1. Introduction

1.1. Results. A tropical curve is a geometric object over the tropical semifield of real numbers \( \mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot) \), where the addition \( \oplus \) is the max-operation in the real field \( \mathbb{R} \), and the multiplication \( \odot \) is the addition of \( \mathbb{R} \). For a tropical curve \( C \) and a divisor \( D \) on \( C \),

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the set \( M = H^0(C, \mathcal{O}_C(D)) \) of the sections of \( D \) has the structure of a \( \mathbb{T} \)-module that is defined as follows.

A \( \mathbb{T} \)-module \( M \) is defined as a module over a semifield. \((M, \oplus, \odot, -\infty)\) is said to be a \( \mathbb{T} \)-module if \((M, \oplus, -\infty)\) is a tropical semigroup, and \( \odot \) is an additive semigroup action on \( M \) by \( \mathbb{T} \). A tropical semigroup is a commutative semigroup with unity such that any element \( v \) satisfies the idempotent condition \( v \oplus v = v \).

A \( \mathbb{T} \)-module \( M \) is analogous to a module over a field. A subset \( S \subset M \) is said to be a basis if it is a minimal system of generators. But the number of elements of a basis of \( M \) is not necessarily equal to the topological dimension of it. We introduce straight \( \mathbb{T} \)-modules in section 2. This class is a generalization of lattice-preserving submodules of the free \( \mathbb{T} \)-module \( \mathbb{T}^n \), where a lattice-preserving submodule is a submodule preserving the infimum of any two elements with respect to the canonical partial order relation on \( \mathbb{T}^n \).

**Theorem 1.1.** Let \( M \) be a finitely generated straight submodule of the free \( \mathbb{T} \)-module \( \mathbb{T}^n \). Then \( M \) is generated by \( n \) elements.

We have four corollaries (Theorem 2.1 2.2 2.3 2.4). The semifield \( \mathbb{T} \) is generalized to a quasi-complete totally ordered rational tropical semifield \( k \). We find a sufficient condition to the existence of a left-inversion of an injective homomorphism of \( k \)-modules (Theorem 2.1). The dimension of a straight reflexive \( k \)-module is defined to be the number of elements of a basis. We show the inequality \( \dim(M) \leq \dim(N) \) for a pair of straight reflexive \( k \)-modules \( M \subset N \) (Theorem 2.2). We show that a finitely generated straight pre-reflexive \( k \)-module is reflexive (Theorem 2.3). Also we consider finiteness of a submodule of a \( k \)-module (Theorem 2.4). The proofs are given in section 3.7.

This result has an application to polytopes in a tropical projective space \( \mathbb{T}\mathbb{P}^n \). By Joswig and Kulas [3], a polytrope (it means a polytope in \( \mathbb{T}\mathbb{P}^n \) that is real convex) is a tropical simplex, and therefore it is the tropically convex hull of at most \( n + 1 \) points. We show a generalization of this result (Theorem 2.5). A polytope \( P \) is the tropically convex hull of at most \( n + 1 \) points if the corresponding submodule \( M \subset \mathbb{T}^{n+1} \) is straight reflexive. Also \( M \) is straight reflexive if \( P \) is a polytrope.

Also we have an application to tropical curves. A Riemann-Roch theorem for tropical curves is proved by Gathmann and Kerber [1]. This theorem states an equality for an invariant \( r(D) \) of the divisor. We see that \( r(D) \) is not an invariant of the \( \mathbb{T} \)-module \( M = H^0(C, \mathcal{O}_C(D)) \) (Example 6.5), and show the inequality \( r(D) \leq \dim(M) - 1 \) (Theorem 2.7).
1.2. **Background.** Surveys of tropical mathematics are found in [4], [7]. Early studies of tropical curves are found in [1], [5], [6]. Tropical varieties are introduced as follows. Let $K = \mathbb{C}[[\mathbb{R}]]$ be the group algebra of power series defined by the group $\mathbb{R}$. We have a multiplicative seminorm

$$||·|| : K \to \mathbb{R}_{\geq 0}$$

defined by

$$||x|| = \exp(-\text{val}(x)),$$

where val means the canonical valuation on $K$. This seminorm induces the amoeba map

$$\mathcal{A} : (K^\times)^n \to \mathbb{R}^n$$

defined by

$$\mathcal{A}(x_1, \ldots, x_n) = (\log ||x_1||, \ldots, \log ||x_n||).$$

The image $\mathcal{A}(V)$ of a variety $V$ in the algebraic torus $(K^\times)^n$ is said to be a tropical variety in the tropical torus $\mathbb{R}^n$.

Tropical algebra is introduced by the map

$$\pi : K \to \mathbb{R} \cup \{-\infty\}$$

defined by

$$\pi(x) = \log ||x||.$$

This map induces a hyperfield homomorphism

$$\pi : K \to X,$$

where $X$ is the tropical hyperfield with underlying set $\mathbb{R} \cup \{-\infty\}$, introduced in [8]. The power set $2^X$ is a semiring with operations induced by multi-operations of $X$.

Now we have the lower-saturation map

$$\nu : X \to 2^X$$

defined by

$$\nu(a) = \{c \in X \mid c \leq a\}.$$ 

The power set $2^X$ has a subsemiring

$$\mathbb{I} = X \cup \nu(X),$$

which is isomorphic to Izhakian’s extended tropical semiring introduced in [2]. The lower-saturation map $\nu$ means the ghost map in [2]. The image $\nu(X)$ means the ghost part, which is isomorphic to the tropical semifield of real numbers $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where operations are defined as follows.

$$a \oplus b = \max\{a, b\},$$

$$a \odot b = a + b.$$
In this paper, the symbol $T$ means the tropical semifield of real numbers. Under the identification $T = \nu(X)$, the canonical homomorphism $\nu: \mathbb{I} \rightarrow T$ is the lower-saturation map.

Section 2 contains definitions and theorems. Section 3 and 4 contain foundation of tropical modules, and the proof of Theorem 2.1, 2.2, 2.3, 2.4, and 2.5. Section 5 and 6 contain foundation of tropical matrices and tropical curves, and the proof of Theorem 2.7. Section 7 is an appendix for tropical plane curves.

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2. Definitions and theorems

A *semigroup* $(M, \oplus)$ is a set $M$ with an associative operation $\oplus$.

**Definition.** $(M, \oplus, -\infty)$ is a *tropical semigroup* if it satisfies the following axioms.

(i) $(M, \oplus)$ is a semigroup.
(ii) $v \oplus w = w \oplus v$.
(iii) $v \oplus -\infty = v$.
(iv) $v \oplus v = v$.

The element $-\infty$ is called the zero element of $M$.

There is a unique partial order relation ‘$\leq$’ on $M$ such that for any $v, w \in M$ it implies

$$\text{sup}\{v, w\} = v \oplus w.$$ 

The proof is given in section 3.1.

**Definition.** A tropical semigroup $M$ is *quasi-complete* if any non-empty subset $S \subset M$ admits the infimum $\inf(S)$ (i.e. it admits the maximum element of the lower-bounds of $S$).

**Definition.** $(A, \oplus, \odot, -\infty, 0)$ is a *tropical semiring* if it satisfies the following axioms.

(i) $(A, \oplus, -\infty)$ is a tropical semigroup.
(ii) $(A, \odot)$ is a semigroup.
(iii) $a \odot b = b \odot a$.
(iv) $a \odot (b \oplus c) = a \odot b \oplus a \odot c$.
(v) $a \odot 0 = a$.
(vi) $a \odot -\infty = -\infty$.

The element $-\infty$ is called the zero element of $A$. The element 0 is called the unity of $A$. 
Definition. \((k, \oplus, \odot, -\infty, 0)\) is a tropical semifield if it satisfies the following axioms.

(i) \((k, \oplus, \odot, -\infty, 0)\) is a tropical semiring.

(ii) For any \(a \in k \setminus \{-\infty\}\) there is an element \(\odot a \in k\) such that \(a \odot (\odot a) = 0\).

Definition. A tropical semifield \(k\) is rational if it satisfies the following conditions.

(i) \(a \in k, m \in \mathbb{N} \Rightarrow \exists b \in k, a = b^m\).

(ii) \(k\) has no maximum element.

The tropical semifield of real numbers \((\mathbb{T}, \oplus, \odot, -\infty, 0)\) is the set \(\mathbb{T} = \mathbb{R} \cup \{-\infty\}\) equipped with addition \(a \oplus b = \max\{a, b\}\) and multiplication \(a \odot b = a + b\) and zero element \(-\infty\) and unity \(0\). \(\mathbb{T}\) is a quasi-complete totally ordered rational tropical semifield.

Let \(k\) be a quasi-complete totally ordered rational tropical semifield.

Definition. \((M, \oplus, \odot, -\infty)\) is a \(k\)-module if it satisfies the following axioms.

(i) \((M, \oplus, -\infty)\) is a tropical semigroup.

(ii) \(\odot\) is a semigroup action \(k \times M \ni (a, v) \mapsto a \odot v \in M\), i.e.

\[i) (a \odot b) \odot v = a \odot (b \odot v).\]

\[ii) 0 \odot v = v.\]

(iii) \((a \oplus b) \odot v = (a \odot v) \oplus (b \odot v).\)

(iv) \(a \odot (v \oplus w) = (a \odot v) \oplus (a \odot w).\)

(v) \(-\infty \odot v = -\infty.\)

(vi) \(a \odot -\infty = -\infty.\)

Definition. A homomorphism \(\alpha: M \to N\) of \(k\)-modules is a map with the following conditions.

(i) \(\alpha(-\infty) = -\infty.\)

(ii) \(\alpha(v \oplus w) = \alpha(v) \oplus \alpha(w).\)

(iii) \(\alpha(a \odot v) = a \odot \alpha(v).\)

Let \(\text{Hom}(M, N)\) denote the \(k\)-module of homomorphisms from \(M\) to \(N\).
The dual module $M^\vee$ is defined by $M^\vee = \text{Hom}(M, k)$. We have the pairing map $\langle \cdot, \cdot \rangle: M \times M^\vee \to k$ defined by

$$\langle v, \xi \rangle = \xi(v).$$

**Definition.** $M$ is **pre-reflexive** if the homomorphism $\iota_M: M \to (M^\vee)^\vee$ is injective. $M$ is **reflexive** if $\iota_M$ is an isomorphism.

**Definition.** A $k$-module $M$ is straight if it is a finitely distributive ordered lattice, i.e. it satisfies the following conditions.

(i) Any two elements $v, w \in M$ admit the infimum $\inf_M \{v, w\}$.
(ii) $v_1, v_2, w \in M \Rightarrow \inf_M \{v_1 \oplus v_2, w\} = \inf_M \{v_1, w\} \oplus \inf_M \{v_2, w\}$.
(iii) $v_1, v_2, w \in M \Rightarrow \inf_M \{v_1 \oplus v_2\} \oplus w = \inf_M \{v_1 \oplus w, v_2 \oplus w\}$.

**Definition.** A homomorphism $\alpha: M \to N$ is **lightly surjective** if for any $w \in N$ there is $v \in M$ such that $w \leq \alpha(v)$.

A homomorphism $\beta: N \to M$ is said to be a left-inversion of $\alpha$ if $\beta \circ \alpha = \text{id}_M$.

**Theorem 2.1.** Let $\alpha: M \to N$ be an injective lightly surjective homomorphism of $k$-modules such that $M$ is straight reflexive. Then $\alpha$ has a left-inversion.

