A more complete version of a minimax theorem

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Abstract. In this paper, we present a more complete version of the minimax theorem established in [7]. As a consequence, we get, for instance, the following result: Let $X$ be a compact, not singleton subset of a normed space $(E, \| \cdot \|)$ and let $Y$ be a convex subset of $E$ such that $X \subseteq \overline{Y}$. Then, for every convex set $S \subseteq Y$ dense in $Y$, for every upper semicontinuous bounded function $\gamma : X \to \mathbb{R}$ and for every $\lambda > \frac{\sup_{x \in X} |\gamma|}{\text{diam}(X)}$, there exists $y^* \in S$ such that the function $x \to \gamma(x) + \lambda \|x - y^*\|$ has at least two global maxima in $X$.

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Here and in what follows, $X$ is a topological space and $Y$ is a convex set in a real Hausdorff topological vector space.

A function $h : X \to \mathbb{R}$ is said to be inf-compact if $h^{-1}(-\infty, r]$ is compact for all $r \in \mathbb{R}$.

A function $k : Y \to \mathbb{R}$ is said to be quasi-concave (resp. quasi-convex)) $k^{-1}([r, +\infty]$) (resp. if $k^{-1}([-\infty, r])$ is convex for all $r \in \mathbb{R}$.

If $S$ is a convex subset of $Y$, we denote by $A_S$ the class of all functions $f : X \times Y \to \mathbb{R}$ such that, for each $y \in S$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact.

Moreover, we denote by $B$ the class of all functions $f : X \times Y \to \mathbb{R}$ such that either, for each $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous, or, for each $x \in X$, the function $f(x, \cdot)$ is concave.

For any $f : X \times Y \to \mathbb{R}$, we set
\[
\alpha_f = \sup_Y \inf_X f
\]
and
\[
\beta_f = \inf_X \sup_Y f .
\]

Also, we denote by $C_f$ the family of all sets $S \subseteq Y$ such that
\[
\inf_X \sup_S f = \inf_X \sup_Y f ,
\]
and by $\tilde{C}_f$ the family of all sets $S \subseteq Y$ such that
\[
\sup_{y \in S} f(x, y) = \sup_{y \in Y} f(x, y )
\]
for all $x \in X$.

In particular, notice that $S \in \tilde{C}_f$ provided, for each $x \in X$, there is a topology on $Y$ for which $S$ is dense and $f(x, \cdot)$ is lower semicontinuous.

Furthermore, we denote by $\tau_f$ the topology on $Y$ generated by the family
\[
\{ \{ y \in Y : f(x, y) < r \}\}_{x \in X, r \in \mathbb{R}} .
\]
So, $\tau_f$ is the weakest topology on $Y$ for which $f(x, \cdot)$ is upper semicontinuous for all $x \in X$.

In [7], we established the following minimax result:

THEOREM A. - For every $g \in A_Y \cap B$, at least one of the following assertions holds:
\[ (j) \sup_Y \inf_X g = \inf_X \sup_Y g . \]
There exists $y^* \in Y$ such that the function $g(\cdot, y^*)$ has at least two global minima.

The relevance of Theorem A resides essentially in the fact that it is a flexible tool which can fruitfully be used to obtain meaningful results of various nature. This is clearly shown by a series of recent papers ([8]-[14]).

So, we believe that it is of interest to present a more complete form of Theorem A: this is just the aim of this paper.

Here is the main abstract result (with the usual rules in $\mathbb{R}$):

**THEOREM 1.** - Let $f : X \times Y \to \mathbb{R}$. Assume that there is a function $\psi : Y \to \mathbb{R}$ such that $f + \psi \in \mathcal{B}$ and

$$\alpha_{f+\psi} < \beta_{f+\psi}.$$  

Then, for every convex set $S \in \mathcal{C}_{f+\psi}$, for every bounded function $\varphi : X \to \mathbb{R}$ and for every $\lambda > 0$ such that $\lambda f + \varphi \in \mathcal{A}_S$ and

$$\lambda > \frac{2\sup_X |\varphi|}{\beta_{f+\psi} - \alpha_{f+\psi}},$$  

there exists $y^* \in S$ such that the function $\lambda f(\cdot, y^*) + \varphi(\cdot)$ has at least two global minima.

