A Local Geoid Determination Based on Ellipsoid Approximation in Western China

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1 Introduction

If $W_0$ is the actual gravity potential on the geoid surface, then in terms of Cartesian coordinates $(x, y, z)$, geoid can be defined as the lattice of points for which the following formula holds:

$$w(x, y, z) = w_0$$

However, in practice, instead of Cartesian coordinates, geoid is presented in terms of surface normal heights of a reference equipotential surface. By selecting reference gravity fields and determined the incremental potential:

$$\delta W(X) = w_0 - w(X)$$

Bruns formula provides us with transformation equation of the incremental potential $\delta W(X)$ in gravity space. For example, if we select a reference field of Somigliana-Fizzetti type, we will end up with an ellipsoid of revolution as the reference equipotential surface, e.g. the International Reference Ellipsoid WGD2000\(^1\). The fixed-free two-boundary value problem in ellipsoidal approximation is tailored to the gravity observables of the type modulus of gravity intensity $\gamma(x)$ (from gravimetry) and gravity potential $w(x)$ or geopotential numbers $c(x)$ (from precise levelling), both with GPS derived positions. These observables satisfy the non-linear Poisson equation. By taking advantage of a normal/reference gravity field, the surface gravity observations can be converted into disturbing quantities of the kind $\delta \gamma(x)$, and $\delta w(x)$. By a proper choice of reference gravity field, which synthesizes the actual gravity field of the earth very closely, the Laplace-Poisson equation for the disturbing quantities can be linearized in gravity space. The basic mathematical models are\(^1\):

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1) Reference potential field

\[ W(\varphi, \eta) = \frac{GM}{\varepsilon} \arccot(\sinh \eta) + \frac{1}{6} \Omega^2 a^2 \left( (3\sinh^2 \eta + 1) \arccot(\sinh \eta) - 3\sinh \eta \right) \]

\[ - \left( (3\sinh^2 \varphi - 1) + \frac{1}{2} \Omega^2 (u^2 + \varepsilon^2) \cos^2 \varphi \right) \]  

(3)

2) Expand geoid's potential into Taylor series

\[ \omega_0 = \omega(x) = \omega(X) + \frac{1}{1!} \left( \nabla \omega(X) \right) h + \frac{1}{2!} \left( \nabla \nabla \omega(X) \right) h^2 \]  

(4)

\[ \delta W(X) = \sum_{n=0}^{\infty} a_n h^n \]  

(5)

3) Bruns Formula

\[ h = \delta W(X) \frac{e^2 \cosh \eta (\cosh^2 \eta - \cos^2 \varphi)^{1/2}}{\epsilon m} \]

\[ (\delta W(X))^2 \left( \frac{e^2 \cosh \eta (\cosh^2 \eta - \cos^2 \varphi)^{1/2}}{\epsilon m} \right)^3 \cdot \]

\[ x \left( \frac{1}{2} \frac{\sinh \eta}{\epsilon^3 \cosh^2 \eta (\cosh^2 \eta - \cos^2 \varphi)^2} \right) \]  

(6)

2 Geoid Determination Methodology

The linearization process leads to linearized fixed-free two-boundary value problem. Our choice of reference field is ellipsoidal eigenvalue/eigenvector expansion of the external gravity field of the Earth up to degree/order 360/360. To remain as close as possible to the actual geometry of the Earth, which in global sense resembles an oblate spheroid/ellipsoid of revolution, we have chosen ellipsoidal coordinates and ellipsoidal Laplace-Poisson equation. In other words, we are using ellipsoidal fixed-free two-boundary value problem to tackle the problem of local high-resolution geoid determination. By eliminating the effect of topographical masses between the surface of the Earth and the reference ellipsoid, disturbing quantities will be left, which are harmonic at the surface of the Earth down to the level of reference ellipsoid. This process can be referred to as terrain reduction. Indeed, in the global sense the Newton Potential generated by topographical masses has to be computed in ellipsoidal coordinates. However, once the effect of global and regional masses is removed, the remaining effect of the local topographical masses can be successfully modeled in planar approximation, extended to a radius of 50 km around the computational points. We refer to this step as the remove step of topographic masses. The harmonic coefficients of EGM96 are compatible with the following model:

\[ U(l, b, \varphi) = \frac{g m}{r} \left( 1 + \sum_{m=2}^{360} \sum_{n=-n}^{n} e_{mn} (l, b, \varphi) \right) \]  

