σ2 YAMABE PROBLEM ON CONIC SPHERES II: BOUNDARY COMPACTNESS OF THE MODULI

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Abstract. In this short note, we prove a convergence theorem on the moduli space of constant σ2 metrics for conic 4-spheres. In particular, we show that when a numerical condition is convergent to the boundary case, the geometry of conic 4-spheres has to converge to the boundary case. As a key step, we establish a Bonnesen type isoperimetric inequality for high dimensions, which is of independent interest.

1. INTRODUCTION

In this paper, we discuss the moduli space of constant σ2 metric on a conic 4-sphere, following our previous work [FW].

We start with notations and some historical comments. Let (M, g) be a smooth manifold and Ric, R be Ricci curvature and scalar curvature respectively. The Schouten tensor is defined as

\[ A_g = \frac{1}{n-2} (Ric - \frac{1}{2(n-1)} R g). \]

Denote \{λ(A_g)\} as the eigenvalue of \( A_g \) respect to \( g \). Define \( k \)-th elementary symmetric function \( σ_k(λ) \) as the following

\[ σ_k(λ) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} λ_{i_1}λ_{i_2}\cdots λ_{i_k}. \]

When \( k = 1 \), up to a constant, \( σ_1(λ(A_g)) \) is the scalar curvature. Define the conformal class \( [g] = \{ g_u = e^{2u}g | u ∈ C^∞(M) \} \).

The classical Yamabe problem is to find a metric in \([g]\) with constant scalar curvature. \( σ_k \) Yamabe problem is raised by Viaclovsky in [V1] to find a metric in \([g]\) with constant \( σ_k \) curvature, which is equivalent to find a solution \( u \) to the following equation

\[ σ_k(λ(A_{g_u})) = c, \]

where \( A_{g_u} = A_g - \nabla^2 u + \nabla u \otimes \nabla u - \frac{[\nabla u]^2}{2} g \) and \( c \) is a constant.

In order to make the equation elliptic, a pointwise positive cone condition is usually considered as below

\[ Γ^+_k := \{ λ = (λ_1, \cdots , λ_n), s.t. σ_1(λ) > 0, \cdots , σ_k(λ) > 0 \}. \]

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In the last 20 years, $\sigma_2$ Yamabe problems in 4-manifolds have been studied intensively due to its variational structure and geometric connection with Gauss-Bonnet-Chern formula. In [CGY1], Chang-Gursky-Yang have constructed a positive $\sigma_2$ curvature metric by exploring the connection between $\sigma_2$ curvature and $Q$ curvature. Furthermore, they have proved that when the manifold is not conformal to sphere, there exists a smooth metric such that its $\sigma_2$ curvature admits any given positive function [CGY2]. In [GS], using a metric structure on the spaces of conformal factors, Gursky-Streets have surprisingly proved that the metric with constant $\sigma_2$ curvature is unique when 4-manifold is not conformal to sphere. For general $\sigma_k$ Yamabe problem, there are abundant results about the existence and regularity of $\sigma_k$ Yamabe problem and we refer to [CLW] [GV2] [GV3] [GV4] [GV5] [GW1] [GW2] [FW] [Tr] and references therein.

As smooth $\sigma_k$ Yamabe problem has achieved huge developments, the singular $\sigma_k$ Yamabe problem can also be considered: given a smooth metric $g_0$ on $M$ of dimension $n$ and $P \subset M$ a lower dimension subset, does there exist a complete or incomplete metric $g = e^{2u}g_0$ on $M \setminus P$ with constant $\sigma_k$ curvature? For $k = 1$, the singular Yamabe problem has been extensively studied from view points of semilinear elliptic equations and geometric sides. For $k = 1$, $n = 2$, or singular Yamabe problem on surfaces, see the seminal works of Troyanov [Tr1, Tr2], see also works of Chen-Li [CL1] [CL2] and Luo-Tian [LT] and Fang-Lai [FL1, FL2, FL3]. For $k = 1$, $n \geq 3$, concerning the positive scalar curvature, in a milestone paper [S], Schoen has given the first construction of solutions with finite number singularities. His proof is further simplified by Mazzeo and Pacard [MP]. Many works have appeared later. Especially, Mazzeo-Pollack-Uhlenbeck [MPU] introduced the moduli space $\mathcal{M}$ consisting of singular solutions for constant scalar curvature and proved that $\mathcal{M}$ is locally a real analytic variety of formal dimension $k$. For $4 \leq 2k < n$, Mazziere-Segatti [LS] have constructed complete locally conformally flat manifold with constant positive $k$-curvature. For $n \geq 5$, the existence of metric admitting singularities with constant $\sigma_2$ curvature on $n$ dimensional manifolds has been proved by Santos [SS] with some additional Weyl conditions. The solutions for singular $\sigma_k$ Yamabe problem are complete metric for $k < \frac{n}{2}$.

Another interesting case of singular Yamabe problem is when we set $k = \frac{n}{2}$. In [CHY], Chang-Han-Yang have studied the radial solution of positive constant $\sigma_k$ curvature equation and especially noticed that when $k = n/2$, the behavior of metric near singularity is cone like. Furthermore, Han-Li-Teixeira in [HLT] have proved that in a punctured ball, the solution is close to a radial solution up to a H"older perturbation, which extends the famous theorem proved by Caffarelli-Gidas-Spruck [CGS] for $k = 1$ case. See also [KMPS]. Combine the results of [HLT] [CHY], the metric with positive constant $\sigma_2$ curvature on 4-sphere can only be conic. With above observations, we have introduced the conic version to $\sigma_2$ Yamabe problem in [FW].

