A lower bound on the acyclic matching number of subcubic graphs

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Abstract

The acyclic matching number of a graph $G$ is the largest size of an acyclic matching in $G$, that is, a matching $M$ in $G$ such that the subgraph of $G$ induced by the vertices incident to an edge in $M$ is a forest. We show that the acyclic matching number of a connected subcubic graph $G$ with $m$ edges is at least $m/6$ except for two small exceptions.

Keywords: Acyclic matching; subcubic graph

1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation. A matching $M$ in a graph $G$ is acyclic \cite{7} if the subgraph of $G$ induced by the set of vertices that are incident to some edge in $M$ is a forest, and the acyclic matching number $\nu_{ac}(G)$ of $G$ is the maximum size of an acyclic matching in $G$. While the ordinary matching number $\nu(G)$ of $G$ is tractable \cite{4}, it has been known for some time that the acyclic matching number is NP-hard for graphs of maximum degree 5 \cite{7,15}. Recently, we \cite{6} showed that just deciding the equality of $\nu(G)$ and $\nu_{ac}(G)$ is already NP-complete when restricted to bipartite graphs $G$ of maximum degree 4. The complexity of the acyclic matching number for cubic graphs is unknown.

In the present paper we establish a tight lower bound on the acyclic matching number of subcubic graphs. Similar results were obtained for the matching number \cite{2,14}, and also for the induced matching number \cite{11,13}. Baste and Rautenbach \cite{1} studied acyclic edge colorings, and showed that the acyclic chromatic index $\chi'_{ac}(G)$ of a graph $G$, that is, the minimum number of acyclic colorings in $G$ into which the edge set of $G$ can be partitioned, is at most $\Delta(G)^2$, where $\Delta(G)$ denotes the maximum degree of $G$. This implies $\nu_{ac}(G) \geq m(G)/\Delta(G)^2$, where $m(G)$ denotes the size of $G$, which, for subcubic graphs, simplifies to $\nu_{ac}(G) \geq m(G)/9$. This latter bound also follows from a lower bound \cite{12} on the induced matching number, which is always at most the acyclic matching number. While the bound is tight for $K_{3,3}$, excluding some small graphs allows a considerable improvement. Let $K_4^+$ be the graph that arises by subdividing one edge of $K_4$ once.

We prove the following.

**Theorem 1** If $G$ is a connected subcubic graph that is not isomorphic to $K_4^+$ or $K_{3,3}$, then $\nu_{ac}(G) \geq m(G)/6$.

Since every subcubic graph $G$ of order $n(G)$ satisfies $m(G) \leq 3n(G)/2$, Theorem 1 is an immediate consequence of the following stronger result. For two graphs $G$ and $H$, let $\kappa_G(H)$ denote the number of components of $G$ that are isomorphic to $H$. 
Theorem 2 If $G$ is a subcubic graph without isolated vertices, then

$$\nu_{ac}(G) \geq \frac{1}{4} \left(n(G) - \kappa_G(K_{2,3}) - \kappa_G(K_{3,3}) - 2\kappa_G(K_{3,3})\right).$$

Note that Theorem 2 is tight; examples are $K_4$, $K_{2,2}$, $K_{1,3}$, or the graph obtained from $K_{1,3}$ by replacing each endvertex with an endblock isomorphic to $K_{2,3}$. The proof of Theorem 2 is postponed to the second section. The reduction arguments within that proof easily lead to a polynomial time algorithm computing acyclic matchings of the guaranteed size.

In a third section, we conclude with some open problems.

2 Proof of Theorem 2

The proof is by contradiction. Therefore, suppose that $G$ is a counterexample to Theorem 2 that is of minimum order $n$. A graph is special if it is isomorphic to $K_{2,3}$, $K_{4}^+$, or $K_{3,3}$. Clearly, $G$ is connected, not special, and $n$ is at least 5. Note that $\nu_{ac}(G) < n/4$.

We derive a contradiction using a series of claims.

Claim 1 No subgraph of $G$ is isomorphic to $K_{4}^+$.

