Analytical solutions for N interacting electron system confined in graph of coupled electrostatic semiconductor and superconducting quantum dots in tight-binding model

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ABSTRACT

Analytical solutions for tight-binding model are presented for position based qubit and N interacting qubits realized by quasi one dimensional network of coupled quantum dots expressed by connected or disconnected graphs of any topology in 2 and 3 dimensions where 1 electron is presented at each separated graphs. Electron(s) quantum dynamical state is described under various electromagnetic circumstances with omission spin degree of freedom. The action of Hadamard and phase rotating gate is given by analytic formulas derived and formulated for any case of physical field evolution preserving the occupancy of two energetic level system. The interface between superconducting Josephson junction and semiconductor position based qubit implemented in coupled semiconductor q-dots is described what can be the base for electrostatic interface between superconducting and semiconductor quantum computer. Modification of Andreev Bound State in Josephson junction by the presence of semiconductor qubit in its proximity and electrostatic interaction with superconducting qubit is spotted by the minimalistic tight-binding model. The obtained results can be generalized for the case of interaction semiconductor qubit interacting electrostatically with Field Induced Josephson junction.

1. INTRODUCTION TO POSITION BASED SEMICONDUCTOR QUBIT

Single electron semiconductor devices has its importance in implementation of quantum computer as well as in implementation of low power consumption CMOS classical computer and was studied by Fujisawa, Petta, Leipold, Giounanlis, Pomorski and many others. On another hand one of the most successful model in condensed matter physics is Hubbard model and its special case known as tight-binding model. We consider two energy level system of position based qubit in tight-binding approach as depicted in Fig.1. Its extensions are given in Fig.4, where one can consider graphs of any shape representing the chain of coupled quantum dots in 2 and 3 dimensions of arbitrary topology. The Hamiltonian of this system is given as

\[ \hat{H}(t) = \begin{pmatrix} E_{p1}(t) & t_{s12}(t) \\ t_{s12}^*(t) & E_{p2}(t) \end{pmatrix} \]

The Hamiltonian \( \hat{H}(t) \) eigenenergies \( E_1(t) \) and \( E_2(t) \) with \( E_2(t) > E_1(t) \) are given as

\[ E_1(t) = \frac{-\sqrt{(E_{p1}(t) - E_{p2}(t))^2 + (t_{s12}(t))^2}}{2} + \frac{E_{p1}(t) + E_{p2}(t)}{2}, \]

\[ E_2(t) = \frac{\sqrt{(E_{p1}(t) - E_{p2}(t))^2 + (t_{s12}(t))^2}}{2} + \frac{E_{p1}(t) + E_{p2}(t)}{2}. \]
and energy eigenstates \(|E_1(t)\rangle\) and \(|E_2(t)\rangle\) have the following form

\[
|E_1(t)\rangle = \left(\frac{(E_{p2}(t) - E_{p1}(t)) + \sqrt{\frac{(E_{p2}(t) - E_{p1}(t))^2}{2} + |t_{s12}(t)|^2}}{-t_{s21}(t) + t_{s12}(t)}\right),
\]

\[
|E_2(t)\rangle = \left(\frac{(E_{p2}(t) - E_{p1}(t)) + \sqrt{\frac{(E_{p2}(t) - E_{p1}(t))^2}{2} + |t_{s12}(t)|^2}}{t_{s21}(t) - t_{s12}(t)}\right).
\]

This Hamiltonian is giving description of 2 coupled quantum wells as depicted in Fig.1. In such situation we have real valued functions \(E_{p1}(t), E_{p2}(t)\) and complex valued functions \(t_{s12}(t) = t_{s21}(t) + it_{s12}(t)\) and \(t_{s21}(t) = t_{s12}(t)\) what is equivalent to knowledge of 4 real valued time-dependent continuous or discontinuous functions \(E_{p1}(t), E_{p2}(t)\), \(t_{s21}(t)\) and \(t_{s12}(t)\). The quantum state is superposition of state localized at node 1 and 2 and therefore is given as

\[
|\psi\rangle_{[x]} = \alpha(t)|0, 1\rangle_x + \beta(t)|0, 0\rangle_x = \alpha(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \(|\alpha(t)|^2\) (|\beta(t)|^2) is probability of finding particle at node 1(2) respectively what brings \(|\alpha(t)|^2 + |\beta(t)|^2 = 1\) and obviously \(\langle 0, 1|_x, |1, 0\rangle_x = 1 = \langle 0, 1|_x, |0, 1\rangle_x\) and \(\langle 1, 0|_x, |0, 1\rangle_x = 0 = \langle 0, 1|_x, |1, 0\rangle_x\). In Schrödinger formalisms states \(|1, 0\rangle_x\) and \(|0, 1\rangle_x\) are Wannier functions that are parametrized by position x. We work in tight-binding approximation and quantum state evolution with time is given as

\[
\frac{i\hbar}{d} \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle = E(t) |\psi(t)\rangle.
\]

Last equation has the analytic solution

\[
|\psi(t)\rangle = \exp\left[\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1) dt_1\right] |\psi(t_0)\rangle = \exp\left[\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1) dt_1\right] \begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix}.
\]
and in quantum density matrix theory we obtain

\[
\hat{\rho}(t) = |\psi(t)\rangle\langle \psi(t)| = \hat{U}(t,t_0)\rho(t_0)\hat{U}(t, t_0)^{-1} = \exp\left[\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1)dt_1\right] |\psi(t_0)\rangle \langle \psi(t_0)| \exp\left[-\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1)dt_1\right] = \exp\left[\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1)dt_1\right] (|\alpha(0)\rangle \langle \alpha(0)| + |\beta(0)\rangle \langle \beta(0)|) \exp\left[-\frac{1}{i\hbar} \int_{t_0}^{t} \hat{H}(t_1)dt_1\right] = \hat{U}(t,t_0) \left( |\alpha(0)\rangle \langle \alpha(0)| + |\beta(0)\rangle \langle \beta(0)| \right) \hat{U}(t_0, t_0).
\]

Having Hermitian matrix \( \hat{A} \) with real valued coefficients \( a_{11}(t), a_{22}(t), a_{12r}(t), a_{12l}(t) \) we observe that

\[
\hat{A}_{2\times2} = \begin{pmatrix} a_{11} & a_{12r} + ia_{12l} \\ a_{12l} - ia_{12r} & a_{22} \end{pmatrix}, \quad \exp\left(\frac{1}{i\hbar} \hat{A}_{2\times2}\right) = \begin{pmatrix} e^{\frac{i}{\hbar} a_{11}} & e^{\frac{i}{\hbar} (a_{12r}+ia_{12l})} \\ e^{\frac{i}{\hbar} (a_{12l}-ia_{12r})} & e^{\frac{i}{\hbar} a_{22}} \end{pmatrix}
\]

and for \( \hat{A}_{N\times N} \) we obtain

\[
\exp\left(\frac{1}{i\hbar} \hat{A}_{N\times N}\right) = \begin{pmatrix} e^{\frac{i}{\hbar} a_{11}} & e^{\frac{i}{\hbar} (a_{12r}+ia_{12l})} & \cdots & e^{\frac{i}{\hbar} (a_{1N}+ia_{11})} \\ e^{\frac{i}{\hbar} (a_{12r}-ia_{12l})} & e^{\frac{i}{\hbar} a_{22}} & \cdots & e^{\frac{i}{\hbar} (a_{2N}+ia_{21})} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{i}{\hbar} (a_{1N}-ia_{11})} & e^{\frac{i}{\hbar} (a_{2N}-ia_{21})} & \cdots & e^{\frac{i}{\hbar} a_{NN}} \end{pmatrix}
\]

Using the above property for matrix of size 2 by 2 we obtain

\[
e^{\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t_1)dt_1} = \hat{U}(t,t_0) = \begin{pmatrix} e^{\frac{i}{\hbar} (f_{01}^t E_{t_1}(t_1)dt_1)} & e^{\frac{i}{\hbar} (f_{02}^t E_{t_2}(t_1)dt_1)} \\ e^{\frac{i}{\hbar} (f_{11}^t E_{t_1}(t_1)dt_1-i f_{12}^t E_{t_2}(t_1)dt_1)} & e^{\frac{i}{\hbar} (f_{22}^t E_{t_2}(t_1)dt_1)} \end{pmatrix}
\]

Final quantum state at time \( t \) evolving from quantum state given at time \( t_0 \) can be written as

\[
|\psi(t)\rangle = \hat{U}(t,t_0) |\psi(t_0)\rangle = \begin{pmatrix} e^{\frac{i}{\hbar} (f_{01}^t E_{t_1}(t_1)dt_1)} \alpha(0) + e^{\frac{i}{\hbar} (f_{02}^t E_{t_2}(t_1)dt_1)} \beta(0) \\ e^{\frac{i}{\hbar} (f_{11}^t E_{t_1}(t_1)dt_1-i f_{12}^t E_{t_2}(t_1)dt_1)} \alpha(0) + e^{\frac{i}{\hbar} (f_{22}^t E_{t_2}(t_1)dt_1)} \beta(0) \end{pmatrix} = \begin{pmatrix} E_1(t) \alpha(0) + E_2(t) \beta(0) \\ E_1(t) \alpha(0) + E_2(t) \beta(0) \end{pmatrix} = \begin{pmatrix} E_1(t) \alpha(0) + E_2(t) \beta(0) \\ E_1(t) \alpha(0) + E_2(t) \beta(0) \end{pmatrix}
\]

where \( |c_{E_1}(t)|^2 \) is probability of occupancy of energetic level \( E_1 \) and \( |c_{E_2}(t)|^2 \) is probability of occupancy of energetic level \( E_2 \) by quantum state \( |\psi(t)\rangle \) and \( |E_1(t)\rangle \langle E_1(t)\rangle + |E_2(t)\rangle \langle E_2(t)\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). We notice that the quantum state norm \( N(t) \in \mathbb{R} \) is changing with time in accordance to formula

\[
N(t) = |\langle \psi(t)| \psi(t)\rangle| = 2(|\alpha(0)|^2 + |\beta(0)|^2) + (\alpha(0)* \beta(0) (e^{\frac{i}{\hbar} (f_{01}^t (E_{t_1}(t_1)-t_1^t(t_1))dt_1)} + e^{\frac{i}{\hbar} (f_{22}^t (E_{t_2}(t_1)-T))dt_1}) + h.c.) = 2 + \alpha(0)* \beta(0) (e^{\frac{i}{\hbar} (f_{01}^t (E_{t_1}(t_1)-t_1^t(t_1))dt_1)} + e^{\frac{i}{\hbar} (f_{22}^t (E_{t_2}(t_1)-T))dt_1}) + \alpha(0)* \beta(0) (e^{\frac{i}{\hbar} (f_{11}^t (E_{t_1}(t_1)+t_1^t(t_1))dt_1)} + e^{\frac{i}{\hbar} (f_{22}^t (E_{t_2}(t_1)+T))dt_1}),
\]

(11)
what brings the probability of finding the quantum state at node 1 as

\[
P_1(t) = \frac{\left| e^{\frac{i}{\hbar} \int_0^t E_p \,(t_1) \,dt_1} \alpha(0) + e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1 + i \int_0^t \,t_{sr} \,(t_1) \,dt_1} \right|^2 \beta(0)}{N(t)}.\]  

(12)

and brings the probability of finding the quantum state at node 2 as

\[
P_2(t) = \frac{\left| e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1} \alpha(0) + e^{\frac{i}{\hbar} \int_0^t E_{psd} \,(t_1) \,dt_1} \beta(0) \right|^2}{N(t)}.\]  

(13)

Phase of wavefunction at node 1 is

\[
Ph(1,t) = Ph\left( \frac{e^{\frac{i}{\hbar} \int_0^t E_{p1} \,(t_1) \,dt_1} \alpha(0) + e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1 + i \int_0^t \,t_{sr} \,(t_1) \,dt_1} \beta(0)}{\sqrt{N(t)}} \right) =
\]

\[
= Ph(\psi(1,t)) = Ph(\hat{\psi}(1,t)|e^{t \xi_1(t)}) = \xi_1(t)\]  

(14)

Phase of wavefunction at node 2 is

\[
Ph(2,t) = Ph\left( \frac{e^{\frac{i}{\hbar} \int_0^t E_{p2} \,(t_1) \,dt_1} \alpha(0) + e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1} \beta(0)}{\sqrt{N(t)}} \right) =
\]

\[
= Ph(\psi(2,t)) = Ph(\hat{\psi}(2,t)|e^{t \xi_2(t)}) = \xi_2(t).\]  

(15)

Phase rotating gate changes \( Ph(\psi(2,t)) - Ph(\psi(1,t)) = \xi(t) \) into value \( Ph(\psi(2,t_2)) - Ph(\psi(1,t_2)) = \xi(t_2) \) at some time \( t_2 > t \), while \( \frac{\psi(1,t)}{\psi(2,t)} = \frac{\psi(1,t)}{\psi(2,t)} \) or when \( \frac{\psi(1,t_2)}{\psi(2,t)} = \frac{\psi(1,t_2)}{\psi(2,t)} \) via given class of functions \( E_{p1}(t'), E_{psr}(t'), t_{sr}(t') \) of certain shape in time \( t' \in (t,t_2) \). Hadamard gate operation can be defined as operation changing quantum state at time \( t \) \( \psi(1,t) = \alpha(0,t) = 1, \beta(0,t) = 0 = \psi(2,t) \) into quantum state at time \( t' \) \( t' > t \) so \( \psi(1,t') = \frac{1}{\sqrt{3}}, \beta(0,t') = \psi(2,t') = \sqrt{\frac{2}{3}} \) and operation changing state \( \psi(1,t) = \alpha(0,t) = 0, \beta(0,t) = 1 = \psi(2,t) \) into quantum state at time \( t_3 \) \( t_3 > t \) so \( \psi(1,t_3) \approx \frac{1}{\sqrt{2}}, \beta(0,t_3) = \psi(2,t_3) = \approx - \frac{1}{\sqrt{2}} \) via given class of functions \( E_{p1}(t''), E_{psr}(t''), t_{sr}(t'') \) of certain shape in time \( t'' \in (t,t_3) \). Having nonorthodox approach we might find class of functions approximately (not exactly) satisfying operation of phase rotating gate or Hadamard gate for the case of 2 energy level system. Obviously going into situation more than 2 energy level system one can find richer class of possible scenarios by considerations of broader class of position based qubit controlling electromagnetic signals. It is thus valid subject of study in pursue for better implementation of quantum technologies. Once solution is found in tight-binding model it can be relatively easily transferred into Schrödinger formalism what brings conclusion on quantum circuit controlling procedures. Before moving to the particular physical examples let us examine the properties of evolution of density matrix with time as

\[
\dot{\rho}(t) = \frac{|\psi(t)\rangle \langle \psi(t)|}{\langle \psi(t)| \langle \psi(t)|} = \hat{U}(t,t_0) (|\psi(t_0)\rangle \langle \psi(t_0)|) \hat{U}(t,t_0)^{-1} = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix} =
\]

\[
\begin{pmatrix}
\frac{|\alpha(0)|^2}{\beta(0)} & \frac{\alpha(0) \beta^{*}(0)}{\beta(0)} \\
\frac{\bar{\alpha}(0) \beta^{*}(0)}{\beta(0)} & \frac{\bar{\beta}(0)^2}{\beta(0)}
\end{pmatrix}
\begin{pmatrix}
\frac{e^{\frac{i}{\hbar} \int_0^t E_{p1} \,(t_1) \,dt_1}}{\sqrt{N(t)}} & \frac{e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1 + i \int_0^t \,t_{sr} \,(t_1) \,dt_1}}{\sqrt{N(t)}} \\
\frac{-\frac{\bar{\beta}(0)^2}{\beta(0)}}{\sqrt{N(t)}} & \frac{\bar{\alpha}(0) \beta^{*}(0)}{\beta(0)}
\end{pmatrix}
\begin{pmatrix}
\frac{e^{\frac{i}{\hbar} \int_0^t E_{p1} \,(t_1) \,dt_1 + i \int_0^t \,t_{sr} \,(t_1) \,dt_1}}{\sqrt{N(t)}} & \frac{e^{\frac{i}{\hbar} \int_0^t E_{psr} \,(t_1) \,dt_1}}{\sqrt{N(t)}} \\
\frac{-\frac{\bar{\beta}(0)^2}{\beta(0)}}{\sqrt{N(t)}} & \frac{\bar{\alpha}(0) \beta^{*}(0)}{\beta(0)}
\end{pmatrix}.
\]

