Manipulating the quantum information of the radial modes of trapped ions: linear phononics, entanglement generation, quantum state transmission and non-locality tests

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Abstract. We present a detailed study on the possibility of manipulating quantum information encoded in the ‘radial’ modes of arrays of trapped ions (i.e. in the ions’ oscillations orthogonal to the trap’s main axis). In such systems, because of the tightness of transverse confinement, the radial modes pertaining to different ions can be addressed individually. In the first part of the paper we show that, if local control of the radial trapping frequencies is available, any linear optical and squeezing operation on the locally defined modes—on single as well as on many modes—can be reproduced by manipulating the frequencies. Then, we proceed to describe schemes apt to generate unprecedented degrees of bipartite and multipartite continuous variable (CV) entanglement under realistic noisy working conditions and even restricting only to a global control of the trapping frequencies. Furthermore, we consider the transmission of the quantum information encoded in the radial modes along the array of ions, and show it to be possible to a remarkable degree of accuracy, for both finite-dimensional and CV quantum states. Finally, as an application, we show that the states which can be generated in this setting allow for the violation of multipartite non-locality

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The last decade saw a boom in the development of experimental capabilities available for quantum information processing. The ability to manipulate the information of discrete variables encoded in polarization, spin and internal atomic degrees of freedom has by now reached very high standards. On the other hand, the control and manipulation of CV quantum information is still almost exclusive to light fields in quantum optical settings. Even though purely optical systems rely on well-established tools and are the natural choice for communication tasks over long distances, they also suffer from significant drawbacks. Notably, the entanglement generation in such systems is strongly limited by the efficiency of parametric processes in nonlinear crystals; moreover, ‘static’ optical degrees of freedom—i.e. light resonating in cavities—are seriously affected by losses and decoherence over their typical dynamical time scales.

Such limitations motivate the question of whether the full potential of infinite-dimensional Hilbert spaces could be better harnessed by ‘massive’ (i.e. related to the position of a massive particle) CV degrees of freedom. Of course, to compete with the so far very successful quantum optical toolbox, such degrees of freedom would have to allow for a range of coherent manipulations at least as exhaustive as what quantum optics currently permits. Besides, to be considered advantageous over quantum optical systems, such degrees of freedom should allow for notable improvements in the generation of quantum entanglement and squeezed states under realistic working conditions.

In the present paper, we argue that trapped ions meet both such requirements, and present an extensive study to substantiate this argument. In particular, we shall focus on the radial
Figure 1. Considered setup: the ions are trapped along the longitudinal direction with the same trapping frequency $\omega_L$. Also, each ion $j$ is trapped along the transverse, ‘radial’ direction (i.e. along the arrows in the drawing) with trapping frequency $\omega_j$. The operator $\hat{X}_j$ stands for the position of ion $j$ along the radial direction: these radial oscillations are described by the modes $\{\hat{X}_j, \hat{P}_j\}$ we shall focus on.

motion of trapped ions, that is on oscillations along a direction orthogonal with respect to the array of ions. These oscillations are described by CV quantum degrees of freedom which we will refer to as radial modes [1, 2]. Radial modes have attracted considerable interest in the last few years, mostly in view of the fact that they allow for a tighter confinement (which also permits one to define the phonons locally).

The paper is organized as follows. After having introduced the description of the physical system (section 2), in section 3 we demonstrate that any linear optical and squeezing operation can be obtained for radial modes of trapped ions by controlling the individual radial trapping frequencies, indicating that trapped ions can at least match the processing capabilities possible for light modes. In section 4, we show that, even restricting to cases where only global control of the trapping potentials is possible, such systems are actually apt to outperform optical modes in the generation of entanglement, both bipartite and multipartite. As applications we consider, in section 5, the propagation of quantum information along the array of ions, at both qubit and CV levels and, in section 6, the violation of multipartite non-locality tests, and show them to be within the reach of current technology. Radial modes will thereby turn out to be promising not only for information processing but also as probes of fundamental physics.

2. The trap

We shall consider the radial modes of $n$ ions of mass $m$ and charge $ze$ in a linear Paul trap [3]. Let $\hat{X}_j$ and $\hat{P}_j$ be the position and momentum operators associated with the radial degree of freedom of the $j$th ion, which is trapped in the radial direction with angular frequency $\omega_j$ (see figure 1). In the following, the longitudinal trapping frequency $\omega_L$ will be the unit of frequency and will set the unit of length as well (equal to $d = \sqrt{z^2e^2/(4\pi \varepsilon_0m \omega_L^2)}$, where $\varepsilon_0$ is the dielectric constant); also, we shall set $\hbar = 1$, so that all the quantities will be dimensionless. The Coulomb interaction affects the local radial oscillation frequencies: for convenience, let us then define the
‘effective’ local radial frequencies
\[ \tilde{\omega}_j = \omega_L \sqrt{\frac{\omega_j^2}{\omega_L^2} - \sum_{l \neq j} \frac{d^3}{|u_j - u_l|^3}}, \]
\{u_j\} being the equilibrium positions of the ions [3]. Rescaling the canonical operators according to \( \hat{x}_j = \sqrt{m\tilde{\omega}_j} \hat{X}_j, \, \hat{p}_j = \hat{P}_j / \sqrt{m \tilde{\omega}_j} \) and grouping them in a vector of operators \( \hat{R} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_j, \ldots, \hat{p}_n)^\top \), allows one to express the global Hamiltonian of the system in the harmonic approximation as
\[ \hat{H} = \frac{i}{2} \hat{R}^\dagger \left( \kappa \oplus \tilde{\omega} \right) \hat{R}, \]
where \( \tilde{\omega} \) is a diagonal matrix: \( \tilde{\omega} = \text{diag} (\tilde{\omega}_1, \ldots, \tilde{\omega}_n) \), whereas the potential matrix \( \kappa \) has diagonal entries \( \kappa_{jj} = \tilde{\omega}_j \) and off-diagonal entries \( \kappa_{jk} = \omega_L^2 d^3 / (\sqrt{\omega_j \tilde{\omega}_k} |u_j - u_k|^3) \) for \( j \neq k \). Notice that, in the harmonic approximation (i.e. at second order in the ions’ displacements) the coupling between radial modes and longitudinal ones vanishes. Throughout the paper, we shall assume a longitudinal trapping frequency \( \omega_L \simeq 1 \text{ MHz} \) and typical radial frequencies ranging from 0.1 to around 10 MHz. Under such conditions, the distance between neighbouring ions is around 5 \( \mu \text{m} \), whereas the radial displacements of the ions are always smaller than 0.1 \( \mu \text{m} \). Hence, fourth and higher order terms in the displacements are at least \((0.1/5)^2 \simeq 4 \times 10^{-4} \) times smaller than the considered second-order terms and can be safely neglected. Our treatment is thus reliable in the parameter region we shall address.

Let us now recall the canonical commutation relations \( \{ \hat{R}_j, \hat{R}_k \} = i \Omega_{jk} \mathbb{I} \), where the \( 2n \times 2n \) matrix \( \Omega \) has entries \( \Omega_{j,k} = \delta_{n,k-j} - \delta_{n,j-k} \) for \( 1 \leq j, k \leq 2n \), and that Gaussian states are defined as states with Gaussian characteristic functions: a Gaussian state \( \varrho \) is thus uniquely determined by its ‘covariance matrix’ (CM) \( \sigma \), with entries given by
\[ \sigma_{jk} \equiv \frac{1}{2} \text{Tr} \left[ \{ \hat{R}_j, \hat{R}_k \} \varrho \right] - \text{Tr} [ \hat{R}_j \varrho ] \text{Tr} [ \hat{R}_k \varrho ] \]
and by the vector of first moments \( R \), with components \( R_j \equiv \text{Tr} [ \hat{R}_j \varrho ] \) [4, 5].

