ERROR ESTIMATES OF A FOURIER INTEGRATOR FOR THE CUBIC SCHRÖDINGER EQUATION AT LOW REGULARITY

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Abstract. We present a new filtered low-regularity Fourier integrator for the cubic nonlinear Schrödinger equation based on recent time discretization and filtering techniques. For this new scheme, we perform a rigorous error analysis and establish better convergence rates at low regularity than known for classical schemes in the literature so far. In our error estimates, we combine the better local error properties of the new scheme with a stability analysis based on general discrete Strichartz-type estimates. The latter allow us to handle a much rougher class of solutions as the error analysis can be carried out directly at the level of $L^2$ compared to classical results in dimension $d$, which are limited to higher-order (sufficiently smooth) Sobolev spaces $H^s$ with $s > d/2$. In particular, we are able to establish a global error estimate in $L^2$ for $H^1$ solutions which is roughly of order $\tau^{1+5/d}$ in dimension $d \leq 3$ ($\tau$ denoting the time discretization parameter). This breaks the “natural order barrier” of $\tau^{1/2}$ for $H^1$ solutions which holds for classical numerical schemes (even in combination with suitable filter functions).

1. Introduction

We consider the cubic nonlinear Schrödinger equation

\begin{equation}
    i\partial_t u = -\Delta u + |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d
\end{equation}

in dimension $d \leq 3$. This equation, and more generally semi-linear Schrödinger equations,

\begin{equation}
    i\partial_t u = \Delta u + \mu |u|^{2p} u, \quad p \in \mathbb{N}, \quad \mu = \pm 1
\end{equation}

are numerically well studied. To approximate the time evolution of (2) various (time) discretization techniques have been proposed in the literature based on, e.g., splitting the right-hand side into the linear and nonlinear part (splitting schemes) or discretizing Duhamel’s formula (exponential integrators), see, e.g., [3, 7, 9, 11, 12, 14, 15, 16, 20, 22, 27, 32] and the references therein.

For smooth solutions the error behaviour of these classical schemes is nowadays well understood and, based on a rigorous error analysis, global error estimates could be established. In the error estimates the regularity of the solution plays a crucial role and convergence (of a certain rate) only holds for sufficiently smooth solutions. One of the reasons for this regularity requirement is the following. Within the construction of all (classical) numerical methods the stiff part (i.e., the term involving the differential operator $-\Delta$) is approximated in a way that the control of the local error requires the boundedness of additional spatial derivatives of the exact solution. Therefore, convergence of a certain order only holds under sufficient additional regularity assumptions on the solution. The severe order reduction of classical numerical schemes in case of non-smooth solutions is nowadays a well established fact in numerical analysis, see, e.g., [13, 14, 23, 28] in case of (non)linear Schrödinger equations.

More precisely, classical schemes for (2) with time step size $\tau$ introduce a local error that behaves roughly like (cf. [18, 27, 28])

\begin{equation}
    \tau^{1+\gamma} (-\Delta)^{\gamma} u(t),
\end{equation}

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$u$ being the exact solution, such that convergence of order $\tau^\gamma$ in $H^s$ requires solutions in $H^{s+2\gamma}$.

Recently, a Fourier integrator for Schrödinger equations was introduced in [28]. The main interesting property of the new scheme lies in the fact that the boundedness of only one additional derivative of the exact solution is required thanks to a local error structure of type

$$\tau^{1+\gamma} |\nabla^\gamma u(t)|$$

such that convergence of order $\tau^\gamma$ in $H^s$ requires solutions only in $H^{s+\gamma}$.

While the new discretization technique presented in [28] allowed us to cut down the regularity assumption in the local error (cf. (3) and (4), respectively), the stability analysis in low regularity spaces remained an open problem. This is due to the fact that the error analysis of low regularity integrators was up to now only based on classical tools. For estimating the nonlinear terms (in the global error) classical bilinear estimates based on Sobolev embedding are exploited. Note that this is a common approach in the error analysis of nonlinear dispersive equations, see, e.g., [13, 14] in case of semi-linear Schrödinger equations and [4, 5, 19, 28] in the context of low regularity integrators. This classical approach easily allows us to prove stability of the numerical scheme at the cost that it requires highly regular solutions. More precisely, the analysis in [28] is restricted in dimension $d$ to higher-order (sufficiently smooth) Sobolev spaces

$$H^s \quad \text{with Sobolev exponent } \quad s > d/2$$

for which $H^s$ is an algebra. The latter assumption allows us to establish the global error estimate

$$\|u(t_n) - u^n\|_s \leq ct^{\gamma} \quad \text{for solutions } u \in H^{s+\gamma} \quad \text{for } s > d/2.$$  

Here, $u^n$ denotes the numerical approximation to the exact solution $u(t)$ at time $t = t_n = n\tau$. While the condition $s > d/2$ is common in classical error analysis of nonlinear problems, it drastically increases the regularity assumptions on the solution: classical convergence estimates (such as (6)) are restricted to the class of solutions in $H^{d/2+\varepsilon+\gamma}$ ($\varepsilon > 0$) which is particularly limiting in higher dimensions $d \geq 2$.

While, from a numerical point of view, the analysis of nonlinear problems at low regularity is still (in large parts) widely open, the difficulty in the control of the nonlinear terms in low regularity spaces could be overcome in many cases at a continuous level. For the Schrödinger equation (2) it is, for instance, a well-established fact (see, for example, the books [8, 26, 31]) that the Cauchy problem for (2) on $\mathbb{R}^d$ is locally well-posed in $L^2$ for $2p \leq d$ and in $H^1$ for $2p \leq \frac{d}{1-\varepsilon}$. The essential tool in the well-posedness analysis in low regularity spaces are Strichartz estimates. In case of the free Schrödinger flow $S(t) = e^{it\Delta}$ on $\mathbb{R}^d$ they take the form

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r} \leq c_{d,q,r}\|u_0\|_2 \quad \text{for } 2 \leq q, r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad (q, r, d) \neq (2, \infty, 2).$$

A natural question is what can we gain from them numerically. In particular, as for parabolic evolution equations, the so-called parabolic smoothing property

$$\|(\nabla^\alpha e^{it\Delta}u_0\|_{L_x^r} \leq c_{d,\tau^{-\alpha}}^{\frac{d}{2}}\|u_0\|_{L_x^r}, \quad \alpha \geq 0$$

is highly exploited in numerical analysis (see, e.g., the recent result [25]). The main difficulty from a numerical point of view is that Strichartz estimates (7) are, in contrast to (5), not pointwise in time and their gain lies in integrability and not differentiability. In particular, Strichartz-like estimates do not hold for the time (nor fully) discrete Schrödinger group $\{e^{i\tau r\Delta}\}_{\tau \in \mathbb{N}}$, see, e.g., the important works [20, 22, 21, 29].

In [20, 22] a new filtered splitting approximation was introduced for the nonlinear Schrödinger equation on $\mathbb{R}^d$, based on filtering the high frequencies in the linear part $S_\tau(t)\varphi = S(t)\Pi_{\tau^{-\frac{1}{2}}\varphi}$ with the filter function

$$\Pi_{\tau^{-\frac{1}{2}}\varphi}(\xi) = \hat{\varphi}(\xi)1_{\{\xi|\leq\tau^{-1/2}\}}, \quad \xi \in \mathbb{R}^d.$$
These filtered groups $S_c(t)$ admit discrete Strichartz-like estimates which are discrete in time and uniform in the time discretization parameter. The latter allows one to show stability of the scheme in the same space where the stability of the PDE is established. For these filtered schemes (of classical order one) error bounds of order one could be established in $L^2$ for solutions in $H^2$ for semilinear Schrödinger equations. In the preprint [10], this result was be extended to the semi discrete (time) analysis of the filtered Lie splitting scheme for $H^1$ solutions at the price of reduced order $\tau^{1/2}$ for time convergence – the natural order barrier of classical numerical schemes at this level of regularity.

Let us also mention the paper [27], where the error of the second-order Strang splitting scheme for nonlinear Schrödinger and Schrödinger–Poisson equations was analysed. In this paper, Lubich’s sophisticated argument allowed for the first time a rigorous second-order convergence bound of Strang splitting for the cubic nonlinear Schrödinger (NLS) equation in $L^2$ for exact solutions in $H^1$ (the natural space of regularity for classical second-order methods). The idea is to first prove fractional convergence of the schemes in a suitable higher-order Sobolev space which implies a priori the boundedness of the numerical scheme in this space. This then allows one to establish error estimates in lower-order Sobolev spaces as classical bilinear estimates can be applied in the stability argument with the numerical solution measured in a stronger norm. As the scaling of dimension and order of convergence play an important role, the argument does, however, not apply to solutions in $H^s$ with $s < d/2$.

In the present work, we introduce a new filtered low-regularity Fourier integrator based on the time discretization technique introduced in [28] and inspired by the filtering of high frequencies [20, 22]. The good properties of the new scheme together with a fine error analysis allow us to establish better convergence rates at low regularity than known in the literature so far, in particular, compared to our previous work [28] on low-regularity integrators which was restricted to sufficiently smooth Sobolev spaces $H^s$ with $s > d/2$. With the aid of general discrete Strichartz-type estimates, we can overcome this limitation and prove $L^2$ estimates for the new scheme for solutions in $H^1$ in dimensions $d \leq 3$.

This approach in particular allows us to break the “natural order barrier” of $\tau^{1/2}$ for $H^1$ solutions. Note that the latter cannot be overcome by classical numerical schemes (not even by introducing suitable filter functions) due to their classical error structure of type $\tau^\delta (-\Delta)^\delta u$, introduced by the leading second order differential operator $-\Delta$.

2. A Fourier integrator for the cubic Schrödinger equation at low regularity, the main theorem and the central idea of the proof

In order to approximate the solution $u(t)$ of (1) at time $t = t_{n+1} = t_n + \tau$ we choose the one-step method

$$
\begin{align*}
  u^{n+1} &= \Phi_K(u^n) := e^{i\tau \Delta} \left( u^n - i\tau \Pi_K \left( (\Pi_K u^n)^2 \varphi_1(-2i\tau\Delta)\Pi_K u^n \right) \right), \\
  u^0 &= \Pi_K u(0)
\end{align*}
$$

(9)

with $\varphi_1(z) = \frac{e^z - 1}{z}$ and the projection operator defined by the Fourier multiplier

$$
\Pi_K = \chi^2 \left( \frac{-i\nabla}{K} \right),
$$

(10)

which in Fourier space reads

$$
\widehat{\Pi_K \phi}(\xi) = \hat{\phi}(\xi) \chi^2 \left( \frac{\xi}{K} \right), \quad \xi \in \mathbb{R}^d.
$$
Here $\chi$ is a smooth radial nonnegative function which is one on $B(0,1)$ and supported in $B(0,2)$, and $K \geq 1$ is considered as a parameter that will depend on $\tau$. Note that, here, we will not restrict ourselves to the choice $K = \tau^{-\frac{d}{2}}$ as in [20], but we allow $K = \tau^{-\frac{d}{2}}$ with some $\alpha \geq 1$. The main reason for this choice is that the introduction of the filter introduces a new term in the error. Indeed, by denoting by $u$ the exact solution of (1) and by $u^n$ the sequence given by the scheme (9), we have the estimate

$$
\|u(t_n) - u^n\|_{L^2} \leq \|u^n - u^K(t_n)\|_{L^2} + \|u^K(t_n) - u(t_n)\|_{L^2},
$$

where $u^K(t)$ denotes the exact solution of the filtered PDE,

$$
i\partial_t u^K = -\Delta u^K + \Pi_K([\Pi_K u^K]^2 \Pi_K u^K), \quad u^K(0) = \Pi_K u(0).$$

We now observe that the scheme (9) is exactly the low-regularity Fourier integrator introduced in [20, 22] and by

$$
\Pi_K u^n = u^n - \Delta u^n + \Pi_K([\Pi_K u^n]^2 \Pi_K u^n), \quad u^n(0) = \Pi_K u(0),
$$

Note that this type of loss of derivative in the Strichartz estimates also occurs in the case of compact solutions the global error is proportional to $\tau^{1/2}$, in general, due to the local error structure (3).

We conclude this section with the main theorem on the precise error estimates for our new scheme.

**Theorem 2.1.** For every $T > 0$ and $u_0 \in H^1$, let us denote by $u \in C([0,T], H^1)$ the exact solution of (1) with initial datum $u_0$ and by $u^n$ the sequence defined by the scheme (9). Then, there exist $\tau_0 > 0$ and $C_T > 0$ such that for every step size $\tau \in (0, \tau_0]$, we have the following error estimates:

- if $d = 1$, with the choice $K = 1/\tau^\frac{d}{2}$,
  $$
  \|u^n - u(t_n)\|_{L^2} \leq C_T \tau^\frac{d}{4}, \quad 0 \leq n \leq N,
  $$

if $d = 2$, with the choice $K = 1/\tau^{\frac{3}{4}}$, 
\[ \|u^n - u(t_n)\|_{L^2} \leq C_T \tau^{\frac{3}{4}}, \quad 0 \leq n \leq N, \]

- if $d = 3$, with the choice $K = 1/\tau^{\frac{2}{3}}$, 
\[ \|u^n - u(t_n)\|_{L^2} \leq C_T \tau^{\frac{2}{3}} |\log \tau|^{\frac{2}{3}}, \quad 0 \leq n \leq N, \]

where $N$ is such that $N\tau \leq T$.

In the above theorem we focused on $H^1$ solutions and optimized the rate of convergence. At the price of allowing a lower rate of convergence, we could handle even rougher data. Note that we have analyzed only the defocusing equation (11). Nevertheless, the same results are true for the focusing one as long as the exact solution remains in $H^1$ (we recall that finite time blow-up in $H^1$ will occur in dimensions $d = 2, 3$).

The rest of the paper is organized as follows.

