Equal-time two-point correlation functions in Coulomb gauge Yang-Mills theory

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Abstract

We apply a new functional perturbative approach to the calculation of the equal-time two-point correlation functions and the potential between static color charges to one-loop order in Coulomb gauge Yang-Mills theory. The functional approach proceeds through a solution of the Schrödinger equation for the vacuum wave functional to order $g^2$ and derives the equal-time correlation functions from a functional integral representation via new diagrammatic rules. We show that the results coincide with those obtained from the usual Lagrangian functional integral approach, extract the beta function and determine the anomalous dimensions of the equal-time gluon and ghost two-point functions and the static potential under the assumption of multiplicative renormalizability to all orders.

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1 Introduction

Finding an accurate (semi-)analytical description of the infrared sector of QCD is still one of the most important challenges of present-day quantum field theory. In this work we concentrate on Yang-Mills theory, QCD without dynamical quarks, since it is in this sector where the peculiar properties of QCD, in particular the confining interaction between quarks, arise. Recently, much of the activity in this area has focused on the formulation and (approximate) solution of Yang-Mills theory in the Coulomb gauge [1]–[7], the primary reason being that the Coulomb gauge Hamiltonian explicitly contains the color-Coulomb potential which furnishes the dominant nonperturbative contribution to the static or heavy quark potential.

Semi-analytical functional approaches to the calculation of gluon and ghost propagators in the infrared, mostly using Dyson-Schwinger equations, have been successful in Landau gauge Yang-Mills theory [8]. In the so-called ghost dominance approximation, even very simple analytical solutions exist in the far infrared [4, 9]. Although the consistency of these solutions is still under discussion, it is natural to inquire whether a similar approach could be useful in the Coulomb gauge. However, the breaking of Lorentz covariance through the Coulomb gauge condition makes the usual Lagrangian functional integral approach quite cumbersome in this gauge, see, e.g., Ref. [10]. For this reason, semi-analytical approaches in Coulomb gauge have mostly used a Hamiltonian formulation. A set of equations similar to Dyson-Schwinger equations is obtained from a variational principle using a Gaussian type of ansatzes for the vacuum wave functional in the Schrödinger representation [1, 3]. In the ghost dominance approximation, furthermore, simple analytical solutions are available for the far infrared [2, 4].

However, the status of the semi-analytical and analytical solutions in the Coulomb gauge is not yet entirely clear, for two reasons: first, two different solutions with an infrared scaling behavior (differing in the infrared exponents) have been found in both the analytical and the semi-analytical approaches [2]–[4], and there is as yet no theoretical guidance to what the physical solution should be; second, the inclusion of the Coulomb form factor (the form factor for the color-Coulomb potential, which measures the deviation of the Coulomb potential from a factorization in terms of ghost propagators) in the set of equations of Dyson-Schwinger type results problematic. In Ref. [3], the equation for the Coulomb form factor has been considered subleading compared to the equations for the gluon and ghost propagators and therefore treated in the tree-level approximation, while in Ref. [11] all equations have been considered to be of the same order and therefore treated on an equal footing, with the result that solutions with infrared scaling behavior cease to exist. It should be emphasized that only solutions with scaling behavior can give rise to a linearly rising Coulomb potential, and that the latest lattice calculations also show a scaling behavior for the equal-time correlation functions in the deep infrared [5]. It is not clear at present how to improve the approximation used in the variational approach in order to arrive at a unique and consistent solution.

On the other hand, an interesting relation between Landau and Coulomb gauge Yang-Mills theory has been pointed out in the ghost dominance approximation in Refs. [2, 4]: the equal-time correlation functions of the Hamiltonian approach in Coulomb gauge are the formal counterparts in three dimensions of the covariant correlation functions in Landau
gauge in four dimensions. Building on this analogy, a possible strategy seems to be to replace the variational principle by a calculation of equal-time correlation functions in the Coulomb gauge and trying to formulate Dyson-Schwinger equations for the latter. In the present work, we take a first step in this direction: we set up a functional integral representation of the equal-time correlation functions (without taking a detour to the space-time correlation functions) that is the precise three-dimensional analogue of the usual functional integral representation of the covariant correlation functions in the Lagrangian approach to Landau gauge Yang-Mills theory. We also develop a diagrammatic representation and a set of Feynman rules for the equal-time correlation functions. We use this new formulation here in order to calculate the equal-time gluon and ghost two-point correlation functions and the potential for static color charges in Coulomb gauge perturbatively to one-loop order. We extract the one-loop beta function and determine the asymptotic ultraviolet behavior of the equal-time two-point functions and the static potential. We also show that our results coincide with those obtained in a Lagrangian functional integral approach \cite{12,13} and use the latter for the renormalization of the equal-time correlation functions and the static potential.

The organization of the paper is as follows: in the next section, we determine the vacuum wave functional perturbatively to order $g^2$ from the solution of the Schrödinger equation. With the vacuum functional determined to the corresponding order, we turn to the calculation of the equal-time gluon and ghost two-point correlation functions in Section 3. We also calculate the one-loop corrections to the static or heavy quark potential (and thus to the Coulomb form factor) in the same section. Although for the latter calculation we need to go beyond the terms that we have calculated for the vacuum functional in Section 2, the relevant additional contributions are quite simply determined. In Section 4, we provide another representation of the equal-time two-point functions by choosing equal times (zero) in the space-time correlation functions determined before in the Lagrangian functional integral representation \cite{12,13} of the theory. The static potential can also be obtained from a two-point function that arises in the Lagrangian approach. We use the alternative representations of the two-point functions and the static potential to perform the renormalization of our results. We show the nonrenormalization of the ghost-gluon vertex in the same section and use it to determine the beta function and the asymptotic ultraviolet behavior of the two-point functions. We also show that the same beta function is found from considering the static potential. Finally, in Section 5, we briefly summarize our findings. We give some details on an important difference that arises between the Lagrangian and the Hamiltonian approach when it comes to the implementation of the Coulomb gauge in the Appendix.

2 Perturbative vacuum functional

It is very simple to write down a functional integral representation of the equal-time correlation functions, given that they are nothing but the vacuum expectation values of products of the field operators. In the Schrödinger representation of Yang-Mills theory in Coulomb gauge, the equal-time $n$-point correlation functions in (3-)momentum space have the follow-
\[ \langle A^a_i(p_1, t = 0) A^b_j(p_2, t = 0) \cdots A^f_r(p_n, t = 0) \rangle = \int D[A] \delta(\nabla \cdot A) \text{FP}(A) A^a_i(p_1) A^b_j(p_2) \cdots A^f_r(p_n) |\psi(A)|^2. \]  

(1)

Here, \( \psi(A) \) is the true vacuum wave functional of the theory. The (absolute) square \( |\psi(A)|^2 \) then plays the role of the exponential of the negative Euclidean classical action in the corresponding representation of the covariant correlation functions (in Euclidean space). \( \text{FP}(A) \equiv \det[-\nabla \cdot D(A)] \), with the covariant derivative in the adjoint representation defined as

\[ D^{ab}(A) = \delta^{ab} \nabla + g f^{abc} A^c, \]  

(2)

is the Faddeev-Popov determinant (in 3 dimensions) which forms a part of the integration measure for the scalar product of states in the Schrödinger representation (see Ref. [14]).

