ASSEMBLAGES AND STEERING IN GENERAL PROBABILISTIC THEORIES

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ABSTRACT. We study steering in the framework of general probabilistic theories. We show that for dichotomic assemblages, steering can be characterized in terms of a certain tensor cross norm, which is also related to a steering degree given by steering robustness. Another contribution is the observation that steering in GPTs can be conveniently treated using Choquet theory for probability measures on the state space. In particular, we find a variational expression for universal steering degree for dichotomic assemblages and obtain conditions characterizing unsteerable states analogous to some conditions recently found for the quantum case. The setting also enables us to rather easily extend the results to infinite dimensions and arbitrary numbers of measurements with arbitrary outcomes.

1. INTRODUCTION

EPR steering was first described by Schrödinger in 1936 [27] as a bipartite scenario where one party can steer the state of a distant party by performing local measurements, in a way that cannot be explained by classical correlations. A precise definition and a systematic treatment was given in [30, 16], where quantum steering was interpreted in operational terms, as the possibility of certifying entanglement when one party is untrusted. In this way, steering became an important resource in quantum information theory and has attracted a lot of attention due to applications as well as for its relations to other nonclassical phenomena, such as entanglement, Bell nonlocality or incompatibility of measurements. In recent years, methods for characterizing and quantifying steering were developed in the literature that can be efficiently evaluated by SDPs, see [11] for an overview. For a recent review of quantum steering see [29].

Quantum steering can be described as follows. Assume Alice and Bob share a bipartite state \( \rho_{AB} \). After Alice performs a measurement \( M_x \) on her side and obtains the result \( a \), Bob is left with the conditional (nonnormalized) state \( \rho_{a|x} = (M_a|x \otimes id)(\rho_{AB}) \). The assemblage \{\( \rho_{a|x} \)\} of conditional states admits a local hidden state (LHS) model if there is an ensemble of states \{\( \lambda(\omega), \rho_\omega \)\} from which \( \rho_{a|x} \) are obtained by a set of conditional probabilities \{\( p(a|x, \omega) \)\}, in this case Bob is not convinced that the state \( \rho_{AB} \) was entangled. If no LHS model exists, Bob can be sure of both entanglement and incompatibility of Alice’s measurements. However, there are entangled states that are unsteerable, which means that the assemblages obtained by any measurement always have a LHS model, this was already observed in [30, 16]. Some steerability criteria were recently obtained in [19, 20, 18] through a geometric approach.

Nonclassical phenomena are often studied in a broader framework of general probabilistic theories (GPTs) [6], which was used for better understanding the operational features of Bell nonlocality, incompatibility of measurements and steering, and their relations [5, 9, 4, 23, 24, 14]. GPTs include the classical and quantum theory, as well as the PR boxes exhibiting maximal violations of Bell inequalities [26], quantum channels [15] or post quantum steering [12]. Another motivation comes from quantum foundations, where the aim is to characterize quantum theory among physical theories.

In the GPT setting, the relations of these phenomena to some mathematical concepts can be revealed. For example, nonclassical correlations can be expressed in terms of tensor cross norms
in Banach spaces [3]. In [7], incompatibility of measurements in GPTs is characterized and quantified using different mathematical points of view: extendability of maps, theory of generalized spectrahedra and tensor cross norms.

The aim of the present work is to study steering in GPTs. We first restrict to dichotomic steering, where Alice uses only dichotomic measurements. In this case, we show that steering is characterized by tensor cross norms in very much the same way as obtained in [7]. In fact, the connection is immediate from the formal relation of the two notions of steering and incompatibility: in the framework of GPTs they can be seen as equivalent. The results then follow straightforwardly from those in [7], but we give different proofs, more clearly related to the structure of tensor products with hypercubic state spaces.

In the second part, we show that steering can be represented using Choquet order on the set of Radon probability measures over the state space. More precisely, the assemblages are represented by sets of (simple) probability measures and a LHS model is given by another probability measure that is an upper bound for the assemblage in Choquet order. We then obtain a convenient description of LHS models and steering using classical results in Choquet theory, which naturally includes infinite assemblages. In particular, we show that we can always use an LHS model concentrated on the extreme boundary of the state space, equivalently, the LHS model is given by a boundary measure. Further, we may assume that this measure is invariant under the group of transformations leaving the assemblage invariant. Using this representation, we find a variational expression for the steering degree for dichotomic assemblages that is independent of the size of the assemblage. In the quantum case, this expression coincides with the quantum steering constants obtained in [8] and relates to the L-summing constant for centrally symmetric state spaces. We also find characterizations of (one way) unsteerable states in a GPT, which are similar to those obtained in [19, 20] in the quantum case.

2. Notations and preliminaries

In this paper we follow the general assumptions and formalism of GPTs used in [17]. See e.g. [25] for another recent exposition with a convenient diagrammatic presentation.

A system of a general probabilistic theory (GPT) is a triple $(V, V^+, \mathbb{1})$, where $V$ is a real vector space of finite dimension, $V^+$ is a closed convex cone which is pointed $(V^+ \cap (-V^+) = \{0\})$ and separating $(V = V^+ - V^+)$ and $\mathbb{1}$ is a distinguished element of the dual vector space $V^* = A$. In $A$, the dual cone $A^+$ is defined as

$$A^+ = \{ f \in A, \langle f, v \rangle \geq 0, \forall v \in V^+ \},$$

here $\langle \cdot, \cdot \rangle$ denotes the duality $A \times V \rightarrow \mathbb{R}$. We will use the notation $\leq$ for the ordering induced by $V^+$ or $A^+$ in their respective spaces. In this ordering, $\mathbb{1}$ is assumed to be an order unit, which means that for all $f \in A$ there is some $\alpha > 0$ such that $\alpha \mathbb{1} \pm f \in A^+$, or, equivalently, that $\mathbb{1}$ is an interior element in $A^+$. The order unit norm in $A$ is defined as

$$\| f \|_1 = \inf \{ \lambda > 0, \lambda \mathbb{1} \pm f \in A^+ \}.$$

The subset

$$K := \{ \rho \in V^+, \langle \mathbb{1}, \rho \rangle = 1 \}.$$

is interpreted as the set of states and called the state space. Clearly, $K$ is a compact convex subset of $V$ and a base of the cone $V^+$. The dual of $\| \cdot \|_1$ in $V$ is the base norm

$$\| v \|_V = \inf \{ \langle \mathbb{1}, v_+ + v_- \rangle, v = v_+ - v_-, v_+, v_- \in V^+ \} = \sup_{\pm f \leq 1} \langle f, v \rangle.$$

Elements of the unit interval $E = \{ f \in A, 0 \leq f \leq 1 \}$ are called effects. The effects can be identified with affine maps $K \rightarrow [0, 1]$ and are interpreted as dichotomic (or yes-no) measurements of the system: the value $\langle f, \rho \rangle \in [0, 1]$ gives the probability that the measurement represented by the effect $f$ gives outcome ”yes” if the system is in the state $\rho \in K$. Similarly, measurements with
n outcomes are represented by collections \( f_1, \ldots, f_n \in E \), \( \sum_i f_i = 1 \), where \( \langle f_i, \rho \rangle \) is interpreted as the probability of i-th outcome in the state \( \rho \).

Given two systems \((V_A, V^+_A, 1_A)\) and \((V_B, V^+_B, 1_B)\), the composite system is \((V_{AB}, V^+_{AB}, 1_{AB})\), where \( V_{AB} = V_A \otimes V_B \), \( 1_{AB} = 1_A \otimes 1_B \) and \( V^+_{AB} \) is a positive cone in \( V_{AB} \) satisfying
\[
V^+_A \otimes_{\min} V^+_B \subseteq V^+_{AB} \subseteq V^+_A \otimes_{\max} V^+_B.
\]

Here \( V^+_A \otimes_{\min} V^+_B \) is the minimal cone
\[
V^+_A \otimes_{\min} V^+_B = \left\{ \sum_i v_{i,A} \otimes v_{i,B}, \; v_{i,A} \in V^+_A, \; v_{i,B} \in V^+_B \right\}
\]
containing all separable states and \( V^+_A \otimes_{\max} V^+_B \) is the maximal cone
\[
V^+_A \otimes_{\max} V^+_B = \{ v_{AB} \in V_{AB}, \; \langle f_A \otimes f_B, v_{AB} \rangle \geq 0, \; \forall f_A \in E_A, \; f_B \in E_B \}
\]
for which all separately prepared measurements are valid. Similarly, the state space \( K_{AB} \) satisfies
\[
K^+_{A \otimes_{\min} B} \subseteq K_{AB} \subseteq K^+_{A \otimes_{\max} B}.
\]
As it was recently proved in [2], the inclusion \( K^+_{A \otimes_{\min} B} \subseteq K^+_{A \otimes_{\max} B} \) is strict, unless \( K_A \) or \( K_B \) is a simplex. The states in \( K^+_{A \otimes_{\max} B} \) are called separable, all other states in \( K^+_{A \otimes_{\min} B} \) are entangled.

Some of the basic examples are described below.

**Example 2.1. Classical theory.** In the classical GPT, the systems have the form \((\mathbb{R}^n, \mathbb{R}^n_+, (1, \ldots, 1))\), where \( \mathbb{R}^n_+ \) is the simplicial cone generated by the positive half-axes. The state space is the simplex \( \Delta_n = \{(x_1, \ldots, x_n), \; x_i \geq 0, \; \sum_i x_i = 1\} \). For any state space \( K \) we have \( \Delta_n \otimes_{\max} K = \Delta_n \otimes_{\min} K = K^{\otimes n} \), the convex direct sum of \( n \)-copies of \( K \).

**Example 2.2. Quantum theory.** Here \( V = M_n^{sa} \) is the space of \( n \) by \( n \) complex hermitian matrices, \( V^+ = M_n^+ \) is the cone of positive semidefinite matrices and \( 1 = I \), the identity matrix. The state space is the set of density matrices \( D_n = \{ \rho \in M_n^+, \; \text{Tr} \rho = 1 \} \). The tensor product of the cones \( M_n^+ \) and \( M_m^+ \) is the cone \( M_m^+ \otimes M_m^+ \) of positive definite matrices in \( M_m^{sa} = M_m^{sa} \otimes M_m^{sa} \).

**Example 2.3.** Note that for any compact convex subset \( S \) in the Euclidean space \( \mathbb{R}^q \) we can construct a triple \((V_S, V^+_S, 1_S)\) with the state space isomorphic to \( S \): put \( V_S = \mathbb{R}^{q+1} \) and \( K = \{(1, x), \; x \in S\} \). We define \( V^+_S \) as the cone generated by \( K \) and \( 1_S = (1, 0) \).

