FREE BRAIDED NONASSOCIATIVE HOPF ALGEBRAS AND SABININ $\tau$-ALGEBRAS

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Abstract. Let $V$ be a linear space over a field $k$ with a braiding $\tau : V \otimes V \to V \otimes V$. We prove that the braiding $\tau$ has a unique extension on the free nonassociative algebra $k\{V\}$ freely generated by $V$ so that $k\{V\}$ is a braided algebra. Moreover, we prove that the free braided algebra $k\{V\}$ has a natural structure of a braided nonassociative Hopf algebra such that every element of the space of generators $V$ is primitive. In the case of involutive braidings, $\tau^2 = \text{id}$, we describe braided analogues of Shestakov-Umirbaev operations and prove that these operations are primitive operations. We introduce a braided version of Sabinin algebras and prove that the set of all primitive elements of a nonassociative $\tau$-algebra is a Sabinin $\tau$-algebra.

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1. Introduction

Lie theory for nonassociative products appeared as a subject of its own in the works of Malcev [4] who constructed the tangent structures corresponding to Moufang loops.

Many of the well-known generalizations of Lie algebras involve only one or two operations: Malcev algebras have one binary bracket, Lie triple systems have one ternary bracket, Bol and Lie-Yamaguti algebras have one binary and one ternary bracket, Akivis algebras have two operations — an antisymmetric binary bracket and a ternary bracket (related to commutator and associator) — with only one identity that relates the two operations and generalizes the Jacobi identity.

For some time Akivis algebras were considered as a possible analogue of Lie algebras for nonassociative product. Nevertheless recent results of Shestakov and Umirbaev [13] demonstrate that there is an infinite set of independent operations, primitive polynomials of a free nonassociative algebra, that must be taken into consideration.

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An important advance in the Lie theory of nonassociative products was the introduction of a hiperalgebra by Mikheev and Sabinin, now called a Sabinin algebra, which is the most general form of the tangent structure for loops, see [6, 11, 12]. Lie, Malcev, Bol, Lie-Yamaguti algebras, and Lie triple systems are specific instances of Sabinin algebras. Sabinin algebras have an infinite set of independent operations. There are three different natural constructions of operations in a Sabinin algebra. Two of those constructions were devised by Sabinin and Mikheev. The third set of operations is precisely the Shestakov-Umirbaev operations.

A construction of universal enveloping algebras, which is very similar in their properties to usual cocommutative Hopf algebras, can be carried out for Bol algebras [9] and, more generally, for all Sabinin algebras [10]. The role of the nonassociative Hopf algebras in the fundamental questions of Lie theory such as integration was clarified in [7, 8], where the authors describe the current understanding of the subject in view of the recent works, many of which use nonassociative Hopf algebras as the main tool.

The aim of the present paper is a developing of the quantum aspects of the nonassociative Lie theory, such as the structure of primitive nonassociative polynomials whose variables form a braided space. To this end, one should consider a free nonassociative algebra as a braided (nonassociative) Hopf algebra. Recall that, in his beautiful paper [10], J.M. Pérez-Izquierdo elaborated a nonassociative analogue of the Hopf algebra concept, he called this an $H$-bialgebra. Instead of the antipode he had to consider two additional bilinear operations, the left division and the right division:

\[
\backslash : H \times H \to H, \quad / : H \times H \to H, \\
(x, y) \mapsto x \backslash y, \quad (x, y) \mapsto x/y,
\]

that satisfy the following “antipode identities”

\[
\sum x(1) \backslash (x(2)y) = \varepsilon(x)y = \sum x(1)(x(2) \backslash y), \\
(y \sum x(1))/x(2) = \varepsilon(x)y = \sum (y/x(1))x(2).
\]

In the case of associative Hopf algebras, the divisions reduce to $x \backslash y = S(x)y$ and $x/y = xS(y)$, where $S$ is the antipode.

The method we used in this paper is based on the analysis of the so-called local action [1] of the braid monoids similar to that given in [3, Section 6.2] for the construction of the free braided associative algebra. In particular, if the ground braiding is involutive ($τ^2 = \text{id}$), then the local action of the braid monoid on the $n$-fold tensor product $V^\otimes n$ reduces to the action of the symmetric group. This allows us to use the following general principle, originally appeared in [2, p. 210] for generalized Gurevich Lie algebras:

If a theorem is valid for ordinary free nonassociative algebras and its statement in a field of braided algebras may be interpreted as a property of the group algebra $k[S_n]$ under the local action, then this theorem is valid for free braided nonassociative algebras as well.

Indeed, if the ground braiding is involutive, then the local action of the braid monoid $B_n$ reduces to the action of the symmetric group $S_n$. The interpretation required in the principle reduces the theorem to a system of equations $F_\lambda(τ_1, τ_2, \ldots, τ_{n-1}) = 0$, where $F_\lambda$
are (associative) polynomials in $\tau_1, \tau_2, \ldots, \tau_{n-1}$. Since the theorem is valid for ordinary free nonassociative algebra, it follows that relations $F_\lambda(\theta_1, \theta_2, \ldots, \theta_{n-1}) = 0$ are valid for the ordinary flip $\theta : x \otimes y \rightarrow y \otimes x$. This implies that $F_\lambda(t_1, t_2, \ldots, t_{n-1}) = 0$ are relations in the group algebra $k[S_n]$. As the local action of $B_n$ reduces to the action of the symmetric group $S_n$, we obtain the system of required equalities $F_\lambda(\tau_1, \tau_2, \ldots, \tau_{n-1}) = 0$.

For example, in this way Shestakov-Umirbaev theorems on primitive operations can be generalized to the braided nonassociative algebra just by suitable interpretation of their statements. We stress, that, nevertheless, the same theorem may have a number of interpretations in terms of relations, consequently, in general, a theorem may have a number of generalizations, not equivalent each other in the field of braided algebra.

In Section 2, we accumulate necessary statements on the calculations inside of the braid monoid. Then, in the third and fourth sections, we interpret the desired properties of a braided nonassociative algebra in terms of relations in braid monoids under the local action. In Section 5, we prove that there exists a unique braided coassociative Hopf algebra structure on the free nonassociative algebra induced by the braiding of the generators space.

In Section 6, we consider in more detail symmetries (involutive braidings). We propose a braided version for the Shestakov-Umirbaev operations and show that they remain primitive polynomials of the braided free nonassociative Hopf algebras ($H$-bialgebra in sense of P. Izquierdo). This allows us to define a symmetric braided version of the Sabinin algebras.

Recall that in [14], M. Takeuchi proposed a noncategorical framework in which braided bialgebras are formulated as algebras and coalgebras with a Yang-Baxter operator (alongside compatibility conditions). This approach is actually more general and convenient than the other approaches (the ground braided space need not be embedded in a braided monoidal category), that is why we apply it here. Nevertheless, an accurate realization of the construction of nonassociative Hopf algebras and (in case of symmety) Sabinin algebras in terms of braided monoidal categories remains an interesting task. All over the paper we suppose the ground field $k$ to be of characteristic zero.

2. Braid relations

The braid monoid $B_n$ is an associative monoid generated by braids $s_1, s_2, \ldots, s_{n-1}$ subject to the relations

\begin{equation}
    s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}, \quad s_i s_j = s_j s_i, \quad 1 \leq k < n - 1, \quad |i - j| > 1.
\end{equation}

Put $[k; k] = 1$ and

\begin{equation}
    [m; k] = s_{k-1} s_{k-2} \cdots s_{m+1} s_m, \quad [k; m] = s_m s_{m+1} \cdots s_{k-2} s_{k-1}, \quad m < k.
\end{equation}

The following relations can be found in [3, p. 28]:

\begin{equation}
    [m; k][r; s] = [r; s][m; k], \quad r \leq s < m \leq k,
\end{equation}

\begin{equation}
    [m; k][r; m] = [r; k], \quad r \leq m \leq k,
\end{equation}

\begin{equation}
    [m - 1; r - 1][k; t] = [k; t][m; r], \quad k \leq m \leq r \leq t.
\end{equation}
For any $1 \leq k \leq r < n$ put
\[
\nu_{r}^{k,n} = [k; r + 1][k + 1; r + 2] \cdots [k + n - r - 1; n].
\]

It is proved in [3] that
\[
\nu_{r}^{k,n} = [n; r][n - 1; r - 1] \cdots [n - r + k; k].
\]

Equalities (2.6) and (2.7) immediately imply that
\[
\nu_{r}^{t,m} \nu_{m}^{m-r+t,n} = \nu_{r}^{t,n} = \nu_{r}^{s+1,n} \nu_{s}^{t,n-r+s}.
\]

Let $S_n$ be the symmetric group on the set of symbols \{1, 2, \ldots, n\}. A permutation $\pi \in S_n$ is called an $r$-shuffle if
\[
\pi(1) < \pi(2) < \ldots < \pi(r), \quad \pi(r + 1) < \pi(r + 2) < \ldots < \pi(n).
\]

Denote by $\text{Sh}_n^r$ the set of all $r$-shuffles in $S_n$.

If $\sigma \in \text{Sh}_n^r$ and $\delta \in \text{Sh}_n^p$, then we define $\sigma \ast \delta \in \text{Sh}_{n+t}^{r+p}$ by
\[
(\sigma \ast \delta)(i) = \sigma(i), \quad 1 \leq i \leq r, \quad (\sigma \ast \delta)(r + i) = s + \delta(i), \quad 1 \leq i \leq p.
\]

Notice that $\sigma \ast \delta$ is well-defined since every element of $\text{Sh}_n^r$ is well-defined by the values on \{1, \ldots, r\}.

