QUASINEUTRAL LIMIT OF THE EULER-POISSON SYSTEM UNDER STRONG MAGNETIC FIELDS

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Abstract. The quasineutral limit of the three dimensional compressible Euler-Poisson (EP) system for ions in plasma under strong magnetic field is rigorously studied. It is proved that as the Debye length and the Larmor radius tend to zero, the solution of the compressible EP system converges strongly to the strong solution of the one-dimensional compressible Euler-equation in the external magnetic field direction. Higher order approximation and convergence rates are also given and detailed studied.

1. Introduction. We consider the following Euler-Poisson system

\begin{align}
\partial_t n + \nabla \cdot (n u) &= 0, \\
\partial_t u + u \cdot \nabla u + T_i \nabla \ln n + \frac{e \times u}{\varepsilon} &= -\nabla \phi, \\
\lambda^2 \Delta \phi &= e^\phi - n,
\end{align}

where $n(t, x)$ and $u(t, x)$ are the density and the velocity vector of the ions in a plasma with magnetic field, $\phi(t, x)$ is the electric potential at time $t > 0$ and $x \in \mathbb{R}^d, d \leq 3$. Here $e = (0, 0, 1)^T$ and $e/\varepsilon$ is the magnetic field with magnitude $1/\varepsilon$ such that if $u = (u_1, u_2, u_3)$ then $e \times u = (-u_2, u_1, 0)$, and $\lambda > 0$ is the scaled Debye length, which is a small quantity compared to the characteristic length of physical interest for typical plasma applications. The parameters in this equation have obvious physical meanings. The parameter $T_i$ is the temperature of the ions, which are called cold when $T_i = 0$. The parameter $\varepsilon$ is proportional to the Larmor radius, which is physically understood as the typical radius of the helix around the axis $x_3$ that the particles follow, due to the intense magnetic field. The Euler-Poisson system and its related models (Vlasov-Poisson, two fluid Euler-Poisson and so on), and their asymptotic behaviors have attracted the interests of mathematicians and physicists during the past two decades, possibly due to their applications to plasmas and semiconductors. In this paper only the quasineutral limit for (1) is concerned.

Since the Debye length is small, the plasma is electrically neutral, i.e., there is no charge separation or electric field, and this limit is widely used in practice. Therefore it is important to justify this ‘zero Debye length’ limit mathematically rigorously, also known as the quasineutral limit. We say the magnetic field is strong in the sense that as the Debye length $\lambda \to 0$, the magnitude of magnetic field $1/\varepsilon \to \infty$. 

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In this paper, we will assume $\lambda^2 = \varepsilon$, so that the Debye length and the Larmor radius ($\sim \varepsilon$) vanish at the same rate, which appears in a lot of physical problems.

Indeed, in recent years there is a large literature studying such quasineutral limit of hydrodynamic models and related kinetic models (such as the Vlasov-Poisson equation) in plasma physics from different point of view. See [1,2,4,6,7,10,11,13–15] to list only a few and the interested readers may refer to the references therein. Most of the results are concerned with the electric type Euler-Poisson systems, which in the quasineutral regime, leads to the incompressible Euler equation. There is a little results concerning the ion type Euler-Poisson systems, which leads to compressible Euler equations as expected. As far as we know, Cordier and Grenier in [2] studied the quasineutral limit for local smooth solutions of the 1D isothermal Euler-Poisson system describing a plasma made of ions and massless electrons, where the electron density is described by the Maxwell-Boltzmann relation with electric potential, for ions with the ion temperature $T_i > 0$ fixed. Recently, Gerard-Varet et al [3] extends their results for the isothermal Euler-Poisson system to $\mathbb{R}^3$ with boundary. Golse and Saint-Raymond [4] studied the quasineutral limit for the Vlasov-Poisson system under strong magnetic field and Han-Kwan [6] studied the quasineutral limit of the Vlasov-Poisson system for ions (i.e., with massless electrons).

In this paper, we consider the quasineutral limit for the Euler-Poisson equation for ions under strong magnetic field. As pointed out above, “strong” means that as the Debye length goes to zero, the magnitude of the magnetic field goes to zero. We will show that, at least for well-prepared initial data, the solutions of the three dimensional Euler-Poisson system (1) converges to solutions of a one-dimensional Euler equation in the $x_3$ direction, parallel to the direction of the magnetic field. See Theorem 2.4 for a precise statement. In the quasineutral (or strong magnetic) regime, no dynamics in the plane perpendicular to the magnetic field are observed, which may be possibly regarded as a confinement result of the strong magnetic field.

For the proof, the main target is then to show uniform (in $\varepsilon$) estimates in some proper norm. We introduce the norm $||| \cdot |||_{s,\varepsilon,T_i}^2$ in (14). It is noted that the proof is indeed independent of the temperature $T_i \geq 0$. This implies that the proof is applicable for the cold ion case $T_i = 0$, where the second estimates in §3.3 are indispensable as in the long wavelength limit for Euler-Poisson system (1) without magnetic field [5, 8, 12]. The proof when $T_i > 0$ could be simpler and the second estimates in Section 3.3 are no more needed. In this case, the usual Sobolev norm $\|(N, U)\|_{H^s}$ suffices. This implies that the pressure term $T_i \nabla \ln n$ plays an important role in the proof, which makes the hyperbolic part of (1) symmetrizable.

We also note that, due to the presence of the strong magnetic field, the convergence rate of the $M^{th}$ order approximation $(n^\text{app}, u^\text{app}, \phi^\text{app})$ is only of $\varepsilon^M$, different from $\varepsilon^{M+1}$ in [7,10].

This paper is organized as follows. In the next section, we show the formal asymptotical expansion, in which the leading order terms satisfy a one-dimensional compressible Euler equation. By cutoff, we reduced (1) to a system for the remainder terms with coefficients depending on the known expansion coefficients. In the third Section, we provide rigorous justification for this formal limit by giving uniform in $\varepsilon$ estimates for the remainder system.

2. Asymptotic analysis and the main results. We take formally the following asymptotic expansion w.r.t. $\varepsilon$ for the solution of (1)

$$
(n^\varepsilon, u^\varepsilon, \phi^\varepsilon) = \sum_{j \geq 0} \varepsilon^j (n^j, u^j, \phi^j),
$$

(2)
where \((n^j, \mathbf{u}^j, \phi^j)\) denotes the velocity vector field of the \(j^{th}\) approximation. Assume accordingly that the initial data \((n^0_0, \mathbf{u}^0_0, \phi^0_0)\) admit such an asymptotic expansion
\[
(n^\varepsilon, \mathbf{u}^\varepsilon, \phi^\varepsilon)(0, x) = \sum_{j \geq 0} \varepsilon^j (n^j_0, \mathbf{u}^j_0, \phi^j_0)(x),
\]
where \((n^j_0, \mathbf{u}^j_0, \phi^j_0)_{j \geq 0}\) are sufficiently smooth in \(T^3\).

### 2.1. Derivation of the Euler equation

Inserting the ansatz (2) into (1) and balance the powers of \(\varepsilon\), we obtain the following systems of equations.

