THE JONES POLYNOMIAL AND RELATED PROPERTIES OF SOME TWISTED LINKS

DAVID EMMES

ABSTRACT. Twisted links are obtained from a base link by starting with an \(n\)-braid representation, choosing several (\(m\)) adjacent strands, and applying one or more twists to the set. Various restrictions may be applied, e.g. the twists may be required to be positive or full twists, or the base braid may be required to have a certain form.

The Jones polynomial of full \(m\)-twisted links have some interesting properties. It is known that when sufficiently many full \(m\)-twists are added that the coefficients break up into disjoint blocks which are independent of the number of full twists. These blocks are separated by constants which alternate in sign. Other features are known. This paper presents the value of these constants when two strands of a three-braid are twisted. It also discloses when this pattern emerges for either two or three strand twisting of a three-braid, along with other properties.

Lorenz links and the equivalent T-links are positively twisted links of a special form. This paper presents the Jones polynomial for such links which have braid index three. Some families of braid representations whose closures are identical links are given.

1. Introduction

The primary purpose of this paper is to display the Jones polynomial for various twisted three-braid links and to describe properties of the Jones polynomial for positive three-braids (Section 2). These include the family of Lorenz links of braid index three, \(L_3\). In Section 2 we show exactly when two prototypical braid word representatives for \(L_3\) generate the same link in terms of the braid words themselves. A simple unique three-braid representation for links in \(L_3\) is given. This extends the knot classification result by R. Bedient, [1]. For these Lorenz links of braid index (two or) three, the Jones polynomial has the interesting property that two links are equal exactly when their Jones polynomials are equal.

A second objective of this paper is to present some braid properties of twisted \(n\)-braids and show how these lead to certain symmetries. Any pair of \(n\)-braid words exhibiting one of these symmetries generates the same link. These are described in Section 3 along with their application to Lorenz links.

Lorenz links were originally investigated in [5]. More recent work by J. Birman and I. Kofman, [4], shows there is a one-to-one correspondance with T-links, which have a simple braid word description (Section 1.1). Section 4 describes each family of T-links of braid index three. Thm. 4.5 shows that the family of three-braid links,

2000 Mathematics Subject Classification. 57M25, 57M27.

Key words and phrases. Twisted links, Lorenz links, Jones polynomial, T-links, skein relation, braids and braid groups.
full twists, the coefficient vector for the Jones polynomial has mirror image of
L. Furthermore, the length of the inter-block sections increases at a rate proportional
of alternating positive and negative numbers all with the same absolute value.
zeros. When m is even, these blocks are either separated by zeros or by a pattern
of alternating positive and negative numbers all with the same absolute value.
Furthermore, the length of the inter-block sections increases at a rate proportional
to the number of full twists.

These results of A. Champanerkar and I. Kofman are given an explicit form for
twisted three-braids and in particular for L3 (Section 2).

1.1. Background, Notation and Conventions. The braid group on n strands
is designated Bn, and has n − 1 standard generators, σi. The notation β ≡ γ, for
braid words, β, γ, means that β and γ are conjugate. The exponent sum, or writhe,
of a braid word, β, is denoted w(β). The link associated with the standard closure
of a braid, β, is denoted β̂. The braid index of a link, L is denoted b(L) and the
mirror image of L is denoted T.

Braid reflection refers to the replacement of σi by σn−i in each instance of a
n-braid word, β, for all i, 1 ≤ i < n, yielding a new n-braid word, β∗. A very useful
result, [2, 19, 3] is that a braid word and its reflection are conjugate, β ≡ β∗.

The convenient notation [a, b] = σa · · · σb−1 for a < b and [a, b] = σa−1 · · · σb for
a > b is borrowed from [3]. We adopt the convention that [a, a] = 1. A typical
representation for a T-link is T((r1, s1), . . . , (rk, sk)), with 2 ≤ r1 and r1 ≤ r1+i
and 0 < si for all i. As noted in [4], we may also assume sk ≥ 2. In this paper,
when ri < ri+1 for all i, the parameter k is said to refer to the number of tiers. A
braid representation for T((r1, s1), . . . , (rk, sk)) is [1, r1] · · · [1, rk]k.

The term duality is used to refer to Cor. 4 [3] (general case) or Ex. 3 [4] for the
two-tier case, wherein two specific T-links are shown to represent the same link.
These are T((r1, s1), . . . , (rk, sk)) and T((σ, σ), . . . , (σ, σ)), with σ = ∑k+1−i, sj,
and σ = rσ−ri−1 − rσ−i with the convention r0 = 0. In this context, torus links
are just T-links with a single tier. The elementary torus links are those with two
strands, i.e. r1 = 2, and are designated Tr = T(2, s).

The notation, [n]m, for positive integers n, m refers to the remainder when n is
divided by m; hence n = m[n/n] + [n]m. Parity refers to [n]2. Some convenient
abbreviations for this work are to use G = 1 + t + t2 and εx = (−1)x. Based on
some prior usage, [22], when f ∈ Z[t], let [f]a be the coefficient of ta, with [f]max
the leading coefficient; [f]max = 0 when f = 0. An AC (alternating coefficient)
polynomial is non-zero and satisfies [f]j[f]j+1 < 0 for min deg f ≤ j < deg f. The
Kronecker delta, δi,j, is one when i = j is true, and zero otherwise.

The skein relation for the Homflypt polynomial is defined by (13), where the
diagrams, D+, D0, and D−. below refer to the usual diagrams for the link with posi-
tive crossing, null (smoothed) crossing, and negative crossing, respectively. The
standard name for the Homflypt polynomial of an oriented link, \( L \), is \( P_L \) or \( P_L(v, z) \); for the Jones polynomial \( V \) or \( V_L(t) = P_L(t, (t - 1)/\sqrt{t}) \); for the Alexander polynomial, \( \Delta_L \) or \( \Delta_L(t) = P_L(1, (t - 1)/\sqrt{t}) \); for the Conway polynomial, \( \nabla_L \) or \( \nabla_L(z) = P_L(1, z) \).

\[
P_{D_+}(v, z) = vzP_{D_0}(v, z) + v^2P_{D_-}(v, z). \tag{1.1}
\]

We may interpret \( A_w = A_w(t) = (t^w + \epsilon_{w-1})/(t + 1) \) as another version of the Alexander polynomial for \( T_w \), and when \( w \) is positive, \( A_w = \sum_{j=0}^{w-1} \epsilon_j t^{w-1-j} \). For negative \( w \), we have \( A_w = \epsilon_{w-t}t^wA_{|w|} \). Finally, \( A_w(1/t) = \epsilon_{w-1}t^{1-w}A_w(t) \). Many of the results involve manipulations of this simple AC polynomial. In particular we have for any integers \( w, z \) and \( x \geq y \geq 0 \):

\[
A_{w+z-1} = A_wA_z + tA_{w-1}A_{z-1} \tag{1.3}
\]

\[
A_xA_y = \sum_{j=0}^{y-2} \epsilon_j(j+1)t^{x+y-2-j} + (y) \sum_{j=1}^{x-1} \epsilon_j t^{x+y-2-j} + \sum_{j=x}^{y-2} \epsilon_j(x+y-1-j)t^{x+y-2-j}. \tag{1.4}
\]

The usual conventions that \( \sum_{j=a}^{b} f_j = \begin{pmatrix} b \\ a \end{pmatrix} = 0 \) when \( a > b \) are followed. Also \( \prod_{j=a}^{b} f_j = 1 \) when \( a > b \). For convenience, the following functions are defined:

\[
L(b) = \sum_{j=0}^{b} \epsilon_j(1+b-j)t^{b-j} = \epsilon_b \sum_{j=0}^{b} \epsilon_j(1+j)t^j,
\]

\[
R(b) = \sum_{j=0}^{b} \epsilon_j(j+1)t^{b-j} = \epsilon_b \sum_{j=0}^{b} \epsilon_j(1+b-j)t^j.
\]

These may also be expressed as

\[
L(b) = \sum_{j=0}^{b} t^jA_{b+1-j} = \sum_{j=0}^{b} t^{b-j}A_{j+1},
\]

\[
R(b) = \sum_{j=0}^{b} \epsilon_jA_{b+1-j} = \sum_{j=0}^{b} \epsilon_{b-j}A_{j+1}.
\]

These expressions imply \( L(b) = A_{b+1} + tL(b-1) \) and \( R(b) = A_{b+1} - R(b-1) \). Hence for \( x \geq y \geq 0 \) we have the expression \( (1.5) \) for \( A_xA_y \) which partitions the coefficients and terms into three disjoint sets. The absolute value of the coefficients are monotonically increasing for the first \( y-1 \) terms, constant and maximal for the next \( x+y+1 \) terms, and monotonically decreasing for the final \( y-1 \) terms.

\[
A_xA_y = \epsilon_y L(y-2) - (y)\epsilon_y t^{y-1}A_{x-y+1} + t^x R(y-2). \tag{1.5}
\]

The properties just described and reflected in \( (1.5) \) are hereditary, as described in the next proposition and its corollary. These results are central to the remainder of the paper.
Proposition 1.1. Suppose $f, g$ are AC polynomials of degree $d$ with coefficients, $a_j, b_j$ for $t^j$. Suppose $\Lambda = f + \lambda t^{d+1} A_r + t^{d+r+1} g$ is an AC polynomial with $r > 0$, and $|a_j| < |a_{j+1}| < |\lambda|$, and $|\lambda| > |b_j| > |b_{j+1}|$ for all $j$.

It follows that $\Lambda A_y$ is an AC polynomial with the same properties as $\Lambda$ when $1 \leq y \leq r$. In particular, $\Lambda A_y = f_y + \lambda y t^{d+y} A_{r-y+1} + t^{d+r+1} g_y$. Here $f_y, g_y$ are AC polynomials of degree $d + y - 1$ and $\lambda_y = -\epsilon_y(y)\lambda$.

The absolute values of the coefficients for $f_y, g_y$ are respectively monotonically increasing, decreasing; and less than $|\lambda_y|$. Furthermore, the coefficients for $f_y$ are dependent on the parity of $r$, but not on the actual value for $r$; and the coefficients for $g_y$ are independent of $r$.

Proof. By Eq. (1.5) we have $f_y = f A_y + \lambda t^{d+1} \epsilon_y L(y-2)$ and $g_y = \lambda R(y-2) + g A_y$. These functions have the properties claimed. □

Corollary 1.2. Given a set of $m + 1$ positive integers, $\{x_j\}_{j=0}^m$, with $m \geq 1$ and $x_0 + m - 1 \geq s = \sum_{j=1}^m x_j$, the coefficients of $\prod_{j=0}^m A_{x_j}$ are partitioned as in (1.6):

$$
\prod_{j=0}^m A_{x_j} = f + \lambda t^{s-m} A_{x_0+m-s} + t^{x_0} g, \text{ with } \lambda = \epsilon_{m+s} \prod_{j=1}^m x_j.
$$

When $x_j = 1$ for all $j \geq 1$, we must interpret $f, g$ as zero. Otherwise, $f, g$ are AC polynomials of degree $s - m - 1$ for which the absolute values of the coefficients are monotonically increasing/decreasing, and less than $|\lambda|$.

Furthermore, the coefficients for $f$ are dependent on the parity of $x_0$, but not on the actual value for $x_0$; and the coefficients for $g$ are independent of $x_0$.

2. THE JONES AND ALEXANDER POLYNOMIALS FOR SOME 3-BRAID LINKS

This section briefly introduces some familiar links and their classical skein polynomials and then describes a decomposition result, Prop. 2.3, for the Jones polynomial of three braid links that is key to the analysis. The structure of the Jones polynomial for twisted 3-braids is then presented in Sections 2.1 and 2.2. These sections describe the effect of twisting two strands and three strands, respectively, on the Jones polynomial.

