DIFFUSION IN ONE DIMENSIONAL RANDOM MEDIUM
AND HYPERBOLIC BROWNIAN MOTION

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Abstract

Classical diffusion in a random medium involves an exponential functional of Brownian motion. This functional also appears in the study of Brownian diffusion on a Riemann surface of constant negative curvature. We analyse in detail this relationship and study various distributions using stochastic calculus and functional integration.

**Keywords**: Brownian motion, random media, negative curvature.
1. Introduction

There is a close link between one dimensional, or quasi-one dimensional, disordered systems and Brownian diffusion on Riemann manifolds of constant negative curvature. Such a correspondence can be traced back to the pioneering work of Gertsenshtein et al. (1) who have shown that statistical properties of reflection and transmission coefficients of waveguides with random inhomogeneities are directly related to some random walk on the Lobachevsky plane. There has been a renewed interest in this approach through the study of mesoscopic systems. The description of quasi-one dimensional mesoscopic wires involves a Fokker-Planck equation giving the probability distribution of the $N$ eigenvalues of the transmission matrix (2). Recently it has been shown that this equation can be interpreted as the diffusion equation on a Riemannian symmetric space (3).

The purpose of this work is to show how the one dimensional classical diffusion of a particle in a quenched random potential $U(x)$ that is itself a Brownian motion, possibly with some constant drift, is directly related to Brownian motion on the hyperbolic plane. Since the latter is the archetype of chaotic systems (4), our work forms a bridge between disordered and chaotic systems.

2. Fundamental random variable for one dimensional classical diffusion in a quenched random potential $U(x)$

A large amount of work has been devoted to random random walks, defined on a lattice by the following Master equation

$$P_n(t + 1) = \alpha_{n-1}P_{n-1}(t) + \beta_{n+1}P_{n+1}$$

(2.1)

where $\alpha_n$ is the random quenched transition rate from site $n$ to site $(n + 1)$, and $\beta_n = 1 - \alpha_n$ is the random quenched transition rate from site $n$ to site $(n - 1)$. From a physical point of view it is convenient to introduce a corresponding random potential $U(n)$ on each site $n$ and to write the ratio of the two transition rates $\alpha_n$ and $\beta_n$ as an Arrhénius factor

$$\sigma_n \equiv \frac{\beta_n}{\alpha_n} = \frac{e^{-\beta[U(n+1)-U(n)]}}{e^{-\beta[U(n)-U(n-1)]}} = e^{\beta[U(n+1)-U(n-1)]}$$

(2.2)

The study of different physical quantities related to this random random walk (5) involves systematically random variables of the following form

$$Z(a, b) = \sum_{n=a}^{b} \prod_{k=a}^{n} \sigma_k = \sigma_a + \sigma_a\sigma_{a+1} + \cdots + \sigma_a\sigma_{a+1}\cdots\sigma_b$$

(2.3)
Their fundamental property is to satisfy the linear random coefficient recurrence relation
\[ Z(a, b) = \sigma_a \left[ 1 + Z(a + 1, b) \right] \] (2.4)

This discrete multiplicative stochastic process also appears for the Ising chain in a random magnetic field \(^6\).

Let us now consider the continuous model of classical diffusion on the line defined by the Fokker-Planck equation for the probability density \(P(x, t \mid x_0, 0)\)
\[ \frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial P}{\partial x} - \beta F(x) P \right) \] (2.5)

where \(\{F(x)\}\) is a quenched random force. In this continuous limit the discrete random variable \(Z(a, b)\) defined in (2.3) becomes the following exponential functional of the random potential \(U(x) = - \int^x F(y)dy\)
\[ \tau(a, b) = \int^b_a dx \ e^{\beta \left[ U(x) - U(a) \right]} = \int^b_a dx \ e^{-\beta \int^x_a F(y)dy} \] (2.6)

The evolution of this functional is governed by the stochastic differential equation
\[ \frac{\partial \tau}{\partial a}(a, b) = \beta F(a) \tau(a, b) - 1 \] (2.7)

which replaces the random coefficient recurrence relation (2.4) satisfied by \(Z(a, b)\). Note that the stochastic term \(F(a)\) appears multiplicatively, so that the fluctuations of the random force are coupled to the values taken by the random process \(\tau(a, b)\).

Let us now make clear how the functional \(\tau(a, b)\) arises in some physical quantities associated with the classical diffusion of a particle in the quenched random environment.

