Rationally trivial quadratic spaces are locally trivial:III

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Abstract

Main results of [Pa], [PaP] are extended to the case of characteristic two. The proof given in the present preprint is "elementary" and is characteristic free.

More precisely, let $R$ be a regular semi-local domain containing a field such that all the residue fields are infinite. Let $K$ be the fraction field of $R$. Let $(R^n, q : R^n \to R)$ be a quadratic space over $R$ such that the quadric $\{q = 0\}$ is smooth over $R$. If the quadratic space $(R^n, q : R^n \to R)$ over $R$ is isotropic over $K$, then there is a unimodular vector $v \in R^n$ such that $q(v) = 0$.

If $\text{char}(R) = 2$, then in the case of even $n$ our assumption on $q$ is equivalent to the one that $q$ is a non-singular space in the sense of [Kn] and in the case of odd $n > 2$ our assumption on $q$ is equivalent to the one that $q$ is a semi-regular in the sense of [Kn].

1 Introduction

Let $k$ be an infinite field, possibly $\text{char}(k) = 2$, and let $X$ be a $k$-smooth irreducible affine scheme, let $x_1, x_2, \ldots, x_s \in X$ be closed points. Let $P$ be a free $k[X]$-module of rank $n > 0$. If $n$ is odd, then let $(P, q : P \to k[X])$ be a semi-regular quadratic module over $k[X]$ in the sense of [Kn] Ch.IV, §3. If $n$ is even, then let $(P, q : P \to k[X])$ be a non-singular quadratic space in the sense of [Kn, Ch.I, (5.3.5)]]. (In both cases it is equivalent of saying that the $X$-scheme $Q := \{q = 0\} \subset \mathbb{P}_X^{n-1}$ is smooth over $X$).

Let $p : Q \to X$ be the projection. For a nonzero element $g \in k[X]$ let $Q_g = p^{-1}(X_g)$. Let $U = \text{Spec}(0_{X,(x_1,x_2,\ldots,x_s)})$. Set $\bar{Q} = U \times_X Q$. For a $k$-scheme $D$ equipped with $k$-morphisms $U \leftarrow D$ and $D \to X_g$ set $DQ = \bar{Q} \times_U D$ and $Q_{D,g} = D \times_{X_g} \bar{Q}_g$.

1.0.1 Proposition. If $n > 1$, then there exists a finite surjective étale $k$-morphism $U \leftarrow D$ of odd degree, a morphism $D \to X_g$ and an isomorphism of the $D$-schemes $DQ \xrightarrow{\phi} Q_{D,g}$.

Given this Proposition we may prove the following Theorem
1.0.2 Theorem (Main). Assume that \( g \in k[X] \) is a non-zero element such that there is a section \( s : X_g \to Q \) of the projection \( Q_g \to X_g \). Then there is a section \( s_U : U \to U \) of the projection \( uU \to U \).

Proof of Main Theorem. We will give a proof of the Theorem only in the local case and left to the reader the semi-local case. So, \( s = 1 \) and we will write \( x \) for \( x_1 \) and \( \mathcal{O}_{X,x} \) for \( \mathcal{O}_{X,x} \). If \( g \in k[X] - m_x \), then there is nothing to prove. Now let \( g \in m_x \) then by Proposition 1.0.1 there is a finite surjective étale \( k \)-morphism \( U \to D \) of odd degree, a morphism \( D \to X_g \) and an isomorphism of the \( D \)-schemes \( D, g \to Q \to Q_{D,g} \).

The section \( s \) defines a section \( s_D = (id, s) : D \to Q \to Q_{D,g} \) of the projection \( Q_{D,g} \to D \). Further \( \Phi \circ s_D : D \to D \) is a section of the projection \( D, g \to Q \to D \). Finally, if \( p_1 : D, g \to D \) is the projection, then \( p_1 \circ \Phi \circ s_D : D \to Q \) is a \( U \)-morphism of \( U \)-schemes. Recall that \( U \to D \) is a finite surjective étale \( k \)-morphism of odd degree and \( U \) is local with an infinite residue field. Whence by a variant of Springer’s theorem proven in [PR] there is a section \( s_U : U \to U \) of the projection \( uU \to U \). (If \( \text{char}(k) = 2 \) the proof a variant of Springer’s theorem given in [PR] works well with a very mild modification). The Theorem is proven.

