Field redefinitions, Weyl invariance and the nature of mavericks

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Abstract

In theories of gravity with non-minimally coupled scalar fields, there are ‘mavericks’—unexpected solutions with odd properties (e.g., black holes with scalar hair in theories with scalar potentials bounded from below). Probably the most famous example is the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) black hole solution in a theory with a scalar field conformally coupled to the gravity, and with a vanishing potential. Its existence naively violates the no-hair conjecture without violating no-hair theorems because of the singular behavior of the scalar field at the horizon. Despite being discovered more than 40 years ago, the nature of the BBMB solution is still the subject of research and debate. We argue here that the key to understanding the nature of maverick solutions is the proper choice of field redefinition schemes in which the solutions are regular. It appears that in such ‘regular’ schemes, mavericks have different physical interpretations; in particular, they are not elementary but composite objects. For example, the BBMB solution is not an extremal black hole, but a collection of a wormhole and a naked singularity. In the process, we show that Weyl-invariant formulation of gravity is a perfect tool for such analyses.

Keywords: black hole solutions, gravity with nonminimally coupled scalars, Weyl-invariance, field redefinitions

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(Some figures may appear in colour only in the online journal)
1. Introduction

Recent developments in cosmology and elementary particle physics imply that it is imperative to improve our understanding of mechanisms for coupling scalar fields to gravity. As a consequence, there has been a revived interest recently in theories with scalar fields non-minimally coupled to gravity. An important and well-known aspect of such theories is that some of them allow new types of solutions which do not have counterparts in standard general relativity with minimally coupled scalars, even in cases in which there are locally well-defined field redefinitions which transform the theory into general relativity with minimally coupled scalars. An interesting example of such solutions are ‘maverick’ black holes—unexpected solutions that may violate the no-hair conjecture by evading standard no-hair theorems. They are characterized by surfaces on which the scalar field is singular, but these singularities are harmless for the properties of classical particle trajectories and tidal accelerations, though some physical quantities are not continuous on these surfaces.

To be specific, let us consider a theory with the scalar field $\chi(x)$ nonminimally coupled to gravity with the following ‘Jordan frame’ action:

$$I = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} \left( R - 2\Lambda - \frac{1}{2} (\partial \chi)^2 - \frac{1}{12} \chi^2 R - V(\chi) \right) \right]$$

where $\kappa = 8\pi G_N$ ($G_N$ being Newton’s constant) and $\Lambda$ is the cosmological constant. For the class of theories $^1$ obtained by taking the scalar potential $V(\chi)$,

$$V(\chi) = -\frac{\kappa \Lambda}{36} \chi^4,$$

maverick black hole solutions have been found. In particular, for the simplest case in which $\Lambda = 0$, this theory has the following solution $^1$, $^2$:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad \chi(r) = \pm \sqrt{\frac{6}{\kappa} \frac{M}{r-M}} \quad (3)$$

where function $f(r)$ is given by

$$f(r) = \left( 1 - \frac{M}{r} \right)^3. \quad (4)$$

As the metric in (3)–(4) is the same as for the extremal Reissner–Nordström solution, it appears that this solution, usually called the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution, describes an asymptotically flat, static, spherically symmetric black hole with mass $M$, an event horizon at $r = M$, and a timelike space-time singularity at $r = 0$. Generalized versions carrying electric and magnetic charge were also constructed $^2$–$^4$, and a generalization to $\Lambda \neq 0$, called the Martinez–Troncoso–Zanelli (MTZ) solution, was obtained in $^5$. Some further generalizations were obtained in $^6$. A BBMB black hole carries (discrete) hair, which means that its existence apparently breaks the no-hair conjecture$^2$. Both dominant and strong energy conditions hold for the BBMB solution and its generalizations.

$^1$ Theory (1), with the scalar potential of the form $V = \lambda \chi^4$, is sometimes referred to have a conformally coupled scalar field, though the gravity part of the action is not conformally invariant. Because of this, and also to prevent confusion for the reader when we introduce (full) Weyl invariance later in the text, we shall avoid using the word ‘conformal’ in this context.

$^2$ Note that there is also a standard Schwarzschild solution with a vanishing everywhere scalar field. For a review and history of no-hair theorems, see $^7$. 
A tricky feature of the BBMB solution is that the scalar field is singular at the horizon. As shown in [14], this singularity is harmless for classical particle trajectories and tidal accelerations, so its existence does not automatically imply that the solution is pathological at the horizon. However, further analyses revealed some potentially problematic aspects of the interpretation of the BBMB solution as a genuine black hole solution. Black hole thermodynamics is not well-defined [15]. Continuity of the equations of motion across the horizon is broken [16]. It is not clear what the proper boundary condition on the horizon for perturbations should be, leading to a controversy regarding (in)stability of the solution under linear spherical perturbations [17, 18]. Also, the perturbatively constructed simple separable rotating solution diverges at $r = 2M$ [19]. One additional strange property of the BBMB solution is that in the region $0 < r < 2M$, the effective Newton’s constant is negative (‘anti-gravity’ behavior).

