ALGEBRAIC AND REGULOUS GEOMETRY
OVER HENSELIAN RANK ONE VALUED FIELDS

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ABSTRACT. This paper develops algebraic geometry over Henselian rank one valued fields \( K \). Several results are given including: the canonical projection \( K^n \times K\mathbb{P}^m \to K^n \) and blow-ups of the \( K \)-points of smooth \( K \)-varieties are definably closed maps; a descent property for blow-ups; a version of the Łojasiewicz inequality for continuous rational functions, curve selection for semialgebraic sets and the theorem on extending continuous hereditarily rational functions, established for the real and \( p \)-adic varieties in our joint paper with J. Kollár. The descent property enables application of desingularization and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. We treat the subject in two different ways according as the valuation is discrete or not. In both cases, we rely on quantifier elimination due to Pas. In the latter, however, we first introduce a certain concept of fiber shrinking for definable sets, which is a relaxed version of curve selection. The last three sections are devoted to the theory of regulous functions and sets over such valued fields. Regulous geometry over the real ground field \( \mathbb{R} \) was developed by Fichou–Huisman–Mangolte–Monnier. The main results here are regulous versions of Nullstellensatz and Cartan’s Theorems A and B.

1. Introduction

In this paper, we deal with Henselian rank one valued fields \( K \) of equicharacteristic zero (with valuation \( v \) of rank one, valuation ring \( R \) and residue field \( k \) of characteristic zero). The \( K \)-points \( X(K) \) of any \( K \)-algebraic variety \( X \) inherit from \( K \) a topology, called the \( K \)-topology. We wish to establish several results concerning algebraic
geometry over such a ground field $K$. Let $L$ be the 3-sorted language of Denef–Pas. First, our purpose is to prove that the projection $K^n \times \mathbb{P}_K^m \to K^n$ is an $L$-definably closed map (the closedness theorem). Further, we shall draw several conclusions, including the theorem that blow-ups of the $K$-points of smooth $K$-varieties are definably closed maps, a descent property for such blow-ups, a version of the Łojasiewicz inequality (Theorem 8.1) for continuous rational functions, curve selection for semialgebraic sets (Theorem 8.3), and the theorem on extending continuous hereditarily rational functions (Theorem 9.2), established for the real and $p$-adic varieties in our joint paper [24] with J. Kollár. The descent property enables application of desingularization and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. We shall treat the subject in two different ways according as the valuation is discrete or not. In both cases, we rely on quantifier elimination due to Pas. In the latter, however, we first introduce a certain concept of fiber shrinking for definable sets, which is a relaxed version of curve selection. Let us mention that this paper comprises our two earlier preprints [35, 36].

The organization of the paper is as follows. In Section 2, we set up notation and terminology including, in particular, the language $L$ of Denef–Pas and the concept of a cell. We recall the theorems on quantifier elimination and on preparation cell decomposition, due to Pas [37]. Finally, when the valuation $v$ is not discrete, we prove the theorem on cell decomposition for $L$-definable sets (Theorem 2.4). In Section 3, we state the closedness theorem and several immediate corollaries, including the descent property. The proof of this theorem, in the case where the ground field $K$ is discretely valued, is given in Section 4.

In Section 5, we study $L$-definable functions of one variable. A result which will play an important role in the sequel is the theorem on existence of the limit (Theorem 5.3). Its proof makes use of Puiseux’s theorem for the local ring of convergent power series.

The next two sections treat the case where the ground field $K$ is not discretely valued. In Section 6, we introduce a certain concept of fiber shrinking for $L$-definable sets (Theorem 6.1). Section 7 contains the proof of the closedness theorem (Theorem 3.1), which makes use of fiber shrinking, cell decomposition (Theorem 2.4) and existence of the limit (Theorem 5.3).

In the subsequent two sections, some further conclusions from the closedness theorems are drawn. Section 8 is devoted to a version of the Łojasiewicz inequality for continuous rational functions and to curve
selection for semialgebraic sets. In Section 9, the theorem on extending continuous hereditarily rational functions, established for the real and $p$-adic varieties in [24], is carried over to the case where the ground field $K$ is a Henselian rank one valued field of equicharacteristic zero. Finally, let us mention that, in real algebraic geometry, applications of hereditarily rational functions and the extension theorem, in particular, are given in papers [25, 26, 27] and [28], which discuss rational maps into spheres and stratified-algebraic vector bundles on real algebraic varieties.

The last three sections are devoted to the theory of regulous functions and sets over Henselian rank one valued fields of equicharacteristic zero. Regulous geometry over the real ground field $\mathbb{R}$ was developed by Fichou–Huisman–Mangolte–Monnier [15]. In Section 10 we set up notation and terminology as well as provide basic results about regulous functions and sets, including the noetherianity of the constructible and regulous topologies. Those results are valid over arbitrary fields with the density property. The next section establishes a regulous version of Nullstellensatz (Theorem 11.4), valid over Henselian rank one valued fields of equicharacteristic zero. The proof relies on a version of the Lojasiewicz inequality (Theorem 8.1). Also drawn are several conclusions, including the existence of a one-to-one correspondence between the radical ideals of the ring of regulous functions and the closed regulous subsets, or one-to-one correspondences between the prime ideals of that ring, the irreducible regulous subsets and the irreducible Zariski closed subsets (Corollaries 11.5 and 11.10). Section 12 provides an exposition of the theory of quasi-coherent regulous sheaves, which generally follows the approach given by Fichou–Huisman–Mangolte–Monnier [15]. It is based on the equivalence of categories between the category of $\mathcal{R}^k$-modules and the category of $\mathcal{R}^k$-modules which, in turn, is a consequence of the one-to-one correspondences mentioned above. The main results are here the regulous versions of Cartan’s Theorems A and B. Finally, we establish a criterion for a continuous function on an affine regulous subvariety to be regulous (Theorem 9.2), which relies on our theorem on extending continuous hereditarily rational functions (Theorem 12.10).

We conclude with the following comment. The metric topology of a non-archimedean field $K$ with a rank one valuation $v$ is totally disconnected. Rigid analytic geometry (see e.g. [6] for its comprehensive foundations), developed by Tate, compensates for this defect by introducing sheaves of analytic functions in a Grothendieck topology.
Another approach is due to Berkovich [3], who filled in the gaps between the points of $K^n$, producing a locally compact Hausdorff space (the so-called analytification of $K^n$), which contains the metric space $K^n$ as a dense subspace if the ground field $K$ is algebraically closed. His construction consists in replacing each point $x$ of a given $K$-variety with the space of all rank one valuations on the residue field $\kappa(x)$ that extend $v$. The theory of stably dominated types, developed in paper [21], deals with non-archimedean fields with valuation of arbitrary rank and generalizes that of tame topology for Berkovich spaces. Currently, various analytic structures over Henselian rank one valued fields are intensively investigated (see e.g. [12] for more information).

2. QUANTIFIER ELIMINATION AND CELL DECOMPOSITION

We begin with quantifier elimination due to Pas in the language $L$ of Denef–Pas with three sorts: the valued field $K$-sort, the value group $\Gamma$-sort and the residue field $k$-sort. The language of the $K$-sort is the language of rings; that of the $\Gamma$-sort is any augmentation of the language of ordered abelian groups (and $\infty$); finally, that of the $k$-sort is any augmentation of the language of rings. We denote $K$-sort variables by $x, y, z, \ldots$, $k$-sort variables by $\xi, \zeta, \eta, \ldots$, and $\Gamma$-sort variables by $k, q, r, \ldots$.

In the case of non-algebraically closed fields, passing to the three sorts with additional two maps: the valuation $v$ and the residue map, is not sufficient. Quantifier elimination due to Pas holds for Henselian valued fields of equicharacteristic zero in the above 3-sorted language with additional two maps: the valuation map $v$ from the field sort to the value group, and a map $ac$ from the field sort to the residue field (angular component map) which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring $R$ of $K$. Note that the only symbols of $L$ connecting the sorts are functions from the main $K$-sort to the auxiliary $\Gamma$-sort and $k$-sort.

Not all valued fields have an angular component map, but it exists whenever the valued field has a cross section or the residue field is $\aleph_1$-saturated (cf. [38]). In general, unlike for $p$-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. For both $p$-adic fields (Denef [13]) and Henselian equicharacteristic zero valued fields (Pas [37]), quantifier elimination was established by means of cell decomposition and a certain preparation theorem (for polynomials in one variable with definable coefficients) combined with each other. In the latter case, however, cells are no longer finite in number, but parametrized by residue field variables.
Finally, let us mention that quantifier elimination based on the sort $RV := K^*/(1+m)$ (where $K^* := K \setminus \{0\}$ and $m$ is the maximal ideal of the valuation ring $R$) was introduced by Besarab [4]. This new sort binds together the value group and residue field into one structure. In paper [20, Section 12], quantifier elimination for Henselian valued fields of equicharacteristic zero, based on this sort, was derived directly from that by Robinson for algebraically closed valued fields. Yet another, more general result, including Henselian valued fields of mixed characteristic, was achieved by Cluckers–Loeser [11] for so-called $b$-minimal structures (from ”ball minimal”); in the case of valued fields, however, countably many sorts $RV_n := K^*/(1+n^m)$, $n \in \mathbb{N}$, are needed.

Below we state the theorem on quantifier elimination due to Pas [37, Theorem 4.1].

**Theorem 2.1.** Let $(K, \Gamma, \kappa)$ be a structure for the 3-sorted language $\mathcal{L}$ of Denef–Pas. Assume that the valued field $K$ is Henselian and of equicharacteristic zero. Then $(K, \Gamma, \kappa)$ admits elimination of $K$-quantifiers in the language $\mathcal{L}$. \hfill $\square$

We immediately obtain the following

**Corollary 2.2.** The 3-sorted structure $(K, \Gamma, \kappa)$ admits full elimination of quantifiers whenever the theories of the value group $\Gamma$ and the residue field $\kappa$ admit quantifier elimination in the languages of their sorts. \hfill $\square$

We now recall the necessary notation and terminology concerning cell decomposition. Consider an $\mathcal{L}$-definable subset $D$ of $K^n \times \kappa^m$, three $\mathcal{L}$-definable functions

$$a(x, \xi), b(x, \xi), c(x, \xi) : D \rightarrow K.$$ 

and a positive integer $\nu$. For each $\xi \in \kappa^m$ set

$$C(\xi) := \{(x, y) \in K^n_x \times K_y : (x, \xi) \in D, \quad v(a(x, \xi)) <_1 v((y - c(x, \xi))^\nu) <_2 v(b(x, \xi)), \quad \overline{\nu c(y - c(x, \xi))} = \xi_1\},$$

where $\nu$ is a positive integer and $<_1, <_2$ stand for $<, \leq$ or no condition in any occurrence. If the sets $C(\xi), \xi \in \kappa^m$, are pairwise disjoint, the union

$$C := \bigcup_{\xi \in \kappa^m} C(\xi)$$

is called a cell in $K^n \times K$ with parameters $\xi$ and center $c(x, \xi)$; $C(\xi)$ is called a fiber of the cell $C$. 
Theorem 2.3. (Preparation Cell Decomposition, [37, Theorem 3.2])

Let
\[ f_1(x,y), \ldots, f_r(x,y) \]
be polynomials in one variable \( y \) with coefficients being \( \mathcal{L} \)-definable functions on \( K^n_x \). Then \( K^n \times K \) admits a finite partition into cells such that for each cell \( A \) with parameters \( \xi \) and center \( c(x,\xi) \) and for all \( i = 1, \ldots, r \) we have:
\[
v(f_i(x,y)) = v\left( \tilde{f}_i(x,\xi)(y - c(x,\xi))^{\nu_i} \right),
\]
\[
\tilde{ac} f_i(x,y) = \xi^{\mu(i)},
\]
where \( \tilde{f}_i(x,\xi) \) are \( \mathcal{L} \)-definable functions, \( \nu_i \in \mathbb{N} \) for all \( i = 1, \ldots, r \), and the map \( \mu : \{1, \ldots, r\} \rightarrow \{1, \ldots, m\} \) does not depend on \( x, y, \xi \). We then say that the functions \( f_1(x,y), \ldots, f_r(x,y) \) are prepared with respect to the variable \( y \).

