Parameterized Complexity of Logic-Based Argumentation in Schaefer’s Framework

Yasir Mahmood
Leibniz Universität Hannover, Institut für Theoretische Informatik, Germany

Arne Meier
Leibniz Universität Hannover, Institut für Theoretische Informatik, Germany

Johannes Schmidt
Jönköping University, Department of Computer Science and Informatics, School of Engineering, Sweden

Abstract

Logic-based argumentation is a well-established formalism modelling nonmonotonic reasoning. It has been playing a major role in AI for decades, now. Informally, a set of formulas is the support for a given claim if it is consistent, subset-minimal, and implies the claim. In such a case, the pair of the support and the claim together is called an argument. In this paper, we study the propositional variants of the following three computational tasks studied in argumentation: ARG (exists a support for a given claim with respect to a given set of formulas), ARG-Check (is a given set a support for a given claim), and ARG-Rel (similarly as ARG plus requiring an additionally given formula to be contained in the support). ARG-Check is complete for the complexity class DP, and the other two problems are known to be complete for the second level of the polynomial hierarchy (Parson et al., J. Log. Comput., 2003) and, accordingly, are highly intractable. Analyzing the reason for this intractability, we perform a two-dimensional classification: first, we consider all possible propositional fragments of the problem within Schaefer’s framework (STOC 1978), and then study different parameterizations for each of the fragment. We identify a list of reasonable structural parameters (size of the claim, support, knowledge-base) that are connected to the aforementioned decision problems. Eventually, we thoroughly draw a fine border of parameterized intractability for each of the problems showing where the problems are fixed-parameter tractable and when this exactly stops. Surprisingly, several cases are of very high intractability (paraNP and beyond).

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms; Computing methodologies → Knowledge representation and reasoning

Keywords and phrases Parameterized complexity, logic-based argumentation, Schaefer’s framework

Funding This work was supported by the German Research Foundation (DFG), project ME 4279/1-2.

Acknowledgements The authors thank the anonymous referees for their valuable feedback.

1 Introduction

Argumentation is a nonmonotonic formalism in artificial intelligence around which an active research community has evolved [2, 1, 32, 3]. Essentially, there exist two branches of argumentation: the abstract [10] and the logic-based [4, 5, 9, 31] approach. The abstract setting mainly focusses on formalizing the argumentative structure in a graph-theoretic way. Arguments are nodes in a directed graph and the ‘attack-relation’ draws which argument eliminates which other. In the logic-based method, one looks for inclusion-minimal consistent sets of formulas Φ (the support) that entail a claim α, modelled through a formula (in the positive case one calls (Φ, α) an argument). In this paper, we focus on the latter formalism and, specifically, study three decision problems. The first, ARG, asks, given a set of formulas Δ (the knowledge-base) and a formula α, whether there exists a subset Φ ⊆ Δ such that (Φ, α) is an argument in Δ. The two further problems of interest are ARG-Check (is a given
set a support for a given claim), and ARG-Rel (ARG plus requiring an additionally given formula to be contained in the support, too).

**Example 1** ([4]). Consider the following two arguments. (A1) Support: Donald is a public person, so we can publicize details about his private life. Claim: We can publicize that Donald plays golf. (A2) Support: Donald just resigned from politics; as a result, he is no longer a public person. Claim: Donald is no longer a public person.

Formalizing these arguments would yield $A_1: \Phi_1 = \{x_{pd} \rightarrow x_{dg}, x_{pd}\}, \alpha_1 = \{x_{dg}\}$, $A_2: \Phi_2 = \{x_{rd} \rightarrow \neg x_{pd}, x_{rd}\}, \alpha_2 = \{\neg x_{pd}\}$, where $x_{pd} \triangleq "\text{Donald is a public person}"$, $x_{dg} \triangleq "\text{Donald plays golf}"$, $x_{rd} \triangleq "\text{Donald resigned from politics}"$. Each argument is supporting its claim, yet together they are conflicting, as $A_2$ attacks $A_1$.

It is rather computationally involved to compute the support of an argument, as ARG was shown to be $\Sigma_p^p$-complete by Parsons et al. [29]. Yet, there have been made efforts to improve the understanding of this high intractability by Creignou et al. [10] [14] in two settings: Schaefer’s [34] as well as Post’s [30] framework. Clearly, such research aims for drawing the fine intractability frontier of computationally involved problems to show for what restrictions there still is hope to reach algorithms running for practical applications. Both approaches mainly focus on restrictions on the logical part of the problem language, that is, restricting the allowed connectives or available constraints.

In this paper, Schaefer’s approach is our focus, that is, the formulas we study are propositional formulas in conjunctive normalform (CNF) whose clauses are formed depending on a fixed set of relations $\Gamma$ (the so-called constraint language, short CL). In this setting, Schaefer’s framework [34] captures well-known classes of CNF-formulas (e.g., Horn, dual-Horn, or Krom). Accordingly, one can see classifications in such a setting as a one-dimensional approach (the dimension is given rise by the considered logical fragments).

We consider a second dimension on the problem in this paper, namely, by investigating its parameterized complexity [13]. Motivated by the claim that the input length is not the only important structural aspect of problems, one studies so-called parameterizations (or parameters) of a problem. The goal of such studies is to identify a parameter that is relevant for practice but also is slowly growing or even of constant value. If, additionally, one is able to construct an algorithm that solves the problem in time $f(k) \cdot |x|^{O(1)}$ for some computable function $f$ and all inputs $(x, k)$, then one calls the problem fixed-parameter tractable. That is why in this case one can solve the problem (for fixed parameter values) in polynomial time. As a result, this complexity class is seen to capture the idea of efficiency in the parameterized sense. While $\text{NP}$-complete problems are considered intractable in the classical setting, on the parameterized level, the complexity class $\text{W}[1]$ is seen to play this counterpart. Informally, this class is characterized via a special kind of satisfiability questions. Above this class an infinite $\text{W}$-hierarchy is defined which culminates in the class $\text{W}[P]$, which in turn is contained in the class $\text{para-NP}$ (problems solvable by NTMs in time $f(k) \cdot |x|^{O(1)}$).

**Contributions**

Our main contributions are the following.

1. We initiate a thorough study of the parameterized complexity of logic-based argumentation.

   We study three parameters: size of the support, of the claim, and of the knowledge-base. We show that the complexity of ARG, regarding the claim as a parameter, varies: $\text{FPT}$, $\text{W}[1]$-$\text{, W}[2]$-$\text{, para-NP}$-$\text{, para-coNP}$-$\text{, as well as para-}\Sigma_p^p\text{-complete cases. For}$

   the same parameter, ARG-Check is $\text{FPT}$ for Schaefer, $\text{para-DP}$-complete otherwise. ARG-Rel is $\text{FPT}$, $\text{para-NP}$-$\text{, or para-}\Sigma_p^p\text{-complete.}$
The size of the knowledge-base as the parameter yields dichotomy results for the two problems ARG and ARG-Rel: \textbf{FPT} versus membership in \textbf{para-coNP} and a lower bound that relates to the implication problem.

Concerning the size of the support as the parameter, we prove a dichotomy: \textbf{FPT} versus \textbf{para-DP}-membership and the same hardness as the implication problem.

2. As a byproduct, we advance the algebraic tools in the context of Schaefer’s framework, and show a list of technical implementation results that are independent of the studied problem and might be beneficial for further research in the constraint context.

3. We classify the parameterized complexity of the implication problem (does a set of propositional formulas $\Phi$ imply a propositional formula $\alpha$?) with respect to the parameter $|\alpha|$ and show that it is \textbf{FPT} if the CL is Schaefer, and \textbf{para-coNP}-complete otherwise.

Related Work

Very recently, Mahmood et al. [26] presented a parameterized classification of abductive reasoning in Schaefer’s framework. Some of their cases, as well as results from Nordh and Zamuttini [25] relate to some of our results. The studies of the implication problem in the frameworks of Schaefer [35] as well as in the one in Post [6] prove a classical complexity landscape. Last year, Hecher et al. [18] conducted a parameterized study of abstract argumentation. The known classical results [10, 28, 35, 6] are partially used in some of our proofs, e.g., showing some parameterized complexity lower bounds. The two mentioned parameterized complexity related papers [18, 26] both are about different formalisms that are slightly related to our setting (the first is about abstract argumentation, the second on abduction).

2 Preliminaries

We assume familiarity with basic notions in complexity theory (cf. [36]) and use the complexity classes $\textbf{P}$, $\textbf{NP}$, $\textbf{coNP}$, $\Sigma^P_2$. For a set $S$, we write $|S|$ for its cardinality. Abusing notation, we will use $|w|$ for a string $w$, to denote its length. If $\varphi$ is a formula, then $\text{Vars}(\varphi)$ denotes its set of variables, and $\text{enc}(\varphi)$ its encoding. W.l.o.g., we assume a reasonable encoding computable in polynomial time that encodes variables in binary. The weight of an assignment $\sigma$ is the number of variables mapped to 1.

Parameterized Complexity

We give a brief introduction to parameterized complexity theory. A more detailed exposition can be found in the textbook of Downey and Fellows (15). A parameterized problem (PP) $\Pi$ is a subset of $\Sigma^* \times \mathbb{N}$, where $\Sigma$ is an alphabet. For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, $k$ is called the parameter. If there exists a deterministic algorithm deciding $\Pi$ in time $f(k) \cdot |x|^{O(1)}$ for every input $(x, k)$, where $f$ is a computable function, then $\Pi$ is \textit{fixed-parameter tractable} (short: \textbf{FPT}).

\begin{definition}
Let $\Sigma$ and $\Delta$ be two alphabets. A PP $\Pi \subseteq \Sigma^* \times \mathbb{N}$ fpt-reduces to a PP $\Theta \subseteq \Delta^* \times \mathbb{N}$, in symbols $\Pi \leq_{\text{FPT}} \Theta$, if the following is true: (i) there is an FPT-computable function $f$, such that, for all $(x, k) \in \Sigma^* \times \mathbb{N}$: $(x, k) \in \Pi \Leftrightarrow f(x, k) \in \Theta$, (ii) there exists a computable function $g$: $\mathbb{N} \to \mathbb{N}$ such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ and $f(x, k) = (y, \ell)$: $\ell \leq g(k)$.

