REFLECTION EQUATION ALGEBRAS, COIDEAL SUBALGEBRAS, AND THEIR CENTRES

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Abstract. Reflection equation algebras and related $U_q(g)$-comodule algebras appear in various constructions of quantum homogeneous spaces and can be obtained via transmutation or equivalently via twisting by a cocycle. In this paper we investigate algebraic and representation theoretic properties of such so called ‘covariantized’ algebras, in particular concerning their centres, invariants, and characters. Generalising M. Noumi’s construction of quantum symmetric pairs we define a coideal subalgebra $B_f$ of $U_q(g)$ for each character $f$ of a covariantized algebra.

The locally finite part $F_l(U_q(g))$ of $U_q(g)$ with respect to the left adjoint action is a special example of a covariantized algebra. We show that for each character $f$ of $F_l(U_q(g))$ the centre $Z(B_f)$ canonically contains the representation ring $\text{Rep}(g)$ of the semisimple Lie algebra $g$. We show moreover that for $g = \mathfrak{sl}_n(\mathbb{C})$ such characters can be constructed from any invertible solution of the reflection equation and hence we obtain many new explicit realisations of $\text{Rep}(\mathfrak{sl}_n(\mathbb{C}))$ inside $U_q(\mathfrak{sl}_n(\mathbb{C}))$. As an example we discuss the solutions of the reflection equation corresponding to the Grassmannian manifold $Gr(m, 2m)$ of $m$-dimensional subspaces in $\mathbb{C}^{2m}$.

Introduction

Originating in the quantum inverse scattering method of the Leningrad school, the theory of quantum groups was to a large extent invented to provide a unified approach to solutions of the quantum Yang-Baxter equation [Dri87]. Without spectral parameter, these solutions are well organised in the structure of a braided monoidal category for each semisimple, finite dimensional, complex Lie algebra $g$. Braided monoidal categories have well-known applications in low dimensional topology and provide representations of the Artin braid group.

If one imposes additional boundary conditions [SkS88], then the quantum Yang-Baxter equation is joined by the so called reflection equation which first appeared in factorised scattering on the half line [Che84]. The notion of a braided monoidal category can be extended to include solutions of the reflection equation [tD98], [tDHO98], however, examples have so far only been constructed by hand for $g = \mathfrak{sl}_n(\mathbb{C})$ [tD99]. The programme outlined in [tD98] aims at applications to braid groups of type $B$ and the affine braid group. Closely related to this programme is the notion of a universal solution of the reflection equation introduced independently in [DKM03]. Again, for quantised universal enveloping algebras, there is no unified construction of examples.

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The reflection equation is also at the heart of certain classes of quantum homogeneous spaces. M. Noumi, T. Sugitani, and M. Dijkhuizen used explicit solutions of the reflection equation to obtain analogues of all classical symmetric pairs as coideal subalgebras of quantised universal enveloping algebras, see e.g. [Nou96], [NS95], [Dij96], [NDS97], [DS99]. Using $t$-operators for the vector representation their construction follows the spirit of [FRT89]. A unified construction of quantum symmetric pairs, more along the lines of V. Drinfeld’s realisation of quantised universal enveloping algebras, was achieved by G. Letzter [Let02]. Central elements in Letzter’s coideal subalgebras lead to solutions of the reflection equation [Kol08].

If one considers the entries of a matrix satisfying the reflection equation as indeterminates then one obtains the so called reflection equation algebra [KS92]. Characters of the reflection equation algebra are the same as numerical solutions of the reflection equation. Such characters were used by J. Donin and A. Mudrov [DM03a] to obtain quantisations of $GL(n)$-orbits in $\text{End}(\mathbb{C}^n)$. Related works by the same authors, e.g. [DKM03], [DM03a], [Mud07], centre around the notion of twisting by a cocycle which goes back to [Dri90] and which transforms FRT-algebras into reflection equation algebras. Twisting by a cocycle, in turn, is also at the heart of S. Majid’s theory of transmutation and covariantized algebras [Maj91, Section 3], [Ma93, Section 4], [Ma95, 7.4]. In this theory the reflection equation algebra occurs as so called braided matrices.

In the present paper we exhibit relations between the five theories referred to above, namely

1. Twisting by a cocycle and quantisation via characters of twisted algebras, e.g. [DKM03], [DM03a], [DM03b], [Mud07],
2. Transmutation and covariantized algebras [Maj91], [Ma93], [Ma95, 7.4],
3. Universal cylinder forms [tD98], [tDHO98], [tD99],
4. Construction of quantum symmetric pairs via solutions of the reflection equation, e.g. [Nou96], [NS95], [Dij96], [NDS97], [DS99],
5. G. Letzter’s construction of quantum symmetric pairs [Let02], [Let03]

which, for the most part, have been developed independently of each other.

Let $G$ be the connected, simply connected affine algebraic group corresponding to the finite dimensional, semisimple, complex Lie algebra $\mathfrak{g}$. Recall that the quantised algebra of functions $k_\mathbb{Q}[G]$ on $G$ is a coquasitriangular Hopf algebra and let $r$ denote its universal $r$-form. We will work in a setting which is tailored to include FRT-algebras and the quantised algebra of functions $k_\mathbb{Q}[G]$ on $G$. More explicitly, we consider any coquasitriangular bialgebra $A$ together with a homomorphism $\Psi : A \rightarrow k_\mathbb{Q}[G]$ of coquasitriangular bialgebras. In this setting, transmutation coincides with twisting by a cocycle and universal cylinder forms for $A$ are, up to translation of conventions, the same as characters of the covariantized algebra $A_r$ of $A$.

By construction, the covariantized algebra $A_r$ is a right comodule algebra over the quantum double of $A$. In our restricted setting, however, $A_r$ also has a left $U_\mathbb{Q}(\mathfrak{g})$-comodule algebra structure. This allows us, for any character $f$ of the covariantized algebra $A_r$, to define a left coideal subalgebra $B_f \subseteq U_\mathbb{Q}(\mathfrak{g})$ in a straightforward manner. We call $B_f$ the Noumi coideal subalgebra corresponding to $f$, because if $A$ is an FRT-algebra then a character of $A_r$ is the same as a solution of the reflection equation and $B_f$ is a coideal of the type constructed in, say [NDS97].

In this paper we investigate algebraic properties of both the covariantized algebra $A_r$ and the Noumi coideal subalgebra $B_f$ in some detail. We are in particular
interested in results concerning the centres $Z(A_r)$ and $Z(B_f)$ as well as characters on $A_r$. We show among other results, that for characters of $A_r$, which are convolution invertible with respect to the coproduct of the bialgebra $A$, the centre $Z(B_f)$ is contained in the locally finite part $F_r(U_q(g))$ of $U_q(g)$ with respect to the right adjoint action. A similar result was obtained by G. Letzter for her quantum symmetric pairs [Let08] Theorem 1.2. The case when $A = k_q[G]$ is of particular interest. We show that in this case the centre $Z(B_f)$ naturally contains a realisation of the representation ring $Rep(g)$ of $g$. This result may seem somewhat surprising. It is well known that $Z(U_q(g))$ is isomorphic to $Rep(g)$ with respect to the grading of $F_r(U_q(g))$, see e.g. [Bau98]. Our constructions show that there are many more natural realisations of $Rep(g)$ inside $U_q(g)$.

Finally, we address the question of how to obtain characters of the covariantized algebra $A_r$. If $A$ is an FRT-algebra then the method devised in [Ko08] provides solutions of the reflection equation via suitable central elements in coideal subalgebras of $U_q(g)$. Here we give a generalised, streamlined presentation of this result. In the case $A = k_q[G]$ we observe that $A_r$ coincides with the left locally finite part $F_l(U_q(g))$. It hence remains to determine all characters of $F_l(U_q(g))$. At this point we restrict to the case $g = sl_n(C)$ and prove that any suitably scaled invertible solution of the reflection equation for the vector representation of $U_q(sl_n(C))$ factors to a character of $F_l(U_q(sl_n(C)))$. Together with the explicit classification in [Mud02] this determines all characters of $F_l(U_q(sl_n(C)))$. It would be desirable to have a classification of characters of $F_l(U_q(g))$ for general $g$.

As an example we follow G. Letzter’s setting [Let03] to discuss the coideal subalgebra $B^s \subseteq U_q(sl_{2m}(C))$ corresponding to the Grassmann manifold $Gr(m, 2m)$ of $m$-dimensional subspaces in $C^{2m}$. Using the structure of $Z(B^s)$ known from [KL08], we show that in this case $Z(B^s)$ is naturally isomorphic to $Rep(sl_n(C))$. Moreover, we calculate the solution of the reflection equation for the element in $Z(B^s)$ corresponding to the vector representation. This allows us to give the corresponding character of the covariantized algebra explicitly. It doesn’t come as a surprise that this character bears close resemblance to the solutions of the reflection equation considered in [NDS97], [tD99], [Mud02], but we avoid painstaking translation of conventions.

We now briefly outline the structure of this paper. In Section 1 we fix notations and conventions for quantised universal enveloping algebras and quantised algebras of functions. The main framework of the paper is outlined in Section 2 in the setting of S. Majid’s theory of transmutation. We establish the relation to Drinfeld twists and discuss transmutation of $k_q[G]$ and FRT-algebras as examples. In Section 3 we begin the investigation of algebraic properties of the covariantized algebra $A_r$. In particular we show that $A_r$ is a domain if and only if $A$ is a domain, which in turn can be used to identify the centre $Z(A_r)$ with the space of $U_q(g)$-invariants in $A_r$. In Subsection 3.4 we collect properties of characters of $A_r$ which allows us in Subsection 3.5 to identify such characters with universal cylinder forms. Section 4 is devoted to the construction and investigation of the Noumi coideal subalgebra $B_f$ for a given character $f$ of $A_r$. In Subsection 4.4 in particular, we establish the realisation of $Rep(g)$ inside $Z(B_f)$ in the case $A = k_q[G]$. Moreover, we show in 4.5 that $Z(B_f) \subseteq F_r(U_q(g))$. The final Section 5 is devoted to the construction of characters of the covariantized algebra $A_r$. First we recall the general construction of solutions of the reflection equation via central elements of coideal subalgebras of $U_q(g)$. From
Subsection 5.2 onwards we restrict to the case $g = sl_n(\mathbb{C})$. In 5.2 we prove that any invertible numerical solution of the reflection equation gives rise to a character of $F_l(U_q(sl_n(\mathbb{C})))$. The last two subsections are devoted to the example $Gr(m, 2m)$.

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1. Notations and conventions

In this introductory section we fix notations and conventions concerning quantised universal enveloping algebras and quantised algebras of functions. All results stated in the first two subsections are well known. Main sources of reference are the monographs [Jan96], [Jos95], and [KS97]. In subsection 1.3 to simplify the presentation of the main thrust of the paper, we recall some possibly less known results concerning universal $r$-forms and $l$-functionals.

Let $\mathbb{Z}$ be the integers, $\mathbb{N}_0$ the non-negative integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{C}$ the complex numbers. Throughout this text, for any coalgebra $C$ we denote the counit by $\varepsilon$ and the coproduct by $\Delta$. We make use of Sweedler notation in the form $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for any $c \in C$, suppressing the summation symbol. We write $C^{\text{cop}}$ to denote the opposite coalgebra with coproduct $c \mapsto c_{(2)} \otimes c_{(1)}$. Similarly, for any algebra $A$ the symbol $A^{\text{op}}$ denotes the opposite algebra with multiplication $a \cdot b \mapsto ba$. If $A$ is a bialgebra then $A^{\text{op}, \text{cop}}$ denotes the bialgebra with both opposite multiplication and opposite comultiplication. For any Hopf algebra $H$ we use the symbol $\sigma$ for the antipode. We write $H^\circ$ to denote the dual Hopf algebra consisting of all matrix coefficients of finite dimensional representations of $H$.

1.1. The quantised universal enveloping algebra $U := \hat{U}_q(g)$. Let $g$ be a finite-dimensional complex semisimple Lie algebra of rank $r$ and let $h$ be a fixed Cartan subalgebra of $g$. Let $\Delta$ denote the root system associated with $(g, h)$. Choose an ordered basis $\pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots for $\Delta$. Let $W$ denote the Weyl group associated to the root system $\Delta$ and let $w_0$ denote the longest element in $W$ with respect to $\pi$. There is a unique $W$-invariant symmetric bilinear form $(\cdot, \cdot)$ on $h^*$ such that $(\alpha, \alpha) = 2$ for all short roots $\alpha \in \Delta$. This form satisfies $(\alpha, \alpha)/2 \in \{1, 2, 3\}$ for all $\alpha \in \Delta$. We write $Q$ for the root lattice and $P$ for the weight lattice associated to the root system $\Delta$. Set $Q^+ = \mathbb{N}_0 \pi$ and let $P^+$ be the set of dominant integral weights with respect to $\pi$. We will denote the fundamental weights in $P^+$ by $\omega_1, \ldots, \omega_r$. Let $\preceq$ denote the dominance partial ordering on $h^*$, so $\mu \preceq \gamma$ if $\gamma - \mu \in Q^+$.

Let $k = \mathbb{C}(q^{1/N})$ denote the field of rational functions in one variable $q^{1/N}$ where $N$ has sufficiently many factors such that $(\lambda, \mu) \in \frac{1}{N}\mathbb{Z}$ for all $\lambda, \mu \in P$. One may for instance choose $N$ to be the order of $P/Q$. We consider here the simply connected quantised universal enveloping algebra $\hat{U}_q(g)$ as the $k$-algebra generated by elements $\{x_i, y_i, \tau(\lambda) \mid i = 1, \ldots, r, \lambda \in P\}$ and relations as given for instance in
Section 3.2.9. In particular the generators satisfy the following relations
\[
\tau(\lambda)x_j = q^{(\lambda,\alpha_j)}x_j\tau(\lambda), \quad \tau(\lambda)y_j = q^{-(\lambda,\alpha_j)}y_j\tau(\lambda),
\]
\[
xy = \delta_{ij}(ti - t_i^{-1})(q_i - q_i^{-1})
\]
where, following common convention, we write \(t_i\) to denote \(\tau(\alpha_i)\) and \(q_i := q^{(\alpha_i,\alpha_i)/2}\).

The generators also satisfy the quantum Serre relations. The algebra \(U\) has the structure of a Hopf algebra with coproduct and antipode given by
\[
\Delta\tau(\lambda) = \tau(\lambda) \otimes \tau(\lambda), \quad \sigma(\tau(\lambda)) = \tau(-\lambda),
\]
\[
\Delta x_i = x_i \otimes 1 + t_i \otimes x_i, \quad \sigma(x_i) = -t_i^{-1}x_i,
\]
\[
\Delta y_i = y_i \otimes t_i^{-1} + 1 \otimes y_i, \quad \sigma(y_i) = -y_it_i.
\]

Note that these conventions also coincide with \[Jan96\] up to renaming of the generators and restriction to the subalgebra generated by \(x_i, y_i, t_i, t_i^{-1}\).

One checks on the generators that the antipode of \(U\) satisfies the relation
\[
\sigma^2(u) = \tau(-2\rho)u\tau(2\rho)
\]
where \(\rho \in Q^+\) denotes the half sum of all positive roots.

We will write \(U^+\) and \(U^-\) to denote the subalgebra of \(U\) generated by \(\{x_1, \ldots, x_r\}\) and \(\{y_1, \ldots, y_r\}\), respectively. Let \(U^0\) be the subalgebra of \(U\) spanned by the elements \(\{\tau(\lambda) | \lambda \in P\}\). Moreover, we will write \(U_q(b^+)\) and \(U_q(b^-)\) to denote the subalgebra of \(U\) generated by \(\{x_i, \tau(\lambda) | i = 1, \ldots, r\}\) and \(\{y_i, \tau(\lambda) | i = 1, \ldots, r, \lambda \in P\}\), respectively.

For any \(U\)-module \(V\) and any \(\lambda \in P\), we call \(\lambda\) a weight of \(V\) if \(\tau(\mu)v = q^{(\mu,\lambda)}v\) for all \(\mu \in P\), and we write \(\text{wt}(v) := \lambda\). For any \(\alpha \in Q\) let \(U_\alpha\) denote the weight space of \(U\) of weight \(\alpha\) with respect to the left adjoint action, more precisely
\[
U_\alpha := \{u \in U | \tau(\lambda)u = q^{(\lambda,\alpha)}u \text{ for all } \lambda \in P\}.
\]

Note that \(U = \bigoplus_{\alpha \in Q} U_\alpha\). Define moreover \(U^+_\alpha := U^+ \cap U_\alpha\) and \(U^-_\alpha := U^- \cap U_\alpha\).

For \(\lambda \in P^+\), let \(V(\lambda)\) be the simple \(U\)-module of highest weight \(\lambda\). In particular, \(V(\lambda)\) is the highest weight module of weight \(\lambda\) such that \(x_i v_\lambda = 0\) for all \(i = 1, \ldots, r\). As usual, we say that a \(U\)-module is of type one if it has a basis consisting of weight vectors with weights in \(P\). We define \(C\) to be the category of all finite dimensional \(U\)-modules of type one. Recall that \(C\) is a rigid, monoidal category via the antipode and the coproduct of \(U\), and that \(C\) is semisimple with simple objects \(V(\lambda)\) for \(\lambda \in P^+\). Moreover, it is well known that \(C\) is a braided monoidal category \[Jan96\] Chapter 7, \[Jos95\] 9.4.7. One hence has a family of \(U\)-module isomorphisms \(R = (R_{V,W} : V \otimes W \to W \otimes V)_{V,W \in \text{Ob}(C)}\) which is natural in both \(V\) and \(W\) and which satisfies the hexagon identities \[Jan96\] 3.18, 3.19. We recall the construction of the braiding \(R\) for \(C\) along the lines of \[Jan96\] Chapter 7 with a slight change of convention. The reason for our choice of conventions will become apparent in Remark \[Jos95\]. For any \(\alpha \in Q^+\), let \(\Theta_\alpha \in U^+_\alpha \otimes U^-_\alpha\) denote the canonical element defined at the beginning of \[Jan96\] 7.1 up to flipping the order of tensor factors. For any \(V, W \in \text{Ob}(C)\) define \(\Theta_{V,W} : V \otimes W \to W \otimes V\) by the action of the formal sum \(\Theta = \sum_{\alpha \in Q^+} \Theta_\alpha\). Moreover, define a \(k\)-linear isomorphism \(f_{V,W} : V \otimes W \to V \otimes W\) by \(f_{V,W}(v \otimes w) = q^{-(\text{wt}(v),\text{wt}(w))}v \otimes w\) for any weight vectors \(v \in V, w \in W\). Now set \(R_{V,W} := P_{12} \circ \Theta_{V,W} \circ f_{V,W} : V \otimes W \to W \otimes V\).
where \( P_{12} \) denotes the flip of tensor factors. It follows from [Jan96] 7.5, 7.8 that \( \bar{R} \) defines a braiding on \( \mathcal{C} \).