Proof of Claim 1: Suppose that $G$ has a subgraph $H$ that is isomorphic to $K_{4}^+$. Let $\nu_1$, $\nu_2$, $\nu_3$, and $\nu_4$ be the vertices of degree 3 in $H$, and let $u$ the vertex of degree 2 in $H$. Let $G' = G - \{\nu_1, \nu_2, \nu_3, \nu_4\}$. Since $G$ is connected, the graph $G'$ is connected. Since $u$ has degree 1 in $G'$, the graph $G'$ is not special. By the choice of $G$, the graph $G'$ is no counterexample to Theorem 2 and, hence, it has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $\nu_1\nu_2$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction.

Claim 2 No endblock of $G$ is isomorphic to $K_{2,3}$.

Proof of Claim 2: Suppose that some endblock $B$ of $G$ is isomorphic to $K_{2,3}$. Let $u$ be the unique cutvertex of $G$ in $B$. Clearly, the vertex $u$ has degree 2 in $B$. The graph $G' = G - (V(B) \setminus \{u\})$ is connected, and, since $u$ has degree 1 in $G'$, it is not special. Therefore, by the choice of $G$, the graph $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding an edge of $B$ that is not incident to $u$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction.

Claim 3 No two vertices of degree 1 have a common neighbor.

Proof of Claim 3: Suppose that $u$ and $v$ are two vertices of degree 1, and that $w$ is their common neighbor. Let $G' = G - \{u, v, w\}$. Since $G'$ is connected and not isomorphic to $K_{3,3}$, the choice of $G$ implies that $G'$ has an acyclic matching $M'$ of size at least $(n(G') - 1)/4 = n/4 - 1$. Since $w$ does not lie on any cycle in $G$, adding the edge $uw$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction.

Claim 4 No vertex of degree 1 is adjacent to a vertex that does not lie on a cycle.

Proof of Claim 4: Suppose that $u$ is a vertex of degree 1 that is adjacent to a vertex $v$ that does not lie on a cycle. By Claim 3 the graph $G' = G - \{u, v\}$ has no isolated vertex. Since $G'$ has at most two components, and no component of $G'$ is isomorphic to $K_{3,3}$, the choice of $G$ implies that $G'$ has an acyclic matching $M'$ of size at least $(n(G') - 2)/4 = n/4 - 1$. Since $v$ does not lie on a cycle, adding the edge $uv$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction.
Claim 5 The minimum degree of $G$ is at least 2.

Proof of Claim: Suppose that $u$ is a vertex of degree 1. By Claim 4, the neighbor $v$ of $u$ lies on a cycle $C$ in $G$. Let $x$ and $w$ be the neighbors of $v$ on $C$.

First, suppose that $w$ has no neighbor of degree 1.

If $G - \{u, v, w\}$ contains an isolated vertex, then this is necessarily the vertex $x$, and $N_G(x) = \{v, w\}$. In this case, let $G' = G - \{u, v, w, x\}$. Clearly, the graph $G'$ is connected and not isomorphic to $K_{3,3}$. If isomorphic to $K_{2,2}$ or $K_{2,3}$, then it follows easily that $\nu_{ac}(G) \geq 3 > 9/4 = n/4$, which is a contradiction. Hence, $G'$ is not special, which implies that $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $uv$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction. Hence, we may assume that $G' = G - \{u, v, w\}$ has no isolated vertex.

Since there are at most three edges between $\{u, v, w\}$ and $V(G')$ in $G$, Claim 2 implies that at most one component of $G'$ is isomorphic to $K_{2,3}$. By the choice of $G$, this implies that $G'$ has an acyclic matching $M'$ of size at least $(n(G') - 1)/4 = n/4 - 1$. Adding the edge $uv$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction. Hence, by symmetry, we may assume that $x$ and $w$ both have a neighbor of degree 1.

Let $y$ be a neighbor $w$ of degree 1. If $x$ and $w$ are adjacent, then $\nu_{ac}(G) = 2 > 6/4 = n/4$, which is a contradiction. Hence, $x$ and $w$ are not adjacent. In view of the cycle $C$, the graph $G' = G - \{u, v, w, y\}$ is connected. Since $G'$ has a vertex of degree 1, it is not special, which implies that $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $uv$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction.

For a set $X$ of vertices of $G$, let $N_G[X] = \bigcup_{u \in X} N_G[u]$.

