THE UNIQUENESS OF INTEGRALS ON HOPF ALGEBRAS
A CATEGORICAL APPROACH

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Abstract. We propose a new method to investigate the dimension of the space of integrals on a Hopf algebra $H$ and other properties of $H$-comodules.

Introduction

The study of integrals is an important part in the theory of Hopf algebras. If the integral does not vanish at the unit, the Hopf algebra is cosemisimple. In general, if the space of integrals does not vanish, many interesting properties of the Hopf algebra can be derived, for instance, the bijectivity of the antipode, the finiteness of injective envelopes, etc...

The study of the space of integrals on a Hopf algebras was initiated by Sweedler, being motivated by the uniqueness of the Haar integral on locally compact groups. Sweedler proved the existence and uniqueness up to a constant of a non-zero integral on any finite dimensional Hopf algebra and asked whether this remains true in infinit dimensional case. This question was answered by Sullivan [7] in 1971. He showed that the space of integrals has dimension not preceeding 1, in other words, if a non-zero integral exists then it is uniquely determined up to a constant. Since then there have been several other proofs of this result [5, 1].

The aim of this paper is to give a new proof of the uniqueness of integrals on Hopf algebras and properties of right $H$-comodules. Our method bases on a theorem of Gabriel-Popescu characterizing Grothendieck categories (see, e.g., [6]). We started with the known fact that a Hopf algebra over a field is completely determined by the category of its, say, right comodules. The question is then whether can we express the properties of the integral space and related properties in terms of the comodule category. It turns out that a non-zero integral on a Hopf algebra $H$ exists if and only if $H$ is a generator in the category $	ext{Comod-}H$ of its right comodules. This category is a Grothendieck category and hence, by the theorem of Gabriel-Popescu, is the quotient of a module category. The latter is indeed the category of (left) $H^*$-modules.

The paper is organized as follows. After recalling some basic fact about Hopf algebras, their comodules and their duals we mention some isomorphisms which play essential role in our work. Then we recall the notion of Grothendieck categories and the theorem of Gabriel-Popescu characterizing these categories (Section 1). We prove in Section 2 the uniqueness of integral. We also mention some interesting consequences of our methods: Corollaries 4 and 7. In Section 3 we describe the structure of the functor $U$ and derive from this the bijectivity of the antipode. Finally we prove that the injective envelope of $k$ is finite dimensional.
Although our main results are known, our method seems to be new and we hope that this method can be generalized to the case of Hopf algebras (or algebroids) over a ring.

1. Preliminaries

1.1. Integrals. Let $k$ be a fixed field and $H$ be a Hopf algebra defined over $k$. A right integral on $H$ is a linear functional $\chi : H \to k$, subject to the following equation

$$\sum_{(a)} \chi(a_{(1)})a_{(2)} = \chi(a), \quad \forall a \in H.$$ 

The space of right integrals on $H$ is denoted by $\int_r$. Consider $k$ as the trivial $H$-comodule by means of the unit map, we see that $\int_r \cong \text{Hom}(H,k)$. Analogously, a left integral on $H$ is a linear map $\phi : H \to k$, satisfying the equation

$$\sum_{(a)} a_{(1)}\chi(a_{(2)}) = \chi(a), \quad \forall a \in H.$$ 

The space of left integrals is denoted by $\int_l$. 

1.2. The dual algebra $H^*$. Let $M$ be a right $H$-comodule with coaction $\rho$ and $X$ be a $k$-space. The coproduct on $H$ induces a coaction on $X \otimes H$, trivial on $X$: $x \otimes h \mapsto \sum_{(h)} x \otimes h_{(1)} \otimes h_{(2)}$. We denote this comodule by $(X) \otimes H$ to emphasize that the coaction does not involve $H$. The map

$$\text{Hom}^H(M,(X) \otimes_k H) \to \text{Hom}_k(M,X), f \mapsto (id \otimes \epsilon) \cdot f$$ 

is an isomorphism with the inverse map given by $h \mapsto (h \otimes id) \cdot \rho$.

