n_4 + n_5) + \cdots + (n_{d-2} + n_{d-1} + n_d) \geq 9(d+1)/3 \),
- if \( d \equiv 0 \mod 3 \), then \( n = (n_0 + n_1 + n_2) + \cdots + (n_{d-2} + n_{d-1} + n_d) + n_d \geq 9d/3 + 1 \), and
- if \( d \equiv 1 \mod 3 \), then \( n = (n_0 + n_1 + n_2) + \cdots + (n_{d-2} + n_{d-1} + n_d) \geq 9d - 1)/3 + 5 \).

In all three cases, we obtain (5).

For the proof of Theorem 3, we need to explain the construction behind (2).

Let \( q \) be an odd prime power, and let \( \mathbb{F}_q \) denote the field of order \( q \). Two non-zero vectors from \( \mathbb{F}_q^3 \) are considered equivalent if they generate the same one-dimensional subspace of \( \mathbb{F}_q^3 \); that is, they are non-zero multiples of each other. The Brown graph \( B(q) \) has as its vertices the equivalence classes \([x]\) of the non-zero vectors \( x \) from \( \mathbb{F}_q^3 \), and two distinct vertices \([x]\) and \([y]\) are adjacent exactly if \( xy^T = 0 \). This graph was proposed independently by Brown [3] and Erdős and Rényi [10]. It is a dense \( C_4 \)-free graph with the following properties [3]:

- The vertex set \( V(B(q)) \) of \( B(q) \) can be partitioned into two sets \( W \) and \( V \), where \( W = \{ [x] : x \in \mathbb{F}_q^3 \setminus \{0\} \text{ with } xx^T = 0 \} \)
  contains the so-called quadric vertices. The vertices in \( W \) have degree \( q \) and the vertices in \( V \) have degree \( q+1 \). Furthermore, the order of \( B(q) \) is \( q^2 + q + 1 \), \( |W| = q + 1 \), and, hence, \( |V| = q^2 \).
- \( W \) is an independent set, and every vertex from \( V \) has exactly two or zero neighbors in \( W \).
- No vertex from \( W \) lies on a triangle.
- Every two non-adjacent vertices as well as every two vertices from \( V \) (adjacent or not) have exactly one common neighbor.
For the construction of the graph $G$ in [2], Erdős et al. [9] modify $B(q)$ as follows: Let $c$ be a quadric vertex of $B(q)$, and let $a$ and $b$ be two neighbors of $c$. Since $c$ is quadric, the vertices $a$ and $b$ are not adjacent and both of degree $q + 1$.

Let

$$A = N_{B(q)}(a) \setminus N_{B(q)}(b) = \{a_0, a_1, \ldots, a_{q-1}\} \quad \text{and} \quad B = N_{B(q)}(b) \setminus N_{B(q)}(a) = \{b_0, b_1, \ldots, b_{q-1}\}$$

be such that $a_0$ and $b_0$ are the second quadric neighbors of $a$ and $b$ distinct from $c$, respectively. Since no quadric vertex lies on a triangle, every vertex in $A$ has exactly one common neighbor with $b$, and every vertex in $B$ has exactly one common neighbor with $a$, the edges of $B(q)$ between $A$ and $B$ form a perfect matching $M$ between these two sets. Possibly by renaming vertices, we may assume that

$$M = \{a_0b_1, a_1b_0\} \cup \{a_2b_2, a_3b_3, \ldots, a_{q-1}b_{q-1}\}.$$ 

For every $i \in [q-1]$, the two non-quadric vertices $a_i$ and $b_i$ have exactly one common neighbor $c_i$. By the properties of $B(q)$, the vertices $c_i$ are all distinct, and do not belong to $\{c\} \cup A \cup B$.

Let

$$C = \{c_1, \ldots, c_{q-1}\}.$$

Let the graph $H$ arise from $B(q)$ by removing the quadric vertex $c$ as well as all edges from $M$, cf. Figure 3.

Figure 3: The structure of $B(q)$ around $\{a, b, c\} \cup A \cup B \cup C$; the figure shows all edges except for possible edges within each of the sets $A$, $B$, and $C$.