**Example 2.4. Centrally symmetric state spaces.** In the previous example, assume that \( S \) is the unit ball of a norm \( \| \cdot \| \) in \( \mathbb{R}^q \). Then \( K = \{(1, x), \; \|x\| \leq 1\} \) and \( V^+_S = \{(t, x), \; \|x\| \leq t\} \). Note that the dual space \( (A_S, A^+_S) \) has the same form for the dual norm \( \| \cdot \|^* \) and the central element \((1, 0)\) is an order unit in both \((V_S, V^+_S)\) and \((A_S, A^+_S)\). In this case we have for the base norm (in \( V_S \)) \( \| (s, x) \|_{V_S} = \max\{|s|, \|x\|\} \) and the order unit norm (in \( A_S \)) becomes \( \| (t, \varphi) \|_{(1, 0)} = |t| + \| \varphi \|^* \).

We now look at the tensor product with any triple \((V, V^+, 1)\). We will use the obvious identifications \( \mathbb{R}^{q+1} \otimes V \cong V^{q+1} \cong V \oplus V^q \), so the first copy is distinguished. Let \((y_0, y) \in V^{q+1}, \; y = (y_1, \ldots, y_q)\), then it is easily checked that \((y_0, y) \in V_S^+ \otimes_{\min} V^+ \) if and only if
\[
y = \sum_j x_j \otimes z_j, \quad \|x_j\| = 1, \; z_j \in V^+, \; \sum_j z_j \leq y_0
\]
and \((y_0, y) \in V_S \otimes_{\max} V^+ \) if and only if
\[
\sum_i \varphi_i y_i \leq y_0, \quad \forall \varphi \in \mathbb{R}^q, \; \| \varphi \|^* = 1.
\]
Note that using this condition for \( \pm \varphi \), we obtain that \( y_0 \in V^+ \) and \( y_0 = 0 \) only if \( y = 0 \).
An important centrally symmetric example is the qubit system \((M_2^{ac}, M_2^+, I_2)\), where the state space is isomorphic to the unit ball in \(\ell_2^2\). We will also frequently use the hypercubic systems obtained from \(\ell_\infty^2\), we will denote the corresponding triple as \((V_g, V_g^+, \mathbb{1}_g)\). The state space

\[
S_g := \{(1, z_1, \ldots, z_g), \ |z_i| \leq 1\}
\]

is isomorphic to the hypercube \([-1, 1]^g\).

We now consider the norms obtained in \(V_{AB}\) from the tensor product of the Banach spaces \(V_A\) and \(V_B\) equipped with their respective base norms. A norm \(\| \cdot \|\) on \(V_{AB}\) is called a (reasonable) cross norm if both \(\| \cdot \|\) and its dual norm are multiplicative on simple tensors. Equivalently,

\[
\| \cdot \|_{\epsilon(A,B)} \leq \| \cdot \| \leq \| \cdot \|_{\pi(A,B)},
\]

where \(\epsilon(A, B)\) denotes the injective cross norm given by

\[
\|v_{AB}\|_{\epsilon(A,B)} = \sup\{(f_A \otimes f_B, v_{AB}), \ 1_A \pm f_A \geq 0, 1_B \pm f_B \geq 0\}
\]

and \(\pi(A, B)\) denotes the projective cross norm

\[
\|v_{AB}\|_{\pi(A,B)} = \inf\{\sum_i \|v_{i,A}\|_{V_A}\|v_{i,B}\|_{V_B}, \ v_{AB} = \sum_i v_{i,A} \otimes v_{i,B}\}.
\]

It was proved in [3] that the base norm for the composite system \((V_{AB}, V_{AB}^+, \mathbb{1}_{AB})\) is a reasonable cross norm, and that it coincides with the projective cross norm \(\pi(A, B)\) for the separable cone \(V_{AB}^+ = V_A^+ \otimes \min V_B^+\). For completeness, we give a proof for the following equivalent formulation of the latter statement.

**Theorem 2.5.** *A bipartite state \(\rho_{AB} \in K_A \otimes_{\max} K_B\) is separable if and only if \(\|\rho_{AB}\|_{\pi(A,B)} \leq 1\).*

**Proof.** Assume \(\rho_{AB}\) is separable, then \(\rho_{AB} = \sum_i \lambda_i \rho_i,A \otimes \rho_i,B\) for \(\rho_i,A \in K_A, \rho_i,B \in K_B\) and probabilities \(\lambda_i\). By definition of the projective cross norm,

\[
\|\rho_{AB}\|_{\pi(A,B)} \leq \sum_i \lambda_i \|\rho_i,A\|_{V_A}\|\rho_i,B\|_{V_B} = 1.
\]

Assume the converse and let \(y_{i,A} \in V_A\) and \(y_{i,B} \in V_B\) be such that \(\rho_{AB} = \sum_i y_{i,A} \otimes y_{i,B}\) and \(\sum_i \|y_{i,A}\|_{V_A}\|y_{i,B}\|_{V_B} \leq 1\). Then \(y_{i,A} = y_{i,A}^+-y_{i,A}^-\) with \(y_{i,A}^+ \in V_A^+\) and \(\|y_{i,A}\|_{V_A} = \|y_{i,A}^+\|_{V_A} + \|y_{i,A}^-\|_{V_A}\), similarly for \(y_{i,B}\). It follows that \(\rho_{AB} = \rho_{AB}^+ - \rho_{AB}^-\), with

\[
\rho_{AB}^+ = \sum_i (y_{i,A}^+ \otimes y_{i,B}^+ + y_{i,A}^- \otimes y_{i,B}^-) \in V_A^+ \otimes \min V_B^+
\]

and

\[
\rho_{AB}^- = \sum_i (y_{i,A}^+ \otimes y_{i,B}^- + y_{i,A}^- \otimes y_{i,B}^+) \in V_A^+ \otimes \min V_B^+
\]

With \(\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B\), we obtain

\[
1 = \langle \mathbb{1}_{AB}, \rho_{AB} \rangle = \langle \mathbb{1}_{AB}, \rho_{AB}^+ - \rho_{AB}^- \rangle \leq \langle \mathbb{1}_{AB}, \rho_{AB}^+ + \rho_{AB}^- \rangle = \sum_i \|y_{i,A}\|_{V_A}\|y_{i,B}\|_{V_B} \leq 1,
\]

whence \(\rho_{AB}^- = 0\) and \(\rho_{AB} = \rho_{AB}^+\) is separable. 

\(\square\)
3. Steering and tensor norms

3.1. Conditional states and assemblages. Let $\sigma_{AB} \in K_{AB}$ be a state of the composite system $(V_{A,B}, V_{A}^+, 1_{AB})$ and let $\{f_{x}\}_{x=1}^{g}$ be a (finite) collection of measurements on the system $(V_{A}, V_{A}^+, 1_{A})$, with effects $f_{a|x}$ and outcomes $a \in \Omega_{x}$. Viewing any $f \in E$ as an affine function over $K$, we may define the conditional states

$$\rho_{a|x} := (f_{a|x} \otimes \text{id}_{B})(\sigma_{AB}) \in V_{B}^+.$$  

For all $x = 1, \ldots, g$, we have $\sum_{a \in \Omega_{x}} \rho_{a|x} = \sigma_{B} = (1_{A} \otimes \text{id}_{B})(\sigma_{AB})$. More generally, any collection

$$\{\rho_{a|x} \in V_{B}^+, \sum_{a} \rho_{a|x} = \sigma_{B}, \ \forall x = 1, \ldots, g\}$$

is called an assemblage with barycenter $\sigma_{B}$. The tuple $x = (k_{1}, \ldots, k_{g})$ with $k_{x} = |\Omega_{x}|$, $x = 1, \ldots, g$ determines the shape of the assemblage. We will say that the assemblage is dichotomic if $k_{x} = 2$ for all $x$, in this case we will use $\Omega_{x} = \{+, -\}$ as the set of labels.

The next result shows that if we do not restrict the choice of the system $V_{A}$ and allow maximal tensor products, all assemblages can be obtained as conditional states for a bipartite state and some collection of measurements on $V_{A}$ of a corresponding shape.

**Theorem 3.1.** Let $x = (k_{1}, \ldots, k_{g})$ and let $S$ be the Cartesian product $S = S_{x} := \Pi_{x=1}^{g} \Delta_{k_{x}}$. For any system $(V, V_{x}^+, 1)$ with state space $K$, the set of all assemblages of shape $x$ can be identified with the tensor product $S \otimes_{\text{max}} K$. In particular, there is a system $(S_{x}, V_{x}^+, 1_{S})$ with state space $S$ and with a canonical set of measurements $\{p_{x}\}$ (identified with projections $p_{x} : S \rightarrow \Delta_{k_{x}}$ onto the $x$-th component), such that for any assemblage $\{\rho_{a|x}\}$ in $V_{x}^+$ of shape $x$, there is a (unique) state $\xi \in S \otimes_{\text{max}} K$ such that

$$\rho_{a|x} = (p_{a|x} \otimes \text{id}_{K})(\xi), \ \forall a, x.$$  

**Proof.** We will give a proof for the case of dichotomic assemblages, mostly to introduce some notations needed later. For the proof in general see [14].

So let $x = (2, \ldots, 2)$. By the isomorphism $(\lambda, 1 - \lambda) \mapsto 2\lambda - 1$, we will identify $S = \Delta_{2}^{g} \simeq [-1, 1]^{g}$. Let $(V_{g}, V_{g}^+, 1_{g})$ be the system with centrally symmetric state space $S_{g} \simeq [-1, 1]^{g}$, see Example 2.4. It is easily seen from (2) that the maximal tensor product $S_{g} \otimes_{\text{max}} K$ can be identified with the subset in $V_{g}^{g+1}$ of elements of the form $(\sigma, y_{1}, \ldots, y_{g})$ with $\sigma \in K$ and $\sigma \pm y_{x} \in V_{x}^{+}$. Clearly, $\{\frac{1}{2}(\sigma \pm y_{x})\}$ is a dichotomic assemblage with barycenter $\sigma$. Conversely, for any assemblage $\{\rho_{a|x}\}$ with barycenter $\sigma$, $(\sigma, y_{1}, \ldots, y_{g})$ with $y_{x} := \rho_{a|x} - \rho_{-a|x}$ is an element of $S_{g} \otimes_{\text{max}} K$ and it is clear that this established a one-to-one correspondence. The effects $p_{a|x}$ have the form $\frac{1}{2}(1, \pm e_{x})$, where $\{e_{x}\}_{x=1}^{g}$ is the standard basis in $R^{g}$.

