RADON MEASURE-VALUED SOLUTIONS
OF FIRST ORDER HYPERBOLIC CONSERVATION LAWS

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Abstract. We study nonnegative solutions of the Cauchy problem
\[
\begin{cases}
  u_t + [\varphi(u)]_x = 0 & \text{in } \mathbb{R} \times (0, T) \\
  u = u_0 \geq 0 & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\]
where \( u_0 \) is a Radon measure and \( \varphi : [0, \infty) \to \mathbb{R} \) is a globally Lipschitz
continuous function. We construct suitably defined entropy solutions in the space
of Radon measures. Under some additional conditions on \( \varphi \), we prove their
uniqueness if the singular part of \( u_0 \) is a finite superposition of Dirac masses.
In terms of the behaviour of \( \varphi \) at infinity we give criteria to distinguish two
cases: either all solutions are function-valued for positive times (an instan-
taneous regularizing effect), or the singular parts of certain solutions persist
until some positive waiting time (in the linear case \( \varphi(u) = u \) this happens for
all times). In the latter case we describe the evolution of the singular parts.

1. Introduction

In this paper we consider the Cauchy problem
\[
\begin{cases}
  u_t + [\varphi(u)]_x = 0 & \text{in } \mathbb{R} \times (0, T) =: S \\
  u = u_0 & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\]
where \( T > 0, u_0 \) is a nonnegative finite Radon measure on \( \mathbb{R} \), and \( \varphi : [0, \infty) \to \mathbb{R}, \varphi(0) = 0 \),
is a Lipschitz continuous function (see assumption \((H_1)\)). Therefore, \( \varphi \)
grows at most linearly.

(a) Problem \((P)\) with a superlinear \( \varphi \) of the type \( \varphi(u) = u^p, p > 1 \) was studied in
\[8\], proving existence and uniqueness of nonnegative entropy solutions (see also \[8\]). By definition, in that paper the solution for positive times takes values in \( L^1(\mathbb{R}) \), although the initial data \( u_0 \) is a finite Radon measure. Interesting, albeit
sparse results concerning \((P)\) with \( \varphi \) at most linear at infinity can be found in
the pioneering paper \[10\], in which the same definition of Radon measure-valued
solutions used below (see equality \((4.8)\)) was proposed.

When \( \varphi(u) = Cu \ (C \in \mathbb{R}) \) problem \((P)\) is the Cauchy problem for the linear
transport equation,
\[
\begin{cases}
  u_t + Cu_x = 0 & \text{in } S \\
  u = u_0 & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\]

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whose solution is trivially the translated of $u_0$ along the lines $x = Ct + x_0$ ($x_0 \in \mathbb{R}$). In particular, the singular part $u_s(\cdot, t)$ of the solution is nonzero for $t > 0$ if and only if the same holds for $t = 0$.

It is natural to ask what happens if $\varphi$ is sublinear. To address this case we must consider solutions of problem $(P)$ which for $t > 0$ possibly are finite Radon measures on $\mathbb{R}$ as the initial data $u_0$. Therefore, throughout the paper we consider solutions of problem $(P)$ as maps from $[0, T]$ to the cone of nonnegative finite Radon measures on $\mathbb{R}$, which satisfy $(P)$ in the following sense: for a suitable class of test functions $\zeta$ there holds

$$\int_S \left[ u_t \zeta_t + \varphi(u) \zeta_x \right] \text{d}x \text{d}t + \int_0^T \langle u_s(\cdot, t), \zeta_\nu(\cdot, t) \rangle_R \text{d}t = -\langle u_0, \zeta(\cdot, 0) \rangle_R$$

(see Definition 3.2). Here the measure $u(t)$ is defined for a.e. $t \in (0, T)$, $u_t \in L^1(S)$ is the density of its absolutely continuous part, $(\cdot, \cdot)_R$ denotes the duality map, and

$$\zeta_\nu := \zeta_t + C_{\varphi} \zeta_x, \quad C_{\varphi} := \lim_{u \to \infty} \frac{\varphi(u)}{u}.$$

Measure-valued entropy solutions are defined similarly (see Definition 3.2).

We use an approximation procedure to construct measure-valued entropy solutions of problem $(P)$ (see Theorem 3.2). In addition, we prove that the singular part $u_s$ of an entropy solution of problem $(P)$ does not increase along the lines $x = x_0 + C_{\varphi} t$ (see Proposition 3.3). In particular, if $C_{\varphi} = 0$ the map $t \mapsto u_s(\cdot, t)$ is nonincreasing.

Concerning the case when $\varphi$ is sublinear, the following example is particularly instructive:

\begin{align*}
\left\{ \begin{array}{ll} 
 u_t + \left[ \varphi(u) \right]_x = 0 & \text{in } S \\
 u = \delta_0 & \text{in } \mathbb{R} \times \{0\}
\end{array} \right.
\end{align*}

(1.1)

with $S := \mathbb{R} \times (0, T)$, $T > 1$ and

\begin{equation}
\varphi(u) = \text{sgn } p \left[ (1 + u)^p - 1 \right] \quad (p < 1, p \neq 0).
\end{equation}

(1.2)

The function in (1.2) is increasing and concave, with $C_{\varphi} = 0$, and belongs to a class for which the constructed entropy solution of problem (1.1)-(1.2) is unique (see Theorem 3.12). Hence the following holds:

**Proposition 1.1.** (i) Let $p < 0$. Let $\xi(t)$ be defined by

$$\xi' = -\left( \frac{|p| \xi^{-1} \frac{\varphi'}{\varphi} - 1}{|p| \xi^{-1} \frac{\varphi'}{\varphi} - 1} \right) \text{ in } (1, T), \quad \xi(1) = 0.$$

Let $A := \{(x, t) \in S | 0 < x \leq |pt|, 0 \leq t \leq 1\} \cup \{(x, t) \in S | \xi(t) \leq x \leq |pt|, 1 < t \leq T\}$, and

\begin{equation}
\begin{split}
u_s(t) := \max \left\{ 1 - t, 0 \right\} \delta_0, \quad u_r(x, t) := \left[ \left( \frac{|pt| x^{-1}}{\varphi'} - 1 \right) \chi_A(x, t) \right] \quad ((x, t) \in S).
\end{split}
\end{equation}

Then $u = u_r + u_s$ is the unique constructed entropy solution of problem (1.1)-(1.2).

(ii) Let $0 < p < 1$. Let $\xi(t)$ be defined by

$$\xi' = \left( \frac{|p| \xi^{-1} \frac{\varphi'}{\varphi} - 1}{|p| \xi^{-1} \frac{\varphi'}{\varphi} - 1} \right) \text{ in } (0, T), \quad \xi(0) = 0.$$

If $B := \{(x, t) \in S | \xi(t) \leq x \leq |pt|, 0 < t \leq T\}$, then

\begin{equation}
\begin{split}
u(x, t) = u_r(x, t) := \left[ \left( \frac{|pt| x^{-1}}{\varphi'} - 1 \right) \chi_B(x, t) \right] \quad ((x, t) \in S)
\end{split}
\end{equation}

is the unique constructed entropy solution of problem (1.1)-(1.2).
Let us define the waiting time \( t_0 \in [0, T] \) for solutions \( u \) of \((P)\):

\[
(1.5) \quad t_0 := \inf \{ \tau \in (0, T] \mid u_s(\cdot, t) = 0, \ u_r(\cdot, t) \in L^\infty(\mathbb{R}) \text{ for a.e. } t \in (\tau, T) \}
\]

(by abuse of language, we call \( t_0 \) “waiting time” even if \( t_0 = T \)). Then by Proposition \ref{prop:1.1}

\[\ast\] Positive waiting times occur in problem \((1.1) - (1.2)\) if and only if \( p < 0 \).

More precisely, if \( p < 0 \) the singular part \( u_s(\cdot, t) \) persists until the waiting time \( t_0 = 1 \) at which it disappears, whereas for \( 0 < p < 1 \) the singular part vanishes for all \( t > 0 \), thus \( t_0 = 0 \) - an instantaneous regularizing effect. Instantaneous regularization also occurs if \( p > 1 \) (see \cite{18} and Remark \ref{rem:5.9}), whereas, as already remarked, in the linear case \( p = 1 \) there holds \( t_0 = T \) if \( u_{0s} \neq 0 \).

Since \( \varphi(u) = \sgn p(1 + u)p - 1 \) \( (p < 1, p \neq 0) \) is bounded if and only if \( p < 0 \), and \( C_{\varphi} = 0 \), statement \(\ast\) could be rephrased as follows:

**Proposition 1.2.** Positive waiting times occur in problem \((1.1)\) if and only if the map \( u \mapsto \varphi(u) - C_{\varphi}u \), with \( \varphi \) as in \((1.2)\), is bounded in \([0, \infty)\).

The above result is generalized to problem \((P)\) by Theorem \ref{thm:3.8} for functions \( \varphi \) which satisfy for \( u \) large a condition implying either concavity or convexity (see assumption \((H_1')\) and Remark \ref{rem:3.5}). The proof of Theorem \ref{thm:3.8} makes use of estimates of the density \( u_r \) of the solution of \((P)\), which are strongly reminiscent of the Aronson-Bénilan inequality for the porous medium equation (see Proposition \ref{prop:6.2}). The main results on the waiting time and the regularity of solutions of \((P)\) are collected in Subsection \ref{subsec:3.3}. The existence and an upper bound, in terms of \( \varphi \) and \( u_0 \), of a waiting time was already pointed out in \cite[Proposition 2.1]{10} (see also Theorem \ref{thm:3.8}-(ii)).

Another interesting feature of the solution of \((1.1) - (1.2)\) with \( p < 0 \) is that for \( t \in (0, 1) \) - i.e., as long as \( u_s(\cdot, t) > 0 \) - there holds

\[
\lim_{x \to -0^+} u_r(x, t) = \infty.
\]

Namely, the regular part \( u_r(\cdot, t) \) diverges when approaching from the right the point \( x_0 = 0 \) where \( u_s(\cdot, t) \) is concentrated. As we shall see below (see \ref{sec:3.25} - \ref{cor:3.20}), this property can be generalized to entropy solutions of a larger class of problems, characterized by the concavity/convexity property on \( \varphi \) mentioned before. In this class a generalized form of this property will also be used as a uniqueness criterion, provided that \( \varphi(u) - C_{\varphi}u \) is bounded in \([0, \infty)\) and \( u_{0s} \) is a finite superposition of Dirac masses (see Proposition \ref{prop:5.7} and Theorem \ref{thm:5.12}). In \cite{10} it was already observed that Kruzkov’s entropy inequalities do not guarantee the uniqueness of solutions (see also Remark \ref{rem:3.8} below), and the formulation of an additional uniqueness criterion was left as an open problem. This problem is addressed in a forthcoming paper where more general compatibility conditions are given, which ensure uniqueness also for non-convex or non-concave functions \( \varphi \) (see \cite{3}).

Apart from the intrinsic mathematical interest of problem \((P)\), it is worth pointing out its connection with a class of relevant models. Ion etching is a common technique for the fabrication of semiconductor devices, also relevant in other fields of metallurgy, in which the material to be etched is bombarded with an ion beam.
(see [15, 23, 24]). Mathematical modelling of the process leads to the Hamilton-Jacobi equation in one space dimension

\[
\begin{aligned}
(HJ) \quad & \begin{cases}
U_t + \varphi(U_x) = 0 & \text{in } \mathbb{R} \times (0, T) \\
U = U_0 & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\end{aligned}
\]

where \( U = U(x,t) \) denotes the thickness of the material and \( \varphi \) is bounded, non-convex and vanishing at infinity. Formal differentiation with respect to \( x \) suggests to describe the problem in terms of the unknown \( u := U_x \), which formally solves \((P)\) with \( u_0 = U_0' \). In this way discontinuous solutions of \((HJ)\) correspond to Radon measure-valued solutions of \((P)\) having a Dirac mass \( \delta_{x_0} \) concentrated at any point \( x_0 \) where \( U(\cdot, t) \) is discontinuous \( (t \in (0,T)) \). A rigorous justification of the above argument, relating discontinuous viscosity solutions of \((HJ)\) to Radon measure-valued entropy solutions of \((P)\), is to our knowledge an open problem (in this connection see [2, 13]).

Let us mention that a number of ideas used in the present paper go back to papers dealing with Radon measure-valued solutions of quasilinear parabolic problems, also of forward-backward type (in particular, see [4, 5, 6, 20, 22, 26]).

The paper is organized as follows. In Section 2 we recall several known results used in the sequel and introduce some notation. In Section 3 we present the main results of the paper. In Section 4 we introduce the approximation procedure needed for the construction of solutions. Sections 5-7 are devoted to the proofs of existence, qualitative properties and uniqueness of solutions.

2. Preliminaries

2.1. Function spaces and Radon measures. We denote by \( \mathcal{M}(\mathbb{R}) \) the Banach space of finite Radon measures on \( \mathbb{R} \), with norm \( \|\mu\|_{\mathcal{M}(\mathbb{R})} := |\mu|_1(\mathbb{R}) \). By \( \mathcal{M}^+(\mathbb{R}) \) we denote the cone of nonnegative finite Radon measures; if \( \mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}) \), we write \( \mu_1 \leq \mu_2 \) if \( \mu_2 - \mu_1 \in \mathcal{M}^+(\mathbb{R}) \). We denote the convex set of probability measures on \( \mathbb{R} \) by \( \mathcal{P}(\mathbb{R}) \subset \mathcal{M}^+(\mathbb{R}) \): \( \|\tau\|_{\mathcal{M}(\mathbb{R})} = \tau(\mathbb{R}) = 1 \) for \( \tau \in \mathcal{P}(\mathbb{R}) \).

We denote by \( C_c(\mathbb{R}) \) the space of continuous real functions with compact support in \( \mathbb{R} \). The space of the functions of bounded variation in \( \mathbb{R} \) is denoted by \( BV(\mathbb{R}) := \{u \in L^1(\mathbb{R}) \mid u' \in \mathcal{M}(\mathbb{R})\} \), where \( u' \) is the distributional derivative of \( u \). It is endowed with the norm \( \|u\|_{BV(\mathbb{R})} := \|u\|_{L^1(\mathbb{R})} + \|u'\|_{\mathcal{M}(\mathbb{R})} \). We say that \( u \in BV_{loc}(\mathbb{R}) \) if \( u \in BV(\Omega) \) for every open bounded subset \( \Omega \subset \mathbb{R} \).

The Lebesgue measure, either on \( \mathbb{R} \) or \( S := \mathbb{R} \times (0, T) \), is denoted by \( |\cdot| \). Integration with respect to the Lebesgue measure on \( \mathbb{R} \) or on \( S \) will be denoted by the usual symbols \( dx \), respectively \( dx dt \). A Borel set \( E \) is null if \( |E| = 0 \). The expression "almost everywhere", or shortly "a.e.", means "up to null sets". For every measurable function \( f \) defined on \( \mathbb{R} \) and \( x_0 \in \mathbb{R} \), we say that \( \text{ess lim}_{x \to x_0} f(x) = l \in \mathbb{R} \) if there is a null set \( E^* \subset \mathbb{R} \) such that \( f(x_n) \to l \) for any sequence \( \{x_n\} \subset \mathbb{R} \setminus (E^* \cup \{x_0\}) \).

We denote the duality map between \( \mathcal{M}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \) by \( \langle \mu, \rho \rangle_B := \int_{\mathbb{R}} \rho d\mu \). By abuse of notation, we extend \( \langle \mu, \rho \rangle_B \) to any \( \mu \)-integrable function \( \rho \). A sequence \( \{\mu_n\} \) converges strongly to \( \mu \) in \( \mathcal{M}(\mathbb{R}) \) if \( \|\mu_n - \mu\|_{\mathcal{M}(\mathbb{R})} \to 0 \) as \( n \to \infty \). A sequence \( \{\mu_n\} \) of (possibly not finite) Radon measures on \( \mathbb{R} \) converges weakly* to a (possibly not finite) Radon measure \( \mu \), \( \mu_n \rightharpoonup^* \mu \), if \( \langle \mu_n, \rho \rangle_B \to \langle \mu, \rho \rangle_B \) for all \( \rho \in C_c(\mathbb{R}) \). Similar definitions are used for (possibly not finite) Radon measures on \( \Omega \times (0, T) \) with \( \Omega \subset \mathbb{R} \).
Every $\mu \in \mathcal{M}(\mathbb{R})$ has a unique decomposition $\mu = \mu_{ac} + \mu_s$, with $\mu_{ac} \in \mathcal{M}(\mathbb{R})$ absolutely continuous and $\mu_s \in \mathcal{M}(\mathbb{R})$ singular with respect to the Lebesgue measure. We denote by $\mu_r \in L^1(\mathbb{R})$ the density of $\mu_{ac}$. Every function $f \in L^1(\mathbb{R})$ can be identified to a finite absolutely continuous Radon measure on $\mathbb{R}$; we shall denote this measure by the same symbol $f$ used for the function.

The restriction $\mu \upharpoonright E$ of $\mu \in \mathcal{M}(\mathbb{R})$ to a Borel set $E \subseteq \mathbb{R}$ is defined by $(\mu \upharpoonright E)(A) := \mu(E \cap A)$ for any Borel set $A \subseteq \mathbb{R}$. Similar notations are used for the spaces of finite Radon measures $\mathcal{M}(\Omega)$ with $\Omega \subseteq \mathbb{R}$, $\mathcal{M}(S)$ and $\mathcal{M}(S \times \mathbb{R})$, where $S := \mathbb{R} \times (0, T)$.

We shall use measures $u \in \mathcal{M}(S)$ which, roughly speaking, admit a parametrization with respect to the time variable:

**Definition 2.1.** We denote by $L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ the set of finite nonnegative Radon measures $u \in \mathcal{M}^+(S)$ such that for a.e. $t \in (0, T)$ there is a measure $u(\cdot, t) \in \mathcal{M}^+(\mathbb{R})$ with the following properties:

(i) if $\zeta \in C([0, T]; C_c(\mathbb{R}))$ the map $t \mapsto \langle u(\cdot, t), \zeta(\cdot, t) \rangle_{\mathbb{R}}$ belongs to $L^1(0, T)$ and

\begin{equation}
\langle u, \zeta \rangle_S = \int_0^T \langle u(\cdot, t), \zeta(\cdot, t) \rangle_{\mathbb{R}} dt ;
\end{equation}

(ii) the map $t \mapsto \|u(\cdot, t)\|_{\mathcal{M}(\mathbb{R})}$ belongs to $L^\infty(0, T)$.

Accordingly, we set

\[ \|u\|_{L^\infty(0, T; \mathcal{M}(\mathbb{R}))} := \text{ess sup}_{t \in (0, T)} \|u(\cdot, t)\|_{\mathcal{M}(\mathbb{R})} \quad \text{for } u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R})). \]

**Remark 2.1.** The definition implies that for all $\rho \in C_c(\mathbb{R})$ the map $t \mapsto \langle u(\cdot, t), \rho \rangle_{\mathbb{R}}$ is measurable, thus the map $u : (0, T) \to \mathcal{M}(\mathbb{R})$ is weakly* measurable (e.g., see [21] Section 6.7). For simplicity we prefer the notation $L^\infty(0, T; \mathcal{M}(\mathbb{R}))$ to the more correct one $L^{ac}_w(0, T; \mathcal{M}(\mathbb{R}))$ which is used in [21].

If $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$, also $u_{ac}, u_s \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ and, by (2.1),

\begin{equation}
\langle u_{ac}, \zeta \rangle_S = \int_S u_{ac} \zeta dx dt, \quad \langle u_s, \zeta \rangle_S = \int_0^T \langle u_s(\cdot, t), \zeta(\cdot, t) \rangle_{\mathbb{R}} dt
\end{equation}

if $\zeta \in C([0, T]; C_c(\mathbb{R}))$. One easily checks that for a.e. $t \in (0, T)$

\begin{equation}
u_{ac}(\cdot, t) = [u(\cdot, t)]_{ac}, \quad u_s(\cdot, t) = [u(\cdot, t)]_s, \quad u_r(\cdot, t) = [u(\cdot, t)]_r,
\end{equation}

where $[u(\cdot, t)]_r$ denotes the density of the measure $[u(\cdot, t)]_{ac}$: for $\rho \in C_c(\mathbb{R})$

\[ \langle [u(\cdot, t)]_{ac} , \rho \rangle_{\mathbb{R}} = \int_{\mathbb{R}} [u(\cdot, t)]_r \rho dx = \int_{\mathbb{R}} u_r(\cdot, t) \rho dx \quad \text{for a.e. } t \in (0, T). \]

In view of (2.2)–(2.3), we shall always identify the quantities which appear on either side of equalities (2.3).

For any $\mu \in \mathcal{M}(\mathbb{R})$ and $a \in \mathbb{R}$, the translated measure $T_a(\mu)$ is defined by

\[ \langle T_a(\mu), \rho \rangle_{\mathbb{R}} := \langle \mu, \rho_{-a} \rangle_{\mathbb{R}} \]

for any $\rho \in C_c(\mathbb{R})$, where $\rho_{-a}(x) := \rho(x + a)$ $(x \in \mathbb{R})$. Clearly, $T_a(\mu) \in \mathcal{M}(\mathbb{R})$ and

\[ [T_a(\mu)]_{ac} = T_a(\mu_{ac}), \quad [T_a(\mu)]_s = T_a(\mu_s). \]
2.2. Young measures. We recall the following result [2].

**Theorem 2.1.** Let \( \Omega \subseteq \mathbb{R}^N \) be Lebesgue measurable, let \( K \subseteq \mathbb{R} \) be closed, and let \( u_n : \Omega \rightarrow \mathbb{R} \) be a sequence of Lebesgue measurable functions such that

\[
\lim_{n \to \infty} |\{ x \in \Omega | u_n(x) \notin U \}| = 0
\]

for any open neighbourhood \( U \) of \( K \) in \( \mathbb{R} \). Then there exist a subsequence \( \{u_j\} \subseteq \{u_n\} \) and a family \( \{\tau_x\} \) of nonnegative measures on \( \mathbb{R} \), depending measurably on \( x \in \Omega \), such that:

(i) \( \|\tau_x\|_{\mathcal{M}(\mathbb{R})} := \int_{\mathbb{R}} d\tau_x \leq 1 \) for a.e. \( x \in \Omega \);

(ii) \( \text{supp} \tau_x \subseteq K \) for a.e. \( x \in \Omega \);

(iii) for every continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfying \( \lim_{|\xi| \to \infty} f(\xi) = 0 \) there holds

\[
f(u_j) \rightharpoonup f^* \quad \text{in} \quad L^\infty(\Omega),
\]

where for a.e. \( x \in \Omega \)

\[
f^*(x) := \langle \tau_x, f \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\xi) d\tau_x(\xi).
\]

Suppose further that \( \{u_j\} \) satisfies the boundedness condition

\[
\lim_{k \to \infty} \sup_j \{|x \in \Omega \cap B_R : |u_j(x)| \geq k \} = 0
\]

for every \( R > 0 \), where \( B_R := \{x \in \mathbb{R}^N ||x| < R\} \).

(iv) \( \tau_x \) is a probability measure for a.e. \( x \in \Omega \);

(v) given any measurable subset \( A \subseteq \Omega \) there holds

\[
f(u_j) \rightharpoonup f^* \quad \text{in} \quad L^1(A)
\]

for all continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \{f(u_j)\} \) is sequentially weakly compact in \( L^1(A) \).

Below we shall always refer to the family \( \{\tau_x\} \) of probability measures given by the previous theorem as the *disintegration of the Young measure* \( \tau \) (or briefly Young measure) associated to the sequence \( \{u_j\} \). We denote the set of Young measures on \( \Omega \times \mathbb{R}^N \) by \( \mathcal{Y}(\Omega; \mathbb{R}) \); in particular, \( \mathcal{Y}(S; \mathbb{R}) \) denotes the set of Young measures on \( S \times \mathbb{R} \) with \( S := \mathbb{R} \times (0, T) \).

**Remark 2.2.** (i) The argument used in the proof of Theorem 2.1 shows that, under hypothesis (2.5), the convergence in (2.6) holds true for Carathéodory functions \( f : A \times \mathbb{R} \to \mathbb{R} \) if \( \{f(\cdot, u_j)\} \) is sequentially weakly relatively compact in \( L^1(A) \).

(ii) Condition (2.5) is very weak. It is equivalent to the statement that for any \( R > 0 \) there is a continuous nondecreasing function \( g_R : [0, \infty) \to \mathbb{R} \), such that

\[
\lim_{\xi \to \infty} g_R(\xi) = \infty, \quad \sup_j \int_{\Omega \cap B_R} g_R(|u_j(x)|) \, dx < \infty.
\]

Therefore, Theorem 2.1 applies to bounded sequences \( \{u_j\} \) in \( L^1(\Omega) \) (in which case \( g_R(\xi) = \xi \)).

If \( \Omega \subseteq \mathbb{R}^N \) is bounded and \( \{u_j\} \) is a bounded but not uniformly integrable sequence in \( L^1(\Omega) \), it is possible to extract a uniformly integrable subsequence “by removing sets of small measure”. This is the content of the following “Biting Lemma” (e.g., see [10, 27] and references therein).
Theorem 2.2. Let \( \{u_n\} \) be a bounded sequence in \( L^1(\Omega) \), where \( \Omega \subset \mathbb{R}^N \) is a bounded open set. Moreover, let \( \{u_j\} \subseteq \{u_n\} \) and \( \{\tau_j\} \) be the subsequence and the Young measure given in Theorem 2.1. Then there exist a subsequence \( \{u_k\} \equiv \{u_{j_k}\} \subseteq \{u_j\} \) and a decreasing sequence of measurable sets \( E_k \subseteq \Omega \) of Lebesgue measure \( |E_k| \to 0 \), such that the sequence \( \{u_k \chi_{\Omega \setminus E_k}\} \) is uniformly integrable and

\[
u_k \chi_{\Omega \setminus E_k} \rightarrow Z := \int_{\mathbb{R}} \xi \, d\tau(\xi) \text{ in } L^1(\Omega),
\]

where \( Z \in L^1(\Omega) \) is called the barycenter of the disintegration \( \{\tau_j\} \).

3. Main results

Throughout the paper we assume that \( u_0 \in \mathcal{M}^*(\mathbb{R}) \). Concerning \( \varphi \), we always suppose that

\[(H_1) \quad \varphi \in C([0, \infty)), \quad \varphi(0) = 0, \quad \varphi' \in L^\infty(0, \infty), \quad \text{there exists } \lim_{u \to \infty} \frac{\varphi(u)}{u} =: C_{\varphi}.
\]

Hence there exists \( M > 0 \) such that

\[
|\varphi'(u)| \leq M, \quad |\varphi(u)| \leq Mu \quad \text{for a.e. } u > 0.
\]

3.1. Definition of solution. In the following definitions we denote by \( \zeta_\nu := \zeta_t + C_{\varphi} \zeta_x \) the derivative of any \( \zeta \in C^1(S) \) along the vector \( \tau \equiv (C_{\varphi}, 1) \).

