Superfluidity in nuclear systems and neutron stars

Armen Sedrakian

Frankfurt Institute for Advanced Studies, Ruth-Moufang str. 1, D-60438 Frankfurt am Main, Germany

John W. Clark†

Department of Physics and McDonnell Center for the Space Sciences, Washington University, St. Louis, Missouri 63130, USA

and

Centro de Investigação em Matemática e Aplicações, University of Madeira, 9020-105 Funchal, Madeira, Portugal

(Dated: July 7, 2018)

Nuclear matter and finite nuclei exhibit the property of superfluidity by forming Cooper pairs due to the attractive component of the nuclear interaction. We review the microscopic theories and methods that are being employed to understand the basic properties of superfluid nuclear systems, with emphasis on the spatially extended matter encountered in neutron stars and in nuclear collisions. Our survey includes techniques which are based on Green functions, correlated basis functions, and Monte Carlo sampling of quantum states. Novel phases arise in nuclear and related systems (such as ultracold atomic gases) when the populations of the fermions that form Cooper pairs are different. Such phases include superfluid states with non-zero total momentum of Cooper pairs, heterogeneous phase-separated states, and phases involving deformation of Fermi surfaces. The phase diagram of imbalanced superfluids is reviewed, focusing especially on the crossover from Bardeen-Cooper-Schrieffer (BCS) pairing to a Bose-Einstein condensate (BEC) of tightly bound dimers as the strength of the attractive interaction increases. The neutron, proton, and hyperonic condensates that may exist under conditions prevailing in neutron stars are discussed, and calculations of weak-interaction rates within the Green functions formalism are examined in detail. We close with a discussion of quantum vortex states in nuclear systems and their dynamics in neutron-star superfluid interiors.
I. INTRODUCTION

Pairing phenomena play an important role in experimental and observational manifestations of neutron stars and finite nuclei. Their theoretical understanding is rooted in the microscopic theory of superconductivity advanced by Bardeen, Cooper and Schrieffer (BCS) (Bardeen et al., 1957). However, strong correlations, which are generic to nuclear systems, and the complex dynamics of finite systems such as nuclei, require developments beyond this theory. The study of nuclear systems is built on our understanding of the underlying nuclear forces and the quantum many-body theory of fermionic systems – both aspects having undergone immense advances during the past several decades. In parallel with these improvements in theoretical and computational techniques, the scope of the problems considered has broadened over the years and now includes such traditionally condensed-matter issues as the crossover from BCS pairing to a Bose-Einstein condensate (BEC), inhomogeneous phases with broken spatial symmetries, pairing in strong magnetic fields and resistive flow of quantum vorticity.

In this review we concentrate on recent developments in the quantum many-body problem associated with nuclear pairing, on novel forms of nuclear pairing that have strong overlap with the physics of ultracold atoms, and on the roles played by pairing in macroscopic manifestations of neutron stars. There exist a number of previous reviews that cover different stages of development of pairing theory in the nuclear context, with emphasis placed on varied aspects of pairing phenomena (Dean and Hjorth-Jensen, 2003; Gezerlis et al., 2013; Lombardo and Schulze, 2001; Page et al., 2013; Takatsuka and Tamagaki, 1993). The reader will benefit from consulting them for an alternative or supplementary exposition of selected topics.

Fundamental insights into nuclear pairing were put forward shortly after the advent of the BCS theory (Bogoliubov, 1958; Bohr et al., 1958; Migdal, 1959). The overwhelming success of the BCS theory in explaining the properties of metallic superconductors provided experimental support of the Cooper pairing mechanism (Cooper, 1956), the fact that two species of fermions interacting via an attractive interaction will form bound states with zero total momentum at sufficiently low temperature. Since the long-range part of the nuclear interaction is attractive, it is natural to conclude that nucleons will form Cooper pairs in nuclei and neutron-star matter, as these systems contain an ensemble of quantum-degenerate states of nucleons bound by either the nuclear force (nuclei) or gravity (neutron stars). In due course, essential aspects of modern quantum many-body theory were introduced, such as the Fermi-liquid theory of nuclear systems (Larkin and Migdal, 1964; Migdal, 1962), quantum vorticity in superfluid neutron matter (Ginzburg and Kirtzhnits, 1965), and type-II superconductivity of the proton component of neutron-star matter (Baym et al., 1969a).

A new impetus to the theory of fermionic pairing was provided by the discovery of pulsars in 1967 (Hewish et al., 1968) and their identification with neutron stars (Gold, 1968). In particular, observations of long time scales for the recovery of regular pulse frequencies following pulsar glitches provided the first evidence of possible superfluidity of neutron star interiors (Baym et al., 1969b). Initial many-body calculations of pairing already predicted the correct magnitude of the gap in neutron and proton fluids of about 1 MeV, although the nuclear interactions available at the time were not very realistic. The initial theoretical treatments indicated that neutron pairing in the inner crust of a neutron star would occur in the $^1S_0$ state (Clark and Yang, 1970, 1971; Kennedy, 1968; Yang and Clark, 1971) and in the $^3P_2$ state (Hoffberg et al., 1970; Takatsuka, 1972; Takatsuka and Tamagaki, 1971) at higher densities present in the stellar core. Because of the low abundance of protons relative to neutrons in $\beta$-stable neutron-star matter, protons were predicted to pair in the $^1S_0$ state over some range of densities within the core (Chao et al., 1972; Takatsuka, 1973).

The uncertainty in the values of the pairing gaps predicted for various models was substantially reduced with the advent of potentials that are realistic in the sense that they provide high-precision fits to the energy dependence of the experimental scattering phase shifts (with $\chi^2 \approx 1$). Within the BCS models of pairing there is in fact a direct relation between scattering phase shifts for nucleonic scattering and the magnitudes of the pairing gaps (Elgarøy and Hjorth-Jensen, 1998; Khodel et al., 1996). Nevertheless, a quantitative microscopic understanding of the way in which gap values are affected by correlations among nucleons produced both by their interactions and Pauli exclusion, is still lacking. We shall examine this situation in some detail.

The last decade has seen impressive advances in both experimental and theoretical research on pairing in the novel fermionic systems realized in ultracold fermionic gases, which exhibit many features in common with nucleonic superfluids. Such atomic systems allow for remarkable control within the relevant parameter space, notably by tuning of the strength of the pairing force via a Feshbach resonance (Bloch et al., 2008; Giorgini et al., 2008; Leggett and Zhang, 2012). Importantly, these systems can provide a testbed for the repertoire of theoretical approaches being used to describe nucleonic pairing at the microscopic level. Existing parallels have been explored in the context of several phenomena, especially the transition from a BCS-paired state to a BEC. Another commonality arises when one studies cases of imbalanced pairing in systems with unequal populations of the fermion species (which is referred to as “imbalance”), as may occur in a nuclear system with unequal numbers of neutrons and protons. Such isospin asymmetry (or flavor asymmetry more generally) can now be simulated in ultracold systems of atoms having unequal populations.
of different hyperfine states. A third parallel involves the quantum vortex states in ultracold atomic gases that can be explored in situ by imaging techniques, thereby providing an analog of neutron vorticity in rotating neutron stars. Accordingly, another focus of this review will be on these three phenomena—the BCS-BEC transition, effects of imbalance of fermionic species, and quantum vorticity—as they occur they occur in nucleonic systems, with the prospect of experimental verification in systems of ultracold atoms.

What can be learned about nuclear superfluidity and pairing from observations of neutron stars? In fact, the observed rotational anomalies in pulsar periods and X-ray measurements of their surface temperatures provide us with significant evidence of superfluidity of their interiors. The pulsed emission of pulsars (with periods of seconds or less) is locked to the rotation period of the star. Pulsars are nearly perfect clocks, with periods increasing gradually over time due to the secular loss of rotational energy. However, some pulsars undergo abrupt increases (“glitches”) in their rotation and spin-down rates that are followed by slow relaxation toward their pre-glitch values, on a time scale of order weeks to years. These recoveries, when they occur, are not perfect in general, i.e., some permanent residual shifts of either sign may remain. Such behavior is attributed to a component within the star that is only weakly coupled to the rigidly rotating normal-matter component responsible for the emission of pulsed radiation (Baym et al., 1969a). A natural candidate for such a phase is the neutron superfluid either in the core (P-wave) or in the crust (S-wave). Furthermore, young neutron stars cool by neutrino emission from their dense interior, and the cooling histories of neutron stars appear to be consistent with the existing data only if the neutrino emission rates incorporate the superfluidity of their interiors (Page et al., 2013; Schaab et al., 1996; Schmitt and Shternin, 2017; Sedrakian, 2007; Yakovlev et al., 2001).

With respect to the phenomenology of neutron stars, this review will focus, in part, on understanding the roles of pairing in their in neutrino and axion emission, as well as quantum vorticity and superfluid dynamics. Naturally this will be supported by our concentration on microscopic many-body methods for computation of the superfluid properties of neutron-star matter.

Radioactive-beam facilities have opened an exciting new arena for nuclear physics—the study of exotic nuclei close to the proton and neutron drip lines. They enable acquisition of vital information on the nature of the pairing in neutron/proton-rich stable and unstable nuclei, which is of great importance in nuclear astrophysics, especially for an understanding of neutron-star crusts (Chamel and Haensel, 2008). Hartree-Fock-Bogolyubov (HFB) theories have evolved into a standard tool that incorporates pairing in the description of medium-to-heavy nuclei (Bennaceur et al., 2017; Bulgac, 2002; Goriely et al., 2013, 2016b; Reinhard, 2017) However, modern HFB codes still employ simplistic pairing interactions that are matched phenomenologically to more rigorous computations in infinite nuclear matter based on realistic nuclear interactions. While some consideration will be given to the role of pairing in exotic nuclei and the neutron-star crust in Sec. V, this will not be a topic of emphasis in the present review.

Natural units \( \hbar = c = k_B \) will be used throughout, unless otherwise indicated.

II. BASIC BCS THEORY FOR NUCLEAR SYSTEMS

A. Pairing Hamiltonian and the gap equation

We start with a brief description of the simplest model of superconductivity, based on Bogolyubov’s method of canonical transformations (Bogolyubov, 1958). This method has served as a prototype for the treatment of pairing in finite nuclei (Ring and Schuck, 1980). Consider a system of fermions with macroscopic number \( N \) described by the pairing Hamiltonian \( \hat{H} \), defined by

\[
\hat{H} - \mu \hat{N} = \sum_{p, \sigma} \epsilon_p \hat{a}_{p\sigma}^\dagger \hat{a}_{p\sigma} - \frac{1}{V} \sum_{p_1 + p_2 = p_3 + p_4} v(p_3; p_4; p_1; p_2) \hat{a}_{p_3, \sigma_1}^\dagger \hat{a}_{p_4, \sigma_2}^\dagger \hat{a}_{p_1, \sigma_1} \hat{a}_{p_2, \sigma_2},
\]

(1)

where \( \hat{a}_{p\sigma}^\dagger \) and \( \hat{a}_{p\sigma} \) are respectively the creation and annihilation operators for particles having spin \( \sigma \) and momentum \( p \), and \( \mu \) is the chemical potential. The first term on the right represents the creation and annihilation operators for particles having spin \( \sigma \) and momentum \( p \) and \( m \) the particle (effective) mass, while the second term represents the potential energy, with \( -v \) denoting the attractive pairing interaction and \( V \) the volume. The method of canonical transformations introduces two new creation and annihilation operators \( \hat{\delta}_{p\sigma}^\dagger \) and \( \hat{\delta}_{p\sigma} \) through

\[
\hat{\delta}_{p\sigma}^\dagger = u_p \hat{a}_{p\sigma}^\dagger + v_p \hat{a}_{-p\sigma}^\dagger \quad \text{and} \quad \hat{\delta}_{p\sigma} = u_p \hat{a}_{p\sigma} - v_p \hat{a}_{-p\sigma}^\dagger,
\]

(2)

These new operators obey the fermionic commutation relations \( \{\hat{\delta}_{p\sigma}, \hat{\delta}_{p'\sigma'}^\dagger\} = \delta_{pp'} \delta_{\sigma\sigma'} \) and \( \{\hat{\delta}_{p\sigma}, \hat{\delta}_{p'\sigma'}\} = \{\hat{\delta}_{p\sigma}, \hat{\delta}_{p'\sigma'}^\dagger\} = 0 \), provided the Bogolyubov amplitudes \( u_p \) and \( v_p \) (which can be chosen to be real in the absence of flow, or for \( S \)-wave pairing) satisfy the normalization condition \( u_p^2 + v_p^2 = 1 \), implying that the thermodynamic potential of the system is a functional of only one amplitude.

---

1 Glitches have been observed since 1969 in about 180 different pulsars with the number of such events exceeding 500. The most prolific glitching pulsar is the Vela pulsar, with typical changes in the spin \( \Delta \nu / \nu \approx 10^{-6} \) and spin-derivative \( \Delta \nu / \nu \approx 10^{-2} \) (Cordes et al., 1988). Smaller glitches with \( \Delta \nu / \nu \approx 10^{-8} \) were observed in the Crab pulsar. For a contemporary review of glitch observations, see Manchester (2018).

---
The thermodynamic potential at zero temperature is given by the standard expression

\[ E - \mu N = \langle \hat{H} - \mu \hat{N} \rangle, \]  

(3)

where \( \langle \ldots \rangle \) denotes the mean value of the operator enclosed in brackets and \( \hat{N} \) is the net particle number operator. The quasiparticle occupation numbers are defined by \( \langle \hat{\alpha}^{\dagger}_{p,\downarrow} \hat{\alpha}_{p,\downarrow} \rangle = n_{p,\downarrow} \) and \( \langle \hat{\alpha}^{\dagger}_{p,\uparrow} \hat{\alpha}_{p,\uparrow} \rangle = n_{p,\uparrow} \). Minimization of the thermodynamic potential (3), which requires \( \delta(E - \mu N)/\delta \epsilon_p = 0 \), leads to the gap equation, which, for the case of an \( S \)-wave pairing interaction \( v_0(p,p') \), takes the form

\[ \Delta_p = \frac{1}{V} \sum_{p'} v_0(p,p') u_p v_{p'} \left( 1 - n_{p',\downarrow} - n_{p',\uparrow} \right), \]  

(4)

with

\[ u_p^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_p}{E_F} \right), \quad v_p^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_p}{E_F} \right). \]  

(5)

The quasiparticle energy in the superconducting state is given by \( E_p = \sqrt{\epsilon_p^2 + \Delta_p^2} \), i.e., the spectrum of a superconductor features an energy gap \( \Delta_p \). Note that the variation \( \delta(E - \mu N)/\delta n_{p,\uparrow} \), with \( u_p \) and \( v_p \), held constant, yields the quantity \( E_p \), confirming its meaning as assigned above. Consequently, fermionic excitations can be created in the system if a Cooper pair breaks, for which an energy of at least \( \Delta_p \) must be supplied to the system. Because the spectrum \( E_p \) has a minimum at the Fermi energy and the minimal value of \( E_p/p \) is not zero and is equal \( \Delta_p/p_F \), where \( p_F \) is the Fermi momentum, the Landau criterion of superfluidity is fulfilled: it is impossible to create excitations for velocities less than \( \Delta_p/p_F \). From the phenomenological standpoint, the Anderson-Bogolyubov modes play a role analogous to that of phonons in liquid \( ^4\)He. These constitute the normal component of the Landau-Tisza two-fluid model of liquid \( ^4\)He, which coexists with the superfluid component, i.e., the Bose condensate of \( ^4\)He atoms.

Although it gives fundamental insights into the nature of pairing and superfluidity in many-fermion systems, the simple pairing model (3)-(5) developed above is not suited to quantitative microscopic description of these phenomena in the nuclear systems that are the subject of this review, for reasons that will become apparent. It is nevertheless of interest to apply this model to the case of spin-1/2 fermions interacting through a contact interaction characterized by a free-space scattering length \( a_\text{f} \), specific examples being cold atomic gases and neutron matter in the dilute gas limit \( |a_\text{f}|p_F \ll 1 \). For neutron matter, the value of the scattering length, \( a_\text{f} \approx -19 \text{ fm} \), implies \( p_F \approx 0.054 \text{ fm}^{-1} \), which translates to a number density \( n \approx 10^{-5}n_0 \), where \( n_0 = 0.16 \text{ fm}^{-3} \) is nuclear saturation density. Therefore, the range of applicability of this model in the case of neutron matter is limited to the asymptotically dilute regime. To proceed, we first note that at finite temperature \( T \), the equilibrium occupation numbers for fermion quasiparticles should be replaced by \( f(p) = \left( e^{E_p/T} + 1 \right)^{-1} \). The gap equation then can be written as

\[ 1 = t_\text{sc} \nu(p_F) \int_0^\infty \frac{dE_p}{2} \left( 1 - 2f(E_p) \frac{1}{\sqrt{\epsilon_p^2 + \Delta^2}} - \frac{1}{\epsilon_p} \right), \]  

(6)

where \( t_\text{sc} = 4\pi |a_\text{f}|/m \) is the magnitude of the two-body scattering matrix (T-matrix), and \( \nu(p_F) = np_F/\pi^2 \) is the density of quasiparticle states summed over spins. In the zero-temperature limit, i.e., when \( f(E) \to 0 \), Eq. (6) can be solved for the gap (Gor’kov and Melik-Barkhudarov, 1961)

\[ \Delta_0 = \tilde{\epsilon} \exp \left( -2/\lambda \right), \]  

(7)
where \( \lambda = 4p_F|a_0|/\pi, \ \hat{\epsilon} = (8\epsilon_F/e^2)\beta_{\text{GM}} \) and the factor \( \beta_{\text{GM}} = (4e)^{-1/3} \) takes into account the in-medium modification of the interaction due to medium polarization.

The result (7) is reminiscent of the BCS weak-coupling formula for the gap in the phonon-mediated electronic pairing model. It reveals a property of BCS pairing that is awkward from the computational standpoint: the exponential sensitivity of the energy gap to variations of the pairing interaction. Studies of pairing in neutron matter within Gor’kov-Melik-Barkhudarov theory (Gor’kov and Melik-Barkhudarov, 1961) and its extensions, especially to finite-range corrections, have been carried out in (Fan et al., 2017; Schulze et al., 2001). For extensions to multicomponent systems and Fermi-Bose mixtures of cold gases see (Heiselberg et al., 2000).

In the asymptotic regime \( T \to T_c \), where \( T_c \) is the critical temperature for destruction of pairing, the gap equation can be linearized by setting \( \Delta = 0 \) in the denominator of Eq. (6). Straightforward integration leads to

\[
T_c = \left( \frac{\gamma}{\pi} \right) \exp \left( -\frac{\pi}{2p_F|a_0|} \right) = \frac{\gamma}{\pi}, \quad \gamma = e^C, \quad C = 0.57 \quad \text{the Euler constant.}
\]

To next-to-leading order in the small and large temperature expansions, one finds

\[
\Delta(T) = \begin{cases} 
\Delta_0 - \sqrt{2\pi}\Delta_0 T \exp(-\Delta_0/T), & T \to 0, \\
2\pi \sqrt{2/7}\zeta(3) [T_c(T_c - T)]^{1/2}, & T \to T_c,
\end{cases} \tag{8}
\]

where \( \zeta(x) \) is the Riemann \( \zeta \)-function. It follows from Eq. (8) that the temperature variations of the gap function at low temperatures are exponentially small, and that upon approach to the critical temperature, the gap closes with infinite slope.

### B. Nucleon-nucleon pairing in different partial waves

The complexity of the problem of pairing in nuclear systems stems largely from the complexity of nuclear interactions. In practice, the assumed interactions divide roughly into those employed in density functional studies and those designed for ab initio computations. With the pairing interaction given directly by the bare nucleon-nucleon (NN) potential, computation of the gap and other superfluid properties of infinite nuclear systems (e.g. neutron stars) has become routine at the mean-field BCS level in the energy range where the interactions are well constrained by the elastic nucleon-nucleon scattering data, i.e., for laboratory energies \( E_{\text{lab}} < 350 \text{ MeV} \). Such calculations performed with NN-interaction models that fit the scattering data with high precision, such as the Argonne \( V_{18} \) (Wiringa et al., 1995), Paris (Lacombe et al., 1980), Nijmegen (Stoks et al., 1994) and Bonn (Machleidt, 2001), converge to nearly identical results for the pairing gaps in the partial waves with \( L \leq 3 \). The low-energy sector of the nuclear force is accurately described by potentials that are based on chiral perturbation theory, in which the interactions are modeled in terms of pion and nucleon fields and are organized in powers of the ratio of a typical momentum scale of the nuclear problem over a cutoff \( \Lambda_{\text{QCD}} \sim 1 \text{ GeV}/c \) provided by the chiral symmetry breaking scale (Machleidt and Entem, 2011).

At sufficiently high order (third or fourth in chiral expansion), the nuclear potentials constructed using chiral effective field theory may have precision comparable to that achieved with the high-precision NN phenomenological potential models mentioned above.

Having presented the simple pairing model studied in II.A as a backdrop, it must be made clear what is considered mean-field BCS theory in the context of actual, strongly interacting many-fermion systems at meaningful densities, especially nuclear matter. Naturally, the restriction to pairing terms in the interaction component of the second-quantized Hamiltonian \( H \), as expressed in Eq. (1), must be lifted, and accommodation must be made for the actual nature of the NN interaction, which can exhibit very strong momentum dependence. Specifically, NN interaction models designed to fit the NN scattering data and deuteron properties contain a strong short-range repulsion in competition with an outer attractive well, plus tensor and spin-orbit components, along with crucial dependence on total spin and isospin \( S \) and \( T \). Consequently, the simple exponential behavior Eq. (7) characteristic of the pairing gap (i) at asymptotically low densities, (ii) for a contact interaction, and (iii) in phonon-mediated electronic pairing, can be misleading when conducting ab initio studies of nuclear systems (Khodel et al., 1996). Even so, the strong sensitivity of predictions of the pairing gap to inputs for the pairing interaction and the density of states persists. An explicit derivation of mean-field BCS theory within the Green function formalism is provided in III.A. Otherwise, the ab initio microscopic approaches to pairing outlined in III have the collective goal of transcending the limitations of mean-field BCS theory in terms of parquet-consistent (Jackson et al., 1982, 1985) irreducible particle-particle vertices and corresponding self-energies.

The pairing patterns in nuclear matter and neutron-star matter can be understood qualitatively on the basis of partial-wave analysis of NN scattering data. Phase shifts derived from this analysis for different partial-wave channels \( 2S+1L_J \) of the two-nucleon scattering problem are identified using standard spectroscopic notation. The relative orbital angular momentum quantum number \( L = 0, 1, 2, \ldots \) is mapped successively to \( S, P, D, F, G, \ldots \), while the total spin quantum number \( S = 0 \), 1 maps to singlet and triplet spin states. The allowed values of the total angular momentum quantum number, \( J = 0, 1, 2, \ldots \), follow from the quantum-mechanical vector sum of the relative orbital and total spin angular momentum operators.

The experimental scattering phases in the range of laboratory energies \( 0 < E_{\text{lab}} \leq 350 \text{ MeV} \) are shown in Fig. 1 for partial waves that are relevant to pairing in nuclear and neutron matter. As discussed in the next subsection, isospin \( T = 1 \) pairing dominates in neutron-rich matter, whereas in symmetrical nuclear matter \( T = 0 \) pairing
At the large isospin asymmetries found in neutron stars, where the neutron number density is about 95% of the total baryonic density, only $T = 1$ Cooper pairs can form (Dean and Hjorth-Jensen, 2003; Lombardo and Schulze, 2001). This will occur at relatively low densities, driven by the attraction in the $^{3}S_0$-partial wave channel. It is seen in Fig. 1 that the attractive $^{3}P_0$ channel remains sub-dominant to the $S$-wave channel in the low-energy regime below $E_{\text{lab}} = 70$ MeV, where this $P$-wave competitor is overtaken by the $^{3}P_2-^{3}F_2$ coupled partial wave as the most attractive $L = 1$ channel. However, it is only at around $E_{\text{lab}} = 170$ MeV that the $^{3}P_2-^{3}F_2$ partial wave starts to dominate the $T = 1$ scattering, as the $^{1}S_0$-wave interaction loses its attractive component and eventually becomes repulsive (having negative phase shifts) for $E_{\text{lab}} > 250$ MeV.

Thus, the dominant $T = 1$ channel above $E_{\text{lab}} = 200$ MeV energy is the coupled $^{3}P_2-^{3}F_2$ partial-wave channel, for which the spatial wave function is antisymmetric, whereas the total spin $S = 1$ and isospin $T = 1$ imply symmetrical components of the wave function in their respective spaces. Accordingly, pairing in the triplet spin-1 channel is allowed by the Pauli principle for two neutrons or protons. In contrast, if the nuclear system has equal populations of neutrons and protons, $S = 1$ and $T = 0$ pairs may be formed in the $^{3}D_2$ channel, which applies exclusively to neutron-proton scattering, being forbidden for like-isospin particles by the Pauli principle.

Note that the $^{1}P_1$ and $^{3}P_1$ partial waves, not shown in Fig. 1, are repulsive within the relevant energy range and are therefore inconsequential for the pairing problem.

Up to this point we have referred to specific features of the nuclear interaction exhibited in two-nucleon scattering over ranges of laboratory energy. How does one translate this behavior into density ranges in neutron stars? This can be done semi-quantitatively by observing that the center-of-mass energy of two scattering fermions, given by $E_{\text{lab}}/2$, should be roughly twice the Fermi energy of the nuclear medium. With applications to neutron stars in mind, we may focus on the high-density, low-temperature regime of highly degenerate nuclear matter. Neutron Fermi energies are of the order $E_{Fn} \approx 60$ MeV in neutron-star matter at the nuclear saturation density, $n_0 = 0.16$ fm$^{-3}$. From this, we can already predict the result, borne out in microscopic many-body calculations, that neutron pairing in the $^{1}S_0$ partial wave will expire at depths slightly above the crust-core interface, where the density is about half $n_0$. The low proton fraction in the neutron-star core, $x_n \approx 5-10\%$, and the correspondingly low proton Fermi energies, imply that proton pairing occurs in the $^{1}S_0$ state up to quite high densities. It is also conceivable that at neutron-star densities in excess of a few times the nuclear saturation density, pairing can occur in higher even-$L$ partial

The nuclear physics aspects of the composition of neutron star interiors is discussed, for example, in the texts by Shapiro and Teukolsky (1983), Glendenning (2000), and Weber (1999).
waves such as the $^1D_2$ channel (not shown in Fig. 1). On the other hand, isospin-symmetric nuclear matter with $n_p = n_n$, where $n_n$ and $n_p$ are the number densities of neutrons and protons, may support pairing in the attractive $^3D_2$ partial wave, with a wave function which is symmetrical in space, antisymmetrical in isospace ($T = 0$) and symmetrical in spin space ($S = 1$). It is conceivable that some models of dense matter would support pairing in the $^3D_2$ partial wave (Alm et al., 1996b). Indeed, the abundances of protons can be equal (or even exceed) that of neutrons if $K^- \text{ condensation takes place (Glendenning and Schaffner-Bielich, 1999; Weber, 1999).}$

In the case of pion-condensed cores of neutron stars, the ground state of matter could be a superposition of neutron-proton quasiparticles filling a single Fermi sphere. Such matter is commonly described by a single type of “nucleonic” quasiparticle (Baym and Flowers, 1974; Campbell et al., 1975; Sawyer and Soni, 1977).

### C. Effects of isospin asymmetry and neutron stars

Much of the research on nuclear pairing is concerned with neutron stars, so it is important to review the state of matter in such objects. The interiors of neutron stars are approximately in equilibrium with respect to the weak interactions during their lifetimes. Small deviations from such equilibrium may be important in some problems, such as the bulk viscosity of matter, but for the most part we will assume strict $\beta$-equilibrium. The resulting disparity between the neutron and proton numbers (breaking the $SU(2)$ symmetry in matter) has profound influence on the pairing patterns in neutron stars.

To illustrate the characteristic asymmetries in stellar interiors, consider the energy density of uniform nuclear matter close to saturation density and the isospin symmetrical limit,

$$\epsilon(n, \alpha) = \epsilon_0(n_0) + \frac{K_0}{2!} \left( \frac{n - n_0}{3n_0} \right)^2 + \epsilon_{\text{sym}}(n) \alpha^2 + O(\alpha^4),$$

$$\epsilon_0(n_0)$$ is the energy density of symmetrical nuclear matter at saturation, $K_0 \simeq 250 \text{ MeV}$ is the nuclear matter compressibility,

$$\alpha = \frac{n_n - n_p}{n_n + n_p},$$

is the isospin asymmetry (or neutron excess), and the symmetry energy close to saturation is given by

$$\epsilon_{\text{sym}}(n) = \epsilon_{\text{sym}}(n_0) + L \left( \frac{n - n_0}{3n_0} \right) + \frac{K_{\text{sym}}}{2!} \left( \frac{n - n_0}{3n_0} \right)^2.$$

The parameters in Eq. (11), i.e., the symmetry energy at saturation $\epsilon_{\text{sym}}(n_0) \simeq 32 \text{ MeV}$ as well as $L \in [50; 90] \text{ MeV}$ and $K_{\text{sym}} \in [-200; 130] \text{ MeV}$ (which define the slope and curvature of the symmetry energy) can be determined empirically. We quote only representative values and ranges for these parameters. [For a discussion of uncertainties in these parameters see Lattimer and Steiner (2014)].

Consider first the simplest case of stellar matter composed of neutrons ($n$), protons ($p$) and electrons ($\epsilon$) in weak equilibrium. The tandem processes of $\beta$ decay and electron capture,

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad p + e^- \rightarrow n + \nu_e$$

establish a certain balance between the abundances of hadrons and leptons, which requires $\mu_e = \mu_n - \mu_p \equiv \delta \mu$, where $\mu$ refers to the chemical potentials of the three particle types involved. Here we set neutrino chemical potentials to zero, because if the temperatures of neutron stars are below several MeV, a condition reached by at most several hours after birth, neutrinos leave the stellar interior without interaction. In addition, charge neutrality requires coincidence of proton and electron number densities, $n_p = n_e$. However, when $\mu_e \geq m_\mu$, where $m_\mu = 105.7 \text{ MeV}$ is the muon rest mass, muons will also be present.

Given the energy density $\epsilon(n, \alpha)$ of asymmetric nuclear matter, the chemical potentials of particles are computed from

$$\mu_i = \frac{\partial \epsilon(n, \alpha)}{\partial n_i} = \frac{\partial(\epsilon(n, \alpha)/n)}{\partial Y_i},$$

where $Y_i = n_i/n$ is the concentration of species $i$. Then, using the expansion (9) we find

$$\delta \mu = \mu_n - \mu_p = 4\epsilon_{\text{sym}}(n)\alpha \geq \Delta(\alpha = 0),$$

where $\Delta(\alpha = 0)$ is the isoscalar (i.e., $S = 1$ and $T = 0$) neutron-proton pairing gap. In fact, $Y_n \sim 0.9$ and $Y_p \sim 0.1$ in neutron-star matter close to nuclear saturation density, so $\alpha \sim 0.8$ and $\mu \sim 100 \text{ MeV}$. This means that the ambient asymmetries are too large for Cooper pairs to form in the isoscalar channel in homogeneous neutron-star matter. Pairing then occurs in the isotriplet channels, specifically $^1S_0$ and $^3P_2-^3F_2$.

