Fast Hybrid Network Algorithms for Shortest Paths in Sparse Graphs

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Abstract

We consider the problem of computing shortest paths in hybrid networks, in which nodes can make use of different communication modes. For example, mobile phones may use ad-hoc connections via Bluetooth or Wi-Fi in addition to the cellular network to solve tasks more efficiently. Like in this case, the different communication modes may differ considerably in range, bandwidth, and flexibility.

We build upon the model of Augustine et al. [SODA ’20], which captures these differences by a local and a global mode. Specifically, the local edges model a fixed communication network in which $O(1)$ messages of size $O(\log n)$ can be sent over every edge in each synchronous round. The global edges form a clique, but nodes are only allowed to send and receive a total of at most $O(\log n)$ messages over global edges, which restricts the nodes to use these edges only very sparsely.

We demonstrate the power of hybrid networks by presenting algorithms to compute Single-Source Shortest Paths and the diameter very efficiently in sparse graphs. Specifically, we present exact $O(\log n)$ time algorithms for pseudotrees (i.e., graphs that contain at most one cycle), and 3-approximations for graphs that have at most $n + O(n^{1/3})$ edges and arboricity $O(\log n)$. For these graph classes, our algorithms provide exponentially faster solutions than the best known algorithms for general graphs in this model.

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1 Introduction

The idea of hybrid networks is to leverage multiple communication modes with different characteristics to deliver scalable throughput, or to reduce complexity, cost or power consumption. In hybrid datacenter networks [18], for example, the server racks can make use of optical switches [22, 51] or wireless antennas [19, 20, 32, 55] to establish direct connections in addition to using the traditional electronic packet switches. Other examples of hybrid communication are combining multipoint with standard VPN connections [49], hybrid WANs [52], or mobile phones using device-to-device communication in addition to cellular networks as in 5G [38]. As a consequence, several theoretical models and algorithms have been proposed for hybrid networks in recent years [20, 36, 7, 8, 14].

In this paper, we focus on the general hybrid network model of Augustine et al. [8]. The authors distinguish two different modes of communication, a local mode, which nodes can use to send messages to their neighbors in an input graph $G$, and a global mode, which
allows the nodes to communicate with any other node of G. The model is parameterized by the number of messages \( \lambda \) that can be sent over each local edge in each round, and the total number of messages \( \gamma \) that each node can send and receive over global edges in a single round. Therefore, the local network rather relates to physical networks, where an edge corresponds to a dedicated connection that cannot be adapted by the nodes, e.g., a cable, an optical connection, or a wireless ad-hoc connection. On the other hand, the global network captures characteristics of logical networks, which are formed as overlays of a shared physical infrastructure such as the internet or a cellular network. Here, nodes can in principle communicate with any other node, but can only perform a limited amount of communication in each round.

Specifically, we consider the hybrid network model with \( \lambda = O(1) \) and \( \gamma = O(\log n) \), i.e., the local network corresponds to the CONGEST model \[45\], whereas the global network is the so-called node-capacitated clique (NCC) \[7, 3, 48\]. Thereby, we only grant the nodes very limited communication capabilities for both communication modes, disallowing them, for example, to gather complete neighborhood information to support their computation. With the exception of a constant factor SSSP approximation, none of the shortest paths algorithms of \[8\], for example, can be directly applied to this very restricted setting, since \[8\] assumes the LOCAL model for the local network. Furthermore, our algorithms do not even exploit the power of the NCC for the global network; in fact, they would also work if the nodes would initially only knew their neighbors in \( G \) and had to learn new node identifiers via introduction (which has recently been termed the NCC\( _0 \) model \[6\]).

As in \[5\], we focus on shortest paths problems. However, instead of investigating general graphs, we present polylogarithmic time algorithms to compute Single-Source Shortest Paths (SSSP) and the diameter in sparse graphs: for pseudotrees, which are graphs that contain at most one cycle, we give exact deterministic \( O(\log n) \) time algorithms for SSSP and the diameter. Furthermore, we present 3-approximate solutions with runtime \( O(\log^2 n) \), w.h.p.\(^1\) for graphs that contain at most \( n + O(n^{1/3}) \) edges and have arboricity \( O(\log n) \). Graphs with bounded arboricity, which include important graph families such as planar graphs, graphs with bounded treewidth, or graphs that exclude a fixed minor, have been extensively studied in the past years. Our algorithms are exponentially faster than the best known algorithms for general graphs for shortest paths problems \[7, 40\].

For the All-Pairs Shortest Paths (APSP) problem, which is not studied in this paper, there is a lower bound of \( \tilde{\Omega}(\sqrt{n}) \) \[8, Theorem 2.5\] that even holds for \( \tilde{O}(\sqrt{n})\)-approximations.\(^2\) Recently, this lower bound has shown to be tight up to polylogarithmic factors \[40\]. The bound specifically also holds for trees, which, together with the results in this paper, shows an exponential gap between computing the diameter and solving APSP in trees. Furthermore, the recent results of \[40\] show that computing (an approximation of) the diameter in general graphs takes time roughly \( \Omega(n^{1/3}) \) (even with unbounded local communication). Therefore, our paper demonstrates that sparse graphs allow for an exponential improvement.

### 1.1 Model and Problem Definition

We consider a hybrid network model in which we are given a fixed node set \( V \) consisting of \( n \) nodes that are connected via local and global edges. The local edges form a fixed, undirected,\(^3\)

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1. An event holds with high probability (w.h.p.) if it holds with probability at least \( 1 - 1/n^c \) for an arbitrary but fixed constant \( c > 0 \).
2. The arboricity of a graph \( G \) is the minimum number of forests into which its edges can be partitioned.
3. The \( \tilde{O} \)-notation hides polylogarithmic factors.
and weighted graph $G = (V, E, w)$ (the local network), where the edge weights are given by $w : E \to \{1, \ldots, W\} \subset \mathbb{N}$ and $W$ is assumed to be polynomial in $n$. We denote the degree of a node $v$ in the local network by $\deg(v)$. Furthermore, every two nodes $u, v \in V$ are connected via a global edge, i.e., the global network forms a clique. Every node $v \in V$ has a unique identifier $id(v)$ of size $O(\log n)$, and, since the nodes form a clique in the global network, every node knows the identifier of every other node. Although this seems to be a fairly strong assumption, our algorithms would also work in the NCC$_0$ model [5] for the global network, in which each node initially only knows the identifiers of its neighbors in $G$, and new connections need to be established by sending node identifiers (which is very similar to the overlay network models of [20, 31, 34]).

We assume a synchronous message passing model, where in each round every node can send messages of size $O(\log n)$ over both local and global edges. Messages that are sent in round $i$ are collectively received at the beginning of round $i + 1$. However, we impose different communication restrictions on the two network types. Specifically, every node can send $O(1)$ (distinct) messages over each of its incident local edges, which corresponds to the CONGEST model for the local network [15]. Additionally, it can send and receive at most $O(\log n)$ many messages over global edges (where, if more than $O(\log n)$ messages are sent to a node, an arbitrary subset of the messages is delivered), which corresponds to the NCC model [7]. Therefore, our hybrid network model is precisely the model proposed in [5] for parameters $\lambda = O(1)$ and $\gamma = O(\log n)$. Note that whereas [5] focuses on the much more promiscuous LOCAL model for the local network, our algorithms do not require nor easily benefit from the power of unbounded communication over local edges.

We define the length of a path $P \subseteq E$ as $w(P) := \sum_{e \in P} w(e)$. A path $P$ from $u$ to $v$ is a shortest path, if there is no path $P'$ from $u$ and $v$ with $w(P') < w(P)$. The distance between two nodes $u$ and $v$ is defined as $d(u, v) := w(P)$, where $P$ is a shortest path from $u$ to $v$.

In the Single-Source Shortest Paths Problem (SSSP), there is one node $s \in V$ and every node $v \in V$ wants to compute $d(s, v)$. In the Diameter Problem, every node wants to learn the diameter $D := \max_{u,v \in V} d(u, v)$. An algorithm computes an $\alpha$-approximation of SSSP, if every node $v \in V$ learns an estimate $\hat{d}(s, v)$ such that $d(s, v) \leq \hat{d}(s, v) \leq \alpha \cdot d(s, v)$. Similarly, for an $\alpha$-approximation of the diameter, every node $v \in V$ has to compute an estimate $\hat{D}$ such that $D \leq \hat{D} \leq \alpha \cdot D$.

1.2 Contribution and Structure of the Paper

The first part of the paper revolves around computing SSSP and the diameter on pseudotrees (i.e., connected graphs that contain at most one cycle). For a more comprehensive presentation, we establish the algorithm in several steps. First, we consider the problems in path graphs (i.e., connected graphs that contain exactly two nodes with degree 1, and every other node has degree 2; see Section 2), then in cycle graphs (i.e., connected graphs in which each node has degree 2, see Section 3), trees (Section 4), and finally examine pseudotrees (Section 5). For every graph, we present deterministic algorithms to solve both problems in $O(\log n)$ rounds, each relying heavily on the results of the previous sections.

In Section 6 we consider a more general class of sparse graphs, namely graphs with at most $n + O(n^{1/3})$ edges and arboricity $O(\log n)$. By using the techniques established in the first part and leveraging the power of the global network to deal with the additional $O(n^{1/3})$ edges, we obtain algorithms to compute $3$-approximations for SSSP and the diameter in time $O(\log^2 n)$, w.h.p. As a byproduct, we also derive a deterministic $O(\log^2 n)$-round algorithm for computing a (balanced) hierarchical tree decomposition of the network.

We remark that our algorithms heavily use techniques from the PRAM literature. For
example, pointer jumping \cite{51}, and the Euler tour technique (e.g., \cite{51 5}), which extends pointer jumping to certain graphs such as trees, have been known for decades, and are also used in distributed algorithms (e.g., \cite{20 9}). As already pointed out in \cite{7}, the NCC in particular has a very close connection to PRAMs. In fact, if $G$ is very sparse, PRAM algorithms can efficiently be simulated in our model even if the edges are very unevenly distributed (i.e., nodes have a very high degree). We formally prove this in Appendix A. This allows us to replace some of our algorithms for path graphs, cycle graphs, and trees by PRAM algorithms (see Section 1.3). The main contribution of this paper thus lies in the algorithms for pseudotrees and sparse graphs. We nonetheless present our distributed solutions without using PRAM simulations, since (1) a direct simulation as in Appendix A only yields randomized algorithms, (2) the algorithms of the later sections heavily build on the basic algorithms of the first sections, (3) a simulation exploits the capabilities of the global network more than necessary. As already pointed out, all of our algorithms would also work in the weaker NCC$_0$ model for the global network, or if the nodes could only contact $\Theta(\log n)$ random nodes in each round. Furthermore, if we restrict the degree of $G$ to be $O(\log n)$, our algorithms can be modified to completely run in the NCC$_0$, i.e., without needing the local network.

We believe that some of the algorithms and techniques presented in this paper may have applications beyond sparse graphs. For example, in combination with sparse spanner constructions (see, e.g., \cite{11}) or skeletons (e.g., \cite{53}), our algorithms may lead to efficient approximations in more general graph classes. Furthermore, the load balancing technique to compute the diameter of a cycle may have implications for routing problems in overlay networks. Finally, our distributed algorithm to construct a hierarchical tree decomposition may be of independent interest, as such constructions are used for example in routing algorithms for wireless networks (see, e.g., \cite{25 37}). It is important to note that in \cite{25 37} the tree decomposition is computed on a minimum spanning tree (MST) of a Unit-Disk graph, which has degree at most 6. Since we consider a more general setting with arbitrary node degrees, the tree decomposition is harder to compute efficiently without violating the communication bounds of the global network. To do so, we make use of a combination of local and global edges.

