ON SOME GENERALIZATIONS OF BATALIN-VILKOVISKY ALGEBRAS

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Abstract

We define the concept of higher order differential operators on a general noncommutative, nonassociative superalgebra $A$, and show that a vertex operator superalgebra (VOSA) has plenty of them, namely modes of vertex operators. A linear operator $\Delta$ is a differential operator of order $\leq r$ if an inductively defined $(r+1)$-linear form $\Phi^{r+1}_{\Delta}$ with values in $A$ is identically zero. These forms resemble the multilinear string products of Zwiebach. When $A$ is a “classical” (i.e. supercommutative, associative) algebra, and $\Delta$ is an odd, square zero, second order differential operator on $A$, $\Phi^2_{\Delta}$ defines a “Batalin-Vilkovisky algebra” structure on $A$. Now that a second order differential operator makes sense, we generalize this notion to any superalgebra with such an operator, and show that all properties of the classical BV bracket but one continue to hold “on the nose”. As special cases, we provide several examples of classical BV algebras, vertex operator BV algebras, and differential BV algebras. We also point out connections to Leibniz algebras and the noncommutative homology theory of Loday. Taking the generalization process one step further, we remove all conditions on the odd operator $\Delta$ to examine the changes in the basic properties of the bracket. We see that a topological chiral algebra with a mild restriction yields a classical BV algebra in the cohomology. Finally, we investigate the quantum BV master equation for (i) classical BV algebras, (ii) vertex operator BV algebras, and (iii) generalized BV algebras, relating it to deformations of differential graded algebras.

1 Introduction

The algebra of Batalin-Vilkovisky (BV) quantization $[BV]$ has been studied from several viewpoints. In its most abstract form, a BV algebra is a supercommutative associative algebra $\mathcal{A}$ with an odd, square zero, second order differential operator $\Delta$. From these axioms follows the existence of an odd Lie bracket $\{,\}$ measuring the failure of $\Delta$ to be a derivation. The definition depends very much on the notion of a second order differential operator, which seems to have been defined so far for operators on supercommutative, associative algebras only. In particular J.-L. Koszul $[Ko]$ proposes a definition of higher order differential operators on such an algebra $\mathcal{A}$ in terms of the vanishing of $r$-linear forms $\Phi^r_{\Delta}$ with values in $\mathcal{A}$, which is not applicable to other types of algebras. At this point we should make the meanings of “super” and “odd” very clear. A superalgebra $\mathcal{A}$ is simply an algebra which has an integer grading preserved by the multiplication, and which is also expected to have identities differing from some ordinary version by certain powers of $-1$, wherever
the product of two homogeneous elements is written in two different ways. For example, a supercommutative (a.k.a. graded-commutative) algebra is a superalgebra where the identity \(ab - (-1)^{|a||b|}ba = 0\) is satisfied for all homogeneous elements \(a, b\). An odd operator \(\Delta\) on \(\mathcal{A}\) is an operator which shifts the grading by an odd integer \(|\Delta|\) (mostly there is an implicit assumption of homogeneity). One may assume this odd integer is \(\pm 1\) in a computation where it only appears as a power (of \(-1\)), as is the case with most of this article. \(\Delta\) is called a superderivation (or simply a derivation) of \(\mathcal{A}\) if it satisfies the usual product rule with modifications; i.e. if \(\Delta(ab) = \Delta(a)b + (-1)^{|a||\Delta|}a\Delta(b)\) for every homogeneous \(a\).

Examples of BV algebras (which we may now call classical) have been floating around: The homology complex \((\wedge g, \partial)\) of a finite dimensional Lie algebra \(g\) is one. Another geometric example involves the Schouten-Nijenhuis bracket on the contravariant antisymmetric tensor fields on a finite dimensional manifold \((\mathcal{Ko})\). Lian and Zuckerman showed in [LZ1] that the cohomology of a BRST complex with the Wick product is a BV algebra, where the BV operator is the weight zero mode of the anti-ghost vertex operator \(b(z)\) (also see the follow-up by Penkava and Schwarz [PS]). In a related construction Bouwknegt and Pilch (in collaboration with McCarthy) showed that a certain semi-infinite (BRST) cohomology of the \(W_3\) algebra has a BV structure, with the “same” BV operator [BP]. Most of these examples are subspaces or subquotients of richer structures, some of which are vertex operator superalgebras (VOSA) or close relatives. Lian and Zuckerman study in [LZ1] several properties of the BV bracket which hold “off-shell”, i.e. on the cochain complex itself, which is a VOSA. One of our goals is to isolate -with a lot of hindsight- the most general properties of a BV bracket and elucidate the algebra implicit in [LZ1].

Some of the problems to be overcome in a broader definition of a BV algebra are the following: The new definition must involve an unambiguous notion of a second order differential operator for the most general types of algebras. The corresponding BV bracket should retain most of its desirable properties. There should be abundant natural examples among VOSA’s, and every major existing example should be a subquotient of one. And finally, these “higher brackets” had better have some geometric or physical significance. Our treatment will fulfill most of these criteria.

In Section 2.1 we propose a definition of higher order differential operators on a superalgebra, which need not be supercommutative or associative. A linear operator \(\Delta\) on \(\mathcal{A}\) is an \(r\)-th order differential operator if and only if a (recursively) well-defined \((r + 1)\)-linear form \(\Phi_{\Delta}^{r+1}(a_1, \ldots, a_{r+1})\) is identically zero. The definition agrees with Koszul’s for classical algebras. Section 2.2 has the important result (Theorem 2.2) that the modes \(u_0, u_1, u_2, \ldots\) of a vertex operator \(u(z) = \sum u_n z^{-n-1}\) are respectively first, second, third, ... order differential operators on the VOSA with respect to the Wick product, so that any odd element \(u\) of a VOSA with a square zero mode \(u_1\) provides us with a BV operator!

Section 3.1 contains the definition of classical Gerstenhaber and BV algebras, and in Section 3.2, we look at examples of classical BV algebras (classical Lie algebra cohomology
and homology complexes with coefficients in a commutative associative algebra, more of the above with multiplication and substitution operators as BV operators, the Weil algebra, the cohomology of a topological chiral algebra, and skew multivector fields with the Schouten-Nijenhuis bracket), supplying simple proofs that the operators under consideration are second order differential operators. Section 3.2.1 provides an answer to Exercise 10 in [LZ4].

We give the definition of a generalized BV algebra (GBVA) and its bracket (namely, an algebra \( A \) with an odd, square zero, second order differential operator \( \Delta \) and \( \{a, b\} = (-1)^{|a|} \Phi^2_\Delta(a, b) \)) in Section 4.1. A GBVA is possibly the most prominent nontrivial example of a “Leibniz algebra”, which is the main ingredient of the noncommutative homology theory of Loday [Lo]. (A Leibniz algebra is a generalized Lie algebra where the superderivation (Leibniz) property of the bracket holds, but skew-symmetry need not.) We define a vertex operator BV algebra (VOBVA) to be a GBVA where \( A \) is a VOSA with the Wick product and \( \Delta \) is some \( u_1 \). Since many known examples of BV algebras can be obtained as the cohomology of a generalized BV algebra, we define a differential BV algebra (DBVA) as a GBVA which exhibits an additional odd, square zero operator \( D \) and a diagonalizable operator \( L \) such that \( D\Delta + \Delta D = L \) (\( L = 0 \) or \( L = L_0 \) of \( Vir \) most of the time), with the condition that the cohomology of \( D \) is also a GBVA. This latter may seem superfluous, but \( D \) need not always be a derivation and hence its cohomology need not be of the form a subalgebra modulo an ideal. We show that all properties of the BV bracket survive these definitions, with the exception of skew-symmetry, which is modified (Proposition 4.8). The extra term \( (-1)^{|a|} \Phi^2_\Delta(a, b) \) associated with the skew-symmetric product \( [a, b] = ab - (-1)^{|a||b|} ba \) vanishes if \( A \) is supercommutative. Next, we remove all restrictions on \( \Delta \) as well and observe that the identities in Proposition 4.8 are modified by certain \( \Phi^2 \) and \( \Phi^3 \) terms, all of which vanish when \( A \) is supercommutative, but not necessarily associative, and \( \Delta \) is square zero and of order two (Proposition 4.13). In Section 4.2 we explore two representatives of VOBVA’s, one of which is the vertex operator Weil algebra (VOWA) with a huge number of BV operators. The second part is a short reminder of the properties of the well-known BRST complex, or more generally, of a topological chiral algebra (TCA). Proposition 4.14 asserts that a TCA where the field \( G(z) \) (definition in 4.2.2) is primary collapses to a classical BV algebra as \( G_0^2 \) becomes zero in the \( Q \)-cohomology.

Section 5.1 deals with the quantum BV master equation for classical BV algebras. We show why \( \Delta(\exp(\frac{i}{\hbar} W)) = 0 \) follows from \( \{W, W\} = 2i\hbar \Delta(W) \). In Section 5.2 we investigate the same equation for VOBVA’s, and finally in 5.3, we discuss the meaning of the master equation for GBVA’s (keeping the conditions on \( \Delta \) intact), as related to the deformation theory of differential graded algebras. A solution of this equation is interpreted as a deformation of the square zero, second order operator \( \Delta \) where all identities are preserved.

We do not give the full definition, nor all the properties, of vertex operator superalgebras, as there are very good (and increasingly accessible) accounts in literature. Apart from the classics [FLM, FHL, DL], and comprehensive reviews like [Geb], we recommend the sequence [LZ2-LZ4] which stresses similarities with supercommutative associative algebras.
and builds up the theory from scratch. It suffices at the moment to say that a VOSA is a \( \mathbb{Z} \)-bigraded vector space \( V \) (one \( L_0 \) and one super grading) where each element \( u \) is represented uniquely (and linearly) by a “vertex operator” \( u(z) = \sum u_n z^{-n-1} \), with \( u_n \in \text{End}(V) \). There is an action of the Virasoro algebra by \( L(z) = \sum L_n z^{-n-2} \); the eigenvalues of \( L_0 \) are bounded from below, and \( L_{-1} \) acts as formal differentiation. The vacuum element \( 1 \) is represented by \( 1 \cdot z^0 \) and is the identity element with respect to the multiplication \( u \times_{-1} v = u_{-1} v \) (the Wick product). If the usual Cauchy-Jacobi identity involving the relations of modes \( u_n \) becomes too oppressive, one may visualize the alternative hidden in [DL], which says that for any \( u, v \) in the VOSA there exists a sufficiently large positive integer \( t \) such that

\[
[u(z_1), v(z_2)](z_1 - z_2)^t = 0
\]

as formal power series. We will frequently make use of the identities (see e.g. [FLM] or [Geb])

\[
(u_m v)_n = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u_{m-i} v_{n+i} - (-1)^{m+|u|} v_{m+n-i} u_i)
\]

and

\[
[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}
\]

for \( m, n \in \mathbb{Z} \) (\( \binom{m}{i} \) is given by \( m(m-1) \cdots (m-i+1)/i! \) if \( i \geq 1 \), and 1 if \( i = 0 \)). Note that a VOSA is not thought of as a super-Virasoro module, as the name sometimes implies in the mathematical physics literature.

We should also mention operadic constructions of BV algebras (such as [Get1], [Hu]). Recently (while this work was in progress) Kimura, Stasheff, and Voronov proposed an abstract partial solution to the problem of lifting the BV algebra structure in [LZ1] to the cochain complex, by introducing “filtered topological gravity” whose state space is a “commutative homotopy algebra”, and stating that such algebras can be obtained as algebras over a certain operad, with some vanishing condition (Thm. 0.1 of [KSV]). We will not study this aspect of BV algebras at all, but point out that there is an obvious way of writing a linear operad whose algebras are exactly the generalized BV algebras as defined in our paper. Finding a topological operad whose cohomology is this linear operad is another matter.

