The Convergence Radius of the Chiral Expansion in the
Dyson-Schwinger Approach

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Abstract

We determine the convergence radius \( m_{\text{conv}} \) for the expansion in the current quark mass using the Dyson-Schwinger (DS) equation of QCD in the rainbow approximation. Within a Gaussian form for the gluon propagator \( D_{\mu\nu}(p) \sim \delta_{\mu\nu} \chi^2 e^{-\frac{p^2}{\Delta}} \) we find that \( m_{\text{conv}} \) increases with decreasing width \( \Delta \) and increasing strength \( \chi^2 \). For those values of \( \chi^2 \) and \( \Delta \), which provide the best known description of low energy hadronic phenomena, \( m_{\text{conv}} \) lies around \( 2\Lambda_{\text{QCD}} \), which is big enough, that the chiral expansion in the strange sector converges. Our analysis also explains the rather low value of \( m_{\text{conv}} \approx 50 \ldots 80 \text{ MeV} \) in the Nambu–Jona-Lasinio model, which as itself can be regarded as a special case of the rainbow DS models, where the gluon propagator is a constant in momentum space.
The chiral perturbation theory (χPT) has turned out to be a rather powerful technique for analyzing hadronic phenomena involving Goldstone bosons (π, K) as well as nucleons at low energies [1–3]. One crucial assumption in this approach is the smallness of the chiral symmetry breaking current mass $m_0$ compared with $\Lambda_{QCD} \approx 200$ MeV. This is surely fulfilled in the $u$-$d$ quark sector ($m_u = 5$ MeV, $m_d = 9$ MeV) but questionable in case of $s$-quarks ($m_s \approx 150$ MeV) [4]. It is therefore important to look at the convergence of the chiral perturbation expansion for values of quark masses in this region. There are mainly two features, which determine the convergence of this series:

1. Non-analytic terms, such as $m_0^{3/2}$ or $\ln m_0$, which arise from Goldstone boson (π, K) loops [5].

2. The convergence of the power series $\sum c_n m_0^n$ itself already on the mean field level, i.e. without any boson loops [6].

In this paper we will focus completely on the second point. This problem has been studied originally in the framework of the chiral σ model by Carruthers and Haymaker (CH) [3] and recently within the Nambu–Jona-Lasinio (NJL) model by Hatsuda [8, 9]. The basic idea as well as the main results are practically the same in both cases [1]. Let us take the NJL model as an example and briefly review the main points. The convergence radius $m_{\text{conv}}$ is determined by the existence of a solution of the classical equation of motion (gap equation) for the constituent quark mass $M$, which reads in Euclidean space:

\[ M = m_0 + 8 N_c G \int_{\Lambda_{UV}} d^4 q \frac{M}{(2\pi)^4 q^2 + M^2} \]

where $G$ denotes the strength of the four fermion point coupling and $\Lambda_{UV}$ the ultraviolet cutoff. Eq. (1) is solved for $M$ with $m_0$ as input, which gives a relation $M = M(m_0)$ (c.f. fig.1). Hereby $G$ and $\Lambda_{UV}$ are fixed and adjusted to reproduce the experimental value of $\Lambda_{QCD}$.

1One should note that the chiral σ model and the NJL model are closely connected through the gradient expansion [10].
the pion decay constant $f_\pi$ as well as a constituent quark mass of $M = 300\text{MeV}$ in the chiral limit ($m_0 = 0$), which can be assumed to be a physically reasonable value. Point (A) in fig.1 corresponds to the vacuum with spontaneously broken chiral symmetry ($m_0 = 0$, $M = 300\text{MeV}$). This is the point around which the chiral expansion in $m_0$ is performed. As one can see the relevant region for the convergence of the series