The continuing interest in the BBMB solution gives a strong motive for understanding the real nature of such maverick solutions and solutions having surfaces on which the scalar field is singular, but these singularities are harmless for classical particle trajectories and tidal accelerations. We suggest here that a key to achieving this goal lies in the proper choice of a field redefinition scheme. We argue that the singularity of the scalar field is a consequence of a breakdown of the Jordan frame scheme at $r = M$, and show that in schemes which are more ‘natural’ (i.e., regular) for describing such solutions, maverick solutions have a different physical interpretation (e.g., the BBMB or the MZT solution is a collection of two objects, a wormhole solution and a naked singularity, separated by an asymptotic region located around $r = M$). We demonstrate that for analyses of this type, it is very convenient to use the Weyl-invariant dilaton formulation of gravity, which in some cases allows easy understanding of the limits of particular schemes and a construction of field redefinition schemes which are more convenient for a given purpose or configuration.

2. Weyl-invariant dilaton gravity

Here, we briefly review the Weyl-invariant formulation of dilaton gravity (WIDG), with the main idea of embedding the theory (1) into this formalism. Our interest here is a gravity theory with a matter sector consisting of one (physical) scalar field, $h(x)^4$. In this case, one can write a manifestly Weyl-invariant action by including an additional scalar field, $\phi(x)$:

$$I_{\text{WIDG}} = \int d^4x \sqrt{-g} \left[ \frac{1}{12} (\phi^2 - h^2)R + \frac{1}{2} (\phi \nabla h)^2 - \frac{1}{2} (\phi h)^2 - V(\phi, h) \right],$$

$$V(\phi, h) = \phi^4 F(h/\phi).$$

This action is invariant on Weyl transformations defined by

$$g^{\mu\nu}(x) \rightarrow \Omega(x)^2 g^{\mu\nu}(x), \quad \phi(x) \rightarrow \Omega(x)\phi(x), \quad h(x) \rightarrow \Omega(x)h(x),$$

where $\Omega(x) \neq 0$ is an otherwise arbitrary function. Weyl invariance by itself does not constrain the function $F$ in (5); however, if we require the scalar potential, $V$, to be regular in the whole $\phi-h$ plane, then it has to be in the form $V(\phi, h) = \sum_{n=0}^{4} c_n h^n \phi^{4-n}$, where $c_n$ are some coupling constants (or, in other words, $F$ should be a polynomial maximally of the order

3 For this reason, the BBMB solution does not violate no-hair theorems. For an analysis of no-hair theorems for theories of the type (1–2), see [8–13]. As we shall see, for the MTZ solution with $\Lambda \neq 0$, the scalar field singularity does not coincide with the event horizon and is inside the black hole.

4 WIDG can naturally encompass much more complicated theories, including the Standard Model of particle physics with minimal or nonminimal Higgs sector. For this, and a more detailed review of WIDG, one can consult [20].

four). If one regards Weyl rescalings as gauge transformations, one scalar field can be gauged away and we are left with the same number of degrees of freedom we started with (i.e., as in (1)). We refer to this theory as WIDG. As is standard in gauge theories, we can deal with Weyl invariance by fixing the gauge. A convenient type of gauge-fixing condition, which keeps manifest diff-covariance and 2-derivative nature of the action, is

$$f(\phi, h) = 0,$$

where \( f \) is some function defining the gauge. Obviously, a gauge-fixing condition of the type (7) defines a curve in the \( \phi-h \) plane. As \( h/\phi \) is gauge-invariant, gauge orbits in the \( \phi-h \) plane are radial straight lines plus the \((0, 0)\) point. It then follows that a gauge will be well-defined in some region of the \( \phi-h \) plane if the curve defined by its gauge condition (7) crosses every orbit from the region once. We now specify three different gauge choices, visualized in figure 1, which we shall refer to later.

### 2.1 Einstein gauge (E-gauge)

The gauge-fixing condition is

$$\phi^2_E - h^2_E = \frac{6}{\kappa}. \quad (8)$$

In this gauge, WIDG action (5) takes the form of the standard general relativity, with Newton’s constant, \( G_N = \kappa/8\pi \), and with one minimally coupled scalar field, \( \sigma \), and a potential, \( V_E \) defined by

$$\sigma = \left[ \frac{6}{\kappa} \tan^{-1}\left( \frac{h}{\phi} \right) \right], \quad V_E(\sigma) = \frac{\kappa^2}{36} \cosh^2 \left( \frac{\kappa}{\sqrt{6}} \sigma \right) F \left( \tanh \left( \frac{\kappa}{\sqrt{6}} \sigma \right) \right). \quad (9)$$

The gauge-fixing condition (8) is obviously singular on the lines \( |\phi| = |h| \), and as a consequence is the (gauge-invariant) field, \( \sigma \). It then follows that every configuration which is regular in the E-gauge must be completely contained inside one of the wedges defined by \( \theta l < \phi l \) (the right and left wedge in figure 1). In conclusion, we see that WIDG theory is equivalent to ordinary general relativity (with scalar field(s)) in the subset of its configuration space.

