Asynchronous Batch and PIR Codes from Hypergraphs

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Abstract—We propose a new model of asynchronous batch codes that allow for parallel recovery of information symbols from a coded database in an asynchronizable manner, i.e. when different queries take different time to process. Then, we show that the graph-based batch codes studied by Rawat et al. are asynchronizable. Further, we demonstrate that hypergraphs of Berge girth at least 4, respectively at least 3, yield graph-based asynchronous batch codes, respectively private information retrieval (PIR) codes. We prove the hypergraph-theoretic proposition that the maximum number of hyperedges in a hypergraph of a fixed Berge girth equals the quantity in a certain generalization of the hypergraph-theoretic (6,3)-problem, first posed by Brown, Erdős, and Sós. We then apply the constructions and bounds by Erdős, Frankl and Rödl about this generalization of the (6,3)-problem, known as the (3r-3,r)-problem, to obtain batch code constructions and bounds on the redundancy of the graph-based asynchronous batch and PIR codes. Finally, we show that the optimal redundancy $\rho(k)$ of graph-based asynchronous batch codes of dimension $k$ with the query size $t = 3$ is $2\sqrt{k}$. Moreover, for a general fixed value of $t \geq 4$, $\rho(k) = O\left(k^{1/(2-\epsilon)}\right)$ for any small $\epsilon > 0$. For a general value of $t \geq 4$, $\lim_{k \to \infty} \rho(k)/\sqrt{k} = \infty$.

Index Terms—primitive linear multiset batch codes, private information retrieval codes, extremal hypergraph theory, Turán theory, packing designs.

I. INTRODUCTION

Batch codes were originally proposed by Ishai et al. [11] for load balancing in distributed systems. One particular class of batch codes that we are interested in is linear (combinatorial) batch codes [16], [20], [31], [26] where the data is viewed as elements of a finite field written as a vector, and it is encoded using a linear transformation of that vector.

Codes for private information retrieval (or PIR codes, in short) were proposed by Fazeli, Vardy and Yaakobi [8]. It was suggested therein to emulate standard private information retrieval protocols using a special layer (code) which maps between the requests of the users and the data which is actually stored in the database.

Linear batch codes and PIR codes have many similarities with locally-repairable codes [6], which are used for repair of lost data in distributed data storage systems. The main difference, however, is that in locally-repairable codes, it is coded symbols that are to be repaired, while in batch codes and PIR codes it is information symbols that are to be restored [23].

II. NOTATION AND PRELIMINARIES

A. Batch and PIR Codes

We denote by $\mathbb{N}$ the set of natural numbers. For $n \in \mathbb{N}$, define $[n] \triangleq \{1, 2, \ldots, n\}$. The notation $I$ is used for an identity matrix. In this work, we consider only (primitive, multiset) batch codes as defined in [20].

Definition II.1 (20). An $(n, k, t)$ batch code $C$ over a finite alphabet $\Sigma$ is defined by an encoding mapping $C : \Sigma^k \to \Sigma^n$, and a decoding mapping $D : \Sigma^n \times [k]^t \to \Sigma^t$, such that

1) For any $x \in \Sigma^k$ and $i_1, i_2, \ldots, i_t \in [k]$,

$$D(y = C(x), i_1, i_2, \ldots, i_t) = (x_{i_1}, x_{i_2}, \ldots, x_{i_t}).$$

2) The symbols in the query $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ can be reconstructed from $t$ respective pairwise disjoint recovery sets of symbols of $y$ (the symbol $x_{i_\ell}$ is reconstructed from the $\ell$-th recovery set for each $\ell, 1 \leq \ell \leq t$).

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a prime power, and $C$ be a linear $[n, k]$ code over $\mathbb{F}$. Denote the redundancy $\rho \triangleq n - k$. For a linear batch code, the encoding of $C$ is given as a multiplication by a $k \times n$ generator matrix $G$ over $\mathbb{F}$ of an information vector $x \in \mathbb{F}^k$.

$$y = x \cdot G; \quad y \in \mathbb{F}^n. \quad (1)$$

A linear batch code with the parameters $n$, $k$, and $t$ over $\mathbb{F}_q$, where $t$ is a number of queried symbols, is denoted as an $[n, k, t]_q$-batch code.