At densities exceeding nuclear saturation by factors of a few, the appearance of heavier baryons with non-zero strangeness becomes energetically favorable in many (but not all) nuclear models.\footnote{Hyperonization of dense nuclear matter was considered early by Ambartsumyan and Saakyan (1960), Pandharipande (1971), and Glendenning (1985) and has been discussed extensively in recent years in attempts to reconcile the astrophysical information on large masses of pulsars with the softening of the equation of state of dense matter caused by hyperons. See Chatterjee and Vidaña (2016); Oertel et al. (2017) and references therein.} To generalize the equilibrium conditions to this situation, one observes that there are two conserved charges in matter – the total baryonic charge and the total electrical charge. (Strangeness is
not conserved in weak interactions, which may convert nucleons into hyperons; hence the strangeness chemical potential is zero.) The chemical potentials are obtained as [see, e.g., (Glendenning, 2000) Eq. (5.52)]

$$\mu_i = \frac{\partial \epsilon(n, \alpha)}{\partial n_i} = \mu_B B_i + \mu_Q Q_i,$$

(15)

where $i$ identifies the baryon and lepton species, $B_i$ and $Q_i$ are their baryonic and electric charges, and $\mu_B$ and $\mu_Q$ are the associated Lagrange multipliers (chemical potentials). Commonly, the chemical potentials of the neutron $\mu_n = \mu_B$ and electron $\mu_e = -\mu_Q$ are taken as independent quantities, in terms of which Eq. (15) determines the chemical potentials of the remaining particles.

In Fig. 2 we illustrate the abundances of various species in a mixture of baryons and leptons in the interior of a neutron star in the case of density-dependent covariant functional theory (Raduta et al., 2018). As already discussed above qualitatively, the neutrons and protons forming the dominant component of matter at low densities are subject to a large disparity in their densities, and hence in their chemical potentials. Therefore, pairing with quantum numbers $S = 1$ and $T = 0$, specifically in the partial-wave channels $^{3}S_1 - ^{3}D_1$ and $^{3}D_2$, is strongly suppressed. Thus the two channels that provide the largest attraction in symmetrical nuclear matter are ineffective in neutron-star matter (see Fig. 1). In other words, within the BCS approximation, pairing in neutron stars is dominated by the $^{1}S_0$ and $^{3}P_{2 - 3}P_2$ partial waves in the $T = 1$ channel at low and high densities, respectively. The disparity in the neutron and proton densities also implies that the transition from $S$-wave to $P$-wave pairing takes place at quite different densities for the neutron and proton components. For neutrons this transition occurs at $n \approx n_0$, whereas for protons the required density is not reached in neutron star interiors in the majority (but not all) models. Notwithstanding the arguments above, it has been argued that in the low-density and low-isospin asymmetry nuclear matter that may be created in low to intermediate heavy-ion collisions, supernova, and proto-neutron-star matter, $^{3}S_1 - ^{3}D_1$ pairing may persist if the pairing interaction does not differ strongly from that in free space. We will address in detail the physics of $^{3}S_1 - ^{3}D_1$ pairing and its suppression in low-density nuclear matter in Sec. IV. In analogy with nucleonic pairing, a hyperonic component of neutron-star interiors will develop BCS condensates when the mutual interaction of hyperons is attractive, as will be discussed in Sec. V.C.

D. Finite nuclei

Although this review is concerned primarily with pairing in infinite nuclear systems, it will be helpful to recapitulate the basic facts about pairing in finite nuclei. Validation of pairing theory in direct terrestrial experiments on accessible nuclides ($N, A = N + Z$) provides a valuable source of constraints and methods potentially relevant to the study of infinite nuclear matter. For in-depth expositions of the pairing in finite nuclei see (Broglia and Zelevinsky, 2013; Dean and Hjorth-Jensen, 2003; Ring and Schuck, 1980).

At the most basic level, pairing correlations in finite nuclei express themselves in the odd-$A$-even-$A$ staggering of the measured binding energies of nuclei. The neutron “pairing gaps” in the cases of odd and even neutron numbers are commonly defined as

$$\Delta_{Z,N}^{\text{odd}} = \frac{1}{2} (E_{Z,N+1} + E_{Z,N-1}) - E_{Z,N}, \text{ (odd } N),$$

(16)

$$\Delta_{Z,N}^{\text{even}} = \frac{1}{2} (E_{Z,N+1} + E_{Z,N-1}) + E_{Z,N}, \text{ (even } N),$$

(17)

where $E_{Z,N}$ is the binding energy of a nucleus with proton number $Z$ and neutron number $N$. The pairing gaps for changes of proton number are defined by the same Eqs. (16) and (17), with the roles of $N$ and $Z$ interchanged. Evaluation of these differences in the case of neutron-number increments shows that odd-$A$ nuclides are less bound than their neighboring nuclides by about 1 MeV on average. Gap values for each fixed $N$ can fluctuate by a factor two. Enhancement of the pairing effect on binding is observed for nuclei having neutron magic numbers $N = 28, 50, 82, \text{ and } 126$. Proton pairing shows the same energetic systematics, with somewhat smaller values of the gap, presumably due to the Coulomb repulsion between protons. The pairing gaps decrease with the mass number of nuclei, a behavior described phenomenologically by fits to gaps. A fit to the neutron and proton

---

4 The basics of covariant density functional theory for nuclear systems are discussed, for example, in (Glendenning, 2000; Meng, 2016; Serot and Walecka, 1997; Weber, 1999).
gaps reads, in MeV units, (Bertsch, 2013)
\[
\Delta_{n,\text{even/odd}}^Z = -a_{\text{even/odd}}^N, \text{ (neutrons)}
\]
\[
\Delta_{p,\text{even/odd}}^Z = (0.96 \pm 0.28)/(1.64 \pm 0.46), \text{ (protons)}
\]
with \(a_{\text{even}} = \pm 0.28 \text{ MeV} \) and \(a_{\text{odd}} = \pm 0.25 \text{ MeV} \).

Differences in the excitation spectra of even-\(N\) and odd-\(N\) nuclei provide another source of evidence for pairing correlations in nuclei. For \(N\) even, the excited states are separated from the ground state by a gap that can be interpreted as the energy needed to break a pair of neutrons, whereas for \(N\) odd, the lowest of the discrete (but dense) energy levels are found well within the range of 1 MeV characteristic of gaps in nuclei. Additionally, it should be noted that the excited states of nuclei may have collective nature that is reminiscent of the phonon modes present in macroscopic superfluids. Because of the finite nature of nuclei, these modes are not necessarily bulk modes, i.e., they could be associated with the lowest-order quadrupolar shape oscillations of the nucleus with angular momentum and parity quantum numbers \(J^z = 2^+\).

Theoretical studies of pairing properties of nuclei in the range of the intermediate and large mass number are generally performed within the framework of density functional theory (DFT) either in non-relativistic (Bennaceur et al., 2017; Bulgac, 2002; Goriely et al., 2013, 2016a; Reinhard, 2017) or relativistic formulations (Kucharek and Ring, 1991; Li et al., 2015; Long et al., 2016, 2010). Such approaches may be based purely on Hartree-Fock (HF) functionals, nuclear properties (energy states and associated densities and currents) being computed in the absence of pairing, with pairing included in a final step within a simplified BCS approach – essentially as outlined in Sec. II.A. Alternatively, nuclear pairing studies may utilize Hartree-Fock-Bogolyubov (HFB) functionals, performing computations that iterate the normal and anomalous states of the system, in a manner that allows for feedback of pairing correlations in the resulting mean fields, guaranteeing a self-consistent solution. The pairing interactions are typically modeled as contact interactions, with the theory outlined in Sec. II.A providing the basis for their treatment. The two parameters of this theory, namely the dimensionless pairing interaction (or coupling) and the energetic range over which the pairing is effective, are adjusted to the phenomenology of the nuclei being considered. Alternatively, these parameters can be chosen so as to reproduce the results of pairing calculations in infinite neutron matter (Goriely et al., 2013, 2016a).

As we shall discuss in the later sections, there are two effects that influence the results obtained with simple two-body contact interactions. First, there could be substantial corrections to the pairing interaction coming from collective modes (polarization effects). Secondly, three-nucleon interactions are non-negligible in nuclear systems, as they have been found to be important in high-precision fits to the properties of light nuclei and to some extent for the saturation of nuclear matter. (Apart from the generic three-body forces originating at the level of quark substructure, there are also “effective” three-body forces generated in diverse theoretical treatments of two-body interactions that feature strong short-range repulsive components.) In addition, the energetic scale \(\tilde{\epsilon}\) of the contact-interaction model is expected to depend on the occupancy of states in the vicinity of the Fermi surface, when the pairing is dominated by the direct (attractive) nuclear interaction. However, when (or if) the pairing is predominantly driven by collective effects, it should reflect the frequency range of collective phonon modes.

As explained in Sec. II.B, at sub-saturation densities the dominant attractive NN interaction is in the $^3S_1–^3D_1$ channel, i.e., the channel supporting a \(np\) bound state in free space – the deuteron. However, the foregoing discussion of pairing in nuclei has involved only isospin-triplet (\(nn\) or \(pp\)), spin-singlet pairing. Noteworthy in this connection is the empirical fact that the binding energies of nuclei on the \(N = Z\) line are larger than those of their neighbors by an amount known as the Wigner energy (Satula et al., 1997). This could be interpreted as evidence for pairing in the $^3S_1–^3D_1$ channel, which is otherwise suppressed for \(N \neq Z\) nuclei by the mismatch in the neutron and proton energy-level occupancies. Obviously, the pairing interaction may be modified by the ambient medium differently in different isospin-spin channels, one consequence being a less attractive force in the $^3S_1–^3D_1$ than in the $^1S_0$ channel. Moreover, the spin-orbit field of the nucleus may affect the spin coupling of nucleonic Cooper pairs differentially, suppressing the $^3S_1–^3D_1$ neutron-proton pairing more than S-wave pairing of like-isospin pairs. Neutron-proton pairing is expected also from Hartree-Fock-Bogolyubov computations of large nuclei (Friedman and Bertsch, 2007).

Besides its influence on static properties of nuclei, pairing and the accompanying superfluidity are known to affect the dynamics of nuclei, including rotation, shape oscillations, and fission. In contrast to neutron stars (addressed intensively in Sec. V), where the effects of nuclear superfluidity extend over macroscopic scales, the characteristic scale of Cooper pairs, i.e., the coherence length, is of the order of the size of the nucleus or somewhat larger. Accordingly, one would expect the breakdown of superfluidity in nuclei to have little or no effect on their global dynamics. Surprisingly, self-consistent cranking HFB models, which reproduce the $2^+$ excitations of nuclei with good accuracy, require moments of inertia which are half the rigid-body value (Delaroche et al., 2010).

In its study, sub-barrier fission offers another tool to access the degree to which various nuclei are superfluid. Specifically, superfluidity enhances the probability of fission, as it produces a larger overlap between different nearly degenerate configurations. Since quantum-mechanical tunneling probability depends exponentially on the energy difference between configurations, one would expect a high sensitivity of the empirical results for the fluid parameters of a given nucleus. In
particular, theoretical interpretation of the fission of $^{234}$U and $^{240}$Pu requires inclusion of an enhancement through superfluidity in order to account for the observed decay lifetimes ([Bertsch, 2013; Sadhukhan et al., 2016]).

### E. Interface between nuclear systems and cold atomic gases

The realization of BCS pairing in ultracold atoms in 2004-2006 ([Greiner et al., 2005; Zwierlein et al., 2006]) was a major development that has considerably enlarged and diversified the scope of fermionic pairing as exemplified in strongly correlated quantum many-body systems. Indeed, prior to this discovery, the domain of application of fermion pairing had been limited to specific examples considered to arise in nature, specifically in nuclei, neutron-star matter, color-superconducting quark matter, liquid $^3$He, and electrons in solids. Quantum gases of fermionic atoms offer the freedom to transcend nature by tuning the interaction between atoms via the Feshbach resonance mechanism ([Bloch et al., 2008; Giorgini et al., 2008; Leggett and Zhang, 2012]), notably to the strongly interacting regime \( p_F |a| \gg 1 \), where \( p_F \) is the Fermi momentum and \( a \) the scattering length of the interaction. In this regime the gas particles can no longer be described as a weakly interacting gas. Remarkably, the maximally strong-coupling regime – the unitary limit corresponding to \( p_F |a| \to \infty \) – has become accessible for ultracold fermions because three-body collisions are strongly suppressed in these systems precisely because of the Pauli principle. (The opposite situation applies for the case of bosonic atoms, where the lifetime of such cold-atom systems tends to zero due to three-body collisions.) This new possibility is of special significance for nuclear physics, because pure neutron matter, having an anomalously large scattering length \( a_n \simeq -19 \text{ fm} \), is close to the unitary regime at very low density.

Furthermore, by adjusting the magnitude of the magnetic field to tune Feshbach resonances ([Chin et al., 2010]), it has become feasible to drive a trapped cold atomic gas experimentally from the weakly interacting BCS regime, where the gas consists of loosely bound Cooper pairs, to the strongly interacting BEC regime of tightly bound dimers. Thus, the theoretical ideas put forward several decades ago in support of a hypothetical BCS-BEC transition ([Bloch et al., 2008; Leggett and Zhang, 2012; Nozières and Schmitt-Rink, 1985]) have been validated in experimental realizations ([Regal and Jin, 2007; Zwierlein et al., 2006]).

The experimental prospects opened by techniques developed to manipulate cold atoms also include the possibility of creating a trapped atomic gas, for example composed of $^6$Li atoms, that has unequal populations of two different hyperfine states – thereby simulating an interacting Fermi gas with unequal numbers of spin-up and spin-down particles. Such systems are expected to exhibit a rich variety of unconventional pairing phases, such as the FFLO phase ([Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1965]) predicted in 1964, which features Cooper pairs with non-zero center-of-mass momentum. Importantly, the combination of these two features – the BCS-BEC crossover and population imbalance – will allow one to explore regimes of strongly interacting paired fermionic matter that have never been accessible in other systems, yet are of high interest for the phenomenological understanding of pairing in asymmetric nuclear matter and spin-polarized neutron matter (see Sec. IV.F and IV.G).

By placing fermions in an optical lattice of suitable design, one is now able to simulate the effect of a periodic potential on the properties of strongly correlated fermions subject to tunable interactions ([Bloch, 2004; Chin et al., 2006; Schreiber et al., 2015; Zwerg, 2003]). So far, experimental studies of quantum many-body systems along such lines has concentrated mainly on properties of Hubbard models ([Chiu et al., 2017]) and the Mott transition ([Esslinger, 2010; Greiner et al., 2002]). With dense-matter astrophysics in mind, one potential application of this new ability is a cold-atom laboratory model of the matter in the crust of a neutron star (see Sec. V). Insight could be gained into the interplay of the periodic potential and pairing in a strongly interacting gas under freely adjustable conditions, including lattice spacing, strength of interaction, various shapes of lattice potentials that may induce non-spherical “nuclei” (pasta phases), etc.\(^5\)

Another area of overlap between the nuclear superfluids in neutron stars and those created in cold-atom traps involves the presence of quantum vortices. Experimental realization of quantum fermionic vortices in trapped gases and their evolution through the BCS-BEC crossover was initially instrumental in proving the very existence of superfluidity in a Fermi gas of $^6$Li ([Zwierlein et al., 2006]). However, the range of phenomena that can be probed experimentally is vast. For example, it embraces studies of: (i) core quasiparticle excitations in different interaction regimes and with respect to imbalance, (ii) mutual friction in superfluid-normal mixtures of gases, (iii) higher-spin vortices, and (iv) mixtures of fermionic superfluids and Fermi-Bose fluids. In fact, vortices were realized recently in mixtures of Fermi-Bose fluids ([Yao et al., 2016]). In anticipation of the aforementioned experimental studies, theoretical work has been carried out on vortex-core quasiparticle excitations in different interaction regimes and with respect to imbalance ([Iskin, 2008; Takahashi et al., 2007; Warringa, 2012; Warringa and Sedrakian, 2011; Yu and Bulgac, 2003]), and on vortex dynamics and pinning ([Bulgac et al., 2013; Bulgac et al., 2013; Bulgac et al., 2013]).

---

\(^5\) Non-spherical nuclear pasta was initially studied in ([Hashimoto et al., 1984; Lorenz et al., 1993; Ravenhall et al., 1983]). Recent advances in studies of these phases in neutron-star crusts are discussed in ([Fattoyev et al., 2017; Schneider et al., 2016; Schnettrumpf and Nazarewicz, 2015]).
Wlazlowski et al., 2016). Macroscopic dynamics of rotating superfluids featuring vortex lattices has been investigated in great detail both theoretically and experimentally (Fetter, 2009). The corresponding studies in ultracold bosonic gases have focused on vortex-lattice oscillations (Tkachenko modes), quadrupolar modes of oscillations, rapid-rotation induced Landau quantization of states, etc.; for reviews see (Cooper, 2008; Verhelst and Tempere, 2017). These experimental studies find analogs in the physics of neutron stars, as will be explained in Section V.

III. METHODS FOR STRONGLY CORRELATED SYSTEMS

A. Green Functions approach and Gor’kov formalism

In this section we outline and discuss the Green functions method for the treatment of superfluid systems. The method was originally introduced by Gor’kov and by Nambu (Gor’kov, 1958; Nambu, 1960). Their formulation is based on thermodynamic Green functions (GF) defined in the imaginary-time formalism. The starting point of this formalism is the set of coupled Dyson-Schwinger equations for the normal and anomalous propagators which contain the self-energies of the system. The self-energies allow for diagrammatic representation which provides a systematic way to account for the correlations in the system in terms of resummations of diagrams in the relevant dynamical channels. A variant of the zero-temperature GF theory of pairing appropriate for nuclear systems was developed by Larkin and Migdal (1964), on the basis of the Landau Fermi-liquid theory for normal systems. Already in this early work a number of important aspects of the fermionic pairing problem were introduced, including wave-function renormalization and summations in the particle-hole and particle-particle channels, with results expressed in terms the phenomenological parameters of the Landau Fermi-liquid theory. This approach was further adapted to finite Fermi systems (nuclei), and a number of nuclear observables were evaluated using the Landau parameters for nuclear systems (Migdal, 1967).

In the following decades the GF method was largely abandoned in the context of nuclear pairing. It was revived in the early 1990s by a number of research groups, specifically in the context of $^3S_1-^3D_1$ pairing in isospin symmetric and asymmetric systems (Baldo et al., 1992a, 1995; Sedrakian et al., 1997), as well as for $^1S_0$ and $^3P_2-^3P_0$ pairing (Baldo et al., 1998; Elgarøy et al., 1996a,b,c). These studies were already based on realistic (i.e., phase-shift equivalent) NN interactions and included single-particle spectra renormalized within Brueckner-type theories of nuclear matter.

Somewhat earlier, the real-time anomalous propagator treatment of nuclear pairing was introduced by Su et al. (1987), but the interactions were treated at the level of the Skyrme effective contact forces commonly used for computations on finite nuclei. The particle-particle and particle-hole resummations in the GF theory are related to the microscopic determination of the Landau Fermi-liquid parameters (see Sec. III.A.3 for details). This task was taken up within GF theory at about the same time in (Ainsworth et al., 1989; Schulze et al., 1996; Wambach et al., 1993).

The class of theories of unpaired matter formulated in terms of GF allows one to deduce only the critical temperature of the superfluid phase transition, as signaled by poles that emerge in the medium-modified scattering matrix of two nucleons (Alm et al., 1993, 1996b; DICKhoff, 1988; Schmidt et al., 1990; Sedrakian et al., 1995; Stein et al., 1995). We relegate to Section III.B the discussion of theories in which pairing is inferred indirectly from instability of the normal state.

An important feature of the GF formulation is that it admits a description beyond the concept of quasiparticles inherent to the Landau Fermi-liquid theory by accounting for the finite width of particle states. This may strongly affect pairing when it is addressed at the level of self-energies (Božek, 2000, 2002, 2003; Božek and CZERESKI, 2002; Müther and Dickhoff, 2005). We relegate the discussion of these theories to subsection III.C.

Excellent reviews of GF methods applied and results obtained up to the turn of the century have been provided in (Dean and Hjorth-Jensen, 2003; Lombardo and Schulze, 2001).

The following two decades have seen wide application of GF theory to superfluid nuclear systems. One approach is to accurately incorporate many-body corrections while enforcing consistency between various ingredients, especially vertex corrections and renormalization of single-particle energies, as has been done for $S$-wave channels (Cao et al., 2006; Sedrakian and Lombardo, 2000; Shen et al., 2005). Another line of development has employed soft effective interactions to account for the resummations in the particle-hole channel in the framework of Landau Fermi-liquid theory, specifically for $S$- and $P$-wave channels (Schwenk and Friman, 2004; Schwenk et al., 2003). The effects of phonons and retardation of the interaction on pairing have also been explored based on effective interactions (Barranco et al., 2005; Sedrakian, 2003). More recently, the following aspects of the problem of nucleonic pairing have been brought into focus: (i) Incorporation of effects on the pairing interaction and self-energies of three-body (3N) forces, either of fundamental origin or generated by the many-body method used to treat strong correlations (Dong et al., 2013; Drischler et al., 2017; Papakonstantinou and Clark, 2017), (ii) calculation of pairing gaps based on a variety of soft, chiral NN interactions (Drischler et al., 2017; Finelli et al., 2015; Maurizio et al., 2014; Srinivas and RAMANAN, 2016), which in part explore the influence of the cutoff of these interactions, and (iii) studies of the effects on pairing of short-range correlations (Ding et al., 2016; Rios et al., 2017), as accounted for in terms of spectral...
functions (to be considered in Sec. III.C). The general trends that emerge from these studies will be discussed at a later stage (see Sec. III.F).

We now turn to an exposition of the Green-function formulation of pairing theory that is applicable to superfluid Fermi systems at finite temperature for finite-range two-body interactions. This formulation, also described by us in an earlier review (Sedrakian and Clark, 2006a) is based on the pioneering work of (Larkin and Migdal, 1964).

1. Green functions formalism

The Gor'kov propagators describing the superfluid state formally obey the Dyson-Schwinger equation

\[
G_{\alpha\beta}(x,x') = G^0_{\alpha\beta}(x,x') + \sum_\gamma \int d^4x''d^4x''' \{\Sigma_{\gamma\beta}(x,x'')G^0_{\alpha\gamma}(x',x''')G_{\alpha\beta}(x'''',x') \times \Sigma_{\gamma\beta}(x'''',x'')G_{\alpha\beta}(x'',x')
\]

(18)

containing the free propagator \(G^0_{\alpha\beta}(x,x')\) and self-energy \(\Sigma_{\alpha\beta}(x,x')\), where \(x\) represents the space-time coordinate and the Greek indices \(\alpha, \beta \ldots\) label spin and isospin states. The most general treatment in real time requires time ordering along the Keldysh-Schwinger contour (Keldysh, 1964; Schwinger, 1961), which generates the well-known 2 \(\times 2\) matrix GF with components representing the causal and acausal GF as well as two additional GF with fixed time arguments on opposite sides of the contour (Botermans and Maffiet, 1990; Danielewicz, 1984; Lifshitz and Pitaevskii, 1981). Below we will be concerned only with the equilibrium physics; hence we do not present the equations for the most general case. Instead, we restrict our attention to the retarded components of the GF and self-energies, which can be obtained through unitary transformation from the initial matrix Keldysh-Schwinger GF (Lifshitz and Pitaevskii, 1981).

In the superfluid state, each retarded GF is a 2 \(\times 2\) matrix in the so-called Nambu-Gor'kov space, with the off-diagonal elements accounting for non-vanishing pairing correlations; explicitly,

\[
iG_{\alpha\beta}(x,x') = \left( \begin{array}{cc} \langle T\psi_\alpha(x)\psi_\beta(x') \rangle & i\langle T\psi_\alpha(x)\bar{\psi}_\beta(x') \rangle \\ i\langle T\bar{\psi}_\alpha(x)\psi_\beta(x') \rangle & \langle T\bar{\psi}_\alpha(x)\bar{\psi}_\beta(x') \rangle \end{array} \right) = i\left( \begin{array}{c} G_{\alpha\beta}(x,x') \\ F_{\alpha\beta}(x,x') \\ -F_{\alpha\beta}^\dagger(x,x') \\ G_{\alpha\beta}(x,x') \end{array} \right),
\]

(19)

where the \(\psi_\alpha(x)\) are baryon field operators, and \(T\) and \(\hat{T}\) signify time-ordering and inverse time-ordering of operator products, respectively. This matrix-propagator obeys the Dyson-Schwinger equation (18), with the free propagator being diagonal in the Gor'kov-Nambu space.

The matrix structure of the self-energy \(\Sigma_{\alpha\beta}(x,x')\) is identical to that of the propagators:

\[
\Sigma_{\alpha\beta}(x,x') = \left( \begin{array}{cc} \Sigma_{\alpha\beta}(x,x') & \Delta_{\alpha\beta}(x,x') \\ \Delta_{\alpha\beta}^\dagger(x,x') & \Sigma_{\alpha\beta}(x,x') \end{array} \right).
\]

(20)

The off-diagonal self-energies \(\Delta\) and \(\Delta^\dagger\) describe the superfluid state and vanish in the unpaired limit.

The Fourier transformation of Eq. (18) with respect to the relative coordinate \(x - x'\) gives a set of coupled equations

\[
G_{\alpha\beta}(p) = G^0_{\alpha\beta}(p) + G^N_{\alpha\gamma}(p)\Sigma_{\gamma\beta}(p)G_{\alpha\beta}(p),
\]

(21)

\[
F^\dagger_{\alpha\beta}(p) = G^0_{\alpha\gamma}(p)\Sigma_{\gamma\beta}(p),
\]

(22)

for the component GF, summation over repeated indices being understood. The dependence of these GF on the center-of-mass coordinate is suppressed and will not be needed until we consider the state where Cooper pairs carry non-zero total momentum in Sec. IV. Equations (21) and (22) represent a coupled system for the normal \(G_{\alpha\beta}(p)\) and anomalous \(F_{\alpha\beta}^\dagger(p)\) propagators. The system is closed once the normal \(\Sigma_{\alpha\beta}(p)\) and anomalous \(\Delta_{\alpha\beta}(p)\) self-energies are expressed in terms of interaction(s) and propagators. Note that similar equations can be written for \(F_{\alpha\beta}(x,x')\) and \(G_{\alpha\beta}(x,x')\), but in most of the cases (e.g. for systems with time-reversal symmetry) these are redundant.

In the normal (unpaired) state, the generalized Dyson-Schwinger equation reduces to a single equation

\[
G^N_{\alpha\beta}(p) = G^0_{\alpha\beta}(p) + G^N_{\alpha\gamma}(p)\Sigma_{\gamma\beta}(p)G^N_{\alpha\beta}(p),
\]

(23)

which has the solution \(G^N_{\alpha\beta}(p) = \delta_{\alpha\beta}[\omega - \varepsilon(p)]^{-1}\), in effect defining the single-particle energies \(\varepsilon(p) = \varepsilon_p + \Sigma(p)\), where \(\varepsilon_p\) is the free single-particle spectrum. Note that the self-energy \(\Sigma(p)\) is diagonal in spin and isospin spaces, given spin-isospin conserving forces. Equations (21) and (22) can now be expressed in a more compact form

\[
G_{\alpha\beta}(p) = G^N_{\alpha\gamma}(p)\left[\delta_{\alpha\beta} + \Delta_{\alpha\beta}(p)F_{\alpha\beta}^\dagger(p)\right],
\]

(24)

\[
F^\dagger_{\alpha\beta}(p) = G^N_{\alpha\gamma}(p)\left[-\delta_{\alpha\beta} + \Delta_{\alpha\beta}^\dagger(p)G_{\alpha\beta}(p)\right],
\]

(25)

having the solution

\[
G_{\alpha\beta}(p) = \frac{\delta_{\alpha\beta}}{\delta_{\alpha\beta} - \omega - [E_A(p) + E_S(p)]/2},
\]

(26)

\[
F^\dagger_{\alpha\beta}(p) = \frac{\Delta_{\alpha\beta}^\dagger(p)}{\delta_{\alpha\beta} - \omega - [E_A(p) - E_S(p)]/2},
\]

(27)

where \(E_{S/A} = [\varepsilon(p) \pm \varepsilon(-p)]/2\) denotes the symmetric (S) and antisymmetric (A) parts of the single-particle spectrum \(\varepsilon(p)\) in the normal state and \(\Delta(p)\) \(\Delta^\dagger(p)\) \(\equiv -\Delta^\dagger(p)\). The GF in Eqs. (26) and (27) share the same poles at

\[
\omega_\pm = E_A(p) \pm \sqrt{E_S^2(p)^2 + \Delta^2(p)}\]

(28)

thereby determining the excitation spectrum. If the normal self-energy is invariant under reflections in space
(i.e. even under \( \mathbf{p} \rightarrow -\mathbf{p} \) and time-reversal invariant (i.e. even under \( \omega \rightarrow -\omega \)), the \( E_A \) component is zero. Thus there is a non-zero energy cost \( \sim 2\Delta \) for creating a fermionic excitation from the ground state of the system – a signature of conventional superconductors leading to the flow of current without resistance (see the discussion in Sec. II). If by some physical mechanism it occurs that \( E_A \neq 0 \), the superconductivity is gapless (Abrikosov, 1988). The realization of this eventuality in nuclear systems will be exemplified in Section IV.

Our methodological considerations have so far been general, in that we have not been explicit about the properties of interactions or correlations in the system under study. Superconductivity is inherently a Fermi-surface phenomenon, so one natural approximation entails an expansion of the self-energy \( \Sigma(\omega, \mathbf{p}) \) around its on-shell value assuming that the off-mass-shell contribution is small. Since the imaginary part of the self-energy vanishes quadratically on the mass shell, the expansion is carried out for the real part by writing

\[
\Re \Sigma(\omega, \mathbf{p}) = \Re \Sigma(\varepsilon_p) + \frac{\partial \Re \Sigma(\omega, \mathbf{p})}{\partial \omega} \mid_{\omega=\varepsilon_p} (\omega - \varepsilon_p),
\]

(29)

where \( \varepsilon_p = \epsilon_p + \Re \Sigma(\varepsilon_p) \) is the on-mass-shell single-particle spectrum in the normal state, i.e., the solution to Eq. (23). Within this approximation, the self-energies in Eqs. (23) contain only on-shell self-energies \( \Sigma(\varepsilon_p, \mathbf{p}) \) and are multiplied by a wave-function renormalization, i.e., \( G_{\alpha\beta} \rightarrow Z(\mathbf{p})G_{\alpha\beta} \) and \( F_{\alpha\beta} \rightarrow Z(\mathbf{p})F_{\alpha\beta} \), where

\[
Z(\mathbf{p})^{-1} = 1 - \frac{\partial \Re \Sigma(\omega, \mathbf{p})}{\partial \omega} \mid_{\omega=\varepsilon_p}.
\]

(30)

A similar expansion may be implemented for the anomalous self-energy. It should be noted, however, that for time-local pairing interactions (essentially all bare or effective soft NN interactions) the gap function is energy-independent. Non-local interactions are naturally generated from local ones, if they are constructed via summations of series, as in models of medium polarization (see Sec. III.A.3.) If the pairing interaction is modeled in terms of boson exchange, the dependence of \( \Delta(\omega) \) on frequency shows significant structure off the mass shell, as will be seen in Sec. III.A.4.

The existence of a Fermi surface also implies an approximation of the momentum dependence of the self-energy, although this approximation can be trivially avoided. Expanding the normal self-energy at the Fermi surface one finds

\[
\varepsilon(p) = v_F(p - p_F) - \mu^*, \quad \frac{m}{m^*} = 1 + \frac{m \partial \Re \Sigma(p)}{p \partial p} \mid_{p=p_F},
\]

(31)

where \( \mu^* \equiv -\epsilon(p_F) + \mu - \Re \Sigma(p_F) \), \( v_F \) is the Fermi velocity, and \( m^* \) is an effective mass. The spectrum (31) now has the proper form for a Fermi liquid, although there are no significant computational gains from this effective-mass approximation.

