We propose an \( A_4 \) extension of the Standard Model with a Lepton Quarticity symmetry correlating dark matter stability with the Dirac nature of neutrinos. The flavor symmetry predicts (i) a generalized bottom-tau mass relation involving all families, (ii) small neutrino masses are induced \textit{a la seesaw}, (iii) CP must be significantly violated in neutrino oscillations, (iv) the atmospheric angle \( \theta_{23} \) lies in the second octant, and (v) only the normal neutrino mass ordering is realized.
I. INTRODUCTION

Probably the number one mystery in particle physics is the understanding of the pattern of fermion masses and mixings from first principles. Indeed, the charged fermion mass pattern is not described in the theory: the Standard Model only allows us the freedom to fit the observed charged fermion masses, while lacking the masses of neutrinos altogether. An approach towards addressing, at least partially, the charged fermion mass problem, is the possibility of relating quarks and lepton masses as a result of a flavor symmetry \cite{1}, i.e.

\[
\frac{m_b}{\sqrt{m_d m_s}} = \frac{m_\tau}{\sqrt{m_e m_\mu}}.
\]

Notice that this mass relation constitutes a consistent flavor-dependent generalization of the conventional bottom-tau SU(5) prediction, but does not require grand-unification. It provides a partial solution to the charged fermion mass problem, which can be shown to hold in some theories of flavor based on the $A_4$ \cite{1,3} and $T_7$ \cite{4} symmetries.

Turning to neutrinos, the origin of their mass, the understanding of their mixing properties and the puzzle of whether they are their own anti-particles continue to defy theorists. Underpinning the solution to such neutrino puzzles may not only write a new chapter of particle physics, but also shed light on astrophysical and cosmological puzzles. One of the latter is the puzzle of Dark Matter, believed to be associated to the existence of a new absolutely or nearly stable neutral particle.

There have been attempts at formulating joint solutions to the above shortcomings of the standard model. For example, in scotogenic models dark matter is introduced as a messenger of radiative neutrino mass generation \cite{5-7} whose stability follows from the radiative nature of the neutrino mass. Several alternative ideas have come out, invoking non-Abelian flavor symmetries \cite{8-10}, such as the $A_4$ symmetry \cite{11-13}. For example, dark matter could be stable as a result of some remnant of the flavor symmetry associated to the pattern of neutrino mixing \cite{14,15}. In all these models neutrinos are Majorana type.

However, recently there has been a renewed interest in Dirac neutrinos \cite{16-31} which may attain naturally small masses in many scenarios. For example, it can happen that the same flavor symmetry which presumably sheds light on the pattern of neutrino oscillation parameters, also implies that neutrinos are Dirac fermions \cite{16}. Specially tantalizing is the idea that the stability of dark matter can be directly traced to the Dirac nature of neutrinos \cite{16,17,19,20}. One way to realize this idea is by means of a $Z_4$ Lepton Quarticity symmetry \cite{16,17}. Within such approach the same $Z_4$ discrete lepton number symmetry ensures the stability of dark matter and the absence of all the Majorana mass terms. Thus owing to Lepton Quarticity, the Dirac nature of neutrinos and the stability of dark matter are intimately related: the breakdown of this symmetry will simultaneously imply loss of dark matter stability as well as the Diracness of neutrinos.

Here we focus on the Lepton Quarticity models of dark matter, along the lines pursued in \cite{16,17}. The plan of the paper is as follows. In Sect. \ref{model} we sketch in some detail the extended particle content required to realize the non-Abelian flavor symmetry of the model, and show how the Dirac nature of neutrinos and the smallness of their seesaw-induced masses both follow from our non-Abelian discrete flavor symmetry. We also briefly discuss the appearance of a viable WIMP dark matter candidate in this model. In Sect. \ref{osc} we present our predictions for the current and future neutrino oscillation experiments. We find that the atmospheric angle $\theta_{23}$ and the CP phase $\delta_{CP}$, whose current experimental determination is still rather poor, are tightly related to each other within our model. Finally we summarize our results in Sect. \ref{conc}.