**PROOF.** Consider the function $g : X \times Y \to \mathbb{R}$ defined by

$$g(x, y) = \lambda(f(x, y) + \psi(y)) + \varphi(x)$$

for all $(x, y) \in X \times Y$. Since $S \in \mathcal{C}_{f+\psi}$, we have

$$\inf_X \sup_Y (f + \psi) = \inf_X \sup_Y (f + \psi).$$  \hspace{1cm} (2)

So, taking (1) and (2) into account, we have

$$\sup_X \inf_Y g \leq \sup_Y \inf_X (f + \psi) \leq \lambda \alpha_{f+\psi} + \sup_X |\varphi|$$

$$< \lambda \beta_{f+\psi} - \sup_X |\varphi| = \lambda \inf_X \sup_Y (f + \psi) - \sup_X |\varphi| \leq \inf_X \sup_Y g.$$ \hspace{1cm} (3)

Now, observe that $g \in \mathcal{A}_S$ since $\lambda f + \varphi \in \mathcal{A}_S$ and, at the same time, $g \in \mathcal{B}$ since $f + \psi \in \mathcal{B}$. As a consequence, we can apply Theorem A to the restriction of the function $g$ to $X \times S$. Therefore, in view of (3), there exists of $y^* \in S$ such that the function $g(\cdot, y^*)$ (and hence $\lambda f(\cdot, y^*) + \varphi(\cdot)$) has at least two global minima, as claimed.

**REMARK 1.** As the above proof shows, Theorem 1 is a direct consequence of Theorem A. However, there are essentially four advantages of Theorem 1 with respect to Theorem A. Namely, suppose that, for a given function $g \in \mathcal{A}_Y$, we are interested in ensuring the validity of assertion $(jj)$. Then, if we apply Theorem A in this regard, we have to show that $g \in \mathcal{B}$ and that assertion $(j)$ does not hold. To the contrary, if we apply Theorem 1, we can ensure the validity of $(jj)$ also in cases where either $g \notin \mathcal{B}$ or $(j)$ holds true too. In addition, Theorem 1 is able to obtain the validity of $(jj)$ even in a remarkably stronger way: not only extending it to suitable perturbations of $g$, but also offering an information on the location of $y^*$.

First, we wish to show how to obtain the very classical minimax theorems in [3] and [6] by means of Theorem 1.

Let $V$ be a real vector space, $A \subseteq V$, $\varphi : A \to \mathbb{R}$. We say that $\varphi$ is finitely lower semicontinuous if, for every finite-dimensional linear subspace $F \subseteq V$, the function $f|_{A \cap F}$ is lower semicontinuous in the Euclidean topology of $F$.

In the next result, the topology of $X$ has no role.

**THEOREM 2.** - Let $X$ be a convex set in a real vector space and let $f \in \mathcal{B}$. Assume that there is a convex set $S \in \mathcal{C}_f$ such that $f(\cdot, y)$ is finitely lower semicontinuous and convex for all $y \in S$. Finally, assume that, for some $x_0 \in X$, the function $f(x_0, \cdot)$ is $\tau_f - \sup$-compact.
Then, one has
\[ \sup_Y \inf_X f = \inf_X \sup_Y f. \]

PROOF. Arguing by contradiction, assume that
\[ \sup_Y \inf_X f < \inf_X \sup_Y f. \]

Denote by \( D \) the family of all convex polytopes in \( X \). Since \( D \) is a filtering cover of \( X \) and \( f(x_0, \cdot) \) is \( \tau_f - \text{sup-compact} \), by Proposition 2.1 of [7], there exists \( P \in D \) such that
\[ \sup_Y \inf_P f < \inf_P \sup_Y f. \]