2. Mean-field BCS theory

The next essential step is to establish the prescription for computing the self-energies. BCS theory is a mean-field theory for the anomalous self-energy, which in its most general form can be written as

\[
\Delta(p) = -2 \int \frac{d^4p'}{(2\pi)^4} \Gamma(p, p') \Im F(p') f(\omega'),
\]

(32)

where \( \Gamma(p, p') \) is a four-point vertex function to be determined from the nucleon-nucleon interaction and \( f(\omega) = [1 + \exp(\beta\omega)]^{-1} \) is the Fermi distribution at inverse temperature \( \beta \). The factor \( 2\Im F(p') f(\omega') \) may be identified as the off-diagonal Green function in the Keldysh-Schwinger formalism. Thus, as expected, the anomalous self-energy is given by a convolution of the interaction vertex and a GF.

Consider next time-local (but space non-local) interactions, in which case the replacement \( \Gamma(p, p') \rightarrow V(p, p') \) can be made and, moreover, \( V(p, p') \) can be expanded in partial waves. Performing wave-function renormalization of the propagator, integrating over the energy variable, and considering a single uncoupled channel, we arrive at an integral equation

\[
\Delta(p) = Z(p) \int \frac{dp'p'^2}{(2\pi)^2} V(p, p') \left( Z(p') \frac{\Delta(p')}{\omega_+(p')} \left[ f(\omega_+(p')) - f(\omega_-(p')) \right] \right).
\]

(33)

for the gap function depending only on the modulus of the momentum, with \( \omega_{\pm}(p) = \pm \sqrt{\varepsilon_p^2 + \Delta(p)^2} \), where \( \varepsilon_p \) is the single-particle spectrum in the normal state. In a number of cases, e.g. in low-density nuclear systems, it is necessary to solve for the density

\[
\rho = -2 \sum_\alpha \int \frac{d^3p}{(2\pi)^3} \Im G(p) f(\omega) = \frac{1}{2} \sum_\alpha \int \frac{d^3p}{(2\pi)^3} Z(p) \sum_{i=+,-} \left( 1 + \frac{\varepsilon_p}{\omega_i} \right) f(\omega_i)
\]

(34)

to obtain the chemical potential, which is modified by the effects of pairing on the single-particle energies. (Here \( \alpha \) denotes a sum over all spin/isospin states.) This “backreaction” of the density on the chemical potential is small in the weak-coupling regime, but becomes important with strong coupling. For an input pairing interaction \( V(p, p') \) and the spectrum \( \varepsilon_p \) in the unpaired state, Eqs. (33) and (34) fully determine the gap and the chemical potential from which all the thermodynamic functions of the system can be computed.

In the case of a momentum-independent pairing interaction, the gap at the Fermi momentum is given by (c.f. Sec. II)

\[
\Delta(p_F) \approx 8\mu^* \exp \left( -\frac{2}{\lambda} \right),
\]

(35)
with effective coupling \( \lambda = \nu(p_F)|V(p_F, p_F)| \) and a density of states \( \nu(p_F) = m^*Z^2(p_F)p_F/\pi^2 \) summed over spins. It is seen that the main effect of wave-function renormalization is to modify the density of states at the Fermi surface. It is important to note that in the case of an NN interaction, the matrix element of the effective pairing interaction \( V(p_F, p_F) \) on the Fermi surface can have the “wrong” sign, implying a dominant repulsive interaction component. However, Eq. (33) may still have a pairing solution because the momentum dependence of \( V(p, p') \) can be sufficiently attractive away from the Fermi surface (Khodel et al., 1996). In mean-field BCS theory as frequently applied to nuclear systems, wave-function renormalization is neglected, i.e., the factor \( Z(p_F) \) is set to unity.

In the foregoing development, we implicitly assumed that the normal self-energy \( \Sigma(p) \) and hence the normal-state spectrum \( \varepsilon(p) \) do not depend on the properties of the paired state, e.g., the gap \( \Delta(p) \). The replacement of \( G(p) \) by \( G^N(p) \) when computing the normal-state spectrum is an approximation (known as the decoupling approximation), which is only valid when the pairing is a small perturbation on the normal ground-state. This approximation should work well for nuclear systems at high densities (implying weak coupling), but might not be adequate at lower densities where the strong-coupling corrections are significant.

Qualitatively, the renormalization of the single-particle spectrum in momentum space (accounted for, in particular, through the effective mass ratio \( m^*/m \) for nucleons) acts to reduce the density of states, therefore the magnitude of the gap, by factors up to two or three, depending on density. Additional reduction comes from the wave-function renormalization \( Z(p) \leq 1 \).

3. Polarization effects

The interaction between nucleons is modified in the nuclear medium. Therefore the replacement \( \Gamma(p, p') \) by the free-space interaction, which describes correctly only the asymptotic states of the nucleons, is an approximation that needs further elaboration. The leading class of modifications of the pairing interaction in the medium arises from “polarization effects” or “screening.” Let us examine this type of modification.

We start with a simple but instructive approach based on ideas from the Landau theory of Fermi liquids. Consider the following integral equation which sums the particle-hole diagrams to all orders

\[
\Gamma(p, p', q) = U(p, p', q) - i \int \frac{d^4p''}{(2\pi)^4} U(p, p'', q) G^N(p'' + q/2) G^N(p'' - q/2) \Gamma(p'', p', q),
\]

(36)

where \( q \) is the momentum transfer. The driving term \( U(p, p', q) \) must be devoid of blocks that contain particle-particle ladders to avoid double summation in the gap equation. In general this driving interaction depends on spin and isospin and can be decomposed as

\[
U(q) = f_q + g_q(\sigma \cdot \sigma') + [f'_q + g'_q(\sigma \cdot \sigma')](\tau \cdot \tau'),
\]

(37)

where \( \sigma \) and \( \tau \) are the vector observables represented by Pauli matrices in the spin and isospin spaces. Equation (37) is written assuming the block \( U(q) \) depends only on the momentum transfer, which is a good approximation for highly degenerate Fermi systems. For illustrative purposes, the tensor component of the interaction and the spin-orbit terms are ignored in Eq. (37). The solution of (36) is given by

\[
\nu(p_F) \Gamma_q = \frac{F_q}{1 + L(q)G_q} + \frac{G_q}{1 + L(q)G_q} (\sigma \cdot \sigma')
\]

\[
+ \left[ \frac{F'_q}{1 + L(q)G_q} + \frac{G'_q}{1 + L(q)G_q} (\sigma \cdot \sigma') \right] (\tau \cdot \tau'),
\]

(38)

where \( F_q = \nu(p_F) f_q, G_q = \nu(p_F) g_q, F'_q = \nu(p_F) f'_q, \) and \( G'_q = \nu(p_F) g'_q \) are the dimensionless particle-hole interactions (Landau parameters), whereas

\[
L(q) = \nu(p_F)^{-1} \int \frac{d^4p''}{(2\pi)^4} G^N(p'' + q/2) G^N(p'' - q/2),
\]

(39)

is the polarization tensor, given in the present case by the Lindhard function (Fetter and Walecka, 1971). The momentum transfer is related to the scattering angle \( \theta \) and Fermi momentum \( p_F \) according to \( q = 2p_F \sin \theta/2 \), assuming that the momenta of particles are restricted to the Fermi surface. The parameters \( F, F', G, \) and \( G' \) can be expanded in spherical harmonics with respect to the scattering angle, writing

\[
\begin{pmatrix} F(q) \\ G(q) \end{pmatrix} = \sum_l \begin{pmatrix} F_l \\ G_l \end{pmatrix} P_l(\cos \theta),
\]

(40)

and similarly for \( F'(q) \) and \( G'(q) \). The Landau parameters \( F_l, G_l, F'_l, \) and \( G'_l \) depend on the Fermi momentum. In neutron matter one has \( \tau \cdot \tau' = 1 \), and the number of independent Landau parameters for each \( q \) or \( l \) can be reduced to two by defining \( F'' = F + F' \) and \( G'' = G + G' \). Keeping the dominant lowest-order harmonics in the expansion (40), the interaction in a singlet pairing state (total spin of the pair \( S = 0 \) and \( \sigma \cdot \sigma' = -3 \)) becomes

\[
\nu(p_F) \Gamma_q = F_0^n \left[ 1 - \frac{L(q)F_0^n}{1 + L(q)F_0^n} \right] - 3G_0^n \left[ 1 - \frac{L(q)G_0^n}{1 + L(q)G_0^n} \right].
\]

(41)

In general, the polarization tensor \( L(q) \) is complex-valued. However, it is real in the limit of zero energy transfer (at fixed momentum) and at zero temperature, being given by

\[
L(q) = -1 + \frac{p_F}{q} \left( 1 - \frac{q^2}{4p_F^2} \right) \ln \left| \frac{2p_F - q}{2p_F + q} \right|.
\]

(42)
The pairing interaction (41) consists of two pieces—the direct part generated by the terms $\propto 1$ inside the square brackets and the remaining induced part, arising from density and spin-density fluctuations, respectively, the term $\propto F_0^2$ and that $\propto G_0^2$.

Given the Landau parameters, the effect of polarization can be assessed by defining a pairing interaction averaged over the momentum transfers and taken at zero-energy transfer

$$\Gamma(q, q') = \frac{1}{2qq'} \int_{|q-q'|}^{q+q'} dp \Gamma(p). \quad (43)$$

The impact of the fluctuations on pairing has been established using the formalism described above for neutron matter below the saturation density $n_0$ by Clark et al. (1976), who showed that the density fluctuations enhance the attraction between neutrons, whereas the spin-density fluctuations reduce it. Using the values of Landau parameters in neutron matter they concluded that the suppression of pairing via spin-density fluctuations is the dominant effect.

We turn now to studies that employ more refined approximations for the induced part of the interaction (Ainsworth et al., 1989; Schulze et al., 1996; Schwenk et al., 2003; Wambach et al., 1993). First, while the structure of Eqs. (41) and (43) remains the same, the replacement

$$F_0^2 - 3G_0^2 \rightarrow V_s - 3V_a = \Gamma_{\text{dir}}(p) \quad (44)$$

is made, with $V_s$ and $V_a$ set equal to the spin symmetrical and anti-symmetrical parts of the bare (phase-shift equivalent) nuclear potential or its low-momentum reduction. Then the induced interaction is written as

$$\nu(p_F)\Gamma_{\text{ind}}(p) = \frac{F(p)^2L(p)}{1+L(p)F(p)} - \frac{3G(p)^2L(p)}{1+L(p)G(p)}. \quad (45)$$

where we have dropped the subscript $n$ on the particle-hole interactions $F(q)$ and $G(q)$, which now depend on the magnitude of the momentum transfer $q$.

The method used to compute the induced interaction was developed in the 1970s and accounts for the mostly repulsive effect of screening on the direct interaction (Babu and Brown, 1973; Bäckman et al., 1985, 1973; Schulze et al., 1996), which by itself contains sufficient attraction to guarantee pairing.

In these approaches the driving term in the series summing up the induced interaction is computed from the Brueckner-Bethe-Goldstone theory of nuclear matter (Bethe, 1971) and is represented by the $G$-matrix. For example, the spin-symmetric interaction $F(q)$ is determined through the coupled integral equations

$$F = G - AG_{\text{ph}}F, \quad A = F + FG_{\text{ph}}A, \quad (46)$$

written for simplicity in operator form), where $A$ represents the particle-hole scattering amplitude and $G_{\text{ph}}$ is the two-body particle-hole propagator. [See (Ainsworth et al., 1989; Wambach et al., 1993) for details.] The spin-antisymmetric channel is treated in complete analogy.

A different scheme for computing the pairing interaction was suggested by Cao et al. (2006), who divide the pairing interaction into three components: (a) the direct bare nuclear interaction; (b) the second Born term in the interaction where vertices are represented by the Brueckner $G$-matrix; (c) the induced part of the interaction, which includes a full resummation of particle-hole series with vertices approximated by Landau parameters derived from the energy functional provided by the $G$-matrix. This study demonstrates the importance of the balance between the self-energy and vertex corrections, which compensate each other in the final result for the pairing gap, resulting in moderately suppressed gaps compared to the BCS result.

Numerical computations of the $1S_0$ gap in neutron matter that include the induced interaction at various levels of sophistication indicate that its dominant repulsive character produces a strong reduction of the gap. The resulting upper bound on the magnitude of the gap is around 1 MeV; however, the density at which the maximum is attained varies substantially (Ainsworth et al., 1989; Cao et al., 2006; Schulze et al., 1996; Schwenk et al., 2003; Wambach et al., 1993).

4. Boson-exchange theories

In reality, the pairing interaction is retarded in time, not only because the mesons, as mediators of the nuclear force, propagate at finite speed, but also because any induced interaction which embodies resummation of a certain class of diagrams is frequency dependent. Such induced pairing interactions can also be framed within a theory of effective phonon exchange between nucleons, as is commonly done in the theories of pairing in finite nuclei. Therefore it is of interest to consider boson-exchange theories in general and leave the nature of bosons arbitrary for the time being. We will comment on possible variants of the theory at the end of this subsection.

Generic theories of pairing based on a boson-exchange model originated in the work of Eliashberg (1960) on electron-phonon superconductivity in metals. The Dyson-Schwinger Eqs. (21) and (22) remain intact in this model. However it is now convenient to split the retarded self-energy into components even ($S$) and odd ($A$) in $\omega$, i.e. $\Sigma(p) = \Sigma_S(p) + \Sigma_A(p)$, and define the wave-function renormalization $Z(p) = 1 - \omega^{-1}\Sigma_A(p)$. The single-particle energy is then renormalized as $E_S = \epsilon_p + \Sigma_S(E_S, p)$. Accordingly, the propagators now take the forms

$$G(p) = \frac{\omega Z(p) + E_S(p)}{(\omega + i\eta)^2Z(p)^2 - E_S(p)^2 - \Delta(p)^2}, \quad (47)$$

$$F(p) = -\frac{\Delta(p)}{(\omega + i\eta)^2Z(p)^2 - E_S(p)^2 - \Delta(p)^2}, \quad (48)$$

where $\Delta\Delta^\dagger \equiv -\Delta^2$. Next we need to specify the pairing
interaction. The time-local part of the interaction appears in the Hartree self-energy (Fig. 3, upper diagrams). The retarded boson-exchange interaction contributes to the Fock self-energy (Fig. 3, lower diagrams).

We do not discuss the Hartree self-energies, as they can be readily calculated from any given nuclear interaction that is local in time (e.g., a phase-shift equivalent nuclear potential). Using the fact that neutron matter is a highly degenerate Fermi system, the normal and anomalous Fock self-energies can be expressed in the following form:

$$\Sigma(p_F, \omega) = -\int_0^\infty d\omega' K(\omega') \left\{ g(\omega') \left[ G(\omega + \omega') + G(\omega - \omega') \right] + \int_{-\infty}^\infty \frac{d\epsilon}{\pi} \Im m \left( G(\epsilon) J(\omega, \omega', \epsilon) \right) \right\} \quad (49)$$

$$\Delta(p_F, \omega) = \int_0^\infty d\omega' K(\omega') \left\{ g(\omega') \left[ F(\omega + \omega') + F(\omega - \omega') \right] + \int_{-\infty}^\infty \frac{d\epsilon}{\pi} \Im m \left( F(\epsilon) J(\omega, \omega', \epsilon) \right) \right\} \quad (50)$$

where $\Sigma(\omega, p_F)$ and $\Delta(\omega, p_F)$ are respectively the normal and anomalous retarded self-energies, while

$$J(\epsilon, \omega, \omega') = \frac{f(\epsilon)}{\epsilon - \omega - \omega' - i\eta} + \frac{1 - f(\epsilon)}{\epsilon - \omega + \omega' - i\eta}, \quad (51)$$

where $g(\omega)$ and $f(\omega)$ and the Bose and Fermi distribution functions. Additionally, we have introduced a momentum-averaged (real) interaction kernel defined by

$$K(\omega) = \frac{m^*}{(2\pi)^3 p_F} \int_0^{2p_F} dq \int_0^{2\pi} d\phi B(q, \omega) \text{Tr} \left\{ \Gamma_0(q) \Gamma(q) \right\}, \quad (52)$$

in which $\Gamma_0$ and $\Gamma$ are the bare and full boson-fermion vertices and $B(\omega, q)$ is the spectral function of the bosons. Eqs. (49) and (50) provide a set of nonlinear coupled integral equations for the complex pairing amplitude and the normal self-energy (or, equivalently the wave-function renormalization). To illustrate some numerical solutions, consider a model in which neutrons interact via soft-pion exchange (Pankratov et al., 2015; Sedrakian, 2003). Given a spectral function for the bosons, the kernel (52) is constructed as input to Eqs. (49) and (50). The input kernel for this specific model is shown in the left panel of Fig. 4, while its right panel shows the zero-temperature solutions of Eqs. (49) and (50). The imaginary component of the gap tends to zero on the mass shell ($\omega = 0$); its real part gives the on-shell value of the gap. For non-zero energies these functions have complex structure that reflects the features of the input kernel $K(\omega)$. Knowledge of the frequency dependence of the pairing gap in nuclear and neutron matter could be important for the analysis of frequency-dependent observables, especially for the description of their dynamical response to various perturbations.

B. T- and G-matrix approaches, Thouless criterion

The onset of pairing correlations, and in particular the critical temperature of the superfluid phase transition, can be determined from properties of the normal (unpaired) state, notably from the scattering matrix, defined here as an extension of the free-space $T$-matrix to a medium of strongly correlated fermions. As considered in more detail below, generalization to the medium can be implemented at different levels. An important class of $T$-matrix theories is obtained when propagation of particles and holes in intermediate states is included symmetrically (Galitikskii, 1958). An alternative exten-
sion, introduced historically in the context of nuclear matter calculations, is based on the $K$-matrix (or reaction matrix) – or in current notation the $G$-matrix – in which only particle-particle propagation is taken into account (Brueckner and Gammel, 1958). The relation between superconductivity and singularities of the $T$- and $G$-matrices was recognized quite early in the development of quantum many-body theory and considered in detail by Emery (1959, 1960). Singular behavior of the $T$-matrix is directly related to the pairing properties of the system, as it can signal the onset of the superfluid phase. In fact, the critical temperature $T_c$ for the onset of pairing in attractive fermionic systems, including nuclear systems, can be extracted as the temperature at which the $T$-matrix of the normal state diverges as $T_c$ is approached from above (i.e., from a higher temperature state). This condition for the determination of the onset of superconductivity is known as Thouless criterion (Thouless, 1960).

In vacuum, both these choices for the scattering matrix reduce trivially to the $T$-matrix of two nucleons interacting in vacuum, which is fitted to the experimental elastic NN phase shifts for laboratory energies below 350 MeV. In the case of $G$-matrix, the singularities are not directly related to the coherently paired state, and it is still meaningful to perform calculations at $T \leq T_c$ without introducing a pairing gap in the fermion energy spectrum (Bethe, 1971).

With the advent of phase-shift equivalent, high-precision NN potentials, $T$-matrix theory was revived and employed to predict the critical temperature of the phase transition to the superfluid state in nuclear matter in the attractive interaction channels (Alm et al., 1993, 1990, 1996b; Röpke et al., 1998; Rubstsova et al., 2017; Schmidt et al., 1990; Sedrakian et al., 1995; Stein et al., 1995). It is interesting that evidence of a di-neutron bound state has been revealed in $G$-matrix calculations that exhibit poles of this quantity lying below the Fermi energy (Arellano et al., 2016; Isaule et al., 2016). The conditions for such singular behavior are analogous to those for $T$-matrix poles, the difference being in the treatment of the intermediate states as we discuss now in some detail.

The integral equation determining the $T$-matrix can be written in momentum space as

$$ T(p, p'; P) = V(p, p') + \int \frac{dp''}{(2\pi)^3} V(p, p'') G_2(p'', P) T(p'', p'; P), \tag{53} $$

where $V(p, p'')$ is the two-particle interaction and the two-particle propagator is given by

$$ G_2(p, P) = \int \frac{d^4p'}{(2\pi)^4} \int \frac{d\omega}{(2\pi)} \left[ G^>(p_+) G^>(p_-) \right] (2\pi)^3 \delta(P - P') \bigg\{ \frac{1}{\Omega - \Omega' + i\eta} \bigg\}, \tag{54} $$

having introduced the four-vectors $p_\pm = P/2 \pm p$, with $P = (P, \Omega)$ denoting the center-of-mass four-momentum. Equation (53) has the familiar form of the Bethe-Salpeter integral equation appearing in scattering theory. The propagators $G^{>\langle}(p)$ are the off-Bethe-Salpeter integral equation appearing in scattering theory. The propagators $G^{>\langle}(p)$ are the off-

$$ -iG^{<}(p) = a(p) f(p), \tag{55} $$

$$ iG^{>}(p) = a(p) [1 - f(p)], $$

where $a(p)$ is the spectral function of fermions and $f(p)$ is the equilibrium Fermi distribution function. The spectral function of quasiparticles (in the normal state) is given by

$$ a(p) = 2\pi Z(p) \delta(\omega - \epsilon(p)), \epsilon(p) = \epsilon_F(p - p_F) - \mu. \tag{56} $$

Here the wave-function renormalization $Z(p)$ is defined in terms of the normal-state self-energy by Eq. (29), while the effective mass and chemical potential are as defined in Eq. (31). With these approximations, Eq. (54) reduces to

$$ G_2(p, P) = Z(p_+)Z(p_-) \frac{Q(p_+, p_-)}{\Omega - \epsilon(p_+) - \epsilon(p_-) + i\eta}, \tag{57} $$

where

$$ Q(p_+, p_-) = [1 - f(p_+)][1 - f(p_-)] - f(p_+)f(p_-). \tag{58} $$

is the Pauli-blocking function, which accounts for the phase-space occupation in the intermediate scattering states of the $T$-matrix. The first and second terms of this expression refer to particle-particle and hole-hole propagations, respectively.

In Brueckner-Bethe-Goldstone theory, a diagrammatic expansion of the normal ground-state energy is carried out in the number of hole lines, and the hole-hole propagation terms are neglected, i.e., one considers a $G$-matrix equation

$$ G(p, p'; P) = V(p, p') + \int \frac{dp''}{(2\pi)^3} V(p, p'') \times \frac{\tilde{Q}(p_+, p_-)}{\Omega - \epsilon(p_+) - \epsilon(p_-) + i\eta} G(p'', p'; P), \tag{59} $$

with $\tilde{Q}(p_+, p_-) = [1 - f(p_+)][1 - f(p_-)]$.

Returning to the $T$-matrix equation (53), we consider the poles that this equation might develop as the temperature is reduced from a temperature $T > T_c$. This can be illustrated analytically by assuming a rank-one separable interaction $V(p, p') = \lambda_0 v(p)v(p')$. The solution of Eq. (53) is then given by

$$ T(p, p'; P) = \frac{V(p, p')}{1 - J(P)}, \tag{60} $$

$$ J(P) = \lambda_0 \int \frac{dp}{(2\pi)^3} v^2(p)G_2(p, P). \tag{61} $$
stant to be determined, reduces the BCS gap equation for pairing, see e.g. (Botermans and Malfliet, 1990; Schnell et al., 1999) and the nucleon-nucleon transport cross-section (Alm et al., 1994).

In case of the $G$-matrix, the absence of hole-hole propagation in the intermediate states breaks particle-hole symmetry. Consequently, the instability of the $T$-matrix that signals the onset of the superfluid state is suppressed and the $G$-matrix can be computed at temperatures below $T_c$, down to $T = 0$. Brueckner-Bethe-Goldstone theory utilizes the $G$-matrix as an effective interaction in generating the perturbative hole-line expansion. That is, the diagrams in the expansion for the energy are ordered according to the number of hole lines present, each hole line implying a convergence factor given roughly by the ratio of the volume, per particle, excluded by the repulsive component of the NN interaction to the mean volume per particle, known as wound parameter. On one hand, this has the apparent virtue of wiping away the instability associated with pairing; on the other, the resulting theory is non-conserving in that it entails self-energies and scattering amplitudes that are asymmetric with respect to interchange of particles and holes. In fact, if conservation laws are to be enforced, any collision integral constructed from scattering amplitudes (or non-equilibrium self-energies in the language of the Keldysh-Schwinger formalism) must vanish in the equilibrium limit. This condition fails to be the case if particle-hole symmetry is broken. A consequence of such non-conservation is seen in violation of the Hugenholz-van Hove theorem (equality of the chemical potential to the Fermi energy in the presence of arbitrarily strong interactions) (Hugenholtz and van Hove, 1958) in theories of nuclear matter based on the $G$-matrix, in which only particle-particle propagation is taken into account (Brueckner and Gammel, 1958). At the same time, as long as the hole-line expansion is valid, such violation ought to be small.

C. Self-consistent Green functions theory

The foundation of self-consistent Green functions (SCGF) theory were established long ago [see for example (Kadanoff and Baym, 1962)]. It can be applied to nuclear matter at finite temperatures above the critical temperature for pairing, see e.g. (Botermans and Malfliet, 1990; Danielewicz, 1984; Dickhoff and van Neck). We start by reviewing the underlying formalism. SCGF theory is a microscopic approach to properties of the normal (unpaired) state in which the interactions between nucleons are accounted for via the two-body $T$-matrix constructed from the bare NN interaction. The single-particle spec-
The two-particle propagator in the superfluid state is given by

$$G_2(p; P) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{\omega + \omega' - i\eta} \tilde{G}^R(p, q) \tilde{G}^A(q, p') \tilde{G}^S(p') \tilde{G}^A(p'),$$

where

$$\tilde{G}^S(p') = \int \frac{d\omega}{2\pi} G_N(-\omega, p') G(\omega, p').$$

The BCS gap equation (33) is recovered if the two-particle propagator $G_2(p)$ is taken in the quasiparticle approximation. Going beyond the quasiparticle approximation within the SCGF theory entails the replacement of the two-particle propagator in the superfluid state $G_2^S(p)$ by its counterpart in the normal state (65). The main advantage of the SCGF approach is that the gap equation is solved while keeping the off-mass-shell information contained in the full spectral function of the normal state.

However, we learned that the $T_{\text{c}}$ matrix from which the spectral function is computed in SCGF theory is divergent below $T_{\text{c}}$. Accordingly, the spectral function apparently needs to be computed at temperatures above $T_{\text{c}}$. This problem is circumvented by extrapolating the imaginary part of the normal-state self-energies to temperatures $T \leq T_{\text{c}}$ using the fact that $3m\Sigma(\omega)$ must vanish on the Fermi surface at $T = 0$. The real part of the self-energy is then computed from the Kramers-Kronig dispersion relation and, in this way, the complete spectral function is constructed below $T_{\text{c}}$ (Frick and M"uther, 2003; Frick et al., 2004; Rios et al., 2006, 2009).

The procedure so outlined allows one to feed information on the spectral function of the nucleonic system, which embodies crucial effects of short-range correlations, into the computation of pairing correlations (Ding et al., 2016; M"uther and Dickhoff, 2005; Rios et al., 2017). We may observe that this procedure relies on the basic strategy of the decoupling approximation: the single-particle properties of the system are computed in the normal state, and the pairing correlations are added in the second step, neglecting their back-reaction on the single-particle energies.

Numerical calculations demonstrate that upon going beyond the approximation that employs on-shell quasiparticles with a renormalized spectrum by adopting the GF given by Eq. (65), the pairing gap is suppressed by about ten percent in the isospin-singlet $^3S_1$ and $^3P_2$ channels, as well as in the isospin-triplet $^3S_1$ and $^3P_2$ channels. These results can be attributed to the shift of some spectral weight from the quasiparticle peak toward other energies, upon implementing full spectral functions (Ding et al., 2016; Rios et al., 2017).
merical results are discussed below in Sec. III.F for the S-wave and in Sec. V.B for the P-wave channels.

D. Correlated Basis Functions Theory

The method of correlated basis functions (CBF) provides a powerful tool for studying strongly correlated fermionic or bosonic quantum systems (Clark, 1979; Clark et al., 1979; Clark and Westhaus, 1966; Fantoni, 1981; Fantoni and Rosati, 1975, 1978; Feenberg, 1969; Krotchesch, 2002; Krotchesch and Clark, 1980; Krotchesch et al., 1981). It was applied to nucleonic pairing at the very early stages of theoretical development of the field (Chao et al., 1972; Clark and Yang, 1970, 1971; Yang and Clark, 1971). Since then it has undergone extensive developments in many areas, in particular in the contexts nuclear systems (Benhar and De Rossi, 2017; Chen et al., 1993, 1986; Fabroicini et al., 2005, 2008; Fan et al., 2017; Pavlou et al., 2017) and the low-density fermionic gas (Fan et al., 2015).

An important feature of CBF theory is that it implements a strategy for building essential normal-state correlations into the description of a strongly interacting interacting Fermi system through the action of a correlation operator $F$. Pairing correlations are then superimposed on the correlated normal ground state, in full analogy to the original approach of BCS theory. In particular, CBF theory is designed for inclusion of the strong short-range correlations produced by repulsive cores in nuclear systems, and the effects of induced long-range interactions can be treated on the same footing. The discussion below will focus on pairing in nuclear matter at zero temperature.

Consider a complete set of correlated normal states defined for each particle number $N$ as

$$\langle \Phi^{(N)}_m \rangle = \frac{F_N \Phi^{(N)}_m}{\langle \Phi^{(N)}_m | F_N F_N | \Phi^{(N)}_m \rangle^{1/2}},$$

(70)

where $| \Phi^{(N)}_m \rangle$ are the eigenstates of the noninteracting Fermi gas, $F_N$ is a correlation operator and $m^{(N)} = \{m_1 \ldots m_N\}$ specifies the set of plane-wave orbitals entering $| \Phi^{(N)}_m \rangle$. The states $| \Psi^{(N)}_m \rangle$ are normalized to unity, but generally not orthogonal. The correlation operator $F_N$ is commonly taken to be of Jastrow-Fennoberg form, depending only on radial distances between pairs of particles,

$$F_N(r_1, \ldots, r_N) = \exp[U_N(r_1, \ldots, r_N)/2],$$

(71)

with

$$U_N = \sum_{i<j} u_{2}(r_{ij}) + \sum_{i<j<k} u_{3}(r_{ij}, r_{jk}, r_{ki}) + \cdots$$

$$+ \sum_{i_1 < \cdots < i_N} u_N(r_{i_1 i_2}, \ldots, r_{i_N i_1}).$$

(72)

This series is usually truncated at the two-body or three-body level. The familiar Jastrow two-body correlation function is $f(r_{ij}) = \exp[u_2(r_{ij})/2]$, with limiting behavior $\lim_{\alpha \to 0} f(r) \to 0$ and $\lim_{r \to \infty} f(r) \to 1$. Dependence on spin and isospin, i.e., state dependence, may also be incorporated, as in

$$F_N = S\{\Pi_{i<j} f(ij)\}, \quad f(ij) = \sum_{\alpha} f_{\alpha}(r_{ij}) O_{\alpha}(ij),$$

(73)

where $S$ is the symmetrization operator and the index $\alpha$ runs over the set of two-body operators $O_{\alpha}(ij)$ entering the NN interaction adopted (or a subset of them), these being formed with appropriate symmetries from spin, isospin, tensor, and spin-orbit operators.

