DISPROOF OF MODULARITY OF MODULI SPACE OF CY 3-FOLDS OF DOUBLE COVERS OF \( \mathbb{P}^3 \) RAMIFIED ALONG EIGHT PLANES IN GENERAL POSITIONS

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Abstract. We prove that the moduli space of Calabi-Yau 3-folds coming from eight planes of \( \mathbb{P}^3 \) in general positions is not modular. In fact we show the stronger statement that the Zariski closure of the monodromy group is actually the whole \( \text{Sp}(20, \mathbb{R}) \). We construct an interesting submoduli, which we call hyperelliptic locus, over which the weight 3 \( \mathbb{Q} \)-Hodge structure is the third wedge product of the weight 1 \( \mathbb{Q} \)-Hodge structure on the corresponding hyperelliptic curve. The non-extendibility of the hyperelliptic locus inside the moduli space of a genuine Shimura subvariety is proved.

1. Introduction

In the study of geometry of moduli space, it is important to characterize those moduli spaces which are locally Hermitian symmetric varieties. We refer the reader to [20], [21], [12] for such a theory based on the Arakelov equality. On the other hand, in order to prove a negative result it is also important to find some necessary conditions, which can be checked quite easily for explicitly given moduli spaces. In this paper, we will work with an interesting moduli space of CY 3-folds, which comes from the hyperplane arrangements in \( \mathbb{P}^3 \) consisting of eight planes in general positions. The aim of our present work is to disprove the modularity of this moduli space by two different methods. Before stating the main theorem, we shall make the meaning of modularity precise, since it could be ambiguous in certain cases. For example, the moduli space of six lines of \( \mathbb{P}^2 \) in general positions, which is identical to the moduli space of six points of \( \mathbb{P}^2 \) in general positions, can be openly embedded either into an arithmetic quotient of type four bounded symmetric domain \( \Pi \Pi \) or into an arithmetic ball quotient \( \Pi, \mathbb{B} \) by different period mappings.

Let \( \mathcal{M} \) be the coarse moduli scheme representing a moduli functor \( \mathcal{M} \) of polarized algebraic manifolds of dimension \( n \). After a finite base change of \( \mathcal{M} \), one obtains a universal family \( f : \mathcal{X} \to S \). The rational primitive middle cohomologies of the

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fibers of \( f \) constitute a \( \mathbb{Q} \)-polarized variation of Hodge structures \( \mathbb{V} \). It induces the period mapping

\[
\phi : S \hookrightarrow \Gamma \backslash G'/K'
\]

where \( G'/K' \) is the classifying space of polarized Hodge structures. If there exists a locally Hermitian symmetric variety \( \Gamma \backslash G/K \) and a locally homogenous PVHS \( \mathbb{W} \) over it such that \( \phi \) factors through the period mapping

\[
\psi : \Gamma \backslash G/K \hookrightarrow \Gamma' \backslash G'/K'
\]

defined by \( \mathbb{W} \) and such that the induced map

\[
\phi : S \to \Gamma \backslash G/K
\]

is an open embedding, then we say that \( f \) is a \textit{modular family} with respect to \( (G/K, \mathbb{W}) \). In the case that the reference locally homogenous PVHS \( \mathbb{W} \) and the Hermitian symmetric space \( G/K \) are clear from the context, we simply say \( f \) is modular family. The moduli space \( \mathfrak{M} \) is said to be \textit{modular} if a certain universal family of \( \mathfrak{M} \) is a modular family. Under this definition, it is clear that if

\[
f : \mathcal{X} \to S
\]

is a modular family with respect to \( (G/K, \mathbb{W}) \), then one has a factorization of the monodromy representation \( \rho \) of \( \mathbb{V} \):

\[
\begin{tikzcd}
\pi_1(S) \ar[r, \rho] & G' \\
& G \\
& G' \ar[u, \rho, hook] \ar[u, \varepsilon]
\end{tikzcd}
\]

where \( \varepsilon : G \to G' \) is the group homomorphism determined by \( \mathbb{W} \).

Now let \( \mathfrak{M}_{\text{CY}} \) be the moduli space of CY 3-folds from eight planes of \( \mathbb{P}^3 \) in general positions. The classifying space of the polarized Hodge structure on the middle cohomology of such a CY 3-fold is

\[
D' = \frac{\text{Sp}(20, \mathbb{R})}{\text{U}(1) \times \text{U}(9)}
\]

The natural Hermitian symmetric space in this case is

\[
D = \frac{\text{SU}(3, 3)}{\text{S(U(3) \times U(3))}}
\]

and the locally homogenous PVHS \( \mathbb{W} \) is the Calabi-Yau like PVHS over \( \Gamma \backslash D \) (cf. [19]), which is induced from the group homomorphism

\[
\bigwedge^3 : \text{SU}(3, 3) \to \text{Sp}(20, \mathbb{R})
\]

The main theorem of this paper is the following

**Theorem 1.1.** Let \( \mathfrak{M}_{\text{CY}} \) and \( (D, \mathbb{W}) \) be as above. Then \( \mathfrak{M}_{\text{CY}} \) is not modular with respect to \( (D, \mathbb{W}) \).
To keep our theorem in perspective, we would like to point out that the analogous moduli spaces of CY \( n \)-folds for \( n \leq 2 \) are modular (cf. [11]). It would be very interesting to extend the present work to the \( n \geq 4 \) cases. At this point, we would like to remind the reader of the early work [18]. They disproved a modularity result similar to Theorem 1.1 in a more general setting. It seems that the theory used in [18] has not been widely accepted within the community of algebraic geometers. We hope our purely Hodge theoretical proof will at least clarify some serious issues about the disproof of modularity. Furthermore, by a study of possible real groups of Hodge type contained in \( \text{Sp}(20, \mathbb{R}) \) and an application of the plethysm method we can deduce a stronger result about the Zariski closure of the monodromy group.

\textbf{Theorem 1.2.} Let \( f : \mathcal{X} \to S \) be a universal family of \( \mathcal{M}_{\text{CY}} \), and \( \mathcal{V} = R^3f_*Q_{\mathcal{X}} \) be the interesting weight 3 VHS. Then the image of the monodromy representation \( \rho : \pi_1(S) \to \text{Sp}(20, \mathbb{R}) \) of \( \mathcal{V} \) is Zariski dense.

In the work of [11], a special submoduli, which is isomorphic to an arithmetic quotient of \( \text{Sp}(4, \mathbb{R})/U(2) \), was constructed. We shall generalize their construction to our case. Since this submoduli arises from the moduli of hyperelliptic curves of genus 3, we simply call it the \textit{hyperelliptic locus}. We show that over the five dimensional hyperelliptic locus the weight 3 VHS of CY 3-folds is isomorphic to the wedge product of weight 1 VHS (cf. Prop. 2.3). It is then natural to ask if one can extend the hyperelliptic locus in \( \mathcal{M}_{\text{CY}} \) to a six dimensional submoduli which is isomorphic to an arithmetic quotient of \( \text{Sp}(6, \mathbb{R})/U(3) \). Using the concept of characteristic subvarieties we arrive at a negative answer of this extension problem.

\textbf{Theorem 1.3.} Let \( \mathcal{H}_{\text{CY}} \) be the hyperelliptic locus of \( \mathcal{M}_{\text{CY}} \). Then there exists no extension of \( \mathcal{H}_{\text{CY}} \) inside \( \mathcal{M}_{\text{CY}} \) such that it is isomorphic to a Zariski open subset of an arithmetic quotient of \( \text{Sp}(6, \mathbb{R})/U(3) \) which is embedded into \( \mathcal{M}_{\text{CY}} \) via the wedge product of the weight 1 VHS of a universal family of abelian 3-folds.

The paper is organized as follows. In \([2]\) we will construct two different Calabi-Yau manifolds from a given hyperplane arrangement, and describe the relation between them. The construction of the hyperelliptic locus concludes the second section. In \([3]\) we will describe our methodology to disprove the modularity. Two different methods will be presented respectively. The actual computations for our moduli space are realized using the theory of Jacobian rings. We have to adapt the current knowledge of Jacobian ring to our case. This is done in the fourth section. Section \([4]\) contains the results of our computations and the proof of the main theorems stated in \([1]\).

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2. Calabi-Yau manifolds from eight planes of \( \mathbb{P}^3 \) in general positions

Let \( H_1, \ldots, H_8 \) denote eight planes of \( \mathbb{P}^3 \) in general position. They sum up to a simple normal crossing divisor

\[
B = H_1 + \cdots + H_8
\]
on \( \mathbb{P}^3 \). Since \( B \) is even, we can form the double cover \( X \) of \( \mathbb{P}^3 \) with branch locus \( B \). Obviously, \( X \) is only a singular variety because \( B \) is singular. Fixing an ordering of the irreducible components \( H_{ij} = H_i \cap H_j \) of singularities of \( B \), we use the canonical resolution of double covers to obtain a smooth model \( \tilde{X} \) of \( X \). Namely, we have the following commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & \tilde{X} \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
\mathbb{P}^3 & \leftarrow & \mathbb{P}^3
\end{array}
\]

where \( \sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) is the composition of the sequence of blows-up with smooth centers (the strict transform of) \( H_{ij} \). The variety \( \tilde{X} \) is a smooth projective CY 3-fold with

\[
h^{2,1}(\tilde{X}) = 9 \quad \text{and} \quad h^{1,1}(\tilde{X}) = 29.
\]

This construction can actually be extended to all \( 2n + 2 \) hyperplane arrangements of \( \mathbb{P}^n \) in general positions, and the Hodge numbers of the primitive middle cohomology of the resulting smooth CY \( n \)-fold \( \tilde{X} \) are

\[
h_{pr}^{n-p}(\tilde{X}) = \left( \frac{n}{p} \right)^2.
\]

For the details, we refer to chapter 3 of [14].

