Plethystics and instantons on ALE spaces

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Abstract

We present an expression of a deformed partition function for $\mathcal{N} = 2$ $U(1)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_k$ by using plethystic exponentials.

Recently $\mathcal{N} = 2$ instantons on ALE spaces were studied in [1, 2] and applied to black hole physics in [3, 4]. In this short note we point out an expression of a deformed partition function for $\mathcal{N} = 2$ $U(1)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_k$ by using plethystic exponentials. Plethystic exponentials have been used in [5] and play a central role in counting the BPS operators.

Let us start with describing the partition function for $\mathcal{N} = 2$ $U(1)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_k$. The partition function was computed in [2] and becomes as follows:

$$Z(\epsilon_1, \epsilon_2; \Lambda, k) = \sum_{\lambda = \mu(\lambda^{(r)})} \Lambda^{2|\lambda|/k} \prod_{\substack{\lambda \in \Lambda \\ h(s) = 0 \mod k}} \frac{1}{(\epsilon_1 l(s) + 1 - \epsilon_2 a(s))(-\epsilon_1 l(s) + \epsilon_2(a(s) + 1))},$$

(1)

where $\epsilon_1, \epsilon_2$ are parameters of the standard torus action on $\mathbb{C}^2/\mathbb{Z}_k$, and $\Lambda$ denotes a scale parameter for the gauge theory. The sum in eq. (1) is a summation over partitions (the Young diagrams). We remark that any partition $\lambda$ can be decomposed into a single partition $\lambda_{\text{core}}$ and

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$k$-tuple partitions $\lambda^{(1)}, \ldots, \lambda^{(k)}$. They are called respectively $k$-core and $k$-quotients. This arises from a division algorithm for the Young diagrams analogous to that for integers. See Appendix A for details. We may write $\lambda = \mu(\lambda_{\text{core}}; \lambda^{(r)})$ to emphasize the decomposition. The sum in eq. (1) is restricted so that only partitions whose $k$-cores are empty contribute to the partition function. The symbol $|\lambda|$ is the number of boxes in the Young diagram: $|\lambda| := \sum_{i=1}^\infty \lambda_i$. For a box $s$ in the Young diagram, $h(s)$, $a(s)$ and $l(s)$ are respectively, as given in (A.2), the hook length, the arm length and the leg length. In eq. (1), the products are taken over boxes whose hook length are multiples of $k$.

We will deform the partition function (1) by using a parameter $R$ in the following manner:

$$Z(\epsilon_1, \epsilon_2; \Lambda, k, R) = \sum_{\lambda = \mu(\phi; \lambda^{(r)})} \nu^{|\lambda|/k} \prod_{s \in \lambda \atop h(s) = 0 \mod k} \frac{1}{(1 - t_1^{h(s)+1}t_2^{-a(s)})(1 - t_1^{-l(s)}t_2^{-a(s)+1})},$$

(2)

where

$$t_1 = e^{\epsilon_1 R}, \quad t_2 = e^{\epsilon_2 R}, \quad \nu = (R\Lambda)^2.$$ (3)

The parameter $R$ is identified in the gauge theory with a circumference of a circle in the fifth direction. For the case of $k = 1$, the deformed partition function is calculated in [8] from the viewpoint of refined topological vertices. It is clear that eq. (2) reduces to eq. (1) at the limit where $R$ goes to zero:

$$\lim_{R \to 0} Z(\epsilon_1, \epsilon_2; \Lambda, k, R) = Z(\epsilon_1, \epsilon_2; \Lambda, k).$$

(4)

Plethystic exponentials appeared in the study of counting the BPS operators [5]. In particular, by using it, counting functions of holomorphic functions (the BPS operators) over the symmetric products $S^N(C^2)$, where the operators are counted weighted by their $U(1) R$ charges (charges of the torus action), are reproduced from the counting function for $N = 1$, that is, $C^2$. For a given function $f(t_1, t_2)$, plethystic exponential $PE(f(t_1, t_2))$ is defined by

$$PE(f(t_1, t_2)) := \exp \left( \sum_{n=1}^\infty \frac{\nu^n}{n} f(t_1^n, t_2^n) \right).$$

(5)

See [9, 10] for details of plethystic exponentials. We point out the following expression of the deformed partition function by means of plethystic exponentials:

$$Z(\epsilon_1, \epsilon_2; \Lambda, k, R) = PE(h_k(t_1, t_2)), \quad PE(h_k(t_1, t_2)) := \frac{1 - t_1^{h_k}t_2^{h_k}}{(1 - t_1^k)(1 - t_2^k)(1 - t_1t_2)}. \quad (6)$$

The RHS of (6) can be simply written as

$$PE(h_k(t_1, t_2)) = \prod_{\substack{m, n = 0 \\ m = n \mod k}} \frac{1}{(1 - \nu t_1^m t_2^n)}.$$ (7)
The function \( h_k(t_1, t_2) \) is a counting function of holomorphic functions on \( \mathbb{C}^2/\mathbb{Z}_k \). The ring of the holomorphic functions is generated by \( X, Y \) and \( Z \) subject to the relation \( XY = Z^k \). The \( U(1) \) charges are read as \((k, 0)\) for \( X \), \((0, k)\) for \( Y \) and \((1, 1)\) for \( Z \). The equality (5) for the case of \( k = 1 \) is due to the identities related with Macdonald’s function \([11]\). It is also understood as one parameter deformation of that made by Fujii and Minabe \([2]\). By letting \( R \to 0 \), (6) reduces to their result:

\[
\lim_{R \to 0} P E (h_k(t_1, t_2)) = \exp \left( \frac{\Lambda^2}{k \epsilon_1 \epsilon_2} \right) = Z(\epsilon_1, \epsilon_2; \Lambda, k). \tag{9}
\]

