Estimates on the Probability of Outliers for Real Random Bargmann-Fock functions.
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Abstract

In this paper we consider the distribution of the zeros of a real random Bargmann-Fock function of one or more variables. For these random functions we prove estimates for two types of families of events, both of which are large deviations from the mean. First, we prove that the probability there are no zeros in \([-r, r]^m \subset \mathbb{R}^m\) decays at least exponentially in terms of \(r^m\). For this event we also prove a lower bound on the order of decay, which we do not expect to be sharp. Secondly, we compute the order of decay for the probability of families of events where the volume of the complex zero set is either much larger or much smaller than expected.

1. Introduction.

Random functions provide techniques to study typical properties of elements of a Hilbert space, and have been used to shed light on the zero set of elements of a Hilbert Space of functions. We will define real random Bargmann-Fock functions as a linear combination of basis functions, where each basis function is weighted with a real standard Gaussian random variable. Alternatively, each random function is a representative of a Gaussian field on the Bargmann-Fock space.

There is a significant body of knowledge concerning the zero set of Gaussian random functions. In particular, in the work of Kac and Rice (\cite{Kac}, \cite{Rice}) techniques were developed to compute the correlation functions for the zeros of Gaussian random functions. This work has more recently been used to compute the two point correlation function for the zeros of model systems of Gaussian Random variables, \((\text{BD}, \text{BSZ2}, \text{Prd})\). Additionally, there is a significant body of results concerning the expected number (or volume) of zeros, the variance of the number of zeros and other results such as a central limit law \((\text{SZ2}, \text{ST2})\).

To complement these results concerning events within the central limit region, recent work has been conducted on families of events which are large deviations from the mean, such as the event where there are no zeros in an interval (or ball) of large radius. This event is called a Hole. As we will be working with random functions on \(\mathbb{C}^m\), whose restriction to \(\mathbb{R}^m\) is real valued, we will need to distinguish between real holes and complex ones.
Definition. We define $\text{Hole}_{r,\mathbb{R}^m}$ to be the event consisting of all real random Bargmann-Fock functions which have no zeros in the interval $[-r, r]^m \subset \mathbb{R}^m$, for a large $r$.

Definition. We define $\text{Hole}_{r,\mathbb{C}^m}$ to be the event consisting of all real random Bargmann-Fock functions which have no zeros in the interval $B(0, r) \subset \mathbb{C}^m$, for a large $r$.

A simple family of events similar to those we will study here is the family of events where a coin is flipped $N$ times but no heads show up, which has probability $= e^{-N\log(2)}$. To facilitate the comparison to this toy system we let $N_r$ be the number of expected zeros (or the expected volume of the zero set) in a ball of radius $r$. A series of results have been shown for the hole probability of complex random functions on various spaces. As in the coin flip model, a general estimate for the order of the decay of the upper bound for any Riemann surface was derived and proven: $\text{Prob}(\text{Hole}_{r,\mathbb{C}^1}) \leq e^{-cN_r}$, \cite{Sod}. For one variable complex Gaussian random functions related to the Hardy space on the disk, the order of the previous estimate was subsequently shown to be sharp: $\text{Prob}(\text{Hole}_{r,\mathbb{C}^1}) = e^{-\pi Nr + o(Nr)}/6$, \cite{PV}. Whereas, for one complex Gaussian random function related to the Bargmann-Fock space in $m$ variables the estimate is not sharp \cite{ST3, Zre1}:

$$e^{-c_2N_r^{1+1/m}} \leq \text{Prob}(\text{Hole}_{N_r,\mathbb{C}^1}) \leq e^{-c_1N_r^{1+1/m}}.$$  

The techniques used in the previous work have also been used to solve results for random SU($m+1$) polynomials \cite{Zre2}, as well as random holomorphic sections of the $N^{th}$ tensor power of a positive line bundle on a compact Kahler Manifold, \cite{SZZ}.

The study of $\text{Hole}_{r,\mathbb{R}^m}$ poses distinctly different challenges and opportunities, and significant strides have been made on one model system: real Gaussian random functions associated to the $L^1$ Hardy Space. In particular, the probability a degree $N$ Kac polynomial has no real zeros has been shown to be $O(N^{-b})$, \cite{DPSZ}. Since this work a comparison inequality for Gaussian processes has been discovered, which is very useful for proving an upper bound for the hole probability \cite{LS}. This comparison inequality will prove to be useful in subsequent sections.

In this work we will contribute to this area by deriving estimates for the real and complex hole probability for (real) Gaussian random functions associated with the Bargmann-Fock Space.

