A CARLEMAN ESTIMATE FOR THE FRACTIONAL HEAT EQUATION
AND ITS APPLICATION IN FINAL STATE OBSERVABILITY

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Abstract. In the paper, we show a global Carleman estimate for the non-local heat equation. To be more precise, let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $O \subset \Omega$ an open subdomain, $s \in (0, 1)$. We show that there exist constants $C_1, C_2, r_0, T_0 > 0$ and a weight function $\alpha : \Omega \to (0, \infty)$ such that any solution $u$ of

\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) + (-\Delta)^s u(x, t) &= f(x, t) \quad \text{for} \quad (x, t) \in \Omega \times (0, \infty), \\
u(x, t) &= 0 \quad \text{for} \quad (x, t) \in \partial \Omega \times (0, \infty),
\end{align*}
\]

satisfies for all $r \geq r_0$ and $T > 0$

\[
\int_0^T \left[ \int_{\Omega} e^{-2r_0^\alpha(x)} |f(x, t)|^2 \, dx \right] \, dt 
\geq C_2 \left[ \int_0^T \int_{\Omega} e^{-2r_0^\alpha(x)} \left\{ |(-\Delta)^s u(x, t)|^2 + \frac{1}{2} \frac{\partial}{\partial t} u(x, t) \right\}^2 + \frac{r}{T_0(T - t)} |u(t, x)|^2 \right] \, dx \, dt.
\]

In order to prove this result, we use the Caffarelli-Silvestre extension procedure. To illustrate the applicability of the result, we prove as a second main result the final state observability of the non-local heat equation.

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1. Introduction

The Carleman estimate was first established in 1939 by Carleman [9]. The original motivation for Carleman was to prove unique continuation theorems, which can informally be stated as follows: Given a partial differential operator $P(x, D)$ of order $m$ in $\mathbb{R}^n$, an oriented hypersurface $\Omega$ in $\mathbb{R}^n$ represented by a level set $\{\rho = 0\}$ of a smooth function $\rho$, and a function $u$ satisfying $P(x, D)u = 0$, then, if $u = 0$ (locally) on one side $\Omega^+ = \{\rho(x) > 0\}$ of $\Omega$, the function $u$ vanishes (locally) on the other side $\Omega^- = \{\rho(x) < 0\}$ as well. This fact can be interpreted in the sense that the complete information about $u$ in $\Omega$ can be retrieved from the information given in $\Omega^+$.

Since then the Carleman estimate became a fundamental tool for different areas in applied mathematics. To highlight some application let us state the following examples coming from inverse problems. Assuming $u$ is a solution of a Cauchy problem of an equation of elliptic, parabolic or hyperbolic type with an unknown coefficient, Bukhgeim and Klibanov [5] proposed

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a method in finding the coefficient given only specific boundary values based on a Carleman estimate. Another example where the Carleman estimate is a powerful tool is showing final state observability of systems arising from hyperbolic or parabolic equations given only observation in a part of the domain, or alternatively, at the boundary. By duality, controllability of these systems can be shown. The first pioneering work was by Fursikov and Imanuvilov [13], who used global Carleman estimates in the context of null controllability, and O. Yu Imanuvilov [24], who popularised the use of Carleman estimates in the context of null controllability. Another relevant contributor was Zuazua, who established global controllability results for some nonlinear systems in [25] for the first time.

Carleman has shown the first estimate for a spatially two dimensional elliptic equation. Since then Carleman estimates exist for a broad class of partial differential equation, e.g. for general linear second-order parabolic equations; However, there does not exist Carleman estimates for non–local operators. To illustrate the physical importance of non-local operators, let us consider the heat diffusion, usually described by the Laplace operator. Using the Laplace operator ignores processes occurring at the microscopic level and is often not sufficiently accurate. Modelling heat conduction in media with a complex internal structure such as amorphous, porous and disordered materials, polymers, glasses, dielectrics and semiconductors, the microscopic level has to be taken into account. In this way, fractional or/and non-local operators of elliptic and parabolic type have caught considerable attention in the recent decades in both pure mathematics and real-world applications, see, e.g. [6, 17]. From a physical point of view, non-local operators play a fundamental role to describe several phenomena. For instance, in image processing, non-local operators were introduced to model interactions between any two points in the image domain, in handling textures and repetitive structures, see Gilboa and Osher [14]. Non-local kernel functions play an important role when studying population dynamics, see e.g. [11], non-local forces govern the movement of objects in gravitational or electromagnetic fields, and, finally non–local operators appears in finance in a natural way as infinitesimal Markovian operators of general L´evy processes, see, e.g. [10, 12].

Here, in our work we are going to prove a Carleman estimate for non–local operators and apply this estimate to obtain final-state observability.

To outline the content of our paper, let us assume that Ω ⊂ R^n is a bounded domain with smooth boundary and f : Ω × [0, T] → R be infinitely often continuously differentiable. In this setting a global Carleman estimate can be now described as follows. Let u be the solution of the heat equation

\[
\begin{align*}
\frac{∂u(x,t)}{∂t} - Δ u(x,t) &= f(x,t) \quad \text{for } (x,t) ∈ Ω × (0,∞), \\
u(x,t) &= 0 \quad \text{for } (x,t) ∈ ∂Ω × (0,∞). \quad \text{(1.1)}
\end{align*}
\]

Given a subdomain O of Ω, one can find a suitable function α : Ω → R_+ such that α is positive and there exist constants C_0 > 0 and r_0 > 0 with (see [22] Theorem 9.4.1)

\[
\int_0^T \int_Ω e^{-\frac{2α(x,t)}{t(T-t)}} \left[ \frac{r}{t(T-t)} |\nabla u(x,t)|^2 + \frac{r^3}{t^3(T-t)^3} |u(x,t)|^2 \right] dx dt 
\leq C_0 \left[ \int_0^T \int_Ω e^{-\frac{2α(x,t)}{t(T-t)}} |f(x,t)|^2 dx dt + \int_0^T \int_O r^3 e^{-\frac{2α(x,t)}{t(T-t)}} |u(x,t)|^2 dx dt \right],
\]

for all r ≥ r_0 and all u ∈ C^2_0(Ω × [0,T]) solving (1.1).
The main purpose of this paper is to extend the Carleman estimate of functions satisfying (1.1) to parabolic equations, where the Laplace operator is replaced by the fractional Laplace, i.e., \((-\Delta)^s\), \(s \in (0, 1)\). In [14] and [23], the Carleman estimate for systems with a fractional time derivative were handled, but, according to the best of author’s knowledge, there is no such estimate for the fractional Laplace operator in a smooth bounded domain of \(\mathbb{R}^n\).

A pivotal tool in our arguments is the well known Caffarelli-Silvestre extension for functions, which allows us to study the fractional Laplace operator by means of a boundary value problem. To explain this extension procedure in more details, let \(\Omega \subset \mathbb{R}^n\) be an open bounded and connected set with boundary \(\partial \Omega\) of class \(C^4\). Let \(u\) be a solution to

\[
\begin{aligned}
(-\Delta)^s u(x) &= f(x), \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(1.3)

Let us consider the cylinder \(\Omega \times (0, \infty) \subset \mathbb{R}^{n+1}\) with boundary \(\partial \Omega \times (0, \infty)\). Note, we denote the points in \(\Omega \times (0, \infty)\) by \((x, x_{n+1})\), \(x = (x_1, \ldots, x_n)\). For any function \(u : \Omega \to \mathbb{R}\) being sufficiently smooth, we define the \(s\)-harmonic extension \(u_{ex} : \Omega \times \mathbb{R}^+ \to \mathbb{R}\) as the solution to the following problem

\[
\begin{aligned}
\text{div}(x^1_{n+1}^{-2s} \nabla u_{ex}(x, x_{n+1})) &= 0 \quad \text{for} \quad (x, x_{n+1}) \in \Omega \times \mathbb{R}^+, \\
u_{ex}(x, x_{n+1}) &= 0 \quad \text{for} \quad (x, x_{n+1}) \in \partial \Omega \times \mathbb{R}^+, \\
u_{ex}(x, 0) &= u(x) \quad \text{for} \quad x \in \Omega.
\end{aligned}
\]

(1.4)

Caffarelli and Silvestre have shown in [7], that the fractional Laplace operator can be seen as a trace operator applied to the harmonic extension. To illustrate this fact, fix \(s \in (0, 1)\), let \(\Omega = \mathbb{R}^n\) and \(u \in H^2_0(\mathbb{R}^n)\). Let \(E_s(u) := u_{ex}\) be its \(s\)-harmonic extension to the upper half space \(\mathbb{R}^{n+1}_+ := \{(x, x_{n+1}) : x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R}^+_+\}\) which is given by the solution to the problem (1.4).