In Section 4, we describe the discrete Strichartz estimates, the proofs are postponed to Section 10. The aim of Section 5 is to analyze the error $\|u^K(t_n) - u^n\|_{L^2}$. A crucial step towards the proof of Theorem 2.1 is performed in Section 6. Indeed, we prove that the exact solution $u^K$ of (12) enjoys discrete Strichartz estimates, see Proposition 6.3, that involve some loss of derivative or loss that is still better than that resulting from straightforward Sobolev embedding. These discrete Strichartz estimates for $u^K$ are needed for two reasons. At first, the structure of the local error described by (4) is a bit sketchy. A more precise description is given by (cf. Corollary 7.2)
\[ \tau^2 |\nabla u^K|^2 u^K(t) \]
so that in order to control the local error in $L^2$ we need at least to control $\|\nabla u^K(t)\|_{L^1}$. Therefore, we need to rely on these discrete Strichartz estimates satisfied by the exact solution $u^K$ of the filtered PDE (12) in order to estimate this part of the local error without using more regularity. The other part, where the estimates of Proposition 6.3 are crucially used, is in the proof of the stability of the scheme at low regularity. Indeed, by defining $e^n = u^n - u^K(t_N)$, we get that $e^n$ solves
\[ e^{n+1} = e^{i\tau\Delta}(e^n - i\tau\Pi_K(\varphi_1(-2i\tau\Delta)\Pi_K e^n(\Pi_K u^K(t_n)^2))) + \cdots \]
where the dots stand for similar or quadratic and cubic terms with respect to $e^n$. Therefore, we get an $L^2$ estimate of the form
\[ \|e^{n+1}\|_{L^2} \leq \|e^n\|_{L^2}(1 + \tau\|u^K(t_n)\|_{L^\infty}^2 + \cdots). \]

In order to prove even boundedness of $e^n$, we need to prove that the expression
\[ \tau \sum_{n=0}^{N} \|u^K(t_n)\|_{L^\infty}^2 \]
is uniformly bounded with respect to $\tau$. This type of estimate will be a consequence of Proposition 6.3. Note that this uniform boundedness in dimension $d \geq 2$ cannot be obtained by using only the fact that $u^K \in C([0, T], H^1)$.

In Sections 7 and 8, we analyze the local error and finally, in Section 9 we prove Theorem 2.1.

3. Notations

Note that the mild solution $u(t) = u(t, \cdot)$ of (1) is given by
\[ u(t_n + \tau) = e^{i\tau\Delta} u(t_n) - ie^{i\tau\Delta} T(u)(\tau, t_n) \]
with the Duhamel operator

\[(14) \quad T(u)(\tau, t_n) = \int_0^\tau e^{-is\Delta} |u(t_n + s)|^2 u(t_n + s) \, ds.\]

Let \( F \) be a function of two variables \((t, x) \in \mathbb{R} \times \mathbb{R}^d\). We use the continuous norms

\[
\| F \|_{L^p L^q} = \left( \int_\mathbb{R} \| F(t, \cdot) \|_{L^q}^p \, dt \right)^{\frac{1}{p}},
\]

\[
\| F \|_{L^p L^q; \tau} = \left( \int_0^\tau \| F(t, \cdot) \|_{L^q}^p \, dt \right)^{\frac{1}{p}}
\]

with the convention that for \( p = \infty \) the integral is replaced by the ess sup.

At the discrete level, for a sequence \((F_k(x))_{k \in \mathbb{Z}}\), we use the notation

\[
\| F \|_{\ell^p_{\tau, \ell^q}} = \| F_k \|_{\ell^p_{\tau, \ell^q}} = \left( \tau \sum_{k \in \mathbb{Z}} \| F_k \|_{L^q}^p \right)^{\frac{1}{p}}
\]

and

\[
\| F_k \|_{\ell^p_{\tau, N} L^q} = \left( \tau \sum_{k=0}^N \| F_k \|_{L^q}^p \right)^{\frac{1}{p}}.
\]

For \( p = \infty \), \( \tau \) times the sum is replaced by the supremum.

Finally, we write \( a \lesssim b \) whenever there is a generic constant \( C > 0 \) such that \( a \leq Cb \).

4. Continuous and discrete Strichartz estimates

Let us first recall the classical Strichartz estimates for the linear Schrödinger equation.

Let us say that \((p, q)\) is admissible if \( p \geq 2, q \geq 2, (p, q, d) \neq (2, \infty, 2) \) and \( \frac{2}{p} + \frac{d}{q} = \frac{d}{2} \). The admissible pair with \( p = 2 \) is called the endpoint. Note that there is no such point in dimensions 1 and 2. As usually, the dual indices of \((p, q)\) will be denoted by \((p', q')\), i.e., \( \frac{1}{p'} + \frac{1}{q'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \).

**Theorem 4.1.** For every \((p, q)\), admissible, there exists \( C > 0 \) such that for every \( f \in L^2 \) and \( F \in L^{p'} L^{q'} \)

\[(15) \quad \| e^{it\Delta} f \|_{L^p L^q} \leq C \| f \|_{L^2} \]

\[(16) \quad \left\| \int_\mathbb{R} e^{-is\Delta} F(s, \cdot) \, ds \right\|_{L^2} \leq C \| F \|_{L^{p'} L^{q'}}. \]

Moreover, for every \((p_1, q_1)\) and \((p_2, q_2)\) admissible, there exists \( C > 0 \) such that for every \( F \in L^{p_1'} L^{q_1'2} \), we have

\[(17) \quad \left\| \int_{-\infty}^t e^{i(t-s)\Delta} F(s, \cdot) \, ds \right\|_{L^{p_1} L^{q_1}} \leq C \| F \|_{L^{p_2'} L^{q_2'2}}. \]

These estimates were proven by Strichartz [30] in a special case and by Ginibre and Velo [17]. The endpoint \( p = 2 \) for \( d \geq 3 \) was proven by Keel and Tao [24].

We shall next study discrete versions of these inequalities for the group

\[(18) \quad S_K(t) = e^{it\Delta} \Pi_K = \Pi_K e^{it\Delta}. \]

We will consider that \( K \geq \tau^{-\frac{1}{2}} \). In the case \( K = \tau^{-\frac{1}{2}} \) such estimates were established in [22]. This is the only choice which ensures estimates without loss. Here, we will allow some loss depending on \( K \) in order to optimize the total error.
Theorem 4.2. For every \((p, q)\) admissible with \(p > 2\), there exists \(C > 0\) such that for every \(K\) and \(\tau\) satisfying \(K\tau^{\frac{1}{p}} \geq 1\) and all \(f \in L^2\), we have
\[
\|S_K(n\tau)f\|_{L^p_t L^q_x} \leq C(K\tau^{\frac{1}{p}})^\frac{2}{p} \|f\|_{L^2}.
\]

For every \((p, q)\) admissible with \(p > 2\), there exists \(C > 0\) such that for every \(K\) and \(\tau\) satisfying \(K\tau^{\frac{1}{p}} \geq 1\) and all \(F \in L^p_t L^q_x\), we have
\[
\left\| \tau \sum_{n \in \mathbb{Z}} S_K(-n\tau)F_n \right\|_{L^2_t} \leq C(K\tau^{\frac{1}{p}})^\frac{2}{p} \|F\|_{L^p_t L^q_x}.
\]

For every \((p_1, q_1), (p_2, q_2)\) admissible with \(p_1 > 2, p_2 > 2\), there exists \(C > 0\) such that for every \(K\) and \(\tau\) satisfying \(K\tau^{\frac{1}{p}} \geq 1\), all \(s \in [-3, 3]\) and all \(F \in L^p_t L^q_x\), we have
\[
\left\| \tau \sum_{k=-\infty}^{n-1} S_K((n-k+s)\tau)F_k \right\|_{L^p_t L^q_x} \leq C(K\tau^{\frac{1}{p}})^\frac{2}{p_1} \|F\|_{L^p_t L^q_x}.
\]

Note that we have excluded the endpoints in the statements of the Strichartz estimates \(19\), \(20\) and \(21\). Also note that in the estimate \(21\), we have added in the definition of the operator a shift \(s\tau\). Though it almost does not change anything in the proof, taking into account this shift will be crucial to get the estimates of Proposition 6.5 and the control of the local error. The proof of Theorem 4.2 is postponed to Section 10.2.

It will be useful to convert the estimates of Theorem 4.2 when \(K = \tau^{-\frac{\alpha}{2}}\) with \(\alpha \geq 1\) (a choice that we will make in order to optimize the error estimate) into estimates with uniformly bounded constants but with loss of derivatives.

Corollary 4.3. For every \((p, q)\) admissible with \(p > 2\), there exists \(C > 0\) such that for every \(0 < \tau \leq 1\) and \(K = \tau^{-\frac{\alpha}{2}}, \alpha \geq 1\), we have
\[
\|S_K(n\tau)f\|_{L^p_t L^q_x} \leq C\|f\|_{H^{\frac{2}{p}}(1, \frac{1}{\alpha})} \quad \text{for all } f \in H^{\frac{2}{p}}(1, \frac{1}{\alpha}).
\]

For every \((p, q)\) admissible with \(p > 2\), there exists \(C > 0\) such that for every \(0 < \tau \leq 1, K = \tau^{-\frac{\alpha}{2}}, \alpha \geq 1\) and \(s \in [-8, 8]\) we have
\[
\left\| \tau \sum_{k=-\infty}^{n-1} S_K((n-k+s)\tau)F_k \right\|_{L^p_t L^q_x} \leq C\|F\|_{L^p_t H^{\frac{2}{p}}(1, \frac{1}{\alpha})} \quad \text{for all } F \in L^p_t H^{\frac{2}{p}}(1, \frac{1}{\alpha}).
\]

Note that, since \(\tau^{\frac{1}{p}}\|\Pi_K f\|_{L^p_t} \leq \|S_K(n\tau)f\|_{L^p_t L^q_x}\) the estimate \(22\) also encodes the modified Sobolev estimate
\[
\tau^{\frac{1}{p}}\|\Pi_K f\|_{L^p_t} \leq C\|f\|_{H^{\frac{2}{p}}(1, \frac{1}{\alpha})}.
\]

The proof of this estimate is postponed to Section 10.3.

5. \(H^1\) Cauchy Problem for (1)

Let us recall the following well-known result for (1). We refer, for example, to the book [26].

**Theorem 5.1.** For \(d \leq 3\) and for every \(u_0 \in H^1\), there exists for every \(T > 0\) a unique solution of (1) in \(C([0, T], H^1)\) such that \(u(0) = u_0\). Moreover, this solution is such that \(u, \nabla u \in L^p_t L^q\) for every admissible \((p, q)\).
Note that in the focusing case, there exists under the same assumptions a maximal $H^1$ solution defined on $[0,T^*)$ (and in this case $T^*$ can be finite) with similar properties. All our convergence estimates thus extend to the focusing case on $[0,T]$ for every $T < T^*$.

Let us now consider a frequency truncated equation

\begin{equation}
\tag{25}
i \partial_t u^K = -\Delta u^K + \Pi_K (|\Pi_K u^K|^2 \Pi_K^* u^K), \quad u^K(0) = \Pi_K u_0.
\end{equation}

As in Theorem 5.1, we can easily get:

**Proposition 5.2.** For $d \leq 3$, $u_0 \in H^1$, and $K \geq 1$, there exists a unique solution of (25) such that $u^K \in C([0,T],H^1)$ for every $T \geq 0$. Moreover $u^K, \nabla u^K \in L^p_t L^q$ for every admissible $(p,q)$. More precisely, for every $T \geq 0$ and every $(p,q)$ admissible, there exists $C_T > 0$ such that for all $K \geq 1$ we have

$$\|u^K\|_{L^p_{t}L^q} \leq C_T.$$

We shall not detail the proof of this proposition that follows exactly the lines of the proof of Theorem 5.1.

**Remark 5.3.** Note that, since $\Pi_{2K} \Pi_K = \Pi_K$, we have that $\Pi_{2K} u_K$ solves the same equation (25) with the same initial data and hence we have by uniqueness that

$$\Pi_{2K} u^K(t) = u^K(t) \quad \text{for all } t \in [0,T].$$

We can also easily get the following corollary.

**Corollary 5.4.** For $d \leq 3$, $u_0 \in H^1$ and every $T > 0$, there exists $C_T > 0$ such that for every $K \geq 1$, we have the estimate

$$\|u - u^K\|_{L^\infty_t L^2} \leq \frac{C_T}{K}.$$

This will allow us to discretize in time the projected equation for $u^K$ only.

**Proof.** Let us first take $M_T$ such that by using Theorem 5.1 and Proposition 5.2, we have

\begin{equation}
\tag{26}
\|u^K\|_{L^p_t W^{1,q}} + \|u\|_{L^p_t W^{1,q}} \leq M_T, \quad K \geq 1
\end{equation}

for some $(p,q)$ admissible with $q$ such that $d < q < 2 + \frac{4}{d-2}$ ($q < \infty$ if $d = 1, 2$) so that $W^{1,q}$ is embedded in $L^\infty$. Note that $M_T$ in general depends on $T$. In the following and more generally, we will denote by $M_T$ a generic constant that depends on $T$.

We further note that (26) in particular yields, by using successively the Sobolev embedding and Hölder's inequality, that

\begin{equation}
\tag{27}
\|u^K\|_{L^\infty_t L^\infty} + \|u\|_{L^\infty_t L^\infty} \lesssim T^{1-\frac{q}{2}} (\|u^K\|_{L^p_t W^{1,q}} + \|u\|_{L^p_t W^{1,q}}) \lesssim T^{1-\frac{q}{2}} M_T,
\end{equation}

where $(p,q)$ is admissible and $q$ is such that $d < q < 2 + \frac{4}{d-2}$ ($q < \infty$ if $d = 1, 2$).

