The Ramsey Number $R(3, K_{10} - e)$
and Computational Bounds for $R(3, G)$

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Abstract

Using computer algorithms we establish that the Ramsey number $R(3, K_{10} - e)$ is equal to 37, which solves the smallest open case for Ramsey numbers of this type. We also obtain new upper bounds for the cases of $R(3, K_k - e)$ for $11 \leq k \leq 16$, and show by construction a new lower bound $55 \leq R(3, K_{13} - e)$.

The new upper bounds on $R(3, K_k - e)$ are obtained by using the values and lower bounds on $e(3, K_l - e, n)$ for $l \leq k$, where $e(3, K_k - e, n)$ is the minimum number of edges in any triangle-free graph on $n$ vertices without $K_k - e$ in the complement. We complete the computation of the exact values of $e(3, K_k - e, n)$ for all $n$ with $k \leq 10$ and for $n \leq 34$ with $k = 11$, and establish many new lower bounds on $e(3, K_k - e, n)$ for higher values of $k$.

Using the maximum triangle-free graph generation method, we determine two other previously unknown Ramsey numbers, namely $R(3, K_{10} - K_3 - e) = 31$ and $R(3, K_{10} - P_3 - e) = 31$. For graphs $G$ on 10 vertices, besides $G = K_{10}$, this leaves 6 open cases of the form $R(3, G)$. The hardest among them appears to be $G = K_{10} - 2K_2$, for which we establish the bounds $31 \leq R(3, K_{10} - 2K_2) \leq 33$.

Keywords: Ramsey number; triangle-free graphs; almost-complete graphs; computation

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1 Notation and Preliminaries

Notation, definitions and tools of this work are analogous to those in our recent study of the classical two-color Ramsey numbers $R(3,k)$ [10], where $R(3,k)$ is defined as the smallest $m$ such that no $m$-vertex triangle-free graph with independence number less than $k$ exists. Using coloring terminology, such graphs can be seen as 2-colorings of the edges of $K_m$, which have no triangles in the first color and no monochromatic $K_k$ in the second color. In this paper we study mainly the case when $K_k - e$, the complete graph of order $k$ with one missing edge, is avoided in the second color.

Let $J_k$ denote the graph $K_k - e$, and hence the Ramsey number $R(K_3, J_k)$ is the smallest $m$ such that every triangle-free graph on $m$ vertices contains $J_k$ in the complement. Similarly as in [10], a graph $F$ will be called a $(G,H; n,e)$-graph, if $|V(F)| = n$, $|E(F)| = e$, $F$ does not contain $G$, and $\overline{F}$ (i.e. the complement of $F$) does not contain $H$. By $\mathcal{R}(G,H; n,e)$ we denote the set of all $(G,H; n,e)$-graphs. We will often omit the parameter $e$, or both $e$ and $n$, or give some range to either of these parameters, when referring to special $(G,H; n,e)$-graphs or sets $\mathcal{R}(G,H; n,e)$. If $G$ or $H$ is the complete graph $K_k$, we will often write $k$ instead of $G$. For example, a $(3, J_k; n)$-graph $F$ is a $(K_3, K_k - e; n,e)$-graph for $e = |E(F)|$.

In the remainder of this paper we will study only triangle-free graphs, and mainly $K_k$ or $J_k$ will be avoided in the complement. Note that for any $G \in \mathcal{R}(3,J_k)$ we have $\Delta(G) < k$, since all neighborhoods of vertices in $G$ are independent sets. $e(3,J_k; n)$ ($= e(K_3, J_k, n)$) is defined as the smallest number of edges in any $(3,J_k; n)$-graph. The sum of the degrees of all neighbors of a vertex $v$ in $G$ will be denoted by $Z_G(v)$. Similarly as in [17] [18] [23], one can easily generalize the tools used in analysis of the classical case $R(3,k)$ [10] [11] [12] [20] [21], as described in the sequel.

Let $G$ be a $(3, J_k ; n,e)$-graph. For any vertex $v \in V(G)$, we will denote by $G_v$ the graph induced in $G$ by the set $V(G) \setminus (N_G(v) \cup \{v\})$. Note that if $d = \deg_G(v)$, then $G_v$ is a $(3, J_{k-1}; n-d-1, e - Z_G(v))$-graph. This also implies that

$$
\gamma(v) = \gamma(v, k, G) = e - Z_G(v) - e(3, J_{k-1}, n-d-1) \geq 0,
$$

(1)

where $\gamma(v)$ is the so called deficiency of vertex $v$ (as in [11]). Finally, the deficiency of the graph $G$ is defined as

$$
\gamma(G) = \sum_{v \in V(G)} \gamma(v, k, G) \geq 0.
$$

(2)

The condition that $\gamma(G) \geq 0$ is often sufficient to derive good lower bounds on $e(3, J_k, n)$, though a stronger condition that all summands $\gamma(v, k, G)$ of (2) are non-negative sometimes implies better bounds. It is easy to compute $\gamma(G)$ just from the degree sequence of $G$ [11] [12] [17]. If $n_i$ is the number of vertices of degree $i$ in a $(3, J_k ; n,e)$-graph $G$, then
\[ \gamma(G) = ne - \sum_{i} n_i(i^2 + e(3, J_{k-1}, n - i - 1)) \geq 0, \]

where \( n = \sum_{i=0}^{k-1} n_i \) and \( 2e = \sum_{i=0}^{k-1} in_i \).

We obtain a number of improvements on lower bounds for \( e(3, J_k, n) \) and upper bounds for \( R(3, J_k) \), summarized at the end of the next section. The main computational result of this paper solves the smallest open case for the Ramsey numbers of the type \( R(3, J_k) \), namely we establish that \( R(3, J_{10}) = 37 \) by improving the previous upper bound \( R(3, J_{10}) \leq 38 \) \cite{17} by one.

Section 3 describes how the algorithms of this work differed from those used by us in the classical case \cite{10}, and how we determined two other previously unknown Ramsey numbers, namely \( R(3, K_{10} - K_3 - e) = 31 \) and \( R(3, K_{10} - P_3 - e) = 31 \), using the maximum triangle-free graph generation method. Section 4 presents progress on \( e(3, J_k, n) \) and \( R(3, J_k) \) for \( k \leq 11 \), and Section 5 for \( k \geq 12 \).

## 2 Summary of Prior and New Results

In 1995, Kim \cite{13} obtained a breakthrough result using probabilistic methods by establishing the exact asymptotics for the classical case, namely \( R(3, k) = \Theta(n^2/\log n) \). The asymptotic behaviour of \( R(3, J_k) \) is clearly the same, since \( K_{k-1} \subset J_k \subset K_k \). The monotonicity of \( e(3, G, n) \) and Ramsey numbers \( R(3, G) \) implies that for all \( n \) and \( k \) we have

\[ e(3, k, n) = e(K_3, K_k, n) \leq e(K_3, J_k, n) \leq e(K_3, K_{k-1}, n), \]
\[ R(3, k - 1) = R(K_3, K_{k-1}) \leq R(K_3, J_k) \leq R(K_3, K_k). \]

For the small cases of \( R(3, J_k) \) much of the progress was obtained by deriving and using good lower bounds on \( e(3, J_k, n) \). Explicit formulas for \( e(3, J_{k+2}, n) \) are known for all \( n \leq 13k/4 - 1 \), and for \( n = 13k/4 \) when \( k = 0 \) mod 4, as follows:

**Theorem 1** \cite{23,20} For all \( n, k \geq 1 \), for which \( e(3, J_{k+2}, n) \) is finite, we have

\[
e(3, J_{k+2}, n) = \begin{cases} 
0 & \text{if } n \leq k + 1, \\
n - k & \text{if } k + 2 \leq n \leq 2k \text{ and } k \geq 1, \\
3n - 5k & \text{if } 2k < n \leq 5k/2 \text{ and } k \geq 3, \\
5n - 10k & \text{if } 5k/2 < n \leq 3k \text{ and } k \geq 6, \\
6n - 13k & \text{if } 3k < n \leq 13k/4 - 1 \text{ and } k \geq 6.
\end{cases}
\]

Furthermore, \( e(3, J_{k+2}, n) = 6n - 13k \) for \( k = 4t \) and \( n = 13t \), and the inequality \( e(3, J_{k+2}, n) \geq 6n - 13k \) holds for all \( n \) and \( k \geq 6 \). All critical graphs have been characterized whenever the equality in the theorem holds for \( n \leq 3k \).
Our main focus in this direction is to obtain new exact values and bounds on \( e(3, J_{k+2}, n) \) for \( n \geq 13k/4 \). This in turn will permit us to prove the new upper bounds on \( R(3, J_k) \), for \( 10 \leq k \leq 16 \).

