Unshackling Linear Algebra from Linear Notation

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Have you ever seen one of those movies where the hero unearths an artifact covered with mysterious symbols, and it takes a brilliant scientist to decipher their meaning? The best of Hollywood creativity goes into devising a plausible system for the symbols. In the movie Contact, for instance, Jodie Foster discovers a datastream of symbols that turn out to be a 3-dimensional blueprint for a device to contact alien life. Hollywood’s tacit (and reasonable) assumption is that the mathematics of a different civilization would look very different.

This paper provides an accessible introduction to trace diagrams, a non-traditional notation for linear algebra that could plausibly have been developed by another civilization. Trace diagrams are a completely different way of looking at vectors and matrices. Vectors are represented by edges in a diagram, and matrices are markings along the edges of the diagram. Instead of representing dot products and cross products by $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$, one glues together the edges corresponding to those vectors. One example is shown in Figure 1.

\[ \text{Figure 1. A trace diagram representation of a matrix determinant.} \]
Surprisingly, the notation is perfectly rigorous, and often leads to proofs more elegant than those written using traditional notation, as we will show for vector identities. While trace diagrams have powerful applications in mathematics and physics \[1 \text{, } 2\], the only prerequisite is an understanding of basic linear algebra and a willingness to work some examples to get used to doing real math with “doodles”.

**On Notation.** Whether one believes that mathematics is created or discovered, *notation* is certainly created. And that notation can direct the course of mathematics. For instance, a matrix encodes the same information as a weighted directed graph, but the two notations inspire completely different questions. Matrices focus attention on concepts such as rank and invertibility, while graphs focus attention on nodes and flow between nodes.

Sometimes notations persist because they are useful and easy to understand, but there also cases where they exist simply because they are easier to write down in a single line of text. For example, permutation cycle notation such as \((2 \ 1 \ 3)\) is precise and easy to write down, but it is nonintuitive and often confuses beginners. The same permutation is much more clearly depicted by the diagram

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\quad \leftrightarrow
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\]

There are many other cases, such as commutative diagrams, graph theory, and knot theory, where any attempt to fit the concepts into a single line would render them incomprehensible.

Outside of a few niche areas, diagrammatic notations have not been widely accepted. This is at least partly due to the difficulty in publishing. In fact, notations similar to trace diagrams have been used by mathematicians and physicists for years. In some cases researchers have “invented” new diagrammatic notations, only to discover that they almost perfectly replicate notation in a long-forgotten manuscript.
In the case of trace diagrams, there is a huge benefit in escaping the confines of a single line of text. The diagrammatic notation focuses attention on different questions, opening up new ideas that are not as readily achieved in the traditional manner. They also provide a refreshing perspective on the traditional theory.

Outline. We begin with the definition of trace diagrams, and move directly into two special cases that help orient the reader to the diagrammatic point-of-view. We then provide an explicit description of how they are calculated. Finally, we provide the diagrammatic perspective on some questions often posed by students seeing vectors and linear algebra for the first time. We also look at some questions inspired by the diagrammatic notation.

We include several examples and exercises throughout, which are particularly important for adjusting to nonstandard notation. The reader is strongly encouraged to follow the examples closely and to complete the exercises.

1. General Trace Diagrams

Definition. Given an underlying vector space of dimension $n$, a trace diagram is a graph whose edges represent vectors. Endpoints of edges may be labeled by vectors, may be left free, or may be joined at ordered nodes of degree $n$. Matrix markings may occur along any edge.

By ordered node, we mean that the edges adjacent to the node are ordered. Matrix markings are directed nodes placed along an edge and labeled by an $n \times n$ matrix.

The free ends of the diagram, along with those labeled by vectors, are typically partitioned into a set of inputs and a set of outputs. Each trace diagram gives rise to a function with corresponding input and output vectors.

If a trace diagram consists of two disconnected pieces, the individual functions are multiplied together to obtain the function of the entire
diagram. Sums of multiple diagrams are also permitted, with the corresponding functions added together.

The next two sections describe special cases of trace diagrams, and are intended to clarify these ideas and to help the reader grow comfortable with diagrammatic reasoning. We will describe functions corresponding to trace diagrams, but will not say why these are the right functions until section 4, where we answer the question of how to compute functions for arbitrary diagrams.

