ERRATUM TO VISCOSITY SOLUTIONS TO COMPLEX MONGE-AMPÈRE EQUATIONS

PHILIPPE EYSSIDIEUX, VINCENT GUEDJ, AHMED ZERIAHI

Abstract. The proof of the comparison principle in [EGZ11] is not complete. We provide here an alternative proof, valid in the ample locus of any big cohomology class, and discuss the resulting modifications.

Introduction

Jeff Streets has informed the authors that the proof of [EGZ11, Theorem 2.14] is not correct as it stands: the localization procedure that we use does not provide enough information to control the Hessian of the penalization function $\varphi_3$ along the diagonal.

We provide here a different approach which yields an alternative proof of the global comparison principle in the ample locus. This is sufficient for constructing unique viscosity solutions to degenerate complex Monge-Ampère equations. The latter are continuous in the ample locus, but the continuity at the boundary remains an open question.

Acknowledgement. We would like to thank Jeff Streets for pointing out the problem in the original proof and interesting exchanges.

1. Comparison principle in the ample locus

We first recall the context. Let $X$ be a compact Kähler manifold of dimension $n$. We consider the complex Monge-Ampère equation on $X$,

$$(1.1) \quad e^\varphi \mu - (\omega + dd^c\varphi)^n = 0,$$

where

- $\omega$ is a continuous closed $(1,1)$–form on $X$ with smooth local potentials such that its cohomology class $\eta := \{\omega\}$ is big,
- $\mu \geq 0$ is a continuous volume form on $X$,
- $\varphi : X \to \mathbb{R}$ is the unknown function,

Theorem 1.1. Let $\varphi$ (resp. $\psi$) be a viscosity subsolution (resp. supersolution) to (1.1) in $X$. Then

$$\varphi \leq \psi \text{ in } \text{Amp}\{\omega\}.$$ 

In particular if $\psi$ is continuous on $X$, then $\varphi \leq \psi$ on $X$.

Date: July 16, 2018.
Recall that the ample locus \( \text{Amp}\{\omega\} \) of the cohomology class of \( \omega \) is the Zariski open subset of points \( x \in X \) such that there exists a positive closed current cohomologous to \( \omega \) which is a Kähler form near \( x \). In particular \( \text{Amp}\{\omega\} = X \) when \( \omega \) is Kähler.

1.1. A refined local comparison principle. We first establish a useful lemma for the local equation

\[
e^u \mu - (dd^c u)^n = 0.\tag{1.2}
\]

Lemma 1.2. Let \( \mu(z) \geq 0, \nu(z) \geq 0 \) be continuous volume forms on some domain \( D \subset \mathbb{C}^n \). Let \( u \) be a subsolution to (1.2) associated to \( \mu \) and let \( v \) be a bounded supersolution to (1.2) associated to \( \nu \) in \( D \). Assume that

(i) the function \( u - v \) achieves a local maximum at some \( x_0 \in D \);

(ii) \( \exists c > 0 \) s.t. \( z \mapsto u(z) - 2c|z|^2 \) is plurisubharmonic near \( x_0 \).

Then \( \nu(x_0) > 0 \) and

\[
e^{u(x_0)} \mu(x_0) \leq e^{v(x_0)} \nu(x_0). \tag{1.3}
\]

Proof. The idea of the proof is to apply Jensen-Ishii’s comparison principle which is obtained from the classical maximum principle by regularizing \( u \) and \( v \) (see [CIL92] [EGZ11]).

We can assume that \( u - v \) achieves a local maximum at 0,

\[ M := \sup_{x \in \mathbb{B}} (u(x) - v(x)) = u(0) - v(0), \]

where \( \mathbb{B} \) is the unit ball in \( \mathbb{C}^n \). The hypothesis (ii) insures that \( u(z) - 2c|z|^2 \) is plurisubharmonic in a neighborhood of \( \mathbb{B} \). Thus for any fixed \( \alpha \in ]0, 1[ \), the function

\[ u_\alpha(z) := u(z) - c\alpha|z|^2 \]

is strictly psh in \( \mathbb{B} \) and \( u_\alpha - v \) achieves a strict maximum at 0 in \( \mathbb{B} \).