We first observe that

$$\|u - \Pi_K u\|_{L^\infty_t L^2} \lesssim \frac{1}{K} \|\nabla u\|_{L^\infty_t L^2} \lesssim \frac{M_T}{K}.$$

By using Duhamel's formula, we have that

\[ u(t) - u^K(t) = e^{it\Delta}(1 - \Pi_K)u_0 - i \int_0^t e^{i(t-s)\Delta} \Pi_K (|u|^2u - |\Pi_K u|^2 \Pi_K u) \, ds \]

\[ - i \int_0^t e^{i(t-s)\Delta} \Pi_K (|\Pi_K u|^2 \Pi_K u - |\Pi_K u|^2 \Pi_K u) \, ds - i \int_0^t e^{i(t-s)\Delta} (1 - \Pi_K)(|u|^2u) \, ds. \]
From standard estimates, we then obtain that for every $T_1 \leq T$,
\[
\|u - u^K\|_{L^\infty_{T_1}} L^2 \leq \frac{C}{K} + C\|u - \Pi_K u\|_{L^\infty_{T_1}} L^2 \left(\|u\|_{L^2_{T_1}}^2 L^\infty + \|\Pi_K u\|_{L^2_{T_1}}^2 L^\infty\right)
+ C\|u - u^K\|_{L^\infty_{T_1}} L^2 \left(\|u\|_{L^2_{T_1}}^2 L^\infty + \|u^K\|_{L^2_{T_1}}^2 L^\infty\right) + \frac{C}{K} \|u\|_{L^\infty_{T_1}} L^2 H^1 \|u\|_{L^2_{T_1}}^2 L^\infty,
\]
where $C > 0$ is a number independent of $T$ and $T_1$. Consequently, by using (26) and (27), we obtain that
\[
\|u - u^K\|_{L^\infty_{T_1}} L^2 \leq \frac{M_T}{K} + CT_1^{2 - \frac{4}{p}} \|u - u^K\|_{L^\infty_{T_1}} L^2 M_T^2,
\]
where $p$ is in particular such that $2/p < 1$. Consequently, we can choose $T_1$ sufficiently small such that
\[
CT_1^{2 - \frac{4}{p}} M_T^2 \leq \frac{1}{2}
\]
and we obtain that
\[
\|u - u^K\|_{L^\infty_{T_1}} L^2 \leq \frac{2M_T}{K}.
\]
This proves the desired estimate on $[0, T_1]$. We can then perform the same argument on $[T_1, 2T_1], \ldots$ to finally get that
\[
\|u - u^K\|_{L^\infty_{T_1}} L^2 \leq \frac{C_T}{K},
\]
where $C_T$ behaves like $e^{CT} M_T$. \hfill $\square$

6. **Discrete Strichartz estimates of the exact solution**

In this section, we shall prove that the sequence $(u^K(t_k))_{0 \leq k \leq N}$ where $u^K$ solves (25) satisfies discrete Strichartz estimates. This will be important in the following to estimate the local error and to control the stability of the scheme.

Let us first notice that by the Sobolev embedding $H^1 \subset L^q$, we have thanks to Proposition 5.2 an estimate $\|u^K(t_k)\|_{L^q_{t_k}} L^\alpha \leq C_T$ for every $(p, q)$ admissible. Nevertheless, this is not sufficient for our purpose. Indeed, for the estimate of the local error, we shall also need discrete Strichartz estimates of $\nabla u^K(t_k)$. Moreover, to prove the stability of the scheme, we shall also need an estimate without loss of the form $\|u^K(t_k)\|_{L^q_{t_k}} L^\infty \leq C_T$ that does not follow from Sobolev embedding in dimensions 2 and 3.

Let us start with an estimate that will ensure a uniform control of $\|u^K(t_k)\|_{L^q_{t_k}} L^\infty$. This will be crucial in the proof of the stability of the scheme.

**Definition 6.1.** Let $K = \tau^{-\frac{\alpha}{2}}$ for some $\alpha \geq 1$. We say that $(p, q, \sigma)$ verifies property (H) if:

\[
(p, q) \text{ is admissible, } \quad p > 2, \quad \sigma q > d, \quad \sigma + \frac{2}{p} \left(1 - \frac{1}{\alpha}\right) \leq 1.
\]

**Remark 6.2.** Let us check that the set of triples $(p, q, \sigma)$ verifying (H) is not empty. In dimension 1, we can clearly take $q = 2$, $p = \infty$ and any $\sigma \in (1/2, 1]$ due to Sobolev embedding. In dimension 2, by taking $\sigma = \frac{2}{q} + \epsilon$, $\epsilon > 0$, (note that it is enough to have the embedding $W^{\sigma, q} \subset L^\infty$), $(p, q, \sigma)$ verifies (H) if $(p, q)$ is admissible, $p > 2$ and $\epsilon - \frac{2}{p\alpha} \leq 0$. It can be satisfied for any $(p, q)$ admissible with $p < \infty$ by taking $\epsilon = \frac{1}{p\alpha}$.

In dimension 3, the set of $(p, q, \sigma)$ verifying (H) is not empty if $\alpha < 2$. Indeed, by taking again $\sigma = \frac{2}{q} + \epsilon$, $\epsilon > 0$, we need to verify $\frac{1}{2} + \epsilon \leq \frac{2}{p\alpha}$ which means that we can find $\epsilon$ if $1 \leq \alpha < \frac{4}{p}$. Consequently, if $\alpha < 2$, we can find $p > 2$ such that this is satisfied.
Proposition 6.3. Let $K = \tau \tilde{\omega}$ for some $\alpha \geq 1$, $\alpha < 2$ in dimension 3. Further, let $(p,q,\sigma)$ verify property (H). Then for every $T > 0$, there exists $C_T$ such that for every $\tau \in (0,1]$ and every $\hat{s} \in [-2\tau,2\tau]$, we have the estimate
\[
\sup_{s \in [0,\tau]} \| e^{i\hat{s}\Delta} u^K(t_k + s) \|_{\tau;1,N-\frac{\hat{s}}{s} L^p} W^{\sigma,q} \leq C_T. \tag{28}
\]

The crucial consequence of this proposition is that, under the above assumptions and in the particular case when $\hat{s} = 0$, we get by Sobolev embedding that
\[
\sup_{s \in [0,\tau]} \| u^K(t_k + s) \|_{\tau;1,N-\frac{s}{\tau} L^2} \leq C_T. \tag{29}
\]
In particular, this implies that
\[
\| u^K(t_k) \|_{\tau;1,N,L^\infty} \leq C_T
\]

for $p > 2$ such that $(p,q,\sigma)$ verifies (H). In dimensions 1 and 2, there is no restriction on $\alpha$. In dimension 3, this only requires that $\alpha < 2$.

Note that another useful consequence of (28) is that, though $\varphi_1(i\tau\Delta)$ is not continuous on $L^q$ for $q \neq 2$ with uniform estimate with respect to $\tau$, we have the following bound.

Corollary 6.4. For $(p,q,\sigma)$ verifying (H), we also obtain that
\[
\| \varphi_1(2i\tau\Delta) u^K(t_k) \|_{\tau;1,N-\frac{s}{\tau} L^p} W^{\sigma,q} \leq C_T. \tag{30}
\]

We will start with the proof of Proposition 6.3.

Proof of Proposition 6.3. We first prove the estimate (28) for $\hat{s} = 0$.

We use Duhamel’s formula to get that for every $0 \leq n \leq N$ and $s \in [0,\tau]$,
\[
u K(t_n + s) = e^{i(t_n + s)\Delta} \Pi_K u_0 - i \int_0^{t_n + s} e^{i(t_n + s - s_1)\Delta} \Pi_K (|\Pi_K u^K|^2 \Pi_K u^K)(s_1) \, ds_1
\]
that we rewrite as
\[
u K(t_n + s) = S_K(t_n + s) u_0 - i \sum_{k=0}^{n-1} \int_0^\tau e^{-i(t_k + \hat{s})\Delta} (|\Pi_K u^K|^2 \Pi_K u^K)(t_k + \hat{s}) \, d\hat{s}
\]
\[- i \int_0^\tau e^{-i(s - \hat{s})\Delta} \Pi_K (|\Pi_K u^K|^2 \Pi_K u^K)(t_n + \hat{s}) \, d\hat{s}.
\]
Therefore,
\[
u K(t_n + s) = S_K(t_n) e^{i\hat{s}\Delta} u_0 - i \sum_{k=0}^{n-1} \int_0^\tau S_K(t_n - k + s - \hat{s})(|\Pi_K u^K|^2 \Pi_K u^K)(t_k + \hat{s}) \, d\hat{s}
\]
\[- i \int_0^\tau S_K(s - \hat{s})(|\Pi_K u^K|^2 \Pi_K u^K)(t_n + \hat{s}) \, d\hat{s}.
\]

Let us fix $M_T$ such that
\[
\| u^K \|_{L^\infty_t H^1} + \| u^K \|_{L^p_t W^{\sigma,q}} \leq M_T \tag{32}
\]
and define $N_1$, $T_1 = N_1 \tau \leq T - \tau$. We shall first prove that we can find $T_1$ sufficiently small depending only on $M_T$ such that
\[
\sup_{s \in [0,\tau]} \| u^K(t_k + s) \|_{\tau;1,N_1-\frac{s}{\tau} L^p} W^{\sigma,q} \leq RM_T \tag{33}
\]
for some $R > 0$ well-chosen.
Let us first observe that by elliptic regularity, for \( q \in (1, \infty) \), we have
\[
\|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \lesssim \|(I - \Delta)^{\frac{\sigma}{2}} u^K(t_k + s)\|_{t^p,q,0,\sigma},
\]
therefore, since \( \Pi_{2K} u^K = u^K \), we can use the modified Sobolev estimate (24) to get
\[
\|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \lesssim \|(I - \Delta)^{\frac{\sigma}{2}} u^K(t_k + s)\|_{t^p,q,0,\sigma} \leq C_0 \|u^K\|_{L^\infty_t H^\sigma} \leq C_0 M_T,
\]
where \( \sigma_1 = \sigma + \frac{2}{p}(1 - \frac{1}{\alpha}) \leq 1 \). We shall thus take \( R = 2C_0 \) in (33). Next, for \( n + 1 \leq N_1 \), assuming that \( \sup_{s \in [0,T]} \|u^K(t_k + s)\|_{t^p,q,1,\sigma,W^{\sigma,q}} \leq 2C_0 M_T \), we get by using (31) and the Strichartz estimate of Corollary 4.3 that
\[
\|u^K(t_k + s)\|_{t^p,q,1,\sigma,W^{\sigma,q}} \leq C_0 \|u_0\|_{H^1} + \frac{C}{T} \int_0^T \|\Pi_K u^K \|_{t_{\sigma,q}}^2 \Pi_K u^K(t_k + \tilde{s})\|_{t_{\sigma,q}} \, d\tilde{s}.
\]
Next, we can use that
\[
\|\Pi_K u^K \|_{t_{\sigma,q}}^2 \Pi_K u^K(t_k + \tilde{s})\|_{t_{\sigma,q}} \leq \|\Pi_K u^K \|_{t_{\sigma,q}}^2 \Pi_K u^K(t_k + \tilde{s})\|_{t_{\sigma,q}} \leq \|u^K\|_{L^\infty_t H^1} \|u^K(t_k + \tilde{s})\|_{t_{\sigma,q}}^2.
\]
Since by the Sobolev and Hölder inequalities, we have
\[
\|u^K(t_k + \tilde{s})\|_{t_{\sigma,q}}^2 \leq T_1^{1 - \frac{2}{p}} \|u^K(t_k + \tilde{s})\|_{t_{\sigma,q}}^2,
\]
we get from the induction assumption that
\[
\|\Pi_K u^K \|_{t_{\sigma,q}}^2 \Pi_K u^K(t_k + \tilde{s})\|_{t_{\sigma,q}} \leq T_1^{1 - \frac{2}{p}} M_T (2C_0 M_T)^2.
\]
In a similar way, we also obtain that
\[
\|\Pi_K u^K \|_{t_{\sigma,q}}^2 \Pi_K u^K(t_k + \tilde{s})\|_{t_{\sigma,q}} \leq T_1^{1 - \frac{2}{p}} M_T \|u^K(t_k + \tilde{s})\|_{t_{\sigma,q}}^2.
\]
and we use that
\[
\|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \leq \|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} + \tau \|u^K(t_{n+1} + s)\|_{W^{\sigma,q}},
\]
which gives from the modified Sobolev embedding (24)
\[
\|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \leq \|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} + M_T.
\]
Consequently, by plugging these estimates into (35), we obtain that
\[
\|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \leq C_0 M_T + C T_1^{1 - \frac{2}{p}} M_T^3 + 8 C T_1^{1 - \frac{2}{p}} C_0^2 M_T^3.
\]
This yields
\[
\sup_{s \in [0,T]} \|u^K(t_k + s)\|_{t^p,q,\sigma,W^{\sigma,q}} \leq 2C_0 M_T
\]
by choosing \( T_1 \) sufficiently small (note that \( T_1 \) depends only on \( M_T \)). This allows one to get by induction that
\[
\sup_{s \in [0,T]} \|u^K(t_k + s)\|_{t^p,q,N_1,\sigma,W^{\sigma,q}} \leq 2C_0 M_T.
\]
Since \( T_1 \) only depends on \( M_T \), we can iterate the argument on \([T_1, 2T_1], \ldots\) to finally get
\[
\sup_{s \in [0,T]} \|u^K(t_k + s)\|_{t^p,q,N,\sigma,W^{\sigma,q}} \leq C_T.
\]
Note that this also yields
\[
\|u^K(t_k)\|_{t^p,q,N,\sigma,W^{\sigma,q}} \leq C_T.
\]
Indeed, we have that
\[
\|u^K(t_k)\|_{p,N}^{\rho,q} \lesssim \|u^K(t_k)\|_{\rho,N-1}^{\rho,q} + \tau^{\frac{1}{p}} \|u^K(t_n)\|_{L^\infty H^1} \leq C_T
\]
since by using the same estimates as in \[\text{(31)}\], we have
\[
\tau^{\frac{1}{p}} \|u^K(t_n)\|_{W^{\sigma,q}} \lesssim \|u^K\|_{L^\infty H^1} \leq M_T.
\]
This proves \[\text{(28)}\] in the case \(\hat{s} = 0\).