2 Higher Order Differential Operators

2.1 Definitions

There is a notion of “higher order differential operators” for (super) commutative, associative algebras, which is consistent with the idea of composites of partial derivatives on an algebra of functions. Let \( \mathcal{A} = \oplus_j \mathcal{A}^j \) be any superalgebra, not necessarily supercommutative
or associative. A linear map $\Delta : A \to A$ is said to be homogeneous of (super)degree $k$, written $|\Delta| = k$, if $\Delta : A^j \to A^{j+k}$ for all $j \in \mathbb{Z}$. The map $\Delta$ is also said to be a first order differential operator (or derivation) on $A$ if

$$\Delta(ab) = \Delta(a)b + (-1)^{|a||\Delta|}a\Delta(b).$$

A second order differential operator is a map $\Delta$ such that the bracket operator $\{ a, b \}$ defined by

$$\{ a, b \} = \Delta(ab) - \Delta(a)b - (-1)^{|a||\Delta|}a\Delta(b)$$

is a derivation for every homogeneous $a$. (The bracket measures the deviation of $\Delta$ from being a derivation.) J.-L. Koszul defines in [Ko] linear maps (equivalently, $r$-linear forms)

$$\Phi^r_\Delta : A^\otimes r \to A \quad (r \geq 1)$$

for every linear operator $\Delta$ on a supercommutative associative algebra $A$ by

$$\Phi^r_\Delta(a_1, \ldots, a_r) = m \circ (\Delta \otimes id_A)\lambda^r(a_1 \otimes \cdots \otimes a_r)$$

where $m(a \otimes b) = ab$ is the multiplication map, and

$$\lambda^r(a_1 \otimes \cdots \otimes a_r) = (a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_r \otimes 1 - 1 \otimes a_r),$$

the right hand side being a product in the supercommutative algebra $A \otimes A$. Then

$$\begin{align*}
\Phi^1_\Delta(a) &= \Delta(a) - \Delta(1)a \\
\Phi^2_\Delta(a, b) &= \Delta(ab) - \Delta(a)b - (-1)^{|a||b|}\Delta(b)a + \Delta(1)ab \\
\Phi^3_\Delta(a, b, c) &= \Delta(abc) - \Delta(ab)c - (-1)^{|a|(|b|+|c|)}\Delta(bc)a \\
& \quad - (-1)^{|c|(|a|+|b|)}\Delta(ca)b + \Delta(ab)c + (-1)^{|a|(|b|+|c|)}\Delta(b)ca \\
& \quad + (-1)^{|c|(|a|+|b|)}\Delta(c)ab - \Delta(1)abc \\
& \quad \vdots
\end{align*}$$

and a map $\Delta$ (of degree $k$) is said to be a differential operator on $A$ of order $\leq r$ (written $\Delta \in \mathcal{D}^k_r$) iff $\Phi^{r+1}_\Delta$ is identically zero. The subspaces $\mathcal{D}^i_r$ of $\text{Lin}(A)$ satisfy

$$\begin{align*}
(i) \quad &\mathcal{D}^1_r \subset \mathcal{D}^2_r \subset \cdots \subset \mathcal{D}^i_r \subset \mathcal{D}^{i+1}_r \subset \cdots \\
(ii) \quad &\mathcal{D}^i_r \mathcal{D}^k_s \subset \mathcal{D}^{i+k}_{r+s} \\
(iii) \quad &\mathcal{D}^i_r, \mathcal{D}^k_s \subset \mathcal{D}^{i+k}_{r+s-1}.
\end{align*}$$

**Remark 2.1** We will consider only differential operators which are “graded”, i.e. which are finite linear combinations of elements of the $\mathcal{D}^i_r$.

Koszul asserts that every $\Phi^{r+1}_\Delta$ can be written in terms of $\Phi^r_\Delta$, such as

$$\Phi^3_\Delta(a, b, c) = \Phi^2_\Delta(a, bc) - \Phi^2_\Delta(a, b)c - (-1)^{|b||c|}\Phi^2_\Delta(a, c)b,$$
so that property (i) above is trivial (no other inductive formula is given).

In order to generalize the notion to a noncommutative, nonassociative superalgebra \( \mathcal{A} \) (e.g. a VOSA) this last formula must be studied. A good inductive definition of \( \Phi'_{\Delta} \) should involve only one multiplication at a time, and no arbitrary change of order of arguments. We propose a new recursive definition\(^1\)

\[
\begin{align*}
\Phi^1_{\Delta}(a) &= \Delta(a) \\
\Phi^2_{\Delta}(a, b) &= \Phi^1_{\Delta}(ab) - \Phi^1_{\Delta}(a)b - (-1)^{|a||\Delta|}a^{\Phi^1_{\Delta}}(b) \\
\Phi^3_{\Delta}(a, b, c) &= \Phi^2_{\Delta}(bc) - \Phi^2_{\Delta}(a, b)c - (-1)^{|b||a|+|\Delta|}b^{\Phi^2_{\Delta}}(a, c) \\
& \quad \vdots \\
\Phi^{r+1}_{\Delta}(a_1, \ldots, a_{r+1}) &= \Phi^r_{\Delta}(a_1, \ldots, a_r a_{r+1}) - \Phi^r_{\Delta}(a_1, \ldots, a_r) a_{r+1} \\
& \quad - (-1)^{|a_r||a_1|+\cdots+|a_{r-1}|+|\Delta|} a_r^{\Phi^r_{\Delta}}(a_1, \ldots, a_{r-1}, a_{r+1}) \\
& \quad \vdots
\end{align*}
\]

which fits the bill. The extra powers of \((-1)\) come from the rearrangement of the symbols \(\Delta, a_1, \ldots, a_{r+1}\). Note that each \(\Phi^{r+1}_{\Delta}(a_1, \ldots, a_{r+1})\) gives the deviation of \(\Phi^r_{\Delta}(a_1, \ldots, a_{r-1}, \ldots)\) (which we can interpret as a higher bracket; see \([\text{LS, Zw, KSV}]\) on “strongly homotopy Lie algebras”, “multilinear string products”, and related notions) from being a derivation, where the order in the definition of the product rule is fixed by \((\text{7})\). Then carefully keeping track of the order of multiplication and of the arguments \(a_i\), we see that

\[
\begin{align*}
\Phi^1_{\Delta}(a) &= \Delta(a) \\
\Phi^2_{\Delta}(a, b) &= \Delta(ab) - \Delta(a)b - (-1)^{|a||\Delta|}a\Delta(b) \\
\Phi^3_{\Delta}(a, b, c) &= \Delta(a(bc)) - (-1)^{|a||\Delta|}a\Delta(bc) \\
& \quad - (-1)^{|b||a|+|\Delta|}b\Delta(ac) - \Delta(ab)c - \Delta(a)(bc) + (\Delta(a)b)c \\
& \quad + (-1)^{|b||a|+|\Delta|}b(\Delta(a)c) + (-1)^{|a||\Delta|}(a\Delta(b))c + (-1)^{|a||b|+|\Delta|}(|a|+|b|)b(a\Delta(c)) \\
& \quad \vdots
\end{align*}
\]

leads to the correct (or plausible, unambiguous, ...) definition of higher order differential operators in a general algebra \(\mathcal{A}\). If \(\mathcal{A}\) happens to have an identity (such as a VOSA and its vacuum element, which is the identity with respect to the Wick product), one may wish to replace the first line in \((7)\) by

\[
\Phi^1_{\Delta}(a) = \Delta(a) - \Delta(1)a
\]

and rewrite \((8)\) accordingly. In the polynomial algebra \(\mathcal{C}[x]\), for example, multiplication by a polynomial \(p(x)\) would be a differential operator of order zero, and would annihilate all

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\(^1\)Prof. Koszul commented in a private communication that there is an older recursive definition due to Grothendieck \([\text{Gr}]\): a linear operator \(\Delta : \mathcal{A} \to \mathcal{A}\) is of order \(\leq r\) if and only if the (super)commutator \([\Delta, m_a]\) is of order \(\leq r-1\) for all \(a \in \mathcal{A}\), where \(m_a\) is left multiplication by \(a\), and operators of order \(\leq -1\) are zero. This definition is different from mine and does not work for VOSA’s.
forms. The first order differential operators would be of the form \( p(x) \frac{d}{dx} \) and annihilate all except possibly the first form, etc.

Definitions (5) and (7)-(9) have been checked to agree up to and including \( \Phi^4_\Delta \) for classical algebras, and it should not be too difficult to give a general proof of equivalence in this special case. The following familiar formula for \( \Phi^4_\Delta \) clearly shows the best way to write any expression \( \Phi^r_\Delta \) in a classical algebra. We apply the operator \( \Delta \) to \( k \) entries out of \( r \) (\( 1 \leq k \leq r \)) in every possible way (up to order) and multiply each of these expressions with the remaining \( r - k \) entries. We then add up all terms, modifying each by \( (-1)^{r-k} \) and by the sign of the permutation of the symbols \( \Delta, a, b, c, \ldots \) under consideration:

\[
\Phi^4_\Delta(a, b, c, d) = \Delta(abcd) - (-1)^{|\Delta||a|} a \Delta(bcd) - (-1)^{|\Delta||a|+|\Delta|} b \Delta(acd) \\
- (-1)^{|\Delta||a|+|b|} c \Delta(abd) - \Delta(abc)d \\
+ (-1)^{|\Delta||a|+|b|} a b \Delta(cd) + (-1)^{|\Delta||a|} a \Delta(bc)d \\
+ (-1)^{|\Delta||a|+|b|} a b c \Delta(ad) + \Delta(abd)c \\
- \Delta(a)bc - (-1)^{|\Delta||a|} a \Delta(b)cd \\
- (-1)^{|\Delta||a|+|b|} a b \Delta(c)d - (-1)^{|\Delta||a|+|b|+|c|} abc \Delta(d).
\]

2.2 Higher Order Differential Operators on VOSA’s

We will show that in a vertex operator superalgebra, the modes \( u_0, u_1, u_2, \ldots \) of any vertex operator

\[
u(z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}, \tag{10}\]

namely the coefficients of \( z^{-1}, z^{-2}, z^{-3}, \ldots \), are differential operators on the VOSA of orders \( \leq 1, 2, 3, \ldots \) respectively (they annihilate the vacuum). Meanwhile all the other coefficients are “left multiplications”, i.e. they annihilate the modified expression (10).

For vertex operators written in the standard form (10), the infinitely many multiplications

\[
a \times_n b = a_n b = a_n b_{-1} 1 \tag{11}\]

are given by

\[
a \times_n b = \text{Res}_z z^n a(z) b,
\]

and \( \times_{-1} \) is the “normal ordered product”, a.k.a. the “Wick product”. It is well known that a residue, \( u_0 \), is a derivation of the VOSA with respect to all products \( \times_n \):

\[
u_0(a_n b) = (u_0 a_n) b + (-1)^{|u||a|} a_n (u_0 b)
\]
(compute \([u_0, a_n]\) from (2) and apply to \(b\)). Then
\[
\Phi^2_{u_0}(a, b) = 0
\]

with respect to all \(\times_n, n \in \mathbb{Z}\), that is, \(u_0 \in \mathcal{D}_1\). However at this point we restrict the
definition of a differential operator to the case of the Wick product, since the other modes
are not as well-behaved as the residue. Let \(ab\) denote \(a \times_{-1} b\) from now on.

**Theorem 2.2** In a VOSA \(V\), the modes \(u_n\) of a vertex operator (10) for \(n \geq 0\) are higher
order differential operators on \(V\), namely,
\[
u_n \in \mathcal{D}_{n+1} \quad \text{for} \quad n \geq 0,
\]
or equivalently,
\[
\Phi^{n+2}_{u_n} \equiv 0 \quad \text{for} \quad n \geq 0.
\]
The remaining modes are left multiplications, that is,
\[
\Phi^k_{u_n} \equiv 0 \quad \forall n \leq -1, \ k \geq 1.
\]

**Remark 2.3** Note that condition (i) of (6) is automatically satisfied.

**Proof of Theorem 2.2.** For \(r \geq 1\), assume \(u_0, \ldots, u_{r-1}\) have been shown to be in \(\mathcal{D}_1, \ldots, \mathcal{D}_r\)
respectively for all \(u\). We start with
\[
\Phi^1_{u_r}(a) = u_ra \quad \text{(definition)},
\]
so that
\[
\Phi^2_{u_r}(a, b) = u_r(a-1)b - (u_a)_{-1}b - (-1)^{|u|a}a_{-1}(u, b)
\]
\[
= -(u_r, a)_{-1}b + [u_r, a_{-1}]b
\]
\[
= -(u_r, a)_{-1}b + \sum_{i=0}^r \binom{r}{i} (u_i a)_{r-1-i}b \quad \text{(from (2))}
\]
\[
= -(u_r, a)_{-1}b + \sum_{i=0}^r \binom{r}{r-i} (u_{r-i} a)_{r-i-1}b
\]
\[
= -(u_r, a)_{-1}b + \sum_{i=0}^r \binom{r}{i} (u_{r-i} a)_{i-1}b \quad \text{(replacing \(i\) by \(r-i\))}
\]
\[
= \sum_{i=1}^r \binom{r}{i} (u_{r-i} a)_{i-1}b \quad \text{(12)}
\]

where the subscripts of the operators applied to \(b\) range from 0 to \(r-1\), and by the induction
step they lie in \(\mathcal{D}_1, \ldots, \mathcal{D}_r\), or simply in \(\mathcal{D}_r\). Hence
\[
\Phi^2_{u_r}(a, b) = \Delta(b) = \Phi^1_{\Delta}(b)
\]
for some $\Delta \in D_r$, keeping $a$ fixed (read $\Delta$ from (12)). Now by an independent and easy induction we can show that
\[ \Phi^k_{ur}(a_1, \ldots, a_k) = \Phi^{k-1}_{\Delta}(a_2, \ldots, a_k) \quad \text{for} \quad k \geq 2 \]
and in particular
\[ \Phi^{r+2}_{ur}(a_1, \ldots, a_{r+2}) = \Phi^{r+1}_{\Delta}(a_2, \ldots, a_{r+2}) \equiv 0, \]
($a_1 = a$, fixed) as $\Delta \in D_r$. Finally, the statement on left multiplications follows from
\[ \Phi^1_{un}(a) = u_n a - (u_n 1)_{-1} a \\
= u_n a - \binom{n}{0} u_n a = 0 \quad \text{(from (1))} \]
and the last Remark. □

**Remark 2.4** Compositions and brackets of modes satisfy properties similar to (ii) and (iii) in (B).

### 3 Classical Batalin-Vilkovisky Algebras

#### 3.1 Definitions and Properties

A Gerstenhaber algebra ([Ger]) is a supercommutative, associative algebra
\[ \mathcal{A} = \bigoplus_j \mathcal{A}^j \]
equipped with an odd bracket $\{ \ , \ \}$ satisfying
\[ \{a, b\} = (-1)^{(|a|+1)(|b|+1)} \{b, a\} \]
(skew-symmetry in the associated, shifted-graded super Lie algebra $\hat{\mathcal{A}} = \bigoplus_j \mathcal{A}^{j+1}$), and
\[ \{a, \{b, c\}\} = \{(a, b), c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\} \]
(the Leibniz rule, or superderivation property, in $\hat{\mathcal{A}}$), as well as the superderivation (Poisson) rule with respect to the multiplication in $\mathcal{A}$:
\[ \{a, bc\} = \{a, b\} c + (-1)^{(|a|+1)|b|} b\{a, c\}. \quad (13) \]
This last condition implies that bracketing with a homogeneous element $a$ is a superderivation on $\mathcal{A}$ which changes the grading by $|a|$ plus an odd integer.

