Erdős-Ginzburg-Ziv type generalizations for linear equations and linear inequalities in three variables

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Abstract

For any linear inequality in three variables \( L \), we determine (if it exist) the smallest integer \( R(L, \mathbb{Z}/3\mathbb{Z}) \) such that: for every mapping \( \chi : [1, n] \rightarrow \{0, 1, 2\} \), with \( n \geq R(L, \mathbb{Z}/3\mathbb{Z}) \), there is a solution \((x_1, x_2, x_3) \in [1, n]^3\) of \( L \) with \( \chi(x_1) + \chi(x_2) + \chi(x_3) \equiv 0 \) (mod 3). Moreover, we prove that \( R(L, \mathbb{Z}/3\mathbb{Z}) = R(L, 2) \), where \( R(L, 2) \) denotes the classical 2-color Rado number, that is, the smallest integer (provided it exist) such that for every 2-coloring of \([1, n]\), with \( n \geq R(L, 2) \), there exist a monochromatic solution of \( L \). Thus, we get an Erdős-Ginzburg-Ziv type generalization for all linear inequalities in three variables having a solution in the positive integers. We also show a number of families of linear equations in three variables \( L \) such that they do not admit such Erdős-Ginzburg-Ziv type generalization, named \( R(L, \mathbb{Z}/3\mathbb{Z}) \neq R(L, 2) \). At the end of this paper some questions are proposed.

1 Introduction

In this paper we investigate colorings of sets of natural numbers. We denote by \([a, b]\) the interval of natural numbers \( \{x \in \mathbb{N} : a \leq x \leq b\} \), and by \([a, b]^k\) the set of vectors \((x_1, x_2, ..., x_k)\) where \( x_i \in [a, b] \) for each \( 1 \leq i \leq k \). An \( r \)-coloring of \([1, n]\) is a function \( \chi : [1, n] \rightarrow [0, r - 1] \). Given an \( r \)-coloring of \([1, n]\), a vector \((x_1, x_2, ..., x_k) \in [1, n]^k\) is called monochromatic if all its entries received the same color, rainbow if all its entries received pairwise distinct colors, and zero-sum if \( \sum_{i=1}^{k} \chi(x_i) \equiv 0 \) (mod \( r \)).

For a Diophantine system of equalities (or inequalities) in \( k \) variables \( L \), we denote by \( R(L, r) \) the classical \( r \)-color Rado number, that is, the smallest integer, provided it exist, such that for every \( r \)-coloring of \([1, n]\), with \( n \geq R(L, r) \), there exist \((x_1, x_2, ..., x_k) \in [1, n]^k\) a solution of \( L \) which is monochromatic. Rado numbers have been widely studied for many years (see for instance [7]). When studying the existence of zero-sum solutions, it is common to refer to an \( r \)-coloring as a \((\mathbb{Z}/r\mathbb{Z})\)-coloring. In this setting, Bialostocki, Bialostocki and Schaal [2] started the study of the parameter \( R(L, \mathbb{Z}/r\mathbb{Z}) \) defined as the smallest integer,
provided it exist, such that for every \((\mathbb{Z}/r\mathbb{Z})\)-coloring of \([1, R(\mathcal{L}, \mathbb{Z}/r\mathbb{Z})]\) there exist a zero-sum solution of \(\mathcal{L}\). Recently, Robertson and other authors studied the same parameter concerning different equations or systems of equations, \([5], [6], [7]\).

We shall note that, if \(\mathcal{L}\) is a system of equalities (or inequalities) in \(k\) variables, then

\[
R(\mathcal{L}, 2) \leq R(\mathcal{L}, \mathbb{Z}/k\mathbb{Z}) \leq R(\mathcal{L}, k),
\]

where the first inequality follows since, in particular, a \((\mathbb{Z}/k\mathbb{Z})\)-coloring that uses only colors 0 and 1 is a 2-coloring where a zero-sum solution is a monochromatic solution; the second inequality of (1) follows since any monochromatic solution of \(\mathcal{L}\) in a \(k\)-coloring of \([1, R(\mathcal{L}, k)]\) is a zero-sum solution too. In view of the Erdős-Ginzburg-Ziv theorem \([8]\), the authors of \([2]\) state that a system \(\mathcal{L}\) admits an EGZ-generalization if \(R(\mathcal{L}, 2) = R(\mathcal{L}, \mathbb{Z}/k\mathbb{Z})\). For example, it is not hard to see that the system

