ASYMPTOTIC ANALYSIS FOR 1D COMPRESSIBLE NAVIER-STOKES-VLASOV EQUATIONS

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Abstract. In this paper, we study the asymptotic analysis of 1D compressible Navier-Stokes-Vlasov equations. By taking advantage of the one space dimension, we obtain the hydrodynamic limit for compressible Navier-Stokes-Vlasov equations with the pressure $P(\rho) = A\rho^\gamma$ ($\gamma > 1$). The proof relies on weak convergence method.

1. Introduction.

1.1. Model and background. In this paper, we are interested in the asymptotic problems of weak solutions for a system of a kinetic equation coupled with compressible isentropic Navier-Stokes equations. This system arise in the description of various industrial applications and combustion phenomena, such as sprays, aerosols, waste water treatment, and so on. For more physical background of this fluid-particle systems, we refer the readers to [3, 11, 15] and the references therein. At the microscopic scale, the cloud of particles interacting with a compressible fluid is described through its distribution function $f(t, x, \xi) \geq 0$, which obeys the following Vlasov equation

$$\partial_t f + \xi \partial_x f + \partial_\xi (F_df) = 0.$$  (1)

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On other hand, the particles evolve in the fluid which is described by macroscopic variables, its density \( \rho(t, x) \geq 0 \) and its velocity field \( u(t, x) \). These quantities verify the following compressible Navier-Stokes equations

\[
\begin{align*}
&\partial_t \rho + \partial_x (\rho u) = 0, \\
&\rho F \left( \partial_t (\rho u) + \partial_x (\rho (u)^2) + \partial_x P(\rho) \right) - \mu \partial_{xx} u = \bar{F},
\end{align*}
\]

with \( \rho_F > 0 \) is a typical value of the fluid mass per unit volume and \( \mu > 0 \) is the kinematic viscosity of the compressible fluid.

In these systems, we suppose that the particles have the same length \( a > 0 \) and mass density \( \rho_P > 0 \) in one dimension. Then the mass of a particle \( M = a \rho_P \).

We denote by

\[
F_d = \frac{6\pi \mu}{\rho_P} \left( u(t, x) - \xi \right).
\]

The right hand side of the fluid equation (2) is the action of the cloud of particles on the fluid, given by

\[
\bar{F} = -\int_{\mathbb{R}} MF_d f(t, x, \xi) d\xi = -6\pi \mu a \int_{\mathbb{R}} (u(t, x) - \xi) f(t, x, \xi) d\xi.
\]

Finally, we will assume that the pressure follows a \( \gamma \)-law:

\[
P(\rho) = A \rho^\gamma, \quad \gamma > 1.
\]

Therefore, we arrive at the following system

\[
\begin{align*}
&\partial_t f + \xi \partial_x f + \frac{6\pi \mu}{\rho_P} \partial_\xi (u(t, x) - \xi) f = 0, \\
&\partial_t \rho + \partial_x (\rho u) = 0, \\
&\rho F \left( \partial_t (\rho u) + \partial_x (\rho (u)^2) + \partial_x P(\rho) \right) - \mu \partial_{xx} u = -6\pi \mu a \int_{\mathbb{R}} (u(t, x) - \xi) f(t, x, \xi) d\xi.
\end{align*}
\]

There is a huge literature on the investigation of global existence of solutions to the fluid-particle system. In general, the analysis of the fluid-particle system is challenging since the density distribution function \( f(t, x, \xi) \) depends on more variables than the macroscopic quantities \( \rho(t, x) \) and \( u(t, x) \). Hamdache [14] studied the global existence of weak solutions and its large-time behavior for the Vlasov-Stokes equations. Next, Boudin, Desvillettes, Grandmont and Moussa [2] considered the existence of global weak solutions to the incompressible Navier-Stokes-Vlasov equations in three-dimensional periodic domains. Later then, Wang and Yu [24, 25] considered the global existence of weak solution to the inhomogeneous Navier-Stokes-Vlasov equations with density-dependent drag force and incompressible Navier-Stokes-Vlasov equations in bounded domain, respectively. Furthermore, Goudon, He, Moussa and Zhang [11] showed the global existence of smooth solutions with small data for Navier-Stokes/Vlasov-Fokker-Planck equations; Carrillo, Duan and Moussa [6] proved the global existence and decay rate of classical solutions close to equilibrium to the Vlasov-Fokker-Planck-Euler system. It is worth mentioning that Mellet and Vasseur [21] proved the global existence of weak solutions to the Vlasov-Fokker-Planck equation coupled to compressible Navier-Stokes equations, it
is as follows:

\[
\begin{cases}
\partial_t f + \xi \cdot \nabla_x f + \text{div}_x (F_d f) = \Delta_{\xi} f, \\
\partial_t \rho + \text{div}_x (\rho u) = 0, \\
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x \rho \gamma - \mu \Delta_x u = \mathcal{F}.
\end{cases}
\]  

(7)

For the global existence and large time behavior of strong solutions close to equilibrium of the above equations, we can see [18].

In recent years, there are many results in the direction of asymptotic analysis (hydrodynamic limit) of weak solutions to the fluid-particle system. For the case of one space dimension, Goudon [10] proved the hydrodynamic limit and stratified limit to the Vlasov equation coupled viscous Burgers equation. For the multi-dimensional case, up to now, all the results are about the Vlasov-Fokker-Planck equation coupled to Navier-Stokes equations. Precisely, by using the method of relative entropy, Goudon, Jabin and Vasseur [12, 13] established two hydrodynamic limits to the Vlasov-Fokker-Planck equation coupled to incompressible Navier-Stokes which have thermal diffusion acting on the particles; Mellet and Vasseur [22] showed the hydrodynamic limit of weak solutions to the Vlasov-Fokker-Planck/compressible Navier-Stokes systems (7). For more results about these directions, we refer the reader to [1, 4, 5, 16, 17, 19] and references therein.