It is well known that any assemblage in quantum theory can be obtained with $V_{A} \simeq V_{B}$. In the general case, this depends on the properties of the system $V_{B}$ as well as the choice of the tensor product. For more information see [5].

3.2. Local hidden state models and steering. As in the quantum case, we say that an assemblage of conditional states $\{\rho_{a|x} = (f_{a|x} \otimes \text{id}_{B})(\sigma_{AB})\}$ admits a local hidden state (LHS) model if there is some (finite) set $\Lambda$, a probability measure $q \in \mathcal{P}(\Lambda)$, conditional probabilities $q(x|a, \lambda)$ and elements $\rho_{\lambda} \in K$ such that

$$\rho_{a|x} = \sum_{\lambda \in \Lambda} q(\lambda)q(a|x, \lambda)\rho_{\lambda}, \ \ a \in \Omega_{x}, \ x = 1, \ldots, g.$$  

We say that a bipartite state $\sigma_{AB}$ is $(A \rightarrow B)$ steerable if there is a set $\{f_{x}\}$ of measurements on the system $A$ such that the assemblage $\{\rho_{a|x}\}$ does not admit a LHS model. If no such collection of measurements exists, the state is unsteerable. We may also restrict the set of measurements, so
we say that, for example, a state is unsteerable by dichotomic measurements if all corresponding dichotomic assemblages admit a LHS.

For a general assemblage \( \{ \rho_{a|x} \} \) satisfying (3) we will also say that the assemblage is classical. We will show later that we may equivalently formulate the condition in (3) with the set \( \Lambda \) replaced by the (possibly infinite) set \( \partial_{e} K \) of extremal points of \( K \) (pure states), common for all LHS models.

**Theorem 3.2.** Let \( \{ \rho_{a|x} \} \) be an assemblage of shape \( k \). Then the assemblage is classical if and only if the corresponding element of \( S_{k} \otimes_{\max} K \) is separable.

**Proof.** We again prove the statement for dichotomic assemblages that we focus on in this section. For a general proof, see [14].

We have seen in the proof of Theorem 3.1 that the element in \( S_{g} \otimes_{\max} K \) corresponding to the assemblage \( \{ \rho_{\pm|x} \} \) is \( (\sigma,y_{1},\ldots,y_{g}) \), with \( y_{x} = \rho_{+|x} - \rho_{-|x} \) and \( \sigma \) the barycenter. Observe that the LHS model (3) for \( \{ \rho_{\pm|x} \} \) is equivalent to

\[
y_{x} = \sum_{\lambda} h_{\lambda}(x) \phi_{\lambda}, \quad x = 1,\ldots,g
\]

for some \( h_{\lambda}(x) \in [-1,1] \) and \( \phi_{\lambda} \in V^{+} \), \( \sum_{\lambda} \phi_{\lambda} = \sigma \). Indeed, if (3) holds then we may put \( h_{\lambda}(x) = q(+|x,\lambda) - q(-|a,\lambda) \) and \( \phi_{\lambda} = q(\lambda) \rho_{x} \). Conversely, we obtain a LHS model from (4) by setting \( q(\pm|x,\lambda) = \frac{1}{2}(1 \pm h_{\lambda}(x)) \) and normalizing the elements \( \phi_{\lambda} \). This amounts to

\[
(\sigma,y) = \sum_{\lambda} h_{\lambda} \otimes \phi_{\lambda}
\]

which can be seen to be equivalent to the characterization of \( S_{g} \otimes_{\min} K \) in (1), see Example 2.4. \( \square \)

Our first characterization of steering by tensor cross norms follows immediately from Theorems 3.1, 3.2 and 2.5.

**Corollary 3.3.** Let \( \{ \rho_{a|x} \} \) be an assemblage and let \( K_{A} = S \). Let \( \xi_{AB} \in K_{A} \otimes K_{B} \) be such that \( (\rho_{a|x} \otimes \text{id})(\xi_{AB}) = \rho_{a|x} \) as in Theorem 3.1. Then the assemblage is classical if and only if \( \|\xi_{AB}\|_{\pi(A,B)} \leq 1 \).

**Remark 3.4.** Note that if \( K_{A} = S \) as in Corollary 3.3, any \( \sigma_{AB} \in K_{A} \otimes K_{B} \) is unsteerable if and only if \( \{ (\rho_{a|x} \otimes \text{id}_{B})(\sigma_{AB}) \} \) admits a LHS model if and only if \( \sigma_{AB} \) is separable. It is well known and immediately seen that a separable state is always unsteerable: indeed, we can view any set of measurements \( \{ f_{x} \} \) as a map \( f_{A} : K_{A} \to S \) such that \( f_{A} \otimes \text{id}_{B} \) maps \( \sigma_{AB} \) to the element in \( S \otimes_{\max} K_{B} \) corresponding to the assemblage of conditional states. Since the map \( f_{A} \) preserves the state spaces, it is a contraction with respect to the base norms and we have from the properties of the projective cross norm that

\[
\| (f_{A} \otimes \text{id}_{B})(\sigma_{AB}) \|_{\pi} \leq \|\sigma_{AB}\|_{\pi} \leq 1.
\]

here \( \pi \) always denotes the projective cross norm for the respective base norms. On the other hand, there are unsteerable entangled quantum states.

3.3. **Dichotomic assemblages and tensor cross norms.** Let \( \sigma \in K \) be an interior element, so that \( \sigma \in \text{int}(V^{+}) \). We will show that classical dichotomic assemblages with barycenter \( \sigma \) can be characterized by tensor cross norms in \( \ell_{\infty}^{g} \otimes V \), if we choose an appropriate norm in \( V \), depending on \( \sigma \). We will denote the set of all such assemblages with \( g \) elements by \( A_{2,g}^{\sigma} \).

Note first that the barycenter \( \sigma \) is an order unit in \( (V,V^{+}) \), so that we can (formally) define a system as \( (A,A^{+},\sigma) \), where \( \sigma \) is the unit effect and the state space becomes

\[
K^{\sigma} = \{ h \in A^{+}, \langle h,\sigma \rangle = 1 \}.
\]
Let us denote the corresponding base norm (in $A$) by $\| \cdot \|_\sigma$ and the dual order unit norm (in $V$) by $\| \cdot \|_\sigma$.

We have seen in the proof of Theorem 3.1 that assemblages in $A_{2,\sigma}^g$ can be identified with elements $y = (y_1, \ldots, y_g) \in V^g \simeq R^g \otimes V$ such that $\pm y_x \leq \sigma$, equivalently, $\| y_x \|_\sigma \leq 1$. The following is an easy observation from the definition of the injective norm.

**Proposition 3.5.** Let $y \in V^g$. Then $\{ \frac{1}{2}(\sigma \pm y_x) \}$ is an assemblage if and only if $\| y \|_{e,\sigma} \leq 1$, where $\| \cdot \|_{e,\sigma}$ is the injective cross norm in the tensor product $\ell_\infty^g \otimes (V, \| \cdot \|_\sigma)$. 

**Proof.** This follows from

$$\| y \|_{e,\sigma} = \max_x \| y_x \|_\sigma.$$ 

We now define a new norm in $\ell_\infty^g \otimes V$: For $y = (y_1, \ldots, y_g) \in V^g \simeq R^g \otimes V$, put

$$\| y \|_{steer,\sigma} := \inf \{ \| \sum_j \phi_j \|_\sigma, \ y = \sum_j z_j \otimes \phi_j, \ \| z_j \|_\infty = 1, \ \phi_j \in V^+ \}.$$ 

**Proposition 3.6.** $\| \cdot \|_{steer,\sigma}$ is a reasonable cross norm on $\ell_\infty^g \otimes (V, \| \cdot \|_\sigma)$.

**Proof.** It is easily seen that $\| \cdot \|_{steer,\sigma}$ is a norm, so it suffices to show that $\| \cdot \|_{e,\sigma} \leq \| \cdot \|_{steer,\sigma} \leq \| \cdot \|_{\pi,\sigma}$, where $\| \cdot \|_{\pi,\sigma}$ is the projective norm. Assume that $y = \sum_j z_j \otimes \phi_j$ with $z_j \in R^g$, $\| z_j \|_\infty = 1$ and $\phi_j \in V^+$. Then $y_x = \sum_j z_{j,x} \phi_j$ so that

$$\| y_x \|_\sigma \leq \sum_j \| \phi_j \|_\sigma \leq \sum_j |z_{j,x}| \| \phi_j \|_\sigma \leq \sum_j \| z_{j,x} \| \| \phi_j \| \leq \sum_j \| z_j \|_\infty \| \phi \|_\sigma$$

hence $\| y_x \|_\sigma \leq \| y \|_{steer,\sigma}$ for all $x$, this implies the first inequality. For the second inequality, let $y = \sum_j \tilde{z}_j \otimes \psi_j$ with $\tilde{z}_j \in \ell_\infty^g$ and $\psi_j \in V$. Put $z_j = \| \tilde{z}_j \|_\infty^{-1} \tilde{z}_j$ and let $\psi_j^+ = \frac{1}{2}(\| \psi_j \|_\sigma \pm \psi_j) \in V^+$. Then $\psi_j = \psi_j^+ - \psi_j^-$ and we have

$$y = \sum_j z_j \| \tilde{z}_j \|_\infty \psi_j^+ + \sum_j (-z_j) \| \tilde{z}_j \|_\infty \psi_j^-$$

and

$$\| y \|_{steer,\sigma} \leq \sum_j \| \tilde{z}_j \|_\infty (\psi_j^+ + \psi_j^-) \|_\sigma \leq \sum_j \| \tilde{z}_j \|_\infty \| \psi_j \|_\sigma.$$ 

This implies the second inequality.

**Theorem 3.7.** Let $y \in V^g$ be such that $\{ \frac{1}{2}(\sigma \pm y_x) \}$ is an assemblage. Then the assemblage is classical if and only if $\| y \|_{steer,\sigma} \leq 1$.

**Proof.** By Theorem 3.2, the assemblage is classical if and only if $(\sigma, y) \in S_g \otimes_{\min} K$. It is immediate from (1) that this is equivalent to $\| y \|_{steer,\sigma} \leq 1$.