Then, in [7] it is shown that \(u_{ex}\) satisfies for all \(x \in \mathbb{R}^n\)

\[
(-\Delta)^s u(x) = \lim_{x_{n+1} \downarrow 0} x^1_{n+1}^{-2s} \nabla_{x_{n+1}} u_{ex}(x, x_{n+1}).
\]

The Caffarelli-Silvestre extension has been used e.g. by Stinga and Torrea [21] to show the Harnack’s inequality for the fractional Laplacian or by Caffarelli and Stinga [8] to reproduce some Caccioppoli type estimates.

In the proof of the Carleman estimate for the fractional Laplace operator, we transfer the nonlocal problem by the Caffarelli-Silvestre extension to a local problem on the half-plane. Here, the first main problem of showing the Carleman estimate is to find an appropriate weight function. We constructed the weight function in in Lemma 4.1 similar to Theorem 9.4.3 and Lemma 9.4.4 of [22]. However, we were faced with some technical problems; due to the extension, we are working on an unbounded domain. The second main problem arises performing the integration by parts formula; here, some boundary terms appears due to a nonlocal setting and also have to be handled.

The structure of the paper is as follows: first, we state in Section 2 the main result, i.e., the Carleman estimate for the nonlocal problem (1.3). In Section 3 we present the finite state observability of the problem (1.3). In Section 4 our main result is proven.

1.1. Notation. For convenience of the readers, we have kept the same notation as in [22]. We denote by \(\mathbb{R}^{n+1}_+\) all \(x \in \mathbb{R}^{n+1}\) by \(x = (x_1, x_{n+1}) \in \mathbb{R}^{n+1}_+\) where \((x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(x_{n+1} \in \mathbb{R}^+_+\). We will write \(\nabla\) if we take the derivative with respect to \(x_1, \ldots, x_n, x_{n+1}\). The same is for the divergence operator div. If we take it only the derivative with respect to \(x_1, \ldots, x_n\), we will write \(\nabla_{x_1, \ldots, x_n}\). Similarly, if we take only the derivative with respect to \(x_{n+1}\), we will write \(\nabla_{x_{n+1}}\).
2. The main result

Let $s \in (0, 1)$ and $A$ be an analytic operator on the domain $D(A)$; the corresponding set of resolvents is contained in $\Sigma_\omega := \{ \delta : \delta \leq \omega \}$, $\omega \in (0, \pi)$. Then, the fractional operator $(-A)^s$ is defined by (see [13] p. 69)

\[
(-A)^s := \frac{\sin(\pi s)}{\pi} \int_C z^{1-s}(-A)(zI - A)^{-1} \, dz,
\]

where the path $C$ runs in the resolvent set from $\infty e^{-i\delta}$ to $\infty e^{i\delta}$, $\omega < \delta < \pi$, avoiding the negative real axis and the origin; $z^{1-s}$ is taken to be positive for real positive values of $z$. In our case, $A$ will be the Laplace operator defined on $\Omega$ with Dirichlet or Neumann boundary conditions. Let $u$ be a solution of the following parabolic equation

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) + (-\Delta)^s u(x, t) = f(x, t) & \text{in } \Omega \times (0, \infty), \\
 u(x, t) = 0 & \text{in } \mathbb{R}^n \setminus \Omega \times (0, \infty).
\end{cases}
\]

(2.1)

Then as mentioned before, one can represent $u$ as the trace on $\{x_{n+1} = 0\}$ of the Caffarelli-Silvestre extension. Let us consider the cylinder $\Omega$ as the domain $\Omega := \{ \delta : \delta \leq \omega \}$, $\omega \in (0, \pi)$. Let $u$ be a solution of the following parabolic equation

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) + (-\Delta)^s u(x, t) = f(x, t) & \text{in } \Omega \times (0, \infty), \\
 u(x, t) = 0 & \text{in } \mathbb{R}^n \setminus \Omega \times (0, \infty).
\end{cases}
\]

Then as mentioned before, one can represent $u$ as the trace on $\{x_{n+1} = 0\}$ of the Caffarelli-Silvestre extension. Let us consider the cylinder $\Omega \times (0, \infty) \subseteq \mathbb{R}^{n+1}$. As before, we denote the points in $\Omega \times (0, \infty)$ by $(x, x_{n+1})$, where $x = (x_1, \ldots, x_n)$. For any function $f : \Omega \times [0, T] \to \mathbb{R}$ being sufficiently smooth, we define the $s$-harmonic extension $u_{ex} : \Omega \times \mathbb{R}^+ \times [0, T] \to \mathbb{R}$ as the solution to the following problem

\[
(P) \quad \begin{cases}
\text{div}(x_{n+1}^{1-2s} \nabla u_{ex}(x, x_{n+1}, t)) = 0 & (x, x_{n+1}) \in \Omega \times \mathbb{R}^+, \\
\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \nabla u_{ex}(x, x_{n+1}, t) = f(t, x) - \frac{\partial}{\partial x} u(x, t) & x \in \Omega, t > 0.
\end{cases}
\]

In our main result using the harmonic extension, we show a Carleman estimate solutions of equation (2.1).

**Theorem 2.1.** Let $\mathcal{O}$ be an open non-empty subset of $\Omega$ and $s \in (0, 1)$. Then, there exists a positive function $\alpha \in C^4(\Omega)$ and constants $T_0, C_1, C_2, r_0 > 0$, depending only on $\Omega$ and $\mathcal{O}$, such that for all

\[
u \in C([0, T]; H^{2s}(\Omega) \cap H_0^s(\Omega)) \cap C^1([0, T]; L^2(\Omega))
\]

and for all $r \geq r_0$ and $T \geq T_0$ we have

\[
\int_0^T \left\{ \int_{\Omega} e^{-\frac{2\alpha(x)}{t^{\frac{n+1}{n-1}}}} \left[ \frac{\partial}{\partial t} u(x, t) + (-\Delta)^s u(x, t) \right]^2 dx + C_1 \int_0^T \int_{\mathcal{O}} \frac{r^2 e^{-\frac{2\alpha(x)}{t^{\frac{n+1}{n-1}}}}}{t^{4(T-t)^{\frac{n}{n-1}}}} |u(x, t)|^2 dx \right\} dt \\
\geq C_2 \int_0^T \int_{\Omega} e^{-\frac{2\alpha(x)}{t^{\frac{n+1}{n-1}}}} \left[ (-\Delta)^s u(x, t) \right|^2 dx + \frac{1}{2} \frac{\partial}{\partial t} u(x, t) \right|^2 + \frac{r}{t^{4(T-t)^{\frac{n}{n-1}}}} |u(x, t)|^2 \right\} dx \, dt.
\]

The proof of Theorem 2.1 is quite technical and needs some preparations. The proof of Theorem 2.1 is given in Section 4.