**Definition.** A **basis** $\{e_\lambda \mid \lambda \in \Lambda\}$ of a $k$-module $M$ is a minimal system of generators (i.e. there is no $\lambda_0 \in \Lambda$ such that the elements $\{e_\lambda \mid \lambda \in \Lambda \setminus \{\lambda_0\}\}$ generate $M$). A subset $S \subset M$ generate $M$ if any element of $M$ is written as a linear combination

$$a_1 \circ v_1 \oplus \cdots \oplus a_r \circ v_r$$

of elements of $S$ over $k$.

**Definition.** An element $e \in M \setminus \{-\infty\}$ is extremal if for any $v_1, v_2 \in M$ such that $v_1 \oplus v_2 = e$ it implies $v_1 = e$ or $v_2 = e$. $M$ is **extremally generated** if $M$ is generated by extremal elements. An extremal ray of $M$ is the submodule generated by an extremal element of $M$.

**Definition.** The **dimension** of a straight reflexive $k$-module $M$ is the number of extremal rays.

The number of extremal rays of $M$ is equal to the number of elements of any basis of $M$. The proof is given in section 3.3.

**Theorem 2.2.** Let $\alpha: M \to N$ be an injective homomorphism of finitely generated straight reflexive $k$-modules. Then

1. $\dim(M) \leq \dim(N)$.
2. If $\dim(M) = \dim(N)$, then $\alpha$ is lightly surjective.
Theorem 2.3. Let $M$ be a finitely generated straight pre-reflexive $k$-module. Then $M$ is reflexive.

Theorem 2.4. Let $\alpha : M \to N$ be an injective homomorphism of straight pre-reflexive $k$-modules. Suppose that $M$ has a basis, and that $N$ is finitely generated. Then $M$ is finitely generated.

Let $P$ be a polytope in $\mathbb{T}P^n$. $P$ is the tropically convex hull of finitely many points $p_1, \ldots, p_r$. Let

$$\varphi : \mathbb{T}^{n+1} \setminus \{-\infty\} \to \mathbb{T}P^n$$

be the canonical projection. Then the subset

$$M = \varphi^{-1}(P) \cup \{-\infty\} \subset \mathbb{T}^{n+1}$$

is a submodule generated by elements $v_1, \ldots, v_r$ such that $\varphi(v_i) = p_i$ ($1 \leq i \leq r$). Also we have an injection

$$\iota : \mathbb{T}^n \to \mathbb{T}P^n$$

defined by $(a_1, \ldots, a_n) \mapsto (0, a_1, \ldots, a_n)$. This map induces an embedding $\mathbb{R}^n \subset \mathbb{T}^n \subset \mathbb{T}P^n$. A polytope $P \subset \mathbb{T}P^n$ is said to be a polytrope if it is a real convex subset of $\mathbb{R}^n$.

Theorem 2.5. Let $P$ be a polytope in $\mathbb{T}P^n$ with the corresponding submodule $M \subset \mathbb{T}^{n+1}$.

1. If $P$ is a polytrope, then $M$ is straight reflexive.
2. If $M$ is straight reflexive, then $P$ is the tropically convex hull of at most $n + 1$ points.

Let $C$ be a tropical curve. Let $D$ be a divisor on $C$. Let $H^0(C, \mathcal{O}_C(D))$ be the set of the sections of $D$. (A section of $D$ is a rational function $f : C \to \mathbb{T}$ such that either $f = -\infty$ or $(f) + D \geq 0$.) For $r \in \mathbb{Z}_{\geq 0}$, let

$$U(D, r) = C^r \setminus S(D, r),$$

$$S(D, r) = \{(P_1, \ldots, P_r) \in C^r \mid H^0(C, \mathcal{O}_C(D - \sum_{1 \leq i \leq r} P_i)) \neq -\infty\}.$$ 

Let $U(D, r) = \emptyset$ if $r = -1$. The following theorem is known.

Theorem 2.6 (Gathmann and Kerber \[1\]). Let $C$ be a compact tropical curve with first Betti number $b_1(C)$. Let $D$ be a divisor on $C$. Let $K$ be the canonical divisor on $C$. Then

$$r(D) - r(K - D) = 1 - b_1(C) + \deg(D),$$

where

$$r(D) = \max\{r \in \mathbb{Z}_{\geq -1} \mid U(D, r) = \emptyset\}.$$
The set $M = H^0(C, \mathcal{O}_C(D))$ is a $\mathbb{T}$-module with addition
\[(f \oplus g)(P) = f(P) \oplus g(P)\]
and scalar multiplication
\[(a \odot f)(P) = a \odot f(P)\]
The dimension of $M$ is defined as follows.

**Definition.** The *dimension* of a $k$-module $M$ is the maximum dimension of the straight reflexive submodules of $M$.

This definition is compatible with the previous one. If $M$ is straight reflexive, then the maximum dimension of the straight reflexive submodules of $M$ equals the dimension of $M$ by Theorem 2.2.

**Theorem 2.7.** Let $C$ be a tropical curve. Let $D$ be a divisor on $C$. Then the inequality
\[r(D) \leq \dim H^0(C, \mathcal{O}_C(D)) - 1\]
is fulfilled.

3. Tropical algebra

3.1. Tropical semigroups, semirings, and semifields.

**Proposition 3.1.** Let $M$ be a tropical semigroup. Then there is a unique partial order relation `$\leq$' such that for any $v, w \in M$ it implies
\[\sup\{v, w\} = v \oplus w.\]

**Proof.** We define a relation `$\leq$' on $M$ as follows.
\[v \leq w \iff v \oplus w = w.\]
This is a partial order relation, because $v \oplus v = v$. The element $v \oplus w$ is the minimum element of the upper bounds of $\{v, w\}$. \hfill \Box

Let $A$ be a tropical semiring.

**Example 3.2.** The *semiring of polynomials* $B = A[x_1, \ldots, x_n]$ is the set of polynomials
\[f = \bigoplus_i a_i \odot x^{\odot i}\]
\[= \bigoplus_{i_1, \ldots, i_n \geq 0} a_{i_1 \ldots i_n} \odot x_1^{\odot i_1} \odot \cdots \odot x_n^{\odot i_n}\]
with coefficients $a_i \in A$, equipped with addition and multiplication of polynomials. $B$ is a tropical semiring. An element $f \in B$ is said to be a tropical polynomial over $A$. The induced map

$$f: A^n \rightarrow A$$

$$(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$$

is said to be a tropical polynomial function.

**Remark 3.3.** We use the notation $ma$ by the meaning of tropical $m$-th power $a \circ^m$. For example, $2(a \oplus b)$ means the second power of $(a \oplus b)$, so we have

$$2(a \oplus b) = 2a \oplus a \circ b \oplus a \circ b \oplus 2b$$

$$= 2a \oplus a \circ b \oplus 2b.$$  

Also a tropical polynomial is written as

$$f = \bigoplus_i a_i \circ ix.$$  

**Proposition 3.4.** Let $A$ be a tropical semiring. Let $f \in A[x_1, \ldots, x_n]$. Then for any $v, w \in A^n$,

$$f(v \oplus w) \geq f(v) \oplus f(w).$$

**Proof.** Assume that

$$f = i_1x_1 \circ \cdots \circ i_nx_n,$$

$$v = (a_1, \ldots, a_n),$$

$$w = (b_1, \ldots, b_n).$$

Then

$$f(v \oplus w) = i_1(a_1 \oplus b_1) \circ \cdots \circ i_n(a_n \oplus b_n)$$

$$\geq (i_1a_1 \circ \cdots \circ i_na_n) \oplus (i_1b_1 \circ \cdots \circ i_nb_n)$$

$$= f(v) \oplus f(w).$$

\[\square\]

Let $k$ be a tropical semifield. Recall that $k$ is said to be rational if it satisfies the following conditions.

(i) $a \in k, m \in \mathbb{N} \Rightarrow \exists b \in k, a = b^\circ_m$.

(ii) $k$ has no maximum element.

**Proposition 3.5.** Let $k$ be a rational tropical semifield. Then for any $a \in k$ it implies

$$\inf_k \{b \in k \mid a < b\} = a.$$
Proof. The case of $a = -\infty$. Suppose that there is an element $c \in k \setminus \{-\infty\}$ such that $k_\geq(c) = k \setminus \{-\infty\}$. Then the element $0 \odot c$ is the maximum element of $k$, which is a contradiction.

The case of $a \neq -\infty$. The condition $a < b$ is fulfilled if and only if $0 < b \odot a$. So we may assume $a = 0$. Suppose that there is an element $c \not\leq 0$ such that $c$ is a lower-bound of the set $\{b \in k \mid 0 < b\}$. There is an element $c' \in k$ such that $c = (c')^2 = 2c'$. Since $0 < 0 \oplus c'$, we have $c \leq 0 \oplus c'$. So we have
\[
2(0 \oplus c') = 0 \oplus c' \oplus 2c' = 0 \oplus c' \oplus c = 0 \oplus c',
\]
so we have $c \leq 0$, which is a contradiction.

\[\square\]

3.2. Modules over a tropical semifield. Let $k$ be a tropical semifield. Let $M$ be a $k$-module.

Definition. A submodule $N$ of $M$ is a subset with the following conditions.

(i) $-\infty \in N$.

(ii) If $v, w \in N$ then $v \oplus w \in N$.

(iii) If $v \in N$ and $a \in k$ then $a \odot v \in N$.

Example 3.6. Suppose that $k$ is totally ordered. Let $q \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $m$. Let $p: k^n \to k$ be a homomorphism of $k$-modules. Then the subset
\[
M = \{v \in k^n \mid mp(v) \leq q(v)\}
\]
is a submodule of $k^n$. Indeed, for $v, w \in M$ and $a \in k$,
\[
mp(a \odot v) = m(a \odot p(v)) = ma \odot mp(v) \leq ma \odot q(v) = q(a \odot v),
\]
\[
mp(v \oplus w) = m(p(v) \oplus p(w)) = \max\{mp(v), mp(w)\} = mp(v) \oplus mp(w) \leq q(v) \oplus q(w).
\]
By Proposition 3.4, we have $q(v) \oplus q(w) \leq q(v \oplus w)$. 

By Proposition 3.4, we have $q(v) \oplus q(w) \leq q(v \oplus w)$.
Example 3.7. A free module $M = k^n$ of finite rank is reflexive. Indeed, there is a pairing map $\langle \cdot, \cdot \rangle : k^n \times k^n \to k$ defined by
\[
\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1 \odot b_1 \oplus \cdots \oplus a_n \odot b_n.
\]
So we have $(k^n)^\vee \cong k^n$.

Recall that $M$ is said to be pre-reflexive if the homomorphism $\iota_M : M \to (M^\vee)^\vee$ is injective.

Proposition 3.8. $M$ is pre-reflexive if and only if there is an injection $M \to F$ for some direct product $F = \prod_{\lambda \in \Lambda} k$.