### 1.3 Related Work

As theoretical models for hybrid networks have only been proposed recently, only few results for such models are known at this point \cite{30 7 8}. Computing an exact solution for SSSP in arbitrary graphs can be done in $\tilde{O}(\sqrt{SPD})$ rounds \cite{8}, where SPD denotes the shortest path diameter. For large SPD, this bound has recently been improved to $\tilde{O}(n^{2/5})$ \cite{10}. The authors of \cite{8} also present several approximation algorithms for SSSP: A $(1+\varepsilon)$-approximation with runtime $\tilde{O}(n^{1/3}/\varepsilon^6)$, a $(1/\varepsilon)O(1/\varepsilon)$-approximation running in $\tilde{O}(n^2)$ rounds and a $2^{O(\sqrt{\log n \log \log n})}$-approximation with runtime $2^{O(\sqrt{\log n \log \log n})}$. For APSP there is an exact algorithm that runs in $\tilde{O}(n^{2/3})$ rounds, a $(1+\varepsilon)$-approximation running in $\tilde{O}(\sqrt{n}/\varepsilon)$ rounds (only for unweighted graphs) and a 3-approximation with runtime $\tilde{O}(\sqrt{n})$ \cite{8}. In \cite{10}, the authors give a lower bound of $\tilde{O}(n^{1/3})$ rounds for computing the diameter in arbitrary graphs in our model. They also give approximation algorithms with approximation factors $(3/2 + \varepsilon)$ and $(1 + \varepsilon)$ that run in time $\tilde{O}(n^{1/3}/\varepsilon)$ and $\tilde{O}(n^{0.397}/\varepsilon)$.

\footnote{We remark that for the algorithms in Section 6 this requires to setup a suitable overlay network like a butterfly in time $O(\log^2 n)$, which can be done using well-known techniques.}
respectively. Even though APSP and the diameter problem are closely related, we show in this paper that the diameter can be computed much faster in our hybrid network model for certain graph classes.

As already pointed out, the global network in our model has a close connection to overlay networks. The NCC model, which has been introduced in [7], mainly focuses on the impact of node capacities, especially when the nodes have a high degree. Since, intuitively, for many graph problems the existence of each edge is relevant for the output, most algorithms in [7] depend on the arboricity $a$ of $G$ (which is, roughly speaking, the time needed to efficiently distribute the load of all edges over the network). The authors present $O(a)$ algorithms for local problems such as MIS, matching, or coloring, a $O(D + a)$ algorithm for BFS tree, and a $O(1)$ algorithm for MST. Recently, $O(\Delta)$-time algorithms for graph realization problems have been presented [6], where $\Delta$ is the maximum node degree; notably, most of the algorithms work in the NCC0 variant. Furthermore, Robinson [45] investigates the information the nodes need to learn to jointly solve graph problems and derives a lower bound for constructing spanners in the NCC. For example, his result implies that spanners with constant stretch require polynomial time in the NCC, and are therefore harder to compute than MSTs.

Our work also relates to the literature concerned with overlay construction [4, 3, 30, 9, 31], where the goal is to transform a low-degree weakly-connected graph into a low-depth overlay network such as a binary tree using node introductions. For example, [31] implies an $O((\log^3 n)^{1/5})$ time overlay construction algorithm for the NCC0 model, if the initial degree is $O(\log n)$. If, additionally, the initial graph is a pseudotree, the algorithm of [4] can be adapted to solve the problem in the NCC0 model in time $O(\log n)$, w.h.p.; our algorithms directly yield a deterministic $O(\log n)$ time alternative for such graphs.

A problem closely related to SSSP is the computation of short routing paths between any given nodes. The problem has, for example, been studied in mobile ad-hoc networks [30], in which constant-competitive routing paths can be computed in $O(\log^2 n)$ rounds [13]. The authors consider a hybrid network model similar to [8], where nodes can communicate using either their WiFi-interface (similar to the local edges) or the cellular infrastructure (similar to global edges).

In the classical CONGEST model there is a lower bound of $\tilde{\Omega}(\sqrt{n} + D)$ rounds to approximate SSSP with a constant factor [30]. This bound is tight, as there is a $(1 + \varepsilon)$-approximation algorithm by Becker et al. that runs in $\tilde{O}(\sqrt{n} + D)$ rounds [12]. The best known algorithms for computing exact SSSP in the CONGEST model are the ones by Ghaeffari and Li [28] and by Forster and Nanongkai [23] which have runtimes of $\tilde{O}(\sqrt{n} \cdot D)$ and $\tilde{O}(\sqrt{n}D^{1/4} + n^{3/5} + D)$, respectively. Computing the diameter can be done in $O(n)$ rounds in the CONGEST model [46], which is also tight to the lower bound [24]. This lower bound even holds for very sparse graphs [1]. In addition to that, the obvious lower bound of $\Omega(D)$ for shortest paths problems also always holds if the graph is sparse. Therefore, algorithms for sparse graphs have been proposed mainly for local problems such as vertex coloring, maximal matching or maximal independent set. There exists an abundance of literature that studies such problems, for example, in graphs with bounded arboricity [29, 10, 39], planar graphs [2, 17, 20, 27] or degree-bounded graphs [43].

Somewhat related to the NCC model, although much more powerful, is the congested clique model, which has received quite some attention in recent years. A $(1 + \varepsilon)$-approximation for SSSP can be computed in $O(\text{polylog}(n))$ rounds in this model [15]. In [16], techniques for faster matrix multiplication in the congested clique model are presented, resulting in a $O(n^{1−2/\omega})$-round algorithm, where $\omega < 2.3728639$ is the exponent of matrix multiplication. Our algorithm for sparse graphs also uses matrix multiplication in order to compute APSP
between $O(n^{1/3})$ nodes in the network in $O(\log^2 n)$ rounds. In general, the results in the congested clique model are of no help in our setting because due to the restriction that a node can only send or receive $O(\log n)$ messages per round via global edges, we cannot effectively emulate congested clique algorithms in the NCC model.

As argued before, by using Lemma 30 in Appendix A for PRAM simulations, we could apply some of the algorithms for PRAMs to our model instead of using native distributed solutions. For example, we are able to use the algorithms of [21] to solve SSSP and diameter in trees in time $O(\log n)$, w.h.p. Furthermore, we can compute the distance between any pair $s$ and $t$ in outerplanar graphs in time $O(\log^3 n)$ by simulating a CREW PRAM. For planar graphs, the distance between $s$ and $t$ can be computed in time $O(\log^3 n (1 + M(q))/n)$, w.h.p., where the nodes know a set of $q$ faces of a planar embedding that covers all vertices, and $M(q)$ is the number of processors required to multiply two $q \times q$ matrices in $O(\log q)$ time in the CREW PRAM.

For graphs with polylogarithmic arboricity, a $(1 + \varepsilon)$-approximation of SSSP can be computed in polylog time using [11] and our simulation framework (with huge polylogarithmic terms). For general graphs, the algorithm can be combined with well-known spanner algorithms for the CONGEST model (e.g., [11]) to achieve constant approximations for SSSP in time $O(n^2)$ time in our hybrid model. This yields an alternative to the constant factor SSSP approximation in [5], which also requires time $O(n^2)$ but has much smaller polylogarithmic factors.

Finally, there also exist randomized $O(\log n)$-time algorithms [11, 33] to compute (minimum) spanning forests in the PRAM model, which could be used instead of the deterministic $O(\log^2 n)$-time algorithm of Gmér et al. [30] in the beginning of Section 6. Again, note that a simulation of PRAM algorithms exploits the capabilities of the NCC and would not work in the weaker NCC$_0$ model for the global network.

## 2 Path Graphs

To begin with an easy example, we first present a simple algorithm to compute SSSP and the diameter of path graphs. The simple idea of our algorithms is to use pointer jumping to select a subset of global edges $S$, which we call shortcut edges, with the following properties: $S$ is a weighted connected graph with degree $O(\log n)$ that contains all nodes of $V$, and for every $u, v \in V$ there exists a path $P \subseteq S$, $|P| = O(\log n)$ (where $|P|$ denotes the number of edges of $P$), such that $w(P) = d(u, v)$, and no path $P$ such that $w(P) < d(u, v)$. Given such a graph, SSSP can easily be solved by performing a broadcast from $s$ in $S$ for $O(\log n)$ rounds: In the first round, $s$ sends a message containing $w(e)$ over each edge $e \in S$ incident to $s$. In every subsequent round, every node $v \in V$ that has already received a message sends a message $k + w(e)$ over each edge $e \in S$ incident to $v$, where $k$ is the smallest value $v$ has received so far. After $O(\log n)$ rounds, every node $v$ must have received $d(s, v)$, and cannot have received any smaller value. Further, the diameter of the line can easily be determined by performing SSSP from both of its endpoints $u, v$, which finally broadcast the diameter $d(u, v)$ to all nodes using the global network.

We construct $S$ using the following simple Introduction Algorithm. $S$ initially contains all edges of $E$. Additional shortcut edges are established by performing pointer jumping: Every node $v$ first selects one of its at most two neighbors as its left neighbor $\ell_1$; if it has two neighbors, the other is selected as $v$’s right neighbor $r_1$. In the first round of our algorithm, every node $v$ with degree 2 establishes $\{\ell_1, r_1\}$ as a new shortcut edge of weight $w(\ell_1, r_1) = w(\ell_1, v) + w(v, r_1)$ by sending the edge to both $\ell_1$ and $r_1$. Whenever a node

\[ \]
v with degree 2 receives shortcut edges \{u, v\} and \{v, w\} from \(\ell_{i-1}\) and \(r_{i-1}\), respectively, at the beginning of some round \(i > 1\), it sets \(\ell_i := u\), \(r_i := w\), and establishes \(\{\ell_i, r_i\}\) by adding up the weights of the two received edges and informing \(\ell_i\) and \(r_i\). The algorithm terminates after \([\log n]\) rounds. Afterwards, for every simple path in \(G\) between \(u\) and \(v\) with \(2^k\) hops for any \(k \leq [\log n] - 1\) we have established a shortcut edge \(e \in S\) with \(w(e) = d(u, v)\). Therefore, \(S\) has the desired properties, and we conclude the following theorem.

\[\textbf{Theorem 1.} \quad \text{SSSP and the diameter can be computed in a path graph in time } O(\log n).\]

3 Cycle Graphs

In cycle graphs, there are two shortest paths between any two nodes that we need to distinguish. For SSSP, this can easily be achieved by performing the SSSP algorithm for path graphs in both directions along the cycle, and let each node choose the minimum of its two computed distances. Formally, let \(v_1, v_2, \ldots, v_n\) denote the \(n\) nodes along a left traversal of the cycle starting from \(s = v_1\) and continuing at \(s\)’s neighbor of smaller identifier, i.e., \(v_2 < v_n\). For any node \(u\), a shortest path from \(s\) to \(u\) must follow a left or right traversal along the cycle, i.e., \((v_1, v_2, \ldots, u)\) or \((v_1, v_n, \ldots, u)\) is a shortest path from \(s\) to \(u\). Therefore, we can solve SSSP on the cycle by performing the SSSP algorithm for the path graph on \(L := (v_1, v_2, \ldots, v_n)\) and \(R := (v_1, v_n, v_{n-1}, \ldots, v_2)\). Thereby, every node \(v\) learns \(d_L(s, v)\), which is the distance from \(s\) to \(v\) in \(L\) (i.e., along a left traversal of the cycle), and \(d_R(s, v)\), which is their distance in \(R\). It is easy to see that \(d(s, v) = \min\{d_L(s, v), d_R(s, v)\}\), as well as its at most two farthest nodes \(u, v\) such that \(d(s, u) = d(s, v) = \text{ecc}(s)\). More precisely, if there is a node \(v \neq s\) such that \(d_L(s, v) = d_R(s, v)\), or if \(s\) is incident to an edge \(\{s, v\}\) such that \(w(\{s, v\}) > W/2\), where \(W = \sum_{e \in E} w(e)\), then \(v\) is the (single) farthest node from \(s\) (note that \(W\) can easily be computed as a byproduct of SSSP). If there is no such node, then the two consecutive nodes \(v_i, v_{i+1} \in L\) such that \(d_L(s, v_i) < d_R(s, v_i)\), but \(d_L(s, v_{i+1}) > d_R(s, v_{i+1})\), are possible candidates for \(v\)’s farthest nodes (we say that \(v_i\) and \(v_{i+1}\) are the left and right farthest nodes of \(s\), respectively). In this case, \(\text{ecc}(s) = \max\{d_L(s, v_i), d_R(s, v_{i+1})\}\).