Observe that \( u_\alpha - (1 - \alpha)u - \alpha c|z|^2 = \alpha(u - 2c|z|^2) \) is psh in \( \mathbb{B} \) hence

\[
(dd^c u_\alpha)^n \geq (1 - \alpha) dd^c u + \alpha c dd^c|z|^2.
\]

It follows that, setting \( dV_{\text{eucl}} := (dd^c|z|^2)^n \),

\[
(dd^c u_\alpha)^n \geq (1 - \alpha)^n (dd^c u)^n + \alpha^n c^n dV_{\text{eucl}}
\]

\[
\geq e^{u_\alpha(z) + n \log(1 - \alpha)} \mu + \alpha^n c^n dV_{\text{eucl}}
\]

in the viscosity sense, noticing that \( u \geq u_\alpha \).

Since \( u_\alpha(0) = u(0) \) we replace in the sequel \( u \) by \( u_\alpha \). We have thus reduced the situation to the case where \( u(z) - c|z|^2 \) is psh in \( \mathbb{B} \), \( u - v \) achieves a strict maximum in \( \mathbb{B} \) at 0 and \( u \) is a subsolution of the equation

\[
e^{u(z) + n \log(1 - \alpha)} \mu + \alpha^n c^n dV_{\text{eucl}} - (dd^c u)^n = 0.
\]

We want to apply Jensen-Ishii’s maximum principle to \( u \) and \( v \) by using the penalty method as in [CIL92]. Fixing \( \varepsilon > 0 \), we want to maximize on \( \mathbb{B} \times \mathbb{B} \) the upper semi-continuous function

\[ w_\varepsilon(x, y) := u(x) - v(y) - (1/2\varepsilon)|x - y|^2. \]
The penalty function forces the maximum of \( w_\varepsilon \) to be asymptotically attained along the diagonal. Since \( w_\varepsilon \) is upper semicontinuous on the compact set \( \overline{B} \times \overline{B} \), there exists \((x_\varepsilon, y_\varepsilon) \in \overline{B} \times \overline{B}\) such that

\[
M_\varepsilon := \sup_{(x,y) \in \overline{B}^2} \left\{ u(x) - v(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\} = u(x_\varepsilon) - v(y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2.
\]

The following result is classical [CIL92, Proposition 3.7]:

**Lemma 1.3.** We have \(|x_\varepsilon - y_\varepsilon|^2 = o(\varepsilon)\). Every limit point \((\hat{x}, \hat{y})\) of \((x_\varepsilon, y_\varepsilon)\) satisfies \(\hat{x} = \hat{y}\), \((\hat{x}, \hat{x}) \in \overline{B} \times \overline{B}\) and

\[
\lim_{\varepsilon \to 0} M_\varepsilon = \lim_{\varepsilon \to 0} (u(x_\varepsilon) - v(y_\varepsilon)) = u(\hat{x}) - v(\hat{x}) = M.
\]

Our hypothesis of strict local maximum guarantees that \(\hat{x} = 0\). Therefore \((x_\varepsilon, y_\varepsilon) \to (0,0)\) as \(\varepsilon \to 0\). Take any sequence \((x_j, y_j) = (x_{\varepsilon_j}, y_{\varepsilon_j}) \to (0,0)\) and \((x_j, y_j) \in \overline{B}^2\) for any \(j > 1\) so that the conditions in the lemma above are satisfied with \(\varepsilon = \varepsilon_j \to 0\).

