A new result for boundedness of solutions to a quasilinear higher-dimensional chemotaxis–haptotaxis model with nonlinear diffusion

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Abstract

This paper deals with a boundary-value problem for a coupled quasilinear chemotaxis–haptotaxis model with nonlinear diffusion

\[
\begin{aligned}
    u_t &= \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) - \xi \nabla \cdot (u\nabla w) + \mu u(1 - u - w), \\
    v_t &= \Delta v - v + u, \\
    w_t &= -vw,
\end{aligned}
\]

in \( N \)-dimensional smoothly bounded domains, where the parameters \( \xi, \chi > 0, \mu > 0 \).

The diffusivity \( D(u) \) is assumed to satisfy \( D(u) \geq C_D u^{m-1} \) for all \( u > 0 \) with some \( C_D > 0 \). Relying on a new energy inequality, in this paper, it is proved that under the conditions

\[
m > \frac{2N}{\left( \frac{\max_{s \geq 1} \lambda_0}{\left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{s}} + 1} \left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right) + \frac{\lambda_0}{N} \left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{s}} - 1 \right) + \frac{\max_{s \geq 1} \lambda_0}{\left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{s}} + 1} \left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right) + \frac{\lambda_0}{N} \left( x + \xi \|w_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{s}} - 1 \right)},
\]

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and proper regularity hypotheses on the initial data, the corresponding initial-boundary problem possesses at least one global bounded classical solution when $D(0) > 0$ (the case of non-degenerate diffusion), while if, $D(0) \geq 0$ (the case of possibly degenerate diffusion), the existence of bounded weak solutions for system is shown. This extends some recent results by several authors.

Key words: Boundedness; Chemotaxis–haptotaxis; Global existence; Logistic source

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1 Introduction

Cancer invasion is a very complex process which involves various biological mechanisms (see [2, 18, 6, 7, 10, 15]). Chemotaxis is the oriented movement of cells along concentration gradients of chemicals produced by the cells themselves or in their environment, and is a significant mechanism of directional migration of cells. A well-known chemotaxis model was proposed by Keller and Segel ([24, 25]) in the 1970s, which describes the aggregation processes of the cellular slime mold Dictyostelium discoideum. Since then, the following quasi-chemotaxis-only model

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u (1 - u), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v + u - v, \quad x \in \Omega, t > 0
\end{align*}
\]

and its variations have been widely studied by many authors, where the main issue of the investigation was whether the solutions to the models are bounded or blow-up (see e.g., Herrero and Velázquez [13], Nagai et al. [33], Winkler et al. [55, 57], the survey [3]). For example, as we all known that all solutions of (1.1) are global in time and bounded when either \( N \geq 3 \) and \( \mu > 0 \) is sufficiently large (see [54] and also [63]), or \( N = 2 \) and \( \mu > 0 \) is arbitrary ([34]). Tello and Winkler ([50]) proved that the global boundedness for parabolic-elliptic chemotaxis-only system (the second equation of (1.1) is replaced by \(-\Delta v + v = u\)) exists under the condition \( \mu > \frac{(N-2)^+}{N} \chi \), moreover, they gave the weak solutions for arbitrary small \( \mu > 0 \). Some recent studies show that nonlinear chemotactic sensitivity functions ([4, 19, 16]), nonlinear diffusion ([26, 9, 38]), or also logistic dampening ([34, 50, 53, 54]) may prevent blow-up of solutions.

One important extension of the classical Keller-Segel model to a more complex cell migration mechanism was proposed by Chaplain and Lolas ([7, 8]) in order to describe processes of cancer invasion. In fact, let \( u = u(x,t) \) denote the density of the tumour cell population, \( v = v(x,t) \) represent the concentration of a matrix-degrading enzyme (MDE) and \( w = w(x,t) \) stand for the density of the surrounding tissue (extracellular matrix (ECM)). Then Chaplain and Lolas ([8]) introduced the following chemotaxis-haptotaxis system as a
model describing the process of cancer invasion

\[
\left\{
\begin{array}{ll}
    u_t = D \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\
    \tau v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\
    w_t = -vw + \eta w(1 - u - w), & x \in \Omega, t > 0,
\end{array}
\right.
\]

(1.2)

where \( \tau \in \{0, 1\} \), \( \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \), \( \partial_x \) denotes the outward normal derivative on \( \partial \Omega \), \( \chi > 0 \) and \( \xi > 0 \) measure the chemotactic and haptotactic sensitivities, respectively. Here \( D > 0 \) as well as \( \mu > 0 \) and \( \eta \geq 0 \) represent the random motility coefficient, the proliferation rate of the cells and the remodeling rate, respectively. Model (1.2) and its analogue have been extensively studied up to now (see \[43, 46, 40, 48, 47, 42, 44, 27, 5, 3, 39, 62\]). In fact, Global existence and asymptotic behavior of solutions to the haptotaxis-only system (\( \chi = 0 \) in the first equation of (1.2)) have been investigated in \[37, 51, 30, 32\] and \[40\] for the case \( \eta = 0 \) and \( \eta \neq 0 \), respectively.

In realistic situations, the renewal of the ECM occurs at much smaller timescales than its degradation (see \[43, 35, 22, 30, 32, 48, 51\]). Therefore, a choice of \( \eta = 0 \) on (1.2) seems justified (see \[43, 35, 22, 30, 32, 48, 51\]). The models mentioned above described the random part of the motion of cancer cells by linear diffusion, however, from a physical point of view migration of the cancer cells through the ECM should rather be regarded like movement in a porous medium, and so we are led to considering the cell motility \( D \) a nonlinear function of the cancer cell density. Inspired by the analysis, in this paper, we consider the following chemotaxis-haptotaxis system with nonlinear diffusion (see also \[44, 3, 52\])

\[
\left\{
\begin{array}{l}
    u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, t > 0, \\
    v_t = \Delta v + u - v, \quad x \in \Omega, t > 0, \\
    w_t = -vw + \eta w(1 - u - w), \quad x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
    u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega
\end{array}
\right.
\]

(1.3)

in a bounded domain \( \Omega \subset R^N (N \geq 1) \) with smooth boundary \( \partial \Omega \). The origin of the system was proposed by Chaplain and Lolas (\[7, 8\]) to describe cancer cell invasion into surrounding
healthy tissue. Here we assume that $D(u)$ is a nonlinear function and satisfies

$$D \in C^2([0, \infty)) \quad \text{and} \quad D(u) \geq C_D u^{m-1} \quad \text{for all } u > 0$$

(1.4)

with some $C_D > 0$ and $m > 0$. Moreover, if, $D(u)$ fulfills

$$D(u) > 0 \quad \text{for all } u \geq 0,$$

(1.5)

so the diffusion is nondegenerate and the solutions may be considered in the sense of classical.

Throughout this paper, the initial data $(u_0, v_0, w_0)$ are assumed that for some $\vartheta \in (0, 1)$

$$\begin{cases}
  u_0 \in C(\bar{\Omega}) \quad \text{with} \quad u_0 \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0 \neq 0, \\
  v_0 \in W^{1,\infty}(\Omega) \quad \text{with} \quad v_0 \geq 0 \quad \text{in} \quad \Omega, \\
  w_0 \in C^{2+\vartheta}(\bar{\Omega}) \quad \text{with} \quad w_0 > 0 \quad \text{in} \quad \bar{\Omega} \quad \text{and} \quad \frac{\partial w_0}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega.
\end{cases}$$

(1.6)

System (1.3) has been widely studied by many authors, where the main issue of the investigation was whether the solutions to the models are bounded or blow-up (see Tao-Winkler [44]). For instance, when $D$ satisfies (1.4)–(1.5), Tao and Winkler ([44]) showed that model (1.3) has global solutions provided that $m > \max\{1, \bar{m}\}$, where

$$\bar{m} := \begin{cases}
  \frac{2N^2+4N-4}{N(N+4)} & \text{if } N \leq 8, \\
  \frac{2N^2+3N+2-\sqrt{6N(N+1)}}{N(N+20)} & \text{if } N \geq 9.
\end{cases}$$

(1.7)

However, they leave a question here: “whether the global solutions are bounded”. If $N \geq 2$, the global boundedness of solutions to (1.3) has been constructed for $m > 2 - \frac{2}{N}$ (see [27, 52]) with the help of the boundedness of $\|\nabla v\|_{L^l(\Omega \times (0,T))}(1 \leq l < \frac{N}{N-1})$. Recently, we ([59]) extended these results to the cases $m > \frac{2N}{N+2}$ by using the boundedness of $\|\nabla v\|_{L^2(\Omega \times (0,T))}$. More recently, if $\mu$ is large enough, Jin [23] (see also [20]) proved that system (1.3) admits a global bounded solution for any $m > 0$. However, we should point that the cases $0 < m \leq \frac{2N}{N+2}$ and small $\mu$ remain unknown even in the case for the chemotaxis-only system (1.3), that is, $w \equiv 0$ in system (1.3). In this paper, we firstly use the boundedness of $\int_{t-1}^t \int_{\Omega} u^{\gamma_0+1}$ (see Lemma 3.6) for some $\gamma_0 > 1$, which is a new result even for chemotaxis-only system (1.3). Then, applying the standard testing procedures, we can derive the uniform boundedness of $\nabla v$ in $L^{l_0}(\Omega)$ for some $l_0 > 2$. We emphasize that the spontaneous boundedness information on
\( \nabla v \) in \( L^0(\Omega) \) (see (3.40)) plays a key role in this process. Using the \( L^0 \)-boundedness of \( \nabla v \) and \( L^1 \)-boundedness of \( u \), we can then acquire the uniform bounds of \( u \) in arbitrary large \( L^p(\Omega) \) provided that the further restriction on \( m \) is satisfied (see the proof of Lemmas 3.10 and 3.14). Finally, combining with Moser iteration method and \( L^p-L^q \) estimates for Neumann heat semigroup, we finally established the \( L^\infty \) bound of \( u \) (see Lemmas 3.15–3.16).

Motivated by the above works, this paper will focus on studying the relationship between the exponent \( m \) and the global existence of solutions to chemotaxis-haptotaxis model (1.3) with nonlinear diffusion. In fact, the aim of the present paper is to study the quasilinear chemotaxis system (1.3) under the conditions (1.4)–(1.5). For non-degenerate and degenerate diffusion both, we will show the existence of global-in-time solutions to system (1.3) that are uniformly bounded. The main results are as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a bounded domain with smooth boundary and \( \chi > 0, \xi > 0, \mu > 0 \). Assume that the nonnegative initial data \((u_0, v_0, w_0)\) fulfill (1.6). Moreover, if \( D \) satisfies (1.4)-(1.5) with

\[
m > \frac{2N}{\left(\frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)}\right)^{1/N} (1 + \frac{2N}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} - 1) - 1)}{N + \left(\frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)}\right)^{1/N} (1 + \frac{2N}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} - 1) - 1)}.
\]

then there exists a triple \((u, v, w)\) in \( C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \) which solves (1.3) in the classical sense. Moreover, both \( u, v \) and \( w \) are bounded in \( \Omega \times (0, \infty) \), that is, there exists a positive constant \( C \) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \quad t > 0.
\]

**Remark 1.1.** (i) Obviously,

\[
\mu_* = \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} > 1 \quad \text{(by using} \quad \mu > 0),
\]

then \( \gamma_{**} = (\mu_* + 1)(N + \mu_* - 1) = \mu_* + 1 + \frac{\mu_*^2 - 1}{N} > \mu_* + 1 > 2 \), hence

\[
\frac{2N}{N + \left(\frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)}\right)^{1/N} (1 + \frac{2N}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} - 1) - 1)} < \frac{2N}{N + 2} \leq 2 - \frac{2}{N},
\]

6
therefore, Theorem 1.1 extends the results of Theorem 1.1 of Zheng ([59]), the results of
Theorem 1.1 of Wang ([52]), the results of as well as of Li-Lankeit ([27]) and partly extends
the results of Theorem 1.1 of Liu et al ([31]). Here the assumption \( m > \frac{2N}{N+2} \) (see [59]) or
\( m > 2 - \frac{2}{N} \) (see [27, 31, 52]) are intrinsically required.

(ii) Obviously, for any \( N \geq 1 \),
\[
\frac{2N}{N + \frac{(\chi \max(1,\lambda_0))}{(\chi \max(1,\lambda_0)^{-\rho})} + (N + \frac{\chi \max(1,\lambda_0))}{(\chi \max(1,\lambda_0)^{-\rho})})} < \tilde{m},
\]
therefore, Theorem 1.1 extends the results of Corollary 1.2 of Tao and Winkler ([44]), who
showed the global existence of solutions the cases \( m > \tilde{m} \), where \( \tilde{m} \) is given by (1.7).

(iii) In the case \( N = 2 \), by using \( \mu > 0 \), then \( \frac{8}{4+\mu+1} < 1 \), our result improves the result
of [44] and [64], in which the assumption \( m = 1 \) or \( m > 1 \) are intrinsically required.

(iv) If \( \mu > \max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)}) \), then by (1.8), we derive that for any \( m > 0 \),
system (1.3) has a classical and bounded solution, which improves the result of [23] as well
as [20] and [5].

(v) The chemotaxis-haptotaxis system therefore has bounded solutions under the same
condition on \( m \) as the pure chemotaxis system with \( w \equiv 0 \) without logistic source (see [45]).
For \( \mu = 0 \) this condition is essentially optimal ([53]).

In the case of possibly degenerate diffusion, system (1.3) admits at least one global
bounded weak solution:

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a bounded domain with smooth boundary and
\( \chi > 0, \xi > 0, \mu > 0 \). Suppose that the initial data \((u_0, v_0, w_0)\) satisfy (1.6). Moreover, if \( D \)
satisfies (1.4) with
\[
m > \frac{2N}{N + \frac{(\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)})^{s+1}) + (N + \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)})^{s+1}}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)})^{s+1}} - 1)}{N + \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)})^{s+1}}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi ||w_0||_{L^\infty(\Omega)})^{s+1}} - 1},
\]
then system (1.3) admits at least one global weak solution \((u, v, w)\) in the sense of definition
4.1 below that exists globally in time and is bounded in the sense that (1.9) holds.
The rest of this paper is organized as follows. In the following section, we recall some preliminary results. Section 3 is devoted to a series of a priori estimates and then prove Theorem 1.1. In Section 4, applying the existence of classical solutions in the non-degenerate case, we will then complete the proof of theorem 1.2 by an approximation procedure in Section 3.