For simplicity, we discuss the standard sphere case here. With the standard stereographic projection, we define $g = e^{2u}g_{E}$ with singularities $p_1, \ldots, p_q$ and

- $u(x) = \beta_i \ln |x - p_i| + v_i(x)$ as $x \to p_i$ for $i = 1, \ldots, q - 1$;
- $u(x) = (-2 - \beta_q) \ln |x| + v_\infty(x)$ as $|x| \to \infty$,

where $v_i(x)$ and $v_\infty(x)$ are bounded in their respective neighborhoods. We call above $g$ conic metric and denote it by $(S^4, D = \sum_{i=1}^{q} \beta_i p_i, g_0)$, where $g_0$ is the standard sphere metric. Thus we have a sequence of singular conformal factors $\{u_i\}$
and we will describe the normalization of $u_i$ in later. We also use $[g_D]$ to denote the conformal class with above conic singularities $D$. The corresponding $\sigma_2$ Yamabe equation in $\mathbb{R}^n$ is

$$\sigma_2(g_E^{-1}A_{ij}) = \frac{3}{2}e^{4u},$$

where

$$A_{ij} = -u_{ij} + u_iu_j - \frac{\nabla u^2}{2}\delta_{ij}.$$

In our previous work [FW], we have studied the conic metric with constant $\sigma_2$ curvature on 4-sphere by constructing a local mass $M(t)$ along level sets of the conformal factor. See Section 2 for more details regarding $M(t)$. Similar to Mazzeo-Pollack-Uhlenbeck [MPU], we use $\mathcal{M}^2(S^4)$ to denote a moduli space as the set of all smooth solutions $u$ to singular $\sigma_2$ Yamabe problem $\sigma_2(\lambda(A_0)) = \frac{1}{2}$ on $S^4 \setminus P$, where $P = \{p_1, p_2, \cdots, p_q\}$ are any fixed singularities.

In [FW], we give the following

Definition 1. Let $(S^4, D, g_0)$ be a conic 4-sphere with the standard round background metric $g_0$. For all $j = 1, \cdots, q$, we denote $\tilde{\beta}_j := \left(\sum_{1 \leq i \neq j \leq q} \tilde{\beta}_i \beta_j^3\right)^{1/3}$.

- We call $(S^4, D)$ subcritical for $\sigma_2$ Yamabe equation if for any $j = 1, \cdots, q$
  $$\frac{2}{3} \beta_j^2 (\beta_j + 2)^2 < \frac{2}{3} \tilde{\beta}_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + \frac{2}{3})(\sum_{1 \leq i \neq j \leq q} \tilde{\beta}_i^2 - \tilde{\beta}_j^2),$$
- We call $(S^4, D)$ critical for $\sigma_2$ Yamabe equation if there exists a $j \in \{1, \cdots, q\}$ such that
  $$\frac{2}{3} \beta_j^2 (\beta_j + 2)^2 = \frac{2}{3} \tilde{\beta}_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + \frac{2}{3})(\sum_{1 \leq i \neq j \leq q} \tilde{\beta}_i^2 - \tilde{\beta}_j^2),$$
- Otherwise, we call $(S^4, D)$ supercritical for $\sigma_2$ Yamabe equation, which means that there exists a $j \in \{1, \cdots, q\}$,
  $$\frac{2}{3} \beta_j^2 (\beta_j + 2)^2 > \frac{2}{3} \tilde{\beta}_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + \frac{2}{3})(\sum_{1 \leq i \neq j \leq q} \tilde{\beta}_i^2 - \tilde{\beta}_j^2).$$

In [FW], we have proved the following theorem

Theorem 2. Let $(S^4, D, g_0)$ be defined as above. Assume that $\lambda(A_0) \in \Gamma^+$. If $(S^4, D)$ is supercritical, then there does not exist a conformal metric $g \in [g_D]$ with positive constant $\sigma_2$ curvature. If $(S^4, D)$ is critical with positive constant $\sigma_2$ curvature, then $(S^4, g)$ is a football.

Here we call $(S^4, g)$ football if $g = e^{2u}g_E$ with two singularities $p, q$ and conformal factor $u$ is radial solution to $\sigma_2(\lambda(A_0)) = constant$ such that for some $-1 < \beta < 0$, $u(x) = \beta \ln|x| + v_1(x)$ near $|x| = 0$ and $u(x) = (-2 - \beta) \ln|x| + v_2(x)$ near infinity.
This football is first described in case (a) of Theorem 1 [CHY].

Theorem 2 gives some partial results which are parallel to the 2-dimensional case. But compared to 2-dimensional case, the 4-dimensional one is fully nonlinear and more complicated. The existence and uniqueness in subcritical condition is still unknown.

In this paper, we want to prove a compactness theorem for the moduli space $\mathcal{M}^2(S^4)$, which can be defined as the collection of all singular metric in $[g_D]$ with constant $\sigma_2$ curvature in the sense of Gromov-Hausdorff convergence, which is a parallel result of [PL] in dimension 2.
First, we define the convergence of conformal divisors. For a sequence of divisors \( D_l = \sum_{i=1}^{q_l} \beta_{l,i}p_{l,i} \), if \( \lim_{l \to \infty} \beta_{l,i} = \beta_i \) and \( \lim_{l \to \infty} p_{l,i} = p_i \) in the smooth topology of \((M, g_0)\), we call 
\[
\lim_{l \to \infty} D_l = D = \sum_{i=1}^{q} \beta_i p_i.
\]
Note that we always assume that \( p_{l,i} \neq p_{l,j} \) for \( i \neq j \). This corresponds to non-merging of singular points of conformal divisors.

**Theorem 3.** Let \( g_l = e^{2u_l}g_E \) for \( l = 1, 2, \cdots \) be a sequence metrics in \( \mathbb{R}^4 \) with singularities \( \sum_{i=1}^{q_l} \beta_{l,i}p_{l,i} \) such that \( \sigma_2(\lambda(A_{g_l})) = \frac{3}{2} \). Suppose \( \tilde{D}_l = \sum_{i=1}^{q} \beta_i p_i \rightarrow \tilde{D}_\infty = \sum_{i=1}^{q} \beta_i p_i \), where \( \tilde{D}_\infty \) is in the critical case. Then there exists a subsequence of \( g_l \) converges in Gromov-Hausdorff sense to \( g_\infty \), which is a conic metric w.r.t \( D_\infty \) with \( \sigma_2(\lambda(g_\infty)) = \frac{3}{2} \) on \( S^4 \setminus D_\infty \). In particular \( D_\infty = \beta(p+q) \), then \( g_\infty \) is a football metric defined by Chang-Han-Yang.