At the order \(O(\varepsilon^{-1})\), we obtain
\[
u_1^0 = \nu_2^0 = 0.
\] (3)

At the order \(O(1)\), we obtain
\[
\begin{aligned}
\frac{\partial}{\partial t} n^0 + \frac{\partial}{\partial x_1} (n^0 \nu_1^0) &= 0, \\
\frac{\partial}{\partial x_2} n^0 + \frac{\partial}{\partial x_3} n^0 &= 0, \\
\frac{\partial}{\partial x_1} n^0 + \frac{\partial}{\partial x_3} n^0 + T_1 \frac{\partial}{\partial x_1} \ln n^0 - \nu^1_2 &= -\partial_1 \phi^0, \\
\frac{\partial}{\partial x_2} n^0 + \frac{\partial}{\partial x_3} n^0 + T_1 \frac{\partial}{\partial x_2} \ln n^0 - \nu^1_3 &= -\partial_2 \phi^0, \\
\frac{\partial}{\partial x_3} n^0 + \frac{\partial}{\partial x_3} n^0 + T_1 \frac{\partial}{\partial x_3} \ln n^0 - \nu^1_2 &= -\partial_3 \phi^0, \\
\phi^0 &= \phi^0,
\end{aligned}
\] (4)

thanks to (3). From (3), (4b) and (4c), we obtain
\[
\begin{aligned}
u^1_2 &= -(T_1 + 1) \frac{\partial}{\partial x_1} \ln n^0, \\
\nu^1_3 &= (T_1 + 1) \frac{\partial}{\partial x_2} \ln n^0.
\end{aligned}
\] (5)

This leads us to consider the compatibility condition of the initial data from (4)
\[
u^1_2|_{t=0} = -(T_1 + 1) \frac{\partial}{\partial x_1} \ln n^0|_{t=0}, \quad \nu^1_3|_{t=0} = (T_1 + 1) \frac{\partial}{\partial x_2} \ln n^0|_{t=0}.
\] (6)

It follows from (4a), (4d) and (4e) that the leading profiles satisfy the following one dimensional compressible Euler equation in the \(x_3\)-direction
\[
\begin{aligned}
\frac{\partial}{\partial t} n^0 + \frac{\partial}{\partial x_3} (n^0 \nu^0_3) &= 0, \\
\frac{\partial}{\partial x_3} n^0 + \frac{\partial}{\partial x_3} n^0 + (T_1 + 1) \frac{\partial}{\partial x_3} \ln n^0 &= 0, \\
n^0|_{t=0} &= n^0_0, \quad \nu^0_3|_{t=0} = \nu^0_{3,0}.
\end{aligned}
\] (7)

This implies that there is no more dynamics in the \((x, y)\) plane, which can be interpreted as a confinement result. We also assume that the plasma is uniform and electrically neutral near infinity. More precisely, let \(\tilde{n}\) be a smooth strictly positive function, constant outside \(x \in [-1, 1]\), satisfying \(\tilde{n} \rightarrow n^\pm > 0\) and \(u \rightarrow 0\) as \(x \rightarrow \pm \infty\). We also assume that the initial conditions \((n^0_0, u^0_{3,0})\) satisfy
\[
u^0_{3,0} \in H^s, \quad (n^0_0 - \tilde{n}) \in H^s, \quad u^0_{3,0} \geq \sigma > 0,
\] (8)
for some constant \(\sigma > 0\) and \(s > 1 + \frac{d}{2}\) sufficiently large. Then the limiting system (7) is hyperbolic symmetrisable, whose classical result for the existence of sufficiently regular solutions in small time can be stated as (see [9])

**Theorem 2.1.** Let \(s > d/2 + 1\) and \((n^0_0, u^0_{3,0})\) be initial data such that (8) holds. Then there exists \(T_* > 0\), maximal time of existence and a unique solution \((n^0(t), u^0_3(t))\) to the Cauchy problem (7) with initial data \((n^0_0, u^0_{3,0})\) such that \((n^0 - \tilde{n}, u^0_{3}) \in C([0, T_*], H^s) \cap C^1([0, T_*], H^{s-1})\).
In what follows, we will work on a time interval $[0, T]$ with $T < T_*$, arbitrary close to $T_*$ in order to insure $0 < \sigma' < n^0(t, x) < \sigma''$ for all $(t, x)$ for some positive constants $\sigma', \sigma''$.

2.2. Determinacy of the profiles $(n^j, u^j, \phi^j)_{j \geq 1}$. Proceeding as above, we obtain at the order $O(\varepsilon)$ that

$$(S_1) \begin{cases} \partial t n^1 + \nabla \cdot (n^1 u^0 + n^0 u^1) = 0, \\ \partial t u^1 + u^0 \cdot \nabla u^1 + u^1 \cdot \nabla u^0 + T_i \nabla (\frac{n^1}{n^0}) + e \times u^2 = -\nabla \phi^1, \\ \Delta \phi^0 = e^{\phi^0} \phi^1 - n^1. \end{cases}$$

(9a)

(9b)

(9c)

Since $u^1_1$ and $u^1_2$ have already been solved from (5), we need only to solve $n^1, u^1_3$ and $\phi^1$ in (9), which reduces to the one-dimensional linearized compressible Euler equation for $(n^1, u^1_3)$

$$(S_1) \begin{cases} \partial_t n^1 + \partial_3 (n^1 u^0_3) + \nabla \cdot (n^0 u^1_3) = 0, \\ \partial_t u^1_3 + u^0_3 \partial_3 u^1_3 + u^1_3 \cdot \nabla u^0_3 + T_i \nabla (\frac{n^1}{n^0}) = -\partial_3 \phi^1, \\ \Delta \phi^0 = e^{\phi^0} \phi^1 - n^1, \\ n^1|_{t=0} = n^0, \ u^1_3|_{t=0} = u^1_{3,0}. \end{cases}$$

(10a)

(10b)

(10c)

(10d)

From (10), we can solve $n^1, u^1_3$ and $\phi^1$, and then by inserting the result to (9), we can further solve $u^1_2$ and $u^1_3$. We also note that the system (10) is self-contained and do not depend on the $(n^j, u^j, \phi^j)$ for $j \geq 2$.

**Theorem 2.2.** Let $s > 1 + d/2$ sufficiently large and $(n^0_0, u^0_3, 0) \in H^{s-3}$. Assume that $(n^0 - \tilde{n}, u^0_3) \in C([0, T_*), H^s) \cap C^1([0, T_*), H^{s-1})$, then there exists a unique solution $(n^1, u^1_3)$ to (10) with initial data $(n^0_0, u^0_{3,0})$ on $[0, T_*)$ such that $(n^1, u^1_3) \in C([0, T_*), H^{s-3}) \cap C^1([0, T_*), H^{s-4})$.

Generally, we obtain the coefficients at order $O(\varepsilon^k)$ for $k \geq 1$ as follows

$$(S_k) \begin{cases} \partial_t n^k + \sum_{0 \leq I, m, l \leq k \atop I + m = k} \nabla \cdot (n^I u^m) = 0, \\ \partial_t u^k + u^0 \cdot \nabla u^k + u^k \cdot \nabla u^0 + T_i \nabla (\frac{n^k}{n^0}) + e \times u^{k+1} + \mathcal{K}^u_{k-1} = -\nabla \phi^k, \\ \Delta \phi^{k-1} = e^{\phi^{k-1}} \phi^k + \mathcal{K}^\phi_{k-1} - n^k, \end{cases}$$

(11a)

(11b)

(11c)

where $\mathcal{K}^u_{k-1} = \mathcal{D}_{k-1} + T \nabla \mathcal{W}_{k-1}$ and $\mathcal{K}^\phi_{k-1}$ are known at order $O(\varepsilon^k)$, and $\mathcal{D}_{k-1}$, $\mathcal{W}_{k-1}$ and $\mathcal{K}^\phi_{k-1}$ are given by, respectively,

$$\mathcal{D}_{k-1} = \sum_{1 \leq l, m \leq k \atop l + m = k} u^l \cdot \nabla u^m,$$

$$\mathcal{W}_{k-1} = -\frac{1}{2} \frac{1}{(n^0)^2} \sum_{l+m=k \atop 1 \leq l, m \leq k} n^l n^m$$

$$+ \frac{1}{3} \frac{1}{(n^0)^3} \sum_{l+m+p=k \atop 1 \leq l, m, p \leq k} n^l n^m n^p + \cdots + \frac{(-1)^{k+1}}{k} \frac{1}{(n^0)^{k}} (n^1)^k,$$
and
\[ H_{k-1}^\phi = e^\phi \left[ \frac{1}{2} \sum_{l+m=k} \phi^l \phi^m + \frac{1}{3!} \sum_{l+m+p=k} \phi^l \phi^m \phi^p + \cdots + \frac{1}{k!} (\phi^1)^k \right] . \]

We note that to the \( k^{th} \) approximations, in (11), \((u_1^{k-1}, u_2^{k-1})\) has already known in the \((k-1)^{th} \) step, and thus the initial conditions for (11) is only given by \((n_0^k, u_3^k, 0)\).