Note that the fields \( A^a_i(p) \) on the left-hand side of Eq. (1) are spatially transverse, \( p \cdot A^a_i(p) = 0 \). We will assume the transversality of the fields \( A^a \) in all of the following formulae, which we could make manifest by introducing a transverse basis in momentum space. However, there is usually no need to do so explicitly.

In order to write down the functional integral for the equal-time correlation functions explicitly, the vacuum wave functional needs to be specified. The analogy with the covariant theory suggests to make an exponential ansatz for this wave functional, in the spirit of the \( e^S \) expansion in many-body physics [15]. We consider a full Volterra expansion of the exponent:

\[ \psi(A) = \exp \left( -\sum_{k=2}^{\infty} \frac{1}{k!} \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_k}{(2\pi)^3} \sum_{i_1,i_2,\ldots,i_k} \sum_{a_1,a_2,\ldots,a_k} f_{k;i_{i_1}i_{i_2}\ldots i_k}^{a_{i_1}a_{i_2}\ldots a_k}(-p_1,\ldots,-p_k) \right. \]

\[ \times A^{a_{i_1}}_i(p_1) \cdots A^{a_k}_{i_k}(p_k)(2\pi)^3 \delta(p_1 + \cdots + p_k). \]  

(3)

Any normalization factor can be conveniently absorbed in the functional integration measure in Eq. (1). Terms linear in \( A \) in the exponent \( (k = 1) \) are excluded by the symmetry of the wave functional under global gauge transformations (in the absence of external color charges). Regarding notation, given that our Hamiltonian formalism is not manifestly covariant, we will denote the contravariant spatial components of 4-vectors by (Latin) \text{\textit{subindices}}.

We insert this ansatz for the vacuum wave functional into the Schrödinger equation

\[ H \psi(A) = E_0 \psi(A), \]  

(4)

where \( H \) is the Christ-Lee Hamiltonian for Coulomb gauge Yang-Mills theory [14],

\[ H = \frac{1}{2} \int d^3 x \left( -\frac{1}{\text{FP}(A)} \frac{\delta}{\delta A^a_i(x)} \text{FP}(A) \frac{\delta}{\delta A^a_i(x)} + B^a_i(x) B^a_i(x) \right) \]

\[ + \frac{g^2}{2} \int d^3 x d^3 y \frac{1}{\text{FP}(A)} \rho^a(x) \text{FP}(A) \langle x, a | (-\nabla \cdot D)^{-1} (-\nabla^2) (-\nabla \cdot D)^{-1} | y, b \rangle \rho^b(y). \]  

(5)
Here,
\[ B^a_i = -\frac{1}{2} \epsilon_{ijk} F^a_{jk} = \left( \nabla \times A^a - \frac{g}{2} f^{abc} A^b \times A^c \right)_i \] (6)
is the chromo-magnetic field, and
\[
\rho^a(x) = \rho^a_0(x) + f^{abc} A^b_\parallel(x) \frac{1}{i} \frac{\delta}{\delta A^c_\parallel(x)}
\] (7)
the color charge density, including external static charges \( \rho_0 \) for later use. Note that we have extracted a factor \( g \) from the color charges in order to simplify the counting of orders of \( g \) in the rest of the paper. The notation \( \langle \mathbf{x}, a|C|\mathbf{y}, b \rangle \) refers to the kernel of the operator \( C \) in an integral representation.

In most of the following perturbative calculation, we will need the Hamiltonian only up to order \( g^2 \), where
\[
H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( -(2\pi)^3 \frac{\delta}{\delta A^a_i(p)}(2\pi)^3 \frac{\delta}{\delta A^a_i(-p)} + A^a_i(-p) p^2 A^a_i(p) \right)
\] (8)
\[
+ \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} A^a_i(-p) \left( \frac{N_c g^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1 - (\mathbf{p} \cdot \mathbf{q})^2}{(p - q)^2} \right) (2\pi)^3 \frac{\delta}{\delta A^a_j(-p)}
\] (9)
\[
+ \frac{g}{3!} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} f^{abc} [\delta_{jk}(p_{1,l} - p_{2,l}) + \delta_{kl}(p_{2,j} - p_{3,j}) + \delta_{lj}(p_{3,k} - p_{1,k})]
\]
\[
	imes A^a_i(p_1) A^a_j(p_2) A^a_k(p_3) (2\pi)^3 \delta(p_1 + p_2 + p_3)
\] (10)
\[
+ \frac{g^2}{4!} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_4}{(2\pi)^3} \left[ f^{abc} f^{cde} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + f^{ace} f^{bde} (\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})
\]
\[
+ f^{ade} f^{bce} (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \right] A^a_i(p_1) A^a_j(p_2) A^a_k(p_3) A^a_l(p_4) (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4)
\] (11)
\[
+ \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho^a(-p) \frac{1}{p^2} \rho^a(p) + \mathcal{O}(g^3).
\] (12)
The term (9) stems from the application of the functional derivative to the Faddeev-Popov determinant. In this term, \( N_c \) stands for the number of colors, \( f^{acd} f^{bcd} = N_c \delta^{ab} \), and \( \hat{p} = \mathbf{p}/|\mathbf{p}| \) denotes a unit vector. In the absence of external charges, we get for the term (12)
\[
\frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho^a(-p) \frac{1}{p^2} \rho^a(p)
\]
\[
= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} A^a_i(-p) \left( N_c g^2 \int \frac{d^3q}{(2\pi)^3} \frac{t_{ij}(q)}{(p - q)^2} \right) (2\pi)^3 \frac{\delta}{\delta A^a_j(-p)}
\] (13)
\[
- \frac{g^2}{4} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_4}{(2\pi)^3} \left( f^{ace} f^{bde} \frac{\delta_{ik}\delta_{jl}}{(p_1 + p_3)^2} + f^{ade} f^{bce} \frac{\delta_{il}\delta_{jk}}{(p_1 + p_4)^2} \right)
\]
\[
	imes (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) A^a_i(p_1) A^a_j(p_2) A^a_k(p_3) A^a_l(p_4) (2\pi)^3 \frac{\delta}{\delta A^a_j(-p_3)} (2\pi)^3 \frac{\delta}{\delta A^a_i(-p_4)}.
\] (14)
In the term (13) on the right-hand side, \( t_{ij}(q) \) denotes the spatially transverse projector or transverse Kronecker delta

\[
t_{ij}(q) \equiv \delta_{ij} - \hat{q}_i \hat{q}_j .
\]  

We shall now show, explicitly up to order \( g^2 \), that there is a unique perturbative solution of the Schrödinger equation (4) for the wave functional \( \psi(A) \) in Eq. (3), if we only suppose that the dominant contribution to the coefficient function \( f_k \) is at least of order \( g^{k-2} \) for \( k \geq 2 \). We will consider the case without external charges to begin with, and include charges \( \rho_q \) later on in the context of the static potential. To order \( g^0 \), the Schrödinger equation reads

\[
(N^2 - 1) \left( \int \frac{d^3 p}{(2\pi)^3} f_2(p) \right) (2\pi)^3 \delta(0) + \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} A_i^a(-p) \left[ p^2 - (f_2(p))^2 \right] A_i^a(p) = E_0 ,
\]

where we have used that

\[
f_{2;ij}(p, -p) = f_2(p)\delta_{ij}\delta^{ab} = f_2(-p)\delta_{ij}\delta^{ab}
\]

