Collimation Mechanism for Atom Lasers

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A mechanism is suggested for creating well-collimated beams of neutral spin-polarized particles by means of magnetic fields. This mechanism can be used in atom lasers for the formation of directed coherent beams of atoms. The directed motion of atoms is achieved only with the help of magnetic fields, no mechanical collimators being involved.

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I. INTRODUCTION

A device emitting a coherent atomic beam, similar to a laser radiating coherent photon rays, is called atom laser [1–7]. Bose atoms can be prepared in the Bose–condensed state by cooling them in a trap [8–10]. The Bose–condensed state is believed to be a coherent state [11], which is supported by the interference experiments [12,13].

It is worth noting that a Bose–condensed state cannot be a pure coherent state, but it can be only partially coherent. This is because a coherent state is not an eigenstate of a Bose system [14]. Another problem is that, if one identifies the Bose–condensed state with a state having broken gauge symmetry, then there appear anomalous fluctuations of the number of particles, with nonthermodynamic behaviour [15,16]. This anomalous behaviour originates from infrared divergences typical of any Bose fluids with broken gauge symmetry [17–19] and is related to the fluctuations of phases of field operators [20].

However, a system of finite number of particles in a trap does not undergo a genuine phase transition with breaking symmetry [21–24], similarly to the case of the Bose gas in restricted geometries [25]. In such cases, the singularities of all thermodynamic functions are rounded off and the latter become analytic, varying smoothly. This means that Bose condensation of a finite number of particles in a trap is a crossover phenomenon that is not accompanied by breaking gauge symmetry. Although breaking the latter can be employed as a technical trick facilitating some calculations, for instance, for considering collective excitations of a trapped Bose gas [26–29]. The usage of this trick is limited by the cases when it does not lead to anomalies, like those for the fluctuations of the number of particles. The Bose condensation in a trap, being a crossover phenomenon, occurs at a crossover temperature that can be defined as a temperature at which at least one of thermodynamic characteristics has a maximum. This definition of a crossover temperature is typical of crossover phenomena [21–25,30,31].

Assume that Bose atoms are trapped and cooled down to experience the Bose condensation, when, at least partly, they become coherent. An important problem in realizing an atom laser is how to form a directional beam of atoms. To make the output highly directional is, actually, the first condition on a laser [7]. However, at the present time there are no mechanisms permitting one to create such well collimated beams of neutral atoms flying out of traps. To form an output coupler, one employs short radio–frequency pulses transferring atoms from trapped states to an untrapped state. But, when escaping from a trap, atoms fly out more or less in all directions [12,13,32], with an anisotropy formed by the gravitational force. This makes the principal difference between the present–day traps and optical lasers. "The major difference between the two devices is that the photons from an optical laser generally emerge in a well–collimated beam, whereas atoms from the atom laser fly out in all directions” [33]. In order to force the escaping atoms to move in one preferable direction, one can employ some mechanical blocking or external laser beams (see discussion in Ref. [34]). For example, to provide a directed output beam, an implementation of the laser scheme using hollow optical fibers was suggested [35], with a momentum kick provided by two laser beams.

In this paper, we consider a new mechanism for creating well–collimated beams of neutral atoms. This mechanism does not require any mechanical blocking or additional laser operation. Collimation of an atomic beam is achieved only by means of magnetic fields of a trap. A particular configuration of the trap magnetic field, leading to a semiconfining regime of motion, has been studied earlier [36–38]. Here we generalize the consideration showing that there exists a large class of magnetic fields permitting one to create directed beams of atoms for atom lasers.

II. EVOLUTION EQUATIONS

When the space variation of magnetic fields is sufficiently smooth, the dynamics of neutral atoms is well described in the semiclassical approximation [36,37] for the quantum–mechanical averages of the real–space coordinate, \( \langle \vec{R} \rangle \equiv \langle R \rangle \equiv \{ R_\alpha \} \), and of the spin operator, \( \vec{S} \equiv \langle \vec{S} \rangle \equiv \{ S_\alpha \} \), where \( \alpha = x, y, z \). Then, for an atom of mass \( m \) and magnetic moment \( \mu_0 \), the equation for the average space variable is

\[
\frac{d^2 R_\alpha}{dt^2} = \frac{\mu_0}{m} \vec{S} \cdot \nabla \frac{\partial B}{\partial R_\alpha}
\]

and the evolution of the average spin is given by the equation

\[
\frac{d \vec{S}}{dt} = \frac{\mu_0}{\hbar} \vec{S} \times \vec{B}
\]

The total magnetic field
\[ \vec{B} = \vec{B}_1 + \vec{B}_2 \]  

(3)

consists of two terms. One is the quadrupole field

\[ \vec{B}_1 = B'_1 \left( R_x \vec{e}_x + R_y \vec{e}_y + \lambda R_z \vec{e}_z \right) , \]