• If the random force has a positive mean \(<F(x) > = F_0 > 0\), the probability distribution of the random functional
\[ \tau_\infty \equiv \lim_{(b-a) \to \infty} \tau(a, b) \] (2.8)
determines the large time anomalous behavior of the Brownian particle position \(^7\). In particular, the velocity defined for each sample as
\[ V = \lim_{t \to \infty} \frac{d}{dt} \int_{-\infty}^{+\infty} dx \ x P(x, t \mid x_0, 0) \] (2.9)
is a self-averaging quantity inversely proportional to the first moment of \(\tau_\infty\) \(^7\)
\[ V = \frac{1}{2 \ <\tau_\infty>} \] (2.10)
When the quenched random force is distributed with the Gaussian measure

\[ DF(x) e^{-\frac{1}{2\sigma} \int dx \ [F(x) - F_0]^2} \]  

the probability distribution \( P_\infty(\tau) \) of the functional \( \tau_\infty \) reads

\[ P_\infty(\tau) = \frac{\alpha}{\Gamma(\mu)} \left( \frac{1}{\alpha \tau} \right)^{1+\mu} e^{-\frac{1}{\alpha \tau}} \propto \tau^{\infty} \]  

where \( \mu = \frac{2F_0}{\beta \sigma} > 0 \) is a dimensionless parameter and \( \alpha = \frac{\sigma \beta^2}{2} \). The algebraic decay at large \( \tau \) accounts for the dynamical phase transitions which occur in this model \( (7) \). In particular, equation (2.10) implies that the value \( \mu = 1 \) separates a phase of vanishing velocity \( V = 0 \) for \( 0 < \mu < 1 \), and a phase of finite velocity \( V > 0 \) for \( \mu > 1 \).

• The functional \( \tau(a, b) \) also arises in the study of transport properties of finite size disordered samples. The stationary current \( J_N \) which goes through a disordered sample of length \( N \) with fixed concentrations \( P_0 \) and \( P_N \) at the boundary can be written in terms of the exponential functional \( \tau_N \equiv \tau(0, N) \) as \( (8) \) \( (9) \)

\[ J_N = \frac{1}{2} \left[ \frac{P_0}{\tau_N} - P_N \frac{\partial \ln \tau_N}{\partial N} \right] \]  

When the end \( x = N \) is a trap described by the boundary condition \( P_N = 0 \), the flux \( J_N \) is simply a random variable inversely proportional to \( \tau_N \). The probability distribution of \( \tau_N \) has been studied for the case of zero mean force \( F_0 = 0 \) \( (8) \), and for the general case with arbitrary mean force \( (9) \) by different methods.

It is also interesting to point out that a lot of mathematical work has recently been devoted to the functional \( \tau_N \) in relation with finance \( (10) \) \( (11) \) \( (12) \). In \( (11) \), Yor pointed out the relation between the functional \( \tau_N \) for the particular case \( \mu = \frac{1}{2} \), and free Brownian motion on the hyperbolic plane. In the following, we first rederive this correspondance, and then generalize it to arbitrary \( \mu \), using some external drift on the hyperbolic plane.

3. Relation with hyperbolic Brownian motion

The upper half-plane \( \{ (x, y) \ , \ y > 0 \} \) endowed with the metric

\[ ds^2 = \frac{dx^2 + dy^2}{y^2} \]  

defines a two-dimensional Riemann manifold of constant negative Gaussian curvature \( R = -1 \). The surface element \( dS \) and the Laplace operator \( \Delta \) are covariantly defined as

\[ dS = \frac{dx \ dy}{y^2} \quad \text{and} \quad \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]
Free Brownian motion on this manifold is defined by the diffusion equation for the Green’s function $G_t(x, y)$

$$
\frac{\partial G}{\partial t} = D \Delta G = D \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G
$$  (3.3)

It is convenient to choose the initial condition at the point $(x = 0, y = 1)$

$$
G_t(x, y) \to \delta(x) \delta(y - 1)
$$  (3.4)

The normalization of the Green function $G_t(x, y)$ then reads for any time $t$

$$
1 = \int dS \ G_t(x, y) = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} dy \ \frac{1}{y^2} G_t(x, y)
$$  (3.5)

Consider the probability density $P_t(x, y)$

$$
P_t(x, y) = \frac{1}{y^2} G_t(x, y)
$$  (3.6)

normalized with respect to the flat measure $dx \ dy$

$$
1 = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} dy \ P_t(x, y)
$$  (3.7)

This probability density satisfies the Fokker-Planck equation

$$
\frac{\partial P}{\partial t} = D \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) y^2 \ P
$$  (3.8)

and the initial condition at $t = 0$

$$
P_t(x, y) \to \delta(x)\delta(y - 1)
$$  (3.9)

We now introduce two independent Gaussian white noises $\eta_1(t)$ and $\eta_2(t)$ and write the stochastic differential equations for the process $\{x(t), y(t)\}$ corresponding to the Fokker-Planck equation (3.8) following respectively the Itô or Stratonovich convention

\begin{align}
\text{Itô} \quad \begin{cases} 
\frac{dx}{dt} = \sqrt{2D} \ y \ \eta_1(t) \\
\frac{dy}{dt} = \sqrt{2D} \ y \ \eta_2(t)
\end{cases} & \quad (3.10) \\
\text{Stratonovich} \quad \begin{cases} 
\frac{dx}{dt} = \sqrt{2D} \ y \ \eta_1(t) \\
\frac{dy}{dt} = -D \ y + \sqrt{2D} \ y \ \eta_2(t)
\end{cases} & \quad (3.11)
\end{align}
Integration of these two coupled differential equations gives

\[
\begin{cases}
  x(t) = \sqrt{2D} \int_0^t dv \eta_1(v) \ e^{-Dv + \sqrt{2D} \int_0^v \eta_2(s) ds} \\
y(t) = e^{-Dt + \sqrt{2D} \int_0^t \eta_2(s) ds}
\end{cases}
\] (3.12)