\[
AP(3) : x + y = 2z, x < y
\]

admits an EGZ-generalization while the Schur equation, \(x + y = z\), does not. More precisely, we have that

\[
9 = R(AP(3), 2) = R(AP(3), \mathbb{Z}/3\mathbb{Z}) = 9,
\]

and

\[
5 = R(x + y = z, 2) < R(x + y = z, \mathbb{Z}/3\mathbb{Z}) = 10,
\]

where \(R(AP(3), 2)\) and \(R(x + y = z, 2)\) are the well known van der Waerden number for 3-term arithmetic progressions concerning two colors and the Schur number concerning two colors respectively, while \(R(x + y = z, \mathbb{Z}/3\mathbb{Z}) = 10\) can be found in \([5]\) and \(R(AP(3), \mathbb{Z}/3\mathbb{Z}) = 9\) can be found in \([6]\). In \([2]\) the authors consider the systems of inequalities

\[
\mathcal{L}_1 : \sum_{i=1}^{k-1} x_i < x_k,
\]

and

\[
\mathcal{L}_2 : \sum_{i=1}^{k-1} x_i < x_k, \quad x_1 < x_2 < \cdots < x_k,
\]

proving that \(\mathcal{L}_2\) admits an EGZ-generalization for \(k\) prime, and \(\mathcal{L}_1\) admits an EGZ-generalization for any \(k\), particularly, \(R(\mathcal{L}_1, 2) = R(\mathcal{L}_1, \mathbb{Z}/k\mathbb{Z}) = k^2 - k + 1\) (see \([2]\)). In this paper we provide analogous results concerning any linear inequality on three variables. More precisely, let \(a, b, c, d \in \mathbb{Z}\), such that \(abc \neq 0\). Then we consider,

\[
\mathcal{L}_3 : ax + by + cz + d < 0.
\]

We prove that \(\mathcal{L}_3\) admits an EGZ-generalization for every set of integers \(\{a, b, c, d\}\) such that the corresponding 2-color Rado number exists. Moreover, we determine, in each case, such Rado numbers (see Theorem 2.2).

Note that, as we investigate linear systems, \(\mathcal{L}\), in three variables, to have an EGZ-generalization means that

\[
R(\mathcal{L}, 2) = R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}),
\]

and the parameter \(R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})\) is defined as the smallest integer, provided it exist, such that for every \(f : [1, R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})] \to \{0, 1, 2\}\) there exist a zero-sum (mod 3) solution of \(\mathcal{L}\).
which, in this case, is either a monochromatic or a rainbow solution of $L$. Therefore, the study of $R(L, \mathbb{Z}/3\mathbb{Z})$ is considered as a canonical Ramsey problem.

The paper is organized as follows. In Section 2, we find the explicit values of $R(L_3, \mathbb{Z}/3\mathbb{Z})$ and $R(L_3, 2)$ (whenever $L_3$ has solutions in the positive integers) in terms of the coefficients of $L_3$. As a corollary, we get that $L_3$ admits an EGZ-generalization in this case. In Section 3 we provide some negative results, that is, we exhibit families of linear equations in three variables which admit no EGZ-generalization. In Section 4, we talk about $r$-regular linear equations and the families $F_k$ and $F_{\mathbb{Z}/k\mathbb{Z}}$. At the end of this section, we give some some problems related with these families.

2 The 2-color Rado numbers for $L_3$

In this section we prove that any linear inequality on three variables, $L_3$, for which the corresponding 2-color Rado number exists, admits an EGZ-generalization. We also determine the value of such Rado numbers depending on the coefficients of $L_3$.

We will repeatedly use the following fact.

**Remark 2.1.** Let $A$ and $B$ be integers such that $A < 0$. Then,

$$A \left(\left\lfloor \frac{B}{-A} \right\rfloor + 1\right) + B < 0 \leq A \left\lfloor \frac{B}{-A} \right\rfloor + B.$$

**Proof.** The first inequality follows since $\frac{B}{-A} - \left\lfloor \frac{B}{-A} \right\rfloor < 1$, equivalently $\frac{B}{-A} < 1 + \left\lfloor \frac{B}{-A} \right\rfloor$ and, multiplying both sides by $A$ (which is negative) we obtain $-B > A \left(\left\lfloor \frac{B}{-A} \right\rfloor + 1\right)$ from which it follows the claim. The second inequality holds true since $\left\lfloor \frac{B}{-A} \right\rfloor \leq \frac{B}{-A}$ and, multiplying both sides by $A$ (which is negative) we obtain $A \left\lfloor \frac{B}{-A} \right\rfloor \geq \frac{B}{-A} = -B$ from which it follows the claim. \qed

Recall that, for integers $a, b, c$ and $d$, such that $abc \neq 0$,

$L_3 : ax + by + cz + d < 0$.