**Theorem 1.1.** *(The decay of the real hole probability)*
If
\[ \psi_\alpha(z_1, z_2, \ldots, z_m) = \sum_j \alpha_j \frac{z_1^{j_1} z_2^{j_2} \cdots z_m^{j_m}}{\sqrt{j_1! \cdots j_m!}}, \]
where \( \alpha_j \) are independent identically distributed real Gaussian random variables, then there exists \( R_m, c_m, C_m \) such that for all \( r > R_m \)
\[ e^{-C_m r^{2m}} < \text{Prob}(\text{Hole}_{r,R_m}) < e^{-c_m r^{2m}} \]

We expect the order of the upper bound to be sharp and thus the lower bound to be improvable in general to be \( e^{-cr^m} \). It is worth remarking that while the upper bound in the theorem is the estimate one would expect based on the coin flip example, the author initially expected a higher order of decay, based on work concerning complex zero sets. Computational troubles, followed by numerical simulations quickly changed the author’s expectations.

This result concerning the real zero set contrasts from the following results for the complex zero sets of real random Bargmann-Fock functions, which are proven in this article using the techniques of Sodin and Tsirelson [ST3].

**Theorem 1.2.** (Probability estimates for over and under crowded zero sets)

If
\[ \psi_\alpha(z_1, z_2, \ldots, z_m) = \sum_j \alpha_j \frac{z_1^{j_1} z_2^{j_2} \cdots z_m^{j_m}}{\sqrt{j_1! \cdots j_m!}}, \]
where \( \alpha_j \) are independent identically distributed real Gaussian random variables, then for all \( \delta > 0 \), there exists \( c_1, c_2, \delta > 0 \) and \( R_m, \delta \) such that for all \( r > R_m, \delta \)
1) \[ e^{-c_1 r^{2m+2}} < \text{Prob} \left( \left\{ n_{\psi_\alpha}(r) - \frac{1}{2} r^2 \geq \delta r^2 \right\} \right) \leq e^{-c_2 r^{2m+2}} \]

where \( n_{\psi_\alpha}(r) \) is the unintegrated counting function for \( \psi_\alpha \), and
2) \[ e^{-c_1 r^{2m+2}} \leq \text{Prob}(\text{Hole}_{r,C_m}) \leq e^{-c_2 r^{2m+2}} \]

In the above theorem both the upper and lower bound are the same (orders) one gets when studying complex random Bargmann-Fock functions in \( m \)-variables, [Zre1].

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2. Background.

Let \( m \in \mathbb{N} \setminus \{0\} \) be the number of variables.

Throughout this paper we use the standard multi-index notations. Specifically, if \( z \in \mathbb{C}^m, \ z = (z_1, z_2, \ldots, z_m) \) and \( j \in \mathbb{N}^m, \ j = (j_1, j_2, \ldots, j_m) \) then

\[
z^j = z_1^{j_1} \cdot z_2^{j_2} \cdot \ldots \cdot z_m^{j_m}
\]

\[
j! = j_1! \cdot j_2! \cdot \ldots \cdot j_m!
\]

\( \zeta \in \mathbb{C}^m, \ z \cdot \zeta = z_1 \zeta_1 + z_2 \zeta_2 + \ldots + z_m \zeta_m \)

\[
P(0, r) = \{ z \in \mathbb{C}^m : |z_1| < r, |z_2| < r, \ldots, |z_m| < r, \}
\]

\[
B(0, r) = \{ z \in \mathbb{C}^m : \left( \sum |z_j|^2 \right)^{\frac{1}{2}} < r \}
\]

The Bargmann-Fock space is defined to be the set

\[
BF = O(\mathbb{C}^m) \cap L^2 \left( \mathbb{C}^m, \frac{1}{\pi^m} e^{-|z|^2} dV(z) \right),
\]

where \( O(\mathbb{C}^m) \) is the set of holomorphic functions on all of \( \mathbb{C}^m \) and \( dV(z) \) is Lebesgue measure on \( \mathbb{C}^m \). With respect to this norm both \( \left\{ \frac{z^j}{\sqrt{j!}} \right\}_{j \in \mathbb{N}^m} \) and \( \left\{ e^{-\frac{1}{2}y^2 + z \cdot \overline{y}} (z - y)^j \right\}_{j \in \mathbb{N}^m} \) are orthonormal bases, for all \( y \in \mathbb{C}^m \). Further, if \( y \in \mathbb{R}^m \) then the restriction of the two previous bases to \( \mathbb{R}^m \) is real valued.

For this paper, we define real random Bargmann-Fock functions as a linear combination real valued basis elements of the Bargmann-Fock space, weighted with real independent identically distributed Gaussian random variables:

\[
\psi_\alpha(x) = \sum_{j \in \mathbb{N}^m} \alpha_j \frac{x^j}{\sqrt{j!}}
\]

As \( \lim \sup |\alpha_j|^{\frac{1}{j}} = 1 \) a.s., \( \psi_\alpha(x) \in O(\mathbb{R}^m) \) a.s. However, \( \psi_\alpha(x) \in (BF)^c \) a.s., since \( \{\alpha_j\} \) is a.s. unbounded.

