Intensionality and Two-steps Interpretations

Zoran Majkić

International Society for Research in Science and Technology
PO Box 2464 Tallahassee, FL 32316 - 2464 USA
majk.1234@yahoo.com
http://zoranmajkic.webs.com/

Abstract. In this paper we considered the extension of the First-order Logic (FOL) by Bealer’s intensional abstraction operator. Contemporary use of the term ‘intension’ derives from the traditional logical Frege-Russell’s doctrine that an idea (logic formula) has both an extension and an intension. Although there is divergence in formulation, it is accepted that the extension of an idea consists of the subjects to which the idea applies, and the intension consists of the attributes implied by the idea. From the Montague’s point of view, the meaning of an idea can be considered as particular extensions in different possible worlds.

In the case of the pure FOL we obtain commutative homomorphic diagram that holds in each given possible world of the intensional FOL, from the free algebra of the FOL syntax, toward its intensional algebra of concepts, and, successively, to the new extensional relational algebra (different from Cylindric algebras). Then we show that it corresponds to the Tarski’s interpretation of the standard extensional FOL in this possible world.

1 Introduction

In “Über Sinn und Bedeutung”, Frege concentrated mostly on the senses of names, holding that all names have a sense. It is natural to hold that the same considerations apply to any expression that has an extension. Two general terms can have the same extension and different cognitive significance; two predicates can have the same extension and different cognitive significance; two sentences can have the same extension and different cognitive significance. So general terms, predicates, and sentences all have senses as well as extensions. The same goes for any expression that has an extension, or is a candidate for extension.

The simplest aspect of an expression’s meaning is its extension. We can stipulate that the extension of a sentence is its truth-value, and that the extension of a singular term is its referent. The extension of other expressions can be seen as associated entities that contribute to the truth-value of a sentence in a manner broadly analogous to the way in which the referent of a singular term contributes to the truth-value of a sentence. In many cases, the extension of an expression will be what we intuitively think of as its referent, although this need not hold in all cases, as the case of sentences illustrates. While Frege himself is often interpreted as holding that a sentence’s referent is its truth-value, this claim is counterintuitive and widely disputed. We can avoid that issue in the present framework by using the technical term ‘extension’. In this context, the claim that the
extension of a sentence is its truth-value is a stipulation.

'Extensional' is most definitely a technical term. Say that the extension of a name is its denotation, the extension of a predicate is the set of things it applies to, and the extension of a sentence is its truth value. A logic is extensional if coextensional expressions can be substituted one for another in any sentence of the logic "salva veritate", that is, without a change in truth value. The intuitive idea behind this principle is that, in an extensional logic, the only logically significant notion of meaning that attaches to an expression is its extension. An intensional logics is exactly one in which substitutivity salva veritate fails for some of the sentences of the logic.

The first conception of intensional entities (or concepts) is built into the possible-worlds treatment of Properties, Relations and Propositions (PRP)s. This conception is commonly attributed to Leibniz, and underlies Alonzo Church’s alternative formulation of Frege’s theory of senses ("A formulation of the Logic of Sense and Denotation" in Henle, Kallen, and Langer, 3-24, and "Outline of a Revised Formulation of the Logic of Sense and Denotation” in two parts, Nous,VI (1973), 24-33, and VIII,(1974),135-156). This conception of PRPs is ideally suited for treating the modalities (necessity, possibility, etc..) and to Montague’s definition of intension of a given virtual predicate \( \phi(x_1, ..., x_k) \) (a FOL open-sentence with the tuple of free variables \( (x_1, ..., x_k) \)) as a mapping from possible worlds into extensions of this virtual predicate. Among the possible worlds we distinguish the actual possible world. For example if we consider a set of predicates of a given Database and their extensions in different time-instances, the actual possible world is identified by the current instance of the time.

The second conception of intensional entities is to be found in in Russell’s doctrine of logical atomism. On this doctrine it is required that all complete definitions of intensional entities be finite as well as unique and non-circular: it offers an algebraic way for definition of complex intensional entities from simple (atomic) entities (i.e., algebra of concepts), conception also evident in Leibniz’s remarks. In a predicate logics, predicates and open-sentences (with free variables) expresses classes (properties and relations), and sentences express propositions. Note that classes (intensional entities) are reified, i.e., they belong to the same domain as individual objects (particulars). This endows the intensional logics with a great deal of uniformity, making it possible to manipulate classes and individual objects in the same language. In particular, when viewed as an individual object, a class can be a member of another class.

The distinction between intensions and extensions is important, considering that extensions can be notoriously difficult to handle in an efficient manner. The extensional equality theory of predicates and functions under higher-order semantics (for example, for two predicates with the same set of attributes \( p = q \) is true iff these symbols are interpreted by the same relation), that is, the strong equational theory of intensions, is not decidable, in general. For example, in the second-order predicate calculus and Church’s simple theory of types, both under the standard semantics, is not even semi-decidable. Thus, separating intensions from extensions makes it possible to have an equational theory over predicate and function names (intensions) that is separate from the extensional equality of relations and functions.

In what follows we denote by \( B^A \) the set of all functions from \( A \) to \( B \), and by \( A^n \) a n-folded cartesian product \( A \times ... \times A \) for \( n \geq 1 \). By \( f, t \) we denote empty set \( \emptyset \) and
singleton set \{< >\} respectively (with the empty tuple < > i.e. the unique tuple of 0-ary relation), which may be thought of as falsity \( f \) and truth \( t \), as those used in the relational algebra. For a given domain \( D \) we define that \( D^0 \) is a singleton set \{< >\}, so that \( \{f, t\} = \mathcal{P}(D^0) \), where \( \mathcal{P} \) is the powerset operator.

2 Intensional FOL language with intensional abstraction

Intensional entities are such concepts as propositions and properties. What make them 'intensional' is that they violate the principle of extensionality; the principle that extensional equivalence implies identity. All (or most) of these intensional entities have been classified at one time or another as kinds of Universals [1].

We consider a non empty domain \( D = D_{-1} \cup D_1 \), where a subdomain \( D_{-1} \) is made of particulars (extensional entities), and the rest \( D_1 = D_0 \cup D_1 \cup D_n \ldots \) is made of universals (\( D_0 \) for propositions (the 0-ary concepts), and \( D_n, n \geq 1 \), for n-ary concepts.

The fundamental entities are intensional abstracts or so called, 'that'-clauses. We assume that they are singular terms; Intensional expressions like 'believe', 'mean', 'assert', 'know', are standard two-place predicates that take 'that'-clauses as arguments. Expressions like 'is necessary', 'is true', and 'is possible' are one-place predicates that take 'that'-clauses as arguments. For example, in the intensional sentence "it is necessary that \( \phi \)" where \( \phi \) is a proposition, the "that \( \phi \)" is denoted by the \( \llangle \phi \rrangle \), where \( \llangle \rrangle \) is the intensional abstraction operator which transforms a logic formula into a term. Or, for example, "x believes that \( \phi \)" is given by formula \( p_2^i(x, \llangle \phi \rrangle) \) (\( p_2^i \) is binary 'believe' predicate).