(7)

where \( e_{mn} (l, b, \varphi) \) are the surface spherical harmonics and \( (l, b, \varphi) \) are the spherical coordinates of the computational point. For the downward continuation, we will use the ellipsoidal Abel-Poisson integral. The downward continuation is necessary to transfer the surface disturbing quantities of the type modulus of gravity intensity and gravity potential from the surface of the Earth to disturbing gravity potential down to the level of reference ellipsoid of World Geodetic Datum 2000. The downward continuation, via the ellipsoidal Abel-Poisson integral, is an improperly-posed problem and can be stabilized by means of Phillips-Tikhonov regularization procedure among the others. Abel-Poisson integral can be written as the compact form:

\[ \int_a^b K(s,t) x(t) \, dt = g(s) \]  

(8)

Any small error in the observations \( g(s) \) will be magnified and transferred into unknowns \( x(t) \). The downward continuation is finally followed by the restore step. In this step on the level of reference ellipsoid we restore the impact of the topographical masses and the effect of the removed reference field. Finally, the potential on the surface of geoid derived from the restore process can be subtracted from the geoid's potential to produce disturbing potential which can be converted to geoidal undulation via Bruns formula (we will use non-linear ellipsoidal Bruns formula). The whole procedure described above can be summarized in terms of a flowchart (Fig. 1). The solution of the boundary value problem is obtained through the following steps.
## 3 Data Preparations

In the previous section the theoretical foundation of a high-resolution geoid determination methodology based on fixed-free two-boundary value problem is established. Here we are going to prepare the data, which will be used in the future to test the derived methodology by calculating a high-resolution local geoid for the Western China and comparing it with Stokes’s gravimetric geoid (Fig. 2). To compute the high-resolution geoid of Western China, based on fixed-free two-boundary value problem (Fig. 3), different types of data are collected and applied.

![Fig. 2 Stokes’s geoid (5° × 5° grid data)](image)

![Fig. 3 Gravity geoid of Western China with respect to WGD 2000 based on 5° × 5° grid data)](image)
These data are:

- modulus of gravity intensity and geopotential numbers at 1,858 stations of the first order leveling network in Western China,
- digital terrain model of Western China,
- ellipsoidal harmonic coefficients of external gravitational field of the Earth up to degree/order 360/360 (Table 1),
- 20 GPS stations of Western China network.

Table 2 gives the first 20 records of these data.