To determine the diameter of \(G\), for every node \(v \in V\) our goal is to compute \(\text{ecc}(v)\); as a byproduct, we will compute \(v\)’s left and right farthest nodes \(v_\ell\) and \(v_r\). If \(v\) only has a single farthest node \(u\), then we denote \(v_\ell = v_r = u\). The diameter can then be computed as \(\max_{v \in V} \text{ecc}(v)\). A simple way to compute these values is to employ a binary-search style approach from all nodes in parallel, and use load balancing techniques from [7] to achieve a runtime of \(O(\log^2 n)\), w.h.p. To come up with a deterministic \(O(\log n)\) time algorithm, however, needs a bit more thought.

Our algorithm works as follows. Let \(s\) be the node with smallest identifier. First, we perform the SSSP algorithm as described above from \(s\) in \(L\) and \(R\), whereby \(s\) learns \(s_\ell\) and \(s_r\) as defined above. Note that \(d_L(s, s_\ell) \leq [W/2]\) and \(d_R(s, s_r) \leq [W/2]\). Let \(L\) be graph that results from removing the edge \(\{s_\ell, s_r\}\) from \(G\) (see Figure 1) (if \(s_\ell = s_r\), then \(L\) ends and begins at \(s_\ell = s_r\) and contains all edges along a right traversal of the graph starting at \(s_\ell\)). Let \(A \subseteq V\) be the set of nodes between \(s_\ell\) and \(s\) (excluding \(s\)), and \(B \subseteq V\) be the set of nodes between \(s\) and \(s_r\) (including \(s\)). In our algorithm, each node \(v \in S\) learns its farthest nodes, which must be nodes of \(C\).

To do so, we assign each node \(v\) a budget \(\phi(v)\), which is \(\left\lceil W/2 \right\rceil - d_L(s, v)\geq 0\), if \(v \in A\), and \(d_R(s, v)\), if \(v \in B\). Roughly speaking, the budget of a node \(v \in A\) determines how far you can move from \(v\) beyond \(s\) along a right traversal of \(G\) until reaching \(v\)’s farthest node \(v_r\). Then, we sort the nodes of \(L\) by their budget. Note that since we consider non-negative edge
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Figure 1: An example of diameter computation in a cycle $G$. The algorithm begins with $s = v_1$. In $L$, $s_r = v_4$ is the farthest node from $s$ along a left traversal of $G$, and $s_r = v_5$ is the farthest node along a right traversal. The white nodes are the nodes of $A$, and the black nodes are $B$. Each node is annotated with its budget. In $I$, the nodes are sorted by their budget, and learn their nearest black nodes. For example, for $v_3$, $x = v_1$ and $y = v_7$.

weights, no two nodes of $A$ and no two nodes of $B$ may have the same budget, but there may be nodes $u \in A$, $v \in B$ with $\phi(u) = \phi(v)$. In this case, we break ties by assuming that $\phi(u) > \phi(v)$. More specifically, the outcome is a sorted list $I$ with first node $s$ that contains all nodes of $L$ and $R$ in the same order they appear in $L$ and $R$, respectively. Such a list can be constructed in time $O(\log n)$, e.g., by using Aspnes and Wu’s algorithm \cite{4}.

Let $v \in A$, and let $x$ be the node of $B$ that is closest to $v$ in $I$ in the direction of $s$ (i.e., in left direction). Since by our sorting $\phi(x)$ is maximal such that $\phi(x) \leq \phi(v)$, we have $x = \arg\max_{u \in B} d_l(s,v) + d_r(s,x) \leq \lfloor W/2 \rfloor$. Furthermore, let $y$ be the nearest node of $B$ in $I$ in the other direction; if that node does not exist (which is the case if $x = s_r$), then $y$ is defined as the successor of $x$ in $R$ (or $s$, if $x$ is the last node of $R$). By performing the Introduction Algorithm on each connected segments of nodes of $A$ in $I$, $v$ learns $x$ and $y$ as well as $d_l(s,x)$ and $d_l(s,y)$.

Lemma 2. We have $x = v_r$. If $d_l(s,v) + d_r(s,x) = W/2$, then $x = v_r$; otherwise, $y = v_l$.

Proof. By the definition of our algorithm, $x$ is the farthest node from $v$ along a right traversal of the cycle such that $d_r(v,x) = d_l(s,v) + d_r(s,x) \leq \lfloor W/2 \rfloor$. Therefore, a shortest path from $v$ to $x$ goes along a right traversal of the cycle. Furthermore, $d_r(v,y) > \lfloor W/2 \rfloor$, but $d_l(v,y) \leq \lfloor W/2 \rfloor$ (i.e., a shortest path from $v$ to $y$ goes along a left traversal). Consequently, $x = v_r$.

If $d_r(v,x) = W/2$, then $d_r(v,x) = d_l(v,x)$, and thus $x = v_r = v_l$. Otherwise, $d_r(v,x) < W/2$, and we must have that $d_l(v,y) > W/2$. Since the path from $v$ to $x$ along a left traversal is not a shortest path, but the path from $v$ to $y$ along a left traversal is a shortest path, the claim follows.

Afterwards, all nodes in $A$ know their farthest nodes as well as their eccentricity. To cover the remaining nodes, we restart the algorithm at node $s_r$ (instead of $s$), whereby, since $d_r(s,s_r) = d_l(s_r,s) \leq \lfloor W/2 \rfloor$, all nodes between $s$ and $s_r$ in $L$ will be covered. Finally, $s_r$ learns its farthest nodes by performing SSSP. We conclude the following theorem.

Theorem 3. SSSP and the diameter can be computed in a cycle graph $G$ in time $O(\log n)$.

Note that the algorithm of \cite{4} is actually randomized. However, since we can easily arrange the nodes as a binary tree, we can replace the randomized pairing procedure of \cite{4} by a deterministic strategy, and, together with the pipelining approach of \cite{4}, also achieve a runtime of $O(\log n)$.
Figure 2 (a) A tree with source node $s$ (black). Each node is labeled with its identifier and each edge is labeled with its weight. (b) The resulting path graph $L$ of virtual nodes. Each node $v_i$ is labeled with its index $i$. (c) A possible orientation with outdegree 3. According to our redistribution rule, for example, all virtual nodes of the central node 2 would be assigned to its neighbors. (d) The edges are assigned weights, and each virtual node is labeled with its distance to $s_L$ (black node).

4 Trees

We now show how the algorithms of the previous sections can be extended to compute SSSP and the diameter on trees. As in the algorithm of Gmyr et al. [30], we adapt the well-known Euler tour technique to a distributed setting and transform the graph into a path $L$ of virtual nodes that corresponds to a depth-first traversal of $G$. More specifically, every node of $G$ simulates one virtual node for each time it is visited in that traversal, and two virtual nodes are neighbors in $L$ if they correspond to subsequent visitations. To solve SSSP, we assign weights to the edges from which the initial distances in $G$ can be inferred, and then solve SSSP in $L$ instead. Finally, we compute the diameter of $G$ by performing the SSSP algorithm twice, which concludes this section.

However, since a node can be visited up to $\Omega(n)$ times in the traversal, it may not be able to simulate all of its virtual nodes in $L$. Therefore, we first need to reassign the virtual nodes to the node’s neighbors such that every node only has to simulate at most 6 virtual nodes using the Nash-Williams forests decomposition technique [42]. More precisely, we compute an orientation of the edges in which each node has outdegree at most 3, and reassign nodes according to this orientation.

Construction and Simulation of $L$. We denote the neighbors of a node $v \in V$ by ascending identifier as $v(0), \ldots, v(\deg(v) - 1)$. Consider the depth-first traversal in $G$ that starts and ends at $s$, and which, whenever it reaches $v$ from some neighbor $v(i)$, continues at $v$’s neighbor $v((i + 1) \mod \deg(v))$. $L$ is the directed path graph of virtual nodes that corresponds to this traversal (see Figure 2a and 2b). The path graph contains a virtual node for each time a node is visited, and a directed edge from each virtual node to its successor in the traversal; however, we leave out the last edge ending at $s$ to break the cycle. More specifically, every node $v$ simulates the nodes $v_0, \ldots, v_{\deg(v) - 1}$, where $v_i$ corresponds to the traversal visiting $v$ from $v(i)$. The first node of $L$ is $s_L := s_{\deg(s) - 1}$, and its last node is the node $v_i$ such that $v = s(\deg(s) - 1)$, and $v((i + 1) \mod \deg(v)) = s$. For every node $v_i$ in $L$ (except the last node of $L$), there is an edge $(v_i, u_j) \in L$ such that $u = v((i + 1) \mod \deg(v))$ and $v = u(j)$. To accordingly introduce each virtual node to its predecessor in $L$, every node $v$ sends $\text{id}(v_i) := \text{id}(v) \circ i$ to $v(i)$ for all $i \in [\deg(v)]$, where $\circ$ denotes the concatenation of two binary strings, and $[k] = \{0, \ldots, k - 1\}$.

It remains to show how the virtual nodes can be redistributed such that each node only has to simulate at most 6 virtual nodes. To do so, we first compute an orientation of $G$, i.e., an assignment of directions to its edges, such that every node has outdegree 3.
Since the arboricity of $G$ is 1, we can use\cite{10} Theorem 3.5 to compute an $H$-partition $H_1, \ldots, H_\ell$ of $G$ with degree 3. The algorithm is based on the Nash-Williams forests decomposition technique\cite{12}: In phase $i \in \{1, \ldots, \ell = O(\log n)\}$, all nodes that have degree at most $(2 + \varepsilon) \cdot a$, where $a$ is the arboricity of $G$, are removed from the graph and join the set $H_i$. We obtain our desired orientation by directing each edge $\{u, v\} \in E$, $u \in H_i$, $v \in H_j$, from $u$ to $v$ if $i < j$, or $i = j$ and $\text{id}(u) < \text{id}(v)$ (see Figure 2\textsuperscript{c} for an example).

Now consider some node $v \in V$ and a virtual node $v_i$ at $v$, and let $u := v(i)$. If $\{v, u\}$ is directed from $v$ to $u$, then $v_i$ is assigned to $v$, and, as before, $v$ takes care of simulating $v_i$. Otherwise, $v_i$ gets assigned to $u$ instead, and $v$ sends the identifier of $v_i$ to $u$. Afterwards, $u$ needs to inform the node $w$ that is responsible for simulating the predecessor of $v_i$ in $L$ that the location of $v_i$ has changed; as $w$ must be either $u$ itself, or a neighbor of $u$, this can be done in a single round. Since in the orientation each node $v$ has at most 3 outgoing edges, for each of which it keeps one virtual node and is assigned one additional virtual node from a neighbor, $v$ has to simulate at most 6 virtual nodes of $L$.

As a byproduct, we obtain that if $G$ is any forest, we can establish separate low-diameter overlays on each of its trees by combining the techniques introduced in this section and the pointer jumping approach of Section 2. For instance, this allows us to efficiently compute aggregates of values stored at each tree’s nodes, as stated in the following remark.

**Remark 4.** Let $H = (V, E)$ be a forest in which every node $v \in V$ stores some value $p_v$, and let $f$ be a distributive aggregate function\footnote{An aggregate function $f$ is called distributive if there is an aggregate function $g$ such that for any multiset $S$ and any partition $S_1, \ldots, S_\ell$ of $S$, $f(S) = g(f(S_1), \ldots, f(S_\ell))$. Classical examples are MAX, MIN, and SUM.}. Every node $v \in V$ can learn $f\{p_u \mid u \in C_v\}$, where $C_v$ is the tree of $H$ that contains $v$, in time $O(\log n)$.

