Interval structures for braid groups $B(e, e, n)$

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Abstract

Complex braid groups are a generalization of Artin-Tits groups. The general goal is to extend what is known for Artin-Tits groups to other complex braid groups. We are interested in Garside structures that derive from intervals. Actually, we construct intervals in the complex reflection group $G(e, e, n)$ which gives rise to Garside groups. Some of these groups correspond to the complex braid group $B(e, e, n)$. For the other Garside groups that appear, we give some of their properties in order to understand these new structures.

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1 Introduction

A reflection is an element \( s \) of \( GL_n(\mathbb{C}) \) with \( n \geq 1 \) such that \( \text{ker}(s-1) \) is a hyperplane and \( s^2 = 1 \). Relaxing the last condition to \( s \) of finite order defines the notion of pseudo-reflection. Let \( W \) be a finite subgroup of \( GL_n(\mathbb{C}) \) and \( \mathcal{R} \) be the set of reflections of \( W \). We say that \( W \) is a complex reflection group if \( W \) is generated by \( \mathcal{R} \). Since every complex reflection group is a direct product of irreducible ones, we restrict our work to irreducible complex reflection groups. These groups have been classified by Shephard and Todd [17] in 1954. The classification consists of two families and 15 exceptions. For \( e, n \geq 1 \), the first family is denoted by \( G(e, e, n) \) and defined as the group of \( n \times n \) matrices consisting of

- monomial matrices (each row and column has a unique nonzero entry),
- with all nonzero entries lying in \( \mu_e \), the group of \( e \)-th roots of unity, and
- for which the product of the nonzero entries is 1.

The second family is denoted by \( G(2e, e, n) \) and defined as the group of \( n \times n \) matrices consisting of monomial matrices, with all nonzero entries lying in \( \mu_{2e} \), and for which the product of the nonzero entries is 1 or \(-1\). For the definition of the exceptional groups, the reader may check [17].

For every complex reflection group \( W \), there exists a corresponding hyperplane arrangement and hyperplane complement: \( \mathcal{A} = \{ \text{Ker}(s-1) \mid s \in \mathcal{R} \} \) and \( X = \mathbb{C}^n \setminus \bigcup \mathcal{A} \). The pure braid group is defined as \( P := \pi_1(X) \) and the braid group (or complex braid group) as \( B := \pi_1(X/W) \). Note that we have the short exact sequence:

\[
1 \longrightarrow P \longrightarrow B \longrightarrow W \longrightarrow 1.
\]

If \( W = G(de, e, n) \) with \( d = 1 \) or 2, then we denote by \( B(de, e, n) \) the associated braid group. This construction of the braid group is also valid for finite complex pseudo-reflection groups. However, using the classification of Shephard and Todd for irreducible complex pseudo-reflection groups and case-by-case results of [3], we may restrict our work to complex reflection groups as far as group-theoretic properties of the braid group are concerned.

The previous definitions are a generalization of the well-known Coxeter and Artin-Tits groups that we recall now. One way of defining a finite Coxeter group \( W \) is by a presentation with

- a generating set \( S \) and
- relations:
  - quadratic relations: \( s^2 = 1 \) for all \( s \in S \) and
  - braid relations: \( \underbrace{sts \cdots}_{m_{st}} = \underbrace{tst \cdots}_{m_{st}} \) for \( s \neq t \in S \) where \( m_{st} \) is the order of \( st \in W \).
The Artin-Tits group $B(W)$ corresponding to $W$ is the group of fractions of the monoid $B^+(W)$ defined by a presentation with generators: a set $\tilde{S}$ in bijection with the set $S$ and the relations are only the braid relations: $\tilde{s}_i \tilde{t} \ldots = \tilde{t} \tilde{s}_i \tilde{t} \ldots$ for $\tilde{s}_i \neq \tilde{t} \in \tilde{S}$.

The seminal example of these groups is when $W = S_n$, the symmetric group with $n \geq 2$. It has a presentation with generators $s_1, s_2, \ldots, s_{n-1}$ and relations:

- $s_i^2 = 1$ for $1 \leq i \leq n - 1$,
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $1 \leq i \leq n - 2$, and
- $s_is_j = s_js_i$ for $|i - j| > 1$.

The Artin-Tits group associated with $S_n$ is the ‘classical’ braid group denoted by $B_n$.

The following diagram presentation encodes all the information about the generators and relations of the presentation of $B_n$.

![Diagram for the presentation of $B_n$.](image)

The link with the first definitions is as follows. Consider $W$ a real reflection group meaning that $W < GL_n(\mathbb{R}) < GL_n(\mathbb{C})$. By a theorem of Coxeter, every real reflection group corresponds to a Coxeter group. Furthermore, by a theorem of Brieskorn, the Artin-Tits group corresponding to $W$ is the braid group $\pi_1(X/W)$ attached to $W$.

It is widely believed that complex braid groups share similar properties with Artin-Tits groups. One would like to extend what is known for Artin-Tits groups to other complex braid groups. For instance, it is shown by Bessis and Corran in 2006, and by Corran and Picantin in 2009 that the complex braid group $B(e, e, n)$ admits some Garside structures. We are interested in constructing Garside structures for $B(e, e, n)$ that derive from intervals in the associated complex reflection group $G(e, e, n)$. For instance, the complex braid group $B(e, e, n)$ admits interval structures that derive from the non-crossing partitions of type $(e, e, n)$. See [1] for a detailed study. The aim of this paper is to provide new interval structures for $B(e, e, n)$. In Section 3, we compute the length of all elements of $G(e, e, n)$ over an appropriate generating set. This allows us to construct intervals in the complex reflection group $G(e, e, n)$ and show that they are lattices which automatically gives rise to Garside structures. This is done in Theorem 5.15. In the last section, we provide group presentations for these Garside structures which we denote by $B^{(k)}(e, e, n)$ for $1 \leq k \leq e - 1$ and identify which of them correspond to $B(e, e, n)$. For the other Garside structures that appear, we give some of their properties in order to understand them. One of the important results obtained is Theorem 6.14:

$B^{(k)}(e, e, n)$ is isomorphic to $B(e, e, n)$ if and only if $k \wedge e = 1$.

In the remaining part of this section, we include the necessary preliminaries to accurately describe these Garside structures.
1.1 Garside monoids and groups

In his PhD thesis, defended in 1965 [12], and in the article that followed [13], Garside solved the Conjugacy Problem for the classical braid group $B_n$ by introducing a sub-monoid $B_n^+$ of $B_n$ and an element $\Delta_n$ of $B_n^+$ that he called fundamental, and then showing that there exists a normal form for every element in $B_n$. In the beginning of the 1970’s, it was realized by Brieskorn and Saito [2] and Deligne [11] that Garside’s results extend to all Artin-Tits groups. At the end of the 1990’s, after listing the abstract properties of $B_n^+$ and the fundamental element $\Delta_n$, Dehornoy and Paris [10] defined the notion of Gaussian groups and Garside groups which leads, in “a natural, but slowly emerging program” as stated in [8], to Garside theory. For a complete study about Garside structures that is still far from complete, we refer the reader to [8].

We start by defining Garside monoids and groups. Let $M$ be a monoid. Under some assumptions about $M$, more precisely the assumptions 1 and 2 of Definition 1.2, one can define a partial order relation on $M$ as follows.

**Definition 1.1.** Let $f, g \in M$. We say that $f$ left-divides $g$ or simply $f$ divides $g$ when there is no confusion, written $f \preceq g$, if $fg' = g$ holds for some $g' \in M$. Similarly, we say that $f$ right-divides $g$, written $f \preceq_r g$, if $g'f = g$ holds for some $g' \in M$.

We are ready to define Garside monoids and groups.

**Definition 1.2.** We say that $M$ is a Garside monoid if

1. $M$ is cancellative, that is $fg = fh \implies g = h$ and $gf = hf \implies g = h$ for $f, g, h \in M$,
2. there exists $\lambda : M \to \mathbb{N}$ s.t. $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $g \neq 1 \implies \lambda(g) \neq 0$,
3. any two elements of $M$ have a gcd and an lcm for $\preceq$ and $\preceq_r$, and
4. there exists an element $\Delta \in M$ such that the set of its left divisors coincides with the set of its right divisors, generate $M$, and is finite.

The element $\Delta$ is called a Garside element of $M$ and the divisors of $\Delta$ are called the simples of the Garside structure.

Assumptions 1 and 3 of Definition 1.2 ensure that Ore’s conditions are satisfied. Hence there exists a group of fractions of the monoid $M$ in which it embeds. This allows us to give the following definition.

**Definition 1.3.** A Garside group is the group of fractions of a Garside monoid.

Note that one of the important aspects of a Garside structure is the existence of a normal form for all elements of the Garside group. Furthermore, many problems like the Word and Conjugacy problems can be solved in Garside groups which makes their study interesting.
1.2 Interval structures

Let $G$ be a finite group generated by a finite set $S$. There is a way to construct Garside structures from intervals in $G$.

We start by defining a partial order relation on $G$.

**Definition 1.4.** Let $f, g \in G$. We say that $g$ is a divisor of $f$ or $f$ is a multiple of $g$, and write $g \preceq f$, if $f = gh$ with $h \in G$ and $\ell(f) = \ell(g) + \ell(h)$, where $\ell(f)$ is the length over $S$ of $f \in G$.

**Definition 1.5.** For $w \in G$, define a monoid $M([1, w])$ by the presentation of monoid with

- generating set $P$ in bijection with the interval $[1, w] := \{ f \in G \mid 1 \preceq f \preceq w \}$ and
- relations: $fg = h$ if $f, g, h \in [1, w]$, $fg = h$, and $f \preceq h$, that is $\ell(f) + \ell(g) = \ell(h)$.

Similarly, one can define the partial order relation on $G$ $g \preceq_r f$ if and only if $\ell(fg^{-1}) + \ell(g) = \ell(f)$, then define the interval $[1, w]_r$ and the monoid $M([1, w]_r)$.

**Definition 1.6.** Let $w$ be in $G$. We say that $w$ is a balanced element of $G$ if $[1, w] = [1, w]_r$.

We have the following theorem (see Section 10 of [15] for a proof).

**Theorem 1.7.** If $w \in G$ is balanced and both posets $([1, w], \preceq)$ and $([1, w]_r, \preceq_r)$ are lattices, then $M([1, w])$ is a Garside monoid with Garside element $w$ and with simples $[1, w]$.

The previous construction gives rise to an interval structure. The interval monoid is $M([1, w])$. When $M([1, w])$ is a Garside monoid, its group of fractions exists and is denoted by $G(M([1, w]))$. We call it the interval group. We will give a seminal example of this structure. It shows that Artin-Tits groups admit interval structures.

**Example 1.8.** Let $W$ be a finite coxeter group

$$W = < S \mid s^2 = 1, \underbrace{sts\cdots}_{m_{xt}} = \underbrace{tst\cdots}_{m_{xt}} \text{ for } s \neq t \in S > .$$

Take $G = W$ and $g = w_0$ the longest element over $S$ in $W$. We have $[1, w_0] = W$. Construct the interval monoid $M([1, w_0])$. We have $M([1, w_0])$ is the Artin-Tits monoid $B^+(W)$. Hence $B^+(W)$ is generated by a copy $\overline{W}$ of $W$ with $\overline{f} g = \overline{h}$ if $fg = h$ and $\ell(f) + \ell(g) = \ell(h)$; $f, g$, and $h \in W$. We have the following result.

**Theorem 1.9.** $B^+(W)$ is a Garside monoid with Garside element $\overline{w_0}$ and with simples $\overline{W}$. 


2 The groups $G(e, e, n)$ and $B(e, e, n)$

In this section, we recall some presentations by generators and relations of $G(e, e, n)$ and $B(e, e, n)$ based on the results of [3] and [6].

2.1 Presentations

Recall that for $e, n \geq 1$, $G(e, e, n)$ is the group of $n \times n$ matrices consisting of monomial matrices, with all nonzero entries lying in $\mu_e$, the $e$-th roots of unity, and for which the product of the nonzero entries is 1. Note that this family includes three families of finite Coxeter groups: $G(1, 1, n)$ is the symmetric group, $G(e, e, 2)$ is the dihedral group, and $G(2, 2, n)$ is type $D_n$.

Define a group by a presentation with generators and relations that can be described by the following diagram.

$$
\begin{align*}
\text{t}_0 & \quad \text{2} \\
\text{t}_1 & \quad \text{2} \\
\text{s}_3 & \quad \text{2} \\
\text{s}_4 & \quad \text{2} \\
\text{s}_{n-1} & \quad \text{2} \\
\text{s}_n & \quad \text{2} \\
\end{align*}
$$

Figure 2: Diagram for the presentation of BMR of $G(e, e, n)$.

The generators of this presentation are $\text{t}_0, \text{t}_1, \text{s}_3, \text{s}_4, \cdots, \text{s}_{n-1}, \text{s}_n$ and the relations are:

- quadratic relations for all generators,
- the relations of the symmetric group for $\text{s}_3, \text{s}_4, \cdots, \text{s}_{n-1},$
- the relation of the dihedral group $I_2(e)$ for $\text{t}_0$ and $\text{t}_1$: $\text{t}_0\text{t}_1\text{t}_0\cdots = \text{t}_1\text{t}_0\text{t}_1\cdots$,
- $\text{s}_3\text{t}_i\text{t}_3 = \text{t}_i\text{s}_3\text{t}_i$ for $i = 0, 1$, and
- $\text{s}_j\text{t}_i = \text{t}_i\text{s}_j$ for $i = 0, 1$ and $4 \leq j \leq n$.

