FACTORIZATION OF COMPLETE INTERSECTIONS IN $\mathbb{P}^5$.

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Abstract. Let $X$ be a complete intersection of two hypersurfaces $F_n$ and $F_k$ in $\mathbb{P}^5$ of degree $n$ and $k$ respectively with $n \geq k$, such that the singularities of $X$ are nodal and $F_k$ is smooth. We prove that if the threefold $X$ has at most $(n + k - 2)(n - 1) - 1$ singular points, then it is factorial.

1. Introduction

In this paper we shall extend to the complete intersection setting a recent theorem of Cheltsov [4], in which he obtained a sharp bound for the number of nodes a threefold hypersurface can have and still be factorial.

Suppose that $X$ is the complete intersection of two hypersurfaces $F_n$ and $F_k$ in $\mathbb{P}^5$ of degree $n$ and $k$ respectively with $n \geq k$, such that $X$ is a nodal threefold. We will prove the following.

Theorem 1.1. Suppose that $F_k$ is smooth. Then the threefold $X$ is $\mathbb{Q}$-factorial, when

$$|Sing(X)| \leq (n + k - 2)(n - 1) - 1.$$  

The next example of a non-factorial nodal complete intersection threefold suggests that the number of nodes, that a hypersurface can have while being factorial, should be strictly less than $(n + k - 2)^2$.

Example 1.2. Let $X$ be the complete intersection in $\mathbb{P}^5$ of two smooth hypersurfaces

$$F = x_3f_1(x_0, x_1, x_2, x_3, x_4, x_5) + x_4f_2(x_0, x_1, x_2, x_3, x_4, x_5) + x_5f_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$G = x_3g_1(x_0, x_1, x_2, x_3, x_4, x_5) + x_4g_2(x_0, x_1, x_2, x_3, x_4, x_5) + x_5g_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

where $f_1, f_2, f_3$ are general hypersurfaces of degree $n - 1$ and $g_1, g_2, g_3$ general hypersurfaces of degree $k - 1$. Then the singular locus $Sing(X)$, which is given by the vanishing of the polynomials

$$x_3 = x_4 = x_5 = f_1g_2 - f_2g_1 = f_1g_3 - f_3g_1 = 0,$$

consists of exactly $(n + k - 2)^2$ nodal points and the threefold $X$ is not factorial.

Therefore, we can expect the following stated in [3] to be true.

Conjecture 1.3. Suppose that $F_k$ is smooth. Then the threefold $X$ is $\mathbb{Q}$-factorial, when

$$|Sing(X)| \leq (n + k - 2)(n + k - 2) - 1.$$  

The assumption of Theorem 1.1 about the smoothness of $F_k$ is essential, as Example 28 in [3] suggests.

In the case of a nodal threefold hypersurface in $\mathbb{P}^4$, namely when $k = 1$, several attempts where made towards proving Theorem 1.1 as one can see in [5] and [12]. However, a complete proof for $k = 1$ was given in [4].

2. Preliminaries

Let $\Sigma$ be a finite subset in $\mathbb{P}^N$. The points of $\Sigma$ impose independent linear conditions on homogeneous forms in $\mathbb{P}^N$ of degree $\xi$, if for every point $P$ of the set $\Sigma$ there is a homogeneous form on $\mathbb{P}^N$ of degree $\xi$ that vanishes at every point of the set $\Sigma \setminus P$ and does not vanish at the point $P$.

The following result, which relates the notion of $\mathbb{Q}$-factoriality with that of independent linear conditions, is due to [6] and was stated in the present form in [3].

I would like to thank Ivan Cheltsov for suggesting the problem to me and for useful comments.
Theorem 2.1. The threefold $X$ is $\mathbb{Q}$-factorial in the case when its singular points impose independent linear conditions on the sections of $H^0(\mathcal{O}_{\mathbb{P}^5}(2n + k - 6)|_G)$.

The following result was proved in [11] and follows from a result of J.Edmonds [9].

