Laws of large numbers for weighted sums of independent random variables: a game of mass

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Abstract

We consider weighted sums of independent random variables regulated by an increment sequence. We provide operative conditions that ensure strong law of large numbers for such sums to hold in both the centered and non-centered case. The existing criteria for the strong law are either implicit or assume some sufficient decay for the sequence of coefficients. In our set up we allow for arbitrary sequence of coefficients, possibly random, provided the random variables regulated by such increments satisfy some mild concentration conditions. In the non-centered case, convergence can be translated into the behavior of a deterministic sequence and it becomes a game of mass provided the expectation of the random variables is a function of the increments. We show how different limiting scenarios can emerge by identifying several classes of increments, for which concrete examples will be offered.

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1 Setup, literature and overview

Let \( X := \{ X_k, k \in \mathbb{N} \} \) be a sequence of independent real valued random variables with finite mean and \( \mathcal{A} \) a Toeplitz summation matrix, i.e., \( \mathcal{A} = (a_{n,k} \in \mathbb{R}_+; n, k \in \mathbb{N}) \) satisfies

\[
\lim_{n} a_{n,k} = 0, \quad \lim_{n} \sum_{k} a_{n,k} = 1, \quad \sup_{n} \sum_{k} |a_{n,k}| < \infty,
\]

where \( \mathbb{R}_+ := [0, \infty) \). In this setup, one seeks conditions on \( X \) and \( \mathcal{A} \) to ensure convergence in probability or almost sure convergence for the sequence \( \{ S_n, n \in \mathbb{N} \} \), where

\[ S_n := \sum_{k} a_{n,k} X_k. \]

This type of questions, known in the literature as weak/strong Law of Large Numbers (LLN), have been investigated since the birth of probability theory, see [4], and has been extensively studied in the XX century, see [10, 7] and references therein for a classical and a more recent account. The quest for operative conditions that apply to a wide range of \( (X, \mathcal{A}) \) and ensure weak/strong convergence of \( S_n \) has been the subject of [6, 8, 11, 9].

When the elements of \( X \) are i.i.d. mean zero random variables, the weak LLN is equivalent to \( \lim_{n} \max_k a_{n,k} = 0 \), see [8, Theorem 1]. In [8, Theorem 2], the following sufficient conditions for the strong LLN are given:

\[
\mathbb{E}[X_1^{1+\frac{1}{\gamma}}] < \infty \quad \text{and} \quad \lim \sup \limits_{n} n^\gamma \max_k a_{n,k} < \infty, \quad \text{for some} \quad \gamma > 0.
\]

For (mean-zero) independent but not identically distributed variables, similar sufficient conditions have been examined in [6, 11, 9]. In particular, in analogy with the two conditions in (1.4), these references require that the variables \( X_k \)'s are stochastically dominated by a random variable \( X^* \) satisfying a moment condition, and that the associated coefficients \( a_{n,k} \) decay sufficiently fast.

Unlike these references, in this paper we impose concentration conditions on \( X \) and obtain sufficient conditions for the weak/strong LLN when \( \lim \sup \limits_{n} \max_k a_{n,k,n} > 0 \). Here, as in [6], we consider a family of weights \( m := \{ m_k \in \mathbb{R}_+, k \in \mathbb{N} \} \), which we will refer to as masses, such that

\[
\sum_{k} m_k = \infty.
\]

Set \( M_n := \sum_{k=1}^{n} m_k \) and

\[
a_{n,k} := \begin{cases} \frac{m_k}{M_n} & \text{if } k \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]

Conditions (1.2) and (1.3) hold true by definition. Also, as (1.5) implies \( \lim \limits_{n} M_n = \infty \) it follows that (1.1) is in force and therefore \( \mathcal{A} \) is a Toeplitz summation matrix. We notice in particular that if its sum in (1.5) is finite, then no LLN can be expected. In fact, if the random variables are not all constant, the limit random variable will have finite yet strictly positive
variance, what precludes convergence to a constant. To describe our results we depart from the set up of [6] and consider \( X_k = X_k(m) \) to be a one parameter family of random variables.

**New contributions and starting motivation:** The first goal of our paper is to look for optimal conditions on \( X \) to ensure that for *any* sequence of positive masses \( m \in \mathbb{R}^+_N \),

\[
S_n = S_n(m) := \sum_{k=1}^{n} \frac{m_k}{M_n} X_k(m_k)
\]

(1.7)

converges to zero as \( n \to \infty \). Due to the nature of the coefficients in (1.6) we will refer to the sum in (1.7) as *incremental sum*.

Our original motivation to look at this type of incremental sums with arbitrary sequence of positive masses \( m \in \mathbb{R}^+_N \) comes from the analysis of the asymptotic speed, and related large deviations, of a certain random walk model in a random environment pursued in [3, 1]. This model is obtained as a perturbation of another process by adding independence through resettings. Such a perturbation in reality gives rise to a slightly more general sum than the incremental one in (1.7). Hence we will prove statements for the above incremental sum but also for the more general one, referred to as *gradual sum*, as defined in (2.1)–(2.2) below.

Furthermore, in the context of [3, 1], \( X \) is in general formed by variables with non-zero mean. Our second goal is to explore in this general non-centered case structural conditions on the masses \( m \) that ensure convergence of the weighted sums. We will in particular identify different classes of masses for which the resulting limit exists and can be characterized, what we will refer to as the *game of mass*.

**Structure of the paper:** In Section 2 we state the general LLNs for centered random variables: Theorems 2.1 and 2.2 respectively, for the weak and the strong laws of the incremental sum; Theorem 2.3 for the more general gradual sum. Section 3 is devoted to the game of mass were we study concrete convergence criteria for non-centered variables. A discussion on the nature of the hypotheses illustrated by counterexamples is presented in Section 4. The concluding Section 5 contains the proofs of the main theorems. Appendix A covers a technical lemma adapted from [6] and used in the proof of Section 5.2.

## 2 LLNs for mean-zero variables

In what follows all random variables are defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \mathbb{E} \) denotes expectation with respect to \( \mathbb{P} \). Let \( X = \{ X_k(m), m \in \mathbb{R}^+_N, k \in \mathbb{N} \} \) be a family of integrable random variables that are independent in \( k \). Our first statement is the the weak LLN for mean-zero independent random variables.

**Theorem 2.1 (Weak LLN).** Assume that \( X \) is such that:

**C** (Centering)

\[
\forall m \in \mathbb{R}^+_N, k \in \mathbb{N}; \quad \mathbb{E}[X_k(m)] = 0.
\]

**W1** (Concentration)

\[
\lim_{m \to \infty} \sup_k \mathbb{P}(|X_k(m)| > \varepsilon) = 0, \quad \forall \varepsilon > 0.
\]

**W2** (Uniform Integrability)

\[
\lim_{A \to \infty} \sup_{k,m} \mathbb{E}\left[|X_k(m)| 1_{|X_k(m)| > A}\right] = 0.
\]
Then, for any sequence \( m \in \mathbb{R}_+^N \) that satisfies (1.5),
\[
\lim_{n \to \infty} \mathbb{P} (|S_n| > \varepsilon) = 0, \quad \forall \varepsilon > 0.
\]

In this centered case, as captured in the next theorem, to obtain a strong LLN for \( \{ S_n, n \in \mathbb{N} \} \) we impose further conditions on \( X \). In particular the concentration condition will be strengthened by requiring a mild polynomial decay and the uniform integrability by a uniform domination.

**Theorem 2.2 (Strong LLN).** Assume that \( X \) satisfies (C) and

\begin{itemize}
  \item [(S1)] (Polynomial decay) \( \exists \delta > 0 : \forall \varepsilon > 0, \exists C = C(\varepsilon) \) such that \( \sup_k \mathbb{P} (|X_k(m)| > \varepsilon) < \frac{C}{m^{\delta}}. \)
  \item [(S2)] (Stochastic domination and moment control) There is a random variable \( X_* \) and \( \gamma > 0 \) such that \( \forall x \in \mathbb{R}, \sup_k \mathbb{P}(X_k(m) > x) \leq \mathbb{P}(X_* > x) \) and \( \mathbb{E}(|X_*|^{2+\gamma}) < \infty. \)
\end{itemize}

Then for any sequence \( m \in \mathbb{R}_+^N \) that satisfies (1.5),
\[
\mathbb{P} \left( \lim_{n \to \infty} S_n = 0 \right) = 1.
\]

As anticipated, motivated by the random walk model in \([3, 1]\), we next focus on a more general sum by considering a time parameter \( t \) that runs on the positive real line partitioned into intervals \( I_k = [M_{k-1}, M_k) \) of size \( m_k: [0, \infty) = \bigcup_k I_k \). As \( t \to \infty \) the increments determined by the partition are gradually completed as captured in definition (2.2) below. For \( m \in \mathbb{R}_+^N \), let \( \ell_t = \ell_t(m) := \inf \{ \ell \in \mathbb{N} : M_{\ell} \geq t \} \), (2.1) and set \( \bar{t} := t - M_{\ell_t-1} \). We define the gradual sum by
\[
S_t = S_t(m) := \sum_{k=1}^{\ell_t-1} \frac{m_k}{t} X_k(m_k) + \frac{\bar{t}}{t} X_{\ell_t}(\bar{t}).
\]
(2.2)

The next theorem, is the extension of Theorem 2.2 for the gradual sum \( S_t \).