1.2. Dimensionless equations. Denoting \(L\) as the space unit, and \(T\) as the time unit. Then the velocity unit \(U = \frac{L}{T}\). We define

\[\mathcal{T}_s = \frac{M}{\rho_p \mu a} = \frac{\rho_p}{6\pi \mu},\]

being the natural relaxation time. The measure of the fluctuation of particles velocity, called the thermal speed is given by

\[\mathcal{V}_{th} = \sqrt{\frac{\kappa \theta_0}{M}},\]

where the constants \(\kappa, \theta_0 > 0\) are the Boltzmann constant and the temperature of the surrounding fluid, respectively. Let us introduce the following new variables and functions

\[t = Tt', \quad x = Lx', \quad \xi = \mathcal{V}_{th} \xi',\]

and

\[\rho'(x',t') = \rho(Lx', Tt'), \quad U'u'(x',t') = u(Lx', Tt'), \quad \frac{\rho_p L^2}{T^2} - P'(x',t') = P(Lx', Tt'), \quad f'(x',t', \xi') = a \mathcal{V}_{th} f(Lx', Tt', \mathcal{V}_{th} \xi').\]

Then the system (6) become

\[
\begin{cases}
\frac{1}{T} \frac{\partial f'}{\partial t'} + \frac{\mathcal{V}_{th}}{L} \xi' \frac{\partial}{\partial x'} f' + \frac{1}{\mathcal{T}_s \mathcal{V}_{th}} \frac{\partial}{\partial \xi'} \left( (U'u' - \mathcal{V}_{th} \xi') f' \right) = 0, \\
\frac{\partial \rho'}{\partial t'} + \frac{\partial (\rho' U')}{\partial x'} = 0, \\
\frac{U}{T} \frac{\partial (\rho' U')}{\partial t'} + \frac{U^2}{L} \frac{\partial (\rho' U'^2)}{\partial x'} + \frac{L}{T^2} \frac{\partial P'}{\partial x'} - \frac{\mu U}{L^2 \rho_F} \frac{\partial^2 u'}{\partial x'^2} \\
\quad = -\frac{\rho_p}{\mathcal{T}_s \rho_F} \int_{\mathbb{R}} (U'u'(x',t') - \mathcal{V}_{th} \xi') f'(x',t', \xi') d\xi'.
\end{cases}
\]  

(8)
Defining the following positive dimensionless quantities:

\[ A = \frac{T}{L} \nu, \quad B = \frac{T}{\rho}, \quad D = B \frac{\rho}{\rho F}, \quad E = \frac{T}{L^2 \rho F}. \]

Hence, dropping the primes of the system (8) for simplicity, we know

\[
\begin{cases}
\partial_t f + A \xi \partial_x f + B \partial_t \left( \frac{1}{A} u - \xi \right) f = 0, \\
\partial_t \rho + \partial_x (\rho u) = 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x P(\rho) - \mu E \partial_{xx} u = -D \int (u - A \xi) f d\xi.
\end{cases}
\]  

(9)

1.3. The main result. As in [4, 12], it is very interesting and significative to discuss the ordering of the quantities with respect to some small parameter \( \epsilon > 0 \) for the compressible Navier-Stokes-Vlasov equations, which leads to singular perturbation problems. In this paper, we consider the case of light particles regime in the system (9)

\[ A = \frac{1}{\epsilon}, \quad B = \frac{1}{\epsilon^2}, \quad D = 1, \quad E = 1. \]

Therefore, the aim of this paper is to study the limit behavior \( (\epsilon \to 0) \) of weak solutions \((f^\epsilon, \rho^\epsilon, u^\epsilon)\) to the following system in a dimensionless form

\[
\begin{cases}
\partial_t f^\epsilon + \frac{1}{\epsilon} \xi \partial_x f^\epsilon + \frac{1}{\epsilon^2} \partial_t \left( (\epsilon u^\epsilon - \xi) f^\epsilon \right) = 0, \\
\partial_t \rho^\epsilon + \partial_x (\rho^\epsilon u^\epsilon) = 0, \\
\partial_t (\rho^\epsilon u^\epsilon) + \partial_x (\rho^\epsilon u^\epsilon)^2 + \partial_x P(\rho^\epsilon) - \mu \partial_{xx} u^\epsilon = -\frac{1}{\epsilon} \int (\epsilon u^\epsilon - \xi) f^\epsilon d\xi,
\end{cases}
\]

(10)

for \((t, x, \xi) \in [0, T] \times \Omega \times \mathbb{R} \), where \( \Omega = (0, 1) \), with the initial data

\[ (f^\epsilon, \rho^\epsilon, u^\epsilon)|_{t=0} = (f_0^\epsilon(x, \xi), \rho_0^\epsilon(x), u_0^\epsilon(x)), \quad x \in \bar{\Omega}, \ \xi \in \mathbb{R}, \]

(11)

and the boundary condition

\[
\begin{cases}
\left\{ \begin{array}{l}
\left. u^\epsilon(t, 0) = u^\epsilon(t, 1) = 0, \quad \text{for} \quad t > 0, \\
\left. f^\epsilon(t, 0, \xi) = f^\epsilon(t, 0, -\xi), \quad \text{for} \quad \xi > 0, \\
\left. f^\epsilon(t, 1, \xi) = f^\epsilon(t, 1, -\xi), \quad \text{for} \quad \xi < 0, \quad t > 0,
\end{array} \right\}
\end{cases}
\]

(12)

where the distribution function \( f^\epsilon(t, x, \xi) \) of particles depends on time \( t \in [0, T] \), the physical position \( x \in \Omega \) and the velocity \( \xi \in \mathbb{R} \) where \( 0 < T < \infty \). Throughout this paper, \( A \) and \( \mu \) will be taken as 1 for simplicity.