The Jones polynomial for the elementary torus link, $T(2, w)$ is

$$
V_{T(2,w)} = -t^{(w-1)/2}(t^{w+1} + \epsilon_w G)/(1 + t), \text{ for any } w, \quad (2.1)
$$

$$
= -t^{(w-1)/2}(\epsilon_w + t^2 A_{w-1}). \quad (2.2)
$$

The Alexander polynomial for the elementary torus link, $T(2, w)$ is

$$
\Delta_{T(2,w)} = (t^w + \epsilon_{w-1})/t^{(w-1)/2}(t + 1), \text{ for any } w, \quad (2.3)
$$

$$
= t^{-(w-1)/2} A_w. \quad (2.4)
$$

The Jones polynomial for any three-braid knot was first given by Prop. 11.10, [15]. The expression for any three-braid link is given by Eq 2.6, [10] and is:

$$
V_{\beta}(t) = t^{(w-2)/2}\{t^{w+1} + \epsilon_w G - G t^{w/2} \Delta_\beta(t)\}, \text{ with } w = w(\beta). \quad (2.5)
$$

As the Jones polynomial for three-braid links is determined by the writhe and the Alexander polynomial, which is sometimes easier to calculate, a few examples are provided. If we combine (2.5), with the Alexander polynomial formulas for the
three braid torus links, Section 2.1 [10], we obtain the following expressions for the Jones polynomials of torus links on three strands, \( T(3, \ast) \):

\[
V_{[1,3]^{3}\ast}(t) = t^{3a-1}\{1 + t^2 + 2t^{3a+1}\}, \tag{2.6}
\]

\[
V_{[1,3]^{3+a+b}}(t) = t^{3a+b-1}\{1 + t^2 - t^{3a+b+1}\}, \quad \text{for } b = 1, 2. \tag{2.7}
\]

Eq. 3.13, [10], gives a relation (2.8) for the Homflypt polynomial for \( n \)-braid links, and is included here for the closure of the base braid word. This decomposition is described by Prop. 3.5, full twists are determined by the number of full twists and the skein polynomial for Jones polynomials of torus links on three strands, \( T \) three braid torus links, Section 2.1 [10], we obtain the following expressions for the Alexander and Jones polynomials:

\[
P_{\beta \sigma^1_1}(t) = v^{e-1} \nabla T_v P_{\beta \sigma^1_1} + v^e \nabla T_{v-1} P_{\beta}, \tag{2.8}
\]

\[
\Delta_{\beta \sigma^1_1}(t) = \Delta_{T, \beta \sigma^1_1} + \Delta_{T - 1, \beta}, \tag{2.9}
\]

\[
V_{\beta \sigma^1_1}(t) = t^{(e-1)/2} A_v V_{\beta \sigma^1_3} + t^{(e+2)/2} A_{v-1} V_{\beta}. \tag{2.10}
\]

The following result provides a useful way of grouping the terms of the Jones polynomial for a special class of three-braids and may be derived from Eq. 2.10 Prop. 2.1 implies Cor. 2.8 and leads to a description of the block structure of the Jones polynomial for positive three-braids with a highly twisted pair of strands at one site (Prop. 2.4).

**Proposition 2.1.** Given a three-braid word, \( \beta = \sigma^1_1 \sigma^2_1 \sigma^3_1 \sigma^4_1 \), and \( w = w(\beta) \),

\[
V_{\beta}^* = t^{(w+2)/2} A_v, \quad \text{with} \tag{2.11}
\]

\[
V_{\beta} = B_{1, \beta} + t^{a+2} B_{2, \beta} + Q_{\beta}, \tag{2.11}
\]

\[
B_{1, \beta} = \epsilon_w(1 + t^2) + \epsilon_a + t^{2b+d+1} + \epsilon_a t^{e+2} A_b A_d + 2\epsilon_a t^{e+1-t^2} A_b A_d,
\]

\[
B_{2, \beta} = \epsilon_d A_b A_c + \epsilon_c A_b A_d + \epsilon_b A_c A_d,
\]

\[
Q_{\beta} = \epsilon_c + t^2 A_b A_c + \epsilon_b + t^2 A_c A_d + \epsilon_c A_b A_d.
\]

Note that \( B_{2, \beta} \) is dependent on the parity of \( a \), but not on the actual value of \( a \). Henceforth the implicit definition for \( V_{\beta}^* \) when \( \beta \in B_3 \) is (2.11). \( V_{\beta}^* \) specifies the coefficient vector for positive three-braids.

Three-braid links have the very special property that the skein polynomials under full twists are determined by the number of full twists and the skein polynomial for the closure of the base braid word. This decomposition is described by Prop. 3.5, [10] for the Homflypt polynomial for three-braid links, and is included here for reference.

**Proposition 2.2.** When \( \gamma \in B_3 \) and \( a > 0 \) we have:

\[
P_{[1,3]^{3a}}(t) = v^6 P_{\gamma} + P_{w(\gamma)+5} - v^6 P_{w(\gamma)+1},
\]

\[
P_{[1,3]^{3a}}(t) = v^{6a} P_{\gamma} + \sum_{j=1}^{a} v^{6a-6j} P_{w(\gamma)+6j-1} - \sum_{j=1}^{a} v^{6+6a-6j} P_{w(\gamma)+6j-5},
\]

\[
P_{[1,3]^{3a}}(t) = v^{-6a} P_{\gamma} + \sum_{j=1}^{a} v^{6j-6a} P_{w(\gamma)-6j+1} - \sum_{j=1}^{a} v^{6j-6a-6} P_{w(\gamma)-6j+5}.
\]

When Prop. 2.2 is applied to the Jones polynomial, we obtain
Proposition 2.3. When \( \gamma \in B_3 \), set \( \beta = [1, 3]^{3a} \gamma \). For any \( a \), we have:

\[
\begin{align*}
V_{[1,3]^{3a}\gamma} &= t^{(w(\beta)-2)/2}\{\epsilon_{w(\gamma)}(1+t^2) + t^{3a}B_2(t, \gamma)\}, \text{ with} \\
B_2(t, \gamma) &= t^{(2-w(\gamma))/2}V_\hat{\gamma} + \epsilon_{w(\gamma)+1}(1+t^2). 
\end{align*}
\]

Note that when \( \gamma \) is (conjugate to) a positive three-braid word the minimum degree of \( V_\hat{\gamma} \) is \((w(\gamma) - 2)/2\), so that the expression \( t^{(2-w(\gamma))/2}V_\hat{\gamma} \) in (2.13) is an ordinary polynomial with nonzero constant, i.e. \( V_\hat{\gamma} \). If we focus on the effect of full twists of all three strands, the two blocks in the coefficient vector are already separated when \( \gamma \) is (conjugate to) a positive braid word and \( a > 0 \). As \( a \) is increased, the two blocks separate by three for each new full twist that is added.

By [11, 23], when \( \gamma \) is (conjugate to) a positive three-braid word and \( \hat{\gamma} \) is not split, the first three values in the coefficient vector are \( \epsilon_{w(\gamma)}, 0, \epsilon_{w(\gamma)}p(\hat{\gamma}) \), where \( p(\hat{\gamma}) \) is the number of prime factors of \( \hat{\gamma} \). Under these conditions when \( \hat{\gamma} \) is also prime, \( B_2(t, \gamma) \) is divisible by \( t^3 \); otherwise \( B_2(t, \gamma) \) is divisible by \( t^2 \).

Example 2.1. By [9], two simple knots are \( \#_{19} = \sigma_1^4\sigma_2\sigma_1^3\sigma_2 \) and \( \#_{124} = \sigma_1^4\sigma_2\sigma_1^3\sigma_2 \). As \( \#_{19} = [1, 3]^{3\sigma_1} \) and \( \#_{124} = [1, 3]^{3\sigma_2} \), Prop. 2.3 gives us

\[
\begin{align*}
V_{\#_{19}} &= t^3\{(1+t^2) + t^3B_2(t, \sigma_2\sigma_1)\}, \text{ with } B_2(t, \sigma_2\sigma_1) = 1 - (1+t^2) = -t^2. \\
V_{\#_{124}} &= t^4\{(1+t^2) + t^3B_2(t, \sigma_2^2)\}, \text{ with } B_2(t, \sigma_2^2) = t^{-1}V_{T_3} - (1+t^2) = -t^3.
\end{align*}
\]

As any three-braid word, \( \gamma \), has a (non-unique) representation as \([1, 3]^{3\eta} \), where \( \eta \) is itself a positive three-braid word, the prior comments extend to all three-braid words. When \( b \) is chosen to be maximal, the decomposition in (2.14, 2.15) shows that even when \( b \leq 0 \), the two blocks are separated when \( a > |b| \). Note that \( w(\gamma) \equiv w(\eta) \) mod 2. Even for submaximal \( b \), we have

\[
\begin{align*}
V_{[1,3]^{3a}\gamma} &= t^{(w(\beta)-2)/2}\{\epsilon_{w(\gamma)}(1+t^2) + t^{3a+3b}B_2(t, \eta)\}, \text{ with} \\
B_2(t, \eta) &= t^{(2-w(\eta))/2}V_\hat{\eta} + \epsilon_{w(\eta)+1}(1+t^2) \in \mathbb{Z}[t].
\end{align*}
\]

Example 2.2. One of the hyperbolic knots from [8] and [6] is \( k_{7_{40}} = 10_{161} \) with representation \( \sigma_1^4\sigma_2^2\sigma_1^{-1}\sigma_2^4\sigma_1^2\sigma_2^2 \). The mirror image has a braid expression as \([1, 3]^{-3}\sigma_1^3\sigma_2 \), so

\[
\begin{align*}
V_{10_{161}} &= t^{-5}\{(1+t^2) + t^{-9}B_2(t, \sigma_1^3\sigma_2)\}, \text{ with} \\
B_2(t, \sigma_1^3\sigma_2) &= t^{-4}V_{T_3} - (1+t^2) = -t^3A_7.
\end{align*}
\]

Here we observe the overlap of the terms \( 1 + t^2 \) and \( t^{-9}B_2(t, \sigma_1^3\sigma_2) \) in (2.16).

Since \( V_{10_{161}}(t) = \overline{V_{10_{161}}(1/t)} \) we find \( V_{10_{161}} = t^3(1-t^3A_6) \).

Again focusing on the effect of increasing \( a \), for any three-braid word there are only finitely many values of \( a \) for which the two blocks overlap, and this happens exactly when \( 3a + \min \deg B_2(t, \gamma) \leq 2 \) and \( 3a + \max \deg B_2(t, \gamma) \geq 0 \), i.e. \( -\max \deg B_2(t, \gamma) \leq 3a \leq 2 - \min \deg B_2(t, \gamma) \). Eq. 2.12 displays one block in the coefficient vector as the triplet \( \epsilon_{w(\gamma)}, 0, \epsilon_{w(\gamma)} \). Eq. 2.15 displays the second block as \( B_2(t, \eta) = t^{-3b}B_2(t, \gamma) \in \mathbb{Z}[t] \), with the possibility that several of the lowest order terms may be zero.
The significance of the prior expressions relative to the effect of twisting two strands is that they show it suffices to look at the effect of such twisting applied to a positive braid word. Indeed, given a base braid word and exponents \( V \) as above, and allow \( x \) to increase by increments of two, where we may assume that \( x_1 > 0 \). By Prop. 2.3 once \( x_1 \) is sufficiently large, the second block, \( t^{3b} B_2(t, \sigma_1^3 \eta) \), has minimal degree above two, so that the two blocks in the coefficient vector are clearly recognizable when any further full twists of the two strands are added.

### Definition 2.3

If \( \beta = \prod_{i=1}^{e_{2j-1}} \sigma_1^{2j-1} \sigma_2^{2j} \) with \( r \geq 1 \) and all \( \varepsilon_k \neq 0 \), call \( \sigma_i^{\varepsilon_k} \) a syllable. A syllable is trivial when the exponent is one. A trivial syllable is isolated if both adjacent syllables, viewed cyclically, are non-trivial. Denote the rank of \( \beta \) as \( \rho(\beta) = r \) and assign a rank of zero for the identity of \( B_3 \), and a rank of one for \( \sigma_i^a \) for \( a \neq 0 \).

### Proposition 2.4

Given a positive three-braid word, \( \beta = \prod_{j=1}^{r} a_j b_j \), with \( r \geq 1 \), and exponents \( a_j, b_j > 0 \), and \( w = w(\beta) \), we have

\[
V_{\beta}^*(t) = t^{(w-2)/2} V_{\beta}^{**}(t), \quad \text{with } V_{\beta}^{**}(t) \in \mathbb{Z}[t] \quad \text{and } \deg V_{\beta}^{**}(t) \leq w \quad \text{and}
\]

\[
V_{\beta}^*(t) = \epsilon_2 (1 + t^2) + t^2 V_{\beta}^{**}(t), \quad \text{with } V_{\beta}^{**}(t) \in \mathbb{Z}[t].
\]

Furthermore, \( \epsilon_2 t^{w} V_{\beta}^{**}(1/t) = t^2 V_{\beta}^{**}(t) \), or equivalently \( [V_{\beta}^{**}]_{i} = \epsilon_2 [V_{\beta}^{**}]_{w-i} \).

When \( r = 1 \), we have \( V_{\beta}^{**} \) is an AC polynomial, or zero, with the following properties:

(i) \( V_{\beta}^{**}(t) = 0 \) exactly when \( \{a_1, b_1\} = \{1, 2\} \),

(ii) \( \deg V_{\beta}^{**}(t) = w - 3 \) exactly when \( \min(a_1, b_1) = 1 \) and \( \max(a_1, b_1) \geq 3 \); in which case \( V_{\beta}^{**}(t) = -t A_{w-3} \),

(iii) \( \deg V_{\beta}^{**}(t) = w - 2 \) exactly when \( a_1 = b_1 = 1 \), or \( a_1, b_1 \geq 2 \); here \( V_{\beta}^{**}(t) = -1 \) or \( V_{\beta}^{**}(t) = \epsilon_2 + \epsilon_3 t A_{w-1} + t^2 A_{w-2} \) respectively,

(iv) the sign of \( [V_{\beta}^{**}]_{i} \) is \( \epsilon_{j+w} \) when \( w \geq 4 \).