**Theorem 2.3 (Generalized strong LLN).** Assume that \( X \) satisfies (C), (S1), (S2) and further:

\begin{itemize}
  \item [(S3)] (Slow relative increment growth) for every \( \varepsilon > 0 \) there is a \( \beta > 1, C_\varepsilon > 0 \) which for every \( t, r > 0 \)
  \[
  \sup_k \mathbb{P} \left( \sup_{s \leq m} |(r + s)X_k(r + s) - rX_k(r)| \geq t\varepsilon \right) \leq \frac{C_\varepsilon m^\beta}{t^{\beta}}.
  \]
\end{itemize}

Then for any sequence \( m \in \mathbb{R}_+^N \) that satisfies (1.5),
\[
\mathbb{P} \left( \lim_{t \to \infty} S_t = 0 \right) = 1.
\]
Remark 2.4. (Continuity assumption for the gradual sum) Assumption (S3) controls the oscillations between the times $M_n$. If the sequence $sX_k(s)$ was a martingale, Doob’s $L^p$ inequality would yield (S3). Also, if the increments

$$(r + s)X_k(r + s) - rX_k(r)$$

were bounded by $f(s)$ then this condition would also follow. Condition (S3) reveals that the one parameter families $\{ X_k(m), m \in \mathbb{R} \}$ we consider here possess some dependence structure or satisfy some increment domination.

If the random variables $X$ are not centered, the convergence of $S_t$ will correspond to the convergence of $\mathbb{E}[S_t]$. This is the content of the next result.

**Corollary 2.5 (Non-centered strong LLN).** Assume that $X$ satisfies (S1) - (S3) and that the sequence $m \in \mathbb{R}^N$ satisfies (1.5). Then, for any increasing sequence $\{ t_k \}_{k \in \mathbb{N}}$ with $\lim_k t_k = \infty$ and $\lim_k \mathbb{E}[S_{t_k}] =: v$,

$$\mathbb{P} \left( \lim_k S_{t_k} = v \right) = 1.$$ 

If $\mathbb{E}[X_k(m)] = v_m$ depends only of $m$, one can relate the convergence of $S_t$ to the structure of $m$. This is what we call the game of mass and explore next.

### 3 The game of mass: operative conditions

As a consequence of Corollary 2.5, it is natural to seek for conditions to be imposed on $(X, m)$ that guarantee convergence of the full sequence $S_t$. In this section, we assume that the expectation of $X_k(m)$ depends only on $m$ and not on $k$, that is:

$$\mathbb{E}[X_k(m)] = v_m \quad \forall k \in \mathbb{N},$$

and that

$$m \mapsto v_m \quad \text{is a bounded continuous function in } \mathbb{R}_+ \cup \{ \infty \}. \quad (3.1)$$

We will classify the mass-sequences $m$ into two classes: regular and non-regular. The notion of regularity will be captured by the existence of the weak limit of the empirical measure associated to a given mass sequence. In Section 3.1 we give the definition of regular masses and show that, contrary to the non-regular ones, the LLN always holds true. We will also investigate other possible notions of regularities and how they related to the above mentioned weak convergence, see Section 3.1.1. Section 3.2 is devoted to examples of bounded masses and their relation to the previously defined regularity notions. In Section 3.3 we identify the regular regime of mass sequences that diverge in the Cesaro sense and provide illustrative examples of how unbounded masses relate to the regularity notions defined in Section 3.1. Finally, in Section 3.4 we investigate what can be said when the mass-sequence $m$ is random, giving in particular further examples of regular masses. The resulting picture of this game of mass is captured at glance in Fig. 1.
Figure 1: Summary of the game of mass \((m, X)\). The above rectangle offers a visual classification of the possible different masses. The region in gray corresponds to masses for which the LLN is valid, that is, \(S_t\) converges. The vertical line divides the masses between regular (left) and irregular (right) ones according to definition 3.1. The horizontal line separates the mass sequences between bounded (down) and unbounded ones for which \(\limsup m_k = \infty\) (up). Among the unbounded masses, those divergent in a Cesaro sense, and in particular those divergent in a classical sense, are always regular. The black and red dotted boxes correspond to those masses for which the related frequencies are asymptotically stable, respectively, in a weak and in a \(L^1\) sense. The roman numbers in each of the different subclasses correspond to the labels of the different illustrative examples from Sections 3.2, 3.3, 3.4.
3.1 Regular mass sequences

Recall (2.1), for a given weight sequence \( m \in \mathbb{R}^N_+ \), define the sequence of empirical mass measures \( \{ \mu_t(\cdot) = \mu_t^{(m)}(\cdot), t \geq 0 \} \) on \( \mathbb{R}_+ \cup \{ \infty \} \) by

\[
\mu_t(\cdot) := \frac{\bar{t}}{t} \delta_t(\cdot) + \sum_{k=1}^{\ell_t-1} \frac{m_k}{t} \delta_{m_k}(\cdot).
\]  

In the sequel, we denote by \( \lambda(f) \) the integral of \( f \) with respect to a generic measure \( \lambda(\cdot) \).

Furthermore, we say that a sequence \( \{ \lambda_t; t \geq 0 \} \) of probability measures on \( \mathbb{R}_+ \cup \{ \infty \} \) converges in the vague sense to a probability measure \( \lambda_* \) on \( \mathbb{R}_+ \cup \{ \infty \} \), denoted by \( \lambda_t \xrightarrow{w} \lambda_* \), when

\[
\lim_t \lambda_t(f) = \lambda_*(f), \quad \text{for every } f \in C_0(\mathbb{R}_+),
\]

where \( C_0(\mathbb{R}_+) \) denotes the space of continuous functions on \([0, \infty)\) that vanish at \( \infty \). We opted for the notation symbol \( w \) in the above vague convergence because it can be seen as weak convergence on \( \mathbb{R}_+ \cup \{ \infty \} \) after proper (Alexandrov’s) compactification. Note that this definition allows for \( \lambda_*(\{\infty\}) := 1 - \lambda_*(\mathbb{R}_+) \) to be strictly positive.

**Definition 3.1 (Regular mass sequence).** We say that \( m \) is a regular mass sequence when there is a probability measure \( \mu_* \) on \( \mathbb{R}_+ \cup \{ \infty \} \) such that

\[
\mu_t \xrightarrow{w} \mu_*.
\]

The following proposition determines the limit of \( S_t \) for regular mass sequences.

**Proposition 3.2 (Limit characterization for regular sequences).** Assume \( X \) satisfies (3.1) and (3.2). Then, for any mass \( m \) and any \( t \geq 0 \):

\[
E[S_t] = \int v_m d\mu_t(m).
\]

In particular, if \( m \) is regular, then

\[
P \left( \lim_t S_t = \int v_m d\mu_*(m) \right) = 1,
\]

else \( S_t \) may or may not converge.

**Proof.**

\[
\int v_m d\mu_t(m) = \frac{\bar{t}}{t} + \sum_{k=1}^{\ell_t-1} v_{m_k} \frac{m_k}{t} = \frac{\bar{t}}{t} X_{\ell_t}(\bar{t}) + \sum_{k=1}^{\ell_t-1} \frac{m_k}{t} X_k(m_k) = E[S_t],
\]

which proves (3.4). As a consequence, if \( m \) is regular, (3.5) follows from Corollary 2.5 and Definition 3.1. When \( m \) is not regular, almost sure convergence is not prevented, in fact, if \( v_m = 0 \) for all \( m \), then by Theorem 2.3, \( S_t \) converges almost surely to 0. On the other hand, Examples XI, XIII presented in Section 3.3.1 below show that almost sure convergence may not hold for irregular masses.
3.1.1 Regularity and stability of empirical frequency

The notion of regularity captured in Definition 3.1 is not the only possible one. For example, instead of looking at asymptotic stability of the empirical measure in (3.3), one may investigate the behavior of the empirical mass frequency \( \{ F_t(\cdot) = F_t^m(\cdot), t \geq 0 \} \) on \( \mathbb{R}_+ \cup \{ \infty \} \) defined as:

\[
F_t(\cdot) := \frac{\delta_t(\cdot)}{\ell_t} + \sum_{k=1}^{t-1} \frac{\delta_{m_k}(\cdot)}{\ell_t}.
\]

We note that, for any \( t \geq 0 \) and any arbitrary function \( f \), the following relation between \( \mu_t \) and \( F_t \) is in force:

\[
\int f(m) \, d\mu_t(m) = \frac{\ell_t}{t} \int m f(m) \, dF_t(m).
\]  (3.6)

In particular, by considering \( f(m) = v_m \) and \( f(m) \equiv 1 \), respectively, we have that

\[
\mathbb{E}[S_t] = \frac{\ell_t}{t} \int v_m m \, dF_t(m),
\]

and

\[
\frac{t}{\ell_t} = \int m \, dF_t(m). \quad (3.7)
\]

The relation in (3.6) may suggest to consider weak convergence of \( F_t \) as a natural alternative notion of regularity. However, as shown in the Proposition 3.2 below, these two notions are not equivalent. We find more convenient to adopt the notion in Definition 3.1 for the following two reasons. First, there are masses for which both \( \nu_t \) and \( F_t \) converge weakly to some \( \nu_\ast \) and \( F_\ast \), respectively, but the limit of \( S_t \) is determined by \( \mu_\ast \) and not by \( F_\ast \), see Examples II, IX, and VII below. Second, among the unbounded masses, those divergent in a Cesaro sense will always be regular while the corresponding \( F_t \) is not guaranteed to admit a limit, see Examples VIII and X.