The following result encapsulates the power of the diagrams:

**Fact.** Topologically equivalent trace diagrams correspond to equal functions, provided (i) the ordering at nodes is preserved, and (ii) the ordering of inputs and outputs is consistent.

Although the diagrams are drawn in the plane, no proof is necessary of this result since trace diagrams are defined combinatorially. We refer the reader to [4] for further details.

2. 3-Diagrams

The two main operations on 3-dimensional vectors are the cross product and the dot product. In 3-diagrams, a special case of trace diagrams that has been used for many years [6], the diagrammatic forms of these operations are

\[
\mathbf{u} \times \mathbf{v} \leftrightarrow \begin{array}{c}
\mathbf{u} \\
\mathbf{v}
\end{array} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{v} \leftrightarrow \begin{array}{c}
\mathbf{u} \\
\mathbf{v}
\end{array}
\]

A 3-diagram is a graph with nodes of degree 3 and edges representing 3-vectors. If an edge is labeled by a particular vector, indicating that it has known value, it is an input to the diagram's function. These are usually drawn at the bottom of the diagram. If an edge is free, meaning it has an unlabeled end, it is an output of the function, and usually drawn at the top of the diagram.

Both diagrams in (1) have input vectors \( \mathbf{u} \) and \( \mathbf{v} \). In the cross product diagram, the third strand is the output, identified with the vector
result \( \mathbf{u} \times \mathbf{v} \). The dot product diagram has no outputs. In this case, we say that the \textit{value} of the diagram is the scalar result \( \mathbf{u} \cdot \mathbf{v} \).

\textbf{Example 1.} Draw the identity

\[
(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w}).
\]

\textit{Solution.} Keeping the vector inputs in the same order, the diagram is:

\[
\begin{align*}
&\text{\includegraphics[width=2cm]{dot_product_diagram1.png}} \\
&\text{\includegraphics[width=2cm]{dot_product_diagram2.png}} \\
&\text{\includegraphics[width=2cm]{dot_product_diagram3.png}} \\
&\text{\includegraphics[width=2cm]{dot_product_diagram4.png}}.
\end{align*}
\]

\textbf{Exercise 2.} What vector identity does this diagram represent?

\[
(2) \quad \begin{align*}
&\text{\includegraphics[width=2cm]{dot_product_diagram5.png}} = \text{\includegraphics[width=2cm]{dot_product_diagram6.png}} = \text{\includegraphics[width=2cm]{dot_product_diagram7.png}} = \text{\includegraphics[width=2cm]{dot_product_diagram8.png}}.
\end{align*}
\]

(The reader is encouraged also to guess the meaning of the fourth term.)

This exercise illustrates the first kind of diagrammatic proof: the \textit{“unproof”}. No work is required to prove \((2)\), since the pictures are topologically equivalent and maintain the same ordering at the vertex. In contrast, here is the traditional direct proof of the first equality. Take \( \mathbf{u} = (u_1, u_2, u_3) \) and a similar notation for \( \mathbf{v} \) and \( \mathbf{w} \). Then

\[
\begin{align*}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \cdot (w_1, w_2, w_3) \\
&= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3;
\end{align*}
\]

\[
\begin{align*}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (u_1, u_2, u_3) \cdot (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \\
&= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1 \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\end{align*}
\]

The proof is trivial, but completely unenlightening. A student might assume the identity is just a coincidence. In truth, the identity exists because of notation rather than the underlying mathematics.

A second type of proof is the \textit{“surgery proof”}, in which we use what we know about simple diagrams to perform manipulations with more
complicated diagrams. For example, we know from Example 1 that

\[(3) \quad \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}} = \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}} - \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}}.\]

Here, stating the identity without labelings indicates that the identity is true for any choice of vector labeling, under the assumption that inputs in the same position are labeled by the same vectors. This identity may be applied at any two adjacent nodes in a 3-diagram.

