Instabilities in Josephson Ladders with Current Induced Magnetic Fields

B. R. Trees and R. A. Murgescu
Department of Physics and Astronomy
Ohio Wesleyan University
Delaware, OH 43015
(November 19, 2018)

We report on a theoretical analysis, consisting of both numerical and analytic work, of the stability of synchronization of a ladder array of Josephson junctions under the influence of current induced magnetic fields. Surprisingly, we find that as the ratio of the mutual to self inductance of the cells of the array is increased a region of unstable behavior occurs followed by reentrant stable synchronization. Analytic work tells us that in order to understand fully the cause of the observed instabilities the behavior of the vertical junctions, sometimes ignored in analytic analyses of ladder arrays, must be taken into account.

PACS No.: 74.50.+r, 05.45.Xt, 05.45.-a

Ladder arrays of Josephson junctions are intriguing systems for a wealth of reasons: the possibility of phase locking, or synchronizing, a maximal subset of junctions suggests their use as microwave sources [1]; they offer rich dynamical behavior, accessible to both theorists and experimentalists, in the field of coupled nonlinear oscillators (with recent interest in the prediction and observation of discrete rotobreathers) [2]; their complexity is between that of better understood one dimensional serial and parallel arrays and full 2D arrays (e. g. square arrays) and thus they offer a nice link between the two geometries; and ladder arrays can, under circumstances that are partially understood, be modeled by the discrete sine-Gordon (DSG) equation [3], which is itself a source of research interest among many [4]. With a desire to understand better the conditions under which stable synchronization can occur, we consider ladder arrays biased with uniform dc bias currents $I_B$ greater than the critical currents of the junctions and include the effects of current induced magnetic fields (CIMF’s) via self and mutual inductances of the cells of the array. (see Fig. 1) Since the array is current-biased above the critical current there will be a nonzero voltage across some subset of junctions in the array. These “active” junctions are synchronized, or phase locked, if their voltages, after some initial transients, are identical functions of time. Furthermore, previous workers have established that the effects of CIMF’s may be important [3] in determining the static and dynamics properties of arrays, and so it is natural to consider the effects of CIMF’s on synchronization as well.

In this Letter we present numerical and analytical evidence that mutual inductance between cells of a ladder array can lead to rich dynamical behavior, including destabilization of synchronization and reentrant synchronization as the relative size of the mutual to self inductance of the cells is increased. The instability results for a finite range of values of the mutual inductance and occurs when, for the resistively and capacitively shunted junction (RCSJ) model, the numerical solutions to the model equations diverge with time. In addition to our numerical solutions we also investigate the dynamics analytically. Namely, the two coupled RCSJ equations for the horizontal and vertical junctions can be approximately decoupled. The simplified equations allow us to calculate analytically the Floquet exponents, which measure the degree of stability of the synchronization. Then, by direct comparison of the analytic results for the Floquet exponents, based on the simplified equations, and the numerical results for the Floquet exponents, based on the full RCSJ equations, we learn valuable information about the roles of the horizontal and vertical junctions in the array. This technique appears to be a powerful way to analyze the relative importance of subsets of junctions that experience different local conditions.

Josephson junctions consist of superconducting islands separated by a thin layer of nonsuperconducting material. In the superconductors, the coherent motion of the paired electrons, or Cooper pairs, leads to a wavefunction of the form $\Psi = |\Psi|e^{i\theta}$, where $\theta$ is the macroscopic phase of the superconductor. The equations describing the dynamics of a single junction depend on the gauge-invariant phase difference, or Josephson phase, across the junction, $\phi = \theta_1 - \theta_2 - (2\pi/\Phi_0) \int_A A \cdot d\ell$, where $A$, the vector potential due to an external magnetic field, is integrated along a path from one side of the junction to the other. $\Phi_0 = \hbar/2e$ is the magnetic flux quantum, where $\hbar$ is Planck’s constant divided by $2\pi$ and $e$ is the electronic charge. We assume in this work that $A = 0$.