The Main Theorem has the following corollaries

1.0.3 Corollary (Main1). Let \( \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \) be the semi-local ring as above and let \( k(X) \) be the rational function field on \( X \). Let \( P \) be a free \( \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \)-module of rank \( n > 1 \) and \( q : \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \to \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \) be a form over \( \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \) as above, that is the \( \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \)-scheme \( Q := \{q = 0\} \subseteq \mathbf{P}^{n-1}_{\mathcal{O}_{X,x}} \) is smooth over \( \mathcal{O}_{X,x} \). If the equation \( q = 0 \) has a non-trivial solution over \( k(X) \), then it has a unimodular solution over \( \mathcal{O}_{X,\{x_1, x_2, \ldots, x_s\}} \).

1.0.4 Corollary (Main2). Let \( R \) be a semi-local regular domain containing a field and \( R \) is such that all the residue fields are infinite. Let \( K \) be the fraction field of \( R \). Let \( P \) be a free \( R \)-module of rank \( n > 1 \) and \( q : P \to P \) be a quadratic form over \( R \) such that the \( R \)-scheme \( Q := \{q = 0\} \subseteq \mathbf{P}^{n-1}_{R} \) is smooth over \( R \). If the equation \( q = 0 \) has a non-trivial solution over \( K \), then it has a unimodular solution over \( R \).

1.0.5 Corollary (Main3). Let \( R \) be a semi-local regular domain containing a field and \( R \) is such that all the residue fields are infinite. Let \( K \) be the fraction field of \( R \). Let \( P \) be a free \( R \)-module of even rank \( n > 0 \) and \( q : P \to R \) be a quadratic form over \( R \) such that the \( R \)-scheme \( Q := \{q = 0\} \subseteq \mathbf{P}^{n-1}_{R} \) is smooth over \( R \). Let \( u \in R^x \) be a unit. If \( u \) is represented by \( q \) over \( K \), then \( u \) is represented by \( q \) already over \( R \).

If \( 1/2 \in R \), then the same holds for a quadratic space of an arbitrary rank.

Proof of Proposition 1.0.1. The following Lemma is a corollary from Lemma 2.2.1 and Proposition 3.1.7. from [Kn]

1.0.6 Lemma. For \( n > 1 \) there exists an affine open subset \( X^0 \) containing \( x \) and a Galois étale cover \( \hat{X}^0 \to X^0 \) such that the \( k[X^0] \)-module \( P \otimes_{k[X]} k[X^0] \) coincides with \( k[X^0]^n \) and \( \pi^*(q) \) is proportional to the quadratic space \( \sum_{i=1}^m T_i \) in the case \( n = 2m \).
and is proportional to the semi-regular quadratic module $\bigoplus_{i=1}^m T_i T_{i+m} \oplus T_n^2$ in the case $n = 2m + 1$.

By this Lemma we may and will assume that $P = k[X]^n$ and that we are given with a Galois étale cover $\pi : X \to X$ such that the quadratic space $\pi^*(q)$ is proportional to a split quadratic space. Let $\Gamma$ be the Galois group of $X$ over $X$. Let $U = \pi^{-1}(U) \subset X$.

Let $U \times X := (U \times X) / \Delta(\Gamma)$. Clearly, $U \times X = (U \times X) / (\Gamma \times \Gamma)$. Let $\rho : U \times X \to U \times X$ be the obvious map.

Let $p_2 : U \times X \to X$ be projection to $X$ and $p_1 : U \times X \to U$ be the projection to $U$. The quadratic spaces $p_1^*(q)$ and $p_2^*(q)$ over $U \times X$ are not proportional in general. However the following Proposition holds (see Appendix, Lemma 2.0.8)

1.0.7 Proposition. The quadratic spaces $\rho^*(p_2^*(q))$ and $\rho^*(p_1^*(q))$ are proportional.

Further by [PSV] Prop. 3.3, Prop. 3.4] and [PaSV] we may find an open $X'$ in $X$ containing $x$ and an open affine $S \subset \mathbf{P}^{d-1}$ (d=dim(X)) and a smooth morphism $f' : X' \to S$ making $X'$ into a smooth relative curve over $S$ with the geometrically irreducible fibres. Moreover we may find $f'$ such that $f'|_{X' \cap Z} : Z' = X' \cap Z \to S$ is finite, where $Z$ is the vanishing locus of $g \in k[X]$. Moreover $f'$ can be written as $pr_S \circ \Pi' = f'$, where $\Pi' : X' \to A^1 \times S$ is a finite surjective morphism. Set $X' = \pi^{-1}(X')$.