Theorem 2.2. The points of $\Sigma$ impose independent linear conditions on homogeneous forms of degree $\xi \geq 2$ if at most $\xi k + 1$ points of $\Sigma$ lie in a $k$-dimensional linear subspace of $\mathbb{P}^N$.

By [1] and [7] we also know the following.

Theorem 2.3. Let $\pi : Y \to \mathbb{P}^2$ be a blow up of distinct points $P_1, \ldots, P_5$ on $\mathbb{P}^2$. Then the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(\xi)) - \sum_{i=1}^5 E_i|$ is base-point-free for all $\xi \leq \max (m(\xi + 3 - m) - 1, m^2)$, where $E_i = \pi^{-1}(P_i)$, $\xi \geq 3$, and $m = \lfloor \frac{\xi + 3}{2} \rfloor$, if at most $k(\xi + 3 - k) - 2$ points of the set $P_1, P_2, \ldots, P_5$ lie on a possibly reducible curve of degree $1 \leq k \leq m$.

What is next is an application, as stated in [12], of the modern Cayley-Bacharach theorem (see [10] or [5]).

Theorem 2.4. Let $\Sigma$ be a subset of a zero-dimensional complete intersection of the hypersurfaces $X_1, X_2, \ldots, X_N$ in $\mathbb{P}^N$ of degrees $d_1, \ldots, d_N$ respectively. Then the points of $\Sigma$ impose dependent linear conditions on homogeneous forms of degree $\sum_{i=1}^N \deg(X_i) - N - 1$ if and only if the equality $|\Sigma| = \prod_{i=1}^N d_i$ holds.

Again due to [1] we have the following.

Theorem 2.5. Let $\Lambda \subseteq \Sigma$ be a subset, let $\phi : \mathbb{P}^r \to \mathbb{P}^m$ be a general projection and let $\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^m}(t)|$ be a linear subsystem that contains all hypersurfaces of degree $t$ that pass through $\Lambda$. Suppose that

- the inequality $|\Lambda| \geq (n + k - 2)t + 1$ holds,
- the set $\phi(\Lambda)$ is contained in an irreducible reduced curve of degree $t$,

where $r > m \geq 2$. Then $\mathcal{M}$ has no base curves and either $m = 2$ or $t > n + k - 2$.

Finally, next is one of our basic tools, a proof of which can be found in [2].

Theorem 2.6. Let $\Sigma$ be a finite subset in $\mathbb{P}^N$ that is a disjoint union of finite subsets $\Lambda$ and $\Delta$, and $P$ be a point in $\Sigma$. Suppose that there is a hypersurface in $\mathbb{P}^N$ of degree $\alpha \geq 1$ that contains all points of the set $\Lambda \setminus P$ and does not contain $P$, and for every point $Q$ in the set $\Delta$ there is a hypersurface in $\mathbb{P}^N$ of degree $\beta \geq 1$ that contains all points of the set $\Sigma \setminus Q$ and does not contain the point $Q$. Then there is a hypersurface in $\mathbb{P}^N$ of degree $\gamma$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$, where $\gamma$ is a natural number such that $\gamma \geq \max(\alpha, \beta)$.

3. Proof of Theorem 2.1

Let us consider the complete intersection $X$ of two hypersurfaces $F_n$ and $F_k$ in $\mathbb{P}^5$ of degrees $n$ and $k$ respectively, with $n \geq k$, such that $X$ is a nodal threefold. Suppose, furthermore, that $F_k$ is smooth and $X$ has at most $(n + k - 2)(n - 1) - 1$ singular points. We denote now by $\Sigma \subset \mathbb{P}^5$ the set of singular points of $X$.

Definition 3.1. We say that the points of a subset $\Gamma \subset \mathbb{P}^r$ have property $\star$ if at most $t(n + k - 2)$ points of the set $\Gamma$ lie on a curve in $\mathbb{P}^r$ of degree $t \in \mathbb{N}$.

For a proof of the following we refer the reader to [3].

