Moments for multi-dimensional Mandelbrot’s cascades

Chunmao HUANG∗

Harbin institute of technology at Weihai, Department of mathematics, 264209, Weihai, China

Abstract

We consider the distributional equation

\[ Z \overset{d}{=} \sum_{k=1}^{N} A_k Z(k), \]

where \( N \) is a random variable taking value in \( \mathbb{N}_0 = \{0, 1, \cdots\} \), \( A_1, A_2, \cdots \) are \( p \times p \) non-negative random matrix, and \( Z, Z(1), Z(2), \cdots \) are i.i.d random vectors in \( \mathbb{R}_p^+ \) with \( \mathbb{R}_p^+ = [0, \infty) \), which are independent of \( (N, A_1, A_2, \cdots) \). Let \( \{Y_n\} \) be the multi-dimensional Mandelbrot’s martingale defined as sums of products of random matrices indexed by nodes of a Galton-Watson tree plus an appropriate vector. Its limit \( Y \) is a solution of the equation above. For \( \alpha > 1 \), we show respectively a sufficient condition and a necessary condition for \( E \|Y\|_\alpha \in (0, \infty) \). Then for a non-degenerate solution \( Z \) of the equation above, we show the decay rates of \( E e^{-t \cdot Z} \) as \( \|t\| \to \infty \) and those of the tail probability \( P(y \cdot Z \leq x) \) as \( x \to 0 \) for given \( y = (y_1, \cdots, y_p) \in \mathbb{R}_p^+ \), and the existence of the harmonic moments of \( y \cdot Z \). As application, these above results about the moments (of positive and negative orders) of \( Y \) are applied to a special multitype branching random walk. Moreover, for the case where all the vectors and matrices of the equation above are complex, a sufficient condition for the \( L^\alpha \) convergence and the \( \alpha \)th-moment of the Mandelbrot’s martingale \( \{Y_n\} \) is also established.

Key words: moments, harmonic moments, Mandelbrot’s martingales, multiplicative cascades, multi-branching random walks

AMS subject classification: 60K37, 60J80

1 Introduction

We consider a multi-dimensional Mandelbrot’s martingale \( \{Y_n\} \) defined as sums of products of random matrices (weights) indexed by nodes of a Galton-Watson tree plus an appropriate vector. We are interested in the existence of the moments of positive and negative orders of its limit \( Y \). For the one-dimensional case, the classical model of Mandelbrot [22] corresponds to the case where the tree is a fixed \( r \)-ary tree (\( r \geq 2 \) being a constant), and all the weights are one-dimensional random variables. This classical model and its variations were studied by many authors in different contexts, see for example: Bingham & Doney [8, 9] for branching processes and general age-dependent branching processes; Kahane & Peyrière [15], Guivarc’h [12] and Barral [3] for multiplicative cascades; Biggins [4] and Biggins & Kyprianou [5] for branching random walks; Durrett & Liggett [11] for some infinite particle systems; Rößler [23] for the Quicksort algorithm. A general one-dimensional model (called Mandelbrot’s cascades) which unifies the study of cascades and branching random walks was presented by Liu [20], where a number of applications were shown. The model considered here is a generalization of the model presented in [20] to the multi-dimensional case. Similar to the one-dimensional case, our model is also corresponding to multi-type branching random walks which attract some authors’ attention recently, see for example Kyprianou & Rahimzadeh Sani [17], Biggins & Rahimzadeh Sani [6] and Biggins [7]. This paper is our first exploration to multi-dimensional Mandelbrot’s cascades. Considering the practicability, we choose to begin with the existence of the moments of \( Y \), which are useful to study the asymptotic properties of \( \{Y_n\} \).

Let’s present our model and problems. We consider the distributional equation of \( Z \):

\[ Z \overset{d}{=} \sum_{k=1}^{N} A_k Z(k), \]
where $N$ is a random variable taking value in $\mathbb{N}_0 = \{0, 1, \cdots\}$, $A_1, A_2, \cdots$ are $p \times p$ non-negative random matrix; $Z, Z(1), Z(2), \cdots$, which are independent of $(N, A_1, A_2, \cdots)$, are i.i.d random vectors in $\mathbb{R}^p_+$ with $\mathbb{R}^+ = [0, \infty)$.  

We say a matrix $A$ is finite if all entries of $A$ are finite, and say $A$ is strictly positive if for some positive integer $n$, all entries of $A^n$ are positive. When a matrix $A$ is finite and strictly positive, the Perron-Frobenius theorem shows that $A$ has a positive maximal eigenvalue $\rho$ and has associated positive right and left eigenvectors $v = (v_1, \cdots, v_p)$ and $u = (u_1, \cdots, u_p)$. Moreover, $u$, $v$ can be normalized so that $\sum_{i=1}^p u_i = \sum_{i=1}^p u_i v_i = 1$. Throughout this paper, we assume that

**Assumption (H).** The matrix $M := \mathbb{E} \sum_{k=1}^N A_k$ is finite and strictly positive with the maximum-modulus eigenvalue 1 and the corresponding left and right eigenvectors $U = (U_1, \cdots, U_p)$, $V = (V_1, \cdots, V_p)$ normalized such that $\sum_{i=1}^p U_i = \sum_{i=1}^p U_i V_i = 1$.

We are interested in the existence of the solution with a moment ($\alpha > 1$) of the equation (E), and furthermore, the existence of its harmonic moments. It is clear that there exists a solution of equation (E).

In fact, we can construct a solution (denoted by $Y$) in the following way. Let $N = \{1, 2, \cdots\}$ and write

$$I = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

for the set of all finite sequences $u = u_1 \cdots u_n$ with $u_i \in N$, where by convention $\mathbb{N}^0 = \{\emptyset\}$ contains the null sequence $\emptyset$. If $u = u_1 \cdots u_n \in I$, we write $|u| = n$ for the length of $u$; if $u = u_1 \cdots u_n, v = v_1 \cdots v_m \in I$, we write $uv = u_1 \cdots u_n v_1 \cdots v_m$ for the sequence obtained by juxtaposition. In particular, $\emptyset u = \emptyset u = u$.

We partially order $I$ by writing $u \leq v$ to mean that for some $u' \in I$, $v = uv'$, and by writing $u < v$ to mean that $u \leq v$ and $u \neq v$.

Let $\{(N_u, A_{u1}, A_{u2}, \cdots)\}$ be a family of independent copies of $(N, A_1, A_2, \cdots)$, indexed by all the finite sequence $u \in I$. For simplicity, we write $(N, A_1, A_2, \cdots)$ for $(N_\emptyset, A_{\emptyset1}, A_{\emptyset2}, \cdots)$. Let $T$ be the Galton-Watson tree with defining elements $(N_u)$ ($u \in I$): (i) $\emptyset \in T$; (ii) if $u \in T$, then $uk \in T$ if and only if $1 \leq k \leq N_u$; (iii) if $uk \in T$, then $u \in T$. Here the null sequence $\emptyset$ is the root of the tree $T$, which can be regarded as the initial particle; $uk$ represents the $k$-th child of $u$; $N_u$ represents the number of offspring of the particle $u$.

Each node of the tree $T$ is marked with the random vector $(N_u, A_{u1}, A_{u2}, \cdots)$. We can imagine that the random matrix $A_{uk}$ is the "weight" associated with the edge $(u, uk)$ linking the nodes $u$ and $uk$ if $u \in T$ and $1 \leq k \leq N_u$; the values $A_{uk}$ for $k > N_u$ are of no influence for our purpose, and will be taken as 0 for convenience.

Let $T_n = \{u \in T : |u| = n\}$ be the set of sequence $u$ in $T$ with length $|u| = n$. Put

$$X_u = A_{u1}A_{u1u2} \cdots A_{u1\cdots u_n} \quad \text{if} \quad u = u_1 \cdots u_n \in I \quad \text{for} \quad n \geq 1,$$

and define

$$Y_0 = V \quad \text{and} \quad Y_n = \sum_{u \in T_n} X_u V \quad \text{for} \quad n \geq 1.$$  \hspace{1cm} (1.1)

It is not difficult to verify that $Y_n = (Y_{n,1}, \cdots, Y_{n,p})$ is a non-negative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma((N_u, A_{u1}, A_{u2}, \cdots) : |u| < n),$$

the $\sigma$-field that contains all information up to generation $n$. We call $\{Y_n\}$ Multi-dimensional Mandelbrot’s martingale. It reduce to the classical Mandelbrot’s martingale when the dimension $p = 1$. Clearly, there exists a non-negative random vector $Y = (Y_1, \cdots, Y_p) \in \mathbb{R}^p_+$ such that

$$Y = \lim_{n \to \infty} Y_n$$

almost surely (a.s.) with $EY_i \leq V_i$ for all $1 \leq i \leq p$ by Fatou’s lemma. Notice that

$$Y_n = \sum_{u \in T_n} A_u Y_{n-1}(u),$$  \hspace{1cm} (1.2)
where \( \{Y_n(u)\} \ (u \in \mathbb{T}_k) \) are independent copies of \( Y_n \) and they are independent of \( \mathcal{F}_k \). Denote \( Y(u) = \lim_{n \to \infty} Y_n(u) \). Letting \( n \to \infty \) in (1.2), we have

\[
Y = \sum_{k=1}^{N} A_k Y(k),
\]

which means that \( Y \) is a solution of the equation (1.3).

**Example 1.1 Multitype branching random walk (MBRW)** A multitype branching random walk (MBRW) with \( p \) types defined as follows. A single particle \( \emptyset \), of type \( i \in \{1,2,\cdots,p\} \) is located at the origin of real line \( \mathbb{R} \). It gives birth to children of the first generation, which are scattered on \( \mathbb{R} \), according to a vector point process \( L_i = (L_{i1}, L_{i2}, \cdots, L_{ip}) \), where \( L_{ij} \) is the point process counting the number of particles of type \( j \in \{1,2,\cdots,p\} \) born to the particle of type \( i \). These particles of the first generation reproduce particles to form the second generation. The displacements of the offsprings of a particle of type \( j \), relative to their parent’s position, are given by the point process \( L_j \). These particles of the second generation reproduce children to form the next generation, and so on. All particles behave independently. We denote the position of a particle \( u \) by \( S_u \) and the type of \( u \) by \( \tau(u) \), then the position of \( uk \), the \( k \)-th child of \( u \) satisfies

\[
S_{uk} = S_u + l_{uk},
\]

where \( l_{uk} \) denotes the displacement of \( uk \) relative to \( u \) whose distribution is determined by \( L_{\tau(uk)} \).