The process \( y(t) \) is therefore simply the exponential of some linear Brownian motion with negative drift. Let us now study more precisely the process \( \{x(t)\} \). Since the white noise \( \eta_1 \) appears linearly in \( x(t) \), all the odd moments therefore vanish

\[
<x^{2n+1}(t) >_{\eta_1} = 0
\] (3.13)

and all the even moments can be rewritten as averages over \( \eta_2 \) only. In particular the second moment reads

\[
<x^2(t) >_{\eta_1,\eta_2} = 2D < \int_0^t dv \ e^{-2Dv + 2\sqrt{2D} \int_0^v \eta_2(s) ds} >_{\eta_2}
\] (3.14)

If we set

\[
\beta \int_0^v F(u) du = 2Dv - 2\sqrt{2D} \int_0^v \eta_2(s) ds
\] (3.15)

the expression between brackets on the right hand side of Eq (3.14) is nothing but the functional

\[
\tau_t = \int_0^t dv \ e^{-\beta \int_0^v F(u) du}
\] (3.16)

which is encountered in the study of classical diffusion in a quenched force \( \{F\} \) distributed as a Gaussian white noise

\[
\mathcal{D} F(x) \ e^{-\frac{1}{2\sigma} \int_{-\infty}^{+\infty} [F(x) - F_0]^2 dx}
\] (3.17)

with parameters \( F_0 = \frac{2D}{\beta} \) and \( \sigma = \frac{D}{\beta^2} \). More generally, all even moments of \( x(t) \) are proportional to moments of the functional \( \tau_t \)

\[
<x^{2n}(t) >_{\eta_1,\eta_2} = (2D)^n < \tau_t^n >_F \mathcal{N}(n)
\] (3.18)

\( \mathcal{N}(n) \) being the combinatorial factor coming from Wick’s theorem that counts the number of ways to pair the \( 2n \) functions \( \eta_1 \). In particular, \( \mathcal{N}(n) \) is equal to the moment of order \( 2n \) of a suitable Gaussian random variable \( \xi \) with variance unity

\[
\mathcal{N}(n) = \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \xi^{2n} e^{-\frac{\xi^2}{2}} = 2^n \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})} = \prod_{k=1}^{n} (2k - 1)
\] (3.19)
These identities between moments allow us to rewrite the process \( x(t) \) in the very simple form suggested by Yor \(^{(11)}\)

\[
x(t) = \sqrt{2D} \int_{0}^{\tau_t} du \eta(u) \tag{3.20}
\]

where \( \eta \) is a new independent Gaussian white noise. The process \( x(t) \) is therefore identical in law to a linear Brownian motion whose effective time is the random variable \( \tau_t \). This result can also be obtained through a functional integration method using an appropriate change of time in path-integral (see Appendix).

To sum up, the free Brownian motion \( \{x(t),y(t)\} \) on the hyperbolic plane can be rewritten in terms of the two independent white noises of measure

\[
\mathcal{D} F(x) \ e^{-\frac{1}{2\sigma} \int_{-\infty}^{+\infty} [F(x) - F_0]^2 dx} \mathcal{D} \eta(t) \ e^{-\frac{1}{2} \int dt \ \eta^2(t)} \tag{3.21}
\]

The process \( y(t) \) is simply the exponential of the Brownian motion with drift \( U(t) = -\int_{0}^{t} F(u)du \)

\[
y(t) = e^{-\frac{\beta}{2} \int_{0}^{t} F(u)du} = e^{\frac{\beta}{2} U(t)} \tag{3.22}
\]

The process \( x(t) \) can be viewed as a linear Brownian motion

\[
x(t) = \sqrt{2D} \int_{0}^{\tau_t} du \ \eta(u) \tag{3.23}
\]

with an effective time \( \tau_t \) that is itself a random process depending on \( y(t) \)

\[
\tau_t = \int_{0}^{t} dv \ e^{\beta U(v)} = \int_{0}^{t} dv \ y^2(v) \tag{3.24}
\]

Note that the dimensionless parameter \( \mu \equiv \frac{2F_0}{\beta \sigma} \) which characterizes the different phases of anomalous diffusion is \( \frac{1}{2} \) for the free hyperbolic Brownian motion. This value represents the natural drift induced by the free hyperbolic Brownian motion. This value is, however, easy to generalize the previous analysis to arbitrary \( \mu \) with the introduction of some external constant drift \( m \) in direction \( y \)

\[
\mu = \frac{1}{2} + m \tag{3.25}
\]

The corresponding stochastic differential equations then read

\[
\text{Itô} \begin{cases}
\frac{dx}{dt} = \sqrt{2D} \ y \ \eta_1(t) \\
\frac{dy}{dt} = -2D \ m \ y + \sqrt{2D} \ y \ \eta_2(t)
\end{cases} \tag{3.26}
\]
Stratonovich\begin{equation}
\begin{cases}
\frac{dx}{dt} = \sqrt{2D} y \eta_1(t) \\
\frac{dy}{dt} = -2D\mu y + \sqrt{2D} y \eta_2(t) 
\end{cases}
\end{equation}

For any $\mu$, there is therefore a direct correspondence through Eqs (3.22-3.23) between the joint stochastic process characterizing the one dimensional diffusion $\{U(t)\}$, exponential functional $\tau_t$ and the Brownian motion $\{x(t), y(t)\}$ on the hyperbolic plane with possibly some external constant drift $m$ along direction $y$. We now consider some consequences of this correspondence.