Replacing notation we write $X$ for $X'$, $\tilde{X}$ for $X'$, $Z$ for $Z'$, $f : X \to S$ for $f' : X' \to S$, $\Pi : X \to A^1 \times S$ for $\Pi' : X' \to A^1 \times S$.

Let $U \times S \tilde{X} := (U \times S \tilde{X}) / \Delta(\Gamma)$. Clearly, $U \times S X = (U \times S \tilde{X}) / (\Gamma \times \Gamma)$. Let

$$\rho_S : U \times S \tilde{X} \to U \times S X$$

be the obvious map.

Let $p_X : U \times S X \to X$ be projection to $X$ and $p_U : U \times S X \to U$ be the projection to $U$. By Proposition 1.0.7 the quadratic spaces $\rho_S^*(p_X^*(q))$ and $\rho_S^*(p_U^*(q))$ are proportional.

Now the pull-back of $\Pi$ be means of the morphism $U \to X \to S$ defines a finite surjective morphism $\Theta : U \times S X \to A^1 \times U$. So, $\Theta \circ \rho_S : U \times S \tilde{X} \to A^1 \times U$ is a finite surjective morphism of $U$-schemes. The $U$-scheme $U \times S \tilde{X}$ is smooth over $U$ since $U \times S X$ is smooth over $U$ and $\rho_S$ is étale. The subscheme $\Delta(U) / \Delta(\Gamma) \subset U \times S X$ projects isomorphically onto $U$. So, we are given with a section $\tilde{\Delta}$ of the morphism

$$U \times S \tilde{X} \xrightarrow{\rho_S} U \times S X \xrightarrow{p_U} U.$$

The recollection from the latter paragraph shows that we are under the hypotheses of Lemma 3.0.9 from Appendix B for the relative $U$-curve $X := U \times S \tilde{X}$ and its closed subset $Z := \rho_S^{-1}(U \times S Z)$. (If to be more accurate, then one should take the connected component $X^c$ of $X$ containing $\tilde{\Delta}(U)$ and the closed subset $Z \cap X^c$ of $X^c$).

By Lemma 3.0.9 there exists an open subscheme $X^0 \to X$ and a finite surjective morphism $\alpha : X^0 \to A^1 \times U$ such that $\alpha$ is étale over $0 \times U$ and $1 \times U$ and $\alpha^{-1}(0 \times U) = \tilde{\Delta}(U) \bigsqcup D_0$. Moreover if we define $D_1$ as $\alpha^{-1}(1 \times U)$, then $D_1 \cap Z = \emptyset$ and $D_0 \cap Z = \emptyset$. One has $[D_1 : U] = [D_0 : U] + 1$. Thus either $[D_1 : U]$ is odd or $[D_0 : U]$ is odd.
Assume $[D_1 : U]$ is odd. Then the morphism $1 \times U \xleftarrow{\alpha|_{D_1}} D_1$, the morphism $D_1 \xrightarrow{px \circ ps} X - Z$ and the isomorphism $\Phi := \Phi|_{D_1}$ satisfy the conclusion of the Proposition [1.0.1] (here $\Phi$ is from the Proposition [1.0.1]. The Proposition is proven.

\[ \square \]

2 Appendix A: Equating Lemma

Let $k$ be a field, $X$ be a $k$-smooth affine scheme, $G$ be a reductive $k$-group, $\mathcal{G}/X$ be a principle $G$-bundle over $X$. Let $\pi : \tilde{X} \to X$ be a finite étale Galois cover of $X$ with a Galois group $\Gamma$ and let $s : \tilde{X} \to \mathcal{G}$ be an $X$-scheme morphism (in other words $\mathcal{G}$ splits over $\tilde{X}$). Let $\overline{X} = (\tilde{X} \times \tilde{X})/(\Gamma \times \Gamma)$.

