Notes on Matrix Valued Paraproducts
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Abstract Denote by $M_n$ the algebra of $n \times n$ matrices. We consider the dyadic paraproducts $\pi_b$ associated with $M_n$ valued functions $b$, and show that the $L^\infty(M_n)$ norm of $b$ does not dominate $\|\pi_b\|_{L^2(\ell_2^n) \rightarrow L^2(\ell_2^n)}$ uniformly over $n$. We also consider paraproducts associated with noncommutative martingales and prove that their boundedness on bounded noncommutative $L^p$—martingale spaces implies their boundedness on bounded noncommutative $L^q$—martingale spaces for all $1 < p < q < \infty$.

1 Introduction
Denote by $M_n$ the algebra of $n \times n$ matrices. Let $(\mathbb{T}, \mathcal{F}_k, dt)$ be the unit circle with Haar measure and the usual dyadic filtration. Let $b$ be an $M_n$ valued function on $\mathbb{T}$.
The matrix valued dyadic paraproduct associated with $b$, denoted by $\pi_b$, is the operator defined as
$$\pi_b(f) = \sum_k (d_k b)(E_{k-1} f), \quad \forall f \in L^2(\ell_2^n).$$
(1.1)

Here $E_k f$ is the conditional expectation of $f$ with respect to $\mathcal{F}_k$, i.e. the unique $\mathcal{F}_k$-measurable function such that
$$\int_F E_k f \, dt = \int_F f \, dt, \quad \forall F \in \mathcal{F}_k.$$ And $d_k b$ is defined to be $E_k b - E_{k-1} b$.

In the classical case (when $b$ is a scalar valued function), paraproducts are usually considered as dyadic singular integrals and play important roles in the proof of the classical T(1) theorem. It is well known that
$$\|\pi_b\|_{L^2 \rightarrow L^2} \preceq \|b\|_{BMO_d},$$
where $BMO_d$ denotes the dyadic BMO norm defined as
$$\|b\|_{BMO_d} = \sup_m \|E_m \sum_{k=m}^{\infty} |d_k b|^2\|_{L^\infty}^{1/2}.$$ And by the Calderón-Zygmund decomposition and the Marcinkiewicz interpolation theorem, we have $\|\pi_b\|_{L^p \rightarrow L^p} \preceq \|\pi_b\|_{L^p \rightarrow L^p} \preceq \|b\|_{BMO_d}$ for all $1 < p < \infty$.

When $b$ is $M_n$ valued, it was proved by Katz ([4]) and independently by Nazarov, Treil and Volberg ([8], see [10] for another proof by Pisier) that
$$\|\pi_b\|_{L^2(\ell_2^n) \rightarrow L^2(\ell_2^n)} \leq c \log(n + 1) \|b\|_{BMO_d}.$$ (1.2)

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Here $\| \cdot \|_{\text{BMO}_c}$ is the column BMO norm defined by

$$\|b\|_{\text{BMO}_c} = \sup_m \left\| E_m \sum_{k=m}^{\infty} (d_kb)^*(d_kb) \right\|_{L^\infty(M_n)}^{1/2},$$

where $(d_kb)^*$ is the adjoint of $d_kb$. Nazarov, Pisier, Treil and Volberg ([7]) proved later that the constant $c \log(n+1)$ in (1.2) is optimal. Thus the $\text{BMO}_c$ norm does not dominate $\| \pi_b \|_{L^2(\ell_2^n) \to L^2(\ell_2^n)}$ uniformly over $n$.

Can we expect something weaker? In particular, does there exist a constant $c$ independent of $n$ such that, for every $n \in \mathbb{N}$,

$$\| \pi_b \|_{L^2(\ell_2^n) \to L^2(\ell_2^n)} \leq c \|b\|_{L^\infty(M_n)}, \tag{1.3}$$

Some known facts made (1.3) look hopeful. For example, the Hankel operator associated with the $M_n$ valued function $b$ has a norm equivalent to $\|b\|_{(H^1(S^1))'}$. Here $\| \cdot \|_{(H^1(S^1))'}$ denotes the dual norm on the trace class valued Hardy space $H^1(S^1)$. And S. Petermichl proved a close relation between $\pi_b$ and the Hankel operators associated with $b$ (see [9]).

In this paper, we prove the following theorem, which shows there does not exist any constant $c$ independent of $n$ such that (1.3) holds.