(to be contracted with spatially transverse fields) as a consequence of the symmetry under the exchange of the arguments, of spatially rotational and global gauge symmetry, and of the fact that \( f_{2;ij}(p_1, p_2) \) is only defined for \( p_1 + p_2 = 0 \). Equation (16) implies that, to the current order,

\[
f_2(p) = |p| ,
\]

\[
E_0 = (N^2 - 1) \left( \int \frac{d^3 p}{(2\pi)^3} |p| \right) (2\pi)^3 \delta(0) .
\]

Generally, the energy \( E_0 \) cancels any field-independent terms multiplying the vacuum functional in the Schrödinger equation to any order in \( g \). Eqs. (18) and (19) represent nothing but the well-known solution of the free \( (g = 0) \) theory. The choice of the sign in Eq. (18) is dictated by the normalizability of the wave functional (3) to order \( g^0 \). As usual, \((2\pi)^3 \delta(0) \) is to be understood as the total volume of space.

To the next (first) order of \( g \), the Schrödinger equation is not much more complicated: it reads

\[
\frac{1}{3!} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \frac{d^3 p_3}{(2\pi)^3} \left\{ i g f^{abc}_{ijk} [\delta_{jk}(p_{1,l} - p_{2,l}) + \delta_{kl}(p_{2,j} - p_{3,j}) + \delta_{lj}(p_{3,k} - p_{1,k})] \right. \\
- 3|p_1| f^{abc}_{3;ijk}(p_1, -p_2, -p_3) A^a_j(p_1) A^b_k(p_2) A^c_i(p_3) (2\pi)^3 \delta(p_1 + p_2 + p_3) = 0 ,
\]

where we have already taken into account the results (18), (19) and the fact that

\[
f^{abc}_{3;ijk}(p_1, p_2, p_3) = f^{abc}_{3;ijk}(p_1, p_2, p_3) ,
\]

which is a consequence of global gauge symmetry and the invariance of the vacuum wave functional under charge conjugation. The unique solution of Eq. (20) with the full symmetry
under the exchange of the arguments is
\[
 f^{abc}_{3;ijk}(p_1, p_2, p_3) = -\frac{ig f^{abc}}{|p_1| + |p_2| + |p_3|} \left[ \delta_{ij}(p_{1,k} - p_{2,k}) + \delta_{jk}(p_{2,i} - p_{3,i}) + \delta_{ki}(p_{3,j} - p_{1,j}) \right].
\] (22)

This equality, and all the following equalities with explicit spatial (Lorentz) indices, are proper equalities only after contracting with the corresponding number of transverse vector fields $A$, or, equivalently, after contracting every external spatial index with a transverse projector, for example in Eq. (22) the index $i$ with $t_i(p_1)$.

Now we will move on to consider the Schrödinger equation to order $g^2$. On the left-hand side, terms with four and two powers of $A$ appear, which have to cancel separately, and an $A$-independent term which must equal $E_0$ to this order. We begin with the term with four powers of $A$. The quartic coupling $\{\}$ in the Hamiltonian has to be cancelled by terms stemming from the second functional derivative in Eq. (8) acting on the vacuum wave functional, and a contribution from Eq. (14). To order $g^2$, the coefficient functions $f_2$ of Eq. (18) and $f_3$ of Eq. (22) contribute, and the function $f_4$ which we will determine. As a result, the coefficient function $f_4$ in the vacuum wave functional takes the following (fully symmetric) form to order $g^2$:

\[
(|p_1| + \ldots + |p_4|) f^{abcd}_{4;ijkl}(p_1, \ldots, p_4)
\]

\[= g^2 \left[ f^{abe} f^{cde} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + f^{ace} f^{bde} (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) + f^{ade} f^{bce} (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) \right] 
\]

\[- \left[ f^{abe}_{3;ijm}(p_1, p_2, -p_1 - p_2) t_{mn}(p_1 + p_2) f^{cde}_{3;kln}(p_3, p_4, p_1 + p_2) + f^{ace}_{3;ikm}(p_1, p_3, -p_1 - p_3) t_{mn}(p_1 + p_3) f^{bde}_{3; jln}(p_2, p_4, p_1 + p_3) + f^{ade}_{3;ilm}(p_1, p_4, -p_1 - p_4) t_{mn}(p_1 + p_4) f^{bce}_{3; jkn}(p_2, p_3, p_1 + p_4) \right] \]

\[- g^2 \left( f^{abe} f^{cde} \delta_{ij} \delta_{kl} \frac{(|p_1| - |p_2|)(|p_3| - |p_4|)}{(p_1 + p_2)^2} + f^{ace} f^{bde} \delta_{ik} \delta_{jl} \frac{(|p_1| - |p_3|)(|p_2| - |p_4|)}{(p_1 + p_3)^2} + f^{ade} f^{bce} \delta_{il} \delta_{jk} \frac{(|p_1| - |p_4|)(|p_2| - |p_3|)}{(p_1 + p_4)^2} \right). \]

(24)

(25)

This result for $f_4$ is represented diagrammatically in Fig. 4. Equation (24), divided by $(|p_1| + \ldots + |p_4|)$, is interpreted as the elementary or “bare” four-gluon vertex. The role of the factor 2 and the signs in Fig. 4 will become clear in the next section. Equation (24) and the second diagram in Fig. 4 represent the contraction of two elementary three-gluon vertices, the latter being given mathematically by Eq. (24). The contraction refers to spatial and color indices and the momenta, with opposite signs. Note that there is no “propagator” factor associated with the contraction (except for a transverse Kronecker delta), and there is a factor $1/(|p_1| + \ldots + |p_4|)$ for the external momenta which is unusual from a diagrammatic point of view. Finally, Eq. (25) and the last diagram in Fig. 4 describe an “elementary” Coulomb interaction between the external gluon lines.
\[2f_4 = - \left( \sum_{\text{perms.}} + 2 \text{ perms.} \right) - \left( \sum_{\text{perms.}} + 2 \text{ perms.} \right)\]

Figure 1: A diagrammatic representation of Eqs. \((23)-(25)\). Every diagram corresponds to precisely one of the Eqs. \((23)-(25)\), in the same order. The “2 perms.” refer to permutations of the external legs.