2 Preliminaries and main results

Before proving our main results, we will give some preliminary lemmas, which play a crucial role in the following proofs. As for the proofs of these lemmas, here we will not repeat them again.

Lemma 2.1. ([11, 21]) Let \( s \geq 1 \) and \( q \geq 1 \). Assume that \( p > 0 \) and \( a \in (0, 1) \) satisfy

\[
\frac{1}{2} - \frac{p}{N} = (1 - a)\frac{q}{s} + a\left(\frac{1}{2} - \frac{1}{N}\right) \quad \text{and} \quad p \leq a.
\]

Then there exist \( c_0, c'_0 > 0 \) such that for all \( u \in W^{1,2}(\Omega) \cap L^s(\Omega), \)

\[
\|u\|_{W^{p,2}(\Omega)} \leq c_0\|\nabla u\|^q_{L^2(\Omega)}\|u\|^{1-a}_{L^s(\Omega)} + c'_0\|u\|_{L^s(\Omega)}.
\]

Lemma 2.2. ([58]) Let \( 0 < \theta \leq p \leq \frac{2N}{N-2} \). There exists a positive constant \( C_{GN} \) such that for all \( u \in W^{1,2}(\Omega) \cap L^p(\Omega), \)

\[
\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|^q_{L^2(\Omega)}\|u\|^{1-a}_{L^s(\Omega)} + \|u\|_{L^s(\Omega)})
\]

is valid with \( a = \frac{N}{\theta} - \frac{N}{p} + \frac{N}{\theta} \in (0, 1). \)

Lemma 2.3. ([14, 61]) Suppose that \( \gamma \in (1, +\infty) \) and \( g \in L^\gamma((0,T);L^\gamma(\Omega)) \). Consider the following evolution equation

\[
\begin{cases}
v_t - \Delta v + v = g, & (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, T), \\
v(x, 0) = v_0(x), & (x, t) \in \Omega.
\end{cases}
\]
For each $v_0 \in W^{2,\gamma}(\Omega)$ such that $\frac{\partial v_0}{\partial \nu} = 0$, there exists a unique solution $v \in W^{1,\gamma}((0,T); L^\gamma(\Omega)) \cap L^\gamma((0,T); W^{2,\gamma}(\Omega))$. In addition, if $s_0 \in [0,T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then there exists a positive constant $\lambda_0 := \lambda_0(\Omega, \gamma, N)$ such that

$$
\int_{s_0}^{T} e^{\gamma s} \|v(\cdot, t)\|_{W^{2,\gamma}(\Omega)} ds \leq \lambda_0 \left( \int_{s_0}^{T} e^{\gamma s} \|g(\cdot, s)\|_{L^\gamma(\Omega)} ds + e^{\gamma s_0} (\|v_0(\cdot, s_0)\|_{W^{2,\gamma}(\Omega)}) \right).
$$

The following local existence result is rather standard; since a similar reasoning in [44, 58], see for example. Therefore, we only give the following lemma without proof.

**Lemma 2.4.** Assume that the nonnegative functions $u_0, v_0, w_0$ satisfies (1.6) for some $\vartheta \in (0, 1)$, $D$ satisfies (1.4) and (1.5). Then there exists a maximal existence time $T_{\text{max}} \in (0, \infty]$ and a triple of nonnegative functions

$$(u, v, w) \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \times C^0((0, T_{\text{max}}); C^2(\bar{\Omega})) \times C^{2,1}(\bar{\Omega} \times [0, T_{\text{max}}))$$

which solves (1.5) classically and satisfies $0 \leq w \leq \|w_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0, T_{\text{max}})$. Moreover, if $T_{\text{max}} < +\infty$, then

$$
\left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right) \to \infty \quad \text{as} \quad t \nearrow T_{\text{max}}. \quad (2.1)
$$

According to the above existence theory, for any $s \in (0, T_{\text{max}})$, $(u(\cdot, s), v(\cdot, s), w(\cdot, s)) \in C^2(\bar{\Omega})$. Without loss of generality, we can assume that there exists a positive constant $K$ such that

$$
\|u_0\|_{C^2(\Omega)} \leq K \quad \text{as well as} \quad \|v_0\|_{C^2(\Omega)} \leq K \quad \text{and} \quad \|w_0\|_{C^2(\Omega)} \leq K. \quad (2.2)
$$

### 3 A priori estimates

The main task of this section is to establish for estimates for the solutions $(u, v, w)$ of problem (1.3). To this end, in straightforward fashion one can check the following boundedness for $u$, which is common in chemotaxis (or chemotaxis–haptotaxis) with logistic source (see e.g. [59, 54, 52, 27]).
Lemma 3.1. There exists $C > 0$ such that the solution of (1.3) satisfies
\[ \int_{\Omega} u + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^l \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}) \quad (3.1) \]
with $l \in [1, \frac{N}{N-1})$.

Since, the third component of (1.3) can be expressed explicitly in terms of $v$. This leads to the following a one-sided pointwise estimate for $-\Delta w$ (see e.g. [49, 41, 44]):

Lemma 3.2. Let $(u, v, w)$ solve (1.3) in $\Omega \times (0, T_{\text{max}})$. Then
\[ -\Delta w(x, t) \leq \|w_0\|_{L^\infty(\Omega)} \cdot v(x, t) + \kappa \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \in (0, T_{\text{max}}), \quad (3.2) \]
where
\[ \kappa := \|\Delta w_0\|_{L^\infty(\Omega)} + 4\|\nabla \sqrt{w_0}\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \quad (3.3) \]

Now we proceed to establish the main step towards our boundedness proof. To this end, let us collect some basic estimates for $u$ and $v$ in comparatively large function spaces. In fact, relying on a standard testing procedure, we derive the following Lemma:

Lemma 3.3. For any $k > 1$, the solution $(u, v, w)$ of (1.3) satisfies that
\[ -\xi \int_{\Omega} u^{k-1} \nabla \cdot (u \nabla w) \leq \frac{(k-1)}{k} \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^k v dx + \kappa \frac{(k-1)}{k} \xi \int_{\Omega} u^k dx, \quad (3.4) \]
where $\kappa$ is the same as (3.3).

Proof. For any $k > 1$, we integrate the left hand of (3.4) and use Lemma 3.2 then get
\begin{align*}
-\xi \int_{\Omega} u^{k-1} \nabla \cdot (u \nabla w) dx &= (k-1) \xi \int_{\Omega} u^{k-1} \nabla u \cdot \nabla w dx \\
&= -\frac{(k-1)}{k} \xi \int_{\Omega} u^k \Delta w dx \\
&\leq \frac{(k-1)}{k} \xi \int_{\Omega} u^k (\|w_0\|_{L^\infty(\Omega)} v + \kappa) dx,
\end{align*}
where $\kappa$ is the same as (3.3). This directly entails (3.4).

Due to the presence of logistic source, some useful estimates for $u$ can be derived.
Lemma 3.4. (see [52, 27, 59]) Assume that $\mu > 0$. There exists a positive constant $K_0$ such that the solution $(u, v, w)$ of (1.3) satisfies

$$\int_{\Omega} u(x, t) dx \leq K_0 \quad \text{for all } t \in (0, T_{max})$$  \hspace{1cm} (3.6)

and

$$\int_{t}^{t+\tau} \int_{\Omega} u^2 \leq K_0 \quad \text{for all } t \in (0, T_{max} - \tau),$$  \hspace{1cm} (3.7)

where we have set

$$\tau := \min \left\{ 1, \frac{1}{6} T_{max} \right\}.$$  \hspace{1cm} (3.8)

In order to establish some estimates for solution $(u, v, w)$, we first recall the following lemma proved in [52] (see also [27, 59]).

Lemma 3.5. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary. Then for all $k > 1$, the solution $(u, v, w)$ of (1.3) satisfies that

$$\frac{1}{k} \frac{d}{dt} \| u \|^k_{L^k(\Omega)} + \frac{(k-1)C_D}{2} \int_{\Omega} u^{m+k-3} |\nabla u|^2 + \mu \int_{\Omega} u^{k+1} \leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{k-1}$$

$$+ \mu \int_{\Omega} u^{k} (1 - u - w)$$

$$\leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{k-1}$$

$$+ \mu \int_{\Omega} u^{k} (1 - u) \quad \text{for all } t \in (0, T_{max})$$  \hspace{1cm} (3.9)

for all $t \in (0, T_{max})$.

Proof. Multiplying (1.3)1 (the first equation of (1.3)) by $u^{k-1}$ and integrating over $\Omega$, we get

$$\frac{1}{k} \frac{d}{dt} \| u \|^k_{L^k(\Omega)} + \frac{(k-1)C_D}{2} \int_{\Omega} u^{m+k-3} |\nabla u|^2 \leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{k-1}$$

$$+ \mu \int_{\Omega} u^{k} (1 - u - w)$$

$$\leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{k-1}$$

$$+ \mu \int_{\Omega} u^{k} (1 - u) \quad \text{for all } t \in (0, T_{max})$$  \hspace{1cm} (3.10)

according to the nonnegativity of $w$. Integrating by parts to the first term on the right hand side of (3.10) and using the Young inequality, we obtain

$$-\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1}$$

$$= (k-1)\chi \int_{\Omega} u^{k-1} \nabla u \cdot \nabla v$$

$$\leq \frac{(k-1)C_D}{2} \int_{\Omega} u^{m+k-3} |\nabla u|^2 + \frac{\chi^2 (k-1)}{2C_D} \int_{\Omega} u^{k+1-m} |\nabla v|^2.$$  \hspace{1cm} (3.11)
On the other hand, due to Lemma 3.3, we have
\[
\begin{align*}
-\xi & \int_{\Omega} \nabla \cdot (u\nabla w) u^{k-1} \\
& \leq \frac{(k-1)}{k} \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^k v dx + \kappa \frac{(k-1)}{k} \xi \int_{\Omega} u^k dx \\
& \leq \frac{(k-1)}{k} \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^k v dx + \kappa \xi \int_{\Omega} u^k dx.
\end{align*}
\]
(3.12)

Furthermore, inserting (3.11)–(3.12) into (3.10), we conclude that for all \(t \in (0, T_{\text{max}})\),
\[
\begin{align*}
\frac{1}{k} \frac{d}{dt} \|u\|_{L^k(\Omega)} + \frac{(k-1)C_D}{2} \int_{\Omega} u^{m+k-3} \vert \nabla u \vert^2 + \mu \int_{\Omega} u^{k+1} \\
& \leq \frac{\chi^2(k-1)}{2C_D} \int_{\Omega} u^{k+1-\gamma_0} \vert \nabla v \vert^2 + \frac{(k-1)}{k} \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^k v + (\mu + \kappa \xi) \int_{\Omega} u^k.
\end{align*}
\]
(3.13)

\[\square\]

We proceed to estimate both integrals on the right of (3.9) in a straightforward manner.

**Lemma 3.6.** Let \((u, v, w)\) be a solution to (1.3) on \((0, T_{\text{max}})\) and
\[
\mu_* = \frac{\max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) - \mu_0^+}.
\]
(3.14)

If \(\mu > 0\), then for all \(1 < \gamma_0 < \mu_*\), there exists a positive constant \(C\) which depends on \(\gamma_0\) such that
\[
\int_{\Omega} u^{\gamma_0}(x, t) \leq C \quad \text{for all } t \in (0, T_{\text{max}})
\]
(3.15)

and
\[
\int_0^t \int_{\Omega} u^{\gamma_0+1}(x, t) \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]
(3.16)

**Proof.** Multiplying (1.3) by \(u^{k-1}\), integrating over \(\Omega\) and using \(w \geq 0\), we get
\[
\begin{align*}
\int \frac{1}{k} \frac{d}{dt} \|u\|_{L^k(\Omega)}^k + C_D(k-1) \int \nabla u^{m+k-3} \vert \nabla u \vert^2 dx \\
& \leq -\chi \int \nabla \cdot (u\nabla v) u^{k-1} dx - \xi \int \nabla \cdot (u\nabla w) u^{k-1} + \mu \int u^k (1 - u - w) \\
& \leq -\chi \int \nabla \cdot (u\nabla v) u^{k-1} dx - \xi \int \nabla \cdot (u\nabla w) u^{k-1} + \mu \int u^k (1 - u).
\end{align*}
\]
(3.17)
We now estimate the right hand side of (3.17) terms by terms. To this end, integrating by parts to the first term on the right hand side of (3.17), we obtain for any $\varepsilon_1 > 0$,

\[
-\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{k-1} = -\frac{(k-1)\chi}{k} \int_{\Omega} u^k \Delta v \\
\leq \frac{(k-1)\chi}{k} \int_{\Omega} u^k |\Delta v| \\
\leq \varepsilon_1 \int_{\Omega} u^{k+1} + \gamma_1 \varepsilon_1^{-k} \int_{\Omega} |\Delta v|^{k+1},
\]

(3.18)

where

\[
\gamma_1 = \frac{1}{k+1} \left( \frac{k+1}{k} \right)^{-k} \left( \frac{(k-1)\chi}{k} \right)^{k+1}.
\]

(3.19)

Due to (3.2) and (3.3), it follows that for any $\varepsilon_2 > 0$

\[
-\xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{k-1} = -\frac{(k-1)\xi}{k} \int_{\Omega} u^k \Delta w \\
\leq \frac{k}{k} \int_{\Omega} u^k + \frac{(k-1)\xi \|w_0\|_{L^\infty(\Omega)}}{k} \int_{\Omega} u^k v \\
\leq \frac{k\xi}{\Omega} \int_{\Omega} u^k + \varepsilon_2 \int_{\Omega} u^{k+1} + \gamma_2 \varepsilon_2^{-k} \int_{\Omega} u^{k+1},
\]

(3.20)

where

\[
\gamma_2 := \frac{1}{k+1} \left( \frac{k+1}{k} \right)^{-k} \left( \frac{(k-1)\xi \|w_0\|_{L^\infty(\Omega)}}{k} \right)^{k+1}
\]

(3.21)

and $\kappa$ is give by (3.3).