**Theorem 1.1** For every $n \in \mathbb{N}$, there exists an $M_n$ valued function $b$ with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that

$$\| \pi_b \|_{L^2(\ell_2^n) \to L^2(\ell_2^n)} \geq c \log(n+1),$$

where $c > 0$ is independent of $n$.

This also gives a new proof that the constant $c \log(n+1)$ in (1.2) is optimal.

Denote by $S^p$ the Schatten $p$ class on $\ell^2$. For $f \in L^p(S^p)$, we define $\pi_b(f)$ as in (1.1) also. As pointed out in [10], it is easy to check that $\| \pi_b \|_{L^2(S^2) \to L^2(S^2)} = \| \pi_b \|_{L^2(\ell^2) \to L^2(\ell^2)}$. For scalar valued $b$, as we mentioned previously, we have $\| \pi_b \|_{L^p \to L^q} \leq \| \pi_b \|_{L^p \to L^q}$. We wonder if this is still true for matrix valued $b$, i.e. if $\pi_b$’s boundedness on $L^p(S^p)$ implies their boundedness on $L^q(S^q)$ for all $1 < p, q < \infty$.

More generally, we can consider paraproducts associated with noncommutative martingales. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized faithful trace $\tau$. For $1 \leq p < \infty$, we denote by $L^p(\mathcal{M})$ the noncommutative $L^p$ space associated with $(\mathcal{M}, \tau)$. Recall the norm in $L^p(\mathcal{M})$ is defined as

$$\| f \|_p = (\tau |f|^p)^{1/p}, \quad \forall f \in L^p(\mathcal{M}),$$

where $|f| = (f^*f)^{1/2}$. For convenience, we usually set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\| \cdot \|_{\mathcal{M}}$. Let $\mathcal{M}_k$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{k \geq 0} \mathcal{M}_k$ generates $\mathcal{M}$ in the weak$^*$ topology. Denote by $E_k$ the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_k$. $E_k$ is a norm 1 projection of $L^p(\mathcal{M})$ onto
$L^p(M_k)$. For $1 \leq p \leq \infty$, a sequence $f = (f_k)_{k \geq 0}$ with $f_k \in L^p(M_k)$ is called a bounded noncommutative $L^p$-martingale, denoted by $(f_k)_{k \geq 0} \in L^p(M)$, if $E_k f_m = f_k, \forall k \leq m$ and

$$\|(f_k)_{k \geq 0}\|_{L^p(M)} = \sup_k \|f_k\|_{L^p(M)} < \infty.$$  

Because of the uniform convexity of the space $L^p(M)$, for $1 < p < \infty$, we can and will identify the space of all bounded $L^p(M)$-martingales with $L^p(M)$ itself. In particular, for any $f \in L^p(M)$, set $f_k = E_k f$, then $f = (f_k)_{k \geq 0}$ is a bounded $L^p(M)$-martingale and $\|(f_k)_{k \geq 0}\|_{L^p(M)} = \|f\|_{L^p(M)}$. Denote by $d_k f = E_k f - E_{k-1} f$.

We say an increasing filtration $M_k$ is “regular” if there exists a constant $c > 0$ such that, for any $m, a \in M_m, a \geq 0$,

$$\|a\|_{\infty} \leq c \|E_{m-1} a\|_{\infty}.$$  

For $M$ with a regular filtration $M_k$, $b \in L^2(M)$, we define paraproducts $\pi_b, \tilde{\pi}_b$ as operators for bounded $L^p(M)$ ($1 < p < \infty$)-martingales $f = (f_k)_{k \geq 0}$ as

$$\pi_b(f) = \sum_k d_k b f_{k-1}, \quad \tilde{\pi}_b(f) = \sum_k f_{k-1} d_k b.$$  

We prove the following result for $\pi_b$ and $\tilde{\pi}_b$:

**Theorem 1.2** Let $1 < p < q < \infty$, if $\tilde{\pi}_b$ and $\pi_b$ are both bounded on $L^p(M)$ then they are both bounded on $L^q(M)$.

We still do not know what happens when $p > q$.

### 2 Proof of Theorem 1.1 and Application to “Sweep” functions.

Denote by $tr$ the usual trace on $M_n$ and $S^p_n(1 \leq p < \infty)$ the Schatten $p$ classes on $\ell^2_n$.

