Deterministic polarization entanglement purification using time-bin entanglement

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Received 17 November 2013, revised 2 June 2014
Accepted for publication 5 June 2014
Published 25 June 2014

Abstract
We present a deterministic entanglement purification protocol (EPP) working with the currently available experiment techniques. In this protocol, we exploit the robust time-bin entanglement to purify the polarization entanglement determinately, which is quite different from the previous EPPs. After purification, the two parties in quantum communication can obtain a maximally entangled photon pair from each transmitted over a polarization-noise channel with a success probability of 100%, in principle. As the maximal polarization entanglement is of utmost importance in long-distance quantum communication, this protocol may be very useful in future applications.

Keywords: entanglement purification, time-bin entanglement, quantum communication

1. Introduction
The distribution of entanglement states between long distant locations is essential for quantum communication [1–3]. For instance, in order to achieve the faithful teleportation of unknown quantum states [3, 4], quantum cryptograph [5–7], or quantum secure direct communication [8–10], people first need to set up a quantum channel with a maximally entangled state. Unfortunately, the source of entanglement is usually fragile. In a practical transmission, the interaction between a quantum entangled system and the innocent noise of the quantum channel always exists, which will make the maximally entangled state degrade and become a mixed state. A degraded quantum channel will make the fidelity of the teleportation degraded and the key in the quantum cryptograph insecure. Therefore, before performing the quantum communication, the parties involved should recover the degraded entangled states into maximally entangled states.

Entanglement purification is used to obtain the maximally entangled states from the less-entangled ones. It is one of the key techniques in quantum repeaters [11–16]. The first entanglement purification protocol (EPP) was proposed by Bennett et al in 1996 [17]. It was used to purify the Werner state [18] with a quantum logical gate, i.e. controlled-NOT (CNOT) gate. This protocol was improved by Deustch et al with similar quantum logical operations, subsequently [19]. The entanglement purification for multiparticle and high dimension systems have also been proposed [20–22]. However, the CNOT gates or similar logical operations are difficult to implement in current experiment.

In long-distance quantum communication, a photon encoded in the polarization degree of freedom (DOF) is a perfect qubit system for its simple manipulation and fast transmission. In 2001, Pan et al proposed an EPP with linear optics [23]. They used two polarization beam splitters (PBSs) to substitute the CNOT gates in [17, 19] to complete the parity-check measurement, which meant that the EPP can be realized easily in experiment. Subsequently, they have demonstrated entanglement purification for general mixed states of polarization-entangled photons [24]. Later, some other EPPs were proposed, such as the EPP based on the cross-Kerr nonlinearity [25], the EPPs for single-photon entanglement [26], quantum-dot and microcavity systems [27, 28], multi-photon systems with a high efficiency based on an entanglement link from subspaces [29], spatial-polarization hyperentangled photon pairs [30], hybrid systems [31] and so on [32–34].
There is another powerful way to realize polarization entanglement purification. It uses the entanglement in other stable DOFs to purify the polarization entanglement. In 2002, Simon and Pan first used the spatial entanglement to purify the polarization entanglement of photon pairs for bit-flip errors [35]. In 2010, Sheng et al proposed the first deterministic EPP to purify both bit-flip and phase-flip errors in photon pairs, resorting to both spatial and frequency entanglement [36]. Subsequently, the simplified deterministic EPPs, which only use the spatial entanglement or frequency entanglement to purify the polarization entanglement, were proposed [37–40].

Interestingly, qubits encoded in the time-bin degree of freedom are particularly suitable for long-distance quantum communication and fundamental experiments. The preparation of time-bin entangled states [41–44], violation of Bell inequalities [45, 46], quantum key distribution [47] and teleportation [48] were widely discussed. Moreover, Humphreys et al discussed the linear optical quantum computation in a single spatial mode [49]. One of the good advantages of the time-bin entanglement is that it is a robust form of optical quantum information, especially for transmission in optical fibers. In 2002, Thew et al demonstrated robust time-bin qubits for distributed quantum communication over 11 km [50]. In 2004, Marcikic et al also reported the experimental distribution of time-bin entangled qubits over 50 km of optical fibers. They demonstrated the violation of the Clauser–Horne–Shimony–Holt–Bell inequality by more than 15 standard deviations without removing the imperfect detectors [51]. Recently, Donohue et al reported their experimental results about the tomographically complete set of time-bin qubit projective measurements and showed that the fidelity of operation is sufficiently high to violate the Clauser–Horne–Shimony–Holt–Bell inequality by more than 6 standard deviations [52].