In particular, for $X = k$ we have an isomorphism $\text{Hom}^H(M,H) \cong \text{Hom}_k(M,k) =: M^*$. For $M = H$ we have an isomorphism $\text{Hom}^H(H,H) \cong H^*$ which is in fact an algebra anti-isomorphism, with respect to the convolution product on $H^*$ is given by

$$(f * g)(h) := \sum_{(h)} f(h_{(1)})g(h_{(2)}).$$

1.3. Rational comodules. For any $N$ in $\text{Comod-} H$ with the coaction map $\rho : N \to N \otimes H, \rho(n) = \sum_{(n)} n_{(1)} \otimes n_{(2)}$. Conctruct an $H^*$-module structure on $N$ as follows:

$$\forall \xi \in H^*, \forall n \in N, \xi * n := \sum_{(n)} n_{(1)} \xi(n_{(2)})$$

In this way every right $H$-comodule has a structure left $H^*$-module. This construction indeed defines a functor from $\text{Comod-} H$ to $H^*-\text{Mod}$ which is fully faithful and exact. An $H^*$-module which is isomorphic to a module obtained in this way is called rational module. It is known that submodules and quotient modules of a rational module are rational, and sum of rational submodules is rational.

Not every $H^*$-module is rational. However, we have a left exact functor $\text{Rat}$ from $H^*-\text{Mod}$ to $\text{Comod-} H$ [8], assigning to any left $H^*$-module $M$ the largest rational submodule of $M$ in $\text{Comod-} H$. 
1.4. **Tensor product and internal homs.** The category of right $H$-comodules is a monoidal category with the usual tensor product over $k$ and the unit object is $k$. An object $X \in \text{Comod-H}$ defines functors $X \otimes -$ and $- \otimes X$ from $\text{Comod-H}$ to itself. The right adjoint functors to these functors are called the left and the right internal hom functors of $X$ and denoted by $\text{rhom}(X, -)$ and $\text{lhom}(X, -)$, respectively. Thus we have natural isomorphisms

$$\text{Hom}_H(X \otimes Y, Z) \cong \text{Hom}_H(Y, \text{rhom}(X, Z))$$

$$\text{Hom}_H(Y \otimes X, Z) \cong \text{Hom}_H(Y, \text{lhom}(X, Z))$$

for all $Y, Z$ in $\text{Comod-H}$.

Let $N$ be a finite dimensional right $H$-comodule then $N^* := \text{Hom}_k(N, k)$ is also a right comodule with the coaction given by the equation

$$\rho(\varphi)(x) := \varphi(x(0)) S(x(1)), x \in N, \varphi \in N^*.$$ 

The map $\text{ev}: M^* \otimes M \to k, \varphi \otimes x \mapsto \varphi(x)$ is a morphism of $H$-comodules. The pair $(M^*, \text{ev})$ is called a left dual to $M$; it is defined uniquely up to isomorphisms. There exists a map $\text{db}: k \to M \otimes M^*$, defined by the conditions $(\text{ev} \otimes \text{id}_{M^*})(\text{id}_M \otimes \text{db}) = \text{id}_{M^*}$ and $(\text{id}_M \otimes \text{ev})(\text{db} \otimes \text{id}_M) = \text{id}_M$, which is also a comodule morphism. The notion of right dual is defined analogously; for instance, $(M, \text{ev})$ is the right dual to $M^*$.

We see that the left dual to a finite dimensional comodule always exists. If the antipode is bijective then the right dual to any finite dimensional comodule also exists. The converse is also true, if the right dual to any finite dimensional comodule exists then the antipode is bijective, see, e.g., [4]. We shall need the following isomorphisms, given by manipulating the morphism $\text{ev}$ and $\text{db}$: for any finite dimensional comodule $N$,

$$\text{Hom}_H(M \otimes N, P) \cong \text{Hom}_H(M, P \otimes N^*)$$

$$\text{Hom}_H(M, N \otimes P) \cong \text{Hom}_H(N^* \otimes M, P)$$

i.e., $\text{lhom}(N, P) \cong P \otimes N^*$ (as $H$-comodules).

1.5. **Doi’s and Sweedler’s isomorphisms.** In the previous subsection, except for the notion of dual comodules we haven’t use the antipode. In fact, the distinguished rôle of the antipode can be expressed by the isomorphisms below. For any right $H$-comodule $V$, Yu. Doi discovered the following isomorphism of $H$-comodules

$$V \otimes H \cong (V) \otimes H$$

via the map $v \otimes h \mapsto \sum v(0) \otimes v(1) h$, and its inverse is given by $v \otimes h \mapsto \sum v(0) \otimes S(v(1)) h$, [2, 1.3 Cor1].