II. THE MODEL SETUP

Here we describe the model in some detail. The particle content of our model along with the $SU(2)_L \otimes Z_4 \otimes A_4$ charge assignments of the particles are given in Table \ref{table}. Note that in Table \ref{table} the $L_i = (\nu_i, l_i)^T$, $i = e, \mu, \tau$ denote the lepton doublets, transforming as indicated under the flavor symmetry.
Apart from the Standard Model fermions, the model also includes three right–handed neutrinos $\nu_{\alpha, R}$ which are singlets under the $\text{SU}(3)_C \otimes \text{SU}(2)_L \otimes \text{U}(1)_{Y}$ gauge group, singlets under $A_4$, but carry charge $z$ under $Z_4$. We also add three gauge singlet Dirac fermions $N_{i, \alpha, L, R}$; $i = 1, 2, 3$ transforming as triplets of $A_4$ and with charge $z$ under $Z_4$, as shown in Table I. Notice that in the scalar sector we have two different sets of fields $\Phi_{i, \alpha}^u, \Phi_{i, \alpha}^d; i = 1, 2, 3$, which are all doublets under the $\text{SU}(2)_L$ gauge group, both sets transforming trivially under $Z_4$. Under the $A_4$ flavor symmetry, $\Phi_{i, \alpha}^d$ transforms as a triplet, while $\Phi_{i, \alpha}^u$ transform as singlets. In addition to the above symmetries we also impose an additional $Z_2$ symmetry $\eta$. Under this $Z_2$ symmetry, all the fields transform as 1 except for $\Phi_{i, \alpha}^d$, $l_{\alpha, R}$ and $d_{\alpha, R}$, which transform as $-1$. The role of this $Z_2$ symmetry is to prevent the Higgs doublets $\Phi_{i, \alpha}^d$ from coupling the up-type quarks and neutrino sector, and the $\Phi_{i, \alpha}^u$ Higgs doublets from the down-type quarks and charged leptons.

In addition we need scalar singlets, for example the $\chi_i, i = 1, 2, 3$. These are gauge singlets transforming as a triplet under the $A_4$ and trivially under $Z_4$. We also add two other gauge singlet scalars $\zeta$ and $\eta$ both of which transform trivially under $A_4$ but carry $Z_4$ charges $z$ and $z^2$ respectively. Notice that, since under the $Z_4$ symmetry the field $\eta$ carries a charge $z^2 = -1$, it follows that $\eta$ can be taken to be real.

As discussed in \cite{16, 17} the lepton quarkity symmetry $Z_4$ serves a double purpose. It not only ensures that neutrinos are Dirac particles, but also guarantees the stability of the scalar particle $\zeta$, making it a viable dark matter WIMP. If the quarkity symmetry is broken either by an explicit soft term or spontaneously, through non-zero vacuum expectation values (vevs) to any of the scalars $\eta, \zeta$ which carry a non-trivial $Z_4$ charge, then both the Dirac nature of neutrinos and stability of dark matter is simultaneously lost.

We now turn our attention to the Yukawa sector of our model. In the neutrino sector the Yukawa terms relevant

---

1 This additional $Z_2$ symmetry is only required in a non-supersymmetric variant. Clearly the model can be easily supersymmetrized, in which case this additional $Z_2$ symmetry is no longer required.
for generating masses for the neutrinos and the heavy neutral fermions $N_L, N_R$ are given by

$$
\mathcal{L}_{\text{Yuk,}\nu} = y_1 \left[ \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}_3 \otimes \begin{pmatrix} N_{1,R} \\ N_{2,R} \\ N_{3,R} \end{pmatrix}_3 \right] \otimes (\Phi^\dagger_I)_1 + y_2 \left[ \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}_3 \otimes \begin{pmatrix} N_{1,R} \\ N_{2,R} \\ N_{3,R} \end{pmatrix}_3 \right] \otimes (\Phi^\dagger)_I + y_3 \left[ \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}_3 \otimes \begin{pmatrix} N_{1,R} \\ N_{2,R} \\ N_{3,R} \end{pmatrix}_3 \right] \otimes (\Phi^\dagger)_I + y_4 \left[ \begin{pmatrix} N_{1,L} \\ N_{2,L} \\ N_{3,L} \end{pmatrix}_3 \otimes \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}_3 \right] \otimes (\nu_e, R)_1
$$

where $3S$ and $3A$ denote the symmetric and antisymmetric $A_4$ triplet combinations obtained from the tensor product of two $A_4$ triplets. Notice also that $3S$ and $3A$ are not two different irreducible representations of $A_4$, which only has one triplet, but simply different contractions with the same transformation rule. Also, $y_i, y'_i, c_1, c_2; i = 1, 2, 3$ are the Yukawa couplings which, for simplicity, are taken to be real. The parameter $M$ is the gauge and flavor-invariant mass term for the heavy leptons. Here we like to highlight the important role played by the $A_4$ flavor symmetry. Owing to the $A_4$ charges of the left and right handed neutrinos, a tree level Yukawa coupling between them of type $\nu_e, \bar{L}_L, \nu_R \Phi^\dagger$ is forbidden. Thus neutrino masses can only appear through type-I Dirac seesaw mechanism as we now discuss.