$$M(m_0) = \sum_n c_n m_0^n$$

are the negative values of $m_0$, i.e. the branch of the curve between the points (A) and (B). If $m_0$ is smaller than $-|m_{\text{conv}}|$ (point (B), where $\frac{\partial m_0}{\partial M} = 0$) no solution of eq.1 exists, that can be reached continuously from A. This means the breakdown of the series (2) for all values of $m_0$ with $|m_0| > |m_{\text{conv}}|$. From fig.1 we can read off $m_{\text{conv}} \approx 70\text{MeV}$. It has been checked that this value does not depend crucially on the form of the regularization scheme (e.g.O(3) and O(4) sharp momentum cutoffs, proper time etc. [11]). From this it follows that in case of the NJL model the strange quark mass $m_s \approx 150\text{MeV}$ lies far beyond $m_{\text{conv}}$. Hatsuda has demonstrated the failure of the convergence of the series (2) explicitly for various observables such as the quark condensate $\langle \bar{q}q \rangle$ or the nucleon sigma term $\Sigma_{\pi N}$ [8,9].

It is the aim of this paper to study this problem within the rainbow Dyson-Schwinger (DS) approach to QCD [12,13] (for a comprehensive review c.f. [14]). This model has been extensively used for describing mesonic and vacuum properties [15–18] and at least tentatively also for the calculation of nucleon observables [19–23]. The general form of the rainbow DS equation for the quark propagator $S(p)$ reads:

$$(-i)\Sigma(p) = \frac{4}{3} g_s^2 \int \frac{d^4k}{(2\pi)^4} [(i\gamma_\mu)(iS(k))(iD^{\mu\nu}(p-k)(i\gamma_\nu))]$$

where $\Sigma(p)$ denotes the quark self energy defined by

$$S^{-1}(p) = p - m_0 - \Sigma(p)$$

and $D^{\mu\nu}(p-k)$ the gluon propagator. After continuation to Euclidean space time and
using the Feynman gauge with the running coupling constant $\alpha(p^2)$ (where $p$ denotes the Euclidean 4-momentum and $p = \sqrt{p \cdot p}$)

$$D^{\mu\nu}(p) = \delta^{\mu\nu}D(p^2) = \delta^{\mu\nu}\frac{4\pi \alpha(p^2)}{g_s^2 p^2}$$

as well as the decomposition of the self energy $\Sigma$

$$\Sigma(p) = p[A(p^2) - 1] + B(p^2) - m_0$$

one obtains the coupled set of equations:

$$p^2[A(p^2) - 1] = \frac{8}{3}g_s^2 \int \frac{d^4q}{(2\pi)^4} D((p - q)^2) \frac{A(q^2)p \cdot q}{q^2 A^2(q^2) + B^2(q^2)}$$

$$B(p^2) = m_0 + \frac{16}{3}g_s^2 \int \frac{d^4q}{(2\pi)^4} D((p - q)^2) \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)}$$

As soon as a form for $D(p^2)$ is specified the eqs. (7) and (8) define a certain model. It is easy to convince oneself that the gap equation of the NJL (1) appears as a special case of eqs. (7) and (8) if the gluon propagator is a constant in momentum space

$$D_{NJL}(p^2) = \frac{3N_c}{2}G$$

or equivalently

$$\alpha_{NJL} = \frac{3N_c}{2}G \frac{g_s}{4\pi} p^2$$

which gives a contact interaction in coordinate space with the momentum independent solutions $A(p^2) = 1$ and $B(p^2) = M$. From a principle point of view none of these models, i.e. none of the gluon propagators is preferred over the other, because none of them can be derived from QCD. On the other hand the general features of QCD are more likely better described with a running coupling $\alpha(p^2)$, which on the one side grows in the infrared region and therefore gives rise to “confined” quarks and on the other side shows asymptotic freedom in the ultraviolet region