Here we will present an intensional FOL with slightly different intensional abstraction than that originally presented in [2]:

Definition 1. The syntax of the First-order Logic language with intensional abstraction \( \llangle \rrangle \), denoted by \( L \), is as follows:

Logic operators (\( \land, \neg, \exists \)): Predicate letters in \( P \) (functional letters are considered as particular case of predicate letters); Variables \( x, y, z, \ldots \) in \( V \); Abstraction \( \llangle \rrangle \), and punctuation symbols (comma, parenthesis). With the following simultaneous inductive definition of term and formula:

1. All variables and constants (0-ary functional letters in \( P \)) are terms.
2. If \( t_1, \ldots, t_k \) are terms, then \( p_k^j(t_1, \ldots, t_k) \) is a formula (\( p_k^j \in P \) is a k-ary predicate letter).
3. If \( \phi \) and \( \psi \) are formulae, then \( (\phi \land \psi), \neg \phi, \text{ and } (\exists x)\phi \) are formulae.
4. If \( \phi(x) \) is a formula (virtual predicate) with a list of free variables in \( x = (x_1, \ldots, x_n) \) (with ordering from-left-to-right of their appearance in \( \phi \)), and \( \alpha \) is its sublist of distinct variables, then \( \llangle \phi \rrangle^\beta_\alpha \) is a term, where \( \beta \) is the remaining list of free variables preserving ordering in \( x \) as well. The externally quantifiable variables are the free variables not in \( \alpha \). When \( n = 0 \), \( \llangle \phi \rrangle \) is a term which denotes a proposition, for \( n \geq 1 \) it denotes a n-ary concept.

An occurrence of a variable \( x_i \) in a formula (or a term) is bound (free) iff it lies (does not lie) within a formula of the form \( (\exists x_i)\phi \) (or a term of the form \( \llangle \phi \rrangle^\alpha_\alpha \) with \( x_i \in \alpha \)).

A variable is free (bound) in a formula (or term) iff it has (does not have) a free occurrence in that formula (or term).
A sentence is a formula having no free variables. The binary predicate letter \( p^2_1 \) for identity is singled out as a distinguished logical predicate and formulae of the form \( p^2_1(t_1, t_2) \) are to be rewritten in the form \( t_1 \equiv t_2 \). We denote by \( R_e \) the binary relation obtained by standard Tarski’s interpretation of this predicate \( p^2_1 \). The logic operators \( \forall, \exists, \Rightarrow \) are defined in terms of \( (\land, \neg, \exists) \) in the usual way.

The universal quantifier is defined by \( \forall = \neg \exists \neg \). Disjunction and implication are expressed by \( \phi \lor \psi = \neg (\neg \phi \land \neg \psi) \), and \( \phi \Rightarrow \psi = \neg \phi \lor \psi \). In FOL with the identity \( = \), the formula \((\exists_1 x)\phi(x)\) denotes the formula \((\exists x)(\forall y)(\phi(x) \land \phi(y) \Rightarrow (x \equiv y))\).

We denote by \( R_e \) the Tarski’s interpretation of \( = \).

In what follows any open-sentence, a formula \( \phi \) with non empty tuple of free variables \((x_1, \ldots, x_m)\), will be called a \( m \)-ary virtual predicate, denoted also by \( \phi(x_1, \ldots, x_m) \).

This definition contains the precise method of establishing the ordering of variables in this tuple: such an method that will be adopted here is the ordering of appearance, from left to right, of free variables in \( \phi \). This method of composing the tuple of free variables is the unique and canonical way of definition of the virtual predicate from a given formula.

The intensional interpretation of this intensional FOL is a mapping between the set \( \mathbb{L} \) of formulae of the logic language and intensional entities in \( \mathcal{D} \), \( I : \mathbb{L} \rightarrow \mathcal{D} \), is a kind of “conceptualization”, such that an open-sentence (virtual predicate) \( \phi(x_1, \ldots, x_k) \) with a tuple of all free variables \((x_1, \ldots, x_k)\) is mapped into a \( k \)-ary concept, that is, an intensional entity \( u = I(\phi(x_1, \ldots, x_k)) \in D_k \), and (closed) sentence \( \psi \) into a proposition (i.e., logic concept) \( v = I(\psi) \in D_0 \) with \( I(\top) = Truth \in D_0 \) for the FOL tautology \( \top \). A language constant \( c \) is mapped into a particular \( a = I(c) \in D_{-1} \) if it is a proper name, otherwise in a correspondent concept in \( \mathcal{D} \).

An assignment \( g : \mathcal{V} \rightarrow \mathcal{D} \) for variables in \( \mathcal{V} \) is applied only to free variables in terms and formulae. Such an assignment \( g \in \mathcal{D}^\mathcal{V} \) can be recursively uniquely extended into the assignment \( g^* : \mathcal{T} \rightarrow \mathcal{D} \), where \( \mathcal{T} \) denotes the set of all terms (here \( I \) is an intensional interpretation of this FOL, as explained in what follows), by:

1. \( g^*(t) = g(x) \in \mathcal{D} \) if the term \( t \) is a variable \( x \in \mathcal{V} \).
2. \( g^*(I(c)) = I(c) \in \mathcal{D} \) if the term \( t \) is a constant \( c \in P \).
3. if \( t \) is an abstracted term \( \langle \phi \rangle^\alpha_\beta \), then \( g^*(\langle \phi \rangle^\alpha_\beta) = I(\phi[\beta/g(\beta)]) \in D_k \), \( k = |\alpha| \) (i.e., the number of variables in \( \alpha \)), where \( g(\beta) = g(y_1, \ldots, y_m) = \langle g(y_1), \ldots, g(y_m) \rangle \) and \( I(\beta/g(\beta)) \) is a uniform replacement of each \( i \)-th variable in the list \( \beta \) with the \( i \)-th constant in the list \( g(\beta) \). Notice that \( \alpha \) is the list of all free variables in the formula \( \phi[\beta/g(\beta)] \).

We denote by \( t/g \) (or \( \phi/g \)) the ground term (or formula) without free variables, obtained by assignment \( g \) from a term \( t \) (or a formula \( \phi \)), and by \( \phi[x/t] \) the formula obtained by uniformly replacing \( x \) by a term \( t \) in \( \phi \).

The distinction between intensions and extensions is important especially because we are now able to have and equational theory over intensional entities (as \( \langle \phi \rangle \)), that is predicate and function “names”, that is separate from the extensional equality of relations and functions. An extensionalization function \( h \) assigns to the intensional elements of \( \mathcal{D} \) an appropriate extension as follows: for each proposition \( u \in D_0 \), \( h(u) \in \{f, t\} \subseteq \mathcal{P}(D_{-1}) \) is its extension (true or false value); for each \( n \)-ary concept \( u \in D_n \), \( h(u) \) is a subset of \( D^n \) (\( n \)-th Cartesian product of \( \mathcal{D} \)); in the case of
Thus, intensional FOL has the simple Tarski first-order semantics, with a decidable constraint (T); they can be considered as abstract or concrete entities, while extensions correspond to various rules that these entities play in different worlds.