On the other hand, in view of $k > 1$, we also derive that

\[
\mu \int_{\Omega} u^k (1 - u) = -\mu \int_{\Omega} u^{k+1} + \left( \mu + \frac{k+1}{k} \right) \int_{\Omega} u^k - \frac{k+1}{k} \int_{\Omega} u^k \\
\leq -\mu \int_{\Omega} u^{k+1} + \left( \mu + 2 \right) \int_{\Omega} u^{k+1} - \frac{k+1}{k} \int_{\Omega} u^k.
\]

(3.22)

Therefore, combined with (3.18), (3.20), (3.17) as well as (3.22) and (1.5), we have

\[
\frac{1}{k} \int_{\Omega} \frac{d}{dt} \|u\|_{L^k(\Omega)}^k + C_D(k-1) \int_{\Omega} u^{m+k-3} |\nabla u|^2 + \frac{k+1}{k} \int_{\Omega} u^k \\
\leq (-\mu + \varepsilon_1 + \varepsilon_2) \int_{\Omega} u^{k+1} + \gamma_1 \varepsilon_1^{-k} \int_{\Omega} |\Delta v|^{k+1} + \gamma_2 \varepsilon_2^{-k} \int_{\Omega} v^{k+1} + C_1 \int_{\Omega} u^k
\]

(3.23)
with $C_1 = \kappa \xi + \mu + 2$. For any $t \in (0, T_{\text{max}})$, applying the Gronwall Lemma to the above inequality shows that

\[
\frac{1}{k} \|u(\cdot, t)\|_{L^k(\Omega)}^k + C_D(k - 1) \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} u^{m+k-3} |\nabla u|^2 \, ds \, dx \\
\leq \frac{1}{k} e^{-(k+1)t} \|u_0\|_{L^k(\Omega)}^k + (\varepsilon_1 + \varepsilon_2 - \mu) \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} \, ds \\
+ \gamma_1 \varepsilon_1^{-k} \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} |\Delta v|^{k+1} \, ds + C_1 \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} u^k \, ds \\
+ \gamma_2 \varepsilon_2^{-k} \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} v^{k+1} \, ds + C_2,
\]

\[ (3.24) \]

where

\[ C_2 := C_2(k) = \frac{1}{k} \|u_0\|_{L^k(\Omega)}^k. \]

Next, a use of Lemma 2.3 and (2.2) leads to

\[
\gamma_1 \varepsilon_1^{-k} \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} |\Delta v|^{k+1} \, ds \\
= \gamma_1 \varepsilon_1^{-k} e^{-(k+1)t} \int_0^t e^{(k+1)s} \int_{\Omega} |\Delta v|^{k+1} \, ds \\
\leq \gamma_1 \varepsilon_1^{-k} e^{-(k+1)t} \lambda_0 \left( \int_0^t \int_{\Omega} e^{(k+1)s} u^{k+1} \, ds \right) \leq \gamma_1 \varepsilon_1^{-k} e^{-(k+1)t} \lambda_0 \left( \|v_0\|_{W^{2,k+1}(\Omega)}^{k+1} \right)
\]

\[ (3.25) \]

and

\[
\gamma_2 \varepsilon_2^{-k} \int_0^t e^{-(k+1)(t-s)} \int_{\Omega} v^{k+1} \, ds \\
= \gamma_2 \varepsilon_2^{-k} e^{-(k+1)t} \int_0^t e^{(k+1)s} \int_{\Omega} v^{k+1} \, ds \\
\leq \gamma_2 \varepsilon_2^{-k} e^{-(k+1)t} \lambda_0 \left( \int_0^t \int_{\Omega} e^{(k+1)s} u^{k+1} \, ds \right) \leq \gamma_2 \varepsilon_2^{-k} e^{-(k+1)t} \lambda_0 \left( \|v_0\|_{W^{2,k+1}(\Omega)}^{k+1} \right)
\]

\[ (3.26) \]

for all $t \in (0, T_{\text{max}})$, where $\lambda_0$ is the same as Lemma 2.3. On the other hand, choosing $\varepsilon_1 = \frac{(k-1)\xi}{k+1} \lambda_0^{-\frac{1}{k+1}}$ and $\varepsilon_2 = \frac{(k-1)\xi}{k+1} \lambda_0^{-\frac{1}{k+1}}$, with the help of (3.19) and (3.21), a simple calculation shows that

\[ \varepsilon_1 + \gamma_1 \lambda_0 \varepsilon_1^{-k} = \frac{(k-1)}{k} \lambda_0^{-\frac{1}{k+1}} \xi \]

and

\[ \varepsilon_2 + \gamma_2 \lambda_0 \varepsilon_2^{-k} = \frac{(k-1)}{k} \lambda_0^{-\frac{1}{k+1}} \xi \|w_0\|_{L^{\infty}(\Omega)}, \]

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so that, substituting (3.25)–(3.26) into (3.24) implies that

\[
\frac{1}{k} \|u(\cdot, t)\|_{L^k(\Omega)}^k + C_D (k-1) \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^{m+k-3} |\nabla u|^2 ds dx \\
\leq (\varepsilon_1 + \gamma_1 \lambda_0 \varepsilon_1^{-k} + \varepsilon_2 + \gamma_2 \lambda_0 \varepsilon_2^{-k} - \mu) \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^{k+1} ds dx \\
+ (\gamma_1 \varepsilon_1^{-k} + \gamma_2 \varepsilon_2^{-k}) e^{-(k+1)(t-s)} \lambda_0 \|v_0\|_{W^{2,k+1}(\Omega)}^{k+1} + C_1 \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^k ds dx + C_2 \\
= \left( \frac{(k-1)}{k} \lambda_0 \frac{1}{\varepsilon_1^{(k+1)}} \chi + \frac{(k-1)}{k} \frac{1}{\lambda_0} \xi \|w_0\|_{L^\infty(\Omega)} - \mu \right) \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^{k+1} ds dx \\
+ (\gamma_1 \varepsilon_1^{-k} + \gamma_2 \varepsilon_2^{-k}) e^{-(k+1)(t-s)} \lambda_0 \|v_0\|_{W^{2,k+1}(\Omega)}^{k+1} + C_1 \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^k ds dx + C_2 \\
\leq \left( \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0 \frac{1}{s^{(k+1)}} \left( \chi + \xi \|w_0\|_{L^\infty(\Omega)} \right) \right) \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^{k+1} ds dx \\
+ C_1 \int_0^t e^{-(k+1)(t-s)} \int_\Omega u^k ds dx + C_3 
\]

(3.27)

with

\[
C_3 = (\gamma_1 \varepsilon_1^{-k} + \gamma_2 \varepsilon_2^{-k}) e^{-(k+1)(t-s)} \lambda_0 \|v_0\|_{W^{2,k+1}(\Omega)}^{k+1} + C_2.
\]

For any \( \varepsilon > 0 \), we choose \( k = \frac{\max_{s \geq 1} \lambda_0 \frac{1}{s^{(k+1)}} \left( \chi + \xi \|w_0\|_{L^\infty(\Omega)} \right)}{(\max_{s \geq 1} \lambda_0 \frac{1}{s^{(k+1)}} \left( \chi + \xi \|w_0\|_{L^\infty(\Omega)} \right) - \mu)} - \varepsilon \). Then

\[
\frac{(k-1)}{k} \max_{s \geq 1} \lambda_0 \frac{1}{s^{(k+1)}} \left( \chi + \xi \|w_0\|_{L^\infty(\Omega)} \right) < \mu.
\]

Thus, by using the Young inequality, we derive that there exists a positive constant \( C_4 \) such that

\[
\int_\Omega u^k(x, t) dx \leq C_4 \text{ for all } t \in (0, T_{\max})
\]

(3.28)

and

\[
\int_0^t \int_\Omega u^{k+1}(x, t) dx \leq C_4 \text{ for all } t \in (0, T_{\max}).
\]

(3.29)

Thereupon, combining with the arbitrariness of \( \varepsilon \) and the Hölder inequality, (3.15) and (3.16) holds. The proof of Lemma 3.6 is completed. \( \square \)

When

\[
\mu \geq \max_{s \geq 1} \lambda_0 \frac{1}{s^{(k+1)}} \left( \chi + \xi \|w_0\|_{L^\infty(\Omega)} \right),
\]

by making use of above lemma, we can derive the following results on the bound \( u \) for in an \( L^k \) space for any \( k > 1 \).
Corollary 3.1. Let \((u, v, w)\) be a solution to (1.3) on \((0, T_{\text{max}})\). If
\[
\mu \geq \max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}),
\]
then for all \(k > 1\), there exists a positive constant \(C\) which depends on \(k\) such that
\[
\int_{\Omega} u^k(x, t) \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\] (3.30)

Proof. This directly results from Lemma 3.6 and the fact that
\[
\max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) = +\infty
\]
by using \(\mu \geq \max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})\). \(\Box\)

In the following, we always assume that
\[
\mu < \max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}),
\]
since, case \(\mu \geq \max_{s \geq 1} \frac{1}{\lambda_0^{s+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})\) has been proved by Corollary 3.1.

Lemma 3.7. Let \((u, v, w)\) be a solution to (1.3) on \((0, T_{\text{max}})\) and \(\Omega \subset \mathbb{R}^N (N \geq 1)\) be a bounded domain with smooth boundary. Then for all \(\beta > 1\), there exists \(\kappa_0 > 0\) such that
\[
\frac{1}{2\beta} \frac{d}{dt} \|\nabla v\|_{L^{2\beta}(\Omega)}^{2\beta} + \frac{(\beta - 1)}{\beta^2} \int_{\Omega} |\nabla |\nabla v|^\beta|^2
\]
\[
+ \frac{1}{2} \int_{\Omega} |\nabla v|^{2\beta - 2} |D^2 v|^2 + \int_{\Omega} |\nabla v|^{2\beta}
\]
\[
\leq \kappa_0 \int_{\Omega} u^2 |\nabla v|^{2\beta - 2} + \kappa_0 \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\] (3.31)

Proof. Using that \(\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2\), by a straightforward computation using
the second equation in (1.3) and several integrations by parts, we find that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^{2\beta}(\Omega)}^{2\beta} = \int_\Omega |\nabla v|^{2\beta-2} \nabla v \cdot \nabla (v - v + u) \\
\frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \Delta |\nabla v|^2 - \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 \\
- \int_\Omega |\nabla v|^{2\beta} - \int_\Omega u \nabla \cdot (|\nabla v|^{2\beta-2} \nabla v) \\
= - \frac{\beta - 1}{2} \int_\Omega |\nabla v|^{2\beta-4} |\nabla |\nabla v|^2|^2 + \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \frac{\partial |\nabla v|^2}{\partial \nu} - \int_\Omega |\nabla v|^{2\beta} \\
- \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 - \int_\Omega u |\nabla v|^{2\beta-2} \Delta v - \int_\Omega u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) \\
= - \frac{2(\beta - 1)}{\beta^2} \int_\Omega |\nabla |\nabla v|^\beta|^2 |\nabla v|^{2\beta-2} + \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \frac{\partial |\nabla v|^2}{\partial \nu} - \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 \\
- \int_\Omega u |\nabla v|^{2\beta-2} \Delta v - \int_\Omega u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) - \int_\Omega |\nabla v|^{2\beta}
\]

for all \(t \in (0, T_{\text{max}})\). Here, since \(|\Delta v| \leq \sqrt{N}|D^2 v|\), by the Young inequality, we can estimate

\[
\int_\Omega u |\nabla v|^{2\beta-2} \Delta v \leq \sqrt{N} \int_\Omega u |\nabla v|^{2\beta-2} |D^2 v| \\
\leq \frac{1}{4} \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 + N \int_\Omega u^2 |\nabla v|^{2\beta-2} \tag{3.33}
\]

for all \(t \in (0, T_{\text{max}})\). As moreover by the Cauchy–Schwarz inequality, we have

\[
- \int_\Omega u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) = -(\beta - 1) \int_\Omega u |\nabla v|^{2(\beta-2)} \nabla v \cdot \nabla |\nabla v|^2 \\
\leq \frac{\beta - 1}{8} \int_\Omega |\nabla v|^{2\beta-4} |\nabla |\nabla v|^2|^2 + 2(\beta - 1) \int_\Omega u^2 |\nabla v|^{2\beta-2} \tag{3.34}
\]

Next we deal with the integration on \(\partial \Omega\). We see from Lemma 2.1 that

\[
\int_{\partial \Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2\beta-2} \leq C_\Omega \int_{\partial \Omega} |\nabla v|^{2\beta} \tag{3.35}
\]

\[
= C_\Omega |\nabla v|^{2\beta} |L^2(\partial \Omega)|.
\]

Let us take \(r \in (0, \frac{1}{2})\). By the embedding \(W^{r+\frac{1}{2}, 2}(\Omega) \hookrightarrow L^2(\partial \Omega)\) is compact (see e.g. Haroske and Triebel [12]), we have

\[
|||\nabla v|^{\beta}||^2_{L^2(\partial \Omega)} \leq C_3 |||\nabla v|^{\beta}||^2_{W^{r+\frac{1}{2}, 2}(\Omega)} \tag{3.36}
\]
In order to apply Lemma 2.1 to the right-hand side of (3.36), let us pick \( a \in (0,1) \) satisfying
\[
a = \frac{1}{2N} + \frac{\beta}{2} + \frac{\gamma}{N} - \frac{1}{2}.
\]
Noting that \( \gamma \in (0,\frac{1}{2}) \) and \( \beta > 1 \) imply that \( \gamma + \frac{\beta}{2} \leq a < 1 \), we see from the fractional Gagliardo–Nirenberg inequality (Lemma 2.1) and boundedness of \(|\nabla v|^{\beta}\) (see Lemma 3.1) that
\[
|\nabla v|_2^\beta \leq c_0|\nabla v|_2^\beta + c_0'\|\nabla v\|_2^\beta + c_4|\nabla v|_2^\beta + C_4.
\]
(3.37)

Combining (3.35) and (3.36) with (3.37), we obtain
\[
\int_{\partial\Omega} \frac{\partial |\nabla v|}{\partial \nu} |\nabla v|^{2\beta-2} \leq C_5 |\nabla v|_2^\beta + C_5.
\]
(3.38)

Now, inserting (3.34)–(3.38) into (3.32) and using the Young inequality we can get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^{2\beta} + \frac{3(\beta - 1)}{4\beta^2} \int_{\Omega} |\nabla v|^{2\beta} + \frac{1}{2} \int_{\Omega} |\nabla v|^{2\beta - 2} |D^2 v|^2 + \int_{\Omega} |\nabla v|^{2\beta} \leq C_6 \int_{\Omega} u^2 |\nabla v|^{2\beta - 2} + C_6 \quad \text{for all } t \in (0,T_{max})
\]
(3.39)

by using the Young inequality.