After symmetry breaking the scalars $\chi_i$ and $\Phi^\dagger_i$ acquire vevs $(\chi_i) = u_i$; $(\Phi^\dagger_i) = v_i^\nu$; $i = 1, 2, 3$. The invariant mass term $M$ can be naturally much larger than the symmetry breaking scales, i.e. $M \gg v_i^\nu, u_i$. In this limit, for any numerical purpose the last two terms in Eq. 2 can be safely neglected. Under this approximation the $6 \times 6$ mass matrix for the neutrinos and the heavy neutral fermions in the basis $(\nu_e, L, \bar{\nu}_e, L, N_{1,L}, N_{2,L}, N_{3,L})$ and $(\nu_e, L, \bar{\nu}_e, L, N_{1,R}, N_{2,R}, N_{3,R})^T$ is given by

$$
M_{\nu,N} = \begin{pmatrix}
0 & 0 & 0 & a_1^1 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2^1 & 0 \\
0 & 0 & 0 & 0 & 0 & a_3^1 \\
y_1^1 u_1 & y_2^1 u_1 & y_3^1 u_1 & M & 0 & 0 \\
y_1^2 u_2 & \omega y_2^1 u_2 & 0 & 0 & M & 0 \\
y_1^3 u_3 & \omega^2 y_2^1 u_3 & \omega y_3^1 u_3 & 0 & 0 & M
\end{pmatrix}
$$

(3)

where $\omega$ is the third root of unity, with $\omega^3 = 1$ and

$$
a_1^1 = y_1 v_1^{\nu_1} + y_2 v_2^{\nu_1} + y_3 v_3^{\nu_1} \\
a_2^1 = y_1 v_1^{\nu_2} + \omega y_2 v_2^{\nu_2} + \omega^2 y_3 v_3^{\nu_2} \\
a_3^1 = y_1 v_1^{\nu_3} + \omega^2 y_2 v_2^{\nu_3} + \omega y_3 v_3^{\nu_3}
$$

(4)

As mentioned before, owing to the $A_4$ symmetry, a direct coupling between $\nu_L$ and $\nu_R$ is forbidden, leading to the vanishing of all entries in the upper left quadrant of Eq. 3. The mass matrix in Eq. 3 can be rewritten in a more
compact form, as
\[
M_{\nu,N} = \begin{pmatrix}
0 & \text{diag}(a'_1, a'_2, a'_3) \\
\text{diag}(u_1, u_2, u_3) \sqrt{3} U_m \text{diag}(y'_1, y'_2, y'_3) & M \text{diag}(1, 1, 1)
\end{pmatrix}
\]
(5)

Where \(U_m\) is the usual magic matrix,
\[
U_m = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}
\]
(6)

Note that, in the limit \(M \gg v_i u_i\), the mass matrix in Eq. (3) can be easily block diagonalized by the perturbative seesaw diagonalization method given in Ref. [32]. The resulting 3 × 3 mass matrix for light neutrinos can be viewed as the Dirac version of the well known type-I seesaw mechanism. The above mass generation mechanism can also be represented diagramatically as shown in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Feynman view of type-I Dirac seesaw mechanism in the model where the indices \(i, j, k, m, l = 1, 2, 3\).}
\end{figure}

The 3 × 3 matrix for the light neutrinos is
\[
M_\nu = \frac{1}{M} \text{diag}(a_1, a_2, a_3) \sqrt{3} U_m \text{diag}(y_1, y_2, y_3) ,
\]
(7)

where \(a_i = a'_i u_i\). We take the alignment \(u_1 = u_2 = u_3 = u\) for the vev of the \(A_4\) triplet scalars \(\chi_i\) similar to [11, 12]. In this alignment limit of \(A_4\) triplet scalars we have
\[
\begin{align*}
a_1 &= (y_1 v_1^u + y_2 v_2^u + y_3 v_3^u) u \\
a_2 &= (y_1 v_1^u + \omega y_2 v_2^u + \omega^2 y_3 v_3^u) u \\
a_3 &= (y_1 v_1^u + \omega^2 y_2 v_2^u + \omega y_3 v_3^u) u
\end{align*}
\]
(8)

which simplifies the notation, although it does not change the form of the neutrino mass matrix in Eq. (7). Notice that we have not imposed any alignment for the vevs of the \(A_4\) singlet scalars \(\Phi_i^u\). The light neutrino mass matrix of Eq. (7) with the simplified \(a_i\) of Eq. (8) can be diagonalized by a bi-unitary transformation
\[
U_\nu^\dagger M_\nu U_\nu = D_\nu ,
\]
(9)

where \(D_\nu\) is diagonal, real and positive. Owing to the \(A_4\) flavor symmetry, the resulting rotation matrix acting on left handed neutrinos \(U_\nu\) in the standard parametrization (for both hierarchies), leads to \(\theta_\nu^{23} = \frac{\pi}{4}\) and \(\delta_\nu = \pm \frac{\pi}{2}\) while the other two angles can be arbitrary. Thus, owing to the \(A_4\) symmetry, \(U_\nu\) in standard parameterization leads to following mixing angles
\[
\begin{align*}
\theta_\nu^{23} &= 45^\circ, \quad \delta_\nu = \pm 90^\circ \\
\theta_\nu^{12} &= \text{arbitrary} \quad \theta_\nu^{13} = \text{arbitrary}
\end{align*}
\]
(10)