### 2.2 Jordan gauge (c-gauge)

The gauge-fixing condition is

$$\phi^2_c = \frac{6}{\kappa}. \quad (10)$$

After defining \( \chi(x) \equiv h_1(x) \) and \( V_c(\chi) \equiv V(\phi, h, \chi) - \Lambda/\kappa \), action in the c-gauge becomes

$$I_c = \int d^4x \sqrt{-g_c} \left[ \frac{1}{2\kappa} (R_c - 2\Lambda) - \frac{1}{2} (\partial\chi)^2 - \frac{1}{12} \chi^2 R_c - V_c(\chi) \right] \quad (11)$$

where subscript \( c \) on the Ricci scalar \( R \) denotes that it is computed from the metric in the c-gauge \( (g_{\mu\nu}^c) \). We see that in the c-gauge, WIDG action (5) takes the form of the action (1). If we specify the WIDG scalar potential to the form

$$V(\phi, h) = \frac{\kappa\Lambda}{36} (\phi^4 - h^4), \quad (12)$$

then in the c-gauge the WIDG action (5) takes exactly the form of the action (1)–(2).
The gauge-fixing condition (10) is singular on the line $\phi = 0$, so it follows that every configuration which is regular in the c-gauge is confined to one of the half-planes, $\phi > 0$ or $\phi < 0$. Obviously, the space of regular configurations in the conformal frame is larger than in the Einstein frame\(^5\). In particular, it allows configurations which cross the $|\phi| = |h|$ line. This observation explains why there are solutions in the c-gauge, such as maverick black holes, which do not have counterparts in the E-gauge. Note that such solutions are necessarily characterized by the property that is negative (‘antigravity region’)\(^6\) in the part of the spacetime effective Newton’s constant $G_{\text{eff}} = 3/4\pi (\phi^2 - h^2)$. But, if a configuration crosses the $\phi = 0$ line, this would manifest as a singular behavior in the c-gauge because $|h| \rightarrow \infty$ when $\phi \rightarrow 0$.

\(^{5}\) The situation where Jordan and Einstein frame spacetimes are not in one-to-one correspondence due to singular conformal factors was originally discussed in [21].

\(^{6}\) Note that effective Newton’s constant, $G_{\text{eff}}$, is singular on the lines $|\phi| = |h|$, which possibly may bring problems, such as instability on small perturbations, to solutions crossing these lines.
2.3 $k$-gauge

The gauge-fixing condition is
\[ \phi_k^2 + h_k^2 = \frac{6}{k}. \]  

(13)

This gauge is regular on the whole $(\phi, h)$ plane except at the Weyl-invariant point $(0, 0)$, so in this sense it is superior to both E- and c-gauges. In particular, it can be used to describe solutions crossing the $\phi = 0$ (with $h \neq 0$) lines, which is the property we shall require later. One convenient parametrization is to introduce the field $\beta(x)$ through
\[ \phi_k = \sqrt{\frac{6}{k}} \cos \beta, \quad h_k = \sqrt{\frac{6}{k}} \sin \beta, \quad \beta = \arctan \frac{h}{\phi}. \]  

(14)

$\beta$ is the (gauge invariant) polar angle in $\phi$–$h$ plane, and so it is periodic with period $2\pi$. After gauge-fixing, the WIDG action becomes
\[ I_k = \frac{1}{2k} \int d^4x \sqrt{-g_k} \cos (2\beta) \left[ R_k - 6 (\partial^2 \beta)^2 - V_k(\beta) \right]. \]
\[ V_k(\beta) = \frac{2k}{\cos(2\beta)} V \left( \phi_k(\beta), h_k(\beta) \right). \]  

(15)

For the choice of the WIDG potential given in (12), one obtains
\[ V_k = 2\Lambda. \]  

(16)

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Let us make two observations here. The first observation is that actions corresponding to different gauge choices are related by field redefinitions. In this language, the E-gauge and the c-gauge correspond to well-known schemes, usually called the Einstein frame and the Jordan frame, respectively. As for the $k$-gauge, we are not aware that the corresponding scheme was studied before, but we have shown that it follows very naturally from the WIDG framework. A connection between any two gauges, denoted by A and B, is in the form
\[ g_{AB}^{\mu\nu} = \Omega_{AB} g_A^{\mu\nu}, \quad \phi_A = \Omega_{AB} \phi_B, \quad h_A = \Omega_{AB} h_B. \]  

(17)

It is easy to show that the factors, $\Omega$, for transitions from the c-gauge to the E-gauge and the $k$-gauge are given, respectively, by
\[ \Omega_{Ec} = \left( 1 - \frac{k}{6} h_c^2 \right)^{\frac{1}{2}}, \quad \Omega_{kc} = \left( 1 + \frac{k}{6} h_c^2 \right)^{\frac{1}{2}}. \]  

(18)

We see that the WIDG formulation offers a convenient, practical tool for dealing with field redefinition schemes through: (i) easy construction of connecting relations between schemes by using equations (6) and (7), (ii) providing a domain of configuration space covered by schemes, and (iii) constructing convenient schemes for particular purposes.