(4)

in which $\vec{e}_\alpha$ are the unit Cartezian vectors and $\lambda$ is an anisotropy parameter. If one requires that $\vec{B}_1$ satisfies the equation $\vec{\nabla} \cdot \vec{B}_1 = 0$, then $\lambda = -2$. But if the field (4) is produced by several different magnetic coils, then one can, in general, realize any anisotropy with an arbitrary parameter $\lambda$. The second term in (3) is a transverse field

\[ \vec{B}_2 = B_2 \vec{h}(t), \quad \vec{h}(t) = h_x \vec{e}_x + h_y \vec{e}_y, \]

(5)

in which $h_\alpha = h_\alpha(t)$ and

\[ |\vec{h}|^2 = h_x^2 + h_y^2 = 1, \quad \vec{h} = \vec{h}(t). \]

It is convenient to work with the dimensionless space variable

\[ \vec{r} \equiv \frac{\vec{R}}{R_0} = \{x, y, z\}, \quad R_0 \equiv \frac{B_2}{B'_1}. \]

(6)

Introduce also the characteristic frequencies

\[ \omega_1^2 = \frac{\mu_0 B'_1}{m R_0}, \quad \omega_2 = \frac{\mu_0 B_2}{h}. \]

(7)

Then equation (1) can be written as

\[ \frac{d^2 \vec{r}}{dt^2} = \omega_1^2 \left( S_x \vec{e}_x + S_y \vec{e}_y + \lambda S_z \vec{e}_z \right). \]

(8)

This equation is to be complemented by initial conditions

\[ \vec{r}(0) = \vec{r}_0 = \{x_0, y_0, z_0\}, \quad \dot{\vec{r}}(0) = \dot{\vec{r}}_0 = \{\dot{x}_0, \dot{y}_0, \dot{z}_0\}, \]

(9)

in which the dot means the time derivative.

The equation (2) for the average spin can be reduced to the form

\[ \frac{d \vec{S}}{dt} = \omega_2 \hat{A} \vec{S}, \]

(10)

where the matrix $\hat{A} = [A_{\alpha\beta}]$ consists of the elements

\[ A_{\alpha\beta} = -A_{\beta\alpha}, \quad A_{\alpha\alpha} = 0 \quad (\alpha, \beta = 1, 2, 3), \]

\[ A_{12} = \lambda z, \quad A_{23} = x + h_x, \quad A_{31} = y + h_y. \]

(11)

The initial condition for (10) is

\[ \vec{S}(0) = \vec{S}_0 = \{S^0_x, S^0_y, S^0_z\}. \]

(12)

Assume that the following inequalities hold true:

\[ |\omega_1| \ll \omega_2, \quad \left| \frac{d \vec{h}}{dt} \right| \ll \omega_2. \]

(13)

Defining an effective frequency $\omega = \omega(t)$ of the transverse field,

\[ \omega \equiv \left| \frac{d \vec{h}}{dt} \right|, \]

(14)

we may rewrite the inequalities (13) as

\[ \left| \frac{\omega_1}{\omega_2} \right| \ll 1, \quad \left| \frac{\omega}{\omega_2} \right| \ll 1. \]

(15)

The existence of these inequalities makes it possible to find approximate solutions to Eqs. (8) and (10) by employing the scale separation approach [39–41].
III. SCALE SEPARATION

The first step in the scale separation approach [39–41] is to classify the functions under consideration into fast and slow. Inequalities (15) suggest that the variables $\vec{r}$ and $\vec{h}$ are slow, as compared to the fast variables $\vec{S}$. It is worth noting that equation (8) for $\vec{r}$ is a differential equation of second order, while that for $\vec{S}$ is a first–order differential equation. Hence, the set of these differential equations does not make what is called the standard form of a dynamical system [42], where all equations are to be first–order differential equations. Then one may ask whether it is admissible to treat the variable $\vec{r}$ as slow with respect to $\vec{S}$, when not their first derivatives are compared. However, it is not difficult to show, by a simple change of notations, as is done in Appendix A, that it is really the case: the variable $\vec{r}$ is low as compared to $\vec{S}$.