3.4. **Steering witnesses.** Let $\xi \in S_k \otimes_{\max} K$ be any element. It is clear from the above results and the definition of the minimal and maximal tensor products that the corresponding assemblage of shape $k$ is classical if and only if $\langle w, \xi \rangle \geq 0$ for any $w \in A_{2,k}^+ \otimes_{\max} A^+$. Therefore any element $w \in A_{2,k}^+ \otimes_{\max} A^+$ defines a steering witness. We say that a steering witness is strict if there is some $\xi \in S_k \otimes_{\max} K$ such that $\langle w, \xi \rangle < 0$, which means that $w \notin A_{2,k}^+ \otimes_{\min} A^+$. Hence the set of all strict steering witnesses is $A_{2,k}^+ \otimes_{\max} A^+ \setminus A_{2,k}^+ \otimes_{\min} A^+$. From now on we restrict to dichotomic ensembles.
Proposition 3.8. An element \((w_0, w_1, \ldots, w_g) \in A^{g+1} \simeq \mathbb{R}^{g+1} \otimes A\) is a steering witness if and only if
\[
\sum_{x=1}^{g} \varepsilon_x w_x \leq w_0, \quad \forall \varepsilon \in \{\pm 1\}^g.
\] (5)

In this case, we always have \(w_0 \in A^+\) and \(w_0 = 0\) implies \(w_x = 0\) for all \(x\). A steering witness is strict if and only if there exist some \(\sigma \in K\) and elements \(y_x \in V, \|y_x\|_\sigma \leq 1\) such that
\[
\langle w_0, \sigma \rangle + \sum_x \langle w_x, y_x \rangle < 0.
\]

Proof. The first part follows from Example 2.4 and the fact that the extremal points \(\varepsilon \in [-1, 1]^g\) are precisely the elements of \(\{\pm 1\}^g\). The last statement follows from the representation of \(S_g \otimes_{\max} K\) in the previous section.

Let now \(\sigma \in \text{int}(V^+)\) and let us restrict to assemblages in \(A_{2,\sigma}^g\). Let \(w = (w_0, w_1, \ldots, w_g)\) be a steering witness. Since \(\sigma\) is an interior point, we have by Proposition 3.8 that \(\langle w_0, \sigma \rangle > 0\) unless \(w_0 = 0\) and in this case \(w = 0\). Therefore we may restrict to witnesses with \(\langle w_0, \sigma \rangle = 1\) and then the value of the witnesses on elements in \(A_{2,\sigma}^g\) is determined by the \(g\)-tuple \((w_1, \ldots, w_g) \in A^g\).

Proposition 3.9. Let \(w \in A^g\). The following are equivalent:

(i) there is some \(w_0 \in A^+\), such that \(\langle w_0, \sigma \rangle = 1\) and \((w_0, w)\) is a steering witness.

(ii) \(\sum_{x=1}^{g} \|\langle w_x, y_x \rangle\| \leq 1\) for all \(y \in V^g\) such that \(\|y\|_{\text{steer}, \sigma} \leq 1\).

Proof. Assume (i) and let \(\|y\|_{\text{steer}, \sigma} \leq 1\). Note that by definition of the norm, we have for any \(\varepsilon \in \{\pm 1\}^g\),
\[
\|\langle \varepsilon_1 y_1, \ldots, \varepsilon_g y_g \rangle\|_{\text{steer}, \sigma} = \|\langle y_1, \ldots, y_g \rangle\|_{\text{steer}, \sigma},
\]
so that \((\sigma, \varepsilon_1 y_1, \ldots, \varepsilon_g y_g)\) corresponds to a classical assemblage. Therefore we have
\[
0 \leq \langle (w_0, w_1, \ldots, w_g), (\sigma, \varepsilon_1 y_1, \ldots, \varepsilon_g y_g) \rangle = 1 + \sum_{x} \varepsilon_x \langle w_x, y_x \rangle, \quad \forall \varepsilon \in \{\pm 1\}^g,
\]
this proves (ii).

For the converse, choose any element \(w_0' \in A\) such that \(\langle w_0', \sigma \rangle = 1\). Let \((\sigma, y) \in S_g \otimes_{\min} K\), then \(\|y\|_{\text{steer}, \sigma} \leq 1\) and by (ii)
\[
\langle (w_0', w_1, \ldots, w_g), (\sigma, y_1, \ldots, y_g) \rangle = 1 + \sum_{x} \langle w_x, y_x \rangle \geq 1 - \sum_{x} |\langle w_x, y_x \rangle| \geq 0.
\]

It follows that \(w' = (w_0', w_1, \ldots, w_g)\) defines a positive functional on the subspace
\[
\mathcal{L} = \{(y_0, y_1, \ldots, y_g) \in V^{g+1}, y_0 \in \mathbb{R}\sigma\}
\]
with the cone \(\mathcal{L} \cap (V^+_g \otimes_{\min} V^+)\). Since this subspace contains the interior element \((\sigma, 0) \in \text{int}(V^+_g \otimes_{\min} V^+)\), \(w'\) extends to an element \(w \in A^+_g \otimes_{\max} A^+\). It is easily checked that \(w = (w_0, w_1, \ldots, w_g)\) with \(w_0 \in A^+\) and
\[
\langle w_0, \sigma \rangle = \langle w, (\sigma, 0) \rangle = \langle w', (\sigma, 0) \rangle = \langle w_0', \sigma \rangle = 1.
\]

This finishes the proof.

We will denote the set of \(w \in A^g\) satisfying the above conditions by \(W_{2,\sigma}^g\), note that this is the unit ball in \(\ell_{\infty}^g \otimes A\) with respect to the cross norm dual to \(\|\cdot\|_{\text{steer}, \sigma}\). By the above result, up to multiplication by a positive constant, \(W_{2,\sigma}^g\) is the set of steering witnesses for assemblages in \(A_{2,\sigma}^g\). We will further say that such a witness is strict if it has a negative value on some assemblage.
in $A_{2,\sigma}^g$. For the following characterization of strict witnesses, recall that the base norm for the formally introduced system $(A, A^+, \sigma)$ has the form
\[ \|h\| = \sup_{y \leq \sigma} \langle h, y \rangle. \]

**Proposition 3.10.** The steering witness $w \in W_{2,\sigma}$ is strict if and only if
\[ \|w\|_{\pi,\sigma} = \sum_{x=1}^{g} \|w_x\|_\sigma > 1, \]
here $\| \cdot \|_{\pi,\sigma}$ is the projective cross norm in the tensor product $\ell_1^g \otimes (A, \| \cdot \|_\sigma)$.

**Proof.** It is easy to check directly that $\|w\|_{\pi,\sigma} = \sum_{x=1}^{g} \|w_x\|_\sigma$. By Proposition 3.8, $w$ is a strict steering witness if and only if there are some $y_x \in V$, $\|y_x\|_\sigma \leq 1$ such that
\[ 0 > 1 + \sum_x \langle w_x, y_x \rangle = 1 - \sum_x \langle w_x, -y_x \rangle \geq 1 - \sum_x \|w_x\|_\sigma. \]

\[ \square \]

### 3.5. Steering degree.

The steering degree of assemblages of shape $k$ can be quantified by the amount of noise that needs to be mixed with the assemblage $\{\rho_{a|x}\}$ in order to obtain a classical assemblage, this is also called steering robustness [22]. The noise is represented by assemblages of the same shape as $\{\rho_{a|x}\}$, see [11, 10] for some variants. Note that the convex structure on the set of assemblages is inherited from $S_k \otimes_{\text{max}} K$. Here we will use a single trivial assemblage, of the form $\{\omega_{a|x}\}$ with $\omega_{a|x} = |\Omega_x|^{-1}$ for all $a \in \Omega_x$. The steering degree of $\{\rho_{a|x}\}$ is defined as
\[ s(\{\rho_{a|x}\}) = \sup\{s \in [0, 1], \{s \rho_{a|x} + (1 - s) |\Omega_x|^{-1}\sigma\} \text{ is classical}\}. \]

We also define $s_{k,\sigma}(K)$ as the infimum of the steering degrees of all assemblages with shape $k$ and barycenter $\sigma$. Since we assume $K$ fixed, we will skip it from the notation. Restricting to dichotomic assemblages, we now show that the steering degree can be expressed using the norm $\| \cdot \|_{\text{steer},\sigma}$. In this case we denote $s_{k,\sigma} = s_{g,\sigma}$.

**Theorem 3.11.** Let $\{\rho_{\pm|x}\} \in A_{2,\sigma}^g$ and let $y_x = \rho_{+|x} - \rho_{-|x}$, $x = 1, \ldots, g$. Then
\[ s(\{\rho_{\pm|x}\}) = \|(y_1, \ldots, y_g)\|_{\text{steer},\sigma}^{-1}. \] (6)

For the overall steering degree, we have
\[ s_{g,\sigma} = \sup_{y \in \mathbb{R}^g \otimes V} \frac{\|y\|_{\epsilon,\sigma}}{\|y\|_{\text{steer},\sigma}} \geq \sup_{y \in \mathbb{R}^g \otimes V} \frac{\|y\|_{\epsilon,\sigma}}{\|y\|_{\pi,\sigma}}. \] (7)

Dually, in terms of the steering witnesses, we obtain
\[ s_{g,\sigma} = \sup\{s \in [0, 1], \sum_{x=1}^{g} s\|w_x\|_\sigma \leq 1, \forall (w_1, \ldots, w_g) \in W_{2,\sigma}^g\}. \] (8)

**Proof.** Let $s \in [0, 1]$, then the tensor element corresponding to the mixed assemblage $\{s \rho_{\pm|x} + (1 - s)\frac{1}{2}\sigma\}$ is $(\sigma, sy_1, \ldots, sy_g)$. By Theorem 3.7 we have
\[ s(\{\rho_{\pm|x}\}) = \sup\{s \in [0, 1], s\|(y_1, \ldots, y_g)\|_{\text{steer},\sigma} \leq 1\} = \|(y_1, \ldots, y_g)\|_{\text{steer},\sigma}^{-1}. \]
The equality (7) now follows from Proposition 3.5 and Proposition 3.6, (8) follows from Proposition 3.9 and the definition of the norm $\| \cdot \|_\sigma$.  

\[ \square \]
In Section 4.2 below we will characterize the universal steering degree for dichotomic assemblages with barycenter $\sigma$:

$$s_{\sigma} := \max \{ s \in [0,1], \ (\sigma, sy) \in S_g \otimes_{\min} K, \ \forall (\sigma, y) \in S_g \otimes_{\max} K, \ \forall g \}$$ (9)

$$= \max \{ s \in [0,1], \ s \sum_i \|w_i\|_\sigma^g \leq 1, \ \forall w \in W_{2,\sigma}^g, \ \forall g \}$$ (10)

$$= \inf_{g \in \mathbb{N}} s_{g,\sigma}. \ \ (11)$$

3.6. Relation to compatibility of measurements. Let $\{f_x\}_{x=1}^g$ be a collection of measurements with outcomes in $\Omega_x$, with effects $f_{a|x}$. We say that the collection is compatible if all $f_x$ are marginals of a joint measurement $h$ with outcomes in $Y = \prod_x \Omega_x$:

$$f_{a|x} = \sum_{(a_1,\ldots,a_g) \in Y,a_x=a} h_{a_1,\ldots,a_g}, \ \ \forall a, x.$$ (12)

The aim of the present section is to remark that such collections of measurements and existence of a joint measurement for them is mathematically equivalent to assemblages and existence of LHS models. This simple observation shows a link to the previously obtained results of a relation of incompatibility and related notions of witnesses and degree to minimal/maximal tensor products of cones and tensor norms, obtained in [7].