The next step is to construct a correlated superfluid ground state residing in Fock space, which allows for a consistent derivation of a gap equation in the presence of both pairing correlations that introduce off-diagonal long-range order and conventional correlations (of short or long range) that preserve $U(1)$ symmetry. Given the [BCS] ground state in terms of Bogolyubov amplitudes,

$$|\text{BCS}\rangle = \prod_{k} \left[ u_{k} + v_{k} \alpha^{\dagger}_{k_{\uparrow}} \alpha^{\dagger}_{k_{\downarrow}} \right] |0\rangle,$$

(74)

a robust choice has proven to be

$$|\text{CBFS}\rangle = \sum_{m,N} |\Psi^{(N)}_{m}\rangle \langle \Phi^{(N)}_{m}| \text{BCS}\rangle,$$

(75)

formed as a superposition of the members of (70). This trial ground state superposes the correlated basis states $|\Psi^{(N)}_{m}\rangle$ with the same amplitudes that the model normal states $|\Phi^{(N)}_{m}\rangle$ have in the corresponding expansion of the original BCS state vector. Given the Ansatz (75) for the correlated superfluid ground state and the two-body Hamiltonian operator $\hat{H}$, the thermodynamic potential of the pair-correlated system can be evaluated with the result (Fan et al., 2015; Krotchesch et al., 1981)

$$\langle \hat{H} - \mu \hat{N} \rangle = H_{00}^{(N)} - \mu N + 2 \sum_{|k|>k_F} u_{k}^{2} \epsilon_{k}$$

$$- 2 \sum_{|k|<k_F} u_{k}^{2} \epsilon_{k} + \sum_{k,k'} V_{kk'} u_{k} v_{k'} u_{k'} v_{k'},$$

(76)

where the $\epsilon_{k}$ are single-particle energies (including the chemical potential), $u_{k}$ and $v_{k}$ are the Bogolyubov amplitudes (2), $\hat{N}$ is the number operator, $H_{00} = \langle \Psi_{0} | \hat{H} | \Psi_{0} \rangle$ is the expectation value of the Hamiltonian in the normal ground state described by $|\Psi_{0}\rangle$, and $V_{kk'}$ is the in-medium effective pairing interaction. This effective pairing interaction has the structure

$$V_{kk'} = W_{kk'} + \left( |\epsilon_{k}| + |\epsilon_{k'}| \right) N_{kk'},$$

(77)

$$W_{kk'} = \langle k \uparrow, k \downarrow | \hat{W} | k' \uparrow, k' \downarrow \rangle_{\alpha},$$

(78)

$$N_{kk'} = \langle k \uparrow, -k \downarrow | \hat{N} | k' \uparrow, -k' \downarrow \rangle_{\alpha},$$

(79)
where the index \( a \) implies antisymmetrization. The two-body operators \( W(1,2) \) and \( N(1,2) \), along with the \( \epsilon_k \), are to be determined from matrix elements \( H_{mn} \) and \( I_{mn} \) of the Hamiltonian and identity through their natural decompositions

\[
I_{mn} \equiv \delta_{mn} + N_{mn}, \quad (80)
\]

\[
H'_{mn} \equiv W_{mn} + \frac{1}{2} (H'_{mm} + H'_{nn}) N_{mn}, \quad (81)
\]

where \( H' = H - H_{00} \).

On the assumption that the energy gap is small compared to the Fermi energy, such that the feedback of pairing on normal-state properties can be neglected, it is justified to consider one Cooper pair at a time in analyzing the correlated BCS state (75). Upon imposing this decoupling approximation, the Bogolyubov amplitudes no longer appear in the gap equation derived by functional minimization of Eq. (76). This then becomes identical in form to the standard mean-field BCS equation, except that the effective pairing interaction \( V_{kk'} \) replaces the bare interaction, and the single-particle energies \( \epsilon_k \) are those of the correlated normal ground state.

We now turn to evaluation of these normal-state properties, the immediate task being determination of the correlation factor \( F_N \). At the level of Jastrow correlations (i.e., having truncated the series (72) for the operator \( U \) at the two-body term \( n = 2 \)) the obligatory Euler-Lagrange (EL) optimization requires that the function \( u_2(r) \) satisfies

\[
\frac{\delta H_{00}}{\delta u_2}(r) = 0, \quad (82)
\]

where \( H_{00} \) is the ground-state energy expectation value. Associated with this optimal energy are a radial distribution function \( g(r) \) and its Fourier partner, the static structure function \( S(k) \).

To proceed further and solve Eq. (82), a reliable method is needed for evaluation of the diagonal and off-diagonal matrix elements in the normal-state correlated basis (70). Initially, cluster expansion techniques have been used to evaluate matrix elements in a basis of correlated states of the Jastrow-Feenberg type, primarily for the ground-state energy, one-body density matrix, and pair distribution functions, but also for off-diagonal matrix elements in a correlated basis. In the simple Jastrow case, these are expansions in the number of correlation bonds \( \eta(r) = f^2(r) - 1 \), or the number of correlated bodies. They are effectively low-density expansions, loosely analogous to the wound-parameter or hole-line expansions of Brueckner-Bethe-Goldstone theory, their terms being given a diagrammatic representation analogous to those for imperfect classical gases (Mayer and Mayer, 1977). Later, methods were developed, originally for the radial distribution function \( g(r_{12}) \), that permitted simultaneous resummation of certain important classes of cluster diagrams, in particular nodal \( (N) \) and non-nodal \( (X) \), and otherwise identified by the direct or exchange involvement of their root points 1,2 [specifically direct-direct \( (dd) \), direct-exchange \( (de) \), exchange-exchange \( (ee) \), or cyclic exchange \( (cc) \)]; see (Clark, 1979; Krotscheck, 2002) for details. Application of this resummation technique to other observables resulted in Fermi-hypernetted chain theory (Fantoni and Rosati, 1975; Krotscheck and Ristig, 1975) for the analysis of the Jastrow-Feenberg correlated normal ground state, subsequently being extended to evaluation of off-diagonal as well as diagonal Hamiltonian matrix elements (Krotscheck and Clark, 1979).

In combination with EL optimization, the simplest nontrivial implementation of FHNC resummation that is consistent in the sense of parquet analysis (Jackson et al., 1982, 1985), named EL-FHNC//0, incorporates both the random-phase approximation and the Bethe-Goldstone equation (i.e., rings and ladders) in a “collective” or averaged-propagator approximation (Fan et al., 2015; Krotscheck, 2002). The latter involves treating particle-particle and hole-hole propagation in the same average way. Adopting the EL-FHNC//0 approximation, the Euler equation (82) takes the form

\[
S(k) \left[ 1 + 2 \frac{S_{0}^2(k)}{S_F(k)} \right]^{1/2} = S_F(k), \quad (83)
\]

where \( S(k) \) and \( S_F(k) \) are respectively the static structure functions of the interacting and noninteracting systems, \( t(k) = \hbar^2 k^2 / 2m \), and

\[
\tilde{V}_{ph}(r) = \left[ 1 + \Gamma_{dd}(r) \right] v(r) + \frac{\hbar^2}{m} \left| \nabla \sqrt{1 + \Gamma_{dd}(r)} \right|^2 + \Gamma_{dd}(r) \tilde{w}_1(r), \quad (84)
\]

where \( v(r) \) is the bare iteration, \( \Gamma_{dd} \) (the FHNC-dressed version of \( f^2(r) - 1 \) of the Jastrow treatment) has Fourier transform

\[
\tilde{\Gamma}_{dd}(k) = \left[ S(k) - S_F(k) \right] / S_F^2(k), \quad (85)
\]

while

\[
\tilde{w}_1(k) = -t(k) \left[ \frac{1}{S_F^2(k)} - \frac{1}{S(k)} \right]^2 \left[ \frac{S(k)}{S_F(k)} + \frac{1}{2} \right], \quad (86)
\]

is an induced interaction.

The two-body operators \( W(1,2) \) and \( N(1,2) \) required for evaluation of the CBF-dressed pairing matrix elements \( V_{kk'} \) of Eq. (77) are given

\[
N(1,2) = N(r_{12}) = \Gamma_{dd}(r_{12}), \quad W(1,2) = W(r_{12}), \quad \tilde{W}(k) = -\frac{t(k)}{S_F^2(k)} \tilde{\Gamma}_{dd}(k), \quad (87)
\]
again in the collective approximation.

The operator $W(1, 2)$ is in practice just the particle-hole interaction given in coordinate space by Eq. (84). It includes a so-called direct interaction consisting of the bare interaction $v(r)$, moderated by dd-dressed two-body correlations, plus a kinetic term caused by the deformation of the wave function at short distances. The induced interaction in the second line of Eq. (84), of long range, accounts for exchange of virtual phonons. Finally, the energies that enter the “energy-numerator” term in Eq. (77) proportional to $N_{kk'}$ reduce to

$$
\epsilon_k = \ell(k) - \mu + \frac{\hat{X}_{cc}(k)}{1 - \hat{X}_{cc}(k)} + \text{const.,}
$$

where the constant is fixed by the condition $\epsilon(k_F) = 0$ and

$$
\hat{X}_{cc}(k) = -\frac{n}{\nu} \int d^3 r e^{ik \cdot r} \Gamma_{dd}(r) \ell(k_F r),
$$

where $X'_{cc}(k)$ is a sum of non-nodal diagrams, with $\ell(x) = (3/x) j_1(x)$ denoting the Slater exchange function, $\nu$ the single-particle degeneracy, and $j_1(x)$ the spherical Bessel function of the first kind.

The expression for $\hat{X}_{cc}(k)$ follows from Eq. (89) on substituting $\Gamma_{dd}(r) \to W(r)$. The closed system of equations (83)-(88) no longer contains any reference to the Jastrow correlation function $f(r) = \exp[iu(r)/2]$. These equations could just as well have been derived in any generic many-body theory, including the GF and coupled-cluster approaches, and especially $T$-matrix theory (Fan et al., 2015).

A concrete implementation of the theory described above has been carried out by Fan et al. (2017) for the $1S_0$ pairing gap in low-density neutron matter using the EL-FHNC/0 approximation for two simplified NN interactions – Argonne $V_{14}'$ and Reid soft core $V_{6}$ (Day, 1981; Reid, 1968), both essentially phase-shift equivalent to Argonne $V_{18}$ in the density range involved. [Earlier calculations within the same framework but implemented in low cluster order were carried out in (Chen et al., 1993, 1986).]

Alternative developments in the CBF/FHNC description of pairing in strongly correlated Fermi systems (Fabrocini et al., 2008), in the formulation developed by Fantoni (1981), take into account non-central terms in the general operator expansion (73) of the two-body correlation $f(ij)$. Notably, the inclusion of its tensor components is a distinct advance. Extensive results from its application to the $1S_0$ neutron gap were reported by Fabrocini et al. (2008) for a variety of generally phase-shift equivalent NN interactions from the Argonne family.\(^8\)

This alternative approach is based on a correlated CBF ground state that differs from that of Eqs. (75) and (70) in the absence of the normalizing denominator in (70). As pointed out in (Krotscheck and Clark, 1980; Krotscheck et al., 1981), this choice is afflicted with divergences for long-range two-body correlations, including the optimal behavior (Krotscheck, 1977) $f(r) \to 1 + \text{const.,}/r^2$ associated with virtual phonon exchange. This deficiency is somewhat mitigated by the fact that the decoupling approximation is not imposed. Another distinction between the two CBF approaches to pairing is that the alternative strategy does not work directly with the BCS ground state, which breaks $U(1)$ symmetry in hosting particle-number fluctuations. Instead, this approach employs the projection of the BCS ket on the $N$-particle subspace of Fock space, which for even $N$ is a Pfaffian. The consequences of this replacement are not explored, except in the recognition that one must seek consistency of the density with the chemical potential.

Applications of CBF theory to nuclear systems are still under development. For example, several studies of the pairing gap in low-density neutron matter within CBF theories find much less suppression of the gap (Benhar and De Rosi, 2017; Fan et al., 2017; Pavlou et al., 2017), compared to those found in the earlier computations (Chen et al., 1993, 1986). Numerical results for pairing in neutron matter from these efforts based on CBF theory are displayed below in Sec. III.F.

### E. Monte Carlo methods

Our survey of the many-body methods for \textit{ab initio} computational exploration of pairing behavior in nuclear systems would not be complete without the important class of stochastic approaches based on Monte Carlo (MC) algorithms. While MC methods have been extensively applied to the normal (unpaired) state of neutron and nuclear matter over an extended period (Gandolfi et al., 2009b), the much more challenging problem of pairing has been addressed in only a handful of studies during the last decade (Fabrocini et al., 2005; Gandolfi et al., 2008, 2009a; Gezerlis and Carlson, 2008, 2010). These studies have focused on phase-shift equivalent interactions, especially the Argonne-Urbana class of potentials. The essence of the MC method is the solution of the non-relativistic Schrödinger equation using stochastic sampling of configurations, as the system is advanced in imaginary time. In practice, an infinite system is simulated in a finite box containing a fixed number of particles subject to periodic boundary conditions. Of specific interest for this review are the Green Function Monte Carlo (GFMC) and auxiliary field Diffusion Monte Carlo (AFDMC) algorithms. The latest GFMC computations of bulk energy and pairing gaps in nuclear matter have been performed for systems of $\sim 60$ nucleons; larger numbers of particles can be accommodated in AFDMC simulations.

AFDMC is a special kind of GFMC method in which spin/isospin configurations are sampled instead of ex-

---

\(^8\) https://www.phy.anl.gov/theory/research/av18/
plicitly summed, allowing extension of the calculation to higher density (Fabrocini et al., 2005). The most recent computations of this kind use a fixed-phase approximation, which resolves the technical difficulties associated with the presence of a tensor interaction (Gandolfi et al., 2008, 2009a; Gezerlis and Carlson, 2008). This work also employs the full bare interaction assumed instead of projecting it on some specific partial-wave channel (e.g., $^1S_0$ for low-density neutron matter). Depending on forms of the starting or trial correlated superfluid and normal states, the energy difference between their evolved versions can be under 4% (Gandolfi et al., 2009a). The starting superfluid state is taken as the product of a Jastrow correlation factor $\Pi_{i<j} f(r_{ij})$ and a token superfluid state consisting of the projection of the BCS state on the $N$-particle Hilbert space of the system. As mentioned in Sec. III.D, for even $N$ the latter is a Pfaffian of pair wave functions $\phi(ij)$ satisfying prescribed boundary conditions. The pair functions were determined from a variational CBF calculation of the energy expectation value using extended FHNC techniques. In the case of odd neutron number, the energy of the unpaired neutron is chosen to minimize the energy.

The standard Green function MC (GFMC) method and the simpler variational MC (VMC) procedure sample only spatial configurations (Gandolfi et al., 2015; Gezerlis and Carlson, 2008, 2010). VMC calculations use Monte Carlo integration to minimize the expectation value of the Hamiltonian, optimizing the trial wave function. In the GFMC approach the Schrödinger equation is brought to the diffusion form with respect to imaginary time and the initial trial wave function is evolved to obtain the lowest energy eigenstate. As in the AFDMC approach, the starting wave function is taken to be the Jastrow-Pfaffian form with a fixed number of particles subject to periodic boundary conditions. The Jastrow part of the wave function is obtained from a lowest-order constrained-variational (LOCV) method.

It should be understood that, of necessity, these methods do not evolve or reach a state with full BCS pairing correlations, which would be a state of indefinite particle number residing in Fock space, but rather its projection onto an $N$-particle subspace. As is done in the case of finite nuclei, the energy gap in pure neutron matter is determined (up to the sign) from the odd-even staggering formula for odd neutron number $N$, thus

$$\Delta(N) = E(N) - \frac{1}{2} [E(N+1) + E(N-1)] . \quad (90)$$

More recent GFMC computations (Gezerlis and Carlson, 2008, 2010) predict gaps which are about 30% smaller than those obtained with AFDMC. Furthermore, the gaps obtained by the two MC methods are suppressed compared to the bare BCS result, as is usually the case with the other methods (SCGF, CBF, etc.) discussed above. One may anticipate that within their error bars the MC computations faithfully account for the strong short-range repulsion of phase-shift equivalent NN interactions. Simulations with larger number of particles may provide further insight into the accuracy of the extrapolations to infinite matter and the role of long-range correlations.

F. Overview of the results

We close this section with an overview of the results obtained for the simplest problem, namely the neutron $^1S_0$ pairing gap; discussion of $^3P_2$-$^3F_2$ pairing is relegated to Sec. V.B. We show in Fig. 22 the results of computations of this quantity, all of which except one are based on versions of the Argonne family of NN potentials, implying that the observed differences are due mainly to differences between the many-body methods applied. All these methods (except the Monte Carlo approaches, which provide data only in the lower-density domain) predict a peak value of the gap $\Delta_F$, i.e., $\Delta_k$ evaluated for $k = k_F$, close to 0.8 fm$^{-1}$, the peak value itself varying in the range 0.8 to 2.5 MeV. The CBF (Fan et al., 2005) and the Monte Carlo calculations faithfully account for the strong short-range repulsion of phase-shift equivalent NN interactions. Simulations with larger number of particles may provide further insight into the accuracy of the extrapolations to infinite matter and the role of long-range correlations.
Cold fermionic atoms, where these systems can be tested as superfluids, a term that has become common in the theory of superfluidity. We shall refer to such systems below as imbalanced superfluids, a term that has become common in the theory of superfluidity. Historically, the studies of imbalanced superfluids began shortly after the advent of BCS theory in the context of electronic materials containing paramagnetic impurities (Chandrasekhar, 1962; Clogston, 1962; Sarma, 1963). The effect of impurity scattering on electrons, on average, was modeled in terms of an effective magnetic field, which then induces an imbalance between the spin-up and spin-down electrons.

The initial studies were carried out in the weak-coupling formalism, where the back-reaction of the pairing on the chemical potential of the system can be ignored. The imbalance was parametrized in terms of the difference \( \delta \mu \) in the chemical potentials of the species, which led to the following picture for the gap \( \Delta \) as a function of \( \delta \mu \) (Chandrasekhar, 1962; Clogston, 1962; Sarma, 1963). The gap is a double-valued function, the upper branch of the two solutions being a constant \( \Delta(\delta \mu) = \Delta(0) \) in the range \( 0 \leq \delta \mu \leq \Delta(0) \) and zero beyond the point \( \delta \mu = \Delta(0) \). The lower branch exists in the range \( \Delta(0)/2 \leq \delta \mu \leq \Delta(0) \), with the gap increasing from zero at the lower limit to \( \Delta(0) \) at the upper limit. Only the portion \( \delta \mu \leq \Delta(0)/\sqrt{2} = \delta \mu_1 \) of the upper branch is stable in the sense that the superconducting state lowers the ground-state energy of the superfluid (Sarma, 1963). In the remaining region of imbalance, the superconducting state is unstable (Sarma instability).

Imbalanced pairing in infinite nuclear systems naturally became of interest in the context of \(^3S_1-^3D_1\) and \(^3D_2\) pairing in isospin asymmetrical nuclear matter (Alm et al., 1993, 1996b), and the critical temperatures in these channels were computed using \(T\)-matrix theory and realistic interactions. The full BCS formulation was applied at about the same time (Sedrakian et al., 1997), and subsequently the single-particle energies \( \epsilon_p \) were renormalized within Brueckner theory, resulting in a major reduction of the gap values and more realistic values of critical isospin asymmetries (Sedrakian and Lombardo, 2000).

The ground state of an imbalanced superfluid may entail breaking of global symmetries, notably translational or rotational symmetries, in some range of parameter space. Breaking of translational invariance was first proposed and studied independently by Fulde and Ferrell (1964) (FF) and Larkin and Ovchinnikov (1965) (LO), (collectively, FFLO), who discovered that the superconducting state where the Cooper pairs carry a finite center of mass (CM) momentum can extend to imbalances beyond those restricted by the Chandrasekhar-Clogston limit. In the weak coupling case, the maximal value of the difference in the chemical potentials of the species for the FFLO type of pairing is \( \delta \mu_2 = 0.755 \Delta(0) \) [\( \delta \mu_1 = 0.707 \Delta(0) \)]. The condensate predicted by Fulde and Ferrell (1964) assumes \( \Delta(r) = \Delta_0 \exp(-iQ \cdot r) \) for the gap function, where \( Q \) is the CM momentum. Larkin and Ovchinnikov (1965) explored various lattice types and concluded that the body-centered-cubic lattice is the most stable configuration near the critical temperature. Imbalanced pairing involving finite momentum of pairs.
of neutrons and protons in infinite nuclear systems has
been studied in $^3S_1$–$^3D_1$ and $^3D_2$ pairing channels, both
within T-matrix theory (Alm et al., 1996b) and in exten-
sions of the BCS theory to account for violation of
spatial symmetries (Mao et al., 2009; Sedrakian, 2001;
Stein et al., 2012, 2014a).

Two alternatives to FFLO phase of imbalanced super-
fluids, proposed later, find their application in nuclear
systems. One involves deformations of the Fermi sur-
faces of the fermion species in population imbalance, with
the prospect of improving the ground-state energy of the
paired system compared with the standard configuration
of Fermi spheres (Müther and Sedrakian, 2002, 2003).
Another possibility is the separation of phases, origi-
nally suggested in the context of cold atomic gases (Be-
daque et al., 2003a), and studied thereafter in infinite nu-
clear systems in the $^3S_1$–$^3D_1$ channel (Stein et al., 2012,
2014a).

Extensive work on imbalanced superfluids was carried
out within the area of ultracold atomic gases starting
shortly after the observation of Bose-Einstein condens-
ates in traps (Bedaque et al., 2003a; Combescot, 2001;
Forbes et al., 2005; Liu and Wilczek, 2003; Schmitt, 2014;
Sedrakian et al., 2005; Stooft et al., 1996). For a review
and further references see (Bloch et al., 2008). These sys-
tems offer unique possibilities for testing the physics of
imbalanced superfluidity under controllable conditions in
laboratory experiments (Partridge et al., 2006; Zwierlein
et al., 2006).

Furthermore, imbalanced superfluids have been exten-
sively studied in the context of color superconductivity
of cold quark matter in compact stars, where the three
lightest flavors of quarks and three quark colors make
the possible patterns of pairing especially interesting: for
reviews and further references see (Alford et al., 2008;
Anglani et al., 2014; Rajagopal and Wilczek, 2001).

A. Formalism

Proceeding to the examination of imbalanced phases
in more detail in terms of the underlying many-body
theory and its application to nuclear matter, we start
with a brief outline of the formalism based on the exten-
sion of the GF method to imbalanced systems within the
imaginary-time Matsubara formalism. [For more details,
see (Stein et al., 2014a)].

Consider a mixture of neutrons ($n$) and protons ($p$) at
some density and temperature. The GF of the superfluid,
written in the Nambu-Gor’kov basis, is given by

$$i\mathcal{R}_{12} = i \begin{pmatrix} G_{12}^+ & F_{12}^- \\ F_{12}^+ & G_{12}^- \end{pmatrix} = \begin{pmatrix} \langle T_r \psi_1' \psi_2^- \rangle & \langle T_r \psi_1 \psi_2 \rangle \\ \langle T_r \psi_1' \psi_2^+ \rangle & \langle T_r \psi_1 \psi_2^\dagger \rangle \end{pmatrix},$$

where $G_{12}^+ \equiv G_{\alpha\beta}^+ (x_1, x_2)$, etc., and $x = (t, r)$ is
the four-dimensional time-space coordinate. The Greek indi-
ces label discrete spin and isospin variables. The op-
erators in (91) can be viewed as bi-spinors, i.e., $\psi_\alpha =
(\psi_n, \psi_p, \psi_{n\dagger}, \psi_{p\dagger})^T$, where the indices $n, p$ label a part-
icle’s isospin and $\uparrow, \downarrow$ label its spin.

The solutions of the Dyson equation for the GF defined
in (91) are

$$G_{n/p}^\pm = \frac{ik_v \pm \epsilon_{p/n}^+}{(ik_v - E_\mp^+/\pm)(ik_v + E_\mp^-/\mp)},$$

$$F_{np}^\pm = -i\Delta (ik_v - E_\mp^+/\mp)(ik_v + E_\mp^-/\pm),$$

$$F_{pn}^\pm = i\Delta (ik_v - E_\mp^+/\pm)(ik_v + E_\mp^-/\mp),$$

where $ik_v$ is the Matsubara frequency,

$$\epsilon_{n/p}^\pm = \frac{1}{2m^*} \left( k \pm \frac{Q}{2} \right)^2 - \mu_{n/p},$$

are the normal state spectra of neutrons and protons,
with $\mu_{n/p}$ denoting their chemical potentials, $m*$
their effective mass, and $Q$ is the center-of-mass momentum
of the Cooper pair. (Note that the difference between
the effective masses of neutrons and protons is small at
the low densities of interest and is neglected.) There are
four branches of the quasiparticle spectrum, which are
given by

$$E_n^\pm = \sqrt{E_S^2 + \Delta^2} + r \delta \mu + a E_A,$$

where $a, r \in \{+, -\}$ and

$$E_S = \frac{Q^2/4 + k^2}{2m^*} - \bar{\mu}, \quad E_A = \frac{k \cdot Q}{2m^*},$$

are the symmetrical and anti-symmetrical parts of the
spectrum, with $\bar{\mu} \equiv (\mu_n + \mu_p)/2$ average of neutron
and proton chemical potentials. Taking $ik_v \rightarrow k_0 + i0^+$,
the GF of Eqs. (92)–(94) are analytically continued to obtain
their retarded counterparts. The densities of neutrons
and protons are then defined in a standard fashion by

$$n_{n/p}(Q) = \frac{2}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_{\nu} G_{n/p}^+ (k_v, k, Q)$$

$$= 2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} \left( 1 + \frac{E_S}{E_S^2 + \Delta^2} \right) f(E_\mp^+) \right.$$

$$+ \left. \frac{1}{2} \left( 1 - \frac{E_S}{E_S^2 + \Delta^2} \right) f(-E_\mp^-) \right].$$

where $k = (k_0, k)$ is the four-momentum, $\beta$ is the inverse
temperature. If the interaction is time-local, the pairing
gap is given by

$$\Delta(k, Q) = \frac{1}{4\beta} \int \frac{d^3k'}{(2\pi)^3} \sum_V V(k, k')$$

$$\times \left[ F_{np}^+ (k_v', Q) + F_{np}^- (k_v', Q) - F_{np}^+ (k_v, Q) - F_{np}^- (k_v, Q) \right].$$
where $V(k,k')$ is the neutron-proton interaction. This interaction could be a bare or effective version, depending on the level of approximation; we will illustrate the physical content of the theory using bare interactions to establish a benchmark.

Performing a partial-wave expansion of the expression in Eq. (99) and an energy integration, one arrives at the gap equation

$$\Delta_l(Q) = \frac{1}{4} \sum_{n,r,l,l'} \int \frac{d^3k'}{(2\pi)^3} V_{l,l'}(k,k') \times \frac{\Delta_{l'}(k',Q)}{2\sqrt{E^2_3(k') + \Delta^2(k',Q)}} \left[1 - 2f(E_n')\right],$$

where $V_{l,l'}(k,k')$ is the interaction in the $^3S_1 - ^3D_1$ partial wave and $\Delta^2 = \sum_3 \Delta^2_3$. Note that the magnitude of the vector $Q$ enters Eqs. (98) and (100) parametrically and should be determined from minimization of the free energy of nuclear matter. Its direction is chosen by the system spontaneously. This minimum condition leads, in fact, to an additional equation for $Q$ that should be solved along with Eqs. (98) and (100).

Quite generally, the ground state of nuclear matter is obtained from the minimization of the respective free energies of the phases

$$F_S = E_S - TS_S, \quad F_N = E_N - TS_N,$$

where indices $S$ and $N$ refer to the superfluid and normal phases, $E$ is the internal energy (statistical average of the system Hamiltonian), and $S$ is the entropy.

The formalism developed to this point also covers the treatment of the heterogeneous, phase-separated (hereafter PS) phase proposed in (Bedaque et al., 2003a) and implemented in the context of nuclear matter in (Stein et al., 2012, 2014a). Allowing for separation of phases implies that we can choose to maximize pairing by having isospin-symmetrical superfluid domains, with all the excess neutrons accommodated in normal regions. Then we already have all the necessary ingredients for evaluating the free energy of such a state, using the simple construction

$$\mathcal{F}(x,\alpha) = (1 - x)F_S(\alpha = 0) +xF_N(\alpha \neq 0), \quad (Q = 0),$$

where $x$ is the filling fraction of the unpaired component. Here $\alpha$ is the isospin asymmetry parameter defined by Eq. (10). In the superfluid phase ($S$), one has by definition $n_{n/p}^{(S)} = n^{(S)}/2$. In the unpaired phase ($N$), the densities of neutrons and protons need not be equal and are assigned the values $n_{n/p}^{(N)}$. Thus, the net densities of neutrons/protons per unit volume are given by $n_{n/p} = (1 - x)n^{(S)} + xn^{(N)}$.

In the preceding discussion of imbalanced phases we considered a particular realization of the FFLO phase with single plane-wave modulation of the gap parameter in space, which corresponds to the original FF phase. This phase in fact breaks only the rotational symmetry along the direction of the vector $Q$, but in many respects this phase is a representative for other realizations of the FFLO phases.

To summarize this subsection, we have surveyed the formalism for imbalanced superfluids and their realization in superfluid nuclear matter in which four possible distinct phases can arise. These can be classified in terms of $Q$, $\Delta$, and $x$ as follows

$$Q = 0, \quad \Delta \neq 0, \quad x = 0, \quad \text{BCS phase},$$
$$Q \neq 0, \quad \Delta \neq 0, \quad x = 0, \quad \text{FFLO phase},$$
$$Q = 0, \quad \Delta \neq 0, \quad x \neq 0, \quad \text{PS phase},$$
$$Q = 0, \quad \Delta = 0, \quad x = 1, \quad \text{unpaired phase}.$$

The competition between these phases is decided on the basis of minimization of the ground state energy.

These four phases of fermionic matter are conceived for two species of fermions having spherically symmetric Fermi surfaces. In Sec. IV.D below we will amend our discussion with yet another phase that breaks the spatial symmetries by deformations of the Fermi surfaces away from spherical shape (Müther and Sedrakian, 2002, 2003).

### B. Homogeneous phase

Equations (98) and (99) have to be solved simultaneously in general. In weakly coupled superfluids, Eq. (98) can be evaluated with the normal state spectrum by setting $\Delta = 0$, in which case it decouples from Eq. (99). This approximation is invalid for strongly coupled systems, in particular for nuclear systems at very low densities.

Once the solutions are found, the free energy can be evaluated and a specific phase can be assigned to a given temperature and density. We start our discussion with the simplest case, the homogeneous imbalanced super-
fluids. Their realization depends essentially on the difference in the chemical potentials of the components, \( \delta \mu = \mu_n - \mu_p \geq 0 \), assuming a neutron excess. A sufficiently large \( \delta \mu \) value will disrupt pairing because fermions lying on different Fermi surfaces cannot overlap to form Cooper pairs, due to the lack of shared phase-space. Finite temperature can counteract the disruptive effect of \( \delta \mu \) because it increases the “diffuseness” of the Fermi surfaces and hence the phase-space overlap between the paired fermions. This physics is illustrated in Fig. 8, where we see that the gap has a maximum as a function of temperature as a consequence of the interplay of two effects: the disruption by \( \delta \mu \) and phase-space expansion by temperature. For large enough asymmetries there exists a lower critical temperature \( T_c^* \) (Sedrakian et al., 1997; Sedrakian and Lombardo, 2000) (not shown in the figure). This phenomenon has also been observed in the context of small superconducting systems (finite nuclei, in fact) and is referred to as “re-entrance” (Balian et al., 1998; Margueron and Khan, 2012). It should be mentioned that the “anomalous” behavior of the BCS gap below the temperature corresponding to the maximum gap (see Fig. 8) leads to a number of anomalies in thermodynamic quantities, such as negative superfluid density or an anomalous jump in the specific heat, which can be a signature of a (metastable) low-temperature homogeneous phase (Sedrakian et al., 2006).

