Properties of Semi-Chiral Superfields

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Abstract
Whenever the $N = (2, 2)$ supersymmetry algebra of non-linear $\sigma$-models in two dimensions does not close off-shell, a holomorphic two-form can be defined. The only known superfields providing candidate auxiliary fields to achieve an off-shell formulation are semi-chiral fields. Such a semi-chiral description is only possible when the two-form is constant. Using an explicit example, hyper-Kähler manifolds, we show that this is not always the case. Finally, we give a concrete construction of semi-chiral potentials for a class of hyper-Kähler manifolds using the duality exchanging a pair consisting of a chiral and a twisted-chiral superfield for one semi-chiral multiplet.
1. Introduction

Starting with Zumino’s discovery that the scalar fields of an $N = 1$ non-linear $\sigma$-model in four dimensions should be viewed as coordinates on a Kähler manifold [1], ample evidence for the interplay between supersymmetry and complex geometry was found.

Requiring more supersymmetry or raising the dimension puts further restrictions on the geometry, while lowering the dimensions relaxes the requirements. Examples of the former statement are well known in 4 dimensions where passing from $N = 1$ to $N = 2$ supersymmetry restricts the geometry of the scalars in vector multiplets to so-called special Kähler manifolds [2]. Similarly, the scalars in hyper-multiplets describe special hyper-Kähler manifolds [3].

Particularly interesting examples of supersymmetric models in lower dimensions are two-dimensional non-linear $\sigma$-models which are used e.g. for the world-sheet description of stringtheory. The closest analog in two dimensions of $N = 1$ supersymmetry in four dimensions is $N = (2, 2)$ supersymmetry. As long as torsion is absent, the target manifold is indeed Kähler [4] and an off-shell description is known. However, once torsion is present, the geometry becomes much richer [5], [6] and an off-shell description is much harder to achieve. Finding an off-shell description has been the subject of numerous studies, [5], [7], [8], [9] and [10], which culminated in [11]. There strong evidence was put forward to support the conjecture that chiral, twisted-chiral and semi-chiral superfields are sufficient to give a manifest off-shell description of these models. These are the only superfields which can be defined by constraints on a general $N = (2, 2)$ superfield which are linear in the fermionic derivatives. Several explicit examples are known. All Kähler-manifolds can be described by chiral superfields, the $SU(2) \times U(1)$ Wess-zumino-Witten model is described either by a chiral and a twisted-chiral field [8] or by a semi-chiral multiplet [11]. Finally, the $SU(2) \times SU(2)$ Wess-Zumino-Witten model requires one semi-chiral and one chiral multiplet [11]. Various models dual to the above mentioned models were constructed in [8], [12], [13].

In the present paper we focus on the case where the $N = (2, 2)$ does not close off-shell in all directions and investigate under which conditions semi-chiral superfields provide an off-shell formulation. Non-closure implies the existence of a holomorphic two-form. A necessary condition for the semi-chiral description to be possible is that there exists a complex coordinate system in which this two-form is constant. Using a particular example, hyper-Kähler manifolds, we are able to show that semi-chiral fields alone are not able to give a full off-shell description thus falsifying the conjecture given in [11]. Semi-chiral potentials do describe hyper-Kähler manifolds provided the potential satisfies a non-linear differential equation.
We end the paper with a short study of duality transformations involving semi-chiral fields. An interesting duality brings a model described by one chiral and one twisted-chiral superfield to a model formulated in terms of one semi-chiral multiplet \cite{12}, \cite{9}. If the original model has an \( N = (4, 4) \) supersymmetry, which is true when the potential satisfies the Laplace equation \cite{5}, then the dual model is a hyper-Kähler manifold. In this way we generate a class of solutions to the non-linear differential equation using solutions of a linear differential equation. This construction is similar in spirit to the construction in \cite{13}, where hyper-Kähler potentials were constructed using the duality between a real linear and a chiral superfield.

2. \( N = (2, 2) \) supersymmetric non-linear \( \sigma \)-models

A bosonic non-linear \( \sigma \)-model in two dimensions is characterized by a manifold, the target manifold, endowed with a metric \( G_{\mu \nu} \) and a closed 3-form \( T_{\mu \nu \rho} \). Locally, the torsion can be written as the exterior derivative of the torsion potential \( B_{\mu \nu} \),

\[
T_{\mu \nu \rho} = -\frac{3}{2} \partial_{[\mu} B_{\nu \rho]} \tag{2.1}
\]

Such a model can be promoted to an \( N = (1, 1) \) supersymmetric model without any additional conditions on the geometry. However, passing from \( N = (1, 1) \) to \( N = (2, 2) \) supersymmetry requires further structure. Two \( (1,1) \) tensors \( J^\mu_\pm \nu \) and \( J^\mu_\mp \nu \) are needed which satisfy

\[
J^\mu_\pm \rho J^\rho_\pm \nu = -\delta^\mu_\nu, \tag{2.2}
\]

\[
N[J_\pm]^{\mu}_{\nu \rho} \equiv J^\mu_{\pm \nu \rho}, J^\rho_{\pm \nu \rho} + J^\rho_{\pm \nu \rho} J^\sigma_{\pm \nu \sigma} = 0, \tag{2.3}
\]

\[
J^\mu_\pm \nu J^\rho_\pm \nu G_{\rho \sigma} = G_{\mu \nu}, \tag{2.4}
\]

\[
\nabla^\pm \mu J^\mu_\pm \nu = 0, \tag{2.5}
\]

where \( \nabla^+ \) and \( \nabla^- \) denote covariant differentiation\footnote{Covariant derivatives are taken as \( \nabla_\nu V^\mu \equiv V^\mu_{,\nu} + \Gamma^\mu_{\rho \nu} V^\rho \) and \( \nabla_\nu V_\mu \equiv V_\mu,\nu - \Gamma^\rho_{\mu \nu} V_\rho \).} using the \( \Gamma^\mu_{+ \nu \rho} \equiv \{ \mu \}_{\nu \rho} + T^\mu_{\nu \rho} \) and \( \Gamma^\mu_{- \nu \rho} \equiv \{ \mu \}_{\nu \rho} - T^\mu_{\nu \rho} \) connections resp. The first two conditions arise from requiring that the supersymmetry algebra is satisfied \textit{on-shell} and the last two conditions follow from the invariance of the action. Eqs. (2.2) and (2.3) imply that both \( J_+ \) and \( J_- \) are complex structures. Eq. (2.4) imposes hermiticity of the metric with respect to both complex structures and eq. (2.5) states that both complex structures are covariantly constant but, when torsion is present, with respect to different connections.