With this result in hand, we can go on to consider the terms quadratic in \(A\) in the Schrödinger equation to order \(g^2\). The relevant contributions originate from Eqs. \((8), (9), (13), \text{ and (14)}\), and involve the functions \(f_2\) and \(f_4\). As a result, we obtain the following equation for the coefficient function \(f_2\) to order \(g^2\):

\[
(f_2(p))^2 \delta^{ab} \delta_{ij} = \left( p^2 - \frac{N_c g^2}{2} |p| \right) \int \frac{d^3q}{(2\pi)^3} \frac{1 - \left( \hat{p} \cdot \hat{q} \right)^2}{(p - q)^2} \delta^{ab} \delta_{ij} \\
+ \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} f^{abc}_{ij;kl}(-p, p, -q, q)t_{kl}(q) - N_c g^2 \delta^{ab} \int \frac{d^3q}{(2\pi)^3} \frac{|p| - |q|}{(p - q)^2} t_{ij}(q). \tag{26}
\]

The explicit result for \(f_2\) to order \(g^2\) is

\[
f_2(p) = \left| p \right| - \frac{N_c g^2}{4} \int \frac{d^3q}{(2\pi)^3} \frac{1 - \left( \hat{p} \cdot \hat{q} \right)^2}{(p - q)^2} + \frac{N_c g^2}{2|p|} \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|p| + 2|q|} \left[ \delta_{ik} p_i + \delta_{kl} q_i - \delta_{il} p_k \right] t_{km}(p - q) t_{in}(q) \tag{27}
\]

\[
- \frac{N_c g^2}{2|p|} \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1 + \left( \hat{p} \cdot \hat{q} \right)^2}{2|p| + 2|q|} \frac{|p| - |q|}{(p - q)^2} \tag{29}
\]

\[
- \frac{N_c g^2}{2|p|} \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{(p - q)^2} \tag{30}
\]

where we have used the contraction of an arbitrary tensor \(T_{ij}(p)\)

\[
\frac{1}{2} T_{ij}(p) t_{ij}(p) = T^t(p) \tag{31}
\]

in order to extract the transverse part. We have presented the diagrams corresponding to Eqs. \((27)-(30)\) in Fig. 2. The first loop integral in Eqs. \((26)\) and \((27)\) results from Eq. \((9)\) and is represented in Fig. 2 as a ghost loop because it stems from the Faddeev-Popov determinant. The following three loop diagrams in Fig. 2 are obtained by contracting two external legs in the diagrams of Fig. 1 see Eq. \((26)\). The last loop integral in Eq. \((26)\), or the integral \((30)\),
\[ 2f_2 = \left( \frac{m}{m} \right)^{-1} - \text{Diagram} - \text{Diagram} - \text{Diagram} - \text{Diagram} - \text{Diagram} - \text{Diagram} \]

Figure 2: The diagrams corresponding to Eqs. (27)–(30). The bare propagator, the inverse \(2|p|\) of which appears in Eq. (27), is marked with an open circle for later use. The first two one-loop diagrams correspond to the integrals in Eq. (27), in the same order. The following diagrams represent Eqs. (28)–(30), respectively. See the text for a motivation of the “crossed” gluon propagator notation in the last loop diagram.

on the other hand, originates from the terms (13) and (14) in the Hamiltonian. Lacking a better notation, we distinguish this contribution from the contraction of Eq. (25), the previous diagram, by marking the gluon propagator with a cross (because there is no term \(|q|\) in the denominator that would indicate the presence of an internal gluon propagator — in fact, the diagram may be interpreted to contain a \(\Pi\Pi\)-correlator, where \(\Pi\) is the momentum conjugate to \(A\)).

We have thus completed the determination of the (exponent of the) perturbative vacuum wave functional to order \(g^2\). The result is given in Eqs. (22), (23)–(25), and (27)–(30), to be substituted in Eq. (3). We can also extract the perturbative vacuum energy to the same order from the \(A\)-independent terms in the Schrödinger equation with the result

\[ E_0 = (N_c^2 - 1) \left( \int \frac{d^3p}{(2\pi)^3} f_2(p) \right) (2\pi)^3 \delta(0) \]  

[cf. Eq. (16)], where Eqs. (27)–(30) have to be substituted for \(f_2(p)\). The explicit expression is not relevant for our purposes. We shall come back to the vacuum energy later in the context of the static potential in the presence of external charges. It should also be clear by now how to take the determination of the perturbative vacuum functional and the vacuum energy systematically to higher orders.

### 3 Equal-time two-point correlation functions

For the calculation of the equal-time correlation functions, we need to include the Faddeev-Popov determinant in the measure of the functional integral, see Eq. (1). For our diagrammatic procedure, it is very convenient to introduce ghost fields and write

\[ FP(A) = \int D[\bar{c}, c] \exp \left( -\int d^3x \bar{c}^a(x) \left( -\nabla \cdot D^{ab}(A) \right) c^b(x) \right) . \]
In our conventions, we have explicitly

$$\int d^3x \, c^a(x)[-\nabla \cdot D^{ab}(A)]c^b(x) = \int \frac{d^3p}{(2\pi)^3} c^a(-p) p^2 c^a(p)$$

(34)

$$+ g \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} i f^{abc} p_{1j} c^a(p_1) c^b(p_2) A^c_j(p_3)(2\pi)^3 \delta(p_1 + p_2 + p_3).$$

(35)

Note that $p_{1j}$ under the integral in Eq. (35) can be replaced by $-p_{2j}$ due to the transversality of $A$.

We now have a representation of the equal-time correlation functions as a functional integral over the transverse components of $A$, the ghost and the antighost fields, see Eq. (1). The integration measure, which would be the exponential of the negative of the Euclidean action in the usual four-dimensional formulation (in Euclidean space), is now given by the exponential in Eq. (33) and the square of Eq. (3). Note that the vacuum functional is real (at least to order $g^2$) because the coefficient functions fulfill the reality condition

$$\left(f^{a_1...a_k}_{i_1...i_k}(-p_1, ..., -p_k)\right)^* = f^{a_1...a_k}_{i_1...i_k}(p_1, ..., p_k).$$

(36)

We shall use the analogy of this representation with the familiar functional integral representation of the covariant correlation functions in the usual four-dimensional formulation for the perturbative determination of the equal-time correlation functions (1). The corresponding Feynman rules are easily identified: the (static) gluon propagator is the inverse of $2|p|$, cf. Eq. (18) (the factor of two is due to the square of the wave functional in the measure), the other contributions $-2\left(f_2(p) - |p| \right)$ and the other coefficient functions $-2f_3(p_1, p_2, p_3)$ and $-2f_4(p_1, ..., p_4)$ determine the two-, three-, and four-gluon vertices. Furthermore, from Eq. (34) we identify the free ghost propagator $1/p^2$ and from Eq. (35) the ghost-gluon vertex.

We consider the gluon equal-time two-point function $\langle A^a_i(p_1) A^b_j(p_2) \rangle$ (with $t = 0$ in the arguments of the gluon fields to be understood) first. One of the contributions to be taken into account is the ghost loop, constructed from two ghost-gluon vertices (35) and two ghost propagators [see Eq. (34)], and furthermore two static gluon propagators from Eq. (18) for the external lines. As it turns out, this contribution is exactly cancelled by the other contribution with the same graph “topology” which arises from contracting one of the two-gluon vertices, (minus twice) the first integral in Eq. (27), with two external gluon propagators. Both contributions are represented diagrammatically in the first line of Fig. 3. The cancellation of the ghost loop contribution from the perturbative gluon two-point function (to one-loop order) is interesting given that this contribution plays a major role in the nonperturbative approaches to the infrared behavior of the equal-time gluon two-point function [2]–[4]. The cancellation of ghost loops was found to be a general feature in the Lagrangian functional integral approach [10, 12]. An alternative way to see the cancellation in our present approach is to write the Faddeev-Popov determinant as

$$\text{FP}(A) = \exp \left[ \text{tr} \ln \left( -\nabla \cdot D(A) \right) \right].$$

(37)

The coefficient of the term quadratic in $A$ in $\text{tr} \ln \left( -\nabla \cdot D(A) \right)$ precisely equals twice the first integral in Eq. (27) and hence cancels out in the exponent.
Figure 3: Diagrammatic representation of the various contributions to the gluonic equal-time two-point function, see Eqs. (37)–(41). The propagators marked with open circles are taken from Eqs. (18) and (34), respectively, while the “direct” contractions without open circles refer to the contractions that appear in the course of the determination of the vacuum wave functional, see Figs. 1 and 2, so that the corresponding parts of the diagrams translate into (minus two times) the mathematical expressions (24)–(25) \[ \frac{1}{(p_1 + \ldots + p_4)} \] and (27)–(30). The notation \( E(\cdot) \) for the sum of all diagrams with the same topology is explained in the text, following Eq. (44).