Our work is motivated by the result of Goudon, Jabin and Vasseur [12], which is about the hydrodynamic limit of incompressible Navier-Stokes/Vlasov-Fokker-Planck system over torus \( \mathbb{T}^2 \), it is as follows:

\[
\begin{cases}
\partial_t f^\epsilon + \frac{1}{\epsilon} \xi \cdot \nabla_x f^\epsilon + \frac{1}{\epsilon^2} \nabla_x \cdot ((\epsilon u^\epsilon - \xi) f^\epsilon) = 0, \\
\partial_t u^\epsilon + \text{div}_x (u^\epsilon \otimes u^\epsilon) + \nabla_x p^\epsilon - \Delta_x u^\epsilon = -\frac{1}{\epsilon} \int R^2 (\epsilon u^\epsilon - \xi) f^\epsilon d\xi, \\
\text{div}_x u^\epsilon = 0, \\
(f^\epsilon)|_{t=0} = f_0^\epsilon, \quad (u^\epsilon)|_{t=0} = u_0^\epsilon,
\end{cases}
\]

(13)
its limit problem is
\[
\begin{align*}
\frac{\partial n}{\partial t} + \text{div}_x (nu - \nabla n) &= 0, \\
\frac{\partial u}{\partial t} + \text{div}_x (u \otimes u + \nabla p) - \Delta_x u &= 0, \\
\text{div}_x u &= 0.
\end{align*}
\]

Precisely, by using the dissipation of particle, Goudon, Jabin and Vasseur obtained $n^\epsilon$ in $n^\epsilon(t, x) = \int_{\mathbb{R}^2} f^\epsilon(t, x, \xi) d\xi$ is bounded in $L^\infty(0, \infty; L^1(\mathbb{T}^2))$, then they can pass to the limit of the nonlinear term $n^\epsilon u^\epsilon$ in dimension two, i.e., $n^\epsilon u^\epsilon$ converges to $nu$ in the sense of distribution. But when there is no dissipation term $\Delta_f f$, this is certainly a great mathematical difficulty when dealing with a rigorous proof of convergence. In this paper, we will establish the limit behavior of weak solutions $(f^\epsilon, \rho^\epsilon, u^\epsilon)$ to the initial boundary value problem of 1D Navier-Stokes-Vlasov equations (10)-(12), there is no dissipation in the kinetic equation and we consider the compressible fluid, it is difficult to obtain the convergence of $n^\epsilon u^\epsilon$ and strong convergence of fluid density $\rho^\epsilon$, we will take advantage of the one space dimension to deal with these difficulties, see Remarks 2.3. For the hydrodynamic limit of fluid-particle coupled models with strong brownian effect and strong drag force (scalings differs from (10) and (13)), we refer to Goudon-Jabin-Vasseur [13] for the incompressible Navier-Stokes/Vlasov-Fokker-Planck system and Mellet-Vasseur [22] for compressible Navier-Stokes/Vlasov-Fokker-Planck system, they used a relative entropy approach to obtain strong convergence of $n^\epsilon, u^\epsilon$ and $n^\epsilon, \rho^\epsilon$ respectively.

Denote

$$n^\epsilon(t, x) = \int_{\mathbb{R}} f^\epsilon(t, x, \xi) d\xi, \quad J^\epsilon(t, x) = \frac{1}{\epsilon} \int_{\mathbb{R}} \xi f^\epsilon(t, x, \xi) d\xi,$$

then

$$- \frac{1}{\epsilon} \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi = J^\epsilon - n^\epsilon u^\epsilon.$$

**Definition 1.1.** We say $(f^\epsilon(t, x, \xi), \rho^\epsilon(t, x), u^\epsilon(t, x))$ is a global weak solution to the initial-boundary value problem (10)-(12), if for any $T > 0$, the following properties hold:

- $\rho^\epsilon \geq 0$, for any $(t, x) \in (0, T) \times \Omega$;
- $\sqrt{\rho^\epsilon} u^\epsilon \in L^\infty(0, T; L^2(\Omega))$, $\rho^\epsilon \in L^\infty(0, T; L^2(\Omega)) \cap C([0, T]; L^1(\Omega))$;
- $u^\epsilon \in L^2(0, T; H^2(\Omega))$, $\rho^\epsilon u^\epsilon \in C([0, T]; L^2_{weak}(\Omega))$;
- $f^\epsilon(t, x, \xi) \geq 0$, for any $(t, x, \xi) \in (0, T) \times \Omega \times \mathbb{R}$;
- $f^\epsilon \in L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}))$, $|\xi|^2 f^\epsilon \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}))$;
- (10)2 and (10)3 hold in the sense of distribution, and the kinetic equation (10)1 holds in the following sense

$$- \int_0^T \int_{\Omega} f^\epsilon (\phi_t + \frac{1}{\epsilon} \xi \partial_x \phi + \frac{1}{\epsilon^2} (\epsilon u^\epsilon - \xi) \partial_x \phi) d\xi dx dt = \int_0^T \int_{\Omega} f^\epsilon (\phi(0, x, \xi) \partial_x \phi) d\xi dx,$$

for any test function $\phi \in C^\infty([0, T] \times \Omega \times \mathbb{R})$, such that $\phi(T, \cdot) = 0$ and $\phi(t, 0, \xi) = \phi(t, 0, -\xi)$ for $\xi > 0$, $\phi(t, 1, \xi) = \phi(t, 1, -\xi)$ for $\xi < 0$;