Let \( \| \cdot \| \) be the Euclidean norm on \( \text{span}(P) \). So, \( \| \cdot \|^2 \) is strictly convex. Now, fix \( \lambda \) so that
\[ \lambda > \frac{2 \sup_{x \in P} \| x \|^2}{\inf_P \sup_Y f - \sup_Y \inf_P f}. \]

Notice that, for each \( y \in S \), the function \( x \to \| x \|^2 + \lambda f(x, y) \) is inf-compact in \( P \) with respect to the Euclidean topology. As a consequence, if we consider \( P \) equipped with the Euclidean topology, we can apply Theorem 1 to the restriction of \( f \) to \( P \times Y \) (recall that \( S \in C_f \)), taking \( \varphi = \| \cdot \|^2 \). Accordingly, there would exist \( y^* \in S \) such that the function \( x \to \| x \|^2 + \lambda f(x, y) \) has at least two global minima in \( P \). But, this is absurd since this function is strictly convex.

\[ \triangle \]

Reasoning exactly as in the proof of Theorem 2 (even in a simplified way, since there is no need to consider the family \( D \)), we also get

THEOREM 3. - Let \( X \) be a compact convex set in a topological vector space such that there exists a lower semicontinuous, strictly convex, bounded function \( \varphi : X \to \mathbb{R} \). Let \( f \in B \). Assume that there is a convex set \( S \in C_f \) such that \( f(\cdot, y) \) is lower semicontinuous and convex for all \( y \in S \).

Then, one has
\[ \sup_Y \inf_X f = \inf_X \sup_Y f. \]

We now revisit two applications of Theorem A in the light of Theorem 1.

The first one concerns the so called farthest points ([1], [4]).

THEOREM 4. - Let \( X \) be compact, not singleton and let \( (E, d) \) be a metric space such that \( X \subseteq E \). Moreover, let \( h : Y \to E \) be such that \( X \subseteq h(Y) \) and let the function \( (x, y) \to f(x, y) := -d(x, h(y)) \) belong to \( B \).

Then, for every convex set \( S \in C_f \), for every bounded function \( \gamma : X \to \mathbb{R} \) and for every \( \lambda > 0 \) such that \( \lambda f - \gamma \in A_S \) and
\[ \lambda > \frac{4 \sup_X |\gamma|}{\text{diam}(X)}, \]
there exists \( y^* \in S \) such that the function \( x \to \gamma(x) + \lambda d(x, h(y^*)) \) has at least two global maxima in \( X \).

PROOF. Since \( X \subseteq h(Y) \), we have
\[ \sup_{x \in X} \inf_{y \in Y} d(x, h(y)) = 0. \]

(4)

Also, for each \( x_1, x_2 \in X, y \in Y \), we have
\[ \frac{d(x_1, x_2)}{2} \leq \max\{d(x_1, h(y)), d(x_2, h(y))\} \]
and so
\[ \frac{\text{diam}(X)}{2} \leq \inf_{\gamma \in Y} \sup_{x \in X} d(x, h(y)) \, . \]  
(5)

Hence, in view of (4) and (5), we have
\[ \sup_{\gamma \in Y} \inf_{x \in X} f \leq -\frac{\text{diam}(X)}{2} < 0 = \inf_{x \in Y} f \, . \]

Now, the conclusion follows directly from Theorem 1 taking \( \varphi = -\gamma \).

\( \triangle \)

Of course, the most natural corollary of Theorem 4 is as follows:

**COROLLARY 1.** - Let \( X \) be a compact, not singleton subset of a normed space \((E, \| \cdot \|)\) and let \( Y \) be a convex subset of \( E \) such that \( X \subseteq Y \).