Lemma 3.2. The points of the set $\Sigma \subset \mathbb{P}^5$ have property $\star$.

According to Theorem 2.1 for any point $P \in \Sigma$ we need to prove that there is a hypersurface of degree $2n + k - 6$, that passes through all the points of the set $\Sigma \setminus P$, but not through the point $P$.

Remark 3.3. As we mentioned, the claim of Theorem 2.1 is true, when $k = 1$ and thus we need only consider the case $k \geq 2$. Furthermore, taking into account the following Lemma, we can assume that $n \geq 5$.  

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Lemma 3.4. The threefold $X$ is $\mathbb{Q}$-factorial, when

$$|\text{Sing}(X)| \leq (n+k-2)(n-1) - 1 \text{ and } k \leq n \leq 4.$$ 

Proof. Indeed, we consider the projection

$$\psi : \mathbb{P}^5 \dashrightarrow \Pi \cong \mathbb{P}^2,$$

from a general plane $\Gamma$ of $\mathbb{P}^5$ to another general plane $\Pi \cong \mathbb{P}^2$, that sends the set $\Sigma$ to $\psi(\Sigma) = \Sigma'$. Choose a point $P \in \Sigma$ and put $P' = \psi(P)$. We have the following cases.

- If $2 = n \geq k = 2$, then $|\Sigma| \leq 1$ and the result holds according to Theorem 2.1.
- If $3 = n \geq k = 2$, then $|\Sigma| \leq 5$ and it imposes independent linear conditions on forms of degree 2.
- If $3 = n \geq k = 3$, then $|\Sigma| \leq 7$ and it imposes independent linear conditions on forms of degree 3.
- If $4 = n \geq k = 2$, then $|\Sigma| \leq 11$ and at most $4t$ points lie on a curve in $\mathbb{P}^5$ of degree $t$. So, the 11 points of $\Sigma$ impose independent linear conditions on forms of degree 4.
- If $4 = n \geq k = 3$, then $|\Sigma| \leq 14$ and at most $5t$ points lie on a curve in $\mathbb{P}^5$ of degree $t$.

If the points of $\Sigma' \subset \Pi$ satisfy property $\star$, then the set $\Sigma' \setminus P'$ satisfies the requirements of Theorem 2.3 for $\xi = 5$ and this implies that the set $\Sigma$ imposes independent linear conditions on forms of degree 5.

Suppose on the contrary that the points $\Sigma'$ do not satisfy Theorem 2.3 for $\xi = 5$. In this case there is a curve $C_2$ of degree 2 in $\Pi$ that passes through at least 11 points of $\Sigma'$. If we take the cone over $C_2$ with vertex $\Gamma$, we obtain a hypersurface $f_2$ in $\mathbb{P}^5$. Denote by $\Lambda_2$ the points of $\Sigma$ that lie on $f_2$. From Theorem 2.4 it follows that the points of $\Lambda_2$ impose independent linear conditions on homogeneous forms of degree 5($2-1$) = 4, since $\Lambda_2$ is a subset of the complete intersection of hypersurfaces of degree 2 in $\mathbb{P}^5$.

The set $|\Sigma \setminus \Lambda_2| \leq 3$ imposes independent linear conditions on forms of degree 2 and, by applying Theorem 2.6 to the two disjoint sets $\Lambda_2$ and $\Sigma \setminus \Lambda_2$, we get that the points of $\Sigma$ impose independent linear conditions on forms of degree 5.

- $4 = n \geq k = 4$. Then $|\Sigma| \leq 17$ and at most $6t$ points lie on a curve $C_t \in \mathbb{P}^5$ of degree $t$.

If the points of $\Sigma' \subset \Pi$ satisfy property $\star$, then the set $\Sigma' \setminus P'$ satisfies the requirements of Theorem 2.3 for $\xi = 6$ and this implies that the set $\Sigma$ imposes independent linear conditions on forms of degree 6.