**Example 3.** Find an alternate expression of \((u \times v) \times w\).

**Solution.** First, represent \((u \times v) \times w\) in diagrammatic form. Then apply (3) in the neighborhood of the two nodes, manipulating the positions of the strands as appropriate:

\[\text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}} = \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}} - \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\node[draw,fill=cyan](3) at (2,0) \{};\node[draw,fill=cyan](4) at (3,0) \{};\end{tikzpicture}}.\]

This proves the following identity:

\[(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.\]

By repeated application of (3), one can write any single expression involving multiple cross products as a sum of expressions with at most one.

### 3. Diagrams with Matrices

In diagrams with matrices, the *matrix markings* are directed nodes placed along edges. We use a chevron symbol to represent matrices, as in the diagrams

\[(4) \quad AB = \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\end{tikzpicture}} = \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\end{tikzpicture}} \quad \text{and} \quad v^T A w = \text{\begin{tikzpicture}[baseline](0,0)\node[draw,fill=cyan](1) at (0,0) \{};\node[draw,fill=cyan](2) at (1,0) \{};\end{tikzpicture}},\]

where \(AB\) is a legal matrix product and \(v^T\) is a row vector. The directionality of the node is required to specify the input and output of the matrix.
One may use this to represent matrix entries diagrammatically, using the expression \( a_{ij} = \hat{e}_i A \hat{e}_j \) for the entry in the \( i \)th row and \( j \)th column of \( A \), where \( \{ \hat{e}_i \} \) and \( \{ \hat{e}_j \} \) are the standard row and column bases for the vector space. The diagram is

\[
(5) \quad a_{ij} = \hat{e}_i A \hat{e}_j.
\]

**Example 4.** Find a diagrammatic representation of the trace \( \text{tr}(A) \).

**Solution.** \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \hat{e}_i \).

**Example 5.** Find a diagrammatic representation of the determinant

\[
(6) \quad \text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.
\]

Here, \( \sigma \in S_n \) is a permutation on \( n \) elements, and \( \text{sgn}(\sigma) \) is the signature of the permutation, defined to be \((-1)^k\), where \( k \) is the number of crossings in a diagram of the permutation.

**Solution.** One approach is to introduce new notation. If \( \begin{array}{c} \vdots \end{array} \) represents a permutation \( \sigma \) on the \( n \) strands, then

\[
\text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \begin{array}{c} A \end{array} \begin{array}{c} \sigma \end{array} \cdots \begin{array}{c} A \end{array} \begin{array}{c} n \end{array}.
\]

For example, if \( n = 2 \), then \( \text{det}(A) = \begin{array}{c} 1 \end{array} \begin{array}{c} A \end{array} \begin{array}{c} 2 \end{array} - \begin{array}{c} 2 \end{array} \begin{array}{c} A \end{array} \begin{array}{c} 1 \end{array} = a_{11}a_{22} - a_{12}a_{21} \).

### 4. Computing Trace Diagram Functions

Let \( \{ \hat{e}_1, \ldots, \hat{e}_n \} \) be an orthonormal basis for an \( n \)-dimensional vector space. A basis diagram is an \( n \)-trace diagram in which every edge has been labeled by one of these basis elements. We use \( i \) as shorthand for \( \hat{e}_i \).
The general process for computing a trace diagram’s function involves two steps:

(1) Express the diagram as a summation over basis diagrams.
(2) Evaluate each basis diagram, and add the results together.

We describe the second step first.

Basis diagrams evaluate to either 0 or a signed product of matrix and vector entries according to the following rules:

Matrix Rule. Each matrix and vector on the diagram contributes an entry to the product, according to the rules

\[ A_{ij} = a_{ij} \] and \[ u_i = u_i. \]

Node Rule. Nodes contribute multipliers of 0, if adjacent labels are repeated, or \( \text{sgn}(\tau) \), if the adjacent labels form a permutation \( \tau \).

In the case where labels are not repeated, a convention is required to read off the permutation precisely. We use a small mark called a ciliation near the node, and read off the permutation in a counter-clockwise fashion starting at that mark. (If the dimension is odd, the counter-clockwise ordering is enough to provide the sign of the permutation, since, for example, \( (1\ 2\ 3) \) and \( (2\ 3\ 1) \) have the same sign. So we can omit ciliations in this case.)

For example, the node

has edges ordered \((2, 4, 1, 3)\), so the permutation is \( \frac{1\ 2\ 3}{2\ 4\ 1\ 3} \). The number of transpositions required to express this permutation is 3, so the node contributes a sign of \(-1\).