The ground state of Hamiltonian \( \hat{H} \) is a Gaussian state with a block diagonal CM \( \sigma_b = (\sigma_s \oplus \sigma_s^{-1}) / 2 \), where
\[ \sigma_s = \tilde{\omega}^{1/2} (\tilde{\omega}^{1/2} \kappa \tilde{\omega}^{1/2})^{-1/2} \tilde{\omega}^{1/2} \]
and vanishing first moments.

Finally, let us remember that the evolution for the time \( t \) of an initial Gaussian state with CM \( \sigma \) under a quadratic Hamiltonian \( \hat{H} = \hat{R}^\dagger H \hat{R} \) (where \( H \) is any symmetric matrix) is a Gaussian state with CM given by
\[ \sigma_t = S_t \sigma S_t^\top, \]
for \( S_t = \exp(\Omega H t) \).

2.1. Dissipation
In our study, we will take into account the decoherence of the radial modes in an environment of phonons with temperature \( T \) and ‘loss rate’ \( \gamma \) (for simplicity assumed to be the same for every mode). Under such conditions, the evolution of the ions’ state \( \varrho \) at frequencies \( \{ \tilde{\omega}_j \} \) is described.
by the following master equation in interaction picture [6]:

\[
\frac{d\rho}{dt} = \gamma \sum_{j=1}^{n} \left[ N_j (2a_j^{\dagger} \rho a_j - a_j a_j^{\dagger} \rho - \rho a_j a_j^{\dagger}) + (N_j + 1) (2a_j \rho a_j^{\dagger} - a_j^{\dagger} a_j \rho - \rho a_j a_j^{\dagger}) \right],
\]

(4)

where the number of phonons in the radial mode \(j\) is given by \(N_j := 1/(\exp(\bar{h}\tilde{\omega}_j/k_B T) + 1)\), according to Bose’s law (\(k_B\) being the Boltzmann constant) and \(a_j := (\hat{X}_j + i\hat{P}_j)/\sqrt{2}\). In accordance with the experimental terminology, we will also refer to the quantity \(\epsilon_j := N_j \gamma\) as the ‘heating rate’, essentially representing the rate at which thermal phonons are injected into mode \(j\).

Equation (4) preserves the Gaussian character of the initial state and can results, for the CM \(\sigma\) of Gaussian states, into the following equation:

\[
\frac{d}{dt} \sigma = \gamma \sigma_\infty - \gamma \sigma
\]

(where \(\sigma_\infty \equiv \sigma' \oplus \sigma'\), \(\sigma' = \text{diag}(N_1 + \frac{1}{2}, \ldots, N_n + \frac{1}{2})\)), with solution

\[
e^{-\gamma t} \sigma_i + (1 - e^{-\gamma t}) \sigma_\infty.
\]

3. Linear phonics and beyond

In this section, we shall assume that the trapping frequencies \(\{\omega_j\}\), and thus \(\{\tilde{\omega}_j\}\), can be controlled locally and changed instantaneously. A local control of the radial trapping frequencies is certainly very challenging to realize at present. There are, nonetheless, several ways in which it might be achieved in a near future, namely by building small, local radial electrodes, by adding local optical standing waves\(^5\), or in arrays of traps (analogous to those proposed for Penning traps [7, 8]). As already mentioned above, we will also assume the changes in trapping frequencies to be instantaneous. In practice, such changes should occur over times at least one order of magnitude shorter than the inverse trapping frequencies we consider (typically ranging between 1 and 10 MHz, requiring switching times around 10–100 ns) for our results to apply. Current setups would struggle to achieve reliable control of the radial trapping frequencies over such times because the radial oscillations are confined by rf-fields. A way to overcome this problem would be to add dc radial trapping fields, which have much faster switching rates, usually one order of magnitude faster than the typical trapping frequency (a limit which could probably be beaten shortly, considering the ongoing miniaturization of the electrodes [9]). However, this would alter the longitudinal trapping frequencies as well, which would then have to be simultaneously adjusted, thus further complicating the experiment. Although certainly extremely challenging, this sort of control could conceivably be achieved within the next few years. Let us also note that a setting formally identical to the one we describe here could be reproduced shortly even for locally defined axial (‘longitudinal’) modes in segmented Paul traps [10]: quite remarkably, this would address both the issue of local control and of switching times.

Our aim is to show how, in principle, local control of the frequencies allows one to perform any arbitrary ‘linear optical’ operation on the radial modes of the ions, that is any unitary

\(^5\) Currently, standing waves realize, at most, trapping potentials of about 1 MHz in experiments where the internal degrees of freedom are controlled. However, the manipulation of radial modes can tolerate much higher scattering rates than that of internal degrees of freedom, and would thus allow for much higher trapping frequencies.
operation under which, in the Heisenberg picture, the vector of operators $\hat{R}$ transforms linearly:

$$\hat{R} \mapsto S\hat{R}.$$ 

The matrix $S$ has to be ‘symplectic’, i.e. such that

$$S^T\Omega S = \Omega$$

(where $\Omega$ is the anti-symmetric symplectic form, defined on page 4), to preserve the canonical commutation relations. Notice that such a class of operators includes squeezing transformations. We should mention that the idea of squeezing the state of a single ion by controlling the trapping frequency was first put forward in the early 1990s [11]. Also, single-mode squeezing of a trapped ion can be achieved by coupling its motion to its internal degrees of freedom [12, 13], and has been realized in experiments [14]. The scope of the present section is however wider, since we set out to prove that any symplectic operation on any number of ions can be implemented by frequency manipulation.

Let us first remark that any symplectic operation $S$ on a system of many canonical degrees of freedom (‘modes’) can be decomposed into a combination of generic single-mode symplectic transformations and two-mode rotations (‘beam splitters’, in the quantum optical terminology) [15, 16]. This fact follows from the Euler decomposition of symplectic operations [15] and from the possibility of decomposing energy preserving operations into a network of beam splitters between pairs of modes [16]. It is therefore sufficient for us to establish the possibility of performing these two subclasses of operations (single-mode symplectic transformations and two-mode beam splitters) on our system of $n$ ions by manipulating the local frequencies. In turn, again because of Euler decomposition, single-mode operations can always be reduced to combinations of squeezings and phase shifts.

### 3.1. Single-mode operations

In what follows, we assume that the original effective frequencies of the ions are different but commensurate, as given by, say, $\tilde{\omega}_j = j\tilde{\omega}$, and that $\tilde{\omega}$ is large enough so that interaction between any two ions is suppressed when the relevant coherent manipulation sets off. More precisely, a close analysis of the normal modes of the system shows that the effect of the Coulomb interaction on the radial modes of ions $j$ and $k$ can be neglected if $\omega_L^2/(\omega_j^2 - \omega_k^2) \ll 1$, i.e. if the squared longitudinal trapping frequency is small with respect to the difference between the squared radial trapping frequencies of the two ions. This is sufficient even if $j$ and $k$ are the nearest neighbours. Later on (section 5), we will directly demonstrate that this approximation is reliable by fixing values for the trapping frequencies and showing that they for highly coherent evolutions between specific pairs of distant ions. Let us then consider the reaction of the system if the frequency of the $j$th ion changes suddenly from $\tilde{\omega}_j$ to $\alpha_j\tilde{\omega}_j$, for some real $\alpha_j$ (clearly,...

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6 In the literature, squeezing transformations have a tendency to be excluded from the class of ‘linear’ transformations, because their implementations require ‘nonlinear’ (third-order) interactions. However, the effective evolution of the modes of interest (‘signal’ and ‘idle’, usually) is actually linear in the sense specified above (linear evolution of the vector $\hat{R}$ in Heisenberg picture), and we will thus generally refer to ‘linear’ transformations as including squeezing.

