HOMOLOGY OF TERNARY ALGEBRAS YIELDING INVARIANTS OF KNOTS AND KNOTTED SURFACES

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Abstract. We define homology of ternary algebras satisfying axioms derived from particle scattering. Its construction involves three presimplicial modules. Adding some simple axioms to the algebras under consideration leads to the degenerate subcomplexes and gives homological invariants of Reidemeister and Roseman moves.

1. Introduction

There has been a substantial amount of work done towards the understanding of (co)homology theories corresponding to labelings of arcs of knot diagrams by elements of binary algebras, see for example [6, 3, 2, 11]. The cocycle invariants obtained from such (co)homology theories proved to be very useful, producing many geometric results concerning knots and knotted surfaces (see [5] for some applications and more literature). On the other hand, labelings of the regions of knot diagrams (and their higher dimensional analogues) were not so actively studied. The arc and region labelings are not completely independent. Under some strong assumptions, including locality, linearity, and labels coming from abelian groups, there is a duality relation between the two labeling schemes (called Wu-Kadanoff duality, in a particular case), see [8]. However, the situation that we consider in this paper is more general: the labels for the regions come from a set equipped with a ternary operation defined in a natural way. We hope to see the homology theory developed here as a basis for construction of knot invariants, just as it was the case with the homology of self-distributive binary structures introduced in [6] and [3].

As in [8], we begin with some motivation from physics. We consider three particles moving with different velocities in 1-dimensional ambient space. They divide it into parts and the state of the vacuum can be different in them (see Fig. 1). When two particles approach each other, they scatter and recede from each other preserving momenta, but the state of the vacuum

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between them can change, and we will assume that the new state is described as $abcT$, where $T: X \times X \times X \to X$ is a ternary operation on the set $X$ of states, and $a$, $b$, and $c$ are the states before scattering, taken in a cyclic clockwise order as in Fig. 2. With three particles, there will be exactly three pairwise scatterings, but their order depends on the initial position of the particles. It is a natural assumption that the states of the vacua after all pairwise scatterings should not depend on this order, and thus two axioms are obtained:

$$(A3L) \quad \forall a, b, c, d \in X \quad (abcT)cdT = [ab(bcdT)T](bcdT)dT$$

and

$$(A3R) \quad \forall a, b, c, d \in X \quad ab(bcdT)T = a(abcT)[(abcT)cdT]T.$$ 

Note that the right side of $A3L$ (resp. $A3R$) is obtained from the left side of $A3L$ (resp. $A3R$) by substitution $c \mapsto bcdT$ (resp. $b \mapsto abcT$).

**Example 1.1.** Let $(G, \cdot)$ be a group. Consider a generalization of conjugation:

$$abcT = a^{-1} \cdot b \cdot c, \quad abcT = a \cdot b \cdot c^{-1}.$$ 

Then $T$ satisfies $A3R$ but not $A3L$, and $\overline{T}$ satisfies $A3L$ but not $A3R$. 

![Figure 1](image1.png)

**Figure 1.**

![Figure 2](image2.png)

**Figure 2.**
Example 1.2. Let \((G, \cdot)\) be a group. The operation

\[\text{abc}_T^2 = a \cdot b^{-1} \cdot c\]

satisfies both A3L and A3R. We can generalize it to the category of extra loops, where

\[\text{abc}_T^2 = (a \cdot b^{-1}) \cdot c\]

satisfies A3L and A3R, see [10] for the proof.

Next, we will define homology theory for algebras \((X, T)\) satisfying A3L and A3R.

2. Homology

First, we recall the definition of a presimplicial module, as we will work with three such modules.

Definition 2.1. Let \(R\) be a commutative unital ring. A presimplicial module is a family \(C = (C_n)\) of \(R\)-modules together with face maps \(d_i^n: C_n \to C_{n-1}\), \(i = 0, 1, \ldots, n\), satisfying

\[d_i^n - d_j^n \circ d_j^n = d_i^{n-1} \circ d_j^n\]

for \(0 \leq i < j \leq n\).

Lemma 2.2. Let \((C_n, d^n)\) be a presimplicial module. Then \((C_n, \partial_n)\), where

\[\partial_n = \sum_{i=0}^n (-1)^i d_i^n\]

is a chain complex \((\partial_n \partial_n = 0)\).

Definition 2.3. Let \((X, T)\) be a ternary algebra satisfying axioms A3L and A3R. Let \(C_n(X) := R\langle X^{n+2}\rangle\) be the \(R\)-module generated freely by \((n + 2)\)-tuples \((x_0, x_1, \ldots, x_n, x_{n+1})\) of elements of \(X\). We define

\[\partial^n_L(x_0, x_1, \ldots, x_n, x_{n+1}) = \sum_{i=0}^n (-1)^i d_i^{n,L}(x_0, x_1, \ldots, x_n, x_{n+1}),\]

where \(d_i^{n,L}\) is defined inductively by

\[d_0^{n,L}(x_0, x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1}),\]

\[d_i^{n,L}(x_0, x_1, \ldots, x_n, x_{n+1}) = d_i^{n,L}(x_0, x_1, \ldots, x_i, x_{i+1})[x_i \mapsto x_i - x_{i-1}x_ix_{i+1}T]\]

for \(i \in \{1, \ldots, n\}\). We also use the second differential:

\[\partial^n_R(x_0, x_1, \ldots, x_n, x_{n+1}) = \sum_{i=0}^n (-1)^i d_i^{n,R}(x_0, x_1, \ldots, x_n, x_{n+1}),\]

where \(d_i^{n,R}\) is defined inductively by

\[d_0^{n,R}(x_0, x_1, \ldots, x_n, x_{n+1}) = (z_0, z_1, \ldots, z_n),\]
where \( z_0 = x_0, \) \( z_i = z_{i-1} x_i x_{i+1}, \) for \( i = 1, \ldots, n, \) and
\[
d_i^{m,R}(x_0, x_1, \ldots, x_n, x_{n+1}) = d_{i-1}^{m,R}(x_0, x_1, \ldots, x_n, x_{n+1})[x_{i-1}x_i x_{i+1} \mapsto x_i]
\]
for \( i \in \{1, \ldots, n\}. \) That is, the formula for \( d_i^{m,R}(x_0, x_1, \ldots, x_n, x_{n+1}) \) is obtained from \( d_{i-1}^{m,R}(x_0, x_1, \ldots, x_n, x_{n+1}) \) by replacing \( x_{i-1}x_i x_{i+1} \) with \( x_i. \)