- The energy inequality

$$\int_\Omega \frac{1}{2} \rho^\epsilon |u^\epsilon|^2 dx + \frac{1}{\gamma - 1} (\rho^\epsilon) \gamma dx \geq \int_\Omega \int_{\mathbb{R}} f^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx$$

$$+ \int_0^T \int_{\Omega} |\partial_x u^\epsilon|^2 dx dt + \frac{1}{\epsilon^2} \int_0^T \int_{\mathbb{R}} |\epsilon u^\epsilon - \xi|^2 d\xi dx dt,$$
Theorem 1.2.
For any fixed $\gamma > \frac{3}{2}$, the boundary value problem (10)-(12) in 1D when $\gamma > \frac{3}{2}$, we notice that the Brownian motion of the particles $\Delta f$ are not crucial in the existence theory of weak solutions. Therefore, using the same method, we can obtain the existence result of the initial-boundary value problem (10)-(12) in 1D when $\gamma > 1$.

**Theorem 1.3.** For any fixed $\epsilon > 0$, assume that the initial data $0 \leq \rho_0 \in L^1(\Omega) \cap L^7(\Omega), \sqrt{\rho_0}u_0^\gamma \in L^2(\Omega), 0 \leq f_0 \in L^\infty(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R})$ and $\|\xi\|^2 f_0 \in L^1(\Omega \times \mathbb{R})$. Then there exists a global weak solution to the problem (10)-(12) in the sense of Definition 1.1.

Our main result in this paper is as follows:

**Theorem 1.3.** Assume that the initial data $0 \leq \rho_0 \in L^1(\Omega) \cap L^7(\Omega), \sqrt{\rho_0}u_0^\gamma \in L^2(\Omega), f_0 \in L^1(\Omega \times \mathbb{R})$ and $|\xi|^2 f_0 \in L^1(\Omega \times \mathbb{R})$, satisfy

\[
\begin{align*}
&\int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{\gamma - 1} (\rho_0)^\gamma \right) \, dx \leq C_0, \\
&\int_{\Omega} \int_{\mathbb{R}} f_0^\gamma (1 + \frac{1}{2} |\xi|^2) d\xi \, dx \leq C_0, \\
&\rho_0 \to \rho_0 \text{ in } L^1(\Omega), f_0 \to f_0 \text{ weakly} * \text{ in } \mathcal{M}^1(\Omega \times \mathbb{R}), \\
&\sqrt{\rho_0}u_0^\gamma \to \sqrt{\rho_0}u_0 \text{ weakly in } L^2(\Omega),
\end{align*}
\]

for some $C_0 > 0$, independent on $\epsilon$. Then, up to a subsequence, the weak solution of the problem (10)-(12) satisfy

\[
\begin{align*}
\rho^\epsilon &\to \rho \text{ in } L^1((0,T) \times \Omega) \text{ and } C([0,T]; L^\infty(\Omega)), \\
u^\epsilon &\to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)), \\
\rho^\epsilon u^\epsilon &\to \rho u \text{ in } C([0,T]; L^{2\gamma}_w(\Omega)), \\
n^\epsilon &\to n \text{ in } C([0,T]; H^{-1}(\Omega)), \text{ and weakly} * \text{ in } L^\infty(0,T; \mathcal{M}^1(\Omega)), \\
J^\epsilon &\to J = nu \text{ weakly in } L^2(0,T; \mathcal{M}^1(\Omega)),
\end{align*}
\]

where $\mathcal{M}^1(\Omega)$ stands for the set of bounded measures on the domain $\Omega$, and $(n, \rho, u)(t, x)$ satisfies

\[
\begin{align*}
\partial_t n + \partial_x (nu) &= 0, \\
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x \rho^\gamma - \partial_{xx} u &= 0,
\end{align*}
\]

with boundary condition

\[
u(t, 0) = \nu(t, 1) = 0, \quad t > 0.
\]

The limit problem consists of the compressible Navier-Stokes equations for the density $\rho$ and velocity $u$, while the macroscopic density $n$ of the particles verifies a transport equation.
Remark 1. The same result in Theorem 1.3 holds for the problem (10)-(11) with periodic boundary condition, i.e., the boundary condition $u = 0$ on $\partial \Omega$ and $f(t, x, \xi) = f(t, x, \xi^*)$ for $x \in \partial \Omega$, $\xi \cdot n(x) < 0$ ($\xi^* = \xi - 2(\xi \cdot n(x))n(x)$) is the specular velocity, $n(x)$ is the outward normal to $\Omega$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain is replaced by periodic boundary condition.

Remark 2. It is more interesting to consider the hydrodynamic limit of the following system of singular equations (asymptotic regime corresponding to a strong Brownian motion and strong drag force). Take
\[
A = 1, \quad B = \frac{1}{\epsilon}, \quad D = \frac{1}{\epsilon}, \quad E = 1,
\]
then the system (9) becomes
\[
\begin{align*}
\partial_t f^\epsilon + \xi \partial_x f^\epsilon + \frac{1}{\epsilon} \partial_\xi ((u^\epsilon - \xi) f^\epsilon) &= 0, \\
\partial_t \rho^\epsilon + \partial_x (\rho^\epsilon u^\epsilon) &= 0, \\
\partial_t (\rho^\epsilon u^\epsilon) + \partial_x (\rho^\epsilon (u^\epsilon)^2) + \partial_x P^\epsilon - \mu \partial_{xx} u^\epsilon &= -\frac{1}{\epsilon} \int_{\mathbb{R}} (u^\epsilon - \xi) f^\epsilon d\xi.
\end{align*}
\]
Since $\frac{1}{2} \int_{\mathbb{R}} (u^\epsilon - \xi) f^\epsilon d\xi$ is not uniformly bounded with respect to $\epsilon$ in $L^2(0, T; L^1(\Omega))$, it is difficult to obtain the uniformly bound of $\int_0^T \int_{\Omega} (\rho^\epsilon)^2 dx dt$ with respect to $\epsilon$ as Lemma 2.2 ($\int_0^T \int_{\Omega} (\rho^\epsilon)^2 dx dt \leq C$ is used to obtain the strong convergence of $\rho^\epsilon$). The third author of this paper and his collaborators used the relative entropy method in [22] to deal with this case, and obtained the strong convergence of $\rho^\epsilon$, see [7].