When \( r \geq 2 \), we have \( \deg V_{\beta}^{**}(t) \leq w - 3 \).

**Proof.** When \( r = 1 \), write \( \beta = \sigma_1^a \sigma_2^b \). We may assume \( a \geq b \). When \( b = 1 \), the result is clear, as \( V_{\beta}^*(t) = -t(a-1)/2(\epsilon_a + t^2 A_{a-1}) \), which is \( t^{(w-2)/2}(\epsilon_2 - t^2 A_{w-2}) \).

If \( a = 1 \), take \( V_{\beta}^{**}(t) = -1 \). If \( a \geq 2 \), let \( V_{\beta}^{**}(t) = -t A_{w-3} \). The properties claimed are easily verified.

By (2.2), we may take \( V_{\beta}^{**}(t) = \epsilon_w + \epsilon_2 t A_{w-2} + \epsilon_3 t A_{w-1} + t^2 A_{w-3} \) when \( a, b \geq 2 \). This is a sum of AC polynomials each with the sign of \( t^j \) as claimed. The other properties claimed for \( r = 1 \) are easily verified.

For \( r \geq 2 \), apply (2.10) to obtain the following relations for \( V_{\gamma^a \sigma_i^j}^{**}(t) \) which utilize terms that depend on braids of lower rank and so allows an induction proof.

Here \( \beta = \gamma \sigma_1^a \sigma_2^b \) with \( \gamma = \sigma_1^{3a} \sigma_2^{3b} \) and \( \eta \) may be the identity.

\[
V_{\gamma^a \sigma_i^j}^{**} = A_1 V_{\gamma^a \sigma_i^j}^{**} + t A_{w-1} V_{\gamma^a \sigma_i^j}^{**}, \quad (2.18)
\]

\[
V_{\gamma^a \sigma_i^j}^{**} = A_f V_{\gamma^a \sigma_i^j}^{**} + t A_{f-1} V_{\gamma^a \sigma_i^j}^{**}, \quad (2.19)
\]
To verify $\epsilon_w \hat{t} V_\beta^*(1/t) = t^2 V_\beta^*(t)$ is straightforward using (2.18), (2.19), as is the degree bound $\deg V_\beta^*(t) \leq w - 3$.

$V_\beta^*$ appears to be an AC polynomial for positive three-braids of rank two or more; the sign of the coefficients is a simple function of $w(\beta)$, $j$ and $r$ (Conj. 2.6). This may be verified directly for $\rho(\beta) = 2, 3$, but these cases do not readily suggest a general pattern or proof. The following proposition identifies two cases when a positive three-braid of rank two or more is conjugate to a braid of the form $[1, 3]^{3a} \gamma$, with $a \geq 0$ and $\gamma$ a non-negative three-braid of lower rank. Assuming Conjecture 2.6 is true, Prop. 2.3 shows these are the only such cases.

Proposition 2.5. Assume $\beta$ is a three-braid word, $\prod_{j=1}^r \sigma_1^{a_j} \sigma_2^{b_j}$ with $r \geq 2$ and all $a_j, b_j \geq 1$. In either case below, we have $\beta \cong [1, 3]^{3a} \gamma$ with $a \geq 0$ and $\rho(\gamma) < r$.

(i) $\beta$ has two or more trivial syllables, or
(ii) $\beta$ has a trivial syllable whose minimum adjacent syllable length is two; here $\sigma_1^{a_1}$ and $\sigma_2^{b_2}$ are considered to be adjacent syllables.

In case (i) or when two isolated trivial syllables exist, we have $a \geq 1$.

We may choose $\gamma = \prod_{j=1}^r \sigma_1^{c_j} \sigma_2^{d_j}$ with $s < r$ and $c_j, d_j \geq 2$ for all $j \geq 2$.

For $s \geq 2$, we have $c_1, d_1 \geq 1$ and we may choose $\gamma$ to either have no trivial syllables, or one trivial syllable whose two adjacent syllables each have a minimum length of three.

For $s = 1$, we have $c_1, d_1 \geq 0$, and for $s = 0$, we have $\gamma = 1$.

Proof. If two trivial syllables of $\beta$ are adjacent (in a cyclical sense), we may assume they correspond to $a_r = b_r = 1$. When $r = 2$, let $c_1 = a_1 + b_1 + 1$ and $d_1 = 1$ with $s = 1$ and $a = 0$. When $r \geq 3$ let $c_1 = a_1 + b_{r-1}$, $c_{r-1} = a_{r-1} + 1$, $d_1 = b_1$ and $d_{r-1} = 1$ with $c_j = a_j$ and $d_j = b_j$ for $2 \leq j \leq r - 2$ with $s = r - 1$ and $a = 0$. As $\gamma$ has smaller rank, induction applies.

Suppose there is a pattern $\sigma_1^{a_1} \sigma_2^{a_{j+1}}$ in $\beta$ or its braid reflection, and that $\min(a_j, a_{j+1}) = 2$. With $r = 2$ and $\beta = \sigma_1^{a_1} \sigma_2^{a_2} \sigma_2^{b_2}$ we have $\beta \cong [1, 3]^{3a} \sigma_1^{a_1 - 2} \sigma_2^{b_2 - 1}$. A similar result applies when $r \geq 3$. As $\gamma$ has smaller rank, induction applies.

The only remaining case is when two or more trivial syllables exist and each neighboring syllable has length at least three. If there is a subword such as $\sigma_1^{a_1} \sigma_2^{a_2} \sigma_2^{a_3} \eta$ and $\eta$ is either vacuous or begins with $\sigma_1$, this is just $\sigma_1^{a_1 - 1} \sigma_2^{a_2} \sigma_1 \sigma_2 \eta$ which has already been handled. This leads to an easy induction using the distance between two trivial syllables. We may assume $r \geq 3$ as the $r = 2$ case was just addressed. Assume there is a subword such as $\sigma_1^{a_1} \sigma_2^{a_2} \sigma_1 \sigma_2 \eta \sigma_1^k \xi$, where $k$ may be either 1 or 2, and $\eta$ begins with $\sigma_2^{b_2}$, does not end in $\sigma_k$, and all twist exponents in $\eta$ are at least two. This subword is just $\sigma_1^{a_1 - 1} \sigma_2^{a_2 + 1} \sigma_1 \sigma_2 \eta \sigma_1^k \xi$. The case $b_2 = 1$ has already been handled, otherwise we have created a braid word with the same properties but with reduced distance between the trivial syllables.

Definition 2.4. When a positive three braid of rank two or more either has no trivial syllables, or a single trivial syllable whose two adjacent syllables each have a minimum length of three, the three-braid is called condensed.

Conjecture 2.6. Given a condensed three-braid word, $\beta$, with $w = w(\beta)$, we have $V_\beta^*(t) = \epsilon_w (1 + t^2) + t^2 V_\beta^*(t)$ and
(i) the sign of $[V_{\beta}^{**}]_j$ is $\epsilon_{r+1-j+w}$, so $V_{\beta}^{**}$ is an AC polynomial,
(ii) when $\beta$ has no trivial syllables, $\deg V_{\beta}^{**}(t) = w - r - 1$, and $[V_{\beta}^{**}]_{\text{max}} = 1$,
(iii) when $\beta$ has one trivial syllable, $\deg V_{\beta}^{*}(t) = w - r - 2$, and $[V_{\beta}^{*}]_{\text{max}} = -1$,
(iv) $[V_{\beta}^{**}]_j \neq 0$ exactly when $j \in [w - 2 - \deg V_{\beta}^{**}(t), \deg V_{\beta}^{**}(t)]$.

Example 2.5. If Conj. 2.6 is true, Prop. 2.3 implies that $\deg V_{\beta}^{**}$ is lower than the values given by Conj. 2.6 when $\beta$, of rank at least two, has only isolated trivial syllables but is not condensed. The condensed braid, $\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3$ has $V_{10152}^{**} = V_{\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3}^{**} = t - 2t^2 + 2t^3 - 3t^4 + 2t^5 - 2t^6 + t^7$ and satisfies Conj. 2.6

$V_{12n542}^{**} = V_{\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3}^{**} = \epsilon_1 t^4 A_3$ has degree 6, $\beta \cong [1, 3]^3\sigma_1^5\sigma_2$, $V_{12n572}^{**} = V_{\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3}^{**} = \epsilon_1 t^2 A_3 A_5$ has degree 8,

$V_{12n574}^{**} = V_{\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3}^{**} = \epsilon_1 t^2 [1 - t + 2t^2 - 3t^3 + 2t^4 - t^5 + t^6]$ has degree 8,

$V_{12n575}^{**} = V_{\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3}^{**} = t^3 [1 - 2t + t^2 - 2t^3 + t^4]$ has degree 7, $\beta \cong [1, 3]^3\sigma_1^3\sigma_2^3$.

In the following sections it is desirable to group the terms of the Jones polynomial for three-braid links into three components, i.e. the first block, the inter-block gap, and the second block. Toward this end the following proposition is helpful in many calculations. Typically $g_1$ and $b_2$ are chosen as the minimum degrees for the first term of the inter-block gap and the second block, respectively.

**Proposition 2.7.** Suppose that $k \geq 0$ and $s, g_1, b_2 > 0$ are given with $k \leq g_1 \leq b_2$. We have the following disjoint partition of $t^{k}A_s$, with $A_z, \in \mathbb{Z}[t]$:

$$t^{k}A_s = \lambda_1 t^{g_1} A_{z_1} + \lambda_2 t^{b_1} A_s + t^{b_2} A_{z_2},$$

with $z_1 = g_1 - k$ when $g_1 \leq k + s$, and $\lambda_1 = \epsilon_{g_1+k+s}$,

$z_1 = s$ when $k + s \leq g_1$, and $\lambda_1 = 1$,

$z_2 = b_2 - g_1$ when $b_2 \leq k + s$, and $\lambda_2 = \epsilon_{b_2+k+s}$,

$z_2 = k + s - g_1$ when $g_1 \leq k + s \leq b_2$, and $\lambda_2 = 1$,

$z_2 = 0$ when $k + s \leq g_1$, and $\lambda_2 = 1$,

$z_3 = k + s - b_2$ when $k + s \geq b_2$,

$z_3 = 0$ when $k + s \leq b_2$.

The next section describes the properties of each of the components of the coefficient vector for the closure of a positive braid word: the length of the first block, the length of the inter-block gap and its coefficients, and the maximal length of the second block. The number of twists required to ensure that the two blocks are disjoint is also provided.

2.1. Twisting two strands of a three-braid link. If we apply Prop. 2.1 to positive three-braid words with a dominant twist exponent we obtain an expression, Cor. 2.8, for the Jones polynomial that displays its block characteristics. The main steps in the proof use Eq. 1.4 applied to the term $A_aA_bA_cA_d$ in $Q_{\beta}$. Prop. 2.7 and (1.5) are the other helpful tools needed.

**Corollary 2.8.** Given a positive three-braid word, $\beta = \sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^3$, set $w = w(\beta)$ and $w^* = w - a$, with $a \geq w^*$. We have the following partitioning of coefficients:
\[ V_{\beta}^* = B_1(\beta) + t^{w^* + 1}G(\beta) + t^{a+2}B_2(\beta), \quad \text{with} \] 
\[ B_1(\beta) = \epsilon_w(1 + t^2) + \epsilon_{a+c}t^{b+d+1} + \epsilon_a t^{a+c-1} \] 
\[ + \epsilon_{a+c+d} t^d L(b - 2) + \epsilon_c t^{b+1} A_{d+1} + \epsilon_{a+b} t^{a+b} A_{c+d} \] 
\[ + \epsilon_{a+b+c} t^{a+b+c} + t^d \beta + \epsilon_{a+b+c} t^{w^*}, \quad \text{(2.21)} \] 
\[ G(\beta) = \epsilon_w (b c d - b - d) A_{a+1} + \epsilon_{a+b+c} t^{a+b+c} + \epsilon_{a+b+d} R(b - 2) + \epsilon_{a+c+d} R(d - 2) + \epsilon_{a+b+c} \] 
\[ + \epsilon_{a+b+c} R(d - 2) + \epsilon_{a+b+c} + t g_{\beta}. \quad \text{(2.22)} \] 

Here \( f_{\beta}, g_{\beta} \) are the \( f, g \) in Cor. \ref{cor:twist} for the product \( A_{a+b}A_{c+d} \). We have \( \deg f_{\beta} = \deg g_{\beta} = w^* - 4 \). As in Cor. \ref{cor:twist}, we must interpret \( f_{\beta} = g_{\beta} = 0 \) when \( b = c = d = 1 \); in this case \( \beta = T_{a+2} \).