So let $\sigma$ be an arbitrary interior point in the state space $K$. As before, we may formally consider the system $(A,A^+,\sigma)$, with the state space $K^\sigma \subseteq A^+$. Note that we always have $1 \in K^\sigma$. The following is rather straightforward.

**Lemma 3.12.** Sets of measurements on $(V,V^+,1)$ correspond precisely to assemblages for $(A,A^+,\sigma)$ with barycenter 1. Moreover, the measurements are compatible if and only if the corresponding assemblage is classical.

Note that the norm $\| \cdot \|_1$ on $A$ is the usual order unit norm and the dual norm $\| \cdot \|_V$ in $V$ is the base norm $\| \cdot \|_V$. The results of the previous sections correspond to the results in [7] for compatibility of dichotomic measurements, in particular the norm $\| \cdot \|_{\text{steer,1}}$ on $\ell_\infty^g \otimes A$ becomes precisely the compatibility norm $\| \cdot \|_c$ as in [7].

The above relations have some immediate consequences, obtained from the results in [7] and duality relations:

1. The steering degree for any $(V,V^+,1)$ with $\dim(V) = d$ is lower bounded by

$$s_{\sigma,\sigma} \geq 1/\min\{g,d\}.$$ (13)

2. In the quantum case, the assemblages are directly related to collections of measurements by the map $\rho \mapsto \sigma^{-1/2} \rho \sigma^{-1/2}$, mapping classical assemblages onto compatible measurements and relating the steering degrees to the compatibility degrees, [28]. Note that if the barycenter $\sigma$ is not a faithful state then we obtain measurements for a quantum system of lower dimension.

3. In the centrally symmetric case, we have for $\sigma = (1,0)$ the tight lower bound

$$s_{(1,0),\sigma} \geq 1/\min\{g,d-1\},$$

attained for the state spaces isomorphic to cross-polytopes (unit balls of the $\ell_1$-norm). In general, for a norm $\| \cdot \|_1$ in $\mathbb{R}^n$ and its unit ball $B$, the steering degrees $s_{k,(1,0)}(B)$ are the same as the compatibility degrees for the dual unit ball $B^\ast$. 


4. Steering and Choquet order

In this section, we show how the properties of probability measures on compact convex sets can be used to characterize steering in GPTs. Although for simplicity the dimension of the systems is assumed to be finite, we remark that most of the results hold as stated here for metrizable convex compacts and with slight technical modifications also for arbitrary compact convex subsets of a locally convex space.

Let \((V,V^+,\mathbb{1})\) be a system with state space \(K\). Let \(C(K)\) denote the Banach space of continuous functions \(f : K \to \mathbb{R}\) with maximum norm. The dual space \((A,A^+)\) to \((V,V^+)\) can be identified with the subspace \(A(\mathbb{K}) \subseteq C(K)\) of affine functions over \(K\), with the cone of positive functions \(A^+ = A(\mathbb{K})^+\), the order unit \(\mathbb{1}\) is the constant unit functional over \(K\). The order unit norm in \(A\) coincides with the maximum norm on \(A(\mathbb{K})\). Let us denote by \(P(K) \subseteq C(K)\) the cone of convex functions in \(C(K)\).

The dual space \(C^*(K)\) is the space of signed Radon measures over \(K\). Let us denote by \(P(K)\) the set of Radon probability measures over \(K\). Then \(P(K)\) is compact in the weak*-topology inherited from the Banach space duality with \(C(K)\), in fact, \(P(K)\) is a Choquet simplex. For any \(\sigma \in K\), the probability measure concentrated in \(\sigma\) belongs to \(P(K)\) and is denoted by \(\delta_\sigma\). A measure of the form \(\sum_{a=1}^n c_i \delta_{\rho_i}, \rho_i \in K, c_i \in \mathbb{R}\) is called simple. For \(\sigma \in K\), we denote by \(P_\sigma(K)\) the subset of probability measures \(\mu \in P(K)\) with barycenter \(\tilde{\mu} = \int_K p d\mu(\rho) = \sigma\).

Recall that the Choquet order on the set of Radon measures over \(K\) is defined as the dual of the ordering in \(C(K)\) obtained from the cone \(P(K)\): we have \(\nu \prec \mu\) if

\[
\int f d\nu \leq \int f d\mu, \quad \forall f \in P(K).
\]

If \(\nu \prec \mu\), then \(\tilde{\mu} = \tilde{\nu}\). A positive measure is maximal with respect to this ordering if and only if it is a boundary measure, that is, concentrated on the extreme boundary \(\partial_\varepsilon K\). We will denote the set of all boundary measures in \(P_\sigma(K)\) by \(P_\sigma^b(K)\). Further, any positive measure is upper bounded by a positive boundary measure, in particular, any element \(\sigma \in K\) is the barycenter of some measure \(\mu \in P_\sigma^b(K)\). For details see e.g. [1, 21].

4.1. Boundary measures and LHS models. Let \(\{\rho_{a|x}\}\) be an assemblage with barycenter \(\sigma\). Observe that a LHS model of the form (3) for \(\{\rho_{a|x}\}\) can be expressed as

\[
\rho_{a|x} = \int_K q(a|x,\rho) d\mu(\rho),
\]

where \(\mu = \sum_{\lambda \in A} q(\lambda) \delta_{\rho_\lambda} \in P_\sigma(K)\).

We will now identify \(\{\rho_{a|x}\}\) with a set of simple probability measures in \(P_\sigma(K)\). Put \(\lambda_{a|x} := \langle 1, \rho_{a|x} \rangle\) and \(\sigma_{a|x} := \lambda_{a|x}^{-1} \rho_{a|x} \in K\) (if \(\lambda_{a|x} = 0\) we may pick any state \(\sigma_{a|x} \in K\)). For any \(x\), put

\[
\mu_x := \sum_{a \in \Omega_x} \lambda_{a|x} \delta_{\sigma_{a|x}} \in P(K), \quad \tilde{\mu}_x = \sum_{a} \lambda_{a|x} \sigma_{a|x} = \sigma_x,
\]

so that we can represent the assemblage as a (finite) set \(\{\mu_x\}_{x=1}^N \subseteq P_\sigma(K)\). Conversely, for any such subset, \(\{\lambda_{a|x} \sigma_{a|x}\}\) is an assemblage with barycenter \(\sigma\). Note that this representation, in contrast with the identification with \(S \otimes_{\text{max}} K\) in the previous section, ignores any permutation of the measurements or relabelling of the outcomes, but in the context of steering these are irrelevant.

Note also that the convex structure induced from \(P_\sigma(K)\) through this representation is different from the one used in the previous paragraphs. We next show that existence of a LHS model can be expressed in terms of the Choquet order in \(P_\sigma(K)\).

**Proposition 4.1.** Let \(\nu \in P_\sigma(K)\) be a simple measure, \(\nu = \sum_a \lambda_{a} \delta_{\sigma_{a}}\). Then \(\nu \prec \mu\) for some \(\mu \in P_\sigma(K)\) if and only if there are measurable functions \(q(a|\cdot) : K \to [0,1]\) such that \(\sum_a q(a|\rho) = 1\).
and
\[ \lambda_a \sigma_a = \int_K \rho q(a | \rho) d\mu(\rho). \]

Proof. Assume that \( \nu < \mu \). Since \( \nu \) is simple, by [1, Cor. I.3.4] this means that there is a convex decomposition \( \mu = \sum_a \lambda_a \mu_a \) such that \( \mu_a \in \mathcal{P}_{\sigma_a}(K) \). Let \( f(a | \cdot) = \frac{\partial q}{\partial \mu} \) and put \( q(a | \cdot) = \lambda_a f(a | \cdot) \), then we may assume that \( q(a | \rho) \in [0, 1] \) and \( \sum_a q(a | \rho) = 1 \) by suitably replacing the values of the functions \( f(a | \cdot) \) on a subset \( K_0 \subseteq K \) with \( \mu(K_0) = 0 \). Then
\[ \lambda_a \sigma_a = \lambda_a \mu_a = \lambda_a \int_K \rho d\mu_a(\rho) = \int_K \rho q(a | \rho) d\mu(\rho). \]

Assume the converse, then \( \lambda_a = \int_K q(a | \rho) d\mu \) so that \( \lambda_a^{-1} q(a | \cdot) d\mu \) defines a probability measure \( \mu_a \) such that \( \sigma_a = \int \rho d\mu_a(\rho) \). For any \( f \in \mathcal{P}(K) \) we have
\[ \int f d\nu = \sum_a \lambda_a f(\sigma_a) \leq \sum_a \lambda_a \int_K f(\rho) d\mu_a = \sum_a \int_K f(\rho) q(a | \rho) d\mu = \int_K f d\mu. \]

We now extend the definition of an assemblage as an arbitrary set \( \{\mu_x\}_{x \in X} \) of simple measures with a common barycenter, so that we no longer assume the parameter set \( X \) to be finite. We say that an assemblage \( \{\sum_a \lambda_{a|x} \delta_{\sigma_a} \}_{x \in X} \subseteq \mathcal{P}(K) \) admits a LHS model (or is classical) if there is some \( \mu \in \mathcal{P}_{\sigma}(K) \) and measurable functions \( q(a | x, \cdot) : K \rightarrow [0, 1], \sum a q(a | x, \cdot) = 1 \) for all \( x \in X \), such that
\[ \lambda_{a|x} \sigma_{a|x} = \int_K \rho q(a | x, \rho) d\mu(\rho), \quad a \in \Omega_x, \ x \in X. \]

The following theorem collects some observations for this definition of a local hidden state model and its relation to Choquet order. Note that the statement (iii) shows that for finite assemblages this definition of a LHS model coincides with the previous one from Section 3.2.

**Theorem 4.2.** Let \( \{\mu_x\}_{x \in X} \subseteq \mathcal{P}_{\sigma}(K) \) be an assemblage. Then

(i) The assemblage is classical if and only if all the measures \( \mu_x \) have a common upper bound in Choquet order. In this case, any measure \( \mu \) such that \( \{\mu_x\} < \mu \) defines some LHS model.