Definition II.2 (23). Linear PIR codes are defined similarly to linear primitive multiset batch codes, with a difference that the supported queries are of the form $(x_{i_1}, x_{i_2}, \ldots, x_{i_t}), i \in [k]$, (and not $(x_{i_1}, x_{i_2}, \ldots, x_{i_t}), i_1, i_2, \ldots, i_t \in [k]$ as in batch codes).

In what follows we only consider linear PIR codes. For constructions of PIR codes see, for example, [13], [25].

B. Graphs and Hypergraphs

Let $W^{(r)}$, $r \geq 2$, denote the set of all unordered $r$-tuples of distinct elements of the set $W$. A graph $G(V, E)$ consists of a finite set $V$, called the vertex set and a finite set $E \subseteq V^{(2)}$
of pairs of vertices, called the edge set. The graph $G(V, E)$ is bipartite with bipartition (or parts) $(A, B)$ if $A \cup B = V$, $A \cap B = \emptyset$, and $|A \cap e| = 1$ and $|B \cap e| = 1$ for every edge $e \in E$. We denote the bipartite graph with distinguished parts $A$ and $B$ as $G(A, B)$ where we call $A$ the left part and $B$ the right part. A $b$-cycle in a graph $G(V, E)$ is a cyclic sequence of $b$ vertices and $b$ edges, alternatingly between vertices and edges, such that each edge consists precisely of the two vertices on either side of it in the sequence. A bipartite graph $G(A, B, E)$ is left-regular if all left degrees $d(a) = \{e \in E : a \in e\}$, where $a \in A$, are equal.

A hypergraph $G(V, E)$ consists of a finite set $V$ of vertices and a finite collection $E$ of subsets of $V$, called (hyper)edges. The hypergraph is $r$-uniform, or an $r$-graph, if each hyperedge consists of the same number $r$ of vertices, that is, $E \subseteq V^r$. Thus, a graph can be viewed as 2-uniform hypergraph. A Berge cycle in a hypergraph is a sequence $(e_1, v_1, e_2, v_2, \ldots, v_b, e_b+1)$ where $e_1, e_2, \ldots, e_b$ are distinct hyperedges, $v_1, v_2, \ldots, v_b$ are distinct vertices, $v_{i-1}, v_i \in e_i$ for all $i$ (we have taken all indices modulo $b$ when defining the sequence) and $e_1 = e_b+1$. A hypergraph is Berge-disconnected if its vertex set $V$ can be partitioned into two non-empty sets $V = V_1 \cup V_2$ such that, for each hyperedge $e$, either $e \cap V_1 = \emptyset$ or $e \cap V_2 = \emptyset$; it is Berge-connected if it is not disconnected.

A hypergraph has Berge girth equal $k$ if (a) it contains a Berge cycle with $k$ hyperedges; (b) it contains no Berge cycles with fewer than $k$ hyperedges. If a subset of vertices is allowed several (a finite number of) times as a hyperedge, we have a multihypergraph. We note that a multi-$r$-graph for $r \geq 2$ with Berge girth at least 3 is necessarily a simple hypergraph, i.e. no subset of vertices appears as an edge several times.

The following definition of the correspondence between bipartite graphs and (multi)hypergraphs will be instrumental.

**Definition II.3.** With a (multi)hypergraph $G(V, E)$ one can associate the bipartite incidence graph $G(E, V, I)$ with left part $E$ and right part $V$ where $\{e, v\}$ is an edge, i.e. $\{e, v\} \subseteq I$ in $G$ if and only if $v \in e \in G$. By going backwards, given a bipartite graph $G(E, V, I)$ we construct a (multi)hypergraph $G'(V, E)$ by identifying each $e \in E$ with the set $\{v \in V \mid \{e, v\} \in I\}$.

Therefore, multihypergraphs are in one-to-one correspondence with bipartite graphs. A multihypergraph is Berge-connected if and only if its incidence graph is connected; there is a one-to-one correspondence between Berge cycles with $k$ hyperedges in the multihypergraph and cycles of length $2k$ in the incidence graph.