Now we combine the above differentials in a standard way, and define
\[
\partial_n(x_0, x_1, \ldots, x_n, x_{n+1}) = \sum_{i=0}^{n} (-1)^i (d_i^{m,L} - d_i^{m,R})(x_0, x_1, \ldots, x_n, x_{n+1}),
\]
that is,
\[
\partial_n = \partial_n^L - \partial_n^R.
\]

**Example 2.4.** We write the differential \( \partial \) in low dimensions:
\[
\partial_0(a, b) = b - a,
\]
\[
\partial_1(a, b, c) = (b, c) - (a, abcT) - (abcT, c) + (a, b),
\]
\[
\partial_2(a, b, c, d) = (b, c, d) - (a, abcT, (abcT)cdT) - (abcT, c, d) + (a, b, bcdT) + (ab(bcdT)T, bcdT, d) - (a, b, c),
\]
\[
\partial_3(a, b, c, d, e) = (b, c, d, e) - (a, abcT, (abcT)cdT, [(abcT)cdT]deT) - (abcT, c, d, e) + (a, b, bcdT, (bcdT)deT) + (ab(bcdT)T, bcdT, d, e) - (a, b, c, cdeT) - (ab[bc(cdeT)T]T, bc(cdeT)T, cdeT, e) + (a, b, c, d).
\]

From now on, we will denote \( (x_0, x_1, \ldots, x_n, x_{n+1}) \) by \( x. \)

We can describe \( d_i^{m,L} \) and \( d_i^{m,R} \) in a different way, defining their coordinates inductively.
\[
d_i^{m,L} = (d_{i,1}^{m,L}, \ldots, d_{i,k}^{m,L}, \ldots, d_{i,n+1}^{m,L})
\]
is calculated from right to left. For \( i \in \{0, \ldots, n\} \) and \( k \in \{1, \ldots, n+1\}, \)
\[
d_{i,k}^{m,L} x = \begin{cases} 
x_{k-1} x_k (d_{i,k+1}^{m,L} x) & \text{if } k \leq i \\
x_k & \text{if } k > i.
\end{cases}
\]
\[
d_i^{m,R} = (d_{i,0}^{m,R}, \ldots, d_{i,k}^{m,R}, \ldots, d_{i,n}^{m,R})
\]
is calculated from left to right. For \( i \in \{0, \ldots, n\} \) and \( k \in \{0, \ldots, n\}, \)
\[
d_{i,k}^{m,R} x = \begin{cases} 
(d_{i,k-1}^{m,R} x) x_k x_{k+1} T & \text{if } k > i \\
x_k & \text{if } k \leq i.
\end{cases}
\]
Theorem 2.5. \((C_n, d^{n,L}_i)\) is a presimplicial module.

Proof. We need to show that
\[
d^{n-1,L}_i d^{n,L}_j x = d^{n-1,L}_{j-1} d^{n,L}_i x
\]
for \(0 \leq i < j \leq n\). The proof will be by induction over \(j - i\). First, let
\(j - i = 1\), so we need to prove that
\[
d^{n-1,L}_i d^{n,L}_{i+1} x = d^{n-1,L}_{i} d^{n,L}_i x.
\]
For \(i = 0\), the validity of equation 4 is immediate \((d^{n,L}_0\) just removes the first input), so assume that \(i > 0\). This part of the proof will show a nice interplay between the axioms A3L and A3R in the differentials. Until the end of this proof, we will denote \((d^{n,L}_{i+1,1} x, \ldots, d^{n,L}_{i+1,k} x, \ldots, d^{n,L}_{i+1,n+1} x)\) by
\((i + 1)\), and \((d^{n,L}_{i+1,1} x, \ldots, d^{n,L}_{i+k} x, \ldots, d^{n,L}_{i+n+1} x)\) by \((i)\). Then
\[
L := d^{n-1,L}_i d^{n,L}_{i+1} x = d^{n-1,L}_{i+1,1} x, \ldots, d^{n,L}_{i+k} x, \ldots, d^{n,L}_{i+n+1} x = d^{n-1,L}_i(i + 1)
\]
for \(0 \leq i < j \leq n\). The proof will be by induction over \(j - i\). First, let
\(j - i = 1\), so we need to prove that
\[
d^{n-1,L}_i d^{n,L}_{i+1} x = d^{n-1,L}_{i} d^{n,L}_i x.
\]
For \(i = 0\), the validity of equation 4 is immediate \((d^{n,L}_0\) just removes the first input), so assume that \(i > 0\). This part of the proof will show a nice interplay between the axioms A3L and A3R in the differentials. Until the end of this proof, we will denote \((d^{n,L}_{i+1,1} x, \ldots, d^{n,L}_{i+1,k} x, \ldots, d^{n,L}_{i+1,n+1} x)\) by
\((i + 1)\), and \((d^{n,L}_{i+1,1} x, \ldots, d^{n,L}_{i+k} x, \ldots, d^{n,L}_{i+n+1} x)\) by \((i)\). Then
\[
L := d^{n-1,L}_i d^{n,L}_{i+1} x = d^{n-1,L}_{i+1,1} x, \ldots, d^{n,L}_{i+k} x, \ldots, d^{n,L}_{i+n+1} x = d^{n-1,L}_i(i + 1)
\]
and
\[
R := d^{n-1,L}_i d^{n,L}_i x = d^{n-1,L}_{i+1,1} x, \ldots, d^{n,L}_{i+k} x, \ldots, d^{n,L}_{i+n+1} x = d^{n-1,L}_i(i)
\]
Before comparing the coordinates in \(L\) and \(R\), recall the definition of \(d^{n-1,L}_{i,k}\), with indices of the inputs beginning with 1:
\[
d^{n-1,L}_{i,k}(y_1, \ldots, y_{n+1}) = \begin{cases} y_k y_{k+1} (d^{n-1,L}_{i,k+1} (y_1, \ldots, y_{n+1})) T & \text{if } k \leq i \\ y_{k+1} & \text{if } k > i \end{cases}
\]
We see that for \(k \geq i + 1\),
\[
d^{n-1,L}_{i,k}(i + 1) = d^{n,L}_{i,k+1} x = x_{k+1} = d^{n,L}_{i,k+1} x = d^{n-1,L}_i(i).
\]
For \(k = i\),
\[
d^{n-1,L}_{i,k}(i) = (d^{n,L}_{i,i} x)(d^{n,L}_{i,i+1} x)(d^{n-1,L}_{i,i+1} (i)) T = (x_{i-1} x_i x_{i+1} T) x_i x_{i+1} x_{i+2} T
\]
and
\[
d^{n-1,L}_{i,k}(i + 1) = (d^{n,L}_{i+1,i} x)(d^{n,L}_{i+1,i+1} x)(d^{n-1,L}_{i+1,i+1} (i + 1)) T = (x_{i-1} x_i x_{i+1} x_{i+2} T) x_i x_{i+1} x_{i+2} T.
\]
Thus, the equality of \(d^{n-1,L}_{i,k}(i)\) and \(d^{n-1,L}_{i,k}(i + 1)\) is exactly the application of the axiom A3L. To prove the equalities of the remaining coordinates, we also need the axiom A3R. Note that
\[
d^{n,L}_{i+1,i+1} x = x_i x_{i+1} x_{i+2} T = x_i x_{i+1} (d^{n-1,L}_{i,i+1} (i)) T.
\]
In general, for $k \leq i + 1$, we have the relation