Therefore one can obtain all density matrix components of position based qubit analytically

\[
\rho_{00}(t) = |\alpha(0)|^2 + |\beta(0)|^2 e^{\frac{i}{\hbar} \int_0^t \,t_{sr} \,(t_1) \,dt_1} + 2|\alpha(0)||\beta(0)| e^{\frac{i}{\hbar} \int_0^t \,t_{sr} \,(t_1) \,dt_1} \cos(t \int_0^t (E_{p1} \,(t_1) + t_{sr} \,(t_1) \,dt_1 + \hbar Ph(\alpha(0)) - Ph(\beta(0))),
\]

\[
\rho_{11}(t) = |\beta(0)|^2 + |\alpha(0)|^2 e^{-\frac{i}{\hbar} \int_0^t \,t_{sr} \,(t_1) \,dt_1} + 2|\alpha(0)||\beta(0)| e^{-\frac{i}{\hbar} \int_0^t \,t_{sr} \,(t_1) \,dt_1} \cos(t \int_0^t (E_{p2} \,(t_1) - t_{sr} \,(t_1) \,dt_1 + \hbar Ph(\alpha(0)) - Ph(\beta(0))),
\]
The normalization of density matrix is achieve by setting \( \frac{\rho(t)}{\rho_{\text{total}}(t) + \rho_{\text{extra}}(t)} = \hat{\rho}(t) \) what guarantess that quantum state is normalized with time. We notice that \( Ph(\alpha(0)) - Ph(\beta(0)) = Ln(\frac{\alpha(0)}{\beta(0)}) \frac{1}{2} \) and non-diagonal parts of density matrix are given by formula

\[
\rho_{10}(t) = \rho_{01}(t) = |\alpha(0)|^2 \beta(0)| \cos(Ph(\alpha(0)) - Ph(\beta(0))) + i \sin(Ph(\alpha(0)) - Ph(\beta(0))) e^{iH \int_0^t dt_1 t_{sr}(t_1)} + \\
|\beta(0)|^2 e^{iH \int_0^t t_{sr}(t_1) dt_1} \left( \cos \left( \frac{1}{\hbar} \int_0^t E_p(t_1) dt_1 \right) + \sin \left( \frac{1}{\hbar} \int_0^t E_p(t_1) dt_1 \right) + \frac{1}{\hbar} \int_0^t t_{sr}(t_1) dt_1 \right) + \\
|\alpha(0)|^2 e^{-iH \int_0^t t_{sr}(t_1) dt_1} \left( \cos \left( \frac{1}{\hbar} \int_0^t E_p(t_1) dt_1 \right) + \sin \left( \frac{1}{\hbar} \int_0^t E_p(t_1) dt_1 \right) + \frac{1}{\hbar} \int_0^t t_{sr}(t_1) dt_1 \right) + \\
|\alpha(0)|^2 |\beta(0)| \left( \cos \left( \frac{1}{\hbar} \hbar Ph(\alpha(0)) - \hbar Ph(\beta(0)) \right) + \frac{1}{\hbar} \int_0^t (E_p(t_1) - E_p(t_1)) dt_1 \right) \right). \tag{16}
\]

2. ACTION OF PHASE ROTATING GATE DESCRIBED ANALYTICALLY

Let us consider the situation of single qubit from Fig[1] when we assume the following dependencies: \( E_p(t) = E_{p2}(t) = E_p = \text{constant} \) and \( t_{sr2}(t) = t_{sr1}(t) = t_s(t) = \text{constant} \). In such we have two time-independent eigenenergies \( E_1 = E_p - t_s \) and \( E_2 + t_s \). For simplicity we assume \( (\alpha(0) \in R), (\beta(0) \in R) \). The probability of finding electron at node 1 is given by angle \( \Theta \) at Bloch sphere expressed as

\[
P_1(t) = |\alpha(t)|^2 = \frac{1}{2} (|\alpha(0)|^2 + |\beta(0)|^2) + \frac{1}{2} (|\alpha(0)|^2 - |\beta(0)|^2) \cos \left( \frac{E_2 - E_1}{\hbar} t \right) = \cos(\Theta(t))^2,
\]

\[
P_2(t) = |\beta(t)|^2 = \frac{1}{2} (|\alpha(0)|^2 + |\beta(0)|^2) - \frac{1}{2} (|\alpha(0)|^2 - |\beta(0)|^2) \cos \left( \frac{E_2 - E_1}{\hbar} t \right) \sin(\Theta(t))^2, \tag{17}
\]

and it oscillates periodically with frequency proportional to distance between energetic levels \( E_2 \) and \( E_1 \) and is given as \( \omega_n = \frac{E_2 - E_1}{\hbar} \). Therefore the same occupancy at node is repeating with periodic time \( t_d = \frac{2\pi}{\omega_n} \) for integer \( n \). Obviously probability of finding of particle at node 2 is \( P_2 = 1 - P_1 \). The phase difference between wavefunctions at node 1 and 2 is denoted as \( \phi(t) \) and can be expressed analytically by formula

\[
\phi(t) = -\text{Arcsin} \left[ \frac{\sin(\frac{E_1 t}{\hbar}) (|\alpha(0)|^2 - |\beta(0)|^2) + \sin(\frac{E_2 t}{\hbar}) (|\alpha(0)|^2 + |\beta(0)|^2)}{\cos(\frac{E_1 t}{\hbar}) (|\alpha(0)|^2 + |\beta(0)|^2) + \cos(\frac{E_2 t}{\hbar}) (|\alpha(0)|^2 + |\beta(0)|^2)} \right] = \\
= -\text{Arcsin} \left[ \frac{1}{2} \left( \frac{1}{2} (1 - \exp(-i \frac{2\pi E_1}{\hbar})) (|\alpha(0)|^2 - |\beta(0)|^2) + \frac{1}{2} (1 + \exp(-i \frac{2\pi E_1}{\hbar})) (|\alpha(0)|^2 + |\beta(0)|^2) \right) = \\
= -\text{Arcsin} \left[ \frac{1}{2} \left( \frac{1}{2} (1 + \exp(-i \frac{2\pi E_2}{\hbar})) (|\alpha(0)|^2 - |\beta(0)|^2) + \frac{1}{2} (1 - \exp(-i \frac{2\pi E_2}{\hbar})) (|\alpha(0)|^2 + |\beta(0)|^2) \right) \right]
\]

\[
\left[ \frac{1}{2} \left( 1 - e^{-i \frac{2\pi E_1}{\hbar}} \right) (|\alpha(0)|^2 - |\beta(0)|^2) + \frac{1}{2} \left( \cos(\theta(t))^2 - \frac{1}{2} |\alpha(0)|^2 - |\beta(0)|^2 |^2 \right) \right] = \\
\left[ \frac{1}{2} \left( 1 + e^{-i \frac{2\pi E_2}{\hbar}} \right) (|\alpha(0)|^2 - |\beta(0)|^2) + \frac{1}{2} \left( \cos(\theta(t))^2 - \frac{1}{2} |\alpha(0)|^2 - |\beta(0)|^2 |^2 \right) \right]
\]

\[
\text{We recognize that three frequencies are involved } \omega_1 = \frac{E_2 - E_1}{\hbar}, \omega_{21m} = \frac{E_2 - E_b}{\hbar}, \omega_{21p} = \frac{E_2 + E_b}{\hbar} \text{ in the dynamics of phase difference of quantum state between nodes 2 and 1. What is more phase difference across position based qubit between nodes 1 and 2 is codependent on the occupancy of the left and right node as given by equation [8] in the case of time-independent Hamiltonian. Such situation is not taking place in most conventional qubits.}
\]
using energy eigenbases to encode information but takes place in position based semiconductor qubit. The ideal phase rotating gate implemented in position based qubit brings desired phase difference between wavefunctions at nodes 2 and 1 is not changing the occupancy of node 1 and 2. If we want to keep the occupancy from time t=0 we need to consider times \( t_g = \frac{2\pi n}{E_2 - E_1} \). At time t=0 phase difference was assumed to be 0. If we want to achieve phase difference \( \phi_d \) the following condition must be meet

\[
|\sin(\phi_d) + \frac{\sin\left(\frac{E_1|\alpha(0)|^2 - |\beta(0)|^2}{E_2 - E_1}\right)}{\cos\left(\frac{E_1|\alpha(0)|^2 - |\beta(0)|^2}{E_2 - E_1}\right)}| = 0.
\]

so positive integer n can be determined in unique way. It simply means that phase imprinted quantum state will have these feature at time \( t = n_s \frac{2\pi}{E_2 - E_1} \). In such way action of phase rotating gate can be implemented in electrostatic position dependent qubit given in Fig.1.

### 3. ACTION OF HADAMARD GATE IN POSITION QUBIT

The Hadamard gate is able to conduct the following unitary transformation on quantum state \( \psi(t) \) and is given as

\[
U_{\text{Hadamard}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

It has property \( U_{\text{Hadamard}}^\dagger U_{\text{Hadamard}} = U_{\text{Hadamard}}U_{\text{Hadamard}}^\dagger = 1 \) so double action of Hadamard gate gives \( U_{\text{Hadamard}}^2 = 1 \).

Let us concentrate on the position dependent qubit with time-independent parameters \( E_{p1}, E_{p2} = E_p = E_p, t_s \in R \). In such case we obtain following eigenenergies \( E_1 = E_p - t_s \) and \( E_2 = E_p + t_s \). From simple calculations we can notice that two eigenergies \( E_1 = E_p - t_s \) and \( E_2 = E_p + t_s \) have corresponding eigenstates

\[
|E_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle_x - |0\rangle_x),
\]

\[
|E_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle_x + |0\rangle_x),
\]

that are orthonormal so \( \langle 0 | 1, 0 \rangle = \langle 0 | 0 | 1 \rangle = 1 \) and \( \langle 1 | 0 | 1 \rangle = \langle 0 | 1 | 0 \rangle = 0 \). At the same time \( \langle E_1 | E_1 \rangle = \langle E_2 | E_2 \rangle = 1 \) and \( \langle E_1 | E_2 \rangle = \langle E_2 | E_1 \rangle = 0 \). We recognize that formula (21) can be written in the compact form as

\[
\begin{pmatrix} |E_2\rangle \\ |E_1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |1\rangle_x \\ |0\rangle_x \end{pmatrix} = U_{\text{Hadamard}} \begin{pmatrix} |1\rangle_x \\ |0\rangle_x \end{pmatrix}
\]

or

\[
\begin{pmatrix} |1\rangle_x \\ |0\rangle_x \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |E_2\rangle \\ |E_1\rangle \end{pmatrix} = U_{\text{Hadamard}}^\dagger \begin{pmatrix} |E_2\rangle \\ |E_1\rangle \end{pmatrix}.
\]

We recognize that quantum transformation is naturally encoded in transformation from position quantum system eigenbases into energy eigenbases. Quantum logical 0 can be spanned (represented) by state \( |1\rangle_x = |0\rangle_L \) (presence of electron in qubit on the left side in Fig.1) and quantum logical 1 can be spanned (represented) by the state \( |0\rangle_x = |1\rangle_R \) (presence of electron in qubit on the right side). Therefore qubit state shall be defined by

\[
|\psi(t)\rangle = \alpha(t) |1\rangle_x + \beta |0\rangle_x = e^{i\Phi(\alpha(t))=i\xi(t)}|\alpha(t)| |1\rangle_x + e^{i\Phi(\beta(t))=i\phi(t)}|\beta(t)| |0\rangle_x = e^{i\xi(t)}|\alpha(t)| |1\rangle_x + e^{i\phi(t)}|\beta(t)| |0\rangle_x.
\]

Action of Hadamard gate requires

\[
|0\rangle_L = |0\rangle_x \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_x + |0\rangle_x) = \frac{1}{\sqrt{2}}(|1\rangle_L + |2\rangle_L),
\]

\[
|0\rangle_R = |0\rangle_x \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_x - |0\rangle_x) = \frac{1}{\sqrt{2}}(|1\rangle_L - |2\rangle_L).
\]
We recognize that quantum logical 0 or presence of state (electron) in left well is achieved when there is equal occupancy (given by \( c_{E1} \)) of energetic level \( E_1 \) and \( E_2 \) so \( |c_{E1}(t)|^2 = |c_{E2}(t)|^2 \). The scheme how to change the complete occupancy of energetic level \( E_1 \) into full occupancy of energetic level \( E_2 \) is given by formula (27) that is associated with time-dependent Hamiltonian applied to position based qubit. The quantum state is given as

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left[ (c_{E1}(t)(|1,0\rangle_x - |0,1\rangle_x)) + (c_{E2}(t)(|1,0\rangle_x + |0,1\rangle_x)) \right] = \\
\frac{1}{\sqrt{2}} \left[ e^{\frac{i}{\hbar} (t-t_0) E_1} c_{E1}(t_0)(|1,0\rangle_x - |0,1\rangle_x)) + (e^{\frac{i}{\hbar} (t-t_0) E_2(t-t_0)} c_{E2}(t)(|1,0\rangle_x + |0,1\rangle_x)) \right] = \\
\frac{1}{\sqrt{2}} \left[ \left( + e^{\frac{i}{\hbar} (t-t_0) E_1} c_{E1}(t_0) \right) (|1,0\rangle_x + |0,1\rangle_x) + \left( - e^{\frac{i}{\hbar} (t-t_0) E_1} c_{E1}(t_0) \right) (|1,0\rangle_x - |0,1\rangle_x) \right].
\]

Such state will evolve after characteristic time from logical state \( |0\rangle_L \) into quantum logical \( |1\rangle_L \) and later into \( |0\rangle_L \) and so on. We can also set logical quantum state in position space parametrized by x and we can read the results of Hadamard operation action in energy space or reversely. Engineer has the choice of setting qubit state in position space (what is more intuitive if one aims to obtain high integration circuits) or in energy space. By setting the quantum state in position space (as by injecting electron from left side into left well of qubit) one needs to read it by energy space or reversely. Reading quantum state after Hadamard operation (or any other quantum operation) in energy space requires either spectroscopy of occupation of energy levels what basically means that we need to use microwaves in order to populate or depopulate given energy level(s). Alternative method for reading the qubit state after Hadamard operation (or any other quantum operation) is determination the state of neighbouring qubit that interacts with measured qubit in electrostatic way as it is depicted in the right side of Fig[1]. Determination of occupancy of energy level \( E_1 \) and \( E_2 \) will give us the information on the qubit state after Hadamard operation (so presence of at least 2 energy levels in physical system is the requirement) and formally we have

\[ |\psi\rangle_{output} = c_{E1} |E_1\rangle + c_{E2} |E_2\rangle = c_{E1} |0\rangle_{L-output} + c_{E2} |1\rangle_{L-output} = U_{\text{Hadamard}}(\alpha |0\rangle_{L-input} + \beta |1\rangle_{L-input}). \] (26)