**Proof of Theorem 1.1.** Let $c(n)$ be the best constant such that

$$\|\pi_b\|_{L^2(\ell^2_n) \to L^2(\ell^2_n)} \leq c(n) \|b\|_{L^\infty(M_n)}, \quad \forall b \in L^\infty(M_n).$$  

Denote by $T$ the triangle projection on $S^1_n$, we are going to show

$$\|T\|_{S^1_n \to S^1_n} \leq c(n).$$  

Once this is proved, we are done since $\|T\|_{S^1_n \to S^1_n} \sim \log(n + 1)$ (see [5]). Note that every $A$ in the unit ball of $S^1_n$ can be written as

$$A = \sum_m \lambda^{(m)} \alpha^{(m)} \otimes \beta^{(m)}$$  

with $\sum_m \lambda^{(m)} \leq 1$, $\sup_m \{||\alpha^{(m)}||_{\ell^2}, ||\beta^{(m)}||_{\ell^2}\} \leq 1$. Therefore, we only need to show
\[
||T(\alpha \otimes \beta)||_{S^1} \leq c(n) ||\alpha||_{\ell^2} ||\beta||_{\ell^2}, \quad \forall \alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell^2_n. \tag{2.4}
\]
Let $D$ be the diagonal $M_n$ valued function defined as
\[
D = \sum_{i=1}^n r_i e_i \otimes e_i
\]
where $r_i$ is the $i$-th Rademacher function on $\mathbb{T}$ and $(e_i)_{i=1}^n$ is the canonical basis of $\ell^2_n$. Given $\alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell^2_n$, let
\[
f = D\alpha, \ g = D\beta.
\]
Then $f, g \in L^2(\ell^2_n)$, and
\[
||f||_{L^2(\ell^2_n)} = ||\alpha||_{\ell^2_n}, \ ||g||_{L^2(\ell^2_n)} = ||\beta||_{\ell^2_n}. \tag{2.5}
\]
It is easy to verify
\[
\sum_k E_{k-1}f \otimes d_kg = D(\sum_{i<j \leq n} \alpha_i \beta_j e_i \otimes e_j)D.
\]
and
\[
\left\| \sum_k E_{k-1}f \otimes d_kg \right\|_{L^1(S^1_n)} = \left\| \sum_{i<j \leq n} \alpha_i \beta_j e_i \otimes e_j \right\|_{S^1_n} = ||T(\alpha \otimes \beta)||_{S^1}. \tag{2.6}
\]
On the other hand, by duality between $L^1(S^1_n)$ and $L^\infty(M_n)$, we have,
\[
\left\| \sum_k E_{k-1}f \otimes d_kg \right\|_{L^1(S^1_n)} = \sup \{ \text{tr} \int \sum_k d_k b (E_{k-1}f \otimes d_k g), \ ||b||_{L^\infty(M_n)} \leq 1 \}
\leq \sup \{ \pi_b(f) ||L^2(\ell^2_n)|| g ||L^2(\ell^2_n)\|, \ ||b||_{L^\infty(M_n)} \leq 1 \}
\leq c(n) ||f||_{L^2(\ell^2_n)} ||g||_{L^2(\ell^2_n)}. \tag{2.7}
\]
Combining (2.7), (2.5) and (2.6) we get (2.4) and the proof is complete. \hfill \blacksquare

Recall that the square function of $b$ is defined as
\[
S(b) = (\sum_k |d_k b|^2)^{\frac{1}{2}}.
\]
The so called “sweep” function is just the square of the square function, for this reason we denote it by $S^2(b)$,
\[
S^2(b) = \sum_k |d_k b|^2.
\]
In the classical case, we know that
\[
||S(b)||_{BMO_d} \leq c||b||_{BMO_d} \tag{2.8}
\]
\[
||S^2(b)||_{BMO_d} \leq c||b||^2_{BMO_d} \tag{2.9}
\]
When considering square functions \(S(b)\) for \(M_n\) valued functions \(b\), a similar result remains true with an absolute constant.