Definition 3.1. By a solution of problem \((P)\) in the sense of Young measures we mean a pair \((u, \tau)\) such that:

(i) \( u \in L^\infty(0, T; \mathcal{M}^*(\mathbb{R})) \), \( \tau \in \mathcal{Y}(S; \mathbb{R}) \);

(ii) \( \text{supp } \tau_{(x,t)} \subseteq [0, \infty) \) for a.e. \((x, t) \in S\), and

\[
u_{x,t}(x, t) = \int_{[0, \infty)} \xi \, d\tau_{(x,t)}(\xi),
\]

where \( \tau_{(x,t)} \in \mathcal{P}(\mathbb{R}) \) is the disintegration of \( \tau \);

(iii) for all \( \zeta \in C^1([0, T]; C^1_{\tau}(\mathbb{R})) \) with \( \zeta(\cdot, T) = 0 \) in \( \mathbb{R} \) there holds

\[
\int_{S} \left[ u_{x,t} \zeta_x + \varphi_{\tau} \zeta_x \right] \, dx \, dt + \int_{0}^{T} \left[ u_{s}(\cdot, t), \zeta(\cdot, t) \right] \, dt = -\left( u_{0}, \zeta(\cdot, 0) \right)_{\mathbb{R}},
\]

where

\[
\varphi_{\tau}(x, t) := \int_{[0, \infty)} \varphi(\xi) \, d\tau_{(x,t)}(\xi) \quad \text{for a.e.} \quad (x, t) \in S.
\]

By an entropy solution of problem \((P)\) in the sense of Young measures we mean a solution in the sense of Young measures such that

\[
\int_{S} \left[ E_{\tau}(x, t) + F_{\tau}(x, t) \right] \, dx \, dt + C_{E} \int_{0}^{T} \left( u_{s}(\cdot, t), \zeta(\cdot, t) \right)_{\mathbb{R}} \, dt + C_{F} \int_{0}^{T} \left( u_{s}(\cdot, t), \zeta(\cdot, t) \right)_{\mathbb{R}} \, dt \geq -\int_{\mathbb{R}} E(u_{0}, \zeta(\cdot, 0)) \, dx - C_{E} \left( u_{0}, \zeta(\cdot, 0) \right)_{\mathbb{R}}.
\]

for all \( \zeta \) as above, \( \zeta \geq 0 \), and for every pair \((E, F)\), \( E, F : [0, \infty) \to \mathbb{R} \), such that

\[
E \text{ convex, } E', F' \in L^\infty(0, \infty), \quad F' = E' \varphi' \text{ in } (0, \infty),
\]

there exist \( \lim_{u \to \infty} \frac{E(u)}{u} =: C_{E}, \lim_{u \to \infty} \frac{F(u)}{u} =: C_{F} \).

In \(3.3\) for a.e. \((x, t) \in S\) we set

\[
E_{\tau}(x, t) := \int_{[0, \infty)} E(\xi) \, d\tau_{(x,t)}(\xi), \quad F_{\tau}(x, t) := \int_{[0, \infty)} F(\xi) \, d\tau_{(x,t)}(\xi).
\]
Entropy subsolutions (respectively supersolutions) of problem (P) in the sense of Young measures are defined by requiring that inequality (3.5) be satisfied for all \( \zeta \) and \((E, F)\) as above, with \( E \) nondecreasing (nonincreasing, respectively).

Observe that choosing \( E(u) = \pm u \) in the entropy inequality (3.5) plainly gives the weak formulation (3.3).

**Remark 3.1.** (i) By (3.3), (3.2) and (3.4),

\[
|\varphi^*(x, t)| \leq M \int_{[0, \infty)} |\varphi(x, t)| \, dx + M u_r(x, t) \quad \text{for a.e.} \quad (x, t) \in S.
\]

Since \( u_r \in L^\infty(0, T; L^1(\mathbb{R})) \), by (3.7) we have that \( \varphi^* \in L^\infty(0, T; L^1(\mathbb{R})) \).

(ii) By (3.3) the functions \( E, F \) have at most linear growth. Arguing as in (i), it follows that \( E^* \) and \( F^* \) belong to \( L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R})) \), and to \( L^\infty(0, T; L^1(\mathbb{R})) \) if \( E(0) = F(0) = 0 \).

**Definition 3.2.** A measure \( u \in L^\infty(0, T; M^*(\mathbb{R})) \) is called a solution of problem (P) if for all \( \zeta \in C^1([0, T]; C^1(\mathbb{R})) \), \( \zeta(\cdot, T) = 0 \) in \( \mathbb{R} \) there holds

\[
\int_S \left[ u_r \zeta_t + \varphi(u_r)\zeta_x \right] \, dx \, dt + \int_0^T \left\langle u_s(\cdot, t), \zeta_t(\cdot, t) \right\rangle_{\mathbb{R}} \, dt = -\left\langle u_0, \zeta(\cdot, 0) \right\rangle_{\mathbb{R}}.
\]

A solution of problem (P) is called an entropy solution, if for all \( \zeta \geq 0 \) as above and for all \((E, F)\) as in (3.6) it satisfies the entropy inequality

\[
\int_S \left[ E(u_r)\zeta_t + F(u_r)\zeta_x \right] \, dx \, dt + C_E \int_0^T \left\langle u_s(\cdot, t), \zeta_t(\cdot, t) \right\rangle_{\mathbb{R}} \, dt + \int_0^T C_F \left\langle u_s(\cdot, t), \zeta_t(\cdot, t) \right\rangle_{\mathbb{R}} \, dt \geq -\int_0^T E(u_r)\zeta(x, 0) \, dx - C_E \left\langle u_0, \zeta(\cdot, 0) \right\rangle_{\mathbb{R}}.
\]

Entropy subsolutions (respectively supersolutions) of problem (P) are defined by requiring (3.9) to be satisfied for all \( \zeta \) and \((E, F)\) as before, with \( E \) nondecreasing (nonincreasing, respectively).

A solution of problem (P) is also a solution in the sense of Young measures. Moreover, it follows from (3.1) that \( \varphi(u_r) \in L^\infty(0, T; L^1(\mathbb{R})) \). Similar remarks hold for entropy solutions, subsolutions and supersolutions.

**Remark 3.2.** (i) If \( C_\varphi = 0 \), equality (3.3) reads

\[
\int_S \left[ u \zeta_t + \varphi(u_r)\zeta_x \right] \, dx \, dt = -\left\langle u_0, \zeta(\cdot, 0) \right\rangle_{\mathbb{R}},
\]

whence \( u_t = -[\varphi(u_r)]_x \) in \( D'(S) \).

(ii) For the Kružkov entropies \( E(u) = |u - k| \), \( F(u) = \text{sgn}(u - k) [\varphi(u) - \varphi(k)] \) \((k \in [0, \infty))\) there holds \( C_E = 1 \), \( C_F = C_\varphi \). Then inequality (3.9) reads, for all \( k \in [0, \infty) \),

\[
\int_S \left[ |u_r - k| \zeta_t + \text{sgn}(u_r - k) [\varphi(u_r) - \varphi(k)]\zeta_x \right] \, dx \, dt + \int_0^T \left\langle u_s(\cdot, t), \zeta_t(\cdot, t) \right\rangle_{\mathbb{R}} \, dt \geq -\int \left| u_0 - k \right| \zeta(x, 0) \, dx - \left\langle u_0, \zeta(\cdot, 0) \right\rangle_{\mathbb{R}}.
\]

The following proposition states that for any solution of (P) in the sense of Young measures the map \( t \mapsto u(t) \), possibly redefined in a null set, is continuous up to \( t = 0 \) with respect to the weak* topology of \( M^*(\mathbb{R}) \). In particular, it explains in which sense the initial condition is satisfied.
Proposition 3.1. Let \((H_1)\) be satisfied, let \((u, \tau)\) be a solution of problem \((P)\) in the sense of Young measures, and let \(\rho \in C_c(\mathbb{R})\). Then
\[
\text{ess lim}_{t \to t_0^+}(u(\cdot,t),\rho)_\mathbb{R} = (u_0,\rho)_\mathbb{R},
\]
\[
\text{ess lim}_{t \to t_0^-}(u(\cdot,t),\rho)_\mathbb{R} = (u(\cdot,t_0),\rho)_\mathbb{R}
\]
for a.e. \(t_0 \in (0,T)\).

The map \(t \mapsto u(t)\) has a representative defined for all \(t \in [0,T]\), such that
\[
\lim_{t \to t_0^+} (u(\cdot,t),\rho)_\mathbb{R} = (u(\cdot,t_0),\rho)_\mathbb{R}
\]
for all \(t_0 \in [0,T]\).

3.2. Existence and monotonicity. The existence of solutions is proven by an approximation procedure. If \(u_0 \in \mathcal{M}^+(\mathbb{R})\), there exist \(u_{0n} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\) such that
\[
u_{0n} \geq 0 \quad \text{in } \mathbb{R}, \quad \|\nu_{0n}\|_{L^1(\mathbb{R})} \leq \|\nu_0\|_{\mathcal{M}(\mathbb{R})},
\]
\[
\nu_{0n} \rightharpoonup \nu_{0}, \quad \nu_{0n} \to \nu_{0r} \quad \text{a.e. in } \mathbb{R}, \quad \|\nu_{0n} - \nu_{0r}\|_{L^\infty[(\mathbb{R}\setminus\text{supp }u_0), \mathbb{R}]} \to 0
\]
(e.g., see [22, Lemma 4.1]). Consider the approximating problem
\[(P_n) \quad \left\{ \begin{array}{ll}
\nu_{nt} + [\varphi(u_n)]_x = 0 & \text{in } S \\
\nu_n = u_{0n} & \text{in } \mathbb{R} \times \{0\} \quad (n \in \mathbb{N}).
\end{array} \right.
\]
Let us recall the definition of entropy solution of problem \((P_n)\) (e.g., see [23]).

Definition 3.3. A function \(u_n \in L^\infty(0,T;L^1(\mathbb{R})) \cap L^\infty(S)\) is called an entropy solution of problem \((P_n)\) if for every \(\zeta \in C^1([0,T];C^1_b(\mathbb{R}))\), \(\zeta(\cdot, T) = 0\) in \(\mathbb{R}\), \(\zeta \geq 0\) and for any couple \((E,F)\) with \(E\) convex, \(F' = E'\varphi'\) there holds
\[
\iint_S [E(u_n)\zeta_t + F(u_n)\zeta_x] \, dx dt \geq -\int_\mathbb{R} E(u_{0n})\zeta(x,0) \, dx.
\]
Entropy solutions are weak solutions: if \(\zeta \in C^1([0,T];C^1_b(\mathbb{R}))\), \(\zeta(\cdot, T) = 0\) in \(\mathbb{R}\)
\[
\iint_S [u_n\zeta_t + \varphi(u_n)\zeta_x] \, dx dt + \int_\mathbb{R} u_{0n}\zeta(x,0) \, dx = 0.
\]
Studying the limiting points of the sequence \(\{u_n\}\) we shall prove the following result.

Theorem 3.2. (i) Let \((H_1)\) be satisfied. Then problem \((P)\) has a solution \(u\), which is obtained as a limiting point of the sequence \(\{u_n\}\) of entropy solutions to problems \((P_n)\). In addition, \(u\) is an entropy solution of problem \((P)\) in the sense of Young measures.

(ii) Let \((H_1)\) and the following assumption:
\[
\varphi \in C^1([0,\infty)), \quad \text{and for every } \bar{a} > 0 \text{ there exist } a, b \geq 0, \ a + b > 0 \quad \text{such that } \varphi' \text{ is strictly monotone in } [\bar{a} - a, \bar{a} + b],
\]
be satisfied. Then \(u\) is an entropy solution of problem \((P)\).

Hypothesis \((\ref{hypothesis})\) fails if for example \(\varphi\) is affine in an interval \((a,b) \subset (0,\infty)\). In that case Proposition \(5.6\, (iii)\), which characterizes the limiting Young measure, gives some additional information.

The following proposition shows that the singular part of an entropy subsolution of \((P)\) does not increase along the lines \(x = C_\varphi t + x_0\).
Proposition 3.3. Let \((H_1)\) be satisfied.
(i) Let \(u\) be an entropy subsolution of problem \((P)\) in the sense of Young measures. Then
\begin{equation}
(3.19) \quad u_s(\cdot, t_2) \leq T_{C, t} u_s(\cdot, t_1) \quad \text{in } M(\mathbb{R}) \quad \text{for a.e. } 0 \leq t_1 \leq t_2 \leq T .
\end{equation}
In particular,
\begin{equation}
(3.20) \quad u_s(\cdot, t) \leq T_{C, t} u_0 \quad \text{in } M(\mathbb{R}) \quad \text{for a.e. } t \in (0, T),
\end{equation}
whence \(\|u_s(\cdot, t)\|_{M(\mathbb{R})} \leq \|u_0\|_{M(\mathbb{R})} \) for a.e. \(t \in (0, T)\).

(ii) Let \(u\) be a solution of problem \((P)\). Then there is conservation of mass:
\begin{equation}
\|u(\cdot, t)\|_{M(\mathbb{R})} = \|u_0\|_{M(\mathbb{R})} \quad \text{for a.e. } t \in (0, T).
\end{equation}

The linear case \(\varphi(u) = u\) shows that equality may hold in (3.19). Moreover, if \(C_\varphi = 0\), it follows from (3.19) that the map \(t \mapsto u_s(\cdot, t)\) is nonincreasing.

3.3. Waiting time and regularity. It is convenient to distinguish two cases: \(C_\varphi = 0\) (sublinear growth at infinity) and \(C_\varphi \neq 0\) (linear growth at infinity), with \(C_\varphi\) defined by \((H_1)\).

3.3.1. Sublinear growth. Beside \((H_1)\), we will use the following assumption:
\begin{align}
(H_2) & \quad \begin{cases}
\varphi \in C^\infty([0, \infty)), C_\varphi = 0; \\
\text{there exist } H \geq -1, K \in \mathbb{R} \text{ such that}
\end{cases} \\
\varphi''(u) [H \varphi(u) + K] \leq -[\varphi'(u)]^2 < 0 \text{ for all } u \in [0, \infty).
\end{align}

By \((H_2)\) the map \(u \mapsto \varphi''(u) [H \varphi(u) + K]\) is strictly negative and continuous in \([0, \infty)\), hence two cases are possible: either (a) \(H \varphi + K > 0\), \(\varphi'' < 0\), or (b) \(H \varphi + K < 0\), \(\varphi'' > 0\) in \([0, \infty)\). In case (a) there holds \(\varphi' > 0\) in \([0, \infty)\), since \(\varphi'' < 0\) and there exists \(\lim_{u \to \infty} \varphi'(u) = C_\varphi = 0\). Similarly, in case (b) there holds \(\varphi' < 0\) in \([0, \infty)\). In particular, in both cases \((H_2)\) implies (3.18). Moreover, if also \((H_1)\) holds, thus \(\varphi(0) = 0\), there holds \(H \varphi + K > 0\) in \([0, \infty)\) if and only if \(H > 0\).

Remark 3.3. The following examples show that all values of \(H \geq -1\) may occur in \((H_2)\):
\begin{align}
\left \{ \begin{array}{ll}
\varphi(u) = \text{sgn} p [(1 + u)^p - 1] & (p < 1, p \neq 0) \\
\varphi(u) = 1 - e^{-\alpha u} & (\alpha > 0)
\end{array} \right.
\quad & \Rightarrow \quad H = \begin{cases}
\frac{p}{1-p} & (-1, 0) \cap (0, \infty),
\end{cases} \quad K = |H|,
\end{align}
\begin{align}
\left \{ \begin{array}{ll}
\varphi(u) = \log(1 + u), \quad \text{or} \quad \varphi(u) = 1 - \frac{1}{\log(e + u)}
\end{array} \right.
\quad & \Rightarrow \quad H = -1, \quad K = 1.
\end{align}

The following property of constructed entropy solutions plays an important role as a uniqueness criterion (see its generalized form given by Proposition 3.4 and Theorem 3.12 below).

Proposition 3.4. Let \((H_1)-(H_2)\) be satisfied, and let \(\varphi\) be bounded in \([0, \infty)\). Then every entropy solution \(u\) of problem \((P)\) given by Theorem 3.2 satisfies for a.e. \(t \in (0, T)\) and all \(x_0 \in \text{supp } u_s(\cdot, t)\):
\begin{equation}
(3.21) \quad \text{ess lim }_{x \to x_0^+} u_r(x, t) = \infty \quad \text{if } \varphi' > 0 \text{ in } [0, \infty),
\end{equation}
\begin{equation}
\text{ess lim }_{x \to x_0^-} u_r(x, t) = \infty \quad \text{if } \varphi' < 0 \text{ in } [0, \infty).
\end{equation}
Theorem 3.5. (i) Let \((H_1)\) be satisfied, let \(u_0, (\{x_0\}) > 0\) for some \(x_0 \in \mathbb{R}\) and let \(u\) be a solution of problem \((P)\). If \(\varphi\) is bounded in \((0, \infty)\), then the waiting time \(t_0\) defined by (1.5) satisfies

\[
(3.22) \quad t_0 \geq \min \left\{ T, \frac{u_0(x_0)}{|\varphi|_{L^\infty(0, \infty)}} \right\} > 0.
\]

(ii) Let \((H_1)-(H_2)\) be satisfied, and let \(u\) be the entropy solution of problem \((P)\) given by Theorem 3.3.

(a) If \(\varphi\) is bounded in \((0, \infty)\), and moreover \(H > -1, |K| < \lim_{u \to \infty} |\varphi(u)| =: \gamma\), then

\[
(3.23) \quad t_0 \leq \min \left\{ T, \frac{(H + 1) \|u_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - |K|} \right\}.
\]

(b) If \(\varphi\) is unbounded in \((0, \infty)\), then \(t_0 = 0\).

Remark 3.4. Concerning the estimates in (3.22) and (3.23), it is worth considering the case in which \(u_0 = \delta_0\) and \(\varphi(u) = 1 - (1 + u)^p, p < 0\). By explicit calculations, in Proposition 1.1 we show that in this case the waiting time defined in (1.5) is \(t_0 = 1\). Hence in this case estimates (3.22), (3.23) are sharp, since

\[
\delta_0(0) = 1 \quad \text{and} \quad \frac{(H + 1) \|\delta_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - |K|} = \frac{(p/(1 - p) + 1) \|\delta_0\|_{\mathcal{M}(\mathbb{R})}}{1 + p/(1 - p)} = 1.
\]

Remark 3.5. In part (ii) of Theorem 3.5 it is enough to require condition \((H_2)\) for large values of \(u\). More precisely (see Remark 3.6), Theorem 3.5(ii) remains valid if instead of \((H_2)\) for some \(k > 0\) there holds:

\[
(H_{2,k}) \quad \text{the function } \varphi_k : [0, \infty) \to \mathbb{R} \quad \text{satisfies } (H_2).
\]

In this connection, observe that the conditions \(H > -1\) and \(|K| < \lim_{u \to \infty} |\varphi(u)|\) exclude the function \(\varphi(u) = 1 - e^{-u}\). The same conditions also exclude the function \(\varphi(u) = 1 - \frac{1}{\log(e+u)}\), where \(K = 1 = \gamma\). However, in this case we can use hypothesis \((H_{2,k})\) for \(k > 0\), which is satisfied with \(H = 0\) and \(K = \log^{-2}(e+k) < \gamma_k = \log^{-1}(e+k)\).

Let us finally mention the following regularization result.

Proposition 3.6. Let \((H_1)-(H_2)\) be satisfied, and let \(\varphi\) be bounded in \([0, \infty)\). Then for a.e. \(t \in (0, T)\) supp \(u_\ast(t)\) is a null set.

Remark 3.6. It suffices to prove Proposition 3.6. Theorem 3.5 and Proposition 3.6 by assuming \(\varphi'' < 0\) in \((H_2)\) (hence, \(K > 0\) by \((H_2)\) and the assumption \(\varphi(0) = 0\). Otherwise it is easily seen that, if \(u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))\) is a solution of problem \((P)\), the map \(\hat{u}\) defined by setting

\[
(\hat{u}, \hat{\varphi})_{S} := \int_{0}^{T} (u(\cdot, t), \varphi(\cdot, t))_{\mathbb{R}} dt
\]

for every \(\zeta \in C([0, T]; \mathcal{C}_c(\mathbb{R}))\) is a solution of the problem

\[
(3.24) \begin{cases} 
\hat{u}_t + [\hat{\varphi}(\hat{u})]_{x} = 0 & \text{in } S \\
\hat{u} = u_0 & \text{in } \mathbb{R} \times \{0\}.
\end{cases}
\]

Here \((\hat{u}_0, \rho)_{\mathbb{R}} := (u_0, \rho(-\cdot))_{\mathbb{R}}\) for all \(\rho \in \mathcal{C}_c(\mathbb{R})\), and the function \(\hat{\varphi} := -\varphi\) satisfies \((H_2)\) with \(\hat{K} := -K\). The same holds for entropy solutions.
3.3.2. Linear growth. Let \( \varphi \) satisfy the following assumption:

\[
(H_2') \quad \begin{cases} 
\varphi \in C^\infty([0, \infty)); \
\varphi''(u) \{ H[\varphi(u) - C_\varphi u] + K \} \leq \frac{1}{2} [\varphi'(u) - C_\varphi]^2 < 0 \text{ for all } u \in [0, \infty) 
\end{cases}
\]

(observe that \((H_2')\) reduces to \((H_2)\) if \(C_\varphi = 0\). If \((H_2')\) holds, the function \(\tilde{\varphi} := \varphi(u) - C_\varphi u\) satisfies \((H_2)\) since \(C_\varphi = 0\).

**Remark 3.7.** It is easily seen that, if \(u\) is a solution (respectively, an entropy solution) of problem \((P)\), then \(v \in L^\infty(0, T; \mathcal{M}^*(\Omega))\) defined by

\[
v(s, t) = T_{h}(u(s, t)) \quad \text{in } \mathcal{M}(\mathbb{R})
\]

for any \(h \in \mathbb{R}\) is a solution (respectively, an entropy solution) of \((P)\) with \(u_0 := T_{-h}(u_0)\). Similarly, \(\tilde{u}(s, t) := T_{-\tilde{c}_\varphi}(u(s, t))\) is a solution (respectively, an entropy solution) of problem \((2.24)\) with \(\tilde{u}_0 = u_0\) and \(\tilde{\varphi}(u) = \varphi(u) - C_\varphi u\).

By Remark 3.7 the above results for the case \(C_\varphi = 0\) can be generalized as follows.

**Proposition 3.7.** Let \((H_1) - (H_2')\) be satisfied, and let \(u \mapsto \varphi(u) - C_\varphi u\) be bounded in \((0, \infty)\). Then every entropy solution \(u\) of problem \((P)\) given by Theorem 3.2 satisfies for a.e. \(t \in (0, T)\) and all \(x_0 \in \text{supp} u_0(s, t)\)

\[
\text{ess lim}_{x \to x_0} u_x(x + C_\varphi t, t) = \infty \quad \text{if } \varphi' > C_\varphi \text{ in } [0, \infty),
\]

\[
\text{ess lim}_{x \to x_0} u_x(x + C_\varphi t, t) = \infty \quad \text{if } \varphi' < C_\varphi \text{ in } [0, \infty).
\]

**Theorem 3.8.** (i) Let \((H_1)\) be satisfied, let \(u_{0_\tilde{a}}(\{x_0\}) > 0\) for some \(x_0 \in \mathbb{R}\) and let \(u\) be a solution of problem \((P)\). If \(u \mapsto \varphi(u) - C_\varphi u\) is bounded in \((0, \infty)\), then

\[
t_0 \geq \min \left\{ T, \frac{u_{0_\tilde{a}}(\{x_0\})}{\| \varphi - C_\varphi u \|_{L^\infty(0, \infty)}} \right\} > 0.
\]

(ii) Let \((H_1)\) and \((H_2')\) be satisfied, and let \(u\) be the entropy solution of problem \((P)\) given by Theorem 3.2

(a) Let \(u \mapsto \varphi(u) - C_\varphi u\) be bounded in \((0, \infty)\). If \(H > -1\) and \(|K| < \lim_{u \to \infty} |\varphi(u) - C_\varphi u| = \tilde{\gamma}\), then

\[
t_0 \leq \min \left\{ T, \frac{(H + 1) \| u_0 \|_{\mathcal{M}(\mathbb{R})}}{\tilde{\gamma} - |K|} \right\}.
\]

(b) Let \(u \mapsto \varphi(u) - C_\varphi u\) be unbounded in \((0, \infty)\). Then \(t_0 = 0\).

Again, Theorem 3.8 (ii) remains valid if for some \(k > 0\) the function \(\varphi_k\) defined in Remark 3.2 satisfies \((H_2')\).

**Proposition 3.9.** Let \((H_1) - (H_2')\) be satisfied, and let \(u \mapsto \varphi(u) - C_\varphi u\) be bounded in \((0, \infty)\). Then for a.e. \(t \in (0, T)\) \(\text{supp} u_x(t)\) is a null set.

3.4. **Uniqueness.** In connection with equality \((3.11)\) observe that, if \(u_{0_\tilde{a}} \neq 0\) and the waiting time \(t_0\) is equal to 0, the map \(t \mapsto u(s, t)\) is not continuous at \(t = 0\) in the strong topology of \(\mathcal{M}(\mathbb{R})\) (otherwise we would have \(\lim_{t \to 0^+} \| u_x(s, t) \|_{\mathcal{M}(\mathbb{R})} = 0 = \| u_0 \|_{\mathcal{M}(\mathbb{R})}\); a contradiction). Instead, continuity along the lines \(x = x_0 + C_\varphi t\) may occur if the waiting time \(t_0\) is positive:
Proposition 3.10. Let \((H_1)\) be satisfied. Let \(u \mapsto \varphi(u) - C u\) be bounded in \((0, \infty)\), and let \(u_0\) satisfy

\[
(3.27) \quad u_0 = \sum_{i=1}^{N} c_i \delta_{x_i} \quad \text{with} \ c_i \in [0, \infty), \ l = 1, \ldots, N \ \text{for some} \ N \in \mathbb{N}.
\]

(i) If condition (3.13) holds, every entropy solution \(u\) of problem \((P)\) given by Theorem 3.12 (ii) satisfies

\[
(3.28) \quad \text{ess lim}_{t \to 0^+} \| T_{C u} (u(\cdot, t)) - u_0 \|_{\mathcal{M}(\mathbb{R})} = 0.
\]

(ii) All entropy solutions \(u\) of problem \((P)\) satisfy \(T_{C u} (u(\cdot, t)) \in C([0, T]; \mathcal{M}(\mathbb{R}))\).

Let us mention that the above statement (ii) holds for any \(u_0 \in \mathcal{M}^+(\mathbb{R})\), if \(\varphi\) satisfies \((H_1) - (H_2')\) (see Proposition 3.2).