Note that \( \tilde{D}_\infty \) and \( D_\infty \) may be different divisors. This theorem describes a geometric merging of conformal divisors when taking the limit.

The proof of this theorem indicates the analytic process about how the singularities affect the shape of the manifolds from subcritical to critical one. For \( k < \frac{n}{2} \), in [W], the second author also gives a compactness theorem for complete manifolds with constant \( k \)-curvature under some natural conditions.

In a subsequent work, authors intend to study the uniqueness of solutions following approach of Gursky-Streets [GS]. We would like to address the existence problem eventually. We would also explore the \( \sigma_2 \) Yamabe problem for general conic 4-manifolds.

We organize this paper as below. In Section 2, we prove a high dimension isoperimetric inequality, which is an extension of the classical Bonnesen inequality in dimension 2. In Section 3, we follow the setup of [FW] to discuss level sets of conformal factors, In Section 4, we will prove the main theorem.

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### 2. High Dimension Bonnesen Type Inequality

In this section we extend the classical 2 dimensional Bonnesen inequality to higher dimensions, which will be used in our proof.

**Theorem 4.** *(Bonnesen)* For a bounded region \( D \subset \mathbb{R}^2 \) with area \( A \) and boundary length \( L \), then

\[
L^2 - 4\pi A \geq \pi^2 (r_e - r_i)^2,
\]

where \( r_e \) and \( r_i \) are, respectively, the radius of minimum circumscribed disc and the radius of maximum inscribed disc of \( D \). Furthermore, the equality holds if and only if \( D \) is a disc.

The extra term of isoperimetric inequality measures the round-ness of \( D \). In particular, when the isoperimetric defect \( L^2 - 4\pi A \) is small, we may conclude that in Gromov-Hausdorff sense, \( D \) is close to a round disc. See [FL1] for an example of application.

For higher dimension, some Bonnesen type isoperimetric inequalities are known, see for example, [ZDC, Z-Z, ZG1]. We list two as follows:
Theorem 5. [ZDC] [Z-Z, ZG1] Let $D$ be a domain in the Euclidean space $\mathbb{R}^n$ with volume $|D| = B$. Let $S$ be the hyper-surface area of $\partial D$. Let $D'$ be its convex hull, which has the corresponding volume $B'$. Let $S'$ be hyper-surface area of $D'$. Then we have $S \geq S'$, and $B' \geq B$. Furthermore,

\[(1.1) \quad S^n - n^n \omega_1 B^{n-1} \geq (S - S')^n,\]

\[(1.2) \quad S^n - n^n \omega_1 B^{n-1} \geq \frac{n^n}{2^{n-2}} \omega_1 (B' - B).\]

Moreover, if $D \subset \mathbb{R}^n$ is convex, then

\[(2.3) \quad \left(\frac{S}{n \omega_1}\right)^{\frac{n}{n-1}} - \frac{B}{\omega_1} \geq \left(\frac{B}{\omega_1}\right)^{1/n} - r_i^n.\]

Here $r_i$ is the radius of maximum inscribed disc of $D$ and $\omega_1 = \frac{1}{n} |S^{n-1}|$, where $S^{n-1}$ is the volume of hyper-surface area of round sphere of dimension $n-1$.

Note that Theorem 5 cannot directly give us the Gromov-Hausdorff type convergence result. In this section, we will deduce a type of Bonnesen inequality that can be applied in geometry more directly.

For our purpose, we first define the isoperimetric deficit for domain $D \subset \mathbb{R}^n$. Let $B$ be the volume of $D$ and let $S = \partial D$. We define

\[\Delta = \left(\frac{|S|}{n \omega_1}\right)^{\frac{n}{n-1}} / \left(\frac{B}{\omega_1}\right) - 1,\]

where $B$ is the volume of $D$, then the classical isoperimetric inequality states that

\[\Delta \geq 0.\]

When $D$ is convex, we define $r_e$ and $r_i$ to be minimum circumscribed disc radius and the radius of maximum inscribed disc radius of $D$, respectively. We define the following quantity to measure the shape of $D$:

\[\tau = \tau(D) = \frac{r_e}{r_i}.\]

It is clear that $\tau \geq 1$ and the equality holds if and only if $D$ is a round disc.

We give our Bonnesen type inequality for convex domain:

Theorem 6. Notations as above. For a convex domain $D \subset \mathbb{R}^n$, if $\Delta \leq \frac{1}{2}$, then we have

\[\Delta \geq f(\tau)\]

where for a universal positive constant $C_n$ which depends only on dimension $n$, and

\[f(\tau) = C_n (\tau - 1)^n \left(\frac{\tau - 1}{\tau + 1}\right)^{(n-1)n}.\]

Proof. Let $O$ be the center of maximum inscribed ball $Q$ of $D$ with radius $r_i$. Let $R = \sup_{x \in D} \text{dist}(O, x)$. We know that, for proper constant $C_1$, which depends only
Proof.

Thus, 

\[ V_{\omega}(D) \geq \omega_1 r_1^n + C_1 (R - r_i) \left( \frac{\sqrt{R - r_i}}{\sqrt{R + r_i}} \right)^{n-1} \]

\[ \geq \omega_1 r_1^n + C_1 (R - r_i) \frac{n+1}{2} \left( \frac{r_i}{\sqrt{R + r_i}} \right)^{n-1} \]

\[ \geq \omega_1 r_1^n + C_1 (R - r_i) \frac{n+1}{2} \left( \frac{r_i}{\sqrt{R + r_i}} \right)^{n-1}. \]

Thus,

\[ V_{\omega}(D)^{1/n} - (\omega_1 r_1^n)^{\frac{1}{n}} \geq C_n (R - r_i) \frac{R - r_i}{R + r_i} \frac{n^{n-1}}{\sum_{i=1}^{n-1} V_{\omega}(D)^{\frac{1}{n}} (\omega_1 r_1^n)^{\frac{n-1}{n}}} \]

If \( \triangle \leq \frac{1}{2} \), then by (2.3),

\[ \left( \frac{1}{2} \right)^{\frac{1}{n}} \geq 1 - \frac{r_i}{(\omega_1)^{\frac{1}{n}}} \]

and for some positive constants \( c_1, c_2 > 0 \),

\[ c_2 r_1^n \geq V_{\omega}(D) \geq c_1 r_1^n. \]