The problems $\Pi$ and $\Theta$ are \textbf{FPT}-equivalent if both $\Pi \leq_{\text{FPT}} \Theta$ and $\Theta \leq_{\text{FPT}} \Pi$ is true. We also use higher classes via the concept of \textit{precomputation on the parameter}.
Definition 3. Let \( C \) be any complexity class. Then \( \text{para-}C \) is the class of all PPs \( \Pi \subseteq \Sigma^* \times \mathbb{N} \) such that there exists a computable function \( \pi : \mathbb{N} \rightarrow \Delta^* \) and a language \( L \subseteq \Pi \in C \) with \( L \subseteq \Sigma^* \times \Delta^* \) such that for all \( (x,k) \in \Sigma^* \times \mathbb{N} \) we have that \( (x,k) \in L \) if and only if \( (x,\pi(k)) \in L \).

Observe that \( \text{para-P} = \text{FPT} \) is true. For a constant \( c \in \mathbb{N} \) and a PP \( \Pi \subseteq \Sigma^* \times \mathbb{N} \), the \( c \)-slice of \( \Pi \), written as \( \Pi_c \), is defined as \( \Pi_c := \{ (x,k) \in \Sigma^* \times \mathbb{N} \mid k = c \} \). Observe that, in our setting, showing \( \Pi \in \text{para-}C \), it suffices to show \( \Pi_c \in C \) for every \( c \in \mathbb{N} \). Consider the following special subclasses of formulas:

\[
\begin{align*}
\Gamma_{0,d} &= \{ \ell_1 \land \ldots \land \ell_c \mid \ell_1,\ldots,\ell_c \text{ are literals and } c \leq d \}, \\
\Delta_{0,d} &= \{ \ell_1 \lor \ldots \lor \ell_c \mid \ell_1,\ldots,\ell_c \text{ are literals and } c \leq d \}, \\
\Gamma_{t,d} &= \{ \bigwedge_{i \in I} \alpha_i \mid \alpha_i \in \Delta_{t-1,d} \text{ for } i \in I \}, \\
\Delta_{t,d} &= \{ \bigvee_{i \in I} \alpha_i \mid \alpha_i \in \Gamma_{t-1,d}, i \in I \}. 
\end{align*}
\]

The parameterized weighted satisfiability problem (p-WSAT) for propositional formulas is defined as below. The problem p-WSAT(\( \Gamma_{t,d} \)) asks, given a \( \Gamma_{t,d} \)-formula \( \alpha \) with \( t,d \geq 1 \) and \( k \in \mathbb{N} \), parameterized by \( k \), is there a satisfying assignment for \( \alpha \) of weight \( k \)?

The classes of the \( \text{W} \)-hierarchy can be defined in terms of these problems.

Proposition 4 ([15]). The problem p-WSAT(\( \Gamma_{t,d} \)) is \( \text{W}[t] \)-complete for each \( t \geq 1 \) and \( d \geq 1 \), under \( \leq \text{FPT} \)-reductions.

Logic-based Argumentation

All formulas in this paper are propositional formulas. We follow the notion of Creignou et al. ([10]).

Definition 5 ([10]). Given a set of formulas \( \Phi \) and a formula \( \alpha \), one says that \( (\Phi,\alpha) \) is an argument (for \( \alpha \)) if (1) \( \Phi \) is consistent, (2) \( \Phi \models \alpha \), and (3) \( \Phi \) is subset-minimal w.r.t. (2). In case of \( \Phi \subseteq \Delta \), \( (\Phi,\alpha) \) is an argument in \( \Delta \). We call \( \alpha \) the claim, \( \Phi \) the support of the argument, and \( \Delta \) the knowledge-base.

In this paper, we consider three problems from the area of logic-based argumentation, namely ARG, ARG-Check, and ARG-Rel. The problem ARG asks, given a set of formulas \( \Delta \) and a formula \( \alpha \), is there a set \( \Phi \subseteq \Delta \) such that \( (\Phi,\alpha) \) is an argument in \( \Delta \)? The problem ARG-Check asks, given a set of formulas \( \Phi \) and a formula \( \alpha \), is \( (\Phi,\alpha) \) an argument? The problem ARG-Rel asks, given a set of formulas \( \Delta \), and formulas \( \psi \in \Delta \) and \( \alpha \), is there a set \( \Phi \subseteq \Delta \) with \( \psi \in \Phi \) such that \( (\Phi,\alpha) \) is an argument in \( \Delta \)?

Turning to the parameterized complexity perspective on the introduced problems, immediate parameters that we consider are \( |\text{enc}(\mathcal{X})| \) (size of the encoding of \( \mathcal{X} \)), \( |\mathcal{X}| \) (number of formulas in \( \mathcal{X} \)), \( |\text{Vars}(\mathcal{X})| \) (number of variables in \( \mathcal{X} \)) for \( \mathcal{X} \in \{ \Delta, \Phi \} \), as well as \( |\text{enc}(\alpha)| \) and \( |\text{Vars}(\alpha)| \). Regarding the parameterized versions of the problems from above, e.g., p-ARG(k), where \( k \) is a parameter, then defines the version of ARG parameterized by \( k \), accordingly.

In the following, we want to formally relate the mentioned notions of encoding length, number of variables, as well as number of formulas. We will see that bounding the encoding length, implies having limited space for encoding variables and, in turn, restricts the number of possible formulas. However, the converse is also true: if one bounds the number of variables, then one also has limited possibilities about defining different formulas. The following definition makes clear what ‘different’ means in our context.
Definition 6 (Formula redundancy). A CNF-formula $\varphi = \bigwedge_{i=1}^{m} C_i$, with $C_i = (\ell_{i,1} \lor \cdots \lor \ell_{i,n_i})$ is redundant if there exist $1 \leq i \neq j \leq m$ such that $\{ \ell_{i,k} \mid 1 \leq k \leq n_i \} = \{ \ell_{j,k} \mid 1 \leq k \leq n_j \}$.

Example 7. The formulas $x \land x$ and $(x \lor x \lor y) \land (x \lor y)$ are redundant. The formulas $x \land y$ and $(x \lor y) \land x$ are not redundant.

Liberatore [24] studied a stronger notion of redundancy in the context of CNF-formulas, namely, on the level of implied clauses. We do not need such a strict notion of redundancy here, as the weaker notion suffices for proving the following Lemma. As a result, in the following, we consider only formulas that are just not redundant. The redundancy (in our context) can be straightforwardly checked in time quadratic in the length of the given formula.

Lemma 8. For any set of CNF-formulas $\Phi$, we have that
1. $|\Phi| \leq 2^{2^{|\text{Vars}(\Phi)|}}$,
2. $f(|\text{Vars}(\Phi)|) \leq |\text{enc}(\Phi)|$, where $f$ is some computable function, and
3. $|\text{enc}(\Phi)| \leq |\Phi|^3$.

Proof. 1. Let $v \in \mathbb{N}$ be a fixed number of variables. As we consider CNF-formulas, a formula consists of clauses of literals. The number of possible clauses then is the number of subsets of possible literals $\{ x_1, \ldots, x_v, \neg x_1, \ldots, \neg x_v \}$, that is, $2^{2^v}$-many. A CNF-formula is a subset of the set of possible clauses. As a result, we have $2^{2^v}$-many possible CNF-formulas.

2. We represent a variable $x_i$ by its binary encoding. Clearly, $|\text{enc}(\Phi)| = \sum_{\varphi \in \Phi} |\text{enc}(\varphi)|$. However,

$$|\text{enc}(\varphi)| \leq |\varphi| \cdot \log(|\text{Vars}(\varphi)|) + |\varphi| = |\varphi| \cdot (\log(|\text{Vars}(\varphi)|) + 1) \leq |\Phi| \cdot (\log(|\text{Vars}(\Phi)|) + 1).$$

As a result, we get

$$|\text{enc}(\Phi)| \leq |\Phi| \cdot \max_{\varphi \in \Phi} |\text{enc}(\varphi)| \leq |\Phi|^2 \cdot (\log(|\text{Vars}(\Phi)|) + 1)$$

As $|\Phi| \leq |\text{enc}(\Phi)|$ is true, we have that

$$|\text{enc}(\Phi)| \leq |\Phi|^2 \cdot (\log(|\text{Vars}(\Phi)|) + 1) \Leftrightarrow (\log(|\text{Vars}(\Phi)|) + 1)^{-1} \leq |\text{enc}(\Phi)|$$

3. As we only consider formulas that are not redundant, the encoding length of a set of formulas contains information about the number of its formulas. We have that $|\text{enc}(\Phi)| \leq |\Phi|^2 \cdot (\log(|\text{Vars}(\Phi)|) + 1)$ (as in (2)). However, $|\Phi| \leq 2^{2^{|\text{Vars}(\Phi)|}}$, and, as a result, we get

$$|\text{enc}(\Phi)| \leq |\Phi|^2 \cdot (\log(2 \cdot |\Phi|) + 1) \leq |\Phi|^3.$$ 

Notice that due to Lemma 8 the problems ARG, ARG-Check, ARG-Rel parameterized with respect to any of the parameters for the respective three (two) variants introduced above are FPT-equivalent. As a result, we will choose the one of the three (two) variants in our results that is technically most convenient. Notice also that the parameter $|\Phi|$ only makes sense for ARG-Check, whereas $|\Delta|$ makes sense only for the other two problems, that is, ARG and ARG-Rel.
2.1 Schaefer’s Framework

For a deeper introduction into Schaefer’s CSP framework, consider the article of Böhler et al. [8].