It is easy to see that the moduli space of ordered eight hyperplane arrangements of \( \mathbb{P}^3 \) in general positions is of dimension 9. Hence, by fixing an ordering of the index set

\[
I = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq 8\},
\]
the above constructions gives rise to a complete moduli scheme \( \mathcal{M}_{CY} \) of smooth CY 3-folds. We note that a different ordering of \( I \) yields a different birational minimal model of the singular CY \( X \).
Now we consider the embedding determined by the starting hyperplane arrangement \((H_1, \ldots, H_8)\), namely
\[
j : \mathbb{P}^3 \hookrightarrow \mathbb{P}^7, \quad x \mapsto (\ell_1(x) : \cdots : \ell_8(x))
\]
where \(\ell_i : \mathbb{C}^4 \rightarrow \mathbb{C}\) denotes a linear form such that \(H_i = \{x \in \mathbb{P}^3 \mid \ell_i = 0\}\) for \(1 \leq i \leq 8\). The defining equations of \(j(\mathbb{P}^3) \subset \mathbb{P}^7\) are the four linearly independent relations among \(\ell_1, \cdots, \ell_8\). Written out explicitly, they are
\[
a_{11}y_1 + \cdots + a_{18}y_8 = 0
\]
where \((y_1 : \cdots : y_8)\) denote homogeneous coordinates of \(\mathbb{P}^7\). We can define a new CY 3-fold \(Y\) which is the complete intersection of four quadrics in \(\mathbb{P}^7\) defined by
\[
a_{11}y_1^2 + \cdots + a_{18}y_8^2 = 0
\]
\[
(2.1)
a_{41}y_1^2 + \cdots + a_{48}y_8^2 = 0.
\]
The variety \(Y\) is smooth since any \(4 \times 4\) minor of the matrix \(A := (a_{ij})\) is nonzero. The two Hodge numbers of \(Y\) are computed to be
\[
h^{2,1}(Y) = 65 \quad \text{and} \quad h^{1,1}(Y) = 1.
\]
In particular, the moduli space of complex structures on \(Y\) is of dimension 65. The covering map
\[
\mathbb{P}^7 \rightarrow \mathbb{P}^7, \quad (y_1 : \cdots : y_8) \mapsto (y_1^2 : \cdots : y_8^2)
\]
restricts to \(p : Y \rightarrow j(\mathbb{P}^3)\). The composite map, denoted again by \(p\),
\[
p : Y \rightarrow j(\mathbb{P}^3) \cong \mathbb{P}^3
\]
events \(Y\) as the Kummer covering of \(\mathbb{P}^3\), branched along \(B\) with degree \(2^7\). Clearly, the Galois group Aut\((Y|\mathbb{P}^3)\) of \(Y\) over \(\mathbb{P}^3\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^7\). There is a canonical surjection
\[
G_1 := (\mathbb{Z}/2\mathbb{Z})^8 \rightarrow \text{Aut}(Y|\mathbb{P}^3), \quad a = (a_1, \ldots, a_8) \mapsto \sigma_a
\]
with \(\sigma_a(y_i) = (-1)^{a_i}y_i\) for \(1 \leq i \leq 8\). Its kernel of order 2 is generated by \((1, \ldots, 1)\). Furthermore, we have a distinguished index 2 normal subgroup \(N_1 \trianglelefteq G_1\) given by
\[
N_1 := \ker \left( G_1 \cong (\mathbb{Z}/2\mathbb{Z})^8 \rightarrow \mathbb{Z}/2\mathbb{Z} \right).
\]
The following proposition reveals the geometric relation between two CY manifolds coming from the same hyperplane arrangement.
Proposition 2.1. Let \((H_1, \ldots, H_8)\) be a hyperplane arrangement of \(\mathbb{P}^3\) in general position. Then one can construct two smooth CY 3-folds \(\tilde{X}\) and \(Y\) as above. One has a natural isomorphism
\[
H^3(\tilde{X}, \mathbb{Q}) \cong H^3(Y, \mathbb{Q})^{N_1}.
\]

Proof. The quotient map \(p\) factors as
\[
Y \xrightarrow{\alpha} Y/N_1 \longrightarrow \mathbb{P}^3.
\]
The degree of \(Y/N_1\) over \(\mathbb{P}^3\) is 2 and one can directly check that \(Y/N_1\) branches exactly along \(B\). Since \(\mathbb{P}^3\) has no torsion element in the second integral cohomology, we have the identification \(X = Y/N_1\). Now let us examine the following commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{\alpha}} & \tilde{Y} \\
\downarrow p & & \downarrow \tilde{p} \\
\mathbb{P}^3 & \xrightarrow{\sigma} & \tilde{\mathbb{P}}^3
\end{array}
\]
where \(\tilde{Y}\) is the normalization of the fiber product of \(Y\) and \(\mathbb{P}^3\) over \(\mathbb{P}^3\). Obviously, \(\tilde{\alpha}\) is a contraction map. Since \(Y\) is smooth, \(\tilde{\alpha}\) induces the isomorphism
\[
H^3(Y, \mathbb{Q}) \cong H^3(\tilde{Y}, \mathbb{Q}).
\]
We put \(\tilde{B}\) to be the strict transform of \(B\) under \(\sigma\). Then the projection \(\tilde{p}\) is the Kummer covering map of degree \(2^7\) with branch locus \(\tilde{B}\). Argued as previously, \(\tilde{p}\) factors as
\[
\tilde{Y} \xrightarrow{\tilde{\alpha}} \tilde{Y}/N_1 = \tilde{X} \xrightarrow{\tilde{p}} \tilde{\mathbb{P}}^3.
\]
Since \(\tilde{\alpha}\) is \(G_1\)-equivariant, we have \(H^3(Y, \mathbb{Q})^{N_1} \cong H^3(\tilde{Y}, \mathbb{Q})^{N_1}\), and as both \(\tilde{X}\) and \(\tilde{Y}\) are smooth,
\[
H^3(\tilde{X}, \mathbb{Q}) \cong H^3(\tilde{Y}, \mathbb{Q})^{N_1}.
\]
Therefore, combining the last two isomorphisms, we obtain the isomorphism stated in the proposition. \(\square\)

We proceed to construct the hyperelliptic locus \(\mathcal{H}_{\text{CY}}\) inside our moduli space \(\mathcal{M}_{\text{CY}}\), generalizing the construction in [11]. We first recall that there is a natural Galois covering
\[
\gamma : (\mathbb{P}^1)^3 \longrightarrow \mathbb{P}^3
\]
with Galois group \(S_3\), the symmetric group of three letters. Explicitly, let
\[
(x_i : y_i), \quad 1 \leq i \leq 3
\]
be the homogenous coordinates of \(i\)-th factor of \((\mathbb{P}^1)^3\), such that the components of the quotient map \(\gamma\) are given by the \(t\)-coefficients of the polynomial \(f(t) = \)
Now we take arbitrary eight distinct points \( p_1, p_2, \ldots, p_8 \in \mathbb{P}^1 \) and construct a hyperplane arrangement from it.

**Lemma 2.2.** For \( 1 \leq i \leq 8 \) let \( H_i \) denote the image of \( \{p_i\} \times \mathbb{P}^1 \times \mathbb{P}^1 \) under \( \gamma \). Then \( H_i \) is a hyperplane in \( \mathbb{P}^3 \), and the hyperplane arrangement \((H_1, H_2, \ldots, H_8)\) is in general position.

**Proof.** Let \( (z_0 : z_1 : z_2 : z_3) \) be the homogenous coordinates of \( \mathbb{P}^3 \), and \( p = (a : b) \) be a point of \( \mathbb{P}^1 \). Then using the expression of \( \gamma \), the defining equation of the image set \( \gamma(\{p\} \times \mathbb{P}^1 \times \mathbb{P}^1) \) is easily seen to be

\[
b^3 z_0 - ab^2 z_1 + a^2 b z_2 - a^3 z_3 = 0.
\]

Therefore, \( H_i \) is obviously a hyperplane in \( \mathbb{P}^3 \). We can choose an appropriate system of coordinates on \( \mathbb{P}^1 \) such that the eight points have coordinates \((-a_i : 1)\) for \( 1 \leq i \leq 8 \). Then the columns of the following matrix give the defining equations of the arrangement \((H_1, H_2, \cdots, H_8)\):

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_8 \\
a_1^2 & a_2^2 & \cdots & a_8^2 \\
a_1^3 & a_2^3 & \cdots & a_8^3
\end{pmatrix}
\]

Now the property of the hyperplane arrangement to be in general position is equivalent to that all \( 4 \times 4 \)-minors of the above matrix are nonzero. Since \( a_1, a_2, \ldots, a_8 \) are distinct from each other by our assumption, all \( 4 \times 4 \)-minors are in Vandermonde form and thus non-zero. The lemma is proved. \( \Box \)

Now let \( C \) be the hyperelliptic curve over \( \mathbb{P}^1 \) branched at \( p_1, \ldots, p_8 \), and let \( q \) denote the corresponding covering map. The Galois group \( G_2 \) of the composition of morphisms

\[
C^3 \xrightarrow{q^3} (\mathbb{P}^1)^3 \xrightarrow{\gamma} \mathbb{P}^3
\]

is isomorphic to the semi-direct product \( N_2 \rtimes S_3 \), where \( N_2 = \langle \iota_1, \iota_2, \iota_3 \rangle \) is the group generated by the hyperelliptic involutions on each factor of \( C \times C \times C \). One observes that there is a distinguished index two subgroup \( G'_2 = N'_2 \rtimes S_3 \) of \( G_2 \), where \( N'_2 \) is the kernel

\[
N'_2 := \ker \left( N_2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \xrightarrow{\Sigma} \mathbb{Z}/2\mathbb{Z} \right)
\]
of the multiplication map. This gives the following commutative diagram of Galois coverings:

\[
\begin{array}{ccc}
C^3 & \xrightarrow{\delta} & X \\
\downarrow q^3 & & \downarrow \pi \\
(\mathbb{P}^1)^3 & \xrightarrow{\gamma} & \mathbb{P}^3
\end{array}
\]

**Lemma 2.3.** The double cover \( \pi : X \to \mathbb{P}^3 \) branches along the union of the hyperplane arrangement \((H_1, H_2, \ldots, H_8)\).

*Proof.* The Galois group of \( \pi \) is generated by \( \iota_1 \) in \( G_2/G'_2 \). By the commutativity of the above diagram, the branch locus of \( \pi \) is the image of the fixed locus of \( \iota_1 \) under the morphism \( \gamma \circ q^3 \). By Lemma 2.2, it is clear that the image is \( \bigcup_{i=1}^8 H_i \). □

By this lemma, the moduli of hyperelliptic curves of genus 3 are embedded into the moduli space \( \mathcal{M}_{CY} \). We call the image \( \mathcal{H}_{CY} \) the hyperelliptic locus, which is five-dimensional. In [11], the analogous submoduli were also characterized as those six lines in general positions tangential to a smooth conic of \( \mathbb{P}^2 \), and it was shown that this submoduli gives the family of Kummer surfaces. The Hodge structure of CY threefold over the hyperelliptic locus is also special in our case.