It is shown in [3] that this group is isomorphic to the complex reflection group $G(e, e, n)$ with $\text{t}_i \mapsto t_i := \left( \begin{array}{ccc} 0 & \zeta_e^{-i} & 0 \\ \zeta_e^i & 0 & 0 \\ 0 & 0 & I_{n-2} \end{array} \right)$ for $i = 0, 1$ and $\text{s}_j \mapsto s_j := \left( \begin{array}{ccc} I_{j-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-j} \end{array} \right)$ for $3 \leq j \leq n$, where $I_k$ is the identity matrix with $1 \leq k \leq n$.

This presentation is called the presentation of BMR (Broué Malle Rouquier) of $G(e, e, n)$. 

6
By [3], removing the quadratic relations gives a presentation of the complex braid group $B(e, e, n)$ attached to $G(e, e, n)$ with diagram as follows.

![Diagram for the presentation of BMR of $B(e, e, n)$](image)

Figure 3: Diagram for the presentation of BMR of $B(e, e, n)$.

Note that the set of generators of this presentation is in bijection with \{t_0, t_1, s_3, s_4, \ldots, s_n\}. This presentation is called the presentation of BMR of $B(e, e, n)$.

### 2.2 Other presentations

Consider the presentation of BMR for the complex braid group $B(e, e, n)$. For $e \geq 3$ and $n \geq 3$, it is shown in [5], p.122, that the monoid defined by this presentation fails to embed in $B(e, e, n)$. Thus, this presentation does not give rise to a Garside structure for $B(e, e, n)$. Considering the interest of the Garside groups given earlier, it is legitimate to look for a Garside structure for $B(e, e, n)$.

Let \( \tilde{t}_i := \tilde{t}_{i-1} \tilde{t}_{i-2} \tilde{t}_{i-1}^{-1} \) for $2 \leq i \leq e - 1$. Consider the following diagram presentation (the kite).

![Diagram for the presentation of CP of $B(e, e, n)$](image)

Figure 4: Diagram for the presentation of CP of $B(e, e, n)$.

The generators of this presentation are \( \tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{e-1}, \tilde{s}_3, \tilde{s}_4, \ldots, \tilde{s}_{n-1}, \tilde{s}_n \) and the relations are:

- the relations of the symmetric group for $\tilde{s}_3, \tilde{s}_4, \ldots, \tilde{s}_{n-1}$,
• the relation of the ‘dual’ dihedral group for $\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{e-1}$: $\tilde{t}_i\tilde{t}_{i-1} = \tilde{t}_j\tilde{t}_{j-1}$ for $i, j \in \mathbb{Z}/e\mathbb{Z}$,

• $\tilde{s}_3\tilde{t}_i\tilde{s}_3 = \tilde{t}_i\tilde{s}_3\tilde{t}_i$ for $0 \leq i \leq e - 1$, and

• $\tilde{s}_j\tilde{t}_i = \tilde{t}_i\tilde{s}_j$ for $0 \leq i \leq e - 1$ and $4 \leq j \leq n$.

It is shown in [6] that the group defined by this presentation is isomorphic to $B(e, e, n)$. We call it the presentation of CP (Corran Picantin) of $B(e, e, n)$. It is also shown in [6] that:

**Proposition 2.1.** The presentation of CP gives rise to a Garside structure for $B(e, e, n)$ with

Garside element: $\Delta = \tilde{t}_1\tilde{t}_0 \tilde{s}_3\tilde{t}_1\tilde{t}_0\tilde{s}_3 \cdots \tilde{s}_n \cdots \tilde{s}_3\tilde{t}_1\tilde{t}_0\tilde{s}_3 \cdots \tilde{s}_n$ and

simples: the elements of the form $\delta_2\delta_3\cdots\delta_n$ where $\delta_i$ is a divisor of $\Delta_i$ for $2 \leq i \leq n$.

It is stated in [6] that if one adds the relations $x^2 = 1$ for all generators $x$ of the presentation of CP, one obtains a presentation of a group isomorphic to $G(e, e, n)$. It is called the presentation of CP of $G(e, e, n)$ with diagram as follows.

![Diagram for the presentation of CP of $G(e, e, n)$](image)

The generators of this presentation belong to the set $X := \{t_0, t_1, \ldots, t_{e-1}, s_3, \ldots, s_n\}$ and the diagram presentation encodes the same relations as the diagram of Figure 2.2 with quadratic relations for all the generators in $X$.

The isomorphism with $G(e, e, n)$ is given by $t_i \mapsto t_i := \begin{pmatrix} 0 & \zeta_e^{-i} & 0 \\ \zeta_e^i & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ for $0 \leq i \leq e - 1$, and $s_j \mapsto s_j := \begin{pmatrix} I_{j-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j} \end{pmatrix}$ for $3 \leq j \leq n$. Denote $X$ the set $\{t_0, t_1, \ldots, t_{e-1}, s_3, \ldots, s_n\}$. 
In the next section, we compute the length over $X$ of each element $w$ in $G(e,e,n)$ by providing a minimal word representative over $X$ of $w$.

## 3 Reduced words in $G(e,e,n)$

In this section, we represent each element of $G(e,e,n)$ by a reduced word over $X$. After some preparations, this is done in Proposition 3.10. Also, we characterize all the elements of $G(e,e,n)$ that are of maximal length over $X$.

Recall that an element $w \in X^\ast$ is called word over $X$. We denote by $\ell(w)$ the length over $X$ of the word $w$.

**Definition 3.1.** Let $w$ be an element of $G(e,e,n)$. We define $\ell(w)$ to be the minimal word length $\ell(w)$ of a word $w$ over $X$ that represents $w$. A reduced expression of $w$ is any word representative of $w$ of word length $\ell(w)$.

We introduce an algorithm that produces a word $RE(w)$ over $X$ for a given matrix $w$ in $G(e,e,n)$. Later on, we prove that $RE(w)$ is a reduced expression over $X$ of $w$.

**Algorithm 1:** A word over $X$ corresponding to an element $w \in G(e,e,n)$.

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**Input**: $w$, a matrix in $G(e,e,n)$.

**Output**: $RE(w)$, a word over $X$.

**Local variables**: $w'$, $RE(w)$, $i$, $U$, $c$, $k$.

**Initialisation**: $U := [1, \zeta_e, \zeta_e^2, \ldots, \zeta_e^{e-1}]$, $s_2 := t_0$, $s_3 := t_0$, $RE(w) = \varepsilon$: the empty word, $w' := w$.

**for** $i$ **from** $n$ **down to** $2$ **do**

$c := 1$; $k := 0$;  
**while** $w'[i,c] = 0$ **do**

$c := c + 1$;  
**end**

*Then* $w'[i,c]$ *is the root of unity on row* $i$;  
**while** $U[k+1] \neq w'[i,c]$ **do**

$k := k + 1$;  
**end**

*Then* $w'[i,c] = \zeta_e^k$.  
**if** $k \neq 0$ **then**

$w' := w's_cs_c-1 \cdots s_3s_2t_k$; *Then* $w'[i,2] = 1$;  
$RE(w) := t_k s_2 s_3 \cdots s_c RE(w)$;  
$c := 2$;  
**end**

$w' := w's_{c+1} \cdots s_i$; *Then* $w'[i,i] = 1$;  
$RE(w) := s_i \cdots s_{c+1} RE(w)$;  
**end**

**Return** $RE(w)$;
Example 3.2. We apply Algorithm 1 to \( w := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in G(3, 3, 4). \)

Step 1 \((i = 4, k = 0, c = 1): \) \( w' := ws_2s_3s_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

Step 2 \((i = 3, k = 1, c = 2): \) \( w' := w's_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \zeta_3^2 & 0 & 0 \\ \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \)

then \( w' := w't_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \) then \( w' := w's_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \)

Step 3 \((i = 2, k = 0, c = 1): \) \( w' := w's_2 = I_4. \)

Hence \( RE(w) = s_2s_3t_1s_2s_4s_3s_2 = t_0s_3t_1t_0s_4s_3t_0. \)

Let \( w_n := w \in G(e, e, n). \) For \( i \) from \( n \) to \( 2, \) the \( i \)-th step of Algorithm 1 transforms the block diagonal matrix \( \begin{pmatrix} \frac{w_i}{0} & 0 \\ 0 & I_{n-i} \end{pmatrix} \) into a block diagonal matrix \( \begin{pmatrix} \frac{w_{i-1}}{0} & 0 \\ 0 & I_{n-i+1} \end{pmatrix} \in G(e, e, n) \) with \( w_1 = I_1. \) Actually, for \( 2 \leq i \leq n, \) there exists a unique \( c \) with \( 1 \leq c \leq n \) such that \( w_i[i, c] \neq 0. \) At each step \( i \) of Algorithm 1 if \( w_i[i, c] = 1, \) we shift it into the diagonal position \( [i, i] \) by right multiplication by transpositions of the symmetric group \( S_n. \) If \( w_i[i, c] \neq 1, \) we shift it into the first column by right multiplication by transpositions, transform it into 1 by right multiplication by an element of \( \{t_0, t_1, \cdots, t_{n-1}\}, \) and then shift the 1 obtained into the diagonal position \( [i, i] \). The following lemma is straightforward.

Lemma 3.3. For \( 2 \leq i \leq n, \) the block \( w_{i-1} \) is obtained by

- removing the row \( i \) and the column \( c \) from \( w_i, \) then by
- multiplying the first column of the new matrix by \( w_i[i, c]. \)

Example 3.4. Let \( w \) be as in Example 3.2. The block \( w_{n-1} \) is obtained by removing the row \( n \) and the column \( 1 \) from \( w_n = w \) to obtain \( \begin{pmatrix} 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{pmatrix}, \) then by multiplying the first column of this matrix by 1. Check that the block \( w_{n-1} \) obtained in Example 3.2 after multiplying \( w \) by \( t_0s_3s_4, \) is the same as the one obtained here. The same can be said for the other blocks \( w_i \) with \( 2 \leq i \leq n - 1. \)

Definition 3.5. At each step \( i \) from \( n \) to \( 2, \)
Theorem 3.6. We have $R E(w) = R E_2(w) R E_3(w) \cdots R E_n(w)$.

Proof. The output $R E(w)$ of Algorithm 1 is a concatenation of the words $R E_2(w), R E_3(w), \ldots, R E_n(w)$ obtained at each step $i$ from $n$ to 2 of Algorithm 1.

Example 3.7. If $w$ is defined as in Example 3.2, we have $R E(w) = t_0 s_3 t_1 t_0 s_3 s_0 t_0$. Hence $R E(w)$ is a word representative over $X$ of $w \in G(e, e, n)$.

Proposition 3.8. The word $R E(w)$ given by Algorithm 1 is a word representative over $X$ of $w \in G(e, e, n)$.

Proof. Algorithm 1 transforms the matrix $w$ into $I_n$ by multiplying it on the right by elements of $X$ and $R E(w)$ is a concatenation (in reverse order) of elements of $X$ corresponding to the matrices of $X$ used to transform $w$ into $I_n$. Hence $R E(w)$ is a word representative over $X$ of $w \in G(e, e, n)$.

The following proposition will prepare us to prove that the output of Algorithm 1 is a reduced expression over $X$ of a given element $w \in G(e, e, n)$.

Proposition 3.9. Let $w$ be an element of $G(e, e, n)$. For all $x \in X$, we have

$$|\ell(R E(xw)) - \ell(R E(w))| = 1.$$ 

Proof. For $1 \leq i \leq n$, there exists unique $c_i$ such that $w[i, c_i] \neq 0$. We denote $w[i, c_i]$ by $a_i$.

Case 1: Suppose $x = s_i$ for $3 \leq i \leq n$.

Since $s_i^2 w = w$, we can assume without restriction that $c_{i-1} < c_i$. Set $w' := s_i w$. Since the left multiplication by the matrix $x$ exchanges the rows $i - 1$ and $i$ of $w$ and the other rows remain the same, by Definition 3.5 and Lemma 3.3 we have:

$$R E_{i+1}(xw) R E_{i+2}(xw) \cdots R E_n(xw) = R E_{i+1}(w) R E_{i+2}(w) \cdots R E_n(w) \text{ and}$$

$$R E_2(xw) R E_3(xw) \cdots R E_{i-2}(xw) = R E_2(w) R E_3(w) \cdots R E_{i-2}(w).$$

Then, in order to prove our property, we should compare $\ell_1 := \ell(R E_{i-1}(w) R E_i(w))$
and $\ell_2 := \ell(\text{RE}_{i-1}(xw)\text{RE}_i(xw))$.

Since $c_{i-1} < c_i$, by Lemma 3.3 the rows $i-1$ and $i$ of the blocks $w_i$ and $w'_i$ are of the form:

$w_i : \begin{array}{c|c|c|c|c|c} & \cdots & c & \cdots & c' & \cdots & i \\ \hline i & b_{i-1} \\
\end{array}$

$w'_i : \begin{array}{c|c|c|c|c|c} & \cdots & c & \cdots & c' & \cdots & i \\ \hline i & a_i \\
\end{array}$

with $c < c'$ and where we write $b_{i-1}$ instead of $a_{i-1}$ since $a_{i-1}$ is likely to change when applying Algorithm 1 if $c_{i-1} = 1$, that is $a_{i-1}$ on the first column of $w$.

We will discuss different cases depending on the values of $a_i$ and $b_{i-1}$.

- **Suppose $a_i = 1$.**
  
  - If $b_{i-1} = 1$, we have $\text{RE}_i(w) = s_i \cdots s_{c'+2}s_{c'+1}$ and $\text{RE}_{i-1}(w) = s_{i-1} \cdots s_{c+2}s_{c+1}$.
    Furthermore, we have $\text{RE}_i(xw) = s_i \cdots s_{c+2}s_{c+1}$ and $\text{RE}_{i-1}(xw) = s_{i-1} \cdots s_{c'+1}s_{c'}$.
    It follows that $\ell_1 = ((i-1) - (c+1) + 1) + (i - (c'+1) + 1) = 2i - c - c' - 1$ and $\ell_2 = ((i-1) - c' + 1) + (i - (c+1) + 1) = 2i - c - c'$ hence $\ell_2 = \ell_1 + 1$.