$$d_{i+1,k}^{n,L}x = x_{k-1}(d_{i,k}^{n,L}x)(d_{i,k}^{n-1,L}(i))T.$$  

We prove it by induction, using A3R:

$$d_{i+1,k}^{n,L}x = x_{k-2}x_{k-1}(d_{i+1,k}^{n,L}x)T = x_{k-2}x_{k-1}[x_{k-1}(d_{i,k}^{n,L}x)(d_{i,k}^{n-1,L}(i))T]T$$

$$= x_{k-2}(d_{i,k}^{n,L}x)((d_{i,k}^{n,L}x)(d_{i,k}^{n-1,L}(i))T)T$$

$$= x_{k-2}(d_{i,k}^{n,L}x)(d_{i,k}^{n,L}(i))T = x_{k-2}(d_{i,k}^{n,L}x)(d_{i,k}^{n-1,L}(i))T.$$  

Now assume that

$$d_{i+1,k}^{n-1,L}(i + 1) = (d_{i+1,k}^{n,L}(i))T$$

and $d_{i,k}^{n-1,L}(i)$, using induction and A3L:

$$d_{i,k-1}^{n-1,L}(i + 1) = (d_{i,k}^{n,L}(i))T$$

for $i'$, $j'$ such that $0 \leq i' < j' \leq n$ and $j' - i' < j - i$, where $j - i \geq 2$. For the rest of the paper, let $x[k]$ denote $(x_0, \ldots, x_{k-1}x_kx_{k+1}T, \ldots, x_{n+1})$. Note that with this new notation $d_{i,k}^{n,L}x = d_{i-1,k}^{n,L}x[i]$ for $i \in \{1, \ldots, n\}$. We have

$$d_{i-1,k}^{n,L}d_{j-1,k}^{n,L} = d_{i,k}^{n,L}d_{j,k}^{n,L}[j] = d_{i,k}^{n-1,L}d_{j-1,k}^{n,L}x[j]$$

$$= d_{i-1,k}^{n-1,L}d_{j-1,k}^{n,L}x[j], \ldots, d_{i-j-2,k}^{n,L}x[j], d_{i-j-1,k}^{n,L}x[j], d_{i-j,k}^{n,L}x[j], d_{i-j+1,k}^{n,L}x[j], \ldots, d_{i,n+1,k}^{n,L}x[j]$$

and

$$d_{j-1,k}^{n-1,L}d_{i,k}^{n,L} = d_{j-1,k}^{n-1,L}(d_{i-1,k}^{n,L}x, \ldots, d_{i,j-1,k}^{n,L}x, d_{i,j,k}^{n,L}x, d_{i,j+1,k}^{n,L}x, \ldots, d_{i,n+1,k}^{n,L}x)$$

$$= d_{j-1,k}^{n-1,L}(d_{i,j,k}^{n,L}x, \ldots, d_{i,j-1,k}^{n,L}x, d_{i,j-1,k}^{n,L}x, d_{i,j,k}^{n,L}x)(d_{i,j+1,k}^{n,L}x, d_{i,j+1,k}^{n,L}x, \ldots, d_{i,n+1,k}^{n,L}x)$$

$$= d_{j-1,k}^{n-1,L}(d_{i,j,k}^{n,L}x, \ldots, d_{i,j-1,k}^{n,L}x, d_{i,j-1,k}^{n,L}x, d_{i,j,k}^{n,L}x)(d_{i,j+1,k}^{n,L}x, d_{i,j+1,k}^{n,L}x, \ldots, d_{i,n+1,k}^{n,L}x).$$

Now the proof ends, since $d_{i,j-1,k}^{n,L}x[j] = x_{j-1}$, so from the formula $\blacksquare$ for $k \leq j - 2$, we have

$$d_{i,k}^{n,L}(x_0, \ldots, x_{j-2}, x_{j-1}, x_{j-1}x_{j+1}T, x_{j+1}, \ldots, x_{n+1})$$

$$= d_{i,k}^{n,L}(x_0, \ldots, x_{j-2}, x_{j-1}, x_{j}, x_{j+1}, \ldots, x_{n+1}).$$
Definition 2.6. Given a ternary operation $T$, let $\hat{T}$ denote the ternary operation defined by $xyz\hat{T} = zyxT$.