Proof. There is an injection $(M^\vee)^\vee \to \prod_{\lambda \in \Lambda} k$, where $\Lambda$ is the set $M^\vee$. Conversely, if there is an injection $M \to F$ for some direct product $F$, then $M$ is pre-reflexive, because $F$ is pre-reflexive. \qed

Lemma 3.9. Suppose that $k$ is rational. Let $M$ be a pre-reflexive $k$-module. Then for any $v \in M$ and any $a \in k$ it implies
\[
\inf_M \{ b \odot v \mid b \in k, a < b \} = a \odot v.
\]

Proof. Let $w \in M$ be a lower-bound of the subset $\{ b \odot v \mid b \in k, a < b \}$. For $\xi \in M^\vee$ and $b \in k$ such that $a < b$, we have
\[
\xi(w) \leq b \odot \xi(v).
\]
By Proposition 3.5, we have
\[
\xi(w) \leq a \odot \xi(v).
\]
Since $M$ is pre-reflexive, we have $w \leq a \odot v$. \qed

Lemma 3.10. Suppose that $k$ is totally ordered. Let $M$ be a pre-reflexive $k$-module. Then for any $v, w \in M$ and any $a \in k$,
\[
v \nleq w, a < 0 \Rightarrow v \nleq w \oplus a \odot v.
\]

Proof. Since $M$ is pre-reflexive, there is an element $\xi \in M^\vee$ such that $\xi(v) \nleq \xi(w)$. Since $k$ is totally ordered, we have $\xi(w) < \xi(v)$. So
\[
\max \{ \xi(w), a \odot \xi(v) \} < \xi(v).
\]
So we have the conclusion. \qed

Example 3.11. Let $G$ be a tropical semigroup with at least two elements. Let $M = (G \times \mathbb{R}) \cup \{-\infty\}$ be the $\mathbb{T}$-module with addition
\[
(v, a) \oplus (w, b) = (v \oplus w, a \oplus b)
\]
and scalar multiplication
\[
c \odot (v, a) = \begin{cases} (v, c \odot a) & \text{if } c \in \mathbb{R} \\ -\infty & \text{if } c = -\infty. \end{cases}
\]
$M$ is a $\mathbb{T}$-module generated by the subset $G \times \{0\}$. $M$ is not pre-reflexive, because it does not satisfy Lemma 3.10. Let $v, w \in G$ be elements such that $v \not\leq w$. Then

$$(v, 0) \not\leq (w, 0),$$

$$(v, 0) \leq (w, 0) \oplus (-1) \odot (v, 0).$$

3.3. Basis and extremal rays. Let $k$ be a totally ordered tropical semifield. Let $M$ be a $k$-module. Recall that an element $e \in M \setminus \{-\infty\}$ is said to be extremal if for any $v_1, v_2 \in M$ such that $v_1 \oplus v_2 = e$ it implies $v_1 = e$ or $v_2 = e$.

**Proposition 3.12.** Let $M$ be a pre-reflexive $k$-module. Then the following are equivalent.

(i) There is a basis of $M$.

(ii) $M$ is extremally generated.

More precisely, a system of generators $E = \{e_\lambda | \lambda \in \Lambda\}$ is a basis if and only if each $e_\lambda$ is extremal and it satisfies $k \odot e_\lambda \neq k \odot e_\mu$ ($\lambda \neq \mu$).

**Proof.** Suppose that there is a basis $E$ of $M$. Let $e_1$ be an element of the basis $E$. Let $v_1, v_2 \in M$ be elements such that $v_1 \oplus v_2 = e_1$. There are elements $e_2, e_3, \ldots, e_r$ of the basis $E$ and elements $a_i, b_i \in k$ such that

$$v_1 = a_1 \odot e_1 \oplus a_2 \odot e_2 \oplus \cdots \oplus a_r \odot e_r,$$

$$v_2 = b_1 \odot e_1 \oplus b_2 \odot e_2 \oplus \cdots \oplus b_r \odot e_r.$$

Since $k$ is totally ordered, we may assume $a_1 \leq b_1$. Then

$$e_1 = b_1 \odot e_1 \oplus w,$$

where

$$w = (a_2 \oplus b_2) \odot e_2 \oplus \cdots \oplus (a_r \oplus b_r) \odot e_r.$$

Since $E$ is a basis, we have $w \neq e_1$. By Lemma 3.10, we have $b_1 = 0$. It means $v_2 \geq e_1$. So we have $v_2 = e_1$. Thus $e_1$ is extremal.

Conversely, let $E$ be a system of generators that consists of extremal elements with different extremal rays. Suppose that $E$ is not a basis. There are elements $e_1, e_2, \ldots, e_r$ of $E$ and elements $a_i, b_i \in k$ such that

$$e_1 = a_2 \odot e_2 \oplus \cdots \oplus a_r \odot e_r.$$

Since $e_1$ is extremal, there is a number $i$ such that $e_1 = a_i \odot e_i$, which is contradiction.

**Proposition 3.13.** Let $\alpha : M \to N$ be a homomorphism of $k$-modules. Let $w \in N$ be an extremal element. Then any minimal element of the subset $\alpha^{-1}(w)$ is extremal.
Proof. Let $e \in M$ be a minimal element of $\alpha^{-1}(w)$. Let $v_1, v_2 \in M$ be elements such that $v_1 \oplus v_2 = e$. Then $\alpha(v_1) \oplus \alpha(v_2) = w$. Since $w$ is extremal, we may assume $\alpha(v_1) = w$. Then $v_1$ is a lower-bound of $e$ in $\alpha^{-1}(w)$. Since $e$ is minimal, we have $v_1 = e$. \qed

3.4. Locators. Let $k$ be a totally ordered tropical semifield. Let $M$ be a $k$-module. For a subset $S \subset M$, the lower-saturation $M_\leq(S)$ is defined by

$$M_\leq(S) = \bigcup_{w \in S} \{v \in M \mid v \leq w\}.$$ 

The set of the lower-bounds $\text{Low}_M(S)$ is defined by

$$\text{Low}_M(S) = \bigcap_{w \in S} \{v \in M \mid v \leq w\}.$$ 

A subset $S \subset M$ is said to be lower-saturated if $M_\leq(S) = S$.

Definition. A locator $S$ of $M$ is a lower-saturated subsemigroup of the semigroup $(M, \oplus)$ that generates the $k$-module $M$.

Let $\text{Loc}(M)$ denote the set of the locators of a $k$-module $M$, equipped with addition

$$S \vee T = S \cap T$$

and scalar multiplication

$$a \circ S = \begin{cases} 
(\otimes a) \circ S & \text{if } a \in k \setminus \{-\infty\} \\
M & \text{if } a = -\infty.
\end{cases}$$

Proposition 3.14. $(\text{Loc}(M), \vee, \circ)$ is a $k$-module with zero element $M$. There is a homomorphism

$$i: M^\vee \longrightarrow \text{Loc}(M)$$

defined by

$$i(\xi) = \{v \in M \mid \langle v, \xi \rangle \leq 0\}.$$ 

Proof. $\text{Loc}(M)$ is a tropical semigroup. Indeed,

$$S \vee S = S \cap S = S.$$ 

$\text{Loc}(M)$ is a $k$-module. Indeed, for $a, b \in k$ such that $a \leq b$, since $S$ is lower-saturated, we have

$$\otimes b \circ S \subset \otimes a \circ S.$$
So we have
\[(a \oplus b) \oplus S = b \circ S \]
\[= (a \circ S) \cap (b \circ S) \]
\[= a \circ S \oplus b \circ S. \]

\(i\) is a homomorphism. Indeed, for \(v \in M\),
\[v \in i(\xi_1 \oplus \xi_2) \iff \langle v, \xi_1 \oplus \xi_2 \rangle \leq 0 \]
\[\iff \langle v, \xi_1 \rangle \oplus \langle v, \xi_2 \rangle \leq 0 \]
\[\iff v \in i(\xi_1) \cap i(\xi_2) \]
\[\iff v \in i(\xi_1) \oplus i(\xi_2). \]

So \(i(\xi_1 \oplus \xi_2) = i(\xi_1) \oplus i(\xi_2)\).

\[v \in i(a \circ \xi) \iff \langle v, a \circ \xi \rangle \leq 0 \]
\[\iff \langle a \circ v, \xi \rangle \leq 0 \]
\[\iff a \circ v \in i(\xi) \]
\[\iff v \in a \circ i(\xi). \]

So \(i(a \circ \xi) = a \circ i(\xi). \)

\[\square \]

**Lemma 3.15.** Suppose that \(k\) is quasi-complete and rational.

1. For any locator \(S \in \text{Loc}(M)\) there is a unique element \(\xi \in M^\vee\) that satisfies the following conditions.
   \[\langle v, \xi \rangle \leq 0 \quad (v \in S), \]
   \[\langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S). \]

2. The mapping \(S \mapsto \xi\) induces a homomorphism
   \[p: \text{Loc}(M) \longrightarrow M^\vee \]
   which satisfies \(p \circ i = \text{id}_{M^\vee}. \)

**Proof.** (1) Let \(\xi: M \rightarrow k\) be the map defined as follows.
\[\xi(v) = \inf_k \{a \in k \mid v \in a \circ S\}. \]

The set in right side is non-empty. (Since \(S\) generates the \(k\)-module \(M\), there are \(s_i \in S\) and \(a_i \in k\) such that
\[v = a_1 \circ s_1 \oplus \cdots \oplus a_r \circ s_r. \]
Let $a$ be the maximum element of $a_1, \ldots, a_r$. Since $S$ is lower-saturated, there are $s'_1 \in S$ such that

$$ v = a \odot (s'_1 \oplus \cdots \oplus s'_r). $$

Since $S$ is a subsemigroup, we have $v \in a \odot S$. For any $v \in M \setminus S$ we have $\xi(v) \leq 0$, because $S$ is lower-saturated. For any $v \in S$, we have $\xi(v) \leq 0$.

We show that $\xi$ is a homomorphism. Since $S$ is lower-saturated, we have

$$ \xi(v) \odot \xi(w) \leq \xi(v \odot w). $$

Suppose that $\xi(v) \odot \xi(w) < \xi(v \odot w)$. There are $a, b \in k$ such that $a \odot b < \xi(v \odot w)$ and $v \in a \odot S$ and $w \in b \odot S$. Then $v \odot w \in (a \odot b) \odot S$. So we have $\xi(v \odot w) \leq a \odot b$, which is contradiction.

We prove uniqueness. Let $\xi \in M^\vee$ be an element that satisfies the following conditions.

$$ \langle v, \xi \rangle \leq 0 \quad (v \in S), $$

$$ \langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S). $$

Then

$$ \langle v, \xi \rangle \leq \inf \{ a \in k \mid v \in a \odot S \} $$

$$ \leq \inf \{ a \in k \mid \langle v, \xi \rangle < a \}. $$

By Proposition 3.5,

$$ \inf \{ a \in k \mid \langle v, \xi \rangle < a \} = \langle v, \xi \rangle. $$

So we have

$$ \langle v, \xi \rangle = \inf \{ a \in k \mid v \in a \odot S \}. $$

(2) We have

$$ \langle v, p(S) \oplus p(T) \rangle \leq 0 \quad (v \in S \cap T), $$

$$ \langle v, p(S) \oplus p(T) \rangle \geq 0 \quad (v \in M \setminus (S \cap T)). $$

It means $p(S) \oplus p(T) = p(S \oplus T)$. So $p$ is a homomorphism. For $\xi \in M^\vee$, let

$$ S = \{ v \in M \mid \langle v, \xi \rangle \leq 0 \}. $$

Then

$$ \langle v, \xi \rangle \leq 0 \quad (v \in S), $$

$$ \langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S). $$

It means $\xi = p(S)$. □
3.5. **Straight modules.** Let $k$ be a totally ordered tropical semifield. Recall that a $k$-module $M$ is said to be straight if it satisfies the following conditions.