1.2. The quantised algebra of functions \( k_q[G] \). Let \( G \) denote the connected, simply-connected affine algebraic group with Lie algebra \( \mathfrak{g} \). We recall the definition of the quantised algebra of functions on \( G \). For any \( V \in \text{Ob}(\mathcal{C}) \) and elements \( v \in V, f \in V^* \) define a linear functional \( c_{f,v} : U \rightarrow k \) by \( c_{f,v}(u) := f(uv) \) for all \( u \in U \). If \( V = V(\lambda) \) then we also write \( \bar{c}_{f,v} = c_{f,v} \) to keep track of the representation \( V(\lambda) \). Let \( C^V := \text{span}\{ c_{f,v} | v \in V, f \in V^* \} \) denote the linear span of all matrix coefficients \( c_{f,v} \) of the representation \( V \). As usual we define \( k_q[G] \) as the Hopf subalgebra of the Hopf dual \( U^\circ \) spanned by the matrix coefficients of all \( V \in \text{Ob}(\mathcal{C}) \).

It is convenient to define a bilinear pairing of Hopf algebras by evaluation

\[
\langle \cdot, \cdot \rangle : k_q[G] \times U \rightarrow k; \quad \langle a, u \rangle := a(u)
\]

for \( a \in k_q[G], u \in U \). The pairing \( \langle \cdot, \cdot \rangle \) is non-degenerate as a consequence of [Jan96; Proposition 5.11]. The quantised algebra of functions \( k_q[G] \) is a left \( U^{\text{cop}} \otimes U \)-module algebra via the action

\[
(X \otimes Y) \cdot a = \langle a(1), \sigma(X) \rangle \langle a(3), Y \rangle a(2).
\]

By construction \( k_q[G] \) has a Peter-Weyl decomposition

\[
k_q[G] = \bigoplus_{\lambda \in P^+} C^V(\lambda)
\]

into irreducible \( U^{\text{cop}} \otimes U \)-modules.

The braiding \( \bar{R} \) of \( \mathcal{C} \) gives rise to the structure of a universal \( r \)-form on \( k_q[G] \). We recall the definition of this notion for the convenience of the reader.

**Definition 1.1.** [KS97; 10.1.1] A coquasitriangular bialgebra \((A, r)\) over a field \( \mathbb{K} \) is a pair consisting of a bialgebra \( A \) over \( \mathbb{K} \) and a convolution invertible linear map \( r : A \otimes A \rightarrow \mathbb{K} \) which satisfies the following relations

\[
r(a(1) \otimes b(1))a(2)b(2) = b(1)a(1)r(a(2) \otimes b(2)),
\]

\[
r(ab \otimes c) = r(a \otimes c(1))r(b \otimes c(2)),
\]

\[
r(a \otimes bc) = r(a(1) \otimes c)r(a(2) \otimes b),
\]

for all \( a, b, c \in A \). The map \( r \) is called a universal \( r \)-form for the bialgebra \( A \).

**Remark 1.2.** In the following, to shorten notation, we will suppress tensor symbols and write \( r(a, b) \) instead of \( r(a \otimes b) \). For later reference note that any universal \( r \)-form satisfies

\[
r(a, 1) = r(1, a) = \varepsilon(a) \quad \text{for all } a \in A.
\]

Note, moreover, that if \( A \) is a Hopf algebra then

\[
r(\sigma(a), \sigma(b)) = r(a, b) \quad \text{for all } a, b \in A
\]

and the convolution inverse \( \bar{r} \) of \( r \) is given by \( \bar{r}(a, b) = r(\sigma(a), b) \) for all \( a, b \in A \). Consult [KS97; 10.1] for more details.
For any coquasitriangular bialgebra \((A, r)\) the linear map \(\bar{r}^{21} : A \otimes A \to \mathbb{K}\) defined by \(\bar{r}^{21}(a, b) = \bar{r}(b, a)\) gives rise to a coquasitriangular bialgebra \((A, \bar{r}^{21})\). We will write \(r'\) to denote either of the universal \(r\)-forms \(r\) or \(\bar{r}^{21}\) for \(A\). For \(A = k_q[G]\) and \(\mathbb{K} = k\) consider the linear map \(r : k_q[G] \otimes k_q[G] \to k\) defined by

\[(1.10) \quad r(c_{f,v}, c_{g,w}) = (g \otimes f)(R_{V,W}(v \otimes w))\]

if \(c_{f,v} \in C^V\) and \(c_{g,w} \in C^W\). It follows from the properties of the braiding \(\hat{R}\) of \(C\) that \((k_q[G], r)\) is a coquasitriangular Hopf algebra. We emphasise that all through this paper the notation \(r\) as universal \(r\)-form on \(k_q[G]\) will always stand for the particular choice \((1.10)\) above.

1.3. Locally finite part and \(l\)-functionals. As any Hopf algebra, the quantised universal enveloping algebra \(U\) is a left and a right module algebra over itself with respect to the left and right adjoint actions defined by

\[\text{ad}_l(u)X := u(1)X\sigma(u(2)), \quad \text{ad}_r(u)X := \sigma(u(1))Xu(2),\]

respectively. The left locally finite part \(F_l(U)\) and the right locally finite part \(F_r(U)\) are defined by

\[F_l(U) := \{u \in U \mid \dim((\text{ad}_l U)u) < \infty\}, \quad F_r(U) := \{u \in U \mid \dim((\text{ad}_r U)u) < \infty\}.\]

Note that results for locally finite parts can be translated from left to right and vice versa using the formulae

\[(1.11) \quad \sigma((\text{ad}_l u)X) = (\text{ad}_r \sigma(u))\sigma(X), \quad \sigma((\text{ad}_r u)X) = (\text{ad}_l \sigma(u))\sigma(X).\]

These formulae imply in particular the relation

\[(1.12) \quad F_r(U) = \sigma(F_l(U)) = \sigma^2(F_r(U)).\]

It was shown in [JL94, Theorem 4.10], [Cal93] that the left locally finite part has a direct sum decomposition

\[(1.13) \quad F_l(U) = \bigoplus_{\lambda \in P^+} (\text{ad}_l U)\tau(-2\lambda)\]

into finite dimensional \(U\)-submodules. Using \((1.12)\) and the relation \(\sigma(\tau(\lambda)) = \tau(-\lambda)\) one obtains a similar decomposition for the right locally finite part

\[(1.14) \quad F_r(U) = \bigoplus_{\lambda \in P^+} (\text{ad}_r U)\tau(2\lambda).\]

The decomposition \((1.13)\) of \(F_l(U)\) is closely related to the Peter-Weyl decomposition \((1.4)\) of \(k_q[G]\) via so called \(l\)-functionals. Recall that \(k_q[G]^p\) denotes the dual Hopf algebra of \(k_q[G]\) and that we write \(r'\) to denote either of the universal \(r\)-forms \(r\) or \(\bar{r}^{21}\) on \(k_q[G]\). Following for instance [KS97, 10.1.3] one obtains linear maps \(l^+_p, l^-_p, l^+_r, l^-_r : k_q[G] \to k_q[G]^p\) defined by

\[(1.15) \quad l^+_p(a) := r'(\cdot, a), \quad l^-_p(a) := r'(\sigma(a), \cdot),\]

\[(1.16) \quad l^+_r(a) := l^-_p(\sigma^{-1}(a(1)))l^+_p(a(2)) = r'(a(1), \cdot)r'(\cdot, a(2)),\]

\[(1.17) \quad l^-_r(a) := l^+_p(a(1))l^-_p(\sigma^{-1}(a(2))) = r'(\cdot, a(1))r'(a(2), \cdot).\]
for any $a \in k_q[G]$. We call these maps $l$-operators and the elements in their image $l$-functionals. Note that the $l$-operators with respect to the universal $r$-forms $r$ and $\tilde{r}$ are related by

$$l^\pm_{\tilde{r}} = l^\mp_r.$$  

The $l$-functionals $l_r$ satisfy commutation relations similar to those of the matrix coefficients in $k_q[G]$. The following statement is proved in [KS97, 10.1.3].

**Lemma 1.3.** For any $a, b \in k_q[G]$ the following relations hold

$$l_r^\pm(ab) = (l_r^\mp(b))_{l_r^\pm}(a),$$

$$l_r^\pm(a(1))l_r^\pm(b(1))r'(a(2), b(2)) = r'(a(1), b(1))l_r^\pm(b(2))(l_r^\pm(a(2))),$$

$$l_r^\pm(a(1))l_r^\pm(b(1))r'(a(2), b(2)) = r'(a(1), b(1))l_r^\pm(b(2))(l_r^\pm(a(2))).$$

Consider $U$ as a subalgebra of $k_q[G]^\circ$ via the non-degenerate Hopf pairing $(1.5)$.

**Lemma 1.4.** $l_r^\pm(k_q[G]) \subseteq U_q(b^\pm)$.  

**Proof.** Fix $V \in Ob(C)$. Choose $v \in V$ of weight $\mu \in P$ and $f \in V^*$ of weight $-\nu \in P$. In the notations introduced at the end of subsection 1.1 one obtains from the definition (1.19) of the universal $r$-form $r$ the relations

$$l_r^+(c_{f,v}) = \sum_{a \in Q^+} (id \otimes c_{f,v})(\Theta_a(\tau(-\mu) \otimes 1)),$$

$$l_r^-(c_{f,v}) = \sum_{a \in Q^+} (\sigma(fv,c) \otimes id)(\Theta_a(1 \otimes \tau(\nu))),$$

which show that $l_r^\pm(k_q[G]) \subseteq U_q(b^\pm)$. \hfill $\square$

**Remark 1.5.** The above lemma explains why we prefer to work with conventions concerning the braiding slightly different from those in [Jan96]. In our conventions we get $l_r^\pm(a) \in U_q(b^\mp)$ for all $a \in k_q[G]$ while the braiding in Jantzen’s book together with definition (1.19) which agrees with [KS97, 10.1.3] would lead to $l_r^\pm(a) \in U_q(b^-)$.

**Remark 1.6.** Let $v_\lambda, v_{w_0}\lambda \in V(\lambda)$ be weight vectors of weights $\lambda$ and $w_0\lambda$ respectively. Fix also $f_\lambda, f_{w_0}\lambda \in V(\lambda)^*$ of weight $-\lambda$ and $w_0\lambda$ such that $f_\lambda(v_\lambda) = 1 = f_{w_0}\lambda(v_{w_0}\lambda)$. By definition (1.17) and (1.22) one obtains

$$\tilde{r}_r(c_{f_\lambda, v_\lambda}) = \tau(-2\lambda), \quad \tilde{r}_{\tilde{r}}(c_{f_{w_0}\lambda, v_{w_0}\lambda}) = \tau(2w_0\lambda).$$

These formulae are central in the proof of the following proposition.

Note that the right adjoint action $ad_r$ of $U$ on itself induces a left action of $U$ on $k_q[G]$, defined by

$$ad_r^*(X)c := \Delta(X) \cdot c = c \circ ad_r(X)$$

for $c \in k_q[G]$ and $X \in U$. The following proposition is in principle contained in [Cal93] (cp. also [Ko08] Proposition 2). We sketch the proof for the convenience of the reader.

**Proposition 1.7.** The $l$-functional $\tilde{r}_r$ defines an isomorphism of left $U$-modules $\tilde{r}_r : k_q[G] \rightarrow F_l(U)$, i.e. $\tilde{r}_r(ad_r^*(X)c) = ad_r(X)\tilde{r}_r(c)$ for all $X \in U$ and $c \in k_q[G]$. Moreover,

$$\tilde{r}_r(C^{V(\lambda)}) = (ad_lU)\tau(-2\lambda)$$
for all $\lambda \in \mathbb{P}^+$, where $\lambda' = \lambda$ if $r' = r$ and $\lambda' = -w_0 \lambda$ if $r' = r_{21}$.

Proof. By (1.18) and by Lemma 1.4 the $l$-functional $\tilde{l}_{r'}$ defines a $k$-linear map $\tilde{l}_{r'} : k_q [G] \to U$. It follows for instance from [KS97, 10.1.3, Proposition 11] that $\tilde{l}_{r'} : k_q [G] \to U$ is a morphism of left $U$-modules (but be aware of the misprint in formula (28) of [KS97, 10.1.3]), hence its image is contained in $F_l(U)$. It now follows from Remark 1.6 that $\tilde{l}_{r'} (C^{(\lambda')}) = (\text{ad}_q U) \tau (-2 \lambda')$. Finally, by [JL94, Theorem 3.5] one has $\dim(C^{(\lambda)}) = \dim((\text{ad}_q U) \tau (-2 \lambda'))$ and hence $\tilde{l}_{r'} : k_q [G] \to F_l(U)$ is an isomorphism.

2. Transmutation

The method of twisting braided bialgebras by a two-cocycle was introduced by V. Drinfeld [Dri90] and extends to module algebras over braided bialgebras. It is at the heart of S. Majid’s construction of covariantized algebras [Maj93] and has been further investigated by J. Donin and A. Mudrov (e.g. [DM03b], [DKM03]).

Here we recall this construction within the framework of S. Majid’s theory of transmutation. We then restrict to a setting, outlined in 2.2, which is tailored to the presentation in [KS97, 10.3] in the setting of coquasitriangular bialgebras. We follow S. Majid’s notion of transmutation as introduced in [Maj93, Section 4] for dual quasitriangular bialgebras (cp. also [Maj95, Section 7.4]). Note that Majid uses the terminology dual quasitriangular instead of coquasitriangular. We follow the presentation in [KS97, 10.3] in the setting of coquasitriangular bialgebras.

Recall [KS97, 10.3.1] that a coquasitriangular bialgebra $(A, r)$ over a field $\mathbb{K}$ is called regular if $r$ is convolution invertible as a $\mathbb{K}$-linear functional on the tensor product coalgebra $A \otimes A^{\text{cop}}$. In this case let $s$ denote the convolution inverse, and note that if $A$ is a coquasitriangular Hopf algebra then $s(a, b) = r(a, \sigma(b))$. The quantum double is a functor from the category of regular coquasitriangular bialgebras to the category of bialgebras. By definition [KS97, 8.2.1, 10.3.1] the quantum double $D(A, r)$ associated to the regular coquasitriangular bialgebra $(A, r)$ coincides with $A^{\text{cop}} \otimes A$ as a coalgebra. The multiplication of $D(A, r)$ is defined by

$$(b \otimes a)(b' \otimes a') = \sum r(a_{(1)}, b_{(2)})s(a_{(3)}, b_{(1)})b'_{(2)}b \otimes a_{(2)}a'$$

for $a, a', b, b' \in A$ (cp. [KS97, 8.2.1]). Hence the canonical linear embeddings $A^{\text{op, cop}} \hookrightarrow D(A, r)$, $a \mapsto a \otimes 1$ and $A \hookrightarrow D(A, r)$, $a \mapsto 1 \otimes a$ are bialgebra homomorphisms. Note that $A$ is a right $D(A, r)$-comodule by

$$(2.1) \quad a \mapsto a_{(2)} \otimes (a_{(1)} \otimes a_{(3)}) \quad \text{for all } a \in A.$$
A can be written in terms of the multiplication of a coquasitriangular bialgebra over $\mathbb{K}$. Transmutation defines a new algebra structure on the vector space $A$, which is a coquasitriangular bialgebra over $\mathbb{K}$.

Throughout this paper we will remain in the following proposition. Proposition-Definition 2.1. \cite[10.3.1, Proposition 30]{KS97} Let $(A, r)$ be a regular coquasitriangular bialgebra over $\mathbb{K}$ with unit 1. The vector space $A$ together with the multiplication $A \otimes A \to A$, $a \otimes b \mapsto a_r b$ defined by

\begin{align}
\theta(b \otimes a) := \sigma(b) a, \quad \text{for all } a, b \in A
\end{align}

defines a surjective morphism $\theta : D(A, r) \to A$ of bialgebras (cp. \cite[10.3.2]{KS97}).

Let $\mathcal{A}$ denote a coquasitriangular bialgebra (cp. \cite[10.3.1]{KS97}). The covariantized algebra $A_r$ is a right $D(A, r)$-comodule algebra with respect to the coaction $\theta$. The multiplication of $A$ can be written in terms of the multiplication of $A_r$ as

\begin{align}
ab = r {\left( a(1), b(1) \right) r(a(3), b(2)) a(2) r b(3)} & \tag{2.5} \\
= r {\left( b(2), a(1) \rangle r(b(3), a(3)) b(1) r a(2) \right)} & \tag{2.6}
\end{align}

for $a, b \in A$.

2.2. Framework. We now outline the general framework in which we will remain for the rest of this paper with the exception of Proposition-Definition 2.5 and Subsection 2.7. Throughout this paper $(\mathcal{A}, \Psi)$ denotes a pair consisting of a coquasitriangular bialgebra $(\mathcal{A}, r_{\mathcal{A}})$ and a fixed homomorphism $\Psi : (\mathcal{A}, r_{\mathcal{A}}) \to (k_q[G], r)$ of coquasitriangular bialgebras where $r$ is the universal $r$-form defined by \eqref{r-form}. Note that in this setting $r_{\mathcal{A}} = r \circ (\Psi \otimes \Psi)$. We will hence, by slight abuse of notation, suppress the subscript $\mathcal{A}$ and denote the universal $r$-form of $\mathcal{A}$ also by $r$. Similarly, we will write $(\mathcal{A}, r_{\mathcal{A}})$ instead of $(\mathcal{A}, r_{\mathcal{A}}) \circ (\Psi \otimes \Psi)$. Note that both $(\mathcal{A}, r)$ and $(\mathcal{A}, r_{\mathcal{A}})$ are regular coquasitriangular bialgebras and we can hence form the covariantized algebras $A_r$ and $A_{r_{\mathcal{A}}}$, as outlined in Proposition-Definition 2.1.