Suppose on the contrary that the points $\Sigma'$ do not satisfy Theorem 2.3 for $\xi = 6$. In this case there is a curve $C_2$ of degree 2 in $\Pi$ that passes through at least 13 points of $\Sigma'$. If we take the cone over $C_2$ with vertex $\Gamma$, we obtain a hypersurface $f_2$ in $\mathbb{P}^5$. Denote by $\Lambda_2$ the points of $\Sigma$ that lie on $f_2$. From Theorem 2.4 it follows that the points of $\Lambda_2$ impose independent linear conditions on homogeneous forms of degree 5($2-1$) = 4, since $\Lambda_2$ is a subset of the complete intersection of hypersurfaces of degree 2 in $\mathbb{P}^5$.

The set $|\Sigma \setminus \Lambda_2| \leq 4$ imposes independent linear conditions on forms of degree 2 and, by applying Theorem 2.6 to the two disjoint sets $\Lambda_2$ and $\Sigma \setminus \Lambda_2$, we get that the points of $\Sigma$ impose independent linear conditions on forms of degree 6.

As we saw above, for $3 \leq n \leq 5$ the points of $\Sigma$ impose independent linear conditions on forms of degree $2n+k-6$, and thus, by Theorem 2.1 the threefold $X$ is $\mathbb{Q}$-factorial. \qed

Lemma 3.5. Suppose that all the singularities of $X$ lie on a plane $\Pi \subset \mathbb{P}^5$. Then for any point $P \in \Sigma$ there is hypersurface of degree $(2n+k-6)$ that contains $\Sigma \setminus P$, but does not contain the point $P$.

Proof. By Remark 3.3 we can see that $\xi = 2n+k-6 \geq 6$. Also, we have

$$|\Sigma \setminus P| \leq \max \left\{ \left\lfloor \frac{2n+k-3}{2} \right\rfloor (2n+k-3 - \left\lfloor \frac{2n+k-3}{2} \right\rfloor) - 1, \left\lfloor \frac{2n+k-3}{2} \right\rfloor^2 \right\},$$

for $k \geq 2$ and $n \geq 5$. In order to show that at most $t(2n+k-3-t) - 2$ points of $\Sigma$ lie on a curve of degree $t$ in $\Pi$, it is enough to show that

$$t(2n+k-3-t) - 2 \geq t(n+k-2) \iff t(n-t-1) \geq 2, \text{ for all } t \leq \frac{2n+k-3}{2}.$$
For $t = 1$ the inequality holds, since $n \geq 5$, and we can assume that $t \geq 2$. It remains to show that $t < n - 1$. Suppose on the contrary that $t \geq n - 1$. The quantity $t(2n + k - 3 - t) - 2$ rises for all $n - 1 \leq t \leq \lfloor \frac{2n+k-3}{2} \rfloor$ and we have

$$|\Sigma \setminus P| \leq (n - 1)(n + k - 2) - 2 \leq t(2n + k - 3 - t) - 2.$$ 

Therefore we see that the requirement of Theorem 2.3 that at most $t$ of $\Sigma$ lie on a curve of degree $t$ in $\Pi$ is satisfied by the set $\Sigma \setminus P$ for all $t \leq \lfloor \frac{2n+k-3}{2} \rfloor$. So there is a hypersurface of degree $(2n + k - 6)$ that contains $\Sigma \setminus P$, but does not contain point $P$. □

Taking into account Theorem 2.3 we can reduce to the case $\Sigma$ is a finite set in $\mathbb{P}^3$, such that at most $(n + k - 2)t$ of its points are contained in a curve in $\mathbb{P}^3$ of degree $t \in \mathbb{N}$. Now fix a general plane $\Pi \in \mathbb{P}^3$ and let

$$\phi: \mathbb{P}^3 \longrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Denote by $\Sigma' = \phi(\Sigma)$ and $P' = \phi(P)$.

**Lemma 3.6.** Suppose that the points of $\Sigma' \subseteq \Pi$ have the property $\star$. Then there is a hypersurface of degree $2n + k - 6$ that contains $\Sigma \setminus P$ and does not contain $P$.