A Batalin-Vilkovisky algebra, or BV algebra, is a Gerstenhaber algebra where the bracket $\{ \ , \ \}$ is obtained from an odd, square zero, second order differential operator $\Delta$ (usually of superdegree $\pm 1$):
\[ \{a, b\} = (-1)^{|a|} \Phi_{\Delta}^2(a, b) \\
= (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta(a)b - a\Delta(b) \quad \text{(14)} \]
We will show in Section 4.1 that the identities above are modified by some $\Phi^2$ and $\Phi^3$ terms when \{a, b\} is defined via (14) for any superalgebra $A$ and an odd operator $\Delta$ with no restrictions.

### 3.2 Classical Examples

We will look at

(i) $(a \otimes \wedge g', d)$: The classical Lie algebra cohomology complex for a finite dimensional Lie algebra $g$, and a commutative algebra $a$ on which $g$ acts by derivations;

(ii) $(a \otimes \wedge g, \partial)$: The classical Lie algebra homology complex for $g$, $a$ as above;

(iii) Multiplication and substitution operators in (i) and (ii);

(iv) $Sg' \otimes \wedge g'$: The Weil algebra on $g$ with several differentials;

(v) The cohomology of a BRST complex, or more generally, the cohomology of a topological chiral algebra;

(vi) The skew multivector fields $A(M)$ on a finite dimensional paracompact differentiable manifold $M$ with the Schouten-Nijenhuis bracket;

and give simple arguments as to why the given operators are second order differential operators. See also [Ko], [AKSZ], [Wi1], and [Sc].

**Remark 3.1** Although most properties apply to a general Lie algebra, we will restrict ourselves to finite dimensional semisimple $g$ in this section.

#### 3.2.1 Classical Lie Algebra Cohomology and Homology

There is a unified “semi-infinite” construction of finite dimensional Lie algebra cohomology and homology complexes ([A1, A2]) which is summarized below. We think of both complexes (i) and (ii) above as modules over the associative superalgebra

$$\mathcal{Y}g = Ug \otimes Cg$$

where $Ug$ is the universal enveloping algebra of $g$, and $Cg$ is the Clifford algebra on $g \oplus g'$, the symmetric bilinear product being given by the pairing between a fixed homogeneous basis \{\iota(e_i)\} of $g$ and its (restricted, i.e. piecewise) dual \{\epsilon(e'_i)\} in $g'$. The superdegrees of the generators of types $\iota$ and $\epsilon$ are taken to be $-1$ and $+1$ respectively. Every (co)homology related operator is realized as an inner derivation of $\mathcal{Y}g$, which then acts on (i) or (ii). Note that $\wedge g'$ is spanned by wedge products of $\epsilon(e'_i)$’s and $\wedge g$ by wedge products of $\iota(e_i)$’s. There is an overall $g$ action given by the operators $\theta(x) \in \text{Der}(\mathcal{Y}g)$, namely

$$\theta(x) = \pi(x) + \rho(x) = \pi(x) + \sum_i: \iota([x, e_i])\epsilon(e'_i) \in \mathcal{Y}g,$$
where $x \in g$, $\pi(x) \in \mathcal{U}g$, $\rho(x) \in \mathcal{C}g$, and the notation : ("normal ordered product") simply means

$$\rho(x) = \left\{ \begin{array}{ll} -\sum_i e(e_i')\iota([x,e_i]) & \text{in case (i)} \\
\sum_i \iota([x,e_i])e(e_i') & \text{in case (ii)}. \end{array} \right.$$  

We say the $\iota$'s are "annihilation operators" whereas the $\epsilon$'s are "creation operators" for the cohomology complex (and vice versa for the homology complex). It was shown in [A1] that there exists a unique derivation $D (= d$ or $\partial$) of $\mathcal{Y}g$ of superdegree $+1$, satisfying

$$D \iota(x) = \theta(x) \quad \forall x \in g.$$  

Then $D$ turns out to be an inner derivation, with formula

$$D = \sum_i \pi(e_i)e(e_i') + \sum_{i<j} : \iota([e_i,e_j])e(e_j')e(e_i') : \in \mathcal{Y}g$$  

$$= \left\{ \begin{array}{ll} d = \sum_i \pi(e_i)e(e_i') + \sum_{i<j} \epsilon(e_i')\epsilon(e_i')\iota([e_i,e_j]) & \text{in case (i)} \\
\partial = \sum_i \pi(e_i)e(e_i') + \sum_{i<j} \iota([e_i,e_j])e(e_j')e(e_i') & \text{in case (ii)}. \end{array} \right. (15)$$  

The characterization of $D$ above translates into the famous Cartan identity

$$D \iota(x) + \iota(x)D = \theta(x)$$

relating operators acting on (i) or (ii). This identity favors $\iota(x)$ over $\epsilon(x')$. We have the additional identities

$$D^2 = 0$$

and

$$[D, \theta(x)] = 0 \quad \forall x \in g.$$  

For both (i) and (ii) (taking $a = C$, the trivial representation, for the moment, so that $\pi(e_i) = 0$), the operator $\rho(x)$ is also a derivation of the associative algebra $\wedge g'$ or $\wedge g$ for all $x \in g$, as

1) $[\rho(x), \ ]$ is a derivation of $\mathcal{C}g$,
2) $\rho(x)$ sends creation operators to creation operators,
3) $\rho(x)1 = 0$.

Of course, from a simpler viewpoint, $\rho$ is nothing but the extension of the (co)adjoint action of $g$ to the exterior algebra by derivations.

How do we get a BV algebra? In case (i), $d$ is a derivation of $\wedge g'$ for similar reasons:

1) $[d, \ ]$ is a derivation of $\mathcal{C}g$,
2) $d$ sends $\epsilon$'s to expressions of type $\sum \epsilon \epsilon$ (see [A1]),
3) $d1 = 0$.

Also $d^2 = 0$, so $d$ is a BV operator on $\wedge g'$ with trivial BV bracket $\{ \ , \ \}$.  

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In case (ii), $\bar{\partial}$ is not a derivation of $\wedge g$, for it fails to send creation operators to (products of) creation operators in $\mathcal{C}_g$:
\[
\partial \iota(x) = \rho(x).
\] (16)

However, $\bar{\partial}$ is a sum of genuine second order differential operators on $\mathcal{C}_g$:
\[
\bar{\partial} = \frac{1}{2} \sum_i \rho(e_i)\epsilon(e'_i)
\]
(both $\rho(e_i)$ and $\epsilon(e'_i)$ are derivations!), hence the square zero operator $\bar{\partial}$ becomes a nontrivial BV operator.

The generalization to a nontrivial module $a$ is easy. Case (i): The second term of $\underline{d}$ in (15) is already a derivation of $\wedge g'$, and the first satisfies
\[
\sum_i \pi(e_i)\epsilon(e'_i)(ab) = \sum_i \epsilon(e'_i)\pi(e_i)(ab)
\]
\[
= \sum_i \epsilon(e'_i)((\pi(e_i)a) + a(\pi(e_i)b))
\]
\[
= (\sum_i \epsilon(e'_i)\pi(e_i))a + a(\sum_i \epsilon(e'_i)\pi(e_i))b,
\]
so that $\underline{d}$ is a derivation.

Case (ii): Once again, $\bar{\partial}$ is not a derivation in general, but is a sum of second order differential operators:
\[
\bar{\partial} = \sum_i (\pi(e_i) + \frac{1}{2} \rho(e_i))\epsilon(e'_i).
\]

3.2.2 Multiplication and Substitution Operators in (i) and (ii)

The odd operators $\iota(x)$ and $\epsilon(x')$ which satisfy the Clifford algebra relations
\[
\iota(x)\iota(y) + \iota(y)\iota(x) = 0
\]
\[
\epsilon(x')\epsilon(y') + \epsilon(y')\epsilon(x') = 0
\]
\[
\epsilon(x')\iota(y) + \iota(y)\epsilon(x') = x'(y) \cdot 1
\]
give rise to -often extremely trivial- BV algebra structures on the complex and/or its (co)homology. Both are square zero for starters. Let us look at $\iota(x)$ first.

In case (i), $[\iota(x), ]$ is a derivation of $\mathcal{C}_g$; $\iota(x)1 = 0$; and $[\iota(x), \epsilon(y')] \in \mathcal{C}$; hence $\iota(x)$ is a derivation of $\wedge g'$ and a (fake) BV operator.

In case (ii), all desired properties hold, including sending the creation operators ($\iota$'s) to oblivion, but now $\iota(x)1 \neq 0$. That’s where we have to pass to the cohomology to get
one of these trivial (zero) BV operators, provided that the homology $H_\ast(g, C)$ does have a supercommutative, associative algebra structure. Since $g$ is semisimple, this condition is satisfied:

$$H_\ast(g, C) = (\wedge g)^g,$$

and also

$$H_1(g, C) = g/[g, g] = 0,$$

so that

$$\iota(x)1 = 0 \quad \forall x \in g \quad \text{in} \quad H_\ast(g, C).$$

Then

$$\iota(x) \equiv 0 \quad \text{on} \quad H_\ast(g, C).$$

**Remark 3.2** The anti-ghost operator $b_0$ in a BRST complex satisfies the Cartan identity like $\iota(x)$, but it combines characteristics of $\partial$ and $\iota(x)$.

Now, about $\epsilon$'s. Case (i): $[\epsilon(x'), \cdot \cdot]$ is a derivation of $\mathfrak{C}g$, $[\epsilon(x'), \epsilon(y')] = 0$, $\epsilon(x')^2 = 0$, but $\epsilon(x')1 \neq 0$. Again for finite dimensional semisimple (even reductive) $g$, we have

$$H^\ast(g, C) = (\wedge g')^g,$$

an algebra,

$$H^1(g, C) = 0,$$

and

$$\epsilon(x')1 = 0.$$

Then

$$\epsilon(x') \equiv 0 \quad \text{on} \quad H^\ast(g, C).$$

Case (ii): $\epsilon(x')$ is similarly shown to be a derivation on $\wedge g$, with $\epsilon(x')^2 = 0$.

### 3.2.3 The Weil Algebra

The classical Weil algebra

$$W\ell = S\ell' \otimes \wedge \ell' \quad (\ell = \text{a finite dimensional Lie algebra})$$

is a well-known object in differential geometry, used for example as a model for connections and curvature, and to define equivariant forms [MQ]. A deep exposition of its algebraic properties can be found in [GHV]. The Weil algebra with one of the differentials, $\mathcal{D}$, is also an example of a classical Lie algebra cohomology complex (type (i) in Section 3.2). We will only point out that we obtain trivial BV brackets on the (cohomology) BV algebras

$$(W\ell, \mathcal{D}), \quad (W\ell, \mathfrak{h}), \quad (W\ell, \kappa), \quad (W\ell, \mathcal{D} + \mathfrak{h})$$

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where the four differentials (resp. the Lie algebra cohomology differential, the Koszul differential, a homotopy operator, and the Weil differential) are all first order differential operators (see [A2]). However, a “change of vacuum” results in the homology complex (case (ii) of 3.2)

\[(W\ell)' = S\ell \otimes \wedge \ell\]

with a reversal of multiplication and substitution operators, and normal ordering changes accordingly (annihilation operators to the right of creation operators, as usual). The $\widehat{\partial}$ homology is

\[H_*(\{W\ell\}', \widehat{\partial}) = (S\ell)^{\ell} \otimes (\wedge \ell)^{\ell}\]

for reductive $\ell$, where $\widehat{\partial}$ is now a second order differential operator giving rise to a nontrivial BV bracket in general.