**Proposition 2.4.** Let \( \tilde{X} \) be the canonical resolution of \( X \). We have an isomorphism of rational polarized Hodge structures

\[ H^3(\tilde{X}, \mathbb{Q}) \cong \bigwedge^3 H^1(C, \mathbb{Q}). \]

*Proof.* As a consequence of Proposition 2.1, we know that

\[ H^3(\tilde{X}, \mathbb{Q}) \cong H^3(X, \mathbb{Q}). \]

So it suffices to prove the isomorphism for \( X \). For this purpose we consider the following commutative diagram

\[
\begin{array}{ccc}
C^3 & \xrightarrow{\delta_1} & S^3(C) \\
\downarrow \delta & & \downarrow \varphi \\
X & \xrightarrow{\delta_2} & \text{Jac}(C)
\end{array}
\]

where \( \varphi \) is the Abel-Jacobi map, \( \delta_1 \) is the quotient map by the subgroup \( S_3 \leq G'_2 \) and \( \delta_2 \) is the projection map. One notes that, since \( S_3 \) is not normal in \( G'_2 \), the map \( \delta_2 \) is only a finite morphism. However, \( \delta \) induces the embedding

\[ \delta^* : H^3(X, \mathbb{Q}) \cong H^3(C^3, \mathbb{Q})^{G'_2} \hookrightarrow H^3(C^3, \mathbb{Q}). \]
Since $\delta = \delta_2 \circ \delta_1$, the pullback $\delta_2^*$ gives the embedding
\[ H^3(X, Q) \xrightarrow{\delta_2^*} H^3(S^3(C), Q). \]
By the Abel-Jacobi theorem, $\varphi$ is a birational morphism and thus induces an isomorphism of Hodge structures on the middle cohomology:
\[ \varphi^* : H^3(Jac(C), Q) \xrightarrow{\cong} H^3(S^3(C), Q). \]
In particular, $\dim_Q H^3(S^3(C), Q) = 20$. Since we computed before that the dimension of $H^3(\tilde{X}, Q)$ is also 20, the map $\delta_2^*$ is in fact an isomorphism. The composition map
\[ H^3(X, Q) \xrightarrow{\delta_2^*} H^3(S^3(C), Q) \xrightarrow{(\varphi^*)^{-1}} H^3(Jac(C), Q) \cong \bigwedge^3 H^1(C, Q) \]
gives the isomorphism required in the proposition. \hfill \square

**Remark 2.5.** It is worthwhile to remark that the same construction and arguments generalize to $n \geq 4$ cases. It will give a $(2n - 1)$-dimensional hyperelliptic locus in the $n^2$-dimensional moduli of CY manifolds, over which the primitive middle dimensional rational Hodge structures are wedge products of weight 1 Hodge structures.

### 3. Characteristic Subvariety and Plethysm

In this section, we will present two different methods to disprove the modularity of $\mathcal{M}_{\text{CY}}$. Our first method is to study a series of invariants of IVHS, introduced in [19], which we call *characteristic subvarieties*. These invariants exploit the geometry of the kernels of iterated Higgs fields of the associated system of Hodge bundles with the given IVHS. In the case of Calabi-Yau like PVHS over bounded symmetric domain, these invariants are proved to be the *characteristic bundles* introduced in [10] by N. Mok, which played a pivotal role in the proof of the metric rigidity theorem of compact quotient of bounded symmetric domains of rank $\geq 2$. The second method uses the idea of *plethysm* in representation theory (cf. [6]). For a fixed simple complex Lie algebra $g$ the plethysm describes the decompositions of representations derived from a given irreducible representation of $g$.

#### 3.1. Characteristic Subvariety

We first recall some results in [19]. The bounded symmetric domain
\[ D = \frac{\text{SU}(3,3)}{S(U(3) \times U(3))} \]
is of rank 3. Let $\mathcal{W}$ be the Calabi-Yau like PVHS over $\Gamma \backslash D$ and $(F, \eta)$ be the associated system of Hodge bundles. By Theorem 3.3 in [19] we have the following

**Lemma 3.1.** For $k = 1, 2$ the $k$-th characteristic subvariety $S_k$ of $(F, \eta)$ coincides with $k$-th characteristic bundle. In particular, for every point $x \in \Gamma \backslash D$,
\[ (S_1)_x \cong \mathbb{P}^2 \times \mathbb{P}^2, \]
and \((\mathcal{S}_2)_x\) is isomorphic to the determinantal hypersurface in \(\mathbb{P}^8\).

Now we take a universal family \(f : \mathcal{X} \to S\) of \(\mathfrak{M}_{\text{CY}}\). Let \(V := R^3f_*Q_\mathcal{X}\) and

\[
E = \bigoplus_{p+q=3} E^{p,q}, \quad \theta = \bigoplus_{p+q=3} \theta^{p,q}
\]

be the corresponding system of Hodge bundles. Since \(V\) is of weight 3, we have also two characteristic subvarieties of \((F, \eta)\), which are denoted by \(\mathcal{R}_k\) for \(k = 1, 2\). If \(f\) is a modular family, then the period mapping \(\phi : S \hookrightarrow \Gamma \setminus D\) will induce an isomorphism

\[
\phi^*W \cong V,
\]

hence an isomorphism \(\phi^*(E, \theta) \cong (F, \eta)\). This implies the isomorphisms

\[
\phi^*\mathcal{S}_k \cong \mathcal{R}_k \quad \text{for} \quad k = 1, 2.
\]

Using Lemma 3.1 we then have the following

**Proposition 3.2.** If there exists a point \(x \in S\) such that

\((\mathcal{R}_1)_x \not\cong \mathbb{P}^2 \times \mathbb{P}^2\)

or \((\mathcal{R}_2)_x\) is not isomorphic to the determinantal hypersurface in \(\mathbb{P}^8\), then \(f\) is not a modular family.

**Remark 3.3.** It was first pointed out by E. Viehweg that the iterated Higgs fields for \((E, \theta)\) are surjective. Namely, the maps

\[
\theta^3 : S^k(T_S) \longrightarrow \text{Hom}(E^{3,0}, E^{3-k,k})
\]

are surjective for \(1 \leq k \leq 3\), where \(T_S\) denotes the tangent bundle over \(S\). If one of these maps were not surjective, then the disproof of modularity of \(\mathfrak{M}_{\text{CY}}\) would have been obtained at this stage already. This phenomenon (or difficulty) actually motivated the two latter authors to study the characteristic subvariety in [19]. It turned out that the present work gives a non-trivial application of the theory of characteristic subvarieties.

**3.2. Plethysm.** The simple real Lie group \(\text{SU}(3, 3)\) is a real form of \(\text{SL}(6, \mathbb{C})\). By Weyl’s unitary trick, one has an equivalence of categories of finite dimensional complex representations of \(\text{SU}(3, 3)\) and finite dimensional complex representations of \(\mathfrak{g} := \mathfrak{sl}(6, \mathbb{C})\). So the plethysm problem for \(\text{SU}(3, 3)\) is transformed into the plethysm problem for \(\mathfrak{g}\).

Let \(V := \mathbb{C}^6\) be the standard representation of \(\mathfrak{g}\). We shall study the plethysm for the fundamental representation \(W := \wedge^3(V)\). In other words, we shall study the decomposition of \(S_\lambda(W)\) for a Schur functor \(S_\lambda\). The two simplest Schur functors are \(S^2\) and \(\wedge^2\). By Exercise 15.32 in [8] we have the following decompositions:

\[
S^2(W) = \Gamma_{10001} \oplus \Gamma_{00200}, \quad \wedge^2 W = \Gamma_{00000} \oplus \Gamma_{01010}.
\]
By formula (15.17) in [6] it is easy to compute that
\[ \dim \Gamma_{0000} = 1, \quad \dim \Gamma_{1001} = 35, \quad \dim \Gamma_{01010} = 189 \]
and \( \dim \Gamma_{00200} = 175 \). However, \( \bigwedge^2 W \) will be of no use for us. That is because, considering \( W \) as \( \mathfrak{sp}(20, \mathbb{C}) \)-representation, one also has a decomposition
\[ \bigwedge^2 W = C \oplus W' \]
where \( C \) is the trivial representation of \( \mathfrak{sp}(20, \mathbb{C}) \) spanned by the symplectic form. On the other hand, \( S^2(W) \) is an irreducible representation of \( \mathfrak{sp}(20, \mathbb{C}) \). It is actually the adjoint representation.

**Proposition 3.4.** Let \( f : \mathcal{X} \to S \) be a universal family of \( \mathfrak{M}_{\text{CY}} \). If \( S^2(E, \theta) \) is not decomposed according to the following pattern, then \( f \) is not a modular family. Explicitly,
\[ S^2(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2) \]
where
\[ E_1 = E_1^{4,2} \oplus E_1^{3,3} \oplus E_1^{2,4} \]
\[ E_2 = E_2^{6,0} \oplus E_2^{5,1} \oplus E_2^{4,2} \oplus E_2^{3,3} \oplus E_2^{2,4} \oplus E_2^{1,5} \oplus E_2^{0,6} \].

Furthermore, the dimensions of Hodge bundles of \( E_1 \) are respectively \( 0, 0, 9, 17, 9, 0, 0 \) and those of \( E_2 \) are \( 1, 9, 45, 65, 45, 9, 1 \).

**Proof.** The modularity of \( f \) will imply a factorization of the monodromy representation
\[ \rho : \pi_1(S) \longrightarrow \text{SU}(3, 3) \longrightarrow \text{Sp}(20, \mathbb{R}). \]
Thus for any Schur functor \( S_\lambda \) the derived PVHS \( S_\lambda(V) \) will decompose into irreducible \( \text{SU}(3, 3) \)-representations. By the formula (3.1) and Deligne [4] Prop. 1.13, we have an decomposition of PVHS
\[ S^2(V) = V_1 \oplus V_2. \]

The system of Hodge bundles \( S^2(E, \theta) \) decomposes into a direct sum of system of Hodge bundles accordingly,
\[ S^2(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2). \]

Since \( W \) is of weight 3, \( S^2(W) \) is of weight 6. One can compute the Hodge numbers of \( (E_i, \theta_i) \) for \( i = 1, 2 \) by restricting the irreducible representations of \( \text{SU}(3, 3) \) to the center \( U(1) \) of its maximal compact subgroup \( S(U(3) \times U(3)) \). If \( \mathbb{I} \) denotes the \( 3 \times 3 \)-identity matrix, then
\[ \left\{ C_z := \begin{pmatrix} z\mathbb{I} & 0 \\ 0 & z^{-1}\mathbb{I} \end{pmatrix} \in \text{GL}_6(\mathbb{C}) \mid z \in U(1) \right\} \]
is the center of $S(U(3) \times U(3))$. We choose the standard basis $(e_1, \ldots, e_6)$ of $V = \mathbb{C}^6$ such that

$$C_z(e_i) = ze_i \text{ for } 1 \leq i \leq 3 \quad \text{and} \quad C_z(e_i) = z^{-1}e_i \text{ for } 4 \leq i \leq 6.$$ 

One notes that $\Gamma_{10001}$ is the unique nontrivial component in $\Gamma_{10000} \otimes \Gamma_{00001}$. It is easy to compute that $C_z$ acts on $\Gamma_{10001}$ with three characters $z^2$, $z^0$, $z^{-2}$, and the dimensions of their eigenspaces are respectively 9, 17, 9. Then the characters of $C_z$ on the other direct component $\Gamma_{00200}$ are $z^6$, $z^4$, $z^2$, $z^0$, $z^{-2}$, $z^{-4}$, $z^{-6}$, and their dimensions of eigenspaces are computed to be 1, 9, 45, 65, 45, 9, 1, respectively. The proof of the proposition is complete. $\square$

4. The Jacobian Ring

In the subsequent part we will carry out the strategies described in section 3 to the special family of CY 3-folds constructed in section 2. For this purpose we let $S$ denote the moduli space of eight planes in $\mathbb{P}^3$ in general positions. Every point $s \in S$ can be determined by a matrix $A \in \mathbb{C}^{4 \times 8}$ with the property that all $(4 \times 4)$-minors of $A$ are non-zero. Furthermore, we let $f : \tilde{X} \rightarrow S$ denote the universal family of $\mathfrak{M}_{CY}$ such that for every fiber $\tilde{X} := \tilde{X}_s$ is obtained by resolution of singularities from the ramified double cover $X \rightarrow \mathbb{P}^3$ associated to a certain matrix $A$ as described in section 2. For our purposes it will be necessary to give an explicit description of the PVHS $V := R^3f_*\mathbb{C}_X$ and the associated system $(E, \theta)$ of Higgs fields in every fiber.