With the slow variables treated as quasi–integrals of motion, the solution of Eq. (10) can be found in the following way. Solve the eigenproblem

$$\hat{A} \vec{b}_i = \alpha_i \vec{b}_i,$$

resulting in the eigenvalues

$$\alpha_{1,2} = \pm i\alpha, \quad \alpha_3 = 0, \quad \alpha^2 = A_{12}^2 + A_{23}^2 + A_{31}^2,$$

and in the eigenvectors

$$\vec{b}_i = \frac{1}{\sqrt{C_i}} \left[ (A_{12}A_{23} - \alpha_i A_{31}) \vec{e}_x + (A_{12}A_{31} + \alpha_i A_{23}) \vec{e}_y + (A_{12}^2 + \alpha_i^2) \vec{e}_z \right],$$

with the normalization constant $C_i$ given by the equation

$$C_i = (A_{12}^2 - |\alpha_i|^2)^2 + (A_{12}^2 + |\alpha_i|^2) (A_{23}^2 + A_{31}^2) .$$

It is possible to check that the eigenvectors $\vec{b}_i$ satisfy the orthonormality condition

$$\vec{b}_i^* \cdot \vec{b}_j = \delta_{ij} .$$

When the transverse field $\vec{h}$ is kept constant, then a particular solution of (10) has the form of $\vec{b}_i \exp(\alpha_i \omega_2 t)$, and the general solution is

$$\vec{S}^{(0)} (t) = \sum_{i=1}^{3} a_i \vec{b}_i \exp(\alpha_i \omega_2 t) \quad (\vec{h} = \text{const}) .$$

Substituting here the dependence of $\vec{h} = \vec{h} (t)$ on time and using the notation

$$\vec{b}_i (t) \equiv \vec{b}_i \left( \vec{h} (t) \right), \quad \alpha_i (t) \equiv \alpha_i \left( \vec{h} (t) \right),$$

we obtain an approximate solution of Eq. (10) in the form

$$\vec{S} (t) = \sum_{i=1}^{3} a_i \vec{b}_i (t) \exp \{ \alpha_i (t) \omega_2 t \} .$$

The coefficients $a_i$ can be defined from the initial condition (12), with the use of the orthonormality condition (17), which gives

$$a_i = \vec{S}_0 \cdot \vec{b}_i (0) .$$

A slightly different presentation of an approximate solution to Eq. (10) can be found as follows. Let us look for a particular solution of the type $\vec{b}_i \exp(\varphi_i)$, keeping in mind that $\vec{b}_i$ and $\varphi_i$ can depend on time. Substituting this particular solution into (10) yields
\begin{equation}
\vec{b}_i \dot{\phi}_i + \dot{\vec{b}}_i = \omega_2 \alpha_i \vec{b}_i ,
\end{equation}

where the dot means the time derivative. Multiplying (21) by \(\vec{b}_i\), we get
\begin{equation}
\dot{\phi}_i = \omega_2 \alpha_i - b^*_i \cdot \vec{b}_i ,
\end{equation}
from where
\begin{equation}
\varphi(t) = \int_0^t \left( \omega_2 \alpha_i - b^*_i \cdot \vec{b}_i \right) dt .
\end{equation}

Substituting (22) back into (21) gives the equation
\begin{equation}
\dot{\vec{b}}_i = \left( b^*_i \cdot \vec{b}_i \right) \vec{b}_i ,
\end{equation}
playing the role of the criterion for the validity of (22). If (23) is valid, then the solution to (10) writes
\begin{equation}
\vec{S}(t) = \sum_{i=1}^3 a_i \vec{b}_i (t) \exp \{ \varphi_i(t) \} .
\end{equation}

Condition (23) imposes a restriction on the behaviour of \(\vec{b}_i\), and it may be not valid exactly for an arbitrary field \(\vec{h}(t)\). However, here we have the parameter \(\varepsilon \equiv |\omega/\omega_2|\), which, according to (15), is small, \(\varepsilon \ll 1\). As far as \(\vec{b}_i\) depends on time through \(\vec{h}(t)\), then \(\vec{b}_i \to 0\) when \(\varepsilon \to 0\). Therefore, equation (23) becomes asymptotically exact when \(\varepsilon \to 0\). And for small \(\varepsilon \ll 1\), equation (23) is asymptotically valid for an arbitrary field \(\vec{h}(t)\) satisfying (15).

Thus, the asymptotic form (24) is an approximate solution to equation (10). With the small parameter \(\varepsilon \ll 1\), the phase (22) is asymptotically equal to
\begin{equation}
\varphi(t) \simeq \omega_2 \alpha_i(t) t \quad (\varepsilon \ll 1) ,
\end{equation}
which reduces solution (24) to the form (19).

In this way, expressions (19) and (24) are both approximate solutions to equation (10), provided that the inequality \(\varepsilon \ll 1\) is valid. When \(\varepsilon \to 0\), these solutions tend to the exact solution corresponding to \(\varepsilon = 0\). Because of this, such solutions are often called asymptotically exact with respect to \(\varepsilon \to 0\).