**Remark 2.1.** In the literature, \( \alpha_j \) is often instead taken to be complex Gaussian random variables, i.e. \( \alpha_j = N(0, \frac{1}{\sqrt{2}}) + \sqrt{-1} \cdot N(0, \frac{1}{\sqrt{2}}) \).

At first glance the definition of a real random Bargmann-Fock function seems to depend on the basis chosen but this is not the case as can be seen in the following two lemmas:
Lemma 2.2. (Real translation invariance law)

If \( \{\alpha_j\} \) is a sequence of i.i.d. real Gaussian random variables, then for all \( y \in \mathbb{R}^m \) there exists a sequence of i.i.d. standard real Gaussian random variables \( \{\beta_j\} \) such that for all \( x \in \mathbb{R}^m \),

\[
\psi_{\alpha}(x) = e^{-\frac{x^2}{2} + y^T x} \psi_{\beta}(x - y)
\]

The above lemma illustrates an important tool that we have at our disposal, but could be reformulated in the terse restatement “that random Bargmann-Fock functions are well defined, independent of basis.” As stated in the above form, the result shows that if we know a result for a random Bargmann-Fock function in one region, we will immediately obtain a similar result on any other region. A proof of this result may be found in [Fer].

If the sequence \( \{\alpha_j\} \) instead was composed of complex Gaussian random variables the complex Gaussian random Bargmann-Fock functions would have a complex translation invariance law (as a Gaussian process). Instead we must be content with the following:

Proposition 2.3. (Complex invariance law)

If \( \{\alpha_j\} \) is a sequence of i.i.d. real Gaussian random variables, then for all \( \zeta \in \mathbb{C}^m \), \( \zeta = y \cdot e^{i \arg(\zeta)} \), where \( y \in \mathbb{R}^m \), there exists a sequence of i.i.d. standard real Gaussian random variables \( \{\beta_j\} \) such that for all \( x \in \mathbb{R}^m \),

\[
\psi_{\alpha}(x) = e^{-\frac{1}{2} |\zeta|^2 + x^T y} \sum_{|j| = 0}^{\infty} \beta_j e^{-j \cdot i \arg(\zeta)} \frac{(xe^{i \arg(\zeta)} - \zeta)^j}{\sqrt{j!}},
\]

where \( (xe^{i \arg(\zeta)} - \zeta)^j = (x_1 e^{i \arg(\zeta_1)} - \zeta_1)^j_1 \cdot \ldots \cdot (x_m e^{i \arg(\zeta_m)} - \zeta_m)^j_m \).

Proof. By the previous lemma,

\[
\psi_{\alpha}(x) = e^{-\frac{x^2}{2} + xy} \sum_{|j| = 0}^{\infty} \beta_j \frac{(x - y)^j}{\sqrt{j!}}
\]

\[
= e^{-\frac{x^2}{2} + xy} \sum_{|j| = 0}^{\infty} \beta_j e^{-j \cdot i \arg(\zeta)} \frac{(e^{i \arg(\zeta)} x - \zeta)^j}{\sqrt{j!}}
\]

These two results (Lemma 2.2 and Corrolary 2.3) will be essential in our subsequent work. Taken together they will allow us to “translate” any result for an open neighborhood of the origin to an open set about any other point.

We need one more technical result:
Proposition 2.4. Let \( \alpha \) be a standard real Gaussian Random Variable, then:

a-i) \( \text{Prob}(\{|\alpha| \geq \lambda\}) \leq \frac{e^{-\lambda^2}}{\sqrt{2\pi}}, \) if \( \lambda \geq 1 \)

a-ii) \( \text{Prob}(\{|\alpha| \leq \lambda\}) \in \left[\lambda \cdot \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}\right], \) if \( \lambda \leq 1 \)

b) If \( \{\alpha_j\}_{j \in \mathbb{N}} \) is a set of independent identically distributed standard Gaussian random variables, then \( \text{Prob}(\{|\alpha_j| < (1+\varepsilon)|j|\}) = c > 0. \)

c) If \( j \in \mathbb{N}^+, n \) then \( \frac{|j|^3}{j} \leq n^3 \)

3. A Lower bound for Probability of Real hole.

In this section we prove a lower bound on the order of decay which we do not expect to be sharp. This bound is however an improvement on the one which could be derived from the inclusion of \( \text{Hole}_{r,\mathbb{C}^m} \) in \( \text{Hole}_{r,\mathbb{R}^m}. \)

Theorem 1.1 (Lower bound)