We also observe in Fig. 8 that at temperatures close to its critical value, homogeneous imbalanced superfluids show the same dependence of the gap on temperature; consequently, the thermodynamics in this regime is analogous to that in the ordinary BCS case. Computation of the free energy of the homogeneous imbalanced phase and its comparison to that of other phases listed in (103) shows that it is preferred in a domain of temperatures adjacent to the critical temperature, where the disruptive effects are small (Stein et al., 2012, 2014a).

### C. FFLO phases

Our next example is the Fulde-Ferrell (FF) phase of nuclear matter paired in the \( ^3S_1^{-3}D_1 \) channel. The physics of this phase can be understood by observing that the finite momentum \( Q \) affects the spectrum of particles in a twofold manner: there is a shift in the symmetric part of the spectrum \( E_S \rightarrow E_S + (Q^2)/8m^* \) and moreover \( E_A = \pm (k \cdot Q)/2m^* \neq 0 \). Thus, in the FF phase there is a positive increase in the quasiparticle kinetic energy \( \propto Q^2 \), which disfavors it relative to the BCS state. However, the anisotropic term \( \propto k \cdot Q \) changes the phase-space overlap of the fermions and promotes pairing in certain directions. Clearly, the FF phase is stabilized when the increase in the kinetic energy loss caused by moving the condensate is overcome by the gain in the potential energy of pairing due to the increase in the phase-space overlap.

The mechanism leading to a stable FF phase signifi-

![FIG. 9 Dependence of the gap on isospin asymmetry in the homogeneous phase (dashed lines) and FFLO phase (solid lines) for several temperature values indicated in the plot. The nuclear-matter density is fixed at \( n = 0.1 \text{ fm}^{-3} \).](image-url)
FIG. 10 Illustration of projected Fermi surfaces (solid traces) in the FF (upper panel) and DFS (lower panel) phases. The concentric (dashed) circles represent projections of spherical Fermi surfaces of neutrons and protons under imbalance in the $x$-$z$ plane.

drakian, 2002, 2003; Sedrakian et al., 2005), in the sense that the original Hamiltonian is O(3) symmetric, but the ground state breaks this symmetry down to a subgroup, for example to O(2). To explore whether deformations lead to improvements of the ground state energy of the system, it is useful to consider spontaneous deformations which are parametrized as

$$E_{r}^{a,a'} = E_{r}^{a} + a'\epsilon_2 P_2(\cos \theta),$$

(104)

where $a' = \pm$, the $E_{r}^{a}$ are given by Eq. (96), $P_2(x)$ is the $n = 2$ Legendre polynomial, and $\theta$ is the angle formed by the particle momentum and the direction of the spontaneous breaking of rotational symmetry.

The gap equation has been solved and the free energy computed for a deformation parameter defined as the relative deformation of the two Fermi surfaces, $\delta\epsilon = (\epsilon_{2, +} - \epsilon_{2, -})/2$. This parameter is the analog of the total momentum $Q$ in the analysis of the FF phase. Computations for nuclear matter and cold atoms show that, in a certain domain of asymmetries, the energy is minimized for non-zero $\delta\epsilon$, i.e., there is a stable minimum corresponding to a state with deformed Fermi surfaces of the components.

The Fermi surfaces in the FF and DFS phases are illustrated in Fig. 10 along with those in an imbalanced homogeneous phase. The intersections of the Fermi surfaces of the two fermionic species in the cases of the FF and DFS phases reveal the mechanisms of the phase-space overlap between the components and the enhancement of pairing correlations achieved in these phases.

As already indicated in Sec. II.E, cold-atom systems offer a playground for testing the theoretical ideas that emerged in the studies of imbalanced system in various contexts. An interesting extension, which we will not discuss in this review, is the study of imbalanced superfluids in periodic external potentials created by optical lattices; for a review see (Kinnunen et al., 2018).

E. BCS-BEC transition

Weakly coupled BCS superfluids form Cooper pairs with characteristic size of the order of the coherence length, which is much larger than the interparticle distance. The pairs are weakly bound, as the scale of binding energy set by the gap $\Delta$ is much smaller than the Fermi energy. It was conjectured long ago that under gradual decrease of density, such BCS superfluids will smoothly evolve into a BEC of tightly bound bosonic dimers, with a size much smaller than the interparticle distance, in what amounts to a strong-coupling limit (Eagles, 1969; Nozières and Schmitt-Rink, 1985). See Fig. 11 for an illustration. This conjecture has been confirmed in experiments on cold atomic gases, where the coupling parameter can be manipulated via tuning the magnetic field to a Feshbach resonance and thus effectively changing the magnitude of the scattering length and its sign (Regal and Jin, 2007; Zwierlein et al., 2006). High-density isospin-symmetric nuclear matter would form a weakly coupled BCS condensate in the dominant $^3S_1-^3D_1$ channel. The BCS-BEC crossover in such a condensate can be
achieved by diluting the system, in which case the reduction of the density of states will eventually lead to a transition to a BEC state, which in this case is a condensate of deuterons (Alm et al., 1993; Baldo et al., 1995; Huang, 2010; Jin et al., 2010; Sedrakian and Clark, 2006b; Stein et al., 1995). This transition occurs smoothly without changes in the condensate wave function; hence it is a crossover in the proper sense. The BCS-BEC crossover is also expected in imbalanced systems, in particular in isospin-asymmetric nuclear matter, unless the pairing is completely disrupted by the mismatch in the Fermi surfaces of protons and neutrons (Lombardo et al., 2001; Mao et al., 2009; Shang and Zuo, 2013; Stein et al., 2012, 2014a,b; Sun and Pan, 2013). In this case phase transitions can be encountered, i.e., the condensate wave function does not evolve smoothly across the BCS-BEC crossover. Therefore, it is more appropriate to speak about a BCS-BEC transition rather than a crossover. The straightforward modification of the original theory of Nozières and Schmitt-Rink (1985) involves adaptation to a gaseous mixture of neutrons and deuterons in the strong-coupling low-density limit. A more subtle issue is the emergence of phase transitions between various phases of imbalanced superfluids discussed above as one moves from weak to strong coupling.

The evidence for isospin-singlet pairing in nuclear phenomenology is scarce. However, it has been conjectured that large enough nuclei may feature spin-aligned \( np \) pairs, based on recent experimental studies of the excited states in \( ^{92} \text{Pd} \) (Cederwall et al., 2011) as well as Hartree-Fock-Bogolyubov computations of large nuclei (Friedman and Bertsch, 2007). Intermediate energy heavy-ion collisions produce large amounts of deuterons in final states, which could be an asymptotic state reached once the initially formed BCS condensate in the \( ^3S_1-^3D_1 \) channel crosses over to a BEC of deuterons (Baldo et al., 1995). It has been also speculated that deuteron condensates can be formed in the dilute nuclear matter found in supernova and hot proto-neutron-star matter at sub-saturation densities; see for example (Clark et al., 2016; Gulminelli and Raduta, 2015; Heckel et al., 2009; Hempel and Schaffner-Bielich, 2010; Oertel et al., 2012; Pais et al., 2015; Raduta and Gulminelli, 2010; Sumiyoshi and Röpke, 2008; Typel et al., 2010; Wu et al., 2017).

The emergence of a BEC in the isospin-singlet channel at asymptotically low densities is straightforward because in the vacuum there is a bound state in this channel — the deuteron. In contrast, the BEC limit in the isomultiplet \( ^3S_1 \) pairing channel is not obvious. Nevertheless, the unusually large scattering length in this channel for neutron-neutron scattering, \( a_{nn} \sim -19 \text{ fm} \), suggests traces of a BEC in neutron-rich systems such as the halo nuclei or neutron matter in compact objects (Abe and Seki, 2009; Isayev, 2008; Kanada-En’yo et al., 2009; Margueron et al., 2007; Matsuo, 2006; Ramanan and Urban, 2013; Salasnich, 2011; Stein et al., 2016; Sun et al., 2010, 2012). We will address this problem below in Sec. IV.G.

1. BCS-BEC transition in the balanced case

Consider first the basics of the BCS-BEC crossover in the \( ^3S_1-^3D_1 \) pairing channel for the case of isospin symmetrical nuclear matter. The equations that are solved in this case for the densities and the gap are respectively (98) and (100) in the symmetrical limit. Results from simultaneous solution of the gap and density equations (Sedrakian and Clark, 2006b) are plotted in Fig. 12 at fixed values of the ratio \( f = n_0/n \). (We recall that \( n_0 = 0.16 \text{ fm}^{-3} \) is the saturation density of symmetrical nuclear matter.) The low- and high-temperature asymptotics of the gap function can be fitted by the BCS-like relations (8): \( \Delta(T) = \Delta(0) - [2\pi c_1\Delta(0)/T]^{1/2} \exp(-\Delta(0)/T) \) for \( T \to 0 \) and \( \Delta(T) = 3.06c_2(T_c(T_c - T))^{3/2} \) for \( T \to T_c \), where \( T_c \) is the critical temperature and \( c_1, c_2 \) are adjustable parameters. The values of the parameters yielding a fit, \( c_1 \simeq 0.2 \) and \( c_2 \simeq 0.9 \), deviate from the predictions \( c_1 = c_2 = 1 \) of BCS theory. As a consequence, the value of the ratio \( \Delta(0)/T_c \) deviates from the BCS prediction of 1.76. Clearly, the discrepancy depends on the inverse density measure \( f \) and reflects the breakdown of the weak-coupling Ansatz.

The transition from the BCS to the BEC regime can be traced in terms of several characteristic quantities. One such parameter is the ratio \( \Delta(0)/|\mu| \). Using this measure, we can now infer from Fig. 12 that the strong-coupling regime characterized by \( \Delta \gg |\mu| \) sets in for \( f \geq 40 \). For small values of \( f \) (~ 20), where \( \Delta \sim |\mu| \), the system is in the transition region intermediate between BCS and BEC. A second signature of the crossover from weak to strong coupling is the sign of the chemical potential of nucleons. Indeed, it changes sign for \( f \simeq 80 \), which
is somewhat below the crossover density between weak-coupling and strong-coupling regimes deduced above. A third method, rather appealing physically, is direct comparison of the size of Cooper pairs, taking the ratio of the coherence length $\xi$ to the interparticle distance $d \sim n^{1/3}$. In the BCS limit, one has by definition $\xi \gg d$; conversely, in the BEC limit $\xi \ll d$. We use this criterion below in identifying the transition parameters.

Finally, we note that in the context of dilute atomic gases, domains of weak and strong coupling are distinguished by the parameter $n|a|^3$, where $a$ is the scattering length. In nuclear matter, the strong-coupling regime was assigned to $f \geq 40$, which translates to $na^3 \gtrsim 0.6 < 1$ if we use $a_{np} = 5.41$ fm for the $n$-$p$ scattering length. Thus one may conclude that symmetrical nuclear matter is indeed in the strong coupling-regime at low densities.

Another interesting feature shown in Fig. 12 is the asymptotic value of the chemical potential, $\mu = -1.1$ MeV at $f \rightarrow \infty$ or $n \rightarrow 0$. Its value is just half the binding energy of the deuteron in free space. Formally, this result can be verified by transforming the gap equation into an eigenvalue problem, in which case it becomes a Schrödinger equation for a two-body bound state described by the anomalous correlation function, with the chemical potential as its energy eigenvalue.

We conclude that the BCS condensate of Cooper pairs in the $^3S_1-^3D_1$ state evolves into a BEC of deuterons under dilution of nuclear matter. The crossover is smooth, taking place without change of symmetry of the many-body wave function in the case of the isospin symmetrical nuclear matter (Alm et al., 1993; Baldo et al., 1995; Huang, 2010; Jin et al., 2010; Sedrakian and Clark, 2006b; Stein et al., 1995).

2. BCS-BEC transition in the imbalanced case

How does the physics of the BCS-BEC crossover change under imbalance between the populations of fermionic species that pair? As we have seen, the condition of imbalance introduces some new and unconventional phases in the BCS limit, and it is natural to ask about their counterparts (if present) in the strong-coupling limit. This problem has been addressed recently in a series of papers (Stein et al., 2012, 2014a,b), in which the equations for the gap and densities [Eqs. (98) and (100)] were solved in a framework that provides for description of both the BCS phase and its low-density BEC counterpart, as well as two unconventional phases which may arise within a range of isospin asymmetries. The FFLO phase was chosen as a representative for phases with broken space symmetries and the collection of phases was supplemented by the heterogeneous phase in which the normal fluid and superfluid occupy separate spatial domains.

Before discussing the phase diagram containing these phases, we survey the intrinsic properties of the BCS-BEC transition$^9$ under isospin imbalance (Lombardo et al., 2001; Stein et al., 2012, 2014a,b). These properties include primarily the Cooper-pair wave function, the occupation probabilities of particles, the coherence length, and the quasiparticle spectra. Their quantitative study provides additional physical insight and understanding of how the system evolves from weak coupling to strong coupling under isospin asymmetry. We note that in the case of phase separation, the only non-trivial phase is the isospin symmetrical BCS phase. Therefore, its intrinsic features, apart from heterogeneity, are identical with those of the standard BCS theory and hence need not be addressed separately.

Recall that in ultracold atomic gases the imbalance is achieved by trapping different amounts of atoms in different hyperfine states, and the transition is achieved by varying their effective interaction strength via the Feshbach mechanism. In contrast, in an extended nuclear system, a BCS-BEC transition is induced by variation of its density and the isospin asymmetry is fixed by the minimization of the energy or the initial conditions, as e.g. in nuclear collisions. As a result, the pairing interaction strength changes, in accord with changes in the relevant energies for in-medium scattering of two nucleons set by the Fermi energy of the system. In consonance, the density of states changes. The BCS-BEC transition in the nuclear system is therefore governed by the combination of these two effects. In contrast to ultracold atoms, it

$^9$ While it is well-established that one deals with a crossover in the proper sense in the case of balanced systems, the imbalance does change the nature of transition. Therefore, we will use transition instead of crossover when dealing with imbalanced systems.
cannot be manipulated at will.

To set the stage, we extract the kernel of the gap equation (100)

$$K(p) = \sum_{a,r} \frac{1 - 2f(E^a_r)}{4\sqrt{E_S(p)^2 + \Delta^2(p,Q)}},$$  \hspace{1cm} (105)

where we recall that \(f(E^a_r)\) is the Fermi distribution function and \(E^a_r\) and \(E_S(p)\) are given by Eqs. (96) and (97).

Physically, \(K(p)\) is the momentum-space wave function of the Cooper pairs, because it obeys a Schrödinger eigenvalue equation in strong coupling. In terms of its configuration-space image, we may write the wave function of a Cooper pair as

$$\Psi(r) = \sqrt{N} \int \frac{d^3p}{(2\pi)^3} [K(p, \Delta) - K(p, 0)]e^{ip \cdot r}. \hspace{1cm} (106)$$

Here \(N\) is a constant determined by the standard normalization of a wave function to unity, and the value \(K(p, 0)\) of the kernel in the normal state is subtracted to regularize the integral, which is otherwise divergent. It is useful also to define the quantities

$$\langle r^2 \rangle = \int d^3r r^2 |\Psi(r)|^2, \quad \xi_{\text{rms}} = \sqrt{\langle r^2 \rangle}, \hspace{1cm} (107)$$

where \(\xi_{\text{rms}}\) is the coherence length, \(\text{i.e., the spatial extension of a Cooper pair. This definition can be contrasted to the weak-coupling BCS analytical formula }\xi_a = k_F/(\pi n^*\Delta). The root-mean-square definition (107) allows one to extend the notion of the coherence length into the strong-coupling regime; therefore, it can be compared to the mean interparticle distance \(d = (3/4\pi n)^{1/3}\) in the entire range of the BCS-BEC transition.

Figure 13 shows the integrand of \(\langle r^2 \rangle\) in Eq. (107) as a function of radial distance \(r\) at densities representative for the three coupling regimes involved in the BCS-BEC transition. At densities corresponding to the weak-coupling regime (labeled WCR, \(\log_{10} n/n_0 = -0.5\)), this function (as well as the wave function \(\Psi(r)\) itself, not shown) has a well-defined oscillatory form of period \(2\pi/k_F\), which persists for multiple tens of fm. The behavior of such a state is commensurate with the long-range order inherent to BCS picture, where the spatial extension of pairs, measured by the coherence length \(\xi\), is much greater than the interparticle distance \(d\). In intermediate- and strong-coupling regimes (ICR and SCR, \(\log_{10} n/n_0 = -1.5\) and \(-2.5\), respectively), the wave function becomes concentrated at the origin, possibly showing a few oscillations indicative of the transition between limiting cases. This behavior is descriptive of particles well-localized in space, a distinctive characteristic of the BEC regime.

In Fig. 14 we show the same quantity \(r^2|\Psi(r)|^2\) as in Fig. 13, but in the FF phase at two different angles \(\theta\), evaluated for an asymmetry \(\alpha = 0.49\) (\(\delta \mu = 6.45\) MeV), where this phase is the ground state of the matter with \(\Delta = 1.27\) MeV and \(Q = 0.4\) fm\(^{-1}\). At \(\theta = 0\) the perfect oscillatory behavior of the BCS case is intact, with a slight modulation due to non-zero \(Q\). At \(\theta = 90^o\), the amplitude of the oscillations is modulated by a second oscillatory mode with period \(2\pi/k_F \sim 20\) fm, in addition to the first mode having period \(2\pi/k_F\). In the FF phase and for \(\theta = 0^o\), the term \(\propto \cos \theta\) renders the quasiparticle spectrum and therefore the Cooper pair wave function close to that expected from ordinary BCS theory. In contrast, for \(\theta = 90^o\), the term \(\propto \cos \theta\) is zero and marked differences are seen, notably damping of the amplitude of oscillations.

Also of central interest are the occupation numbers \(N_{n/p}(k)\) of proton and neutron states, which are identified as the integrands of Eq. (98). At zero temperature and in unpaired matter, the functions \(N_{n/p}(k)\) are discontinuous at the Fermi surface. Numerical results for balanced and imbalanced superfluids are shown in the three coupling regimes of interest in Fig. 15. A key feature of this figure that is universal for imbalanced superfluids and nuclear systems is the appearance of a “breach” (Forbes et al., 2005; Gubankova et al., 2003, 2006; Liu and Wilczek, 2003) or “blocking region” (Lombardo et al., 2001) for large asymmetries. These designations refer to the entire expulsion of the minority component (in this case the protons, \(N_p = 0\)) from a region around the Fermi momentum of the balanced system, accompanied by maximal occupancy \((N_n/2 = 1)\) of the majority component (here the neutrons). Examination of \(N_{n/p}(k)\) in the FF phase for different angles shows that for small enough \(\theta\) the breach disappears and the occupation numbers resemble each other in shape. This reflects the fact that for certain directions the effects of asymmetry are mitigated by the non-zero \(Q\).

The ICR (middle panel) is characterized by loss of the Fermi character of the occupation numbers and vanishing of the breach. In addition, for large enough \(\alpha\), the occu-
FIG. 15 Dependence of the neutron and proton occupation numbers on momentum $k$ (in units of Fermi momentum) at densities $\log_{10} n/n_0 = -0.5, -1.5$ and $-2.5$ corresponding to the three coupling regimes WCR, ICR, and SCR, respectively, for a range of asymmetries $\alpha$ indicated in the legend. The boundaries of the exclusion regions seen in the WCR (left panel) are smooth due to the non-zero value of temperature.

Quasiparticle spectra provide further insight into the nature of superfluid states. In the balanced case, one finds a dispersion relation with a minimum $E^+ = E^- = \Delta$ for $k = k_F$ [the spectra with lower $\pm$ indices being degenerate; see Eq. (96)]. For non-zero asymmetries one has $E^+ = E_2 \pm \delta \mu$, which induces a shift in the minima. For protons the spectrum becomes gapless: no energy is required to create excitations of two modes (say $k_1$ and $k_2$) for which the dispersion relation intersects the zero-energy axis. This phenomenon is referred as gapless superconductivity (Abrikosov, 1988). The momentum interval $k_1 \leq k \leq k_2$ is in fact where the “breach” in the occupation of the minority component exists.

Finally, in the SCR, the balanced limit corresponds simply to a gas of deuterons, and the dispersion relation has a minimum at the origin that corresponds to the (average) chemical potential, which in the low density limit tends to the value $-1.1$ MeV, as discussed above. Imbalance changes the position of the average chemical potential downwards and separates the quasiparticle spectra by an amount $\delta \mu$. Because there is unique minimum, the dispersion relation crosses zero only once at a finite $k$.

Upon introducing the FF phase, if one again considers different values of the angle $\theta$, it turns out that for $\theta = 0^\circ$, two of the four branches of quasiparticle spectra closely resemble the spectrum of the ordinary BCS phase. For large $\theta \leq 90^\circ$, the dispersion relations supported by this phase are close to those of the imbalanced BCS case, which implies strong suppression of pairing. This behavior again points to the key mechanism by which the FF phase enhances pairing – the restoration of pairing correlations through an improved overlap between the Fermi surfaces of neutrons and protons in certain directions.

F. Toward a complete phase diagram

A central problem in the physics of imbalanced many-fermion systems is the concrete realization of their phase diagram in the parameter space spanned by the density (or in cold-atom physics by the scattering length), the temperature, and the degree of imbalance. While the details of the phase diagram will certainly depend on the specifics of the interactions (contact vs. finite range, purely central or complicated by tensor and spin-orbit components), the generic structure of the phase diagram should be universal. It is also expected to exhibit universal features across diverse systems including cold atomic gases, nuclear systems, and dense quark matter.

Results of a detailed study of the phase diagram of the imbalanced systems presented by generally asymmetric nuclear matter, which admits the four phases listed in Eq. (103) and discussed above, were reported in a series of two papers (Stein et al., 2012, 2014a). The resulting phase diagram is shown in Fig. 16. The phases are arranged in the temperature-density plane, and the phase boundaries have been computed for several values of isospin asymmetry $\alpha$. The generic structure of the phase diagram is as follows. (a) Above the critical temperature $T_{c0}(n)$ for the normal-to-superfluid phase transition at $\alpha = 0$, the nuclear matter is in the unpaired phase. (b) At low temperatures, high densities, and moderate to large asymmetries, the FFLO phase forms the ground state in the triangular regions indicated in Fig. 16. (c) Moving to stronger couplings (lower densities), one finds the domain of phase separation (PS) at sufficiently low temperatures. (d) The ordinary BCS phase with isospin asymmetry intervenes at higher temperatures.

As seen in Fig. 16, the extreme low-density (strong coupling) limit features two counterparts of the weakly coupled phases: first, the BCS phase at intermediate temperatures evolves into the BEC phase of deuterons; second, the PS-BCS phase transforms into the PS-BEC phase, in which the superfluid domains contain a BEC of deuterons. These transitions are indicated in the figure by phase boundaries, although we should stress that the BCS-BEC transition and the PS-BCS to PS-BEC transition are smooth crossovers. The transition to the normal state and the phase transitions between the superfluid phases are generally of second order and are indicated by thin solid lines in Fig. 16. The only exception is the

\[ \theta = 0^\circ \]
FIG. 16 Phase diagram of dilute nuclear matter in the temperature-density plane for several isospin asymmetries $\alpha$, where the density is normalized by the nuclear saturation density $n_0$. For each fixed $\alpha$ there are two tri-critical points, the point bordering the FFLO phase being always a Lifshitz point (Hornreich, 1980). In the special case $\alpha = 0.255$, these tri-critical points merge into a tetra-critical point for $\log_{10}(n/n_0) = -0.22$ and $T = 2.85$ MeV (square dot). The FFLO phase completely disappears to the left of the point $\log_{10}(n/n_0) = -0.65$ and $T = 0$ (shown by the triangle) for $\alpha = 0.62$. The transition from BCS pairing to BEC, identified by the change of sign of the chemical potential $\mu$, occurs on the vertical lines located around $\log_{10}(n/n_0) = -2$.

PS-BCS to FFLO phase transition, which is of first order. We further notice that at finite asymmetry there is a locus where three of the four phases meet, corresponding to a tri-critical point. For each fixed $\alpha$ there are two tri-critical points. Of these, one is a Lifshitz point, since one of the adjacent phases represents a modulated phase (Hornreich, 1980). We observe also that at low temperatures, the ordinary BCS-BEC crossover, which is a smooth crossover to an asymptotic state corresponding to a mixture of a Bose condensate of deuterons and a gas of excess neutrons, is replaced by a new type of transition in which the fragmented superfluid contains a deuteron BEC surrounded by a phase containing neutron-rich unpaired nuclear matter.

G. Spin polarized neutron matter

Another important scenario in which unconventional nuclear superfluidity arises is spin-polarized neutron matter in strong magnetic fields. Strongly magnetized neutron stars, known as magnetars, are characterized by surface fields of order $B \sim 10^{15}$ G (Thompson and Duncan, 1995; Turolla et al., 2015) and may feature fields that are larger by factors of a few in their interiors (Bocquet et al., 1995; Chatterjee et al., 2015). Magnetic fields of this magnitude can suppress the pairing of neutrons and protons in the $S$-wave state (Sinha and Sedrakian, 2015; Stein et al., 2016), but the mechanisms of suppression for charged and neutral condensates are different. The proton $S$-wave pairing is quenched because of the Landau diamagnetic currents of protons induced by the field; this happens once the Larmor radius of a proton in the magnetic field becomes of the order of the coherence length of the proton condensate. The neutron pairing is suppressed when the $S$-wave neutron gap becomes of the order of the Pauli-paramagnetic interaction of the neutron spin with the magnetic field $B$. The magnitude of this interaction is $|\mu_N|B$, the neutron spin magnetic moment being given by $\mu_N = g_n(m_n/m_e)\mu_N$, where $g_n = -1.91$ is the neutron $g$ factor, $m_n^*$ its effective mass, and $\mu_N = e\hbar/2m_n$ the nuclear magneton.

Thus, the physics of neutron matter in strong magnetic fields would parallel that of the $^{3}\S_1\cdot^{3}D_1$ condensate discussed in Sec. IV, with the paramagnetic interaction playing the role of the isospin asymmetry. This possibility was anticipated for the FFLO phase (Sedrakian, 2001), and the phase-separated phase of neutron matter has been investigated in detail in (Gezerlis and Sharma, 2012). More recently, signatures of the BCS-BEC crossover in spin-polarized neutron matter and the emergence of dineutron correlations in the presence of a magnetic field have received attention (Stein et al., 2016), generalizing the previous studies of this phenomenon in unmagnetized neutron matter (Abe and Seki, 2009; Isayev, 2008; Kanada-En’yo et al., 2009; Margueron et al., 2007; Matsuo, 2006; Salasnich, 2011; Sun et al., 2010, 2012).

The critical field of unpairing of $S$-wave superfluidity in neutron matter is of great phenomenological interest for the physics of magnetar crusts. The magnitude of the magnetic field $B$ in the crust and outer-core regions of magnetars cannot be determined directly from observations. One may anticipate that their interior fields are somewhat larger than the surface fields $B \sim 10^{15}$ G based on the modeling of magnetar equilibrium figures. Some magnetar models suggest that strong toroidal $B$-fields are confined to the crust. Therefore, if local fields are larger than the critical field for unpairing, neutron superfluidity will be absent. Computation of the critical field (Stein et al., 2016) indicates that at a temperature $T = 0.05$ MeV characteristic of neutron stars, it is of order $10^{16}$ G at the base of the inner crust, i.e., at $\log_{10}(n/n_0) = -3$, and increases up to $10^{17}$ G for densities one order of magnitude larger.

Thus, in contrast to the analogous case of asymmetric nuclear matter, the interest in spin-polarized neutron matter lies primarily in the dependence of pairing on the magnetic field, rather on the spin polarization per se. Figure 17 shows the magnitude of the field needed to generate a prescribed polarization in neutron matter; it is seen that at low densities lower magnetic fields are required. It is also to be noted that the magnetic field required to produce a specified polarization increases with decreasing temperature. The combined effect of variation of the gap and the polarization with density produces critical magnetic fields that are maximal at about...
spin-polarized neutron condensate – which include the Cooper-pair wave function, occupation numbers, and quasiparticle spectra – shows that their behavior runs parallel to those of asymmetrical nuclear matter, already discussed in some detail. In particular, one finds (Stein et al., 2016) that the Cooper-pair wave functions and the function $r^2 |\Psi(r)|^2$ exhibit oscillatory behavior characteristic of long-range order, the wave vector of the oscillation being $2 \pi / k_F n$, where $k_F n$ is the neutron Fermi wave number. The quasiparticle occupation numbers likewise display a breach around the Fermi momentum $k_F$, which is most pronounced in the high-density and low-temperature limit where the matter is highly degenerate. Furthermore, the feature of gapless superfluidity is again observed in this case: at large polarizations, the energy spectrum of the minority-spin particles crosses the zero-energy level, where modes can be excited without any energy cost.

As argued above, neutron matter exhibits the features of a BCS-BEC crossover, although at asymptotically low densities two neutrons are not bound. Evidence of this transition is clearly seen in Table I by comparing (a) the first row (high-density entry) showing $\xi_{\text{rms}} / d > 1$ and $\Delta / \mu \ll 1$, which is characteristic to the BCS phase, with the third row (low-density entry), where $\xi_{\text{rms}} / d < 1$ and $\Delta / \mu \sim 1$. The behavior at low density is interpreted as a precursor state to a (non-existent) dineutron BEC.

V. ASTROPHYSICAL MANIFESTATIONS OF PAIRING IN NEUTRON STARS

A. Pairing patterns in neutron stars

So far we have concentrated on the microscopic physics of superfluidity in extended nuclear systems in a general setting, over broad ranges of density, temperature and isospin/spin asymmetry. Our next task is to adapt these considerations to the conditions prevailing in neutron stars. The matter in neutron stars is characterized by conserved charges, specifically baryon number and electrical charge. Also, at the microscopic level, the interior of a neutron star is in approximate electro-weak equilib-

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
\log_{10} (n/n_0) & $k_{Fn}$ & $\Delta$ & $m^*/m$ & $\mu_n$ & $d$ & $\xi_{\text{rms}}$ \\
[1em]
& [fm$^{-1}$] & [MeV] & [MeV] & [fm] & [fm] \\
\hline
-1.0 & 0.78 & 2.46 & 0.967 & 12.94 & 2.46 & 4.87 \\
-1.5 & 0.53 & 1.91 & 0.989 & 5.65 & 3.61 & 3.55 \\
-2.0 & 0.36 & 1.07 & 0.997 & 2.49 & 5.30 & 2.36 \\
\hline
\end{tabular}
\caption{Parameters of the $^1S_0$ condensate for $T = 0.25$ MeV and $\alpha = 0$ at selected values of the total particle density $n$ (in units of $n_0$). Other table entries are the Fermi momentum $k_F = (3 \pi^2 n)^{1/3}$, pairing gap $\Delta$, effective mass (in units of the bare mass), chemical potential $\mu_n$, interparticle distance $d$, and coherence length $\xi_{\text{rms}}$.}
\end{table}
This combination, with charge conservation, determines the phase and composition of neutron-star matter at any given depth in the star (cf. Sec.II.C).  

Figure 19 shows a schematic cross-section of the interior of a neutron star of mass $M = 1.4 M_\odot$. Among the multitude of possible phases occurring at different densities, the locations of nucleonic superfluid and superconducting phases are indicated in order of increasing depth in the star, along with possible hypernuclear and quark domains in the core. 