**Remark 2.1.** Usually, in the parabolic setting one does not get any condition on the time $T$. However, we could only proof the estimate for $T$ being sufficiently large. It would be interesting if the condition on $T$ could be weakened.
3. Final State Observability

Let $n \geq 1, T > 0,$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. The state of a trajectory of a linear time–invariant system governed by an operator $A : D(A) \hookrightarrow L^2(\Omega) \to L^2(\Omega)$ is defined by a homogenous differential equation of the form

$$
\begin{cases}
\frac{\partial}{\partial t} u(x,t) + Au(x,t) = 0 & \text{for all } (x,t) \in \Omega \times (0, \infty), \\
u(x,t) = 0 & \text{for all } (x,t) \in \mathbb{R}^n \setminus \Omega \times (0, \infty), \\
u(x,0) = u_0(x) & \text{for all } x \in \Omega.
\end{cases}
$$

(3.1)

A question, appearing, e.g., in engineering is, can one predict the state $u(T)$ at a given time $T > 0$ of the system (3.1), if the initial data is unknown only having the partial information of $u$. Has one to sample the whole system, or is it sufficient only to track some part of the system? This concept leads to the definition of observability which can be formulated as follows.

Let $u = \{u(t) : 0 \leq t \leq T\}$ be a solution to the system (3.1) such that $u \in C_b(0,T; L^2(\Omega))$. Let $C \in L(L^2(\Omega), Y)$, (where $Y = L^2(\Omega)$) be a bounded linear operator and let $y = Cx$. Given $\{y(t) : 0 \leq t \leq T\}$, is it possible to reconstruct $u(T)$?

Let us assume that, we have only access to a part $O$ of the domain. In particular, we only observe $\chi_O u(t)$ for $0 \leq t \leq T$. Here, the operator $C$ is given by

$$(C u)(x,t) := \chi_O(x)u(x,t).$$

Now, can we reconstruct from $\{C u(t) : 0 \leq t \leq T\}$ the final state $u(T)$?

It can be shown that this problem is equivalent to the question: Does there exists for any $T > 0$ a constant $c_T > 0$ such that (see [22, Definition 6.1.1, p. 173])

$$|u(T)|^2_{L^2(\Omega)} \leq c_T^2 \int_0^T |C u(t)|^2_{L^2(\Omega)} dt,$$

where $u$ is a solution to (3.1). In our example where the leading operator is the Laplace or a nondegenerate elliptic partial differential operator of second order, one can find results in the literature, see [22, Theorem 9.5.1, p. 313]. In case the leading operator is the fractional Laplace operator less is known.

Let $n \geq 1, T > 0$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let us consider the fractional heat equation:

$$
\begin{align*}
\frac{\partial}{\partial t} u(x,t) + (\Delta)^s u(x,t) &= 0 & &\text{for all } (x,t) \in \Omega \times (0, \infty), \\
\frac{\partial}{\partial t} u(x,t) &= 0 & &\text{for all } (x,t) \in \mathbb{R}^n \setminus \Omega \times (0, \infty), \\
\frac{\partial}{\partial t} u(x,0) &= u_0(x) & &\text{for all } x \in \Omega.
\end{align*}
$$

Fix a nonempty open set $O$ of $\Omega$ and let $C$ be given by

$$(C u) = u \chi_O, \quad u \in L^2(\Omega),$$

where $\chi_O$ is the characteristic function of $O$. By means of Theorem 2.1 the following Theorem can be proven.

**Theorem 3.1.** There exists a $T_0 > 0$ such that for any $T \geq T_0$ there exists a constant $C(T_0) > 0$ such that any function $u : \Omega \times [0,T] \to \mathbb{R}$ being a solution to (P) satisfies

$$C(T) \int_\Omega |u(x,T)|^2 dx \leq \int_0^T \int_\Omega |u(x,t)|^2 dx dt.$$
Proof. The proof consists of two steps. First we show that for any $r > 0$ the $L^2$-norm of $u(T)$ and $u(0)$ can be estimated by $u(T - r)$ and $u(r)$, respectively. In a second step, we show that $u(T)$ can be estimated by $\int_0^T \|u(x,t)\|^2 dx dt$.

Step 1: Let us note that the semigroup $(S(t))_{t \geq 0}$ of the operator $(-\Delta)^s$ is an analytic semigroup. In particular, there exists a constant $M > 0$ such that for any $\tau > 0$ we have for any $\alpha \in (0,1)$

$$|(\Delta)^{\alpha \beta} S(\tau)v|_{L^2} \leq e^{M \tau - \alpha \beta} |v|_{L^2},$$

and, since $I - S(\tau) = \int_0^\tau (-\Delta)^{\alpha \beta} S(t) dt$, we know that there exists a $\gamma \in (0,1]$ such that for any $\tau > 0$ we have

$$|(S(t) - I)v|_{L^2} \leq C t^{1-\alpha} |(-\Delta)^{\alpha \beta} v|_{L^2}^2, \quad \forall t \in [0, \tau], \ v \in \text{dom}((-\Delta)^{\frac{s}{2}}).$$

Hence, for $\tau = \frac{T}{\tau}$, there exists constant $C > 0$ such that

$$|u(T)|_{L^2}^2 \leq \tau^{-1-\alpha} e^{M \tau} |(-\Delta)^{\alpha \beta} u(T - \tau)|_{L^2}^2 + |u(T - t)|_{L^2}^2, \quad \forall t \in [T - \tau, T].$$

Now, we have for $\alpha = \frac{1}{2}$

$$|u(T)|_{L^2}^2 \leq C |(-\Delta)^{s} u(T - \tau)|_{L^2}^2 \leq \frac{C}{T - 2\tau} \int_\tau^{T - \tau} \tau^{-1-\alpha} e^{M \tau} |(-\Delta)^{\frac{s}{2}} u(T - \tau)|_{L^2}^2 dt \leq \frac{C}{T - 2\tau} \int_\tau^{T - \tau} S(T - \tau - t)(-\Delta)^{\frac{s}{2}} u(t)|_{L^2}^2 dt.$$ 

Due to the fact that

$$|(-\Delta)^{\frac{s}{2}} S(t)v|_{L^2} \leq e^{M \tau} |(-\Delta)^{\frac{s}{2}} v|_{L^2}, \quad \forall v \in L^2(\Omega),$$

we can write

$$|u(T)|_{L^2}^2 \leq \frac{C}{T - 2\tau} e^{M(T - \tau)} \int_\tau^{T - \tau} \tau^{-1-\alpha} e^{M \tau} |(-\Delta)^{\frac{s}{2}} u(t)|_{L^2}^2 dt.$$ (3.2)

Step 2: We have shown in Corollary 2.1 that under the conditions of Theorem 3.1 there exists a strictly positive function $\alpha$ and constants $C_1, r_0 > 0$ such that we have for the solution $u$ of (3.1) and all $0 < r \leq r_0$,

$$\int_0^T \int_\Omega e^{\frac{\alpha}{2(t - r)}} \left[ \frac{r}{(t - r)^3} \right] |(-\Delta)^{\frac{s}{2}} u(x,t)|^2 dx dt \leq C_1 \int_0^T \int_\Omega \frac{r^{\alpha}}{(t - r)^3} |u(x,t)|^2 dx dt.$$ (3.3)

Observe, we can find a constant $C' = C'(\tau) > 0$ such that we have

$$\frac{r}{(t - r)^3} e^{\frac{\alpha}{2(t - r)}} \geq C'(\tau), \quad \forall t \in (\tau, T - \tau).$$