If \( \alpha_j \) is a sequence of real Gaussian random variables centered about \( \{a_j\} \in \ell^1, \) and \( \psi_\alpha(z) \) is a Gaussian random function associated to the Bargmann-Fock space then there exist \( C_m, R_m > 0 \) such that for all \( r > R_m \)

\[
e^{-c_m r^2 m} < \text{Prob}(\{\alpha : \forall x \in [-r, r]^m, \psi_\alpha(z) \neq 0\})
\]

Proof. As the zero set of a random function associated with the Bargmann-Fock space is translationally invariant, it suffices to show, after rescaling, that there are no zeros in the interval \((0, r)^m.\)

Let \( \Omega_r \) be the event where:

i) \( \alpha_j \geq E_m + 1, \) \( \forall j : 0 \leq \|j\|_{\ell^1} \leq \left[48mr^2\right] = [(m \cdot 2 \cdot 12)(2r)^2] \)

ii) \( \|\alpha_j\|_{\ell^1} \leq 2^{|j|_{\ell^1}} \cdot \|j\|_{\ell^1} > \left[48mr^2\right] \geq 48mr^2 \)

Hence, \( \text{Prob}(\Omega_r) \geq C(e^{-cm r^{2m}}) \), by independence and Proposition 2.4.

Given \( \alpha \in \Omega_r \) and \( x \in (0, r)^m \) we now show that \( \alpha \) must belong to \( \text{Hole}_{r,\mathbb{R}^m}: \)

\[
\psi_\alpha(x) \geq \sum_{\|j\|_{\ell^1} = 0}^{\|j\|_{\ell^1} \leq [24mr^2]} \alpha_j \frac{x^j}{\sqrt{j!}} - \sum_{|j| > [24mr^2]} |\alpha_j| \frac{r^j}{\sqrt{j!}} \]

\[
= \sum_{1}^{1} - \sum_{2}^{2}
\]

\[
\sum_{1}^{1} \geq \alpha_0 \geq E_m + 1
\]
\[
\sum_{|j| > 24mr^2} \leq \sum_{|j| > 24mr^2} 2^{\frac{\|j\|_{\ell^1}}{24n}} \left( \frac{\|j\|_{\ell^1}}{24m} \right)^{\frac{\|j\|_{\ell^1}}{24m}} \frac{1}{\sqrt{|j|!}}, \text{ as } r < \sqrt{\frac{\|j\|_{\ell^1}}{24m}} \\
\leq c \sum_{|j|_{\ell^1} > 24mr^2} 2^{\frac{\|j\|_{\ell^1}}{24n}} \left( \frac{\|j\|_{\ell^1}}{24m} \right)^{\frac{\|j\|_{\ell^1}}{24m}} \prod_{k=1}^{k=m} \left( \frac{e}{j_k} \right)^{\frac{j_k}{2}}, \text{ by Stirling’s formula} \\
= c \sum_{|j|_{\ell^1} > 24mr^2} \left( \frac{\|j\|_{\ell^1}}{24n} \right)^{\frac{\|j\|_{\ell^1}}{24m}} \left( \frac{e}{12} \right)^{\frac{\|j\|_{\ell^1}}{2}} \\
\leq c \sum_{l=1}^{l=m} \left( \frac{1}{2} \right)^l l^m \leq E_m 
\]

Hence, if \( \alpha \in \Omega_r \) then for all \( x \in [0, r]^m \), \( |\psi_\alpha(x)| \geq 1 \)

\[\square\]

4. An upper bound for the Real hole probability.

Given that \( \alpha \in \text{Hole}_{r,\mathbb{R}^m} \) then either the random function is strictly positive (for \( x \in (-r, r)^m \)) or strictly negative. Further, the values at any two distant points are almost independent. Together, these two observations form the basis of the intuition behind why one can expect an exponential decay for the hole probability of a random Bargmann-Fock function, as at lattice points the values are nearly independent (for a coarse lattice).

To make this argument rigorous we will use the following result of Li and Shao. \([LS]\). This comparison also invites comparison with the coin flip model mentioned in the introduction.

**Theorem 4.1.** let \( n \geq 3 \) and let \((X_j)_{1 \leq j \leq n}\) be a sequence of standard jointly normal random variables with \( E[X_jX_k] = a_{i,j} \), then

\[2^{-n} \leq \text{Prob} \left( \bigcap_{j=1}^{n} \{ X_j \leq 0 \} \right) \leq 2^{-n} \exp \left\{ \sum_{1 \leq k < j \leq n} \log \left( \frac{\pi}{\pi - 2 \arcsin(a_{k,j})} \right) \right\} \]

This lemma is a useful generalization of Slepian’s lemma, as it gives both an upper and a lower bound. In the case of one variable the following result may in fact be proven by using Slepian’s lemma, and comparing discretized
real random Bargmann-Fock functions with the Ornstein-Uhlenbeck process.