We proceed to establish the main step towards our boundedness proof. The following lemma can be used to improve our knowledge on integrability of \( \nabla v \), provided that \( \mu > 0 \). Its repeated application will form the core of our regularity proof.

**Lemma 3.8.** Let \((u,v,w)\) be a solution to (1.3) on \((0,T_{max})\) and \( \mu > 0 \). Then for any \( 1 < \gamma_0 < \mu_* \), there exists \( C > 0 \) such that
\[
|\nabla v(\cdot,t)|_{L^{(\gamma_0+1)\frac{N+\gamma_0-1}{N}}^{\frac{N+\gamma_0-1}{\gamma_0}}(\Omega)} \leq C \quad \text{for all } t \in (0,T_{max}),
\]
(3.40)

where \( \mu_* \) is given by (3.14).

**Proof.** Let \( \gamma_0 \) and \( \mu_* \) be same as Lemma 3.6. For the above \( 1 < \gamma_0 < \mu_* \), we choose \( \beta = \frac{\gamma_0+1}{2} \) in (3.31). Then by using the Young inequality, we derive that for some positive constant \( C_1 \),
\[
\kappa_0 \int_{\Omega} u^2 |\nabla v|^{2\beta-2} = \kappa_0 \int_{\Omega} u^2 |\nabla v|^{\gamma_0-1}
\leq \frac{1}{2} \int_{\Omega} |\nabla v|^{\gamma_0+1} + C_1 \int_{\Omega} u^{\gamma_0+1} \quad \text{for all } t \in (0,T_{max}).
\]
(3.41)
Here $\kappa_0$ is the same as (3.31). Inserting (3.41) into (3.31), we conclude that there exists a positive constant $C_2$ such that

\[
\frac{1}{\gamma_0 + 1} \frac{d}{dt} \| \nabla v \|_{L^{\gamma_0 + 1}(\Omega)}^{\gamma_0 + 1} + \frac{3(\gamma_0 + 1)}{(\gamma_0 + 1)^2} \int_{\Omega} \left| \nabla \| \nabla v \|_{\gamma_0 + 1} \right|^2 \\
+ \frac{1}{2} \int_{\Omega} \left| \nabla v \right|^{\gamma_0 - 1} |Dv|^2 + \frac{1}{2} \int_{\Omega} \left| \nabla v \right|^{\gamma_0 + 1} \\
\leq C_1 \int_{\Omega} u^{\gamma_0 + 1} + C_2 \text{ for all } t \in (0, T_{\text{max}}),
\]

which combined with (3.16) implies that

\[
\int_{\Omega} \left| \nabla v \right|^{\gamma_0 + 1}(x,t) \, dx \leq C_3 \text{ for all } t \in (0, T_{\text{max}})
\]

by an ODE comparison argument. On the other hand, for any $\beta > 1$, it then follows from Lemma 2.2 that there exist positive constants $\kappa_1$ and $\kappa_2$ such that

\[
\| \nabla v \|_{L^2, \frac{2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}^{\frac{2\beta + 2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}(\Omega) = \| \left| \nabla \right|^{\beta} \|_{L^2, \frac{2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}^{\frac{2\beta + 2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}(\Omega) \\
\leq \kappa_1 (\left| \nabla \right| \| \nabla v \|^{\beta} \|_{L^2(\Omega)}^{2(\gamma_0 + 1)} \left| \nabla v \right|^{\beta} \|_{L^2(\Omega)}^{\frac{2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}(\Omega) + \left| \nabla \right| \| \nabla v \|^{\beta} \|_{L^2(\Omega)}^{\frac{2(\gamma_0 + 1)}{N(\gamma_0 + 1)}}(\Omega) \\
\leq \kappa_2 (\left| \nabla \right| \| \nabla v \|^{\beta} \|_{L^2(\Omega)}^{2(\gamma_0 + 1)} + 1)
\]

by using (3.43). Next, picking $\beta = \frac{(\gamma_0 + 1)(N+\gamma_0-1)}{2N}$ in (3.31), then $\beta > 1$, so that, by (3.31), we derive that

\[
\frac{1}{(\gamma_0 + 1)(N+\gamma_0-1)} \frac{d}{dt} \| \nabla v \|_{L^\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{N}(\Omega)}^{\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{N}}(\Omega) \\
+ 3(\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{2N} - 1) \int_{\Omega} \left| \nabla \| \nabla v \|^{\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{2N}} \right|^2 \\
+ \frac{1}{2} \int_{\Omega} \left| \nabla v \right|^{\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{N}} |Dv|^2 + \int_{\Omega} \left| \nabla v \right|^{\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{N}} \\
\leq \kappa_0 \int_{\Omega} u^2 \left| \nabla v \right|^{\frac{(\gamma_0 + 1)(N+\gamma_0-1)}{N}} + \kappa_0 \\
\leq \frac{(\gamma_0 + 1)(N+\gamma_0-1)}{2N} \| \nabla v \|^2 \| \nabla v \|^\beta \|_{L^\frac{2(\gamma_0 + 1)}{N}(\Omega)}^{\frac{2\beta + 2(\gamma_0 + 1)}{N}}(\Omega) + C_4 \int_{\Omega} u^{\gamma_0 + 1} + C_5 \text{ for all } t \in (0, T_{\text{max}}),
\]

(3.45)
where $\kappa_2$ is the same as (3.44). Therefore, collecting (3.44) and (3.45), we have

$$\frac{1}{N} \frac{d}{dt} \| \nabla v \|_{L^{(\gamma_0+1)(N+\gamma_0-1)/N}(\Omega)}$$

$$+ 2(\frac{(\gamma_0+1)(N+\gamma_0-1)}{N})^2 \int_{\Omega} |\nabla v|^{(\gamma_0+1)(N+\gamma_0-1)/N} \leq C_4 \int_{\Omega} u^{\gamma_0+1} + C_5$$

for all $t \in (0, T_{\text{max}})$, therefore, in view of (3.16), by using an ODE comparison argument again, we have

$$\| \nabla v \|_{L^{(\gamma_0+1)(N+\gamma_0-1)/N}(\Omega)} \leq C_6$$

with some positive constant $C_6$, which yields (3.40), and hence completes the proof.

Lemma 3.9. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary. Then for all $\beta > 1$ and $k > 1$, the solution of (1.3) from Lemma 2.4 satisfies

$$\frac{d}{dt} \left( \frac{1}{k} \| u \|_{L_k(\Omega)} \right) + \frac{1}{2\beta} \| \nabla v \|_{L^2(\Omega)}^{2\beta} + \frac{3(\beta-1)}{4\beta^2} \int_{\Omega} |\nabla |\nabla v|^{\beta}|^2 + \frac{\mu}{2} \int_{\Omega} u^{k+1}$$

$$+ \frac{1}{2} \int_{\Omega} |\nabla v|^{2\beta-2} |D^2 v|^2 + \int_{\Omega} |\nabla v|^{2\beta} + \frac{(k-1)m}{4} \int_{\Omega} u^{m+k-3} |\nabla u|^2$$

$$\leq C \left( \frac{(k-1)^2}{2C_D} \right) \int_{\Omega} u^{k+1-m} |\nabla v|^2 + \int_{\Omega} u^2 |\nabla v|^{2\beta-2} + \int_{\Omega} v^{k+1} + C,$$

where $C$ is a positive constant.

Proof. Collecting Lemma 3.5 and Lemma 3.8 we can derive (3.48) by using the Young inequality.

We next plan to estimate the right-hand sides in the above inequalities appropriately by using a priori information provided by Lemma 3.8 and Lemma 3.1. Here the following lemma will play an important role in making efficient use of the known $L^{(\gamma_0+1)(N+\gamma_0-1)/N}(\Omega)$ bound for $\nabla v$. The following lemma provides some elementary material that will be essential to our bootstrap procedure.

Lemma 3.10. Let

$$\tilde{H}(y) = \frac{2N^2}{N^2+(y+1)(N+y-1)} - \left[ 1 + \frac{[N^2-(y+1)(N+y-1)/N]}{N(y+1)(N+y-1)} \right]$$

(3.49)
with \((y + 1)(N + y - 1) > N^2\), for any \(y > 1\) and \(N \geq 2\). Then we have
\[
\min_{y>1} \tilde{H}(y) \geq 0. \tag{3.50}
\]

Proof. It is easy to verify that \(N^2 < (y + 1)(N + y - 1)\) and \(y > 1\) and \(N \geq 2\) implies that
\[
y > -\frac{N + \sqrt{N^2 - 4N + 4}}{2} = \frac{-N + \sqrt{4N^2 + (N - 2)^2}}{2} \geq \frac{N}{2}. \tag{3.51}
\]

On the other hand, by some basic calculation, one has
\[
\frac{2N^2}{N^2 + (y + 1)(N + y - 1)} - [1 + \frac{[N^2 - (y + 1)(N + y - 1)]y}{N(y + 1)(N + y - 1)}] = \frac{[(y + 1)(N + y - 1) - N^2][\frac{y}{N(y + 1)(N + y - 1)} - \frac{1}{N^2 + (y + 1)(N + y - 1)}]}{N(y + 1)(N + y - 1)[N^2 + (y + 1)(N + y - 1)]}h_1(y)
\]
with
\[
h_1(y) = \left[ yN^2 + y(y + 1)(y + N - 1) - N(y + 1)(N + y - 1) \right] = y^3 + (N - 1)y - N^2 + N. \tag{3.52}
\]

Now, by some basic calculation, one has,
\[
h_1'(y) = 3y^2 + (N - 1) > 0.
\]
by using \(N \geq 2\) and \(y > 1\). Therefore, by (3.51), we have
\[
h_1(y) \geq h_1\left(\frac{N}{2}\right) = \frac{N^3}{8} - \frac{N^2}{2} + \frac{N}{2} \tag{3.53}
\]
\[
= : \tilde{h}_1(N).
\]

Since, \(\tilde{h}_1'(N) = \frac{3N^2}{8} - N + \frac{1}{2} > 0\) by using \(N \geq 2\). Thus, \(\tilde{h}_1(N) > \tilde{h}_1(2) = 0\), so that, inserting (3.53)–(3.54) into (3.52) and \((y + 1)(N + y - 1) > N^2\), we obtain that
\[
\frac{2N^2}{N^2 + (y + 1)(N + y - 1)} > [1 + \frac{[N^2 - (y + 1)(N + y - 1)]y}{N(y + 1)(N + y - 1)}]. \tag{3.55}
\]

Now, we can make use of Lemma 3.4 as well as Lemma 2.2 and Lemma 3.8 to estimate the integrals on the right-hand sides of (3.31) and (3.9) (or (3.48)). To this end, we will establish bounds for \(\int_{\Omega} u^k dx\) with any \(k > 1\) by Lemma 3.4 as well as Lemma 2.2 and Lemma 3.10.
Lemma 3.11. Assume that $m > \frac{2N}{N+\gamma_*}$ with $N = 2$, where

$$
\gamma_* = \frac{(\mu_* + 1)(N + \mu_* - 1)}{N}
$$

(3.56)

and $\mu_*$ is same as (3.14). Then for all $k > 1$, there exists $C > 0$ such that

$$
\|u(\cdot, t)\|_{L^k(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
$$

(3.57)

Proof. Next, due to (3.56) and $N = 2$ and $m > \frac{2N}{N+\gamma_*}$ implies that

$$
m > \frac{8}{4 + [\mu_* + 1]^2},
$$

(3.58)

where $\mu_*$ is given by (3.14). Now, in view of $\mu > 0$ implies that

$$
\mu_* > 1
$$

and

$$
4 < (\mu_* + 1)^2,
$$

therefore, employing Lemma 3.10, we have

$$
m > 1 + \frac{[1 - \frac{(\mu_* + 1)^2}{4}][\mu_*]}{2 \times \frac{(\mu_* + 1)^2}{4}}.
$$

(3.59)

Thus, we may choose $q_0 \in (1, \mu_*)$ which is close to $\mu_*$ such that

$$
m > 1 + \frac{[1 - \frac{(q_0 + 1)^2}{4}][q_0]}{2 \times \frac{(q_0 + 1)^2}{4}}.
$$

(3.60)

Next, observing that $\frac{(q_0 + 1)^2}{4} \in (1, \frac{(\mu_* + 1)^2}{4})$, thus, in light of Lemma 3.8, we derive that there exists a positive constant $C_1$ such that

$$
\|\nabla v(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}}),
$$

(3.61)

where

$$
p_0 = \frac{(q_0 + 1)^2}{4} > 1
$$

(3.62)

by using $q_0 > 1$. 
Now, choosing $k > \max\{3, |1 - m| + q_0 \frac{p_0 - 1}{p_0}\}$ in (3.60), then, we have

\[
\frac{1}{k} \frac{d}{dt} \|u\|^{k}_{L^{k}(\Omega)} + \frac{(k - 1)C_{D}}{2} \int_{\Omega} u^{m+k-3} \nabla u |^{2} + \mu \int_{\Omega} u^{k+1} \\
\leq \frac{\chi^{2}(k-1)}{2C_{D}} \int_{\Omega} u^{k+1-m} \nabla v |^{2} + C_{2} \int_{\Omega} u^{k}(v + 1)
\]

(3.63)

with $C_{2} = \max\{\xi \|w_{0}\|_{L^{\infty}(\Omega)}, \mu + \kappa \xi\}$. We estimate the rightmost integral by means of the Hölder inequality according to

\[
\frac{\chi^{2}(k-1)}{2C_{D}} \int_{\Omega} u^{k+1-m} \nabla v |^{2} \\
\leq \frac{\chi^{2}(k-1)(\int_{\Omega} u^{\frac{m}{m-1}(k+1-m)} \frac{p_{0}-1}{p_{0}}) \frac{p_{0}-1}{p_{0}} (\int_{\Omega} |\nabla v|^{2p_{0}})}{C_{3}\chi^{2}(k-1)} \leq \frac{C_{3}}{2C_{D}} \|u\|_{L^{\infty}(\Omega)}^{k+1-m} \nabla v |^{2} \|u\|_{L^{\infty}(\Omega)}^{2k-1} \frac{1}{p_{0}}
\]