The \( N = (2, 2) \) models characterized by eqs. (2.2-2.5) realize the \( N = (2, 2) \) supersymmetry algebra \textit{on-shell} only. One can show that the off-shell non-closing terms in the algebra, are proportional to the commutator of the two complex structures, \( [J_+, J_-] \) \cite{11}. The construction of a manifest off-shell supersymmetric version of these
model was the subject of intense investigations [3], [7], [8], [9], [11]. Locally, the cotangent space can be decomposed as
\[ \ker[J_{+}, J_{-}] \oplus (\ker[J_{+}, J_{-}])^\perp = \ker(J_{+} - J_{-}) \oplus \ker(J_{+} + J_{-}) \oplus (\ker[J_{+}, J_{-}])^\perp. \] (2.6)

In [3], it was shown that \( \ker(J_{+} - J_{-}) \) and \( \ker(J_{+} + J_{-}) \) are integrable to chiral and twisted-chiral superfields resp. These superfields count as many components as \( N = (1,1) \) superfields. Indeed, as the algebra closes off-shell in these directions, one does not expect that any new auxiliary fields are needed.

Chiral and twisted chiral superfields separately describe Kähler manifolds. However when both of them are simultaneously present, the resulting manifold exhibits a product structure which projects on two Kähler subspaces [3]. The complete manifold is not Kähler, which can be seen from the fact that it has torsion.

In [11], it was shown that the dimension of \( (\ker[J_{+}, J_{-}])^\perp \) is a multiple of four (see also further in this section). Furthermore, off-shell closure of the algebra requires additional auxiliary fields compared to the manifestly \( N = (1,1) \) formulation of the model. Only one class of superfields defined by constraints linear in the derivatives satisfies these requirements: the semi-chiral superfields [7]. This, combined with several non-trivial examples, led to the conjecture [11] that semi-chiral superfields are sufficient to describe \( (\ker[J_{+}, J_{-}])^\perp \). In the present paper, we will give a class of explicit examples disproving the conjecture.

From now on we focus our attention on the case where \( \ker[J_{+}, J_{-}] = \emptyset \). Denoting \( C \equiv [J_{+}, J_{-}] \), we construct a non-degenerate two-form \( \omega(U, V) \equiv G(U, C^{-1}V) \) or in local coordinates
\[ \omega_{\mu\nu} = G_{\mu\rho}(C^{-1})^{\rho}_{\nu}. \] (2.7)

The inverse of the commutator \( C^{-1} \) can be written as the formal power series
\[ C^{-1} = \sum_{n \geq 0} (J_{+}J_{-})^{2n+1} = -\sum_{n \geq 0} (J_{+}J_{-})^{2n+1}. \] (2.8)

The two-form satisfies \( \omega(J_{+}U, J_{+}V) = -\omega(U, V) \).

Introducing complex coordinates \( z^\alpha \) and \( \bar{z}^{\bar{\alpha}} = (z^\alpha)^* \), we diagonalize \( J_{+} \): \( J_{+}^\alpha_{\beta} = i\delta^\alpha_{\beta} \) and \( J_{+}^\alpha_{\beta} = 0 \). In complex coordinates, only \( \omega_{\alpha\bar{\beta}} \) and its complex conjugate are non-vanishing. Eqs. (2.3) and (2.4) for \( J_{+} \) imply that \( T_{\alpha\beta\gamma} = 0 \) and \( G_{\alpha\beta} = 0 \) resp. Finally eq. (2.5) for \( J_{+} \) yields \( \Gamma_{+\alpha\beta\gamma} = \Gamma_{-\alpha\gamma\beta} = 0 \). Combining this with eq. (2.5) for \( J_{-} \) gives
\[ \partial_\alpha \omega_{\beta\gamma} = 0. \] (2.9)

Finally, let us give a very short proof that \( \ker[J_{+}, J_{-}] = \emptyset \) implies that \( d = 4n \) with \( n \in \mathbb{N} \). We view \( \omega_{\alpha\beta} \) as the components of an anti-symmetric \( d/2 \times d/2 \) matrix.
From $\ker[J_+, J_-] = \emptyset$ combined with the non-degeneracy of the metric, we get that its determinant is non-vanishing implying that $d/2$ should be even.