Unless otherwise stated, we shall assume in the sequel that the ground field \( K \) is a Henselian rank one valued field of equicharacteristic zero. Under the additional condition that the ground field \( K \) is not discretely valued, the following result can be deduced.

Theorem 2.4. (Cell decomposition) If, in addition, the field \( K \) is not discretely valued, then every \( \mathcal{L} \)-definable subset \( B \) of \( K^n \times K \) is a finite union of cells.

Proof. By elimination of \( K \)-quantifiers, \( B \) is a finite union of sets defined by conditions of the form
\[
v(f_1(x,y), \ldots, f_r(x,y)) \in P,
\]
\[
(\tilde{ac} g_1(x,y), \ldots, \tilde{ac} g_s(x,y)) \in Q,
\]
where \( f_i, g_j \in K[x,y] \) are polynomials, and \( P \) and \( Q \) are \( \mathcal{L} \)-definable subsets of \( \Gamma^r \) and \( \mathbb{K}^s \), respectively (since \( a = 0 \) iff \( \tilde{ac} a = 0 \)). Thus we may assume that \( B \) is such a set.

The sort \( \Gamma \) admits quantifier elimination in the language of ordered groups, because it is an ordered subgroup of the ordered additive group \( (\mathbb{R}, +, <) \) without minimal positive element (see e.g. [10]). Therefore we can assume that the set \( P \) is defined by finitely many inequalities of the form
\[
\{ \alpha \in \Gamma^r : k_1\alpha_1 + \cdots + k_r\alpha_r + \beta \lessgtr 0 \},
\]
where \( k_1, \ldots, k_r \in \mathbb{Z} \), \( \beta \in \Gamma \) and \( \lessgtr \) stands for \( < \) or \( = \).

But there exists a finite partition of \( K^n \times K \) into cells, which prepares the polynomials
\[ f_1(x,y), \ldots, f_r(x,y) \quad \text{and} \quad g_1(x,y), \ldots, g_s(x,y). \]
On each cell \( C \) of this partition (of the form considered before), we thus have

\[
v(f_i(x, y)) = v\left( \tilde{f}_i(x, \xi)(y - c(x, \xi))^{\nu_i} \right),
\]

\[
\text{ac} f_i(x, y) = \xi_{\mu(i)},
\]

and

\[
v(g_i(x, y)) = v\left( \tilde{g}_i(x, \xi)(y - c(x, \xi))^{\theta_i} \right),
\]

\[
\text{ac} g_i(x, y) = \xi_{\eta(i)}.
\]

Hence the intersection \( B \cap C(\xi) \) is defined by finitely many conditions of the form

\[
v \left( \prod_{i=1}^{r} \tilde{f}_i(x, \xi)^{k_i}(y - c(x, \xi))^{k_i \nu_i} \right) + \beta < 0
\]

and

\[
(\xi_{\eta(1)}, \ldots, \xi_{\eta(s)}) \in Q.
\]

After renumbering the integers \( k_i \), we may assume that \( k_1, \ldots, k_p, \) with \( p \leq r \), are non-negative integers and \( k_{p+1}, \ldots, k_r \) are negative integers. Take \( b \in K \) such that \( v(b) = \beta \). Then the conditions of the first form are equivalent to

\[
v \left( b \prod_{i=1}^{p} \tilde{f}_i(x, \xi)^{k_i}(y - c(x, \xi))^{k_i \nu_i} \right) < v \left( \prod_{i=p+1}^{r} \tilde{f}_i(x, \xi)^{-k_i}(y - c(x, \xi))^{-k_i \nu_i} \right).
\]

We can therefore partition the cell \( C \) into a finite number of finer cells with the same center \( c(x, \xi) \), so that the intersection \( B \cap C(\xi) \) is the union of some of them. This finishes the proof. \( \square \)

**Remark 2.5.** A more careful analysis of cells in the above proof leads to the sharpening of Theorem 2.4 stated below:

*Every \( \mathcal{L} \)-definable subset \( B \) of \( K^n \times K \) can be partitioned into a finite union of cells.*

In Section 4, we shall apply Theorem 2.4 to establish fiber shrinking over Henselian, rank one, not discretely valued fields (Theorem 6.1).

### 3. Closedness theorem

One of the most important results of this paper is the following

**Theorem 3.1.** (Closedness theorem) Let \( D \) be an \( \mathcal{L} \)-definable subset of \( K^n \). Then the canonical projection

\[
\pi : D \times R^m \longrightarrow D
\]

is definably closed in the \( K \)-topology, i.e. if \( B \subset D \times R^m \) is an \( \mathcal{L} \)-definable closed subset, so is its image \( \pi(B) \subset D \).
We shall provide two different proofs for this theorem according as the ground field \( K \) is discretely valued or not. While the former case is considered in the next section, the latter in Section 7. When the ground field \( K \) is locally compact, the theorem holds by a routine topological argument.

We immediately obtain four corollaries stated below.

**Corollary 3.2.** Let \( K\mathbb{P}^m \) be the projective space of dimension \( m \) over \( K \) and \( D \) an \( \mathcal{L} \)-definable subset of \( K^n \). Then the canonical projection
\[
\pi : D \times K\mathbb{P}^m \to D
\]
is definably closed. \( \square \)

**Corollary 3.3.** Let \( \phi_i, i = 0, \ldots, m \), be regular functions on \( K^n \), \( D \) be an \( \mathcal{L} \)-definable subset of \( K^n \) and \( \sigma : Y \to K\mathbb{A}^n \) the blow-up of the affine space \( K\mathbb{A}^n \) with respect to the ideal \((\phi_0, \ldots, \phi_m)\). Then the restriction
\[
\sigma : Y(K) \cap \sigma^{-1}(D) \to D
\]
is a definably closed quotient map.

*Proof.* Indeed, \( Y(K) \) can be regarded as a closed subvariety of \( K^n \times K\mathbb{P}^m \) and \( \sigma \) the canonical projection. \( \square \)

Since the problem is local with respect to the target space, the above corollary immediately generalizes to the case where the \( K \)-variety \( Y \) is the blow-up of a smooth \( K \)-variety \( X \).

**Corollary 3.4.** Let \( X \) be a smooth \( K \)-variety, \( \phi_i, i = 0, \ldots, m \), regular functions on \( X \), \( D \) be an \( \mathcal{L} \)-definable subset of \( X(K) \) and \( \sigma : Y \to X \) the blow-up of the ideal \((\phi_0, \ldots, \phi_m)\). Then the restriction
\[
\sigma : Y(K) \cap \sigma^{-1}(D) \to D
\]
is a definably closed quotient map. \( \square \)

**Corollary 3.5.** (Descent property) Under the assumptions of Corollary 3.4, every continuous \( \mathcal{L} \)-definable function \( g : \sigma^{-1}(D) \to K \) that is constant on the fibers of the blow-up \( \sigma \) descends to a (unique) continuous \( \mathcal{L} \)-definable function \( f : D \to K \). \( \square \)

4. **Proof of Theorem 3.1 when the valuation is discrete**

Through the transfer principle of Ax–Kochen–Ershov (see e.g. [9]), it suffices to prove Theorem 3.1 for the case where the ground field \( K \) is a complete, discretely valued field of equicharacteristic zero. Such fields are, by virtue of Cohen’s structure theorem, the quotient fields \( K = \mathbb{k}(t) \) of formal power series rings \( \mathbb{k}[[t]] \) in one variable \( t \) with
coefficients from a field $k$ of characteristic zero. The valuation $v$ and the angular component $\overline{ac}$ of a formal power series are the degree and the coefficient of its initial monomial, respectively.

The additive group $\mathbb{Z}$ is an example of ordered $\mathbb{Z}$-group, i.e. an ordered abelian group with a (unique) smallest positive element (denoted by 1) subject to the following additional axioms:

$$\forall k \; k > 0 \Rightarrow k \geq 1$$

and

$$\forall k \; \exists q \; \bigvee_{r=0}^{n-1} k = nq + r$$

for all integers $n > 1$. The language of the value group sort will be the Presburger language of ordered $\mathbb{Z}$-groups, i.e. the language of ordered groups $\{<,+,-,0\}$ augmented by 1 and binary relation symbols $\equiv_n$ for congruence modulo $n$ subject to the axioms:

$$\forall k,r \; k \equiv_n r \iff \exists q \; k - r = nq$$

for all integers $n > 1$. This theory of ordered $\mathbb{Z}$-groups in the Presburger language has quantifier elimination and definable Skolem (choice) functions. The above two countable axiom schemas can be replaced by universal ones when we augment the language by adding the function symbols $\left[\frac{k}{n}\right]$ (of one variable $k$) for division with remainder, which fulfil the following postulates:

$$\left[\frac{k}{n}\right] = q \iff \bigvee_{r=0}^{n-1} k = nq + r$$

for all integers $n > 1$. The theory of ordered $\mathbb{Z}$-groups admits therefore both quantifier elimination and universal axioms in the Presburger language augmented by division with remainder. Thus every definable function is piecewise given by finitely many terms and, moreover, is piecewise linear (cf. [10]).

In the residue field sort, we can add new relation symbols for all definable sets and impose suitable postulates. This enables quantifier elimination for the residue field in the augmented language. In this fashion, we have full quantifier elimination in the 3-sorted structure $(K,\mathbb{Z},k)$ with $K = k((t))$.

Now we can readily pass to the proof of Theorem 3.1 which, of course, reduces easily to the case $m = 1$. So let $B$ be an $\mathcal{L}$-definable closed (in the $K$-topology) subset of $D \times R_y \subset K^n_x \times R_y$. It suffices to prove
that if \( a \) lies in the closure of the projection \( A := \pi(B) \), then there is a point \( b \in B \) such that \( \pi(b) = a \).

Without loss of generality, we may assume that \( a = 0 \). Put

\[ \Lambda := \{(v(x_1), \ldots, v(x_n)) \in \mathbb{Z}^n : x = (x_1, \ldots, x_n) \in A\} \]

The set \( \Lambda \) contains points all coordinates of which are arbitrarily large, because the point \( a = 0 \) lies in the closure of \( A \). Hence and by definable choice, \( \Lambda \) contains a set \( \Lambda_0 \) of the form

\[ \Lambda_0 = \{(k, \alpha_2(k), \ldots, \alpha_n(k)) \in \mathbb{N}^n : k \in \Delta\} \subset \Lambda, \]

where \( \Delta \subset \mathbb{N} \) is an unbounded definable subset and

\[ \alpha_2, \ldots, \alpha_n : \Delta \rightarrow \mathbb{N} \]

are increasing unbounded functions given by a term (because a function in one variable given by a term is either increasing or decreasing). We are going to recursively construct a point \( b = (0, w) \in B \) with \( w \in \mathbb{R} \) by performing the following algorithm.