**Theorem 1.1** *(Upper bound for Theorem 1.1)*

If $\alpha_j$ is a sequence of real Gaussian random variables centered about $\{a_j\} \in \ell^1$, and $\psi_\alpha(z)$ is a Gaussian random function associated to the Bargmann-Fock space then there exist $C_m, R_m > 0$ such that for all $r > R$

$$\text{Prob}\{\{\alpha : \forall x \in [-r, r]^m \subset \mathbb{R}^m, \psi_\alpha(z) \neq 0\}\} < e^{-c_2 r_m}$$

**Proof.** Let $Y_t = e^{-\frac{1}{2} x^2} \psi_\alpha(x)$.

$E[Y_t Y_s] = a_{t,s} = e^{-\frac{1}{2}(s-t)^2}$

Let $J = [-r, r]^m \cap (2N)^m$.

For real Gaussian random functions:

$$\text{Hole}_{r, \mathbb{R}^m} = \{Y_t > 0, t \in [-r, r]^m\} \cup \{Y_t < 0, t \in [-r, r]^m\}.$$  

$$\text{Hole}_{r, \mathbb{R}^m} \subset \{Y_t > 0, t \in J\} \cup \{Y_t < 0, t \in J\}.$$  

By symmetry $\{Y_t > 0, t \in J\}$ and $\{Y_t < 0, t \in J\}$ have the same probability and it thus suffice to prove that either of these have exponential decay. By theorem 4.1 and elementary estimates we may now finish the proof:

$$\text{Prob} \left( \bigcap_{j \in J} \{Y_j \leq 0\} \right) \leq 2^{-|J|} \exp \left\{ \frac{1}{2} \sum_{k \neq j, k \in J} \log \left( \frac{\pi}{\pi - 2 \arcsin(a_{k,j})} \right) \right\}$$

$$\leq 2^{-|J|} \exp \left\{ \sum_{j=-|r|}^{|r|} \frac{j^m}{2} \log \left( \frac{\pi}{\pi - 2 \arcsin(e^{-2j^2})} \right) \right\}$$

$$\leq 2^{-|J|} \exp \left\{ \sum_{j=-|r|}^{|r|} \frac{j^m}{2} \log \left( \frac{\pi}{\pi - 3e^{-2j^2}} \right) \right\}$$

$$\leq 2^{-|J|} \exp \left\{ \sum_{j=-|r|}^{|r|} \frac{j^m}{2} \log \left( 1 + \frac{6}{\pi} e^{-2j^2} \right) \right\}$$

$$\leq 2^{-|J|} \exp \left\{ \sum_{j=-|r|}^{|r|} \frac{3j^m}{\pi} e^{-2j^2} \right\}$$

$$\leq 2^{-|J|} C_m = C_m 2(2|r|+1)^m$$

5. Over and under crowded complex zero sets.

We now restrict ourselves to exploring the event where there are no complex zeros for a real random Bargmann-Fock function. This is solved
by using techniques developed and used by Sodin and Tsirelson [ST3] to prove a similar result for complex random Bargmann-Fock functions in one variable. These techniques were also generalized to the case of $m$ variable complex random Bargmann-Fock functions ([Zre1]) and may further be generalized to include $m$ variable real random Bargmann-Fock functions which we do now. The crucial difference between the argument presented here and that which was presented in [ST3] and [Zre1] is that real random Bargmann-Fock functions are not translationally invariant, but Lemma 2.3 will be more than adequate to make up for this.

We begin this with the following statement about the rate of growth of a random function.

**Lemma 5.1.** If $\{\alpha_j\}$ is a sequence of independent Gaussian random variables, and if $\theta_j \in [0, 2\pi]^m$, then for all $\delta > 0$ there exists $c_\delta, R_\delta > 0$ such that for all $r > R_\delta$,

$$\text{Prob} \left( \left\{ \log \left( \max_{B(0,r)} \left| \sum \alpha_j e^{\theta_j i z_j} \sqrt{\frac{j^2}{j!}} \right| \right) - \frac{1}{2} r^2 \right\} \geq \delta r^2 \right) \leq e^{-c_\delta r^{2n+2}}$$

Only minor modifications for the argument presented in [Zre1] are needed to prove this result in this form. These modifications are needed as previous versions of this lemma have only involved complex Gaussian random variables and these modifications are minor as the steps used in the proof are only dependent on the norm of the random variables. A sketch of the proof is included here for completeness sake.