  - If $b_{i-1} = \zeta^k$ with $1 \leq k \leq e - 1$, we have $\text{RE}_i(w) = s_i \cdots s_{c'+2}s_{c'+1}$ and $\text{RE}_{i-1}(w) = s_{i-1} \cdots s_{3t_kt_0s_3s_{c}}$.
    Furthermore, we have $\text{RE}_i(xw) = s_i \cdots s_{3t_kt_0s_3s_{c}}$ and $\text{RE}_{i-1}(xw) = s_{c'} \cdots s_{i-1}$.
    It follows that $\ell_1 = (((i-1) - 3 + 1) + 2 + (c - 3 + 1)) + (i - (c'+1) + 1) = 2i + c - c' - 3$ and $\ell_2 = ((i-1) - c'+1) + ((i-3+1) + 2 + (c-3+1)) = 2i + c - c' - 2$ hence $\ell_2 = \ell_1 + 1$.

It follows that

\[
\text{if } a_i = 1, \text{ then } \ell(\text{RE}(s_iw)) = \ell(\text{RE}(w)) + 1. \quad (a)
\]

- **Suppose now that $a_i = \zeta^k$ with $1 \leq k \leq e - 1$.**
  
  - If $b_{i-1} = 1$, we have $\text{RE}_i(w) = s_i \cdots s_{3t_kt_0s_3s_{c'}}$ and $\text{RE}_{i-1}(w) = s_{i-1} \cdots s_{c+1}$.
    Also, we have $\text{RE}_i(xw) = s_i \cdots s_{c+1}$ and $\text{RE}_{i-1}(xw) = s_{i-1} \cdots s_{3t_kt_0s_3s_{c'-1}}$.
    It follows that $\ell_1 = ((i-1) - (c+1) + 1) + ((i-3+1) + 2 + (c' - 3 + 1)) = 2i - c + c' - 5$ and $\ell_2 = ((i-1) - 3 + 1) + 2 + ((c' - 1) - 3 + 1) + (i - (c+1) - 1) = 2i - c + c' - 6$ hence $\ell_2 = \ell_1 - 1$. 

\[12\]
If \( b_{i-1} = \zeta^k \) with \( 1 \leq k \leq e - 1 \), we have \( \text{RE}_i(w) = s_i \cdots s_3 t_k t_0 s_3 s_c \) and
\[
\text{RE}_{i-1}(w) = s_i \cdots s_3 t_k t_0 s_3 s_c.
\]
Also, we have \( \text{RE}_i(xw) = s_i \cdots s_3 t_k t_0 s_3 s_c \) and
\[
\text{RE}_{i-1}(xw) = s_i \cdots s_3 t_k t_0 s_3 s_c - 1.
\]
It follows that \( \ell_1 = ((i - 1) - 3) + 2 + (c - 3) + 2 + (c' - 3) + 2 = 2i + c + c' - 5 \) and \( \ell_2 = ((i - 1) - 3 + 1) + 2 + ((c' - 1) - 3 + 1) + (i - 3 + 1) + 2 + (c - 3 + 1) = 2i + c + c' - 6 \) hence \( \ell_2 = \ell_1 - 1 \).

It follows that
\[
\text{if } a_i \neq 1, \text{ then } \ell(\text{RE}(s_i w)) = \ell(\text{RE}(w)) - 1. \tag{b}
\]

Case 2: Suppose \( x = t_i \) for \( 0 \leq i \leq e - 1 \).

Set \( w' := t_i w \). By definition of the left multiplication by \( t_i \), we have that the last \( n - 2 \) rows of \( w \) and \( w' \) are the same. Hence, by Definition \( 3.5 \) and Lemma \( 3.3 \) we have:
\[
\text{RE}_3(xw) \text{RE}_4(xw) \cdots \text{RE}_n(xw) = \text{RE}_3(w) \text{RE}_4(w) \cdots \text{RE}_n(w).
\]
In order to prove our property in this case, we should compare \( \ell_1 := \ell(\text{RE}_2(w)) \) and \( \ell_2 := \ell(\text{RE}_2(xw)) \).

- Consider the case where \( c_1 < c_2 \).

Since \( c_1 < c_2 \), by Lemma \( 3.3 \), the blocks \( w_2 \) and \( w'_2 \) are of the form:
\[
w_2 = \begin{pmatrix} b_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad w'_2 = \begin{pmatrix} 0 & \zeta^{-i} a_2 \\ \zeta^i b_1 & 0 \end{pmatrix}
\]
with \( b_1 \) instead of \( a_1 \) since \( a_1 \) is likely to change when applying Algorithm \( \Pi \) if \( c_1 = 1 \).

  - Suppose \( a_2 = 1 \), we have \( b_1 = 1 \) necessarily hence \( \ell_1 = 0 \). Since \( \text{RE}_2(xw) = t_i \), we have \( \ell_2 = 1 \). It follows that when \( c_1 < c_2 \),
    \[
    \text{if } a_2 = 1, \text{ then } \ell(\text{RE}(t_i w)) = \ell(\text{RE}(w)) + 1. \tag{c}
    \]

  - Suppose \( a_2 = \zeta^k \) with \( 1 \leq k \leq e - 1 \), we have \( \text{RE}_2(w) = t_k t_0 \). Thus \( \ell_1 = 2 \). We also have \( \ell_2 = 1 \) for any \( b_1 \). It follows that when \( c_1 < c_2 \),
    \[
    \text{if } a_2 \neq 1, \text{ then } \ell(\text{RE}(t_i w)) = \ell(\text{RE}(w)) - 1. \tag{d}
    \]

- Now, consider the case where \( c_1 > c_2 \).

Since \( c_1 > c_2 \), by Lemma \( 3.3 \), the blocks \( w_2 \) and \( w'_2 \) are of the form:
\[
w_2 = \begin{pmatrix} 0 & a_1 \\ b_2 & 0 \end{pmatrix} \quad \text{and} \quad w'_2 = \begin{pmatrix} \zeta^{-i} b_2 & 0 \\ 0 & \zeta^i a_1 \end{pmatrix}
\]
with \( b_2 \) instead of \( a_2 \) since \( a_2 \) is likely to change when applying Algorithm \( \Pi \) if \( c_2 = 1 \).
Suppose $a_1 \neq \zeta_e^{-1}$. By Proposition 3.9, we have $\ell = 1$ necessarily for any $b_2$, and since $\zeta_e^i a_1 \neq 1$, we have $\ell = 2$.

Hence when $c_1 > c_2$,

- if $a_1 \neq \zeta_e^{-1}$, then $\ell(\text{RE}(t_iw)) = \ell(\text{RE}(w)) + 1$. (e)

Suppose $a_1 = \zeta_e^{-1}$, we have $\ell = 1$ and $\ell = 0$ for any $b_2$. Hence when $c_1 > c_2$,

- if $a_1 = \zeta_e^{-1}$, then $\ell(\text{RE}(t_iw)) = \ell(\text{RE}(w)) - 1$. (f)

This finishes our proof.

**Proposition 3.10.** Let $w$ be an element of $G(e, c, n)$. The word $\text{RE}(w)$ is a reduced expression over $X$ of $w$.

**Proof.** We must prove that $\ell(w) = \ell(\text{RE}(w))$.

Let $x_1x_2 \cdots x_r$ be a reduced expression over $X$ of $w$. Hence $\ell(w) = \ell(x_1x_2 \cdots x_r) = r$. Since $\text{RE}(w)$ is a word representative over $X$ of $w$, we have $\ell(\text{RE}(w)) \geq \ell(x_1x_2 \cdots x_r) = r$. We prove that $\ell(\text{RE}(w)) \leq r$. Observe that we can write $w$ as $x_1, x_2, \ldots, x_r$, where $x_1, x_2, \ldots, x_r$ are the matrices of $G(e, c, n)$ corresponding to $x_1, x_2, \ldots, x_r$.

By Proposition 3.9, we have:

\[ \ell(\text{RE}(w)) = \ell(\text{RE}(x_1x_2 \cdots x_r)) \leq \ell(\text{RE}(x_2x_3 \cdots x_r)) + 1 \leq \ell(\text{RE}(x_3 \cdots x_r)) + 2 \leq \cdots \leq r. \]

Hence $\ell(\text{RE}(w)) = r = \ell(w)$ and we are done.

The following proposition is useful in the other sections. It summarizes the proof of Proposition 3.9.

**Proposition 3.11.** Let $w$ be an element of $G(e, c, n)$. Denote by $a_i$ the unique nonzero entry $w[i, c_i]$ on the row $i$ of $w$ where $1 \leq i, c_i \leq n$.

1. For $3 \leq i \leq n$, we have:
   (a) if $c_{i-1} < c_i$, then $\ell(s_{i}w) = \ell(w) - 1$ if and only if $a_i \neq 1$.
   (b) if $c_{i-1} > c_i$, then $\ell(s_{i}w) = \ell(w) - 1$ if and only if $a_{i-1} = 1$.

2. If $c_1 < c_2$, then $\forall 0 \leq k \leq e - 1$, we have $\ell(t_kw) = \ell(w) - 1$ if and only if $a_2 \neq 1$.

3. If $c_1 > c_2$, then $\forall 0 \leq k \leq e - 1$, we have $\ell(t_kw) = \ell(w) - 1$ if and only if $a_1 = \zeta_e^{-k}$.

**Proof.** The claim 1(a) is deduced from (a) and (b), 2 is deduced from (c) and (d), and 3 is deduced from (e) and (f) where (a), (b), (c), (d), (e), and (f) are given in the proof of Proposition 3.9. Since $s_i^2 = 1$, 1(b) can be deduced from 1(a).
The next proposition is about elements of $G(e, e, n)$ that are of maximal length.

**Proposition 3.12.** The maximal length of an element of $G(e, e, n)$ is $n(n - 1)$. It is realized for diagonal matrices $w$ such that $w[i, i]$ is an $e$-th root of unity different from 1 for $2 \leq i \leq n$. A minimal word representative of such an element is then of the form $t_k t_0 s_3 t_k t_0 s_3 \cdots s_3 t_k t_0 s_3 \cdots s_n$ with $1 \leq k_2, \cdots, k_n \leq (e - 1)$, and the number of elements that are of maximal length is $(e - 1)^{(n-1)}$.

**Proof.** By Algorithm 1 an element $w$ in $G(e, e, n)$ is of maximal length when $w[i, i] = \zeta^k_e$ for $2 \leq i \leq n$ and $\zeta^k_e \neq 1$. By Lemma 3.3 this condition is satisfied when $w$ is a diagonal matrix such that $w[i, i]$ is an $e$-th root of unity different from 1 for $2 \leq i \leq n$. A minimal word representative given by Algorithm 1 for such an element is of the form $t_k t_0 s_3 t_k t_0 s_3 \cdots s_3 t_k t_0 s_3 \cdots s_n$ ($1 \leq k_2, \cdots, k_n \leq (e - 1)$) which is of length $2 + 4 + \ldots + 2(n - 1) = n(n - 1)$. The number of elements of this form is $(e - 1)^{(n-1)}$.

**Example 3.13.** Consider $\lambda := \begin{pmatrix} (\zeta^{-1}_e)^{(n-1)} \\ \ldots \\ \zeta_e \\ \ldots \\ \zeta_e \\ \end{pmatrix} \in G(e, e, n)$. We have $RE(\lambda) = t_1 t_0 s_3 t_1 t_0 s_3 \cdots s_3 t_1 t_0 s_3 \cdots s_n$. Hence $\ell(\lambda) = n(n - 1)$ which is the maximal length of an element of $G(e, e, n)$.

Note that $\lambda$ is the image in $G(e, e, n)$ of $\Lambda$, the Garside element of the presentation of Corran and Picantin of the complex braid group $B(e, e, n)$.

Once the length function over $X$ is understood, we can construct intervals in $G(e, e, n)$ and characterize the balanced elements that are of maximal length.

## 4 Balanced elements of maximal length

The aim of this section is to prove that the only balanced elements of $G(e, e, n)$ that are of maximal length are $\lambda^k$ with $1 \leq k \leq e - 1$, where $\lambda$ was given in Section 3. This is done by characterizing the intervals of the elements that are of maximal length. We start by recalling the partial order relations on $G(e, e, n)$.

**Definition 4.1.** Let $w, w' \in G(e, e, n)$. We say that $w'$ is a divisor of $w$ or $w$ is a multiple of $w'$, and write $w' \leq w$, if $w = w'w''$ with $w'' \in G(e, e, n)$ and $\ell(w) = \ell(w') + \ell(w'')$. This defines a partial order relation on $G(e, e, n)$.

Similarly, we have another partial order relation on $G(e, e, n)$.

**Definition 4.2.** Let $w, w' \in G(e, e, n)$. We say that $w'$ is a left divisor of $w$ or $w$ is a left multiple of $w'$, and write $w' \leq_l w$, if there exists $w'' \in G(e, e, n)$ such that $w = w''w'$ and $\ell(w) = \ell(w'') + \ell(w')$.

The following lemma is straightforward.
Lemma 4.3. Let \( w, w' \in G(e, e, n) \) and let \( x_1 x_2 \cdots x_r \) be a reduced expression of \( w' \) over \( X \). We have \( w' \leq w \) if and only if \( \forall \ 1 \leq i \leq r, \ell(x_i x_{i-1} \cdots x_1 w) = \ell(x_{i-1} \cdots x_1 w) - 1 \).