(i) Any two elements $v, w \in M$ admit the infimum $\inf_M \{v, w\}$.
(ii) $v_1, v_2, w \in M \Rightarrow \inf_M \{v_1 \oplus v_2, w\} = \inf_M \{v_1, w\} \oplus \inf_M \{v_2, w\}$.
(iii) $v_1, v_2, w \in M \Rightarrow \inf_M \{v_1, v_2\} \oplus w = \inf_M \{v_1 \oplus w, v_2 \oplus w\}$.

**Proposition 3.16.** The above conditions (ii), (iii) are equivalent.

**Proof.** (ii) $\Rightarrow$ (iii).

$$
\inf_M \{v_1 \oplus w, v_2 \oplus w\} = \inf_M \{v_1, v_2\} \oplus \inf_M \{v_1, w\} \oplus \inf_M \{w, v_2\} \oplus w
$$

(iii) $\Rightarrow$ (ii) is similar. $\square$

**Definition.** A homomorphism $\alpha : M \to N$ of $k$-modules is lattice-preserving if for any $v, w \in M$ and any lower-bound $x \in \text{Low}_N(\alpha(v), \alpha(w))$ there is a lower-bound $y \in \text{Low}_M(v, w)$ such that $x \leq \alpha(y)$. If $M, N$ are ordered lattices, $\alpha$ is lattice-preserving if and only if it preserves the infimum of any two elements.

**Proposition 3.17.** Let $\alpha : M \to N$ be a lattice-preserving injective homomorphism of $k$-modules such that $N$ is straight. Then $M$ is straight.

**Proof.** For $v, w \in M$, let $x = \inf_M \{\alpha(v), \alpha(w)\}$. There is a lower-bound $y$ of $\{v, w\}$ such that $x \leq \alpha(y)$. Then $y = \inf_M \{v, w\}$. (Let $y' \in M$ be a lower-bound of $\{v, w\}$. Then $\alpha(y') \leq x \leq \alpha(y)$. Since $\alpha$ is injective, we have $y' \leq y$.) $\alpha(y)$ is a lower-bound of $\{\alpha(v), \alpha(w)\}$. So we have $x = \alpha(y)$. $M$ is finitely distributive, because $\alpha$ preserves the infimum of any two elements. $\square$

**Proposition 3.18.** Suppose that $k$ is quasi-complete and rational. Let $M$ be a straight $k$-module. Then $M^\vee$ and $\text{Loc}(M)$ are straight.

**Proof.** We show that $\text{Loc}(M)$ is straight. For $S, T \in \text{Loc}(M)$, let

$$
U = S \oplus T = \{s \oplus t \mid s \in S, t \in T\}.
$$

$U$ is lower-saturated. (Let $v \in M$ and $s \in S$ and $t \in T$ be elements such that $v \leq s \oplus t$. Then

$$
v = \inf_M \{v, s \oplus t\}
$$

$$
= \inf_M \{v, s\} \oplus \inf_M \{v, t\}.
$$
So we have \( v \in U \).) \( U \) is a locator of \( M \), and we have
\[
U = \inf_{\text{Loc}(M)} \{ S, T \}.
\]

\( \text{Loc}(M) \) is finitely distributive. Indeed,
\[
(S_1 \cap S_2) \oplus T = (S_1 \oplus T) \cap (S_2 \oplus T).
\]
(\( \text{Let } v \) be an element of right side. There are \( s_1 \in S_1 \) and \( s_2 \in S_2 \) and \( t_1, t_2 \in T \) such that
\[
v = s_1 \oplus t_1 = s_2 \oplus t_2.
\]
Then
\[
v = \inf_M \{ s_1 \oplus t_1, s_2 \oplus t_2 \}
= \inf_M \{ s_1, s_2 \} \oplus \inf_M \{ s_1, t_2 \} \oplus \inf_M \{ t_1, s_2 \} \oplus \inf_M \{ t_1, t_2 \}.
\]
So we have \( v \in (S_1 \cap S_2) \oplus T \).

We show that \( M^\vee \) is straight. For \( \xi_1, \xi_2 \in M^\vee \), let \( S_1, S_2 \in \text{Loc}(M) \) be the induced element. There is a unique element \( \eta \in M^\vee \) that satisfies the following conditions (Lemma 3.13).
\[
\langle v, \eta \rangle \leq 0 \ (v \in S_1 \oplus S_2),
\langle v, \eta \rangle \geq 0 \ (v \in M \setminus (S_1 \oplus S_2)).
\]

We have \( \eta = \inf_{M^\vee} \{ \xi_1, \xi_2 \} \). So the canonical injection \( i: M^\vee \to \text{Loc}(M) \) is lattice-preserving. Since \( \text{Loc}(M) \) is straight, \( M^\vee \) is straight (Proposition 3.17).

\[\text{□}\]

\textbf{Proposition 3.19.} Let \( M \) be a \( k \)-module. Let \( \eta: M \to k \) be a lattice-preserving homomorphism. Then \( \eta \) is an extremal element of \( M^\vee \).

\textbf{Proof.} Suppose that \( \eta \) is not extremal. There are elements \( \xi_1, \xi_2 \in M^\vee \) and elements \( v_1, v_2 \in M \) such that \( \xi_1 \oplus \xi_2 = \eta \) and \( \langle v_1, \xi_1 \rangle < \langle v_1, \eta \rangle \) and \( \langle v_2, \xi_2 \rangle < \langle v_2, \eta \rangle \). We may assume \( \langle v_1, \eta \rangle = \langle v_2, \eta \rangle = 0 \). Since \( \eta \) is lattice-preserving, there is a lower-bound \( w \) of \( \{ v_1, v_2 \} \) such that \( \langle w, \eta \rangle = 0 \). Then
\[
0 = \langle w, \eta \rangle
= \langle w, \xi_1 \oplus \xi_2 \rangle
\leq \langle v_1, \xi_1 \rangle \oplus \langle v_2, \xi_2 \rangle
< \langle v_1, \eta \rangle \oplus \langle v_2, \eta \rangle
= 0,
\]
which is contradiction. \[\text{□}\]

\textbf{Definition.} A \emph{dual element} \( \eta \in M^\vee \) of an element \( e \in M \) is an element with the following conditions.
\[ (i) \langle e, \eta \rangle = 0. \]
\[ (ii) v \in M, \xi \in M^\vee \Rightarrow \langle v, \eta \rangle \odot \langle e, \xi \rangle \leq \langle v, \xi \rangle. \]

**Proposition 3.20.** The dual element of an element \( e \in M \) is unique.

*Proof.* Let \( \eta \) be a dual element of \( e \). Then
\[
\eta = \min \{ \xi \in M^\vee \mid \langle e, \xi \rangle = 0 \},
\]
because
\[
\eta \odot \langle e, \xi \rangle \leq \xi
\]
for any \( \xi \in M^\vee \).

**Proposition 3.21.** Let \( e \in M \) be an element of a pre-reflexive \( k \)-module \( M \). Suppose that \( e \) has the dual element \( \eta \in M^\vee \). Then

1. \( e \) is an extremal element.
2. \( \eta : M \to k \) is a lattice-preserving homomorphism (therefore is an extremal element of \( M^\vee \)).

*Proof.* (1) Let \( v_1, v_2 \in M \) be elements such that \( v_1 \oplus v_2 = e \). Then
\[
\langle v_1, \eta \rangle \oplus \langle v_2, \eta \rangle = \langle e, \eta \rangle = 0.
\]
We may assume \( \langle v_1, \eta \rangle = 0 \). For \( \xi \in M^\vee \),
\[
\langle e, \xi \rangle = \langle v_1, \eta \rangle \odot \langle e, \xi \rangle \\
\leq \langle v_1, \xi \rangle.
\]
Since \( M \) is pre-reflexive, we have \( e \leq v_1 \). So we have \( e = v_1 \).

(2) Let \( v_1, v_2 \in M \) be elements such that \( \langle v_1, \eta \rangle \leq \langle v_2, \eta \rangle \). Let \( w = \langle v_1, \eta \rangle \odot e \). For \( \xi \in M^\vee \), we have \( \langle w, \xi \rangle \leq \langle v_i, \xi \rangle \). Since \( M \) is pre-reflexive, we have \( w \leq v_i \). So \( w \) is a lower-bound of \( \{ v_1, v_2 \} \) such that \( \langle w, \eta \rangle = \langle v_1, \eta \rangle \). Thus \( \eta \) is lattice-preserving. By Proposition 3.19, \( \eta \) is an extremal element of \( M^\vee \).

**Lemma 3.22.** Suppose that \( k \) is quasi-complete and rational. Let \( M \) be a straight pre-reflexive \( k \)-module. Then any extremal element of \( M \) has the dual element.

*Proof.* Let \( e \in M \) be an extremal element. The subset
\[
S = \{ v \in M \mid e \nless v \}
\]
is a subsemigroup. (Let \( v_1, v_2 \in M \) be elements such that \( e \leq v_1 \oplus v_2 \). Then
\[
e = \inf_M \{ e, v_1 \oplus v_2 \} \\
= \inf_M \{ e, v_1 \} \oplus \inf_M \{ e, v_2 \}.
\]
We may assume $e = \inf_M \{e, v_1\}$. Then $e \leq v_1$. Also $S$ generates the $k$ module $M$. (Let $v \in M$ be any element. By Lemma 3.9, we have
\[
\inf_M \{b \odot v \mid b \in k \setminus \{-\infty\}\} = -\infty.
\]
So there is $b \in k \setminus \{-\infty\}$ such that $e \not\leq b \odot v$. Thus $S$ is a locator of $M$. By Lemma 3.15 there is a unique element $\eta \in M^\vee$ that satisfies the following conditions.
\[
\langle v, \eta \rangle \leq 0 \quad (v \in S), \\
\langle v, \eta \rangle \geq 0 \quad (v \in M \setminus S).
\]
Then
\[
\langle v, \eta \rangle \leq \inf_k \{a \in k \mid a \odot e \not\leq v\} \\
\leq \inf_k \{a \in k \mid \langle v, \eta \rangle < a\} \\
= \langle v, \eta \rangle.
\]
So we have
\[
\langle v, \eta \rangle = \inf_k \{a \in k \mid a \odot e \not\leq v\}.
\]
So
\[
\langle e, \eta \rangle = \inf_k \{a \in k \mid 0 < a\} \\
= 0.
\]
Also, for any $a \in k$ such that $0 < a$, we have
\[
(\langle v, \eta \rangle \odot a) \odot e \leq v.
\]
By Lemma 3.9 we have
\[
\langle v, \eta \rangle \odot e \leq v.
\]
Thus $\eta$ is the dual element of $e$. □

**Lemma 3.23.** Let $M$ be a finitely generated pre-reflexive $k$-module. Let $\beta: k^n \to M$ be the surjection defined by a basis $\{e_1, \ldots, e_n\}$ of $M$. Suppose that $e_i$ has the dual element $\eta_i$ ($1 \leq i \leq n$). Then

1. $M$ is straight.
2. The homomorphism $\alpha: M \to k^n$ defined by the elements $\eta_1, \ldots, \eta_n$ is a right-inversion of $\beta$, i.e. $\beta \circ \alpha = \text{id}_M$.
3. $\alpha$ is the unique right-inversion of $\beta$.