Furthermore,

\[ V_{\omega}(D)^{1/n} - (\omega_1 r_1^n)^{\frac{1}{n}} \geq C_n (R - r_i) \frac{R - r_i}{R + r_i} \frac{n^{n-1}}{\sum_{i=1}^{n-1} V_{\omega}(D)^{\frac{1}{n}} (\omega_1 r_1^n)^{\frac{n-1}{n}}} \]

\[ \geq C_3 (R - r_i) \left( \frac{R - r_i}{R + r_i} \right)^{\frac{n-1}{2}} \]

Therefore, using (2.3), we have

\[ \triangle \geq \frac{[V_{\omega}(D)^{\frac{1}{n}} - (\omega_1 r_1^n)]^n}{V_{\omega}(D)} \geq C_4 \left( \frac{R}{r_i} - 1 \right)^n \left( \frac{R - r_i}{R + r_i} \right)^{n-1}. \]

As \( \frac{R - r_i}{R + r_i} \) is increasing and the obvious fact that \( R \geq r_e \), we have

\[ \triangle \geq C_4 \left( \frac{r_e}{r_i} - 1 \right)^n \left( \frac{r_e/r_i - 1}{r_e/r_i + 1} \right)^{n-1}. \]

We have thus finished the proof. \( \square \)

Note this is not the sharpest estimate for convex domain. However this is sufficient to conclude geometric results, especially for Gromov-Hausdorff convergence. Here we have the following geometric application.

**Theorem 7.** Suppose there exists a sequence convex sets \( D_k \subset \mathbb{R}^n \) such that \( \triangle(D_k) \to 0 \). Then, up to a subsequence, \( D_k \) converges to a ball in Hausdorff sense.

**Proof.** As \( \triangle(D_k) \to 0 \) and Theorem 6 \( \frac{r_{e,k}}{r_{i,k}} \to 1 \), where \( r_{e,k} \) and \( r_{i,k} \) to be minimum circumscribed disc radius and the radius of maximum inscribed disc radius of \( D_k \). We can then pick a subsequence such that \( r_{i,k} \to r \in (0, \infty) \), the convergence result then follows naturally. \( \square \)

In this paper, we just need the 4-dimensional result. We list the result for general dimension, which is of independent interest.
3. Analysis on the level set

In this section, we recall some definition and results of [FW] for our discussion.

For a conic metric \((S^4, e^{2u}g_E, D = \sum_{i=1}^{q} \gamma_ip_i)\), assume that \(\sigma_2(\lambda(g_u)) = \text{constant}\) on \(S^4 \setminus D\). We begin with the following definition about the level sets of the conformal factor \(u\):

\[
L(t) = \{x : u = t\} \subset M,
\]
\[
S(t) = \{x : u \geq t\} \subset M.
\]

For a fixed \(t\), it is clear that \(L(t)\) is smooth at a point \(P \in L(t)\) if and only if \(\nabla u \neq 0\).

Define

\[
S = \{x \in \mathbb{R}^4 \setminus \{p_1, \cdots, p_q\} | \nabla u(x) = 0\}.
\]

In [FW], we have proved the following lemma, which implies the properness of the integral on the level set.

**Lemma 8.** [FW] For \(U \subset \mathbb{R}^4\), if \(\sigma_2(g_u) \neq 0\) in \(U\), then \(\mathcal{H}^3(S \cap U) = \mathcal{H}^4(S \cap U) = 0\), where \(\mathcal{H}^3, \mathcal{H}^4\) is the 3-dimensional and 4-dimensional Hausdorff measure.

We adopt the local coordinates in [FW]. Denote

\[
A(t) = \int_{S(t)} e^{4u} \, dl,
\]
\[
B(t) = \int_{S(t)} dl,
\]
\[
C(t) = e^{4t}B(t),
\]
and

\[
z(t) = -\left(\int_{L(t)} |\nabla u|^3 \, dl\right)^{1/3}.
\]

and finally,

\[
D(t) = \frac{1}{4} (\int_{L(t) \setminus S} \{2H|\nabla u|^2 - 2|\nabla u|^3\} \, dl).
\]

Here \(H\) is the mean curvature of the level set \(L(t)\), and \(dl\) is the induced 3-dimensional measure on \(L(t)\). In this paper, \(\int\) means \(\int_{S^3} \int\) and \(\omega_1 = |B_1| = \frac{1}{4} |S_3|\).

To see how singularities affect the shape of a manifold, we introduce a mass \(M(t)\) and some important inequalities, which are proved in our previous paper in [FW].

Define

\[
M(t) = \frac{2}{3}D(t) + \frac{4}{9}D(t)z(t) + \frac{1}{36}z^4(t) - C(t).
\]

**Theorem 9.** [FW] Let \(u\) be the solution to (1.1), we have

\[
\int_{\mathbb{R}^4} e^{4u} = \frac{2}{3}|S_3|(2 - \sum_{i=1}^{q} \gamma_i^3 + 3 \gamma_1^2),
\]

(3.1)

\[
M'(t) \geq 0,
\]

(3.2)

\[
E = M' + 4C,
\]
\( (3.3) \quad \frac{1}{3} \frac{d}{dt} (z^3) = \int_{L(t)} \left( \frac{H}{3} |\nabla u| - \nabla_{nn} u \right) |\nabla u|, \)

and

\( (3.4) \quad (A')^2 \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u| \int_{L(t)} \left( \frac{H}{3} |\nabla u| - \nabla_{nn} u \right) |\nabla u| dl \geq \frac{3}{2} t^{-12t} |L(t)|^4 \frac{1}{|S_3|^4}, \)

where

\[ E = \frac{1}{3} (2z A' + \frac{2}{3} z') \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u|. \]

The proof of (3.1) (3.2) is given in Theorem 14 [FW]; (3.3) is from Lemma 12 in [FW]; (3.4) is contained in the proof of Lemma 14 in [FW].