Assume that \( N_i := \sum_{j=1}^{p} Z_{ij}(\mathbb{R}) \) has the same distribution for all \( 1 \leq i \leq p \), which means that all particles produce offspring according to the same distribution if we don’t care the type. Under this assumption, all particles \( u \in I \) associated with the numbers of their offspring \( N_u \) form a Galton-Watson tree \( \mathbb{T} \) described above. We remark that this assumption is not necessary in a usual MBRW, so the example presented here is just a special case of MBRW. For more information and more results about the usual MBRW, cf [6,7,17].

For \( t \in \mathbb{R} \), define the matrix \( \tilde{M}(t) = (\tilde{M}_{ij}(t)) \) as

\[
\tilde{M}_{ij}(t) = \sum_{u \in \mathbb{N}_i, \tau(u) = j} e^{-tS_u} \quad (\tau(\emptyset) = i).
\]

Assume that \( \tilde{M}(t) \) defined above is finite and strictly positive. Denote the positive maximal eigenvalue of \( \tilde{M}(t) \) by \( \tilde{\rho}(t) \) and the associated normalized positive left and right eigenvectors by \( \tilde{U}(t) = (\tilde{U}_1(t), \cdots, \tilde{U}_p(t)) \) and \( \tilde{V}(t) = (\tilde{V}_1(t), \cdots, \tilde{V}_p(t)) \) respectively. For each \( i = 1, 2, \cdots, p \), let

\[
W_{n,i}(t) := \sum_{u \in \mathbb{N}_i, \tau(u) = j} \tilde{V}_{\tau(u)}(t) e^{-tS_u} \tilde{U}_i(t) / \tilde{\rho}(t)^n \quad (\tau(\emptyset) = i).
\]

It is known that for each \( i = 1, 2, \cdots, p \), \( \{W_{n,i}(t)\} \) forms a non-negative martingale with mean one, hence it converges a.s. to a non-negative random variable \( W^i(t) \) with \( \mathbb{E}W_i(t) \leq 1 \). Write

\[
Y_n = \left( W_{n,1}(t) \tilde{V}_1(t), \ W_{n,2}(t) \tilde{V}_2(t), \cdots, W_{n,p}(t) \tilde{V}_p(t) \right).
\]

We can see that the martingale \( \{Y_n\} \) is just the Mandelbrot’s martingale defined in (1.1) if we put the random matrix \( A_k = ((A_k)_{ij}) \), where

\[
(A_k)_{ij} = \frac{e^{-tS_k}}{\tilde{\rho}(t)} 1_{\{\tau(k) = j\}} \quad (\tau(\emptyset) = i).
\]

Indeed, with \( A_k \), we have \( M = \frac{\tilde{M}(t)}{\tilde{\rho}(t)} \), so that \( V = \tilde{V}(t) \). Notice that for \( u = u_1 \cdots u_n \),

\[
(A_{u_1 \cdots u_n})_{ij} = \frac{e^{-tS_u}}{\tilde{\rho}(t)^n} 1_{\{\tau(u) = j\}} \quad (\tau(\emptyset) = i).
\]
Thus by (1.1), for each \( i = 1, 2, \ldots, p \), with \( \tau(0) = i \),

\[
Y_{n,i} = \sum_{u \in \mathbb{F}_n} e^{-tS_u} \rho(t)^n \bar{V}_{\tau(u)}(t) = W_{n,i}(t) \tilde{V}_i(t)
\]

Therefore, the limit of \( Y_n \), namely,

\[
Y = \left( W_1(t) \tilde{V}_1(t), \ W_2(t) \tilde{V}_2(t), \ \cdots, W_p(t) \tilde{V}_p(t) \right)
\]

satisfies (1.3).

2 Main results

Let \( Y \) be the limit of the Mandelbrot’s martingale \( \{Y_n\} \). We first discuss the existence of the \( \alpha \)-th moment (\( \alpha > 1 \)) of \( Y \), which implies its non-degeneracy.

For \( t \in \mathbb{R} \) fixed, define the random matrix \( A_k^{(t)} = ((A_k^{(t)})_{ij}) \) as \( (A_k^{(t)})_{ij} := |(A_k)_{ij}|^\alpha \). Let

\[
M(t) := \mathbb{E} \sum_{k=1}^N A_k^{(t)}
\]

When \( M(t) \) is finite and strictly positive, we denote its positive maximal eigenvalue by \( \rho(t) \) and the corresponding positive left and right eigenvectors by \( U(t) = (U_1(t), \ldots, U_p(t)) \) and \( V(t) = (V_1(t), \ldots, V_p(t)) \) normalized such that \( \sum_{i=1}^p U_i(t) = \sum_{i=1}^p U_i(t)V_i(t) = 1 \). Define

\[
X_n^{(t)} = A_{u_1}^{(t)}A_{u_2}^{(t)} \cdots A_{u_n}^{(t)} \quad \text{if} \quad u = u_1 \ldots u_n \in I \quad \text{for} \quad n \geq 1,
\]

\[
\rho_1(t) \leq \rho(t) < \infty
\]

Clearly, \( Y_n^{(t)} = (Y_{n,1}^{(t)}, \ldots, Y_{n,p}^{(t)}) \) is a non-negative martingale with mean \( V(t) \), so it converges a.s. to a random vector \( Y^{(t)} = (Y_1^{(t)}, \ldots, Y_p^{(t)}) \). In particular, when \( t = 1 \), we have \( X_{u}^{(1)} = X_u, \rho(1) = 1 \) and \( V(1) = V \), hence \( Y_n^{(1)} = Y_n \) and \( Y^{(1)} = Y \).

Further more, define the matrix \( M_n(t) = ((M_n(t))_{ij}) \) as

\[
(M_n(t))_{ij} := \mathbb{E} \sum_{u \in \mathbb{T}_n} [(X_u)_{ij}]^\alpha
\]

with the maximum-modulus eigenvalue denoted by \( \rho_n(t) \) and the corresponding normalized positive left and right eigenvectors by \( U_n(t) = (U_{n,1}(t), \ldots, U_{n,p}(t)) \) and \( V_n(t) = (V_{n,1}(t), \ldots, V_{n,p}(t)) \). In particular, \( \rho_1(t) = \rho(t) \).

We denote that throughout this paper the notation norm \( \|A\| \) represents any one of the matrix norms if \( A \) is a matrix, and \( \|u\| = \sum_{j=1}^p |u_j| \) is the \( L^1 \)-norm of \( u = (u_1, \ldots, u_p) \) if \( u \) is a vector.

Theorem 2.1 (Moments). Let \( \alpha > 1 \).

(a) If \( \mathbb{E} \sum_{k=1}^N A_k \|_{\|A_k\|^\alpha} < \infty \) and \( \rho^{(\alpha-1)} \rho_0(\alpha) < 1 \) for some positive integer \( n \), then

\[
0 < \mathbb{E}\|Y\|^\alpha < \infty \quad \text{and} \quad \mathbb{E}Y = V.
\]

(b) Conversely, if \( 0 < \mathbb{E}\|Y\|^\alpha < \infty \), then \( \mathbb{E} \sum_{k=1}^N A_k \|_{\|A_k\|^\alpha} < \infty \) and \( \rho_0(\alpha) \leq 1 \) for all \( n \). If additionally

\[
P(\forall k \in \{1, 2, \ldots, N\}, \ A_k \ has \ a \ positive \ column \ vector) > 0,
\]

then \( \rho_0(\alpha) < 1 \) for all \( n \).
Remark 2.1. (i) For $\alpha > 1$, under Assumption (H), the condition $\mathbb{E}\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$ ensures that $M(\alpha)$ is finite and strictly positive, so that $\rho(\alpha)$ exists. Notice that for each $t \in \mathbb{R}$ fixed,

$$[M(t)]^n \leq M_n(t) \leq p^{(\alpha-1)(n-1)}[M(t)]^n,$$

where for two matrix $A = (a_{ij}), B = (b_{ij})$, the inequality $A \preceq B$ means that $a_{ij} \leq b_{ij}$ for all $i, j$. Thus the existences of $\rho(t)$ and $\rho_n(t)$ are equivalent, and we moreover have for each $t \in \mathbb{R}$ fixed,

$$\rho(t)^n \leq \rho_n(t) \leq p^{(\alpha-1)(n-1)}\rho(t)^n.$$ 

Therefore, under Assumption (H) and the condition $\mathbb{E}\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$, $\rho_n(t)$ exists for all $t \in [1, \alpha]$ and for all $n$. Besides, we remark that the condition $\mathbb{E}\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$ is equivalent to $\mathbb{E}\|Y\|^\alpha < \infty$.

(ii) Under Assumption (H), $\mathbb{E}\|Y\|^\alpha > 0$ is equivalent to $\mathbb{E}(Y_i)^\alpha > 0$ for all $i \in \{1, \cdots, p\}$. Indeed, by (1.3), one can see that $\mathbb{E}Y$ is a an eigenvector associated to the eigenvalue 1. If it is non-trivial, i.e. $\mathbb{E}Y \neq 0$, then $\mathbb{E}Y = cV$ for some constant $c > 0$, which implies that $\mathbb{E}Y$ is positive.