$$\alpha(p^2) \overset{p^2 \to \infty}{=} \frac{d\pi}{\ln(p^2/\Lambda_{QCD}^2)}$$
(with \( d = \frac{12}{33-2N_f} \)), rather than with a form like \( \alpha_{NJL}(p^2) \sim p^2 \), which gets small in the IR region but grows for high \( p^2 \) and has to be cut off at \( \Lambda_{QCD} \). In refs. [15,17] a superposition of a Gaussian form with strength \( \chi^2 \) and width \( \Delta \) (dominating at small \( p^2 \)) and the standard asymptotic form (dominating at high \( p^2 \))

\[
\alpha(p^2) = \frac{3\pi\chi^2}{4} \left( \frac{p^2}{\Delta^2} \right) e^{-p^2/\Delta} + \frac{d\pi}{\ln(\tau + p^2/\Lambda^2)}
\]  

with \( \Lambda = \Lambda_{QCD} = 190\text{MeV}, \chi = 1.14\text{GeV}, \Delta = 0.002\text{GeV}^2 \) and \( \tau = 3.0 \) has been shown to provide a reasonable description of low energy hadronic phenomena. Furthermore it turned out that the strength of \( \alpha(p^2) \) in the IR domain is large enough that the model shows spontaneously broken chiral symmetry, i.e. the DS equation (7,8) has a nontrivial solution \( B(p^2) \neq 0 \) if \( m_0 = 0 \), as well as “confinement”, in the sense that the quark propagator \( S(p) \) does not have a pole at timelike \( p^2 \).

For our study of the convergence of the chiral expansion in this model we will first consider a pure Gaussian ansatz

\[
\alpha(p^2) = \frac{3\pi\chi^2}{4} \left( \frac{p^2}{\Delta^2} \right) e^{-p^2/\Delta}
\]  

and vary the strength \( \chi^2 \) as well as the width \( \Delta \). For the determination of \( m_{\text{conv}} \) we have again to look at negative \( m_0 \). In order to do so we write the set of the DS equations (7,8) in the form:

\[
p^2[A(p) - 1] = \frac{4}{3} \int_0^\infty dq q A(q^2) \frac{A(q^2)}{q^2 A(q^2) + B^2(q)} \cdot K_A(p, q)
\]

\[
B(p) = B(0) + \frac{4}{3} \int_0^\infty dq q^3 \frac{B(q)}{q^2 A(q^2) + B^2(q)} \cdot [K_B(p, q) - K_B(0, 0)]
\]

where the integral kernels \( K_A \) and \( K_B \) are given by:

\[
K_A(p, q) = \int d\Omega \frac{\alpha(p - q)^2}{(p - q)^2} (pq) = \frac{3\chi^2}{4\Delta} e^{-\frac{p^2 + q^2}{\Delta}} \left[ \frac{I_0 \left( \frac{2pq}{\Delta} \right)}{2} + \frac{I_2 \left( \frac{2pq}{\Delta} \right)}{2} - \frac{I_1 \left( \frac{2pq}{\Delta} \right)}{\left( \frac{2pq}{\Delta} \right)} \right]
\]

and

\[
K_B(p, q) = \int d\Omega \frac{\alpha(p - q)^2}{(p - q)^2} = 3\chi^2 e^{-\frac{p^2+q^2}{\Delta}} \frac{I_1 \left( \frac{2pq}{\Delta} \right)}{\left( \frac{2pq}{\Delta} \right)}
\]
respectively. Hereby \( I_n \) denotes the Bessel function of order \( n \). The set of integral equations (14) and (15) is formally independent of \( m_0 \) but instead depends on the initial value \( B(0) \). It can be uniquely solved for any given \( B(0) \) and the corresponding \( m_0 \) can then be extracted from the solution \( B(p) \) at high \( p \)

\[
\lim_{p \to \infty} B(p) = m_0
\]  

(18)

This renders a relationship between \( B(0) \) and \( m_0 \), i.e. \( m_0 = m_0[B(0)] \), analogous to the one displayed in fig.1. The convergence radius \( m_{\text{conv}} \) is then determined by the minimum of this curve, i.e. the point:

\[
\frac{\partial m_0}{\partial B(0)} = 0
\]  

(19)

The set of eqs.(14) and (15) is solved by a selfconsistent procedure in the interval \([0, 1000\Lambda]\) with logarithmic grid points.