A logical relation of arity \( k \in \mathbb{N} \) is a relation \( R \subseteq \{0, 1\}^k \), and a constraint \( C \) is a formula \( C = R(x_1, \ldots, x_k) \), where \( R \) is a \( k \)-ary logical relation, and \( x_1, \ldots, x_k \) are (not necessarily distinct) variables. If \( V \) is a set of variables and \( u \) a variable, then \( C[V/u] \) denotes the constraint obtained from \( C \) by replacing every occurrence of every variable of \( V \) by \( u \). An assignment \( \theta \) satisfies \( C \), if \( (\theta(x_1), \ldots, \theta(x_k)) \in R \). A constraint language (CL) \( \Gamma \) is a finite set of logical relations, and a \( \Gamma \)-formula is a conjunction of constraints over elements from \( \Gamma \). Eventually, a \( \Gamma \)-formula \( \varphi \) is satisfied by an assignment \( \theta \), if \( \theta \) simultaneously satisfies all constraints in it. In such a case \( \theta \) is also called a model of \( \varphi \). Whenever a \( \Gamma \)-formula or a constraint is logically equivalent to a single clause or term or literal, we treat it as such. We say that a \( k \)-ary relation \( R \) is represented by a formula \( \phi \) in CNF if \( \phi \) is a formula over \( k \) distinct variables \( x_1, \ldots, x_k \) and \( \phi \equiv R(x_1, \ldots, x_k) \). Moreover, we say that \( R \) is

- **Horn** (resp., **dual-Horn**) if \( \phi \) contains at most one positive (negative) literal per each clause.
- **Bijunctive** if \( \phi \) contains at most two literals per each clause.
- **Affine** if \( \phi \) is a conjunction of linear equations of the form \( x_1 \oplus \ldots \oplus x_n = a \) where \( a \in \{0, 1\} \).
- **Essentially negative** if every clause in \( \phi \) is either negative or unit positive. \( R \) is **essentially positive** if every clause in \( \phi \) is either positive or unit negative.
- **1-valid** (resp., **0-valid**) if every clause in \( \phi \) contains at least one positive (negative) literal. Furthermore, we say a relation is **Schaefer** if it is Horn, dual-Horn, bijunctive, or affine. We say that a relation is **\( \varepsilon \)-valid** if it is 1- or 0-valid or both. Finally, for a property \( P \) of a relation, we say that a CL \( \Gamma \) is \( P \) if all relations in \( \Gamma \) are \( P \).

**Definition 9.** 1. The set \( \langle \Gamma \rangle \) is the smallest set of relations that contains \( \Gamma \), the equality constraint, and which is closed under primitive positive first order definitions, that is, if \( \phi \) is an \( \Gamma \) \( \cup \{=\} \)-formula and \( R(x_1, \ldots, x_n) \equiv \exists y_1 \ldots \exists y_l \phi(x_1, \ldots, x_n, y_1, \ldots, y_l) \), then \( R \in \langle \Gamma \rangle \). In other words, \( \langle \Gamma \rangle \) is the set of relations that can be expressed as an \( \Gamma \) \( \cup \{=\} \)-formula with existentially quantified variables.

2. The set \( \langle \Gamma \rangle_{\neq} \) is the set of relations that can be expressed as a \( \Gamma \)-formula with existentially quantified variables (no equality relation is allowed).

3. The set \( \langle \Gamma \rangle_{\neq, \neq} \) is the set of relations that can be expressed as a \( \Gamma \)-formula (neither the equality relation nor existentially quantified variables are allowed).

The set \( \langle \Gamma \rangle \) is called a relational clone or a co-clone with base \( \Gamma \) [7]. Notice that for a co-clone \( C \) and a CL \( \Gamma \) the statements \( \Gamma \subseteq C \), \( \langle \Gamma \rangle \subseteq C \), \( \langle \Gamma \rangle_{\neq} \subseteq C \) and \( \langle \Gamma \rangle_{\neq, \neq} \subseteq C \) are equivalent. Throughout the paper, we refer to different types of Boolean relations and corresponding co-clones following Schaefer’s terminology [33]. For a tabular overview of co-clones, relational properties, and bases, we refer the reader to Table 1. Note that \( \langle \Gamma \rangle_{\neq} \subseteq \langle \Gamma \rangle \) is true by definition. The other direction is not true in general. However, if \( (x = y) \in \langle \Gamma \rangle_{\neq} \), then we have that \( \langle \Gamma \rangle_{\neq} = \langle \Gamma \rangle \).

**Example 10.** Let \( R(x_1, x_2, x_3) := (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \). Then

\[
(x_1 \lor x_2) \land (x_2 \lor x_3 = 0) \equiv \exists y (R(x_1, x_2, y) \land F(y) \land (x_2 = x_3)),
\]

where \( F = \{0\} \). This implies that \((x_1 \lor x_2) \land (x_2 \lor x_3 = 0) \in \langle \{R, F\} \rangle \).
| co-clone | base | clause type | name/indication |
|----------|------|-------------|----------------|
| BR (H₆) | 1-IN-3 = \{001,010,100\} | all clauses | all Boolean relations |
| I₀ | x → y | at least one positive literal per clause | 1-valid |
| I₁ | x → y | at least one negative literal per clause | 0-valid |
| I₂ | EVEN⁴, x → y | at least one negative and one positive literal per clause | 1- and 0-valid |
| I₃ | NAE = \{0,1\} \setminus \{000,111\} | cf. previous column | comprehensive |
| I₄ | DIP = \{0,1\} \setminus \{010,001\} | cf. previous column | comprehensive and 1- and 0-valid |
| I₅ | x ∧ y → z, x, ¬z | clauses with at most one positive literal | Horn |
| I₆ | x ∧ y → z, x | clauses with exactly one positive literal | definite Horn |
| I₇ | x ∧ y → z | \((-x₁ ∧ x₂) ∨ \cdots ∨ \neg xₙ\), n ≥ 2 | Horn and 0-valid |
| I₈ | x → y → z | Horn and 1- and 0-valid |
| I₉ | x ∧ y → ¬z | DualHorn and 1- and 0-valid |
| I₁₀ | x ∧ y → z, x, ¬z | clauses with at most one negative literal | dualHorn and 1-valid |
| I₁₁ | x ∧ y → ¬z | clauses with exactly one negative literal | definite dualHorn |
| I₁₂ | x ∧ y → ¬z | \((-x₁ ∧ x₂) ∨ \cdots ∨ \neg xₙ\), n ≥ 2 | dualHorn and 1- and 0-valid |
| I₁₃ | EVEN⁴, x, ¬z | all affine clauses (all linear equations) | affine |
| I₁₄ | EVEN⁴, x | \((x₁ \cdots xₙ = 0), n ≥ 0, n = n\) (mod 2) | affine and 1-valid |
| I₁₅ | EVEN⁴, ¬z | affine and 0-valid |
| I₁₆ | EVEN⁴, x ⊕ y | \((x₁ \cdots xₙ = 0), n \text{ even}, n \in \{0,1\}\) | |
| I₁₇ | EVEN⁴, x ⊕ y | all affine and 1- and 0-valid | |
| I₁₈ | x ⊕ y → z | clauses of size 1 or 2 | 2-affine |
| I₁₉ | x ⊕ y | affine clauses of size 2 | strict 2-affine |
| I₂₀ | x → y, x, ¬z | \((x₁ → x₂), (x₁), (¬x₁)\) | implicative and 1-valid |
| I₂₁ | x → y, x | \((x₁ → x₂), (x₁)\) | implicative and 0-valid |
| I₂₂ | x → y → z | \((x₁ → x₂), (¬x₁)\) | implicative and 1- and 0-valid |
| S₁₀ | cf. next column | \((x₁), (x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | IHS-l |
| S₁₅ | cf. next column | \((x₁), (x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | IHS-l | of width k |
| S₁₆ | cf. next column | \((x₁), (¬x₁ ∨ \cdots ∨ ¬xₙ), n = 0, (x₁ → x₂)\) | essentially negative |
| S₁₇ | cf. next column | \((x₁), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0, (x₁ → x₂)\) | essentially negative of width k |
| S₁₈ | cf. next column | \((x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | - |
| S₁₉ | cf. next column | \((x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | - |
| S₂₀ | cf. next column | \((¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0, (x₁ → x₂)\) | negative |
| S₂₁ | cf. next column | \((¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0, (x₁ → x₂)\) | negative of width k |
| S₂₂ | cf. next column | \((¬x₁), (x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | IHS-l+ |
| S₂₃ | cf. next column | \((¬x₁), (x₁ → x₂), (¬x₁ ∨ \cdots ∨ ¬xₙ), n ≥ 0\) | IHS-l+ | of width k |
| S₂₄ | cf. next column | \((¬x₁), (x₁ ∨ \cdots ∨ xₙ), n ≥ 0\) | essentially positive |
| S₂₅ | cf. next column | \((¬x₁), (x₁ ∨ \cdots ∨ xₙ), n ≥ 0, (x₁ → x₂)\) | essentially positive of width k |
| S₂₆ | cf. next column | \((x₁ → x₂), (x₁ ∨ \cdots ∨ xₙ), n ≥ 0\) | - |
| S₂₇ | cf. next column | \((x₁ → x₂), (x₁ ∨ \cdots ∨ xₙ), n ≥ 0\) | - |
| S₂₈ | cf. next column | \((x₁ → x₂), (x₁ ∨ \cdots ∨ xₙ), n ≥ 0\) | positive |
| S₂₉ | cf. next column | \((x₁ ∨ \cdots ∨ xₙ), n ≥ 0, (x₁ → x₂)\) | positive of width k |
| R₂₀ | x₁, ¬x₂ | \((x₁), (¬x₁), (x₁ → x₂)\) | - |
| R₂₁ | x₁ | \((x₁)\) | - |
| R₂₂ | ¬x₁ | \((¬x₁), (x₁ → x₂)\) | - |
| R₂₃ | (B/B) | \((x₁ → x₂)\) | - |

Table 1 Overview of bases and clause descriptions for co-clones, where \(\text{EVEN}^4 = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus 1\).
2.2 Technical Implementation Results

We say a Boolean relation $R$ is strictly essentially positive (resp., strictly essentially negative) if it can be defined by a conjunction of literals and positive clauses (resp., negative clauses) only. Note that the only difference to essentially positive (resp., essentially negative) is the absence of the equality relation (see Table 1). We abbreviate in the following essentially positive by “ess.pos.” and essentially negative by “ess.neg.”.