In this letter, we present a deterministic EPP for the polarization entanglement of photon pairs, resorting to the time-bin entanglement. Compared with conventional EPPs, the two parties in quantum communication can obtain a genuine pure entangled pair without consuming less-entangled photon pairs. We do not require the initial state to be the hyperentangled one evolution of the time-bin entanglement is more robust and can be well manipulated, which makes this EPP more useful in quantum communication applications and distributed quantum computation, compared with those based on the spatial entanglement or the frequency entanglement.

2. Deterministic EPP using time-bin entanglement

Now we start to explain this protocol by discussing a simple example. Interestingly, in this EPP, we do not require the polarization part to be entangled. Before the parties, Alice and Bob, share the polarization entangled pair, they first encode the initial pure polarization entanglement \[ |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|H\rangle|H\rangle + |V\rangle|V\rangle) \] into the time-bin entanglement. \(|H\rangle\) and \(|V\rangle\) denote the horizontal and vertical polarizations of the single photon, respectively. As shown in figure 1, the polarization beam splitter (PBS) can transmit the \(|H\rangle\) polarization photon and reflect the \(|V\rangle\) polarization photon, respectively. After the photons pass through PBS1 and PBS2, the initial state \(|\Phi^{+}\rangle_{AB}\) evolves as

\[
|\Phi^{+}\rangle_{AB} = \frac{1}{\sqrt{2}} \left( |H\rangle_{A} |H\rangle_{B} + |V\rangle_{A} |V\rangle_{B} \right) \\
= |H\rangle_{A} |H\rangle_{B} \otimes \frac{1}{\sqrt{2}} \left( |S\rangle_{A} |S\rangle_{B} + |L\rangle_{A} |L\rangle_{B} \right) \\
= |H\rangle_{A} |H\rangle_{B} \otimes \frac{1}{\sqrt{2}} \left( |S\rangle_{A} |S\rangle_{B} + |L\rangle_{A} |L\rangle_{B} \right) . \tag{1}
\]

The subscripts A and B mean that the photons transmit to Alice and Bob, respectively. The superscripts S and L denote the different spatial modes. L is the long arm and S is the short arm. After the photons pass through two 50:50 beam splitters (BSs), the state of the two-photon system \(|\Phi^{+}\rangle_{AB}\) becomes

\[
|\Phi^{+}\rangle'_{AB} = \frac{1}{2} \left( |H\rangle_{a1} |H\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1} |S\rangle_{a2} + |L\rangle_{a1} |L\rangle_{a2} \right) \\
+ \frac{1}{2} \left( |H\rangle_{a1} |H\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1} |S\rangle_{a2} - |L\rangle_{a1} |L\rangle_{a2} \right) \\
+ \frac{1}{2} \left( |H\rangle_{b1} |H\rangle_{b2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{b1} |S\rangle_{b2} - |L\rangle_{b1} |L\rangle_{b2} \right) \\
+ \frac{1}{2} \left( |H\rangle_{b1} |H\rangle_{b2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{b1} |S\rangle_{b2} + |L\rangle_{b1} |L\rangle_{b2} \right) . \tag{2}
\]

The initial state will be in the different spatial modes a1a2, a1b2, b1a2 and b1b2 with the same probability of 1/4. Now we only discuss the transmission in the channels a1a2. The same method can be used for the states in channels a1b2, a2b1 and b1b2 with or without a little modification.

The initial state with \(|H\rangle_{a1} |H\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1} |S\rangle_{a2} + |L\rangle_{a1} |L\rangle_{a2} \) can be rewritten as

\[
\rho = \rho_{P} \otimes \rho_{T} . \tag{3}
\]

Here \(\rho_{P}\) is the polarization part of the initial state with \(\rho_{P} = \langle HH | HH \rangle\) and \(\rho_{T}\) in the time-bin part with \(\rho_{T} = |\Phi^{+}\rangle_{\Phi^{+}}\).

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The photon pair transmits in the noisy channel, the polarization DOF of the photons is incident to be influenced by the vibration, the thermal fluctuation, or the distort of the fiber [53]. These effects...
will make the polarization entanglement be degraded and make the pure state \( \rho \) become a mixed state \( \rho' \) with

\[
\rho' = F(\Phi^+) (\Phi^+ + a|\Phi^-)(\Phi^- + b|\Psi^+)(\Psi^+ + c|\Phi^-)(\Phi^-). \tag{4}
\]

Here \( F = a + b + c = 1 \) and

\[
|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|H\rangle|H\rangle - |V\rangle|V\rangle),
\]
\[
|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle \pm |V\rangle|H\rangle). \tag{5}
\]

Fortunately, the time-bin entanglement is more robust than the polarization entanglement. After the photons pass through the noisy channel, the whole state of the photon system becomes

\[
\rho' = \rho \otimes \rho_T,
\]

and then the photons arrive at the setup for purification, as shown in figure 2.