**Assigning Weights.** To assign appropriate weights to the edges of $L$ from which we can infer the node’s distances in $G$, we first have to transform $G$ into a rooted tree. To do so, we simply perform SSSP from $s_L$ (the first node in $L$) in the (unweighted) version of $L$. Thereby, every virtual node $x$ learns its traversal distance, i.e., how many steps the depth-first traversal takes until it reaches $x$. Further, every node $v$ can easily compute which of its virtual nodes $v_i$ is visited first by taking the minimum traversal distance of its virtual nodes\footnote{Since the virtual nodes of $v$ might actually be assigned to neighbors of $v$, their traversal distances first have to be sent to $v$ using the local network.}. Let $v_i$ be the virtual node of $v$ that has smallest traversal distance, and let $u_j$ be the predecessor of $v_i$ in $L$. It is easy to see that $u$ is the parent of $v$ in the rooted tree, which implies the following lemma.

**Lemma 5.** Any tree $G$ can be rooted in $O(\log n)$ time.

For each virtual node $v_j$ of $v$ (except the first node $s_L$), to the edge $(u_i, v_j) \in L$, $v$ assigns the weight

$$w(u_i, v_j) = \begin{cases} w(\{u, v\}) & \text{if } u \text{ is } v\text{'s parent} \\ -w(\{u, v\}) & \text{if } v \text{ is } u\text{'s parent.} \end{cases}$$

If $v_j$ is assigned to a neighbor of $v$, it informs that neighbor about the weight (see Figure 2\textsuperscript{d}).

**Lemma 6.** Let $v \in V$, and let $d_L(s_L, v_i)$ denote the distance from $s_L$ to a virtual node $v_i$ at $v$ in the (weighted) graph $L$. We have that $d_L(s_L, v_i) = d(s, v)$.
Proof. Consider the path \( P \) from \( s \) to \( v \) in \( G \). The depth-first traversal from \( s \) to \( v \) traverses every edge of \( P \) from parent to child, i.e., for every edge in \( P \) there is a directed edge with the same weight between \( s \) and \( v \) in \( L \). However, at some of the nodes of \( P \) (including \( s \) and \( v \)) the traversal may take detours into other subtrees before traversing the next edge of \( P \). As every edge of \( L \) that corresponds to an edge in the subtree is visited, and the weights of all those edges sum up to 0, the distance from \( s \) to \( v \) equals the sum of all edges in \( P \), which is \( d(s,v) \).

By the above lemma, we can compute SSSP in \( G \) by solving SSSP in \( L \) using Theorem\(^8\) which leads to the following theorem.

**Theorem 7.** SSSP can be computed in a tree in time \( O(\log n) \).

We further remark that similar techniques can also be used to compute aggregates of the subtrees of any node of a tree, which is a fact we use in Section\(^6\).

**Remark 8.** Let \( H = (V,E) \) be a forest and assume that each node \( v \in V \) stores some value \( p_v \). The goal of each node \( v \) is to compute the value \( \sum_{u \in C_v} p_u \) for each of its neighbors \( u \), where \( C_v \) is the connected component \( C \) of the subgraph \( H' \) of \( H \) induced by \( V \setminus \{v\} \) that contains \( u \). The problem can be solved in time \( O(\log n) \).

Proof. Let \( s \in V \) be the node that has smallest identifier in \( V \), which can easily be computed using Remark\(^4\). We construct \( L \) exactly as described in the algorithm for computing SSSP with source \( s \) on trees, but choose the weights of the edges differently. More precisely, to every edge \( (u,v) \) of \( L \) we assign the weight \( w((u,v)) := p_u \), if \( v \) is \( u \)'s parent, and 0, otherwise. Further, we assign a value \( d(s_L) := p_s \) to \( s_L \) (the first node of \( L \)). With these values as edge weights, we perform the SSSP algorithm on \( L \) from \( s_L \), whereby every virtual node \( v_i \) learns the value \( d(v_i) := d(s_L) + d_L(s_L,v_i) \). The sum of all values \( M := \sum_{v \in V} p_v \) can be computed and broadcast to every node of \( H \) in time \( O(\log n) \) using Remark\(^4\).

The problem can now be solved as follows. Consider some node \( v \), let \( u \) be a neighbor of \( v \), and let \( i \) be the value such that \( u = v(i) \) (recall that \( v(i) \) is the neighbor of \( v \) that has the \( i \)-th highest identifier, \( 0 \leq i \leq \deg(v) - 1 \)). If \( u \) is the parent of \( v \) in the tree rooted at \( s \), then \( \sum_{w \in C_u} p_w = M - (d(v - 1 \mod \deg v) - d(v_i)) \). If otherwise \( u \) is a child of \( v \) in the tree rooted at \( s \) (unless \( v = s \) and \( i = \deg(s) - 1 \), which is a special case), then \( \sum_{w \in C_u} p_w = d(v_i) - d(v - 1 \mod \deg v) \). Finally, if \( v = s \) and \( i = \deg(s) - 1 \), we have \( \sum_{w \in C_u} p_w = M - d(v - 1 \mod \deg v) \).

**Lemma 9.** Let \( G \) be a tree, \( s,v \in V \) such that \( d(s,v) \) is maximal. Then ecc(\( v \)) = \( D \).

Proof. Assume to the contrary that there is a node \( w \in V \) such that ecc(\( u \)) \( > \) ecc(\( v \)). Then there must be a node \( w \in V \) such that \( d(u,w) = ecc(u) > ecc(v) \geq d(v,w) \). Note that \( d(u,w) > d(u,v) \), as otherwise ecc(\( u \)) \( \leq \) ecc(\( v \)), which would contradict our assumption. Let \( P_1 \) be the path from \( s \) to \( v \), \( P_2 \) be the path from \( u \) to \( w \), and let \( t \) be the node that is closest to \( s \), i.e., \( t = \arg \min_{x \in P_2} d(s,x) \).

If \( t \notin P_1 \), then let \( x \) be the node farthest from \( s \) that lies on \( P_1 \), and also on the path from \( s \) to \( t \) (\( x \) might be \( s \)). Then \( d(u,w) \leq d(u,x) + d(x,w) \leq d(v,x) + d(x,w) = d(v,w) \), which contradicts \( d(u,w) > d(v,w) \). Otherwise, \( t \) must lie on a path from \( v \) to \( u \) or on a path from \( v \) to \( w \). In the first case, \( d(u,w) = d(u,t) + d(t,w) \leq d(u,t) + d(t,v) = d(u,v) \), which

---

\(^8\) Note that for the algorithm to work in the directed path graph \( L \), shortcuts must be established in the bidirected version of \( L \), whereas the subsequent broadcast from \( s_L \) uses only the directed edges of \( L \).
implies \( \text{ecc}(v) \geq \text{ecc}(u) \); the second case analogously implies \( \text{ecc}(v) \geq \text{ecc}(w) \). Therefore, both cases lead to a contradiction with the assumption that \( \text{ecc}(v) < \text{ecc}(u) = \text{ecc}(w) \). ▶

By the well-known lemma above, for the diameter it suffices to perform SSSP once from the node \( s \) that has smallest identifier, then choose a node \( v \) with maximum distance to \( s \), and perform SSSP from \( v \). Since \( \text{ecc}(v) = D \), the node with maximum distance to \( v \) yields the diameter. Together with Remark 4, we conclude the following theorem.

▶ Theorem 10. The diameter can be computed in a tree in time \( O(\log n) \).

5 Pseudotrees

Recall that a pseudotree is a graph that contains at most one cycle. We define a cycle node to be a node that is part of a cycle, and all other nodes as tree nodes. For each cycle node \( v \), we define its tree \( T_v \) to be the connected component that contains \( v \) in the graph in which \( v \)'s two adjacent cycle nodes are removed. Before we show how SSSP and the diameter can be computed, we describe how the cycle can be identified, if it exists.

For this, we use the same approach as for the construction of the path \( L \) in the tree. We let each node \( v \) simulate \( \deg(v) \) virtual nodes \( v_0, \ldots, v_{\deg(v)-1} \) and connect the virtual nodes according to the same rules as described in Section 4, with the exception that we do not leave out the last edge ending at \( s \). If there is no cycle, then this yields a single cycle of virtual nodes, in which case we can use our previous algorithms. Otherwise, this will create two cycles of virtual nodes with the property that every cycle node must have at least one of its virtual nodes in each virtual cycle. See Figure 3 for an example. Note that since nodes may have a high degree, we also need to redistribute the virtual nodes using the same approach as in Section 4. Since the arboricity of a pseudotree is at most 2, we can compute an orientation with outdegree 6 [10, Theorem 3.5], and thus after redistributing the virtual nodes every node simulates at most 12 virtual nodes.

To differentiate the at most two cycles of virtual nodes from each other, we first establish shortcuts by performing the Introduction Algorithm on the virtual nodes. Afterwards, every virtual node \( v_i \) broadcasts the value \( \text{id}(v) \circ i \) along all of its shortcuts; by repeatedly letting each node broadcast the highest value received so far for \( O(\log n) \) rounds, each virtual node learns the maximum of all values in its cycle. Any node whose virtual nodes learned different
maxima must be a cycle node; if there exists no such node, which can easily be determined using Remark 4 in $G$, there is no cycle in $G$. We conclude the lemma below.

- **Lemma 11.** After $O(\log n)$ rounds every node $v \in V$ knows whether there is a cycle, and, if so, whether it is a cycle node.

**Proof (Sketch).** We sketch the correctness of our construction by showing that if $G$ contains one cycle, then (1) the virtual nodes of each tree node are contained in the same cycle, (2) each cycle node has two virtual nodes contained in different cycles. For (1), let $v$ be a cycle node and $\{v, w\}$ be an edge to some tree node $w$. By our construction, there is exactly one virtual node $v_i$ of $v$ that is connected to a virtual node of $w$ and there is exactly one virtual node $w_j$ of $w$ that is connected to a virtual node $v_j$ of $v$. As presented in Section 4 this yields a path of virtual nodes starting at $v_i$, that traverses the subtree with root $w$ in a depth-first-search manner and ends at $v_j$, which implies (2).

Specifically, this shows that the tree nodes do not introduce additional cycles to our construction; therefore, we can disregard them and assume that $G$ forms a single cycle that does not contain any tree nodes. For this cycle it has to hold by our construction that every cycle node $v$ has exactly two virtual nodes $v_0$ and $v_1$ that are not directly connected to each other. This immediately implies that the virtual nodes have to form exactly two distinct cycles of virtual nodes, since in case they would form one or more than two cycles, there has to exist a cycle node whose virtual nodes are connected to each other.

Since we already know how to compute SSSP and the diameter on trees, for the remainder of this section we assume that $G$ contains a cycle. In order to solve SSSP, we first perform our SSSP algorithm for tree graphs from $s$ in the tree $T_v$ in which $s$ lies (note that $s$ may be $v$ itself). Thereby, every node in $T_v$ learns its distance to $s$. Specifically, $v$ learns $d(s, v)$, and can make this value known to all nodes by using Remark 4. After performing SSSP with source $v$ on the cycle nodes only, every cycle node $u \neq v$ knows $d(s, v) + d(v, u) = d(s, u)$, and can inform all nodes in $T_v$ about it using Remark 4. Finally, $u$ performs SSSP in $T_v$ with source $u$, whereby each node $w \in T_u$ learns $d(s, u) + d(u, v) = d(s, w)$. Together with Theorems 7 we obtain the following theorem.

- **Theorem 12.** SSSP can be computed in a pseudotree in time $O(\log n)$.

We now describe how to compute the diameter in a pseudotree. In our algorithm, every cycle node $v$ contributes up to two candidates for the diameter. The first candidate for a node $v$ is the diameter of its tree $D(T_v)$. If $\text{ecc}(v) > d(v)$, where $d(v) := \text{ecc}_{T_v}(v)$ is the depth of $T_v$, then $v$ also contributes the value $\text{ecc}(v) + d(v)$ as a candidate. We first show how the values can be computed, and then prove that the maximum of all candidates, which can easily be determined using Remark 4, is the diameter of $G$.