### 3.2.4 The Cohomology of a Topological Chiral Algebra

The reader may want to come back to this section after reading the vertex operator version in Section 4.2.2. We will assume the definitions and properties of topological chiral algebras and BRST complexes in 4.2.2 for now, as well as

**Lemma 3.3 ([Li])** Let

\[V = \oplus_{n \geq 0} V[n]\]

be a VO(S)A with nonnegative weights only. Then $V[0]$ is a (super)commutative, associative algebra under the Wick product.

On a BRST complex $V$ with differential $Q$, we have the “semi-infinite Cartan identity”

\[Qb_0 + b_0Q = L_0,\]  

where $b_0$, $L_0$ are not residues, but coefficients of $z^{-2}$ of the vertex operators

\[b(z) = \sum b_n z^{-n-2}\] (anti-ghost)

and

\[L(z) = \sum L_n z^{-n-2}\] (stress-energy tensor; Virasoro),

written in a non-standard way when compared with (10). The operator $b_0$ plays the role of $\iota(L_0)$ and is odd, square zero, and most importantly a second order differential operator on $V$, as was shown. The cohomology of this complex, namely $\mathcal{H}^* = \text{Ker}(Q)/\text{Im}(Q)$, was first shown by Lian and Zuckerman to be a classical BV algebra with $\Delta = b_0$. We provide here a short proof.

**Proposition 3.4 ([LZ1])** The cohomology $\mathcal{H}^*$ of a BRST complex is a classical BV algebra under the induced Wick product and the operator $b_0$.  

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Proof. $H^*$ is a VOSA as $Q$ is a residue and $L(z) \in \text{Ker}(Q)$. Thanks to (17), $H^*$ is also confined to weight zero. But then $H^*$ is a VOSA with trivial $L_0$ grading, hence by Lemma 3.3 it is a supercommutative associative algebra with respect to the Wick product. The operator $b_0$, a second order differential operator on the VOSA according to Theorem 2.2, acts on $H^*$ as explained later in Proposition 4.7. $\square$

Remark 3.5 The commutative subalgebra $H^0$ is what Witten calls the ground ring of the theory [Wi2].

The cohomology of a topological chiral algebra is similarly a classical BV algebra provided that the operator $G_0$ (analogue of $b_0$) is square zero, at least in the cohomology. We will discuss conditions on $G(z)$ in 4.2.2.

We will see that at least algebraically it is more natural to consider the whole BRST complex as a “vertex operator BV algebra”, and to obtain $(H^*, b_0)$ as a classical subquotient of $(V, b_0)$. Note that in Section 4.2.2 we denote a BRST complex by $V \otimes \wedge^*$ and not by $V$. In both [LZ1] and [PS] there are indications that “higher brackets” on the original complex would be interesting.

3.2.5 The Schouten-Nijenhuis Bracket on Skew Multivector Fields

The Schouten-Nijenhuis bracket $[,]_{SN}$ ([Sn, Ni, Ko, Mi2]) is a differential invariant (concomitant) of skew multivector fields which generalizes the Lie bracket of two vector fields. The bracket satisfies skew-symmetry as well as the Jacobi and Poisson identities (with respect to the properly shifted gradings) and makes

$$A(M) = \text{Sec}(\wedge TM) = \wedge_{C^\infty(M)}\text{Sec}(TM)$$

into a Gerstenhaber algebra ($M$ is an $n$-dimensional paracompact differentiable manifold). The definition is given by

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]_{SN} = \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q \text{ for } p, q \geq 1$$

and

$$[f, u]_{SN} = \iota(df) u \text{ for } f \in A^0(M) = C^\infty(M), \quad u \in A(M).$$

Koszul shows that $(A(M), [\cdot, \cdot]_{SN})$ can moreover be made into a BV algebra:

**Proposition 3.6 ([Ko])** For every torsion free linear connection $\nabla$ in $TM$, there exists a unique differential operator $D_{\nabla}$ which generates $[,]_{SN}$ via $\Phi^2_{D_{\nabla}}$, such that

$$D_{\nabla}(X) = -\text{div}_{\nabla}(X) = -\text{Tr}(Y \mapsto [X, Y] - \nabla_X(Y)).$$
This map is the composition of two maps: One sending $\nabla$ to the induced connection in $\wedge^n T M$, and the other the canonical bijection of the affine space of linear connections in $\wedge^n T M$ onto the set of differential operators which generate the Schouten-Nijenhuis bracket. In this bijection the images of connections of zero curvature in $\wedge^n T M$ are square zero operators.

We will omit the details of the proof; however, we will give a local expression for $D_\nabla$ (on an open set $U \subset M$):

$$D_U = -\sum_{i=1}^n \iota(\alpha_i) \nabla_{X_i},$$

where $X_1,\ldots,X_n$ is a basis for the module $A^1(U)$ of vector fields on $U$, $\alpha_1,\ldots,\alpha_n$ is the dual basis of 1-forms, and $\iota(\alpha_i)$ are “substitution operators”, or “interior products” (analogues of the substitution operators $\iota(X)$ acting on differential forms, i.e. on $\text{Sec}(\wedge^* T^* M)$). We obtain a differential operator $D_\nabla$ of degree −1 and order ≤ 2 by patching. There are two obvious examples where $D_\nabla$ is square zero. One occurs when $\nabla$ is the Levi-Civita connection associated to a Riemannian metric on $M$. The second one is the case of an orientable manifold $M$ with a nonvanishing section $v$ of $\wedge^n T M$, which then determines a flat connection in $\wedge^n T M$ whose image is a square zero operator generating the Schouten-Nijenhuis bracket.

It would be of interest to know which bilinear differential concomitants (such as the brackets studied by Frölicher, Nijenhuis, Richardson, and Schouten, and the generalized Poisson bracket mentioned at the end of Section 4.1) are BV brackets, even though the differential forms under consideration may have other than a supercommutative, associative algebra structure. We recommend [KMS] for background and references.

4 Generalized BV Algebras

4.1 Definitions and Properties

The existence of higher order differential operators and higher brackets on a VOSA $V$ leads naturally to the concept of a vertex operator Batalin-Vilkovisky algebra. Even more generally,

**Definition 4.1** A generalized Batalin-Vilkovisky algebra (GBVA) is a superalgebra $\mathcal{A}$ (possibly with identity, $1_\mathcal{A}$) with an odd, square zero, second order differential operator $\Delta$ (which annihilates the identity, if any). The generalized BV bracket $\{\ , \}$ is given by

$$\{a, b\} = (-1)^{|a|} \Phi^2_\Delta(a, b)$$

(18)

where

$$\Phi^2_\Delta(a, b) = \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b).$$
We will later specialize to the case of

**Definition 4.2** A vertex operator Batalin-Vilkovisky algebra (VOBVA) is a GBVA \((\mathcal{A}, 1_\mathcal{A}, \Delta)\) where \(\mathcal{A}\) is a VOSA with the Wick product, \(1_\mathcal{A}\) is the vacuum \(1\), and \(\Delta\) is an odd, square zero mode \(u_1\).

**Remark 4.3** A supercommutative associative algebra is a trivial example of a VOBVA, since it is already a VOSA, as explained in [LZ2].

Some of our examples will be differential BV algebras:

**Definition 4.4** A differential Batalin-Vilkovisky algebra (DBVA) \((\mathcal{A}, 1_\mathcal{A}, \Delta, D, L)\) consists of

(i) a superalgebra \(\mathcal{A}\) with identity \(1_\mathcal{A}\),

(ii) an odd, square zero, second order differential operator \(\Delta \ (\Delta 1_\mathcal{A} = 0)\) making \((\mathcal{A}, \Delta)\) into a generalized BV algebra,

(iii) an odd, square zero operator \(D \ (D 1_\mathcal{A} = 0)\) and a diagonalizable operator \(L\) on \(\mathcal{A}\) such that

\[ [D, \Delta] = D\Delta + \Delta D = L, \quad (19) \]

and such that the cohomology \(H = \text{Ker}(D)/\text{Im}(D)\) of the complex \((\mathcal{A}, D)\) is also a GBVA, with the induced product and BV operator \(\Delta\). There are no restrictions on the eigenvalues of \(L\), which we’ll call *weights*. In most examples \(D\) and \(\Delta\) have zero weight as operators, but we’ll assume only that they are homogeneous (with opposite weights if \(L_0 \neq 0\)).

**Remark 4.5** (i) One usually thinks of \(L\) as the Virasoro operator \(L_0\), but sometimes we’ll just take \(L \equiv 0\).

(ii) After finishing the preliminary manuscript I became aware of Getzler’s paper [Get2] in which he defines a differential BV algebra in a similar fashion. In fact, I was persuaded by Jim Stasheff that this is a better name than “cohomology BV algebra”!

**Lemma 4.6** In a VOSA \(V\), \(L_0 - (n + 1) \cdot id\) is a derivation of the product \(\times_n\) defined in Eq. (11). In particular, \(L_0\) is a derivation with respect to the Wick product.

**Proof.**

If \(a(z) = \sum a_n z^{-n-1}\) is an \(L_0\)-homogeneous element with \(L_0(a) = \text{wt}_a \cdot a\), then it follows from the axioms of a VOSA that

\[ [L_0, a_n] = (\text{wt}_a - n - 1) \cdot a_n \quad ([\text{FHL}]). \]

Then for homogeneous \(a, b\),

\[ (L_0 - (n + 1) \cdot id)(a_n(b)) = L_0(a_n(b)) - (n + 1)a_n(b) \]
\[ = (L_0a_n)(b) - (n + 1)a_n(b) \]

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\[(a_nL_0)(b) + [L_0, a_n](b) - (n + 1)a_n(b) = a_n(L_0(b)) + (\text{wt}_a - n - 1)a_n(b) - (n + 1)a_n(b) = (\text{wt}_b + \text{wt}_a - n - 1 - n - 1)a_n(b) = (\text{wt}_a + \text{wt}_b - 2n - 2)a_n(b)\]

and

\[(L_0 - (n + 1) \cdot \text{id})(a), (b) + a_n((L_0 - (n + 1) \cdot \text{id})(b)) = (\text{wt}_a - n - 1)a_n(b) + (\text{wt}_b - n - 1)a_n(b) = (\text{wt}_a + \text{wt}_b - 2n - 2)a_n(b).\]

Extend to \(V\) by bilinearity. □

**Proposition 4.7** In a DBVA we have

\[[L, D] = [L, \Delta] = 0,\]

and \(\Delta\) descends to the cohomology \(\mathcal{H}\), which has zero \(L\)-weight.

**Proof.**

\[[L, D] = [\Delta D + D\Delta, D] = \Delta D^2 + D\Delta D - D\Delta D - D^2 \Delta = 0.\]

Similarly \([L, \Delta] = 0\). Then both \(D\) and \(\Delta\) preserve eigenspaces of \(L\), and anticommute on the kernel. Moreover, by virtue of (19) the cohomology can only be realized in weight zero: if \(a\) is a homogeneous element with \(L(a) = \lambda a\) and \(D(a) = 0\), then

\[D(\Delta(a)) + \Delta(D(a)) = L(a) \Rightarrow D(\Delta(a)) = \lambda a.\]

If \(\lambda \neq 0\), we have \(a \in \text{Im}(D)\) and \(a = 0\) in \(\mathcal{H}\). □

Let us discuss the properties of the generalized BV bracket. Parts of the following computations have appeared in other forms in [Get1], [Ko], [LZ1], [PS]. We define a new “Lie” bracket on \(\mathcal{A}\) which is bilinear and skew-symmetric: Let

\[[a, b] = ab - (-1)^{|a||b|}ba\]

for homogeneous \(a, b\). Let \(\Phi^r_\Delta\) denote the usual \(r\)-form and \(\tilde{\Phi}^r_\Delta\) the one defined with respect to the bracket above. This bilinear product was also mentioned in [LZ1]. Note that if \(\mathcal{A}\) is supercommutative and associative then \([\ ,\ ]\) is trivial, and so are all \(\tilde{\Phi}^r_\Delta\). Finally, recall that \(\hat{\mathcal{A}} = \oplus_j \mathcal{A}_j^{j+1}\).