Remark 3. In this paper, we take full advantage of the Sobolev embedding inequality $H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$ in one-dimensional space, then we can obtain the key estimates (30)-(31). Furthermore, $H^1_0(\Omega) \hookrightarrow \hookrightarrow C(\overline{\Omega})$ holds in $1D$, we can deduce the convergence of $u^\epsilon$ in $C([0, T]; H^{-1}(\Omega))$, which is used to deal with the convergence of the nonlinear term $n^\epsilon u^\epsilon$, see (40).

Notations: In the following, $C$ from line to line denote the generic positive constants depending on the initial data and $T$, but independent of $\epsilon$.

2. A priori estimates. In this section, we derive some a priori estimates of the initial-boundary value problem (10)-(12), which will help us to do asymptotic analysis of the weak solution. Firstly, we give the basic energy estimates.

Lemma 2.1. Let $(f^\epsilon, \rho^\epsilon, u^\epsilon)$ be the solution to the problem (10)-(12). Then, for any $t \in (0, T)$, it holds that
\[
\int_{\Omega} \int_{\mathbb{R}} (1 + |\xi|^2) f^\epsilon d\xi dx + \int_{\Omega} (\rho^\epsilon |u^\epsilon|^2 + (\rho^\epsilon)^2) dx + \int_0^T \int_{\Omega} |\partial_x u^\epsilon|^2 dx dt \leq C, \quad (25)
\]
\[
\int_0^T \int_{\Omega} \int_{\mathbb{R}} |\epsilon u^\epsilon - \xi|^2 d\xi dx dt \leq C \epsilon^2, \quad (26)
\]

Proof: Since $(f^\epsilon, \rho^\epsilon, u^\epsilon)$ be the weak solution to the problem (10)-(12), then integrating the kinetic equation (10)_1 with respect to $\xi$ and $x$ over $\mathbb{R} \times \Omega$ leads to the following conservation relation:
\[
\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}} f^\epsilon(t, x, \xi) d\xi dx = 0.
\]
Multiplying the kinetic equation \((10)_1\) by \(\frac{1}{2} |\xi|^2\) and integrating the result with respect to \(\xi\) and \(x\), we have
\[
\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}} \frac{1}{2} |\xi|^2 f^\epsilon(t,x,\xi) d\xi dx - \frac{1}{\epsilon^2} \int_{\Omega} \int_{\mathbb{R}} \xi (\rho^\epsilon - \xi) f^\epsilon d\xi dx = 0.
\]
Integrating the fluid equation \((10)_3\) multiplied by \(u^\epsilon\) gives
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho^\epsilon |u^\epsilon|^2 + \frac{1}{\gamma - 1} (\rho^\epsilon)^\gamma \right) dx + \int_{\Omega} |\partial_x u^\epsilon|^2 dx + \frac{1}{\epsilon^2} \int_{\Omega} \int_{\mathbb{R}} \epsilon u^\epsilon (\epsilon u^\epsilon - \xi) f^\epsilon d\xi dx = 0,
\]
where we have used the fact \((10)_2\). Therefore, by adding the three previous relations, we have
\[
\frac{d}{dt} \left\{ \int_{\Omega} \int_{\mathbb{R}} (1 + \frac{1}{2} |\xi|^2) f^\epsilon d\xi dx + \int_{\Omega} \left( \frac{1}{2} \rho^\epsilon |u^\epsilon|^2 + \frac{1}{\gamma - 1} (\rho^\epsilon)^\gamma \right) dx \right\}
+ \int_{\Omega} |\partial_x u^\epsilon|^2 dx + \frac{1}{\epsilon^2} \int_{\Omega} \int_{\mathbb{R}} f^\epsilon |\epsilon u^\epsilon - \xi|^2 d\xi dx = 0,
\]
and this gives \((25)-(26)\). Thus we finish the proof of Lemma 2.1.

Next, we will discuss some estimates associated to the distribution function \(f^\epsilon(t,x,\xi)\).

**Corollary 1.** Let the assumptions of Theorem 1.3 be fulfilled. Then
\[
\| n^\epsilon \|_{L^\infty(0,T; L^1(\Omega))} \leq C, \tag{27}
\]
\[
\| n^\epsilon u^\epsilon \|_{L^2(0,T; L^1(\Omega))} + \| n^\epsilon |u^\epsilon|^2 \|_{L^1(0,T; L^1(\Omega))} \leq C, \tag{28}
\]
and
\[
\left\| \frac{1}{\epsilon} \int_\mathbb{R} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi \right\|_{L^2(0,T; L^1(\Omega))} + \| J^\epsilon \|_{L^2(0,T; L^1(\Omega))} \leq C. \tag{29}
\]