Theorem 2.1(a) shows a sufficient condition for the existence of the $\alpha$th-moment ($\alpha > 1$) of $Y$, or equivalently, the $L^\alpha$ convergence of the martingale $\{Y_n\}$ to its limit $Y$. If $\mathbb{E}(Y_i)^\alpha < \infty$, it is obvious that $\mathbb{E}Y_i = Y_i$ and $P(Y_i > 0) > 0$. As $Y$ is a solution of the equation (E), Theorem 2.1(a) in fact also gives the existence of a non-trivial solution of equation (E).

Moreover, if $p^{(\alpha-1)}\rho_\alpha(\alpha) < 1$ for some positive integer $\alpha$, Theorem 2.1 implies that $0 < \mathbb{E}\|Y\|^\alpha < \infty$ if and only if $\mathbb{E}\|Y_1\|^\alpha < \infty$, which reveals that $Y_1$ and $Y$ would have the same asymptotic properties. In particular, for $p = 1$, if $P(\forall k \in \{1, 2, \cdots, N\}, A_k > 0) > 0$, Theorem 2.1 says that

$$0 < \mathbb{E}Y_1^\alpha < \infty \quad \text{if and only if} \quad \mathbb{E}Y_1^\alpha < \infty \quad \text{and} \quad \rho(\alpha) < 1.$$ 

This result was obtained by Liu (20, Theorem 2.1) with the help of a size-biased measure. Here our proof will present a different idea based on inequalities for martingale. Our method, which is available for both $p = 1$ and $p > 1$, also avoids the trouble of finding an convenient size-biased measure for the case where $p > 1$. We mention that this method can also be used to the complex case where $A_k$ are complex random matrices and $Z$ and $Z(k)$ are complex random vectors, see Section 6.

Now we consider the existence of harmonic moments of $Y$, i.e., $\mathbb{E}(Y_i)^{-\lambda} < \infty$, for each $i \in \{1, 2, \cdots, p\}$, where $\lambda > 0$. We shall deal with a more general case, with a general non-trivial solution of equation (E), denoted still by $Z$, instead of $Y$.

Let $Z$ be a non-trivial solution of equation (E). Then we have $P(Z > 0) > 0$, where $Z > 0$ means that $Z_i > 0$ for all $i = 1, 2, \cdots, p$. Assume that (2.2) holds, and

$$P(N = 0) = 0, \quad P(N = 1) < 1. \quad (2.3)$$

In fact, assumption (2.2) is object to ensure that the probability $P(Z = 0)$ is a solution of the equation $f(q) = q$, where $f(s) = Es^N$ $(0 \leq s \leq 1)$ is the generating function of $N$. Since $P(Z = 0) < 1$, under assumptions (2.3), by the unity of solution, we have $P(Z = 0) = 0$, or namely, $P(Z > 0) = 1$. Let

$$\phi(t) = \mathbb{E}e^{-tZ}, \quad t = (t_1, \cdots, t_p) \in \mathbb{R}^p_+,$$

be the Laplace transform of $Z$, where we write $u \cdot v = \sum_{j=1}^{p} u_jv_j$ for the inner product of two vectors $u$ and $v$. We are interested in the decay rate of $\phi(t)$ as $\|t\| \to \infty$ and that of the tail probability $P(y \cdot Z \leq x)$ as $x \to 0$, for given $y = (y_1, \cdots, y_p) \in \mathbb{R}_+^p$, as well as the harmonic moment $\mathbb{E}(y \cdot Z)^{-\lambda}$ for $\lambda > 0$. Set

$$m := \text{essinf} \ N$$

Then $m \geq 1$, since $P(N = 0) = 0$. We have the following result.
Theorem 2.4 (Harmonic moments). Assume \( (2.2) \) and \( (2.3) \). Write \( a_{ij} = (A_1)_{ij} \). If

\[
\mathbb{E} \left( \min_{i} \sum_{j=1}^{p} a_{ij} \right)^{-\lambda} < \infty \quad \text{and} \quad \mathbb{E} \left[ \left( \min_{i} \sum_{j=1}^{p} a_{ij} \right)^{-\lambda} 1_{\{N=1\}} \right] < 1
\]

for some \( \lambda > 0 \), then

\[
\phi(t) = O(\|t\|^{-\lambda}) \quad (\|t\| \to \infty),
\]

and for every fixed non-zero \( y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+, \)

\[
P(y \cdot Z \leq x) = O(x^{\lambda}) \quad (x \to 0), \quad \mathbb{E}(y \cdot Z)^{-\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).
\]

If additionally \( m \geq 1 \) and \( \mathbb{E} \left[ \prod_{k=1}^{m} \left( \min_{i} \sum_{j=1}^{p} (A_1)_{ij} \right)^{-\lambda} \right] < \infty \), then

\[
\phi(t) = O(\|t\|^{-\lambda m}) \quad (\|t\| \to \infty), \quad P(y \cdot Z \leq x) = O(x^{\lambda m}) \quad (x \to 0), \quad \mathbb{E}(y \cdot Z)^{-\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).
\]

From Theorem 2.2, we can deduce similar results for each component \( Z_i \) of \( Z \). Let \( \phi_i(t) = \mathbb{E} e^{-tZ_i} \) \((t > 0)\) be the Laplace transform of \( Z_i \). Denote by \( e_i \) the vector which the \( i \)-th component is 1 and the others are 0. Then \( \phi_i(t) = \phi(t e_i) \) and \( e_i \cdot Z = Z_i \). Applying Theorem 2.2 to \( \phi(t e_i) \) and \( e_i \cdot Z \), we immediately get the following corollary.

Corollary 2.3. Under the conditions of Theorem 2.2 we have for each \( i \in \{1, 2, \cdots, p\} \),

\[
\phi_i(t) = O(t^{-\lambda}) \quad (t \to \infty), \quad P(Z_i \leq x) = O(x^{\lambda}) \quad (x \to 0), \quad \mathbb{E}(Z_i)^{-\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).
\]

If additionally \( m \geq 1 \) and \( \mathbb{E} \left[ \prod_{k=1}^{m} \left( \min_{i} \sum_{j=1}^{p} (A_1)_{ij} \right)^{-\lambda} \right] < \infty \), then

\[
\phi_i(t) = O(t^{-\lambda m}) \quad (t \to \infty), \quad P(Z_i \leq x) = O(x^{\lambda m}) \quad (x \to 0), \quad \mathbb{E}(Z_i)^{-\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).
\]

For \( p = 1 \), Theorem 2.2 (or Corollary 2.3) coincides with the results of Liu [21], Theorems 2.1 and 2.4. But when \( p > 1 \), to find the critical value for the existence of harmonic moments like [21] seems difficult. Similar to (21), Theorem 2.5, we also have result below about the exponential decay rate of \( \phi(t) \).

Theorem 2.4 (The exponential case). Assume that \( (2.3) \) holds, \( m \geq 2 \) and \( \min_{i,j} (A_1)_{ij} \geq \underline{a} \) a.s. for some constant \( \underline{a} > 0 \) and all \( 1 \leq k \leq m \).

(a) If \( P(N = m) > 0 \), then there exists a constant \( C_1 > 0 \) such that for all \( \|t\| > 0 \) large enough,

\[
\phi(t) \leq \exp\{-C_1 \|t\|^{-\gamma}\},
\]

and for every fixed non-zero \( y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+, \) there exists a constant \( C_{1,y} > 0 \) such that for all \( x > 0 \) small enough,

\[
P(y \cdot Z \leq x) \leq \exp\{-C_{1,y} x^{-\gamma/(1-\gamma)}\},
\]

where \( \gamma = -\log \underline{a} / \log (\underline{a} p) \in (0, 1) \).

(b) For some \( \varepsilon > 0 \) satisfying \( (\underline{a} + \varepsilon)p m < 1 \), if \( P\left( N = m, \max_{i,j} (A_1)_{ij} \leq \underline{a} + \varepsilon \text{ for all } 1 \leq k \leq m \right) > 0 \), then there exists a constant \( C_2 > 0 \) such that for all \( \|t\| > 0 \) large enough,

\[
\phi(t) \geq \exp\{-C_2 \|t\|^{-\gamma\varepsilon}\},
\]

and for every fixed \( y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+, \) there exists a constant \( C_{2,y} > 0 \) such that for all \( x > 0 \) small enough,

\[
P(y \cdot Z \leq x) \geq \exp\{-C_{2,y} x^{-\gamma\varepsilon/(1-\gamma\varepsilon)}\},
\]

where \( \gamma(\varepsilon) = -\log \underline{a} / \log (\underline{a} + \varepsilon)p \in (0, 1) \).
Finally, as applications of the above moment results for the limit \( Y \) of Mandelbrot’s martingale \( \{ Y_n \} \), we consider the MBRW described in Example 1.1 and show the sufficient conditions for the existence of moments (of positive and negative orders) of \( W_i(t) \), for each \( i = 1, 2, \cdots, p \) and for \( t \in \mathbb{R} \) fixed. For MBRW, it is obvious that (2.2) is satisfied. Notice that \( M(\alpha) = \frac{M(\alpha)}{p(t)^\alpha} \), which leads to \( \rho(\alpha) = \frac{\rho(\alpha)}{p(t)^\alpha} \). Applying Theorem 2.1 yields the result for the moments of positive orders, and Theorem 2.2 yields the one for the moments of negative orders.

**Corollary 2.5 (Application to MBRW).** We consider the MBRW described in Example 1.1.

(a) Let \( \alpha > 1 \). If \( \max_i \mathbb{E}(W_{1,i}(t))^\alpha < \infty \) and \( p^\alpha - 1 \frac{\rho(\alpha)}{p(t)^\alpha} < 1 \), then \( \max_i \mathbb{E}[W_i(t)]^\alpha < \infty \).

(b) Assume (2.3). Denote by \( S_i^1 \) the displacement of the first child of the initial particle \( \emptyset \) of type \( i \in \{1, \cdots, p\} \). Let \( \lambda > 0 \). If \( \max_i \mathbb{E} e^{-(\lambda+\epsilon)\lambda S_i^1} < \infty \) and \( \max_i \mathbb{E} e^{-(\lambda+\epsilon)\lambda S_i^1} 1_{\{\lambda S_i^1 < 1\}} < 1 \) for some \( \epsilon > 0 \), then \( \max_i \mathbb{E}[W_i(t)]^{-\lambda} < \infty \).