Yet, it is interesting to look at the LLN from the perspective of masses with “well-behaving” asymptotic frequencies. In particular, the next proposition clarify how the relation between \( \mu_t \) and \( F_t \) expressed in (3.6) behaves in the limit. In particular it shows how to relate the behavior of the empirical frequencies and empirical masses under different modes of convergence, which we next define.

In the following statement, we write \( F_t \xrightarrow{L^1} F_\ast \), if there exists a measure \( F_\ast(\cdot) \) on \( \mathbb{R}_+ \) for which

\[
F_t \xrightarrow{w} F_\ast \quad \text{and} \quad \int m \, dF_t(m) \to \int m \, dF_\ast(m) < \infty.
\]

In a somewhat dual manner, we write \( \mu_t \xrightarrow{w^+} \mu_\ast \), if

\[
\mu_t \xrightarrow{w} \mu_\ast \quad \text{and} \quad \int \frac{1}{m} \, d\mu_t(m) \to \int \frac{1}{m} \, d\mu_\ast(m) < \infty.
\]

**Proposition 3.3 (Regularity and stable frequencies).** Assume \( X \) satisfies (3.1) and (3.2), consider an arbitrary mass sequence \( m \) and assume that \( A := \lim_{t \to \infty} \frac{t}{\ell_t} \in [0, \infty] \) exists. Then:
a if $F_t \xrightarrow{L^1} F_\ast \neq \delta_0 \Rightarrow \mu_t \xrightarrow{w} \mu_\ast$ with $\mu_\ast(f) := A \int m f(m) \, dF_\ast(m)$ and $A \in (0, \infty)$,

b if $\mu_t \xrightarrow{w^*} \mu_\ast \neq \delta_\infty \Rightarrow F_t \xrightarrow{w^*} F_\ast$ with $F_\ast(f) := \frac{1}{A} \int \frac{1}{m} f(m) \, d\mu_\ast(m)$ and $1/A \in (0, \infty)$.

**Proof.** For item a, by (3.7) and the assumption that $F_t(\cdot)$ converges in $L^1$ to $F_\ast \neq \delta_0$, we have that

\[
\frac{t}{\ell_t} = \int m \, dF_t(m) \to \int m \, dF_\ast(m) \in (0, \infty).
\]

The above relation together with (3.6) implies that

\[
\mu_t(f) = \frac{\ell_t}{t} \int m f(m) \, dF_t(m) \to A \int m f(m) \, dF_\ast(m).
\]

We are left with the proof of item b. From (3.7), the vague convergence of $\mu_t(f)$ applied to the function $f(m) = 1/m$ and the assumption that $\mu_\ast \neq \delta_\infty$ we have that

\[
\frac{\ell_t}{t} = \int \frac{1}{m} \, d\mu_t(m) \to \int \frac{1}{m} \, d\mu_\ast(m) = A \in (0, \infty).
\] (3.8)

Therefore, for any bounded continuous function $f$, by combining (3.6) with the regularity assumption and (3.8) we conclude that

\[
F_t(f) = \frac{t}{\ell_t} \int \frac{1}{m} f(m) \, d\mu_t(m) \to \frac{1}{A} \int \frac{1}{m} f(m) \, d\mu_\ast(m).
\]

Proposition 3.3 explains part of the different relations depicted in Fig. 1 among the dotted boxes corresponding to masses for which $F_t$ converge weakly and in $L^1$. In what follows, with the help of examples, we explore more how these notions of weak and $L^1$ convergence for $F_t$ relate to the regularity of $\mu_t$. The examples are organized in the following sections, and in particular they clarify how Figure 1 emerges. First we examine bounded masses, Section 3.2. Second we examine Cesàro divergent masses, Section 3.3.1. Third unbounded masses that are not Cesàro divergent, Section 3.3.2. Finally we examine i.i.d. random masses, Section 3.4.

### 3.2 Examples of bounded masses

If the sequence of masses $m$ is bounded then weak convergence of $F_t$ implies $L^1$ convergence of $F_t$. We also remark that when the sequence is regular the limit of $S_t$ exists and is given by $v = \int v_m \, d\mu_\ast(m)$. The following examples show how the notion of regular masses relates with the notion of weak convergent frequency.

**I (regular + $L^1$-lim $F_t$).** When $\sup_k m_k < \infty$, $L^1$ convergence follows from weak convergence of the empirical frequency plus uniform integrability. If $F_\ast(m) \neq \delta_0$, then the formula for the limit of $S_t$ is given in terms of $F_\ast$. Indeed, by item a of Proposition 3.3 we conclude that

\[
\mathbb{P} \left( \lim_{t \to \infty} S_t = v \right) = 1 \quad \text{where} \quad v = A \int v_m \, dF_\ast(m), \quad A := \lim_{t} \frac{\ell_t}{t}.
\]
II (regular + $L^1$-lim $F_t$). This example shows that if the limit $F_t = \delta_0$ then $\mu_t$ may not be given by the expression in item (a) of Proposition 3.3.

Consider the triangular array $\{ a_{i,j}, i,j \in \mathbb{N}, j \leq i \}$ defined by $a_{i,1} := 1$ and for $1 < j \leq i$, $a_{i,j} := 2^{-i}$, represented below

1,
1, 2\(^{-1}\),
1, 4\(^{-1}\), 4\(^{-1}\),
1, 8\(^{-1}\), 8\(^{-1}\), 8\(^{-1}\),
...

For the sequence of increment take $m_k$ to be the $k$-th term of this array, more precisely let $i(k)$ be such that

$$\frac{(i(k) - 1)i(k)}{2} \leq k \leq \frac{i(k)(i(k) + 1)}{2}$$

and $j(k) := \frac{k - \frac{(i(k) - 1)i(k)}{2}}{2}$. Let $m_k := a_{i(k),j(k)}$. In this example, $F_t \overset{L^1}{\rightarrow} \delta_0$ while $\mu_t \overset{w}{\rightarrow} \delta_1$. This shows that the $L^1$ limit of $F_t$ is not sufficient to describe the limit of $\mathcal{S}_t$, which is given by $v_1 = \int v_m \, d\mu_\ast(m)$.

III (regular + $\#$ lim $F_t$). Let us now move to an example of bounded regular mass sequence such that the limit of $F_t$ does not exist. Consider the sequence $m$ defined as follows:

(i) Set $m_1 = 1$,

(ii) while $F_{m(k)}(1) > 1/4$ set $m_k = 2^{-k}$ else, go to (iii),

(iii) while $F_{m(k)}(1) < 3/4$ set $m_k = 1$ else, go to (ii).

In this case, $\mu_t \overset{w}{\rightarrow} \delta_1$ and $F_t$ does not converge.

Note that if $m$ is not regular, then depending on $\{ v_m, m \in \mathbb{R}_+ \}$, $\mathbb{E}[S_t]$ may or may not converge. If there are $K, L \in \mathbb{R}_+$ such that $v_K < v_L$, as in the example below, it is simple to construct a sequence $m$ for which $\mathbb{E}[S_t]$ does not converge.