Claim 6 No subgraph of $G$ is isomorphic to $K_{2,3}$.

Proof of Claim: Suppose that $G$ has a subgraph $H$ that is isomorphic to $K_{2,3}$. Claim 4 implies that $H$ is an induced subgraph of $G$. Let $u_1$, $u_2$, and $u_3$ be the vertices of degree 2 in $H$, and let $v_1$ and $v_2$ be the vertices of degree 3 in $H$.

First, suppose that $u_1$ has degree 2 in $G$. Since $G$ is not special, we may assume that $u_2$ has degree 3 in $G$. By Claim 5, the graph $G' = (V(H) \setminus \{u_2\})$ has no isolated vertex, and, since $u_2$ has degree 1 in $G'$, it is not special. It follows that $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $u_1v_1$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction. Hence, by symmetry, we may assume that all vertices in $U = \{u_1, u_2, u_3\}$ have degree 3 in $G$.

Next, suppose that $u_1$ and $u_2$ have a common neighbor $u$ that is distinct from $v_1$ and $v_2$. Let $G' = G - N_G[U]$. Note that there are at most 3 edges between $N_G[U]$ and $V(G')$ in $G$. By Claim 5, the graph $G'$ has at most one isolated vertex, and, by Claim 1 at most one component of $G'$ is isomorphic to $K_{2,3}$. Furthermore, the graph $G'$ does not have an isolated vertex as well as a component isomorphic to $K_{2,3}$. This implies that $G'$ has an acyclic matching $M'$ of size at least $(n(G') - 1)/4 = n/4 - 2$. Adding the two edges $u_1u_3$ and $u_3v_1$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction. Hence, by symmetry, no two vertices in $U$ have a common neighbor that is distinct from $v_1$ and $v_2$.

The graph $G'$ that arises by contracting all edges of $H$ is simple and connected. If $G'$ is special, then $G$ has order at most 11, and an acyclic matching consisting of the three edges between $N_G[U]$ and $V(G) \setminus N_G[U]$ in $G$, which is a contradiction. Hence, $G'$ is not special, which implies that $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Let $M''$ be the acyclic matching in $G$ corresponding to $M'$. Since $M''$ covers at most one vertex
in $U$, say $u_1$, adding the edge $u_2v_1$ to $M'$ yields an acyclic matching in $G$ of size at least $n/4$, which is a contradiction. \hfill \Box

Claim 1 Claim 6 and the choice of $G$ imply that every proper induced subgraph $G'$ of $G$ with $i(G')$ isolated vertices has an acyclic matching $M'$ such that

$$|M'| \geq \frac{n(G') - i(G')}{4}. \quad (1)$$

Claim 7 No two vertices of degree 2 are adjacent.

Proof of Claim 7: Suppose that $u$ and $v$ are adjacent vertices of degree 2, and that $w$ is the neighbor of $u$ distinct from $v$. By Claim 5, the graph $G' = G - \{u, v, w\}$ has at most one isolated vertex, and, hence, by (1), it has an acyclic matching $M'$ of size at least $(n(G') - 1)/4 = n/4 - 1$. Adding the edge $uv$ to $M'$ yields a contradiction. \hfill \Box

Claim 8 No vertex of degree 2 lies on a triangle.

Proof of Claim 8: Suppose that $u_1u_2u_3u_4$ is a cycle in $G$ such that $u_1$ has degree 2. By Claim 7 the vertices $u_2$ and $u_3$ have degree 3. Since $n \geq 5$, the graph $G' = G - \{u_1, u_2, u_3\}$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M'$ of size at least $n(G')/4 > n/4 - 1$. Adding the edge $u_1u_2$ to $M'$ yields a contradiction. \hfill \Box

Claim 9 No vertex of degree 2 lies on a cycle of length 4.

Proof of Claim 9: Suppose that $u_1u_2u_3u_4u_1$ is a cycle in $G$ such that $u_1$ has degree 2. By Claims 7 and 8, the vertices $u_2$ and $u_4$ have degree 3, and are not adjacent. By Claims 8 and 8, the graph $G' = G - \{u_1, u_2, u_3, u_4\}$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $u_1u_2$ to $M'$ yields a contradiction. \hfill \Box

Claim 10 No cycle of length 5 contains two vertices of degree 2.