The general method we use is first to compute, if feasible, the exact value of \( e(3, J_k, n) \), or to derive a lower bound using a combination of equalities (3) and (6), and computations. Better lower bounds on \( e(3, J_{k-1}, m), m < n \), often lead to better lower bounds on \( e(3, J_k, n) \). If we show that \( e(3, J_k, n) = \infty \), then we obtain an upper bound \( R(3, J_k) \leq n \).

Full enumeration of the sets \( R(3, J_k) \) for \( k \leq 6 \) was completed in [18], all such graphs for \( k = 7 \) were uploaded by Fidytek at a website [8], and they were confirmed in this work. Radziszowski computed the values of \( e(3, J_7, n) \) and \( e(3, J_8, n) \) in [18]. Some of the values and bounds for \( k = 9 \) and \( k = 10 \), beyond those given by Theorem 1, were obtained by McKay, Piwakowski and Radziszowski in [17]. In this paper we complete this census for all cases of \( n \) with \( k \leq 10 \), and give new lower bounds for some higher parameters.

A \((3, J_k; n)\)-graph is called critical for a Ramsey number \( R(3, J_k) \) if \( n = R(3, J_k) - 1 \). In [17], McKay et al. determined that there are at least 6 critical triangle Ramsey graphs for \( J_9 \). Using the maximum triangle-free method (see Section 3), we find one more such graph and thus determine that there are exactly 7 critical graphs for \( R(3, J_9) \). They can be downloaded from the House of Graphs [1] by searching for the keywords “critical ramsey graph for \( R(3,K9-e) \”).

There is an obvious similarity between Theorem 1 and the results for \( e(3, k, n) \) obtained in [22] as summarized in Theorem 2 in [10], though also note that there are some differences. In particular, various cases are now restricted to \( k > c \). The graphs showing that these restrictions are necessary are listed in [23].

Our new results on \( R(3, J_k) \) are marked in bold in Table 1, which presents the values and best known bounds on the Ramsey numbers \( R(3, J_k) \) and \( R(3, K_k) \) for \( k \leq 16 \). The new upper bounds for \( J_{10} \) and \( J_{11} \) improve on the bounds given in [17] by 1 and 2, respectively. Other upper bounds in bold are recorded for the first time.

| \( k \) | \( R(3, J_k) \) | \( R(3, K_k) \) | \( k \) | \( R(3, J_k) \) | \( R(3, K_k) \) |
|-----|----------------|----------------|-----|----------------|----------------|
| 3   | 5              | 6              | 10  | 37             | 40–42          |
| 4   | 7              | 9              | 11  | 42–45          | 47–50          |
| 5   | 11             | 14             | 12  | 47–53          | 52–59          |
| 6   | 17             | 18             | 13  | 55–62          | 59–68          |
| 7   | 21             | 23             | 14  | 59–71          | 66–77          |
| 8   | 25             | 28             | 15  | 69–80          | 73–87          |
| 9   | 31             | 36             | 16  | 73–91          | 82–98          |

**Table 1:** Ramsey numbers \( R(3, J_k) \) and \( R(3, K_k) \), for \( k \leq 16, J_k = K_k - e \).

The results \( R(3, 11) \geq 47 [6] \) and \( R(3, 16) \geq 82 [7] \) were recently obtained by Exoo. Our recent work [10], after 25 years of no progress, improved the upper bound on \( R(3, 10) \)
from 43 \[21\] to 42, similarly as all other upper bounds for \(R(3,k)\) in the last column of Table 1. The references for all other bounds and values, and the previous bounds, are listed in \[19, 10\].

In a related cumulative work, Brinkmann, Goedgebeur and Schlage-Puchta \[3\] completed the computation of all Ramsey numbers of the form \(R(3,G)\) for graphs \(G\) on up to 10 vertices, except 10 cases. The exceptions included \(K_{10}, J_{10}\), and 8 other graphs close to \(K_{10}\). The complements of these 10 graphs are depicted in Figure 1. In fact, the authors of \[3\] showed in their article that the Ramsey number for all of these remaining cases is at least 31. In addition to \(J_{10}\), two of these cases are solved in this work, namely \(R(3,K_{10} - K_3 - e) = R(3,K_{10} - P_3 - e) = 31\). Hence for graphs \(G\) on 10 vertices, besides \(G = K_{10}\), this leaves 6 other open cases of the form \(R(3,G)\). The hardest among them appears to be \(G = K_{10} - 2K_2\), for which we establish the bounds

\[
31 \leq R(3, K_{10} - 2K_2) \leq 33.
\]

Figure 1: The complements \(\overline{G}\) of the 10 remaining graphs of order 10 which have \(R(3,G) \geq 31\), for which Brinkmann et al. \[3\] were unable to determine the Ramsey number. Graphs which must have the same Ramsey number are grouped by [ ] and ] (see \[3, 9\]).

We note that in three cases of Table 1, namely for \(k = 12, 14\) and 16, the best known lower bounds for \(R(3,J_k)\) are the same as for \(R(3,K_{k-1})\), and thus likely they can be improved. Our new bounds on \(R(3,J_k)\) are summarized in the following theorem.

**Theorem 2**  
(a) \(R(3,J_{10}) = 37\),  
(b) \(R(3,J_{11}) \leq 45\), \(R(3,J_{12}) \leq 53\), \(R(3,J_{13}) \leq 62\), \(R(3,J_{14}) \leq 71\), \(R(3,J_{15}) \leq 80\), \(R(3,J_{16}) \leq 91\), and (c) \(R(3,J_{13}) \geq 55\).

**Proof.** The lower bound \(R(3,J_{10}) \geq 37\) was established in \[17\]. The remaining sections describe our computational methods, and intermediate values and bounds on \(e(3,J_k,n)\). These imply the upper bound for (a) in Section 4, and the bounds for (b) in Section 5. Result (c) follows from a circular \((3,J_{13};54)\)-graph which we constructed. It has arc distances \(\{2,3,9,16,20,24\}\). Note that all of these distances have a nontrivial gcd with 54, and thus this graph cannot be transformed to an isomorphic circulant by a modular multiplier, so that 1 appears as one of the distances. \(\Box\)
Problem (Erdős-Sós, 1980 [4, 5]) Let $\Delta_k = R(3, k) - R(3, k - 1)$. Is it true that $\Delta_k \rightarrow \infty$? $\Delta_k/k \rightarrow 0$?

Only easy bounds $3 \leq \Delta_k \leq k$ are known. The results of this paper don’t give any general improvement on the bounds for $\Delta_k$, however we note that better understanding of the behavior of $R(3, J_k)$ relative to $R(3, K_k)$ may lead to such improvements since

$$
\Delta_k = (R(3, K_k) - R(3, J_k)) + (R(3, J_k) - R(3, K_{k-1})).
$$

(7)

The new results on $R(3, G)$ for some of the open cases listed in Figure 1 are as follows.