Now we treat the question of how to express diagrams as sums of basis diagrams. For now, we assume that there are no free edges. The summation uses the following rule:
Inferred Summation Rule. Express each edge in the diagram that is not already labeled by a basis element as a sum over basis labels:

(7) \[ \mathcal{I} = \sum_{i=1}^{n} \mathcal{I}_i. \]

This rule is the analog of the expression of the identity matrix $I$ as the sum $I = \sum_{i=1}^{n} \hat{e}_i \hat{e}^i$.

The following example computes the dot product diagram in (1).

Example 6. Show that $\mathcal{U} \mathcal{V} = \mathbf{u} \cdot \mathbf{v}$.

Solution. Use inferred summation over the interior edge.

\[ \mathcal{U} \mathcal{V} = \sum_{i=1}^{n} \mathcal{U}_i \mathcal{V}_i = \sum_{i=1}^{n} u_i v_i = \mathbf{u} \cdot \mathbf{v}. \]

The next example justifies the terminology ‘trace’ diagrams:

Example 7. Evaluate the diagram $\mathcal{A}$.

Solution. Infer summation over the $n$ basic labels on the interior edge:

(8) \[ \mathcal{A} = \sum_{i=1}^{n} \mathcal{A}_i = \sum_{i=1}^{n} a_{ii} = \text{tr}(A). \]

The rule $\mathcal{T} \mathcal{F} = \mathcal{A} \mathcal{B}$ for matrix products follows by inferred summation on the strand connecting the two matrices:

\[ \mathcal{T} \mathcal{F} = \sum_{k=1}^{n} \mathcal{A}_k \mathcal{B}_k = \sum_{k=1}^{n} a_{ik} b_{kj} \equiv (AB)_{ij} = \mathcal{A} \mathcal{B}. \]

A similar approach allows vectors in diagrams to be reduced directly, in the sense that

(9) \[ \mathcal{V} = \sum_{i=1}^{n} v_i \mathcal{I}_i. \]

This vector summation rule is the analog of the equation $\mathbf{v} = \sum_{i=1}^{n} v_i \hat{e}_i$, and may be proven by applying (7) on the edge in $\mathcal{V}$. 
Example 8. Given \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \), evaluate \( \mathbf{u} \cdot \mathbf{v} \).

Solution. Using (9) and the node rule:

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} u_i v_j = u_1 v_2 + u_2 v_1 = u_1 v_2 - u_2 v_1 = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.
\]

Exercise 9. Show, as suggested in Exercise 2, that

\[
\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\]

Exercise 10. (Harder) Given an \( n \times n \) matrix \( \mathbf{A} \), show that

\[
\begin{vmatrix} \mathbf{A} \cdots \mathbf{A} \end{vmatrix} = \det(\mathbf{A})
\]

and that

\[
\begin{vmatrix} \mathbf{A} \cdots \mathbf{A} 
\end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} n! \det(\mathbf{A}).
\]

We have not yet described how to compute functions for diagrams with free (unlabeled) edges. These edges correspond to outputs of the underlying functions, and the rule for computing them is as follows:

**Free Edge Rule.** A diagram with a single free edge corresponds to a function with output \( \mathbf{w} = (w_1, w_2, \ldots, w_n) \), where \( w_i \) is found by labeling the free strand by the basis label \( i \) and evaluating the result.

For example, \( \begin{array}{c} \mathbf{u} \\
\mathbf{v} \end{array} \) has a single free edge, corresponding to output \( \mathbf{w} = (w_1, w_2, w_3) \). The vector summation and node rules give

\[
w_1 = \mathbf{w} + \mathbf{u} = u_2 v_3 - u_3 v_2.
\]

Exercise 11. Compute \( w_2 \) and \( w_3 \) in the above example. Then explain why \( \mathbf{u} \times \mathbf{v} \).
The rules discussed in this section can be used to evaluate any trace diagram with a scalar or vector output. We refer the reader to [4] for an explanation of how to compute diagrams with multiple outputs.