Set \(\phi_j(x, y) := \frac{1}{2\varepsilon_j}|x - y|^2\). The function \((x, y) \mapsto u(x) - v(y) - \phi_j(x, y)\) achieves its maximum in \(\overline{B}^2\) at the interior point \((x_j, y_j) \in \overline{B}^2\). We can thus apply Jensen-Ishii’s maximum principle and obtain the following estimates:

**Lemma 1.4.** For any \(\gamma > 0\), we can find \((p^+, Q^+), (p_-, Q_-) \in \mathbb{C}^n \times \operatorname{Sym}_R^2(\mathbb{C}^n)\) such that

1. \((p^+, Q^+) \in \mathcal{J}^{2+}u(x_j)\), \((p_-, Q_-) \in \mathcal{J}^{2-}v(y_j)\), where

\[
p^+ = \frac{(x_j - y_j)}{2\varepsilon_j} + p_-.
\]

2. The block diagonal matrix with entries \((Q^+, Q_-)\) satisfies:

\[
-(\gamma^{-1} + \|A\|)I \leq \begin{pmatrix} Q^+ & 0 \\ 0 & -Q_- \end{pmatrix} \leq A + \gamma A^2,
\]

where \(A = D^2\phi_j(x_j, y_j) = \gamma^{-1} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\), and \(\|A\|\) is the spectral radius of \(A\).

We choose \(\gamma = \varepsilon_j\). Thus

\[
-(2\varepsilon_j^{-1})I \leq \begin{pmatrix} Q^+ & 0 \\ 0 & -Q_- \end{pmatrix} \leq \frac{3}{\varepsilon_j} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

Looking at the upper and lower diagonal terms we deduce that the eigenvalues of \(Q^+, Q_-\) are \(O(\varepsilon_j^{-1})\). Evaluating the inequality on vectors of the form \((Z,Z)\) we deduce that \(Q^+ \preceq Q_-\) in the sense of quadratic forms.

For a fixed \(Q \in \operatorname{Sym}_R^2(\mathbb{C}^n)\), denote by \(H = Q^{1,1}\) its \((1,1)\)-part. It is a hermitian matrix. Since \((p^+, Q^+) \in \mathcal{J}^{2+}u(x_j)\), we deduce from the viscosity differential inequality satisfied by \(u\) that \(H^+\) is positive definite (see [EGZ11]).
The inequality $H^+ \leq H_-$ forces $H_- > 0$, thus $0 \leq H^+ \leq H_-$ and $\det H^+ \leq \det H_-$. 

The viscosity differential inequalities satisfied by $u$ and $v$ yield 
\[ e^{u(x_j) + n \log(1-\alpha)} \mu(x_j) + \alpha^n e^n dV_{\text{eucl}}(x_j) \leq \det H^+ \leq \det H_- \leq e^{v(y_j)} \nu(y_j), \]

hence
\[ e^{u(x_j) + n \log(1-\alpha)} \mu(x_j) + \alpha^n e^n dV_{\text{eucl}}(x_j) \leq e^{v(y_j)} \nu(y_j). \]

Proof. Assume that $\varphi$ is a subsolution to the complex Monge-Ampère equation (1.1) associated to $\mu$ and $\psi$ is a supersolution to the complex Monge-Ampère equation (1.1) associated to $\nu$ in $X$. 

Fix $\varepsilon \in ]0, 1[$ and set 
\[ \tilde{\varphi}(x) := (1 - \varepsilon)\varphi(x) + \varepsilon \rho(x), \]

where $\rho \leq \varphi$ is a $\omega$-psh function satisfying $\omega + dd^c \rho \geq \beta$, where $\beta$ is a Kähler form on $X$. Such a function exists since the cohomology class $\eta$ of $\omega$ is big. One can moreover impose $\rho$ to be smooth in the ample locus $\Omega := \text{Amp}\{\omega\}$, with analytic singularities, and such that $\rho(x) \to -\infty$ as $x \to \partial X = X \setminus \Omega$. 