The idea behind the construction of $H$ is to destroy all paths in $B(q)$ of length at most two between $a$ and $b$ increasing their distance to four. Now, the graph $G$ considered for (2) arises from the disjoint union of copies $H_1, \ldots, H_k$ of $H$ by adding edges between the vertex $b$ from $H_i$ and the vertex $a$ from $H_{i+1}$ for every $i \in [k-1]$. The graph $G$ has minimum degree $q-1$, diameter $5k-1$, and order $(q^2 + q)k$.
which implies \( \square \). Note that \( G \) contains bridges for \( k \geq 2 \), that is, while its minimum degree is large, it is not even 2-edge-connected regardless of the value of \( q \).

For the proof of Theorem \( \square \), we consider the edge-connectivity of \( H \).

**Lemma 4.** The graph \( H \) is \((q - 6)\)-vertex-connected.

**Proof.** We show the existence of at least \( q - 6 \) internally vertex-disjoint paths between any two distinct vertices \( x \) and \( y \) of \( H \). In \( H \), the two vertices \( a \) and \( b \) are connected by the \( q - 1 \) internally vertex-disjoint paths \( a a_1 c_1 b_1, \ldots, a a_{q-1} c_{q-1} b_{q-1} b \), cf. Figure \( \square \)

Now, let \( \{x, y\} \neq \{a, b\} \). We consider different cases.

First, we assume that \( x \) and \( y \) are not adjacent. By symmetry, we may assume that \( d_{B(q)}(x) \leq d_{B(q)}(y) \), and that \( x \notin \{a, b\} \). By the properties of \( B(q) \), the set \( N_{B(q)}(x) \setminus N_{B(q)}(y) \) contains a set \( X \) of \( q - 1 \) vertices \( x_1, \ldots, x_{q-1} \), and each vertex \( x_i \) from \( X \) has exactly one common neighbor \( y_i \) with \( y \). Let \( Y = \{y_1, \ldots, y_{q-1}\} \). By the properties of \( B(q) \), the set \( Y \) is disjoint from \( \{x, y\} \cup X \), and the \( q - 1 \) elements of \( Y \) are all distinct. In \( B(q) \), there are the \( q - 1 \) internally vertex-disjoint paths

\[
xx_1y_1y, \ldots, xx_{q-1}y_{q-1}y
\]

between \( x \) and \( y \). At most one of these paths contains the vertex \( c \). If four more of these paths contain an edge from \( M \), then three vertices from \( X \) are incident with edges from \( M \). Since the edges in \( M \) connect neighbors of \( a \) with neighbors of \( b \), this implies that at least two vertices from \( X \) are neighbors of either \( a \) or \( b \). Now, the vertex \( x \notin \{a, b\} \) shares two neighbors with \( a \) or \( b \), which is a contradiction to the properties of \( B(q) \). Hence, at most three of the above \( q - 1 \) paths contain an edge from \( M \), and, in \( H \), there are at least \( q - 1 - 3 \) internally vertex-disjoint paths between \( x \) and \( y \).

For the rest of the proof, we may assume that \( x \) and \( y \) are adjacent.

Next, we assume that \( x, y \notin \{a, b\} \). By symmetry, we may assume that \( d_{B(q)}(x) \leq d_{B(q)}(y) \), in particular, the vertex \( y \) is not quadric. By the properties of \( B(q) \), regardless of whether \( x \) is quadric or not,

- the set \( N_{B(q)}(x) \setminus N_{B(q)}[y] \) contains a set \( X \) of \( q - 1 \) vertices \( x_1, \ldots, x_{q-1} \),
- the set \( N_{B(q)}(y) \setminus N_{B(q)}[x] \) contains a set \( Y \) of \( q - 1 \) vertices \( y_1, \ldots, y_{q-1} \),
- there is no edge between \( X \) and \( Y \), and,
- for every \( i \in [q - 1] \), the vertex \( x_i \) and the vertex \( y_i \) have a unique common neighbor \( z_i \).

