POINCARÉ BUNDLE FOR THE FIXED DETERMINANT MODULI SPACE ON A NODAL CURVE

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Abstract. Let \( Y \) be an integral nodal projective curve of arithmetic genus \( g \geq 2 \) with \( m \) nodes defined over an algebraically closed field \( k \) and \( x \) a nonsingular closed point of \( Y \). Let \( n \) and \( d \) be coprime integers with \( n \geq 2 \). Fix a line bundle \( L \) of degree \( d \) on \( Y \). Let \( U_Y(n,d,L) \) denote the (compactified) “fixed determinant moduli space”. We prove that the restriction \( U_L,x \) of the Poincare bundle to \( x \times U_Y(n,d,L) \) is stable with respect to the polarisation \( \theta_L \) and its restriction to \( x \times U'_Y(n,d,L) \), where \( U'_Y(n,d,L) \) is the moduli space of vector bundles of rank \( n \) and determinant \( L \), is stable with respect to any polarisation. We show that the Poincaré bundle \( U_L \) on \( Y \times U'_Y(n,d,L) \) is stable with respect to the polarisation \( a\alpha + b\theta_L \) where \( \alpha \) is a fixed ample Cartier divisor on \( Y \) and \( a, b \) are positive integers.

1. Introduction

Let \( Y \) be an integral projective curve of arithmetic genus \( g \geq 2 \) defined over an algebraically closed field \( k \) with at most nodes as singularities. Let \( n \) and \( d \) be coprime integers with \( n \geq 2 \). Let \( U_Y(n,d) \) denote the moduli space of stable torsion free sheaves of rank \( n \) and degree \( d \) on \( Y \) and \( U'_Y(n,d) \) its open dense sub variety corresponding to stable vector bundles. For a fixed line bundle \( L \) of degree \( d \) on \( Y \), let \( U'_Y(n,d,L) \) denote the sub variety of \( U'_Y(n,d) \) consisting of vector bundles with determinant isomorphic to \( L \). Let \( U_Y(n,d,L) \) be the closure of \( U'_Y(n,d,L) \) in \( U_Y(n,d) \) (with reduced structure).

There exits a Poincaré sheaf

\[
U_L: Y \times U_Y(n,d,L) \to E
\]

such that the restriction \( U_L|_{Y \times [E]} \cong E \) for any \( E \in U_Y(n,d,L) \) [12, Theorem 5.12]. Let \( x \) denote a nonsingular point of \( Y \). Then the restriction \( U_L,x \), of the Poincaré sheaf to \( x \times U_Y(n,d,L) \), is a vector bundle on \( x \times U_Y(n,d,L) \cong U_Y(n,d,L) \). There is a canonically defined ample line bundle \( \theta \) on \( U_Y(n,d,L) \) (the determinant of cohomology line bundle), let \( \theta_L: U_Y(n,d,L) \to U_Y(n,d,L) \) be its restriction. One has Pic \( U'_Y(n,d,L) \cong \mathbb{Z} \) [3, Theorem 1] and the restriction of \( \theta_L \) to \( U'_Y(n,d,L) \) is the generator of Pic \( U'_Y(n,d,L) \).

We do not know if Pic \( U_Y(n,d,L) \cong \mathbb{Z} \). We show that any normalisation of \( U_Y(n,d,L) \) is locally factorial and has Picard group isomorphic to \( \mathbb{Z} \) (subsection 2.4). We deduce

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that the scheme $R$ of $(g - 1)$-dimensional subspaces of $k^{2g+2}$ which are isotropic for the pencil of quadrics with Segre symbol $[2 \, 1 \, \cdots \, 1]$ is locally factorial.

By the (semi)stability of a sheaf on a polarised variety, we mean the slope (semi)stability with respect to the polarisation. In case $X$ is smooth, the semistability of $U_{L,x}$ with respect to the polarisation $\theta_L$ and the stability of $U_L$ with respect to suitable polarisations were proved in \cite{2} using Higgs bundles. In this small note, we generalise their results to nodal curves. We give a different and incredibly simple proof without using the spectral curves and Higgs bundles. Our main results are the following theorems.

**Theorem 1.1.** (Theorem 3.3) Let $x \in Y$ be a nonsingular closed point.

1. The restriction $U_{L,x}$ of the Poincare sheaf to $x \times U_Y(n,d,L)$ is a vector bundle, it is stable with respect to the polarisation $\theta_L$.
2. The restriction of $U_{L,x}$ to $x \times U'_Y(n,d,L)$ is stable with respect to any polarisation.

Theorem 1.1 generalises \cite{2}, Proposition 1.4 to nodal curves. In fact, it improves \cite{2}, Proposition 1.4 even in the case $Y$ is smooth as \cite{2}, Proposition 1.4 only proves semistability of $U_{L,x}$ in the smooth case.