Let $q_i : \tilde{X} \times \tilde{X} \to \tilde{X}$ be projection to the $i$-th factor and let $p_i : X \times X \to X$ be projection to the $i$-th factor. The principal $G$ bundles $\mathcal{G}_1 := p_1^* (\mathcal{G})$ and $\mathcal{G}_2 := p_2^* (\mathcal{G})$ over $X \times X$ are not isomorphic in general. However the following Proposition holds

**2.0.8 Lemma.** The principal $G$-bundles $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ are isomorphic and moreover there is such an isomorphism $\Phi : \pi^*(\mathcal{G}_1) \to \pi^*(\mathcal{G}_2)$ that the restriction of $\Phi$ to the subscheme $X = \Delta(X)/(\Gamma \times \Gamma) \subset \overline{X} \times \overline{X}$ is the identity isomorphism.

**Proof.** The morphism $s : \tilde{X} \to \mathcal{G}$ gives rise to a 1-cocycle $a : \Gamma \to G(\tilde{X})$ defined as follows:

given $\gamma \in \Gamma$ consider the composition $s \circ \gamma$ and set $a_\gamma \in G(\tilde{X})$ to be a unique element with $a_\gamma \cdot s = s \circ \gamma$ in $G(\tilde{X})$.

It’s straight forward to check that the 1-cocycle corresponding to the principal $G$ bundle $\pi^*(\mathcal{G}_1)$ and the morphism $\overline{X} \times \overline{X} \xrightarrow{q_1} \tilde{X} \xrightarrow{s} \mathcal{G}$ coincides with the one

$$\Gamma \xrightarrow{a} G(\overline{X}) \xrightarrow{q_1} G(\tilde{X} \times \tilde{X}).$$

Similarly the 1-cocycle corresponding to the principal $G$ bundle $\pi^*(\mathcal{G}_2)$ and the morphism $\overline{X} \times \overline{X} \xrightarrow{q_2} \tilde{X} \xrightarrow{s} \mathcal{G}$ coincides with the one

$$\Gamma \xrightarrow{a} G(\overline{X}) \xrightarrow{q_2} G(\tilde{X} \times \tilde{X}).$$

Let $b \in G(\tilde{X} \times \tilde{X})$ be an element defined by the equality $b \cdot (s \circ q_2) = s \circ q_1$. To prove that the principal $G$ bundles $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ are isomorphic it suffices to check that for every $\gamma \in \Gamma$ the following relation holds in $G(\tilde{X} \times \tilde{X})$

$$\gamma b \cdot q_2^*(a)(\gamma) \cdot b^{-1} = q_1^*(a)(\gamma),$$

where $q_i^*(a)(\gamma) := q_i^* \circ a$ for $i = 1, 2$.

To prove the relation (1) it suffices to check the following one in $\mathcal{G}(\tilde{X} \times \tilde{X})$

$$(\gamma b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = q_1^*(a)(\gamma) \cdot (s \circ q_1).$$
One has the following chain of relations

\[(\gamma b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = (\gamma b \cdot q_2^*(a)(\gamma)) \cdot (s \circ q_2) = \gamma b \cdot \gamma (s \circ q_2) = \gamma (s \circ q_1) = s \circ q_1 \circ (\gamma \times \gamma)\]

The first one follows from the definition of the element \(b\), the second one follows from the commutativity of the diagram

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\nu} & G \\
(\gamma, s) \downarrow & & \downarrow s \\
X & \xrightarrow{\gamma} & \tilde{X} \\
q_2 \downarrow & & \downarrow q_2 \\
\tilde{X} \times \tilde{X} & \xrightarrow{\gamma \times \gamma} & \tilde{X} \times \tilde{X},
\end{array}
\]

the third one follows from the commutativity of the diagram

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\nu} & G \\
(b, s \circ q_2) \downarrow & & \downarrow s \\
\tilde{X} \times \tilde{X} & \xrightarrow{q_2} & \tilde{X} \\
\gamma \times \gamma \downarrow & & \downarrow q_2 \\
\tilde{X} \times \tilde{X} & \xrightarrow{\gamma \times \gamma} & \tilde{X} \times \tilde{X}.
\end{array}
\]

Thus \((\gamma b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = s \circ q_1 \circ (\gamma \times \gamma)\). The right hand side of the relation (2) is equal to \(s \circ q_1 \circ (\gamma \times \gamma)\) as well, as follows from the commutativity of the diagram

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\nu} & G \\
(\gamma, s) \downarrow & & \downarrow s \\
X & \xrightarrow{\gamma} & \tilde{X} \\
q_1 \downarrow & & \downarrow q_1 \\
\tilde{X} \times \tilde{X} & \xrightarrow{\gamma \times \gamma} & \tilde{X} \times \tilde{X}.
\end{array}
\]

So, the relation (2) holds. Whence the relation (1) holds. Whence the principal \(G\) bundles \(\pi^*(G_1)\) and \(\pi^*(G_2)\) are isomorphic.