4. Marginal laws of the processes $\tau_t$, $x(t)$ and $y(t)$

The marginal law $Y_t(y)$ of the process $y(t)$ reads according to eq (3.22)

\begin{equation}
Y_t(y) = \int_{U(0)=0} DU(s) e^{-\frac{1}{2\sigma} \int_0^t \left( \frac{dU}{ds} + F_0 \right)^2 ds} \delta \left( y - \alpha e^{\frac{\beta}{2} U(t)} \right)
\end{equation}

We therefore get after some algebra the following log-normal distribution

\begin{equation}
Y_t(y) = \frac{1}{y \sqrt{4\pi Dt}} e^{-\frac{1}{4Dt} \left[ \ln(y) + 2\mu Dt \right]^2}
\end{equation}

where $\mu = \frac{2F_0}{\beta \sigma}$ and $D = \frac{\beta^2 \sigma}{8}$. In the case of free Brownian motion ($\mu = \frac{1}{2}$), this marginal law tends to a $\delta$ distribution in the limit $t \to \infty$

\begin{equation}
Y_\infty(y) = \delta(y)
\end{equation}

The Brownian particle is therefore attracted to the $y = 0$ axis as a result of the curvature of the hyperbolic plane. Note that this axis represents infinity on this plane. This limit law remains unchanged as long as $\mu \equiv \left( \frac{1}{2} + m \right) > 0$. However, when the constant external drift $m$ along $y$ is negative enough to overcome the natural drift of the hyperbolic plane ($m < -\frac{1}{2}$), there is no equilibrium distribution for the process $y(t)$. In fact for $\mu > 0$, the existence of this stationary distribution for the process $y(t)$ governs the existence of a stationary distribution for the process $x(t)$ that we now construct.

The process $x(t)$ is a Brownian motion of effective time $\tau_t$ which is a functional of the process $y(t)$

\begin{equation}
x(t) = \sqrt{2D} \int_0^t du \eta(u) \quad \text{with} \quad \tau_t = \int_0^t dv e^{\beta U(v)} = \int_0^t dv y^2(v)
\end{equation}
The statistical independence of $\eta(t)$ and $\tau_t$ allows us to write the marginal law $X_t(x)$ of the process $x(t)$ in terms of the probability distribution $\psi_t(\tau)$ of the functional $\tau_t$

$$X_t(x) = \int_0^\infty d\tau \, \psi_t(\tau) \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{x^2}{4D\tau}} \quad (4.5)$$

In a previous work \(^{(9)}\), we showed that $\psi_t(\tau)$ satisfies a Fokker-Planck equation, associated with the stochastic differential equation (2.7) which in the Stratonovich prescription reads

$$\frac{\partial \psi_t(\tau)}{\partial t} = \frac{\partial}{\partial \tau} \left[ \alpha \tau^2 \frac{\partial \psi_N(\tau)}{\partial \tau} + \left( (\mu + 1)\alpha \tau - 1 \right) \psi_N(\tau) \right] \quad (4.6)$$

It has to be supplemented by the initial condition $\psi_{t=0}(\tau) = \delta(\tau)$. We recall the solution found in terms of an expansion in the Fokker-Planck eigenvectors basis

$$\psi_t(\tau) = \sum_{0 \leq n < \frac{\mu}{2}} e^{-\alpha n(\mu-n)} \frac{(-1)^n(\mu-2n)}{\Gamma(1+\mu-n)} \left( \frac{1}{\alpha \tau} \right)^{1+\mu-n} L_n^{\mu-2n} \left( \frac{1}{\alpha \tau} \right) e^{-\frac{1}{\alpha \tau}}$$

$$+ \frac{\alpha}{4\pi^2} \int_0^\infty ds \, e^{-\frac{\alpha}{4\pi}(\mu^2+s^2)} s \sinh \pi s \left| \Gamma \left( -\frac{\mu}{2} + i \frac{s}{2} \right) \right|^2 \left( \frac{1}{\alpha \tau} \right)^{\frac{\mu}{2}} W_{\frac{\mu}{2} + i \frac{s}{2}} \left( \frac{1}{\alpha \tau} \right) e^{-\frac{1}{2\alpha \tau}} \quad (4.7)$$