4. Proof of main theorem

In this section, we are ready to give a proof of our main theorem. Note that we have a sequence of conformal metrics representing constant \( \sigma_2 \) metric with prescribed conic singularities. Each can be written as a conformal factor on \( \mathbb{R}^4 \). We will need to fix the conformal gauge properly to obtain proper convergence. First of all, note that the limiting divisor \( \tilde{D}_\infty \) is critical, which means that there exists \( j \) such that

\[ \frac{3}{8} \beta_j^2 (\beta_j + 2)^2 = \frac{3}{8} \tilde{\beta}_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + \frac{3}{2} \sum_{1 \leq i \neq j \leq q} \beta_i^2 - \tilde{\beta}_j^2) \]

with \( \tilde{\beta}_j := \left( \sum_{1 \leq i \neq j \leq q} \beta_i^3 \right)^{1/3} \). Without loss of generality, we assume \( j = q \) and \( p_q = \infty \). Note that now we have fixed infinity, we still have the freedom to choose an origin and a corresponding coordinate system, which are to be determined later.

By the definition of convergence of conformal divisors, and (3.1) of Theorem 9 we have

\[ \lim_{l \to \infty} \left[ M_l(+\infty) - M_l(-\infty) \right] = 0, \]

where

\[ M_l(t) = \frac{2}{3} D_l(t) + \frac{4}{9} D_l(t) z_l(t) + \frac{1}{36} z_l^4(t) - C_l(t). \]

By Lemma 11 in [FW] for a fixed \( l \),

\[ \lim_{t \to -\infty} M_l(t) = \frac{1}{4} \beta_{l,q}^2 (\beta_{l,q} + 2)^2 \]

and

\[ \lim_{t \to +\infty} M_l(t) = \frac{1}{4} \tilde{\beta}_l (\tilde{\beta}_l + 2)^2 + \frac{2}{3} \tilde{\beta}_l^2 + 1 \left( \sum_{1 \leq i \neq q} \beta_i^2 - \tilde{\beta}_l^2 \right). \]

Theorem 10. For \( t \in \mathbb{R} \) almost everywhere, after passing to a subsequence,

\[ \lim_{l \to \infty} (B_l(t) - B_l(t)) = 0. \]
Furthermore
\[ \lim_{t \to \infty} e^{4t}(B_t^*(t) - B_t(t)) = 0. \]
Here \(|S_t^*(t)|, B_t^*(t)\) are the surface area and volume of the convex hull for \(S_t = \{u_t > t\}\) respectively.

**Proof.** We do the following computation

\[ (M'_t)^3 + 12(M'_t)^2C_t + 48C'_t^2M'_t \]
\[ = M'_t(t)\{(M'_t + 4C'_t)^2 + 4C'_t(t)[M'_t(t) + 4C'_t(t)] + (4C'_t)^2\} \]
\[ = (E_t(t) - 4C'_t(t))(E_t^2 + 4C'_t(t)E_t + (4C'_t)^2) \]
\[ = \left( \frac{1}{2}(2z_tA'_t + \frac{2}{3}z_t\int_{L_t(t)} \sigma_1(\tilde{A})|\nabla u_t|) \right)^3 - (4C'_t(t))^3 \]
\[ \geq \frac{2}{3}(A'_t)^2 \int_{L_t(t)} \sigma_1(\tilde{A})|\nabla u_t| \int_{L_t(t)} (\frac{H}{3}|\nabla u_t| - \nabla_{nn}u_t)|\nabla u_t|dt - (4C'_t(t))^3 \]
\[ (4.1) \geq e^{12t}|L_t(t)|^4 \frac{1}{|S_t^2(t)|^4} - e^{12t}(4B_t(t))^3 \]

where
\[ E_t = \frac{1}{3}(2z_tA'_t + \frac{2}{3}z_t\int_{L_t(t)} \sigma_1(\tilde{A})|\nabla u_t|). \]

In (4.1), the fourth inequality holds due to Hölder inequality and (3.3), while the last inequality is due to (4.3).

Therefore by (2.1) (2.2) and (4.1), we have

\[ (M'_t)^3 + 12C'_t(M'_t)^2 + 48C'_t^2M'_t \geq \frac{e^{12t}}{|S_t^1|} (|L_t(t)| - |L_t^1(t)|)^4, \]

and

\[ (M'_t)^3 + 12C'_t(M'_t)^2 + 48C'_t^2M'_t \geq \frac{e^{12t}}{|S_t^2|} 4^4 \omega_{1}(B_t^*(t) - B_t(t))^3. \]

Furthermore,

\[ (4.4) \int_{t_0}^{+\infty} ((M'_t(t))^3 + 12C'_t(M'_t)^2 + 48C'_t^2M'_t)^{1/3}dt \]
\[ \leq \int_{t_0}^{+\infty} M'_t(t) + (12C'_t)^{1/3}(M'_t)^{2/3} + (48C'_t^2M'_t)^{1/3}dt \]
\[ \leq \int_{t_0}^{+\infty} M'_t(t)dt + (\int_{t_0}^{+\infty} M'_t(t)^{2/3}(\int_{t_0}^{+\infty} 12C'_t(t)dt)^{1/3} + (48)^{1/3}(\int_{t_0}^{+\infty} M'_t(t)dt)^{1/3}(\int_{t_0}^{+\infty} C'_t(t))^{2/3}. \]

As \( \int_{-\infty}^{+\infty} 4C_t(t)dt = \int_{R^+} e^{4u_t} < C < \infty \) by Theorem 9, with (4.2) (4.4), we get
\[ \int_{t_0}^{\infty} M'_t dt + (\int_{t_0}^{\infty} M'_t)^{2/3} (\int_{t_0}^{\infty} 12C dt)^{1/3} + (48)^{1/3} (\int_{t_0}^{\infty} M'_t dt)^{1/3} (\int_{t_0}^{\infty} C dt)^{2/3} \]
\[ \geq \int_{t_0}^{\infty} \left( \frac{e^{12t}}{|S_3|^4} \right) (|L_t(t)| - |L'_t(t)|)^{1/3} dt \]

and
\[ \int_{t_0}^{\infty} M'_t dt + (\int_{t_0}^{\infty} M'_t)^{2/3} (\int_{t_0}^{\infty} 12C dt)^{1/3} + (48)^{1/3} (\int_{t_0}^{\infty} M'_t dt)^{1/3} (\int_{t_0}^{\infty} C dt)^{2/3} \]
\[ \geq \int_{t_0}^{\infty} \left( \frac{e^{12t}}{|S_3|^4} \right) 4\omega_1 (B^*_t(t) - B_t(t))^3^{1/3} dt. \]