The found solution for the fast variable \(\vec{S}\) has to be substituted into the equation (8) for the slow variable \(\vec{r}\), averaging the right–hand side of this equation over time according to the rule
\begin{equation}
\langle f \rangle \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(\vec{r},t) dt ,
\end{equation}
with the slow variable \(\vec{r}\) kept fixed. This averaging procedure transforms (8) to the equation
\begin{equation}
\frac{d^2 \vec{r}}{dt^2} = \vec{F} ,
\end{equation}
with the averaged force
\begin{equation}
\vec{F} \equiv \omega_1^2 \langle S_x \vec{e}_x + S_y \vec{e}_y + \lambda S_z \vec{e}_z \rangle .
\end{equation}

Note that any form of \(\vec{S}\), either (19) or (24), can be used, since in both these cases we come to the equality
\begin{equation}
\langle \vec{S} \rangle = a_3 \langle \vec{b}_3 \rangle ,
\end{equation}
in which
\begin{equation}
a_3 = \frac{(x + h_0^\alpha)S_x^\alpha + (y + h_0^\alpha)S_y^\alpha + \lambda x z S_z^\alpha}{(x + h_x^\alpha)^2 + (y + h_y^\alpha)^2 + \lambda^2 z^2} , \quad b_3 = \frac{(x + h_x^\alpha \vec{e}_x + (y + h_y^\alpha) \vec{e}_y + \lambda \vec{e}_z}{(x + h_x^\alpha)^2 + (y + h_y^\alpha)^2 + \lambda^2 z^2} ,
\end{equation}
with \(h_0^\alpha \equiv h_\alpha(0)\) and \(\alpha = x, y, z\). In this way, for the averaged force (27) we have
\begin{equation}
\vec{F} = \omega_1^2 a_3 \langle \vec{b}_3 \rangle \vec{e}_x + b_3 \langle \vec{b}_3 \rangle \vec{e}_y + \lambda b_3 \langle \vec{b}_3 \rangle \vec{e}_z ,
\end{equation}
where \(b_3^\alpha\), with \(\alpha = x, y, z\), are the components of the vector \(\vec{b}_3 = \{b_3^\alpha\}\).
IV. ROTATING FIELD

To proceed further, let us concretize the transverse field $\vec{h}$ ($t$) in (5). Take this field in the form of the rotating field with

$$h_x = \cos \omega t, \quad h_y = \sin \omega t,$$

which is employed in some magnetic traps [43,44]. Then, using $h_x^0 = 1$, $h_y^0 = 0$, and the equality

$$< x \cos \omega t + y \sin \omega t > = 0,$$

for the force (30) we find

$$\vec{F} = \frac{\omega^2 [(1 + x)S_x^0 + yS_y^0 + \lambda z S_z^0] (x e_x + y e_y + 2\lambda^2 z e_z)}{2(1 + 2x + x^2 + y^2 + \lambda^2 z^2)(1 + x^2 + y^2 + \lambda^2 z^2)^{1/2}}.$$

(32)

If we take for the initial spin polarization the standard initial condition $S_x^0 \neq 0$, $S_y^0 = S_z^0 = 0$ that is used for confining atoms in a trap, then, for $S_x^0 < 0$, the force (32) really provides confinement, when atoms oscillate in a nearly harmonic potential. This type of the confined oscillating motion does not change much if we add to (32) the gravitational force. A vertical field gradient supplies the levitating force to support the atom against gravity. The combination of the magnetic field and gravity produces a very nearly harmonic confining potential within the trap volume in all three dimensions, so that the atom oscillates around an effective equilibrium position [45,46]. The role of the gravitational force is discussed in Appendix B.

We take here a different type of the initial spin polarization corresponding to the initial condition

$$S_x^0 = S_y^0 = 0, \quad S_z^0 = S \neq 0.$$  

(33)

The choice of initial conditions, as is well known from quantum mechanics, is not prescribed a priori but can be realized in any desirable way. Some more details on the possibility of realizing different initial conditions for the spin polarization are considered in Appendix C.

To simplify the evolution equations, it is convenient to measure time in units of $\omega_0^{-1}$, where

$$\omega_0^2 \equiv |S\omega^2|.$$  

(34)

In what follows, we shall deal with this dimensionless time. To return to the dimensional time, we need to set $t \rightarrow \omega_0 t$.

In order to take into account the finite size of a trap, we introduce the shape factor

$$\varphi(\vec{r}) = \exp \left\{-\frac{1}{L^2} (x^2 + y^2 + \varepsilon^2 z^2) \right\},$$

(35)

in which $L$ is the characteristic radius of the device in the radial direction and $L/\varepsilon$ is the length of the device in the axial direction. And let us define the function

$$f(\vec{r}) = \frac{\varphi(\vec{r})}{[(1 + 2x + x^2 + y^2 + \lambda^2 z^2)(1 + x^2 + y^2 + \lambda^2 z^2)]^{1/2}}.$$  

(36)

With these definitions, equation (26), for the initial spin polarization (33), results in the system of equations for the components

$$\frac{d^2 x}{dt^2} = \frac{\lambda}{2} (\text{sgn}S) fxz, \quad \frac{d^2 y}{dt^2} = \frac{\lambda}{2} (\text{sgn}S) fyz, \quad \frac{d^2 z}{dt^2} = \lambda^3 (\text{sgn}S) f z^2.$$  