**Theorem 2.3.** Let \( s > 1 + d/2 \) be sufficiently large and \((n_0^k, u_3^k, 0)\) \( \in H^{s-3k} \). Assume also that \((n_0^0 - n_0, u_1^0) \in C([0, T_*], H^s) \cap C^1([0, T_*], H^{s-1})\), \((n_1^0, u_2^0) \in C([0, T_*], H^{s-3j}) \cap C^1([0, T_*], H^{s-3j-1})\) for \( 1 \leq j \leq k - 1 \). Then there exists a unique solution \((n_k^0, u_k^0) \in C([0, T_*], H^{s-3k}) \cap C^1([0, T_*], H^{s-3k-1})\) on \([0, T_*]\) for the problem (11) with initial data \((n_0^k, u_3^k, 0)\).

### 2.3. Main results.

To show that \((n^\varepsilon, u^\varepsilon, \phi^\varepsilon)\) converges to the solution of the compressible Euler equation, we need to make the above procedure rigorous. Let \((n^\varepsilon, u^\varepsilon, \phi^\varepsilon)\) be a solution of (1) of the following expansion
\[
(n^\varepsilon, u^\varepsilon, \phi^\varepsilon) = \sum_{j=0}^{M} \varepsilon^j (n^R_j, u^R_j, \phi^R_j) + \varepsilon^M (n^R, u^R, \phi^R),
\]
where \((n^{app}, u^{app}, \phi^{app})\) denotes the approximation of the solution \((n^\varepsilon, u^\varepsilon, \phi^\varepsilon)\) to the \( M^{th} \) order, \((n^R, u^R, \phi^R)\) denotes the remainder terms of the approximation, \(u^{app} = (u_1^{app}, u_2^{app}, u_3^{app})\) and \( u^R = (u_1, u_2, u_3) \). Inserting the expression (12) into (1), and subtracting the equations satisfied by \((n^{app}, u^{app}, \phi^{app})\) ((4), (11)) for \( 1 \leq k \leq M \), we obtain the remainder system for \((n^R, u^R, \phi^R)\):

\[
\begin{align*}
\partial_t n^R + \nabla \cdot (n^R u^{app}) + \nabla \cdot (n^{app} u^R) + \varepsilon^M \nabla \cdot (n^R u^R) + R_n &= 0, \\
\partial_t u^R + u^{app} \cdot \nabla u^R + u^R \cdot \nabla u^{app} + \varepsilon^M u^R \cdot \nabla u^R + \frac{\varepsilon^M}{\varepsilon} u^R &= 0, \\
\varepsilon \Delta \phi^R &= n^0 \phi^R - n^R + \sqrt{\varepsilon} R_\phi \sqrt{\varepsilon} \phi^R + F_\phi,
\end{align*}
\]

where
\[
R_n = \sum_{0 \leq l+m \leq M} \varepsilon^{l+m-M} \nabla \cdot (n^l u^m) \sim O(1),
\]
\[
R_\phi (\varepsilon \phi^R) = R_\phi (n^0, \phi^1, \cdots, \phi^M, \varepsilon \phi^R),
\]
\[
R_p (n^R) = \frac{\nabla n^R}{n^\varepsilon} + \frac{B}{n^\varepsilon} - \frac{b n^R}{n^\varepsilon},
\]
\[
F_\phi = F_\phi (\phi^0, \phi^1, \cdots, \varepsilon^{M-1} \phi^M) \sim O(1),
\]
with
\[
B = \{ \nabla n^{app} - b n^{app} / \varepsilon^M \} \sim O(1), \quad R_u \sim O(1)
\]
\[
b = \{ \text{finite collection of terms } n^1, n^2, \cdots, n^M \} \sim O(1).
\]
To state the main results, we define
\[
\|(N, U)\|_{s, ε, T_i}^2 = \|N\|^2_{H^s} + ε\|U\|^2_{H^s} + εT_i\|\nabla N\|^2_{H^s},
\]
\[
\|(N, U, Φ)\|_{n, ε, T_i}^2 = \|(U, N)\|^2_{H^s} + \|Φ\|^2_{H^s} + ε^2\|\nabla Φ\|^2_{H^s}.
\]
(14)

Let \(u_0^i = (0, 0, u_3^0)\) for \(0 \leq j \leq M\). The main result in this paper is the following

**Theorem 2.4.** Let \(s > \frac{d}{2} + 1\) be fixed and integer \(M \geq s - 2\). Let \((n_0^0, u_0^0, u_3^0)\) be initial data such that (8) holds and \((n_0^0(t), u_0^0(t))\) be the solution of the Euler system (7) on \([0, T]\) given by Theorem 2.1, and \((n^j(t), u^j(t), φ^j(t))\) are solutions of (11) with initial data \((n_0^0, u_0^0)\) given by Theorem 2.3 for \(1 \leq j \leq M\). Assume that the initial data \((n_0^0, u_3^0)\) satisfy the compatibility conditions (4), (9) and (11) for all \(0 \leq k \leq M\) and

\[
\|(n_0^0 - n_0^0 - \sum_{j=1}^{M} ε^j n_0^j, u_0^0 - u_0^0 - \sum_{j=1}^{M} ε^j u_0^j)\|_{s, ε, T_i} \leq Cε^M.
\]

Then there exists \(ε_0 > 0\) and solutions \((n^ε(t, x), u^ε(t, x), φ^ε(t, x))\) of (1) with initial data \((n_0(t), 0, u_3(t))\) on \([0, T^ε]\) with \(\liminf_{ε→0} T^ε \geq T\). Moreover, for every \(T' < T\) and for all \(0 < ε < ε_0\), there holds

\[
\|(n^ε - n^{app}, u^ε - u^{app}, φ^ε - φ^{app})\|_{s, ε, T_i} \leq Cε^M,
\]

(15)

where the constant \(C\) is independent of \(ε\) and \(\{n^j, u^j, φ^j\}_{0 \leq j \leq M}\) are solutions to the problems (7) and (11).

**Remark 1.** When \(M = 1\), we obtain

\[
\|(n^ε - n^0, u^ε - u^0, φ^ε - φ^0)\|_{s, ε} \leq Cε^s.
\]

This implies that as \(ε \to 0\), the solution to the Euler-Poisson system (1) converges in the \(H^s\)-norm to the solution of the Euler equation (7) with the same initial data.

**Remark 2.** The estimate in (15) is independent of the temperature \(T_i\) for \(T_i\) small. Precisely, there exists \(T_{em} > 0\) such that the constant \(C\) is independent of \(T_i \in [0, T_{em}]\). This implies that the quasineutral limit result in Theorem 2.4 also applies for all \(T_i \geq 0\).

3. **Rigorous justifications.** This section is dedicated to proof of Theorem 2.4. Before we prove the main results, we first give two calculus inequalities, which will be frequently used throughout.

**Lemma 3.1.** Suppose that \(s > 0\) and \(p \in (1, +∞)\). There hold

\[
\|∂^α f, g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_3}} \|g\|_{W^{s-1, p_2}} + \|f\|_{W^{s, p_3}} \|g\|_{L^{p_4}})
\]

and

\[
\|∂^α (fg)\|_{L^p} \leq C(\|f\|_{L^{p_3}} \|g\|_{W^{s, p_2}} + \|f\|_{W^{s, p_3}} \|g\|_{L^{p_4}})
\]

with \(p_2, p_3 \in (1, +∞)\) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},
\]

and \(f, g\) are such that the right hand side terms make sense.
In particular, when $s \geq 2$, $H^s$ is an algebra. Then if $f, g \in H^s$ then $fg \in H^s$ with
\[
\|\partial^\alpha (fg)\| \leq C \|f\|_{H^s} \|g\|_{H^s}
\]
and $[\partial^\alpha, fg] \in L^2$ by Sobolev inequality with
\[
\|\partial^\alpha (fg)\| \leq C \|f\|_{H^s} \|g\|_{H^{s-1}},
\]
where we may take $p_1 = 3, p_2 = 6, p_3 = 2, p_4 = \infty$ and $p = 2$ for example.