Next we turn to the tadpole contribution which is obtained from the elementary four-gluon vertex extracted from Eq. (23) appropriately contracted with three static gluon propagators. Again, there is a second contribution with the same “topology” given by the two-gluon vertex from the last integral in Eq. (27) contracted with two external propagators, cf. the second line in Fig. 3. The sum of these two contributions to the gluon equal-time two-point function is (with \( p \equiv p_1 \))

\[
- \frac{2 N_c g^2}{(2|p|)^3} \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|p| + 2|q|} \frac{1}{2|q|} - \frac{2 N_c g^2}{(2|p|)^3} \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|p| + 2|q|} \]

\[= \frac{2 N_c g^2}{(2|p|)^3} \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|}, \quad (38)\]

to be multiplied with \( \delta^{ab} t_{ij}(p) \).

The most complicated contribution to the two-point function comes from diagrams with the gluon loop topology (with two three-gluon vertices). There are three different diagrams of this type, represented in the third line of Fig. 3, the first from contracting two three-gluon vertices extracted from Eq. (22) with four static gluon propagators (two internal and two external), the second from contracting the part of the four-gluon vertex given by Eq.
Performing the contractions explicitly. A straightforward, but somewhat tedious calculation gives

\[
\begin{align*}
\frac{2N_c g^2}{(2|p|)^3} & \int \frac{d^3q}{(2\pi)^3} \frac{2|p| + 2|p - q|}{(|p| + |q| + |p - q|)^2 |q| |2p - q|} \\
& \times \left( \delta_{km} p_n + \delta_{mn} q_k - \delta_{nk} p_m \right) t_{mr}(p - q) t_{ns}(q) \left( \delta_{lr} p_s + \delta_{rs} q_l - \delta_{sl} p_r \right) t_{kl}(p) \quad (39)
\end{align*}
\]

(again, to be multiplied with \(\delta^{ab} t_{ij}(p)\), and \(p \equiv p_1\)). The tensor structure in this expression is invariant under the transformation \(q \rightarrow p - q\), a fact we can use to replace \(2|p| + 2|p - q|\) in the numerator with \(2|p| + |q| + |p - q|\). The tensor structure itself can be simplified by performing the contractions explicitly. A straightforward, but somewhat tedious calculation gives

\[
\begin{align*}
\left( \delta_{km} p_n + \delta_{mn} q_k - \delta_{nk} p_m \right) t_{mr}(p - q) t_{ns}(q) \left( \delta_{lr} p_s + \delta_{rs} q_l - \delta_{sl} p_r \right) t_{kl}(p) \\
= (1 - (\hat{p} \cdot \hat{q})^2) \left( 2p^2 + 2q^2 + \frac{p^2 q^2 + (p \cdot q)^2}{(p - q)^2} \right). \quad (40)
\end{align*}
\]

At last, we turn to the contributions that involve the (non-Abelian) Coulomb potential. There are, again, three such terms, represented in the last line of Fig. 3, the first from the four-gluon vertex derived from Eq. (25) contracted with three static gluon propagators, and the other two using the two-gluon vertices corresponding to the two integrals (29) and (30) contracted with two gluon propagators each. The sum of these terms is

\[
\begin{align*}
\frac{2N_c g^2}{(2|p|)^3} & \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|} \left( |p| - |q| \right)^2 + \frac{2N_c g^2}{(2|p|)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|} \left( |p| - |q| \right)^2 \\
& = \frac{2N_c g^2}{(2|p|)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|} \left( |p| - |q| \right)^2 \quad (41)
\end{align*}
\]

to be multiplied with \(\delta^{ab} t_{ij}(p)\) as before, and \(p \equiv p_1\). On the left-hand side of Eq. (41), we have added up the contributions from the contraction of the four-gluon vertex and the two-gluon vertex in Eq. (29) to give the first loop integral. The left-hand side of Eq. (41) is represented diagrammatically as the right-hand side of the last line in Fig. 3.

Putting it all together, the result for the equal-time gluon two-point function is, to order \(g^2\),

\[
\langle A_i^a(p_1) A_j^b(p_2) \rangle = \left\{ \begin{array}{l}
\frac{1}{2|p_1|} - \frac{2N_c g^2}{(2|p_1|)^3} \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|} \\
+ \frac{2N_c g^2}{(2|p_1|)^3} \frac{2}{2|p_1|} \int \frac{d^3q}{(2\pi)^3} \frac{(1 - (\hat{p}_1 \cdot \hat{q})^2)(2|p_1| + |q| + |p_1 - q|)}{|p_1| + |q| + |p_1 - q|)^2 |q| |2p_1 - q|} \\
\times \left( 2p_1^2 + 2q^2 + \frac{p_1^2 q^2 + (p_1 \cdot q)^2}{(p_1 - q)^2} \right)
\end{array} \right. \quad (42)
\]

\[
\begin{align*}
\end{align*}
\]
\[
\langle \bar{c} c \rangle = \ldots + \ldots \quad (45)
\]

Figure 4: Diagrammatic representation of Eq. (45).

\[
+ \frac{2N_c g^2}{(2|p_1|)^3} \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1 + (\hat{p}_1 \cdot \hat{q})^2}{2|q| (p_1 - q)^2} \frac{p_1^2 - q^2}{(p_1 - q)^2} \delta^{ab} t_{ij}(p_1)(2\pi)^3 \delta(p_1 + p_2). \quad (44)
\]

Looking at the diagrams on the left-hand sides of the first three lines in Fig. 3, it is clear that there is precisely one diagram among those of the same topology that is constructed exclusively from the elementary vertices (22), (23), and (35) and the “bare” propagators taken from Eqs. (18) and (34). We will call this kind of diagram an “F-diagram”. Interestingly, the sum of all diagrams with the same topology which we will refer to as an “E-diagram”, can be constructed from the corresponding F-diagram by a formal operation that we call the “E-operator”.