Since $\tilde{\varphi} - \psi$ is bounded from above on $X$, tends to $-\infty$ when $x \to \partial X$, and is upper semicontinuous in $\Omega$, the maximum of $\tilde{\varphi} - \psi$ is achieved at some point $x_0 \in \Omega$.

\[ M := \sup_{x \in X} (\tilde{\varphi}(x) - \psi(x)) = \tilde{\varphi}(x_0) - \psi(x_0). \]

Observe that $\tilde{\varphi}$ satisfies $(\omega + dd^c \tilde{\varphi})^n \geq (1 - \varepsilon)^n (\omega + dd^c \varphi)^n + \varepsilon^n \beta^n$, in the viscosity sense in $\Omega$. Now $\tilde{\varphi} \leq \varphi$ since $\rho < \varphi$ hence
\[ (\omega + dd^c \varphi)^n \geq e^{\tilde{\varphi}} \{(1 - \varepsilon)^n \mu(x) + e^{-C} \varepsilon^n \beta^n\}, \]

where $\tilde{\varphi} \leq \varphi \leq C$. 

The idea is to localize near $x_0$ and use Lemma 1.2. Choose complex coordinates $z = (z^1, \ldots, z^n)$ near $x_0$ defining a biholomorphism identifying a closed neighborhood of $x_0$ to the closed complex ball $\bar{B}_2 := B(0, 2) \subset \mathbb{C}^n$ of radius 2, sending $x_0$ to the origin in $\mathbb{C}^n$. 

Theorem 2.5). The inequality $H^+ \leq H_-$ forces $H_- > 0$, thus $0 \leq H^+ \leq H_-$ and $\det H^+ \leq \det H_-$. 

The viscosity differential inequalities satisfied by $u$ and $v$ yield 
\[ e^{u(x_j) + n \log(1-\alpha)} \mu(x_j) + \alpha^n e^n dV_{\text{eucl}}(x_j) \leq \det H^+ \leq \det H_- \leq e^{v(y_j)} \nu(y_j), \]

hence 
\[ e^{u(x_j) + n \log(1-\alpha)} \mu(x_j) + \alpha^n e^n dV_{\text{eucl}}(x_j) \leq e^{v(y_j)} \nu(y_j). \]
We let $h_\omega(x)$ be a smooth local potential for $\omega$ in $B_2$, i.e. $dd^c h_\omega = \omega$ in $B_2$. Setting $u := \tilde{\varphi} \circ z^{-1} + h_\omega \circ z^{-1}$ in $B_2$ we obtain
\begin{equation}
(1.5) \quad e^u \tilde{\mu} \leq (dd^c u)^n, \quad \text{in } B_2,
\end{equation}
where
\[ \tilde{\mu} = e^{-h_\omega \circ z^{-1}} z^* \{ (1 - \varepsilon)^n \mu(x) + e^{-C \varepsilon^n \beta n} \} > 0 \]
is a continuous volume form on $B_2$.

Similarly the lower semi-continuous function $v := \psi \circ z^{-1} + h_\omega \circ z^{-1}$ satisfies the viscosity differential inequality
\begin{equation}
(1.6) \quad e^v \tilde{\nu} \geq (dd^c v)^n, \quad \text{in } B_2,
\end{equation}
where $\tilde{\nu} := e^{-h_\omega \circ z^{-1}} z^*(\nu) > 0$ is a continuous volume form on $B_2$.

Our hypothesis guarantees
\begin{equation}
(1.7) \quad M = \sup_X \{ \tilde{\varphi} - \tilde{\psi} \} = \max_{\overline{B}} \{ u(\zeta) - v(\zeta) \} = u(0) - v(0),
\end{equation}
i.e. $u - v$ achieves its maximum at the interior point $0 \in \mathbb{B}$. Moreover $dd^c u = \omega + dd^c \varphi \geq \varepsilon \beta$, i.e. $u$ is 2c-strictly psh in $B_2$ for some $c = c(\varepsilon) > 0$.