(3.64)

by using (3.61), where $C_{3} > 0$. Since, $k > |1 - m| + q_0 \frac{p_0 - 1}{p_0}$, we have

\[
\frac{q_0}{k + m - 1} \leq \frac{p_{0}(k + 1 - m)}{(p_0 - 1)(k + m - 1)} < +\infty,
\]

so that, the Gagliardo-Nirenberg inequality (Lemma 2.2) indicates that

\[
\frac{\chi^{2}(k-1)}{2C_{D}} \|u\|_{L^{k+1-m}}^{k+1-m} \nabla v |^{2} \|u\|_{L^{\infty}(\Omega)}^{2(k+1-m)} \frac{1}{p_{0}} \left(\frac{\int_{\Omega} |\nabla v|^{2p_{0}}}{\int_{\Omega} u^{(p_0 - 1)(k+1-m)}} \right) \\
\leq C_{4}(\|\nabla u\|^{k+1-m} \|u\|_{L^{2}(\Omega)}^{k+1-m} \|u\|_{L^{\infty}(\Omega)}^{2(k+1-m)} \frac{1}{p_{0}}) \\
\leq C_{5}(\|\nabla u\|^{k+1-m} \|u\|_{L^{2}(\Omega)}^{k+1-m} \|u\|_{L^{\infty}(\Omega)}^{2(k+1-m)} \frac{1}{p_{0}} + 1)
\]

(3.65)

with some positive constants $C_{4}$ as well as $C_{5}$, where $q_0$ is the same as (3.60). Here we have used $L^{1}(\Omega)$ boundedness for $u$ (see Lemma (3.1)). Due to (3.60), one has

\[
\frac{2(k + 1 - m)}{k + m - 1} - \frac{2(p_0 - 1)q_0}{p_0(k + m - 1)} < 2,
\]

so that, applying the Young inequality implies that there exists a positive constant $C_{7}$ such that

\[
\frac{1}{k} \frac{d}{dt} \|u\|_{L^{k}(\Omega)}^{k} + \frac{(k - 1)C_{D}}{4} \int_{\Omega} u^{m+k-3} \nabla u |^{2} + \mu \int_{\Omega} u^{k+1} \\
\leq C_{2} \int_{\Omega} u^{k}(v + 1) + C_{7} \text{ for all } t \in (0, T_{\text{max}}),
\]

(3.66)

which together with the Young inequality implies that

\[
\frac{1}{k} \frac{d}{dt} \|u\|_{L^{k}(\Omega)}^{k} + \frac{(k - 1)C_{D}}{4} \int_{\Omega} u^{m+k-3} \nabla u |^{2} + \mu \int_{\Omega} u^{k+1} \\
\leq C_{9} \int_{\Omega} (v + 1)^{k} + C_{8} \text{ for all } t \in (0, T_{\text{max}})
\]

(3.67)
for some positive constants $C_8$ and $C_9$. Now, in view of $2p_0 > 2$ (see (3.76)) and $N = 2$, then by Sobolev imbedding theorems, we derive from (3.61) that

$$\|v\|_{L^\infty(\Omega)} \leq C_{10}\|\nabla v\|_{L^{2p_0}(\Omega)},$$

so that, combined with (3.67) implies that

$$\frac{1}{k}\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + \frac{(k-1)C_D}{4}\int_\Omega u^{m+k-3}|\nabla u|^2 + \mu\int_\Omega u^{k+1} \leq C_{11} \text{ for all } t \in (0, T_{max}),$$

(3.68)

whereas a standard ODE comparison argument shows that (3.57) holds. \qed

**Lemma 3.12.** Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary. Furthermore, assume that $m > \frac{2N}{N + \gamma_s}$ with $N \geq 1$, where $\gamma_s$ is given by (3.56). If

$$\frac{(\mu_s + 1)(N + \mu_s - 1)}{2N} > \frac{N}{2},$$

(3.69)

then for any $k > 1$, there exists a positive constant $C$ such that

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq C \text{ for all } t \in (0, T_{max}),$$

(3.70)

where $\mu_s$ is given by (3.14).

**Proof.** Due to

$$m > \frac{2N}{N + \frac{(\mu_s + 1)(N + \mu_s - 1)}{2N}}.$$

Now, in view of $\mu > 0$ implies that

$$\mu_s > 1,$$

which together with Lemma 3.10 results in

$$m > 1 + \frac{\left[N - 2\frac{(\mu_s + 1)(N + \mu_s - 1)}{2N}\right]\mu_s}{2N \times \frac{(\mu_s + 1)(N + \mu_s - 1)}{2N}}.$$

(3.71)

Thus by (3.69), we may choose $q_{0,*} \in (1, \mu_s)$ which is close to $\mu_s$ such that

$$m > 1 + \frac{\left[N - 2\frac{(q_{0,*} + 1)(N + q_{0,*} - 1)}{2N}\right]q_{0,*}}{2N \times \frac{(q_{0,*} + 1)(N + q_{0,*} - 1)}{2N}}.$$

(3.72)
and
\[
(q_0, + 1)(N + q_0, - 1) > \frac{N}{2}
\]  \tag{3.73}

Next, observing that \( \frac{(q_0, + 1)(N + q_0, - 1)}{2N} \in (1, \frac{1}{2}) \), thus, in light of Lemma 3.8, we derive that there exists a positive constant \( C_1 \) such that
\[
\|\nabla v(\cdot, t)\|_{L^{2p_0,} (\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}}),
\]  \tag{3.74}

where
\[
p_0, = \frac{(q_0, + 1)(N + q_0, - 1)}{2N} > \frac{N}{2}.
\]  \tag{3.75}

Now, choosing \( k > \max\{N + 1, |1 - m| + q_0, p_0, - 1, \frac{1 - m}{2p_0, - N(2p_0, N - 2p_0,)}\} \) in (3.9), then, we have
\[
\frac{1}{k - 1} \int \frac{d}{dt} \|u\|_{L^k (\Omega)}^k + \frac{(k - 1)C_D}{2} \int u^{m+k-3} \nabla u \| \nabla u \| + \frac{\mu}{2} \int u^{k+1} \leq \frac{\chi^2(k - 1)}{2m} \int u^{k+1-m} \nabla u \| \nabla u \| + C_2 \int u^k \| v + 1 \| \text{ for all } t \in (0, T_{\text{max}})
\]  \tag{3.76}

with \( C_2 = \max\{\|v\|_{L^{\infty}}(\Omega), \mu + \kappa \xi\} \). According to the estimate of \( \nabla v \) in (3.74) along with the Hölder inequality we derive that there exists a positive constant \( C_3 > 0 \) such that
\[
\frac{\chi^2(k - 1)}{2C_D} \int \|u\|_{L^k (\Omega)}^{k+1-m} \| \nabla u \|^2 \leq \frac{\chi^2(k - 1)}{2C_D} \left( \int \frac{p_0,}{p_0, - (k+1-m)} \frac{p_0, - 1}{p_0,} \left( \int \|\nabla u\|_{L^{2p_0,} (\Omega)} \right)^{\frac{1}{p_0,}} \right) \leq C_3 \frac{\chi^2(k - 1)}{2C_D} \|u\|_{L^{k+1-m} (\Omega)}^{k+1-m} \| \nabla u \|^2
\]  \tag{3.77}

In view of \( k > \max\{1 - m\| + q_0, p_0, - 1, \frac{1 - m}{2p_0, - N(2p_0, N - 2p_0,)}\} \), we have
\[
q_0, \frac{k + m - 1}{k + m - 1} \leq \frac{p_0, (k + 1 - m)}{p_0, - 1(k + m - 1)} < \frac{N}{N - 2}.
\]

An application of the Gagliardo-Nirenberg inequality (see Lemma 2.2) implies that for some positive constants \( C_4 \) as well as \( C_5 \) such that
\[
\frac{\chi^2(k - 1)}{2C_D} \|u\|_{L^2 (\Omega)}^{k+1-m} \| \nabla u \|^2 \leq C_4 \left( \int \frac{2^{k+1-m}}{2^{k+1-m} + \frac{2q_0,}{2q_0, - 1} + \frac{2q_0,}{2q_0, - 1} + \frac{2q_0,}{2q_0, + 1}} \|u\|_{L^2 (\Omega)}^{k+1-m} \| \nabla u \|^2 + \|u\|_{L^2 (\Omega)}^{k+1-m} \| \nabla u \|^2 \right)
\]  \tag{3.78}
by using Lemma 3.1 where $q_{0,*}$ is the same as (3.72). Due to (3.72), one has
\[ \frac{N(k + 1 - m - q_{0,*} + \frac{q_{0,*}}{p_{0,*}})}{N(k + m - 1 - q_{0,*}) + 2q_{0,*}} < 2, \]
so that, applying the Young inequality implies that
\begin{align*}
1 \frac{d}{dt} \|u\|_{L^k(\Omega)}^k &+ \frac{(k - 1)C_D}{4} \int_{\Omega} u^{m+k-3}|\nabla u|^2 + \frac{\mu}{2} \int_{\Omega} u^{k+1} \\
&\leq C_2 \int_{\Omega} u^k(v+1) + C_7 \text{ for all } t \in (0, T_{max}),
\end{align*}
(3.79)
which together with the Young inequality again yields to
\begin{align*}
1 \frac{d}{dt} \|u\|_{L^k(\Omega)}^k &+ \frac{(k - 1)C_D}{4} \int_{\Omega} u^{m+k-3}|\nabla u|^2 + \mu \int_{\Omega} u^{k+1} \\
&\leq C_9 \int_{\Omega} (v+1)^k + C_8 \text{ for all } t \in (0, T_{max})
\end{align*}
(3.80)
and some positive constants $C_8$ and $C_9$. Now, in view of $2p_0 > N$ (see (3.75)), then by Sobolev imbedding theorems, we derive from (3.74) that
\[ \|v\|_{L^\infty(\Omega)} \leq C_{10} \|\nabla v\|_{L^{2p_0}(\Omega)}, \]
so that, combined with (3.80) implies that
\begin{align*}
1 \frac{d}{dt} \|u\|_{L^k(\Omega)}^k &+ \frac{(k - 1)C_D}{4} \int_{\Omega} u^{m+k-3}|\nabla u|^2 + \mu \int_{\Omega} u^{k+1} \\
&\leq C_{11} \text{ for all } t \in (0, T_{max}),
\end{align*}
(3.81)
whereas a standard ODE comparison argument shows that (3.70) holds.

\[ \square \]

**Lemma 3.13.** Let $\Omega \subset \mathbb{R}^N (N \neq 2)$ be a bounded domain with smooth boundary. Furthermore, assume that $m > \frac{2N}{N + \gamma_*}$ with $N \geq 1$, where $\gamma_*$ is given by (3.36). If
\[ \frac{(\mu_* + 1)(N + \mu_* - 1)}{2N} \leq \frac{N}{2}, \]
(3.82)
then for any $k > 1$, there exists a positive constant $C$ such that
\[ \|u(\cdot, t)\|_{L^k(\Omega)} \leq C \text{ for all } t \in (0, T_{max}). \]
(3.83)
Proof. Let
\[
\bar{\beta} = \max\{\frac{4N^{(\mu_1+1)(N+\frac{\chi_{\max(1,\lambda_0)}(1)}{\chi_{\max(1,\lambda_0)}(1)}+1)}}{N + (\mu_1+1)(N+\frac{\chi_{\max(1,\lambda_0)}(1)}{\chi_{\max(1,\lambda_0)}(1)}+1)} - 2, \frac{2}{N} - m, 16, 8N + 2\}.
\]

Due to
\[
m > \frac{2N}{N + (\mu_1+1)(N+\frac{\chi_{\max(1,\lambda_0)}(1)}{\chi_{\max(1,\lambda_0)}(1)}+1)},
\]
as well as (3.82) and Lemma 3.8, we may choose \(q_0,*** \in (1, \mu_*)\) which is close to \(\mu_*\) such that
\[
\frac{2N(m-1)q_0,***}{N-2q_0,***}(\frac{2}{N} - 1 + \frac{\beta}{q_0,***}) + 2 - m - \frac{2}{N} > \frac{N(1 - \frac{4}{\beta})}{1 + \frac{4N}{2q_0,***} - \frac{4N}{\beta}}(\frac{2}{N} - 1 + \frac{\beta}{q_0,***}) + 2 - m - \frac{2}{N}
\]
for all \(\beta \geq \bar{\beta}\). Therefore, we can choose
\[
k \in \left(\frac{N(1 - \frac{4}{\beta})}{1 + \frac{4N}{2q_0,***} - \frac{4N}{\beta}}(\frac{2}{N} - 1 + \frac{\beta}{q_0,***}) + 2 - m - \frac{2}{N}, \frac{2N(m-1)q_0,***}{N-2q_0,***}(\frac{2}{N} - 1 + \frac{\beta}{q_0,***}) + 2 - m - \frac{2}{N}\right).
\]

Now for the above \(k\), by the H"older inequality, we have
\[
J_1 : = \frac{\chi^2(k-1)}{2C_D} \int_\Omega u^{k+1-m}|\nabla v|^2 \\
\leq \frac{\chi^2(k-1)}{2C_D} \left(\int_\Omega u^{N(k+1-m)}\right)\frac{N^2}{N} \left(\frac{\int |\nabla v|^{N}}{\int |\nabla v|^{N}}\right)^{\frac{2}{N}} \\
= \frac{\chi^2(k-1)}{2C_D} \left\| u^{\frac{k+1-m}{2}} \right\|_{2N\frac{k+1-m}{k+1-m}} \left\| \nabla v \right\|_{L^N(\Omega)}^{2}.
\]

Next, by (3.82), we derive that \(m \geq 1\), so that, in view of \(N \geq 3\), we have
\[
\frac{1}{k+m-1} \leq \frac{k+1-m}{k+m-1} \frac{N}{2} \leq \frac{N}{(N-2)_{+}},
\]
so that, by using Gagliardo-Nirenberg interpolation inequality (Lemma 2.2) and \(L^1(\Omega)\) boundedness for \(u\) (see Lemma 3.1), we have
\[
\frac{\chi^2(k-1)}{2C_D} \left\| u^{\frac{k+1-m}{2}} \right\|_{2N\frac{k+1-m}{k+1-m}} \left\| \nabla v \right\|_{L^{N}(\Omega)}^{2} \leq C_1(\|\nabla u^{\frac{k+1-m}{2}}\|_{L^1(\Omega)}^{\mu_1} u^{\frac{k+1-m}{2}}\|_{L^{2N}(\Omega)}^{1-\mu_1} + \| u^{\frac{k+1-m}{2}}\|_{L^2(\Omega)}^{2} \left\| \nabla v \right\|_{L^{N}(\Omega)}^{2} \leq C_2(\|\nabla v\|_{L^2(\Omega)}^{2} + 1)\]