Let \( Z = \{z_1, \ldots, z_{q-1}\} \). By the properties of \( B(q) \), the \( q - 1 \) elements of \( Z \) are all distinct, and \( Z \) is disjoint from \( \{x, y\} \cup X \cup Y \). In \( B(q) \), there are the \( q - 1 \) internally vertex-disjoint paths

\[
xx_1z_1y_1y, \ldots, xx_{q-1}z_{q-1}y_{q-1}y
\]

between \( x \) and \( y \). At most one of these paths contains the vertex \( c \). If five of these paths contain an edge from \( M \), then three vertices from either \( X \) or \( Y \) are incident edges from \( M \). Similarly as before, this implies that one of the two vertices \( x \) and \( y \) has two common neighbors with one of the two vertices \( a \) and \( b \), which is a contradiction to the properties of \( B(q) \). Hence, at most four of the above \( q - 1 \) paths contain an edge from \( M \), and, in \( H \), there are at least \( q - 1 - 4 \) internally vertex-disjoint paths between \( x \) and \( y \).

For the rest of the proof, we may assume, by the symmetry between \( a \) and \( b \), that \( y = a \).
As $x$ and $y$ are adjacent, this implies $x \in A$.

First, we assume that $x = a_0$, that is, the vertex $a_0$ is the unique second quadric neighbor of $a$ distinct from $c$. The vertex $x = a_0$ has no neighbor in $\{b, c\} \cup (B \setminus \{b_1\}) \cup C$. Let $X = N_{B(q)}(x) \setminus \{a, b_1\} = \{x_2, \ldots, x_{q-1}\}$ and $Y = \{a_2, \ldots, a_{q-1}\} \subseteq A$. For every $i \in \{2, \ldots, q-1\}$, the vertices $x_i$ and $a_i$ have a unique common neighbor $z_i$. By the properties of $B(q)$, the vertices $z_2, \ldots, z_{q-1}$ are all distinct. Note that $z_i$ may coincide with $b_i$ or $c_i$ but is distinct from $c$ and $c_j$ for $j \neq i$, cf. Figure 4 for an illustration.

If $z_i = b_i$, then let $P_i$ be the path $xx_i b_i c_i a_i y$ avoiding the edge $a_i b_i$ from $M$, otherwise, let $P_i$ be the path $xx_i z_i a_i y$. In $H$, there are the $q - 2$ internally vertex-disjoint paths $P_2, \ldots, P_{q-1}$ between $x$ and $y$.

Finally, we assume that $x = a_i$ for some $i \in [q-1]$. The vertex $x = a_i$ has exactly one neighbor $b_j$ in $B$, exactly one neighbor $c_i$ in $C$, exactly one neighbor in $A$ — its unique common neighbor with $a$, and no neighbor in $\{b, c\}$. Hence, the set $X = N_{B(q)}(x) \setminus (\{a, b_j, c_i\} \cup A)$ contains $q - 3$ vertices $x_3, \ldots, x_{q-1}$. Let $Y = \{a_3, \ldots, a_{q-1}\} \subseteq A$. For every $i \in \{3, \ldots, q-1\}$, the vertices $x_i$ and $a_i$ have a unique common neighbor $z_i$. By the properties of $B(q)$, the vertices $z_3, \ldots, z_{q-1}$ are all distinct. Again, the vertex $z_i$ may coincide with $b_i$ or $c_i$ but is distinct from $c$ and $c_j$ for $j \neq i$. Defining the paths $P_3, \ldots, P_{q-1}$ as above, we obtain that, in $H$, there are the $q - 3$ internally vertex-disjoint paths between $x$ and $y$.

This completes the proof.

Adapting the proof of Lemma 4 it is not difficult to show that $B(q)$ is actually $q$-vertex-connected.

Proof of Theorem 3. The corresponding graph arises from the disjoint union of copies $H_1, \ldots, H_k$ of $H$ by identifying the vertex $b$ from $H_i$ with the vertex $a$ from $H_{i+1}$ for every $i \in [k-1]$. 

\[ \Box \]
3 Conclusion

Our results motivate several research problems. In Theorem 1, one could determine the best possible value for the additive constant. In fact, we believe that the graphs in Figure 1 are extremal. In Theorem 2, one could determine the best possible factor. While, unfortunately, this seems to require a detailed and tedious case analysis in combination with a non-local argument, we believe that our approach from Theorem 1 could be adapted. Finally, and most interestingly, one should determine the smallest possible constant $c$ such that $d \leq \frac{\lambda}{\lambda+O(\lambda)}$ for every $C_4$-free graph of order $n$, diameter $d$, and edge-connectivity $\lambda$. Note that (1) and Theorem 3 imply $4 \leq c \leq 5$.

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