**Theorem 1.2.** (Theorem 3.5) The Poincare sheaf $U_L$ on $Y \times U_Y(n,d,L)$ is stable with respect to the polarisation $a\alpha + b\theta_L$ where $\alpha$ is a fixed ample Cartier divisor on $Y$ and $a, b$ are positive integers.

Theorem 1.2 generalises \cite{2}, Theorem 1.5 to nodal curves. Our proofs of these two theorems were inspired by the ideas in \cite{9}. Recently, the results of \cite{9} are generalised to nodal curves by myself and my collaborators \cite{1}. We use some computations in this paper in our proofs in \cite{1}.

## 2. Preliminaries

In this section, we set up the notation and recall some definitions and results needed.

### 2.1. Notation.

Let $Y$ be an integral projective curve with only $m$ nodes (ordinary double points) as singularities defined over an algebraically closed field. Let $g = h^1(Y, \mathcal{O}_Y)$ be the arithmetic genus of $Y$, we assume that $g \geq 2$. Let

$p : X \longrightarrow Y$

be the normalisation map. For a node $y_j \in Y$, let $x_j$ and $z_j$ denote the points of $X$ lying over $y_j$. For each $j = 1, \cdots, m$, let $D_j = x_j + z_j$ denote the divisor on $X$.

For a torsion free sheaf $F$ on $Y$, let $r(F)$ denote the (generic) rank of $F$ and $d(F) = \chi(F) - r(F)\chi(\mathcal{O}_Y)$ denote the degree of $F$. Let $\mu(F) = d(F)/r(F)$ denote the slope of $F$.

Let $H$ be an ample line bundle on a variety $Z$ of dimension $m \geq 2$ and $F$ a torsionfree sheaf on $Z$. Then degree of $F$ with respect to $H$, denoted by $d(F)$, is the degree of the
restriction of $F$ to a general complete intersection curve on $Z$ of the form $D^{m-1}$ for some divisor $D \in |H|$. We remark that if the singular set of $Z$ has codimension at least 2, then the general complete intersection curve can be chosen to lie in the set of nonsingular points of $Z$. Hence if $Z' \subset Z$ is an open subset contained in the set of nonsingular points of $Z$ such that its complement has codimension at least 2, the degree of $F$ with respect to $H$ on $Z$ is the same as the degree of $F|_{Z'}$ with respect to $H|_{Z'}$.

If $Z$ is projective, then we have

$$d(F) = c_1(F).c_1(H)^{m-1}[Z],$$

where $[Z]$ denotes the fundamental class of $Z$. In case Pic $Z \cong \mathbb{Z}$ with $c_1(H)$ the ample generator, one can write $c_1(F) = \lambda c_1(H)$ for some integer $\lambda$ so that $d(F) = \lambda N, N = c_1(H)^m[Z]$. If $Z$ is a projective space $\mathbb{P}$, then we take $H$ to be the tautological line bundle $\mathcal{O}_\mathbb{P}(1)$. Thus $N = 1$ for the projective space $\mathbb{P}$.

2.2. Moduli spaces.

Let $J(Y) := \text{Pic}^d(Y)$ be the generalised Jacobian of $Y$ i.e., the moduli space of lines bundles of degree $d$ on $Y$. Let $U_Y(n,d)$ denote the moduli space of stable torsion free sheaves of rank $n$ and degree $d$ on $Y$. It is irreducible [14], it is a seminormal projective variety of dimension $n^2(g-1)+1$ [16, Theorem 4.2]. Let $U_Y'(n,d)$ be its open dense subvariety corresponding to stable vector bundles on $Y$. From the remark on p.167, section 7 of [12], one deduces that $U_Y'(n,d)$ is a normal quasi-projective variety, being the GIT quotient of a nonsingular variety $R'$ by $\text{PGL}(N)$ for some $N >> 0$. Since $n$ and $d$ are coprime, the points of $U_Y'(n,d)$ correspond to stable vector bundles. Since the automorphisms of stable bundles are scalars, $R'$ is a principal $\text{PGL}(N)$-bundle over $U_Y'(n,d)$, it follows that $U_Y'(n,d)$ is a normal quasi-projective variety. For a fixed line bundle $L$ on $Y$, let $U_Y'(n,d,L)$ denote the sub variety of $U_Y'(n,d)$ consisting of vector bundles with determinant isomorphic to $L$. There is a smooth determinant morphism from $U_Y'(n,d)$ onto the nonsingular variety $J(Y)$. It follows that $U_Y'(n,d,L)$ is a nonsingular variety of dimension $(n^2-1)(g-1)$. Let $U_Y(n,d,L)$ denote the closure of $U_Y'(n,d,L)$ in $U_Y(n,d)$ (with reduced structure).