The same rule applies to the second line in Fig. 3 or Eq. (38), only that the propagator that starts and ends at the same vertex has to be counted twice in the sum over (internal and external) |k|. Similarly, the E-operator can be used to sum the first two diagrams on the left-hand side of the last line in Fig. 3. We have to consider the Coulomb interaction as an elementary vertex given by (25) to this end, and count the gluon propagator that starts and ends at this vertex twice in the sum over |k| just as in the case of the other elementary four-gluon vertex. The same rule for the generation of the E-diagrams (given the elementary vertices and propagators) has been shown to hold up to two-loop order for the equal-time two-point function and to one-loop order for the four-point function in the context of a scalar $\phi^4$ theory [16]. Note that two contributions, the second diagram in the first line of Fig. 3 and the third diagram in the last line in the same figure corresponding to the first loop integral in Eq. (27) and to Eq. (39), respectively, do not fit into this general scheme.

The calculation of the equal-time ghost two-point function from the graphical rules is much simpler. One obtains directly

\[
\langle c^a(p_1)\bar{c}^b(p_2) \rangle = \left( \frac{1}{p_1^2} + \frac{N_c g^2}{p_1^2} \right) \int \frac{d^3q}{(2\pi)^3} \frac{1 - (\hat{p}_1 \cdot \hat{q})^2}{(p_1 - q)^2 (2|q|)} \frac{p_1^2 - q^2}{2|q|} \delta^{ab} (2\pi)^3 \delta(p_1 + p_2), \quad (45)
\]

corresponding to the diagrams in Fig. 4. Note that one of the factors $1/p_1^2$ for the external ghost propagators cancels against the momentum dependence of the ghost-gluon vertices.
We shall close this section with a calculation of the static (heavy quark) potential, the energy for a configuration of static external color charges \( \rho_q(x) \). To this end, we introduce charges \( \rho_q(x) \) into the Hamiltonian, see Eqs. \((7)\) and \((12)\). Compared to the Coulomb term \((13)–(14)\) which was calculated in the absence of external charges, there are two new terms of order \( g^2 \):

\[
\frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho_q^o(-p) \frac{1}{p^2} \rho_q^o(p),
\]

(46)

which is \(A\)-independent and hence only contributes to the vacuum energy but leaves the vacuum wave functional unchanged, and

\[
g^2 \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} f_{abc} \frac{1}{p_1} (2\pi)^3 \delta(p_1 + p_2 + p_3) \rho_q^o(p_1) A^c_j(p_1) \frac{1}{(2\pi)^3} \frac{\delta}{\delta A^c_j(-p_3)}. \quad (47)
\]

The latter term, when applied to the vacuum wave functional, generates the following (properly symmetrized) expression to order \( g^2 \),

\[
-\frac{g^2}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} f_{abc} \frac{1}{p_1^2} \frac{1}{|p_2|} \frac{1}{|p_3|} \rho_q^o(p_1) A^b_j(p_2) A^c_j(p_3) (2\pi)^3 \delta(p_1 + p_2 + p_3), \quad (48)
\]

which implies that the vacuum wave functional to order \( g^2 \) has to be modified in order to fulfill the Schrödinger equation with this new term.

A term cancelling the expression \((48)\) in the Schrödinger equation can only result from the second derivative term in the Hamiltonian [Eq. \((5)\)]. It is then simple to see that we have to add the expression

\[
-\frac{g^2}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} f_{abc} \frac{1}{|p_2|} \frac{1}{|p_3|} \rho_q^o(p_1) A^b_j(p_2) A^c_j(p_3) (2\pi)^3 \delta(p_1 + p_2 + p_3), \quad (49)
\]

to the negative of the exponent of the vacuum wave functional in order to satisfy the Schrödinger equation to order \( g^2 \). This term describes the back-reaction of the vacuum to the presence of the external charges to order \( g^2 \). Observe that due to the presence of the external charges \( \rho_q(p) \), the coefficient function of \( A^b_j(p_1) A^c_j(p_2) \) in the vacuum wave functional ceases to be of the form \( f_2(p_2) \delta^{ab} \delta_{ij} (2\pi)^3 \delta \left( \mathbf{p}_1 + \mathbf{p}_2 \right) \). Furthermore, contrary to the terms found before, the contribution \((49)\) is imaginary. The rest of the vacuum wave functional determined in the previous section remains without change.

We now have to calculate the vacuum energy in the presence of the external charges. To order \( g^2 \), the result is the former one, Eq. \((32)\), without any contribution from the new term \((49)\), plus Eq. \((46)\) which is the part of the energy that depends on the external charges and hence defines the potential to this order. Of course, this is just the well-known Coulomb potential of electrodynamics. We are really interested in the first quantum corrections to this “bare” potential, which are of order \( g^4 \).

In general, the vacuum energy is given by

\[
E_0 = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho_q^o(-p) \frac{1}{p^2} \rho_q^o(p) - \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{\delta}{\delta A^c_j(p)} \frac{\delta}{\delta A^c_j(-p)} \psi(A) \bigg|_{A=0}, \quad (50)
\]

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which reduces to Eq. (52) in the absence of external charges. A contribution of order $g^4$ can hence only originate from the terms in the vacuum wave functional that are quadratic in $\mathbf{A}$. As long as we are only interested in the potential between static sources, we can concentrate on terms that contain precisely two powers of $\rho_q$. We then start by identifying all the contributions to the Schrödinger equation of order $g^4$ that contain two powers of $\mathbf{A}$ and two powers of $\rho_q$. One of these contributions results from expanding the Coulomb kernel

$$
\langle \mathbf{x}, a | (-\nabla \cdot \mathbf{D})^{-1} (-\nabla^2) (-\nabla \cdot \mathbf{D})^{-1} | \mathbf{y}, b \rangle
$$

in Eq. (5) to second order in $\mathbf{A}$ for $\rho = \rho_q$. The result is the term

$$
-\frac{3}{4} g^4 \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_4}{(2\pi)^3} \left( f_{\text{face}} f_{\text{bde}} \rho_{1,i} \rho_{2,j} \frac{p_{1,i} p_{2,j}}{p_1^2 p_2^2 (p_1 + p_2)^2} + f_{\text{face}} f_{\text{bce}} \rho_{1,i} \rho_{2,i} \frac{p_{1,i} p_{2,i}}{p_1^2 p_2^2 (p_1 + p_2)^2} \right) 
$$

with

$$
\rho_{1,i} \rho_{2,i} \mathcal{A}_{1}^i(p_3) \mathcal{A}_{1}^d(p_4)(2\pi)^3 \delta(p_1 + \ldots + p_4)
$$

on the left-hand side of the Schrödinger equation.

Another contribution of the same type arises from the second functional derivative in Eq. (8) acting (twice) on the term (19), which gives the contribution

$$
\frac{g^4}{4} \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_4}{(2\pi)^3} \left( f_{\text{face}} f_{\text{bde}} t_{ij}(p_1 + p_3) \frac{|p_1 + p_3| - |p_3|}{|p_1 + p_3| + |p_3|} \frac{|p_1 + p_3| - |p_4|}{|p_1 + p_3| + |p_4|} 
$$

with

$$
\rho_{1,i} \rho_{2,i} \mathcal{A}_{1}^i(p_3) \mathcal{A}_{1}^d(p_4)(2\pi)^3 \delta(p_1 + \ldots + p_4)
$$

to the Schrödinger equation. The last contribution of the same type comes from the “mixed” term where the operator (17) acts upon the expression (49) in the wave functional. The result is

$$
-\frac{g^4}{4} \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_4}{(2\pi)^3} \left( f_{\text{face}} f_{\text{bde}} t_{ij}(p_1 + p_3) \left( \frac{|p_1 + p_3| - |p_3|}{|p_1 + p_3| + |p_3|} + \frac{|p_1 + p_3| - |p_4|}{|p_1 + p_3| + |p_4|} \right) 
$$

with

$$
\rho_{1,i} \rho_{2,i} \mathcal{A}_{1}^i(p_3) \mathcal{A}_{1}^d(p_4)(2\pi)^3 \delta(p_1 + \ldots + p_4)
$$

to be included in the Schrödinger equation. It can be shown that no other contributions quadratic in $\mathbf{A}$ and in $\rho_q$ exist to order $g^4$.