Substituting the estimate above into (3.2), we get

$$|u(T)|_{L^2}^2 \leq \frac{C}{T - 2\tau} e^{M(T - \tau)} \int_\tau^{T - \tau} \frac{r}{(t - r)^3} e^{\frac{\alpha}{2(t - r)}} |(-\Delta)^{\frac{s}{2}} u(t)|_{L^2}^2 ds \leq \frac{C}{T - 2\tau} e^{M(T - \tau)} \int_0^T \frac{r}{(t - r)^3} e^{\frac{\alpha}{2(t - r)}} |(-\Delta)^{\frac{s}{2}} u(t)|_{L^2}^2 ds.$$
Estimating the RHS by the LHS of (33), we know for a constant $\hat{C} > 0$ that
\[
|u(T)|_{L^2}^2 \leq \frac{\hat{C}}{T - 2\tau} e^{M(T-\tau)} \int_0^T \int_{\Omega} e^{\frac{2\alpha(x)}{n}(T-t)} |u(x,t)|^2 dx dt.
\]
Now, since there exists a constant $C'' = C''(T) > 0$ such that
\[
e^{-\frac{2\alpha(x)}{n}(T-t)} \leq C''(T), \quad t \in [0,T], \ x \in \Omega,
\]
we have for a constant $\hat{C} > 0$
\[
|u(T)|_{L^2}^2 \leq \frac{\hat{C}}{T - 2\tau} e^{M(T-\tau)} \int_0^T \int_{\Omega} |u(x,t)|^2 dx dt,
\]
which is the assertion. \hfill $\square$

### 4. Proof of Theorem 2.1

The proof uses the $s$–harmonic extension and is done in several step. First we will construct a weight function $\rho$ on the extended space $\Omega \times \mathbb{R}^+_0$. Secondly, we will prove for the weight function $\rho$ and a function $\psi$ defined by the product $\psi := u_{ex} \rho^s$ a kind of Carleman estimate. Then we transfer the Carleman estimate for $\psi$ to a Carleman estimate for $u$. Here, we use the representation of $(\Delta)^s u(x) = -\lim_{s_{n+1} \to 0} x_{n+1}^{1-2s} \nabla x_{n+1} u_{ex}(x,x_{n+1}), \ x \in \Omega$ and $x_{n+1} > 0$.

Before starting with the actual proof let us observe the following property. From the maximum principle, we know that if $w(x) \geq 0$ for all $x \in \Omega$, then $u_{ex} \geq 0$ for all $(x,x_{n+1}) \in \Omega \times \mathbb{R}^+_0$. Let us remind, it follows by the maximum principle, that if $w$ is bounded, its harmonic extension $w_{ex}$ is also bounded.

Since we need it later, let us shortly show that the fractional Laplace also satisfies the Poincaré inequality.

**Proposition 4.1.** Let $\Omega$ be a bounded domain with smooth boundary and $w$ be a positive weight function bounded from below and above, i.e. there exists constants $k,K > 0$ such that
\[
k \leq w(x) \leq K, \quad x \in \Omega.
\]
Then, we have
\[
\int_{\Omega} w(x)|u(x,t)|^2 dx \leq \frac{K}{k} \int_{\Omega} w(x)|(\Delta)^s u(x,t)|^2 dx.
\]

**Proof.** This follows by interpolation. In fact, first note that $|(\Delta)^s u|_{L^2} = |u|_{[H_0,H_1,s]}$, where $[.]$ denotes the interpolation functor (see [4]), and $H_0 = L^2(\Omega), H_1 = \{v \in L^2(\Omega) : (I - \Delta)v \in L^2(\Omega)\}$. Fix $u \in L^2(\Omega)$. Next, by the definition of the interpolation, see the K-method in [4, p. 38, Chapter 3.1]), for any $\epsilon$ we know that there exists functions $u_0 : \mathbb{N} \to H_0$ and $u_1 : \mathbb{N} \to H_1$, $u = u_0 + u_1$, such that
\[
\epsilon + |(\Delta)^s u|_{L^2} \geq \sum_{n \in \mathbb{N}} n^\theta \left((u_0(n)|_{H_0} + \frac{1}{n} |u_1(n)|_{H_1}\right).
\]
By the Poincaré inequality for the Laplace operator we have
\[
\epsilon + |(\Delta)^s u|_{L^2} \geq \sum_{n \in \mathbb{N}} n^\theta \left((u_0(n)|_{H_0} + \frac{1}{n} |u_1(n)|_{H_0}\right).
\]
By the definition of the interpolation space we have
\[ \epsilon + \|(-\Delta)^s u\|_{L^2} \geq \|u\|_{H_0^s, H_0^s} \geq \|u\|_{L^2}. \]
It remains to include the weight function \( w \). Here, note that
\[ \|(-\Delta)^s u\|_{L^2(w)} \geq k \|(-\Delta)^s u\|_{L^2} \geq k\|u\|_{L^2} \geq \frac{k}{K}\|u\|_{L^2(w)}. \]
The last line and the fact that \( \epsilon > 0 \) was arbitrary, shows the assertion. \( \square \)

**Step I:** In the first step we define the weight function. First let us suppose \( \gamma : \Omega \to \mathbb{R} \) is a function such that \( \gamma \in C_b^2(\Omega) \), \( \gamma(x) = 0 \) on \( \Omega \setminus \mathcal{O} \), and \( \gamma(x) > 0 \) for all \( x \in \mathcal{O}^0 \), the interior of \( \mathcal{O} \).

Let \( \eta = (-\Delta)^{-s} \gamma \) and let \( \eta_{\text{ex}} \) be the s-harmonic extension of \( \eta \). By fractional elliptic regularity theory [10], we know, if \( \eta \) is bounded, then \( \eta_{\text{ex}} \) is bounded. Let
\[ F_0 = 4 \max_{(x,x_{n+1}) \in \overline{\Omega} \times \mathbb{R}_0^+} \eta_{\text{ex}}(x, x_{n+1}) \] (4.1)
and let
\[ \alpha(x, x_{n+1}) = e^{\lambda F_0} - e^{\lambda \eta_{\text{ex}}(x, x_{n+1})} \text{ for all } (x, x_{n+1}) \in \overline{\Omega} \times \mathbb{R}_0^+. \] (4.2)

Observe, by the construction of \( \alpha \), there exist constants \( K_{\alpha}, k_{\alpha} > 0 \) such that
\[ k_{\alpha} \leq \alpha(x, x_{n+1}) \leq K_{\alpha}, \quad \forall (x, x_{n+1}) \in \Omega \times \mathbb{R}_0^+. \] (4.3)

Let \( \beta \) be given by
\[ \beta(x, x_{n+1}, t) = \frac{\alpha(x, x_{n+1})}{t(T-t)}, \quad (x, x_{n+1}, t) \in \Omega \times \mathbb{R}_0^+ \times [0, T]. \]

Observe, by the definition of \( \beta \) we have
\[ \nabla_{x_{n+1}} \beta(x, x_{n+1}, t) = \frac{\nabla_{x_{n+1}} \alpha(x, x_{n+1})}{t(T-t)} = \frac{-\lambda e^{\lambda \eta_{\text{ex}}(x, x_{n+1})}}{t(T-t)} \nabla_{x_{n+1}} \eta_{\text{ex}}(x, x_{n+1}) \] (4.4)
and
\[ \frac{\partial}{\partial t} \beta(x, x_{n+1}, t) = \frac{2t - T}{t^2(T-t)^2}. \] (4.5)

Finally, let us define the weight function
\[ \rho(x, x_{n+1}, t) := e^{\beta(x,x_{n+1}, t)} \text{ for all } (x, x_{n+1}, t) \in \overline{\Omega} \times \mathbb{R}_0^+ \times (0, T). \] (4.6)