**Remark:** (Tarski’s constraint) This semantics has to preserve Tarski’s semantics of the FOL, that is, for any formula $\phi \in \mathcal{L}$, any assignment $g \in \mathcal{D}^V$, and every $h \in \mathcal{E}$ it has to be satisfied that:

\[
\text{(T)} \quad h(I(\phi/g)) = t \iff (g(x_1),...,g(x_k)) \in h(I(\phi)).
\]

Thus, intensional FOL has the simple Tarski first-order semantics, with a decidable unification problem, but we need also the actual world mapping which maps any intensional entity to its actual world extension. In what follows we will identify a possible world by a particular mapping which assigns to intensional entities their extensions in such possible world. That is direct bridge between intensional FOL and possible worlds representation [4-5,6,7,8,9], where intension of a proposition is a function from possible worlds $\mathcal{W}$ to truth-values, and properties and functions from $\mathcal{W}$ to sets of possible (usually not-actual) objects.

Here $\mathcal{E}$ denotes the set of possible extensionalization functions that satisfy the constraint (T); they can be considered as possible worlds (as in Montague’s intensional semantics for natural language [6,8]), as demonstrated in [10,11], given by the bijection $\iota : \mathcal{W} \simeq \mathcal{E}$.

Now we are able to define formally this intensional semantics [9]:

**Definition 2. Two-step Intensional Semantics:**
Let $\mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{P}(\mathcal{D}^k) = \sum_{k \in \mathbb{N}} \mathcal{P}(\mathcal{D}^k)$ be the set of all $k$-ary relations, where $k \in \mathbb{N} = \{0, 1, 2, ...\}$. Notice that $\{f, t\} = \mathcal{P}(\mathcal{D}^0) \in \mathcal{R}$, that is, the truth values are extensions in $\mathcal{R}$.

The intensional semantics of the logic language with the set of formulae $\mathcal{L}$ can be represented by the mapping

\[
\mathcal{L} \rightarrow_I \mathcal{D} \rightarrow_{w \in \mathcal{W}} \mathcal{R},
\]

where $\rightarrow_I$ is a fixed intensional interpretation $I : \mathcal{L} \rightarrow \mathcal{D}$ and $\rightarrow_{w \in \mathcal{W}}$ is the set of all extensionalization functions $h = \iota(w) : \mathcal{D} \rightarrow \mathcal{R}$ in $\mathcal{E}$, where $\iota : \mathcal{W} \rightarrow \mathcal{E}$ is the mapping from the set of possible worlds to the set of extensionalization functions.

We define the mapping $I_n : \mathcal{L}_{op} \rightarrow \mathcal{R}^{\mathcal{W}}$, where $\mathcal{L}_{op}$ is the subset of formulae with free variables (virtual predicates), such that for any virtual predicate $\phi(x_1, ..., x_k) \in \mathcal{L}_{op}$, the mapping $I_n(\phi(x_1, ..., x_k)) : \mathcal{W} \rightarrow \mathcal{R}$ is the Montague’s meaning (i.e., intension) of this virtual predicate [4,5,6,7,8], that is, the mapping which returns with the extension of this (virtual) predicate in every possible world in $\mathcal{W}$.
We adopted this two-step intensional semantics, instead of well known Montague’s semantics (which lies in the construction of a compositional and recursive semantics that covers both intension and extension) because of a number of its weaknesses.

**Example:** Let us consider the following two past participles: ‘bought’ and ‘sold’ (with unary predicates $p_1^1(x)$, ‘$x$ has been bought’, and $p_2^1(x)$, ‘$x$ has been sold’). These two different concepts in the Montague’s semantics would have not only the same extension but also their intension, from the fact that their extensions are identical in every possible world.

Within the two-steps formalism we can avoid this problem by assigning two different concepts (meanings) $u = I(p_1^1(x))$ and $v = I(p_2^1(x))$ in $D_1$. Notice that the same problem we have in the Montague’s semantics for two sentences with different meanings, which bear the same truth value across all possible worlds: in the Montague’s semantics they will be forced to the same meaning.

□

Another relevant question w.r.t. this two-step interpretations of an intensional semantics is how in it is managed the extensional identity relation $\doteq$ (binary predicate of the identity) of the FOL. Here this extensional identity relation is mapped into the binary concept $Id = I(\doteq (x, y)) \in D_2$, such that $(\forall w \in \mathcal{V})(is(w)(Id) = R_{=} = 0)$, where $\doteq(x, y)$ (i.e., $p_1^2(x, y)$) denotes an atom of the FOL of the binary predicate for identity in FOL, usually written by FOL formula $x \doteq y$ (here we prefer to distinguish this formal symbol $\doteq \in P$ of the built-in identity binary predicate letter in the FOL from the standard mathematical symbol ‘$=$’ used in all mathematical definitions in this paper).

In what follows we will use the function $f_{<>}: \mathfrak{R} \to \mathfrak{R}$, such that for any $R \in \mathfrak{R}$, $f_{<>}(R) = \{<>\}$ if $R \neq \emptyset$; $\emptyset$ otherwise. Let us define the following set of algebraic operators for relations in $\mathfrak{R}$:

1. binary operator $\bowtie_S: \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, such that for any two relations $R_1, R_2 \in \mathfrak{R}$, the $R_1 \bowtie_S R_2$ is equal to the relation obtained by natural join of these two relations $\mathfrak{R}$ $\bowtie$ $S$ is a non empty set of pairs of joined columns of respective relations (where the first argument is the column index of the relation $R_1$ while the second argument is the column index of the joined column of the relation $R_2$); otherwise it is equal to the cartesian product $R_1 \times R_2$. For example, the logic formula $\phi(x_1, x_j, x_k, x_l, x_m) \land \psi(y_1, y_i, y_j, y_l)$ will be traduced by the algebraic expression $R_1 \bowtie_S R_2$ where $R_1 \in \mathcal{P}(D^3)$, $R_2 \in \mathcal{P}(D^4)$ are the extensions for a given Tarski’s interpretation of the virtual predicate $\phi, \psi$ relatively, so that $S = \{(4, 1), (2, 3)\}$ and the resulting relation will have the following ordering of attributes: $(x_1, x_j, x_k, x_l, y_1, y_i)$.

2. unary operator $\neg: \mathfrak{R} \to \mathfrak{R}$, such that for any k-ary (with $k \geq 0$) relation $R \in \mathcal{P}(D^k) \subset \mathfrak{R}$ we have that $\neg (R) = D^k \setminus R \in D^k$, where \`\` is the substraction of relations. For example, the logic formula $\neg \phi(x_1, x_j, x_k, x_l, x_m)$ will be traduced by the algebraic expression $D^k \setminus R$ where $R$ is the extensions for a given Tarski’s interpretation of the virtual predicate $\phi$.

3. unary operator $\pi_m: \mathfrak{R} \to \mathfrak{R}$, such that for any k-ary (with $k \geq 0$) relation $R \in \mathcal{P}(D^k) \subset \mathfrak{R}$ we have that $\pi_m(R)$ is equal to the relation obtained by elimination of the $m$-th column of the relation $R$ if $1 \leq m \leq k$ and $k \geq 2$; equal to $f_{<>}(R)$ if $m = k = 1$; otherwise it is equal to $R$. For example, the logic
formulas \((\exists x_k)\phi(x_i, x_j, x_k, x_l, x_m)\) will be translated by the algebraic expression 
\(\pi_{-3}(R)\) where \(R\) is the extensions for a given Tarski’s interpretation of the virtual predicate \(\phi\) and the resulting relation will have the following ordering of attributes:
\((x_i, x_j, x_l, x_m)\).