**Step 1.** Let

\[ \Xi_1 := \{(v(x_1), \ldots, v(x_n), v(y)) \in \Lambda_0 \times \mathbb{N} : (x, y) \in B\}, \]

and

\[ \beta_1(k) := \sup \{l \in \mathbb{N} : (k, \alpha_2(k), \ldots, \alpha_n(k), l) \in \Xi_1\} \in \mathbb{N} \cup \{\infty\}, \quad k \in \Lambda_0. \]

If \( \limsup_{k \to \infty} \beta_1(k) = \infty \), there is a sequence \((x^{(\nu)}, y^{(\nu)}) \in B, \nu \in \mathbb{N}, \)

such that

\[ v(x_1^{(\nu)}), \ldots, v(x_n^{(\nu)}), v(y^{(\nu)}) \to \infty \]

when \( \nu \to \infty \). Since the set \( B \) is a closed subset of \( D \times R_y \), we get

\[ (x^{(\nu)}, y^{(\nu)}) \to 0 \in B \quad \text{when} \quad \nu \to \infty, \]

and thus \( w = 0 \) is the point we are looking for. Here the algorithm stops.

Otherwise

\[ \Lambda_1 \times \{l_1\} \subset \Xi_1 \]

for some infinite definable subset \( \Lambda_1 \) of \( \Lambda_0 \) and \( l_1 \in \mathbb{N} \). The set

\[ \{(v(x_1), \ldots, v(x_n); \overline{ac}(y)) \in \Lambda_1 \times \mathbb{R} : (x, y) \in B, \ v(y) = l_1\} \]

is definable in the language \( L \). By quantifier elimination, it is given by a quantifier-free formula with variables only from the value group \( \Gamma \)-sort and the residue field \( k \)-sort. Therefore there is a finite partitioning of \( \Lambda_1 \) into definable subsets over each of which the fibres of the above set are constant, because quantifier-free \( L \)-definable subsets of the product \( \mathbb{Z}^n \times k \) of the two sorts are finite unions of the Cartesian products of
definable subsets in $\mathbb{Z}^n$ and in $k$, respectively. One of those definable subsets, say $\Lambda_1'$, must be infinite. Consequently, for some $\xi_1 \in k$, the set
\[
\Xi_2 := \{(v(x_1), \ldots, v(x_n), v(y - \xi_1 t^{l_1})) \in \Lambda_1' \times \mathbb{N} : (x, y) \in B\}
\]
contains points of the form $(k, l) \in \mathbb{N}^{n+1}$, where $k \in \Lambda_1'$ and $l > l_1$.

Step 2. Let
\[
\beta_2(k) := \sup \{l \in \mathbb{N} : (k, \alpha_2(k), \ldots, \alpha_n(k), l) \in \Xi_2 \} \in \mathbb{N} \cup \{\infty\}, \quad k \in \Lambda_1'.
\]
If $\limsup_{k \to \infty} \beta_2(k) = \infty$, there is a sequence $(x^{(\nu)}, y^{(\nu)}) \in B$, $\nu \in \mathbb{N}$, such that
\[
v(x_1^{(\nu)}), \ldots, v(x_n^{(\nu)}), v(y^{(\nu)} - \xi_1 t^{l_1}) \to \infty
\]
when $\nu \to \infty$. Since the set $B$ is a closed subset of $D \times R_y$, we get
\[
(x^{(\nu)}, y^{(\nu)}) \to (0, \xi_1 t^{l_1}) \in B \quad \text{when} \quad \nu \to \infty,
\]
and thus $w = \xi_1 t^{l_1}$ is the point we are looking for. Here the algorithm stops.

Otherwise
\[
\Lambda_2 \times \{l_2\} \subset \Xi_3
\]
for some infinite definable subset $\Lambda_2$ of $\Lambda_1'$ and $l_2 > l_1$. Again, for some $\xi_2 \in k$, the set
\[
\Xi_3 := \{(v(x_1), \ldots, v(x_n), v(y - \xi_1 t^{l_1} - \xi_2 t^{l_2})) \in \Lambda_2' \times \mathbb{N} : (x, y) \in B\}
\]
contains points of the form $(k, l) \in \mathbb{N}^{n+1}$, where $k \in \Lambda_2'$, $\Lambda_2'$ is an infinite definable subset of $\Lambda_2$ and $l > l_2$.

Step 3 is carried out in the same way as the previous ones; and so on.

In this fashion, the algorithm either stops after a finite number of steps and then yields the desired point $w \in R$ (actually, $w \in k[t]$) such that $(0, w) \in B$, or it does not stop and then yields a formal power series
\[
w := \xi_1 t^{l_1} + \xi_2 t^{l_2} + \xi_3 t^{l_3} + \ldots, \quad 0 \leq l_1 < l_2 < l_3 < \ldots
\]
such that for each $\nu \in \mathbb{N}$ there exists an element $(x^{(\nu)}, y^{(\nu)}) \in B$ for which
\[
v(y^{(\nu)} - \xi_1 t^{l_1} - \xi_2 t^{l_2} - \ldots - \xi_\nu t^{l_\nu}) \geq l_\nu + 1 \geq \nu, \quad v(x_1^{(\nu)}), \ldots, v(x_n^{(\nu)}) \geq \nu.
\]
Hence $v(y^{(\nu)} - w) \geq \nu$, and thus the sequence $(x^{(\nu)}, y^{(\nu)})$ tends to the point $b := (0, w)$ when $\nu$ tends to $\infty$. Since the set $B$ is a closed subset of $D \times R$, the point $b$ belongs to $B$, which completes the proof. \hfill $\square$
5. Definable functions of one variable

In this section, $K$ is a Henselian rank one valued field of equicharacteristic zero. We begin with the following

**Proposition 5.1.** Let $f : A \to K$ be an $\mathcal{L}$-definable function on a subset $A$ of $K^n$. Then there is a finite partition of $A$ into $\mathcal{L}$-definable sets $A_i$ and irreducible polynomials $P_i(x, y), i = 1, \ldots, k$, such that for each $a \in A_i$ the polynomial $P_i(a, y)$ in $y$ does not vanish and

$$P_i(a, f(a)) = 0 \quad \text{for all } a \in A_i, \ i = 1, \ldots, k.$$

**Proof.** By elimination of $K$-quantifiers, as in the proof of Theorem 2.4, the graph of $f$ is a finite union of sets $B_i, i = 1, \ldots, k$, defined by conditions of the form

$$(v(f_1(x, y)), \ldots, v(f_r(x, y))) \in P, \quad (\overline{ac} g_1(x, y), \ldots, \overline{ac} g_s(x, y)) \in Q,$$

where $f_i, g_j \in K[x, y]$ are polynomials, and $P$ and $Q$ are $\mathcal{L}$-definable subsets of $\Gamma^r$ and $k^s$, respectively. Each set $B_i$ is the graph of the restriction of $f$ to an $\mathcal{L}$-definable subset $A_i$. Since, for each point $a \in A_i$, the fibre of $B_i$ over $a$ consists of one point, the above condition imposed on angular components includes one of the form $\overline{ac} g_j(x, y) = 0$ or, equivalently, $g_j(x, y) = 0$, for some $j = 1, \ldots, s$, which may depend on $a$, where the polynomial $g_j(a, y)$ in $y$ does not vanish. This means that the set

$$\{(\overline{ac} g_1(x, y), \ldots, \overline{ac} g_s(x, y)) : (x, y) \in B_i\}$$

is contained in the union of hyperplanes $\bigcup_{j=1}^{s} \{\xi_j = 0\}$ and, furthermore, that, for each point $a \in A_i$, there is an index $j = 1, \ldots, s$ such that the polynomial $g_j(a, y)$ in $y$ does not vanish and $g_j(a, f(a)) = 0$. Clearly, for any $j = 1, \ldots, s$, this property of points $a \in A_i$ is $\mathcal{L}$-definable. Therefore we can partition the set $A_i$ into subsets each of which fulfils the condition required in the conclusion with some irreducible factors of the polynomial $g_j(x, y)$. \hfill \Box

Consider a complete rank one valued field $L$. For every non-negative integer $r$, let $L\{x\}_r$ be the local ring of all formal power series

$$\phi(x) = \sum_{k=0}^{\infty} a_k x^k \in L[[x]]$$

in one variable $x$ such that $v(a_k) + kr$ tends to $\infty$ when $k \to \infty$; $L\{x\}_0$ coincides with the ring of restricted formal power series. Then the local
ring
\[ L \{ x \} := \bigcup_{r=0}^{\infty} L \{ x \}_r \]
is Henselian, which can be directly deduced by means of the implicit function theorem for restricted power series in several variables (see [7, Chap. III, § 4], [16] and also [17, Chap. I, § 5]).

Now, let \( L \) be the completion of the algebraic closure \( \overline{K} \) of the ground field \( K \). Clearly, the Henselian local ring \( L \{ x \} \) is closed under division by the coordinate and power substitution. Therefore it follows from our paper [34, Section 2] that Puiseux’s theorem holds for \( L \{ x \} \). We still need an auxiliary lemma.

**Lemma 5.2.** The field \( K \) is a closed subspace of its algebraic closure \( \overline{K} \).

**Proof.** Indeed, Denote by \( \text{cl} (E, F) \) the closure of a subset \( E \) in \( F \), and let \( \widehat{K} \) be the completion of \( K \). We have
\[ \text{cl} (K, \overline{K}) = \text{cl} (K, L) \cap \overline{K} = \widehat{K} \cap \overline{K}. \]
But, through the transfer principle of Ax-Kochen–Ershov (see e.g. [9]), \( K \) is an elementary substructure of \( \widehat{K} \) and, a fortiori, is algebraically closed in \( \widehat{K} \). Hence
\[ \text{cl} (K, \overline{K}) = \widehat{K} \cap \overline{K} = K, \]
as asserted. \( \square \)

Consider an irreducible polynomial
\[ P(x, y) = \sum_{i=0}^{d} p_i(x)y^i \in K[x, y] \]
in two variables \( x, y \) of \( y \)-degree \( d \geq 1 \). Let \( Z \) be the Zariski closure of its zero locus in \( \overline{K} \times \overline{K}^{\mathbb{P}^1} \). Performing a linear fractional transformation over the ground field \( K \) of the variable \( y \), we can assume that the fiber \( \{ w_1, \ldots, w_s \} \), \( s \leq d \), of \( Z \) over \( x = 0 \) does not contain the point at infinity, i.e. \( w_1, \ldots, w_s \in \overline{K} \). Then \( p_d(0) \neq 0 \) and \( p_d(x) \) is a unit in \( L \{ x \} \). Via Hensel’s lemma, we get the Hensel decomposition
\[ P(x, y) = p_d(x) \cdot \prod_{j=1}^{s} P_j(x, y) \]
of \( P(x, y) \) into polynomials
\[ P_j(x, y) = (y - w_j)^{d_j} + p_{j1}(x)(y - w_j)^{d_{j-1}} + \cdots + p_{jd_j}(x) \in L \{ x \} [y - w_j] \]
which are Weierstrass with respect to $y - w_j$, $j = 1, \ldots, s$, respectively. By Puiseux’s theorem, there is a neighbourhood $U$ of $0 \in \overline{K}$ such that the trace of $Z$ on $U \times \overline{K}$ is a finite union of sets of the form

$$Z_{\phi_j} = \{(x^q, \phi_j(x)) : x \in U\} \text{ with some } \phi_j \in L\{x\}, \; q \in \mathbb{N}, \; j = 1, \ldots, s.$$ 

Obviously, for $j = 1, \ldots, s$, the fiber of $Z_{\phi_j}$ over $x \in U$ tends to the point $\phi_j(0) = w_j$ when $x \to 0$.

If $\phi_j(0) \in \overline{K} \setminus K$, it follows from Lemma 5.2 that

$$Z_{\phi_j} \cap ((U \cap K) \times K) = \emptyset,$$

after perhaps shrinking the neighbourhood $U$.