Then, for every convex set \( S \subseteq Y \) dense in \( Y \), for every upper semicontinuous bounded function \( \gamma : X \rightarrow \mathbb{R} \) and for every \( \lambda > \frac{\sup_X |\gamma|}{\text{diam}(X)} \), there exists \( y^* \in S \) such that the function \( x \rightarrow \gamma(x) + \lambda \|x - y^*\| \) has at least two global maxima in \( X \).

In turn, from Corollary 1, we clearly get

**COROLLARY 2.** - Let \( X \) be a compact subset of a normed space \((E, \| \cdot \|)\) and let \( Y \) be a convex subset of \( E \) such that \( X \subseteq Y \). Assume that there exist a sequence \( \{S_n\} \) of convex subsets of \( Y \) dense in \( Y \) and a sequence \( \{\gamma_n\} \) of upper semicontinuous bounded real-valued functions on \( X \), with \( \lim_{n \to \infty} \sup_X |\gamma_n| = 0 \), such that, for each \( n \in \mathbb{N} \) and for each \( y \in S_n \), the function \( x \rightarrow \gamma_n(x) + \|x - y\| \) has a unique global maximum in \( X \).

Then, \( X \) is a singleton.

**REMARK 2.** - Notice that Corollary 2 improves Theorem 1.1 of [14] which, in turn, extended a classical result by Klee ([5]) to normed spaces. More precisely, Theorem 1.1 of [14] agrees with the particular case of Corollary 2 in which each \( S_n \) is equal to \( \text{conv}(X) \) and each \( \gamma_n \) is equal to 0.

The second application concerns the calculus of variations.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary and let \( p > 1 \). On the Sobolev space \( W^{1,p}(\Omega) \), we consider the norm
\[ \|u\| = \left( \int_{\Omega} |\nabla u(x)|^p dx + \int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} \, . \]

If \( n \geq p \), we denote by \( \mathcal{E} \) the class of all continuous functions \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[ \sup_{\xi \in \mathbb{R}} \frac{\sigma(\xi)}{1 + |\xi|^q} < +\infty \, , \]
where \( 0 < q < \frac{pn}{n-p} \) if \( p < n \) and \( 0 < q < +\infty \) if \( p = n \). While, when \( n < p \), \( \mathcal{E} \) stands for the class of all continuous functions \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \).

Recall that a function \( h : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R} \) is said to be a normal integrand ([15]) if it is \( \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m) \)-measurable and \( h(x, \cdot) \) is lower semicontinuous for a.e. \( x \in \Omega \). Here \( \mathcal{L}(\Omega) \) and \( \mathcal{B}(\mathbb{R}^m) \) denote the Lebesgue and the Borel \( \sigma \)-algebras of subsets of \( \Omega \) and \( \mathbb{R}^m \), respectively.

Recall that if \( h \) is a normal integrand then, for each measurable function \( u : \Omega \rightarrow \mathbb{R}^m \), the composite function \( x \rightarrow h(x, u(x)) \) is measurable ([15]).

We denote by \( \mathcal{F} \) the class of all normal integrands \( h : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( h(x, \xi, \cdot) \) is convex for all \( (x, \xi) \in \Omega \times \mathbb{R} \) and there are \( M \in L^1(\Omega) \), \( b > 0 \) such that
\[ M(x) - b(|\xi| + |\eta|) \leq h(x, \xi, \eta) \]
for all \( (x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \).

Let us also recall two results proved in [9].
PROPOSITION 1. - Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $p > 1$ and let $h : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be normal integrand such that, for some $c, d > 0$, one has

\[ c|\eta|^p - d \leq h(x, \xi, \eta) \]

for all $(x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and

\[ \lim_{|\xi| \to +\infty} \inf_{(x, \eta) \in \Omega \times \mathbb{R}^n} h(x, \xi, \eta) = +\infty . \]

Then, in $W^{1,p}(\Omega)$, one has

\[ \lim_{\|u\| \to +\infty} \int_{\Omega} h(x, u(x), \nabla u(x)) dx = +\infty . \]

PROPOSITION 2. - Let $X, Y$ be two non-empty sets and $I : X \to \mathbb{R}$, $J : X \times Y \to \mathbb{R}$ two given functions. Assume that there are two sets $A, B \subseteq X$ such that:

(a) $\sup_A I < \inf_B I$ ;
(b) $\sup_Y \inf_A J(x, y) \leq 0$ ;
(c) $\inf_B \sup_Y J(x, y) \geq 0$ ;
(d) $\inf_X \sup_Y J(x, y) = +\infty$.