Corollary 2.5(a) gives a sufficient condition for the existence of \( \alpha \)-th moment of \( W_i(t) \). In fact, if we deal with the martingale \( \{ W_{n,i}(t) \} \) directly according to the ideas in the proof of Theorem 2.1, the condition \( p^\alpha - 1 \frac{\rho(\alpha)}{p(t)^\alpha} < 1 \) can be weaken to \( \frac{\rho(\alpha)}{p(t)^\alpha} < 1 \) (see Huang [14], where we show that \( \max_i \mathbb{E}(W_{1,i}(t))^\alpha < \infty \) and \( \frac{\rho(\alpha)}{p(t)^\alpha} < 1 \) is a necessary and sufficient condition for \( \max_i \mathbb{E}[W_i(t)]^\alpha < \infty \).

The rest part of the paper is arranged as follows. In next section, we shall establish two auxiliary inequalities for the martingale \( \{ Y_n(t) \} \), which will be used in Section 4 for the proof of Theorem 2.1. In Section 5, we shall prove Theorems 2.2 and 2.4. Finally, we shall consider the complex case in Section 6, where we shall show sufficient conditions for the \( L^\alpha \) convergence and the \( \alpha \)-th moment of the Mandelbrot’s martingale \( \{ Y_n \} \).

### 3 The martingale \( \{ Y_n(t) \} \)

The critical idea of the proof of Theorem 2.1 is to notice the double martingale structure (cf [2] for more information) of the martingale \( \{ Y_n(t) \} \) and apply the inequality of martingale (Burkholder’s inequality) to it. We shall go along the proof of Theorem 2.2 according to the lines of Huang & Liu [13] or Alsmeyer et al. [1]. In this section, we show two lemmas (inequalities) to the martingale \( \{ Y_n(t) \} \) which will be used in the proof of Theorem 2.1.

**Lemma 3.1.** Let \( \alpha > 1 \). Fix \( t \in \mathbb{R} \). If \( \max_i \mathbb{E} \left[ Y_{1,i}^{(t)} \right]^\alpha < \infty \), then for each \( i = 1, \cdots, p \),

(a) for \( \alpha \in (1, 2) \),

\[
\mathbb{E} \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^{\alpha} \leq C p^{\alpha - 1} \left[ \frac{\rho(t)^{\alpha}}{p(t)^\alpha} \right]^n \;
\]

(b) for \( \alpha > 2 \),

\[
\mathbb{E} \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^{\alpha} \leq C p^{\alpha/2} \left[ \frac{\rho(t)^{\alpha/2}}{p(t)^\alpha} \right]^n \mathbb{E} \left[ Y_{1,i}^{(t)} \right]^{\alpha/2},
\]

where \( C \) is a constant depending on \( \alpha, p, t \).

**Proof.** We can decompose \( Y_{n,i}^{(t)} \) as

\[
Y_{n+1,i}^{(t)} = \frac{1}{p(t)} \sum_{j=1}^{p} \sum_{u \in \mathcal{T}_{n-1}} (X_{u,j}^{(t)})_{ij} V_j(t) = \frac{1}{p(t)} \sum_{j=1}^{p} \sum_{u \in \mathcal{T}_{n-1}} (X_{u,j}^{(t)})_{ij} Y_{1,j}^{(t)}(u),
\]

where \( Y_{1,j}^{(t)}(u) \) is a version of \( Y_{1,j}^{(t)} \) at root \( u \). Hence

\[
Y_{n+1,i}^{(t)} - Y_n^{(t)} = \frac{1}{p(t)^n} \sum_{j=1}^{p} \sum_{u \in \mathcal{T}_{n-1}} (X_{u,j}^{(t)})_{ij} \left[ Y_{1,j}^{(t)}(u) - V_j(t) \right].
\]
By Burkholder’s inequality (see for example [10]),

\[
E \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^\alpha \leq \frac{p^{\alpha-1}}{\rho(t)^{\alpha m}} \sum_{j=1}^{p} E \left[ \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij} \left| Y_{1,j}^{(t)}(u) - V_j(t) \right| \right]^\alpha \\
\leq \frac{C}{\rho(t)^{\alpha m}} \sum_{j=1}^{p} \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^2 \left| Y_{1,j}^{(t)}(u) - V_j(t) \right|^2 \right)^{\alpha/2}.
\]

Noticing the fact that

\[
E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^2 \left| Y_{1,j}^{(t)}(u) - V_j(t) \right|^2 \right)^{\alpha/2} \leq \begin{cases} 
E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^\alpha \right) E \left| Y_{1,j}^{(t)} - V_j(t) \right|^\alpha, & \text{for } \alpha \in (1, 2], \\
E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^2 \right)^{\alpha/2} E \left| Y_{1,j}^{(t)} - V_j(t) \right|^\alpha, & \text{for } \alpha > 2,
\end{cases}
\]

we have

\[
E \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^\alpha \leq \begin{cases} 
\frac{C}{\rho(t)^{\alpha m}} \sum_{j=1}^{p} E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^\alpha \right) E \left| Y_{1,j}^{(t)} - V_j(t) \right|^\alpha, & \text{for } \alpha \in (1, 2], \\
\frac{C}{\rho(t)^{\alpha m}} \sum_{j=1}^{p} \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^2 \right)^{\alpha/2} E \left| Y_{1,j}^{(t)} - V_j(t) \right|^\alpha, & \text{for } \alpha > 2.
\end{cases}
\]

(3.3)

Note that for \( u \in \mathbb{T}_n \),

\[
(X_u^{(t)})_{ij}^\alpha \leq p^{(\alpha-1)(n-1)}(X_u^{(at)})_{ij}, \quad \forall \alpha > 1,
\]

and

\[
\sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij} \leq \frac{\rho(t)^n}{V_j(t)} Y_{n,i}^{(t)}.
\]

Thus

\[
E \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^\alpha \leq p^{(\alpha-1)(n-1)} E \sum_{u \in \mathbb{T}_n} (X_u^{(at)})_{ij} \leq \frac{V_j(at)}{V_j(t)} p^{(\alpha-1)(n-1)} \rho(at)^n.
\]

Applying this inequality to the first inequality of (3.3), and noticing that \( \max_i E \left[ Y_{1,i}^{(t)} \right]^\alpha < \infty \), we obtain (3.1). To get (3.2), we only need to see that

\[
E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(t)})_{ij}^2 \right)^{\alpha/2} \leq p^{\alpha(n-1)/2} E \left( \sum_{u \in \mathbb{T}_n} (X_u^{(2t)})_{ij}^\alpha \right)^{\alpha/2} \leq V_j(2t)^{-\alpha/2} p^{\alpha(n-1)/2} \rho(2t)^{\alpha(n-1)/2} E \left[ Y_{n,i}^{(2t)} \right]^\alpha/2,
\]

and combing this inequality with the second inequality of (3.3).

\[\square\]

**Lemma 3.2.** Let \( \alpha > 1 \). Fix \( t \in \mathbb{R} \). If \( \max_i E[Y_{1,i}^{(t)}]^{\alpha} < \infty \), then for each \( i = 1, \ldots, p \),

\[
E \left[ Y_{n,i}^{(t)} \right]^{\alpha} \leq C n^{1 + \frac{\alpha m}{2\alpha}} \left[ \max \{1, p^{\alpha-1} \frac{\rho(at)}{\rho(t)^{\alpha}}, p^{\alpha(n-1)/2} \frac{\rho(2t)^{\alpha(n-1)/2}}{\rho(t)^{\alpha}}, l = 1, 2, \ldots, m \} \right]^n
\]

(3.6)

for \( \alpha \in (2^m, 2^{m+1}] \), where \( m \geq 0 \) is an integer.
Proof. At first, for \( m = 0, \alpha \in (1, 2] \). Applying Burkholder’s inequality to the martingale \( \{Y_{n,i}^{(t)}\} \) and by Lemma 3.1,

\[
E \left| Y_{n+1,i}^{(t)} - 1 \right|^\alpha \leq C \sum_{k=0}^{n-1} E \left| Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)} \right|^\alpha \leq C \sum_{k=0}^{n-1} p^{(\alpha-1)k} \left( \frac{\rho(\alpha t)}{\rho(t)} \right)^k \leq Cn \left( \max\{1, p^{\alpha-1} \frac{\rho(\alpha t)}{\rho(t)} \} \right)^\alpha.
\]

Thus

\[
E \left[ Y_{n,i}^{(t)} \right]^\alpha \leq Cn \left[ \max\{1, p^{\alpha-1} \frac{\rho(\alpha t)}{\rho(t)} \} \right]^\alpha.
\]

So (3.6) holds for \( m = 0 \). Now suppose that (3.6) holds for some \( m \geq 0 \), we shall prove it still holds for \( m + 1 \). For \( \alpha \in (2^{m+1}, 2^{m+2}) \), we have \( \alpha/2 \in (2^m, 2^{m+1}) \). Since \( \max_i E \left[ Y_{i}^{(t)} \right]^\alpha \leq \infty \) ensures that \( \max_i E \left[ Y_{i}^{(2t)} \right]^\alpha/2 \leq \infty \), by induction, we have

\[
E \left[ Y_{i}^{(2t)} \right]^\alpha/2 \leq Ck^{1+\frac{2m+1}{2m+2} \alpha} \left[ \max\{1, p^{\alpha/2-1} \frac{\rho(\alpha t)}{\rho(2^{m+1}t)} \frac{\rho(2^{m+1}t)^{\alpha/2+1}}{\rho(2^{m+1}t)} \}, l = 1, \ldots, m \} \right]^k.
\] (3.7)

