Non-perturbative method to compute thermal correlations in one-dimensional systems

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We develop a highly efficient method to numerically simulate thermal fluctuations and correlations in non-relativistic continuous bosonic one-dimensional systems. The method is suitable for arbitrary local interactions as long as the system remains dynamically stable. We start by proving the equivalence of describing the systems through the transfer matrix formalism and a Fokker-Planck equation for a distribution evolving in space. The Fokker-Planck equation is known to be equivalent to a stochastic differential (Itô) equation. The latter is very suitable for computer simulations, allowing the calculation of any desired correlation function. As an illustration, we apply our method to the case of two tunnel-coupled quasi-condensates of bosonic atoms. The results are compared to the predictions of the sine-Gordon model for which we develop analytic expressions directly from the transfer matrix formalism.

I. INTRODUCTION

One-dimensional (1D) quantum systems attract much attention because their dynamics is strongly affected by the restricted phase space available for scattering\textsuperscript{[13]}. As a result, such systems may exhibit a number of non-trivial properties. For example, one can observe integrability as well as its breakdown. The most well-known 1D theoretical models include the Lieb-Liniger model\textsuperscript{[4,5]}, the quantum Luttinger liquid theory\textsuperscript{[6]}, and the sine-Gordon model\textsuperscript{[7,9]}. Experimentally, 1D quantum systems are realized by freezing the motion of particles in two tightly confined dimensions. Available 1D systems range from ultracold atomic gases\textsuperscript{[10,11]} to slow-light polaritons\textsuperscript{[12]} as well as superfluid \textsuperscript{4}He atoms adsorbed in nanometer-wide pores\textsuperscript{[13]}. Several recent experimental studies underline the importance of 1D systems as a testbed for theoretical ideas, in and out of equilibrium\textsuperscript{[14–17]}.

The rapid progress in the preparation, manipulation and characterization of experimental systems, especially in the realm of ultracold atoms, also leads to a need for ever improving theoretical descriptions. In particular, the recent measurement of higher-order correlation functions in 1D quasi-condensates\textsuperscript{[17]} calls for novel theoretical methods beyond the perturbative approach.

This paper presents a detailed description of a versatile, non-perturbative method to calculate thermal correlations of multicomponent bosonic fields in the mean-field approximation in 1D. The search for the method has been inspired by the observation that thermal Gaussian fluctuations in a 1D system of bosons with quadratic effective Hamiltonian can be described by the Ornstein-Uhlenbeck stochastic process\textsuperscript{[18]}, where the spatial coordinate plays the role usually taken by time in conventional applications of random processes. The method developed in Ref.\textsuperscript{[18]} has been applied to the analysis of experimental data for 1D systems\textsuperscript{[19,20]} and, in its extended version\textsuperscript{[21]}, even for two-dimensional systems\textsuperscript{[22]}. These works relied on the absence of interactions between quasiparticles in the theoretical description, i.e., quadratic Hamiltonians. We extended our previous approach to systems where such interactions have to be taken into account, i.e., to non-quadratic Hamiltonians. It is surprising that the description via a stochastic process along the 1D co-ordinate is also possible in this case. The stochastic process provides highly efficient numerical sampling of classical fields with the statistics given by the thermal equilibrium. We present the method and apply it to the case of two tunnel-coupled 1D quasi-condensates\textsuperscript{[23–24]}.

We consider a system of bosonic particles effectively confined to a 1D geometry. On the quantum level, the particles are described by field operators obeying standard bosonic commutation rules\textsuperscript{[1–3]}. Different components \(\hat{\psi}_j\) of the field may correspond, for example, to atoms of the same kind trapped in different atomic waveguides (this case will be considered in detail in Sec.\textsuperscript{[IV]}). Another example would be the case of different atomic species or isotopes, or of different spin states of the same isotope trapped in the same waveguide. However, in the experimentally accessible range of temperatures and densities the equilibrium properties of ultracold gases of bosonic atoms in 1D are often dominated by classical thermal fluctuations\textsuperscript{[18]}. Therefore we may describe the system within the mean-field approximation, replacing the operators \(\hat{\psi}_j\) with classical complex fields \(\psi_j\).

Let us now introduce the notation used in the paper. The co-ordinate in 1D is denoted by \(z\). Instead of the \(M/2\) complex, \(z\)-dependent components \(\psi_j\) (\(M\) is an even integer number) we can also use \(M\) real components

\[ q_{2j-1}(z) = \text{Re}\, \psi_j(z), \quad q_{2j}(z) = \text{Im}\, \psi_j(z), \]

\(j = 1, 2, \ldots, M/2\). \hfill(1)

In the following we will not indicate the \(z\)-dependence of the fields explicitly. Without loss of generality, we assume that all the components are characterized by the same...
mass $m$. The Hamiltonian function of the system is then
\[
H = \int_{-L/2}^{L/2} dz \left\{ \sum_{j=1}^{M} \left[ \frac{\hbar^2}{2m} \left( \frac{\partial q_j}{\partial z} \right)^2 - \mu_j q_j^2 \right] + V(q_1, \ldots, q_M) \right\}. \tag{2}
\]

Here $\mu_j$ is the chemical potential for the $j$th component. Since $q_{2j-1}$ and $q_{2j}$ are the real and imaginary part of the same complex field, there are only $M/2$ independent chemical potentials. $V(q_1, \ldots, q_M)$ represents the local interaction energy density, which does not explicitly depend on $z$ (homogeneous system).

We consider the thermodynamic limit $L \to \infty$, while the mass density remains constant. We therefore don’t have to specify boundary conditions, as will become clear later. In the following, we will refer to the integrand in Eq. (2) as energy density $h(q_1, \ldots, q_M)$.

The paper is organized as follows. In Section II we recall the basics of the transfer matrix formalism. In Section III we demonstrate the equivalence of this formalism to the description via an Itô stochastic ordinary differential equation. The full mean-field model describing both the phase and density fluctuations in two tunnel-coupled quasicondensates is introduced in Section IV. In the same Section we discuss the sine-Gordon (SG) model as an approximation describing phase fluctuations only. Within the SG model, we derive analytic expressions for the second and fourth moments of the relative phase difference, see Section V. In Section VI we present the results for the full model obtained numerically using our computational method based on the stochastic process. We compare these results with those of the SG model and discuss their similarities and differences. Section VII is devoted to conclusive remarks.

II. THE TRANSFER MATRIX FORMALISM

In this section, we briefly recapitulate some basics of the transfer matrix formalism. We are interested in equal-time correlations of the observables $\mathcal{F}^{(i)}(z_i) = \mathcal{F}^{(i)}[q_1(z_i), \ldots, q_M(z_i)]$ measured at different points $z_i, i = 1, 2, \ldots, l$ at equilibrium with temperature $T$. We consider the general case where all these selected observables $\mathcal{F}^{(i)}$ may be different.