We now discuss briefly the key phases that exist inside a neutron star, beginning just below its surface and moving toward the center. At densities $\rho \approx 10^6$ g cm$^{-3}$, neutron-star matter is fully ionized, being composed of ions of $^{56}$Fe and relativistic electrons. Such matter, much like that in a white dwarf involving heavier isotopes, solidifies below a melting temperature $T_m \approx 10^9$–$10^{10}$ K, such that we anticipate a correlated solid phase in mature neutron stars. We note that this solid phase may have a very thin blanket (several cm in total) made up of lighter elements including H, He, etc., in ionized, atomic, or molecular form. The composition of this enveloping blanket can, in principle, be extracted from observations of photoemission from the surface of the star.

Under the constraints of neutrality and equilibrium, the matter becomes more neutron-rich as the depth and hence the density increases. The outer crust of the star, which spans the density range $10^7 \leq \rho \leq 10^{11}$ g cm$^{-3}$, is made up of a sequence of nuclei, their neutron fraction increasing with depth, a characteristic sequence being $^{62}$Ni, $^{86}$Kr, $^{84}$Se, $^{82}$Ge, $^{80}$Zn, $^{124}$Mo, $^{122}$Zr, $^{120}$Sr, and their neutron-rich isotopes. The lattice formed by these nuclei may not be perfect and may contain nuclei with mass numbers different from those predicted for the ground state. Nucleonic superfluidity in the outer crust exists inside the individual bound finite nuclei and can be described using standard methods, such Hartree-Fock plus BCS or Hartree-Fock-Bogolyubov theories. 

At a density $\rho \approx 4 \times 10^{11}$ g cm$^{-3}$, neutrons drip out of the nuclei and start filling continuum states. Consequently a degenerate neutron gas occupies the space between the nuclear clusters. The resulting phase, which features neutron-rich nuclei immersed in an electronic background and a dilute neutron gas, occupies the inner crust of a neutron star and extends in density up to half the saturation density of symmetrical nuclear matter, $\rho_0 = 2.8 \times 10^{14}$ g cm$^{-3}$. Here one finds sequences of nuclei that are neutron-rich isotopes of Zr and Sn, which have proton numbers $Z = 40$ and 50 respectively. As a rule, the mass number of the nuclei increases with density and lies in the range $100 \leq A \leq 1500$. Since the unbound low-energy neutrons tend to fill a Fermi sphere and their interaction in the $^1S_0$ channel is attractive, they form a superfluid, which is the main object of applications of the theories discussed in Sec. III. The neutron condensate in the inner crust plays a fundamental role in theories of neutron-star cooling (Page et al., 2013; Weber, 1999; Yakovlev et al., 2001) as well as in theories of their rotational dynamics [for a recent review and further references see (Haskell and Sedrakian, 2017)].

The inner crust terminates with a first-order phase transition at the point where the clusters merge together to form a continuum. The new phase, occupying the outer core of the star, is a fluid mixture of neutrons ($n$), protons ($p$), and electrons ($e$), along with muons ($\mu$) appearing at somewhat higher densities.

As depicted in Fig. 19, the actual phase structure of matter that exists in the densest part of the core ($\rho > 3\rho_0$) is uncertain. A number of conjectured phases have been explored. One possibility is hyperonization of matter, which has attracted much attention in recent years. The mechanism driving the hyperonization process is the Pauli exclusion principle: once the Fermi energies of neutrons and electrons (including their rest masses) become of the order of the in-medium masses of $\Sigma^{+\Lambda}$, $\Lambda$, or $\Xi^{0\Xi}$ hyperons, their formation becomes energetically more favorable with increasing density, than further increase in the Fermi energies of the neutron and electron constituents. If the hyperons experience mutual attractive interactions, they will form condensates by the same

---

10 More detailed expositions of the composition and structure of compact stars can be found, e.g., in the texts by Glendenning (2000); Shapiro and Teukolsky (1983) and Weber (1999) and recent review articles (Alford et al., 2008; Blaschke and Chamel, 2013; Lattimer and Prakash, 2004; Oertel et al., 2017; Page et al., 2013; Sedrakian, 2007).
BCS mechanism that operates for non-strange baryons – a possibility that will be considered below.

Depending on the equation of state of the matter, the central densities in the most massive neutron stars can lie in the range 5-10 times \( \rho_0 \), and deconfinement of hadrons into quarks becomes a plausible outcome. Deconfinement may set in after the hyperonization of matter or even before, depending on the unknown density for the deconfinement transition. We will not discuss quark color superconductivity in this review, although it may have profound implications for neutron-star phenomenology; the interested reader is referred to the reviews (Alford et al., 2008; Anglani et al., 2014; Rajagopal and Wilczek, 2001). Medium-modification of meson properties may also lead to their Bose-Einstein condensation and, consequently, their superfluidity. For pions, such condensation can arise through an instability of the particle-hole nucleonic excitations in the medium having pion quantum numbers. The interplay between pion condensation and nucleonic pairing has been covered extensively in the literature; see the reviews (Takatsuka and Tamagaki, 1981, 1993).

Turning to the superfluid phases within neutron stars identified in Fig. 19, we first concentrate on neutron and proton condensates at low (partial) densities (respectively in the inner crust and outer core), for which more reliable computations can be made. We recall the uncertainties involved in the many-body methods outlined in Sec. III, in particular the fact that many-body calculation of pairing gaps are usually simplified by adopting the decoupling approximation, i.e., by computing the single-particle energies in the normal state.

Figure 20 displays the neutron and proton pairing gaps as function of the baryon density, for the composition shown in the same figure. The bands for neutrons are chosen to show the range of reasonable values bounded by actual computations; the upper bound corresponds to the CBF calculations of Fan et al. (2017), whereas the lower bound corresponds to Fermi-liquid computations of Wambach et al. (1993). Neutron S-wave superfluidity occurs at lower densities corresponding to the neutron crust. It is described essentially by the result for pure neutron matter, because at these densities the protons are confined in the nuclear clusters. However, the coupling between the neutron fluid and crustal phonon modes or band structure induced by the lattice can alter the value of the gap (see Sec. V.F for a discussion). The \( ^1S_0 \) neutron gap closes in the vicinity of crust-core interface. In the outer core, where the proton density becomes comparable to the neutron density in the inner crust, the protons also pair in the \( ^1S_0 \) state, with a gap of order of 1 MeV. The upper and lower bounds for protons are shown for computations based on the BCS theory with single-particle renormalization from Brueckner theory. The upper bound is obtained when the three-body (3B) force is neglected in the solution of the gap equation and the equation of state of matter (Baldo et al., 1992b), whereas the lower bound is obtained when the three-body forces are included in both (Baldo and Schulze, 2007). Neutrons in the core pair with much smaller gaps of order of 100 keV. The gap obtained in SCGF theory (Ding et al., 2016) provides the lower bound. The upper bound is based on BCS theory with screening corrections and single-particle spectrum obtained from Brueckner theory (Baldo et al., 1998). Stronger suppression by polarizations effects has been found due to spin-dependence of the effective interaction in neutron matter (Schwenk and Friman, 2004). We will discuss the \( P \)-wave gap computations in more detail in the next subsection.

Figure 21 shows the pairing gaps of protons obtained with the phase-shift equivalent Argonne (Baldo et al., 1992b; Baldo and Schulze, 2007) and Bonn potentials (Elgarøy et al., 1996a). The computations were carried in the BCS approximation by using the bare two-body or two- plus three-body interactions. The single-particle energies in all computations were obtained from the Brueckner-Hartree-Fock theory of nuclear matter. These were then used to obtain the effective mass of the quasi-particles in the gap equation. The results BS and BCLL
FIG. 21 Dependence of the \(1S_0\) pairing gap of protons on their Fermi momentum from BCS theory with phase-shift equivalent interactions: the BS result is based on the Argonne \(V_{18}\) two-body interaction, whereas BS+3B includes in addition the Urbana UIX three-body interaction (Baldo and Schulze, 2007); the EEHO result was obtained with the Bonn two-body interaction (Elgarøy et al., 1996a); BCLL refers to an earlier computation with the Argonne two-body interaction (Baldo et al., 1992b). The vertical line indicates the Fermi momentum corresponding to the crust-core transition.

were obtained with the Argonne \(V_{18}\) two-body interaction; the BS+3B result also includes the Urbana UIX three-body interactions. It is seen that the three-body forces reduce the gap by 25%. If in addition to this the wave-function renormalization given by Eq. (30) is taken into account, no pairing solutions are found (Baldo and Schulze, 2007). The absence of proton superconductivity, if confirmed by future computations, would have profound implications for the physics of compact stars. We also note that in all models the proton pairing gap attains its maximum (almost) at the crust-core boundary; this may have some interesting implications on the type of proton superconductivity (i.e., type-II vs. type-I) throughout the core of the star and, consequently, on the formation of flux tubes vs. superconducting domains across the core (Sinha and Sedrakian, 2015) and references therein.

Clearly, the dependence of gaps on the density of stellar matter could strongly depend on the underlying model that provides the equation of state and the particle fractions. Nevertheless, the general arrangement of the pairing gaps depicted in Fig. 20, such as the transition from \(S\)- to \(P\)-wave neutron pairing around the crust-core interface, and larger proton \(S\)-wave than neutron \(P\)-wave gap in the core, is rather robust.

B. Pairing in higher partial waves

From the preceding overview of neutron-star structure it is clear that, if present, \(^3P_2\) pairing in neutron matter is phenomenologically important, since the neutron fluid in the core of the star occupies a large volume fraction. We now review the computations of this version of triplet pairing, as it involves a number of new aspects relative to \(^1S_0\) pairing (Baldo et al., 1992b, 1998; Khodel et al., 2004, 2001a,b; Maurizio et al., 2014; Papakonstantinou and Clark, 2017; Schwenk and Friman, 2004; Srinivas and Ramanan, 2016; Zverev et al., 2003). In this case there can be competition between states involving various projections \(M\) of the orbital angular momentum \(L\). Additional complications arise from a dominant spin-orbit interaction and the tensor coupling of the \(^3P_2\) partial wave to the \(^3F_2\) state.

To solve the gap equation in the \(P\)-wave channel one starts with the expansion of the pairing interaction in partial waves,

\[
V(p, p') = 4\pi \sum_L (2L + 1) P_L(\hat{p} \cdot \hat{p}') V_L(p, p'),
\]

where the \(P_L\) are Legendre polynomials, and an associated expansion of the gap function

\[
\Delta(p) = \sum_{L,M} \sqrt{\frac{4\pi}{2L + 1}} Y_{LM}(\hat{p}) \Delta_{LM}(p)
\]

in spherical harmonics \(Y_{LM}\), where \(L\) is the orbital quantum number and \(M\) the corresponding magnetic quantum number. It is apparent that the non-linearity of the gap equation couples the various gap components of \(\Delta_{LM}(p)\). Regarding the \(M\) dependence, this problem is simplified by performing an angle average within the denominator of the kernel of the gap equation, by focusing on real solutions and replacing \(\sqrt{\epsilon^2(p) + \Delta^2(p)}\) by \(\sqrt{\epsilon^2(p)^2 + D^2(p)}\), where the “angle-averaged” gap is given by

\[
D^2(p) \equiv \int \frac{d\Omega}{4\pi} \Delta^2(p) = \sum_{L,M} \frac{1}{2L + 1} |\Delta_{LM}(k)|^2
\]

and \(\epsilon(p)\) is a single-particle energy. Such angle-averaging may be performed with confidence in the case of pure \(^3P_2\) pairing for two reasons. First, the near-degeneracy of the five existing real solutions of the gap equation involving different combinations of magnetic substates is a universal feature of this problem (Khodel et al., 1998, 2001b; Takatsuka and Tamagaki, 1993; Zverev et al., 2003). Additionally, numerical solutions of the gap equation for neutron matter generally yield small gap values \(\Delta\) compared to the Fermi energy \(\epsilon_F\). The degeneracy is lifted somewhat by the tensor force in the coupled-channel problem \(^3P_2\)\(^3F_2\), where 13 real solutions arise (Zverev et al., 2003), but the condition \(\Delta/\epsilon_F \ll 1\) still holds.

With this approximation the angular integrals are trivial, and one finds a one-dimensional gap equation for the \(L\)-th component of the gap:

\[
\Delta_L(p) = -\int_0^\infty \frac{dp'}{\pi} \frac{V_L(p, p')}{\sqrt{\epsilon(p')^2 + \sum_L |\Delta_L(p')|^2}} \Delta_L(p').
\]
In a first approximation one may neglect terms in the sum on the right-hand side of (111) having \( L' \neq L \), based on the common assumption that a specific pairing channel is dominant over the density range concerned, an assumption that gains credence from the argument in Sec. II.B relating densities (more precisely \( k_F \) ranges) to in-medium collision energies, and hence to the dominant NN phase shift. However, a “specific pairing channel” may involve coupling of different \( L \) states through spin dependence of the interaction. In the present case there is a strong coupling to the \(^3F_2\) wave due to the tensor components of the pairing force, which must be included to obtain quantitative results. Thus the gap equation to be solved takes the form of coupled equations for \( P \)- and \( F \)-wave components of the gap, \( \Delta \),

\[
\begin{pmatrix}
\Delta_L \\
\Delta_{L'}
\end{pmatrix} = \int_0^\infty \frac{dp'p'^2}{\pi E(p')} \begin{pmatrix}
-V_{LL} & V_{L'L} \\
-V_{L'L'} & V_{L'L'}
\end{pmatrix} \begin{pmatrix}
\Delta_L \\
\Delta_{L'}
\end{pmatrix},
\]

(112)

where \( E(p) = \sqrt{e^2(p) + D^2(p)} \) with \( D^2(p) = |\Delta_L(p)|^2 + |\Delta_{L'}(p)|^2 \) with \( L = P \) and \( L' = F \). Note that this coupling is analogous to that in the \(^3S_1-^3D_1\) channel discussed in Sections II.B and III.B.

Quantitative understanding of \( P \)-wave pairing, or more precisely \(^3P_2-^3F_2\) pairing, is further complicated by the presence of three-body forces, which play an increasingly important role at the high baryon densities of the outer-core region of a neutron star. (At the lower densities where \(^1S_0\) neutron pairing dominates, the three-body force can be safely neglected.) In the case of phase-shift equivalent NN potentials, the models of the three-nucleon (3N) force in use have been constrained by the physics of light nuclei, as is the case for the Urbana UIX 3N interaction (Pudliner et al., 1997). This model has been used in conjunction with the Argonne \( V_{18} \) NN interaction to estimate the \(^3P_2-^3F_2\) pairing gap (Zuo et al., 2008). However, a readjustment of the parameters of the UIX interaction, rendering it less repulsive, was required to guarantee agreement with the empirical nuclear-matter saturation properties within Brueckner-Hartree-Fock theory, which in turn was used to obtain the single-particle spectrum entering the gap equation (Zhou et al., 2004). Within this scheme, the triplet gap was found to be slightly reduced from the result for \( V_{18} \) alone, to a maximum value of the order 0.5 MeV. (We note that upon neglecting the single-particle renormalization, the assumed 3N interaction causes an enhancement of the gap.) The Bonn-B meson-exchange phase-shift equivalent potential including the two- and three-body terms has also been used to estimate the effect of 3N interactions on \(^3P_2-^3F_2\) pairing (Dong et al., 2013). The maximum of the gap in this study is about 0.6 MeV, but is attained at slightly higher densities (higher neutron \( k_F \) values). The authors also considered the effects of wave-function renormalization, as in Eq. (30), and found a suppression of the gap by an order of magnitude.

The \(^3P_2-^3F_2\) gap in neutron matter has also been studied for chiral two-nucleon (2N) and 3N interactions (Maurizio et al., 2014). Assuming a free single-particle energy spectrum, the 3N force was found to produce an enhancement in third order of the chiral expansion. Introduction of the single-particle renormalization again leads to a moderate suppression of the gap, to a maximal value of order 0.4 MeV. In another recent exploratory study of the effects of a 3N interaction (Papakonstantinou and Clark, 2017), strong sensitivity of the \(^3P_2\) gap to the choice of the interactions and the mandatory consistency between the 2N and 3N forces was demonstrated.

In summary, these and other similar computational efforts have shown the importance of including 3N interactions for accurate determination of the pairing gaps in the \(^3P_2-^3F_2\) coupled channel. Consistent extrapolations of the NN and 3N interactions from nuclear saturation to higher densities, such that these forces are properly constrained by empirical data, will be required, along with microscopic many-body theories that are reliable at high density.

This brings us to remaining aspect of the problem of spin-triplet pairing in high density matter that demands further attention, namely the influence of correlations that are not included in BCS theory. Long-range correlations have been found to induce a strong suppression of the gap when non-central Landau interactions are used in conjunction with the weak-coupling formula (Schwenk et al., 1998). GMZ: uses the same theory as GM, but accounts for wave-function renormalization of the quasiparticle spectrum (lower maximum curve) plus three-body forces (higher maximum curve) (Dong et al., 2013). SCGF: self-consistent GF computations including only short-range correlations (higher maximum curve) and both short- and long-range correlations (lower maximum curve) (Ding et al., 2016). The results BCS, GM and SCGF were obtained with the Argonne \( V_{18} \) potential, the GMZ – using Bonn-B potential.
A more recent computation (Ding et al., 2016) indicates that the suppression of pairing by short-range correlations is counteracted by enhancement due to long-range correlations, i.e., these two factors tend to compensate each other.

At high densities, isospin-symmetrical nuclear matter should pair in the $^3D_2$ channel, by forming isospin-singlet pairs, the attractive interaction in this channel being stronger than that in the $^3P_2$ channel, as seen in Fig. 1. In much in the same way as one expects a transition from $^3S_1-^3D_1$ pairing to $^3S_0$ pairing with rising isospin asymmetry, nuclear matter at still higher densities should undergo a transition from $^3D_2$ pairing to $^3P_2-^3F_2$ pairing as isospin asymmetry increases from zero to larger values. However, computations show that already a small imbalance destroys the D-wave pairing in nuclear matter (Alm et al., 1996b), so it can be realized only in nearly symmetrical nuclear matter.

C. Hyperonic pairing

The inner core of a neutron star may contain a hyperonic component, because the rise of neutron and electron energies with density can be compensated by the onset of lower-energy hyperons. Just as was the case with protons, a relatively low fraction of hyperons implies that they will pair in the $^1S_0$ channel if there is a sufficiently attractive component available in their interaction at low energies (Balberg and Barnea, 1998; Raduta et al., 2016; Takatsuka et al., 2001; Takatsuka et al., 2002, 2006; Vidaña and Tolós, 2004; Wang and Shen, 2010; Xu et al., 2014).

We can exclude from the outset the possibility of hyperon-nucleon pairing and pairing between non-identical hyperons, due to the imbalance between the chemical potentials of baryon components of the core [however, see (Zhou et al., 2005)]. In contrast to the purely nucleonic case considered in Sec. IV, additional imbalance will arise because of the substantial disparity in the masses of different hyperons as well the large difference between the masses of nucleons and hyperons. It cannot be ruled out that the Fermi surfaces of non-identical particles may, for some density, be close enough to support cross-species pairing, but this can only occur in a rather limited density range and depends significantly on the underlying model of the composition of matter.

Before discussing the pairing gaps in hypernuclear matter, we need to address the ambient composition and the single-particle energies of hyperons. The complication here is that the non-relativistic theories we have thus far considered fail to account for the measured two-solar mass pulsars (Balberg and Gal, 1997; Baldo et al., 2000; Burgio et al., 2011; Rijken and Schulze, 2016). Relativistic covariant density-functional (hereafter DF) theory allows for modeling of the properties of hypernuclear matter consistent with the astrophysical constraint on masses of hypernuclear compact stars, as well as with laboratory constraints on the depths of potentials in (hyper)nuclear matter. Adopting relativistic DF, the single-particle energies of hyperons (collectively denoted $Y$ below) are expressed as

$$E^Y(k) = \sqrt{k^2 + m^2_Y + g_\omega Y \omega + g_\phi Y \phi + g_\rho Y \tau_3 Y \rho + \Sigma_R},$$

(113)

where $\Sigma_R$ represents the rearrangement term entering the models with density-dependent couplings, $\mu_Y = E^Y(k_F)$ stands for the chemical potential, and $m^*_Y = m_Y - g_{\sigma Y} \sigma - g_{\rho Y} \rho \tau_3$. $\sigma$, $\rho$, $\omega$, and $\phi$ refer to the mesonic fields, while the $g_{\alpha Y}$ with $\alpha \in (\sigma, \sigma^*, \rho, \omega)$ are the hyperon-meson couplings.

A method for computing the pairing gaps in relativistic DF theories that has been validated in studies of finite nuclei within relativistic Hartree-Fock-Bogolyubov theory is based on solution of the non-relativistic BCS equation for a given two-nucleon potential using single-particle energies and particle composition computed by the relativistic DF method (Kucharek and Ring, 1991). While there is a clear inconsistency in treating the unpaired matter and pairing correlations in different theories, this approach is close in the spirit to the decoupling approximation widely applied in the non-relativistic theories discussed previously. In the BCS approximation, it becomes straightforward to solve the gap equation for the pairing of a given type of hyperons. It can be written as

$$\Delta_Y(k) = -\frac{1}{4\pi^2} \int dk' k'^2 \frac{V_{YY}(k,k') \Delta_Y(k')}{\sqrt{(E^Y(k')-\mu_Y)^2 + \Delta_Y^2(k')}}.$$

(114)

where $E^Y(k)$ is the single-particle energy of hyperon $Y$ given by Eq. (113). The pairing interaction, in the BCS approximation, is given by the interaction in the $^1S_0$ channel,

$$V_{YY}(k,k') = 4\pi \int dr r^2 j_0(kr) V_{YY}(r) j_0(k'r),$$

(115)

where $j_0(kr) = \sin(kr) / (kr)$ is the spherical Bessel function of order zero and $V_{YY}(r)$ is the $YY$ interaction in coordinate space.

Recently, hypernuclear pairing in compact stars was addressed in the context of their cooling (Raduta et al., 2018) using the strategy outlined above. For the $\Lambda\Lambda$ pairing interaction, the configuration space parameterization of the ESC00 potential (Rijken, 2001) proposed by Filikhin and Gal (2002) was adopted. For the $\Xi^-\Xi^-$

11 The fundamentals of covariant density functional theory for nuclear systems is discussed, for example, in (Glendenning, 2000; Meng, 2016; Serot and Walecka, 1997; Weber, 1999). Recent applications of covariant density functional theory to hypernuclear matter are reviewed in Chatterjee and Vidaña (2016); Oertel et al. (2017).
and $\Xi^0\Xi^0$ interactions, a model presented in (Garcilazo et al., 2016) was selected, specifically the one that corresponds to the Nijmegen Extended Soft Core potential ESC08c (Rijken et al., 2013). These potentials were chosen to maximize the attraction in the respective channels, in order to obtain an upper bound on the pairing gap within BCS theory.

Figure 23 shows the dependence of the pairing gaps for $\Lambda$ and $\Xi^{-,0}$ hyperons on the baryon density (Raduta et al., 2018). Because this involves input for the composition of matter and self-energy effects, the results depend on the chosen DFs, which are labeled as DDME2 (Fortin et al., 2016), GM1A (Gusakov et al., 2014), and SWL (Spinella, 2017). It is interesting that the $\Lambda$ pairing is restricted to densities $n_b \leq 0.55$ fm$^{-3}$; therefore at higher densities there may exist regions of unpaired $\Lambda$ matter in the most massive stars, provided there is no significant attraction and therefore pairing in higher partial waves. The $\Xi^-$ hyperons remain paired up to the highest densities considered in (Raduta et al., 2018).

D. Overview of neutrino radiation from compact stars

Theoretical modeling of the thermal evolution (cooling) of neutron stars tests neutron-star interior composition predicted by microscopic theories of dense matter. Such models are confronted with observations of the X-ray emission from the surfaces of nearby neutron stars. The cooling evolution is roughly divided into three stages (Page et al., 2013; Sedrakian, 2007; Weber, 1999; Yakovlev et al., 2001): (a) There is an initial transient stage during which the core temperature drops to about 0.1 MeV ($1$ MeV $= 1.16 \times 10^{10}$ K); the subsequent thermal history of the star does not depend on this initial stage. (b) The neutrino cooling era lasts for $t \lesssim 10^5$ yr, the main cooling mechanism being neutrino radiation from the stellar interior. This phase controls theoretical predictions of the surface temperatures of the observed thermally emitting stars. The importance of studies of the thermal history of neutron stars lies in its strong dependence on the neutrino emission rates from dense matter during the neutrino cooling era. These rates, in turn, depend crucially on the particle content and superfluidity of the interior components. (c) The photon cooling era $t \gtrsim 10^5$ yr is dominated by radiation of photons from the surface of the star and heating due to the dissipation of rotational and magnetic energy (González-Jiménez et al., 2015; Schaab et al., 1999).

In this subsection, we briefly review the main processes responsible for neutrino radiation. More detailed surveys can be found in (Page et al., 2013; Schaab et al., 1996; Schmitt and Shternin, 2017; Sedrakian, 2007; Yakovlev et al., 2001). In particular, we will focus on the effects of superfluidity on these processes to set the stage for a more detailed discussion of microscopic calculations of neutrino emission rates from superfluid matter in the next section.

The various processes or reactions generating neutrinos can be classified according to the number of fermions involved (Shapiro and Teukolsky, 1983). The rationale of this classification is that each fermion participating in such a reaction introduces a factor $T/\epsilon_F$, which is a small parameter when the temperature is much less than the Fermi energy involved. Thus, the processes of leading order in powers of $T/\epsilon_F$ are given by

\[ n \rightarrow p + e + \bar{\nu}_e, \quad p + e \rightarrow n + \nu_e, \quad (116) \]
\[ N \rightarrow N + \nu_f + \bar{\nu}_f \quad (\text{forbidden}), \quad (117) \]

where $N \in (n,p)$ refers to a nucleon, $\nu$ and $\bar{\nu}$ to neutrino and antineutrino and index $f = e, \mu, \tau$ to neutrino flavors. In dealing with the quasiparticle states of nucleons having an infinite lifetime, the rates of these reactions are constrained kinematically. The second process (117), known as neutral-current neutrino pair bremsstrahlung, is forbidden by energy and momentum conservation. The Urca reaction (116) is kinematically allowed in the matter under $\beta$-equilibrium if the proton fraction $Y_p \gtrsim 11 - 14\%$ (Boguta, 1981; Lattimer et al., 1991). The processes having two baryons in the initial (and final) states obtained from (116) and (117)
by adding a nucleon, \( i.e., \)
\[
N + n \rightarrow N + p + e + \bar{\nu}_f, \tag{118}
\]
\[
N + N' \rightarrow N + N' + \nu_f + \bar{\nu}_f, \tag{119}
\]
are allowed kinematically but are suppressed by an extra factor \((T/\epsilon_F)^2\). Estimates of emissivity (power of energy radiated per unit volume) for the three relevant processes above are \(\tilde{\epsilon}_{\text{Urca}} \sim 10^{27} \times T_9^4\) for reaction (116), \(\tilde{\epsilon}_{\text{mod. Urca}} \sim 10^{21} \times T_9^4\) for reaction (118) and its inverse, and \(\tilde{\epsilon}_{\nu\nu} \sim 10^{19} \times T_9^4\) for reaction (119). Here \(T_9\) is the temperature in units \(10^9\) K. In addition to these nucleonic processes, bremsstrahlung by electrons scattering off nuclei and impurities in the crust contributes to the neutrino radiation, but we will not discuss these mechanisms as they are independent of the baryonic superfluidity.

The hyperon-rich matter will emit neutrinos via the hyperonic Urca processes (Prakash et al., 1992)
\[
\Lambda \rightarrow p + l + \bar{\nu}_l, \tag{120}
\]
\[
\Sigma^- \rightarrow \left( \begin{array}{c} n \\ \Lambda \\ \Sigma^0 \end{array} \right) + l + \bar{\nu}_l, \tag{121}
\]
\[
\Xi^- \rightarrow \left( \begin{array}{c} \Xi^0 \\ \Sigma^0 \end{array} \right) + l + \bar{\nu}_l, \tag{122}
\]
\[
\Xi^0 \rightarrow \Sigma^+ + l + \bar{\nu}_l, \tag{123}
\]
where \(l\) stands for a lepton, either electron or muon, and \(\bar{\nu}_l\) is the associated antineutrino. The hyperonic Urca thresholds are much lower than those for nucleons. Once the relative abundances of hyperons exceed a few percent, the hyperonic Urca processes are kinematically allowed. Consequently, the reactions (120)-(123) will operate provided the relevant species of hyperons are present in matter. Since the hyperon abundances increase rapidly once they become energetically favorable, the corresponding threshold densities practically coincide with the onset densities for hyperons.

For completeness, we point out here that pion or kaon BECs will radiate via the reactions
\[
N + \langle \pi^- \rangle \rightarrow N + e^- + \bar{\nu}_e, \tag{124}
\]
\[
N + \langle K^- \rangle \rightarrow N + e^- + \bar{\nu}_e, \tag{125}
\]
where \(\langle \pi^- \rangle\) and \(\langle K^- \rangle\) refer to the pion and kaon coherent states. The corresponding emissivities are large compared to those of the baryonic processes above, but the BEC of pions and kaons is not guaranteed to occur in neutron stars; their discussion is beyond the scope of this review (Migdal et al., 1990; Muto et al., 2003; Tatsumi, 2013). Finally, once deconfinement takes place, the quark cores of neutron stars will radiate neutrinos predominantly through the quark counterparts of the Urca process (116) (Alford et al., 2008; Anglani et al., 2014; Rajagopal and Wilczek, 2001).

Neutrino radiation is suppressed once superfluid phases are formed in neutron stars, exponentially at asymptotically low temperature by a Boltzmann factor \(\exp[-\Delta_{\text{max}}(0)/T]\) for processes of the Urca type (116), where \(\Delta_{\text{max}}(0)\) is the largest of the gaps of nucleons. A more detailed discussion of this suppression will be given below in Sec. V.G. Similarly, the processes (118) and (119) are suppressed at low temperatures by factors \(\exp[-\Delta_N(0) + \Delta_{N'}(0)/T]\), where \(N\) and \(N'\) label the pair of initial (or final) baryons.

Somewhat counterintuitively, superfluidity opens a new channel of neutrino emission, which is due to the process of neutrino-pair bremsstrahlung via the neutral-current Cooper pair-breaking and formation (PBF) process:
\[
\{NN\} \rightarrow N + N + \nu_f + \bar{\nu}_f, \quad N + N \rightarrow \{NN\} + \nu_f + \bar{\nu}_f, \tag{126}
\]
where \(\{NN\}\) stands for a nucleonic Cooper pair. The neutrino emission by these two reactions was computed initially neglecting vertex corrections (Flowers et al., 1976; Kaminker et al., 1999; Voskresensky and Senatorov, 1987), which were considered later in a series of works (Kolomeitsev and Voskresensky, 2008, 2010; Leinson and Pérez, 2006; Leinson and Pérez, 2012; Sedrakian et al., 2007; Steiner and Reddy, 2009).

The first reaction in (126) can be viewed as a breakup of nucleonic Cooper pairs that is accompanied by neutrino-pair emission. The second reaction involves recombination of nucleonic excitations into a Cooper pair. Although at low temperatures the PBF processes are again suppressed exponentially, they are very effective in cooling neutron stars at temperatures not far below the critical temperature \(T_c\) of a relevant nucleonic condensate (Page et al., 2009; Schaab et al., 1997). These processes are operative also in hyperonic condensates, as discussed below.