Proof of Claim 10: Suppose that the cycle $u_1u_2u_3u_4u_5u_1$ contains two vertices of degree 2. By Claim 7 we may assume that $u_1$ and $u_4$ have degree 2, and that $u_2$, $u_3$, and $u_5$ have degree 3. Let $G' = G - (N_G(u_5) \cup \{u_2, u_3\})$. Since there are at most 4 edges between $N_G(u_5) \cup \{u_2, u_3\}$ and $V(G')$ in $G$, the graph $G'$ has at most two isolated vertices, and, hence, by (1), it has an acyclic matching $M'$ of size at least $(n(G') - 2)/4 = n/4 - 2$. Adding the edges $u_1u_2$ and $u_4u_5$ to $M'$ yields a contradiction. \hfill \Box

Claim 11 $G$ is cubic.

Proof of Claim 11: Suppose that $u$ is a vertex of degree 2. By Claims 7 8 and 9, the neighbors of $u$, say $v$ and $w$, have degree 3, are not adjacent, and have no common neighbor except for $u$. Let $x$ be a neighbor of $v$ distinct from $u$. By Claims 8 and 10 the graph $G' = G - \{u, v, w, x\}$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $uv$ to $M'$ yields a contradiction. \hfill \Box

Claim 12 $G$ is triangle-free.

Proof of Claim 12: Suppose that $u_1u_2u_3u_1$ is a triangle in $G$. By Claims 1 and 11 the graph $G' = G - N_G(u_1)$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - 1$. Adding the edge $u_1u_2$ to $M'$ yields a contradiction. \hfill \Box

Let $C : u_1u_2 \ldots u_gu_1$ be a shortest cycle in $G$. For $i \in [g]$, let $v_i$ be the neighbor of $u_i$ not on $C$. By Claim 12 we have $g \geq 4$. 

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Claim 13 \( g \geq 5 \).

Proof of Claim 13: Suppose that \( g = 4 \). By Claims 9 and 12, the vertices \( v_1, v_2, v_3, \) and \( v_4 \) are distinct. Let \( w_1 \) and \( w_2 \) be the neighbors of \( v_1 \) distinct from \( u_1 \).

First, suppose that \( w_1 = w_2 \). By Claim 11, the graph \( G' = G - (N_G[v_1] \cup \{u_2, u_3, u_4\}) \) has at most one isolated vertex, and, hence, by (11), it has an acyclic matching \( M' \) of size at least \((n(G') - 1)/4 = n/4 - 2\). Adding the edges \( u_1v_1 \) and \( u_2u_3 \) to \( M' \) yields a contradiction. Hence, we may assume, by symmetry, that \( \{v_1, v_2, v_3, v_4\} \) is independent.

Next, suppose that \( \exists x \) such that \( G' = G - (\{u_1, u_2, u_4\}) \) has at most two isolated vertices, and, hence, by (1), it has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 3\). Adding the edges \( xw_1, u_1v_1, \) and \( u_2u_3 \) to \( M' \) yields a contradiction. Hence, we may assume that the graph \( G' = G - N_G(\{u_1, u_2, u_3\}) \) has no isolated vertex. By (11), \( G' \) has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 2\). Adding the edges \( u_1v_1 \) and \( u_2u_3 \) to \( M' \) yields a contradiction. \( \square \)

Claim 14 \( g \geq 6 \).

Proof of Claim 14: Suppose that \( g = 5 \). By Claim 13, the vertices \( v_1, v_2, v_3, v_4, \) and \( v_5 \) are distinct. Suppose that \( \exists x \) such that \( G' = G - (\{u_1, u_2, u_3, u_4\}) \) has at most two isolated vertices, and, hence, by (11), it has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 3\). Adding the edges \( xw_1, u_1v_1, \) and \( u_2u_3 \) to \( M' \) yields a contradiction. Hence, we may assume that the graph \( G' = G - N_G(\{u_1, u_2, u_3, u_4\}) \) has no isolated vertex. By (11), the graph \( G' \) has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 2\). Adding the edges \( u_1v_1 \) and \( u_2u_3 \) to \( M' \) yields a contradiction. \( \square \)

Claim 15 \( g \geq 7 \).