**Theorem 2.2.** Let $a, b, c, d \in \mathbb{Z}$, such that $abc \neq 0$, $a \leq b \leq c$, and define $\sigma = a + b + c + d$. If $L_3$ has a solution in the positive integers, then

$$R(L_3, 2) = R(L_3, \mathbb{Z}/3\mathbb{Z}) = \begin{cases} 1 & \text{if } \sigma < 0, \\ \left\lfloor \frac{d}{-a-b-c} \right\rfloor + 1 & \text{if } \sigma \geq 0 \text{ and } a \leq b \leq c < 0, \\ c\left(\left\lfloor \frac{\frac{a}{b}d + 1}{-a-b} \right\rfloor + 1\right) & \text{if } \sigma \geq 0 \text{ and } a \leq b < 0 < c, \\ \left(\frac{b+c}{-a}\right)\left(\left\lfloor \frac{\frac{a}{b}c + d}{-a} \right\rfloor + 1\right) + 1 & \text{if } \sigma \geq 0 \text{ and } a < 0 < b \leq c, \end{cases}$$
Proof. Let \{a, b, c, d\} be a set of integers such that \(L_3\) has some (integer) positive solution. If \(a + b + c + d < 0\) then \((1, 1, 1)\) is a monochromatic solution of \(L_3\) and so \(R(L_3, 2) = R(L_3, \mathbb{Z}/3\mathbb{Z}) = 1\). If \(a + b + c + d \geq 0\), then necessarily some of the coefficients, \(a, b\) or \(c\), must be negative (otherwise, for all \(x, y, z\) positive integers, \(ax + by + cz + d \geq a + b + c + d \geq 0\) and \(L_3\) would have no solutions in the positive integers). Thus, assuming that \(a + b + c + d \geq 0\), we consider three cases.

Case 1. Assume that \(a \leq b \leq c < 0\). Define \(k_0 = \left\lfloor -\frac{d}{a-b-c} \right\rfloor + 1\). First note that, since \(a + b + c + d \geq 0\) and \(-a - b - c > 0\), then \(k_0 > 1\). Observe now that, for any \(x, y, z \in [1, k_0 - 1]\),

\[
ax + by + cz + d \geq a(k_0 - 1) + b(k_0 - 1) + c(k_0 - 1) + d = (a + b + c) \left\lfloor \frac{d}{-a-b-c} \right\rfloor + d \geq 0,
\]

where the last inequality follows by taking \(A = a + b + c < 0\) and \(B = d\) in Remark 2.1. Then, we conclude that \(L_3\) has no solution in \([1, k_0 - 1]\). On the other hand,

\[
ax + by + cz + d = (a + b + c) \left\lfloor \frac{d}{-a-b-c} \right\rfloor + 1 + d < 0,
\]

where the inequality follows by Remark 2.1 (taking again \(A = a + b + c < 0\) and \(B = d\)). From (2), we conclude that \((k_0, k_0, k_0)\) is a solution of \(L_3\), and so any coloring of \([1, k_0]\) will contain a monochromatic (zero-sum) solution of \(L_3\). Hence, \(R(L_3, 2) = R(L_3, \mathbb{Z}/3\mathbb{Z}) = k_0\).