The second observation is that domains in the configuration space may also be limited by the possible singularities of the scalar potential, $V(\phi, h)$. As an example, consider a theory in the E-gauge (Einstein frame). First note that Weyl invariance by itself does not restrict a form of the function $F$ in (5), and as a consequence the potential in the Einstein frame, $V_E(\sigma)$, is also arbitrary. However, if one requires for the WIDG potential, $V(\phi, h)$, to be regular in the whole $\phi$–$h$ plane, then $F$ must be a polynomial of (maximal) order 4, which then severely restricts $V_E(\sigma)$. If we want to have a standard polynomial potential in the Einstein frame
\[ V_2(\sigma) = \sum_{j=0}^3 \lambda_j \sigma^j, \quad (19) \]

where \( \lambda_j \) are some coupling constants, which look ‘natural’ from the Einstein frame perspective, then from (9) it follows that the corresponding WIDG potential, \( V(\phi, h) \), is singular at \( l \phi l = l h l \). This may also prevent the existence of regular configurations from crossing these lines in other gauges; note that configuration space in the Einstein frame is limited by these lines by definition. Another example of a singular potential is obtained by allowing \( F \) to be a polynomial of an order higher than 4. In this case, the potential is singular on the line \( \phi = 0 \), which can put a boundary on regular solutions in general gauges; the configuration space in the Jordan frame is by definition limited by the \( \phi = 0 \) line. A similar argument applies to couplings of the scalar fields to the matter sector, when they exist (we ignore them in this paper). In the rest of this paper, we analyze solutions in theories with regular WIDG potentials in the whole \( \phi-h \) plane, so this issue will not appear in our analyses here.

3. The BBMB solution

We now turn our attention to the BBMB solution (3)–(4). As the metric is the same as for the extremal Reisner–Nordstrom solution with the mass, \( M \), it apparently describes a spherically symmetric, asymptotically flat extremal black hole with an event horizon located at \( r = M \). The BBMB black hole is a solution of the action (1)–(2) with \( \Lambda = 0 \). We have shown in the previous section that this theory may be described as WIDG (5) with the potential, \( V = 0 \), written in the c-gauge. In this language, the BBMB solution is

\[
\begin{align*}
\text{ds}_c^2 &= -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left( \sin^2 \theta d\phi^2 + d\theta^2 \right), \\
\phi_c(r) &= -\sqrt{\frac{6}{\kappa}} \frac{M}{r}.
\end{align*}
\]

(20)

The ‘ugly’ property of the BBMB solution is the singular behavior of the scalar field, \( h_c \), at \( r = M \).

At \( r = 2M \), one has \( \phi = h \), while for \( r \to 0 \), one gets \( l hl \to l \phi l \). It follows that in the region \( r < 2M \), one has \( l hl < l \phi l \). As discussed in section 2, the E-gauge cannot accommodate such antigravity behavior, and this immediately explains why there is not a corresponding regular solution in the Einstein frame covering the \( r > M \) region.

A more intriguing aspect of the BBMB solution is the singular behavior of the field \( h_c \) at \( r = M \). Using the WIDG picture (and figure 1), it is easy to understand the nature of this singularity—as \( h_c \to \infty \) this signals that the c-gauge breaks down, which can only happen when \( \phi \to 0 \) in regular gauges (i.e., regular field redefinition schemes). There are essentially two possible scenarios: (a) \( h \neq 0 \) at \( r = M \), or (b) \( h = 0 \) at \( r = M \).

Let us envisage scenario (a) first. In this case we can use a field redefinition scheme corresponding to the k-gauge (13), as it implies \( h_k^2 = 6/\kappa > 0 \) for \( \phi_k = 0 \). The action in the k-gauge is by (15)

\[ I_k = \frac{1}{2\kappa} \int d^3 \sqrt{-G_k} \cos (2\beta) \left[ R_k - 6 (\partial \beta)^2 \right]. \]

(21)

We do not have to solve corresponding field equations, as the transition rule from the c-gauge to the k-gauge is already given in (17) and the second equation of (18). Inserting (20) into
(17)–(18), we obtain the expression for the BBMB solution in the k-gauge

\[ g_k^{\mu\nu} = \Omega_k^2 g_c^{\mu\nu}, \quad \phi_k = \Omega_k \phi_c, \quad h_k = \Omega_k h_c, \]

\[ \Omega_k = (r - M)(r - M^2 + M^2)^{1/2}. \tag{22} \]

Explicitly written, the BBMB solution in the k-gauge scheme is

\[ ds_k^2 = \left[ \left( 1 - \frac{M}{r} \right)^2 + \frac{M^2}{r^2} \right] \left( -dt^2 + \frac{dr^2}{\left( 1 - \frac{M}{r} \right)^2} + \frac{r^2}{\left( 1 - \frac{M}{r} \right)^2} (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

\[ \beta(r) = \arctan \frac{M}{r - M}, \tag{23} \]

where the ‘physical’ k-gauge scalar field, \( \beta \), is obtained from (14).