### E. Pair-breaking and formation processes

We now turn to a more detailed description of the PBF processes introduced in the previous section. The initial calculations, which did not include vertex corrections, led to the conclusion that the neutrino emission via neutral vector currents is large compared to that via axial vector currents (Flowers et al., 1976; Kaminker et al., 1999; Voskresensky and Senatorov, 1987). More recent work has shown that the vertex corrections substantially suppress the emission via vector currents while they leave the axial vector emission unaffected (Kolomeitsev and Voskresensky, 2008, 2010; Leinson and Pérez, 2006; Sedrakian, 2012; Sedrakian et al., 2007; Steiner and Reddy, 2009). Accordingly, the axial vector current emission is the dominant one.

The low-energy neutral weak-current interaction Lagrangian describing the interaction of the neutrino field
ψ and baryonic current \( j_\mu \) is given by
\[
\mathcal{L}_W = - \frac{G_F}{\sqrt{2}} j_\mu \bar{\psi}_B \gamma^\mu (1 - \gamma^5) \psi, \tag{127}
\]
where \( G_F \) is the Fermi coupling constant. The baryon current for each \( B \)-baryon is
\[
\bar{\psi}_B \gamma_\mu (c^{(B)}_V - c^{(A)}_A) \gamma^\mu \psi_B, \tag{128}
\]
where \( \bar{\psi}_B \) is the quantum field of the baryon and \( c^{(B)}_V \) and \( c^{(A)}_A \) are its vector and axial vector couplings, respectively.

The rate at which neutrinos are radiated from matter (the neutrino emissivity) can be obtained either by using the optical theorem in finite-temperature field theory (Voskresensky and Senatorov, 1987) or directly from the kinetic equation for neutrinos formulated in terms of real-time propagators (Sedrakian and Dieperink, 2000). Both methods lead to the following expression for the neutral-current neutrino pair bremsstrahlung emissivity
\[
\varepsilon_{\nu\bar{\nu}} = -2 \left( \frac{G_F}{\sqrt{2}} \right)^2 \int d^4 q \ g(\omega) \omega \sum_{i=1,2} \int \frac{d^3 q_i}{(2\pi)^3 2\omega_i} \times \text{Im}[L^{\mu\lambda}(q_i) \Pi_{\mu\lambda}(q)] \delta^{(4)}(q - \sum_i q_i), \tag{129}
\]
where \( q_i = (\omega_i, q_i) \), with \( i = 1, 2 \), are the neutrino momenta, \( g(\omega) = [\exp(\omega/T) - 1]^{-1} \) is the Bose distribution function, \( \Pi_{\mu\lambda}(q) \) is the retarded polarization tensor of baryons, and
\[
L^{\mu\nu}(q_1, q_2) = 4 \left[ q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - (q_1 \cdot q_2) g^{\mu\nu} - i\epsilon^{\alpha\beta\mu\nu} q_1^\alpha q_2^\beta \right] \tag{130}
\]
is the leptonic trace. Here the emissivity is defined per neutrino flavor, i.e., the full rate of neutrino radiation through weak neutral currents is larger by a factor \( N_f \), the number of neutrino flavors. We will assume \( N_f = 3 \) massless neutrino flavors.

The central quantity in (129) is the polarization tensor \( \Pi_{\mu\lambda}(q) \), which describes the response of the superfluid to electroweak vector and axial currents. A microscopic approach for calculating the response function (or polarization tensor) in superfluid matter was first developed by Abrikosov et al. (1963) in the context of the electrodynamics of superconductors. In this theory, the response of superconductors to external probes is expressed in the language of propagators at non-zero temperature and density, with contact interactions that do not distinguish among the particle-hole and particle-particle channels. It is equivalent to the theories initially advanced for metallic superconductors (Anderson, 1958; Bogolyubov et al., 1958), which are based on equations of motion for second-quantized operators. A more general approach was subsequently developed within the Fermi-liquid theory for superconductors and superfluids (Larkin and Migdal, 1964; Leggett, 1966). The latter method implements wave-function renormalization of the quasiparticle spectrum and higher-order harmonics in the interaction channels, and postulates particle-hole (ph) and particle-particle (pp) interactions having different strengths and/or signs.

Computation of the polarization tensor proceeds in three steps. In the first step, one solves the coupled integral equation for the three-point vertices (shown in Fig. 24) in the superfluid matter. Next, the four polarization tensors shown in Fig. 25 are resummed to obtain the full response function. Finally, this function is expanded, to first non-vanishing order, in the small parameter \( v_F/c \ll 1 \), where \( v_F \) is the Fermi velocity of nucleons and \( c \) is the speed of light. In the case of vector-current response, a non-zero contribution is obtained at order \( v_F^2 \), whereas in the case of axial vector coupling, one finds a non-zero contribution at order \( v_F^4 \) (here and below we again set \( c = 1 \)). Next, the phase-space integrals in the emissivity (129) are computed after contracting the polarization tensor with the trace over leptonic currents. The final result for three neutrino flavors (\( N_f = 3 \)) can be cast in the form (Kolomeitsev and Voskresensky, 2008; Leinson and Pérez, 2006; Sedrakian, 2012; Steiner and...
neutrons, being confined to a narrow band ∼ from the dimensional analysis (Shapiro and Teukolsky, 1983). The temperature dependence of the pair-breaking processes can be understood from the dimensional analysis (Shapiro and Teukolsky, 1983). First, we observe that the initial- and final-state neutrons, being confined to a narrow band ∼ T around the Fermi surface, each contribute a factor T, while the final-state neutrino and antineutrino contribute each a factor T^3. Energy and momentum conservation provide an additional factor T^{-2} (the momentum exchange being thermal because of thermal neutrinos). Another factor T arises due to the fact that one computes the energy production rate.

The rate as given by Eqs. (133) and (134) is applicable for the 1S0 neutron condensate in the neutron-star crust and the proton condensate in the core of the star. A calculation similar to that described above can be carried out for the 3P_{2\rightarrow 3}F_{2} condensate (Leinson, 2013; Yakovlev et al., 1999). The main difference from the S-wave case is that the leading-order contribution arises already at first order (∝ 1) in the small-v_F expansion, with other factors and temperature dependence being the same.

The hyperonic S-wave condensates introduced above will also contribute to the neutrino radiation through pair-breaking processes, in full analogy to their nucleonic counterparts. These processes were initially considered without vertex corrections (Jaiikumar and Prakash, 2001; Yakovlev et al., 1999) and revised later to account for them in (Raduta et al., 2018). The inclusion of vertex corrections implies that the contribution from vector-current coupling is negligible for S-wave paired hyperons, compared to that from the axial-vector coupling. By the same argument as made for nucleons, the former contribution scales as v_F^4, whereas the latter scales as v_F^2. This last contribution is given, in analogy with the nucleon case, by Eqs. (133) and (134), but with nucleonic quantities replaced by their hyperonic analogues (Raduta et al., 2018) including the electro-weak coupling constants given in (Savage and Walden, 1997).

F. Collective modes and entrainment

The set of polarization tensors shown in Fig. 25 also determines the collective modes of the fermionic superfluid. Indeed the vector and axial-vector responses are associated in the non-relativistic limit with the vertices

\[ \Gamma_{0}^{D\mu} = (1, v_F), \quad \Gamma_{0}^{S\mu} = (\sigma \cdot v_F, \sigma), \]

which are the same as for the density and density-current (subscript D) and the spin-current and spin-density perturbations (subscript S); here μ is the Dirac index. By definition, the dispersion relations of the collective modes are obtained from the poles of the polarization tensor (Larkin and Migdal, 1964; Leggett, 1966). For single-component neutral superfluids, one finds two branches of density modes: the Anderson-Bogolyubov mode, which is an acoustic mode having dispersion \( \omega = c_s k \), with \( \omega \) and \( k \) the mode energy and momentum. At low temperatures, the mode velocity is given by

\[ c_s = \frac{v_F}{\sqrt{3}} (1 + F_1)^{1/2}, \]

where \( v_F \) is the Fermi velocity and \( F_1 \) is the l = 1 Landau parameter defined in Eq. (40). The second mode
is the so-called Higgs mode and has a finite threshold as $k \to 0$ of the order of the pair-breaking energy $2\Delta$. These modes in the nuclear matter were studied long ago (Larkin and Migdal, 1964), but the case of neutron matter below saturation density has been considered only recently (Keller and Sedrakian, 2013b; Kolomeitsev and Voskresensky, 2011; Martin and Urban, 2014).

An approximation based on pure neutron matter may not be accurate in a number of problems associated with the physics of neutron star crusts. The lattice of nuclei in which the neutron fluid is embedded affects its properties in a number of ways. The elementary excitations of the lattice are phonons, and they will affect the neutron spectrum via neutron-phonon coupling contributing to the neutron effective mass as well as to the pairing interaction (Sedrakian, 1996). Furthermore, the Anderson-Bogolyubov mode (136) of the neutron superfluid couples to the phonon modes of the lattice, in which case there is a mixing among the modes (Chamel et al., 2013; Cirigliano et al., 2011; Urban and Oertel, 2015).

Of particular interest for the phenomenology of neutron stars is the hydrodynamical limit, in which case a crustal layer can be considered as a two-fluid system consisting of the unbound neutron superfluid and plasma comprising crustal nuclei and electrons (Carter et al., 2006; Chamel, 2012, 2013, 2017; Martin and Urban, 2016; Watanabe and Pethick, 2017). The normal fluid is locked into the motion of the star by the magnetic field. Its identification is not unambiguous: there are neutrons inside and outside of nuclear clusters, and there is a “transfusion” from one to another under non-stationary conditions. The coupling between the neutron superfluid (labeled $n$) and crustal plasma (labeled $p$) is reflected at the hydrodynamical level in the entrainment effect, which states that the (mass) currents $\mathbf{p}$ of the fluids are given by

$$
\mathbf{p}_n = \rho_{nn} \mathbf{v}_n + \rho_{np} \mathbf{v}_p, \quad \mathbf{p}_p = \rho_{np} \mathbf{v}_n + \rho_{pp} \mathbf{v}_p,
$$

(137)

where $\rho_{nn}$, $\rho_{pp}$ are diagonal and $\rho_{np} = \rho_{pn}$ are off-diagonal densities, which form a $2 \times 2$ entrainment matrix and $\mathbf{v}_i \equiv (1/2m_i) \nabla \phi_i - (e_i/m_i) \mathbf{A}$, with $i \in n, p$, where $\phi_i$ is the phase of the pairing amplitude, $\mathbf{A}$ is the vector potential, and $e_i$ and $m_i$ stand for the charge and the mass of component $i$. Note that the vector $\mathbf{v}$ transforms as a co-vector and should not be confused with the proper velocity of the fluid, which is a contravariant vector (Carter et al., 2006; Chamel, 2012). The off-diagonal terms $\rho_{np} = \rho_{pn}$ account for the fact that the mass-current of a given component is not aligned with the gradient of the phase amplitude.

Before proceeding, we note that entrainment was originally introduced in the context of mixtures of superfluid phases of He (Andreev and Bashkin, 1976). It was then applied in the context of neutron stars to describe the mixtures of neutron and proton superfluids in the star’s core (Alpar et al., 1984; Sedrakyan and Shakhabasyan, 1980; Vardanyan and Sedrakyan, 1981). See the discussion in Sec. VI.D. The elements of the entrainment matrix are related to each other by Galilean invariance, so it is sufficient to determine, for example, only the ratio $\rho_{pn}/\rho_{pp}$, known as the entrainment coefficient. Relations (137) demonstrate that static computations of the density of neutrons outside of nuclear clusters cannot be used as a measure of the density of the superfluid neutron component.

The classical hydrodynamical interaction between neutron superfluid and crustal nuclei was studied extensively (Epstein, 1988; Magierski and Buliga, 2004; Martin and Urban, 2016; Sedrakian, 1996). One considers the flow of a neutron superfluid past a nucleus, which induces a backflow and endows the nucleus with a hydrodynamic mass. The amount of “free” (i.e., moving with velocity $v_n$) neutron superfluid determined in this manner can be expressed through the ratio (Chamel, 2017; Martin and Urban, 2016)

$$
\frac{\rho_{nn}}{\rho_n} = 1 + 3 \frac{V_A}{V_{cell}} \frac{\delta - \gamma}{\delta + 2\gamma},
$$

(138)

where $\gamma$ is the density ratio of the neutrons outside and inside of the nucleus in the static limit, $\delta$ is the fraction of superfluid neutrons within a nuclear cluster of volume $V_A$, and $V_{cell}$ is the volume of the Wigner-Seitz cell. Consider for illustration the case $\delta = 0$, which corresponds to the limit of an impenetrable cluster; then the ratio (138) is independent of $\gamma$ and one finds a lower bound $\rho_{nn}/\rho_n = 1 - (3/2)(V_A/V_{cell})$ with $V_A/V_{cell} \ll 1$. We see that $\rho_{nn} \simeq \rho_n$, i.e., the entrainment is weak, which is confirmed by detailed computations (Martin and Urban, 2016). However, band-structure calculations (analogous to those in the theory of solids) of the same quantity predict a strong entrainment, with the density of superfluid neutrons reduced by an order of magnitude (Chamel, 2012, 2013, 2017). Such quantum mechanical calculations are more reliable than their hydrodynamical counterparts, which assume that the coherence length of the neutron superfluid is much smaller than other scales in the problem, in particular, the size of the nucleus. While these results were obtained in the limit where the pairing can be neglected compared to other scales, specifically the depth of the lattice potential, a semi-analytical model (Watanabe and Pethick, 2017) which includes pairing correlations suggests a rather weak entrainment. It is in the range predicted by the hydrodynamical models. The depth of the potential in this study is of the order of the pairing gap, so it cannot be neglected. Further studies of this problem are needed in order to resolve this discrepancy. Phenomenologically, strong entrainment would imply that there is not enough moment of inertia in the superfluid component of the crust to account for pulsar glitch dynamics (Andersson et al., 2012; Chamel, 2013; Watanabe and Pethick, 2017).

In the high-density region, corresponding to the quantum-liquid core of the star, the nature of superfluid modes is affected by the fact that there are plasma modes associated with the oscillations of the proton-electron component. Moreover, there is a coupling between the
neutron and proton fluids that have different Fermi velocities and gaps (Baldo and Ducoin, 2009, 2017). In addition, the $P$-wave nature of the neutron superfluid admits modes associated with the spin of the Cooper pairs (Bedaque and Nicholson, 2013; Bedaque et al., 2015; Bedaque and Reddy, 2014; Bedaque et al., 2003b; Leinson, 2011, 2012). The small-amplitude hydrodynamical oscillation modes of the fluids (in addition to the usual first and second sound) display new modes which arise due to the entrainment and coupling to the plasma oscillations of the electron-proton component (Epstein, 1988; Vardanyan and Sedrakyan, 1981).

Phenomenologically, as new degrees of freedom, the collective modes contribute to the thermodynamics of the superfluid. The specific heat is of particular interest for cooling of neutron stars (di Gallo et al., 2011; Keller and Sedrakian, 2013b; Martin and Urban, 2014). The collective modes (phonons) can lose energy by neutrino emission (Bedaque et al., 2003b; Leinson, 2011) and can contribute to the transport (Aguilera et al., 2009; Mannarelli et al., 2013; Manuel et al., 2014; Manuel and Tolos, 2011).

G. Urca process in superfluid phases

Urca processes involving nucleons (116) or hyperons (120)-(123) are operative at high densities for many models of the equation of state of dense matter. Consequently, it is important to understand how the rates of these processes are affected by superfluidity of the baryons. At asymptotically low temperatures $T \ll \min(\Delta_n, \Delta_p)$, where $\Delta_{n/p}$ are the neutron/proton gaps, the neutrino radiation is suppressed by a Boltzmann factor $\exp(-\Delta_{\text{max}}/T)$, with $\Delta_{\text{max}} = \max(\Delta_n, \Delta_p)$ the larger of the neutron and proton gaps (Lattimer et al., 1991). However, a substantial part of the neutrino cooling of a neutron star occurs in the temperature range $0.2 \leq T/T_c \leq 1$, where $T_c$ is the relevant critical temperature for either neutron or proton pairing. Hence a more accurate description of the suppression of Urca processes is required. Because pairing mainly affects the phase space of nucleons, an initial step is to introduce the BCS spectrum in the distribution functions of nucleons when computing the rate of the Urca process (Levenfish and Yakovlev, 1994). In principle, the matrix element of the process is also modified because one is dealing with a coherent state composed of a superposition of particles and holes as expressed by the coherence factors $u_p/v_p$. It is convenient to carry out the computation using the Green functions for baryons (Sedrakian, 2005, 2007). To lowest order (i.e., neglecting the vertex corrections discussed above) the one-loop contribution to the Urca process is given by the diagram shown in Fig. 27. A new feature in such computation that keeps $u_p \neq 1$ and $v_p \neq 0$, is the emergence of the pair-breaking process in the Urca channel. The polarization tensor computed from the diagram in Fig. 27 contains contributions not only from the scattering processes $\propto \left[ f_n(p) - f_p(p + q) \right]$, where $f_{p/n}$ are the proton and neutron distribution functions and $q$ is the momentum transfer, but also from processes $\propto \left[ 1 - f_n(p) - f_p(p + q) \right]$ that are due to the breaking of neutron and proton Cooper pairs. Close to the critical temperature $0.5 \leq T/T_c \leq 1$ the scattering contribution is dominant, but at lower temperatures, the pair-breaking contribution becomes comparable to the scattering contribution, without changing cooling behaviour qualitatively.

Neutron stars are seismically active bodies. Density oscillations (more specifically first sound in a superfluid) can induce variations in the chemical potentials of species which can modify the Urca process rate in the superfluid phases. Large enough density oscillations can displace the Fermi seas of nucleons and bridge the gap (Alford and Pangeni, 2017; Alford et al., 2012b). This superthermal effect may strongly enhance the rate of the Urca process in the superfluid up to levels comparable to that of the normal state. In hadronic matter, the relative amplitudes of the density oscillations required for this effect to be operative are of the order of $\Delta n/n \sim 10^{-3}$. Consequently, an (unstable) growth of oscillation amplitude in a superfluid can saturate due to the dissipation of the energy of oscillations via neutrino emission (Alford et al., 2012a). Out-of-equilibrium Urca processes in the superfluid phases are also important for understanding the coupled rotational and chemical evolution of neutron stars (González-Jiménez et al., 2015; Petrovich and Reisenegger, 2011).

H. Axion radiation from superfluid phases

Superfluid phases of neutron stars may radiate not only the three neutrino flavors encountered in the Standard Model (SM), but hypothetical particles that have been conjectured in various extension of the SM. Confrontation of theoretical tracks of neutron star cooling with measurements of X-ray flux from suitable neutron-star candidates thus can constrain the properties of such particles and their coupling to the SM sector. We discuss this possibility using the specific example of axions, which were originally introduced by Wilczek (1978) and Weinberg (1978) to solve the strong-CP problem in QCD (Peccei and Quinn, 1977; 't Hooft, 1976).

Stellar physics has indeed been widely used to put con-
strains on the models of particle physics beyond SM. As non-SM particles can be produced in stellar environments, they can contribute to transport and losses of energy. This allows setting constraints on the strength of coupling of these particles to SM matter, by requiring that their existence does not introduce contradictions in estimates of stellar lifetimes and energy-loss rates (Gianotti et al., 2017; Raffelt, 1996, 2008). This kind of astrophysical limit was obtained from the physics of the Sun, red giants and horizontal-branch stars in globular clusters, white dwarfs, neutron stars and from the duration of the neutrino burst of the supernova SN1987A (Olive et al., 2014). In the case of neutron stars, we need to assume that the axion emission, which carries additional energy away from the stellar interior, does not significantly alter the agreement between theoretical cooling models and observations.

The computation of the pair-breaking process

\[ \{NN\} \rightarrow N + N + a, \quad N + N \rightarrow \{NN\} + a, \]

(139)

involving the emission of an axion \( a \), is analogous to the axial-current neutrino emission, since the axion couples to the nucleonic axial current; the required response function is represented by Fig. 26 where now instead of a \( Z_0 \) gauge boson, an axion is attached to the nucleonic loop.

To set the notation, we start with the interaction Lagrangian

\[ \mathcal{L}_{int}^{(B)} = \frac{1}{f_a} B^\mu A_\mu, \]

(140)

in which \( f_a \) is the axion decay constant, and the baryon and axion currents are given by

\[ B^\mu = \sum_N \frac{C_N}{2} \bar{\psi}_N \gamma^\mu \gamma_5 \psi_N, \quad A_\mu = \partial_\mu a, \]

(141)

where \( C_N \) is the Peccei-Quinn (PQ) charge of a baryonic current and we denote nucleons collectively by \( N \in n, p \). The dimensionless Yukawa coupling can be defined as

\[ g_{aNN} = C_N m_N/f_a, \]

from which it follows that the axionic “fine-structure constant” is \( \alpha_{aNN} = g_{aNN}^2/4\pi \).

The charges introduced above are given by generalized Goldberger-Treiman relations

\[ C_p = (C_u - \eta) \Delta_u + (C_d - \eta z) \Delta_d + (C_s - \eta w) \Delta_s, \]

(142)

\[ C_n = (C_u - \eta) \Delta_d + (C_d - \eta z) \Delta_u + (C_s - \eta w) \Delta_s, \]

(143)

where \( \eta = (1 + z + w)^{-1} \), with \( z = m_u/m_d, w = m_s/m_d \) and \( \Delta_u = 0.84 \pm 0.02, \Delta_d = -0.43 \pm 0.02, \) and \( \Delta_s = -0.09 \pm 0.02 \). The main uncertainty is associated with \( z = m_u/m_d = 0.35 \pm 0.6 \). While there are numerous models of axions, a particularly useful model is the hadronic axion model (Kim, 1979; Shifman et al., 1980) with \( C_{u,d,s} = 0 \); in this model, the nucleonic charges vary in the range

\[ -0.51 \leq C_p \leq -0.36, \quad -0.05 \leq C_n \leq 0.1. \]

The axion mass is related to \( f_a \) by

\[ m_a = \frac{z^{1/2} f_a m_\pi}{1 + z} \frac{\text{eV}}{f_a/10^3 \text{ GeV}} \]

(145)

in terms of the pion mass \( m_\pi = 135 \text{ MeV} \) and the decay constant \( f_\pi = 92 \text{ MeV} \), having adopted the value \( z = 0.56 \) from the range of \( z \) values quoted. Equation (145) translates a lower bound on \( f_a \) to an upper bound on the axion mass.

Computations analogous to those for neutrinos lead to the axion emissivity from \( S \)-wave condensates (Keller and Sedrakian, 2013a),

\[ \epsilon_{aNN}^S = \frac{2C_N^2}{3\pi} f_a^{-2} v_N(0) v_{FN}^2 T^5 I_{aNN}^S, \]

(146)

where

\[ I_{aNN}^S = z_N^5 \int_1^{\infty} dy \frac{y^3}{\sqrt{y^2 - 1}} f_F^2 (z_N y) \]

(147)

and \( z_N = \Delta_N^S(T)/T \); here \( \Delta_N^S \) refers to the \( S \)-wave \( N \)-nucleonic gap. A qualitative bound on \( m_a \) can now be obtained by requiring that the axion cooling does not overshadow the neutrino cooling (which is assumed to be dominated by the \( S \)-wave neutrino radiation), i.e.,

\[ \frac{\epsilon_a^S}{\epsilon_{\nu\nu}^S} = \frac{59.2C_N^2}{4f_F^2 G_F^2 \Delta_N^S(T)^2} r(z) \leq 1, \]

(148)

where \( r(z) \) is the ratio of the phase-space integral for axions (147) to its counterpart (134) for neutrinos and is numerically bound from above by \( r(z) \leq 1 \). Hence this factor can be dropped from the bound on \( f_a \). Substituting into Eq. (148) the value of the Fermi coupling constant \( G_F = 1.166 	imes 10^{-5} \text{ GeV}^{-2} \), we may convert this bound to

\[ \frac{f_a/10^{10} \text{ GeV}}{C_N} > 0.038 \left[ \frac{1 \text{ MeV}}{\Delta_N^S(T)} \right]. \]

(149)

Using Eq. (145), this translates to an upper bound on the axion mass of

\[ m_a \leq 0.163 \text{ eV} \left[ \frac{\Delta_N^S(T)}{1 \text{ MeV}} \right]. \]

(150)

The nucleon pairing gap on the right-hand side can, in fact, be replaced by the critical temperature \( T_c \), because in the temperature range which is important for pair-breaking processes, i.e., \( 0.5 \leq T/T_c < 1 \), BCS theory predicts \( \Delta(T) \simeq T_c \).

As explained previously, the neutron condensate in neutron-star cores is paired in the \(^3\!P_2–^3\!F_2\) channel, i.e., in a state which features an anisotropic gap (Zverev et al.,
The corresponding axion emissivity is found to be (Sedrakian, 2016)

$$\epsilon_{an}^P = \frac{2C_n^2}{3\pi} f_{a}^{-2}\nu_n(0) T^5 I_{an}^P,$$

(151)

where

$$I_{an}^P = \int \frac{d\Omega}{4\pi} z_N^2 \int_1^\infty dy \frac{y^3}{\sqrt{y^2-1}} f_P^2(z_N y).$$

(152)

Here $d\Omega$ denotes the integration over the solid angle, and $z_N = \Delta_P^P(T, \theta)/T$ depends on the polar angle $\theta$, where $\Delta_P^P(T, \theta)$ is the pairing gap in the $P$-wave channel. Note that $C_n = 0$ is not excluded; i.e., it is conceivable that axions are not emitted by the neutron $P$-wave condensate.

The axion emissivities (146) and (152) scale with temperature as $\propto T^5$. This scaling differs from its neutrino counterpart (133), which is $\propto T^7$. Accordingly, axionic cooling processes would change the slope of the cooling curves in the temperature-age diagram. Detailed numerical simulations of axionic cooling (Sedrakian, 2016) yield the regions of exclusion of axion masses and couplings illustrated in Fig. 28. As seen in this figure, the results from axion cooling simulations of neutron stars and their comparison with the X-ray data on thermally emitting neutron stars, which depend crucially on the axion emission by superfluid phases, are compatible with other constraints derived from stellar physics.

### VI. QUANTUM VORTICITY

#### A. Motivation

The motivation for the study of vorticity in nuclear systems derives from the fact that neutron stars are rotating and that neutrons, which form a neutral superfluid, must rotate by forming quantized rotational vortices. Although it has been conjectured that vortex states exist in finite nuclei, the coherence length of the nucleonic condensate, which sets the size of the vortex core, is of the order or larger than the nuclear radius. In neutron stars, rotation at angular velocity $\Omega$ induces a mesh of neutron vortices with number density

$$n_n = \frac{2\Omega}{\kappa}, \quad \kappa = \frac{\pi}{m_n},$$

(153)

where $\kappa$ is the quantum of circulation and $m_n$ is the neutron mass. In the parameter range where the proton superconductor in neutron stars is of type II, electromagnetic vortices are formed with a density

$$n_p = \frac{B}{\phi_0}, \quad \phi_0 = \frac{\pi}{e},$$

(154)

where $\phi_0$ is the flux quantum and $B$ is the mean magnetic-field induction. The Abrikosov vortex lattices of neutron and proton superfluids are triangular with the lengths of basis vectors given by

$$d_n = \left(\frac{\kappa}{\sqrt{3}\Omega}\right)^{1/2}, \quad d_p = \left(\frac{2\phi_0}{\sqrt{3}B}\right)^{1/2},$$

(155)
which are of order $10^{-4}$ cm and $10^{-3}$ cm respectively, for rotation periods of the order of a fraction of second and fields $B \sim 10^{12}$ G. The latter scale $d_p$ is larger than the penetration depth of the magnetic field, $\lambda \approx 10^{-11}$ cm, set by the Meissner mass of a photon inside the proton superconductor.

These length scales define a new mesoscopic scale for the description of neutron-star superfluids and superconductors, since an averaging over a large number of vortices is required to obtain the hydrodynamical fluid velocity and the macroscopic value of the magnetic field. The microscopic scale is set by the size of the vortex core, which for charged and neutral fermionic superfluids alike is given by the coherence length $\xi$. Within the region $r \leq \xi$, where $r$ is the radial cylindrical coordinate, the order parameter of the superfluid is suppressed linearly for $r \to 0$, vanishing at its center. From the microscopic point of view, the core of a vortex contains a new type of excitation – a quasiparticle bound state that emerges from the solution of the microscopic Bogolyubov-De Gennes (BdG) theory (de Gennes, 1999).

This section is devoted to the physics of these excitations and their interactions with matter, which give rise to mutual friction. The primary motivation for studies of mutual friction in neutron stars is a deeper understanding of the non-stationary dynamics of neutron-star rotation, in particular, the phenomena of glitches and post-glitch relaxation in pulsars; for a recent review and further references see (Haskell and Sedrakian, 2017).

### B. Vortex core quasiparticles

The microscopic theory of bound states of a fermionic vortex was initially developed by Caroli et al. (1964) for a vortex in a type-II superconductor. Their approach is based on the solution of the BdG equations for the pairing amplitudes $u(r)$ and $v(r)$, as introduced Sec. II, but in configuration space. These early results were soon adapted to neutron vortices, so as to obtain the coefficients of mutual friction in the core of a neutron star in terms of interactions of the neutron quasiparticles bound in the vortex core with ambient electrons (Feibelman, 1971).

An isolated neutron vortex was studied in (de Blasio and Elgarøy, 1999; Elgarøy and De Blasio, 2001; Yu and Bulgac, 2003) by solving the BdG equations in neutron matter. Substantial depletion in the region of the vortex core was found in (Yu and Bulgac, 2003), a feature uncharacteristic of condensed-matter vortices. Density depletion in vortex cores is important since it allows the vortices to be detected experimentally in ultracold atomic gases (Zwierlein et al., 2006). The vortex profile in an ultracold atomic gas was investigated in a population-imbalanced gas (Iskin, 2008; Takahashi et al., 2007; Warringa, 2012; Warringa and Sedrakian, 2011) and across the BCS-BEC crossover (Chien et al., 2006; Machida and Koyama, 2005; Machida et al., 2006). As discussed in Sec. IV, population imbalances are typically too large in neutron stars to be relevant. However, they could be relevant in super-strong magnetic fields where neutrons become spin-polarized (Gezerlis and Sharma, 2012; Stein et al., 2016).