We are convinced of the equality (6) by comparing the first few coefficients in the Taylor expansions with respect to \( \nu \) of the both hand sides. As an example, let us see the case of \( k = 3 \). We will compare the coefficients of \( \nu^2 \). The Young diagrams which contribute to the coefficient of \( \nu^2 \) in (2) are those of six boxes as shown in Figure 1. In each Young diagram, only the boxes filled with black circles contribute to the products in (2). We then see

\[
\text{Coeff}_{\nu^2} (Z(\epsilon_1, \epsilon_2; \Lambda, 3, R)) = \frac{t^2 (q^2 + t q + t^2) (q^3 + t^4 q^4 + t^2 q^3 + t^3 q^2 + t q + t^5)}{(q^3 - 1)^2 (q^3 + 1) (t^3 - 1)^2 (t^3 + 1)}
\]

\[
= \frac{1}{2} h_3(t_1^2, t_2^2) + \frac{1}{2} (h_3(t_1, t_2))^2
\]

\[
= \text{Coeff}_{\nu^2} (P E (h_k(t_1, t_2))). \tag{10}
\]

Figure 1: Young diagrams which contribute to the coefficient of \( \nu^2 \) in eq. (2) for the case of \( k = 3 \). The boxes filled with black circles contribute to the products in eq. (2). 3-tuple partitions attached below each Young diagram are the corresponding 3-quotients.
A Partitions

A partition \( \lambda \) is a nonincreasing sequence of non-negative integers: \( \lambda = (\lambda_1, \lambda_2, \cdots) \). Partitions are often identified with the Young diagrams. Take a box \( s = (i, j) \in \lambda \) and let \( H_s \) be the hook.

\[
H_{s=(i,j)} = \{(k, l) \in \lambda | k = i, l \geq j, \text{ or } l = j, k > i\}.
\] (A.1)

The arm length, the leg length and the hook length are

\[
a(s) = \lambda_i - j, \quad l(s) = \lambda_j^t - i, \quad h(s) = a(s) + l(s) + 1,
\] (A.2)

where \( \lambda' = (\lambda_1^t, \lambda_2^t, \cdots) \) is the dual partition. The hook \( H_s \) is called \( r \)-hook if \( h(s) = r \).

Fix a positive integer \( k \). A partition \( \lambda \) is called \( k \)-core if it does not contain any \( k \)-hook. Otherwise, another partition \( \lambda' \) is obtained from \( \lambda \) by removing one of the \( k \)-hooks. The procedure can be continued until we obtain a \( k \)-core. This partition turns out to be independent from the way \( k \)-hooks are removed and is denoted by \( \lambda_{\text{core}} \). The set of removed \( k \)-hooks constitutes \( k \)-tuple of partitions \( \lambda^{(1)}, \cdots, \lambda^{(k)} \). These partitions are obtained uniquely from \( \lambda \) and are called \( k \)-quotients. In the case of \( \lambda_{\text{core}} = \phi \), we can read the \( k \)-quotients \( \lambda^{(1)}, \cdots, \lambda^{(k)} \) from the following relation:

\[
\{\lambda_i - i; i \geq 1\} = \bigcup_{r=1}^{k} \left\{k(\lambda_{i_r}^{(r)} - i_r) + r - 1; i_r \geq 1\right\}.
\] (A.3)

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References

[1] F. Fucito, J. F. Morales and R. Poghossian, “Multi instanton calculus on ALE spaces,” Nucl. Phys. B 703, 518 (2004) [arXiv:hep-th/0406243].

[2] S. Fujii and S. Minabe, “A Combinatorial Study on Quiver Varieties,” arXiv:math.ag/0510455.

[3] L. Griguolo, D. Seminara, R. J. Szabo and A. Tanzini, “Black holes, instanton counting on toric singularities and q-deformed two-dimensional Yang-Mills theory,” arXiv:hep-th/0610155.

[4] F. Fucito, J. F. Morales and R. Poghossian, “Instanton on toric singularities and black hole countings,” arXiv:hep-th/0610154.

[5] S. Benvenuti, B. Feng, A. Hanany and Y. H. He, “Counting BPS operators in gauge theories: Quivers, syzygies and plethystics,” arXiv:hep-th/0608050.
[6] G. James and A. Kerber, “The Representation Theory of the Symmetric Group,” Addison-Wesley, Reading, MA 1981.

[7] J. B. Olsson, “Combinatorics and Representations of Finite Groups,” Lecture Notes, University of Essen, vol. 20(1993).

[8] H. Awata and H. Kanno, “Instanton counting, Macdonald functions and the moduli space of D-branes,” JHEP 0505, 039 (2005) [arXiv:hep-th/0502061].

[9] E. Getzler, M.M. Kapranov, “Modular operads,” dg-ga/9408003.

[10] J. M. F. Labastida and M. Marino, “A new point of view in the theory of knot and link invariants,” arXiv:math.qa/0104180.

[11] I. G. Macdonald, “Symmetric Functions and Hall Polynomials,” Clarendon Press, 1995.