Notice that the ordering of attributes of resulting relations corresponds to the method used for generating the ordering of variables in the tuples of free variables adopted for virtual predicates.

Analogously to Boolean algebras which are extensional models of propositional logic, we introduce an intensional algebra for this intensional FOL as follows.

**Definition 3.** Intensional algebra for the intensional FOL in Definition 1 is a structure 
\(\mathcal{A}_{int} = (\mathcal{D}, f, t, \text{Id}, \text{Truth}, \{\conjs\}_{S \in \mathcal{P}(\mathcal{N})}, \neg, \{\exists\}_{n \in \mathcal{N}})\) with binary operations \(\conjs : D_1 \times D_1 \to D_1\), unary operation \(\neg : D_1 \to D_1\), unary operations \(\exists n : D_1 \to D_1\), such that for any extensionalization function \(h \in \mathcal{E}\), and \(u \in D_k, v \in D_j, k, j \geq 0\),
1. \(h(\text{Id}) = R_n\) and \(h(\text{Truth}) = \{<>\}\).
2. \(h(\conjs(u, v)) = h(u) \bowtie_S h(v)\), where \(\bowtie_S\) is the natural join operation defined above and \(\conjs(u, v) \in D_m\) where \(m = k + j - |S|\) if for every pair \((i_1, i_2) \in S\) it holds that \(1 \leq i_1 \leq k, 1 \leq i_2 \leq j\) (otherwise \(\conjs(u, v) \in D_{k+j}\)).
3. \(h(\neg(u)) = \sim (h(u)) = D^\emptyset \setminus \{h(u)\}\), where \(\sim\) is the operation defined above and \(\neg(u) \in D_k\) if \(1 \leq n \leq k\) (otherwise \(\neg(u)\) is the identity function).

Notice that for \(u \in D_0\), \(h(\neg(u)) = \sim (h(u)) = D^\emptyset \setminus \{h(u)\}\) unless \(n \leq k\) (otherwise \(\neg(u)\) is the identity function).

We define a derived operation \(\text{union} : (\mathcal{P}(D_i) \setminus \emptyset) \to D_i, i \geq 0\), such that, for any \(B = \{u_1, ..., u_n\} \in \mathcal{P}(D_i)\) we have that \(\text{union}(\{u_1, ..., u_n\}) \triangleq u_1\) if \(n = 1\);
\(\neg(u_1, ..., \neg(u_n))\), where \(S = \{(l, l) \mid 1 \leq l \leq i\}\), otherwise. Then we obtain that for \(n \geq 2\):
\(h(\text{union}(B)) = h(\neg(\conjs(\neg(u_1), \conjs(..., \neg(u_n))))) = D^\emptyset \setminus (D^\emptyset \setminus h(u_1)) \circlearrowleft_S ... \circlearrowleft_S (D^\emptyset \setminus h(u_n))) = D^\emptyset \setminus (D^\emptyset \setminus h(u_1)) \cap ... \cap (D^\emptyset \setminus h(u_n))) = \bigcup \{h(u_j) \mid 1 \leq j \leq n\} = \bigcup \{h(u) \mid u \in B\}.

Intensional interpretation \(I : \mathcal{L} \to \mathcal{D}\) satisfies the following homomorphic extension:

1. The logic formula \(\phi(x_i, x_j, x_k, x_l, x_m) \land \psi(x_l, y_i, x_j, y_j)\) will be intensionally interpreted by the concept \(u_1 \in D_7\), obtained by the algebraic expression \(\conjs(u, v)\) where \(u = I(\phi(x_i, x_j, x_k, x_l, x_m)) \in D_5, v = I(\psi(x_l, y_i, x_j, y_j)) \in D_4\) are the concepts of the virtual predicates \(\phi, \psi\), relatively, and \(S = \{(4, 1), (2, 3)\}\). Consequently, we have that for any two formulae \(\phi, \psi \in \mathcal{L}\) and a particular operator \(\conjs\) uniquely determined by tuples of free variables in these two formulae, \(I(\phi \land \psi) = \conjs(I(\phi), I(\psi))\).
2. The logic formula \(\neg\phi(x_i, x_j, x_k, x_l, x_m)\) will be intensionally interpreted by the concept \(u_1 \in D_7\), obtained by the algebraic expression \(\neg(u)\) where \(u = I(\phi(x_i, x_j, x_k, x_l, x_m)) \in D_5\) is the concept of the virtual predicate \(\phi\). Consequently, we have that for any formula \(\phi \in \mathcal{L}\), \(I(\neg \phi) = \neg(I(\phi))\).
I follow: for any abstracted term given in the version of Bealer \[12\], as follows:

For example, \(\{\phi\} = T \rightarrow D\) is equal to the application of this function to the value \(I(\phi)\)

where \(\beta\) denotes the set of free variables in \(\phi\) (otherwise \(n = 0\) if this quantified variable is not a free variable in \(\phi\)). \(I(\exists x)\phi = \text{exists}_n(I(\phi))\)

Once one has found a method for specifying the interpretations of singular terms of \(L\) (take in consideration the particularity of abstracted terms), the Tarski-style definitions of truth and validity for \(L\) may be given in the customary way. What is being south specifically is a method for characterizing the intensional interpretations of singular terms of \(L\) in such a way that a given singular abstracted term \(\langle \phi \rangle_{\gamma}^\beta\) will denote an appropriate property, relation, or proposition, depending on the value of \(m = |\alpha|\). Thus, the mapping of intensional abstracts (terms) into \(D\) we will define differently from that given in the version of Bealer \[12\], as follows:

**Definition 4.** An intensional interpretation \(I\) can be extended to abstracted terms as follows: for any abstracted term \(\langle \phi \rangle_{\gamma}^\beta\) we define that,

\[I(\langle \phi \rangle_{\gamma}^\beta) = \text{union}(\{I(p_{i}^{m}(x_{1},\ldots,x_{m}) | g \in D_{\overline{\gamma}}\})\),\]

where \(\beta\) denotes the set of elements in the list \(\beta\), and the assignments in \(D_{\overline{\gamma}}\) are limited only to the variables in \(\overline{\gamma}\).