First we give a description of $V$ as a local system of graded $\mathbb{C}$-vector spaces. Let $\mathcal{O}_S$ denote the sheaf of holomorphic functions on $S$ and $a_{ij} \in \Gamma(S, \mathcal{O}_S)$ the coordinate functions for $1 \leq i \leq 4$ and $1 \leq j \leq 8$. Furthermore, we let $\mathcal{R} := \mathcal{O}_S[x_1, \ldots, x_8, y_1, \ldots, y_4]$ denote the free $\mathcal{O}_S$-algebra in 12 indeterminates. For $p \in \mathbb{N}_0$ we define $\mathcal{R}^p$ to be the $\mathcal{O}_S$-submodule of elements which have total degree $\deg_x = 2p$ in the variables $x_j$ and total degree $\deg_y = p$ in the variables $y_i$. We define a global sections $f_i, f \in \Gamma(S, \mathcal{R})$ by $f_i := \sum_{j=1}^8 a_{ij} x_j^2$ for $1 \leq i \leq 4$ and $F := \sum_{i=1}^4 y_i f_i$. The twelve partial derivatives

$$\frac{\partial F}{\partial x_j} \text{ for } 1 \leq j \leq 8 \quad \text{and} \quad \frac{\partial F}{\partial y_i} \text{ for } 1 \leq i \leq 4$$

generate an ideal sheaf in $\mathcal{R}$ which we denote by $\mathcal{I}$. Finally, we let the group $G_1$ from section 2 act on the sheaf $\mathcal{R}$ by sending $a = (a_1, \ldots, a_8) \mapsto \sigma_a$ with

$$\sigma_a(x_i) = (-1)^{a_i} x_i \text{ for } 1 \leq i \leq 8 \quad \text{and} \quad \sigma_a(y_j) = y_j \text{ for } 1 \leq j \leq 4.$$
Then obviously $\sigma_a(\mathcal{I}) \subseteq \mathcal{I}$ holds for all $a \in G_1$. Now we obtain the following explicit description of our PVHS $V$.

**Proposition 4.1.** There is a canonical isomorphism of local systems

$$V \otimes_{\mathcal{O}} \mathcal{O}_S \cong (\mathcal{A}/\mathcal{I})^{N_1}$$

which maps $V^{3-p,p} \otimes_{\mathcal{O}} \mathcal{O}_S$ onto the submodule generated by $\mathcal{A}^p$ for $0 \leq p \leq 3$.

**Proof.** Let $g : Y \to S$ denote the family of intersections of four quadrics in $\mathbb{P}^7$ as constructed in section 2, i.e. for every $s = A = (a_{ij}) \in S$ the fiber $Y_s$ in the intersection of quadrics given by the equations (2.1). Furthermore, by $\mathcal{W} := R^3 g_* C_Y$ we denote the associated PVHS. According to Proposition 2.1 we have a canonical isomorphism $V \cong \mathcal{W}^{N_1}$, so that it remains to establish the isomorphism

$$\mathcal{W}^{N_1} \otimes_{\mathcal{O}} \mathcal{O}_S \cong (\mathcal{A}/\mathcal{I})^{N_1}.$$

First we show that $\mathcal{W} \otimes_{\mathcal{O}} \mathcal{O}_S \cong \mathcal{A}/\mathcal{I}$. This is a special case of Proposition 2.2.10 in [13], and although it is stated only for individual varieties, the result carries over to algebraic families. Here we just sketch the essential steps. Let $\mathbb{P}^7_S$ denote the projective 7-space over $S$ on which the coherent sheaf $E := \mathcal{O}_{\mathbb{P}^7_S}(2) \oplus 4$ is defined, and let $P := \mathbb{P}(E)$ denote the associated projective bundle. Then $P$ contains a toric hypersurface $\hat{Y}$ given by the equation $F = \sum_{i=1}^{4} y_i f_i$ from above. Let $\pi : P \to \mathbb{P}^7_S$ denote the canonical projection, extend $g$ to a map $\hat{g} : \mathbb{P}^7_S \to S$ and let $h := \hat{g} \circ \pi$. Then the embedding $\pi^{-1}(Y) \hookrightarrow \hat{Y}$ induces a natural isomorphism

$$R^9 h_* C_{\hat{Y}} \cong \mathcal{W} \otimes H^6(\mathbb{P}^3, \mathcal{O})$$

of PVHS on $S$, the right part of the tensor product being constant of rank one.

Now let $V := P \setminus Y$ denote the open complement of $Y$. Then the Gysin sequence relating the PVHS’s of $P$, $\hat{Y}$ and $V$ gives rise to an isomorphism $R^9 h_* C_{\hat{Y}} \cong R^{10} h_* C_Y$ of PVHS. In order to compute the latter, we make use of de Rham’s theorem which enables us to describe the cohomology

$$R^{10} h_* C_Y \otimes_{\mathcal{O}} \mathcal{O}_S \cong \mathbb{R}^{10} h_* \Omega_{V|S} \cong \mathbb{R}^{10} h_* \Omega_{P|S}(\ast \hat{Y})$$

in terms of the sheaf $\Omega_{P|S}(\ast \hat{Y})$ of relative differentials on $P$ with poles along $\hat{Y}$, where the functor $\mathbb{R}^{10} h_*$ denotes hypercohomology. Since the sheaves $\Omega_{P|S}^i(m \hat{Y})$ are acyclic for $i, m > 0$, it can be computed by taking global sections. That is, if $\mathcal{Z}_{P|S} \subseteq \mathcal{O}_{P|S}^{10}(\ast \hat{Y})$ denotes the subsheaf of closed differentials and $\mathcal{B}_{P|S}$ the subsheaf of exact ones, then simply

$$\mathbb{R}^{10} h_* \Omega_{P|S}(\ast \hat{Y}) \cong (h_* \mathcal{Z}_{P|S})/(h_* \mathcal{B}_{P|S}).$$
The sections of $h_\ast \Omega^0_{\mathcal{P}|S}(\ast \hat{\mathcal{Y}})$ can be described in terms of the $\mathcal{O}_S$-algebra $\mathcal{R}$. Namely, let $\omega_0$ denote the homogeneous differential form
\[ \hat{\Omega} := dx_1 \wedge \cdots \wedge dx_8 \wedge dy_1 \wedge \cdots \wedge dy_4 \]
and define the vector fields $\vartheta_i := \partial/\partial x_i$ and $\lambda_j := \partial/\partial y_j$ for $1 \leq i \leq 8$ and $1 \leq j \leq 4$. If we put
\[ \theta_1 := \sum_{j=1}^{4} y_j \lambda_j, \quad \theta_2 := \sum_{i=1}^{8} x_i \vartheta_i - 2 \sum_{j=1}^{4} y_j \lambda_j \]
and $\Omega := \theta_1 \theta_2(\hat{\Omega})$, then every section $\omega$ of $h_\ast \Omega^0_{\mathcal{P}|S}(\ast \hat{\mathcal{Y}})$ can be written in the form
\[(4.2) \quad \omega = \frac{H \Omega}{F^{p+1}} \quad \text{where } H \text{ is a section of } \mathcal{R}^p.\]
In degree 9, any section $\psi$ of $h_\ast \Omega^9_{\mathcal{P}|S}(\ast \hat{\mathcal{Y}})$ can be written as
\[ \psi = \sum_{i=1}^{8} G_i \Omega_i - \sum_{j=1}^{4} H_j \Omega_j \]
where $\Omega_i := \theta_1 \theta_2 \vartheta_i$, $\Omega_j := \theta_1 \theta_2 \lambda_j$ and $G_i$, $H_j$ are sections of $\mathcal{R}$ such that $\deg_X(G_i) = 2p + 1$, $\deg_Y(G_i) = p$ and $\deg_X(H_j) = 2p$, $\deg_Y(H_j) = p + 1$ for $1 \leq i \leq 8$ and $1 \leq j \leq 4$. Its exterior derivative is $d\psi = H \Omega / F^{p+5}$ where
\[ H = 2 \sum_{i=1}^{8} \frac{\partial F}{\partial x_i} G_i + 2 \sum_{j=1}^{4} \frac{\partial F}{\partial y_j} H_j - F \left( \sum_{i=1}^{8} \frac{\partial G_i}{\partial x_i} + \sum_{j=1}^{4} \frac{\partial H_j}{\partial y_j} \right). \]
We see that $\omega$ can be reduced to lower pole order if and only if the section $h$ is a section of the ideal sheaf $\mathcal{I}$. This shows that
\[ (h_\ast \mathcal{P}|S)/(h_\ast \mathcal{R}|S) \cong \mathcal{R}/\mathcal{I}. \]
Combing all isomorphisms, the desired assertion $\mathcal{W} \otimes_{\mathcal{C}} \mathcal{O}_S \cong \mathcal{R}/\mathcal{I}$ follows. Observing that the action of $G_1$ on $\mathcal{W} \otimes_{\mathcal{C}} \mathcal{O}_S$ is compatible with the action defined above on $\mathcal{R}/\mathcal{I}$, we obtain $\mathcal{W}^{N_1} \otimes_{\mathcal{C}} \mathcal{O}_S \cong (\mathcal{R}/\mathcal{I})^{N_1}$.