**Proposition 2.3** For any \(n \in \mathbb{N}\), and any \(M_n\) valued function \(b\), we have
\[
||S(b)||_{BMO_c} \leq \sqrt{2}||b||_{BMO_c}
\]

**Proof.** Since we are in the dyadic case, we have
\[
||S(b)||^2_{BMO_c} \leq 2 \sup_m ||E_m[(S(b) - E_mS(b))^*(S(b) - E_mS(b))]||_{L^\infty(M_n)}
\]
\[= 2 \sup_m ||E_mS^2(b) - (E_mS(b))^2||_{L^\infty(M_n)}
\]
Note
\[
E_mS^2(b) - \sum_{k=1}^m |d_k|b|^2 \geq E_mS^2(b) - (E_mS(b))^2 \geq 0.
\]
We get
\[
||S(b)||^2_{BMO_c} \leq 2 \sup_m ||E_mS^2(b) - \sum_{k=1}^m |d_k|b|^2||_{L^\infty(M_n)}
\]
\[= 2 \sup_m ||E_m \sum_{k=m+1}^m |d_k|b|^2||_{L^\infty(M_n)}
\]
\[\leq 2||b||^2_{BMO_c}. \blacksquare
\]
Matrix valued sweep functions have been studied in [1], [2] etc. Unlike in the case of square functions, it is proved in [1] that the best constant \(c_n\) such that
\[
||S^2(b)||_{BMO_c} \leq c_n||b||^2_{BMO_c} \tag{2.10}
\]
is \(c \log(n + 1)\). The following result shows that the best constant \(c_n\) is still \(c \log(n + 1)\) even if we replace \(|| \cdot ||_{BMO_c}\) by the bigger norm \(|| \cdot ||_{L^\infty(M_n)}\) in the right side of (2.10).

**Theorem 2.4** For every \(n \in \mathbb{N}\), there exists an \(M_n\) valued function \(b\) with \(||b||_{L^\infty(M_n)} \leq 1\) but such that
\[
||S^2(b)||_{BMO_c} \geq c \log(n + 1).
\]
Proof. Consider a function \( b \) that works for the statement of Theorem 1.1. Then \( \|b\|_{L^\infty(M_n)} \leq 1 \) and there exists a function \( f \in L^2(S_n^2) \), such that \( \|f\|_{L^2(S_n^2)} \leq 1 \) and

\[
\left\| \sum_k d_k b E_{k-1} f \right\|_{L^2(S_n^2)} \geq c \log(n + 1). \tag{2.11}
\]

We compute the square of the left side of (2.11) and get

\[
\left\| \sum_k d_k b E_{k-1} f \right\|^2_{L^2(S_n^2)} = \text{tr} \int \sum_k |d_k b|^2 E_{k-1} f E_{k-1} f^* \]

\[
= \text{tr} \int \sum_k |d_k|^2 (\sum_{i \leq k} |d_i f^*|^2 + \sum_{i > k} E_{i-1} f d_i f^* + \sum_{i > k} d_i f E_{i-1} f^* ) \]

\[
= \text{tr} \int \sum_i (\sum_{k > i} |d_k|^2 |d_i f^*|^2 + \text{tr} \sum_i (\sum_{k > i} |d_k|^2 ) (E_{i-1} f d_i f^* + d_i f E_{i-1} f^* ) \]

\[
= I + II
\]

For \( I \), note \( |d_i f^*|^2 \) is \( \mathcal{F}_i \) measurable, we have

\[
I = \text{tr} \int \sum_i E_i (\sum_{k > i} |d_k|^2 ) |d_i f^*|^2 \]

\[
\leq \sup_i \|E_i (\sum_{k > i} |d_k|^2 )\|_{L^\infty(M_n)} (\text{tr} \int \sum_i |d_i f^*|^2 ) \]

\[
\leq \|b\|^2_{BMO_c} \|f\|^2_{L^2(S_n^2)} \leq 4
\]

For \( II \), note \( E_{i-1} f d_i f^* + d_i f E_{i-1} f^* \) is a martingale difference and \( \sum_{k \leq i} |d_k|^2 \) is \( \mathcal{F}_{i-1} \) measurable since we are in the dyadic case, we get

\[
II = \text{tr} \int \sum_i S^2(b) (E_{i-1} f d_i f^* + d_i f E_{i-1} f^* ) \]

\[
= \text{tr} \int \sum_i d_i (S^2(b)) (E_{i-1} f d_i f^* + d_i f E_{i-1} f^* ) \]

\[
\leq 2 \| \sum_i d_i (S^2(b)) E_{i-1} f \||L^2(S_n^2)||f||_{L^2(S_n^2)} \]

\[
\leq 2 \| \pi_{S^2(b)} \||L^2(S_n^2) \rightarrow L^2(S_n^2) \]

\[
\leq 2 c \log(n + 1) \|S^2(b)||_{BMO_c}.
\]