IV (irregular + $\#$ lim $F_t$). Let $m$ be the sequence composed of $A_i$ increments of size $K$ followed by $B_i$ increments of size $L$ where the sequences $(A_i)_i, (B_i)_i$ will be determined later. More formally, define $\tau_0 := 0, \tau_n = \tau_{n-1} + A_n + B_n$ and set

$$m_k = \begin{cases} K & \text{if } k \in (\tau_n, \tau_n + A_{n+1}] \text{ for some } n \geq 0, \text{ and} \\ L & \text{if } k \in (\tau_n + A_{n+1}, \tau_{n+1}] \text{ for some } n \geq 0. \end{cases} \quad (3.9)$$

Choose $(A_i, B_i : i \in \mathbb{N})$ such that for all $n \in \mathbb{N}$, $A_n < A_{n+1}$, $B_n < B_{n+1}$ and

$$\frac{L(B_1 + \ldots + B_n)}{K(A_1 + \ldots + A_{n+1})} \leq \frac{1}{n} \quad \text{and} \quad \frac{K(A_1 + \ldots + A_{n+1})}{L(B_1 + \ldots + B_{n+1})} \leq \frac{1}{n}.$$

If $v_K < v_L$ then $\mathbb{E}[\mathcal{S}_t]$ does not converge as

$$\limsup_t \mathbb{E}[\mathcal{S}_t] = v_L \neq v_K = \liminf_t \mathbb{E}[\mathcal{S}_t].$$
V (irregular + $L^1$-lim $F_i$). By combining the sequence defined in Example II with the one defined in Example IV we find an irregular sequence for which $F_i \overset{w}{\rightarrow} F_*$. More precisely, let $m'_k$ be the sequence defined in Example IV and consider a triangular array $a_{i,j}$ defined by $a_{i,1} := m'_i$ and for $1 < j \leq i$, set $a_{i,j} := 2^{-i}$. To conclude, set $m_k := a_{i(k), j(k)}$ with $i(k), j(k)$ as defined in Example II. Note that this sequence is irregular even though $F \overset{w}{\rightarrow} \delta_0$.

As we look back to item (b) of Proposition 3.3 we see that $w^+$ convergence can not occur in any of the examples of bounded regular mass for which the empirical frequency does not converge. Indeed, all those example have a significant amount of increments of negligible mass, and as such, they modify the empirical frequency without affecting the limit of the mass sequence. We now move to the study of unbounded masses.

3.3 Unbounded masses

3.3.1 Divergent and Césaro’s divergent masses

We say that a sequence of masses $m$ is Césaro’s divergent when

$$
\lim_{n} \frac{m_1 + \ldots + m_n}{n} \to \infty
$$

(3.10)

In this case one has that $\mu \overset{w}{\rightarrow} \delta_\infty$. Therefore the Césaro divergent sequences are always regular and by Proposition 3.2 it follows that

$$
P(\lim_t S_t = v_\infty) = 1.
$$

(3.11)

A particular case of Césaro divergence is given by the divergent masses as captured in the next example.

VI (Divergent mass + $w$-lim $F_i$). We say that a sequence of masses $m$ is divergent when

$$
\lim_{k \to \infty} m_k = \infty,
$$

(3.12)

in which case (3.10) holds true and hence (3.11).

The regime captured in (3.12) is treated in [1, Theorem 1.10]. Theorem 2.3 can actually be seen as a generalization of the latter. As briefly mentioned at the end of Section 4, the present proofs could actually cover even more general variants, for example when relaxing the assumption in Equation (3.1). The following example shows that in the Césaro divergent regime, the empirical sequence may converge, but may not be able to capture the limit of $S_t$.

VII (Césaro divergent mass + $w$-lim $F_i$). Consider the following mass sequence $m$

$$
m_k := \begin{cases} 
1 & \text{if } k \text{ is odd, and} \\
k & \text{if } k \text{ is even.}
\end{cases}
$$

Informally, half the increments are 1, and the other half is divergent. More precisely

$$
F_t \overset{w}{\rightarrow} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_\infty.
$$
As such, one might be tempted to say that $E[S_t] \to \frac{1}{2}v_1 + \frac{1}{2}v_\infty$ as $t \to \infty$. This is not the case because one has to take into account the relative weights of the sequences. As it turns out, the mass of increments of size 1 for this particular sequence vanishes in the limit. Indeed, note that the sum of the first $2k$ increments, $M_{2k}$ is

$$M_{2k} = \frac{k(k-1)}{2} + k = \frac{k^2 + k}{2}.$$  

Now note that $\frac{k}{M_{2k}} \to 0$ and therefore

$$E[S_{M_{2k}}] = \frac{k}{M_{2k}}v_1 + \frac{1}{M_{2k}} \sum_{i=1}^{k} iv_i \to v_\infty.$$  

Also in this example, if $v_1 \neq v_\infty$, then the weak limit of $F_t$ does not determine the limit of $S_t$, even if it is well defined.

As in the bounded case, see Example III, also Cesaro divergent sequences may not have well behaving empirical frequencies, as shown in the next example.

VIII (Cesaro divergent mass + $\not\exists w$-lim $F_t$). It is possible to construct a sequence $m$ that is regular but such that $F_t$ does not converge weakly. Take an irregular sequence such as the one defined in (3.9) and insert it with a huge increment so that it diverges in the Cesaro sense.

3.3.2 Unbounded sequences that do not diverge in the Cesaro sense

When dealing with unbounded masses that are not Cesaro divergent, then the sequence is not granted to be regular and more subtle scenarios emerge, as the following examples illustrate. We start with an example of a regular sequence allowing an asymptotic positive mass of increments of finite size and positive mass at infinity.

IX (Regular $\liminf m_k < \infty + w$-lim $F_1$). Let $m \in \mathbb{R}_+^N$ be such that $m_1 := 1, m_2 := 2, m_3 := 1$. If $m_k := j > 1$ then the next $j-1$ increments will be of size 1 after that $m_{k+j} := j+1$. The sequence of increment sizes can be arranged in a triangular array $\{a_{i,j}\}_{i,j \geq 1}$, where $m_k := a_{i(k),j(k)}$ with $i(k), j(k)$ as in Example III.

$$1, 2, 1, 3, 1, 1, \ldots$$

In this case $F_t \overset{w}{\to} \delta_1$ but $\mu_t \overset{w}{\to} \frac{1}{2} \delta_1 + \frac{1}{2} \delta_\infty$ and by Proposition (3.2)

$$P\left(\lim_{t \to \infty} S_t = \frac{1}{2}v_1 + \frac{1}{2}v_\infty\right) = 1.$$  

We notice in particular that if $v_1 > v_\infty$, the above mass sequence is another example of a regular sequence for which the weak limit of $F_t$ does not determine the limit of $S_t$, even when it exists.

The next example shows a regular sequence with unbounded increments and for which the Frequency does not converge.
X (regular + \( \not\exists \) w-lim \( F_t \)). Take \( m \) as in Example III but replace the \( k \)-th increment of mass 1 by the \( k \)-th increment of the sequence defined in Example IX. For this example,

\[
\limsup_t F_t(1) - \liminf_t F_t(1) \geq 1/2,
\]

what precludes convergence of \( F_t \) in the weak sense. Furthermore, since the total mass on increments smaller than 1 is finite, \( \mu_t \xrightarrow{\text{w}} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_\infty \) and the sequence is regular.

XI (irregular + w-lim \( F_t \)). Only weak convergence of the empirical measure \( F_t \) does not imply convergence of \( S_t \). Indeed, assume that \( v_1 > v_\infty \), let \( m \) be such that it alternates \( N_i \) increments of size 1 with one increment of size \( K_i \). More precisely, for \( n \in (j + N_1 + \ldots + N_j, j + N_1 + \ldots + N_j + N_{j+1}] \) set \( m_n = 1 \) and for \( n = j + N_1 + \ldots + N_j \) set \( m_n = K_j \). Now, choose \((N_i, K_i)\) such that

\[
\frac{N_1 + \ldots + N_i}{K_i} \leq \frac{1}{i} \quad \text{and} \quad \frac{K_1 + \ldots + K_i}{N_{i+1}} \leq \frac{1}{i}.
\]

Note that \( F_t \xrightarrow{\text{w}} \delta_1 \), but

\[
\limsup_t \mathbb{E}[S_t] = v_1 > v_\infty = \liminf_t \mathbb{E}[S_t].
\]

XII (irregular + \( L^1 \)-lim \( F_t \)). In this example we construct an unbounded irregular sequence for which \( F_t \) converges in \( L^1 \). In particular from item (m) of Proposition 3.3 it follows that this limit must be \( \delta_0 \). Take the sequence defined in Example IV and replace the \( B_k \) increments of size \( L \) by a single increment of size \( LB_k \). As \( B_k \to \infty \), the resulting mass sequence is unbounded.

XIII (irregular + \( \not\exists \) w-lim \( F_t \).). It is also possible to construct a sequence \( m \) that is irregular but such that \( F_t \) does not converge weakly. Take the sequence defined in Example III and replace the \( k \)-th increment of size 1 by the \( k \)-th increment of the sequence defined in XII.