If $\pi \in \text{Sh}_n^r$, then put
\[
[\pi] = [1; \pi(1)][2; \pi(2)] \cdots [r; \pi(r)],
\]
where $[\pi] = 1$ if $r = 0$.

Let $f = f(s_1, \ldots, s_{n-1})$ be an arbitrary element of the monoid algebra of $B_n$. Then put
\[
f(\iota) = f(s_{t+1}, \ldots, s_{t+n-1}).
\]

Notice that $f(\iota)$ belongs to the monoid algebra of $B_{t+n}$. For example, if $\pi \in \text{Sh}_n^r$, then
\[
[\pi](\iota) = [t + 1; t + \pi(1)][t + 2; t + \pi(2)] \cdots [t + r; t + \pi(r)] \in B_{n+t}.
\]

**Lemma 2.1.** If $\pi \in \text{Sh}_n^r$, then
\[
[\pi] \nu_{r}^{1,n} = \nu_{r}^{1,n}[\pi](n-r).
\]

**Proof.** Consider $[i; \pi(i)]$, where $1 \leq i \leq s$. Then $1 \leq i < i + 1 \leq \pi(i) + 1 \leq r + 1$.

Applying (2.5), we get
\[
[i; \pi(i)] \nu_{r}^{1,n} = [i; \pi(i)][1; r + 1][2; r + 2] \cdots [n - r; n]
= [i; \pi(i)][1; r + 1][2; r + 2] \cdots [n - r; n][i + n - r; \pi(i) + n - r]
= \nu_{r}^{1,n}[i + n - r; \pi(i) + n - r].
\]

Hence,
\[
[\pi] \nu_{r}^{1,n} = [1; \pi(1)] \cdots [s; \pi(s)] \nu_{r}^{1,n}
= \nu_{r}^{1,n}[1 + n - r; \pi(1) + n - r] \cdots [s + n - r; \pi(s) + n - r] = \nu_{r}^{1,n}[\pi](n-r). \quad \square
\]

**Lemma 2.2.** Let $\sigma \in \text{Sh}_s^r$ and $\delta \in \text{Sh}_t^p$. Then
\[
[s][\delta](\iota) \cdot \nu_{s}^{r+1,p+s} = [\sigma \ast \delta].
\]
Applying (2.3), we can write
\[ [\delta]_{(s)} \nu^{r+1,p+s}_s = [s + 1; s + \delta(1)] \cdots [s + p; s + \delta(p)][r + 1; s + 1] \cdots [r + p; s + p]. \]
Applying (2.4), we get
\[ [\delta]_{(s)} \nu^{r+1,p+s}_s = ([s + 1; s + \delta(1)][r + 1; s + 1]) \cdots ([s + p; s + \delta(p)][r + p; s + p]). \]
Moreover, applying (2.4), we get
\[ [\delta]_{(s)} \nu^{r+1,p+s}_s = [r + 1; s + \delta(1)] \cdots [r + p; s + \delta(p)]. \]

Consequently,
\[ [\sigma][\delta]_{(s)} \cdot \nu^{r+1,p+s}_s = [1; \sigma(1)] \cdots [r; \sigma(r)][r + 1; s + \delta(1)] \cdots [r + p; s + \delta(p)] = [\sigma \ast \delta] \]
by the definition (2.9) of \( \sigma \ast \delta \).

A linear space \( V \) over a field \( k \) is called a braided space if there is a fixed linear mapping \( \tau : V \otimes V \rightarrow V \otimes V \) (in general not necessarily invertible) that satisfies the braid relation:
\[ (\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) = (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau). \]

**Example 1.** If \( x_1, x_2, \ldots, x_n \) is a basis of a linear space \( V \), then for arbitrary parameters \( q_{is} \in k \), \( 1 \leq i, s \leq n \), the linear mapping defined by
\[ \tau : x_i \otimes x_s \mapsto q_{is} \cdot x_s \otimes x_i \]
is a braiding and is called a diagonal braiding.

Let \( V \) be a linear space with a braiding \( \tau : V \otimes V \rightarrow V \otimes V \). Consider the linear mappings
\[ \tau_i = \text{id} \otimes (i-1) \otimes \tau \otimes \text{id} \otimes (n-i-1) : V^\otimes n \rightarrow V^\otimes n, \quad 1 \leq i < n. \]
Due to (2.12), the mappings \( \tau_i \) satisfy all defining relations of the braid monoid (2.1):
\[ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad 1 \leq i < n - 1; \quad \tau_i \tau_j = \tau_j \tau_i, \quad |i - j| > 1. \]
Therefore, \( u \cdot s_i = u \tau_i \) is a well-defined action of the braid monoid \( B_n \) on \( V^\otimes n \). Following D. Gurevich \( \square \) we call this action a local action.

It is well-known that the symmetric group \( S_n \) as a monoid is defined by the generators \( t_i = (i, i + 1) \) (transpositions), \( 1 \leq i < n \), and the relations
\[ t_i^2 = 1, \quad t_k t_{k+1} t_k = t_{k+1} t_k t_{k+1}, \quad t_i t_j = t_j t_i, \quad |i - j| > 1. \]
A braiding \( \tau : V \otimes V \rightarrow V \otimes V \) is called involutive if \( \tau^2 = \text{id} \). In this case \( \tau_i^2 = \text{id} \) and the local action of \( B_n \) on \( V^\otimes n \) induces a local action of \( S_n \) on \( V^\otimes n \).

The best known example of an involutive braiding is the ordinary flip \( \theta \) defined by \( \theta(x \otimes y) = y \otimes x \) for all \( x, y \in V \).

**Remark 2.1.** If a braiding \( \tau \) is involutive then every relation
\[ F(\tau_1, \tau_2, \ldots, \tau_{n-1}) = 0, \]
where \( F \) is an (associative) polynomial, holds if it holds for the ordinary flip \( \tau_i = \theta_i \), \( 1 \leq i < n \).
In fact, in the case of the ordinary flip \( \tau \) the monoid generated by \( \tau_1, \tau_2, \ldots, \tau_{n-1} \) is isomorphic to \( S_n \) and in the case of an arbitrary involutive braiding \( \tau \) this monoid is a homomorphic image of \( S_n \).

All definitions of this section given in the language of \( s_1, \ldots, s_{n-1} \) without any reminders will be applied to the elements \( \tau_1, \ldots, \tau_{n-1} \).

3. Braided algebras and bialgebras

Let \( V \) and \( V' \) be spaces with braidings \( \tau \) and \( \tau' \), respectively. A linear mapping \( \varphi : V \to V' \) is called a homomorphism of braided spaces if it respects the braidings; that is,

\[
\tau(\varphi \otimes \varphi) = (\varphi \otimes \varphi)\tau'.
\]

Let \( (a \otimes b)\tau = \sum b_i \otimes a_i \) and \( (a' \otimes b')\tau = \sum b'_i \otimes a'_i \). In this case the definition of the homomorphism takes the form

\[
\sum \varphi(b_i) \otimes \varphi(a_i) = \sum \varphi(b'_i) \otimes \varphi(a'_i),
\]

or, informally, \( \varphi(a_i) = \varphi(a)_i \).

An algebra \( R \) with a multiplication \( m : R \otimes R \to R \) is called a braided algebra if it is a braided space and

\[
(3.1) \quad (m \otimes \text{id})\tau = \tau_2 \tau_1 (\text{id} \otimes m), \quad (\text{id} \otimes m)\tau = \tau_1 \tau_2 (m \otimes \text{id}).
\]

In these formulas, as above, we use the so-called “exponential notation” for actions of the operators; that is, the operators in a superposition act from the left to the right.

The identity element of an algebra is usually denoted by 1, but sometimes we shall use the unit mapping as well:

\[
\eta : k \to R, \quad \eta(\alpha) = \alpha \cdot 1.
\]

A homomorphism of braided algebras is a linear mapping that is both a homomorphism of algebras and braided spaces.

A coalgebra is a triple \((C, \Delta, \varepsilon)\), where \( C \) is a vector space, \( \Delta : C \to C \otimes C \) (coproduct or comultiplication) and \( \varepsilon : C \to k \) (counit) are linear mappings satisfying

\[
(3.2) \quad \Delta(\varepsilon \otimes \text{id}) = \text{id} = \Delta(\text{id} \otimes \varepsilon).
\]

A coalgebra is coassociative if it satisfies

\[
(3.3) \quad \Delta(\Delta \otimes \text{id}) = \Delta(\text{id} \otimes \Delta).
\]

A coalgebra \((C, \Delta^b, \varepsilon)\) is called braided if it is a braided space and

\[
(3.4) \quad \tau(\varepsilon \otimes \text{id}) = (\text{id} \otimes \varepsilon)\tau, \quad \tau(\text{id} \otimes \varepsilon) = (\varepsilon \otimes \text{id})\tau;
\]

\[
(3.5) \quad \tau(\text{id} \otimes \Delta^b) = (\Delta^b \otimes \text{id})\tau_2 \tau_1, \quad \tau(\Delta^b \otimes \text{id}) = (\text{id} \otimes \Delta^b)\tau_1 \tau_2.
\]

A homomorphism of braided coalgebras \( \varphi : V \to V' \) is a homomorphism of coalgebras that respects the braidings, i.e., \( \varphi \) satisfies the relations

\[
(3.6) \quad \Delta^b(\varphi(a)) = \sum_{(a)} \varphi(a^{(1)}) \otimes \varphi(a^{(2)}), \quad \varepsilon(\varphi(a)) = \varepsilon(a).
\]
A braided bialgebra is a braided algebra and a braided coalgebra \( H \) (with the same braiding) where the coproduct is an algebra homomorphism
\[
\Delta^b : H \to H \otimes H.
\]
Here, \( H \otimes H \) is the ordinary tensor product of spaces with a new multiplication
\[
(a \otimes b)(c \otimes d) = \sum_i (ac_i \otimes b_i d), \quad \text{where} \quad (b \otimes c)\tau = \sum_i c_i \otimes b_i.
\]

A homomorphism of braided bialgebras is a homomorphism of coalgebras and braided algebras.