Then, one has

\[ \sup_{Y} \inf_{X} (I + J) \leq \sup_{X} I < \inf_{X} I \leq \inf_{Y} \sup_{X} (I + J) . \]

Furthermore, let us also recall the following classical fact:

PROPOSITION 3. - Let $A \subseteq \mathbb{R}^n$ be any open set and let $v \in L^1(\Omega) \setminus \{0\}$. Then, one has

\[ \sup_{\alpha \in C^\infty_0(\Omega)} \int_{\Omega} \alpha(x)v(x) dx = +\infty . \]

After these preliminaries, we can prove the following result:

THEOREM 5. - Let $h, k \in \mathcal{F}$ and let $\sigma \in \mathcal{E}$ be a strictly monotone function. Assume that:

(i) there are $c, d > 0$ such that

\[ c|\eta|^p - d \leq h(x, \xi, \eta) \]

for all $(x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and

\[ \lim_{|\xi| \to +\infty} \inf_{(x, \eta) \in \Omega \times \mathbb{R}^n} h(x, \xi, \eta) = +\infty ; \]

(ii) for each $\xi \in \mathbb{R}$, the function $h(\cdot, \xi, 0)$ lies in $L^1(\Omega)$ ;

(iii) there are $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, with $\xi_1 < \xi_2 < \xi_3$, such that

\[ \max \left\{ \int_{\Omega} h(x, \xi_1, 0) dx, \int_{\Omega} h(x, \xi_3, 0) dx \right\} < \int_{\Omega} h(x, \xi_2, 0) dx . \]

Then, for every sequentially weakly closed set $V \subseteq W^{1,p}(\Omega)$, containing the constants, for every convex set $T \subseteq L^\infty(\Omega)$ dense in $L^\infty(\Omega)$, for every non-decreasing, continuous, bounded function $\omega : U \to \mathbb{R}$, where $U := \{ \int_{\Omega} k(x, u(x), \nabla u(x)) dx : u \in W^{1,p}(\Omega) \}$, and for every $\lambda$ satisfying

\[ \lambda > \frac{2 \sup_{\Omega} |\omega|}{\int_{\Omega} h(x, \xi_2, 0) dx - \max \{ \int_{\Omega} h(x, \xi_1, 0) dx, \int_{\Omega} h(x, \xi_3, 0) dx \} ,} \]

(6)
there exists $\gamma \in T$ such that the restriction to $V$ of the functional
\[ u \to \lambda \int_{\Omega} h(x, u(x), \nabla u(x)) dx + \int_{\Omega} \gamma(x)\sigma(u(x)) dx + \omega \left( \int_{\Omega} k(x, u(x), \nabla u(x)) dx \right) \]
has at least two global minima. The same conclusion holds also with $T = C_0^\infty(\Omega)$.