Hence combining (3.7) with (3.2) we get

\[
E \left| Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)} \right|^\alpha \leq Ck^{1+\frac{m}{2m+2} \alpha} \left[ \max\{1, p^{\alpha/2-1} \frac{\rho(\alpha t)}{\rho(t)} \frac{\rho(2^{m}t)^{\alpha/2+1}}{\rho(2^{m}t)} \}, l = 1, \ldots, m \} \right]^k.
\] (3.8)

By Burkholder’s inequality and Minkowski’s inequality, and applying (3.8),

\[
E \left| Y_{n,i}^{(t)} - 1 \right|^\alpha \leq C \left( \sum_{k=0}^{n-1} E \left| Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)} \right|^{\alpha/2} \right)^{\alpha/2} \leq C \left( \sum_{k=0}^{n-1} k^{(1+\frac{m}{2m+2} \alpha) \cdot \frac{\alpha}{2}} \left[ \max\{1, p^{\alpha/2-1} \frac{\rho(\alpha t)}{\rho(t)} \frac{\rho(2^{m}t)^{\alpha/2+1}}{\rho(2^{m}t)} \}, l = 1, \ldots, m \} \right]^{2k/\alpha} \right)^{\alpha/2} \leq Cn^{1+\frac{m}{2m+2} \alpha} \left[ \max\{1, p^{\alpha/2-1} \frac{\rho(\alpha t)}{\rho(t)} \frac{\rho(2^{m}t)^{\alpha/2+1}}{\rho(2^{m}t)} \}, l = 1, \ldots, m \} \right],
\]

which implies that (3.6) holds for \( m + 1 \). This completes the proof. \( \square \)

Remark 3.1. Lemmas 3.1(b) and 3.2 also holds with \( \beta \) in place of 2 for any \( \beta \in (1, 2] \). To see this fact, observing that

\[
E \left( \sum_{u \in T_n} \left[ X_{u}^{(t)} \right]_{ij}^2 \right)^{\alpha/2} \leq E \left( \sum_{u \in T_n} \left[ X_{u}^{(t)} \right]_{ij}^\beta \right)^{\alpha/\beta}
\]

in the proof of Lemma 3.1 one just need to repeat the proofs of Lemmas 3.1(b) and 3.2 with \( \beta \) in place of 2 for the case where \( \alpha > 2 \).

4 Proof of Theorem 2.1

Now we give the proof of Theorem 2.1 by using the inequalities for the martingale \( \{Y_n^{(t)}\} \) (Lemmas 3.1 and 3.2) which are obtained in Section 3.
Proof of Theorem 2.1. The proof of (a) is composed by two steps.

Step 1: we will show that if $E\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$ and $p^{(\alpha - 1)} \rho(\alpha) < 1$, then for each $i$, $EY_i = V_i$ and $E[Y_i]^\alpha < \infty$, which implies that $EY = V$ and $0 < E\|Y\|^\alpha < \infty$. In fact, it suffices to prove that

$$\sup_n E[Y_n, i]^\alpha < \infty$$

for each $i$, which is equivalent to $Y_n \rightarrow Y_i$ in $L^\alpha$, so that $EY_i = V_i$ and $0 < E[Y_i]^\alpha < \infty$. The condition $E\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$, or equivalently, $\max_{i} E[Y_{1, i}]^\alpha < \infty$, ensures the finiteness of $M(t)$ for all $t \in [1, \alpha]$. Moreover, since $M(1) = M$ is strictly positive, by the log-convexity of $(M(t))_{i, j}$, we have $M(t)$ is strictly positive for all $t \in [1, \alpha]$, so $\rho(t)$ exists for all $t \in [1, \alpha]$. For $\alpha \in (1, 2]$, by Burkholder's inequality and Lemma 3.1,

$$\sup_n E[Y_n, i]^\alpha \leq C \sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2} \leq C \sum_{n=0}^{\infty} p^{(\alpha - 1)n} \rho(\alpha)^n < \infty.$$

For $\alpha > 2$, by Burkholder’s inequality and Minkowski’s inequality,

$$\sup_n E[Y_n, i]^\alpha \leq C \left( \sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2} \right)^{\alpha/2} < \infty.$$

We shall show the series $\sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2} < \infty$. Observing that $\max_i E[Y_{1, i}]^\alpha < \infty$ since $M_{1, i} = 1 + \sum_{j=1}^{\alpha} 1 < \infty$, by Lemma 3.2, we have for $\alpha \in [2, 2^{m+1}]$ ($m \geq 1$ is an integer),

$$\mathbb{E}\left[\left(\sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2}\right)^{\alpha/2}\right] = \mathbb{E}\left[\left(\sum_{n=0}^{\infty} (E[Y_{n+2, i} - Y_{n+1, i}]^\alpha)^{\alpha/2}\right)^{\alpha/2}\right] \leq Cn^{\gamma} \left(\max\{1, \rho(\alpha)^{\alpha/2}, \rho^{(2^{m+1})\alpha/2}\}ight)^{\alpha/2},$$

where $\gamma = 1 + \frac{m-1}{2^m} \rho < \frac{\alpha}{2}$. By Lemma 3.1 and (4.1),

$$\mathbb{E}\left[\left(\sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2}\right)^{\alpha/2}\right] \leq \left(\sum_{n=0}^{\infty} \max\{1, p^{\alpha/2} - \rho(\alpha)^{\alpha/2}, \rho^{(2^{m+1})\alpha/2}\}ight)^{\alpha/2} \leq Cn^{\alpha/2} \left(\max\{p^{\alpha/2} - \rho(\alpha)^{\alpha/2}, \rho^{(2^{m+1})\alpha/2}\} \right)^{\alpha/2}.$$

Therefore,

$$\sum_{n=0}^{\infty} (E[Y_{n+1, i} - Y_{n, i}]^\alpha)^{\alpha/2} \leq C \sum_{n=0}^{\infty} \left(\max\{p^{\alpha/2} - \rho(\alpha)^{\alpha/2}, \rho^{(2^{m+1})\alpha/2}\} \right)^{\alpha/2}.$$

The series in the right side of the inequality above converges if and only if

$$\max\{p^{\alpha/2} - \rho(\alpha)^{\alpha/2}, \rho^{(2^{m+1})\alpha/2}\} < 1.$$

Note that $\rho(t)$ is log-convex since $(M(t))_{i, j}$ is log-convex (Kingman 1961). We have $\forall \beta \in (1, \alpha)$, $\rho(\beta) \leq \rho(\alpha)^{\beta/(\alpha - 1)}$. Thus

$$p^{\frac{\alpha}{\alpha - 1}} \rho^{\alpha/\beta} \leq \rho^{\alpha/\alpha - 1} < 1,$$

and so (4.2) is true from this fact.

Step 2: we will prove that if $E\|\sum_{k=1}^{N} A_k\|^\alpha < \infty$ and $p^{(\alpha - 1)} \rho(\alpha) < 1$ for some $r$, then for each $i$, $EY_i = V_i$ and $E[Y_i]^\alpha < \infty$. Let $N := N_r$ be the population of the $r$-generation and $A_i := X_w$, where $w^r$ denotes the $i$-th particle of the $r$-generation. We consider $(N, \tilde{A}_1, \tilde{A}_2, \cdots)$. Clearly, $\tilde{M} := \mathbb{E}\sum_{i=1}^{N} \tilde{A}_i$ is finite and strictly positive with the maximum-modulus eigenvalue 1 and the corresponding eigenvectors $\tilde{U} = \tilde{U}, \tilde{V} = \tilde{V}$. Let $\tilde{T}$ be the corresponding Galton-Watson tree and $\tilde{T}_n = \{u \in \tilde{T} : |u| = n\}$. Define

$$\tilde{Y}_n := \sum_{u \in \tilde{T}_n} \tilde{X}_n V \text{ with } \tilde{X}_u := A_{u_1} \cdots A_{u_{1\cdots n}} \text{ for } u \in \tilde{T}_n.$$
Similarly, we define $\mathbf{M}(t)$, $\mathbf{Y}_n(t)$, $\hat{\rho}(t)$ and $\mathbf{V}(t)$ like Section 2. It is easy to see that $\mathbf{Y}_n$ has the same distribution as $\mathbf{Y}_{nr}$, therefore, $\mathbf{Y} := \lim_{n \to \infty} \mathbf{Y}_n$ a.s. has the same distribution as $\mathbf{Y}$. To get $\mathbb{E}Y_i = V_i$, and $\mathbb{E}[Y_i]^\alpha < \infty$, by Step 1, we only need to verify $\mathbb{E}\|\sum_{i=1}^{N} \hat{A}_i\|^\alpha < \infty$ and $\rho^{(\alpha-1)}(\alpha) < 1$. The latter is obvious since $\mathbf{M}(t) = \mathbf{M}_r(t)$ and so $\hat{\rho}(\alpha) = \rho_r(\alpha)$. To verify the former, we notice that $\mathbb{E}\|\sum_{i=1}^{N} \hat{A}_i\|^\alpha < \infty$ is equivalent to $\max \mathbb{E}[\hat{Y}_{1,i}]^\alpha < \infty$, which is true by Lemma 3.2, since $\mathbb{E}[\hat{Y}_{1,i}]^\alpha = \mathbb{E}[Y_{r,i}]^\alpha$.