In 2007 J. Pérez-Izquierdo \cite{10} elaborated the concept of a nonassociative Hopf algebra and he called it an \( H \)-bialgebra. An \( H \)-bialgebra has two additional (left and right) division operations instead of the antipode. Associative \( H \)-bialgebras are exactly associative Hopf algebras. For this reason nonassociative \( H \)-bialgebras are called also nonassociative Hopf algebras \cite{8}.

A braided nonassociative Hopf algebra is a braided coassociative bialgebra \( H \) with two extra bilinear operations, the left and right divisions,
\[
\backslash : H \times H \to H, \quad / : H \times H \to H,
\]
\[
(x, y) \mapsto x \backslash y, \quad (x, y) \mapsto x / y,
\]
such that
\[
\sum x_{(1)} \backslash (x_{(2)} y) = \varepsilon(x)y = \sum x_{(1)} (x_{(2)} \backslash y),
\]
\[
(y \sum x_{(1)}) / x_{(2)} = \varepsilon(x)y = \sum (y / x_{(1)}) x_{(2)}.
\]

A homomorphism of braided nonassociative Hopf algebras is a homomorphism \( \varphi \) of braided bialgebras that satisfies
\[
\varphi(x \backslash y) = \varphi(x) \backslash \varphi(y), \quad \varphi(x / y) = \varphi(x) / \varphi(y).
\]

4. Free braided nonassociative algebra

Let \( V \) be a linear space with a braiding \( \tau : V \otimes V \to V \otimes V \). We fix a linear basis \( X = \{ x_i | i \in I \} \) for \( V \). The set \( X^* \) of all associative words in \( X \) is a linear basis for the free associative algebra \( k \langle X \rangle \). This algebra is isomorphic to the tensor algebra \( T(V) = \bigoplus_{i=0}^{\infty} V^\otimes i \) of the linear space \( V \) with the concatenation product. We identify the words of length \( m \) in \( X \) with the corresponding tensors from \( V^\otimes m \). We set \( V^\otimes 0 = k \cdot 1 \) and \( 1 \otimes v = v \otimes 1 = v \).

The product in the tensor algebra \( T(V) \) will be denoted by \( m' \). We have \( (u \otimes' v)m' = u \otimes v \), where the sign \( \otimes' \) is the same tensor product \( \otimes \) with one additional function that separates a pair of tensors to which the product \( m' \) is applied.

V. Kharchenko proved that the braiding \( \tau \) has a unique extension \( \tau' \) on the free algebra \( k \langle X \rangle \) so that \( k \langle X \rangle \) is a braided algebra \cite[Chapter 6]{3]. For any \( 0 \leq r \leq n \) denote by \( \theta_r \) the linear mapping
\[
\theta_r : V^\otimes n \to V^\otimes r \otimes' V^\otimes (n-r)
\]
defined by
\[(z_1z_2\cdots z_n)\theta_r = z_1z_2\cdots z_r \otimes z_{r+1}\cdots z_n, \quad z_i \in X.\]
The braiding \(\tau'\) is defined in [3] by
\[(4.1) \quad (u \otimes v)\tau' = (u \otimes v)\nu^1_{r,m} \theta_{n-r}, \quad u \in V^\otimes r, \quad v \in V^{\otimes (n-r)}\]
(if \(r = 0\) or \(r = n\), then this definition means that \((1 \otimes v)\tau' = v \otimes 1\) or \((u \otimes 1)\tau' = 1 \otimes u\), respectively).

Moreover, the algebra \(k\langle X \rangle\) has a natural structure of a braided Hopf algebra such that every element of \(X\) is primitive [3, Theorem 6.2].

Denote by \(k\langle X \rangle = k\langle V \rangle\) the free nonassociative algebra over \(k\) freely generated by the set \(X\), where as above \(X\) is a fixed basis of the space \(V\). The product on \(k\langle X \rangle\) will be denoted by
\[m : k\langle X \rangle \otimes k\langle X \rangle \to k\langle X \rangle.\]

Recall that a nonassociative word is a word where the parenthesis are arranged to show how the multiplication applies. Sometimes it is more convenient to variate a designation of the parenthesis, for example instead of \((xy)z\) one may write \(x \cdot y \cdot z\), whereas \(((z(xy))t)v\) takes the form \(\{(z \cdot xy)t\}v\). A right-normed nonassociative word,
\[u = (((...((x_1x_2)x_3)\ldots)x_m),\]
has a simplified notation without parenthesis,
\[u = x_1x_2x_3...x_m.\]
The formal definition of a nonassociative word is given in [15, Chapter 1, Section 1].

The set of all nonassociative words in the alphabet \(X\) forms a linear basis for \(k\langle X \rangle\). Every nonassociative word of length \(m\) has a unique representation \(uR = u \cdot R\), where \(u\) is an associative word of length \(m\) and \(R\) is an arrangement of parenthesis. Because every associative word \(u \in X^*\) of length \(m\) is identified with an element of \(V^\otimes m\), we can consider an arrangement of parenthesis \(R\) as a linear mapping.

\[R : V^\otimes m \to k\langle V \rangle.\]
We call this function a parenthesis function of \(m\) arguments. For example, if \(uR = (x_1x_2)(x_3(x_4x_5))\), then \(R = (\cdots)\cdot(\cdots)\).

Let \(P\) be a free nonassociative monoid of all nonassociative words in one free variable \(y\). Obviously, every arrangement of parenthesis \(R\) uniquely defines a nonassociative word in \(y\), and we may identify by \(R\) with that word. In this way we consider every element of \(P\) as a parenthesis function. Denote by \(P_m\) the set of all parenthesis functions of \(m\) arguments. We can linearly extend the action of \(R \in P_m\) on \(k\langle V \rangle\) by \(V^\otimes n \otimes R = 0\) if \(n \neq m\). Moreover, we can extend the action of parenthesis defined on \(k\langle V \rangle\) to the action of the monoid algebra \(kP\) by linearity. Obviously,
\[k\langle V \rangle \otimes kP \to k\langle V \rangle \quad (a \otimes p) \mapsto a \cdot p\]
is an isomorphism of linear spaces, because an associative word and a parenthesis function uniquely defines a nonassociative word. Below we call the monoid \(P\) the parenthesis monoid and \(kP\) the parenthesis algebra over \(k\). Notice that \(kP\) is a free nonassociative algebra in one free variable \(y\).
A record of the form \( uR \), where \( u \in X^* \) and \( R \in P \), usually means that \( u \) is an associative word of length \( m \) and \( R \in P_m \). Under these conditions, we have

\[
(uR)(vL) = (uv)(RL), \quad u, v \in X^*, R, L \in P.
\]

**Theorem 4.1.** The braiding \( \tau : V \otimes V \to V \otimes V \) has a unique extension on the free nonassociative algebra \( k\{X\} \) so that \( k\{X\} \) is a braided algebra.

**Proof.** Define a linear mapping

\[
\tau^* : k\{X\} \otimes k\{X\} \to k\{X\} \otimes k\{X\}
\]

by

\[
(uR \otimes vL)\tau^* = (u \otimes v')\tau'(L \otimes R),
\]

where \( \tau' \) is the braiding of \( k\{X\} \), \( u, v \in X^* \), and \( R, L \in P \) defined by (4.1). This definition is equivalent to the operator equality

\[
(R \otimes L)\tau^* = \tau'(L \otimes R).
\]

Using this equality several times, we obtain

\[
(R \otimes L \otimes M)\tau^*_1 \tau^*_2 \tau^*_1 = \tau'_1(L \otimes R \otimes M)\tau^*_2 \tau^*_1 = \tau'_1 \tau'_2(L \otimes M \otimes R)\tau^*_1 = \tau'_1 \tau'_2 \tau'_1(M \otimes L \otimes R).
\]

Similarly,

\[
(R \otimes L \otimes M)\tau^*_1 \tau^*_2 \tau^*_1 = \tau'_2 \tau'_1 \tau'_2(M \otimes L \otimes R).
\]

The braid relation (2.12) holds for \( \tau' \), see [3, Theorem 6.2]. Consequently,

\[
(R \otimes L \otimes M)\tau^*_1 \tau^*_2 \tau^*_1 = (R \otimes L \otimes M)\tau^*_2 \tau^*_1 \tau^*_2
\]

and

\[
\tau^*_1 \tau^*_2 \tau^*_1 = \tau^*_2 \tau^*_1 \tau^*_2;
\]

that is, \( \tau^* \) is a braiding.