In order to introduce the transfer operator, we will first start with the partition function. In classical field approximation, the partition function of the system can be expressed as the functional integral
\[
Z = \int Dq_1 \cdots \int Dq_M e^{-\beta H}, \tag{3}
\]
where $\beta = 1/(k_B T)$. Recall the definition of the functional integral as the limit of the statistical sum of a model on a grid with the spacing $\Delta z \to 0$. We consider the two adjacent grid points $z_0$ and $z_0 + \Delta z$. The fields at this points are denoted by $\{q_j\}$ and $\{q_j\}$, respectively. Then Eq. (3) can be rewritten as
\[
Z = \int d^M q \; Z_{>|z_0+\Delta z|} \int d^M q' e^{-\beta \Delta z h_{id}(\{q_j\}, \{q_j\})} Z_{<|z_0|}, \tag{4}
\]
where $\int d^M q$ stands for $\int_{-\infty}^{\infty} dq_1 \cdots \int_{-\infty}^{\infty} dq_M$ and we have introduced the finite-difference representation of the energy density
\[
h_{id}(\{q_j\}, \{q_j\}) = \left\{ \sum_{j=1}^{M} \left[ \frac{\hbar^2}{2m} \left( \frac{q_j - q_j'}{\Delta z} \right)^2 - \mu_j q_j^2 \right] + V(q_1, \ldots, q_M) \right\}. \tag{5}
\]

The quantities $Z_{<|z_0|}$ and $Z_{>|z_0+\Delta z|}$ are the partition functions for the subregions $z < z_0$ and $z > z_0 + \Delta z$, respectively. We have
\[
Z_{<|z_0|} = \int Dq_1 \cdots \int Dq_M e^{-\beta \int_{-L/2}^{L/2} dz h}, \tag{6}
\]
where the functional integral is only over fields in the region $[-L/2, z_0]$, not including the fields at the point $z_0$ itself. Analogously we define
\[
Z_{>|z_0+\Delta z|} = \int Dq_1 \cdots \int Dq_M e^{-\beta \int_{z_0+\Delta z}^{L/2} dz h}, \tag{7}
\]
where the functional integral is only over fields in the region $(z_0 + \Delta z, L/2]$, again not including the fields at the point $z_0 + \Delta z$ itself. In Eq. (4), $Z_{<|z_0|}$ is therefore a function of the fields $\{q_j\}$ at point $z_0$ and $Z_{>|z_0+\Delta z|}$ is a function of $\{q_j\}$.

Let us now define the transfer integral operator $\hat{T}_{\Delta z}$ over the infinitesimally small distance $\Delta z$ through its action on an arbitrary function $\Psi$ of the field variables $q_j$ as
\[
\hat{T}_{\Delta z} \Psi = C \int d^M q' e^{-\beta \Delta z h_{id}(\{q_j\}, \{q_j\})} \Psi(\{q_j\}). \tag{8}
\]
The constant $C = [\hbar^2/(\pi \Delta z m k_B T)]^{M/2}$ is introduced for the normalization of the Gaussian integrals and is not important for the rest of the derivation. Using the definition of $\hat{T}_{\Delta z}$ in Eq. (4) we can write
\[
Z = \int d^M q \; Z_{>|z_0+\Delta z|} \hat{T}_{\Delta z} Z_{<|z_0|}, \tag{9}
\]
Applying the transfer operator $N$ times, we obtain
\[
Z = \int d^M q \; Z_{>|z_0+N\Delta z|} \left( \hat{T}_{\Delta z} \right)^N Z_{<|z_0|}. \tag{10}
\]
In general, we can define the transfer integral operator for a finite distance $|z' - z|$ as the product of infinitely
many transfer integral operators for infinitesimally small distances:
\[
\hat{T}_|z′ - z| = \lim_{N \to \infty} \left( \hat{T}_|z′ - z|/N \right)^N.
\]

Since we have assumed a homogeneous system, all transfer integral operators \( \hat{T}_{\Delta z} \) for the same infinitesimally small distance \( \Delta z \) are identical.

To further simplify the equations we have to find the eigenfunctions of \( \hat{T}_{\Delta z} \). In order to do so, we expand \( \Psi(q_i) \) in Eq. (5) around \( \{q_i\} \) up to second order in \( q_j' - q_j \) and perform the integration. The result is \( \hat{T}_{\Delta z} \Psi \approx (1 - \Delta z \hat{K}) \Psi \) for \( \Delta z \to 0 \). As this is true for an arbitrary function \( \Psi \), we can simply write
\[
\hat{T}_{\Delta z} \approx (1 - \Delta z \hat{K})
\]
for \( \Delta z \to 0 \). Here an auxiliary operator
\[
\hat{K} = \sum_{j=1}^{M} \left(-D \frac{\partial^2}{\partial q_j^2} - \frac{\mu_j}{k_B T} q_j^2\right) + \frac{V(q_1, \ldots, q_M)}{k_B T}
\]
was introduced.

The auxiliary operator \( \hat{K} \) has the structure of a quantum-mechanical Hamiltonian for a single particle in an external potential \( U(q_1, \ldots, q_M) = [V(q_1, \ldots, q_M) - \sum_{j=1}^{M} \mu_j q_j^2]/(k_B T) \) in an \( M \)-dimensional space. Therefore, \( \hat{K} \) has all the conventional properties of a single-particle Hamiltonian. In what follows, we consider only “potentials” that yield a spectrum that is bounded from below. This is not the case for dynamically unstable systems, e.g., for an atomic Bose-gas with attractive interactions \( [29] \). A stable system is characterized by an interaction that increases (or, at least, does not decrease) when the absolute value of any of the field components grows infinitely.

To solve the eigenvalue problem
\[
\hat{K} \Psi_{\nu}(q_1, \ldots, q_M) = \kappa_{\nu} \Psi_{\nu}(q_1, \ldots, q_M)
\]
for a system characterized by an interaction that increases unlimitedly for \( q_j \to \pm \infty \), we set the boundary conditions
\[
\Psi_{\nu}|_{q_j \to \pm \infty} = 0, \quad j = 1, 2, \ldots, M.
\]

If the particular form of \( V(q_1, \ldots, q_M) \) allows for the existence of a continuous spectrum, we can require, instead of Eq. (16), that the eigenfunctions are finite at \( q_j \to \pm \infty \).