Doing so for the different \(A_4\) singlet scalar vevs is not very natural. Indeed, unlike the case of \(A_4\) triplet scalars, a priori the vevs of different \(A_4\) singlet scalars have no reason to obey any mutual alignment.
Similar features of maximal $\theta_{23}$ and $\delta$ have been obtained previously in the context of Majorana neutrinos [12, 33]. Although the angles $\theta_{12}^{\nu}$ and $\theta_{13}^{\nu}$ can take arbitrary values they are strongly correlated with each other.

We have performed an extensive numerical scan for both type of hierarchies in the whole parameter range taking all Yukawa couplings in the perturbative range of $[-1, 1]$. We find that in the whole allowed range for either type of hierarchy, one cannot simultaneously fit both $\theta_{12}^{\nu}$ and $\theta_{13}^{\nu}$ in the current global experimental range obtained from neutrino oscillation experiments [34]. This implies that in our model $U_{\nu}$ alone cannot explain the current neutrino oscillation data.

However, the lepton mixing matrix $U_{LM}$ which is probed by neutrino oscillation experiments is the product of the charged lepton rotation matrix $U_l$ with the neutrino transformation matrix $U_{\nu}$ [35], i.e.

$$U_{LM} = U_l^\dagger U_{\nu}$$

(11)

In our model the charged lepton mixing matrix $U_l$ is also non-trivial and contributes to the full leptonic mixing matrix $U_{LM}$. We now move to discuss the structure of mass matrices and mixing matrices for charged leptons as well as the up and down type quarks.

We now turn to the discussion with up type quark mass matrix. The invariant Yukawa Lagrangian relevant to generating up type quark mass matrix is given by

$$\mathcal{L}_{\text{Yuk, }u} = y_1^u \left\{ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{3} \otimes \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}_{3} \right\} \otimes (\Phi_1^u)_{1}^{\prime} + y_2^u \left\{ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{3} \otimes \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}_{3} \right\} \otimes (\Phi_2^u)_{1}^{\prime}$$

$$+ y_3^u \left\{ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{3} \otimes \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}_{3} \right\} \otimes (\Phi_3^u)_{1}^{\prime} + h.c.$$  

(12)

where $y_i^u; i = 1, 2, 3$ are the Yukawa couplings which for simplicity we take to be all real. After spontaneous symmetry breaking Eq. (12) leads to a diagonal mass matrix given by

$$M_u = \begin{pmatrix} y_1^u v_1^u + y_2^u v_2^u + y_3^u v_3^u & 0 & 0 \\ 0 & y_1^u v_1^u + \omega y_2^u v_2^u + \omega^2 y_3^u v_3^u & 0 \\ 0 & 0 & y_1^u v_1^u + \omega^2 y_2^u v_2^u + \omega y_3^u v_3^u \end{pmatrix}.$$  

(13)

On the other hand, the Yukawa Lagrangian relevant to down type quarks mass generation is given by

$$\mathcal{L}_{\text{Yuk, }d} = y_1^d \left\{ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{3} \otimes \begin{pmatrix} q_d,R \\ q_s,R \\ q_b,R \end{pmatrix}_{3} \otimes (\Phi_1^d)_{3}^{\prime} \right\} + y_2^d \left\{ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{3} \otimes \begin{pmatrix} q_d,R \\ q_s,R \\ q_b,R \end{pmatrix}_{3} \otimes (\Phi_2^d)_{3}^{\prime} \right\} + h.c.$$  

(14)

where $y_i^d; i = 1, 2$ are the Yukawa couplings which for simplicity are taken to be real. The resulting mass matrix for down type quarks after spontaneous symmetry breaking is given by

$$M_d = \begin{pmatrix} 0 & a_d \alpha & b_d \\ b_d \alpha & 0 & a_d r \\ a_d & b_d r & 0 \end{pmatrix}.$$  

(15)

where $(\Phi_i^d) = \nu_i^d, i = 1, 2, 3$ and $a_d = (y_1^d - y_2^d) v_2^d, b_l = (y_1^d + y_2^d) v_2^d$. Moreover, $\alpha$ and $r$ are ratios of the vevs of $\Phi_i^d$ and are given as $\alpha = v_2^d/v_2^d$ and $r = v_1^d/v_2^d$. 