After $v$ has identified itself as a cycle node, it can easily compute its depth $d(v)$ in time $O(\log n)$ by performing SSSP on $T_v$ from $v$ using Theorem 7 and then computing the maximum distance $d(v, u)$ of any node $u$ in $T_v$ using Remark 4. Furthermore, $D(T_v)$ can be computed in time $O(\log n)$ via an application of Theorem 10.

It remains to show how $v$ can learn $\text{ecc}(v)$. We define $m_l(v) := \max_{u \in V} d(u) - d_l(v, u)$, and $m_r(v) := \max_{u \in V} d(u) - d_r(v, u)$ (recall that $d_l(v, u)$ and $d_r(v, u)$ denote the distances from $v$ to $u$ along a left or right traversal of the cycle, respectively). To compute these values, the nodes first establish shortcuts along a left and right traversal of the cycle using the
Introduction algorithm\footnote{This time, each node \( v \) participates, and the initial left (right) neighbor of \( v \) is \( v \)'s successor (predecessor) along a left traversal.} Afterwards, every cycle node \( v \) computes \( m_\ell(v) \) (and, analogously, \( m_r(v) \)) in the following way. \( v \) maintains a value \( x_v \), which will obtain the value \( m_\ell(v) \) after \( O(\log n) \) rounds. Initially, \( x_v := d(v) \). In the first round, every cycle node \( v \) sends \( x_v - w(\{v, r_i\}) \) to its right neighbor \( r_i \). When \( v \) receives a value \( x \) at the beginning of round \( i \), it sets \( x_v := \max\{x_v, x\} \) and sends \( x_v - w(\{v, r_i\}) \) to \( r_i \).

\textbf{Lemma 13.} At the end of round \(|\log n| + 1 \), \( x_v = m_\ell(v) \).

\textbf{Proof.} We show that at the end of round \( i \geq 1 \),

\[
x_v = \max_{u \in V_l(v,i)} (d(u) - d_\ell(v,u)),
\]

where \( V_l(v,i) \) contains node \( u \in V \) if the (directed) path from \( v \) to \( u \) in \( G_\ell \) contains at most \( 2^{i-1} - 1 \) hops. The lemma follows from the fact that \( V_l(v, |\log n| + 1) = V \).

At the end of round 1, \( x_v = d(v) \), which establishes the inductive base since \( v \) is the only node within 0 hops from \( v \). By the induction hypothesis, at the beginning of round \( i > 1 \) we have that \( x_v = \max_{u \in V_l(v,i-1)} (d(u) - d_\ell(v,u)) \). Furthermore, \( v \) receives

\[
x = \max_{u \in V_l(\ell, i-1)} (d(u) - d_\ell(\ell, i-1, u)) - w(\{\ell, i-1, v\})
\]

\[
= \max_{u \in V_l(\ell, i-1)} (d(u) - d_\ell(v,u)).
\]

Since \( V_l(v, i-1) \cup V_l(\ell, i-1, i-1) = V_\ell(v, i) \), we conclude that \( \max\{x_v, x\} = \max_{u \in V_l(v,i)} (d(u) - d_\ell(v,u)) \). \hfill \blacktriangleleft

Now, every cycle node \( v \) performs the diameter algorithm for the cycle of Theorem 3. Thereby, \( v \) computes its left and right farthest nodes \( v_l \) and \( v_r \) such that either \( v_l = v_r \), or \( \{v_l, v_r\} \in E \). We have the following lemma.

\textbf{Lemma 14.} Let \( v \in V \) be a cycle node. \( \text{ecc}(v) = \max\{d_\ell(v, v_l) + m_\ell(v_l), d_r(v, v_r) + m_\ell(v_r)\} \).

\textbf{Proof.} Let \( t \in V \) such that \( d(v, t) = \text{ecc}(v) \), and let \( u \) be a cycle node such that \( t \) is a node of \( T_u \). W.l.o.g., assume that \( u \) lies on the right side of \( v \), i.e., \( d_r(v, u) \leq d_\ell(v, u) \). As in Section 3 we define \( d_r \) and \( d_\ell \) to be \( d_\ell(v, v_l) \) and \( d_r(v, v_r) \), respectively. We show that (1) \( d_r + m_\ell(v_l) \geq \text{ecc}(v) \), and that (2) \( d_\ell + m_\ell(v_r) \leq \text{ecc}(v) \) and \( d_r + m_\ell(v_r) \leq \text{ecc}(v) \). Both statements together immediately imply the claim.

For (1), note that \( v_r \) will consider \( u \) as a cycle node for the computation of \( m_\ell(v_r) \), and thus \( m_\ell(v_r) \geq d(u) - d_\ell(v_r, u) \). Therefore, we have that

\[
d_r + m_\ell(v_r) \geq d_r - d_\ell(v_r, u) + d(u) = d_\ell(v, u) + d(u) = d(v, t).
\]

For (2), we only show that \( d_\ell + m_\ell(v_r) \leq \text{ecc}(v) \); the other side is analogous. Let \( w \) be the node such that \( m_\ell(v_l) = d(w) - d_\ell(v_l, w) \). First, assume that \( w \) lies on the left side of \( v \), i.e., \( d_\ell(v, w) \leq d_\ell(v, w) \). In this case, we have that \( d_\ell(v, w) = d_\ell - d_\ell(v, w) \), which implies

\[
m_r(v_l) = d(w) - d_\ell(v_l, w)
\]

\[
= d(w) + d_\ell(v, w) - d_\ell
\]

\[
\leq \text{ecc}(v) - d_r.
\]
where \( E \) which concludes the proof.

More specifically, as many other distributed algorithms for MST computation, requests have been sent from one supernode to another form a spanning tree. Supernode remains. As a byproduct, Observation 4 remarks that the edges over which merge overlay in time.

The algorithm computes an MST of \( G \) deterministically in time \( O(\log^2 n) \).

Proof. The Overlay Construction Algorithm presented in [30] constructs a low-diameter overlay in time \( O(\log n) \) by alternatingly grouping and merging supernodes until a single supernode remains. As a byproduct, Observation 4 remarks that the edges over which merge requests have been sent from one supernode to another form a spanning tree.

To obtain an MST, we change the way a supernode \( u \) chooses a neighboring supernode to merge with. More specifically, as many other distributed algorithms for MST computation,

\[ m_e(v) = d(w) - d_e(v, w) = d(w) - d_e(v, w) - d_e \leq d(w) + d_e(v, w) - d_e \leq \text{ecc}(v) - d_e, \]

which concludes the proof.

Using the previous results, the nodes can now compute their candidates and determine the maximum of all candidates. It remains to show the following lemma, from which we obtain Theorem 16.

**Lemma 15.** Let \( C \) be the set of all candidates. \( \max_{c \in C} \{c\} = D. \)

**Proof.** First, note that since every candidate value corresponds to the length of a shortest path in \( G \), \( c \leq D \) for all \( c \in C \). Let \( s, t \in V \) be two nodes such that \( D = d(s, t) \), and let \( T_v \) and \( T_w \) with cycle nodes \( v \) and \( w \) be the trees of \( s \) and \( t \), respectively. We show that \( v \) or \( w \) compute \( D \) as one of their candidates. First, note that if one of the two nodes \( s \) and \( t \), say \( s \), is a cycle node, then \( D = \text{ecc}(v) = \text{ecc}(v) + d(v) \), and \( \text{ecc}(v) > d(v) = 0 \); therefore, \( v \) chooses \( D \) as a candidate.

Therefore, assume that both \( s \) and \( t \) are tree nodes. If \( s \) and \( t \) belong to the same tree, i.e., \( v = w \), we have that \( d(s, t) = D(T_v) \), which is a candidate of \( v \). Otherwise, \( D = \text{ecc}(v) + d(v) = \text{ecc}(w) + d(w) \). We only have to show that \( \text{ecc}(v) > d(v) \) or \( \text{ecc}(w) > d(w) \). Assume to the contrary that \( \text{ecc}(v) = d(v) \) and \( \text{ecc}(w) = d(w) \) (note that \( \text{ecc}(u) \geq d(u) \) for every cycle node \( u \)). Therefore, \( \text{ecc}(v) = d(v, w) + d(w) = d(v) \), and \( \text{ecc}(w) = d(v, w) + d(v) = d(w) \), which implies that \( d(v, w) = 0 \). However, this contradicts the assumption that \( v \neq w \). \( \blacksquare \)

**Theorem 16.** The diameter can be computed in a pseudotree in time \( O(\log n) \).

### 6 Sparse Graphs

In this final section, we present constant factor approximations for SSSP and the diameter in graphs that contain at most \( n + O(n^{1/3}) \) edges and that have arboricity at most \( O(\log n) \). Our algorithm for such graphs relies on a minimum spanning tree (MST) \( M = (V, E') \) of \( G \), where \( E' \subseteq E \). \( M \) can be computed in time \( O(\log^2 n) \) using [30]. Observation 4, in a slightly modified way.

**Lemma 17.** The algorithm computes an MST of \( G \) deterministically in time \( O(\log^2 n) \).

**Proof.** The Overlay Construction Algorithm presented in [30] constructs a low-diameter overlay in time \( O(\log n) \) by alternatingly grouping and merging supernodes until a single supernode remains. As a byproduct, Observation 4 remarks that the edges over which merge requests have been sent from one supernode to another form a spanning tree.

To obtain an MST, we change the way a supernode \( u \) chooses a neighboring supernode to merge with. More specifically, as many other distributed algorithms for MST computation,
our modification directly mimics the classic approach of Borůvka [43]. Instead of choosing the adjacent supernode \( v \) that has the highest identifier, and sending a merge request if \( v \)'s identifier is higher than \( u \)'s identifier, \( u \) determines the outgoing edge (i.e., the edge incident to a node of \( u \) whose other endpoint is not in \( u \)) with smallest weight, breaking ties by choosing the edge with smallest identifier (where the identifier of an edge \( \{x, y\} \), \( \text{id}(x) < \text{id}(y) \), is given by \( \text{id}(x) \circ \text{id}(y) \)). It is well-known that the edges chosen in this way form an MST.

Compared to the grouping stage described in [30], this yields components of supernodes that form pseudotrees with a cycle of length 2 (see, e.g., [35]). However, such cycles can easily be resolved locally by the supernodes such that the resulting components form trees, which allows us to perform the merging stage of [30] without any further modifications.

We call any edge \( e \in E \setminus E' \) that does not lie in \( M \) a non-tree edge. Further, we call a node shortcut node if it is adjacent to a non-tree edge, and define \( S \subseteq V \) as the set of shortcut nodes. Clearly, after computing \( M \) every node \( v \in S \) knows that it is a shortcut node, i.e., if one of its incident edges has not been added to \( M \).

In the remainder of this section, we will compute approximate distances by (1) computing the distance from each node to its closest shortcut node in \( G \), and (2) determining the distance between any two shortcut nodes in \( G \). Our algorithms rely on a balanced decomposition tree \( T_M \), which allows us to quickly determine pairwise distances between nodes in \( G \), and which is presented in Section 6.1. In Section 6.2, \( T_M \) is extended by a set of edges that allow us to solve (1) by performing a distributed multi-source Bellman-Ford algorithm for \( \Theta(n) \) rounds. For (2), in Section 6.3 we first compute the distance between any two shortcut nodes in \( M \), and then perform matrix multiplications to obtain the pairwise distances between shortcut nodes in \( G \). By exploiting the fact that \( |S| = O(n^{1/3}) \), and using techniques of [7], we are able to distribute the \( \Theta(n) \) operations of each of the \( \Theta(n) \) multiplications efficiently using the global network. In Section 6.4, we finally show how the information can be used to compute 3-approximations for SSSP and the diameter.

### 6.1 Hierarchical Tree Decomposition

We next present an algorithm to compute a hierarchical tree decomposition of the MST \( M \), resulting in a balanced decomposition tree \( T_M \). \( T_M \) will enable us to compute distances between nodes in \( M \) in logarithmic time, despite the fact that the diameter and degree of \( M \) may be very high.