Consider a ladder array of underdamped junctions with $N$ cells and periodic boundary conditions, as shown in Fig. 1. Each junction has a McCumber parameter $\beta_c \equiv 2\pi CI_e R^2/\Phi_0$, where $C(R)$ is the junction’s capacitance.
An analysis based on the RCSJ equations and including the effect of induced magnetic flux leads to a pair of coupled equations for the Josephson phases of the horizontal and vertical junctions,

\[ \beta_c \phi'' + \phi' + \sin \phi + \frac{1}{\beta_L} \text{Tr} \cdot X^{-1} \cdot (Z \cdot \phi + 2\psi) = 0, \]

\[ \beta_c \psi'' + \psi' + \alpha \sin \psi + \frac{1}{\beta_L} \text{Tr} \cdot X^{-1} \cdot (Z \cdot \phi + 2\psi) = 0, \]

where \( \beta_L \equiv 2\pi LI_{cx}/\Phi_0 \) is the dimensionless self-inductance of a given cell, and \( \alpha \equiv I_{cy}/I_{cx} \) is the critical current anisotropy. The prime symbols denote differentiation with respect to dimensionless time, \( \tau \equiv t/\tau_c \) where \( \tau_c \equiv \hbar/2eI_{cx}R \).

These equations are compactly represented in matrix notation, where \( \phi \) and \( \psi \) are \( N \)-dimensional vectors representing the Josephson phases of the horizontal and vertical junctions, respectively. \( X \) is an \( N \times N \) matrix that depends on geometry, and \( X^{-1} \) is the dimensionless inductance matrix, also \( N \times N \) in size. The diagonal terms of \( X \) represent the self-inductance of a given cell, \( \text{diag}(X) = \pm \mu_L \), where \( \mu_L \equiv M/L \) is the dimensionless mutual inductance. All other elements of \( X \) are zero. Eqs. 1 and 2 were solved numerically for \( \phi \) and \( \psi \) via a fourth-order Runge-Kutta algorithm as a function of the parameters \( N, \beta_c, \beta_L, \mu_L, \) and \( \delta_B \equiv I_B/I_{cx} \) (the dimensionless bias current). The starting configuration consisted of random voltages and zero Josephson phases.

As described elsewhere [8], a stability analysis of the solutions to Eqs. 1 and 2 follows by letting \( \phi = \phi_0 + \eta \) and \( \psi = \psi_0 + \delta \), where \( \phi_0 \) and \( \psi_0 \) are solutions to these equations. Eqs. 1 and 2 are linearized with respect to \( \eta \) and \( \delta \). The time dependence of the perturbations has the form \( \eta \sim e^{\lambda t} \) and \( \delta \sim e^{\mu t} \), where \( \lambda \) and \( \mu \) represent the Floquet exponents for the horizontal (vertical) junctions. If \( \text{Re}(\lambda) > 0 \) or \( \text{Re}(\mu) > 0 \) we expect unstable behavior of the array for the given set of circuit parameters. In fact, we are interested in that exponent whose magnitude is closest to zero, as that describes the stability of the longest-lived mode of the array.