**Proof.** Let $\psi_{\alpha,\theta} = \sum \alpha_j e^{\theta_j i z_j} \sqrt{\frac{j^2}{j!}}$

Let $M_{r,\alpha,\theta} = \max_{z \in B(0,r)} |\psi_{\alpha,\theta}(z)|$

Let $\Gamma_r = \{\alpha : \frac{\log(M_{r,\alpha,\theta})}{r^2} \geq \frac{1}{2} + \delta\}$

The proof that $\text{Prob}(\Gamma_r) \leq e^{-c_\delta,1 r^{2m+2}}$ is extremely easy as $\max_{|j| < N} |\alpha_j|$ is expected to grow polynomially and would need to grow exponentially to be in this event. Rigorously,

Let $\Omega_r$ be the event where:

1) $|\alpha_j| \leq e^{\delta r^2}$, $|j| \leq 4e \cdot m \cdot r^2$

2) $|\alpha_j| \leq 2^{\delta r^2}$, $|j| > 4e \cdot m \cdot r^2$

$$\text{Prob}(\Omega_r^c) \leq e^{-e \cdot r^2} \leq e^{-cr^{2m+2}}$$, by Proposition 2.3
If \( \alpha \in \Omega_r \), then for all \( z \in B(0, r) \), then:

\[
M_{r, \alpha, \theta} \leq \max_{z \in B(0, r)} \left( \sum_{|j| \leq 4e^{-m} \left( \frac{1}{2} r^2 \right)} |\alpha_j| \frac{|z|^j}{\sqrt{j!}} + \sum_{|j| > 4e^{-m} \left( \frac{1}{2} r^2 \right)} |\alpha_j| \frac{|z|^j}{\sqrt{j!}} \right)
\]

\[
\leq e^{\left( \frac{1}{2} + \frac{\delta}{2} \right) r^2},
\]

by a series of standard estimates presented in [Zre1].

Thus \( \Gamma_r \subset \Omega_{r} \), proving half the result.

Let \( M'_{r, \alpha, \theta} = \max_{P(0, r)} |\psi_\alpha| \)

Let \( E_{\delta, r, \theta} = \{ \alpha : \log(M'_{r, \alpha, \theta} < \left( \frac{1}{2} - \mathbf{\delta} \right)r^2 \} \)

If \( \alpha \in E_{\delta, r, \theta} \) then by Cauchy's Integral Formula:

\[
\left| \frac{\partial^j \psi_\alpha}{\partial z^j} \right| (0) = |\alpha_j| \sqrt{j!}.
\]

Further, by direct computation:

\[
\left| \frac{\partial^j \psi_\alpha}{\partial z^j} \right| (0) = |\alpha_j| \sqrt{j!}.
\]

When these estimates are combined with elementary estimates and Stirling’s formula, it can be shown that \( \exists \Delta > 0 \) such that \( \forall \delta \leq \Delta \) if \( j \in \left[ 1 - \sqrt{\delta \over n}, 1 + \sqrt{\delta \over n} \right] \) then

\[|\alpha_j| \leq e^{-\delta \over 4}.
\]

For more details see [Zre1]. Thus,

\[
\text{Prob}(\{ \log(M'_{r, \alpha, \theta} \leq \left( \frac{1}{2} - \delta \right)r^2 \})
\]

\[
\leq \text{Prob}(\{ |\alpha_j| \leq e^{-\frac{\delta}{4}} : j_k \in \left( \sqrt{\delta \over m}, 1 + \sqrt{\delta \over m} \right) \})
\]

\[
\leq e^{-c_{m, \delta} r^{2m+2}},
\]

by Proposition 2.4.

The result then follows by subadditivity as

\[
M_{r, \alpha, \theta} \geq M'_{r, \alpha, \theta} \Rightarrow \left( \frac{1}{2} - \delta \right)r^2 \subset E_{\delta, r, \theta}
\]

As a consequence of this previous result and Proposition 2.3 (the complex invariance law), we have the following result:

**Corollary 5.2.** For all \( \delta > 0 \) there exists \( R_\delta \) such that for all \( r > R_\delta \), if \( z_0 \in B(0, r) \setminus B(0, \frac{1}{2} r) \) then

\[
\text{Prob}(\{ \exists \zeta \in B(z_0, \delta r) \ s.t. \ \log|\psi_\alpha(\zeta)| > \left( \frac{1}{2} - \delta \right)|z_0|^2 \}) \leq e^{-c_{\delta} r^{2m+2}}
\]

This result has been known for complex random Bargmann-Fock functions, and the key difference between this proof and its predecessors is that
real random Bargmann-Fock functions are not “complex translation” invariant, and as such we have to allow for multiplication of our random variables by $e^{i\theta}$, $\theta \in \mathbb{R}^m$.

Proof. It suffices to prove the result for any small delta. Let $\delta < \frac{1}{4}$. Let $y e^{i\theta} = w$, where $y \in \mathbb{R}^m$. Let $\psi_{\alpha,\theta}(x) = \sum_{|j|=0}^{\infty} \alpha_j e^{i\theta j} \frac{x^j}{\sqrt{j!}}$

We restrict ourselves to the following event:

$$\max_{z \in \partial B(0,\delta r)} e^{-\frac{(\delta r)^2}{2}} |\psi_{\alpha,\theta}(z)| \geq -\delta^3 r^2,$$

whose complement, by Lemma 5.1, occurs with an appropriately small probability.