To get the estimate in the general case, we apply \(e^{i\hat{s}\Delta}\) to \(\text{(31)}\) to get
\[
e^{i\hat{s}\Delta}u^K(t_n + s) = S_K(t_n)e^{i(s+\hat{s})\Delta}u_0 - i \sum_{k=0}^{n-1} \int_0^\tau S_K(t_n-k+s+\hat{s}-\tilde{s})(\|\Pi_K u^K|^2 \Pi_K u^K)(t_k+\tilde{s}) \, d\tilde{s}
\]
\[
- i \int_0^s S_K(s+\tilde{s}-(\Pi_K u^K|^2 \Pi_K u^K)(t_n+\hat{s}) \, d\tilde{s}.
\]
From the same use of the Strichartz estimates of Corollary \[\text{(4.3)}\] as above, we obtain that
\[
\|e^{i\hat{s}\Delta}u^K(t_k+s)\|_{\rho,N-1}^{\rho,q} \leq C_0\|u_0\|_{H^1} + M_T\tau^{1-\frac{1}{p}} \sup_{s \in [0,\tau]} \|u^K(t_k+s)\|_{\rho,N-1}^{\rho,q}.
\]
Since we have already proved the estimate \[\text{(28)}\] for \(\hat{s} = 0\), this proves the estimate in the general case. Note that we can use the same trick as above to get the estimate for \(\|e^{i\hat{s}\Delta}u^K(t_k)\|_{\rho,N}^{\rho,q}\). □

It remains to prove \[\text{(30)}\].

\textbf{Proof of Corollary 6.4.} We first note that we can decompose
\[
\varphi_1(2i\tau\Delta)u^K(t_k) = \varphi_1(2i\tau\Delta)(1-\Pi_{\tau^{-\frac{1}{2}}})u^K(t_k) + \varphi_1(2i\tau\Delta)\Pi_{\tau^{-\frac{1}{2}}} u^K(t_k).
\]
By using Lemma \[\text{(11.1)}\] we have that the multiplier \(\varphi_1(2i\tau\Delta)\Pi_{\tau^{-\frac{1}{2}}}\) is continuous on \(L^q\) for every \(q\) with norm uniform in \(\tau\). Therefore, we get from Proposition \[\text{(6.3)}\] that
\[
\|\varphi_1(2i\tau\Delta)\Pi_{\tau^{-\frac{1}{2}}} u^K(t_k)\|_{\rho,N}^{\rho,q} \leq C\|u^K(t_k)\|_{\rho,N}^{\rho,q} \leq C_T.
\]
To estimate the remaining part, we just observe that
\[
\varphi_1(2i\tau\Delta)(1-\Pi_{\tau^{-\frac{1}{2}}}u^K(t_k)) = \frac{1-\Pi_{\tau^{-\frac{1}{2}}}}{2i\tau\Delta} e^{2i\tau\Delta} u^K(t_k) - \frac{1-\Pi_{\tau^{-\frac{1}{2}}}}{2i\tau\Delta} u^K(t_k).
\]
Again, the multiplier \(\frac{1-\Pi_{\tau^{-\frac{1}{2}}}}{2i\tau\Delta}\) is continuous on \(L^q\) for every \(q\) with norm uniform in \(\tau\), see \[\text{(10.2)}\] in Lemma \[\text{(11.1)}\]. Therefore, we obtain that
\[
\|\varphi_1(2i\tau\Delta)(1-\Pi_{\tau^{-\frac{1}{2}}}u^K(t_k))\|_{\rho,N}^{\rho,q} \leq C\left(\|e^{2i\tau\Delta} u^K(t_k)\|_{\rho,N}^{\rho,q} + \|u^K(t_k)\|_{\rho,N}^{\rho,q}\right)
\]
and the result follows by using again Proposition \[\text{(6.3)}\]. □

\textbf{Proposition 6.5.} For every \(T \geq 0\), \(u_0 \in H^1\) and for every \((p,q)\) admissible with \(p > 2\), there exists \(C_T > 0\) such that for every \(K, \tau\) as in Proposition \[\text{(6.3)}\] with \(\alpha < 2\) in dimension 3, we have uniformly in \(s \in [-2\tau, 2\tau]\) the estimate
\[
\|e^{i\hat{s}\Delta} \nabla u^K(t_k)\|_{\rho,N}^{\rho,q} \leq C_T (K^{\frac{1}{2}})\hat{s}^2.
\]
Note that the above proposition gives in particular an estimate of \(\|\nabla u^K(t_k)\|_{\rho,N}^{\rho,q}\) in the special case \(s = 0\).
By using again (31), we write
\begin{equation}
\begin{aligned}
e^{is\Delta} \nabla u_K(t_n) &= S_K(n\tau)(e^{is\Delta} \nabla u_0) \\
&= -i \int_0^\tau \sum_{k=0}^{n-1} S_K(t_{n-k} + s - \bar{s}) \nabla (\|\Pi_K u_k^K\|^2 \Pi_K u_k^K)(t_k + \bar{s}) \, d\bar{s}.
\end{aligned}
\end{equation}

We can then use Theorem 4.2 (note that (s - \bar{s})/\tau is uniformly bounded in [-3, 3]) to get
\begin{equation}
\begin{aligned}
\|e^{is\Delta} \nabla u_K(t_n)\|_{p,t,N,L} &\leq C \left((K\tau^\frac{1}{2})^\frac{2}{p} \|e^{is\Delta} \nabla u_0\|_{L} + \int_0^\tau \sup_{\bar{s} \in [0,\tau]} \|\nabla (\|\Pi_K u_k^K\|^2 \Pi_K u_k^K)(t_k + \bar{s})\|_{p_t,N,L} \, d\bar{s}\right).
\end{aligned}
\end{equation}

To estimate the last term in the above estimate, we use that
\begin{equation}
\begin{aligned}
\sup_{\bar{s} \in [0,\tau]} \|\nabla (\|\Pi_K u_k^K\|^2 \Pi_K u_k^K)(t_k + \bar{s})\|_{p_t,N,L} &\leq \sup_{\bar{s} \in [0,\tau]} \|\nabla u_k^K(t_k + \bar{s})\|_{L} \|\Pi_K u_k^K(t_k + \bar{s})\|_{L_\infty}^2 \|_{p_t,N} \\
&\leq \|\nabla u_k^K\|_{L_\infty,L} \sup_{\bar{s} \in [0,\tau]} \|\Pi_K u_k^K(t_k + \bar{s})\|_{p_t,N,L}^2.
\end{aligned}
\end{equation}

To conclude, we can use the estimate (29) which holds even in dimension 3 with the assumption that \( \alpha < 2 \) by using Remark 6.2 and Proposition 6.3. \( \square \)

7. Local error analysis

We shall now study the time discretization (9) of (25). By using Duhamel’s formula, we get that
\begin{equation}
u^K(t_n + \tau) = e^{i\tau \Delta} u^K(t_n) - ie^{i\tau \Delta} \Pi_K T(\Pi_K u^K)(\tau, t_n),
\end{equation}
where
\begin{equation}
T(\Pi_K u^K)(\tau, t_n) = \int_0^\tau e^{-is\Delta} \left[\|\Pi_K u^K(t_n + s)\|^2 \Pi_K u^K(t_n + s)\right] ds.
\end{equation}

Iterating Duhamel’s formula (38), i.e., plugging the expansion
\begin{equation}
u^K(t_n + s) = e^{is\Delta} u^K(t_n) - ie^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n)
\end{equation}
(which follows by replacing \( \tau \) with \( s \) in (38)) into (38), furthermore yields that
\begin{equation}
\begin{aligned}
u^K(t_n + \tau) &= e^{i\tau \Delta} u^K(t_n) \\
&- ie^{i\tau \Delta} \Pi_K \int_0^\tau e^{-is\Delta} \left((e^{is\Delta} \Pi_K u^K(t_n) - ie^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n))^2 \\
&\cdot (e^{-is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n))\right) ds \\
&= e^{i\tau \Delta} u^K(t_n) - ie^{i\tau \Delta} \Pi_K \int_0^\tau e^{-is\Delta} \left((e^{is\Delta} \Pi_K u^K(t_n))^2 (e^{-is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n))\right) ds \\
&+ ie^{i\tau \Delta} \Pi_K \int_0^\tau e^{-is\Delta} \left[T_1 + T_2 + T_3 + T_4 + T_5\right](s, t_n) ds
\end{aligned}
\end{equation}
with
\[
T_1(s, t_n) = -i (e^{is\Delta} \Pi_K u^K(t_n))^2 e^{-is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n)
\]
\[
T_2(s, t_n) = -2 (e^{is\Delta} \Pi_K u^K(t_n)) \ | e^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n) |^2
\]
(41)
\[
T_3(s, t_n) = i | e^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n) |^2 e^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n)
\]
\[
T_4(s, t_n) = 2i | e^{is\Delta} \Pi_K u^K(t_n) |^2 e^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n)
\]
\[
T_5(s, t_n) = (e^{-is\Delta} \Pi_K \bar{u}^K(t_n)) (e^{is\Delta} \Pi_K T(\Pi_K u^K)(s, t_n))^2 .
\]

In the following we set
\[
E_1(u^K, \tau, t_n) = i \int_0^\tau e^{-is\Delta} \left[ T_1 + T_2 + T_3 + T_4 + T_5 \right](s, t_n) \, ds
\]
such that by (40) we have that
\[
u^K(t_n + \tau) = e^{i\tau\Delta} u^K(t_n) - ie^{i\tau\Delta} \Pi_K \int_0^\tau e^{-is\Delta} \left[ (e^{is\Delta} \Pi_K u^K(t_n))^2 (e^{-is\Delta} \Pi_K \bar{u}^K(t_n)) \right] \, ds + e^{i\tau\Delta} \Pi_K E_1(u^K, \tau, t_n).
\]
(43)

To compare the exact solution (43) with the numerical solution (9) we need the following Lemma.

**Lemma 7.1.** It holds that
\[
-e^{is\Delta} \left( e^{is\Delta} w \right)^2 (e^{-is\Delta} \bar{w}) - w^2 (e^{-2is\Delta} \bar{w})
\]
\[
= -2i \int_0^s e^{-is\Delta} \left[ \nabla (e^{is\Delta} w)^2 \nabla \left( e^{i(s_1-2s)\Delta} \bar{w} \right) + (\nabla e^{is\Delta} w)^2 \left( e^{i(s_1-2s)\Delta} \bar{w} \right) \right] \, ds_1,
\]
where we set \(\nabla f\nabla g = \sum_{i=1}^d (\partial_i f)(\partial_i g)\) and \((\nabla f)^2 = \nabla f \nabla f\).

**Proof.** With the aid of the (inverse) Fourier transform
\[
u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{\nu}(\xi) e^{ix\cdot\xi} \, d\xi
\]
we obtain with the notation \(\xi_j \xi_\ell = (\xi_j, \xi_\ell)\) that
\[
\mathcal{F} \left(-2i \int_0^s e^{-is\Delta} \left[ \nabla (e^{is\Delta} w)^2 \nabla \left( e^{i(s_1-2s)\Delta} \bar{w} \right) + (\nabla e^{is\Delta} w)^2 \left( e^{i(s_1-2s)\Delta} \bar{w} \right) \right] \, ds_1 \right)(\xi)
\]
\[
= 2i (2\pi)^{-3d/2} \int_{\mathbb{R}^d} \nabla_{\xi_1, \xi_2, \xi_3} \hat{\nu}(\xi_1, \xi_2, \xi_3) \hat{\nu}(\xi_2, \xi_3) e^{2is\xi_1^2} \times
\]
\[
\int_0^s (-\xi_1(\xi_2 + \xi_3) + \xi_2\xi_3) e^{is(\xi_2 + \xi_3)} e^{-is(\xi_1^2 + \xi_2^2 + \xi_3^2)} \, ds_1
\]
\[
= (2\pi)^{-3d/2} \int_{\mathbb{R}^d} \nabla_{\xi_1, \xi_2, \xi_3} \hat{\nu}(\xi_1, \xi_2, \xi_3) \hat{\nu}(\xi_2, \xi_3) e^{2is\xi_1^2} \times
\]
\[
\int_0^s 2i (-\xi_1(\xi_2 + \xi_3) + \xi_2\xi_3) e^{2is(\xi_1^2 + \xi_2^2 + \xi_3^2)} \, ds_1
\]
\[
= \mathcal{F} \left(-e^{is\Delta} \left( e^{is\Delta} w \right)^2 \left( e^{i(s_1-2s)\Delta} \bar{w} \right) \right)_{s_1=0} \left(\xi\right).
\]
(45)

This proves the desired relation. \(\square\)

With the aid of the above lemma we get an alternative expression of the exact solution (43).
Corollary 7.2. The solution of (25) can be expressed as follows
\[ u^K(t_{n+1}) = e^{i\tau \Delta} u^K(t_n) - \tau S_K(\tau) \left( (\Pi_K u^K(t_n))^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^K(t_n) \right) \]
\[ + i S_K(\tau) \left( E_1(u^K, \tau, t_n) + E_2(u^K, \tau, t_n) \right), \]
where \( S_K = \Pi_K e^{i\tau \Delta} \) is defined in (18), \( E_1 \) given in (42) and \( E_2 \) reads
\[ E_2(u^K, \tau, t_n) = -2 \int_0^\tau \int_0^s e^{-i s_1 \Delta} \left[ \nabla (e^{i \tau_1 \Delta} \Pi_K u^K(t_n))^2 \nabla \left( e^{i(s_1-2\tau)\Delta} \Pi_K u^K(t_n) \right) \right] \]
\[ + \left( \nabla e^{i \tau_1 \Delta} \Pi_K u^K(t_n) \right)^2 \left( e^{i(s_1-2\tau)\Delta} \Pi_K u^K(t_n) \right) \] \[ ds_1 ds. \]