Let $N = n^R$, $U = u^R$ and $\Phi = \phi^R$. We can rewrite (13) in terms of $(N, U, \Phi)$ as
\[
\begin{aligned}
\partial_t N + \nabla \cdot (Nu^{app}) + \nabla \cdot (n^{app} U) + \varepsilon M \nabla \cdot (NU) + R_n = 0, \\
\partial_t U + u^{app} \cdot \nabla U + U \cdot \nabla u^{app} + \varepsilon M U \cdot \nabla U + \frac{c \times U}{\varepsilon} \\
+ T_i R_p(N) = -\nabla \Phi + R_u, \\
\varepsilon \Delta \Phi = n^R \Phi - N + \sqrt{\varepsilon} R_\Phi(\sqrt{\varepsilon} \Phi) + \mathcal{F}_\phi.
\end{aligned}
\] (16)

For fixed $T_i \geq 0$, we will give uniform (in $\varepsilon$) estimates for the solution $(N, U, \Phi)$ to (16). To simplify the representation slightly, we assume that (1) has smooth solutions in very small time $T^\varepsilon$ dependent on $\varepsilon$. This fact can be proved by classical arguments by adapting the following proof. Let $\bar{C}$ be a constant to be determined later, much larger than the bound of $\|(N_0, U_0, \Phi_0)\|_s$, such that
\[
\sup_{[0,T^\varepsilon]} \|(N, U, \Phi)\|_s \leq \bar{C}.
\] (17)

Hence, there exists some $\varepsilon_1 > 0$ depending possibly on $\bar{C}$ such that
\[
\sigma'/2 < n^\varepsilon < 2\sigma'', \quad \forall 0 < \varepsilon < \varepsilon_1.
\]

In what follows, we will simplify $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ as $(n, u, \phi)$ by omitting the superscript $\varepsilon$. The readers should not be confused.

3.1. Elliptic estimates. Local well-posedness of the pressureless Euler-Poisson system (1) can be found in [8, Theorem 10.1].

**Lemma 3.2.** Let $M \geq 1$ and $\Phi \in L^2 \cap L^\infty$, then
\[
\|R_\Phi(\sqrt{\varepsilon} \Phi)\|_{H^k} \leq C_1(1 + \|\Phi\|_{H^k}), \quad \forall k \geq 0,
\]
where $C_1 = C_1(\varepsilon^{1/4} \|\Phi\|_{L^\infty})$ depends on $\varepsilon^{1/4} \|\Phi\|_{L^\infty}$.

Here, the constant $C_1(\cdot)$ is non-decreasing in its argument. Hence, by continuity assumption (17), we may thus fix $C_1 = C_1(\varepsilon_0^{1/4} \bar{C})$ hereafter.

**Proof.** We only consider the case $M = 1$, while the cases $M \geq 2$ can be proved exactly in the same spirit. From the construction of $R_\Phi(\sqrt{\varepsilon} \Phi)$, we have
\[
\begin{aligned}
\sqrt{\varepsilon} R_\Phi(\sqrt{\varepsilon} \phi^R) + \mathcal{F}_\phi &= \{e^{\phi^0 + \varepsilon \phi^R} - (e^{\phi^0} + \varepsilon e^{\phi^0} \phi^R)\}/\varepsilon \\
&= e^{\phi^0} \{e^{\varepsilon \phi^1 + \varepsilon \phi^R} - (1 + \varepsilon \phi^1 + \varepsilon \phi^R)\}/\varepsilon + e^{\phi^0} \phi^1 \\
&= \frac{1}{\varepsilon} \sum_{n \geq 2} \frac{(\varepsilon \phi^1 + \varepsilon \phi^R)^n}{n!} + e^{\phi^0} \phi^1 = I + II.
\end{aligned}
\]
The last term $II$ is independent of the unknown remainder $\Phi$ and is classified into $\mathcal{F}_\phi$. The first term $I$ depends on $\Phi$ and is put into $\sqrt{\varepsilon} R_\phi(\sqrt{\varepsilon} \phi^R)$ and can be bounded by

$$
|I| = |(\phi^1 + \phi^R) \sum_{n \geq 2} (\varepsilon \phi^1 + \varepsilon \phi^R)^{n-1} \frac{1}{n!}|
$$

$$
\leq \varepsilon^{3/4} |(\phi^1 + \phi^R) \sum_{n \geq 2} |(\varepsilon^{1/4} \phi^1 + \phi^R)|_{L^\infty}^{n-1} \frac{1}{n!}|
$$

$$
\leq \varepsilon^{3/4} |(\phi^1 + \phi^R)| \sum_{n \geq 2} |(\varepsilon^{1/4} \phi^1 + \phi^R)|_{L^\infty} \frac{1}{n!}|
$$

$$
\leq \varepsilon^{1/2} C(\varepsilon^{1/4} |\phi^R|_{L^\infty}) |\phi^1 + \phi^R|,
$$

where we suppress the dependence of $C$ on $\|\phi^1\|_{L^\infty}$. By taking the $L^2$ norm, we have

$$
\|\sqrt{\varepsilon} R_\phi(\sqrt{\varepsilon} \phi^R)\|_{L^2} \leq \sqrt{\varepsilon} C(\|\varepsilon^{1/4} \phi^R\|_{L^\infty})(1 + \|\Phi\|_{L^2}).
$$

The same treatments will lead to the estimates for all $M \geq 2$. The greater $M$ is, the easier the proof is.

On the other hand, repeating this process, by Moser’s inequality and Sobolev inequalities, we can obtain the estimates in higher Sobolev norms

$$
\|\sqrt{\varepsilon} R_\phi(\sqrt{\varepsilon} \phi^R)\|_{H^k} \leq \sqrt{\varepsilon} C(\|\varepsilon^{1/4} \phi^R\|_{L^\infty})(1 + \|\Phi\|_{H^k})
$$

where $k \geq 1$ are integers.

With this estimate, the following elliptic estimates for the Poisson equation (16c) can be easily achieved.

**Lemma 3.3.** Let $M \geq 1$ and any multi-indices $\alpha \geq 0$, there exists some $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$,

$$
\|\partial^\alpha \phi^R \| + \varepsilon \|\partial^\alpha \nabla \phi^R\| + \varepsilon\|\Delta \partial^\alpha \phi^R\| \leq 1 + \|\partial^\alpha n^R\|
$$

$$
\leq 1 + \|\phi^R\|_{H^k} + \sqrt{\varepsilon}\|\partial^\alpha \nabla \phi^R\| + \varepsilon\|\Delta \phi^R\|_{H^k}.
$$

We note that the two norms $\|\partial^\alpha \phi^R\| + \varepsilon\|\partial^\alpha \nabla \phi^R\| + \varepsilon\|\partial^\alpha \Delta \phi^R\|$ and $\|\partial^\alpha \phi^R\| + \varepsilon\|\partial^\alpha \Delta \phi^R\|$ are equivalent by interpolation.

### 3.2. First estimates

We will make estimates by taking inner product of (16b) with $U - \varepsilon \Delta U$. But since the proof is too long to be readable in that way, we will split the estimates into two subsections. First, we make estimates with $U$ in this subsection and then with $-\varepsilon \Delta U$ in the next subsection.