**Proposition 4.8** Let \((\mathcal{A}, \Delta)\) be a generalized BV algebra. The generalized BV bracket satisfies
(i) Modified skew-symmetry in \((\hat{A}, \{, \})\):
\[
\{a, b\} + (-1)^{|a|+1} |b|+1 \{b, a\} = (-1)^{|a|} \tilde{\Phi}_2^3(a, b).
\]

(ii) Leibniz rule (as opposed to the cyclically symmetric version of the Jacobi identity) in \((\hat{A}, \{, \})\):
\[
\{x, \{a, b\}\} = \{\{x, a\}, b\} + (-1)^{|x|+1} |a|+1 \{a, \{x, b\}\}.
\]

(iii) Poisson rule (superderivation property of \(\{a, \}\) on \(A\)):
\[
\{a, bc\} - \{a, b\}c - (-1)^{|a|+|b|} b\{a, c\} = (-1)^{|a|} \tilde{\Phi}_3^3(a, b, c) = 0.
\]

(iv) Derivation rule for \(\Delta\) on \((\hat{A}, \{, \})\):
\[
\Delta(\{a, b\}) - \{\Delta(a), b\} - (-1)^{|a|+1} \{a, \Delta(b)\} = 0.
\]

Proposition 4.9 If \((A, 1_A, \Delta, D, L)\) is a differential BV algebra where both \(D\) and \(L\) are derivations of \(A\), then just like \(\Delta\), the operator \(D\) is a derivation of \((\hat{A}, \{, \})\). Namely,
\[
D(\{a, b\}) = \{D(a), b\} + (-1)^{|a|+1} \{a, D(b)\}.
\]

Remark 4.10 This result was proven in [LZ1] in the special case of a BRST complex with \(\Delta = b_0\) \((b(z) = \sum b_n z^{-n-2})\) and \(D = Q = \text{BRST operator}\).

Remark 4.11 A variant of property (i) in Proposition 4.8 was written in [LZ1] in the form
\[
\{u, v\} + (-1)^{|u|+1} |v|+1 \{v, u\} = (-1)^{|u|-1} (Qm'(u, v) - m'(Qu, v) - (-1)^{|u|} m'(u, Qu))
\]
(Eq. (2.22)) where \(m'\) is a bilinear operation. Since \(Q\) is a derivation of the Wick product, we are justified in calling (20) “skew-symmetry up to homotopy”. But identity (i) is valid in any GBVA, and we need not have the additional DBVA structure. Although \(\Delta\) is a derivation of the bracket it defines, we will not go so far as to say GBVA’s are homotopy BV algebras (HBVA), because \(\Delta\) is not a derivation of the original product on \(A\); but a DBVA is a step towards an HBVA. One could make up several definitions for an HBVA depending on which identities (including supercommutativity, associativity, and \(\Delta^2 = 0\)) hold on the nose and which up to homotopy and higher homotopies. As was noted in [LZ1], algebraic identities holding only up to homotopy are discussed in Stasheff’s work [St1]. Also see [LS], [LZ1], [Get1], and [GV].
Lie algebras and generalized BV algebras (with the generalized BV bracket and shifted grading) are two large classes of examples of what J.-L. Loday calls Leibniz algebras:

**Definition 4.12 [Lo]** A Leibniz algebra \( \mathcal{A} = \bigoplus_j \mathcal{A}_j \) is a vector space with super \( \mathbb{Z} \)-grading and a bilinear superbracket
\[
[\ ,\ ] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}
\]
satisfying
\[
[x, [y, z]] = [[x, y], z] + (-1)^{|x|y}|[y, [x, z]]
\]
for all homogeneous \( x, y \).

Then a Leibniz algebra module \( M \) is just a Lie algebra module over the Lie algebra \( \bar{\mathcal{A}} \), which is the quotient of \( \mathcal{A} \) by the two-sided ideal generated by all \([x, x]\), \( x \in \mathcal{A} \). The classical (Chevalley-Eilenberg) homology complex can now be replaced by the complex
\[
(T\mathcal{A} \otimes M, \partial)
\]
where \( T\mathcal{A} \) is the tensor algebra on \( \mathcal{A} \) and \( \partial \) is an operator which has almost the same formula as the classical one. Showing \( \partial \) is square zero requires nothing more than the Leibniz property of the bracket in \( \mathcal{A} \). This variant of Lie algebra (co)homology is explained in [Lo].

For an interesting Leibniz algebra which is not a Lie algebra, see [Mil]. Michor defines an odd generalization \( \{\ ,\ \}^2 \) of the Poisson bracket to \( \Omega^*(M) \) from \( \Omega^0(M) = C^\infty(M) \), where \( M \) is a symplectic manifold. The new bracket satisfies the Leibniz identity, but not skew-symmetry (Lemma 4.3). One has to take the quotient of \( \Omega^*(M) \) by the subspace \( B^*(M) \) of exact forms in order to restore skew-symmetry.

Here are the proofs of the two main propositions.

**Proof of Proposition 4.8**

(i) \[ a, b \] + (-1)^{(|a|+1)(|b|+1)}\{b, a\}
\[
= (-1)^{|a|} \Phi_\Delta^2 (a, b) + (-1)^{|a||b|+|a|+1} \Phi_\Delta^2 (b, a)
\]
\[
= (-1)^{|a|}(\Delta(ab) - \Delta(a)b - (-1)^{|a|a}\Delta(b)) + (-1)^{|a||b|+|a|+1}(\Delta(ba) - \Delta(b)a - (-1)^{|b|b}\Delta(a))
\]
\[
= (-1)^{|a|}\Delta(ab) - (-1)^{|a||b|}ba - (-1)^{|a|}(\Delta(a)b - (-1)^{|b|(|a|+1)b}\Delta(a))
\]
\[
-(-1)^{|a|}(-1)^{|a|}(a\Delta(b) - (-1)^{|a|(|b|+1)\Delta(b)a})
\]
\[
= (-1)^{|a|}(\Delta([a, b]) - [\Delta(a), b] - (-1)^{|a|[a, \Delta(b)]})
\]
\[
= (-1)^{|a|} \Phi_\Delta^2 (a, b).
\]

(iii) \(-1)^{|a|}\{a, bc\} - \{a, b\}c - (-1)^{|b|(|a|+1)b \{a, c\})
\[= \Phi^2_\Delta(a, bc) - \Phi^3_\Delta(a, b)c - (-1)^{|b|(|a|+1)}b\Phi^2_\Delta(a, c)\]
\[= \Phi^3_\Delta(a, b, c) \equiv 0.\]

(iv) \((-1)^{|a|}\Delta(\{a, b\}) - \{\Delta(a), b\} - (-1)^{|a|+1}\{a, \Delta(b)\}\]
\[= \Delta(\Phi^2_\Delta(a, b)) + \Phi^2_\Delta(\Delta(a), b) + (-1)^{|a|}\Phi^2_\Delta(a, \Delta(b))\]
\[= \Delta(\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)) + \Delta(\Delta(a)b) - \Delta(\Delta(a))b - (-1)^{|a|+1}\Delta(a)\Delta(b)\]
\[+ (-1)^{|a|}\Delta(a\Delta(b)) - (-1)^{|a|}\Delta(a)\Delta(b) - a\Delta(\Delta(b))\]
\[= 0.\]

(ii) We will make use of

(A) \(\{x, ab\} - \{x, a\}b - (-1)^{|x|+1}a\{x, b\} = 0.\)

(B) \(\{x, \Delta(a)b\} - \{x, \Delta(a)\}b - (-1)^{|x|+1}(|a|+1)\Delta(a)\{x, b\} = 0.\)

(C) \(\{x, a\Delta(b)\} - \{x, a\}\Delta(b) - (-1)^{|x|+1}a\{x, \Delta(b)\} = 0.\)

(D) \(\Delta(\{x, a\}) - \{\Delta(x), a\} - (-1)^{|x|+1}\{x, \Delta(a)\} = 0.\)

(E) \(\Delta(\{x, b\}) - \{\Delta(x), b\} - (-1)^{|x|+1}\{x, \Delta(b)\} = 0.\)

(F) \(\Delta(\{x, ab\}) = \{\Delta(x), ab\} + (-1)^{|x|+1}\{x, \Delta(ab)\}\]
\[= \{\Delta(x), a\}b + (-1)^{|a||x|}a\{\Delta(x), b\} + (-1)^{|x|+1}\{x, \Delta(ab)\}.\]

Now:
\[\{\{x, a\}, b\} + (-1)^{|x|+1}(|a|+1)\{x, a\}\{x, b\}\]
\[= (-1)^{|a|+|x|+1}\Phi^2_\Delta(\{x, a\}, b) + (-1)^{|a||x|+|x|+1}\Phi^2_\Delta(\{x, b\})\]
\[= (-1)^{|a|+|x|+1}\Delta(\{x, a\}b) - \Delta(\{x, a\})b - (-1)^{|a|+|x|+1}\{x, a\}\Delta(b)\]
\[+ (-1)^{|a||x|+|x|+1}\Delta(a\{x, b\}) - \Delta(a\{x, b\}) - (-1)^{|a|}\Delta(\{x, b\})\]
\[= (-1)^{|a|+|x|+1}\Delta(\{x, a\}b) + (-1)^{|a|+|x|}\Delta(\{x, a\})b - \{x, a\}\Delta(b)\]
\[+ (-1)^{|a||x|+|x|+1}\Delta(a\{x, b\}) + (-1)^{|a||x|+|x|+1}\Delta(a\{x, b\}) + (-1)^{|a||x|+|x|+1}\Delta(a\{x, b\})\]
\[= (-1)^{|a|+|x|+1}\Delta(\{x, ab\} + (-1)^{|a||x|+|a|+1}a\{x, b\}) \text{ by (A)}\]
\[+ (-1)^{|a|+|x|}(\{\Delta(x), a\} + (-1)^{|x|+1}\{x, \Delta(a)\})b \text{ by (D)}\]
\[\{x, a\}\Delta(b) + (-1)^{|a||x|+|x|+1}\Delta(a\{x, b\}) + (-1)^{|a||x|+|x|}\Delta(a\{x, b\})\]
\[+ (-1)^{|a||x|+|a|+|x|}a(\{\Delta(x), b\} + (-1)^{|x|+1}\{x, \Delta(b)\}) \text{ by (E)}.\]
\[
= (-1)^{|a|+|x|+1} \Delta \{x, ab\} + (-1)^{|a|+|x|} \{\Delta(x), a\} b + (-1)^{|a|+1} \{x, \Delta(a)\} b - \{x, a\} \Delta(b)
\]
\[
+(-1)^{|a|+|x|+|z|} \Delta(a) \{x, b\} + (-1)^{|a||x|+|a|+|x|} a \{\Delta(x), b\} + (-1)^{|a||x|+|a|+1} a \{x, \Delta(b)\}
\]
\[
= (-1)^{|a|+|x|+1} \Delta \{x, ab\} + (-1)^{|a|+|x|} \{\Delta(x), a\} b
\]
\[
+(((-1)^{|a|+1} \{x, \Delta(a)\} b + (-1)^{|a||x|+|a|} \Delta(a) \{x, b\})
\]
\[
+(-\{x, a\} \Delta(b) + (-1)^{|a||x|+|a|+1} a \{x, \Delta(b)\}) + (-1)^{|a||x|+|a|+1} a \{\Delta(x), b\}
\]
\[
= (-1)^{|a|+|x|+1} \Delta \{x, ab\} + (-1)^{|a|+|x|} \{\Delta(x), a\} b + (-1)^{|a||x|+|a|+|x|} a \{\Delta(x), b\}
\]
\[
+(-1)^{|a|+1} \{x, \Delta(a)\} b \text{ by (B)}
\]
\[
-\{x, a\} \Delta(b) \text{ by (C)}
\]
\[
= (-1)^{|a|+|x|} (-\Delta \{x, ab\} + \{\Delta(x), a\} b + (-1)^{|a|} a \{\Delta(x), b\})
\]
\[
+(-1)^{|a|+1} \{x, \Delta(a)\} b - \{x, a\} \Delta(b)
\]
\[
= (-1)^{|a|} \{x, \Delta(ab)\} \text{ by (F)}
\]
\[
+(-1)^{|a|+1} \{x, \Delta(a)\} b - \{x, a\} \Delta(b)
\]
\[
= (-1)^{|a|} \{x, \Delta(ab) - \Delta(a)b - (-1)^{|a|} a\} \Delta(b)\}
\]
\[
= \{x, (-1)^{|a|} \Phi^2_{2a}(a, b)\}
\]
\[
= \{x, \{a, b\}\}. \quad \square
\]

**Proof of Proposition 4.9.**

\[
D(\{a, b\}) - \{D(a), b\} - (-1)^{|a|+1} \{a, D(b)\}
\]
\[
= (-1)^{|a|} D(\Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b))
\]
\[
+(-1)^{|a|} (\Delta(D(a)b) - (\Delta D)(a)b + (-1)^{|a|} D(a)\Delta(b))
\]
\[
+\Delta(aD(b)) - \Delta(a)D(b) + (-1)^{|a|} a(\Delta D)(b)
\]
\[
= (-1)^{|a|} (D\Delta)(ab) - (-1)^{|a|} (D\Delta)(a)b + \Delta(a)D(b)
\]
\[
-\Delta(a)D(b) - (-1)^{|a|} a(\Delta D)(b) + (-1)^{|a|} \Delta(D(a)b)
\]
\[
-(-1)^{|a|}(\Delta D)(a)b + D(a)\Delta(b) + \Delta(aD(b))
\]
\[
-\Delta(a)D(b) - (-1)^{|a|} a(\Delta D)(b)
\]