7 Notice also that the linear scaling of the coupling we assumed is not a stringent requirement. Because the coupling between the ions $j$ and $k$ falls off like $|j - k|^{-3}$, one can safely assume, say, a ‘triangle’ profile (alternately increasing and decreasing) for the couplings in long chains of ions. Note also that, for a linear scaling, the population of the levels of ion $j$ due to the interaction with $k$ roughly scales as $1/(1 + (\tilde{\omega}/\omega_L^2)(j - k)^3)$: a frequency step $\tilde{\omega} \simeq 20\omega_L$ is already enough to make the effects of all the interactions essentially negligible.
such changes are completely governed by changes in the \( \omega_j \)'s). The Heisenberg equation of motion for \( \hat{x}_j \) and \( \hat{p}_j \) can be immediately integrated in such a case, resulting into a symplectic transformation \( S_j(t) \) given by

\[
S_j(t) = \begin{pmatrix}
\alpha_j^{1/2} & 0 \\
0 & \alpha_j^{-1/2}
\end{pmatrix} \begin{pmatrix}
\cos(\tilde{\omega}_j \alpha_j t) & \sin(\tilde{\omega}_j \alpha_j t) \\
-\sin(\tilde{\omega}_j \alpha_j t) & \cos(\tilde{\omega}_j \alpha_j t)
\end{pmatrix} \begin{pmatrix}
\alpha_j^{-1/2} & 0 \\
0 & \alpha_j^{1/2}
\end{pmatrix}.
\]

The first and last matrix of this decomposition corresponds to ‘squeezing’ operations in the quantum optical terminology, whereas the second factor is known as a ‘phase shift’ (i.e. a rotation in the single-mode phase space). Combinations of squeezings and phase shifts make up any possible single-mode symplectic operation: we thus need to show that such operations can be implemented individually on any ion of the system in a controllable manner (as pointed out above, this is sufficient because of the Euler decomposition of single-mode operations [15]).

3.1.1. Phase shift. To realize a phase-shift operation on the \( k \)th ion, without any squeezing, it is sufficient to change the frequencies of all the other ions in the same way, such that, in the notation defined above, \( \alpha_k = 1 \) and \( \alpha_j = \alpha \neq 1 \) for \( j \neq k \) (\( \alpha_j \) being the factor by which the frequency of ion \( j \) is multiplied by manipulating the local trapping potential). As apparent from equation (5), after a time \( t_a = 2\pi/(\tilde{\omega}_k \alpha) \) from the change one has \( S_j = \mathbb{1}_2 \) for \( j \neq k \): all the ions which undergo a change in trapping frequency are back to their initial state (let us recall that \( \tilde{\omega}_j = j \tilde{\omega} \) by assumption, so that after \( t_a \) the central rotation of equation (5) reduces to the identity and the two opposite squeezing transformations just nullify each other). On the other hand, the transformation \( S_k \) will be equal to the central rotation (with no squeezing, as \( \alpha_k \) is kept equal to 1) of angle \( \varphi_k = 2\pi k/\alpha \): the oscillation of the \( k \)th ion will have acquired such a phase, analogous to the ‘optical phase’ of light modes. If the frequencies are switched back to the original values after a time \( t_a \), the net effect of the evolution is then analogous to an ‘optical’ phase shift of angle \( \varphi_k \) on the ion \( k \).

3.1.2. Squeezing. In order to squeeze the state of ion \( k \), leaving all the other ions unaffected, one can conversely change only the pertinent frequency, so that \( \alpha_k \neq 1 \) and \( \alpha_j = 1 \) for \( j \neq k \). Then, after a time period \( t = 2\pi/\tilde{\omega}_k \), all the other ions will have returned to the initial state, while ion \( k \) will be squeezed and phase shifted according to equation (5). The phase shift can always be arbitrarily corrected by applying the strategy to obtain ‘pure’ phase shifts described in section 3.1.1, which conclusively shows that any linear operation on a single mode can be implemented by controlling the local trapping frequencies.

The degree of squeezing achievable with such a strategy depends crucially on the phase shift operation since, as shown by equation (5), the two squeezing operations act along orthogonal directions and are the inverse of each other. In the case \( \alpha_k = (\frac{1}{4} + h)/k \) for \( h \in \mathbb{N} \), the two squeezings act effectively along the same phase space direction: in this instance the phase shift can be balanced by a counter rotation of \( \pi/4 \) in phase space and the final squeezing operation is a diagonal matrix given by \( \text{diag}(\alpha_k, \alpha_k^{-1}) \). In principle, the value of \( \alpha_k \) is only limited by the stability of the system and the breakdown of the harmonic approximation [2], occurring when the squeezing is comparable to the ratio between the size of the wave packet

\[8 \text{ In fact, arbitrary adjustment of the phase allows for ‘pure’ squeezing operations along the phase space directions } x \text{ and } p, \text{ and thus implies the possibility to construct any linear operation, because of the symplectic Euler decomposition previously discussed.} \]
and the distance between the atoms. In actual experiments, the considered setup would thus allow for \( \sqrt{\alpha_k} \ll 500 \). Notice that such values are by far out of reach in optical systems, where squeezings corresponding to \( \alpha_k \approx 8 \) were recently reported [17]. Also notice that, by placing the ions inside cavities, the squeezing of the massive degrees of freedom could be transferred to light, so that radial modes could act as an effective source of squeezing (and potentially even entanglement) for optical systems as well [18], analogously to what has already been proposed for trapped atoms [19].

3.1.3. Beam splitters. To extend our proof to any linear operation, on any number of ions, let us now turn to ‘beam splitting’ operations between the radial modes of any two ions in the array. Such operations are achieved by bringing the two modes in question (hereafter labelled by \( j \) and \( k \)) to the same frequency \( \tilde{\omega} = \tilde{\omega}_j = \tilde{\omega}_k \), so that the Coulomb interaction between them is no longer suppressed. Switching to the interaction picture, one has the following coupling Hamiltonian between the two modes:

\[
\hat{H}_I = \kappa_{jk} \hat{x}_j(t) \hat{x}_k(t) = \kappa_{jk} (a_j e^{-i\tilde{\omega}t} + a_j^\dagger e^{i\tilde{\omega}t})(a_k e^{-i\tilde{\omega}t} + a_k^\dagger e^{i\tilde{\omega}t}),
\]

where the ladder operators are defined as \( \hat{x}_j = (a_j + a_j^\dagger) \) and \( \kappa_{jk} \) is an entry of the matrix \( \kappa \) defined in equation (1). If the frequency \( \tilde{\omega} \) is sufficiently large the rotating wave approximation reliably applies and the rotating terms can be neglected to yield\(^9\)

\[
\hat{H}_I = \kappa_{jk} (a_j a_k^\dagger + a_j^\dagger a_k).
\]

This Hamiltonian realizes exactly the desired beam splitter-like evolution, resulting into a symplectic transformation which mixes \( \hat{x}_j \) with \( \hat{x}_k \) and \( \hat{p}_j \) with \( \hat{p}_k \) (rotating such pairs equally, by the angle \( \kappa_{jk} t \)). For instance, a ‘50 : 50’ beam splitter is achieved after a time \( t = \pi/(4\kappa_{jk}) \). Since the interaction requires a change of the local frequencies it includes automatically in it a local operation, which may however be corrected before or after the ‘beam-splitting’ procedure, as detailed above.

Summing up, the combined arguments presented in sections 3.1.1–3.1.3 show that any ‘linear optical’ operation, including squeezing operations and entangling operations acting on multiple different modes, can be implemented for radial modes of trapped ions by properly tuning the frequencies of the radial microtraps of the individual ions (assuming the frequencies’ changes are realized over times much shorter than \( \omega_j^{-1} \)).