The composite \(\tilde{X} \xrightarrow{\Delta} \tilde{X} \times \tilde{X} \xrightarrow{\varphi_2} \tilde{X} \xrightarrow{\gamma} G\) equals \(s\) and equals the composite \(s \circ q_1 \circ \Delta\). Whence \(\Delta^*(b) = 1 \in G(\tilde{X})\). This shows that the restriction to \(X = \Delta(X)/\Delta(\Gamma)\) of the isomorphism \(\pi^*(G_1)\) and \(\pi^*(G_2)\) corresponding to the element \(b\) is the identity isomorphism. The Lemma is proved.

\(\square\)
3 Appendix B: a variant of geometric lemma

Let $k$ be an infinite field, $Y$ be a $k$-smooth algebraic variety, $y \in Y$ be a point, $\mathcal{O} = \mathcal{O}_{Y,y}$ be the local ring, $U = \text{Spec}(\mathcal{O})$. Let $X/U$ be a $U$-smooth relative curve with geometrically connected fibres equipped with a finite surjective morphism $\pi : X \to A^1 \times U$ and equipped with a section $\Delta : U \to X$ of the projection $p : X \to U$. Let $\mathcal{Z} \subset X$ be a closed subset finite over $U$. The following Lemma is a variant of Lemma 5.1 from [OP].

3.0.9 Lemma. There exists an open subscheme $X^0 \to X$ and a finite surjective morphism $\alpha : X^0 \to A^1 \times U$ such that $\alpha$ is étale over $0 \times U$ and $1 \times U$ and $\alpha^{-1}(0 \times U) = \Delta(U) \bigsqcup D_0$. Moreover if we define $D_1$ as $\alpha^{-1}(1 \times U)$, then $D_1 \cap \mathcal{Z} = \emptyset$ and $D_0 \cap \mathcal{Z} = \emptyset$.

Proof. Let $\bar{X}$ be the normalization of the scheme $P^1 \times U$ in the function field $k(\bar{X})$ of $X$. Let $\bar{\pi} : \bar{X} \to P^1 \times U$ be the morphism. Let $\mathcal{X}_{\infty} = \pi^{-1}(\infty \times U)$ be the set theoretic preimage of $\infty \times U$. Choose a hyperplane $P$ such that $\mathcal{Z} \cap \mathcal{X}_{\infty} = \emptyset$. Define a Cartier divisor $(\mathcal{U}, H)$ over $P$.

Let $L' = \bar{\pi}^*(\mathcal{O}_{P^1 \times U}(1))$, $L'' = \mathcal{O}_X(\Delta(U))$. Let $D_{\infty} = (\pi^*)(\infty \times U)$ be the pull-back of the Cartier divisor $\infty \times U \subset P^1 \times U$. Choose and fix a closed embedding $i : \bar{X} \hookrightarrow P^n \times U$ of $U$-schemes. Set $L = i^*(\mathcal{O}_P \times U(1))$.

The sheaf $L$ is very ample. Thus the sheaf $L'' \otimes L$ is very ample as well. So, there exists a closed embedding $i'' : \bar{X} \hookrightarrow P^n' \times U$ of $U$-schemes such that $L'' \otimes L = (i'')^*(\mathcal{O}_{P^n' \times U}(1))$. Using Bertini theorem choose a hyperplane $H'' \subset P^n' \times U$ such that $(a'') H'' \cap \Delta(U) = \emptyset$, $H'' \cap \mathcal{Z} = \emptyset$, $H'' \cap D_{\infty} = \emptyset$.

Define a Cartier divisor $D''$ on $\bar{X}$ as the the closed subscheme $H'' \cap \bar{X}$ of $\bar{X}$.

Regard $D'' = D'' \bigsqcup D_{\infty}$ as a Cartier divisor on $\bar{X}$. Clearly, one has $\mathcal{O}_{\bar{X}}(D'') = L'' \otimes L \otimes L'$.