Recall that \( m' \) denotes the product in the associative algebra \( k\{X\} \) and \( m \) denotes the product in \( k\{X\} \). Applying (4.1), we obtain

\[
(uR \otimes vL \otimes wM)(m \otimes id)\tau^* = (uR \cdot vL \otimes wM)\tau^* = (uv \otimes w)(RL \otimes M)\tau^* = (uv \otimes w)\tau'(M \otimes RL) = (u \otimes v \otimes w)(m' \otimes id)\tau'(M \otimes RL).
\]

Using that \( m' \) and \( \tau' \) satisfy (4.1), we have

\[
(uR \otimes vL \otimes wM)(m \otimes id)\tau^* = (u \otimes v \otimes w)\tau'_2 \tau'_1(id \otimes m')(M \otimes RL).
\]

Applying (1.2) several times, we also obtain

\[
(uR \otimes vL \otimes wM)\tau^*_2 \tau^*_1(id \otimes m) = (u \otimes v \otimes w)(R \otimes L \otimes M)\tau^*_2 \tau^*_1(id \otimes m)
\]

\[
= (u \otimes v \otimes w)\tau'_2 \tau'_1(M \otimes R \otimes L)(id \otimes m) = (u \otimes v \otimes w)\tau'_2 \tau'_1(M \otimes RL).
\]

Consequently,

\[
(uR \otimes vL \otimes wM)(m \otimes id)\tau^* = (uR \otimes vL \otimes wM)\tau^*_2 \tau^*_1(id \otimes m)
\]

and

\[
(m \otimes id)\tau^* = \tau^*_2 \tau^*_1(id \otimes m);
\]

that is, the first of the relations (3.1) holds for \( \tau^* \). One may check the second of the relations (3.1) similarly. So, \( k\{X\} \) is a braided \( \tau^* \)-algebra.
The uniqueness of \( \tau^* \) easily follows from the relations (3.1). In fact, for all \( a, b, c \in k\{X\} \), we have

\[
(a \otimes b \otimes c)(m \otimes \text{id})\tau^* = (ab \otimes c)\tau^*,
\]

and

\[
(a \otimes b \otimes c)\tau^* (\text{id} \otimes m) = (a \otimes b \otimes c)\tau^* (\text{id} \otimes m) = \sum_{i,j} c_{ij} \otimes a_{ij} b_i,
\]

where \( (b \otimes c)\tau^* = \sum_i c_i \otimes b_i \) and \( (a \otimes c_i)\tau^* = \sum_j c_{ij} \otimes a_{ij} \). Consequently, we may write the first of the relations (3.1) in the form

\[
(ab \otimes c)\tau^* = \sum_{i,j} c_{ij} \otimes a_{ij} b_i.
\]

This implies the uniqueness of \((ab \otimes c)\tau^*\) modulo \((b \otimes c)\tau^*\) and \((a \otimes c_i)\tau^*\). The second of the relations (3.1) implies the uniqueness of \((a \otimes bc)\tau^*\) in the same way. \( \Box \)

5. Braided Hopf algebra structure on \( k\{X\} \)

The equalities

(5.1) \[ \Delta^b(x) = x \otimes 1 + 1 \otimes x, \quad x \in X, \]

uniquely define a homomorphism \( \Delta^b : k\{X\} \to k\{X\} \otimes k\{X\} \) because \( k\{X\} \) is freely generated by \( X \). We are going to convert \( k\{X\} \) into a nonassociative Hopf algebra.

Since \( \varepsilon : k\{X\} \to k \) is a homomorphism of algebras, it follows that (3.2) gives \( x \varepsilon(1) + 1 \varepsilon(x) = x \) for all \( x \in X \). Consequently, \( \varepsilon(x) = 0 \) because \( \varepsilon(1) = 1 \). Thus, \( \Delta^b \) uniquely defines the counit (5.2) \[ \varepsilon : k\{X\} \to k, \quad x \mapsto 0, \quad x \in X. \]

Let \( R \in P_n \) be an arbitrary \( n \)-ary parenthesis function and let \( u = x_1 x_2 \ldots x_n \) be an arbitrary associative word of length \( n \). Let \( \alpha = \{ \alpha_1 < \ldots < \alpha_r \} \) be a subset of \( \{1, 2, \ldots, n\} \). Denote by \( ^\alpha(uR) \) the nonassociative word obtained from \( uR \) by substituting 1 instead of all \( x_i, i \neq \alpha_j, 1 \leq j \leq r \). Also \( ^\alpha R \) denotes the parenthesis function corresponding to the arrangement of parenthesis in \( ^\alpha(uR) \).

If \( \pi \in \text{Sh}_n \), then we set

\[ \pi(uR) = (\pi(1) < \ldots < \pi(r))(uR), \quad (uR)^\pi = (\pi(r+1) < \ldots < \pi(n))(uR), \]

and

\[ ^\pi R = \{ \pi(1) < \ldots < \pi(r) \} R, \quad R^\pi = \{ \pi(r+1) < \ldots < \pi(n) \} R. \]

For example, if \( uR = (x_1 x_2)(x_3 x_4)(x_5 x_6) \), then \( R = (\cdots)((\cdots)(\cdots)). \) If

\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \in \text{Sh}_6^2, \]

then

\[ ^\pi R = (12)(45)(36)(21), \quad (uR)^\pi = (13)(24)(56)(65). \]
then \( \pi(uR) = x_2x_6, \pi R = (\cdot), (uR)^\pi = (x_1((x_3x_4)x_5)), \text{ and } R^\pi = (\cdot((\cdot)\cdot)). \) Of course, one may consider an arbitrary nonassociative word instead of \( uR. \) For example, if \( w = ((x_6x_1)x_1)((x_3x_4)x_3), \) then \( \pi w = x_1x_3, \) and \( w^\pi = (x_6x_1)(x_3x_4). \)

**Lemma 5.1.** Let \( R \) be an arbitrary parenthesis function of \( n \) arguments. Then

\[
(5.3) \quad R\Delta^b = \sum_{r=0}^{n} \sum_{\sigma \in Sh^r} [\pi](\tau R \otimes R^\pi).
\]

**Proof.** Put \( u = x_1x_2 \ldots x_n. \) If \( n = 1, \) then \( R = (\cdot) \) and (5.3) is obviously true. If \( n = 2, \) then \( R = (\cdot). \) Using (5.1), we get

\[
u \cdot R\Delta^b = ((x_1x_2))\Delta^b = x_1x_2 \otimes 1 + x_1 \otimes x_2 + (x_1 \otimes x_2) \tau + 1 \otimes x_1x_2.
\]

The right-hand side of (5.3) applied to \( u \) gives

\[
u \cdot (\text{id}(1 \otimes (\cdot)) + [1; 1]((\cdot) \otimes (\cdot)) + [1; 2](\cdot \otimes (\cdot)) + [1; 1][2; 2](\cdot \otimes 1))
\]

\[
= 1 \otimes x_1x_2 + x_1 \otimes x_2 + (x_1 \otimes x_2) \tau + x_1x_2 \otimes 1.
\]

This proves (5.3) for \( n = 2. \)

We prove (5.3) by induction on \( n. \) If \( n \geq 2, \) then \( R \) has a unique representation in the form \( R = LM, \) and, respectively, \( uR = vL \cdot wM \) for some nontrivial associative words \( v \) and \( w. \) Suppose that \( L \) and \( M \) are functions of \( s \) and \( t \) arguments, respectively. We may assume that (5.3) is true for \( L \) and \( M \) since \( s, t < n. \) Consequently,

\[
u \cdot R\Delta^b = \Delta^b(uR) = \Delta^b(vL) \otimes \Delta^b(wM)
\]

\[
= (\sum_{r=0}^{s} \sum_{\sigma \in Sh^r} v[\sigma](\tau L \otimes L^\sigma)) \otimes (\sum_{p=0}^{t} \sum_{\delta \in Sh^p} w[\delta](\delta M \otimes M^\delta)).
\]

Suppose that

\[
v[\sigma] = \sum_i (v_1 \otimes v_{2i}), \quad w[\delta] = \sum_j (w_1 \otimes w_{2j}).
\]

In this case

\[
v[\sigma](\tau L \otimes L^\sigma) = \sum_i (v_1(\tau L) \otimes v_{2i}L^\sigma), \quad w[\delta](\delta M \otimes M^\delta) = \sum_j (w_1(\delta M) \otimes w_{2j}M^\delta).
\]

By the definition of the product in \( k\{X\} \otimes k\{X\}, \) we get

\[
(v_1(\tau L) \otimes v_{2i}L^\sigma) \otimes (w_1(\delta M) \otimes w_{2j}M^\delta) = (v_1(\tau L))(v_{2i}L^\sigma \otimes w_{1j}(\delta M)) \tau^*(w_{2j}M^\delta),
\]

where \( a(b \otimes c)d \) equals the element \( ab \otimes cd. \) Moreover, we have

\[
(v_{2i}L^\sigma \otimes w_{1j}(\delta M))^\tau^* = (v_{2i} \otimes w_{1j})^\tau^*(\delta M \otimes L^\sigma)
\]

by the definition of \( \tau^*. \) Therefore,

\[
(v_1(\tau L) \otimes v_{2i}L^\sigma) \otimes (w_1(\delta M) \otimes w_{2j}M^\delta) = (v_1 \otimes (v_{2i} \otimes w_{1j})) \tau^* \otimes w_{2j})(\tau^*(\delta L)(\delta M \otimes L^\sigma)).
\]