Let us analyze this solution more closely. By construction, the scalar field singularity, \( r = M \), is gone and all k-gauge scalar fields (including \( \beta \)) are regular in the whole interval, \( 0 \leq r < \infty \). The field, \( \beta \), is 0 at \( r \to \infty \), \( \pi/2 \) at \( r = M \), and \( 3\pi/4 \) (which means \( h = -\phi \)) at \( r = 0 \). As the factor \( \Omega_k \) behaves as

\[ \Omega_k(r) \xrightarrow{r \to 0} \frac{1}{\sqrt{2}} + O(r), \quad \Omega_k(r) \xrightarrow{r \to \infty} 1 + O\left( \frac{1}{r^2} \right), \tag{24} \]

the interpretations of the limits, \( r \to 0 \) and \( r \to \infty \) are essentially unchanged by the field redefinition. What remains is to analyze the metric (23) in the \( r \approx M \) region. It can be shown that components of the Riemann tensor, \( R^{\mu\nu}_{\alpha\beta} \), all vanish at \( r = M \) (as well as all curvature invariants constructed out of the metric, the Riemann tensor, and covariant derivatives).

To understand the region \( r > M \), first note that the ‘Schwarzschild’ radius, \( r/\Omega_k(r) \), has its minimum at \( r = 2M \). It is convenient to pass to the new radial coordinate, \( \mathcal{R} \), defined by

\[ \mathcal{R}^2 + 8M^2 \equiv r^2 \Omega_k(r)^{-2}. \tag{25} \]

The coordinate, \( \mathcal{R} \), is defined to go from \(-\infty\) to \(+\infty\) as \( r \) goes from \( M \) to \(+\infty\). In the new coordinate system, the metric in the asymptotic regions, \( \mathcal{R} \to \pm \infty \), behaves as

\[ ds_k^2 = -\left( 1 - \frac{2M}{|\mathcal{R}|} + O\left( \mathcal{R}^{-2} \right) \right) dt^2 + \left( 1 + \frac{2M}{|\mathcal{R}|} + O\left( \mathcal{R}^{-2} \right) \right) dr^2 \]

\[ + \left( \mathcal{R}^2 + 8M^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{26} \]

Taking all of this together, we have established that the patch of spacetime defined by \( M < r < \infty \) describes a wormhole with two asymptotically flat infinities, \( r \to M \) and \( r \to \infty \), connected by the throat centered at \( r = 2M \) (\( \mathcal{R} = 0 \)) and with its proper radius equal to \( 2\sqrt{2}M \).

As for the \( 0 < r < M \) region, its nature is best understood by using it as a new (‘Schwarzschild’) radial coordinate, \( \mathcal{R} \equiv r/\Omega_k \), with a domain \( 0 < \mathcal{R} < \infty \). In the limit \( r \to M^- \) (\( \mathcal{R} \to \infty \)), the metric in the k-gauge is again of the form (26), except that now \( \mathcal{R}^2 + 8M^2 \to \mathcal{R}^2 \). We conclude that the metric in the region \( 0 < r < M \) describes an asymptotically flat spacetime with a naked singularity.

It is not hard to see that these conclusions are not exclusive to the k-gauge, but are valid in all regular gauges in which scalar fields are regular everywhere and \( h \neq 0 \) at \( r = M \). Now we can come back to scenario (b) (i.e., to gauges in which \( h = 0 \) at \( r = M \)). It is rather easy to see that in such gauges the metric is singular at \( r = M \), so these are not regular gauges for the
BBMB solution. A final conclusion is that in all regular field redefinition schemes, the BBMB solution does not describe a single black hole but a collection of an asymptotically flat spacetime with a naked singularity and an asymptotically flat wormhole.

4. The MTZ solution

The BBMB solution is a special case of a larger class of solutions of the actions (1)–(2), parametrized by the value of the cosmological constant, \( \Lambda \), beside the mass parameter, \( M \). This class, known as the MTZ solution, is again given by (3), but with the function, \( f \), now being

\[
f(r) = \left(1 - \frac{M}{r}\right)^2 - \frac{\Lambda}{3} r^2. \tag{27}\]

For \( \Lambda = 0 \), one gets the BBMB solution. If \( \Lambda \neq 0 \), then an event horizon is present only if

\[
\Lambda > 0 \quad \text{and} \quad 0 < M < \frac{\ell}{4} \equiv \frac{1}{4} \sqrt{\frac{3}{\Lambda}}. \tag{28}\]