Here \( M_t \) is increasing respect to \( t \) by Theorem 9 and
\[ \int_{-\infty}^{+\infty} M'_t(t) dt = M_t(+\infty) - M_t(-\infty) \to 0 \]
as \( l \to \infty \). Thus, \( \int_{t_0}^{\infty} M'_t(t) dt \to 0 \). Then \( (e^{12t} 4^3 \omega_1 |S_3|^4)^{1/3} \int_{t_0}^{\infty} B^*_t(t) - B_t(t) dt \to 0 \)
as \( l \to \infty \). We get that \( B^*_t(t) - B_t(t) \geq 0 \) converges to 0 in \( L^1_{(t_0, \infty)} \) norm. Actually \( e^{4t} (B^*_t(t) - B_t(t)) \) converges to 0 in \( L^1_{(t_0, \infty)} \) norm as \( l \to \infty \).

Thus after passing to a subsequence \( B^*_t(t) - B_t(t) \geq 0 \) converges to 0 almost everywhere for \( t \geq t_0 \). Repeating the same argument for a sequence of \( t_i \to -\infty \)
and using a diagonal argument, it follows that \( B^*_t(t) - B_t(t) \) converges to 0 almost everywhere on \( \mathbb{R} \). Thanks to Sard’s theorem, we have proved the theorem.

For convenience, we denote \( U \) as the set of \( t \in \mathbb{R} \) satisfying Theorem 10.

Similarly,

Lemma 11. For \( t \in U \),
\[ \lim_{l \to \infty} e^{4t} \triangle(t)|B_l(t)| = 0, \]
and
\[ \lim_{l \to \infty} (B^*_t(t) - \omega_1 r^*_t) = 0. \]
Here \( r^*_t \) is the radius of maximum inscribed disc of \( S^*_t \), which is the convex hull of \( S_t \).

Proof. By Theorem 9
\[ M'_t(t) = E_t(t) - 4C_t(t) \]
\[ \geq \frac{1}{3} (A_t)^2 \int_{L_t(t)} \sigma_t(\mathbf{A})|\nabla u| \int_{L_t(t)} \left( \frac{H}{3} |\nabla u| - |\nabla_n u| \right) |\nabla u| dt^{1/3} - 4C_t(t) \]
\[ \geq (e^{12t} |L_t(t)|^{4/3} / |S_3|^4)^{1/3} - e^{4t} (4B_t(t)) \geq e^{4t} \triangle(t) \frac{|B^*_t(t)|}{\omega_1}, \]
where \( \triangle(t) = (\frac{|L'_t(t)|}{\omega_1})^{4/3} / (\frac{|B^*_t(t)|}{\omega_1}) - 1 \) and \( L_t(t) \geq L^*_t(t), B_t(t) \leq B^*_t(t) \).

Like the argument as above, we have a subset \( U_1 \) with \( |\mathbb{R}\setminus U_1| = 0 \), such that for \( t \in U_1 \),
\[ \lim_{l \to \infty} e^{4t} \triangle(t)B^*_t(t) = 0. \]
For generic \( t \in U_1 \), by Theorem 10

\[
0 = \lim_{l \to \infty} e^{4t} \Delta_i(t) \frac{B_{l}(t)}{\omega_1} = \lim_{l \to \infty} e^{4t} \left( \frac{|L_{l}|}{4\omega_1} \right)^{4/3} - \frac{e^{4t} B_{l}^+}{\omega_1} \geq \lim_{l \to \infty} \left( \frac{B_{l}^+}{\omega_1} \right)^{1/4} - r_i^* \right)^4 e^{4t}.
\]

So we have \( \lim_{l \to \infty} \left( \frac{B_{l}^+}{\omega_1} \right)^{1/4} - r_i^* \right)^4 = 0 \) and also \( \lim_{l \to \infty} (B_{l}^+ - \omega_1 r_i^*)^4 = 0 \). Without confusion, we still denote \( U \) as \( U \cap U_1 \).

Now, we want to introduce a crucial Harnack-type inequality, which has been proved by Li-Li [LL1].

**Theorem 12.** [LL1] For \( n \geq 3, 1 \leq k \leq n \) and \( R > 0 \), let \( B_{3R} \subset \mathbb{R}^n \) be a ball of radius \( 3R \), \( u \in C^2(B_{3R}) \) and \( g = e^{2u} g_E \) be a positive solution of \( \sigma_k(\lambda(A_g)) \in 1 \) in \( B_{3R} \) satisfying \( \lambda(A_g) \in \Gamma_k \) in \( B_{3R} \). Then

\[
\max_{B_{3R}} e^u \min_{B_{3R}} e^u \leq C(n)R^{-2}.
\]

Let \( \Sigma_i(t) \) be the connected component of \( S_i(t) \) with largest area. Now let us further fix the normalization of \( u_t \). As the scaling and translation do not change the conic metric, we pick a \( t_0 \in U \), \( r_0 > 0 \) and choose appropriate scaling and translation for \( u_t \) such that,

\[
S_i(t_0) = \{|u_t > t_0| = B_0 = \omega_1 r_0^3.
\]

Furthermore, we require that the centroid of \( S_i(t_0) \) is the origin. This important step of conformal gauge fixing allows us to derive the exact geometric consequence.