The morphism $\Psi$ allows us to transfer key structures of $k_q[G]$ to $\mathcal{A}$. These structures involve the Hopf pairing \eqref{Hopf-pairing} and the antipode of $k_q[G]$.

Lemma 2.2. The algebra $\mathcal{A}$ is a left $U^\text{cop} \otimes U$-module algebra with respect to the action defined by

\begin{equation}
(X \otimes Y) \cdot a = \langle \Psi(a_{(1)}), \sigma(X) \rangle \langle \Psi(a_{(3)}), Y \rangle a_{(2)}
\end{equation}

for $X, Y \in U$ and $a \in \mathcal{A}$. With respect to this action $\mathcal{A}$ is a direct sum of finite dimensional, simple, type one $U \otimes U$-modules.

Proof. The first claim is verified in the same way as one proves that \eqref{Hopf-pairing} defines a $U^\text{cop} \otimes U$-module algebra structure on $k_q[G]$. To verify the second statement observe that $\mathcal{A}$ is a locally finite $U \otimes U$-module since any element $a \in \mathcal{A}$ is contained in a finite dimensional subcoalgebra of $\mathcal{A}$ \cite[Theorem 2.2.1]{Swe69}. Moreover, $\mathcal{A}$ is a type one $U \otimes U$-module since $k_q[G]$ consists of the matrix coefficients of type one $U$-modules.
Recall that all through this paper we use the notation $r'$ to denote either $r$ as defined by (1.10) or $\tilde{r}$. In the next lemma we transfer the left $U$-module structure of $k_q[G]$ defined by (1.23) to the covariantized algebra $A_{r'}$.

**Lemma 2.3.** (i) The covariantized algebra $A_{r'}$ is a right $k_q[G]$-comodule algebra with the coaction $\delta : A_{r'} \rightarrow A_{r'} \otimes k_q[G]$ defined by
\[
\delta(a) = a_{(2)} \otimes \sigma(\Psi(a_{(1)}))\Psi(a_{(3)}).
\]

(ii) The covariantized algebra $A_{r'}$ is a left $U$-module algebra by
\[
\text{ad}^*_r(X)a := \Delta(X) \cdot a = \langle \sigma(\Psi(a_{(1)}))\Psi(a_{(3)}), X \rangle a_{(2)}
\]
for $X \in U$ and $a \in A$.

**Proof.** The bialgebra map $\Psi$ allows us the push the right $D(A, r'_A)$-comodule algebra structure on $A_{r'}$ to a right $D(k_q[G], r')$-comodule algebra structure
\[
a \mapsto a_{(2)} \otimes (\Psi(a_{(1)}) \otimes \Psi(a_{(3)}))
\]
on $A_{r'}$. Composing this map with the bialgebra homomorphism (2.2) for $A = k_q[G]$ one obtains the right $k_q[G]$-comodule algebra structure $\delta$ on $A_{r'}$. This proves (i) and (ii) follows immediately from the fact that (1.2) is a Hopf pairing. \hfill \Box

At the beginning of this subsection we agreed to write $r(a, b)$ for $a, b \in A$ instead of $r_A(a, b) = r(\Psi(a), \Psi(b))$. To simplify notations in later calculations, we now push this convention further and suppress the symbol $\Psi$ as far as possible.

**Convention:** We will allow elements of $A$ inside the arguments of the universal $r$-form $r$ for $k_q[G]$, the $l$-operators $l^+_r, l^-_r, l'_r, \tilde{l}'_r$, and inside the first entry of the pairing (1.2). Whenever an element $a \in A$ occurs in one of these places the homomorphism $\Psi$ has to be applied to $a$ first. Similarly, we write $\sigma(a)$ instead of $\sigma(\Psi(a))$ for any $a \in A$. For example, $\tilde{l}'_r(\sigma(a)b)$ with $a, b \in A$, should be read as $\tilde{l}'_r(\sigma(\Psi(a))\Psi(b))$.

It is important to remember that $A$ is only a bialgebra and not necessarily a Hopf algebra. Note that in our conventions the multiplication in the covariantized algebra $A_{r'}$ can be written as
\[
a r' b = r'(a_{(1)}, b_{(2)})r'(a_{(3)}, \sigma(b_{(1)}))a_{(2)}b_{(3)}
\]
\[
= r'(a_{(2)}, b_{(3)})r'(a_{(3)}, \sigma(b_{(1)}))b_{(2)}a_{(1)}
\]
for $a, b \in A$, while on the other hand,
\[
ab = r'(\sigma(a_{(1)}), b_{(1)})r'(a_{(3)}, b_{(2)})a_{(2)}b_{(3)}
\]
\[
= r'(b_{(2)}, a_{(1)})r'(\sigma(b_{(3)}), a_{(3)})b_{(1)}r'a_{(2)}
\]
for $a, b \in A$.

We end this subsection with a discussion of the image of the map $\Psi$. Note that $\Psi(A)$ is a subbialgebra of $k_q[G]$. As we couldn’t pinpoint the classification of subbialgebras of $k_q[G]$ in the literature we provide the following proposition for the convenience of the reader. For simplicity we assume that $g$ is a simple Lie algebra.

Recall that for any lattice $L \subset h^*$ with $Q \subseteq L \subseteq P$ there exists a uniquely determined affine algebraic group $G_L$ with Lie algebra $g$ and fundamental group $P/L$. [Hum73, Chapter XI]. We define $k_q[G_L]$ to be the subalgebra of $k_q[G]$ generated by all matrix coefficients of type one representations of $U$ with highest weight in $L$.

**Proposition 2.4.** Assume that $g$ is simple and let $B \neq k_l$ be a subbialgebra of $k_q[G]$. Then $B = k_q[G_L]$ for some lattice $L \subset h^*$ with $Q \subseteq L \subseteq P$. 


Proof. It follows from the fact that the coalgebra \( k_q[G] \) is cosemisimple that
\[
\mathcal{B} = \bigoplus_{\lambda \in M} C^V(\lambda)
\]
for some subset \( M \subseteq P^+ \). For any \( V \in \text{Ob}(\mathcal{C}) \) of dimension \( N \) the \( U \)-module \( V^{\otimes (N-1)} \) contains a copy of the dual representation \( V^* \). This implies that \( \mathcal{B} \) is a Hopf subalgebra of \( k_q[G] \). Hence all formulae involving \( R \) are twists for \( \mathcal{U} = \mathcal{U}(\mathfrak{g}) \). Since \( \mathcal{H} \) is a Hopf subalgebra of \( k_q[G] \). Hence the span of all matrix coefficients of \( \mathfrak{g} \)-modules with highest weight in \( M \) is a Hopf subalgebra of the coordinate ring \( \mathbb{C}[G] \) of the affine algebraic group \( G \). It now follows from the classification of semisimple affine algebraic groups [Hum75, Chapter XI] that \( M = L \cap P^+ \) for some lattice \( L \subseteq \mathfrak{h}^* \) with \( Q \subseteq L \subseteq P \). \( \square \)

2.3. Transmutation and Drinfeld twists. We give an alternative construction of the covariantized algebra \( \mathcal{A}_F \) using Drinfeld twists [Dri90]. While this is not indispensable for the understanding of this paper, it allows us to relate later results to the work of J. Donin and A. Mudrov (e.g. [DM03a], [DKM03], [Mud07]). For the convenience of the reader we recall the construction of twisted algebras in the following proposition. We denote any bialgebra \( H \) with unit \( 1_H \), multiplication \( \mu : H \otimes H \to H \), counit \( \varepsilon \), and coproduct \( \Delta \) by \( (H, 1_H, \mu, \varepsilon, \Delta) \). Moreover, we freely make use of the well known leg notation: For any unital algebra \( B \) and any \( i, j, n \in \mathbb{N} \) let \( \phi_{ij} : B \otimes B \to B^{\otimes n} \) denote the algebra homomorphism defined by \( \phi_{ij}(x \otimes y) = 1 \otimes \ldots \otimes x \otimes \ldots \otimes y \otimes \ldots \otimes 1 \) with \( x \) and \( y \) in the \( i \)-th and \( j \)-th position, respectively. For any element \( F \in B \otimes B \) we write \( F_{ij} := \phi_{ij}(F) \) with \( n \) being understood from the context.

**Proposition-Definition 2.5.** (i) [Dri90] Let \( (H, 1_H, \mu, \varepsilon, \Delta) \) be a bialgebra and let \( F \in H \otimes H \) be an invertible element such that
\[
(\Delta \otimes \text{Id})(F)_{12} = (\text{Id} \otimes \Delta)(F)_{23},
\]
\[
(\epsilon \otimes \text{Id})(F) = 1_H = (\text{Id} \otimes \epsilon)(F).
\]
For all \( x \in H \) define
\[
\Delta_F(x) = F^{-1} \Delta(x) F.
\]
Then \( H_F := (H, 1_H, \mu, \varepsilon, \Delta_F) \) is a bialgebra. We call \( F \) a twist for \( H \).

(ii) [DM03a], [DKM03] Proposition 7] Let \( H \) be a bialgebra and \( F \in H \otimes H \) a twist for \( H \). Let \( (A, 1_A, m) \) be a unital left \( H \)-module algebra with multiplication \( m : A \otimes A \to A \). Define a linear map
\[
m_F : A \otimes A \to A, \quad m_F(a \otimes b) = m(F(a \otimes b)).
\]
Then \( A_F := (A, 1_A, m_F) \) is an algebra. The left \( H \)-module structure on \( A \) turns \( A_F \) into a left \( H_F \)-module algebra. To simplify notation we define \( a_F b := m_F(a \otimes b) \).

(iii) [DM03a], [DKM03] Proposition 3] Let \( (\mathcal{U}, 1, \mu, \varepsilon, \Delta, \mathcal{R}) \) be a braided bialgebra with universal \( \mathcal{R} \)-matrix \( \mathcal{R} \) and \( H := \mathcal{U}^{\text{cop}} \otimes \mathcal{U} \) the product bialgebra. Then the two elements
\[
F := R_{13} R_{23} \in H \otimes H, \quad \overline{F} := R_{24}^{-1} R_{14}^{-1} \in H \otimes H
\]
are twists for \( H \).

We now return to the setting of Subsection 2.2. We want to apply the above proposition for \( \mathcal{U} = \mathcal{U} \) and \( A = \mathcal{A} \). Here \( \mathcal{R} \) is given formally via the braiding of \( \mathcal{C} \) as the family \( \mathcal{R} = \{ P_{12} \circ \hat{R}_{Y,W} : V \otimes W \to V \otimes W \mid V, W \in \text{Ob}(\mathcal{C}) \} \) of linear maps. Hence all formulae involving \( \mathcal{R} \) have to be interpreted as actions on suitable
tensor products of finite dimensional $U$-modules. Recall from Lemma 2.24 that $A$ is a $U^{\text{cop}} \otimes U$-module algebra. The algebras $(U^{\text{cop}} \otimes U)_F$ and $(U^{\text{cop}} \otimes U)_\overline{F}$ do not exist, but in view of the second half of Lemma 2.24 it is still possible to form the algebras $A_F$ and $A_{\overline{F}}$.

**Lemma 2.26.** The identity map of $A$ defines isomorphisms $A_r \simeq A_F$ and $A_{\overline{F}_r} \simeq (A_F)^{\text{cop}}$ of algebras.

**Proof.** By the definition of the universal r-form $r$ and by the explicit expressions (2.17) of $F$ and $\overline{F}$ we have

\begin{align}
(2.18) & \quad a_F b = r(a_{(3)}, \sigma(b_{(1)})) r(a_{(1)}, b_{(2)}) a_{(2)} b_{(3)}, \\
(2.19) & \quad a_{\overline{F}} b = r(\sigma(a_{(3)}), b_{(2)}) r(\sigma^2(a_{(1)}), b_{(3)}) a_{(2)} b_{(1)}
\end{align}

for $a, b \in A$. Comparing with (2.11) and (2.12) we conclude that $a_F b = a_r b$ and $b_{\overline{F}a} = a_{\overline{F}_r} b$ for all $a, b \in A$. \hfill \Box

**Remark 2.7.** In our setting the analogue of the $H_F$-module algebra structure from Proposition-Definition 2.3 (ii) is provided by the right $D(k_q[G], r)$-comodule algebra structure (2.10) of $A_r$.

2.4. **Transmutation of $k_q[G]$.** The first important class of examples for the general setting outlined in the Subsection 2.2 consists of the bialgebras $A := k_q[G_L]$ for a lattice $L \subset \mathfrak{h}^*$ with $Q \subseteq L \subseteq P$. In this case the map $\Psi$ is just the embedding. In this subsection we discuss the special case where $A = k_q[G]$ with universal r-form $r$. Recall from Lemma 2.23 (ii) that $k_q[G]_r$ is a left $U$-module algebra with respect to the action $\text{ad}_r^*\sigma$. The left locally finite part $F_1(U)$, on the other hand, is a left $U$-module algebra with respect to the left adjoint action $\text{ad}_l^*$.

**Proposition 2.8.** The l-functional $\tilde{l}_r^*: k_q[G]_{r'} \rightarrow F_1(U)$ defines an isomorphism of left $U$-module algebras.

**Proof.** We have already seen in Proposition 1.7 that $\tilde{l}_r^*: k_q[G] \rightarrow F_1(U)$ is an isomorphism of left $U$-modules. By [KS97, 10.1.3 (30)] the map $\tilde{l}_r^*: k_q[G]_{r'} \rightarrow F_1(U)$ is also an algebra homomorphism. \hfill \Box

**Remark 2.9.** It was pointed out in [Mud07] that in the case $A = k_q[G]$ the twisted algebra $A_F = A_r$ is a quantisation of the coordinate ring $\mathbb{C}[G]$ of the affine algebraic group $G$ with respect to the so called Semenov-Tian-Shansky Poisson bracket. We may hence view the locally finite part $F_1(U)$ as a quantisation of the coordinate ring of $G$ considered as a $G$-space under the action by conjugation.

Moreover, from this perspective the analogue of the Kostant-Richardson theorem obtained in [Mud07] in the h-adic setting corresponds to the separation of variables theorem obtained in [JL93] for $U$.

2.5. **FRT algebras and reflection equation algebras.** We now discuss the second important class of examples for the general setting outlined in Subsection 2.2. For any $V \in \text{Ob}(\mathcal{C})$ the braiding of $\mathcal{C}$ defined in Subsection 1.1 provides a linear automorphism $R_{V,V} := P_{12} \circ \tilde{R}_{V,V}$ of $V \otimes V$, where $P_{12}$ denotes the flip of tensor factors. This automorphism satisfies the quantum Yang-Baxter equation and hence can be used to perform the FRT-construction [FRT89, KS97 9.1]. One obtains a coquasitriangular bialgebra $A(R_{V,V})$ which is generated as an algebra by the linear space $V^* \otimes V$. In this case the map $\Psi$ is induced by the canonical map $f \otimes v \mapsto c_{f,v}$ of $V^* \otimes V$ onto the linear space of matrix coefficients $C^V \subset k_q[G]$. 

**Remark 2.10.**
To make this construction more explicit assume \( \dim(V) = N \) and choose a basis \( \{v_1, \ldots, v_N\} \) of \( V \) with dual basis \( \{f_1, \ldots, f_N\} \). We denote the generator of \( \mathcal{A}(R_{V,V}) \) corresponding to \( f_i \otimes v_j \) by \( t_{ij} \). Let \( R \) denote the \((N^2 \times N^2)\)-matrix corresponding to \( R_{V,V} \) with respect to the chosen bases and consider the generators \( t_{ij} \) as entries of an \((N \times N)\)-matrix \( T \). Then the defining relations of \( \mathcal{A}(R_{V,V}) \) may be written as

\[
RT_1T_2 = T_2T_1R
\]

(2.20)

where \( T_1 = T \otimes \text{id}, \ T_2 = \text{id} \otimes T \) are \((N^2 \times N^2)\)-matrices with entries in \( \mathcal{A}(R_{V,V}) \).

It is well known [KS97, 9.1.1, 10.1.2] that \( \mathcal{A}(R_{V,V}) \) has the structure of a coquasitriangular bialgebra with coproduct

\[
\Delta(t_{ij}) = \sum_{m=1}^{N} t_{im}^i \otimes t_{jm}^m
\]

and universal \( r \)-form defined by \( r_{\mathcal{A}(R_{V,V})}(t_{ij}^i, t_{jm}^m) = r(c_{f_i,v_j}, c_{f_m,v_n}) \). In this case the homomorphism of coquasitriangular bialgebras \( \Psi : \mathcal{A}(R_{V,V}) \to k_q[G] \) is given by \( \Psi(t_{ij}) = c_{f_i,v_j} \).

**Proposition 2.10.** [Man95, p. 334], [KS97, 10.3.1, Example 18], [DM03a, Prop. 4.11]

In the above setup, the covariantized algebra \( \mathcal{A}(R_{V,V})^r \) is isomorphic to the unital \( k \)-algebra with generators \( s_{ij}^j \), for \( 1 \leq i, j \leq N \), and relations

\[
S^1_{2R_2R_1}S_1R_{21} = R_{21}S_1R_{21}S_2.
\]

(2.21)

where \( S \) is the \( N \times N \) matrix with entries \( s_{ij}^j \) and \( R' = R \) if \( r' = r \) while \( R' = R_{21}^{-1} \) if \( r' = \bar{r}_{21} \).

**Proof.** By (2.13) and (2.14), the generators \( s_{ij}^j := t_{ij}^j \) of \( \mathcal{A}(R_{V,V})^r \) satisfy the relations (2.21). The fact that (2.21) are the defining relations for \( \mathcal{A}(R_{V,V})^r \) in terms of the algebraic generators \( s_{ij}^j := t_{ij}^j \) follows along the lines of say [Mud07, Appendix A]. \( \square \)

The covariantized algebra \( \mathcal{A}(R_{V,V})^r \) is called the reflection equation algebra associated to \( R' \). It is also referred to as the braided matrix algebra associated to \( R' \).