We used (1.2) in the last step. Combining this with (2.11), we get

\[
c \log(n + 1) \leq \left\| \sum_k d_k b E_{k-1} f \right\|^2_{L^2(S_n^2)} \leq 4 + 2 c \log(n + 1) \|S^2(b)||_{BMO_c}
\]
Thus
\[ \|S^2(b)\|_{\text{BMO}_c} \geq c \log(n + 1). \]
This completes the proof. \( \blacksquare \)

3 Proof of Theorem 1.2.

We keep the notations introduced in the end of Section 1. Recall BMO spaces of noncommutative martingales are defined for \( x = (x_k) \in L^2(\mathcal{M}) \) as below (see [?, ??]):

\[
\text{BMO}_c(\mathcal{M}) = \{ x : ||x||_{\text{BMO}_c(\mathcal{M})} = \sup_n \left\| \sum_{k=n}^\infty d_k x \right\|^{\frac{1}{2}}_\mathcal{M} < \infty \};
\]

\[
\text{BMO}_r(\mathcal{M}) = \{ x : ||x||_{\text{BMO}_r(\mathcal{M})} = ||x^*||_{\text{BMO}_c(\mathcal{M})} < \infty \};
\]

\[
\text{BMO}_{cr}(\mathcal{M}) = \{ x : ||x||_{\text{BMO}_{cr}(\mathcal{M})} = \max\{ ||x||_{\text{BMO}_c(\mathcal{M})}, ||x||_{\text{BMO}_r(\mathcal{M})} \} < \infty \}.
\]

When \( \mathcal{M} = L^\infty(M_n) \), \( \text{BMO}_c(\mathcal{M}) \) is just \( \text{BMO}_c \) considered in Section 1 and 2. In this section, for noncommutative martingale \( b \), we consider \( \pi_b \) and \( \tilde{\pi}_b \) as operators on bounded noncommutative \( L^p \)-martingale spaces introduced in Section 1. We will need the following interpolation result and the John-Nirenberg theorem for noncommutative martingales proved by Junge and Musat recently (see [3], [6]).

**Theorem 3.5** (Musat) For \( 1 \leq p \leq q < \infty \),

\[
(\text{BMO}_{cr}(\mathcal{M}), L^p(\mathcal{M}))_\theta = L^q(\mathcal{M}), \text{ with } \theta = \frac{p}{q}.
\]

**Theorem 3.6** (Junge, Musat) For any \( 1 \leq q < \infty \) and any \( g = (g_k)_k \in \text{BMO}_{cr}(\mathcal{M}) \), there exist \( c_q, c'_q > 0 \) such that

\[
c'_q ||g||_{\text{BMO}_{cr}} \leq \sup_{m \in \mathbb{N}} \sup_{a \in \mathcal{M}_m, \tau(||a||^2) \leq 1} \{ ||\sum_{k \geq m} d_k ga||_{L^q(\mathcal{M})}, ||\sum_{k \geq m} ad_k g||_{L^q(\mathcal{M})} \} \leq c_q ||g||_{\text{BMO}_{cr}}.
\]

In fact, the formula above is proved for \( q \geq 2 \) in [3]. It is not hard to show that it is also true for \( 1 \leq q < 2 \). In the following, we give a simpler proof of it in the tracial case.

**Proof.** Note for any \( g \in \text{BMO}_{cr}(\mathcal{M}) \),

\[
||g||_{\text{BMO}_{cr}(\mathcal{M})} = \sup_{m \in \mathbb{N}} \sup_{a \in \mathcal{M}_m, \tau(||a||^2) \leq 1} \{ ||\sum_{k \geq m} d_k ga||_{L^2(\mathcal{M})}, ||\sum_{k \geq m} ad_k g||_{L^2(\mathcal{M})} \}.
\]

We get \( c_2 = c'_2 = 1 \). Note for \( p, r, s \) with \( 1/p = 1/r + 1/s \) and \( a \in L^p(\mathcal{M}) \), \( ||a||_{L^p(\mathcal{M})} \leq 1 \), there exist \( b, c \) such that \( a = bc \) and \( ||b||_{L^r(\mathcal{M})} \leq 1, ||c||_{L^s(\mathcal{M})} \leq 1 \). By Hölder’s inequality we then get \( c_q = 1 \) for \( 1 \leq q < 2 \) and \( c'_q = 1 \) for \( 2 < q < \infty \). Thus for \( 2 < q < \infty \), we
only need to prove the second inequality of (3.12). And, for $1 \leq q < 2$, we only need to prove the first inequality of (3.12). Fix $g \in BMO_{cr}(\mathcal{M})$, $m \in \mathbb{N}$, consider the left multiplier $L_m$ and the right multiplier $R_m$ defined as

$$L_m(a) = \sum_{k \geq m} d_k g a \quad \text{and} \quad R_m(a) = \sum_{k \geq m} a d_k g, \quad \forall a \in \mathcal{M}_m.$$ 