Our algorithm constructs \( T_M = (V, E_T) \) as a rooted tree \( T_M = (V, E_T) \) of depth \( \Theta(n) \) with root \( r \in V \) by selecting a set of global edges \( E_T \). Each node \( v \in V \) knows its parent \( p_T(u) \in V \). To each edge \( \{u, v\} \in E_T \) we assign a weight \( w(\{u, v\}) \) that equals the sum of the weights of all edges on the (unique) path from \( u \) to \( v \) in \( M \). Further, each node \( v \in V \) is assigned a distinct label \( l(v) \in \{0, 1\}^{\Theta(n)} \) such that \( l(v) \) is a prefix of \( l(u) \) for all children \( u \) of \( v \) in \( T_M \), and \( l(r) = \varepsilon \) (the empty word).

From a high level, the algorithm works as follows. Starting with \( M \), within \( \Theta(n) \) iterations \( M \) is divided into smaller and smaller components until each component consists of a single node. More specifically, in iteration \( i \), every remaining component \( A \) handles one recursive call of the algorithm independently from the recursive calls executed in other components. The goal of \( A \) is to select a split node \( x \), which becomes a node at depth \( i - 1 \) in \( T_M \), and whose removal from \( M \) divides \( A \) into components of size at most \( |A|/2 \). The split node \( x \) then recursively calls the algorithm in each resulting component; the split nodes that are selected in each component become the children of \( x \) in \( T_M \) (see Figure 4).
(a) The MST $M$ after the first step of the tree decomposition. The black node is the root $r$, and the grey nodes are the first split nodes chosen for each of $r$’s subtrees. The algorithm will recursively be called in each connected components of white nodes. (b) The resulting balanced decomposition tree $T_M$.

When the algorithm is called at some node $v$, it is associated with a label parameter $l \in \{0, 1\}^{O(\log n)}$ and a parent parameter $p \in V$. The first recursive call is initiated at the node with smallest identifiers with parameters $l = \varepsilon$ and $p = \emptyset$. Assume that a recursive call is issued at $v \in V$, and let $A$ be the component of $M$ in which $v$ lies. Using Remark 8, every node $u$ in $A$ can easily compute the number of nodes that lie in each of its adjacent subtrees (i.e., the size of the resulting components of $A$ after removing $u$). It is easy to see that there must be a node $x$ whose removal divides $A$ into components of size at most $|A|/2$ (see, e.g., [8, Lemma 4.1]); if there are multiple such nodes, let $x$ be the one that has smallest identifier. $v$ selects $x$ as a split node using Remark 4 and informs $x$ about $p$ and $l$ using the global network. If $p \neq \emptyset$, in which case we will have that $\{v, p\} \in E$, $v$ also sends $w(\{v, p\})$ to $x$. By performing the SSSP algorithm of Theorem 7 with source $v$ in $A$, $x$ learns $d_M(x, v)$, and becomes a node of $T_M$ with $p_T(x) = p$, $l_T(x) = l$, and, if $p \neq \emptyset$, $w(\{x, p\}) = d_M(x, v) + w(\{v, p\})$.

Afterwards, $x$ instructs each of its $k$ (remaining) neighbors in $M$ to continue the recursion. More specifically, for $1 \leq j \leq k$, $x$ calls the algorithm with parameters $p = x$ and $l = l_T(x) \circ b_j$ at its neighbor $u$ with $j$-th largest identifier, where $b_j$ is the binary representation of $j - 1$. To do so, it simply sends a message to $u$ over the local edge $\{x, u\} \in E$.

\begin{theorem}
A balanced decomposition tree $T_M$ for $M$ can be computed in time $O(\log^2 n)$.
\end{theorem}

\begin{proof}
It is easy to see that our algorithm constructs a correct balanced decomposition tree.

It remains to analyse the runtime of our algorithm. In each recursive call we need $O(\log n)$ rounds to compute the sizes of all subtrees for any node (Remark 8) and $O(\log n)$ rounds to find a split node (Remark 4). Computing the weight of a global edge chosen to be in $E_T$ takes $O(\log n)$ rounds (Theorem 7). Since the component’s sizes at least halve in every iteration, the algorithm terminates after $O(\log n)$ iterations. This proves the theorem.
\end{proof}

It is easy to see that one can route a message from any node $s$ to any node $t$ in $O(\log n)$ rounds by following the unique path in the tree from $s$ to $t$, using the node labels to find the next node on the path; however, the sum of the edge’s weights along that path may be higher than the actual distance between $s$ and $t$ in $M$.

\subsection{Finding Nearest Shortcut Nodes}

To efficiently compute the nearest shortcut node for each node $u \in V$, we extend $T_M$ to a distance graph $D_T = (V, E_D)$, $E_D \supseteq E_T$, by establishing additional edges between the nodes.
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![Figure 5](image.png) Example for the construction of additional edges (indicated by the dashed lines) going into the node u. Grey edges are edges of M, straight black edges are edges in T_M, u is v’s parent in T_M, with x, y and z being in the subtree of v in T_M. We always choose the node in the subtree that goes in the direction back to u in M, so we add edges \( \{y, u\} \) and \( \{x, u\} \). We do not add an edge \( \{z, u\} \) because its subtree follows the opposite direction of u from the perspective of y. The descendants of u in D_T are marked grey.

of T_M. Specifically, unlike T_M, the distance between any two nodes in D_T will be equal to their distance in M, which allows us to employ a distributed Bellman-Ford approach.

We describe the algorithm to construct D_T from the perspective of a fixed node \( u \in V \) (for an illustration, see Figure 5). For each edge \( \{u, v\} \in E_T \) for which there does not exist a local edge \( \{u, v\} \in E_M \), we know that the edge \( \{u, v\} \) "skips" the nodes on the unique path between u and v in M (for example, the edge \( \{r, a\} \) in Figure 4 skips the nodes b and c). Consequently, these nodes must lie in a subtree of v in T_M. Therefore, to compute the exact distance from u to a skipped node w, we cannot just simply add up the edges in E_T on the path from u to w, as this sum must be larger than the actual distance d(u, w).

To circumvent this problem, u’s goal is to establish additional edges to some of these skipped nodes. Let \( x \in V \) be the neighbor of u in M that lies on the unique path from u to v in M. Note that x is the node that v recursively called with parent parameter u and label parameter l(v) in the algorithm of Theorem 18 and which then selected v as a split node. To initiate the construction of edges in each of its subtrees, u needs to send messages to all of its children in T_M that skipped some nodes; however, since u may have a high degree, it may be unable to do so using the global network. Instead, for v, x adopts this task and sends a message containing l(x), id(u), id(x), and w(\( \{u, v\} \)) to v. Upon receiving the call from x, v contacts its child node y in T_M whose label is a prefix of l(x), forwarding u’s identifier, l(x) and the (updated) weight w(\( \{y, u\} \)) = w(\( \{u, v\} \)) - w(\( \{v, y\} \)): as before, v also defers this task to the neighbor that lies on the unique path from v to y. Then, v also defers this task to the set E_D by informing x about it (again, it cannot reach u directly due to u’s potentially high degree). Then, v continues the recursion at its child in T_M that lies in u’s direction, until the process reaches x itself. Since the depth of T_M is O(log n), x learns at most O(log n) additional edges for u.

Note that since the process from u propagates down the tree level by level, we can perform the algorithm at all nodes in parallel, whereby the separate construction processes follow each other in a pipelined fashion without causing too much communication. Together with Theorem 18 we obtain the following lemma.

**Lemma 19.** The distance graph D_T = (V, E_T) for M can be computed in time O(log^2 n).

From the way we construct the node’s additional edges in E_D, and the fact that the edges in E_T preserve distances in M, we conclude the following lemma.
Lemma 20. For any edge \( \{u,v\} \in E_D \) it holds \( w(\{u,v\}) = d_M(u,v) \), where \( d_M(u,v) \) denotes the distance between \( u \) and \( v \) in \( M \).

The next lemma is crucial for showing the correctness of the algorithms that follow.

Lemma 21. For every \( u,v \in V \) we have that (i) every path from \( u \) to \( v \) in \( D_T \) has length at least \( d_M(u,v) \), and (ii) there exists a path \( P \) with \( w(P) = d_M(u,v) \) and \( |P| \in O(\log n) \) that only contains nodes of the unique path from \( u \) to \( v \) in \( T_M \).

Proof. For (i), assume to the contrary that there is a path \( P \) from \( u \) to \( v \) with length less than \( d_M(u,v) \). As no path in \( M \) from \( u \) to \( v \) can be shorter than \( d_M(u,v) \), \( P \) contains at least one edge from \( E_D \). However, by the way we construct the weight of the edges in the distance graph \( D_T \), it holds for any edge \( \{x,y\} \in E_D \) that \( w(\{x,y\}) = d_M(x,y) \). As \( P \) can be any arbitrary path from \( u \) to \( v \), we get \( w(P) \geq d_M(u,v) \), which is a contradiction.

For (ii), let \( P_T = (u = x_0, x_1, x_2, \ldots, x_m = v) \) be the path from \( u \) to \( v \) consisting of edges \( E_T \). By the construction of \( T \), \( |P_T| = O(\log n) \). In case \( w(P_T) = d_M(u,v) \) we are done, so let us assume that \( w(P_T) > d_M(u,v) \). We show that we can replace subpaths of \( P_T \) by single edges out of \( E_D \setminus E_T \) until we arrive at the path with the desired properties.

Let \( w \) be the node in \( P_T \) that has smallest depth in \( T_M \), i.e., the highest common ancestor of \( u \) and \( v \) in \( T_M \). We follow the right subpath \( P_r = (w = x_i, \ldots, x_m = v) \) of \( P_T \) from \( w \) to \( v \) (the left subpath \( P_l \) from \( u \) to \( w \) is analogous). Starting at \( w \), we sum the weights of the edges of \( T_M \) on \( P_r \) until we reach a node \( x_j \) such that the sum is higher than \( d_M(w,x_j) \). In this case, the edge \( \{x_{j-2}, x_{j-1}\} \) must have skipped the node \( x_j \), i.e., \( x_j \) lies on the unique path from \( x_{j-2} \) to \( x_{j-1} \) in \( M \). We now follow \( P_r \) as long as we only move in the direction of \( x_{j-2} \) in \( M \), i.e., we move to the next node if that node is closer to \( x_{j-2} \) in \( M \) than the previous one, until we stop at a node \( x_k \). By the definition of our algorithm, there must be an edge \( \{x_{j-2}, x_k\} \in E_D \) with \( w(\{x_{j-2}, x_k\}) = d_M(\{x_{j-2}, x_k\}) \). We replace the subpath of \( P_r \) from \( x_{j-2} \) to \( x_k \) by this edge, after which the length of the subpath of \( P_r \) from \( x_i \) to \( x_k \) equals \( d_M(x_i, x_k) \). We continue the process starting at \( x_k \) until we reach \( x_m \) and obtain that \( w(P_r) = d_M(x_i, x_k) \). After we have performed the same process at \( P_r \) (in the other direction), we have that \( w(P_r) = d_M(u,v) \). Finally, note that \( |P_T| = O(\log n) \), so \( P_T \) has all the desired properties of the path \( P \) from the statement of the lemma.

We are now ready to compute the nearest shortcut node \( u \in V \) of each node \( v \in V \). Specifically, we make sure that \( v \) gets to know both \( u \)’s identifier and \( d(u,v) \). For this, each shortcut node \( v \) performs a distributed version of the Bellman-Ford algorithm. From an abstract level, the algorithm works as follows. In the first round, every shortcut node sends a message containing its identifier and distance value \( 0 \) to itself. In every subsequent round, every node \( v \in V \) chooses the message with smallest distance value \( d \) received so far, and sends a message containing \( d + w(\{v,u\}) \) to each neighbor \( u \in D_T \). After \( O(\log n) \) rounds, every node \( v \) knows the distance \( d_M(u,v) \) to its closest shortcut node \( u \) in \( M \). Since for any closest shortcut node \( w \) in \( G \) there must be a shortest path from \( v \) to \( w \) that only contains edges of \( M \), this implies that \( u \) must also be closest to \( v \) in \( G \), and \( d_M(u,v) = d(u,v) \).