**Proof.** The points of the set $\Sigma'$ satisfy the requirements of Theorem 2.3, following the proof of Lemma 3.5. Thus, there is a curve $C$ in $\Pi$ of degree $2n + k - 6$, that passes through all the points of the set $\Sigma' \setminus P'$, but not through the point $P'$. By taking the cone in $\mathbb{P}^3$ over the curve $C$ with vertex $O$, we obtain the required hypersurface. □

We may assume then, that the points of the set $\Sigma' \subseteq \Pi$ do not have property $\star$. Then there is a subset $\Lambda_1^r \subseteq \Sigma$ with $|\Lambda_1^r| > r(n + k - 2)$, but after projection the points

$$\phi(\Lambda_1^r) \subseteq \Sigma' \subseteq \Pi \cong \mathbb{P}^2$$

are contained in a curve $C_r \subseteq \Pi$ of degree $r$. Moreover, we may assume that $r$ is the smallest natural number, such that at least $(n + k - 2)r + 1$ points of $\Sigma'$ lie on a curve of degree $r$, which implies that the curve $C_r$ is irreducible and reduced.

By repeating how we constructed $\Lambda_1^r$, we obtain a non-empty disjoint union of subsets

$$\Lambda = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \subseteq \Sigma,$$

such that $|\Lambda_j^i| > j(n + k - 2)$, the points of the set

$$\phi(\Lambda_j^i) \subseteq \Sigma'$$

are contained in an irreducible curve in $\Pi$ of degree $j$, and the points of the subset

$$\phi(\Sigma \setminus \Lambda) \subseteq \Sigma' \subseteq \Pi \cong \mathbb{P}^2$$

have property $\star$, where $c_j \geq 0$. Let $\Xi_j^i$ be the base locus of the linear subsystem in $|O_{P_1}(j)|$ of all surfaces of degree $j$ passing through the set $\Lambda_j^i$. Then according to Theorem 2.3 the base locus $\Xi_j^i$ is a finite set of points and we have $c_r > 0$ and

$$|\Sigma \setminus \Lambda| < (n - 1)(n + k - 2) - \sum_{i=r}^{l} i(n + k - 2)c_i = (n + k - 2) \left(n - 1 - \sum_{i=r}^{l} ic_i \right).$$

**Corollary 3.7.** The inequality $\sum_{i=r}^{l} ic_i \leq n - 2$ holds.

Put $\Delta = \Sigma \cap (\bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i)$. Then $\Lambda \subseteq \Delta \subseteq \Sigma$.