Proposition 11. Let $\Gamma$ be a CL that is neither ess.pos., nor ess.neg. Then, we have that $(x = y) \in (\Gamma)_{\not\not\not}$ and $(\Gamma) = (\Gamma)_{\not\not\not}$.

With the following implementation result we can strengthen this statement to Lemma 13.

Lemma 12. Let $\Gamma$ be a CL that is not $\varepsilon$-valid. If $\Gamma$ is ess.neg. and not strictly ess.neg. or ess.pos. and not strictly ess.pos. then we have that $(x = y) \land t \land \neg f \in (\Gamma)_{\not\not\not}$.

Proof. We prove the statement for $\Gamma$ that is ess.pos. but not strictly ess.pos. The other case can be treated analogously. W.l.o.g., let $\Gamma = \{R\}$, thus $R$ is ess.pos. but not strictly ess.pos. Furthermore, $R$ is neither 1-valid nor 0-valid. Let $R$ be of arity $k$ and let $V = \{x_1, \ldots, x_k\}$ be a set of $k$ distinct variables. By definition of ess.pos. (cf. IS$_{02}$ in Table 1), $R$ can be written as conjunction of negative literals, positive clauses and equalities.

If $R$ can be written without any equality, then $R$ is strictly ess.pos., a contradiction. As a result, any representation of $R$ as conjunction of negative literals, positive clauses and equalities requires at least one equality. Suppose, w.l.o.g., that $R(x_1, \ldots, x_k) \not\models (x_1 = x_2)$, while $R(x_1, \ldots, x_k) \not\models x_1$. We define the following three subsets of $V$: $W = \{x_i \mid R(x_1, \ldots, x_k) \models (x_2 = x_1)\}$, $N = \{x_i \mid R(x_1, \ldots, x_k) \models \neg x_1\}$, and $P = V \setminus (W \cup N)$

By construction the three sets provide a partition of $V$. Then, $W$ is nonempty by construction, $N$ is nonempty since $R$ is not 1-valid and $P$ is nonempty since $R$ is not 0-valid. Denote by $C$ the $\{R\}$-constraint $C = R(x_1, \ldots, x_k)$. Consider the constraint $M(x_1, x_2, t, f) = C[W/x_2, P/t, N/f]$. One verifies that $M(x_1, x_2, t, f) \equiv (x_1 = x_2) \land t \land \neg f$.

Lemma 13. Let $\Gamma$ be a CL that is neither strictly ess.pos., nor strictly ess.neg. Then $(x = y) \in (\Gamma)_{\not\not\not}$ and $(\Gamma) = (\Gamma)_{\not\not\not}$.

Proof. If $\Gamma$ is not ess.pos. and not ess.neg. the statement follows from Proposition 11. Note that this lemma’s statement implies that $\Gamma$ is not $\varepsilon$-valid. If $\Gamma$ is ess.pos. or ess.neg., by Lemma 12 we have $(x = y) \land t \land \neg f \in (\Gamma)_{\not\not\not}$. Conclude by noticing that $(x = y) \equiv \exists t \forall f (x = y) \land t \land \neg f \in (\Gamma)_{\not\not\not}$.

Lemma 14. Let $\Gamma$ be a CL that is neither $\varepsilon$-valid, nor ess.pos., nor ess.neg. Then, if $\Gamma$ is 1. not Horn, not dual-Horn, and not complementive, then $(x \neq y) \land t \land \neg f \in (\Gamma)_{\not\not\not}$, 2. not Horn, not dual-Horn, and complementive, then $(x \neq y) \not\in (\Gamma)_{\not\not\not}$, and 3. Horn or dual-Horn, then $(x = y) \land t \land \neg f \not\in (\Gamma)_{\not\not\not}$.

Proof. This follows immediately from the proof of Proposition 11. The proof given in [26, 25, Lemma 7] makes a case distinction according to whether $\Gamma$ is 0-valid and/or 1-valid. In the case of non $\varepsilon$-valid $\Gamma$ a further case distinction is made according to whether $\Gamma$ is Horn and/or dualHorn. Here the statements 1., 2., and 3. are proven.

Let us denote by $T/F$ the unary relations that implement true/false. That is, $T = \{(1)\}$ and $F = \{(0)\}$. The following implementation results are folklore.
Proposition 15 (Creignou et al. [11]). If Γ is a CL that is
1. complementive and not $\varepsilon$-valid, then $(x \neq y) \in (\Gamma)_{\not\in \Phi}$,
2. neither complementive, nor $\varepsilon$-valid, then $(t \land \bar{f}) \in (\Gamma)_{\not\in \Phi}$,
3. $1$-valid and not $0$-valid, then $T \in (\Gamma)_{\not\in \Phi}$,
4. $0$-valid and not $1$-valid, then $F \in (\Gamma)_{\not\in \Phi}$, and
5. $0$-valid and $1$-valid, then $(x = y) \in (\Gamma)_{\not\in \Phi}$.

2.3 Parameterized Implication Problem

In this subsection, we consider the parameterized complexity of the implication problem (IMP). The problem IMP(Γ) asks, given a set of Γ-formulas $\Phi$ and a Γ-formula $\alpha$, is $\Phi \vDash \alpha$ true? For p-IMP, the parameterized version of IMP, we consider the parameter $k \in \{|\Phi|,|\alpha|\}$, and also write p-IMP($\Gamma$, $k$). The following corollary is due to Schnoor and Schnoor [35, Theorem 6.5]. They study a restriction of our problem IMP, where $|\Phi| = 1$.

Corollary 16. Let Γ be a CL. IMP(Γ) is in P when Γ is Schaefer and coNP-complete otherwise.

Consequently, the parameterized problem p-IMP(Γ, $k$) is FPT when Γ is Schaefer and $k \in \{|\Phi|,|\alpha|\}$. We consider the cases when Γ is not Schaefer. In the following, we differentiate the restrictions on $\Phi$ from the ones on $\alpha$. That is, we introduce a technical variant, IMP($\Gamma'$, Γ) of the implication problem. An instance of IMP($\Gamma'$, Γ) is a tuple ($\Phi$, $\alpha$), where $\Phi$ is a set of $\Gamma'$-formulas and $\alpha$ is a $\Gamma$-formula. The following corollary also follows from the work of Schnoor and Schnoor [35, Theorem 6.5].

Corollary 17. Let $\Gamma$ and $\Gamma'$ be non-Schaefer CLs. If $\Gamma' \subseteq \langle \Gamma \rangle$ then IMP($\Gamma'$, $\Gamma$) $\leq^P_m$ IMP($\Gamma$).

Regarding non-Schaefer CLs, it turns out that the parameter $\alpha$ does not make the problem any easier. One possible explanation for this hardness is that the formulas in $\Phi$ and $\alpha$ do not necessarily share a set of variables.

Lemma 18. The problem p-IMP($\Gamma$, $|\alpha|$) is para-coNP-complete when the CL $\Gamma$ is not Schaefer.

Proof. Membership follows because the classical problem is in coNP. To achieve the lower bound, we reduce from the unsatisfiability problem. That is, given a formula $\Phi$, the question is whether $\Phi$ is unsatisfiable. Moreover, checking unsatisfiability is coNP-complete for non-Schaefer languages (follows by Schaefer’s [31] SAT classification).

We will inherently use Corollary [17] and make a case distinction as whether ($\Phi$, $\alpha$) is $1$-valid, $0$-valid or complementive.

Case 1. Let $\Gamma$ be $1$-valid and not $0$-valid. We prove that for some well chosen $1$-valid language $\Gamma'$ and a $\Gamma'$-formula $\Phi$, the problem p-IMP($\Gamma'$, $\Gamma$, $|\alpha|$) is para-coNP-hard. According to item [3] in Proposition [15], $T \in (\Gamma')_{\not\in \Phi}$. Let $\alpha = T(x)$ and $\Phi$ be a $\Gamma'$-formula where $x$ does not appear. Then $\Phi \vDash \alpha$ if and only if $\Phi$ is unsatisfiable. This is because, if $\Phi$ is satisfiable then there is an assignment $s$ such that $s \vDash \psi$. This gives a contradiction because the assignment $s'$ that extends $s$ by $s'(x) = 0$ satisfies that $s' \not\vDash \Phi$ and $s' \not\vDash \alpha$.

