QUANTUM DYNAMICS WITH INTERMEDIATE MEASUREMENTS IN AGREEMENT WITH THE CLASSICAL DYNAMICS

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ABSTRACT

The effect of repetitive measurement for quantum dynamics of driven by an intensive external force of the simple few-level systems as well as of the multilevel systems that exhibit the quantum localisation of classical chaos is investigated. Frequent measurement of the simple system yields to the quantum Zeno effect while that of the suppressed quantum system, which classical counterpart exhibits chaos, results in the delocalisation of the quantum suppression. From the analysis we may conclude that continuously observable quasiclassical system evolves essentially classically-like.

1. Introduction

Dynamics of a quantum system, when it is not being observed, may be described by the Schrödinger equation, by the quantum Liouville equation for the density matrix or by the equation for evolution of the Wigner function. In general, the quantum and classical realms are related by the correspondence principle: physical characteristics of the highly exited quantum systems with large quantum numbers are close to those of its classical counterpart. However, recent researches have suggested that the use of the correspondence principle for strongly driven nonlinear systems is not so straightforward: a quantum interference effect suppresses the classical diffusion-like chaotic motion and results in the interference of macroscopic systems’ states. The exponential instability of classical motion characterised by the positive maximal Lyapunov exponent destroys the deterministic image of the classical physics and results in the unpredictability of the trajectory. Such stochasticity implies a continuous spectrum of the motion. On the other hand, the frequency spectrum of any quantum system, which motion is bounded in the phase space, is always discrete. Accordingly, the motion of such a system is regular. Therefore, the question: can (and if yes, how) quantum mechanics give chaos as a limiting behaviour, is open until now [1].

It is postulated in the von Neumann axiomatics of the quantum mechanics that any measurement changes abruptly the systems under consideration state and projects it to an eigenstate of the measured observation. The measurement process follows the irreversible dynamics and results to the disappearance of coherence of the sys-
tem’s state: to the decay of the off-diagonal matrix elements of the density matrix or randomization of the phases of the wave function’s amplitudes.

Quantum system undergoes relatively slow evolution at an early period after preparation or measurement [2]. Therefore, the repetitive frequent observation of the quantum system can inhibit the decay of unstable or dynamics of the driven by an external driven field system. This phenomenon is called the quantum Zeno effect [3]. It should be noticed, that until now the quantum Zeno effect has mostly been intensively analysed for the purely quantum systems consisting only of the few (usually two or three) quantum states. However, it is of interesting to investigate the influence of the repetitive frequent measurement on the evolution of the multilevel quasiclassical systems, the classical counterparts of which exhibit chaos. It has been established that chaotic dynamics of such systems, e.g. dynamics of strongly driven by periodic external field nonlinear systems, is suppressed of the quantum interference effect and results in the quantum localisation of the classical dynamics in the energy space of the system (see, e.g. [4-5]). Thus, the quantum localisation phenomenon strongly limits the quantum motion of the unobservable system.

As it was mentioned above, the repeated frequent measurement or continuous observation of quantum system can freeze its dynamics too. It is natural to expect that frequent measurement of the suppressed system will result in the additional freezing of the system’s state. However, as we will see late, such supposition is wrong.

Here we show how the quantum Zeno effect in a two-level-system may be described by the wave function without the density matrix formalism. Then we use such a method for analysis of dynamics of the multilevel systems affected by repeated measurement. We reveal that the repetitive measurement of the multilevel systems with quantum suppression of classical chaos results in the delocalisation of the states superposition and acceleration of the chaotic dynamics.