Proof. On the one hand, we have \( w'w'' = w \) with \( w'' = x_r x_{r-1} \cdots x_1 w \) and the condition \( \forall \ 1 \leq i \leq r, \ell(x_i x_{i-1} \cdots x_1 w) = \ell(x_{i-1} \cdots x_1 w) - 1 \) implies that \( \ell(w'') = \ell(w) - r \). So we get \( \ell(w'') + \ell(w') = \ell(w) \). Hence \( w' \leq w \).

On the other hand, since \( x^2 = 1 \) for all \( x \in X \), we have \( \ell(xw) = \ell(w) \pm 1 \) for all \( w \in G(e, e, n) \). If there exists \( i \) such that \( \ell(x_i x_{i-1} \cdots x_1 w) = \ell(x_{i-1} \cdots x_1 w) + 1 \) with \( 1 \leq i \leq r \), then \( \ell(w'') = \ell(x_r x_{r-1} \cdots x_1 w) > \ell(w) - r \). It follows that \( \ell(w') + \ell(w'') > \ell(w) \). Hence \( w' \not\leq w \).

Consider the homomorphism \( - : X^* \to G(e, e, n) : x \mapsto x := x \in X \). If \( RE(w) = x_1 x_2 \cdots x_r \) with \( w \in G(e, e, n) \) and \( x_1, x_2, \ldots, x_r \in X \), then \( RE(w) = x_1 x_2 \cdots x_r = w \) where \( x_1, x_2, \ldots, x_r \in X \).

**Definition 4.4.** We define \( D \) to be the set
\[
\{ w \in G(e, e, n) \text{ s.t. } RE_i(w) \leq RE_i(\lambda) \text{ for } 2 \leq i \leq n \}.
\]

**Proposition 4.5.** The set \( D \) consists of elements \( w \) of \( G(e, e, n) \) such that for all \( 2 \leq i \leq n \), \( RE_i(w) \) is one of the following words
\[
\begin{align*}
& s_i \cdots s_3 t_1 t_0 s_3 \cdots s_{i'} \quad \text{with } 3 \leq i' \leq i, \\
& s_i \cdots s_3 t_1 t_0, \\
& s_i \cdots s_3 t_k \quad \text{with } 0 \leq k \leq e - 1, \text{ and} \\
& s_i \cdots s_{i'} \quad \text{with } 3 \leq i' \leq i.
\end{align*}
\]

Proof. We have \( RE_i(\lambda) = s_i \cdots s_3 t_1 t_0 s_3 \cdots s_i \). Let \( w \in G(e, e, n) \). Note that \( RE_i(w) \) is necessarily one of the words given in the first column of the following table. For each \( RE_i(w) \), there exists unique \( w' \in G(e, e, n) \) with \( RE(w') \) given in the second column, such that \( RE(w') w' = RE(\lambda) \). In the last column, we compute \( \ell(RE_i(w')) + \ell(w') \) that is equal to \( \ell(RE_i(\lambda)) = 2(i - 1) \) only for the first four cases. This result follows immediately.

| \( RE_i(w) \) | \( RE(w') \) | \( \ell(RE_i(w')) + \ell(w') \) |
|----------------|----------------|---------------------|
| \( s_i \cdots s_3 t_1 t_0 s_3 \cdots s_{i'} \) with \( 3 \leq i' \leq i \) | \( s_{i'+1} \cdots s_i \) | \( 2(i - 1) \) |
| \( s_i \cdots s_3 t_1 t_0 \) | \( s_3 \cdots s_i \) | \( 2(i - 1) \) |
| \( s_i \cdots s_3 t_k \) with \( 0 \leq k \leq e - 1 \) | \( t_k \cdots s_3 \cdots s_i \) | \( 2(i - 1) \) |
| \( s_i \cdots s_{i'} \) with \( 3 \leq i' \leq i \) | \( s_{i'-1} \cdots s_3 t_1 t_0 s_3 \cdots s_i \) | \( 2(i - 1) \) |
| \( s_i \cdots s_3 t_k t_0 s_3 \cdots s_{i'} \) with \( 2 \leq k \leq e - 1 \) | \( t_0 t_k \cdots s_3 \cdots s_i \) | \( 2i \) |
| \( s_i \cdots s_3 t_k t_0 s_3 \cdots s_{i'} \) with \( 2 \leq k \leq e - 1 \), and \( 3 \leq i' \leq i \) | \( s_i \cdots s_3 t_0 t_{k-1} s_3 \cdots s_i \) | \( 2(i - 1) \) |

The next proposition characterizes the divisors of \( \lambda \) in \( G(e, e, n) \).

**Proposition 4.6.** The set \( D \) is equal to the interval \([1, \lambda]\) where
\[
[1, \lambda] = \{ w \in G(e, e, n) \text{ s.t. } 1 \leq w \leq \lambda \}.
\]
Proposition 3.11, it follows that the left multiplication of $t$ decreases the length. Note that by these left multiplications, the length. Hence $w$ by $\lambda$. Since $\lambda[i,i] = \zeta_{e}$ ($\neq 1$), by 1(a) of Proposition 3.11, the left multiplication of $\alpha_{i}\lambda$ by $s_{3}\cdots s_{i}$ decreases the length. Also, by 2 of Proposition 3.11 the left multiplication of $s_{3}\cdots s_{i}\alpha_{i}\lambda$ by $t_{1}$ decreases the length. Note that by these left multiplications, $\lambda[i,i] = \zeta_{e}$ is shifted to the first row then transformed to $\zeta_{e}\zeta_{e}^{-1} = 1$. Hence, by 1(b) of Proposition 3.11 the left multiplication of $t_{1}s_{3}\cdots s_{i}\alpha_{i}\lambda$ by $s_{i}\cdots s_{3}t_{0}$ decreases the length, as desired.

Suppose that $RE_{i}(w) = s_{i}\cdots s_{3}t_{k}$ with $0 \leq k \leq e - 1$. We have $RE_{i}(w) = t_{k}s_{3}\cdots s_{i}$. Since $\lambda[i,i] = \zeta_{e}$ ($\neq 1$), by 1(a) of Proposition 3.11 the left multiplication of $\alpha_{i}\lambda$ by $s_{3}\cdots s_{i}$ decreases the length. By 2 of Proposition 3.11 the left multiplication of $s_{3}\cdots s_{i}\alpha_{i}\lambda$ by $t_{k}$ also decreases the length. Hence, applying Lemma 4.3 we have $w \prec \lambda$.

Conversely, suppose that $w \notin D$, we prove that $w \notin \lambda$. If $RE(w) = x_{1}\cdots x_{r}$, by Lemma 4.3 we show that there exists $1 \leq i \leq r$ such that $\ell(x_{i},x_{i-1}\cdots x_{1}\lambda) = \ell(x_{i-1}\cdots x_{1}\lambda) + 1$. Since $w \notin D$, by Proposition 4.5 we may consider the first $RE_{i}(w)$ that appears in $RE(w) = RE_{2}(w)\cdots RE_{n}(w)$ such that $RE_{i}(w) = s_{i}\cdots s_{3}t_{k}t_{0}$ or $s_{i}\cdots s_{3}t_{k}t_{0}s_{3}\cdots s_{i}$, with $2 \leq k \leq e - 1$ and $3 \leq i' \leq i$. Thus, we have $RE_{i}(w) = t_{0}t_{k}s_{3}\cdots s_{i}$ or $s_{i}\cdots s_{3}t_{0}t_{k}s_{3}\cdots s_{i}$, respectively. Since $\lambda[i,i] = \zeta_{e}$ ($\neq 1$), by 1(a) of Proposition 3.11 the left multiplication of $\alpha_{i}\lambda$ by $s_{3}\cdots s_{i}$ decreases the length. By 2 of Proposition 3.11 the left multiplication of $s_{3}\cdots s_{i}\alpha_{i}\lambda$ by $t_{k}$ also decreases the length. Note that by these left multiplications, $\lambda[i,i] = \zeta_{e}$ is shifted to the first row then transformed to $\zeta_{e}\zeta_{e}^{-k} = \zeta_{e}^{1-k}$. Since $2 \leq k \leq e - 1$, we have $\zeta_{e}^{1-k} \neq 1$. By 3 of Proposition 3.11 it follows that the left multiplication of $t_{k}s_{3}\cdots s_{i}\alpha_{i}\lambda$ by $t_{0}$ increases the length. Hence $w \not< \lambda$.

We want to recognize if an element $w \in G(e,e,n)$ is in the set $D$ directly from its matrix form. For this purpose, we start by describing a matrix form for each element $w \in G(e,e,n)$. Since $w$ is a monomial matrix, there exists nonzero entries that we refer to as bullets such that the upper left-hand side of $w$ with respect to the bullets, that we denote by $Z(w)$, have all its entries zero. We denote the other side of the matrix by $Z'(w)$.
Conversely, let \( R \) be the \( \rho \). We have \( Z \) entries of \( \rho \). Let \( \rho \) be the \( s \) of \( \rho \). \( \rho \) is an element \( \rho \) with \( 3 \leq i' \leq i \). By \( \rho \) we have \( \rho[i, i'] = \rho[i, c] \) for \( 1 < i' \leq i \). Hence, by \( \rho \) we have \( \rho[i, c] = 1 \) or \( \rho \).

Conversely, let \( \rho[i, c] \in Z'(\rho) \). We have \( \rho[i, c] = 1 \) or \( \rho \). Then we have again \( RE_i(\rho) = s_i \cdots s_3 t_1 t_0, s_i \cdots s_3 t_1 t_0 s_3 \cdots s_{i'}, \) or \( s_i \cdots s_{i'} \) with \( 3 \leq i' \leq i \). If \( \rho[i, c] \) is a bullet of \( \rho \), by \( \rho \) we have \( \rho[i, 1] = \rho k \) for \( 0 \leq k \leq e - 1 \), for which case \( RE_i(\rho) = s_i \cdots s_3 t_k \). Hence, by \( \rho \) we have \( \rho \in D \).

\[ 
\begin{pmatrix} 
0 & 0 & 0 & \zeta_3^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & \zeta_3 & 0 \\
1 & 0 & 0 & 0 \\
0 & \zeta_3^2 & 0 & 0 \\
\end{pmatrix} \in G(3, 3, 5). \] 

The bullets are the encircled elements and the drawn path separates \( Z(\rho) \) from \( Z'(\rho) \).

**Remark 4.8.** Let \( \rho[i, c] \) be one of the bullets of \( \rho \in G(e, e, n) \). We have

\[ \rho[i, c] \in Z(\rho) \text{ and } \rho[i, c - 1] \in Z(\rho). \]

Also the bullets are the only nonzero entries of \( \rho \) that satisfy this condition.

The following proposition gives a nice description of the divisors of \( \lambda \) in \( G(e, e, n) \).

**Proposition 4.9.** Let \( \rho \in G(e, e, n) \). We have that \( \rho \in D \) if and only if all nonzero entries of \( Z'(\rho) \) are either \( 1 \) or \( \zeta_e \).

**Proof.** Let \( \rho \in D \) and let \( \rho[i, c] \) be a nonzero entry of \( Z'(\rho) \). Since \( \rho \in D \), by Proposition 4.8 we have \( RE_i(\rho) = s_i \cdots s_3 t_1 t_0, s_i \cdots s_3 t_1 t_0 s_3 \cdots s_{i'}, \) or \( s_i \cdots s_{i'} \) with \( 3 \leq i' \leq i \). By Lemma 3.3 we have \( \rho[i, i'] = \rho[i, c] \) for \( 1 < i' \leq i \). Hence, by Definition 3.3 we have \( \rho[i, c] = 1 \) or \( \zeta_e \).

Conversely, let \( \rho[i, c] \in Z'(\rho) \). We have \( \rho[i, c] = 1 \) or \( \rho \). Then we have again \( RE_i(\rho) = s_i \cdots s_3 t_1 t_0, s_i \cdots s_3 t_1 t_0 s_3 \cdots s_{i'}, \) or \( s_i \cdots s_{i'} \) for \( 3 \leq i' \leq i \). If \( \rho[i, c] \) is a bullet of \( \rho \), by Lemma 3.3 we have \( \rho[i, 1] = \rho k \) for \( 0 \leq k \leq e - 1 \), for which case \( RE_i(\rho) = s_i \cdots s_3 t_k \). Hence, by Proposition 4.9 we have \( \rho \in D \).

\[ 
\begin{pmatrix} 
0 & 0 & \zeta_3^2 \\
0 & \zeta_3 & 0 \\
\zeta_3 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \in G(3, 3, 4). \] 

Since \( Z'(\rho) \) contains \( \zeta_3^2 \), it follows immediately that \( \rho \notin D \).

Our description of the interval \( [1, \lambda] \) allows us to prove easily that \( \lambda \) is balanced. Let us recall the definition of a balanced element.

**Definition 4.11.** A balanced element in \( G(e, e, n) \) is an element \( \rho \) such that \( \rho' \preceq \rho \) holds precisely when \( \rho \preceq \rho' \).

The next lemma is obvious. It will be useful later.

**Lemma 4.12.** Let \( \rho \) be a balanced element and let \( \rho, \rho' \in [1, g] \). If \( \rho' \preceq \rho \), then \( (\rho')^{-1} \rho \preceq g \).

In order to prove that \( \lambda \) is balanced, we first check the following.

**Lemma 4.13.** If \( \rho \in D \), we have \( \rho^{-1} \lambda \in D \) and \( \lambda \rho^{-1} \in D \).
Proof. We show that \( w^{-1} \lambda = \overline{t_w} \lambda \in D \) and \( \lambda w^{-1} = \lambda \overline{t_w} \in D \), where \( \overline{t_w} \) is the complex conjugate of the transpose \( t_w \) of the matrix \( w \). We use the matrix form of an element of \( D \). Actually, by Remark 4.8, the bullets of \( t_w \) correspond to the complex conjugate of the bullets of \( w \). Also, all nonzero entries of \( w^{-1} \), apart from its bullets, are in \( \{1, \zeta_3^{-1}\} \). Multiplying \( w^{-1} \) by \( \lambda \), we have that all nonzero entries of \( w^{-1} \lambda \) and of \( \lambda w^{-1} \), apart from their bullets, are in \( \{ \zeta_3, 1 \} \). Hence \( w^{-1} \lambda \in D \) and \( \lambda w^{-1} \in D \).

\[ \square \]

Example 4.14. We illustrate the idea of the proof of Lemma 4.13.