Observe, straightforward calculations give for \( r > 0 \),
\[ \nabla_{x_{n+1}} \rho^r(x, x_{n+1}, t) = r \rho^r(x, x_{n+1}, t) \nabla_{x_{n+1}} \beta(x, x_{n+1}, t) \] (4.7)
\[ = \frac{r \lambda}{t(T-t)} \rho^r(x, x_{n+1}, t) e^{\lambda \eta_{\text{ex}}(x,x_{n+1})} \nabla_{x_{n+1}} \eta_{\text{ex}}(x, x_{n+1}). \]

**Step II:** Let us assume that a function \( u_{\text{ex}} \) solves problem (P). In particular, \( u_{\text{ex}} \) is a solution to
\[
\begin{cases}
\text{div}(\rho^{1-2s}_{x_{n+1}} \nabla u_{\text{ex}}(x, x_{n+1}, t)) = 0 & (x, x_{n+1}) \in \Omega \times \mathbb{R}^+, \\
uex(x, x_{n+1}, t) = 0 & (x, x_{n+1}) \in \partial\Omega \times \mathbb{R}^+, \\
l_{x_{n+1}} \nabla_{x_{n+1}} u_{\text{ex}}(x, x_{n+1}, t) = -(\Delta)^s u(x, t) & (x, t) \in \Omega \times (0, T).
\end{cases}
\] (4.8)

For \( r > 0 \), let us define the function
\[ \psi(x, x_{n+1}, t) := \rho^{-r}(x, x_{n+1}, t) u_{\text{ex}}(x, x_{n+1}, t), \quad (x, x_{n+1}, t) \in \Omega \times \mathbb{R}_0^+ \times [0, T]. \] (4.9)
In this step we will prove a kind of Carleman estimate for the function \( \psi \), which is the key tool to prove our main result.

**Lemma 4.1.** There exists constants \( C_1, C_2, r_0, \lambda_0 > 0, k > 0 \) (depending on \( \Omega, \mathcal{O}, T \)) such that for all \( u \), for all \( r \leq r_0 \), and all \( \lambda \geq \lambda_0 \) the following estimate holds

\[
\int_0^T \int_{\Omega} e^{-\frac{2\alpha(x_n+1)}{(n+1)r \beta(x_n+1)}} \left| \frac{\partial}{\partial t} + (-\Delta)^s \right| u(x,t) \, dx \, dt + \int_0^T \int_{\mathcal{O}} K \lambda r^r \psi^2(x,0,t) \alpha(x,0) e^{\lambda \eta(x,0)} \left( \frac{r|2t-T|}{t(T-t)} + \frac{r \lambda \alpha(x,0) e^{\lambda \eta(x,0)} \gamma(x)}{\epsilon} + |2t-T| \right) \gamma(x) \, dx \\
+ \int_0^T \int_{\Omega} \frac{r T^2}{\beta^4(t-T)^3} \psi^2(x,0,t) \alpha(x,0) \alpha(x_n+1) \, dx \, dt \\
\geq \lim_{x \to +\infty} \int_0^T \int_{\Omega} \frac{1}{8} |x^{-2s} \nabla x_{n+1} \psi(x,0,t)|^2 \, dx \, dt + \int_0^T \frac{1}{2} \left| \frac{\partial}{\partial t} \psi(x,0,t) \right|_{L^2(\Omega)}^2 \, dt \\
+ \int_0^T \int_{\Omega} \frac{\lambda^2 r^2}{\beta^4(t-T)^3} \left| \psi(x,0,t) \alpha(x,0) e^{\lambda \eta(x)} \gamma(x) \right|^2 \, dx \, dt \\
+ \int_0^T \int_{\Omega} 2r(2t-T)^2 \psi^2(x,0,t) \alpha^2(x,0) \, dx \, dt,
\]

where \( u_{ex} \) is the extension of \( u \), i.e., \( u_{ex} \) solves problem (4.8).

**Proof.** For \((x, x_{n+1}, t) \in \Omega \times \mathbb{R}_0^+ \times [0, T]\) let us put

\[
f_r(x, x_{n+1}, t) := \rho^{-r}(x, x_{n+1}, t) \left\{ \frac{\partial}{\partial t} - x^{-2s} \nabla x_{n+1} \right\} u_{ex}(x, x_{n+1}, t).
\]

Since \( \psi \rho^r = u_{ex} \), straightforward calculations give

\[
f_r(x, 0, t) = \lim_{x \to +\infty} \rho^{-r}(x, x_{n+1}, t) \frac{\partial}{\partial t} \rho^r(x, x_{n+1}, t) \psi(x, x_{n+1}, t) \\
- \lim_{x \to +\infty} x^{-2s} \nabla x_{n+1} \left( \rho^r(x, x_{n+1}, t) \psi(x, x_{n+1}, t) \right)
\]

\[
= \lim_{x \to +\infty} \rho^{-r}(x, x_{n+1}, t) \left\{ \rho^r(x, x_{n+1}, t) \frac{\partial}{\partial t} \psi(x, x_{n+1}, t) \\
+ r \psi(x, x_{n+1}, t) \rho^r(x, x_{n+1}, t) \frac{\partial}{\partial t} \beta(x, x_{n+1}, t) \right\}
\]

\[
- \lim_{x \to +\infty} \rho^{-r}(x, x_{n+1}, t) x^{-2s} \left\{ \rho^r(x, x_{n+1}, t) \nabla x_{n+1} \psi(x, x_{n+1}, t) \\
+ r \rho^r(x, x_{n+1}, t) \psi(x, x_{n+1}, t) \nabla x_{n+1} \beta(x, x_{n+1}, t) \right\}
\]

\[
= \lim_{x \to +\infty} \left\{ \frac{\partial}{\partial t} \psi(x, x_{n+1}, t) + r \psi(x, x_{n+1}, t) \frac{\partial}{\partial t} \beta(x, x_{n+1}, t) \right\}
\]

\[
- \lim_{x \to +\infty} x^{-2s} \left\{ \nabla x_{n+1} \psi(x, x_{n+1}, t) + r \psi(x, x_{n+1}, t) \nabla x_{n+1} \beta(x, x_{n+1}, t) \right\}.
\]
Due to the identities (4.1) and (4.5), we get

\[
\begin{aligned}
f_r(x, 0, t) &= \lim_{x_{n+1} \to 0} \left\{ \frac{\partial}{\partial t} \psi(x, x_{n+1}, t) + r\psi(x, x_{n+1}, t)\alpha(x, x_{n+1}) \frac{(2t - T)}{t^2(T - t)^2} \
&\quad - x_{n+1}^{-2} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t) + \frac{\lambda r}{t(T - t)} \psi(x, x_{n+1}, t)e^{\lambda \eta_{ex} (x, x_{n+1})} x_{n+1}^{-2} \nabla_{x_{n+1}} \eta_{ex} (x, x_{n+1}) \right\},
\end{aligned}
\]

Let us put

\[
\begin{aligned}
M_1(x, x_{n+1}, t) &:= \frac{r(2t - T)}{t^2(T - t)^2} \psi(x, x_{n+1}, t)\alpha(x, x_{n+1}), \\
M_2(x, x_{n+1}, t) &:= x_{n+1}^{-2} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t), \\
M_3(x, x_{n+1}, t) &:= \frac{\lambda r}{t(T - t)} \psi(x, x_{n+1}, t)e^{\lambda \eta_{ex} (x, x_{n+1})} x_{n+1}^{-2} \nabla_{x_{n+1}} \eta_{ex} (x, x_{n+1}).
\end{aligned}
\]