An $r$-graph $G'(V', E')$ is a sub-$r$-graph of an $r$-graph $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq \{e \in E \mid e \subseteq V'\}$. We say that the sub-$r$-graph is induced by the vertex set $V'$, if in addition, we have $E' = \{e \in E \mid e \subseteq V'\}$. Similarly we say that a subset of hyperedges $E'$ induces the vertex set $\bigcup_{e \in E'} e$.

### C. Graph-based Batch and PIR Codes

Let $C$ be an $[n, k, t]_q$ batch (PIR) code defined by a systematic encoding matrix $G = [I | A]$. Take $y \in C$. The symbols of $y$ corresponding to the systematic part of $G$ are called information symbols, and the remaining symbols are called parity symbols. The following bipartite graph representation was proposed in [20]: let $G(A, B, E)$ be a bipartite graph, where $A$ is the set of the information symbols, $B$ is the set of the parity symbols, and $E = \{\{u, v\} : u \in A, v \in B, u \text{ participates in parity } v\}$.

**Theorem II.1.** ([20] Theorem 1 and Lemma 2) Let $C$ be an $[n, k, t]$ systematic code represented by the bipartite graph $G(A, B, E)$. Assume that there exists an induced subgraph $H(A', B', E')$ of $G$, that is, $B' \subseteq B$ and $E' = \{e \in E : |e \cap B'\mid = 1\}$, such that:

(i) Each node in $A$ has degree at least $t-1$ in the bipartite graph $H$.

(ii) The graph $H$ has girth $\geq 8$ (respectively, $\geq 6$).

Then, $C$ is an $[n, k, t]$ batch code (respectively, PIR code).

It follows from Theorem II.1 that constructions of left-regular bipartite graphs without short cycles yield constructions of batch and PIR codes. In what follows, we use this approach in order to construct batch and PIR codes with good parameters. Specifically, we use known constructions of good hypergraphs, which can be mapped to bipartite graphs without short cycles, in order to construct good codes.

### III. Asynchronous Batch Codes

In this section, we introduce a new special family of batch codes, termed asynchronous batch codes. Assume that $C$ is a linear $[n, k, t]$ batch code as in Definition III.1 used for retrieving a batch of $t$ symbols $(x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_t})$, $\ell_i \in [k]$, in parallel from a coded database that consists of $n$ servers, such that at most one symbol is retrieved from each server.

To this end assume that the response time of servers for different requests varies, and thus some symbol $x_{\ell_j}$ can be retrieved faster than the other symbols. In asynchronous retrieval mode, once $x_{\ell_j}$ was retrieved, it is possible to retrieve any other request $x_{\ell_{j+1}}$, $\ell_{j+1} \in [k]$, in parallel to retrieving of $(x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_{j-1}}, x_{\ell_{j+1}}, \ldots, x_{\ell_t})$, without reading more than one symbol from each server. In that way, the asynchronous batch codes support (asynchronous) retrieval of $t$ symbols in parallel. We proceed with a formal definition.

**Definition III.1.** An asynchronous (linear primitive multiset) $[n, k, t]$ batch code $C$ is a (linear primitive multiset) batch code with the additional property that for any legal query $(x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_t})$, for all $\ell_i \in [k]$, it is always possible to replace $x_{\ell_j}$ by some $x_{\ell_{j+1}}$, $\ell_{j+1} \in [k]$, such that $x_{\ell_{t+1}}$ is retrieved from the servers not used for retrieval of $(x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_{j+1}}, x_{\ell_{j+1}}, \ldots, x_{\ell_t})$, without reading more than one symbol from each server.

**Example III.1.** Consider the systematic $[8, 4, 3]_2$ batch code $C$ generated by the matrix $G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$.
The query \((x_1, x_1, x_1)\) can be retrieved from the following disjoint sets of symbols: \(x_1 = y_1, x_1 = y_2 + y_5, x_1 = y_3 + y_7\).

Assume that the first queried symbol \(x_1\) has already been retrieved (while the last two queries are still being served), and the new query \(x_2\) has arrived. Then, we can use the recovery \(x_2 = y_4 + y_6\) for the newcomer \(x_2\), without affecting the recovery sets of the other two queries.