The moduli varieties $U_Y'(n,d)$ and $U_Y'(n,d,L)$ are locally factorial ([3, Theorem I], [4, Theorem 3A]). One has

$$\text{Pic} U_Y'(n,d,L) \cong \mathbb{Z}$$

(for $g \geq 2$) [31 Theorem I]. There is a canonically defined ample line bundle $\theta$ on $U_Y(n,d)$, let

$$\theta_L \rightarrow U_Y(n,d,L)$$

be its restriction. The restriction of $\theta_L$ to $U_Y'(n,d,L)$ is the generator of Pic $U_Y'(n,d,L)$ [3 Proposition 3.2].

2.3. $(l,m)$-stability for torsionfree sheaves.

We make the following definition generalising [13, Definition 5.1].
Definition 2.1. Let $l$ and $m$ be integers. A torsionfree sheaf $F$ on $Y$ is $(l,m)$-stable if, for every proper subsheaf $G$ of $F$ (with a torsionfree quotient), one has

$$\frac{d(G) + l}{r(G)} < \frac{d(F) + l - m}{r(F)}.$$ 

We remark that a torsionfree sheaf $F$ is stable if and only if it is $(0,0)$-stable.

Proposition 2.2. [6, Proposition 2.2] The $(0,1)$-stable vector bundles $F$, of rank $n$ and determinant $L'$ of degree $d'$ with $n$ coprime to $d' - 1$, form an open subset of the moduli space $U^*_L(n,d')$ of stable torsionfree sheaves.

2.4. Local factoriality.

We do not know if $U_Y(n, d, L)$ is locally factorial or normal. We make a few observations.

Lemma 2.3.

(1) $\text{Cl}(U_Y(n, d, L)) \cong \mathbb{Z}$.

(2) Let $\tilde{U}_Y(n, d, L)$ denote a normalisation of $U_Y(n, D, L)$ and $\pi : \tilde{U}_Y(n, d, L) \to U_Y(n, d, L)$ the normalisation map. Then $\tilde{U}_Y(n, d, L)$ is locally factorial and $\text{Pic} \; \tilde{U}_Y(n, d, L) \cong \mathbb{Z}$.

Proof. (1) As $U_Y(n, d, L)$ is nonsingular in codimension 1, by [10, Proposition 6.5, p.133], we have an isomorphism of class groups

$$\text{Cl}(U'_Y(n, d, L)) \cong \text{Cl}(U_Y(n, d, L)).$$

Since $U'_Y(n, d, L)$ is locally factorial, one has $\text{Cl}(U'_Y(n, d, L)) \cong \text{Pic} \; U'_Y(n, d, L) \cong \mathbb{Z}$. Hence

$$\text{Cl}(U_Y(n, d, L)) \cong \mathbb{Z}.$$ 

(2) There is a commutative diagram

$$\begin{array}{ccc}
\text{Pic}(\tilde{U}_Y(n, d, L)) & \xrightarrow{\phi} & \text{Cl}(\tilde{U}_Y(n, d, L)) \\
\downarrow \text{res}_P & & \downarrow \text{res}_C \\
\text{Pic} \; \pi^{-1}(U'_Y(n, d, L)) & \xrightarrow{\phi'} & \text{Cl}(\pi^{-1}U'_Y(n, d, L))
\end{array}$$

As $\pi^{-1}U'_Y(n, d, L) \cong U'_Y(n, d, L)$, $\tilde{U}_Y(n, d, L)$ is nonsingular in codimension 1. Hence by [10, Proposition 6.5, p.133] the map $\text{res}_C$ is an isomorphism. Since $\pi^{-1}U'_Y(n, d, L)$ is locally factorial, $\phi'$ is an isomorphisms. The map $\text{res}_P$ is a surjection as $\text{Pic} \; \pi^{-1}U'_Y(n, d, L)$ is generated by the restriction of the line bundle $\pi^*\theta_L$ on $\tilde{U}_Y(n, d, L)$. Since $\tilde{U}_Y(n, d, L)$ is normal and $\pi$ being finite, the codimension of the complement of $\pi^{-1}(U'_Y(n, d, L))$ in $\tilde{U}_Y(n, d, L)$ is at least 2, the map $\text{res}_P$ is injective. Then $\text{res}_P$ is an isomorphism. From the commutativity of the diagram, it follows that $\phi$ is an isomorphism and hence $\tilde{U}_Y(n, d, L)$ is locally factorial with Picard group isomorphic to the group of integers.