Remark 2.7. $(X, T)$ satisfies A3R if and only if $(X, \hat{T})$ satisfies A3L. $(X, T)$ satisfies A3L if and only if $(X, \hat{T})$ satisfies A3R.

Let $y^r$ denote reversing the order of the elements of the tuple $y$; we will also use the linear extension of this operator, denoting it with the same symbol. When two or more operators are considered, we will add their symbols to the differentials, as in the following lemma.

Lemma 2.8. $d^{n.R,T}_i x = (d^{n.L,T}_{n-i} x^r)^r$ for $i \in \{0, \ldots, n\}$.

Proof. Proof by induction over $n - i$. For $i = n$:

$$
\begin{align*}
  d^{n.R,T}_n x &= (x_0, \ldots, x_n) = (x_n, \ldots, x_0)^r = (d^{0.L,T}_{0} (x_{n+1}, \ldots, x_0))^r = (d^{n.L,T}_n x^r)^r.
\end{align*}
$$

Suppose that the equality in this lemma is true for some $i \leq n$. We will show it for $i - 1$. We note that $d^{n.R,T}_{i-1} x$ is obtained from $d^{n.R,T}_i x$ by substitution $x_i \mapsto x_{i-1}x_{i+1}T$. It follows that

$$
\begin{align*}
  d^{n.R,T}_{i-1} x &= d^{n.R,T}_i (x_0, \ldots, x_{i-1}, x_{i-1}x_{i+1}T, x_{i+1}, \ldots, x_{n+1}) \\
  &= (d^{n.L,T}_{n-i} (x_0, \ldots, x_{i-1}, x_{i-1}x_{i+1}T, x_{i+1}, \ldots, x_{n+1})^r)^r \\
  &= (d^{n.L,T}_{n-i} (x_{n+1}, \ldots, x_{i+1}, x_{i+1}x_{i-1}T, x_{i-1}, \ldots, x_0))^r \\
  &= (d^{n.L,T}_{n-i+1} (x_{n+1}, \ldots, x_i, \ldots, x_0))^r = (d^{n.L,T}_{n-(i-1)} x^r)^r.
\end{align*}
$$

Theorem 2.9. $(C_n, d^{n.R}_i)$ is a presimplicial module.

Proof. If $T$ satisfies the axioms A3L and A3R, then $\hat{T}$ satisfies them also, and the equation

$$
\begin{align*}
  d^{n-1,L,T}_i d^{n.L,T}_j &= d^{n-1,L,T}_{j-1} d^{n.L,T}_i
\end{align*}
$$

holds for $0 \leq i < j \leq n$. Then, for $0 \leq i < j \leq n$, we have

$$
\begin{align*}
  d^{n-1.R,T}_i d^{n.R,T}_j x &= d^{n-1.R,T}_i (d^{n.L,T}_{n-j} x^r)^r = (d^{n-1,L,T}_i ((d^{n.L,T}_{n-j} x^r)^r)^r \\
  &= (d^{n-1,L,T}_n d^{n.L,T}_{n-j} x^r)^r = (d^{n-1,L,T}_{n-j-1} d^{n.L,T}_{n-i} x^r)^r \\
  &= (d^{n-1,L.T}_{n-i} ((d^{n.L,T}_{n-i} x^r)^r)^r = (d^{n-1,L,T}_{n-i} d^{n.R,T}_j x^r)^r \\
  &= d^{n-1.R,T}_i d^{n.R,T}_j x.
\end{align*}
$$

Theorem 2.10. $(C_n, d^n_i = d^{n.L}_i - d^{n.R}_i)$ is a presimplicial module.
Proof. We need to show that for $0 \leq i < j \leq n$:
\[
d_i^{m-1}d_j^m = (d_i^{m-1,L} - d_i^{m-1,R})(d_j^{m,L} - d_j^{m,R})
\]
\[
= d_i^{m-1,L}d_j^{m,L} - d_i^{m-1,L}d_j^{m,R} - d_i^{m-1,R}d_j^{m,L} + d_i^{m-1,R}d_j^{m,R}
\]
is equal to
\[
d_{j-1}^{m-1}d_i^m = (d_{j-1}^{m-1,L} - d_{j-1}^{m-1,R})(d_i^{m,L} - d_i^{m,R})
\]
\[
= d_{j-1}^{m-1,L}d_i^{m,L} - d_{j-1}^{m-1,L}d_i^{m,R} - d_{j-1}^{m-1,R}d_i^{m,L} + d_{j-1}^{m-1,R}d_i^{m,R}.
\]
We can make use of Theorems 2.5 and 2.9, and then what remains to be shown is:
\[
d_i^{m-1,R}d_j^{m,L} = d_{j-1}^{m-1,L}d_i^{m,R}
\]
and
\[
d_i^{m-1,L}d_j^{m,R} = d_{j-1}^{m-1,R}d_i^{m,L}.
\]
These equalities will be proven in the next two lemmas. \qed

Lemma 2.11.
\[
d_i^{m-1,R}d_{j}^{m,L} = d_{j-1}^{m-1,L}d_i^{m,R}
\]
for $0 \leq i < j \leq n$.