The following uniqueness result will be proven in Section 7.

Theorem 3.11. Let \((H_1)\) be satisfied and let \(u \mapsto \varphi(u) - C u\) be bounded and monotonic in \((0, \infty)\). Let \(u_0\) satisfy (3.27). Then there exists at most one entropy solution \(u\) of problem \((P)\) which satisfies either (3.25) or (3.26), and the condition

\[
(3.29) \quad \text{ess lim}_{t \to 0^+} \| u_r(\cdot, t) - u_0 \|_{L^1(\mathbb{R})} = 0.
\]

By Propositions 3.7, 3.10 and Theorem 3.11 we have the following existence and uniqueness result (observe that \((H_2')\) implies (3.13)).

Theorem 3.12. Let \((H_1) - (H_2')\) be satisfied, and let \(u \mapsto \varphi(u) - C u\) be bounded in \((0, \infty)\). Let \(u_0\) satisfy (3.27). Then there exists a unique entropy solution of problem \((P)\) which satisfies (3.25) or (3.26).

Remark 3.8. Conditions (3.25) - (3.26) in Theorem 3.12 cannot be omitted. In fact, there exist entropy solutions of problem \((P)\) which do not satisfy either (3.25) or (3.26), depending on \(\varphi\). Therefore, by Proposition 3.7 they are different from those given by Theorem 3.2, thus uniqueness fails.

For example, let \(u_0\), \(u_0_s \neq 0\) and \(u_0_r \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\). Let \(u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))\) be defined

\[
(3.30) \quad u(\cdot, t) := u_r(\cdot, t) + T_{C u} (u_0_r)
\]

for a.e. \(t \in (0, T)\), where \(u_r \in C([0, T]; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R})\) is the unique entropy solution of problem \((P)\) with \(u_0\) replaced by \(u_0_r\). Since \(u(\cdot, 0) = u_r(\cdot, 0) + u_0_s = u_0_r + u_0_s = u_0\), one easily checks that (3.8) - (3.10) are satisfied, thus \(u\) is an entropy solution of \((P)\). On the other hand \(u_r \in L^\infty(S)\), so \(u_r(\cdot, t) \in L^\infty(\mathbb{R})\) for a.e. \(t \in (0, T)\) and (3.25) - (3.26) fails.

Remark 3.9. If \(u \mapsto \varphi(u) - C u\) is unbounded and satisfies assumptions \((H_1) - (H_2')\), by [18] Theorem 1.1 and Theorem 3.8 for every \(u_0 \in \mathcal{M}^+(\mathbb{R})\) there exists a unique entropy solution of problem \((P)\) with waiting time \(t_0\) equal to 0. In fact, every entropy solution \(u\) given by Theorem 3.8 is a solution according to [18]. This follows if we show that

\[
(3.31) \quad u = u_r \in L^\infty(\mathbb{R} \times (\tau, T)) \quad \text{for every} \ \tau > 0
\]

and \(\text{ess lim}_{t \to 0^+} u(\cdot, t) = u_0\) narrowly in \(\mathcal{M}(\mathbb{R})\), i.e., \(\text{ess lim}_{t \to 0^+} \{u(\cdot, t), \rho\} = \{u_0, \rho\} \) for all bounded \(\rho \in C(\mathbb{R})\). The latter follows from (3.11) and Proposition 3.3(ii) (see [18] Proposition 2, p. 38).

To prove (3.31) we fix \(\tau > 0\). By (3.14) we may assume that \(u_r(\cdot, \tau) \in L^\infty(\mathbb{R})\) and \(u(\cdot, t) = u_r(\cdot, t)\) for all \(t \geq \tau\). By standard approximation arguments, we may substitute in the entropy inequality (3.9) \(E(u) = \int_{\mathbb{R}} (s-k_r)_+ \), with \(k_r = \|u_r(\cdot, \tau)\|_{L^\infty(\mathbb{R})}\), and \(\zeta(x, t) = \chi_{[\tau, t]}(t)\). Hence \(\int_{\mathbb{R}} [u_r(\cdot, t) - k_r]_+ \, dx \leq 0\) for a.e. \(t \geq \tau\) and (3.31) follows.
4. Approximating problems

In this section we consider problem \((P_n)\). Let \(u_{0n} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\) satisfy \((5.14)\) and let \(\{u_{0n}^\varepsilon\} \subseteq C_c^\infty(\mathbb{R})\), \(u_{0n}^\varepsilon \geq 0\) be any sequence such that

\[
\|u_{0n}^\varepsilon\|_{L^1(\mathbb{R})} \leq \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{M(\mathbb{R})}, \quad \|u_{0n}^\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})},
\]

(4.1)

\[
\begin{align*}
&u_{n}^\varepsilon \rightharpoonup u_{0n} \text{ in } L^1(\mathbb{R}), \quad u_{n}^\varepsilon \rightharpoonup u_{0n} \text{ in } L^\infty(\mathbb{R}).
\end{align*}
\]

(4.2)

Let \(\eta \in C_c^\infty(\mathbb{R})\) be a standard mollifier, let \(\eta_{\varepsilon}(u) := \frac{1}{\varepsilon} \eta\left(\frac{u}{\varepsilon}\right)\) for \(\varepsilon > 0\), and set

\[
\varphi_{\varepsilon}(u) := \left(\eta_{\varepsilon} * \overline{\varphi}\right)(u) - \left(\eta_{\varepsilon} * \overline{\varphi}\right)(0) = \int_R \eta_{\varepsilon}(u-v) \overline{\varphi}(v) dv - \int_R \eta_{\varepsilon}(-v) \overline{\varphi}(v) dv \quad (u \in \mathbb{R})
\]

(here \(\overline{\varphi}(u) = \varphi(u)\) for \(u \geq 0\) and \(\overline{\varphi}(u) = 0\) for \(u < 0\)). The regularized problem associated with \((P_n)\),

\[
\begin{align*}
\left\{ \begin{aligned}
u_{nt}^\varepsilon + [\varphi_{\varepsilon}(u_{n}^\varepsilon)]_x &= \varepsilon u_{nxx} & \text{in } S \\
u_{n}^\varepsilon &= \varepsilon u_{0n} & \text{in } \mathbb{R} \times \{0\}
\end{aligned} \right.
\]

(\(Q_n^\varepsilon\))

(4.3)

(4.4)

(4.5)

(where \(\varepsilon > 0\), \(n \in \mathbb{N}\), has a unique strong solution \(u_{n}^\varepsilon \in C([0,T]; H^2(\mathbb{R})) \cap L^\infty(S), \quad u_{n}^\varepsilon \in L^2(S)\) (e.g., see [13]). Some properties of the family \(\{u_{n}^\varepsilon\}\) are collected in the following lemmata. Up to minor changes the proof is standard (e.g., see [9]), thus is omitted.

**Lemma 4.1.** Let \(u_{n}^\varepsilon\) be the solution of problem \((Q_n^\varepsilon)\). Then for every \(n \in \mathbb{N}\) and \(\varepsilon > 0\)

\[
u_{n}^\varepsilon \geq 0 \quad \text{in } S, \quad \|u_{n}^\varepsilon\|_{L^\infty(S)} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})},
\]

\[
\int_R u_{n}^\varepsilon(x,t) \, dx = \int_R u_{0n}^\varepsilon(x) \, dx \quad (t \in (0,T)),
\]

(4.3)

\[
\sup_{t \in (0,T)} \|u_{n}^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{M(\mathbb{R})},
\]

(4.4)

\[
\sup_{t \in (0,T)} \|u_{n}^\varepsilon(\cdot + h,t) - u_{n}^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_{0n}(\cdot + h) - u_{0n}\|_{L^1(\mathbb{R})} \quad \text{for any } h \in \mathbb{R}.
\]

(4.5)

**Lemma 4.2.** Let \(\varphi\) satisfy \((3.1)\). Then there exists \(C > 0\), which only depends on \(\|u_0\|_{M(\mathbb{R})}\), such that for all \(n \in \mathbb{N}\), \(\varepsilon \in (0,1)\) and \(p \in (0,1)\)

\[
\varepsilon \int_S (1 + u_{n}^\varepsilon)^{p-2} (u_{nxx}^\varepsilon)^2 \, dx dt \leq \frac{C}{p(1-p)}.
\]

(4.6)

**Proof.** Let \(U \in C^2([0,\infty)), U' \geq 0\ in \ (0,\infty),\) and set

\[
\Theta_{U,\varepsilon}(u) := \int_0^u U'(s) \varphi'(s) \, ds + \theta_U \quad (\theta_U \in \mathbb{R}).
\]

(4.7)

Multiplying the first equation in \((Q_n^\varepsilon)\) by \(U'(u_{n}^\varepsilon)\) gives

\[
[U(u_{n}^\varepsilon)]_x + \left[\Theta_{U,\varepsilon}(u_{n}^\varepsilon)\right]_x = \varepsilon [U(u_{n}^\varepsilon)]_{xx} - \varepsilon U''(u_{n}^\varepsilon)(u_{nxx}^\varepsilon)^2 \quad \text{in } S.
\]

(4.8)

Hence for all \(\zeta \in C^1([0,T]; C^2(\mathbb{R}))\)

\[
\varepsilon \int_S U''(u_{n}^\varepsilon)(u_{nxx}^\varepsilon)^2 \zeta \, dx dt + \int_R U(u_{n}^\varepsilon(x,T))\zeta(x,T) \, dx = \int_R U(u_{0n})\zeta(x,0) \, dx + \int_S \{U(u_{n}^\varepsilon) \zeta_t + \Theta_{U,\varepsilon}(u_{n}^\varepsilon) \zeta_x + \varepsilon U(u_{n}^\varepsilon) \zeta_{xx}\} \, dx dt.
\]

(4.9)
By (3.1) and the definition of the function $\varphi_\varepsilon$, for all $u \geq 0$
\begin{equation}
|\Theta_{U,\varepsilon}(u)| \leq \int_0^u U'(s)|\varphi_\varepsilon'(s)| \, ds + |\theta_U| \leq M \left[ U(u) - U(0) \right] + |\theta_U|.
\end{equation}
Choose $\theta_U = 0$, $U(u) = (1 + u)^p - 1$, with $p \in (0, 1)$, and
\[\zeta = \rho_k := \chi_{[\varepsilon/2,\varepsilon]} + \rho(-k)\chi_{(\varepsilon/2-k,\varepsilon-k)} + \rho(+k)\chi_{(-k+1/2,\varepsilon-k)} \quad (k \in \mathbb{N}),\]
with any $\rho \in C^2_c((-1,1))$ such that $\rho(0) = 1$, $0 \leq \rho \leq 1$ and the derivatives $\rho'$, $\rho''$ vanish at $\{0\}$. Then $0 \leq U(u) \leq u$ for $u \geq 0$ and, by (4.1), (4.9) and (4.10),
\[\varepsilon \rho(1-p) \int_S (1 + u_n)^{p-2}(u_n^*)^2 \rho_k \, dx \leq \int u_n^*(x) \, dx + \int_S \{M u_n^*|\rho_k'| + \varepsilon u_n^*|\rho_k''|\} \, dx \leq \left\{ 1 + (M + 1)T \right\} \|\varphi_\varepsilon\|_{C^2((-1,1))} =: C\]
for all $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$. Passing to the limit $k \to \infty$ we obtain (4.6). \qed

\textbf{Lemma 4.3.} Let $\varphi$ satisfy (3.1) and let $U \in C^2([0,\infty))$ be such that
\begin{equation}
|U''(u)| \leq K (1 + u)^{p-2} \quad \text{for all } u \in [0,\infty), \text{ for some } K \geq 0 \text{ and } p \in (0, 1).
\end{equation}
Then there exists $C_p > 0$ such that for all $n \in \mathbb{N}$ and $\varepsilon > 0$
\begin{equation}
\varepsilon \int_S |U''(u_n^*)|(u_n^*)^2 \, dx \leq C_p.
\end{equation}
If moreover $U' \in L^\infty(0,\infty)$, the family $\{U_{n,\varepsilon}\}$, where
\begin{equation}
U_{n,\varepsilon}(t) := \int_{\mathbb{R}} U(u_n^*)(x) \rho(x) \, dx \quad (t \in (0, T))
\end{equation}
and $\rho \in C^2_c(\mathbb{R})$, is bounded in $BV(0,T)$.

\textbf{Proof.} Inequality (4.12) follows immediately from (4.9) and (4.11). To prove that $\{U_{n,\varepsilon}\}$ is bounded in $BV(0, T)$, observe that by (4.8)
\begin{equation}
(U_{n,\varepsilon})'(t) = \int_{\mathbb{R}} (\Theta_{U,\varepsilon}(u_n^*) + \varepsilon U(u_n^*)\rho'' - \varepsilon U''(u_n^*) (u_n^*)^2 \rho')(x, t) \, dx.
\end{equation}
Since $U' \in L^\infty(0,\infty)$, there exists $N > 0$ such that $|U'(u)| \leq N (1 + u)$ for $u \geq 0$.
Then $|U'(u_n^*)| \leq N (1 + u_n^*)$ and by (4.7), (3.1) and the definition of $\varphi_\varepsilon$, there holds
\[|\Theta_{U,\varepsilon}(u_n^*)| \leq \|\varphi_\varepsilon U''\|_{L^\infty((0,\infty))}|u_n^*| + |\theta_U| =: \tilde{M} u_n^* + |\theta_U|.
\]
Then it follows from (4.11) that
\begin{align*}
\left| (U_{n,\varepsilon})' \right|(t) &\leq \|\rho\|_{C^2_c(\mathbb{R})} \int_{\mathbb{R}} \left[ (\tilde{M} + \varepsilon N)u_n^*(x, t) + \varepsilon N + |\theta_U| \right] \, dx + \\
&\quad + \varepsilon \|\rho\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left[ |U''(u_n^*)|(u_n^*)^2 \right](x, t) \, dx,
\end{align*}
and, by (4.11) and (4.12), there exists a constant $C_{p,\varepsilon} > 0$ such that
\begin{equation}
\int_0^T \left| (U_{n,\varepsilon})' \right|(t) \, dt \leq \|\rho\|_{C^2_c(\mathbb{R})} \left\{ (\tilde{M} + N)T |u_0|_{M(\mathbb{R})} + C_{p,\varepsilon} \right\}.
\end{equation}
On the other hand, by (4.3) and since $|U(u_n^*)| \leq N (1 + u_n^*)$, there holds
\begin{equation}
\int_0^T |U_{n,\varepsilon}|(t) \, dt \leq NT \|\rho\|_{L^\infty(\mathbb{R})} \left( |u_0|_{M(\mathbb{R})} + |\text{supp } \rho| \right),
\end{equation}
whence the result follows. \qed

From the above lemmata we get the following convergence results.
Lemma 4.4. (i) Let $\varphi \in C([0, \infty))$. Then there exist a subsequence $\{u_{n_m}^m\} \subseteq \{u_n^m\}$ and $u_n \in L^\infty(S) \cap L^\infty(0, T; L^1(\mathbb{R}))$ such that as $\varepsilon_m \to 0$

$$u_{n_m}^m \rightharpoonup u_n \text{ in } L^\infty(S), \quad u_{n_m}^m \to u_n \text{ and } \varphi_{\varepsilon_m}(u_{n_m}^m) \to \varphi(u_n) \text{ a.e. in } S,$$

Moreover, $u_n \geq 0$ a.e. in $S$, $\|u_n\|_{L^\infty(S)} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})}$, and

$$\sup_{t \in (0, T)} \|u_n(t)\|_{L^1(\mathbb{R})} \leq \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}.$$  

(ii) Let $\varphi$ satisfy [3.1], let $\rho \in C_c^2(\mathbb{R})$, and let $U \in C^2([0, \infty))$, with $U' \in L^\infty(0, \infty)$, satisfy (4.11). Let $U_{n, \rho}^m$ be defined by (4.13) and set

$$U_{n, \rho}(t) := \int_\mathbb{R} U(u_n(x, t)) \rho(x) \, dx \quad (t \in (0, T)).$$

Then

$$U_{n, \rho}^m \rightharpoonup U_{n, \rho} \text{ in } L^1(0, T) \text{ and a.e. in } (0, T).$$

Proof. By (4.3), $u_{n_m}^m \rightharpoonup u_n$ in $L^\infty(S)$, where $u_n \in L^\infty(S)$, $\|u_n\|_{L^\infty(S)} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})}$ and $u_n \geq 0$ a.e. in $S$. The a.e.-convergence of $u_{n_m}^m$ and part (ii) follow from (4.18), and since $\varphi_{\varepsilon}$ converges uniformly to the continuous function $\varphi$ on compact subsets of $\mathbb{R}$, we also obtain the a.e.-convergence of $\varphi_{\varepsilon_m}(u_{n_m}^m)$.

It remains to prove (4.18) and (4.19). We claim that that for a.e. $t \in (0, T)$

$$u_{n_m}^m(\cdot, t) \rightharpoonup u_n(\cdot, t) \text{ in } L^\infty(\mathbb{R}) \text{ as } \varepsilon_m \to 0.$$  

Set $I_{n, \rho}^m(t) := \int_\mathbb{R} u_{n_m}^m(x, t) \rho(x) \, dx$ for $t \in (0, T)$ and let $\rho \in C_c^2(\mathbb{R})$. By Lemma 4.3 with $U(u) = u$, the sequence $\{I_{n, \rho}^m\}$ is bounded in $BV(0, T)$ and has a subsequence (not relabelled) $\{I_{n_m, \rho}^m\}$ such that

$$I_{n_m, \rho}^m \rightharpoonup I_{n, \rho} \text{ in } L^1(0, T) \text{ as } \varepsilon_m \to 0$$

for some $I_{n, \rho} \in BV(0, T)$. Since $u_{n_m}^m \rightharpoonup u_n$ in $L^\infty(S)$,

$$\lim_{m \to \infty} \int_0^T I_{n, \rho}^m(t) \, dt = \int_S u_n(x, t) \rho(x) \, dx \, dt = \int_0^T \left( \int_\mathbb{R} u_n(x, t) \rho(x) \, dx \right) \, dt,$$

whence $I_{n, \rho} = \int_\mathbb{R} u_n(x, t) \rho(x) \, dx$ for a.e. $t \in (0, T)$ and the convergence in (4.23) is satisfied along the whole sequence $\{I_{n_m, \rho}^m\}$. Hence for all $\rho \in C_c^2(\mathbb{R})$ there exists a null set $N \subset (0, T)$ such that

$$\lim_{\varepsilon_m \to 0} \int_\mathbb{R} u_{n_m}^m(x, t) \rho(x) \, dx = \int_\mathbb{R} u_n(x, t) \rho(x) \, dx \quad \text{for all } t \in (0, T) \setminus N.$$  

Since $C_c^2(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ and $L^1(\mathbb{R})$ is separable, the choice of the set $N$ can be made independent of $\rho$. Hence we have proven (4.22).

By (4.2), (4.4), (4.5), and the Fréchet-Kolmogorov Theorem, $\{u_{n_m}^m(\cdot, t)\}$ is relatively compact in $L^1((-L, L))$ for all $t \in (0, T)$ and $L > 0$. Hence, by (4.22),

$$u_{n_m}^m(\cdot, t) \to u_n(\cdot, t) \text{ in } L^1((-L, L)) \text{ as } \varepsilon_m \to 0 \text{ for } L > 0 \text{ and a.e. } t \in (0, T),$$

and (4.19) follows from (4.4). Finally (4.18) follows from (4.4), (4.24) and the Dominated Convergence Theorem. □
Proposition 4.5. Let $\varphi \in C([0, \infty))$. For all $n \in \mathbb{N}$ problem $(P_n)$ has an entropy solution $u_n$, which is unique if $\varphi$ is locally Lipschitz continuous. For a.e. $t \in (0, T)$ there holds

\[ \|u_n(\cdot + h, t) - u_n(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0(\cdot + h) - u_0\|_{L^1(\mathbb{R})} \quad \text{for any } h \in \mathbb{R}, \]

(4.25) \[ \int_{\mathbb{R}} u_n(x, t) \, dx = \int_{\mathbb{R}} u_0(x) \, dx. \]

Moreover, given $\rho \in C^2(\mathbb{R})$ and $U \in C(\mathbb{R})$ with $U' \in L^\infty(0, \infty)$, satisfying

(4.14), the sequence $(U_{n, \rho})$ defined by (4.20) is bounded in $BV(0, T)$.

Proof. Let $\zeta$ and $E$ be as in Definition 3.3 and $F'_\varepsilon = E'\varphi'_\varepsilon$. Then

\[ \int_S \{ E(u_n^\varepsilon) \left( \zeta + \varepsilon \zeta_x \right) + F'_\varepsilon(u_n^\varepsilon) \zeta_x \} \, dxdt + \int_{\mathbb{R}} E(u_0^\varepsilon)(\zeta(x, 0)) \, dx \geq 0, \]

where $u_n^\varepsilon$ is defined by Lemma 4.4. By (4.3), it is not restrictive to assume that $E(u) = |u - k|$, $F'_\varepsilon(u) = \text{sgn}(u - k) [\varphi'_\varepsilon(u) - \varphi'_\varepsilon(k)]$ $(k \in [0, \infty))$. By (4.3),

\[ \|\varphi_m(u_n^\varepsilon)\|_{L^\infty(S)} \leq \sup_{|v| \leq \varepsilon} |\varphi_m(v)| \leq \sup_{|v| \leq \varepsilon} |\varphi(v)|. \]

(4.26) Since $\varphi_m(u_n^\varepsilon) \to \varphi(u_n)$ a.e. in $S$ (see (4.17)), it follows from (4.18) and the Dominated Convergence Theorem that

\[ \int_S F_m(u_n^\varepsilon) \zeta_x \, dxdt \to \int_S F(u_n)(\zeta_x) \, dxdt \quad \text{as } \varepsilon \to 0. \]

The remaining terms in (4.27) (with $\varepsilon = \varepsilon_m$) are dealt with similarly. Letting $\varepsilon_m \to 0$, we obtain (4.16), so $u_n$ is an entropy solution of problem $(P_n)$. Its uniqueness follows from Kružkov’s Theorem (25).

Inequality (4.26) follows from (4.5) and (4.21). Concerning (4.26), it follows from (3.17) that for all $\rho \in C^1(\mathbb{R})$ and a.e. $t \in (0, T)$

\[ \int_{\mathbb{R}} u_n(x, t) \rho(x) \, dx - \int_{\mathbb{R}} u_0(x) \rho(x) \, dx = \int_0^t \int_{\mathbb{R}} \varphi(u_n)(x, s) \rho'(x) \, dxds. \]

(4.28) Let $\{\rho_k\} \subseteq C^1(\mathbb{R})$ be such that $\rho_k(x) = 1$ for $x \in [-k, k]$, $\rho_k(x) = 0$ if $|x| \geq k + 1$, and $\|\rho'_k\|_{L^\infty(\mathbb{R})} \leq 2$. Setting $\rho = \rho_k$ in (4.28), and taking $k \to \infty$, we get

\[ \left| \int_0^t \int_{\mathbb{R}} \varphi(u_n)(x, s) \rho_k(x) \, dxds \right| \leq 2M \int_0^t \int_{|x| \leq k} |u_n(x, s)| \, dxds \to 0, \]

since $u_n \in L^1(S)$. On the other hand, by the Monotone Convergence Theorem,

\[ \int_{\mathbb{R}} u_n(x, t) \rho_k(x) \, dx \to \int_{\mathbb{R}} u_n(x, t) \, dx, \quad \int_{\mathbb{R}} u_0(x) \rho_k(x) \, dx \to \int_{\mathbb{R}} u_0(x) \, dx, \]

and (4.26) follows from (4.28).

Finally, let us show that $(U_{n, \rho})$ is bounded in $BV(0, T)$. By (4.16) and (4.21)

\[ \int_0^T |U_{n, \rho}(t)| \, dt = \lim_{\varepsilon_m \to 0} \int_0^T |U_{n, \rho}^{\varepsilon_m}(t)| \, dt \leq N|\rho|_{L^\infty(\mathbb{R})} \left( T \|u_0\|_{M(\mathbb{R})} + |\text{supp } \rho| \right), \]

and, by (4.13) and the lower semicontinuity of the total variation in $L^1(0, T)$ ([13, Theorem 1, Subsection 5.2.1]), we get

\[ |U_{n, \rho}|_{M(0, T)} \leq \|\rho\|_{C^2(\mathbb{R})} \left( T\bar{M} + N \right) \left( T \|u_0\|_{M(\mathbb{R})} + C_{p, \rho} \right) \]

with $C_{p, \rho} > 0$ as in (4.13). This completes the proof. \qed
5. Existence and monotonicity: Proofs

Proposition 5.1. Let \((H_1)\) hold and let \(u_n\) be the entropy solution of problem \((P_n)\). Then there exist a sequence \(\{u_{n_j}\}\) and \(u \in L^\infty(0, T; \mathcal{M}^*(\mathbb{R}))\) such that

\[
\tag{5.1} u_{n_j} \overset{\ast}{\rightharpoonup} u \quad \text{in} \quad \mathcal{M}(S).
\]

For all \(L > 0\) there exists a decreasing sequence \(\{E_j\} \subset (-L, L) \times (0, T)\) of Lebesgue measurable sets with \(|E_j| \to 0\) as \(j \to \infty\), such that

\[
\tag{5.2} u_{n_j} \chi_{((-L, L) \times (0, T)) \setminus E_j} \rightharpoonup u_b := \int_{(0, \infty)} \xi \, d\tau(\xi) \quad \text{in} \quad L^1((-L, L) \times (0, T)),
\]

where \(\tau \in \mathcal{Y}(S; \mathbb{R})\) is the Young measure associated with \(\{u_{n_j}\}\), and

\[
\tag{5.3} u_{n_j} \chi_{E_j} \overset{\ast}{\rightharpoonup} \mu := -u_b \quad \text{in} \quad \mathcal{M}((-L, L) \times (0, T)).
\]

Proof. By \cite[(10)]{15} there exist \(u \in \mathcal{M}^*(S)\) and a sequence \(\{u_{n_j}\}\) such that \(u_{n_j} \overset{\ast}{\rightharpoonup} u\) in \(\mathcal{M}(S)\). Arguing as in \cite[Proposition 4.2]{20} we obtain that \(u \in L^\infty(0, T; \mathcal{M}^*(\mathbb{R}))\).