(37)

These equations are invariant with respect to the transformations $\lambda \rightarrow -\lambda$, $S \rightarrow -S$ as well as to the transformations $S \rightarrow -S, z \rightarrow -z$. Therefore, it is sufficient to study only one case for which the sign of $\lambda S$ is fixed, since the change $\lambda S \rightarrow -\lambda S$ leads to a picture that is mirror symmetric with respect to the $x-y$ plane. Take, for instance,

$$\lambda S > 0, \quad |\lambda| \equiv \beta.$$  

(38)

Changing $\lambda S$ by $-\lambda S$ would invert the whole picture according to the tranformation $z \rightarrow -z$. We may also notice that the system of equations (37) possesses the integral of motion
\[ x(t) \frac{d}{dt} \ln \frac{y}{x} = \text{const}, \]  
(39)

which shows that the motion along the \( x \) and \( y \) axes are similar to each other, so that, under the same initial conditions for \( x \) and \( y \), the laws of motion \( x(t) \) and \( y(t) \) coincide. Because of this, we consider in what follows only the \( x \) component. Thus, taking account of (38), from (37) we have

\[ \frac{d^2 x}{dt^2} = \frac{\beta}{2} f x z, \quad \frac{d^2 z}{dt^2} = \beta^3 f z^2. \]  
(40)

At the initial stage of the process, when \( |\vec{r}| \ll 1 \) and \( f(\vec{r}) \simeq 1 \), the equations (40) can be solved analytically. Then the second equation from (40) writes

\[ \frac{d^2 z}{dt^2} = \beta^3 z^2. \]  
(41)

This can be integrated once yielding

\[ \left( \frac{dz}{dt} \right)^2 = \frac{2}{3} \beta^3 \left( z^3 - z_0^3 + \zeta \right), \]  
(42)

where

\[ \zeta = \frac{3}{2\beta^3} \zeta_0^2. \]  
(43)

Eq. (42) is the Weierstrass equation whose solution

\[ z(t) = \frac{6}{\beta^3} P(t - t_0) \]  
(44)

is expressed through the Weierstrass function [47]. The time \( t_0 \) plays the role of the escape time [36,37] and is defined from the initial condition \( z(0) = z_0 \), which gives

\[ t_0 = \sqrt{\frac{3}{2\beta^3} \int_{z_0}^{\infty} \frac{dz}{\sqrt{z^3 - z_0^3 + \zeta}}}. \]  
(45)

The first equation from (40) takes the form

\[ \frac{dx^2}{dt^2} = \frac{3}{\beta^2} P(t - t_0) x, \]  
(46)

which is the Lamé equation [48] whose degree is defined by the equality \( n(n + 1) = 3/\beta^2 \), with \( n > 0 \), which yields

\[ n = \frac{1}{2} \left( \sqrt{1 + \frac{12}{\beta^2}} - 1 \right). \]  
(47)

The solution to (46) can be expressed through the Lamé functions [48].

From the properties of the Weierstrass functions [47] it follows that the axial motion is bounded from below by the minimal value

\[ z_{\text{min}} = (z_0^3 - \zeta)^{1/3}. \]  
(48)

When \( t \to t_0 \), then

\[ x(t) \sim |t - t_0|^{-1/2}, \quad z(t) \sim |t - t_0|^{-2}. \]  
(49)

The aspect ratio

\[ \frac{x(t)}{z(t)} \sim |t - t_0|^{3/2} \to 0 \]
shows that the atom trajectories are stretched in the axial direction.

The asymptotic expressions (49) give, of course, only a qualitative understanding of the collimation process, since equations (41) and (44) are valid only for the initial stage of atomic motion, when \( x \) and \( z \) are small. To consider the dynamics of atoms for arbitrary times, we have to solve Eqs. (40) numerically. We accomplished such a numerical investigation for the parameter \( \beta = 2 \) and \( \varepsilon = 1 \). The initial positions of atoms are taken in the center of a trap, with varying initial velocities \( |\dot{x}_0| \leq 1 \) and \( |\dot{z}_0| \leq 1 \). The atomic trajectories in the \( x - z \) plane are presented in Figs. 1 to 3. The phase portraits for the velocities

\[
v(t) \equiv \frac{dx}{dt}, \quad w(t) \equiv \frac{dz}{dt}
\]

are shown in Figs. 4 to 6. These figures demonstrate that the bunch of atomic trajectories is essentially squeezed in the radial direction forming a well collimated beam.