The system of eigenfunctions is complete, orthogonal and normalized, like for a standard quantum-mechanical Hamiltonian problem. Note that the dimensionality of the eigenvalues \( \kappa_{\nu} \) of the Hamiltonian-like operator \( \hat{K} \) is inverse length and not energy. The operator \( \nu \) is invariant with respect to complex conjugation. Therefore the eigenfunctions of \( \hat{K} \) can always be chosen to be real. In what follows we always assume \( \Psi_{\nu}^* = \Psi_{\nu} \).

We assign the index \( \nu = 0 \) to the lowest eigenvalue of \( \hat{K} \) and, correspondingly, to the respective function. In what follows we refer to \( \kappa_0 \) and \( \Psi_0 \) as to the “ground state” eigenvalue and eigenfunction, respectively. We assume that this ground state is not degenerate. If the operator \( \hat{K} \) possesses a symmetry that makes the ground state degenerate, one can resort to a standard scheme of symmetry breaking \( [30] \).

Let us now go back to simplifying the expressions for the partition function Eq. (10). In the following, we will use the bra-ket notation
\[
\Psi_{\nu}(q_1, \ldots, q_M) \equiv \langle q_1, \ldots, q_M|\nu\rangle.
\]
As one can easily see from Eq. (12), the eigenfunction of \( \hat{K} \) is also eigenfunctions of \( \hat{T}_{\Delta z} \). In the limit \( \Delta z \to 0 \) we have
\[
\hat{T}_{\Delta z} \Psi_{\nu} \approx (1 - \Delta z \kappa_{\nu}) \Psi_{\nu} \approx \exp(-\Delta z \kappa_{\nu}) \Psi_{\nu}.
\]

Using the spectral representation of \( \hat{T}_{\Delta z} \) and the orthonormality of the eigenfunctions \( |\nu\rangle \) we can simplify Eq. (11) for the transfer integral operator for a finite distance to
\[
\hat{T}_|z′ - z| = \sum_{\nu} |\nu\rangle \langle \nu| \exp(-\kappa_{\nu}|z′ - z|).
\]

One can see that for large distances \( |z′ - z| \) the operator is dominated by the ground state \( |0\rangle \).

By construction, \( Z_{<}|z_0| = \hat{T}_{|z_0 - z|} Z_{<}|z_0| \), where \( z_0 < z_{-1} \). Therefore, \( Z_{<}|z_0| \) is dominated by the “ground state” eigenfunction, i.e.,
\[
Z_{<}|z_0| = \text{const} \Psi_0
\]
for a point \( z_0 \) far from the end points \( z = \pm L/2 \). The same holds true for \( Z_{>}|z_0| \). In the thermodynamic limit we can therefore write the partition function, up to an unimportant numerical factor, as
\[
Z = \langle 0|\hat{T}_{|z - z_0|}|0\rangle = \exp(-\kappa_0|z′ - z|).
\]

Let us now use the introduced quantities to calculate correlation functions. For simplicity, we will first focus on two-point functions. The generalization to a higher number of points can easily be done. In the quantum description, a two-point correlation function is
\[
\langle \hat{F}^{(1)}(z_1) \hat{F}^{(2)}(z_2) \rangle = \text{Tr} \left[ \hat{\Phi}^{(1)}(z_1) \hat{\Phi}^{(2)}(z_2) \hat{\rho} \right],
\]
where \( \hat{\rho} \) is the density matrix of the many-body quantum system. The operators \( \hat{\Phi}^{(j)}(z_i) \) of the local observables are composed of normally ordered products of the field operators \( \hat{\phi}_j^+(z_i) \) and \( \hat{\phi}_j(z_i) \) that create or, respectively, annihilate a particle at the point \( z_i \).
In classical field approximation, the density matrix is replaced by the phase-space distribution function and the local observables become functions of the classical fields at the respective points. In thermal equilibrium, the correlation function of two observables \( \mathcal{F}^{(1)} \) and \( \mathcal{F}^{(2)} \) at the points \( z_1 \) and \( z_2 \) is

\[
\langle \mathcal{F}^{(1)} \rangle_{z_1} \langle \mathcal{F}^{(2)} \rangle_{z_2} = \frac{1}{Z} \int Dq_1 \cdots \int Dq_M e^{-\beta H} \mathcal{F}^{(1)}|_{z_1} \mathcal{F}^{(2)}|_{z_2}.
\]  

(22)

For the sake of definiteness, we assume the ordering of the spatial points \( z_2 > z_1 \) and can write

\[
\langle \mathcal{F}^{(1)} \rangle_{z_1} \langle \mathcal{F}^{(2)} \rangle_{z_2} = \frac{1}{Z} \int d^Mq \mathcal{Z}_{z_2} \mathcal{F}^{(2)}|_{z_2} \mathcal{T}_{z_2-z_1} \mathcal{F}^{(1)}|_{z_1} \mathcal{Z}_{z_1}.
\]  

(23)

Using Eqs. (18, 20) we get

\[
\langle \mathcal{F}^{(1)} \rangle_{z_1} \langle \mathcal{F}^{(2)} \rangle_{z_2} = \left\{ \exp \left[ -\kappa_0 (z_2 - z_1) \right] \right\}^{-1} \times \sum_{\nu} \langle 0 | \mathcal{F}^{(2)} | \nu \rangle e^{-\kappa_0 (z_2 - z_1)} \langle \nu | \mathcal{F}^{(1)} | 0 \rangle
\]

\[
= \sum_{\nu} \langle 0 | \mathcal{F}^{(2)} | \nu \rangle e^{-\kappa_0 (z_2 - z_1)} \langle \nu | \mathcal{F}^{(1)} | 0 \rangle.
\]  

(24)

Here

\[
\langle \nu | \mathcal{F}^{(1)} | \nu \rangle = \int d^Mq \Psi_\nu \mathcal{F}^{(1)} \Psi_\nu
\]

is a standard quantum-mechanical matrix element (in a basis of real functions).

Eq. (24) can be easily generalized to an \( l \)-point correlation function that, in a general case, can contain \( l \) different observables \( \mathcal{F}^{(i)} \), \( i = 1, 2, \ldots, l \):

\[
\langle \mathcal{F}^{(1)} \rangle_{z_1} \langle \mathcal{F}^{(2)} \rangle_{z_2} \cdots \langle \mathcal{F}^{(l)} \rangle_{z_l} = \sum_{\nu_1, \ldots, \nu_{l-1}} \langle 0 | \mathcal{F}^{(1)} | \nu_{l-1} \rangle \cdots
\]

\[
\times \langle \nu_2 | \mathcal{F}^{(2)} | \nu_1 \rangle \langle \nu_1 | \mathcal{F}^{(1)} | 0 \rangle \prod_{i=1}^{l-1} e^{-\kappa_0 (z_i - z_{i+1})}.
\]  

(25)

Here the spatial points are ordered as \( z_1 > \cdots > z_2 > z_1 \).