Since by \cite[(10)]{15} the sequence \(\{u_{n_j}\}\) is bounded in \(L^1(S)\), by Theorem \cite[Proposition 4.2]{20} there exist a subsequence of \(\{u_{n_{j_l}}\}\) (not relabeled) and a Young measure \(\tau \in \mathcal{Y}(S; \mathbb{R})\) such that:

\(\text{i}\) for every measurable set \(A \subset S\), \cite[(2.21)]{20} are valid for any \(f \in C(\mathbb{R})\) such that the sequence \(\{f(u_{n_j})\}\) is sequentially weakly relatively compact in \(L^1(A)\);

\(\text{ii}\) \(\text{supp} \tau(x, t) \subset [0, \infty)\) for a.e. \((x, t) \in S\) (here \(\tau(x, t)\) is the disintegration of \(\tau\)). Then the result follows by Theorem \cite[Proposition 4.2]{20} and a standard diagonal procedure. \(\square\)

Remark 5.1. The function \(u_b\) in \cite[(5.2)]{15} is defined for a.e. \((x, t) \in S\), since \(\tau\) is globally defined in \(S\). In addition, by \cite[(10)]{15} and the arbitrariness of \(L\) in Proposition \cite[Remark 4.1]{15}, a routine proof shows that \(u_b \in L^\infty(0, T; L^1(\mathbb{R}))\) and \(u_b \geq 0\) a.e. in \(S\). Therefore the Radon measure \(\mu \geq 0\) (see \cite[(5.3)]{15}) is defined on \(S\), \(\mu \in L^\infty(0, T; \mathcal{M}^*(\mathbb{R}))\), and

\[
\tag{5.4} \mu = u - u_b \Rightarrow u = u_b + \mu \quad \text{in} \quad \mathcal{M}(S).
\]

Proposition 5.2. Let \((H_1)\) hold, let \(\mu\) be as in \cite[(5.1)]{15} and let \(U \in C([0, \infty))\). If

\[
\tag{5.5} \lim_{u \to \infty} \frac{U(u)}{u} =: C_U \in [0, \infty),
\]

for all \(L > 0\)

\[
\tag{5.6} U(u_{n_j}) \overset{\ast}{\rightharpoonup} U^* + C_U \mu \quad \text{in} \quad \mathcal{M}((-L, L) \times (0, T)),
\]

where \(U^* \in L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R}))\) is defined by

\[
U^*(x, t) := \int_{(0, \infty)} U(\xi) \, d\tau(x, t)(\xi) \quad \text{for a.e.} \quad (x, t) \in S.
\]

Remark 5.2. If \(U \in C([0, \infty))\) satisfies \cite[(5.5)]{15}, there exists \(N > 0\) such that

\[
\tag{5.7} |U(u)| \leq N(1 + u) \quad \text{for} \quad u \geq 0.
\]

Moreover \(U^* \in L^\infty(0, T; L^1(\mathbb{R}))\) if \(|U(u)| \leq N u\), since \(u_b \in L^\infty(0, T; L^1(\mathbb{R}))\) and

\[
|U^*(x, t)| \leq \int_{(0, \infty)} |U(\xi)| \, d\tau(x, t)(\xi) \leq N \int_{(0, \infty)} \xi \, d\tau(x, t)(\xi) = N u_b(x, t) \quad \text{for a.e.} \quad (x, t) \in S.
\]
Proof of Proposition 5.2. For all \( \varepsilon > 0 \) there exist \( m_{\varepsilon} > 0 \) such that
\[
-\varepsilon u < U(u) - C_U u < \varepsilon u \quad \text{if } u > m_{\varepsilon}.
\]
For any \( m \in \mathbb{N}, m > m_{\varepsilon} \) let \( l_{1m}, l_{2m} \in C([0, \infty)) \) be such that \( 0 \leq l_{1m} \leq 1, 0 \leq l_{2m} \leq 1, l_{1m} + l_{2m} = 1 \) in \([0, \infty]\), \( \supp l_{1m} \subseteq [0, m + 1] \), \( \supp l_{2m} \subseteq [m, \infty) \). Then, by (5.8),
\[
|U(u_{n_j}) - [U(u_{n_j}) l_{1m}(u_{n_j}) + C_U u_{n_j} l_{2m}(u_{n_j})]| < \varepsilon u_{n_j} l_{2m}(u_{n_j}) \quad \text{for } j \in \mathbb{N}.
\]
Since \( \sup_{S \subseteq [0, m+1]} |U(u)| < \infty \), \( \{U(u_{n_j}) l_{1m}(u_{n_j})\} \) is uniformly integrable in \((-L, L) \times (0, T)\). Hence, by Theorem 2.1 for all \( L > 0 \)
\[
U(u_{n_j}) l_{1m}(u_{n_j}) \rightarrow U_{1m}^{*} := \int_{[0, \infty)} U(\xi) l_{1m}(\xi) \, d\tau(\xi)
\]
in \( L^1((-L, L) \times (0, T)) \). Here \( U_{1m}^{*} \) belongs to \( L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R})) \) since, by (5.7),
\[
|U_{1m}^{*}| \leq \int_{[0, \infty)} |U(\xi)| l_{1m}(\xi) \, d\tau(\xi) \leq N \int_{[0, \infty)} (1 + \xi) \, d\tau(\xi) \leq N(1 + u_b).
\]
Similarly, by (5.10), (5.11) and (5.12) with \( U(u) = u \),
\[
\begin{align*}
\int_{[0, \infty)} l_{1m}(\xi) \, d\tau(\xi) &= u_b - \int_{[0, \infty)} \xi l_{1m}(\xi) \, d\tau(\xi) + \mu = \int_{[0, \infty)} [1 - l_{1m}(\xi)] \, d\tau(\xi) + \mu = \\
&= \int_{[0, \infty)} \xi l_{2m}(\xi) \, d\tau(\xi) + \mu =: l_{2m}^*(\xi) \quad \text{in } \mathcal{M}((-L, L) \times (0, T)).
\end{align*}
\]
From (5.9) - (5.12) for any \( \xi \in C_c((-L, L) \times (0, T)) \), \( \xi \geq 0 \), and \( m \) as above we get
\[
\begin{align*}
&\int_{(-L, L) \times (0, T)} U_{1m}^{*} + (C_U - \varepsilon) l_{2m}^* \xi \, dx dt + (C_U - \varepsilon) \langle \mu, \zeta \rangle_{(-L, L) \times (0, T)} \\
&\leq \liminf_{n_j \to \infty} \int_{(-L, L) \times (0, T)} U(u_{n_j}) \xi \, dx dt \leq \limsup_{n_j \to \infty} \int_{(-L, L) \times (0, T)} U(u_{n_j}) \xi \, dx dt \\
&\leq \int_{(-L, L) \times (0, T)} U_{1m}^{*} + (C_U + \varepsilon) l_{2m}^* \xi \, dx dt + (C_U + \varepsilon) \langle \mu, \zeta \rangle_{(-L, L) \times (0, T)}.
\end{align*}
\]
Since \( U_{1m}^{*} \in L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R})) \),
\[
0 \leq l_{2m}^*(\xi) = \int_{[0, \infty)} \xi \, d\tau(\xi) \leq u_b \in L^\infty(0, T; L^1(\mathbb{R}))
\]
and
\[
\lim_{\varepsilon \to 0} l_{2m}^*(x, t) = 0, \quad \lim_{\varepsilon \to 0} U_{1m}^{*}(x, t) = U^{*}(x, t) \quad \text{for a.e. } (x, t) \in S,
\]
letting \( m \to \infty \) in (5.13) we get plainly
\[
\begin{align*}
&\int_{(-L, L) \times (0, T)} U^{*} \xi \, dx dt + (C_U - \varepsilon) \langle \mu, \zeta \rangle_{(-L, L) \times (0, T)} \\
&\leq \liminf_{n_j \to \infty} \int_{(-L, L) \times (0, T)} U(u_{n_j}) \xi \, dx dt \leq \limsup_{n_j \to \infty} \int_{(-L, L) \times (0, T)} U(u_{n_j}) \xi \, dx dt \\
&\leq \int_{(-L, L) \times (0, T)} U^{*} \xi \, dx dt + (C_U + \varepsilon) \langle \mu, \zeta \rangle_{(-L, L) \times (0, T)},
\end{align*}
\]
whence
\[
0 \leq \lim_{n_j \to \infty} \sup_{S \subseteq [0, m+1]} U(u_{n_j}) \xi \, dx dt - \liminf_{n_j \to \infty} \int_{(-L, L) \times (0, T)} U(u_{n_j}) \xi \, dx dt \\
\leq 2\varepsilon \langle \mu, \zeta \rangle_{(-L, L) \times (0, T)}.
\]
From the above inequalities the conclusion follows. \( \square \)
Proposition 5.3. Let \((H_1)\) hold. Let \(\mu, U\) and \(U^*\) be as in Proposition 4.5. Then

\[
\int_0^T \left| \int_{\mathbb{R}} U(u_{i,j})(x,t) \rho(x) \, dx - \int_{\mathbb{R}} U^*(x,t) \rho(x) \, dx - C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right| \, dt \to 0
\]

as \(j \to \infty\) for \(\rho \in C_c(\mathbb{R})\). Moreover, for all \(L > 0\) there exist a null set \(N \subset (0,T)\) and a subsequence of \(\{u_{i,j}\}\) (not relabelled), such that for all \(t \in (0,T) \setminus N\)

\[
U(u_{i,j})(\cdot, t) \xrightarrow{\ast} U^*(\cdot, t) + C_U \mu(\cdot,t) \quad \text{in } \mathcal{M}((-L,L)).
\]

Remark 5.3. Choosing \(U(u) = u\) in (5.15), we obtain that

\[
u_{i,j}(\cdot, t) \xrightarrow{\ast} u(\cdot, t) \quad \text{in } \mathcal{M}((-L,L)) \quad \text{for a.e. } t \in (0,T) \text{ and } L > 0.
\]

If \(U \in C([0,\infty))\) satisfies (5.15), \(U^* \in L^\infty(0,T;L^1_{\text{loc}}(\mathbb{R}))\) and \(\{U(u_{i,j})\}\) is bounded in \(L^\infty(0,T;L^1_{\text{loc}}(\mathbb{R}))\) (see (4.19) and (5.7). Since every \(\zeta \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\) can be uniformly approximated in bounded sets by finite sums \(\sum_{i=1}^p f^{i,p}(x)g^{i,p}(t)\) with \(f^{i,p}, g^{i,p}\) bounded and continuous functions \((1 \leq i \leq p; \text{ e.g., see [11 Théorème D.1.1]}\)), it follows from (5.14) that, as \(j \to \infty\), for all \(\zeta \in C([0,T];C_c(\mathbb{R}))\)

\[
\int_0^T \left| \int_{\mathbb{R}} U(u_{i,j})\zeta(x,t) \, dx - \int_{\mathbb{R}} U^*\zeta(x,t) \, dx - C_U \langle \mu(\cdot,t), \zeta(\cdot,t) \rangle_{\mathbb{R}} \right| \, dt \to 0.
\]

Proof of Proposition 5.3. (i) Let us first prove (5.14) if \(U \in C^2([0,\infty))\), with \(U^* \in L^\infty(0,\infty)\), and satisfies (4.11) and (5.3). Let \(\rho \in C_c(\mathbb{R})\), \(h \in C_c(0,T)\) and fix any \(L > 0\) such that \(\text{supp } \rho \subset (-L,L)\). Then by (5.6)

\[
\int_0^T \int_{\mathbb{R}} U_{i,j}(x,t) h(t) \, dx \, dt \to \int_0^T U^*_\rho(t) h(t) \, dt + C_U \int_0^T h(t) \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \, dt,
\]

where \(U_{i,j,\rho}\) is defined by (4.20) and \(U^*_\rho(t) := \int_{\mathbb{R}} U^*(x,t) \rho(x) \, dx\). Since, by Proposition 4.5, \(\{U_{i,j,\rho}\}\) is bounded in \(BV(0,T)\) if \(\rho \in C^2_c(\mathbb{R})\), there exists a subsequence which converges in \(L^1(0,T)\). Combined with (5.15) this yields that \(U_{i,j,\rho} \to U^*_\rho + C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}}\) in \(D(0,T)\) and in \(L^1(0,T)\) for all \(\rho \in C^2_c(\mathbb{R})\). Since the sequence \(\{U(u_{i,j})\}\) is bounded in \(L^\infty(0,T;L^1((-L,L)))\) and \(U^* \in L^\infty(0,T;L^1((-L,L)))\), the condition \(\rho \in C^2_c(\mathbb{R})\) may be relaxed to \(\rho \in C_c(\mathbb{R})\), and we have found (5.14).

(ii) Next we prove (5.14) for all \(U \in C([0,\infty)) \cap L^\infty(0,\infty)\) (in this case \(C_U = 0\)). We set \(U_k : [0,\infty) \to [0,\infty)\), \(U_k(u) := \langle U \chi_{[0,k]} \ast \theta_k \rangle(u)\) for any \(u \geq 0\), where \(\theta_k \geq 0\) is a sequence of standard mollifiers \((k \in \mathbb{N})\). Then \(\{U_k\} \subset C^2_c([0,\infty))\), \(U_k \to U\) uniformly on compact subsets of \([0,\infty)\) and \(\|U_k\|_{L^\infty(\mathbb{R})} \leq \|U\|_{L^\infty(\mathbb{R})}\). By part (i)
and \([4.14]\), for all \(\rho \in C_c(\mathbb{R})\) and \(k \in \mathbb{N}, M > 0\)

\[
\limsup_{j \to \infty} \int_0^T dt \left| \int_{\mathbb{R}} U(u_{n_j}) \rho(x) \, dx - \int_{\mathbb{R}} U^*(x,t) \rho(x) \, dx \right| \\
\leq \limsup_{j \to \infty} \int_{\{0 \leq u_{n_j} \leq M\}} |U(u_{n_j}) - U_k(u_{n_j})| |\rho| \, dx dt + \\
\int_{\{u_{n_j} > M\}} |U(u_{n_j}) - U_k(u_{n_j})| |\rho| \, dx dt + \int_S |U^* - U_k^*| |\rho| \, dx dt \leq \\
\|\rho\|_{\infty} \sup_{\{\xi \leq M\}} |U_k(\xi) - U(\xi)| d\tau(x,t)(\xi) \leq \\
\|U_k - U\|_{L^\infty(0,M)} + \|U\|_{L^\infty(\mathbb{R})} \int_{\{\xi > M\}} d\tau(x,t)(\xi).
\]

Letting \(k \to \infty\) we obtain, since \(U_k \to U\) uniformly on compact sets in \([0, \infty)\),

\[
(5.19) \quad \limsup_{j \to \infty} \int_0^T dt \left| \int_{\mathbb{R}} U(u_{n_j}) \rho(x) \, dx - \int_{\mathbb{R}} U^*(x,t) \rho(x) \, dx \right| \\
\leq 2 \|\rho\|_{C(\mathbb{R})} \|U\|_{L^\infty(\mathbb{R})} \frac{T \|\mu_0\|_{M(\mathbb{R})}}{M} + \int_{\{\xi > M\}} d\tau(x,t)(\xi).
\]

Since \(\tau(x,t)\) is a probability measure, there holds \(\int_{\{\xi > M\}} d\tau(x,t)(\xi) \to 0\) as \(M \to \infty\) for a.e. \((x,t) \in S\), thus by the Dominated Convergence Theorem

\[
\int_{\supp(0,T)} d\tau(x,t)(\xi) \to 0 \quad \text{as} \quad M \to \infty.
\]

Then letting \(M \to \infty\) in \((5.19)\) we obtain \((5.14)\).

\((iii)\) Now let \(U \in C([0, \infty))\) be any function satisfying \((5.5)\). Arguing as in the proof of Proposition \((4.2)\), let \(l_{1_m}, l_{2_m} \in C^2([0, \infty)) \quad (m \in \mathbb{N})\) satisfy \(l_{1_m}, l_{2_m} \geq 0\) and \(l_{1_m} + l_{2_m} = 1\) in \([0, \infty)\), \(\sup l_{1_m} \leq [0, m+1]\), and \(\sup l_{2_m} \in [m, \infty)\). Then

\[
(5.20) \quad U(u_{n_j}) = U(u_{n_j}) l_{1_m}(u_{n_j}) + U(u_{n_j}) l_{2,m}(u_{n_j}),
\]

and, by \((4.8)\), for all \(\varepsilon > 0\) and \(m > m_\varepsilon\)

\[
(5.21) \quad (C_U - \varepsilon) u_{n_j} l_{2_m}(u_{n_j}) \leq U(u_{n_j}) l_{2_m}(u_{n_j}) \leq (C_U + \varepsilon) u_{n_j} l_{2_m}(u_{n_j}).
\]

Since \(\|U_{1_m}\|_{L^\infty(\mathbb{R})} \leq \|U\|_{C([0,m+1])} < \infty\), the function \(U_{1_m}\) belongs to \(C([0, \infty)) \cap L^\infty(\mathbb{R})\). Then by part \((ii)\)

\[
(5.22) \quad \int_0^T \left| \int_{\mathbb{R}} \left[ U(u_{n_j}) l_{1_m}(u_{n_j}) \right](x,t) \rho(x) \, dx - \int_{\mathbb{R}} U_{1_m}^*(x,t) \rho(x) \, dx \right| \, dt \to 0
\]
as \( j \to \infty \), where \( \rho \in C_c(\mathbb{R}) \) and \( U_{1m}^* \) is defined by (5.10). By (5.21) and (4.13)

\[
\int_0^T \left| \int_{\mathbb{R}} \left( U(u_{n_j}) - C_U l_{2m}(u_{n_j}) \right)(x,t) \rho(x) \, dx \right| \, dt \leq 
\leq \varepsilon \int_S |u_{n_j}| |\rho(x)| \, dx \leq \varepsilon T \|u_0\|_{\mathcal{M}(\mathbb{R})} \|\rho\|_{\infty}.
\]

Then we obtain that

\[
\int_0^T \left| \int_{\mathbb{R}} \left( U(u_{n_j}) - C_U l_{2m}^*(x,t) \rho(x) \, dx - C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right) \right| \, dt \leq 
\leq \varepsilon T \|u_0\|_{\mathcal{M}(\mathbb{R})} \|\rho\|_{\infty} + C_U \int_0^T \left| \int_{\mathbb{R}} \left( u_{n_j} l_{2m}(u_{n_j}) - l_{2m}^* \right)(x,t) \rho(x) \, dx - \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right| \, dt,
\]

with \( l_{2m}^* \) defined as in (5.12). The map \( u \mapsto ul_{2m}(u) \) belongs to \( C^2([0,\infty)) \), has bounded derivative and satisfies (4.11) and (5.5), with \( C_U = 1 \). Then by part (i), (5.20) and (5.22)

(5.23)

\[
\limsup_{j \to \infty} \int_0^T \left| \int_{\mathbb{R}} \left( U(u_{n_j}) - U_{1m}^* - C_U l_{2m}^*(x,t) \rho(x) \, dx - C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right. \right| \, dt \leq 
\leq \varepsilon T \|u_0\|_{\mathcal{M}(\mathbb{R})} \|\rho\|_{\infty} \quad \text{if } m > m_\varepsilon.
\]

To complete the proof of (5.14) we show that

(5.24)

\[
\lim_{m \to \infty} \int_S |U^* - U_{1m}^* - C_U l_{2m}^*(x,t) |\rho(x)| \, dx \, dt = 0.
\]

By (5.21),

\[
|U^* - U_{1m}^* - C_U l_{2m}^*(x,t)| \leq \int_{[0,\infty)} |U(\xi) - U(\xi)| \xi_1 m(\xi) - C_U |l_{2m}(\xi)| \, d\tau(x,t)(\xi) = 
= \int_{[0,\infty)} |U(\xi) - C_U |l_{2m}(\xi)| \, d\tau(x,t)(\xi) \leq \varepsilon \int_{[m,\infty)} \xi \, d\tau(x,t)(\xi) \leq \varepsilon b(x,t)
\]

for a.e. \((x,t) \in S\). Since \( u_0 \in L^0(0,T;L^1(\mathbb{R})) \) and \( \int_{[m,\infty)} \xi \, d\tau(x,t)(\xi) \to 0 \) as \( m \to \infty \) for a.e. \((x,t) \in S\), (5.24) follows from the Dominated Convergence Theorem.

Letting \( m \to \infty \) in (5.23), it follows from (5.24) that

\[
\limsup_{j \to \infty} \int_0^T \left| \int_{\mathbb{R}} \left( U(u_{n_j}) - U^* \right)(x,t) \rho(x) \, dx - C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right| \, dt \leq 
\leq \limsup_{m \to \infty} \left\{ \limsup_{j \to \infty} \int_0^T \left| \int_{\mathbb{R}} \left( U(u_{n_j}) - U_{1m}^* - C_U l_{2m}^* \right) \rho \, dx - C_U \langle \mu(\cdot,t), \rho \rangle_{\mathbb{R}} \right| \, dt \right\} \leq 
\leq \varepsilon T \|u_0\|_{\mathcal{M}(\mathbb{R})} \|\rho\|_{\infty},
\]

and (5.14) follows from the arbitrariness of \( \varepsilon \).

Finally (5.15) follows from (5.14), the separability of \( C_c(\mathbb{R}) \) and a diagonal argument; we leave the details to the reader. \qed

**Proposition 5.4.** Let \((H_1)\) hold. Then \((5.4)\) is the Lebesgue decomposition of \( u\):

(5.25)

\[
u_b = u_\ast \text{ a.e. in } S, \quad \mu = u_s \text{ in } \mathcal{M}(S).
\]

**Proof.** Let \( U \) be a convex function with \( U(0) = 0 \) and \( U' \in L^\infty(0,\infty) \). By (5.16),

(5.26)

\[
\int_{\mathbb{R}} U(u_{n_j})(x,t) \zeta(x,t) \, dx \leq \int_{\mathbb{R}} U(u_{0n_j})(x) \zeta(x,0) \, dx \leq 
\leq \int_{\mathbb{R} \times (0,T)} \left\{ U(u_{n_j}) \zeta_t + \Theta_U(u_{n_j}) \zeta_x \right\} \, dx \, dt
\]
for all $\zeta \in C^1([0,T];C^1_c(\mathbb{R}))$ and a.e. $\tilde{t} \in (0,t)$, where

$$\Theta_U(u) := \int_0^u U'(s) \varphi'(s) \, ds + \theta_U \quad (\theta_U \in \mathbb{R}).$$

Let $U_m(u) = (u - m)\chi_{[m,\infty)}(u)$ and $\theta_{U_m} = 0$ ($m \in \mathbb{N}$). Since $U_m(u)/u \to C_{U_m} = 1$ and $\Theta_{U_m}(u)/u \to C_{\varphi}$ as $u \to \infty$ (with $C_{\varphi}$ as in $(H_1)$), it follows from (5.17) that

$$\int_0^t \left| \int_{\mathbb{R}} [U_m(u_n)](x,t) \, dx - \int_{\mathbb{R}} [U^*_m(x,t)](x,t) \, dx - \langle \mu(\cdot, t), \zeta(\cdot, t) \rangle \right| \, dt \to 0$$

and

$$\int_0^t \left| \int_{\mathbb{R}} [\Theta_{U_m}(u_n)\zeta_x](x,t) \, dx - \int_{\mathbb{R}} [\Theta^*_m(x,t)](x,t) \, dx - C_{\varphi} \langle \mu(\cdot, t), \zeta(\cdot, t) \rangle \right| \, dt \to 0$$

as $j \to \infty$, where

$$U^*_m(x,t) := \int_{\{0,\infty\}} U_m(\xi) \, d\tau(x,t)(\xi), \quad \Theta^*_m(x,t) := \int_{\{0,\infty\}} \Theta_{U_m}(\xi) \, d\tau(x,t)(\xi)$$

belong to $L^\infty(0,T;L^1_{loc}(\mathbb{R}))$. In particular, setting $\zeta^\prime := \zeta_t + C_{\varphi} \zeta_x$, we have that

$$\int_{\mathbb{R}^2 \times (0,t)} \left\{ U_m(u_n) \zeta_t + \Theta_{U_m}(u_n) \zeta_x \right\} \, dx \, dt \to$$

$$\int_{\mathbb{R}^2 \times (0,t)} \left\{ U^*_m \zeta_t + \Theta^*_m \zeta_x \right\} \, dx \, dt + \int_0^t \langle \mu(\cdot, t), \zeta(\cdot, t) \rangle \, dt.$$

By (5.15) and a diagonal argument, there exist a null set $N \subset (0,T)$ and a subsequence, denoted again by $\{u_{nj}\}$, such that for all $\tilde{t} \in (0,T) \setminus N$ and $m \in \mathbb{N}$

$$\lim_{n_j \to \infty} \int_{\mathbb{R}} U_m(u_{nj})(x,\tilde{t}) \zeta(x,\tilde{t}) \, dx = \int_{\mathbb{R}} U^*_m(x,\tilde{t}) \zeta(x,\tilde{t}) \, dx + \langle \mu(\cdot, \tilde{t}), \zeta(\cdot, \tilde{t}) \rangle_{\mathbb{R}}.$$

Since $\{U_m(u_{0nj}) - u_{0nj}\}$ is bounded in $L^\infty(\mathbb{R})$ and converges a.e. to $U_m(u_{0r}) - u_{0r}$, it follows from (5.15) that

$$\lim_{n_j \to \infty} \int_{\mathbb{R}} U_m(u_{0nj})(x) \zeta(x,0) \, dx = \int_{\mathbb{R}} U_m(u_{0r})(x) \zeta(x,0) \, dx + \langle u_{0s}, \zeta(\cdot, 0) \rangle_{\mathbb{R}}.$$

Setting $U = U_m$ in (5.20) and letting $j \to \infty$, we obtain from (5.28) that

$$\int_{\mathbb{R}} U^*_m(x,\tilde{t}) \zeta(x,\tilde{t}) \, dx + \langle \mu(\cdot, \tilde{t}), \zeta(\cdot, \tilde{t}) \rangle_{\mathbb{R}} \leq \int_{\mathbb{R}^2 \times (0,t)} \left\{ U^*_m \zeta_t + \Theta^*_m \zeta_x \right\} \, dx \, dt +$$

$$\int_0^t \langle \mu(\cdot, t), \zeta(\cdot, t) \rangle_{\mathbb{R}} \, dt + \int_{\mathbb{R}} U_m(u_{0r})(x) \zeta(x,0) \, dx + \langle u_{0s}, \zeta(\cdot, 0) \rangle_{\mathbb{R}}$$

for all $\tilde{t} \in (0,T) \setminus N$ and $m \in \mathbb{N}$. Since for all $u \geq 0$ (see (5.14))

$$0 \leq U_m(u) \leq u \chi_{[m,\infty)}(u), \quad |\Theta_{U_m}(u)| = |\varphi(u) - \varphi(m)| \chi_{[m,u)}(u) \leq M u \chi_{[m,\infty)}(u)$$

we have that $|U^*_m| \leq u_b$, $|\Theta^*_m| \leq M u_b$, $U^*_m \to 0$ and $\Theta^*_m(x,\tilde{t}) \to 0$ (as $m \to \infty$) a.e. in $S$. Thus, by the Dominated Convergence Theorem and (5.31), for all $\tilde{t} \in (0,T) \setminus N$

$$\langle \mu(\cdot, \tilde{t}), \zeta(\cdot, \tilde{t}) \rangle_{\mathbb{R}} \leq \int_0^t \langle \mu(\cdot, t), \zeta(\cdot, t) \rangle_{\mathbb{R}} \, dt + \langle u_{0s}, \zeta(\cdot, 0) \rangle_{\mathbb{R}}.$$

Let $\rho \in C^1_c(\mathbb{R})$ and $\zeta(x,t) = \rho(x - C_{\varphi} t)$, so $\zeta_t \equiv 0$. By (5.32), $\langle \mu(\cdot, \tilde{t}), \rho(\cdot - C_{\varphi} \tilde{t}) \rangle_{\mathbb{R}} \leq \langle u_{0s}, \rho \rangle_{\mathbb{R}}$. Hence $\mu(\cdot, \tilde{t})$ is singular with respect to the Lebesgue measure and, since $\mu(\cdot, \tilde{t}) = \{\mu(\cdot, t)\}_t = \mu_x(\cdot, \tilde{t})$ for a.e. $\tilde{t} \in (0,T)$ (see (2.24)), (5.35) follows from the uniqueness of the Lebesgue decomposition. 