By the definition of \( \tau^*, \) we have

\[
(v_{2i} \otimes w_{1j})^\tau^* = (v_{2i} \otimes w_{1j})v_1^{1,p+s-r} \theta_p.
\]

Then

\[
v_{1i} \otimes (v_{2i} \otimes w_{1j})(\theta^r \otimes w_{2j}) = (v_{1i} \otimes v_{2i} \otimes w_{1j} \otimes w_{2j})v_1^{p+1,p+s} \theta_{r+p}.
\]
and
\[(v_{i_1}(\sigma L) \otimes v_{j_2} L^\sigma) \otimes (w_{i_1j}(\delta M) \otimes w_{j_2} M^\delta)) = (v_{i_1} \otimes v_{i_2} \otimes w_{i_1j} \otimes w_{j_2}) \nu_s^{r+1,p+s}(\sigma L)(\delta M) \otimes L^\sigma M^\delta).\]

Consequently,
\[v[\sigma](\sigma L \otimes L^\sigma) \otimes w[\delta](\delta M \otimes M^\delta) = \sum_{i,j}(v_{i_1}(\sigma L) \otimes v_{i_2} L^\sigma) \otimes (w_{i_1j}(\delta M) \otimes w_{j_2} M^\delta)\]
\[= \sum_{i,j}(v_{i_1} \otimes v_{i_2} \otimes w_{i_1j} \otimes w_{j_2}) \nu_s^{r+1,p+s}(\sigma L)(\delta M) \otimes L^\sigma M^\delta).\]

Furthermore,
\[\sum_{i,j}(v_{i_1} \otimes v_{i_2} \otimes w_{i_1j} \otimes w_{j_2}) = (\sum_i v_{i_1} \otimes v_{i_2}) \otimes (\sum_j w_{i_1j} \otimes w_{j_2})\]
\[= v[\sigma] \otimes w[\delta] = v[1；\sigma(1)] \cdots [r；\sigma(r)] \otimes w[1；\delta(1)] \cdots [p；\delta(p)]\]
\[= (v \otimes w)[1；\sigma(1)] \cdots [r；\sigma(r)] [s + \delta(1)] \cdots [s + p；s + \delta(p)]\]
\[= (v \otimes w)[\sigma][\delta](s),\]

and
\[\sum_{i,j}(v_{i_1} \otimes v_{i_2} \otimes w_{i_1j} \otimes w_{j_2}) \nu_s^{r+1,p+s} = (v \otimes w)[\sigma][\delta](s) \nu_s^{r+1,p+s} = (v \otimes w)[\sigma \ast \delta]\]
by Lemma 2.2. Hence,
\[v[\sigma](\sigma L \otimes L^\sigma) \otimes w[\delta](\delta M \otimes M^\delta) = (v \otimes w)[\sigma \ast \delta](\sigma L)(\delta M) \otimes L^\sigma M^\delta).\]

Consequently,
\[u \cdot R \Delta^b = \Delta^b(uR) = \sum_{r=0}^{s} \sum_{p=0}^{t} \sum_{\sigma \in \text{Sh}_r^s} \sum_{\delta \in \text{Sh}_r^p} (v \otimes w) \cdot [\sigma \ast \delta](\sigma L)(\delta M) \otimes L^\sigma M^\delta).\]

Notice that
\[u = v \otimes w, (\sigma L)(\delta M) = \sigma \ast \delta (LM), L^\sigma M^\delta = (LM)^{\sigma \ast \delta}.\]

Finally,
\[u \cdot R \Delta^b = \sum_{r=0}^{s} \sum_{p=0}^{t} \sum_{\sigma \in \text{Sh}_r^s} \sum_{\delta \in \text{Sh}_r^p} u \cdot [\sigma \ast \delta](\sigma \ast \delta (LM) \otimes (LM)^{\sigma \ast \delta}).\]

Consider an arbitrary shuffle $\pi \in \text{Sh}_n^q$. For any given $s,t$ such that $s + t = n$, we can uniquely define integers $r, p$ and shuffles $\sigma \in \text{Sh}_r^s, \delta \in \text{Sh}_r^p$ such that $\pi = \sigma \ast \delta$ and $q = r + p$. In fact, $r$ is the greatest integer such that $r \leq q$ and $\pi(r) \leq s$, whereas $\sigma$ is uniquely defined by $\sigma(i) = \pi(i), 1 \leq i \leq r$. Then $p = q - r$, and $\delta(i) = \pi(r + i), 1 \leq i \leq p$.

Using this decomposition of shuffles, we can replace $\sum_{r=0}^{s} \sum_{p=0}^{t} \sum_{\sigma \in \text{Sh}_r^s} \sum_{\delta \in \text{Sh}_r^p}$ with $\sum_{r=0}^{s} \sum_{\pi \in \text{Sh}_r^s}$ and $\sigma \ast \delta$ with $\pi$ in the last formula for $u \cdot R \Delta^b$. This gives (5.3). □

**Lemma 5.2.** The coproduct $\Delta^b$ is coassociative.
Proof. We have to check that (3.3) holds. Let \( R \in P \) be an arbitrary parenthesis function of \( n \) arguments. By (5.3), we have

\[
R\Delta^b(\Delta^b \otimes \text{id}) = \sum_{p=0}^{n} \sum_{\pi \in \text{Sh}_p^n} [\pi](\pi R \otimes (\pi R)\pi) (\Delta^b \otimes \text{id})
\]

\[
= \sum_{p=0}^{n} \sum_{\pi \in \text{Sh}_p^n} [\pi](\pi R \Delta^b \otimes \pi R) = \sum_{p=0}^{n} \sum_{\pi \in \text{Sh}_p^n} [\pi](\sum_{r=0}^{p} [\sigma](\sigma (\pi R) \otimes (\pi R)\sigma) \otimes \pi R)
\]

\[
= \sum_{p=0}^{n} \sum_{r=0}^{p} \sum_{\sigma \in \text{Sh}_r} [\pi][\sigma](\sigma (\pi R) \otimes \pi R)\sigma R
\]

Similarly,

\[
R\Delta^b(\text{id} \otimes \Delta^b) = \sum_{r=0}^{n} \sum_{\theta \in \text{Sh}_r^n} [\theta](\theta R \otimes \theta R\theta) (\text{id} \otimes \Delta^b)
\]

\[
= \sum_{r=0}^{n} \sum_{\theta \in \text{Sh}_r^n} [\theta](\theta R \otimes R\theta \Delta^b) = \sum_{r=0}^{n} \sum_{\theta \in \text{Sh}_r^n} [\theta](\theta R \otimes (\theta R\theta) \otimes R\theta)
\]

\[
= \sum_{r=0}^{n} \sum_{s=0}^{r} \sum_{\delta \in \text{Sh}_s^n} [\theta][\delta](\theta R \otimes \pi \otimes (R\theta) \otimes \pi)
\]

In order to prove the statement of the lemma, we need to prove (5.4)

\[
\sum_{r+s \leq n} \sum_{\pi \in \text{Sh}_r^n} \sum_{\sigma \in \text{Sh}_s^n} [\pi][\sigma](\sigma (\pi R) \otimes (\pi R)\pi)
\]

\[
= \sum_{r+s \leq n} \sum_{\theta \in \text{Sh}_r^n} \sum_{\delta \in \text{Sh}_s^n} [\theta][\delta](\theta R \otimes (R\theta) \otimes (R\theta))
\]

Let \( \alpha = \{\alpha_1 < \ldots < \alpha_r\} \), \( \beta = \{\beta_1 < \ldots < \beta_s\} \), and \( \gamma = \{\gamma_1 < \ldots < \gamma_t\} \) be a partition of \( \{1, 2, \ldots, n\} \) in three subsets. Define \( \pi \in \text{Sh}_r^{n+s} \) by

\[
\{\pi(1) < \ldots < \pi(r+s)\} = \alpha \cup \beta.
\]

Obviously, \( R^\pi = \gamma R \). Let \( i_1 < \ldots < i_r \) be integers such that