2. Dynamics of the two-level systems

The simplest time evolution of the two-state wave function $\Psi = a_1 |1\rangle + a_2 |2\rangle$ from time moment $t_k = k\tau$ to $t_{k+1} = (k + 1)\tau$ may be represented as

$$
\begin{pmatrix}
a_{1}(k+1) \\
a_{2}(k+1)
\end{pmatrix} = A 
\begin{pmatrix}
a_{1}(k) \\
a_{2}(k)
\end{pmatrix},
$$

(2.1)

$$
A = \begin{pmatrix}
\cos \varphi & i \sin \varphi \\
i \sin \varphi & \cos \varphi
\end{pmatrix}, \quad \varphi = \frac{1}{2} \Omega \tau
$$

(2.2)

where $\Omega$ is the Rabi frequency. Evidently the evolution of the amplitudes from time $t = 0$ till $t = n\tau$ may be expressed as

$$
\begin{pmatrix}
a_{1}(n) \\
a_{2}(n)
\end{pmatrix} = A^n \begin{pmatrix}
a_{1}(0) \\
a_{2}(0)
\end{pmatrix}.
$$

(2.3)
One can calculate matrix $A^n$ by the method of diagonalization of the matrix $A$. The result naturally is

$$ A^n = \begin{pmatrix} \cos n\varphi & i \sin n\varphi \\ i \sin n\varphi & \cos n\varphi \end{pmatrix}. \quad (2.4) $$

For the time interval $T = n\tau = \pi/\Omega$ a certain (with the probability 1) transition between the states takes place.

Measurement of the system’s state in the time moment $t = k\tau$ projects the system to the state $|1\rangle$ with the probability $p_1(k) = |a_1(k)|^2$ or to the state $|2\rangle$ with the probability $p_2(k) = |a_2(k)|^2$. After each of the measurement the phases of the amplitudes $a_1(k)$ and $a_2(k)$ are random which results in the absence of the interference terms in the expressions for the probabilities. The evolution from the time $t = 0$ until $t = n\tau$ with the $(n-1)$ intermediate measurement is described by the equation

$$ \begin{pmatrix} p_1(n) \\ p_2(n) \end{pmatrix} = M^n \begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix}. \quad (2.5) $$

Matrix $M^n$ calculated by the diagonalization method is

$$ M^n = \frac{1}{2} \begin{pmatrix} 1 + \cos^n 2\varphi & 1 - \cos^n 2\varphi \\ 1 - \cos^n 2\varphi & 1 + \cos^n 2\varphi \end{pmatrix}. \quad (2.6) $$

From eqs. (2.5) and (2.6) we get the quantum Zeno effect [3]: if initially the system is in state $|1\rangle$, the outcome of evolution until the time $T = n\tau = \pi/\Omega$ with the intermediate measurements will be given by the probabilities $p_1(T) = (1 + \cos^n 2\varphi)/2 \to 1$ and $p_2(T) = (1 - \cos^n 2\varphi)/2 \to 0$, $\ (n \to \infty)$. This result represents the inhibition of the quantum dynamics by measurements and confirms the proposition that act of measurement may be expressed as randomisation of the amplitudes’ phases.

### 3. Dynamics of the multilevel systems

In general the Schrödinger equation for strongly driven multilevel systems can not be solved analytically. However, the mapping form of quantum equations of motion greatly facilitates investigation of stochasticity and quantum–classical correspondence for the chaotic dynamics. From the standpoint of an understanding of manifestation of the measurements for the dynamics of the multilevel systems the region of large quantum numbers is of greatest interest. Here we may use the quasiclassical approximation with the convenient variables angle $\theta$ and action $I$. The simplest system in which the dynamical chaos and quantum localisation of states may be observed is a system with one degree of freedom described by the nonlinear Hamiltonian, $H_0(I)$, and driven by the periodic $V(\theta, t) = k \cos \theta \sum_j \delta(t - j\tau)$ kicks [4-5]. Integrating the Schrödinger equation over the period $\tau$ we obtain a map [5-7]

$$ a_m(t_{j+1}) = e^{-i\beta_m} \sum_n a_n(t_j) J_{m-n}(k), \quad \beta_m = H_0(m)\tau, \quad t_j = j\tau \quad (3.1) $$
for the amplitudes $a_m(t_j)$ before the appropriate $j$-kick in expansion of the state function $\Psi(\theta, t)$ through the eigenfunctions, $\varphi_m = i^{-m} e^{im\theta}/\sqrt{2\pi}$, of the action $I = -i \frac{\partial}{\partial \theta}$. Here $J_m(k)$ is the Bessel function.