The following result is based on the concept of compensated compactness (e.g., see [12]).
Proposition 5.5. Let $(H_1)$ hold. Then $\varphi(u_r) = \int_{[0,\infty)} \varphi(\xi) \, d\tau(\xi)$ a.e. in $S$.

Proof. Let $U, V \in C^2([0,\infty)) \cap L^\infty((0,\infty))$ satisfy (5.11), and assume that $\Theta_U, \Theta_V$, defined by (5.27), belong to $L^\infty((0,\infty))$. By (5.12) there holds

$$\varepsilon \|U''(u_n^\varepsilon)(u_n^\varepsilon)^2\|_{L^1(S)} \leq C_p \quad \text{and} \quad \varepsilon \|V''(u_n^\varepsilon)(u_n^\varepsilon)^2\|_{L^1(S)} \leq C_p$$

for all $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, and up to a subsequence

$$(5.33) \quad \varepsilon U''(u_n^\varepsilon)(u_n^\varepsilon)^2 \xrightarrow[*]{\varepsilon \to 0} \lambda_n, \quad \varepsilon V''(u_n^\varepsilon)(u_n^\varepsilon)^2 \xrightarrow[*]{\varepsilon \to 0} \mu_n$$

in $\mathcal{M}(S)$ as $\varepsilon \to 0$, for some $\lambda_n, \mu_n \in \mathcal{M}(S)$. By the lower semicontinuity of the norm,

$$(5.34) \quad \|\lambda_n\|_{\mathcal{M}(S)} \leq C_p, \quad \|\mu_n\|_{\mathcal{M}(S)} \leq C_p$$

for $n \in \mathbb{N}$.

Let $\zeta \in C^2_c(S)$. Then (see (4.18))

$$(5.35) \quad \varepsilon \iint_S U''(u_n^\varepsilon)(u_n^\varepsilon)^2 \zeta \, dx \, dt = \iint_S \{U(u_n^\varepsilon)\zeta_t + \Theta_{U,\varepsilon}(u_n^\varepsilon)\zeta_x + \varepsilon U(u_n^\varepsilon)\zeta_{xx}\} \, dx \, dt,$$

where $\Theta_{U,\varepsilon}(u) = \int_0^u U'(s) \varphi'(s) \, ds + \theta_U$, $\theta_U \in \mathbb{R}$, By (5.11) and (5.13), for all $n \in \mathbb{N}$

$$\|\Theta_{U,\varepsilon}(u_n^\varepsilon)\| \leq \int_0^{1u_{n\varepsilon}} |U'(s)| \varphi'(s) \, ds + |\theta_U| \leq M \iint_0^{1u_{n\varepsilon}} |U'(s)| \, ds + |\theta_U| \leq \gamma_{n,U}$$

for some $\gamma_{n,U} \geq 0$, so for fixed $n \in \mathbb{N}$ the family $\{\Theta_{U,\varepsilon}(u_n^\varepsilon)\}$ is uniformly bounded in $L^\infty(S)$. Similar results hold for $V$ and $\Theta_{V,\varepsilon}(u) = \int_0^u V'(s) \varphi'(s) \, ds + \theta_V$, and letting $\varepsilon \to 0$ in (5.35) along some subsequence $\{\varepsilon_m\}$ (see the proof of Proposition 4.3), it follows from by (5.35) that for all $n \in \mathbb{N}$ and $\zeta \in C^2_c(S)$

$$(5.36) \quad \iint_S \{U(u_n)\zeta_t + \Theta_U(u_n)\zeta_x\} \, dx \, dt = (\lambda_n, \zeta)_S, \quad \iint_S \{V(u_n)\zeta_t + \Theta_V(u_n)\zeta_x\} \, dx \, dt = (\mu_n, \zeta)_S$$

where $u_n$ is the entropy solution of the approximating problem $(P_n)$ (see (4.17)).

Let $A \subset S$ be a bounded open set and let $Y_n, Z_n : A \to \mathbb{R}^2$ be defined by

$$Y_n := (\Theta_U(u_n), U(u_n)), \quad Z_n := (V(u_n), -\Theta_V(u_n)).$$

By (5.30),

$$(5.37) \quad \text{div} Y_n = -\lambda_n, \quad \text{curl} Z_n = -\mu_n \quad \text{in} \ D'(A).$$

Since $U, \Theta_U, V, \Theta_V$ are bounded in $(0,\infty)$, the sequences $U(u_n), \Theta_U(u_n), V(u_n)$ and $\Theta_V(u_n)$ are bounded in $L^1(A)$ and uniformly integrable, and, by Theorem 2.1

$$U(u_n) \to U^* := \int_{[0,\infty)} U(\xi) \, d\tau(\xi) \quad \text{in} \ L^1(A), \quad \Theta_U(u_n) \to \Theta_U^* := \int_{[0,\infty)} \Theta_U(\xi) \, d\tau(\xi) \quad \text{in} \ L^1(A),$$

$$V(u_n) \to V^* := \int_{[0,\infty)} V(\xi) \, d\tau(\xi) \quad \text{in} \ L^1(A), \quad \Theta_V(u_n) \to \Theta_V^* := \int_{[0,\infty)} \Theta_V(\xi) \, d\tau(\xi) \quad \text{in} \ L^1(A)$$

in $L^1(A)$, where $\tau(\xi)$ denotes the disintegration of the Young measure $\tau$ associated with $\{u_n\}$. Since the sequences $U(u_n), \Theta_U(u_n), V(u_n) \text{ and } \Theta_V(u_n)$ are bounded in $L^\infty(A) \subset L^2(A)$, they also converge weakly in $L^2(A)$, so

$$Y_n \to Y^* := (\Theta_U^*, U^*), \quad Z_n \to Z^* := (V^*, -\Theta_V^*) \quad \text{in} \ L^2(A).$$

By a similar argument

$$(5.38) \quad Y_n \cdot Z_n := \Theta_U(u_n) V(u_n) - \Theta_V(u_n) U(u_n) \to$$

$$- \int_{[0,\infty)} [\Theta_U(\xi)V(\xi) - \Theta_V(\xi)U(\xi)] \, d\tau(\xi) \quad \text{in} \ L^2(A).$$
By (5.34) and (5.37), \( \{\text{div} Y_\alpha\} \) and \( \{\text{curl} Z_\alpha\} \) are precompact in \( W^{-1,2}(A) \) (see [12, Chapter 1, Corollary 1]) and, by the div-curl lemma,

\[
Y_\alpha \cdot Z_\alpha \to Y^* \cdot Z^* = \Theta^*_U V^* - \Theta^*_V U^* \quad \text{in } D'(A).
\]

By (5.38) and (5.39),

\[
\int_{[0,\infty)} [\Theta_U(\xi) - \Theta^*_U] V(\xi) d\tau(\xi) = \int_{[0,\infty)} [U(\xi) - U^*] \Theta_V(\xi) d\tau(\xi) \quad \text{a.e. in } A.
\]

For every \( U \) as above with \( U' > 0 \) in \( (0,\infty) \), by a standard approximation argument we may choose \( V(u) = |U^* - U(u)| \), so \( \Theta_V(u) = \text{sgn}(U(u) - U^*)[\Theta_U(u) - \Theta_U(U^{-1}(U^*))] \) and, by (5.40),

\[
\left[ \Theta^*_U - \Theta_U(U^{-1}(U^*)) \right] \int_{[0,\infty)} |U^* - U(\xi)| d\tau(\xi) = 0.
\]

Let \( U_k \in C^2([0,\infty)) \cap L^\infty((0,\infty)) \) satisfy (1.11) and

\[
U_k(0) = 0, \quad 0 < U'_k \leq U'_{k+1} \leq 1 \quad \text{in } [0,\infty), \quad U'_k(u) \to 1 \text{ for } u \geq 0 \text{ as } k \to \infty.
\]

By (5.41), \( |\Theta_{U_k}(u)| \leq \int_0^u U'_{k}(s)|\varphi'(s)|ds + |	heta_{U_k}| \leq MU_k(u) + |\theta_{U_k}|, \) thus \( \Theta_{U_k} \) is bounded in \( (0,\infty) \) for every \( k \in \mathbb{N} \). We claim that, as \( k \to \infty \),

\[
U^*_k := \int_{[0,\infty)} U_k(\xi) d\tau(\xi) \to u_r \quad \text{a.e. in } A,
\]

where \( \Theta^*_U := \int_{[0,\infty)} \Theta_{U_k}(\xi) d\tau(\xi) \) (recall that \( \varphi \in L^1([0,\infty); d\tau(x,t)) \), see Remark 5.21). By (5.43) and the Dominated Convergence Theorem, for a.e. \( (x,t) \in A \)

\[
\int_{[0,\infty)} |U^*_k(x,t) - U_k(\xi)| d\tau(x,t)(\xi) \to \int_{[0,\infty)} |u_r(x,t) - \xi| d\tau(x,t)(\xi) \quad \text{as } k \to \infty,
\]

since \( 0 \leq U_k(\xi) \leq \xi \) for all \( k \in \mathbb{N} \) and \( I(\xi) := \xi \) belongs to \( L^1([0,\infty); d\tau(x,t)) \) (recall that, by (5.22), the definition of \( u_0 \) in (5.22), \( u_r(x,t) = \int_{[0,\infty]} \xi d\tau(x,t)(\xi) < \infty \) for a.e. \( (x,t) \in S \). Letting \( k \to \infty \) in (5.41), with \( U = U_k \), we obtain that for a.e. \( (x,t) \in A \)

\[
\left[ \int_{[0,\infty)} \varphi(\xi) d\tau(x,t)(\xi) - \varphi(u_r)(x,t) \right] \int_{[0,\infty)} |u_r(x,t) - \xi| d\tau(x,t)(\xi) = 0,
\]

and Proposition 5.3 follows from the arbitrariness of \( A \).

It remains to prove (5.38) and (5.41). By (5.39) and the Monotone Convergence Theorem, \( U_k(\xi) \to \xi \) for any \( \xi \in [0,\infty) \), and (5.33) follows (recall that \( I(\xi) = \xi \in L^1((0,\infty),d\tau)) \). Concerning (5.41) we observe that

\[
\Theta^*_U - \Theta_U(U^{-1}(U^*)) = \int_{[0,\infty)} \left( \int_0^\xi U'_k(s) \varphi'(s) ds \right) d\tau(\xi) - \int_0^{U^{-1}(U^*)} U'_k(s) \varphi'(s) ds.
\]

Since \( U'_k(\xi) \to 1 \) and \( |U'_k(\xi)\varphi'(\xi)| \leq M \) for \( \xi \geq 0 \) (see (5.42) and (3.11)), it follows from the Dominated Convergence Theorem that

\[
\int_{[0,\infty)} \left( \int_0^\xi U'_k(s) \varphi'(s) ds \right) d\tau(x,t)(\xi) \to \int_{[0,\infty)} \varphi(\xi) d\tau(x,t)(\xi).
\]
On the other hand,
\begin{equation}
\int_0^{U_k^{-1}(U_k^*(x,t))} U_k'(s) \varphi'(s) \, ds - \varphi(u_r)(x,t) = 
\int_0^{u_r(x,t)} [U_k'(s) - 1] \varphi'(s) \, ds + \int_{u_r(x,t)}^{U_k^{-1}(U_k^*(x,t))} U_k'(s) \varphi'(s) \, ds.
\end{equation}

Arguing as before one shows that the first term in the right-hand side of (5.44) vanishes as \( k \to \infty \). As for the second term we observe that, by (5.42) and (5.43),
\[ \left| \int_{u_r(x,t)}^{U_k^{-1}(U_k^*(x,t))} U_k'(s) \varphi'(s) \, ds \right| \leq M \left| u_r(x,t) - U_k^{-1}(U_k^*(x,t)) \right| \leq M \left( \left| u_r(x,t) - U_k^{-1}(u_r(x,t)) \right| + \sup_{s \in I_{u_r(x,t)}} \frac{1}{U_k'(s)} \left| u_r(x,t) - U_k^*(x,t) \right| \right) \]
for some \( \delta > 0 \) and all \( k \in \mathbb{N} \) sufficiently large, where \( I_k(q) \equiv (q - \delta, q + \delta) \). Hence
\begin{equation}
\int_0^{U_k^{-1}(U_k^*(x,t))} U_k'(s) \varphi'(s) \, ds \to \varphi(u_r)(x,t) \quad \text{for a.e. } (x,t) \in A,
\end{equation}
and we obtain (5.44) from (5.45), (5.46) and (5.48).

To prove the second part of Theorem 3.2 we need the following result, which characterizes the disintegration of the Young measure \( \tau \).

**Proposition 5.6.** Let \((H_1)\) hold and \( \varphi \in C^1([0,\infty)) \) satisfy for all \( \bar{u} > 0 \) either (3.18) or (5.49)
\[ \exists \ a > 0, \ b \in (0,\infty) \text{ such that } \varphi' \text{ is constant in } I_{a,b} = [\bar{u} - a, \bar{u} + b] \text{ and, if } b < \infty, \ \varphi' \text{ is strictly monotone in } [\bar{u} + b, \bar{u} + b] \text{ and } [\bar{u} - a, \bar{u} - a] \text{ for some } b > b \text{ and } \bar{a} \in (a, \bar{u}). \]

Then for a.e. \((x,t) \in S\) the following holds:

(i) if \( u_r(x,t) = 0 \), then \( \tau_{(x,t)} = \delta_0 \);

(ii) if \( \varphi' \) is strictly monotone in \( I_{a,b} = [u_r(x,t) - a, u_r(x,t) + b] \) with \( a, b \geq 0, a + b > 0 \),
\begin{equation}
\tau_{(x,t)} = \delta_{u_r(x,t)}.
\end{equation}

(iii) if \( \varphi' \) is constant in the above interval \( I_{a,b} \) for some \( a > 0, b > 0 \), then
\begin{equation}
\sup \tau_{(x,t)} \subseteq I_{(x,t)} \quad \text{for a.e. } (x,t) \in S,
\end{equation}
where \( I_{(x,t)} \supseteq I_{a,b} \) is the maximal interval where \( \varphi'(\cdot) \equiv \varphi'(u_r(x,t)) \).

**Proof.** Let \((x,t) \in S\) be fixed. If \( u_r(x,t) = 0 \) it follows from (5.2a) and the definition of \( u_0 \) in (5.2) that \( \int_{[0,\infty)} \xi \, d\tau_{(x,t)}(\xi) = 0 \), which implies part (i): \( \tau_{(x,t)} = \delta_0 \).

So let \( u_r(x,t) > 0 \). Let \( l_1 := u_r(x,t) \), \( l_2 > l_1 \) and
\[ V_k(u) := k(u - l_1)\chi_{(l_1,l_1+\bar{\epsilon})}(u) + \chi_{[l_1+\bar{\epsilon},l_2]}(u) + k \left( l_2 + \frac{1}{k} - u \right) \chi_{[l_2,l_2+\bar{\epsilon})}(u) \]
for \( u \geq 0 \) and sufficiently large \( k \in \mathbb{N} \). Then \( V_k(u) \to \chi_{(l_1,l_2]}(u) \) as \( k \to \infty \), and
\[ \Theta V_k(u) = \int_0^u V_k'(s) \varphi'(s) \, ds \to \varphi' \left( \chi_{(l_1,l_2]} \right) (u) + \left[ \varphi'(l_1) - \varphi'(l_2) \right] \chi_{(l_2,\infty)}(u) \quad (u \geq 0). \]
By standard approximation arguments, (5.10) is satisfied with \( U = U_k \) and \( V = V_k \), where \( \{U_k\} \) is the sequence in the proof of Proposition 5.5 (see (5.42)):
\[ \int_{[0,\infty)} [\Theta V_k(\xi) - \Theta V_k(x,t)] V_k(\xi) \, d\tau_{(x,t)}(\xi) = \int_{[0,\infty)} \left[ U_k(\xi) - U_k^*(x,t) \right] \Theta V_k(\xi) \, d\tau_{(x,t)}(\xi). \]
Letting $k \to \infty$ and arguing as in the proof of Proposition 5.5, we obtain that

$$U_k(x,t) - U_k^*(x,t) \to \xi - \int_{[0,\infty)} \xi d\tau(x,t) \Rightarrow \xi - u_r(x,t) = \xi - l_1,$$

$$\Theta_{U_k}(x,t) - \Theta_{U_k^*}(x,t) \to \int_{[0,\infty)} \varphi(\xi) d\tau(x,t) \Rightarrow \varphi(\xi) - \varphi(u_r(x,t)) = \varphi(l_1) - \varphi(\xi) = \varphi(\xi) - \varphi(\xi)$$

for all $\xi \geq 0$ (see (5.28) and Proposition 5.6). This implies that

$$\int_{[0,\infty)} [\Theta_{U_k}(x,t) - \Theta_{U_k^*}(x,t)] V_k(\xi) d\tau(x,t)(\xi) \to \int_{[l_1,l_2]} [\varphi(\xi) - \varphi(l_1)] d\tau(x,t)(\xi),$$

$$\int_{[0,\infty)} [U_k(x,t) - U_k^*(x,t)] \Theta_{V_k}(\xi) d\tau(x,t)(\xi) \to \int_{[l_1,l_2]} \varphi(l_1)(\xi - l_1) d\tau(x,t)(\xi) + \int_{(l_2,\infty)} (\xi - l_1) d\tau(x,t)(\xi),$$

whence

$$(5.52)$$

$$\int_{(l_1,l_2]} [\varphi(\xi) - \varphi(l_1)] d\tau(x,t)(\xi) = [\varphi(l_1) - \varphi(l_2)] \int_{[l_1,l_2]} (\xi - l_1) d\tau(x,t)(\xi).$$

Similarly, let $l_0 \in (0,l_1)$ and set

$$\tilde{V}_k(u) := k(u - l_0)\chi_{[l_0,l_0 + l_1)}(u) + \chi_{[l_0 + l_1,\infty)}(u) + k(1 - u)\chi_{[1,\infty)}(u).$$

Then $\tilde{V}_k(u) \to \chi_{[l_0,l_1]}(u)$, and

$$\Theta_{\tilde{V}_k}(u) = \int_{l_1}^u \tilde{V}_k(s) \varphi'(s) d s \to \varphi'(l_1)\chi_{[l_0,l_1]}(u) + [\varphi'(l_1) - \varphi'(l_0)] \chi_{[0,l_0]}(u) \quad (u \geq 0).$$

Letting $k \to \infty$ in (5.40) with $U = U_k$ as above and $V = \tilde{V}_k$, we obtain that

$$(5.53)$$

$$\int_{(l_1,l_2]} [\varphi(\xi) - \varphi(l_1)] d\tau(x,t)(\xi) = [\varphi'(l_1) - \varphi'(l_0)] \int_{[0,l_0]} (\xi - l_1) d\tau(x,t)(\xi).$$

By (5.18) and (5.40), we can distinguish two cases.

(a) If $\varphi$ is strictly convex or strictly concave in $[l_1,l_2]$, it follows from (5.52) that

$$\int_{(l_1,l_2]} [\varphi(\xi) - \varphi(l_1)] d\tau(x,t)(\xi) = [\varphi'(l_1) - \varphi'(l_0)] \int_{[0,l_0]} (\xi - l_1) d\tau(x,t)(\xi),$$

where

$$\chi_{(l_1,l_2]}(\xi)|\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)| > 0 \quad \text{and} \quad |\varphi'(l_1) - \varphi'(l_2)| > 0.$$ 

This implies that $\text{supp} \tau(x,t) \subseteq [0,l_1]$. Since $\tau(x,t)$ is a probability measure and $l_1 := u_r(x,t)$,

$$u_r(x,t) = \int_{[0,u_r(x,t)]} \xi d\tau(x,t)(\xi) = \int_{[0,u_r(x,t)]} (\xi - u_r(x,t)) d\tau(x,t)(\xi) + u_r(x,t)$$

(see (5.2) and (5.25)), thus

$$\int_{[0,u_r(x,t)]} (\xi - u_r(x,t)) d\tau(x,t)(\xi) = \tau(x,t)([0,u_r(x,t)]) = 0.$$ 

Hence $\text{supp} \tau(x,t) = \{u_r(x,t)\}$ and (5.50) follows since $\tau(x,t)$ is a probability measure.

Similarly, if $\varphi$ is strictly convex or strictly concave in $(l_0,l_1)$, it follows from (5.53) that $\tau(x,t)([l_0,l_1]) = 0$ (we omit the details). Thus, $\text{supp} \tau(x,t) \subseteq [l_1,\infty)$ and arguing as above we obtain (5.31).

(b) If $\varphi$ is affine in $[l_1 - c,l_1 + c]$ for some $c > 0$, let $I = [\tilde{l}_0,\tilde{l}_2]$ be the maximal interval containing $l_1$ where $\varphi'(\xi) = \varphi'(l_1)$. If $I = [0,\infty)$, (5.51) is satisfied. If $\tilde{l}_2 < \infty$, by
For every \( \tilde{\varphi} \) of Proposition 5.6: in fact, (3.9) follows from (3.5) and (5.50). By Proposition 5.1, it is also a solution in the sense of Definition 3.2. This (by Proposition 5.1) is an entropy solution of problem \( \mathcal{P} \).

Thus the function \( \tilde{\tau} \) obtain (5.51): \( \text{supp} \{ \tilde{l}, 0 \} \) > 0 and \( |\varphi'(1) - \varphi'(2)| > 0 \).

It follows that \( \tau(x, t) \in [\tilde{\varphi}(x, t), \bar{\varphi}(x, t)] \), whence \( \tau(x, t) \in [l_0, \bar{l}_0] \). Similarly, if \( l_0 > 0 \), by (5.49) and the maximality of \( I \), \( \varphi \) is strictly convex (or concave) in \( [l_0 - a, \bar{l}_0] \) for some \( a > 0 \) and affine in \( [l_0, \bar{l}_0] \). Arguing as before, we obtain from (5.53), with \( l_0 \in (l_0 - a, \bar{l}_0) \), that \( \tau(x, t) \in [l_0, \bar{l}_0] \) (we omit the details). Summing up we obtain (5.51): \( \text{supp} \tau(x, t) \subseteq [l_0, \bar{l}_0] = I \).

**Remark 5.4.** If (3.18) is satisfied for all \( \bar{u} > 0 \), it follows from (5.50) and standard properties of narrow convergence of Young measures (see (27)) that \( \{ u_{n_j} \} \) is the subsequent in Proposition 5.1. Therefore, up to a subsequence, \( u_{n_j} \to u_r \) a.e. in \( S \). Hence, if \( \varphi \) is bounded, it follows from the Dominated Convergence Theorem that \( \varphi(u_{n_j}) \to \varphi(u_r) \) in \( L^1((-L, L) \times (0, T)) \) for all \( L > 0 \).

Now we can prove Theorem 3.2.

**Proof of Theorem 3.2.** Let \( \zeta \in C^1([0, T]; C_c^1(\mathbb{R})) \), \( \zeta(\cdot, T) = 0 \) in \( \mathbb{R} \), and let \( L > 0 \) be such that \( \text{supp} \zeta \subseteq (-L, L) \times [0, T] \). By (5.17), with \( U(u) = u \) and \( U(u) = \varphi(u) \),

\[
\begin{align*}
\int_S u_{n_j} \zeta_t \, dx \, dt &= \int_S u_r \zeta_t \, dx \, dt + \int_0^T \langle u_s(\cdot, t), \zeta(\cdot, t) \rangle \, dt, \\
\int_S \varphi(u_{n_j}) \zeta_x \, dx \, dt &= \int_S \varphi^* \zeta_x \, dx \, dt + C \varphi \int_0^T \langle u_s(\cdot, t), \zeta(\cdot, t) \rangle \, dt
\end{align*}
\]

(see also (3.22)). Letting \( j \to \infty \) in (3.17), with \( n = n_j \), we obtain (3.3). Inequality (3.5) is proven similarly, since by arguing as in Proposition 5.2 we get

\[
E(u_{n_j}) \to E(u_{n_j}) + C_E u_{n_j} \quad \text{in } \mathcal{M}(\mathbb{R})
\]

(in this regard, see also (3.15)). Thus the function \( u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R})) \) given by Proposition 5.1 is an entropy solution of problem \( (P) \) in the sense of Young measures. By Proposition 5.3 it is also a solution in the sense of Definition 3.2. This proves the first part of the theorem. The second part is an immediate consequence of Proposition 5.6 in fact, (3.5) follows from (3.4) and (3.6).

Let us end this section by proving Proposition 5.3.