3. Semi-chiral parametrization

We denote the semi-chiral coordinates by $r^a$, $\bar{r}^\bar{a}$, $s^\hat{a}$ and $s^{\hat{\bar{a}}}$, $a, \bar{a}, \hat{a}, \hat{\bar{a}} \in \{1, \cdots n\}$, and we introduce a real function $K(r, \bar{r}, s, \bar{s})$, the semi-chiral potential. It is determined modulo the transformation $K \propto K + f(r) + \bar{f}(\bar{r}) + g(s) + \bar{g}(\bar{s})$ with $f(r)$ and $g(s)$ arbitrary holomorphic functions of $r$ and $s$ resp. The potential is the Lagrange density in $N = (2, 2)$ superspace. Passing to $N = (1, 1)$ superspace and eliminating the auxiliary fields through their equations of motion yields explicit expressions for the metric, torsion potential and the complex structures \cite{11}. In order to facilitate the notation, we introduce the $2n \times 2n$ matrices $L$, $N$, $M$, $\hat{N}$ and $\hat{M}$.

\begin{align*}
L & \equiv \begin{pmatrix} K_{\hat{a}b} & K_{\hat{a}\bar{b}} \\ K_{\bar{a}\hat{b}} & K_{\bar{a}\bar{b}} \end{pmatrix}, \\
N & \equiv \begin{pmatrix} K_{ab} & K_{a\bar{b}} \\ K_{\bar{a}b} & K_{\bar{a}\bar{b}} \end{pmatrix}, \\
\hat{N} & \equiv \begin{pmatrix} K_{\hat{a}\hat{b}} & K_{\hat{a}\bar{b}} \\ K_{\bar{a}\hat{b}} & K_{\bar{a}\bar{b}} \end{pmatrix},
\end{align*}

\begin{align*}
M & \equiv \begin{pmatrix} 0 & K_{\hat{a}b} \\ K_{\bar{a}\hat{b}} & 0 \end{pmatrix}, \\
\hat{M} & \equiv \begin{pmatrix} 0 & K_{\hat{a}\bar{b}} \\ K_{\bar{a}\hat{b}} & 0 \end{pmatrix},
\end{align*}

(3.1)

where e.g. $K_{\hat{a}b}$ stands for $K_{\hat{a}b} \equiv \frac{\partial^2 K}{\partial r^a \partial \bar{r}^\bar{b}}$. Finally we also need the matrix $P$, defined by

\begin{equation}
P \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}

(3.3)

In terms of these matrices, the complex structures are given by\cite{11}

\begin{align*}
J_+ & = \begin{pmatrix} iP & 0 \\ -2iL^{-1}TP & iL^{-1}PPL^T \end{pmatrix}, \\
J_- & = \begin{pmatrix} iL^{-1}PL & 2iL^{-1}P\hat{M} \\ 0 & iP \end{pmatrix}.
\end{align*}

(3.4)

The metric and torsion potential have simple expressions in terms of the complex structures,

\begin{align*}
G & = \frac{1}{2} \begin{pmatrix} 0 & +L^T \\ -L & 0 \end{pmatrix} [J_+, J_-] \\
B & = \frac{1}{2} \begin{pmatrix} 0 & +L^T \\ -L & 0 \end{pmatrix} \{J_+, J_-\}.
\end{align*}

(3.5) (3.6)

Eq. (3.5) clearly shows that the vanishing of $\ker[J_+, J_-]$ is necessary and sufficient for the existence of a non-degenerate metric. Furthermore, the potential should be

\footnote{Rows and columns are labeled as $r\bar{r}s\bar{s}$ and we rescaled $J_-$ by a factor $-1$.}
such that $\det L \neq 0$. Eq. (3.5) gives the explicit form for the two-form $\omega$,

$$\omega_{ab} = \frac{1}{2} K_{ab}, \quad \omega_{\hat{a} \hat{b}} = \frac{1}{2} K_{\hat{a} \hat{b}}. \quad (3.7)$$

Quite remarkable is the existence of simple coordinate transformations which diagonalize either $J_+$ or $J_-$. Consider

$$r^a \rightarrow z^a = r^a,$$
$$s^\hat{a} \rightarrow w_\hat{a} = K_{\hat{a}}, \quad (3.8)$$

then $J_+ \rightarrow J'_+$ with

$$J'_+ = \left( \begin{array}{cc} iP & 0 \\ 0 & iP \end{array} \right), \quad (3.9)$$

and $G \rightarrow G'$ and $B \rightarrow B'$ where

$$G' = \frac{1}{2} \left( \begin{array}{cc} 0 & +1 \\ -1 & 0 \end{array} \right) [J'_+, J'_-],$$
$$B' = \frac{1}{2} \left( \begin{array}{cc} 0 & +1 \\ -1 & 0 \end{array} \right) \{J'_+, J'_-\}. \quad (3.10)$$

Rows and columns are labeled as $z, \bar{z}, w$ and $\bar{w}$. Note that there is also a simple coordinate transformation which diagonalizes $J_-$ which is obtained by reversing the roles of $r$ and $s$ in the previous expressions.

Eq. (3.10) now gives the two-form $\omega$ in complex coordinates,

$$\omega_\hat{a} \hat{b} = \frac{1}{2} \delta_\hat{a} \hat{b}. \quad (3.11)$$

So we reach the conclusion that a necessary condition for a semi-chiral parametrization to be possible is the existence of a complex coordinate system in which the two-form $\omega$ is constant! Note that for $d = 4$, we can always find a holomorphic coordinate transformation which makes the two-form $\omega$ constant. For $d > 4$ this is not the case anymore.

Finally, for the sake of completeness, let us mention that these models are 1-loop UV finite provided they are Ricci flat where the Ricci tensor is computed using the connection with torsion. In [16] the one-loop $\beta$-function was directly computed in $N = (2,2)$ superspace. Its vanishing yields a constraint on the potential which requires the existence of a holomorphic function of $r$, $f(r)$ and a holomorphic function of $s$, $g(s)$, such that

$$\frac{\det \mathcal{N}_2}{\det \mathcal{N}_1} = (-1)^n |f(r)|^2 |g(s)|^2, \quad (3.12)$$
where \( n \) is the number of semi-chiral multiplets and \( N_1 \) and \( N_2 \) are \( 2n \times 2n \) matrices given by

\[
N_1 \equiv \begin{pmatrix} K_{\dot{a}b} & K_{\dot{a}\dot{b}} \\ K_{\dot{a}b} & K_{\dot{a}\dot{b}} \end{pmatrix}, \quad N_2 \equiv \begin{pmatrix} K_{\dot{a}b} & K_{\dot{a}\dot{b}} \\ K_{\dot{a}b} & K_{\dot{a}\dot{b}} \end{pmatrix}.
\]