### 2.6. Quantum adjoint orbits.

For any unital \( k \)-algebra \( B \) we denote by \( B^\wedge \) the set of characters of \( B \). More explicitly, \( B^\wedge \) is the set of nonzero algebra homomorphisms \( f : B \to k \). Note in particular that a character \( f : B \to k \) automatically preserves the units.

For any character \( f \in \mathcal{A}^\wedge \), the coaction \( \delta \) defined by (2.8) allows us to construct an algebra homomorphism

\[
\delta_f : \mathcal{A}^r \to k_q[G], \quad \delta_f(a) := (f \otimes \text{id})\delta(a).
\]

We define

\[
k_q[\mathcal{O}_f] := \delta_f(\mathcal{A}^r)
\]

and call \( k_q[\mathcal{O}_f] \) the quantum adjoint orbit associated to the character \( f \).

**Lemma 2.11.** For any character \( f \in \mathcal{A}^\wedge \), the following hold:

1. \( k_q[\mathcal{O}_f] \) is a right coideal subalgebra of \( k_q[G] \).
2. \( k_q[\mathcal{O}_f] = \{ \sigma(\Psi(a_{(1)}))f(a_{(2)})\Psi(a_{(3)}) \mid a \in \mathcal{A} \} \)
Proof. The first statement of the lemma holds because $\delta_f : \mathcal{A}_{\mathbf{r}'} \rightarrow k_q[G]$ is a homomorphism of right $k_q[G]$-comodule algebras. The second statement holds by definition of $\delta_f$. □

Remark 2.12. Recall from Remark 2.9 that if $\mathcal{A} = k_q[G]$ then $\mathcal{A}_{\mathbf{r}'}$ is a quantum analogue of the algebra of functions on $G$ considered as a $G$-space with respect to the adjoint action. Hence in this case $k_q[G]$ is a quantum analogue of the algebra of functions on the conjugacy class of the classical point corresponding to the character $f \in \mathcal{A}$. If, on the other hand, $\mathcal{A} = \mathcal{A}(R_{V,V})$ is an FRT algebra as in Subsection 2.3 then $k_q[G]$ is a quantum analogue of the functions on the orbit of an element in $\text{End}(V)$ under conjugation by $G$.

Remark 2.13. Quantum adjoint orbits for characters of the reflection equation algebra were defined and investigated in [DM03a]. In that paper, for $G = SL(N)$ and the vector representation, these algebras are given explicitly in terms of generators and relations based on the classification in [Mud92]. Moreover, their semiclassical limits are determined.

2.7. Universal cylinder forms. In this subsection we give another motivation for the investigation and construction of characters of covariantized algebras. The notion of a universal cylinder form for a coquasitriangular bialgebra $(\mathcal{A}, \mathbf{r}_A)$ was introduced in [tDHO98]. It appeared as a necessary ingredient in order to find representations of the braid group of type $B_n$ coming from quantum groups. Examples were constructed in [tDHO98] for $A = k_q[SL(2)]$ and in [tD99] for $A = k_q[SL(N)]$.

We recall the main definition from [tDHO98, (1.1)].

Definition 2.14. Let $(\mathcal{A}, \mathbf{r}_A)$ be a coquasitriangular bialgebra over a field $\mathbb{K}$. A linear functional $f : \mathcal{A} \rightarrow \mathbb{K}$ is called a universal cylinder form for $(\mathcal{A}, \mathbf{r}_A)$ if it is convolution invertible and satisfies

$$f(ab) = f(a(1))\mathbf{r}_A(b(1), a(2))f(b(2))\mathbf{r}_A(a(3), b(3))$$

$$= r_A(b(1), a(1))f(b(2))r_A(a(2), b(3))f(a(3))$$

for all $a, b \in \mathcal{A}$. We denote by $\text{CF}(\mathcal{A}, \mathbf{r}_A)$ the set of universal cylinder forms for $(\mathcal{A}, \mathbf{r}_A)$.

We will explain in subsection 3.5 that in our setting a universal cylinder form is the same as a character of a covariantized algebra. In view of the similarity of Relations (2.13), (2.14) and Relations (2.22), (2.23), this is nearly obvious, however, conventions in [tDHO98] slightly differ from ours.

3. Algebraic properties of covariantized algebras

We now begin the general investigation of the covariantized algebra $\mathcal{A}_{\mathbf{r}'}$ in the framework outlined in Subsection 2.2. In this section we collect algebraic properties of $\mathcal{A}_{\mathbf{r}'}$ which are obtained via the $\mathcal{U}_{\text{cop}} \otimes \mathcal{U}$-module algebra structure (2.7) of $\mathcal{A}$ and the $\mathcal{U}$-module algebra structure $\mathbf{a}^*_\mathcal{U}$ of $\mathcal{A}_{\mathbf{r}'}$ defined in (2.9). In particular, we obtain results about zero divisors in $\mathcal{A}_{\mathbf{r}'}$ and relate the centre of $\mathcal{A}_{\mathbf{r}'}$ to the space of $\mathbf{a}^*_\mathcal{U}$-invariant elements. Moreover, we collect properties of characters of covariantized algebras and establish the relation to universal cylinder forms.
3.1. **Zero divisors.** In this subsection we show that the covariantized algebra \( \mathcal{A}_r \) is a domain if and only if the algebra \( \mathcal{A} \) is a domain. We start with the following convenient way to express the multiplication \((2.11)−(2.14)\) of \( \mathcal{A}_r \) in terms of the adjoint action \((2.9)\) of the \( l^+ \)-functionals \( l_r^+ \) on \( \mathcal{A} \).

**Lemma 3.1.** For all \( a, b \in \mathcal{A} \) the following relations hold:

\[
\begin{align*}
(3.1) \quad a_r b &= (\text{ad}^*_{\bar{r}}[l_r^+(\sigma(b_{(1)}))] a) b_{(2)} = (\text{ad}^*_{\bar{r}}[l_r^-(\sigma^{-1}(a_{(2)}))] b) a_{(1)}, \\
(3.2) \quad ab &= b_{(1)} \bar{r}’ (\text{ad}^*_{\bar{r}}[l_r^+(b_{(2)}))] a) = (\text{ad}^*_{\bar{r}}[l_r^+(b_{(1)}))] a) \bar{r} r b_{(2)}.
\end{align*}
\]

Recall that a weight vector \( v \) in a \( U \)-module is called a highest weight vector, if \( X v = 0 \) for all \( X \in U^+ \). For highest weight vectors with respect to the action \((2.9)\) the gauged multiplication takes a very simple form.

**Lemma 3.2.** Let \( a \in \mathcal{A} \) be a highest weight vector of weight \( \lambda \in P^+ \) with respect to the action \( \text{ad}^*_r \). Then the following relations hold

\[
\begin{align*}
a_r b &= a[(\tau(-\lambda) \otimes 1) \cdot b], \\
b_{\mathbb{F}_2} a &= a[(1 \otimes \tau(\lambda)) \cdot b]
\end{align*}
\]

for all \( b \in \mathcal{A} \).

**Proof.** It follows from \((1.22)\) that

\[
\text{ad}^*_r(l_r^+(b)) a = \langle b, \tau(-\lambda) \rangle a
\]

for all \( b \in \mathcal{A} \). If \( \bar{r}’ = r \), then \((3.1)\) implies

\[
a_r b = (\text{ad}^*_r[l_r^+(\sigma(b_{(1)}))] a) b_{(2)} = a\langle \sigma(b_{(1)}), \tau(-\lambda) \rangle b_{(2)} = a[(\tau(-\lambda) \otimes 1) \cdot b].
\]

If \( \bar{r}’ = \mathfrak{r}_{21} \), then \((1.18)\) and \((3.1)\) give

\[
b_{\mathfrak{r}_{21}} a = (\text{ad}^*_r[l_r^+(\sigma^{-1}(b_{(2)}))] a) b_{(1)} = a\langle \sigma^{-1}(b_{(2)}), \tau(-\lambda) \rangle b_{(1)} = a[(1 \otimes \tau(\lambda)) \cdot b].
\]

\(\Box\)

**Proposition 3.3.** The algebra \( \mathcal{A} \) is a domain if and only if \( \mathcal{A}_r \) is a domain.

**Proof.** Suppose that \( \mathcal{A}_r \) has zero divisors. Recall that \( \mathcal{A}_r \) is a type one, locally finite \( U \)-module algebra with respect to the action \( \text{ad}^*_r \). Hence there exist \( \text{ad}^*_r(U) \) highest weight vectors \( a, b \in \mathcal{A}_r \) satisfying \( a_r b = 0 \) (cp. the argument given in the proof of [Jos95, 9.1.9(1)]). By Lemma 3.2 we conclude that \( a \) or \( b \) is a zero divisor in \( \mathcal{A} \).

Conversely, suppose that \( \mathcal{A} \) has zero divisors. Considering this time \( \mathcal{A} \) as the type one, locally finite \( U^{\text{op}} \otimes U \)-module algebra with respect to the action \((2.7)\), there exist \( U \otimes U \) highest weight vectors \( a, b \in \mathcal{A} \) satisfying \( ab = 0 \). Note that in this case \( (\tau(-\lambda) \otimes 1) \cdot b \) and \( (1 \otimes \tau(\lambda)) \cdot b \) are nonzero scalar multiples of \( b \) for any \( \lambda \in P \). Moreover, \( a \) and \( b \) are also highest weight vectors for the action \( \text{ad}^*_r \) and hence Lemma 3.2 implies that \( a_r b = 0 \) and \( b_{\mathbb{F}_2} a = 0 \) \(\Box\).

3.2. **Invariants and centres.** For any left \( U \)-module \( \mathcal{B} \) define the space of invariants \( \mathcal{B}^{\text{inv}} \) by

\[
\mathcal{B}^{\text{inv}} := \{ b \in \mathcal{B} \mid X b = \varepsilon(X) b \text{ for all } X \in U \}.
\]

Note that if \( \mathcal{B} \) is a \( U \)-module algebra, then \( \mathcal{B}^{\text{inv}} \subseteq \mathcal{B} \) is a subalgebra. We remain in the setting of Subsection 2.2. In the present subsection we describe properties...
of the subspace \( A^{\text{inv}} \) of invariants of \( A \) with respect to the \( U \)-action \( \text{ad}^*_\rho \) defined by (2.4). Note first that in terms of the coaction \( \delta \) defined by (2.8) one has

\[
A^{\text{inv}} = \{ a \in A \mid \delta(a) = a \otimes 1 \}.
\]

As \( A \) and the covariantized algebra \( A_r \) coincide as vector spaces, we may consider \( A^{\text{inv}} \) alternatively as a subspace of \( A \) or \( A_r \). By Lemma 2.3(ii) \( A_r \) is a \( U \)-module algebra with respect to the action \( \text{ad}^*_\rho \), and hence \( A^{\text{inv}} \subseteq A_r \) is a subalgebra. In the following proposition and in Proposition 3.6 we relate \( A^{\text{inv}} \) to the centre \( Z(A_r) \) of the covariantized algebra.

**Proposition 3.4.** The following relations hold:

1. \( a_r b = ab = b_r a \) for \( a \in A^{\text{inv}} \) and \( b \in A \).
2. \( A^{\text{inv}} \subseteq Z(A_r) \).
3. \( A^{\text{inv}} \) is a commutative subalgebra of \( A \).
4. \( A^{\text{inv}} = \{ a \in A \mid (1 \otimes X) \cdot a = (\sigma(X) \otimes 1) \cdot a \text{ for all } X \in U \} \).
5. For any \( a \in A^{\text{inv}} \) and \( X, Y \in U \) one has
   \[
a_{(1)} \otimes ((X \otimes Y) \cdot a_{(2)}) = ((\sigma(Y) \otimes \sigma(X)) \cdot a_{(1)}) \otimes a_{(2)}
   \]
   and in particular, \( a_{(1)} \otimes \text{ad}^*_\rho(X)a_{(2)} = (\text{ad}^*_\rho(\sigma(X))a_{(1)}) \otimes a_{(2)} \).
6. \( A^{\text{inv}} = \{ a \in A \mid a_{(1)} \otimes \Psi(a_{(2)}) = a_{(2)} \otimes \sigma^2(\Psi(a_{(1)})) \} \).

**Proof.** Property (4) follows from Lemma 3.4 and the fact that \( \varepsilon(l_\rho^\pm(a)) = \varepsilon(a) \) for all \( a \in A \). Properties (2) and (3) follow immediately from property (4).

To verify (4) note that by definition of the action \( \text{ad}^*_\rho \) the set on the right-hand side of (4) is contained in \( A^{\text{inv}} \). To obtain the converse inclusion one verifies for any \( a \in A^{\text{inv}} \) the relation

\[
(1 \otimes X) \cdot a = (\sigma(X_{(1)}) \otimes 1)(X_{(2)} \otimes X_{(3)}) \cdot a = (\sigma(X) \otimes 1) \cdot a.
\]

This completes the proof of (4) which also implies property (6) via the non-degenerate Hopf pairing \( \langle \cdot, \cdot \rangle \) between \( U \) and \( k_q[G] \).

It remains to prove (5). Let \( a \in A \) and \( X \in U \). Then

\[
a_{(1)} \otimes ((X \otimes 1) \cdot a_{(2)}) = (\sigma(a_{(2)}), X)a_{(1)} \otimes a_{(3)} = ((1 \otimes \sigma(X)) \cdot a_{(1)}) \otimes a_{(2)}.
\]

Hence it suffices to prove (5.3) for \( X = 1 \). Let \( a \in A^{\text{inv}} \) and \( Y \in U \), then the desired formula follows by applying the comultiplication \( \Delta \) of \( A \) to the identity \( (1 \otimes Y) \cdot a = (\sigma(Y) \otimes 1) \cdot a \) from (4). \( \square \)

**Example 3.5.** For any \( V \in \text{Ob}(C) \) define a quantum trace \( \text{tr}_q,V \in \text{CV} \) by

\[
\text{tr}_q,V(X) := \text{tr}_V(X\tau(2\rho))
\]

where \( \text{tr}_V \) denotes the trace on the linear space \( V \) and the argument \( X\tau(2\rho) \) is considered as an endomorphism of \( V \). It follows from (1.1) that \( \text{tr}_q,V \in k_q[G]^{\text{inv}} \) where \( k_q[G] \) is a left \( U \)-module via the action \( \text{ad}^*_\rho \) defined by (1.23). The Peter-Weyl decomposition (1.4) and the fact that \( \text{CV}(\lambda) \cong V(\lambda)^* \otimes V(\lambda) \) as left \( U \)-modules imply that the quantum traces \( \{ c_\lambda := \text{tr}_q,V(\lambda) \mid \lambda \in P^+ \} \) form a linear basis of \( k_q[G]^{\text{inv}} \).

One may ask if the inclusions in Proposition 3.4(2) are equalities. This does indeed hold if one assumes \( A \) to be a domain.

**Proposition 3.6.** If \( A \) is a domain then \( A^{\text{inv}} = Z(A_r) \).
In view of Proposition 3.3 and Proposition 3.4(2) the identity $A^{\text{inv}} = Z(A_r)$ is verified by applying the following lemma to the locally finite, type one, left $U$-module algebra $A_r$.

**Lemma 3.7.** Let $B$ be a locally finite, type one, left $U$-module algebra and assume that $b^2 \neq 0$ for all nonzero elements $b \in B$. Then $Z(B) \subseteq B^{\text{inv}}$.

**Proof.** Without loss of generality we may assume that $U = U_q(\mathfrak{sl}_2(C))$. During this proof, to simplify notation, we write $x$, $y$, and $t$ instead of $x_1$, $y_1$, and $t_1 = \tau(2\alpha_1)$, respectively. As $B^{\text{inv}}$ is the isotypical component of $B$ corresponding to the trivial representation of $U$ one has

$$B^{\text{inv}} \cap xB = \{0\}.$$ (3.4)

Consider now a central element $a \in Z(B)$. As all weight components of $a$ are themselves central we may assume that $ta = ca$ for some $c \in \{q^m \mid m \in \mathbb{Z}\}$. Choose $n \in \mathbb{N}_0$ minimal such that $x^n a \neq 0$ and $x^{n+1} a = 0$. We want to show that $n = 0$. Assume on the contrary that $n \neq 0$. The centrality of $a$ implies

$$a(x^n a) = (x^n a)a.$$ (3.5)

Acting by $x^n$ from the left and using $\Delta(x) = x \otimes 1 + t \otimes x$ one obtains

$$(x^n a)(x^n a) = (q^{2n} c^n)(x^n a)(x^n a).$$

By assumption the nonzero element $b := x^n a$ satisfies $b^2 \neq 0$ and hence $c = q^{-2n}$ since $n \neq 0$. One thus obtains $tx^n a = x^n a$ which, together with $x^{n+1} a = 0$ and the locally finiteness of $B$ implies $y(x^n a) = 0$. Hence $x^n a \in B^{\text{inv}} \setminus \{0\}$ which by (3.4) is a contradiction to $n \neq 0$. In the same way one shows $ya = 0$ and hence $c = 1$ since $B$ is a locally finite, type one $U$-module. This implies $a \in B^{\text{inv}}$. \hfill \Box

Observe that in the setting of Lemma 3.7 we do not necessarily have the inclusion $B^{\text{inv}} \subseteq Z(B)$.

**Example 3.8.** By Proposition 2.8 one has an isomorphism $\hat{I}_r : k_q[G]^r \rightarrow F_l(U)$ of $U$-module algebras. Hence Proposition 3.4(2) for $A = k_q[G]$ implies the well known fact that $Z(F_l(U)) = F_l(U)^{\text{inv}} = \hat{I}_r(k_q[G]^r)^{\text{inv}}$.

**Example 3.9.** Let $V$ be the vector representation of $U_q(\mathfrak{sl}_n)$ and let $A(R_{V,V})$ be the corresponding FRT algebra considered in subsection 2.2. It is well known that $A(R_{V,V})$ is a domain. Hence Proposition 3.7 implies $Z(A(R_{V,V})^r) = A(R_{V,V})^{\text{inv}}$ for the corresponding reflection equation algebra $A(R_{V,V})^r$.

### 3.3. The comultiplication as an algebra homomorphism of $A^{\text{inv}}$.

The following proposition links the invariant subalgebra $A^{\text{inv}}$ in a different way with the covariantized algebras $A_r$ and $A_{r_1}$. Let $\mu := \Delta|_{A^{\text{inv}}}$ denote the restriction of the comultiplication $\Delta$ to $A^{\text{inv}}$. We identify its range with the product algebra $A_r \otimes A_{r_1}^{\text{op}}$.