Proof. We use induction over $j - i$. First we prove that
\[
d_i^{m-1,L}d_{i+1}^{m,L} = d_i^{m-1,L}d_{i}^{m,R}
\]
for $i \in \{0, \ldots, n - 1\}$. This equality follows from equalities
\[
L := d_{i+1,k}^{m-1,L}(d_{i+1,1}^{m,L}, \ldots, d_{i+1,n+1}^{m,L}) = d_{i,k}^{m-1,L}(d_{i,0}^{m,R}, \ldots, d_{i,n}^{m,R}) =: R,
\]
that we will show for $k \in \{1, \ldots, n\}$. For $1 \leq k \leq i$ we have:
\[
L = d_{i+1,k}^{m,L}x = x_{k-1}x_k(d_{i+1,k+1}^{m,L})T
= x_{k-1}x_k(\ldots(x_{i-1}x_i(x_i x_{i+1}x_{i+2}T)T)\ldots)T
\]
and
\[
R = (d_{i,k-1}^{m,R})(d_{i,k}^{m,R})x(d_{i,k+1}^{m-1,L}(d_{i,0}^{m,R}, \ldots, d_{i,n}^{m,R}))T
= x_{k-1}x_k(d_{i,k+1}^{m-1,L}(d_{i,0}^{m,R}, \ldots, d_{i,n}^{m,R}))T
= x_{k-1}x_k(\ldots(x_{i-1}x_i(d_{i+1,k}^{m,R})T)\ldots)T
= x_{k-1}x_k(\ldots(x_{i-1}x_i((d_{i,i}^{m,R})x_{i+1}x_{i+2}T)T)\ldots)T
= x_{k-1}x_k(\ldots(x_{i-1}x_i(x_i x_{i+1}x_{i+2}T)T)\ldots)T.
\]
If $k = i + 1$, then
\[
L = d_{i+1,i+1}^{m,L}x = x_i x_{i+1}x_{i+2}T.
\]
We check that the inputs for comparing $k > i$

\[ \text{For } k > i + 1: \]

\[ \begin{align*}
L &= (d_{i,k-2}^{m-1,R}(d_{i+1,1}^L, \ldots, d_{i+1,n+1}^L))((d_{i+1,j}^L)(d_{i+1,k+1}^L)T = d_{i+1,k+2}^R(d_{i+1,j}^L)T \\
R &= d_{i,k}^{m,R} = (d_{i,k-2}^{m,R}(d_{i+1,1}^R, \ldots, d_{i+1,n+1}^R))((d_{i+1,j}^R)(d_{i+1,k}^R)T = d_{i+1,k+2}^R(d_{i+1,j}^R)T.
\end{align*} \]

We see that the coordinate that makes a difference between $x$ and $x[j]$ is not used. Now we consider the $j$-th input in $R$:

\[ \begin{align*}
(d_{i,j-2}^{m,R})(d_{i,j-1}^{m,R})(d_{i+1,j}^{m,R})T &= (d_{i,j-2}^{m,R})(d_{i,j-1}^{m,R}(x_{j-1}x_{j+1}T)T \\
&= (d_{i,j-2}^{m,R})(d_{i+1,j-2}^{m,R}x_{j-1}x_{j+1}T)T \\
&= (d_{i,j}^{m,R})x_{j-1}(x_{j-1}x_{j+1}T)T
\end{align*} \]

and the equal $j$-th input in $L$:

\[ \begin{align*}
d_{i,j}^{m,R}x[j] &= (d_{i,j}^{m,R}x[j])(x_{j-1}x_{j+1}T)
\end{align*} \]

We will also look at $(j + 1)$-st inputs; in $L$ it is

\[ \begin{align*}
d_{i,j}^{m,R}x[j] &= (d_{i,j}^{m,R}x[j])(x_{j-1}x_{j+1}T)x_{j+1}T \\
&= [(d_{i,j}^{m,R}x[j])x_{j-1}(x_{j-1}x_{j+1}T)T](x_{j-1}x_{j+1}T)x_{j+1}T \\
&= ((d_{i,j}^{m,R}x[j])x_{j-1}x_{j+1}T)x_{j}x_{j+1}T,
\end{align*} \]
and in $R$ it is equal expression:

$$d^n_{i,j} x = (d^n_{i,j-1} x)x_jx_{j+1}T = ((d^n_{i,j-2} x)x_{j-1}x_j T)x_jx_{j+1}T.$$ 

The equalities of the later inputs follow inductively from the equality that we have just checked.

**Lemma 2.12.**

$$d^{n-1,L}_{i} d^n_{j} = d^{n-1,R}_{j} d^n_{i}$$

for $0 \leq i < j \leq n$.

**Proof.** First, from the fact that $d^n_0$ and $d^{n-1,L}_0$ only remove the left-most input, we see that

$$d^{n-1,L}_0 d^n_j = d^{n-1,R}_{j-1} d^n_0$$

for $j \in \{1, \ldots, n\}$. Now we use the induction over $n - (j - i)$ (with 0 corresponding to the case $i = 0$, $j = n$ that we have just considered). Let

$$R := d^{n-1,R}_{j-1} d^n_i x = d^{n-1,R}_{j-1} d^n_{i-1} x[i] = d^{n-1,L}_{i-1} d^n_x[i]$$

$$= d^{n-1,L}_{i-1} (d^n_{j-1} x[i], \ldots, d^n_{j, i-1} x[i]),$$

$$L := d^{n-1,L}_{j-1} d^n_i x$$

$$= d^{n-1,L}_{j-1} (d^n_{j,0} x, \ldots, d^n_{j, i-1} x, (d^n_{j, i} x)(d^n_{j, i+1} x)T, d^n_{j, i+1} x, \ldots, d^n_{j, n} x).$$

Since $d^n_{j,k}(y_0, \ldots, y_{n+1}) = y_k$ for $k \leq i + 1 \leq j$, we see the equality of the first corresponding $i + 2$ inputs for $d^{n-1,L}_{i-1}$ in $R$ and $L$. The equality of the inputs with greater indices follows from the recursive definition of the coordinates of $d^n_j$. \hfill \Box

**Definition 2.13.** [1] A precubical module $D$ is a sequence of modules $D_n$, $n \geq 0$, together with face maps $d^0_i, d^1_i : D_n \rightarrow D_{n-1}$, for $n \geq 1$, $i = 1, \ldots, n$, satisfying

$$d^0_i d^1_j = d^1_{j-1} d^0_i$$

for all $i < j$ and $\epsilon, \delta \in \{0, 1\}$.