This leads after integration with the Gaussian kernel (4.5) and the transposition $\alpha = 4D$ to an expansion presenting the same time relaxation spectrum

$$X_t(x) = \sum_{0 \leq n < \frac{\mu}{2}} e^{-4Dn(\mu-n) t} \frac{(-1)^n(\mu-2n)}{n!\Gamma(1+\mu-n)} \frac{\Gamma(\mu + \frac{1}{2} - n)}{\Gamma(\frac{1}{2} - n)} \left( \frac{1}{1+x^2} \right)^{\mu+\frac{1}{2}}$$

$$\times F \left( -n, n - \mu, \frac{1}{2}; -x^2 \right)$$

$$+ \frac{1}{4\pi^2 \sqrt{\pi}} \int_0^\infty ds \, e^{-D(\mu^2+s^2) t} s \sinh \pi s \left| \Gamma \left( -\frac{\mu}{2} + i \frac{s}{2} \right) \right|^2 \left| \Gamma \left( \frac{\mu+1}{2} + i \frac{s}{2} \right) \right|^2$$

$$\times F \left( \frac{\mu+1}{2} + i \frac{s}{2}, \frac{\mu+1}{2} - i \frac{s}{2}, \frac{1}{2}; -x^2 \right) \quad (4.8)$$

where $F(a, b, c; z)$ denotes the hypergeometric function of parameters $(a, b, c)$.

For $\mu > 0$, there exists an equilibrium distribution $X_\infty(x)$

$$X_\infty(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu)} \left( \frac{1}{1+x^2} \right)^{\mu+\frac{1}{2}} \quad (4.9)$$
The existence of this limit law for hyperbolic Brownian motion reveals a "localization" phenomenon in direction $x$. This effect comes from the attraction towards the axis $y = 0$. Note that for the free case ($\mu = \frac{1}{2}$) the asymptotic marginal law $X_\infty(x)$ is simply a Lorentzian

$$X_\infty(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$$  \hspace{1cm} (4.10)

In summary, there exist equilibrium distributions for $Y_\infty$ and $X_\infty$ as long as $\mu = (\frac{1}{2} + m) > 0$

$$\begin{align*}
Y_\infty(y) &= \delta(y) \\
X_\infty(x) &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu)} \left( \frac{1}{1 + x^2} \right)^{\mu + \frac{1}{2}}
\end{align*}$$  \hspace{1cm} (4.11)

The joint law of the processes $\{x(t), y(t)\}$ therefore presents the following factorized form in the limit $t \to \infty$

$$P_\infty(x, y) = X_\infty(x)\delta(y)$$  \hspace{1cm} (4.12)

However, as long as time $t$ is finite, the two processes $x(t)$ and $y(t)$ remain coupled. The study of their joint law is then needed to get a complete description of hyperbolic Brownian motion.

5. **Joint laws of the processes $\{\tau_t, U(t)\}$ and $\{x(t), y(t)\}$**

Path-integral methods allow us to obtain very simply the joint law $\psi_t(\tau \parallel u)$ of the random variables

$$\tau_t = \int_0^t dx \ e^{\beta U(x)} \quad \text{and} \quad U(t)$$

Let us generalize our previous approach (9). The $\tau$-Laplace transform of the joint law

$$E \left( e^{-p\tau_t} \parallel u \right) = \int_0^\infty d\tau \ e^{-p\tau} \ \psi_t(\tau \parallel u)$$  \hspace{1cm} (5.1)

can be written as a path-integral over the random potential

$$E \left( e^{-p\tau_t} \parallel u \right) = \int_{U(0)=0}^{U(t)=u} DU(x) \ e^{-\frac{1}{2\sigma} \int_0^t \left( \frac{dU}{dx} + F_0 \right)^2 dx - p \int_0^t dx \ e^{\beta U(x)}}$$

$$= e^{-\frac{F_0^2 t}{2\sigma}} \ e^{-\frac{F_0}{\sigma} \int_{U(0)=0}^{U(t)=u} DU(x) \ e^{-\frac{1}{2\sigma} \int_0^t \left( \frac{dU}{dx} \right)^2 dx - p \int_0^t dx e^{\beta U(x)}}}$$  \hspace{1cm} (5.2)
The remaining path-integral is simply the Euclidean quantum mechanics Green’s function $\langle u | e^{-tH} | 0 \rangle$ associated with the Liouville Hamiltonian

$$
H = -\frac{\sigma}{2} \frac{d^2}{du^2} + p \ e^\beta u
$$

(5.3)

We therefore get

$$
E \left( e^{-p\tau_t} \parallel u \right) = e \frac{F_0^2}{2\sigma} \frac{F_0}{ \sigma} u \ < u | e^{-tH} | 0 >
$$

(5.4)

The expansion of the Green’s function $\langle u | e^{-tH} | 0 \rangle$ in the basis of eigenfunctions $\psi_k(u)$ of the Hamiltonian $H$

$$
\psi_k(u) = 2 \sqrt{\frac{\beta k}{\alpha \pi}} \sinh \frac{2k\pi}{\sqrt{\alpha}} K_{2ik/\sqrt{\alpha}} \left( 2 \sqrt{\frac{p}{\alpha}} e^\beta u/2 \right)
$$

(5.5)

gives

$$
< u | e^{-tH} | 0 > = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \psi_k(u) \psi_k^*(0) \ e^{-k^2t}
$$

(5.6)