Finally, the invariant Yukawa terms for the charged leptons is given by
\[
\mathcal{L}_\text{Yuk,1} = y_1 \left[ \begin{pmatrix} L_1 \\ \bar{L}_2 \\ L_3 \end{pmatrix} \otimes \begin{pmatrix} l_{e,R} \\ l_{\mu,R} \\ l_{\tau,R} \end{pmatrix} \otimes \begin{pmatrix} \Phi_1^d \\ \Phi_2^d \\ \Phi_3^d \end{pmatrix} \right] + y_2 \left[ \begin{pmatrix} \bar{L}_1 \\ L_2 \\ L_3 \end{pmatrix} \otimes \begin{pmatrix} l_{e,R} \\ l_{\mu,R} \\ l_{\tau,R} \end{pmatrix} \otimes \begin{pmatrix} \Phi_1^d \\ \Phi_2^d \\ \Phi_3^d \end{pmatrix} \right] + h.c. \quad (16)
\]
where \( y_1, i = 1, 2 \), are the Yukawa couplings which, for simplicity, we take to be real. After symmetry breaking the charged lepton mass matrix is given by
\[
M_l = \begin{pmatrix} 0 & a_l \alpha & b_l \\ b_l \alpha & 0 & a_l r \\ a_l & b_l r & 0 \end{pmatrix} \quad (17)
\]
where, just as in the down quark case, here also \( a_l = (y_1 - y_2)\nu_1^2, b_l = (y_1 + y_2)\nu_2^2 \). The parameters \( \alpha, r \) which are the ratios of the vevs of \( \Phi_i^d \) i.e. \( \alpha = \nu_1^2/\nu_2^2 \) and \( r = \nu_2^2/\nu_3^2 \) are the same as those defined after Eq. 15. This matrix is completely analogous to the down-type quark mass matrix. Note that while \( \alpha \) and \( r \) are the same both in the quark and in the lepton sector, as they are simply ratios between the vevs of \( \Phi_i^d \), while \( a_f \) and \( b_f, f \in \{ l, q \} \), are different.

These mass matrices for charged leptons and down-type quarks correspond to those discussed in [1-4] and lead to the generalized bottom-tau relation of [1]. In section III we show that there is enough freedom to fit the charged lepton and down type quark masses within their 1-\( \sigma \) range. Apart from fitting all the masses as well as leading to the generalized bottom-tau relations, the charged lepton mass matrix [17] also leads to non-trivial charged lepton rotation matrix \( U_l \). As we show in section III this non-trivial contribution from \( U_l \) results in a lepton mixing matrix \( U_{LM} \) consistent with the current global fits to neutrino oscillation data [34]. The lepton mixing matrix obtained from our model also implies normal hierarchy for neutrino masses and leads to an interesting correlation between the atmospheric mixing angle and CP violating phase, the two most ill-determined parameters in leptonic mixing matrix.

III. FLAVOR PREDICTIONS: NUMERICAL RESULTS

In this section we discuss the phenomenological implications of our model. The important predictions emerging in our model are: a) the flavor-dependent bottom-tau unification mass relation of Eq. (1) b) a correlation between the two poorly determined oscillation parameters: the atmospheric angle \( \theta_{23} \) and \( \delta_{CP} \) and c) a normal hierarchy for the neutrinos. In this section we discuss these numerical predictions in some detail, given the experimentally measured “down-type” fermion masses, solar and reactor mixing angles as well as neutrino squared mass differences.

A. Charged lepton and down-type quark masses

We start our discussion by looking in more detail at the down type quark and charged lepton mass matrices discussed previously in Eqs. 15 and 17. This structure for the down type quark and charged lepton mass matrices has been previously discussed in several works [1-4]. In this section for illustration purpose we first discuss the results obtained in previous works by closely following the approach taken in previous works like in [36]. Subsequently, we will generalize the analysis of previous works and discuss how the same results can be obtained using a more general setup and more detailed considerations.

We start from the charged lepton mass matrix obtained in Eq. 17. The correct charged lepton masses are reproduced if the vevs of the \( A_4 \) triplet fields \( \Phi_i^d \) satisfy the alignment limit \( \nu^d = (1, \epsilon_1, \epsilon_2) \), where \( \nu^d \gg \epsilon_1, \epsilon_2 \). Then, in similar notation and spirit as in Ref. [36], we extract the three invariants of the Hermitian matrix \( S = M_l M_l^\dagger \): \( Det(S), Tr(S) \)
and $Tr(S^2) - Tr(S^2)$. We then compute their values in the diagonal basis in terms of the charged lepton masses, $m_e$, $m_\mu$ and $m_\tau$. The equations are

$$
\begin{align*}
\text{Det}S &= (m_e m_\mu m_\tau)^2 \\
\text{Tr}S &= m_e^2 + m_\mu^2 + m_\tau^2 \\
(\text{Tr}S)^2 - \text{Tr}S^2 &= 2m_\mu^2 m_\tau^2 + 2m_e^2 m_\tau^2 + 2m_\mu^2 m_e^2 \\
\end{align*}
$$

