Discrete series of representations for
$\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$

Pavel Šťovíček

Department of Mathematics, Faculty of Nuclear Science
Czech Technical University
Trojanova 13, 120 00 Prague
Czech Republic

Abstract

A possible generalization of the method of orbits to $SL_q(2, \mathbb{R})$ is discussed.

1 Introduction

There are not so many results concerning non-compact real quantum groups. Naturally a major attention has been paid to groups with some physical interpretation (see, for example, [1]). Even in the case of real groups of low dimension the treatment seems to be quite complicated [2]. Recently the quantum group $SL_q(2, \mathbb{R})$ attracted some attention; particular efforts have been made to describe and classify its quantum homogeneous spaces [3]. In this contribution we focus on a possible generalization of the method of orbits to $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$.

In the paper [4] there was described a construction of representations for a Hopf algebra $\mathcal{U}$ which may be considered as a generalization of the method of orbits due to Kostant and Kirillov. In fact, the construction concentrates on just one step of the method of orbits. It is already supposed that there is given a left $\mathcal{U}$-module algebra $\mathcal{C}$ with the action denoted by $\xi$ and fulfilling two conditions: $\xi(X) \cdot 1 = \varepsilon(X) 1$ and Leibniz rule

$$\xi(X) \cdot (fg) = (\xi(X_1)) \cdot f(\xi(X_2)) \cdot g, \quad \forall X \in \mathcal{U}, \forall f, g \in \mathcal{C}. \quad (1)$$
We use Sweedler’s notation: $\Delta X = X_{(1)} \otimes X_{(2)}$ where $\Delta$ denotes the comultiplication.

Classically, $\mathcal{U}$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, $C$ is an algebra of functions living on a coadjoint orbit, or a reduction of such an algebra via a polarization, and $\xi$ is nothing but the infinitesimal action. So $\xi(X)$, $X \in \mathfrak{g}$, are vector fields. A critical step of the method of orbits is a replacement of vector fields $\xi(X)$ by first order differential operators of the form $\nabla(\xi(X)) + 2\pi i \lambda(X)$ where $\nabla$ is an appropriate covariant derivative and $\lambda(X)$ is a function depending linearly on $X$. And this is where the construction attempts a generalization applicable also to quantum groups. It results in a modification of the left action $\xi$. The new action will be denoted simply by a dot.

The construction is also known in somewhat different form as the twisted adjoint action [1].

The simplest example is $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ ($q$ is not a root of unity) with the generators $K$, $K^{-1}$, $E$ and $F$, and the defining relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quadKF = q^{-1}FK,$$

$$[E,F] = \frac{1}{q-q^{-1}}(K^2 - K^{-2}).$$

(2)

The comultiplication, the counite and the antipode are the usual ones.

Let $A = A_q(SL(2, \mathbb{C}))$ be the dual Hopf algebra of quantum functions on $SL(2, \mathbb{C})$ with the generators $a, b, c, d$, and set $C = \mathbb{C}[z]$. Then

$$L : C \to A \otimes C, \quad L(z) = (c \otimes 1 + d \otimes z)(a \otimes 1 + b \otimes z)^{-1},$$

(3)

is a left coaction (defined in fact rather formally). Using the standard pairing between $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ and $A_q(SL(2, \mathbb{C}))$ one introduces a right action, and with the aid of the antipode $S$ one can pass to a left action,

$$\xi(x) \cdot f = \langle SX, f_{(1)} \rangle f_{(2)}, \quad \forall X \in \mathcal{U}, \forall f \in C.$$

(4)

After some rescaling of the complex variable $z$ one arrives at the action

$$\xi(K) \cdot z^j = q^j z^j, \quad \xi(E) \cdot z^j = [j]_q z^{j+1}, \quad \xi(F) \cdot z^j = -[j]_q z^{j-1},$$

(5)

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$  

(6)

The mentioned construction is applicable to this case and yields a one-parameter family of modified actions

$$K \cdot z^j = q^{(2j-\sigma)/2} z^j, \quad E \cdot z^j = q^{-\sigma/2} [j - \sigma]_q z^{j+1}, \quad F \cdot z^j = -q^{\sigma/2} [j]_q z^{j-1}.$$ 

(7)

The action can be restricted to the two real forms of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$, namely $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$. The latter one is determined by the involution

$$K^* = K, \quad E^* = -q^{-1} E, \quad F^* = -q F,$$

(8)
with \( q \) being a complex unite (\( q^* = q^{-1} \)).

