Parametrically amplified radiation in a cavity with an oscillating wall

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Abstract

We introduce a time-dependent perturbation method to calculate the number of created particles in a 1D cavity with an oscillating wall of the frequency $\Omega$. This method makes it easy to find the dominant part of the solution which results from the parametric resonance. The maximal number of particles are created at the mode frequency $\Omega/2$. Using the Floquet theory, we discuss the long-time behavior of the particle creation.

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I. INTRODUCTION

The particle creation by the parametric resonance is a very important phenomenon to understand the mechanism of reheating after inflation in the early universe [1]. The photon production by the parametric resonance in a cavity with an oscillating wall is another important phenomenon to be observed in the experimental situation. Recently, the photon creation in an empty cavity with oscillating boundaries has attracted much attention [2–5]. It was also proposed that the high-
\( Q \)
emagnetic cavities may provide a possibility to detect the photons produced in the nonstationary Casimir effect [6,7]. Therein, they considered the vibrating wall with the frequency \( \Omega = 2\omega_1 \) and found the resonance excitation of the electromagnetic modes.

In this paper we consider the quantum (electromagnetic) field in a cavity with an oscillating boundary of the frequency \( \Omega \). We calculate the number of particles (photons) produced by the parametric resonance. For the calculation of time-evolution of quantum fields, we introduce a time-dependent perturbation method that makes it possible to calculate the photon number for any \( \Omega \) and to observe clearly the effect of the parametric resonance. For a long-time behavior of the quantum field we use the Floquet theory.

The organization of this paper is as follows. In Sec. II we review the scheme of the field quantization in the case of moving boundaries. In Sec. III we introduce a new perturbation method to find the time evolution of the quantum field. Here we write the dominant part of the solution of wave equation which results from the parametric resonance. We calculate the number of particles created by the vibration of the boundary [8]. In Sec. IV we use the Floquet theory in the perturbation expansion to examine the long-time behavior of the solutions. We develeop the method to find the characteristic exponent of the solution and the periodic part of the corresponding solution that is linear combinations of the mode functions. We get the three term recurrence relations between the coefficients of the mode functions. The last section is devoted to the summary and discussion.

II. QUANTUM FIELDS IN A 1D CAVITY WITH A MOVING BOUNDARY

Let us consider a quantum field obeying the wave equation \((c = 1)\)

\[
\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = 0
\]

with time-dependent boundary conditions:

\[
A(0, t) = 0 = A(L(t), t).
\]

The field operator in the Heisenberg representation \(A(x, t)\) can be expanded as

\[
A(x, t) = \sum_n \left[ b_n \psi_n(x, t) + b_n^\dagger \psi_n^*(x, t) \right],
\]

where \(b_n^\dagger\) and \(b_n\) are the creation and the annihilation operators and \(\psi_n(x, t)\) is the corresponding mode function which satisfies the boundary condition \(\psi_n(0, t) = 0 = \psi_n(L(t), t)\).
For an arbitrary moment of time, following the approach of Refs. [9–11], we expand the mode function as

$$\psi_n(x, t) = \sum_k Q_{nk}(t) \varphi_k(x, t)$$  \hspace{1cm} (2.4)$$

with the instantaneous basis

$$\varphi_k(x, L(t)) = \sqrt{\frac{2}{L(t)}} \sin \frac{\pi k x}{L(t)}.$$  \hspace{1cm} (2.5)$$

Here $Q_{nk}(t)$ obeys an infinite set of coupled differential equations [12]:

$$\ddot{Q}_{nk} + \omega_k^2(t)Q_{nk} = 2\lambda \sum_j g_{kj} \dot{Q}_{nj} + \dot{\lambda} \sum_j g_{kj} Q_{nj} + \lambda^2 \sum_{j,l} g_{jk} g_{jl} Q_{nl}$$  \hspace{1cm} (2.6)$$

where $\lambda = \dot{L}/L$ and

$$g_{kj} = \begin{cases} 
(-1)^{k-j} \frac{2kj}{j^2 - k^2} & (j \neq k) \\
0 & (j = k)
\end{cases}.$$  \hspace{1cm} (2.7)$$