In this case, the metric is the same as for the Reissner–Nordström–de Sitter black hole with the special value of charge (\( Q = M \) in standard conventions)\(^7\). We focus on this case, which we refer to as the MTZ solution in the rest of the paper\(^8\). The equation \( f(r_0) = 0 \) then has three solutions

\[
\begin{align*}
    r_- &= \frac{\ell}{2} \left( \sqrt{1 + 4 \frac{M}{\ell}} - 1 \right), \\
    r_+ &= \frac{\ell}{2} \left( 1 - \sqrt{1 + 4 \frac{M}{\ell}} \right), \\
    r_{++} &= \frac{\ell}{2} \left( 1 + \sqrt{1 + 4 \frac{M}{\ell}} \right)
\end{align*} \tag{29}\]

satisfying

\[
0 < r_- < M < r_+ < 2M < \frac{\ell}{2} < r_{++} < \ell, \tag{30}\]

where \( r_+ \) is the outer (event) horizon, \( r_- \) is the inner (Cauchy) horizon, and \( r_{++} \) is the cosmological horizon. There are important differences between BBMB and MTZ solutions. The MTZ solution is nonextremal and asymptotically de Sitter. In addition, the singularity of the scalar field, located at \( r = M \), is inside the MTZ solution, so the event horizon is completely regular\(^9\). Our aim here is to show that despite these differences, the analysis applied to the BBMB solution can be straightforwardly repeated here.

We start by observing that the singularity of the scalar field at \( r = M \) appears because that is where the field redefinition scheme breaks. In WIDG language, this happens because \( \phi = 0 \) at \( r = M \), and this cannot be regularly represented in the c-gauge. The k-gauge does not have

\(^7\) For such RNdS black holes, temperatures of the event horizon and the cosmological horizon are the same, so they are thermodynamically stable. For detailed analysis, see [22].

\(^8\) In the rest of the parameter space, the MTZ solution describes naked singularity, which is asymptotically de Sitter or AdS depending on the sign of \( \Lambda \). Our analysis can be easily extended to these cases.

\(^9\) The reason why the MTZ solution does not violate no-hair theorems is that the scalar potential (2) is not bounded from below for \( \Lambda > 0 \). There are many known black hole solutions with scalar hair in theories with the potential unbounded from below (including those in standard general relativity with minimally coupled scalar fields) which are completely regular except for the central singularity. So, strictly speaking, one could say that the MTZ is not a maverick solution.
the problem with $\phi = 0$ if $h \neq 0$ (13), so we can use it to obtain a regular description of the MTZ solution. Equations (20)–(24) with the corresponding consequences apply again here, with the only difference that the function $f(r)$ is now given in (27). The MTZ solution in a field redefinition scheme corresponding to the k-gauge, in which action is given in (15)–(16), is then given by

$$ds_k^2 = \frac{(r - M)^2 + M^2}{(r - M)^2} \left\{ - \left[ (r - M)^2 - \frac{\Lambda}{3} r^4 \right] \frac{dt^2}{r^2} + \frac{r^2 dr^2}{(r - M)^2 - \frac{\Lambda}{3} r^4} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\}$$

$$\beta(r) = \arctan \frac{M}{r - M}$$

(31)

Let us analyze this solution. Again, $r \to 0$ and $r \to \infty$ behavior is unchanged by passing to the k-gauge. However, a behavior of the metric near $r = M$ is different than in the c-gauge.

For $r > M$, it is convenient to switch to a new coordinate $\mathcal{R}$ defined in (25), which goes from $-\infty$ to $\infty$ as $r$ goes from $M$ to $\infty$. The metric in the k-gauge for $r \to M$ ($\mathcal{R} \to -\infty$) is then approximately given by

$$ds_k^2 \approx \frac{\Lambda}{3} \mathcal{R}^2 dt^2 - \frac{3}{\Lambda} \frac{d\mathcal{R}^2}{\mathcal{R}^2} + \mathcal{R}^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad |\mathcal{R}| \gg M.$$  

(32)

which is an asymptotic behavior of de Sitter space. Both asymptotic regions, $\mathcal{R} \to \pm \infty$, have the same de Sitter radius equal to $\ell = \sqrt{3/\Lambda}$. The region $r > M$ can be called an asymptotically de Sitter wormhole with a throat located at $r = 2M$ ($\mathcal{R} = 0$) with a proper radius $2\sqrt{2} M$ and cosmological horizons located at $r = r_+$ and $r = r_-$. 

As for the region $0 < r < M$, a better choice for the radial coordinate is $\mathcal{R} \equiv r/\Omega_+(r)$, which uniformly goes from $0$ to $\infty$ as $r$ goes from $0$ to $M$. In the limit $r \to M$ ($\mathcal{R} \to \infty$), the metric (31) behaves as in (32), which is an asymptotically de Sitter spacetime with a radius $\mathcal{R}$. In this region, the MTZ solution in the k-gauge describes a spherically symmetric, asymptotically de Sitter spacetime with a naked singularity with the cosmological horizon located at $r = r_-$. 

In conclusion, we have shown that the MTZ solution, when represented in a regular field redefinition scheme, is not a black hole at all, but instead describes a collection of two spherically symmetric, asymptotically de Sitter objects—a wormhole and a naked singularity. Again, a singularity surface ($r = M$) is in fact a new asymptotic region from the perspective of regular schemes.