Then

\[
f_r(x, 0, t) = \lim_{x_{n+1} \to 0} \left\{ \frac{\partial}{\partial t} \psi(x, x_{n+1}, t) + M_1(x, x_{n+1}, t) - M_2(x, x_{n+1}, t) + M_3(x, x_{n+1}, t) \right\},
\]

and

\[
\begin{aligned}
\int_0^T \int_\Omega \lim_{x_{n+1} \to 0} |f_r(x, x_{n+1}, t)|^2 \, dt \\
&= \int_0^T \lim_{x_{n+1} \to 0} \left\{ \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t) + M_1(\cdot, x_{n+1}, t) - M_2(\cdot, x_{n+1}, t) - M_3(\cdot, x_{n+1}, t) \right\}^2_{L^2(\Omega)} \, dt \\
&\geq \int_0^T \lim_{x_{n+1} \to 0} \left\{ \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t) \right\}^2_{L^2(\Omega)} + |M_1(\cdot, x_{n+1}, t) - M_2(\cdot, x_{n+1}, t) - M_3(\cdot, x_{n+1}, t)|^2_{L^2(\Omega)} \\
&\quad + 2 \left\langle \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_1(\cdot, x_{n+1}, t) - M_2(\cdot, x_{n+1}, t) + M_3(\cdot, x_{n+1}, t) \right\rangle_{L^2(\Omega)} \\
&\quad - 2 \left\langle M_1(\cdot, x_{n+1}, t), M_2(\cdot, x_{n+1}, t) + M_3(\cdot, x_{n+1}, t) \right\rangle_{L^2(\Omega)} + 2 \left\langle M_2(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \right\rangle_{L^2(\Omega)} \, dt.
\end{aligned}
\]

First, note that

\[
\begin{aligned}
|M_1(\cdot, x_{n+1}, t)|^2_{L^2(\Omega)} &= \frac{r^2(2t - T)}{t^4(T - t)^4} \int_\Omega |\psi(x, x_{n+1}, t)\alpha(x, x_{n+1})|^2 \, dx, \\
|M_2(\cdot, x_{n+1}, t)|^2_{L^2(\Omega)} &= \int_\Omega |x_{n+1}^{-2} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 \, dx,
\end{aligned}
\]

and

\[
|M_3(\cdot, x_{n+1}, t)|^2_{L^2(\Omega)} = \frac{\lambda^2 r^2}{t^2(T - t)^2} \int_\Omega |\psi(x, x_{n+1}, t)e^{\lambda \eta_{ex} (x, x_{n+1})} x_{n+1}^{-2} \nabla_{x_{n+1}} \eta_{ex} (x, x_{n+1})|^2 \, dx.
\]

We note that \(|M_1(\cdot, x_{n+1}, t)|^2_{L^2(\Omega)} = 0\) at \(t = \frac{T}{2}\).

As next, we tackle the terms where the time derivative occurs. Since we have for a nice function \(a(t)\)

\[
\frac{1}{2} \frac{\partial}{\partial t} (\psi^2(x, x_{n+1}, t)a(t)) = \psi(x, x_{n+1}, t) \frac{\partial}{\partial t} \psi(x, x_{n+1}, t)a(t) + \frac{1}{2} \psi^2(x, x_{n+1}, t) \frac{d}{dt} a(t),
\]
and taking into account
\[
\lim_{t \to 0} \frac{r(2t - T)}{t^2(T - t)^2} \psi(x, x_{n+1}, t) = \lim_{t \to T} \frac{r(2t - T)}{t^2(T - t)^2} \psi(x, x_{n+1}, t) = 0,
\]
we obtain
\[
\langle \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_1(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} = -\frac{1}{2} \int_{\Omega} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) \, dx \frac{\partial}{\partial t} \left( \frac{r(2t - T)}{t^2(T - t)^2} \right).
\]
Note that
\[
\frac{\partial}{\partial t} \left( \frac{r(2t - T)}{t^2(T - t)^2} \right) \leq \frac{2(3t^2 - 3tT + T^2)}{t^3(T - t)^3},
\]
and \( (3t^2 - 3tT + T^2) \leq T^2 \) we get
\[
\int_0^T \left( \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_1(\cdot, x_{n+1}, t) \right)_{L^2(\Omega)}(t) \, dt
\]
\[
= -r \int_0^T \int_{\Omega} \frac{2(3t^2 - 3tT + T^2)}{t^3(t - T)^3} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) \, dx \, dt
\]
\[
\geq -r \int_0^T \int_{\Omega} \frac{2T^2}{t^3(t - T)^3} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) \, dx \, dt.
\]
The Young’s inequality gives next
\[
\left| \int_0^T \langle \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_2(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)}(t) \right|
\leq \int_0^T \left( \frac{1}{2} \frac{\partial}{\partial t} \psi^2(\cdot, x_{n+1}, t) \right)_{L^2(\Omega)} + \frac{1}{2} \left| \psi(\cdot, x_{n+1}, t) \right|^2_{L^2(\Omega)} \, dt.
\]
Next, we investigate \( \langle \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} \). The chain role gives
\[
\frac{\partial}{\partial t} \left( \frac{\lambda r}{t(T - t)} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{ex}(x, x_{n+1}) \right)
\]
\[
= \frac{\lambda r}{t^2(T - t)} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{ex}(x, x_{n+1})
\]
\[
+ \frac{\lambda r}{t(T - t)} \psi(x, x_{n+1}, t) \frac{\partial}{\partial t} \psi(x, x_{n+1}, t) \alpha(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{ex}(x, x_{n+1}).
\]
Taking into account that we have
\[
\lim_{t \to 0} \frac{1}{t(T - t)} \psi^2(x, x_{n+1}, t) = \lim_{t \to T} \frac{1}{t(T - t)} \psi^2(x, x_{n+1}, t) = 0,
\]
we get
\[
\left| \int_0^T \langle \frac{\partial}{\partial t} \psi(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)}(t) \right|
\leq \int_0^T \int_{\Omega} \frac{\lambda r^2(2t - T)}{t^2(T - t)^2} \psi^2(\cdot, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda_{ex}(x, x_{n+1})} x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{ex}(x, x_{n+1}) \, dx \, dt.
\]
Calculating the term \( \langle M_1(\cdot, x_{n+1}, t), M_2(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} \) we obtain,
\[
\langle M_1(\cdot, x_{n+1}, t), M_2(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} = \frac{r(2t - T)}{t^2(T - t)} \int_{\Omega} \psi(x, x_{n+1}, t) \alpha(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t) \, dx.
\]
The Young inequality gives for any $\epsilon_0 > 0$

$$\left| \langle M_1(\cdot, x_{n+1}, t), M_2(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} \right| \leq \epsilon_0 \int_{\Omega} |x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 dx + \frac{\lambda^2 (2t - T)^2}{4 \epsilon_0 t^4 (T - t)^4} \int_{\Omega} |\psi(x, x_{n+1}, t) \alpha(x, x_{n+1})|^2 dx.$$

Calculating the term $\langle M_1(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)}$ we obtain

$$\langle M_1(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} = \frac{\lambda r^2 (2t - T)}{t^3 (T - t)^3} \int_{\Omega} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{\text{loc}}(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{\text{loc}}(x, x_{n+1})} dx.$$

Calculating the term $\langle M_2(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)}$ we obtain

$$\langle M_2(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} = \frac{\lambda r}{t(T - t)} \int_{\Omega} \psi(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{\text{loc}}(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{\text{loc}}(x, x_{n+1})} dx.$$