Another isomorphism was established by Sweedler [8, 5.1.3]:

$$\int_v \otimes H \cong \text{Rat}(H^*).$$

given by $\varphi \otimes h \mapsto \varphi_h : \varphi_h(a) = \varphi(a S(h))$. A similar isomorphism exists for right integrals.
Given an element $\varphi \in \int_l$, define the map $\varphi^* : H \to H^*$, $\varphi^*(h)(a) = \varphi(hS(a))$. This is a morphism of right $H^*$-modules. Indeed, for all $\xi \in H^*$:

$$\varphi^*(h \xi)(a) = \sum_{(a)} \xi(h_{(1)}) \varphi(h_{(2)} S(a)) = \sum_{(a)} \varphi(hS(a_{(1)}) \xi(a_{(2)}) = (\varphi^*(h) \xi)(a).$$

Thus, if $\int_l \neq 0$, we can chose a left integral $\varphi$ such that $\varphi^*$ is non-zero. Consequently $H^*$ considered as right module on itself contains non-zero rational submodule. Using Sweedler’s isomorphism above (for right integrals), we conclude that $\int_r \neq 0$. We therefore have proved (bearing in mind the symmetry between the notions of left and right integrals)

**Lemma 1.** Let $H$ be a Hopf algebra over a field $k$. Then $\int_r \neq 0$ if and only if $\int_l \neq 0$.

1.6. **$H$-Comod is a Grothendieck category.** By definition, a Grothendieck category is an abelian category in which direct limits exist and preserve left exact sequence and there exists a generator. An example of Grothendieck category is the category of modules over a ring. In fact, a theorem of Gabriel-Popescu [6, 4.1] states that a Grothendieck category is the quotient of a module category. More explicitly, the following is known. Let $C$ be a generator in the Grothendieck category $C$. Let $A := \text{End}(C)$. The functor $T = \text{Hom}(C, -) : C \to \text{Mod-A}$ is then left exact and fully faithful. There exists a left adjoint to this functor, say $U$, which is exact and by means of which $C$ is a quotient category of $\text{Mod-A}$. The adjointness is

$$\text{Hom}_C(U(M), X) \cong \text{Hom}_A(M, \text{Hom}(C, X)).$$

where $M$ is an $A$-module and $X \in C$. It follows from the faithfulness of $\text{Hom}(C, -)$ that

$$U(\text{Hom}(C, X)) \cong X, \quad \forall X \in C.$$

2. **The uniqueness of the integral**

Our method of proving the uniqueness of integrals is first to show that the category of $H$-comodules is a Grothendieck category with $H$ being a generator, then to apply the theorem of Gabriel-Popescu to construct an exact functor $U$ from the category of $H^*$-modules to the category of $H$-comodules.

**Lemma 2.** Let $H$ be a Hopf algebra over a field $k$ with a non-zero right integral. Then $H$ is a generator in $\text{Comod-H}$.

**Proof.** We have according to (1) and (2)

$$\text{Hom}^H(H, M) \cong \text{Hom}^H(M^* \otimes H, k)$$

$$\cong \text{Hom}^H((M^*) \otimes H, k)$$

$$\cong M^{**} \otimes \text{Hom}^H(H, k)$$

Thus $H$ is a generator in $\text{Comod-H}$. $\blacksquare$
The category $\text{Comod-}H$ is a cocomplete abelian category, in which direct limits are exact. According to Lemma 2, $\text{Comod-}H$ is a Grothendieck category with $H$ being a generator. Putting $A := \text{Hom}^H(H, H)$, we have functor

$$T : \text{Comod-}H \longrightarrow \text{Mod-}A, X \mapsto \text{Hom}^H(H, X).$$

$T$ is full and faithful and according to Gabriel-Popescu’s Theorem for $\text{Comod-}H$, it has a left adjoint functor $U : \text{Mod-}A \longrightarrow \text{Comod-}H$, which is exact:

$$\text{Hom}^H(U(M), N) \cong \text{Hom}_A(M, T(N)).$$

Remember that $A$ and $H^*$ are anti-isomorphic, so every right $A$-module is a left $H^*$-module and conversely. The explicit action of $H^*$ on $\text{Hom}^H(H, N)$ is given by

$$\xi \ast f(h) := \sum_{(h)} \xi(h_{(1)}) f(h_{(2)}), \text{ for } \xi \in H^*, f \in \text{Hom}^H(H, N), h \in H$$