In order to prove the refined statement on the grading, notice that by the above construction
\[ \mathcal{W}^{3-p,p} \cong R^{6-p,3+p} h_\ast \mathcal{C}_\mathcal{Y} \cong R^{7-p,3+p} h_\ast \mathcal{C}_\mathcal{Y}. \]
By the comparison of Hodge and pole filtration, the part $(F^{3+p} h_\ast \mathcal{C}_\mathcal{Y}) \otimes_{\mathcal{C}} \mathcal{O}_S$ coincides with the subsheaf of $R^{10} h_\ast \Omega^0_{\mathcal{P}|S}(\ast \hat{\mathcal{Y}})$ generated by differentials of pole order $\geq p + 4$. This shows that $\mathcal{W}^{3-p,p} \otimes_{\mathcal{C}} \mathcal{O}_S$ corresponds to the subsheaf of $\mathcal{R}/\mathcal{I}$ generated by $\mathcal{R}^p$. \hfill $\Box$

The description of the local system $\mathcal{V}$ in terms of the Jacobian ring $\mathcal{R}/\mathcal{I}$ admits an explicit computation the Gauss-Manin connection and the Higgs field in one-parameter families. Let $h : \hat{\mathcal{Y}} \to S$ denote the family of toric hypersurfaces that we used in the proof of Proposition 4.1. Furthermore, let $T$ denote an open subset of $\mathbb{A}^1$ and $h : \hat{\mathcal{Y}}_T \to T$ be the family obtained by restriction. Over $T$ the defining
equation of \( \hat{Y} \) inside the toric variety \( P \) is given by an equation \( F = 0 \) with \( F \in \mathbb{C}(t)[x_1, ..., y_4] \), and the Gauss-Manin connection
\[
\nabla : R^0 h_* \mathcal{C}_\hat{Y} \rightarrow R^0 h_* \mathcal{C}_\hat{Y} \otimes_{\mathcal{O}_T} \Omega^1_T
\]
acts on the de Rham cohomology of the complement by
\[
(4.3) \quad \omega = \frac{H_\Omega}{F^{p+5}} \mapsto - (p + 4)(\partial_t F) \frac{H_\Omega}{F^{p+5}} \otimes dt
\]
provided that the section \( H \) of \( \mathcal{R}_T \) is chosen such that \( \partial_t H = 0 \). Thus if one fixes a local basis of \( (\mathcal{R}/\mathcal{I})^{N_1} \) given by polynomials over the function field \( \mathbb{C}(t) \), one can compute a representation matrix of \( \nabla \) by applying the map \((4.3)\) to all basis elements and reducing them with respect to the basis. A representation matrix for the Higgs field
\[
(4.4) \quad \theta : R^0 h_* \mathcal{C}_\hat{Y} \rightarrow R^0 h_* \mathcal{C}_\hat{Y} \otimes_{\mathcal{O}_T} \Omega^1_T
\]
is obtained by projecting the images of the basis elements inside the \( \mathcal{R}^p \)-part onto the subspace generated by \( \mathcal{R}^{p+1} \), for \( 0 \leq p \leq 3 \). By the canonical isomorphism of Proposition 4.1 this also yields a local representation matrix of \( \theta : \nabla_T \rightarrow \nabla_T \otimes_{\mathcal{O}_T} \Omega^1_T \), or equivalently, of
\[
(4.4) \quad \theta : T_T \otimes_{\mathcal{O}_T} \nabla_T \rightarrow \nabla_T.
\]
For our purposes it will be sufficient to compute the map \((4.4)\) in the infinitesimal neighborhood of a point \( x \in T \), which turns out to be much easier. Let \( \hat{X} := \hat{X}_x \) denote the fiber at \( x \). We have an exact sequence of vector bundles over \( \hat{X} \) given by
\[
(4.5) \quad 0 \rightarrow T_{\hat{X}} \rightarrow T_{\hat{X}|\hat{X}} \rightarrow f^*(T_S)|_{\hat{X}} \rightarrow 0
\]
where the vertical bars mean restriction. The bundle on the right hand side is trivial with generic fiber \( T_{S,x} \), the tangent space of \( S \) at \( x \). Since \( \hat{X} \) is compact, all sections of the trivial bundle are constant, so that
\[
T_{S,x} = H^0(X, f^*(T_S)|_{\hat{X}})
\]
holds. Now the long exact cohomology sequence associated to the short exact sequence \((4.5)\) yields a map
\[
\rho : T_{S,x} \rightarrow H^1(\hat{X}, T_{\hat{X}})
\]
the \textit{Kodaira-Spencer map}. It is known to be an isomorphism.

Let \( R \) denote the stalk of the local system \((\mathcal{R}/\mathcal{I})^{N_1} \) at \( x \), and by \( R_p \) the stalks of the images of \( \mathcal{R}_p \), for \( 0 \leq p \leq 3 \). Then \( R = \bigoplus_{p=0}^3 R_p \) is a finite-dimensional \( \mathbb{C} \)-algebra.

\textbf{Lemma 4.2.} \textit{There is a canonical isomorphism} \( R_1 \cong H^1(\hat{X}, T_{\hat{X}}) \).
Proof. Since \( \tilde{X} \) is a Calabi-Yau manifold, the canonical bundle \( \mathcal{K}_{\tilde{X}} = \Omega^3_{\tilde{X}} \) is trivial, which gives rise to a natural identification

\[
\mathcal{T}_{\tilde{X}} = (\Omega^1_{\tilde{X}})^* \cong \Omega^2_{\tilde{X}}.
\]

It implies that \( H^1(\tilde{X}, \mathcal{T}_{\tilde{X}}) \) is isomorphic to \( H^{2,1}(\tilde{X}) = H^1(\tilde{X}, \Omega^2_{\tilde{X}}) \). On the other hand, if we specialize the isomorphism from Proposition 4.1 to the stalks at \( x \), we obtain \( H^{2,1}(\tilde{X}) \cong R_1 \).

□

Proposition 4.3. For \( 0 \leq p \leq 2 \) there is a commutative diagram

\[
\begin{array}{ccc}
T_{S,x} \otimes H^{3-p,p}(\tilde{X}) & \xrightarrow{\theta} & H^{2-p,p+1}(\tilde{X}) \\
\downarrow \cong & & \downarrow \cong \\
R_1 \otimes R_p & \xrightarrow{\mu} & R_{p+1}
\end{array}
\]

where the vertical arrows are induced by the Kodaira-Spencer map and Proposition [4,7] and where the lower horizontal arrow denotes multiplication on the graded \( C \)-algebra \( R \).

Proof. It is known that the derivation of a cohomology class in \( H^{3-p,p}(\tilde{X}) \) with respect to a tangent direction \( v \in T_{S,x} \) is given by the cup product

\[
H^1(\tilde{X}, \mathcal{T}_{\tilde{X}}) \otimes H^q(\tilde{X}, \Omega^p_{\tilde{X}}) \xrightarrow{\rho(v)} H^{q+1}(\tilde{X}, \Omega^{p-1}_{\tilde{X}})
\]

with the Kodaira-Spencer class \( \rho(v) \) (see e.g. [2], Lemma 5.3.3). In the de Rham cohomology of the toric hypersurface \( \tilde{Y} \), the cup product between cohomology classes corresponds to the wedge product between differential forms. Furthermore, we have seen in (4.2) that every differential is defined by a polynomial in \( R \). It can be checked easily that the multiplication of polynomials corresponds to the wedge product of the corresponding differential forms.

□

For later use we need an explicit, fiberwise description of the characteristic subvarieties \( \mathcal{R}_k \) introduced in section 3 associated to our special universal family \( f : \tilde{X} \to S \). To this end we introduce the symmetric algebra \( S(R_1^*) \) over the dual of \( R_1 \), which is the homogeneous coordinate ring of \( \mathbb{P}(R_1^*) \). Taking the multiplication map to its dual, we obtain a linear map

\[
\mu^* : R_2^* \rightarrow S^2(R_1^*)
\]

and we let \( a_1 \) denote the ideal generated by the image of \( \mu^* \). Similarly, we let \( a_2 \) denote the ideal generated by the image of the dualized multiplication map \( \mu^* : R_3^* \rightarrow S^3(R_1^*) \). Then the fibers of the characteristic varieties are obtained in the following way.

Lemma 4.4. For a point \( x \in S \) as above and \( k = 1, 2 \), the fiber of the \( k \)-th characteristic subvariety \( (\mathcal{R}_k)_x \) is isomorphic to the projective subvariety \( Z_k := \text{Proj}(A_k) \) of \( \mathbb{P}(R_1^*) \), where \( A_k \) denotes the graded quotient ring \( S(R_1^*)/a_k \).
Proof. We recall the definition of the \( k \)-th characteristic subvariety as given in [19]. For our system \((E, \theta)\) of Hodge bundles, the \((k + 1)\)-st iterated Higgs field defines a map

\[
\theta^{k+1} : S^{k+1}(T_S) \rightarrow \text{Hom}(E^{3,0}, E^{2-k,k+1})
\]

whose kernel we denote by \( \mathcal{S}_k \). Then \( \mathcal{R}_k = \text{Proj}(\mathcal{S}_k) \) as a subvariety of \( \mathbb{P}(T_S) \). For \( k = 1 \) the stalk \((\mathcal{S}_1)_x\) at \( x \in S \) is the kernel of

\[
\theta^2 : S^2(T_{S,x}) \cong S^2(R_1) \rightarrow \text{Hom}(R_3, R_2) \cong R_2 ,
\]

the first isomorphism coming from Lemma 4.2 and the Kodaira-Spencer map, the second being a consequence of the fact that \( R_3 \) is one-dimensional. If we dualize this map, up to a non-zero constant we obtain \( \mu^* \), and the kernel of \( \theta^2_x \) is isomorphic to \( S^2(R_1^*)/a_1 \), the cokernel of \( \mu^* \). Since this quotient generates \( A_1 \), we obtain \( Z_1 \cong (\mathcal{R}_1)_x \). The proof for \( k = 2 \) is similar. \( \Box \)

5. PROOFS OF THE MAIN THEOREMS

We recall some basic notions from computational commutative algebra. Let \( K \) be a field and \( R := K[x_1, \ldots, x_n] \) the polynomial ring in \( n \) indeterminates. A monomial ordering is a total ordering \( \prec \) on the set of monomials in \( R \) such that \( f \prec g \) implies \( fh \prec gh \) for monomials \( f, g, h \in R \). In our computations we will use the graded lexicographical ordering, which is defined as follows: First one fixes an ordering on the set of indeterminates by requiring \( x_1 \succ x_2 \succ \cdots \succ x_n \). Now let

\[
f = y_1 y_2 \cdots y_r \quad \text{and} \quad g = z_1 z_2 \cdots z_s \quad \text{with} \quad y_i, z_i \in \{x_1, \ldots, x_n\} \quad \text{for all} \quad i
\]

such that \( y_i \succ y_j \) or \( y_i = y_j \) for \( i \leq j \), and similarly for the factors of \( g \). Then by definition \( f \succ g \) if either \( r \succ s \) or \( r = s \) and there is an \( m \in \mathbb{N}_0 \) such that \( y_i = z_i \) for \( 1 \leq i \leq m \) and \( y_{m+1} \succ z_{m+1} \).

The total ordering on the monomials extends to a partial ordering on \( R \) by defining \( f \prec g \) iff the maximal monomial of \( f \) is smaller than the maximal monomial of \( g \). Furthermore, zero is defined to be the least element in \( R \). If \( a \subseteq R \) is an ideal, then we say that an element \( f \in R \) is in normal form with respect to \( a \) and write \( f = \text{NF}(f) \) if \( f \) is minimal inside the coset \( f + a \). It can be shown that the normal form is unique; in particular, \( \text{NF}(f) = 0 \) if and only if \( f \in a \).