Note that since nodes may have a high degree in \( D_T \), we must again relay messages using local neighbors. Specifically, if \( v \) wants to send a message to a node \( u \) that is a descendant of \( v \) in \( T_M \), it instructs its local neighbor \( x \) that lies on the path from \( v \) to \( u \) in \( M \) to send the message over the global network. If otherwise \( u \) is an ancestor of \( v \), it must know a reference to \( x \), which can then deliver the message to \( v \). By our construction, every node will only receive \( O(\log n) \) messages over the global network in each round.
Lemma 22. After $O(\log n)$ rounds, each node $v \in V$ knows $id(u)$ of its nearest shortcut node $u$ in $G$ and its distance $d(u,v)$ to it.

Proof. We first show that $v$ learns $id(u)$ and $d_M(u,v)$ of its closest shortcut node $u$ in $M$. Let $P_M = (v = x_0, x_1, \ldots, x_k = u)$ be the shortest path from $v$ to $u$ in $M$. Due to Lemma 21 (i), we know no message associated with $u$ ever has a smaller distance value than $d_M(u,v)$. Also, due to Lemma 21 (ii), there is a path $P$ of length $O(\log n)$ from $u$ to $v$ with $w(P) = d_M(u,v)$. We claim that a message from $u$ traverses the whole path $P$ until it arrives at $v$ after $O(\log n)$ rounds. Assume to the contrary that this is not the case. Then there must exist a node $x_i$ on the path $P$ that does not send a message with $id(u)$ and distance value $d_M(u,x_{i+1})$ to $x_{i+1}$. This can only happen if $x_i$ knows a shortcut node $z$ with $d_M(z,x_i) < d_M(u,x_i)$ such that $d_M(z,x_i) < d_M(v,x_i)$. This implies that $z$ is a shortcut node with $d_M(u,z) < d_M(u,v)$, contradicting the fact that $v$ is $u$’s nearest shortcut node.

Finally, we show that $v$ is also the closest shortcut node to $v$ in $G$. Assume to the contrary that there is a shortcut node $w \in V$ such that $d(v, w) < d(v, u)$. If there is a shortest path from $v$ to $w$ in $G$ that does contain different shortcut nodes, then $w$ cannot be closest in $G$. Otherwise, the path contains only edges of $M$, which implies that $w$ is also closer to $v$ in $M$ than $u$, which contradicts our choice of $u$.

Finally, the following remark, which we will use later, implies that for each node $v$ there is at most one additional edge in $D_T$ to an ancestor in $T_M$.

Remark 23. If our algorithm creates an additional edge from $s$ to $t$, where $s$ is an ancestor of $t$ in $T_M$, then no node on the path from $s$ to $t$ in $T_M$ creates an additional edge to $t$.

Proof. Assume there exists an edge $\{s,t\} \in E_D \setminus E_T$ and let $(s,v_1,\ldots,v_k,t)$ be the unique path from $s$ to $t$ in $T_M$.

Then, by our algorithm it has to hold that $d_M(s,t) < d_T(s,v_1) + d_T(v_1,t)$, where $d_T(u,v)$ denotes the distance between nodes $u$ and $v$ in $T_M$. As $s$ has an additional edge to $t$, it has to hold for all nodes $v_i \in \{v_1,\ldots,v_k\}$ that $d_M(s,t) = d_T(s,v_1) - d_T(v_1,v_i) - d_T(v_i,t)$ and therefore $d_T(v_i,t) = d_T(v_k,t) + \sum_{j=i}^{k-1} d_T(v_j,v_{j+1}) = d_M(v_i,t)$. This implies that our algorithm does not generate additional edges $\{v_i,t\}$ for all $i \in \{1,\ldots,k\}$.

6.3 Computing APSP between Shortcut Nodes

In this section, we first describe how the shortcut nodes can compute their pairwise distances in $M$ by using $D_T$. Then, we explain how the information can be used to compute all pairwise distances between shortcut nodes in $G$ by performing matrix multiplications.

Compute Distances in $M$. First, each node learns the total number of shortcut nodes $n_k := |S|$, and each shortcut node is assigned a unique identifier from $[n_k]$. The first part can easily achieved using Remark 4. For the second part, consider the patricia trie $P$ on the node’s identifiers, which, since each node knows all identifiers, is implicitly given to the nodes. By performing a convergecast in $P$ (where each inner node is simulated the leaf node in its subtree that has smallest identifier), every inner node of $P$ can learn the number of shortcut nodes in its subtree in $P$. This allows the root of $P$ to assign intervals of labels to its children in $P$, which further divide the interval according to the number of shortcut nodes in their children’s subtrees, until every shortcut node is assigned a unique identifier.

Note that it is impossible for a shortcut node to explicitly learn all the distances to all other shortcut nodes in polylogarithmic time, since it may have to learn $\Omega(n^{1/3})$ many bits. However, if we could distribute the distances of all $O(n^{2/3})$ pairs of shortcut nodes uniformly
We conclude the following lemma. Technically, note that sufficient shared randomness can be achieved in our model by broadcasting Theorem 2.3. Then, each inner node of Lemma 24. Then, each inner node of groups, the parameters of Theorem 2.3 are chosen uniformly and independently at random using a (pseudo)-random hash function $h : [n_c]^2 \rightarrow V$ that is known to all nodes and that satisfies $h(i,j) = h(j,i)$ \[12\] The goal of $h(i,j)$ is to infer $d_M(i,j)$ from learning all the edges on the path from $i$ to $j$ in $D_T$.

To do so, the representative $h(i,j)$ first has to retrieve the labels of both $i$ and $j$ in $T_M$. However, $i$ cannot send this information directly, as it would have to reach the representatives. To that end, $h(i,j)$ first joins the multicast groups of $g(i)$ and $g(j)$. Technically, it participates in the construction of multicast trees in a simulated $\lfloor \log n \rfloor$-dimensional butterfly network towards them. More precisely, the nodes of the $i$-th column of the butterfly are simulated by the node with $i$-th highest identifier, and each source is a node of the butterfly’s bottom level chosen uniformly and independently at random using a (pseudo)-random hash function $g$. Since the construction is very technical, we leave out the details and defer the interested reader to [7]. By applying [7] Theorem 2.3 as a black box with parameters $\ell := O(\log n)$ and $L := O(n^{2/3})$ (since each node acts for at most $O(\log n)$ of the $n^{2/3}$ representatives and each representative joins two multicast groups), we obtain multicast trees with congestion $C = O(\log n)$ in time $O(\log n)$, w.h.p. We then use [7] Theorem 2.4 to let each shortcut node $i$ multicast its label $l(i)$ to all representatives $h(i,k)$. With parameter $\hat{\ell}$ as the maximum number of representatives simulated by the same node, multiplied by 2 (which can easily be computed using Remark 4 on $M$), and congestion $C$, the theorem gives a runtime of $O(\log n)$, w.h.p.

From the knowledge of $l(i)$ and $l(j)$, $h(i,j)$ can easily infer the labels of all nodes on the path $P$ from $i$ to $j$ in $T_M$. Specifically, it knows the label $l(x)$ of the highest ancestor $x$ of $i$ and $j$ in $T_M$, which is simply the longest common prefix of $l(i)$ and $l(j)$\[13\] The goal of $h(i,j)$ is to retrieve the edge from each node $v \in P \setminus \{x\}$ to its parent in $T_M$, as well as $v$’s additional edge to an ancestor in $D_T$, of which there can be at most one by Remark 23. Since by Lemma 23 these edges contain a shortest path from $i$ to $j$ that preserves the distance in $M$, $h(i,j)$ can easily compute $d_M(i,j)$ using this information.

To retrieve the edges, $h(i,j)$ joins the multicast groups of all nodes of $P \setminus \{x\}$ using [7] Theorem 2.3. Then, each inner node of $T_M$ performs a multicast using [7] Theorem 2.4 to inform all nodes in its multicast group about its at most two edges. Since each node acts on behalf of at most $O(\log n)$ representatives, and each representative joins $O(\log n)$ multicast groups, the parameters of [7] Theorem 2.3 are $\ell := O(\log n)$ and $L := O(n^{2/3} \log n)$, and for [7] Theorem 2.4 we have $\hat{\ell} := O(\log n)$; therefore, all can be done in $O(\log n)$ rounds, w.h.p.

We conclude the following lemma.

Lemma 24. Every representative $h(i,j)$ learns $d_M(i,j)$ in time $O(\log n)$, w.h.p.

Compute Distances in $G$. Let $A \in \{1,\ldots,W\}^{n_c \times n_c}$ be the distance matrix of the
shortcut nodes, where
\[
A_{i,j} = \begin{cases} 
\min\{w([i,j], d_M(i,j)) & \text{if } \{i,j\} \in E \\
 d_M(i,j) & \text{otherwise.}
\end{cases}
\]

Our goal is to square \( A \) for \([\log n]\) many iterations in the min-plus semiring. More precisely, we define \( A^1 = A \), and for \( t \geq 1 \) we have that \( A_{i,j}^{2t} = \min_{k \in [n]} (A_{i,k}^{t} + A_{k,j}^{t}) \).

**Lemma 25.** For \( t \geq 0 \), we have that \( A_{i,j}^{2t} = \min_{P \in \mathcal{P}(i,j,t)} w(P) \), where \( \mathcal{P}(i,j,t) \) is the set of all paths from \( i \) to \( j \) in \( G \) that contain at most \( [2^{t-1}] \) non-tree edges apart from \( \{i,j\} \), i.e., edges \( \{x,y\} \) such that \( x, y \in S \) and, if \( \{i,j\} \in E \), \( \{x,y\} \neq \{i,j\} \).

**Proof.** We prove by induction on \( t \). For \( t = 0 \), and since there are no negative cycles in \( G \), \( P(i,j,0) \) contains paths that do not contain any non-tree edge apart from \( \{i,j\} \), of which there are at most two: The path from \( i \) to \( j \) in \( M \), and, if \( \{i,j\} \) exists, the path that only contains \( \{i,j\} \). Since \( A_{i,j}^{0} \) is defined as the minimum of \( d_M(i,j) \) and, if \( \{i,j\} \) exists, \( w(\{i,j\}) \), the base case holds.

Now consider \( t > 1 \). We have that
\[
A_{i,j}^{2t} = \min_{k \in [n]} (A_{i,k}^{2t-1} + A_{k,j}^{2t-1}).
\]

Let \( P^* \in P(i,j,t) \) be a path between \( i \) and \( j \) such that \( w(P^*) = \min_{P \in \mathcal{P}(i,j,t)} w(P) \). Since \( P^* \) contains at most \([2^{t-1}] = 2^{t-1}\) many non-tree edges apart from \( \{i,j\} \), it can be divided into two subpaths \( P_1^* \) and \( P_2^* \), where \( P_1^* \) is a path from \( i \) to some shortcut node \( k \) (which might be \( i \) or \( j \)) with \( P_1^* \in P(i,k,t-1) \), and \( P_2^* \) is a path from \( k \) to \( j \) such that \( P_2^* \in P(k,j,t-1) \). Note that \( w(P_1^*) = \min_{P \in P(i,k,t-1)} w(P) \) and \( w(P_2^*) = \min_{P \in P(k,j,t-1)} w(P) \), as otherwise \( P^* \) would not be optimal. We conclude that
\[
A_{i,j}^{2t} = A_{i,k}^{2t-1} + A_{k,j}^{2t-1} = w(P_1^*) + w(P_2^*) = w(P^*) = \min_{P \in \mathcal{P}(i,j,t)} w(P),
\]

where the first equality holds because \( A_{i,j}^{2t} < A_{i,k}^{2t-1} + A_{k,j}^{2t-1} \) and the induction hypothesis would imply the existence of a path in \( P(i,k,t-1) \) shorter than \( P_1^* \) (or a path in \( P(k,j,t-1) \) shorter than \( P_2^* \)).