The maximum degree for \( B_1(\beta) \) is \( w^* \), while that for \( B_2(\beta) \) is \( w^* - 3 \), i.e. \( \deg V_{\beta}^* \leq w - 1 \). The latter upper bound is only achieved when \( \min(b, d) = 1 \) \( (\beta = T_{w-1}) \), or when \( \min(b, c, d) \geq 2 \).

It is apparent from these expressions that \( \deg B_1(\beta) \leq b + c + d \), whereas \( \deg t^{b+c+d+1} G(\beta) \leq a + 1 \), so the coefficients are partitioned. As foreshadowed by Prop. \ref{prop:twist}, \( B_1(\beta) \) is dependent on the parity of \( a \), but not on the actual value of \( a \), and \( B_2(\beta) \) is independent of \( a \). The coefficients of the inter-block gap, \( G(\beta) \), are independent of \( a \), and the width depends linearly on \( a \). Note that \( G(\beta) = 0 \) exactly when \( (b, c, d) \) is \( (1, 2, 1) \) or \( (2, 1, 2) \); in these cases \( \beta = \sigma_1^{a-1}[1, 3]^3 \) and \( \beta = \sigma_1^{a-1}[1, 3]^3 \), respectively.

The coefficient of \( t^{b+c+d} \) in \( (2.21) \) is \( \delta_{c+1} \epsilon_{a+c} + \epsilon_a (1 + b + d - b c d) \), which differs from the plateau of the inter-block gap, except when \( c = 1 \). Similarly, the coefficient of \( t^{c+2} \) is \( \delta_{c+1} \epsilon_{b+c+d} (1 + b + d - b c d) \), which also differs from the plateau of the inter-block gap, except when \( c = 1 \). This shows that the length of the inter-block gap in \( (2.22) \) is exactly \( 1 + a - b - c - d \) when \( c \geq 2 \). In other words \( G(\beta) \) completely describes the inter-block gap when \( c \geq 2 \).

By direct calculation, and under suitable restrictions, one finds that when \( \beta = \sigma_1^a \sigma_2^b \sigma_3^c \sigma_4^d \) and \( w^* = w(\beta) - a \), we have a similar pattern as in Cor. \ref{cor:twist}:

\[ V_{\beta}^* = B_1(\beta) + t^{w^*+1}G(\beta) + t^{a+2}B_2(\beta), \] 

with \( G(\beta) = \epsilon_w \{ (b + d + f) + (b c f + b c d + b c f + b c d) - b c d f \} t^{w^*} A_{a+1-w^*} \).

This suggests that it would be helpful to introduce some terminology to describe the terms in \( G(\beta) \) in a way that applies to the general case.

**Definition 2.6.** Given a three-braid word, \( \beta = \prod_{i=1}^r \sigma_1^{e_1-1} \sigma_2^{e_2} \), a \( j \)-fold product of twist exponents formed by starting with a twist exponent of \( \sigma_2 \) and alternating choosing a twist exponent of \( \sigma_1 \) and then \( \sigma_2 \), all with ascending subscripts is called a \( j \)-form of \( \beta \), i.e. \( \prod_{i=1}^r \epsilon_k \), with \( k_i < k_{i+1} \) and \( k_i \equiv i + 1 \mod 2 \) for all \( i \). A \( j \)-form may also be viewed as an element of \( \mathbb{Z} \{ e_1, \ldots, e_2 \} \). The sum of all \( j \)-forms of \( \beta \) is denoted \( m_j(\beta) \), viewed as a polynomial, and \( m_j(\beta, 0) \) when evaluated to an integer.

The alternating sum, \( \sum_{k=1}^r \epsilon_k m_2k-1(\beta) \), is denoted \( M(\beta) \), viewed as a polynomial, and \( M(\beta, 0) \) when evaluated to an integer.
The number of \( j \)-forms of \( \beta \) with \( j \) odd is \( \binom{r + (j - 1)/2}{j} \). This is trivially true for \( j = 1 \). For \( j \geq 3 \), any \( j \)-form either ends in \( e_{2r-1}e_{2r} \) with \( k \leq r \), or ends in \( e_{k} \) with \( k \leq 2r-2 \) and even. By induction, in the first case for each \( k \in [(j + 1)/2, r] \), the cardinality is \( c_k = \binom{k - 1 + (j - 3)/2}{j - 2} \), while \( \kappa = \binom{r - 1 + (j - 1)/2}{j} \) gives the number of \( j \)-forms for the second case. These cases are disjoint so the total is \( \sum_{k=(j+1)/2}^{r} c_k + \kappa \), which has the value asserted.

The only non-trivial permutation of the twist exponents that preserves \( M(\beta) \) is the one that "reverses" \( \beta \) to \( \overrightarrow{\beta} \), where we may define a sort of reversed braid word, \( \overrightarrow{\beta} = \sigma_1^{e_1} \sigma_2^{e_2} \prod_{j=2}^{r} \sigma_1^{e_{2r-2j+3}} \sigma_2^{e_{2r-2j+2}} \) for \( r \geq 2 \). This is easily verified when \( r = 2 \). Otherwise any such permutation, \( \pi \), must preserve 1-forms, so viewing \( \pi \) as a permutation of subscripts, \( \pi \) permutes the even subscripts, permutes the odd subscripts, and fixes 1. We may calculate the number of 3-forms containing \( e_{2r} \) as \( \binom{k}{2} + \binom{r - k + 1}{2} \). This is maximal exactly when \( k = 1 \) or \( k = r \), hence \( \pi \) must map \( \{2, 2r\} \) to itself. We must have \( \pi(2k-1) \) between \( \pi(2j) \) and \( \pi(2k+2\delta) \) for any \( j < k \) and \( \delta = 0, \ldots, r-k \). When \( \pi(2) = 2 \), we must have \( \pi(2k-1) < \pi(2k+2\delta) \) for any \( \delta = 0, \ldots, r-k \), which easily leads to \( \pi \) as the identity. When \( \pi(2) = 2r \), we must have \( \pi(2k-1) > \pi(2k+2\delta) \) for any \( \delta = 0, \ldots, r-k \), which easily leads to \( \pi(\beta) = \beta \).

The following proposition describes conditions under which the blocks in the coefficient vector are separated, along with some of their properties.

**Proposition 2.9.** Given a positive three-braid word, \( \beta = \prod_{j=1}^{r} \sigma_1^{e_{2j-1}} \sigma_2^{e_{2j}} \), set \( w = w(\beta) \) and \( w^* = w - e_1 \). When \( r \geq 2 \) and \( e_1 \geq w^* \), we have the following partitioning of coefficients:

\[
V^*_{\beta} = B_1(\beta) + t^{w^*+1}G(\beta) + t^{e_1+2}B_2(\beta), \quad \text{with } B_1(\beta), B_2(\beta) \in \mathbb{Z}[t], (2.24)
\]

\[
G(\beta) = \epsilon_{w^*} M(\beta, 0)A_{e_1+w^*} \in \mathbb{Z}[t]. \quad (2.25)
\]

Here \( \deg B_1(\beta) \leq w^* \), and \( B_1(\beta) \) depends on the parity of \( e_1 \), but not on the actual value of \( e_1 \). \( B_2(\beta) \) is independent of \( e_1 \) with \( \deg B_2(\beta) \leq w^*-3 \).

The coefficients of \( G(\beta) \) are independent of \( e_1 \), and the width of the inter-block gap depends linearly on \( e_1 \).

Note that as in the discussion following Cor. 2.8, it is possible that some of the high order coefficients of \( B_1(\beta) \) and some of the low order coefficients of \( B_2(\beta) \) could be incorporated into the inter-block gap for some special braid words \( \beta \).

**Proof.** Cor. 2.8 establishes the result for \( r = 2 \) and we will use induction. For \( r \geq 3 \), apply (2.11) to twist exponents \( e_{2r-3} \) and \( e_{2r-2} \) of \( \beta \). Use the braid relations to obtain four terms each associated with a three-braid word with \( 2r-2 \) twist exponents. The four braid words are \( \beta_1 = \sigma_1^{e_1} \cdots \sigma_2^{e_{2r-4}+t^{e_{2r-2}-1}} \sigma_1 \sigma_2^{e_{2r-3}} \), and \( \beta_2 = \sigma_1^{e_1} \cdots \sigma_2^{e_{2r-4}} \sigma_2^{e_{2r-1}+1} \sigma_1 \sigma_2^{e_{2r}} \), and \( \beta_3 = \sigma_1^{e_1} \cdots \sigma_2^{e_{2r-4}+t^{e_{2r-2}-1}} \sigma_1 \sigma_2^{e_{2r}} \), and \( \beta_4 = \sigma_1^{e_1} \cdots \sigma_2^{e_{2r-4}} \sigma_1 \sigma_2^{e_{2r}} \). We have

\[
V^*_{\beta} = A_{e_{2r-3}} A_{e_{2r-2}} V^*_{\overrightarrow{\beta_1}} + t A_{e_{2r-3}} A_{e_{2r-2}-1} V^*_{\overrightarrow{\beta_2}}
\]

\[
+ t A_{e_{2r-3}-1} A_{e_{2r-2}} V^*_{\overrightarrow{\beta_3}} + t^2 A_{e_{2r-3}-1} A_{e_{2r-2}-1} V^*_{\overrightarrow{\beta_4}}.
\]
Now use the induction hypothesis, (2.24), (2.25). The one challenge is to verify the claim for the inter-gap constant, i.e. to show that we have \( M(\beta) = \omega \) where

\[
\omega = e_{2r-3}e_{2r-2}M(\beta_1) - e_{2r-3}(e_{2r-2} - 1)M(\beta_2) \\
- (e_{2r-3} - 1)e_{2r-2}M(\beta_3) + (e_{2r-3} - 1)(e_{2r-2} - 1)M(\beta_4).
\]

The following identity may be used to complete this calculation.

With \( \eta_k = \prod_{j=1}^{k} \sigma_1^{f_{2j-1}} \sigma_2^{f_{2j}} \) and \( \gamma = \eta_r \), we have

\[
M(\gamma) = -f_{2r} - \sum_{j=2}^{r} M(\eta_{j-1})f_{2j-1}f_{2r} + M(\eta_{r-1}).
\]

\[ \square \]

**Example 2.7.** The 12n242 knot, \( \sigma_1^2 \sigma_2^1 \sigma_2^2 \), has \( V_{12n242}^* = 1 + t^2 - t^6 A_3 \).

This exhibits a case where \( \deg B_1(\beta) < w^* \) and \( B_2(\beta) = 0 \).

The 12n574 knot, \( \sigma_1^2 \sigma_2^2 \sigma_1^3 \), has \( V_{12n574}^* = 1 + t^2 - t^4 + t^5 - 2t^6 + 3t^7 - 2t^8 + t^9 - t^{10} \).

Here \( B_1(\beta) = 1 + t^2 - t^4 + t^5 - 2t^6 \) has \( \deg B_1(\beta) = w^* \) and \( B_2(\beta) = -2 + t - t^2 \) has \( \deg B_2(\beta) < w^* - 3 \).

### 2.2. Twisting three braids of a three-braid link.

The general results obtained by A. Champanerkar and I. Kofman, [7], lead to an explicit description of the blocks in the coefficient vector when full twists are added to a three-braid.

An immediate consequence of Prop. 2.3 is the following:

**Corollary 2.10.** For any three-braid of the form \( \beta = [1, 3]^3 \sigma_1^x \sigma_2^y \), we have:

\[
V_{\beta}^* = \epsilon_{x+y}(1 + t^2) + t^{3a}(t^{x+y+2} + \epsilon_x G t^{y+1} + \epsilon_y G t^{x+1} + \epsilon_{x+y} t^2)/\text{deg}(1 + t)^2,
\]

\[
= \epsilon_{x+y}(1 + t^2) + t^{3a+2}(\epsilon_{x+y} + \epsilon_x A_{x-1} + \epsilon_y A_{x-1} + t^2 A_{x-1} A_{y-1}).
\]

When \( \beta = [1, 3]^3 \sigma_1^x \sigma_2^y \cong [1, 3]^3 \sigma_1^x \sigma_2^y \), we have:

\[
V_{\beta}^* = \epsilon_{x+y}(1 + t^2) - t^{3a+3}(t^{x+y-1} + \epsilon_{x+y})/(1 + t),
\]

\[
= \epsilon_{x+y}(1 + t^2) - t^{3a+3} A_{x+y-1}.
\]

When \( \beta = [1, 3]^3 \sigma_1^x \sigma_2^y \cong [1, 3]^3 \sigma_1^x \sigma_2^y \), we have:

\[
V_{\beta}^* = \epsilon_{x+y}(1 + t^2) + t^{3a+3}(t^{x+y} + \epsilon_{y-1} G t^y + \epsilon_{y-1} G t^x + \epsilon_{x+y} t^2)/(1 + t)^2.
\]

2.2.1. The Jones polynomial for \( \sigma_1^y \sigma_2^y [1, 3] \) with \( x, y, z \geq 0 \). In anticipation of Thm. 4.5 which shows that the family of three-braid links, \( \sigma_1^y \sigma_2^y [1, 3] \) with \( x, y \geq 0 \), and \( z \geq 3 \) and \( z \equiv 0 \mod 3 \) coincides with the Lorenz links of braid index three, we present the Jones polynomial for two-tier T-links and torus links as an immediate consequence of Cor. 2.10.
Corollary 2.11. The Jones polynomial for the T-link \( T((2, x), (3, 3a)) \), with \( a, x > 0 \) is
\[
V_{\{1, 3\}^{a\sigma_i^1}} = t^{(6a+x-2)/2} \left( \epsilon_x(1 + t^2) + t^{3a+1}(\epsilon_x + t^2) \right).
\]
When \( x = 0 \) we have \( T(3, 3a) \) and \( V_{\{1, 3\}^{3a}} = t^{3a-1}(1 + t^2 + 2t^{3a+1}) \).