We finally obtain that the $\tau$-Laplace transform of the joint law $\phi_t(\tau \parallel y)$ of the random variables

$$
\tau_t = \int_0^t dx \ e^\beta U(x) \quad \text{and} \quad y(t) = e^\frac{\beta}{2} U(t)
$$

reads after some changes of variables

$$
E \left( e^{-p\tau_t} \parallel y \right) = \int_0^{+\infty} d\tau e^{-p\tau} \phi_t(\tau \parallel y)
$$

$$
= e^{-\frac{\alpha t}{4} \mu^2} \frac{1}{y^{1+\mu}} \int_{-\infty}^{+\infty} dq \ e^{-\frac{\alpha t}{4} q^2} q \sinh \pi q K_{iq} \left( 2y \sqrt{\frac{p}{\alpha}} \right) K_{iq} \left( 2 \sqrt{\frac{p}{\alpha}} \right)
$$

(5.7)

Let us now compute the $x^2$-Laplace transform of the joint law $Q_t(x, y)$ of the hyperbolic Brownian motion starting from the series of moments of $x^2(t)$, with $y(t)$ being fixed as $y$

$$
E \left( e^{-qx^2(t)} \parallel y \right) = \int_{-\infty}^{+\infty} dx \ e^{-qx^2} Q_t(x, y)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} E \left( x^{2n}(t) \parallel y(t) = y \right)
$$

(5.8)

Eq (3.23) allows us to write the following relation between moments of $x^2(t)$ and moments of $\tau_t$ when $y(t)$ is fixed as $y$

$$
E \left( x^{2n}(t) \parallel y \right) = (2D)^n \ N(n) E \left( \tau_t^n(t) \parallel y \right)
$$

(5.9)
with the combinatorial factor $\mathcal{N}(n)$ introduced in (3.18-3.19). Using the integral representation

$$
\mathcal{N}(n) = \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \xi^{2n} e^{-\frac{\xi^2}{2}}
$$

we can resum the series of moments of $\tau_t$ under the integral

$$
E \left( e^{-q x^2(t) \parallel y} \right) = \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} E \left( e^{-q 2D \xi^2 \tau_t \parallel y} \right) 
$$

Using Eq (5.7) and the correspondence $\alpha = 4D$ we get

$$
E \left( e^{-q x^2(t) \parallel y} \right) = \frac{1}{\pi^2 \sqrt{\pi q}} \left( \frac{1}{y} \right)^{\mu+1} \int_{0}^{\infty} dk \ e^{-\frac{k^2}{4q}} \int_{-\infty}^{+\infty} d\nu \ e^{-D t \left( \mu^2 + \nu^2 \right)} \nu \sinh \pi \nu \ K_{i\nu} (ky) K_{i\nu}(k)
$$

For the free case ($\mu = \frac{1}{2}$), this result can readily be recovered from the expression of the Green’s function $G_t(x,y)$ on the hyperbolic plane (14) through

$$
E \left( e^{-q x^2(t) \parallel y} \right) = \int_{-\infty}^{+\infty} dx \ e^{-q x^2} \frac{1}{y^2} G_t(x,y)
$$

Let us finally mention that an alternative expression of the joint law $\phi_t(\tau \parallel y)$ given in Eq (5.7) has been obtained by Yor (12) through the time Laplace transform

$$
\int_{0}^{\infty} dt \ e^{-st} \phi_t(\tau \parallel y) = \frac{1}{\tau y^{1+\mu}} e^{-\left( \frac{1+\mu^2}{\tau y^2} \right)} I_{\nu} \left( \frac{z}{\alpha \tau} \right) \quad \text{where} \quad \nu = \sqrt{\mu^2 + 4 \frac{s}{\alpha}}
$$

More generally, we refer to the mathematical literature (10) (11) (12) for different expressions related with probability distribution of the functional $\tau \{ T_s \}$ where $T_s$ is an independent time, exponentially distributed with parameter $s$.

6. Conformal mapping from Poincaré half-plane to unit-disk

We introduce the conformal mapping from the Poincaré half upper-plane \{ $z = x + iy, y > 0$ \} to the unit disk \{ $w = re^{i\theta}, |r| \leq 1$ \}

$$
w = \frac{iz + 1}{z + i}
$$

12
The radial coordinate $r$ is directly related to the hyperbolic distance $d$ on the Poincaré half-plane between the arbitrary point $(x, y)$ and the point \{\(x = 0, y = 1\)\} that we choose as the initial point of Brownian motion (3.9)

\[
r = \tanh \left( \frac{d}{2} \right)
\]

The circle at infinity $r = 1$ corresponds to the axis $y = 0$. The unit disk is very well suited to study the free hyperbolic Brownian motion since it contains explicitly the rotational invariance in the angle $\theta$. Let us write in the new coordinates the metric

\[
ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{4}{(1 - r^2)^2} (dr^2 + r^2 d\theta^2)
\]

and the Laplace operator

\[
\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{(1 - r^2)^2}{4} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right]
\]

In the free case, the Fokker-Planck equation for the probability density $Q_t(r, \theta)$ reads

\[
\frac{\partial Q}{\partial t} = \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} \left( \frac{(1 - r^2)^2}{4r} Q \right) + \frac{(1 - r^2)^2}{4r^2} \frac{\partial^2}{\partial \theta^2} Q \right]
\]