**Proposition 3.10.** The map $\mu := \Delta|_{A^{\text{inv}}}$ defines an injective algebra homomorphism

$$\mu : A^{\text{inv}} \rightarrow A_r \otimes A_{r_1}^{\text{op}}.$$ (3.5)

**Proof.** Assume that $a, b \in A^{\text{inv}}$. By Proposition 3.4(3) we have

$$(b_{(1)} \otimes b_{(2)} \otimes \Psi(b_{(3)}) = b_{(2)} \otimes b_{(3)} \otimes a^2(\Psi(b_{(1)}))).$$
Using Lemma 3.1 one now calculates
\[ a_{(1)} r_{(1)} b_{(1)} \otimes b_{(2)F_2}, a_{(2)} \]
\[ = (ad^*_r [l^+_r (\sigma(b_{(1)}))] a_{(1)}) b_{(2)} \otimes b_{(3)F_2}, a_{(2)} \]
\[ = a_{(1)} b_{(1)} \otimes b_{(2)F_2}, (ad^*_r [l^+_r (b_{(3)}))] a_{(2)}) \]
\[ = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \]
where we used Proposition 3.4(5) for the third equality and relations (3.7) and (3.2) to obtain the last equality. This completes the proof of the proposition. □

Remark 3.11. The theory of transmutation as introduced by S. Majid includes the construction of a new algebra structure on \( \mathcal{A}_{\psi}^{\otimes 2} \) which turns the comultiplication map \( \Delta \) of \( \mathcal{A} \) into and algebra homomorphism \( \Delta : \mathcal{A}_{\psi} \to \mathcal{A}_{\psi}^{\otimes 2} \). In Majid’s terminology this turns \( \mathcal{A}_{\psi} \) into a so-called braided bialgebra (not to be confused with the notion of a quasitriangular bialgebra). The algebra homomorphism in Proposition 3.10 bears no immediate relation to the braided bialgebra structure on \( \mathcal{A}_{\psi} \).

3.4. Elementary properties of characters. To simplify later considerations on characters of the covariantized algebra \( \mathcal{A}_{\psi} \), we first collect some general properties of algebra homomorphisms from \( \mathcal{A}_{\psi} \) to an arbitrary algebra.

Lemma 3.12. Let \( \mathcal{B} \) be a \( k \)-algebra and \( f : \mathcal{A} \to \mathcal{B} \) a linear map.

(i) The following are equivalent:
   (1) The map \( f : \mathcal{A}_{\psi} \to \mathcal{B} \) is an algebra homomorphism.
   (2) For all \( a, b \in \mathcal{A} \) one has
   \[ f(ab) = r^\sigma(a_{(1)}), b_{(1)}) r^\sigma(a_{(3)}, b_{(2)}) f(a_{(2)}) f(b_{(3)}) \]
   (3) For all \( a, b \in \mathcal{A} \) one has
   \[ f(ab) = r^\sigma(b_{(2)}, a_{(1)}) r^\sigma(b_{(3)}, a_{(3)}) f(b_{(1)}) f(a_{(2)}) \]

(ii) If \( f : \mathcal{A}_{\psi} \to \mathcal{B} \) is an algebra homomorphism then the relations
   \[ f(a) f(b) = \langle \sigma(a_{(1)}) a_{(3)}, l^\sigma_r (b_{(3)}) \sigma^{-1}(l^\sigma_r (b_{(1)})) \rangle f(b_{(2)}) f(a_{(2)}) \]
   \[ = \langle \sigma(b_{(1)}) b_{(3)}, l^\sigma_r (a_{(3)}) \sigma^{-1}(l^\sigma_r (a_{(1)})) \rangle f(b_{(2)}) f(a_{(2)}) \]
hold for all \( a, b \in \mathcal{A} \).

Proof. (i) Using properties (1.6), (1.7), and (1.9) of the universal \( r^\sigma \) one verifies that condition (2) is equivalent to the relation
\[ r^\sigma(a_{(1)}), b_{(2)}) r^\sigma(a_{(3)}, \sigma^{-1}(l^\sigma_r (b_{(1)}))) f(a_{(2)}) b_{(3)} = f(a) f(b) \]
for all \( a, b \in \mathcal{A} \). In view of (2.11) this relation is equivalent to \( f : \mathcal{A}_{\psi} \to \mathcal{B} \) being an algebra homomorphism. The equivalence of properties (1) and (3) follows in the same way from (2.12).

(ii) Note that (2.11) can be rewritten as
\[ a_{\psi} b = \langle \sigma(a_{(1)}) a_{(3)}, \sigma^{-1}(l^\sigma_r (b_{(1)}))) a_{(2)} b_{(2)} \]
and (3.7) as
\[ f(ab) = \langle \sigma(a_{(1)}) a_{(3)}, l^\sigma_r (b_{(2)}) \rangle f(b_{(1)}) f(a_{(2)}) \].
Combining these expressions we get for \(a, b \in \mathcal{A}\) the relation
\[
f(a)f(b) = f(a \cdot b) = (\sigma(a_{(1)})a_{(3)}, l_{\mathcal{K}}(b_{(3)})\sigma^{-1}(l_{\mathcal{K}}^{-1}(b_{(1)})))f(b_{(2)})f(a_{(2)}),
\]
which coincides with (3.8). Formula (3.9) is verified in a similar manner, now using (2.12) and (3.10) instead of (2.11) and (3.8).

Recall that we denote the set of characters of a unital \(k\)-algebra \(B\) by \(B^\wedge\). For any two linear functionals \(f, g\) on a coalgebra \(D\) we denote their convolution product by \(f \ast g\). Recall that the convolution product turns the linear dual space \(D^*\) into a unital algebra. In the following lemma we view characters of \(\mathcal{A}^\vee\) as linear functionals on the coalgebra \(\mathcal{A}\).

**Lemma 3.13.** (i) Suppose \(f \in \mathcal{A}^*\) is convolution invertible with convolution inverse \(\tilde{f} \in \mathcal{A}^*\). Then \(f \in \mathcal{A}_C^\vee\) if and only if \(\tilde{f} \in \mathcal{A}_C^\wedge\).

(ii) If \(\mathcal{A}\) is a Hopf algebra then any \(f \in \mathcal{A}_C^\vee\) is convolution invertible with convolution inverse \(\tilde{f}\) satisfying
\[
\tilde{f}(a) = r'(\sigma(a_{(1)}), a_{(3)})f(\sigma(a_{(2)}))r'(\sigma^2(a_{(3)}), a_{(4)})
= r'(\sigma^2(a_{(2)}), a_{(3)})f(\sigma^{-1}(a_{(4)}))r'(\sigma(a_{(1)}), a_{(5)}) \quad \text{for all } a \in \mathcal{A}.
\]

**Proof.** (i) Let \(f \in \mathcal{A}^*\) be convolution invertible with convolution inverse \(\tilde{f}\). By Lemma 3.12 (i) the functional \(f\) is a character of \(\mathcal{A}_r\) if and only if \(f(1) = 1\) and
\[
(3.10) \quad f \circ m = \tilde{f} \ast (f \otimes \varepsilon) \ast r \ast (\varepsilon \otimes f)
\]
where \(m\) denotes the multiplication of \(\mathcal{A}\) and \(\ast\) is the convolution product of \((\mathcal{A} \otimes \mathcal{A})^*\). Note that \(f\) satisfies (3.10) if and only if \(\tilde{f}\) satisfies the relation
\[
\tilde{f} \circ m = (\varepsilon \otimes \tilde{f}) \ast \tilde{r} \ast (\tilde{f} \otimes \varepsilon) \ast r.
\]
The latter relation is equivalent to
\[
(3.11) \quad \tilde{f}(ab) = \tilde{f}(b_{(1)})\mathcal{F}_{21}(b_{(2)}, a_{(1)})\tilde{f}(a_{(2)})\mathcal{F}_{21}(\sigma(b_{(3)}), a_{(3)})
\]
for all \(a, b \in \mathcal{A}\), which again by Lemma 3.12 (i) holds if and only if \(\tilde{f} : \mathcal{A}_{F_{21}} \to k\) is an algebra homomorphism. Moreover, \(f(1) = 1\) if and only if \(\tilde{f}(1) = 1\).

(ii) If \(\mathcal{A}\) is a Hopf algebra and \(f \in \mathcal{A}_C^\vee\) then we can define two functionals \(\tilde{f}_r\) and \(\tilde{f}_l\) on \(\mathcal{A}\) by
\[
\tilde{f}_r(a) = r'(\sigma^2(a_{(2)}), a_{(3)})f(\sigma^{-1}(a_{(4)}))r'(\sigma(a_{(1)}), a_{(5)}),
\]
\[
\tilde{f}_l(a) = r'(\sigma(a_{(1)}), a_{(3)})f(\sigma(a_{(2)}))r'(\sigma^2(a_{(3)}), a_{(4)})
\]
for all \(a \in \mathcal{A}\). Applying (3.7) to \(f(\sigma^{-1}(a_{(2)})a_{(1)})\) one obtains that \(\tilde{f}_r\) is a right convolution inverse of \(f\). Similarly, applying (3.6) to \(f(\sigma(a_{(1)})a_{(2)})\) one obtains that \(\tilde{f}_l\) is a left convolution inverse of \(f\). Hence \(f\) is convolution invertible with inverse \(\tilde{f} = f_r = f_l\). 

### 3.5. Characters and universal cylinder forms.

As an application of Lemma 3.12 (i) we now explain that in our setting a universal cylinder form is the same as a character of a covariantized bialgebra. Recall the notion of a universal cylinder form defined in subsection 2.4. Restrict again to the case where \(K = k\) and \((\mathcal{A}, r_{\mathcal{A}})\) is of the form considered in Subsection 2.2. To obtain the desired identification we need the linear twist functional \(u : k_q[G] \to k\) defined by
\[
u(c) := q^{-\langle k + 2\rho, l \rangle} \varepsilon(c)
\]
for all $c \in C^V(\mu)$, $\mu \in P^+$. Note that by construction

\[(3.12)\quad c(1)u(c(2)) = u(c(1))c(2)\]

for all $c \in k_q[G]$. Moreover, it is a well known analogue of [KS97, 8.4.3, Proposition 22] that in our setting the relation

\[(3.13)\quad r(d(1), c(1))r(c(2), d(2))u(c(3)d(3)) = u(c)u(d)\]

holds for all $c, d \in k_q[G]$. Following the general conventions of Subsection 2.2 we consider $u$ as a functional on $A$ and again suppress the homomorphism $\Psi$ in our notation. We are now able to formulate the desired identification. Let $A^\times_r$ denote the set of convolution invertible characters on $A_r$. Note that $(A^\times_r, r^21)$ is a coquasitriangular bialgebra.

**Proposition 3.14.** The map

\[(3.14)\quad CF(A^{op}, r^21) \to A^\times_r, \quad f \mapsto f \ast u\]

is a bijection.

**Proof.** Note first that $u$ is convolution invertible with inverse $\bar{u}$ defined by

\[\bar{u}(c) := q^{[\mu+2\rho, \mu]}\xi(c) \quad \text{for all } c \in C^V(\mu), \mu \in P^+.\]

Given $f \in CF(A^{op}, r^21)$ define $g := f \ast u$ and note that $g$ is convolution invertible because both $f$ and $u$ are. Using (2.22) and (3.13) one calculates

\[g(ab) = f(a(1)b(1))u(a(2)b(2))\]
\[= f(b(1))r_{21}(a(1), b(2))f(a(2))r_{21}(b(3), a(3))\]
\[\times r(\sigma(a(4)), b(4))r(\sigma(b(5)), a(5))u(a(6))u(b(6))\]
\[= f(b(1))u(b(2))r(b(3), a(1))f(a(2))u(a(3))r(\sigma(b(4)), a(4))\]
\[= g(b(1))r(b(2), a(1))g(a(2))r(\sigma(b(3)), a(3)).\]

Hence Lemma 3.12(i) implies $g \in A^\times_r$. One checks analogously to the above calculation that the map $A^\times_r \to CF(A^{op}, r^21)$, $g \mapsto g \ast \bar{u}$ is well defined. Hence (3.14) is indeed a bijection.\[\square\]

4. Noumi coideal subalgebras of $U$

The essential ingredient in Noumi’s construction of quantum symmetric pairs (e.g. [Nou96, NS95, Dj96, NDS97, DS99]) is a solution of the reflection equation. We have seen in Subsection 2.5 that a solution of the reflection equation is the same as a character of the reflection equation algebra, which in turn is obtained from the FRT algebra via transmutation. In this section we formulate a generalised Noumi type construction of coideal subalgebras of $U$ in terms of characters of the covariantized algebra $A_r^\times$. 

4.1. U-comodule algebra structure on $A_r$. The left locally finite part $F_1(U)$ is a left coideal subalgebra of $U$. Using the $l$-functionals from Subsection 1.3 and Proposition 1.7 the coproduct of elements in $F_1(U)$ for all $a \in k_q[G]$ 

\begin{equation}
\Delta(l_r^+(a)) = l_r^+(a(1)) \sigma(l_r^-(a(3))) \otimes l_r^+(a(2))
\end{equation}

for all $a \in k_q[G]$. The left $U$-comodule structure $\Delta|_{F_1(U)}$ of $F_1(U)$ can be lifted to a left $U$-comodule structure on the covariantized algebra $A_r$. The map $\tilde{l}_r : A_r \to F_1(U)$ will turn out to be $U$-comodule algebra homomorphisms. Recall our conventions concerning Subsection 2.2 and define a linear map \n
\begin{equation}
d_r : A_r \to U \otimes A_r,
\end{equation}

It is checked that $d_r$ defines a $U$-comodule structure on $A$. According to the following lemma, this structure is compatible with the algebra structure of $A_r$.

**Lemma 4.1.** The left coaction $d_r$ turns $A_r$ into a left $U$-comodule algebra. The map $\tilde{l}_r : A_r \to F_1(U)$ is a homomorphism of left $U$-comodule algebras.

**Proof.** Note that the second statement follows immediately from the first statement and from comparison of (111) with (122). We now verify that $(A_r, d_r)$ is a comodule algebra. For any $a, b \in A$ one calculates

\begin{align}
d_r(a \otimes b) &= \\frac{2.11}{=} l_r^+(a(2)) l_r^-(a(3)) \sigma(l_r^-(a(4)) b(5)) l_r^+(a(1)) \sigma(l_r^+(a(2))) a(1) b(2) \otimes a(3) b(4) \\
\frac{1.19}{=} l_r^+(a(3)) l_r^+(a(2)) \sigma(l_r^+(a(4))) l_r^-(a(1)) \sigma(l_r^-(b(5))) a(1) b(2) \otimes a(3) b(4) \\
\frac{1.20}{=} l_r^+(a(3)) l_r^+(a(2)) \sigma(l_r^+(a(4))) l_r^-(a(1)) \sigma(l_r^-(b(5))) a(1) b(2) \otimes a(3) b(4) \\
\frac{1.21}{=} l_r^+(a(3)) l_r^+(a(2)) \sigma(l_r^+(a(4))) l_r^-(a(1)) \sigma(l_r^-(b(5))) a(1) b(2) \otimes a(3) b(4) \\
\frac{2.11}{=} l_r^+(a(3)) l_r^+(a(2)) \sigma(l_r^+(a(4))) l_r^-(a(1)) \sigma(l_r^-(b(5))) a(1) b(2) \otimes a(3) b(4)
\end{align}

which completes the proof of the lemma. \hfill \Box

4.2. Coideal subalgebras of $U$. Consider $f \in A_r^\circ$. Analogously to the construction of $k_q[O_f]$ in subsection 2.2 one can use the coaction $d_r$ defined by (1.2) to obtain an algebra homomorphism

\begin{equation}
d_f : A_r \to U, \quad d_f(a) := (\text{id} \otimes f)d_r(a)
\end{equation}

We define

\begin{equation}
B_f := d_f(A_r).
\end{equation}

Note that $d_f$ and hence $B_f$ depend on our choice of universal $r$-form $r' = r$ or $r' = \bar{r}_{21}$. This dependence is only implicit in our notation via the choice of the character $f$.

**Lemma 4.2.** For any character $f \in A_r^\circ$ the following hold:

1. $B_f$ is a left coideal subalgebra of $U$.
2. $B_f = \{l_r^+(a(1)) f(a(2)) \sigma(l_r^-(a(3))) \mid a \in A\}$. 


Lemma 4.3. Assume that \( k \equiv k \oplus g \). We give a slight generalisation of this fact which will be useful in the proof of Proposition 4.1. To this end note that if \( g \) is a semisimple Lie algebra then there is a tensor product decomposition of Hopf algebras \( U \cong U_1 \otimes U_2 \) where \( U_i \cong U_q(g_i) \) for \( i = 1, 2 \). Moreover, in this case \( k_q[G] \cong k_q[G_1] \otimes k_q[G_2] \) for the semisimple, simply connected affine algebraic groups \( G_1 \) and \( G_2 \) corresponding to \( g_1 \) and \( g_2 \), respectively. Finally, if \( r_1 \) and \( r_2 \) denote the universal \( r \)-forms of \( k_q[G_1] \) and \( k_q[G_2] \), respectively, then \( r = r_1 \oplus r_2 \).

Lemma 4.4. For any \( f \in A \) the following hold:

1. \( a(X) = \varepsilon(a) \varepsilon(X) \) for all \( X \in B_f \) and \( a \in k_q[O_f] \).
2. \( f(ad^r(X)a) = \varepsilon(X)f(a) \) for all \( X \in B_f \) and \( a \in A \).