(18)

The expressions in Eq. 18 can be readily solved in the vev alignment limit $\nu^d(1, \epsilon_1, \epsilon_2)$ discussed before. This amounts to the approximation

$$
\begin{align*}
v_1^d &\gg v_3^d \quad \text{and} \quad \frac{v_1^d}{v_2^d} \gg \frac{y_1^d + y_2^d}{y_1^d - y_2^d} \gg 1, \\
or equivalently:
\end{align*}
$$

$$
\begin{align*}
r &\gg \alpha \quad \text{and} \quad r \gg \frac{b_l}{a_l} \gg 1.
\end{align*}
$$

The solutions for $r$, $a_l$ and $b_l$ are given as

$$
\begin{align*}
r &= \frac{m_\tau}{\sqrt{m_e m_\mu}} \sqrt{\alpha} \\
a_l &= \frac{m_\mu}{m_\tau} \sqrt{\frac{m_e m_\mu}{\alpha}} \\
b_l &= \sqrt{\frac{m_e m_\mu}{\alpha}}
\end{align*}
$$

(19) (20) (21)

Owing to the $A_4$ symmetry, the charged lepton mass matrix in Eq. 17 and down type quark mass matrix in Eq. 15 have the same structure. As a result the down quark mass matrix can also be decomposed using equations analogous to Eq. 18. For down type mass matrix of 15 we obtain

$$
\begin{align*}
r &= \frac{m_b}{\sqrt{m_d m_s}} \sqrt{\alpha} \\
a_d &= \frac{m_s}{m_b} \sqrt{\frac{m_d m_s}{\alpha}} \\
b_d &= \sqrt{\frac{m_d m_s}{\alpha}}
\end{align*}
$$

(22) (23) (24)

Note that the parameters $\alpha$ and $r$ are common for both the charged lepton sector as well as in the down-type quark sector, as they are simply ratios between vevs of the fields $\Phi^d_i$. Thus comparing Eqs. 19 and 22 we obtain the following mass relation

$$
\frac{m_\tau}{\sqrt{m_e m_\mu}} = \frac{m_b}{\sqrt{m_d m_s}}
$$

(25)

The procedure sketched above can be performed in a more general way by solving the equations numerically. The relevant equations for the case of charged leptons are

$$
\begin{align*}
(m_e m_\mu m_\tau)^2 &= a_t^6 r^2 \alpha^2 + 2a_t^3 b_l^2 r^2 \alpha^2 + b_l^6 r^2 \alpha^2 \\
m_e^2 + m_\mu^2 + m_\tau^2 &= (a_t^2 + b_l^2)(1 + r^2 + \alpha^2) \\
2m_\mu^2 m_\tau^2 + 2m_e^2 m_\tau^2 + 2m_e^2 m_\tau^2 &= (a_t^2 + b_l^2)^2(1 + r^2 + \alpha^2)^2 - (a_t^2 + b_l^2 r^2)^2 - (b_l^6 + a_t^2 \alpha^2)^2 - (a_t^2 r^2 + b_l^2 \alpha^2)^2
\end{align*}
$$

(26)

Taking as input parameters the best fit values (at $M_Z$ scale) for the charged lepton masses [37] and imposing $r > \frac{b_l}{a_t}$, there is a one-parameter family of solutions to these equations. These are related to the approximate solution described before. We build the functions $r(\alpha)$, $a_l(\alpha)$ and $b_l(\alpha)$ taking $\alpha$ as a free parameter. In the correct range for
the parameter $\alpha$, the unique solution is found to be near the limit $r \gg b \gg 1$ and therefore it again leads to the mass relation in Eq. 1. Since the $(\alpha, r, a_l, b_l)$ are solutions of Eqs. 26, the charged lepton masses are fitted exactly to their best-fit values. In order to underpin the relevant solution for down type quark masses, we also need to take into account not only the mass relation 1 and the charged lepton masses, but also the constraints for the experimental measurements (along with renormalization group evolution to $M_Z$ scale) of all the down-type quark masses \[37\]. Here, we will impose the rather stringent 1-$\sigma$ bound\[37\] on the down-type quarks masses at $Z$ boson energy scale \[37\].

Then, for each valid $(\alpha, r, a_l, b_l)$ we take $a_q$ and $b_q$ as

\[ a_q = \frac{m_s}{m_b} \sqrt{\frac{m_d m_s}{\alpha}} (1 + \epsilon_1) \]

\[ b_q = \sqrt{\frac{m_d m_s}{\alpha}} (1 + \epsilon_2) , \]

where $\epsilon_1$ and $\epsilon_2$ are expected to be small.