**PROOF.** Fix $V, T, \omega, \lambda$ as in the conclusion. Since $\sigma \in \mathcal{E}$, in view of the Rellich-Kondrachov theorem, for each $u \in W^{1,p}(\Omega)$, we have $\sigma \circ u \in L^1(\Omega)$ and, for each $\gamma \in L^\infty(\Omega)$, the functional $u \to \int_{\Omega} \gamma(x)\sigma(u(x)) dx$ is sequentially weakly continuous. Moreover, since $h, k \in \mathcal{F}$ the functionals $u \to \int_{\Omega} h(x, u(x), \nabla u(x)) dx$ and $u \to \int_{\Omega} k(x, u(x), \nabla u(x)) dx$ (possibly taking the value $+\infty$) are sequentially weakly lower semicontinuous ([2], Theorem 4.6.8). Hence, since $\omega$ is non-decreasing and continuous, the functional $u \to \omega \left( \int_{\Omega} k(x, u(x), \nabla u(x)) dx \right)$ is sequentially weakly lower semicontinuous too. Set
\[ X = \left\{ u \in V : \int_{\Omega} h(x, u(x), \nabla u(x)) dx < +\infty \right\}. \]
By (ii), the constants belong to $X$. Fix $\gamma \in L^\infty(\Omega)$. By (i), there is $\delta > 0$ such that
\[ h(x, \xi, \eta) - 2\|\gamma\|_{L^\infty(\Omega)}\|\sigma(\xi)\| \geq 0 \]
for all $(x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ with $|\xi| > \delta$. So, we have
\[ \int \frac{c}{2} |\eta|^p - d - \|\gamma\|_{L^\infty(\Omega)} \sup_{|\xi| \leq \delta} |\sigma(\xi)| \leq h(x, \xi, \eta) + \gamma(x)\sigma(\xi) \]
for all $(x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and, of course,
\[ \lim_{|\xi| \to +\infty} \inf_{(x, \eta) \in \Omega \times \mathbb{R}^n} (h(x, \xi, \eta) + \gamma(x)\sigma(\xi)) = +\infty. \]
Consequently, in view of Proposition 2.1, we have, in $W^{1,p}(\Omega)$,
\[ \lim_{\|u\| \to +\infty} \left( \int_{\Omega} h(x, u(x), \nabla u(x)) dx + \int_{\Omega} \gamma(x)\sigma(u(x)) dx \right) = +\infty. \]
This implies that, for each $r \in \mathbb{R}$, the set
\[ \left\{ u \in V : \int_{\Omega} h(x, u(x), \nabla u(x)) dx + \int_{\Omega} \gamma(x)\sigma(u(x)) dx \leq r \right\} \]
is weakly compact by reflexivity and by Eberlein-Smulyan’s theorem. Of course, we also have
\[ \left\{ u \in V : \int_{\Omega} h(x, u(x), \nabla u(x)) dx + \int_{\Omega} \gamma(x)\sigma(u(x)) dx \leq r \right\} = \left\{ u \in X : \int_{\Omega} h(x, u(x), \nabla u(x)) dx + \int_{\Omega} \gamma(x)\sigma(u(x)) dx \leq r \right\}. \]
Now, observe that, if we put
\[ A = \{ \xi_1, \xi_3 \} \]
and
\[ B = \{ \xi_2 \}, \]
and define $I : X \to \mathbb{R}, J : X \times L^\infty(\Omega) \to \mathbb{R}$ by
\[ I(u) = \int_{\Omega} h(x, u(x), \nabla u(x)) dx, \]
\[ J(u, \gamma) = \int_{\Omega} \gamma(x)(\sigma(u(x)) - \sigma(\xi_2)) \, dx \]

for all \( u \in X, \gamma \in L^\infty(\Omega) \), we clearly have

\[
\inf_{u \in B} \sup_{\gamma \in L^\infty(\Omega)} J(u, \gamma) = 0
\]

and, by (iii),

\[
\sup_A I < \inf_B I .
\]

Since \( \sigma \) is strictly monotone, the numbers \( \sigma(\xi_1) - \sigma(\xi_2) \) and \( \sigma(\xi_3) - \sigma(\xi_2) \) have opposite signs. This clearly implies that

\[
\sup_{\gamma \in L^\infty(\Omega)} \inf_{u \in A} J(u, \gamma) \leq 0 .
\]

Furthermore, if \( u \in X \setminus \{ \xi_2 \} \), again by strict monotonicity, \( \sigma \circ u \neq \sigma(\xi_2) \), and so we have

\[
\sup_{\gamma \in L^\infty(\Omega)} J(u, \gamma) = +\infty .
\]