(3rd, 4th, 8th, 10th terms cancel out. Expand 1st term, join 2nd and 7th, also 5th and 11th, and 6th and 9th.)

\[
= -(-1)^{|a|} \Delta(D(ab)) + (-1)^{|a|} L(ab) - (-1)^{|a|} (D\Delta + \Delta D)(a)b
\]
\[
-(-1)^{|a|} a(D\Delta + \Delta D)(b) + (-1)^{|a|} \Delta(D(a)b) + (-1)^{|a|} aD(b))
\]

(Join 1st and 5th, also 2nd, 3rd, and 4th terms.)

\[
= -(-1)^{|a|} \Delta(D(ab) - D(a)b - (-1)^{|a|} aD(b)) + (-1)^{|a|}(L(ab) - L(a)b - aL(b))
\]
= -(-1)^{|a|} \Delta (\Phi_D^2(a, b)) + (-1)^{|a|}\Phi_L^2(a, b) = 0. \quad \Box

Although the level of generalization in this section seems adequate for most examples, it is instructive to study the bracket \{a, b\} on a superalgebra \(A\) with a plain odd operator \(\Delta\), not necessarily square zero, or of order two. We find that Proposition 4.8 is replaced by

**Proposition 4.13** For a superalgebra \(A\) and an odd operator \(\Delta\), the bracket \{ , \} defined by (18) satisfies the following identities:

(i) Modified skew-symmetry:

\[
\{a, b\} + (-1)^{(|a|+1)(|b|+1)}\{b, a\} = (-1)^{|a|}\Phi_D^2(a, b).
\]

(ii) Modified Leibniz rule:

\[
\{\{x, a\}, b\} + (-1)^{(|x|+1)(|a|+1)}\{a, \{x, b\}\} - \{x, \{a, b\}\} = (-1)^{|a|}(\Delta(\Phi_\Delta^3(x, a, b)) - \Phi_\Delta^3(x, a, b) + \Phi_\Delta^3(\Delta(x), a, b) \]
\[
+ (-1)^{|x|}\Phi_\Delta^3(x, \Delta(a), b) + (-1)^{|x|+|a|}\Phi_\Delta^3(x, a, \Delta(b))).
\]

(iii) Modified Poisson rule:

\[
\{a, bc\} - \{a, b\}c - (-1)^{(|a|+1)|b|}b\{a, c\} = (-1)^{|a|}\Phi_\Delta^3(a, b, c).
\]

(iv) Modified derivation rule for \(\Delta\):

\[
\Delta(\{a, b\}) - \{\Delta(a), b\} - (-1)^{|a|+1}\{a, \Delta(b)\} = (-1)^{|a|}\Phi_\Delta^2(a, b).
\]

It is clear that we obtain a classical BV algebra when \(A\) is supercommutative, \(\Delta\) is of order two, and \(\Delta^2\) is zero (or of order one). The associativity condition on classical BV algebras turns out to be superfluous. The proof of Proposition 4.13 can be obtained from that of Proposition 4.8 mostly by retaining terms (such as those containing \(\Delta^2\) and \(\Phi_\Delta^3\)) which were formerly discarded. Part (ii) has a strong resemblance to Lemma 1.5 in [Ko], and (iv) is exactly his Equation (1.8).

If we now add an odd, square zero, first order differential operator \(D\) and a diagonalizable, first order differential operator \(L\) to this picture, such that \(D\Delta + \Delta D = L\) as before, Proposition 4.9 remains unchanged. The result is already in terms of \(\Delta(\Phi_D^2(a, b))\) and \(\Phi_L^2(a, b)\), allowing more general operators. The fact that \(D^2 = 0\) is not used at all. The need for unadorned algebras and operators will become clear in Section 4.2.2, where we introduce topological chiral algebras.

### 4.2 Examples of Vertex Operator BV Algebras

#### 4.2.1 The Vertex Operator Weil Algebra
The semi-infinite Weil complex $W^{\infty/2}g$ associated to a tame Lie algebra $g$ (i.e. $g = \oplus g_n$, $[g_m, g_n] \subset g_{m+n}$, $\dim g_n < \infty$) was first considered by Feigin and Frenkel in [FF]. They in particular computed the semi-infinite cohomology for $g = \text{Witt}$. Next, the vertex operator Weil algebra (VOWA) $W^{\infty/2}g$ on a loop algebra

$$W^{\infty/2}g = S^{\infty/2}g \otimes \mathbb{C} \wedge^{\infty/2}g$$

with bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n}$$

was studied by the author in [A2, A3]. We refer the reader to [A3] for details.

The VOWA is graded by the eigenvalues 0, 1, 2, ... of $L_0$, and $W^{\infty/2}g[0]$ is exactly the classical Weil algebra

$$W\ell = S\ell \otimes \wedge\ell$$

considered in Section 3.2.3. Most semi-infinite operators restrict to the classical subspace. In [A3] it was noted that $W^{\infty/2}g$ is a topological chiral algebra (see Section 4.2.2) because of an identity

$$hk + kh = -L_0$$

involving the semi-infinite Koszul differential, a new semi-infinite homotopy operator, and the Virasoro operator $L_0$. If we don’t mind taking $L = 0$ in (19), we have several VOBVA and DBVA structures on $A = W^{\infty/2}g$ ($Q$ is the semi-infinite cohomology operator):

(i) $(A, \Delta, D, L) = (W^{\infty/2}g, k, h, -L_0), \quad (H, \Delta) = (C, 0)$

(ii) $(A, \Delta, D, L) = (W^{\infty/2}g, Q, h, 0), \quad (H, \Delta) = (C, 0)$

(iii) $(A, \Delta, D, L) = (W^{\infty/2}g, h, Q, 0), \quad (H, \Delta) = (H(W^{\infty/2}g, Q), h)$

(iv) $(A, \Delta, D, L) = (W^{\infty/2}g, k, Q, 0), \quad (H, \Delta) = (H(W^{\infty/2}g, Q), k)$.

Now, the space $H = H(W^{\infty/2}g, Q)$ is not a supercommutative, associative algebra, but it contains the classical space $H[0] = H(\ell, d)$, which is. As was shown in [A2], $H$ is infinitely richer than $H(\ell, d)$. The operators $h$ and $k$ induce nontrivial first and second order differential operators on $H$, and case (iv) even has a (possibly) nontrivial BV bracket on the cohomology. For reductive $\ell$, two more vertex operators and their weight zero modes $r, t$ can be exploited [A2, A3]. These additional operators also satisfy

$$rt + tr = -L_0.$$
(v) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, t, r, -L_0), \quad (\mathcal{H}, \Delta) = (H(W\ell, r), t)\)

(vi) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, Q, r, 0), \quad (\mathcal{H}, \Delta) = (H(W\ell, r), Q)\)

(vii) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, r, Q, 0), \quad (\mathcal{H}, \Delta) = (H(W^{\infty/2} \tilde{\ell}, Q), r)\)

(viii) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, t, Q, 0), \quad (\mathcal{H}, \Delta) = (H(W^{\infty/2} \tilde{\ell}, Q), t)\)

(once again, (v) and (viii) have a potentially nonzero BV bracket as \(t\) is a genuine second order operator) and also

(ix) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, r, h, 0), \quad (\mathcal{H}, \Delta) = (C, 0)\)

(x) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, t, h, 0), \quad (\mathcal{H}, \Delta) = (C, 0)\)

(xi) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, h, r, 0), \quad (\mathcal{H}, \Delta) = (H(W\ell, r), h)\)

(xii) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, k, r, 0), \quad (\mathcal{H}, \Delta) = (H(W\ell, r), k)\).

While \(h\) has a classical analogue, \(k, r,\) and \(t\) don’t, and as noted above, \(k\) and \(t\) may give rise to nonzero BV brackets in the cohomology. So far we have been using a restricted version of Definition 4.1 where the operator \(D\) is a derivation, which is very common. Relaxing this condition (say, allowing \(D\) to be a second order differential operator), we would get several additional examples. One of the DBVA’s we can obtain from \(W^{\infty/2} \tilde{\ell}\) in this fashion is

(xiii) \((A, \Delta, D, L) = (W^{\infty/2} \tilde{\ell}, Q, k, 0), \quad (\mathcal{H}, \Delta) = (W\ell, d),\)

so that the classical Weil algebra of 3.2.3 is the cohomology of a BV algebra (with trivial BV bracket).

**4.2.2 Topological Chiral Algebras and the BRST Complex**

We take the following definition from [LZ1]. A topological chiral algebra (TCA) consists of

(i) A VOSA,

(ii) A weight one even field \(F(z) = \sum F_n z^{-n-1}\) whose residue (charge) \(F_0\) is the “fermion number operator”, or “ghost number operator”,

(iii) A weight one primary (Virasoro-singular) field \(J(z) = \sum J_n z^{-n-1}\) with fermion number one and a square zero charge \(Q = J_0\),

(iv) A weight two primary field \(G(z) = \sum G_n z^{-n-2}\) with fermion number \(-1\), satisfying

\[
[Q, G(z)] = L(z), \quad (21)
\]
where $L(z) = \sum L_n z^{-n-2}$ is the “stress-energy (Virasoro) field”. By (21), the Virasoro algebra acts trivially on the cohomology, hence the name “topological”.

The above definition is tailored for the BRST complex

$$V \otimes \wedge^*$$

where $V$ is any VOA with central charge 26 and $\wedge^*$ is the “ghost system”, i.e. a simple $bc$-system generated by two fields

$$b(z) = \sum b_n z^{-n-2}, \quad c(z) = \sum c_n z^{-n+1}$$

with stress-energy field

$$L^\wedge(z) = - : (d/dz) b(z) c(z) : -2 : b(z) d/dz c(z) : .$$

From the viewpoint of semi-infinite cohomology, (22) is a cohomology complex where

$$\wedge^* = \wedge^{\infty/2+*}(\text{Witt})' ,$$

the Witt algebra being the quotient of Vir by its center. The central element of Vir acts by $-26$ on (23), so in order to have a closed action of Witt and a square zero differential we have to tensor with a VOA of central charge 26 (“cancellation of anomalies”). The choice of vacuum is given by

$$b_n 1 = c_m 1 = 0 \quad \text{for } n \geq -1, m \geq 2,$$

because (23) is a VOSA only in this vacuum. The Virasoro (Witt) action is given by

$$\rho(z) = L^\wedge(z) \quad \text{on } \wedge^*$$

and

$$\theta(z) = L(z) = L^V(z) + L^\wedge(z) \quad \text{on } V \otimes \wedge^* .$$

We also have

$$\iota(L_n) = b_n = G_n, \quad \epsilon(L'_n) = c_{-n}, \quad d = Q = J_0, \quad J(z) = : c(z)(L^V(z) + \frac{1}{2} L^\wedge(z)) : ,$$

so that (24) is just the Cartan identity in disguise. The same equation tells us that the vertex operator $L(z)$ is $Q$-exact (not true, for example, for the VOWA). Finally,

$$F(z) = : c(z) b(z) : .$$

The vertex operator Weil algebra is also a TCA and a DBVA with

$$[h, k(z)] = - L(z)$$

26
and

\[ F(z) = \sum_u : c'(z) b^u(z) : \]

(see [A3]). If \( \ell \) is semisimple, it is also true that

\[ [r, t(z)] = -L(z). \]

Topological chiral algebras where \( G^2_0 = 0 \) are both VOBVA’s and DBVA’s, with

\[ \Delta = Q, \quad D = G_0, \quad L = L_0 \]  \hspace{1cm} (24)

or

\[ \Delta = G_0, \quad D = Q, \quad L = L_0. \]

In the case of a BRST complex, (24) yields zero cohomology for \( D = b_0 \) (because \( b_0 c_0 + c_0 b_0 = 1 \)), and in any case the BV bracket on the complex is trivial. The second choice is the DBVA which was implicit in [LZ1].

There seems to be a demand for TCA’s without the condition \( G^2_0 = 0 \) as well, as was explained to the author by José M. Figueroa-O’Farrill after the first preprint of this article appeared (see [Ka, Get2, IR, Fi, FS]). It is once again desirable to have a classical BV algebra as the \( Q \)-cohomology. Even without the “primary” condition on \( G(z) \) (see (21)), one can show that Proposition 4.7 still holds for a TCA: We have

\[ [L_0, Q] = [L_0, G_0] = 0 \]

because \( Q \) and \( G_0 \) are weight zero modes, and all the arguments in the proof are valid. Then the cohomology \( \mathcal{H} \), being a VOSA of weight zero, is a supercommutative, associative algebra with a second order, odd differential operator \( G_0 \). The only remaining issue is to impose minimal conditions on \( G(z) \) so that \( G^2_0 = 0 \) in \( \mathcal{H} \). In [Ka] and [Get2] we find Kazama algebras in which \( G_0 \) satisfies an identity

\[ G^2_0 = [Q, P] \]

where \( P \) is an odd, third order operator, making \( G^2_0 = 0 \) in \( \mathcal{H} \). We observe that the above condition will be satisfied if \( G(z) \) is primary:

**Proposition 4.14** Let \( V \) be a topological chiral algebra with a primary field \( G(z) \). Then \( \mathcal{H} = H^*(V, Q) \) is a classical BV algebra.