It can be shown that for any initial selection of the recovery sets, and for any finished query and new query, there is always a way to select disjoint recovery sets. Therefore, \(\mathcal{C}\) is an asynchronous \([8, 4, 3/2]\) batch code.

It is straightforward to see that any asynchronous \([n, k, t]\) batch code is an \([n, k, t]\) batch code. The opposite, however, does not always hold.

**Example III.2.** Consider batch codes, which are obtained by taking simplex codes as suggested in [24]. For example, \(\mathcal{C}\), formed by the matrix

\[
G = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

is a \([7, 3, 4/2]\) batch code. Assume that the query \((x_1, x_1, x_1, x_1)\) was submitted by the users. Then, one copy of \(x_1\) is retrieved from \(y_1\), and for each of the remaining three copies of \(x_1\), at least two symbols of \(y\) have to be used. Assume that the query that uses \(y_1\) was served, but the remaining queries are still being served. If the next query \(x_2\) arrives, it is impossible to serve it without accessing one of the servers containing \(y_2, \cdots, y_7\) at least twice. Therefore, \(\mathcal{C}\) is not an asynchronous \([7, 3, 4/2]\) batch code.

It turns out, that the conditions in Theorem **II.1** yield asynchronous batch codes. More formally:

**Theorem III.1.** Let \(\mathcal{C}\) be an \([n, k]\) systematic code represented by the bipartite graph \(G(A, B, E)\). Assume that there exists an induced subgraph \(H(A', B', E')\) of \(G\), that is, \(B' \subseteq B\) and \(E' = \{e \in E : |e \cap B'| = 1\}\), such that:

(i) Each node in \(A\) has degree at least \(t - 1\) in the bipartite graph \(H\).

(ii) The graph \(H\) has girth at least 8.

Then, \(\mathcal{C}\) is an asynchronous \([n, k, t]\) batch code.

The omitted proof follows from [20, Lemma 2]:

**Lemma III.2.** Let \(\mathcal{C}\) be an \([n, k]\) systematic code represented by the bipartite graph \(G(A, B, E)\). Assume that there exists an induced subgraph \(H(A', B', E')\) of \(G\), that is, \(B' \subseteq B\) and \(E' = \{e \in E : |e \cap B'| = 1\}\), such that:

(i) Each node in \(A\) has degree at least \(t - 1\) in the bipartite graph \(H\).

(ii) The graph \(H\) has girth at least 8.

Then, each message symbol has at least \(t\) disjoint repair group (including the group formed by the information symbol itself). Moreover, for any \(i, j \in [k], i \neq j\), any one of the disjoint repair groups for the message symbol \(x_i\) has common symbols with at most one of the disjoint repair groups for the message symbol \(x_j\).

**Definition III.2.** An asynchronous \([n, k, t]\) batch code, which can be represented as in Theorem **III.1** is called a graph-based asynchronous batch code.

**IV. Case \(t = 3\)**

In this section, we start by considering a simple case \(t = 3\). We derive a tight upper bound on the optimal redundancy of graph-based asynchronous batch codes.

**Theorem IV.1.** Let \(\mathcal{C}\) be a graph-based asynchronous \([n, k, t \geq 3]\) batch code. Then, its redundancy is \(\rho \geq 2\sqrt{k}\).

**Proof.** Let \(\hat{\mathcal{C}} = (A, B, \hat{E})\) be a bipartite graph that corresponds to the code \(\mathcal{C}\). Then, the girth of \(\hat{\mathcal{C}}\) is \(\geq 8\), and \(d(a) \geq 2\) for \(a \in A\). Also, \(k = |A|, n - k = |B|, and t \geq 3\).

First, we delete edges of \(\hat{\mathcal{C}}\) such that after deletion \(d(a) = 2\) for \(a \in A\), and denote the new graph \(G\) (note that we change the code). We construct a new (non-bipartite) graph, \(G' = (V', E')\), from \(G\), by following the correspondence in Definition **II.3**. Since the left degree of \(G\) is \(2\), the result is indeed a graph (rather than hypergraph). Specifically, take \(V' = B\). For each \(u \in A\), replace \(u\) and two edges \(\{u, v_1\}\) and \(\{u, v_2\}\) incident with it by a new edge \(e_u = \{v_1, v_2\}\). The construction implies that there is a cycle of length \(2t\) in \(G\) if and only if there is a cycle of length \(t\) in \(G'\). Thus, \(G\) has girth \(\geq 8\) if and only if \(G'\) has girth \(\geq 4\).