Let \( f : \mathcal{X} \rightarrow S \) denote the family of CYs defined at the beginning of section 4. In order to prove the theorems from section 4, it suffices to consider one particular fiber of this family. Let \( \lambda_j := j \) for \( 1 \leq j \leq 8 \) and define the matrix \( A \in \mathbb{C}^{4 \times 8} \) by \( a_{ij} := \lambda^j \) for \( 1 \leq i \leq 4 \) and \( 1 \leq j \leq 8 \). We define \( x_0 \in S \) to be the point corresponding to the matrix \( A \), and let \( \tilde{X} := \mathcal{X}_{x_0} \) denote its fiber. For \( p = 0, \ldots, 3 \) let \( R_p \) be the ring defined before Lemma 4.2. By Proposition 4.1, \( R_p^{N_1} \) is isomorphic to \( H^{3-p,p}(\tilde{X}) \) for \( 0 \leq p \leq 3 \).
Lemma 5.1. The following elements constitute a basis of $R_p^{N_1}$ for $p = 0, ..., 3$.

\begin{align*}
p = 0 & \quad 1 \\
p = 1 & \quad x_2^1 y_2, x_2^1 y_3, x_2^1 y_4, x_6^2 y_2, x_6^2 y_3, x_6^2 y_4, x_7^2 y_2, x_7^2 y_4 \\
p = 2 & \quad x_6^4 y_2, x_6^4 y_3, x_6^4 y_4, x_6^2 x_7^2 y_3, x_6^2 x_7^2 y_4, x_7^4 y_2, x_7^4 y_3, x_7^4 y_4 \\
p = 3 & \quad x_7^6 y_4
\end{align*}

Proof. For each $p$ we list all monomials with $\deg_x = 2p$ and $\deg_y = p$. If we let $e_1, ..., e_8$ denote the canonical basis of $(\mathbb{Z}/2\mathbb{Z})^8$, then $N_1$ is generated by the set

$$B := \{e_i + e_{i+1} \mid 1 \leq i \leq 7\} \cup \{e_1 + e_8\}.$$ 

We remove all elements $g$ from the list with $\sigma_a(g) \neq g$ for some $a \in B$ or with $\text{NF}(g) \neq g$. By uniqueness and linearity of the normal form, the remaining elements are linearly independent in $R_p^{N_1}$. Since the Betti numbers of $X$ are $1, 9, 9, 1$, respectively, the assertion follows.

Proposition 5.2. The fiber $(R_1)_{x_0}$ of the first characteristic subvariety at $x_0$ is two-dimensional.

Proof. By Lemma 4.4 we have to compute the ideal $a_1 \subset S'(R_1^*)$ which is generated by the image of the dual multiplication map $\mu^* : R_2^* \rightarrow S^2(R_1^*)$. If we define the basis of $R_1$ and $w_1, ..., w_9$ the basis of $R_2$ as defined in [4.1]. Furthermore, we fix a bijection

$$\varphi : \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq j \leq 9\} \overset{\sim}{\longrightarrow} \{1, ..., 45\}$$

and put $u_{\varphi(i,j)} := v_i v_j$. The first step is to compute a representation matrix of the multiplication map

$$\mu : S^2(R_1) \longrightarrow R_2$$

with respect to the basis $u_1, ..., u_{45}$ and $w_1, ..., w_9$. By computing the normal forms of elements with respect to the Jacobian ideal $\mathcal{J}_{x_0} \subseteq R$, we determine $c_{\varphi(i,j)k} \in \mathbb{Q}$ such that

$$\text{NF}(v_i v_j) = \sum_{k=1}^{9} c_{\varphi(i,j)k} w_k \quad \text{for} \quad 1 \leq i, j \leq 9.$$ 

Then $C := (c_{\ell k}) \in \mathbb{C}^{45,9}$ is the desired representation matrix. Its transpose represents $\mu^*$ with respect to the dual basis $w_1^*, ..., w_9^*$ and $u_1^*, ..., u_{45}^*$. Notice that $(v_i v_j)^* = 2 v_i^* v_j^*$ for $1 \leq i, j \leq 9$. Thus if we define

$$\tilde{c}_{\ell k} := \begin{cases} 
2 c_{\ell k} & \ell = \varphi(i,j), \quad i = j \\
2 c_{\ell k} & \ell = \varphi(i,j), \quad i \neq j
\end{cases}$$

then $\tilde{C} \in \mathbb{C}^{9,45}$ is a representation matrix of $\mu^*$ with respect to $w_1^*, ..., w_9^*$ and $\tilde{u}_1, ..., \tilde{u}_{45}$, where $\tilde{u}_{\varphi(i,j)} := v_i^* v_j^*$. Each row corresponds to one generator of $a_1$ in $S^2(R_1^*)$. Furthermore, the choice of a basis $v_1^*, ..., v_9^*$ admits a natural identification

$$\text{Proj}(S(R_1^*)) \cong \mathbb{P}^8.$$
Let $z_1,\ldots, z_9$ denote a new set of indeterminates. If we define $f_1,\ldots, f_9$ by
\[ f_\ell := \sum_{i=1}^{9} \sum_{j=i}^{9} \tilde{c}_{\varphi(i,j)} \ell z_i z_j \]
then the variety in $\mathbb{P}^8$ defined by $f_1 = \cdots = f_9 = 0$ is isomorphic to $(\mathcal{R}_1)_{x_0}$. Since the matrix $\tilde{C}$ is known in explicit term, we can use computer algebra to compute its dimension. We obtain $\dim(\mathcal{R}_1)_{x_0} = 2$. \hfill □

In order to prove the non-modularity of $f$, by Proposition 3.2 it is sufficient to determine a single points $x \in S$ such that the fiber $(\mathcal{R}_1)_x$ is not isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. By Proposition 4.4, the fiber $(\mathcal{R}_1)_{x_0}$ is only two-dimensional. Thus both $f$ and $\mathcal{M}_{\text{CY}}$ cannot be modular, and Theorem 1.1 is proved.

Proof of Theorem 1.3 The proof will be achieved by contradiction. Let $\mathcal{S}_{\text{CY}} \subset \mathcal{S}'_{\text{CY}} \subset \mathcal{M}_{\text{CY}}$ be an extension as described in the theorem, and let $f : \tilde{X} \to S$ denote a universal family. Let $Z$ be the sublocus in $S$ mapping to $\mathcal{S}'_{\text{CY}}$, and $g = f|_Z : \tilde{X}|_{f^{-1}(Z)} \to Z$ be the corresponding subfamily. Then $g$ is a modular family with respect to $(\text{Sp}(6,\mathbb{R})/U(3), \mathbb{W})$ where $\mathbb{W}$ is the Calabi-Yau like PVHS over $\text{Sp}(6,\mathbb{R})/U(3)$ (cf. [19]). Let $(F, \eta)$ be the corresponding Higgs bundle of the subfamily $g$. By Theorem 3.3 in [19], the Higgs bundle $(F, \eta)$ has two characteristic subvarieties and the fibers of the first characteristic subvariety are all isomorphic to $\mathbb{P}^2$. Take one point $x \in Z$, and denote by $(\mathcal{R}'_1)_x$ be the fiber over $x$ of the first characteristic subvariety of $(F, \eta)$. By the geometric description of the characteristic subvariety in Lemma 3.2 [19], we know that in $\mathbb{P}(T_{S,x})$ the equality
\[ (\mathcal{R}'_1)_x = (\mathcal{R}_1)_x \cap \mathbb{P}(T_{Z,x}) \]
holds. Now if $\dim(\mathcal{R}_1)_x = 2$, then we necessarily have an isomorphism
\[ (\mathcal{R}_1)_x = (\mathcal{R}'_1)_x \simeq \mathbb{P}^2. \]
Since our computation is local, we simply take the point $x$ to be the same point as used in the above proof of Theorem 1.1. The arithmetic genus of $(\mathcal{R}_1)_x$ is calculated to be -41, whereas the arithmetic genus of $\mathbb{P}^2$ is 0. So $(\mathcal{R}_1)_x$ is non-isomorphic to $\mathbb{P}^2$. Therefore such an extension does not exist. \hfill □

Now we give a second proof of Theorem 1.1 which is based on the plethysm method described in subsection 3.2. As before by $(E, \theta)$ we denote the Hodge bundle associated to the family $f : \tilde{X} \to S$. The Higgs field
\[ \theta_{x_0} : T_{S,x_0} \otimes E_{x_0}^{3,0} \longrightarrow E_{x_0}^{2,1} \]
induces in a natural way a linear map
\[ S^2(\theta_{x_0}) : T_{S,x_0} \otimes S^2(E_{x_0})^{6,0} \longrightarrow S^2(E_{x_0})^{5,1} \]
on the symmetric 2-space. By threefold iteration we obtain
\[ S^2(\theta^3_{x_0}) : S^3(T_{S,x_0}) \otimes S^2(E_{x_0})^{6,0} \rightarrow S^2(E_{x_0})^{3,3}. \]

**Proposition 5.3.** The image of \( S^2(\theta^3_{x_0}) \) is 78-dimensional.

**Proof.** By Proposition 4.3 it is sufficient to compute the image of the linear map
\[ S^2(\mu^3) : S^3(R_1) \otimes S^2(R_0) \rightarrow S^2(R) \]
induced by the multiplication map \( \mu : R_1 \otimes R_0 \rightarrow R_1 \). Let \( v_1, \ldots, v_{20} \) denote the basis of \( R = \oplus_{p=0}^3 R_p \) specified in Lemma 5.1. In particular, we assume that the elements \( w_k := v_{k+1} \) span the subspace \( R_1 \), where \( 1 \leq k \leq 9 \). For each \( k \) we determine the coefficients \( c^{(k)}_{ij} \in \mathbb{C} \) such that
\[ v_i w_k = \sum_{j=1}^{20} c^{(k)}_{ij} v_j \quad \text{for } 1 \leq i \leq 20 \text{ and } 1 \leq j \leq 9. \]
This is done by reduction modulo \( \mathfrak{A}_{x_0} \) as in the proof of Theorem 5.2. Then \( C_k := c^{(k)}_{ij} \) is a representation matrix of the linear map
\[ \mu_{w_k} : R \rightarrow R, \quad v \mapsto vw_k \]
with respect to \( v_1, \ldots, v_{20} \). With these matrices it now an easy task to compute the induced action of \( \mu_{w_k} \) on \( S^2(R) \). Fix a bijection
\[ \varphi : \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq j \leq 20\} \rightarrow \{1, \ldots, 210\} \]
and define a basis \( u_1, \ldots, u_{210} \) of \( S^2(R) \) by \( u_{\varphi(i,j)} := v_i v_j \) for \( 1 \leq i \leq j \leq 20 \). Then \( S^2(\mu_{w_k}) \) acts on this basis by
\[ S^2(\mu_{w_k})(u_{\varphi(i,j)}) := \sum_{\ell=1}^{20} c^{(k)}_{i\ell} v_{\ell} v_j + \sum_{\ell=1}^{20} c^{(k)}_{j\ell} v_i v_\ell. \]
The subspace \( U^{6,0} \) of \( S^2(R) \) of degree zero is one-dimensional and generated by \( u_{\varphi(0,0)} \). Now the space \( S^2(\mu)(U^{6,0}) \) is generated by the images of all maps \( S^2(\mu_{w_k}) \) \( (k = 1, \ldots, 9) \) applied to \( u_{\varphi(0,0)} \). By 5.1 and computational linear algebra it turns out to be 9-dimensional and thus all of \( U^{5,1} \). Applying all maps \( S^2(\mu_{w_k}) \) to \( U^{5,1} \) we obtain a subspace of \( S^2(R) \) of dimension 45 contained in \( U^{4,2} \), and a third application yields a 78-dimensional subspace of \( U^{3,3} \). \( \square \)

Now we explain how Proposition 5.3 implies Theorem 1.1. If \( f : \tilde{X} \rightarrow S \) were modular, by Proposition 5.4 there would be a decomposition of Hodge bundles
\[ S^2(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2), \]
such that each graded piece \( E^{p-p,p}_I \) has a specific dimension. This decomposition would exist in any fiber. In particular, the image of \( S^2(E_{2,x_0})^{6,0} = S^2(E_{x_0})^{6,0} \) under the iterated Higgs field
\[ S^2(\theta^3_{x_0}) : S^3(T_{S,x_0}) \otimes S^2(E_{x_0})^{6,0} \rightarrow S^2(E_{x_0})^{3,3} \]
would be contained in $S^2(E_{2,x_0})^{3,3}$ and thus be at most 65-dimensional. But since the image of $S^2(\theta_{3,x_0}^3)$ has dimension 78, the decomposition cannot exist.