Figure 2(a) shows the minimum Floquet exponent as a function of \( \mu_L \) for a 5-cell ladder. The three different sets of symbols represent three different sets of initial values for the voltages, \( \phi' \) and \( \psi' \). The data show that the exponents do depend, to some degree, on the values of the starting voltages, although all the runs follow the same trend. As \( \mu_L \) is increased from zero towards 0.5, the stability of phase-locking increases, as shown by the negative exponent of growing magnitude, while the degree of stability decreases for increasing \( \mu_L \) greater than approximately 0.6. Even more interesting is the behavior of the ladder in the range 0.5 \( \lesssim \mu_L \lesssim 0.6 \). For these values of the mutual inductance the ladder is actually unstable! This is evidenced by very rapidly growing phases and voltages with time as Eqs. 1 and 2 are numerically integrated. For \( N = 5 \), the lower limit of this instability region is \( \mu_L^{(1)} = 0.5 \) independent of other circuit parameters such as \( \beta_c \) and \( \beta_L \). The upper limit of this region, which we denote by \( \mu_L^{(2)} \), depends on such quantities as the value of the starting voltages as well as on the value of \( \beta_L \). For example, for a fixed set of starting voltages, we find that \( \mu_L^{(2)} \) is a decreasing function of \( \beta_L \). Also, it is interesting to note that this instability region does not appear at all if both the phases and the voltages are initialized to zero! (See discussion below for the reason for this behavior.)

Physically, this instability is due to a competition between the self-inductance of a given loop (say loop \( j \)), which wishes to have a current with a given sense of circulation, and the mutual inductance of the two neighboring loops \( (j \pm 1) \), which wish to have the current in loop \( j \) flow in the opposite sense. We have also looked at ladders with \( N = 6, 7, 8, \) and 9. All show this instability in the vicinity of \( \mu_L = 0.5 \). Indeed, we would expect this competition-induced instability to be independent of ladder size for the case of nearest-neighbor mutual inductance in that the onset of the instability should always occur at \( \mu_L = 0.5 \).
Interestingly, ladders with $N = 7$, 8, and 9 also exhibit a second instability region that the 5-cell ladder does not exhibit. Fig. 3(b) shows the Floquet exponents for $N = 7$. This second instability region has an onset at a value of $\mu_L^{(3)} > \mu_L^{(2)}$ that is dependent on ladder size. (See cross-hatched region in Fig. 3(b).) We now turn to an analytic calculation of the Floquet exponents, which helps us understand the source of these instabilities.

A reasonable starting approximation is to ignore the coupling between the horizontal and vertical junctions but otherwise not to ignore the effects of the vertical junctions. That is, we let $\psi = 0$ in eq. 3 and $\phi = 0$ in eq. 2. An analysis like that described in Ref. 3 applied to eq. 3 leads to a Mathieu equation describing the time dependence of the perturbations to the horizontal Josephson phases. In such a case, the corresponding Floquet exponents for the horizontal junctions can be calculated analytically. The result is

$$\text{Re} (\lambda_m t_c) = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4 \omega_m^{(N)} (\frac{\beta_c}{\beta_L})^2},$$

where we can think of $\omega_m^{(N)}$ as the effective normal-mode frequencies of the ladder. For example, $\omega_m^{(5)} = (4 \sin^2(\pi m/7) + 2 \mu_L \{\cos(2 \pi m/7) - \cos(4 \pi m/7)\})/(\mu_L^2 - \mu_L - 1)$, where $0 \leq m \leq N - 1$. Note that $\omega_m^{(N)}$ is a function of $\mu_L$.

Eq. 3 was used to produce the solid curves in Figs. 2(a) and (b). Note that if, for particular values of $\omega_m^{(N)}$, $\beta_c$ and $\beta_L$, the argument of the square root in Eq. 3 is positive and larger than one, then at least one of the Floquet exponents will be positive, signaling unstable phase-locked solutions. In fact, since $\beta_c > 0$ and $\beta_L > 0$ such an instability will occur if $\omega_m^{(N)} < 0$! Plots of $\omega_1^{(N)}$ versus $\mu_L$ for $N = 5$ and 7 (not shown here) demonstrate that $\omega_2^{(5)} > 0$ for $0 \leq \mu_L \leq 1$, but $\omega_1^{(7)}$ is negative for $\mu_L > 0.8$. We have checked that $\omega_m^{(N)} > 0$ for $m \neq 1$ and for $N = 5$ and 7. Thus the cause of this instability in the 7-cell ladder for $\mu_L > \mu_L^{(3)}$ is the $m = 1$ normal mode. That is, this instability is a geometrical effect, in that it does not occur for $N = 5$ for example, and it is triggered by an effective normal mode frequency of the horizontal junctions becoming negative.