Case 2. Let $\Gamma$ be $0$-valid and not $1$-valid. According to item [4] in Proposition [15], $F \in (\Gamma')_{\not\in \Phi}$. This case is similar to Case 1, as we take $\alpha = F(x)$ and $\Phi$ a $\Gamma'$-formula not containing $x$. 
Case 3. Let \( \Gamma \) be complementable but not \( \varepsilon \)-valid. We prove that for any chosen complementary language \( \Gamma' \) and \( \Gamma \)-formula \( \alpha \), the problem \( p\text{-IMP}(\Gamma', \Gamma, |\alpha|) \) is \( \text{para-coNP} \)-hard. According to item (1) in Proposition 15, \( x \neq y \in (\Gamma')_{\exists \neq \forall} \). Then, \( \text{coNP} \)-hardness follows, as for any set of \( \Gamma' \)-formulas \( \Phi \), \( \Phi \models (x \neq x) \) if and only if \( \Phi \) is unsatisfiable.

Case 4. Let \( \Gamma \) be \( 0 \)- and \( 1 \)-valid. By Lemma 14 (1)/(2), we have access to \( \neq \). We can state a reduction from the complement of \( \text{SAT} \) to \( p\text{-IMP}(\Gamma, |\alpha|) \) as in Case 3. That is, \( \Phi \) is unsatisfiable if and only if \( \Phi \models x \neq x \) for a fresh variable \( x \).

Note that regarding the parameter \(|\Phi|\), the problem \( p\text{-IMP}(\Gamma, |\Phi|) \) is \( \text{FPT} \) if \( \Gamma \) is Schaefer. Otherwise, only \( \text{coNP} \)-membership is clear.

3 Parameter: Size of the Claim \( \alpha \)

In this section we discuss the complexity results regarding the parameter \( \alpha \), that is, the number of variables and the encoding size of \( \alpha \). It turns out that the computational complexity of the argumentation problems is hidden in the structure of the underlying CL. That is, in many cases, considering the claim size as a parameter does not lower the complexity. This is proved by noting that certain slices of the parameterized problems already yield hardness results.

**Theorem 19.** \( p\text{-ARG}(\Gamma, |\alpha|) \), for a CL \( \Gamma \), is
1. \( \text{FPT} \) if \( \Gamma \) is Schaefer and \( \varepsilon \)-valid,
2. \( \text{para-NP} \)-complete if \( \Gamma \) is Schaefer and neither \( \varepsilon \)-valid, nor strictly ess.pos., nor strictly ess.neg.,
3. in \( \text{W}[1] \) if \( \Gamma \) is strictly ess.neg. and strictly ess.pos.,
4. in \( \text{W}[2] \) if \( \Gamma \) is strictly ess.neg. or strictly ess.pos.,
5. \( \text{para-coNP} \)-complete if \( \Gamma \) is not Schaefer and \( \varepsilon \)-valid, and
6. \( \text{para-} \Sigma^p_2 \)-complete if \( \Gamma \) is not Schaefer and not \( \varepsilon \)-valid.

**Proof.** (1) The classical problem \( \text{ARG}(\Gamma) \) is already in \( \text{P} \) for this case [10 Thm 5.3]. (2) The upper bound follows because the unparameterized problem \( \text{ARG}(\Gamma) \) is in \( \text{NP} \) [10 Prop 5.1]. The lower bound is proven in Lemmas 21, 22 and 23. (3) is proven in Lemma 24.

For (5) (resp., (6)), the membership follows because the classical problem is in \( \text{coNP} \) (resp., \( \Sigma^p_2 \)) [10 Thm 5.3]. For hardness of \( p\text{-ARG}(\Gamma, |\alpha|) \) when \( \Gamma \) is \( \varepsilon \)-valid, notice that, since \( \Delta \) is \( \varepsilon \)-valid, an instance \( (\Delta, \alpha) \) of \( p\text{-ARG} \) admits an argument if and only if \( \Delta \models \alpha \). The result follows from Lemma 15 because the implication problem is still \( \text{para-coNP} \)-hard. Finally, when \( \Gamma \) is not Schaefer and not \( \varepsilon \)-valid, in the proofs of Creignou et al. [10 Prop 5.2] the constructed reductions define \( \alpha \) whose length is 2 or 3. Accordingly, either the 2-slice or the 3-slice is \( \Sigma^p_2 \)-hard. This gives the desired hardness result.

For technical reasons we introduce the following variant of the argumentation existence problem. The problem \( \text{ARG}(\Gamma, R) \) asks, given a set of \( \Gamma \)-formulas \( \Delta \) and an \( R \)-formula \( \exists \Phi \subseteq \Delta \) s.t. \( (\Phi, \alpha) \) is an argument in \( \Delta \)?

**Lemma 20.** Let \( \Gamma, \Gamma' \) be two CLs and \( R \) a Boolean relation. If \( \Gamma' \subseteq (\Gamma)_{\exists \neq \forall} \) and \( R \in (\Gamma)_{\exists \neq \forall} \), then \( p\text{-ARG}(\Gamma', R) \leq^\text{log} \text{ARG}(\Gamma) \).

**Proof.** Let \( (\Delta, \alpha) \) be an instance of the first problem, where \( \Delta = \{ \delta_i \mid i \in I \} \) and \( \alpha = R(x_1, \ldots, x_k) \). We map this instance to \( (\Delta', \alpha') \), where \( \Delta' = \{ \delta'_i \mid \delta_i \in \Delta \} \) and \( \alpha' \) is a \( \Gamma \)-formula equivalent to \( R(x_1, \ldots, x_k) \) (which exists because \( R \in (\Gamma)_{\exists \neq \forall} \)). For \( i \in I \) we
obtain $\delta'_i$ from $\delta_i$ by replacing $\delta_i$ by an equivalent $\Gamma$-formula with existential quantifiers (such a representation exists since $\Gamma' \subseteq \langle \Gamma \rangle_{\phi}$) and deleting all existential quantifiers.

Note that the previous result is only used to show lower bounds for specific slices and, accordingly, is stated in the classical setting.

\textbf{Lemma 21.} If the CL $\Gamma$ is neither affine, nor $\varepsilon$-valid, nor ess.pos., nor ess.neg., then p-ARG($\Gamma, |\alpha|$) is para-NP-hard.

\textbf{Proof.} We give a reduction from the NP-complete problem Pos-1-In-3-Sat such that $|\alpha|$ is constant. An instance of Pos-1-In-3-Sat is a 3CNF-formula with only positive literals, the question is to determine whether there is a satisfying assignment which maps exactly one variable in each clause to true. We make a case distinction according to the case (1) and (3) in Lemma 14 (case (2) can not occur for $\phi$). We then show that the other two cases can be treated with minor modifications of the procedure.

Let $\varphi$ be an instance of Pos-1-In-3-Sat. We first reduce $\varphi$ to and instance $(\Delta, \alpha)$ of ARG($\{T, F, =\}, (x = y) \land t \land \neg f$), and then conclude with Lemmas 14 and 20. Given $\varphi = \bigwedge_{i=1}^{k} (x_i \lor y_i \lor z_i)$, an instance of Pos-1-In-3-Sat and let $t, f, c_1, \ldots, c_{k+1}$ be fresh variables. We let $\Delta$ and $\alpha$ as following.

$$
\Delta = \bigcup_{i=1}^{k} \{x_i \land \neg y_i \land \neg z_i \land (c_i = c_{i+1}) \land t \land \neg f\}
\cup \bigcup_{i=1}^{k} \{\neg x_i \land y_i \land \neg z_i \land (c_i = c_{i+1}) \land t \land \neg f\}
\cup \bigcup_{i=1}^{k} \{\neg x_i \land \neg y_i \land z_i \land (c_i = c_{i+1}) \land t \land \neg f\},
\alpha = (c_1 = c_{k+1}) \land t \land \neg f.
$$

Note that any formula in $\Delta$ is expressible as $\Gamma$-formula since $\{T, F, =\} \subseteq \mathcal{M}_2 \subseteq \langle \Gamma \rangle_{\phi}$ (cf. Table 1). Since by Lemma 13, $(\langle \Gamma \rangle_{\phi}) = (\Gamma)$ and by construction $(x = y) \land t \land \neg f \in (\langle \Gamma \rangle_{\phi})$, we have, by Lemma 20 the desired reduction to p-ARG($\Gamma, |\alpha|$). Note that in the reduction of Lemma 20 the size of $\alpha$ is always constant.

For case (1) of Lemma 14 we have that $(x \neq y) \land t \land \neg f \in (\langle \Gamma \rangle_{\phi})$. To cope with this change in the reduction we introduce one additional variable $d$ and replace $\alpha$ by $(c_1 \neq d) \land (d \neq c_{k+1}) \land t \land \neg f$.

\textbf{Lemma 22.} If the CL $\Gamma$ is affine, neither $\varepsilon$-valid, nor ess. pos., nor ess.neg., then p-ARG($\Gamma, |\alpha|$) is para-NP-hard.

\textbf{Proof.} We proceed analogously to the proof of Lemma 21. We give a reduction from the NP-complete problem Pos-1-In-3-Sat such that $|\alpha|$ is constant. We make a case distinction according to case (1) and (2) in Lemma 14 (case 3. can not occur for $\Gamma$ is affine and not ess.pos.). First, we treat the second case, that is, we have that $(x \neq y) \in (\langle \Gamma \rangle_{\phi})$. Then, we show that the first case can be treated with minor modifications of the procedure.