**Proof of Theorem 1.2** Let $\rho : \pi_1(S) \to \text{Sp}(20, \mathbb{R})$ be the monodromy representation. We know that $\rho$ is irreducible since the VHS $V$ is irreducible. Let $G$ be the Zariski closure of the monodromy group in $\text{Sp}(20, \mathbb{R})$. Then we have a factorization:

$$
\rho : \pi_1(S) \longrightarrow G \longrightarrow \text{Sp}(20, \mathbb{R}),
$$

and $\rho : G \to \text{Sp}(20, \mathbb{R})$ is irreducible. If $G$ is not the whole group, $G$ must be a proper Lie subgroup of $\text{Sp}(20, \mathbb{R})$. We will now derive a contradiction by a sequence of steps.

**Step 1.** Differentiating $\rho$ we pass to the real Lie algebra monomorphism

$$
\chi : \mathfrak{g} \longrightarrow \mathfrak{sp}(20, \mathbb{R})
$$

where $\mathfrak{g} = \text{Lie}(G)$. By Deligne [3] Cor. 4.2.9 we know that $\mathfrak{g}$ is semi-simple. We then complexify $\chi$ to obtain $\chi_C : \mathfrak{g}_C \to \mathfrak{sp}(20, \mathbb{C})$, which is irreducible in the sense that after composition with the natural representation

$$
\mathfrak{sp}(20, \mathbb{C}) \longrightarrow \mathfrak{gl}(20, \mathbb{C})
$$

$\chi_C$ is an irreducible representation of the semi-simple complex Lie algebra $\mathfrak{g}_C$.

**Step 2.** In this step we classify all possible complex Lie algebra monomorphism $\chi_C : \mathfrak{g}_C \to \mathfrak{sp}(20, \mathbb{C})$ where $\mathfrak{g}_C$ is semi-simple and $\chi_C$ is irreducible in the sense described above. In order to classify $(\mathfrak{g}_C, \chi_C)$, we observe that it suffices to consider all 20-dimensional irreducible representations of complex semi-simple Lie algebras $\mathfrak{g}_C$. Actually, an irreducible representation $\mathfrak{g}_C \to \mathfrak{gl}(V_C)$ with $\dim(V_C) > 20$ admits no factorization

$$
\mathfrak{g}_C \longrightarrow \mathfrak{sp}(20, \mathbb{C}) \longrightarrow \mathfrak{gl}(V_C).
$$

The reason is that, since $\mathfrak{g}_C$ is mapped onto a proper subspace of $\mathfrak{sp}(20, \mathbb{C})$, the composition must decompose and hence is reducible. We can list all such possibilities. Our method is first to find all 20-dimensional representation of a semi-simple Lie algebra, and then exclude those whose images do not lie in $\mathfrak{sp}(20, \mathbb{C})$.

**Case 1.** $\mathfrak{g}_C$ has only one simple factor:

(a) $(A_1, [19])$,
(b) $(A_5, [0, 0, 1, 0, 0])$,
(c) $(C_2, [3, 0])$.

**Case 2.** $\mathfrak{g}_C$ has two simple factors:

(a) $(A_1 \oplus C_2, [1] \otimes [2, 0])$,
(b) $(A_1 \oplus C_2, [4] \otimes [1, 0])$,
(c) $(A_1 \oplus D_5, [1] \otimes [1, 0, 0, 0, 0])$,
(d) \((C_2 \oplus C_2, [1, 0] \otimes [0, 1])\).

A 20-dimensional representation has image in \(\mathfrak{sp}(20, \mathbb{C})\) if and only if there exists an one-dimensional component in the irreducible decomposition of the second wedge power. Here is an example. The pair \((A_1 \oplus A_1 \oplus A_4, [1] \otimes [1] \otimes [1, 0, 0, 0])\) associates to the semi-simple Lie algebra \(A_1 \oplus A_1 \oplus A_4\) a 20-dimensional representation with the highest weight \([1] \otimes [1] \otimes [1, 0, 0, 0]\). One easily checks that in the irreducible decomposition

\[
\bigwedge^2 ([1] \otimes [1] \otimes [1, 0, 0, 0]) \simeq [0] \otimes [2] \otimes [2, 0, 0, 0] \oplus [2] \otimes [0] \otimes [2, 0, 0, 0] \\
\oplus [0] \otimes [0] \otimes [0, 1, 0, 0] \oplus [2] \otimes [2] \otimes [0, 1, 0, 0],
\]

there is no one dimensional component.

**Step 3.** All possible simple real groups of Hodge types are listed in §4 [16]. Based on this and the classification given in the last step we can now discuss them case by case by applying the plethysm method. However, the following general result about the \(\mathbb{C}\)-PVHS structures on a tensor product will simplify our argument to a large extent. Since this result is of interest in itself, we would like to include a proof in this paper.

Let \(V\) be an irreducible \(\mathbb{C}\)-PVHS over a quasi-projective manifold \(X \setminus S\) and with unipotent local monodromy around \(S\). Let

\[\rho : \pi_1(X \setminus S) \to \text{GL}(V)\]

be the corresponding representation of the fundamental group and \(G\) be the Zariski closure of \(\rho\). Assume \(G = G_1 \times G_2\) with \(G_i\) simple. Then according to Schur’s lemma \(V\) is decomposed into

\[V \simeq V_1 \otimes V_2,\]

where \(V_i\) corresponds to a representation

\[\rho_i : \pi_1(X \setminus S) \to G_i.\]

**Proposition 5.4.** The \(\mathbb{C}\)-PVHS on \(V\) factors into PVHS’s on each factor \(V_i\), i.e. each \(V_i\) admits a \(\mathbb{C}\)-PVHS structure such that their tensor product on \(V_i\) coincides with the \(\mathbb{C}\)-PVHS on \(V\).

**Proof of Proposition 5.4:** We write \(\dim V_i = n_i\) for \(i = 1, 2\) and we assume that \(n_1 \geq n_2\) without lose of generality. We first need the following lemma.

**Lemma 5.5.** Each factor \(\rho_i\) has quasi-unipotent local monodromy around \(S\).

**Proof.** By choosing a base point in \(S\), the tensor product decomposition of \(V\) gives the tensor product decomposition of the vector space \(V \simeq V_1 \otimes V_2\) with group action, and since \(G_i\) is simple, \(G_i \subset \text{SL}(V_i)\) for \(i = 1, 2\). Now we apply \(\bigwedge^{n_2}\) on the above
isomorphism. Ex. 6.11(b) in [6] tells us that, for $V$ considered as a representation space of $\text{SL}(V_1) \times \text{SL}(V_2)$, there exists an irreducible component

$$S^{n_2}(V_1) \subset \bigwedge^{n_2}(V).$$

Since $V$ is of unipotent local monodromy, each direct component of $\bigwedge^{n_2}(V)$ is of unipotent local monodromy, too. In particular, $S^{n_2}(V_1)$ is of unipotent local monodromy. Let $T$ be one of local monodromy operators of $\rho_1$, and $\lambda$ be one of eigenvalues of $T$. Then clearly, $\lambda^{n_2}$ is one of eigenvalues of $T$ on $S^{n_2}(V_1)$, hence is equal to one. This proves that $\rho_1$ is of quasi-unipotent local monodromy. And by the unipotency of $\rho$, $\rho_2$ is of quasi-unipotent local monodromy as well. This completes the proof of Lemma 5.5. □

Since $\rho_i : \pi_1(X \setminus S) \to G_i$ is a Zariski dense representation into the simple algebraic group $G_i$ and with quasi-unipotent local monodromy around $S$, by Jost-Zuo [8] there exists a pluri-harmonic metric on the flat bundle $V_i$ with finite energy, which makes $V_i$ into a Higgs bundle $(E, \theta)^i_i$ over $X \setminus S$. Furthermore, T. Mochizuki [9] has analyzed the singularity of this harmonic metric in detail and has shown that $(E, \theta)^i_i$ admits a logarithmic extension $(\tilde{E}, \tilde{\theta})^i_i$ over $X$, i.e. $\tilde{E}_i$ is an extension of $E_i$, $\tilde{\theta}_i$ is an extension of $\theta_i$ and such that

$$\tilde{\theta} : \tilde{E}_i \to \tilde{E}_i \otimes \Omega^1_X(\log S).$$

Such a pluri-harmonic metric is called tame. In this case the residue of $\tilde{\theta}$ along $S$ is nilpotent.

From the proof of Lemma 5.5, we know that, by applying the Schur functor $\bigwedge^{n_2}$, one finds a direct factor of $\bigwedge^{n_2}(V_1 \otimes V_2)$ of the form

$$S^{n_2}(V_1) \otimes \text{det}(V_2),$$

and $S^{n_2}$ is non-trivial. Since $G_2$ is simple, $\text{det}(V_2)$ is the trivial representation.

We consider $G_1$ as a simple algebraic subgroup of $\text{GL}(V_1)$. Since the Schur functor $S^{n_2}$ is non-trivial and $G_1$ is a simple algebraic group, the representation

$$S^{n_2} : G_1 \to \text{GL}(S^{n_2}(V_1))$$

is faithful. Since $\rho_1$ is Zariski dense in $G_1$, $S^{n_2}(\rho_1)$ is irreducible. Since $\bigwedge^{n_2}(V_1 \otimes V_2)$ is semi-simple, there exists a decomposition

$$\bigwedge^{n_2}(V_1 \otimes V_2) = \bigoplus_i S_i \otimes W_i,$$

where $S_i$ are irreducible and $W_i$ are trivial. By Deligne's Prop. 1.13 in [4], there exists uniquely $\mathbb{C}$-PVHS on $S_i$ and $\mathbb{C}$-HS on $W_i$ such that the direct sum of the
tensor products of them coincides with the $C$-PVHS on $\bigwedge^{n_2}(V_1 \otimes V_2)$. So, in particular, there exists a $C$-PVHS on $S^{n_2}(V_1)$.