Thus we have

$$\text{Hom}^H(U(M), N) \cong \text{Hom}_{H^*}(M, \text{Hom}(H, N)).$$

for any left $H^*$-module $M$ and right $H$-comodule $N$. In particular, for any $N$ in $\text{Comod-}H$,

$$U(\text{Hom}^H(H, N)) \cong N.$$

We have proved

**Proposition 3.** Let $H$ be a Hopf algebra with a non-zero right integral and

$$T : \text{Comod-}H \longrightarrow H^*-\text{Mod}, X \mapsto \text{Hom}^H(H, X)$$

be a functor from $\text{Comod-}H$ to $H^*-\text{Mod}$. Then there exists a functor $U : H^*-\text{Mod} \longrightarrow \text{Comod-}H$, which is a left adjoint functor of $T$ and $U$ is exact. □

Since $H$ is injective in $\text{Comod-}H$, using standard arguments we deduce

**Corollary 4.** With the assumption of Proposition 3, $H^*$ is injective in the category $H^*-\text{Mod}$.

On the other hand, setting $N = H$ in (11), we have

$$\text{Hom}^H(U(M), H) \cong \text{Hom}_{H^*}(M, \text{Hom}^H(H, H))$$

$$\cong \text{Hom}_{H^*}(M, H^*),$$

that is, for any $M$ in $\text{Comod-}H$,

$$\text{Hom}^H(U(M), H) \cong \text{Hom}_{H^*}(M, H^*).$$

For $M$ in $\text{Comod-}H$, consider it as an $H^*$-module, (13) has the following form

$$\text{Hom}^H(U(M), H) \cong \text{Hom}^H(M, \text{Rat}(H^*))$$

$$\cong \text{Hom}^H(M, (\int l) \otimes H)$$

$$\cong \text{Hom}_k(M, \int l),$$

where the second isomorphism follows from Sweedler’s isomorphism (6). Consequently, if $M$ has finite dimension

$$U(M)^* \cong \int l \otimes M^*$$
(as vector spaces). Since $\int_I \neq 0$, $U(M) \neq 0$ for any $M \neq 0$. If there exists a comodule $M$ such that $\dim_k U(M) \leq \dim_k M$ then $\int_I$ is one-dimensional.

**Lemma 5.** Let $H$ be a Hopf algebra over a field $k$ with a non-zero right integral then there exists a finite dimensional comodule $N$ such that $\dim_k U(N) \leq \dim_k N$.

**Proof.** Let $C$ be a finite dimensional subcoalgebra of $H$. Then the action of $H^*$ on $\text{Hom}_{H^*}(H, k)$ induces in the natural way an action on $\text{Hom}_{H^*}(C, k)$:

$$\xi * g(c) = \sum_{(c)} \xi(c(1))g(c(2)), \quad c \in C, g \in \text{Hom}_{H^*}(C, k).$$

Moreover, there exists a left $H^*$-module morphism from $\text{Hom}_{H^*}(H, k)$ to $\text{Hom}_{H^*}(C, k)$ given by restriction:

$$\varphi_C : \text{Hom}_{H^*}(H, k) \to \text{Hom}_{H^*}(C, k), f \mapsto \varphi_C(f) = f|_C.$$

We show that $\text{Hom}_{H^*}(C, k)$ is rational. Consider $C^*$ as a right $H$-comodule with the coaction given by condition

$$\sum_{(g)} g(0)(c) \otimes g(1) = \sum_{(c)} c(1) \otimes g(c(2)), \quad \forall c \in C, g \in C^*.$$ 

Thus $C^*$ is a rational left $H^*$-module, with the explicit action

$$(\xi * g)(c) = \sum_{(c)} \xi(c(1))g(c(2)).$$

We see that the natural inclusion $\text{Hom}_{H^*}(C, k) \to \text{Hom}_k(C, k) = C^*$ is compatible with the actions of $H^*$, thus $\text{Hom}_{H^*}(C, k)$ is an $H^*$-subcomodule of $C^*$, whence rational.