Therefore, the sets \( A, B \) and the functions \( I, J \) satisfy the assumptions of Proposition 2 and hence we have

\[
\sup_{L^\infty(\Omega)} \inf_X (I + J) \leq \max\left\{ \int_{\Omega} h(x, \xi_1, 0) \, dx, \int_{\Omega} h(x, \xi_3, 0) \, dx \right\} < \int_{\Omega} h(x, \xi_2, 0) \, dx \leq \inf_X \sup_{L^\infty(\Omega)} (I + J) . \quad (7)
\]

Now, we can apply Theorem 1 considering \( X \) equipped with the weak topology and taking

\[
Y = L^\infty(\Omega) ,
\]

\[
f = I + J ,
\]

\[
\psi = 0 ,
\]

\[
S = \frac{1}{\lambda} T
\]

and

\[
\varphi(u) = \omega \left( \int_{\Omega} k(x, u(x), \nabla u(x)) \, dx \right) .
\]

Notice that, in view of (7), inequality (1) holds thanks to (6), and the conclusion follows. When \( T = C^\infty_0(\Omega) \) the same proof as above holds in view of Proposition 3.

REMARK 4. - Notice that condition (iii) holds if and only if the function \( \int_{\Omega} h(x, \, \cdot \, , 0) \) is not quasi-convex.

REMARK 5. - For \( \omega = 0 \), Theorem 5 reduces to Theorem 1.2 of [9].

We conclude presenting an application of Theorem 5 to the Neumann problem for a Kirchhoff-type equation.

Given \( K : [0, \infty[ \to \mathbb{R} \) and a Carathéodory function \( \psi : \Omega \times \mathbb{R} \to \mathbb{R} \), consider the following Neumann problem

\[
\begin{cases}
-K \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right) \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \psi(x, u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega ,
\end{cases}
\]

where \( \nu \) is the outward unit normal to \( \partial \Omega \).
Let us recall that a weak solution of this problem is any \( u \in W^{1,p}(\Omega) \) such that, for every \( v \in W^{1,p}(\Omega) \), one has \( \psi(\cdot, u(\cdot)) v(\cdot) \in L^1(\Omega) \) and

\[
K \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right) \int_{\Omega} |\nabla v(x)|^p - 2 |\nabla u(x)| \nabla v(x) \, dx - \int_{\Omega} \psi(x, u(x)) v(x) \, dx = 0 .
\]

THEOREM 6. - Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two \( C^1 \) functions lying in \( \mathcal{E} \) and satisfying the following conditions:

(a) the function \( g' \) has a constant sign and \( \text{int}((g')^{-1}(0)) = \emptyset \);

(b) \( \lim_{|x| \to +\infty} \frac{f(|x|)}{|x|} = +\infty \);

(c) there are \( \xi_1, \xi_2, \xi_3 \in \mathbb{R} \), with \( \xi_1 < \xi_2 < \xi_3 \), such that

\[
\max\{f(\xi_1), f(\xi_3)\} < f(\xi_2).
\]

Then, for every \( a > 0 \), for every \( \beta \in L^\infty(\Omega) \), with \( \inf_{\Omega} \beta > 0 \), for every convex set \( T \subseteq L^\infty(\Omega) \) dense in \( L^\infty(\Omega) \), for every \( C^1 \), non-decreasing, bounded function \( \chi : [0, +\infty[ \to \mathbb{R} \), and for every \( \lambda \) satisfying

\[
\lambda > \frac{2 \sup_{[0, +\infty[} |\chi|}{p(f(\xi_2) - \max\{f(\xi_1), f(\xi_3)\})} \int_{\Omega} \beta(x) \, dx
\]

there exists \( \gamma \in T \) such that the problem

\[
\begin{aligned}
&- (a + \chi' \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)) \text{div}(|\nabla u|^{p-2} \nabla u) = \gamma(x) g'(u) - \lambda \beta(x) f'(u) \quad \text{in } \Omega \\
&\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

has at least two weak solutions.