(ii) If \( \{\mu_x\} < \mu \), then we may always assume that \( \mu \in \mathcal{P}_{\sigma}(K) \).

(iii) If \( X \) is a finite set and \( \{\mu_x\} < \mu \), then we may assume that \( \mu \) is simple.

(iv) The assemblage is classical if and only if any finite sub-assemblage \( \{\mu_x\}_{x \in F}, F \subseteq X, |F| < \infty \) is classical.

(v) Assume that \( \{\mu_x\} < \mu \). If the assemblage is invariant under an affine bijection \( T : K \rightarrow K \) (that is, for all \( x \in X, \mu_T = \mu_x \) for some \( x' \in X \)), then we may assume that \( \mu \) is invariant under \( T \).

Proof. The statement (i) follows immediately from Proposition 4.1, (ii) follows from the fact that any measure is upper bounded (in the Choquet order) by a boundary measure. For (iii), let \( \{\mu_x = \sum_a \lambda_{a|x} \delta_{\sigma_a} \}_{x=1}^g \) and assume a LHS model (12) for the assemblage with some measure \( \mu_0 \). As in the proof of Proposition 4.1, let \( \mu_{a|x} = \lambda_{a|x}^{-1} q(a | x, \cdot) d\mu_0 \in \mathcal{P}_{\sigma_a}(K) \). Then for all \( x = 1, \ldots, g \) we have a convex decomposition \( \sum_a \lambda_{a|x} \mu_{a|x} = \mu_0 \). Since \( \mathcal{P}(K) \) is a Choquet simplex, all the decompositions have a common refinement: there are probability measures \( \mu_\omega \) indexed by \( \omega \in \Omega = \Omega_1 \times \cdots \times \Omega_g \) and some \( q \in \mathcal{P}(\Omega) \) such that \( \mu_0 = \sum_\omega q(\omega) \mu_\omega \) and
\[ \lambda_{a|x} \mu_{a|x} = \sum_{\omega, \omega_x = a} q(\omega) \mu_\omega = \sum_\omega d(a | x, \omega) q(\omega) \mu_\omega, \]
where \( d(a|x, \omega) = 1 \) if \( \omega_x = a \) and is 0 otherwise. Put \( \rho_\omega := \bar{\mu}_\omega \), then we obtain
\[
\rho_{a|x} = \lambda_{a|x} \sigma_{a|x} = \lambda_{a|x} \bar{\mu}_{a|x} = \sum_\omega d(a|x, \omega)q(\omega)\rho_\omega,
\]
which is a LHS model with a simple measure \( \mu := \sum_\omega q(\omega)\delta_{\rho_\omega} \).

To prove (iv), assume that any finite sub-assemblage \( \{\mu_x\}_{x \in F} \) is classical. By (i), this is equivalent to the fact that for any finite \( F \subseteq X \), the subset
\[
M_F := \{ \mu \in \mathcal{P}_S(K), \mu_x < \mu, \forall x \in F \}
\]
is nonempty and it is easily seen from the definition of Choquet order that \( M_F \) is also closed, in the topology of \( \mathcal{P}(K) \). Moreover, since for any finite collection \( F_i \subseteq A, i = 1, \ldots, n \), we have \( \bigcap_i M_{F_i} = M_{\bigcup_i F_i} \), we see that
\[
\{ M_F, F \subseteq X \text{ is finite} \}
\]
is a collection of closed subsets in \( \mathcal{P}(K) \) with the finite intersection property. The statement now follows by compactness of \( \mathcal{P}(K) \).

To prove (v), let \( \{\mu_x\} < \mu \) and let \( T : K \to K \) be an affine bijection preserving \( \{\mu_x\}_{x \in X} \), then for any \( f \in P(K) \) and \( x \in X \),
\[
\int_K f d\mu_x = \int_K f(T^{-1}(\rho))d\mu^T_\rho(\rho) \leq \int_K f \circ T^{-1} d\mu = \int_K f d\mu^T
\]
so that \( \mu_x < \mu^T \). The set of all affine bijections \( K \to K \) form a compact group (in the topology of pointwise convergence) of which the elements preserving the assemblage form a compact subgroup \( G \). It is easily checked that the map \( G \to \mathcal{P}(K) \) given by \( S \mapsto \mu^S \) is continuous. Let \( m \) be the Haar measure for \( G \) and let \( \mu_m = \int_G \mu^S dm(S) \), then \( \mu_m \) is invariant under \( T \) and we have for any \( f \in P(K) \) and \( x \in X \):
\[
\int_K f d\mu_m = \int_G \int_K f(\rho) d\mu_m^S(\rho) d\mu(S) \geq \int f d\mu_x.
\]

We now give some further characterization of the Choquet order in the case of simple measures.

**Proposition 4.3.** Let \( \mu, \nu \in \mathcal{P}(K) \) and assume that \( \nu = \sum_{a=1}^k \lambda_a \delta_{\sigma_a} \) is simple. Then \( \nu < \mu \) if and only if for all \( g_1, \ldots, g_k \in A \) we have
\[
\sum_a \lambda_a \langle g_a, \sigma_a \rangle \leq \int_K (g_1 \vee \cdots \vee g_k)(\rho) d\mu.
\]

**Proof.** Assume the inequality holds for all \( g_1, \ldots, g_k \in A \). Let \( f \in P(K) \), then there are affine functions \( g_a \in A(K) = A, a = 1, \ldots, k \), such that \( g_a \leq f \) and \( f(\sigma_a) = g_a(\sigma_a) \). Then we have \( \bigvee_a g_a \leq f \) and therefore
\[
\int f d\nu = \sum_a \lambda_a f(\sigma_a) = \sum_a \lambda_a \langle g_a, \sigma_a \rangle \leq \int_K (\bigvee_a g_a)(\rho) d\mu \leq \int_K f d\mu.
\]

For the converse, assume that \( \nu < \mu \), then
\[
\sum_a \lambda_a \langle g_a, \sigma_a \rangle \leq \sum_a \lambda_a \langle \bigvee_{a'} g_{a'}, \sigma_a \rangle = \int_K (\bigvee_{a'} g_{a'})(\rho) d\nu \leq \int_K (\bigvee_{a'} g_{a'})(\rho) d\mu,
\]
the last inequality follows from the fact that the maximum of affine functions is convex. \( \square \)
4.2. Dichotomic assemblages and steering degree. We now restrict our attention to dichotomic assemblages \( \{\mu_x\}_{x \in X} \subseteq \mathcal{P}_{2,\sigma}(K) \), where \( \mathcal{P}_{2,\sigma}(K) \) denotes the subset of measures in \( \mathcal{P}_\sigma(K) \) supported in two points. In this case we obtain a simpler characterization of the Choquet order.

**Lemma 4.4.** For \( \nu \in \mathcal{P}_{2,\sigma}(K) \) and \( \mu \in \mathcal{P}_\sigma(K) \), we have \( \nu \prec \mu \) if and only if
\[
\int_K |\langle h, \rho \rangle| d\nu(\rho) \leq \int_K |\langle h, \rho \rangle| d\mu(\rho), \quad \forall h \in A.
\]

**Proof.** Let \( g_+ \in A, \rho \in K \). Note that
\[
(g_+ \lor g_-)(\rho) = \max\{\langle g_+, \rho \rangle, \langle g_-, \rho \rangle\} = \frac{1}{2}(\langle g_+ - g_-, \rho \rangle + \langle g_+ + g_-, \rho \rangle).
\]
If the inequality in the lemma is satisfied, then we have
\[
\lambda_+ \langle g_+, \sigma_+ \rangle + \lambda_- \langle g_-, \sigma_- \rangle \leq \int_K (g_+ \lor g_-)(\rho) d\nu = \frac{1}{2}\left(\int_K |\langle g_+ - g_-, \rho \rangle| d\nu(\rho) + \langle g_+ + g_-, \sigma \rangle \right)
\]
\[
\leq \frac{1}{2}\left(\int_K |\langle g_+ - g_-, \rho \rangle| d\mu(\rho) + \langle g_+ + g_-, \sigma \rangle \right) = \int_K (g_+ \lor g_-)(\rho) d\mu
\]
By Proposition 4.3, this implies \( \nu \prec \mu \). The converse holds since \( \rho \mapsto |\langle h, \rho \rangle| \) is convex for any \( h \in A \).

**Lemma 4.5.** For any \( h \in A \) and \( \mu \in \mathcal{P}_\sigma(K) \), we have
\[
\int_K |\langle h, \rho \rangle| d\mu \leq ||h||^\sigma = \max\{\nu \in \mathcal{P}_{2,\sigma}(K)\} \int_K |\langle h, \rho \rangle| d\nu.
\]

**Proof.** Let \( K_\pm = \{\rho \in K, \pm \langle h, \rho \rangle \geq 0\} \), then
\[
\int_K |\langle h, \rho \rangle| d\mu(\rho) = \int_{K_+} \langle h, \rho \rangle d\mu - \int_{K_-} \langle h, \rho \rangle d\mu = \langle h, \mu_+ - \mu_- \rangle
\]
where \( \mu_\pm = \int_{K_\pm} \rho d\mu(\rho) \). Since \( \mu_+ + \mu_- = \sigma \), we have \( \pm (\mu_+ - \mu_-) \leq \sigma \), so that \( \langle h, \mu_+ - \mu_- \rangle \leq ||h||^\sigma \).
Since \( || \cdot ||^\sigma \) is a base norm with respect to the order unit \( \sigma \), there are some \( y_\pm \in V^+ \) such that \( y_+ + y_- = \sigma \) and
\[
||h||^\sigma = \langle h, y_+ - y_- \rangle = |\langle h, y_+ \rangle| + |\langle h, y_- \rangle| = \int_K |\langle h, \rho \rangle| d\nu(\rho),
\]
where \( \nu \in \mathcal{P}_{2,\sigma}(K) \).

We also have an alternative characterization of the witness set \( \mathcal{W}_{2,\sigma}^g \).