The Cauchy Schwarz and Young inequality gives for any $\epsilon > 0$

$$\left| \langle M_2(\cdot, x_{n+1}, t), M_3(\cdot, x_{n+1}, t) \rangle_{L^2(\Omega)} \right| \leq \epsilon \int_{\Omega} |x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 dx + \frac{\lambda^2 r^2}{4 \epsilon t^2 (T - t)^2} \int_{\Omega} |\psi(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{\text{loc}}(x, x_{n+1}) x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{\text{loc}}(x, x_{n+1})}|^2 dx.$$
Collecting altogether, we get
\[
\int_0^T \int_\Omega x_{n+1} \left[ \left| f_r(x, x_{n+1}, t) \right|^2 dx \right] dt \\
\geq \lim_{x_{n+1} \downarrow 0} \left\{ \int_0^T \frac{1}{2} \frac{\partial}{\partial t} \psi(t, x_{n+1}, t) \right\}^2_{L^2(\Omega)} + \int_\Omega \left( 1 - \frac{1}{2} - 2\epsilon_0 - 2\epsilon \right) x_{n+1}^{-2s} \nabla x_{n+1} \psi(x, x_{n+1}, t)\right|^2 dx dt \\
+ \int_0^T \int_\Omega \frac{\lambda^2 r^2}{t^4(t - T)^2} \psi(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{x_0} (x, x_{n+1})} x_{n+1}^{-2s} \nabla x_{n+1} \eta_{x_0} (x, x_{n+1})^2 dx dt \\
+ \int_0^T \int_\Omega \frac{r^2(2t - T)^2}{t^4(T - t)^4} \psi^2(x, x_{n+1}, t) \alpha^2(x, x_{n+1}) dx \\
- \int_0^T \int_\Omega \frac{2r^2 T^2}{t^3(T - t)^3} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) dx dt \\
- \int_0^T \int_\Omega \frac{\lambda^2 r^2}{t^5(T - t)^5} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{x_0} (x, x_{n+1})} x_{n+1}^{-2s} \nabla x_{n+1} \eta_{x_0} (x, x_{n+1})^2 dx \\
- \int_0^T \int_\Omega \frac{1}{4} \frac{\lambda^2 r^2}{t^2(T - t)^2} \psi^2(x, x_{n+1}, t) \alpha(x, x_{n+1}) e^{\lambda \eta_{x_0} (x, x_{n+1})} x_{n+1}^{-2s} \nabla x_{n+1} \eta_{x_0} (x, x_{n+1})^2 dx \\
= I_1 + I_2 + I_3 + I_4 - I_5 - I_6 - I_7 - I_8 - I_9.
\]

Note, for \( \epsilon_0 \in (\frac{1}{4}, \frac{3}{4}) \) we have \( I_4 - I_6 = (1 - \frac{1}{4\epsilon_0})I_4 \). As next, let us remind that
\[
x_{n+1}^{-2s} \nabla x_{n+1} \eta_{x_0} (x, x_{n+1}) \longrightarrow (-\Delta)^s \eta(x)
\]
for \( x_{n+1} \downarrow 0 \) and \( \gamma(x) = 0 \) for all \( x \in \Omega \setminus \partial \). Taking partially the limit \( x_{n+1} \downarrow 0 \) and subtracting \( I_5, I_6, \) and \( I_8 \) on both sides, we get
\[
\int_0^T \int_\Omega x_{n+1} \left[ x_{n+1}^{-2s} \nabla x_{n+1} \eta_{x_0} (x, x_{n+1}) \right] dx dt \\
= \lim_{x_{n+1} \downarrow 0} \left\{ \int_0^T \frac{1}{2} \frac{\partial}{\partial t} \psi(t, 0, t) \right\}^2_{L^2(\Omega)} + \int_\Omega \left( 1 - \frac{1}{2} - 2\epsilon_0 - 2\epsilon \right) x_{n+1}^{-2s} \nabla x_{n+1} \psi(x, x_{n+1}, t)\right|^2 dx dt \\
+ \int_0^T \int_\Omega \frac{\lambda^2 r^2}{t^4(t - T)^2} \psi(x, 0, t) \alpha(x, 0) dx dt \\
+ \int_0^T \int_\Omega \frac{r^2(2t - T)^2}{t^4(T - t)^4} \psi^2(x, 0, t) \alpha^2(x, 0) dx dt \\
+ C(\epsilon_0) \int_0^T \int_\Omega \frac{r^2(2t - T)^2}{t^4(T - t)^4} \psi^2(x, 0, t) \alpha^2(x, 0) dx dt.
\]

From this estimate Lemma 4.1 follows.
Step III: In the last step we replace $\psi$ by $\rho^{-r} u$ to get the estimate for $u$. First, note that we have
\[
x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t) = \rho^{-r}(x, x_{n+1}, t) x_{n+1}^{1-2s} \nabla_{x_{n+1}} u_{\text{ex}}(x, x_{n+1}, t) + \frac{\lambda r}{t(T-t)} u_{\text{ex}}(x, x_{n+1}, t) \rho^{-r}(x, x_{n+1}, t) e^{\lambda \eta(x)(x, x_{n+1})} x_{n+1}^{1-2s} \nabla_{x_{n+1}} \eta_{\text{ex}}(x, x_{n+1})
\]
\[= J_1(x, x_{n+1}, t) \nabla_{x_{n+1}} u_{\text{ex}}(x, x_{n+1}, t) + \lambda r J_2(x, x_{n+1}, t) u_{\text{ex}}(x, x_{n+1}, t).
\]
Using the elementary inequality
\[|a - r \lambda b|^2 \geq \frac{a^2}{2} - r^2 \lambda^2 b^2
\]
we get
\[
\lim_{x_{n+1} \to 0} \int_0^\Omega |x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 \, dx 
\geq \lim_{x_{n+1} \to 0} \int_0^\Omega |J_1(x, x_{n+1}, t) \nabla_{x_{n+1}} u_{\text{ex}}(x, x_{n+1}, t)|^2 - \lambda^2 \rho^{-2r} \lim_{x_{n+1} \to 0} K_1 \int_0^\Omega |J_2(x, x_{n+1}, t) u(x, x_{n+1}, t)|^2 \, dx.
\]
Taking the limit gives
\[
\lim_{x_{n+1} \to 0} \int_0^\Omega |x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 \, dx \geq \int_\Omega \rho^{-2r}(x, 0, t) |(-\Delta)^s u(x, t)|^2 \, dx 
- \int_\Omega \frac{\lambda^2 \rho^{-2r}(x, 0, t)}{t^2(T-t)^2} |u(x, t)|^2 e^{\lambda \eta(x)} |(-\Delta)^s \eta(x)|^2 \, dx.
\]
Due to the definition of $\eta$, we know that $(-\Delta)^s \eta = \gamma = 0$ for all $x \in \mathcal{O} \subset \Omega$. In addition, we know that $e^{\lambda \eta(x)} |(-\Delta)^s \eta(x)|$ is bounded from below and above. Hence, there exists a constants $C_1, C_2 > 0$ such that
\[
\lim_{x_{n+1} \to 0} \int_0^\Omega |x_{n+1}^{1-2s} \nabla_{x_{n+1}} \psi(x, x_{n+1}, t)|^2 \, dx 
+ C_1 \int_\mathcal{O} \rho^{-2r}(x, 0, t) \frac{\lambda^2 \rho^{-2r}}{t^2(T-t)^2} |u(x, t)|^2 \, dx \geq C_2 \int_\Omega \rho^{-2r}(x, 0, t) |(-\Delta)^s u(x, t)|^2 \, dx.
\]
Next, by (4.5) we get
\[
\frac{1}{2} \left\| \frac{\partial}{\partial t} \psi(x, 0, t) \right\|_{L^2(\Omega)}^2 = \frac{1}{2} \left\| \rho^{-r}(x, 0, t) \frac{\partial}{\partial t} u(x, x_{n+1}, t) + u(x, x_{n+1}, t) \frac{\partial}{\partial t} \rho^{-r}(x, 0, t) \right\|_{L^2(\Omega)}^2 \geq \frac{1}{2} \left\| \rho^{-r}(x, 0, t) \frac{\partial}{\partial t} u(x, t) - ru(x, t) \rho^{-r}(x, 0, t) \frac{\alpha(x, 0)(t - T - 2t)}{t^2(T-t)^2} \right\|_{L^2(\Omega)}^2.
\]
Using (4.12) we get
\[
\frac{1}{2} \left\| \frac{\partial}{\partial t} \psi(x, 0, t) \right\|_{L^2(\Omega)}^2 + r^2 \int_\Omega \rho^{-2r}(x, 0, t) \left| u(x, t) \frac{\alpha(x, 0)(t - T - 2t)}{t^2(T-t)^2} \right|^2 \, dx \geq \rho^{-2r}(x, 0, t) \left\| \frac{\partial}{\partial t} u(x, t) \right\|_{L^2(\Omega)}^2.
\]
Finally, due to the definition of $f_r$ we have
\[
\rho^{-r}(x, 0, t) \left[ \frac{\partial}{\partial t} + (-\Delta)^s \right] u = f_r(x, 0, t),
\]
Therefore, from Lemma 4.1 follows that there exist some constants $C_1, \ldots, C_6 > 0$ such that the following estimate is valid