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with some positive constants $C_1$ as well as $C_2$ and
\[ \mu_1 = \frac{N[k+m-1]}{2} - \frac{N(k+m-1)}{2N(k+1-m)} = [k + m - 1] \frac{N}{2} - \frac{N}{N-2(k+1-m)} \in (0, 1). \]

On the other hand, due to Lemma 3.2 and the fact that $\beta \geq \bar{\beta} > \frac{N}{2}$ and $N > 2$, we have
\[
\|\nabla v\|_{L^2(\Omega)}^2 \leq C_5(\|\nabla |\nabla v|^{\beta}\|_{L^2(\Omega)}^2 + 1)(\|\nabla v\|^2_{L^2(\Omega)} + 1) \quad (3.88)
\]
\[
\leq C_4(\|\nabla |\nabla v|^{\beta}\|_{L^2(\Omega)}^2 + 1),
\]
where some positive constants $C_3$ as well as $C_4$ and
\[ \mu_2 = \frac{N\beta - \frac{N\beta}{2q_{0,***}}}{1 - \frac{N}{2} + \frac{N\beta}{2q_{0,***}}} = \frac{N}{1 - \frac{N}{2} + \frac{N\beta}{2q_{0,***}}} - \frac{N}{(0, 1)}. \]

Inserting (3.87) into (3.86) as well as by means of the Young inequality and (3.85) we see that for any $\delta > 0$,
\[
J_1 \leq C_5(\|\nabla u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^2(\Omega)}^2 + 1)(\|\nabla v\|^2_{L^2(\Omega)} + 1)
= C_5(\|\nabla u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^2(\Omega)}^2 + 1)(\|\nabla v\|^2_{L^2(\Omega)} + 1) \quad (3.89)
\]
\[
\leq \delta \int_{\Omega} |\nabla u_{k+1-m}^{\frac{k-m-1}{2}}|^2 + \delta \|\nabla v\|^2_{L^2(\Omega)} + C_6 \quad \text{for all } t \in (0, T_{max})
\]
for all $\beta \geq \bar{\beta}$. Here we have use the fact that $k < \frac{2N(m-1)q_{0,***}}{N-2q_{0,***}}(\frac{2}{N} - 1 + \frac{\beta}{q_{0,***}}) + 2 - m - \frac{2}{N}$ and $k > 2 - \frac{2}{N} - m$. Next, due to the H"older inequality and $\beta \geq \bar{\beta} > 16$, we have
\[
J_2 : = \int_{\Omega} u^2|\nabla v|^{2\beta-2}
\leq \left( \int_{\Omega} u^2 \right)^{\frac{\beta}{\beta-8}} \left( \int_{\Omega} |\nabla v|^{\beta(2\beta-2)} \right)^{\frac{8}{\beta-8}}
\leq \left( \int_{\Omega} u^2 \right)^{\frac{\beta}{\beta-8}} \left( \int_{\Omega} |\nabla v|^{\beta(2\beta-2)} \right)^{\frac{8}{\beta-8}} \quad (3.90)
\]
\[
= \|u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^{\frac{2q_{0,***}}{1-m}}(\Omega)} \|\nabla v\|_{L^\frac{2q_{0,***}}{1-m}(\Omega)}^{(2\beta-2)} \quad (3.91)
\]

On the other hand, with the help of $k > 1 - m + \frac{N-2}{4N} \beta$, $\beta \geq \bar{\beta} > 4$ and Lemma 2.2 we conclude that
\[
\|u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^{\frac{2q_{0,***}}{1-m}}(\Omega)} \leq C_7(\|\nabla u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^2(\Omega)} + \|u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^{\frac{2q_{0,***}}{1-m}}(\Omega)})^{\frac{4}{k-m-1}} \quad (3.91)
\]
\[
\leq C_8(\|\nabla u_{k+1-m}^{\frac{k-m-1}{2}}\|_{L^2(\Omega)} + 1)
\]
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with some positive constants $C_7$ as well as $C_8$ and

$$
\mu_3 = \frac{N(k+m-1)/2}{1 - \frac{N}{2} + \frac{N(k+m-1)}{2}} = [k + m - 1] - \frac{N}{2} \left( 1 - \frac{N}{2} + \frac{N(k+m-1)}{2} \right) \in (0, 1).
$$

On the other hand, again, it infers by the Gagliardo-Nirenberg inequality (Lemma 2.2) that there are $C_9 > 0$ and $C_{10} > 0$ ensuring

$$
\| \nabla v \|_{L^{(2\beta-2)/(\beta-2)}_\beta(\Omega)}^{(2\beta-2)/\beta(\Omega)} \leq C_9 (\| \nabla |\nabla v|^\beta \|_{L^2(\Omega)}^{(2\beta-2)/(1-\mu_4)} \| \nabla v|^\beta \|_{L^{2(\beta-2)/(1-\mu_4)}_\beta(\Omega)}^{(2\beta-2)/(1-\mu_4)} + \| \nabla v|^\beta \|_{L^{(2\beta-2)/\beta(\Omega)}_\beta(\Omega)}^{(2\beta-2)/\beta(\Omega)} ) \tag{3.92}
$$

for any $\beta \geq \bar{\beta} > \frac{7N+2}{2}$, where

$$
\mu_4 = \frac{N\beta}{2q_0,***} - \frac{N\beta}{2q_0,***} = \frac{N\beta}{2q_0,***} - \frac{N(1-\bar{\beta})}{2q_0,***} \in (0, 1).
$$

Inserting (3.91), (3.92) into (3.90) and using $k > \frac{N(1-\bar{\beta})}{2q_0,***} \left( \frac{2}{\bar{\beta}} - 1 + \frac{\bar{\beta}}{q_0,***} \right) + 2 - m - \frac{2}{N}$ and $k > 2 - \frac{2}{N} - m$ and Lemma 2.2, we derive that for the above $\delta > 0$,

$$
J_2 \leq C_{11} (\| \nabla u_m \|_{L^2(\Omega)}^{(1+N(1-k)/2)} + 1) (\| \nabla |\nabla v|^\beta \|_{L^2(\Omega)}^{(1+N(1-k))} + 1) \tag{3.93}
$$

for all $t \in (0, T_{\max})$.

Finally, with the help of (3.1) and by the Sobolev inequality, the Young inequality and (3.85), we conclude that for the above $\delta > 0$, there exist positive constants $C_{13}$, $C_{14}$ as well as $C_{15}$ and $C_{16}$ such that

$$
J_3 = \int_\Omega v^{k+1} \leq C_{13} \| v \|_{L^2(\Omega)}^{k+1} \leq C_{14} (\| \nabla v \|_{L^2(\Omega)}^{k+1} + 1) \tag{3.94}
$$

for all $\beta \geq \bar{\beta} > N + 1$. Now, inserting (3.89), (3.93)–(3.94) into (3.38) and using the Young
inequality and choosing δ small enough yields to

\[
\frac{d}{dt} \left( \frac{1}{k} \| u \|_{L_k^k(\Omega)}^k + \frac{1}{2\beta} \| \nabla v \|_{L^{2\beta}(\Omega)}^{2\beta} \right) + \frac{3(\beta - 1)}{8\beta^2} \int_{\Omega} |\nabla|\nabla v|^{\beta}|^2 + \frac{\mu}{2} \int_{\Omega} u^{k+1} \\
+ \frac{1}{5} \int_{\Omega} |\nabla v|^{2\beta - 2} |D^2 v|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^{2\beta} + \frac{(k - 1)\mu}{8} \int_{\Omega} u^{m+k-3} |\nabla u|^2
\leq C_{16} \text{ for all } t \in (0, T_{\text{max}})
\]

and some positive constant $C_{16}$. Therefore, letting $y := \int_{\Omega} u^k + \int_{\Omega} |\nabla v|^{2\beta}$ in (3.95) yields to

\[
\frac{d}{dt} y(t) + C_{17} y(t) \leq C_{18} \text{ for all } t \in (0, T_{\text{max}}).
\]

Thus a standard ODE comparison argument implies boundedness of $y(t)$ for all $t \in (0, T_{\text{max}})$. Clearly, $\| u(\cdot, t) \|_{L_k^k(\Omega)}$ and $\| \nabla v(\cdot, t) \|_{L^{2\beta}(\Omega)}$ are bounded for all $t \in (0, T_{\text{max}})$. Obviously, $\lim_{\beta \to +\infty} \frac{N(1 - \frac{4}{\beta})}{1 + \frac{N}{2\beta} - \frac{4N}{\beta}} (2N - 1 + \frac{\beta}{q_0,**}) = \lim_{\beta \to +\infty} \frac{2N(m - 1)q_0,**}{N - 2q_0,**} (\frac{2N - 1 + \frac{\beta}{q_0,**}}{N}) + 2 - m - \frac{2}{N} = +\infty$, hence, the boundedness of $\| u(\cdot, t) \|_{L_k^k(\Omega)}$ and the Hölder inequality implies the results.

Employing Lemmas 3.12–3.13, we can prove the following lemma.

**Lemma 3.14.** Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary. Furthermore, assume that $m > \frac{2N}{N + \gamma_*}$ with $N \geq 1$, where $\gamma_*$ is given by (3.56). Then for any $k > 1$, there exists a positive constant $C$ such that

\[
\| u(\cdot, t) \|_{L_k^k(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]

Along with the Duhamel’s principle and $L^p-L^q$ estimates for Neumann heat semigroup, the above lemma yields the following Lemma.

**Lemma 3.15.** Let $(u, v, w)$ be the solution of the problem (1.3). Then

\[
\| v(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]

**Proof.** By Duhamel’s principle, we see that the solution $v$ can be expressed as follows

\[
v(t) = e^{t(\Delta - 1)} v_0 + \int_0^t e^{(t-s)(\Delta - 1)} v(s) ds, \ t \in (0, T_{\text{max}}),
\]
where \((e^{t\Delta})_{t \geq 0}\) is the Neumann heat semigroup in \(\Omega\). Using Lemma 3.14, we follow the \(L^p-L^q\) estimates for Neumann heat semigroup to obtain for any \(t \in (0, T_{max})\), we obtain
\[
\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq e^{-t} \|\nabla v_0\|_{L^{\infty}(\Omega)} + \int_0^t (t-s)^{-
abla^2(t-s)} \|u(\cdot, s)\|_{L^2N(\Omega)} ds
\]
(3.99)
This lemma is proved.

Applying Lemma 3.14 and Lemma 3.15, a straightforward adaptation of the well-established Moser-type iteration procedure [1] allows us to formulate a general condition which is sufficient for the boundedness of \(u\).

**Lemma 3.16.** Let \((u, v, w)\) be the solution of the problem (1.3). Then there exists a positive constant \(C\) such that
\[
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{max}).
\]
(3.100)

**Proof.** Firstly, by Lemma 3.14 we obtain that for any \(k > 0\),
\[
\|u(\cdot, t)\|_{L^k(\Omega)} \leq \alpha_k \text{ for all } t \in (0, T_{max}),
\]
(3.101)
where \(\alpha_k\) depends on \(k\). Multiplying the first equation of (1.3) by \(ku^{k-1}\) with \(k \geq \max\{2m, N+2\}\), integrating it over \(\Omega\), then using the boundary condition \(\partial u / \partial \nu = 0\) and combining with Lemma 3.15, we obtain
\[
\frac{d}{dt}\|u\|_{L^k(\Omega)}^k + k(k-1)C_D \int_\Omega (u+\varepsilon)^{m-1} u^{k-2} |\nabla u|^2 + \mu k \int_\Omega u^{k+1} + \int_\Omega u^k
\leq k(k-1)C_D \int_\Omega u^{k-1} \nabla u \cdot \nabla v - \xi \int_\Omega \nabla \cdot (u \nabla w) u^{k-1} + (\mu k + 1) \int_\Omega u^k
\leq k(k-1)C_D \int_\Omega u^{k-1} \nabla u \cdot \nabla v + \frac{(k-1)}{k} \xi \|w_0\|_{L^\infty(\Omega)} \int_\Omega u^k dx + \frac{(k-1)}{k} \xi \int_\Omega u^k dx + (\mu k + 1) \int_\Omega u^k
\leq \frac{k(k-1)C_D}{2} \int_\Omega (u+\varepsilon)^{m-1} u^{k-2} |\nabla u|^2 + C_1 k \int_\Omega u^k + C_2 k^2 \int_\Omega u^k (u+\varepsilon)^{1-m},
\]
(3.102)
by using (3.12), where \(C_1 > 0, C_2 > 0\), as all subsequently appearing constants \(C_i (i = 3, 4, \ldots)\) are independent of \(k\). In what follows, we estimate the last two terms of (3.102).
When \(m \geq 1\), then,
\[
(u+\varepsilon)^{1-m} \leq u^{1-m} \text{ and } (u+\varepsilon)^{m-1} \geq u^{m-1},
\]

\[
(3.102)
\]
so that, by (3.102), we have
\[
\frac{d}{dt} \|u\|_{L^k(\Omega)}^k + \frac{k(k-1)C_D}{2} \int_{\Omega} u^{m+k-3} |\nabla u|^2 + \int_{\Omega} u^k \leq C_1 k \int_{\Omega} u^k + C_2 k^2 \int_{\Omega} u^{k+1-m}.
\] (3.103)