Case 2. Assume that \(a \leq b < 0 < c\). Define the function

\[
\psi : \mathbb{Z} \to \mathbb{Z}, \quad \psi(x) = \left\lfloor \frac{cx + d}{-a-b} \right\rfloor + 1,
\]

and set \(k_1 = \psi(1)\) and \(k_2 = \psi(k_1)\). First note that, since \(a + b + c + d \geq 0\) and \(-a - b > 0\), then \(k_1 > 1\) and, as \(\psi\) is a nondecreasing function, then \(1 < k_1 \leq k_2\). From (1), it suffices to show that

\[
k_2 \leq R(L_3, 2)
\]

and

\[
R(L_3, \mathbb{Z}/3\mathbb{Z}) \leq k_2.
\]

To show (4), we exhibit a 2-coloring of \([1, k_2 - 1]\) without monochromatic solutions of \(L_3\). Define \(\chi_1 : [1, k_2 - 1] \to \{0, 1\}\) as

\[
\chi_1(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq k_1 - 1, \\
1 & \text{if } k_1 \leq x \leq k_2 - 1.
\end{cases}
\]

Note that, for all \(x, y, z \in [1, k_1 - 1]\),

\[
ax + by + cz + d \geq a(k_1 - 1) + b(k_1 - 1) + c + d = (a + b) \left\lfloor \frac{c + d}{-a-b} \right\rfloor + c + d \geq 0,
\]
where the last inequality follows by taking \( A = a + b < 0 \) and \( B = c + d \) in Remark 2.1. From (5) and (6), we conclude that there are no monochromatic solutions of \( \mathcal{L}_3 \) with respect to \( \chi \), which completes the proof of (3).

Now we prove (4). Let \( \chi : [1, k_2] \to \{0, 1, 2\} \) be an arbitrary coloring, and assume that \( \chi \) contains no zero-sum solutions of \( \mathcal{L}_3 \). We will use two times the first inequality of Remark 2.1. First take \( A = a + b < 0 \) and \( B = c + d \) to obtain

\[
\begin{align*}
    ak_1 + bk_1 + c + d &= (a + b) \left( \left\lfloor \frac{c + d}{-a - b} \right\rfloor + 1 \right) + c + d < 0.
\end{align*}
\]

Now, take \( A = a + b < 0 \) and \( B = ck_1 + d \) to obtain

\[
\begin{align*}
    ak_2 + bk_2 + ck_1 + d &= (a + b) \left( \left\lfloor \frac{ck_1 + d}{-a - b} \right\rfloor + 1 \right) + ck_1 + d < 0.
\end{align*}
\]

By (7) we know that \((k_1, k_1, 1)\) is a solution of \( \mathcal{L}_3 \) which, by assumption, cannot be zero-sum. Suppose, without loss of generality, that \( \chi(1) = 0 \) and \( \chi(k_1) = 1 \). Next, we prove that \( \chi(k_2) \) cannot be 0, 1 or 2.

- By (8), we know that \((k_2, k_2, k_1)\) is a solution of \( \mathcal{L}_3 \), and so \( \chi(k_2) \neq \chi(k_1) = 1 \).
- Since \( c > 0 \) and \( k_1 > 1 \) then \( ak_2 + bk_2 + c + d \leq ak_2 + bk_2 + ck_1 + d \), which together with (8) implies that \((k_2, k_2, 1)\) is a solution of \( \mathcal{L}_3 \). Thus, \( \chi(k_2) \neq \chi(1) = 0 \).
- Since \( a < 0 \) and \( k_2 > k_1 \) then \( ak_2 + bk_1 + c + d \leq ak_1 + bk_1 + c + d \), which together with (7) implies that \((k_2, k_1, 1)\) is a solution of \( \mathcal{L}_3 \). Thus, \( \chi(k_2) \neq 2 \).

This contradiction implies the existence of a zero-sum solution in any \((\mathbb{Z}/3\mathbb{Z})\)-coloring of \([1, k_2]\), and we completed the proof of (4).

**Case 3.** Assume that \( a < 0 < b \leq c \). Define the function

\[
    \phi : \mathbb{Z} \to \mathbb{Z}, \quad \phi(x) = \left\lfloor \frac{(b+c)x + d}{-a} \right\rfloor + 1,
\]

and set \( k_3 = \phi(1) \) and \( k_4 = \phi(k_3) \). First note that, since \( a + b + c + d \geq 0 \), then \( k_3 > 1 \) and, as \( \phi \) is a nondecreasing function, then \( 1 < k_3 \leq k_4 \). From (11), it is enough to show that

\[
    k_4 \leq R(\mathcal{L}_3, 2) \tag{9}
\]

and

\[
    R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) \leq k_4. \tag{10}
\]
To show \(9\), we exhibit a 2-coloring of \([1, k_4 - 1]\) without monochromatic solutions of \(L_3\). Define \(\chi_2 : [1, k_4 - 1] \rightarrow \{0, 1\}\) as

\[
\chi_2(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq k_3 - 1, \\
1 & \text{if } k_3 \leq x \leq k_4 - 1.
\end{cases}
\]

Note that, for all \(x, y, z \in [1, k_3 - 1]\),

\[
ax + by + cz + d \geq a(k_3 - 1) + b + c + d
\]

where the last inequality follows by taking \(A = a < 0\) and \(B = b + c + d\) in Remark 2.1.