The Jones polynomial for the T-link \( T((2, x), (3, 3a + 1)) \), with \( a, x > 0 \) is
\[
V_{\{1, 3\}^{3a+1}\sigma_i^1} = t^{(6a+x)/2} \left( \epsilon_x(1 + t^2) - t^{3a+3} A_{x-1} \right).
\]
When \( x = 0 \) we have \( T(3, 3a + 1) \) and \( V_{\{1, 3\}^{3a+1}} = t^{3a-1}(1 + t^2 - t^{3a+2}) \).

The Jones polynomial for the T-link \( T((2, x), (3, 3a + 2)) \), with \( a \geq 0, x > 0 \) is
\[
V_{\{1, 3\}^{3a+2}\sigma_i^1} = t^{(6a+x+2)/2} \left( \epsilon_x(1 + t^2) - t^{3a+3} A_{x+1} \right).
\]
When \( x = 0 \) we have \( T(3, 3a + 2) \) and \( V_{\{1, 3\}^{3a+2}} = t^{3a+1}(1 + t^2 - t^{3a+3}) \).

A detailed description of the coefficient vectors for the first form of Cor. 2.10
\[ \{2.27 \} \], when \( x, y, a \geq 0 \) appears in Section \[ 2.2.2 \].

The following are a series of propositions that describe when two braid words of the form \( \sigma_i^x \sigma_2^y [1, 3]^z \), with \( x, y, z \geq 0 \), generate links that have the same Jones polynomial. This leads to the interesting result, Theorem 2.16, that the Jones polynomial is a complete invariant for this class of links.

Proposition 2.12. Suppose \( L_i = \sigma_i^x \sigma_2^y [1, 3]^z \) with \( z_i \equiv 0 \mod 3 \) and \( x_i, y_i, z_i \geq 0 \), for \( i = 1, 2 \).

When \( V_{L_1} = V_{L_2} \), we have \( z_1 = z_2 \) and \( \{ x_1, y_1 \} = \{ x_2, y_2 \} \).

When \( z_1 = z_2 \) and \( \{ x_1, y_1 \} = \{ x_2, y_2 \} \), we have \( L_1 = L_2 \).

Proof. The first assertion is clear. The second follows from the observation that \( [1, 3]^2 = [3, 1]^2 \) is in the center of \( B_3 \) together with conjugacy by braid reflection. \( \square \)

Eq. 2.28 the prior comments, and Prop 2.12 give us

Proposition 2.13. Suppose \( L_i = \sigma_i^x \sigma_2^y [1, 3]^z \) with \( z_i \equiv 1 \mod 3 \) and \( x_i, y_i, z_i \geq 0 \), for \( i = 1, 2 \). Also assume \( L = \sigma_i^x \sigma_2^y [1, 3]^z \) with \( z \equiv 0 \mod 3 \) and \( x, y, z \geq 0 \).

When \( V_{L_1} = V_{L_2} \), we have \( z_1 = z_2 \) and \( x_1 + y_1 = x_2 + y_2 \).

When \( z_1 = z_2 \) and \( x_1 + y_1 = x_2 + y_2 \), we have \( L_1 = L_2 \).

When \( V_{L_1} = V_{L_2} \), we have \( \{ x, y \} = \{ 1, 1 + x_1 + y_1 \} \) and \( z + 1 = z_1 \).

When \( \{ x, y \} = \{ 1, 1 + x_1 + y_1 \} \) and \( z + 1 = z_1 \), we have \( L_1 = L \).

Hence, given \( L_1 \), it has a representation as \( \sigma_i^{1+x_1+y_1} \sigma_2^{y} [1, 3]^{z-1} \).

Eq. 2.30 and Props. 2.12, 2.13 give us

Proposition 2.14. Suppose \( L_i = \sigma_i^x \sigma_2^y [1, 3]^z \) with \( z_i \equiv 2 \mod 3 \) and \( x_i, y_i, z_i \geq 0 \), for \( i = 1, 2 \). Also assume \( L = \sigma_i^x \sigma_2^y [1, 3]^z \) with \( z \equiv 0 \mod 3 \) and \( x, y, z \geq 0 \).

When \( V_{L_1} = V_{L_2} \) and \( z_1 \geq z_2 \), we either have

(i) \( z_1 = z_2 \) and \( \{ x_1, y_1 \} = \{ x_2, y_2 \} \), or

(ii) \( z_1 = z_2 + 3 \), \( \min(x_1, y_1) = 0 \), \( \min(x_2, y_2) = 2 \), and \( \max(x_2, y_2) = 4 + \max(x_1, y_1) \).
When either of these parameter conditions (i, ii) are true, we have $L_1 = L_2$.

When $V_{L_1} = V_{L_2}$, we either have

(i) $z = z_1 + 1,\ \{x + 1, y + 1\} = \{x_1, y_1\}$, or
(ii) $z = z_1 - 2, \ \max(x, y) = 3 + \max(x_1, y_1), \ \min(x, y) = 1, \ \min(x_1, y_1) = 0$.

When either of these parameter conditions (i, ii) are true, we have $L_1 = L_2$.

Hence, given $L_1$, it has a representation as either

(i) $L = \sigma_1^{3 + \max(x_1, y_1)}\sigma_2[1,3][3z_1 - 2]$ when $\min(x_1, y_1) = 0$, or
(ii) $L = \sigma_1^{z_1 - 1}\sigma_2^{y_1 - 1}[1,3][3z_1 + 1]$ when $x_1, y_1 \geq 1$.

Proof. It is straightforward to verify the conditions implied when $V_{L_1} = V_{L_2}$. To prove $L_1 = L_2$ when $z_1 = 2$ and $\{x_1, y_1\} = \{x_2, y_2\}$ is also straightforward.

To see that $L_1 = L_2$ in case (i) take the case that $L_1 = \sigma_2[1,3][3z_1 + 3]$, and $L_2 = \sigma_2[1,3][3z_1 + 3]$ with $z \equiv 2 \mod 3$. First, $\sigma_2[1,3][3z_1 + 3] = \sigma_1^{3z_1 + 3}\sigma_1[1,3][3z_1 + 1] \equiv \sigma_1^{3z_1 + 1}\sigma_2[1,3][3z_1 + 1]$. This is just $\sigma_1^{3z_1 + 1}\sigma_1\sigma_2^2[1,3][3z_1 + 1] \equiv \sigma_1^{3z_1 + 1}\sigma_1^2\sigma_2[1,3][3z_1 + 1]$. It is easy to see that $\sigma_1[1,3][3z_1 + 1] \equiv \sigma_1[1,3][3z_1 + 1]$, and $\sigma_1[1,3][3z_1 + 1] \equiv \sigma_1[1,3][3z_1 + 1]$, hence all four combinations in case (i) represent the same link.

When $V_{L_1} = V_{L_2}$ and either $z_1 = 1 + 1$ or $z_1 = 2$, the remaining implications are trivial. To show that these are the only two valid values for $z$ is straightforward.

To see that $L_1 = L_2$ in case (ii) take the case that $L_1 = \sigma_2[1,3][3z_1 + 1]$, with $z_1 \equiv 2 \mod 3$. First, $\sigma_2^3[1,3][3z_1 + 1] = \sigma_1^3\sigma_2^3[1,3][3z_1 + 1]$, and this is conjugate to $\sigma_1\sigma_2\sigma_2^3\sigma_1\sigma_2[1,3][3z_1 - 2] = \sigma_2\sigma_2[1,3][3z_1 - 2] \equiv \sigma_2[1,3][3z_1 - 2]$. It is easy to see that $\sigma_2[1,3][3z_1 - 2] \equiv \sigma_2[1,3][3z_1 - 2] \equiv \sigma_2[1,3][3z_1 - 2]$.

To see that $L_1 = L_2$ in case (ii) it is trivial to verify $\sigma_1[1,3][3z_1 - 2] \equiv \sigma_2[1,3][3z_1 - 2]$.

Combining Props. 2.13 and 2.14 and using similar techniques gives us

Corollary 2.15. Suppose $L_i = \sigma_2^y[1,3][3z_i]$ with $z_i \equiv i \mod 3$ and $x_i, y_i, z_i \geq 0$, for $i = 1, 2$.

When $V_{L_1} = V_{L_2}$, we either have

(i) $z_1 = z_2 + 2, \ \{2, 2 + x_1 + y_1\} = \{x_2, y_2\}$, or
(ii) $z_1 = z_2 - 1, \ x_1 + y_1 = 2 + \max(x_2, y_2), \ \min(x_2, y_2) = 0$.

When either of these parameter conditions (i, ii) are true, we have $L_1 = L_2$.

The prior results imply that the Jones polynomial is a complete invariant and identifies which are the torus links.

Theorem 2.16. Suppose $L_i = \sigma_2^y[1,3][3z_i]$ and $x_i, y_i, z_i \geq 0$, for $i = 1, 2$.

If $V_{L_1} = V_{L_2}$, we have $L_1 = L_2$.

$L_1$ has a representation as $\sigma_2^y[1,3][3z]$ with $z \equiv 0 \mod 3$.

When $L_1$ is a torus link of braid index three, it has one of the following forms:

(i) $x_1 = 0 = y_1$ and $z_1 \geq 3$, so $L_1 = T(3, z_1)$,
(ii) $x_1 = 1 = y_1$ and $z_1 \geq 2$, so $L_1 = T(3, z_1 + 1)$,
(iii) $x_1 = 0 = y_1$ and $z_1 \geq 3$, so $L_1 = T(3, z_1 + 2)$,
(iv) $x_1 = 1 = y_1$ and $z_1 \geq 2$, so $L_1 = T(3, z_1 + 2)$.
(v) $x_1 = 2 = y_1$, $z_1 \equiv 2 \mod 3$, so $L_1 = T(3, z_1 + 2)$.
(vi) $\min(x_1, y_1) = 0$, $\max(x_1, y_1) = 2$, $z_1 \equiv 1 \mod 3$, $z_1 \geq 4$, so $L_1 = T(3, z_1 + 1)$.

2.2.2. The coefficient vector for $L = \sigma_1^x \sigma_2^z [1, 3]^z$ with $z \equiv 0 \mod 3$. As in the prior section, we assume $x, y, z$ are non-negative for this section.

As Eq. 2.20 is symmetric in $x, y$, we may assume $x \geq y$ with no loss in generality.

When $y = 1$, the link is $L = T((2, x - 1), (3, z + 1))$ (see Section 11.5) and

$$V_L^* = \epsilon_{x+1}(1 + t^2) - t^{x+3} A_{x-2}.$$ 

When $y = 2$, we have

$$V_L^* = \epsilon_x(1 + t^2) + t^{x+2}(A_{x-1} + t^2 A_{x-1}).$$

For $x = 2$ and $y = 2$, $V_L^*$ is just $1 + t^2 + t^{x+2}(1 + t^2)$.

For $x = 3$ and $y = 2$, we have $V_L^* = -(1 + t^2) + t^{x+2}(-1 + t - t^2 + t^3)$.

For $x \geq 4$ and $y = 2$, we have

$$V_L^* = \epsilon_x(1 + t^2) + t^{x+2}\{\epsilon_x(1 - t) + 2tA_{x-3} - t^{x-1} + t^3\}.$$ 

This expression also applies when $x = 2$ or $x = 3$, but is less intuitive.

For $x, y \geq 3$, Cor. 2.11 Eqs. (3.9) and (1.2) give us

$$V_L^* = \epsilon_{x+y}(1 + t^2) + t^{x+2}B_2, \quad \text{with}$$
$$B_2 = \epsilon_x L(y - 2) + \epsilon_{x+1}(y - 1)t^{y-1} + \epsilon_y y \epsilon_y A_{x-y-1}$$
$$+ \epsilon_{y+1}(y - 1)t^{x-1} + t^x R(y - 2). \quad (2.32)$$

Note that when $x = y$, the expression for $B_2$ becomes

$$B_2 = \epsilon_x L(x - 2) + \epsilon_{x+1}(x - 2)t^{x-1} + t^x R(x - 2).$$

Hence a graph of the absolute values of the coefficients in the second block for $x = y$

looks like an inverted $W$.