**Proof:** First, using by \((25)\) and the definition of \(n^\epsilon\), we have \((27)\). Next, it follows from \((27)\) and Sobolev embedding \(H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)\), we have
\[
\int_0^T \left( \int_{\Omega} n^\epsilon |u^\epsilon|^2 dx \right)^2 dt \leq \int_0^T \| u^\epsilon(t,\cdot) \|_{L^\infty(\Omega)}^2 \left( \int_{\Omega} n^\epsilon dx \right)^2 dt \leq C \| u^\epsilon \|_{L^2(0,T; H^1_0(\Omega))}^2 \leq C;
\]
where we have used \((25)\) in the last inequality.
Similarly, we can obtain
\[
\int_0^T \int_{\Omega} n^\epsilon |u^\epsilon|^2 dx dt \leq \int_0^T \| u^\epsilon(t,\cdot) \|_{L^\infty(\Omega)}^2 \left( \int_{\Omega} n^\epsilon dx \right) dt \leq C \| u^\epsilon \|_{L^2(0,T; H^1_0(\Omega))}^2 \leq C.
\]
Finally, by using the Hölder inequality, (25) and (26), one deduces that
\[
\left\| \frac{1}{\epsilon} \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi \right\|^2_{L^2(0,T;L^1(\Omega))} = \frac{1}{\epsilon^2} \int_0^T \left( \int_{\Omega} \left| \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi \right| dx \right)^2 dt
\leq \frac{1}{\epsilon^2} \int_0^T \left( \int_{\Omega} \int_{\mathbb{R}} |\epsilon u^\epsilon - \xi|^2 f^\epsilon d\xi dx \cdot \int_{\Omega} \int_{\mathbb{R}} f^\epsilon d\xi dx \right) dt
\leq C.
\]
Noticing that
\[
J^\epsilon = n^\epsilon u^\epsilon - \frac{1}{\epsilon} \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi,
\]
which together with (28) yields the bound of \(J^\epsilon\) in \(L^2(0,T;L^1(\Omega))\). This completes the proof of Lemma 2.1.

Finally, in order to obtain the strong convergence of \(\rho^\epsilon\), we need the higher order integrability of \(\rho^\epsilon\).

**Lemma 2.2.** Let \((f^\epsilon, \rho^\epsilon, u^\epsilon)\) be the solution to the problem (10)-(12). Then, it holds that
\[
\int_0^T \int_{\Omega} (\rho^\epsilon)^{2\gamma} dx dt \leq C. \tag{32}
\]

**Proof.** Since \((f^\epsilon, \rho^\epsilon, u^\epsilon)\) is a global weak solution to the initial-boundary value problem (10)-(12), then (10) holds in the sense of distribution, i.e.,
\[
\int_{\Omega} \rho^\epsilon u^\epsilon \varphi dx|_{t=0}^{t=t} - \int_0^t \int_{\Omega} \rho^\epsilon u^\epsilon \varphi_1 dx ds - \int_0^t \int_{\Omega} \rho^\epsilon |u^\epsilon|^2 \varphi_2 dx ds
\]
\[
- \int_0^t \int_{\Omega} P(\rho^\epsilon) \varphi_2 dx ds + \int_0^t \int_{\Omega} u^\epsilon_\varphi \varphi_2 dx ds
\]
\[
= \int_0^t \int_{\Omega} (J^\epsilon - n^\epsilon n^\epsilon) \varphi dx ds,
\]
for any test function \(\varphi(x,t) \in C^\infty([0,T] \times \bar{\Omega})\). Motivated by [9, 20], take \(\varphi(x,t) = \int_0^x (\rho^\epsilon)^{\gamma}(y,s) dy - x \int_{\Omega} (\rho^\epsilon)^{\gamma}(y,s) dy\), then we have
\[
\int_0^t \int_{\Omega} (\rho^\epsilon)^{2\gamma} dx ds
\]
\[
= \int_{\Omega} \rho^\epsilon u^\epsilon \left( \int_0^x (\rho^\epsilon)^{\gamma} dy - x \int_{\Omega} (\rho^\epsilon)^{\gamma} dy \right) dx
\]
\[
- \int_{\Omega} \rho_0^\epsilon u_0^\epsilon \left( \int_0^x (\rho_0^\epsilon)^{\gamma} dy - x \int_{\Omega} (\rho_0^\epsilon)^{\gamma} dy \right) dx
\]
\[
- \int_0^t \int_{\Omega} \rho^\epsilon u^\epsilon \left( \int_0^x [(\rho^\epsilon)^{\gamma}] dy - x \int_{\Omega} [(\rho^\epsilon)^{\gamma}] dy \right) dx ds
\]
\[
- \int_0^t \int_{\Omega} \rho^\epsilon |u^\epsilon|^2 \left( (\rho^\epsilon)^{\gamma} - \int_{\Omega} (\rho^\epsilon)^{\gamma} dy \right) dx ds
\]
\[
+ \int_0^t \left( \int_{\Omega} (\rho^\epsilon)^{\gamma} dx \right)^2 ds + \int_0^t \int_{\Omega} u_x^\epsilon \left( (\rho^\epsilon)^{\gamma} - \int_{\Omega} (\rho^\epsilon)^{\gamma} dy \right) dx ds
\]
We shall estimate each term in the right-hand side of (34). First, it follows from Cauchy's inequality and (25) that
\[
R_1 + R_2 \leq 2 \int_\Omega \rho^\gamma |u^\epsilon| dx \int_\Omega (\rho^\gamma)^\gamma dx + 2 \int_\Omega \rho_0^\gamma |u_0^\epsilon| dx \int_\Omega (\rho_0^\gamma)^\gamma dx
\leq C \int_\Omega \rho^\gamma |u^\epsilon| dx + C \int_\Omega \rho_0^\gamma |u_0^\epsilon| dx
\leq C \int_\Omega \rho^\gamma |u^\epsilon|^2 dx + C \int_\Omega \rho^\gamma dx + C \int_\Omega \rho_0^\gamma |u_0^\epsilon|^2 dx + C \int_\Omega \rho_0^\gamma dx
\leq C.
\]