1. Let us first look at solutions with various widths within the Gaussian parameterization (13) keeping \( \chi^2 \) fixed at \( \chi = 8.0\Lambda \). The forms of the corresponding running coupling constants \( \alpha(p) \) are displayed in fig.2, where for comparison we have included the one of the NJL eq.(10) in addition. In fig.3 we compare the solutions \( B(p) \) for a fixed value of \( B(0) = 7.75 \), whereas fig.4 shows the dependence \( m_0 = m_0[B(0)] \). It is interesting to look at the limes \( \Delta \to 0 \), which means, in fact, that the gluon propagator approaches a \( \delta \)-function in momentum space (dotted line in fig.4):

\[
D(p - q) = (2\pi)^4 \frac{3}{16} \frac{1}{g_s^2} \chi^2 \delta^{(4)}(p - q)
\]  

(20)

In this case we obtain an algebraic relation between \( m_0 \) and \( B(0) \) from eq.(8):

\[
m_0 = B(0) - \frac{\chi^2}{B(0)}
\]  

(21)

From figs.3 and 4 we can clearly see, that \( m_{\text{conv}} \) rises with decreasing \( \Delta \). For small values of \( \Delta \) (e.g. \( \Delta = 0.2 \Lambda^2 \), full line in fig.4) one faces the problem that the self
consistent iteration procedure for the solution of the system (14) and (15) does not converge but ends up switching between two or more configurations, if the value of \( B(0) \) deviates too much from the one in the chiral limit \( (m_0 = 0) \). This is a typical feature encountered in many nonlinear systems and does not mean that there exists no solution at all, but only that the simple selfconsistent procedure is not able to find it. It might be possible that more elaborate methods are successful in this case. On the other side we are clearly able to give at least a minimum value for \( m_{\text{conv}} \) from the corresponding curve in fig.4, which is sufficient for our analysis. Furthermore we recognize that for those small \( \Delta \) the curve, as far as it can be calculated from the self consistent procedure, lies very close to the one obtained with a \( \delta \)-function gluon propagator \( (\Delta = 0, \text{dotted line in fig.4}) \).

2. For \( \Delta = 0 \) it is clear from eq.(21) that

\[
\{m_0[B(0)]\}_{\chi_1} > \{m_0[B(0)]\}_{\chi_2}
\]

if \( \chi_1 < \chi_2 \). We have checked that this relation holds generally if the value of \( \Delta \). From this we conclude that \( m_{\text{conv}} \) increases with increasing strength \( \chi^2 \).

3. Finally we have considered also gluon propagators of the form (12), which have in addition to the Gaussian form in the IR region have the logarithmic tail at high \( p^2 \) leading to asymptotic freedom. In this case the momentum integrals in eqs.(5),(8), (14) and (15) run from 0 to the “renormalization point” \( \mu^2 \) and the \( m_0 \) is considered as the running mass \( m_0(\mu) \). One can convince oneself, that the DS equations eqs.(7) and (8) or (14) and (15) are consistent with the renormalization group results [24–26,13]:

\[
B(p^2)^{p^2 \to \infty} \sim \left[ m_0(\mu) \left( \ln \frac{\mu^2}{\Lambda^2} \right)^d \left( \ln \frac{p^2}{\Lambda^2} \right)^{-d} \right] + \left[ -\frac{4\pi^2d}{3} \langle \bar{q}q \rangle(\mu) \left( \ln \frac{\mu^2}{\Lambda^2} \right)^{-d} \right] \frac{1}{p^2} \left( \ln \frac{p^2}{\Lambda^2} \right)^{d-1}
\]

and

\[
(23)
\]
\[ m_0(\mu) = m_0(\mu) \left( \frac{\ln \frac{\mu^2}{\Lambda^2}}{\ln \frac{\mu^2}{\Lambda^2}} \right)^d \] (24)