3.4 Random masses

In this section we conclude this game of mass by considering random mass sequences \( m \). More specifically, we let \( m_k \) be an i.i.d. sequence of random variables, independent of \( X \), each distributed according to a measure \( \nu \) on \( \mathbb{R}_+ \). There are two cases depending on whether \( \nu \) has finite or infinite mean. For notational ease, we model \((m_k, k \in \mathbb{N})\) as i.i.d. random variables in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

XIV (Regular + (un) bounded + \( L^1 \)-lim \( F_t \)). Assume that \( \nu \neq 0 \), \( \int \nu(m \, dm) < \infty \), and define the increments \( m_k \) to be sampled independently from \( \nu \). By the Glivenko-Cantelli Theorem [5, Theorem 2.4.7] it follows that almost surely \( F_t(x) \) converges in \( L^1 \) to \( \nu \). By item (m) of Proposition 3.3 it follows that

\[
\mathbb{P}(\mu_t \xrightarrow{\text{w}} \nu) = 1.
\]

Therefore, almost surely, the sequence \( m_k \) is regular and

\[
\mathbb{P}\left( \lim_t \mathbb{E}[S_t] = \int v_x d\nu(x) \right) = 1.
\]
**XV (Regular + Cesaro + \( \exists \) w-lim \( F_t \)).** Now, assume that \( \int m \, d\nu(m) = \infty \) and define the increments \( m_k \) to be sampled independently from \( \nu \). In this case

\[
P\left( \frac{m_1 + \ldots + m_k}{k} \to \infty \right) = 1. \tag{3.13}
\]

Then note that after \( k \) increments, the mass of increments of size smaller than \( a > 0, \mu_t([0,a]) \), is bounded by \( \frac{ka}{m_1 + \ldots + m_k} \) and therefore, by (3.13), for any \( a > 0 \), almost surely \( \mu_t([0,a]) \to 0 \). This implies that \( \mu_t \to \delta_\infty \) and therefore

\[
P\left( \lim_t S_t = v_\infty \right) = 1.
\]

### 4 Hypothesis and counterexamples

#### 4.1 Weak LLN: necessity of (W1) and (W2).

Booth conditions (W1) (W2) are necessary for the weak LLN. The necessity for condition (W1) is due to [6, Theorem 1]. We show below that condition (W2) is necessary by means of a counter-example.

**Counter-example:** Consider a sequence \( \{U_k, k \in \mathbb{N}\} \) of i.i.d. uniform random variables on \((0,1)\) and \( X_k(m) := V_m(U_k) \), where

\[
V_m(u) = \begin{cases} 
A_m & \text{if } u \in (0,g(m)), \\
-A_m & \text{if } u \in (g(m),2g(m)), \\
0 & \text{else.}
\end{cases}
\]

with this definition, it follows that

\[
P(|X_m(k)| > 0) = 2g(m).
\]

Assume that \( g \) is a strictly decreasing continuous function such that \( \lim_{m \to \infty} g(m) = 0 \). Let \( m_k := \inf\{ m : g(m) > 1/k \} \). This implies that \( m_k \to \infty \) as \( k \to \infty \) and so (1.5) is satisfied. Furthermore by the definition of \( X_k(m_k) \), the assumptions (C) and (W1) in Theorem 2.1 are verified. Now choose \( \{A_{m_k}, k \in \mathbb{N}\} \) to be such that

\[
\frac{m_n}{M_N(n)} A_{m_n} > 1 + \sum_{k=1}^{n-1} A_{m_k},
\]

where \( N(n) \) is such that

\[
P(\exists n \leq j \leq N(n) : X_j(m_j) \neq 0) > \frac{1}{2}.
\]

Such an \( N(n) \) exists and is finite since by the second Borel-Cantelli Lemma and the continuity of probability measures:

\[
1 = \mathbb{P}(\exists j \geq n : X_j(m_j) \neq 0) = \lim_{N \to \infty} \mathbb{P}(\exists n \leq j < N : X_j(m_j) \neq 0).
\]
With this choice of $A_{m_n}$ it follows that if there is a $j$, $i \leq j \leq N(i)$ for which $|X_j(m_j)| > 0$ then $|S_N(i)| > 1$. Therefore for any $i \in \mathbb{N}$,

$$\mathbb{P}( |S_N(i)| > 1 ) > \frac{1}{2}.$$ 

As $\mathbb{P}(S_n > 0 \mid |S_n| > 0) = \frac{1}{2}$ we conclude that the weak LLN does not hold.

4.2 Strong LLN: near optimality of (S1).

One could try to improve the condition in (S1) by requiring a decay smaller than polynomial, that is:

$$\mathbb{P}( |X_k(m)| > \varepsilon ) < \frac{C_\varepsilon}{f(m)},$$

for some sub-linear $f(m)$. When we look for a scale that grows slower than any polynomial, $f(m) = \log(m)$ is a natural candidate. However, as illustrated next, this already allows for counterexamples.

**Counter-example:** Let $\{ U_k, k \in \mathbb{N} \}$ be a sequence of i.i.d. uniform random variables on $(0,1)$ and let $X_k(m) := g_m(U_k)$ where

$$g_m(x) := \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2\log_2 m}), \\ -1 & \text{if } x \in [\frac{1}{2\log_2 m}, \frac{1}{\log_2 m}), \\ 0 & \text{else}. \end{cases}$$

Note that $X$ fulfills assumptions (C)–(S2)–(S3) and instead of (S1) it satisfies

$$\mathbb{P}( |X_k(m)| > \varepsilon ) = \frac{1}{\log_2 m}.$$ 

Now take $m$ with $m_k = 4^k$. For such an $m$ we see that the incremental sum $S_n$ does not satisfy the strong LLN. Indeed, as

$$\mathbb{P}( |X_k(m_k)| = 1 ) = \frac{1}{2k},$$

by the second Borel Cantelli lemma,

$$\mathbb{P}( |X_k(m_k)| = 1, \text{ i.o} ) = 1.$$ 

Therefore, by (1.7) it follows that there is an $\varepsilon > 0$ for which

$$\mathbb{P}( |S_n - S_{n-1}| > \varepsilon \text{ i.o.} ) = 1,$$

which means that almost surely $S_n$ does not converge.

In light of the above example, we see that the condition (S1) is near to optimal. Indeed, to improve it, we would need to find $f(m)$ satisfying

$$\log^k(m) << f(m) << m^\delta \quad \forall k \in \mathbb{N}, \delta > 0.$$
4.3 Possible variants and other remarks

- **Independence.** Our proofs rely on the independence in $k$ of $\{X_k(m), m \in \mathbb{R}_+, k \in \mathbb{N}\}$. However, for certain choices of well-behaving mass sequences $m$, it seems possible to adapt our arguments and still obtain a strong LLN in presence of “weak enough dependence”, though the notion of “weak enough dependence” would very much depend on the weight sequence and this is why we did not pursue this line of investigation.

- **Relaxing condition (3.1).** In the game of mass exposed in Section 3, for simplicity we restrict our analysis to variables with mean independent of $k$, as captured in assumption (3.1). We note that such a restriction is not really needed, as the reader can easily check for example by considering $X_k(m)$’s with mean, say, $v_m$ and $v'_m \neq v_m$ depending on the parity of $k$. Yet, the resulting analysis would branch into many different regimes depending on how exactly condition (3.1) is violated.

- **Fluctuations and large deviations.** It would be natural to inquire “higher order asymptotics”, such as large deviations or scaling limit characterizations, for the sums in (1.16) or (2.2). However, this type of questions heavily rely on the specific distribution of the sequence of variables $X$ thus preventing a general self-contained treatment. Still, it is interesting to note that these other questions can give rise to many subtleties and anomalous behavior. This is well illustrated by the specific model in random media introduced in [3] that motivated the present paper, we refer the interested reader to [2] for results on crossovers phenomena in related fluctuations, and to [2] for stability results of large deviations rate functions.

5 Proofs

5.1 Weak law of large numbers

In this section we prove Theorem 2.1 by implementing a truncation argument along the line of [6]. For each $K > 0$, let $S^K_n$ represent the contribution to $S_n$ coming from the increments larger than $K$, i.e.

$$S^K_n := \sum_{k=1}^{n} \frac{m_k}{M_n} X_k(m_k) 1_{m_k > K}.$$ 

Now note that due to (W1) and (W2) it follows that

$$\lim_{K \to \infty} \sup_{m>K} \sup_k \mathbb{E}[|X_k(m)|] = 0.$$  \hspace{1cm} (5.1)

Indeed, for any $\varepsilon > 0$ and any $A > \varepsilon$

$$\mathbb{E}[|X_k(m)|] \leq \varepsilon + A \mathbb{P}(\varepsilon < |X_k(m)| \leq A) + \mathbb{E}[|X_k(m)| 1_{|X_k(m)| > A}].$$

the right hand side above can be bounded by $3\varepsilon$ using (W1) and (W2), and since $\varepsilon > 0$ is arbitrary, (5.1) follows. Now let $\bar{S}^K_n := S_n - S^K_n$ be the contribution to $S_n$ coming from the increments smaller than $K$. By the triangle inequality and the union bound it follows that

$$\mathbb{P}\left(|S_n| > \varepsilon\right) \leq \mathbb{P}\left(|S^K_n| + |\bar{S}^K_n| > \varepsilon\right) \leq \mathbb{P}\left(|S^K_n| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|\bar{S}^K_n| > \frac{\varepsilon}{2}\right).$$
As $\sum_{k=1}^{n} \frac{m_k}{M_n} I_{m_k > K} \leq 1$, and Markov’s inequality imply $\limsup_{n \to \infty} P \left( S_n^K > \varepsilon \right) = 0$, and therefore

$$\limsup_{n \to \infty} P \left( |S_n| > \varepsilon \right) \leq \limsup_{n \to \infty} P \left( |S_n^K| > \frac{\varepsilon}{2} \right)$$

It remains to prove that the right-hand side above goes to zero for arbitrary $\varepsilon > 0$. Now consider the truncated random variables

$$Y_k(m_k) := X_k(m_k) I_{|X_k(m_k)| \leq \frac{M_n}{m_k}} I_{m_k < K}$$

and notice that as $M_n \to \infty$,

$$\lim \max_{1 \leq k \leq n} \frac{m_k}{M_n} I_{m_k < K} = 0. \quad (5.2)$$

Set $\bar{s}_n^K := \sum_{k=1}^{n} Y_n(m_k)$. We will first argue that this truncated sum $\bar{s}_n^K$ approximate well $S_n^K$, and then show that the variance of truncation vanishes. To perform these two steps we will need the following lemma, whose proof is postponed at the end of this section.