**Remark 2.14.** If we take $D_n = C_{n-1}(X)$, and start numbering differentials $d^{n-1,L}_i$ and $d^{n-1,R}_i$ from 1 instead of 0, then $(D_n, d^{n-1,L}_i, d^{n-1,R}_i)$ is a precubical module.

**Definition 2.15.** We write $H^L(X)$, $H^R(X)$ and $H(X)$ for the homology of $(C_n(X), \partial^L_n)$, $(C_n(X), \partial^R_n)$ and $(C_n(X), \partial_n)$, respectively. We call the first two: left homology and right homology, respectively.
3. Degenerate subcomplex

Definition 3.1. Let $C^D_n(X)$ denote the $R$-module generated freely by $(n+2)$-tuples $(x_0, x_1, \ldots, x_n, x_{n+1})$ of elements of a ternary algebra $(X, T)$ such that $x_i = x_{i+2}$ for some $i$.

Definition 3.2. We define additional axioms, needed for the degenerate part:

(A1) $\forall_{a,b \in X} \ abaT = b$,

(A2L) $\forall_{a,b,c \in X} \ ab(bacT)T = c$,

(A2R) $\forall_{a,b,c \in X} \ (cabT)baT = c$.

Remark 3.3. $\hat{T}$ satisfies A2R if and only if $T$ satisfies A2L.

Lemma 3.4. Let $(X, T)$ satisfy A1 and A2L. Then

$$\partial^L_n(C^D_n) \subset C^D_{n-1}.$$ 

Proof. We note that the last $n - i + 1$ coordinates of $d^{n,L}_i x$ are the same as the last $n - i + 1$ coordinates of $x$. That is, $d^{n,L}_i x$ contains at the end the sequence $x_{i+1}, \ldots, x_{n+1}$. Suppose that $x$ is such that $x_j = x_{j+2}$, for some $j$. Then this repetition occurs also in all $d^{n,L}_i x$ with $i \in \{0, \ldots, j-1\}$. Now we consider $i = j$ and $i = j + 1$:

$$d^{n,L}_{j+1} x = d^{n,L}_j (x_0, \ldots, x_j x_{j+1} x_{j+2} T, \ldots, x_{n+1}) = d^{n,L}_j (x_0, \ldots, x_{j+1}, \ldots, x_{n+1}) = d^{n,L}_j x.$$

However, in $\partial^L_n$, $d^{n,L}_j x$ and $d^{n,L}_{j+1} x$ appear with opposite signs. Now let $j + 2 \leq i \leq n$:

$$d^{n,L}_{i,j+1} x = x_j x_{j+1} x_{j+2} (d^{n,L}_{i,j+2} x) T = x_j x_{j+1} (x_{j+1} x_{j+2} (d^{n,L}_{i,j+3} x) T) T$$

$$= x_j x_{j+1} x_{j+3} (d^{n,L}_{i,j+3} x) T = d^{n,L}_{i,j+3} x.$$

Thus, there is a repetition in $d^{n,L}_i x$ on coordinates $j + 1$ and $j + 3$. \hfill \square

Lemma 3.5. Let $(X, T)$ satisfy A1 and A2R. Then

$$\partial^R_n(C^D_n) \subset C^D_{n-1}.$$ 

Proof. From Lemma 2.8 it follows that

$$\partial^R_n x = (-1)^n (\partial^L_n \hat{T}_n x)^r.$$ 

Lemma now follows from Remark 3.3 and Lemma 3.4. \hfill \square

Corollary 3.6. Let $(X, T)$ satisfy A1, A2L and A2R. Then

$$\partial_n(C^D_n) \subset C^D_{n-1}.$$ 

Definition 3.7. Now that we have proved that, with the addition of suitable axioms, \((C_n^D(X), \partial_n^L), (C_n^D(X), \partial_n^R)\) and \((C_n^D(X), \partial_n)\) are chain subcomplexes of \((C_n(X), \partial_n^L), (C_n(X), \partial_n^R)\) and \((C_n(X), \partial_n)\), respectively, we call their homology left degenerate, right degenerate and degenerate, and denote it by \(H^{LD}(X), H^{RD}(X)\) and \(H^{D}(X)\), respectively.

We define quotient complexes \((C_n^N(X), \partial_n^L) = (C_n(X)/C_n^D(X), \partial_n^L), (C_n^N(X), \partial_n^R) = (C_n(X)/C_n^D(X), \partial_n^R)\) and \((C_n^N(X), \partial_n) = (C_n(X)/C_n^D(X), \partial_n)\) with induced differentials (and the same notation). We call the homology of these complexes left normalized, right normalized and normalized, and denote it by \(H^{LN}(X), H^{RN}(X)\) and \(H^{N}(X)\), respectively.

4. Knot invariants from homology

In [10] we defined various ternary algebras giving knot invariants via assignment of algebra elements to the regions of a knot diagram. In this first paper on homology of such structures, we will focus on a simple invariant that uses one operator.

First, we narrow the definition of a (combinatorial) \(n\)-quasigroup given in [12] to the case of ternary quasigroups.

Definition 4.1. A ternary quasigroup is a set \(X\) equipped with a ternary operation \(T: X^3 \to X\) such that for a quadruple \((x_1, x_2, x_3, x_0)\) of elements of \(X\) satisfying \(x_1x_2x_3T = x_0\), specification of any three elements of the quadruple determines the remaining one uniquely.

Definition 4.2. Let \((X, T)\) be a ternary algebra, and let \(D\) be an unoriented classical link diagram. A ternary coloring of \(D\) with \((X, T)\) is an assignment of elements of \(X\) to the regions of \(D\) satisfying a rule illustrated in Fig. 3 at every crossing. Specifically, if \(x, y, z, w \in X\) are the colors at a crossing, then \(w = xyzT\), where \(w\) and \(x\) are assigned to the regions separated by the over-arc, and \(x, y\) and \(z\) are taken cyclically.
Again referring to the Fig. 3, we see that every color around a crossing is determined uniquely by the other three colors, so the structures used for ternary colorings of link diagrams should be ternary quasigroups.