Since every shortest path in \( P \) contains at most \( n-1 \) non-tree edges, after squaring \( A \) for \([\log n]\) many iterations, we obtain the distance \( A_{i,j}^{[\log n]} = d(i,j) \) for every two shortcut nodes \( i \) and \( j \).

We now describe how the matrix can efficiently be multiplied. As an invariant to our algorithm, we show that at the beginning of the \( t \)-th multiplication, every representative \( h(i,j) \) stores \( A_{i,j}^{2t-1} \). Thus, for the induction basis we first need to ensure that every representative \( h(i,j) \) learns \( A_{i,j} \). By Lemma 24, \( h(i,j) \) already knows \( d_M(i,j) \), thus it only needs to retrieve \( w(\{i,j\}) \), if that edge exists. To do so, we first compute an orientation with outdegree \( O(\log n) \) in time \( O(\log n) \) using [10] Corollary 3.12 in the local network. For every edge \( \{i,j\} \) that is directed from \( i \) to \( j \), \( i \) sends a message containing to \( w(\{i,j\}) \) to \( h(i,j) \); since the arboricity of \( G \) is \( O(\log n) \), every node only has to send at most \( O(\log n) \) messages.

The \( t \)-th multiplication is then done in the following way. We use a (pseudo-)random hash function \( h : [n]^3 \to V \), where \( h(i,j,k) = h(j,i,k) \). First, every node \( h(i,j,k) \in V \) needs to learn \( A_{i,j}^{2t-1} \). To do so, \( h(i,j,k) \) joins the multicast group of \( h(i,j) \) using [7] Theorem
With the help of [7, Theorem 2.4], $h(i,j)$ can then multicast $A_{i,j}^{t-1}$ to all $h(i,j,k)$. Since there are $L \leq (n_c)^2 = O(n)$ nodes $h(i,j,k)$ that each join a multicast group, and each node needs to send and receive at most $t = O(\log n)$ values, w.h.p., the theorems imply a runtime of $O(\log n)$, w.h.p.

After $h(i,j,k)$ has received $A_{i,k}^{t-1}$, it sends it to both $h(i,k,j)$ and $h(j,k,i)$. It is easy to see that thereby $h(i,j,k)$ will receive $A_{i,k}^{t-1}$ from $h(i,k)$ and $A_{j,k}^{t-1}$ from $h(j,k)$, and can compute $A_{i,k}^{t-1} + A_{j,k}^{t-1}$. Afterwards, $h(i,j,k)$ then sends the value $A_{i,k}^{t-1} + A_{j,k}^{t-1}$ to $h(i,j)$ by participating in an aggregation using [7, Theorem 2.2] and the minimum function, whereby $h(i,j)$ receives $A_{i,j}^t$. By the same arguments as before, $L = O(n)$, and $t = O(\log n)$, which implies a runtime of $O(\log n)$, w.h.p. We conclude the following lemma.

**Lemma 26.** After $[\log n]$ many matrix multiplications, $h(i,j)$ stores $d_G(i,j)$ for every $i, j \in [n_c]$. The total number of rounds is $O(\log^2 n)$, w.h.p.

### 6.4 Approximating SSSP and the Diameter

We are now all set in order to compute approximate distances between any two nodes $s, t \in V$. Specifically, we approximate $d(s,t)$ by $\tilde{d}(s,t) = \min \{d_M(s,t), d(s,v_s) + d(v_s,v_t) + d(v_t,t)\}$, where $v_s$ is the shortcut node with minimum distance to $s$ and $v_t$ is the shortcut node with minimum distance to $t$ in $G$. The following lemma shows that $\tilde{d}(s,t)$ gives a 3-approximation for $d(s,t)$.

**Lemma 27.** Let $s, t \in V$ and $d(s,t)$ be the length of the shortest path from $s$ to $t$. It holds that $d(s,t) \leq \tilde{d}(s,t) \leq 3d(s,t)$.

**Proof.** If the shortest path between $s$ and $t$ does not contain any shortcut node, then $d(s,t) = d_M(s,t) = d(s,t)$. Assume the shortest path $P$ from $s$ to $t$ contains at least one non-tree edge $(x,y)$, such that $d(s,t) = d(s,v_s) + d(v_s,v_t) + d(v_t,t)$. Then $P$ contains at least two shortcut nodes $x$ and $y$ with $d(s,t) = d(s,x) + d(x,y) + d(y,t)$. Consider Figure 6 for an illustration.

Obviously, $\tilde{d}(s,t)$ represents the distance of a path from $s$ to $t$, so $\tilde{d}(s,t) \geq d(s,t)$ holds. Since $v_s$ is the nearest shortcut node of $s$ we get $d(s,v_s) \leq d(s,x)$ and, analogously, $d(v_t,t) \leq d(y,t)$. Also, it holds

$$d(v_s,v_t) \leq d(v_s,x) + d(s,x) + d(x,y) + d(y,t) + d(t,v_t)$$

$$= d(s,v_s) + d(s,t) + d(t,v_t).$$
Putting all pieces together, we get
\[
\tilde{d}(s, t) = d(s, v_s) + d(v_s, v_t) + d(v_t, t) \\
\leq d(s, v_s) + d(s, v_s) + d(s, t) + d(t, v_t) + d(v_t, t) \\
= 2(d(s, v_s) + d(v_t, t)) + d(s, t) \\
\leq 2(d(s, x) + d(y, t)) + d(s, t) \\
\leq 2d(s, t) + d(s, t) \\
= 3d(s, t).
\]

To approximate SSSP, every node \(v\) needs to learn \(\tilde{d}(s, v)\) for a given source \(s\). To do so, the nodes first have to compute \(d_M(s, v)\), which can be done in time \(O(\log n)\) by performing SSSP in \(M\) using Theorem \(7\). Then, the nodes construct \(D_T\) in time \(O(\log^2 n)\) using Lemma \(19\). With the help of \(D_T\) and Lemma \(22\), \(s\) can compute \(d_M(s, v_s)\), which is then broadcast to all nodes in time \(O(\log n)\) using Remark \(4\). Then, we compute all pairwise distances in \(G\) between all shortcut nodes in time \(O(\log^2 n)\), w.h.p., using Lemma \(26\). Specifically, every shortcut node \(v\) learns \(d(v_s, v)\). By performing a slight variant of the algorithm of Lemma \(22\), we can make sure that every node \(t\) not only learns its closest shortcut node \(v_t\) in \(M\), but also retrieves \(d(v_s, v_t)\) from \(v_t\) within \(O(\log n)\) rounds. Since \(t\) is now equipped with all information necessary to compute \(\tilde{d}(s, t)\), we conclude the following theorem.

\begin{theorem}
A 3-approximation of the diameter can be computed in graphs that contain at most \(n + O(n^{1/3})\) edges and have arboricity \(O(\log n)\) in time \(O(\log^2 n)\), w.h.p.
\end{theorem}

For a 3-approximation of the diameter, consider \(\bar{D} = 2\max_{x \in V} d(s, v_x) + \max_{x,y \in S} d(x,y)\), where \(v_x\) is the closest shortcut node to \(s\) in \(G\), and \(S\) is the set of shortcut nodes. \(\bar{D}\) can easily be computed using Lemmas \(19, 22\), and \(26\), and by using Remark \(4\) on \(M\) to determine the maxima of the obtained values. Therefore, by the triangle inequality, we have that \(D \leq \bar{D}\). Further, since \(d(s, v_x) \leq D\) and \(\max_{x,y \in S} d(x,y) \leq D\), we have that \(\bar{D} \leq 3D\), which implies the following theorem.

\begin{theorem}
A 3-approximation of the diameter can be computed in graphs that contain at most \(n + O(n^{1/3})\) edges and have arboricity \(O(\log n)\) in time \(O(\log^2 n)\), w.h.p.
\end{theorem}
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A PRAM Simulation

Let \( \mathcal{A} \) be a PRAM algorithm that solves a graph problem on \( G \) using \( N \) processors with depth \( T \). Obviously, the total size of the input is \( O(|E|) \).

Lemma 30. An EREW PRAM algorithm \( \mathcal{A} \) can be simulated in the hybrid model in time \( O(a/(\log n) + T \cdot (N/(n \log n) + 1) + \log n) \), w.h.p. A CRCW PRAM algorithm \( \mathcal{A} \) can be simulated in time \( O(a/(\log n) + T \cdot (N/n + \log n)) \), w.h.p.

Proof. Since in a PRAM the processes work over a set of shared memory cells \( M \), we first need to map all of these cells uniformly onto the nodes. The total number of memory cells \(|M|\) is arbitrary but polynomial and each memory cell is identified by a unique address \( x \) and is mapped to a node \( h(x) \), where \( h : M \rightarrow V \) is a pseudo-random hash function. For this, we need shared randomness. It suffices to have \( \Theta(\log n) \)-independence, for which only \( \Theta(\log^2 n) \) bits suffice. Broadcasting these \( \Theta(\log^2 n) \) bits to all nodes takes time \( O(\log n) \).

To deliver \( x \) to \( h(x) \), the nodes compute an \( O(a) \)-orientation in time \( O(\log n) \) [10] Corollary 3.12. Note that each edge in \( G \) can be represented by a constant amount of memory cells. When the edge \( \{v, w\} \) that corresponds to \( v \)'s memory cell with address \( x \) is directed towards \( v, v \) fills in the part of the input that corresponds to \( \{v, w\} \) by sending messages to all nodes that hold the corresponding memory cells (of which there can only be constantly many). Since each node has to send at most \( O(a) \) messages, it can send them out in time \( O(a/\log n) \) by sending them in batches of size \( \lceil \log n \rceil \).

We are now able to describe the simulation of \( \mathcal{A} \): Let \( k = n/\lceil \log n \rceil \). Each step of \( \mathcal{A} \) is divided into \( N/k \) sub-steps, where in sub-step \( t \) the processors \( (t - 1)k + 1, (t - 1)k + 2, \ldots, \min\{N, tk\} \) are active. Each node simulates \( O(\log n) \) processors. Specifically, node \( i \) simulates the processors \( (t - 1)k + (i - 1)|\log n| + 1 \) to \( \min\{N, (t - 1)k + i|\log n|\} \). When a processor attempts to access memory cell \( x \) in some sub-step, the node that simulates it sends a message to the node \( h(x) \), which returns the requested data in the next round. Since each node simulates \( O(\log n) \) processors, each node only sends \( O(\log n) \) requests in each sub-step. Also, in each sub-step at most \( n|\log n| \) requests to distinct memory cells are sent in total as at most \( n|\log n| \) are active in each sub-step. These requests are stored at positions chosen uniformly and independently at random, so each node only has to respond to \( O(\log n) \) requests, w.h.p.

In an EREW PRAM algorithm, the requests and responses can be sent immediately, since each memory location will only be accessed by at most one processor at a time. In this case, one round of the simulation takes time \( O(N/(n \log n) + 1) \).

In a CRCW PRAM algorithm, it may happen that the same cell is read or written by multiple processors. Thus, the processors cannot send requests directly, but need to participate in aggregations towards the respective memory cells using techniques from [7]. In case of a write, the aggregation determines which value is actually written; in case of a read, the aggregation is used to construct a multicast tree which is used to inform all nodes that are interested in the particular memory cell about its value. Since there can be only \( O(n \log n) \) members of aggregation/multicast groups, and by the argument above each node only participates and is target of \( O(\log n) \) aggregations (at most one for each processor it simulates), performing a sub-step takes time \( O(\log n) \), w.h.p., by [7]. Thus, each step can be performed in time \( O(N/n + \log n) \), w.h.p. (note that the additional \( \log n \)-overhead stems from the fact in case \( N > n \), one single node still needs time \( O(\log n) \) to simulate a sub-step).