Translating this result using Lemma 2.3 gives the following:

$$\max_{z \in B(0,\delta r)} e^{-\frac{(\delta r)^2}{2}} |\psi_{\alpha,\theta}(z)| = \max_{z \in B(0,\delta r)} e^{-\frac{|z|^2}{2}} |\psi_{\alpha,\theta}(z)| = \max_{z \in B(0,\delta r)} e^{-\frac{|z-y|^2}{2}} |\psi_{\beta}(z-w)|$$

$$\max_{z \in B(0,\delta r)} \log(|\psi_{\beta}(z-w)|) - \frac{1}{2}|z-y|^2 = \max_{z \in B(w,\delta r)} \left( \log(|\psi_{\beta}(z)|) - \frac{1}{2}|x|^2 \right)$$

$$\leq \max_{z \in B(w,\delta r)} (\log(|\psi_{\beta}(z)|)) - \frac{1}{2}|y-\delta r|^2$$

$$\leq \max_{z \in B(w,\delta r)} (\log(|\psi_{\beta}(z)|)) - \frac{1}{2}|y|^2 + \delta r|y| - \frac{1}{2}\delta^2 r^2$$

Thus,

$$\max_{z \in B(w,\delta r)} \log |\psi_{\beta}(z)| \geq \frac{1}{2}|y|^2 + \frac{1}{2}\delta^2 r^2 - \delta r|y| - \delta^3 r^2 \geq \frac{1}{2}|y|^2 - \frac{\delta}{4}|y|^2$$

□

This result allows us to prove the following lemma concerning the average of $\log |\psi_{\alpha}|$ with respect to the rotationally invariant Haar probability measure on the sphere of radius $r$, $d\sigma_r$:

Lemma 5.3. For all $\delta > 0$, there exists $c_m > 0$ such that for all $r > R_m$

$$\text{Prob} \left( \left\{ \frac{1}{r^2} \int_{z \in \partial B(0,r)} \log |\psi_{\alpha}| d\sigma_r(z) \leq \frac{1}{2} - \Delta \right\} \right) \leq e^{-cr^{2m+2}}$$

Using Corollary 5.2, Proposition 2.3 and the same steps as those in [Zre1], this result may be proven.
Proof. (Sketch).

The proof uses regularity properties of subharmonic functions, and thus we begin by fixing a constant $\kappa_\delta$ which is near but less than 1. This is done to avoid singularities of the Poisson kernel, and still be able to accurately estimate the average of $\log(|\psi_\alpha|)$. We then choose a disjoint partition, $\{I_{j,\delta}^{r}\}$, of $S_{\kappa r} = \partial B(0, r)$ by projecting even $2m - 1$ cubes so that $\text{diam}(I_{j}^{r}) \leq c_{\delta,m}r$.

Let $\sigma_j = \sigma_{\kappa r}(I_{j}^{r})$, which does not depend on $r$, and for all $j$ fix a point $x_j \in I_{j}^{r}$. By Corollary 5.2, for each $j$ there exists a $\zeta_j \in B(x_j, \delta r)$ such that

$$\log(|\psi_\alpha(\zeta_j)|) > \left(\frac{1}{2} - 3\delta\right)|x_j|^2 = \left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2$$

except for $N$ different events each of which has probability less than $e^{-c' \delta r^{2m+2}}$, and thus the union of all of these also has probability less than $e^{-c r^{2m+2}}$.

As we have the same estimate for each $j$ for $|\psi_\alpha(\zeta_j)|$, and $\sum \sigma_j = 1$ we have:

$$\left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2 \leq \sum_{j=1}^{N} \sigma_j \log(|\psi_\alpha(\zeta_j)|)$$

$$\leq \int_{\partial B(0,r)} \left( \sum_{j} \sigma_j \psi_\alpha(\zeta_j, z) \log(|\psi_\alpha(z)|) \right) d\sigma_r(z)$$

$$= \int_{\partial B(0,r)} \left( \sum_{j} \sigma_j (\psi_\alpha(\zeta_j, z) - 1) \right) \log(|\psi_\alpha(z)|) d\sigma_r(z)$$

$$+ \int_{\partial B(0,r)} \log(|\psi_\alpha(z)|) d\sigma_r(z)$$

Hence,

$$\int_{\partial B(0,r)} \log(|\psi_\alpha|) d\sigma_r \geq \left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2 - \int |\log|\psi_\alpha|| d\sigma_r \cdot \max_{z} \left| \sum_{j} \sigma_j (\psi_\alpha(\zeta_j, z) - 1) \right|$$