Next, for \( R_3 \), it follows from (10)\(_2\) and the boundary condition that
\[
R_3 = \int_0^t \int_\Omega \rho^\gamma u^\epsilon \int_0^x [(\rho^\gamma)^\gamma u^\epsilon_x] dy dx ds - \int_0^t \int_\Omega x \rho^\gamma u^\epsilon \int_0^x [(\rho^\gamma)^\gamma u^\epsilon_x] dy dx ds
= \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + (\gamma - 1) \int_0^t \int_\Omega \rho^\gamma u^\epsilon \int_0^x (\rho^\gamma)^{\gamma} u^\epsilon_x dy dx ds
- (\gamma - 1) \int_0^t \left( \int_\Omega x \rho^\gamma u^\epsilon dx \right) \left( \int_\Omega (\rho^\gamma)^{\gamma} u^\epsilon_x dy \right) ds,
\]
which together with Cauchy's inequality and (25) yields that
\[
R_3 \leq \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + C \left( \int_0^t \int_\Omega \rho^\gamma |u^\epsilon| dx \right) \left( \int_\Omega (\rho^\gamma)^{\gamma} |u^\epsilon_x| dy \right) ds
\leq \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + C \int_0^t \int_\Omega (\rho^\gamma)^{\gamma} |u^\epsilon_x| dx ds
\leq \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + \frac{1}{4} \int_0^t \int_\Omega (\rho^\gamma)^{2\gamma} dx ds + C.
\]

For \( R_4-R_6 \), the combination of (25) and Cauchy's inequality lead to
\[
R_4 = - \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + \int_0^t \left( \int_\Omega \rho^\gamma |u^\epsilon| dx \right) \left( \int_\Omega (\rho^\gamma)^{\gamma} dy \right) ds
\leq - \int_0^t \int_\Omega (\rho^\gamma)^{\gamma+1} |u^\epsilon|^2 dx ds + C.
\]
\[
R_5 = \int_0^t \left( \int_\Omega (\rho^\gamma)^{\gamma} dx \right)^2 ds \leq C.
\]
\[
R_6 = \int_0^t \int_\Omega u_x^\epsilon \left( (\rho^\gamma)\gamma - \int_\Omega (\rho^\gamma)^{\gamma} dy \right) dx ds
\leq \int_0^t \int_\Omega |u_x^\epsilon| (\rho^\gamma)^{\gamma} dx ds + C \int_0^t \int_\Omega |u_x^\epsilon| dx ds \leq \frac{1}{4} \int_0^t \int_\Omega (\rho^\gamma)^{2\gamma} dx ds + C.
\]
Finally, due to (25), (28) and (29), one has

\[
R_7 = - \int_0^T \int_\Omega (J' - n'u') \cdot \left( \int_0^x (\rho')^\gamma dy - x \int_\Omega (\rho')^\gamma dy \right) dx ds
\]

\[
\leq C \|J' - n'u'\|_{L^1(0,T;L^1(\Omega))} \cdot \sup_{0 \leq t \leq T} \int_\Omega (\rho')^\gamma dx
\]

\[
\leq C \|J' - n'u'\|_{L^2(0,T;L^1(\Omega))} \leq C.
\]

This completes the proof of Lemma 2.2.

3. Passing to the limit. In this section, we will prove our main theorem. We first present two auxiliary compactness lemmas which be used in the following proof.

**Lemma 3.1 ([23]).** Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow E \). Then the following imbeddings are compact:

(i) \( \{ \varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y) \} \hookrightarrow L^q(0,T;E) \), if \( 1 \leq q \leq \infty \);

(ii) \( \{ \varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y) \} \hookrightarrow C([0,T];E) \), if \( 1 < r \leq \infty \).

**Lemma 3.2 ([20]).** Let \( X \) be a separable reflexive Banach space, \( Y \) is a Banach space such that \( X \hookrightarrow Y \), \( Y' \) is separable and dense in \( X' \). Assume \( \{ g^n \} \) satisfies: \( \|g^n\|_{L^\infty(0,T;X)} \leq C, \|\partial_t g^n\|_{L^p(0,T;Y)} \leq C \), for some \( 1 < p \leq \infty \). Then \( g^n \) is relatively compact in \( C([0,T];X_{weak}) \).

From the estimates discussed above, we can deal with a subsequence such that

\[
\begin{align*}
\rho' & \rightharpoonup \rho \text{ weakly } * \text{ in } L^\infty(0,T;L^\gamma(\Omega)), \\
\rho' & \rightharpoonup \rho \text{ weakly in } L^2(\Omega), \\
(\rho')^\gamma & \rightharpoonup \overline{\rho} \text{ weakly in } L^2(\Omega), \\
u' & \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_0(\Omega)), \\
Lu & \rightharpoonup f \text{ weakly } * \text{ in } L^\infty(0,T;M^1(\Omega)), \\
n' & \rightharpoonup n \text{ weakly } * \text{ in } L^\infty(0,T;M^1(\Omega)), \\
J' & \rightharpoonup J \text{ weakly in } L^2(0,T;M^1(\Omega)).
\end{align*}
\]

**Step 1. The convergence of \( n' \).** Integrating the kinetic equation (10) with respect to \( \xi \), one obtains that

\[
\partial_t n' + \partial_x J' = 0,
\]

then by the convergence in (35), we can obtain

\[
\partial_t n + \partial_x J = 0, \text{ in } D'((0,T) \times \Omega).
\]