| n  | m  | \( U_{n,m} \) | \( U_{n,-m} \) |
|----|----|---------------|---------------|
| 0  | 0  | 1.001 119 102 96\( \times 10^6 \) | 0.000 000 000 00\( \times 10^6 \) |
| 1  | 0  | 0.000 000 000 00\( \times 10^6 \) | 0.000 000 000 00\( \times 10^6 \) |
| 1  | 1  | 0.000 000 000 00\( \times 10^6 \) | 0.000 000 000 00\( \times 10^6 \) |
| 2  | 0  | 5.159 938 192 97\( \times 10^{-4} \) | 0.000 000 000 00\( \times 10^6 \) |
| 2  | 1  | -1.877 061 378 78\( \times 10^{-10} \) | 1.199 872 996 52\( \times 10^{-9} \) |
| 2  | 2  | 2.444 998 019 19\( \times 10^{-6} \) | -1.403 527 553 10\( \times 10^{-6} \) |
| 3  | 0  | 9.629 817 356 54\( \times 10^{-7} \) | 0.000 000 000 00\( \times 10^6 \) |
| 3  | 1  | 2.041 374 034 89\( \times 10^{-6} \) | 2.499 069 473 06\( \times 10^{-7} \) |
| 3  | 2  | 9.086 848 019 95\( \times 10^{-7} \) | 6.218 021 235 49\( \times 10^{-7} \) |
| 3  | 3  | 7.229 576 213 28\( \times 10^{-7} \) | 1.418 053 554 46\( \times 10^{-6} \) |
| 4  | 0  | -2.523 416 950 41\( \times 10^{-7} \) | 0.000 000 000 00\( \times 10^6 \) |
| 4  | 1  | -5.402 634 301 10\( \times 10^{-7} \) | -4.769 145 686 02\( \times 10^{-7} \) |
| 4  | 2  | 3.597 251 791 39\( \times 10^{-7} \) | 6.630 404 686 04\( \times 10^{-7} \) |
| 4  | 3  | 9.956 204 817 04\( \times 10^{-7} \) | -2.019 116 803 03\( \times 10^{-7} \) |
| 4  | 4  | -1.890 794 009 83\( \times 10^{-7} \) | 3.097 026 073 41\( \times 10^{-7} \) |
| 5  | 0  | 7.490 516 045 96\( \times 10^{-8} \) | 0.000 000 000 00\( \times 10^6 \) |
| 5  | 1  | -5.113 403 351 53\( \times 10^{-8} \) | -9.386 792 850 03\( \times 10^{-8} \) |
| 5  | 2  | 6.621 459 330 33\( \times 10^{-7} \) | -3.289 712 992 83\( \times 10^{-7} \) |
| 5  | 3  | 4.528 368 826 66\( \times 10^{-7} \) | -2.181 781 690 88\( \times 10^{-7} \) |
| 5  | 4  | -2.986 304 971 82\( \times 10^{-7} \) | 4.992 302 023 78\( \times 10^{-8} \) |
| 5  | 5  | 1.754 697 117 71\( \times 10^{-7} \) | -6.712 884 212 25\( \times 10^{-7} \) |
| 6  | 6  | 9.704 355 702 36\( \times 10^{-9} \) | -2.378 831 392 82\( \times 10^{-7} \) |
| 7  | 7  | 1.095 090 712 86\( \times 10^{-9} \) | 2.451 408 242 72\( \times 10^{-8} \) |
| 8  | 8  | -1.244 658 282 65\( \times 10^{-7} \) | 1.208 957 925 30\( \times 10^{-7} \) |
| 9  | 9  | -4.789 280 382 00\( \times 10^{-8} \) | 9.693 530 236 12\( \times 10^{-8} \) |
| 10 | 10 | 1.008 473 197 43\( \times 10^{-7} \) | -2.408 857 810 42\( \times 10^{-7} \) |
| 20 | 20 | 4.027 357 310 45\( \times 10^{-9} \) | -1.208 369 174 88\( \times 10^{-7} \) |
| 36 | 36 | 4.616 529 606 55\( \times 10^{-9} \) | -5.961 908 637 84\( \times 10^{-7} \) |
| 60 | 60 | 4.244 680 515 3\( \times 10^{-10} \) | 3.942 842 098 53\( \times 10^{-10} \) |
| 120| 120| -4.583 227 879 54\( \times 10^{-10} \) | -1.596 659 345 69\( \times 10^{-9} \) |
| 180| 180| -4.079 328 759 77\( \times 10^{-10} \) | -5.896 923 325 81\( \times 10^{-10} \) |
| 240| 240| -2.315 537 245 10\( \times 10^{-10} \) | -4.624 018 991 46\( \times 10^{-11} \) |
| 300| 300| -5.040 211 591 93\( \times 10^{-11} \) | -1.016 150 944 89\( \times 10^{-10} \) |
| 360| 360| -4.490 176 873 90\( \times 10^{-25} \) | -8.330 101 280 12\( \times 10^{-11} \) |

4 Conclusions

Geoid determination is based on ellipsoidal approximation in geometry and gravity space. Geoid is solved via fixed-free two-boundary value problem of physical geodesy. The solution technique produces a harmonic incremental potential field by means of remove-restore methodology for the centrifugal potential, the topographic masses (terrain effect) and the higher order (degree/order 360/360) ellipsoidal harmonic expansion. The innovative method is now being used in the current computation for high resolution regional geoid for Western China and to compare the geoid with Stokes’s gravimetric geoid (Fig. 2). A higher order ellipsoidal reference potential field with respect to the WGD 2000 is implemented avoiding any datum bias with respect to the traditional spherical approach. The calculated quasi-geoid for GPS station based
on minimum distance mapping of the physical surface of the earth to the Somigliana-Pizzetti deviates from Stokes's geoid by $(1.93 \pm 1.452)$ m on average. This difference may be mainly due to the interpolations process involved in providing the GPS stations with geopotential numbers.