In order to trace the situation in physical system we start from consideration of time-independent Hamiltonian and quantum state dynamics given by

\[
|\psi(t)\rangle = |\psi(t)\rangle_E = (c_{E1,t} |E_1\rangle + c_{E2,t} |E_2\rangle) = (e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_1\rangle} + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} |E_2\rangle} = \\
= e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_1\rangle} + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} |E_2\rangle} = e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0}} c_{E2,t_0} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0}} (E_{1-t_0} |E_{2-t_0}\rangle) = \\
= \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} (|1,0\rangle_x - |0,1\rangle_x) + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} (|1,0\rangle_x + |0,1\rangle_x)} = \\
= \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_{1-t_0}\rangle} c_{E2,t_0} (|1,0\rangle_x + |0,1\rangle_x) + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} (|1,0\rangle_x - |0,1\rangle_x)} = \\
= \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_{1-t_0}\rangle} + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} |E_{2-t_0}\rangle} (|1,0\rangle_x + |0,1\rangle_x) + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} (|1,0\rangle_x - |0,1\rangle_x)} = \\
= \frac{1}{\sqrt{2}} (\alpha |1,0\rangle_x + \beta |0,1\rangle_x) = \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_{1-t_0}\rangle} + e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} |E_{2-t_0}\rangle} |1,0\rangle_x = \frac{1}{\sqrt{2}} (\alpha |1,0\rangle_x + \beta |0,1\rangle_x).
\]

(27)

where \( |c_{E1,t}|^2 = \text{constans} = |c_{E1,t_0}|^2 \) (since \( c_{E1,t} = e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0}} \) is the probability of occupancy of first energetic level \( E_1 \) and \( |c_{E2,t}|^2 = \text{constans} = |c_{E2,t_0}|^2 \) (since \( c_{E2,t} = e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0}} \) is the probability of occupancy of second energetic level \( E_2 \) so \( |c_{E1,t}|^2 + |c_{E2,t}|^2 = 1 \). We have obtained oscillations of occupancy at node 1 and node 2 given as

\[
\alpha(t) = \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E1,t_0} |E_{1-t_0}\rangle} \text{ and } \beta(t) = \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar} H_{t-t_0} c_{E2,t_0} |E_{2-t_0}\rangle}.
\]

(28)
and those results are the most simple case of solutions for the case of 2 energy level system and can be compared with more general formula 16. We can spot that phase different between coefficients \( \alpha(t) \) and \( \beta(t) \) as well as phase difference \( c_{E1}(t) \) and \( c_{E2}(t) \) has sinusoidal periodicity with frequency of oscillation proportional to difference between eigenvalues \( \omega_0 = \frac{E_2 - E_1}{\hbar} \). Normalization condition for coefficients \( \alpha_t \) and \( \beta_t \) given by \( |\alpha_t|^2 + |\beta_t|^2 = 1 \) automatically implies normalization condition for \( c_{E1,t} \) and \( c_{E2,t} \) given by \( |c_{E1,t}|^2 + |c_{E2,t}|^2 = 1 \) and reversely. Such reasoning can be easily extended for more than 2 energetic levels as given by Hamiltonian 4 by 4 describing position dependent qubit. We refer to the situation from Fig[4] with presence of microwave field that comes from external antenna or from AC voltage component applied to 3 gates controlling the state of position based qubit. In case of lack of time-dependent fields the occupancy of energetic levels E1 and E2 are unchanged with time. The Hamiltonian able to

\[
H = E_1(t) |E_1(t)\rangle \langle E_1(t)| + E_2(t) |E_2(t)\rangle \langle E_2(t)| + f_1(t) |E_1(t)\rangle \langle E_2(t)| + f_2(t) |E_2(t)\rangle \langle E_1(t)|
\]

(29)

where \( f_1(t) \) and \( f_2(t) \) are time-dependent functions.

Evaluating expressions \( f_1(t) |E_2\rangle \langle E_1| \) and

\[
f_1(t) |E_2\rangle \langle E_1| = f_1(t) \frac{1}{2} (|1,0\rangle_x + |0,1\rangle_x)(|1,0\rangle_x - |0,1\rangle_x) = f_1(t) \frac{1}{2} (|1,0\rangle_x^2 - |0,1\rangle_x^2 - |1,0\rangle_x |0,1\rangle_x + |0,1\rangle_x |1,0\rangle_x),
\]

\[
f_2(t) |E_1\rangle \langle E_2| = f_2(t) \frac{1}{2} (|1,0\rangle_x - |0,1\rangle_x)(|1,0\rangle_x + |0,1\rangle_x) = f_2(t) \frac{1}{2} (|1,0\rangle_x |0,1\rangle_x - |0,1\rangle_x |1,0\rangle_x + |1,0\rangle_x |0,1\rangle_x - |0,1\rangle_x |1,0\rangle_x)
\]

(30)

Final Hamiltonian under microwave field or time-dependent voltages applied to 3 gates controlling qubit becomes

\[
\hat{H}(t) = \begin{pmatrix} E_p + \frac{1}{2} (f_1(t) + f_2(t)) & |t_s| + \frac{1}{2} (f_2(t) - f_1(t)) \\ |t_s| - \frac{1}{2} (f_2(t) - f_1(t)) & E_p - \frac{1}{2} (f_1 + f_2) \end{pmatrix}.
\]

(31)

Hamiltonian \( \hat{H}(t) \) is Hermitian what implies that \( f_1(t) = f_a(t) + i f_b(t) \) and \( f_1(t) = f_a(t) - i f_b(t) \) \( f_a \in \mathbb{R}, f_b \in \mathbb{R} \) if one generalizes \( f_1(t) \) and \( f_2(t) \) functions to be complex valued. Therefore finally Hamiltonian can be written in terms of real-valued functions

\[
\hat{H}(t) = \begin{pmatrix} E_p + \frac{1}{2} (f_a(t) + f_b(t)) & |t_s| - i f_b(t) \\ |t_s| + i f_b(t) & E_p - \frac{1}{2} (f_a(t) + f_b(t)) \end{pmatrix}.
\]

(32)

The obtained eigenenergies are

\[
E_1(t) = \frac{1}{2} (E_p - \sqrt{\frac{f_a(t) + f_b(t)}{2}^2 + |f_b(t)|^2 + |t_s|^2}),
\]

\[
E_2(t) = \frac{1}{2} (E_p + \sqrt{\frac{f_a(t) + f_b(t)}{2}^2 + |f_b(t)|^2 + |t_s|^2}).
\]

(33)

and energy eigenstate

\[
|E_1, t\rangle = \left( \frac{f_a(t) + f_b(t)}{t_s + i f_b(t)} - \sqrt{\frac{f_a(t) + f_b(t)}{t_s + i f_b(t)}^2 + |f_b(t)|^2 + |t_s|^2} \right) - 1, \]

\[
|E_2, t\rangle = \left( \frac{f_a(t) + f_b(t)}{t_s + i f_b(t)} + \sqrt{\frac{f_a(t) + f_b(t)}{t_s + i f_b(t)}^2 + |f_b(t)|^2 + |t_s|^2} \right) + 1.
\]

(34)
Heating up of any quantum system can be considered as the increase of occupancy of higher energy levels and depopulation of occupancy of lower energetic levels. In the case of position based qubit described in Fig.1 it simply means that the occupancy of ground energy state $E_1$ is decreased while occupancy of first excited energy state $E_2$ is increased. In such situation of constant coefficients $E_{p1}, E_{p2}, t_{s12} = t_{s21}$ with time the occupancy of each among two energy levels remains constant with time.

The Hamiltonian increasing the occupancy of eigenenergy state $E_2$ ("heating up") can be described as

$$\hat{H}(t) = \hat{H}_0 + f_1(t) |E_2\rangle \langle E_1| = (E_1 |E_1\rangle \langle E_1| + E_2 |E_2\rangle \langle E_2|)_0 + f_1(t) |E_2\rangle \langle E_1| = \begin{pmatrix} E_2 & f_1(t) \\ 0 & E_1 \end{pmatrix},$$

(35)

where $\hat{H}_0$ is time independent Hamiltonian and time-dependent part of Hamiltonian is proportional to $f_1(t)$. We obtain the equations of motion

$$i\hbar \frac{d}{dt} c_{E_1}(t) = E_1 c_{E_1}(t)$$

(36)

and

$$i\hbar \frac{d}{dt} c_{E_2}(t) = E_2 c_{E_2}(t) + f_1(t) c_{E_1}(t).$$

(37)

In case of lack of "heating up" or "cooling down" of quantum state we set $f_1 = 0$ and consequently

$$i\hbar \frac{d}{dt} c_{E_2}(t) = E_2 c_{E_2}(t), i\hbar \frac{d}{dt} c_{E_1}(t) = E_1 c_{E_1}(t).$$

(38)

what brings $c_{E_1}(t) = c_{E_1}(0)e^{-i\frac{E_1 t}{\hbar}}$ and $c_{E_2}(t) = c_{E_2}(0)e^{-i\frac{E_2 t}{\hbar}}$ what simplify means occupancy of energetic levels

$$P(t) = |c_{E_2}(t)|^2 = P_0(t) - probability of occupancy of the excited state$.
\[ |c_{E1}(t)|^2 = constant_1, \quad |c_{E2}(t)|^2 = constant_2. \]

Dynamics of quantum state "being heated" follows equation

\[ (\hat{H}_0 + f_1 |E_2\rangle \langle E_1|)(c_{E1}(t) |E_1\rangle + c_{E2}(t) |E_2\rangle) = ((c_{E2}(t)E_2 + f_1c_{E1}(t)) |E_2\rangle + \beta E_1c_{E1}(t) |E_1\rangle) = \hbar i [\frac{d}{dt}c_{E1}(t) |E_1\rangle + \frac{d}{dt}(t)c_{E2}(t) |E_2\rangle]. \]

what concludes that the population of \(|E_2\rangle\) state increases while population of \(|E_1\rangle\) is constant. The only way of avoiding interpretation difficulties is by assumption that state normalization is artifact and what really matters is the ratio between occupancy of excited and occupancy of ground state. This ratio gives the probability of having the excited quantum state. One can determine the one equation whose solutions are present in the second equation by the relations

\[ +\hbar i \frac{d}{dt}c_{E1}(t) = E_1c_{E1}(t)(t), \]
\[ +\hbar i \frac{d}{dt}c_{E2}(t) = (c_{E2}(t)E_2 + f_1(t)c_{E1}(t)) \]

that could be reduced to one equation

\[ +\hbar i \frac{d}{dt}c_{E2}(t) = (c_{E2}(t)E_2 + f_1(t)c_{E1}(0)e^{\frac{i}{\hbar}tE_1}). \]

Setting \(f_1 = constant\) and such that \(f_1 = 0\) for \(t < 0\) and non-zero otherwise we can make first guess as \(c_{E2}(t)(t) = c_1e^{\frac{itE_1}{\hbar}} + c_2e^{\frac{itE_2}{\hbar}} + c_0\) and we obtain

\[ \hbar i \frac{d}{dt}c_{E2}(t) = E_1c_1e^{\frac{itE_1}{\hbar}} + E_2c_2e^{\frac{itE_2}{\hbar}} \]
\[ - (E_2 - E_1)c_1 = E_2c_{E2}(t) - (E_2 - E_1)c_1 e^{\frac{itE_2}{\hbar}} \]

(41)

that brings \(+f_1c_{E1}(t)(0) = -(E_2 - E_1)c_1\) and finally we have \(c_1 = -\frac{f_1c_{E2}(0)}{E_2 - E_1}\) what implies \(c_{E2}(t) = c_2e^{\frac{itE_2}{\hbar}} - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{itE_1}{\hbar}}\) and \(c_0 = e^{\frac{itE_2}{\hbar}} [c_2 - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{itE_1}{\hbar}}] + c_0\). At \(t=0\) we have \(c_{E2}(0) = \sqrt{1 - |c_{E1}(0)|^2}\) what brings \(c_2 = c_{E2}(0)\) and finally we obtain

\[ c_{E2}(t)(t) = e^{\frac{itE_2}{\hbar}} [c_{E2}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{itE_1}{\hbar}}]. \]

(42)

and its derivative is

\[ i\hbar \frac{d}{dt}c_{E2}(t) = E_2[c_{E1}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1}] + f_1c_{E1} = E_2c_{E2}(0^+) + f_1c_{E1}(0^+). \]

(43)

Suddenly turning on function \(f_1\) at \(t=0\) bring discontinuity to the first derivative \(\frac{d}{dt}c_{E2}(t)(t = 0)\) of \(c_{E2}(t)\), but preserving continuity of the occupancy function. It is quite straightforward to evaluate the probability of occupancy of excited state \(E_2\) what is expressed by formula

\[ P_e(t) = \frac{|c_{E2}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{it(E_1 - E_2)}{\hbar}} e^{\frac{itE_2}{\hbar}}|^2}{|c_{E2}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{it(E_1 - E_2)}{\hbar}}|^2 + |c_{E1}(0)|^2} = \frac{|c_{E2}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{it(E_1 - E_2)}{\hbar}}|^2}{|c_{E2}(0) - \frac{f_1c_{E1}(0)}{E_2 - E_1} e^{\frac{it(E_1 - E_2)}{\hbar}}|^2 + 1} \]

(44)

One can refer to different scenarios of \(P_e(t)\) dependence on time as given by Fig.2. For small values of \(f_1\) we can convert the obtained results to the Hamiltonian matrix in position representation in the following form of

\[ \hat{H}(t) = \begin{pmatrix} E_p & t_s & t_s \\ t_s & E_p & t_s \\ t_s & t_s & E_p \end{pmatrix} + \frac{1}{2} \begin{pmatrix} f_1(t) & -f_1(t) & +f_1(t) \\ -f_1(t) & f_1(t) & -f_1(t) \\ +f_1(t) & -f_1(t) & f_1(t) \end{pmatrix}. \]

(45)

We recognize that in case of both real and complex values of function \(f_1\) such matrix is not Hermitian. It simply means that it can be interpreted as the Hamiltonian describing dissipation of quantum state in phenomenological way.
4. RABI OSCILLATIONS IN GENERAL CASE FOR 2 ENERGY LEVEL SYSTEM

In general case during heating up of q-state or during cooling down of q-state we need to consider the Hamiltonian as \( H = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2| + f_1(t) |E_2\rangle\langle E_1| + f_2(t) |E_1\rangle\langle E_2|. \) If we want to have time-dependent only \( E_1(t) \) and only \( E_2(t) \) states we need to consider \( H = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2| + f_1(t) |E_2\rangle\langle E_2| + f_2(t) |E_1\rangle\langle E_1| + f_2(t) |E_1\rangle\langle E_2|). \) Let us spot the dynamics of quantum state with time so we have \( f_1(t), f_2(t) = 0 \) for \( t < 0 \) and constant non-zero otherwise \((f_1(t) = f_1 = \text{const}_1, f_2(t) = f_2 = \text{const}_2)\) so one obtains the equation

\[
+\hbar \frac{d}{dt} c_{E_1}(t) = (c_{E_1}(t) E_1 + f_2(t) c_{E_2}(t)), +\hbar \frac{d}{dt} c_{E_2}(t) = (c_{E_2}(t) E_2 + f_1(t) c_{E_1}(t)).
\]

From first equation we have \( \frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) = c_{E_2}(t) \) and we obtain the second equation

\[
+\hbar \frac{d}{dt} (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) = (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 + f_1(t) c_{E_1}(t).
\]

what gives

\[
\frac{d}{dt} (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) = \frac{d f_2}{dt} \frac{1}{f_2(t)} (-\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) + (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)))
\]

\[
= \frac{1}{i \hbar} (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 + \frac{1}{i \hbar} f_1(t) c_{E_1}(t)(t).
\]

and it gives

\[
\frac{d^2 c_{E_2}(t)}{dt^2} \frac{1}{f_2(t)} \frac{d}{dt} c_{E_2}(t) \left[-\frac{d f_2}{dt} \frac{1}{f_2(t)} (-\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) + (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 \right] + \frac{1}{i \hbar} f_1(t)f_2(t)) = 0.
\]

After multiplication by \( \frac{f_2(t)}{\hbar} \) the last equation gives

\[
\frac{d^2 c_{E_2}(t)}{dt^2} \frac{1}{f_2(t)} \frac{d}{dt} c_{E_2}(t) \left[-\frac{d f_2}{dt} \frac{1}{f_2(t)} (-\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) + (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 \right] + \frac{1}{h^2} f_1(t)f_2(t)) = 0.
\]