Consider \( w \in D \) as follows and show that \( \overline{t_w} \lambda \in D \): \( w = \begin{pmatrix} 0 & 0 & 0 & 0 & \zeta_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \zeta_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \). Let \( \lambda w \in D \) where \( \lambda = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \).

Proposition 4.15. We have that \( \lambda \) is balanced.

Proof. Suppose that \( w \preceq \lambda \). We have \( w \in D \) then \( \lambda w^{-1} \in D \) by Lemma 4.13. Hence \( \lambda = (\lambda w^{-1}) w \) satisfies \( \ell(\lambda w^{-1}) + \ell(w) = \ell(\lambda) \), namely \( w \preceq_r \lambda \). Conversely, suppose that \( w \preceq_r \lambda \). We have \( \lambda = w^r w \) with \( w^r \in G(e, e, n) \) and \( \ell(w^r) + \ell(w) = \ell(\lambda) \). It follows that \( w^r \in D \) then \( w^r \lambda \in D \) by Lemma 4.13. Since \( w = w^{-1} \lambda \), we have \( w \in D \), namely \( w \preceq \lambda \).

\[ \square \]

We state without proof a similar result for the powers of \( \lambda \) in \( G(e, e, n) \).

Definition 4.16. For \( 1 \leq k \leq e-1 \), let \( D_k \) be the set of \( w \in G(e, e, n) \) such that all nonzero entries of \( Z'(w) \) are either 1 or \( \zeta_3^k \).

Proposition 4.17. For \( 1 \leq k \leq e-1 \), we have \( [1, \lambda^k] = D_k \) and \( \lambda^k \) is balanced.

Example 4.18. Let \( w = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \zeta_3^2 & 0 & 0 & 0 & 0 \end{pmatrix} \in G(3, 3, 5) \). We have \( w \in [1, \lambda^2] \).
More generally, let \( w \in G(e, e, n) \) be an element of maximal length, namely by Proposition 3.12 a diagonal matrix such that for \( 2 \leq i \leq n \), \( w[i, i] \) is an \( e \)-th root of unity different from 1. As previously, we can prove that a divisor \( w' \) of \( w \) satisfies that for all \( 2 \leq i \leq n \), if \( w'[i, i] \neq 0 \) is not a bullet of \( w' \), then \( w'[i, i] = 1 \) or \( w[i, i] \).

Suppose that \( w \) is of maximal length such that \( w[i, i] \neq w[j, j] \) with \( 2 \leq i \neq j \leq n \). We have \((i, j) \preceq w\) where \((i, j)\) is the transposition matrix. Hence \( w' := (i, j)^{-1}w = (i, j)w \preceq w \). We have \( w'[j, i] = w[i, i] \). Thus, \( w'[j, i] \) that is not a bullet of \( w' \) is different from 1 and \( w[j, i] \), since it is equal to \( w[i, i] \). Hence \( w' \not\equiv w \). We thus get the following.

**Proposition 4.19.** The balanced elements of \( G(e, e, n) \) that are of maximal length over \( X \) are precisely \( \lambda^k \) where \( 1 \leq k \leq e - 1 \).

We are ready to study the interval structures associated with the intervals \([1, \lambda^k]\) where \( 1 \leq k \leq e - 1 \).

## 5 Interval structures

In this section, we construct the monoid \( M([1, \lambda^k]) \) associated with each of the intervals \([1, \lambda^k]\) constructed in Section 4 with \( 1 \leq k \leq e - 1 \). By Proposition 4.19 \( \lambda^k \) is balanced. Hence, by Theorem 1.7, in order to prove that \( M([1, \lambda^k]) \) is a Garside monoid, it remains to show that both posets \(([1, \lambda^k], \preceq)\) and \(([1, \lambda^k], \preceq_{e})\) are lattices. This is stated in Corollary 5.13. The interval structure is given in Theorem 5.15.

Let \( 1 \leq k \leq e - 1 \) and let \( w \in [1, \lambda^k] \). For each \( 1 \leq i \leq n \) there exists a unique \( c_i \) such that \( w[i, c_i] \neq 0 \). We denote \( w[i, c_i] \) by \( a_i \). We apply Lemma 4.3 to prove the following lemmas. The reader is invited to write the matrix form of \( w \) to illustrate each step of the proof.

**Lemma 5.1.** Let \( t_i \preceq w \) where \( i \in \mathbb{Z}/e\mathbb{Z} \).

- If \( c_1 < c_2 \), then \( t_k t_0 \preceq w \) and
- if \( c_2 < c_1 \), then \( t_j \not\equiv w \) for all \( j \) with \( j \neq i \).

Hence if \( t_i \preceq w \) and \( t_j \preceq w \) with \( i, j \in \mathbb{Z}/e\mathbb{Z} \) and \( i \neq j \), then \( t_k t_0 \preceq w \).

**Proof.** Suppose \( c_1 < c_2 \).

Since \( w \in [1, \lambda^k] \), by Proposition 4.19 \( a_2 = 1 \) or \( \zeta_e^k \). Since \( t_i \preceq w \), we have \( \ell(t_i w) = \ell(w) - 1 \). Hence by 2 of Proposition 3.11 we get \( a_2 \neq 1 \). Hence \( a_2 = \zeta_e^k \). Again by 2 of Proposition 3.11 since \( a_2 \neq 1 \), we have \( \ell(t_k w) = \ell(w) - 1 \). Let \( w' := t_k w \). We have \( w'[1, c_2] = \zeta_e^{-k} a_2 = \zeta_e^{-k} \zeta_e^k = 1 \). Hence by 3 of Proposition 3.11 \( \ell(t_0 w') = \ell(w') - 1 \). It follows that \( t_k t_0 \preceq w \).

Suppose \( c_2 < c_1 \).

Since \( t_i \preceq w \), we have \( \ell(t_i w) = \ell(w) - 1 \). Hence by 3 of Proposition 3.11 we have \( a_1 = \zeta_e^{-i} \). If there exists \( j \in \mathbb{Z}/e\mathbb{Z} \) with \( j \neq i \) such that \( t_j \preceq w \), then \( \ell(t_j w) = \ell(w) - 1 \). Again by 3 of Proposition 3.11 we have \( a_1 = \zeta_e^{-j} \). Thus, \( i = j \) which is not allowed.

The last statement of the lemma follows immediately. □
Lemma 5.2. If $t_i \leq w$ with $i \in \mathbb{Z}/e\mathbb{Z}$ and $s_3 \leq w$, then $s_3 t_i s_3 = t_i s_3 t_i \leq w$.

Proof. Set $w' := s_3 w$ and $w'' := t_i w$.

Suppose $w \in [1, \lambda^k]$, by Proposition 4.9, we have $a_2 = 1$ or $\zeta_e^k$. Since $t_i \leq w$, we have $\ell(t_i w) = \ell(w) - 1$. Thus, by 2 of Proposition 3.11, we get $a_2 \neq 1$. Hence $a_2 = \zeta_e^k$. Suppose that $c_3 < c_2$. Since $s_3 \leq w$, we have $\ell(s_3 w) = \ell(w) - 1$. Hence by 1(b) of Proposition 3.11, $a_2 = 1$ which is not allowed. Then, assume $c_2 < c_3$. Since $w \in [1, \lambda^k]$, we have $a_3 = 1$ or $\zeta_e^k$. By 1(a) of Proposition 3.11, we have $a_3 \neq 1$. Hence $a_3 = \zeta_e^k$.

Now, we prove that $s_3 t_i s_3 \leq w$ by applying Lemma 5.3. Indeed, since $a_3 \neq 1$, by 2 of Proposition 3.11, $\ell(t_i w') = \ell(w') - 1$, and since $a_2 \neq 1$, by 1(a) of Proposition 3.11, we have $\ell(s_3 w'') = \ell(w'') - 1$.

Suppose $c_2 < c_1$.

Since $\ell(t_i w) = \ell(w) - 1$, by 3 of Proposition 3.11, we have $a_1 = \zeta_e^{-i}$.

- Assume $c_2 < c_3$.
  Since $\ell(s_3 w) = \ell(w) - 1$, by 1(a) of Proposition 3.11, we have $a_3 \neq 1$. We have $\ell(t_i w') = \ell(w') - 1$ for both cases $c_1 < c_3$ and $c_3 < c_1$. Actually, if $c_1 < c_3$, since $a_3 \neq 1$, by 2 of Proposition 3.11, we have $\ell(t_i w'' = \ell(w') - 1$, and if $c_3 < c_1$, since $a_1 = \zeta_e^{-i}$, by 3 of Proposition 3.11, $\ell(t_i w') = \ell(w') - 1$. Now, since $\zeta_e a_1 = \zeta_e^{-i} = 1$, by 1(b) of Proposition 3.11, we have $\ell(s_3 w'') = \ell(w'') - 1$.

- Assume $c_3 < c_2$.
  Since $a_1 = \zeta_e^{-i}$, by 3 of Proposition 3.11, $\ell(t_i w') = \ell(w') - 1$. Since $\zeta_e a_1 = 1$, by 1(b) of Proposition 3.11, we have $\ell(s_3 w'') = \ell(w'') - 1$.

Lemma 5.3. If $t_i \leq w$ with $i \in \mathbb{Z}/e\mathbb{Z}$ and $s_j \leq w$ with $4 \leq j \leq n$, then $t_i s_j = s_j t_i \leq w$.

Proof. We distinguish four different cases: case 1: $c_1 < c_2$ and $c_j - 1 < c_j$, case 2: $c_1 < c_2$ and $c_j < c_j - 1$, case 3: $c_2 < c_1$ and $c_j - 1 < c_j$, and case 4: $c_2 < c_1$ and $c_j < c_j - 1$. The proof is similar to the proofs of Lemmas 5.1 and 5.2, so we prove that $s_j t_i \leq w$ only for the first case. Suppose that $c_1 < c_2$ and $c_j - 1 < c_j$. Since $t_i \leq w$, we have $\ell(t_i w) = \ell(w) - 1$. Hence, by 2 of Proposition 3.11, we have $a_2 \neq 1$. Also, since $s_j \leq w$, we have $\ell(s_j w) = \ell(w) - 1$. Hence, by 1(a) of Proposition 3.11, we have $a_j \neq 1$. Set $w' := s_j w$. Since $a_2 \neq 1$, then $\ell(t_i w') = \ell(w') - 1$. Hence $s_j t_i \leq w$.

The proof of the following lemma is similar to the proofs of Lemmas 5.2 and 5.3 and is left to the reader.

Lemma 5.4. If $s_i \leq w$ and $s_{i+1} \leq w$ for $3 \leq i \leq n - 1$, then $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \leq w$, and if $s_i \leq w$ and $s_j \leq w$ for $3 \leq i, j \leq n$ and $|i - j| > 1$, then $s_i s_j = s_j s_i \leq w$.

The following proposition is a direct consequence of all the preceding lemmas.

Proposition 5.5. Let $x, y \in X = \{t_0, t_1, \cdots, t_{e-1}, s_3, \cdots, s_n\}$. The least common multiple in $([1, \lambda^k], \preceq)$ of $x$ and $y$, denoted by $x \vee y$, exists and is given by the following identities:
• $t_i \lor t_j = t_k t_0 = t_i t_{i-k} = t_j t_{j-k}$ for $i \neq j \in \mathbb{Z}/e\mathbb{Z}$,

• $t_i \lor s_3 = t_i s_3 t_i = s_3 t_i s_3$ for $i \in \mathbb{Z}/e\mathbb{Z}$,

• $t_i \lor s_j = t_i s_j = s_j t_i$ for $i \in \mathbb{Z}/e\mathbb{Z}$ and $4 \leq j \leq n$,

• $s_i \lor s_{i+1} = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $3 \leq i \leq n-1$, and

• $s_i \lor s_j = s_i s_j = s_j s_i$ for $3 \leq i \neq j \leq n$ and $|i-j| > 1$.

We have a similar result for $([1, \lambda^k], \preceq_r)$.

**Proposition 5.6.** Let $x, y \in X$. The least common multiple in $([1, \lambda^k], \preceq_r)$ of $x$ and $y$, denoted by $x \lor r y$, exists and is equal to $x \lor y$.

**Proof.** Define an antihomomorphism $\phi : G(e, e, n) \to G(e, e, n) : t_i \mapsto t_{-i}, s_j \mapsto s_j$ with $i \in \mathbb{Z}/e\mathbb{Z}$ and $3 \leq j \leq n$. It is obvious that $\phi^2$ is the identity. Also, if $w \in G(e, e, n)$, we have $\ell(\phi(w)) = \ell(w)$. Let $x, y \in X$ and $w \in [1, \lambda^k]$. We prove that if $x \preceq_r w$ and $y \preceq_r w$, then $x \lor y \preceq_r w$. Actually, we have $w = vxw$ and $w = w'y$ with $v, v' \in G(e, e, n)$. Thus, $\phi(w) = \phi(x)\phi(v)$ and $\phi(w) = \phi(y)\phi(v')$. Since $\phi \circ \phi \circ \phi = \phi$, $\phi$ respects the length over $X$ in $G(e, e, n)$, we have $\phi(x) \preceq \phi(w)$ and $\phi(y) \preceq \phi(w)$. Hence $\phi(x) \lor \phi(y) \preceq \phi(w)$. Moreover, one can check that $\phi(x) \lor \phi(y) = \phi(x \lor y)$ for all $x, y \in X$. Thus, $\phi(x \lor y) \preceq \phi(w)$, that is $\phi(w) = \phi(x \lor y)u$ for some $u \in G(e, e, n)$. We get $w = \phi(u)(x \lor y)$ and since $\phi$ respects the length function, we have $x \lor y \preceq_r w$. \hfill \Box

Note that Propositions 5.5 and 5.6 are important to prove that both posets $([1, \lambda^k], \preceq)$ and $([1, \lambda^k], \preceq_r)$ are lattices. Actually, they will make possible an induction proof of Proposition 5.12 below. For now, let us state some general properties about lattices that will be useful in our proof. Let $(S, \preceq)$ be a finite poset.