\[
\{\pi(i_1) < \ldots < \pi(i_r)\} = \alpha.
\]
Define $\sigma \in \text{Sh}_{r+s}^{r}$ by
\[
\{\sigma(1) < \ldots < \sigma(r)\} = \{i_1 < \ldots < i_r\}.
\]
It is easy to see that $\sigma(\pi R) = \alpha R$ and $(\pi R)\sigma = \beta R$. Notice also that
\[
(\pi \sigma)(i) = \alpha_i, \; 1 \leq i \leq r; \quad (\pi \sigma)(r + j) = \beta_j, \; 1 \leq j \leq s.
\]
Consider the increasing sequence $\pi(1), \ldots, \pi(r + s)$. This sequence has two increasing subsequences $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$. We turn $\pi(1), \ldots, \pi(r+s)$ into $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ by moving the elements $\beta_s, \beta_{s-1}, \ldots, \beta_1$ in the line to the right, i.e., first move $\beta_s$ to the rightmost position applying transpositions, then $\beta_{s-1}$ to the second rightmost position, and so on. Let $j_i$ be the number of transpositions applied to $\beta_i$ during this procedure. We are going to prove that
\[
(\pi \sigma)(r + j_i) = \beta_i, \quad 1 \leq i \leq s.
\]
We demonstrate how to calculate $[1; (\pi \sigma)(1)]$ using (2.3) and change $[\sigma(1); (\pi \sigma)(1)]$ by $[\sigma(1); (\pi \sigma)(1)]$ using (2.4). Then we move $[1; (\pi \sigma)(1)]$ to the leftmost position by (2.5). We obtain an equality of the form
\[
[1; (\pi \sigma)(1)] \cdot \pi \sigma = [1; (\pi \sigma)(1)][2; d_2] \cdots [r + s; d_{r+s}] = [1; (\pi \sigma)(1)] [\sigma(1)] \cdot \pi \sigma = \pi \sigma.
\]
We have $2 \leq d_2 < \ldots < d_{r+s}$ despite replacing of some $[u; v]$ by $[u + 1; v + 1]$ using (2.5) because $[\sigma(1); (\pi \sigma)(1)]$ is disappeared in $[2; d_2] \cdots [r + s; d_{r+s}]$. Moreover, all multipliers of $[1; (\pi \sigma)(1)] \cdot \pi \sigma = [1; (\pi \sigma)(1)]$ remain unchanged in $[2; d_2] \cdots [r + s; d_{r+s}]$. This allows us to calculate $[2; d_2] \cdots [r + s; d_{r+s}] [2; (\pi \sigma)(2)]$ in the same manner. Continuing these discussions we obtain
\[
[\pi][\sigma] = [1; (\pi \sigma)(1)] \cdot \pi \sigma = [1; (\pi \sigma)(1)][2; d_2] \cdots [r + s; d_{r+s}]
\]
where $r + 1 \leq \mu_{r+1} < \ldots < \mu_{r+s} \leq n$. Notice that $[r + i; \mu_{r+i}]$ appeared instead of $[\sigma(r+i); (\pi \sigma)(r+i)] = [\sigma(r+i); \beta_i]$. In order to move $[\sigma(r+i); \beta_i]$ to the place of $[r+i; \mu_{r+i}]$ we apply relations of the form (2.5) exactly $j_i$ times. Consequently, $\mu_{r+i} = \beta_i + j_i$. This proves (5.5).

Define $\theta \in \text{Sh}_n^r$ by $\{\theta(1) < \ldots < \theta(r)\} = \alpha$. Then $\theta R = \alpha R$ and $\{\theta(r+1) < \ldots < \theta(n)\} = \beta \cup \gamma$. Let $k_1 < \ldots < k_s$ be integers such that
\[
\{\theta(r+k_1) < \ldots < \theta(r+k_s)\} = \{\beta_1 < \ldots < \beta_s\}.
\]
Define $\delta \in \text{Sh}_{s+r}^s$ by $\{\delta(1) < \ldots < \delta(s)\} = \{k_1 < \ldots < k_s\}$. It is easy to see that $\delta(R^\theta) = \beta R$ and $\pi(R^\theta) = \gamma R$.

Notice that there are $k_i - 1$ elements in $\beta \cup \gamma$ less than $\beta_i$ by the definition of $k_i$. There are $r - j_i$ elements in $\alpha$ less than $\beta_i$ by the definition of $j_i$. Therefore,
\[
\beta_i = k_i - 1 + r - j_i + 1 = k_i + r - j_i, 1 \leq i \leq s.
\]
Then,
\[
[\delta](r) = [r + 1; r + \delta(1)] \cdots [r + s; r + \delta(s)] = [r + 1; r + k_1] \cdots [r + s; r + k_s]
\]
\[
= [r + 1; \beta_1 + j_1] \cdots [r + s; \beta_s + j_s].
\]
and
\[ [\theta][\delta]_r = [1; \alpha_1] \cdots [r; \alpha_r][r+1; \beta_1 + j_1] \cdots [r+s; \beta_s + j_s] = [\pi][\sigma] \]
by \((5.5)\).

Notice that \(\alpha, \beta, \gamma\) uniquely define \((\pi, \sigma) \in Sh_{r+s}^r \times Sh_{r+s}^s\) and \((\theta, \delta) \in Sh_{r+s}^r \times Sh_{r+s}^s\). This establishes a one-to-one correspondence between \(Sh_{r+s}^r \times Sh_{r+s}^s\) and \(Sh_{r+s}^r \times Sh_{r+s}^s\).

Consider
\[ A = [\pi][\sigma](^\pi R) \otimes (^\sigma R) \]
and
\[ A' = [\theta][\delta]_r (^\theta R) \otimes (^\delta R). \]

We already have proved that \(A = A'\). Notice that \(A\) is a summand of the left hand side of \((5.4)\) and \(A'\) is a summand of the right hand side of \((5.4)\). The correspondence between the pairs \((\pi, \sigma)\) and \((\theta, \delta)\) for all \(r, s\) gives a one-to-one correspondence between the summands \(A\) and \(A'\) of the left and right hand sides of \((5.4)\). This proves that \((5.4)\) is valid. \(\square\)

**Lemma 5.3.** The braided nonassociative algebra \(k\{X\}\) with the coproduct \(\Delta^b\) defined by \((5.7)\) and the counit \(\varepsilon\) define by \((5.2)\) is a braided bialgebra.

**Proof.** Let's check the first of the relations \((3.5)\). Let \(R \in P_r\) and \(L \in P_{n-r}\). Then
\[
(R \otimes L)\tau^*(\text{id} \otimes \Delta^b) = \nu_r^{1,n}(L \otimes R)(\text{id} \otimes \Delta^b) = \nu_r^{1,n}(L \otimes R \Delta^b)
\]
\[
= \nu_r^{1,n}(L \otimes \sum_{s=0}^{r-1} \sum_{\pi \in Sh^s} [\pi]((^\pi R \otimes R^\pi)) = \sum_{s=0}^{r-1} \nu_r^{1,n}[\pi]_{(n-r)}(L \otimes R \otimes R^\pi)
\]
and
\[
(R \otimes L)(\Delta^b \otimes \text{id}) \tau_2^* \tau_1^* = (R \Delta^b \otimes L) \tau_2^* \tau_1^* = \left( \sum_{s=0}^{r-1} \sum_{\pi \in Sh^s} [\pi]((^\pi R \otimes R^\pi) \otimes L) \tau_2^* \tau_1^*ight)
\]
\[
= \left( \sum_{s=0}^{r-1} \sum_{\pi \in Sh^s} [\pi]((^\pi R \otimes R^\pi) \otimes L) \tau_2^* \tau_1^*ight).
\]

Notice that
\[
(\pi R \otimes R^\pi \otimes L) \tau_2^* \tau_1^* = (\pi R \otimes (R^\pi \otimes L) \tau_1^*) = (\pi R \otimes R^\pi \otimes L) \tau_1^* = \nu_r^{1,n}(L \otimes R)^{1,n-s}(L \otimes R^\pi)) \tau_1^*
\]
\[
= \nu_r^{s+1,n}(\pi R \otimes R^\pi \otimes L \tau_1^*) = \nu_r^{s+1,n}(\pi R \otimes L) \tau_1^* \otimes R^\pi
\]
\[
= \nu_r^{s+1,n}(\nu_s^{1,n-s}(L \otimes R \otimes R^\pi) = \nu_s^{1,n+s}(L \otimes R \otimes R^\pi)
\]
by \((2.7)\). Consequently,
\[
(R \otimes L)(\Delta^b \otimes \text{id}) \tau_2^* \tau_1^* = \sum_{s=0}^{r-1} \sum_{\pi \in Sh^s} [\pi] \nu_r^{1,n}(L \otimes R \otimes R^\pi).
\]

Notice that \(\nu_r^{1,n} = [1; r+1][2; r+2] \cdots [n-r; n]\). Consider \([i; \pi(i)]\), where \(1 \leq i \leq s\). Then \(1 \leq i < i+1 \leq \pi(i) + 1 \leq r + 1\). Consequently, applying Lemma 2.1, we get
\[
[i; \pi(i)] \nu_r^{1,n} = \nu_r^{1,n}[i + n - r; \pi(i) + n - r].
\]
Hence
\[ [\pi] \nu_{1,n}^{1,n} = [1; \pi(1)] \cdots [s; \pi(s)] \nu_{1,n}^{1,n} \]
\[ = \nu_{1,n}^{1,n} [1 + n - r; \pi(1) + n - r] \cdots [s + n - r; \pi(s) + n - r] = \nu_{1,n}^{1,n} [\pi]_{n-r}^{1} \]

This proves that
\[ (R \otimes L) \tau^*(\text{id} \otimes \Delta_b) = (R \otimes L)(\Delta_b \otimes \text{id}) \tau^*_2 \tau^*_1; \]
that is,
\[ \tau^*(\text{id} \otimes \Delta_b) = (\Delta_b \otimes \text{id}) \tau^*_2 \tau^*_1. \]

Similarly one can check the second of the relations (3.5). The relations (3.2) and (3.4) hold trivially. □

**Theorem 5.1.** There is a unique braided nonassociative Hopf algebra structure on \( k\{X\} \) such that all elements of the generator set \( X \) are primitive.

**Proof.** It remains only to show the existence and uniqueness of the left and right divisions. We consider only the left one. We shall prove that the first equation of (3.9) uniquely defines \( \backslash \). In fact, \( 1 \backslash y = y \) since \( \Delta^b(1) = 1 \otimes 1 \) and \( \varepsilon(1) = 1 \).