**Lemma 5.1.** Consider the setup of Theorem 2.1 then

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} \frac{M_n}{m_k} P \left( |X_k(m)| I_{m_k < K} \geq \frac{M_n}{m_k} \right) = 0, \quad (5.3)$$

and

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} \frac{m_k}{M_n} E \left[ Y_k^2(m_k) \right] = 0. \quad (5.4)$$

By the union bound, the definition of $Y_k(m_k)$, using that $\sum_{k=1}^{n} \frac{m_k}{M_n} \leq 1$ we have that

$$\limsup_{n \to \infty} P \left( \bar{s}_n^K \neq \bar{s}_n^K \right) \leq \limsup_{n \to \infty} \sum_{k=1}^{n} P \left( X_k(m_k) I_{m_k < K} = Y_k(m_k) \right)$$

$$= \limsup_{n \to \infty} \sum_{k=1}^{n} P \left( |X_k(m)| I_{m_k < K} \geq \frac{M_n}{m_k} \right)$$

$$\leq \limsup_{n \to \infty} \max_{1 \leq k \leq n} \frac{M_n}{m_k} P \left( |X_k(m)| I_{m_k < K} \geq \frac{M_n}{m_k} \right) \sum_{k=1}^{n} \frac{m_k}{M_n}$$

$$\leq \limsup_{n \to \infty} \max_{1 \leq k \leq n} \frac{M_n}{m_k} P \left( |X_k(m)| I_{m_k < K} \geq \frac{M_n}{m_k} \right),$$

the latter can be made arbitrary small via (5.3). Hence it suffices to consider $\bar{s}_n^K$ instead of $S_n^K$. We next control the mean and the variance of $\bar{s}_n^K$.

**The mean.** As $X_k(m_k)$ is uniformly integrable family of centered random variables, by (5.2) it follows that $\limsup_{n} \sup_k E \left[ |Y_k(m_k)| \right] = 0$, from which it follows that $\lim_{n} E \left( \bar{s}_n^K \right) = 0$.

**The Variance.** Similarly, by independence and (5.2) we can estimate

$$\limsup_{n} \text{Var} \left( \bar{s}_n^K \right) = \limsup_{n} \sum_{k=1}^{n} \frac{m_k^2}{M_n^2} \text{Var}(Y_k(m_k))$$

$$\leq \limsup_{n} \sum_{k=1}^{n} \frac{m_k}{M_n} \max_{1 \leq k \leq n} \frac{m_k}{M_n} \text{Var}(Y_k(m_k)) \quad (5.5)$$

$$\leq \limsup_{n} \max_{1 \leq k \leq n} \frac{m_k}{M_n} E \left[ Y_k^2(m_k) \right] = 0.$$
Finally, \( \lim_n E \left( \bar{s}_n^K \right) = 0 \) together with (5.5) and Chebyshev’s inequality yield

\[
\limsup_n P \left( \bar{s}_n^K \geq \varepsilon \right) \leq \limsup_n \frac{4}{\varepsilon^2 \Var \left( \bar{s}_n^K \right)} = 0.
\]

Proof of Lemma 5.1 Since \( \lim_n \inf_{1 \leq k \leq n} \frac{M_n}{m_k} = \infty \), equation (5.3) follows from (W2) as

\[
\lim_n \frac{M_n}{m_k} P \left( |X_k(m)| \geq \frac{M_n}{m_k} \right) \leq \lim_n \mathbb{E} \left[ |X_k(m)| \mathbb{1}_{|X_k(m)| \geq \inf_{1 \leq k \leq n} \frac{M_n}{m_k}} \right] = 0.
\]

To prove (5.4), let \( F_{k,m}(a) := P(|X_k(m)| < a) \) and note first that integration by parts yields

\[
\int_0^T x^2 dF_{k,m}(x) = T^2 P(|X_k(m)| < T) - 2 \int_0^T x P(|X_k(m)| < x) \, dx
\]

\[
= T^2 [1 - P(|X_k(m)| \geq T)] - 2 \int_0^T x [1 - P(|X_k(m)| \geq x)] \, dx
\]

\[
= -T^2 P(|X_k(m)| \geq T) + 2 \int_0^T x P(|X_k(m)| \geq x) \, dx.
\]

(5.6)

Observe further that

\[
\frac{M_n}{m_k} \int_0^{m_k} x^2 dF_{k,m}(x) = \mathbb{E} \left[ Y_k^2(m_k) \right].
\]

(5.7)

Since \( \lim_n \max_{1 \leq k \leq n} \frac{m_k}{M_n} = 0 \) and \( \lim_{T \to \infty} \sup_{k,m} T P(|X_k(m)| \geq T) = 0 \), by (5.7) and (5.6), it follows that

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T x^2 dF_{k,m}(x) = 0.
\]

5.2 Strong law of large numbers for the incremental sum

As in \( \mathbb{I} \), our proof here relies on an iterative decomposition into “small” and “big” increments and we rely on a multi-scale decomposition. At each scale, the small contribution is defined as the truncated sum that, thanks to the stochastic domination assumption (S2), can be dealt with the techniques of \( \mathbb{I} \). What is left, classified as “big”, splits again into a “small” and a “big” part. At this level, the small one is controlled in the same way as before. The iteration proceeds until reaching a scale where the condition (S1) is sufficient to ensure convergence. The proof is organized as follows. We first iteratively decompose the sum \( S_n \) into a finite number of sums of relatively small increments and one sum of large increments. Next we show that the large increment sum converges to zero almost surely. Finally we prove that each of the small increments also converge to zero almost surely.

The recursive decomposition. Fix \( K \in \mathbb{N} \) such that

\[
\delta K > 1, \quad \text{and} \quad \frac{K}{K - 1} < 1 + \gamma
\]
with $\delta$ from (S1) and $\gamma$ from (S2). Define $k_j^0 := j$ let \( \{ k_j^{0,s} : j \in \mathbb{N} \} := \{ k_j^0 : m_{k_j^0} \leq 1 \} \) and set

\[
\{ k_j^1 : j \in \mathbb{N} \} := \{ k_j^0 : j \in \mathbb{N} \} \setminus \{ k_j^{0,s} : j \in \mathbb{N} \}.
\]

For $i \geq 1$, given \( \{ k_i^1, k_i^2, \ldots \} \), define the (i)-st small increments by

\[
\{ k_i^{j,s} : j \in \mathbb{N} \} = \{ k_i^j \in \mathbb{N} : m_{k_i^j} < j^{i/K} \},
\]

and define the $k_{ij}^{i+1}$ (large) increments by

\[
\{ k_{ij}^{i+1}, k_{ij}^{i+1}, \ldots \} = \{ k_i^j, k_i^2, \ldots \} \setminus \{ k_i^{i,s}, k_i^{i,s}, \ldots \}.
\]

Now, denote by $J(i, s; n)$ the cardinality of \( \{ k_i^{j,s} : m_{k_i^j} \leq n \} \) and by $J(i; n)$ the cardinality of \( \{ k_i^j : M_{k_i^j} \leq n \} \). To ease the notation set $X_k := X_k(m_k) a_{j,n}^i := \frac{m_{k_i^j}}{M_n}$, and $a_{j,n}^{i,s} = \frac{m_{k_i^{i,s}}}{M_n}$.