**Remark:** Here we can make the question if there is a sense to extend the interpretation also to (abstracted) terms, because in Tarski's interpretation of FOL we do not have any interpretation for terms, but only the assignments for terms, as we defined previously by the mapping \(g^* : T \rightarrow D\). The answer is positive, because the abstraction symbol \(\langle \_ \rangle_{\gamma}^\beta\) can be considered as a kind of the unary built-in functional symbol of intensional FOL, so that we can apply the Tarskiian interpretation to this functional symbol into the fixed mapping \(I(\langle \_ \rangle_{\gamma}^\beta) : L \rightarrow D\), so that for any \(\phi \in L\) we have that \(I(\langle \phi \rangle_{\gamma}^\beta)\) is equal to the application of this function to the value \(\phi\), that is, to \(I(\langle \_ \rangle_{\gamma}^\beta)(\phi)\). In such an approach we would introduce also the typed variable \(X\) for the formulae in \(L\), so that the Tarskiian assignment for this functional symbol with variable \(X\), with \(g(X) = \phi \in L\), can be given by:

\[g^*(\langle \_ \rangle_{\gamma}^\beta(X)) = I(\langle \_ \rangle_{\gamma}^\beta)(g(X)) = I(\langle \_ \rangle_{\gamma}^\beta)(\phi)\]

= \langle \_ \rangle_{\gamma}^\beta(\alpha) \in D_\gamma, \text{ if } \{\overline{\gamma}\} \text{ is not equal to the set of free variables in } \phi; \]

\[= \text{union}(\{I(p_{i}^{m}(g(x_{1}),\ldots,g(x_{m})) | g \in D_{\overline{\gamma}}\}) \in D_{\overline{\gamma}}, \text{ otherwise.}\]

Notice than if \(\beta = \emptyset\) is the empty list, then \(I(\langle \phi \rangle_{\gamma}^\beta) = I(\phi)\). Consequently, the denotation of \(\langle \phi \rangle\) is equal to the meaning of a proposition \(\phi\), that is, \(I(\langle \phi \rangle) = I(\phi)\) in \(D_0\). In the case when \(\phi\) is an atom \(p_{i}^{m}(x_{1},\ldots,x_{m})\) then \(I(\langle p_{i}^{m}(x_{1},\ldots,x_{m}) \rangle_{x_{1},\ldots,x_{m}}\rangle) = I(p_{i}^{m}(x_{1},\ldots,x_{m})) \in D_{m}, \text{ while}\)

\[I(\langle p_{i}^{m}(x_{1},\ldots,x_{m}) \rangle_{x_{1},\ldots,x_{m}}\rangle = \text{union}(\{I(p_{i}^{m}(g(x_{1}),\ldots,g(x_{m})) | g \in D_{\{x_{1},\ldots,x_{m}\}}\}) \in D_{0}, \text{ with } h(I(\langle p_{i}^{m}(x_{1},\ldots,x_{m}) \rangle_{x_{1},\ldots,x_{m}}\rangle) = h(I(\langle p_{i}^{m}(x_{1},\ldots,x_{m}) \rangle_{x_{1},\ldots,x_{m}}\rangle) \in \{f, t\}.\]

For example,

\[h(I(\langle p_{i}^{1}(x_{1}) \wedge \neg p_{i}^{1}(x_{1}) \rangle_{x_{1}})) = h(I(\langle \exists x_{1}(\langle p_{i}^{1}(x_{1}) \wedge \neg p_{i}^{1}(x_{1}) \rangle_{x_{1}})) = f.\]
The interpretation of a more complex abstract $<\phi>^\beta_\alpha$ is defined in terms of the interpretations of the relevant syntactically simpler expressions, because the interpretation of more complex formulae is defined in terms of the interpretation of the relevant syntactically simpler formulae, based on the intensional algebra above. For example, $I(p_1^1(x) \land p_2^1(x)) = \text{conj}((1,1))(I(p_1^1(x)),I(p_2^1(x)))$, $I(\neg \phi) = \text{neg}(I(\phi))$, $I(\exists x_1)\phi(x_1, x_2, x_3, x_k) = \text{exist}_3(I(\phi))$.

Consequently, based on the intensional algebra in Definition 3 and on intensional interpretations of abstracted term in Definition 4, it holds that the interpretation of any formula in $\mathcal{L}$ (and any abstracted term) will be reduced to an algebraic expression over intensional interpretations of abstracted term in Definition 4 it holds that the interpretation of any formula in $\mathcal{L}$ (and any abstracted term) will be reduced to an algebraic expression over intensional in-interpretations of abstracted term in Definition 4, it holds that the interpretation of any formula in $\mathcal{L}$ (and any abstracted term) will be reduced to an algebraic expression over intensional interpretations of abstracted term in Definition 4, it holds that the interpretation of any formula in $\mathcal{L}$ (and any abstracted term) will be reduced to an algebraic expression over interpretations of primitive atoms in $\mathcal{L}$. This obtained expression is finite for any finite formula (or abstracted term), and represents the meaning of such finite formula (or abstracted term).

The extension of abstracted terms satisfy the following property:

**Proposition 1** For any abstracted term $<\phi>^\beta_\alpha$ with $|\alpha| \geq 1$ we have that $h(I(<\phi>^\beta_\alpha)) = \pi_{-\beta}(h(I(\phi)))$.

where $\pi_{-\{y_1,\ldots,y_k\}} = \pi_{-y_1} \circ \ldots \circ \pi_{-y_k}$ is the sequential composition of functions, and $\pi_{-\beta}$ is an identity.

**Proof:** Let $\mathbf{x}$ be a tuple of all free variables in $\phi$, so that $\mathbf{x} = \mathcal{P} \cup \mathcal{A}$, $\alpha = (x_1, \ldots, x_k)$, then we have that $h(I(<\phi>^\beta_\alpha)) = h(\text{union}(I(\phi[\beta/g(\beta)])) | g \in D^\mathcal{P}))$, from Def. 4,

\[
= \{ h(I(\phi[\beta/g(\beta)])) | g \in D^\mathcal{P} \}
\]

\[
= \{ g_1(\alpha) | g_1 \in D^\mathcal{A} \text{ and } h(I(\phi/g_1)) = t \} | g \in D^\mathcal{P} \}
\]

\[
= \pi_{-\beta}(\{ g_1(\alpha) | g_1 \in D^\mathcal{A} \text{ and } h(I(\phi/g_1)) = t \})
\]

\[
= \pi_{-\beta}(\{ g_1(\mathbf{x}) | g_1 \in D^\mathcal{A} \text{ and } h(I(\phi)) \})
\]

\[
= \pi_{-\beta}(h(I(\phi))).
\]

We can connect $E$ with a possible-world semantics. Such a correspondence is a natural identification of intensional logics with modal Kripke based logics.

**Definition 5.** (Model): A model for intensional FOL with fixed intensional interpretation $I$, which express the two-step intensional semantics in Definition 2 is the Kripke structure $\mathcal{M}_{int} = (\mathcal{W}, D, V)$, where $\mathcal{W} = \{ is^{-1}(h) | h \in E \}$, a mapping $V: \mathcal{W} \times P \rightarrow \bigcup_{n<\omega} (t, f)^D^n$, with $P$ a set of predicate symbols of the language, such that for any world $w = is^{-1}(h) \in \mathcal{W}$, $p_i^n \in P$, and $(u_1, \ldots, u_n) \in D^n$ it holds that $V(w,p_i^n)(u_1, \ldots, u_n) = h(p_i^n(u_1, \ldots, u_n))$. The satisfaction relation $\models_{w,g}$ for a given $w \in \mathcal{W}$ and assignment $g \in D\mathcal{W}$ is defined as follows:

1. $\mathcal{M} \models_{w,g} p_i^n(x_1, \ldots, x_k)$ iff $V(w,p_i^n)(g(x_1), \ldots, g(x_k)) = t$,
2. $\mathcal{M} \models_{w,g} \varphi \land \psi$ iff $\mathcal{M} \models_{w,g} \varphi$ and $\mathcal{M} \models_{w,g} \psi$,
3. $\mathcal{M} \models_{w,g} \neg \varphi$ iff not $\mathcal{M} \models_{w,g} \varphi$.
4. $\mathcal{M} \models_{w,g} (\exists x) \phi$ iff
4.1. $\mathcal{M} \models_{w,g} \phi$, if $x$ is not a free variable in $\phi$;
4.2. exists $u \in D$ such that $\mathcal{M} \models_{w,g} \phi[x/u]$, if $x$ is a free variable in $\phi$. 