and the time-dependent mode frequency is

$$\omega_k(t) = \frac{k\pi}{L(t)}.$$  \hspace{1cm} (2.8)$$

For $L(t \leq 0) = L_0$, the right hand side of Eq. (2.6) vanishes and the solution in this region is chosen to be

$$Q_{nk}(t) = e^{-i\omega_k t} \delta_{nk}.$$  \hspace{1cm} (2.9)$$

so that the field (2.3) can be written as

$$A(x, t \leq 0) = \sum_n [b_n e^{-i\omega_n t} \varphi_n(x, L_0) + H.c.]$$  \hspace{1cm} (2.10)$$

where $\omega_n = \frac{\pi n}{L_0}$. For the scalar field or the electromagnetic field, the Hamiltonian has the form $H = \sum_n \omega_n (b_n^\dagger b_n + \frac{1}{2})$, and we can interpret $b_n^\dagger b_n$ as the number operator associated with the particle with the frequency $\omega_n$.

After the change of the boundary, we assume $L(t \geq T) = L_0$, then the solution of Eq. (2.6) with the initial condition (2.9), can be written as

$$Q_{nk}(t \geq T) = \alpha_{nk} e^{-i\omega_k t} + \beta_{nk} e^{i\omega_k t}.$$  \hspace{1cm} (2.11)$$

From (2.3) and (2.4), we have
\[ A(x, t \geq T) = \sum_n a_n e^{-i\omega_n t} \sqrt{2\omega_n} \varphi_n(x, L_0) + \text{H.c.}, \quad (2.12) \]

where
\[
a_k = \sum_n [b_n \alpha_{nk} + b_n^\dagger \beta_{nk}^*] \\
a_k^\dagger = \sum_n [b_n^\dagger \alpha_{nk}^* + b_n \beta_{nk}]. \quad (2.13)
\]

Further, it follows from \( H = \sum_n \omega_n (a_n^\dagger a_n + \frac{1}{2}) \) that \( a_n^\dagger a_n \) is the new number operator at \( t \geq T \).

If we start with a vacuum state \( |0_b\rangle \) such that \( b |0_b\rangle = 0 \), the expectation value of the new number operator is
\[
N_k = \langle 0_b | a_k^\dagger a_k | 0_b \rangle = \sum_{n=1}^{\infty} |\beta_{nk}|^2 , \quad (2.14)
\]
which is the number of created particles. (Note that the quantum state does not evolve in time in the Heisenberg picture.)

III. TIME EVOLUTION OF THE QUANTUM FIELD IN A CAVITY WITH AN OSCILLATING BOUNDARY

In this section we find the time evolution of quantum field operator \( (2.3) \) by solving Eq. \( (2.6) \) with the motion of the wall given by
\[
L(t) = L_0[1 + \epsilon \sin(\Omega t)]. \quad (3.1)
\]
Here \( \Omega = \gamma \omega_1 = \gamma \pi / L_0 \) and \( \epsilon \) is a small parameter characterized by the displacement of the wall. For \( \epsilon \ll 1 \), having in mind that \( \lambda(t) \sim \epsilon \) and taking the first order of \( \epsilon \) in the mode frequency \( (2.8) \)
\[
\omega_k(t) = \frac{k\pi}{L_0} [1 + \epsilon \sin(\Omega t)]^{-1} , \quad (3.2)
\]
we can replace Eq. \( (2.7) \) by a pair of coupled first-order differential equations
\[
\dot{Q}_{nk} = P_{nk} \]
\[
\dot{P}_{nk} = -\omega_k^2 (1 - 2\epsilon \sin \Omega t) Q_{nk} + 2 \frac{\dot{L}}{L} \sum_j g_{kj} P_{nj} + \frac{\ddot{L}}{L} \sum_j g_{kj} Q_{nj} + O(\epsilon^2). \quad (3.3)
\]
Introducing the new dynamical variables
\[
X_{n,k\mp} = \sqrt{\frac{\omega_k}{2}} \left( Q_{nk} \pm \frac{i}{\omega_k} P_{nk} \right) \quad (3.4)
\]
and the vector notation

\[
\vec{X}_n(t) = \begin{pmatrix}
X_{n,1-} \\
X_{n,1+} \\
X_{n,2-} \\
\vdots
\end{pmatrix},
\]  