**Lemma 3.8.** The points of the set $\Delta$ impose independent linear conditions on forms of degree $2n + k - 6$. 

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Proof. We have the exact sequence
\[ 0 \rightarrow \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n + k - 6) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2n + k - 6) \rightarrow \mathcal{O}_\Delta \rightarrow 0 , \]
where \( \mathcal{I}_\Delta \) is the ideal sheaf of the closed subscheme \( \Delta \) of \( \mathbb{P}^3 \). Then the points of \( \Delta \) impose independent linear conditions on forms of degree \( 2n + k - 6 \), if and only if
\[ h^1 \left( \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n + k - 6) \right) = 0 . \]
We assume on the contrary that \( h^1 \left( \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n + k - 6) \right) \neq 0 \). Let \( \mathcal{M} \) be a linear subsystem in \( |\mathcal{O}_{\mathbb{P}^3}(n - 2)| \) that contains all surfaces that pass through all points of the set \( \Delta \). Then the base locus of \( \mathcal{M} \) is zero-dimensional, since \( \sum_{i=r}^l ic_i \leq n - 2 \) and
\[ \Delta \subseteq \bigcup_{i=r}^l \bigcup_{j=1}^{c_i} \Xi^i_j , \]
but \( \Xi^i_j \) is a zero-dimensional base locus of a linear subsystem of \( |\mathcal{O}_{\mathbb{P}^3}(j)| \). Let \( \Gamma \) be the complete intersection
\[ \Gamma = M_1 \cdot M_2 \cdot M_3 , \]
of three general surfaces \( M_1, M_2, M_3 \) in \( \mathcal{M} \). Then \( \Gamma \) is zero-dimensional and \( \Delta \) is closed subscheme of \( \Gamma \). Let
\[ \mathcal{I}_\Gamma = \text{Ann} \left( \mathcal{I}_\Delta / \mathcal{I}_\Gamma \right) . \]
Then
\[ 0 \neq h^1 \left( \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n + k - 6) \right) = h^0 \left( \mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n - k - 4) \right) - h^0 \left( \mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n - k - 4) \right) . \]
Therefore \( h^0 \left( \mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n - k - 4) \right) \neq 0 \) and there is a surface \( F \in |\mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n - k - 4)| \). We have
\[ (n - k - 4)(n - 2)^2 = F \cdot M_2 \cdot M_3 \geq h^0(\mathcal{O}_{\mathcal{I}_\Gamma}) = h^0(\mathcal{O}_{\mathcal{I}_\Gamma}) - h^0(\mathcal{O}_{\mathcal{I}_\Delta}) = (n - 2)^3 - |\Delta| , \]
which implies \( |\Delta| \geq (k + 2)(n - 2)^2 \). But \( |\Delta| \leq |\Sigma| < (n - 1)(n + k - 2) \), which is impossible since \( k \geq 2 \) and \( n \geq 5 \). \( \square \)

We see that \( \Delta \nsubseteq \Sigma \). Put \( \Gamma = \Sigma \setminus \Delta \) and \( d = 2n + k - 6 - \sum_{i=r}^l ic_i \).

Lemma 3.9. The inequality \( d \geq 3 \) holds.

Proof. Suppose that \( d \leq 2 \). Since \( \sum_{i=r}^l ic_i \leq n - 2 \) due to Corollary 3.7, we have
\[ 2 \geq d = 2n + k - 6 - \sum_{i=r}^l ic_i \geq 2n + k - 6 - (n - 2) = n + k - 4 \geq 3 , \]
which is impossible. \( \square \)

For the number of points of the set \( \Gamma' \) we have
\[ |\Gamma'| = |\Gamma| \leq |\Sigma \setminus \Delta| \leq (n + k - 2) \left( n - 1 - \sum_{i=r}^l ic_i \right) - 2 , \]
and for \( d = 2n + k - 6 - \sum_{i=r}^l ic_i \), since \( n \geq 5 \) and \( k \geq 2 \), we get
\[ |\Gamma'| \leq (n + k - 2) \left( n - 1 - \sum_{i=r}^l ic_i \right) - 2 \leq \max \left\{ \left[ \frac{d + 3}{2} \right] \left( d + 3 - \left[ \frac{d + 3}{2} \right] \right) - 1, \left[ \frac{d + 3}{2} \right]^2 \right\} . \]

Lemma 3.10. If the points of the set \( \Gamma \) impose dependent linear conditions on forms of degree \( d \), then at most \( d \) points of the set \( \Gamma' \) lie on a line in \( \Pi \cong \mathbb{P}^2 \).

Proof. Let us assume on the contrary that there is a line that contains at least \( d + 1 \) points of \( \Gamma \). Since the points of \( \Gamma \) satisfy property \( \star \), at most \( n + k - 2 \) of its points lie on a line, thus
\[ n + k - 2 \geq d + 1 = 2n + k - 6 - \sum_{i=r}^l ic_i + 1 , \]
which along with Corollary 3.7 implies that $$n - 3 \leq \sum_{i=r}^{t} ic_i \leq n - 2.$$ If $$\sum_{i=r}^{t} ic_i = n - 2,$$ then $$|\Gamma| \leq n + k - 4$$ and we get a contradiction as no more than $$n + k - 4 < d + 1$$ points can lie on a line. If $$\sum_{i=r}^{t} ic_i = n - 3,$$ then $$|\Gamma| \leq 2(n + k - 3)$$ and according to Theorem 2.2 the points of $$\Gamma$$ impose independent linear conditions on forms of degree $$d = n + k - 3,$$ which contradicts our assumption. By Theorem 2.5 the number of points of $$\Gamma'$$ that can lie on a line $$\Pi \cong \mathbb{P}^2$$ is at most $$d.$$ \[\square\]