□
Corollary 2.4. Consider a singular pencil of quadrics with Segre symbol $[2 1 \cdots 1]$ given by
\[ q_1 = \sum_{i=1}^{2g} X_i^2 + 2X_0Y_0, \quad q_2 = \sum_{i=1}^{2g} a_i X_i^2 + (X_0^2 + 2a_0 X_0 Y_0), \]
with $a_i, i = 0, 1, \ldots, 2g$, distinct scalars. Let $R$ be the scheme of $(g-1)$-dimensional subspaces of $k^{2g+2}$ which are isotropic for this pencil. Then $R$ is locally factorial. Let $R_0$ be the subscheme of $R$ consisting of those subspaces which contain the unique singular point of the intersection of quadrics of the pencil. Let $R' := R - R_0$. Then
\[ \Pic R = \Pic R' = \mathbb{Z}. \]

Proof. Let $X$ be an irreducible reduced projective hyperelliptic curve of arithmetic genus $g$ with a single ordinary node as its only singularity, the node is a ramification point of $X$. To such a curve one can associate a singular pencil of quadrics with Segre symbol $[2 1 \cdots 1]$ [5]. Then $R$ is a normalisation of $U_Y(2, d, L)$, $d$ odd [5, Theorem 1.2]. More precisely, there is a morphism $f$ from the scheme $R$ onto $U_Y(2, d, L)$ such that the restrictions of $f$ to $R'$ and $R_0$ are isomorphisms onto $U'_Y(2, d, L)$ and $U_Y(2, d, L) - U'_Y(2, d, L)$. Hence the corollary follows from Lemma 2.3.

We note that in particular, if the arithmetic genus $g = 2$, then the intersection $R$ of quadrics in $\mathbb{P}^5$ is the normalisation of $U_Y(2, d, L)$.

\[ \square \]

Remark 2.5. Possibly $Cl(U_Y(n, d, L)) \not\cong \Pic U_Y(n, d, L)$ as the following example (communicated by V. Srinivas) shows. Let $X$ be the variety obtained by identifying two distinct points $p_1$ and $p_2$ of $\mathbb{P}^2$. Then $Cl(X) \cong Cl(\mathbb{P} - p_1 - p_2) \cong \mathbb{Z}$. However, the Picard group of $X$ is an extension of $\mathbb{Z}$ by $G_m$ as a line bundle on $X$ is obtained identifying the fibres $L_{p_1}$ and $L_{p_2}$ of a line bundle $L$ on $\mathbb{P}_2$.

3. Stability of the bundles $U_{L,x}$ and $U_L$

Since $n$ and $d$ are mutually coprime, there exists a Poincaré sheaf
\[ U_L \to Y \times U_Y(n, d, L), \]
such that the restriction $U_L|_{Y \times [E]} \cong E$ for any $E \in U_Y(n, d, L)$ [12, Theorem 5.12']. Any two such sheaves differ by the pull back of a line bundle on $U_Y(n, d, L)$. Since any torsionfree sheaf on $Y$ is locally free on the subset $Y'$ of nonsingular points of $Y$, the restriction of $U_L$ to $Y' \times U_Y(n, d, L)$ is locally free. It follows that for a nonsingular point $x$ of $Y$, the restriction $U_{L,x}$ of the Poincare sheaf to $x \times U_Y(n, d, L)$, is a vector bundle on $x \times U_Y(n, d, L) \cong U_Y(n, d, L)$. In this section, we study the stability of the restriction $U_{L,x}$ (for $x$ a nonsingular point of $Y$) with respect to the polarisation $\theta_L$ and the stability of $U_L$ with respect to suitable polarisations.
3.1. The morphism $\psi_{F,x}$.

We fix a nonsingular point $x$ of $Y$ and a $(0,1)$-stable vector bundle $F$ of rank $n$ and determinant $L(x), d(L) = d$ on $Y$. Let $k_x$ denote the torsion sheaf of length 1 supported at $x$. Denote by $F_x$ the fibre of $F$ at $x$ and by $P := P(F_x^*)$ the projective space of lines in $F_x^*$. For every nonzero element $\phi \in P(F_x^*)$, we have a nonzero homomorphism $\phi : F \to k_x$ giving an exact sequence

\[(3.1) 0 \to E \to F \to k_x \to 0.\]

Since $x$ is a nonsingular point of $Y$ and $F$ is locally free, it follows that $E$ is a locally free sheaf of rank $n$ and determinant $L$. Hence we have the following exact sequence on $Y \times P(F_x^*)$ with $E$ a vector bundle.

\[(3.2) 0 \to E \to p_1^*F \to O_{x \times P}(1) \to 0.\]

As in [13, Lemma 5.5], one can see that $(0,1)$-stability of $F$ implies that $E \in U_Y'(n,d)$. By the universal property of $U_Y(n,d,L)$, we have a morphism

$$
\psi_{F,x} : P(F_x^*) \to U_Y(n,d,L),
$$

such that, for some integer $j$, we have an isomorphism

\[(3.3) E \cong (id \times \psi_{F,x})^*U_L \otimes p_2^*(O_P(-j)).\]

**Lemma 3.1.** $\psi_{F,x}$ is an isomorphism onto its image.

**Proof.** This can be proved exactly as [13, Lemma 5.9] ([13, lemma 5.6] and [8, Lemma 3] for injectivity). We note that $\psi_{F,x}$ maps into $U_Y'(n,d)$. $\square$

Let $E_x = E|_{x \times P}$. There is an exact sequence

\[(3.4) 0 \to O_P(1) \to E_x \to \Omega^1_P(1) \to 0,\]

[9, Lemma 3.1].