Now we prove the converse (b). Suppose that $0 < \mathbb{E}\|\mathbf{Y}\|^\alpha < \infty$, which implies that $\max \mathbb{E}[Y_i]^\alpha < \infty$ and $\mathbf{Y}$ is non-degenerate. As $\mathbf{Y}$ is a non-trivial solution of the equation (E), we have $\mathbb{E}\mathbf{Y} = \mathbf{ME}$ with $\mathbb{E}\mathbf{Y} \neq 0$, which means that $\mathbb{E}\mathbf{Y}$ is a non-trivial eigenvector corresponding to the eigenvalue 1, and so $\mathbb{E}\mathbf{Y} = c \mathbf{Y}$ for some constant $c > 0$. By equation (E), for each $i$,

$$Y_i = \sum_{k=1}^{N} (A_k Y(k))_i = \sum_{k=1}^{N} P \sum_{j=1}^{p} (A_k)_{ij} Y_i(k).$$

By Jensen’s inequality, for each $i$,

$$\mathbb{E}[Y_i]^\alpha \geq \mathbb{E} \left[ \mathbb{E} \left( \sum_{k=1}^{N} \sum_{j=1}^{p} (A_k)_{ij} Y_j(k) | F_i \right) \right]^\alpha = \mathbb{E} \mathbb{E} \left( \sum_{k=1}^{N} \sum_{j=1}^{p} (A_k)_{ij} Y_j \right)^\alpha = c^\alpha \mathbb{E} \mathbb{E} \left( \sum_{k=1}^{N} (A_k) V_i \right)^\alpha = c^\alpha \mathbb{E}[Y_{1,i}]^\alpha.$$

Thus $\max \mathbb{E}[Y_i]^\alpha < \infty$ implies $\max \mathbb{E}[Y_{1,i}]^\alpha < \infty$, or equivalently, $\mathbb{E}\|\sum_{i=1}^{N} \hat{A}_i\|^\alpha < \infty$.

Next, we consider $\rho_n(t)$. Since

$$[Y_i]^\alpha = \left( \sum_{u \in T_n} \sum_{j=1}^{p} (X_u)_{ij} Y_j(u) \right)^\alpha \geq \sum_{j=1}^{p} \sum_{u \in T_n} [(X_u)_{ij}]^\alpha [Y_j(u)]^\alpha, \quad (4.3)$$

we obtain

$$\mathbb{E}[Y_i]^\alpha \geq \sum_{j=1}^{p} \mathbb{E} \sum_{u \in T_n} [(X_u)_{ij}]^\alpha \mathbb{E}[Y_j]^\alpha = \sum_{j=1}^{p} (M_n(\alpha))_{ij} \mathbb{E}[Y_j]^\alpha. \quad (4.4)$$

Thus

$$\sum_{j=1}^{p} \rho_n(\alpha) \mathbb{E}[Y_i]^\alpha \geq \sum_{j=1}^{p} \mathbb{E}[Y_j]^\alpha \sum_{i=1}^{p} U_{n,i}(\alpha)(M_n(\alpha))_{ij} = \rho_n(\alpha) \sum_{j=1}^{p} U_{n,j}(\alpha) \mathbb{E}[Y_j]^\alpha, \quad (4.5)$$

which leads to $\rho_n(\alpha) \leq 1$. If additionally (2.2) holds, then for each $i$, $P \left( \sum_{j=1}^{p} \sum_{u \in T_n} 1_{(X_u)_{ij} > 0} = 0 \text{ or } 1 \right) < 1$.

Hence the strictly inequality in (4.3) holds with positive probability, and so both (4.4) and (4.5) are strictly inequalities, which leads to $\rho_n(\alpha) < 1$.

\section{Proof of Theorems 2.2 and 2.4}

We will prove Theorems 2.2 and 2.4 based on the equation (E), with ideas from Liu [21]. Recall that $\phi(t) = \mathbb{E}e^{-t \mathbf{Z}}$ is the Laplace transform of the non-trivial solution $\mathbf{Z}$ to the equation (E). By (E), $\phi(t)$ satisfies the functional equation

$$\phi(t) = \mathbb{E} \prod_{k=1}^{N} \phi(t A_k). \quad (5.1)$$
Our proofs are based on this equation.

To prove Theorem 2.2, the two lemmas below are necessary.

**Lemma 5.1.** Let \( \phi : \mathbb{R}^p_+ \rightarrow \mathbb{R}_+ \) be a bounded function, and \( \mathbf{A} = (a_{ij}) \) be a non-zero matrix such that for some \( 0 < q < 1, t_\varepsilon > 0 \) and all \( t \) satisfying \( \|t\| > t_\varepsilon \),

\[
\phi(t) \leq q\mathbb{E}\phi(t\mathbf{A}).
\]

(5.2)

If \( q\mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda} < 1 \), then \( \phi(t) = O(\|t\|^{-\lambda}) \) (\( \|t\| \rightarrow \infty \)).

**Proof.** Assume that \( \phi \) is bounded by a constant \( K \). For \( t \neq 0 \), if \( \|t\| \leq t_\varepsilon \), then \( \phi(t) \leq K t_\varepsilon^\lambda \|t\|^{-\lambda} \), which yields by (5.2)

\[
\phi(t) \leq q\mathbb{E}\phi(t\mathbf{A}) + C\|t\|^{-\lambda},
\]

for all \( t \neq 0 \),

(5.3)

where \( C \) is a general positive constant. Let \( \{\mathbf{A}_k\} \) be a family of i.i.d copies of \( \mathbf{A} \). By induction on (5.3),

\[
\phi(t) \leq q^n\mathbb{E}\phi(t\mathbf{A}_1 \cdots \mathbf{A}_n) + C\left(\sum_{k=1}^{n-1} (q^k - 1)\mathbb{E}\|t\mathbf{A}_1 \cdots \mathbf{A}_{k-1}\|^{-\lambda} + \|t\|^{-\lambda}\right),
\]

for all \( t \neq 0 \).

(5.4)

Note that for any matrix \( \mathbf{A} = (a_{ij}) \) and vector \( t \), we have

\[
\|t\mathbf{A}\| = \sum_j (t\mathbf{A})_j = \sum_j t_i \min_i \sum_j a_{ij} = \|t\| \left(\min_i \sum_j a_{ij}\right).
\]

Thus by the independency of \( \{\mathbf{A}_k\} \),

\[
\mathbb{E}\|t\mathbf{A}_1 \cdots \mathbf{A}_k\|^{-\lambda} \leq \|t\|^{-\lambda}\mathbb{E}\left(\prod_{i=1}^{k} \left(\min_j \sum_i (\mathbf{A}_i)_{ij}\right)^{-\lambda}\right) = \|t\|^{-\lambda}\left[\mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda}\right]^k.
\]

(5.5)

Combing (5.5) with (5.4) and letting \( n \rightarrow \infty \) leads to \( \phi(t) = O(\|t\|^{-\lambda}) \) (\( \|t\| \rightarrow \infty \)).

**Lemma 5.2** (19, Lemma 4.4). Let \( X \) be a positive random variable. For \( 0 < a < \infty \), consider the following statements:

\( i) \) \( \mathbb{E}X^{-a} < \infty \); \( \quad \)
\( ii) \) \( \mathbb{E}e^{-tX} = O(t^{-a})(t \rightarrow \infty) \); \( \quad \)
\( iii) \) \( \mathbb{P}(X \leq x) = O(x^a)(x \rightarrow 0) \); \( \quad \)
\( iv) \) \( \forall b \in (0, a), \mathbb{E}X^{-b} < \infty \).

Then the following implications hold: \( i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \).

**Proof of Theorem 2.2** Let \( N_\delta = \sum_{k=1}^{N} 1_{\{\min\sum_j (\mathbf{A}_k)_{ij} > \delta\}} \) for \( \delta > 0 \). Then \( N_\delta \uparrow N \), as \( \delta \downarrow 0 \). Since \( \phi(t) = \mathbb{E}e^{-t\mathbf{Z}} \rightarrow 0 \) as \( \|t\| \rightarrow \infty \), there exists \( t_\varepsilon > 0 \) such that for \( \|t\| > t_\varepsilon \), \( \phi(t) < \varepsilon \). For \( \|t\| > t_\varepsilon/\delta \), if \( \min_i \sum_j (\mathbf{A}_k)_{ij} > \delta \), we have \( \|t\mathbf{A}_k\| \geq \|t\| \min_j \sum_i (\mathbf{A}_k)_{ij} > t_\varepsilon \). By equation (5.1),

\[
\phi(t) \leq \mathbb{E}\phi(t\mathbf{A}) \left(\mathbb{E}^{N_\delta^{-1}}1_{\{N_\delta \geq 1\}} + 1_{\{N_\delta = 0\}}\right) = q_{\varepsilon, \delta}\mathbb{E}\phi(t\tilde{\mathbf{A}}),
\]

where \( q_{\varepsilon, \delta} = \mathbb{E}\left(\mathbb{E}^{N_\delta^{-1}}1_{\{N_\delta \geq 1\}} + 1_{\{N_\delta = 0\}}\right) \) and \( \tilde{\mathbf{A}} = (\tilde{a}_{ij}) \) is a random matrix whose distribution is determined by \( \mathbb{E}\phi(\mathbf{A}) = \frac{1}{q_{\varepsilon, \delta}}\mathbb{E}\phi(t\mathbf{A}) \left(e^{N_\delta^{-1}}1_{\{N_\delta \geq 1\}} + 1_{\{N_\delta = 0\}}\right) \) for all bounded and measurable function \( g \) on \( \mathbb{R}_+^p \). We can see that by the dominated convergence theorem,

\[
q_{\varepsilon, \delta} \mathbb{E}N^{-1}1_{\{N \geq 1\}} \xrightarrow{\varepsilon, 0} \mathbb{P}(N = 1) < 1,
\]

12
and since $E\left(\min_i \sum_j a_{ij}\right)^{-\lambda} < \infty$,

$$q_{\varepsilon,\delta} E\left(\min_j \sum_i \tilde{a}_{ij}\right)^{-\lambda} = E\left(\min_i \sum_j a_{ij}\right)^{-\lambda} (e^{N_i-1}1_{\{N_i \geq 1\}} + 1_{\{N_i = 0\}})$$

$$\delta_{\delta, \theta} E\left(\min_j \sum_i a_{ij}\right)^{-\lambda} 1_{\{N_i \geq 1\}} \rightarrow_{\delta, \theta} E\left(\min_i \sum_j a_{ij}\right)^{-\lambda} 1_{\{N_i = 1\}} < 1.$$

By Lemma 5.1, $\phi(t) = O(||t||^{-\lambda}) (||t|| \rightarrow \infty)$. Thus for given non-zero $y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+$, $E e^{-t y \cdot Z} = O(t^{-\lambda})(t \rightarrow \infty)$, so that by Lemma 5.2, $P(y \cdot Z \leq x) = O(x^\lambda)(x \rightarrow 0)$ and $E(y \cdot Z)^{-\lambda_1, \infty}$, $\forall 0 < \lambda_1 < \lambda$.