Linear optical operations, complemented by displacements (see section 3.2), correspond to all the unitary transformations that preserve the Gaussian character of the initial state. Therefore, all the developments based on Gaussian states in quantum optics, in particular concerning entanglement manipulation [4] and information protocols [20], can be carried over to radial modes of ion traps if local control is achieved. Arguably, the framework we have outlined for the implementation of linear optical operations on the local oscillations of distinct ions, to which one may refer as ‘linear phononics’, could offer further practical advantages over its quantum optical counterparts, essentially related to the static nature of the quantum information processed (for instance, ‘mode matching’, which hampers substantially the implementation of linear optics, is not a problem here).

\(^9\) This always holds at the trapping frequencies we shall consider in specific examples, going from 1 to few tens of MHz, for which stability around \( 10^{-3} \) are achievable.
3.2. Further manipulations and measurements

Beside the linear operations treated so far, implemented through varying trapping potentials and Coulomb interactions, the motional degrees of freedom of trapped ions allow for other controlled manipulations, some of which take advantage of the coupling between motion and the internal degrees of freedom that can be engineered by applying standing- or traveling-wave pulses to the ions. In order to give a complete account of the possibilities offered by radial modes, let us here review such strategies, and briefly comment about their applicability to radial modes. As a general remark let us mention that, because of the tighter confinement they allow for, radial modes easily meet the Lamb–Dicke condition (depending on the width of the ground state’s wavepacket), which means that the coupling with the internal degrees of freedom can be tailored to a high degree of accuracy (generally better than for longitudinal modes)\(^\text{10}\). Also, individual ions can be addressed in such manipulations as, at spacings of some micrometers and assuming pulses’ waists of the order of 1 \(\mu\)m, less than 1% of the central laser power would be shined on neighbouring ions with respect to the central one.

Displacement operations, which shift the operators \(\hat{R}_j\) by real numbers, can be realized in several ways: by classical driving fields, by standing waves, or by shifting the radial equilibrium positions of the ions [14]. In particular, transverse driving fields could be applied to displace the radial modes, with no particular hurdles. In the following, the unitary displacing the canonical operators of mode \(j\) by, respectively, \(x_j\) and \(p_j\) will be denoted by \(\hat{D}_j(x_j, p_j)\).

Notably, even non-Gaussian states can be engineered in this setup with relative ease (with respect to quantum optics, where non-Gaussian manipulations require higher order nonlinearities, usually extremely weak), either by entering the nonlinear regime of the Coulomb interactions or by coupling the internal degrees of freedom of the ions. The experimental realization of a cross-Kerr coupling for the longitudinal oscillations of trapped ions has been recently reported [2].

The coupling to internal degrees of freedom also allows for Gaussian and non-Gaussian measurements on individual ions. The measurement of quadrature operators, corresponding to homodyning, was proposed in [21]–[23]. Quantum non-demolition measurement of local number states and parity could be measured as well, by applying the scheme suggested and realized for cavity QED in [24], based on the dispersive coupling of the number of oscillations to two internal levels of the ion. Quite remarkably, such a scheme would allow one to measure the phonon number’s parity on a single copy of the state and run of the apparatus. As we will see in section 6, this possibility is consequential for the violation of Bell-like inequalities. Notice also that such non-Gaussian measurements pose a considerable technological challenge for light fields, where resolving photon numbers with high detector efficiency is still daunting despite recent progress [25, 26]. On the other hand, the schemes recalled above are bound to be comparatively slow, requiring to wait for half a period of coherent interaction between motions and internal levels and then subsequent readout of the internal levels by, e.g. fluorescence. When internal degrees of freedom are involved, radial modes require longer times, roughly on the order of tens of microseconds, but achieve remarkable precision.

Finally, concerning the preparation of initial states for coherent manipulations, let us recall that cooling of ion oscillations to their ground state can be achieved very efficiently by sideband-Raman pulses [27, 28] (or more recent variations over such a strategy [29, 30]). Note that,\(^\text{10}\) The price to pay for such an accuracy is longer operation times, which could be ultimately reduced if stronger lasers became viable.
since in the described scenario the initial local potential is much larger than the interaction, simultaneous cooling of all the chain could be done by cooling each ion to its local ground state, which is to a good approximation the global ground state. The local single excitations on such ground states may be prepared by addressing the first ion with a proper sequence of blue and red sideband pulses, as detailed in [14].

In the remainder of the paper, we shall demonstrate the potential of linear operations on radial modes in specific applications, with a particular focus on settings fully accessible to current experiments.

4. Entanglement generation

The set of operations described in section 3, including local squeezing and two-mode beam splitters between distant modes, allows for the generation of entanglement between the radial modes of two ions in the chain. In fact, as is well known, two single-mode states squeezed along orthogonal quadratures, entering a beam splitter, give rise to an entangled state for the outgoing modes (this is a standard procedure to create CV entanglement in quantum optics). In principle, for the system of trapped ions under examination, the local control of the transverse trapping frequencies allows for the creation of tailored entangled states, where the entanglement can be built up between any two ions in the chain. Moreover, and possibly even more intriguingly, the possibility of controlling a system of \( n \) parties (the ions) all constantly interacting with each other, paves the way for the creation of multipartite entanglement between the distinct ions.

In this section, we show that unprecedented degrees of bipartite entanglement, as well as interesting and robust multipartite entangled states, can be obtained in settings accessible to current experiments, starting from a ground state of the system and requiring only global control of the trapping potential (where, i.e. the trapping frequencies \( \{ \omega_j \} \) are the same for all ions at any time, and are only changed simultaneously). Further, we will briefly discuss the possibilities of a local control of the frequencies that would open up for the controlled dynamics of multipartite entanglement.

The specific situation we shall address starts off from the ground state \( \rho_g \) of Hamiltonian \( \hat{H} \) [see equation (1)]—with all frequencies being equal, i.e. \( \omega_j = \omega_i \) for \( 1 \leq j \leq n \)—as the initial state. Next, the trapping frequencies are changed to the common value \( \omega_f \), so that the state \( \rho_g \) will not be stationary anymore under the modified Hamiltonian. For large \( \omega_f \), the initial state \( \rho_g \) contains very little entanglement but, if the frequencies are suddenly changed, entanglement builds up during the subsequent evolution (see [31] for an analogous scheme in chains of nanomechanical oscillators). Entanglement between different subsystems will be quantified by the logarithmic negativity \( \mathcal{E}_N \equiv \log_2 \| \tilde{\rho} \|_1 \), where \( \| \tilde{\rho} \|_1 \) stands for the trace norm of the ‘partially transposed’ density matrix of the subsystem [5, 32]. Different partitions will be denoted by superscripts referring to the ions: for instance, the logarithmic negativity between ions 1–2 and 3–4 will be denoted by \( \mathcal{E}_{12}^{34} \). The intuition behind this entanglement generation method is that operating faster than the speed of sound in the system, i.e. essentially, faster than the inverse of the energy gaps, is analogous to operating locally\(^{11} \). Thus, by changing the local

\(^{11}\) The speed of sound in a chain of ions can be defined by assuming the system to be infinite and translationally invariant, while keeping the same typical interaction strengths. The analysis of small chains of ions shows that a speed of sound defined in such a way still accounts for the propagation of excitations in finite systems (see, e.g. figure 4 of [33]) and turns out to be proportional to the trap size times the lowest energy gap of the system. For the thermodynamical properties of long chains of ions, see also [34].
Figure 2. Dynamics of the entanglement (in ebits of logarithmic negativity) between the radial modes of the ions, in a trap containing $n = 4$ ions and with longitudinal trapping frequency $\omega_L = 1$ MHz. At time $t = 0$ the system starts off from the ground state for transverse trapping frequency $\omega_t = 5$ MHz and then evolves under a trapping frequency $\omega_f = 2$ MHz. The plots on the left show the logarithmic negativity shared by all the possible bipartitions of the four modes (because of the spatial symmetry, one has only five distinct cases in the chain: for instance, one has $E_{N}^{1,234} = E_{N}^{4,123}$). On the right, the logarithmic negativity shared by pairs of modes is plotted (again, because of symmetry, only four distinct cases occur). The continuous (red) curves refer to a case with no dissipation, whereas the dashed (green) curves refer to a case with heating rate $\epsilon = 2$ kHz (resulting from a coupling to the bath $\gamma = 10^{-4}$ Hz and a temperature $T = 294$ K).

potentials (in the same way for each ion) fast enough one generates local squeezing, which is then ‘converted’ into entanglement by the time evolution through the harmonic coupling between the ions, analogous to a set of beam splitters.