**Proof.** The corollary follows by applying Lemma 7.1 in the integral in (33). \( \square \)

8. Global error analysis

Note that we can write our scheme (9) in the form
\[ u^{n+1} = e^{i\tau \Delta} u^n - \tau S_K(\tau) \left( (\Pi_K u^n)^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^n \right), \]
and that the exact solution \( u^K(t) \) of the projected equation is given by (46). Let \( e^n = u^K(t_n) - u^n \) denote the error, i.e., the difference between numerical and exact solution. The errors thus satisfies the following recursion
\[ e^{n+1} = e^{i\tau \Delta} e^n - \tau S_K(\tau) \left( (\Pi_K u^K(t_n))^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^K(t_n) - (\Pi_K u^n)^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^n \right) \]
\[ + i S_K(\tau) \left( E_1(u^K, \tau, t_n) + E_2(u^K, \tau, t_n) \right) \]
with \( e^0 = 0 \). Therefore, by solving this recursion, we obtain that
\[ e^n = \tau \sum_{k=0}^{n-1} S_K(t_{n-k}) \left( (\Pi_K u^K(t_k))^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^K(t_k) - (\Pi_K u^k)^2 \varphi_1(-2i\tau \Delta) \Pi_K \bar{u}^k \right) \]
\[ + i \sum_{k=0}^{n-1} S_K(t_{n-k}) \left( E_1(u^K, \tau, t_k) + E_2(u^K, \tau, t_k) \right). \]

Let us set
\[ F_1^n = \sum_{k=0}^{n-1} S_K(t_{n-k}) E_1(u^K, \tau, t_k), \quad F_2^n = \sum_{k=0}^{n-1} S_K(t_{n-k}) E_2(u^K, \tau, t_k). \]

Then, we have the following estimates

**Lemma 8.1.** For every \( T > 0 \) and \((p,q)\) admissible with \( p > 2 \), there exists \( C_T > 0 \) such that for every \( K, \tau \) as in Proposition 6.3 with \( \alpha < 2 \) in dimension 3, we have the estimates
\[ \| F_1^n \|_{L^q}^p \lesssim (K \tau^{\frac{1}{4}})^{\frac{3}{2}} \tau C_T, \quad \| F_2^n \|_{L^q}^p \lesssim K (K \tau^{\frac{1}{4}})^{\frac{5}{2}} \tau C_T. \]

The second part of the estimate (50) is very rough, but will be enough for our purpose. Note that, by using Sobolev embedding, we deduce from the above estimates that in dimension 3, we have
\[ \| F_1^n \|_{L^4} \lesssim \| F_1^n \|_{W^{1,4}}^{\frac{3}{2}} \lesssim \| F_1^n \|_{L^3}^{\frac{3}{2}} \| F_1^n \|_{L^3}^{\frac{1}{2}} \lesssim \tau K^{\frac{1}{4}} (K \tau^{\frac{1}{4}})^{\frac{1}{2}}. \]

As we will see below, \( F_1^n \) is the best part of the error in the sense that the above estimates yield an error of order \( \tau \) in \( L^4 \).
Proof. In the proof, \( C_T \) will stand for a number that depends only on \( T \) and on the estimates of Proposition 5.2 of the exact solution. In particular, it is independent of \( \tau \) and \( K \). We first write by using the discrete Strichartz estimates

\[
\| \mathcal{F}_n \|_{l^p_{r,N} L^q} \leq (K \tau)^{\frac{2}{p}} \tau^{-1} \| E_1(u^K, \tau, t_k) \|_{l^p_{r,N} L^2}
\]

(52)

\[
\leq C_T (K \tau)^{\frac{2}{p}} \sup_{s \in [0,\tau]} \left( \| T_1(t_n, s) \|_{l^p_{r,N} L^2} + \| T_2(t_n, s) \|_{l^p_{r,N} L^2} + \| T_3(t_n, s) \|_{l^p_{r,N} L^2} + \| T_4(t_n, s) \|_{l^p_{r,N} L^2} + \| T_5(t_n, s) \|_{l^p_{r,N} L^2} \right).
\]

Next, by using (51), we get that

\[
\| T_1(t_n, s) \|_{L^2} \lesssim \left( \| e^{is\Delta} u^K (t_n) \|_{L^2} \right) e^{i\sigma \Delta} T(\Pi K^\tau u^K)(s, t_n) \|_{L^6} \lesssim \| e^{is\Delta} u^K (t_n) \|_{L^6} e^{i\sigma \Delta} T(\Pi K^\tau u^K)(s, t_n) \|_{L^6}.
\]

Next, we have by Sobolev embedding that

\[
\| e^{is\Delta} u^K (t_n) \|_{L^6} \lesssim \| e^{is\Delta} u^K (t_n) \|_{H^1} \lesssim \| u^K (t_n) \|_{H^1}
\]

and since

\[
e^{is\Delta} T(\Pi K^\tau u^K)(s, t_n) = \int_0^s e^{i(s-\tilde{s})\Delta} |\Pi K^\tau u^K (t_n + \tilde{s})|^2 \Pi K^\tau u^K (t_n + \tilde{s}) \, d\tilde{s},
\]

we obtain by Sobolev embedding that

\[
\| e^{is\Delta} T(\Pi K^\tau u^K)(s, t_n) \|_{L^6} \lesssim \| T(\Pi K^\tau u^K)(s, t_n) \|_{H^1}.
\]

Consequently,

\[
\| T(\Pi K^\tau u^K)(s, t_n) \|_{H^1}
\]

(54)

\[
\lesssim \int_0^\tau \left( \| \nabla \Pi K^\tau u^K (t_n + \tilde{s}) \|_{L^2} + \| \Pi K^\tau u^K (t_n + \tilde{s}) \|_{L^2} \right) d\tilde{s}
\]

\[
\lesssim \| u^K \|_{L^\infty H^1} \int_0^\tau \| u^K (t_n + \tilde{s}) \|_{L^\infty}^2 d\tilde{s}
\]

which yields

\[
\| T(\Pi K^\tau u^K)(s, t_n) \|_{l^p_{r,N} H^1} \lesssim \| u^K \|_{L^\infty H^1} \sup_{\tilde{s} \in [0,\tau]} \| u^K (t_n + \tilde{s}) \|_{l^p_{r,N} L^\infty}^2.
\]

We thus obtain that

\[
\| T_1(t_n, s) \|_{l^p_{r,N} L^2} \leq \tau \| u^K \|_{L^\infty H^1} \sup_{\tilde{s} \in [0,\tau]} \| u^K (t_n + \tilde{s}) \|_{l^p_{r,N} L^\infty}^2.
\]

By using Proposition 6.3 that yields (29) thanks to Remark 6.2 (with \( \alpha < 2 \) in dimension 3), we finally obtain

(56)

\[
\| T_1(t_n, s) \|_{l^p_{r,N} L^2} \leq \tau C_T.
\]

In a similar way, we obtain that

\[
\| T_2(s, t_n) \|_{L^2} \lesssim \| e^{is\Delta} u^K (t_n) \|_{L^6} \| e^{i\sigma \Delta} T(\Pi K^\tau u^K)(s, t_n) \|_{L^6}
\]

\[
\lesssim \| u^K (t_n) \|_{H^1} \| T(\Pi K^\tau u^K)(s, t_n) \|_{H^1}^2
\]

and hence, by using again (55), we get

\[
\| T_2(s, t_n) \|_{l^p_{r,N} L^2} \lesssim \| u \|_{L^\infty H^1} \| T(\Pi K^\tau u^K)(s, t_n) \|_{l^p_{r,N} H^1} \| T(\Pi K^\tau u^K)(s, t_n) \|_{l^p_{r,N} H^1}^2.
\]
We can use again (55) to estimate \( \| (\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \). Therefore, we only need to estimate \( \| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \). By using again (54), we get that
\[
\| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{H^{1}} \lesssim \int_{0}^{T} \| u^{K}(t_{n} + s) \|_{H^{1}} \| u^{K}(t_{n} + s) \|_{L^{\infty}} ds \lesssim \| u^{K} \|_{L_{r,s}^{1,1} L^{1}} \| u^{K} \|_{L_{r,s}^{1,1} L^{1}}
\]
and, therefore,
\[
(57) \quad \| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \leq \| u^{K} \|_{L_{r,s}^{1,1} L^{1}} \| u^{K} \|_{L_{r,s}^{1,1} L^{1}} \leq C_{T}
\]
since \( u^{K} \) satisfies the continuous Strichartz estimates (27). We thus finally obtain that
\[
(58) \quad \| T_{2}(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \lesssim \tau C_{T}.
\]
Finally, from the same arguments as above, we have that
\[
\| T_{3}(s, t_{n}) \|_{L^{2}} \lesssim \| e^{i \Delta T(\Pi_{K}^{n}u^{K})(s, t_{n})} \|_{L^{6}}^{3} \lesssim \| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{H^{1}}^{3}.
\]
Consequently,
\[
\| T_{3}(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \lesssim \| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}}^{2} \| T(\Pi_{K}^{n}u^{K})(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}}
\]
and therefore, by using (57) and (55), we also obtain that
\[
(59) \quad \| T_{3}(s, t_{n}) \|_{l_{r,s}^{1,1} L^{1}} \leq \tau C_{T}.
\]
The term \( T_{4} \) is estimated in the same way as \( T_{1} \), the term \( T_{5} \) in the same way as \( T_{2} \). Consequently, by combining (56), (58), (59) with (52), we finally obtain that
\[
\| F_{1}^{n} \|_{l_{r,s}^{1,1} L^{1}} \lesssim (K \tau^{1/2})^{2} \tau C_{T}.
\]
Since \( F_{1}^{n} = \Pi_{2K} F_{1}^{n} \) we also readily obtain that
\[
\| F_{1}^{n} \|_{l_{r,s}^{1,1} W^{1,1}} \lesssim K \| F_{1}^{n} \|_{l_{r,s}^{1,1} L^{1}} \lesssim K (K \tau^{1/2})^{2} \tau C_{T}.
\]
Indeed, the first above estimate, is a consequence of the fact that we can write
\[
(60) \quad \Pi_{2K} F_{1}^{n} = \rho_{\epsilon} \ast f, \quad \rho_{\epsilon}(x) = \frac{1}{\epsilon^{d}} \rho \left( \frac{x}{\epsilon} \right), \quad \epsilon = \frac{1}{2K}, \quad \rho = \mathcal{F}^{-1}(\chi^{2}) \in S(\mathbb{R}^{d})
\]
and standard convolution inequalities that thus yield
\[
\| \nabla F_{1}^{n} \|_{L^{1}} \lesssim \frac{1}{\epsilon} \| F_{1}^{n} \|_{L^{1}}.
\]
This ends the proof of (60).

We shall now analyze the second part of the error.

**Lemma 8.2.** For every \( T > 0 \) and \( (p, q) \) admissible with \( p > 2 \), there exists \( C_{T} > 0 \) such that for every \( K, \tau \) as in Proposition 7.3, with \( \alpha < 2 \) in dimension 3, we have the estimates
\[
(61) \quad \| F_{2}^{n} \|_{l_{r,s}^{1,1} L^{1}} \lesssim C_{T} \tau (K \tau^{1/2})^{2 + \frac{2}{p}} \left( \log K \right)^{\frac{3}{2}}, \quad \text{if } d = 1,
\]
\[
(62) \quad \| F_{2}^{n} \|_{l_{r,s}^{1,1} L^{1}} \lesssim C_{T} \tau (K \tau^{1/2})^{1 + \frac{2}{p}}, \quad \text{if } d = 2,
\]
\[
(63) \quad \| F_{2}^{n} \|_{l_{r,s}^{1,1} L^{1}} \lesssim C_{T} \tau (K \tau^{1/2})^{2 + \frac{2}{p}} \left( \log K \right)^{\frac{3}{2}}, \quad \text{if } d = 3.
\]
Moreover, in dimension 3, we also have the estimate
\[
(64) \quad \| F_{2}^{n} \|_{l_{r,s}^{1,1} L^{4}} \lesssim C_{T} \tau (K \tau^{1/2})^{\frac{5}{2}} K^{-\frac{1}{2}} \left( \log K \right)^{\frac{3}{2}}, \quad \text{if } d = 3.
\]
Proof. At first, we observe that using the expressions \((47), (49)\), we can write that

\[
\mathcal{F}_2^n = 2 \int_0^\tau \int_0^\tau e^{-is_1\Delta} \sum_{k=0}^{n-1} S_K(t_{n-k}) G(s, s_1, t_k) \, ds_1 \, ds \\
= 2 \int_0^\tau \int_0^\tau \sum_{k=0}^{n-1} S_K(t_{n-k} - s_1) G(s, s_1, t_k) \, ds_1 \, ds,
\]

(65)

where

\[
G(s, s_1, t_k) = -\nabla (e^{is_1\Delta} \Pi_K u^K(t_k))^2 \nabla (e^{i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)) + (\nabla e^{is_1\Delta} \Pi_K u^K(t_k))^2 (e^{i(s_1 - 2s)\Delta} \Pi_K u(t_k))
\]

and we observe that \(s/\tau, s_1/\tau, (s_1 - 2s)/\tau\) are uniformly bounded in \([-2, 1]\) so that we will be able to use Theorem 4.2 and Propositions 6.3 and 6.5. We first estimate

\[
\|\mathcal{F}_2^n\|_{p, r, N L^q} \lesssim \tau^2 \sup_{0 \leq s_1 \leq s \leq \tau} |\sum_{k=0}^{n-1} S_K(t_{n-k} - s_1) G(s, s_1, t_k)|_{l^p_{r, N} L^q}.
\]

(66)

Then, using discrete Strichartz estimates, we obtain that

\[
\|\mathcal{F}_2^n\|_{p, r, N L^q} \lesssim \tau (K r^{\frac{1}{s}}) \sup_{0 \leq s_1, s \leq \tau} \|G(s, s_1, t_k)\|_{l^p_{r, N} L^2}.
\]

We shall then use slightly different arguments depending on the dimension. In dimension \(d \leq 2\), we use Hölder’s inequality to get

\[
\|G(s, s_1, t_k)\|_{L^2} \lesssim \|
abla e^{-i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)\|_{L^4} \|
abla e^{i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)\|_{L^4} \|e^{i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)\|_{L^\infty} + \|
abla e^{i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)\|_{L^4} \|e^{i(s_1 - 2s)\Delta} \Pi_K u^K(t_k)\|_{L^\infty}
\]

and therefore,

\[
\|\mathcal{F}_2^n\|_{p, r, N L^q} \lesssim \tau (K r^{\frac{1}{s}}) \left( \sup_{\tilde{s} \in [-2\tau, \tau]} \|\nabla e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^p_{r, N} L^4} \right)^2 \sup_{\tilde{s} \in [-2\tau, \tau]} \|e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^2_{r, N} L^\infty}.
\]

(67)

Next, we use the estimate

\[
\sup_{\tilde{s} \in [-2\tau, 2\tau]} \|e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^2_{r, N} L^\infty} \leq C_T
\]

from (28) of Proposition 6.5. Indeed, as noticed after Proposition 6.3, in dimensions 1 and 2, this estimate is true without further restriction on \(\alpha \geq 1\). Moreover, for all \(\tilde{s} \in [-2\tau, 2\tau]\), we have the estimate

\[
\|\nabla e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^p_{r, N} L^4} \leq C_T (K r^{\frac{1}{s}})^\frac{1}{4}
\]

(69)

from Proposition 6.5 in dimension \(d \leq 2\). Indeed for \(d = 1\), using Hölder and (36), we have

\[
\|\nabla e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^4_{r, N} L^4} \leq C_T \|\nabla e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^8_{r, N} L^4} \leq C_T (K r^{\frac{1}{s}})^\frac{1}{4},
\]

while for \(d = 2\), we can use directly the fact that \((4, 4)\) is an admissible Strichartz pair to get

\[
\|\nabla e^{i\tilde{s} \Delta} \Pi_K u^K(t_k)\|_{l^4_{r, N} L^4} \leq C_T (K r^{\frac{1}{s}})^\frac{1}{4}.
\]

Consequently, by combining (67), (68), (69), we get the desired estimate

\[
\|\mathcal{F}_2^n\|_{p, r, N L^q} \lesssim \tau (K r^{\frac{1}{s}})^\frac{1}{2} (K r^{\frac{1}{s}})^\frac{d}{2}
\]

(18)
for $d \leq 2$.