**Proof.** Since \( G(z) \) is a primary field of weight two, we have the identity

\[ [L_m, G_n] = (m - n) G_{m+n} \]  \hspace{1cm} (25)
(see [FLM]), in particular
\[ [L_{-2}, G_{-2}] = 0. \]
Then the field \( G(z)G(z) \), or the associated element \((G_{-2})^21\), is a cocycle:
\[
Q (G_{-2})^21 = [Q, G_{-2}]G_{-2}1 - G_{-2}[Q, G_{-2}]1 = L_{-2}G_{-2}1 - G_{-2}L_{-2}1 = 0.
\]
Also, since \( Q \) commutes with all \( L_n \), we have
\[
Q L_3(G_{-2})^21 = 0.
\]
But then by the identity
\[
QG_0 + G_0Q = L_0,
\]
both \((G_{-2})^21\) and \(L_3(G_{-2})^21\) (of weights 4 and 1 respectively) are coboundaries. Recall that since \( Q \) is a residue, every coboundary vertex operator \( Q R(z) \) is a \( Q \)-commutator, namely
\[
Q R(z) = [Q, R(z)] = \sum [Q, R_{(n)}]z^{-n-1}.
\]
The proof will be complete when we show that \( G_0^2 \) is a linear combination of the weight zero modes of the fields corresponding to \((G_{-2})^21\) and \(L_3(G_{-2})^21\). By identity (1) we find that the weight zero mode of \( G(z)G(z) \) is
\[
(G_{-2}G)_0 = (G_{-1})_{(3)} = \sum_{i \geq 0} (-1)^i(-1)^i(G_{(-1-i)}G_{(3+i)} - G_{(2-i)}G_{(i)})
\]
\[
= -G_{(1)}G_{(1)} - G_{(2)}G_{(0)} - G_{(0)}G_{(2)} = -G_0^2 - [G_1, G_{-1}],
\]
where we translate between the two conventions in subscripts by
\[
R(z) = \sum R_nz^{-n-\text{wt}(R)} = \sum R_{(n)}z^{-n-1}.
\]
Next, we observe that
\[
L_3(G_{-2})^21 = [L_3, G_{-2}]G_{-2}1 + G_{-2}[L_3, G_{-2}]1 = 5G_1G_{-2}1 + 5G_{-2}G_11 = 5G_1G_{-2}1,
\]
by (23) and the fact that coefficients of \( z^n \) for \( n < 0 \) annihilate the vacuum. We calculate the zero mode of \( \frac{1}{5}L_3(G_{-2})^21 \) as follows:
\[
(G_1G)_0 = (G_{(2)}G)_{(0)} = 2(-2G_{(1)}^2 + [G_{(2)}, G_{(0)}]) = 2(-2G_0^2 + [G_1, G_{-1}]),
\]
and \( G_0^2 \) is indeed a linear combination of \((G_{-2}G)_0\) and \((G_1G)_0\). \( \square \)

This result and several similar ones are reported to be known to J. M. Figueroa-O’Farrill and T. Kimura. Their proof seems to involve an identity similar to (2), and a modified version would read as follows:
\[
2G_0^2 = [G_0, G_0] = [G_{(1)}, G_{(1)}] = (G_{-1}G)_0 + (G_0G)_0
\]
28
by (2), where once again
\[ Q G_{-1}G = Q G_0G = 0 \]
is a consequence of (25).

5 BV Master Equation

5.1 Quantum BV Master Equation for Classical BV Algebras

In [BV] it is proposed that we look for an action (a bosonic function of fields and antifields)
\[ W = S + \sum_{p=1}^{\infty} \hbar^p M_p \] (26)
satisfying the quantum master equation
\[ \{W, W\} = 2i\hbar \Delta(W), \] (27)
or equivalently,
\[ \{S, S\} = 0 \]
\[ \{M_1, S\} = i\Delta(S) \]
\[ \{M_p, S\} = i\Delta(M_{p-1}) - \frac{1}{2} \sum_{q=1}^{p-1} \{M_q, M_{p-q}\} \quad \text{for } p \geq 2, \] (28)
so that
\[ \Delta(\exp(\frac{i}{\hbar}W)) = 0. \] (29)

Note that the classical part of \( W \), shown by \( S \), satisfies the classical master equation
\[ \{S, S\} = 0. \] (30)

Infinite sums will be assumed to have a purely formal meaning, or else to be finite when interpreted in some context. Our task will be to understand why condition (29) follows from (27) (a result often cited in literature without proof) when \( \mathcal{A} \) is a supercommutative, associative algebra.

Assume
\[ \{W, W\} = \lambda \Delta(W) \] (31)
for some even element \( W \) of a classical BV algebra and some complex constant \( \lambda \). Note that for even \( V, W \) we have
\[ \{V, W\} = \Phi^2_\Delta(V, W). \]
Lemma 5.1 For $k \geq 0$,
\[ \{W, W^k\} = k\lambda \Delta(W)W^{k-1}. \] (32)

Proof. Induction on $k$. For $k = 1$, we have $\{W, W\} = \lambda \Delta(W)$. Assume (32) holds for 1, 2, ..., $k$.
\[ \{W, W^{k+1}\} = \{W, W \cdot W^k\} = \{W, W\}W^k + W\{W, W^k\} \]
\[ = \lambda \Delta(W)W^k + W(k\lambda \Delta(W)W^{k-1}) \] by (31) and (32)
\[ = (k + 1)\lambda \Delta(W)W^k. \]

Lemma 5.2 For $k \geq 0$,
\[ \Delta(W^k) = \frac{k(k-1)}{2} \lambda \Delta(W)W^{k-2} + k\Delta(W)W^{k-1}. \] (33)

Proof. Induction on $k$. The statement holds for $k = 0, 1$. Assume (33) holds for 1, 2, ..., $k$.
\[ \Delta(W^{k+1}) = \Delta(W \cdot W^k) = \Phi^2_\Delta(W, W^k) + \Delta(W)W^k + W\Delta(W^k) \]
\[ = \{W, W^k\} + \Delta(W)W^k + W\Delta(W^k) \]
\[ = k\lambda \Delta(W)W^{k-1} + \Delta(W)W^k + W\left(\frac{k(k-1)}{2}\lambda \Delta(W)W^{k-2} + k\Delta(W)W^{k-1}\right) \] (by Lemma 5.1 and induction step)
\[ = \left(\frac{k(k-1)}{2} + k\right)\lambda \Delta(W)W^{k-1} + (k + 1)\Delta(W)W^k \]
\[ = \frac{k(k+1)}{2} \lambda \Delta(W)W^{k-1} + (k + 1)\Delta(W)W^k. \]

Proposition 5.3 $\Delta(exp(\frac{i}{\hbar}W)) = (\frac{i\lambda}{2\hbar} + 1)\Delta(\frac{i}{\hbar}W)exp(\frac{i}{\hbar}W)$.

Proof. We will replace $\frac{i}{\hbar}W$ by $V$, and condition (31) by
\[ \{V, V\} = \mu \Delta(V), \]
where $\mu = i\lambda/\hbar$, and prove the simpler identity
\[ \Delta(exp(V)) = (\frac{\mu}{2} + 1)\Delta(V)exp(V). \]

Thus
\[ \Delta(exp(V)) \]
\[ = \Delta(V) + \sum_{k=2}^{\infty} \frac{k(k-1)}{2k!} \mu \Delta(V)V^{k-2} + \sum_{k=2}^{\infty} \frac{k}{k!} \Delta(V)V^{k-1} \]
\[ = \Delta(V) + \frac{\mu}{2} \Delta(V)\sum_{k=0}^{\infty} \frac{V^k}{k!} + \Delta(V)\sum_{k=0}^{\infty} \frac{V^k}{k!} \]
\[ = \frac{\mu}{2} \Delta(V)\sum_{k=0}^{\infty} \frac{V^k}{k!} + \Delta(V)\sum_{k=0}^{\infty} \frac{V^k}{k!} = (\frac{\mu}{2} + 1)\Delta(V)exp(V). \]
As a result, we have

**Corollary 5.4** If (34) holds for even $W$ with $\lambda = 2i\hbar$, then

$$\Delta(\exp(\frac{i}{\hbar}W)) = 0.$$ 

**Remark 5.5** A recent formulation of Eq. (29) in terms of the forms $\Phi_k^\Delta$ was given by J. Alfaro and P. H. Damgaard [AD]. This is an identity I completely missed in my preprints, and I am grateful to P.H. Damgaard for explaining the equivalence to me. One can show, from the remarks at the end of Section 2.1 and by induction, that

$$\Delta(W^k) = \sum_{j=1}^{k} \binom{k}{j} W^{k-j} \Phi_j^\Delta(W, \ldots, W)$$

for every $k \geq 1$, with the assumptions that $W$ is an even element in a supercommutative, associative algebra, and $\Delta$ is any odd operator. Then it is a matter of simple algebra to show

$$\Delta(\exp(\frac{i}{\hbar}W)) = \exp(\frac{i}{\hbar}W) \sum_{k=1}^{\infty} \frac{1}{k!} (\frac{i}{\hbar})^k \Phi_k^\Delta(W, \ldots, W),$$

under the same assumptions. This last identity allows us to replace (29) by

$$\sum_{k=1}^{\infty} \frac{1}{k!} (\frac{i}{\hbar})^k \Phi_k^\Delta(W, \ldots, W) = 0,$$

which would work very well even for more general types of algebras! Note that if $\Delta$ is of order two, then only the first two terms survive, and we obtain the quantum BV master equation (34).

5.2 Quantum BV Master Equation for Vertex Operator BV Algebras

A question raised in [LZ1] is the meaning of the equations (27) and (30) in a conformal field theory, and their relations to deformations of the theory. It is stated that

**Proposition 5.6** (Proposition 3.3 in [LZ1]) The first order pole of the operator product expansion (OPE) $(b_{-1}S)(z)(b_{-1}S)(w)$, where $S$ is an even, weight zero, BRST-invariant element, vanishes if and only if the bracket $\{S, S\}$ defined via the BV operator

$$\Delta = b_0, \quad \text{where} \quad b(z) = \sum b_n z^{-n-2}$$

vanishes.

Such an element will correspond to a “first order deformation” (perturbation) of the BRST complex, and if all poles vanish, one obtains deformations of all orders. The authors then determine all solutions of $\{S, S\} = 0$ in the BRST cohomology of the “c=1 model”.
Let us give a proof of Proposition 5.6.

**Proof.** The $n$-th order pole of the OPE $u(z)v(z)$ is given by $u_{n-1}v$ (see [LZ3]). Replace $b_n$ by $u_{n+1}$ to achieve standard notation. We want to prove that if $S$ is even, with

$$L_0S = QS = 0,$$  \hspace{1cm} (35)

then

$$(u_0S)_0u_0S = 0 \iff (u_0S)_0S = 0.$$  

Equivalently, we want

$$\{S, u_0S\} = 0 \iff \{S, S\} = 0.$$  

One direction is quite generally true:

$$(u_0S)_0S = 0 \implies u_0(u_0S)_0S = 0 \implies -(u_0S)_0u_0S + [u_0, (u_0S)_0]S = 0.$$  

But

$$[u_0, (u_0S)_0] = [u_0, [u_0, S_0]] = [[u_0, u_0], S_0] - [u_0, [u_0, S_0]]$$  

(where $u_0$ is odd, square zero)

$$\implies [u_0, [u_0, S_0]] = 0 \implies (u_0S)_0u_0S = 0.$$  

Conversely, assume that

$$(u_0S)_0u_0S = 0,$$  

or, as we showed above,

$$u_0(u_0S)_0S = 0.$$  

This last equation translates to

$$b_{-1}\{S, S\} = 0 \implies Qb_{-1}\{S, S\} = 0 \implies -b_{-1}Q\{S, S\} + L_{-1}\{S, S\} = 0.$$  

But recall that we are working with a CBVA and $Q$ is a derivation. By virtue of (35) and Proposition 4.9, we have

$$Q\{S, S\} = 0,$$  

leading to

$$L_{-1}\{S, S\} = 0.$$  \hspace{1cm} (36)

By a fundamental axiom of VOSA’s, condition (36) means that the vertex operator $\{S, S\}(z)$ is a constant, and

$$[\{S, S\}_{-1}, v(z)] = 0$$

for any $v(z)$, as

$$\{S, S\}_iv = 0 \cdot v = 0 \quad \forall i \geq 0$$  

(see (3)). This in turn implies that

$$v_0\{S, S\} = [v_0, \{S, S\}_{-1}]1 = 0 \quad \forall v.$$  

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In particular for $v_0 = F_0$, the ghost number (superdegree) operator, we have

$$F_0\{S,S\} = -\{S,S\} = 0.$$  

\[ \Box \]

**Remark 5.7** Note that we did not need the condition $L_0 S = 0$.

Since the proof is valid in any TCA, we may choose one with overall nonnegative grading, so that all poles except the first two vanish when $S$ is of weight zero. The VOWA provides examples of such TCA’s. The two basic identities are

$$[h, k(z)] = -L(z) \quad \text{and} \quad [r, t(z)] = -L(z),$$

the latter being defined for semisimple $\ell$ only.