**Lemma 4.6.** Let \( w \in A^g \). Then \( w \in \mathcal{W}_{2,\sigma}^g \) if and only if for all \( \mu \in \mathcal{P}_\sigma(K) \),
\[
\sum_x \int_K |\langle w_x, \rho \rangle| d\mu(\rho) \leq 1.
\]

**Proof.** Assume that \( (w_1, \ldots, w_d) \in \mathcal{W}_{2,\sigma}^g \), then by Proposition 3.8 there is some \( w_0 \in A^+, \langle w_0, \sigma \rangle = 1 \) such that \( \sum_x \epsilon_x w_x \leq w_0 \) for all \( \epsilon \in \{\pm 1\}^g \). For any \( \rho \in K \), there is some \( \epsilon \in \{\pm 1\}^g \) such that
\[
\sum_x |\langle w_x, \rho \rangle| = \sum_x \epsilon_x |\langle w_x, \rho \rangle| \leq |\langle w_0, \rho \rangle|.
\]
For any \( \mu \in \mathcal{P}_\sigma(K) \) we obtain
\[
\sum_x \int_K |\langle w_x, \rho \rangle| d\mu \leq \int_K |\langle w_0, \rho \rangle| d\mu = \langle w_0, \sigma \rangle = 1.
\]
For the converse, let \( \{\mu_x = \lambda \delta_{x+} + (1 - \lambda) \delta_{x-}\}_{x=1}^g \) be a classical dichotomic assemblage. Note that the corresponding element in \( S_g \otimes \min K \) has the form \((\sigma, y)\) with \( y_x = \lambda \sigma_{x+} - (1 - \lambda) \sigma_{x-}\). Let \( \mu \in \mathcal{P}_\sigma(K) \) be such that \( \mu_x < \mu \) for \( x = 1, \ldots, g \), then by the definition of Choquet order
\[
\sum_x |\langle w_x, y_x \rangle| \leq \sum_x \lambda |\langle w_x, \sigma_{x+} \rangle| + (1 - \lambda) |\langle w_x, \sigma_{x-} \rangle| = \sum_x \int_K |\langle w_x, \rho \rangle| d\mu_x \leq \sum_x \int_K |\langle w_x, \rho \rangle| d\mu \leq 1.
\]
The assertion now follows from Prop. 3.9.

We now obtain an expression for the universal steering degree \( s_\sigma \).

**Theorem 4.7.** For \( \mu \in \mathcal{P}_\sigma(K) \), let
\[
c_\mu := \inf_{h \in A, \|h\|^\sigma = 1} \int_K |\langle h, \rho \rangle| d\mu(\rho).
\]
Then \( c_\mu \leq s_\sigma \). There exists a boundary measure \( \mu \in \mathcal{P}_\sigma(K) \) such that \( c_\mu = s_\sigma \), invariant under any affine bijection \( K \rightarrow K \) that preserves \( \sigma \).

*Proof.* Let \( \mu \in \mathcal{P}_\sigma(K) \). Note that \( c_\mu \) is the largest \( c \in [0,1] \) such that \( c \|h\|^\sigma \leq \int_K |\langle h, \rho \rangle| d\mu(\rho) \), for all \( h \in A \). Let \( (w_1, \ldots, w_g) \in V_2^{\sigma} \), then by Lemma 4.6 we have
\[
c_\mu \sum_x \|w_x\|^\sigma \leq \sum_x \int_K |\langle w_x, \rho \rangle| d\mu(\rho) \leq 1,
\]
so that \( c_\mu \leq s_\sigma \) by (10).

We now prove existence of the measure such that equality is attained. Let \( s = s_\sigma \) and let \( \nu = \lambda \delta_{\sigma^+} + (1 - \lambda) \delta_{\sigma^-} \in \mathcal{P}_{2,\sigma}(K) \). Put \( \rho_\sigma|_{\nu} = \lambda \sigma_{\sigma^+}, \rho_\sigma|_{\nu} = (1 - \lambda) \sigma_{\sigma^-} \) and let \( \mu_{s,\nu} \in \mathcal{P}_{2,\sigma}(K) \) be the measure corresponding to \( \{s \rho_\sigma|_{\nu} + (1 - s) \frac{\mu_{\sigma}}{2}\} \). By definition of \( s_\sigma \) and Theorem 4.2 (iv), we see that \( \{\mu_{s,\nu}\}_{\nu \in \mathcal{P}_{2,\sigma}(K)} \) is a classical dichotomic assemblage, moreover, it is clearly invariant under the group \( G_\sigma \) of affine bijections that preserve \( \sigma \). Using again Theorem 4.2, we see that there is some measure \( \mu \in \mathcal{P}_{\sigma}(K) \), invariant under \( G_\sigma \) and such that
\[
\mu_{s,\nu} \prec \mu, \quad \forall \nu \in \mathcal{P}_{2,\sigma}(K).
\]
Let \( h \in A \) and assume that \( h \notin \pm A^+ \). Then there are some \( \rho_\pm \in V^+ \), \( \rho_+ + \rho_- = \sigma \) such that \( \|h\|^\sigma = |\langle h, \rho_+ \rangle| + |\langle h, \rho_- \rangle| = \int |\langle h, \rho \rangle| d\nu \) and \( |\langle h, \rho_\pm \rangle| \geq 0 \), here \( \nu \in \mathcal{P}_{2,\sigma}(K) \) is the corresponding measure. We then have
\[
\int_K |\langle h, \rho \rangle| d\mu \geq \int_K |\langle h, \rho \rangle| d\mu_{s,\nu} = |\langle h, s(\rho_+ - \rho_-) \rangle| = s \|h\|^\sigma
\]
If \( h \in \pm A^+ \), then
\[
\int_K |\langle h, \rho \rangle| d\mu = |\langle h, \sigma \rangle| = \|h\|^\sigma \geq s \|h\|^\sigma.
\]
It follows that \( c_\mu \geq s \). By the first part of the proof, we now have \( c_\mu = s_\sigma \). 

*Example 4.8.* For a quantum system \( (M_n^\sigma, M_n^+ I) \), \( s_\sigma \) is the same for all faithful states \( \sigma \). This follows by using the map \( \sigma^{-1/2} \cdot \sigma^{-1/2} + 1 \) in point (2) on p. 10 to map assemblages to measurements and the relation to compatibility degree, see also [8]. So we may choose \( \sigma = n^{-1}I \) and then \( \|h\|^\sigma = n^{-1} \|h\|^2 \), where \( \|h\|^2 \) is the trace norm. The universal dichotomic steering degree (and compatibility degree) is then given by
\[
s_\sigma = \inf_{\|h\|^2 = n} \int_{\mathcal{P}_n} |\langle \psi, h \psi \rangle| d\mu(\psi),
\]
where \( \mu \) is the unique unitarily invariant measure on the set \( \mathcal{P}_n \) of pure states. This corresponds to the results of [8], where the infimum was also evaluated.
Example 4.9. Let \((V, V^+, 1)\) be the centrally symmetric system given by the norm \(\| \cdot \|\) in \(\mathbb{R}^n\) and let \(\sigma\) be the central element \(\sigma = (1, 0)\). Note that in this case the dichotomic assemblages are the same as sets of dichotomic measurements for the centrally symmetric system given by the norm \(\| \cdot \|^*\), so by the results of [7] we have that \(s_{(0,1)} = \pi_1^{-1}\) where \(\pi_1\) is the 1-summing constant for the norm \(\| \cdot \|^*\).

We now check this in our setting. The base \(K^\sigma\) of \(A^+\) is the dual state space isomorphic to the unit ball of the dual norm

\[ K^{(1,0)} = K^* = \{(1, \psi), \|\psi\|^* \leq 1\} \]

and the base norm is then \(\|(t, \varphi)\|^{(1,0)} = \max\{|t|, \|\varphi\|^*\}\). The boundary measures \(\nu \in \mathcal{P}_{(1,0)}^b(K)\) which are invariant under affine bijections preserving \((1, 0)\) correspond to regular Borel probability measures on the set \(C = \partial_e B\) that are invariant under isometries of \(\| \cdot \|\). Let \((t, \varphi) \in A, \|(t, \varphi)\|^{(1,0)} = 1\), then for any such \(\nu\),

\[
1 \geq \int_K |\langle (t, \varphi), (1, x) \rangle|d\nu = \int_B |t + \langle \varphi, x \rangle|d\nu = \frac{1}{2} \int_B |t + \langle \varphi, x \rangle| + |t - \langle \varphi, x \rangle|d\nu
= \int_B \max\{|t|, |\langle \varphi, x \rangle|\}d\nu \geq \int_B |\langle \varphi, x \rangle|d\nu = \int_K |\langle (0, \varphi), (1, x) \rangle|d\nu
\]

(the second equality holds since \(\nu\) is invariant under the map \(x \mapsto -x\)). If \(\|\varphi\|^* \leq |t|\), then we have \(|\langle \varphi, x \rangle| \leq |t|\) for all \(x \in B\) and the integral is equal to \(|t| = 1\). If \(\|\varphi\|^* \geq |t|\), then \(\|\varphi\|^* = 1\). It follows that infimum in the definition of \(c_\nu\) is attained at an element with \(t = 0\) and we have

\[
c_\nu = \inf_{\|\varphi\|^* = 1} \int_C |\langle \varphi, x \rangle|d\mu \leq \pi_1^{-1}.
\]

Here the inequality follows from [13, Thm. 1], moreover, equality is attained for some invariant probability measure \(\nu_0\) on \(C\). It follows from our results that there is some invariant boundary measure \(\mu\) such that

\[
\pi_1^{-1} = c_{\nu_0} \leq s_{(1,0)} = c_\mu \leq \pi_1^{-1},
\]

so that, indeed, \(\pi_1^{-1} = s_{(1,0)}\).

5. Unsteerable states in GPTs

Let \((V_A, V_A^+, 1_A)\) and \((V_B, V_B^+, 1_B)\) be two system and let \(\sigma_{AB} \in K_A \otimes_{\max} K_B\). For any measurement \(f\) on the system \(V_A\), let \(\nu_f \in \mathcal{P}_{\sigma_B}(K_B)\) denote the simple measure given by the conditional states \(\{(fa \otimes id_B)(\sigma_{AB})\}\). By Theorem 4.2, \(\sigma_{AB}\) is \((A \rightarrow B)\) unsteerable (by a specified type of measurements) if and only if the assemblage of conditional states \(\{\nu_f\}\), parametrized by the set of all measurements \(f\) (of the specified type) is classical: there is some measure \(\mu \in \mathcal{P}_{\sigma_B}^a(K_B)\) such that for all \(f\),

\[
\{\nu_f\} \prec \mu.
\]

Assume that \(U_A : K_A \rightarrow K_A\) and \(U_B : K_B \rightarrow K_B\) are affine bijections such that \(U_A \otimes U_B\) preserves \(\sigma_{AB}\). If also \(U_A\) preserves the specified family of measurements, then since

\[
U_B((fa \otimes id_B)(\sigma_{AB})) = (fa \circ U_A^{-1} \circ U_A \otimes U_B)(\sigma_{AB}) = (fa \circ U_A^{-1} \otimes id_B)(\sigma_{AB}),
\]

the assemblage is invariant under \(U_B\), so we may assume that \(\mu\) is an invariant boundary probability measure.