The rules of stochastic calculus (13) give the corresponding stochastic differential equations

\[
\text{Itô} \left\{ \begin{array}{l}
\frac{dr}{dt} = D \frac{(1 - r^2)^2}{4r} + \sqrt{2D} \frac{1 - r^2}{2} \eta_r(t) \\
\frac{d\theta}{dt} = \sqrt{2D} \frac{1 - r^2}{2r} \eta_\theta(t)
\end{array} \right.
\]

\[
\text{Stratonovich} \left\{ \begin{array}{l}
\frac{dr}{dt} = D \frac{1 - r^4}{4r} + \sqrt{2D} \frac{1 - r^2}{2} \eta_r(t) \\
\frac{d\theta}{dt} = \sqrt{2D} \frac{1 - r^2}{2r} \eta_\theta(t)
\end{array} \right.
\]

These equations cannot be integrated straightforwardly to give the processes $\{r(t), \theta(t)\}$ as functionals of the white noises $\{\eta_r(t), \eta_\theta(t)\}$, unlike the representation (3.12) for the processes $\{x(t), y(t)\}$. However, the asymptotic probability distribution $Q_\infty(r, \theta)$ in the limit $t \to \infty$ can be written directly from symmetry considerations. As expected, it is simply the uniform measure on the unit circle

\[
Q_\infty(r, \theta) = \frac{1}{2\pi} \delta(r - 1)
\]

For $\mu \neq \frac{1}{2}$, the external constant drift $m = (\mu - \frac{1}{2})$ along direction $y$ breaks the rotational invariance in the angle $\theta$, and the unit disk is not particularly well suited anymore. Nevertheless, it is easy to write the asymptotic probability distribution $Q_\infty(r, \theta)$
from the asymptotic law (4.12) for \( P_\infty(x, y) \) and from the expression of the Jacobian \( J\{(r, \theta)/(x, y)\} \)

\[
Q_\infty(r, \theta) = \frac{4r}{(r^2 + 1 - 2r \sin \theta)^2} P_\infty \left( \frac{2r \cos \theta}{r^2 + 1 - 2r \sin \theta}, \frac{1 - r^2}{r^2 + 1 - 2r \sin \theta} \right)
\]

(6.9)

We obtain after some algebra the generalisation of (6.8)

\[
Q_\infty(r, \theta) = \delta(r - 1) \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \mu + \frac{1}{2} \right)}{\Gamma(\mu)} \frac{1}{2^\mu + \frac{1}{2}} (1 - \sin \theta)^{\mu - \frac{1}{2}}
\]

(6.10)

7. Conclusion

We have explored in great details the relations existing between, on the one hand, the exponential functional \( \tau_t \) that governs most of the transport properties of one dimensional classical diffusion in a random potential \( U(x) \), distributed as a Brownian motion possibly with some drift, and on the other, free or biased hyperbolic Brownian motion. As discussed in the introduction, this has to be considered as a special example of the more general link that connects one dimensional, or quasi-one dimensional, disordered systems, which usually admit some multiplicative stochastic structure, and Brownian diffusion on symmetric spaces.

It is interesting to point out that the time relaxation spectrum found for the probability distributions \( \psi_t(\tau) \) and \( X_t(x) \) (Eqs 4.7 and 4.8) also appears for the quantum spectrum of a particle subject to a constant magnetic field \( B \) on the hyperbolic plane \(^{(15)}\). For the hyperbolic geometry, constant magnetic field is defined as the flux per covariant surface element \( dS = \frac{1}{y^2} dx dy \). The two spectra coincide if we choose the following correspondance between magnetic field \( B > 0 \) and drift \( \mu \)

\[
B = \frac{1 + \mu}{2}
\]

In this context, the existence of bound states in a strong enough magnetic field \( B > \frac{1}{2} \) corresponds to the presence of closed classical orbits \(^{(15)}\). It would be particularly interesting to understand more deeply this correspondance between spectra at the level of stochastic processes themselves.

As a final remark, let us mention that the Liouville Hamiltonian that we encountered in the path-integral formalism (5.3), and that is closely related to hyperbolic geometry, also appears in the study of refined properties of one dimensional quantum localisation for the Schrödinger Hamiltonian \( H = -\frac{d^2}{dx^2} + V(x) \) where \( V(x) \) is a Gaussian white noise potential \(^{(16)}\). Kolokolov uses a path-integral method to compute correlation functions of eigenstates and distribution function of inverse participation ratio in
the high energy limit. In this formalism, Liouville Hamiltonian shows up in an effective action of path-integral. Expansion of this path-integral in a basis of eigenstates then gives expressions very similar to the one we obtained for the probability distribution of the joint law of the processes \( \{ \tau_t, U(t) \} \).

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**Appendix : Path-integral method to prove the identity in law (3.20)**

In paragraph 3 we derived the identity in law (3.20) mentioned by Yor for the free hyperbolic Brownian motion \((11)\) through a moments calculation. As explained before, this identity can easily be generalized to any \( \mu \) by the same method. Let us now derive it through a more straightforward path-integral method using an appropriate stochastic reparametrisation of time in path-integral. This tool has already proven to be very useful in other contexts \((17)\).