**Proof.** For $v \in M$, we have
\[
v \geq \langle v, \eta_1 \rangle \odot e_1 \oplus \cdots \oplus \langle v, \eta_n \rangle \odot e_n.
\]
It means $\beta \circ \alpha \leq \text{id}_M$. Also, for $1 \leq i \leq n$, we have
\[
\beta \circ \alpha (e_i) \geq \langle e_i, \eta_i \rangle \circ e_i = e_i.
\]
Since $M$ is generated by $\{e_1, \ldots, e_n\}$, we have $\beta \circ \alpha = \text{id}_M$. So $\alpha$ is injective. Also $\alpha$ is lattice-preserving (Proposition 3.21). Since $k^n$ is straight, $M$ is straight (Proposition 3.17).

We prove uniqueness. Let $\eta'_1, \ldots, \eta'_n \in M^\vee$ be elements such that the induced homomorphism $M \to k^n$ is a right-inversion of $\beta$. Then we have
\[
v = \langle v, \eta'_1 \rangle \circ e_1 \oplus \cdots \oplus \langle v, \eta'_n \rangle \circ e_n.
\]
So
\[
e_i = \langle e_i, \eta'_i \rangle \circ e_i \oplus w_i,
\]
where
\[
w_i = \bigoplus_{j \neq i} \langle e_i, \eta'_j \rangle \circ e_j.
\]
Since $\{e_1, \ldots, e_n\}$ is a basis, we have $w_i \neq e_i$ and
\[
e_i = \langle e_i, \eta'_i \rangle \circ e_i
\]
(Proposition 3.12). So we have $\langle e_i, \eta'_i \rangle = 0$. Thus $\eta'_i$ is the dual element of $e_i$. $\square$

3.6. Existence of inversions. Let $k$ be a totally ordered tropical semifield. Let $\alpha : M \to N$ be a homomorphism of $k$-modules.

Definition. An element $\xi \in M^\vee$ dominates an element $w \in N$ if there is an element $v \in M$ such that $\langle v, \xi \rangle \leq 0$ and $w \leq \alpha(v)$.

Proposition 3.24. Let $\xi_i \in M^\vee$ be an element that dominates $w_i \in N$ ($i = 1, 2$). Then any lower-bound $\xi \in \text{Low}_{M^\vee}(\xi_1, \xi_2)$ dominates $w_1 \oplus w_2$.

Proof. There are elements $v_1, v_2 \in M$ such that $\langle v_i, \xi_i \rangle \leq 0$ and $w_i \leq \alpha(v_i)$. Then
\[
\langle v_1 \oplus v_2, \xi \rangle \leq \langle v_1, \xi_1 \rangle \oplus \langle v_2, \xi_2 \rangle \leq 0.
\]
Also we have $w_1 \oplus w_2 \leq \alpha(v_1 \oplus v_2)$.

Recall that a homomorphism $\alpha : M \to N$ is said to be lightly surjective if for any $w \in N$ there is $v \in M$ such that $w \leq \alpha(v)$.

Lemma 3.25. Let $\alpha : M \to N$ be an injective lightly surjective homomorphism of $k$-modules. Suppose that $M^\vee$ is straight.
(1) There is a homomorphism \( \gamma: N \to \text{Loc}(M') \) that satisfies the following condition. For any \( w \in N \) the locator \( \gamma(w) \) is the subsemigroup of \( M' \) generated by the elements that dominates the element \( w \).

(2) The diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow & & \downarrow \gamma \\
(M')^\lor & \xrightarrow{i} & \text{Loc}(M')
\end{array}
\]

commutes, i.e. for any \( v \in M \) and any \( \xi \in M' \) the condition \( \langle v, \xi \rangle \leq 0 \) is fulfilled if and only if \( \xi \in \gamma(\alpha(v)) \).

**Proof.** (1) For \( w \in N \), let \( \gamma(w) \subset M' \) be the subsemigroup of \( M' \) generated by the elements that dominates the element \( w \). \( \gamma(w) \) is lower-saturated. (Let \( \xi \in M \) and \( \xi' \in \gamma(w) \) be elements such that \( \xi \leq \xi' \). There are elements \( \xi_1, \ldots, \xi_r \in M' \) such that \( \xi_i \) dominates \( w \) and

\[
\xi' = \xi_1 \oplus \cdots \oplus \xi_r.
\]

Then

\[
\xi = \inf_{M'} \{ \xi, \xi_1 \oplus \cdots \oplus \xi_r \} \\
= \inf_{M'} \{ \xi, \xi_1 \} \oplus \cdots \oplus \inf_{M'} \{ \xi, \xi_r \}.
\]

So \( \xi \in \gamma(w) \). Also \( \gamma(w) \) generates the \( k \)-module \( M' \). (Let \( \xi \in M' \) be any element. Since \( \alpha \) is lightly surjective, there is \( v \in M \) such that \( w \leq \alpha(v) \). Let \( a \in k \setminus \{-\infty\} \) be an element such that \( \langle v, \xi \rangle \leq a \). Then \( \otimes a \circ \xi \) dominates \( w \).) So \( \gamma(w) \) is a locator of \( M' \).

We show that \( \gamma \) is a homomorphism. For \( w_1, w_2 \in N \), we have

\[
\gamma(w_1 \oplus w_2) \subset \gamma(w_1) \cap \gamma(w_2).
\]

Let \( \xi \) be an element of right side. There are elements \( \xi_{i,j} \in M' \) (1 \( \leq i \leq 2 \), 1 \( \leq j \leq r \)) such that \( \xi_{i,j} \) dominates \( w_i \) and

\[
\xi = \xi_{1,1} \oplus \cdots \oplus \xi_{1,r} = \xi_{2,1} \oplus \cdots \oplus \xi_{2,r}.
\]

Then

\[
\xi = \inf_{M'} \{ \xi_{1,1} \oplus \cdots \oplus \xi_{1,r}, \xi_{2,1} \oplus \cdots \oplus \xi_{2,r} \}
\]

\[
= \bigoplus_{i,j} \eta_{i,j},
\]

where

\[
\eta_{i,j} = \inf_{M'} \{ \xi_{i,j} \}.\]
η\textsubscript{i,j} dominates \(w_1 \oplus w_2\) (Proposition 3.24). So we have \(\xi \in \gamma(w_1 \oplus w_2)\).

(2) Let \(\xi \in M^\vee\) be an element that dominates \(\alpha(v)\). There is an element \(v' \in M\) such that \(\langle v', \xi \rangle \leq 0\) and \(\alpha(v) \leq \alpha(v')\). Since \(\alpha\) is injective, we have \(v \leq v'\). So we have

\[
\langle v, \xi \rangle \leq \langle v', \xi \rangle \leq 0.
\]

Let

\[
T = \{\xi \in M^\vee \mid \langle v, \xi \rangle \leq 0\}.
\]

Now we have \(\xi \in T\). Since \(T\) is a subsemigroup, we have \(\gamma(\alpha(v)) = T\). □

3.7. **Straight reflexive modules.** Let \(k\) be a quasi-complete totally ordered rational tropical semifield. Recall that the dimension of a straight reflexive \(k\)-module \(M\) is the number of extremal rays. By Proposition 3.12, the number of elements of any basis of \(M\) is \(\dim(M)\).

**Proof of Theorem 2.1.** We have an isomorphism \(i_M : M \to (M^\vee)^\vee\) and a homomorphism \(\gamma : N \to \text{Loc}(M^\vee)\) defined in Lemma 3.25. There is a left-inversion \(p\) of the homomorphism \(i : (M^\vee)^\vee \to \text{Loc}(M^\vee)\) (Lemma 3.15). By the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{i_M} & & \downarrow{\gamma} \\
(M^\vee)^\vee & \xrightarrow{i} & \text{Loc}(M^\vee)
\end{array}
\]

we have \(i_M^{-1} \circ p \circ \gamma \circ \alpha = \text{id}_M\). □

**Proof of Theorem 2.2.** By Lemma 3.22 and Lemma 3.23, there is an injection \(N \to k^n\), where \(n = \dim(N)\). Let \(N'\) be the lower-saturation of the image of \(M \to k^n\). \(N'\) is a free module of rank \(n' \leq n\). If \(n' = n\), then \(\alpha\) is lightly surjective. Now we may assume that \(N = k^n\) and that \(\alpha\) is lightly surjective. By Theorem 2.1 \(\alpha\) has a left-inversion \(\beta : N \to M\). Since \(\beta\) is surjective, we have \(\dim(M) \leq \dim(N)\). □

**Proof of Theorem 2.3.** By Lemma 3.22 and Lemma 3.23, there is a right-inversion \(\alpha : M \to k^n\) of the surjection \(\beta : k^n \to M\). By the commutative diagram

\[
\begin{array}{ccc}
k^n & \xrightarrow{\beta} & M \\
\downarrow{\gamma} & & \downarrow{i_M} \\
k^n(\beta^\vee)^\vee & \xrightarrow{(\beta^\vee)^\vee} & \text{Loc}(M^\vee)^\vee
\end{array}
\]
we have $\iota_M^{-1} = \beta \circ (\alpha^\vee)^\vee$.

Proof of Theorem 2.4. By Theorem 2.3 $N$ is reflexive. Similarly to the proof of Theorem 2.2 we may assume that $N = k^n$ and that $\alpha$ is lightly surjective. We have a homomorphism $\gamma: N \to \text{Loc}(M^\vee)$ defined in Lemma 3.25. There is a left-inversion $p$ of the homomorphism $i: (M^\vee)^\vee \to \text{Loc}(M^\vee)$ (Lemma 3.15). There is a homomorphism $\delta: M^\vee \to N^\vee$ such that for any $w \in N$ and any $\xi \in M^\vee$ it implies

$$\langle w, \delta(\xi) \rangle = \langle p(\gamma(w)), \xi \rangle.$$ 

By the commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\iota_M \downarrow & & \downarrow \gamma \\
(M^\vee)^\vee & \xrightarrow{i} & \text{Loc}(M^\vee)
\end{array}
$$

for any $v \in M$ we have

$$\langle \alpha(v), \delta(\xi) \rangle = \langle v, \xi \rangle.$$ 

So $\alpha^\vee \circ \delta = \text{id}_{M^\vee}$. So we have

$$\dim(M^\vee) \leq \dim(N^\vee) = n.$$ 

By Lemma 3.22 and Proposition 3.21 there is an injection from the set of the extremal rays of $M$ to the set of the extremal rays of $M^\vee$. So we have $\dim(M) \leq n$. □

Example 3.26. There is an example of straight submodule $M \subset T^2$ that is not finitely generated. Let

$$M = \{(a, b) \in T^2 \mid b \neq -\infty\} \cup \{-\infty\}.$$ 

$M$ is a submodule of $T^2$. $M$ is straight, because it is lattice-preserving.