By Mantel’s Theorem [17] (see also: Turán’s Theorem [24]), this implies that the number of edges \(|E'|\) satisfies \(|E'| \leq |V'|^2/4\). Since \(|A| = k\) and \(|B| = n - k\), we obtain that \(|V'| = n - k\) and \(|E'| = k\). Therefore, the redundancy \(\rho = n - k \geq 2\sqrt{k}\). The redundancy of the original code is at least as large.

This bound is in fact tight. For example, consider a complete bipartite graph \(G'\) with a vertex set \(V' = A' \cup B'\), \(A' \cap B' = \emptyset\), \(|A'| = |B'|\). This graph has \(|V'|^2/4\) edges in total, and girth 4. Moreover, this graph has the largest possible number of edges for any girth-4 graph with \(|V'|\) vertices, as seen by Mantel’s Theorem [17].

Next, we convert this graph into a bipartite graph \(G\) by using the inverse of the above mapping. Namely, each edge is replaced by a triple “edge, vertex, edge”. We obtain that \(G\) is a left regular bipartite graph of left degree 2 with \(|A| = |V'|^2/4\) and \(|B| = |V'|. The graph \(G\) has girth 8 and hence it yields an asynchronous batch code having length \(n = |V'|^2/4 + |V'|\), number of information symbols \(k = |V'|^2/4\), redundancy \(\rho = 2\sqrt{k} = |V'|\), and \(t = 3\).

We remark that the lower bound \(\rho \geq \sqrt{2k} + O(1)\) on the optimal redundancy of PIR codes (for \(t \geq 3\)) was recently obtained by Rao and Vardy in [19], and their result implies the same lower bound on the redundancy of linear batch codes. Moreover, they show that this bound is tight for PIR codes. In this section, we showed a tighter lower bound \(\rho \geq 2\sqrt{k}\) for graph-based asynchronous batch codes (for all \(t \geq 3\), and
presented an explicit construction of asynchronous batch codes for \( t = 3 \) that attain this bound. As we show in the sequel, for graph-based asynchronous batch codes with \( t > 3 \) the lower bound on \( \rho \) can be further tightened. We consider a modified code also for general \( t \).

\[ \text{V. Hypergraph Theory} \]

In their 1973 papers, Brown, Erdős and Sós [3, 4] pose the following extremal combinatorial problems on \( r \)-graphs.

**Problem V.1.** Let \( f^{(r)}(\eta; \kappa, s) \) denote the smallest \( m \) such that every \( r \)-graph on \( \eta \) vertices with \( m \) edges contains at least one sub-\( r \)-graph on \( \kappa \) vertices with \( s \) edges. What are the bounds on \( f^{(r)}(\eta; \kappa, s) \) for fixed \( r, \kappa \) and \( s \)?

Observe that \( F^{(r)}(\eta; \kappa, s) \triangleq f^{(r)}(\eta; \kappa, s) - 1 \) is the maximum size (number of edges) of an \( r \)-graph with no set of \( \kappa \) vertices containing \( s \) or more hyperedges. The resolution of the case \( f^{(3)}(\eta; 6, 3) \), known as the \((6, 3)\)-problem, by Ruzsa and Szemerédi [22] is a classical result in extremal combinatorics. Erdős, Frankl and Rödl [7] extended this result to any fixed \( r \), also giving an easier construction for the lower bound, solving the so-called \((3r - 3, 3)\)-problem. There are various later generalisations of [22] and [7], see for example [1] and the references therein, and the survey [9].

In what follows, we show that finding the maximum number of hyperedges of a hypergraph with a given Berge girth is essentially a generalization of the \((6, 3)\)-problem, by Ruzsa and Szemerédi [22] is a classical result in extremal combinatorics. Then, we apply the resolution of the \((3r - 3, 3)\) problem by [7] to batch codes.