V. CONSTANT FIELD

The limiting case of a transverse field satisfying the second inequality in (15) corresponds to a constant field with

\[ h_x = \text{const}, \quad h_y = \text{const}. \]

Then the effective frequency (14) is zero. The matrix \( \hat{A} \) in (10), with the elements (11), does not depend explicitly on time. Under \( \rightarrow \hat{r} \) fixed, the solution (19) becomes an exact solution to equation (10). For the average force (30), we find

\[
\rightarrow \vec{F} = \omega \left( x + h_x \right) \frac{S^0_x + (y + h_y) S^0_y + \lambda z S^0_z}{\left( x + h_x \right)^2 + \left( y + h_y \right)^2 + \lambda^2 z^2} \left[ \left( x + h_x \right) \vec{e}_x + (y + h_y) \vec{e}_y + \lambda z \vec{e}_z \right].
\]

Note that for the initial spin polarization \( S^0_x \neq 0 \), \( S^0_y = S^0_z = 0 \) the force (52) does not provide confinement, contrary to the average force (32) related to the rotating field (31). However, confinement is not our concern here. We wish to find a regime of semiconfinement, when the atoms move predominantly in one direction.

Let us take for the initial spin polarization the initial condition (33). Then the average force (52) becomes

\[
\rightarrow \vec{F} = \omega^2 \lambda (\text{sgn} S) z \frac{(x + h_x) \vec{e}_x + (y + h_y) \vec{e}_y + \lambda z \vec{e}_z}{\left( x + h_x \right)^2 + \left( y + h_y \right)^2 + \lambda^2 z^2},
\]

where the notation (34) is used. For the trap form factor, we accept the same expression (35). But, instead of (36), we now have

\[
f(\hat{r}) = \frac{\varphi(\hat{r})}{\left( x + h_x \right)^2 + \left( y + h_y \right)^2 + \lambda^2 z^2}.
\]

Passing to the dimensionless time measured in units of \( \omega^{-1} \), we obtain as the equation of motion (26) the system of equations

\[
\frac{dx^2}{dt^2} = \lambda (\text{sgn} S) (x + h_x) f z, \quad \frac{dy^2}{dt^2} = \lambda (\text{sgn} S) (y + h_y) f z,
\]

\[
\frac{dz^2}{dt^2} = \lambda^3 (\text{sgn} S) f z^2.
\]

Eqs. (55), though are different form (37), possess the same invariance property with respect to the transformations \( \lambda \to -\lambda \), \( S \to -S \) and \( S \to -S \), \( z \to -z \). Thence, we can again accept condition (38). The integral of motion (39) exists for (55) only if \( h_x = h_y \).

The system of equations (55) also provides the semiconfining regime of motion. For example, at the initial stage, when \( |\hat{r}| \ll 1 \) and \( f(\hat{r}) \simeq 1 \), assuming condition (38), we have

\[
\frac{dx^2}{dt^2} = \beta h_x z, \quad \frac{dz^2}{dt^2} = \beta^3 z^2.
\]
The equation for the $y$–component is the same as for the variable $x$, with the change of $h_x$ by $h_y$. The equation for $z$ coincides with (41), thus, having the same solution (44), demonstrating that the atomic motion is locked from below by the minimal value (48). When the time approaches the escape time (45), then instead of (49), we now have

$$x(t) \sim \ln |t - t_0|, \quad z(t) \sim |t - t_0|^{-2},$$

which shows that, with the transverse constant field, the collimation is even better than with the rotating field. At the late stage of the process, when $|x/h_x| \gg 1$ and $|y/h_y| \gg 1$, the factors (54) and (36) are asymptotically equal, and the evolution equations (55) and (37) become equivalent.

In this way, the semiconfining regime of motion can be realized for different magnetic fields satisfying condition (15), under the initial spin polarization (33). This regime is clearly illustrated by numerical calculations for the rotating transverse field. The numerical investigation of motion for the constant transverse field will be given in a separate paper. The semiconfining regime, we described, can be used for creating well–collimated beams from atom lasers.

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Appendix A. Standard Form

Equations (8) and (10) can be written in the standard form, as a system of differential equations of first order,
\[
\frac{d\vec{r}}{dt} = \vec{p}, \quad \frac{d\vec{p}}{dt} = \omega_1^2 f_1, \quad \frac{d\vec{S}}{dt} = \omega_2 f_2,
\]
with finite functions \(f_1\) and \(f_2\). By means of the notation
\[
t' \equiv \omega_2 t, \quad \frac{\vec{p}}{\omega_1} = \epsilon \vec{p'}, \quad \epsilon \equiv \frac{\omega_1}{\omega_2},
\]
the above differential equations can be presented as
\[
\frac{d\vec{r}}{dt'} = \epsilon \vec{p'}, \quad \frac{d\vec{p'}}{dt'} = \epsilon f_1, \quad \frac{d\vec{S}}{dt'} = f_2.
\]
Condition (15) tells us that \(\epsilon\) is a small parameter, \(|\epsilon| \ll 1\). Then from the latter system of equations it follows immediately that \(\vec{r}'\) is a slow variable with respect to \(\vec{S}\). The same concerns the effective velocity \(\vec{p}'\).