Let \( u \) be an arbitrary nonassociative word of length \( m \geq 1 \). By Lemma 5.1 \( u \Delta_b \) can be uniquely written in the form
\[ u \Delta_b = u \otimes 1 + \sum \alpha_i a_i \otimes c_i, \]
where \( a_i, c_i \) are nonassociative words and length of each \( a_i \) is less than \( m \). Using the first equality of (3.9) we obtain
\[ u \backslash y = - \sum \alpha_i a_i \backslash (c_i y) \]
because \( \varepsilon(u) = 0 \). This provides a recursive definition of \( u \backslash y \) by the length of \( u \).

The vector space \( \text{Hom}(k\{X\}, \text{End}(k\{X\})) \) with the convolution product
\[ (\varphi \ast \psi) = \sum \varphi(x(1))\psi(x(2)) \]
is an associative algebra with the identity \( \iota : x \mapsto \varepsilon(x)\text{Id} \) [10, p. 839].

Consider
\[ \mathcal{L} : k\{X\} \to \text{End}(k\{X\}), \ x \mapsto \mathcal{L}_x, \ \mathcal{L}_x(a) = xa, \]
and
\[ \backslash : k\{X\} \to \text{End}(k\{X\}), \ x \mapsto \backslash_x, \ \backslash_x(a) = x \backslash a. \]
The first equality of (3.9) exactly means that \( \backslash \ast \mathcal{L} = \iota \); that is, \( \backslash \) is a left inverse to \( \mathcal{L} \).

Using the second equality of (3.9) we can similarly define \( \backslash' \) and check that \( \mathcal{L} \ast \backslash' = \iota \), i.e., \( \backslash' \) is a right inverse to \( \mathcal{L} \). Consequently, \( \backslash = \backslash' \). □
6. Primitive elements

Let $B$ be an arbitrary braided (nonassociative) Hopf algebra with the braiding $\tau : B \otimes B \to B \otimes B$. Put
\[
m_i = \text{id} \otimes (i-1) \otimes \text{id} \otimes (n-i-1) : B^\otimes n \to B^\otimes n-1, \quad 1 \leq i \leq n-1,
\]
where $m : B \otimes B \to B$ is the multiplication.

For example, if $u = u_1 \otimes u_2 \otimes \ldots \otimes u_k \in B^k$, then
\[
u m_1^{k-1} = (\ldots (u_1 u_2) \ldots u_k)
\]
is the right-normed product of the elements $u_1, u_2, \ldots, u_k$. In this section we denote by $R_k$ the parenthesis function such that
\[
u R_k = (\ldots (u_1 u_2) \ldots u_k);
\]
that is, $R_k$ is the right-normed $k$-ary parenthesis function.

An $n$-ary operation $T$ on $B$ is called primitive if $T(u_1, \ldots, u_n)$ is primitive for any primitive elements $u_1, \ldots, u_n \in B$. For example, the commutator $[x, y] = xy - yx$ and the associator $(x, y, z) = xyz - xy(z)$ are primitive operations [3, Lemma 8.2] in the case of the ordinary flip $(\tau(x \otimes y) = (y \otimes x))$. I.P. Shestakov and U.U. Umirbaev [13, p. 539] introduced a system of primitive operations $p_{m,n}$ for all $m, n \geq 1$ in the following way.

Given $m, n \geq 1$, let $U = (u_1, u_2, \ldots, u_m)$ and $V = (v_1, v_2, \ldots, v_n)$ be sequences of nonassociative polynomials, and let $U = u_1 u_2 \ldots u_m, V = v_1 v_2 \ldots v_n$ be the corresponding right-normed products. The operations are defined inductively:
\[
p(U; V; w) = (U, V, w) - \sum U(1) V(1) : p(U(2); V(2); w)
\]
where $(U, V, w)$ is the associator. Here Sweedler’s notation is extended so as to mean that the sum is taken over all partitions of the sequences $U$ and $V$ into pairs of subsequences, $U = U(1) \cup U(2)$ and $V = V(1) \cup V(2)$ such that $|U(1)| + |V(1)| \geq 1$, $U(2) \neq \emptyset$, $V(2) \neq \emptyset$; the expressions $U(1)$ and $V(1)$ are the right-normed products of the elements of $U(1)$ and $V(1)$ respectively.

Our next step is to define $\tau$-analogues of these operations when $\tau^2 = \text{id}$.

Denote by
\[
[x, y]_\tau = (x \otimes y) m - (y \otimes x) \tau m
\]
the braided commutator.

Lemma 6.1. If $\tau$ is involutive, then the braided commutator $[x, y]_\tau$ is a primitive operation.

Proof. Let $y_1, y_2$ be arbitrary primitive elements of $B$. Suppose that $(y_1 \otimes y_2)^\tau = \sum z(2) \otimes z(1)$, where $z(1), z(2) \in B$. Then
\[
[y_1, y_2]_\tau = (y_1 \otimes y_2) m - (y_1 \otimes y_2)^\tau m = y_1 y_2 - \sum z(2) z(1),
\]
\[
([y_1, y_2]_\tau) b = y_1 y_2 \otimes 1 + y_1 \otimes y_2 + (y_1 \otimes y_2)^\tau + 1 \otimes y_1 y_2
\]
\[
- \sum (z(2) z(1) \otimes 1 + z(2) \otimes z(1) + (z(2) \otimes z(1))^\tau + 1 \otimes z(2) z(1)).
\]
Notice that $\sum (z_{(2)} \otimes z_{(1)})^\tau = ((y_1 \otimes y_2)^\tau)^\tau = (y_1 \otimes y_2)^{\tau^2} = y_1 \otimes y_2$ since $\tau$ is involutive. Consequently,

$$([y_1, y_2]^\tau) \Delta^b = [y_1, y_2]^\tau \otimes 1 + 1 \otimes [y_1, y_2]^\tau;$$

that is, the braided commutator is a primitive operation. □

We define the following system of operations

$$P = P_{m,n} : B^\otimes m \otimes B^\otimes n \otimes B = B^\otimes (m+n+1) \to B, \ m, n \geq 1,$$

by induction on $m + n$. Put

$$P_{1,1} = (m_1 - m_2)m.\label{6.1}$$

Notice that $P_{1,1}$ is the associator, i.e., $P_{1,1}(x, y, z) = (x, y, z)$. If $m + n > 1$, then a recursive formula for $P_{m,n}$ is given by

$$P_{m,n} = (R_m \otimes R_n \otimes \text{id})P_{1,1} - \sum_{r+s>0} \sum_{\pi \in S_{m+1}^r} \sum_{\sigma \in S_{n+1}^s} [\pi][\sigma](m)\nu_{m}^{r+1,m+s}(R_r \otimes R_s \otimes P_{m-r,n-s})m_1 m,$$

where $\sum_{r+s>0}$ means the summation over all $r, s$ such that $r \geq 0, s \geq 0, m - r \geq 0, n - s \geq 0$, and $r + s > 0$. Moreover, in this case we can assume that $m - r \geq 1, n - s \geq 1$ because $P_{i,j} = 0$ if $i + j < 2$.

It is not difficult to check that in the case of the ordinary flip the operations $P_{m,n}$ coincide with the above primitive operations $p_{m,n}$. The main result of [13] states that the set of all operations $p_{m,n}, m, n \geq 1$, together with the commutator $[\cdot, \cdot]$ represents a complete and an independent system of primitive operations in nonassociative algebras.

**Theorem 6.1.** If $\tau$ is involutive, then the operations $P_{m,n}$ for all $m, n \geq 1$ are primitive.

**Proof.** Let $B$ be an arbitrary braided nonassociative Hopf algebra with braiding $\tau$. Let $V$ be the space of primitive elements of $B$. Denote by $\Delta^b$ the coproduct of $B$. Notice that $P_{m,n}$ is a primitive operation if and only if the equality

$$(6.1) \quad P_{m,n} \Delta^b = P_{m,n} \otimes 1 + 1 \otimes P_{m,n}$$

holds on $V^\otimes (m+n+1)$. Using (5.3), this equality can be rewritten in the form

$$(6.2) \quad F_1 P_1 + F_2 P_2 + \ldots + F_s P_s = 0,$$

where $F_i$ are elements of the monoid algebra of the monoid generated by $\tau_1, \ldots, \tau_{m+n}$ for all $i$ and $P_1, P_2, \ldots, P_s$ are independent tensor products of parenthesis functions. Consequently, (6.2) holds on $V^\otimes (m+n+1)$ if and only if

$$(6.3) \quad F_1 = F_2 = \ldots = F_s = 0$$

holds on $V^\otimes (m+n+1)$.

Notice that if $\tau$ is an ordinary flip, then $P_{m,n}$ exactly becomes the primitive operation $p_{m,n}$ described in [13]. Moreover, every algebra can be considered as a braided algebra with the ordinary flip. This means that the equalities (6.3) hold for the ordinary flip. Taking into account the principle (see Remark 2.1), this implies that (6.3) is true for any involutive braiding $\tau$. Consequently, $P_{m,n}$ is a primitive operation. □
Let $\tau$ be an involutive braiding. D. Gurevich [1] introduced the concept of a “Lie $\tau$-algebra”, which is a wide generalization of the notions of Lie, super Lie, and color Lie algebras, as follows.

A braided algebra $L$ with involutive braiding $\tau$ and multiplication $m : L \otimes L \to L$ is called a Lie $\tau$-algebra if

$$m + \tau m = 0,$$

$$(\text{id} + \tau_1 \tau_2 + \tau_2 \tau_1)(m \otimes \text{id})m = 0.$$

These identities are braided analogues of the antisymmetric identity and the Jacobi identity.