**Lemma 3.2.** ([9, Lemma 3.2]) Let $W \subset E_x$ be a non-zero coherent subsheaf of $E_x$ such that:

1. the quotient $E_x/W$ is torsionfree, and
2. $d(W)/r(W) \geq d(E_x)/r(E_x)$.

Then $W$ contains the line subbundle $O_P(1)$ of $E_x$.

3.2. Stability of $U_{L,x}$.

Let $H$ be an ample line bundle on a variety $Z$ of dimension $m \geq 2$ and $F$ a torsionfree sheaf on $Z$. Then the degree of $F$ with respect to $H$, denoted by $d(F)$, is the degree of the restriction of $F$ to a general complete intersection curve on $Z$ rationally equivalent to $H^{m-1}$. We recall that if the singular set of $Z$ has codimension at least 2, then the general complete intersection curve can be chosen to lie in the set of nonsingular points of $Z$. 
If \( Z \) is projective, then 
\[
d(F) = (c_1(F).H^{m-1})[Z]
\]
where \([Z]\) denotes the fundamental class of \( Z \).

Our aim in this section is to prove the following theorem.

**Theorem 3.3.** Let \( x \in Y \) be a nonsingular closed point.

1. The restriction \( \mathcal{U}_{L,x} \) of the Poincare sheaf to \( x \times U_Y(n,d,L) \) is stable with respect to the polarisation \( \theta_L \).
2. The restriction of \( \mathcal{U}_{L,x} \) to \( x \times U'_Y(n,d,L) \) is stable with respect to any polarisation.

We recall some constructions and results from [6].

### 3.3. Each point of the projective bundle

\[
P_x := \mathbb{P}(\mathcal{U}_{L,x})
\]
corresponds to a pair \((E,\ell)\) where \( E \in U_Y(n,d,L) \) and \( \ell \in \mathbb{P}(E_x) = \mathbb{P}(\text{Ext}^1(k_x,E)) \) and hence determines an exact sequence of type \((3.1)\) and thus a torsionfree sheaf \( F \). Let \( H_x \) be the open subset of \( P_x \) defined by

\[
H_x := \{(E,\ell) \in P_x \mid F \in U_Y(n,d+1,L(x)) \text{ is } (0,1)-\text{stable}\}.
\]

We have maps

\[
\begin{array}{ccc}
H_x & \rightarrow & V \\
\downarrow & & \downarrow \psi \\
U_Y(n,d,L) & & U_Y(n,d+1,L(x))
\end{array}
\]

Here \( p \) is the projection \( p : H_x \to U_Y(n,d,L) \) defined by \( p(E,\ell) = E \). We have a morphism \( q : H_x \to U_Y(n,d+1,L(x)) \) defined by \( (E,\ell) \mapsto F \) with image the nonempty open subset \( V \subset U_Y(n,d+1,L(x)) \) of \((0,1)\)-stable vector bundles (Proposition \(2.2\)). The fibre of \( q \) over \( F \in V \) is \( \mathbb{P}(F_x^*) \). The restriction of the projection map \( p \) to the fibre \( \mathbb{P}(F_x^*) \) is precisely \( \psi_{F,x} \), hence the fibre \( P(F_x^*) \) maps isomorphically onto its image \( P(F,x) := \psi_{F,x}(\mathbb{P}(F_x^*)) \) (Lemma \(3.1\)).

**Lemma 3.4.** Let \( \mathcal{U}' \subset \mathcal{U}_{L,x} \) be a subsheaf of rank \( r \) with \( 0 < r < r(\mathcal{U}_{L,x}) \). Let \( x_1, \ldots, x_p \in Y \) be nonsingular points.

1. The singular set \( S \) of \( \mathcal{U}' \) has codimension at least 2 in \( U_Y(n,d,L) \).
2. There is a nonempty open set \( U \subset U_Y(n,d,L) \) such that for \( E \in U \),
   a) \( \mathcal{U}' \) is locally free at \( E \),
   b) the homomorphism of fibres \( \mathcal{U}'_E \to (\mathcal{U}_{L,x})_E \) is injective,
   c) for all \( x_i \) and for the generic line \( \ell \) in \( E_{x_i} \), the vector bundle \( F \) associated to \( (E,\ell) \) is \((0,1)\)-stable and \( \mathcal{U}' \) is locally free at every point of \( P(F,x_i) \) outside a subvariety of codimension at least 2.