The case \( l = 1 \) yields the thermal average of an arbitrary observable

\[
\langle \mathcal{F}^{(1)} \rangle = \langle 0 | \mathcal{F}^{(1)} | 0 \rangle.
\]  

(26)

An important particular case of Eq. (26) is the expression for the equilibrium distribution \( W_{eq}(q_1, \ldots, q_M) \) of the local values of the fields

\[
W_{eq}(q_1, \ldots, q_M) = \int d^Mq | \Psi_0(q_1', \ldots, q_M') |^2
\]

\[
\times \prod_{j=1}^{M} \delta(q_j' - q_j)
\]

\[
= | \Psi_0(q_1, \ldots, q_M) |^2.
\]  

(27)

With this we finish the recollection of the basics of the transfer matrix formalism [26, 28]. In the next Section we present our proof of its equivalence to a description based on a certain stochastic process.

III. FROM THE TRANSFER MATRIX TO THE STOCHASTIC PROCESS

Calculating correlation functions directly from Eq. (25) might be a challenging task, because of the need to know many eigenvalues and eigenvectors of \( \hat{K} \). Therefore we have developed a method of numerical calculation of correlation functions in 1D that is based on the transfer matrix formalism but requires the knowledge of the ground state function \( \psi_0 \) only.

Assume that we know \( \psi_0 \) and, hence, \( W_{eq} \). This task can be accomplished in many ways, from the diagonalization of \( \hat{K} \) in a suitable basis or propagation of the corresponding Schrödinger-type equation in imaginary time to the variational method. Consider then a Fokker-Planck equation (FPE)

\[
\frac{\partial W(q_1, \ldots, q_M)}{\partial z} = \sum_{j=1}^{M} \left\{ D \frac{\partial^2}{\partial q_j^2} W(q_1, \ldots, q_M) - \frac{\partial}{\partial q_j} \left[ A_j(q_1, \ldots, q_M) W(q_1, \ldots, q_M) \right] \right\}
\]  

(28)

for the distribution function \( W(q_1, \ldots, q_M) \) of the local values of the fields with the co-ordinate \( z \) playing the role of time in the conventional version of the FPE. The diffusion coefficient \( D \) is defined by Eq. (14). We then require \( W_{eq} \) defined by Eq. (27) to be the stationary solution of Eq. (28). This determines the drift coefficients

\[
A_j = D \frac{\partial \ln W_{eq}(q_1, \ldots, q_M)}{\partial q_j}
\]

\[
= 2D \frac{\partial \ln | \Psi_0(q_1, \ldots, q_M) |}{\partial q_j}.
\]  

(29)

Note, that \( \psi_0 \) possesses all the standard properties of a ground state function of a Hamiltonian problem, in particular, for all finite \( q_j \)’s it is non-zero and, hence, the coefficients \( A_j \) have no singularities. This also means that we can always choose \( \psi_0 \) to be real and positive, what we will assume for the rest of the paper.

In the following we will show that the correlation functions described by this FPE are identical to the ones we get from Eq. (25). In order to do this we will use the well-known spectral method of solving the FPE [31]. We introduce a new unknown function \( \Xi(q_1', \ldots, q_M, z) \) via the relation

\[
W = \sqrt{W_{eq}} = \psi_0 \Xi.
\]  

(30)

Substituting Eqs. (29, 30) into Eq. (28), we obtain

\[
\frac{\partial \Xi}{\partial z} = \sum_{j=1}^{M} \left( D \frac{\partial^2}{\partial q_j^2} D \frac{\partial^2}{\partial q_j^2} \psi_0 \right) \Xi.
\]  

(31)
Eq. (31) can be transformed into
\[
\frac{\partial \Xi}{\partial z} = -\hat{K} \Xi + \Xi \frac{1}{\Psi_0} \hat{K} \Psi_0.
\] (32)

By definition, \( \hat{K} \Psi_0 = \kappa_0 \Psi_0 \), where \( \hat{K} \) is given in Eq. (13). With this we get
\[
\frac{\partial \Xi}{\partial z} = -(\hat{K} - \kappa_0) \Xi.
\] (33)

The general solution of Eq. (33) is
\[
\Xi = \sum_\nu c_\nu \Psi_\nu(q_1, \ldots, q_M) e^{-(\kappa_\nu - \kappa_0)z},
\] (34)
where \( c_\nu \) are the constant coefficients.

Let us now look at the probability density \( W_c(q_1, z_2) \) for \( q_1, \ldots, q_M \) at \( z = z_2 \) under the condition that the field values at \( z = z_1 < z_2 \) are \( q_1^\prime, \ldots, q_M^\prime \). This conditional probability density is a solution of Eq. (28) with the initial condition
\[
W_c(q_1, z_1|q_1^\prime, z_1) = \prod_{j=1}^M \delta(q_j - q_j^\prime).
\] (35)

Using the completeness of the set of eigenfunctions \( \Psi_\nu \)
\[
\prod_{j=1}^M \delta(q_j - q_j^\prime) = \sum_\nu \Psi_\nu(q_j) \Psi_\nu(q_j^\prime)
\] we find the particular values of the coefficients \( c_\nu \) in Eq. (34). We obtain for \( z_2 > z_1 \)
\[
W_c(q_1, z_2|q_1^\prime, z_1) = \frac{\Psi_0(q_j)}{\Psi_0(q_j^\prime)} \times \sum_\nu \Psi_\nu(q_j) e^{-(\kappa_\nu - \kappa_0)(z_2 - z_1)} \Psi_\nu(q_j^\prime).
\] (36)

The equilibrium distribution of local values of the fields is given by the stationary solution of Eq. (28), which is \( W_{eq} \). The two-point correlation function can therefore be written as
\[
\langle \mathcal{F}^{(1)}(z_1) \mathcal{F}^{(2)}(z_2) \rangle = \int d^Mq_1 \mathcal{F}^{(2)}(q_j) \times \int d^Mq_2 \mathcal{F}^{(1)}(q_j) W_{eq}(q_j^\prime).
\] (37)

Using the solution Eq. (35), one can easily see the equivalence to Eq. (24). Convolving the conditional probability densities subsequently, one can show that the \( l \)-point correlation function for the stochastic process described by the FPE is given in the most general case by Eq. (25). We can therefore conclude that the transfer matrix formalism and the description by the FPE are equivalent, since the correlation functions following from either of these methods are identical.