We continue with our proof. For \( t < t_0 \), we know \( B_{l}(t) > B_{l}(t_0) \). We need to control the volume \( B_{l}(t) \). Note that

\[
S_i(t) \setminus S_i(t_0) = \{|x, t < u_t < t_0|\}
\]

For large \( l \), we assume that \( |\Sigma_i(t_0)| \to B_0 \) for some constant \( B_0 \). Then we denote the largest inscribed ball in \( S_i(t) \) as \( B_{r_1}(z) \). Then by Theorem [LL1],

\[
e^{2t} \leq \max_{B_{r_1}(z)} e^u \min_{B_{r_1}(z)} e^u \leq c(n)R_{r_1}^{-2}\]

and therefore \( r_1(t_0) \leq r_1(t) \leq c(n)e^{-t} \). We claim that \( \lim_{l \to \infty} r_l^* \) is bounded. Otherwise, by \( R_l(t) = R_l(t_0) \), for some positive constants \( c, c_1, c_2 \),

\[
\lim_{l \to \infty} c(R_l^*)^3 > \lim_{l \to \infty} B_{l}(t) = \lim_{l \to \infty} B_{l}^+(t) \geq \lim_{l \to \infty} c_1(R_l^*)^3 R_l^*
\]

and

\[
\lim_{l \to \infty} (R_l^* - c_2(R_l^*)^3/2) > 0,
\]

which is contradicted to \( \lim_{l \to \infty} (B_{l}^* - \omega_1 r_l^*^4) = 0 \). As \( \lim_{l \to \infty} r_l^* \) is bounded and \( r_l^* \geq r_1 \), \( B_{l}^* \) and \( \lim_{l \to \infty} R_l^* \) is bounded.

**Corollary 13.** For \( t \in U \), \( \lim_{l \to \infty} R_l^*(t)/r_l^*(t) = 1 \) and \( \lim_{l \to \infty}(R_l(t)/r_l(t)) = 1 \). Furthermore, \( \lim_{l \to \infty} S_l^* \) and \( \lim_{l \to \infty} S_l \) are round balls. Here \( R_l^* \) is the minimum outer ball of \( S_l \).
**Lemma 14.** $(1 + t)^{4/3} \geq 1 + t^{4/3} + \frac{4}{3}(2^{1/3} - 1)t$ for $0 < t < 1$.

**Proof.** Define $f(t) = (1 + t)^{4/3} - 1 - t^{4/3} - \frac{4}{3}(2^{1/3} - 1)t$. It is easy to check that $f'(t) \geq 0$ and $f(0) = 0$. We have finished the proof.

**Lemma 15.** For generic $t$, there exists a $B_\infty(t)$ such that

\[
\Sigma(t) \to B_\infty(t),
\]

and

\[
|S(t) \setminus \Sigma(t)| \to 0,
\]

as $t \to \infty$.

**Proof.** Denote $S^1_l(t) = \Sigma_l(t)$ and $S^2_l(t) = S_l(t) \setminus \Sigma_l(t)$. If $\liminf_{t \to \infty} |S_l(t) \setminus \Sigma_l(t)| \geq \delta_1$ and $\liminf_{t \to \infty} |\Sigma_l(t)| \geq \delta_2$, (without loss of generality, we assume $|\partial S^1_l(t)| > |\partial S^2_l(t)|$), then by Lemma 14

\[
\Delta_l(t) = (|\partial S^1_l(t)| + |\partial S^2_l(t)|)^{4/3}\left(\frac{1}{4\omega_1}\right)^{4/3} - \frac{|S^1_l(t)| + |S^2_l(t)|}{\omega_1} \geq \frac{4}{3}(2^{1/3} - 1)\frac{1}{\omega_1}|\partial S^1_l(t)|^{4/3}|\partial S^2_l(t)|
\]

This is a contradiction to Lemma 11. We have finished the proof.

Finally, we start the proof of the Main theorem as below.

Take an increasing sequence $\{t_i\}_{i \in \mathbb{Z}} \subset U$. By Lemma 14, for any $0 < \lambda << 1/2$ and every $t_i$, there exists a positive integer $L_i$ such that for all $l > L_i$,

\[
|\Sigma_l(t_i)| \geq 1 - \lambda.
\]

By a diagonal argument, we may pick a subsequence of $\{u_l\}$ (which still is denoted as $\{u_l\}$) and assume that (4.5) holds for all $t$. Notice that for $t_i > t_j$, we have

\[
|u_l(t_i)| \geq 1 - \lambda.
\]

\[
\Sigma_l(t_i) \subset \Sigma_l(t_j).
\]
Note that we have already chosen $t_0 \in U$ such that the centroid of $S_i(t_0)$ is the origin. It thus follows from $\lim_{t \to \infty} \Delta_i(t_0) = 0$ that a subsequence of $\Sigma_i(t_0)$ converges in Hausdorff distance to $B_{r_0}(0)$ with $r_0 = (\frac{B_i}{\omega_i})^{1/4}$.

By (4.6),

$$\Sigma_i(t_i) \supset \Sigma_i(t_0) \supset \Sigma_i(t_j), \quad \text{for } t_i < t_0 < t_j.$$  

It thus follows that each fixed $t_i$ the centroid of $\Sigma_i(t_i)$ is contained in a bounded set. For each $t_i$, by passing to a subsequence if necessary, we know that $|\Sigma_i(t_i)| \to B_{\infty}(t_i)$ and $\{|\Sigma_i(t_i)|\}$ converges to a ball in Gromov-Hausdorff sense as $l \to \infty$. We denote the corresponding limit by $B_{r_i}(z_i)$. As $\Sigma_i(t_i) \to \mathbb{R}^4$ as $t_i \to -\infty$, we actually have $\lim_{i \to -\infty} B_{r_i}(z_i) = \mathbb{R}^4$. In summary, we have proved the following

**Lemma 16.** There exists a sequence of descending balls

$$\cdots \supset B_{r_{i-1}}(z_i) \supset B_{r_i}(z_i) \supset B_{r_{i+1}}(z_{i+1}) \supset \cdots$$

with $\lim_{i \to -\infty} r_i = \infty$ and $\lim_{i \to \infty} r_i = 0$, such that $\Sigma_i(t_i)$ converges to $B_{r_i}(z_i)$ in Hausdorff distance.

As $p_{i,t}$ are singularities, we shall assume that $\lim_{i \to \infty} p_{i,t} = p_{\infty,i}$ for $i \geq 2$. Furthermore, we have $\{z_i\}$ is in a bounded domain. Here $\lim_{i \to -\infty} B_{r_i}(p_i) = \mathbb{R}^4$.