The state \( \rho' \) can be described as follows: with the probability of \( F \), the state is \( |\Phi^\rangle|\Phi^\rangle \). With the probability of \( a \), the photon pair is in the state \( |\Phi^\rangle|\Phi^\rangle \). With the probability of \( b \) and \( c \), the photon pair is in the states \( |\Psi^\rangle|\Phi^\rangle \) and \( |\Phi^\rangle|\Psi^\rangle \) respectively.

We consider the case with \( |\Phi^\rangle|\Phi^\rangle \). It can be written as

\[
\frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_2 + |V\rangle_a|V\rangle_2) (|S\rangle_a|S\rangle_2 + |L\rangle_a|L\rangle_2).
\]

We first discuss the item \( |H\rangle_a|H\rangle_2 \otimes \frac{1}{\sqrt{2}}(|S\rangle_a|S\rangle_2 + |L\rangle_a|L\rangle_2) \). After the photons pass through PBS3 and PBS4, respectively, the whole state evolves as

\[
|H\rangle_{a1}|H\rangle_{a2} \otimes \frac{1}{\sqrt{2}}(|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2})
\]
\[
+ |H\rangle_{k1}|H\rangle_{k2} \otimes \frac{1}{\sqrt{2}}(|S\rangle_{k1}|S\rangle_{k2} + |L\rangle_{k1}|L\rangle_{k2})
\]
\[
- \frac{1}{\sqrt{2}} (|V\rangle_{k1}|V\rangle_{k2} \otimes |S\rangle_{k1}|S\rangle_{k2} + |H\rangle_{k1}|H\rangle_{k2} \otimes |L\rangle_{k1}|L\rangle_{k2})
\]
\[
= \frac{1}{\sqrt{2}} (|H\rangle_{a1}|H\rangle_{a2} + |V\rangle_{a1}|V\rangle_{a2}). \tag{7}
\]

Here \( P_{CA} \) is a Pockels cell [54]. It precedes the polarization interferometers and coordinates by the time reference. The parties in quantum communication makes the \( P_{CA} \) work only at the time corresponding to the scheduled arrival of the early state of the photons, performing the transformation \( |V\rangle \rightarrow |H\rangle \).

Finally, after the second Mach–Zehnder (M–Z) interferometer, the whole state of the photon pair becomes

\[
\frac{1}{\sqrt{2}} (|H\rangle_{a1}|H\rangle_{a2} + |V\rangle_{a1}|V\rangle_{a2})
\]
\[
+ \frac{1}{\sqrt{2}} (|H\rangle_{k1}|H\rangle_{k2} + |V\rangle_{k1}|V\rangle_{k2})
\]
\[
= \frac{1}{\sqrt{2}} (|H\rangle_{d1}|H\rangle_{d2} + |V\rangle_{d1}|V\rangle_{d2}). \tag{8}
\]

It is shown that they will obtain the maximally entangled state \( |\Phi^+\rangle \) by post selecting the photons in spatial modes \( D1 \) and \( D2 \). The item \( |V\rangle_{a1}|V\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2}) \) will be reflected by two PBSs and become \( |H\rangle_{a1}|H\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2}) \). It can also become the maximally entangled state \( |\Phi^+\rangle \) and finally be post selected by \( D3 \) and \( D4 \).

The other states \( |\Phi^\rangle|\Phi^\rangle, |\Psi^\rangle|\Phi^\rangle, \) and \( |\Phi^\rangle|\Psi^\rangle \) can evolve in the same way. For example, in the first item of \( |\Phi^\rangle|\Phi^\rangle \), after the setup shown in figure 2, the item \( |H\rangle_{a1}|V\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2}) \) becomes

\[
|H\rangle_{a1}|V\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2})
\]
\[
- |H\rangle_{k1}|V\rangle_{k2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{k1}|S\rangle_{k2} + |L\rangle_{k1}|L\rangle_{k2})
\]
\[
+ \frac{1}{\sqrt{2}} (|V\rangle_{k1}|V\rangle_{k2} \otimes |S\rangle_{k1}|S\rangle_{k2} + |H\rangle_{k1}|H\rangle_{k2} \otimes |L\rangle_{k1}|L\rangle_{k2})
\]
\[
= \frac{1}{\sqrt{2}} (|H\rangle_{d1}|V\rangle_{d2} + |V\rangle_{d1}|H\rangle_{d2}). \tag{9}
\]
and D4. The item $|V\rangle_{a1}|H\rangle_{a2} \otimes \frac{1}{\sqrt{2}} (|S\rangle_{a1}|S\rangle_{a2} + |L\rangle_{a1}|L\rangle_{a2})$ can also be used to obtain the maximally entangled state $|\Phi^+\rangle$ in the spatial modes D2 and D3, respectively. Here we only discuss the case that the state is in the mode of a1a2 with the probability of 1/4. Other cases can be discussed in the same way. In this way, the parties in quantum communication can purify the polarization entangled state deterministically.