$$
\int_0^T \int_{\Omega} \rho^{-2r}(x,0,t) \left( \left[ \frac{\partial}{\partial t} + (-\Delta)^s \right] u \right)^2_{L^2(\Omega)} dt
$$

$$
+ \int_0^T \int_{\Omega} \left( \frac{C_0 \lambda^2 r^2 + C_1}{t^2(T-t)^2} + \frac{C_2 \lambda r^2}{t^3(T-t)^3} + \frac{C_3 \lambda^2 r^2}{t^4(T-t)^4} \right) \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{C_4 r^2 (2t - T)^2}{t^4(T-t)^4} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{C_5 r T^2}{t^3(T-t)^3} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
\geq \lim_{x_n \to 0} \int_0^T \int_{\Omega} \left( \frac{1}{2} \frac{\partial}{\partial t} \psi(x,0,t) \right)^2_{L^2(\Omega)} dt + \int_0^T \int_{\Omega} \frac{1}{8} |x_{n+1} - 2x_n \nabla x_{n+1} \psi(x_{n+1},t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{C_0 \lambda^2 r^2}{t^2(T-t)^2} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{C_4 r^2 (2t - T)^2}{t^4(T-t)^4} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{C_6 \lambda^2 r^2}{t^2(T-t)^2} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{r^2 C_7 (2t - T)^2}{t^4(T-t)^4} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt,
$$

First let us tackle the fourth term on the left hand side, i.e.

$$
\int_0^T \int_{\Omega} \frac{C_5 r T^2}{2t^3(T-t)^3} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx dt.
$$

Here, we split the integral over $t$ for some $t_0 = \kappa T \in (0, T)$ into the part $[0, t_0] \cup [T - t_0, T]$ and $[t_0, T - t_0]$. If $t \in [0, t_0] \cup [T - t_0, T]$ and

$$
\frac{C_7 r [2\kappa - 1]}{\kappa(1 - \kappa)} \geq C_5 T,
$$

we can cancel the fourth term by

$$
\int_0^T \int_{\Omega} \frac{C_7 r^2 (2t - T)^2}{t^4(T-t)^4} \rho^{-2r}(x,0,t) |u(x,t)|^2 dx.
$$

If $t \in [t_0, T - t_0]$ and $T$ is sufficiently large, i.e.,

$$
\frac{C_4 r}{4\kappa^8(1 - \kappa)^3 T^3} \leq \frac{1}{16},
$$
we can use Proposition 4.1 and cancel the term by
\[\int_{\Omega} \frac{1}{8} \rho^{-2r}(x,0,t)|(-\Delta)^s u(x,t)|^2 \, dx.\]
Here it is important that \(t \in [t_0, T - t_0]\), in particular, that the weight function can be bounded from below and above. In this way, a factor
\[e^{\frac{1}{4 \pi (1-\kappa)}}\]
appears. Observe, if \(T \to \infty\) the factor tends to one and can be neglected. The last calculation gives the assertion of Theorem 2.1.

References

[1] A. Astolfi, D. Karagiannis, and R. Ortega. Nonlinear and adaptive control with applications. Communications and Control Engineering. London: Springer, (2008).
[2] T. Bayen, A. Rapaport. Optimal control. New trends in applications to bioprocesses. Amsterdam: Elsevier, (2019).
[3] A. Belmiloudi. Stabilization, optimal and robust control. Theory and applications in biological and physical sciences. Communications and Control Engineering. London: Springer, (2008).
[4] J. Bergh and J. Löfström. Interpolation Spaces - An introduction. Grundlehren der mathematischen Wissenschaften, Springer, (1967).
[5] Bukhgeǐm, A. L., and Klibanov, M. V. Uniqueness in the large of a class of multidimensional inverse problems. Dokl. Akad. Nauk SSSR 260, 2 (1981), 209–272.
[6] Bucur, C., and Valdinoci, E. Nonlocal diffusion and applications., vol. 20. Cham: Springer; Bologna: UMI, (2016).
[7] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32, 7-9 (2007), 1245–1260.
[8] Caffarelli, L., and Sting, P. Fractional elliptic equations, Caccioppoli estimates and regularity. Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2007), 767–807.
[9] Carleman, T. Sur un problème d’unicité pur les systèmes d’équations aux dérivées partielles à deux variables indépendantes. Ark. Mat., Astr. Fys. 26, 17 (1939), 9 pages.
[10] Cont, R., and Tankov, P. Financial modelling with jump processes. Boca Raton, FL: Chapman and Hall/CRC, (2004).
[11] Coville, J., and Dupaigne, L. On a non-local equation arising in population dynamics. Proc. R. Soc. Edinb., Sect. A, Math. 137, 4 (2007), 727–755.
[12] P.W. Fernand, E. Hausenblas. On Markovian semigroups of Lévy driven SDEs, symbols and pseudodifferential operators. [arXiv:1904.09114]
[13] Fursikov, A. V., and Imanuvilov, O. Y. Controllability of evolution equations, vol. 34 of Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, (1996).
[14] Gilboa, G., and Osher, S. Nonlocal operators with applications to image processing. Multiscale Model. Simul. 7, 3 (2008), 1005–1028.
[15] Huang, X., Li, Z., and Yamamoto, M. Carleman estimates for the time-fractional advection-diffusion equations and applications. Inverse Problems 35 (2019), 045003, 36 pages.
[16] Khapalov, A. *Mobile point sensors and actuators in the controllability theory of partial differential equations*. Springer, Cham, (2017).

[17] Nicola, F., and Rodino, L. *Global pseudo-differential calculus on Euclidean spaces*. Pseudo-Differential Operators. Theory and Applications. Birkhäuser Verlag, Basel, (2010).

[18] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, (1983).

[19] Ros-Oton, X., and Serra, J. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9) 101*, 3 (2014), 275–302.

[20] Silling, S. A. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids 48*, 1 (2000), 175–209.

[21] Stinga, P. and Torrea, J. Extension problem and Harnack’s inequality for some fractional operators. *Comm. Partial Differential Equations 35*, (2010), 2092–2122.

[22] Tucsnak, M., and Weiss, G. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, (2009).

[23] Xu, X., Cheng, J., and Yamamoto, M. Carleman estimate for a fractional diffusion equation with half order and application. *Appl. Anal.*, 90 (2011), 1355–1371.

[24] Yu., O. Controllability of parabolic equations. *Mat. Sb. 186*, 6 (1995), 109–132.

[25] Zuazua, E. Exact boundary controllability for the semilinear wave equation. In *Nonlinear partial differential equations and their applications*. Collège de France Seminar, Vol. X (Paris, 1987–1988), vol. 220 of *Pitman Res. Notes Math. Ser.* Longman Sci. Tech., Harlow, 1991, pp. 357–391.

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