The case portrayed in figure 2 represents four ions starting from the ground state for $\omega_t = 5$ MHz, which then evolves under the frequency $\omega_f = 2$ MHz for 20 $\mu$s (for a transverse trapping frequency $\omega_L = 1$ MHz). As can be seen, the global state gets entangled under any possible bipartition of the four modes, a situation which is referred to in the literature as ‘complete inseparability’ [35]. A more extensive inspection shows that this is a general property of the Hamiltonian at hand, even for larger number of ions. Of course, as shown in figure 3, complete inseparability also occurs for three ions: in section 6 we shall see how the multipartite entanglement for three ions analysed in figure 3 could serve to violate Bell-like inequalities. Our plots also show that such an entanglement exhibits considerable resilience under realistic heating rates. In this respect, notice that the heating rate for the dashed (green) curve plotted in the graphs is $\epsilon = 2$ kHz, and that heating rates as low as $\simeq 10$ Hz have been recently observed in ion traps (even though for a single ion only [36, 37]): robust multipartite entanglement (between 5 and 10 ebits of logarithmic negativity) between the ions can thus be created. Such degrees of entanglement are inconceivable for multipartite systems in photonic systems where, furthermore, the manipulation of several modes tends to become awkward due to the increasing number of mode-matching conditions to be fulfilled.
Figure 3. Dynamics of the entanglement (in ebits of logarithmic negativity) between the radial modes of the ions, in a trap containing \( n = 3 \) ions and with longitudinal trapping frequency \( \omega_L = 1 \text{ MHz} \). At time \( t = 0 \) the system starts off from the ground state for transverse trapping frequency \( \omega_T = 20 \text{ MHz} \) and then evolves under a trapping frequency \( \omega_f = 2 \text{ MHz} \). The plots on the left show the logarithmic negativity shared by all the possible bipartitions of the four modes (because of the spatial symmetry, one has only two distinct cases in the chain). On the right, the logarithmic negativity shared by pairs of modes is plotted (again, because of symmetry, there occur only two distinct cases). The continuous (red) curves refer to a case with no dissipation, the dotted (blue) curves refer to a case with heating rate \( \epsilon = 200 \text{ Hz} \) (resulting from a coupling to the bath \( \gamma = 10^{-5} \text{ Hz} \) and a temperature \( T = 294 \text{ K} \)), whereas the dashed (green) curves refer to a case with heating rate \( \epsilon = 2 \text{ KHz} \) (resulting from a coupling to the bath \( \gamma = 10^{-4} \text{ Hz} \) and a temperature \( T = 294 \text{ K} \)).

In the plots, we also report the entanglement of pairs of modes in the traps for four, three and two ions. For bipartite entanglement as well, as shown in figure 4, the degree of robust entanglement achievable is by far beyond the maximal values obtained in quantum optical systems\(^{12}\), even considering rather modest frequency jumps. The case portrayed in figure 4(b) starts from the ground state for \( \tilde{\nu}_I = 10 \text{ MHz} \). Afterwards, the frequency is switched to \( \omega_f = 2 \text{ MHz} \) for 5 \( \mu \text{s} \) and then back to \( \omega_I \) for 25 \( \mu \text{s} \): such a cycle realizes a highly entangling operation because it is tailored such that the squeezings act always in the same phase-space directions and, in principle, can be iterated at will to obtain any desired degree of entanglement. In practice, decoherence and dissipation will degrade such an entanglement after some iterations. However, as is apparent from the plot, stable and robust entanglement (around 10 ebits of logarithmic negativity) between two ions could be created under realistic conditions.

Also, in the traps with more than two ions, the dynamics of the entanglement between pairs of modes displays a ‘monogamous’ behaviour [39, 40]: for four ions, when the entanglement

\(^{12}\)To the best of our knowledge, the highest measured value for the logarithmic negativity in optical systems (inferred from state reconstruction) is \( E_N \simeq 1.6 \text{ ebits} \) [38]. A simple evaluation also shows that, exploiting the degree of squeezing reported in [17] and assuming perfect beam-splitting operations, one could achieve at most \( E_N \lesssim 3 \text{ ebits} \).
between ions 1 and 2 fades, the entanglement between 1 and 3 starts raising and then is replaced by entanglement between 1 and 4, which in turn fades out before the revival of the quantum correlations between 1 and 2. The entanglement between 2 and 3—strongly favoured by the fact that such ions are both nearest neighbours and at perfect resonance with each other (not true in general, as the Coulomb corrections on the trapping frequencies depend on the position in the trap)—is always positive but, significantly, its peaks and dips follow those of the entanglement between 1 and 4, as one would heuristically expect by virtue of monogamy. As recalled above, the reported values of $\omega_i$ and $\omega_f$ represent the ‘bare’ trapping frequencies, not taking into account the corrections due to Coulomb repulsion: therefore, all ions are not at resonance during the evolution. Now, if one assumes that local control were available, one could correct for such a mismatch by applying different bare frequencies to ions in different positions along the trap’s longitudinal axis. We applied such corrections in the four ions case considered above and found that, quite interestingly, one can spread correlations more evenly in this way and get closer to the ultimate bounds imposed by monogamy: after $10 \mu s$ (all the parameters being the same aside from the corrections) one ends up with a state where each radial mode is individually entangled with all the other radial modes (a situation which never arises without corrections, see figure 2).

The CV entanglement between two radial modes of the ions could be swapped to light if cavities were added to the setup, with the potential to achieve unprecedented degrees of optical entanglement, as the parametric processes are definitely outperformed by the strategy outlined...
above based on the trapping frequency’s control: this possibility will be the subject of a detailed investigation in a forthcoming paper [18].

5. Propagation of quantum information

The locally defined radial modes of the ions, interacting via the Hamiltonian (1), are an example of a harmonic chain, where the coupling between different oscillators can be, to a very good extent, controlled. The propagation of quantum information in such a setting has been proposed and analysed theoretically in [41], but not yet realized in practice since mechanical oscillators have yet to enter the quantum regime (due to the technical difficulties still encountered in controlling both the couplings and the decoherence and dissipation of chains of nano-oscillators [42]). Trapped ions could thus provide a very promising alternative to implement harmonic chains, realize quantum data buses [43] and test the results predicted by theoretical studies. More generally, such a demonstration would be a further application of ion traps as quantum simulators, as already proposed in several past studies [44, 45]. In this section, we shall present an overview of the possibilities offered by the radial modes as harmonic chains for the propagation of quantum states, with quantitative investigations, in scenarios similar to those detailed in the preceding sections.