In dimension 3, the estimate (67) is not sufficient to conclude since (4, 4) is not an admissible pair. We write in place the estimate
\[
\|F_2^n\|_{\tau, N}^{p} \lesssim \tau (K\tau^{\frac{1}{2}})^{\frac{2}{p}} \left( \sup_{\delta \in [-2, \tau]} \|\nabla e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{\frac{2}{p},N}L^1} \right)^2 \sup_{\delta \in [-2, \tau]} \|e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{4,N}L^\infty}.
\]
and therefore, we get from (36) that
\[
\|F_2^n\|_{\tau, N}^{p} \lesssim \tau (K\tau^{\frac{1}{2}})^{\frac{2}{p}} C_{T} \sup_{\delta \in [-2, \tau]} \|e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{4,N}L^\infty}.
\]
Here we cannot use anymore Proposition 6.3 in order to estimate $\|e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{4,N}L^\infty}$ without loss unless we take $\alpha = 1$, which would yield a non optimal total error. We are thus forced to use Sobolev embedding and (36). Thanks to Lemma 11.1.2
\[
\|e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{4,N}L^\infty} \leq \log K \|e^{i\delta \Delta} \Pi_K u^K(t_k)\|_{l^{4,N}W^{1,3}} \leq C_{T}(\log K)^{\frac{2}{3}} (K\tau^{\frac{1}{2}})^{\frac{2}{3}}.
\]
This finally yields
\[
\|F_2^n\|_{\tau, N}^{p} \lesssim \tau (K\tau^{\frac{1}{2}})^{\frac{2}{p}} (K\tau^{\frac{1}{2}})^{2} C_{T}(\log K)^{\frac{2}{3}},
\]
which is the desired estimate in dimension 3.

To get (63), we just observe that since $\Pi_{2N} F_2^n = F_2^n$, we can thus write $F_2^n = \rho_{e} * F_2^n$ with $\rho_{e}$ as in (60) and use Youngs inequality to obtain
\[
\|F_2^n\|_{l^{4,N}L^1} \lesssim K^{\frac{1}{2}} \|F_2^n\|_{l^{4,N}L^3}.
\]
Since (4, 3) is an admissible pair in dimension 3, we can use (63) to get the desired estimate. \qed

9. Proof of Theorem 2.1

At first, we use Corollary 5.4, to write that
\[
\|u(t_n) - u^n\|_{L^2} \leq \|u(t_n) - u^K(t_n)\|_{L^2} + \|u^K(t_n) - u^n\|_{L^2} \leq \frac{C_{T}}{K} + \|e^n\|_{L^2}.
\]
To estimate $e^n$ we shall use equation (48). Note that the consistency error on the right-hand side can be estimated thanks to Lemma 8.1 and Lemma 8.2. We shall choose our parameter $K$ in an optimal way so that the contribution of the consistency error in $L^2$ is of order $1/K$ in order to get contributions of the same order in the two terms of (70). This choice will depend on the dimension since the estimates of Lemma 8.2 depend on the dimension.

**Dimension** $d \leq 2$. In dimension $d \leq 2$, by using Lemma 8.1 and Lemma 8.2 we have that
\[
\|F_1^n\|_{l^{\infty,N}L^2} + \|F_2^n\|_{l^{\infty,N}L^2} \leq C_{T}(\tau + K^{\frac{d}{2}} \tau^{1+\frac{d}{4}}).
\]
We thus choose $K$ such that $K^{\frac{d}{2}} \tau^{1+\frac{d}{4}} = \frac{1}{K}$ which gives
\[
K = \tau^{-\frac{1+\frac{d}{2}}{\frac{1}{2}+d}}.
\]
Note that this choice gives in particular that
\[
K^{\tau^{\frac{1}{2}}} = \tau^{-\frac{1}{2+\frac{d}{2}}},
\]
an expression that will be useful in future computations. Under this-CFL type condition, we get that
\[
\|F_1^n\|_{l^{\infty,N}L^2} + \|F_2^n\|_{l^{\infty,N}L^2} \leq C_{T} \tau^{\frac{1+\frac{d}{2}}{2+\frac{d}{2}}}
\]
and more generally that for every \((p, q)\) admissible, \(p > 2\),
\[
\|F_1^n\|_{p, N L^q} + \|F_2^n\|_{p, N L^q} \leq C_T T^\frac{1}{2} \frac{\|e\|}{(4 + d - \frac{4}{p})}. 
\] (73)

Let us define \(N_1\) such that \(N_1 T = T_1 \leq T\). We shall first prove by induction that \(e^n\) verifies the estimate
\[
\|e^n\|_{X_{r,k}} := \frac{1}{\tau^\frac{d}{2}} \|e^n\|_{v_{r,k}^\tau L^2} + \frac{1}{\tau^\frac{d}{2}} \|e^n\|_{l_{n,k}^\tau L^4} \leq 8 C_T, \quad 0 \leq k \leq N_1, 
\] (74)
for \(T_1\) and \(\tau\) sufficiently small compared to \(C_T\). Note that the control of the above norm gives that we propagate an estimate of order \(T^\frac{1}{2} \frac{\|e\|}{(4 + d - \frac{4}{p})}\) for the norm \(\|e^n\|_{l_{n,k}^\tau L^4}\). This is less than \(T^\frac{1}{2} \frac{\|e\|}{(4 + d - \frac{4}{p})}\) that one would expect in view of estimate (73). This would nevertheless be sufficient to close the following argument. One of the reasons for this choice is the control of terms involving the filter function \(\varphi_1(2i\tau\Delta)\). Indeed, this operator is not uniformly bounded on \(L^p\) for \(p \neq 2\). Nevertheless, we get by Sobolev embedding and (103) that
\[
\|\varphi_1(-2i\tau\Delta)e^n\|_{v_{r,k}^\tau L^4} \leq \|\varphi_1(-2i\tau\Delta)e^n\|_{l_{n,k}^\tau L^4} \leq C_T T^\frac{1}{2} \|e^n\|_{v_{r,k}^\tau L^2} \leq C_T T^\frac{1}{2} \|e^n\|_{X_{r,k}}. 
\]

Consequently, since \(T^\frac{1}{2} \frac{\|e\|}{(4 + d - \frac{4}{p})} \geq \frac{1}{T^\frac{d}{2}} \|e^n\|_{X_{r,k}}\) when \(d \leq 2\), we get that
\[
\|\varphi_1(-2i\tau\Delta)e^n\|_{X_{r,k}} \leq C_T \|e^n\|_{X_{r,k}}. 
\] (75)

Let us rewrite (48) as
\[
e^n = \sum_{k=0}^{n-1} S_K(t_{n-k})G_k + F_1^n + F_2^n
\] (76)
where
\[
G_k = \Pi_K e^k (\Pi_K u^K(t_k) + \Pi_K u^k) \varphi_1(-2i\tau\Delta)\Pi_K u^K(t_k) + (\Pi_K u^k)^2 \varphi_1(-2i\tau\Delta)\Pi_K u^k. 
\]

Note that by substituting \(u^k = u^K(t_k) - e^k\), we can write
\[
G_k = G_k^1 + G_k^2 + G_k^3, 
\] (77)
where
\[
G_k^1 = 2(\Pi_K u^K(t_k))(\varphi_1(-2i\tau\Delta)\Pi_K u^K(t_k))(\Pi_K e^k) + (\Pi_K u^K(t_k))^2 \varphi_1(-2i\tau\Delta)\Pi_K e^k, 
\]
\[
G_k^2 = -(\varphi_1(-2i\tau\Delta)\Pi_K u^K(t_k))(\Pi_K e^k)^2 - 2(\Pi_K u^K(t_k))(\Pi_K e^k)\varphi_1(-2i\tau\Delta)\Pi_K e^k, 
\]
\[
G_k^3 = (\Pi_K e^k)^2 \varphi_1(-2i\tau\Delta)\Pi_K e^k. 
\]

To estimate \(e^n\), we use the discrete Strichartz inequalities of Theorem 4.2 and our choice (71). In the following \(C\) is again a generic number independent of \(T_1, T, \tau\) and \(K\). We first get that
\[
\|e^n\|_{l_{n,k}^\tau L^2} \leq C T^\frac{1}{2} \frac{\|e\|}{(4 + d - \frac{4}{p})} + C \|G_n^1\|_{l_{n,k}^\tau L^2} + C \|G_n^2\|_{l_{n,k}^\tau L^2} + C \frac{1}{T^\frac{d}{2}} \|G_n^3\|_{l_{n,k}^\tau L^4}. 
\] (78)
To estimate the right-hand side, we first use that
\[
\|G_n^1\|_{l_{n,k}^\tau L^2} \leq C \|e^n\|_{l_{n,k}^\tau L^2} \left(\|u^K(t_n)\|_{l_{n,k}^\tau L^\infty}^2 + \|\varphi_1(-2i\tau\Delta)\Pi_K u^K(t_n)\|_{l_{n,k}^\tau L^\infty}^2\right). 
\]
If \(d = 1\), the above right-hand side can be easily estimated since
\[
\|u^K(t_n)\|_{l_{n,k}^\tau L^\infty}^2 + \|\varphi_1(-2i\tau\Delta)\Pi_K u^K(t_n)\|_{l_{n,k}^\tau L^\infty}^2 \leq C T_1 \|u^K\|_{l_{n,k}^\tau L^\infty}^2 H^1 \leq T_1 C_T^2. 
\]
If \( d = 2 \), we can use Remark [6.2] to obtain
\[
\|u^K(t_n)\|_{r,k,L^\infty} + \|\varphi_1(-2i\tau\Delta)\Pi_Ku^K(t_n)\|_{r,k,L^\infty} \leq T_1^{\frac{1}{8}} \left( \|u^K(t_n)\|_{L^2} + \|\varphi_1(-2i\tau\Delta)\Pi_Ku^K(t_n)\|_{L^2} \right) \leq T_1^{\frac{1}{4}} C_T
\]
for some suitable choice of \( \sigma \) slightly larger than \( 1/2 \). This thus yields, by using Proposition [6.3] and Corollary [6.3]
\[
(79) \quad \|G_n^1\|_{r,k,L^2} \leq T_1^{\frac{1}{2}} C_T^2 \|e^n\|_{r,k,L^2}.
\]
Let us now estimate \( G_n^2 \). From similar arguments, we obtain that
\[
\|G_n^2\|_{r,k,L^2} \leq C \left( \|e^n\|_{r,k,L^4}^2 + \|\varphi_1(-2i\tau\Delta)e^n\|^2_{r,k,L^4} \right) \left( \|u^K(t_n)\|_{r,k,L^\infty} + \|\varphi_1(-2i\tau\Delta)\Pi_Ku^K(t_n)\|_{r,k,L^\infty} \right)
\]
which yields
\[
(80) \quad \|G_n^2\|_{r,k,L^2} \leq C_T(\|e^n\|_{r,k,L^4}^2 + \|\varphi_1(-2i\tau\Delta)e^n\|^2_{r,k,L^4}).
\]
Finally, to estimate the last term in the right-hand side of (78), we use that
\[
\|G_n^3\|_{r,k,L^2} \leq C_T \left( \|e^n\|_{r,k,L^4}^3 + \|\varphi_1(-2i\tau\Delta)e^n\|_{r,k,L^4}^3 \right).
\]
Consequently, by plugging (79), (80) and (81) into (78) and by using the observation (75), we get that
\[
(82) \quad \|e^n\|_{r,k+1,L^2} \leq C_T^\frac{d+4}{d+2} + T_1^{\frac{1}{2}} C_T^2 \|e^n\|_{r,k,L^2} + C_T^\frac{d+4}{d+2} \|e^n\|_{r,k,L^4}^2 + C_T^\frac{1}{d+2} \|e^n\|_{r,k,L^4}^3.
\]
In a similar way, by using again the discrete Strichartz inequalities, we find that
\[
\|e^n\|_{r,k+1,L^4} \leq C_T^\frac{d+4}{d+2} + \frac{C_T^\frac{d+4}{d+2}}{\|G_n^1\|_{r,k,L^2}^2 + \|G_n^2\|_{r,k,L^2}} + \frac{C_T^\frac{1}{d+2}}{C_T^\frac{1}{d+2} + \|G_n^3\|_{r,k,L^2}^2}.
\]
Consequently, by using again (79), (80), (81) and (75), we find that
\[
(83) \quad \|e^n\|_{r,k+1,L^4} \leq C_T^\frac{d+4}{d+2} + \frac{1}{T_1^{\frac{d+4}{d+2}}} + \frac{T_1^{\frac{1}{2}} C_T^2 \|e^n\|_{r,k,L^2}}{\|G_n^1\|_{r,k,L^2}^2 + \|G_n^2\|_{r,k,L^2}} + \frac{C_T^\frac{1}{d+2}}{\|G_n^3\|_{r,k,L^2}^2} \|e^n\|_{r,k,L^4}^3.
\]
By combining (83) and (82), and by using that \( \|e^n\|_{r,k} \) satisfies (74), we obtain that
\[
\|e^n\|_{r,k} \leq 2C_T + T_1^{\frac{1}{2}} C_T^2 \|e^n\|_{r,k} + C_T^\frac{3}{d+2} \|e^n\|_{r,k}^3 + C_T^\frac{3}{2+2}\|e^n\|_{r,k}^3 \leq 2C_T + T_1^{\frac{1}{2}} C_T^2 \|e^n\|_{r,k} + C_T^\frac{3}{2+2}\|e^n\|_{r,k}^3.
\]
Consequently, by taking \( T_1 \) sufficiently small so that \( T_1^{\frac{1}{2}} C_T^2 \leq \frac{1}{2} \), we get that
\[
\|e^n\|_{r,k} \leq 4C_T + CC_T^\frac{3}{2+2}\|e^n\|_{r,k}^3 \leq 8C_T
\]
for \( \tau \) sufficiently small. This proves that
\[
\|e^n\|_{r,N_I} \leq SL_T.
\]
We can then iterate the estimates on \([T_1, 2T_1]\),... to finally obtain after a finite number of steps
\[
\|e^n\|_{X_{r,N}} \leq \tilde{C}_T.
\]
This proves the error estimate in dimension \(d \leq 2\).