Also, for all \(x, y, z \in [k_3, k_4 - 1]\),

\[
ax + by + cz + d \geq a(k_4 - 1) + bk_3 + ck_3 + d
\]

where the last inequality follows by taking \(A = a < 0\) and \(B = bk_3 + ck_3 + d\) in Remark 2.1.

From (11) and (12), we conclude that there are no monochromatic solutions of \(L_3\) with respect to \(\chi_2\), which completes the proof of \(9\).

Now we prove \(10\). Let \(\chi : [1, k_4] \rightarrow \{0, 1, 2\}\) be an arbitrary coloring, and assume that \(\chi\) contains no zero-sum solutions of \(L_3\). We will use two times the first inequality of Remark 2.1. First take \(A = a < 0\) and \(B = b + c + d\) to obtain

\[
ak_3 + b + c + d = a \left( \left\lfloor \frac{b + c + d}{a} \right\rfloor + 1 \right) + b + c + d < 0.
\]

Now take \(A = a < 0\) and \(B = (b + c)k_3 + d\) to obtain

\[
ak_4 + bk_3 + ck_3 + d = a \left( \left\lfloor \frac{(b + c)k_3 + d}{a} \right\rfloor + 1 \right) + (b + c)k_3 + d < 0.
\]

By (13) we know that \((k_3, 1, 1)\) is a solution of \(L_3\) which, by assumption, cannot be zero-sum. Suppose, without loss of generality, that \(\chi(1) = 0\) and \(\chi(k_3) = 1\). Next, we prove that \(\chi(k_4)\) cannot be 0, 1 or 2.

- By (14), we know that \((k_4, k_3, k_3)\) is a solution of \(L_3\), and so \(\chi(k_4) \neq \chi(k_3) = 1\).

- Since \(c \geq b > 0\) and \(k_3 > 1\) then \(ak_4 + b + c + d \leq ak_4 + bk_3 + ck_3 + d\), which together with (14) implies that \((k_4, 1, 1)\) is a solution of \(L_3\). Thus, \(\chi(k_4) \neq \chi(1) = 0\).

- Since \(c > 0\) and \(k_3 > 1\) then \(ak_4 + bk_3 + c + d \leq ak_4 + bk_3 + ck_3 + d\), which together with (14) implies that \((k_4, k_3, 1)\) is a solution of \(L_3\). Thus, \(\chi(k_4) \neq 2\).

This contradiction implies the existence of a zero-sum solution in any \((\mathbb{Z}/3\mathbb{Z})\)-coloring of \([1, k_4]\), and we completed the proof of \(10\).

As an immediate consequence of Theorem 2.2 we conclude the following.

**Corollary 2.3.** Let \(a, b, c, d \in \mathbb{Z}\), such that \(abc \neq 0\). If \(L_3 : ax + by + cz + d < 0\) has a solution in the positive integers, then \(L_3\) admits an EGZ-generalization.
3 Negative results

In this section we exhibit different families of linear equations in three variables which admit no \( \text{EGZ}\)-generalization. In other words, we study equations, \( \mathcal{L} \), where \( R(\mathcal{L}, 2) \neq R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \). Naturally, we focus our attention in equations such that both \( R(\mathcal{L}, 2) \) and \( R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \) exist. Although Rado’s Theorem characterizes the equations \( \mathcal{L} \) such that \( R(\mathcal{L}, 2) \) exists, there is a small number of families of equations where the value \( R(\mathcal{L}, 2) \) is explicitly known, see [4], [8]. In this section we develop some ideas to compare \( R(\mathcal{L}, 2) \) and \( R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \) for some equations, and then we get some applications to show that \( \mathcal{L} \) does not admit \( \text{EGZ}\)-generalization.