**Proof of Proposition 5.3.** For every \( \tilde{\zeta} \in C^1([0, T]; C_c^1(\mathbb{R})) \), \( \zeta(\cdot, T) = 0 \), we set \( E(u) = U_m(u) = (u - m) \chi_{\{u \geq m\}}(u) \) and \( F(u) = F_m(u) = \int_0^u U_m(\xi) \varphi'(\xi) \, d\xi = (\varphi(u) - \varphi(m)) \chi_{\{u \geq m\}}(u) \) in the entropy inequalities (5.5) (\( m \in \mathbb{N} \)). Then we get

\[
\begin{align*}
\int_S \{ U_m \tilde{\zeta}_t + F_m \tilde{\zeta}_x \} \, dx \, dt + \int_0^T \langle u_s(\cdot, t), \tilde{\zeta}(\cdot, t) \rangle \, dt \\
+ C \varphi \int_0^T \langle u_s(\cdot, t), \tilde{\zeta}(\cdot, t) \rangle \, dt &\geq - \int \mathcal{E}_m(u_{n_j}) \zeta(x, 0) \, dx - \{ u_{n_j}, \zeta(\cdot, 0) \} \, dt,
\end{align*}
\]

where, for a.e. \( (x, t) \in S \),

\[
U_m(x, t) := \int_{(0, \infty)} U_m(\xi) \, d\tau(x, t)(\xi), \quad F_m(x, t) := \int_{(0, \infty)} F_m(\xi) \, d\tau(x, t)(\xi).
\]
As in the proof of Proposition [5.4] there holds \( \int_R U_m(u_0)\zeta(x,0)\,dx \to 0 \) as \( m \to \infty \), whence

\[
(5.54) \quad \int_0^T \left\{ u_s(t), \zeta(t) \right\}_R\,dt + C_\varphi \int_0^T \left\{ u_s(t), \zeta(t) \right\}_R\,dt \geq -\left\{ u_{0s}, \zeta(\cdot,0) \right\}_R.
\]

Let \( \zeta \in C([0,T];C_c(\mathbb{R})) \). By definition of \( L^\infty(0,T;M(\mathbb{R})) \) (see Definition 2.1), the map \( t \mapsto \left\{ u_s(t), \zeta(\cdot,t) \right\}_R \) belongs to \( L^\infty(0,T) \). Hence

\[
(5.55) \quad \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t+h} \left\{ u_s(\cdot,t), \zeta(\cdot,t) \right\}_R\,dt = \left\{ u_s(\cdot,\bar{t}), \zeta(\cdot,\bar{t}) \right\}_R \quad \text{for every } \bar{t} \in (0,T) \setminus N
\]

for some null set \( N \subset (0,T) \) (by separability arguments, \( N \) is independent of \( \zeta \); see the proof of [22] Lemma 3.1). Let \( t_1, t_2 \in (0,T) \setminus N, 0 < t_1 < t_2 < T \). By standard approximation arguments we can choose \( \zeta(x,t) = g_h(t)\zeta(x,t) \) in (5.54), where

\[
(5.56) \quad g_h(t) := \frac{1}{h}(t-t_1)\chi_{\{t_1 \leq s \leq t+h\}}(t) + \frac{1}{h}(t_2-h-t)\chi_{\{t_2 \leq s \leq t+h\}}(t)
\]

and \( h \in (0,\min\{t_2-t_1, T-t_2\}) \). Letting \( h \to 0 \) in (5.54) we obtain that

\[
(5.57) \quad \left\{ u_s(\cdot,t_2), \zeta(\cdot,t_2) \right\}_R \leq \int_{t_1}^{t_2} \left\{ u_s(\cdot,t), \zeta(\cdot,t) \right\}_R\,dt + \left\{ u_s(\cdot,t_1), \zeta(\cdot,t_1) \right\}_R.
\]

Similarly, let \( f_h(t) := \chi_{\{0 \leq t \leq t_1\}}(t) + \frac{1}{h}(t_2-h-t)\chi_{\{t_2 \leq t \leq t+h\}}(t) \). Setting \( \bar{\zeta}(x,t) = f_h(t)\zeta(x,t) \) in (5.57) and letting \( h \to 0^+ \) we obtain that

\[
(5.58) \quad \left\{ u_s(\cdot,t), \zeta(\cdot,t) \right\}_R \leq \int_0^t \left\{ u_s(\cdot,\tau), \zeta(\cdot,\tau) \right\}_R\,d\tau + \left\{ u_{0s}, \zeta(\cdot,0) \right\}_R.
\]

Arguing as in the last part of the proof of Proposition [5.4] we obtain \( (5.20) \) and \( (3.11) \) from respectively (5.57) and (5.58) (we omit the details).

\([ii] \) It follows from (3.8) that for a.e. \( \tau \in (0,T) \) and \( m \in \mathbb{N} \)

\[
(5.59) \quad \left\{ u(\cdot,\tau), \rho_m \right\}_R - \left\{ u_0, \rho_m \right\}_R = \int_0^\tau \left\{ \int_{\Omega_m} \varphi'(u_s(x,t))\rho'_m \,dx + C_\varphi \left\{ u_s(\cdot,t) \right\}_R \right\} \,dt,
\]

where \( \{\rho_m\} \subset C^1_c(\mathbb{R}) \) is such that \( \rho_m = 1 \) in \([-m,m]\), \( \text{supp}\rho_m \subset [-m-1,m+1] \), 0 \( \leq \rho_m \leq 1 \) and \( |\rho'_m| \leq 2 \) in \( \mathbb{R} \), and \( \Omega_m := [-m-1,-m] \cup [m,m+1] \). Since \( u_s \in L^\infty(0,T;M^*(\mathbb{R})) \) and \( \varphi(u_s) \in L^\infty(0,T;L^1(\mathbb{R})) \), a routine proof shows that

\[
\lim_{m \to \infty} \int_0^\tau \varphi(u_s(x,t))\rho'_m(x)\,dx = \lim_{m \to \infty} \int_0^\tau \left\{ u_s(\cdot,t) \right\}_R \,dt = 0.
\]

Since \( \rho_m(x) \to 1 \) for all \( x \in \mathbb{R} \), we also get that \( \left\{ u(\cdot,\tau), \rho_m \right\}_R \to \left\| u(\cdot,\tau) \right\|_{M(\mathbb{R})} \) and \( \left\{ u_0, \rho_m \right\}_R \to \left\| u_0 \right\|_{M(\mathbb{R})} \) as \( m \to \infty \). Letting \( m \to \infty \) in (5.59) we obtain claim \( (ii) \). \( \Box \)

6. Regularity: Proofs

The first regularity result which we prove is Proposition [3.1]. Hence we need

**Lemma 6.1.** Let \((H_1)\) be satisfied. Let \((u,\tau)\) be a Young measure solution of problem \((P)\). Then there exists a null set \( F^* \subset (0,T) \) such that for every \( t_0, t_1 \in (0,T) \setminus F^* \), \( t_0 < t_1 \) and any \( \rho \in C^1_c(\mathbb{R}) \) there holds

\[
(6.1) \quad \left\{ u(\cdot,t_1), \rho \right\}_R - \left\{ u(\cdot,t_0), \rho \right\}_R = \int_{t_0}^{t_1} \left\{ \int_{\mathbb{R}} \varphi^*(x,t)\rho'(x) \,dx + C_\varphi \left\{ u_s(\cdot,t), \rho' \right\}_R \right\} \,dt,
\]

\[
(6.2) \quad \left\{ u(\cdot,t_1), \rho \right\}_R - \left\{ u(\cdot,t_0), \rho \right\}_R = \int_{t_0}^{t_1} \left\{ \int_{\mathbb{R}} \varphi^*(x,t)\rho'(x) \,dx + C_\varphi \left\{ u_s(\cdot,t), \rho' \right\}_R \right\} \,dt.
\]

\]
Proof. Since \( u \in L^\infty(0,T;\mathcal{M}^*(\mathbb{R})) \), there exists a null set \( F_0 \subseteq (0,T) \) such that the spatial disintegration \( u(\cdot,t) \in \mathcal{M}^*(\mathbb{R}) \) is defined for every \( t \in (0,T) \setminus F_0 \). Arguing as in the proof of [22] Lemma 3.1 we can show that there exists a null set \( F^* \subseteq (0,T) \), \( F_0 \subseteq F^* \), such that for every \( \rho \in C_c(\mathbb{R}) \) and \( t \in (0,T) \setminus F^* \)
\begin{equation}
\lim_{q \to \infty} \left\{ 2q \int_{t^{-}}^{t^{+}} \left| (u(\cdot,s),\rho)_{\mathbb{R}} - (u(\cdot,t),\rho)_{\mathbb{R}} \right| ds \right\} = 0. \tag{6.3}
\end{equation}

The proof of (6.1) is based on (3.3) and (6.3). Let \( \rho \in C_c^1(\mathbb{R}) \) and \( t_1 \in (0,T) \setminus F^* \).
By standard regularization arguments we can set \( \zeta = \rho(x)k_q(t) \) in (3.3), with \( q \geq \frac{1}{t-t_1} + 1 \) \( q \in \mathbb{N} \) and \( k_q(t) := \min\{1,q(t_1 + \frac{1}{q} - t)\} \to \chi_{(0,t_1]} \) in \( (0,T) \) as \( q \to \infty \):
\begin{equation}
q \int_{t_1}^{t_1+\frac{1}{q}} (u(\cdot,t),\rho)_{\mathbb{R}} dt - \langle u_0,\rho \rangle_{\mathbb{R}} = \int_0^T \left\{ \int_{\mathbb{R}} \varphi^* (x,t) \rho' (x) dx + C_{\varphi} \langle u_0 (\cdot,t), \rho' \rangle_{\mathbb{R}} \right\} k_q(t) dt
\end{equation}
Letting \( q \to \infty \) we obtain (6.1) from (3.7) and (6.3). Subtracting from (6.1) the same inequality with \( t_1 \) replaced by \( t_0 \), we obtain (6.2).

Proof of Proposition 6.7. Let \( F^* \subseteq (0,T) \) be the null set given by Lemma 6.1.
Let \( \{\tau_n\} \subseteq (0,T) \setminus F^*, \tau_n \to 0^+ \) as \( n \to \infty \). Since, by (3.7), \( u \in L^\infty(0,T;\mathcal{M}^*(\mathbb{R})) \) and \( \varphi^* \in L^\infty(0,T;L^1(\mathbb{R})) \), it follows from (6.1) that \( (u(\cdot,\tau_n),\rho)_{\mathbb{R}} \to (u_0,\rho)_{\mathbb{R}} \) for all \( \rho \in C_c^1(\mathbb{R}) \). Since, by Definition 2.4.(ii), \( \sup_n \|u(\cdot,\tau_n)\|_{\mathcal{M}(\mathbb{R})} \leq C \), there exist \( \mu_0 \in \mathcal{M}^*(\mathbb{R}) \) and a subsequence \( \{\tau_{n_k}\} \) such that \( u(\cdot,\tau_{n_k}) \rightharpoonup \mu_0 \) in \( \mathcal{M}(\mathbb{R}) \) as \( k \to \infty \). By standard density arguments, this implies that \( \mu_0 = u_0 \). Hence \( u(\cdot,\tau_n) \rightharpoonup u_0 \) along the whole sequence \( \{\tau_n\} \), and (6.11) follows from (6.1) and the arbitrariness of \( \{\tau_n\} \).
Similarly, it follows from (6.2) that \( (u(\cdot,\tau_n),\rho)_{\mathbb{R}} \to (u(\cdot,t_0),\rho)_{\mathbb{R}} \) for all \( \rho \in C_c^1(\mathbb{R}) \) as \( \tau_n \to t_0 \) if \( t_0,\tau_n \in (0,T) \setminus F^* \) and we obtain (6.12).
To prove (6.13) we observe that, given \( t_0 \in [0,T] \) and two sequences \( \tau_n^1 \) and \( \tau_n^2 \) contained in \( (0,T) \setminus F^* \) and converging to \( t_0 \), \( (u(\cdot,\tau_n^1) - u(\cdot,\tau_n^2),\rho)_{\mathbb{R}} \to 0 \) for all \( \rho \in C_c(\mathbb{R}) \). Hence, if \( t_0 \notin F^* \), the continuous extension of \( u(\cdot,t) \) from \( (0,T) \setminus F^* \) with respect to the weak* topology is well-defined.

Let us now prove the results of Subsection 8.3. As explained there, replacing \( x \) by \( -x \) and \( \varphi \) by \( -\varphi \), it suffices to prove Proposition 6.3 and Proposition 6.4. Moreover, replacing \( x \) by \( -x \) and \( \varphi \) by \( -\varphi \), it suffices to do so by assuming that \((H_2)\) is satisfied with \( \varphi'' < 0, \varphi' > 0 \) in \( (0,\infty) \) (see Remark 3.6). Therefore, we make use of the following assumption:
\begin{equation}
(\tilde{H}_2) \begin{cases}
\varphi \in C_c^\infty([0,\infty)), \ C_\varphi = 0; \\
\varphi''(u) < 0, \text{ and there exist } H \geq -1, K > 0 \text{ such that } \varphi''(u) [H \varphi(u) + K] \leq -[\varphi'(u)]^2 < 0 \text{ for all } u \in [0,\infty) \end{cases}
\end{equation}
(see 1).}

First we prove some estimates of the constructed entropy solutions. As already said, these estimates are analogous to the Aronson-Bénilan inequality for the convex case \( u^p, p > 1 \) (see [1]).

Proposition 6.2. Let \((H_1)\) and \((\tilde{H}_2)\) be satisfied, and let \( u \) be an entropy solution of problem (P) given by Theorem 7.3 Then for a.e. \( 0 < t_1 < t_2 \leq T \)
\begin{equation}
\varphi(u_r)(\cdot,t_2) + \frac{K}{H} \leq \left( \frac{t_2}{t_1} \right)^H \varphi(u_r)(\cdot,t_1) + \frac{K}{H} \quad \text{a.e. in } \mathbb{R} \text{ if } H \neq 0, \tag{6.4}
\end{equation}
\[
\varphi(u_\varepsilon)(\cdot,t_2) - K \log(t_2) \leq \varphi(u_\varepsilon)(\cdot,t_1) - K \log t_1 \quad \text{a.e. in } \mathbb{R} \quad \text{if } H = 0.
\]

Moreover, if there exists \( L > 0 \) such that
\[
(H_3) \quad H \varphi(u) + K \leq L(1 + u) \varphi'(u) \quad \text{for } u \geq 0,
\]
then \( u_t \in \mathcal{M}(\Omega \times (\tau,T)) \), \([\varphi(u_\varepsilon)]_t \in \mathcal{M}(\Omega \times (\tau,T))\), and \( u \in C((0,T];\mathcal{M}(\Omega)) \) for every bounded open set \( \Omega \subset \mathbb{R} \) and \( \tau > 0 \).

**Remark 6.1.** If \( \varphi(u) = \text{sgn} \, p \, [(1 + u)^p - 1] \quad (p < 1, p \neq 0) \), (6.4) becomes
\[
u_t(\cdot,t_2) \leq \left( \frac{t_2}{t_1} \right)^{\frac{1}{p-1}}[1 + u(\cdot,t_1)] - 1 \quad \text{a.e. in } \mathbb{R}, \text{ for a.e. } 0 < t_1 \leq t_2 \leq T
\]
(see Remark 3.3). Similarly, if \( \varphi(u) = \log(1 + u) \), (6.5) becomes
\[
u_t(\cdot,t_2) \leq \left( \frac{t_2}{t_1} \right)[1 + u(\cdot,t_1)] - 1 \quad \text{a.e. in } \mathbb{R}, \text{ for a.e. } 0 < t_1 \leq t_2 \leq T.
\]

Let \( \hat{H}_2 \) hold. To prove Proposition 6.2 we use a different regularization of \( (P_n) \):
\[
(V^c_n) \quad \begin{cases} y_{n,t}^c + [\varphi'(y_n^c)]x = \varepsilon \varphi(y_n^c) & \text{in } S \\ y_n^c = u_{0n}^c & \text{in } \mathbb{R} \times \{0\}, \end{cases}
\]
where \( \{u_{0n}^c\} \) satisfies (4.1)-(4.2). The existence, uniqueness and regularity results recalled in Section 3 for problem \( (Q^c_n) \), as well as the a priori estimates in Lemma 4.4 and the convergence results in Lemma 4.3 (i), continue to hold for solutions of \( (V^c_n) \) (see 171). In particular, there exist a sequence \( \{y_{n}^{m^c}\} \) and \( y_n \in L^\infty(S) \cap L^\infty(0,T;L^1(\mathbb{R})) \) such that \( y_{n}^{m^c} \rightharpoonup y_n \) in \( L^\infty(S) \) and for all \( L > 0 \)
\[
y_{n}^{m^c} \to y_n \quad \text{in } L^1((-L,L) \times (0,T)) \quad \text{as } m \to 0.
\]

From \( (V^c_n) \), for every \( E \) convex, \( F' = E' \varphi' \), and \( \zeta \) as in Definition 3.3 we get
\[
\int_S \{ E(y_{n}^{m^c}) \zeta + F(y_{n}^{m^c}) \zeta_x \} \, dx \, dt + \int_\mathbb{R} E(u_{0n}^c) \zeta(x,0) \, dx \geq \varepsilon_m \int_S F'(y_{n}^{m^c}) y_{n}^{m^c} \zeta_x \, dx \, dt.
\]
Arguing as in the proof of Proposition 4.3 and letting \( \varepsilon_m \to 0 \), we obtain that
\[
\int_S \{ E(y_n) \zeta + F(y_n) \zeta_x \} \, dx \, dt \geq - \int_\mathbb{R} E(u_{0n}^c) \zeta(x,0) \, dx.
\]
So \( y_n \) satisfies (3.10) and, by Kružkov’s uniqueness theorem, \( y_n = u_n \). Hence we have shown:

**Lemma 6.3.** Let \( (H_1) \) and \( \hat{H}_2 \) be satisfied, and let \( u_n \) be the unique entropy solution of the problem \( (P_n) \) given by Proposition 4.3. Then there exists a subsequence \( \{y_{n}^{m^c}\} \) of solutions of \( (V^c_n) \) such that \( y_{n}^{m^c} \rightharpoonup u_n \) in \( L^\infty(S) \) and satisfies (6.6).

**Lemma 6.4.** Let \( (H_1) \) and \( \hat{H}_2 \) be satisfied. Then
\[
\frac{\partial}{\partial t} \left[ \frac{H \varphi'(y_n^c)(\cdot,t) + K}{t^H} \right] \leq 0 \quad \text{in } \mathbb{R} \quad \text{if } H > 0,
\]
\[
\frac{\partial}{\partial t} \left[ \varphi(y_n^c)(\cdot,t) - K \log t \right] \leq 0 \quad \text{in } \mathbb{R} \quad \text{if } H = 0.
\]
for all \( t \in (0,T) \), \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). Moreover, if \( (H_3) \) is satisfied, then
\[
ty_{n,t}^c \leq L(1 + y_n^c) \quad \text{in } S.
\]
Proof. For convenience we set $A = \varepsilon \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}$, thus $y_{nt}^\varepsilon = A[\varphi(y_{n_t}^\varepsilon)]$ in $S$. Let

$$z_{nt}^\varepsilon := ty_{nt}^\varepsilon - g(y_{n_t}^\varepsilon), \quad \text{where} \quad g(y_{n_t}^\varepsilon) := \frac{H\varphi(y_{n_t}^\varepsilon) + K}{\varphi'(y_{n_t}^\varepsilon)} \quad (n \in \mathbb{N}).$$

It follows from $(\tilde{H}_2)$ and a straightforward calculation that

$$z_{nt}^\varepsilon = A[\varphi'(y_{n_t}^\varepsilon)]z_{nt}^\varepsilon + \left[H + 1 - g'(y_{n_t}^\varepsilon)\right]\frac{z_{nt}^\varepsilon + g(y_{n_t}^\varepsilon)}{t} \leq A[\varphi'(y_{n_t}^\varepsilon)]z_{nt}^\varepsilon + \left[H + 1 - g'(y_{n_t}^\varepsilon)\right]\frac{z_{nt}^\varepsilon}{t}$$

in $S$. Since $z_{nt}^\varepsilon = -g(u_{n_t}^\varepsilon) \leq 0$ in $\mathbb{R} \times \{0\}$, it follows from the comparison principle for parabolic equations that $z_{nt}^\varepsilon \leq 0$ in $S$ for all $n \in \mathbb{N}$. Hence $ty_{nt}^\varepsilon(\cdot, t) \leq g(y_{n_t}^\varepsilon)(\cdot, t)$ in $\mathbb{R}$ for all $t \in (0, T)$, which implies (6.7) and, if $(H_3)$ is satisfied, (6.9). □

Proof of Proposition 6.2. Let $\{y_{n_t}^m\}$ be as in the proof of Lemma 6.3. By (6.7)–(6.8),

$$\varphi(y_{n_t}^m)(\cdot, t_2) + \frac{K}{H} \leq \left(\frac{t_2}{t_1}\right)^H \left[\varphi(y_{n_t}^m)(\cdot, t_1) + \frac{K}{H}\right] \quad \text{in} \; \mathbb{R} \; \text{if} \; H \neq 0,$$

$$\varphi(y_{n_t}^m)(x, t_2) - K \log(t_2) \leq \varphi(y_{n_t}^m)(x, t_1) - K \log t_1 \quad \text{in} \; \mathbb{R} \; \text{if} \; H = 0$$

for all $0 < t_1 \leq t_2 \leq T$ and $n \in \mathbb{N}$. Hence, by Lemma 6.3,

$$\varphi(u_n)(\cdot, t_2) + \frac{K}{H} \leq \left(\frac{t_2}{t_1}\right)^H \left[\varphi(u_n)(\cdot, t_1) + \frac{K}{H}\right] \quad \text{a.e. in} \; \mathbb{R} \; \text{if} \; H \neq 0,$$

$$\varphi(u_n)(\cdot, t_2) - K \log(t_2) \leq \varphi(u_n)(\cdot, t_1) - K \log t_1 \quad \text{a.e. in} \; \mathbb{R} \; \text{if} \; H = 0$$

for a.e. $0 < t_1 \leq t_2 \leq T$. Since $\varphi'$ is strictly decreasing in $[0, \infty)$ (recall that $\varphi$ is concave by assumption $(\tilde{H}_2)$), possibly extracting another subsequence (denoted again by $\{u_n\}$), $\varphi(u_n) \rightarrow \varphi(u_r)$ a.e. in $S$ (see Remark 5.4). Letting $j \rightarrow \infty$ in (6.10)–(6.11) (with $n = n_j$) we obtain (6.4)–(6.5).

Let $\Omega = (L, L)$. If $(H_3)$ is satisfied, it follows from (6.9) and (6.3) that

$$t \int_\Omega [y_{n_t}^m]^+ (x, t) \, dx \leq L |\Omega| + |u_0|_{\mathcal{M}(\mathbb{R})} \quad \text{for all} \; t \in (0, T].$$

Since $|y_{n_t}^m|^+ = 2[y_{n_t}^m]^+ - y_{n_t}^m$ a.e. in $S$, there exists $C_1 > 0$ such that

$$\int_T^T \int_\Omega |y_{n_t}^m|^+ (x, t) \, dx \, dt \leq 2(2T - \tau) \frac{L|\Omega| + |u_0|_{\mathcal{M}(\mathbb{R})}}{\tau} + \int_\Omega \left\{y_{n_t}^m(\cdot, t) - y_{n_t}^m(\cdot, T)\right\} \leq C_1 \frac{\tau}{T}$$

for all $\tau > 0$, $\varepsilon > 0$ and $n \in \mathbb{N}$, and, by (6.11),

$$\int_\Omega \int_T^T |y_{n_t}^m|^+ \, dx \, dt \leq \frac{C_1}{\tau}, \quad \int_\Omega \int_T^T [\varphi(y_{n_t}^m)]^+ \, dx \, dt \leq \frac{MC_1}{\tau}.$$

Let $\{\varepsilon_m\}$ and $\{n_j\}$ be as in Lemma 6.3 and (6.1). Then

$$\lim_{n_j \to \infty} \lim_{\varepsilon_m \to 0} \left(y_{n_t}^m, \zeta\right)_{\Omega \times (\tau, T)} = \left(u, \zeta\right)_{\Omega \times (\tau, T)} \quad \text{for all} \; \zeta \in C^1_c(\Omega \times (\tau, T)),$$

whence, by (6.13) and the lower semicontinuity of the total variation,

$$|u_t|_{\mathcal{M}(\Omega \times (\tau, T))} \leq C_1 \frac{\tau}{T}.$$

Similarly, by (6.10), (6.25) and Proposition 6.5,

$$\lim_{n_j \to \infty} \lim_{\varepsilon_m \to 0} \left(\varphi(y_{n_t}^m), \zeta_t\right)_{\Omega \times (\tau, T)} = \int_\Omega \int_T \varphi(\zeta_t) \, dx \, dt \quad \text{for all} \; \zeta \in C^1_c(\Omega \times (\tau, T)).$$
and, by (6.13) and the lower semicontinuity of the total variation,
\[ \| \varphi(u_t) \|_{\mathcal{M}([\tau,T])} \leq \frac{MC_{\Omega}}{\tau}. \]

It remains to prove that \( u \in C((0,T]; \mathcal{M}(\Omega)) \). Observe that for all \( t_1, t_2 \in (0,T] \), \( 0 < \tau < t_1 < t_2 \), and \( \rho \in C^2_c(\mathbb{R}) \), \( 0 \leq \rho \leq 1 \) in \( \Omega \),
\[ \int_{\Omega} |y_n^\varepsilon(x,t_2) - y_n^\varepsilon(x,t_1)| \, dx \leq \int_{\mathbb{R}} |y_n^\varepsilon(x,t_2) - y_n^\varepsilon(x,t_1)| \rho(x) \, dx \leq \int_{t_1}^{t_2} \int_{\mathbb{R}} |y_n^\varepsilon| \rho \, dx \, dt = \int_{t_1}^{t_2} \int_{\mathbb{R}} (2 |y_n^\varepsilon| - y_n^\varepsilon) \rho \, dx \, dt = 2 \int_{t_1}^{t_2} \int_{\mathbb{R}} (\varphi(y_n^\varepsilon) \rho'(x) + \varepsilon \varphi'(y_n^\varepsilon)) \, dx \, dt \leq 2 \int_{t_1}^{t_2} \int_{\mathbb{R}} (\varphi(y_n^\varepsilon) \rho'(x) + \varepsilon \varphi'(y_n^\varepsilon)) \, dx \, dt, \]
where we have used (6.12). We let \( \varepsilon = \varepsilon_m \to 0 \) and use (5.1) and (4.19):
\[ \int_{\Omega} |u_n(x,t_2) - u_n(x,t_1)| \, dx \leq 2 \int_{t_1}^{t_2} \int_{\mathbb{R}} (\varphi(u_n) \rho'(x) + \varepsilon \varphi'(u_n)) \, dx \, dt \leq \frac{C}{\tau} |t_2 - t_1|. \]

By (3.16) and the lower semicontinuity of the total variation,
\[ \| u(\cdot,t_2) - u(\cdot,t_1) \|_{\mathcal{M}(\Omega)} \leq \frac{C}{\tau} |t_1 - t_2| \quad \text{for a.e. } 0 < \tau < t_1 < t_2 \leq T. \]
So we may define \( u(\cdot,t) \) for all \( t \in [\tau,T] \) such that \( u \in C([\tau,T]; \mathcal{M}(\Omega)) \). Since \( \tau > 0 \) is arbitrary, the proof is complete. \( \square \)

To prove Proposition 3.4, we need the following lemma.

**Lemma 6.5.** Let \( (H_1) \) be satisfied, and let \( u \) be the solution of problem (P) given by Theorem 3.2. Let \( \{ u_{n_k} \} \) be as in the proof of Theorem 3.2. Then for a.e. \( t \in (0,T) \) and all \( x_0 \in \text{supp} u_\cdot \), there exists a sequence \( \{ x_{0k} \} \subset \mathbb{R} \) and a subsequence \( \{ u_{n_k} \} \) of \( \{ u_{n_k} \} \) such that \( x_{0k} \to x_0 \) and \( u_{n_k}(x_{0k},t) \to \infty \) as \( k \to \infty \).