**Dimension** \(d = 3\). For \(d = 3\), following the same scheme of proof, we observe that
\[
\|F^n\|_{l^8_{r,N}L^2} + \|F^n\|_{l^8_{r,N}L^2} \leq C_T \log \tau^{\frac{3}{2}} \tau^{\frac{3}{2}},
\]
To optimize the total error, we thus choose \(K\) such that \(\tau^2 K^2 = \frac{1}{K}\), which yields
\[
(84) \quad K = \tau^{-\frac{2}{3}}
\]
and therefore
\[
\alpha = \frac{4}{3} < 2, \quad K^{\frac{2}{3}} = \tau^{-\frac{4}{3}}.
\]
The error thus verifies in particular thanks to Lemmas [8.1], [8.2] and (51) that
\[
(85) \quad \|F^n\|_{l^8_{r,N}L^2} + \|F^n\|_{l^8_{r,N}L^2} \leq C_T \log \tau^{\frac{3}{2}} \tau^{\frac{3}{2}}, \quad \|F^1\|_{l^8_{r,N}L^4} + \|F^2\|_{l^8_{r,N}L^4} \leq C_T \log \tau^{\frac{3}{2}} \tau^{\frac{7}{2}}.
\]
By using the same approach as before, we first prove by induction that for all \(0 \leq k \leq N_1\)
\[
(86) \quad \|e^n\|_{X_{r,k}} := \frac{1}{|\log \tau|^{\frac{3}{2}} \tau^{\frac{3}{2}}} \|e^n\|_{l^8_{r,N}L^2} + \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l^4_{r,N}L^4} \leq 8C_T.
\]
Note that we propagate only the rate \(\frac{19}{2}\) for the \(l^4_{r,N}L^4\) norm as we would expect \(\tau^{\frac{7}{2}} \log \tau^{\frac{2}{3}}\) from the estimate of the source term (85). This is needed in order to close the argument below with this choice of norms. Moreover, as before this allows us to get by Sobolev embedding and (103) that
\[
(87) \quad \|\varphi_1(-2i\tau \Delta) e^n\|_{l^8_{r,N}L^4} \lesssim \|\varphi_1(-2i\tau \Delta) e^n\|_{l^8_{r,N}H^\frac{1}{2}} \lesssim \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l^8_{r,N}L^2} \lesssim \tau^\frac{7}{2} \log \tau^{\frac{2}{3}} \|e^n\|_{X_{r,k}}.
\]
From the same arguments as above, we get from (70) and the discrete Strichartz estimates that
\[
(88) \quad \|e^n\|_{l^8_{r,N+1}L^2} \leq C_T \tau^{\frac{3}{2}} |\log \tau|^{\frac{3}{2}} + C \|G^1\|_{l^4_{r,N}L^2} + C \|G^2\|_{l^4_{r,N}L^2} + C \frac{1}{\tau^{\frac{2}{3}}} \|G^3\|_{l^4_{r,N}L^4}.
\]
To estimate \(\|G^1\|_{l^4_{r,N}L^2}\), we just use Hölder to get as before
\[
\|G^1\|_{l^4_{r,N}L^2} \leq C \|e^n\|_{l^8_{r,N}L^2} \left(\|u^K(t_n)\|_{l^\infty_{r,k}L^\infty} + \|\varphi_1(-2i\tau \Delta) \Pi_K u^K(t_n)\|_{l^\infty_{r,k}L^\infty}\right).
\]
Next, the crucial observation is that since \(\alpha = \frac{4}{3}\), we can use Remark 6.2 to get that
\[
\|u^K(t_n)\|_{l^2_{r,k}L^\infty} + \|\varphi_1(-2i\tau \Delta) \Pi_K u^K(t_n)\|_{l^2_{r,k}L^\infty} \lesssim T_1^{\frac{10}{15}} \left(\|u^K(t_n)\|_{l^2_{r,k}W^{\sigma, \frac{20}{3}}} + \|\varphi_1(-2i\tau \Delta) \Pi_K u^K(t_n)\|_{l^2_{r,k}W^{\sigma, \frac{20}{3}}}\right)
\]
for \(\sigma \in (21/30, 24/30)\). This allows us to use Proposition 6.3 and Corollary 6.4 to obtain that
\[
(89) \quad \|u^K(t_n)\|_{l^2_{r,k}L^\infty} + \|\varphi_1(-2i\tau \Delta) \Pi_K u^K(t_n)\|_{l^2_{r,k}L^\infty} \lesssim C_T T_1^{\frac{1}{10}}
\]
and therefore
\[
(90) \quad \|G^1\|_{l^4_{r,N}L^2} \leq C_T T_1^{\frac{1}{10}} \|e^n\|_{l^\infty_{r,N}L^2}.
\]
For the estimate of \( \|G_n^2\|_{l_{r,k}^2} \), we can still write
\[
\|G_n^2\|_{l_{r,k}^2} \leq C \left( \|e^n\|_{l_{r,k}^4}^2 \| \varphi_1(-2i\tau \Delta)e^n\|_{l_{r,k}^4} + \|e^n\|_{l_{r,k}^4}^2 \| \varphi_1(-2i\tau \Delta)\Pi_K e^n\|_{l_{r,k}^4} \right).
\]
Consequently, by using again (89), we obtain that (91)
\[
\|G_n^2\|_{l_{r,k}^2} \leq C_T \left( \|e^n\|_{l_{r,k}^4}^2 \| \varphi_1(-2i\tau \Delta)e^n\|_{l_{r,k}^4} \right) \leq C_T (\frac{19}{16} + \frac{31}{32} |\log \tau|^{\frac{2}{3}})\|e^n\|_{X_{r,k}}^2,
\]
where we have used (87) and the fact that
\[
\| \varphi_1(-2i\tau \Delta)e^n\|_{l_{r,k}^4} \leq T\| \varphi_1(-2i\tau \Delta)e^n\|_{l_{r,k}^4}
\]
to get the last estimate.

It remains to estimate \( \|G_n^3\|_{l_{r,k}^4} \). From Hölder’s inequality, we get
\[
\|G_n^3\|_{l_{r,k}^4} \leq C \left( \|e^n\|_{l_{r,k}^4}^3 \| \varphi_1(-2i\tau \Delta)e^n\|_{l_{r,k}^4} \right).
\]
By using the reverse inclusion rule for the discrete \( l_p \) spaces,
\[
\|f\|_{l_p X} \leq \frac{1}{\tau^{\frac{q}{p}}} |f\|_{l_q X}, \quad p > q,
\]
we get
\[
\|e^n\|_{l_{r,k}^4}^3 \leq \left( \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l_{r,k}^4} \right)^3 \leq \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l_{r,k}^4}^3.
\]
This yields by using again (87)
\[
\|e^n\|_{l_{r,k}^4}^3 \leq C_T \left( \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l_{r,k}^4}^3 \right) \leq \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l_{r,k}^4}^3.
\]
Consequently, we deduce from (88) and (90), (91), (92) and by using the induction assumption that
\[
\|e^n\|_{X_{r,k+1}}^2 \leq C_T + C_T T \frac{1}{\tau^{\frac{2}{3}}} \|e^n\|_{l_{r,k+1}^4} \|e^n\|_{l_{r,k+1}^4} + CC_T \frac{1}{\tau^{\frac{2}{3}}} + CC_T \frac{1}{\tau^{\frac{2}{3}}} T \frac{1}{\tau^{\frac{2}{3}}}.
\]
In a similar way, we can estimate \( \|e^n\|_{l_{r,k+1}^4} \). By using as previously that we have the frequency localization \( \Pi_{2K} e^n = e^n \) and the discrete Strichartz estimates, we get that
\[
\|e^n\|_{l_{r,k+1}^4} \leq C_T \|e^n\|_{l_{r,k+1}^4} + C_T\left( \frac{1}{\tau^{\frac{2}{3}}} \right)^{\frac{1}{2}} \left( \frac{1}{\tau^{\frac{2}{3}}} \right)^{\frac{1}{2}} \left( \|G_n^1\|_{l_{r,k+1}^2} + \|G_n^2\|_{l_{r,k+1}^2} + \|G_n^3\|_{l_{r,k+1}^4} \right).
\]
The additional loss \( \left( \frac{1}{\tau^{\frac{2}{3}}} \right)^{\frac{1}{2}} \) comes from the fact that we need to use first the estimate
\[
\left\| \sum_{k=0}^{n-1} S_K(t_{n-k})G_k \right\|_{l_{r,k+1}^4} \leq \left( \frac{1}{\tau^{\frac{2}{3}}} \right)^{\frac{1}{2}} \tau \sum_{k=0}^{n-1} S_K(t_{n-k})G_k \right\|_{l_{r,k+1}^4}.
before using the discrete Strichartz estimates since (4, 4) is not admissible in dimension 3. By using again (90), (91), (92), we therefore obtain that
\[
\|e^n\|_{L^4_{\tau,k+1}} + \|e^n\|_{L^4_{\tau,k}} \leq C_T \log \tau \frac{2}{2^{1/3}} + C_T T^{1/2} \|e^n\|_{X_{\tau,k}} + CC_3 T^{1/2} \|e^n\|_{L^4_{\tau,k}} + CC_3 T^{1/2} \|e^n\|_{L^4_{\tau,k}}.
\]
By combining (93) and the last estimate, we obtain that
\[
\|e^n\|_{X_{\tau,k+1}} \leq \frac{2}{C_T} + \frac{C_T}{2} \|e^n\|_{X_{\tau,k}}. \tag{94}
\]
for some \(\delta > 0\). Therefore, we can finish the proof as above.

10. Proof of the discrete Strichartz estimates.

10.1. Dispersive estimates. Let us start with the proof of a dispersive inequality.

**Lemma 10.1.** There exists \(C > 0\) such that for every \(K \geq 1\), every \(p \in [2, \infty]\), every \(t \in \mathbb{R}\), and every \(f \in L^p\), we have the estimate
\[
\|S_K(t)f\|_{L^p} \leq C \frac{K^{d(1 - \frac{2}{p})}}{1 + |t|^{\frac{d}{2}(1 - \frac{2}{p})}} \|f\|_{L^{p'}}.
\]

**Proof.** In this proof \(C > 0\) will stand for a number independent of \(K\). Let us observe that with the choice of \(\Pi_K\) as in (10), we can write
\[
S_K(t)f = \rho_{\epsilon} * (e^{it\Delta}(\rho_{\epsilon} * f))
\]
where \(\rho_{\epsilon} = \frac{1}{\epsilon^d} \rho \left( \frac{x}{\epsilon} \right), \epsilon = \frac{1}{K}, \rho(x) = F^{-1}(\chi)(x) \in L^1\). From Young’s inequality for convolutions and the standard dispersive estimate for \(e^{it\Delta}\), we thus get that
\[
\|S_K(t)f\|_{L^\infty} \leq C \|\rho_{\epsilon}\|_{L^1} \frac{1}{|t|^{\frac{d}{2}}} \|\rho_{\epsilon}\|_{L^1} \|f\|_{L^1} \leq C \frac{1}{|t|^{\frac{d}{2}}} \|f\|_{L^1}, \quad t \neq 0.
\]
For \(|t| \leq 1\), we use the estimate
\[
\|S_K(t)f\|_{L^\infty} \leq \|\rho_{\epsilon}\|_{L^2} \|e^{it\Delta}(\rho_{\epsilon} * f)\|_{L^2} \leq \|\rho_{\epsilon}\|_{L^2} \|\rho_{\epsilon} * f\|_{L^2} \leq \|\rho_{\epsilon}\|_{L^2}^2 \|f\|_{L^1} \leq CK^d \|f\|_{L^1}.
\]
By combining the two inequalities, we get that
\[
\|S_K(t)f\|_{L^\infty} \leq C \frac{K^d}{(1 + |t|^{\frac{d}{2}})} \|f\|_{L^1}.
\]
Since we also have that
\[
\|S_K(t)f\|_{L^2} \leq \|f\|_{L^2},
\]
we get the desired estimate by complex interpolation. \(\square\)
10.2. **Proof of Theorem 4.2** By a scaling argument, it is sufficient to study the case \( \tau = 1 \). Indeed, we have that

\[
S_K(t)\phi(x) = \left( S_{K\tau^{\frac{1}{2}}} \left( \frac{t}{\tau} \right) \phi\left( \frac{x}{\tau^{\frac{1}{2}}} \right) \right).
\]