Let us mention that if

$$\phi_j(0) \in K \quad \text{and} \quad \phi_j \in L\{x\} \setminus \hat{K}\{x\},$$

then

$$Z_{\phi_j} \cap ((U \cap K) \times K) = \{(0, \phi_j(0))\}$$

after perhaps shrinking the neighbourhood $U$. Indeed, let

$$\phi_j(x) = \sum_{k=0}^{\infty} a_k x^k \in L[[x]]$$

and $p$ be the smallest positive integer with $a_p \in L \setminus \hat{K}$. Since $\hat{K}$ is a closed subspace of $L$, we get

$$\sum_{k=p}^{\infty} a_k x^k = x^p \left( a_p + x \cdot \sum_{k=p+1}^{\infty} a_k x^{k-(p+1)} \right) \notin \hat{K}$$

for $x$ close enough to 0, and thus the assertion follows.

Suppose now that an $L$-definable function $f : A \to K$ satisfies the equation

$$P(x, f(x)) = 0 \quad \text{for} \quad x \in A$$

and 0 is an accumulation point of the set $A$. It follows immediately from the foregoing discussion that the set $A$ can be partitioned into a finite number of $L$-definable sets $A_j$, $j = 1, \ldots, r$ with $r \leq s$, such that, after perhaps renumbering of the fiber $\{w_1, \ldots, w_s\}$ of the set $\{P(x, f(x)) = 0\}$ over $x = 0$, we have

$$\lim_{x \to 0} f|A_j(x) = w_j \quad \text{for each} \quad j = 1, \ldots, r.$$ 

Hence and by Proposition 5.1, we immediately obtain the following
Theorem 5.3. (Existence of the limit) Let \( f : A \to K \) be an \( \mathcal{L} \)-definable function on a subset \( A \) of \( K \) and suppose 0 is an accumulation point of \( A \). Then there is a finite partition of \( A \) into \( \mathcal{L} \)-definable sets \( A_1, \ldots, A_r \) and points \( w_1, \ldots, w_r \in K \mathbb{P}^1 \) such that
\[
\lim_{x \to 0} f|_{A_j}(x) = w_j \quad \text{for} \ j = 1, \ldots, r.
\]
Moreover, there is a neighbourhood \( U \) of 0 such that the definable set
\[
\{(v(x), v(f(x))) : x \in (E \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}) \subset \mathbb{R} \times (\mathbb{R} \cup \{\infty\})
\]
is contained in a finite number of affine lines with non-negative rational slopes and \( \mathbb{R} \times \{\infty\} \). \(\blacksquare\)

Remark 5.4. In the above theorem, the existence of the limits could also be established through the lemma on the continuity of roots of a monic polynomial, which can be found in e.g. [6, Chap. 3, § 3]). The second conclusion, in turn, follows from Puiseux’s parametrization.

6. Fiber shrinking for definable sets

In this section, we additionally assume that the ground field \( K \) is not discretely valued. Let \( A \) be an \( \mathcal{L} \)-definable subset of \( K^n \) with accumulation point \( a = (a_1, \ldots, a_n) \in K^n \), and \( E \) an \( \mathcal{L} \)-definable subset of \( K \) with accumulation point \( a_1 \). We call an \( \mathcal{L} \)-definable family of sets
\[
\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A
\]
an \( \mathcal{L} \)-definable \( x_1 \)-fiber shrinking for the set \( A \) at \( a \) if
\[
\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),
\]
i.e. for any neighbourhood \( U \) of \( (a_2, \ldots, a_n) \in K^{n-1} \), there is a neighbourhood \( V \) of \( a_1 \in K \) such that \( \emptyset \neq \Phi_t \subset U \) for every \( t \in V \cap E \), \( t \neq a_1 \). When \( n = 1 \), \( A \) is itself a fiber shrinking for the subset \( A \) of \( K \) at an accumulation point \( a \in K \). This concept is a relaxed version of curve selection, available in the non-archimedean geometry treated in this paper.

Theorem 6.1. (Fiber shrinking) Every \( \mathcal{L} \)-definable subset \( A \) of \( K^n \) with accumulation point \( a \in K^n \) has, after a permutation of the coordinates, an \( \mathcal{L} \)-definable \( x_1 \)-fiber shrinking at \( a \).

Proof. We proceed with induction with respect to the dimension of the ambient affine space \( n \). The case \( n = 1 \) is trivial. So assuming the assertion to hold for \( n \), we shall prove it for \( n + 1 \). We may, of course, assume that \( a = 0 \). Let \( x = (x_1, \ldots, x_{n+1}) \) be coordinates in \( K^n \).

If 0 is an accumulation point of the intersections
\[ A \cap \{x_i = 0\}, \quad i = 1, \ldots, n + 1, \]
we are done by the induction hypothesis. Thus we may assume that the intersection
\[ A \cap \bigcup_{i=1}^{n+1} \{x_i = 0\} = \emptyset \]
is empty. Then the definable (in the \( \Gamma \)-sort) set
\[ P := \{(v(x_i), \ldots, v(x_{n+1})) \in \Gamma^{n+1} : x \in A\} \]
contains an accumulation point \((\infty, \ldots, \infty)\).

Since we work under the assumption that \( \Gamma \) is an ordered subgroup of the ordered additive group \((\mathbb{R}, +, <)\) without minimal positive element, \( \Gamma \) admits quantifier elimination in the language of ordered groups and, moreover, every definable function is piecewise \( \mathbb{Q} \)-linear (see e.g. [10]). Consequently, the set \( P \), being a finite union of semi-linear subsets, contains an affine semi-line
\[ L := \left\{ \frac{1}{s} \cdot (r_1k + \gamma_1, \ldots, r_nk + \gamma_n, r_0k + \gamma_0) \in \Gamma^{n+1} : k \in \Gamma, k > \beta \right\}, \]
where \( s, r_1, \ldots, r_{n+1} \) are positive integers, \( \gamma_1, \ldots, \gamma_{n+1}, \beta \in \Gamma \). Now, it is easy to check that the set
\[ \Phi := \{x \in A : (v(x_i), \ldots, v(x_{n+1})) \in L\} \]
is an \( \mathcal{L} \)-definable \( x_1 \)-fiber shrinking for the set \( A \) at 0. This finishes the proof.

7. Proof of Theorem 3.1 when the valuation is not discrete

The proof reduces easily to the case \( m = 1 \). We must show that if a point \( a \) lies in the closure of \( A := \pi(B) \), then there is a point \( b \) in the closure of \( B \) such that \( \pi(b) = a \). We may obviously assume that \( a = 0 \notin A \). By Theorem 6.1, there exists, after a permutation of the coordinates, an \( \mathcal{L} \)-definable \( x_1 \)-fiber shrinking \( \Phi \) for \( A \) at \( a \):
\[ \Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A, \quad \lim_{t \to 0} \Phi_t = 0; \]
here \( E \) is the canonical projection of \( A \) onto the \( x_1 \)-axis. Put
\[ B^* := \{(t, y) \in K \times R : \exists u \in \Phi_t (t, u, y) \in B\}; \]
it easy to check that if a point \((0, w) \in K^2\) lies in the closure of \(B^*\),
then the point \((0, w) \in K^{n+1}\) lies in the closure of \(B\). The problem is
thus reduced to the case \(n = 1\) and \(a = 0 \in K\).

By Theorem 2.4 (on cell decomposition), we can assume that \(B\) is a cell
\[ C \subset K_x \times R \subset K_x \times K_y \]
with parameters \(\xi \in k^m\) and center \(c(x, \xi)\), i.e. \(B\) is the disjoint union
of sets \(C_\xi\) of the form (cf. Section 1):
\[ C(\xi) := \{(x, y) \in K_x \times K_y : (x, \xi) \in D, \]
\[ v(a(x, \xi)) <_1 v((y - c(x, \xi))^\nu) <_2 v(b(x, \xi)), \quad \overline{ac}(y - c(x, \xi)) = \xi_1\}. \]
But the set
\[ \{(v(x), \xi) \in \Gamma \times k^m : \exists y \in R \ (x, y) \in C_\xi\} \]
is an \(\mathcal{L}\)-definable subset of the product \(\Gamma \times k^m\) of the two sorts, and
thus it is a finite union of the Cartesian products of definable subsets
in \(\Gamma\) and in \(k^m\), respectively. It follows that 0 is an accumulation point
of the projection \(\pi(C(\xi'))\) of the fiber \(C(\xi')\) for a parameter \(\xi' \in k^m\).

We are thus reduced to the case where \(B\) is the fiber
\[ C(\xi') := \{(x, y) \in K_x \times K_y : (x, \xi') \in D, \]
\[ v(a(x, \xi')) <_1 v((y - c(x, \xi'))^\nu) <_2 v(b(x, \xi')), \quad \overline{ac}(y - c(x, \xi')) = \xi'_1\} \]
of the cell for a parameter \(\xi'\). We abbreviate \(c(x, \xi'), a(x, \xi'), b(x, \xi')\) to
\(c(x), a(x), b(x)\) for simplicity.

In the statement of Theorem 3.1, we may equivalently replace \(R\) with
the projective line \(K\mathbb{P}^1\), because the latter is the union of two open and
closed charts biregular to \(R\). By Theorem 5.3, we can thus assume that
the limits, say \(c(0), a(0), b(0)\), of \(c(x), a(x), b(x)\) when \(x \to 0\) exist in
\(K\mathbb{P}^1\). Performing a linear fractional transformation of the coordinate
\(y\), we get
\[ c(0), a(0), b(0) \in K. \]
The role of the center \(c(x)\) is immaterial. We can assume, without loss
of generality, that it vanishes, \(c(x) \equiv 0\), for if a point \(b = (0, w) \in K^2\)
lies in the closure of the cell with zero center, the point \((0, w + c(0))\)
lies in the closure of the cell with center \(c(x)\).

Observe now that if
\[ \lim_{x \to 0} f(x) =: f(0) \in K \quad \text{and} \quad f(0) \neq 0, \]
then \(v(f(x)) = v(f(0))\) near \(0 \in K\). Therefore, it is straightforward
to verify the cases where only \(<_2\) occurs, and where \(<_1\) occurs and
\(a(0) \neq 0\). Thus it remains to consider the case where \(<_1\) occurs and
a(0) = 0. But then the point \( b = (0,0) \) is an accumulation point of \( B \) lying over \( a = 0 \) and the set \( B \) is itself an \( x \)-fiber shrinking at \( (0,0) \). This completes the proof of the closedness theorem. \( \square \)

8. Łojasiewicz inequality and curve selection

From now on, we shall again deal with the general case where the ground field \( K \) is a Henselian rank one valued field of equicharacteristic zero. Consider a smooth \( K \)-variety \( X \) (it would be sufficient to assume that \( X \) is smooth at all \( K \)-rational points of \( X(K) \)). By a continuous rational function \( f \) on \( X(K) \) we mean a rational function \( f \) on \( X \) that extends continuously to \( X(K) \).

**Theorem 8.1.** (Łojasiewicz inequality) Let \( f, g \) be two continuous rational functions on \( X(K) \) and \( U \) be an \( \mathcal{L} \)-definable open (in the \( K \)-topology) subset of \( X(K) \). If

\[
\{x \in U : g(x) = 0\} \subset \{x \in U : f(x) = 0\},
\]

then there exist a positive integer \( s \) and a continuous rational function \( h \) on \( U \) such that \( f^s(x) = h(x) \cdot g(x) \) for all \( x \in U \).