**Proof.** This can be proved as in Lemma [6 Lemma 4.2] using [6 Proposition 2.1]. \( \square \)
Proof of Theorem 3.3

Proof. (1) Let \( \mathcal{U}' \subset \mathcal{U}_{L,x} \) be a torsionfree subsheaf of rank \( r \) with \( 0 < r < r(\mathcal{U}_{L,x}) \) and with a torsionfree quotient. Since \( U_Y(n, d, L) - U_Y'(n, d, L) \) is of codimension at least 3 by [6] Proposition 2.1 (or [7] Theorem 1.3) for codimension at least 2) and \( U_Y'(n, d, L) \) is nonsingular, there is an open subset \( Z' \subset U_Y'(n, d, L) \), with \( S := U_Y(n, d, L) - Z' \) of codimension at least 2 in \( U_Y(n, d, L) \), such that \( \mathcal{U}' \) is locally free on \( Z' \) and \( \mathcal{U}'_E \to (\mathcal{U}_{L,x})_E \) is injection for all \( E \in Z' \) (Lemma 3.4). Then \( p^{-1}Z' \subset H_x \) is a Zariski open subset with \( \operatorname{codim} H_x - p^{-1}Z' \geq 2 \) so that \( \dim S \leq \dim U_Y(n, d, L) + n - 3 = \dim U_Y(n, d + 1, L(x)) + n - 3 \). The image \( V \) of \( q \) has dimension equal to the dimension of \( U_Y(n, d + 1, L(x)) \), hence the general fibre of \( q \) intersects \( S \) in a closed subset of dimension at most \( n - 3 \). Therefore for a general \((0, 1)\)-stable vector bundle \( F \in U_{L(x)}'(n, d + 1) \), the complement of \( \psi_{F,x}^{-1}(Z') \) has codimension at least 2. By generality of \( F \), we can assume that \( F \) is defined by a pair \((E, \ell)\) and \( \ell \) is not in the fibre \( P(\mathcal{U}')_E \) i.e. the line determined by \( \ell \) is not contained in the fibre of \( \mathcal{U}' \) at \( E \) (as in [9] Proof of theorem 3.6).

For a sheaf \( N \) on \( P \), define \( N(-j) := N \otimes \mathcal{O}_P(-j) \). By equation (3.3), one has
\[
\psi_{F,x}^*(\mathcal{U}_{L,x})(-j) \cong \mathcal{E}_x,
\]
where \( \mathcal{E}_x \) is as in (3.4). Hence
\[
\psi_{F,x}^*(\mathcal{U}')(-j)|_{\psi_{F,x}^{-1}Z'} \subset \mathcal{E}_x|_{\psi_{F,x}^{-1}Z'}.
\]
The condition that the line determined by \( \ell \) is not contained in the fibre of \( \mathcal{U}' \) at \( E \) implies that \( \psi_{F,x}^*(\mathcal{U}')(-j)|_{\psi_{F,x}^{-1}Z'} \) does not contain the line subbundle \( \mathcal{O}_P(1)|_{\psi_{F,x}^{-1}Z'} \) of \( \mathcal{E}_x \).

For a general \( F \), the complement of \( \psi_{F,x}^{-1}(Z') \) in \( P \) has codimension at least 2, hence \( \psi_{F,x}^*(\mathcal{U}')(-j)|_{\psi_{F,x}^{-1}Z'} \) can be extended to \( P \). By Lemma 3.2 (applied to an extension of \( \psi_{F,x}^*(\mathcal{U}')(-j)|_{\psi_{F,x}^{-1}Z'} \) to \( P \), we have
\[
\mu(\psi_{F,x}^*(\mathcal{U}')(-j)) < \mu(\mathcal{E}_x) = \mu(\psi_{F,x}^*(\mathcal{U}_{L,x})(-j))
\]
so that
\[
\mu(\psi_{F,x}^*(\mathcal{U}')) < \mu(\psi_{F,x}^*(\mathcal{U}_{L,x})). \tag{3.5}
\]
One has \( c_1(\mathcal{U}'|_{Z'}) = c_1(\mathcal{U}'|_{U_Y'(n, d, L)}) = \lambda_{U'}c_1(\theta_L|_{U_Y'(n, d, L)}) \) for some scalar \( \lambda_{U'} \). Since \( d(\mathcal{U}') = d(\mathcal{U}'|_{U_Y'(n, d, L)}) \), this implies that \( d(\mathcal{U}') = \lambda_{U'}d(\theta_L) \), where \( d(\theta_L) = \theta_L^{(n^2-1)(g-1)}[U_Y(n, d, L)] \). Similarly, \( d(\mathcal{U}_{L,x}) = \lambda_{U_{L,x}}d(\theta_L) \). We have \( d(\psi_{F,x}^*(\mathcal{U}')) = \lambda_{U'}d(\psi_{F,x}^*(\theta_L)) \). Since \( \theta_L \) is an ample line bundle, by Lemma 3.1, \( \psi_{F,x}^*(\theta_L) \) is an ample line bundle on \( P \). As \( \operatorname{Pic} P = \mathbb{Z} \) is generated by \( \mathcal{O}_P(1) \), we have \( \psi_{F,x}^*(\theta_L) = \mathcal{O}_P(\delta) \) for some \( \delta > 0 \). Hence one gets \( d(\psi_{F,x}^*(\mathcal{U}')) = \lambda_{U'}\delta \). Similarly, \( d(\psi_{F,x}^*(\mathcal{U}_{L,x})) = \lambda_{U_{L,x}}\delta \). Thus the equation (3.3) is equivalent to
\[
\frac{\lambda_{U'}r(\mathcal{U}')} {r(\mathcal{U}'|_{Z'})} < \frac{\lambda_{U_{L,x}}r(\mathcal{U}_{L,x})} {r(\mathcal{U}_{L,x})}.
\]
Hence
\[
\frac{\lambda_{U'}d(\theta_L) r(\mathcal{U}')} {r(\mathcal{U}'|_{Z'})} < \frac{\lambda_{U_{L,x}}d(\theta_L) r(\mathcal{U}_{L,x})} {r(\mathcal{U}_{L,x})}.
\]
Thus \( \mu(U') < \mu(U_{L,x}) \), proving the stability of \( U_{L,x} \).