(3.13)

4. Hyper-Kähler manifolds

As is clear from eq. (3.5), the semi-chiral parametrization is well defined, provided \( \ker[J_+, J_-] = \emptyset \). The most familiar class of complex manifolds satisfying this are the hyper-Kähler manifolds. A hyper-Kähler manifold has three complex structures \( J_i \), \( i \in \{1, 2, 3\} \) which satisfy

\[
J_i J_j = -\delta_{ij} \mathbf{1} + \varepsilon_{ijk} J_k,
\]

(4.1)

and which are such that the manifold is Kähler with respect to all three of them. It is easy to see that a hyper-Kähler manifold has a two-sphere worth of complex structures. Indeed \( \vec{x} \cdot \vec{J} = \sum_{i=1}^3 x_i J_i \) is a complex structure provided that \( \vec{x} \cdot \vec{x} = 1 \). Choosing e.g. \( J_+ = J_1 \), and requiring that \( \ker[J_+, J_-] = \emptyset \), we find that \( J_- \) can be any element of the two-sphere, except for the north- and southpole, \( \vec{x} = (\pm 1, 0, 0) \).

Choosing for \( J_- \) the north-pole, we obtain the description of the manifold in terms of chiral superfields, while choosing the south-pole we get the parametrization in terms of twisted-chiral superfields. Clearly, these are the only two choices where \( J_+ \) and \( J_- \) commute. In other words, there is a cylinder worth of choices for \( J_- \) where the algebra does not close off-shell and which can potentially be described by semi-chiral coordinates. In order for this to work, we need at least that the torsion vanishes, \( T = 0 \). Indeed, choosing one element of the cylinder \( J_- = \vec{x} \cdot \vec{J} \), we get \( \{J_+, J_-\} = -2x_1 \mathbf{1} \). From eq. (3.6), one obtains that for a generic element of the cylinder, the torsion potential differs from zero, but the torsion still vanishes, \( T = dB = 0 \). Turning to the two-form \( \omega \) introduced in section 2, one easily shows that, for the present choice for \( J_+ = J_1 \) and \( J_- = \vec{x} \cdot \vec{J} \), it is given by

\[
\omega = \frac{1}{2(x_2^2 + x_3^2)} (x_3 \omega_2 - x_2 \omega_3),
\]

(4.2)

where \( \omega_i \) are the fundamental two-forms of the hyper-Kähler manifold, defined by \( \omega_i(U, V) \equiv G(U, J_i V), \ i \in \{1, 2, 3\} \). As was shown in the previous section, a semi-chiral parametrization is only possible, if complex coordinates exist, diagonalizing \( J_+ \), where \( \omega \) is constant. In this case \( \omega \) is given as a linear combination of the the two fundamental two-forms \( \omega_2 \) and \( \omega_3 \). Several explicit examples are known of higher dimensional hyper-Kähler manifolds where this is not the case [13].

As was already mentioned, there is still the simplest case \( d = 4 \), where both \( \omega_2 \) and \( \omega_3 \) can be made constant through a holomorphic coordinate transformation. However
we find, as we will see in next section, that the metric satisfies the Monge-Ampère equation after passing from semi-chiral to complex coordinates. To achieve this on a hyper-Kähler manifold, one already needs a holomorphic coordinate transformation. The residual holomorphic coordinate transformations can now only turn the two-form to a constant provided the two-form was originally a phase, which as far as we know, is not necessarily true. Nonetheless, many interesting examples are of this kind. In particular, the four dimensional hyper-Kähler manifolds constructed in \[13\] are of this form. This includes familiar examples such as the multi-Eguchi-Hanson \[14\] and Taub-NUT \[15\] self-dual instantons.

5. The four-dimensional case

The simplest hyper-Kähler manifolds are the four dimensional ones. We will choose $J_+ = J_1$ and $J_- = J_2$. Requiring $\{J_+, J_-\} = 0$ we find using eq. (3.4) that the semi-chiral potential should satisfy

$$|K_{rs}|^2 + |K_{r\bar{s}}|^2 = 2K_{r\bar{r}}K_{s\bar{s}}. \tag{5.1}$$

Performing the coordinate transformation eq. (3.8), we find that the metric eq. (3.10) satisfies the Monge-Ampère equation,

$$G'_{z\bar{w}} G'_{w\bar{z}} - G'_{z\bar{w}} G'_{w\bar{z}} = 1, \tag{5.2}$$

iff. eq. (5.1) holds. Indeed, in four dimensions one finds that eq. (3.12) yields the following non-vanishing components of the metric (and their complex conjugates),

$$
\begin{align*}
G_{rr} &= 4\Omega^{-1}K_{rs}K_{r\bar{s}}, & G_{r\bar{r}} &= 2\Omega^{-1}K_{r\bar{r}} \left(|K_{rs}|^2 + |K_{r\bar{s}}|^2\right), \\
G_{rs} &= 2\Omega^{-1}K_{r\bar{s}} \left(K_{r\bar{r}}K_{s\bar{s}} + |K_{rs}|^2\right), & G_{r\bar{s}} &= 2\Omega^{-1}K_{r\bar{s}} \left(K_{r\bar{r}}K_{s\bar{s}} + |K_{rs}|^2\right), \\
G_{ss} &= 4\Omega^{-1}K_{s\bar{s}}K_{rs}K_{r\bar{s}}, & G_{s\bar{s}} &= 2\Omega^{-1}K_{s\bar{s}} \left(|K_{rs}|^2 + |K_{r\bar{s}}|^2\right),
\end{align*}
$$

where $\Omega \equiv |K_{rs}|^2 - |K_{r\bar{s}}|^2$. After the coordinate transformation eq. (3.8), the components of the metric are given by

$$
\begin{align*}
G'_{w\bar{w}} &= 2\Omega^{-1}K_{s\bar{s}}, & G'_{w\bar{w}} &= 2\Omega^{-1} \left(K_{rs}K_{r\bar{s}} - K_{r\bar{r}}K_{s\bar{s}}\right), \\
G'_{z\bar{z}} &= G'_{w\bar{w}} \left(|G_{z\bar{w}}|^2 + 1\right) - (2\Omega K_{s\bar{s}})^{-1} \left(|K_{rs}|^2 + |K_{r\bar{s}}|^2 - 2K_{r\bar{r}}K_{s\bar{s}}\right)^2. \tag{5.4}
\end{align*}
$$