Using this procedure gives sets of parameters $(\alpha, r, a_l, b_l, a_q, b_q)$ which give at the same time best fit values for charged lepton masses, down-type quarks inside the 1$\sigma$ range and the mass relation in Eq. 1. In figure 2 we show the family-dependent bottom-tau mass prediction of our model for the s and d masses, along with their allowed 1-$\sigma$ ranges.

Figure 2. Prediction for the s and d quark masses (at $M_Z$ scale) in our model. The dark blue area is the allowed region from our model for the s and d quark masses, while varying the mass of the b quark in its 1-$\sigma$ range. The light blue area is the allowed 1-$\sigma$ range (at $M_Z$ scale) for the mass of the quarks s and d \[37\].

B. The charged piece of the lepton mixing matrix $U_l$

The charged lepton mass matrix Eq. 17 not only leads to correct lepton masses but also to non-trivial charged lepton rotation matrix $U_l$ as we discuss now. Just like the neutrino mass matrix, the charged lepton mass matrix can

\[ ^3 \text{Imposing 1-$\sigma$ is in fact rather stringent, and can easily be relaxed to a more conservative criterium e.g. 3-$\sigma$. We have deliberatively imposed the stringent 1-$\sigma$ bound in other to highlight the high precision obtained from our results.} \]

\[ ^3 \]
also be diagonalized by a bi-unitary transformation as

\[ U_l^\dagger M_l V = \text{diag}(m_e, m_\mu, m_\tau) \] (29)

The charged lepton mixing matrix \( U_l \) in standard parameterization can be written as

\[ U_l = P U_{23}(\theta_{23}^l, 0) U_{13}(\theta_{13}^l, \delta_l^l) U_{12}(\theta_{12}^l, 0) P' \] (30)

where \( P \) and \( P' \) are diagonal matrix of phases and \( U_{ij} \) is the usual complex rotation matrix appearing in the symmetrical parametrization of fermion mixing given in [35], e.g.

\[
U_{12}(\theta_{12}, \delta) = \begin{pmatrix}
\cos \theta_{12} & e^{-i \delta} \sin \theta_{12} & 0 \\
-e^{i \delta} \sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\] (31)

with an analogous definitions for \( U_{13} \) and \( U_{23} \). For the charged lepton mass matrix \( m_l \) we find that

\[
\sin \theta_{12}^l = \frac{m_e}{m_\mu} \frac{1}{\sqrt{\alpha}} + \mathcal{O}(\frac{1}{\alpha^2}) \approx \mathcal{O}(\lambda_C)
\]
\[
\sin \theta_{13}^l = \frac{m_\mu}{m_\tau} \frac{1}{\sqrt{\alpha}} + \mathcal{O}(\frac{1}{\alpha^2}) \approx \mathcal{O}(10^{-5})
\]
\[
\sin \theta_{23}^l = \frac{m_e m_\mu}{m_\tau^2} \frac{1}{\alpha} + \mathcal{O}(\frac{1}{\alpha^2}) \approx \mathcal{O}(10^{-7})
\] (32)

Where \( \lambda_C \approx 0.22 \) is the sine of the Cabbibo angle. In order to reproduce adequate values for the CKM matrix elements we may introduce a vector-like quark mixing with the up-type quarks, along the lines followed recently in [38].

The diagonal phases in \( P \) and \( P' \) are all exactly 0 except for one which is \( \pi \). Performing the numerical computation reconfirms the results obtained in Eq. 32 for the charged lepton mass matrix i.e. \( \theta_{12}^l \) is finite and its value depends on the value of \( \alpha \) in an inverse way, while \( \theta_{13}^l \) and \( \theta_{23}^l \) are both negligible (in particular, \( \theta_{13}^l \sim 10^{-5} \) and \( \theta_{23}^l \sim 10^{-7} \). Then, the charged lepton mixing matrix for our model is given as

\[
U_l \approx \begin{pmatrix}
\cos \theta_{12}^l & \sin \theta_{12}^l & 0 \\
-\sin \theta_{12}^l & \cos \theta_{12}^l & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \times \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\] (33)

Thus in our model the lepton mixing matrix \( U_{LM} = U_l^\dagger U_\nu \) receives significant charged lepton corrections which have interesting phenomenological consequences as we discuss in next section.