It is easy to show that the satisfaction relation $|=\,$ for this Kripke semantics in a world $w = is^{-1}(h)$ is defined by, $\mathcal{M} |= w, g \phi$ iff $h(I(\phi/g)) = t$.

We can enrich this intensional FOL by another modal operators, as for example the "necessity" universal operator $\Box$ with an accessibility relation $\mathcal{R} = W \times W$, obtaining the S5 Kripke structure $\mathcal{M}_{int} = (W, \mathcal{R}, D, V)$, in order to be able to define the following equivalences between the abstracted terms without free variables $<\phi>_{\alpha}^1/g$ and $<\psi>_{\alpha}^2/g$, where all free variables (not in $\alpha$) are instantiated by $g \in D^V$ (here $A \equiv B$ denotes the formula $(A \Rightarrow B) \land (B \Rightarrow A)$:

- **(Strong) intensional equivalence (or equality) "$\approx$" is defined by:**
  
  $<\phi>_{\alpha}^1/g \approx <\psi>_{\alpha}^2/g$ iff $\Box(\phi[\beta_1/g(\beta_1)] \equiv \psi[\beta_2/g(\beta_2)])$,

  with $\mathcal{M} |= w, g' \Box \phi$ iff for all $w' \in W$, $(w, w') \in \mathcal{R}$ implies $\mathcal{M} |= w', g' \phi$.

  From Example 1 we have that $<p_1(x)>_x \approx <p_2(x)>_x$, that is 'x has been bought' and 'x has been sold' are intensionally equivalent, but they have not the same meaning (the concept $I(p_1(x)) \in D_1$ is different from $I(p_2(x)) \in D_1$).

- **Weak intensional equivalence "$\equiv$" is defined by:**
  
  $<\phi>_{\alpha}^1/g \equiv <\psi>_{\alpha}^2/g$ iff $\Box[\phi[\beta_1/g(\beta_1)] \equiv \psi[\beta_2/g(\beta_2)]]$.

  The symbol $\Diamond \equiv \neg \Box$ is the correspondent existential modal operator.

This weak equivalence is used for P2P database integration in a number of papers [13-14-15-16-17-18-19].

Notice that we do not use the intensional equality in our language, thus we do not need the correspondent operator in intensional algebra $\mathcal{A}_{int}$ for the logic "necessity" modal operator $\Box$.

This semantics is equivalent to the algebraic semantics for $\mathbf{L}$ in [2], for the case of the conception where intensional entities are considered to be equal if and only if they are necessarily equivalent. Intensional equality is much stronger that the standard extensional equality in the actual world, just because requires the extensional equality in all possible worlds, in fact, if $<\phi>_{\alpha}^1/g \equiv <\psi>_{\alpha}^2/g$ then $h(I(<A>^{\alpha}_i/g)) = h(I(<\phi>_{\alpha}^1/g))$ for all extensionalization functions $h \in \mathcal{E}$ (that is possible worlds is $is^{-1}(h) \in \mathcal{W}$).

It is easy to verify that the intensional equality means that in every possible world $w \in \mathcal{W}$ the intensional entities $u_1$ and $u_2$ have the same extensions.

Let the logic modal formula $\Box[\phi[\beta_1/g(\beta_1)]]$, where the assignment $g$ is applied only to free variables in $\beta_1$ of a formula $\phi$ not in the list of variables in $\alpha = (x_1, ..., x_n)$, $n \geq 1$, represents a $n$-ary intensional concept such that $I(\Box[\phi[\beta_1/g(\beta_1)]] \in D_n$ and $I(\phi[\beta_1/g(\beta_1)]) = I(<\phi>_{\alpha}^1/g) \in D_n$. Then the extension of this $n$-ary concept is equal to (here the mapping $necess : D_i \mapsto D_i$ for each $i \geq 0$ is a new operation of the intensional algebra $\mathcal{A}_{int}$ in Definition[3]):

\[
h(I(\Box[\phi[\beta_1/g(\beta_1)])) = h(necess(I(\phi[\beta_1/g(\beta_1)]))) = \]

\[
= \{(g'(x_1), ..., g'(x_n)) \mid \mathcal{M} |= w, g' \Box[\phi[\beta_1/g(\beta_1)] \land g' \in D^V) \}
\]

\[
\in \bigcap_{h_1 \in \epsilon} \ h_1(I(\phi[\beta_1/g(\beta_1)])) \).
\]

While,

\[
h(I(\Diamond[\phi[\beta_1/g(\beta_1)])) = h(I(\neg \Box[\phi[\beta_1/g(\beta_1)]])
\]

\[
= h(neg(necess(I(\neg \phi[\beta_1/g(\beta_1)])))))
\]
Thus, two concepts are intensionally abstraction operator $h$.

Consequently, the concepts $\bigcup h \in h_1(I(\phi |\beta_1 / g(\beta_1)))$

= $D_n \setminus h(necess(I(\neg \phi |\beta_1 / g(\beta_1))))$

= $D_n \setminus (\bigcap h_1 \in h_1(I(\neg \phi |\beta_1 / g(\beta_1))))$

= $D_n \setminus (\bigcap h_1 \in h_1(neg(I(\phi |\beta_1 / g(\beta_1))))$

= $D_n \setminus (\bigcap h_1 \in \bigcup (h_1(I(\phi |\beta_1 / g(\beta_1))))$

= $\bigcup h_1(I(\phi |\beta_1 / g(\beta_1))))$

Consequently, the concepts $\Box \phi |\beta_1 / g(\beta_1)$ and $\Diamond \phi |\beta_1 / g(\beta_1)$ are the built-in (or rigid) concept as well, whose extensions does not depend on possible worlds.

Thus, two concepts are intensionally equal, that is, $\langle \phi \rangle_{\alpha} / g \equiv \langle \psi \rangle_{\beta} / g$, iff $h(I(\phi |\beta_1 / g(\beta_1))) = h(I(\psi |\beta_2 / g(\beta_2)))$ for every $h$.

Moreover, two concepts are weakly equivalent, that is, $\langle \phi \rangle_{\alpha} / g \equiv \langle \psi \rangle_{\beta} / g$, iff $h(I(\Diamond \phi |\beta_1 / g(\beta_1))) = h(I(\Diamond \psi |\beta_2 / g(\beta_2)))$.

3 Application to the intensional FOL without abstraction operator

In the case for the intensional FOL defined in Def. 1 without the Bealer’s intensional abstraction operator $\ll$, we obtain the syntax of the standard FOL but with intensional semantics as presented in [9].

Such a FOL has a well known Tarski’s interpretation, defined as follows:

- An interpretation (Tarski) $I_T$ consists in a non empty domain $D$ and a mapping
  that assigns to any predicate letter $p^k \in P$ a relation $R = I_T(p^k) \subseteq D^k$, to any
  functional letter $f^k \in F$ a function $I_T(f^k) : D^k \rightarrow D$, or, equivalently, its graph
  relation $R = I_T(f^k) \subseteq D^{k+1}$ where the $k+1$-th column is the resulting function’s
  value, and to each individual constant $c \in F$ one given element $I_T(c) \in D$.