5. A NEW LOOK AT SOME OLD QUESTIONS

The following questions are often asked by advanced students in multivariable calculus and linear algebra courses:

Question 1: Is there any pattern underlying vector identities with dot and cross products?
Question 2: Why is the cross product defined “only” in three dimensions?
Question 3: Is the rule for computing cross products using a determinant just a coincidence?
Question 4: Why do the trace and determinant show up in the discussion of eigenvalues and the det(A − λI) = 0 equation?

Some readers may have ready answers for these questions, but to a student seeing vectors or matrices for the first time, the answers may not be readily understood.

Trace diagrams are a good source of intuition for these problems, as it is often easier to generalize visual patterns than algebraic ones.

Question 1 has already been addressed. The diagrammatic rule (3) unifies all vector identities involving more than one cross product. Finding the identity is as easy as finding two adjacent nodes in a diagram.

For Question 2, the visual cross product easily extends in dimension $n$ to

$$\begin{align*}
\begin{array}{c}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\vdots \\
\mathbf{u}_{n-1}
\end{array}
\rightarrow
\mathbf{u}_1 \times \mathbf{u}_2 \times \cdots \times \mathbf{u}_{n-1}.
\end{align*}$$

From a visual perspective, there is zero coincidence in Question 3. Trace diagrams that differ only in placement of inputs and outputs represent the same underlying concept. In the correspondence with linear algebra, these inputs and outputs play almost identical roles in computation. So the determinant $\begin{array}{c}
\mathbf{u} \\
\mathbf{v} \\
\mathbf{w}
\end{array} = \det[\mathbf{u \ v \ w}]$ and the cross
product $u \times v$ are merely two different windows into one idea. This visual perspective also begs the question of what the related diagrams $\text{and} \ \text{represent algebraically. As the answer involves}

terminology beyond the scope of this paper, we refer the interested reader to [4] to uncover the answer.

Before getting to Question 4, consider this combinatorial question: what are the simplest closed diagrams marked by a matrix $A$? The simplest case is $\text{A} = \text{tr}(A)$, with no nodes. If the diagram has one node and is closed, $n$ must be even. Moreover, any unmarked edge in the diagram would produce a multiplier of 0, so the only case is

\[(10)\]

Two-node diagrams include the family

\[(11)\]

The following result gives a partial answer to Question 4:

**Theorem** (Corollary 20 in [5]). The diagrams in (11) are the coefficients of the characteristic polynomial $\text{det}(A - \lambda I)$, up to a multiple that depends only on $n$ and the number of marked edges.

In this case, the diagrammatic pattern indicates a new way to understand the characteristic polynomial. The trace and determinant, which are constant multiples of the first and last diagrams in (11), show up as special cases of a family of diagrams representing all coefficients of the polynomial. The traditional notation for these coefficients is $\text{tr}_i(A)$, where $i = 1, \ldots, n$ is the number of marked edges.

There is more to say about closed diagrams marked by a matrix $A$, and many unanswered questions. What is the function underlying (10)? (We believe it to be the Pfaffian, but have not yet proven this
fact.) What are the other diagrams with two nodes, and what are their functions? What can be said about diagrams with more than two nodes?

6. Concluding Remarks

Trace diagrams are particularly suited to recognizing and generalizing patterns in vector and matrix algebra. In many cases, they lead to simple or even “trivial” proofs. Moreover, linear algebra is decidedly nonlinear, in the sense that matrix algebra does not naturally play out along a 1-dimensional line of text. One can reasonably argue that the diagrammatic notation for the cross product, the trace, and the determinant is a better approximation of the underlying “truth” of these concepts than the standard $\mathbf{u} \times \mathbf{v}$, $\text{tr}(A)$, and $\text{det}(A)$.

Trace diagrams also have a strong potential to help advance certain areas of mathematics. In invariant theory, they simplify the process of generating and understanding trace relations, which is fundamental for recognizing structures of interest in the theory. In another interesting connection, the four-color theorem may be expressed entirely in terms of these diagrams [3]. Independent of these applications, there are many unanswered questions about the diagrams themselves, particularly when matrices are restricted to certain groups.

Trace diagrams are not likely to revolutionize linear algebra, but they provide a refreshing perspective and a new way to understand the subject. And one cannot help but wonder what linear algebra would now look like if trace diagrams had been as easy to typeset as rows and columns of numbers.

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