In analogical way we obtain

\[
\frac{d^2 c_{E_2}(t)}{dt^2} \frac{1}{f_1(t)} \frac{d}{dt} c_{E_2}(t) \left[-\frac{d f_1}{dt} \frac{1}{f_1(t)} (-\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) + (\frac{1}{f_1(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 \right] + \frac{1}{h^2} f_1(t)f_2(t)) = 0.
\]

Boundary conditions are given as

\[
\frac{i \hbar}{dt} c_{E_1}(t_0) = E_1 c_{E_2}(t_0) + f_2(t_0^+) c_{E_1}(t_0)
\]

\[
\frac{i \hbar}{dt} c_{E_2}(t_0) = E_2 c_{E_1}(t_0) + f_1(t_0^+) c_{E_2}(t_0)
\]

\[
\frac{i \hbar}{dt} c_{E_2}(t_0^+) = E_2 c_{E_1}(t_0^+) + f_1(t_0^+) c_{E_2}(t_0^+)
\]

\[
\frac{i \hbar}{dt} c_{E_1}(t_0^+) = E_1 c_{E_2}(t_0^+) + f_2(t_0^+) c_{E_1}(t_0^+).
\]

From later considerations it turns out that \( f_2(t)^* = f_2(t) \) so \( f_1(t) = f_a(t) + i f_b(t) \) and \( f_2(t) = f_a(t) - i f_b(t) \), where \( f_a(t) \) and \( f_b(t) \) are real valued functions. Therefore we can write the equations of motion as

\[
\frac{d^2 c_{E_1}(t)}{dt^2} + \frac{d}{dt} c_{E_1}(t) \left[-\frac{d f_2}{dt} \frac{1}{f_2(t)} (-\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t)) + (\frac{1}{f_2(t)} (+\hbar \frac{d}{dt} c_{E_1}(t) - E_1 c_{E_1}(t))) E_2 \right] + \frac{1}{i \hbar} f_1(t)f_2(t)) = 0.
\]

(56)
In analogical way we obtain

$$ \frac{d^2}{dt^2} c_{E_2}(t) + \frac{d}{dt} c_{E_2}(t_0)[\frac{1}{\hbar} \left( E_1 + E_2 \right) - c_{E_1}(t)[\frac{E_1 E_2}{\hbar^2} - \frac{i}{\hbar} \frac{df_1}{dt} E_2 + \frac{1}{\hbar^2} (f_a(t)^2 + f_b(t)^2)] = 0. \quad (57)$$

Boundary conditions are given as

$$ i\hbar \frac{d}{dt} c_{E_2}(t_0^+) = E_1 c_{E_2}(t_0^+) + (f_a(t_0) - if_b(t_0)) c_{E_1}(t_0) \quad (58)$$

$$ i\hbar \frac{d}{dt} c_{E_1}(t_0^+) = E_2 c_{E_1}(t_0^+) + (f_a(t_0) + if_b(t_0)) c_{E_2}(t_0) \quad (59)$$

$$ c_{E_2}(t_0^+) = c_{E_2}(t_0) \quad (60)$$

$$ c_{E_1}(t_0^+) = c_{E_1}(t_0) \quad (61)$$

Very special case is when

$$ f_1(t) = a \exp(\text{ct}) + ib \exp(\text{ct}), \ f_2(t) = a \exp(\text{ct}) - ib \exp(\text{ct}) \quad (62)$$

where c, a and b are real valued. In such cases we obtain the equations for the occupancy of energy state $E_1$ and $E_2$ expressed as

$$ \frac{d^2}{dt^2} c_{E_2}(t) + \frac{d}{dt} c_{E_2}(t)[c + \frac{i}{\hbar} (E_1 + E_2)] + c_{E_2}(t)[\frac{E_1 E_2}{\hbar^2} - \frac{i}{\hbar} E_1 c + \frac{1}{\hbar^2} (a^2 + b^2) \exp(2\text{ct})] = 0. \quad (63)$$

First case is $c = 0, \ h = 1$ and solution is

$$ c_{E_1}(t) = e^{-\frac{i}{2}(E_1 + E_2 - i\sqrt{4a^2 - 4b^2 - E_1^2 + 2E_1 E_2 - E_2^2})t} g_1 + e^{\frac{i}{2}(-i(E_1 + E_2) + \sqrt{4a^2 - 4b^2 - E_1^2 + 2E_1 E_2 - E_2^2})t} g_2, \quad (64)$$

where $g_1$ and $g_1$ are complex values. Having non-zero c we obtain solutions

$$ c_{E_1}(t) = c_1 \exp \left( \frac{1}{2} t \left( -\sqrt{-4a^2 e^{2\text{ct}} - 4b^2 e^{2\text{ct}} + c^2 - 2icE_1 + 2icE_2 - E_1^2 + 2E_1 E_2 - E_2^2} + c - iE_1 - iE_2 \right) \right) + c_2 \exp \left( \frac{1}{2} t \left( \sqrt{-4a^2 e^{2\text{ct}} - 4b^2 e^{2\text{ct}} + c^2 - 2icE_1 + 2icE_2 - E_1^2 + 2E_1 E_2 - E_2^2} + c - iE_1 - iE_2 \right) \right), \quad (65)$$

$$ c_{E_2}(t) = c_1 \exp \left( \frac{1}{2} t \left( -\sqrt{-4a^2 e^{2\text{ct}} - 4b^2 e^{2\text{ct}} + c^2 + 2icE_1 - 2icE_2 - E_1^2 + 2E_1 E_2 - E_2^2} + c - iE_1 - iE_2 \right) \right) + c_2 \exp \left( \frac{1}{2} t \left( \sqrt{-4a^2 e^{2\text{ct}} - 4b^2 e^{2\text{ct}} + c^2 + 2icE_1 - 2icE_2 - E_1^2 + 2E_1 E_2 - E_2^2} + c - iE_1 - iE_2 \right) \right), \quad (66)$$

The simplified case of last formula can be given as

$$ c_{E_2}(t) = -g_4 \exp \left( -\frac{1}{2} it \left( -i \sqrt{-E_1^2 + 2E_1 E_2 - E_2^2 - 4 + E_1 + E_2} \right) \right) \left( -1 + \exp \left( \frac{1}{2} it \left( -i \sqrt{-E_1^2 + 2E_1 E_2 - E_2^2 - 4 + E_1 + E_2} \right) + \frac{1}{2} it \left( \sqrt{-E_1^2 + 2E_1 E_2 - E_2^2 - 4 - i(E_1 + E_2)} \right) \right) \right) \quad (67)$$

and the numerical example of its dependence on time is depicted in Fig.2, where initially energy level $E_1$ was completely populated and with time the full population of energy level $E_2$ was achieved while energy level $E_1$ was completely depopulated. Such dependence can be used for example in the action of Hadamard gate implemented in electrostatic position dependent qubit. If $f_1(t)$ and $f_2(t)$ functions have small values one can assume $|E_1\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle_x - |0, 1\rangle_x)$ and $|E_2\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle_x + |0, 1\rangle_x)$ and

$$ \hat{H}(t)_x = \begin{pmatrix} \frac{E_p}{2} & t_s & \frac{1}{2}(f_1(t) + f_2(t)) \\ t_s & \frac{E_p}{2} & \frac{1}{2}(f_1(t) - f_2(t)) \\ \frac{1}{2}(f_1(t) + f_2(t)) & \frac{1}{2}(f_1(t) - f_2(t)) & -f_1(t) - f_2(t) \end{pmatrix}. \quad (68)$$

Hermicity of last Hamiltonian requires that $f_1(t) = f_2(t)^*$. 

coefficients describing energy localized at 2 nodes

The probability of presence of electron at node 1 is

probability of presence of electron at node 2 is

such case the quantum state for

only concentrate on the system with electrons confiment by some effective potential. It requires introduction of

wells controlled electrostatically. Quite obviously we are omitting continuum spectrum of eigenenergies and we

of time-dependent Hamiltonian. In most general case we have

5. EXTENSION OF 2-ENERGY TIGHT BINDING MODEL INTO N ENERGETIC LEVELS FOR POSITION BASED QUBIT IN ARBITRARY ELECTROMAGNETIC ENVIRONMENT

Picture presented before as in equation [1] with N=2 energetic levels can be easily extended for arbitrary number
of energy levels \( E_1 < E_2 < \ldots < E_{2N_1} = N \), what is valid in time-independent case. It is worth mentioning that very
last chain of inequalities between time dependent eigenenergies does not need to be always valid in the general case
of time-dependent Hamiltonian. In most general case we have \( N = 2N_1 \) energetic levels among 2 coupled quantum
wells controlled electrostatically. Quite obviously we are omitting continuum spectrum of eigenenergies and we
only concentrate on the system with electrons confinement by some effective potential. It requires introduction of
2\( N_1 \) orthogonal Wannier function bases such that \( |x_1\rangle \ldots |x_{N_1}\rangle, |x_2\rangle \ldots |x_{N_2}\rangle \) = \((1,0)_{E_{1-2}} \ldots \)

\((1,0)_{E_{N_1-1}-E_{N_1}} \ldots (0,1)_{E_1-E_2} \ldots (0,1)_{E_{N_1-1}-E_{N_1}} \) and such that \( \langle x_1| (|x_m\rangle) = 0 \) for any \( m \) different than \( k \). In
such case the quantum state for \( N_1 = 3 \) \((N = 2N_1)\) is described as

\[
|\psi\rangle(t) = \gamma_{E_1-E_2,p_1(t)} \langle 1 | E_1, E_2 + \gamma_{E_1-E_4,p_1(t)} \langle x_1 | E_3, E_4 + \gamma_{E_5-E_6,p_1(t)} \langle x_1 | E_5, E_6 +
\gamma_{E_5-E_6,p_2(t)} \langle x_2 | E_5, E_6 + \gamma_{E_3-E_4,p_2(t)} \langle x_2 | E_3, E_4 + \gamma_{E_1-E_2,p_2(t)} \langle x_2 | E_1, E_2 =
\]

\[
= \frac{1}{\sqrt{N}} \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
+ \gamma_{E_3-E_4,p_1(t)}
+ \gamma_{E_5-E_6,p_2(t)}
+ \gamma_{E_1-E_2,p_2(t)}
\]

\[
= \begin{pmatrix}
\gamma_{E_1-E_2,p_1(t)} \\
\gamma_{E_5-E_6,p_1(t)} \\
\gamma_{E_3-E_4,p_2(t)} \\
\gamma_{E_1-E_2,p_2(t)}
\end{pmatrix}
\].

(69)

The probability of presence of electron at node 1 is \( P_1(t) = |\gamma_{E_1-E_2,p_1(t)} + \gamma_{E_3-E_4,p_1(t)} + \gamma_{E_5-E_6,p_2(t)}|^2 \) and the probability of presence of electron at node 2 is \( P_2(t) = |\gamma_{E_1-E_2,p_2(t)} + \gamma_{E_3-E_4,p_2(t)} + \gamma_{E_5-E_6,p_2(t)}|^2 \). The act
of measurement on position based qubit is represented by the operator

\[
P_{Left} = |1,0\rangle_{E_1,E_2} \langle 1,0 | E_1, E_2 + |1,0\rangle_{E_1,E_4} \langle 1,0 | E_3, E_4 + |1,0\rangle_{E_5,E_6} \langle 1,0 | E_5, E_6 ,
\]

(70)

\[
P_{Right} = |0,1\rangle_{E_1,E_2} \langle 0,1 | E_1, E_2 + |0,1\rangle_{E_1,E_4} \langle 0,1 | E_3, E_4 + |0,1\rangle_{E_5,E_6} \langle 0,1 | E_5, E_6 .
\]

(71)

Let us review the Hamiltonian describing system with \( N = 2N_1 \) energy levels. Essentially we have \( 2N_1 \) coefficients describing energy localized at 2 nodes \( E_{p_1,1}, E_{p_1,2}, \ldots, E_{p_1,N_1}, E_{p_2,1}, E_{p_2,2}, \ldots, E_{p_2,N_1}, \) so we are dealing
with $E_{pu,m}$ coefficients, where $m=1..N_1$, $u$ is 1 or 2 and we have taken into account existence of all $N = 2N_1$ energetic levels. Let us set $N_1 = 3$ and in such case the quantum state Hamiltonian in the case of lack of transition between energetic levels corresponding to Fig.4. can be written as

$$
\hat{H} = \begin{pmatrix}
E_{1,p1} & 0 & 0 & 0 & 0 & t_{2,p1\rightarrow p2} \\
0 & E_{2,p1} & 0 & 0 & t_{3,p1\rightarrow p2} & 0 \\
0 & 0 & E_{3,p1} & t_{3,p1\rightarrow p2} & 0 & 0 \\
0 & 0 & t_{3,p2\rightarrow p1} & E_{3,p2} & 0 & 0 \\
t_{2,p2\rightarrow p1} & 0 & 0 & 0 & E_{2,p2} & 0 \\
t_{1,p1\rightarrow p2} & 0 & 0 & 0 & 0 & E_{1,p2}
\end{pmatrix}
$$

$$
= E_{1,t} |E_{1,t}\rangle \langle E_{1,t}| + E_{2,t} |E_{2,t}\rangle \langle E_{2,t}| + E_{3,t} |E_{3,t}\rangle \langle E_{3,t}| + E_{4,t} |E_{4,t}\rangle \langle E_{4,t}| + E_{5,t} |E_{5,t}\rangle \langle E_{5,t}| + E_{6,t} |E_{6,t}\rangle \langle E_{6,t}|. \quad (72)
$$

It is important to mention that in the case of lack of time-dependent Hamiltonian having any among frequency components $\frac{E_k - E_l}{\hbar}$ for $k \neq l$ such that $(k, l) = 1..6$ there is no possibility for the occurrence of resonant state and change of probability of occupancy among different energetic levels. In such case $\langle 1,0|E_{1,0}E_{1,0}\rangle |1,0\rangle = 0$. However it is not true if there exists resonant state and if for example Hamiltonian consists following non-zero components with frequencies $\left( \frac{E_1-E_3}{\hbar}, \frac{E_1-E_4}{\hbar}, \frac{E_2-E_3}{\hbar}, \frac{E_2-E_4}{\hbar} \right)$.

Now we are moving towards the situation of system with position based qubit with 5 energetic levels, two-different potential minima and one occupied localized state on the right side as depicted in Fig.5. We have Hamiltonian of the form

$$
\hat{H} = \begin{pmatrix}
E_{2,p1} & 0 & 0 & t_{2,p1\rightarrow p2} & 0 \\
0 & E_{3,p1} & t_{3,p1\rightarrow p2} & 0 & 0 \\
0 & t_{3,p2\rightarrow p1} & E_{3,p2} & 0 & 0 \\
t_{2,p2\rightarrow p1} & 0 & 0 & E_{2,p2} & 0 \\
0 & 0 & 0 & 0 & E_{1,p1}
\end{pmatrix}
$$

$$
= E_{1}(t) |E_{1}(t)\rangle \langle E_{1}(t)| + .. + E_{5}(t) |E_{5}(t)\rangle \langle E_{5}(t)|. \quad (73)
$$
with corresponding quantum given as

$$|\psi, t\rangle_x = \gamma_{E_5,E_4,p_1}(t) |1,0\rangle_{E_5,E_4} + \gamma_{E_3,E_2,p_1}(t) |1,0\rangle_{E_3,E_2} + \gamma_{E_5,E_4,p_2}(t) |0,1\rangle_{E_5,E_4} + \gamma_{E_3,E_2,p_2}(t) |0,1\rangle_{E_3,E_2} +$$

$$+ \gamma_{E_1,p_2}(t) |0,1\rangle_{E_1} = \begin{pmatrix} \gamma_{E_5,E_4,p_1}(t) \\ \gamma_{E_3,E_2,p_1}(t) \\ \gamma_{E_5,E_4,p_2}(t) \\ \gamma_{E_3,E_2,p_2}(t) \\ \gamma_{E_1,p_2}(t) \end{pmatrix}. \quad (74)$$

The energetic states parametrized by $E_5, E_4$ or $E_3, E_2$ can move freely between node 1 and 2 so they are delocalized while the state numerated by $E_1$ is the particular localized ground state. Specified Hamiltonian structure implies that the ground state cannot be moved to excited states and reversely excited states cannot be moved into ground state.