5.1. Transmission of two-dimensional quantum states

The transmission of quantum information encoded in finite-dimensional quantum states (most often qubit states) through a chain of interacting quantum systems has drawn much attention in recent years [46], as such systems are envisaged as possible quantum buses, linking different parts of future quantum processors. However, despite considerable efforts, finding systems where such a transmission can be realized with sufficiently small level of noise has proven to be very challenging. Therefore, good practical candidates for such tasks are still of interest. Proposals in this context have so far focused on the transmission of finite-dimensional states in chains of interacting (effective or proper) ‘spins’. Here, instead, we consider qubit states encoded in the bosonic Hilbert space of the radial modes of trapped ions and, guided by the results of section 3, we show that such modes are able to send qubit states through chains of trapped ions.

As a preliminary remark, let us notice that, if the ratio between the radial and the longitudinal trapping frequencies is very large, \( \omega_j / \omega_L \gg 1 \forall j \), then the effect of the Coulomb interaction on the Hamiltonian governing the radial modes is negligible\(^{13}\) and the Hamiltonian (1) reduces to \( \hat{H} \simeq \frac{1}{2} \hat{R}_j \hat{\omega} + \hat{\omega} \hat{R}_j \). That is to say, the radial oscillations of the individual ions are decoupled, with local normal frequencies approximately given by \( \{ \omega_j \} \). In this situation, the ground state of the system is just the ordinary, non-squeezed, vacuum: a Gaussian state with CM equal to the identity and vanishing first moments. For instance, for \( \omega_j = 50 \omega_L \forall j \) the relative discrepancy between the actual ground state and the vacuum is only \( 10^{-3} \) in terms of covariances. Starting from such a ground state, we will consider a scenario where the sender has its qubit state encoded in the single excitation sector of the oscillations of the first ion (i.e. on the subspace spanned by the phonon states |0⟩ and |1⟩) and wants to send the excitation

\(^{13}\)See page 4 and notice that, quite remarkably, the ratios \( d / |\mu_j - \mu_k| \) does not depend on the trapping frequencies nor on the masses of the ions but only on the total number of ions \( n \) (see [3] for details).
through the chain to a receiver who owns the last ion. Notice that, as recalled in section 3.2, a (non-Gaussian) single qubit state of this kind could be prepared, in practice, by coupling the motion of an ion to the internal degrees of freedom through a proper sequence of blue and red sideband pulses.

Neglecting, for the time being, thermal noise (see below), one has that the state $|0\rangle$ can be sent perfectly through the chain. Therefore, the overall fidelity of the transmission depends solely on the fidelity with which $|1\rangle$ can be sent from the first to the last ion. The amplitude of such a transfer can be determined analytically by tracking the evolution of the initial operator $a_1^\dagger$ (creating the excitation in the first ion) in the Heisenberg picture. Because the system is harmonic the evolution of $a_1^\dagger$ results in a linear combination of field operators, which can be determined with standard techniques. We could thus consider different situations. The histograms we plot show the probability amplitude $P$ for finding a single excitation at different sites in the chain. The resulting transmission fidelity (averaged over the Haar measure of a single qubit space) is given by $1 - \frac{1}{2}(1 - \sqrt{P})^2 + \frac{3}{4}\sqrt{P}(1 - \sqrt{P})$, monotonically increasing with $P$. We will consider a chain of $n = 10$ ions and an initial ground state for trapping frequencies $\omega_j = 50\omega_L \forall j$.

Figure 5(a) shows the case where the trapping frequencies are left unchanged after the initial preparation. Because the interacting terms, while very small, are still present, the excitation propagates because the local mode of the first ion is not exactly a normal mode of the system. After an evolution time $t = 375\omega_L^{-1}$, 0.76 of the probability amplitude is transferred to the final state. However, the propagation is rather dispersive, as each mode is on resonance with all the others.

In order to boost the probability of transmitting the excitation, sender and receiver can switch the local trapping frequencies of their respective ions from the initial value $\omega_i = 50\omega_L$ down to a common value $\omega_f$: in this way they can realize the beam-splitting operation at distance described in section 3.1.3 which, in principle, would allow for perfect swapping. This specific example thus also illustrates our previous general discussion and shows with which precision and over which operating times are linear operations actually possible over a chain of ions. As one can see from figure 5(b), where $\omega_f = 5\omega_i$ has been assumed, the beam-splitting operation is virtually perfect: all the probability amplitude is gradually transferred to the final ion, whereas the other ions are never involved in the process. This quality in the transfer comes at the expense of transfer time: over ten ions the beam splitting operation takes roughly an order of magnitude more ($t \simeq 5600\omega_i^{-1}$) than the imperfect transfer considered in figure 5(a). This is simply due to the fact that, by keeping the middle ions on resonance, one takes advantage of the interaction between nearby ions, which are clearly stronger (as the interaction decays like the cube of the distance) and propagate the excitation through the chain. For given trapping frequencies and $n \simeq 10$, the time needed to achieve the beam splitter between radial modes at the two ends of a chain of $n$ ions scales as $n^2$ (this is just due to the fact that the difference between the equilibrium positions of the two ions at the ends of the array, as determined by the Coulomb interactions, scales, for $n \simeq 10$, like $n^{2/3}$, whereas the coupling $k_{1n}$ is inversely proportional to the cube of the distance). Notice also, as a side remark, that since the Coulomb corrections to the local trapping frequencies are symmetric with respect to the longitudinal centre of the trap, the transfer between extremal ions is somewhat favoured in practice, as equal trapping potentials will result in equal effective trapping frequencies for such ions.

Finally, in figure 5(c), we consider a ‘compromise’ between the two cases analysed above: all the frequencies are changed to the frequency $10\omega_L$ obtaining, after a time $t = 625\omega_L^{-1}$,
a transferred probability amplitude of 0.85. It is important to remark that the slower beam-splitting operation, where the quantum information does not disperse through the chain at all, is not only more precise than the other two options examined, but also much more stable with respect to imperfections in the allowed interaction times.

Let us now turn to inspect the effect of decoherence: clearly, thermal phonons are deleterious for the transfer of information encoded in single excitation sectors. The analytical estimates for the transfer times we just presented allow us to determine the restrictions on the heating rates that would allow such systems to transfer information in practical situations. Assuming a longitudinal trapping frequency of 1 MHz and a (less ambitious) chain of five ions...
ions, one would realize a beam-splitting operation (perfect transfer) between extremal ions in $t \simeq 1.4$ ms. In such a case, heating rates as low as $\epsilon \simeq 0.1$ kHz would be needed for a coherent transfer to take place. For such heating rates, the action of the thermal phonons on the fidelity between initial and final states can be assumed to be linear resulting, after average over the Haar measure, into a mean fidelity $F \simeq 1 - 3\epsilon t/2 \simeq 0.8$. Such heating rates are certainly demanding for arrays of traps, but not inconceivable, considering the heating rates achieved for single traps and the fact that current systems all operate at room temperature (thus, in principle, two orders of magnitude could be gained on heating rates).

5.2. The role of the internal degrees of freedom

In the preceding section, while discussing the transfer of finite-dimensional quantum information, we started off from a situation where a qubit state is encoded in the first two number states of the first ion. As already mentioned above, one of the most expedient ways to create such a finite-dimensional state for the motional degrees of freedom is creating the desired state for the ‘internal’ degrees of freedom of the ion (embodied by its internal electronic levels) and then coupling them to the local radial modes to achieve the swapping. In order to provide the reader with a more comprehensive treatment and give him/her a flavour of the way the internal degrees of freedom enter in such dynamics, we briefly give account of a specific example where the internal degrees of freedom are involved throughout the whole transmission of quantum information.