5. The Anabalon–Cisterna solution

In [23], it was shown that the MTZ solution is a special case of a larger class of solutions, parametrized with an additional parameter, $\xi$, corresponding to the Jordan frame action (1) with the potential, $V_\xi$, given by

$$V_\xi(\chi) = -\frac{\kappa \Lambda}{36} \chi^4 + \frac{\Lambda \sqrt{6} \xi}{9} \frac{\xi}{\xi^2 + 1} \left( \frac{6}{\kappa} - \chi^2 \right) \chi.$$  

(33)
The solutions are given by

\[
\begin{align*}
\text{ds}^2 &= \frac{(r - (\xi + 1)M)^2}{(r - M)^2} \left\{ (r - M)^2 - \frac{\lambda}{3} r^4 \right\} \frac{dr^2}{r^2} \\
&\quad + \frac{r^2 dr^2}{(r - M)^2 - \frac{\lambda}{3} r^4} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)
\end{align*}
\]

\[
\chi(r) = \sqrt{\frac{6}{\kappa}} \frac{(\xi + 1)M - \xi r}{r - (\xi + 1)M}, \quad \lambda \equiv \Lambda \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right)^2
\]

(34)

and there is also the second branch obtained by applying \(\chi \rightarrow -\chi\) and \(\xi \rightarrow -\xi\) on (34) simultaneously. For the sake of clarity, let us restrict ourselves\(^1\) to \(|\xi| < 1, \Lambda > 0, \text{ and } 0 < M < L/\lambda\). In this case, the metric in (34) is asymptotically de Sitter as \(r \rightarrow \infty\) with the radius \(L\), has a singularity at \(r = 0\), and three Killing horizons \(r_+\) and \(r_{++}\) defined by (29) (with substitution \(\ell \rightarrow L\), which satisfy \(0 < r_- < M < r_+ < 2M < \frac{L}{\lambda} < r_{++} < L\). At \(r = (\xi + 1)M\), the scalar field, \(\chi\), is singular, while at \(r = 2M\) it takes the value \(\chi = \sqrt{6/\kappa}\). All these properties look rather similar to those obtained for the MTZ black hole in the Jordan frame (up to some ‘deformations’ caused by \(\xi \neq 0\)). In addition, for \(\xi \neq 0\) in the limit \(r \rightarrow M\), one has \(R^\mu_\nu = \frac{\kappa}{\xi^2} \delta^\mu_\nu\), which indicates that \(r = M\) is also an asymptotic region with the de Sitter radius equal to \(\xi L\). It follows that (34) describes two separate asymptotically de Sitter configurations: one defined in \(r < M\) (a naked singularity) and the other in \(r > M\) (a wormhole-like)\(^1\). Note that we obtained this before for the MTZ black hole, but only after we passed to some regular scheme (e.g., the k-gauge).

It is important to observe two additional singular properties of the solution (34).

(i) For \(\xi \neq 0\), there is an additional space-time singularity at \(r = (\xi + 1)M\) (exactly where the scalar field \(\chi\) is singular) appearing either in the wormhole (for \(\xi > 0\)) or in the naked singularity (for \(\xi < 0\)) subsolution.

(ii) The solution with \(\xi = 0\) has the metric which is well-defined across \(r = M\), because of a cancellation of the numerator and the denominator of the factor in (34). This is in fact the MTZ solution studied in the previous section. We have here an example in which a maverick solution appears as a singularity in the continuous set of solutions belonging to theories with different potentials.

Armed with the experience from previous sections, we can now easily understand what is happening here by using the WIDG language. First, note that the action (1) with the potential (33) can be obtained from the WIDG action (5) with a scalar potential

\[
V(\phi, h) = \frac{\kappa \Lambda}{36} \left( \phi^2 - h^2 \right) \left( \phi^2 + h^2 + \frac{4 \xi}{\xi^2 + 1} \phi h \right)
\]

(35)

by using the c-gauge (10) and defining \(\chi = h\). Let us follow the solution (34) in the \(\phi-h\) plane, which at \(r = \infty\) starts in the right wedge (see figure 1), then passes \(\phi = h\) at \(r = 2M\) and enters the upper wedge, which is the antigravity region. After that, the behavior depends on the sign of \(\xi\). For \(\xi < 0\), the solution then reaches the second asymptotic region at \(r = M\) where \(h/\phi = 1/|\xi| > 0\), which guarantees that the c-gauge is regular. As a consequence, the solution in the c-gauge for \(r > M\) is regular. However, as we follow the c-gauge solution in

\(^1\) The following analysis straightforwardly extends to other sectors of the parameter space.

\(^1\) For \(\Lambda = 0\) the only difference is that solutions are asymptotically flat instead of de Sitter.
the $r < M$ region, we hit the surface $r = (\xi + 1)M$ at which both the metric and $h_i$ are singular. But we can now anticipate what is going on—in regular gauges for $r = (\xi + 1)M$ one hits $\phi = 0$ which is a singular line for the c-gauge. We conclude that there is not a problem with the solution, but with the gauge (field redefinition scheme), and if we use a regular gauge this singularity should go away. Let us mention that if we took $\xi > 0$ instead, the only difference in the argument is that 'singularity' $r = (\xi + 1)M$ is on the $r > M$ side (in the wormhole solution).