**Lemma 3.11.** At most $$(n + k - 2)t$$ points of the set $$\Gamma'$$ lie on a curve in $$\Pi \cong \mathbb{P}^2$$ of degree $$t,$$ for every $$t \leq \frac{d + 3}{2}.$$

**Proof.** We need to check the condition that at most $$(n + k - 2)t$$ points of $$\Gamma'$$ lie on a curve of degree $$t$$ only for $$2 \leq t \leq \frac{d + 3}{2},$$ such that

$$t(d + 3 - t) - 2 < |\Gamma'|.$$ Because the set $$\Gamma'$$ satisfies property $$\star,$$ at most $$(n + k - 2)t$$ of its points can lie on a curve of degree $$t$$ and therefore it is enough to prove that

$$t(d + 3 - t) - 2 \geq (n + k - 2)t \iff t \left(n - 1 - \sum_{i=r}^{t} ic_i \right) \geq 2,$$ for all $$2 \leq t \leq \frac{d + 3}{2}.$$ As we saw Lemma 3.10 implies that $$t \geq 2$$ and we only need to show that $$t < n - 1 - \sum_{i=r}^{t} ic_i.$$ Suppose that

$$n - 1 - \sum_{i=r}^{t} ic_i \leq t \leq \frac{d + 3}{2},$$

then

$$(n - 1 - \sum_{i=r}^{t} ic_i)(n + k - 2) = (n - 1 - \sum_{i=r}^{t} ic_i)(d + 3 - (n - 1 - \sum_{i=r}^{t} ic_i)) - 2 \leq t(d + 3 - t) - 2,$$ since the quantity $$t(d + 3 - t) - 2$$ increases, as $$t \leq \frac{d + 3}{2}$$ increases. But then

$$(n - 1 - \sum_{i=r}^{t} ic_i)(n + k - 2) - 2 \leq t(d + 3 - t) - 2 < |\Gamma'| \leq (n - 1 - \sum_{i=r}^{t} ic_i)(n + k - 2) - 2,$$ which is a contradiction. \[\square\]

**Lemma 3.12.** The points of the set $$\Sigma$$ impose independent linear conditions on homogeneous forms of degree $$2n + k - 6.$$

**Proof.** According to Lemma 3.9 and Lemma 3.11 all the requirements of Theorem 2.3 for $$\xi = d$$ are satisfied and thus, the points of $$\Gamma$$ impose independent linear conditions on homogeneous forms of degree $$d.$$ Hence, for any point $$Q$$ in $$\Gamma,$$ there is a hypersurface $$G_Q$$ of degree $$d,$$ such that $$G_Q(\Gamma \setminus Q) = 0$$ and $$G_Q(Q) \neq 0.$$ Furthermore, by the way the set $$\Delta$$ was constructed, there is a form $$F$$ of degree $$\sum_{i=r}^{t} ic_i$$ in $$\mathbb{P}^3,$$ that vanishes at every point of the set $$\Delta,$$ but does not vanish at any point of the set $$\Gamma.$$ Therefore, for any point $$Q \in \Gamma$$ we obtain a hypersurface $$FG_Q$$ of degree $$2n + k - 6,$$ such that $$FG_Q(\Sigma) = 0$$ and $$FG_Q(Q) \neq 0.$$ Also, by Lemma 3.8 for any point $$R \in \Delta$$ there is a hypersurface of degree $$2n + k - 6$$ that passes through all points of $$\Delta \setminus R,$$ except for the point $$R.$$ By applying Theorem 2.6 to the two disjoint sets $$\Delta$$ and $$\Gamma,$$ we prove the Lemma. \[\square\]
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