We shall consider a trap with four ions, with $\omega_j = 10\omega_L$ for $1 \leq j \leq 4$. To keep the example down to earth, we will not assume any capability of local control nor any change during the evolution for the radial frequencies here. The only manipulation required will be the switching on and off of the coupling between the local phonons and the two levels of each ion where the initial qubit state is encoded. Denoting such levels $|e_j\rangle$ and $|g_j\rangle$, and setting $\sigma_{x,j} := |e_j\rangle\langle g_j| + |g_j\rangle\langle e_j|$, we shall assume the following interaction Hamiltonian $\hat{H}_{j}^{\text{int}}$ for each ion $j$:

$$\hat{H}_{j}^{\text{int}} = g_j \sigma_{x,j} \hat{X}_j,$$

which describes appropriately the coupling realized in experiments (see, e.g. [14]). For simplicity, the interaction strengths $g_j$ will be all set to $\omega_L$ (notice that, because the oscillators’ frequency are set to $10\omega_L$, the interaction reduces, up to a very good approximation, to a rotating wave one). As initial state, at $t = 0$, we assume a Bell state between the internal degrees of freedom of ions 2 and 3, and the ground state for the remainder $f$ the system:

$$|g\rangle_1 \otimes (|g\rangle_2 \otimes |e\rangle_3 + |e\rangle_2 \otimes |g\rangle_3) \otimes |g\rangle_4 \otimes \varrho_g,$$

where $\varrho_g$ is the ground state of the radial modes of the ions. As a signature for the transmission of quantum information, we will consider the evolution of the entanglement between the internal and motional degrees of freedom of ions 2–3 (initially entangled) and ions 1–4 (which get entangled through the process), which is reported in figure 6 in terms of logarithmic negativity.

14 The effect of losses and thermal phonons for small $\epsilon$ (so that $\epsilon t \ll 1$ over the interesting time scales) can be easily estimated by letting the master equation, equation (4), for $N \gg 1$ act at first order on an otherwise perfectly transferred generic state $a|0\rangle + \beta|1\rangle$, and by setting $\epsilon = \gamma N$. The fidelity between initial and final states (corresponding to the overlap, for pure initial state) can be then determined and averaged over the Haar measure of a single qubit Hilbert space, to obtain the mean value $\mathcal{F} \simeq 1 - 3\epsilon t/2$. New Journal of Physics 11 (2009) 023007 (http://www.njp.org/)
Entanglement—Internal degrees of freedom ions 2,3

\[ E_{N_2,3} \] (ebits)

Entanglement—Internal degrees of freedom ions 1,4

\[ E_{N_1,4} \] (ebits)

Entanglement—Motional degrees of freedom ions 2,3

\[ E_{N_2,3} \] (ebits)

Entanglement—Motional degrees of freedom ions 1,4

\[ E_{N_1,4} \] (ebits)

**Figure 6.** Propagation of the entanglement (logarithmic negativity) between internal and motional degrees of freedom. The initial state is a Bell singlet between the internal degrees of freedom of the second and the third ions; the transverse trapping frequency is \( 10\omega_L \). The upper (lower) plot on the left column shows the logarithmic negativity between the internal (motional) degrees of freedom of ions 2 and 3. The upper (lower) plot on the right column shows the logarithmic negativity between the internal (motional) degrees of freedom of ions 1 and 4. The dynamics is divided in to three stages. In the first stage (for \( 0 \leq t \leq 1.22\pi/\omega_L \)), the coupling between internal and motional degrees of freedom of ions 2 and 3 is on, and the entanglement is swapped the former to the latter. In the next stage (for \( 1.22\pi/\omega_L \leq t \leq 6.22\pi/\omega_L \)), all the internal and motional degrees of freedom couplings are off, and the entanglement propagates through the chain via the motional degrees of freedom. Finally, for \( t \geq 6.22\pi/\omega_L \), the coupling between internal and motional degrees of freedom of ions 1 and 4 is on, and the final swapping takes place.

The dynamics is then split into three stages: from time \( t = 0 \) to \( 1.2 \times 2\pi/\omega_L \), the internal and motional degrees of freedom of ions 2 and 3 are coupled and the entanglement is dynamically swapped from the former to the latter. Next, from \( t = 1.2 \times 2\pi/\omega_L \) to \( 6.2 \times 2\pi/\omega_L \), the coupling is switched off and the entanglement propagates through the chain via the motional degrees of freedom, and the radial modes of ions 1 and 4 get entangled. Finally, from \( t = 6.2 \times 2\pi/\omega_L \) to \( 10 \times 2\pi/\omega_L \), the entanglement is swapped to the internal degrees of freedom of ions 1 and 4. One can see that, though the quantum information partially disperses under such conditions (essentially due to the stringent restrictions we put, in this instance, on the control of the local frequencies), the process is capable of transferring the entanglement from the internal degrees of freedom of ions 2 and 3 to those of ions 1 and 4 through the radial motional degrees of freedom.
5.3. Propagation of CV states and entanglement

Clearly, quantum information (and entanglement) can be propagated through the chains under examination also at a CV level, when populating the whole infinite-dimensional Hilbert space. The study of information and entanglement propagation over harmonic chains has been proposed and discussed in detail in [41, 43]. Here, we shall investigate cases of such propagations along the chains of ions we are considering, addressing specific instances which can be realized in the laboratory. Our aim is pointing out at what degree and under what conditions can the theoretical schemes based on harmonic chains be implemented on radial modes of trapped ions.

As argued in [41], the capacity of transmitting quantum states between distant parties is closely related to the capacity of transmitting, or ‘swapping’, entanglement between them. In fact, the latter also critically requires the transfer to happen coherently through the whole process. In view of this fact, we will limit ourselves to consider the propagation of CV entanglement (rather than the fidelity between sent and received quantum states). Also, in order to focus on a feasible scenario, we shall consider a common initial condition: the chain of \( n \) ions starts off from the (completely separable) ground state for \( \omega_j = \omega_0 \forall j \) and, then, entanglement between the first and the second ion is created, as described in section 4, by switching their frequencies to the same frequency \( \omega_e \). From now on the first ion is left off-resonance at frequency \( \omega_e \) (and thus effectively isolated from the chain), and the aim is transmitting its initial entanglement with the second ion through the chain over to the final \( n \)th ion. Two options will be examined:

(i) the entanglement is swapped by a chain of beam splitters between neighbouring ions (achieved by setting such neighbouring ions at the original frequency while putting the others off-resonance);

(ii) the entanglement is swapped directly by a beam splitter between the ions at the far ends of the chain.

Notice that in case (i) the beam splitters operate at the original frequency \( \omega_0 \) so that no local squeezing takes place, as evident from equation (5). Local squeezing would affect the entanglement between the modes but also, crucially, alter the original state to be transmitted and is thus undesirable in the present context.