It is easy to check that

$$\sup_m \|L_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})} = \|g\|_{BMO_{cr}},$$

$$\sup_m \|L_m\|_{L^\infty(\mathcal{M}_m) \rightarrow BMO_{cr}} \leq \|g\|_{BMO_{cr}},$$

$$\sup_m \|R_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})} = \|g\|_{BMO_{cr}},$$

$$\sup_m \|R_m\|_{L^\infty(\mathcal{M}_m) \rightarrow BMO_{cr}} \leq \|g\|_{BMO_{cr}}.$$ 

Thus $L_m, R_m$ extend to bounded operators from $L^2(\mathcal{M}_m)$ to $L^2(\mathcal{M})$, as well as from $L^\infty(\mathcal{M}_m)$ to $BMO_{cr}(\mathcal{M})$. By Musat’s interpolation result Theorem 3.5, we get $L_m$ and $R_m$ are bounded from $L^q(\mathcal{M}_m)$ to $L^q(\mathcal{M})$ and their operator norms are smaller than $c_q \|g\|_{BMO_{cr}}$, for all $2 \leq q < \infty$. By taking supremum over $m$, we prove the second inequality of (3.12) for $q \geq 2$.

For $1 \leq q < 2$, by interpolation again, for $\theta = \frac{q}{2}$ and some $c_q'' > 0$,

$$\|L_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})} \leq c_q'' \|L_m\|_{L^\theta(\mathcal{M}_m) \rightarrow L^\theta(\mathcal{M})} \|L_m\|_{L^{1-\theta}(\mathcal{M}_m) \rightarrow BMO_{cr}},$$

$$\|R_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})} \leq c_q'' \|R_m\|_{L^\theta(\mathcal{M}_m) \rightarrow L^\theta(\mathcal{M})} \|R_m\|_{L^{1-\theta}(\mathcal{M}_m) \rightarrow BMO_{cr}}.$$ 

Thus

$$\|g\|_{BMO_{cr}} = \max \{\sup_m \|L_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})}, \sup_m \|R_m\|_{L^2(\mathcal{M}_m) \rightarrow L^2(\mathcal{M})}\} \leq c_q'' \|g\|_{BMO_{cr}} \sup_m \{\|L_m\|_{L^\theta(\mathcal{M}_m) \rightarrow L^\theta(\mathcal{M})}, \|R_m\|_{L^\theta(\mathcal{M}_m) \rightarrow L^\theta(\mathcal{M})}\}.$$ 

This gives the first inequality of (3.12) with $c_q = (c_q'')^{-\frac{1}{2}}$ for $1 \leq q < 2$. □

Recall that we say a filtration $\mathcal{M}_k$ is “regular” if, for some $c > 0$, $\|a\|_\infty \leq c\|E_{m-a}\|_\infty, \forall m \in \mathbb{N}, a \geq 0, a \in \mathcal{M}_m$.

**Lemma 3.7** For any regular filtration $\mathcal{M}_k$, we have

$$\|b\|_{BMO_{cr}(\mathcal{M})} \leq c_p \max \{\|\pi_b\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})}, \|\widetilde{\pi}_b\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})}\}, \forall 1 \leq p < \infty. \quad (3.13)$$

**Proof.** Note, for any $b \in BMO_{cr}(\mathcal{M})$ with respect to the regular filtration $\mathcal{M}_k$,

$$\|b\|_{BMO_{cr}(\mathcal{M})} \leq c \sup_{m \in \mathbb{N}} \sup_{\tau a \leq 1, a \in \mathcal{M}_m} \{\|\sum_{k_m \geq m} d_k ba\|_{L^2(\mathcal{M})}, \|\sum_{k_m \geq m} a d_k b\|_{L^2(\mathcal{M})}\}.$$ 

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Similar to the proof of Theorem 3.6, we can get,
\[ c_q ||b||_{BMO_c} \leq \sup_{m \in \mathbb{N}} \sup_{a \in \mathcal{M}, |\tau|a|^q \leq 1} \left\{ || \sum_{k \geq m} d_k ba ||_{L^q(\mathcal{M})}, || \sum_{k \geq m} a d_k b ||_{L^q(\mathcal{M})} \right\} \leq c_q ||b||_{BMO_c}. \tag{3.14} \]