The coupling between ground state and first excited state at node 2 occurs in the case of modified Hamiltonian of the following form as

$$\hat{H} = \begin{pmatrix} E_{5,2} & 0 & 0 & t_{2,1} & 0 \\ 0 & E_{3,1} & t_{3,1} & 0 & 0 \\ 0 & t_{3,2} & E_{3,2} & 0 & 0 \\ 0 & 0 & 0 & E_{3,2} & 0 \\ 0 & 0 & 0 & 0 & E_{1} \end{pmatrix} = E_1(t) |E_1(t)\rangle \langle E_1(t)| + E_2(t) |E_2(t)\rangle \langle E_2(t)| + \ldots + E_5(t) |E_5(t)\rangle \langle E_5(t)| + f_1(t) |E_2\rangle \langle E_1| + f_2(t) |E_2\rangle \langle E_2| + f_3(t) |E_3\rangle \langle E_3| + f_4(t) |E_3\rangle \langle E_3|. \quad (75)$$

In particular state it is allowed for the wave-packet in the right-well to undergo transition from energetic state $E_1$ to $E_2$ and $E_3$ and reversely. Better picture can be obtained from Schroedinger equation. Last Hamiltonian implies presence of time-dependent component in matrix that has $\omega_{21} = \frac{E_2 - E_1}{\hbar}$ and $\omega_{31} = \frac{E_3 - E_1}{\hbar}$ frequency components.

In such case the projectors $(|0,1\rangle_{E_1,E_2} \langle 0,1|_{E_1,E_2})(|0,1\rangle_{E_1,E_3} \langle 0,1|_{E_3,E_1})$ are different from zero because of existence of resonant states characterized by frequencies $\omega_{21}$ and $\omega_{31}$. Now we are moving from position based Hamiltonian representation into energy based that is by identity transformation

$$\hat{H}(t) = \begin{pmatrix} E_{5,2} & 0 & 0 & 0 & 0 \\ 0 & E_{4,1} & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & E_1 \end{pmatrix} \begin{pmatrix} E_{5,2} & 0 & 0 & 0 & 0 \\ 0 & E_{4,1} & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & E_1 \end{pmatrix} = \begin{pmatrix} E_{5,2} & 0 & 0 & 0 & 0 \\ 0 & E_{4,1} & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & E_1 \end{pmatrix} \begin{pmatrix} E_{5,2} & 0 & 0 & 0 & 0 \\ 0 & E_{4,1} & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & E_1 \end{pmatrix}$$

Now we need to specify the energy eigenstates introducing $\hat{E} = \text{diag}(E_5, E_4, E_3, E_2, E_1)$ and we obtain $\hat{E}$ acting on

$$\begin{pmatrix} \frac{E_{5,2}}{E_5} & 0 & 0 & \frac{t_{2,1}}{E_5} & 0 \\ 0 & \frac{E_{4,1}}{E_4} & \frac{t_{3,1}}{E_4} & 0 & 0 \\ 0 & \frac{t_{3,2}}{E_3} & \frac{E_{3,2}}{E_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{E_{2,2}}{E_2} & \frac{t_{2,1}}{E_1} \\ 0 & 0 & 0 & \frac{t_{2,1}}{E_1} & \frac{E_{1,1}}{E_1} \end{pmatrix} \begin{pmatrix} \gamma_{E_3,E_2,p_1} \\ \gamma_{E_4,E_2,p_1} \\ \gamma_{E_4,E_2,p_1} \\ \gamma_{E_3,E_2,p_2} \\ \gamma_{E_1,p_2} \end{pmatrix} = \hat{E} \begin{pmatrix} \gamma_{E_3,E_2,p_1}(t) \frac{E_{5,2}}{E_5} + \frac{t_{2,1}}{E_5} \gamma_{E_3,E_2,p_2}(t) \\ \gamma_{E_4,E_2,p_1}(t) \frac{E_{4,1}}{E_4} + \frac{t_{3,1}}{E_4} \gamma_{E_4,E_2,p_2}(t) \\ \gamma_{E_4,E_2,p_1}(t) \frac{E_{3,2}}{E_3} + \frac{t_{3,2}}{E_3} \gamma_{E_4,E_2,p_2}(t) \\ \gamma_{E_3,E_2,p_2}(t) \frac{E_{2,2}}{E_2} + \frac{t_{2,1}}{E_2} \gamma_{E_3,E_2,p_2}(t) \\ \gamma_{E_1,p_2}(t) \frac{E_{1,1}}{E_1} \end{pmatrix} \quad (76)$$
It is noticeable to recognize that the ground state eigenvector from localized state was converted into delocalized state by the presence of non-zero $\gamma_{E1,p2}(t)E_{1,p1}$ term in the Hamiltonian.

$$
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

Also second energy level eigenvector was changed.

$$
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

The element $t_{2\rightarrow1,p2\rightarrow p2}$ is responsible for heating up or cooling down of the localized state. We notice that all other eigenenergy vectors were not changed by the presence of non-zero elements $t_{2\rightarrow1,p2\rightarrow p2} = t'_{1\rightarrow2,p2\rightarrow p2}$ in the Hamiltonian.

It might happen that potential minima (bottom) in position based qubit can have arbitrary depth so more than one eigenenergy state can be localized. Number of localized states can be arbitrary big both on the left and the right side. In considered example we have only localized on the right state. Localized states can be heated up or cool down so one localized state is transferring into another localized state in the same quantum well. In general k states (as $k=2$ in reference to the matrix) can be localized on the right side among $k+m$ all energetic states (where $m=4$ is number of delocalized eigenenergy states) so total number of Hamiltonian eigenenergy state $k+m$ is $4+2=6$.

$$
\hat{H} =
\begin{pmatrix}
E_{2,p1} & 0 & 0 & t_{2,p1\rightarrow p2} & 0 & 0 \\
0 & E_{3,p1} & t_{3,p1\rightarrow p2} & 0 & 0 & 0 \\
0 & t_{3,p2\rightarrow p1} & E_{3,p2} & 0 & 0 & 0 \\
t_{2,p2\rightarrow p1} & 0 & 0 & E_{2,p2} & t_{1\rightarrow2,p2\rightarrow p2} & t_{0\rightarrow2,p2\rightarrow p2} \\
0 & 0 & 0 & t_{2\rightarrow1,p2\rightarrow p2} & E_{1,p2} & t_{0\rightarrow1,p2\rightarrow p2} \\
0 & 0 & 0 & t_{2\rightarrow0,p2\rightarrow p2} & t_{1\rightarrow0,p2\rightarrow p2} & E_{0,p2}
\end{pmatrix}
$$
We can recognize that term $t_{1\rightarrow 0,p_2\rightarrow p_2}$ is able to heat up and cool down the localized q-state between 0 and 1 energetic level in q-well $p_2$ and term $t_{2\rightarrow 0,p_2\rightarrow p_2}$ is describing interaction between 0 and 2 energy level in q-well $p_2$, while term $t_{2\rightarrow 1,p_2\rightarrow p_2}$ describes the interaction between 1st and 2nd energetic level in second quantum well $p_2$.

Now we are describing the situation of 3 localized state in the left well (associated with matrix coefficients in green) and 2 localized states in the right wells (associated with matrix coefficients in red) and 4 states that are delocalized so we are dealing with matrix of 9 states.

$$
\begin{pmatrix}
E_{-1,p_1} & t_{0\rightarrow -1,p_1\rightarrow p_1} & t_{1\rightarrow -1,p_1\rightarrow p_1} & 0 & 0 & 0 & 0 & 0 & t_{0\rightarrow -1,p_1\rightarrow p_2} \\
t_{-1\rightarrow 0,p_1\rightarrow p_1} & E_{0,p_1} & t_{1\rightarrow 0,p_1\rightarrow p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
t_{-1\rightarrow 1,p_1\rightarrow p_1} & t_{0\rightarrow 1,p_1\rightarrow p_1} & E_{1,p_1} & t_{2\rightarrow 1,p_1\rightarrow p_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_{1\rightarrow 2,p_1\rightarrow p_1} & E_{2,p_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_{3,p_2\rightarrow p_1} & E_{3,p_1} & t_{3,p_2\rightarrow p_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{2,p_2\rightarrow p_1} & E_{2,p_2} & t_{0\rightarrow 2,p_2\rightarrow p_2} & t_{0\rightarrow 1,p_2\rightarrow p_2} \\
0 & 0 & 0 & 0 & 0 & 0 & t_{2\rightarrow 1,p_2\rightarrow p_2} & E_{1,p_2} & t_{0\rightarrow 1,p_2\rightarrow p_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0\rightarrow 2,p_2\rightarrow p_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{1\rightarrow 0,p_2\rightarrow p_2} & 0 \\
\end{pmatrix}
$$

(80)

Heating up and cooling down of the localized quantum state in the left q-well is controlled by Hamiltonian coefficients $t_{0\rightarrow -1,p_1\rightarrow p_1}$, $t_{1\rightarrow 0,p_1\rightarrow p_1}$, $t_{1\rightarrow -1,p_1\rightarrow p_1}$ and its conjugate counterparts $t_{-1\rightarrow 0,p_1\rightarrow p_1}$, $t_{-1\rightarrow 1,p_1\rightarrow p_1}$, $t_{-1\rightarrow 1,p_1\rightarrow p_1}$. Moving delocalized q-state in the left q-well $p_1$ into delocalized q-state in the left $p_2$ well is by non-zero $t_{1\rightarrow 2,p_1\rightarrow p_1}$ and its conjugate $t_{2\rightarrow 1,p_1\rightarrow p_1}$ in orange color. From the point of view of q-mechanics it is also possible to transfer one q-state localized in the left q-well into the q-state localized in the right q-well. It is achieved by the non-zero coefficient $t_{0\rightarrow -1,p_1\rightarrow p_2}$ and its conjugate $t_{-1\rightarrow 0,p_2\rightarrow p_1}$ in brown color. All these transfer between states of different energies requires microwave field or AC voltage components. In case of matrix 9 by 9 we can spot $(9^2 - 9)/2 = 21$ time dependent processes of transfer from one energetic state into another energetic state in the same q-well or into opposite q-well. In general for N by N matrix one has $(N^2 - N)/2$ such processes. More detailed knowledge about this processes might be only extracted from Schroedinger formalism in 1, 2 or 3 dimensions. In most general case in the case of system with 9 energetic levels are depicted in Fig.6.

Now we are describing the most general situation for the system preserving 6 energy levels where position of potential minima and maxima can change in time so localized states can change into delocalized or reversely. It is thus describing the system is placed in outside time-dependent electromagnetic field of any dependence so the matrix of position-based qubit $\hat{H}(t)$ can be written as

$$
\hat{H}(t) =
\begin{pmatrix}
E_{1,p_1} & t_{2\rightarrow 1,p_1\rightarrow p_1} & t_{3\rightarrow 1,p_1\rightarrow p_1} & t_{3\rightarrow 1,p_2\rightarrow p_1} & t_{2\rightarrow 1,p_2\rightarrow p_1} & t_{1,p_2\rightarrow p_1} \\
1_{1\rightarrow 2,p_1\rightarrow p_1} & E_{2,p_1} & t_{3\rightarrow 2,p_1\rightarrow p_1} & t_{3\rightarrow 2,p_2\rightarrow p_1} & t_{2,p_2\rightarrow p_1} & t_{1\rightarrow 2,p_2\rightarrow p_1} \\
1_{1\rightarrow 3,p_1\rightarrow p_1} & t_{2\rightarrow 3,p_1\rightarrow p_1} & E_{3,p_1} & t_{3,p_2\rightarrow p_1} & t_{2\rightarrow 3,p_2\rightarrow p_1} & t_{1\rightarrow 3,p_2\rightarrow p_1} \\
1_{1\rightarrow 3,p_1\rightarrow p_2} & t_{2\rightarrow 3,p_1\rightarrow p_2} & E_{3,p_2} & t_{3,p_1\rightarrow p_2} & t_{2\rightarrow 3,p_2\rightarrow p_2} & t_{1\rightarrow 3,p_2\rightarrow p_2} \\
1_{1\rightarrow 2,p_2\rightarrow p_2} & t_{2,p_2\rightarrow p_1} & t_{3,p_2\rightarrow p_2} & E_{2,p_2} & t_{1\rightarrow 2,p_2\rightarrow p_2} & t_{1\rightarrow 2,p_2\rightarrow p_2} \\
1_{1,p_1\rightarrow p_2} & t_{2\rightarrow 1,p_1\rightarrow p_2} & t_{3,p_1\rightarrow p_2} & t_{3,p_2\rightarrow p_2} & E_{2,p_2} & 0 \\
\end{pmatrix}
$$

(81)

Such matrix is Hermitian so $t_{k,s,p_2\rightarrow p_1} = t_{k,s,p_1\rightarrow p_2}$ for $k$ and $s$ among 1, 2 and 3 and for $p_1$ and $p_2$ having value $p_1$ (presence of electron in left quantum well) or $p_2$ (presence of electron in right quantum well) and having real-valued diagonal elements. The meaning of non-diagonal coefficients is non-trivial.