**Proof.** Clearly, the functions \( f, g \) are quotients of some regular functions on \( X \):

\[
f = f_1/f_2, \quad g = g_1/g_2.
\]

We shall now apply a strong form of Hironaka’s transformation to a normal crossing (see e.g. [22, Chap. III] for references and relatively short proofs). Thus we can take a finite composite \( \sigma : Y \rightarrow X \) of blow-ups along smooth centers such that the pull-backs

\[
f_1^\sigma = f_1 \circ \sigma, \quad f_2^\sigma = f_2 \circ \sigma, \quad g_1^\sigma = g_1 \circ \sigma, \quad g_2^\sigma = g_2 \circ \sigma,
\]

are normal crossing divisors on \( Y \) ordered with respect to divisibility relation. Since the quotients

\[
f^\sigma = f_1^\sigma/f_2^\sigma \quad \text{and} \quad g^\sigma = g_1^\sigma/g_2^\sigma
\]

are continuous on \( Y(K) \), they are normal crossing divisors on \( Y(K) \) too. Therefore, since

\[
\{y \in \sigma^{-1}(U) : g^\sigma(y) = 0\} \subset \{y \in \sigma^{-1}(U) : f^\sigma(y) = 0\},
\]

there is a positive integer \( s \) and a normal crossing divisor \( H \) on \( Y(K) \cap \sigma^{-1}(U) \) vanishing on \( \{y \in Y(K) \cap \sigma^{-1}(U) : f^\sigma(y) = 0\} \) such that

\[
(f^\sigma)^s(y) = H(y) \cdot g^\sigma(y) \quad \text{for all} \quad y \in Y(K) \cap \sigma^{-1}(U).
\]

Clearly, the regular function \( H \) descends to the quotient \( f^s/g \) over the set \( \{x \in U : g(x) \neq 0\} \). On the other hand, \( H \) vanishes on the fibres \( \sigma^{-1}(x) \) if \( x \in U \) and \( g(x) = 0 \). Hence and by the descent property
(Corollary 3.5), \( H \) descends to a continuous rational function \( h \) on \( U \). Then \( f^*(x) = h(x) \cdot g(x) \) for all \( x \in U \), as asserted. \( \square \)

**Remark 8.2.** For the real version of the above theorem, we refer the reader to [5, Thm. 2.6.6]. The version over \( p \)-adic fields and their finite extensions is valid, because these fields are locally compact, and thus descent is available by a purely topological argument.

We now pass to curve selection over fields under study, which are not locally compact. While the real version of curve selection goes back to papers [8, 41] (see also [31, 32, 5]), the \( p \)-adic one was achieved in papers [40, 14].

By a semialgebraic subset of \( K^n \) we mean a (finite) Boolean combination of sets of the form
\[
\{ x \in K^n : v(f(x)) \leq v(g(x)) \},
\]
where \( f \) and \( g \) are regular functions on \( K^n \). We call a map \( \varphi \) semialgebraic if its graph is a semialgebraic set.

**Theorem 8.3. (Curve Selection Lemma)** Let \( A \) be a semialgebraic subset of \( K^n \). If \( 0 \) lies in the closure (in the \( K \)-topology) of \( A \setminus \{ a \} \), then there is a semialgebraic map \( \varphi : R \rightarrow K^n \) given by restricted power series such that
\[
\varphi(0) = a \quad \text{and} \quad \varphi(R \setminus \{ 0 \}) \subset A \setminus \{ a \}.
\]

**Proof.** It is easy to check that every semialgebraic set is a finite union of sets of the form
\[
\{ x \in K^n : v(f_i(x)) \leq_1 v(g_1(x)) \} \cap \ldots \cap \{ x \in K^n : v(f_r(x)) \leq_r v(g_r(x)) \},
\]
where \( f_1, \ldots, f_r, g_1, \ldots, g_r \) are regular functions and \( \leq_1, \ldots, \leq_r \) stand for \( \leq \) or \( < \). We may assume, of course, that \( A \) is a set of this form and \( a = 0 \). As before, we take a finite composite
\[
\sigma : Y \rightarrow K^\infty
\]
of blow-ups along smooth centers such that the pull-backs of the coordinates \( x_1, \ldots, x_n \) and
\[
f_1^\sigma, \ldots, f_r^\sigma \quad \text{and} \quad g_1^\sigma, \ldots, g_r^\sigma
\]
are normal crossing divisors ordered with respect to divisibility relation, unless one of those functions vanishes. Since the restriction \( \sigma : Y(K) \rightarrow K^n \) is definably closed (Corollary 3.4), there is a point \( b \in Y(K) \cap \sigma^{-1}(a) \) which lies in the closure of
\[
B := Y(K) \cap \sigma^{-1}(A \setminus \{ a \}).
\]
Further, we get
\[Y(K) \cap \sigma^{-1}(A) = \{v(f_1^K(y)) <_1 v(g_1^K(y))\} \cap \ldots \cap \{v(f_n^K(y)) <_r v(g_n^K(y))\},\]
and thus \(\sigma^{-1}(A)\) is in suitable local coordinates \(y = (y_1, \ldots, y_n)\) near \(b = 0\) a finite intersection of sets of the form
\[\{v(y^r) \leq v(u(y))\}, \{v(u(y)) \leq v(y^r)\}, \{v(y^\beta) < \infty\} \text{ or } \{\infty = v(y^\gamma)\},\]
where \(\alpha, \beta, \gamma \in \mathbb{N}^n\) and \(u(y)\) is a regular function which vanishes nowhere.

The first case cannot occur because \(b = 0\) lies in the closure of \(B\); the second case holds in a neighbourhood of \(b\); the third and fourth cases are equivalent to \(y^\beta \neq 0\) and \(y^\gamma = 0\), respectively. Consequently, since the pull-backs of the coordinates \(x_1, \ldots, x_n\) are monomial divisors too, \(B\) contains the set \((R \setminus \{0\}) \cdot c\) when \(c \in B\) is a point sufficiently close to \(b = 0\). Then the map
\[\varphi : R \rightarrow K^n, \quad \varphi(z) = \sigma(z \cdot c)\]
has the desired properties. \(\square\)

9. Extending continuous hereditarily rational functions

We first recall an elementary lemma from [24].

**Lemma 9.1.** If the ground field \(K\) is not algebraically closed, then there are polynomials \(G_r(x_1, \ldots, x_r)\) in any number of variables whose only zero on \(K^r\) is \((0, \ldots, 0)\).

**Proof.** Indeed, take a polynomial
\[g(t) = t^d + a_1 t^{d-1} + \cdots + a_d \in K[t], \quad d > 1,\]
which has no roots. Then its homogenization
\[G_2(x_1, x_2) = x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d\]
is a polynomial in two variables we are looking for. Further, we can recursively define polynomials \(G_r\) by putting
\[G_{r+1}(x_1, \ldots, x_r, x_{r+1}) := G_2(G_r(x_1, \ldots, x_r), x_{r+1}). \quad \square\]

We keep further the assumption that \(K\) is a Henselian rank one valued field of equicharacteristic zero and, additionally, that it is not algebraically closed. We thus have at our disposal the descent property (Corollary 3.5) and the \\Lojasiewicz inequality (Theorem 8.1) (which hold also over locally compact fields by a purely topological argument). We are therefore able, by adapting mutatis mutandis its proof, to carry over Proposition 11 from [24] to the case of such non-archimedean fields. \(\square\)
**Theorem 9.2.** *(Extending continuous hereditarily rational functions)*

Let $X$ be a smooth $K$-variety and $W \subset Z \subset X$ closed subvarieties. Let $f$ be a continuous hereditarily rational function on $Z(K)$ that is regular at all $K$-points of $Z(K) \setminus W(K)$. Then $f$ extends to a continuous hereditarily rational function $F$ on $X(K)$ that is regular at all $K$-points of $X(K) \setminus W(K)$.

**Sketch of the Proof.** We shall keep the notation from paper [24]. The main modification of the proof in comparison with paper [24] is the definition of the functions $G$ and $F_{dn}$ which improve the rational function $P/Q$. Now we need the following corrections:

$$G := \frac{P}{Q} \cdot \frac{Q^d}{G_2(Q, H)}$$

and

$$F_{dn} := G \cdot \frac{Q_{dn}^d}{G_2(Q^n, H)} = \frac{P}{Q} \cdot \frac{Q^d}{G_2(Q, H)} \cdot \frac{Q_{dn}^d}{G_2(Q^n, H)},$$

where the positive integer $d$ and the polynomial $G_2$ are taken from Lemma 9.1 and its proof. It is clear that the restriction of $F_{dn}$ to $Z \setminus W$ equals $f_2$, and thus Theorem 9.2 will be proven once we show that the rational function $F_{dn}$ restricts to a continuous function $\Phi_{dn}$ on $X(K)$ for $n \gg 1$.

We work on the variety $\pi : X_1(K) \longrightarrow X(K)$ obtained by blowing up the ideal $(PQ^{d-1}, G_2(Q, H))$. Equivalently, $X_1(K)$ is the Zariski closure of the graph of $G$ in $X(K) \times K\mathbb{P}^1$. Two open charts are considered:

- a Zariski open neighbourhood $U^*$ of the closure (in the $K$-topology) $Z^*$ of $\pi^{-1}(Z(K) \setminus W(K))$;
- an open (in the $K$-topology) set $V^* := X_1(K) \setminus Z^*$, which is an $\mathcal{L}$-definable subset of $X_1(K)$.

The subtlest analysis is on the latter chart, on which $F_{dn} \circ \pi$ can be written in the form

$$F_{dn} \circ \pi = (P \circ \pi) \cdot \left( \frac{Q_{dn-1}^d}{H^d} \circ \pi \right) \cdot \left( \frac{Q^d}{G_2(Q, H)} \circ \pi \right) \cdot \left( \frac{H^d}{G_2(Q^n, H)} \circ \pi \right).$$

By the Lojasiewicz inequality (Theorem 8.1), the second factor extends to a continuous rational function on $V^*$ for $n \gg 1$.

Via the descent property, it suffices to show that the rational function $F_{dn} \circ \pi$ (with $n \gg 1$) extends to a continuous function on $X_1(K)$ that vanishes on

$$E := \pi^{-1}(W(K)).$$
The proof of this fact goes now along the same line of reasoning as in paper [24], once we know that the factors
\[
\frac{Q^{dn}}{G_2(Q^n, H)} \quad \frac{Q^d}{G_2(Q, H)} \quad \text{and} \quad \frac{H^d}{G_2(Q^n, H)}
\]
are regular functions off \(W(K)\) whose valuations are bounded from below. But this follows immediately from an auxiliary lemma:

**Lemma 9.3.** Let \(g\) be the polynomial from the proof of Lemma 9.1. Then the set of values
\[
v\left(\frac{t^d}{g(t)}\right) \in \mathbb{Z}, \quad t \in K,
\]
is bounded from below.

In order to prove this lemma, observe that
\[
v\left(\frac{t^d}{g(t)}\right) = v\left(1 + \frac{a_1}{t} + \cdots + \frac{a_d}{t^d}\right)
\]
for \(t \in K, t \neq 0\). Hence the values under study are zero when \(iv(t) < v(a_i)\) for all \(i = 1, \ldots, d\). Therefore, we are reduced to analysing the case where
\[
v(t) \geq k := \min \left\{ \frac{v(a_i)}{i} : i = 1, \ldots, d \right\}.
\]
Denote by \(\Gamma\) the valuation group of \(v\). Thus we must show that the set of values \(v(g(t)) \in \Gamma\) when \(v(t) \geq k\) is bounded from above.

Take elements \(a, b \in R, a, b \neq 0\), such that \(aa_i \in R\) for all \(i = 1, \ldots, d\), and \(bt \in R\) whenever \(v(t) \geq k\). Then
\[
(ab)^d g(t) = (ab)^d + aba_1(ab)^{d-1} + (ab)^2a_2(ab)^{d-2} + \cdots + (ab)^d a_d =: h(abt),
\]
where \(h\) is a monic polynomial with coefficients from \(R\) which has no roots in \(K\). Clearly, it is sufficient to show that the set of values \(v(h(t)) \in \Gamma\) when \(t \in R\) is bounded from above.