Let $s \geq 4$ be an integer, and $\alpha$ be a multi-index with $|\alpha| \leq s$ and set

$$
N_\alpha = \partial^\alpha N, \ U_\alpha = \partial^\alpha U, \ \Phi_\alpha = \partial^\alpha \Phi.
$$

Differentiating the remainder system (16b) with $\partial^\alpha$, taking inner product with $U_\alpha$ and by integration by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|U_\alpha\|^2 = -\frac{1}{2} \int \nabla \cdot u U_\alpha |^2 - \int |\partial^\alpha u| \nabla U U_\alpha - \int |\partial^\alpha (U \cdot \nabla u)_{app}| U_\alpha
$$

$$
- \int |\partial^\alpha R_p(N) U_\alpha + \int \Phi_\alpha \nabla \cdot U_\alpha - \int |\partial^\alpha R_n U_\alpha| = \sum_{i=1}^6 I_i.
$$

(18)
For the term $I_1 \sim I_3$ and $I_6$, it is easy to show that

$$|I_1| + |I_3| \lesssim (1 + \varepsilon^M \| \nabla U \|_{L^\infty}) \| U \|_{H^s} \| U_\alpha \|,$$

$$|I_2| \lesssim (1 + \varepsilon^M \| U \|_{H^s}) \| \nabla U \|_{H^{s-1}} \| U_\alpha \|,$$

$$|I_6| \lesssim 1 + \| U_\alpha \|^2,$$

thanks to the calculus inequalities in Lemma 3.1 and $R_\alpha$ is of $O(1)$.

**Estimate of $I_4$.** By integration by parts, $I_4$ in (18) can be rewritten as

$$I_4 = -T_t \int \partial^\alpha \left( \frac{\nabla N}{n} \right) U_\alpha - T_t \int \partial^\alpha \left( \frac{B + bN}{n} \right) U_\alpha = I_{411} + I_{412},$$

where $R_\alpha(N) = \{ \nabla N + B - bN \}/n$ and $n = n^{app} + \varepsilon^M N$. We also note that $B$ and $b$ are both of order $O(1)$ and hence

$$|I_{42}| \lesssim T_t (1 + \varepsilon^M \| N \|_{H^s}) (1 + \| U \|_{H^s}^2 + \| N \|_{H^s}^2).$$

For the term $I_{411}$, we have by integration by parts

$$I_{411} = -T_t \int \left[ \partial^\alpha \left( \frac{\nabla N}{n} \right) \cdot U_\alpha - T_t \int \left[ \partial^\alpha \left( \frac{B + bN}{n} \right) \right] U_\alpha \right]$$

$$= T_t \int \left[ \partial^\alpha \left( \frac{B + bN}{n} \right) \cdot U_\alpha + T_t \int \nabla \left( \frac{1}{n} \right) \partial^\alpha N \cdot U_\alpha - T_t \int \left[ \partial^\alpha \left( \frac{B + bN}{n} \right) \right] U_\alpha \right]$$

$$= I_{4111} + I_{4112} + I_{4113}.$$ From calculus inequalities in 3.1, the last two terms can be bounded by

$$|I_{4112}| + |I_{4113}| \lesssim T_t (1 + \varepsilon^M \| \nabla N \|_{H^s}) (1 + \| N \|_{H^s}^2 \| U \|_{H^s}).$$

In what follows, we treat the first term on the RHS of $I_{411}$. By (20), we have

$$I_{4111} = -T_t \int \frac{N_\alpha}{n^2} \left( \partial_t N_\alpha + \left[ \partial^\alpha \nabla, n \right] U + \nabla \partial^\alpha \cdot (N u^{app}) + \partial^\alpha R_\alpha \right)$$

$$= -T_t d \int \int \frac{|N_\alpha|^2}{n^2} + T_t \int \partial_t \left( \frac{1}{n^2} \right) |N_\alpha|^2 - T_t \int \frac{N_\alpha}{n^2} \left[ \partial^\alpha \nabla, U \right]$$

$$- T_t \int \frac{u^{app}}{n^2} N_\alpha \nabla N_\alpha - T_t \int \frac{N_\alpha}{n^2} \left[ \partial^\alpha \nabla, u^{app} \right] N - T_t \int \frac{N_\alpha}{n^2} \partial^\alpha R_\alpha = \sum_{i=1}^6 I_{411i}.$$ From Lemma 3.1, it is easy to show that

$$|I_{4112}| + |I_{4115}| + |I_{4116}| \lesssim T_t (1 + \varepsilon^M \| \partial_t N \|_{L^\infty}) (1 + \| N \|_{H^s}^2).$$

Moreover, by integration by parts, we have

$$|I_{4113}| + |I_{4114}| \lesssim T_t (1 + \varepsilon^M \| N \|_{H^s}^2) \| U \|_{H^s} \| U \|_{H^s}.$$ Therefore, we have the estimates for $I_4$ in (18)

$$I_4 \lesssim -T_t \int \frac{|N_\alpha|^2}{n^2} + T_t (1 + \varepsilon^M \| N \|_{H^s}^2) (1 + \| N \|_{H^s}^2).$$

(19)
Estimates of $I_5$. Differentiating the equation (16a), we obtain
\[
\partial_t N_\alpha + n \nabla \cdot u_\alpha + [\partial^\alpha \nabla, n] U + \nabla \partial^\alpha \cdot (Nu^{opp}) + \partial^\alpha R_\alpha = 0. \tag{20}
\]
Inserting this equation into $I_5$, we obtain
\[
I_5 = -\int \frac{\Phi_\alpha}{n} \partial_t N_\alpha - \int \frac{\Phi_\alpha}{n} [\partial^\alpha \nabla, n] U - \int \frac{\Phi_\alpha}{n} \nabla \partial^\alpha \cdot (Nu^{opp}) - \int \frac{\Phi_\alpha}{n} \partial^\alpha R_\alpha = I_{51} + \cdots + I_{54}. \tag{21}
\]
We first treat the term $I_{52}$ in (21). By integration by parts,
\[
I_{52} = -\int \frac{\Phi_\alpha}{n} \partial^\alpha \nabla n u - \sum_{\beta \neq \alpha} C_\beta \int \frac{\Phi_\alpha}{n} \partial^\beta \nabla n \cdot \partial^{\alpha-\beta} U
= -\int \frac{\Phi_\alpha}{n} \partial^\alpha \nabla n^{opp} u + \epsilon^M \int \nabla \cdot \left( \frac{\Phi_\alpha}{n} U \right) \partial^\alpha N - \sum_{\beta \neq \alpha} \cdots,
\]
which implies that
\[
|I_{52}| \lesssim \|\Phi_\alpha\| \|U\| + \epsilon^{M-\frac{3}{4}}\|\partial^\alpha N\| \left\| \nabla \left( \frac{\sqrt{\Phi_\alpha} U}{n} \right) \right\| + (1 + \epsilon^M\|N\|_{H^1})\|\Phi_\alpha\| \|U\|_{H^2}
\lesssim (1 + \epsilon^{2M-1}\|N\|_{H^2})(\|\Phi_\alpha\|^2 + \|U\|^2_{H^2} + \|N\|^2 + \epsilon\|\nabla\Phi_\alpha\|^2),
\]
thanks to the fact that $H^2$ is an algebra.
We now treat $I_{53}$. Differentiating (16c) with $\partial^\alpha$, we have
\[
\nabla N_\alpha = n^0 \nabla \Phi_\alpha + [\nabla \partial^\alpha, n^0] \Phi - \epsilon \nabla \Delta \Phi_\alpha + \sqrt{\epsilon} \nabla \partial^\alpha R_\phi(\cdot) + \nabla \partial^\alpha F_\phi, \tag{22}
\]
which enables us to rewrite $I_{53}$ as
\[
I_{53} = -\int \frac{\Phi_\alpha}{n} \nabla \partial^\alpha \cdot u^{opp} - \int \frac{\Phi_\alpha}{n} [\nabla \partial^\alpha, u^{opp}] N
= -\int \frac{n^0 \Phi_\alpha}{n} \nabla \partial^\alpha \Phi \cdot u^{opp} - \int \frac{\Phi_\alpha}{n} [\nabla \partial^\alpha, n^0] \Phi \cdot u^{opp} + \int \frac{\epsilon \Phi_\alpha}{n} \Delta \Phi_\alpha \cdot u^{opp}
- \int \frac{\sqrt{\epsilon} \Phi_\alpha}{n} \nabla \partial^\alpha R_\phi(\sqrt{\epsilon}) \cdot u^{opp} - \int \frac{\Phi_\alpha}{n} \nabla \partial^\alpha F_\phi \cdot u^{opp} - \int \frac{\Phi_\alpha}{n} [\nabla \partial^\alpha, u^{opp}] N
= I_{531} + \cdots + I_{536}.
\]
Since by continuity assumption, $\sigma'/2 < \|n\|_{L^\infty} < 2\sigma''$, we obtain by integration by parts
\[
|I_{531}| = \left| \frac{1}{2} \int \nabla \cdot \left( \frac{n^0 u^{opp}}{n} \right) \Phi_\alpha \right|^2 \leq C(1 + \epsilon^M\|\nabla N\|_{L^\infty})\|\Phi_\alpha\|^2.
\]
Similarly, by calculus inequalities in Lemma 3.1
\[
|I_{532}| \lesssim \|\Phi_\alpha\| \|\nabla \partial^\alpha, n^0] \Phi \| \lesssim \|\Phi_\alpha\|^2_{H^2},
\]
Moreover, by integration by parts twice, we obtain
\[
I_{533} = -\epsilon \int \nabla \Phi_\alpha \nabla \Phi_\alpha \cdot \frac{u^{opp}}{n} - \frac{\epsilon}{2} \int \Phi_\alpha \Delta \Phi_\alpha \nabla \cdot \left( \frac{u^{opp}}{n} \right)
= \frac{3\epsilon}{2} \int \nabla \cdot \left( \frac{u^{opp}}{n} \right) \|\nabla \Phi_\alpha\|^2 + \epsilon \int \Phi_\alpha \nabla \Phi_\alpha \nabla \cdot \left( \frac{u^{opp}}{n} \right).
\]
Hence, by Hölder inequality, we obtain
\[
I_{533} \lesssim (1 + \epsilon^M\|N\|^2_{H^2})(\|\Phi_\alpha\|^2 + \epsilon\|\nabla\Phi_\alpha\|^2).
\]
Similarly, we have
\[ I_{534} = \int \sqrt{\varepsilon} \nabla \Phi_\alpha \partial^\alpha R_\Phi (\sqrt{\varepsilon} \Phi) \cdot \frac{u^{app}}{n} + \int \sqrt{\varepsilon} \Phi_\alpha \partial^\alpha R_\Phi (\sqrt{\varepsilon} \Phi) \nabla \cdot \frac{u^{app}}{n}, \]
which follows from Lemma 3.2 that
\[ |I_{534}| \lesssim \sqrt{\varepsilon} C_1 (1 + \| \Phi \|_{H^s}) \| \nabla \Phi_\alpha \| + (1 + \varepsilon^M \| \nabla N \|_{L^\infty}) (1 + \| \Phi \|^2_{H^s}) \lesssim (1 + \varepsilon^M \| \nabla N \|_{L^\infty}) (1 + \| \Phi \|^2_{H^s} + \varepsilon \| \nabla \Phi_\alpha \|^2). \]