As a result we get that eq. (5.1) is indeed a necessary and sufficient condition on the potential so that it describes a hyper-Kähler potential. Note that, as expected, eq. (3.12) is satisfied with $f(r) = g(s) = 1$.

In [13], a powerful method was developed to construct solutions to eq. (5.2), the Legendre transformation construction. In the remainder of this section we will discuss various duality transformations which involve semi-chiral fields and we will
give a construction of solutions to eq. (5.1) analogous to the Legendre transform construction in [13].

In [12] (see also [9]) various duality transformations in N = (2, 2) superspace were catalogued. The simplest ones are those which do not need any isometries. They arise by passing to a first order action in superspace. I.e. the superfield constraints are imposed through Lagrange multipliers. The best known example is the duality between chiral and complex linear superfields (for a recent account see e.g. [17]). Both describe Kähler geometry and the former gives the minimal description while the latter is a non-minimal description. The two potentials are related through a simple Legendre transformation.

Similar duality transformations exist for semi-chiral superfields. One can perform a Legendre transformation either with respect to r or with respect to s or with respect to both of them. Given a potential \( K(r, \bar{r}, s, \bar{s}) \), we can construct three potentials

\[
K_1(r, \bar{r}, s, \bar{s}) = K(r', \bar{r}', s, \bar{s}) - r'r - \bar{r}'\bar{r},
\]

\[
K_2(r, \bar{r}, s, \bar{s}) = K(r, \bar{r}, s', \bar{s}') - s's - \bar{s}'\bar{s},
\]

\[
K_3(r, \bar{r}, s, \bar{s}) = K(r', \bar{r}', s', \bar{s}') - r'r - \bar{r}'\bar{r} - s's - \bar{s}'\bar{s},
\]

(5.5)

where in the first case \( r = K_{r'} \), in the second case \( s = K_{s'} \) and in the last case \( r = K_{r'} \) and \( s = K_{s'} \) hold. One verifies immediately that if \( K \) satisfies eq. (5.1) then so do \( K_1 \), \( K_2 \) and \( K_3 \). These three duality transformations simply shuffle around the auxiliary field content of the system and act as mere coordinate transformations on the physical fields.

In case isometries are present more interesting duality transformations become possible. The most typical semi-chiral example is the one which interchanges one semi-chiral multiplet for one chiral and one twisted chiral multiplet. The geometry obviously changes now. In the present case the semi-chiral coordinates describe a hyper-Kähler manifold, while the chiral/twisted chiral combination describes a manifold with a product structure which has e.g. a non-trivial torsion. At the chiral/twisted-chiral side the model shows a simple enhancement of the supersymmetry to \( N = (4, 4) \), provided the potential satisfies the Laplace equation. The dual potential turns out to describe a hyper-Kähler manifold. By this construction, one obtains immediately the semi-chiral parametrization of the hyper-Kähler manifold. The advantage of this construction is that the non-linear differential equation (5.1) gets replaced by a linear differential equation, the Laplace equation.

The starting point is a real prepotential \( F(x, u, \bar{u}) \), where \( x \in \mathbb{R} \) and \( u \in \mathbb{C} \), which satisfies the Laplace equation

\[
F_{xx} + F_{u\bar{u}} = 0.
\]

(5.6)
This combines two requirements: the chiral/twisted-chiral potential, which is precisely the prepotential under consideration, simultaneously exhibits an Abelian isometry and it has $N = (4, 4)$ supersymmetry. Full details can be found in the appendix. The present coordinates $x$ and $u$ are related to the chiral and twisted-chiral coordinates $z$ and $w$ by

$$x = -i(z - \bar{z} + w - \bar{w}), \quad u = z + \bar{w} \quad \text{and} \quad \bar{u} = u^*.$$  

The semi-chiral potential $K(r - \bar{r}, r + \bar{s}, \bar{r} + s)$ is obtained from the prepotential through a Legendre transformation,

$$K(r - \bar{r}, r + \bar{s}, \bar{r} + s) = F(x, u, \bar{u}) - \frac{u}{2}(r + \bar{r} + 2\bar{s}) - \frac{\bar{u}}{2}(r + \bar{r} + 2s) - \frac{ix}{2}(r - \bar{r}), \quad (5.7)$$

where

$$F_x = \frac{i}{2}(r - \bar{r}), \quad F_u = \frac{1}{2}(r + \bar{r} + 2\bar{s}), \quad F_{\bar{u}} = \frac{1}{2}(r + \bar{r} + 2s). \quad (5.8)$$