The integration of stochastic differential equations (3.27) gives the process \( x(t) \) as the following functional of the two white noises \( \{ \eta_1, \eta_2 \} \)

\[
x(t) = \sqrt{2D} \int_0^t dv \, \eta_1(v) \, e^{-2D\mu v + \sqrt{2D} \int_0^v \eta_2(s)ds}
\]

(A.1)

The marginal law \( X_t(x) \) of the process \( x(t) \) therefore reads by definition

\[
X_t(x) = \int \mathcal{D}\eta_1(u) \, \mathcal{D}\eta_2(u) \, e^{-\frac{1}{2} \int_0^t du \, (\eta_1^2(u) + \eta_2^2(u))} \delta\left(x - \sqrt{2D} \int_0^t dv \, \eta_1(v) \, e^{-2D\mu v + \sqrt{2D} \int_0^v \eta_2(s)ds}\right)
\]

(A.2)

Let us first change from \( \eta_2(u) \) to \( U(u) \) defined by \( \frac{d}{dv} U(v) = -2D\mu v + \sqrt{2D} \int_0^v \eta_2(s)ds \)

\[
X_t(x) = \int \mathcal{D}\eta_1(u) \, e^{-\frac{1}{2} \int_0^t du \, \eta_1^2(u)} \int_{U(0)=0} \mathcal{D}U(u) e^{-\frac{1}{4D} \int_0^t du \, \left( \frac{\beta}{2} \frac{dU}{dv} + 2D\mu \right)^2}
\]
\begin{align}
\delta \left( x - \sqrt{2D} \int_0^t dv \, \eta_1(v) e^{\frac{\beta}{2} U(v)} \right) \quad (A.3)
\end{align}

And now from \( \eta_1(u) \) to \( x(u) \equiv \sqrt{2D} \int_0^u dv \, \eta_1(v) e^{\frac{\beta}{2} U(v)} \)

\begin{align}
X_t(x) &= \int_0^x \mathcal{D}x(u) \int_{U(0)=0}^U \mathcal{D}U(u) e^{-\frac{1}{4D} \int_0^t du \left( \frac{\beta}{2} \frac{dU}{dv} + 2D\mu \right)^2} \\
&\quad - \frac{1}{4D} \int_0^t du \left( \frac{dx}{du} e^{-\frac{\beta}{2} U(u)} \right)^2 \\
&= \int_0^\infty d\tau \, 1 \sqrt{\frac{1}{4\pi D\tau}} e^{-\frac{x^2}{4Dt}} \psi(t(\tau)) \quad (A.4)
\end{align}

Let us now perform a time-reparametrisation of the trajectories \( x(u) \) in order to recover the Wiener measure

\begin{align}
\int_0^t du \left( \frac{dx}{du} \right)^2 e^{-\beta U(u)} = \int_0^\tau ds \left( \frac{dx}{ds} \right)^2
\end{align}

where

\( Jds = e^{\beta U(u)} du \) and \( \tau\{U(u)\} = \int_0^t e^{\beta U(u)} du \) \quad (A.5)

The new final time \( \tau\{U(u)\} \) is not fixed anymore, but depends on the realization of random potential \( U(u) \). To take into account this constraint, we can insert the identity

\begin{align}
1 = \int_0^\infty d\tau \, \delta \left( \tau - \int_0^t e^{\beta U(u)} du \right) \quad (A.6)
\end{align}

to obtain

\begin{align}
X_t(x) &= \int_0^\infty d\tau \int_0^x \mathcal{D}x(u) \int_{U(0)=0}^U \mathcal{D}U(u) e^{-\frac{1}{4D} \int_0^t du \left( \frac{\beta}{2} \frac{dU}{dv} + 2D\mu \right)^2} \\
&\quad - \frac{1}{4D} \int_0^\tau ds \left( \frac{dx}{ds} \right)^2 \delta \left( \tau - \int_0^t e^{\beta U(u)} du \right) \\
&= \int_0^\infty d\tau \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{x^2}{4Dt}} \psi(t(\tau)) \quad (A.7)
\end{align}

Let us now perform the Gaussian path-integral on \( x(u) \)

\begin{align}
\int_0^x \mathcal{D}x(u) e^{-\frac{1}{4D} \int_0^\tau ds \left( \frac{dx}{ds} \right)^2} = \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{x^2}{4Dt}} \quad (A.8)
\end{align}

to get

\begin{align}
X_t(x) &= \int_0^\infty d\tau \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{x^2}{4Dt}} \psi(t(\tau)) \quad (A.9)
\end{align}
where

$$\psi_t(\tau) = \int_{U(0)=0} DU(u)e^{-\frac{1}{4D} \int_0^t du \left( \frac{\beta}{2} \frac{dU}{dv} + 2D\mu \right)^2} \delta \left( \tau - \int_0^t e^{\beta U(u)} du \right)$$ (A.10)

is by definition the probability distribution of the functional \( \tau_t = \int_0^t e^{\beta U(u)} du \).

Eq (A.9) is just the translation in terms of probability distributions of the identity in law between the process \( x(t) \) and a linear Brownian motion of stochastic time \( \tau_t \)

$$x(t) = \sqrt{2D} \int_0^{\tau_t} du \eta(u)$$

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