For the last two terms, it is easy to show that
\[ |I_{535}| + |I_{536}| \lesssim 1 + \| N \|_{H^s} + \| \Phi_\alpha \|^2. \]

Therefore, the term \( I_{53} \) in (21) is bounded by
\[ |I_{53}| \lesssim (1 + \varepsilon^M \| \nabla \|_{H^s}) (1 + \| N \|^2_{H^s} + \| \Phi \|^2_{H^s} + \varepsilon \| \nabla \Phi_\alpha \|^2). \]

Since \( R_\alpha \) is of \( O(1) \), it is easy to show that
\[ |I_{54}| \lesssim \| \Phi_\alpha \| \lesssim 1 + \| \Phi_\alpha \|^2. \]

Finally, we treat the term \( I_{51} \) in (21). Using (16c) again, we have
\[ \partial_t \partial^\alpha N = n^0 \partial_t \partial^\alpha \Phi + [\partial_t \partial^\alpha, n^0] \Phi - \varepsilon \partial_t \Delta \Phi_\alpha + \sqrt{\varepsilon} \partial_t \partial^\alpha R_\Phi (\sqrt{\varepsilon} \Phi) + \partial_t \partial^\alpha F_\Phi. \]

On the other hand, since \( 0 < \sigma' < n^0 < \sigma'' \) is bounded from below and above
\[ I_{51} = - \int \frac{\Phi_\alpha}{n} \partial_t N_\alpha - \int \frac{n^0 \Phi_\alpha}{n} \partial_t n_\alpha = - \int \frac{\partial^\alpha (n^0 \Phi)}{n} \partial_t N_\alpha + \int \frac{[\partial^\alpha, n^0] \Phi}{n} \partial_t n_\alpha = I_{511} + I_{512}. \]

For the last term on the right hand side, we have by integration by parts
\[ I_{512} = - \int \partial \left( \frac{[\partial^\alpha, n^0] \Phi}{n} \right) \partial_t \partial^\alpha^{-1} N. \]

and hence
\[ |I_{512}| \lesssim \| \partial \left( \frac{[\partial^\alpha, n^0] \Phi}{n} \right) \| \| \partial_t \partial^\alpha^{-1} N \|. \]

From (16a), we obtain
\[ \| \partial_t \partial^\alpha^{-1} N \| \lesssim (1 + \varepsilon^M \| N \|_{H^s}) (1 + \| N \|_{U} \|_{H^s}). \]

Moreover, since
\[ \| \partial \left( \frac{[\partial^\alpha, n^0] \Phi}{n} \right) \| \lesssim (1 + \varepsilon^M \| N \|_{H^s}) \| \Phi \|_{H^s}, \]
we have
\[ |I_{512}| \lesssim (1 + \varepsilon^2M \| N \|^2_{H^s}) (1 + \| N \|_{U} \|_{H^s} \|_{H^s}). \]

The estimate of \( I_{511} \) is left to the next subsection.

Therefore, we arrive at the following
\[ |I_5| \lesssim - \int \frac{\partial^\alpha (n^0 \Phi)}{n} \partial_t N_\alpha + (1 + \varepsilon^2M^{-1} \| N \|_{U} \|_{H^s}) (1 + \| \Phi \|_{U} \|_{H^s} \|^2 + \varepsilon \| \nabla \Phi_\alpha \|^2_{H^s}). \]
Proposition 1. We arrive at the following

\[ \frac{1}{2} \frac{d}{dt} \| U_\alpha \|^2 + \frac{T_i}{2} \frac{d}{dt} \int |N_{\alpha 1}|^2 \frac{dt}{n^2} \lesssim - \int \frac{\partial^\alpha (u^0 \Phi)}{n} \partial_i N_\alpha \]

\[ + (1 + \varepsilon^{4M-2} \| N, U \|_{H^s}) (1 + \| N, U, \Phi \|_{H^s}^2 + \varepsilon \| \nabla \Phi \|_{H^s}^2). \]

3.3. Second estimates. Now, we differentiate the remainder system (16b) with \( \partial^\alpha \), then take the inner product of (16b) with \(-\varepsilon \Delta U_\alpha\), and by integration by parts twice, we obtain

\[ \varepsilon \frac{d}{dt} \| \nabla U_\alpha \|^2 = \frac{\varepsilon}{2} \int \nabla \cdot u [\nabla U_\alpha]^2 - 2 \nabla u \nabla U_\alpha \nabla U_\alpha + \varepsilon \int \partial^\alpha (u_\alpha) \nabla U \Delta U_\alpha \]

\[ + \varepsilon \int \partial^\alpha (u_\alpha \Delta \nabla u) \Delta U_\alpha + \varepsilon T_i \int \partial^\alpha R_p(N) \Delta U_\alpha \]

\[ + \varepsilon \int \nabla \Phi_\alpha \Delta U_\alpha + \varepsilon \int \partial^\alpha R_\alpha \Delta U_\alpha =: \sum_{i=1}^6 J_i, \tag{25} \]

where \( \nabla u \nabla U_\alpha \nabla U_\alpha = \sum_{i,j,k} \partial_j u^i \partial_i U_\alpha^j \partial_j U_\alpha^k \). By integration by parts, it is easy to show that

\[ |J_1| + |J_3| + |J_6| \lesssim (1 + \varepsilon^M \| \nabla U \|_{L^\infty})(\| U_\alpha \|^2 + \varepsilon \| \nabla U_\alpha \|^2). \]