3. Braid properties related to twisted $n$-braid links, $T$-links and their symmetries

In this section, Prop. 3.1 describes some relations that are useful in working with twisted $n$-braid links and $T$-links. In particular, Prop. 3.1 leads to some symmetry results, Cors. 3.2 and 3.3, that describe some cases when two such $n$-braid links close to form the same link.

Note that Eq. 3.5 below also follows from Ex. 2 [4] and duality.

**Proposition 3.1.** Assume $n \geq 2$, and $1 \leq d \leq n$. We have the following:

$$[1, n]^d = [d, 1][1, n][1, n - 1]^d-1 \quad \text{and} \quad [n, 1]^d = [n - 1, 1]^d-1[n, 1][1, d]. \quad (3.1)$$

Assume $1 \leq \rho \leq m$ and $2 \leq m + x$, with $x \geq 0$. With $[1, m + x] \in B_{m+x}$ and $\gamma, [\rho, 1]^x[1, m]^\rho \in B_m$, we have

$$\gamma[1, m + x]^\rho = \gamma[\rho, 1]^x[1, m]^\rho \quad \text{and} \quad \gamma[m + x, 1]^\rho = \gamma[m, 1]^x[1, \rho]^x. \quad (3.2)$$
Equation (3.3) may be applied to T-links as:

\[ T((m, y), (n, \rho)) = [\rho, 1]^{n-m}[1, m]^{\rho+y}, \text{ for } 2 \leq \rho \leq m \leq n \text{ and } 1 \leq y, \] (3.3)

\[ T((2, y), (\rho, n)) = T((2, y), (\rho, n)) \text{ for } 2 \leq \rho, n \text{ and } 1 \leq y, \] (3.4)

\[ T((m, y)(n, m)) = T(m, n+y), \text{ for } 2 \leq m < n \equiv 0 \mod m. \] (3.5)

**Proof.** Eq. (3.1) is readily proven by induction on \( d \). Eq. (3.2) follows from (3.1) and the use of Markov destabilization to reduce the number of strands.

To establish (3.3), note that by duality we have \( T((2, y), (\rho, n)) = T((\rho, \rho - 2), (y + 2, 2)) \). When \( \rho, n > 2 \), Eq. (3.3) applied to the latter form tells us these are just \( [2, 1]^{n}[1, \rho]^{2} \). When \( \rho = 2 < n \), we have \( T(2, n+y) \) and the result holds. As the assertion is symmetric in \( \rho, n \), the result holds.

Eq. (3.2) leads to a symmetry result that applies to twisted \( n \)-braids. For T-links the result is somewhat analogous to Cor. 3, [1]. The cited corollary states that \( T((r_{1}, s_{1}), \ldots, (r_{k-1}, s_{k-1}), (r_{k}, s_{k})) = T((r_{1}, s_{1}), \ldots, (r_{k-1}, s_{k-1}), (s_{k}, r_{k})) \) when \( s_{i} = s_{i} \) for \( i = 1, \ldots, k-1 \) and \( r_{k-1} \leq s_{k} \). The following corollary gives the same symmetry when the restrictions are removed from the low index parameters and are transferred to the final pair of parameters.

**Corollary 3.2.** When \( \gamma \in B_{a} \), with \( a \leq s \leq r \) and \( s \mid r \), we have \( \gamma[1, r]^{s} = \gamma[1, s]^{r} \).

It follows that when \( r_{k-1} \leq s_{k} \leq r_{k} \) and \( s_{k} \mid r_{k} \), we have

\[ T((r_{1}, s_{1}), \ldots, (r_{k-1}, s_{k-1}), (r_{k}, s_{k})) = T((r_{1}, s_{1}), \ldots, (r_{k-1}, s_{k-1}), (s_{k}, r_{k})). \]

A further set of symmetry results implied by Prop. 3.1 is the following

**Corollary 3.3.** When \( \gamma \in B_{a} \), with \( a \leq m \leq n_{1}, n_{2} \) and \( n_{1} + y_{1} = n_{2} + y_{2} \) and \( [n_{1}]_{m} = [n_{2}]_{m}, \) we have \( \gamma[1, m]^{y_{1}}[1, n_{1}]^{m} = \gamma[1, m]^{y_{2}}[1, n_{2}]^{m}. \) Applied to T-links, \( L_{i} = T((r_{1}, s_{1}), \ldots, (r_{k-2}, s_{k-2}), (m, y_{i}), (n_{i}, m)) \) for \( i = 1, 2 \) and \( r_{k-2} \leq m, \) we have \( L_{1} = L_{2}. \) In particular, for two-tier T-links, we have

\[ T((m, y_{1}), (n_{1}, m)) = T((m, y_{2}), (n_{2}, m)). \] (3.6)

When \( n_{1} + y_{1} = n_{2} + y_{2} \) and instead \( [n_{1}]_{m} = [y_{2}]_{m}, \) we also have

\[ T((m, y_{1})(n_{1}, m)) = T((m, y_{2})(n_{2}, m)). \] (3.7)

In particular, when \( m < y \) we have \( T((m, y)(n, m)) = T((m, n)(y, m)). \) By Cor. 3, [1], this is also true when \( m = y. \)

**Proof.** By (3.2), we have \( \gamma[1, m]^{y_{1}}[1, n_{1}]^{m} = \gamma[1, m]^{y_{2}}[1, n_{2}]^{m} \), and this is \( \gamma[1, m]^{y_{1}}[m, 1]^{n_{1}-y_{1}+n_{1}-[n_{1}]_{m}} \), which is thus the same link for \( i = 1, 2. \)

Eq. (3.7) tells us that for a fixed choice of \( S \) and \( \delta \in [0, m - 1] \) the family of T-links given by \( \{ T((m, S - \lambda m - \delta), (\lambda m + \delta, m)) : \lambda \in [1, \left[ (S - \delta - 1)/m \right] \} \) represent a single link.

Furthermore \( T((m, S - \lambda m - \delta), (\lambda m + \delta, m)) = T((m, \lambda m + \delta), (S - \lambda m - \delta, m)) \) for \( \lambda \leq \left[ (S - \delta - m)/m \right] \) by Eq. (3.7) or by the prior observation plus duality. These
two families have disjoint parameter values except when \( S - \lambda m - \delta = \lambda' m + \delta \), i.e. \( S - 2\delta = m(\lambda + \lambda') \) for suitable choices of \( \lambda, \lambda' \in [1, [(S - \delta - m)/m]] \).

**Example 3.1.** To illustrate the first observation, suppose \( S = 37, m = 7, \delta = 3 \). The set \( \{T((7, 27), (10, 7)): T((7, 20), (17, 7)); T((7, 13), (24, 7)); T((7, 6), (31, 7))\} \) represents a single T-link.

The second observation shows that \( T((7, 27), (10, 7)) = T((7, 10), (27, 7)) \) and \( T((7, 20), (17, 7)) = T((7, 17), (20, 7)) \) and \( T((7, 13), (24, 7)) = T((7, 24), (13, 7)) \).

These two families are disjoint, since \( 31 = 7(\lambda + \lambda') \) has no integral solution.

Cor. 8, [4], gives a formula for the braid index of a T-link in terms of the form, \( L = T((r_1, s_1), \ldots, (r_k, s_k)) \) and its dual form \( T((\overline{r}_1, \overline{s}_1), \ldots, (\overline{r}_k, \overline{s}_k)) \). With \( i_0 = \min \{ i : r_i \geq r_{k-1} \} \), and \( j_0 = \min \{ j : \overline{r}_j \geq \overline{r}_{k-1} \} \), Cor. 8, [4] gives \( b(L) = \min(b_{i_0}, b_{j_0}) \).

The following is a simple but useful observation:

**Proposition 3.4.** Suppose \( L = T((r_1, s_1), \ldots, (r_k, s_k)) \).

When \( r_{i_0} = \overline{r}_{k-i_0} \), we have \( j_0 = k - i_0 \) and \( b(L) = r_{i_0} \).

When \( r_{i_0} > \overline{r}_{k-i_0} \), we have \( j_0 = 1 + k - i_0 \) and \( b(L) = \min(r_{i_0}, r_{1+k-i_0}) \).

Furthermore, \( r_i \geq i + 1 \) and \( \overline{r}_i \geq i + 1 \).

**Corollary 3.5.** Suppose \( L = T((r_1, s_1), \ldots, (r_k, s_k)) \). We have \( k \leq 2b(L) - 2 \).

When \( k = 2b(L) - 2 \) we have \( i_0 = b(L) - 1 = j_0 = b(L) - 1 \) and \( r_{b(L)-1} = b(L) = \overline{r}_{b(L)-1} \); indeed \( r_i = i + 1 = \overline{r}_i \) for \( i \leq b(L) - 1 \).

**Proof.** We must have \( i_0 \leq b(L) - 1 \) or \( j_0 \leq b(L) - 1 \) by Cor. 8, [4] and Prop. 3.4.

First consider the case \( j_0 \geq b(L) \), so that \( i_0 \leq b(L) - 1 \). As \( b(L) = r_{i_0} \geq \overline{r}_{k-i_0} \geq k - i_0 \), we have \( k \leq 2b(L) - 2 \). Now \( k = 2b(L) - 2 \) implies \( i_0 = b(L) - 1 \) and \( b(L) = \overline{r}_{k-i_0} \), so \( j_0 = k - i_0 \) by Prop. 3.4. Thus \( j_0 = b(L) - 1 \) contrary to assumption. A similar argument applies when \( i_0 \geq b(L) \), so we are left with the cases that \( i_0, j_0 \leq b(L) - 1 \).

When \( r_{i_0} > \overline{r}_{k-i_0} \), we have \( j_0 = 1 + k - i_0 \), i.e. \( k = i_0 + j_0 - 1 \leq 2b(L) - 3 \).

When \( r_{i_0} = \overline{r}_{k-i_0} \), we have \( j_0 = k - i_0 \), so that \( k \leq 2b(L) - 2 \), with equality exactly when \( i_0 = j_0 = b(L) - 1 \). \( \square \)

Cor. 3.5 shows that T-links with braid index two must have two or fewer tiers; those with braid index three must have four or fewer tiers. For a fixed choice of braid index, \( b \), the T-link with the maximum number of tiers, \( 2b - 2 \), has the form \( T((2, s_1), (3, s_2), \ldots, (b, s_{b-1}), (r_b, 1), \ldots, (r_{2b-3}, 1), (r_{2b-2}, 2)) \).

4. The Jones polynomial and other properties of T-links

4.1. The Jones polynomial for two tier T-links with \( b(L) \leq 3 \). By Cor. 8 [4], in order for the braid index to be two, a two tier T-link must have the form \( T((2, y), (n, 2)) \). Eq. 3.4 tells us that these are just \( T(2, n + y) \).

In order for the braid index to be three, a two tier T-link must have one of the following forms, \( T((m, y), (n, \rho)) \):

(i) \( T((2, y), (3, \rho)) \), with \( 3 \leq \rho \), (see Section 4.1.3),
(ii) \( T((2, y), (n, 3)) \), (see Section 4.1.1),
(iii) \( T((3, y), (n, 3)) \), (see Section 4.1.3),
(iv) \( T((3, y), (n, 2)) \), (see Section 4.1.2),
(v) \( T((m, 1), (n, 2)) \), with \( 3 < m \), (see Section 4.1.4),
4.1.1. The Jones polynomial for two tier T-links with $m = 2$, $\rho = 3$. Eq. (3.3) applied to $T((2, y), (n, 3))$ tells us these are just $T((2, y), (3, n))$. This is analyzed in Section 4.1.5.

4.1.2. The Jones polynomial for two tier T-links with $m = 3$, $\rho = 2$. By duality we have $T((3, y), (n, 2)) = T((2, n - 3), (y + 2, 3))$. Eq. (3.3) applied to the second form tells us these are just $T((2, n - 3), (3, y + 2))$. This is analyzed in Section 4.1.5.

Example 4.1. $k_{739} = T((3, 3), (11, 2))$ is a hyperbolic Lorenz knot, [4], [8].

4.1.3. The Jones polynomial for two tier T-links with $m = 3$, $\rho = 3$. By duality we have $T((3, y), (n, 3)) = T((3, n - 3), (y + 3, 3))$. Eq. (3.3) tells us these are just $T(3, n + y)$ when $3|n$ or $3|y$.

Restating (2.6) and (2.7) for the case that $3 \equiv 0$ mod 3, note that (3.3) tells us that $T((3, y), (n, 3)) = [3, 1]^{n-3}[1, 3]^{y+n-3}$. The braid properties discussed in Section 3 allow us to write $T((3, y), (n, 3)) = [3, 1]^{n}[1, 3]^{y+n-3}$.

When $[n]_3 = 1 = [y]_3$, we have $T((3, y), (n, 3)) = [3, 1][1, 3]^{y+n-3}$, and this is $\sigma_1^2\sigma_2^3[1, 3]^{y+n-[n]_3}$, so we may apply (2.26) with $w = 2y + 2n$ to get

$$V_{T((3, y), (n, 3))}(t) = t^{y+n-1}\{(1 + t^2) + t^{y+n}(1 + t^2)\}.$$

This class contains no torus links.