**Definition 5.7.** Say that $(S, \preceq)$ is a meet-semilattice if and only if $f \land g := \text{gcd}(f, g)$ exists for any $f, g \in S$.

Equivalently, $(S, \preceq)$ is a meet-semilattice if and only if $\bigwedge T$ exists for any finite nonempty subset $T$ of $S$.

**Definition 5.8.** Say that $(S, \preceq)$ is a join-semilattice if and only if $f \lor g := \text{lcm}(f, g)$ exists for any $f, g \in S$.

Equivalently, $(S, \preceq)$ is a join-semilattice if and only if $\bigvee T$ exists for any finite nonempty subset $T$ of $S$.

**Proposition 5.9.** $(S, \preceq)$ is a meet-semilattice if and only if for any $f, g \in S$, either $f \lor g$ exists, or $f$ and $g$ have no common multiples.

**Proof.** Let $f, g \in S$ and suppose that $f$ and $g$ have at least one common multiple. Let $A := \{ h \in S \mid f \preceq h \text{ and } g \preceq h \}$ be the set of the common multiples of $f$ and $g$. Since $S$ is finite, $A$ is also finite. Since $(S, \preceq)$ is a meet-semilattice, $\bigwedge A$ exists and $\bigwedge A = \text{lcm}(f, g) = f \lor g$.

Conversely, let $f, g \in S$ and let $B := \{ h \in S \mid h \preceq f \text{ and } h \preceq g \}$ be the set of all common divisors of $f$ and $g$. Since all elements of $B$ have common multiples, $\bigvee B$ exists and we have $\bigvee B = \text{gcd}(f, g) = f \land g$. \hfill \Box
**Definition 5.10.** The poset \((S, \preceq)\) is a lattice if and only if it is both a meet- and join-semilattice.

The following lemma is a consequence of Proposition 5.9.

**Lemma 5.11.** If \((S, \preceq)\) is a meet-semilattice such that \(\bigvee S\) exists, then \((S, \preceq)\) is a lattice.

We will prove that \(([1, \lambda^k], \preceq)\) is a meet-semilattice by applying Proposition 5.9. On occasion, for \(1 \leq m \leq \ell(\lambda^k)\) with \(\ell(\lambda^k) = n(n - 1)\), we introduce

\[
([1, \lambda^k])_m := \{w \in [1, \lambda^k] \text{ s.t. } \ell(w) \leq m\}.
\]

**Proposition 5.12.** Let \(0 \leq k \leq e - 1\). For \(1 \leq m \leq n(n - 1)\) and \(u, v\) in \([1, \lambda^k])_m\), either \(u \vee v\) exists in \([1, \lambda^k])_m\), or \(u\) and \(v\) do not have common multiples in \([1, \lambda^k])_m\).

**Proof.** Let \(1 \leq m \leq n(n - 1)\). We make a proof by induction on \(m\). By Proposition 5.3, our claim holds for \(m = 1\). Suppose \(m > 1\). Assume that the claim holds for all \(1 \leq m' \leq m - 1\). We want to prove it for \(m' = m\). The proof is illustrated in Figure 6 below. Let \(u, v\) be in \([1, \lambda^k])_m\) such that \(u\) and \(v\) have at least one common multiple in \([1, \lambda^k])_m\), which we denote by \(w\). Write \(u = xu_1\) and \(v = yv_1\) such that \(x, y \in X\) and \(\ell(u) = \ell(u_1) + 1\), \(\ell(v) = \ell(v_1) + 1\). By Proposition 5.3, \(x \vee y\) exists. Since \(x \preceq w\) and \(y \preceq w\), \(x \vee y\) divides \(w\). We can write \(x \vee y = xy_1 = yx_1\) with \(\ell(x \vee y) = \ell(x_1) + 1 = \ell(y_1) + 1\). By Lemma 5.12, we have \(x_1, y_1 \in [1, \lambda^k]\). Also, we have \(\ell(x_1) < m\), \(\ell(y_1) < m\) and \(x_1, y_1\) have a common multiple in \([1, \lambda^k])_{m-1}\). Thus, by the induction hypothesis, \(x_1 \vee y_1\) exists in \([1, \lambda^k])_{m-1}\). Similarly, \(y_1 \vee u_1\) exists in \([1, \lambda^k])_{m-1}\). Write \(x_1 \vee y_1 = x_1y_1 = x_1\) with \(\ell(x_1 \vee y_1) = \ell(x_1) + \ell(y_1) + \ell(x_1) + \ell(y_1)\). Also, we have \(\ell(u_2) < m\), \(\ell(v_2) < m\) and \(u_2, v_2\) have a common multiple in \([1, \lambda^k])_{m-1}\). Thus, by the induction hypothesis, \(u_2 \vee v_2\) exists in \([1, \lambda^k])_{m-1}\). Write \(u_2 \vee v_2 = u_2v_3 = u_2v_3\) with \(\ell(u_2 \vee v_2) = \ell(u_2) + \ell(v_2) = \ell(u_2) + \ell(v_2)\). Since \(uy_2v_3 = vx_2u_3\) is a common multiple of \(u\) and \(v\) that divides every common multiple \(w\) of \(u\) and \(v\), we deduce that \(u \vee v = uy_2v_3 = vx_2u_3\) and we are done.

\[
\begin{tikzpicture}
\node (x) at (0,0) {}; \node (y) at (0,1) {}; \node (u1) at (-1,0) {}; \node (u2) at (-1,1) {}; \node (u3) at (1,1) {}; \node (v1) at (1,0) {}; \node (v2) at (1,1) {}; \node (y1) at (0,0) {}; \node (y2) at (0,1) {}; \node (y3) at (0,2) {}; \node (w) at (2,2) {}; \draw (x) -- (u1) -- (u2) -- (u3) -- (v1) -- (v2) -- (w); \draw (x) -- (y1); \draw (x) -- (y2); \draw (y1) -- (y2); \draw (y2) -- (y3); \draw (y1) -- (v1); \draw (y1) -- (v2); \draw (y2) -- (v1); \draw (y2) -- (v2); \draw (v1) -- (v3); \draw (v2) -- (v3); \draw (v1) -- (w); \draw (v2) -- (w); \end{tikzpicture}
\]

**Figure 6:** The proof of Proposition 5.12

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Similarly, applying Proposition 5.6, we obtain the same results for \([1, \lambda^k], \preceq_r\).

We thus proved the following.

**Corollary 5.13.** Both posets \([1, \lambda^k], \preceq\) and \([1, \lambda^k], \preceq_r\) are lattices.

**Proof.** Applying Proposition 5.9, \((1, \lambda^k], \preceq\) is a meet-semilattice. Also, by definition of the interval \([1, \lambda^k]\), we have \(\bigvee [1, \lambda^k] = \lambda^k\). Thus, applying Proposition 5.9, \((1, \lambda^k], \preceq\) is a lattice. The same can be done for \((1, \lambda^k], \preceq_r\).

We are ready to define the interval monoid \(M([1, \lambda^k])\).

**Definition 5.14.** Let \(D_k\) be a set in bijection with \([1, \lambda^k]\) with

\[ [1, \lambda^k] \to D_k : w \mapsto w. \]

We define the monoid \(M([1, \lambda^k])\) by the following presentation of monoid with

- generating set: \(D_k\) (a copy of the interval \([1, \lambda^k]\)) and
- relations: \(w = w' w''\) whenever \(w, w'\) and \(w'' \in [1, \lambda^k]\), \(w = w' w''\) and \(\ell(w) = \ell(w') + \ell(w'')\).

We have that \(\lambda^k\) is balanced. Also, by Corollary 5.13 both posets \((1, \lambda^k], \preceq\) and \((1, \lambda^k], \preceq_r\) are lattices. Hence, by Theorem 1.7 we have:

**Theorem 5.15.** \(M([1, \lambda^k])\) is an interval monoid with Garside element \(\lambda^k\) and with simples \(D_k\). Its group of fractions exists and is denoted by \(G(M([1, \lambda^k]))\).

These interval structures have been implemented into GAP3, Contribution to the chevie package (see [16]). The next section is devoted to the study of these interval structures.

## 6 About the interval structures

In this section, we provide a new presentation of the interval monoid \(M([1, \lambda^k])\). Furthermore, we prove that \(G(M([1, \lambda^k]))\) is isomorphic to the complex braid group \(B(e, e, n)\) if and only if \(k \wedge e = 1\). When \(k \wedge e \neq 1\), we describe these new structures and show some of their properties.

### 6.1 Presentations

Our first aim is to prove that the interval monoid \(M([1, \lambda^k])\) is isomorphic to the monoid \(B^{\oplus k}(e, e, n)\) defined as follows.

**Definition 6.1.** For \(1 \leq k \leq e - 1\), we define the monoid \(B^{\oplus k}(e, e, n)\) by a presentation of monoid with

- generating set: \(\tilde{X} = \{\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_{e-1}, \tilde{s}_3, \cdots, \tilde{s}_n\}\) and
\[\tilde{s}_i \tilde{s}_j \tilde{s}_i = \tilde{s}_j \tilde{s}_i \tilde{s}_j \quad \text{for } |i - j| = 1,\]
\[\tilde{s}_i \tilde{s}_j = \tilde{s}_j \tilde{s}_i \quad \text{for } |i - j| > 1,\]

- relations: 
\[\tilde{s}_3 t_i \tilde{s}_3 = t_i \tilde{s}_3 t_i \quad \text{for } i \in \mathbb{Z}/ce\mathbb{Z},\]
\[\tilde{s}_j \tilde{t}_i = \tilde{t}_i \tilde{s}_j \quad \text{for } i \in \mathbb{Z}/ce\mathbb{Z} \text{ and } 4 \leq j \leq n, \text{ and}\]
\[\tilde{t}_i \tilde{t}_{i-k} = \tilde{t}_j \tilde{t}_{j-k} \quad \text{for } i, j \in \mathbb{Z}/ce\mathbb{Z}.\]

Note that the monoid \(B^{\oplus 1}(e, e, n)\) is the monoid \(B^{\oplus}(e, e, n)\) of Corran and Picantin introduced in Section 2.

The following result is similar to Matsumoto’s property in the case of real reflection groups.

**Proposition 6.2.** There exists a map \(F : [1, \lambda^k] \rightarrow B^{\oplus k}(e, e, n)\) defined by
\[F(w) = \tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_r \text{ whenever } x_1 x_2 \cdots x_r \text{ is a reduced expression over } X \text{ of } w, \text{ where } \tilde{x}_i \in \tilde{X} \text{ for } 1 \leq i \leq r.\]

**Proof.** Let \(w_1\) and \(w_2\) be in \(X^*\). We write \(w_1 \xrightarrow{B} w_2\) if \(w_2\) is obtained from \(w_1\) by applying only the relations of the presentation of \(B^{\oplus k}(e, e, n)\) where we replace \(\tilde{t}_i\) by \(t_i\) and \(\tilde{s}_j\) by \(s_j\) for all \(i \in \mathbb{Z}/ce\mathbb{Z}\) and \(3 \leq j \leq n\).

Let \(w\) be in \([1, \lambda^k]\) and suppose that \(w_1\) and \(w_2\) are two reduced expressions over \(X\) of \(w\). We prove that \(w_1 \xrightarrow{B} w_2\) by induction on \(\ell(w_1)\).

The result holds vacuously for \(\ell(w_1) = 0\) and \(\ell(w_1) = 1\). Suppose that \(\ell(w_1) > 1\). Write
\[w_1 = x_1 w'_1 \text{ and } w_2 = x_2 w'_2, \text{ with } x_1, x_2 \in X.\]

If \(x_1 = x_2\), we have \(x_1 w'_1 = x_2 w'_2\) in \(G(e, e, n)\) from which we get \(w'_1 = w'_2\). Then, by the induction hypothesis, we have \(w'_1 \xrightarrow{B} w'_2\). Hence \(w_1 \xrightarrow{B} w_2\).

If \(x_1 \neq x_2\), since \(x_1 \leq w\) and \(x_2 \leq w\), we have \(x_1 \vee x_2 \leq w\) where \(x_1 \vee x_2\) is the lcm of \(x_1\) and \(x_2\) in \(([1, \lambda^k], \leq)\) given in Proposition 5.5. Write \(w = (x_1 \vee x_2) w'\). Also, write \(x_1 \vee x_2 = x_{1V1}\) and \(x_1 \vee x_2 = x_{2V2}\) where we can check that \(x_{1V1} \xrightarrow{B} x_{2V2}\) for all possible cases for \(x_1\) and \(x_2\). All the words \(x_{1V1}, x_{2V2}, x_{1V1}, x_{2V2}\) represent \(w\). In particular, \(x_{1V1}\) and \(x_{1V1} w'\) represent \(w\). Hence \(w'_1 = v_1 w'\), and by the induction hypothesis, we have \(w'_1 \xrightarrow{B} v_1 w'\). Thus, we have
\[x_{1V1} w'_1 \xrightarrow{B} x_{1V1} w'.\]

Similarly, since \(x_{2V2} w'_1\) and \(x_{2V2} w'\) represent \(w\), we get
\[x_{2V2} w' \xrightarrow{B} x_{2V2} w'_2.\]

Since \(x_{1V1} w' \xrightarrow{B} x_{2V2} w'\), we have
\[x_{1V1} w' \xrightarrow{B} x_{2V2} w'.\]

We obtain:
\[w_1 \xrightarrow{B} x_{1V1} w'_1 \xrightarrow{B} x_{1V1} w' \xrightarrow{B} x_{2V2} w' \xrightarrow{B} x_{2V2} w'_2 \xrightarrow{B} w_2.\]

Hence \(w_1 \xrightarrow{B} w_2\) and we are done. \(\square\)
By the following proposition, we provide an alternative presentation of the interval monoid \( M([1, \lambda^k]) \) given in Definition 5.14.