In order to take full advantage of the dissipated quantities appearing on the left-hand side herein, for any \(\delta > 0\), we first invoke the Gagliardo-Nirenberg inequality which provides \(C_3 > 0, C_4 > 0\) as well as \(C_5 > 0\) and \(C_6 > 0\) such that
\[
C_1 k \|u\|_{L^k(\Omega)}^k = C_1 k \|u\|_{L^k(\Omega)}^{k+m-1} \left\| \frac{2k}{k+m-1} \right\|_{L^{k+m-1}(\Omega)}^{2Nk/(k+m-1)} \|u\|_{L^{k+m-1}(\Omega)}^{k+m-1} \leq C_2 k \|u\|_{L^k(\Omega)}^k \delta \|\nabla u\|_{L^2(\Omega)}^2 + C_4 k \|u\|_{L^k(\Omega)}^{k+m-1} \left\| \frac{2k}{k+m-1} \right\|_{L^{k+m-1}(\Omega)}^{2Nk/(k+m-1)} \|u\|_{L^{k+m-1}(\Omega)}^{k+m-1} + C_5 k \|u\|_{L^k(\Omega)}^k \delta \|\nabla u\|_{L^2(\Omega)}^2 + C_6 k \|u\|_{L^k(\Omega)}^k \left\| \frac{2k}{k+m-1} \right\|_{L^{k+m-1}(\Omega)}^{2Nk/(k+m-1)} \|u\|_{L^{k+m-1}(\Omega)}^{k+m-1}.
\] (3.104)

and
\[
C_2 k^2 \|u\|_{L^{k+1-m}(\Omega)}^{k+1-m} = C_2 k^2 \|u\|_{L^{k+1-m}(\Omega)}^{k+m-1} \left\| \frac{2k}{2(k+1-m)} \right\|_{L^{2(k+1-m)}(\Omega)}^{2Nk/(2(k+1-m))} \|u\|_{L^{2(k+1-m)}(\Omega)}^{k+m-1} \leq C_5 k^2 \|\nabla u\|_{L^2(\Omega)}^2 + C_6 k \|u\|_{L^k(\Omega)}^{k+m-1} \left\| \frac{2k}{2(k+1-m)} \right\|_{L^{2(k+1-m)}(\Omega)}^{2Nk/(2(k+1-m))} \|u\|_{L^{2(k+1-m)}(\Omega)}^{k+m-1} + C_5 k^2 \|u\|_{L^k(\Omega)}^k \delta \|\nabla u\|_{L^2(\Omega)}^2 + C_6 k \|u\|_{L^k(\Omega)}^{k+m-1} \left\| \frac{2k}{2(k+1-m)} \right\|_{L^{2(k+1-m)}(\Omega)}^{2Nk/(2(k+1-m))} \|u\|_{L^{2(k+1-m)}(\Omega)}^{k+m-1}.
\] (3.105)

Taking \(\delta\) appropriately small, and substituting the above two inequalities into (3.103), we obtain
\[
\frac{d}{dt} \|u\|_{L^k(\Omega)}^k + \int_{\Omega} u^k \leq C_7 k^N \|u\|_{L^k(\Omega)}^{k+2} + C_8 k^{N+2} \|u\|_{L^k(\Omega)}^{k+2} + C_9 k \|u\|_{L^k(\Omega)}^k + C_{10} k^2 \|u\|_{L^k(\Omega)}^{k+1-m}.
\] (3.106)

Notice that
\[
\frac{k[k+2(m-1)]}{(k+2N(m-1))} \leq \max\{ \frac{k[2k+N(m-1)]}{(2k+2N(m-1))}, k+1-m \} \leq k.
\]
we further obtain
\[
\frac{d}{dt} \|u\|_{L^k(\Omega)}^k + \int_{\Omega} u^k \leq C_{11} k^{N+2} \|u\|_{L^k(\Omega)}^{k(k+2N(m-1))} + C_{12} k^{N+2} \|u\|_{L^k(\Omega)}^k.
\] (3.107)

In what follows, we use Moser iteration method to show the \( L^\infty \) estimate of \( u \). Take \( k_i = 2k_{i-1} = 2^ik_0, k_0 = 2m, M_i = \sup_{t \in (0,T_{\text{max}})} \int_{\Omega} u_{i}^{k_i} \), then when \( m \geq 1 \), we have
\[
M_i \leq \max \{\lambda_i M_{i-1}^2, \|1 + u_0\|_{L^\infty(\Omega)}^k\}.
\] (3.108)

with some \( \lambda > 1 \). Now if \( \lambda_i M_{i-1}^2 \leq \|1 + u_0\|_{L^\infty(\Omega)}^{k_i} \) for infinitely many \( i \geq 1 \), we get (3.33) with \( C = \|1 + u_0\|_{L^\infty(\Omega)} \). Otherwise \( M_i \leq \lambda_i M_{i-1}^2 \) for all \( i = 0, 1, \ldots \), so \( \ln M_i \leq i \ln \lambda + 2 \ln M_{i-1} \).

By induction, we get
\[
\ln M_i \leq (i + 2) \ln \lambda + 2^i (\ln M_0 + 2 \ln \lambda)
\]
and thus
\[
M_i \leq \lambda^{i+2+2^i} M_0^{2^i}.
\]

From this, it follows that (3.100) is valid with some positive constant. When \( 0 < m < 1 \), noticing that \( u^k(u + \varepsilon)^{1-m} \leq u^{k+1-m} + \varepsilon^{1-m} u^k \), then by (3.101) and \( \varepsilon \in (0, 1) \), we derive from Lemma 3.15 that
\[
\frac{d}{dt} \|u\|_{L^k(\Omega)}^k + \int_{u \geq \varepsilon} u^{m+k-3} |\nabla u|^2 + \mu k \int_{\Omega} u^{k+1} + \mu k \int_{\Omega} u^k
\]
\[
\leq C_{13} k^2 \int_{u \leq \varepsilon} u^k + C_{14} k^2 \int_{u \geq \varepsilon} u^{k+1-m}\]
\[
\leq C_{13} k^2 \int_{u \leq \varepsilon} u^k + C_{14} k^2 \int_{u \geq \varepsilon} u^{k+1-m} + C_{14} k^2 \int_{u \geq \varepsilon} u^{k+1-m}.
\] (3.109)

Denote \( J(t) = \{x \in u > \varepsilon\} \). By virtue of the Gagliardo-Nirenberg interpolation inequality and (3.101), we obtain
\[
\|u\|_{L^k(J(t))}^k \leq \|u\|_{L^3(J(t))} \|u\|_{L^3(\Omega)}^{k-1} \leq C_{15} \|u\|_{L^3(\Omega)}^{k-1}.
\]
and
\[
\|u\|_{L^{k+1-m}(J(t))} \leq \|u\|_{L^{(2-m)}(J(t))} \|u\|_{L^{3(2-m)}(\Omega)}^{k-1} \leq C_{16} \|u\|_{L^{3(2-m)}(\Omega)}^{k-1}.
\]
Substituting the above two inequalities into (3.102), we obtain
\[
\frac{d}{dt} \|u\|^k_{L^k(\Omega)} + k(k - 1)C_D \int_{J(t)} (u + \varepsilon)^{m-1} u^{k-2} |\nabla u|^2 + \mu k \int_{\Omega} u^{k+1} + \int_{\Omega} u^k \\
\leq C_{17} k^2 \int_{\Omega} u^k + C_{18} k^2 \|u\|^{k-1}_{L^{\frac{3(k-1)}{2}}(J(t))}.
\]  
(3.110)

Using the Gagliardo-Nirenberg interpolation inequality again, it follows
\[
C_{19} k^2 \|u\|^{k-1}_{L^{\frac{3(k-1)}{2}}(J(t))} \leq C_{19} k^2 \|\nabla u\|^k_{L^k(\Omega)} + C_{20} k^2 \left( \int_{J(t)} 2^{(N-2)k + 2N(m-1)} \|u\|^k_{L^k(\Omega)} \right)^{\frac{k}{k-1}} \leq C_{19} k^2 \|u\|^{k-1}_{L^k(\Omega)}.
\]

substituting this inequality into (3.110) gives
\[
\frac{d}{dt} \|u\|^k_{L^k(\Omega)} + k(k - 1)C_D \int_{J(t)} (u + \varepsilon)^{m-1} u^{k-2} |\nabla u|^2 + \mu k \int_{\Omega} u^{k+1} + \int_{\Omega} u^k \\
\leq C_{21} k^2 \int_{\Omega} u^k + C_{23} k^2 \left( \int_{J(t)} 2^{(N-2)k + 2N(m-1)} \|u\|^k_{L^k(\Omega)} \right)^{\frac{k}{k-1}} + C_{22} k^2 \|u\|^{k-1}_{L^k(\Omega)}.
\]  
(3.111)

Similarly to the case \(m \geq 1\), and we obtain (3.100).

\(\square\)

By virtue of (2.1) and Lemmas 3.15 and 3.16, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.**

*Proof.* From (2.1) and Lemmas 3.15 and 3.16, we derive that there exists a constant \(C > 0\) independent of \(\varepsilon\) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\]  
(3.112)

Suppose on the contrary that \(T_{\text{max}} < \infty\), then (4.8) contradicts to the blow-up criterion (2.1), which implies \(T_{\text{max}} = \infty\). Therefore, the classical solution \((u, v, w)\) is global in time and bounded.

\(\square\)
4 Proof of Theorem 1.2

The goal of this section is to prove theorem 1.2. In the absence of (1.5), the first equation in system (1.3) may be degenerate, so system (1.3) might not have classical solutions. Our goal is to construct solutions of (1.3) as limits of solutions to appropriately regularized problems. To this end, we approximate the diffusion coefficient function in (1.3) by a family \((D_{\varepsilon})_{\varepsilon \in (0,1)}\) of functions

\[ D_{\varepsilon} \in C^2((0, \infty)) \text{ such that } D_{\varepsilon}(u) \geq \varepsilon \text{ for all } u > 0 \]

and \(D(u) \leq D_{\varepsilon}(u) \leq D(u) + 2\varepsilon \text{ for all } u > 0 \text{ and } \varepsilon \in (0,1).\)

Therefore, for any \(\varepsilon \in (0,1),\) the regularized problem of (1.3) is presented as follows

\[
\begin{aligned}
    u_{\varepsilon t} &= \nabla \cdot (D_{\varepsilon}(u_{\varepsilon}) \nabla u_{\varepsilon}) - \chi \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) - \xi \nabla \cdot (u_{\varepsilon} \nabla w_{\varepsilon}) + \mu u_{\varepsilon}(1 - u_{\varepsilon} - w_{\varepsilon}), & x \in \Omega, t > 0, \\
    v_{\varepsilon t} &= \Delta v_{\varepsilon} + u_{\varepsilon} - v_{\varepsilon}, & x \in \Omega, t > 0, \\
    w_{\varepsilon t} &= -v_{\varepsilon} w_{\varepsilon}, & x \in \Omega, t > 0, \\
    \frac{\partial u_{\varepsilon}}{\partial \nu} = \frac{\partial v_{\varepsilon}}{\partial \nu} = \frac{\partial w_{\varepsilon}}{\partial \nu} &= 0, & x \in \partial \Omega, t > 0, \\
    u_{\varepsilon}(x,0) &= u_0(x), v_{\varepsilon}(x,0) = v_0(x), w_{\varepsilon}(x,0) = w_0(x), & x \in \Omega.
\end{aligned}
\]  

(4.1)

We are now in the position to construct global weak solutions for (1.3). Before going into details, let us first give the definition of weak solution.

**Definition 4.1.** Let \(T \in (0, \infty],\) and \(\Omega \subset R^N\) be a bounded domain with smooth boundary. A triple \((u,v,w)\) of nonnegative functions defined on \(\Omega \times (0, T)\), is called a weak solution to (1.3) on \([0, T)\) if

(i) \(u \in L^2_{\text{loc}}([0, T); L^2(\Omega)), \quad v \in L^2_{\text{loc}}([0, T); W^{1,2}(\Omega)), \quad w \in L^2_{\text{loc}}([0, T); W^{1,2}(\Omega));\)  

(ii) \(H(u) \in L^1_{\text{loc}}(\Omega \times [0, T)), \quad u \nabla v \quad \text{and} \quad u \nabla w \in L^1_{\text{loc}}(\Omega \times [0, T); R^N);\)  

(iii) \((u,v,w)\) satisfies (1.3) in the sense that for every \(\varphi \in C_0^\infty(\Omega \times [0, T))\)

\[
\begin{aligned}
    & \quad - \int_0^T \int_{\Omega} u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) \\
    &= \int_0^T \int_{\Omega} H(u) \Delta \varphi + \chi \int_0^T \int_{\Omega} u \nabla v \cdot \nabla \varphi + \xi \int_0^T \int_{\Omega} u \nabla w \cdot \nabla \varphi + \int_0^T \int_{\Omega} \mu(u - u - w) \varphi(t, \cdot); \\
\end{aligned}
\]  

(4.4)
holds as well as
\begin{equation}
- \int_0^T \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^T \int_\Omega v \varphi + \int_0^T \int_\Omega u \varphi.
\end{equation}
(4.5)

and
\begin{equation}
- \int_0^T \int_\Omega w \varphi_t - \int_\Omega w_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega vw \varphi,
\end{equation}
(4.6)

where we let
\begin{equation}
H(s) = \int_0^s D(\sigma) d\sigma \quad \text{for} \quad s \geq 0.
\end{equation}
(4.7)

In particular, if \( T = \infty \) can be taken, then \((u, v, w)\) is called a global-in-time weak solution to \((1.3)\).

We proceed to establish the main step towards the boundedness of weak solutions to \((1.3)\). To this end, firstly, from \((2.1)\) and Lemmas \(3.15-3.16\), we can easily derive the following estimates for \(u_\varepsilon\) and \(v_\varepsilon\), which plays an important role in proving Theorem \(1.2\).

**Lemma 4.1.** Let \( m > \frac{2N}{N + \gamma_*} \) with \( N \geq 1 \), where \( \gamma_* \) is given by \((3.56)\). Then one can find \( C > 0 \) independent of \( \varepsilon \in (0, 1) \) such that
\begin{equation}
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \quad t \in (0, \infty),
\end{equation}
(4.8)

and
\begin{equation}
\|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, \infty).
\end{equation}
(4.9)

Now with the above boundedness information at hand, we may invoke standard parabolic regularity to obtain the Hölder regularity properties.

**Lemma 4.2.** Let \( m > \frac{2N}{N + \gamma_*} \) with \( N \geq 1 \), where \( \gamma_* \) is given by \((3.56)\). Then for any \( \varepsilon \in (0, 1) \), one can find \( \mu \in (0, 1) \) such that for some \( C > 0 \)
\begin{equation}
\|v_\varepsilon(\cdot, t)\|_{C^{\mu, \frac{N}{2}}(\Omega \times [t, t+1])} \leq C \quad \text{for all} \quad t \in (0, \infty),
\end{equation}
(4.10)

and such that for any \( \tau > 0 \) there exists \( C(\tau) > 0 \) fulfilling
\begin{equation}
\|\nabla v_\varepsilon(\cdot, t)\|_{C^{\mu, \frac{N}{2}}(\Omega \times [t, t+1])} \leq C \quad \text{for all} \quad t \in (\tau, \infty).
\end{equation}
(4.11)
Proof. Firstly, by Lemma 4.1, we derive that \( g_\varepsilon \) is bounded in \( L^\infty(\Omega \times (0, \infty)) \), where \( g_\varepsilon(x, t) := -v_\varepsilon(x, t) + u_\varepsilon(x, t) \) for all \((x, t) \in \Omega \times (0, \infty)\). Therefore, in view of the standard parabolic regularity theory to the second equation of (4.1), one has (4.10) and (4.11) hold.