We recall the well-known equivalence of the FPE and the stochastic differential Itô equation
\[
dq_j = A_j dq_j + \sqrt{2D_j} dX_j, \quad (37)
\]
where \( dq_j \) are infinitesimally small, mutually uncorrelated, random terms obeying Gaussian statistics with zero mean and the variance equal to \( dz_j \). The bar denotes averaging over the ensemble of realizations of the stochastic process. The initial values (say, at \( z = 0 \)) of the fields for each realization are obtained by (pseudo)random sampling their equilibrium distribution \( W_{eq} \). The subsequent numerical integration of Eq. (37) and averaging over many realizations yields the correlation functions.

Eq. (37) is therefore a generalization of our previous method to simulate the classical thermal fluctuations in a system described by a quadratic Hamiltonian using the Ornstein-Uhlenbeck stochastic process \[18\] to the case of the arbitrary local interaction \( V \). The main advantage of the stochastic ordinary differential Eq. (37) is that its numerical integration is much simpler and less resource-consuming than the integration of the partial differential Eq. (28) on a multidimensional grid. The main computational difficulty is now reduced to the precise determination of \( \Psi_0 \). The determination of all other eigenfunctions and eigenvalues appearing in Eq. (25) is actually not necessary.

IV. TUNNEL-COUPLED QUASI-CONDENSATES IN 1D: THE FULL MODEL VS. THE SINÉ-GORDON MODEL

We apply our method to the calculation of thermal phase and density fluctuations of two tunnel-coupled 1D quasi-condensates of ultracold bosonic atoms. The quasi-condensates in the right (R) or in the left (L) 1D atomic waveguide are described in the mean-field approximation by complex classical fields \( \psi_\nu = q_1 + iq_2 \) and \( \psi_L = q_3 + iq_4 \), respectively. Alternatively, it is possible to express these complex fields \( \psi_\nu = \sqrt{n_\nu} e^{i\eta_\nu} \), \( \varsigma = R, L \), through the quasi-condensate atom-number densities \( n_{R,L} \) and phases \( \theta_{R,L} \). This system is described by Eq. (2) with the interaction term \[23, 24\]
\[
V(q_1, q_2, q_3, q_4) = \frac{g}{2} \left[ (q_1^2 + q_2^2)^2 + (q_3^2 + q_4^2)^2 \right] - 2\hbar J (q_1 q_3 + q_2 q_4), \quad (38)
\]
where \( g > 0 \) is the strength of the contact interaction of atoms in 1D and \( J \) is the single-particle tunneling rate. The tunneling provides exchange of atoms between the two waveguides, therefore atoms in both of them have the same chemical potential \( \mu_R = \mu_L = \mu \). The mean 1D atom-number density in each of the waveguides that corresponds to this chemical potential is denoted by \( n_{1D} \).

It is convenient to parametrize the density variables as
\[ n_\tau = r^2 n_{1D} \]. Then we obtain
\[
\hat{K} = \frac{1}{\lambda_T} \left[ \hat{K}_R + \hat{K}_L - 2b \tau_R \tau_L \cos(\theta_R - \theta_L) \right], \tag{39}
\]
where
\[
\hat{K}_\zeta = -\left( \frac{\partial^2}{\partial \zeta^2} + \frac{1}{r_\zeta} \frac{\partial}{\partial r_\zeta} + \frac{1}{r_\zeta^2} \frac{\partial^2}{\partial \phi^2} \right) + \alpha r_\zeta^2 \left[ \frac{r_\zeta}{2} - 2 \left( \tilde{\mu} - \frac{b}{2\alpha} \right) \right]. \tag{40}
\]
The dimensionless parameters of the problem are
\[
\alpha = \frac{\lambda_T^2}{4\xi_h^2}, \quad b = \frac{\lambda_T^2}{8\xi_h^2}, \tag{41}
\]
where \( \xi_h = \hbar / \sqrt{a n_{1D} m} \) is the quasi-condensate healing length, \( \lambda_T = 2\hbar^2 n_{1D} / (m_k b T) \) is the thermal coherence length and \( l_\zeta = \sqrt{\hbar/(4m J)} \) is the typical length of the relative phase locking \[13,35\]. The parameters \( \alpha \) and \( b \) can be understood as the ratio of the energies of the mean-field repulsion and of the tunnel coupling, respectively, to the kinetic energy of an atom localized at the length scale of the order of \( \lambda_T \). Note that \( \tilde{\mu} = \mu / (gn_{1D}) \) is not a free parameter, but has to be chosen such that the average 1D density equals \( n_{1D} \) in both waveguides, i.e., \( \langle r_\zeta^2 \rangle = \langle r_L^2 \rangle = 1 \). Therefore the eigenstates of \( \hat{K} \) depend on \( \alpha \) and \( b \) only. Since \( D = n_{1D}/\lambda_T \), the solution of the Itô equation \[37\] also depends on the scaled distance \( z/\lambda_T \).

In the following we will focus on discussing the relative phase fluctuations \( \dot{\theta}_- = \theta_R - \theta_L \), because they can be accessed experimentally through matter-wave interferometry \[36\]. While it only makes sense to discuss the phase \( \dot{\theta}_-(z) \) modulo \( 2\pi \) for a single point, the unbound phase differences \( \dot{\theta}_-(z) - \dot{\theta}_-(z') \) between two different points \( z \) and \( z' \) have a physical meaning. We obtain continuous phase profiles from the numerical samples of \( \psi_{R,L} \) through phase unwrapping, i.e., by assuming that \( \dot{\theta}_- \) between neighbouring points on the numerical grid does not differ by more then \( \pi \). The same procedure has been applied to experimental data in Ref. \[17\].

We will compare the results for the two coupled quasi-condensates to the predictions of the SG model
\[
H_{SG} = \int_{-L/2}^{L/2} dz \left[ gn^2 + \frac{\hbar^2 n_{1D}}{4m} \left( \frac{\partial \theta_-}{\partial z} \right)^2 - 2b \hbar J n_{1D} \cos \theta_- \right], \tag{42}
\]
where \( n_- = (n_R - n_L) / 2 \). The SG model was proposed as an effective model for the coupled quasi-condensates \[37\]. Its validity in a certain parameter regime was recently confirmed experimentally \[17\]. Since Eq. \[42\] does not contain terms coupling \( n_- \) to \( \dot{\theta}_- \), the relative density fluctuations can be integrated out and the relative phase correlations are fully determined by the eigensystem of the auxiliary Hermitian operator for the SG model \[35\]
\[
\hat{K}^{SG} = \frac{1}{\lambda_T} \left( -2 \frac{\partial^2}{\partial \theta_-^2} - 2b \cos \theta_- \right), \tag{43}
\]
which can be formally obtained from Eq. \[39\] by setting \( r_L = r_R = 1 \), \( \partial / \partial r_{R,L} \equiv 0 \), and \( \partial^2 / \partial \theta_{R,L}^2 \equiv \partial^2 / \partial \theta_-^2 \). Note that Eq. \[43\] does not contain the parameter \( \alpha \), i.e., the equal-time phase correlations in the SG model at finite temperature do not depend on the atomic interaction strength.