In general case the eigenvalues of described matrix cannot be determined analytically unless there are some preimposed symmetries as for example $E_{k,p_1} = E_{k,p_2}$ for $k=1,2$ and 3 and in such case eigenvalues are determined by the roots of polynomial of 3rd order in analytical way. Very last reasoning can be conducted also for the system with 8 energetic levels when one deals with roots of polynomial of 4th order. By proper electromagnetic engineering the system with 6 energetic levels can be controlled by $((36 - 6)/2) + 6 = 15 + 6 = 21$ time dependent parameters. In most general case the system of position based qubit having 2 coupled quantum dots with 6 energy levels can be parametrized by 36 real valued functions that are time-dependent. Quite obviously the same system with 2N energetic levels can be parametrized by $(2N)^2$ real valued functions under the assumption.
that occupancy of electron is distributed among $2N$ energetic levels. We introduce notation $\gamma_{1,p1} = \gamma_{E1-E2,p1}$, $\gamma_{2,p1} = \gamma_{E3-E4,p1}$, $\gamma_{3,p1} = \gamma_{E5-E6,p1}$, $\gamma_{3,p2} = \gamma_{E5-E6,p2}$, $\gamma_{2,p2} = \gamma_{E3-E4,p2}$, $\gamma_{1,p2} = \gamma_{E1-E2,p2}$. The last matrix can be written in energy bases by using the last matrix of Hamiltonian with identity $\hat{H}(t) \hat{\psi}(t) = \hat{H}(t) \hat{\psi}(t)$.
It is worth noticing that having knowledge on all eigenvalues $|E_1, t\rangle \langle E_1, t| \delta_{k,l}$ and

$$
|E_{2, t}\rangle = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
\quad
|E_{2, t}\rangle \langle E_{2, t}| + \sum_{k=3}^{6} E_k = 1
$$

It is worth noticing that having knowledge on all eigenvalues $E_1(t), \ldots, E_N(t)$ with time we can determine the eigenenergy occupancy with time from position occupancy in unique way. From above considerations the
following relations takes place

\[
\begin{pmatrix}
E_{1,p_1} & t_{1-1,p_1-p_1} & t_{2-1,p_1-p_1} & t_{3-1,p_2-p_1} & t_{3-1,p_2-p_1} & t_{4-1,p_2-p_1} & t_{1-2,p_1-p_2} \\
E_{2,p_1} & E_{1,p_1} & E_{3,p_1} & E_{4,p_1} & E_{5,p_1} & E_{6,p_1} & E_{6,p_2} \\
E_{3,p_1} & E_{2,p_1} & E_{1,p_1} & E_{5,p_1} & E_{6,p_1} & E_{6,p_2} & E_{6,p_2} \\
E_{4,p_1} & E_{3,p_1} & E_{2,p_1} & E_{1,p_1} & E_{5,p_1} & E_{6,p_1} & E_{6,p_2} \\
E_{5,p_1} & E_{4,p_1} & E_{3,p_1} & E_{2,p_1} & E_{1,p_1} & E_{5,p_1} & E_{6,p_1} \\
E_{6,p_1} & E_{5,p_1} & E_{4,p_1} & E_{3,p_1} & E_{2,p_1} & E_{1,p_1} & E_{5,p_1} \\
E_{6,p_2} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} \\
E_{6,p_2} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} & E_{6,p_1} \\
\end{pmatrix} \times \begin{pmatrix}
\gamma_{E1-E2,p_1(t)} \\
\gamma_{E3-E4,p_1(t)} \\
\gamma_{E5-E6,p_1(t)} \\
\gamma_{E5-E6,p_2(t)} \\
\gamma_{E3-E4,p_2(t)} \\
\gamma_{E1-E2,p_2(t)} \\
\end{pmatrix} = \begin{pmatrix}
\hat{c}_{E_{1,p_1}(t)} \\
\hat{c}_{E_{2,p_1}(t)} \\
\hat{c}_{E_{3,p_1}(t)} \\
\hat{c}_{E_{3,p_2}(t)} \\
\hat{c}_{E_{4,p_2}(t)} \\
\hat{c}_{E_{1,p_2}(t)} \\
\end{pmatrix}
\]

By proper controlling matrix in position representation we can achieved desired occupancy of energetic levels with time expressed by \( \hat{c}_{E_{1,p_1}(t)}, \ldots, \hat{c}_{E_{1,p_2}(t)} \) coefficients. On another hand preimposing dependence of occupancy of energetic levels by quantum state expressed in \( \hat{c}_{E_{1,p_1}(t)}, \ldots, \hat{c}_{E_{1,p_2}(t)} \) with time one can achieve desired dependence of electrons positions \( \gamma_{E1,p_1(t)}, \gamma_{E1,p_2(t)} \) by using relation \( |\psi, t\rangle_x = \hat{g}(t) = \hat{A}(t)^{-1} \hat{c}_E(t) = \hat{A}(t)^{-1} |\psi, t\rangle_E \).

### 6. CASE OF ELECTROSTATIC QUBIT INTERACTION

We consider most minimalist model of electrostatically interacting two position-based qubits that are double quantum dots A (with nodes 1 and 2 and named as U-upper qubit) and B (with nodes 1’ and 2’ and named as L-lower qubit) with local confinement potentials as given in the right side of Fig.1. By introducing notation \( |1, 0\rangle_z = |1\rangle, |0, 1\rangle_z = |2\rangle, |1', 0\rangle_z = |1'\rangle, |0', 1\rangle_z = |1'\rangle \) the minimalistic Hamiltonian of the system of electrostatically interacting position based qubits can be written as

\[
\hat{H} = (t_{s21}(t) |2\rangle \langle 1| + t_{s12}(t) |1\rangle \langle 2|)\hat{I}_d + (\hat{I}_d(t_{s21}(t) |2\rangle \langle 1'| + t_{s12}(t) |1\rangle \langle 2'|) + (E_{p1}(t) |1\rangle \langle 1| + E_{p2}(t) |2\rangle \langle 2|)\hat{I}_b + \hat{I}_a(E_{p1}(t) |1\rangle \langle 1'| + E_{p2}(t) |2\rangle \langle 2'|) + \frac{q^2}{d_{11'}} |1, 1\rangle \langle 1, 1'| + \frac{q^2}{d_{22'}} |2, 2\rangle \langle 2, 2'| + \frac{q^2}{d_{12'}} |1, 2\rangle \langle 1, 2'| + \frac{q^2}{d_{12'}} |2, 1\rangle \langle 2, 1'| \quad (84)
\]

described by parameters \( E_{p1}(t), E_{p2}(t), E_{p1}(t), E_{p2}(t), t_{s12}(t), t_{s12}'(t) \) and distances between nodes k and k1:

\( d_{11'}, d_{22'}, d_{22'}, d_{12'}, d_{12'} \). In such case q-state of the system is given as

\[
|\psi, t\rangle = \gamma_{1,t} |1\rangle_U |1\rangle_U |1\rangle_U + 2\gamma_{2,t} |1\rangle_U |0\rangle_U |0\rangle_L + \gamma_{3,t} |0\rangle_U |1\rangle_U |0\rangle_L + 4\gamma_{4,t} |0\rangle_U |1\rangle_U |0\rangle_L , \quad (85)
\]

where normalization condition gives \( |\gamma_{1,t}|^2 + \ldots + |\gamma_{4,t}|^2 \). Probability of finding electron in upper system at node 1 is by action of projector \( \hat{P}_{U} = |1\rangle_U |1\rangle_U |1\rangle_U + |0\rangle_U |0\rangle_L |0\rangle_L \) on q-state \( \hat{P}_{U} |\psi\rangle \) so it gives probability amplitude \( |\gamma_{1,t}|^2 + |\gamma_{3,t}|^2 \).

On another hand probability of finding electron from qubit A (U) at node 2 and electron from qubit B(L) at node 1 is obtained by projection \( \hat{P}_{U,L} = |0\rangle_U |1\rangle_L + |1\rangle_U |0\rangle_L \) acting on q-state giving \( \langle 0\rangle_U |1\rangle_L |\psi\rangle \) that gives probability amplitude \( |\gamma_{2,t}|^2 \). Referring to picture from Fig.1 we set distances between nodes as

\( d_{11'} = d_{22'} = d_{12'}, d_{22'} = \sqrt{(a + b)^2 + d_{12'}^2} \) and assume Coulomb electrostatic energy to be of the form

\( E_c(k, l) = \frac{q^2}{d_{12'}} \) and hence we obtain the matrix Hamiltonian given as

\[
\hat{H}(t) = \begin{pmatrix}
E_{p1}(t) + E_{p1}(t) + \frac{q^2}{d_{11'}} & t_{s12}(t) & t_{s12}'(t) & 0 & 0 \\
t_{s12}(t)^* & E_{p1}(t) + E_{p2}(t) + \frac{q^2}{(d_{11'})^2(a + b)^2} & 0 & t_{s12}(t) & 0 \\
t_{s12}'(t)^* & t_{s12}'(t) & 0 & E_{p2}(t) + E_{p1}(t) + \frac{q^2}{(d_{11'})^2(a + b)^2} & t_{s12}(t) \\
0 & 0 & 0 & t_{s12}'(t)^* & E_{p2}(t) + E_{p2}(t) + \frac{q^2}{d_{12'}} \\
\end{pmatrix}
\]

We can introduce notation \( E_{c1} = \frac{q^2}{d_{12'}} \) and \( E_{c2} = \frac{q^2}{d_{12'}} \). In most general case of 2 qubit electrostatic interaction one has 4 different Coulomb terms on matrix diagonal \( E_{c1} = \frac{q^2}{d_{12'}}, E_{c2} = \frac{q^2}{d_{12'}}, E_{c3} = \frac{q^2}{d_{12'}} \) and \( E_{c4} = \frac{q^2}{d_{22'}} \). In case of 2 qubit interacting electrostatically in any geometrical configuration all eigenenergies and eigenstates can be determined analytically since one can establish matrix 4 by 4 eigenvalues and eigenvectors. Two electrostatically coupled qubits are entangled if hopping elements in each qubit are non-zero. It is worth mentioning that
completely localized states in each of the qubit are not entangled. More details on the case of 2 interaction Single Electron Lines can be found in work of Pomorski et al. \[13\] where 2 position-based qudits are electrostatically interacting. Coming back to the electrostatic interaction of position based qubits we can express the quantum state at any by operator $e^{\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t')dt'} = \hat{U}(t, t_0) =$

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

$|\psi, t\rangle = \hat{U}(t, t_0) |\psi, t_0\rangle$. \[86\]

The example of function of eigenenergy spectra of 2 electrostatically interacting qudits on distance is given by Fig[6].

Importing observation is that any element of matrix $\hat{H}(t')$ for $t' \in (t_0, t)$ denoted as $H_{k,l}(t')$ is transferred to element $\hat{U}_{k,l}(t, t_0) = e^{\frac{i}{\hbar} \int_{t_0}^{t} dt' (H_{k,l}(t'))}$ of matrix $\hat{U}(t, t_0)$. In particular all zeros of matrix $H_{k,l}(t')$ are are 1 values in $\hat{U}(t, t_0)$. We can easily generalize the presented reasoning for the system of N electrostatically coupled electrons confined by some local potentials. However we need to know the position dependent Hamiltonian eigenstate at the initial time $t_0$. In case $N > 2$ finding such eigenstate is the numerical problem since analytical solutions for roots of polynomials of one variable for higher order than 4 does not exist. Using numerical eigenstate at time instance $t_0$ we can compute the system quantum dynamics in analytical way. This give us the strong and relatively simple mathematical tool giving full determination of quantum dynamical state at the any instance of time. The act of measurement on position based qubit is represented by the operator $P_{Left} = \{1, 0\}_E_1, E_2 \{1, 0\}_E_1, E_2$, and $P_{Right} = \{0, 1\}_E_1, E_2 \{0, 1\}_E_1, E_2$.

### 6.1 Simplified picture of symmetric Q-Swap gate

Now we need to find system 4 eigenvalues and eigenstates(4 orthogonal 4-dimensional vectors) so we are dealing with matrix eigenvalue problem which is the subject of classical algebra. Let us assume that 2 double quantum dot systems are symmetric and biased by the same voltages generating potential bottoms $V_s$ so we have $E_{p1} = E_{p2} = E_{p1'} = E_{p2'} = E_p = V_s$ and that $t_{s12} = t_{s12'} = t_s$. Denoting $E_c(1, 1') = E_c(2, 2') = E_{c1}$ and $E_c(2, 1') = E_c(2, 1') = E_{c2}$ we are obtaining 4 orthogonal Hamiltonian eigenvectors

\[
|E_1\rangle = \begin{pmatrix}
-1 \\
0 \\
0 \\
+1
\end{pmatrix} = -|1, 0\rangle_U |1, 0\rangle_L + |0, 1\rangle_U |0, 1\rangle_L \neq (a_1 |1, 0\rangle_U + a_2 |0, 1\rangle_U)(a_3 |1, 0\rangle_U + a_4 |0, 1\rangle_U), \tag{87}
\]
\[ |E_2 \rangle = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = -|1,0 \rangle_U |0,1 \rangle_L + |0,1 \rangle_U |1,0 \rangle_L \neq (a_1 |1,0 \rangle_U + a_2 |0,1 \rangle_U)(a_3 |1,0 \rangle_U + a_4 |0,1 \rangle_U), \]  
(88)

\[ |E_{3(4)} \rangle = \begin{pmatrix} \frac{1}{4t_s} \pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} \\ \frac{1}{4t_s} \pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} \\ \frac{1}{4t_s} \pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} \\ 1 \end{pmatrix} = |1,0 \rangle_U |1,0 \rangle_L + |0,1 \rangle_U |0,1 \rangle_L + c(|1,0 \rangle_U |0,1 \rangle_L + |0,1 \rangle_U |1,0 \rangle_L) = \\
= |(1,0) \rangle_U + |0,1 \rangle_U)(|1,0 \rangle_L + |0,1 \rangle_L) + (c - 1)(|1,0 \rangle_U |0,1 \rangle_L + |0,1 \rangle_U |1,0 \rangle_L) \\
\neq (a_1 |1,0 \rangle_U + a_2 |0,1 \rangle_U)(a_3 |1,0 \rangle_U + a_4 |0,1 \rangle_U), \]  
(89)

where \(c = \frac{4t_s}{\pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2}} \). First two \( |E_1 \rangle \) and \( |E_2 \rangle \) energy eigenstates are always entangled, while \( |E_3 \rangle \) and \( |E_4 \rangle \) eigenenergies are only partially entangled if \( \pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} \neq 1 \). If \( c = 1 = \frac{4t_s}{\pm (E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2}} \) last two energy eigenstates are not entangled. Situation of \( c = 1 \) takes place when \( E_{c1} = E_{c2} \) so when two qubits are infinitely far away so when they are electrostatically decoupled. Situation of \( c = 0 \) is interesting because it means that \( |E_3 \rangle \) and \( |E_4 \rangle \) are maximally entangled and it occurs when \( t_s = 0 \) so when two electrons are maximally localized in each of the qubit so there is no hopping between left and right well.

The obtained eigenenergy states correspond to 4 eigenenergies

\[ E_1 = E_{c1} + 2V_s, E_2 = E_{c2} + 2V_s, E_1 > E_2 \]

\[ E_3 = \frac{1}{2}((E_{c1} + E_{c2}) - \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} + 4V_s) = \\
\frac{1}{2}((q^2 \frac{1}{d_1} + \frac{1}{\sqrt{d_1^2 + (a + b)^2}}) - \sqrt{(q^2 \frac{1}{d_1} - \frac{1}{\sqrt{d_1^2 + (a + b)^2}})^2 + 16t_s^2} + 4V_s), \]

\[ E_4 = \frac{1}{2}((E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2} + 4V_s) = \\
\frac{1}{2}((q^2 \frac{1}{d_1} + \frac{1}{\sqrt{d_1^2 + (a + b)^2}}) + \sqrt{(q^2 \frac{1}{d_1} - \frac{1}{\sqrt{d_1^2 + (a + b)^2}})^2 + 16t_s^2} + 4V_s), E_4 > E_3. \]

We also notice that the eigenenergy states \( |E_1 \rangle, |E_2 \rangle, |E_3 \rangle, |E_4 \rangle \) do not have its classical counterpart since upper electron exists at both positions 1 and 2 and lower electron exists at both positions at the same time. We observe that when distance between two systems of double quantum dots goes into infinity the energy difference between quantum state corresponding to \( |E_3 \rangle \) and \( |E_4 \rangle \) goes to zero. This makes those two entangled states to be degenerated.

Normalized 4 eigenvectors of 2 interacting qubits in SWAP Q-Gate configuration are of the following form

\[ |E_1 \rangle_n = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |E_2 \rangle_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \]

\[ |E_3 \rangle_n = \sqrt{\frac{4t_s}{(E_{c2} - E_{c1}) + 8t_s - \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2}}} \begin{pmatrix} 1 \\ -1 \\ -\frac{4t_s}{(E_{c1} + E_{c2}) + \sqrt{(E_{c1} - E_{c2})^2 + 16t_s^2}} \end{pmatrix}, \]

\[ (90) \]
to cool down (or heat-up) the system of interacting qubits to the energy $E_1$ (or to energy $E_2$). Otherwise we might also wish to disentangle two electrostatically interacting qubits. In such way one of the scenario is to bring the quantum system either to energy $E_3$ or $E_4$ so only partial entanglement will be achieved. Other scenario would be by bringing the occupancy of different energetic levels so net entanglement is reduced. One can use the entanglement witness in quantifying the existence of entanglement. One of the simplest q-state entanglement measurement is von Neumann entanglement entropy as it is expressed by formula 105 that requires the knowledge of q-system density matrix with time. Such matrix can be obtained analytically for the case of 2 electrostatically interacting qubits.