To get the invariance under the third Reidemeister move, the axioms A3L and A3R are needed (see Fig. 4). Substitutions in the equations around the crossing in Fig. 3 give axioms A2L, A2R and the axiom

\[(A2M) \quad \forall_{a,b,c \in X} \quad a(bcaT)bT = c,\]

from which follows the invariance under the second Reidemeister move, see Fig. 5.

**Definition 4.3.** For the sake of brevity, we will call an algebra \((X, T)\) satisfying the axioms A1, A2L, A2M, A2R, A3L and A3R a **tern**.
Example 4.4. $X = \mathbb{R}^n$ with $pqrT = p + q - r$ is a tern. Geometrically, $T$ reflects the point $r$ through the middle of the interval connecting the points $p$ and $q$.

Example 4.5. A tern useful for computations, due to its simplicity: $R_n = (\{0, \ldots, n-1\}, T)$, where $pqrT = p + q - r \mod n$.

More generally:

Example 4.6. Let $(G, \cdot)$ be a group. Then $(G, T)$, where $xyzT = x \cdot z^{-1} \cdot y$, is a tern.

To define tern colorings in general, we use the following description of diagrams of manifold knots given in [9].

Definition 4.7. [9] Let $M$ be a closed $n$-manifold embedded piecewise linearly (or smoothly) in $\mathbb{R}^{n+2}$ and let $p: \mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be the projection. $M$ is assumed to be in general position with respect to the projection $p$. The singularity set $\Delta$ that is the closure of the multiple point set \{ $y \in \mathbb{R}^{n+1} | |p^{-1}(y) \cap M| > 1$ \} in $\mathbb{R}^{n+1}$ is regarded as an $(n-1)$-dimensional stratified complex in which $(n-1)$-dimensional strata consist of transverse double points; they are called double point strata. By a diagram of $M$ we mean the image $p(M)$ equipped with the under-over information at each transverse double point.

Definition 4.8. Let $D$ be a diagram of $M$, and let $(X, T)$ be a tern. Let $R$ denote the set of regions of $D$, that is, the components of $\mathbb{R}^{n+1} - p(M)$. For each double point stratum, there are four regions around it. A tern coloring of $D$ is a function $c: R \to X$ satisfying the following condition for every double point stratum: if $R_1, R_2, R_3, R_4 \in R$ are the regions around the stratum, then $c(R_i)$, for $i \in \{1, 2, 3, 4\}$, can be expressed in terms of the colors of the other three regions as $c(R_i) = c(R_j)c(R_k)c(R_l)T$, where $R_i$ and $R_j$ are separated by an over-sheet (part of $p(M)$ that was higher before the
projection with respect to the direction of the projection), and $R_j$ and $R_k$ are separated by an under-sheet.

For knotted surfaces, we use broken diagrams (see [4] for details on broken surface diagrams), and the above definition is illustrated in Fig. 6.

Remark 4.9. One can check that the tern whose generators are with one-to-one correspondence with the regions of the knotted surface diagram, and relations come from the double point curves, and are as in Definition 4.8, is invariant, up to isomorphism, under Roseman moves. To do that, one can use Tietze transformations (they are applicable for presentations of such algebras). In a similar way, we can assign a tern to a classical knot diagram, and its isomorphism class will be an invariant of Reidemeister moves.

Remark 4.10. The axioms A3L and A3R can be also seen on a colored triple point. They are needed for the consistency of the coloring; see Fig. 7, where a triple point is in the middle of a sphere containing three intersecting disks.

Definition 4.11. [4, 8] Suppose that the surface $F$ is oriented. We can give a co-orientation to the complement of the branch point set $Br$ on $p(F)$ as follows. For a point $x \in p(F) - Br$, choose vectors $v_1, v_2$ that are tangent to $p(F)$ in $\mathbb{R}^3$, so that the oriented frame $(v_1, v_2)$ matches the orientation of $F$. Then a normal vector $v(x)$ in $\mathbb{R}^3$ is chosen so that the ordered triple $(v, v_1, v_2)$ matches the orientation of 3-space.
The sign of a triple point is defined as follows. Let $v_T$, $v_M$ and $v_B$ be co-orientation normal vectors to the top, middle, and bottom sheets, respectively, that intersect at a triple point. If the oriented frame $(v_T, v_M, v_B)$ coincides with the right hand orientation of 3-space, the triple point is said to be positive, otherwise it is negative.

For a triple point, a source region is the region near the triple point such that $v_T$, $v_M$ and $v_B$ point away from it; a target region is the region such that $v_T$, $v_M$ and $v_B$ point towards it. In the same way, we define source and target regions for the double points (crossings) of oriented classical knot diagrams using the standard co-orientation vectors.

Definition 4.12. For a triple point of an oriented broken surface diagram, a term ascending path will mean a sequence of regions $R_0|R_1|R_2|R_3$ starting with the source region and ending with the target region such that the regions $R_0$ and $R_1$ are separated by the bottom (the most broken) sheet, $R_1$ and $R_2$ by the middle sheet, and $R_2$ and $R_3$ by the top sheet.

For a colored diagram and the ascending path as above, the sequence of colors $(c(R_0), c(R_1), c(R_2), c(R_3)) \in X^4$ will be called a colored ascending path.

We use similar definitions in other dimensions; for a given crossing of a colored oriented diagram of a classical knot, an ascending path is a triple of regions $(r_1, r_2, r_3)$, where $r_1$ is the source region, $r_1$ and $r_2$ are separated by an under-arc, and $r_3$ is the target region (separated from $r_2$ by the over-arc); then a colored ascending path is a triple $(c(r_1), c(r_2), c(r_3)) \in X^3$.

Definition 4.13. Given a tern $(X, T)$, by tern homology we mean the homology of $(C_n(X)/C_n^D(X), \partial_n)$, and we will denote it by $H_T(X)$.

Lemma 4.14. For a tern-colored oriented knot (resp. surface) diagram, a sum of the colored ascending paths over all double (resp. triple) points,
Figure 9.