| St No. | Geopotential/kGal·m | Latitude | Longitude | Gravity/mGal | Height/m |
|-------|---------------------|----------|-----------|--------------|----------|
| 1     | 915.535 24          | 35.385   | 100.312   | 978 781.145  | 935.383  |
| 2     | 929.251 58          | 35.362   | 100.432   | 978 773.611  | 949.404  |
| 3     | 936.367 17          | 35.341   | 100.461   | 978 770.444  | 956.677  |
| 4     | 938.236 51          | 35.574   | 97.534    | 978 770.325  | 958.587  |
| 5     | 955.933 5           | 35.581   | 97.464    | 978 762.15   | 976.676  |
| 6     | 959.184 12          | 35.425   | 100.270   | 978 755.314  | 980.004  |
| 7     | 959.833 82          | 35.585   | 97.225    | 978 772.076  | 980.651  |
| 8     | 959.938 6           | 35.421   | 100.245   | 978 756.167  | 980.774  |
| 9     | 960.328 03          | 35.331   | 100.463   | 978 753.054  | 981.175  |
| 10    | 961.414 46          | 35.411   | 100.253   | 978 754.065  | 982.284  |
| 11    | 961.900 05          | 35.473   | 100.101   | 978 757.187  | 982.777  |
| 12    | 962.988 83          | 35.455   | 100.161   | 978 755.495  | 983.886  |
| 13    | 963.584 15          | 35.411   | 100.194   | 978 752.841  | 984.502  |
| 14    | 964.122 13          | 35.455   | 100.225   | 978 778.355  | 985.026  |
| 15    | 965.123 3           | 35.425   | 100.183   | 978 753.42   | 986.074  |
| 16    | 965.764 6           | 35.530   | 100.065   | 978 761.573  | 986.721  |
| 17    | 967.980 42          | 35.341   | 100.433   | 978 745.642  | 989.001  |
| 18    | 968.388 14          | 35.473   | 100.190   | 978 761.054  | 989.402  |
| 19    | 968.714 68          | 35.430   | 100.135   | 978 752.703  | 989.741  |
| 20    | 971.441 41          | 35.554   | 100.091   | 978 768.472  | 992.514  |

Acknowledgements

Thank Professor Grafarend (University of Stuttgart, Germany) for the beneficial discussions I have had with him and for using his software.

References

1. Grafarend E. W., Ardalan A. A. (1999) World geodetic datum 2000. Journal of Geodesy, 73: 611-623
2. Forsberg R. (1998) Geoid tayloring to GPS with example of a 1 cm geoid of Denmark. Second Continental Workship on the Geoid in Europe, Budapest, Hungary.
3. Grafarend E. W., Ardalan A. A. (1997) W_0: an estimate in the finnish height datum N60. Epoch 1993. 4, from 25 GPS points of the Baltic Sea Level Project. Journal of Geodesy, 71: 673-679
4. Forsberg R. (1994) Terrain effects in geoid computations. The International School for the Determination and Use of the Geoid, Milano, Italy.
5. Grafarend E. W., Ardalan A. A., Sideris M. G. (1999) The spheroidal fixed-free two-boundary value problem for geoid determination (the spheroidal Bruns transform). Journal of Geodesy, 73: 513-533
6. Gharem E., Li, J. C. (2000) Discussion on some FFT problems to determine the geoid. Geo-spatial Information Science, 3(2): 53-57
7. Gharem E., Li, J. C., Liu J. (2000) A preliminary investigations for a new Western China gravimetric geoid determination. EOS, Trans. Am. Geophy. Un. Suppl.ment, 81 (S168)
8. Heiskanen W. A., Moritz H. (1967) Physical geodesy. San Francisco-London: Freeman & Company.
9. Rapp R. H., Nemer R. S. (1994) A joint GSFC/DMA project for improving the model of the Earth’s gravitational field. International Association of Geodesy Symposium No. 113, Graz, Austria.