Proof. Let

\[
X = d_f(b) = l_r^+(b(1))f(b(2))\sigma(l_r^-(b(3)))
\]

\[
a = \delta_f(c) = \sigma(\psi(c(1)))f(c(2))\psi(c(3))
\]

for some \( b, c \in A \). Then formula (3.3) implies

\[
a(X) = \langle \sigma(c_1) c(3), l_r^+(b(1))\sigma(l_r^-(b(3))) f(c(2)) f(b(2)) \rangle = f(b)f(c) = \varepsilon(a)\varepsilon(X)
\]

and hence (1) holds. Moreover, for \( X \) as above and any \( a \in A \) one calculates

\[
f(ad^r(X)a) = \langle \sigma(a(1)) a(3), X f(a(2)) \rangle
\]

\[
= \langle \sigma(a(1)) a(3), l_r^+(b(1))\sigma(l_r^-(b(3))) f(a(2)) f(b(2)) \rangle
\]

\[
= f(b)f(a)
\]

which proves (2).
Note that by construction \( k_q[G]^B \) is a right coideal of \( k_q[G] \). Moreover, if \( B \) is a (left or right) coideal of \( U \) then \( k_q[G]^B \) is a right coideal subalgebra of \( k_q[G] \).

**Corollary 4.5.** For any \( f \in \mathcal{A}_r^\text{inv} \) one has \( k_q[\mathcal{O}_f] \subseteq k_q[G]^B_f \).

**Proof.** For \( a \in k_q[\mathcal{O}_f] \) and \( X \in B_f \) Lemma [4.3](#) implies

\[
(a_{(1)}, X) a_{(2)} = \varepsilon(a_{(1)}) \varepsilon(X) a_{(2)} = \varepsilon(X) a
\]

because \( k_q[\mathcal{O}_f] \) is a right coideal subalgebra of \( k_q[G] \).

**Remark 4.6.** It would be desirable to have a general condition for characters \( f \in \mathcal{A}_r^\text{inv} \) which implies the equality \( k_q[\mathcal{O}_f] = k_q[G]^B_f \). For quantum Grassmann manifolds this equality holds by [NDS97, Proposition 2.4] which was given without proof.

### 4.4. The centre of Noumi coideal subalgebras.

We give a second construction of the central subalgebra \( d_f(\mathcal{A}^\text{inv}) \) of \( B_f \) obtained in Lemma [4.2](#). Recall from Proposition [3.10](#) that the coproduct can be used to define an algebra homomorphism

\[
\mu : \mathcal{A}^\text{inv} \rightarrow \mathcal{A}_r \otimes \mathcal{A}_r^\text{op}
\]

Recall moreover that \( \sigma(F_l(U)) = F_r(U) \) and that by Proposition [2.8](#) the map \( \sigma \circ \tilde{l}_{21} : \mathcal{A}_r^{op} \rightarrow F_r(U) \) is an algebra homomorphism. For \( f \in \mathcal{A}_r^\text{inv} \) we now define an algebra homomorphism \( \Phi_f : \mathcal{A}^\text{inv} \rightarrow F_r(U) \) as the composition

\[
\Phi_f : \mathcal{A}^\text{inv} \rightarrow \mathcal{A}_r \otimes \mathcal{A}_r^{op} \rightarrow \mathcal{A}_r \otimes \mathcal{A}_r^{op} \rightarrow F_r(U),
\]

or more explicitly

\[
\Phi_f(a) := f(a_{(1)}) \sigma(\tilde{l}_{21}(a_{(2)}))
\]

for all \( a \in \mathcal{A}^\text{inv} \).

**Lemma 4.7.** (i) For all \( c \in \mathcal{A}^\text{inv} \) and \( f \in \mathcal{A}_r^\text{inv} \) one has \( d_f(c) = \Phi_f(c) \).

(ii) For all \( f, g \in \mathcal{A}_r^\text{inv} \) one has \( d_f(\mathcal{A}^\text{inv}) \subseteq Z(B_f) \cap F_r(U) \).

**Proof.** Applying the coproduct to the relation in Proposition [3.10](#) one obtains

\[
c_{(1)} \otimes c_{(2)} \otimes \Psi(c_{(3)}) = c_{(2)} \otimes c_{(1)} \otimes \sigma^2(\Psi(c_{(1)}))
\]

for all \( c \in \mathcal{A}^\text{inv} \). By definition of \( \Phi_f \) and \([1.17], [1.18]\) one now gets

\[
\Phi_f(c) = f(c_{(1)}) \sigma^2(l^r_r(c_{(3)})) \sigma(l^r_l(c_{(2)})) = \frac{4 \Delta}{\Delta} l^r_r(c_{(1)}) f(c_{(2)}) \sigma(l^r_l(c_{(3)})) = d_f(c)
\]

for all \( c \in \mathcal{A}^\text{inv} \) which proves (i). Claim (ii) follows from (i) and the inclusion \( \mathcal{A}^\text{inv} \subseteq Z(\mathcal{A}_r) \).

The subalgebra \( d_f(\mathcal{A}^\text{inv}) \) of \( Z(B_f) \cap F_r(U) \) is of particular interest if the character \( f \) factors through \( \Psi \) or, more explicitly, if \( \text{ker}(\Psi) \subseteq \text{ker}(f) \). In this case, if in addition \( \mathfrak{g} \) is simple, we may assume that \( \mathcal{A} = k_q[G_L] \) for some lattice \( L \subseteq \mathfrak{h}^* \) such that \( Q \subseteq L \subseteq P \) by Proposition [2.3](#). Throughout this subsection, for arbitrary semisimple \( \mathfrak{g} \), we denote by \( L \subseteq \mathfrak{h}^* \) a lattice such that \( Q \subseteq L \subseteq P \).

**Lemma 4.8.** If \( \mathcal{A} = k_q[G_L] \) and \( f \in \mathcal{A}_r^\text{inv} \) then the map

\[
d_f|_{\mathcal{A}^\text{inv}} : \mathcal{A}^\text{inv} \rightarrow Z(B_f) \cap F_r(U)
\]

is an injective algebra homomorphism.
Proof. Assume \( df(a) = 0 \) for some \( a \in A^{inv} \). This implies \( f(a_{(1)})\sigma(\bar{r}_{F_1}(a_{(2)})) = 0 \) by Lemma 4.7(i). By Proposition 4.7 the map \( \sigma \circ \bar{r}_{F_1} \) is injective and hence \( f(a_{(1)})a_{(2)} = 0 \). We now use the fact that \( f \) is convolution invertible from Lemma 4.13(ii) to obtain

\[
a = f(a_{(1)})f(a_{(2)})a_{(3)} = (\hat{f} \otimes \text{id})\Delta(f(a_{(1)})a_{(2)}) = 0.
\]
Hence \( df|_{A^{inv}} \) is indeed injective. The fact that \( df|_{A^{inv}} \) is an algebra homomorphism follows from Proposition 3.34(1),(2) and from the fact that \( d_r \) is an algebra homomorphism by Lemma 4.8. \( \square \)

Let \( \text{Rep}(g) \) denote the representation ring of \( g \), i.e. the \( \mathbb{C} \)-algebra with basis \( \{r_\lambda\}_{\lambda \in P^+} \) and product

\[
r_{\lambda r_\mu} = \sum_{\nu \in P^+} m_{\lambda, \mu}^{\nu} r_{\nu}, \quad \text{where} \quad m_{\lambda, \mu}^{\nu} := \dim \left( \text{Hom}_U(V(\nu), V(\lambda) \otimes V(\mu)) \right).
\]
For any lattice \( L \subset g^* \) such that \( Q \subset L \subset P \) let \( \text{Rep}(g)_L \) denote the subalgebra of \( \text{Rep}(g) \) with basis \( \{r_\lambda \mid \lambda \in L \cap P^+ \} \). It was proved for instance in [JL92, 8.6] that there exists a basis \( \{z_\lambda\}_{\lambda \in P^+} \) of the centre \( Z(U) \) with \( z_\lambda \in (\text{ad}_r U)\tau(2\lambda) \) such that the map

\[
Z(U) \rightarrow \text{Rep}(g), \quad z_\lambda \mapsto r_\lambda
\]
defines an isomorphism of algebras (cp. also [Bau98]). By Propositions 1.7, 2.8 and 3.6 the elements

\[
c_\lambda' := \bar{r}_{F_1}^{-1}(\sigma^{-1}(z_\lambda)) \in C^V(-w_0\lambda) \cap k_q[G]^{inv}
\]
also satisfy \( c_\lambda' c_\mu' = \sum_{\nu \in P^+} m_{\lambda, \mu}^{\nu} c_\nu' \) and hence yield a realisation of \( \text{Rep}(g) \) inside \( k_q[G]^{inv} \). Note that \( c_\lambda' \) is \( (\text{ad}_r U) \)-invariant and hence coincides with the quantum trace \( c_{-w_0\lambda} \) defined in Example 3.3 up to a scalar factor. The following theorem is now an immediate consequence of the fact that for \( f \in k_q[G]_r^\lambda \) the map \( df|_{k_q[G]^{inv}} \) is an injective algebra homomorphism by Lemma 4.8.

**Theorem 4.9.** If \( A = k_q[G_L] \) and \( f \in A_r^\lambda \) then the map

\[
\text{Rep}(g)_L \rightarrow Z(B_f) \cap F_r(U), \quad r_\lambda \mapsto df(c_\lambda')
\]
is an injective homomorphism of algebras.

**Remark 4.10.** Note that by Proposition 1.7 and Lemma 4.7 one has for any \( \lambda \in L \cap P^+ \) the relation

\[
df(c_\lambda') = \Phi_f(c_\lambda') \in \sigma (\text{ad}_r U)\tau(2\lambda) = (\text{ad}_r U)\tau(2\lambda).
\]

Theorem 4.9 hence states that the centre of the Noumi algebra \( B_f \) contains a canonical subalgebra \( df(A^{inv}) \) which is naturally (with respect to the grading coming from \( F_r(U) \)) isomorphic to the representation ring \( \text{Rep}(g)_L \).

**4.5. Local finiteness.** In view of Lemma 4.4(ii), Lemma 4.8 and Theorem 4.9 it is natural to ask if \( Z(B_f) \) is contained in \( F_r(U) \) for any Noumi coideal subalgebra \( B_f \). We attack this question in this subsection using a convenient criterion to determine whether an element in \( U \) is contained in \( F_r(U) \). We first recall the following preparatory lemma, which is valid for an arbitrary Hopf algebra \( H \).
Lemma 4.11. Let $B \subseteq H$ be a left coideal subalgebra and $C \subseteq H$ a right coideal subalgebra of a Hopf algebra $H$. Then
\begin{align}
Z(B) &= \{ b \in B \mid (\text{ad}_x)b = \varepsilon(x)b \text{ for all } x \in B \}, \\
Z(C) &= \{ c \in C \mid (\text{ad}_x)c = \varepsilon(x)c \text{ for all } x \in C \}.
\end{align}
where $(\text{ad}_x)b = x(1)h\sigma(x(2))$ and $(\text{ad}_x)c = \sigma(x(1))hx(2)$ for $x, h \in H$.

Proof. This result is proved in complete analogy to [Jos95, Lemma 1.3.3]. To see that the right hand sides of (4.7) and (4.8) are contained in $Z(B)$ and $Z(C)$, respectively, one uses
\[ hx = x(1)((\text{ad}_{x(2)})h), \quad xh = ((\text{ad}_{x(1)})h)x(2) \]
for any $x, h \in H$. \hfill \Box

Proposition 4.12. For $i = 1, \ldots, r$ let $C_i \in U$ be elements such that
\[(4.9) \quad C_i \in x_i\tau(\mu_i) + \bigoplus_{\alpha \leq 0} U_{\alpha}
\]
for some $\mu_i \in P$. If $u \in U$ satisfies $(\text{ad}_i C_i)u = \varepsilon(C_i)u$ for all $i = 1, \ldots, r$ then $u \in F_i(U)$. Similarly, if $u \in U$ satisfies $(\text{ad}_i C_i)u = \varepsilon(C_i)u$ for all $i = 1, \ldots, r$ then $u \in F_i(U)$.

Proof. Note that if $x \in U$ satisfies $\dim((\text{ad}_i U_q(b^+))x) < \infty$ then $x \in F_i(U)$. The proof of this fact is nicely written up as the first step of the proof of [FM98, Lemma 3.1.1]. We now use this fact to prove the first statement of the proposition. Decompose the element $u = \sum_{\gamma \in \Omega} u_{\gamma}$ where $u_{\gamma} \in U_\gamma$. It suffices to show by induction on $\gamma$ that $\dim((\text{ad}_i U_q(b^+))u_{\gamma}) < \infty$. Fix $\beta \in Q$ and assume that $\dim((\text{ad}_i U_q(b^+))u_{\gamma}) < \infty$ for all $\gamma > \beta$. The relation $(\text{ad}_i C_i)u = \varepsilon(C_i)u$ and the special form (4.9) of the elements $C_i$ imply
\[ \varepsilon(C_i)u_{\beta + \alpha_i} \in q^{(\mu_i, \beta)}(\text{ad}_i x_i)u_{\beta} + \sum_{\gamma \geq \beta + \alpha_i} (\text{ad}_i U_{\beta - \gamma + \alpha_i})u_{\gamma} \]
and hence
\[ (\text{ad}_i x_i)u_{\beta} \in \sum_{\gamma > \beta} (\text{ad}_i U)u_{\gamma} \]
for $i = 1, \ldots, r$. By induction hypothesis the right hand side of the above expression is contained in $F_i(U)$ and hence $\dim((\text{ad}_i U_q(b^+))u_{\beta}) < \infty$. This completes the proof of the first statement. Using the relations \[(4.11) \quad x_i \tau(\mu_i),e_i \in U_{\mu_i} \]
one immediately obtains the second claim of the proposition. \hfill \Box

Remark 4.13. The second half of the above proof resembles an argument given in the proof of [Let97, Lemma 4.4]. Following G. Letzter’s more general setting one can even show that any element $u \in U$ such that $\text{span}\{(\text{ad}_i C_i^m)u \mid m \in \mathbb{N}_0\}$ is finite-dimensional for all $i = 1, \ldots, r$, belongs to $F_i(U)$.

Lemma 4.11 and Proposition 4.12 now imply the following local finiteness result for Noumi coideal subalgebras in $U$.

Proposition 4.14. Let $f \in \mathcal{A}_b^\bullet$ be convolution invertible. Then $Z(B_f) \subseteq F_i(U)$. 

Proof. By Lemma 4.11 any \( b \in Z(B_f) \) satisfies \( \text{ad}_r((\beta - \varepsilon(\beta))X)b = 0 \) for all \( \beta \in B_f \) and \( X \in U \). In particular for any \( a \in A \) one obtains

\[
\text{ad}_r([df(a(1)) - f(a(1))] r^{-}(a(2)) \bar{f}(a(3)))b = 0.
\]

This relation may be rewritten as

\[
\text{ad}_r( l^+(a) - f(a(1)) l^{-}(a(2)) \bar{f}(a(3)))b = 0
\]

for all \( a \in A \). Take any generator \( x_j \in U \). As above Lemma 4.3 we can decompose the Hopf algebra \( U \) in the form \( U \cong U_1 \otimes U_2 \) where \( x_j \in U_1 \) and and \( U_i \cong U(g_i) \), \( i = 1, 2 \), with simple \( g_1 \) and semisimple or trivial \( g_2 \). By Lemma 4.3 we may assume that the left or right regular action of \( U_1 \) on \( \Psi(A) \) is non-trivial. Hence there exists \( \lambda \in P^+ \) such that \( U_1 \) acts non-trivially on \( V(\lambda) \) and \( C_{V(\lambda)}^\vee \subset \Psi(A) \). It follows from (1.22) that there exist weight vectors \( v \in V(\lambda) \) and \( f \in V(\lambda)^* \) such that \( l^+_r(c_{f,v}^\lambda) = x_j \tau(-\text{wt}(v)) \). Choose \( a \in A \) such that \( \Psi(a) = c_{f,v}^\lambda \) and consider the element

\[
C_j := l^+_r(a) - f(a(1)) l^{-}(a(2)) \bar{f}(a(3)).
\]

By construction \( C_j \) is of the form (1.9), and by (4.10) we have \( \text{ad}_r C_j b = \varepsilon(C_j)b \). As this construction is possible for any \( x_j \) one may now apply Proposition 4.12 to obtain \( b \in F_r(U) \).

5. Constructing characters of \( A_r \)

In Section 4 we explained the relevance of characters of the covariantized algebra \( A_r \). They are the main ingredient in the construction of Noumi coideal subalgebras and quantum adjoint orbits. As explained in Section 3.5 universal cylinder forms coincide with characters of covariantized algebras. The case when \( A = k_q[G] \) is of particular interest because it leads to realisations of \( \text{Rep}(g) \) inside \( U \). We now address the immediate question of how to obtain such characters.

5.1. Solutions of the reflection equation from central elements. Consider an element \( C \in F_r(U) \). It follows from the direct sum decomposition (1.14) that there exists a finite subset \( P_C^+ \) of \( P^+ \) such that \( C \in \bigoplus_{\mu \in P_C^+} (\text{ad}_r U) \tau(-2w_0 \mu) \). Define \( c_C := \sum_{\mu \in P_C^+} c_\mu \in k_q[G]^{\text{inte}} \) as sum of the quantum traces \( c_\mu \). Note that by definition of quantum traces in Example 3.3 and by Proposition 4.12 there exists a uniquely determined linear functional \( f_C : k_q[G] \rightarrow k \) such that

\[
C = f_C(c_C(1)) \sigma(\bar{f}_r(c_C(2)))
\]

and \( f_C(C^{V(\mu)}) = 0 \) for all \( \mu \notin P_C^+ \). Note, moreover, that \( f_C \) depends on \( C \) only and not on the choice of \( P_C^+ \). We use the functional \( f_C \) to reformulate and generalise the observation made in [Ko08, Corollary 2] that suitable central elements in coideal subalgebras of \( U \) lead to solutions of the reflection equation.

Proposition 5.1. Let \( B \subseteq U \) be a left coideal subalgebra and \( C \in Z(B) \cap F_r(U) \). Then the functional \( f_C : k_q[G] \rightarrow k \) defined above satisfies the relation

\[
f_C(b(1)) r(b(2), a(1)) f_C(a(2)) r(\sigma(b(3)), a(3)) = r(\sigma(a(1)), b(1)) f_C(a(2)) r(a(3), b(2)) f_C(b(3))
\]

for all \( a, b \in k_q[G] \).
To make the proof of the above proposition more manageable we separate the main technical step in the following lemma.