**Theorem 3.1.** Let \( a,b,c,d \) be integers where \( a,b,c \) are odd and \( d \) is even, such that both \( R( ax + by + cz = \frac{d}{2}, 2) \) and \( R( ax + by + cz = d, \mathbb{Z}/3\mathbb{Z}) \) exist. Then
\[
2R \left( ax + by + cz = \frac{d}{2}, 2 \right) \leq R( ax + by + cz = d, \mathbb{Z}/3\mathbb{Z}) .
\]

**Proof.** Abbreviate writing \( R := R( ax + by + cz = \frac{d}{2}, 2) \). Let \( \chi_0 : [1, R - 1] \rightarrow \{0,1\} \) be a coloring such that \( ax + by + cz = \frac{d}{2} \) has not monochromatic solutions with respect to \( \chi_0 \). Define
\[
\chi : [1,2R - 1] \rightarrow \{0,1,2\}, \quad \chi(n) = \begin{cases} 
\chi_0 \left( \frac{n}{2} \right) & \text{if } n \text{ is even} \\
2 & \text{if } n \text{ is odd}
\end{cases}
\]
To prove the claim of the theorem, it is enough to show that \( ax + by + cz = d \) has no zero-sum solutions with respect to \( \chi \). Let \( (x_0, y_0, z_0) \) be a solution of \( ax + by + cz = d \). Since \( d \) is even and \( a,b,c \) are odd, we have that either the 3 entries of \( (x_0, y_0, z_0) \) are even or exactly one of the entries is even.

First assume that the 3 entries of \( (x,y,z) \) are even. Then \( \chi(x_0) = \chi_0 \left( \frac{x_0}{2} \right), \chi(y_0) = \chi_0 \left( \frac{y_0}{2} \right) \) and \( \chi(z_0) = \chi_0 \left( \frac{z_0}{2} \right) \); since \( ax + by + cz = \frac{d}{2} \) has not monochromatic solutions with respect to \( \chi_0 \), \( \left( \frac{x_0}{2}, \frac{y_0}{2}, \frac{z_0}{2} \right) \) is not monochromatic. Therefore \( (x_0, y_0, z_0) \) is not a zero-sum solutions with respect to \( \chi \).

Now assume that exactly one of the entries of \( (x,y,z) \) is even; without loss of generality assume that \( x_0 \) is even. Then \( \chi(x_0) = \chi_0 \left( \frac{x_0}{2} \right), \chi(y_0) = 2 \) and \( \chi(z_0) = 2 \). This means that \( (x_0, y_0, z_0) \) is not a zero-sum solutions with respect to \( \chi. \)

The next result is an immediate corollary of Theorem [3.1]

**Corollary 3.2.** Let \( \mathcal{L} \) be the equation \( ax + by + cz = 0 \), and assume that \( R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \) exist. Then, \( \mathcal{L} \) admits no \( \text{EGZ}\)-generalization if \( a,b \) and \( c \) are odd integers.

**Proof.** By taking \( d = 0 \) in Theorem [3.1] we conclude that \( 2R(\mathcal{L}, 2) \leq R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \), and so \( R(\mathcal{L}, 2) \neq R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \).

Also Theorem [3.1] provides some applications for non-homogeneous linear equations.

**Corollary 3.3.** Let \( d \) be a negative even integer. Then, the equation \( x + y - z = d \) admits no \( \text{EGZ}\)-generalization.
Proof. From [4, Thm. 9.14], we have that
\[ R(x + y - z = d, 2) = 5 - 4d, \]
and
\[ R \left( x + y - z = \frac{d}{2}, 2 \right) = 5 - 2d. \]

On the other hand, Theorem 3.1 leads to
\[ 2R \left( x + y - z = \frac{d}{2}, 2 \right) \leq R(x + y - z = d, \mathbb{Z}/3\mathbb{Z}). \]

Hence
\[
R(x + y - z = d, \mathbb{Z}/3\mathbb{Z}) \geq 2R \left( x + y - z = \frac{d}{2}, 2 \right) \\
= 2(5 - 2d) \\
> 5 - 4d \\
= R(x + y - z = d, 2). 
\]

Corollary 3.4. Let \( d \) be a positive integer congruent to 6, 8 or 0 modulo 10. Then, the equation \( x + y - z = d \) admits no EGZ-generalization.