Since $N = \bigcup_{i=0}^{K^2} \{ k_i^{j,s} : j \in \mathbb{N} \} \cup \{ k_i^{K^2}, j \in \mathbb{N} \}$, we have that

\[
S_n = \sum_{i=1}^{K^2} \sum_{j=1}^{J(i; n)} a_{j,n}^i X_{k_i^j} + \sum_{j=1}^{J(K^2; n)} a_{j,n}^{K^2} X_{k_i^{K^2}}
\]

\[
= \sum_{i=1}^{K^2} S_{i,n}^s + S_n^{K^2}.
\]

In what follows we show that

\[
\mathbb{P} \left( \limsup_{n} S_{n}^{K^2} = 0 \right) = 1, \quad (5.8)
\]

\[
\mathbb{P} \left( \limsup_{n} S_{n}^{s} = 0 \right) = 1 \text{ for } i \in \{0, 1, \ldots, K^2\}. \quad (5.9)
\]

**The large increments sum.** To prove (5.8) it is enough to show that for any $\varepsilon > 0$

\[
\mathbb{P} \left( \limsup_{n} S_{n}^{K^2} \leq \varepsilon \right) = 1. \quad (5.10)
\]

By (S1), and the fact that $m_{k_{j}^{K^2}} \geq j^K$,

\[
\mathbb{P}(X_{k_{j}^{K^2}} > \varepsilon) \leq \frac{C \varepsilon}{(m_{k_{j}^{K^2}}) \delta} \leq \frac{C \varepsilon}{j^K \delta}.
\]

Since $K\delta > 1$, it follows that

\[
\sum_{j=1}^{\infty} \mathbb{P}(X_{k_{j}^{K^2}} > \varepsilon) < \infty,
\]

and by the Borel Cantelli Lemma, we have that

\[
\mathbb{P} \left( \limsup_{j} X_{k_{j}^{K^2}} \leq \varepsilon \right) = 1.
\]

As $M_n \to \infty$ and $\sum_{j=1}^{J(K^2, n)} m_{k_{j}^{K^2}} \leq M_n$, we conclude that (5.10) holds.
The small increment sums. The proof of (5.9) will be split in two parts, first we prove it for \( i \geq 1 \) and then we treat the case \( i = 0 \). For notation ease, set for any \( J \in \mathbb{N} \),

\[
\tilde{m}_j := m_{k,j}, \quad \tilde{M}_J := \sum_{j=1}^{J} \tilde{m}_j, \quad \tilde{a}_{j,J} := \frac{\tilde{m}_j}{\tilde{M}_J}, \quad \text{and let} \quad \tilde{S}_J = \sum_{j=1}^{J} \tilde{a}_{j,J} X_{k,j}.
\]

Now note that for any \( n \)

\[
S_{n}^{i,s} = \frac{\tilde{M}_{J(i,s;n)}}{M_{n}} \tilde{S}_{J(i,s;n)}.
\]

As \( \frac{\tilde{M}_{J(i,s;n)}}{M_{n}} \leq 1 \), it follows that \( \limsup_n |S_{n}^{i,s}| \leq \limsup_J |\tilde{S}_J| \). Therefore, it suffices to show that

\[
P \left( \lim \tilde{S}_J = 0 \right) = 1. \tag{5.11}
\]

Since for \( i \geq 1 \), \( m_{k,j} \in [j^{(i-1)/K}, j^{i/K}] \), we have the following bounds

\[
\tilde{M}_J \geq cJ^{1+(i-1)/K}, \quad \tilde{a}_J \leq \frac{1}{cJ^{K-1}}. \tag{5.12}
\]

Note that, by (5.12) there is \( C > 0 \) for which

\[
\max_{j \leq J} \tilde{a}_{j,J} \leq \frac{C}{J^{K-1}}. \tag{5.13}
\]

Now, as \( \lim_J \tilde{a}_{j,J} = 0, \sum_J \tilde{a}_{j,J} = 1 \), condition (5.13) and (S2) hold, one can apply Theorem 2 in [9] with \( \nu = \frac{K-1}{K} \) to obtain (5.11) and therefore (5.9) for \( i \geq 1 \). To conclude the proof of Theorem 2.2 it remains to verify that \( S_{n}^{0,s} \) converges to 0 almost surely. The proof is an adaptation of Theorem 4 in [6] and is postponed to Appendix A.

\[
\Box
\]

5.3 Strong law for the gradual sum

In this section we prove Theorem 2.3. Recall the decomposition of \( S_t \) from (2.2). As \( S_t \) is a convex combination of \( S_{\ell_t} \) and the boundary term \( X_{\ell_t}(\bar{t}) \) with with \( \bar{t} = t - M_{\ell_t} \), the proof of 2.3 follows from the proof of (2.2) and the fact that that the contribution of the boundary term \( \frac{\bar{t}}{t} X_{\ell_t}(\bar{t}) \) converges to 0 almost surely.

To prove Theorem 2.3 it remains suffices to show

\[
P \left( \lim \frac{\bar{t}}{t} X_{\ell_t}(\bar{t}) = 0 \right) = 1. \tag{5.14}
\]

We divide the proof of (5.14) in two steps. First we show (5.14) for a properly defined small increments, \( m_{k+1} < (1 + \alpha_k)M_k \). Then we show (5.14) for the complement set that we refer to as the set of large increments.
5.3.1 The small Increments

Let $V_n = \sup \{ \frac{s}{(M_n + s)} | X_k(s) : s \in [0, m_k) \}$ and note that

$$\limsup_{\nu} \frac{1}{\nu} X_\nu(\bar{t}) = \limsup_{n} V_n. \quad (5.15)$$

Thanks to condition $\text{(S3)}$, we can control the oscillations $V_n$ for small increments that satisfy a growth condition defined as follows. Fix $\beta > 1$ as in $\text{(S3)}$ and let $\alpha_j = \frac{1}{j^a}$ with $a \in (1/\beta, 1)$. The first small increment is defined by

$$k'_1 := \inf \{ k \in \mathbb{N} : m_{k+1} < (1 + \alpha_1)M_k \},$$

and define recursively for $j$-th small increment by

$$k'_{j+1} := \{ k \in \mathbb{N} : k > k'_j, m_{k+1} < (1 + \alpha_{j+1})M_k \}.$$ 

As for any $k'_j, m_{k'_j+1} < \alpha_j M_{k'_j}$, by $\text{(S3)}$ it follows that for any $\varepsilon > 0$

$$\mathbb{P}(V_{k'_j} > \varepsilon) \leq \mathbb{P} \left[ \sup_{s : \leq \alpha_j M_{k'_j}} |s| X_{k'_j}(s)| > \varepsilon M_{k'_j} \right] \leq \alpha_j^\beta C \varepsilon.$$ 

As $\sum_j \alpha_j^\beta < \infty$, by the Borel-Cantelli lemma we conclude that

$$\mathbb{P} \left( \limsup_{j} V_{k'_j} \leq \varepsilon \right) = 1, \quad (5.16)$$

and since $\varepsilon > 0$ it follows that

$$\mathbb{P} \left( \lim_{j} V_{k'_j} = 0 \right) = 1. \quad (5.17)$$

5.3.2 The large increments

By $(5.17)$ we can restrict our attention to $\{ k'_1, k'_2, \ldots \} = \mathbb{N} \setminus \{ k'_1, k'_2, \ldots \}$. Note that since $\alpha_j \leq 1$

$$(1 + \alpha_j) \geq C \exp(\alpha_j/2), \quad (5.18)$$

for some $C > 0$. Therefore, the terms in the sequence $\{ M_{k'_j} \}$ satisfy the following growth condition:

$$M_{k'_j} \geq C \prod_{j=1}^{i} (1 + \alpha_j)M_1 \geq \exp \left( \sum_{j=1}^{i} \frac{\alpha_j}{2} \right) M_1 \geq \exp(c a^{1-a})M_1. \quad (5.19)$$

The proof now proceeds in two steps, we first show that the boundary term $\frac{1}{\nu} X_\nu(\bar{t})$ converges to zero along a subsequence $\{ t_{i,j} : i, j \in \mathbb{N} \cup \{ 0 \} \}$, what we call pinning, and then based on this result we show that the full sequence converges to zero as we bound its oscillations on the intervals $[t_{i,j}, t_{i,j+1}]$. 

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**Pinning.** For the boundary increments \( k \in \{ k_1^*, k_2^*, \ldots \} \) consider the following pinning procedure. Let \( k_0^* = i(k_0^*) = 0 \) and define recursively for \( n \in \mathbb{N} \)

\[
i(k_n^*) := \inf\{i > i(k_{n-1}^*) : M_{k_n^*+1} \prod_{j=i(k_{n-1}^*)}^i (1 + \alpha_j) M_{k_n^*} > M_{k_n^*+1}\}.
\]

We note that (5.18) together with \( \sum_j \alpha_j = \infty \) imply that \( i(k_n^*) < \infty \) for all \( n \). Now to define the pinning sequence let \( t_{i,0} := k_i^* \) and for \( j \in \{ i, \ldots, i(k_i^*) - i(k_{i-1}^*) \} \) set

\[
t_{i,j} := \begin{cases} (1 + \alpha(k_{i-1}^*)+j) t_{i,j-1} & \text{if } j < i(k_i^*) - i(k_{i-1}^*) \\ M_{k_{i+1}^*} & \text{if } j = i(k_i^*) - i(k_{i-1}^*). \end{cases}
\]

Now it follows from the definition of \( \bar{t} \) that

\[
\bar{t}_{i,j} = t_{i,j} - M_{k_i^*} = M_{k_i^*} \prod_{n=1}^j (1 + \alpha_{j+i(k_{i-1}^*)})^n - 1. \tag{5.20}
\]

By the polynomial decay in (S1) it follows that for any \( \varepsilon > 0 \), and \( i, j > 0 \)

\[
\mathbb{P}\left[ \left| \frac{\bar{t}_{i,j} - t_{i,j}}{t_{i,j}} X_{k_i^*}(\bar{t}_{i,j}) \right| \geq \varepsilon \right] \leq \mathbb{P}\left[ \left| X_{k_i^*}(\bar{t}_{i,j}) \right| \geq \varepsilon \right] \leq \frac{C_\varepsilon}{(\bar{t}_{i,j})^\delta}.
\]

By (5.20) and (5.19), the sum over \( i, j > 0 \) of the above probability is finite and therefore for any \( \varepsilon > 0 \)

\[
\mathbb{P}\left[ \left| \frac{\bar{t}_{i,j} - t_{i,j}}{t_{i,j}} \right| X_{k_i^*}(\bar{t}_{i,j}) \right] \geq \varepsilon \text{ for infinitely many } i, j \right] = 0,
\]

which implies that

\[
\mathbb{P}\left[ \limsup_{i,j} \left| \frac{\bar{t}_{i,j} - t_{i,j}}{t_{i,j}} \right| X_{k_i^*}(\bar{t}_{i,j}) = 0 \right] = 1.
\]

It remains to understand the behaviour of the boundary term in \( [t_{i,j}, t_{i,j+1}] \).