We are now ready to prove Theorem 3.

**Proof.** (of Theorem 3) Let $z_0 = \cap_{i \in \mathbb{Z}} B_{r_i}(z_i)$ be the point given by Lemma 16.

We will show that $u_l$ is uniformly bounded on any compact subset $K \subset \mathbb{R}^4 \setminus \{p_2, \cdots p_q-1, z_0\}$. For any given $\varepsilon > 0$, we have a uniform $C_1 \in \mathbb{N}$ such that for any $l > C_1$,

$$d(p_{l,i}, K) > \varepsilon.$$  

Clearly, for such a compact set $K$, there exist $r_i > r_j$ such that

$$K \subset B_{r_i}(z_i) \setminus B_{r_j}(z_j).$$

For $\delta > 0$ small enough, we have

$$K \subset \mathcal{N}_\delta(B_{r_i}(p_i)) \setminus \mathcal{N}_\delta(B_{r_j}(p_j)),$$

where $\mathcal{N}_\delta(\cdot)$ is the $\delta$–neighborhood. By Lemma 16, there exists $C_2 > 0$ such that for $l > L_2$,

$$K \subset \Sigma_l(t_i) \setminus \Sigma_l(t_j).$$

It follows that

$$u_l(x) \geq t_i, \quad \text{for } x \in K \text{ and } l > L_2.$$  

It remains to show a uniform upper bound for $u_l$. By Lemma 16 for any $\epsilon$, there exists $C_3 \in \mathbb{N}$ such that for all $l > C_3$, any connected component $\Omega$ of $S_l(t_i) \setminus \Sigma_l(t_i)$ satisfies

$$|\Omega| < \left(\frac{\epsilon}{2}\right)^3, |\partial \Omega| \leq \left(\frac{\epsilon}{2}\right)^3.$$  

As the singular point $d(p_{l,i}, K) > \epsilon$, any component $\Omega$ of $S_l(t_i) \setminus \Sigma_l(t_i)$ containing any singular point can not intersect $K$, otherwise $|\partial \Omega| > \left(\frac{\epsilon}{2}\right)^3$. For $K \subset \Sigma_l(t_i) \setminus \Sigma_l(t_j)$, $\Sigma_l(t_j)$ does not intersect $K$.

For $K' \subset \Sigma_l(t_i) \setminus \Sigma_l(t_j)$, $\Sigma_l(t_j)$ does not intersect $K' \supset K$. So $u_l$ are $C^2(K')$. As Theorem 12, we know

$$\max_{K} e^{u_l} \min_{K'} e^{u_l} \leq C(K, K', d(K, K')).$$
As
\[
\min_{K} e^{u_l} \geq e^{t_1}
\]
\[\max_{K} \exp(u_l) \leq \frac{C}{l_n}.\] So \(u_l\) is uniform bounded in \(K\) for \(l > \max\{C_1, C_2, C_3\}\).

Then by the classical \(C^1, C^2\) estimates, we have \(|u_l|_{C^{2,\alpha}(K)} \leq C(K)\) for some \(0 < \alpha < 1\). Furthermore, we have higher regularity estimate:
\[|u_l|_{C^\infty(K)} \leq C.\]

Then \(u_l \to u_\infty\) in \(C^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{p_1, \ldots, p_{q-1}, z_0\})\). Then \(\sigma_2(\lambda(A_{g_\infty})) = \frac{3}{2}\) in \(\mathbb{R}^4 \setminus \{p_2, \ldots, p_{q-1}, z_0\}\).

For \(t \in U, D, z\) is defined on the level set and we will have \(D_l(t) \to D_\infty(t)\) and \(z_l(t) \to z_\infty(t)\) in \(C^0_{\text{loc}}(U)\). By Corollary 10 and Lemma 15, \(\lim_{t \to \infty} S_i\) is ball and
\[
\lim_{t \to \infty} e^{4t} B_i(t) = e^{4t} B_\infty(t) = e^{4t}|\{u_\infty > t\}|.
\]

Actually by Lemma 11 we also have
\[
\lim_{l \to \infty} e^{4t} B_l(t) = \lim_{l \to \infty} e^{4t}(\frac{L_l}{4\omega_1})^{4/3} = e^{4t}(\frac{1}{4\omega_1})^{4/3} L^{4/3}_\infty(t).
\]

The isoperimetric equality holds and then we know \(g_{u_\infty}\) is a solution to \(\sigma_2(\lambda(A_{g_\infty})) = \frac{3}{2}\) on \(\mathbb{R}^n \setminus \{z_0, \infty\}\) and \(\{z_0, \infty\}\) is two singular points. For \(z_0 = \cap_{i \in \mathbb{Z}} B_{r_i}(z_i),\) \(z_0 = 0\).

We have
\[
\lim_{t \to \infty} M_l(t) = \lim_{t \to \infty} \frac{2}{3} D_l(t) + \frac{4}{9} D_l(t) z_l(t) + \frac{1}{36} z_l^2(t) - C_l(t)
\]
\[
= \frac{2}{3} D_\infty(t) + \frac{4}{9} D_\infty(t) z_\infty(t) + \frac{1}{36} z_\infty^2(t) - e^{4t} B_\infty(t).
\]

Also for every \(l\),
\[
\frac{1}{4} \beta_{l,q}^2 (\beta_{l,q} + 2) = \lim_{t \to \infty} M_l(t) \leq M_l(t)
\]
\[
\leq \lim_{t \to \infty} M_l(t) = \frac{1}{4} \beta_{l,q}^2 (\beta_{l,q} + 2) + \frac{2}{3} \beta_{l,q} + 1)(\sum_{1 \leq i \neq j \leq q} \beta_{l,i}^2 - \beta_{l,q}^2).
\]

As the divisor \(D_l \to D_\infty\) and \(D_\infty\) is critical divisor, \(M_l(t)\) converges to \(\lim_{t \to \infty} M_l(t)\) uniformly. And \(M_\infty(t) = \frac{1}{4} \beta_{l,q}^2 (\beta_{l,q} + 2)^2\). Thus implies that \(M_\infty(t) = 0\) which means that \(g_\infty\) is the standard football.

\[\square\]

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