The result may be completed as it is shown in [Zre1] that:

$$\max_{z \in \partial B(0,r)} \left| \sum_{j} \sigma_j (\psi_\alpha(\zeta_j, z) - 1) \right| \leq C_n \delta \frac{1}{2^{2m-1}}$$

and that Lemma 5.1 $\Rightarrow \exists c_m, R_m$, such that for all $r > R_m$,

$$\text{Prob} \left( \{ ||\psi_\alpha||_{L^1(S_r,\sigma)} < C_m r^2 \} \right) < e^{-c_m r^{2m+2}}.$$
By combining lemmas 5.1 and 5.3 and an $m$-dimensional analog of Jensen’s formula we get an upperbound for the probability of a large class of events: 

**Theorem 1.2 (Probability estimates for over and under crowded zero sets)**

If

\[
\psi_\alpha(z_1, z_2, \ldots, z_m) = \sum_j \alpha_j \frac{z_1^{j_1} z_2^{j_2} \cdots z_m^{j_m}}{\sqrt{j_1! \cdot j_m!}},
\]

where $\alpha_j$ are independent identically distributed real Gaussian random variables, then for all $\delta > 0$, there exists $c_1, c_{2, \delta} > 0$ and $R_{m, \delta}$ such that for all $r > R_{m, \delta}$

1) $e^{-c_1 r^{2m+2}} \leq \text{Prob} \left( \left\{ \left| n_{\psi_\alpha}(r) \right| \geq \frac{1}{2} r^2 \right\} \right) \leq e^{-c_{2, \delta} r^{2m+2}}$

where $n_{\psi_\alpha}(r)$ is the unintegrated counting function for $\psi_\alpha$, and

2) $e^{-c_1 r^{2m+2}} \leq \text{Prob} (\text{Hole}_{r, Cm}) \leq e^{-c_{2, \delta} r^{2m+2}}$

**Proof.** To obtain the upper probability estimate we need only apply Lemmas 5.1 and 5.3 to the $m$-dimensional analog of Jensen’s inequality. For further details on this argument consult the one present in [Zre1] which may be adapted word for word.

To finish proving the result it suffices to show that the event where there is a hole in the ball of radius $r$ contains an event whose probability is larger than $e^{-c r^{2n+2}}$.

Let $\Omega_r$ be the event where:

i) $|\alpha_0| \geq E_m + 1$,

ii) $|\alpha_j| \leq e^{-(1 + \frac{2}{12}) r^2}$, $\forall j : 1 \leq |j| \leq \lceil 24mr^2 \rceil = \lceil (n \cdot 2 \cdot 12)r^2 \rceil$

iii) $|\alpha_j| \leq 2^{\frac{|j|}{12}}$, $|j| > \lceil 24mr^2 \rceil \geq 24mr^2$

$\text{Prob}(\Omega_r) \geq C (e^{-c_m r^{2m+2}}) \geq e^{-c r^{2m+2}}$, by independence and Proposition 2.4.

If $\alpha \in \Omega_r$ then

\[
|\psi_\alpha(z)| \geq |\alpha_0| - \sum_{|j| \leq \lceil 24mr^2 \rceil} |\alpha_j| \frac{r^{|j|}}{\sqrt{|j|!}} - \sum_{|j| > \lceil 24mr^2 \rceil} |\alpha_j| \frac{r^{|j|}}{\sqrt{|j|!}} = |\alpha_0| - \sum - \sum
\]
\[
\sum_{|j| > 24mr^2} \leq e^{-(1+m^2)r^2} \sum_{|j|=1} \frac{e^{r|j|}}{\sqrt{j!}} \leq e^{-(1+m^2)r^2} \sqrt{(2mr^2 + 1)^m} \sqrt{(e^m)}, \text{ by Cauchy-Schwarz inequality.}
\]
\[
\leq C_m r^m e^{-r^2} \leq ce^{-0.9r^2} < \frac{1}{2} \text{ for } r > R_m
\]
\[
\sum_{|j| > 24mr^2} \leq \sum_{|j| > 24mr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24m}\right)^{\frac{|j|}{24m}} \frac{1}{\sqrt{j!}} \leq c \sum_{|j| > 24mr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24m}\right)^{\frac{|j|}{24m}} \prod_{k=1}^{k=m} \left(\frac{e}{j_k}\right)^{\frac{|j|}{2}} \frac{1}{j_k}
\]
\[
= c \sum_{|j| > 24mr^2} \left(\frac{1}{4}\right)^{\frac{|j|}{2}} \prod_{k=1}^{k=m} \frac{j_k}{j_k^2} \frac{1}{j_k} \leq c \sum_{|j| > 1} \left(\frac{1}{4}\right)^{\frac{|j|}{2}} \leq E_m
\]

Hence, \(|\psi_\alpha(z)| \geq E_m + 1 - \sum - \sum \geq \frac{1}{2}\). \qed

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