**Proof.** Let \( x_0 \in \text{supp} u_\cdot \). We may assume that the convergence in (6.16) is satisfied for this \( t \). Since \( x_0 \in \text{supp} u_\cdot \), there is no neighbourhood \( I_\delta(x_0) \) such that the sequence \( \{ u_{n_k} \} \) lies in a bounded subset of \( L^\infty(I_\delta(x_0)) \). Otherwise, up to a subsequence, \( u_{n_k} \to f_t \) in \( L^\infty(I_\delta(x_0)) \) for some \( f_t \in L^\infty(I_\delta(x_0)) \), \( f_t \geq 0 \). However, this would imply that \( u_\cdot \to 0 \) in \( I_\delta(x_0) \), a contradiction.

Setting \( \delta = 1/k \), we obtain that \( \sup_{n_k \in \mathbb{N}} \| u_{n_k} \|_{L^\infty(I_{1/k}(x_0))} = \infty \) for all \( k \in \mathbb{N} \). Hence for all \( k \in \mathbb{N} \) there exists \( x_{0k} \in I_{1/k}(x_0) \) such that \( u_{n_k}(x_{0k},t) \geq k \).

**Proof of Proposition 3.4.** As pointed out above, it suffices to prove equality (3.21) by assuming \((H_2)\). Let \( \{ u_{n_j} \} \) be as in the proof of Lemma 6.5. By Lemma 6.3 for every \( n_j \in \mathbb{N} \) there exists \( \varepsilon_m \to 0 \) such that
\[ y_{n_j}^{\varepsilon_m}(\cdot,t) \to u_{n_j}(\cdot,t) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}) \quad \text{as } \varepsilon_m \to 0 \quad \text{for a.e. } t \in (0,T). \]
By the proof of Lemma 6.5 for all $t \in (0, T)$

$$
\varepsilon_m \frac{\phi(y_{n_j}^m)(\cdot, t) - \phi(y_{n_j}^m)(\cdot, t)}{t} = \frac{g(y_{n_j}^m)(\cdot, t)}{t} \in \mathbb{R},
$$
where $g(u) = \frac{\phi(u) + K}{\phi(u)} > 0$. For every $\varepsilon < T$, let $\rho \in C^2_c((\bar{x}, \bar{x}))$, $\rho \geq 0$. Multiplying (6.16) by $\rho/\phi(y_{n_j}^m(\cdot, t))$, integrating by parts and setting $\Psi(y) = \int_0^y \phi(u) du$, we find that

$$
\int_{\bar{x}}^{\bar{x}} \Psi(y_{n_j}^m)(x, t)[\varepsilon_m \rho''(x) + \rho'(x)] dx \leq \frac{1}{t} \int_{\bar{x}}^{\bar{x}} \rho(x) dx - \varepsilon_m \int_{\bar{x}}^{\bar{x}} \frac{\phi'(y_{n_j}^m) g'(y_{n_j}^m)(y_{n_j}^m)^2}{\phi(y_{n_j}^m)^2} dx \rho(x) dx \leq \frac{1}{t} \int_{\bar{x}}^{\bar{x}} \rho(x) dx
$$
(observe that by $(\tilde{H}_2)$ there holds $g'(u) \geq H + 1 \geq 0$ and $\Psi$ is bounded). Hence, by (6.14),

$$
\int_{\bar{x}}^{\bar{x}} \Psi(y_{n_j}^m)(x, t) \rho'(x) dx \leq \frac{1}{t} \int_{\bar{x}}^{\bar{x}} \rho(x) dx.
$$

Let $x_0 \in \text{supp} u_n(\cdot, t)$, and let $\{u_{n_k}\} \in \mathbb{R}$, $\{u_{n_k}\}$ be as in Lemma 6.5 for a.e. $t \in (0, T)$. Let $\bar{x} > x_0$ be fixed. Since $x_{n_k} \to x_0$, there exists $k \in \mathbb{N}$ such that $\bar{x} > x_{n_k}$ for all $k > \bar{k}$. Consider any sequence $\{\rho_{n_k}\} \subset C^1_c((x_{n_k}, \bar{x}))$, $0 \leq \rho_{n_k} \leq 1$, $\rho_{n_k} \to \chi_{(x_{n_k}, \bar{x})}$ in $\mathbb{R}$. Without loss of generality, we may assume that both $x_{n_k}$ and $\bar{x}$ are Lebesgue points of $u_{n_k}(\cdot, t)$ for all $k \in \mathbb{N}$. Setting $\rho = \rho_{n_k}$ and $x = x_{n_k}$ in (6.16), letting $m \to \infty$ we find that

$$
\Psi(u_{n_k})(x_{n_k}, t) \leq \Psi(u_{n_k})(\bar{x}, t) + \frac{1}{t}(\bar{x} - x_{n_k}) \quad \text{for all } n_k.
$$
Since $\Psi$ is continuous, by Lemma 6.5 and Remark 5.4 (recall that $\phi$ satisfies (3.18) since $\phi$ is strictly concave by assumption $(\tilde{H}_2)$), letting $n_k \to \infty$ gives

$$
\Psi(u)(\bar{x}, t) + \frac{1}{t}(\bar{x} - x_0) \geq \Psi(\infty) \quad \text{for a.e. } \bar{x} > x_0,
$$
whence by the invertibility of $\Psi$

$$
u_\tau(\bar{x}, t) \geq \Psi^{-1}\left(\Psi(\infty) - \frac{1}{t}(\bar{x} - x_0)\right) \quad \text{for a.e. } \bar{x} > x_0.
$$

Letting $\bar{x} \to x_0^+$ in the previous inequality we obtain (3.21). $\square$

To prove Theorem 6.5 we need the following result.

**Proposition 6.6.** Let $(H_1)$ be satisfied. Let $C_\phi = 0$, and let $u$ be a solution of problem (P). Then for a.e. $0 \leq t_1 \leq t_2 \leq T$:

1. the map $x \mapsto \Phi(x, t_1, t_2) = \int_{t_1}^{t_2} \phi(u_r)(x, t) dt$ belongs to $BV(\mathbb{R})$;
2. for all $x_0, x_1 \in \mathbb{R}$, $x_0 \leq x_1$,

$$
u(\cdot, t_2)((x_0, x_1]) - \nu(\cdot, t_1)((x_0, x_1]) = \Phi(x_0^+, t_1, t_2) - \Phi(x_0^-, t_1, t_2),
$$
$$
u(\cdot, t_2)((x_0, x_1]) - \nu(\cdot, t_1)((x_0, x_1]) = \Phi(x_0^-, 0, t_2) - \Phi(x_1^+, 0, t_2).
$$

**Remark 6.2.** It is easily seen that, if $C_\phi \not= 0$, equalities (6.18)-(6.19) are replaced by

$$
\nu(\cdot, t_2)((x_0, x_1]) - \tau_{C_\phi(t_2-t_1)}(\nu(\cdot, t_1))((x_0, x_1]) = \Phi(x_0^+, t_1, t_2) - \Phi(x_0^-, t_1, t_2),
$$
$$
\nu(\cdot, t_2)((x_0, x_1]) - \tau_{C_\phi t_2}(\nu_0)((x_0, x_1]) = \Phi(x_0^-, 0, t_2) - \Phi(x_1^+, 0, t_2),
$$

where $\nu_0$ is defined as in (6.18).
where now
\[
\Phi(x, t_1, t_2) := \int_{t_1}^{t_2} [\varphi(u_r) - C_r u_r](x + C_r(t - t_1), t) \, dt.
\]

**Proof of Proposition 6.6.** (i) By (3.1), \( \int_{t_1}^{t_2} \varphi(u_r)(x, t) \, dt \leq M \int_{t_1}^{t_2} u_r(x, t) \, dt \in L^1(\mathbb{R}) \). We argue as in the proof of Proposition 3.3 (see (5.55)): there exists a null set \( N \subset (0, T) \) such that
\[
\lim_{h \to 0} \frac{1}{h} \int_{t_1}^{t_1+h} (u(\cdot, t), \rho)_R \, dt = (u(\cdot, \bar{t}), \rho)_R \quad \text{for all } \rho \in C_c(\mathbb{R}) \text{ and } \bar{t} \in (0, T) \smallsetminus N.
\]
Let \( t_1, t_2 \in (0, T) \smallsetminus N, \, 0 < t_1 < t_2 < T, \, \rho \in C^1_c(\mathbb{R}), \) and \( \zeta(x, t) = g_h(t)\rho(x) \), with \( g_h \) as in (5.30). Since \( C_0 = 0 \), we obtain from (5.30) that
\[
\frac{1}{h} \int_{t_1}^{t_1+h} (u(\cdot, t), \rho)_R \, dt - \frac{1}{h} \int_{t_2}^{t_2+h} (u(\cdot, t), \rho)_R \, dt + \int_0^{T} g_h(t) \rho'(x) \varphi(u_r)(x, t) \, dx \, dt = 0.
\]
Letting \( h \to 0 \), it follows from (6.22) that
\[
\langle u(\cdot, t_2), \rho \rangle_R - \langle u(\cdot, t_1), \rho \rangle_R = \int_R \Phi(x, t_1, t_2) \rho'(x) \, dx.
\]
Hence the distributional derivative \( \Phi_x(x, t_1, t_2) \) belongs to \( \mathcal{M}(\mathbb{R}) \).

(ii) We set, for \( m \in \mathbb{N} \) and \( x \in \mathbb{R} \),
\[
\rho_m(x) := m \left( -x - x_0 + \frac{1}{m} \right) \chi_{[x_0 - \frac{1}{m}, x_0]}(x) + \chi_{(x_0, x_1]}(x) + m \left( -x - x_1 + \frac{1}{m} \right) \chi_{[x_1, x_1 + \frac{1}{m}]}.\]

By standard regularization arguments we can choose \( \rho = \rho_m \) in (6.23):
\[
\langle u(\cdot, t_2), \rho_m \rangle_R - \langle u(\cdot, t_1), \rho_m \rangle_R = m \int_{x_0 - \frac{1}{m}}^{x_0} \Phi(x, t_1, t_2) \, dx - m \int_{x_1}^{x_1 + \frac{1}{m}} \Phi(x, t_1, t_2) \, dx.
\]
By the Dominated Convergence Theorem, \( \langle u(\cdot, t_i), \rho_m \rangle_R \to u(\cdot, t_i)([x_0, x_1]) \) as \( m \to \infty \) \( (i = 1, 2) \), whereas, by part (i),
\[
m \int_{x_0 - \frac{1}{m}}^{x_0} \Phi(x, t_1, t_2) \, dx \to \Phi(x_0^-, t_1, t_2), \quad m \int_{x_1}^{x_1 + \frac{1}{m}} \Phi(x, t_1, t_2) \, dx \to \Phi(x_1^+, t_1, t_2).
\]
Hence (6.18) follows from (6.24). The proof of (6.19) is similar. \( \square \)

**Remark 6.3.** Observe that, by (3.19) and (6.20) with \( x_0 = x_1 = x \), all entropy solutions of problem (P) satisfy for a.e. \( 0 \leq t_1 \leq t_2 \leq T \)
\[
\Phi(x^-, t_1, t_2) \leq \Phi(x^+, t_1, t_2) \quad \text{for all } x \in \mathbb{R}
\]
with \( \Phi \) defined by (6.21).

Now we are ready to prove Theorem 3.5 and Proposition 3.6. As pointed out at the beginning of this section, in doing so it is not restrictive to assume that \( \dot{H}_2 \) holds.

**Proof of Theorem 3.5.** (i) By (6.19), for a.e. \( 0 \leq t \leq T \)
\[
u_s(t)(\{x_0\}) = u_{0s}(\{x_0\}) + \Phi(x^0_0, 0, t) - \Phi(x^0_0, 0, t) \geq u_{0s}(\{x_0\}) - \|\varphi\|_{L^\infty(0, \infty)} t,
\]
whence \( u_s(t)(\{x_0\}) > 0 \) if \( t \in \left( 0, \frac{u_{0s}(\{x_0\})}{\|\varphi\|_{L^\infty(0, \infty)}} \right) \). Hence (6.22) follows.
(ii) Let $u_n$ be the entropy solution of problem $(P_n)$ given by Proposition 4.3. We argue as in the proof of Proposition 6.6 for all $n \in \mathbb{N}$ the map $x \mapsto \Phi_n(x, t_1, t_2) := \int_{t_1}^{t_2} \varphi(u_n)(x, t) \, dt$ belongs to $BV(\mathbb{R})$, and for a.e. $0 \leq t_1 \leq t_2 \leq T$ and a.e. $x_0 \leq x_1 \in \mathbb{R}$
\[
\int_{x_0}^{x_1} u_n(x, t_2) \, dx - \int_{x_0}^{x_1} u_n(x, t_1) \, dx = \Phi_n(x_0, t_1, t_2) - \Phi_n(x_1, t_1, t_2).
\]

Letting $x_1 \to \infty$, it follows from (6.20) and (8.14) that
\[
\int_{t_1}^{t_2} \varphi(u_n)(x, t) \, dt \leq \|u_0\|_{\mathcal{M}(\mathbb{R})} \quad \text{for } n \in \mathbb{N} \text{ and a.e. } x \in \mathbb{R}.
\]

Let $\{y_{m}^{\varepsilon}\}$ be the subsequence used in the proof of Lemma 6.3. By (6.7) and (6.8), for every $0 < t_1 \leq t \leq T$ and $x \in \mathbb{R}$
\[
\int_{t_1}^{t} \varphi(y_{m}^{\varepsilon})(x, s) \, ds = \frac{1}{H} \int_{t_1}^{t} \frac{H \varphi(y_{m}^{\varepsilon})(x, s)}{s^H} s^H \, ds - \frac{K}{H} (t - t_1) \geq \frac{H \varphi(y_{m}^{\varepsilon})(x, t) + K}{H} \frac{t + 1}{H + 1} - \frac{K}{H} (t - t_1) \quad \text{if } H \neq 0,
\]
\[
\int_{t_1}^{t} \varphi(y_{m}^{\varepsilon})(x, s) \, ds = \int_{t_1}^{t} [\varphi(y_{m}^{\varepsilon})(x, s) - K \log s] \, ds + K \int_{t_1}^{t} \log s \, ds \geq [\varphi(y_{m}^{\varepsilon})(x, t) - K \log t] (t - t_1) + K [t \log t - t] - K [t_1 \log t_1 - t_1] \quad \text{if } H = 0.
\]

Letting $\varepsilon_m \to 0$, by (6.20) we obtain that for a.e. $t \in (t_1, T)$ and a.e. $x \in \mathbb{R}$
\[
\|u_0\|_{\mathcal{M}(\mathbb{R})} \geq \Phi_n(x, t_1, t) \geq \frac{H \varphi(u_n)(x, t) + K}{H} \frac{t + 1}{H + 1} - \frac{K}{H} (t - t_1) \quad \text{if } H \neq 0,
\]
\[
[\varphi(u_n)(x, t) - K] (t - t_1) + K t_1 \log \frac{t}{t_1} \quad \text{if } H = 0.
\]

Letting $t_1 \to 0^+$ we find in both cases that
\[
(6.26) \quad \varphi(u_n)(x, t) \leq \frac{(H + 1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{t} + K \quad \text{for a.e. } t \in (t_1, T) \text{ and a.e. } x \in \mathbb{R}
\]

(recall that we have assumed $H > -1$ if $\varphi$ is bounded; otherwise, if $\varphi$ is unbounded, there holds $H \geq 0$ since $\varphi' > 0$ and $H \varphi + K > 0$ in $[0, \infty)$ by $(\hat{H}_2)$). If $\lim_{u \to \infty} \varphi(u) = \gamma < \infty, K < \gamma$ and $H > -1$, the sequence $\{u_n(\cdot, t)\}$ lies in a bounded subset of $L^\infty(\mathbb{R})$ (thus, by (5.10) $u_\gamma(\cdot, t) = 0$, and $u_\gamma(\cdot, t)$ is a bounded subset of $L^\infty(\mathbb{R})$) for a.e. $t \in (0, T)$ such that
\[
(H + 1)\|u_0\|_{\mathcal{M}(\mathbb{R})} + K \gamma < t \quad \Leftrightarrow \quad t > \frac{(H + 1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - K}.
\]

This proves claim (ii)-(a).

If $\gamma = \infty$, there holds $H \geq 0$ since $H \varphi + K > 0$ in $[0, \infty)$ (see $(\hat{H}_2)$). Then by (6.20) the sequence $\{u_n(\cdot, t)\}$ lies in a bounded subset of $L^\infty(\mathbb{R})$ for a.e. $t \in (0, T)$, hence by (5.10) as $n \to \infty$ we obtain that $t_0 = 0$. Hence claim (ii)-(b) follows. This completes the proof.

\[ \Box \]

**Remark 6.4.** As we claimed in Remark 5.5 in Theorem 5.5 (ii) we may relax hypothesis $(H_2)$ to $(H_{2,k})$, with $k > 0$. To prove this, for every $u_0 \in \mathcal{M}'(\Omega)$ let $\{u_{0n}\}$ be any sequence as in (3.14)-(3.15), and let $u_n$ be the entropy solution of problem $(P_n)$. Set $v_{0n} := G_k(u_{0n})$, where $G_k(u) := (u - k)^+$ for every $u \geq 0$, and let $v_n$ be the entropy solution of problem
\[
\begin{cases}
  v_{nt} + [\varphi_k(v_n)]_x = 0 & \text{in } S \\
v_n = v_{0n} & \text{in } \mathbb{R} \times \{0\}
\end{cases}
\]
\((\varphi_k(u) = \varphi(u + k) - \varphi(k))\). A standard calculation shows that \(G_k(u_n)\) is an entropy subsolution of the above problem, whence

\[
G_k(u_n) \leq v_n \text{ a.e. in } S.
\]

Following the proof of Theorem 3.3, the sequence \(\{v_n\}\) converges to an entropy solution \(v\) of problem \((P)\) with initial datum \(v_0 = u_{0s} + G_k(u_{0r})\). Moreover, by assumption \((H_{2,k})\), \(\varphi_k\) satisfies \((H_2)\) and we may apply Theorem 3.5(ii) to \(v\). Therefore the conclusion follows from (6.27).

**Proof of Proposition 3.6.** By the proof of Proposition 3.4 inequality (6.17) is satisfied for a.e. \(t \in (0,T)\) and all \(x_0 \in \text{supp } u_s(\cdot,t)\). We fix such \(t\). Let \(x_1 \in \text{supp } u_s(\cdot,t)\) and set \(I_1 := (x_1 - \varepsilon, x_1 + \varepsilon)\) with \(\varepsilon > 0\). By (6.17),

\[
\int_{I_1} u_r(x,t) \, dx \geq \int_{x_1}^{x_1 + \varepsilon} \Psi^{-1}\left(\Psi(\infty) - \frac{1}{t}(x-x_1)\right) \, dx = \int_0^\varepsilon \Psi^{-1}\left(\Psi(\infty) - \frac{\varepsilon}{t}\right) \, dy =: B_\varepsilon.
\]

If \(\text{supp } u_s(\cdot,t) \notin I_1\), let \(x_2 \in \text{supp } u_s(\cdot,t) \cap I_1\) and set \(I_2 := (x_2 - \varepsilon, x_2 + \varepsilon)\). Since \((x_1, x_1 + \varepsilon) \cap (x_2, x_2 + \varepsilon) = \emptyset\) we have that

\[
\int_{I_1 \cup I_2} u_r(x,t) \, dx \geq \int_{x_1}^{x_1 + \varepsilon} u_r(x,t) \, dx + \int_{x_2}^{x_2 + \varepsilon} u_r(x,t) \, dx \geq 2B_\varepsilon.
\]

We continue this construction recursively as long as \(\text{supp } u_s(\cdot,t) \notin I_1 \cup \cdots \cup I_{n-1}\), with \(I_{n-1} := (x_{n-1} - \varepsilon, x_{n-1} + \varepsilon)\): there exists \(x_n \in \text{supp } u_s(\cdot,t) \cap (I_1 \cup \cdots \cup I_{n-1})\) such that, setting \(I_n := (x_n - \varepsilon, x_n + \varepsilon)\),

\[
nB_\varepsilon \leq \int_{I_1 \cup \cdots \cup I_n} u_r(x,t) \, dx \leq \|u_0\|_{M(\mathbb{R})}.
\]

Hence this construction stops at some \(n = n_\varepsilon\), and \(n_\varepsilon B_\varepsilon \leq \|u_0\|_{M(\mathbb{R})}\). Therefore,

\[
\text{supp } u_s(\cdot,t) \in I_1 \cup \cdots \cup I_{n_\varepsilon}, \quad |\text{supp } s(\cdot,t)| \leq |I_1 \cup \cdots \cup I_{n_\varepsilon}| \leq 2n_\varepsilon \varepsilon \leq \frac{2\varepsilon}{B_\varepsilon} \|u_0\|_{M(\mathbb{R})}.
\]

Since \(B_\varepsilon/\varepsilon \to \infty\) as \(\varepsilon \to 0\), the claim follows. \(\square\)

**7. Uniqueness: Proofs**

Again, without loss of generality we may assume that \(C_\varphi = 0\) in the following proofs (see Remark 3.17).

**Proof of Proposition 3.10 (i)** The first step of the proof consists in showing that

\[
\text{ess lim}_{t \to 0^+} \|u_r(\cdot,t) - u_{0r}\|_{L^1(\mathbb{R})} = 0.
\]

Let \(\{u_n^\varepsilon\}\) be the sequence of solutions to problems \((Q_n^\varepsilon)\) considered in Section A and let \(\{x_l\} (l = 1, \ldots, N)\) be as in (3.27). We set \(I_l := (x_l, x_{l+1})\), \(Q_l := I_l \times (0, \tau)\) \((l = 1, \ldots, N - 1)\), \(I_1 := (\infty, x_1)\), \(I_N := (x_N, \infty)\), and \(Q_0 := I_1 \times (0, \tau)\).

Let \(1 \leq l \leq N - 1\) and \(\rho \in C_0^1(I_l), \rho \geq 0\). Let \(h_0 > 0\) be such that \(x + \varepsilon \in I_l\) if \(\varepsilon \in \text{supp } \rho\) and \(|h| < h_0\). Let \(\delta > 0\). Setting \(y_n^\varepsilon(x,t) := u_n^\varepsilon(x + h, t)\) and \(z := (u_n^\varepsilon - u_n^\varepsilon)(\rho + \delta)\), we apply the \(L^1\)-contraction property to the parabolic equation

\[
\int_{x_l}^{x_{l+1}} \left[ R + \frac{2\varepsilon \rho'}{\rho + \delta} \right] \frac{dx}{x} - \varepsilon z_{xx} = \left( \frac{R \rho'}{\rho + \delta} + \varepsilon \frac{\rho''}{\rho + \delta} \right) z = (\varphi_k(u_n^\varepsilon) - \varphi_k(u_n^\varepsilon)) \rho' + \varepsilon [u_n^\varepsilon - u_n^\varepsilon] \rho''.
\]
where \( R := \frac{\varphi(u_n^\varepsilon(x,t)) - \varphi(u_n^\varepsilon(x,t))}{u_n^\varepsilon(x,t)} \) if \( u_n^\varepsilon \neq v_n^\varepsilon \), and \( R := \varphi'_e(u_n^\varepsilon) \) otherwise. Hence
\[
\int_{I_0} |z(x,\tau)|_1 dx \leq \int_{I_0} |z(x,0)|_1 dx + \int_{0}^{\tau} \int_{I_0} |\varphi_e(u_n^\varepsilon(x,t)) - \varphi_e(u_n^\varepsilon(x+h,t))| \rho'(x) dx dt + \varepsilon \int_{I_0} \int_{I_0} |u_n^\varepsilon(x,t) - u_n^\varepsilon(x+h,t)| |\rho''(x)|_1 dx dt \quad \text{for } \tau \in (0,T).
\]

First we let \( \delta \to 0 \) and then \( \varepsilon = \varepsilon_m \to 0 \), where \( \{\varepsilon_m\} \) is as in Lemma 4.4. Hence
\[
(7.2) \quad \int_{I_0} |u_n(x,\tau) - u_n(x+h,\tau)|_1 \rho(x) dx \leq \int_{I_0} |u_{0n}(x) - u_{0n}(x+h)|_1 \rho(x) dx + \int_{I_0} \int_{I_0} |\varphi_e(u_n(x,t)) - \varphi_e(u_n(x+h,t))| \rho'(x) dx dt \quad \text{for a.e. } \tau \in (0,T),
\]
where \( u_n \) is the entropy solution of problem \((P_n)\) \((n \in \mathbb{N})\). Since, by (3.16), \( u_n(\cdot, t) \to u(\cdot, t) \) in \( M(I_t) \) for a.e. \( t \in (0,T) \) and, by (3.20) and (3.27), \( u_n(\cdot, t) \) \( L_1(I_t) \leq u_{0n} \leq I_t = 0 \), the lower semicontinuity of the total variation implies that for a.e. \( \tau \in (0,T) \)
\[
\int_{I_0} |u_r(x,\tau) - u_r(x+h,\tau)|_1 \rho(x) dx \leq \liminf_{n \to \infty} \int_{I_0} |u_n(x,\tau) - u_n(x+h,\tau)|_1 \rho(x) dx.
\]
By (3.15), \( \int_{I_0} |u_{0n}(x) - u_{0n}(x+h)|_1 \rho(x) dx \to \int_{I_0} |u_r(x) - u_r(x+h)|_1 \rho(x) dx \). In addition \( \varphi(u_{0n}) \to \varphi(u_r) \) in \( L^1(Q) \) for a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) (see Remark 5.4). Letting \( n = n_k \to \infty \) in (7.2), we obtain that for a.e. \( \tau \in (0,T) \)
\[
(7.3) \quad \int_{I_0} |u_r(x,\tau) - u_r(x+h,\tau)|_1 \rho(x) dx \leq \int_{I_0} |u_r(x) - u_r(x+h)|_1 \rho(x) dx + \int_{I_0} \int_{I_0} |\varphi_e(u_r(x,t)) - \varphi_e(u_r(x+h,t))| \rho'(x) dx dt.
\]
Let \( \{\tau_n\} \subset (0,T) \) be any sequence such that \( \tau_n \to 0^+ \) and (7.3) is satisfied with \( \tau = \tau_n \). Since \( u_{0r} \in L^1(\mathbb{R}) \) and \( \varphi(u_r) \in L^1(S) \), it follows from (7.3) and the Fréchet-Kolmogorov Theorem that the sequence \( \{u_r(\cdot, \tau_n)\} \rho \) is relatively compact in \( L^1(\mathbb{R}) \). Then, by (3.11) and a standard argument,
\[
(7.4) \quad u_r(\cdot, \tau_n) \rho \to u_{0r} \rho \quad \text{in } L^1(\mathbb{R}).
\]
It follows from (3.3) and (3.13) that for each \( n \in \mathbb{N} \)
\[
(7.5) \quad \int_{I_0} \int_{I_0} |u_r(x,\tau_n) - u_r(x)|_1 \rho(x) dx = \int_{0}^{\tau_n} \int_{I_0} \varphi(u_r(x,t)) \rho'(x) dx dt.
\]
For sufficiently small \( \delta > 0 \), the characteristic function \( \chi_{(x_1,x_1+\delta)} \) \( \chi_{(x_1,x_1+\delta)} \) can be approximated by functions \( \rho_k \in C_c^2(I_t) \), \( \rho_k \geq 0 \) such that \( \int_{I_t} |\rho'(x)|_1 dx \leq 4 \) for all \( k \in \mathbb{N} \). Setting \( \rho = \rho_k \) in (7.5) and letting \( k \to \infty \), we find that
\[
(7.6) \quad \int_{x_1}^{x_1+\delta} u_r(x,\tau_n) dx + \int_{x_1+\delta}^{x_1+\delta} u_r(x,\tau_n) dx \leq \int_{x_1}^{x_1+\delta} u_{0r}(x) dx + \int_{x_1+\delta}^{x_1+\delta} u_{0r}(x) dx + 4 \| \varphi \|_{L^\infty(0,\infty)} \tau_n.
\]
Since \( u_{0r} \in L^1(\mathbb{R}) \), for every \( \sigma > 0 \) there exists \( \delta > 0 \) such that
\[
(7.7) \quad \int_{x_1}^{x_1+\delta} u_{0r}(x) dx + \int_{x_1+\delta}^{x_1+\delta} u_{0r}(x) dx \leq \sigma.
\]
If \( \rho \in C_c(I_t) \) is such that \( 0 \leq \rho \leq 1 \) in \( I_t \), \( \rho = 1 \) in \( [x_1 + \delta, x_1 + \delta] \), then
\[
|u_r(\cdot, \tau_n) - u_{0r}| = |u_r(\cdot, \tau_n) - u_{0r}| + |u_r(\cdot, \tau_n) - u_{0r}|(1 - \rho) \chi_{(x_1,x_1+\delta)} \chi_{(x_1+\delta,x_1+\delta)}.
\]
in $I_l$. Hence, by (7.1) and (7.2),

$$
\int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}(\cdot) \right| dx \leq 2 \left\{ \int_{I_l} \left| u_{0r}\right| dx + \int_{I_l} \left| u_{0r}\right| dx \right\} + 4 \left\| \varphi \right\|_{L^{\infty}(0, \infty)} \tau_n +
$$

$$
+ \int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}\right| dx \leq \int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}\right| dx + 4 \left\| \varphi \right\|_{L^{\infty}(0, \infty)} \tau_n + 2 \sigma.
$$

Letting $n \to \infty$ in the above inequality, by (7.4) we obtain that $\limsup \frac{1}{n} \int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}\right| dx \leq 2 \sigma$, whence, by the arbitrariness of $\sigma$,

$$
\lim_{n \to \infty} \int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}(\cdot) \right| dx = 0 \quad (l = 1, \ldots, N - 1).
$$

A similar argument shows that $\int_{I_l} \left| u_r(\cdot, \tau_n) - u_{0r}(\cdot) \right| dx \to 0$ as $n \to \infty$, thus (7.1) follows.