The sheaf $L$ is very ample. Thus the sheaf $L' \otimes L$ is very ample as well. So, there exists a closed embedding $i' : \bar{X} \hookrightarrow P^n' \times U$ of $U$-schemes such that $L' \otimes L = (i')^*(\mathcal{O}_{P^n' \times U}(1))$. Using Bertini theorem choose a hyperplane $H' \subset P^n' \times U$ such that $(a') H' \cap \Delta(U) = \emptyset$, $H' \cap \mathcal{Z} = \emptyset$, $H' \cap D_0' = \emptyset$; $(b')$ the scheme theoretic intersection $H' \cap \bar{X}$ is a $(k)$-smooth scheme.

Define a Cartier divisor $D'$ on $\bar{X}$ as the closed subscheme $D' = H' \cap \bar{X}$ of $\bar{X}$.

Regard $D'_1 := D' \bigsqcup \Delta(U)$ as a Cartier divisor on $\bar{X}$. Clearly, one has $\mathcal{O}_{\bar{X}}(D'_1) = L' \otimes L \otimes L''$.

Observe that $D'$ is an essentially $k$-smooth scheme finite and étale over $U$. Let $s'$ and $s''$ be global sections of $L' \otimes L \otimes L''$ such that the vanishing locus of $s'$ is the Cartier divisor $D'_1$ and the vanishing locus of $s''$ is the Cartier divisor $D''$. Clearly $D'_1 \cap D'' = \emptyset$. Thus $f = \{s', s''\} : \bar{X} \to P^1$ is a regular morphism of $U$-schemes. Set $\bar{\alpha} = (f, \bar{p}) : \bar{X} \to P^1 \times U$.

Clearly, $\bar{\alpha}$ is a finite surjective morphism. Set $X^0 = \bar{\alpha}^{-1}(A^1 \times U)$ and $\alpha = \bar{\alpha}|_{X^0} : X^0 \to A^1 \times U$. 

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Clearly, $\alpha$ is a finite surjective morphism and $X^0$ is an open subscheme of $X$. Since $\alpha$ is a finite surjective morphism and $X^0$, $A^1 \times U$ are regular schemes the morphism $\alpha$ is flat by a theorem of Grothendieck. Since $D'_1$ is finite étale over $U$ the morphism $\alpha$ is étale over $0 \times U$. So, we may choose a point $1 \in P^1$ such that the $\alpha$ is étale over $1 \times U$ and $(\alpha)^{-1}(1 \times U) \cap Z = \emptyset$. If we set $D_0 = D'_1$, then $\alpha^{-1}(0 \times U) = \Delta(U) \bigsqcup D_0$ and $D_0 \cap Z = \emptyset$. The Lemma is proven.

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References

[A]  M. Artin. Comparaison avec la cohomologie classique: cas d’un préschéma lisse. Lect. Notes Math., vol. 305, Exp XI.

[OP] M. Ojanguren, I. Panin. A purity theorem for the Witt group. Ann. Sci. Ecole Norm. Sup. (4) 32 (1999), no. 1, 71–86.

[PR] I. Panin, U. Rehmann. A variant of a Theorem by Springer. Algebra i Analyz, vol. 19 (2007), 117–125. [www.math.uiuc.edu/K-theory/0671/2003]

[Pa] I. Panin. Rationally isotropic quadratic spaces are locally isotropic. Invent. math. 176, 397-403 (2009).

[PaP] I. Panin, K. Pimenov. Rationally isotropic quadratic spaces are locally isotropic:II. Documenta Mathematica, Vol. Extra Volume: 5. Andrei A. Suslin’s Sixtieth Birthday , 515–523, (2010).

[P] D. Popescu. General Néron desingularization and approximation, Nagoya Math. Journal, 104 (1986), 85–115.

[PaSV] I. Panin, A. Stavrova, N. Vavilov. Grothendieck—Serre conjecture I: Appendix, Preprint October, 2009, [http://arxiv.org/abs/0910.5465](http://arxiv.org/abs/0910.5465).

[PSV] I. Panin, A. Stavrova, N. Vavilov. On Grothendieck—Serre’s conjecture concerning principal $G$-bundles over reductive group schemes I, Preprint (2009), [http://www.math.uiuc.edu/K-theory/0929/](http://www.math.uiuc.edu/K-theory/0929/).

[Kn] M. Knus. Quadratic and hermitian forms over rings. Springer Verlag, 1991.