Appendix B. Gravitational Force

Since an atom has a mass, then, contrary to photon lasers, the existence of gravity may influence the performance of atom lasers. The influence of gravity is not essential for the confining regime, when we start with the initial spin polarization \(S^0_x < 0, S^0_y = S^0_z = 0\). Then, adding to the evolution equation (8) the gravitational force \(-mg \vec{e}_z\), in the case of the rotating field, we come to the equations
\[
\frac{d^2x}{dt^2} = -\frac{1}{2}|S^0_x|\omega_1^2 x, \quad \frac{d^2z}{dt^2} = -\lambda^2 |S^0_z|\omega_1^2 z - \frac{g}{R_0},
\]
describing the motion of atoms inside the trap, where \(|\vec{r}| \ll 1\). We do not write down the equation for \(y\), which is similar to that for \(x\). These equations, as is evident, define an oscillatory motion in all three dimensions. The sole thing that is changed, when taking account of gravity, is the shift of the equilibrium position on the \(z\)-axis from zero to \(z_{eq} = -g(\lambda^2 |S^0_z|\omega_1^2 R_0)^{-1}\). So that the center of harmonic oscillations is shifted from the center of coordinates to the point \(\{0, 0, z_{eq}\}\). Certainly, if the quadrupole field is switched off, the atoms fall down because of gravity. But while the quadrupole field is sufficient for confinement, the gravitational force does not change principally the regime of motion corresponding to simple harmonic oscillations. Recall that the constant transverse field does not provide confinement in any case, so that the gravity is again of no importance.

In the case of the semiconfining regime, with the initial spin polarization (33), the axis \(z\) is an axis of the device, but not necessary the vertical axis. Since the device can be oriented arbitrarily, the gravitational force can also be directed along different axes. For instance, assume that this force is \(-mg \vec{e}_x\). Then, for the rotating transverse field, instead of (40), we have
\[
\frac{d^2x}{dt^2} = \frac{\beta}{2} f_{xz} - \gamma, \quad \frac{d^2z}{dt^2} = \beta^3 f z^2,
\]
where \(\gamma \equiv g/\omega_1^2 R_0\). To estimate the value of \(\gamma\), let us take the parameters of the magnetic fields as in the quadrupole traps with the rotating transverse field [43,44]. Then \(\omega_1 \sim 10^2 - 10^3\text{s}^{-1}\) and \(R_0 \sim 0.1 - 0.5\text{cm}\). With \(g \approx 10^3\text{cm s}^{-2}\), this gives \(\gamma \sim 10^{-3} - 1\). The equation for \(z\) is the same as earlier and describes the semiconfining motion along the \(z\)-axis. The motion in the radial direction is also semiconfined, with the downward deviation of an atomic beam because of gravity. Thus, the regime of motion does not change principally.

If the device is oriented so that its \(z\)-axis is in the vertical direction, then, instead of (40), we get
\[
\frac{d^2x}{dt^2} = \frac{\beta}{2} f_{xz}, \quad \frac{d^2z}{dt^2} = \beta^3 f z^2 - \gamma.
\]
Integrating once the equation for \( z \), we have
\[
\left( \frac{dz}{dt} \right)^2 = \frac{2}{3} \beta^3 \left[ z^3 - z_0^3 - \frac{3 \gamma}{\beta^2} (z - z_0) + \zeta \right].
\]

The latter equation can be reduced to the Weierstrass form
\[
\left( \frac{dP}{dt} \right)^2 = 4P^3 - g_2 P - g_3
\]
with the Weierstrass invariants
\[
g_2 = \frac{12 \gamma}{\beta^4}, \quad g_3 = \frac{2}{3} \beta^3 z_0^3 - 2 \gamma z_0 - z_0^2.
\]

For \( z \), then, we find
\[
z(t) = \frac{6}{\beta^3} P(t - t_0).
\]

When \( g_2^3 < 27g_3^2 \), then the motion along the \( z \)-axis is, as earlier, semiconfined. But if \( g_2^3 > 27g_3^2 \), then the motion of atoms can become confined depending on initial conditions for \( z \). Recall again, that we are not obliged to align the device axis along the gravitational force. Therefore the semiconfining regime of motion can always be realized.