Using eqs. (5.6), (5.7) and (5.8), we get

$$K_{rs} = \frac{1}{2\Lambda}(F_{xu}^2 - |F_{xu}|^2 + iF_{xx}F_{xu} - F_{xx}^2 + iF_{uu}F_{x\bar{u}} - F_{xx}F_{uu}),$$

$$K_{r\bar{s}} = \frac{1}{2\Lambda}(F_{x\bar{u}}^2 - |F_{x\bar{u}}|^2 + iF_{xx}F_{x\bar{u}} - F_{xx}^2 + iF_{\bar{u}\bar{u}}F_{x\bar{u}} - F_{xx}F_{\bar{u}\bar{u}}),$$

$$K_{r\bar{r}} = \frac{1}{4\Lambda}((F_{xu} - F_{x\bar{u}})^2 - |F_{uu} + F_{xx}|^2),$$

$$K_{s\bar{s}} = -\frac{1}{\Lambda}(F_{xx}^2 + |F_{xx}|^2), \quad (5.9)$$

where

$$\Lambda = -F_{xx}^3 + F_{xx}|F_{uu}|^2 - 2F_{xx}|F_{xu}|^2 - F_{xx}^2F_{x\bar{u}} - F_{xx}^2F_{\bar{u}\bar{u}}. \quad (5.10)$$

Using this, one immediately checks that the resulting semi-chiral potential satisfies eq. (5.1). In other words, dualizing a chiral/twisted-chiral potential having an $N = (4, 4)$ supersymmetry and an Abelian isometry yields a semi-chiral potential which describes a hyper-Kähler manifold! The resulting potential obviously still has an Abelian isometry, $\delta r = \varepsilon$ and $\delta s = -\varepsilon$ with $\varepsilon \in \mathbb{R}$.

This construction is strikingly similar to the Legendre transform construction in [13], which follows from the duality between an $N = (4, 4)$ model described by a real linear and a chiral superfield and an $N = (4, 4)$ model described by two chiral superfields. Again this duality requires an Abelian isometry. There too, eq. (5.6) was the starting point for the construction of hyper-Kähler potentials directly in local
complex coordinates. From a real prepotential $\hat{F}(x, v, \bar{v})$, satisfying $\hat{F}_{xx} + \hat{F}_{v\bar{v}} = 0$, one obtains the Kähler potential, $K_{\text{Kähler}}$, through the Legendre transformation

$$K_{\text{Kähler}}(v, \bar{v}, z, \bar{z}) = \hat{F}(x, v, \bar{v}) - (z + \bar{z})x,$$  \hspace{1cm} (5.11)$$

where

$$\hat{F}_x = z + \bar{z}. \hspace{1cm} (5.12)$$

This allows for explicit expressions for the metric in terms of the prepotential,

$$G'_{\bar{z}z} = -\hat{F}_{xx}^{-1}, \quad G'_{z\bar{v}} = \hat{F}_{x\bar{v}}\hat{F}_{xx}^{-1}, \hspace{1cm} (5.13)$$

and $G'_{\bar{z}v} = G'_{z\bar{v}}^*$ and $G'_{v\bar{v}}$ follows from the Monge-Ampère equation. The extra fundamental two-forms are constant.

Comparing both constructions is possible if we pass from semi-chiral to complex coordinates. For simplicity, we use the coordinate transformation which diagonalizes $J_-$,

$$r \rightarrow \bar{w} = K_s, \quad s \rightarrow \bar{z} = s, \hspace{1cm} (5.14)$$

and from eq. (5.7), one gets the identification $w = -u$. The metric is then given by eq. (5.4) in which $r$ and $s$ are interchanged. Combining this with eq. (5.3) yields the expression for the metric in complex coordinates,

$$G'_{\bar{z}z} = i(F_{zu} - F_{z\bar{u}})^{-1}, \quad G'_{z\bar{v}} = i(F_{uu} + F_{xx})(F_{zu} - F_{z\bar{u}})^{-1}, \hspace{1cm} (5.15)$$

where once more we obtain $G'_{w\bar{w}}$ from the Monge-Ampère equation. Comparing eq. (5.13) to eq. (5.15) we get, after identifying $u = -iv$, a relation between $F(x, v, \bar{v})$ and $\hat{F}(x, v, \bar{v})$,

$$\hat{F}_x = F_v + F_{\bar{v}} + \alpha, \hspace{1cm} (5.16)$$

with $\alpha$ a real constant. Given either $F$ or $\hat{F}$, this allows the construction of $\hat{F}$ and $F$ resp. The requirement that the resulting prepotential satisfies the Laplace equation fully determines the prepotential $\hat{F}$, once $F$ is given. However, given $\hat{F}$, $F$ is only determined modulo a function $G(v - \bar{v}, x)$ which satisfies $G_v + G_{\bar{v}} = 0$ and $G_{xx} = G_{vv}$. This has no influence on both eqs. (5.16) and (5.6) but it might be needed in order to have a well-defined Legendre transformation in eq. (5.8) which is equivalent to requiring that $\Lambda$ in eq. (5.10) does not vanish.

We give a few examples where each time both the complex and the semi-chiral prepotential are given. Each time it is straightforward to check that both satisfy the Laplace equation and that they are related by eq. (5.16).
The four-dimensional special hyper-Kähler manifolds [3] are described by

\[
\hat{F} = -\left( v\bar{f}(\bar{v}) + \bar{v}f(v) \right) + \frac{x^2}{2} \left( \partial_v f(v) + \partial_{\bar{v}} \bar{f}(\bar{v}) \right),
\]

\[
F = x \left( f(v) + \bar{f}(\bar{v}) \right) + \frac{a}{2} x^2 + \frac{a}{2} (v - \bar{v})^2,
\tag{5.17}
\]

where \( f(v) \) is a holomorphic function of \( v \) and \( \bar{f}(\bar{v}) \) its complex conjugate and \( a \) is an arbitrary real constant.