To achieve the convergence result, we need to derive some regularity properties of time derivatives.

Lemma 4.3. Let \( m > \frac{2N}{N+\gamma_}\) with \( N \geq 1 \), where \( \gamma_\) is given by (3.30). Moreover, let \( \varsigma > m \) and \( \varsigma \geq 2(m-1) \). Then for all \( T > 0 \) and \( \varepsilon \in (0, 1) \), there exists \( C(T) > 0 \) such that

\[
\int_0^T \| \partial_t u_\varepsilon^{\varsigma}(\cdot, t) \|_{W^{2,\varsigma}(\Omega)^*} dt \leq C(T) \tag{4.12}
\]

and

\[
\| w_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C(T). \tag{4.13}
\]

Proof. Firstly, with the help of Lemma 4.1, for all \( \varepsilon \in (0, 1) \), we can fix a positive constants \( C_1 \) such that

\[
u_\varepsilon \leq C_1 \quad \text{and} \quad |\nabla v_\varepsilon| \leq C_1 \quad \text{in} \quad \Omega \times (0, \infty). \tag{4.14}
\]

Next, we will prove (4.12). To this end, for any fixed \( \psi \in C_0^\infty(\Omega) \), multiplying the first equation by \( u_\varepsilon^{\varsigma-1} \psi \), we have

\[
\begin{align*}
\frac{1}{\varsigma} & \int_\Omega \partial_t u_\varepsilon^{\varsigma}(\cdot, t) \cdot \psi \\
= & \int_\Omega u_\varepsilon^{\varsigma-1} \left[ \nabla \cdot (D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon) - \chi \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) - \xi \nabla \cdot (u_\varepsilon \nabla w_\varepsilon) + \mu u_\varepsilon(1 - w_\varepsilon - u_\varepsilon) \right] \cdot \psi \\
= & -(\varsigma - 1) \int_\Omega u_\varepsilon^{\varsigma-2} D_\varepsilon(u_\varepsilon) |\nabla n_\varepsilon|^2 \psi - \int_\Omega u_\varepsilon^{\varsigma-1} D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \psi \\
& + (\varsigma - 1) \chi \int_\Omega u_\varepsilon^{\varsigma-1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \psi + \chi \int_\Omega u_\varepsilon^{\varsigma-1} \nabla v_\varepsilon \cdot \nabla \psi \\
& + (\varsigma - 1) \xi \int_\Omega u_\varepsilon^{\varsigma-1} \nabla u_\varepsilon \cdot \nabla w_\varepsilon \psi + \xi \int_\Omega u_\varepsilon^{\varsigma-1} \nabla w_\varepsilon \cdot \nabla \psi \\
& + \mu \int_\Omega u_\varepsilon^\varsigma \psi - \mu \int_\Omega u_\varepsilon w_\varepsilon \psi - \mu \int_\Omega u_\varepsilon^{\varsigma+1} \psi \quad \text{for all} \quad t \in (0, \infty).
\end{align*}
\tag{4.15}
\]

Next, we will estimate the right-hand sides of (4.15). To this end, assuming that \( p := \varsigma - m + 1 \), then \( \varsigma > m \) and \( \varsigma \geq 2(m-1) \) yield to \( p > 1 \) and \( p \geq m - 1 \). Since (4.14), we
integrate (3.9) with respect to \( t \) over \((0, T)\) for some fixed \( T > 0 \) and then have
\[
\frac{1}{p} \int_0^T u_\varepsilon^p(\cdot, T) + \frac{CD(p-1)}{2} \int_0^T \int_\Omega u_\varepsilon^{m+p-3}|\nabla u_\varepsilon|^2 \leq \frac{(p-1)\chi^2}{2CD} \int_\Omega u_\varepsilon^k v_\varepsilon + \mu \int_\Omega u_\varepsilon^{p+1-m} |\nabla u_\varepsilon|^2 + \left(\frac{(k-1)}{k}\xi\right)\|w_0\|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon^k v_\varepsilon
\]
(4.16)

On the other hand, by \( p = \varsigma - m + 1 \), we have
\[
\int_0^T \int_\Omega u_\varepsilon^{\varsigma-2} |\nabla u_\varepsilon|^2 = \int_0^T \int_\Omega u_\varepsilon^{m+p-3} |\nabla u_\varepsilon|^2 \leq C_2(1 + T)
\]
(4.17)
for some positive constant \( C_2 \). Next, by (4.14), we also derive that
\[
\mu \int_\Omega u_\varepsilon^{\varsigma} \psi - \mu \int_\Omega u_\varepsilon^{\varsigma} w_\varepsilon^\varsigma \psi - \mu \int_\Omega u_\varepsilon^{\varsigma+1} \psi \leq C_1^\varsigma |\Omega| (\mu + \mu \|w_0\|_{L^\infty(\Omega)} + \mu C_1) \|\psi\|_{L^\infty(\Omega)}
\]
(4.18)
for all \( \varepsilon \in (0, 1) \). Moreover, by (4.14)--(4.18) and the Young inequality, we conclude that there exists \( C_3 > 0 \) such that
\[
| \int_\Omega \partial_t u_\varepsilon^{\varsigma}(\cdot, t) \cdot \psi | \leq C_3(\int_\Omega u_\varepsilon^{\varsigma-2} |\nabla u_\varepsilon|^2 + 1) \|\psi\|_{W^{1,\infty}(\Omega)}
\]
(4.19)
Due to the embedding \( W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \) for \( q > N \), we deduce that there exists \( C_4 > 0 \) such that
\[
\|\partial_t u_\varepsilon^{\varsigma}(\cdot, t)\|_{(W^{2,q}(\Omega))^*} \leq C_4(\int_\Omega u_\varepsilon^{\varsigma-2} |\nabla u_\varepsilon|^2 + 1) \text{ for all } t \in (0, \infty) \text{ and any } \varepsilon \in (0, 1).
\]
(4.20)
Now, combining (4.17) and (4.20), we can get (4.12). Now, observing that the third equation of (4.1) is an ODE, we derive that for any \((x, t) \in \Omega \times (0, \infty),\)
\[
w_\varepsilon(x, t) = w_0(x)e^{-\int_0^t v_\varepsilon(x, s)ds}.
\]
(4.21)
Hence, by a basic calculation, we conclude that for any \((x, t) \in \Omega \times (0, \infty),\)
\[
\nabla w_\varepsilon(x, t) = \nabla w_0(x)e^{-\int_0^t v_\varepsilon(x, s)ds} - w_0(x)e^{-\int_0^t v_\varepsilon(x, s)ds} \int_0^t \nabla v_\varepsilon(x, s)ds,
\]
(4.22)
which together with (1.6) and (4.14) implies (4.13).
We are now in the position to prove our main result on global weak solvability.

**Lemma 4.4.** Assume that \( m > \frac{2N}{N + \gamma_*} \) with \( N \geq 1 \), where \( \gamma_* \) is given by (3.56). Then there exists \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) such that \( \varepsilon_j \to 0 \) as \( j \to \infty \) and that

\[
\begin{align*}
 u_\varepsilon &\to u \text{ a.e. in } \Omega \times (0, \infty), \\
u_\varepsilon &\rightharpoonup u \text{ weakly star in } L^\infty(\Omega \times (0, \infty)), \\
\nabla v_\varepsilon &\to \nabla v \text{ in } C^0_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \\
\nabla w_\varepsilon &\to \nabla w \text{ weakly star in } L^\infty(\Omega \times (0, \infty))
\end{align*}
\]

as well as

\[
u_\varepsilon \to v \text{ in } C^0_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \tag{4.25}
\]

\[
\nabla v_\varepsilon \to \nabla v \text{ in } L^\infty(\Omega \times (0, \infty)) \tag{4.27}
\]

\[
w_\varepsilon \rightharpoonup w \text{ weakly star in } L^\infty(\Omega \times (0, \infty)) \tag{4.28}
\]

and

\[
\nabla w_\varepsilon \rightharpoonup \nabla w \text{ weakly star in } L^\infty_{\text{loc}}(\Omega \times (0, \infty)) \tag{4.29}
\]

with some triple \((u, v, w)\) which is a global weak solution of (4.1) in the sense of Definition 4.1.

**Proof.** Firstly, due to Lemma 4.1 and Lemma 4.3, for each \( T > 0 \), we can find \( \varepsilon \)-independent constant \( C(T) \) such that for all \( t \in (0, T) \),

\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|w_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C(T) \tag{4.30}
\]

as well as

\[
\int_0^T \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-3} |\nabla u_\varepsilon|^2 \leq C(T) \text{ for any } p > 1 \text{ and } p \geq m - 1. \tag{4.31}
\]

Now, choosing \( \varphi \in W^{1,2}(\Omega) \) as a second function in the first equation in (4.1) and using (4.30), we have

\[
\left| \int_\Omega (v_\varepsilon, t) \varphi \right| \\
= \left| \int_\Omega [\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon] \varphi \right| \\
= \left| \int_\Omega [-\nabla v_\varepsilon \cdot \nabla \varphi + v_\varepsilon \varphi + u_\varepsilon \varphi] \right| \\
\leq \{ \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{L^2(\Omega)} \} \times \|\varphi\|_{W^{1,2}(\Omega)}
\]
for all \( t > 0 \). Along with (4.30), further implies that
\[
\int_0^T \| \partial_t v_\varepsilon (\cdot, t) \|_{(W^{1,2}(\Omega))'}^2 \, dt \\
\leq \int_0^T \left\{ \| \nabla u_\varepsilon \|_{L^2(\Omega)} + \| v_\varepsilon \|_{L^2(\Omega)} + \| n_\varepsilon \|_{L^2(\Omega)} \right\}^2 \, dt \\
\leq C_1 (T),
\]
where \( C_1 \) is a positive constant independent of \( \varepsilon \). Combining estimates (4.30)–(4.32) and the fact that \( w_\varepsilon \leq \| w_0 \|_{L^\infty(\Omega)} \), we can pick a sequence \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) with \( \varepsilon = \varepsilon_j \searrow 0 \) as \( j \to \infty \) such that (4.24)–(4.29) are valid with certain limit functions \( u, v \) and \( w \) belonging to the indicated spaces. We next fix \( \varsigma > m \) satisfying \( \varsigma \geq 2(m-1) \) and set \( p := 2\varsigma - m + 1 \), then by (4.31) implies that for each \( T > 0 \), \((u_\varsigma_\varepsilon)_{\varepsilon \in (0,1)}\) is bounded in \( L^2 ((0,T); W^{1,2}(\Omega)) \). With the help of Lemma 4.3, we also show that \((\partial_t u_\varsigma_\varepsilon)_{\varepsilon \in (0,1)}\) is bounded in \( L^1 ((0,T); (W^{2,q}(\Omega))^*) \) for each \( T > 0 \)
and some \( q > N \). Hence, an Aubin-Lions lemma (see e.g. [36]) applies to the above inequality we have the strong precompactness of \((u_\varepsilon)_{\varepsilon \in (0,1)}\) in \( L^2 (\Omega \times (0,T)) \). Therefore, we can pick a suitable subsequence such that \( u_\varepsilon \to z \) for some nonnegative measurable \( z : \Omega \times (0,\Omega) \to \mathbb{R} \).

In light of (4.24) and the Egorov theorem, we have \( z = u \) necessarily, so that (4.23) is valid.

Next we shall prove that \((u,v,w)\) is a weak solution of problem (1.3). To this end, multiplying the first equation as well as the second equation and third equation in (4.1) by \( \phi \in C^\infty_0 (\Omega \times [0,\infty)) \), we obtain
\[
- \int_0^\infty \int_\Omega u_\varepsilon \phi_t - \int_\Omega u_0 \phi (\cdot, 0) = \int_0^\infty \int_\Omega H(u_\varepsilon + \varepsilon) d\sigma \Delta \phi + \chi \int_0^\infty \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \phi \\
+ \xi \int_0^\infty \int_\Omega u_\varepsilon \nabla w_\varepsilon \cdot \nabla \phi + \int_0^\infty \int_\Omega (\mu u_\varepsilon - \mu u_\varepsilon w_\varepsilon - \mu u_\varepsilon^2) \phi
\]
(4.33)
as well as
\[
- \int_0^\infty \int_\Omega v_\varepsilon \phi_t - \int_\Omega v_0 \phi (\cdot, 0) = - \int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \phi - \int_0^\infty \int_\Omega v_\varepsilon \phi + \int_0^\infty \int_\Omega v_\varepsilon \phi 
\]
(4.34)
and
\[
- \int_0^\infty \int_\Omega w_\varepsilon \phi_t - \int_\Omega w_0 \phi (\cdot, 0) = - \int_0^\infty \int_\Omega v_\varepsilon w_\varepsilon \phi.
\]
(4.35)
for all $\varepsilon \in (0, 1)$, where $H$ is given by (4.7). Then (4.23)–(4.27), and the dominated convergence theorem enables us to conclude

$$-\int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = \int_0^\infty \int_\Omega H(u) \Delta \varphi + \chi \int_0^\infty \int_\Omega u \nabla v \cdot \nabla \varphi + \xi \int_0^\infty \int_\Omega u \nabla w \cdot \nabla \varphi + \int_0^\infty \int_\Omega (\mu u - \mu vw - \mu u^2) \varphi$$

as well as

$$-\int_0^\infty \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega v \varphi + \int_0^\infty \int_\Omega u \varphi$$

and

$$-\int_0^\infty \int_\Omega w \varphi_t - \int_\Omega w_0 \varphi(\cdot, 0) = -\int_0^\infty \int_\Omega vw \varphi$$

by a limit procedure. The proof of Lemma 4.4 is completed.

We can now easily prove our main result.

**The proof of Theorem 1.2**

A combination of Lemma 4.1 and Lemma 4.4 directly leads to our desired result.

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