Example 3.27. There is an example of extremally generated submodule $M \subset T^3$ that is not finitely generated. Let

$$M = \left\{ (a, b, c) \in T^3 \mid (-1) \odot a \oplus c \leq b, \quad 2b \leq a \odot c \right\}.$$ 

$M$ is a submodule of $T^3$ (Example 3.6). For $0 \leq t \leq 1$, let

$$e(t) = (2t, t, 0) \in M.$$ 

e(t) is extremal. (Proposition 3.13) Indeed $e(t)$ is a minimal element of the subset

$$S_t = \{(a, b, c) \in M \mid b = t\}.$$
So it is extremal.) So $M$ is not finitely generated. \{e(t) \mid 0 \leq t \leq 1\} is a basis of $M$. Indeed, for any $(a, b, c) \in M$,

$$(a, b, c) = c \odot e(b \odot c) \oplus (2b \odot a) \odot e(a \odot b).$$

$M$ is not straight. Indeed, let

$$v_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$
$$v_2 = (1, \frac{1}{2}, 0),$$
$$w = (1, 0, 0).$$

Then we have

$$\inf_M \{v_1, v_2\} = \left(\frac{1}{2}, \frac{1}{4}, 0\right),$$
$$\inf_M \{v_1, v_2\} \oplus w = (1, \frac{1}{4}, 0),$$
$$\inf_M \{v_1 \oplus w, v_2 \oplus w\} = (1, \frac{1}{2}, 0).$$

So $M$ is not straight.

### 3.8 Free modules.

Let $k$ be a totally ordered tropical semifield. Let $F = k^n$ be the free module with the basis \{e_1, \ldots, e_n\}. Let $F^*$ be the set of the linear combinations of \{e_1, \ldots, e_n\} with coefficients in $k^* = k \setminus \{-\infty\}$. Let \{e_1^\vee, \ldots, e_n^\vee\} be the dual basis in $F^\vee$. We have a bijective map

$$\psi: F^* \longrightarrow (F^\vee)^*$$

defined by

$$\psi(a_1 \odot e_1 \oplus \cdots \oplus a_n \odot e_n) = (\odot a_1) \odot e_1^\vee \oplus \cdots \oplus (\odot a_n) \odot e_n^\vee.$$

For $v, w \in F^*$, the condition $v \leq w$ is fulfilled if and only if

$$\langle v, \psi(w) \rangle \leq 0.$$

For $w \in F^*$ and $1 \leq i \leq n$, let

$$M(w, i) = \{v \in F \mid \langle v, e_j^\vee \rangle \odot \langle e_j, \psi(w) \rangle \geq \langle v, e_i^\vee \rangle \odot \langle e_i, \psi(w) \rangle\}.$$

$M(w, i)$ is a submodule of $F$ (Example 3.6). It is easy to see that $M(w, i)$ is lattice-preserving in $F$, i.e. the inclusion $M(w, i) \rightarrow F$ preserves the infimum of any two elements. For $\eta \in F^\vee$ and $1 \leq i \leq n$, let

$$N(\eta, i) = \{v \in F \mid \langle v, \eta \rangle = \langle v, e_i^\vee \rangle \odot \langle e_i, \eta \rangle\}$$
$$= \{v \in F \mid \langle v, e_j^\vee \rangle \odot \langle e_j, \eta \rangle \leq \langle v, e_i^\vee \rangle \odot \langle e_i, \eta \rangle\}.$$

$N(\eta, i)$ is also a lattice-preserving submodule of $F$. 
Proposition 3.28. Let $M$ be a submodule of $F$ with a basis $\{w_1, \ldots, w_r\}$. Suppose that $w_h \in F^*$ $(1 \leq h \leq r)$. Then the following are equivalent.

(i) $M$ is lattice-preserving in $F$.
(ii) For any $i \in \{1, \ldots, n\}$, there is the minimum element of $M \cap V_i$, where

$$V_i = \{v \in F \mid \langle v, e_i^\vee \rangle = 0\}.$$

(iii) There is a surjective map

$$s: \{1, \ldots, n\} \longrightarrow \{1, \ldots, r\}$$

such that

$$M = \bigcap_{1 \leq i \leq n} M(w_{s(i)}, i).$$

(iv) There is a surjective map

$$s: \{1, \ldots, n\} \longrightarrow \{1, \ldots, r\}$$

such that

$$M = \bigcap_{1 \leq i \leq n} N(\eta_{s(i)}, i),$$

where $\eta_h$ is the dual element of $w_h$.

Proof. (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) are easy.

Since $F^\vee$ is also a free module, for $\eta \in (F^\vee)^*$ and $1 \leq i \leq n$ we have the lattice-preserving submodule $M(\eta, i)$ of $F^\vee$. The bijective map

$$\psi: F^* \longrightarrow (F^\vee)^*$$

induces bijective maps

$$\psi': M \setminus \{-\infty\} \longrightarrow M^\vee \setminus \{-\infty\},$$

$$\psi'': N(\eta, i) \setminus \{-\infty\} \longrightarrow M(\eta, i) \setminus \{-\infty\}.$$

So we have only to prove that conditions (i), (ii), (iii) are equivalent.

(i) $\Rightarrow$ (ii). Let

$$v_i = \inf_F \{a_{h,i} \odot w_h \mid 1 \leq h \leq r\},$$

where

$$a_{h,i} = \langle w_h, e_i^\vee \rangle.$$

Then $v_i$ is the minimum element of $M \cap V_i$.

(ii) $\Rightarrow$ (iii). Let $v_i$ be the minimum element of $M \cap V_i$. $v_i$ is an extremal element of $M$. The extremal ray $k \odot v_i$ is generated by an element of the basis $\{w_1, \ldots, w_r\}$ (Proposition 3.12). There is a number $s(i)$ such that $k \odot v_i = k \odot w_{s(i)}$.

We show that $s$ is surjective. For $h \in \{1, \ldots, r\}$, we have

$$w_h = a_{h,1} \odot v_1 \oplus \cdots \oplus a_{h,n} \odot v_n.$$
Since \( w_h \) is extremal (Proposition 3.12), there is a number \( i \) such that
\[
w_h = a_{h,i} \odot v_i.
\]
So we have \( h = s(i) \).

We show the equality
\[
M = \bigcap_{1 \leq i \leq n} M(w_{s(i)}, i).
\]

For \( v \in F \), let \( x_i = \langle v, e_i^\vee \rangle \). The condition \( v \in M(w_{s(i)}, i) \) is fulfilled if and only if for any \( j \) it implies
\[
\odot a_{s(i),j} \odot x_j \geq \odot a_{s(i),i} \odot x_i.
\]

For \( 1 \leq h \leq r \) and \( 1 \leq i \leq n \), we have
\[
\odot a_{s(i),i} \odot w_{s(i)} = v_i
\leq \odot a_{h,i} \odot w_h.
\]

For \( 1 \leq j \leq n \), we have
\[
\odot a_{s(i),i} \odot a_{s(i),j} \leq \odot a_{h,i} \odot a_{h,j}.
\]

It means \( w_h \in M(w_{s(i)}, i) \). Since \( M \) is generated by \( \{w_1, \ldots, w_r\} \), we have
\[
M \subset \bigcap_{1 \leq i \leq n} M(w_{s(i)}, i).
\]

Let \( v \) be an element of right side. Then
\[
v = \bigoplus_i \odot a_{s(i),i} \odot x_i \odot w_{s(i)}.
\]
(Indeed,
\[
\langle v, e_i^\vee \rangle = x_i
= \langle \odot a_{s(i),i} \odot x_i \odot w_{s(i)}, e_i^\vee \rangle.
\]
So
\[
v \leq \bigoplus_i \odot a_{s(i),i} \odot x_i \odot w_{s(i)}.
\]

The converse is easy.) So we have \( v \in M \). \( \square \)
4. Polytopes in a tropical projective space

Let $F = \mathbb{T}^{n+1}$ be the free module with coordinates $(x_1, \ldots, x_{n+1})$ over $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. Let $F^* = \mathbb{R}^{n+1}$.

**Proposition 4.1.** Let $M$ be a submodule of $F$ generated by finitely many elements of $F^*$. Then the following are equivalent.

(i) $M$ is lattice-preserving in $F$.
(ii) $M \setminus \{ -\infty \}$ is a real convex subset of $\mathbb{R}^{n+1}$.

**Proof.** (i) $\Rightarrow$ (ii). By Proposition 3.28, $M$ is defined by inequalities $x_j \geq x_i - c_{i,j}$ $(i, j \in \{1, \ldots, n+1\})$ for some $c_{i,j} \in \mathbb{R}$. So $M \setminus \{ -\infty \}$ is real convex.

(ii) $\Rightarrow$ (i). Let $\pi_1: F \to \mathbb{T}^n$ and $\pi_2: F \to \mathbb{T}$ be projections defined as follows.

\[
\pi_1(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n), \\
\pi_2(x_1, \ldots, x_{n+1}) = x_{n+1}.
\]

For $a \in \mathbb{R}$, let $N_i(a) \subset F$ be the submodule defined as follows.

\[
N_i(a) = \{ v = (x_1, \ldots, x_{n+1}) \in F \mid x_{n+1} = x_i + a \}.
\]

By induction on $n$, we may assume that modules $\pi_1(M)$, $\pi_2(M)$, $M \cap N_i(a)$ are lattice-preserving. Suppose that $M$ is not lattice-preserving. By Proposition 3.28, there is a number $i$ such that there is no minimum element of $M \cap V_i$, where

\[
V_i = \{ v = (x_1, \ldots, x_{n+1}) \in F \mid x_i = 0 \}.
\]

We may assume $i \leq n$. Let $w_1, w_2$ be minimal elements of $M \cap V_i$ such that $\pi_1(w_1)$ is the minimum element of $\pi_1(M \cap V_i)$ and that $\pi_2(w_2)$ is the minimum element of $\pi_2(M \cap V_i)$. Let $a \in \mathbb{R}$ be an element such that

\[
\pi_2(w_2) < a < \pi_2(w_1).
\]

There is the minimum element $v(a)$ of $M \cap N_i(a) \cap V_i$. Since $M \cap V_i$ is real convex, $v(a)$ is a minimal element of $M \cap V_i$. (Let $v' \in M \cap V_i$ be an element such that $v' < v(a)$. The real line segment combining $v'$ and $w_1$ contains an element $v'' \in M \cap N_i(a) \cap V_i$ such that $v'' \neq v'$. Since $\pi_1(w_1) < \pi_1(v') \leq \pi_1(v(a))$, we have $v'' < v(a)$. ) So $M$ has infinitely many extremal rays, which is contradiction. $\square$

Let

\[
\varphi: \mathbb{T}^{n+1} \setminus \{ -\infty \} \longrightarrow \mathbb{T} \mathbb{P}^n
\]
be the canonical projection to the tropical projective space $\mathbb{T}\mathbb{P}^n$. We identify $\varphi(\mathbb{R}^{n+1})$ with $\mathbb{R}^n$. A subset $P \subset \mathbb{T}\mathbb{P}^n$ is said to be tropically convex if the subset

$$M = \varphi^{-1}(P) \cup \{-\infty\} \subset \mathbb{T}^{n+1}$$

is a submodule. A subset $P \subset \mathbb{T}\mathbb{P}^n$ is said to be a tropical polytope if it is the tropically convex hull of finitely many points of $\mathbb{R}^n$.

**Proof of Theorem 2.5.** (1) Suppose that $P$ is a polytrope. Then $P$ is real convex. By Proposition 4.1, $M$ is lattice-preserving in $\mathbb{T}^{n+1}$. So $M$ is straight. By Theorem 2.3, $M$ is reflexive.

(2) Suppose that $M$ is straight reflexive. Let $\{v_1, \ldots, v_r\}$ be a basis of $M$. By Theorem 2.2 we have $r \leq n+1$. Let $p_i = \varphi(v_i)$. Then $P$ is the tropically convex hull of $\{p_1, \ldots, p_r\}$. \qed

5. **Square matrices over a tropical semifield**

Let $k$ be a totally ordered rational tropical semifield. A square matrix of order $n$ over $k$ is a homomorphism $A: k^n \to k^n$. Let $\{e_1, \ldots, e_n\}$ be the basis of $k^n$. The coefficient $\langle A \otimes e_j, e_i^\vee \rangle$ is simply written as $A_{ij}$. Let $E_n: k^n \to k^n$ be the identity.