3. Discussion

So far, we have discussed the whole purification process. It is straightforward to extend this EPP to the case for a multipartite state $\frac{1}{\sqrt{2}}(|HH\ldots H\rangle + |VV\ldots V\rangle)$. The parties in quantum communication first generate the state $|HH\ldots H\rangle \otimes \frac{1}{\sqrt{2}} (|SS\ldots S\rangle + |LL\ldots L\rangle)$ and then all the parties use the same setup shown in figure 2 to purify the polarization part. Finally, they will obtain the maximally entangled state in polarization with the same success probability of 100%. From above explanation, the realization of this EPP is essentially based on the robust time-bin entanglement. Several experiments described above showed that it is indeed robust for the transmission in an optical fiber. Actually, photons have the same transmission speed in the optical channel. The state $\frac{1}{\sqrt{2}}(|S\rangle|L\rangle + |L\rangle|S\rangle)$ coming from a bit-flip error never occurs. On the other hand, the phase-flip error also does not exist. In the practical transmission, the phase-flip error mainly comes from the fluctuation of the path length. For example, in the DLCZ protocol for quantum repeaters, the single-photon entanglement is in the form of $\frac{1}{\sqrt{2}}(|01\rangle + e^{i\theta}|10\rangle)$ [12]. The phase $\theta_{AB} = \theta_A - \theta_B$ denotes the difference of the phase shifts in the left and the right sides of the channel [15, 26]. Here, path length fluctuations do not lead to the phase flip. Because in a practical experiment, the time-of-flight difference of the M–Z interferometer is set to be on the order of a few nanoseconds, which is much less than the time of fluctuation in the fiber. That is to say, the photons in the S and L arms will suffer from the same noise, which is the collective noise [53]. Generally speaking, if a phase fluctuation occurs, it will make the two photons in the $|S\rangle|S\rangle$ time-bin become $|S\rangle \otimes e^{i\theta}|S\rangle$, the same effect will be on the later time-bin state, which make $|L\rangle|L\rangle \rightarrow |L\rangle \otimes e^{i\theta}|L\rangle$. The phase $\theta$ will become the global phase for the whole state and can be omitted. Existing experiments for time-bin entanglement also showed that it is more robust for distribution of tens of kilometers [50–52].

Local operations and classical communication cannot increase entanglement. Entanglement purification can be regarded as the transformation of the entanglement. In previous conventional EPPs, they need two pairs in the less-entangled states in each purification process, which means that the entanglement is transformed from the target pair to the source pair. It can be considered as the transformation between the same kind of DOF. Using the entanglement encoded in other degrees of freedom provides us a good way to perform the purification [36–39]. Interestingly, one of the significant features of these kinds of EPPs is that it does not require the initial polarization part of the state to be entangled. It makes these kinds of EPPs more like the protocol of entanglement distribution [53].

It is interesting to compare the present protocol with the performances of the previous EPPs. The present protocol has several advantages. First, only one pair of states is required, while in the conventional EPPs, they always need two photon pairs in less-entangled states. Second, we can get a maximally entangled pair and the other conventional EPPs only improve the quality of the mixed state. They should consume a large number of low quality mixed states to obtain a small number of high quality mixed states. As shown by Deng in [39], deterministic entanglement purification can be completed with the entanglement in only one DOF, not the hyperentanglement. Compared with some of the other deterministic EPPs, this protocol does not resort to the hyperentanglement. Moreover, the time-bin entanglement is more robust than the entanglement encoded in the other DOFs, which makes this protocol extremely suitable in practical quantum communication application and distributed quantum computation.

Finally, let us briefly discuss the possible realization of the present EPP in experiment. We explained this protocol based on the ideal entanglement source. Actually, the spontaneous parametric down-conversion (SPDC) source with the form of an ultrashort wave-packet (~ 100 fs) is also suitable for this protocol. Certainly, we should also require the difference of the M–Z interferometer and the PC are on the order of a few nanoseconds, respectively [54]. Current available PC can meet the requirements for the order of a few nanoseconds. In particular, Zhang et al showed that the efficiency of plasma PC at the whole aperture is better than 99% [55].

4. Conclusion

In conclusion, we have presented a determinate EPP assisted by the time-bin entanglement of photon systems. The success probability of this protocol is 100%, in principle. We do not need large pairs of less-entangled states. As the time-bin entanglement is more robust than the entanglement encoded in other DOFs, this EPP may have its practical application in current long-distance quantum communication and distributed quantum computation.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11104159 and 11347110), Qing Lan Project, Jiangsu Province 1311 Talent Plan, Nanjing University of Posts and Telecommunications and the Priority Academic Development Program of Jiangsu Higher Education Institutions, China.

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