**Theorem 3** (a) $31 \leq R(3, K_{10} - 2K_2) \leq 33$,
(b) $R(3, K_{10} - K_3 - e) = R(3, K_{10} - P_3 - e) = 31$.

**Proof.** The lower bound of 31 for each of the three cases was established in [3].

The upper bound of 33 for (a) was obtained using essentially the same method as the one used in the main case of this paper for $J_{10}$ (see Section 3 for more details). This improves over the trivial bound of 37 implied by Theorem 2(a). Table 14 in Appendix 1 contains information about data used to derive the new bound: the values of $e(3, K_9 - 2K_2; n)$ and the counts of corresponding graphs. Using only degree sequence analysis and the values of $e(3, K_9 - 2K_2; n)$ one obtains the bound $R(3, K_{10} - 2K_2) \leq 35$. Further computations using the neighbour gluing algorithm were required to obtain the upper bound 33.

The computations applying the maximum triangle-free method from [3] were enhanced as described in Section 3, and gave the upper bounds needed for (b). \qed

Interestingly, contrary to our initial intuition, the case of $K_{10} - 2K_2$ appears to be significantly more difficult than $J_{10}$. The computational effort which was required to prove $R(3, K_{10} - 2K_2) \leq 33$ was similar to the computational effort to prove $R(3, J_{10}) \leq 37$, but it looks like it is computationally infeasible to improve the upper bound for $R(3, K_{10} - 2K_2)$ any further by our current algorithms. Our numerous attempts to improve the lower bound failed, and consequently we conjecture that $R(3, K_{10} - 2K_2) = 31$. If true, this would imply that for each of the 6 remaining open cases of $G$ on 10 vertices (except $K_{10}$), we have $R(3, G) = 31$.

Finally, we would like to note that we performed exhaustive searches for circulant graphs on up to 61 vertices in an attempt to improve lower bounds for $R(3, k)$, $R(3, J_k)$, and for $R(3, G)$ for the remaining graphs of order 10. If any of these lower bounds can still be improved, it must be by using graphs which are not circulant.
3 Algorithms

Similarly as in [10], we use two independent techniques to determine triangle Ramsey numbers: the maximum triangle-free method and the neighborhood gluing extension method. These methods are outlined in the following subsections.

Maximum Triangle-Free Method

Generating maximal triangle-free graphs

A maximal triangle-free graph (in short, an mtf graph) is a triangle-free graph such that the insertion of any new edge forms a triangle. It is easy to see that there exists a \((3, J_k; n)\)-graph if and only if there is an mtf \((3, J_k; n)\)-graph. Brinkmann, Goedgebeur and Schlage-Puchta [3] developed an algorithm to exhaustively generate mtf graphs and mtf Ramsey graphs efficiently. They implemented their algorithm in a program called \textit{triangleramsey} [2]. We refer the reader to [3, 9] for more details about the algorithm.

Using this program they determined the triangle Ramsey number \(R(3, G)\) of nearly all graphs \(G\) of order 10. The complements of the 10 graphs for which they were unable to determine the Ramsey number are shown in Figure 1 in Section 2.

It is computationally infeasible to use \textit{triangleramsey} to determine all mtf \((3, J_{10})\)-graphs. However, we executed \textit{triangleramsey} on a large computer cluster and were able to determine all mtf Ramsey graphs for \(K_{10} - P_3 - e\) up to 31 vertices (where \(P_k\) is the path with \(k\) vertices). This is one of the remaining graphs whose Ramsey number could not be determined by Brinkmann et al. The new computations took approximately 20 CPU years and the result is that there are 4 mtf Ramsey graphs with 30 vertices for \(K_{10} - P_3 - e\) and no mtf Ramsey graphs with 31 vertices. Thus, this proves that \(R(3, K_{10} - P_3 - e) = 31\). By monotonicity of Ramsey numbers we have \(R(3, K_{10} - K_3 - e) \leq R(3, K_{10} - P_3 - e)\), and thus the lower bound of 31 for both cases [3, 9] implies that \(R(3, K_{10} - K_3 - e) = R(3, K_{10} - P_3 - e) = 31\).

We also performed sample runs with \textit{triangleramsey} for the other remaining graphs, but it looks like it will be computationally infeasible to complete this task by this method. E.g., we estimate that approximately 144 CPU years would be required to determine \(R(3, K_{10} - P_4)\) by running \textit{triangleramsey}, and sample tests for \(R(3, K_{10} - K_4)\) indicate that this case will take much longer than 200 CPU years.

Generating complete sets of triangle Ramsey graphs

In order to determine \(e(3, J_k, n)\) also non-maximal triangle-free Ramsey graphs are required. Given all mtf \((3, J_k; n)\)-graphs, we can obtain all \((3, J_k; n)\)-graphs by recursively removing edges in all possible ways and testing if the obtained graphs are still \((3, J_k; n)\)-graphs. We used \textit{nauty} [15, 16] to make sure no isomorphic copies are output. We generated, amongst others, the full sets \(R(3, J_9; 28), R(3, J_9; 29)\) and \(R(3, J_9; 30)\) (see Appendix 1 for detailed results) using this method.
This mtf method is too slow for generating all \((3, \text{J}_{10}; n)\)-graphs for \(n\) which were needed in this work. Nevertheless, we used this method to verify the correctness of our other programs for smaller parameters. The results agreed in all cases in which more than one method was used (see Appendix 2 for more details).

**Neighborhood Gluing Extension Method**

The main method we used to improve upper bounds for \(R(3, \text{J}_k)\) is the neighborhood gluing extension method. In this method our extension algorithm takes a \((3, \text{J}_k; m)\)-graph \(H\) as input and produces all \((3, \text{J}_{k+1}; n, e)\)-graphs \(G\), often with some specific restrictions on \(n\) and \(e\), such that for some vertex \(v \in V(G)\) graph \(H\) is isomorphic to \(G_v\). The program also gets an expansion degree \(d = n - m - 1\) as input. Thus it connects, or *glues*, the \(d\) neighbors of a vertex \(v\) to \(H\) in all possible ways. Note that each neighbor of \(v\) is glued to an independent set, otherwise the extended graph would contain triangles. Similarly as in [10], various optimizations and bounding criteria are used to speed up the algorithm.

For example, suppose that we are aiming to construct \((3, \text{J}_{k+1})\)-graphs. Note that the complement of a graph \(G\) contains \(\text{J}_k\) if and only if \(G\) contains a spanning subgraph of \(\overline{\text{J}_k}\) as an induced subgraph. If two neighbors \(u_1\) and \(u_2\) of \(v\) have already been connected to independent sets \(S_1\) and \(S_2\) in \(H\) and \(H[V(H) \setminus (S_1 \cup S_2)]\) contains a spanning subgraph of \(\overline{\text{J}_{k-1}}\) as induced subgraph, we can abort the recursion, since this cannot yield any \((3, \text{J}_{k+1})\)-graphs.

There is, however, also one optimization which is specific to \(\text{J}_k\). Namely, we do not have to connect the neighbors of \(v\) to independent sets \(S\) for which \(H[V(H) \setminus S]\) induces an independent set of order \(k - 1\), since otherwise this graph would contain \(\overline{\text{J}_{k+1}}\) as induced subgraph (an independent set of order \(k - 1\) together with the disjoint edge \(\{u, v\}\)).

For more details about the general gluing algorithm, we refer the reader to [10, 9].

Most values and new bounds for \(e(3, \text{J}_k, n)\), which are listed in Section 4, were obtained by the gluing extension method. In Appendix 2 we describe how we tested the correctness of our implementation.

The strategy we used to determine if the parameters of the input graphs to which our extender program was applied are sufficient, i.e. that it is guaranteed that all \((3, \text{J}_{k+1}; n, e)\)-graphs are generated, is the same as in [10] and is outlined in the next subsection.