Let $\Delta(A)$, $\overline{\Delta}(A)$ be square matrices of order $n$ defined as follows.

$$\Delta(A)_{ij} = \delta_{ij} \otimes A_{ij},$$

$$\overline{\Delta}(A)_{ij} = \overline{\delta}_{ij} \otimes A_{ij},$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases}$$

$$\overline{\delta}_{ij} = \begin{cases} -\infty & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The determinant $\det(A)$ is the sum of elements $A_{1s(1)} \otimes \cdots \otimes A_{ns(n)}$ for all permutations $s \in S(n)$.

**Lemma 5.1.** Let $A$ be a square matrix of order $n$ over $k$. Suppose that $\Delta(A) = E_n$ and $\det(A) = 0$. Then $A^{\otimes n} = A^{\otimes n-1}$.

**Proof.** Since $E_n \leq A$, we have $A^{\otimes r} \leq A^{\otimes r+1}$ for any $r \geq 0$. $(A^{\otimes n})_{ij}$ is the sum of elements

$$b = A_{h(0)h(1)} \otimes A_{h(1)h(2)} \otimes \cdots \otimes A_{h(n-1)h(n)}$$
for all maps \( h: \{0, \ldots, n\} \rightarrow \{1, \ldots, n\} \) such that \( h(0) = i \) and \( h(n) = j \). \( h \) is not injective. So there are numbers \( l, m \) and a cyclic permutation \( s \in S(n) \) such that
\[
s: h(l) \mapsto h(l + 1) \mapsto \cdots \mapsto h(m - 1) \mapsto h(m) = h(l).
\]
Since \( \Delta(A) = E_n \), we have
\[
A_{h(l)h(l+1)} \circ \cdots \circ A_{h(m-1)h(m)} \leq \det(A).
\]
So we have
\[
A \circ n \leq \det(A) \circ A^{\circ n-1}.
\]
Since \( \det(A) = 0 \), we have the conclusion. \( \Box \)

**Lemma 5.2.** Let \( A \) be a square matrix of order \( n \) over \( k \). Then either (i) or (ii) is fulfilled.

(i) There are an element \( v \in (k \setminus \{-\infty\})^n \) and an element \( \varepsilon > 0 \) such that
\[
(A \oplus \varepsilon \oplus \overline{\Delta}(A)) \oplus v = \Delta(A) \oplus v.
\]

(ii) There is an element \( v \in k^n \setminus \{-\infty\} \) such that
\[
A \circ v = \overline{\Delta}(A) \circ v.
\]

**Proof.** Let \( e(A) \) be the sum of elements \( A_{1s(1)} \circ \cdots \circ A_{ns(n)} \) for all \( s \in S(n) \setminus \{id\} \). Let
\[
c(A) = \det(\Delta(A)) = A_{11} \circ \cdots \circ A_{nn}.
\]
We show that the condition (i) is fulfilled if \( e(A) < c(A) \). Replacing \( A \) by \( \circ(\Delta(A)) \circ A \), we may assume \( \Delta(A) = E_n \). There is an element \( \varepsilon \in k \) such that \( \varepsilon > 0 \) and
\[
e(A) \circ n\varepsilon \leq c(A).
\]
Let
\[
B = A \oplus \varepsilon \oplus \overline{\Delta}(A).
\]
Then we have \( e(B) \leq c(B) \). By Lemma 5.1 we have \( B^{\circ n} = B^{\circ n-1} \).
Let \( w \in (k \setminus \{-\infty\})^n \) be any element. Let \( v = B^{\circ n-1} \circ w \). Then we have \( B \circ v = v \).

We show that the condition (ii) is fulfilled if \( c(A) \leq e(A) \). We may assume \( \Delta(A) = E_n \). (If \( A_{ii} = -\infty \), then the element \( v = e_i \) satisfies the conclusion.) There is a cyclic permutation \( s \in S(n) \setminus \{\text{id}\} \) and a map \( h: \{0, \ldots, l\} \rightarrow \{1, \ldots, n\} \) such that
\[
s: h(0) \mapsto h(1) \mapsto \cdots \mapsto h(l - 1) \mapsto h(l) = h(0),
\]
\[
A_{h(0)h(1)} \circ \cdots \circ A_{h(l-1)h(l)} \geq 0.
\]
Let
\[ v = \bigoplus_{1 \leq m \leq l} (A_{h(m)}h(m+1) \circ \cdots \circ A_{h(l-1)}h(l)) \circ e_{h(m)}. \]

Then
\[ \nabla(A) \circ v \geq v. \]

So we have the conclusion. \( \square \)

6. Tropical curves

Let \( A = \mathbb{T}[x_1, -x_1, \ldots, x_n, -x_n] \) be the semiring of Laurent polynomials over \( T = \mathbb{R} \cup \{-\infty\} \) (where \( -x_i \) means \( \ominus x_i \)). Let
\[ f = \bigoplus_{i_1, \ldots, i_n \in \mathbb{Z}} c_{i_1 \ldots i_n} \ominus i_1 x_1 \ominus \cdots \ominus i_n x_n \]
be any element of \( A \). The induced map
\[ f : \mathbb{R}^n \longrightarrow \mathbb{T} \]
\[ (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n) \]
is said to be a Laurent polynomial function over \( T \). If \( f \) is a monomial, then \( f \) is a \( \mathbb{Z} \)-affine function, i.e. there are \( c \in \mathbb{R} \) and \( i_1, \ldots, i_n \in \mathbb{Z} \) such that
\[ f = c + i_1 x_1 + \cdots + i_n x_n. \]

In general case, \( f \) is the supremum of finitely many \( \mathbb{Z} \)-affine functions, which is a locally convex piecewise-\( \mathbb{Z} \)-affine function.

Let \( \Gamma_n \subset \mathbb{R}^n \) be the subset defined as follows.
\[ \Gamma_n = E_0 \cup E_1 \cup \cdots \cup E_n, \]
\[ E_0 = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | \forall i, \forall j, a_j = a_j \geq 0\}, \]
for \( 1 \leq i \leq n, \)
\[ E_i = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | a_i \leq 0, \forall j \neq i, a_j = 0\}. \]
\( \Gamma_n \) has a \((n+1)\)-valent vertex \( P = (0, \ldots, 0) \). Also \( \Gamma_n \) is equipped with Euclidean topology on \( \mathbb{R}^n \).

**Definition.** A function \( f : \Gamma_n \rightarrow \mathbb{T} \) is regular if it is induced by a locally Laurent polynomial function \( f : \mathbb{R}^n \rightarrow \mathbb{T} \).

Let \( \mathcal{O}_{\Gamma_n} \) be the sheaf of the regular functions on \( \Gamma_n \). \( \mathcal{O}_{\Gamma_n} \) is a sheaf of semirings. Let \( R \) be the stalk of \( \mathcal{O}_{\Gamma_n} \) at the vertex \( P \).

**Proposition 6.1.** Let \( f \in R \setminus \{-\infty\} \) be any element. Then there are a unique number \( r \in \mathbb{Z}_{\geq 0} \) and a unique Laurent monomial \( h \in R \) such that
\[ f = h \odot r(x_1 \oplus 0). \]
Proof. $f$ is the sum of Laurent monomials $f_1, \ldots, f_m$. If $f_j(P) < f(P)$, then $f_j < f$ on a neighborhood of $P$. So we may assume $f_j(P) = f(P)$. Then $f$ is $\mathbb{Z}$-affine on $E_i$ ($1 \leq i \leq n$). So there is $a_i \in \mathbb{Z}$ such that $f = f(P) \odot a_i x_i$ on $E_i$. Let
\[ h = f(P) \odot a_1 x_1 \odot \cdots \odot a_n x_n. \]
Then $f = h$ on $E_1 \cup \cdots \cup E_n$. $f \odot h$ is the sum of monomials $g_1, \ldots, g_m$ such that $g_j(P) = 0$. There are $b_{ij} \in \mathbb{Z}_{\geq 0}$ such that $g_j = b_{1j} x_1 \odot \cdots \odot b_{nj} x_n$.

Then $g_j = (b_{1j} + \cdots + b_{nj}) x_1$
on $E_0$. So we have $f \odot h = rx_1$ on $E_0$, where
\[ r = \bigoplus_{1 \leq j \leq m} \sum_{1 \leq i \leq n} b_{ij}. \]

The number $r$ in the above statement is called the order of $f$ at $P$, and denoted by \text{ord}(f, P).

For $0 \leq i \leq n$, let $X_i f$ be the partial differential of $f$ at $P$ with direction $E_i$ (i.e. $X_i f = a_i$ if and only if $f = f(P) - a x_i$ on $E_i$ ($1 \leq i \leq n$). $X_0 f = a$ if and only if $f = f(P) + ax_1$ on $E_0$.)

**Proposition 6.2.** Let $f \in R \setminus \{-\infty\}$ be any element. Then
\[ \text{ord}(f, P) = \sum_{0 \leq i \leq n} X_i f. \]

Proof. Let $h$ be a Laurent monomial written as follows.
\[ h = c \odot a_1 x_1 \odot \cdots \odot a_n x_n. \]

Then
\[ X_i h = -a_i \quad (1 \leq i \leq n), \]
\[ X_0 h = a_1 + \cdots + a_n. \]

So we have
\[ \sum_{0 \leq i \leq n} X_i h = 0. \]

Also we have
\[ \sum_{0 \leq i \leq n} X_i (x_1 \oplus 0) = 1. \]

So we have the conclusion. \qed

**Proposition 6.3.** Let $f, g \in R \setminus \{-\infty\}$ be any elements.

(1) $\text{ord}(f \odot g, P) = \text{ord}(f, P) + \text{ord}(g, P)$. 


(2) If \( g(P) \leq f(P) \), then \( \text{ord}(f, P) \leq \text{ord}(f \oplus g, P) \).

Proof. (1) is easy.

(2) If \( g(P) < f(P) \), then \( f \oplus g = f \). So we may assume \( g(P) = f(P) \). Then we have

\[ X_i(f \oplus g) = X_i f \oplus X_i g. \]

By Proposition 6.2, we have the conclusion. \( \square \)

A function \( f : \Gamma_n \to \mathbb{T} \) is said to be rational if locally

\[ f = g_1 - g_2 = g_1 \odot g_2 \]

for regular functions \( g_1, g_2 \). By Proposition 6.1 there is a number \( m \geq 0 \) such that the function \( m(x_1 \odot 0) \odot f \) is regular at \( P \). The order of \( f \) at \( P \) is defined as follows.

\[ \text{ord}(f, P) = \text{ord}(m(x_1 \odot 0) \odot f, P) - m. \]

Let \( Q \in \Gamma_n \) be a point such that \( Q \neq P \). Then a neighborhood of \( Q \) is embedded in \( \Gamma_1 = \mathbb{R} \). So we can define the order of \( f \) at \( Q \) similarly.

Definition. \((C, \mathcal{O}_C)\) is a tropical curve if for any \( P \in C \) there are a neighborhood \( U \) of \( P \) and a number \( n \geq 1 \) such that \((U, \mathcal{O}_U)\) is embedded in \((\Gamma_n, \mathcal{O}_{\Gamma_n})\).