On the other hand, by considering \( \pi_b(a), \tilde{\pi}_b(a) \) for \( a \in \mathcal{M}, ||a||_{L^p(\mathcal{M})} \leq 1 \), we have
\[
\sup_{a \in \mathcal{M}, |\tau|a|^q \leq 1} \left\{ || \sum_{k \geq m} d_k ba ||_{L^p(\mathcal{M})}, || \sum_{k \geq m} a d_k b ||_{L^p(\mathcal{M})} \right\} \leq 2 \max \{ ||\pi_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})}, ||\tilde{\pi}_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} \}.
\]
Taking supremum over \( m \) in the inequality above, we get (3.13) by (3.14).

**Lemma 3.8** For \( 1 < p < \infty \), we have
\[
||\pi_b||_{L^\infty(\mathcal{M}) \to BMO_c(\mathcal{M})} \leq c_p (||\pi_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} + ||b||_{BMO_c(\mathcal{M})}). \tag{3.15}
\]
\[
||\tilde{\pi}_b||_{L^\infty(\mathcal{M}) \to BMO_c(\mathcal{M})} \leq c_p (||\tilde{\pi}_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} + ||b||_{BMO_c(\mathcal{M})}). \tag{3.16}
\]

**Proof.** We prove (3.15) only. Fix a \( f \in L^\infty(\mathcal{M}) \) with \( ||f||_{L^\infty(\mathcal{M})} \leq 1 \). We have
\[
\left\| E_m \sum_{k \geq m} |d_k ba E_{k-1} f|^2 \right\|_{L^\infty(\mathcal{M})} = \sup \{ \tau E_m \sum_{k \geq m} |d_k ba E_{k-1} f|^2 a, a \in \mathcal{M}, a \geq 0, |\tau a| \leq 1 \}
\]
\[
= \sup \{ \tau \sum_{k \geq m} (d_k ba E_{k-1} f a^\frac{1}{2})^\ast (d_k ba E_{k-1} f a^\frac{1}{2}), a \in \mathcal{M}, a \geq 0, |\tau a| \leq 1 \}
\]
\[
\leq \sup_a \left\| d_m ba E_{m-1} f a^\frac{1}{2} + \sum_{k \geq m} d_k ba E_{k-1} (f a^\frac{1}{2}) \right\|_{L^p(\mathcal{M})} \left\| \sum_{k \geq m} d_k ba E_{k-1} f a^\frac{1}{2} \right\|_{L^q(\mathcal{M})}
\]
Note \( ||d_m ba E_{m-1} f a^\frac{1}{2}||_{L^p(\mathcal{M})} \leq ||d_m b||_{\mathcal{M}} \leq ||b||_{BMO_c} \). By (3.12) we get
\[
\left\| E_m \sum_{k \geq m} |d_k ba E_{k-1} f|^2 \right\|_{L^\infty(\mathcal{M})} \leq c_q (||b||_{BMO_c} + ||\pi_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} ||\pi_b(f)||_{BMO_c(\mathcal{M})}). \tag{3.17}
\]
Taking supremum over \( m \) in (3.17), we get
\[
||\pi_b(f)||_{BMO_c(\mathcal{M})}^2 \leq c_q (||b||_{BMO_c} + ||\pi_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} ||\pi_b(f)||_{BMO_c(\mathcal{M})}).
\]
On the other hand, since \((E_{m-1} f)(E_{m-1} f)^\ast \leq 1\), we have
\[
||\pi_b(f)||_{BMO_c(\mathcal{M})} \leq ||b||_{BMO_c(\mathcal{M})}.
\]
Thus,
\[ \|\pi_b(f)\|_{BMO_c(M)}^2 \leq (c_q + 1)(\|\pi_b\|_{L^p(M)} \to L^p(M) + \|b\|_{BMO_c(M)}) \|\pi_b(f)\|_{BMO_c(M)}. \]
Therefore
\[ \|\pi_b\|_{L^\infty(M) \to BMO_c(M)} \leq (c_q + 1)(\|\pi_b\|_{L^p(M)} \to L^p(M) + \|b\|_{BMO_c(M)}). \]