It is interesting to spot the dependence of eigenenergies on distance between interacting qubits in the general case as it is depicted in Fig.6. Now we are moving towards description the procedure of cooling down or heating up in Q-Swap gate. The procedure was discussed previously in the case of single qubit. Now it is exercised in the case of 2-qubit electrostatic interaction. For the sake of simplicity we will change the occupancy of the energy levels to $E_1$ and $E_2$ and keep the occupancy of other energy levels unchanged. We can write the $|E_2\rangle \langle E_1|$ as

$$|E_2\rangle_n \langle E_1|_n = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We are introducing $f_1$ and $f_2$ real valued functions of small magnitude $f(t) = f_1(t) = f_2(t), (|f_1|, |f_2| << \langle E_1, E_2\rangle)$ and we are considering the following Hamiltonian having $\hat{H}_0$ that is time-independent and other part dependent part as

$$\hat{H} = \hat{H}_0 + f_1(t) |E_2\rangle_n \langle E_1|_n + f_2(t) |E_1\rangle_2 \langle E_2|_n = E_1 |E_1\rangle \langle E_1| + E_2 |E_2\rangle \langle E_2| + f_1(t) |E_2\rangle_2 \langle E_2|_n + f_2(t) |E_1\rangle_2 \langle E_2|_n =$$

$$= \begin{pmatrix} 2E_p + \frac{q^2}{4t^2} & t_s + \frac{q^2}{\sqrt{(d_1)^2 + (b+a)^2}} & t_s & 0 \\ \frac{q^2}{\sqrt{(d_1)^2 + (b+a)^2}} & 0 & 2E_p + \frac{q^2}{\sqrt{(d_1)^2 + (b+a)^2}} & t_s \\ t_s & 0 & t_s & 2E_p + \frac{q^2}{4t^2} \\ 0 & t_s & 0 & \frac{q^2}{\sqrt{(d_1)^2 + (b+a)^2}} \end{pmatrix} +$$

$$+ \frac{1}{2} \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} + f_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 2E_p + \frac{q^2}{4t^2} & t_s + \frac{1}{2} f(t) & t_s - \frac{1}{2} f(t) & 0 \\ \frac{1}{2} f(t) & 0 & t_s & \frac{1}{2} f(t) \\ t_s - \frac{1}{2} f(t) & 0 & 2E_p + \frac{q^2}{\sqrt{(d_1)^2 + (b+a)^2}} & t_s + \frac{1}{2} f(t) \\ 0 & t_s + \frac{1}{2} f(t) & 2E_p + \frac{q^2}{4t^2} & 0 \end{pmatrix} = \hat{H}(t)|_{E_1 \rightarrow E_2, Q-Swap}.$$
Initially we have established the following parameters of tight-binding model as \( t_{s12} = t_{s12}' \). Changing \( t_{s12} \)
into \( t_{s12} - \frac{t(t)}{2} \) and \( t_{s12}' \) into \( t_{s12}' + \frac{t(t)}{2} \) while keeping other parameters of tight-binding model unchanged will result in the heating up (cooling down) of q-state of SWAP gate so population of energy level \( E_1 \) and \( E_2 \) are time-depenent, while populations of energy levels \( E_3 \) and \( E_4 \) are unchanged. Practically our results mean that we need to keep all our confiment potential bottoms constant, while changing barrier height between neighbouring q-dots in each of position based qubits. In such way we have established the procedure of perturbative cooling (heating up) of q-state. Non-perturbative approach is absolutely possible but it requires full knowledge of time dependent eigenstates and eigenenergies (solutions of eigenenergies of 4th order polynomial are very lengthy in general case) and therefore corresponding expression are very lengthy. In similar fashion we can heat up or cool down two coupled Single Electron Lines as in Fig.1 or any other q-system having \( N \) interacting q-bodies that can be represented by the system of \( N \)-interacting position based qubits.

### 7. ANALYTIC EXTENSIONS OF TOPOLOGY OF CHAIN OF COUPLED QUANTUM DOTS

Since we have electrostatic control of interaction between quantum dots we can turn on coupling between two chains of quantum dots as it is depicted in Fig.7. where Coulomb electrostatic interaction occurs between \( m \) and \( n' \) node of two separated chain and is given by \( E_c(m,n') = f(m,n') = \frac{q^2}{\delta_{m,n'}} \). The quantum state of right-system is given as

\[
|\psi\rangle = \gamma_{1,1'}(t) |1\rangle |1'\rangle + \gamma_{1,2'}(t) |1\rangle |2'\rangle + \gamma_{1,3'}(t) |1\rangle |3'\rangle + \gamma_{1,4'}(t) |1\rangle |4'\rangle \\
+ \gamma_{2,1'}(t) |2\rangle |1'\rangle + \gamma_{2,2'}(t) |2\rangle |2'\rangle + \gamma_{2,3'}(t) |2\rangle |3'\rangle + \gamma_{2,4'}(t) |2\rangle |4'\rangle \\
+ \gamma_{3,1'}(t) |3\rangle |1'\rangle + \gamma_{3,2'}(t) |3\rangle |2'\rangle + \gamma_{3,3'}(t) |3\rangle |3'\rangle + \gamma_{3,4'}(t) |3\rangle |4'\rangle .
\]

(96)

where \( \sum_{k,l'} |\gamma_{k,l'}|^2 = 1 \). After extension by 2 elements the quantum state of left system is given as

\[
|\psi\rangle = \gamma_{1,1'}(t) |1\rangle |1'\rangle + \gamma_{1,2'}(t) |1\rangle |2'\rangle + \gamma_{1,3'}(t) |1\rangle |3'\rangle + \gamma_{1,4'}(t) |1\rangle |4'\rangle \\
+ \gamma_{2,1'}(t) |2\rangle |1'\rangle + \gamma_{2,2'}(t) |2\rangle |2'\rangle + \gamma_{2,3'}(t) |2\rangle |3'\rangle + \gamma_{2,4'}(t) |2\rangle |4'\rangle \\
+ \gamma_{3,1'}(t) |3\rangle |1'\rangle + \gamma_{3,2'}(t) |3\rangle |2'\rangle + \gamma_{3,3'}(t) |3\rangle |3'\rangle + \gamma_{3,4'}(t) |3\rangle |4'\rangle \\
+ \gamma_{4,1'}(t) |4\rangle |1'\rangle + \gamma_{4,2'}(t) |4\rangle |2'\rangle + \gamma_{4,3'}(t) |4\rangle |3'\rangle + \gamma_{4,4'}(t) |4\rangle |4'\rangle .
\]

(97)

where again \( \sum_{s,w'} |\gamma_{s,w'}|^2 = 1 \).

The Hamiltonian of the system before extension is

\[
\hat{H} = \begin{pmatrix}
H_1 & H_2 \\
H_3 & H_4
\end{pmatrix}
\]

(98)

and after extension into system depicted in Fig.7 (left side) is

\[
\hat{H}_{ext}(t) = \begin{pmatrix}
H_1 & H_2 & H_{e1} \\
H_3 & H_4 & H_{e2} \\
H_{e5} & H_{e4} & H_{e3}
\end{pmatrix}
\]

(99)

with matrix subcomponents \( \hat{H}_{1}(t) = \)

\[
\begin{pmatrix}
E_{p1} + E_{p1'} + \frac{q^2}{\delta_{1,1'}} & t_{s21,1'} & 0 & 0 & 0 & 0 & 0 & 0 \\
t_{s21,1'} & E_{p1} + E_{p2'} + \frac{q^2}{\delta_{1,2'}} & t_{s21,2'} & 0 & 0 & 0 & 0 & 0 \\
0 & t_{s21,2'} & E_{p1} + E_{p3'} + \frac{q^2}{\delta_{1,3'}} & t_{s21,3'} & 0 & 0 & 0 & 0 \\
0 & 0 & t_{s21,3'} & E_{p1} + E_{p4'} + \frac{q^2}{\delta_{1,4'}} & t_{s21,4'} & 0 & 0 & 0 \\
0 & 0 & 0 & t_{s21,4'} & E_{p1} + E_{p5'} + \frac{q^2}{\delta_{1,5'}} & t_{s21,5'} & 0 & 0 \\
0 & 0 & 0 & 0 & t_{s21,5'} & E_{p1} + E_{p6'} + \frac{q^2}{\delta_{1,6'}} & t_{s21,6'} & 0 \\
0 & 0 & 0 & 0 & 0 & t_{s21,6'} & E_{p1} + E_{p7'} + \frac{q^2}{\delta_{1,7'}} & t_{s21,7'} \\
0 & 0 & 0 & 0 & 0 & 0 & t_{s21,7'} & E_{p1} + E_{p8'} + \frac{q^2}{\delta_{1,8'}}
\end{pmatrix}
\]
\[ \hat{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{s2,3} & 0 \\ t_{s2',3'} & 0 & 0 & 0 & t_{s2,3} \end{pmatrix}, \]

\[ \hat{H}_3 = \begin{pmatrix} 0 & t_{s2,1} & 0 & 0 & t_{s3',2'} \\ 0 & 0 & t_{s2,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{s3,2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \hat{H}_4(t) = \begin{pmatrix} E_{p3}(t) + E_{p3'}(t) + \frac{t_{s2,1}^2}{2,3'} & t_{s3',4'}(t) & 0 & 0 & 0 \\ t_{s4',3'}(t) & E_{p3}(t) + E_{p3'}(t) + \frac{t_{s2,1}^2}{2,3'} & 0 & 0 & 0 \\ 0 & 0 & E_{p3}(t) + E_{p3'}(t) + \frac{t_{s2,1}^2}{2,3'} & t_{s3',4'}(t) & 0 \\ 0 & 0 & t_{s2',1'}(t) & E_{p3}(t) + E_{p3'}(t) + \frac{t_{s2,1}^2}{2,3'} & t_{s3',4'}(t) \\ 0 & 0 & 0 & t_{s2',1'}(t) & E_{p3}(t) + E_{p3'}(t) + \frac{t_{s2,1}^2}{2,3'} \end{pmatrix} \]

We can determine inductive step of quantum dot graph extension by adding matrices \( \hat{H}_{e1}, \ldots, \hat{H}_{e5} \) to the formula [99] in the form as given

\[ \hat{H}_{e1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & t_{s2',5'} & 0 & 0 & 0 \end{pmatrix}, = \hat{H}_{e5}, \]

\[ \hat{H}_{e2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, = \hat{H}_{e4}, \]

\[ \hat{H}_{e3} = \]

\[ \left( \begin{array}{cccccc} E_{p1} + E_{p5'} + \frac{t_{s1,2}^2}{3,5'} & t_{s1,2} & 0 & 0 & 0 \\ t_{s2,1} & E_{p2} + E_{p5'} + \frac{t_{s2,3}^2}{3,5'} & t_{s2,3} & 0 & 0 \\ 0 & t_{s3,2} & E_{p3} + E_{p5'} + \frac{t_{s2,3}^2}{3,5'} & 0 & 0 \\ t_{s4',5'} & 0 & 0 & E_{p1} + E_{p6'} + \frac{t_{s2,1}^2}{3,6'} & t_{s1,2} \\ 0 & 0 & t_{s6',5'} & 0 & t_{s2,1} \\ 0 & 0 & t_{s6',5'} & 0 & t_{s2,1} \end{array} \right) \]

Similarly as before having knowledge of quantum state at \( t_0 \) we can evaluate the state at time \( t \) by computing \( \exp\left(\int_{t_0}^{t} -\frac{1}{\hbar} \hat{H}_{ext}(t)dt'\right) |\psi, t_0\rangle = |\psi, t\rangle \) what bases on the same method already presented before in Equation [8]. We can also perform the procedure of heating up or cooling down of the quantum state in the way as it was described before.
8. ELECTROSTATIC INTERACTION OF JOSEPHSON JUNCTION QUBIT WITH SEMICONDUCTOR ELECTROSTATIC QUBIT

The state of Josephson junction is well described by Bogoliubov-de Gennes (BdG) equation\(^{12}\) pointing the correlation between electron and holes as

\[
\begin{pmatrix}
H_0 & \Delta(x) \\
\Delta^*(x) & -H_0
\end{pmatrix}
\begin{pmatrix}
u_n(x) \\
v_n(x)
\end{pmatrix}
= E_n
\begin{pmatrix}
u_n(x) \\
v_n(x)
\end{pmatrix},
\]

where \(H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\) is free electron Hamiltonian with self-consistency relation \(\Delta(x) = \sum_n (1 - 2f(E_n)) u_n(x) v_n^*(x)\), where \(\Delta(x)\) is the superconducting order parameter and \(f(E_n) = \frac{1}{1 + e^{\frac{E_n - \gamma}{T}}}\) is Fermi-Dirac distribution function and \(u_n(x)\) and \(v_n(x)\) are electron and hole wavefunctions. In case of bulk superconductor with constant superconducting order parameter we obtain \(E_n = \pm \sqrt{|H_0|^2 + |\Delta|^2}\). In later considerations we are going to omit the self-consistency relation assuming the dependence of superconducting order parameter as step-like function. It shall be underlined that BdG equation is mean field equation that is derived basing on BCS theory of superconductivity. It it thus naturally valid for the case of many particles. Semiconductor single electron line with 2 nodes can be regarded as electrostatic position dependent qubit and can be described by

\[
H_{\text{semi}} = t_{s1,2} |1 \rangle \langle 2| + t_{s2,1} |2 \rangle \langle 1| + E_{p1} |1 \rangle \langle 1| + E_{p2} |2 \rangle \langle 2|,
\]

We refer to the physical situation depicted in Fig.8. We can express coupling of 2 systems assuming 4 nodes for electron or hole and 2 nodes for electron confined in semiconductor so we have eigenvector having 16 components \((|0\rangle_e |1\rangle_s, |0\rangle_e |2\rangle_s, |1\rangle_e |1\rangle_s, |1\rangle_e |2\rangle_s, |2\rangle_e |1\rangle_s, |2\rangle_e |2\rangle_s, |3\rangle_e |1\rangle_s, |2\rangle_e |3\rangle_s, |1\rangle_h |2\rangle_s, |2\rangle_h |1\rangle_s, |2\rangle_h |2\rangle_s, |3\rangle_h |1\rangle_s, |3\rangle_h |2\rangle_s, |3\rangle_h |3\rangle_s)\) where \(s\) refers to semiconductor qubit whose quantum state is superposition of \(|1\rangle_s\) and \(|2\rangle_s\) and states \(|0\rangle_e, \ldots, |3\rangle_e\), \(|0\rangle_h, \ldots, |3\rangle_h\) characterizes the state of electron and hole respectively in ABS [Andreev Bound State when electron moving in normal (non-superconducting) region between superconductors is reflected as hole when it comes into superconducting area and when hole moving in normal region is reflected as electron when it meets superconductor etc..] of Josephson junction. This time the quantum state of the system can be written as

\[
|\psi, t\rangle = \gamma_1(t) |0\rangle_e |1\rangle_s + \gamma_2(t) |0\rangle_e |2\rangle_s + \gamma_3(t) |1\rangle_e |1\rangle_s + \gamma_4(t) |1\rangle_e |2\rangle_s + \gamma_5(t) |2\rangle_e |1\rangle_s + \gamma_6(t) |2\rangle_e |2\rangle_s + \\
+ \gamma_7(t) |2\rangle_e |1\rangle_s + \gamma_8(t) |2\rangle_e |2\rangle_s + \gamma_9(t) |0\rangle_h |1\rangle_s + \gamma_{10}(t) |0\rangle_h |2\rangle_s + \gamma_{11}(t) |1\rangle_h |1\rangle_s + \\
+ \gamma_{12}(t) |1\rangle_h |2\rangle_s + \gamma_{13}(t) |2\rangle_h |1\rangle_s + \gamma_{14}(t) |2\rangle_h |2\rangle_s + \gamma_{15}(t) |2\rangle_e |1\rangle_s + \gamma_{16}(t) |2\rangle_e |2\rangle_s.
\]