**Lemma 5.2.** Let \( B_r \subseteq U \) be a right coideal subalgebra and \( D \in Z(B_r) \cap F_1(U) \). Then the relation

\[
\mathfrak{r}(\sigma(m_{(1)})m_{(3)}, \sigma^{-1}(n_{(2)})) m_{(2)} \otimes n_{(1)} = \mathfrak{r}(\sigma^2(n_{(1)}), \sigma(m_{(1)})m_{(3)}) m_{(2)} \otimes n_{(2)}
\]

holds for \( m = n = \tilde{1}_{r^2}(D) \in k_q[G] \).

**Proof.** It follows from Lemma 1.11 and the coideal property \( \Delta(B_r) \subseteq B_r \otimes U \) that \( (\mathrm{ad} \, D_{(1)}) D \otimes D_{(2)} = D \otimes D \). In view of relations 1.1 and 1.18 this can be rewritten as

\[
\mathrm{ad}_l(\tilde{1}_r(n_{(1)}) \sigma(\tilde{1}_r(n_{(3)}))) D \otimes \tilde{1}_{r^2}(n_{(2)}) = D \otimes \tilde{1}_{r^2}(n).
\]

In view of Proposition 1.7 the above relation implies

\[
\mathrm{ad}_l(\tilde{1}_r(n_{(1)}) \sigma(\tilde{1}_r(n_{(4)}))) D \otimes n_{(2)} \otimes n_{(3)} = D \otimes n_{(1)} \otimes n_{(2)}
\]

and hence

\[
\mathrm{ad}_l(\sigma(\tilde{1}_r(n_{(2)}))) D \otimes n_{(1)} = \mathrm{ad}_l(\sigma^{-1}(\tilde{1}_r(n_{(1)}))) D \otimes n_{(2)}.
\]

Inserting \( D = \tilde{1}_{r^2}(m) \) and using the fact that \( \tilde{1}_{r^2} : k_q[G] \rightarrow F_1(U) \) is an isomorphism of left \( U \)-modules one obtains

\[
(\sigma(m_{(1)})m_{(3)}, \sigma(\tilde{1}_r(n_{(2)}))) m_{(2)} \otimes n_{(1)} = (\sigma(m_{(1)})m_{(3)}, \sigma^{-1}(\tilde{1}_r(n_{(1)}))) m_{(2)} \otimes n_{(2)}.
\]

By definition of \( \tilde{1}_r^+ \) and \( \tilde{1}_r^- \) this is equivalent to the desired formula. \( \square \)

**Proof of Proposition 5.1.** We apply Lemma 5.2 to the right coideal subalgebra \( B_r := \sigma^{-1}(B) \) of \( U \) and to the element \( D := \sigma^{-1}(C) \). To simplify notation we define \( f := f_C \), \( e := c_C \), and \( c := c_C \). Note that in the notation of Lemma 5.2 one has \( m = f(c_{(1)})c_{(2)} \) and \( n = f(e_{(1)})e_{(2)} \) and hence one gets

\[
\begin{align*}
  f(c_{(1)}) f(e_{(1)}) & \mathfrak{r}(\sigma(c_{(4)}), e_{(3)}) \mathfrak{r}(\sigma^2(c_{(2)}), e_{(4)}) c_{(3)} \otimes e_{(2)} \\
  & = f(c_{(1)}) f(e_{(1)}) \mathfrak{r}(\sigma(e_{(3)}), c_{(2)}) c_{(3)} \otimes e_{(4)}.
\end{align*}
\]

Using Proposition 3.3 we now apply

\[
(c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \otimes c_{(4)} = c_{(2)} \otimes c_{(3)} \otimes c_{(4)} \otimes \sigma^2(c_{(1)}))
\]

to both sides of (5.3), and we apply

\[
e_{(1)} \otimes e_{(2)} \otimes e_{(3)} \otimes e_{(4)} = e_{(3)} \otimes e_{(4)} \otimes \sigma^2(e_{(1)}) \otimes \sigma^2(e_{(2)})
\]

to the left hand side of (5.3). One obtains

\[
\begin{align*}
  f(c_{(2)}) f(e_{(3)}) & \mathfrak{r}(\sigma(c_{(1)}), e_{(1)}) \mathfrak{r}(e_{(3)}, e_{(2)}) c_{(4)} \otimes e_{(4)} \\
  & = f(c_{(2)}) f(e_{(1)}) \mathfrak{r}(e_{(2)}, c_{(1)}) \mathfrak{r}(\sigma(e_{(3)}), c_{(3)}) c_{(4)} \otimes e_{(4)}.
\end{align*}
\]

In view of the special form of quantum trace \( c = e = c_C \) this proves (5.2) for all \( a, b \in \bigoplus_{\mu \in P^+ C} C^V(\mu) \) and hence for all \( a, b \in k_q[G] \). \( \square \)

Proposition 5.1 provides characters of the reflection equation algebra from Subsection 2.4 via suitable central elements in any coideal subalgebra of \( U \). Recall that for any \( V \in \mathrm{Ob}(C) \) the corresponding FRT algebra \( \mathcal{A}(R_{V, V}) \) is generated by the linear space \( V^* \otimes V \) as an algebra. Let \( l^h_v \in \mathcal{A}(R_{V, V}) \) denote the generator corresponding to \( h \otimes v \in V^* \otimes V \). The following Corollary is a direct consequence of Proposition 2.10 and Proposition 5.1.
Corollary 5.3. Let $B \subseteq U$ be a left coideal subalgebra and let $C \in Z(B) \cap F_r(U)$. For any $V \in \text{Ob}(C)$ there exists a unique character $g_{C,V} \in \mathcal{A}(R_{V,V})^C_r$ such that $g_{C,V}(t^h) = f_C(c_{h,v})$ for all $h \in V^*, v \in V$.

To make the Noumi coideal subalgebra corresponding to the character $g_{C,V}$ more explicit, we will use the following auxiliary observation. For any $\mu \in P^+$ let $\mu : F_r(U) \to (\text{ad}_r U) \tau(2\mu)$ denote the projection map with respect to the direct sum decomposition $(1.14)$.

Lemma 5.4. Let $B \subseteq U$ be a left coideal subalgebra and $\mu \in P^+$. Then
\[ p_{\mu}(B \cap F_r(U)) \subseteq B \quad \text{and} \quad p_{\mu}(Z(B) \cap F_r(U)) \subseteq Z(B). \]

Proof. Note that $p_{\mu}$ is a homomorphism of right coideals in $U$. Together with the fact that $B$ is a left coideal subalgebra this implies the first inclusion. The second inclusion is now an immediate consequence of Lemma 4.11. \qed

By the above lemma any $C \in Z(B) \cap F_r(U)$ can be written as a finite sum
\[ (5.4) \quad C = \sum_{\mu \in P^+_C} C_\mu \quad \text{with} \ C_\mu \in (\text{ad}_r U) \tau(-2w_0\mu) \cap Z(B). \]

The Noumi coideal subalgebra $B_{g_{C,V}}$ corresponding to the character from Corollary 5.3 is obtained from the central elements $C_\mu$ as follows.

Proposition 5.5. Let $B \subseteq U$ be a left coideal subalgebra and $C \in Z(B) \cap F_r(U)$. For $V \in \text{Ob}(C)$ let $g_{V,C} \in \mathcal{A}(R_{V,V})^C_r$ be the character obtained in Corollary 5.3. Let $C = \sum_{\mu \in P^+_C} C_\mu$ be the decomposition $(5.4)$. Then the Noumi coideal subalgebra $B_{g_{C,V}}$ is the left coideal subalgebra of $U$ generated (as a left coideal subalgebra) by the elements $C_\mu$ for all $\mu \in P^+_C$ with $\text{Hom}_V(V(\mu),V) \neq \{0\}$. In particular, one has $B_{g_{C,V}} \subseteq B$.

Proof. Let $B_\mu \subseteq U$ denote the left coideal generated by the element $C_\mu$ and as before let $f_C$ be the linear functional defined by $(5.1)$. The relation $C_\mu = f_C(c_{\mu(1)})\sigma(\bar{t}_{\mathbb{F}_2}(c_{\mu(2)}))$ implies in view of relation $(4.1)$ that
\[ \Delta(C_\mu) = f_C(c_{\mu(1)})\sigma(\bar{t}_{\mathbb{F}_2}(c_{\mu(3)})) \otimes \sigma^2(\bar{t}_{\mathbb{F}_2}(c_{\mu(4)}))\sigma(\bar{t}_{\mathbb{F}_2}(c_{\mu(2)})) = \sigma^{-1}(\bar{t}_{\mathbb{F}_2}(c_{\mu(1)})) \otimes \bar{t}_{\mathbb{F}_2}(c_{\mu(2)})f_C(c_{\mu(3)})\sigma(\bar{t}_{\mathbb{F}_2}(c_{\mu(4)})) \]
where we used Proposition 3.4 $(6)$ and $(1.18)$ for the last equality. Hence we obtain
\[ B_\mu = \{l^+_r(e_{(1)})f_C(e_{(2)})\sigma(l^-_r(e_{(3)})) \mid e \in C^V(\mu) \}. \]

On the other hand, by Lemma 4.2 $(2)$ the algebra $B_{g_{C,V}}$ is generated as an algebra by the subspace
\[ \{l^+_r(e_{(1)})f_C(e_{(2)})\sigma(l^-_r(e_{(3)})) \mid e \in C^V \}. \]
This subspace coincides with the span of all $B_\mu$ for $\mu \in P^+_C$ such that $V(\mu)$ occurs as a direct summand in $V$. \qed

Remark 5.6. G. Letzter’s family of quantum symmetric pair coideal subalgebras $B$ of $U$ ([Let02]) is a very interesting class of examples to which Propositions 5.1 and 5.5 apply. The centre of these left coideal subalgebras was determined in [KL08].
It follows from [KL08, Footnote to Corollary 8.3] that for each of these left coideal subalgebras \( B \) of \( U \) there exists a subset \( P^+_{Z(B)} \subseteq P^+ \) such that

\[
\dim \left( Z(B) \cap (\text{ad}_r U) \tau(-2w_0 \mu) \right) = \begin{cases} 
1 & \text{if } \mu \in P^+_{Z(B)}, \\
0 & \text{else}
\end{cases}
\]

and

\[
Z(B) = \bigoplus_{\mu \in P^+_{Z(B)}} \left( Z(B) \cap (\text{ad}_r U) \tau(-2w_0 \mu) \right).
\]

Note that this subset was denoted by \( P_{Z(B)} \) in [KL08]. The set \( P^+_{Z(B)} \) is explicitly determined in [KL08, Proposition 9.1]. In many cases, in particular if \( g \) has no diagram automorphisms, one has \( P^+_{Z(B)} = P^+ \). Moreover, \( P^+_{Z(B)} \) is invariant under taking dual weights. Proposition 9.1 implies that for any quantum symmetric pair coideal subalgebra \( B \) and any \( \mu \in P^+_{Z(B)} \) one obtains a solution of the reflection equation (5.2.4) for \( V = V(\mu) \). By [Kol08, Proposition 4] this solution is non-diagonal and hence no multiple of the identity. In Subsection 5.3 we will explicitly discuss the quantum symmetric pair corresponding to the Grassmannian manifold \( Gr(m,2m) \) of \( m \)-dimensional subspaces in \( \mathbb{C}^{2m} \).

5.2. Characters of \( F_l (U_q (\mathfrak{sl}_n)) \). For the rest of this section we restrict to the case where \( g = \mathfrak{sl}_n = \mathfrak{sl}_n(\mathbb{C}) \) and \( V = V(\omega_1) \) is the vector representation of \( U = U_q (\mathfrak{sl}_n) \). Note that \( r = \text{rank}(\mathfrak{sl}_n) = n - 1 \). We choose the root system for \( \mathfrak{sl}_n \) and the simple roots \( \{ \alpha_1, \ldots, \alpha_r \} \) as in [Hum72, 12.1]. Recall that \( V \) has a basis \( \{ v_1, \ldots, v_n \} \) such that

\[
x_i v_j = \delta_{i,j-1} v_{j-1}, \quad y_i v_j = \delta_{i,j+1} v_{j+1}, \quad t_i v_j = q^{-\delta_{i+1,j}+\delta_{i,j}} v_j.
\]

As in previous sections we let \( \{ f_1, \ldots, f_n \} \) denote the basis of \( V^* \) dual to \( \{ v_1, \ldots, v_n \} \). To shorten notation we define \( e_{i,j} := c_{i,j} v_j \).

As in subsection 2.5 let \( R := R_{V,V} = P_{12} \circ \hat{R}_{V,V} \) be the \( R \)-matrix corresponding to \( V \). If we define an \((n^2 \times n^2)\)-matrix \((R_{ij}^{kl})\) by \( R_{ij}^{kl} = R(c_{i,k}, c_{j,l}) \) then one has \( R(v_k \otimes v_l) = \sum_{i,j=1}^{n} v_i \otimes v_j R_{ij}^{kl} \). It follows from [Jan96, 3.15] translated to our conventions that

\[
R_{ij}^{kl} = q^{1/n} \begin{cases} 
q^{-1} & \text{if } i = j = k = l, \\
1 & \text{if } i = k \neq j = l, \\
q^{-1} - q & \text{if } i = l \neq j = k, \\
0 & \text{else}.
\end{cases}
\]

Note that this matrix coincides with the matrix in [KS97, 8.4 (60)] up to the overall factor and taking the inverse of the transpose.

Recall the FRT-algebra \( A(R) \) considered in Subsection 2.5. As in that subsection we denote the generators \( \{ t^i_j \} \) of \( A(R) \) by \( \{ s^i_j \} \) if we consider them as generators of the reflection equation algebra \( A(R)_r \). For any element \( \sigma \) in the symmetric group \( S_n \) let \( l(\sigma) \) be its length. Recall that the quantum determinant

\[
\det_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{\sigma(1)}^1 \ldots t_{\sigma(n)}^n
\]

is a central element in \( A(R) \) such that \( A(R)/(\det_q - 1) \cong k_q[SL(n)] \) (cp. [KS97, 9.2.3, 11.2.3]). If one considers \( \det_q \) as an element in the covariantized algebra
Lemma 5.7. There is a one-to-one correspondence between the set $k_q[SL(n)]_r^\wedge$ and the set \{ $f \in A(R)_r^\wedge$ | $f(\text{det}_q) = 1$ \}.

If $f \in A(R)_r^\wedge$ is a character such that $f(\text{det}_q) = \beta^n$ for some $\beta \in k \setminus \{0\}$ then the map $g : k_q[SL(n)]_r \to k$ defined by $g(c_{ij}) := \beta^{-1} f(s^i_j)$ defines a character of $k_q[SL(n)]_r \cong F_l(U_q(sl_n))$.

Proof. For any character $f \in A(R)_r^\wedge$ such that $f(\text{det}_q) = \beta^n$ one defines a character $g \in A(R)_r$ by $g(s^i_j) := \beta^{-1} f(s^i_j)$. Note that $g$ is well-defined because all relations of $A(R)_r$ are homogeneous. Moreover, one has $g(\text{det}_q) = \beta^{-n} f(\text{det}_q) = 1$ and hence $g$ factors to a character of $k_q[SL(n)]_r$.

By Lemma 5.7 a character $f$ of the reflection equation algebra $A(R)_r$ gives rise to a character of the locally finite part $F_l(U)$ if and only if $f(\text{det}_q) \neq 0$ and the base field contains an $n$-th root of $f(\text{det}_q)$. The following proposition provides an elementary yet somewhat surprising criterion for the first condition to hold.

Proposition 5.8. For any character $f \in A(R)_r^\wedge$ the following are equivalent:

1. $f(\text{det}_q) \neq 0$.
2. $f : A(R) \to k$ is convolution invertible.
3. The matrix $M := (f(s^i_j))$ is invertible.

Proof. Note first that $(2)$ implies $(3)$ by definition of the coproduct of the bialgebra $A(R)$. We now show that $(1)$ implies $(2)$. Assume that $f(\text{det}_q) \neq 0$ and let $\beta$ be an $n$-th root of $f(\text{det}_q)$ in a field extension $k'$ of $k$. For now we consider all algebras to be defined over $k'$, yet the convolution inverse of $f$ we construct will be defined over $k$. The character $g \in A(R)_r$ defined by $g(s^i_j) := \beta^{-1} f(s^i_j)$ factors to a character $g' \in k'_q[SL(n)]_r$ because $g(\text{det}_q) = 1$. By Lemma 3.13.(2) the functional $g'$ is convolution invertible, and we denote its convolution inverse by $\bar{g}'$. Define $\bar{g}(a) := \bar{g}'(\Psi(a))$ where $\Psi : A(R) \to k'_q[SL(n)]$ denotes the canonical bialgebra homomorphism from Subsection 2.5. One checks that $g : A(R) \to k'$ is convolution invertible with convolution inverse $\bar{g}$. For $m \in \mathbb{N}$ let $A(R)_m \subseteq A(R)$ denote the homogeneous component of degree $m$ of $A(R)$ defined over $k$. Note that $A(R)_m$ is a $k$-subcoalgebra of $A(R)$ and hence $g(A(R)_m) \subseteq \beta^{-m}k$ implies $\bar{g}(A(R)_m) \subseteq \beta^m k$. Indeed, any element $a \in A(R)_m$ is contained in a subcoalgebra $A = \text{span}_k \{a_{ij} \mid 1 \leq i, j \leq N\}$ for some $a_{ij} \in A(R)_m$ with $\Delta(a_{ij}) = \sum_{h=1}^N a_{ih} \otimes a_{hj}$. The implication $\bar{g}(a) \in \beta^m k$ now follows from the fact that the matrix $(\bar{g}(a_{ij}))$ is the inverse of the matrix $(g(a_{ij})) \in \text{Mat}_N(\beta^{-m} k)$. Define now a functional $\bar{f} : A(R) \to k$ by $\bar{f} |_{A(R)_m} = \beta^{-m} \bar{g} |_{A(R)_m}$ and linearity. Note that $\bar{f}$ is defined over $k$ and not only over $k'$. By construction $\bar{f}$ is a convolution inverse of $f$.

It remains to show that $(3)$ implies $(1)$. To this end we first recall some well known properties of $\text{det}_q$ and related expressions in $A(R)$ (compare e.g. [KS97, 9.2.1, 9.2.2]).