(2) Note that \( \psi_{F,x}(P) \subset U'_Y(n,d,L) \). Hence Part (2) can be proved similarly as Part (1) using the facts that the singular set \( U_Y(n,d,L) \) is of codimension at least 2 in \( U_Y(n,d,L) \) and Pic \( U'_Y(n,d,L) \cong \mathbb{Z} \), with generator \( \theta_L \).

\[ \square \]

3.4. Stability of the Poincaré bundle on \( Y \times U_Y(n,d,L) \).

The moduli space \( U_Y(n,d,L) \) is unirational [7, Lemma 3.5(1)]. Hence there is no non-constant map from \( U_Y(n,d,L) \) to Pic \( Y \). Using the see-saw theorem [11, Corollary 6, p. 54], one sees that every line bundle on \( Y \times U_Y(n,d,L) \) is the tensor product of the pull back of a line bundle on \( U_Y(n,d,L) \) and the pull back of a line bundle on \( Y \). Hence we take the polarisation on \( Y \times U_Y(n,d,L) \) represented by a divisor of the form \( a \alpha + b \theta_L \), where \( \alpha \) is a fixed ample Cartier divisor on \( Y \) and \( a, b \) are positive integers.

**Theorem 3.5.** The Poincaré bundle \( U_L \) on \( Y \times U_Y(n,d,L) \) is stable with respect to the polarisation \( a \alpha + b \theta_L \) with \( a, b \) positive integers.

**Proof.** Let \( Q \) be a torsionfree quotient sheaf of \( U_L \). It suffices to check that

\[ r(U_L)d(Q) - r(Q)d(U_L) > 0. \]

Let \( E \in U'_Y(n,d,L) \) be a general element. Let \( Q_Y = Q|_{Y \times E} \). Then \( Q_Y \) is a quotient of the locally free sheaf \( E \). The quotient \( Q_Y \) may not be torsionfree, let \( Q'_Y := Q_Y/tor \) be the quotient of \( Q_Y \) by its torsion subsheaf. Let \( Y' \) be the set of smooth points of \( Y \).

Since \( Y' \times U'_Y(n,d,L) \) is smooth, there is an open subset \( Z' \subset Y' \times U'_Y(n,d,L) \) whose complement in \( Y' \times U'_Y(n,d,L) \) is of codimension at least 2 such that \( Q \) is locally free over \( Z' \). For a general \( E \in U'_Y(n,d,L) \), \( Z' \) contains the curve \( Y' \times E \). This implies that \( Q_Y \) has torsion at most at nodes of \( Y \) and \( r(Q) = r(Q_Y) = r(Q'_Y) \). Since \( E \) is stable, one has \( d(Q') \geq d(Q'_Y) \) on \( Y \).

Since \( E \) is stable, one has \( r(E)d(Q') - r(Q'_Y)d(E) > 0 \). This implies that

\[ r(U_L)d(Q) - r(Q)d(U_L) > 0. \]

(3.6)

Since the complement of \( Z' \) in \( Y' \times U'_Y(n,d,L) \) is of codimension at least 2, for a general \( x \in Y' \), \( Z' \cap (x \times U'_Y(n,d,L)) \) has complement of codimension at least 2 \( x \times U'_Y(n,d,L) \).