Quantum dynamics represented by map (3.1) with the non-linear Hamiltonian $H_0(I)$ is similar to the classical one only for some finite time $t^*$ after which it reveals an essential decrease of the diffusion rate asymptotically resulting in the exponential localisation of the system’s state with the localisation length $\lambda \sim k^2/2 [1,4,5]$.

Each measurement of the system’s state between $(j-1)$ and $j$ kicks projects it to one of the state $\varphi_m$ with the probability $P_m(t_j) = |a_m(t_j)|^2$. After such a measurement the phase of the amplitude $a_m(t_j)$ is random. Therefore, the influence of the measurements for further dynamics of the system may be expressed as replacement of the amplitudes $a_m(t_j)$ by the amplitudes $\exp\left[i2\pi g_m(t_j)\right]a_m(t_j)$, where $g_m(t_j)$ is a random number in case of measurement of the $\varphi_m$-state’s population before the $j$ kick and equals zero in absence of such a measurement. So we may analyse the influence on the dynamics of measurements performed after every kick, after every $N$ kicks or of the measurements just of some states, e.g. only of the initial state, and observe the reduction of the quantum localisation effect in a degree depending on the extent and frequency of the measurement [7]. In the case of measurement of all states after every kick we have the uncorrelated transitions between the states and diffusion-like motion with the quantum diffusion coefficient in the $n$-space

$$B(n) = \frac{1}{2\tau} \sum_m (m-n)^2 J_{m-n}^2(k) = \frac{k^2}{4\tau}$$

which coincides with the classical one. Therefore, the quantum evolution of frequently observable chaotic system is more classical-like than dynamics of the isolated system.

To facilitate the comparison between quantum and classical dynamics it is convenient to employ the Wigner representation, $\rho_W(x, p, t)$, of the density matrix. The Wigner function of the system evolves according to equation

$$\frac{\partial \rho_W}{\partial t} = \{H, \rho_W\}_M \equiv \{H, \rho_W\} + \sum_{n \geq 1} \frac{\hbar^{2n} (-1)^n}{2^{2n} (2n+1)!} \frac{\partial^{2n+1} V}{\partial x^{2n+1}} \frac{\partial^{2n+1} \rho_W}{\partial p^{2n+1}}$$

where by $\{\ldots\}_M$ and $\{\ldots\}$ are denoted the Moyal and the Poisson brackets, respectively, while the Hamiltonian of the system is of the form $H = p^2/2m + V(x, t)$. The terms in eq. (3.3) containing Planck’s constant and higher derivatives give the quantum corrections to the classical dynamics generated by the Poisson brackets. In the region of regular dynamics one can neglect the quantum corrections for very long time if the characteristic actions of the system are large. For classically chaotic motion the exponential instabilities lead to the development of the fine structure of the Wigner function and exponential growth of its derivatives. As a result, the quantum corrections become significant after relatively short time even for macroscopic bodies [8]. The extremely small additional diffusion-like terms in eq. (3.3), which reproduce
the effect of interaction with the environment or frequent measurement, prohibits development of the Wigner function’s fine structure and removes barriers posed by classical chaos for the correspondence principle.

4. Conclusion

The quantum-classical correspondence problem caused of the chaotic dynamics is closely related with the old problem of measurement in quantum mechanics. Even the simplest detector follows irreversible dynamics due to the coupling to the multitude of vacuum modes which results in the randomisation of the quantum amplitudes’ phases, decay of the off-diagonal matrix elements of the density matrix or to the smoothing of the fine structure of the Wigner distribution function—what we need to obtain the classical equations of motion. The repetitive measurement of the multilevel systems with quantum suppression of classical chaos results in delocalisation of the states superposition and acceleration of the chaotic dynamics which is opposite to the quantum Zeno effect in driven systems. In the limit of the frequent full measurement or unpredictable interaction with the environment the quantum dynamics of such quasiclassical systems approaches the classical motion.

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