In analogy to the determination of the expression (19) from Eq. (18), the three contributions (52)–(54) to the Schrödinger equation are taken care of by including the following expression in the negative exponent of the vacuum wave functional [in addition to (22),
\[ -3 \left( \text{Diagram 1} + 1 \text{ perm.} \right) - 2 \left( \text{Diagram 2} + 1 \text{ perm.} \right) \]

Figure 5: Diagrammatic representation of the contributions (55) (multiplied by 2) to the vacuum wave functional.

\[ (23) - (25), (27) - (30), \text{ and (49)} \]

\[ - \frac{g^4}{4} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_4}{(2\pi)^3} \frac{1}{\hat{p}_1 \hat{p}_2 (|p_3| + |p_4|)} \left\{ \int f^{ace} f^{bde} \left[ \frac{3p_{1,i}p_{2,j}}{(p_1 + p_3)^2} - t_{ij}(p_1 + p_3) \right] \right. \\
\times \left[ \frac{p_1 + p_3 - |p_3|}{|p_1 + p_3| + |p_4|} - \frac{p_1 + p_3 - |p_4|}{|p_1 + p_3| + |p_4|} \right] \\
+ f^{ade} f^{fbc} \left[ \frac{3p_{1,i}p_{2,j}}{(p_1 + p_4)^2} - t_{ij}(p_1 + p_4) \right] \left( \frac{p_1 + p_4 - |p_3|}{|p_1 + p_4| + |p_3|} - \frac{p_1 + p_4 - |p_4|}{|p_1 + p_4| + |p_4|} \right) \}
\]

\[ \rho^a_q(p_1) \rho^b_q(p_2) A^a_{i_1}(p_3) A^b_j(p_4) (2\pi)^3 \delta(p_1 + \ldots + p_4). \quad (55) \]

This result (multiplied by 2) is represented diagrammatically in Fig. 5, where we have denoted the “Coulomb propagator” $1/p^2$ as a double line. The first diagram (and its permutation) corresponds to the expression (52), while the second diagram (plus its permutation) corresponds to the sum of the expressions (53) and (54). From Eq. (54), we find the contribution to the vacuum energy

\[ \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho^a_q(-p) \frac{1}{(p^2)^2} \left\{ N_c g^2 \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|q|} \left[ 3 \frac{p^2 - (p \cdot \hat{q})^2}{(p - q)^2} \right. \right. \\
+ t_{ij}(q)t_{ij}(p - q) \left( \frac{|p - q| - |q|}{|p - q| + |q|} \right)^2 - 2 \frac{|p - q| - |q|}{|p - q| + |q|} + 3 \frac{p^2 - (p \cdot \hat{q})^2}{(p + q)^2} \\
+ t_{ij}(q)t_{ij}(p + q) \left( \frac{|p + q| - |q|}{|p + q| + |q|} \right)^2 - 2 \frac{|p + q| - |q|}{|p + q| + |q|} \right) \left\} \rho^a_q(p). \quad (56) \]

This latter expression can be simplified by shifting $q \rightarrow q - p$ in the last two terms (in the round bracket). Together with Eq. (46), we find for the part of the vacuum energy that is quadratic in the external static charge $\rho_q$,

\[ E^{(\rho_q, 2)}_0 = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \rho^a_q(-p)V(p)\rho^a_q(p), \quad (57) \]
the following result for the static potential to order $g^2$

\[
V(p) = \frac{1}{p^2} + \frac{N_c g^2}{p^2} \int \frac{d^3q}{(2\pi)^3} \frac{1 - (\hat{p} \cdot \hat{q})^2}{2|q|(p - q)^2}
\]

\[
- \frac{N_c g^2}{(p^2)^2} \int \frac{d^3q}{(2\pi)^3} \left( 1 + \frac{(p - q) \cdot q}{(p - q)^2 q^2} \right) \frac{(|p - q| - |q|)^2}{2|p - q| 2|q| (|p - q| + |q|)}.
\]

Note that to the proper color Coulomb potential (see, e.g., Ref. [19]), only the antiscreening term, the integral in Eq. (58), contributes, while the full static potential also contains the screening contribution (59). The semi-analytical variational approaches [3, 11] have only considered the proper color Coulomb potential so far.

We can associate diagrams with the different contributions in Eqs. (58)–(59) in a natural way. The vertex that joins two Coulomb (double) lines and one gluon line corresponds to the same mathematical expression as the ghost-gluon vertex since both objects originate from the Faddeev-Popov operator ($-\nabla \cdot D$) (or its inverse). On the other hand, the vertex with two gluon lines and one Coulomb line translates to the expression

\[
igf_{abc} \frac{|p_2| - |p_3|}{|p_2| + |p_3|} \delta_{jk},
\]

where the gluon lines carry the momenta $p_2$ and $p_3$, (spatial) Lorentz indices $j$ and $k$, and color indices $b$ and $c$ [cf. Eq. (49)]. Note that the “elementary” Coulomb interaction (25) is different from the contraction of two such vertices with a Coulomb propagator. With these conventions, we can represent the static potential as in Eq. (60). The E-operator in Fig. 6 exclusively refers to the internal gluon propagators and thus amounts to multiplying the integrand with $|p - q| + |q|$.

We hence have succeeded in calculating the equal-time gluon and ghost two-point functions and the static potential to one-loop order in our functional perturbative approach, with the results (42)–(45) and (58)–(59). The same results can be obtained from a straightforward application of Rayleigh-Schrödinger perturbation theory [17]. Compared to these latter calculations, we have here developed a functional integral and diagrammatical approach that is potentially advantageous in higher-order perturbative calculations. We have also described a set of simplified diagrammatic rules (the “E-operator”) for the determination of equal-time correlation functions that is expected to carry over to higher perturbative orders and is hoped to eventually lead to nonperturbative equations for the equal-time correlation functions analogous to Dyson-Schwinger equations.
4 Lagrangian approach and renormalization

Naive power counting shows that the results of the preceding section, Eqs. (42)–(45) and (58)–(59), are ultraviolet (UV) divergent and need to be renormalized. However, some of the denominators occurring in the loop integrals are of a different type from those that usually appear in covariant perturbation theory, and efficient techniques for the handling of these terms have yet to be developed. These remarks apply in particular to Eqs. (43) and (59).