**Degree Sequence Feasibility**

This method is based on the same principles as in the classical case [10, 21, 14]. Suppose we know the values or lower bounds on \(e(3, \text{J}_k, m)\) for some fixed \(k\) and we wish to know all feasible degree sequences of \((3, \text{J}_{k+1}; n, e)\)-graphs. We construct the system of integer constraints consisting of \(n = \sum_{i=0}^{k} n_i, 2e = \sum_{i=0}^{k} in_i,\) and the inequality (3). If it has no solutions then we can conclude that no such graphs exist. Otherwise, we obtain solutions for \(n_i\)’s which include all potential degree sequences.
4 Progress on Computing Small $e(3, J_k, n)$

| vertices $n$ | $k$  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|--------------|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 3            |      | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 4            |      | 4  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |
| 5            |      | $\infty$ | 4  | 2  |    |    |    |    |    |    |    |    |    |    |    |
| 6            |      | 6  | 3  | 2  |    |    |    |    |    |    |    |    |    |    |    |
| 7            |      | $\infty$ | 6  | 3  | 2  |    |    |    |    |    |    |    |    |    |    |
| 8            |      | 8  | 4  | 3  | 2  |    |    |    |    |    |    |    |    |    |    |
| 9            |      | 12 | 7  | 4  | 3  | 2  |    |    |    |    |    |    |    |    |    |
| 10           |      | 15 | 10 | 5  | 4  | 3  | 2  |    |    |    |    |    |    |    |    |
| 11           |      | $\infty$ | 14 | 8  | 5  | 4  | 3  | 2  |    |    |    |    |    |    |    |
| 12           |      | 18 | 11 | 6  | 5  | 4  | 3  | 2  |    |    |    |    |    |    |    |
| 13           |      | 24 | 15 | 9  | 6  | 5  | 4  | 3  | 2  |    |    |    |    |    |    |
| 14           |      | 30 | 19 | 12 | 7  | 6  | 5  | 4  | 3  | 2  |    |    |    |    |    |
| 15           |      | 35 | 24 | 15 | 10 | 7  | 6  | 5  | 4  | 3  | 2  |    |    |    |    |
| 16           |      | 40 | 30 | 20 | 13 | 8  | 7  | 6  | 5  | 4  | 3  | 2  |    |    |    |
| 17           |      | $\infty$ | 37 | 25 | 16 | 11 | 8  | 7  | 6  | 5  | 4  | 3  |    |    |    |
| 18           |      | 43 | 30 | 20 | 14 | 9  | 8  | 7  | 6  | 5  | 4  |    |    |    |    |
| 19           |      | 54 | 37 | 25 | 17 | 12 | 9  | 8  | 7  | 6  | 5  |    |    |    |    |
| 20           |      | 60 | 44 | 30 | 20 | 15 | 10 | 9  | 8  | 7  | 6  |    |    |    |    |
| 21           |      | $\infty$ | 51 | 35 | 25 | 18 | 13 | 10 | 9  | 8  | 7  |    |    |    |    |
| 22           |      | 59 | 42 | 30 | 21 | 16 | 11 | 10 | 9  | 8  |    |    |    |    |    |
| 23           |      | 70 | 49 | 35 | 25 | 19 | 14 | 11 | 10 | 9  |    |    |    |    |    |
| 24           |      | 80 | 56 | 40 | 30 | 22 | 17 | 12 | 11 | 10 |    |    |    |    |    |
| 25           |      | $\infty$ | 65 | 46 | 35 | 25 | 20 | 15 | 12 | 11 |    |    |    |    |    |
| 26           |      | 73 | 52 | 40 | 30 | 23 | 18 | 13 | 12 |    |    |    |    |    |    |
| 27           |      | 81 | 61 | 45 | 35 | 26 | 21 | 16 | 13 |    |    |    |    |    |    |
| 28           |      | 95 | 68 | 51 | 40 | 30 | 24 | 19 | 14 |    |    |    |    |    |    |
| 29           |      | 106 | 77 | 58 | 45 | 35 | 27 | 22 | 17 |    |    |    |    |    |    |
| 30           |      | 117 | 86 | 66 | 50 | 40 | 30 | 25 | 20 |    |    |    |    |    |    |
| 31           |      | $\infty$ | 95 | 73 | 56 | 45 | 35 | 28 | 23 |    |    |    |    |    |    |

Table 2: Exact values of $e(3, J_k, n)$, for $3 \leq k \leq 16$, $3 \leq n \leq 31$.

Most of the values of $e(3, J_k, n)$ collected in Table 2 are implied by Theorem 1, others were obtained in [18, 17, 23], and those in bold are the result of this work. The bottom-left blank part covers cases where the graphs with corresponding parameters do not exist, while all entries in the top-right blank area indicate 0 edges. By Theorem 1 and [22], for fixed $k$, $e(3, J_k, n)$ is equal to $e(3, K_{k-1}, n)$ for most small $n$, and from the data presented in the following it looks like that this equality persists further as $n$ grows. Only sporadic
counterexamples to such behavior for \( n \) not much larger than \( 13k/4 \) are known: seven such cases are listed in [23] for \( k \leq 7 \), and another one can be noted in Table 4 for \( k = 11, n = 32 \). In other words, the second inequality of (4) seems to be much closer to equality than the first, and the opposite seems to hold in (5). If true, we can expect that the first part of \( \Delta_k \) in (7) is significantly larger than the second part.

**Exact values of** \( e(3, J_9, n) \)

The values of \( e(3, J_9, \leq 21) \) are determined by Theorem 1. The values of \( e(3, J_9, n) \) for \( 22 \leq n \leq 30 \) were obtained by computations, mostly by the gluing extender algorithm which is outlined in Section 3, and they are presented in Table 2. These values improve over previously reported lower bounds [23, 17]. We note that \( e(3, J_9, n) = e(3, K_8, n) \) for all \( 9 \leq n \leq 26 \).

**Exact values of** \( e(3, J_{10}, n) \)

The values of \( e(3, J_{10}, \leq 26) \) are determined by Theorem 1. The values for \( 27 \leq n \leq 37 \) were obtained by the gluing extender algorithm of Section 3, and they are presented in Table 3. These values improve over previously reported lower bounds [23, 17]. We note that \( e(3, J_{10}, n) = e(3, K_9, n) \) for all \( 10 \leq n \leq 35 \) (see [10]).

| \( n \) | \( e(3, J_{10}, n) \) | previous bound/comments |
|--------|---------------------|------------------------|
| 26     | 52                  | Theorem 1              |
| 27     | 61                  | 58                     |
| 28     | 68                  | 65                     |
| 29     | 77                  | 72                     |
| 30     | 86                  | 81                     |
| 31     | 95                  | 90                     |
| 32     | 104                 | 99                     |
| 33     | 118                 | 110                    |
| 34     | 129                 | 121                    |
| 35     | 140                 | 133                    |
| 36     | 156                 | 146, maximum 162       |
| 37     | \( \infty \)       | hence \( R(3, J_{10}) \leq 37 \), Theorem 2(a) |

**Table 3:** Values of \( e(3, J_{10}, n) \), for \( n \geq 26 \).

**Values and lower bounds on** \( e(3, J_{11}, n) \)

Table 4 presents what we know about \( e(3, J_{11}, n) \) beyond Theorem 1, which determines the values of \( e(3, J_{11}, \leq 28) \). The values and bounds for \( 29 \leq n \leq 41 \) were obtained by the gluing extender algorithm outlined in Section 3. The lower bounds on \( e(3, J_{11}, \geq 42) \) are based on solving integer constraints (2) and (3), using the exact values of \( e(3, J_{10}, n) \) listed in Table 3. We note that \( e(3, J_{11}, n) = e(3, K_{10}, n) \) for all \( 11 \leq n \leq 34 \), except
for $n = 32$ (see [10]). Four seemingly exceptional $(3, J_{11}; 32, 80)$-graphs can be obtained from the *House of Graphs* [1] by searching for the keywords “exceptional minimal ramsey graph”. One of them is the Wells graph, also called the Armanios-Wells graph. It is a double cover of the complement of the Clebsch graph. One of the other special graphs is formed by two disjoint copies of the Clebsch graph itself. This works, since the Clebsch graph is the unique $(3, J_6; 16, 40)$-graph [18] (denoted $G_4$ in the latter paper).