(3.5)

the above equation (3.3) can be written as

\[
\frac{d}{dt} \vec{X}_n(t) = V^{(0)} \vec{X}_n(t) + \epsilon V^{(1)} \vec{X}_n(t)
\]  

(3.6)

where \( V^{(0)} \) and \( V^{(1)} \) are matrices given by

\[
V^{(0)}_{k\sigma,j\sigma'} = i\omega_k \sigma \delta_{kj} \delta_{\sigma\sigma'}
\]  

(3.7)

and

\[
V^{(1)}_{k\sigma,j\sigma'} = \sum_{s=\pm} \omega_1 v^s_{k\sigma,j\sigma'} e^{si\omega_1 t},
\]  

(3.8)

where

\[
v^s_{k\sigma,j\sigma'} = \sigma \gamma g_{kj} \sqrt{\frac{j}{k}} \left( \frac{\sigma'}{2} + s \frac{\gamma}{4j} \right) - s\sigma \frac{k}{2} \delta_{kj}
\]  

(3.9)

with \( s, \sigma, \sigma' = +, - \). Here we used \( \Omega = \gamma \omega_1 \) and \( \omega_k = k\omega_1 \).

To find the solution of Eq. (3.6), we introduce a perturbation expansion:

\[
\vec{X}_n = \vec{X}^{(0)}_n + \epsilon \vec{X}^{(1)}_n + \epsilon^2 \vec{X}^{(2)}_n + \cdots.
\]  

(3.10)

By inserting (3.10) into Eq. (3.6), identifying powers of \( \epsilon \) yields a series of equations:

\[
\frac{d}{dt} \vec{X}^{(0)}_n = V^{(0)} \vec{X}^{(0)}_n
\]  

(3.11)

\[
\frac{d}{dt} \vec{X}^{(1)}_n = V^{(1)} \vec{X}^{(0)}_n + V^{(0)} \vec{X}^{(1)}_n.
\]  

(3.12)

From the initial condition (2.9), we have the solution to zeroth order equation (3.11)

\[
X^{(0)}_{n,k\sigma} = \delta_{nk} \delta_{\sigma \sigma} e^{-i\omega_k t},
\]  

(3.13)

and to the first order equation (3.12)

\[
X^{(1)}_{n,k\sigma}(t) = \omega_1 e^{\sigma' \omega_1 t} \int_0^t dt' v^s_{k\sigma,n} e^{-i(\sigma' k + \sigma n)\omega_1 t'}.
\]  

(3.14)

When the exponent of exponential function in the integrand of (3.14) vanishes, we have terms proportional to \( \omega_1 t \) which are the effects of parametric resonance. In the usual situation, since \( \omega_1 t \gg 1 \), only the resonance terms are dominant and the solution becomes by retaining only them:
After some time interval $T$ the wall stops at $x = L_0$, then the solution is described by (2.11). By comparing (2.11) with (3.13), the Bogoliubov coefficient $\beta_{nk}$ can be read from the solution $Q_{nk}$ to the leading order in $\epsilon$

$$\beta_{nk} = \epsilon \omega_1 T v^+_{k+,n} - \delta_{k,\gamma-n},$$

which is the coefficient of negative frequency mode function in (3.13).

Using (2.7) and (3.9), finally we have

$$|\beta_{nk}|^2 = \frac{1}{4} nk (\epsilon \omega_1 T)^2 \delta_{k,\gamma-n}. \quad (3.17)$$

Therefore the total number of particles created in the $k$ th mode from the empty cavity is

$$N_k = \sum_{n=1}^{\infty} |\beta_{nk}|^2 = \begin{cases} \frac{1}{4} (\gamma - k) k (\epsilon \omega_1 T)^2 & k < \gamma, \\ 0 & \text{otherwise}. \end{cases} \quad (3.18)$$

This result is a generalization of Ref. [7] in the short time limit ($\epsilon \omega_1 T \ll 1$) and it agrees with that result for $\gamma = 2$ and $k = 1$. It should also be noted that the maximal number of photons are created at the mode frequency

$$k = \frac{\gamma}{2} \text{ or } \omega_k = \frac{\Omega}{2}. \quad (3.19)$$

for $\gamma = \text{even}$ and at its nearest neighbor frequencies $k = (\gamma \pm 1)/2$ for $\gamma = \text{odd}$.