Since $H$ is the union of its finite dimensional subcoalgebras, for a non-zero right integral $\chi$, there exists a finite dimensional coalgebra $C$ such that $\chi|_C \neq 0$. Thus there exists an exact sequence

$$0 \to K \to \text{Hom}_{H^*}(H, k) \to \Gamma \to 0 \tag{16}$$

with $K = \text{Ker}\varphi_C, \Gamma = \text{Im}\varphi_C$. $\Gamma$ is a rational $H^*$-module, being a submodule of $\text{Hom}_{H^*}(C, k)$. $\Gamma \neq 0$ since it contains $\varphi_C(\chi)$. Since $U$ is exact, the following sequence is also exact

$$0 \to U(K) \to U(\text{Hom}_{H^*}(H, k)) \to U(\Gamma) \to 0 \tag{17}.$$ 

By (12), $U(\text{Hom}_{H^*}(H, k)) \cong k$, hence

$$0 \to U(K) \to k \to U(\Gamma) \to 0 \tag{18}. $$

Thus, $\dim_k U(\Gamma) = 1 \leq \dim_k \Gamma$. 

**Theorem 6.** Assume that the Hopf algebra $H$ possesses a non-zero right integral then it is uniquely determined up to a constant.

**Proof.** According to Lemma 5 and the discussion preceding it, $\int_I$ is one-dimensional. Using the symmetry between the notions of left and right integrals, we see that $\int_r$ is also one-dimensional. 

Corollary 7. The space of right integrals $\int_r \cong \text{Hom}^H(H,k) \cong \text{Hom}_{H^*}(H,k)$ with the action of $H^*$ given in (11) is a rational module.

Proof. In the exact sequence (11) $\text{Hom}_{H^*}(H,k)$ is one-dimensional and $\Gamma \neq 0$, hence is one-dimensional, too. Thus $K = 0$ and $\Gamma \cong \text{Hom}_{H^*}(H,k)$, whence $\text{Hom}_{H^*}(H,k)$ is rational.

Note that this one dimensional module induces a distinguished group-like element introduced by Radford [3]. Call this group-like element $\gamma$, we have by definition, for any right integral $\chi$ on $H$, $\delta(\chi) = \chi \otimes \gamma$. Since $\delta$ is induced from the action of $H^*$ on $\text{Hom}_{H^*}(H,k)$, we have the following equation for $\gamma$:

$$\xi \ast \chi(a) = \xi(\gamma)\chi(a) = \sum_a \xi(a(1))\chi(a(2)).$$

or equivalently

$$(19) \quad \chi(a)\gamma = \sum_a a(1)\chi(a(2)).$$

3. The bijectivity of the antipode

In this section we prove that if $\int_r \neq 0$ then the antipode is bijective.

Lemma 8. Let $H$ be a Hopf algebra, $N$ be a right $H$-comodule of finite dimension. Then $\text{Hom}^H(H,N)$ is a rational $H^*$-module and isomorphic to $N^{**} \otimes \Gamma$ as $H$-comodules.

Proof. According to (11), we have, for $N$ finite dimensional,

$$(20) \quad \text{Hom}^H(H,N) \cong N^{**} \otimes \text{Hom}^H(H,k).$$

It is more convenient to describe the inverse of this isomorphism, which is given by $\varphi : n \otimes f \mapsto f_n, f_n(h) := \sum_{(n)} n_{(0)}f(S(n_{(1)})h)$.

According to Corollary 7, $N^{**} \otimes \text{Hom}^H(H,k)$ is a right $H$-comodule, hence a left $H^*$-module with the action:

$$\xi \ast (n \otimes f) := \sum_{(n)} \xi(S^2(n_{(1)}))n_{(0)} \otimes f$$

for any $\xi$ in $H^*$ and $n \otimes f$ in $N^{**} \otimes \text{Hom}^H(H,k)$.