Now, we reduce Pos-1-In-3-Sat to ARG($\{=, \neq\}, \{\neq\}$), and then conclude with Lemmas 14 and 20. We give the following reduction. Let $\varphi = \bigwedge_{i=1}^{k} (x_i \lor y_i \lor z_i)$ be an instance of Pos-1-In-3-Sat and let $t, d, c_1, \ldots, c_{k+1}$ be fresh variables. We map $\varphi$ to $(\Delta, \alpha)$, where

$$
\Delta = \bigcup_{i=1}^{k} \{(x_i = t) \land (y_i \neq t) \land (z_i \neq t) \land (c_i = c_{i+1})\}
\cup \bigcup_{i=1}^{k} \{(x_i \neq t) \land (y_i = t) \land (z_i \neq t) \land (c_i = c_{i+1})\}
\cup \bigcup_{i=1}^{k} \{(x_i \neq t) \land (y_i \neq t) \land (z_i = t) \land (c_i = c_{i+1})\},
\alpha = (c_1 \neq d) \land (d \neq c_{k+1}).
$$
Note that any formula in $\Delta$ is expressible as $\Gamma$-formula since $\{=,\neq\} \subseteq ID \subseteq (\Gamma)$ (cf. [26 Table 1]). Since by Lemma 13 $(\Gamma)_{\neq} = (\Gamma)$ and by construction $(x \neq y) \in (\Gamma)_{\neq}$, we have by Lemma 20 the desired reduction to $p$-ARG($\Gamma$). Note that in the reduction of Lemma 20 the size of $\alpha$ is always constant.

For case (1) of Lemma 14 we have that $(x \neq y) \wedge t \wedge \neg f \in (\Gamma)_{\neq}$. To cope with this change in the reduction, we introduce one additional variable $f$ and add the constraints $t \land \neg f$ to $\alpha$ as well as to every formula in $\Delta$.

**Lemma 23.** Let $\Gamma$ be a CL that is not $e$-valid. If $\Gamma$ is ess.pos. and not strictly ess.pos. or ess.neg. and not strictly ess.neg., then $p$-ARG($\Gamma, |\alpha|$) is para-NP-hard.

**Proof.** We can use exactly the same reduction as in Lemma 21 except we do not require a case distinction. Note that, by Proposition 15, we have that $(t \land \neg f) \in (\Gamma)_{\neq}$. Since $\exists f (t \land \neg f) \equiv T(t)$ and $\exists t (t \land \neg f) \equiv F(f)$, we conclude that $T, F \in (\Gamma)_{\neq}$. Further, by Lemma 13 we have that $(x = y) \in (\Gamma)_{\neq}$. Together we have $\{T, F, =\} \subseteq (\Gamma)_{\neq}$, and thus any formula in $\Delta$ is expressible as $\Gamma$-formula with existential quantifiers but without equality. By Lemma 12 it follows that $(x = y) \wedge t \wedge \neg f \in (\Gamma)_{\neq}$. Hence we can apply Lemma 20 to conclude.

**Lemma 24.** Let $\Gamma$ be a CL that is strictly ess.neg. and strictly ess.pos., then $p$-ARG($\Gamma, |\alpha|$) $\in W[1]$.

**Proof.** We give a reduction to the $W[1]$-complete problem CLIQUE. Note that by definition of strictly ess.pos. and strictly ess.neg., any $\Gamma$-formula can be written as a $\{T, F\}$-formula. Let $(\Delta, \alpha)$ be an instance of $\text{ARG}([T, F], |\alpha|)$. Note that $\Delta$ is a set of terms and $\alpha$ is also a term. Let $\Delta = \{t_1, \ldots, t_n\}$ and $\alpha = l_1 \land \cdots \land l_k$. Then we have the following two observations.

**Observation 1:** There is a support for $\alpha$ if there is a support of cardinality at most $k$: If $l_i$ can be explained at all, then one $\varphi \in \Delta$ is sufficient. In other words, there is no $l_i$ in $\alpha$ such that a combination of two terms from $\Delta$ is necessary to explain $l_i$.

**Observation 2:** If a set of terms is pairwise consistent, then the whole set is consistent. If a set of terms is inconsistent, then there are two terms which are pairwise inconsistent.

For each literal $l_i \in \alpha$, form the sets $L^+_i = \{t \in \Delta \mid l_i \in t\}$ and $L^-_i = \{t \in \Delta \mid \neg l_i \in t\}$. That is, each $t \in L^+_i$ is a candidate support, whereas, no $t \in L^-_i$ can be in the support, for every $i \leq k$. Let $N = \bigcup_{i \leq k} L^-_i$ and denote $L_i = L^+_i \setminus N$. It is important to notice that there is a support only if $L_i \neq \emptyset$ for each $i \leq k$. Otherwise, for some $i$, the support $\Phi$ can not contain a term $t$ supporting $l_i$ such that $\Phi$ is consistent. It remains to determine whether $\Phi$, that includes one $t$ from each $L_i$ is consistent. The consistency still needs to be checked because terms in $\Delta$ may contain literals not in $\alpha$. That is, care should be taken when selecting which terms to include in the support.

Consider the following graph $G = (V, E)$. There is one node corresponding to each element $t_i$ and each set $L_i$. By slightly abusing the notation, we write $V = \bigcup_{i \leq k} L_i$. It is worth mentioning that if a term appears in two different sets, say $L_r$ and $L_s$, then there are distinct nodes for each term. We will explain later why this is required. Finally, the edge relation denotes the pairwise consistency of terms. That is, there is an edge between a term $t_{r,i} \in L_i$ and $t_{s,j} \in L_j$ if $t_{r,i} \land t_{s,j}$ is consistent. However, there is no edge between $t_{r,i}$ and $t_{s,j}$. That is, if the two terms belong to the same set $L_i$, or if two terms belong to two different sets $L_i$ and $L_j$ but these are pairwise inconsistent, then there is no edge.

We first prove that the reduction is indeed FPT. The sets $L^+_i$ and $L^-_i$ can be computed in polynomial time. The size of $V$ is $O(k \cdot n^2)$, because there are $k$ sets of the form $L_i$, each
contains at most \( n \) terms of size at most \( n \) (where \( n \) is the input size). To draw the edges, for each element \( t \in L_1 \), one needs to check the pairwise consistency for each of the remaining \( k - 1 \) sets, each of size \( O(n^2) \). This gives \( O(k \cdot n^3) \) time for one element of \( L_1 \). To determine edges for each element of \( L_1 \), it requires \( n \cdot O(k \cdot n^3) = O(k \cdot n^4) \) time. Finally, to repeat this for each set \( L_i \), it requires \( O(k^2 \cdot n^4) \) time. This proves that the reduction can be performed in \( \text{FPT} \)-time.

Now we prove that the reduction preserves the cliques of \( G \) of size \( k \) and the supports of \( (\Delta, \alpha) \).

\( \triangleright \) **Claim 25.** \((V, E)\) admits a clique of size \( k \) if and only if \((\Delta, \alpha)\) admits a support.

**Proof of Claim 25.** \( \Rightarrow \). Let \( S \subseteq V \) be a clique of size \( k \) in \((V, E)\). Since there are no edges between the two elements from the same set \( L_i \), this implies \( S \) contains exactly one term from each \( L_i \). Furthermore, the fact that \( S \) is a clique implies that the set of terms is consistent. This provides a support for \( \alpha \).

\( \Leftarrow \). Let \( \Phi \) be a support for \( \alpha \) in \( \Delta \). According to observation 1, for each \( l_i \in \alpha \), there is a term \( t \in \Delta \) such that \( t \models l_i \). The set \( L_i \) contains every such a \( t \in \Delta \). Moreover, this holds for each \( l_i \in \alpha \), this implies that \( \Phi \) contains at least one \( t \) from \( L_i \) for each \( i \leq k \). Finally, since \( \Phi \) is consistent, this implies that every pair of nodes corresponding to the terms in \( \Phi \) contains an edge. This gives a clique in \((V, E)\). \( \triangleright \)

It might happen that there is one term \( t \in \Delta \) such that \( t \models l_i \land l_j \), and due to our construction, \( t \in L_i \cap L_j \). However, the graph contains two separate nodes for each occurrence of \( t \). This is required to ensure that a support of size smaller than \( k \) also guarantees a clique of size \( k \) for \((V, E)\). \( \triangleright \)

\( \triangleright \) **Lemma 26.** Let \( \Gamma \) be a CL that is strictly ess.neg. or strictly ess.pos., then \( \text{p-ARG}(\Gamma, |\alpha|) \in \text{W}[2] \).

**Proof.** We only prove the statement for \( \Gamma \) that is strict ess.pos. The other case is proven analogously. Since we consider only finite constraint languages we have that \( \Gamma \subseteq \text{IS}_{02}^r \) for some \( r \geq 2 \). Therefore, any \( \Gamma \)-formula can be written as a conjunction of positive or negative literals and positive clauses of size at most \( r \).

Let \( \Delta = \{c_1, \ldots, c_n\} \) and \( \alpha = a_1 \land \cdots \land a_k \), where each \( a_i \) is either a positive or negative literal, or a positive clause of size \( \leq r \). Then we have the following three claims.

\( \triangleright \) **Claim 27.** If \( a_i \) can be explained at all, then at most \( r \) formulas from \( \Delta \) are sufficient. Consequently, there is a support for \( \alpha \) iff there is a support of size at most \( r \cdot k \).