**Theorem V.1.** Let \( B^{(r)}(\eta; \kappa) \) be the maximum number of hyperedges in an \( r \)-graph with \( \eta \) vertices and Berge girth at least \( \kappa + 1 \). Then \( F^{(r)}(\eta; \kappa r - \kappa, \kappa) = B^{(r)}(\eta; \kappa) \).

We prove this theorem in Lemmas V.2 - V.4.

**Lemma V.2.** For a Berge-connected hypergraph \( \mathcal{G}(V, E) \) with \(|V| \geq 2 \) we have:

1) \( \sum_{e \in E}(|e| - 1) \geq |V| - 1 \).
2) \( \mathcal{G}(V, E) \) contains no Berge cycles (is a Berge tree) if and only if \( \sum_{e \in E}(|e| - 1) = |V| - 1 \).
3) \( \mathcal{G}(V, E) \) contains exactly one Berge cycle if and only if \( \sum_{e \in E}(|e| - 1) = |V| \).

Lemma V.2 can be proved using properties of the incidence graphs, the proof is omitted.

**Definition V.1.** A hypergraph satisfies the \((\kappa r - \kappa, \kappa)\)-condition if no set of \( \kappa r - \kappa \) of its vertices contains \( \kappa \) or more hyperedges.

**Lemma V.3.** An \( r \)-graph of Berge girth at least \( \kappa + 1 \) satisfies the \((\kappa r - \kappa, \kappa)\)-condition.

**Proof.** Consider any \( \kappa \) hyperedges of this graph. They induce no Berge cycle. For each of the Berge-connected components (maximal connected subhypergraphs) \( \mathcal{G}'(V', E') \) of the hypergraph induced by these \( \kappa \) hyperedges, we have \( \sum_{e \in E'}(|e| - 1) = |V'| - 1 \) by Condition 2 of Lemma V.2. Therefore \( \sum_{e \in E}(|e| - 1) = \kappa(r - 1) = |V| - c \) for the hypergraph induced by these \( \kappa \) hyperedges, where \( c \geq 1 \) is the number of Berge-connected components. Therefore the number of vertices induced by these \( \kappa \) hyperedges is \( \kappa(r - 1) + c > \kappa r - \kappa \). Thus the hypergraph satisfies the \((\kappa r - \kappa, \kappa)\)-condition.

**Lemma V.4.** An \( r \)-graph that satisfies the \((\kappa r - \kappa, \kappa)\)-condition can be changed (its hyperedges can be re-wired) so that it still has the same number of hyperedges, still satisfies the \((\kappa r - \kappa, \kappa)\)-condition and has Berge girth at least \( \kappa + 1 \).

**Proof.** If an \( r \)-graph satisfies the \((\kappa r - \kappa, \kappa)\)-condition, then from Definition V.1 the total number of vertices used by any \( \kappa \) hyperedges is at least \( \kappa(r - 1) + 1 \). Consider two cases.

Case 1: If the graph induced by these hyperedges were connected, by Condition 2 of Lemma V.2 it contains no Berge cycles. In that case, there is no cycle with \( \leq \kappa \) hyperedges.

Case 2: If the graph induced by these hyperedges were disconnected, consider a Berge-connected component which has some small Berge-cycles. This component has fewer than \( \kappa \) hyperedges, since otherwise there are \( \kappa \) of its connected hyperedges violating the \((\kappa r - \kappa, \kappa)\)-condition.

Next, take a vertex \( v \) in two adjacent hyperedges \( e \) and \( e' \) of a cycle of \( \leq \kappa \) hyperedges, and re-wire \( e \) by deleting \( v \) from it, and adding into \( e \) another vertex from outside the connected component. This procedure strictly reduces the number of connected components with less than \( \kappa \) hyperedges (an isolated vertex is a connected component by itself), therefore we can only repeat it a finite number of times, and eventually, we will have no Berge cycles on \( \kappa \) or fewer hyperedges (see Lemma V.2). \( \square \)

VI. PIR codes from hypergraphs of girth \( \geq 3 \)

**Definition VI.1.** A \( \tau-(\eta, r, \lambda) \) packing design is an \( r \)-graph of \( \eta \) vertices (called points) and of edges (called blocks) such that each \( \tau \)-tuple of vertices (points) is contained in at most \( \lambda \) edges (blocks).