Consider a splitting field \(\tilde{K} = K(u_1, \ldots, u_d)\) of the polynomial \(h\), where \(u_1, \ldots, u_d\) are the roots of \(h\). Let \(\tilde{v}\) be a (unique) extension to \(\tilde{K}\) of the valuation \(v\), \(\tilde{R}\) be its valuation ring and \(\tilde{\Gamma}, \Gamma \subset \tilde{\Gamma}\), its valuation group (see e.g. [42, Chap. VI, § 11] for valuations of algebraic field extensions). Then
\[
u_1, \ldots, u_d \in \tilde{R} \setminus R \quad \text{and} \quad h(t) = \prod_{i=1}^{d} (t - u_i).
Since $R$ is a closed subring of $\tilde{R}$ by Lemma 5.2, there exists an $l \in \tilde{\Gamma}$ such that $\tilde{v}(t-u_i) \leq l$ for all $i = 1, \ldots, d$ and $t \in R$. Hence $v(h(t)) \leq dl$ for all $t \in R$, and thus the lemma follows. □

In this fashion, we have demonstrated how to adapt the proof of Proposition 11 from paper [24] to the case of Henselian rank one valued fields of equicharacteristic zero. Note that the proofs of all remaining results from paper [24] work over general topological fields with the (DP) property and thus, in particular, over all Henselian rank one valued fields.

10. Regulous functions and sets

In these last three sections, we shall carry the theory of regulous functions over the real ground field $\mathbb{R}$, developed by Fichou–Huisman–Mangolte–Monnier [15], over to non-archimedean algebraic geometry over Henselian rank one valued fields $K$ of equicharacteristic zero. We assume that the ground field $K$ is not algebraically closed. (Otherwise, the notion of a regulous function on a normal variety coincides with that of a regular function and, in general, the study of continuous rational functions leads to the concept of seminormality and seminormalization; cf. [1, 2] or [23, Sec.10.2] for a recent treatment.) Every such field is a topological field which enjoys the density property (cf. [24]), i.e. the following equivalent conditions hold:

(1) If $X$ is a smooth, irreducible $K$-variety and $\emptyset \neq U \subset X$ is a Zariski open subset, then $U(K)$ is dense in $X(K)$ in the $K$-topology.

(2) If $C$ is a smooth, irreducible $K$-curve and $\emptyset \neq C^0 \subset C$ is Zariski open, then $C(K)$ is dense in $C(K)$ in the $K$-topology.

(3) If $C$ is a smooth, irreducible $K$-curve, then $C(K)$ has no isolated points.

The $K$-points $X(K)$ of any algebraic $K$-variety $X$ inherit from $K$ a topology, called the $K$-topology.

In this section, we deal with the ground fields $K$ with the density property. Observe first that if $f$ is a rational function on an affine $K$-variety $X$ which is regular on a Zariski open subset $U$, then there exist two regular functions $p, q$ on $X$ such that

$$f = \frac{p}{q} \quad \text{and} \quad q(x) \neq 0 \quad \text{for all} \quad x \in U(K).$$

When $X \subset K\mathbb{A}^n$, then $p, q$ can be polynomial functions. Every rational function $f$ is regular on the largest Zariski open subset of $X$, called its
regular locus and denoted by \( \text{dom}(f) \). Further, assume that \( Z \) is a closed subvariety of a \( K \)-variety \( X \). Then every rational function \( f \) on \( Z \) that is regular on \( Z(K) \) extends to a rational function \( F \) on \( X \) that is regular on \( X(K) \). Both the results can be deduced via Lemma 9.1 (cf. [24], the proof of Lemma 15 on extending regular functions).

Suppose now that \( X \) is a smooth affine \( K \)-variety or, at least, an affine \( K \)-variety that is smooth at all \( K \)-points \( X(K) \). We say that a function \( f \) on \( X(K) \) is \( k \)-regulous, \( k \in \mathbb{N} \cup \{\infty\} \), if it is of class \( C^k \) and there is a Zariski dense open subset \( U \) of \( X \) such that the restriction of \( f \) to \( U(K) \) is a regular function. A function \( f \) on \( X(K) \) is called regulous if it is 0-regulous. Denote by \( R^k(X(K)) \) the ring of \( k \)-regulous functions on \( X \). The ring \( R^{\infty}(X(K)) \) of \( \infty \)-regulous functions on \( X \) coincides with the ring \( \mathcal{O}(X(K)) \) of regular functions on \( X \). This follows easily from the faithful flatness of the formal power series ring \( K[[x_1, \ldots, x_n]] \) over the local ring of regular function germs at 0 \( \in K^n \). Therefore we shall most often restrict ourselves to the case \( k \in \mathbb{N} \).

Whenever \( K \) is a Henselian rank one valued field of equicharacteristic zero, the technique of transformation to a normal crossing by blowing up along smooth centers and the descent property (Corollary 3.5) enable the following characterization: given a smooth algebraic \( K \)-variety \( X \), a function \( f : X(K) \to K \) is regulous iff there exists a finite composite \( \sigma : \tilde{X} \to X \) of blow-ups with smooth centers such that the pull-back \( f^\sigma := f \circ \sigma \) is a regular function on \( \tilde{X}(K) \).

We say that a subset \( V \) of \( K^n \) is \( k \)-regulous closed if it is the zero set of a family \( E \subset R^k(K^n) \) of \( k \)-regulous functions:

\[
V = Z(E) := \{ x \in K^n : f(x) = 0 \text{ for all } f \in E \}.
\]

A subset \( U \) of \( K^n \) is called \( k \)-regulous open if its complement \( K^n \setminus U \) is \( k \)-regulous closed. The family of \( k \)-regulous open subsets of \( K^n \) is a topology on \( K^n \), called the \( k \)-regulous topology on \( K^n \).

If \( f \neq 0 \) is a \( k \)-regulous function on \( K^n \) with regular locus \( U = \text{dom}(f) \), then

\[
f = \frac{p}{q} \quad \text{where } p, q \in K[x_1, \ldots, x_n], \quad Z(q) = K^n \setminus U(K),
\]

and \( p, q \) are coprime polynomials. Clearly, \( Z(q) \subset Z(p) \) and it follows, by passage to the algebraic closure of \( K \), that the zero set \( Z(q) \) is of codimension \( \geq 2 \) in \( K^n \). Thus the complement \( K^n \setminus \text{dom}(f) \) is of codimension \( \geq 2 \) in \( K^n \). Consequently, every \( k \)-regulous function on
$K$ is regular and every $k$-regulous function on $K^2$ is regular at all but finitely many points.

We now recall some results about algebraic sets over arbitrary fields $F$. Let $V$ be an affine $F$-variety. Then the regular locus $\text{Reg}(V)$ of $V$ is a non-empty, Zariski open subset of $V$ and, moreover, its trace on the set $V(F)$ of $F$-rational points of $V$ is non-empty; cf. [29], Chap. VI, Corollary 1.17 to the Jacobian criterion for regular local rings and the remark preceding it. Therefore, for every affine $F$-subvariety $V$ of $FA^n$, the set $V(F)$ of its $F$-rational points is a finite (disjoint) union of smooth, Zariski locally closed subsets of pure dimension.

We call a subset $E$ of $K^n$ constructible if it is a (finite) Boolean combination of Zariski closed subsets of $K^n$. Every such set $E$ is, of course, a finite union of Zariski locally closed subsets. Moreover, by the above observation, $E$ is a finite union of smooth, Zariski locally closed subsets of pure dimension with irreducible Zariski closure. Thus it follows immediately from the density property that every closed (in the $K$-topology) constructible subset $E$ is a finite union of a (unique) family $\Sigma(E)$ of subsets each of which is the regular locus $\text{Reg}(V) \cap K^n$ of an irreducible affine $K$-subvariety $V$ of $KA^n$; obviously, $\text{Reg}(V) \cap K^n$ is a smooth, Zariski locally closed subset of pure dimension $\dim V$.

Below we introduce the constructible topology on $K^n$. We shall see in the next section that the $k$-regulous topology coincides with the constructible topology for all $k \in \mathbb{N}$.

**Theorem 10.1.** If $K$ is a topological field with the density property, then the family of all closed (in the $K$-topology) constructible subsets of $K^n$ is the family of closed sets for a topology, called the constructible topology on $K^n$. Furthermore, this topology is noetherian, i.e. every descending sequence of closed constructible subsets of $K^n$ stabilizes.

**Proof.** Clearly, it suffices to prove only the last assertion. We shall follow the reasoning from our paper [33] which showed that the quasi-analytic topology is noetherian. For any closed (in the $K$-topology) constructible subset $E$ of $K^n$, let $\mu_i(E)$ be the number of elements from the family $\Sigma(E)$, constructed above, of dimension $i$, $i = 0, 1, \ldots, n$, and put

$$\mu(E) = (\mu_n(E), \mu_{n-1}(E), \ldots, \mu_0(E)) \in \mathbb{N}^{n+1}.$$  

Consider now a descending sequence of closed constructible subsets

$$K^n \supset E_1 \supset E_2 \supset E_3 \supset \ldots.$$  

It is easy to check that for any two closed (in the $K$-topology) constructible subsets $D \subset E$, we have $\mu(D) \leq \mu(E)$ and, furthermore,
$D = E \iff \mu(D) = \mu(E)$. Hence we get the decreasing (in the lexicographic order) sequence of multi-indices
\[\mu(E_1) \geq \mu(E_2) \geq \mu(E_3) \geq \ldots,\]
which must stabilize for some $N \in \mathbb{N}$:
\[\mu(E_N) = \mu(E_{N+1}) = \mu(E_{N+2}) = \ldots\]
Then
\[E_N = E_{N+1} = E_{N+2} = \ldots,\]
as desired.

**Proposition 10.2.** Suppose $K$ is a field with the density property. Then there is a one-to-one correspondence between the irreducible closed constructible subsets $E$ of $K^n$ and the irreducible Zariski closed subsets $V$ of $K^n$:
\[\alpha : E \mapsto \overline{E}^Z \quad \text{and} \quad \beta : V \mapsto \overline{\text{Reg}(V)}^c,\]
where $\overline{E}^Z$ stands for the Zariski closure of $E$ and $\overline{A}$ for the closure of $A$ in the constructible topology.

**Proof.** We have $\alpha \circ \beta = \text{Id}$, because $\overline{\text{Reg}(V)}^Z = V$ for every irreducible Zariski closed subset $V$ of $K^n$. In view of the foregoing discussion, the assignment $\beta$ is surjective. Therefore every irreducible closed constructible subset $E$ of $K^n$ is of the form $E = \beta(V) = \overline{\text{Reg}(V)}^c$ for an irreducible Zariski closed subset $V$ of $K^n$. Hence
\[(\beta \circ \alpha)(E) = (\beta \circ \alpha)(\beta(V)) = (\beta \circ \alpha \circ \beta)(V) = (\beta \circ \text{Id})(V) = \beta(V) = E,\]
and thus $\beta \circ \alpha = \text{Id}$, which finishes the proof.

Below we recall Proposition 8 from paper [24] which holds over any topological fields with the density property.

**Proposition 10.3.** Let $X$ be an algebraic $K$-variety and $f$ a rational function on $X$ that is regular on $X^0 \subset X$. Assume that $f|_{X^0(K)}$ has a continuous extension $f^c : X(K) \to K$. Let $Z \subset X$ be an irreducible subvariety that is not contained in the singular locus of $X$. Then there is a Zariski dense open subset $Z^0 \subset Z$ such that $f^c|_{Z^0(K)}$ is a regular function.

We immediately obtain two corollaries:

**Corollary 10.4.** Let $X$ be an algebraic $K$-variety that is smooth at all $K$-rational points $X(K)$ and $f$ a rational function on $X$ that is regular on $X^0 \subset X$. Assume that $f|_{X^0(K)}$ has a continuous extension $f^c : X(K) \to K$. Then there is a sequence of closed subvarieties
\[\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X\]
such that for \( i = 0, \ldots, n \) the restriction of \( f \) to \( X_i(K) \setminus X_{i-1}(K) \) is regular. Moreover, we can require that each set \( X_i \setminus X_{i-1} \) be smooth of pure dimension \( i \).