PROOF. Fix \( a, \beta, T, \chi \) and \( \lambda \) as in the conclusion. We are going to apply Theorem 5, defining \( h, k, \sigma \) by

\[
h(x, \xi, \eta) = \frac{a}{p\lambda} |\eta|^p + \beta(x) f(\xi),
\]

\[
k(\eta) = |\eta|^p,
\]

\[
\sigma(\xi) = -g(\xi)
\]

for all \( (x, \xi, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \). It is immediate to realize that, by (a) \(- (a, 3)\), the above \( h, k, \sigma \) satisfy the assumptions of Theorem 5. Then, applying Theorem 5 with \( \omega = \frac{1}{p\lambda} \chi \), we get the existence of \( \gamma \in T \) such that the functional

\[
u \to \lambda \left( \frac{a}{p\lambda} \int_{\Omega} |\nabla u(x)|^p \, dx + \int_{\Omega} \beta(x) f(u(x)) \, dx \right) - \int_{\Omega} \gamma(x) g(u(x)) \, dx + \frac{1}{p\lambda} \chi \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)
\]

has at least two global minima in \( W^{1,p}(\Omega) \). But, by classical results (recall that \( f, g \in \mathcal{E} \)), such a functional is \( C^1 \) and its critical points (and so, in particular, its global minima) are weak solutions of problem \((P)\). The proof is complete.

A challenging problem is as follows:

PROBLEM 1. - Does the conclusion of Theorem 6 hold with \textit{three} instead of \textit{two}?
References

[1] S. COBZAȘ, Geometric properties of Banach spaces and the existence of nearest and farthest points, Abstr. Appl. Anal., 2005, n. 3, 259-285.

[2] Z. DENKOWSKI, S. MIGÓRSKI and N. S. PAPAGEORGIOU, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic Publishers, 2003.

[3] K. FAN, Minimax theorems, Proc. Nat. Acad. Sci. U.S.A., 39 (1953), 42-47.

[4] J.-B. HIRIART-URRUTY, La conjecture des points les plus éloignés revisitée, Ann. Sci. Math. Québec 29 (2005), 197-214.

[5] V. L. KLEE, Convexity of Chebyshev sets, Math. Ann., 142 (1960/1961), 292-304.

[6] H. KNESER, Sur un théorème fondamental de la théorie des jeux, C. R. Acad. Sci. Paris 234 (1952), 2418-2420.

[7] B. RICCERI, On a minimax theorem: an improvement, a new proof and an overview of its applications, Minimax Theory Appl., 2 (2017), 99-152.

[8] B. RICCERI, Another multiplicity result for the periodic solutions of certain systems, Linear Nonlinear Anal., 5 (2019), 371-378.

[9] B. RICCERI, Miscellaneous applications of certain minimax theorems II, Acta Math. Vietnam., 45 (2020), 515-524.

[10] B. RICCERI, A class of equations with three solutions, Mathematics (2020), 8, 478.

[11] B. RICCERI, An invitation to the study of a uniqueness problem, in “Nonlinear Analysis and Global Optimization”, Th. M. Rassias and P. M. Pardalos eds., 445-448, Springer, 2021.

[12] B. RICCERI, A class of functionals possessing multiple global minima, Stud. Univ. Babeş-Bolyai Math., 66 (2021), 75-84.

[13] B. RICCERI, An alternative theorem for gradient systems, Pure Appl. Funct. Anal., 6 (2021), 373-381.

[14] B. RICCERI, On the applications of a minimax theorem, Optimization, to appear.

[15] R. T. ROCKAFELLAR, Integral functionals, normal integrands and measurable selections, Lecture Notes in Math., Vol. 543, 157-207, Springer, Berlin, 1976.

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