**Oscillations.** Now we use (S3) to compute the oscillations between the pinned values of the boundary. Fix \( \varepsilon > 0 \) and consider the event \( \Omega_{i_0} \) defined by

\[
\Omega_{i_0} := \left\{ \sup_{j} \left| \frac{\bar{t}_{i,j} - t_{i,j}}{t_{i,j}} X_{k_i^*}(\bar{t}_{i,j}) \right| \leq \varepsilon \text{ for } i > i_0 \right\}.
\]

Therefore, on \( \Omega_{i_0} \) for \( t \in [t_{i,j}, t_{i,j+1}] \) and \( j \geq 1 \)

\[
\frac{\bar{t}}{t} X_k(t) - \frac{\bar{t}_{k,j}}{t_{k,j}} X_k(\bar{t}_{k,j}) = \frac{1}{t} [\bar{t} X_k(t) - \bar{t}_{k,j} X_k(\bar{t}_{k,j})] + \left( \frac{\bar{t}}{t} - \frac{\bar{t}_{k,j}}{t_{k,j}} \right) X_k(\bar{t}_{k,j}) \leq \frac{1}{t} [\bar{t} X_k(t) - \bar{t}_{k,j} X_k(\bar{t}_{k,j})] + (C_\alpha - 1) \varepsilon
\]

where

\[
C_\alpha = \sup_{i,j} \sup_{t \in [t_{i,j}, t_{i,j+1}]} \frac{\bar{t}}{t_{k_j^*}} < \infty
\]

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Let $s := t - t_{i,j}$. By (S3) it follows that on $\Omega_{i0}$

$$
\mathbb{P} \left[ \sup_{s \leq t_{i,j} + 1 - t_{i,j}} \left| \frac{\tilde{f}_{i,j} + s}{t_{i,j} + s} X_{k_i^*}(\tilde{f}_{i,j} + s) - \frac{\tilde{f}_{i,j}}{t_{i,j}} X_{k_i^*}(\tilde{f}_{i,j}) \right| > C_\alpha \varepsilon \right] \leq C_{\alpha} \varepsilon .
$$

As the sum over $i, j \in \mathbb{N}$ is finite it follows by that

$$
\mathbb{P} \left[ \limsup_{k} V_k \leq C_\alpha \varepsilon \right] \geq \limsup_{k_0} \mathbb{P}(\Omega_{k_0}) = 1 \quad (5.21)
$$

Since $\varepsilon > 0$ is arbitrary, from (5.15), (5.16) and (5.21) we conclude that (5.14) holds.

## A Bounded increments

To deal with the case $i = 0$ if $\lim \sum_{i=1}^{n} m_{i,0}^k < \infty$ it follows that $S_n^{i,0}$ converges to 0. For this reason assume without loss of generality that $m_k = m_k^{i,0}$ and that

$$
\lim_n M_n = \lim_{n} \sum_{i=1}^{n} m_{i,0}^n \to \infty.
$$

We next consider the truncated versions of $X_k$

$$
Y_k := X_k 1_{\{|X_k| \leq \frac{M_k}{m_k}\}}.
$$

The proof proceeds in two steps: first we show that

$$
\mathbb{P}(Y_k \neq X_k \text{ i.o.}) = 0. \quad (A.1)
$$

This implies that the limit of $S_n$ equals the limit of $\tilde{S}_n := \sum_{k=1}^{n} a_{n,k} Y_k$. The proof will be complete once we prove that

$$
\mathbb{P} \left( \lim_n \tilde{S}_n = 0 \right) = 1. \quad (A.2)
$$

**Proof of (A.1).** Let $N(x) := \{k : \frac{M_k}{m_k} \leq x\}$, $F^*(a) := \mathbb{P}(|X^*| < a)$, and note that by the stochastic domination (S2)

$$
\sum_k \mathbb{P}(Y_k \neq X_k) \leq \sum_k \mathbb{P}(|X_k| \geq \frac{M_k}{m_k}) \leq \sum_j \mathbb{P}(|X^*| \geq \frac{M_k}{m_k}) \leq \sum_j \int_{|x| \geq \frac{M_k}{m_k}} dF^*(x) = \int N(x) dF^*(x) = \mathbb{E}[N(|X^*|)].
$$

To obtain (A.1) it remains to prove that

$$
\mathbb{E}[N(|X^*|)] < \infty.
$$

This step follows from Lemma 2 of [6] which states that

$$
\limsup \frac{N(x)}{x \log x} \leq 2. \quad (A.3)
$$

By (A.3) it follows that $N(x) \leq Cx^{1+\gamma}$ and therefore (S2) implies that $\mathbb{E}[N(|X^*|)] < \infty$. □
proof of (A.2) As
\[ \lim_n \mathbb{E} [\bar{S}_n] = 0, \]
to prove (A.2) it suffices to show that
\[ \sum_k \frac{m_k^2}{M_k} \text{Var}(Y_k) < \infty. \]  
(A.4)

As \( \frac{M_k}{m_k} \to \infty \) it follows that there is a \( C \) such that
\[ \mathbb{E} [Y_k^2] \leq C \int_{|x| \leq \frac{M_k}{m_k}} x^2 dF^*(x). \]

The sum in (A.4) can be bounded by
\[ C \sum_k \frac{m_k^2}{M_k^2} \int_{|x| < \frac{M_k}{m_k}} x^2 dF(x) = C \int x^2 \sum_{k: \frac{M_k}{m_k} \geq |x|} \frac{m_k^2}{M_k^2} dF^*(x). \]

To complete the proof it remains to show that the right-hand side above is finite. This follows from the following claims whose proofs are given right after:
\[ \sum_{k: \frac{M_k}{m_k} \geq |x|} \frac{m_k^2}{M_k^2} \leq 2 \int_{y \geq |x|} \frac{N(y)}{y^3} \, dy, \]  
(A.5)
and
\[ \int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} \, dy \, dF^*(x) < \infty. \]  
(A.6)

proof of (A.5) Observe that by the definition of \( N \) and integration by parts
\[ \sum_{k: |x| < \frac{M_k}{m_k} \leq z} \frac{m_k^2}{M_k^2} = \int_{|x| < y < z} \frac{dN(y)}{y^2} = \frac{N(z)}{z^2} - \frac{N(|x|)}{x^2} + 2 \int_{|x| < y < z} \frac{N(y)}{y^3} \, dy, \]
furthermore, since \( N(z) \leq N(y) \) for \( z \leq y \) and \( \frac{1}{z^2} = 2 \int_z^\infty \frac{1}{y^2} \, dy \)
\[ \frac{N(z)}{z^2} \leq 2 \int_z^\infty \frac{N(y)}{y^3} \, dy \to 0 \]
and so
\[ \sum_{k: |x| < \frac{M_k}{m_k} \leq z} \frac{m_k^2}{M_k^2} = \lim_z \sum_{k: |x| < \frac{M_k}{m_k} < z} \frac{m_k^2}{M_k^2} \int_{|x| < y < z} \frac{dN(y)}{y^2} \leq 2 \int_{|x| < y} \frac{N(y)}{y^3} \, dy, \]
\[ \square \]
proof of (A.6) Again by (A.3) it follows that

\[ \int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} \, dy \, dF^*(x) \leq \int x^2 \int_{y \geq |x|} \frac{Cy^{1+\gamma}}{y^3} \, dy \, dF^*(x) \]
\[ = \int x^2 \int_{y \geq |x|} \frac{C}{y^2 - \gamma} \, dy \, dF^*(x) = \int x^{1+\gamma} \frac{C}{(1-\gamma)x^{1-\gamma}} \, dF^*(x) \]
\[ = \frac{C}{1-\gamma} \mathbb{E} \left[ |X^*|^{1+\gamma} \right] < \infty. \]

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