**taken with the sign of the crossing (resp. triple point) is a cycle in tern homology of the tern used to color the diagram.**

**Proof.** Calculating the differential on a colored ascending path of a diagram of an \(n\)-dimensional knot corresponds to taking a sum of suitably signed lower dimensional colored ascending paths. For knotted surfaces, a colored ascending path is of length four, and each triple obtained after taking the differential of this path corresponds to two intersecting sheets dividing the space into four regions, see Fig. 9. One of these regions is a source region with respect to the co-orientation of this selected pair of sheets. Then, the triple is a colored ascending path starting with the color of this lower-degree source region. Each triple of colors has its twin triple on the opposite side of the third sheet. The sign convention (in all dimensions) is that behind (co-orientationwise) the most broken sheet of the configuration there is a negative tuple, the one in front of it is positive, and then this convention alternates as we consider successively less broken sheets. In particular, for knotted surfaces, behind the middle sheet the tuple is positive, and again negative behind the top sheet. Because of the sign conventions, each such triple corresponding to the two intersecting sheets and a triple point will cancel out with the triple at the next triple point to which these two intersecting sheets connect, or if they form a branch point, then the triple is a degenerate cycle of the form \(\pm(x, y, x)\).

For colored classical knots, ascending paths are of length three, and each pair of colors of regions obtained after taking the differential corresponds to an edge, and is a colored ascending path with respect to this edge. The pairs

\[
\partial(a, b, c, d) = (b, c, d) - (a, e, g) \\
- (e, c, d) + (a, b, h) \\
+ (f, h, d) - (a, b, c)
\]
behind the under-arc get a negative sign, and the ones behind the over-arc are positive, see Fig. 8. Because of the sign conventions, each such pair of elements corresponding to an edge and a crossing will cancel out with a pair corresponding to the same edge and its next crossing.

□

Lemma 4.15. The tern homology class of a cycle assigned to an oriented and colored knot diagram is invariant under Reidemeister moves.

Proof. The first Reidemeister move adds or removes a degenerate cycle \((a, b, a)\), so it does not change the tern homology class. The contributions coming from the two crossings in the second Reidemeister move cancel out, because the crossings have opposite signs. Now consider the third Reidemeister move with all the crossings positive, see Fig. 4. The contributions from the crossings after the move, minus the contributions before the move are equal to the boundary

\[
\partial_2(a, b, c, d) = (b, c, d) - (a, abcT, (abcT)cdT) \\
- (abcT, c, d) + (a, b, bcdT) \\
+ (ab(bcdT)T, bcdT, d) - (a, b, c).
\]

□

Lemma 4.16. The tern homology class of a cycle assigned to an oriented and colored broken surface diagram is invariant under Roseman moves.
Proof. We only need to consider the Roseman moves that involve triple points. The fifth Roseman move, Fig. 10 is similar to the second Reidemeister move in that the contributions from the two triple points with opposite signs cancel out. The sixth Roseman move, Fig. 11 produces (or

Figure 12.
deletes) a degenerate chain. As for the seventh Roseman move (also called the tetrahedral move), it is illustrated via movies (see [4] and [3] for the definition and the use of movies of knotted surfaces). The contributions from the triple points before the move, see Fig. 12 minus the contributions
after the move, see Fig. 13, are equal to the boundary
\[ \partial_3(a, b, c, d, e) = (b, c, d, e) - (a, abcT, (abcT)cdT, [(abcT)cdT]deT) \]
\[ - (abcT, c, d, e) + (a, b, bcdT, (bcdT)deT) \]
\[ + (ab(bcdT)T, bcdT, d, e) - (a, b, c, cdeT) \]
\[ - (ab|bc(cdeT)T|T, bc(cdeT)T, cdeT, e) + (a, b, c, d). \]

\[ \square \]

**Example 4.17.** To illustrate the above definitions, we finish with a simple example. The left side of Fig. 14 shows an abstract tern coloring of a diagram of a trefoil knot. The only requirement is that \( b = a(a(abcT)cT)cT \).

The cycle associated with this oriented diagram is
\[ (a, b, c) + (a, abcT, c) + (a, a(abcT)cT, c). \]

If the tern used for the coloring is \( R_3 = (\{0, 1, 2\}, T) \), with \( pqrT = p + q - r \) (mod 3), then one of the possible colorings is \( a = 0, b = 1, c = 2 \) (see the right side of Fig. 14). Calculations with GAP show that the above cycle, which now becomes \( (0, 1, 2) + (0, 2, 2) + (0, 0, 2) \), represents \( \mathbb{Z}_3 \) in the torsion part of \( H^T_1(R_3) \).

We define tern cohomology groups for the homology described in this paper in a standard way, so the cocycles used for knot diagrams are functions \( f : X \times X \times X \to A \), where \( A \) is an abelian group, satisfying two conditions:
\[ \forall_{a,b \in X} f(a, b, a) = 0, \]
and
\[ \forall_{a,b,c,d \in X} f(b, c, d) - f(a, abcT, (abcT)cdT) \]
\[ - f(abcT, c, d) + f(a, b, bcdT) \]
\[ + f(ab(bcdT)T, bcdT, d) - f(a, b, c) = 0. \]
An example of a cocycle for $X = \mathbb{R}^3$, with values in $\mathbb{Z}_3$, that evaluates nontrivially on the cycle $(0,1,2) + (0,2,2) + (0,0,2)$, is the following sum of the characteristic functions:

$$\chi(2,0,1) + \chi(0,2,1) + \chi(1,0,2) + \chi(1,1,0) + \chi(2,1,0) - \chi(0,1,2) - \chi(0,0,2).$$

Such cocycles can be used for state-sum invariants, similar to the ones defined in [3]: we consider the cycles corresponding to all possible tern colorings of a diagram and then evaluate a given cocycle on all these cycles (using the multiplicative notation for the abelian group $A$). The sum of these evaluations is in the group ring $\mathbb{Z}[A]$ and is an invariant of Reidemeister moves (analogous construction works for knotted surfaces and Roseman moves).

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