**Proposition 6.3.** The monoid \( B^\oplus k(e, e, n) \) is isomorphic to \( M([1, \lambda^k]) \).

**Proof.** Consider the map \( \rho: D_k \rightarrow B^\oplus k(e, e, n) : w \mapsto F(w) \) where \( F \) is defined in Proposition 6.2. Let \( w = w'w'' \) be a defining relation of \( M([1, \lambda^k]) \). Since \( \ell(w) = \ell(w') + \ell(w'') \), a reduced expression for \( w'w'' \) is obtained by concatenating reduced expressions for \( w' \) and \( w'' \). It follows that \( F(w'w'') = F(w')F(w'') \). We conclude that \( \rho \) has a unique extension to a monoid homomorphism \( M([1, \lambda^k]) \rightarrow B^\oplus k(e, e, n) \), which we denote by the same symbol.

Conversely, consider the map \( \rho' : \tilde{X} \rightarrow M([1, \lambda^k]) : \tilde{x} \mapsto x \). In order to prove that \( \rho' \) extends to a unique monoid homomorphism \( B^\oplus k(e, e, n) \rightarrow M([1, \lambda^k]) \), we have to check that \( w_1 = w_2 \) in \( M([1, \lambda^k]) \) for any defining relation \( \tilde{w}_1 = \tilde{w}_2 \) of \( B^\oplus k(e, e, n) \). Given a relation \( \tilde{w}_1 = \tilde{w}_2 = \tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_r \) of \( B^\oplus k(e, e, n) \), we have \( w_1 = w_2 = x_1x_2 \cdots x_r \), a reduced word over \( X \). On the other hand, applying repeatedly the defining relations in \( M([1, \lambda^k]) \) yields to \( w = x_1x_2 \cdots x_r \) if \( w = x_1x_2 \cdots x_r \) is a reduced expression over \( X \). Thus, we can conclude that \( w_1 = w_2 \), as desired.

Hence we have defined two homomorphisms \( \rho: D_k \rightarrow B^\oplus k(e, e, n) \) and \( \rho' : \tilde{X} \rightarrow M([1, \lambda^k]) \) such that \( \rho \circ \rho' = id_{B^\oplus k(e, e, n)} \) and \( \rho' \circ \rho = id_{M([1, \lambda^k])} \). It follows that \( B^\oplus k(e, e, n) \) is isomorphic to \( M([1, \lambda^k]) \).

We deduce that \( B^\oplus k(e, e, n) \) is a Garside monoid and we denote by \( B^{(k)}(e, e, n) \) its group of fractions.

Fix \( k \) such that \( 1 \leq k \leq e - 1 \). A diagram presentation for \( B^{(k)}(e, e, n) \) is the same as the diagram corresponding to the presentation of Corran and Picantin of \( B(e, e, n) \) given in Figure 2.2 with a dashed edge between \( \tilde{t}_i \) and \( \tilde{t}_{i-k} \) and between \( \tilde{t}_j \) and \( \tilde{t}_{j-k} \) for each relation of the form \( \tilde{t}_i\tilde{t}_{i-k} = \tilde{t}_j\tilde{t}_{j-k}, i, j \in \mathbb{Z}/e\mathbb{Z} \). For example, the diagram corresponding to \( B^{(2)}(8, 8, 2) \) is as follows.

![Figure 7: Diagram for the presentation of \( B^{(2)}(8, 8, 2) \).](image-url)
6.2 Identifying $B(e, e, n)$

Now, we want to check which of the monoids $B^\oplus_k(e, e, n)$ are isomorphic to $B^\oplus(e, e, n)$. Assume there exists an isomorphism $\phi : B^\oplus_k(e, e, n) \rightarrow B^\oplus(e, e, n)$ for a given $k$ with $0 \leq k \leq e - 1$. We start with the following lemma.

**Lemma 6.4.** The isomorphism $\phi$ fixes $\tilde{s}_3, \tilde{s}_4, \ldots, \tilde{s}_n$ and permutes the $\tilde{t}_i$ where $i \in \mathbb{Z}/e\mathbb{Z}$.

**Proof.** Let $f$ be in $\tilde{X}^*$. We have $\ell(f) \leq \ell(\phi(f))$. Thus, we have $\ell(\tilde{x}) \leq \ell(\phi(\tilde{x}))$ for $\tilde{x} \in \tilde{X}$. Also, $\ell(\phi(\tilde{x})) \leq \ell(\phi^{-1}(\phi(\tilde{x}))) = \ell(\tilde{x})$. Hence $\ell(\tilde{x}) = \ell(\phi(\tilde{x})) = 1$. It follows that $\phi(\tilde{x}) \in \{\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{e-1}, \tilde{s}_3, \ldots, \tilde{s}_n\}$.

Furthermore, the only generator of $B^\oplus_k(e, e, n)$ that commutes with all other generators except for one of them is $\tilde{s}_n$. On the other hand, $\tilde{s}_n$ is the only generator of $B^\oplus(e, e, n)$ that satisfies the latter property. Hence $\phi(\tilde{s}_n) = \tilde{s}_n$. Next, $\tilde{s}_{n-1}$ is the only generator of $B^\oplus_k(e, e, n)$ that does not commute with $\tilde{s}_n$. The only generator of $B^\oplus(e, e, n)$ that does not commute with $\tilde{s}_n$ is also $\tilde{s}_{n-1}$. Hence $\phi(\tilde{s}_{n-1}) = \tilde{s}_{n-1}$.

Next, the only generator of $B^\oplus_k(e, e, n)$ different from $\tilde{s}_n$ and that does not commute with $\tilde{s}_{n-1}$ is $\tilde{s}_{n-2}$ and so on, we get $\phi(\tilde{s}_j) = \tilde{s}_j$ for $3 \leq j \leq n$. It remains that $\phi(\{\tilde{t}_i \mid 0 \leq i \leq e - 1\}) = \{\tilde{t}_i \mid 0 \leq i \leq e - 1\}$.

**Proposition 6.5.** The monoids $B^\oplus_k(e, e, n)$ and $B^\oplus(e, e, n)$ are isomorphic if and only if $k \wedge e = 1$.

**Proof.** Assume there exists an isomorphism $\phi$ between the monoids $B^\oplus_k(e, e, n)$ and $B^\oplus(e, e, n)$. By Lemma 6.3 we have $\phi(\tilde{s}_j) = \tilde{s}_j$ for $3 \leq j \leq n$ and $\phi(\{\tilde{t}_i \mid 0 \leq i \leq e - 1\}) = \{\tilde{t}_i \mid 0 \leq i \leq e - 1\}$. A diagram presentation $\Gamma_k$ of $B^\oplus_k(e, e, 2)$ for $1 \leq k \leq e - 1$ can be viewed as the same diagram presentation of $B^\oplus(e, e, 2)$ given earlier. The isomorphism $\phi$ implies that we have only one connected component in $\Gamma_k$ or, in another words, a closed chain. Hence $k$ is a generator of the cyclic group $\mathbb{Z}/e\mathbb{Z}$. Then $k$ satisfies the condition $k \wedge e = 1$.

Conversely, let $1 \leq k \leq e - 1$ such that $k \wedge e = 1$. We define a map $\phi : B^\oplus(e, e, n) \rightarrow B^\oplus_k(e, e, n)$ where $\phi(\tilde{s}_j) = \tilde{s}_j$ for $3 \leq j \leq n$ and $\phi(\tilde{t}_0) = \tilde{t}_0$, $\phi(\tilde{t}_1) = \tilde{t}_1$, $\phi(\tilde{t}_2) = \tilde{t}_2$, $\ldots$, $\phi(\tilde{t}_{e-1}) = \tilde{t}_{e-1}$. The map $\phi$ is a well-defined monoid homomorphism, which is both surjective (as it corresponds a generator of $B^\oplus(e, e, n)$ to a generator of $B^\oplus_k(e, e, n)$) and injective (as it is bijective on the relations). Hence $\phi$ defines an isomorphism of monoids.

When $k \wedge e = 1$, since $B^\oplus_k(e, e, n)$ is isomorphic to $B^\oplus(e, e, n)$, we have the following.

**Corollary 6.6.** $B^\oplus(e, e, n)$ is isomorphic to the complex braid group $B(e, e, n)$ for $k \wedge e = 1$.

The reason that the proof of Proposition 6.5 fails in the case $k \wedge e \neq 1$ is that we have more than one connected component in $\Gamma_k$ that link $\tilde{t}_0, \tilde{t}_1, \ldots$, and $\tilde{t}_{e-1}$ together, as we can see in Figure 7. Actually, it is easy to check that the number of connected components that link $\tilde{t}_0, \tilde{t}_1, \ldots$, and $\tilde{t}_{e-1}$ together is the number of cosets of the subgroup of $\mathbb{Z}/e\mathbb{Z}$ generated by the class of $k$, that is equal to $k \wedge e$, and each of these cosets have $e' = e/k \wedge e$ elements. This will be useful in the next subsection.
6.3 New Garside groups

When \( k \land e \neq 1 \), we describe \( B^{(k)}(e, e, n) \) as an amalgamated product of \( k \land e \) copies of the complex braid group \( B(e', e', n) \) with \( e' = e/e \land k \), over a common subgroup which is the Artin-Tits group \( B(2, 1, n - 1) \). This allows us to compute the center of \( B^{(k)}(e, e, n) \). Finally, using the Garside structure of \( B^{(k)}(e, e, n) \), we compute its first and second integral homology groups using the Dehornoy-Lafont complex [9] and the method used in [4].

By an adaptation of the results of Crisp in [7] as in Lemma 5.2 of [4], we have the following embedding. Let \( B := B(2, 1, n - 1) \) be the Artin-Tits group defined by the following diagram presentation.

\[
\begin{array}{ccccccc}
q_1 & q_2 & q_3 & \cdots & q_{n-2} & q_{n-1} \\
\circ & \circ & \circ & \cdots & \circ & \circ
\end{array}
\]

**Proposition 6.7.** The group \( B \) injects in \( B^{(k)}(e, e, n) \).

**Proof.** Define a monoid homomorphism \( \phi : B^+ \rightarrow B^{\oplus k}(e, e, n) : q_1 \mapsto \tilde{t}_i \tilde{t}_{i-k}, q_2 \mapsto \tilde{s}_3, \ldots, q_{n-1} \mapsto \tilde{s}_n \). It is easy to check that for all \( x, y \in \{q_1, q_2, \ldots, q_{n-1}\} \), we have \( \text{lcm}(\phi(x), \phi(y)) = \phi(\text{lcm}(x, y)) \). Hence by applying Lemma 5.2 of [4], \( B(2, 1, n - 1) \) injects in \( B^{(k)}(e, e, n) \).

We construct \( B^{(k)}(e, e, n) \) as follows.

**Proposition 6.8.** Let \( B^{(1)} := B(e', e', n) \ast_B B(e', e', n) \) be the amalgamated product of two copies of \( B(e', e', n) \) over \( B = B(2, 1, n - 1) \) with \( e' = e/e \land k \). Define \( B^{(2)} := B(e', e', n) \ast_B (B(e', e', n) \ast_B B(e', e', n)) \) and so on until defining \( B((e \land k) - 1) \). We have \( B((e \land k) - 1) = B^{(k)}(e, e, n) \).

**Proof.** Due to the presentation of \( B^{(k)}(e, e, n) \) given in Definition 6.1 and to the presentation of the amalgamated products (see Section 4.2 of [14]), one can deduce that \( B((e \land k) - 1) \) is equal to \( B^{(k)}(e, e, n) \).
Example 6.9. Consider the case of $B^{(2)}(6,6,3)$. It is an amalgamated product of $k \wedge e = 2$ copies of $B(e',e',3)$ with $e' = e/(k \wedge e) = 3$ over the Artin-Tits group $B(2,1,2)$. Consider the following diagram of this amalgamation.

The presentation of $B(3,3,3) \ast B(3,3,3)$ over $B(2,1,2)$ is as follows:
- the generators are the union of the generators of the two copies of $B(3,3,3)$,
- the relations are the union of the relations of the two copies of $B(3,3,3)$ with the additional relations $\tilde{s}_3 = \tilde{s}'_3$ and $\tilde{t}_2\tilde{t}_0 = \tilde{t}_3\tilde{t}_1$

This is exactly the presentation of $B^{(2)}(6,6,3)$ given in Definition 6.1.

Proposition 6.10. The center of $B^{(k)}(e,e,n)$ is infinite cyclic isomorphic to $\mathbb{Z}$.

Proof. By Corollary 4.5 of [14] that computes the center of an amalgamated product, the center of $B^{(k)}(e,e,n)$ is the intersection of the centers of $B$ and $B(e',e',n)$. Since the center of $B$ and $B(e',e',n)$ is infinite cyclic [3], the center of $B^{(k)}(e,e,n)$ is infinite cyclic isomorphic to $\mathbb{Z}$. 