**Proof of Claim 27.** If \( a_i \) is a negative literal, then at most 1 formula from \( \Delta \) is sufficient (one that contains \( a_i \)). If \( a_i \) is a positive literal, then at most \( r \) formulas from \( \Delta \) are sufficient (worst case: one \( e \in \Delta \) contains a (positive) clause which contains \( a_i \), then we need at most \( r - 1 \) more \( e \)'s in order to force all other variables in the clause to 0). If \( a_i \) is a positive clause, then it suffices to explain one variable from that clause, as a result, in the previous case, at most \( r \) formulas from \( \Delta \) are sufficient. In other words, there is no \( a_i \) in \( \alpha \) such that more than \( r \) formulas from \( \Delta \) are necessary to explain \( a_i \). \( \triangleright \)

\( \triangleright \) **Claim 28.** Let \( \Phi \subseteq \Delta \). If all subsets of \( \Phi \) of size \( r + 1 \) are consistent, then \( \Phi \) is consistent. In contra position: If \( \Phi \) is inconsistent, then it contains a subset of size at most \( r + 1 \) which is inconsistent.
Proof of Claim 28. Similar to the previous proof: the worst case to create an inconsistency is to take a formula containing a positive clause of size \( r \) and then \( r \) formulas forcing together all variables from the positive clause to 0.

Let \( U = \{ u_1, \ldots, u_m \} \) be a collection of fresh variables, where each variable will represent a different subset of \( \Delta \) of size at most \( r \), that is, \( m \leq r \cdot |\Delta|^r \). For each \( u_i \) denote by \( S(u_i) \) the subset of \( \Delta \) it represents. For \( V \subseteq U \) define \( S(V) = \bigcup_{u_i \in V} S(u_i) \). Define

\[
L_i = \bigvee_{S(u_i) = a_i} u_j
\]

and

\[
\varphi = \bigwedge_{i=1}^k L_i \land \bigwedge_{V \subseteq U. s.t. |V| \leq r+1 \text{ and } S(V) = \emptyset} \left( \bigvee_{u_i \in V} \neg u_i \right)
\]

The role of each \( L_i \) is to make sure that each \( a_i \) is explained. The role of the negative clauses in \( \varphi \) is to make sure that inconsistent explanations are forbidden.

\[\blacktriangleleft\]

Claim 29. \( (\Delta, \alpha) \) admits a support iff \( \varphi \) is satisfiable iff \( \varphi \) has a model of weight at most \( k \).

Proof of Claim 29. Be \( \Phi \subseteq \Delta \) a support for \( \alpha \). Since \( \Phi \) explains each \( a_i \), by observation 1 there is a set \( E(a_i) \subseteq \Phi \) of size at most \( r \) such that \( E(a_i) \models a_i \). By construction each \( E(a_i) \) corresponds to a \( u_j \in U \). One verifies that these \( u_j \)'s constitute a model of \( \varphi \) of weight at most \( k \) (if \( i \neq j \) it can happen that \( E(a_i) = E(a_j) \), therefore at most).

For the other direction let \( \varphi \) be satisfiable. By construction of \( \varphi \) there is a model of weight at most \( k \) (in each \( L_i \) it is sufficient to have at most one positive literal). Be \( W \subseteq U \) such a model of weight at most \( k \). By construction of \( \varphi \) the set \( S(W) \) is consistent and explains each \( a_i \). Therefore, \( S(W) \) constitutes a support for \( \alpha \).

Our final map is \( (\Delta, \alpha) \mapsto \varphi \land \bigvee_{i=1}^{k+1} x_i \), where \( x_i \) are fresh variables. We conclude by observing that \( \varphi \) is satisfiable if and only if \( \varphi \land \bigvee_{i=1}^{k+1} x_i \) is \((k+1)\)-satisfiable.

\[\blacktriangleleft\]

Theorem 30. \( \text{p-ARG-Check}(\Gamma, |\alpha|) \), for a CL \( \Gamma \), is (1.) \( \text{FPT} \) if \( \Gamma \) is Schaefer, and (2.) \( \text{para-DP} \)-complete otherwise.

Proof. 1. This follows from [10, Theorem 6.1] as classically \( \text{ARG-Rel}(\Gamma) \in \text{P} \).

2. Here, the membership follows as classically \( \text{ARG-Rel}(\Gamma) \in \text{DP} \). Furthermore, the reduction in the proof of [10, Propositions 6.3 and 6.4] always uses a fixed size of the claim \( \alpha \). As a consequence, certain slices of \( \text{ARG-Check}(\Gamma) \) are \( \text{DP} \)-hard, giving the desired results.

\[\blacktriangleleft\]

Theorem 31. \( \text{p-ARG-Rel}(\Gamma, |\alpha|) \), for a CL \( \Gamma \), is

1. \( \text{FPT} \) if \( \Gamma \) is either positive or negative.
2. \( \text{para-NP} \)-complete if \( \Gamma \) is Schaefer but neither strictly ess.neg. nor strictly ess.pos.
3. \( \text{para-} \Sigma_2^P \)-complete if \( \Gamma \) is not Schaefer.

Proof. 1. This follows as classically \( \text{ARG-Rel}(\Gamma) \in \text{P} \) by [10, Prop. 7.3].

2. Here, the membership follows because the classical problem is in \( \text{NP} \). We make a case distinction as whether \( \Gamma \) is \( \varepsilon \)-valid or not.

Case 1. Let \( \Gamma \) be Schaefer and \( \varepsilon \)-valid, but neither positive nor negative. The hardness follows because the 2-slice of the problem is already \( \text{NP} \)-hard [10, Proposition 7.6].
Case 2. Let \( \Gamma \) be Schaefer but neither \( \varepsilon \)-valid, nor strictly ess.neg. or strictly ess.pos. The hardness follows from Theorem [19]. This is due to the reason that p-ARG-Rel is always harder than p-ARG via the reduction \((\Delta, \alpha) \mapsto (\Delta \cup \{\psi\}, \psi, \alpha)\).

3. In this case, the membership is true because the classical problem is in \( \Sigma^P_2 \). Hardness follows from a result of [10, Prop. 7.7]. Notice that, while proving the hardness for each sub case, the claim \( \alpha \) has fixed size in each reduction. This implies that certain slices in each case are \( \Sigma^P_2 \)-hard, consequently, giving the desired hardness results.

\[
\text{Case 2. Let } \Gamma \text{ be Schaefer but neither } \varepsilon \text{-valid, nor strictly ess.neg. or strictly ess.pos.}
\]

\[
The hardness follows from Theorem [19]. This is due to the reason that p-ARG-Rel is always harder than p-ARG via the reduction \((\Delta, \alpha) \mapsto (\Delta \cup \{\psi\}, \psi, \alpha)\).
\]

3. In this case, the membership is true because the classical problem is in \( \Sigma^P_2 \). Hardness follows from a result of [10, Prop. 7.7]. Notice that, while proving the hardness for each sub case, the claim \( \alpha \) has fixed size in each reduction. This implies that certain slices in each case are \( \Sigma^P_2 \)-hard, consequently, giving the desired hardness results.

4 Parameters: Size of Support, Knowledge-Base

Regarding these parameters, we will always show a dichotomy: for the Schaefer cases, the problem is \( \text{FPT} \), otherwise we have a lower bound by the implication problem.

Recall that the collection \( \Delta \) of formulas is not assumed to be consistent.

\[
\text{Theorem 32. p-ARG}(\Gamma, |\Delta|) \text{ and p-ARG-Rel}(\Gamma, |\Delta|), \text{ for CLs } \Gamma, \text{ are (1.) } \text{FPT} \text{ if } \Gamma \text{ is Schaefer, and (2.) } \text{p-IMP}(\Gamma, |\Phi|) \text{-hard and in para-coNP otherwise.}
\]

\[
\text{Proof. 1. Notice that the number of subsets of } \Delta \text{ is bounded by the parameter. Consequently, one simply checks each subset of } \Delta \text{ as a possible support } \Phi \text{ for } \alpha. \text{ Moreover, the size of each support } \Phi \text{ is also bounded by the parameter, as a result, one can determine the satisfiability and entailment in FPT-time. This is because, the satisfiability and entailment for Schaefer languages is in } \text{P}.
\]

\[
\text{2. For the lower bound, we have } \text{p-IMP}(\Gamma, |\Phi|) \leq \text{FPT-ARG}(\Gamma, |\Delta|) \leq \text{FPT-p-ARG-Rel}(\Gamma, |\Delta|) \text{ by identities.}
\]

For membership, we make case distinction as whether \( \Gamma \) is \( \varepsilon \)-valid or not.

\[
\text{Case 1. } \Gamma \text{ is } \varepsilon \text{-valid. The membership follows because the unparameterized problem } \text{ARG}(\Gamma) \text{ is in coNP when } \Gamma \text{ is } \varepsilon \text{-valid.}
\]

\[
\text{Case 2. } \Gamma \text{ is neither } 0 \text{-valid nor } 1 \text{-valid. The membership follows because for each candidate } \Phi, \text{ one needs to determine whether } \Phi \text{ is consistent and } \Phi \models \alpha. \text{ The consistency can be checked in FPT-time because } |\Phi| \text{ is bounded by the parameter. The entailment problem for non-Schaefer, non } \varepsilon \text{-valid languages is still in para-coNP when } |\Phi| \text{ is the parameter.}
\]

\[
\text{For p-ARG-Rel}(\Gamma, |\Delta|) \in \text{para-coNP}, \text{ try all the subsets of } \Delta \text{ that contain } \psi, \text{ as a candidate support.}
\]

When the support size \( |\Phi| \) is considered as a parameter, the problems ARG and ARG-Rel become irrelevant. Consequently, we only consider the problem ARG-Check.

\[
\text{Corollary 33. p-ARG-Check}(\Gamma, |\Phi|), \text{ for a CL } \Gamma, \text{ is (1.) } \text{FPT} \text{ if } \Gamma \text{ is Schaefer, and (2.) } \text{p-IMP}(\Gamma, |\Phi|) \text{-hard and in para-DP otherwise.}
\]

5 Conclusion and Outlook

In this paper, we performed a two dimensional classification of reasoning in logic-based argumentation. On the one side, we studied syntactical fragments in the spirit of Schaefer’s framework of co-clones. On the other side, we analysed a list of parameters and classified the parameterized complexity of three central reasoning problems accordingly.