**IV. LONG-TIME BEHAVIOR OF THE SOLUTION: PERTURBATION APPROACH USING FLOQUET THEORY**

In this section we discuss the long-time behavior of the solution to the differential equation (3.6). Although the time-dependent perturbation method developed in the previous section gives the method to calculate the higher order solution, it does not provide the convergence of the solution. So it is difficult to examine the long-time behavior of the solution. Here we develop another perturbation method using the Floquet theory.

Consider a $\tau$-periodic system of differential equations such as (3.6):

$$\frac{d}{dt} \vec{X}(t) = [V(0) + \epsilon V(1)] \vec{X}(t) \quad (4.1)$$

with the periodic condition

$$[V(0) + \epsilon V(1)](t + \tau) = [V(0) + \epsilon V(1)](t), \quad (4.2)$$
where $\tau = 2\pi/\Omega$. The Floquet theory states that the solution of (4.1) should be of the form

$$\bar{X}(t) = e^{\mu t} \tilde{Z}(t)$$

(4.3)

where $\tilde{Z}(t)$ is $\tau$-periodic. For simplicity of the indices of the matrix, we will slightly change the notation:

$$\bar{X}(t) = \begin{pmatrix}
\vdots \\
X_{-2} \\
X_{-1} \\
X_1 \\
X_2 \\
\vdots
\end{pmatrix},$$

(4.4)

$$V^{(0)}_{kj} = ik\omega_1 \delta_{kj},$$

(4.5)

and

$$V^{(1)}_{kj} = \sum_{s=\pm} \omega_1 v^s_{k,j} e^{is\gamma \omega_1 t},$$

(4.6)

where

$$v^s_{k,j} = \gamma g_{k,j} \sqrt{\frac{j}{k}} \left(\frac{1}{2} + s\frac{\gamma}{4j}\right) - \frac{k}{2} \delta_{|k||j|}$$

(4.7)

and $k$ and $j$ are nonzero integers.

To find the solution of Eq. (4.1), we introduce a perturbation expansion:

$$\bar{X}(t) = e^{(\epsilon \mu_1 + \epsilon^2 \mu_2 + \ldots) \omega_1 t}$$

$$\times [\tilde{Z}^{(0)}(t) + \epsilon \tilde{Z}^{(1)}(t) + \epsilon^2 \tilde{Z}^{(2)}(t) + \ldots].$$

(4.8)

The zeroth order equation

$$\frac{d}{dt} \tilde{Z}^{(0)}(t) = V^{(0)} \tilde{Z}^{(0)}(t)$$

(4.9)

and the first order equation

$$\frac{d}{dt} \tilde{Z}^{(1)}(t) + \mu_1 \tilde{Z}^{(0)}(t) = V^{(0)} \tilde{Z}^{(1)}(t) + V^{(1)} \tilde{Z}^{(0)}(t)$$

(4.10)

can be easily solved as:

$$Z^{(0)}_k = C_k e^{i\omega_k t}$$

(4.11)

and
\[ Z^{(1)}_k = e^{ik\omega t} \int_0^t dt' e^{-ik\omega t'} \times \sum_j \left[ \sum_s v^s_{k,j} e^{i\gamma \omega t'} - \mu_1 \delta_{kj} \right] C_j e^{ij\omega t}. \] (4.12)

When the exponent of the exponential function in the integrand of (4.12) vanishes, the integration gives the term proportional to \( t \). This contradicts the periodicity condition of \( Z(t) \), therefore the coefficient of such term should vanish. Thus, we have the following recurrence relation

\[ v_{k,k+\gamma}^+ C_{k+\gamma}^+ - \mu_1 C_k + v^+_{k,k-\gamma} C_{k-\gamma} = 0. \] (4.13)

Note that the coefficients are coupled to the \( \gamma \)th neighbor modes. This three term recurrence relation can be written as the following linear equation

\[ M(\mu_1) \vec{C} = 0 \] (4.14)