Furthermore, (29) and (36) imply

\[
\|\partial_t n'\|_{L^2(0,T;W^{-1,1}(\Omega))} \leq C.
\]

Since \( L^1(\Omega) \subset M^1(\Omega) = (C(\Omega))^' \hookrightarrow H^{-1}(\Omega) \subset W^{-1,1}(\Omega) \), then by (27) and Lemma 3.1, we can obtain

\[
n' \rightharpoonup n \text{ in } C([0,T];H^{-1}(\Omega)),
\]

which together with the convergence \( u' \) in (35), imply

\[
n'u' \rightharpoonup nu \text{ in } D'((0,T) \times \Omega).
\]
Next, multiplying the Vlasov equation (10) by $\xi$ and integrating the resulting equality over $\mathbb{R}$, one deduces that
\[ \epsilon^2 \partial_t J^\epsilon + \partial_x \left( \int_{\mathbb{R}} |\xi|^2 f^\epsilon d\xi \right) = n^\epsilon u^\epsilon - J^\epsilon. \] (41)

Since
\[ \int_{\mathbb{R}} |\xi|^2 f^\epsilon d\xi = -\int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi + \epsilon \int_{\mathbb{R}} u^\epsilon f^\epsilon \xi d\xi. \] (42)

By using (25) and (26), we have
\[ \left\| \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi \right\|_{L^1((0,T) \times \Omega)} \leq \left( \int_0^T \int_{\Omega} |\epsilon u^\epsilon - \xi|^2 f^\epsilon d\xi dx dt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Omega} |\xi|^2 f^\epsilon d\xi dx dt \right)^{\frac{1}{2}} \leq C \epsilon, \] (43)

and we have from (25) and (28) that
\[ \left\| \int_{\mathbb{R}} u^\epsilon f^\epsilon \xi d\xi \right\|_{L^1((0,T) \times \Omega)} = \int_0^T \int_{\Omega} \left| \int_{\mathbb{R}} u^\epsilon f^\epsilon \xi d\xi \right| dx dt \leq \int_0^T \int_{\Omega} \int_{\mathbb{R}} |u^\epsilon|^2 f^\epsilon \xi d\xi dx dt \leq \left( \int_0^T \int_{\Omega} f^\epsilon |u^\epsilon|^2 dx dt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Omega} |\xi|^2 f^\epsilon d\xi dx dt \right)^{\frac{1}{2}} \leq C. \] (44)

Then (42)-(44) imply
\[ \int_{\mathbb{R}} |\xi|^2 f^\epsilon d\xi \rightharpoonup 0, \text{ in } D'(\mathbb{R} \times \Omega). \] (45)

Returning to (41), we have, as $\epsilon \to 0$,
\[ J = nu, \text{ in } D'((0,T) \times \Omega). \] (46)

**Step 2. The convergence of the fluid parts.** Since we have obtained the estimates $\int_0^T \int_{\Omega} (\rho^\epsilon)^2 dx dt \leq C$, and
\[ \left\| \frac{1}{\epsilon} \int_{\mathbb{R}} (\epsilon u^\epsilon - \xi) f^\epsilon d\xi \right\|_{L^2(0,T;L^1(\Omega))} \leq C, \] (47)

then following the same ideas in [8, 9, 20], we can obtain
\[ \rho^\epsilon \rightharpoonup \rho \text{ in } L^1((0,T) \times \Omega) \text{ and } C([0,T];L^\gamma_{\text{weak}}(\Omega)), \] (48)

Then by (32), the strong convergence of $\rho^\epsilon$, and the interpolation inequality, we have
\[ \begin{cases} \rho^\epsilon \to \rho \text{ in } L^p((0,T) \times \Omega), \text{ for any } 1 \leq p < 2\gamma, \\ (\rho^\epsilon)^\gamma \to \rho^\gamma \text{ in } L^1((0,T) \times \Omega). \end{cases} \] (49)
Moreover, from (25), one obtains
\[ \|\rho^\epsilon u^\epsilon\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \leq C. \]  
(50)

From (10), we get
\[ \frac{\partial}{\partial t}(\rho^\epsilon u^\epsilon) \text{ is bounded in } L^\infty(0,T;W^{-1,1}(\Omega)) \]
\[ + L^2(0,T;H^{-1}(\Omega)) + L^2(0,T;L^1(\Omega)) \]
\[ \subset L^2(0,T;W^{-1,1}(\Omega)), \]  
(51)

and \( L^{\frac{2\gamma}{\gamma-1}}(\Omega) \hookrightarrow L^1(\Omega) \subset M^1(\Omega) = (C(\bar{\Omega}))' \hookrightarrow H^{-1}(\Omega) \subset W^{-1,1}(\Omega) \), then by Lemmas 3.1-3.2, one deduces that
\[ \rho^\epsilon u^\epsilon \rightarrow \rho u \text{ in } C([0,T];L^{\frac{2\gamma}{\gamma-1}}(\Omega)), \]
\[ \rho^\epsilon u^\epsilon \rightarrow \rho u \text{ in } C([0,T];H^{-1}(\Omega)). \]  
(52)

From the convergence \( u^\epsilon(t,x) \) in (35) and (52), we have
\[ \rho^\epsilon(u^\epsilon)^2 \rightharpoonup \rho u^2 \text{ in } D'((0,T) \times \Omega), \]  
(53)

With the above convergence (35), (40), (49), (52) and (53), we can obtain
\[ \begin{cases}
\partial_t \rho + \partial_x(\rho u) = 0, \\
\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x\rho^\gamma - \partial_{xx}u = 0,
\end{cases} \]  
(54)

in \( D'((0,T) \times \Omega) \). This together with (37) completes the proof of Theorem 1.3.

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