In [13], two different representations of flat space were given

\[
\hat{F}_1 = v\bar{v} - \frac{1}{2} x^2
\]

\[
F_1 = -\frac{x}{2} (v + \bar{v}) + \frac{1}{2} x^2 + \frac{1}{2} (v - \bar{v})^2,
\tag{5.18}
\]

and

\[
\hat{F}_2 = r - x \ln(x + r) + \frac{1}{2} x \ln(4\bar{v}v),
\]

\[
F_2 = \frac{1}{8} (v - \bar{v} + x) \ln(v - \bar{v} - x) - \frac{1}{8} (v - \bar{v} - x) \ln(v - \bar{v} + x) + \frac{1}{4} (v + \bar{v}) \ln(4\bar{v}v) - \frac{1}{2} x \ln(v + \bar{v} + r) - \frac{1}{2} (v + \bar{v}) \ln(x + r) + \frac{1}{4} (v - \bar{v}) \ln \left( \frac{v 2(v - \bar{v})\bar{v} + x(x + r)}{\bar{v} 2(v - \bar{v})v - x(x + r)} \right),
\tag{5.19}
\]

where

\[
r \equiv \sqrt{x^2 + 4\bar{v}v}.
\tag{5.20}
\]

Other prepotentials for hyper-Kähler manifolds can now be constructed by superimposing \( F_1 \) and \( F_2 \) or equivalently \( \hat{F}_1 \) and \( \hat{F}_2 \). In this way one obtains the multi-Eguchi-Hanson manifolds [14, 15, 13] by superimposing \( F_2 \) or \( \hat{F}_2 \) about different points \( \vec{\rho}_A \),

\[
F_{EH} = \sum_{A=1}^{m+1} F_2(\vec{r} - \vec{\rho}_a),
\tag{5.21}
\]

or the Taub-NUT manifolds [13, 13] by adding an \( F_1 \) to this

\[
F_{TN} = F_1(\vec{r}) + \sum_{A=1}^{m+1} F_2(\vec{r} - \vec{\rho}_a),
\tag{5.22}
\]

and similar expressions where the complex prepotentials are replaced by semi-chiral prepotentials.
6. Conclusions

In [11], it was conjectured that chiral, twisted-chiral and semi-chiral superfields are sufficient to give a full off-shell, manifest supersymmetric description of \( N = (2, 2) \) supersymmetric non-linear \( \sigma \)-models in two dimensions. The chiral and twisted-chiral superfields do give a complete description of the directions along which the supersymmetry closes [9] while the semi-chiral superfields were expected to introduce the necessary auxiliary fields for those directions where no off-shell closure was achieved.

In the present paper we showed that this is not true. Non-closure of the \( N = (2, 2) \) supersymmetry implies the existence of a holomorphic two-form. Moving from semi-chiral coordinates to complex coordinates, one gets that this two-form is constant. Hyper-Kähler manifolds provide particularly interesting examples. Choosing left- and right-complex structures to be anti-commuting, we do get full non-closure of the algebra. The above mentioned two-form is the fundamental two-form associated with the “third” complex structure which is the product of the left with the right complex structure. A necessary condition for the semi-chiral parametrization to be possible is that this fundamental two-form is a constant, which is not the case for an arbitrary hyper-Kähler manifold!

As a result, the problem of finding a manifest supersymmetric description of \( N = (2, 2) \) non-linear \( \sigma \)-models is once more open. Chiral, twisted-chiral and semi-chiral superfields exhaust the superfields which can be defined by constraints linear in the superderivatives. What remains are constraints which are higher order in the derivatives. These have not been systematically studied, but past experience shows that most often they give non-minimal descriptions of known super-multiplets. Finally, there is a last possibility which would certainly work but which involves harmonic superspace. A drawback of this approach is that it is extremely hard to extract explicit expressions for metric and torsion.

Finally, we presented a systematic way to construct \( d = 4 \) hyper-Kähler manifolds starting from an intriguing duality transformation between an \( N = (4, 4) \) model described by one chiral and one twisted chiral superfield and a \( d = 4 \) hyper-Kähler manifold. In particular, this implies that well-known hyper-Kähler manifolds such as multi-Eguchi-Hanson and Taub-NUT have a dual which is a complex manifold with a product structure. The consequences of this duality transformation and the relation, if any, with the non-abelian T-duals of these models given in [18], certainly merit further study.
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Appendix: $N = (2,2)$ superspace

In this appendix we summarize some properties of $N = (2,2)$ superspace and superfields, together with some aspects of duality transformations. The fermionic coordinates which parametrize the $N = (2,2)$ superspace are denoted by $\theta^+$, $\theta^+ \equiv -(\theta^+)\dagger$, $\theta^-$ and $\theta^- \equiv -(\theta^-)\dagger$ and the bosonic coordinates by $x^+$ and $x^-$. The fermionic derivatives $D_+$, $D_+ \equiv (D_+)\dagger$, $D_-$ and $D_- \equiv (D_-)\dagger$ satisfy

$$\{D_+, D_+\} = 2i\partial_+, \quad \{D-, D_-\} = 2i\partial_-,$$

(A.1)

with all other (anti-)commutators between derivatives vanishing. The $N = (1,1)$ superderivatives are given by the real part of the $N = (2,2)$ fermionic derivatives,

$$\hat{D}_+ \equiv \frac{1}{2}(D_+ + D_+), \quad \hat{D}_- \equiv \frac{1}{2}(D_- + D_-),$$

(A.2)

while the extra supersymmetry generators are then proportional to the imaginary part of the $N = (2,2)$ superderivatives,

$$\hat{Q}_+ \equiv \frac{i}{2}(D_+ - D_+), \quad \hat{Q}_- \equiv \frac{i}{2}(D_- - D_-).$$

(A.3)

Consider a set of general $N = (2,2)$ superfields, $X^{\mu}$, $\mu \in \{1, \cdots, d\}$. The most general constraints linear in the derivatives are

$$\hat{Q}_+ X^{\mu} = J_{\mu \nu}^{+}(X)\hat{D}_+ X^{\nu}, \quad \hat{Q}_- X^{\mu} = J_{\mu \nu}^{-}(X)\hat{D}_- X^{\nu}.$$ 

(A.4)

A detailed analysis of the integrability conditions following from eq. (A.4) yields that both $J_+$ and $J_-$ are complex structures which mutually commute [11]. Through a suitable coordinate transformation they can be diagonalized resulting in two classes of superfields, chiral and twisted-chiral [3] superfields.