| $n$ | $e(3, J_{11}, n) \geq$ | comments |
|-----|------------------------|----------|
| 28  | 51                     | exact, Theorem 1 |
| 29  | 58                     | exact     |
| 30  | 66                     | exact     |
| 31  | 73                     | exact     |
| 32  | 80                     | exact, $e(3, 10, 32) = 81$ |
| 33  | 90                     | exact     |
| 34  | 99                     | exact     |
| 35  | 107                    | extender  |
| 36  | 117                    | extender  |
| 37  | 128                    | extender  |
| 38  | 139                    | extender  |
| 39  | 151                    | extender  |
| 40  | 161                    | extender  |
| 41  | 172                    | extender  |
| 42  | 185                    | $e(3, 10, 42) = \infty$ |
| 43  | 201                    |          |
| 44  | 217                    | maximum 220 |
| 45  | $\infty$              | hence $R(3, J_{11}) \leq 45$, Theorem 2(b) |

**Table 4**: Values and lower bounds on $e(3, J_{11}, n)$, for $n \geq 28$.

## 5 Progress on $e(3, J_k, n)$ and $R(3, J_k)$ for Higher $k$

The results in Tables 3 and 4 required computations of our gluing extender algorithm. We did not perform any such computations in an attempt to improve the lower bounds on $e(3, J_k, n)$ for $k \geq 12$, because such computations would be hardly feasible. The results presented in this section depend only on the degree sequence analysis described in Section 3, using constraints (2), (3) and the results for $k \leq 11$ from the previous section.

### Lower bounds on $e(3, J_{12}, n)$

Beyond the range of equality in Theorem 1 (for $n \geq 32$), the lower bounds we obtained for $e(3, J_{12}, n)$ are the same as for $e(3, K_{11}, n)$, for all $33 \leq n \leq 49$, except for $n = 38$ (see
and they are presented in Table 5. They were obtained by using constraints (2) and (3) as described in Section 3. In one case, for $n = 39$, we can improve the lower bound by one as in the following lemma.

**Lemma 4** $e(3, J_{12}, 39) \geq 117$.

**Proof.** Suppose that $G$ is a $(3, J_{12}; 39, e)$-graph with $e \leq 116$. Using (2) and (3) with the bounds of Table 4 gives no solutions for $e < 116$ and two feasible degree sequences $n_i$ for $e = 116$: $n_4 = 1, n_6 = 38$ and $n_5 = 2, n_6 = 37$. If $\deg(v) = 4$ then $Z_G(v) = 24$, and if $\deg(v) = 5$ then $Z_G(v) \geq 29$. In both cases this contradicts inequality (1), and thus we have $e(3, J_{12}, 39) \geq 117$. $\square$

| $n$  | $e(3, J_{12}, n) \geq$ | comments               |
|------|-------------------------|------------------------|
| 31   | 56                      | exact, Theorem 1        |
| 32   | 62                      | Theorem 1               |
| 33   | 68                      | Theorem 1               |
| 34   | 75                      |                        |
| 35   | 83                      |                        |
| 36   | 92                      |                        |
| 37   | 100                     |                        |
| 38   | 108                     | $e(3, 11, 32) \geq 109$|
| 39   | 117                     | improvement by Lemma 4 |
| 40   | 128                     |                        |
| 41   | 138                     |                        |
| 42   | 149                     |                        |
| 43   | 159                     |                        |
| 44   | 170                     |                        |
| 45   | 182                     |                        |
| 46   | 195                     |                        |
| 47   | 209                     |                        |
| 48   | 222                     | unique solution $n_7 = 36, n_8 = 12$ |
| 49   | 237                     |                        |
| 50   | 252                     | $e(3, 11, 50) = \infty$ |
| 51   | 266                     |                        |
| 52   | 280                     | maximum 286             |
| 53   | $\infty$               | hence $R(3, J_{12}) \leq 53$, Theorem 2(b) |

**Table 5:** Lower bounds on $e(3, J_{12}, n)$, for $n \geq 31$. 

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Lower bounds on $e(3, J_{13}, n)$

Beyond the range of equality in Theorem 1 (for $n \geq 35$), the lower bounds we obtained for $e(3, J_{13}, n)$ are the same as for $e(3, K_{12}, n)$, for all $35 \leq n \leq 58$, except for $n = 45, 46$ (see [10]), and they are presented in Table 6. They were obtained by using constraints (2) and (3) as described in Section 3.

| $n$ | $e(3, J_{13}, n) \geq$ | comments |
|-----|-------------------------|----------|
| 34  | 61                      | exact, Theorem 1 |
| 35  | 67                      | Theorem 1 |
| 36  | 73                      | Theorem 1 |
| 37  | 79                      | Theorem 1 |
| 38  | 86                      |          |
| 39  | 93                      |          |
| 40  | 100                     | unique solution $n_5 = 40$ |
| 41  | 109                     | unique solution $n_5 = 28, n_6 = 13$ |
| 42  | 119                     |          |
| 43  | 128                     |          |
| 44  | 138                     |          |
| 45  | 147                     | $e(3, 12, 45) \geq 148$ |
| 46  | 157                     | $e(3, 12, 46) \geq 158$ |
| 47  | 167                     |          |
| 48  | 179                     |          |
| 49  | 191                     |          |
| 50  | 203                     |          |
| 51  | 216                     |          |
| 52  | 229                     |          |
| 53  | 241                     |          |
| 54  | 255                     |          |
| 55  | 269                     |          |
| 56  | 283                     | unique solution $n_{10} = 50, n_{11} = 6$ |
| 57  | 299                     |          |
| 58  | 316                     |          |
| 59  | 333                     | $e(3, 12, 59) = \infty$ |
| 60  | 350                     | maximum 360 |
| 61  | 366                     | must be regular $n_{12} = 61$ |
| 62  | $\infty$               | hence $R(3, J_{13}) \leq 62$, Theorem 2(b) |

Table 6: Lower bounds on $e(3, J_{13}, n)$, for $n \geq 34$. 
Lower bounds on $e(3, J_{14}, n)$

Beyond the range of equality in Theorem 1 (for $n \geq 40$), the lower bounds we obtained for $e(3, J_{14}, n)$ are the same as for $e(3, K_{13}, n)$, for all $41 \leq n \leq 67$, except for $n = 54, 55$ (see [10]), and they are presented in Table 7. They were obtained by using constraints (2) and (3) as described in Section 3.

| $n$ | $e(3, J_{14}, n)$ $\geq$ | comments                      |
|-----|-------------------------|-------------------------------|
| 39  | 78                      | exact, Theorem 1              |
| 40  | 84                      | Theorem 1                     |
| 41  | 91                      |                               |
| 42  | 97                      |                               |
| 43  | 104                     |                               |
| 44  | 112                     |                               |
| 45  | 120                     |                               |
| 46  | 128                     |                               |
| 47  | 136                     |                               |
| 48  | 146                     |                               |
| 49  | 157                     |                               |
| 50  | 167                     |                               |
| 51  | 177                     |                               |
| 52  | 189                     |                               |
| 53  | 200                     |                               |
| 54  | 210                     | $e(3, 13, 54) \geq 212$       |
| 55  | 222                     | $e(3, 13, 55) \geq 223$       |
| 56  | 234                     |                               |
| 57  | 247                     |                               |
| 58  | 260                     |                               |
| 59  | 275                     |                               |
| 60  | 289                     |                               |
| 61  | 303                     |                               |
| 62  | 319                     |                               |
| 63  | 334                     |                               |
| 64  | 350                     |                               |
| 65  | 365                     |                               |
| 66  | 381                     |                               |
| 67  | 398                     |                               |
| 68  | 416                     | $e(3, 13, 68) = \infty$       |
| 69  | 434                     |                               |
| 70  | 451                     | maximum 455                   |
| 71  | $\infty$                | hence $R(3, J_{14}) \leq 71$, Theorem 2(b) |