Proof of Claim 15: Suppose that \( g = 6 \). Let \( w_1 \) and \( w_2 \) be the neighbors of \( v_1 \) distinct from \( u_1 \). By Claim 14, the vertices \( v_1 \) for \( i \in [6] \setminus \{4\}, \) \( w_1, \) and \( w_2 \) are distinct. Suppose that \( \exists x \) such that \( G' = G - (\{v_1, v_3, v_4, v_5\}) \) has at most two isolated vertices, and, hence, by (11), it has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 4\). Adding the edges \( xw_1, u_1v_1, \) and \( u_2v_3 \) to \( M' \) yields a contradiction. Hence, we may assume that the graph \( G' = G - N_G(\{v_1, v_3, v_5, v_6\}) \) has no isolated vertex. By (11), the graph \( G' \) has an acyclic matching \( M' \) of size at least \((n(G') - 2)/4 = n/4 - 3\). Adding the edges \( u_1v_1, u_2v_3, \) and \( u_3v_6 \) to \( M' \) yields a contradiction. \( \square \)

We are now in a position to complete the proof.

First, suppose that \( g \) is odd. If the graph \( G' = G - N_G(\{u_1, \ldots, u_{g-2}\}) \) has an isolated vertex, then, by Claim 11, there is a cycle of length at most \( \frac{g}{2} + 4 \). Since the last expression is less than \( g \) for odd \( g \) at least 7, it follows that \( G' \) has no isolated vertex. By (1), the graph \( G' \) has an acyclic matching \( M' \) of size at least \( n(G')/4 = n/4 - (g - 1)/2 \). Adding the edges in \( \{u_{g-1}u_2 : i \in [(g - 1)/2]\} \) to \( M' \) yields a contradiction. Hence, we may assume that \( g \) is even. Let \( w_1 \) and \( w_2 \) be the neighbors of \( v_1 \) distinct from \( u_1 \). By the choice of \( C \), the vertices \( v_i \) for \( i \in [g], \) \( w_1, \) and \( w_2 \) are distinct. If the graph \( G' = G - N_G(\{v_1, u_1, \ldots, u_{g-2}\}) \) has an
isolated vertex, then, by Claim 11, there is a cycle of length at most $\left\lfloor \frac{g}{3} \right\rfloor + 5$. Since the last expression is less than $g$ for even $g$ at least 8, it follows that $G'$ has no isolated vertex. By 11, the graph $G'$ has an acyclic matching $M'$ of size at least $n(G')/4 = n/4 - g/2$. Adding the edges in $\{u_1v_1\} \cup \{u_{2i}u_{2i+1} : i \in [(g - 2)/2]\}$ to $M'$ yields a contradiction, which completes the proof.

3 Conclusion

We believe that Theorem 2 can be improved as follows.

**Conjecture 3** There is a constant $c$ such that $\nu_{ac}(G) \geq \frac{3n(G)}{11} - c$ for every connected subcubic graph $G$.

Conjecture 3 would be asymptotically best possible. If $H$ arises from a copy of $K_{1,2}$, where $u(H)$ denotes the vertex of degree 2, by replacing each endvertex with an endblock isomorphic to $K_{2,3}$, and, for some positive integer $k$, the connected subcubic graph $G_k$ arises from $k$ disjoint copies $H_1, \ldots, H_k$ of $H$ by adding, for every $i \in [k-1]$, an edge between $u(H_i)$ and some vertex of degree 2 in $H_{i+1}$ that is distinct from $u(H_{i+1})$, then $\nu_{ac}(G_k) = 3n(G_k)/11$.

For general maximum degree, we pose the following conjecture motivated by [13].

**Conjecture 4** If $G$ is a graph of maximum degree $\Delta$ without isolated vertices, then

$$\nu_{ac}(G) \geq \min \left\{ \frac{2n(G)}{\left\lceil \frac{\Delta}{2} \right\rceil + 1}, \frac{n(G)}{2\Delta} \right\}.$$

There should be better lower bounds on the acyclic matching number for graphs of large girth, and methods from [3,5,10] might be useful. Moreover, a lower bound as Conjecture 4 which is essentially tight for all possible densities of a graph $G$ of bounded maximum degree, would be interesting, yet very challenging.

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