Proof. On the one hand, we have from [4, Thm. 9.15] that
\[ R(x + y - z = d, 2) = d - \left\lceil \frac{d}{5} \right\rceil + 1, \]
and
\[ R \left( x + y - z = \frac{d}{2}, 2 \right) = \frac{d}{2} - \left\lceil \frac{d}{5} \right\rceil + 1. \]

On the other hand, Theorem 3.1 leads to
\[ 2R \left( x + y - z = \frac{d}{2}, 2 \right) \leq R(x + y - z = d, \mathbb{Z}/3\mathbb{Z}). \]

Thus, since \( d \) is congruent to 6, 8 or 0 modulo 10, we get that
\[
R(x + y - z = d, \mathbb{Z}/3\mathbb{Z}) \geq 2R \left( x + y - z = \frac{d}{2}, 2 \right) \\
= 2 \left( \frac{d}{2} - \left\lceil \frac{d}{5} \right\rceil + 1 \right) \\
> d - \left\lceil \frac{d}{5} \right\rceil + 1 \\
= R(x + y - z = d, 2). \]

\[ \square \]
4 Other directions

A linear homogenous equation is called \( r \)-regular if every \( r \)-coloring of \( \mathbb{N} \) contains a monochromatic solution of it (equivalently, an equation \( L \) is called \( r \)-regular if \( R(L, r) \) exist). A linear homogenous equation is called regular if it is \( r \)-regular for all positive integers \( r \). Denote by \( F_r \) the family of linear homogenous equations which are \( r \)-regular. For equations on \( k \geq 3 \) variables, Rado completely determined \( F_2 \): it is the set of equations, \( \sum_{i=1}^{k} c_i x_i = 0 \) for which there exist \( i, j \in \{1, \cdots, k\} \) such that \( c_i < 0 \) and \( c_j > 0 \) (see, for instance \([4]\)). For other values of \( r \in \mathbb{Z}^+ \), the family \( F_r \) is not characterized. Rado’s Single Equation Theorem states that a linear homogenous equation on \( k \geq 2 \) variables, \( \sum_{i=1}^{k} c_i x_i = 0 \) (\( c_i \)'s are non-zero integers), is regular if and only if there exist a non-empty \( D \subseteq \{1, \ldots, k\} \) such that \( \sum_{d \in D} c_d = 0 \). Naturally, \( F_{r+1} \subseteq F_r \) for all \( r \in \mathbb{Z}^+ \). In his Ph.D. dissertation, Rado conjectured that, for all \( r \in \mathbb{Z}^+ \), there are equations that are \( r \)-regular but not \((r+1)\)-regular. This conjecture was solved by Alexeev and Tsimerman in 2010 \([1]\), where they confirm that \( F_{r+1} \nsubseteq F_r \). For any \( k \in \mathbb{N} \), define \( F_{\mathbb{Z}/k\mathbb{Z}} \) to be the family of linear homogeneous equations in \( k \) variables, \( \mathcal{L} \), for which \( R(\mathcal{L}, \mathbb{Z}/k\mathbb{Z}) \) exist. By \([1]\) we know that

\[
F_3 \subseteq F_{\mathbb{Z}/3\mathbb{Z}} \subseteq F_2.
\]

We will show that

\[
F_3 \nsubseteq F_{\mathbb{Z}/3\mathbb{Z}}. \tag{15}
\]

For all \( n \in \mathbb{N} \), denote by \( \text{ord}_2(n) \) the maximum \( m \in \mathbb{Z} \) such that \( 2^m \) divides \( n \). First note that the equation \( x + 2y - 4z = 0 \) is not in \( F_3 \) since the coloring

\[
\chi : \mathbb{N} \to \{0, 1, 2\}, \quad \chi(n) = \begin{cases} 
0 & \text{if } \text{ord}_2(n) \equiv 0 \mod 3, \\
1 & \text{if } \text{ord}_2(n) \equiv 1 \mod 3, \\
2 & \text{if } \text{ord}_2(n) \equiv 2 \mod 3.
\end{cases}
\]

has not monochromatic solutions of it. The next proposition implies that \( x + 2y - 4z = 0 \) is in \( F_{\mathbb{Z}/3\mathbb{Z}} \) and therefore \((15)\) holds.