Table 7: Lower bounds on $e(3, J_{14}, n)$, for $n \geq 39$. 
Lower bounds on $e(3, J_{15}, n)$
The lower bounds we obtained for $e(3, J_{15}, n)$ are the same as for $e(3, K_{14}, n)$ for all $71 \leq n \leq 76$ (see [10]), and they are presented in Table 8. They were obtained by using constraints (2) and (3) as described in Section 3.

| $n$ | $e(3, J_{15}, n) \geq$ | comments |
|-----|-----------------|----------|
| 71  | 398             |          |
| 72  | 415             |          |
| 73  | 432             |          |
| 74  | 449             | 451 needed for $R(3, J_{16}) \leq 90$ |
| 75  | 468             |          |
| 76  | 486             | 473 sufficient for $R(3, J_{16}) \leq 91$ |
| 77  | 505             | $e(3, 14, 77) = \infty$ |
| 78  | 524             |          |
| 79  | 543             | maximum 553 |
| 80  | $\infty$       | hence $R(3, J_{15}) \leq 80$, Theorem 2(b) |

Table 8: Lower bounds on $e(3, J_{15}, n)$, for $n \geq 71$.

A 15-regular $(3, J_{16}; 90, 675)$-graph $G$ is feasible when for every vertex $v$ its $G_v$ is a $(3, J_{15}; 74, 450)$-graph. Constraints (2) and (3) have no feasible solution for $(3, J_{16}; 91)$-graphs, and thus $R(3, J_{16}) \leq 91$.

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Appendix 1: Graph Counts

Tables 9–13 below contain all known exact counts of $(3, J_k; n, e)$-graphs for specified $n$, for $k = 7, 8, 9, 10$ and 11, respectively. All graph counts were obtained by the algorithms described in Section 3. Empty entries indicate 0. In all cases, the maximum number of edges is bounded by $\Delta(G)n/2 \leq (k - 1)n/2$. All $(3, J_k; n, e(3, J_k, n))$-graphs, $k \leq 10$, which were constructed by our programs can be obtained from the House of Graphs [1] by searching for the keywords “minimal ramsey graph * Kk-e”.

All $(3, J_7)$-graphs were previously determined by Fidytek [8]. We include the counts of $(3, J_7)$-graphs in Table 9 for completeness and more uniform presentation, since Fidytek provided statistics for $J_7$-free graphs whose complement does not contain a $K_3$, while we list triangle-free graphs whose complement does not contain a $J_7$. The latter is similar to the tables with data about $(3, J_k)$-graphs for $k \leq 6$ in [18].

Table 14 contains all known exact counts of $(3, K_9 - 2K_2; n, e)$-graphs. These graph counts were also obtained by the algorithms described in Section 3. All edge minimal $(3, K_9 - 2K_2; n, e(3, K_9 - 2K_2, n))$-graphs which were constructed by our programs can be obtained from the House of Graphs [1] by searching for the keywords “minimal ramsey graph * K9-2K2”.
| edges | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 3     | 1 |   |    |    |    |    |    |    |    |    |    |    |    |
| 4     | 6 | 1 |    |    |    |    |    |    |    |    |    |    |    |
| 5     | 14| 2 | 1  |    |    |    |    |    |    |    |    |    |    |
| 6     | 31| 14| 1  |    |    |    |    |    |    |    |    |    |    |
| 7     | 51| 44| 5  |    |    |    |    |    |    |    |    |    |    |
| 8     | 69| 108|27 | 1  |    |    |    |    |    |    |    |    |    |
| 9     | 76| 195|102| 3  |    |    |    |    |    |    |    |    |    |
| 10    | 66| 291|327| 29 |    |    |    |    |    |    |    |    |    |
| 11    | 41| 329|771| 131| 1  |    |    |    |    |    |    |    |    |
| 12    | 22| 302|1355|5239|398 |    |    |    |    |    |    |    |    |
| 13    | 9 | 204|1778|2158|47  |    |    |    |    |    |    |    |    |
| 14    | 3 | 117|1808|5239|398 |    |    |    |    |    |    |    |    |
| 15    | 2 | 53 |1439|8961|2434|1  |    |    |    |    |    |    |    |
| 16    | 1 | 25 |918 |11150|9872|16 |    |    |    |    |    |    |    |
| 17    | 9 | 492|11072|26586|241 |    |    |    |    |    |    |    |    |
| 18    | 4 | 231|8505|49752|2665 |    |    |    |    |    |    |    |    |
| 19    | 1 | 99 |5260|67226|16313|1  |    |    |    |    |    |    |    |
| 20    | 1 | 44 |2794|68351|60891|13 |    |    |    |    |    |    |    |
| 21    | 19| 1294|54124|145552|300 |    |    |    |    |    |    |    |    |
| 22    | 7 | 578|34707|238525|397  |    |    |    |    |    |    |    |    |
| 23    | 3 | 233|18757|280341|28889 |    |    |    |    |    |    |    |    |
| 24    | 2 | 101|8976|247162|117123|2  |    |    |    |    |    |    |    |
| 25    | 1 | 41 |3942|169011|291706|14 |    |    |    |    |    |    |    |
| 26    | 18| 1669|93503|477543|305  |    |    |    |    |    |    |    |    |
| 27    | 6 | 693|43149|543488|4521 |    |    |    |    |    |    |    |    |
| 28    | 3 | 289|17392|451296|32828 |    |    |    |    |    |    |    |    |
| 29    | 1 | 115|6217|286635|121140 |    |    |    |    |    |    |    |    |
| 30    | 1 | 52 |2073|146341|256923|3  |    |    |    |    |    |    |    |
| 31    | 21| 626|63112|338238|22  |    |    |    |    |    |    |    |    |
| 32    | 10| 190|24207|296128|361 |    |    |    |    |    |    |    |    |
| 33    | 4 | 50 |8505|181637|3251 |    |    |    |    |    |    |    |    |
| 34    | 3 | 14 |2841|83169|14968 |    |    |    |    |    |    |    |    |
| 35    | 2 | 1  |884 |30257|35296 |    |    |    |    |    |    |    |    |
| 36    | 1 | 1 | 275|9648 |45855 |    |    |    |    |    |    |    |    |
| 37    | 75| 2865|34944 |    |    |    |    |    |    |    |    |    |    |
| 38    | 22| 883 |16583 |54  |    |    |    |    |    |    |    |    |    |
| 39    | 5 | 273|5269 |349 |    |    |    |    |    |    |    |    |    |
| 40    | 2 | 94 |1334 |1070 |    |    |    |    |    |    |    |    |    |
| 41    | 32| 350 |1501 |    |    |    |    |    |    |    |    |    |    |
| 42    | 11| 134 |1174 |    |    |    |    |    |    |    |    |    |    |
| 43    | 4 | 50 |522 |2  |    |    |    |    |    |    |    |    |    |
| 44    | 1 | 25 |147 |8  |    |    |    |    |    |    |    |    |    |
| 45    | 1 | 8 | 26 |38 |    |    |    |    |    |    |    |    |    |
| 46    | 4 | 6 | 61 |    |    |    |    |    |    |    |    |    |    |
| 47    | 1 | 1 | 36 |    |    |    |    |    |    |    |    |    |    |
| 48    | 1 | 1 | 17 |    |    |    |    |    |    |    |    |    |    |
| 49    | 1 | 4 |    |    |    |    |    |    |    |    |    |    |    |
| 50    | 1 |    |    |    |    |    |    |    |    |    |    |    |    |
| 51    | 1 |    |    |    |    |    |    |    |    |    |    |    |    |
| 54    | 1 |    |    |    |    |    |    |    |    |    |    |    |    |
| 55-59 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 60    |    |    |    |    |    |    |    |    |    |    |    |    |    |

| [\mathcal{K}(3, J_7; n)] | 392  | 1697 | 9430 | 58522 | 348038 | 1323836 | 2447170 | 1358974 | 158459 | 4853 | 225 | 1  | 1  |