This analytic work, however, does not point to the horizontal junctions as the cause of the instability near $\mu_L = 0.5$. To appreciate this behavior it is crucial to look to the vertical junctions. A procedure similar to that which led to eq. 3 leads to a set of effective Floquet exponents for the vertical junctions

$$\text{Re} (\Lambda_m t_c) = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4 \gamma_m^{(N)} (\frac{\alpha \cos \psi_0 + \frac{2 \gamma_m^{(N)}}{\beta_L}}{(\alpha \beta_c \cos \psi_0)}/2)},$$

where the geometrical factor $\gamma_m^{(N)}$ is also a function of $\mu_L$ and is similar but not identical to $\omega_m^{(N)}$. In this case, the vertical junctions will exhibit an exponentially growing Josephson phase if $\gamma_m^{(N)} < -\frac{1}{2}$ now a plot of $\gamma_m^{(5)}$ versus $\mu_L$ (see inset of Fig. 3(a)) shows that the function abruptly becomes negative at $\mu_L = 0.5$ and asymptotically approaches zero from the negative side as $\mu_L$ is increased further. (We have checked that $\gamma_m^{(5)} > 0$ for $m \neq 0$. Also, we see similar behavior for the 7-cell ladders.) If we assume that $\cos \psi_0 > 0$, then the vertical junctions will be unstable for $\gamma_m^{(N)} < 0$. Based on the behavior of $\gamma_0^{(5)}$ an instability region will exist for a range of $\mu_L$ values, $\mu_L^{(1)} \leq \mu_L \leq \mu_L^{(2)}$ where $\mu_L^{(1)} = 0.5$ and $\mu_L^{(2)}$ will depend on $\alpha$, $\beta_L$, and $\cos \psi_0$. For example, as $\beta_L$ increases we expect that $\mu_L^{(2)}$ will decrease, i.e. approach a value of 0.5. This agrees with the behavior of the numerical results for the Floquet exponents. Also, the inequality $\gamma_m^{(N)} < -\frac{1}{2}$ should depend on the value of $\cos \psi_0$. This is relevant to the numerical results in Fig. 3 where we see that the value of $\mu_L^{(2)}$ does indeed on the choice of the starting configuration of phases and voltages. In general, then, it is clear that the instability near $\mu_L = 0.5$ originates with the vertical junctions and would thus be missed by an analysis that was based solely on the horizontal junctions. It is also clear why this instability does not appear numerically when both the Josephson phases and the voltages across the junctions are initialized to zero. In such a scenario, although the horizontal junctions may be active, the only possible solution for the vertical junctions is to keep zero voltages and Josephson phases for all times. Since we know this instability region is triggered by the vertical junctions, the vertical junctions have no chance to “go unstable” and thus the instability never appears.

We conclude that mutual inductance between cells of an underdamped ladder array has the effect of destabilizing synchronization for ranges of values of $\mu_L$, the ratio of the mutual to self inductance. These specific ranges of $\mu_L$ that
lead to unstable behavior are geometry dependent. An analytic calculation of the Floquet exponents based on the horizontal junctions agrees with the numerical exponents, based on the full RCSJ equations, for those values of $\mu_L$ for which stable phase locking occur. To understand the cause of all the observed instabilities, however, it is crucial in the analytic work to consider the behavior of the vertical junctions.

Although some values of the mutual inductance used in these simulations can not be obtained in simple ladder arrays, this work suggests that experimentalists may wish to attempt fabrication of arrays that enhances the mutual over the self-inductance, perhaps making it possible to look for the rich dynamical behavior predicted here. Certainly researchers working on the problem of coherent emission from Josephson junction arrays should be aware this potential for unstable behavior exists.