Due to the simpler nature of the model we can obtain results directly from the transfer matrix formalism, without numerical implementation of Eq. \[57\]. This will be the topic of the next Section \[VI\]. The numerical results for the full model will be presented in Section \[VI\].

V. RESULTS FOR THE SINE-GORDON MODEL

In contrast to the previous work \[35\], we consider the moments of the relative phase difference \( \dot{\theta}_-(z) - \dot{\theta}_-(z') \) itself and not the correlation function \( \langle \exp[i(\dot{\theta}_-(z) - \dot{\theta}_-(z'))] \rangle \) for its imaginary exponent. The system is translationally invariant, therefore we can set \( z' = 0 \). We also assume \( z > 0 \). Eq. \[43\] is invariant against inversion of the sign of \( \dot{\theta}_- \). Therefore only even moments \( \langle [\dot{\theta}_-(z) - \dot{\theta}_-(0)]^k \rangle, k = 2, 4, 6, \ldots \), are non-zero. We express them as
\[
\langle [\dot{\theta}_-(z) - \dot{\theta}_-(0)]^k \rangle = (-i)^k \lim_{\varepsilon \to 0} \frac{\partial^k}{\partial \varepsilon^k} C(\varepsilon, z), \tag{44}
\]
where
\[
C(\varepsilon, z) = \langle \exp[i\varepsilon(\dot{\theta}_-(z) - \dot{\theta}_-(0))] \rangle. \tag{45}
\]
Note that Eq. \[43\] is equivalent to a Hamiltonian with a periodic potential. Bloch’s theorem therefore tells us that the eigenfunctions must fulfill quasi-periodic boundary conditions, i.e., they must be Bloch waves. However, when evaluating \( \langle \exp[i\varepsilon(\dot{\theta}_-(z) - i\dot{\theta}_-(0))] \rangle \) using Eq. \[24\], only the eigenfunctions satisfying periodic boundary conditions over \( 2\pi \) are relevant \[35\]. In contrast, the direct evaluation of \( C(\varepsilon, z) \) with \( \varepsilon < 1 \) also requires knowledge of the eigenfunctions satisfying quasi-periodic boundary conditions over \( 2\pi \). However, there is a trick that allows us to avoid this demanding calculation and use the \( 2\pi \)-periodic eigenfunctions only.

We observe that applying Eq. \[24\] yields
\[
C(\varepsilon, z) = \langle 0 | e^{i\varepsilon \theta_-} \exp[-(\hat{K}^{SG} - \kappa_0) z] | e^{-i\varepsilon \theta_-} \langle 0 \rangle \rangle = 0. \tag{46}
\]
Note that the lowest eigenvalue \( \kappa_0 \) corresponds to the state \( | \theta_- \rangle \) that is periodic over \( 2\pi \). We now observe that \( e^{i\varepsilon \theta_-} \) and its inverse, \( e^{-i\varepsilon \theta_-} \), perform a unitary
transformation of the operator $\exp[-(\hat{K}^{SG} - \kappa_0)z]$, yielding

$$C(\varepsilon, z) = \langle 0 | \exp \left\{ - \frac{2}{\lambda_T} \left( \frac{\partial}{\partial \theta_-} - i \varepsilon \right)^2 - \frac{2b}{\lambda_T} \cos \theta_- - \kappa_0 \right\} z | 0 \rangle$$

$$= \exp \left( - \frac{2b}{\lambda_T} \cos \theta_- - \kappa_0 \right) z | 0 \rangle \tag{47}$$

where

$$\mathcal{J}(\varepsilon, z) = \langle 0 | \exp \left\{ - \left( \frac{2i\varepsilon}{\lambda_T} \frac{\partial}{\partial \theta_-} + \hat{K}^{SG} - \kappa_0 \right)z \right\} | 0 \rangle. \tag{48}$$

The evaluation of the correlations via Eqs. (44, 45) is then reduced to expanding the exponential in Eq. (48), differentiating Eq. (47) over $\varepsilon$ and setting the limit $\varepsilon \to 0$. The resulting expression is an infinite sum of powers of $\hat{K}^{SG} - \kappa_0$ and the derivative $\partial / \partial \theta_-$. Only the latter operator has off-diagonal matrix elements in the basis of eigenstates of $\hat{K}^{SG}$. When we calculate the $k$th moment of the relative phase difference, the operator $\partial / \partial \theta_-$ appears in each term of the infinite sum maximum $k$ times. If it acts on an eigenfunction that is periodic over $2\pi$ then the resulting function is also periodic. Therefore, we avoid the necessity to calculate quasi-periodic eigenfunctions as the inner product of a periodic and a quasi-periodic functions vanishes. Assembling the series yields for the second and fourth moments

$$\langle [\theta_-(z) - \theta_-(0)]^2 \rangle = \frac{4z}{\lambda_T} - \left( \frac{4z}{\lambda_T} \right)^2 \sum_\nu |\psi_{\nu0}|^2 F_1(1; 3; - (\kappa_\nu - \kappa_0)z), \tag{49}$$

$$\langle [\theta_-(z) - \theta_-(0)]^4 \rangle = 3 \langle [\theta_-(z) - \theta_-(0)]^2 \rangle^2 + 3 \left( \frac{4z}{\lambda_T} \right)^4 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \frac{(-1)^{j_1+j_2+j_3}}{(j_1+j_2+j_3+4)!} \sum_{\nu_1, \nu_2, \nu_3} |\psi_{\nu_1}|^2 |\psi_{\nu_2}|^2 |\psi_{\nu_3}|^2 \times \sum_\nu |\psi_{\nu}|^2 F_1(1; 3; - (\kappa_\nu - \kappa_0)z)^2 \right\}. \tag{50}$$

Here

$$|\psi_{\nu\nu'} = \int_{-\pi}^{\pi} d\theta \psi_\nu(\theta) \frac{d\psi_{\nu'}(\theta)}{d\theta},$$

where $\psi_\nu$ is the $\nu$th eigenfunction of the operator $\hat{K}^{SG}$ satisfying periodic boundary conditions on the interval from $-\pi$ to $\pi$. The set of $\psi_\nu(\theta_-)$ can be found by the numerical diagonalization of $\hat{K}^{SG}$ in the Fourier (i.e., cosine and sine) basis. Furthermore, we expressed the series

$$\sum_{j=0}^{\infty} \frac{2x^j}{(j+2)!} = \frac{2(e^x - 1 - x)}{x^2} = F_1(1; 3; x)$$

via the confluent hypergeometric function $F_1(\alpha; \beta; x)$. In the next section we will present the results of the full model and compare them to Eqs. (49, 50).