The general behavior of the relationship \( m_0(\mu) = m_0(\mu)[B(0)] \) is the same as without this asymptotic tail. With the parameter set from refs. [14,17] mentioned above we obtain a convergence radius of at least \( m_{\text{conv}}(\mu = 1000\Lambda) = 1.6\Lambda \). Due to eq.(24) this corresponds at a typical hadronic scale of \( \mu \approx 1\text{GeV} \) to a value of \( m_{\text{conv}}(\mu \approx 1\text{GeV}) \approx 3\Lambda \), which lies clearly above the strange quark mass.

Our results can be summarized as follow:

1. In the framework of the rainbow DS approach, where the form of the gluon propagator \( D(p^2) \) is taken as input, the convergence radius of the chiral perturbation expansion \( m_{\text{conv}} \) turns out to be crucially dependent on the infrared behavior of the gluon propagator.

2. \( m_{\text{conv}} \) gets larger if in the IR domain the overall strength of the gluon propagator increases or its width decreases.

3. Using a parameterization \( D(p^2) \) which provides a good description of hadronic phenomena at low energies we obtain a value for \( m_{\text{conv}} \) which is clearly larger than the strange quark mass and therefore the chiral expansion in the strange quark sector converges.

4. Especially we are now able to explain the rather low value of \( m_{\text{conv}} \) and therefore the poor convergence of the chiral expansion in the strange quark sector in case of the NJL or the chiral \( \sigma \) model, where the corresponding gluon propagator is a “small” constant in momentum space. Our results indicate that this small value of \( m_{\text{conv}} \approx 50\ldots80\text{MeV} \) in these models arises more likely due to the special form of a contact interaction than it is a general feature of QCD and the early breakdown of the chiral expansion does not occur in a theory with infrared slavery.
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FIGURES

FIG. 1. Determination of $m_{\text{conv}}$ in the NJL with proper-time regularization, $\Lambda_{UV} = 634\text{MeV}$, $G = 30\Lambda_{UV}^{-2}$.

FIG. 2. Comparison between running coupling constants $\alpha(p)$: Gaussian form (eq. (13) with $\chi = 8\Lambda$ and $\Lambda = 200\text{MeV}$) for various values of $\Delta$ and the NJL (eq. (10) with $\Lambda_{UV} = 634\text{MeV}$, $G = 30\Lambda_{UV}^{-2}$).

FIG. 3. The solution $B(p)$ of the DS equations (14) and (15) for a Gaussian running coupling $\alpha(p)$ with various widths $\Delta$ ($\chi = 8\Lambda$, $B(0) = 7.75\Lambda$, $\Lambda = 200\text{MeV}$).

FIG. 4. The dependence $m_0 = m_0[B(0)]$ obtained from the solutions of the DS equation (13) using a Gaussian running coupling compared with the $\delta$-fct.limes $\Delta = 0$ (eq. (20)) and the NJL. The parameters are the same as in fig. 2.
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http://arxiv.org/ps/hep-ph/9405289v1
This figure "fig1-2.png" is available in "png" format from:

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\[ \Delta = 0.20 \cdot \Lambda^2 \]

\[ \Delta = 0.85 \cdot \Lambda^2 \]

\[ \Delta = 2.50 \cdot \Lambda^2 \]

\[ \text{NJL} \]
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\[ \frac{B}{\Lambda} = 0.20 \cdot \Lambda^2 \]

\[ \frac{B}{\Lambda} = 0.50 \cdot \Lambda^2 \]

\[ \frac{B}{\Lambda} = 0.85 \cdot \Lambda^2 \]

\[ \frac{B}{\Lambda} = 2.50 \cdot \Lambda^2 \]
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$\Delta = 0$ (δ-fct.)
$\Delta = 0.20 \Lambda^2$
$\Delta = 0.85 \Lambda^2$
$\Delta = 2.50 \Lambda^2$
NJL