Consider an \( r \)-graph \( \mathcal{G}(V, E) \), where \( V \) is a point set and \( E \) is a block set, \(|V| = \eta\), \( \mathcal{G}(V, E) \) of Berge girth at least 3 is equivalently an \( \tau-(\eta, r, 1) \) packing design. The maximum size (number of blocks) \( D(\eta, r) \) of a packing design, is upper-bounded by the well-known improved 1st and 2nd Johnson bounds [12], see also [13]. It follows from Keevash’s result on the existence of designs [13], which uses pseudorandomness arguments, that for all large enough \( \eta \), there is a packing design attaining either the improved 1st or 2nd Johnson bound (see also [10] referring to an earlier version of [13]).

An interesting special case is when each pair of points is contained in a unique block. In that case we obtain a Steiner 2-design, also known as a combinatorial \( 2-(\eta, r, 1) \) block design, or a \( (\eta, r, 1)-\text{BIBD} \) (balanced incomplete block design). Compared to packing designs, Steiner 2-designs are much more rare, as they are simultaneously covering designs. Fazeli, Vardy and Yaakobi [8] use Steiner 2-designs to construct PIR codes. The construction works verbatim if one starts...
with a packing design. Following Wilson [28–30], it obtain
asymptotic redundancy of PIR codes \( \rho = \Theta(\sqrt{k}) \). Wilson in [28–30] is concerned only with the asymptotics, while concrete packing designs will produce concrete PIR codes.

VII. Batch Codes from Hypergraphs of Girth \( \geq 4 \)

Bounds and constructions for \( r \)-graphs \( G(V, E) \) on \( \eta \) vertices of Berge girth at least 4 can be given via the \((3r-3, 3)\)-problem in the language of the \((6, 3)\)-problem, as seen from Theorem [27]. Bounds apply directly, while constructions may need to be modified slightly to lose small Berge cycles.

In [27], Erdős, Frankl and Rödl address precisely the \((3r-3, 3)\)-problem for \( r \geq 3 \). By modifying the construction of Behrend [2], they construct \( r \)-graphs on \( \eta \) vertices with the number of hyperedges asymptotically greater than \( \eta^{2-\epsilon} \) for any \( \epsilon > 0 \). The construction produces hypergraphs of Berge girth at least 4. The authors prove an upper bound \( O(\eta^2) \) on the maximum number of hyperedges, using an early version of the Szemerédi’s Regularity Lemma (see [14] and [21]).

By mapping the hypergraph \( G(V, E) \) constructed in [27] back onto \( G(E, V, I) \), and by using notations for batch codes, we obtain a bipartite graph with \((n-k)^{2-\epsilon}\) left vertices and \( n-k \) right vertices of girth 8. The corresponding graph-based asynchronous batch code has \( k = (n-k)^{2-\epsilon} \), and so its redundancy is bounded from above by \( \rho(k) = n-k = O\left( k^{1/(2-\epsilon)} \right) \) for any \( \epsilon > 0 \), and for any fixed \( t \).

We note that the upper bound in [27] similarly yields the lower bound

\[
\lim_{k \to \infty} \left( \rho(k)/\sqrt{k} \right) \to \infty \tag{2}
\]

for the optimal redundancy \( \rho(k) \) of the graph-based asynchronous codes, for any fixed \( t \geq 4 \).

We compare these results with their counterparts in [26], where it was shown that for any \( t \geq 5 \) the optimal redundancy of general (multiset primitive) linear batch codes behaves as \( O(\sqrt{k \log k}) \), while for \( t \in \{3, 4\} \) the corresponding redundancy is \( O(\sqrt{k}) \). It is worth mentioning that for \( t = 4 \) there is a gap between the optimal redundancy \( O(\sqrt{k}) \) of the codes studied in [26] and the lower bound (2) for the graph-based asynchronous batch codes. It remains an open question what is the exact asymptotics of the optimal redundancy for the graph-based asynchronous batch codes for \( t \geq 5 \), and whether the lower bound (2) actually matches the upper bound \( O(\sqrt{k \log k}) \) obtained in [26], or there is a gap between the optimal redundancy of these two families of codes.

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