We now proceed as before by taking the k-gauge as a representative of regular gauges for this situation. It is easy to show that for the potential (35), the action in the k-gauge is

$$I_k = \frac{1}{2\kappa} \int d^4x \sqrt{-g_k} \cos (2\beta) \left\{ R_k - 6 (\partial \phi)^2 - 2\Lambda \left[ 1 + \frac{2\xi}{\xi^2 + 1} \sin (2\beta) \right] \right\}, \quad (36)$$

We see that the scalar potential in the k-gauge is rather simple, and this may possibly explain why analytic solutions are obtainable. The easiest way to obtain a solution in the k-gauge is by Weyl-rescaling the c-gauge solution (34) by the factor $\Omega_{kc}$ calculated from (18). The result is

$$ds_k^2 = \left( \frac{\xi^2 + 1}{\xi^2} \right) r^2 - \frac{2(\xi + 1)^2 M(r - M)}{(r - M)^2} \left\{ \frac{(r - M)^2 - \frac{\lambda}{3} r^2}{r^2} dt^2 + \frac{dr^2}{(r - M)^2 - \frac{\lambda}{3} r^4} + r^2 d\Omega_2 \right\},$$

$$\beta(r) = \arctan \left( \frac{(\xi + 1)M - \xi r}{r - (\xi + 1)M} \right). \quad (37)$$

A simple analysis reveals that the solution in the k-gauge consists of two asymptotically de Sitter pieces; the $0 < r < M$ part describes a naked singularity while the $r > M$ part describes a wormhole with a throat located at $r = 2M$ with a radius equal to $2\sqrt{2} (1 - \xi) M$. de Sitter radia of asymptotic regions $r \to \infty$ and $r = 2M$, are now equal and given by $L_+ \sqrt{1 + \xi^2}$. We now emphasize the following results of our analysis of the Anabalon-Cisterna solution.

- In the k-gauge, as in all regular field redefinition schemes, there are no singularities aside from the one in $r = 0$. For solutions which are regular in the c-gauge, the physical interpretation is the same in the c-gauge (Jordan frame) and the k-gauge.
- In the k-gauge, the solution for $\xi \neq 0$ (the MTZ solution) is not special and has essentially the same physical interpretation as other solutions with $l \xi < 1$.

These observations strengthen our claim that the proper physical interpretation of the BBMB and the MTZ solutions are obtained in regular field redefinition schemes, such as the k-gauge.

6. Discussion and conclusion

We have analyzed the BBMB solution, and its generalizations (such as the MTZ solution), which are strange ('maverick') solutions with scalar hair in the theory with the scalar field conformally coupled to gravity. The metric part of the solution is the same as for a particular
Reissner–Nordstom black hole, but the scalar field develops a singularity which may compromise the black hole interpretation. Our observation is that the singularity appears exactly at the place where one expects a breakdown of the field redefinition scheme (the Jordan frame in this case) in which the solution is originally constructed and presented. After passing to a regular field redefinition scheme, the solution becomes regular but the interpretation changes. Instead of describing a black hole, the BBMB/MTZ solution describes a collection of two solutions with different physical interpretations—a naked singularity and a wormhole. This is because the singular surface is in fact a new asymptotic region when solutions are described in regular field redefinition schemes. Our conclusion is also supported by viewing the BBMB/MTZ solution as a special case of a broader class of solutions, constructed in [23]. In regular schemes, the interpretation is the same for the larger class of solutions, contrary to the Jordan frame scheme where the BBMB/MTZ solutions are exceptions and have singular interpretations. It is trivial to show that our conclusions extend to all known generalizations of these solutions (e.g., electrically charged and so on). Our procedure may in principle be generalized to all solutions with surfaces on which the scalar field is singular, with singularities being harmless for the properties of classical particle trajectories and tidal accelerations.

Embedding of the original Lagrangian into the framework of WIDG gave us a perfect tool to study the domain of configuration space, and consequently to locate the source of a singularity to the breakdown of the field redefinition scheme. WIDG formulation allowed us to construct a new scheme (in which we nicked the k-gauge) which accommodates a larger configuration space than in the Jordan or the Einstein frame, in which solutions can be regularly represented. We believe that this scheme may find uses more broadly in searches for new solutions, either in cosmological, black hole/wormhole, or other contexts. We note that the theory for which analytic solutions are found in [23] looks simpler when expressed in the k-gauge frame instead of the Jordan frame, and this could be a part of the answer why closed-form analytic solutions were found at all for such non trivial scalar potentials. It should be mentioned that the WIDG formulation attracted some attention in recent years in analyses of new cyclic solutions [20] or inflationary models [25, 26] in cosmology, high-energy behavior of the Standard Model [27, 28], and black hole singularities in [24]. Here we emphasized its usefulness as a tool in the analyses of the connections and differences between different field redefinition schemes, a topic which still occasionally leads to confusion [29].

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