For the term \( J_2 \), we have

\[ |J_2| \lesssim \varepsilon \| \nabla U_\alpha \| \| \nabla (\partial^\alpha, u) \nabla \nabla \|
\lesssim \varepsilon \| \nabla U_\alpha \| (\| \partial^\alpha \nabla u \nabla U \| + \| \partial^\alpha u \nabla \|)
\lesssim \varepsilon \| \nabla U_\alpha \| (\| \nabla^2 u \nabla \|_{L^3} \| \nabla U \|_{W^{s-1,6}} + \| \nabla u \|_{H^s} \| \nabla \|_{L^\infty}
\lesssim \| \partial^\alpha \|_{H^s}(\| \nabla U \|^2_{H^s} + \varepsilon \| \nabla \|_{H^s}^2). \]

The estimate of \( J_4 \). The term \( J_4 \) in (25) can be treated exactly as \( I_4 \) since

\[ J_4 = -\varepsilon T_i \int \partial^\alpha \nabla \left( \frac{\nabla N}{n} \right) \nabla U_\alpha - \varepsilon T_i \int \partial^\alpha \nabla \left( \frac{B + bN}{n} \right) \nabla U_\alpha = J_{41} + J_{42}. \]

Since \( B \) and \( b \) are both of order \( O(1) \), we have

\[ |J_{42}| \lesssim \varepsilon T_i (1 + \varepsilon^+ \| N \|_{H^s}^2) (1 + \| \nabla \|_{H^s}^2 + \varepsilon \| \nabla \|_{H^s}^2). \]

For the term \( J_{41} \), we have by integration by parts twice

\[ J_{41} = \varepsilon T_i \int \frac{\partial^\alpha \nabla N}{n} \nabla \nabla \cdot U_\alpha + \varepsilon T_i \int \left( \frac{\nabla}{n} \right) \partial^\alpha \nabla \nabla \cdot U_\alpha \]

\[ - \varepsilon T_i \int \partial^\alpha \nabla \left( \frac{1}{n} \right) \nabla N \nabla U_\alpha = J_{411} + J_{412} + J_{413}. \]

The last two terms can be bounded by Sobolev inequality and calculus inequalities

\[ |J_{412} + J_{413}| \lesssim \varepsilon T_i (1 + \varepsilon^{(s+1)} M (\| N \|_{H^s}^{s+1})) \| \nabla N \|_{H^s} \| \nabla \|_{H^s}. \]
For the term $J_{411}$, we have by (20), we have

$$J_{411} = -\varepsilon T_i \int \frac{\nabla N_\alpha}{n} \nabla \left( \frac{\partial N_\alpha}{n} + [\partial^\alpha \nabla, n] U + \nabla \partial^\alpha \cdot (N u^{app}) + \partial^\alpha R_n \right)$$

$$= -\varepsilon T_i \int \frac{\nabla N_\alpha}{n^2} \nabla \partial N_\alpha - \varepsilon T_i \int \frac{\nabla N_\alpha}{n} \partial N_\alpha \nabla \left( \frac{1}{n} \right) - \varepsilon T_i \int \frac{\nabla N_\alpha}{n} \nabla (\cdots)$$

$$= -\frac{\varepsilon T_i}{2} \frac{d}{dt} \int \frac{|\nabla N_\alpha|^2}{n^2} + \frac{\varepsilon T_i}{2} \int \frac{\partial (\frac{1}{n^2}) |\nabla N_\alpha|^2}{n^2} - \varepsilon T_i \int \frac{\nabla N_\alpha}{n} \partial N_\alpha \nabla \left( \frac{\partial^\alpha \cdot (N u^{app})}{n} \right)$$

$$- \varepsilon T_i \int \frac{\nabla N_\alpha}{n} \nabla \left( \frac{\partial^\alpha R_n}{n} \right) = \sum_{i=1}^{6} J_{411i}.$$
Using (16c) once more and by integration by parts, we obtain
\[ J_4 \leq \frac{\varepsilon T_i}{2} \int \frac{|\nabla N_0|^2}{n^2} dt + \varepsilon T_i (1 + \varepsilon^M \|N, U\|_{H^s})(1 + \|N, U\|_{H^s}^2 + \|\nabla N, \nabla U\|_{H^s}^2). \]  
(26)

**Estimate of \( J_5 \).** By integration by parts and (20), we have
\[ J_5 = -\varepsilon \int \Delta \Phi \nabla \cdot U_{\alpha} + \int \frac{\varepsilon \Delta \Phi}{n} \partial_t N_{\alpha} + \int \frac{\varepsilon \Delta \Phi}{n} [\partial^\alpha \nabla, n] U + \int \frac{\varepsilon \Delta \Phi}{n} \partial^\alpha R_{\alpha} = J_{51} + \cdots + J_{54}. \]  
(27)

First, recalling \( J_{51} \), we can rewrite \( J_{51} \) as
\[ J_{51} = \int \frac{\partial^\alpha (\varepsilon \Delta \Phi - (n^0 \Phi))}{n} \partial_t N_{\alpha} - I_{511} =: K - I_{511}. \]  
(28)

Using (16c) once more and by integration by parts, we obtain
\[ K = -\int \frac{N_{\alpha}}{n} \partial_t N_{\alpha} + \sqrt{\varepsilon} \int \frac{\partial^\alpha R_{\alpha}}{n} \partial_t N_{\alpha} + \int \frac{\partial^\alpha F_{\phi}}{n} \partial_t N_{\alpha} \]
\[ = -\frac{1}{2} \frac{d}{dt} \int \frac{|N_{\alpha}|^2}{n} + \frac{1}{2} \int \partial_t (\frac{1}{n}) |N_{\alpha}|^2 + \sqrt{\varepsilon} \int \frac{\partial^\alpha R_{\alpha}}{n} \partial_t N_{\alpha} + \int \frac{\partial^\alpha F_{\phi}}{n} \partial_t N_{\alpha} \]
\[ = \sum_{i=1}^4 K_i. \]

For \( K_2 \), it is easy to show that
\[ |K_2| \lesssim (1 + \varepsilon^M \|\partial_t N\|_{L^\infty}) \|N_{\alpha}\|^2. \]

For \( K_3 \), by integration by parts, we obtain
\[ |K_3| \lesssim \sqrt{\varepsilon} \|\partial_t \partial^\alpha R_{\alpha}\|_{L^2} \|\partial_t \partial^\alpha - 1 N\|_{L^2} \]
\[ \lesssim \|\partial_t \partial^\alpha - 1 N\|^2_{L^2} + \varepsilon (1 + \varepsilon^M \|\nabla N\|_{H^s})(1 + \|\Phi\|_{H^{s+1}}^2). \]

Similarly, for \( K_4 \), we obtain by integration by parts that
\[ |K_4| \lesssim (1 + \varepsilon^M \|\nabla N\|_{H^s})(1 + \|\partial_t \partial^\alpha - 1 N\|^2_{L^2}). \]

On the other hand, from (20), we easily obtain the estimates
\[ \|\partial_t \partial^\alpha - 1 N\|_{L^2} \lesssim (1 + \varepsilon^M \|N\|_{L^\infty}) \|\nabla N\|_{L^2} \]
\[ + (1 + \varepsilon^M \|\nabla N\|_{L^\infty}) \|\partial^\alpha N\|_{L^2} + \|N_{\alpha}\|_{L^2} + 1 \]
\[ \lesssim (1 + \varepsilon^M \|N, U\|_{H^s})(1 + \|N, U\|_{H^s}). \]