As shown in [4], the construction is applicable to any compact simple Lie group from the four principal series and yields all irreducible representations. In the present paper we concentrate, however, on the non-compact real form \( U_q(sl(2, \mathbb{R})) \).

2 Classical method of orbits for \( sl(2, \mathbb{R}) \)

Let us identify \( sl(2, \mathbb{R}) \) with \( \mathbb{R}^3 \):

\[
\mathbb{R}^3 \ni (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_3 & x_1 + x_2 \\ -x_1 + x_2 & -x_3 \end{pmatrix} \in sl(2, \mathbb{R}).
\]  

(9)

Relating to every matrix \( X \in sl(2, \mathbb{R}) \) a functional \( f_X \), \( \langle f_X, Y \rangle = \text{Tr} \, X^t \, Y \), we identify \( sl(2, \mathbb{R}) \) with its dual. The coadjoint orbits are either the origin or the quadrics

\[
x_1^2 - x_2^2 - x_3^2 = c = \text{const}.
\]  

(10)

The orbits contributing to the discrete series are characterized by \( c = k^2, \ k > 0 \). We choose the leaf with \( \text{sgn} \, x_1 > 0 \). Then it is natural to use \( (x_2, x_3) \) as coordinates on the orbit, with \( x_1 = (k^2 + x_2^2 + x_3^2)^{1/2} \).

However it is more convenient to introduce complex coordinates \((z, \bar{z})\) as follows:

\[
z = \frac{x_1 + x_2}{x_3 - ik}.
\]  

(11)

Then the orbit is identified with the upper complex half-plane, \( \text{Im} \, z > 0 \), and the coadjoint action takes the form

\[
g \cdot z = \frac{c + dz}{a + bz} \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).
\]  

(12)

The infinitesimal action reads \((E, F, H)\) is the usual basis)

\[
\xi_E = z^2 \partial_z + \text{c.c.}, \quad \xi_F = -\partial_z + \text{c.c.}, \quad \xi_H = 2z \partial_z + \text{c.c.}
\]  

(13)

There exist two polarizations which are nothing but the mutually conjugate complex structures \( \partial_z \) and \( \partial_{\bar{z}} \). We choose the former one and so the constructed representation should act in a space of holomorphic functions on the upper complex half-plane.

The method of orbits leads to the following modification of the infinitesimal action \([13]\):

\[
\varrho(E) = z^2 \partial_z + 4\pi k, \quad \varrho(F) = -\partial_z, \quad \varrho(H) = 2z \partial_z + 4\pi k.
\]  

(14)

This representation of the Lie algebra can be integrated to a representation of the Lie group, and the result reads

\[
\varrho(g) \cdot f(z) = (d - bz)^{-4\pi k} f \left( \frac{-c + a z}{d - bz} \right) \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]  

(15)
The representation \( \varrho \) depends on \( g \) continuously and unambiguously if and only if
\[
4\pi k = n \in \mathbb{N}.
\]
(16)
This constraint is caused by the nontrivial topology of \( SL(2, \mathbb{R}) \). Moreover, the representation \( \varrho \) is unitary with respect to the \( L^2 \)-norm
\[
\|f\|^2 = \int_{\mathbb{R} \times \mathbb{R}_+} |f(u,v)|^2 v^{n-2} \, du \, dv
\]
where \( z = u + iv \). Because of the divergence at \( v = 0 \) we have to impose another constraint, namely \( n \geq 2 \).