### Appendix C. Initial Conditions

The evolution equations (1) and (2), we started with, are the equations for the averages of the position and spin operators. The average of an operator \( \hat{Q} \) of an observable quantity is the scalar product \(< \hat{Q} \equiv (\psi, \hat{Q}\psi) \), where \( \psi = \psi(t) \) is the wave function satisfying the time–dependent Schrödinger equation. The evolution equation for such an average reads
\[
\frac{i\hbar}{\partial t} < \hat{Q} >= [\hat{Q}, \hat{H}],
\]
where \( \hat{H} \) is the Hamiltonian (see e.g. [49,50]). The initial state \( \psi_0 = \psi(0) \) of the time–dependent Schrödinger equation is, in general, an arbitrary function. Consequently, the initial value \( \psi(t, t_0) = \psi_0 = \hat{Q}\psi_0 \) is also arbitrary. One tells that the initial state can be prepared [50,51]. The evolution equation for the average \( \hat{S} = \hat{S}\) (\( t = < \hat{S} > \) of the spin operator \( \hat{S} \), is equation (2) describing the spin precession from an arbitrary prepared initial value \( \hat{S}_0 = \hat{S}(0) \), as is discussed, e.g., in Ref. [52]. In this way, according to the basic principles of Quantum Mechanics, the initial state can, in general, be chosen arbitrarily, resulting in an arbitrary value of \( \hat{S}_0 \).

It is a different question how this or that particular state can be realized in experiment. One distinguishes two limiting cases of preparing initial conditions, adiabatic and nonadiabatic [53,54]. In the first case, the time–dependent Schrödinger equation with a Hamiltonian \( \hat{H}(t) \) is ascribed an initial state \( \psi_0 \) that is an eigenstate of \( \hat{H}(0) \), i.e., \( \hat{H}(0)\psi_0 = E(0)\psi_0 \). In the second case, a chosen initial state \( \psi_0 \) is not an eigenstate of \( \hat{H}(0) \). This implies that the state \( \psi_0 \) could be prepared, for \( t \leq 0 \), as an eigenstate of another Hamiltonian, say \( \hat{H}_0 \), and then, at the time \( t = 0 \), some external field is suddenly turned on, instantaneously changing the Hamiltonian from \( \hat{H}_0 \) to \( \hat{H}(0) \), so that \( \hat{H}_0 \neq \hat{H}(0) \). These two types of initial conditions correspond to different physical realizations of an adiabatically slow changing field and of a sudden switch on of an external field [53,54]. Both these cases can, in principle, be realized in experiment. Of course, when one aims at trapping atoms, the adiabatic motion is necessary [55], with adiabatic initial conditions. But this is not mandatory when one’s aim is to obtain a specific nonconfined regime. Then one should try nonadiabatic initial conditions. Our conditions (33) for the initial spin polarization are such nonadiabatic conditions providing the semiconfining regime of motion which can be used for creating well–collimated beams of atom lasers. This kind of initial conditions can be realized in practice in different ways. For example, one can confine atoms in a trap with a vertical bias field [45], where all atoms become spin–polarized along the \( z \)-axis. After this, the bias field in the \( z \)-direction is suddenly switched off and, at the same time, a transverse bias field, as in experiments [43,44], is suddenly switched on. Another possibility could be to prepare spin–polarized atoms in one trap and to suddenly load them into another trap with the desired field configuration. We are not going to plunge into technical details of
preparing such initial conditions for realizing the semiconfined motion. This is neither our aim nor our speciality. To do this is the challenge for experimentalists. We think that this semiconfined motion can be realized, since there are no principal theoretical obstacles for it.

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Figure Captions

**Fig. 1.** The trajectories of atoms at the initial stage of the collimation process, $0 \leq t \leq 5$, for a trap with $L = 100$: (a) isotropic initial conditions, $|\dot{x}_0| \leq 1$, $|\dot{z}_0| \leq 1$; (b) slightly anisotropic initial velocities, $|\dot{x}_0| \leq 0.25$, $|\dot{z}_0| \leq 1$.

**Fig. 2.** Atomic trajectories for the time $0 \leq t \leq 20$ for two different traps: (a) $L = 10$; (b) $L = 100$.

**Fig. 3.** Atomic trajectories for long times, $0 \leq t \leq 100$, for a trap with $L = 100$.

**Fig. 4.** The velocities of atoms in the radial, $v(t)$, and axial, $w(t)$, directions, with initial velocities $|\dot{x}_0| \leq 1$ and $|\dot{z}_0| \leq 1$ for two cases: (a) $0 \leq t \leq 100$, $L = 100$, (b) $0 \leq t \leq 200$, $L = 10000$.

**Fig. 5.** Atomic velocities for the initial conditions $|\dot{x}_0| \leq 0.1$ and $|\dot{z}_0| \leq 0.1$ for the following cases: (a) $0 \leq t \leq 100$, $L = 100$; (b) $0 \leq t \leq 200$, $L = 1000$.

**Fig. 6.** The velocities of atoms for the initial conditions $|\dot{x}_0| \leq 0.1$ and $|\dot{z}_0| \leq 0.1$ for $L = 10000$, at different stages: (a) initial stage, $0 \leq t \leq 10$; (b) long times, $0 \leq t \leq 200$. For $t > 200$, the picture practically does not change.