As a take-away message we get that \( \alpha \) as a parameter does not help to reach tractable fragments of p-ARG.
The case for p-ARG-Rel(Γ, |α|) when Γ is strictly ess.neg. or strictly ess.pos. is still open. Also, few tight complexity results have to be found and the implication problem regarding the parameter |Φ| has to be understood.

It is worth noting that for some CLs, e.g., those that are ε-valid, the problem p-ARG-Check is harder than p-ARG. This is because the problem p-ARG under consideration is the decision problem. Having the identity reduction from p-ARG-Check to p-ARG shows that the minimality is checked by solving the problem p-ARG, already. This shows that computing a minimal support is potentially harder than deciding whether such a support exists, unless the complexity classes DP and coNP coincide. We pose as an interesting open problem to classify the function version of ARG, in both, the classical and the parameterized setting.

Regarding other parameters, treewidth [33] is a quite promising structural property that led to several FPT-results in the parameterized setting: artificial intelligence [22], knowledge representation [20], abduction in Datalog [21], and databases [23]. Fellows et al. [17] show that abductive reasoning benefits from this parameter as well. Using a reduction between abduction and argumentation [10] might yield FPT-results in our setting. Furthermore, we plan to give a precise classification of p-IMP.

As further future work, we plan investigating the (parameterized) enumeration complexity [19, 13, 12, 27] of reasoning in this setting.

References

1. Leila Amgoud and Henri Prade. Using arguments for making and explaining decisions. *Artif. Intell.*, 173(3-4):413–436, 2009.
2. Katie Atkinson, Pietro Baroni, Massimiliano Giacomin, Anthony Hunter, Henry Prakken, Chris Reed, Guillermo Ricardo Simari, Matthias Thimm, and Serena Villata. Towards artificial argumentation. *AI Mag.*, 38(3):25–36, 2017. doi:10.1609/aimag.v38i3.2704
3. Pietro Baroni, Dov Gabbay, Massimiliano Giacomin, and Leendert van der Torre, editors. *Handbook of Formal Argumentation*. College Publications, 2018.
4. Philippe Besnard and Anthony Hunter. A logic-based theory of deductive arguments. *Artif. Intell.*, 128(1-2):203–235, 2001.
5. Philippe Besnard and Anthony Hunter. *Elements of Argumentation*. MIT Press, 2008.
6. Olaf Beyersdorff, Arne Meier, Michael Thomas, and Heribert Vollmer. The complexity of propositional implication. *Inf. Process. Lett.*, 109(18):1071–1077, 2009. doi:10.1016/j.ipl.2009.06.015
7. Elmar Böhler, Steffen Reith, Henning Schnoor, and Heribert Vollmer. Bases for boolean co-clones. *Inf. Process. Lett.*, 96(2):59–66, 2005. doi:10.1016/j.ipl.2005.06.003
8. Elmar Böhler, Nadia Creignou, Steffen Reith, and Heribert Vollmer. Playing with boolean blocks, part ii: Constraint satisfaction problems. *ACM SIGACT-Newsletter*, 35, 2004.
9. Carlos Iván Chesñevar, Ana Gabriela Maguitman, and Ronald Prescott Loui. Logical models of argument. *ACM Comput. Surv.*, 32(4):337–383, 2000.
10. Nadia Creignou, Uwe Egli, and Johannes Schmidt. Complexity classifications for logic-based argumentation. *ACM Trans. Comput. Log.*, 15(3):19:1–19:20, 2014. doi:10.1145/2629421
11. Nadia Creignou, Sanjeev Khanna, and Madhu Sudan. Complexity classifications of Boolean constraint satisfaction problems, volume 7 of SIAM monographs on discrete mathematics and applications. SIAM, 2001.
12. Nadia Creignou, Raïda Ktari, Arne Meier, Julian-Steffen Müller, Frédéric Olive, and Heribert Vollmer. Parameterised enumeration for modification problems. *Algorithms*, 12(9):189, 2019. doi:10.3390/a12090189
13. Nadia Creignou, Arne Meier, Julian-Steffen Müller, Johannes Schmidt, and Heribert Vollmer. Paradigms for parameterized enumeration. *Theory Comput. Syst.*, 60(4):737–758, 2017. doi:10.1007/s00224-016-9702-4
Nadia Creignou, Johannes Schmidt, Michael Thomas, and Stefan Woltran. Complexity of logic-based argumentation in post’s framework. *Argument & Computation*, 2(2-3):107–129, 2011.

Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artif. Intell.*, 77(2):321–358, 1995.

Michael R. Fellows, Andreas Pfandler, Frances A. Rosamond, and Stefan Rümmele. The parameterized complexity of abduction. In Jörg Hoffmann and Bart Selman, editors, *Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence, July 22-26, 2012, Toronto, Ontario, Canada*. AAAI Press, 2012. URL: http://www.aaai.org/ocs/index.php/AAAI/AAAI12/paper/view/5048.

Johannes Klaus Fichte, Markus Hecher, and Arne Meier. Counting complexity for reasoning in abstract argumentation. In *The Thirty-Third AAAI Conference on Artificial Intelligence, AAAI 2019, The Thirty-First Innovative Applications of Artificial Intelligence Conference, IAAI 2019, The Ninth AAAI Symposium on Educational Advances in Artificial Intelligence, EAAI 2019, Honolulu, Hawaii, USA, January 27 - February 1, 2019*, pages 2827–2834. AAAI Press, 2019. doi:10.1609/aaai.v33i01.33012827.

Fedor V. Fomin and Dieter Kratsch. *Exact Exponential Algorithms*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2010. doi:10.1007/978-3-642-16533-7.

Georg Gottlob, Reinhard Pichler, and Fang Wei. Bounded treewidth as a key to tractability of knowledge representation and reasoning. In *Proceedings, The Twenty-First National Conference on Artificial Intelligence and the Eighteenth Innovative Applications of Artificial Intelligence Conference, July 16-20, 2006, Boston, Massachusetts, USA*, pages 250–256. AAAI Press, 2006.

Georg Gottlob, Reinhard Pichler, and Fang Wei. Efficient datalog abduction through bounded treewidth. In *AAAI*, pages 1626–1631, 2007.

Georg Gottlob and Stefan Seizier. Fixed-Parameter Algorithms For Artificial Intelligence, Constraint Satisfaction and Database Problems. *The Computer Journal*, 51(3):303–325, 09 2007. doi:10.1093/comjnl/bxm056.

Martin Grohe. The complexity of homomorphism and constraint satisfaction problems seen from the other side. *J. ACM*, 54(1), March 2007. doi:10.1145/1206035.1206036.

Paolo Liberatore. Redundancy in logic I: CNF propositional formulae. *Artif. Intell.*, 163(2):203–232, 2005.

Yasir Mahmood, Arne Meier, and Johannes Schmidt. Parameterised complexity for abduction. *CoRR*, abs/1906.00703, 2019. URL: http://arxiv.org/abs/1906.00703 [arxiv:1906.00703]

Yasir Mahmood, Arne Meier, and Johannes Schmidt. Parameterised complexity of abduction in Schaefer’s framework. In *Logical Foundations of Computer Science - International Symposium, LFCS 2020, Deerfield Beach, FL, USA, January 4-7, 2020, Proceedings*, pages 195–213, 2020. doi:10.1007/978-3-030-36755-8_13.

Arne Meier. *Parametrised enumeration*. Habilitation thesis, Leibniz Universität Hannover, 2020. doi:10.15488/9427.

Gustav Nordh and Bruno Zanuttini. What makes propositional abduction tractable. *Artif. Intell.*, 172(10):1245–1284, 2008. doi:10.1016/j.artint.2008.02.001.

Simon Parsons, Michael J. Wooldridge, and Leila Amgoud. Properties and complexity of some formal inter-agent dialogues. *J. Log. Comput.*, 13(3):347–376, 2003.

Emil L. Post. The two-valued iterative systems of mathematical logic. *Annals of Mathematical Studies*, 5:1–122, 1941.

Henry Prakken and Gerard Vreeswijk. *Logics for Defeasible Argumentation*, pages 219–318. Springer Netherlands, Dordrecht, 2002.

Antonio Rago, Oana Cocarascu, and Francesca Toni. Argumentation-based recommendations: Fantastic explanations and how to find them. In *IJCAI*, pages 1949–1955. ijcai.org, 2018.
Neil Robertson and Paul D. Seymour. Graph minors. III. planar tree-width. *J. Comb. Theory, Ser. B*, 36(1):49–64, 1984. doi:10.1016/0095-8956(84)90013-3.

Thomas J. Schaefer. The complexity of satisfiability problems. In Richard J. Lipton, Walter A. Burkhard, Walter J. Savitch, Emily P. Friedman, and Alfred V. Aho, editors, *Proceedings of the 10th Annual ACM Symposium on Theory of Computing, May 1-3, 1978, San Diego, California, USA*, pages 216–226. ACM, 1978. doi:10.1145/800133.804350.

Henning Schnoor and Ilka Schnoor. Partial polymorphisms and constraint satisfaction problems. In *Complexity of Constraints - An Overview of Current Research Themes [Result of a Dagstuhl Seminar]*, pages 229–254, 2008. doi:10.1007/978-3-540-92800-3_9.

Michael Sipser. *Introduction to the theory of computation*. PWS Publishing Company, 1997.