**Proposition 4.1.** The equation \( x + 2y - 4z = 0 \) satisfies \( R(x + 2y - 4z = 0, \mathbb{Z}/3\mathbb{Z}) = 8 \).

**Proof.** Let \( \mathcal{L} \) be the equation \( x + 2y - 4z = 0 \). First we show that \( R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \geq 8 \). Let \( \chi : [1, 8] \to \{0, 1, 2\} \) be a coloring. We assume that there are not zero-sum solutions of \( \mathcal{L} \) with respect to \( \chi \), and we will get a contradiction. Assume without loss of generality that \( \chi(1) = 0 \). Since \( (2, 1, 1) \) is a solution of \( \mathcal{L} \), \( \chi(2) \neq 0 \); assume without loss of generality that \( \chi(2) = 1 \). Note that \( (2, 3, 2) \), \( (4, 2, 2) \) and \( (4, 4, 3) \) are solutions of \( \mathcal{L} \). Thus either \( \chi(3) = 0 \) and \( \chi(4) = 2 \), or \( \chi(3) = 2 \) and \( \chi(4) = 0 \). Notice that \( (6, 3, 3) \) and \( (4, 6, 4) \) are solutions of \( \mathcal{L} \) so \( \chi(6) = 1 \). Since \( (6, 5, 4) \) and \( (2, 5, 3) \) are solutions of \( \mathcal{L} \), we get \( \chi(5) = 1 \). For any value of \( \chi(8) \), we obtain a zero-sum solution inasmuch as \( (8, 2, 3) \), \( (8, 4, 4) \) and \( (8, 6, 5) \) are solutions of \( \mathcal{L} \) and this is the desired contradiction.

On the other hand, \( R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) > 7 \) since there are not zero-sum solutions with respect to the coloring

\[
\chi : [1, 7] \to \{0, 1, 2\}, \quad \chi(n) = \begin{cases} 
0 & \text{if } n \in \{1, 4, 7\}, \\
1 & \text{if } n \in \{2, 5, 6\}, \\
2 & \text{if } n = 3.
\end{cases}
\]

and this completes the proof. \(\blacksquare\)
From the previous discussion, we know that $\mathcal{F}_3 \subseteq \mathcal{F}_{\mathbb{Z}/3\mathbb{Z}} \subseteq \mathcal{F}_2$. A natural question arises from this chain.

**Problem 1.** Is it true that $\mathcal{F}_{\mathbb{Z}/3\mathbb{Z}} \subseteq \mathcal{F}_2$?

From (1) we know that $\mathcal{F}_k \subseteq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$ for all $k \geq 3$. Thus it would be interesting to know if there are $k \in \mathbb{N}$ such that the equality is achieved.

**Problem 2.** For all $k \geq 3$, $\mathcal{F}_k \subseteq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$?

Finally we know that $\mathcal{F}_k \subseteq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$ and $\mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ for all $k \geq 3$. However we do not know whether there is a relation between $\mathcal{F}_{k-1}$ and $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$.

**Problem 3.** For all $k \geq 3$, $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}} \subseteq \mathcal{F}_{k-1}$?

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References

[1] B. Alexeev and J. Tsimerman, *Equations resolving a conjecture of Rado on partition regularity*, J. Combin. Theory Ser. A 117 (2010), 1008-1010.

[2] Arie Bialostocki, Guy Bialostocki and Daniel Schaal. *A zero-sum theorem* J. Combin. Theory Ser. A 101 (2003), 147-152.

[3] J. Fox, R. Radoićić, *The axiom of choice and the degree of regularity of equations over the reals*, preprint, December 2005.

[4] B. M. Landman, A. Robertson. *Ramsey Theory on the Integers*, Student Mathematical Library Vol. 73 AMS (2015).

[5] A. Robertson *Zero-sum generalized Schur numbers*, J. Comb. Number Theory 10 (2018), 51-62.

[6] A. Robertson, *Zero-sum analogues of van der Waerden’s theorem on arithmetic progressions*, J. Comb. 11 (2020), 231-248.

[7] A. Robertson, B. Roy, S. Sarkar, *The determination of 2-color zero-sum generalized Schur numbers*, Integers 18 (2018), Paper No. A96, 7 pp.

[8] S. D. Adhikari, L. Boza, S. Eliahou, J. M. Marín, M. P. Revuelta, and M. I. Sanz, *On the n-color Rado number for the equation $x_1 + x_2 + \ldots + x_k + c = x_{k+1}$*, Math. Comput. 85 (2016), 2047-2064.