**Corollary 10.5.** If \( f \) is a regulous function on \( K^n \), then there is a sequence of Zariski closed subsets
\[
\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_n = K^n
\]
such that for \( i = 0, \ldots, n \) the restriction of \( f \) to \( E_i \setminus E_{i-1} \) is regular. Moreover, we can require that each set \( E_i \setminus E_{i-1} \) be smooth of pure dimension \( i \).

**Remark 10.6.** Given a finite number of regulous functions \( f_1, \ldots, f_p \), there is a filtration
\[
\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_n = K^n
\]
as in the above corollary such that for \( i = 0, \ldots, n \) the restriction of each function \( f_j, j = 1, \ldots, p \), to \( E_i \setminus E_{i-1} \) is regular.

Now, three further consequences of the above corollary will be drawn. We say that a map
\[
f = (f_1, \ldots, f_p) : K^n \to K^p
\]
is \( k \)-regulous if all its components \( f_1, \ldots, f_p \) are regulous functions on \( K^n \).

**Corollary 10.7.** If two maps
\[
g : K^m \to K^n \quad \text{and} \quad f : K^n \to K^p
\]
are \( k \)-regulous, so is its composition \( f \circ g \).

**Proof.** Indeed, let \( U \) be the common regular locus of the components \( g_1, \ldots, g_n \) of the map \( g \):
\[
U := \text{dom}(g_1) \cap \ldots \cap \text{dom}(g_n) \subset \mathbb{R}^m.
\]
Take a filtration
\[
\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_n = K^n
\]
described in the remark after Corollary 10.5 for the functions \( f_1, \ldots, f_p \). Then \( U \) is the following union of Zariski locally closed subsets
\[
U = \bigcup_{i=0}^n \left( U \cap g^{-1}(E_i \setminus E_{i-1}) \right).
\]
Clearly, one of these sets, say \( U \cap g^{-1}(E_{i_0} \setminus E_{i_0-1}) \) must be a Zariski dense open subset of \( U \) and of \( K^m \) too. Hence \( f \circ g \) is a regular function on \( U \cap g^{-1}(E_{i_0} \setminus E_{i_0-1}) \), which is the required result.
Corollary 10.8. The zero set \( Z(f) \) of a reguluous function \( f \) on \( K^n \) is a closed (in the \( K \)-topology) constructible subset of \( K^n \).

Corollary 10.9. The zero set \( Z(f_1, \ldots, f_p) \) of a finite number of reguluous functions \( f_1, \ldots, f_p \) on \( K^n \) is a closed (in the \( K \)-topology) constructible subset of \( K^n \).

Proof. This follows directly from Corollary 10.8 and Lemma 9.1.

Hence and by Theorem 10.1, we immediately obtain

Proposition 10.10. The \( k \)-regulous topology on \( K^n \) is noetherian.

Corollary 10.11. Every \( k \)-regulous closed subset of \( K^n \) is the zero set \( Z(f) \) of a \( k \)-regulous function \( f \) on \( K^n \), and thus is a closed (in the \( K \)-topology) constructible subset of \( K^n \). Hence every \( k \)-regulous open subset of \( K^n \) is of the form

\[
D(f) := K^n \setminus Z(f) = \{x \in K^n : f(x) \neq 0\}
\]

for a \( k \)-regulous function \( f \) on \( K^n \).

Corollaries 10.11 and 10.7 yield the following

Corollary 10.12. Every \( k \)-regulous map \( f : K^n \to K^m \) is continuous in the \( k \)-regulous topology.

11. Regulous Nullstellensatz

We keep further the assumption that the ground field \( K \) is a Henselian rank one valued field of equicharacteristic zero that is not algebraically closed. Throughout this section, \( k \) will be a non-negative integer. We begin with a consequence of Theorem 8.1 (Lojasiewicz inequality).

Proposition 11.1. Let \( f, g \) be rational functions on \( KA^n \) such that \( f \) extends to a continuous function on \( K^n \) and \( g \) extends to a continuous function on the set \( D(f) \). Then the function \( f^s g \) extends, for \( s \gg 0 \), by zero through the set \( Z(f) \) to a continuous rational function on \( K^n \).

Proof. We can find a finite composite \( \sigma : M \to KA^n \) of blow-ups along smooth centers such that the pull-backs

\[
f^\sigma := f \circ \sigma \quad \text{and} \quad g^\sigma := g \circ \sigma
\]

are regular functions at all \( K \)-rational points on

\[
M(K) \quad \text{and} \quad M(K) \setminus \sigma^{-1}(Z(f)),
\]

respectively. Then there are regular functions \( p, q \) on \( M \) such that

\[
g^\sigma = \frac{p}{q} \quad \text{and} \quad Z(q) := \{y \in M(K) : q(y) = 0\} \subset Z(f^\sigma).
\]
It follows immediately from Theorem 8.1 that the rational function

\( \frac{(f^\sigma)^s}{q}, \) for \( s \gg 0, \)

extends by zero through the set \( \mathcal{Z}(f^\sigma) \) to a continuous function on \( M(K) \), whence so does the rational function \( (f^\sigma)^s \cdot g^\sigma \). By the descent property (Corollary 3.5), the continuous function \( (f^\sigma)^s \cdot g^\sigma \) descends to a continuous function on \( K^n \) that vanishes on \( \mathcal{Z}(f) \). This is the required result. \( \square \)

The two corollaries stated below are verbatim counterparts of Lemmata 5.1 and 5.2 from paper [15], established over the real ground field \( \mathbb{R} \). Nevertheless, the proofs given here seem to be simpler.

**Corollary 11.2.** Let \( f \) be a \( k \)-regulous function on \( K^n \) and \( g \) a \( k \)-regulous function on the open subset \( \mathcal{D}(f) \). Then the function \( f^s g \) extends, for \( s \gg 0 \), by zero through the zero set \( \mathcal{Z}(f) \) to a \( k \)-regulous function on \( K^n \). Hence the ring of \( k \)-regulous functions on \( \mathcal{D}(f) \) is the localization \( \mathcal{R}^k(K^n)_f \).

**Proof.** The case \( k = 0 \) is just Proposition 11.1. If \( s \) is an exponent large enough so that \( f^s g \) extends to a continuous function on \( K^n \) vanishing on \( \mathcal{Z}(f) \), then \( f^{s+k} g \) is \( k \) flat on \( \mathcal{Z}(f) \), and thus is \( k \)-regulous on \( K^n \), as desired. \( \square \)

**Corollary 11.3.** Let \( U \) be a \( k \)-regulous open subset of \( K^n \), \( f \) a \( k \)-regulous function on \( U \) and \( g \) a \( k \)-regulous function on the open subset \( \mathcal{D}(f) \subset U \). Then the function \( f^s g \) extends, for \( s \gg 0 \), by zero through the zero set \( \mathcal{Z}(f) \subset U \) to a \( k \)-regulous function on \( U \).

**Proof.** By Corollary 10.11, \( U = \mathcal{D}(h) \) for a \( k \)-regulous function on \( K^n \). From the above corollary we get

\( f h^s \in \mathcal{R}^k(K^n) \) for \( s \gg 0 \).

Hence

\( g \in \mathcal{R}^k(K^n)_{fh^s} \)

for integers \( s \) large enough, and thus the conclusion follows. \( \square \)

Now we can readily pass to a regulous version of Nullstellensatz, which follows directly from Corollary 11.2 and the fact that the \( k \)-regulous topology is noetherian.

**Theorem 11.4.** If \( I \) is an ideal in the ring \( \mathcal{R}^k(K^n) \) of \( k \)-regulous functions on \( K^n \), then

\( \text{Rad}(I) = \mathcal{I}(\mathcal{Z}(I)), \)
where
\[ I(E) := \{ f \in R^k(K^n) : f(x) = 0 \text{ for all } x \in E \} \]
for a subset \( E \) of \( K^n \).

**Proof.** The inclusion \( \text{Rad}(I) = I(Z(I)) \) is obvious. For the converse inclusion, observe that, since the \( k \)-regulous topology is noetherian (Proposition 10.10), there is a function \( g \in I \) such that \( Z(I) = Z(g) \). Then \( Z(g) \subset Z(f) \) for any \( f \in I(Z(I)) \), and thus the function \( 1/g \) is \( k \)-regulous on the set \( \mathcal{D}(f) \). By Corollary 11.2, we get
\[ \frac{f^s}{g} \in R^k(K^n) \]
for \( s \gg 0 \) large enough. Hence
\[ f^s \in g \cdot R^k(K^n) \subset I, \]
concluding the proof. \( \square \)

**Corollary 11.5.** There is a one-to-one correspondence between the radical ideals of the ring \( R^k(K^n) \) and the \( k \)-regulous closed subsets of \( K^n \). Consequently, the prime ideals of \( R^k(K^n) \) correspond to the irreducible \( k \)-regulous closed subsets of \( K^n \), and the maximal ideals \( m \) of \( R^k(K^n) \) correspond to the points \( x \) of \( K^n \) so that we get the bijection
\[ K^n \ni x \mapsto m_x := \{ f \in R^k(K^n) : f(x) = 0 \} \in \text{Max}(R^k(K^n)). \]
The resulting embedding
\[ \iota : K^n \ni x \mapsto m_x \in \text{Spec}(R^k(K^n)) \]
is continuous in the \( k \)-regulous and Zariski topologies, and induces a one-to-one correspondence between the \( k \)-regulous closed subsets of \( K^n \) and the subsets of \( \text{Spec}(R^k(K^n)) \) closed in the Zariski topology.

**Proof.** The embedding \( \iota \) is continuous by the very definition of the Zariski topology. The last assertion follows immediately from the Nullstellensatz and the fact that the closed subsets of \( \text{Spec}(R^k(K^n)) \) are precisely of the form
\[ \{ p \in \text{Spec}(R^k(K^n)) : p \supset I \}, \]
where \( I \) runs over all radical ideals of \( \text{Spec}(R^k(K^n)). \) \( \square \)

The above corollary along with Proposition 10.10 and Corollary 10.11 yield immediately
Corollary 11.6. With the above notation, the space $\text{Spec}(\mathcal{R}^k(K^n))$ with the Zariski topology is noetherian, and the embedding $\iota$ induces a one-to-one correspondence between the $k$-regulous open subsets of $K^n$ and the subsets of $\text{Spec}(\mathcal{R}^k(K^n))$ open in the Zariski topology. In particular, every open subset of $\text{Spec}(\mathcal{R}^k(K^n))$ is of the form

$$\mathcal{U}(f) := \{p \in \text{Spec}(\mathcal{R}^k(K^n)) : f \notin p\}, \quad f \in \mathcal{R}^k(K^n),$$

corresponding to the subset $D(f)$ of $K^n$.

Remark 11.7. As demonstrated in paper [15] (see also [30, Ex. 6.11]), the ring $\mathcal{R}^k(K^n)$ is not noetherian for all $k, n \in \mathbb{N}, n \geq 2$.

From Theorem 11.4 and Corollary 10.11, we immediately obtain

Corollary 11.8. Every radical ideal of $\mathcal{R}^k(K^n)$ is the radical of a principal ideal of $\mathcal{R}^k(K^n)$.

Finally, we return to the comparison of the regulous and constructible topologies. Below we state the non-archimedean version of [15, Theorem 6.4] by Fichou–Huisman–Mangolte–Monnier, which says that those topologies coincide in the real algebraic geometry. The proof relies on their Lemmata 5.1 and 5.2, and can be repeated verbatim in the case of the ground fields $K$ studied in our paper.

Theorem 11.9. The $k$-regulous closed subsets of $K^n$ are precisely the closed (in the $K$-topology) constructible subsets of $K^n$.