Lie algebras first appeared as tangent algebras of Lie groups. In the case of simply connected Lie groups, Lie algebras determine the corresponding groups up to isomorphism. It was shown by P.O. Miheev and L.V. Sabinin [5] that a simply connected local analytic loop is determined up to isomorphism by a more sophisticated analogue of a tangent algebra with a series of multilinear operations. These algebras are now called Sabinin algebras [8, 7, 10].

We are going to give a definition of Sabinin $\tau$-algebras. First we define analogues of the relations (3.1). Let $\tau$ be a braiding on a vector space $R$.

A vector space $A$ with an involutive braiding $\tau$ is called a Sabinin $\tau$-algebra if it is endowed with the multilinear braided operations

$$S_{m,2}(x_1, \ldots, x_m, y, z), \ m \geq 0,$$

$$\Phi_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n), \ m \geq 1, n \geq 2,$$

that satisfy the identities:

$$(7.2) \quad S_{m,2} + \tau_{m+1}S_{m,2} = 0,$$

$$(7.3) \quad S_{m+2,2} - \tau_{r+1}S_{m+2,2} + \sum_{k=0}^{r} \sum_{\pi \in \text{Sh}_k^r} [\pi](\text{id} \otimes k \otimes S_{r-k,2} \otimes \text{id} \otimes (m-r+2))S_{m-r+k+1,2} = 0$$

for all $1 \leq r < m + 2$,

$$(7.4) \quad (1 + \tau_{m+1} \tau_{m+2} + \tau_{m+2} \tau_{m+1})S_{m+1,2} + \sum_{k=0}^{r} \sum_{\pi \in \text{Sh}_k^r} [\pi](\text{id} \otimes k \otimes S_{m-k,2} \otimes \text{id})S_{k,2} = 0,$$

and

$$(7.5) \quad \Phi_{m,n} - \tau_r \Phi_{m,n} = 0$$

for all integers $r$ such that $1 \leq r < m + n$ and $r \neq m$.

If $\tau = \theta$ is the ordinary flip, then every Sabinin $\tau$-algebra is a Sabinin algebra [10, Section 1], [13, Section 4].
Let $V$ be an arbitrary (nonassociative) braided algebra over a field $k$ of characteristic zero with involutive braiding $\tau$. For any $x_1, x_2, \ldots, x_m, y, z \in A$ we put
\begin{equation}
S_{0,2}(y, z) = -[y, z]_{\tau}
\end{equation}
and
\begin{equation}
S_{m,2}(x_1, x_2, \ldots, x_m, y, z) = u(-P_{m,1} + \tau_{m+1}P_{m,1}),
\end{equation}
where $u = x_1 \otimes x_2 \otimes \ldots \otimes x_m \otimes y \otimes z$, $m \geq 1$.

Let $T_n$ be the monoid generated by $\tau_1, \tau_2, \ldots, \tau_{n-1}$. Since $\tau$ is involutive, we have a homomorphism from the symmetric group $S_n$ to $T_n$ such that $t_i \mapsto \tau_i$ for all $1 \leq i < n$. We will denote the image of any $\pi \in S_n$ in $T_n$ by $\bar{\pi}$, $n \geq 1$.

If $m \geq 1, n \geq 2$, then for any $x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n \in V$ we put
\begin{equation}
\Phi_{m,n}(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n) = \frac{1}{m! n!} \sum_{\pi \in S_m, \sigma \in S_n} u\bar{\pi}\bar{\sigma}(m)P_{m,n-1},
\end{equation}
where $u = x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n$.

**Theorem 7.1.** Let $B$ be a braided algebra with involutive braiding $\tau$ over a field $k$ of characteristic zero. The set $\mathfrak{A}$ is a braided Sabinin $\tau$-algebra with respect to the operations (7.6), (7.7), and (7.8).

**Proof.** First we check that the operations (7.6), (7.7), and (7.8) are braided operations. Let $M$ be one of these operations. Suppose that $M$ is an $m$-ary operation. Using the relations (3.1), we can rewrite the relations (7.1) in the form (6.2) where $F_i$ are elements of the monoid algebra of the monoid generated by $\tau_1, \ldots, \tau_{m-1}$ for all $i$ and $P_1, P_2, \ldots, P_s$ are independent tensor products of parenthesis functions. Consequently, (6.2) holds if and only if (6.3) holds. It is easy to check that the relations (7.1) hold for any $m$-ary operation if $\tau$ is the ordinary flip. This means that (6.2) and (6.3) also hold for the ordinary flip. The principle (see Remark 2.1) implies that (6.3) is true for an arbitrary involutive braiding.

Notice that each of the identities (7.2)-(7.4) also can be written in the form (6.2), where $P_1, P_2, \ldots, P_s$ are independent parenthesis functions. Consequently, this identity is equivalent to (6.3). Moreover, all identities (7.2)-(7.4) hold for the ordinary flip [13, Section 4]. This means that (6.3) holds for the ordinary flip. Remark 2.1 again implies that the identities (7.2)-(7.4) hold for any involutive braiding. □

**Corollary 7.1.** Let $B$ be a braided (nonassociative) Hopf algebra with involutive braiding $\tau$ over a field $k$ of characteristic zero. The set $\mathfrak{A}$ of all primitive elements of $B$ is a braided Sabinin $\tau$-algebra with respect to the operations (7.6), (7.7), and (7.8).

**Proof.** By the theorem above the algebra $B$ itself is a $\tau$-Sabinin algebra with respect to the operations $S_{m,2}$ and $\Phi_{m,n}$. By theorem 6.1 these operations are primitive, consequently the set $\mathfrak{A}$ is closed with respect to all Sabinin $\tau$-operations. □

Finally, we formulate two interesting open problems.

1. Let $k\{V\}$ be a free nonassociative algebra over a braided space $V$ with involutive braiding. Does $V$ generate the space Prim($k\{V\}$) of all primitive nonassociative polynomials as a $\tau$-Sabinin algebra?
This problem has positive solution for nonassociative product with ordinary flip (see [13, Corollary 3.3] or [3, Theorem 8.2]) and also for the case of associative product with arbitrary involutive braiding (see [3, Theorem 7.6]).

For each nonassociative braided algebra $B$ with an involutive braiding, $B(\tau)$ denotes the $\tau$-Sabinin algebra $B$ with respect to the operations (7.6), (7.7), and (7.8).

2. Is it true that each $\tau$-Sabinin algebra can be isomorphically embedded into a $\tau$-Sabinin algebra $B(\tau)$ for a suitable $\tau$-algebra $B$?

Recall that a similar problem has a positive solution for Sabinin algebras (when $\tau$ is the ordinary flip, see [10]). The problem has an affirmative answer in the case of associative product with arbitrary symmetry $\tau$ because each $\tau$-Lie algebra is a subagebra of its associative universal enveloping $\tau$-algebra (see [3, Theorem 7.3]).

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References

[1] D.I. Gurevich, Generalized translation operators on Lie groups. (Russian) Izv. Akad. Nauk Armyan. SSR Ser. Mat. v. 18 (1983), no. 4, 305–317.

[2] V. K. Kharchenko, Braided Version of Shirshov-Witt Theorem, J. Algebra, v. 294, 2005, 196–225.

[3] V. Kharchenko, Quantum Lie theory. A multilinear approach. Lecture Notes in Mathematics, 2150. Springer, Cham, 2015. xiii+302 pp.

[4] A.I. Malcev, Analytic loops, Mat. Sb. N.S. v. 36(78), 1955, 569–576.

[5] P.O. Miheev, L.V. Sabinin, On the infinitesimal theory of local analytic loops (Russian), Dokl. Akad. Nauk SSSR 297, no. 4 (1987), 801–804. English translation: Soviet Math. Dokl. v. 36, no. 3 (1988), 545–548.

[6] P.O. Miheev and L.V. Sabinin, Quasigroups and differential geometry. In: O. Chein; H. Pflugfelder and J.D.H. Smith (Eds.) Quasigroups and Loops: Theory and Applications, pp. 357–430. Heldermann Verlag, Berlin, 1990.

[7] J. Mostovoy, J.M. Pérez Izquierdo, Formal multiplications, bialgebras of distributions and non-associative Lie theory. Transform. Groups, v. 15, 2010, 625–653.

[8] J. Mostovoy, J.M. Pérez-Izquierdo, I.P. Shestakov, Hopf algebras in non-associative Lie theory, Bulletin of Mathematical Sciences, v. 4, no. 1, 2014, 129–173.

[9] J.M. Pérez Izquierdo, An envelope for Bol algebras, J. Algebra, v. 284, 2005, 480–493.

[10] J.M. Pérez Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops, Adv. Math. v. 208, 2007, 834–876.

[11] L.V. Sabinin, Smooth Quasigroups and Loops, Mathematics and its Applications, v. 429, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.

[12] L.V. Sabinin, Smooth quasigroups and loops: forty-five years of incredible growth, Commentat. Math. Univ. Carol., v. 41(2000), 377–400.

[13] I.P. Shestakov, U.U. Umirbaev, Free Akivis Algebras, Primitive Elements, and Hyperalgebras. J. Algebra v. 250 (2002), 533–548.

[14] 221. M. Takeuchi, Survey of braided Hopf algebras, in: New Trends in Hopf Algebra Theory, Contemp. Math., v. 267, AMS, Providence RI, 2000, pp. 301–324.

[15] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, Rings that are nearly associative, Academic Press, N.Y., 1982.