Since \( U_Y(n,d,L) - U'_Y(n,d,L) \) has codimension at least 3 ([8, Proposition 2.1]), it follows that the open subset \( Z' \cap (x \times U_Y(n,d,L)) \) has complement of codimension at least 2 in \( x \times U_Y(n,d,L) \). Since \( x \) is a non-singular point, the sheaf \( Q_x := Q_{x \times U_Y(n,d,L)} \) is locally free on \( Z' \cap (x \times U_Y(n,d,L)) \) and \( d(Q_x) = d(Q_x|_{Z \cap (x \times U'_Y(n,d,L))}) \). For a non-singular point, \( U_{L,x} = U|_{x \times U_Y(n,d,L)} \) is a stable vector bundle by Theorem 3.3. Hence,

\[ r(U_{L,x})d(Q_x) - r(Q_x)d(U_{L,x}) > 0, \]

i.e.

\[ r(U_L)d(Q) - r(Q)d(U_{L,x}) > 0. \]
Let \( m := \dim U_Y(n, d, L) \). One has
\[
d(U_L) = (c_1(U_L) \cdot (b \theta_L + a \alpha)^m)[Y \times U_Y(n, d, L)]
\]
\[
= c_1(U_L) \cdot (m a b^{m-1} \alpha \theta_L^{m-1} + b^m \theta_L^m)[Y \times U_Y(n, d, L)]
\]
\[
= m a b^{m-1} \alpha [Y] d(U_{L,x}) + b^m d(E) \theta_L^m[U_Y(n, d, L)]
\]
Similarly,
\[
d(Q) = m a b^{m-1} \alpha [Y] d(Q_x) + b^m d(Q_Y) \theta_L^m[U_Y(n, d, L)]
\]
Then
\[
r(U_L)d(Q) - r(Q)d(U_L) = m a b^{m-1} \alpha [Y] (r(U_L)d(Q_x) - r(Q)d(U_{L,x})) + b^m \theta_L^m[U_Y(n, d, L)](r(U_L)d(Q_Y) - r(Q)d(E))
\]
> 0,
by (3.6) and (3.7).

This completes the proof of the theorem. \( \square \)

**Corollary 3.6.** The Poincaré bundle \( U_L' \) on \( Y \times U_Y(n, d, L) \) is stable with respect to any polarisation.

**Proof.** This follows as in Theorem 3.5 using Theorem 3.3(2) and the facts that the Picard group of \( U_Y(n, d, L) \) is isomorphic to \( \mathbb{Z} \) and it is generated by \( \theta_L \) [3, Theorem I]. \( \square \)

**References**

[1] Arusha C., Bhosle Usha N., Singh Sanjay Kumar: Projective Poincaré and Picard bundles for moduli spaces of vector bundles on a nodal curve, to appear in Bull. Sci. Math.
[2] Balaji V., Brambila-Paz L., Newstead, P.E.: Stability of the Poincaré bundle, Math. Nachr., 188, 5 - 15 (1997).
[3] Bhosle Usha N.: Picard groups of moduli of vector bundles, Math. Ann. 314, 245 - 263 (1999).
[4] Bhosle Usha N.: Picard groups of the moduli spaces of semistable sheaves I, Proc. Indian Acad. Sci. (Math. Sci.) 114 (2), 107 - 122 (2004).
[5] Bhosle Usha N.: Moduli of torsionfree sheaves of rank 2 and odd degree on a nodal hyperelliptic curve. Beitr Algebra Geom (Contributions to algebra and geometry), 54 (1) (2013) 155-179. DOI 10.1007/s13366-012-0107-5.
[6] Bhosle Usha N.: Picard bundle on the fixed determinant moduli space. Preprint 2018.
[7] Bhosle Usha N., Sanjay Kumar Singh: Fourier-Mukai transform on a compactified Jacobian. Published online (20 June 2018), DOI: 10/1093/imrn/rny136/5040022.
[8] Biswas I., Brambila-Paz L., Gomez T.L., Newstead P.E.: Stability of the Picard bundle, Bull. London Math. Soc., 34, 561 - 568, (2002).
[9] Biswas I., Brambila-Paz L., Newstead P.E.: Stability of projective Poincaré and Picard bundles, Bull. London Math. Soc., 41, 458 - 472 (2009).
[10] Hartshorne R., Algebraic Geometry, GTM 52, Springer, Berlin, (1977).
[11] Mumford D.: Abelian Varieties, Tata Institute of Fundamental Research Studies in Mathematics 5, Oxford University Press, London (1970).
[12] Newstead P.E.: Introduction to moduli problems and orbit spaces, Springer-Verlag, (1978).
[13] Narasimhan M. S., Ramanan S.: Geometry of Hecke cycles I, C.P. Ramanujan - a tribute, Studies in Math. 8, Tata Inst. Fund. Res., (1978).
[14] Rego C. J. : Compactification of the space of vector bundles on a singular curve, Comm. Math. Helv. 57, 226 - 236, (1982).
[15] Seshadri C. S.: Fibrés vectoriels sur les courbes algébriques. Asterisque 96 (1982).
[16] Sun X.: Degeneration of moduli spaces and generalised theta functions, J. Algebraic Geom., 9 , 459- 527, (2000).