For the second part, notice that we have obtained $\phi(t) < C||t||^\lambda$ for all $||t|| > 0$ in the first part, where $C$ is a positive constant. By equation (5.1),

$$\phi(t) \leq C \prod_{k=1}^m \phi(tA_k) \leq C \prod_{k=1}^m ||tA_k||^{-\lambda} \leq C \prod_{k=1}^m ||t||^{-\lambda} \prod_{k=1}^m \left(\min_i \sum_j (A_{k})_{ij}\right)^{-\lambda}.$$

The rest results follow by Lemma 5.2.

Proof of Theorem 2.4. We only prove the results for $\phi(t)$. The assertions for $P(y \cdot Z \leq x)$ follow from that about $E e^{-t y \cdot Z}$ and the Tauberian Theorem of exponential type (cf [18]).

We first prove (a). By equation (5.4),

$$1 = \sum_{j=1}^p V_j = E \sum_{k=1}^N \sum_{j,l=1}^p (A_k)_{jl} V_j > E \sum_{k=1}^m \sum_{j,l=1}^p (A_k)_{jl} V_j \geq \sum_{j=1}^p \sum_{i,j=1}^p (A_k)_{ij} V_j.$$ 

The strict inequality holds because of (2.2) and $P(N = m) > 0$. Therefore, we have $\gamma = -\log m / \log (ap) \in (0, 1)$. Since for all $k$,

$$\phi(tA_k) = E \exp \left\{ -\sum_{i,j=1}^p t_i (A_k)_{ij} Z_j \right\} \leq E \exp \{ -a ||t|| e \cdot Z \} = \phi(a ||t|| e),$$

where $e = (1, 1, \cdots, 1)$, by equation (5.1),

$$\phi(t) \leq E \prod_{k=1}^m \phi(tA_k) \leq \phi(a ||t|| e)^m. \quad (5.6)$$

Applying (5.6) with $t = a ||t|| e$, we have $\phi(a ||t|| e) \leq \phi(a^2 p ||t|| e)^m$, so that $\phi(t) \leq \phi(a^2 p ||t|| e)^m$. By iteration, we get

$$\phi(t) \leq \phi(a^2 p^{k-1} ||t|| e)^m. \quad (5.7)$$

As $ap < 1$, for $||t|| \geq p$, there exists an integer $k \geq 0$ such that $p/(ap)^k \leq ||t|| < p/(ap)^{k+1}$. So this $k$ satisfies

$$\frac{\log p - \log ||t||}{\log (ap)} - 1 < k \leq \frac{\log p - \log ||t||}{\log (ap)}.$$ 

(5.8)

For any $x \geq 1$, one can see that

$$\phi(xe) = E \exp \left\{ -x \sum_{j=1}^p Z_j \right\} \leq E \exp \{ -x \sum_{j=1}^p Z_j \} = \phi(e) < 1. \quad (5.9)$$

13
Since \( a^k p^{k-1} \|t\| \geq 1 \), by (5.7), (5.9) and (5.21), we have

\[
\log \phi(t) \leq m^k \log \phi(e) \leq \exp \left\{ \frac{\log m}{\log(a^k p)} (\log p - \log \|t\|) \right\} = -C_1 \|t\|^\gamma,
\]

where \( C_1 = -p^{-\gamma} \log \phi(e) > 0 \).

We then prove (b). The proof is similar to that of (a). If \( \max_{ij}(A_k)_{ij} \leq a + \varepsilon \), then

\[
\phi(t A_k) \geq \phi((a + \varepsilon) \|t\| e).
\]

By equation (5.1)

\[
\phi(t) \geq \prod_{k=1}^m \phi(t A_k) 1_{\{N = m, \max_{ij}(A_k)_{ij} \leq a + \varepsilon, \forall k\}} \geq P \left[ \phi((a + \varepsilon) \|t\| e) \right]^{m},
\]

where \( P = P(N = m, \max_{ij}(A_k)_{ij} \leq a + \varepsilon, \forall k) < 1 \). By iteration, we get

\[
\phi(t) \geq \rho^{\sum_{j=1}^{k} m^j} [\phi((a + \varepsilon) \|t\| e)]^{m^k}.
\]

As \( (a + \varepsilon) p < 1 \), for \( \|t\| \geq p \), there exists an integer \( k \geq 1 \) such that \( p/((a + \varepsilon) p)^{k-1} \leq \|t\| < p/((a + \varepsilon) p)^k \). Since \( \phi(x e) \geq \phi(e) \) for any \( x < 1 \) and \( (a + \varepsilon) \|t\| e < 1 \), we have

\[
\phi((a + \varepsilon) \|t\| e) \geq \phi(e).
\]

Therefore, (5.11) yields

\[
\log \phi(t) \geq m^k \left( \log \phi(e) + m^{-k} \sum_{j=0}^{k-1} m^j \log \rho \right) \\
\geq m^k \left( \log \phi(e) + \frac{\log \rho}{m - 1} \right) \\
\geq \exp \left\{ \frac{\log m}{\log((a + \varepsilon) p)} (\log p - \log \|t\|) + \log m \right\} \left( \log \phi(e) + \frac{\log \rho}{m - 1} \right) \\
= -C_2 \|t\|^\gamma \varepsilon,
\]

where \( C_2 = -p^{-\gamma(\varepsilon)} m \left( \log \phi(e) + \frac{\log \rho}{m - 1} \right) > 0 \).

\[ \square \]

6 Moments for the complex case

In this section, we consider the complex case, where in equation (E), all the matrix \( A_k \) and the vectors \( Z, Z(k) \) are complex (with \( \mathbb{C} \) in place of \( \mathbb{R} \)). Here we are still interested in the existence of the \( \alpha \)-th moment (\( \alpha > 1 \)) solution, or in other words, the \( L^\alpha \) convergence and the \( \alpha \)-th moment of the Mandelbrot’s martingale \( \{Y_n\} \) defined by (1.1).

Besides Assumption (H), we assume moreover that

\[
\tilde{M} := \mathbb{E} \sum_{i=1}^N \hat{A}_k \quad \text{with} \quad (\hat{A}_k)_{ij} := |(A_k)_{ij}|
\]

is finite and strictly positive. For \( t \in \mathbb{R} \) fixed, let

\[
\tilde{M}(t) := \mathbb{E} \sum_{i=1}^N \hat{A}_k^{(t)} \quad \text{with} \quad (\hat{A}_k^{(t)})_{ij} := (A_k)^{t}_{ij},
\]

where \( \hat{A}_k \) is the complex matrix defined by (1.1). Here we are interested in the existence of the \( \alpha \)-th moment (\( \alpha > 1 \)) of \( \tilde{M}(t) \).
whose maximum-modulus eigenvalue is denoted by \( \hat{\rho}(t) \) and the corresponding normalized left and right positive eigenvectors by \( \hat{U}(t), \hat{V}(t) \). Define

\[
\hat{Y}_n^{(t)} := \sum_{u \in T_n} \hat{X}_u^{(t)} \hat{V}(t) \quad \text{with} \quad \hat{X}_u^{(t)} := \hat{A}_{u_1}^{(t)} \cdots \hat{A}_{u_r}^{(t)} \quad \text{for} \, u \in T_n.
\]

Obviously, \( \{\hat{Y}_n^{(t)}\} \) has the same structure as the martingale \( \{Y_n^{(t)}\} \) of the real case for which we have established inequalities in Section 3, therefore, we can apply these results (Lemmas 3.1 and 3.2) to the martingale \( \{\hat{Y}_n^{(t)}\} \).

Following similar arguments to the proof of Theorem 2.1, we reach the following result for the complex case.

**Theorem 6.1** (Complex case). Assume that all the matrix \( A_k \) and the vectors \( Z, Z(k) \) are complex. Let \( \alpha > 1 \). If \( \mathbb{E} \| \sum_{i=1}^{N} A_i \|^\alpha < \infty \) and either of the following assertions holds:

(i) \( \alpha \in (1, 2] \) and \( \hat{\rho}(\alpha) < 1 \);

(ii) \( \alpha > 2 \) and \( \max \{p^{\alpha-1} \hat{\rho}(\alpha), p^{\alpha/\beta} \hat{\rho}(\beta)\} < 1 \) for some \( \beta \in (1, 2] \),

then \( \sup \mathbb{E}\|Y_n\|^\alpha < \infty \), and \( \{Y_n\} \) converges a.s. and in \( L^\alpha \) to a random vector \( Y \), so that \( \mathbb{E}Y = \mathbf{V} \) and \( 0 < \mathbb{E}\|Y\|^\alpha < \infty \).

In particular, for the case \( p = 1 \), it is easy to see that \( V = 1 \) and \( \hat{\rho}(t) = \hat{m}(t) = \mathbb{E} \sum_{i=1}^{N} |A_i|^\alpha \).

**Corollary 6.2** (case \( p=1 \)). Let \( p = 1 \) and \( \alpha > 1 \). If \( \mathbb{E} \left( \sum_{i=1}^{N} |A_i| \right)^\alpha < \infty \) and either of the following assertions holds:

(i) \( \alpha \in (1, 2] \) and \( \hat{\rho}(\alpha) < 1 \);

(ii) \( \alpha > 2 \) and \( \max \{\hat{\rho}(\alpha), \hat{\rho}(\beta)\} < 1 \) for some \( \beta \in (1, 2] \),

then \( \sup \mathbb{E}|Y_n|^\alpha < \infty \) and \( \{Y_n\} \) converges a.s. and in \( L^\alpha \) to a random variable \( Y \), so that \( \mathbb{E}Y = 1 \) and \( 0 < \mathbb{E}Y|^\alpha < \infty \).