**Table 9:** Number of \((3, J_7; n, e)\)-graphs, for \(n \geq 8\).
| edges $e$ | number of vertices $n$ |
|----------|-----------------|
| 15       | 1               |
| 16       | 2               |
| 17       | 18              |
| 18       | 188             |
| 19       | ?               |
| 20       | ? 2             |
| 21       | ? 17            |
| 22       | ? 358           |
| 23       | ? 10659         |
| 24       | ? ?             |
| 25       | ? ? 2           |
| 26       | ? ? 44          |
| 27       | ? ? 2576        |
| 28       | ? ? 117474      |
| 29       | ? ? ?           |
| 30       | ? ? ? 2         |
| 31       | ? ? ? 22        |
| 32       | ? ? ? 1175      |
| 33       | ? ? ? 79025     |
| 34-36    | ? ? ? ?         |
| 37       | ? ? ? ? ? 20    |
| 38       | ? ? ? ? ? 2031  |
| 39       | ? ? ? ? ? 130297|
| 40       | ? ? ? ? ? 393909|
| 41-43    | ? ? ? ? ?       |
| 44       | ? ? ? ? ? 169   |
| 45       | ? ? ? ? ? 8231  |
| 46       | ? ? ? ? ? 310400|
| 47       | ? ? ? ? ? 5839714|
| 48-50    | ? ? ? ? ?       |
| 51       | ? ? ? ? ? 7     |
| 52       | ? ? ? ? ? 375   |
| 53       | ? ? ? ? ? 14141 |
| 54       | ? ? ? ? ? 255635|
| 55       | ? ? ? ? ? 2262269|
| 56-58    | ? ? ? ? ?       |
| 59       | ? ? ? ? ? 2     |
| 60       | ? ? ? ? ? 13    |
| 61       | ? ? ? ? ? 162   |
| 62       | ? ? ? ? ? 1630  |
| 63       | ? ? ? ? ? 9101  |
| 64       | ? ? ? ? ? 26611 |
| 65       | ? ? ? ? ? 42700 |
| 66       | ? ? ? ? ? 41455 |
| 67       | ? ? ? ? 26459   |
| 68       | ? ? ? 11716     |
| 69       | ? ? 3657        |
| 70       | ? ? 957 1       |
| 71       | ? ? 208 2       |
| 72       | ? ? 42 8        |
| 73       | ? ? 10 6        |
| 74       | ? ? 2 4         |
| 75       | 1               |
| 76       | 1               |
| 77       | 13              |
| 78-79    | 1               |
| 80       | 0               |
| 81-83    | 0               |
| 84       | 8               |

| $|R(3,J_8;n)|$ | 164725 | 36 | 9 |

**Table 10:** Number of $(3, J_8; n, e)$-graphs, for $n \geq 15$. 
| edges | number of vertices n |
|-------|---------------------|
|       | 20 21 22 23 24 25 26 27 28 29 30 |
| 30    | ? 21 |
| 31    | 64 |
| 32    | 2073 |
| 33-34 | ? |
| 35    | ? 1 |
| 36    | ? 20 |
| 37    | ? 951 |
| 38    | ? 396353 |
| 39-41 | ? ? |
| 42    | ? ? 21 |
| 43    | ? ? 1592 |
| 44    | ? ? 86833 |
| 45    | ? ? 396353 |
| 46-48 | ? ? |
| 49    | ? ? ? 103 |
| 50    | ? ? ? 9102 |
| 51    | ? ? ? 514099 |
| 52-55 | ? ? |
| 56    | ? ? ? 54 |
| 57    | ? ? ? 3639 |
| 58    | ? ? ? 173608 |
| 59-64 | ? ? ? |
| 65    | ? ? ? ? 547 |
| 66    | ? ? ? ? 48964 |
| 67    | ? ? ? ? 2538589 |
| 68-72 | ? ? ? |
| 73    | ? ? ? ? 62 |
| 74    | ? ? ? ? 1857 |
| 75    | ? ? ? ? 36799 |
| 76    | ? ? ? ? 755052 |
| 77-80 | ? ? ? |
| 81    | ? ? ? ? 4 |
| 82    | ? ? ? ? 24 |
| 83    | ? ? ? ? 197 |
| 84    | ? ? ? ? 1126 |
| 85    | ? ? ? ? 6206 |
| 86    | ? ? ? ? 42468 |
| 87    | ? ? ? ? 384398 |
| 88    | ? ? ? ? 2843005 |
| 89-94 | ? ? ? |
| 95    | ? ? ? ? 1 |
| 96    | ? ? ? ? 14 |
| 97    | ? ? ? ? 107 |
| 98    | ? ? ? ? 1062 |
| 99    | ? ? ? ? 5182 |
| 100   | ? ? ? ? 16588 |
| 101   | ? ? ? ? 34077 |
| 102   | ? ? 50241 |
| 103   | ? ? 51686 |
| 104   | ? ? 39702 |
| 105   | ? ? 21621 |
| 106   | ? 9370 |
| 107   | ? 2864 |
| 108   | ? 843 |
| 109   | 158 |
| 110   | 49 |
| 111   | 7 |
| 112   | 91 |
| 113-115 | 0 |
| 116   | 1 |
| 117   | 1 |
| 118   | 1 |
| 119   | 1 |
| 120   | 1 |

\[ |R[3, J_9; n]| \]

Table 11: Number of \((3, J_9; n, e)\)-graphs, for \(n \geq 20\).
Table 12: Number of $(3, J_{10}; n, e)$-graphs, for $n \geq 24$. 

| edges | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $e$   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 40    | 2  |    |    |    |    |    |    |    |    |    |    |    |    |
| 41    | 32 |    |    |    |    |    |    |    |    |    |    |    |    |
| 42    | 2089 |    |    |    |    |    |    |    |    |    |    |    |    |
| 43-45 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 46    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 47    | 39 |    |    |    |    |    |    |    |    |    |    |    |    |
| 48    | 4113 |    |    |    |    |    |    |    |    |    |    |    |    |
| 49-51 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 52    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 53    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 54    | 444 |    |    |    |    |    |    |    |    |    |    |    |    |
| 55    | 58550 |    |    |    |    |    |    |    |    |    |    |    |    |
| 56-60 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 61    | 700 |    |    |    |    |    |    |    |    |    |    |    |    |
| 62    | 95185 |    |    |    |    |    |    |    |    |    |    |    |    |
| 63    | 6531339 |    |    |    |    |    |    |    |    |    |    |    |    |
| 64-67 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 68    | 126 |    |    |    |    |    |    |    |    |    |    |    |    |
| 69    | 17223 |    |    |    |    |    |    |    |    |    |    |    |    |
| 70    | 1204171 |    |    |    |    |    |    |    |    |    |    |    |    |
| 71-76 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 77    | 1342 |    |    |    |    |    |    |    |    |    |    |    |    |
| 78    | 156982 |    |    |    |    |    |    |    |    |    |    |    |    |
| 79-85 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 86    | 1800 |    |    |    |    |    |    |    |    |    |    |    |    |
| 87    | 147407 |    |    |    |    |    |    |    |    |    |    |    |    |
| 88-94 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 95    | 560 |    |    |    |    |    |    |    |    |    |    |    |    |
| 96    | 35154 |    |    |    |    |    |    |    |    |    |    |    |    |
| 97-103|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 104   | 39 |    |    |    |    |    |    |    |    |    |    |    |    |
| 105   | 952 |    |    |    |    |    |    |    |    |    |    |    |    |
| 106-117|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 118   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 119   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 120   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 121-128|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 129   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 130   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 131   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 132-139|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 140   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 141   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 142-146|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 147   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 148   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 149   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 150   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 151   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 152   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 153   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 154-155|    |    |    |    |    |    |    |    |    |    |    |    |    |
| 156   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 157   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 158   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 159   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 160   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 161   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 162   |    |    |    |    |    |    |    |    |    |    |    |    |    |