We also make the following observation by comparing weights on both sides of the master equation:

**Proposition 5.8** In a VOBVA, any even solution $W$ of the quantum master equation (31) with $\lambda \neq 0$ and $\Delta(W) \neq 0$ which is $L_0$-homogeneous has to be of weight zero. In fact, if $W$ is any solution with a homogeneous component $W'$ of nonzero highest (or lowest) weight, then $W'$ satisfies the classical master equation $\{W', W'\} = 0$.

5.3 Quantum BV Master Equation for Generalized BV Algebras

A classical problem: If $(L^*, \delta)$ is a $\mathbb{Z}$-graded super Lie algebra with an odd derivation $\delta$ of the Lie bracket (say of degree 1 and square zero), for which odd elements $a$ of $L$ will the addition of the inner derivation $\text{ad}(a)$ result in an odd, square zero derivation $\tilde{\delta}$? We first note that $\delta + \text{ad}(a)$ is still a derivation of the Lie bracket. Next, we observe

$$\tilde{\delta}^2 = 0 \quad \Leftrightarrow \quad \delta^2 + [\delta, \text{ad}(a)] + \text{ad}(a)^2 = 0$$

$$\Leftrightarrow \quad \text{ad}(\delta(a)) + \frac{1}{2}\text{ad}([a,a]) = 0 \quad \Leftrightarrow \quad \text{ad}(\delta(a) + \frac{1}{2}[a,a]) = 0.$$  

This last condition holds when, for example,

$$\delta(a) + \frac{1}{2}[a,a] = 0.  \tag{38}$$

If $[,]$ is an odd Lie bracket, as in a classical BV algebra, we seek an even element $a$ instead. We may call (38) a deformation equation (see [NR, Ger]). In a (classical) differential BV algebra $(\mathcal{A}, 1_\mathcal{A}, \Delta, D, L)$ where $D$ and $L$ are derivations of the associative product in $\mathcal{A}$, we have two deformation equations, namely

$$D(a) + \frac{1}{2}\{a,a\} = 0  \tag{39}$$
and
\[ \Delta(a) + \frac{1}{2}\{a, a\} = 0, \]  
(40)
where the second one also deserves the name “quantum BV master equation”. Note that although \( \Delta \) is in general a second order differential operator on \( \mathcal{A} \), it is a derivation of \( (\hat{\mathcal{A}}, \{ , \}) \). Then solving the quantum BV master equation means finding deformations of the BV operator \( \Delta \) by inner derivations (see [St2]). Addition of an inner derivation, of course, does not change the BV bracket.

In the case of a generalized BV algebra \( \mathcal{A} \) with bracket \( \{ , \} \), \( (\hat{\mathcal{A}}, \{ , \}) \) is only a Leibniz algebra. But all we need in (37) is the Leibniz property and the fact that \( \Delta \) is a derivation of the bracket, so that the even solutions of Eq. (40) still give us deformations of \( \Delta \). In other words, \( \Delta + \{a, \} \) is again an odd, square zero, second order differential operator on \( \mathcal{A} \) which induces the bracket \( \{ , \} \) associated with \( \Delta \). If (40) holds in a DBVA, the commutator of \( \Delta + \{a, \} \) with \( D \) is still \( L \), provided that \( a \) is in the kernel of \( D \).

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REFERENCES.

[A1] F. AKMAN, A characterization of the differential in semi-infinite cohomology, *J. Algebra* **162** (1993), 194-209; preprint [hep-th/9302144](http://arxiv.org/abs/hep-th/9302144).

[A2] F. AKMAN, “The semi-infinite Weil complex of a graded Lie algebra”, Ph.D. Thesis, Yale University, 1993.

[A3] F. AKMAN, “Some cohomology operators in 2-D field theory”, in the Proceedings of the Conference on Quantum Topology, Kansas State University, Manhattan, KS, 24-28 March 1993, ed. David N. Yetter, World Scientific, Singapore, 1994; preprint [hep-th/9307153](http://arxiv.org/abs/hep-th/9307153).

[AKSZ] M. ALEXANDROV, M. KONTSEVITCH, A. SCHWARZ, AND O. ZABORONSKY, The geometry of the master equation and topological quantum field theory, preprint [hep-th/9502010](http://arxiv.org/abs/hep-th/9502010).

[AD] J. ALFARO AND P.H. DAMGAARD, Non-abelian antibrackets, preprint [hep-th/9511066](http://arxiv.org/abs/hep-th/9511066).

[BV] I.A. BATALIN AND G.A. VILKOVISKY, Gauge algebra and quantization, *Phys. Lett.* **102B** (1981), 27-31.
I.A. BATALIN AND G.A. VILKOVISKY, Quantization of gauge theories with linearly dependent generators, *Phys. Rev.* **D28** (1983), 2567-2582.

[BP] P. BOUWKNEGT AND K. PILCH, The BV-algebra structure of $W_3$ cohomology, to appear in the Proceedings of “Gürsey Memorial Conference I: Strings and Symmetries”, eds. M. Serdaroğlu et al., Springer-Verlag, Berlin; preprint USC-94/17.

[DL] C. DONG AND J. LEPOWSKY, “Generalized vertex algebras and relative vertex operators”, Progress in Mathematics v. 112, Birkhäuser, Boston, 1993.

[Fi] JOSÉ M. FIGUEROA-O’FARRILL, Are all TCFT’s obtained by twisting N= 2 SCFT’s?, talk given at the workshop on Strings, Gravity, and Related Topics, held at the ICTP (Trieste, Italy) on June 29-30, 1995; preprint [hep-th/9507024](http://arxiv.org/abs/hep-th/9507024).

[FS] JOSÉ M. FIGUEROA-O’FARRILL AND SONIA STANCIU, Nonreductive WZW models and their CFT’s, preprint [hep-th/9506151](http://arxiv.org/abs/hep-th/9506151), QMW-PH-95-16.

[FHL] I.B. FRENKEL, Y.-Z. HUANG, AND J. LEPOWSKY, On axiomatic approaches to vertex operator algebras and modules, *Memoirs AMS* (1992).

[FLM] I.B. FRENKEL, J. LEPOWSKY, AND A. MEURMAN, “Vertex operator algebras and the Monster”, Academic Press, New York, 1988.
[Geb] R.W. GEBERT, Introduction to vertex algebras, Borcherds algebras, and the Monster Lie algebra, preprint hep-th/9308151 and DESY 93-120.

[Ger] M. GERSTENHABER, On the deformation of rings and algebras, *Ann. Math.* 79 (1964), 59.
M. GERSTENHABER, The cohomology structure of an associative ring, *Ann. Math.* 78 (1962), 267.

[GV] M. GERSTENHABER AND A.A. VORONOV, Homotopy G-algebras and moduli space operad, preprint MPI/94-71, Max-Planck-Institut in Bonn 1994, hep-th/9409063.

[Get1] E. GETZLER, Batalin-Vilkovisky algebras and two-dimensional topological field theories, *Commun. Math. Phys.* 159 (1994), 265-285; preprint hep-th/9212043.

[Get2] E. GETZLER, Manin pairs and topological field theory, *Ann. Phys.* 237 (1995), 161-201.

[GHV] W. GREUB, S. HALPERIN, AND R. VANSTONE, “Connections, curvature, and cohomology”, v.3, Academic Press, New York, 1972-1976.

[Gr] A. GROTHENDIECK, “Eléments de Géometrie Algébrique IV, Etude locals des schémas et des morphismes de schémas”, Pub. Math. IHES # 32, 1967 (Prop. 16.8.8 on p.42).

[Hu] Y.-Z. HUANG, Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras, *Commun. Math. Phys.* 164 (1994), 105-144.

[IR] J.M. ISIDRO AND A.V. RAMALLO, Topological current algebras in two dimensions, *Phys. Lett.* 316B (1993) 488-495; preprint hep-th/9307176, US-FT-7/93.

[Ka] Y. KAZAMA, Novel topological field theories, *Mod. Phys. Lett.* A6 (1991), 1321-1332.

[KSV] T. KIMURA, J. STASHEFF, AND A.A. VORONOV, Homology of moduli spaces of curves and commutative homotopy algebras, preprint alg-geom/9502006.

[KMS] I. KOLÁŘ, P.W. MICHOR, AND J. SLOVÁK, “Natural Operations in Differential Geometry”, Springer-Verlag, Berlin, 1993.

[Ko] J.-L. KOSZUL, Crochet de Schouten-Nijenhuis et cohomologie, *Astérisque* (1985), 257-271.

[LS] T. LADA AND J.D. STASHEFF, Introduction to sh Lie algebras for physicists, preprint hep-th/9209099, UNC-MATH-92/2.
[Lo] J.-L. LODAY, “Cyclic Homology”, Grundlehren der mathematischen Wissenschaften 301, Springer-Verlag, Berlin, 1992.

[Li] B. H. LIAN, On the classification of simple vertex operator algebras, Commun. Math. Phys. 163 (1994), 307-357.

[LZ1] B. H. LIAN AND G. J. ZUCKERMAN, New perspectives on the BRST-algebraic structure of string theory, Commun. Math. Phys. 154 (1993), 613-646; preprint hep-th/9211072; MR 94e:81333.

[LZ2] B. H. LIAN AND G. J. ZUCKERMAN, Some classical and quantum algebras, in “Lie Theory and Geometry”, eds. Brylinski et al, Progress in Mathematics 123, Birkhäuser, Boston, 1994, 509-529; preprint hep-th/9404010.

[LZ3] B. H. LIAN AND G. J. ZUCKERMAN, Commutative quantum operator algebras, to appear in J. Pure Appl. Alg. 100 (1995); preprint q-alg/9501014.

[LZ4] B. H. LIAN AND G. J. ZUCKERMAN, Moonshine cohomology, preprint q-alg/9501015.

[MQ] V. MATHAI AND D. QUILLEN, Superconnections, Thom classes, and equivariant differential forms, Topology 25 (1986), 85-110.

[Mi1] P.W. MICHOR, A generalization of Hamiltonian mechanics, Journal of Geometry and Physics 2 (1985), 67-82.

[Mi2] P.W. MICHOR, Remarks on the Schouten-Nijenhuis bracket, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II 16 (1987), 207-215.

[Ni] A. NIJENHUIS, Jacobi-type identities for bilinear differential concomitants of certain tensor fields I, Koninklijke Nederlandse Akademie van Wetenschappen, Series A, Proceedings 58 (1955), 390-403 (journal later continued as Indagationes Math.).

[NR] A. NIJENHUIS AND R.W. RICHARDSON, JR., Deformations of Lie algebra structures, Journal of Mathematics and Mechanics 17 (1967), 89-105.

[PS] M. PENKAVA AND A. SCHWARZ, On some algebraic structures arising in string theory, preprint hep-th/9212072.

[Sn] J.A. SCHOUTEN, Über Differentialkonkomitanten zweier kontravarianter Größen, Koninklijke Nederlandse Akademie van Wetenschappen, Series A, Proceedings 2 (1940), 449-452. Also see
J.A. SCHOUTEN, On the differential operators of first order in tensor calculus, in Convegno Internazionale di Geometria Differenziale, Italy, Sept. 20-26 1923, Edizioni Cremonese delle Casa Editrice Perrella, Rome, 1954.
[Sc] A. SCHWARZ, Geometry of Batalin-Vilkovisky quantization, preprint hep-th/9205088.

[St1] J. STASHEFF, Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras, ...

J. STASHEFF, Homotopy associativity of $H$-spaces I and II, *AMS Trans.* 108 (1963), 275-292 and 293-312.

[St2] J. STASHEFF, Homological reduction of constrained Poisson algebras, to appear in *J. Diff. Geom.*

[Wi1] E. WITTEN, A note on the anti-bracket formalism, preprint IASSNS-HEP-90/9.

[Wi2] E. WITTEN, Ground ring of the two dimensional string theory, *Nucl. Phys.* B373 (1992), 187.

[Zw] B. ZWIEBACH, Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation, *Nucl. Phys.* B390 (1993), 33-152.