To complete the proof of (7.28) observe that by (7.20) there holds $u_s(\cdot, t) \leq u_{0s}$ in $\mathcal{M}(\mathbb{R})$ (recall that $C_{\varphi} = 0$ by assumption). Hence

$$
\langle u_{0u} - u_s(\cdot, t), \rho \rangle_{\mathbb{R}} \geq \left\| u_s(\cdot, t) - u_{0s}\right\|_{\mathcal{M}(\mathbb{R})}
$$

for all $\rho \in C_c(\mathbb{R})$ such that $\rho(x) = 1$ for every $x \in \text{supp } u_{0s}$. From the previous inequality, (3.11) and (7.1) we get

$$
\text{ess lim}_{t \to 0^+} \left\| u_s(\cdot, t) - u_{0s}\right\|_{\mathcal{M}(\mathbb{R})} \leq \text{ess lim}_{t \to 0^+} \langle u_{0u} - u_s(\cdot, t), \rho \rangle_{\mathbb{R}} =
$$

$$
= \text{ess lim}_{t \to 0^+} \left\{ \left( u_{0u} - u_s(\cdot, t)ight) - \int_{\mathbb{R}} \left( u_r(x, t) - u_{0r}\right) \rho(x) dx \right\} = 0.
$$

Then (7.28) follows.

(ii) Let $\zeta^+ \in C^1_c(Q_1)$, $\zeta^+ \geq 0$, and for every $1 \leq l \leq N - 1$ let $\zeta_l \in C^1_c(Q_l)$, $\zeta_l \geq 0$. Let $h_0 > 0$ be such that $(x + h, t) \in Q_l$ (respectively, $(x + h, t) \in Q_0$) if $(x, t) \in \text{supp } \zeta_l$ (respectively, if $(x, t) \in \text{supp } \zeta_0$) and $|h| < h_0$.

Let $u$ be an entropy solution of problem $(P)$, thus $v(\cdot, t) = T_h(u(\cdot, t))$ is an entropy solution of problem $(P)$ with $u_0$ replaced by $v_0 := T_{-h}(u_0)$ (see Remark 4.7). We shall prove that for all $l = 1, \ldots, N - 1$ and $\zeta_l$ as above

$$
\iint_{Q_l} \left\{ \left| v_r - u_r \right| \zeta_l - \text{sgn} \left( v_r - u_r \right) \left[ \varphi(v_r) - \varphi(u_r) \right] \zeta_l \right\} \, dx \, dt \geq 0,
$$

and for all $\zeta^+$ as above

$$
\iint_{Q_l} \left\{ \left| v_r - u_r \right| \zeta^+_l - \text{sgn} \left( v_r - u_r \right) \left[ \varphi(v_r) - \varphi(u_r) \right] \zeta^+_l \right\} \, dx \, dt \geq 0.
$$

Relying on (7.9) and (7.10) we can conclude the proof by an argument similar to that used in (i). Let $\rho \in C^1_c(I_l)$, $0 \leq \rho \leq 1$, be such that $x + h \in I_l$ if $x \in \text{supp } \rho$ and $|h| < h_0$.

By a proper choice of the function $\zeta_l$ in (7.34), for a.e. $0 < t_0 < t_1 \leq T$ we get

$$
\int_{I_l} \left| u_r(x, t_1) - v_r(x, t_1) \right| \rho(x) \, dx \leq \int_{I_l} \left| u_r(x, t_0) - v_r(x, t_0) \right| \rho(x) \, dx +
$$

$$
+ \int_{t_0}^{t_1} \left| \varphi(u_r) - \varphi(v_r) \right| \, dx \, dt.
$$

Let $t_0 > 0$ be fixed. Then for every $\tau \in (t_0, T]$ there exists a sequence $\tau_n \to \tau$ such that $\tau_n \in (t_0, T]$ and the above inequality holds true with $t_1 = \tau_n$ for every $n$:

$$
(7.11) \iint_{I_l} \left| u_r(x + h, \tau_n) - u_r(x, \tau_n) \right| \rho(x) \, dx \leq \int_{I_l} \left| u_r(x + h, t_0) - u_r(x, t_0) \right| \rho(x) \, dx +
$$

$$
+ \left\| \rho' \right\|_\infty \int_0^T \left| \varphi(u_r(x + h, t)) - \varphi(u_r(x, t)) \right| \, dx \, dt.
$$
Since \( \varphi(u_r) \in L^1(S) \) and \( u_r(\cdot, t_0) \in L^1(\mathbb{R}) \), inequality (7.11) and the Fréchet-Kolmogorov Theorem imply that the sequence \( \{u_r(\cdot, \tau) \rho\} \) is relatively compact in \( L^1(\mathbb{R}) \), whence, by Proposition 3.1 and a standard argument,

\[
(7.12) \quad u_r(\cdot, \tau) \rho \to u_r(\cdot, \tau) \rho \quad \text{in} \quad L^1(\mathbb{R}).
\]

Moreover, by arguing as in (7.6) and (7.7) with \( u_{0r} \) replaced by \( u_r(\cdot, \tau) \), for every \( \sigma > 0 \) there exists \( \delta > 0 \) such that

\[
(7.13) \quad \int_{x_i}^{x_{i+1}} u_r(x, \tau_n) \, dx + \int_{x_{i+1} - \delta}^{x_i + \delta} u_r(x, \tau_n) \, dx \leq \sigma + 4\|\varphi\|_{L^\infty(0, \infty)} |\tau_n - \tau|.
\]

As in the proof of claim (i), combining (7.12) and (7.13) gives

\[
\lim_{n \to \infty} \int_I |u_r(x, \tau_n) - u_r(x, \tau)| \, dx = 0 \quad (i = 1, \ldots, N - 1)
\]

(by a similar argument, \( \int_I |u_r(x, \tau_n) - u_r(x, \tau)| \, dx \to 0 \) as \( n \to \infty \)), whence

\[
\text{ess lim}_{t \to \tau} \|u_r(\cdot, t) - u_r(\cdot, \tau)\|_{L^1(\mathbb{R})} = 0.
\]

Since \( C_{\varphi} = 0 \) it follows from (3.19) that \( u_s(\cdot, t_2) \leq u_s(\cdot, t_1) \) in \( M(\mathbb{R}) \) if \( t_2 > t_1 \), whence by arguing as in (7.8) we also obtain

\[
\text{ess lim}_{t \to \tau^+} \|u_s(\cdot, t) - u_s(\cdot, \tau)\|_{M(\mathbb{R})} = \text{ess lim}_{t \to \tau^-} \|u_s(\cdot, t) - u_s(\cdot, \tau)\|_{M(\mathbb{R})} = 0
\]

and claim (ii) follows.

Finally, it remains to prove (7.9) (the proof of (7.10) is analogous). Let \( 1 \leq l \leq N - 1 \) and \( \zeta \in C^1_c(Q_1) \), \( \zeta \geq 0 \), be fixed as above. Since \( C_{\varphi} = 0 \), it follows from (3.20) and (7.27) that \( u_s(\cdot, t) = v_s(\cdot, t) = 0 \) on \( \text{supp} \, \zeta(\cdot, t) \) for a.e. \( t \in (0, T) \), and from (3.10) that, for \( k \in [0, \infty) \),

\[
(7.14) \quad \int_{Q_1} \{ |u_r - k| \zeta t + \text{sgn}(u_r - k) [\varphi(u_r) - \varphi(k)] \zeta t \} \, dx \, dt \geq 0,
\]

\[
(7.15) \quad \int_{Q_1} \{ |v_r - k| \zeta t + \text{sgn}(v_r - k) [\varphi(v_r) - \varphi(k)] \zeta t \} \, dx \, dt \geq 0.
\]

We apply Kružkov’s method of doubling variables. Let \( Z_t = Z_t(x, t, y, s) \in C^1_c(Q_t \times Q_t) \), \( Z_t \geq 0 \). It follows from (7.14) and (7.15) that

\[
\int_{Q_1} \{ |u_r(x, t) - v_r(y, s)| Z_t(x, t, y, s) + \text{sgn}(u_r(x, t) - v_r(y, s)) [\varphi(u_r(x, t)) - \varphi(v_r(y, s))] Z_{t_x}(x, t, y, s) \} \, dx \, dt \geq 0
\]

and

\[
\int_{Q_1} \{ |v_r(y, s) - u_r(x, t)| Z_{t_y}(x, t, y, s) + \text{sgn}(v_r(y, s) - u_r(x, t)) [\varphi(v_r(y, s)) - \varphi(u_r(x, t))] Z_{t_y}(x, t, y, s) \} \, dy \, ds \geq 0,
\]

whence

\[
\int_{Q_1} \left\{ |u_r(x, t) - v_r(y, s)| (Z_t + Z_{t_x}) (x, t, y, s) + \text{sgn}(u_r(x, t) - v_r(y, s)) [\varphi(u_r(x, t)) - \varphi(v_r(y, s))] (Z_{t_x} + Z_{t_y}) (x, t, y, s) \right\} \, dx \, dt \, dy \, ds \geq 0.
\]

We choose

\[
Z_t(x, t, y, s) = Z_t^f (x, t, y, s) := \zeta(t, x) \zeta_s (x - y, t - s) \quad (\epsilon > 0),
\]
where $\zeta_\epsilon$ is a smooth approximation of the Dirac mass $\delta_{(0,0)}$,

$$\zeta_\epsilon(x,y) = \frac{1}{\epsilon^2} \theta\left(\frac{x}{\epsilon}\right) \eta\left(\frac{y}{\epsilon}\right) \geq 0, \quad \text{with supp}\theta \subseteq (-1,1), \text{supp}\eta \subseteq (-1,1).$$

Then $Z_{lt} + Z_{lx} = \zeta_{lt} \zeta_\epsilon$ and $Z_{lx} + Z_{ly} = \zeta_{lx} \zeta_\epsilon$, whence, for sufficiently small $\epsilon$,

$$\lim_{\epsilon \to 0^+} \int_{Q_1 \times Q_1} \left|u_r(x,t) - v_r(y,s)\right| \zeta_{lt}(x,t) +$$

$$+ \text{sgn}(u_r(x,t) - v_r(y,s)) \left[\varphi(u_r(x,t)) - \varphi(v_r(y,s))\right] \zeta_{lx}(x,t) \zeta_{lx}(x-y,t-s) \, dx \, dy \, ds \geq 0.$$ 

Now (7.16) follows by letting $\epsilon \to 0^+$: we claim that

$$(7.16) \quad \lim_{\epsilon \to 0^+} \int_{Q_1 \times Q_1} \left|u_r(x,t) - v_r(y,s)\right| \zeta_{lt}(x,t) \zeta_{n}(x-y,t-s) \, dx \, dy \, ds =$$

$$= \int_{Q_1} \left|u_r(x,t) - v_r(x,t)\right| \zeta_{lt}(x,t) \, dx \, dt.$$

Analogously, it can be proven that

$$\lim_{\epsilon \to 0^+} \int_{Q_1 \times Q_1} \text{sgn}(u_r(x,t) - v_r(y,s)) \left[\varphi(u_r(x,t)) - \varphi(v_r(y,s))\right] \zeta_{lx}(x,t) \zeta_{n}(x-y,t-s) \, dx \, dy \, ds =$$

$$= \int_{Q_1} \text{sgn}(u_r(x,t) - v_r(y,s)) \left[\varphi(u_r(x,t)) - \varphi(v_r(x,t))\right] \zeta_{lx}(x,t) \, dx \, dt.$$ 

In order to prove (7.17), for every sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$, we set

$$F_n(x,t) := \int_{Q_1} \left|u_r(x,t) - v_r(y,s)\right| \zeta_{n}(x-y,t-s) \, dy \, ds \quad \text{for } (x,t) \in K_1 := \text{supp}\zeta_l,$$

and observe that $F_n \to |u_r - v_r| \text{ a.e. in } (x,t) \in K_1$ and

$$|F_n(x,t)| \leq |u_r(x,t)| + \int_{Q_1} |v_r(y,s)| \zeta_{n}(x-y,t-s) \, dy \, ds$$

$$= |u_r(x,t)| + |v_r(x,t)| \to |u_r(x,t)| + |v_r(x,t)| \quad \text{in } L^1(K_1).$$

Thus, by a variant of the Dominated Convergence Theorem (e.g., see [14, Theorem 4, Section 1.3]), $F_n \to |u_r - v_r|$ in $L^1(K_1)$, and we obtain (7.16). This completes the proof of (7.16), thus the result follows.

Proof of Theorem 7.17. Without loss of generality we may assume that $\varphi$ is nondecreasing (see Remark 3.6). By Theorem 3.3(i),

$$\tau := \sup \{t \in (0,T) \mid u_{is}(\cdot,t)(\{x_l\}) > 0 \forall i = 1,\ldots,N; \, l = 1,2\} > 0.$$ 

Let us first prove that

$$(7.17) \quad u_{1r} = u_{2r} \quad \text{a.e. in } \mathbb{R} \times (0,\tau).$$

To this aim, let $x_1,\ldots,x_N$ be the points in (3.27). Set $I_l := (x_l, x_{l+1})$, $Q_l := I_l \times (0,\tau)$ $(l = 1,\ldots,N-1)$, and $I_0 := (-\infty,x_1)$, $I_N := (x_N,\infty)$, $Q_0 := I_0 \times (0,\tau)$. By arguing as in the last part of the proof of Proposition 3.10(ii) (in particular, see the proof of (7.16) and (7.17), it follows that for all $l = 1,\ldots,N-1$ and $\zeta_l \in C^1_c(Q_l)$, $\zeta_l \geq 0$,

$$(7.18) \quad \int_{Q_l} \left\{|u_{lr} - u_{2r}| \zeta_l + |\varphi(u_{lr}) - \varphi(u_{2r})|(x,t)\right\} \zeta_l \, dx \, dt \geq 0$$

and, for all $\zeta^\pm \in C^1_c(Q_N)$, $\zeta^\pm \geq 0$,

$$(7.19) \quad \int_{Q_N} \left\{|u_{lr} - u_{2r}| \zeta_l^\pm + |\varphi(u_{lr}) - \varphi(u_{2r})| \zeta_l^\pm(x,t)\right\} \, dx \, dt \geq 0$$

Therefore, $u_{1r} = u_{2r}$ a.e. in $\mathbb{R} \times (0,\tau).$
(recall that \( \varphi \) by assumption is increasing). We must show that (7.18) and (7.19) imply (7.17). Let \( h \in C^1([0, \tau), h \geq 0, \) and 
\[
\begin{align*}
\rho_{i,p}(x) &= p \left( x - x_i - \frac{1}{p} \right)^+ \chi_{[x_i+1/p, x_i+2/p]}(x) + \chi_{[x_i+2/p, x_{i+1}-2/p]}(x) - \\
&\quad - p \left( x - x_{i+1} + \frac{1}{p} \right)^+ \chi_{[x_{i+1}-2/p, x_{i+1}-1/p]}(x) (l = 1, \ldots, N - 1),
\end{align*}
\]
with \( p \in \mathbb{N} \) sufficiently large. By standard approximation arguments we may choose \( \zeta_i = \zeta_{i,p} := \rho_{i,p}(x)h(t) \) in (7.18): 
\[
(7.20) \quad 0 \leq \int_{Q_1} \left\{ |u_{1r} - u_{2r}| \rho_{i,p}(x)h'(t) + |\varphi(u_{1r}) - \varphi(u_{2r})| \rho'_{i,p}(x)h(t) \right\} \, dx \, dt.
\]
By the Dominated Convergence Theorem, as \( p \to \infty, \)
\[
\int_{Q_1} \left\{ |u_{1r} - u_{2r}| \rho_{i,p}(x)h'(t) \right\} \, dx \, dt \to \int_0^\tau \int_{x_i}^{x_{i+1}} |u_{1r} - u_{2r}| \, dx \, dt.
\]
Since \( \rho'_{i,p}(x) = p\chi_{(x_i, x_i+1/p)}(x) - p\chi_{(x_i+2/p, x_{i+1}-1/p)}(x) \) and \( \varphi \) is bounded, it follows from (3.21) and the Dominated Convergence Theorem that 
\[
\lim sup_{p \to \infty} \int_{Q_1} |\varphi(u_{1r}) - \varphi(u_{2r})| \rho'_{i,p}(x)h(t) \, dx \, dt \<=\frac{}{}\int_0^\tau h(t) \left( \lim sup_{p \to \infty} \int_{x_i}^{x_{i+\frac{1}{p}}} |\varphi(u_{1r}) - \varphi(u_{2r})| \, dx \right) dt = 0.
\]
Hence, by (7.21),
\[
\int_0^\tau \int_{x_i}^{x_{i+1}} |u_{1r}(x,t) - u_{2r}(x,t)| \, dx \, dt \geq 0
\]
and, by a proper choice of \( h, \)
\[
(7.22) \quad \|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(I_l)} \leq \|u_{1r}(\cdot,t_1) - u_{2r}(\cdot,t_1)\|_{L^1(I_l)} \quad \text{for every } 0 < t_1 \leq t \leq \tau
\]
(recall that \( u_{ir} \in C(\{0, \tau\}; M(\mathbb{R})), i = 1, 2, \) by Proposition 6.10 (ii)). Letting \( t_1 \to 0^+ \) it follows from (3.20) that \( \|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(I_l)} = 0 \) for a.e. \( t \in (0, \tau) \) and all \( l = 1, \ldots, N - 1. \) The proof that \( \|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(\mathbb{R} \times I_l)} = 0 \) for a.e. \( t \in (0, \tau) \) is similar, so we have proven (7.17).

Next, let us prove that 
\[
(7.22) \quad u_1 = u_2 \quad \text{in } M(\mathbb{R} \times (0, \tau)).
\]
By (6.18) and (7.17), for every \( \xi \in C^1([0, \tau]; C^1(\mathbb{R})), \zeta(\xi, 0) = 0 \) in \( \mathbb{R} \)
\[
\int_0^T \langle u_{1s}(\cdot,t) - u_{2s}(\cdot,t), \zeta(\cdot,t) \rangle \, dt = \int S \left\{ (u_{1r} - u_{2r})(\cdot,t) \right\} \, dt = 0.
\]
We argue as in the proof of Lemma 6.1, there exists a null set \( F_0 \subset (0, \tau) \) such that 
\( \langle u_{1s}(\cdot,t) - u_{2s}(\cdot,t), \rho \rangle = 0 \) for all \( t \in (0, \tau) \setminus F_0 \) and \( \rho \in C^1(\mathbb{R}). \) Hence \( u_1 = u_2 \) in \( L^\infty(0, \tau; M(\mathbb{R})), \) and, by (7.17), equality (7.22) follows.

If \( \tau = T \) the proof is complete. Otherwise, there exist \( N_1 < N \) different points \( x_{i_k} \in \{x_1, \ldots, x_N\} \) such that \( u_{1i_k}(\cdot, \tau)(\{x_{i_k}\}) > 0 \) for every \( k = 1, \ldots, N_1 \) and \( i = 1, 2, \) moreover, for every point \( x_i \in \{x_1, \ldots, x_N\}, x_i \neq x_{i_k}, \) it follows from (6.19), with \( x_0 = x_1 = x_1, \) that \( u_{1i}(\cdot, \tau)(\{x_{i}\}) = u_{2i}(\cdot, \tau)(\{x_{i}\}) = 0, \) since \( \varphi(u_{1r}) = \varphi(u_{2r}) \) in \( \mathbb{R} \times (0, \tau) \) by (6.18). Then we set 
\[
\tau_1 := \sup \{ t \in [\tau, T) \mid u_{1i}(\cdot,t)(\{x_{i_k}\}) > 0 \ \forall k = 1, \ldots, N_1; \ i = 1, 2 \}.
\]
Arguing as in the proof of (7.17) we obtain inequality (7.21) for every \( \tau < t_1 \leq t \leq \tau_1 \). Since \( u_\nu \in C((0, T]; \mathcal{M}(\mathbb{R})) \), \( i = 1, 2 \) (see Proposition 3.10 (ii)), and \( u_{1\nu}(\cdot, \tau) = u_{2\nu}(\cdot, \tau) \), letting \( t_1 \to \tau^+ \) we get \( u_{1\nu} = u_{2\nu} \) a.e. in \( \mathbb{R} \times (\tau, \tau_1) \) (whence, also \( u_1 = u_2 \) in \( \mathcal{M}(\mathbb{R} \times (\tau, \tau_1)) \)) and the proof is completed in a finite number of steps. \( \square \)

Let us finally prove Proposition 7.1.

**Proof of Proposition 7.1.** A calculation proves that the solution defined by (1.13) if \( p < 0 \), respectively by (1.14) if \( 0 < p < 1 \) is an entropy solution of problem (1.11)–(1.12).

If \( p < 0 \), the solution also satisfies (3.21) for \( 0 < t < 1 \) and (3.20), so claim (i) follows from the uniqueness result in Theorem 3.12. If \( 0 < p < 1 \), uniqueness of entropy solutions such that \( u_\nu(t) = 0 \) for \( t > 0 \) and \( u_\nu \in L^\infty(\mathbb{R} \times (\tau, T)) \) for \( \tau \in (0, T) \) can be used (the proof of this uniqueness result is very similar to that given in [18], thus we omit the details; see also Remark 7.9). Hence claim (ii) follows. \( \square \)

**Remark 7.1.** It is instructive to describe the approximation procedure which gives the solutions mentioned in Proposition 7.1. Consider the family of approximating problems

\[
(R_n) \quad \begin{cases}
  u_{nt} + [\varphi(u_n)]_x = 0 & \text{in } S \\
  u_n = \frac{1}{2} \chi(\cdot + \frac{1}{n}) & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\]

with \( \varphi \) given by (1.2). It is easily seen that the entropy solution of \((R_n)\) is

\[
u_n(x, t) = \begin{cases}
0 & \text{if } x \geq |p|t + \frac{1}{n}, \\
\left( \frac{n|p|}{n|x| - 1} \right)^{\frac{1}{1-p}} - 1 & \text{if } |p|t + \frac{1}{n} > x \geq \left( \frac{2}{n|x| + 1} \right)^{1-p} |p|t + \frac{1}{n}, \\
\frac{1}{n} & \text{if } \left( \frac{2}{n|x| + 1} \right)^{1-p} |p|t + \frac{1}{n} > x \geq \frac{2 \left( \frac{2}{n|x| + 1} \right)^{1-p} - 1}{t - \frac{1}{n}}, \\
0 & \text{if } \frac{2 \left( \frac{2}{n|x| + 1} \right)^{1-p} - 1}{t - \frac{1}{n}} > x.
\end{cases}
\]

for \( 0 \leq t \leq t_n := \frac{1}{\varphi(\frac{1}{n}) - \frac{1}{n} \varphi(\frac{1}{n})} \). At \( t = t_n \) a shock \( x = \xi(t) \) stems from \( x = x_n := \frac{1}{n} \varphi(\frac{1}{n}) \). Which solves the problem

\[
\begin{align*}
\xi_n(t) &= \frac{\varphi(u_n(\xi_n(t), t))}{u_n(\xi_n(t), t)} = \text{sgn} p \left( \frac{n|p|}{n\xi - 1} \right)^{\frac{1}{1-p}} - 1 & \text{if } t > t_n, \\
\xi_n(t_n) &= x_n.
\end{align*}
\]

Hence for \( t > t_n \) the entropy solution of \((R_n)\) is

\[
u_n(x, t) = \begin{cases}
0 & \text{if } x \geq |p|t + \frac{1}{n}, \\
\left( \frac{n|p|}{n|x| - 1} \right)^{\frac{1}{1-p}} - 1 & \text{if } |p|t + \frac{1}{n} > x \geq \xi_n(t), \\
0 & \text{if } \xi_n(t) \geq x.
\end{cases}
\]

Letting \( n \to \infty \) we obtain the entropy solution defined in parts (i) (if \( p < 0 \)) and (ii) (if \( 0 < p < 1 \)) of Proposition 7.1.

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