When $[n]_3 = 1$ and $[y]_3 = 2$, we have $T((3, y), (n, 3)) = \sigma_2^3[1, 3]^{y+n-[n]_3}$, so we may apply (2.26) with $w = 2y + 2n$ to get

$$V_{T((3, y), (n, 3))}(t) = t^{y+n-1}\{(1 + t^2) - t^{y+n}(1 + t^2)\}.$$

The same result applies when $[n]_3 = 2$ and $[y]_3 = 1$ by Eq. (3.3) This class contains no torus links.

When $[n]_3 = 2 = [y]_3$, we have $T((3, y), (n, 3)) = \sigma_1^3[1, 3]^{y+n-3}$, so we may apply (2.26) with $w = 2y + 2n$ to get

$$V_{T((3, y), (n, 3))}(t) = t^{y+n-1}\{(1 + t^2) + t^{y+n}(1 + t^2)\}.$$

This class contains no torus links.

Hence, in all cases $T((3, y), (n, 3))$ has a representation as $\sigma_1^a\sigma_2^b[1, 3]^z$ with $z \equiv 0$ mod 3. The detailed description is given in the following proposition.

Proposition 4.1. The family of two tier T-links $T((3, y), (n, 3))$ is partitioned into six families according to their representation by $\sigma_1^a\sigma_2^b[1, 3]^z$ with $z \equiv 0$ mod 3 and $a \geq 3$ and $b \geq 0$:

(i) $a = b = 0$ are the torus links $T(3, z) = T((3, x), (z - x, 3))$ whenever $x \equiv 0$ mod 3 and $3 \leq x \leq z - 6$,

(ii) $a = b = 1$ are the torus links $T(3, z + 1)$ which is the same as

(a) $T((3, 1 + x), (x, 3))$ whenever $x \equiv 0$ mod 3 and $0 \leq x \leq z - 6$,

(b) $T((3, x), (x, 1 + z - x, 3))$ whenever $x \equiv 0$ mod 3 and $3 \leq x \leq z - 3$,

(iii) $a = 3$ and $b = 1$ are the torus links $T(3, z + 2)$ which is the same as


Example 4.2. The following are hyperbolic Lorenz knots, [4]:

(i) \( k_{33} = T((2, 4), (3, 4)) \leftrightarrow \sigma_1^2 \sigma_2[1, 3]^3 \),
(ii) \( k_{34} = T((2, 2), (3, 8)) \leftrightarrow \sigma_1 \sigma_2[1, 3]^6 \),
(iii) \( k_{55} = T((2, 2), (3, 11)) \leftrightarrow \sigma_1^3 \sigma_2[1, 3]^9 \),
(iv) \( k_{511} = T((2, 6), (3, 4)) \leftrightarrow \sigma_1^2 \sigma_2[1, 3]^3 \),
(v) \( k_{65} = T((2, 2), (3, 14)) \leftrightarrow \sigma_1^4 \sigma_2[1, 3]^12 \),
(vi) \( k_{619} = T((2, 6), (3, 5)) \leftrightarrow \sigma_1^4 \sigma_2[1, 3]^3 \),
(vii) \( k_{75} = T((2, 4), (3, 16)) \leftrightarrow \sigma_1^5 \sigma_2[1, 3]^15 \),
(viii) \( k_{768} = T((2, 6), (3, 10)) \leftrightarrow \sigma_1^5 \sigma_2[1, 3]^9 \),
(ix) \( k_{790} = T((2, 6), (3, 8)) \leftrightarrow \sigma_1^6 \sigma_2[1, 3]^6 \).

Note that \( k_{739} = T((3, 3), (11, 2)) = T((2, 8), (5, 3)) = T((2, 8), (3, 5)) \) and \( T((2, 8), (3, 5)) \leftrightarrow \sigma_1 \sigma_2[1, 3]^3 \). Also \( \sigma_1^3 \sigma_2[1, 3]^{30} = T((3, 3a - 1), (4, 3)) \) by Prop. 4.1 case [v].

4.2. The Jones polynomial for 3-tier T-links with braid index 3. The 3-tier T-links have the form \( T((r_1, s_1), (r_2, s_2), (r_3, s_3)) \), with dual form \( T((s_3, r_3 - r_2), (s_3 + s_2, r_2 - r_1), (s_3 + s_2 + s_1, r_1)) \), [4].

First consider the case when \( r_1 = s_3 \). Using Cor. 8, [4], we have \( i_0 = \min\{i : r_1 \geq 73 - i\} = 2 \), and \( j_0 = \min\{j : r_7 \geq r_3 - j\} = 2 \). To obtain a braid index of 3 we need \( 3 = \min(r_2, 72) \). This implies \( r_1 = 2 \), with either \( r_2 = 3 \) or \( s_3 = 1 \). In the former case we have \( T((2, s_3), (3, s_3), (r_3, s_3)) \) with dual form \( T((2, r_3 - 3), (2 + s_3, r_2), (s_3 + s_2 + s_1, 2)) \). In the latter case we have \( T((2, s_3), (r_2, 1), (r_3, 2)) \), with dual form \( T((r_2, r_3 - 2), (3, s_3 + s_2 + s_1), 2)) \).

As these four forms are readily seen to be interchangeable, we study the form \( T((2, s_3), (3, s_2), (r_3, s_3)) = [1, 2]^r_3[1, 3]^r_3[2, 1]^{r_3 - 3}[1, 3]^2 \), by Prop. 5.3. This is just

(i) \( \sigma_1^{r_3 + r_3} \sigma_2[1, 3]^{r_3}, \) i.e. \( T((2, s_3 + r_3 - 1), (3, s_2 + 1)) \) when \( s_3 \equiv 0 \) mod 3,
(ii) \( \sigma_1^4 \sigma_2^{3-2} [1, 3]_{s+2}^2 \), when \( s_2 \equiv 1 \mod 3 \),
  (a) this is the T-link \( T((2, s_1 - 1), (3, s_2 + 3)) \) when \( r_3 = 4 \),
  (b) this is the T-link \( T((2, r_3 - 4), (3, s_2 + 3)) \) when \( s_1 = 1 \),
  (iii) \( \sigma_1^{3+1} \sigma_2^{3-2}[1, 3]_{s+1}^2 \) when \( s_2 \equiv 2 \mod 3 \).

Hence \( T((2, s_1), (3, s_2), (r_3, 2)) \) always has a representation as \( \sigma_1^4 \sigma_2^2[1, 3]_{z} \) with \( z \equiv 0 \mod 3 \). The Jones polynomial for case III when \( s_1 > 1 \) and \( r_3 > 4 \) is described by (2.27) and its coefficient vector is described in Section 2.2.2; this also applies to case III. The following result summarizes all the non-trivial forms that represent the same link.

**Proposition 4.2.** The following is an exhaustive list of the 3-tier T-links with \( r_1 = s_3 \) and braid index 3 which have no 2-tier representation. These have a braid representation \( \sigma_1^4 \sigma_2^2[1, 3]_{z} \), or \( \sigma_1^4 \sigma_2^2[1, 3]_{z} \), where \( x, y \geq 2 \) and \( (x, y) \neq (2, 2) \) and \( z \equiv 0 \mod 3 \) with \( z \geq 3 \).

(i) \( T((2, x), (3, z - 2), (3, y, 2)) \) or \( T((2, y), (3, z - 2), (3, x, 2)) \) from [18]
(ii) dual to prior item: \( T((2, y), (z, 1), (x + z, 2)) \) or \( T((2, x), (z, 1), (y + z, 2)) \),
(iii) \( T((2, x - 1), (3, z - 1), (2 + y, 2)) \) or \( T((2, y - 1), (3, z - 1), (2 + x, 2)) \) from [18]
(iv) dual: \( T((2, y - 1), (z + 1, 1), (x + z, 2)) \) or \( T((2, x - 1), (z + 1, 1), (y + z, 2)) \).

Note that if we map \( T((2, r_3 - r_2), (3, r_2 - 2), (3 + s_1, 2)) \) to \( \sigma_1^4 \sigma_2^2[1, 3]_{z} \), we obtain \( T((2, y), (z, 1), (x + z, 2)) \), which appears in the list above, as does the dual form. The form for mapping \( T((2, r_3 - r_2), (3, r_2 - 2), (3 + s_1, 2)) \) to \( \sigma_1^4 \sigma_2^2[1, 3]_{z} \), and its dual, also appear in the list above.

**Example 4.3.** \( k_{799} = T((2, 2), (3, 2), (5, 2)) \) is a hyperbolic Lorenz knot, [4]. By case III preceding Prop. 4.2 it is generated by \( \sigma_1^4 \sigma_2^2[1, 3]_{3} \).

Now consider the case when \( r_1 > s_3 \), hence \( r_1 \geq 3 \) and \( i_0 = 1 \) or 2. Prop. 3.3 says that in case \( i_0 = 1 \) and \( r_1 > \frac{3}{2} \) we have \( j_0 = 3 \). In order for the braid index to be three, we must have \( 3 = r_1 \), as \( \frac{3}{2} \geq 4 \). Thus \( s_3 = 2 \) and \( \sigma_7 = s_3 + s_2 \geq 3 \), which contradicts the assumption \( r_1 > \frac{3}{2} \). If \( i_0 = 2 \), we must have \( j_0 = 2 \) as \( j_0 = 1 \) implies \( \sigma_7 = s_3 \geq r_2 \). In order for the braid index to be three, we must have \( \sigma_2 = 3 \) and \( \sigma_7 = s_3 + 2 \). This contradicts the assumption \( i_0 = 2 \), as \( 3 \leq r_1 = \frac{3}{2} \).

The prior paragraph shows that the only viable case is \( i_0 = 1 \) and \( r_1 = \frac{3}{2} \), so that \( j_0 = 2 \). In order for the braid index to be three, we must have \( s_3 = 2 \), \( s_2 = 1 \) and \( r_1 = 3 \). This gives rise to the T-link \( T((3, s_1), (r_2, 1), (r_3, 2)) \), with dual form \( T((2, r_3 - r_2), (3, r_2 - 3), (3 + s_1, 3)) \). Prop. 3.1 shows the latter may be represented by the braid word \( [1, 2]^{r_3-r_2}[1, 3]^{r_2-3}[3, 1]^{s_1}[1, 3]^{3} \). We have the following representations:

(i) \( [1, 2]^{r_3-r_2}[1, 3]^{s_1+r_2+2} \), i.e. \( T((2, r_3 - r_2), (3, s_1 + r_2)) \) when \( s_1 \equiv 0 \mod 3 \),
(ii) \( [1, 2]^{r_3-r_2+2}[1, 3]^{s_1+r_2-1} \), i.e. \( T((2, r_3 - r_2 + 2), (3, s_1 + r_2 - 1)) \) when \( s_1 \equiv 2 \mod 3 \),
(iii) when \( s_1 \equiv 1 \mod 3 \),
  (a) \( \sigma_1^{r_3-r_2+1} \sigma_2[1, 3]^{s_1+r_2-1} \), i.e. \( T((2, r_3 - r_2), (3, s_1 + r_2)) \) when \( r_2 \equiv 0 \mod 3 \),
  (b) \( \sigma_1^{r_3-r_2+2} \sigma_2[1, 3]^{s_1+r_2-2} \) when \( r_2 \equiv 1 \mod 3 \),
  (c) \( \sigma_1^{r_3-r_2+5} \sigma_2[1, 3]^{s_1+r_2-3} \), i.e. \( T((2, r_3 - r_2 + 4), (3, s_1 + r_2 - 2)) \) when \( r_2 \equiv 2 \mod 3 \).
To see the result when $s_1 \equiv 2 \mod 3$, first note that $[1, 2]^{r_1 - r_2}[1, 3]^{s_1 + r_2 - 2}[3, 1]^2$ is a braid representative. For the case $s_1 \equiv 1 \mod 3$ and $r_2 \equiv 2 \mod 3$, note that $[1, 2]^{r_1 - r_2}((\sigma_1 \sigma_2)^3[1, 3]^{s_1 + r_2 - 2})$ is a representative, which is equivalent to $[1, 2]^{r_1 - r_2 + \sigma_1 \sigma_2 M_2[1, 3]^{s_1 + r_2 - 3}}$

The case when $r_1 < s_3$ gives rise to a dual form with $r_1 < r_2$, in which case the prior analysis applies to the dual form. Thus the 3-tier T-links of the form $T((r_1, s_1), (r_2, s_2), (r_3, s_3))$ with braid index three always have a representation as $\sigma_1^2 \sigma_2^2[1, 3]^2$ with $z \equiv 0 \mod 3$. The Jones polynomial for case (ii) is described by $\sigma_1^2 \sigma_2^2[1, 3]^2$ and its coefficient vector is described in Section 2.2.2. The following result summarizes all the true 3-tier forms, item (ii) that represent the same link.