VI. RESULTS FOR THE FULL MODEL AND DISCUSSION

A. The ground state

Finding the ground state of the operator $\hat{K}$ for the full model [Eqs. (39, 40)] is a formidable task. However, the general structure of the operator [13] [and, hence, of Eq. (39)] that contains only local pairwise interactions admits for a solution. First of all, we notice that Eq. (39) is invariant with respect to simultaneously shifting both the angles $\theta_R$ and $\theta_L$ by the same value. Therefore, the (non-degenerate) ground state $\Psi_0$ must be independent of $\theta_\perp = \theta_R + \theta_L$.

We search the ground state in the form

$$\Psi_0(r_R, r_L, \theta_\perp) = \sum_{\ell, \ell' = 0}^{\ell_{\text{max}}} \sum_{n, n' = 0}^{n_{\text{max}}} c_{\ell, n, n'} \Phi_{n, \ell}(r_R) \Phi_{n', \ell}(r_L) \times \cos(\ell_\perp) \sqrt{\pi} \frac{1 + \delta_{\ell,0}}{\sqrt{1 + \delta_{\ell,0}}}, \tag{51}$$

where $\Phi_{n, \ell}(r_\varsigma)$, $\varsigma = R, L$, are the eigenfunctions of the operator

$$\hat{K}^{(\ell)}_\varsigma = - \left( \frac{\partial^2}{\partial r_\varsigma^2} + \frac{1}{r_\varsigma} \frac{\partial}{\partial r_\varsigma} \right) + \frac{\ell^2}{r_\varsigma^2} + \alpha \left( \frac{3}{2} \right)^2 \left( \frac{\mu - b}{2\alpha} \right), \tag{52}$$

labelled by the number $n_\ell$ of its nodes at $0 < r_\varsigma < \infty$. Eq. (52) is obtained from Eq. (40) by replacing $\partial^2 / \partial \theta_-^2$ with $-\ell^2$. The normalization condition for the (real)
eigenfunctions of $\hat{K}_c^{(f)}$ is
\[
\int_0^\infty dr \, (\Phi_n, r) (\Phi_n, r) = \delta_{n', n_r}.
\]

The most convenient basis to diagonalize $\hat{K}_c^{(f)}$ is given by the radial eigenfunctions of a two-dimensional harmonic oscillator \[28\, 39\]: we expand $\Phi_n, r(r_c)$ in the functions
\[
R_{n\ell} = \eta \sqrt{2n!/(n+\ell)!} (y r_c) \ell \exp(-\eta^2 r_c^2/2) L_n^\ell (\eta^2 r_c^2),
\]
with $n = 0, 1, \ldots, n_{\text{max}}$. The limiting number $n_{\text{max}}$ for the expansion of $\Phi_n, r(r_c)$ should not be confused with $n_{r_{\text{max}}}$ in Eq. \[27\]. In the equation, $L_n^\ell$ is the generalized Laguerre polynomial with $\ell \geq 0$ and $\eta$ is a numerical scaling factor. Experience showed that the best convergence is attained for $\eta \approx 5$. The key advantage of this choice of basis is that the matrix elements of the operator $\hat{K}_c^{(f)}$, consisting of the Laplacian in the polar co-ordinates and the powers $r_c^2$, $r_c^4$, can be found analytically. Analytic expressions for the matrix elements of $r_c \exp(\pm i\theta_c)$ also exist in this basis. All these matrix elements needed for the diagonalization of the complete operator $\hat{K}$ can be easily computed by expressing them through the rotated bosonic creation and annihilation operators for a two-dimensional harmonic oscillator \[10\, 31\].

The ground state $\Psi_0$ is found from the diagonalization of $\hat{K}$ as the lowest-eigenvalue solution. The diagonalization happens in the basis of the constituent functions given in Eq. \[27\]. Theoretically, one would have to diagonalize an infinitely large matrix, practically one only has to diagonalize a submatrix which size is determined by the values for $\ell_{\text{max}}$ and $n_{r_{\text{max}}}$. They are chosen to be large enough to ensure convergence. For the results presented in this paper, we chose $\ell_{\text{max}} = 20$ and $n_{r_{\text{max}}} = 17$. Convergence was checked by comparing to the results for $\ell_{\text{max}} = 13$ and $n_{r_{\text{max}}} = 10$. We begin the diagonalization with the initial guess $\hat{\mu} = 1 - b/(2\alpha)$ for the chemical potential. After obtaining $\Psi_0$, we can compute $\langle r_c^2 \rangle_L$. For symmetry reasons, it does not matter whether we look at $R$ or $L$. By assumption, this value $\langle r^2 \rangle$ must be equal to 1, but the first try will yield some value $\langle r^2 \rangle \neq 1$. Therefore, we rescale $\hat{\mu}$ to the value $\hat{\mu} + 1 - \langle r^2 \rangle$ and repeat the procedure until we obtain $\langle r^2 \rangle = 1$ with the required precision. For the data presented in this paper, we required $|\langle r^2 \rangle - 1| < 10^{-4}$.

Note that a quick evaluation of $\Psi_0$ [Eq. \[31\]] is paramount for an efficient solution of Eq. \[37\]. Pre-calculating $\Psi_0$ on a three dimensional grid is not an option due to the large number of grid-points that would be required. Calculating everything from scratch is also inefficient as the functions $\Phi_n, r(r_c)$ themselves consist of sums with many terms. The best option therefore seems to be to pre-calculate each of the functions $\Phi_n, r(r_c)$ and also $\cos(\ell \theta_\gamma)$ on the respective one-dimensional grids, then evaluating the sum in Eq. \[31\] every time we need a certain value $\Psi_0(r_R, r_L, \theta_-)$. The grids used for the data presented in this paper had 1024 points. Convergence was checked by comparison with the results for only 512 grid-points. To save computational time we consider only the terms with $c(n_n, n_n') > 10^{-5}$ in the sum Eq. \[51\]. Again we checked for convergence by comparing to the results for a cutoff of only $10^{-3}$.

### B. Correlation functions

After having obtained the ground state, we can integrate Eq. \[37\] by the forward Euler method, using a pseudo-random generator to simulate the random terms. The step size in $z$-direction was chosen as $\delta z/\lambda_T = 1/4000$. We checked for convergence by comparing to $\delta z/\lambda_T = 1/1000$. The presented results have been calculated from $1.2 \times 10^5$ numerical realizations.