C. The lepton mixing matrix and neutrino mass ordering

As mentioned before in Section 11, the light neutrino mass matrix in Eq. 7 leads to the neutrino mixing matrix \( U_\nu \) which in standard parameterization [35] leads to

\[ U_\nu = P U_{23}(\pi/4, 0) U_{13}(\theta_{13}, \pi/2) U_{12}(\theta_{12}^l, 0) P' \] (34)

As mentioned before, owing to the \( A_4 \) symmetry, we have that \( \theta_{23}^\nu = 45^\circ \) and \( \delta_{CP}^\nu = 90^\circ \) for both types of mass ordering: normal hierarchy (NH) or inverted hierarchy (IH). Since neutrinos in our model are Dirac fermions, the phases in the right in Eq. 84 i.e. \( P' \), are unphysical, while \( \theta_{13}^\nu \) and \( \theta_{12}^\nu \) are strongly correlated between each other. This result is completely general and follows from the \( A_4 \) symmetry, independently of the mass hierarchy, NH or IH. However, the behavior of the correlation between \( \theta_{12}^\nu \) and \( \theta_{13}^\nu \) does depend on the choice of NH or IH.

Taking into account the results in the previous sections, the lepton mixing matrix is

\[
U_{LM} = U_l^\dagger U_\nu = U_{12}(\theta_{12}^l, 0)^\dagger U_\nu
\] (35)
One can regard the matrix $U_l$ as a correction to the neutrino mixing parameters obtained just by diagonalizing the neutrino mass matrix. For the NH case, the angle $\theta_{12}^l$ has to be big enough ($\sim 15^\circ$) so as to account for the correct mixing angles of the lepton mixing matrix, but at the same time it has to remain controlled ($< 20^\circ$) otherwise the down-type quark masses cannot be fitted. This means that the parameter $\alpha$ has to be between 0.04 and 0.08. This lepton mixing matrix can fit the neutrino oscillation parameters within $3\sigma$ at the same time as the mass matrices fit the down-type quarks and the neutrino squared mass differences in the $1\sigma$ range and the charged lepton masses exactly. Once the lepton mixing matrix is written in the standard parametrization, two interesting features arise. On the one hand, $\theta_{23} > 45^\circ$ and, on the other, a strong correlation appears between the atmospheric angle $\theta_{23}$ and $\delta_{CP}$, as shown in figure 3.

![Figure 3. CP violation and $\theta_{23}$ predictions within the model. Left panel: $\delta_{CP}$ vs $\theta_{23}$. The green regions are the $1\sigma$ (dark) and $3\sigma$ (light) regions for $\theta_{23}$ from current oscillation fit. Right panel: Same correlation, now showing $J_{CP}$ vs $\sin^2 \theta_{23}$ and zooming in the region allowed by the model, fully consistent in the $2\sigma$ experimental range.](image)

For IH, a different scenario arises. As in the case for NH, lepton corrections cannot be very big otherwise the down-type quark masses will not be fitted. However, the structure of the correlation between $\theta_{12}^\nu$ and $\theta_{13}^\nu$ implies that for allowed charged lepton corrections, the reactor angle $\theta_{13}$ is always outside the $3\sigma$ allowed range. Note that the model does not include any a priori theoretical bias in favour of normal hierarchy but it is a prediction of the model once one impose experimental constraints.

IV. DISCUSSION AND SUMMARY

We have proposed a $A_4 \otimes Z_4 \otimes Z_2$ flavor extension of the Standard Model with naturally small Dirac neutrino masses. Our lepton quarticity symmetry simultaneously forbids Majorana mass terms and provides dark matter stability. The flavor symmetry plays a multiple role, providing : (i) a generalized family-dependent bottom-tau mass relation, Eq. (1) and Fig. 2, (ii) a natural realization of the type-I seesaw mechanism for Dirac neutrino masses, as the tree level Dirac Yukawa term between left and right handed neutrinos is forbidden, (iii) a very predictive flavor structure to the lepton mixing matrix. The latter directly correlates the CP phase $\delta_{CP}$ and the atmospheric angle $\theta_{23}$, as shown in Fig. 3. This implies that (iv) CP must be significantly violated in neutrino oscillations, and the atmospheric angle $\theta_{23}$ lies in the second octant, (v) only the normal neutrino mass ordering is realized.

Our approach provides an adequate pattern of neutrino mass and mixing as well as a viable stable dark matter. This is achieved while providing testable predictions concerning the currently most relevant oscillation parameters, the atmospheric angle $\theta_{23}$ and the CP phase $\delta_{CP}$, as well as a successful family generalization of bottom-tau unification, despite the absence of an underlying Grand Unified Theory. Our lepton quarticity approach also leads to other
interesting phenomena such as neutrinoless quadruple beta decay ($0\nu4\beta$), which has recently been probed by the NEMO collaboration [39]. The intimate connection between the Dirac nature of neutrinos and dark matter stability constitutes a key feature of our model. Other phenomenological implications will be taken up elsewhere.

ACKNOWLEDGMENTS

This research is supported by the Spanish grants FPA2014-58183-P, Multidark CSD2009-00064, SEV-2014-0398 (MINECO) and PROMETEOII/2014/084 (Generalitat Valenciana). The feynman diagram in Fig.1 was drawn using Jaxodraw [40].

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