The proof of Theorem 6.1 is similar to that of Theorem 2.1. We first show several lemmas for the martingale \( \{\hat{Y}_n^{(t)}\} \).

**Lemma 6.3.** Let \( \alpha > 1 \). Fix \( t \in \mathbb{R} \). Assume that \( \max_i \mathbb{E}|Y_{1,i}^{(t)}|^\alpha < \infty \). Then for each \( i = 1, 2, \ldots, p \),

(a) if \( \alpha \in (1, 2] \),

\[
\mathbb{E} \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^\alpha \leq C p^{(\alpha-1)n} \left( \frac{\hat{\rho}(\alpha t)}{\left| \rho(t) \right|^\alpha} \right)^n; \quad (6.1)
\]

(b) if \( \alpha > 2 \), for any \( \beta \in (1, 2] \),

\[
\mathbb{E} \left| Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} \right|^\alpha \leq C p^{\alpha n/2} \left( \frac{\hat{\rho}(\beta t)^{\alpha/\beta}}{\left| \rho(t) \right|^\alpha} \right)^n \mathbb{E} \left| \hat{Y}_{n,i}^{(\beta t)} \right|^{\alpha/\beta}, \quad (6.2)
\]

where \( C \) is a constant depending on \( \alpha, p, t \).
The log-convexity of \(\hat{\rho}\) then for \(\alpha > 1\), we remark that \(\hat{\rho}(t)\) is derivable on \((1, \alpha)\) with derivative

\[
\frac{d}{dt}\hat{\rho}(t) = \frac{d}{dt}\log(\hat{\rho}(t)) = \frac{1}{\hat{\rho}(t)} \frac{d}{dt}\hat{\rho}(t).
\]

Combing Lemmas 6.3 and 6.4 leads to Lemma 6.5 below.

**Lemma 6.5.** Let \(\alpha > 1\). Assume that \(E\|\sum_{k=1}^{N} \hat{A}_k\|^\alpha < \infty\). Then for each \(i = 1, 2, \cdots, p\),

(a) if \(\alpha \in (1, 2)\),

\[
E|Y_{n+1,i} - Y_{n,i}|^\alpha \leq C(p^{\alpha-1}\hat{\rho}(\alpha))^n;
\]

(b) if \(\alpha > 2\), for any \(\beta \in (1, 2)\),

\[
E|Y_{n+1,i} - Y_{n,i}|^\alpha \leq Cn^{\alpha/\beta}\max\{p^{\alpha-1}\hat{\rho}(\alpha),\beta^{\alpha/\beta}\hat{\rho}(\beta)^{\alpha/\beta}\}^n,
\]

where \(C\) is a constant depending on \(\alpha, p, t\).

**Proof.** Firstly, we remark that \(\hat{\rho}(t)\) exists for all \(t \in [1, \alpha]\) since \(E\|\sum_{k=1}^{N} \hat{A}_k\|^\alpha < \infty\) and \(M(1) = M\) is finite and strictly positive. Furthermore, \(E\|\sum_{k=1}^{N} \hat{A}_k\|^\alpha < \infty\) implies that \(\max E|Y_{1,i}|^\alpha < \infty\) and \(\max E|\hat{Y}_{1,i}|^\alpha < \infty\). So (6.3) is directly from (6.1). For \(\alpha > 2\), by Lemma 6.3, we have

\[
E|Y_{n+1,i} - Y_{n,i}|^\alpha \leq Cn^{\alpha/\beta}\max\{p^{\alpha-1}\hat{\rho}(\alpha),\beta^{\alpha/\beta}\hat{\rho}(\beta)^{\alpha/\beta}\}^n,
\]

if \(\alpha \in (\beta^{m}, \beta^{m+1})\) \((m \geq 1)\) is an integer. Let

\[
g(x) := \log(p^{1-1/x}\hat{\rho}(x)^{1/x}) = (1 - \frac{1}{x}) \log p + \frac{1}{x} \log \hat{\rho}(x).
\]

Clearly, \(g(x)\) is derivable on \((1, \alpha)\) with derivative

\[
g'(x) = \frac{h(x)}{x},\quad \text{where } h(x) := \log p + \frac{x \hat{\rho}'(x)}{\hat{\rho}(x)} - \log \hat{\rho}(x).
\]

The log-convexity of \(\hat{\rho}(x)\) implies that \(h(x)\) is increasing, hence \(g(x)\) reaches its maximum on a closed interval at the extremity points. We have

\[
\sup_{\beta \leq x \leq \alpha} \{p^{1-1/x}\hat{\rho}(x)^{1/x}\} = \max\{p^{1-1/\alpha}\hat{\rho}(\alpha)^{1/\alpha}, p^{1/\beta}\hat{\rho}(\beta)^{1/\beta}\}.
\]

The proof is complete.
Now we prove Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.5 we can obtain the series \( \sum_{n} (E|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} < \infty \). Observing that \( (E|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} \leq n - 1 \sum_{k=n}^{\infty} (E|Y_{k+1,i} - Y_{k,i}|^\alpha)^{1/\alpha} + 1 \), we immediately get

\[
\sup_{n} E|Y_{n,i}|^\alpha \leq \left( \sum_{n=0}^{\infty} (E|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} + 1 \right)^{\alpha} < \infty.
\]

Notice that \( E \sum_{n} |Y_{n+1,i} - Y_{n,i}| \leq \sum_{n} (E|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} < \infty \). This fact leads to the a.s. convergence of the series \( \sum_{n} |Y_{n+1,i} - Y_{n,i}| \), which show that \( \{Y_{n,i}\} \) is a Chauchy sequence in the sense a.s., so there exists a random variable \( Y_i \) such that \( Y_{n,i} \to Y_i \) a.s.. By Fatou’s Lemma, we have

\[
E|Y_{n,i} - Y_i|^\alpha = E \lim_{l \to \infty} |Y_{n+l,i} - Y_{n,i}|^\alpha \leq \liminf_{l \to \infty} E|Y_{n+l,i} - Y_{n,i}|^\alpha \leq \left( \sum_{k=n}^{\infty} (E|Y_{k+1,i} - Y_{k,i}|^\alpha)^{1/\alpha} \right)^{\alpha n \to \infty} 0.
\]

Thus \( Y_{n,i} \to Y_i \) in \( L^\alpha \), so that \( EY_i = V_i \) and \( 0 < E(Y_i^\alpha) < \infty \). □

References

[1] G. Alsmeyer, A. Iksanov, S. Polotsky, U. Rösler, Exponential rate of \( L_p \)-convergence of intrinsic martingales in supercritical branching random walks. Theory Stoch. Process 15 (2009), 1-18.

[2] G. Alsmeyer and D. Kuhlbusch. Double martingale structure and existence of \( \phi \)-moments for weighted branching processes. Münster J. Math. 3 (2010), 163-212.

[3] J. Barral, Moments, continuité, et analyse multifractale des martingales de Mandelbrot. Probab. Theory Relat. Fields 113 (1999), 535-569.

[4] J.D. Biggins, Martingale convergence in the branching random walk. J. Appl. Probab. 14 (1977), 25-37.

[5] J.D. Biggins, A.E. Kyprianou, Seneta-Heyde norming in the branching random walk. Ann. Probab. 25 (1997), 337-360.

[6] J.D. Biggins, A. Rahimzadeh Sani, Convergence results in multitype, multivariate, branching random walk. Adv. Appl. Probab. 37 (2005), 681-705.

[7] J.D. Biggins, Spreading speeds in reducible multitype branching random walk. Ann. Appl. Probab. 22 (2012), 1778-1821.

[8] N.H. Bingham, R.A. Doney, Asymptotic properties of supercritical branching processes I: The Galton-Watson processes. Adv. Appl. Prob. 6 (1974), 711-731.

[9] N.H. Bingham, R.A. Doney, Asymptotic properties of supercritical branching processes II: Crump-Mode and Jirina processes. Adv. Appl. Prob. 7 (1975), 66-82.

[10] Y.S. Chow, H. Teicher, Probability theory: Independence, Interchangeability and Martingales. Springer-Verlag, New York, 1988.

[11] R. Durrett, T. Liggett, Fixed points of the smoothing transformation. Z. Wahrsch. verw. Gebeite 64 (1983), 275-301.

[12] Y. Guivarc’h, Sur une extension de la notion de loi semi-stable. Ann. IHP 26 (1990), 261-185.

[13] C. Huang, Q. Liu, Convergence in \( L^p \) and its exponential rate for a branching process in a random environment. Available at http://arxiv.org/pdf/1011.0533.pdf
[14] C. Huang, Théorèmes limites et vitesses de convergence pour certains processus de branchement et des marches aléatoires branchantes. PH. D Thesis (2010).

[15] J.P. Kahane, J. Peyrière, Sur certaines martingales de Benoît Mandelbrot. Adv. Math. 22 (1976), 131-145.

[16] J.F.C. Kingman, A convexity property of positive matrices. Quart. Jnl. Math. Oxford. 12 (1961), 283-4.

[17] Y. Kyprianou, A. Rahimzadeh Sani, Martingale convergence and the functional equation in the multitype branching random walk. Bernoulli 7 (2001), 593-604.

[18] Q. Liu, The exact hausdorff dimension of a branching set. Probab. Theory Related Fields 104 (1996), 515-538.

[19] Q. Liu, Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks. Stoch. Proc. Appl. 82 (1999), 61-87.

[20] Q. Liu, On generalized multiplicascades. Stoc. Proc. Appl. 86 (2000), 263-286.

[21] Q. Liu, Asymptotic properties and absolute continuity of laws stable by random weighted mean. Stoch. Proc. Appl. 95 (2001), 83-107.

[22] B. Mandelbrot, Multiplications aléatoires et distributions invariantes par moyenne pondérée aléatoire. C. R. Acad. Sci. Paris 278 (1974), 289-292,355-358.

[23] U. Rösler, A fixed point theorem for distribution. Stoch. Proc. Appl. 42 (1992), 195-214.