The two strategies (i) and (ii) are compared in figure 7 for a chain of four ions. Such a small system already permits one to highlight all the essential features of the two strategies. As apparent, a complete dynamical swapping of the entanglement can be achieved in both cases, with remarkable accuracy. A chain of beam splitters allows for a faster transfer, as one should expect since the coupling \( \kappa_{jk} \) between distant ions is inversely proportional to their cubed distance: in general, the time needed to send information across \( n \) ions by a ‘relay’ of beam splitters roughly scales like \( n \), whereas the time needed to achieve a beam-splitting operation between first and last ion scales like \( n^3 \). A ‘relay’, taking less time, is thus less sensitive to decoherence and dissipation. On the other hand, a single beam-splitting operation between distant ions only requires a single adjustment of the trapping frequencies (after the initial entanglement is created), whereas a chain of beam splitters requires \( n - 2 \) manipulations, which would be possibly difficult to master in practice and could involve more errors and imperfections. Therefore, we have investigated the effect of imperfections in frequencies and evolution times on the transfer for both cases: the series of beam splitters turns out to be more
Figure 7. Transfer of the entanglement with the first ion of a chain (measured by ebits of logarithmic negativity $E_{12}^N$) from the second ion to the last one, in a chain of four ions. From top to bottom, the plots show the entanglement between ions 1 and 2 ($E_{12}^N$), 1 and 3 ($E_{13}^N$) and 1 and 4 ($E_{14}^N$). On the left side (a), the entanglement is transferred by a relay of beam splitters obtained by switching the radial trapping frequencies of ions 2 and 3 to $\sqrt{98.8} \omega_L$ for a time $46 \omega_L^{-1}$ while keeping the other ions off-resonant and then, finally, by switching the trapping frequencies of ions 3 and 4 to $\sqrt{98.8} \omega_L$ (with the other ions off-resonant). On the right side (b), the entanglement is transferred directly by a beam splitter between ions 2 and 4, obtained by setting their frequencies to $\sqrt{98.8} \omega_L$ while keeping the others off resonance. In both cases, the state at time $t = 0$ is the ground state for all trapping frequencies equal to $\sqrt{98.8} \omega_L$ (such a value takes into account the corrections due to Coulomb repulsions) and, during the initial 2 $\mu$s, the entanglement between ions 1 and 2 is built up by bringing their ‘bare’ trapping frequencies down to 2 $\mu$s.

Non-locality tests

The presence of strong Gaussian multipartite entanglement highlighted in section 4 can be experimentally demonstrated and put to use in testing quantum non-locality. Central to this
endeavour is the capability of performing non-Gaussian measurements on the motional state of the ions, pointed out in section 3.2. In particular, we already recalled that phonon-number parity measures are possible on single copies of the system, and so are displacement operations. In this section, we will explore the possibility of violating multipartite Bell-like inequalities (the so-called ‘Klyshko’ inequalities [47]) by measuring ‘displaced parity’ observables, as proposed for generic CV systems by Banaszek and Wodkiewicz [48]. Although bound to be subject to the locality loophole (considering that the distances between the ions are typically of the order of 1 µm), such an experiment would be a remarkable test of quantum non-locality with massive particles, which is still lacking.

To fix ideas and address a situation within the reach of current experiments, we shall study the test on the three-mode Gaussian state whose entanglement is described in figure 3, setting the evolution time to \( t = 5 \mu s \) and the heating rate to \( \epsilon = 200 \) Hz (resulting from a coupling to the bath \( \gamma = 10^{-4} \) Hz and a temperature \( T = 294 \) K), whose CM will be denoted by \( \sigma_3 \). The family of (non-Gaussian) local, bounded, dichotomic observables for the displaced parity test is given by \( \Pi_j(x_j, p_j) \equiv \hat{D}_j(x_j, p_j) (\Pi_1 - 1) n_j \hat{D}_j(x_j, p_j) \), where \( \hat{D}_j \) and \( n_j \) are, respectively, the displacement and number of phonon operators of ion \( j \). The three observers, pertaining to the three ions, randomly apply two different displacements \([\hat{D}_j(x_j, p_j) \text{ and } \hat{D}_j(x'_j, p'_j)]\) on their ions and then measure parity locally. Such a measurement is clearly non-Gaussian, and allows one to violate Bell inequalities with the Gaussian states. The expectation value of the operator \( \Pi(R) \equiv \Pi_1(x_1, p_1) \otimes \Pi_2(x_2, p_2) \otimes \Pi_3(x_3, p_3) \) is simply proportional to the Wigner function \( W(R) \) of the composite system evaluated in the point \( R = (x_1, x_2, x_3, p_1, p_2, p_3)^T \): \( \langle \Pi(R) \rangle = (2/\pi)^3 W(R) \) [49]. For a three-mode Gaussian state with CM \( \sigma_3 \) one has

\[
W(R) = e^{-(1/2)R^T \sigma_3^{-1} R} / (\pi^3 \sqrt{\text{Det}|\sigma_3|}).
\]

The Bell–Klyshko inequality finally reads

\[
\begin{align*}
B_3 &\equiv 8 \left| W(x_1, x_2, x'_3, p_1, p_2, p'_3) + W(x_1, x'_2, x_3, p_1, p'_2, p_3) \\
&+ W(x'_1, x_2, x_3, p'_1, p_2, p_3) - W(x'_1, x'_2, x'_3, p'_1, p'_2, p'_3) \right| \leq 2.
\end{align*}
\]

Quantum mechanics allows for \( 2 \leq B_3 \leq 4 \).

Figure 8 shows a region in the space of displacements where the violation of the inequality is substantial and remarkably stable. It is apparent that, for such choices of displacements, the tolerable error on the displacement operation needed to maintain the violation is around 1.5 nm, which is within reach of current experimental capabilities. Also, as shown in the plot on the right, heating rates of 200 Hz are still compatible with a violation of the inequalities.

This preliminary study reveals very promising perspectives concerning the violation of Bell inequalities with massive degrees of freedom. Even if subject to a locality loophole, this endeavour would still stand out as a major, not yet probed, testing ground for fundamental quantum mechanics [33], and epitomises the considerable promise radial modes hold for quantum information and fundamental investigations alike.

7. Summary and outlook

We demonstrated how the local control of the trapping frequencies would allow one to reproduce any linear optical manipulation on radial modes of trapped ions. Drawing from previous studies on similar settings, we pointed out that phonon detection and homodyne detection as well as the
Figure 8. Function $B_3$ for $p_1 = p_2 = p_3 = p'_1 = p'_2 = p'_3 = 0$, $x'_1 = x'_3 = 6\, \text{nm}$, $x'_2 = -4\, \text{nm}$, $-7\,\text{nm} \leq x_1 = x_3 \leq 3\,\text{nm}$ and $-3\,\text{nm} \leq x_2 \leq 7\,\text{nm}$ for the three-mode Gaussian state obtained as detailed in the caption of figure 3 after a time $t = 5\,\mu s$. On the left-hand side no dissipation is accounted for, whereas on the right-hand side a heating rate $\epsilon \simeq 200\,\text{Hz}$ is introduced. Dimensions were reintroduced assuming $\omega_L = 1\,\text{MHz}$ and Ca ions. The plane $B_3 = 2$, above which violation occurs, is the bottom of the plots; in the displayed region, the functions reach maxima of $\approx 2.45$ for the pure state case and of $\approx 2.28$ with dissipation.

implementation of non-Gaussian operations is possible in this setting. Next, we emphasized that, even restricting to global control, such manipulations enjoy a high efficiency in entanglement generation and a considerable resilience in the face of currently achievable dissipation rates. We then manifested that radial modes could be used for the transmission of quantum information, stored in finite-dimensional subspaces of the bosonic Hilbert space as well as in the full CV domain. Finally, we showed that, through (achievable) measurements of the displaced parity operator, Bell-like inequalities can be violated under realistic decoherence and dissipation rates. As a rule of thumb, applying to systems with $n \simeq 6$ ions or less and longitudinal trapping frequencies $\omega_L \simeq 1\,\text{MHz}$, our study indicates that heating rates around 1 kHz are sufficiently low for coherent manipulations and robust entanglement generation in the CV regime, whereas non-locality tests and transmission of finite-dimensional quantum dimensional information (stored in single phonons) are more delicate, requiring heating rates around 100 Hz to be carried out efficiently.

In light of the above, the experimental pursuit of the presented programme holds considerable promise, concerning both technological developments, such as the storage and manipulation of quantum information or the efficient generation of entanglement, and tests of fundamental physical aspects, as in the non-locality test for massive degrees of freedom here discussed. As a first step to push this analysis further, a study is currently under way to examine the possibility of swapping the remarkable entanglement produced in the traps to light modes, to ultimately beat parametric processes in the generation of quantum optical CV entanglement [18].
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