Despite having obtained the ground state $\Psi_0$ in polar coordinates, we perform the random process in Cartesian coordinates $q_j$ to avoid numerical pitfalls. The derivatives of $\ln(W_{\text{eq}})$ needed for the drift coefficients $A_{ij}$ [Eq. \[29\]] are calculated as finite differences in polar coordinates, then transformed to Cartesian coordinates in the usual way. Calculating for example the derivative in $r_L$, we first find the two points on the radial grid that are closest to the actual value of $r_L$. For the values of the two remaining variables we accept the grid points closest to the actual value of $r_R$ and $\theta_-$, respectively. We then evaluate $\ln(W_{\text{eq}})$ on this two points of the three parameter grid which differ only in $r_L$, take the difference and divide by the step of the radial grid.

We will compare the results to the predictions of the SG model with the rescaled parameters $\lambda_T = \lambda_T/(1/r_c^2)_{\text{reg}}$ and $b = b (r_R r_L)/(1/r_c^2)_{\text{reg}}$. Here $(1/r_c^2)_{\text{reg}}$ represents the regularized mean inverse density (in dimensionless units), for symmetry reasons the expectation value is the same for $\zeta = L, R$. The regularization is necessary, because otherwise the $\ell = 0$ component of the ground state would yield a logarithmic divergence of the integral. Different regularizations have been tested and all yielded very close results. In the end we chose to simply exclude a very small region around $r_c = 0$.

The reason for the rescaling of the parameters becomes clear when looking at the operator $\hat{K}$ given in Eqs. \[39\, 10\]. We average it over the ground state fluctuations of $r_{R,L}$ (the need for regularization of $1/r_c^2$ should be kept in mind). This averaging yields an operator

\[
\hat{K}_{\text{rescaled}} = \frac{1}{\lambda_T} \left( \frac{1}{r_c^2} \right)_{\text{reg}} \times \left( -2 \frac{\partial^2}{\partial \theta_-^2} - b (r_R r_L) \left( \frac{1}{r_c^2} \right)_{\text{reg}} \cos \theta_- \right).
\]

The rescaling of $\lambda_T$ and $b$ can be directly read from the comparison of Eqs. \[49\] and \[53\].

Fig. \[1\] shows the results for $\alpha = 100$. For $b = 1$ (subfigure (a)), which corresponds to intermediate phase locking, one sees good agreement between the results for
FIG. 1. (Color online.) Second and fourth moment of the relative phase difference between two points along the 1D direction $z$. Results for $\alpha = 100$ and (a) $b = 1$, (b) $b = 5$. (a) In the intermediate phase-locking regime ($\langle \cos(\theta - \theta_0) \rangle = 0.58$) we observe good agreement between the two coupled 1D quasi-condensates (solid blue lines) and the sine-Gordon model with the rescaled parameters (dashed orange lines). Clear deviations from the sine-Gordon model without rescaling of the parameters (dashed orange lines) are visible. (b) For strong phase locking ($\langle \cos(\theta - \theta_0) \rangle = 0.83$) we get clear deviations also for the rescaled sine-Gordon theory. The same is true for small phase locking (not shown). For stronger phase-locking $b = 5$ deviations are clearly visible [Fig. 1(b)]. For higher values of $\alpha$ (lower temperatures or higher densities) the agreement between the full theory and the rescaled SG model holds even for strong phase locking (see Fig. 2 for $\alpha = 500$). One can understand the different behavior for the different values of $\alpha$ from the amount of density fluctuations being present in the system. The higher $\alpha$ (i.e., the more pronounced the effect of interatomic repulsion), the more suppressed are the density fluctuations. The accuracy of the SG description is thus increased. This explains the good agreement of the experimental data of Ref. 17 with the SG model. The value of $\alpha$ there is rather high ($\alpha \approx 600$). Repeating the measurements for $\alpha = 100$ would be a challenging task due to the finite resolution of the imaging system.

Note that, for the parameters in Fig. 1(b) ($\alpha = 100$ Locking (see Fig. 2 for $\alpha = 500$). One can understand the different behavior for the different values of $\alpha$ from the amount of density fluctuations being present in the system. The higher $\alpha$ (i.e., the more pronounced the effect of interatomic repulsion), the more suppressed are the density fluctuations. The accuracy of the SG description is thus increased. This explains the good agreement of the experimental data of Ref. 17 with the SG model. The value of $\alpha$ there is rather high ($\alpha \approx 600$). Repeating the measurements for $\alpha = 100$ would be a challenging task due to the finite resolution of the imaging system.

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from the predictions of the SG model for the underlying distribution of $\theta_-$ which is a measure for the non-Gaussianity of the under-
locking (intermediate values of $\langle \cos \theta_- \rangle$) and strong phase-locking ($\langle \cos \theta_- \rangle \approx 1$) has different physical ori-
observations. For intermediate phase-locking, $W_{\text{eq}}(r_R, r_L, \theta_-)$ as a function of $\theta_-$ for fixed $r_R, r_L$ is non-Gaussian in the relevant range of $r_R$ and $r_L$ (close to 1). For strong phase-locking this is not the case any more. The distribution of $\theta_-$ for different points $r_R, r_L$ is approximately Gaussian, with the variance depending on $r_R, r_L$ (see Fig. 3). Therefore, averaging over different points leads to an overall distribution for $\theta_-$ which is non-Gaussian.

VII. CONCLUSION

To summarize, we have developed a versatile method for calculating thermal expectation values for non-relativistic 1D bosonic systems. To be treatable by our method, a 1D system should satisfy the following criteria. First of all, it must be at thermal equilibrium. Secondly, the system must be dominated by classical thermal fluctuations; the quantum fluctuations should be negligible compared to them, which is often the case in weakly interacting atomic quasicondensates [15]. Of course, strongly interacting quantum systems (such as the Tonks-Girardeau gas) cannot be modeled by our method. The third criterion requires the dynamical stability of the system, the thermodynamic limit should exist, which is the case for atomic gases with repulsive in-
teractions. The last condition is the locality of interactions This is not too restrictive, since short-range interactions are well approximated by a contact potential; tunnel coupling between waveguides and coherent electromagnetically driven transitions between spin states also comply with this restriction.

We applied the method to the case of two tunnel-coupled 1D quasi-condensates and compared the results to the predictions of the simpler sine-Gordon model for which we obtained analytical results. We identified the cases when the two descriptions agree and when their predictions differ.

Our non-perturbative method is based on the stochastic ordinary differential equation (Itô equation). The main advantage of the presented method is its computational efficiency. Calculating the $1.2 \times 10^5$ realizations used for Fig. 1 and 2 takes around 2 hours on a desktop computer, which is at least by an order of magnitude shorter than what more traditional methods like stochast-

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We applied the method to the case of two tunnel-coupled 1D quasi-condensates and compared the results to the predictions of the simpler sine-Gordon model for which we obtained analytical results. We identified the cases when the two descriptions agree and when their predictions differ.

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