The above theorem along with Proposition 10.2 and Corollary 11.5 yield the following

Corollary 11.10. There are one-to-one correspondences between the prime ideals of the ring $\mathcal{R}^k(K^n)$, the irreducible closed constructible subsets of $K^n$ and the irreducible Zariski closed subsets of $K^n$.

Corollary 11.11. The dimension of the topological space $K^n$ with the regulous topology and the Krull dimension of the ring $\mathcal{R}^k(K^n)$ is $n$.

12. Quasi-coherent regulous sheaves

The concepts of quasi-coherent $k$-regulous sheaves on $K^n$ and $k$-regulous affine varieties, $k \in \mathbb{N} \cup \{\infty\}$, can be introduced over valued fields studied in this paper, similarly as by Fichou–Huisman–Mangolte–Monnier [15] over the real ground field $\mathbb{R}$. Also, the majority of their results concerning these concepts carry over to the non-archimedean geometry with similar proofs. For the sake of completeness, we provide an exposition of the theory of quasi-coherent regulous sheaves. Here we shall deal only with $k$-regulous functions with a non-negative integer $k$. 
because for $k = \infty$ we encounter the classical case of regular functions and quasi-coherent algebraic sheaves.

Consider an affine scheme $Y = \text{Spec}(A)$ with structure sheaf $\mathcal{O}_Y$. Any $A$-module $M$ determines a quasi-coherent sheaf $\tilde{M}$ on $Y$ (cf. [19, Chap. II]). The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of $A$-modules and the category of quasi-coherent $\mathcal{O}_Y$-modules. Its inverse is the global sections functor

$$\mathcal{F} \mapsto H^0(Y, \mathcal{F})$$

(op.cit., Chap. II, Corollary 5.5).

Denote by $\tilde{R}^k$ the structure sheaf of the affine scheme $\text{Spec}(\mathcal{R}^k(K^n))$ and by $\mathcal{R}^k$ the sheaf of $k$-regulous function germs (in the $k$-regulous topology equal to the constructible topology) on $K^n$. It follows directly from Corollaries 11.5 and 11.2 that the restriction $\iota^{-1}\tilde{R}^k$ of $\tilde{R}^k$ to $K^n$ coincides with the sheaf $\mathcal{R}^k$; conversely, $\iota_* \mathcal{R}^k = \tilde{R}^k$.

By a $k$-regulous sheaf $\mathcal{F}$ we mean a sheaf of $\mathcal{R}^k$-modules. Through Corollaries 11.5 and 11.6, the functor $\iota^{-1}$ of restriction to $K^n$ gives an equivalence of categories between $\tilde{R}^k$-modules and $\mathcal{R}^k$-modules. Its inverse is the direct image functor $\iota_*$. We say that $\mathcal{F}$ is a quasi-coherent $k$-regulous sheaf on $K^n$ if it is the restriction to $K^n$ of a quasi-coherent $\tilde{R}^k$-module. Thus the functor $\iota^{-1}$ induces an equivalence of categories between quasi-coherent $\tilde{R}^k$-modules and quasi-coherent $\mathcal{R}^k$-modules, whose inverse is the direct image functor $\iota_*$. For any $\mathcal{R}^k(K^n)$-module $\mathcal{M}$, we shall denote by $M$ both the associated sheaf on $\text{Spec}(\mathcal{R}^k(K^n))$ and its restriction to $K^n$. This abuse of notation does not lead to confusion. We thus obtain the following version of Cartan’s Theorem A:

**Theorem 12.1.** The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of $\mathcal{R}^k(K^n)$-modules and the category of quasi-coherent $\mathcal{R}^k$-modules. Its inverse is the global sections functor

$$\mathcal{F} \mapsto H^0(K^n, \mathcal{F}).$$

In particular, every quasi-coherent sheaf $\mathcal{F}$ is generated by its global sections $H^0(K^n, \mathcal{F})$. $\square$

The regulous version of Cartan’s Theorem B, stated below, follows directly from the version for affine (not necessarily noetherian in view of Remark 11.7) schemes (cf. [18, Theorem 1.3.1]) via the discussed equivalence of categories (being the functor $\iota^{-1}$ of restriction to $K^n$).
**Theorem 12.2.** If $\mathcal{F}$ is a quasi-coherent $k$-regulous sheaf on $K^n$, then 
\[ H^i(K^n, \mathcal{F}) = 0 \text{ for all } i > 0. \]

**Corollary 12.3.** The global sections functor 
\[ \mathcal{F} \mapsto H^0(K^n, \mathcal{F}) \]
on the category of quasi-coherent $k$-regulous sheaves on $K^n$ is exact.

Let $V$ be a $k$-regulous closed subset of $K^n$ and $\mathcal{I}(V)$ the sheaf of those $k$-regulous function germs on $K^n$ that vanish on $V$. It is a quasi-coherent sheaf of ideals of $\mathcal{R}^k$, the sheaf $\mathcal{R}^k/\mathcal{I}(V)$ has support $V$ and is generated by its global sections (Theorem A); moreover 
\[ H^0(K^n, \mathcal{R}^k/\mathcal{I}(V)) = H^0(K^n, \mathcal{R}^k)/H^0(K^n, \mathcal{I}(V)) \]
(Theorem B). The subset $V$ inherits the $k$-regulous topology from $K^n$ and constitutes, together with the restriction $\mathcal{R}^k_V$ of the sheaf $\mathcal{R}^k/\mathcal{I}(V)$ to $V$, a locally ringed space of $K$-algebras, called an affine $k$-regulous subvariety of $K^n$. More generally, by an affine $k$-regulous variety we mean any locally ringed space of $K$-algebras that is isomorphic to an affine $k$-regulous subvariety of $K^n$ for some $n \in \mathbb{N}$.

We can define in the ordinary fashion the category of quasi-coherent $\mathcal{R}^k_V$-modules. Each such module extends trivially by zero to a quasi-coherent $\mathcal{R}^k$-module on $K^n$. The sections $\mathcal{R}^k_V(V)$ of the structure sheaf $\mathcal{R}^k_V$ are called $k$-regulous functions on $V$. It follows from Cartan’s Theorem B that each $k$-regulous function on $V$ is the restriction to $V$ of a $k$-regulous function on $K^n$. Hence we immediately obtain the following two results.

**Proposition 12.4.** Let $W$ and $V$ be two affine $k$-regulous subvarieties of $K^m$ and $K^n$, respectively. Then the following three conditions are equivalent:

1) $f : W \to V$ is a morphism of locally ringed spaces;

2) $f = (f_1, \ldots, f_n) : W \to K^n$ where $f_1, \ldots, f_n$ are $k$-regulous functions on $W$ such that $f(W) \subset V$;

3) $f$ extends to a $k$-regulous map $K^m \to K^n$. □

We then call $f : W \to V$ a $k$-regulous map.

**Corollary 12.5.** Let $W$, $V$ and $X$ be affine $k$-regulous subvarieties of $K^m$, $K^n$ and $K^p$, respectively. If two maps 
\[ g : W \to V \text{ and } f : V \to X \]
are $k$-regulous, so is its composition $f \circ g$. □
It is clear that Cartan’s theorems remain valid for quasi-coherent $k$-regulous sheaves on affine $k$-regulous varieties $V$.

**Theorem 12.6.** The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of $\mathcal{R}_V^k(V)$-modules and the category of quasi-coherent $\mathcal{R}_V^k$-modules. Its inverse is the global sections functor $\mathcal{F} \mapsto H^0(V, \mathcal{F})$.

In particular, every quasi-coherent sheaf $\mathcal{F}$ is generated by its global sections $H^0(V, \mathcal{F})$. \hfill \Box

**Theorem 12.7.** If $\mathcal{F}$ is a quasi-coherent $k$-regulous sheaf on $V$, then $H^i(V, \mathcal{F}) = 0$ for all $i > 0$. \hfill \Box

**Corollary 12.8.** The global sections functor $\mathcal{F} \mapsto H^0(V, \mathcal{F})$ on the category of quasi-coherent $k$-regulous sheaves on $V$ is exact. \hfill \Box

Note that every non-empty $k$-regulous open subset $U$ of $K^n$ is an affine $k$-regulous variety. Indeed, if $U = D(f)$ for a $k$-regulous function $f$ on $K^n$ (Corollary 10.11), then $U$ is isomorphic to the affine $k$-regulous subvariety $V := Z(yf(x) - 1) \subset K_x^n \times K_y$.

**Remark 12.9.** Consider a smooth algebraic subvariety $X$ of the affine space $KA^n$. We may look at the set $V := X(K)$ of its $K$-points both as an algebraic variety $X(K)$ and as a $k$-regulous subvariety $V$ of $K^n$. Every function $f : V \to K$ that is $k$-regulous on $V$ in the second sense remains, of course, $k$-regulous on $X(K)$ in the sense of the definition from the beginning of Section 10.

**Open problem.** The problem whether the converse implication is true for $k > 0$ is unsolved as yet.

For $k = 0$ the answer is in the affirmative, which follows immediately from Theorem 9.2: given an arbitrary algebraic subvariety $X$ of $KA^n$, every continuous hereditarily rational function $f$ on $X(K)$ extends to a continuous rational function on $K^n$, whence $f$ is regulous on $V$. This theorem was proved for real and $p$-adic varieties in paper [24].

Finally, we wish to give a criterion for a continuous function to be regulous. It relies on Theorem 9.2 on extending continuous hereditarily rational functions on algebraic $K$-varieties.
Theorem 12.10. Let $V$ be an affine regulous subvariety of $K^n$ and $f : V \rightarrow K$ a function continuous in the $K$-topology. Then a necessary and sufficient condition for $f$ to be a regulous function is the following:

(*) For every Zariski closed subset $Z$ of $K^n$ there exist an open Zariski dense subset $U$ of the Zariski closure of $V \cap Z$ in $K^n$ and a regular function $g$ on $U$ such that

$$f(x) = g(x) \quad \text{for all } x \in V \cap Z \cap U.$$  

Proof. By Corollary 10.5, the necessary condition is clear, because $f$ is the restriction to $V$ of a regulous function on $K^n$ (Corollary 12.3 to Cartan’s Theorem B).

In order to prove the sufficient condition, we proceed with induction with respect to the dimension $d$ of the set $V$ which is a closed (in the $K$-topology) constructible subset of $K^n$. The case $d = 0$ is trivial. Assuming the assertion to hold for dimensions less than $d$, we shall prove it for $d$. So suppose $V$ is of dimension $d$. By Proposition 10.2, the Zariski closure $W$ of $V$ in $K^n$ is of dimension $d$ and we have

$$W^0 := \{ x \in W : W \text{ is smooth of dimension } d \text{ at } x \} \subset V.$$  

Obviously, $W^* := W \setminus W^0$ is a Zariski closed subset of $K^n$ of dimension less than $d$. Therefore $Y := V \cap W^*$ is a regulous closed subset of $V$ of dimension less that $d$ and

$$W = W^0 \cup W^* \subset V \cup W^* \subset W.$$  

Since the restriction $f|Y$ satisfies condition (*), it is a regulous function on $Y$ by the induction hypothesis. It is thus the restriction to $Y$ of a regulous function $F$ on $K^n$ (Corollary 12.3 to Cartan’s Theorem B).

Further, the function $f$ and the restriction $F|W^*$ can be glued to a function

$$g : W = V \cup W^* \rightarrow K, \quad g(x) = \begin{cases} f(x) : x \in V \\ F(x) : x \in W^* \end{cases}$$

which satisfies condition (*) as well. Now, it follows from Theorem 9.2 that $g$ extends to a regulous function $G$ on $K^n$. Since $f$ is the restriction to $V$ of the function $G$ which is regulous, so is $f$, as required. \qed

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