We prove that $\varphi$ is a morphism of left $H^*$-modules (with respect to the above actions). We have

$$\varphi(\xi \ast (n \otimes f))(h) = \varphi(n_{(0)} \otimes f(\xi \ast (S^2(n_{(1)})))(h) = \xi(S^2(n_{(1)}))f_{n_{(0)}}(h) = \sum_{(n)} n_{(0)}f(S(n_{(1)})h)\xi(S^2(n_{(2)}))$$

and

$$\xi \ast (\varphi(f \otimes n))(h) = \sum_{(h)(n)} \xi(h_{(1)})n_{(0)}f(S(n_{(1)})h_{(2)}).$$
According to (19), we have
\[ f(S(n(1))h) \gamma = \sum_{(n),(h)} S(n(2)) h_1 f(S(n(1))h_2), \]
whence
\[ \sum_{(n)} n_0 f(S(n(1))h) = \sum_{(n)} S^2(n(2)) \gamma f(S(n(1))h). \]

Therefore,
\[ \sum_{(n)} n_0 f(S(n(1))h) \xi(S^2(n(2))) = \sum_{(h)(n)} \xi(h_1) n_0 f(S(n(1))h_2). \]
Thus \( \varphi \) is a morphism of left \( H^* \)-modules, that is
\[ \text{(21)} \quad Hom^H(H, N) \cong N^{**} \otimes Hom^H(H, k) \]
as left \( H^* \)-modules. Since \( N^{**} \otimes Hom^H(H, k) \) is a rational \( H^* \)-module, so is the module \( Hom^H(H, N) \).

**Proposition 9.** Let \( H \) be a Hopf algebra with a non-zero integral and \( U \) be the functor defined as in Proposition 3. Then for any finite dimensional comodule \( N \), we have \( U(N)^{**} \cong N \otimes \Gamma \).

**Proof.** From (11) we have, for finite dimensional right \( H \)-comodules \( M, N \):
\[ Hom^H(U(N), M) \cong Hom^H(N, Hom^H(H, M)) \]
From (21),
\[ Hom^H(N, Hom^H(H, M)) \cong Hom^H(N, M^{**} \otimes \Gamma). \]
We have \( \Gamma \) is invertible \( (\Gamma^* \otimes \Gamma \cong k \cong \Gamma^* \otimes \Gamma) \), hence
\[ Hom^H(M \otimes \Gamma, N \otimes \Gamma) \cong Hom^H(M, N). \]
Therefore
\[ Hom^H(N, M^{**} \otimes \Gamma) \cong Hom^H(N \otimes \Gamma^*, M^{**}). \]
Using (3) and (4), we have, for any finite dimensional comodule \( M \),
\[ Hom^H(U(N)^{**} \otimes M^*, k) \cong Hom^H(U(N), M) \]
\[ \cong Hom^H(N, M^{**} \otimes \Gamma) \]
\[ \cong Hom^H(N \otimes \Gamma^*, M^{**}) \]
\[ \cong Hom^H(N \otimes \Gamma \otimes M^*, k) \]

hence \( U(N)^{**} \cong N \otimes \Gamma \).

**Corollary 10.** Let \( H \) be a Hopf algebra over a field \( k \) with antipode \( S \), and assume \( \int_r \neq 0 \). Then \( S \) is bijective.

**Proof.** We have \( (\Gamma \otimes U(N)^*)^* \cong U(N)^{**} \otimes \Gamma^* \cong N \), i.e., every finite dimensional right \( H \)-comodule \( N \) has right dual. Hence the antipode is bijective.

**Corollary 11.** Let \( H \) possess a non-zero integral. Then the injective envelope of \( k \) is finite dimensional.
Proof. Let \( \varphi \) be a non-zero right integral on \( H \). According to the proof of Theorem \( \Box \) there exists a finite dimensional right ideal \( J \) of \( H \) such that the restriction of \( \varphi \) on it is non-zero. Choose such a right ideal \( J \) with minimal dimension. We show that \( J \) is a direct summand of \( H \) as \( H \)-right comodule. Let \( \varphi_J \) be the restriction on \( J \): \( \varphi_J : J \longrightarrow k \).

According to Proposition \( \Box \) the functor \( \text{Hom}^H(H,-) \) is exact on finite dimensional comodules, hence there exists a morphism \( \pi : H \longrightarrow J \), such that \( \varphi = \varphi_J \circ \pi \). Hence \( \varphi_J = \varphi \circ \pi|_J \). By the minimality of \( J \), \( \pi \) should be surjective, whence \( \pi|_J \) should be bijective since \( J \) is finite dimensional. Therefore \( J \) is a direct summand of \( H \), whence \( J \) is projective with respect to finite dimensional comodules. Since each comodule is the union of its finite dimensional subcomodules, we conclude that \( J \) is projective in \( \text{Comod-H} \). Hence \( J^* \) is injective, and thus, the injective envelope of \( k \).

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