(a) For any $\mu \in S_n$ the relation

$$\sum_{\sigma \in S_n} (-q)^{l(\sigma)} \mu^{(1)}_{\sigma(1)} \cdots \mu^{(n)}_{\sigma(n)} = (-q)^{l(\mu)} \text{det}_q$$

holds in $A(R)$. 

\[ REFLECTION EQUATION ALGEBRAS AND CENTRES 31 \]
(b) For any elements $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}$ such that $i_k = i_l$ for some $k \neq l$ the relation
\[
\sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{\sigma(1)}^{i_1} \cdots t_{\sigma(n)}^{i_n} = 0
\]
holds in $\mathcal{A}(R)$.

Assume that $M$ is invertible. We consider $M$ as an endomorphism of $V$ defined by $M(v_i) = \sum_j v_j f(s^i_j)$. For $k = 1, \ldots, n$ define automorphisms of $V^\otimes n$ by
\[
M_k := \text{id}_{V^\otimes n-k} \otimes (R_{V,V^\otimes k-1}^{-1}(M \otimes \text{id}_{V^\otimes k-1})R_{V,V^\otimes k-1})
\]
and set
\[
D := M_n \ldots M_1.
\]
Note that by construction and relations (3.6) one has
\[
D(v_{i_1} \otimes \ldots \otimes v_{i_n}) = \sum_{j_1, \ldots, j_n} v_{j_1} \otimes \ldots \otimes v_{j_n} f(t_{i_1}^{j_1} \cdots t_{i_n}^{j_n}).
\]
Applying this to the element
\[
Y := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \in V^\otimes n
\]
and using properties (a) and (b) above one obtains
\[
D(Y) = \sum_{j_1, \ldots, j_n} v_{j_1} \otimes \ldots \otimes v_{j_n} f \left( \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{\sigma(1)}^{j_1} \cdots t_{\sigma(n)}^{j_n} \right)
\]
\[
= f(\det_q) \sum_{\mu \in S_n} (-q)^{l(\mu)} v_{\mu(1)} \otimes \ldots \otimes v_{\mu(n)}.
\]
As $D$ is injective one obtains $f(\det_q) \neq 0$. \qed

Remark 5.9. The above proof is inspired by the arguments in [D99, Section 3]. Note, however, that in his paper T. tom Dieck restricts to a class of bottom-right triangular block-matrices.

Remark 5.10. It would be interesting to write the quantum determinant explicitly in terms of the generators $\{s^i_j\}$ of the reflection equation algebra $\mathcal{A}(R)_r$. This seems to be a non-trivial combinatorial task.

Remark 5.11. Characters of $\mathcal{A}(R)_r$ were classified explicitly in [Mud02]. In view of Lemma 5.7 and Proposition 5.8 this also gives the classification of characters of $F_l(U_q(\mathfrak{sl}_n))$.

5.3. An example: quantum Grassmann manifolds $Gr(m,2m)$. Let coideal subalgebras $B$ of $U$ corresponding to symmetric pairs of Lie algebras are given explicitly in [Let03, Section 7]. As an example we consider here the symmetric pair $(\mathfrak{sl}_{2m}, \mathfrak{sl}_{2m} \cap (\mathfrak{gl}_m \oplus \mathfrak{gl}_m))$ labelled by AII(case 2) in the general classification [Ara02]. The corresponding symmetric space is the Grassmann manifold $Gr(m,2m)$ of $m$-dimensional subspaces in $\mathbb{C}^{2m}$. We prefer here to work with the corresponding right coideal subalgebra obtained via the antipode.

Assume that $n = 2m$ is even. Fix a parameter $s \in \mathbb{k}$ and consider the subalgebra $B^s$ of $U$ generated by the following elements:

(i) $\tau(\omega_i - \omega_{s(i)})$ for $i \neq m$, 

\[
\omega_{s(i)} := \omega_{i} + \sum_{j \neq i} s_{i,j} \omega_j - s_{j,i} \omega_i
\]
(ii) \( B_i = y_i + t_i^{-1} x_{p(i)} \) for \( i \neq m \),
(iii) \( B_m = y_m + t_m^{-1} x_m + s t_m^{-1} \)
where \( p(i) := n - i \). One verifies that \( B^* \) is a right coideal subalgebra of \( U \). The pair \( (U, B^*) \) is a quantum analogue of the pair \((U(\mathfrak{sl}_n), U(\mathfrak{sl}_n \cap (\mathfrak{gl}_m \oplus \mathfrak{gl}_m)))\). Note that the family \( \sigma(B^*) \), \( s \in k \), coincides with the family of left coideal subalgebras for AsAII (case 2) given in [Le03, p. 284], up to extension by elements in \( U^0 \). Note moreover, that the subalgebra of \( U^0 \) generated by the elements in (i) coincides with \( \text{span}_k \{ \tau(\lambda) | \lambda \in P, w_0 \lambda = \lambda \} \) and hence \( t_i t_i^{-1} \) in \( B^* \) for all \( 1 \leq i \leq r \).

Recall that \( r = \text{rank}(\mathfrak{sl}_n) = n - 1 \) and let \( \omega_r := -w_0 \omega_1 \) be the fundamental weight such that \( V(\omega_r) \) is dual to the vector representation \( V(\omega_1) \). Recall the structure of centre of the left coideal subalgebra \( \sigma(B^*) \) described in Remark 5.6.

By [KL08, Proposition 9.1] one has \( P^2_{Z(\sigma(B^*))} = P^* \) and hence
\[
\text{dim}(Z(B^*) \cap (\text{ad}_U) \tau(-2\omega_r)) = 1.
\]

Let \( D \in Z(B^*) \cap (\text{ad}_U) \tau(-2\omega_r) \) be a nonzero element and set \( C := \sigma(D) \). By the results of Subsection 5.1 there exists a linear functional \( f_C : C^V \to k \) such that
\[
C = f_C(c_{(1)}) \sigma(\tilde{\tau}_{x_2}(c_{(2)}))
\]
where \( c = c_{\omega_1} \) is the quantum trace of \( V(\omega_1) \). By Corollary 5.9 one obtains a corresponding character \( g_{C,V} \) of the reflection equation algebra \( \mathcal{A}(R)_\tau \). In particular the matrix \( M := (g_{C,V}(s^j_i)) \) is a numerical solution of the reflection equation (2.21) with \( R \) given by (5.8). We will prove the following lemma in the next subsection where we determine \( M \) explicitly.

**Lemma 5.12.** The matrix \( M := (g_{C,V}(s^j_i)) \) is invertible.

We now summarise the results obtained about \( B^* \), its centre, the element \( C = \sigma(D) \) and the corresponding character \( g_{C,V} \in \mathcal{A}(R)_\tau^\lambda \). In the following theorem we consider all algebras to be defined over a suitable field extension \( k' \) of \( k \).

**Theorem 5.13.** Let \( k' \) be a field extension of \( k \) which contains an element \( \beta \) with \( \beta^n = g_{C,V}(\det_q) \). Then the following hold:

1. \( g(c_{(j)}) := \beta^{-1} g_{C,V}(s^j_i) \) defines a character of \( k'_q(SL(n)) \), \( \cong F_l(U_q(\mathfrak{sl}_n)) \).
2. There exists a basis \( \{ D_\lambda | \lambda \in P^+ \} \) of the centre \( Z(B^*) \) such that the following hold for all \( \mu, \lambda \in P^+ \):
   (a) \( D_\lambda \in Z(B^*) \cap (\text{ad}_U) \tau(-2\lambda) \).
   (b) \( D_\lambda D_\mu = \sum_{\nu \in P_+} m^\nu_{\lambda, \mu} D_\nu \)
   where \( m^\nu_{\lambda, \mu} := \text{dim}(\text{Hom}_V(V(\nu), V(\lambda) \otimes V(\mu))) \).

**Proof.** By Lemma 5.12 the matrix \( (g_{C,V}(s^j_i)) \) is invertible and thus \( g_{C,V}(\det_q) \neq 0 \) by Proposition 5.8. Hence we may apply Lemma 5.7 to obtain the desired character of \( k'_q(SL(n)) \), \( \cong F_l(U_q(\mathfrak{sl}_n)) \) which proves (1).

To verify part (2) define \( D_\lambda := \tau^{-1}(d_\lambda(c'_\lambda)) \) for all \( \lambda \in P^+ \), where \( c'_\lambda \in k'_q(SL(n))^{|n\nu} \) is the invariant element defined in (1.5). Theorem 4.9 and (4.6) imply that
\[
0 \neq D_\lambda \in (\text{ad}_U) \tau(-2\lambda) \cap Z(\sigma^{-1}(B_{g_{C,V}})).
\]

Note that up to a nonzero scalar factor the element \( D_\lambda \), defined in this way coincides with the element \( D \). The elements \( \{ D_\lambda | \lambda \in P^+ \} \) satisfy the relations in (b) because the elements \( \{ c'_\lambda | \lambda \in P^+ \} \) do, as explained in subsection 4.4.
It remains to show that the elements \( \{ D_\lambda \mid \lambda \in P^+ \} \) form a basis of \( Z(B^*) \). By Proposition 5.5, one has \( D_\lambda \in B^* \) for all \( \lambda \in P^+ \). Moreover, as \( D \in Z(B^*) \) and (b) holds, all elements \( D_\lambda \) for \( \lambda \in P^+ \) are invariant under the left adjoint action of \( B^* \).

By Lemma 4.11 and (5.10) one hence obtains
\[
D_\lambda \in Z(B^*) \cap (\text{ad}_i U)\tau(-2\lambda) \setminus \{0\}
\]
for all \( \lambda \in P^+ \). In view of (5.5) and (5.6) this implies that \( \{ D_\lambda \mid \lambda \in P^+ \} \) is a basis of \( Z(B^*) \).

\( \Box \)

**Remark 5.14.** With some additional technical effort Lemma 5.12 and hence Theorem 5.13 can also be shown to hold for quantum symmetric pairs corresponding to arbitrary Grassmannian manifolds \( Gr(m,n) \) where \( 2m \leq n \). The corresponding quantum symmetric pair coideal subalgebras are defined in [Let03, p. 284] as AIH\((\text{case 1)})/AIV.

### 5.4. An explicit solution of the reflection equation

In this subsection we will prove Lemma 5.12 and explicitly determine the numerical solution \( M = (g_{C,V}(s^i_j)) \) of the reflection equation for the functional \( f_C \) defined in (5.9). For further calculations we first provide some explicit formulae. Recall the basis \( \{ v_1, \ldots, v_n \} \) with dual basis \( \{ f_1, \ldots, f_n \} \) chosen in Subsection 5.2. Note that \( \text{wt}(v_i) = \omega_1 - \alpha_1 - \cdots - \alpha_{i-1} = -\omega_{i-1} + \omega_i \) where we have set \( \omega_0 = \omega_n = 0 \). Hence one obtains
\[
(5.11) \quad (2p, \text{wt}(v_i)) = n - 2i + 1.
\]
It follows from (5.1) that the matrix coefficients \( c_{i,j} := c_{f_i,v_j} \) satisfy the relations
\[
\begin{align*}
(\text{ad}_{x_i}^* x_i)c_{k,l} &= \delta_{i,k} q^{-1} c_{k+1,l} + \delta_{i,k} q^{-1} c_{k-1,l} - \delta_{i,l} q^{-1} c_{k,l-1}, \\
(\text{ad}_{y_i}^* y_i)c_{k,l} &= -\delta_{i,k} q^{1} c_{k+1,l} + \delta_{i,k} q^{1} c_{k-1,l} + \delta_{i,l} c_{k,l-1}, \\
(\text{ad}_{t_i}^* t_i)c_{k,l} &= q^{-1} \delta_{i,k} q^{1} c_{k+1,l} + \delta_{i,l} c_{k,l} - \delta_{i,k} q^{-1} c_{k,l}.
\end{align*}
\]
(5.12)

Define an \((n \times n)\)-matrix \( \Omega \) by
\[
(5.13) \quad D = \sum_{i,j=1}^{n} \Omega_{ij} \tilde{I}_{21}(c_{i,j}).
\]
Note that by definition (5.9) of the functional \( f_C \) and by (5.11) we have
\[
(5.14) \quad g_{C,V}(s^i_j) = f_C(c_{i,j}) = q^{-2(p,\text{wt}(v_i))} \Omega_{ji} = q^{2n-2-1} \Omega_{ji}.
\]
Hence it suffices to determine \( \Omega \) in order to determine \( g_{C,V} \). Define an involutive automorphism of the weight lattice \( \Theta : P \to P, \Theta(\mu) = w_{0i} \mu \). Note that \( \Theta(\alpha_i) = -\alpha_{n-i} \). By [Kol08, Lemma 5] the central element \( D \in Z(B^*) \cap (\text{ad}_i U)\tau(-2\omega_r) \) satisfies
\[
(5.15) \quad D \in \sum_{\zeta, \xi \in Q^+} (\text{ad}_i U^\zeta_\zeta)(\text{ad}_i U^-^\xi_\xi)\tau(-2\omega_r).
\]
Recall from Remark 1.6 that \( \hat{I}_{\mathbb{F}_3}(c_{n,n}) = \tau(-2\omega_r) \). By Proposition 1.4 the map \( \hat{I}_{\mathbb{F}_3} \) is an isomorphism of left \( U \)-modules and hence relations (5.15) and (5.12) imply that \( \Omega_{i,j} = 0 \) if \( n - i + 1 > j \). Hence we can write the matrix \( \Omega \) in the form

\[
\Omega = \begin{pmatrix}
0 & F \\
G & H
\end{pmatrix}
\]

where each entry \( F, G, \) and \( H \) is an \((m \times m)\)-matrix. It follows from the \( \text{ad}_l(t_i t_{p(i)}^{-1}) \)-invariance of \( D \) for \( 1 \leq i < m \) that both \( F \) and \( G \) are codiagonal and that \( H \) is diagonal. To determine the remaining entries of \( F, G, \) and \( H \) one uses the following formulae which immediately follow from (5.12).

**Lemma 5.15.** For \( 1 \leq i < m \) and \( 1 \leq j, k \leq m \) the following relations hold:

\[
\begin{align*}
(\text{ad}_l^* B_i) c_{k,n-k+1} & = -\delta_{i,k-1} q c_{k-1,n-k+1} + \delta_{i,k} q c_{k,n-k}, \\
(\text{ad}_l^* B_i) c_{n-k+1,k} & = \delta_{i,k} c_{n-k+1,k+1} - \delta_{i,k-1} c_{n-k+2,k}, \\
(\text{ad}_l^* B_i) c_{n-j+1,n-j+1} & = -\delta_{i,j-1} q^{-1} c_{n-j+2,n-j+1} + \delta_{i,j} q c_{n-j+1,n-j}.
\end{align*}
\]

The \( \text{ad}_l(B_i) \)-invariance of \( D \) for \( 1 \leq i < m \) now implies that all codiagonal entries of \( F \) and \( G \) are the same, respectively, and moreover that

\[
\Omega_{i,i} = q^2 \Omega_{i+1,i+1},
\]

if \( m + 1 \leq i \leq n - 1 \). It remains to determine the relation between \( \Omega_{m+1,m+1} \) and the entries of \( F \) or \( G \). To this end one calculates

\[
\begin{align*}
(\text{ad}_l^* B_m) c_{m+1,m+1} & = -q^2 c_{m,m+1} + q^{-1} c_{m+1,m} + s c_{m+1,m+1}, \\
(\text{ad}_l^* B_m) c_{m+1,m} & = -c_{m,m} + c_{m+1,m+1} + s q^{-2} c_{m+1,m}, \\
(\text{ad}_l^* B_m) c_{m,m+1} & = -q^{-1} c_{m+1,m+1} + q^{-1} c_{m,m} + s q^2 c_{m,m+1}.
\end{align*}
\]

Comparing coefficients in the equality \( \text{ad}_l(B_m) D = sD \) with respect to the basis elements \( l_{\mathbb{F}_3}(c_{i,j}) \) for \( (i,j) = (m+1,m), (m+1,m+1) \) one obtains

\[
F = qG
\]

and

\[
\Omega_{m+1,m+1} = s(q-q^{-1}) \Omega_{m+1,m}.
\]

The relations (5.20), (5.21), and (5.22) determine \( \Omega \) up to an overall scalar factor \( \lambda \in k \setminus \{0\} \). One obtains

\[
\Omega_{i,j} = \lambda \begin{cases} 
q & \text{if } i = n - j + 1 \leq m, \\
1 & \text{if } i = n - j + 1 \geq m, \\
(s(q-q^{-1})q^{-2(i-m-1)} & \text{if } i = j \geq m + 1, \\
0 & \text{else.}
\end{cases}
\]

The character \( g_{C,V} \) is obtained from (5.14). We summarise the result of our calculation in the following proposition which also proves Lemma 5.12.

**Proposition 5.16.** Up to a nonzero scalar multiple the numerical solution \( M = (g_{C,V}(s^2)) \) of the reflection equation corresponding to the central element \( C = \)
\[ \sigma(D) \in Z(\sigma(B^c)) \cap (\text{ad}_{\tau}) \tau(2\omega_r) \] is given by

\[
g_{C,V}(s_i^j) = \begin{cases} 
q^{2i-n} & \text{if } j = n - i + 1 \leq m, \\
q^{2i-n-1} & \text{if } j = n - i + 1 \geq m, \\
s(q^2 - 1) & \text{if } i = j + m + 1, \\
0 & \text{else.}
\end{cases}
\]

**Remark 5.17.** Note the structural similarity between the matrix \( M = (g_{C,V}(s_i^j)) \) and the matrix defined in [NDS97, (2.14)], [DS99, (6.13)]. In principal, this similarity is not surprising. One argues along the lines of [Let99, Section 6] that up to a Hopf algebra automorphism of \( U \) and painstaking translation of conventions the coideal subalgebra constructed in [NDS97] is a subalgebra of \( B^c \). Since the subspaces consisting of \( \text{ad}_{\tau} \)-invariant elements in \( F_1(U) \simeq k[G] \) with respect to the two coideal subalgebras coincide ([NDS97, Theorem 2.6], [DS99, Theorem 6.6] and [Let02, Theorem 7.7]) the centres of the two coideal subalgebras also coincide. This in turn links the corresponding solutions of the reflection equation.

**Remark 5.18.** The universal cylinder forms constructed in [tD99] also have the same structure as the character \( g_{C,V} \) constructed for the symmetric pair AIII(case 2) above.

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