Therefore, we obtain
\[ |J_{511}| \lesssim (1 + \varepsilon^M \|N, U\|_{H^s}^3)(1 + \|N, U\|_{H^s}^2 + \|\Phi\|_{H^{s+1}}^2) - I_{511}. \]  
(29)
Next, we treat the term $J_{52}$ in (28)

$$J_{52} = \int \frac{\varepsilon \Delta \Phi_\alpha}{n} [\partial^\alpha \nabla, n] U$$

$$= \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \partial^\alpha \nabla n U + \sum_{\beta \neq \alpha} C^\beta_\alpha \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \partial^\beta \nabla n \cdot \partial^{\alpha - \beta} U$$

$$= \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \partial^\alpha \nabla n^{app} U + \varepsilon M \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \partial^\alpha \nabla N U$$

$$+ \sum_{\beta \neq \alpha} C^\beta_\alpha \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \partial^\beta \nabla n \cdot \partial^{\alpha - \beta} U = J_{521} + J_{522} + J_{523}.$$  \hspace{1cm} (30)

For the first two terms on the RHS of (30), we have

$$|J_{521}| + |J_{523}| \lesssim (1 + \varepsilon M \|(U, N)\|_{H^s})(\varepsilon^2 \|\Delta \Phi_\alpha\|^2 + \|(N, U)\|_{H^s}^2).$$

The estimate of $J_{522}$ is similar to that of $J_{53}$ in (31). By using (22) and after a series estimates, one finally obtains

$$|J_{522}| \lesssim (1 + \varepsilon^2 M \|(U, N)\|_{H^s})(1 + \|N, \Phi\|_{H^s}^2, + \varepsilon \|\nabla \Phi\|_{H^s}^2 + \varepsilon^2 \|\Delta \Phi_\alpha\|^2).$$

We first consider $J_{53}$. Similar to $J_{53}$, using (22), we have

$$J_{53} = \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \nabla \partial^\alpha N \cdot u^{app} + \int \frac{\varepsilon \Delta \Phi_\alpha}{n} [\nabla \partial^\alpha, u^{app}] N$$

$$= \int \frac{\varepsilon n^0 \Delta \Phi_\alpha}{n} \nabla \Phi_\alpha \cdot u^{app} + \int \frac{\varepsilon \Delta \Phi_\alpha}{n} [\nabla \partial^\alpha, n^0] \Phi \cdot u^{app}$$

$$- \int \frac{\varepsilon^2 \Delta \Phi_\alpha}{n} \nabla \Phi_\alpha \cdot u^{app} + \int \frac{\varepsilon^3/2 \Delta \Phi_\alpha}{n} \nabla \partial^\alpha \beta \Phi_\beta \cdot u^{app}$$

$$+ \int \frac{\varepsilon \Delta \Phi_\alpha}{n} \nabla \partial^\alpha \Phi^{\alpha'} \cdot u^{app} + \int \frac{\varepsilon \Delta \Phi_\alpha}{n} [\nabla \partial^\alpha, u^{app}] N = \sum_{i=1}^6 J_{53i}. \hspace{1cm} (31)$$

By continuity assumption, $\sigma'/2 < n < 2\sigma''$ and integration by parts, we obtain the following estimates

$$|J_{531}| = \left| \frac{\varepsilon}{2} \int \nabla \cdot \left( \frac{n^0 u^{app}}{n} \right) \nabla \Phi_\alpha \right|^2 \lesssim \varepsilon (1 + \varepsilon M \|\nabla N\|_{L^\infty}) \|\nabla \Phi_\alpha\|^2.$$ 

Similarly, by calculus inequalities in Lemma 3.1

$$|J_{532}| \lesssim \varepsilon \|\Delta \Phi_\alpha\| \|\nabla \partial^\alpha, n^0 \nabla \Phi\| \lesssim \|\Phi\|_{H^s}^2 + \varepsilon^2 \|\Delta \Phi_\alpha\|^2.$$ 

Moreover, by integration by parts, we obtain

$$|J_{533}| = \left| \frac{1}{2} \int \varepsilon^2 |\Delta \Phi_\alpha|^2 \nabla \cdot \left( \frac{u^{app}}{n} \right) \right| \lesssim \varepsilon^2 (1 + \varepsilon M \|\nabla N\|_{L^\infty}) \|\Delta \Phi_\alpha\|^2.$$ 

Similarly, we have by Hölder inequality

$$|J_{534}| \lesssim \varepsilon^{3/2} \|\Delta \Phi_\alpha\| \|\nabla \partial^\alpha \beta \Phi_\beta \cdot u^{app} \| \lesssim \varepsilon^2 \|\Delta \Phi_\alpha\|^2 + \varepsilon C_1 (1 + \|\Phi\|_{H^{s+1}}).$$

For the last two term, it is easy to show that

$$|J_{535}| + |J_{536}| \lesssim \varepsilon \|\Delta \Phi_\alpha\| \lesssim 1 + \|N\|_{H^s}^2 + \varepsilon^2 \|\Delta \Phi_\alpha\|^2.$$
Note that we can make $C_1$ in Lemma 3.2 bounded by a uniform constant, say 1, when $0 < \varepsilon < \varepsilon_0$ for some small $\varepsilon_0$, and hence

$$|J_{53}| \lesssim (1 + \varepsilon^M \|\nabla N\|_{L^\infty})(1 + \|N, \Phi\|^2_{H^s} + \varepsilon^2 \|\Delta \Phi\|^2_{H^s}).$$

Finally, it is easy to show that

$$|J_{54}| \lesssim \varepsilon \|\Delta \Phi_\alpha\| \lesssim 1 + \varepsilon^2 \|\Phi_\alpha\|^2.$$

Finally, we obtain

$$J_5 \lesssim (1 + \varepsilon^{3M} \|(N, U)\|^2_{H^s})(1 + \|\nabla \Phi\|^2_{H^s} + \varepsilon \|\nabla \Phi\|^2_{H^s} + \varepsilon^2 \|\Delta \Phi\|^2_{H^s}) - I_{511}.$$ 

Combining all these estimates together, we obtain

**Proposition 2.** We arrive at the following

$$\frac{\varepsilon}{2} \frac{d}{dt} \|\nabla U_\alpha\|^2 + \frac{1}{2} \frac{d}{dt} \int \frac{|N_\alpha|^2}{n} + \varepsilon T_i \frac{d}{dt} \int \frac{|\nabla N_\alpha|^2}{n^2} \lesssim \int \frac{\partial^\alpha (n^0 \Phi)}{n} \partial_t N_\alpha + \left(1 + \varepsilon^{(s+1)M} \|N, U\|_{H^{s+1}}\right) \times \left(1 + \|N, U, \Phi\|^2_{H^s} + \varepsilon (1 + T_i) \|\nabla U, \nabla \Phi\|^2_{H^s} + \varepsilon T_i \|\nabla N\|^2_{H^s} + \varepsilon^2 \|\Delta \Phi\|^2_{H^s}\right).$$

**3.4. Proof of the Theorem 2.4.** Now, we denote

$$E_{s, T_i}^2 = \|N, U\|^2_{H^s} + \varepsilon \|\nabla U\|^2_{H^s} + T_i \|N\|^2_{H^s} + \varepsilon T_i \|\nabla N\|^2_{H^s}.$$ 

Using Lemma 3.3, Proposition 1 and Proposition 2 (adding them together for $0 \leq |\alpha| \leq s$), we arrive at the following inequality

$$\frac{d}{dt} E_{s, T_i}^2 \lesssim C(1 + \varepsilon^{(s+1)M} \|(N, U)\|_{H^{s+1}})(1 + E_{s, T_i}^2).$$

Gronwall inequality then ends the proof of Theorem 2.4. Note also that the argument in $C(\cdot)$ depends on $\varepsilon^{(s+1)M} \|(N, U)\|_{H^{s+1}}$, we get an existence time of (16) which equals $T'$ for $\varepsilon$ small enough, for every $T'$ arbitrary close to $T$.

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