- Chiral superfields, $z$ and $\bar{z} \equiv z^\dagger$,
  $$D_+ z = D_- z = 0 \text{ or } Q_+ z = +i\hat{D}_+ z \text{ and } Q_- z = +i\hat{D}_- z.$$

- Twisted chiral superfields, $w$ and $\bar{w} \equiv w^\dagger$,
  $$D_+ w = D_- w = 0 \text{ or } Q_+ w = +i\hat{D}_+ w \text{ and } Q_- w = -i\hat{D}_- w.$$

On a chiral superfield $dz$, both $J_+$ and $J_-$ have eigenvalue $+i$. On a twisted chiral superfield $dw$, one finds that $J_+$ has eigenvalue $+i$ while $J_-$ has eigenvalue $-i$. Chiral
and twisted chiral fields have the same number of components as a general \( N = (1, 1) \) superfield, consistent with the fact that the algebra closes in the directions where the complex structures commute. A weaker set of constraints is still possible where only one chirality gets constrained. A detailed analysis shows that, in order to get non-trivial dynamics, they should occur in pairs, the members of which have constraints of opposite chirality. This results in semi-chiral superfields \[7\].

- Semi-chiral superfields, \( r, \bar{r} \equiv r^\dagger \), \( s \) and \( \bar{s} \equiv s^\dagger \),
  \begin{equation}
  D_\pm r = D_\pm s = 0 \text{ or } \hat{Q}_\pm r = +i\hat{D}_\pm r \text{ and } \hat{Q}_\pm s = -i\hat{D}_\pm s.
  \end{equation}

Semi-chiral superfields contain twice as many components as an \( N = (1, 1) \) superfield, however, half of them turn out to be auxiliary. On \( dr \), \( J_+ \) is diagonal with eigenvalue \(+i\), while on \( ds \), \( J_- \) is diagonal with eigenvalue \(+i\). The precise action of \( J_- \) on \( dr \) and \( J_+ \) on \( ds \) is model dependent and can only be obtained after elimination of the auxiliary fields.

Other constraints, linear in the derivatives, are still possible, but they imply restrictions on the dependence of the superfields on the bosonic coordinates (see e.g. \[19\]). We do not consider this here as it does not seem relevant to the present case.

Finally, we comment on duality transformations involving semi-chiral superfields. It is well known that in the presence of an abelian isometry a chiral field can be dualized to a twisted-chiral superfield and vice-versa. Similarly \[3, 11\], when a specific abelian isometry is present, a pair consisting of a chiral and a twisted-chiral superfield can be dualized to a semi-chiral field and vice-versa. Consider a model described by a chiral/twisted chiral potential \( F(z, \bar{z}, w, \bar{w}) \), which is such that an abelian isometry exists,

\begin{equation}
\delta z = \varepsilon, \quad \delta w = -\varepsilon,
\end{equation}

with \( \varepsilon \) a real constant. We introduce prepotentials \( V \) and \( W \) which are complex unconstrained superfields and one set of semi-chiral superfields \( r \) and \( s \). We consider the first order action

\begin{equation}
S = \int d^2xd^4\theta F(W - \bar{W}, V + \bar{V}, \bar{V} + W) \\
\quad \quad \quad - r(V + W) - \bar{r}(\bar{V} + \bar{W}) - s(\bar{V} + W) - \bar{s}(V + \bar{W}).
\end{equation}

If we first integrate over the semi-chiral fields, we recover the original model in terms of a chiral and a twisted-chiral field. Indeed, the equations of motion for \( r \) and \( s \),

\begin{equation}
D_+(V + W) = D_+(\bar{V} + W) = D_-(\bar{V} + W) = D_-(V + W),
\end{equation}

imply that

\begin{equation}
D_\pm D_-(V - \bar{V}) = D_\pm D_-(\bar{V} - V) = D_\pm D_-(W - \bar{W}) = D_\pm D_-(W - \bar{W}) = 0.
\end{equation}
which are solved by putting $V = z$ and $W = w$. The dual model is obtained by first integrating over the pre-potentials which yields

$$F_V = r + \tilde{s}, \quad F_W = r + s,$$  \hspace{1cm} (A.9)

which can be solved for the prepotentials $V(r, \bar{r}, s, \bar{s})$ and $W(r, \bar{r}, s, \bar{s})$. The semichiral potential, $K(r - \bar{r}, r + \bar{s}, \bar{r} + s)$, is simply the Legendre transform of $F$,

$$K(r - \bar{r}, r + \bar{s}, \bar{r} + s) = F(W - \bar{W}, V + \bar{W}, \bar{V} + W) - (W - \bar{W})(r - \bar{r}) - (V + \bar{W})(r + \bar{s}) - (\bar{V} + W)(\bar{r} + s).$$  \hspace{1cm} (A.10)

In [5] it was shown that a system with one chiral and one twisted-chiral superfield allows for a full $N = (4, 4)$ supersymmetry,

$$\begin{align*}
\delta z &= \eta^+ D_+ \bar{w} + \eta^- D_- w, \\
\delta w &= -\eta^+ D_+ \bar{z} - \eta^- D_- z, \\
\delta \bar{z} &= \eta^+ D_+ w + \eta^- D_- \bar{w}, \\
\delta \bar{w} &= -\eta^+ D_+ z - \eta^- D_- \bar{z},
\end{align*}$$  \hspace{1cm} (A.11)

provided the potential $F(z, \bar{z}, w, \bar{w})$, satisfies the Laplace equation,

$$F_{zz} + F_{\bar{w}w} = 0.$$  \hspace{1cm} (A.12)

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