By the uniqueness of such pluri-harmonic metric, $S^{n_2}(\bar{E}, \bar{\theta})_1$ coincides with the $C$-PVHS on $S^{n_2}(V_1)$. Hence $S^{n_2}(\bar{E}, \bar{\theta})_1$ is a fixed point of the $C^\times$-action. The representation $G_1 \to \text{GL}(S^{n_2}(V_1))$ induces a morphism

$$\phi_{S^{n_2}} : M(\pi_1(X \setminus S), G_1)^{s.s} \to M(\pi_1(X \setminus S), \text{GL}(S^{n_2}(V_1)))^{s.s}$$

between the corresponding moduli spaces of semi-simple representations. By Simpson’s Cor. 9.16 in [17], $\phi_{S^{n_2}}$ is finite.

If $S = \emptyset$, then $C^\times$ acts on both moduli spaces continuously via Hermitian-Yang-Mills metrics on poly-stable Higgs bundles $(E, t\theta)$. And this action is compatible with $\phi_{S^{n_2}}$. Since $S^{n_2}(\rho_1)$ is a fixed point of $C^\times$-action, $\rho_1$ is a fixed point of $C^\times$-action. This just means that $(E, \theta)_1$ is a fixed point of $C^\times$-action. Hence $(E, \theta)_1$ is a $C$-PVHS on $V_1$. In general $S \neq \emptyset$. We take a curve $C \setminus S \subset X \setminus S$, which is a complete intersection of ample hypersurfaces. Taking the restrictions

$$\rho_1|_{C \setminus S} \in M(\pi_1(C \setminus S), G_1)^{s.s},$$

we have

$$S^{n_2}(\rho_1)|_{C \setminus S} \in M(\pi_1(C \setminus S), \text{GL}(S^{n_2}(V_1)))^{s.s}.$$

We consider the map

$$\phi_{S^{n_2}} : M(\pi_1(C \setminus S), G_1)^{s.s} \to M(\pi_1(C \setminus S), \text{GL}(S^{n_2}(V_1)))^{s.s}.$$ 

By Simpson’s main theorem in [15], there exist Hermitian-Yang-Mills metrics on poly-stable Higgs bundles on $C$ with logarithmic pole of the Higgs field on $S$. And the $C^\times$-action can be defined on both spaces of semi-simple representations on $C \setminus S$ via Hermitian-Yang-Mills metric on $(\bar{E}, \bar{\theta})$. Applying the same argument as above to the compact case, we show that the pulled back Higgs bundle $(\bar{E}, \bar{\theta})_1$ to $C \setminus S$ is a fixed point of the $C^\times$-action. If we choose $C \setminus S$ sufficiently ample, then $(\bar{E}, \bar{\theta})_1$ is also a fixed point of the $C^\times$-action. (The isomorphism $(\bar{E}, \bar{\theta})_1|_C \simeq (\bar{E}, \bar{\theta})_1|_C$ extends to an isomorphism $(\bar{E}, \bar{\theta})_1 \simeq (E, t\theta)_1|_C$ if $C$ is sufficiently ample.) Again by Simpson, $(\bar{E}, \bar{\theta})_1$ is a $C$-PVHS on $V_1$.

Similarly, we also show that $V_2$ admits a $C$-PVHS. The tensor product of $C$-PVHS on $V_1$ and on $V_2$ is a $C$-PVHS on $V_1 \otimes V_2$. By Deligne’s uniqueness theorem on $C$-PVHS on irreducible local systems, this tensor product coincides with the original $C$-PVHS on $V_1 \otimes V_2$. Proposition [14] is completed. \(\square\)

Now we start with the analysis of case 2. By the above proposition, we know that in this case we have an isomorphism

$$(E, \theta) \simeq (E_1, \theta_1) \otimes (E_2, \theta_2),$$
where each \((E_i, \theta_i)\) is a system of Hodge bundles. Because the Hodge numbers of \(E\) are \(1, 9, 9, 1\), it is not difficult to see that, up to permutation of factors, the Hodge numbers of \((E_1, \theta_1)\) are \(1, 1, 9, 9, 1\). Since \((E_1, \theta_1)\) comes from a \(\mathbb{R}\)-PVHS, \((E_2, \theta_2)\) also comes from a \(\mathbb{R}\)-PVHS. This implies that, if \(g = g_1 \oplus g_2\), then up to permutation of factors, \(g_1 = \mathfrak{su}(1, 1)\) and \(g_2 \subset \mathfrak{so}(2, 8)\).

From the list in §4 \[16\], we know that the possible real forms of \(\mathfrak{sp}(4, \mathbb{C})\) are \(\mathfrak{sp}(1, 1)\) and \(\mathfrak{sp}(4, \mathbb{R})\), and those of \(\mathfrak{so}(10, \mathbb{C})\) are \(\mathfrak{so}(2, 8)\) and \(\mathfrak{so}(4, 6)\), respectively. It is straightforward to check that the only possibility of case 2 is

- case (2c) \((\mathfrak{su}(1, 1) \oplus \mathfrak{so}(2, 8), \text{id} \otimes \text{id})\).

Obviously, case (1a) is impossible since it is of weight 19. For those real forms of Hermitian types in case 1 we can again compare the Hodge numbers. The only possibilities are

- case (1b) \((\mathfrak{su}(3, 3), \Lambda^3)\);
- case (1c) \((\mathfrak{sp}(1, 1), S^3)\).

Note that case (1c) is of non-Hermitian type.

**Step 4.** We have already excluded case (1b) using the method of plethysm. In the last step, we apply the method further in order to exclude the left two cases. The argument for case (2c) is similar. We give the analogous statement of Proposition \[3.4\] as follows:

\[
S^2(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2) \oplus (E_3, \theta_3)
\]

where

\[
E_1 = E_1^{4.2} \oplus E_1^{3,3} \oplus E_1^{2.4}
\]

\[
E_2 = E_2^{4.2} \oplus E_2^{3,3} \oplus E_2^{2.4}
\]

\[
E_3 = E_3^{6.0} \oplus E_3^{5,1} \oplus E_3^{4,2} \oplus E_3^{3,3} \oplus E_3^{2.4} \oplus E_3^{1.5} \oplus E_3^{0.6}
\]

The Hodge numbers of \(E_1\) are 0, 0, 1, 1, 1, 0, 0, respectively, those of \(E_2\) are 0, 0, 8, 29, 8, 0, 0 and those of \(E_3\) are 1, 9, 45, 52, 45, 9, 1. So by the computational result in Prop. \[5.3\] we see that case (2c) is impossible. For case (1c), the corresponding result is the following:

\[
S^2(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2) \oplus (E_3, \theta_3) \oplus (E_4, \theta_4)
\]

where the respective dimensions of \(E_i\) are 10, 35, 81, 84. But we are unable to obtain further information on the Hodge numbers of \(E_i\), because \(\text{Sp}(1, 1)\) is of non-Hermitian type. But fortunately we can still get a contradiction by the actual computation. The argument works as follows. The first Hodge bundle of \(S^2(E, \theta)\) is of dimension 1, it must lie in one of \((E_i, \theta_i)\), and hence the rank of the Higgs subsheaf generated by the first Hodge bundle will not exceed 84. Over the point
used in Prop. 5.3, the rank of the stalk of the Higgs subsheaf generated by the first Hodge bundle is not less than

\[ 1 + 9 + 45 + 78 = 133. \]

This gives the desired contradiction for case (1c). The proof is complete. \(\square\)

References

[1] Allcock, D.; Carlson, J.; Toledo, D.; The complex hyperbolic geometry of the moduli space of cubic surfaces. J. Algebraic Geometry. 11 (2002), no. 4, 659-724.

[2] Carlson, J.; Müller-Stach, S.; Peters, C.; Period Mappings and Period Domains, Cambridge Studies in Advanced Mathematics, Cambridge University Press 2003.

[3] Deligne, P.; Théorie de Hodge II, Publ.Math.I.H.E.S. 40(1971), 5-57.

[4] Deligne, P.; Un théorème de finitude pour la monodromie, Discrete Groups in Geometry and Analysis, Birkhauser 1987, 1-19.

[5] Dolgachev, I.; van Geemen, B.; Kondo, S.; A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces. J. Reine Angew. Math. 588 (2005), 99-148.

[6] Fulton, W.; Harris, J.; Representation theory, A first course, GTM 129.

[7] Griffiths, P.; Periods of integrals on algebraic manifolds II. Local study of the period mapping, American J. Math. (90), 1968 805-865.

[8] Jost, J.; Zuo, K.; Harmonic maps and \(\text{SL}(r,C)\)-representations of fundamental groups of quasiprojective manifolds. J. Algebraic Geometry. 5 (1996), no. 1, 77-106.

[9] Mochizuki, T.; Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure. J. Differential Geometry. 62 (2002), no. 3, 351-559.

[10] Mok, N.; Uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domains. Annals of Mathematics. 125(1987), No.1, 105-152.

[11] Matsumoto, K.; Sasaki, T.; Yoshida, M.; The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type \((3,6)\), International Journal of Mathematics, Vol 3, No.1, 1-164, 1992.

[12] Moeller, M.; Viehweg, E.; Zuo, K.; Stability of Hodge bundles and a numerical characterization of Shimura varieties, arXiv AG/07063462.

[13] Nagel, J.; The image of the Abel-Jacobi map for complete intersections, Ph.D Thesis, Rijksuniversiteit Leiden 1997.

[14] Sheng, M.; On the geometric realizations of Hermitian symmetric domains, Ph.D Thesis, The Chinese University of Hong Kong 2005.

[15] Simpson, C.; Harmonic bundles on noncompact curves. J. Amer. Math. Soc. 3 (1990), no. 3, 713-770.

[16] Simpson, C.; Higgs bundles and local systems, Publ. Math. I.H.E.S., No.75, 5-95, 1992.

[17] Simpson, C.; Moduli of representations of the fundamental group of a smooth projective varieties II, Publ. Math. I.H.E.S., No.80, 5-79, 1994.
[18] Sasaki, T; Yamaguchi, K; Yoshida, M; On the rigidity of differential systems modelled on Hermitian symmetric spaces and disproofs of a conjecture concerning modular interpretations of configuration spaces. Advanced Studies in Pure Mathematics 25, 318-354, 1997.

[19] Sheng, M; Zuo, K; Calabi-Yau like PVHS and characteristic subvariety over bounded symmetric domains, arXiv: AG/07053779.

[20] Viehweg, E; Zuo, K; A characterization of certain Shimura curves in the moduli stack of abelian varieties, J. Differential Geometry, 66, No.2, 233-287, 2004.

[21] Viehweg, E; Zuo, K; Arakelov inequalities and the uniformization of certain rigid Shimura varieties, To be appeared in J. Differential Geometry.

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