The authors wish to thank Barbara Andereck, Tom Dillman, Steve Herbert, Mark Jarrell, and David Stroud for useful conversations. This research was funded by the Howard Hughes Medical Institute Undergraduate Biological Sciences Education Program grant #71196-529503 to Ohio Wesleyan University.

[1] P. Barbara, A. B. Cawthorne, S. V. Shitov, and C. J. Lobb, Phys. Rev. Lett. 82, 1963 (99); M. Basler, W. Krech, and K. Platov, Phys. Rev. B 58, 3409 (98); K. Wiesenfeld, S. P. Benz, and P. A. A. Booij, J. Appl. Phys. 76, 3832 (94).
[2] P. Binder, D. Abramov, A. V. Ustinov, S. Flach, and Y. Zolotaryuk, Phys. Rev. Lett. 84, 745 (00); E. Trías, J. J. Mazo, and T. P. Orlando, Phys. Rev. Lett. 84, 741 (00).
[3] S. Ryu, W. Yu, and D. Stroud, Phys. Rev. E 53, 2190 (96).
[4] S. F. Mingaleev, Yu. B. Gaididei, E. Majerníková, and S. Shpyrka, Phys. Rev. E 61, 4454 (00) and the references therein.
[5] J. J. Mazo and J. C. Ciria, Phys. Rev. B 54, 16068 (96); D. Domínguez and J. V. José, Phys. Rev. B 53, 11692 (96); A. Majhofer, T. Wolf, and W. Dieterich, Phys. Rev. B 44, 9634 (91); J. R. Phillips, H. S. J. van der Zant, J. White, and T. P. Orlando, Phys. Rev. B 50, 9387 (94).
[6] J. R. Phillips, H. S. J. van der Zant, J. White, and T. P. Orlando, Phys. Rev. B 47, 5219 (93).
[7] M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1996).
[8] B. R. Trees and N. Hussain, Phys. Rev. E 61, 6415 (00).

FIG. 1. Ladder array of Josephson junctions with periodic boundary conditions. The horizontal junctions, along the rungs of the ladder, are parallel to the $x$ axis, while the vertical junctions are parallel to the $y$ axis. A dc bias current, $I_B$, is injected at each node on one side and extracted from the opposite side. The Josephson phase for the horizontal (vertical) junction in the $j$th plaquette is $\phi_j$ ($\psi_j$).

FIG. 2. Minimum Floquet exponent for periodic ladders versus the dimensionless, nearest-neighbor mutual inductance. (The results correspond to $i_B = 10$, $\beta_c = 10$, and $\beta_L = 100$.) The three different sets of symbols represent three different starting configurations of voltages across the junctions, while the Josephson phase differences were always initialized to zero. The solid line represents an analytic result (Eq. [3]) based on the horizontal junctions. (a) $N = 5$. The analytic result predicts stable phase-locked solutions for $0 \leq \mu_L \leq 1$. The numerical results exhibit an instability, however, for $\mu_L^{(1)} \leq \mu_L \leq \mu_L^{(2)}$ where $\mu_L^{(1)} = 0.5$ and $\mu_L^{(2)}$ is dependent on the starting configuration of phases and voltages, as well as on the value of $\beta_L$. This instability originates with the vertical junctions. INSER T: geometric quantity $\gamma_0^{(5)}$ versus $\mu_L$. (see eq. [3]) (b) $N = 7$. In this case, the geometry of the ladder leads to an instability for $\mu_L > 0.8$ that originates with the horizontal junctions. This “large $\mu_L$” instability is marked in the figure as a cross-hatched region. The instability due to the vertical junctions near $\mu_L = 0.5$ still exists but is narrower than for the 5-cell ladder.
This figure "fig2a.jpg" is available in "jpg" format from:

http://arxiv.org/ps/cond-mat/0008278v1
This figure "fig2b.jpg" is available in "jpg" format from:

http://arxiv.org/ps/cond-mat/0008278v1