A divisor \( D \) on a tropical curve \( C \) is an element of the free abelian group \( \text{Div}(C) \) generated by all the points of \( C \). For a rational function \( f : C \to \mathbb{T} \), the divisor \( (f) \in \text{Div}(C) \) is defined as follows.

\[ (f) = \sum_{P \in C} \text{ord}(f, P)P. \]

\( f \) is said to be a section of \( D \) if either \( f = -\infty \) or \( (f) + D \geq 0 \). Let \( \mathcal{O}_C(D) \) be the sheaf of the sections of \( D \).

Proposition 6.4. The set \( M = H^0(C, \mathcal{O}_C(D)) \) is a \( \mathbb{T} \)-module.

Proof. Let \( f, g \in M \setminus \{-\infty\} \) be any elements. By Proposition 6.3, for \( P \in C \) we have

\[ \text{ord}(f \oplus g, P) \geq \min\{\text{ord}(f, P), \text{ord}(g, P)\}. \]

So

\[ (f \oplus g) + D \geq \inf_{\text{Div}(C)} \{(f), (g)\} + D \geq 0. \]

So we have \( f \oplus g \in M \). \( \square \)

Recall that

\[ r(D) = \max\{r \in \mathbb{Z}_{\geq -1} \mid U(D, r) = \emptyset\}. \]
Proof of Theorem 2.7. Note that \( r(D) = s(D) - 1 \), where 
\[
s(D) = \min\{r \in \mathbb{Z}_{\geq 0} \mid U(D, r) \neq \emptyset \}.
\]
Let \( m = s(D) \). We show that there is a straight reflexive submodule \( N \subset M = H^0(C, \mathcal{O}_C(D)) \) with dimension \( m \). Let \( P_1, \ldots, P_m \in C \) be points such that 
\[
H^0(C, \mathcal{O}_C(D - E)) = -\infty,
\]
where 
\[
E = P_1 + \cdots + P_m.
\]
There is an element 
\[
f_i \in H^0(C, \mathcal{O}_C(D - E + P_i))
\]
such that \( f_i \neq -\infty \). Let 
\[
\alpha : \mathbb{T}^m \rightarrow M
\]
be the homomorphism defined by \( \alpha(e_i) = f_i \). Let 
\[
\beta : M \rightarrow \mathbb{T}^m
\]
be the homomorphism defined by 
\[
\beta(g) = g(P_1) \odot e_1 \oplus \cdots \oplus g(P_m) \odot e_m.
\]
Let \( A \) be the square matrix induced by \( \beta \circ \alpha : \mathbb{T}^m \rightarrow \mathbb{T}^m \).

Now we suppose that there is an element 
\[
v = a_1 \odot e_1 \oplus \cdots \oplus a_m \odot e_m \in \mathbb{T}^m \setminus \{-\infty\}
\]
such that \( A \odot v = \overline{\Delta}(A) \odot v \). Then there is a map \( h : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \) such that \( h(i) \neq i \) and 
\[
\alpha(v)(P_i) = a_{h(i)} \odot f_{h(i)}(P_i).
\]
Then 
\[
\text{ord}(\alpha(v), P_i) \geq \text{ord}(f_{h(i)}, P_i)
\]
(Proposition 6.3). So \( \alpha(v) \) is a section of \( D - E \) such that \( \alpha(v) \neq -\infty \), which is contradiction.

So there is no element \( v \in \mathbb{T}^m \setminus \{-\infty\} \) such that \( A \odot v = \overline{\Delta}(A) \odot v \). By Lemma 5.2 there are an element \( v \in \mathbb{R}^m \) and an element \( \varepsilon > 0 \) such that 
\[
(A \oplus \varepsilon \odot \overline{\Delta}(A)) \odot v = \Delta(A) \odot v.
\]
Let \( L(v, \varepsilon) \subset \mathbb{T}^m \) be the submodule defined as follows. 
\[
L(v, \varepsilon) = \mathbb{T} \odot \{w \in \mathbb{T}^m \mid v \leq w \leq \varepsilon \odot v\}.
\]
\( L(v, \varepsilon) \) is a straight reflexive \( \mathbb{T} \)-module with dimension \( m \). We have 
\[
A|_{L(v, \varepsilon)} = \Delta(A)|_{L(v, \varepsilon)}.
\]
So $\alpha$ is injective on $L(v, \varepsilon)$. The image $N = \alpha(L(v, \varepsilon))$ is a submodule of $M$ such that $N \cong L(v, \varepsilon)$. □

**Example 6.5.** The mapping $D \mapsto r(D)$ is not an invariant of a $\mathbb{T}$-module. We show that there are tropical curves $C, C'$ and divisors $D, D'$ such that

$$H^0(C, \mathcal{O}_C(D)) \cong H^0(C', \mathcal{O}_{C'}(D')),$$

$$r(D) \neq r(D').$$

Let $C$ be a tropical curve with genus 1 with a vertex $V$ and an edge $E$. Let $P$ be an interior point of $E$. Let $D = V + P$. Then $H^0(C, \mathcal{O}_C(D))$ is isomorphic to the submodule of $\mathbb{T}^2$ generated by $(0, 0)$ and $\left(0, \frac{a}{2}\right)$, where $a$ is the lattice length of $E$. We have $r(D) = 1$.

Let $C'$ be a tropical curve with genus 2 with vertices $V_1, V_2$ and edges $E_1, E_2, E_3$ such that the boundary of $E_i$ is $\{V_1, V_2\}$ ($1 \leq i \leq 3$). Let $P$ be an interior point of $E_1$. Let $D' = V_1 + P$. Then for any interior point $Q$ of $E_2 \cup E_3$ we have

$$H^0(C', \mathcal{O}_{C'}(D' - Q)) = -\infty.$$

So $H^0(C', \mathcal{O}_{C'}(D'))$ is isomorphic to the submodule of $\mathbb{T}^2$ generated by $(0, 0)$ and $(0, \frac{b}{3})$, where $b$ is the lattice length of the path from $V_1$ to $P$ contained in $E_1$. We have $r(D') = 0$. In the case of $a = b$, the required condition is fulfilled.

## 7. Tropical plane curves

### 7.1. Tropicalization

It is well known that some example of tropical curve is given by tropicalization of a family of affine complex curves.

First, we define tropical plane curves. Let $f \in \mathbb{T}[x, -x, y, -y]$ be a Laurent polynomial over $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. The subset

$$V(f) = \{(a, b) \in \mathbb{R}^2 \mid -f \text{ is not locally convex at } (a, b)\}$$

is called the algebraic subset defined by $f$. The morphism $C_f \to \mathbb{R}^2$ parametrizing $V(f)$ with a tropical curve $C_f$ is called the tropical plane curve defined by $f$. The genus of $C_f$ is defined to be the first Betti number $b_1(C_f)$.

A tropical plane curve is a dequantization of complex amoebas in following way. For $t > 1$, let

$$\mathcal{A}_t : (\mathbb{C}^\times)^2 \longrightarrow \mathbb{R}^2$$

be the homomorphism of groups defined by

$$\mathcal{A}_t(a, b) = \left(\frac{\log |a|}{\log(t)}, \frac{\log |b|}{\log(t)}\right).$$
\( \mathcal{A}_t \) is called the complex amoeba map. Let
\[
g_t \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}] \quad (t > 1)
\]
be a family of complex Laurent polynomials such that each coefficient is a Laurent polynomial of \( t^{-1} \). This family is written as an element of a valuation field \( K \). We use the group algebra \( K = \mathbb{C}[[\mathbb{R}]] \) of power series defined by the group \( \mathbb{R} \). The indeterminate is denoted by \( t^{-1} \), and the valuation is defined to be the maximum index of \( t \) multiplied by \(-1\). So, \( \text{val}(t^a) = -a \). The family \( \{g_t \mid t > 1\} \) is written as an element
\[
g \in K[z_1, z_1^{-1}, z_2, z_2^{-1}].
\]
The amoeba map over \( K \)
\[
\mathcal{A}: (K^\times)^2 \longrightarrow \mathbb{R}^2
\]
is defined as follows.
\[
\mathcal{A}(a, b) = (-\text{val}(a), -\text{val}(b)).
\]
The affine curve \( V(g) \subset (K^\times)^2 \) is the family of affine complex curves \( V(g_t) \subset (\mathbb{C}^\times)^2 \). Taking \( t \to +\infty \), the family of complex amoebas \( \mathcal{A}_t(V(g_t)) \) converges to the amoeba \( \mathcal{A}(V(g)) \) over \( K \). Also, the amoeba over \( K \) is the algebraic subset defined by a tropical Laurent polynomial. Let
\[
\mathcal{A}: K[z_1, z_1^{-1}, z_2, z_2^{-1}] \longrightarrow \mathbb{T}[x, -x, y, -y]
\]
be the map defined as follows.
\[
\mathcal{A}(g) = f,
\]
\[
g = \sum_{i,j \in \mathbb{Z}} c_{ij} z_1^i z_2^j,
\]
\[
f = \bigoplus_{i,j \in \mathbb{Z}} -\text{val}(c_{ij}) \odot ix \odot jy.
\]
Then we have
\[
\mathcal{A}(V(g)) = V(f).
\]
This construction is called the tropicalization of a family of affine complex curves.
7.2. Examples.

Example 7.1. For $a, b, c \in \mathbb{C}^*$, let

$$g = a + bz_1 + cz_2.$$  

Then

$$f = \mathcal{A}(g) = 0 \oplus x \oplus y.$$  

The tropical plane curve $C_f$ is said to be a tropical projective line. We have $b_1(C_f) = 0$.

Example 7.2. For $r, s \in \mathbb{N}$ and $a_i, b_j \in \mathbb{R}$, let

$$f = f_1 \odot f_2,$$

$$f_1 = a_0 \oplus a_1 \odot x \oplus a_2 \odot 2x \oplus \cdots \oplus a_r \odot rx,$$

$$f_2 = b_0 \oplus b_1 \odot y \oplus b_2 \odot 2y \oplus \cdots \oplus b_s \odot sy.$$  

Assume that

$$2a_i > a_{i-1} + a_{i+1},$$

$$2b_j > b_{j-1} + b_{j+1}.$$  

Then $b_1(C_f) = (r - 1)(s - 1)$.

REFERENCES

[1] A. Gathmann and M. Kerber. A Riemann-Roch theorem in tropical geometry. Mathematische Zeitschrift 259, Number 1 (2008), 217-230.

[2] Zur Izhakian. Tropical Algebraic Sets, Ideals and An Algebraic Nullstellensatz. International Journal of Algebra and Computation 18 (2008), 1067-1098.

[3] M. Joswig and K. Kulas. Tropical and ordinary convexity combined. Preprint, arXiv:0801.4835.

[4] Grigori Litvinov. The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. Preprint, arXiv:math.GM/0501038.

[5] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. J. Amer. Math. Soc. 18 (2005), 313-377.

[6] G. Mikhalkin and I. Zharkov. Tropical curves, their Jacobians and Theta functions. Preprint, arXiv:math.AG/0612267.

[7] Yen-lung Tsai. Working with Tropical Meromorphic Functions of One Variable. Preprint, arXiv:1101.2703.

[8] Oleg Viro. Hyperfields for Tropical Geometry I. Hyperfields and dequantization. Preprint, arXiv:1006.3034.