Note that the state \(\sigma_{AB}\) determines an affine map \(S_{A\rightarrow B} : K_A^{\sigma_A} \rightarrow K_B\) by

\[
\langle h_A \otimes h_B, \sigma_{AB} \rangle = \langle h_B, S_{A\rightarrow B}(h_B) \rangle, \quad h_A \in K_A^{\sigma_A}, \ h_B \in E_B.
\]

Similarly, there is an affine map \(S_{B\rightarrow A} : K_B^{\sigma_B} \rightarrow K_A\). Here the marginal states are not necessarily interior points of the state space, in that case we can restrict to the generated faces of \(K_A\) and \(K_B\), the corresponding base normed spaces and their duals. Below we assume \(\sigma_A \in \text{int}(K_A)\) and \(\sigma_B \in \text{int}(K_B)\) for simplicity. We then have \((fa \otimes id_B)(\sigma_{AB}) = S_{A\rightarrow B}(fa)\). By Lemma 3.12, the
family of measurements \( \{ f_a \} \) is (formally) and assemblage which admits a LHS model if and only if the family is compatible. The following is now immediate.

**Proposition 5.1.** The state \( \sigma_{AB} \) is unsteerable by a family of measurements \( \mathcal{F} \) if and only if \( S_{A \rightarrow B} \) is \( \mathcal{F} \)-incompatibility breaking: it maps \( \mathcal{F} \) to an assemblage with a LHS model.

Combining with Proposition 4.3, we obtain the following condition (this should be compared with the condition in [19, Theorem 1] in the quantum case).

**Theorem 5.2.** Let \( \mathcal{F} \) be a set of measurements with \( k \) outcomes. A state \( \sigma_{AB} \) is \( \mathcal{F} \)-unsteerable if and only if for any boundary measure \( \mu \in \mathcal{P}_{\sigma_B(K)}^b \), any measurement \( f \in \mathcal{F} \) and any \( g_1, \ldots, g_k \in A \), we have

\[
\sum_a \langle g_a, S_{A \rightarrow B}(f_a) \rangle \leq \int (g_1 \vee \cdots \vee g_k) d\mu.
\]

If \( \mathcal{F} \) is the set of dichotomic measurements, we obtain the following characterization by properties of the map \( S_{B \rightarrow A} \), or its extension to a positive map \( A_B \rightarrow V_A \). Note that the inequality is similar to the principal radius [29, Eq.(320)] for qubit systems.

**Theorem 5.3.** A state \( \sigma_{AB} \in K_A \otimes_{\text{max}} K_B \) is unsteerable by dichotomic measurements if and only if there is a boundary measure \( \mu \in \mathcal{P}_{\sigma_B(K_B)}^b \) such that for all \( h \in A_B \),

\[
\| S_{B \rightarrow A}(h) \|_{V_A} \leq \int |\langle h, \rho \rangle| d\mu(\rho).
\]

In particular, this is true if \( \| S_{B \rightarrow A}(h) \|_{V_A} \leq s_{\sigma_B} \| h \|_{\sigma_B} \) for all \( h \).

**Proof.** Let \( \{ f_{\pm} \} \) be a dichotomic measurement (that is, \( f_\pm \in E_B \) and \( f_+ + f_- = 1_B \)) and let \( \nu_f \in \mathcal{P}_{2,\sigma_B}^b(K_B) \) be the measure corresponding to \( \{(f_\pm \otimes id)(\sigma_{AB})\} \). By Lemma 4.4, we see that \( \sigma_{AB} \) is unsteerable if and only if there is some \( \mu \in \mathcal{P}_{\sigma_B}^b(K_B) \) such that for all \( h \in A_B \) and all measurements \( \{ f_{\pm} \} \),

\[
\int |\langle h, \rho \rangle| d\nu_f \leq \int |\langle h, \rho \rangle| d\mu.
\]

The integral on the left has the form

\[
\int |\langle h, \rho \rangle| d\nu_f = |\langle f_+ \otimes h, \sigma_{AB} \rangle| + |\langle f_- \otimes h, \sigma_{AB} \rangle| = |\langle f_+, S_{B \rightarrow A}(h) \rangle| + |\langle f_-, S_{B \rightarrow A}(h) \rangle|
\]

and the supremum over all dichotomic measurements is equal to \( \| S_{B \rightarrow A}(h) \|_{V_A} \). The last statement follows from Theorem 4.7.

\[\Box\]

**6. Conclusions**

We have studied steering in the setting of general probabilistic theories. For dichotomic measurements, we proved that steering can be characterized and quantified in terms of certain Banach space tensor cross norms, analogously to compatibility of dichotomic measurements. In the general case, we have shown that steering can be conveniently treated using the classical Choquet theory for probability measures on the compact and convex state space.

We used this setting for some alternative characterization of LHS models. For dichotomic assemblages with a fixed barycenter, we found a variational expression for the universal steering degree that generalizes the expressions known from quantum systems and centrally symmetric systems. We also considered characterizations of bipartite states that are unsteerable and obtained conditions similar to those recently proved for quantum systems.

Our results can be immediately applied to the study of compatible measurements, in particular a similar formula can be found for the \( g \)-independent incompatibility degree for dichotomic measurements that was only lower bounded by the 1-summing constant in [7]. Observe also that similar
results hold also for compact convex subsets in arbitrary (infinite dimensional) locally convex spaces and can be easily extended beyond finite outcome measurements.

Acknowledgement

This work was supported by the grant VEGA 1/0142/20 and the Slovak Research and Development Agency grant APVV-20-0069.

References

[1] Erik M. Alfsen. *Compact convex sets and boundary integrals*. Springer, 1971.
[2] Guillaume Aubrun, Ludovico Lami, Carlos Palazuelos, and Martin Plávala. Entangleability of cones. *Geometric and Functional Analysis*, 31(2):181–205, 2021.
[3] Guillaume Aubrun, Ludovico Lami, Carlos Palazuelos, Stanislaw J. Szarek, and Andreas Winter. Universal gaps for XOR games from estimates on tensor norm ratios. *arXiv preprint arXiv:1809.10616*, 2018.
[4] Manik Banik. Measurement incompatibility and Schrödinger-Einstein-Podolsky-Rosen steering in a class of probabilistic theories. *Journal of Mathematical Physics*, 56(5):052101, 2015.
[5] Howard Barnum, Carl Philipp Gaebler, and Alexander Wilce. Ensemble steering, weak self-duality, and the structure of probabilistic theories. *Foundations of Physics*, 43(12):1411–1427, 2013.
[6] J. Barrett. Information processing in generalized probabilistic theories. *Physical Review A*, 75:03230, 2007.
[7] Andreas Bluhm, A. Jenčová, and I. Nechita. Incompatibility in general probabilistic theories, generalized spectrahedra, and tensor norms. *arXiv:2011.06797*, 2020.
[8] Andreas Bluhm and Ion Nechita. Maximal violation of steering inequalities and the matrix cube. *arXiv preprint arXiv:2105.11302*, 2021.
[9] P. Busch and N. Stevens. Steering, incompatibility, and Bell inequality violations in a class of probabilistic theories. *Phys. Rev. A*, 86:022123, 2014.
[10] D. Cavalcanti and P. Skrzypczyk. Quantitative relations between measurement incompatibility, quantum steering, and nonlocality. *Phys. Rev. A*, 93:052112, May 2016.
[11] Daniel Cavalcanti and Paul Skrzypczyk. Quantum steering: a review with focus on semidefinite programming. *Reports on Progress in Physics*, 80(2):024001, 2016.
[12] Paulo J. F. Cavalcanti, John H. Selby, Jamie Sikora, Thomas D. Galley, and Ana Belén Sainz. Wittworld: A generalised probabilistic theory featuring post-quantum steering. *arXiv:2102.06581*, 2021.
[13] Yehoram Gordon. On p-absolutely summing constants of Banach spaces. *Israel Journal of Mathematics*, 7(2):151–163, 1969.
[14] Anna Jenčová. Incompatible measurements in a class of general probabilistic theories. *Physical Review A*, 98(1):012133, 2018.
[15] A. Jenčová and Martin Plávala. Conditions on the existence of maximally incompatible two-outcome measurements in general probabilistic theory. *Phys. Rev. A*, 96, 2017, arXiv:1703.09447.
[16] Steve James Jones, Howard Mark Wiseman, and Andrew C. Doherty. Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering. *Physical Review A*, 76(5):052116, 2007.
[17] Ludovico Lami. Non-classical correlations in quantum mechanics and beyond. *PhD thesis. arXiv preprint arXiv:1803.02902*, 2018.
[18] H Chau Nguyen and Otfried Gühne. Some quantum measurements with three outcomes can reveal nonclassicality where all two-outcome measurements fail to do so. *Physical Review Letters*, 125(23):230402, 2020.
[19] H. Chau Nguyen, Antony Milne, Thanh Vu, and Sania Jevtic. Quantum steering with positive operator valued measures. *Journal of Physics A: Mathematical and Theoretical*, 2018.
[20] H Chau Nguyen, Huy-Viet Nguyen, and Otfried Gühne. Geometry of Einstein-Podolsky-Rosen Correlations. *Physical review letters*, 122(24):240401, 2019.
[21] Robert R Phelps. *Lectures on Choquet’s theorem*. Springer Science & Business Media, 2001.
[22] Marco Piani and John Watrous. Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, 114:060404, Feb 2015.
[23] Martin Plávala. All measurements in a probabilistic theory are compatible if and only if the state space is a simplex. *Physical Review A*, 94(4):042108, 2016.
[24] Martin Plávala. Conditions for the compatibility of channels in general probabilistic theory and their connection to steering and Bell nonlocality. *Physical Review A*, 96:052127, 2017.
[25] Martin Plávala. General probabilistic theories: An introduction. *arXiv preprint arXiv:2103.07469*, 2021.
[26] S. Popescu and D. Rohrlich. Quantum nonlocality as an axiom. *Found. Phys.*, 24:379–385, 1994.
[27] E. Schrödinger. Probability relations between separated systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, 32(3):446–452, 1936.
[28] Roope Uola, Costantino Budroni, Otfried Gühne, and Juha-Pekka Pellonpää. One-to-one mapping between steering and joint measurability problems. Physical review letters, 115(23):230402, 2015.

[29] Roope Uola, Ana CS Costa, H Chau Nguyen, and Otfried Gühne. Quantum steering. Reviews of Modern Physics, 92(1):015001, 2020.

[30] H. M. Wiseman, S. J. Jones, and A. C. Doherty. Steering, entanglement, nonlocality, and the EPR paradox. Phys. Rev. Lett., 98:140402, 2007.

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