The restriction of \( \varrho \) to the compact subgroup \( SO(2) \subset SL(2, \mathbb{R}) \) can be diagonalized and the eigen-functions form an orthogonal basis in the carrier Hilbert space. The element \( E - F \) is a basis element in \( \mathfrak{so}(2) \). Fix \( n \in \mathbb{N}, n \geq 2 \), and set
\[
\psi_m(z) = (z - i)^m (z + i)^{-m-n}, \quad m = 0, 1, 2, \ldots \tag{18}
\]
Then
\[
\varrho(E - F) \psi_m = i(2m + n) \psi_m \tag{19}
\]
and
\[
\begin{aligned}
\varrho(E) \psi_m &= \frac{1}{2} m \psi_{m-1} + \frac{1}{2} \cdot (2m + n) \psi_m + \frac{1}{2} i(m + n) \psi_{m+1}, \\
\varrho(F) \psi_m &= \frac{1}{2} i m \psi_{m-1} - \frac{1}{2} i(2m + n) \psi_m + \frac{1}{2} i(m + n) \psi_{m+1}, \\
\varrho(H) \psi_m &= m \psi_{m-1} - (m + n) \psi_{m+1}. \tag{20-22}
\end{aligned}
\]
Note, however, that the eigen-functions \( \psi_m \) are not normalized. The norm is
\[
\|\psi\| = 2^{-n+1} \left( \frac{\pi \frac{m!}{(m+n)!}}{(m+n-1)!} \right)^{1/2} \tag{23}
\]

3 \( q \)-deformation of the discrete series

Here we come to the main goal of the present paper, namely to a \( q \)-deformation of the discrete series of representations of \( SL(2, \mathbb{R}) \). Fix \( n \in \mathbb{Z}_+ \) and set \( \sigma = -n \) in (7). The relations (7) imply that
\[
\begin{aligned}
K \cdot f(z) &= q^{n/2} f(qz), \\
E \cdot f(z) &= \frac{z}{q - q^{-1}} (q^n K \cdot f(z) - K^{-1} \cdot f(z)), \\
F \cdot f(z) &= -\frac{1}{(q - q^{-1}) z} (q^{-n} K \cdot f(z) - K^{-1} \cdot f(z)). \tag{24}
\end{aligned}
\]
Introduce the functions
\[ \psi_m(z) = \prod_{j=0}^{m-1} (q^{2(j+n)} z - i) / \prod_{j=0}^{m+n-1} (q^{2(j-m)} z + i). \]  

(25)

Then it holds
\[ (q^{2n} EK - FK) \cdot \psi_m = i q^n [2m + n]_q \psi_m \]

(26)

and
\[
EK^{-1} \cdot \psi_m = \frac{q^{4m+n-1} (1 + q^2)}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} [m]_q \psi_{m-1}
\]
\[ + i \frac{q^{4m+2} (1 + q^{2(n-1)})}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} [2m + n]_q \psi_m \]

(27)

\[
FK^{-1} \cdot \psi_m = \frac{q^{8m+5n-5} (1 + q^2)}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} [m]_q \psi_{m-1}
\]
\[ - i \frac{q^{4m+2n} (1 + q^{2(n-1)})}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} [2m + n]_q \psi_m \]
\[ + i \frac{q^{4m+2n} (1 + q^2)}{(1 + q^{4m+2n}) (1 + q^{4m+2n})} [m + n]_q \psi_{m+1}, \]

(28)

\[
K^{-2} \cdot \psi_m = - (q^2 - q^{-2}) \frac{q^{6m+3n-2}}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} [m]_q \psi_{m-1}
\]
\[ + i \frac{q^{4m+2} (1 + q^{2(n-1)}) (1 + q^2)}{(1 + q^{4m+2n-2}) (1 + q^{4m+2n})} \psi_m \]
\[ + (q^2 - q^{-2}) \frac{q^{2m+n}}{(1 + q^{4m+2n}) (1 + q^{4m+2n})} [m + n]_q \psi_{m+1}. \]

(29)

Actually, relations (25-29) can be considered as $q$-deformations of relations (18-22).

References

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