We apply Lemma 1.2 and conclude that $\tilde{\nu}(0) > 0$ and
\[ e^u(0) \tilde{\mu}(0) \leq e^v(0) \tilde{\nu}(0), \]
Going back to $\varphi$ and $\psi$ we obtain $\nu(x_0) > 0$ and
\[ (1 - \varepsilon)^n e^{\varphi(x_0)} \mu(x_0) \leq e^{\psi(x_0)} \nu(x_0). \]
When $\mu = \nu$ we can divide by $\nu(x_0) = \mu(x_0) > 0$ and obtain
\[ (1 - \varepsilon) \varphi(x) + \varepsilon \rho(x) \leq \psi(x) - n \log(1 - \varepsilon), \]
for all $x \in X$ and $0 < \varepsilon < 1$. Letting $\varepsilon \to 0$, we infer $\varphi \leq \psi$, in $X \setminus \{ \rho = -\infty \} = \Omega$. The set $\{ \rho = -\infty \}$ has Lebesgue measure 0 hence the inequality $\varphi \leq \psi$ holds on $X$ if $\psi$ is continuous on $X$. \qed

2. Further modifications

2.1. Statements of [EGZ11]. The definitions and statements have to be modified as follows when working on compact complex manifolds:

- a viscosity subsolution is bounded from above on $X$, u.s.c. in $\text{Amp}\{\omega\}$ where it satisfies the corresponding differential inequalities;
- a viscosity supersolution is bounded from below, l.s.c. in $\text{Amp}\{\omega\}$ where it satisfies the corresponding differential inequalities;
- a viscosity solution is both a subsolution and a supersolution, in particular it is bounded on $X$ and continuous in $\text{Amp}(\{\omega\})$.

To construct the unique viscosity solution we proceed as previously done, using the Perron method: the family of subsolution is non empty, it is uniformly bounded from above by a continuous supersolution (e.g. a constant).

Since the comparison principle is only shown to hold in the ample locus, the solutions we construct are continuous in the ample locus rather than in all of $X$. Thus the statements of Theorem A, Corollary B, Theorem C (see
also Corollary 3.4, Corollary 3.5, Theorem 3.6 and Corollary 3.7) have to be modified accordingly, replacing "continuous" by "continuous in the ample locus" or "bounded on $X$ and continuous in the ample locus", etc.

2.2. Continuous approximation of quasi-psh functions. The proof of the main result of [EGZ15] has to be modified similarly. It provides an approximation process by $\omega$-psh functions which are merely continuous in the ample locus. It is an interesting open problem to decide whether these approximants are actually globally continuous on $X$ (see [EGZ09, Definition 2.2]). It follows from [CGZ13] that this is the case when $\omega = \pi^* \omega_Y$ is the pull-back of a Hodge form on a singular projective variety $Y$, under a desingularization $\pi : X \to Y$.

2.3. Parabolic theory. A similar problem occurs in the localization technique used in [EGZ16] to prove the parabolic viscosity comparison principle. The method proposed in this note can be adapted to the parabolic setting and yields an alternative proof of the parabolic comparison principle valid in the ample locus. We will give the details elsewhere.

References

[CGZ13] Coman, D., Guedj, V., Zeriahi, A. Extension of plurisubharmonic functions with growth control. Journal für die reine und angewandte Mathematik 676 (2013), 33-49.

[CIL92] Crandall, M., Ishii, H., Lions, P.L. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. 27 (1992), 1-67.

[EGZ09] Eyssidieux, P., Guedj, V., Zeriahi, A. Singular Kähler-Einstein metrics. J. Amer. Math. Soc. 22 (2009), 607-639.

[EGZ11] Eyssidieux, P., Guedj, V., Zeriahi, A. Viscosity solutions to degenerate complex Monge-Ampère equations. Comm. Pure and Applied Math. 64 (2011), 1059-1094.

[EGZ15] P. Eyssidieux, V. Guedj, A. Zeriahi: Continuous approximation of quasi-plurisubharmonic functions. Cont. Mathematics Volume 644 (2015), 67-78.

[EGZ16] Eyssidieux, P. Guedj, V., Zeriahi, A. Weak solutions to degenerate complex Monge-Ampère flows II. Advances in Math. 293 (2016), 37-80.

Université Joseph Fourier et Institut Universitaire de France
E-mail address: Philippe.Eyssidieux@ujf-grenoble.fr

Institut de Mathématiques de Toulouse et Institut Universitaire de France
E-mail address: vincent.guedj@math.univ-toulouse.fr

Institut de Mathématiques de Toulouse, France
E-mail address: zeriahi@math.univ-toulouse.fr