From the condition for the existence of the nontrivial solution, we can find the characteristic exponents by solving

\[ \det M(\mu_1) = 0. \] (4.15)

Then we have the eigenvalues \( \mu^A_1 \) and the corresponding eigenvectors \( \vec{C}^A \), where we introduced the superscript \( A \) to distinguish the eigenvectors. Then the characteristic solution is

\[ X^A_k(t \geq 0) = e^{\mu^A_1 \omega_1 t} C^A_k e^{ik\omega_1 t}. \] (4.16)

Note that the amplitude (the coefficient of the harmonic function) is exponentially increasing when the real part of the characteristic exponent \( \mu_1 \) is positive. This is the effect of the parametric resonance.

By linear combinations of the characteristic solutions, we find the solutions

\[ X_{n,k}(t) = \sum_A d_{nA} X^A_k(t) \] (4.17)

that satisfy the initial conditions

\[ X_{n,k}(t \leq 0) = e^{ik\omega_1 t} \delta_{nk} \theta(-n), \] (4.18)

where \( \theta \) is the Heaviside unit step function. From the initial conditions, the coefficients \( d_{nA} \) are obtained by solving the linear equations

\[ \sum_A d_{nA} C^A_k = \delta_{nk} \theta(-n). \] (4.19)

Since the matrix equation is infinite dimensional, it is difficult to find the full solution in a closed form. However, it is helpful to consider the truncated matrix in order to understand the long-time behavior of the solution. As the simplest model \( (\gamma = 2) \), we consider only the
mode frequency to the next neighboring frequency, e. g. to the $\omega_3$ for $\omega_1$ mode, we have the following eigenvalue equation:

$$
\begin{pmatrix}
-\mu_1 & -\sqrt{3}/2 & 0 & 0 \\
\sqrt{3}/2 & -\mu_1 & -1 & 0 \\
0 & -1 & -\mu_1 & \sqrt{3}/2 \\
0 & 0 & -\sqrt{3}/2 & -\mu_1
\end{pmatrix}
\begin{pmatrix}
C_{-3} \\
C_{-1} \\
C_1 \\
C_3
\end{pmatrix} = 0,
$$

(4.20)

then this equation has the following eigenvalues

$$
\mu_1 = \pm(1 \pm i\sqrt{2})/2
$$

(4.21)

and the corresponding eigenvectors

$$
\vec{C}(\mu_1) = \begin{pmatrix}
1 \\
-2\mu_1/\sqrt{3} \\
\sqrt{3}/2 + 2\mu_1^2/\sqrt{3} \\
\mu_1 + 4\mu_1(\mu_1^2 - 1)/3
\end{pmatrix}.
$$

(4.22)

In the long-time behavior, the solutions of which characteristic exponent is positive will be dominant.

**V. DISCUSSION**

We developed a perturbation method to find the time-evolution of the field in a cavity with an oscillating boundary. This method makes it possible to calculate the particle number for any oscillation frequency of the boundary and to observe clearly the effect of the parametric resonance. The results show that the effect of parametric resonance is the largest at the half of the frequency of the oscillating boundary ($\omega_k = \Omega/2$). This can be understood by considering the Mathieu equation

$$
\ddot{x} + \mu^2(1 + \epsilon \cos \Omega t)x = 0,
$$

(5.1)

where the parametric resonance takes place most strongly for $\Omega = 2\mu$. In addition, in the case $\gamma > 2$, we see the other resonance effects in addition to $\omega_k = \Omega/2$, which is due to the effect of couplings with other mode frequencies in the cavity.

We used the Floquet theory to examine the long-time behavior of the solutions and introduced the general scheme to find the solution that is available in the long time. For the simplest case, we found the characteristic exponents and the corresponding eigenvectors. In the parametric system, the stability-unstability structure is important because the resonance condition, $\omega_n = \Omega - \omega_k$ in (3.16), is hardly satisfied exactly in the experimental situation. In fact the condition of parametric resonance admits some discrepancy as seen from the solutions of the Mathieu equation. Therefore it is expected that the above simplest case is a good model to study the stability-unstability structure in the system of coupled parametric oscillators. We hope to report on this structure in a future paper. Finally we would like to mention that it is remained to solve the three term recurrence relation (4.13) in a future study.
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