Therefore, it suffices to prove the estimates

\[
\left\| S_{K\tau^{\frac{1}{2}}}(-n)f \right\|_{L^p L^q} \leq C(K\tau^{\frac{1}{2}})^{\frac{2}{p}}\|f\|_{L^2},
\]

\[
\sum_{n\in\mathbb{Z}} S_{K\tau^{\frac{1}{2}}}(-n)F_n \leq C(K\tau^{\frac{1}{2}})^{\frac{2}{p}}\|F\|_{L^p L^q},
\]

\[
\sum_{k\in\mathbb{Z}} S_{K\tau^{\frac{1}{2}}}((n-k+s)F_k) \leq C(K\tau^{\frac{1}{2}})^{\frac{2}{p_1}+\frac{2}{p_2}}\|F\|_{L^p L^q},
\]

where \( L^p \) now stands for the usual discrete norms on sequences \( \|u\|_{L^p X} = (\sum_{n\in\mathbb{Z}} \|u_n\|_{X}^p)^{\frac{1}{p}} \). These estimates are equivalent through the usual \( \mathcal{T}\mathcal{T}^* \) argument. If we define \( (\mathcal{T}f)_n = S_{K\tau^{\frac{1}{2}}}(-n)f \). Then

\[
\mathcal{T}^* F = \sum_{k\in\mathbb{Z}} S_{K\tau^{\frac{1}{2}}}(-k)F_k,
\]

\[
(\mathcal{T}\mathcal{T}^*)_n = \sum_{k\in\mathbb{Z}} S_{K\tau^{\frac{1}{2}}}((n-k+s)F_k)
\]

and

\[
\|\mathcal{T}\|_{L^2 \rightarrow L^p L^q} = \|\mathcal{T}^*\|_{L^p L^q \rightarrow L^2} = \|\mathcal{T}\mathcal{T}^*\|_{L^p L^q \rightarrow L^p L^q}^{\frac{1}{2}}.
\]

Note that the estimate (97) corresponds to an estimate of \( \mathcal{T}e^{is\Delta}\mathcal{T}^* \) so that the estimate of \( \mathcal{T}\mathcal{T}^* \) is a special case with \( s = 0 \). We shall first prove the estimate for \( \mathcal{T}e^{is\Delta}\mathcal{T}^* \). We write that uniformly for \( s \in [-8, 8] \),

\[
\|\mathcal{T}e^{is\Delta}\mathcal{T}^* F\|_{L^p L^q} \leq \left\| \sum_{k\in\mathbb{Z}} S_{K\tau^{\frac{1}{2}}}((n-k+s)F_k) \right\|_{L^q} \leq C \sum_{k\in\mathbb{Z}} \frac{(K\tau^{\frac{1}{2}})^{\frac{d(1-\frac{2}{p})}{4}}}{1 + |n-k|^{\frac{d(1-\frac{2}{p})}{4}}} \|F_k\|_{L^q},
\]

where the last inequality comes from Lemma [10.1] applied to \( S_{K\tau^{\frac{1}{2}}} \) for \( K\tau^{\frac{1}{2}} \geq 1 \). From a discrete version of the Hardy–Littlewood–Sobolev inequality (see again [20]), we then obtain that

\[
\|\mathcal{T}e^{is\Delta}\mathcal{T}^*\|_{L^p L^q}^{\frac{1}{2}} \leq C(K\tau^{\frac{1}{2}})^{\frac{d(1-\frac{2}{p})}{4}} = C(K\tau^{\frac{1}{2}})^{\frac{2}{p}}
\]

by using the admissibility relation as long as \( p > 2 \). This yields (95) and (96). To get the general form of (97), it suffices to estimate \( \mathcal{T}e^{is\Delta}\mathcal{T}^* \) by composing the estimate for \( \mathcal{T} \), the \( L^2 \) continuity of \( e^{is\Delta} \) and the estimate for \( \mathcal{T}^* \). Once we have (97), the truncated version comes from the discrete Christ–Kiselev lemma as in [22] except in the case that \((p_1, q_1)\) and \((p_2, q_2)\) are the endpoint, but we excluded it for these estimates. One could also use a classical interpolation argument.

10.3. **Proof of Corollary 4.3** We shall use the Littlewood–Paley decomposition in order to convert the loss in the estimates of Theorem 4.2 into a loss of derivative. Let us recall some basic facts, we refer to the book [1] for the proofs. We take a partition of unity of the form

\[
1 = \varphi_1(\xi) + \sum_{k \geq 0} \varphi_k(\xi)
\]

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where \( \varphi_{-1} \) is supported in the ball \( \overline{B}(0,1) \) and each \( \varphi_k(\xi) = \varphi(\xi/2^k), \ k \geq 0 \) is supported in the annulus \( 2^{k-1} \lesssim |\xi| \lesssim 2^{k+1} \). We can then decompose any tempered distribution as

\[
    u = \sum_{k \geq -1} u_k, \quad F(u_k)(\xi) = \varphi_k(\xi) \hat{u}(\xi).
\]

We shall only use the following facts:

- **Bernstein inequality.** For every \( \sigma \geq 0 \) and every \( p \in [1, \infty] \), there exist constants \( c > 0 \) and \( C > 0 \) such that for every \( k \geq 0 \), we have

\[
    c2^{\sigma k} \| (\varphi_k(-i\nabla))u \|_{L^p} \leq \| -i\nabla\|_\sigma (\varphi_k(-i\nabla)u) \|_{L^p} \leq C2^{\sigma k} \| (\varphi_k(-i\nabla)u) \|_{L^p}. \tag{98}
\]

- **Characterization of \( L^q \) spaces.** For \( q \geq 2 \), the \( L^q \) norm of a function is equivalent to the norm

\[
    \left\| \left( \sum_{k \geq -1} |u_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q} := \| (u_k) \|_{L^q(\mathbb{Z})}. \tag{99}
\]

Note that when \( q = 2 \), we can invert the order of summation so that

\[
    \|u\|_{L^2} \sim \left( \sum_{k \geq -1} \|u_k\|_{L^2}^2 \right)^{\frac{1}{2}} = \| (u_k) \|_{L^2(\mathbb{Z})},
\]

where \( \sim \) denotes the equivalence of norms. Further, by Minkowski’s inequality, we have that

\[
    \|u\|_{L^q} \lesssim \| (u_k) \|_{L^q(\mathbb{Z})}.
\]

Let us first prove \( (22) \). By using the Littlewood–Paley decomposition, we first note that thanks to Minkowski’s inequality, we have

\[
    \|S_K(n\tau)u\|_{L^p} \lesssim \left( \sum_{k \geq -1} \|S_K(n\tau)u_k\|_{L^q}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{k \geq -1} \|S_K(n\tau)u_k\|_{L^p}^2 \right)^{\frac{1}{2}}, \tag{100}
\]

since \( p \geq 2 \). To estimate the terms inside the sum, we observe that

\[
    S_K(n\tau)u_k = S_{2^k}(n\tau)\Pi_Ku_k.
\]

Note that, because of the truncation \( \Pi_K \), the sum is actually finite. We sum only over the \( k \) such that \( 2^k \lesssim K = \tau^{-\frac{1}{2}} \).

If \( 2^k \tau^{-\frac{1}{2}} \lesssim 1 \), we can also write

\[
    S_K(n\tau)u_k = S_{\tau^{-\frac{1}{2}}}(n\tau)\Pi_Ku_k.
\]

Therefore, by Theorem \( 4.2 \) we obtain the estimate without loss

\[
    \|S_K(n\tau)u_k\|_{L^p} \leq C\|u_k\|_{L^2}.
\]

If \( \tau^{-\frac{1}{2}} \leq 2^k \leq \tau^{-\frac{1}{2}} \), we obtain that

\[
    \|S_K(n\tau)u_k\|_{L^p} \leq C(2^k \tau^{-\frac{1}{2}})^{\frac{2}{p}}\|u_k\|_{L^2}.
\]

Consequently, from the two sides of the Bernstein inequality, we obtain

\[
    \|S_K(n\tau)u_k\|_{L^p} \leq C(\tau^{\frac{1}{2}})^{\frac{2}{p}}\|u_k\|_{L^2} \leq C\|u_k\|_{H^{\frac{2}{p}(1-\frac{1}{p})}}.
\]
This yields thanks to (100)
\[ \|S_K(n\tau)u\|_{L^p} \lesssim \left( \sum_{k \geq -1} \|u_k\|_{H^{\frac{3}{p}(1-\frac{1}{p})}}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{H^{\frac{3}{p}(1-\frac{1}{p})}}, \]
which gives (22).

The proof of (23) follows exactly the same lines.

11. Some technical estimates

11.1. Properties of the filter function.

Lemma 11.1. We have the following properties:

- For every \( p \in [1, \infty] \), there exists \( C > 0 \) such that for every \( \tau \in (0, 1] \),
  \[ \| \varphi_1(-2i\tau \Delta)\Pi_{\tau^{-\frac{1}{2}}} f \|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all} \quad f \in L^p. \]  
  \[ (101) \]

- For every \( p \in (1, \infty) \), there exists \( C > 0 \) such that for every \( \tau \in (0, 1] \)
  \[ \left\| \frac{1 - \Pi}{2i\tau \Delta} f \right\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all} \quad f \in L^p. \]  
  \[ (102) \]

- For every \( s \in [0, 2] \), there exists \( C > 0 \) such that for every \( \tau \in (0, 1] \)
  \[ \|\varphi_1(-2i\tau \Delta)f\|_{H^s} \leq \frac{C}{\tau^{\frac{s}{2}}}\|f\|_{L^2} \quad \text{for all} \quad f \in L^2. \]  
  \[ (103) \]

Proof. We first prove (101). Let us set by \( L_\tau = \varphi_1(-2i\tau \Delta)\Pi_{\tau^{-\frac{1}{2}}} \). We first observe that \( \quad L_\tau f = \left( L_1(f(\tau^{-\frac{1}{2}})) \left( \frac{-}{\tau^{\frac{1}{2}}} \right) \right). \)

Therefore, by scaling, it suffices to prove the estimate (101) for \( L_1 \). Next, we can also write that \( \quad L_1 f = \Phi * f \)

where \( \Phi = \mathcal{F}^{-1}m_1 \) with \( m_1(\xi) = \varphi_1(2i|\xi|^2)\chi^2(\xi) \). Since \( \chi \) is compactly supported and \( \varphi_1 \) is smooth, we have that \( m_1 \) and therefore \( \Phi \) are in the Schwartz class, therefore we get in particular that \( \Phi \in L^1 \)
and the result follows from standard properties of convolutions.

By the same scaling argument, to prove (102), it suffices to prove the estimate with \( \tau = 1 \). We observe again that this amounts to prove the \( L^p \) continuity of the Fourier multiplier by \( m_2(\xi) = \frac{1 - \chi^2(\xi)}{2i|\xi|^2} \). We observe that \( m_2 \) is a smooth bounded function that satisfies in addition the estimate

\[ |\partial^\alpha m_2(\xi)| \leq \frac{C_\alpha}{|\xi|^{\alpha}} \quad \text{for all} \quad \xi \in \mathbb{R}^d \]

for every \( \alpha \in \mathbb{N}^d \). Consequently, the result follows from the Hörmander–Mikhlin Theorem.

To get (103), it suffices to observe that the function \( \varphi_1(2i\tau|\xi|^2)(1 + \tau|\xi|^2)^{\frac{s}{2}} \) is uniformly bounded by a constant independent of \( \tau \) and to use the Bessel identity. \[ \square \]
11.2. A localized critical Sobolev embedding. We have the following classical borderline Sobolev estimate for frequency localized functions in dimension 3.

**Lemma 11.2.** The exists $C > 0$ such that for every $u \in W^{1,3}(\mathbb{R}^3)$ with $\text{Supp} \ u \subset B(0, 4K)$, $K \geq 1$, we have

$$\|u\|_{L^\infty} \leq C (\log K)^{\frac{2}{3}} \|u\|_{W^{1,3}}.$$  

**Proof.** By using the Littlewood–Paley decomposition introduced in the previous section and the triangular inequality, we have that

$$\|u\|_{L^\infty} \leq \sum_{2^k \leq 4K} \|u_k\|_{L^\infty}.$$  

Note that the sum is finite thanks to the assumption on the support of the Fourier transform of $u$. Next, since $u_k = \Pi_{4^k} u$, we get from Young’s inequality for convolutions that

$$\|u_k\|_{L^\infty} \lesssim 2^k \|u_k\|_{L^3}.$$  

Therefore, by using the Bernstein inequality, we get that

$$\|u\|_{L^\infty} \lesssim \sum_{2^k \leq 4K} 2^k \|u_k\|_{L^3} \lesssim \|u\|_{L^3} + \sum_{1 \leq 2^k \leq 4K} \|\nabla u_k\|_{L^3}.$$  

Next, from Hölder’s inequality and Fubini we get that

$$\sum_{1 \leq 2^k \leq 4K} \|\nabla u_k\|_{L^3} \lesssim (\log K)^{\frac{2}{3}} \left( \sum_{k \geq -1} \|\nabla u_k\|_{L^3}^3 \right)^\frac{1}{3} \lesssim (\log K)^{\frac{2}{3}} \left( \sum_{k \geq -1} |\nabla u_k|^3 \right)^\frac{1}{3}.$$  

Since, by the embedding of discrete $l^p$ spaces, we have that

$$\left( \sum_{k \geq -1} |\nabla u_k|^3 \right)^\frac{1}{3} \lesssim \left( \sum_{k \geq -1} |\nabla u_k|^2 \right)^\frac{1}{2},$$  

we finally obtain that

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^3} + (\log K)^{\frac{2}{3}} \left( \sum_{k \geq -1} |\nabla u_k|^2 \right)^\frac{1}{2} \lesssim (\log K)^{\frac{2}{3}} \|u\|_{W^{1,3}},$$  

where the final estimate comes from.

\[\square\]

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