Consequently, from the intensional point of view, an interpretation of Tarski is a possible world in the Montague’s intensional semantics, that is $w = I_T \in W$. The correspondent extensionalization function if $h = is(w) = is(I_T)$.

We define the satisfaction for the logic formulae in $L$ and a given assignment $g : \mathcal{V} \rightarrow D$ inductively, as follows:

If a formula $\phi$ is an atomic formula $p^k(t_1, ..., t_k)$, then this assignment $g$ satisfies $\phi$ iff $(g^*(t_1), ..., g^*(t_k)) \in I_T(p^k)$; $g$ satisfies $\neg \phi$ iff it does not satisfy $\phi$; $g$ satisfies $\phi \land \psi$ iff $g$ satisfies $\phi$ and $g$ satisfies $\psi$; $g$ satisfies $(\exists x_i)\phi$ iff exists an assignment $g' \in D^V$ that may differ from $g$ only for the variable $x_i \in \mathcal{V}$, and $g'$ satisfies $\phi$.

A formula $\phi$ is true for a given interpretation $I_T$ iff $\phi$ is satisfied by every assignment $g \in D^V$. A formula $\phi$ is valid (i.e., tautology) iff $\phi$ is true for every Tarski’s interpretation $I_T \in \mathcal{I}_T$. An interpretation $I_T$ is a model of a set of formulae $\Gamma$ iff every formula $\phi \in \Gamma$ is true in this interpretation. We denote by $\text{FOL}(\Gamma)$ the FOL with a set of assumptions $\Gamma$, and by $\mathcal{I}_T(\Gamma)$ the subset of Tarski’s interpretations that are models of $\Gamma$, with $\mathcal{I}_T(\emptyset) = \mathcal{I}_T$. A formula $\phi$ is said to be a logical consequence of $\Gamma$, denoted by $\Gamma \vdash \phi$, iff $\phi$ is true in all interpretations in $\mathcal{I}_T(\Gamma)$. Thus, $\vdash \phi$ iff $\phi$ is a tautology.

The basic set of axioms of the FOL are that of the propositional logic with two additional axioms: (A1) $(\forall x)(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow (\forall x)\psi)$, ($x$ does not occur in $\phi$ and it is not bound in $\psi$), and (A2) $(\forall x)\phi \Rightarrow \phi[t/x]$, (neither $x$ nor any variable in $t$ occurs bound in $\phi$). For the FOL with identity, we need the proper axiom (A3) $\forall x_1 \equiv x_2 \Rightarrow (x_1 \equiv x_3 \Rightarrow x_2 \equiv x_3)$. 
The inference rules are Modus Ponens and generalization (G) "if \( \phi \) is a theorem and \( x \) is not bound in \( \phi \), then (\( \forall x \))\( \phi \) is a theorem".

The standard FOL is considered as an extensional logic because two open-sentences with the same tuple of variables \( \phi(x_1, ..., x_m) \) and \( \psi(x_1, ..., x_m) \) are equal iff they have the same extension in a given interpretation \( I_T \), that is iff \( I_T^*(\phi(x_1, ..., x_m)) = I_T^*(\psi(x_1, ..., x_m)) \), where \( I_T^* \) is the unique extension of \( I_T \) to all formulae, as follows:

1. For a (closed) sentence \( \phi/g \) we have that \( I_T^*(\phi/g) = t \) iff \( g \) satisfies \( \phi \), as recursively defined above.

2. For an open-sentence \( \phi \) with the tuple of free variables \( (x_1, ..., x_m) \) we have that
   \[
   I_T^*(\phi(x_1, ..., x_m)) = \text{def } \{ (g(x_1), ..., g(x_m)) | g \in \mathcal{D}^k \text{ and } I_T^*(\phi/g) = t \}.
   \]
   It is easy to verify that for a formula \( \phi \) with the tuple of free variables \( (x_1, ..., x_m) \),
   \[
   I_T^*(\phi(x_1, ..., x_m))/g = t \iff (g(x_1), ..., g(x_m)) \in I_T^*(\phi(x_1, ..., x_m)).
   \]

This extensional equality of virtual predicates can be generalized to the extensional equivalence when both predicates \( \phi, \psi \) has the same set of free variables but their ordering in the tuples of free variables are not identical: such two virtual predicates are equivalent if the extension of the first is equal to the proper permutation of columns of the extension of the second virtual predicate. It is easy to verify that such an extensional equivalence corresponds to the logical equivalence denoted by \( \phi \equiv \psi \).

This extensional equivalence between two relations \( R_1, R_2 \in \mathcal{R} \) with the same arity will be denoted by \( R_1 \equiv R_2 \), while the extensional identity will be denoted in the standard way by \( R_1 = R_2 \).

Let \( \mathcal{A}_{FOL} = (\mathcal{L}, \land, \top, \neg, \exists) \) be a free syntax algebra for "First-order logic with identity \( = \)", with the set \( \mathcal{L} \) of first-order logic formulae, with \( \top \) denoting the tautology formula (the contradiction formula is denoted by \( \neg \top \)), with the set of variables in \( \mathcal{V} \) and the domain of values in \( \mathcal{D} \). It is well known that we are able to make the extensional algebraization of the FOL by using the cylindric algebras [19] that are the extension of Boolean algebras with a set of binary operators for the FOL identity relations and a set of unary algebraic operators ("projections") for each case of FOL quantification (\( \exists x \)). In what follows we will make an analog extensional algebraization over \( \mathcal{R} \) but by interpretation of the logic conjunction \( \land \) by a set of natural join operators over relations introduced by Codd’s relational algebra [3][20] as a kind of a predicate calculus whose interpretations are tied to the database.

**Corollary 1** **Extensional FOL Semantics:**

Let us define the extensional relational algebra for the FOL by,

\[ \mathcal{A}_R = (\mathcal{R}, R_n = \{ < > \}, \{ > \}_{S \in \mathcal{P}(\mathbb{N})}, \sim, \{ \pi_{-n} \}_{n \in \mathbb{N}}), \]

where \( \{ < > \} \in \mathcal{R} \) is the algebraic value correspondent to the logic truth, and \( R_n \) is the binary relation for extensionally equal elements. We will use ‘\( = \)’ for the extensional identity for relations in \( \mathcal{R} \).

Then, for any Tarski’s interpretation \( I_T \) its unique extension to all formulae \( I_T^* \) : \( \mathcal{L} \to \mathcal{R} \) is also the homomorphism \( I_T^* : \mathcal{A}_{FOL} \to \mathcal{A}_R \) from the free syntax FOL algebra into this extensional relational algebra.

**Proof:** In [9]

\( \square \)
Consequently, we obtain the following Intensional/extensional FOL semantics [9]:

For any Tarski’s interpretation \( I_T \) of the FOL, the following diagram of homomorphisms commutes,

\[
\begin{array}{ccc}
A_{int} (\text{concepts/meaning}) & \xleftarrow{\text{int}} & A_{FOL} (\text{syntax}) \\
I (\text{intensional interpretation}) & \xrightarrow{\text{Frege/Russell}} & I^* (\text{Tarski's interpretation}) \\
A_R (\text{denotation}) & \xrightarrow{\text{extensionalization}} & A_{int} (\text{extensional interpretation}) \\
\end{array}
\]

where \( h = is(w) \) where \( w = I_T \in W \) is the explicit possible world (extensional Tarski’s interpretation).