**Proposition 4.3.** The following is an exhaustive list of the 3-tier T-links with $r_1 \neq s_3$ and braid index 3 which have no 2-tier representation. These have a braid representation $\sigma_1^2 \sigma_2^2[1, 3]^2$, or $\sigma_1^2 \sigma_2^2[1, 3]^2$, where $x, y \geq 2$ and $z \equiv 0 \mod 3$ with $z \geq 3$. In fact, $\min(x, y) = 2$, and $3 \leq \max(x, y) = M$. For any choice of $r_2 \equiv 1 \mod 3$ with $4 \leq r_2 \leq z + 1$ we have:

(i) $T((2, M - 2), (r_2 - 3), (z + 5 - r_2, 3))$, or $T((3, z + 2 - r_2), (r_2, 1), (M + r_2 - 2, 2))$.

Note that these are a subset of the links described by Prop. 4.2.

### 4.3. The Jones polynomial for 4-tier T-links with braid index three.

Cor. 3.3 shows that $T((2, s_1), (3, s_2), (r_3, 1), (r_4, 2))$ is the general form for 4-tier T-links with braid index 3. Proposition 3.3 tells us this has a representation as $[1, 2]^{s_1}[1, 3]^{s_2}[1, r_3][2, 1]^{s_4 - r_3}[1, r_3]^{2} = [1, 2]^{s_1}[1, 3]^{s_2}\sigma_2^{s_4 - r_3}[1, r_3]^3$. Invoke Prop. 3.3 again to obtain $[1, 2]^{s_1}[1, 3]^{s_2}\sigma_2^{s_4 - r_3}[1, 3]^{2}$, and this may be rewritten as $\sigma_1^2[1, 3]^{s_2}\sigma_2^{s_4 - r_3}[1, 3]^{2}$.

We have the following representations when $s_2 \equiv 0 \mod 3$:

(i) $\sigma_1^2 \sigma_2^{s_4 - r_3}[1, 3]^{s_2 + r_3}$, when $r_3 \equiv 0 \mod 3$,

(a) this is $T((2, s_1 + r_3 - 1), (3, s_2 + r_3 + 1))$ when $s_1 = 1$, (b) this is $T((2, s_1 + r_3 - 1), (3, s_2 + r_3 + 1))$ when $s_2 = r_3 - 1$,

(ii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 1}[1, 3]^{s_2 + r_3 - 1}$, when $r_3 \equiv 1 \mod 3$,

(iii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 3}[1, 3]^{s_2 + r_3 - 2}$, i.e. $T((2, s_1 + r_4 - 3), (3, s_2 + r_3 - 1))$ when $r_3 \equiv 2 \mod 3$.

To see the result when $r_3 \equiv 0 \mod 3$, note that $\sigma_1^2 \sigma_2^{s_4 - r_3}[1, 3]^{2}[1, 3]^{2}$ is just $\sigma_1^2 \sigma_2^{s_4 - r_3}[1, 3]^{2}$.

We have the following representations when $s_2 \equiv 1 \mod 3$:

(i) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 1}[1, 3]^{s_2 + r_3 - 1}$, when $r_3 \equiv 0 \mod 3$,

(ii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 3}[1, 3]^{s_2 + r_3 - 1}$, when $r_3 \equiv 1 \mod 3$,

(iii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 5}[1, 3]^{s_2 + r_3 - 2}$, i.e. $T((2, s_1 + r_4 - 3), (3, s_2 + r_3))$ when $r_3 \equiv 2 \mod 3$.

We have the following representations when $s_2 \equiv 2 \mod 3$:

(i) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 3}[1, 3]^{s_2 + r_3 - 2}$, $T((2, s_1 + r_4 - 3), (3, s_2 + r_3 - 1))$ when $r_3 \equiv 0 \mod 3$,

(ii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 5}[1, 3]^{s_2 + r_3 - 1}$, i.e. $T((2, s_1 + r_4 - 3), (3, s_2 + r_3 - 1))$ when $r_3 \equiv 1 \mod 3$,

(iii) $\sigma_1^2 \sigma_2^{s_4 - r_3 + 7}[1, 3]^{3} + s_2 + r_3 - 2$, i.e. $T((2, s_1 + r_4 - 3), (3, s_2 + r_3 - 1))$ when $r_3 \equiv 2 \mod 3$. 

---

The above analysis provides a comprehensive understanding of the Jones polynomial and related properties for twisted links, encompassing various cases and representations, allowing for a deeper insight into the algebraic and geometric properties of these links.
The Jones polynomial for cases which are not identified as two-tier T-links are described by (2.26) and the coefficient vector is described in Section 2.2.2. The prior calculations are summarized next. Recall that \( s^T = r_k - r_{k-1} \).

**Proposition 4.4.** A 4-tier T-link with braid index 3 has a 2-tier or 3-tier form. Suppose \( L_i = T((2, s_{1,i}), (3, s_{2,i}), (r_{3,i}, 1), (r_{4,i}, 2)) \) for \( i = 1, 2 \) and \( \{s_{1,1}, s_{1,1}\} = \{s_{1,2}, s_{1,2}\} \) and \( \{s_{2,1}, 3, r_{3,1}, 3\} = \{s_{2,2}, 3, r_{3,2}, 3\} \) and \( s_{2,1} + r_{3,1} = s_{2,2} + r_{3,2} \).

It follows that \( L_1 = L_2 \).

4.4. Classification result for three-braid T-links. The results in the prior sections may be combined to give us the following result, which extends the classification result for knots in \( L_3 \) by R. Bedient, [1].

**Theorem 4.5.** Suppose \( \beta(x, y, z) = \sigma_1^x \sigma_2^y [1, 3]^z \) with \( z \equiv 0 \mod 3 \), and \( z \geq 3 \) and \( x \geq y \geq 0 \).

If \( L \) is a T-link of braid index three, then it has a unique representation as some \( \beta(x, y, z) \) and \( L \) has a T-link representation with three or fewer tiers:

(i) when \( x = y = 0 \), we have \( L = T(3, z) \),
(ii) when \( x > y = 0 \), we have \( L = T((2, x), (3, z)) \), with no torus links present,
(iii) when \( x = y = 1 \), we have \( L = T(3, z + 1) \),
(iv) when \( x = 3 \) and \( y = 1 \), we have \( L = T(3, z + 2) \),
(v) when \( x > y = 1 \), we have \( L = T((2, x - 1), (3, z + 1)) \), with no torus links present for \( x \neq 3 \),
(vi) when \( x = y = 2 \), we have \( L = T((3, z - 2), (4, 3)) \), which is not a torus link,
(vii) when \( x, y \geq 2 \), we have \( L = T((2, x - 1), (3, z - 1), (2 + y, 2)) \), with no torus links present. The only 2-tier T-link present is when \( x = y = 2 \).

The closure of any such braid word, \( \beta(x, y, z) \), is a T-link of braid index three.

Note that we cannot allow \( z = 0 \), since such a braid word represents the connected sum of two elementary torus links, \( T_x \# T_y \), while Cor. 1, [4], indicates that T-links are prime, and of course in this setting \( x \neq 1 \) and \( y \neq 1 \).

It remains open to find what characteristics of a general T-link lead to a Jones polynomial with coefficients which are both "sparse" and "low" in value. "Sparse" would refer to relatively few non-zero values within each block, as [7] shows that the blocks separate as the number of full twists increases. It would be interesting to know when the blocks in the coefficient vector are disjoint, what is their length, when are the inter-block coefficients non-zero when full twists on an even number of strands are added, and what is their value, even if the full Jones polynomial could not be calculated.

The family of Lorenz links of braid index three has the interesting property that the Jones polynomial distinguishes distinct links within this family. This is not the general behavior across Lorenz links as there are distinct Lorenz knots that have the same Jones polynomial (see [4]). It was shown in Prop. 2.25, [10], that the Jones polynomial distinguishes distinct connected sums of elementary torus links.

**Remark 4.1.** It would be interesting to know whether the Jones polynomial distinguishes distinct links within the family \( \{(1, n)\# \prod_{i=1}^{n-1} \sigma_i^{z_i} : z \equiv 0 \mod n, \text{ and } z, x_i \geq 0 \text{ for all } i\} \), as this is true for \( z = 0 \), or \( n = 2, 3 \).

**Remark 4.2.** Cor. 3.5 shows that T-links with braid index \( b \) must have \( 2b - 2 \) or fewer tiers. However for both \( b = 2, 3 \) the T-link with the maximum number of tiers,
$T((2, s_1), (3, s_2), \ldots, (b, s_{b-1}), (r_b, 1), (r_{2b-3}, 1), (r_{2b-2}, 2))$, has a representation with fewer tiers. It would be interesting to establish whether this is always true. A related question is whether there is always some link that has a representation with $2b - 3$ tiers but no fewer tiers suffice as this is true for $b \leq 3$.

**Acknowledgments**

The author would like to thank J. Birman for her suggestion to investigate the Jones polynomial of Lorenz links, which leads to many other interesting questions. I. Kofman made the helpful suggestion to look at R. Bedient’s paper. [1].

**References**

[1] R.E. Bedient, Classifying 3-trip Lorenz knots, *Topology and its Applications*, 20 (1985), 89–96.
[2] J.S. Birman, Non-conjugate braids can define isotopic knots, *Comm. Pure Appl. Math.*, 22 (1969), 239–242.
[3] J.S. Birman and T.E. Brendle, Braids: A Survey, *arXiv:math.GT/0109205v2* (2005).
[4] J.S. Birman and I. Kofman, A new twist on Lorenz links, *Topology*, 2 (2009), 227–248.
[5] J. S. Birman and R. F. Williams, Knotted Periodic Orbits in Dynamical Systems-I: Lorenz’s Equations, Topology 22, No. 1 (1983), 47–82.
[6] J.C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants, [http://www.indiana.edu/~knotinfo](http://www.indiana.edu/~knotinfo) 2010
[7] A. Champanerkar and I. Kofman, On the Mahler measure of Jones polynomials under twisting, *Algebr. Geom. Topol.*, 5 (2005), 1–22.
[8] A. Champanerkar, I. Kofman, E. Patterson, The next simplest hyperbolic knots, *J. Knot Theory Ramifications*, 13 (2004), 965–987.
[9] P. Cromwell, Positive braids are visually prime. *Proc. London Math. Soc.* (3) 67 (1993), 384–424.
[10] D. Emmes, An Expression for the Homflypt polynomial and some related properties, *arXiv:math.GT/1009.5056v1* (2010).
[11] T. Fiedler, On the degree of the Jones polynomial, *Topology* 30 (1991), 1–8.
[12] J. Franks and R.F. Williams, Braids and the Jones polynomial, *Trans. Amer. Math. Soc.*, 303 (1987), 97–108.
[13] P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, and A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.*, 12 (1985), 183–312.
[14] V.F.R. Jones, A Polynomial Invariant of Knots and Links via von Neumann Algebras *Bull. Amer. Math. Soc.*, 12 (1985), 103–111.
[15] V.F.R. Jones, Hecke Algebra Representations of Braid Groups and Link Polynomials *Ann. of Math.*, 126 (1987), 335–388.
[16] W.B.R. Lickorish and K.C. Millett, A polynomial invariant for oriented links, *Topology* 26(1) (1987), 107–141.
[17] K. Murasugi, On Closed 3-braids *Mem. of Amer. Math. Soc.*, 151 (1974), American Mathematical Society, Providence, RI.
[18] K. Murasugi, *Knot Theory & Its Applications* (Birkhauser, Boston, 1996).
[19] K. Murasugi and R.S.D. Thomas, Isotopic Closed Nonconjugate Braids *Proc. of Amer. Math. Soc.*, 33 (1972), American Mathematical Society, Providence, RI.
[20] J.H. Przytycki and P. Traczyk, Invariants of links of Conway type, *Kobe J. Math.* 4 (1987), 115–139.
[21] A. Stoimenow, On the Crossing Number of Positive Knots and Braids and Braid Index Criteria of Jones and Morton-Williams-Franks, *Trans. Amer. Math. Soc.*, 354 (2002) 3927–3954.
[22] A. Stoimenow, Properties of Closed 3-Braids and Other Link Braid Representations *arXiv:math.GT/0606435v2* (2007).
[23] A. Stoimenow, On Polynomials and Surfaces of Variously Positive Links, *arXiv:math.GT/0202226* (2003).
[24] M. C. Sullivan, Factoring families of positive knots on Lorenz-like templates, *J. Knot Theory Ramifications*, 17 (2008), 1175–1187.
[25] M. C. Sullivan, Factoring positive braids via branched manifolds, Preprint http://galileo.math.siu.edu/~msulliva/Preprints/ (2005).

[26] M. C. Sullivan, Positive braids with a half twist are prime, J. Knot Theory Ramifications, 6 (1997), 405–415.

Unaffiliated

E-mail address: davidemmes@gmail.com