Table 12: Number of $(3, J_{10}; n, e)$-graphs, for $n \geq 24$. 

| number of vertices n | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |
|----------------------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $n$                  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| $e$ | 29 | 30 | 31 | 32 | 33 | 34 |
|-----|----|----|----|----|----|----|
| 58  | 5  |    |    |    |    |    |
| 59  | 1364 |    |    |    |    |    |
| 60-65 | ?  |    |    |    |    |    |
| 66  | ?  | 5084 |    |    |    |    |
| 67-72 | ?  | ?  |    |    |    |    |
| 73  | ?  | ?  | 2657 |    |    |    |
| 74-79 | ?  | ?  | ?  |    |    |    |
| 80  | ?  | ?  | ?  | 4  |    |    |
| 81  | ?  | ?  | ?  | 6601 |    |    |
| 82-89 | ?  | ?  | ?  | ?  |    |    |
| 90  | ?  | ?  | ?  | ?  | 57099 |    |
| 91-98 | ?  | ?  | ?  | ?  | ?  |    |
| 99  | ?  | ?  | ?  | ?  | ?  | $\geq 1$ |
| $\geq 100$ | ?  | ?  | ?  | ?  | ?  | ?  |

Table 13: Number of $(3, J_{11}; n, e)$-graphs, for $29 \leq n \leq 34$. 
| edges | number of vertices n |
|-------|---------------------|
| 45    | 1                   |
| 46    | 2                   |
| 47    | 61                  |
| 48    | 3743                |
| 49    | 408410              |
| 50-53 | ?                   |
| 54    | ? 2                 |
| 55    | ? 299               |
| 56    | ? 20314             |
| 57    | ? 985296            |
| 58    | ? 23618486          |
| 59-60 | ? ?                 |
| 63    | ? ? 9               |
| 64    | ? ? 528             |
| 65    | ? ? 24860           |
| 66    | ? ? 566836          |
| 67    | ? ? 5830123         |
| 68-72 | ? ?                 |
| 72    | ? ? ? 104           |
| 73    | ? ? ? 1068          |
| 74    | ? ? ? 7913          |
| 75    | ? ? ? 31134         |
| 76    | ? ? ? 84634         |
| 77    | ? ? ? 160815        |
| 78    | ? ? ? 215365        |
| 79    | ? ? ? 207752        |
| 80    | ? ? ? 172746 18     |
| 81    | ? ? ? 142474 97     |
| 82    | ? ? ? 121641 333    |
| 83    | ? ? ? 107869 516    |
| 84    | ? ? ? 108001 416    |
| 85    | ? ? ? 110938 158    |
| 86    | ? ? ? 101090 30     |
| 87    | ? ? ? 72665 5       |
| 88    | ? ? ? 40935 1       |
| 89    | ? ? ? 17722 0       |
| 90    | ? ? 6262 0 3        |
| 91    | ? ? 1779 0 2        |
| 92    | ? ? 522 5 1         |
| 93    | ? 129 16 0          |
| 94    | 46 35 0             |
| 95    | 8 34 0              |
| 96    | 5 19 0              |
| 97    | 6 0                 |
| 98    | 2 0                 |
| 99    | 1 0                 |
| 100   | 1 0                 |
| 101-103 | ? ? 1713017 1693 8 |

Table 14: Number of $(3, K_9 - 2K_2; n, e)$-graphs, for $n \geq 21$. 
Appendix 2: Testing correctness

Since most results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified. Below we describe how we tested the correctness of our programs.

Correctness

- For every \((3, J_k)\)-graph which was output by our programs, we verified that it does not contain a spanning subgraph of \(J_k\) as induced subgraph by using an independent program.

- Every Ramsey graph for \(K_k\) is also a Ramsey graph for \(J_{k+1}\). Therefore, we verified that the complete lists of \((3, J_{k+1}; n, e)\)-graphs which were generated by our programs include all known \((3, k; n, e)\)-graphs which we had found in [10].

- For every \((3, J_k; n, e(3, J_k, n))\)-graph which was generated by our programs, we verified that dropping any edge results in a graph which contains a spanning subgraph of \(J_k\) as induced subgraph.

- For various \((3, J_k; n, \leq e)\)-graphs we added up to \(f\) edges in all possible ways to obtain \((3, J_k; n, \leq e + f)\)-graphs. For the cases where we already had the complete set of \((3, J_k; n, \leq e + f)\)-graphs, we verified that no new \((3, J_k; n, \leq e + f)\)-graphs were obtained. We used this, amongst other cases, to verify that no new \((3, J_{10}; 26, \leq 55), (3, J_{10}; 28, \leq 70), (3, J_{10}; 30, \leq 87)\) or \((3, J_{11}; 32, \leq 81)\)-graphs were obtained.

- For various \((3, J_k; n, \leq e + f)\)-graphs we dropped one edge in all possible ways and verified that no new \((3, J_k; n, \leq e + f - 1)\)-graphs were obtained. We used this technique, amongst other cases, to verify that no new \((3, J_{10}; 26, \leq 54), (3, J_{10}; 28, \leq 69), (3, J_{10}; 30, 86)\) or \((3, J_{11}; 32, 80)\)-graphs were obtained.

- For various sets of \((3, J_{k+1}; n, \leq e)\)-graphs we took each member \(G\) and constructed from it all \(G_v\)'s. We then verified that this did not yield any new \((3, J_k; n - \deg(v) - 1, \leq e - Z(v))\)-graphs for the cases where we have all such graphs. We performed this test, amongst other cases, on the sets of \((3, J_8; 28, \leq 70)\) and \((3, J_{10}; 32, \leq 81)\)-graphs.

Various sets of graphs can be obtained by both the maximum triangle-free and neighborhood gluing extension method. Therefore, as a test for the correctness of our implementations, we applied both methods for the generation of several sets of graphs. We also compared our results with known results. In each case, the results were in complete agreement. More details are given below:

- The sets of \((3, J_8; 19, \leq 38), (3, J_8; 20, \leq 46), (3, J_8; 21, \leq 54)\) and \((3, J_9; 27, \leq 86)\)-graphs were obtained by both the maximum triangle-free method and the neighborhood gluing extension method. The results were in complete agreement.
• The counts of all $(3, J_7)$-graphs are confirmed by [8].

• The counts of all $(3, J_8; 19, 37)$, $(3, J_8; 20, 44)$, $(3, J_8; 21, \leq 52)$, $(3, J_8; 22, \leq 60)$, $(3, J_8; 23, \leq 71)$ and $(3, J_8; 24, \leq 81)$-graphs are confirmed by [18].

• The counts of $(3, J_8; 22, \leq 65)$, $(3, J_8; 23)$ and $(3, J_8; 24)$-graphs are confirmed by [17].

Since our results are in complete agreement with previous results and since all of our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations and results.

**Computation Time**

We implemented the extension algorithms described in Section 3 in C. Most computations were performed on a cluster with Intel Xeon L5520 CPU’s at 2.27 GHz, on which a computational effort of one CPU year can be usually completed in about 8 elapsed hours. The overall computational effort which was required to improve the upper bounds of $R(3, J_k)$ is estimated to be about 40 CPU years. This includes the time used by a variety of programs. The most CPU-intensive task was the computation to determine all $(3, J_9; \geq 28)$-graphs with the maximum triangle-free method. This took approximately 13 CPU years. Also the computations to determine new lower bounds on $e(3, J_{11}, n)$ took relatively long. For example, it took nearly 5 CPU years using the neighborhood gluing method to prove that $e(3, J_{11}, 39) \geq 151$.

The CPU time needed to complete the computations of Section 5 was negligible.