This homomorphic diagram formally express the fusion of Frege’s and Russell’s semantics [21,22,23] of meaning and denotation of the FOL language, and renders mathematically correct the definition of what we call an “intuitive notion of intensionality”, in terms of which a language is intensional if denotation is distinguished from sense: that is, if both a denotation and sense is ascribed to its expressions. This notion is simply adopted from Frege’s contribution (without its infinite sense-hierarchy, avoided by Russell’s approach where there is only one meaning relation, one fundamental relation between words and things, here represented by one fixed intensional interpretation \( I \)), where the sense contains mode of presentation (here described algebraically as an algebra of concepts (intensions) \( A_{int} \), and where sense determines denotation for any given extensionalization function \( h \) (correspondent to a given Traski’s interpretation \( I_T \)). More about the relationships between Frege’s and Russell’s theories of meaning may be found in the Chapter 7, “Extensionality and Meaning”, in [12].

As noted by Gottlob Frege and Rudolf Carnap (he uses terms Intension/extension in the place of Frege’s terms sense/denotation [24]), the two logic formulae with the same denotation (i.e., the same extension for a given Tarski’s interpretation \( I_T \)) need not have the same sense (intension), thus such co-denotational expressions are not substitutable in general.

In fact there is exactly one sense (meaning) of a given logic formula in \( \mathcal{L} \), defined by the uniquely fixed intensional interpretation \( I \), and a set of possible denotations (extensions) each determined by a given Tarski’s interpretation of the FOL as follows from Definition 2:

\[
\mathcal{L} \rightarrow_I \mathcal{D} \iff h = is(I_T) & k I_T \in W = \triangledown_I (I') \mathcal{R}.
\]

Often ‘intension’ has been used exclusively in connection with possible worlds semantics, however, here we use (as many others; as Bealer for example) ’intension’ in a more wide sense, that is as an algebraic expression in the intensional algebra of meanings (concepts) \( A_{int} \) which represents the structural composition of more complex concepts (meanings) from the given set of atomic meanings. Consequently, not only the denotation (extension) is compositional, but also the meaning (intension) is compositional.
4 Conclusion

Semantics is the theory concerning the fundamental relations between words and things. In Tarskian semantics of the FOL one defines what it takes for a sentence in a language to be true relative to a model. This puts one in a position to define what it takes for a sentence in a language to be valid. Tarskian semantics often proves quite useful in logic. Despite this, Tarskian semantics neglects meaning, as if truth in language were autonomous. Because of that the Tarskian theory of truth becomes inessential to the semantics for more expressive logics, or more 'natural' languages.

Both, Montague’s and Bealer’s approaches were useful for this investigation of the intensional FOL with intensional abstraction operator, but the first is not adequate and explains why we adopted two-step intensional semantics (intensional interpretation with the set of extensionalization functions).

At the end of this work we defined an extensional algebra for the FOL (different from standard cylindric algebras), and the commutative homomorphic diagram that express the generalization of the Tarskian theory of truth for the FOL into the Frege/Russell’s theory of meaning.

References

1. G. Bealer, “Universals,” *The Journal of Philosophy*, vol. 90, pp. 5–32, 1993.
2. G. Bealer, “Theories of properties, relations, and propositions,” *The Journal of Philosophy*, vol. 76, pp. 634–648, 1979.
3. E. F. Codd, “A relational model of data for large shared data banks,” *Communications of the ACM*, vol. 13, no. 6, pp. 377–387, 1970.
4. D. K. Lewis, “On the plurality of worlds,” *Oxford: Blackwell*, 1986.
5. R. Stalnaker, “Inquiry,” *Cambridge, MA: MIT Press*, 1984.
6. R. Montague, “Universal grammar.” *Theoria*, vol. 36, pp. 373–398, 1970.
7. R. Montague, “The proper treatment of quantification in ordinary English.” *Approaches to Natural Language*, in J. Hintikka et al. (editors), Reidel, Dordrecht, pp. 221–242, 1973.
8. R. Montague, “Formal philosophy. selected papers of Richard Montague,” in *R. Thomas (editor)*, Yale University Press, New Haven, London, pp. 108–221, 1974.
9. Z. Majkić, “First-order logic: Modality and intensionality,” *arXiv: 1103.0680v1*, 03 March, pp. 1–33, 2011.
10. Z. Majkić, “Intensional first-order logic for P2P database systems,” *Journal of Data Semantics* (JoDS XII), *LNCS 5480*, Springer-Verlag Berlin Heidelberg, pp. 131–152, 2009.
11. Z. Majkić, “Intensional semantics for RDF data structures,” *12th International Database Engineering & Applications Systems (IDEAS08)*, Coimbra, Portugal, 10-13 September, 2008.
12. G. Bealer, “Quality and concept.” *Oxford University Press*, USA, 1982.
13. Z. Majkić, “Weakly-coupled ontology integration of P2P database systems,” *1st Int. Workshop on Peer-to-Peer Knowledge Management (P2PKM)*, August 22, Boston, USA, 2004.
14. Z. Majkić, “Intensional P2P mapping between RDF ontologies,” *6th International Conference on Web Information Systems (WISE-05)*, November 20-22, New York. In M. Kitsuregawa (Eds.) *LNCS 3806*, pp. 592–594, 2005.
15. Z. Majkić, “Intensional semantics for P2P data integration,” *Journal on Data Semantics (JoDS)* VI, *Special Issue on 'Emergent Semantics', LNCS 4090*, pp. 47–66, 2006.
16. Z. Majkić, “Non omniscient intensional contextual reasoning for query-agents in P2P systems,” *3rd Indian International Conference on Artificial Intelligence (IJCAI-07)*, December 17-19, Pune, India, 2007.
17. Z.Majkić, “Coalgebraic specification of query computation in intensional P2P database systems,” *Int. Conference on Theoretical and Mathematical Foundations of Computer Science (TMFCS-08), Orlando FL, USA, July 7-9, 2008.*
18. Z.Majkić, “RDF view-based interoperability in intensional FOL for Peer-to-Peer database systems,” *International Conference on Enterprise Information Systems and Web Technologies (EISWT-08), Orlando FL, USA, July 9-11, 2008.*
19. L.Henkin, J.D.Monk, and A.Tarski, “Cylindric algebras I,” *North-Holland*, 1971.
20. A.Pirotte, “A precise definition of basic relational notions and of the relational algebra,” *ACM SIGMOD Record, Vol.13, no.1*, pp. 30–45, 1982.
21. G.Frege, “Über Sinn und Bedeutung,” *Zeitschrift für Philosophie und Philosophische Kritik*, pp. 22–50, 1892.
22. B.Russell, “On Denoting,” *Mind, XIV, Reprinted in Russell, Logic and Knowledge*, pp. 479–493, 1905.
23. A.N.Whitehead and B.Russell, “Principia Mathematica,” *Vol. I, Cambridge*, 1910.
24. R.Carnap, “Meaning and Necessity,” *Chicago*, 1947.