Normalization condition implies \(|\gamma_1(t)|^2 + |\gamma_2(t)|^2 + \ldots + |\gamma_{16}(t)|^2 = 1\) at any instance of time \(t\). Such system has 16 eigenenergies. The probability of find electron at node 1 under any presence of electron in semiconductor qubit at node 1 or 2 is obtained by applying projection of \(|1\rangle_e |1\rangle_s + |1\rangle_e |2\rangle_s\) so \(|\langle 1|_e \langle 1|_s + \langle 1|_e \langle 2|_s |\psi, t\rangle|^2\) is probability of finding electron at node 1 in Josephson junction. We obtain the following structures of matrices
corresponding to $H_0$ part of BdGe equation in the forma as $\hat{H}_{0[e]} =$

$$
\begin{pmatrix}
E_{p1} + E_{e0} & t_s & te_{(1,0)} & 0 & t_{e(2,0)} & 0 & t_{e(3,0)} & 0 \\
t_s^* & E_{p2} + E_{e0} & 0 & t_{e(1,0)} & 0 & t_{e(2,0)} & 0 & t_{e(3,0)} \\
te_{(1,0)} & 0 & E_{p1} + \frac{q^2}{a} + E_{e1} & t_s & t_{e(2,1)} & 0 & t_{e(3,1)} & 0 \\
0 & t_{e(1,0)}^* & t_s^* & E_{p2} + E_{e1} + \frac{q^2}{b} & 0 & t_{e(2,1)} & 0 & t_{e(3,1)} \\
te_{(2,0)} & 0 & t_{e(2,1)}^* & 0 & E_{p2} + E_{e2} + \frac{q^2}{b} & t_s & t_{e(3,2)} & 0 \\
0 & t_{e(2,0)}^* & 0 & t_{e(2,1)}^* & E_{p2} + E_{e2} + \frac{q^2}{a} & 0 & t_{e(3,2)} & t_s \\
te_{(3,0)} & 0 & t_{e(3,1)}^* & 0 & E_{p1} + E_{3e} & t_s^* & t_s & E_{p2} + E_{3e} \\
0 & t_{e(3,0)}^* & 0 & t_{e(3,1)}^* & 0 & t_{e(3,2)} & t_s^* & 0 \\
\end{pmatrix}
$$

Parameters $E_{p1}$, $E_{p2}$, $t_s$ correspond to semiconductor position based qubit and distance between semiconductor qubit and Josephson junction is given by a and b. Other parameters $E_{e0}, E_{e1}, E_{e2}, E_{e3}$, $E_{h0}, E_{h1}, E_{h2}, E_{h3}$ describes localization energy of electron and hole at nodes 0, 1, 2 and 3 of Josephson junction. In analogous way we can write $H_{0[k]} =$

$$
\begin{pmatrix}
E_{p1} + E_{h0} & t_s & th_{(1,0)} & 0 & th_{(2,0)} & 0 & th_{(3,0)} & 0 \\
t_s^* & E_{p2} + E_{h0} & 0 & th_{(1,0)} & 0 & th_{(2,0)} & 0 & th_{(3,0)} \\
th_{(1,0)} & 0 & E_{p1} - \frac{q^2}{a} + E_{h1} & th_{(1,0)} & 0 & th_{(2,1)} & 0 & th_{(3,1)} \\
0 & th_{(1,0)}^* & th_{(2,1)}^* & E_{p2} + E_{h1} - \frac{q^2}{b} & 0 & th_{(2,1)} & 0 & th_{(3,1)} \\
th_{(2,0)} & 0 & th_{(2,1)}^* & 0 & E_{p1} + E_{h2} - \frac{q^2}{b} & th_{(2,1)} & 0 & th_{(3,2)} \\
0 & th_{(2,0)}^* & 0 & th_{(2,1)}^* & E_{p2} + E_{h2} - \frac{q^2}{a} & th_{(2,2)} & 0 & th_{(3,2)} \\
th_{(3,0)} & 0 & th_{(3,1)}^* & 0 & E_{p1} + E_{3h} & th_{(3,2)} & th_{(3,2)} & 0 \\
0 & th_{(3,0)}^* & 0 & th_{(3,1)}^* & 0 & th_{(3,2)} & th_{(3,2)} & 0 \\
\end{pmatrix}
$$
and two other matrices

\[
\hat{\Delta}_1 = \\
\begin{pmatrix}
\Delta(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Delta(0) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Delta(1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta(1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta(1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta(2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta(2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta(3)
\end{pmatrix}
, \quad \hat{\Delta}_2 = \hat{\Delta}_1^\dagger. 
\] 

(103)

Finally we obtain the following structure of tight-binding Bogoliubov-de Gennes equations including the interaction of semiconductor qubit with Josephson junction described in the minimalistic way in the form

\[
\hat{H}_{\text{eff}} = \begin{pmatrix}
\hat{H}_{0[\text{c}]} \\
\hat{\Delta}_1 \\
\hat{H}_{0[\text{h}]}
\end{pmatrix}.
\] 

(104)

Similarly as before having knowledge of quantum state at \( t_0 \) we can evaluate the state at time \( t \) by computing \( \exp(\int_{t_0}^t \frac{1}{\hbar} \hat{H}_{\text{ext}}(t) dt') \langle \psi(t) | \psi(t_0) \rangle \) what bases on the same method already presented before in Equation \([\text{8}]\). We can also perform the procedure of heating up or cooling down of the quantum state in the way as it was described before or we can regulate the population of pointed energetic level(s). In most minimalistic tight-binding model of Josephson junction Sc-I-Sc (Superconductor-Insulator-Superconductor) we set \( \Delta(1) = \Delta(2) = 0 \) what corresponds to the simplest form of Andreev Bound State in Tunneling Josephson junction. However in weak-links and in the Field Induced Josephson junctions all diagonal elements are non-zero and \( |\Delta| \) has maximum at \( \Delta(0) \) and \( \Delta(3) \) that can be considered as superconducting state of bulk superconductors. Quite naturally Field Induced Josephson junction\([12]\) can have special profile of dependence of superconducting order parameter \( \Delta(x) \) on position \( x \) with presence of built-in magnetic fields in area of junction. It will also have special complex valued hopping constants for electron and hole in area of superconductor that will incorporate the profile of magnetic field present across Josephson junction. Specified Hamiltonian describing electrostatic interface between superconducting Josephson junction and semiconductor position-based qubit has the following parameters describing the state of position based semiconductor qubit \( E_{p1}, E_{p2}, t_s = t_{sr} + it_{is} \) (4 real valued time dependent functions), and parameters describing the state of Josephson junction \( E_{e0}, E_{e1}, E_{e2}, E_{e3}, E_{h0}, E_{h1}, E_{h2}, E_{h3}, \Delta(0), \Delta(1), \Delta(2), \Delta(3), t_{c(1,0)}, t_{c(2,1)}, t_{c(2,3)}, t_{e(3,0)}, t_{h(3,0)}, t_{h(2,1)}, t_{h(2,3)}, t_{h(3,0)} \) as well as geometrical parameters describing electrostatic interaction between semiconductor JJ and semiconductor qubit by a and b. It is worth mentioning that electrostatic interaction taken into account is only between nodes 1-1s, 1-2s, 2-1s, 2-2s what means 4 channels for Coulomb interaction and simplifies the model greatly so one can find analytical solutions as well. The assumption with 4 channels of electrostatic interaction is physically justifiable if one assumes that \( \Delta(0) \neq 0, \Delta(3) \neq 0 \) and \( (\Delta(1), \Delta(2)) \rightarrow 0 \). Therefore formally we have omitted the following channels of electrostatic interaction \( 0-1s, 3-1s, 0-2s, 3-2s \). It is commonly known that superconducting state especially with strong superconductivity as in case of bulk superconductor is not supporting and shielding itself from the external and internal electrostatic field of certain strength as it naturally protects its ground superconducting macroscopic state. Having established the mathematical structure describing the electrostatic interaction between semiconductor position based qubit and Josephson junction we can move into first analytical and numerical calculations. First simplification is that \( \Delta(1) = \Delta(2) = 0 \) and \( \Delta = \Delta(0) = \Delta(3) \in R \) so it means that there is no net electric current flowing via Josephson junction since electric current flow imposes the condition of phase difference among superconducting order parameter \( \Delta(0) \) and \( \Delta(3) \) and in such case superconducting order parameter is complex valued scalar. Also it implies that there is no magnetic field in our system since magnetic field brings phase imprint between \( \Delta(0) \) and \( \Delta(3) \). Second simplification is that \( E_{p1} = E_{p2} = E_p, t_s \in R \). Third simplification is that \( E_{c0} = E_{c1} = E_{c2} = E_{c3} = -E_{h0} = -E_{h1} = -E_{h2} = -E_{h3} = V \) so it implies electron-hole symmetry in area of ABS that is the middle of Josephson junction. In such way all hole eigenenergies are corresponding to electron eigenenergies with - sign. Last assumption is that electron or hole hopping in the area of ABS in between nearest neighbours is such that \( t_{c(k,k+1)} \neq 0 \) and \( t_{h(k,k+1)} \neq 0 \) and is 0 otherwise. One can name such feature of transport in Josephson junction as diffusive and not ballistic what brings the mathematical simplifications. Having established such facts we can move into analytical and numerical
calculations. The Hamiltonian of physical system has such structure that allows analytic determination of all eigenenergies since Hamiltonian matrix has many symmetries. In particular we can obtain the spectrum of eigenenergies in dependence on the distance \( a \) as depicted in Fig. 9 and spectrum of eigenenergies in dependence of superconducting order parameter as given in Fig. 10. One can recognize certain similarities with Fig. 2. It simply means that increase of superconducting order parameter strenght brings similar effect as increase of distance between interaction of semiconductor position based qubit and Josephson junction. One of the most interesting feature is tuning the landscape of eigenenergies by applying small voltage (below the size \( 2e\Delta \)) to non-superconducting region of Josephson junction. In such case one obtains the features as described in Fig. 11. In the described considerations the spin degree of freedom was omitted in case of Josephson junction as well as in case of semiconductor position based qubit. However they could be easily included but it would increase the size of matrix describing interaction between superconductor Josephson junction and semiconductor electrostatic qubit from 16 by 16 to the size 8*4=32 so one obtains matrix 32 by 32. Adding strong spin-orbit interaction to the Hamiltonian of Josephson junction under the presence of magnetic field allows to describe topological Josephson junction. In such way we can obtain the effective 32 by 32 Hamiltonian for interaction between semiconductor position based qubit and topological Josephson junction in minimalistic way. It shall be also underlined that so far we have used BdGe formalism that is suitable for mean field theory domain. However in our case we have considered very special interactions between individual (electrons, holes) present in area of Josephson junction and specific individual electron present in area of semiconductor qubit. Usage of BdGe formalism is therefore first level of possible approximation and further more detailed study can be attempted in determination of microscopic processes present interacting Josephson junction with semiconductor qubit in more detailed way. It is sufficient to mention that in our case superconductors shall have relatively small size so we are dealing with relatively small number of electrons and holes in non-superconducting area. More detailed considerations are however beyond the scope of this work and requires Density Functional Theory (DFT) methods, etc.

9. CONCLUSIONS

The obtained results have its meaning in development of single electron electrostatic quantum neural networks, quantum gates as CNOT, SWAP, Toffoli and Fredkin gates as well as any other types of quantum gates with \( N \) inputs and \( M \) outputs. Single electron semiconductor devices can be attractive from point of view of power consumption and they can approach similar performance as Rapid Single Quantum Flux superconducting circuits having much smaller dimensions than superconducting circuits. In conducted computations spin-degree of freedom was neglected. However it can be added in straightforward way doubling the size of Hilbert space. The obtained results allow us to obtain the entanglement of qubit \( A \) (for example) using biparticle von Neumann entropy \( S(t)_{A} \) of qubit \( A \) in two electrostatically interacting qubits with time as given by formula

\[
S(t) = -Tr[\rho(t)\log(\rho(t))],
\]

where \( Tr[.] \) is matrix trace operator and \( \hat{\rho}_{A} = Tr_{B}[\hat{\rho}] \) is the reduced density matrix of \( A \) qubit after presence of \( B \) qubit was traced out with partial trace \( Tr_{B}[.] \). The obtained results can be mapped to Schroedinger formalism.
Figure 10. Eigenenergies of semiconductor qubit coupled to Josephson junction in dependence on superconducting order parameter in minimalistic approach.

Figure 11. Tunning the spectrum of eigenenergies in electrostatic qubit interacting with Josephson junction while we are changing the chemical potential of insulator region in Josephson junction at all nodes 0, 1, 2 and 3 in the same time.
in order to obtain higher accuracy and resolution in description of quantum state dynamics. One can use the obtained results in determination of quantum transport in the single electron devices or arbitrary topology what can helpful in optimization of devices functionality and sequence of controlling sequences shaping the electron confinement potential. Topological phase transitions as described by Sachdev\textsuperscript{9} Choi\textsuperscript{8} Belzig\textsuperscript{11} are expected to take place in arrays of coupled electrostatic qubits due to the similarity of tight-binding applied in semiconductor coupled quantum well model to Josephson model in Cooper pair box superconducting qubits. All results are quite straightforward to be generalized for electrons and holes confined in net of coupled quantum dots (what changes only sign of electrostatic energy so $q^2 \rightarrow -q^2$) under the assumption that recombination processes do not occur. What is more the interaction between electrostatic position based qubit and Josephson junction was formulated and solved in tight-binding model. In quite straightforward way one obtains the electrostatically coupled networks of graphs interacting with single Josephson junction in analytical way. It shall have its importance in development of interface between semiconductor CMOS quantum computer and already developed superconducting computer.

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REFERENCES

1. D. Leipold, Controlled Rabi Oscillations as foundation for entangled quantum aperture logic, Seminar at Berkley Lab, 25th July 2018
2. P. Giounanlis, E. Blokhina, K. Pomorski, D. R. Leipold, R. B. Staszewski, Modeling of Semiconductor Electrostatic Qubits Realized Through Coupled Quantum Dots, 10.1109/ACCESS.2019.2909489, IEEE Access, 2019
3. Krzysztof Pomorski, Panagiotis Giounanlis, Elena Blokhina, Dirk Leipold, Pawel Peczkowski, Robert Bogdan Staszewski, From two types of electrostatic position-dependent semiconductor qubits to quantum universal gates and hybrid semiconductor-superconducting quantum computer, Proc. SPIE 11054, Superconductivity and Particle Accelerators 2018, 110540M, 2019
4. T. Fujisawa, T. Hayashi, HD Cheong, YH Jeong, and Y. Hirayama, Rotation and phase-shift operations for a charge qubit in a double quantum dot. Physica E: Low-dimensional Systems and Nanostructures, 21(2-4):10461052, 2004.
5. K. D. Petersson, J. R. Petta, H. Lu, and A. C. Gossard. Quantum coherence in a one-electron semiconductor charge qubit. Phys. Rev. Lett., 105:246804, 2010.
6. Jozef Spalek, Wstep do fizyki materii skondensowanej, PWN, 2015.
7. K. Pomorski, H. Akaike, A. Fujimaki, and K. Rusek. Relaxation method in description of ram memory cell in rsfq computer, COMPEL, 38(1):395-414, 2019.
8. M. S. Choi, J. Yi, M. Y. Choi, J. Choi, and S. I. Lee. Quantum phase transitions in josephson-junction chains. Phys. Rev. B, 57:R716-R719, 1998.
9. S. Sachdev. Quantum phase transitions. Cambridge Univ. Press, 2011.
10. H. Q. Xu. Method of calculations for electron transport in multiterminal quantum systems based on real-space lattice models. Phys. Rev. B, 66:165305.
11. D. Maile, S. Andergassen, and W. Belzig. Quantum phase transition with dissipative frustration. Phys. Rev. B, 97, 2018.
12. K. Pomorski, P. Prokopow, Possible existence of field-induced Josephson junctions, Vol.249, No. 9, Physica Status Solidi B, 2012
13. Krzysztof Pomorski, Panagiotis Giounanlis, Elena Blokhina, Dirk Leipold, Pawel Peczkowski, Robert Bogdan Staszewski, Analytic view on Coupled Single-Electron Lines, ArXiv: 2674524, 2019