**Proof of Theorem 1.2.** By Lemma 3.7 and Lemma 3.8 we get immediately that
\[
\max \{\|\pi_b\|_{L^\infty(M) \to BMO_c}, \|\pi_b\|_{L^\infty(M) \to BMO_c}\} 
\leq c_p \max \{\|\pi_b\|_{L^p(M) \to L^p(M)}, \|\pi_b\|_{L^p(M) \to L^p(M)}\} 
\]
By the interpolation results of noncommutative martingales (Theorem 3.5), we get
\[
\max \{\|\pi_b\|_{L^\infty(M) \to L^\infty(M)}, \|\pi_b\|_{L^\infty(M) \to L^\infty(M)}\} 
\leq c_p \max \{\|\pi_b\|_{L^p(M) \to L^p(M)}, \|\pi_b\|_{L^p(M) \to L^p(M)}\}, 
\]
for all $1 < p < q < \infty$.

**Question:** Assume $\pi_b, \pi_b$ are of type $(p, p)$, are they of weak type $(1, 1)$? More precisely, assume $\|\pi_b\|_{L^p(M) \to L^p(M)} + \|\pi_b\|_{L^p(M) \to L^p(M)} < \infty$, does there exist a constant $C > 0$ such that, for any $f \in L^1(M)$, $\lambda > 0$, there is a projection $e \in M$ such that
\[
\tau(e^\perp) \leq C \frac{\|f\|_{L^1(M)}}{\lambda} \quad \text{and} \quad \|e\pi_b(f)e\|_{L^\infty(M)} + \|e\pi_b(f)e\|_{L^\infty(M)} \leq \lambda? 
\]

We have the following corollary by applying results of this section to matrix valued dyadic paraproducts discussed in Section 1 and Section 2. Note $M_n$ valued dyadic martingales on the unit circle are noncommutative martingales associated with the von Neumann algebra $\mathcal{M} = L^\infty(\mathbb{T}) \otimes M_n$ and the filtration $\mathcal{M}_k = L^\infty(\mathbb{T}, \mathcal{F}_k) \otimes M_n$.

**Corollary 3.9** Let $1 < p < \infty$, denote by $c_p(n)$ the best constant such that
\[
\|\pi_b\|_{L^p(S_n^1) \to L^p(S_n^1)} \leq c_p(n) \|b\|_{L^\infty(M_n)}, \forall b. 
\]
Then
\[
c_p(n) \sim \log(n + 1). 
\]

**Proof.** Note in the proof of Theorem 1.1, if we see $f$ as a column matrix valued function and $g$ as a row matrix valued function, we will have
\[
\|f\|_{L^p(S_n^1)} = \|\alpha\|_{\ell_2^n}, \|g\|_{L^p(S_n^1)} = \|\beta\|_{\ell_2^n}. 
\]
By the same method, we can prove $c_p(n) \geq c \log(n + 1)$ for all $1 < p < \infty$. For the inverse relation, by (1.2) we have $c_2(n) \leq c \log(n + 1)$. Then, by (3.15), we get
\[\|\pi_b\|_{L^\infty(M_n) \rightarrow \text{BMO}_{cr}} \leq c_2(c_2(n) \|b\|_{L^\infty(M_n)} + \|b\|_{\text{BMO}_{cr}}) \leq c \log(n + 1) \|b\|_{L^\infty(M_n)}, \quad \forall b \in L^\infty(M_n)\] (3.18)

Denote by $\pi_b^*$ the adjoint operator of the dyadic paraproduct $\pi_b$, then
\[\pi_b^*(f) = \sum_k (d_k b)^* E_k f.\]

Note we have the decomposition
\[\pi_b^*(f) = b^* f - \pi_{b^*}(f) - (\pi_{f^*}(b))^*.\]

By (3.18), we get
\[\|\pi_b^*\|_{L^\infty(M_n) \rightarrow \text{BMO}_{cr}} \leq \|b^*\|_{L^\infty(M_n)} + c \log(n + 1) \|b^*\|_{L^\infty(M_n)} + c \log(n + 1) \|b\|_{L^\infty(M_n)} \leq c \log(n + 1) \|b\|_{L^\infty(M_n)}.\] (3.19)

By (3.18), (3.19) and the interpolation result Theorem 3.5, we get
\[\|\pi_b\|_{L^p(S^n_{\mu}) \rightarrow L^p(S^n_{\mu})} \leq c_p \log(n + 1) \|b\|_{L^\infty(M_n)}, \quad \forall 1 < p < \infty.\]

Therefore, we can conclude $c_p(n) \sim \log(n + 1)$. □

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