Figure 8: The construction of $B^{(k)}(e,e,n)$. 
Since the center of $B(e,e,n)$ is also isomorphic to $\mathbb{Z}$ (see [3]), in order to distinguish these structures from the braid groups $B(e,e,n)$, we compute their first and second integral homology groups. We recall the Dehornoy-Lafont complex and follow the method in [4] where the second integral homology group of $B(e,e,n)$ is computed.

We order the elements of $\tilde{X}$ by considering $\tilde{s}_n < \tilde{s}_{n-1} < \cdots < \tilde{s}_3 < \tilde{i}_0 < \tilde{i}_1 < \cdots < \tilde{i}_{e-1}$. For $f \in B^\oplus k(e,e,n)$, denote by $d(f)$ the smaller element in $\tilde{X}$ which divides $f$ on the right. An $r$-cell is an $r$-tuple $[x_1, \ldots, x_r]$ of elements in $\tilde{X}$ such that $x_1 < x_2 < \cdots < x_r$ and $x_i = d(lcm(x_i, x_{i+1}, \ldots, x_r))$. The set $C_r$ of $r$-chains is the free $\mathbb{Z}B^\oplus k(e,e,n)$-module with basis $\tilde{X}_r$, the set of all $r$-cells with the convention $\tilde{X}_0 = \{[\emptyset]\}$. We provide the definition of the differential $\partial_r : C_r \to C_{r-1}$.

**Definition 6.11.** Let $[\alpha, A]$ be an $(r + 1)$-cell, with $\alpha \in \tilde{X}$ and $A$ an $r$-cell. Denote $\alpha/A$ the unique element of $B^\oplus k(e,e,n)$ such that $(\alpha/A)\lcm(A) = \lcm(\alpha, A)$. Define the differential $\partial_r : C_r \to C_{r-1}$ recursively through two $\mathbb{Z}$-module homomorphisms $s_r : C_r \to C_{r+1}$ and $u_r : C_r \to C_r$ as follows.

$$\partial_{r+1}[\alpha, A] = \alpha/A[A] - u_r(\alpha/A[A]),$$

with $u_{r+1} = s_r \circ \partial_{r+1}$ where $u_0([f]) = [\emptyset]$, for all $f \in B^\oplus k(e,e,n)$, and $s_r([\emptyset]) = 0$, $s_r([x[A]]) = 0$ if $\alpha := d(lcm(A))$ coincides with the first coefficient in $A$, and otherwise $s_r([x[A]]) = y([\alpha, A]) + s_r(yu_r(\alpha/A[A]))$ with $x = y\alpha/A$.

We provide the final result of the computation of $\partial_1$, $\partial_2$, and $\partial_3$ for all 1, 2, and 3-cells, respectively. For all $x \in \tilde{X}$, we have

$$\partial_1[x] = (x - 1)[\emptyset],$$

for all $1 \leq i \leq e - 1$,

$$\partial_2[\tilde{i}_0, \tilde{i}_1] = \tilde{i}_{i+k} - \tilde{i}_0 - \tilde{i}_1,$$

for $x, y \in \tilde{X}$ with $xy = yx$,

$$\partial_2[x, y] = (xy + 1 - x)[y] + (y - xy - 1)[x],$$

and for $x, y \in \tilde{X}$ with $xy = yx$,

$$\partial_2[x, y] = (x - 1)[y] - (y - 1)[x].$$

For $j \neq -k \mod e$, we have:

$$\partial_3[\tilde{s}_3, \tilde{t}_0, \tilde{t}_j] = (\tilde{s}_3\tilde{t}_0\tilde{s}_3 - \tilde{t}_0\tilde{s}_3 + \tilde{t}_j + 2k\tilde{s}_3)[\tilde{t}_0, \tilde{t}_j] - \tilde{t}_j + 2k\tilde{s}_3\tilde{t}_j + [\tilde{s}_3, \tilde{t}_j] + (\tilde{t}_j + 2k - \tilde{s}_3\tilde{t}_j + 2k)[\tilde{s}_3, \tilde{t}_j + 2k] + (\tilde{s}_3 - \tilde{t}_j + 2k\tilde{s}_3 - 1)\tilde{t}_0\tilde{t}_j + (\tilde{s}_3\tilde{t}_2k - \tilde{t}_2k\tilde{s}_3\tilde{t}_j) + (\tilde{t}_2k\tilde{s}_3 + 1 - \tilde{t}_3\tilde{t}_j)\tilde{t}_0\tilde{t}_j + [\tilde{s}_3, \tilde{t}_j + 2k] + \tilde{t}_2k\tilde{s}_3\tilde{t}_j\tilde{t}_0\tilde{t}_j - [\tilde{s}_3, \tilde{t}_2k]$$

and

$$\partial_3[\tilde{s}_3, \tilde{t}_0, \tilde{t}_{-k}] = (\tilde{s}_3\tilde{t}_0\tilde{s}_3 - \tilde{t}_0\tilde{s}_3 + \tilde{t}_k\tilde{s}_3)[\tilde{t}_0, \tilde{t}_{-k}] - \tilde{t}_k\tilde{s}_3\tilde{t}_0\tilde{t}_3\tilde{t}_{-k} + (\tilde{t}_2k + \tilde{s}_3\tilde{t}_2k)[\tilde{s}_3, \tilde{t}_0] + (\tilde{t}_k - \tilde{s}_3\tilde{t}_k + \tilde{t}_2k\tilde{s}_3\tilde{t}_0)[\tilde{s}_3, \tilde{t}_0] - [\tilde{s}_3, \tilde{t}_2k].$$

Also, for $1 \leq i \leq e - 1$ and $4 \leq j \leq n$, we have:
\[ \partial_3[\tilde{s}_j, \tilde{t}_0, \tilde{t}_i] = (\tilde{s}_j - 1)[\tilde{t}_0, \tilde{t}_i] - \tilde{t}_{i+k}[\tilde{s}_j, \tilde{t}_i] + \tilde{t}_k[\tilde{s}_j, \tilde{t}_0] - [\tilde{s}_j, \tilde{t}_{i+k}] + [\tilde{s}_j, \tilde{t}_k], \]

for \( x, y, z \in \tilde{X} \) with \( xyx = yxy, xz = zx, \) and \( yzy = zyz, \)

\[ \partial_3[x, y, z] = (z + xyz - yz - 1)[x, y] - [x, z] + (xz - z + 1 - yxz)y[x, z] + (x - 1 - xy + yzx)[y, z], \]

for \( x, y, z \in \tilde{X} \) with \( xyx = yxy, xz = zx, \) and \( yzy = zyz, \)

\[ \partial_3[x, y, z] = (1 - x + xy)[y, z] + (y - 1 - xy)[x, z] + (z - 1)[x, y], \]

for \( x, y, z \in \tilde{X} \) with \( xy = yx, xz = zx, \) and \( yzy = zyz, \)

\[ \partial_3[x, y, z] = (1 + yz - z)[x, y] + (y - 1 - zy)[x, z] + (x - 1)[y, z], \]

and

\[ \partial_3[x, y, z] = (1 - y)[x, z] + (z - 1)[x, y] + (x - 1)[y, z]. \]

Let \( d_r = \partial_r \otimes_{\mathbb{Z}B^{**}(e,e,n)} \mathbb{Z} : C_r \otimes_{\mathbb{Z}B^{**}(e,e,n)} \mathbb{Z} \rightarrow C_{r-1} \otimes_{\mathbb{Z}B^{**}(e,e,n)} \mathbb{Z} \) be the differential with trivial coefficients. For example, for \( d_2, \) we have: for all \( 1 \leq i \leq e - 1, \)

\[ d_2[\tilde{t}_0, \tilde{t}_i] = [\tilde{t}_i] - [\tilde{t}_0] - [\tilde{t}_k] + [\tilde{t}_{i+k}], \]

for \( x, y \in \tilde{X} \) with \( xyx = yxy, \)

\[ d_2[x, y] = [y] - [x], \]

and

\[ d_2[x, y] = 0. \]

The same can be done for \( \partial_3. \)

**Definition 6.12.** Define the integral homology group of order \( r \) to be

\[ H_r(B^{(k)}(e, e, n), \mathbb{Z}) = \ker(d_r)/\text{Im}(d_{r+1}). \]

We are ready to compute the first and second integral homology groups of \( B^{(k)}(e, e, n). \)

Using the presentation of \( B^{(k)}(e, e, n) \) given in 6.1, one can check that the abelianization of \( B^{(k)}(e, e, n) \) is isomorphic to \( \mathbb{Z}. \) Since \( H_1(B^{(k)}(e, e, n), \mathbb{Z}) \) is isomorphic to the abelianization of \( B^{(k)}(e, e, n), \) we deduce that \( H_1(B^{(k)}(e, e, n), \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}. \)

Since \( H_1(B(e, e, n), \mathbb{Z}) \) is also isomorphic to \( \mathbb{Z} \) (see 3), the first integral homology group does not give any additional information whether these groups are isomorphic to some \( B(e, e, n) \) or not.

Recall that by 4, we have

- \( H_2(B(e, e, 3), \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \) where \( e \geq 2, \)
- \( H_2(B(e, e, 4), \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) when \( e \) is odd and \( H_2(B(e, e, 4), \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \) when \( e \) is even, and
- \( H_2(B(e, e, n), \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) when \( n \geq 5 \) and \( e \geq 2. \)
In order to compute $H_2(B^{(k)}(e, e, n), \mathbb{Z})$, we follow exactly the proof of Theorem 6.4 in [4]. Using the same notations as in [4], we set

$$v_i = [\tilde{t}_0, \tilde{t}_i] + [\tilde{s}_3, \tilde{t}_0] + [\tilde{s}_3, \tilde{t}_k] - [\tilde{s}_3, \tilde{t}_i] - [\tilde{s}_3, \tilde{t}_{i+k}]$$

where $1 \leq i \leq e - 1$, and we also have

$$H_2(B^{(k)}(e, e, n), \mathbb{Z}) = (K_1/d_3(C_1)) \oplus (K_2/d_3(C_2)).$$

We have $d_3[\tilde{s}_3, \tilde{t}_0, \tilde{t}_j] = v_j - v_{j+k} + v_k$ if $j \neq -k$, and $d_3[\tilde{s}_3, \tilde{t}_0, \tilde{t}_{-k}] = v_{-k} + v_k$. Denote $u_i = [\tilde{s}_3, \tilde{t}_0, \tilde{t}_i]$ for $1 \leq i \leq e - 1$. We define a basis of $C_2$ as follows. For each coset of the subgroup of $\mathbb{Z}/e\mathbb{Z}$ generated by the class of $k$, say $\{\tilde{t}_x, \tilde{t}_{x+k}, \ldots, \tilde{t}_{x-k}\}$ such that $1 \leq x \leq e - 1$, we define $w_{x+i} = u_{x+k} + u_{x+(i+1)k} + \cdots + u_{x-k}$ for $0 \leq i \leq e - 1$, and when $x = 0$, we define $w_{i} = u_{ik} + u_{(i+1)k} + \cdots + u_{-k}$ for $1 \leq i \leq e - 1$. In the $\mathbb{Z}$-basis $(w_k, w_{2k}, \ldots, w_{-k}, w_1, w_{1+k}, \ldots, w_{1-k}, \ldots, w_x, w_{x+k}, \ldots, w_{x-k}, \ldots)$ and $(v_k, v_{2k}, \ldots, v_{-k}, v_1, v_{1+k}, \ldots, v_{1-k}, \ldots, v_x, v_{x+k}, \ldots, v_{x-k}, \ldots)$, $d_3$ is in triangular form with $(e \land k) - 1$ diagonal coefficients that are zero, all other diagonal coefficients are equal to 1 except one of them that is equal to $e' = e/(e \land k)$. In this case, we have $H_2(B^{(k)}(e, e, 3), \mathbb{Z}) = \mathbb{Z}^{(e \land k) - 1} \times \mathbb{Z}/e'\mathbb{Z}$. The rest of the proof is essentially similar to the proof of Theorem 6.4 in [4]. When $n = 4$, we get $[\tilde{s}_4, \tilde{t}_i] \equiv [\tilde{s}_4, \tilde{t}_{i+2k}]$ for every $i$ and since $2[\tilde{s}_4, \tilde{t}_i] = 0$ for every $i$, we get $K_2/d_3(C_2) \simeq (\mathbb{Z}/2\mathbb{Z})^c$, where $c$ is the number of cosets of the subgroup of $\mathbb{Z}/e\mathbb{Z}$ generated by the class of $2k$.

In the following proposition, we provide the second integral homology group of $B^{(k)}(e, e, n)$.

**Proposition 6.13.** Let $n \geq 3$ and $e \geq 2$.

- When $n = 3$, we have $H_2(B^{(k)}(e, e, 3), \mathbb{Z}) \simeq \mathbb{Z}^{(e \land k) - 1} \times \mathbb{Z}/e'\mathbb{Z}$,
- when $n = 4$, $H_2(B^{(k)}(e, e, 4), \mathbb{Z}) \simeq \mathbb{Z}^{(e \land k) - 1} \times \mathbb{Z}/e'\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^c$, where $c$ is the number of cosets of the subgroup of $\mathbb{Z}/e\mathbb{Z}$ generated by the class of $2k$, and
- when $n \geq 5$, $H_2(B^{(k)}(e, e, n), \mathbb{Z}) \simeq \mathbb{Z}^{(e \land k) - 1} \times \mathbb{Z}/e'\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Comparing this result with $H_2(B(e, e, n), \mathbb{Z})$, one can check that if $k \land e \neq 1$, $B^{(k)}(e, e, n)$ is not isomorphic to a complex braid group of type $B(d, d, n)$ with $d \geq 2$. Thus, we conclude by the following theorem.

**Theorem 6.14.** $B^{(k)}(e, e, n)$ is isomorphic to $B(e, e, n)$ if and only if $k \land e = 1$.

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