The BdG theory can be derived using the Green functions formalism introduced in Sec. III.A, with specialization to configuration space. In this case, the Dyson-Schwinger equation for the Nambu-Gor’kov propagator takes the form

$$G^{-1}(X, X') = -\left(\frac{\partial}{\partial \tau} + H\right)\delta(X - X'),$$

where $X = (r, \tau)$ is the four-coordinate including the imaginary time $\tau$, while

$$H = \left(\begin{array}{cc} h(\Omega) - \mu_\uparrow + gn_\uparrow(r) & \Delta(r) \\ \Delta^*(r) & -h(\Omega)^* + \mu_\downarrow - gn_\downarrow(r) \end{array}\right).$$

In this expression, $h(\Omega)$ denotes the single-particle Hamiltonian in the frame rotating with frequency $\Omega$, and $\uparrow, \downarrow$ refer to spin-down and spin-up particles with chemical potentials $\mu_\uparrow, \downarrow$ and densities $n_\uparrow, \downarrow$, $g$ being the strength of the assumed four-fermion contact interaction. The solutions of the Dyson-Schwinger equation are obtained by inversion of Eq. (156). This is done by solving the BdG equation

$$H\begin{pmatrix} u_i(r) \\ v_i(r) \end{pmatrix} = E_i \begin{pmatrix} u_i(r) \\ v_i(r) \end{pmatrix}$$

for the amplitudes $u_i(r)$ and $v_i(r)$, where the index $i$ refers to the particle’s spin state and the energies $E_i$ are the eigenvalues of the BdG equation. The functions $u_i(r)$ and $v_i(r)$ are normalized by $\int d^3r \left[|u_i(r)|^2 + |v_i(r)|^2\right] = 1$. The densities of up-spin and down-spin fermions, expressed as

$$n_\uparrow(r) = \sum_i f(E_i)|u_i(r)|^2,$$

$$n_\downarrow(r) = \sum_i f(-E_i)|v_i(r)|^2$$

in terms of the Fermi-Dirac distribution function $f(E)$, are to be determined simultaneously with the solution of the BdG equation. The gap function $\Delta(r)$ is obtained from the anomalous component of the GF, which is given by

$$G_{\uparrow\downarrow}(r, \tau; r', \tau') = \sum_i f(E_i)u_i(r)v_i^*(r')$$

in the limit $r' \to r$. One finds a relation between the gap and the GF in Eq. (161) of the following form

$$G_{\uparrow\downarrow}(r, \tau; r', \tau) = -\frac{m\Delta(\vec{x})}{4\pi} \frac{1}{|r - r'|} + G_{\uparrow\downarrow}^{\text{reg}}(r, \tau; r, \tau),$$

for $r'$ close to $r$.
where $m$ is the (effective) mass of fermions and the regular part $G_{\tau\tau}^{\text{reg}}(r, \tau; r, \tau)$ of the GF can be found elsewhere (Yu and Bulgac, 2003). The Helmholtz free energy $F$ can be now evaluated using the solutions of the BdG equation, according to

$$
F = -\sum_i \left[ \frac{|E_i|}{2} + \frac{1}{\beta} \log \left( 1 + \exp^{-\beta|E_i|} \right) \right] + \sum_i \epsilon_i - \int d^3x G_{\tau\tau}(r, \tau; r, \tau)^* \Delta(\vec{x}) - g \int d^3x n_\tau(r)n_{\tau}(r) + \mu_\tau N_\tau + \mu_\tau N_{\tau},
$$

(163)

where $\epsilon_i$ are the eigenvalues of the Hartree-Fock Hamiltonian $H_{HF} = H(\Omega = 0) - \mu + gn(r)$.

It should be mentioned that some of the individual terms in the last equation are ultraviolet-divergent, but their sum, and hence the Helmholtz free energy, is ultraviolet-finite. Eq. (163) allows one to determine the parameter space spanned in the phase diagram by the coupling $g$, the population imbalance, the rotation frequency, and relevant thermodynamic quantities.

We now present approximate solutions of BdG equations that provide insight into recent numerical work. The states of the vortex core can be approximated as (Caroli and Matricon, 1965)

$$
\begin{pmatrix}
  u_{\mu \tau}(r) \\
  v_{\mu \tau}(r)
\end{pmatrix}
= e^{i\phi_{\mu \tau}} \left( e^{i\theta (\mu-\frac{1}{2})} e^{i\theta (\mu+\frac{1}{2})} \right) \begin{pmatrix}
  u_{\mu}(r) \\
  v_{\mu}(r)
\end{pmatrix},
$$

(164)

where the vector $r = (r, \theta, z)$ has been decomposed into cylindrical coordinates with the $z$-axis along the vortex circulation, $\|$ and $\perp$ being its components parallel and perpendicular to the vortex circulation. Here $\mu$ labels the azimuthal quantum number, which assumes half-integer positive values. It is seen that the vortex-core states are plane waves along the vortex circulation, but are quantized in the orthogonal direction. The radial part of the wave function is given by

$$
\begin{pmatrix}
  u_{\mu}(r) \\
  v_{\mu}(r)
\end{pmatrix}
= 2 \left( \frac{2}{\pi p_\perp} \right)^{1/2} e^{-K(r)} \begin{pmatrix}
  \cos \left( p_\perp r - \frac{\pi \mu}{2} \right) \\
  \sin \left( p_\perp r - \frac{\pi \mu}{2} \right)
\end{pmatrix},
$$

(165)

where $p_\perp = \sqrt{p^2 - p_F^2}$, $p_F$ being the neutron Fermi momentum. The function in the exponent is

$$
K(r) = \frac{p_F}{\pi p_\perp} \Delta_\infty \int_0^r \Delta(r')dr' \approx \frac{p_F r}{\pi p_\perp \xi} \left( 1 + \frac{\xi e^{-r/\xi}}{r} \right),
$$

(166)

where $\Delta_\infty$ is the asymptotic value of the gap far from the vortex core, while $\xi$ is the coherence length. The eigenstates of neutrons in the core of a vortex for small momentum have energies given by

$$
\epsilon_\mu(p) \approx \frac{\pi \mu \Delta_\infty^2}{2\xi_F^2} \left( 1 + \frac{p^2}{2p_F^2} \right),
$$

(167)

where $\epsilon_F$ is the Fermi energy.

C. Vortex dynamics and pinning

Following the suggestion by Anderson and Itoh (1975) that the neutron superfluid dynamics is driven by the interaction of vortices with the nuclear lattice in the inner crust of a neutron star, many calculations have been performed in efforts to understand the pinning-type interactions between vortices and nuclei. This is, in general, a time-dependent problem, but the static interactions are of great interest as well. Indeed, the stationary minimum energy state of a neutron vortex could require its pinning to a nucleus (with geometrical overlap), or, alternatively, its pinning in the space between nuclei if the vortex-nucleus interaction is repulsive.

Stationary studies of the pinning in neutron stars compare the energy difference between a configuration where nucleus and vortex are well separated with a configuration in which they intersect. A naive picture suggests that the energy required to create the vortex core quasi-particles out of the condensate is gained if the vortex passes through the nucleus (Alpar et al., 1984; Pines et al., 1980). A more flexible and quantitative basis is offered by Ginzburg-Landau theory (Epstein and Baym, 1988), as other contributions to the Ginzburg-Landau functional besides the condensation energy can play key roles. In this approach, whether the vortices pin on nuclei or in between them depends on the density; typically high densities favor pinning to nuclei. Similar conclusions have also been reached in semi-classical models that assume a realistic Argonne interaction (Donati and Pizzochero, 2004, 2006); however, the magnitude of the pinning energy or force is smaller by an order of magnitude compared to what is found in the Ginzburg-Landau models. Microscopic solutions of the BdG equations for the pinning problem exist (Avogadro et al., 2008), but the results for pinning energies are not conclusive.

A number of time-dependent formulations of the vortex-nucleus interaction go beyond static considerations that simply compare the energy differences between stationary pinned and unpinned configurations. Dynamical studies have included (i) purely hydrodynamical modeling (Link, 2009; Sedrakian, 1995), (ii) modeling based on Gross-Pitaevskii-like equations (Bulgac et al., 2013) and, most recently, (iii) application of time-dependent superfluid density functional theory (Wlazłowski et al., 2016). The last study captures most of the microphysics, and it concludes that nuclei repel vortices in the neutron-star crust, i.e., if pinned, vortices reside in between the nuclear clusters.

D. Mutual friction

Mutual friction arises through the interaction of vortices with the ambient non-superfluid components of neutron star matter. Analogous phenomena have been investigated extensively in the context of liquid He-II hydrodynamics (Sonin, 2016), but the context of neutron stars
is unique because both the ambient fluid and the vorticity are of fermionic nature. We next review the microphysics and kinetics of particle interactions with the bound states in the vortex cores of quantized vortices. Electrons will couple to the core quasiparticles of the neutron vortex via the interaction of the electron charge $e$ with the neutron magnetic moment $\mu_n = -1.913\mu_N$, where (restoring $\hbar$) $\mu_N = e\hbar/2m_n$ is the nuclear magneton (Feibelman, 1971). The relaxation timescale for the electron momentum due to scattering by neutron vortex-core quasiparticles is given by (Bildsten and Epstein, 1989)

$$\tau_{\mu\nu} = \frac{1.6 \times 10^3}{\Omega} \frac{\Delta}{\epsilon_{F}} \left( \frac{\epsilon_{F}}{\epsilon_{F\mu}} \right)^2 \times \left( \frac{\epsilon_{F\mu}}{2m_n} \right)^{1/2} \exp \left( \frac{\epsilon_0^{1/2}}{T} \right),$$  \hspace{1cm} (168)

where $\epsilon_{F\mu}/\epsilon_{F\nu}$ are the electron/neutron Fermi energies, $\Delta$ is the $S$-wave neutron pairing gap, $\epsilon_0^{1/2}$ is given by Eq. (167) with $\mu = 1/2$, and $\Omega$ is the angular velocity of the superfluid, which enters through the number of scattering centers per cm$^2$ according to formula (153). We see that the relaxation time is proportional to the Boltzmann factor that measures the probability of finding core quasiparticle states at a given temperature.

The electron dynamics in the stellar core is strongly affected by the proton component, but we assume for the time being that electrons interact exclusively with neutron vortices in a $P$-wave superfluid. The order parameter in the $P$-wave case has a tensor character and can be written as a traceless and symmetric function $A_{\mu\nu}$, $\mu, \nu = 1, 2, 3$. This function can be decomposed in cylindrical coordinates $(r, \phi, z)$ as (Sauls et al., 1982)

$$A_{\mu\nu} = \frac{\Delta}{\sqrt{2}} e^{i\phi} \left\{ [f_1 \hat{r}_\mu \hat{r}_{\nu} + f_2 \hat{\phi}_\mu \hat{\phi}_{\nu}] - (f_1 + f_2) \hat{z}_\mu \hat{z}_{\nu} + ig(r\hat{r}_\mu \hat{\phi}_{\nu} + r\hat{r}_\nu \hat{\phi}_{\mu}) \right\},$$  \hspace{1cm} (169)

where $g(r)$ and $f_{1,2}(r)$ are the radial functions describing the vortex profile and $\Delta$ is the average value of the gap in the $^3P_2$ channel. The new aspect emerging in the case of a neutron star is that $P$-wave vortices possess intrinsic magnetization because the relevant Cooper pairs are spin-1 objects. Their magnetization is given by (Sauls et al., 1982)

$$M_V(r) = \frac{\gamma_n \hbar}{2} = g_n \mu_N \frac{\sigma(r)}{2},$$  \hspace{1cm} (170)

where $\gamma_n = g_n \mu_N \hbar^{-1}$ is the gyromagnetic ratio of the neutron with $g_n = 3.826$. For a $P$-wave vortex, the spin density $\hbar \sigma/2$ may be estimated as

$$\sigma(r) = \frac{\mu_n \Delta_n^2}{3} \ln \left( \frac{\Delta}{T_c} \right) g(r) [f_1(r) - f_2(r)],$$  \hspace{1cm} (171)

where $\Delta$ is a cutoff parameter and $\mu_n$ is the neutron density of states at the Fermi surface. The magnitude of the vortex magnetization that follows from Eq. (171) is in turn estimated as

$$|M_V(^3P_2)| = \frac{g_n \hbar \mu_N}{2} n_n \left( \frac{\Delta}{\epsilon_{F\mu}} \right)^2 \approx 10^{11} \text{ G}.$$  \hspace{1cm} (172)

The interaction of electrons with $P$-wave superfluid vortices is driven by the interaction $-e\gamma \cdot A(r)$, with $A(r) = A(r)\hat{\phi}$ derived from the magnetization according to

$$A(r) = \frac{1}{r} \int_0^r |M_V(r')| r' dr'.$$  \hspace{1cm} (173)

Finally, the relaxation time for the electron-vortex scattering is obtained as (Sauls et al., 1982)

$$\tau_{\mu\nu} \simeq \frac{7.91 \times 10^8}{\Omega} \frac{\left( k_{F\mu}/\text{fm} \right)}{\left( \frac{\text{MeV}}{\Delta_n} \right)} \left( \frac{n_n}{n_p} \right)^{2/3}.$$  \hspace{1cm} (174)

In contrast to the case of scattering off the quasiparticles, the relaxation time (174) is nearly independent of temperature, the only temperature-dependent quantity being the gap. The result (174) sets a lower limit on the scattering rate at low temperatures ($T \ll \Delta$), where the relaxation time $\tau_{\mu\nu}$ of Eq. (168) is exponentially suppressed.

Allowing now for a proton component, we identify additional interaction channels, which turn out to be dominant in most cases. Let us first consider the case of non-superfluid protons, since at sufficiently high densities the proton $^1S_0$ gap closes. The neutron quasiparticles in the cores of vortices will then couple to proton excitations, in much the same way as they coupled to the electron component (Feibelman, 1971). However, an important distinction is that the protons will couple to neutrons by the strong nuclear force, instead of the much weaker electromagnetic interaction. The corresponding relaxation time becomes (Sedrakian, 1998)

$$\tau_{\mu\nu} = \frac{0.71}{\Omega_s} \frac{m_n m_p}{m_n m_p} \left( \frac{\epsilon_{F\mu}}{\epsilon_{Fp}} \right)^2 \frac{\epsilon_{0}^{1/2}}{T} \times \exp \left( \frac{\epsilon_0^{0/2}}{T} \right),$$  \hspace{1cm} (175)

where $\mu_{pn} = m_p m_n/(m_n^2 + m_p^2)$ is the reduced mass of the neutron-proton system (entering the relation between the cross-section and the scattering amplitude squared), $\epsilon_{0}^{0/2}$ is the lowest energy of vortex-core excitations according to Eq. (167), and $\langle \sigma_{np} \rangle$ can be viewed as an average neutron-proton cross-section. Eq. (175) suggests a much stronger coupling between the electron-proton plasma and the neutron vortices than implied by any of the previously quoted timescales.

Consider next the case of superconducting protons, in which no quasiparticle excitations are available for coupling to vortex-core quasiparticles. Nevertheless, in this
case, there is an entrainment effect that induces a new type of magnetization of the neutron vortex (Alpar et al., 1984; Vardanyan and Sedrakian, 1981). In effect, neutron vortices carry a non-integral multiple of the flux quantum,

$$\phi^* = k\phi_0, \quad k = \frac{m_p^*}{m_p},$$

(176)

which leads to a magnetic field larger by four orders of magnitude than that due to the spontaneous magnetization predicted by Eq. (172). The relaxation timescales are correspondingly shorter. It is now convenient to define the relaxation time in terms of a zero-range counterpart given by

$$\tau_0^{-1} = \frac{2n_v}{k_{eF}} \left( \frac{\pi^2 \phi^2_s}{4\phi_0^2} \right).$$

(177)

The term in parentheses is an approximation to the exact Aharonov-Bohm scattering result, in which $\sin^2(\pi/2)(\phi_s/\phi_0)$ appears instead (Alford and Wilczek, 1989). The finite range result then can be written as (Alpar et al., 1984)

$$\tau_{\phi}^{-1} = \frac{3\pi}{32} \left( \frac{\epsilon_{Fe}}{m_p} \right) \frac{\tau_0^{-1}}{k_{eF} \lambda},$$

(178)

where $\lambda$ is the penetration depth. We call attention to the weak dependence of the scattering relaxation time on the temperature, reflecting the fact that the coupling is to the magnetic field and not to the thermally excited quasiparticles.

A more complete discussion of mutual friction requires consideration of the interaction between neutron and proton vortices and their intertwined dynamics, which however is beyond the scope of our focus on microphysics. We refer the reader to a recent review (Haskell and Sedrakian, 2017) for such a discussion.

VII. CONCLUSIONS

This review has covered a range of topics on nucleonic superfluidity with an emphasis on extended systems such as neutron stars and matter created in nuclear collisions. The pairing problem at the level of mean-field BCS theory, in which the pairing interaction is extracted directly from free-space nuclear interactions, is essentially solved within the density range corresponding to energies where the scattering phase shifts are known. There still exist discrepancies between various methods for microscopic many-body calculation of pairing properties, notably in relation to the issue of suppression of $S$-wave pairing in neutron matter by long-range collective fluctuations in the nuclear medium. Theories that incorporate such fluctuation corrections as well as the effects of short-range correlations due to the repulsion of the two-nucleon potential at short distances have been emerging in recent years. The goal of achieving convergent results for pairing in low-density nuclear matter appears to be within sight. Other important objectives that arise at higher densities are harder to achieve. These include especially the challenge of accurate evaluation of pairing gaps in the $^3P_2 - ^3F_2$ channel, which is complicated by their characteristically small magnitude, high sensitivity to the two-body pairing interaction, which is not well constrained theoretically, and the increasingly important role of the three-nucleon forces. Additionally, the off-shell behavior of the pairing gap and its impact on the phenomenology of nucleonic superfluids remain largely unexplored.

Superfluid phases with broken space-time symmetries have received much attention from theorists during the past two decades. Recent experimental realization of imbalanced superfluids in ultracold fermionic atomic gases has created the possibility of laboratory tests of the predictions of the many-body theory under highly controlled conditions. There are excellent prospects for future cross-fertilization of nuclear theory and experimental activity in cold atomic gases, especially in identifying the phases of imbalanced superfluids and in exploring the physics of the BCS-BEC crossover. The phase diagram of imbalanced superfluids, as outlined in this review, offers a rich arena for mutual interaction and enrichment of quantum many-body theories and experimental studies of trapped atomic gases.

As discussed in detail in this review, the physics of the thermal evolution of neutron stars is a sensitive probe of their interior physics, particularly their composition. Accurate weak-interaction rates in the superfluid phases of neutron stars are of great importance for reliable modeling of neutron-star cooling. The quantum many-body methods involved in computations of these rates, some of which existed already in the 1960s, have been recently applied to compute the weak response of nucleonic superfluids, thereby providing accurate rates of neutrino emission from nucleonic and hyperonic superfluids. Future observational progress in measuring and modeling the surface radiation of neutron stars, in conjunction with improved theoretical input and simulations of neutron stars, can yield further clues on their interior composition and on the couplings of non-standard-model particles (e.g. axions) to matter.

Quantum vortex states, reviewed in the last section, are fundamental to an understanding of the rich spectrum of observed rotational anomalies in pulsars. This is yet another area in which models and theories developed for nuclear systems can be tested in laboratory experiments on ultracold atomic gases. Further theoretical studies of vortex dynamics, combined with pulsar timing observations, can be expected to shed new light on the internal structure of the superfluid phases of neutron stars, especially on the microphysics of mutual friction as surveyed in this review.
Acknowledgments

We are grateful to our colleagues, too numerous to name them all, who have helped us to shape our views on the topics covered in this review. A.S. acknowledges the support by the DFG (Grants No. SE 1836/3-2 and No. SE 1836/4-1), by the Helmholtz International Center for FAIR, and by the NewCompStar COST Action MP1304. J.W.C. acknowledges support from the McDonnell Center for the Space Sciences and is grateful for the hospitality of the Centro de Investigação em Matemática e Aplicações, University of Madeira, Funchal, Portugal.

References

Abe, T., and R. Seki, 2009, Phys. Rev. C 79(5), 054002.

Abrikosov, A., 1988, Fundamentals of the Theory of Metals (North-Holland, Amsterdam).

Abrikosov, A., L. Gorkov, and I. Dzyaloshinski, 1963, Methods of quantum field theory in statistical physics (Dover, New York, N.Y.).

Aguilera, D. N., V. Cirigliano, J. A. Pons, S. Reddy, and R. Sharma, 2009, Phys. Rev. Lett. 102(9), 091101.

Ainsworth, T. L., J. Wambach, and D. Pines, 1989, Phys. Lett. B 222, 173.

Alford, M. G., S. Mahmoodifar, and K. Schwenzer, 2012a, Phys. Rev. D 85(4), 044051.

Alford, M. G., and K. Pangeni, 2017, Phys. Rev. C 95(1), 015802.

Alford, M. G., S. Reddy, and K. Schwenzer, 2012b, Phys. Rev. Lett. 108(11), 111102.

Alford, M. G., A. Schmitt, K. Rajagopal, and T. Schäfer, 2008, Rev. Mod. Phys. 80, 1455.

Alford, M. G., and F. Wilczek, 1989, Phys. Rev. Lett. 62, 1071.

Alm, T. B. L. Friman, G. Röpke, and H. Schulz, 1993, Nucl. Phys. A 551, 45.

Alm, T., G. Röpke, and M. Schmidt, 1990, Zeitschrift für Physik A Hadrons and Nuclei 337, 355.

Alm, T., G. Röpke, and M. Schmidt, 1994, Phys. Rev. C 50, 31.

Alm, T., G. Röpke, A. Schnell, N. H. Kwong, and H. S. Köhler, 1996a, Phys. Rev. C 53, 2181.

Alm, T., G. Röpke, A. Sedrakian, and F. Weber, 1996b, Nucl. Phys. A 604, 491.

Alpar, M. A., D. Pines, P. W. Anderson, and J. Shaham, 1984, ApJ 276, 325.

Ambartsumyan, V. A., and G. S. Saakyan, 1960, Soviet Ast. 4, 187.

Anderson, P. W., 1958, Phys. Rev. 112, 1900.

Anderson, P. W., and N. Itoh, 1975, Nature 256, 25.

Andersson, N., K. Glampedakis, W. C. G. Ho, and C. M. Espinoza, 2012, Phys. Rev. Lett. 109(24), 241103.

Andreev, A. F., and E. P. Bashkin, 1976, Sov. Phys. JETP 42, 164.

Anglani, R., R. Casalbuoni, M. Ciminale, N. Ippolito, R. Gatto, M. Mammarella, and M. Ruggieri, 2014, Rev. Mod. Phys. 86, 509.

Arellano, H. F., F. Isaule, and A. Rios, 2016, Eur. Phys. J. A 52, 299.

Avogadro, P., F. Barranco, R. A. Broglia, and E. Vigezzi, 2008, Nucl. Phys. A 811, 378.

Balberg, S., and N. Barnea, 1998, Phys. Rev. C 57, 409.

Baldo, M., I. Bombaci, and U. Lombardo, 1992a, Phys. Lett. B 283, 8.

Baldo, M., J. Cugnon, A. Lejeune, and U. Lombardo, 1992b, Nucl. Phys. A 536, 349.

Benhar, O., and G. De Rosi, 2017, J. Low Temp. Phys. 187, 164.

Bogolyubov, N. N., 1995, Soviet Physics Doklady 3, 279.

Bogoliubov, N. N., 1958, Nuov. Cim. 7, 794.

Bogolyubov, N. N., V. V. Tolmachev, and D. V. Shirkov, 1958, Fortsch. Phys. 6, 605.

Boguta, J., 1981, Phys. Lett. B 106, 255.
Fabrocini, A., S. Fantoni, A. Y. Illarionov, and K. E. Schmidt, 2008, Nucl. Phys. A 803, 137.
Fan, H.-H., E. Krotoscheck, and J. W. Clark, 2017, J. Low Temp. Phys. 189, 470.
Fan, H. H., E. Krotoscheck, T. Lichtenegger, D. Mateo, and R. E. Zillich, 2015, Phys. Rev. A 92(2), 023640.
Fantoni, S., 1981, Nucl. Phys. A 363, 381.
Fantoni, S., and S. Rosati, 1975, Nuov. Cim. A 25, 593.
Fantoni, S., and S. Rosati, 1978, Nuov. Cim. A 43, 413.
Fattoyev, F. J., C. J. Horowitz, and B. Schuetrumpf, 2017, Phys. Rev. C 95(5), 055804.
Feenberg, E., 1969, Theory of quantum fluids (Academic Press, New York, N.Y.).
Feibelman, P. J., 1971, Phys. Rev. D 81.
Fetter, A. L., and J. D. Walecka, 1971, Quantum theory of many-particle systems (McGraw-Hill).
Fetter, A. L., and D. W. Walecka, 1971, Quantum theory of many-particle systems (McGraw-Hill).
Filikhin, I., and A. Gal, 2002, Nucl. Phys. A 707(3), 491.
Flowers, E., M. Ruderman, and P. Sutherland, 1976, Rev. Mod. Phys. 81, 647.
Forbes, M. M., E. Gubankova, W. V. Liu, and F. Wilczek, 2005, Phys. Rev. Lett. 94(1), 017001.
Frick, T., and H. Mütter, 2003, Phys. Rev. C 68(3), 034310.
Frick, T., H. Mütter, and A. Polls, 2004, Phys. Rev. C 69(5), 054305.
Friedman, W. A., and G. F. Bertsch, 2007, Phys. Rev. C 76(5), 057301.
Fulde, P., and R. A. Ferrell, 1964, Phys. Rev. 135, 550.
Galitski, V. M., 1958, Sov. Phys. JETP 7, 104.
Gandolfi, S., A. Gezerlis, and J. Carlson, 2015, Ann. Rev. Nucl. Part. Sci. 65, 303.
Gandolfi, S., A. Y. Illarionov, S. Fantoni, F. Pederiva, and K. E. Schmidt, 2008, Phys. Rev. Lett. 101(13), 132501.
Gandolfi, S., A. Y. Illarionov, F. Pederiva, K. E. Schmidt, and S. Fantoni, 2009a, Phys. Rev. C 80(4), 045802.
Gandolfi, S., A. Y. Illarionov, K. E. Schmidt, F. Pederiva, and S. Fantoni, 2009b, Phys. Rev. C 79(5), 054005.
Garcilazo, H., A. Valcarce, and J. Vijande, 2016, Phys. Rev. C 94(2), 024002.
de Gennes, P.-G., 1999, Superconductivity of Metals and Alloys (Advanced Book Program, Perseus Books, New York, N.Y.).
Gezerlis, A., and J. Carlson, 2008, Phys. Rev. C 77(3), 032801.
Gezerlis, A., and J. Carlson, 2010, Phys. Rev. C 81(2), 025803.
Gezerlis, A., C. J. Pethick, and A. Schwenk, 2013, in Novel Superfluids, edited by K. H. Bennemann and J. B. Ketterson (Oxford University Press, Oxford, UK), International Series of Monographs on Physics, p. 580.
Gezerlis, A., and R. Sharma, 2012, Phys. Rev. C 85(1), 015806.
Gianotti, M., I. G. Irastorza, J. Redondo, A. Ringwald, and K. Saikawa, 2017, J. Cosmo. Astropart. Phys. 10, 010.
Ginzburg, V. L., and D. A. Kirzhnits, 1965, Sov. Phys. JETP 20, 1346.
Giorgini, S., L. P. Pitaevskii, and S. Stringari, 2008, Rev. Mod. Phys. 80, 1215.
Glendenning, N. K., 1985, ApJ 293, 470.
Glendenning, N. K., 2000, Compact stars: nuclear physics, particle physics, and general relativity (Springer, New York, N.Y.).
Glendenning, N. K., and J. Schaffner-Bielich, 1999, Phys. Rev. C 60(2), 025803.
Gold, T., 1968, Nature 218, 731.
González-Jiménez, N., C. Petrovich, and A. Reisenegger, 2015, MNRAS 447, 2073.
Goriely, S., N. Chamel, and J. M. Pearson, 2013, Phys. Rev. C 88(6), 061302.
Goriely, S., N. Chamel, and J. M. Pearson, 2016a, Phys. Rev. C 93(3), 034337.
Goriely, S., N. Chamel, and J. M. Pearson, 2016b, in Journal of Physics Conference Series, volume 665 of Journal of Physics Conference Series, p. 012038.
Gor'kov, L. P., 1958, Sov. Phys. JETP 9(6), 505.
Gor'kov, L. P., and T. K. Melik-Barkhudarov, 1961, Sov. Phys. JETP 13, 1018.
Greiner, M., O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, 2002, Nature 415, 39.
Greiner, M., C. A. Regal, and D. S. Jin, 2005, Phys. Rev. Lett. 94(7), 070403.
Gubankova, E., W. V. Liu, and F. Wilczek, 2003, Phys. Rev. Lett. 91(3), 032001.
Gubankova, E., A. Schmitt, and F. Wilczek, 2006, Phys. Rev. B 74(6), 064505.
Gulminelli, F., and A. R. Raduta, 2015, Phys. Rev. C 92(5), 055803.
Gussakov, M. E., P. Haensel, and E. M. Kantor, 2014, MN-RAS 439(1), 318.
Hashimoto, M., H. Seki, and M. Yamada, 1984, Prog. Theor. Phys. 71, 320.
Haskell, B., and A. Sedrakian, 2017, eprint 1709.10340.
He, L., M. Jin, and P. Zhuang, 2006, Phys. Rev. B 74(21), 214516.
Heckel, S., P. P. Schneider, and A. Sedrakian, 2009, Phys. Rev. C 80(1), 015805.
Heiselberg, H., C. J. Pethick, H. Smith, and L. Viverit, 2000, Physical Review Letters 85, 2418.
Hempel, M., and J. Schaffner-Bielich, 2010, Nucl. Phys. A 837, 210.
Hewish, A., S. J. Bell, J. D. H. Pilkington, P. F. Scott, and R. A. Collins, 1968, Nature 217, 709.
Hoffberg, M., A. E. Glassgold, R. W. Richardson, and M. Ruderman, 1970, Phys. Rev. Lett. 24, 775.
Hornerich, R. M., 1980, Journal of Magnetism and Magnetic Materials 15, 387.
Huang, X.-G., 2010, Phys. Rev. C 81(3), 034007.
Hugenholz, N. M., and L. van Hove, 1958, Physica 24, 363.
Ismail, F., H. F. Arelanno, and A. Rios, 2016, Phys. Rev. C 94(3), 034004.
Isayev, A. A., 2008, Phys. Rev. C 78(1), 014306.
Isak, M., 2008, Phys. Rev. A 78(2), 021604.
Jackson, A. D., A. Lande, and R. A. Smith, 1982, Phys. Rep. 86, 55.
Jackson, A. D., A. Lande, and R. A. Smith, 1985, Phys. Rev. Lett. 54, 1469.
Jaikumar, P., and M. Prakash, 2001, Phys. Lett. B 516, 345.
Jin, M., L. He, and P. Zhuang, 2007, Int. J. Mod. Phys. E 16, 2363.
Jin, M., M. Urban, and P. Schuck, 2010, Phys. Rev. C 82(2), 024911.
Kadanoff, L., and G. Baym, 1962, Quantum statistical mechanics: Green’s function methods in equilibrium and...
Wu, X.-H., S.-B. Wang, A. Sedrakian, and G. Röpke, 2017, J. Low Temp. Phys. 189, 133.
Xu, R., C. Wu, and Z. Ren, 2014, Int. J. Mod. Phys. E 23, 1450078.
Yakovlev, D. G., A. D. Kaminker, O. Y. Gnedin, and P. Haensel, 2001, Phys. Rep. 354, 1.
Yakovlev, D. G., A. D. Kaminker, and K. P. Levenfish, 1999, A&A 343, 650.
Yang, C.-H., and J. W. Clark, 1971, Nucl. Phys. A 174, 49.
Yao, X.-C., H.-Z. Chen, Y.-P. Wu, X.-P. Liu, X.-Q. Wang, X. Jiang, Y. Deng, Y.-A. Chen, and J.-W. Pan, 2016, Phys. Rev. Lett. 117(14), 145301.
Yu, Y., and A. Bulgac, 2003, Phys. Rev. Lett. 90(16), 161101.
Zhou, X.-R., G. F. Burgio, U. Lombardo, H.-J. Schulze, and W. Zuo, 2004, Phys. Rev. C 69(1), 018801.
Zhou, X.-R., H.-J. Schulze, F. Pan, and J. P. Draayer, 2005, Phys. Rev. Lett. 95(5), 051101.
Zuo, W., C. X. Cui, U. Lombardo, and H.-J. Schulze, 2008, Phys. Rev. C 78(1), 015805.
Zverev, M. V., J. W. Clark, and V. A. Khodel, 2003, Nucl. Phys. A 720, 20.
Zwerger, W., 2003, Journal of Optics B: Quantum and Semiclassical Optics 5, S9.
Zwierlein, M. W., A. Schirotzek, C. H. Schunck, and W. Ketterle, 2006, Science 311, 492.