The equal-time correlation functions we have been calculating are a special or limiting case of the usual space-time correlation functions, whence we naturally obtain the representation

$$\langle A^a_i(p_1, t = 0) A^b_j(p_2, t = 0) \rangle = \int_{-\infty}^{\infty} \frac{dp_{1,4}}{2\pi} \frac{dp_{2,4}}{2\pi} \langle A^a_i(p_1, p_{1,4}) A^b_j(p_2, p_{2,4}) \rangle$$

(61)

(with the space-time correlation functions written in Euclidean space-time). Since the regularization and renormalization program has been developed for the space-time correlation functions, this representation is quite useful for our purposes. In the case of Coulomb gauge Yang-Mills theory, however, covariance is explicitly broken through the gauge condition, and the calculation of the space-time correlation functions in the usual Lagrangian functional integral approach represents a difficulty by itself. Techniques have been developed to overcome these difficulties and applied in Ref. [12] to the calculation of the two-point correlation functions to one-loop order.

Before properly considering the renormalization of our results, we will verify that these coincide on a formal level with the expressions obtained via Eq. (61), taking for the space-time correlation functions the formulas derived in the Lagrangian functional integral approach in Ref. [12]. In our notation,

$$\langle A^a_i(p_1) A^b_j(p_2) \rangle = \left( \frac{1}{p_1^2} - \frac{N_c g^2}{(p_1^2)^2} \right) \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \delta_{a,b} \delta_{i,j} (p_1 + p_2)$$

(62)

$$+ \frac{N_c g^2}{(p_1^2)^2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \left( \delta_{km} p_{1,n} + \delta_{mn} q_k - \delta_{nk} p_{1,m} \right) t_{mr}(p_1 - q) t_{ns}(q)$$

$$\times \frac{\left( \delta_{lr} p_{1,s} + \delta_{rs} q_l - \delta_{sl} p_{1,r} \right) t_{kl}(p_1)}{(p_1 - q)^2}$$

(63)

$$+ \frac{N_c g^2}{(p_1^2)^2} \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{t_{kl}(p_1) t_{kl}(q) (p_{1,4}^2 - q_4^2)}{q^2 (p_1 - q)^2} \delta^{ab} \delta_{ij} (p_1)(2\pi)^4 \delta(p_1 + p_2),$$

(64)

with $p_1^2 \equiv p_1^2 + p_{1,4}^2$ in Euclidean space-time. Formal integration of $p_{1,4}$ and $p_{2,4}$ as in Eq. (61) and of the component $q_4$ of the loop momentum, most easily using the residue theorem, leads to our equal-time correlation function (42)–(44). In Eq. (62), we have included the tadpole diagram in order that the correspondence with the equal-time gluon two-point function (42)–(44) be term by term. The tadpole diagram was not considered explicitly in Refs. [12, 13] because it vanishes in dimensional regularization.
The formal expression for the space-time correlation function is \[12\] in the Lagrangian functional integral approach, as was first pointed out by Zwanziger \[19\].

A translation, the Weyl gauge is the following important difference between the two approaches: in the Lagrangian formulation, it is also not trivial. Concerning the gauge fixing procedure, there of Refs. \[10, 12, 13\] gives the same results for the equal-time correlation functions as our condition over \((p_{1,4} + p_{2,4})\) instead [more symmetrically, one may integrate over \(p_{1,4}\) and \(p_{2,4}\) or putting \(t_1\) and \(t_2\) to zero then results in a factor \(\delta(0)\). In this case, in order to reproduce Eq. \[45\], we integrate either over \(p_{1,4}\) or over \(p_{2,4}\), which just eliminates the delta function for energy conservation [more symmetrically, one may integrate over \((p_{1,4} + p_{2,4})\) instead]. Performing the integral over \(q_4\) then converts Eq. \[65\] to Eq. \[45\].

Although it is certainly not surprising that the Lagrangian functional integral approach of Refs. \[10\], \[12\], \[13\] gives the same results for the equal-time correlation functions as our Hamiltonian approach, it is also not trivial. Concerning the gauge fixing procedure, there is the following important difference between the two approaches: in the Lagrangian formulation, the Weyl gauge \(A_0 \equiv 0\) cannot be implemented in addition to the Coulomb gauge condition \(\nabla \cdot A \equiv 0\) \[18\]. Indeed, in the first-order Lagrangian formalism, integrating out the \(A_0\)-field rather than setting \(A_0\) to zero yields an expression in the exponent of the measure for the functional integral that resembles the Christ-Lee Hamiltonian \[10\]. The derivation of the Hamilton operator \[5\] by Christ and Lee \[14\], on the other hand, relies on the existence of a gauge transformation that makes any gauge field \(A_\mu\) satisfy both the Coulomb and Weyl gauge conditions. We discuss the possibility of simultaneously implementing the Weyl and Coulomb gauges in the Hamiltonian and the Lagrangian approaches in the Appendix.

Although not defined as an equal-time correlation function in our approach, it turns out that the static potential \[58\]—\[59\] is related to the space-time two-point function \(\langle A^a_0(p_1)A^b_0(p_2)\rangle\) in the Lagrangian functional integral approach, as was first pointed out by Zwanziger \[19\]. The formal expression for the space-time correlation function is \[12\]

\[
\langle A^a_0(p_1)A^b_0(p_2)\rangle = \left(\frac{1}{\mathbf{p}_1^2} + \frac{N_c g^2}{(\mathbf{p}_1^2)^2}\right) \int \frac{d^4 q}{(2\pi)^4} \frac{p_{1,i} p_{1,j} t_{ij}(q)}{q^2 (\mathbf{p}_1 - \mathbf{q})^2} \delta^{ab} (2\pi)^4 \delta(p_1 + p_2), \tag{65}
\]

and the fact that the dependence on \(p_{1,4}\) and \(p_{2,4}\) is exclusively through the delta function for energy conservation implies that \(\langle c^a(p_1, t_1) \bar{c}^b(p_2, t_2)\rangle\) contains the factor \(\delta(t_1 - t_2)\). Integrating over \(p_{1,4}\) and \(p_{2,4}\) or putting \(t_1\) and \(t_2\) to zero then results in a factor \(\delta(0)\). In this case, in order to reproduce Eq. \[45\], we integrate either over \(p_{1,4}\) or over \(p_{2,4}\), which just eliminates the delta function for energy conservation [more symmetrically, one may integrate over \((p_{1,4} + p_{2,4})\) instead]. Performing the integral over \(q_4\) then converts Eq. \[65\] to Eq. \[45\].

Integrating over either \(p_{1,4}\) or \(p_{2,4}\) and over the energy component \(q_4\) of the loop momentum, we obtain the antiscreening contribution \[58\] to the static potential from Eq. \[66\] because the latter is already instantaneous. In order to find Eq. \[59\] starting from Eq. \[67\], we have to put the respective other energy component, \(p_{2,4}\) or \(p_{1,4}\), to zero in addition [this is not necessary in the cases of Eqs. \[65\] and \[66\], because there the result of integrating over one of the energy components is independent of the other]. For \(\langle A^a_0(p_1, t_1)A^b_0(p_2, t_2)\rangle\), this procedure amounts to integrating over the relative time \(t_1 - t_2\), which is, in fact, intuitively quite appealing for a non-instantaneous contribution to the potential between static sources.
We shall now use the representation (61) of the equal-time gluon two-point function and the corresponding representations of the ghost two-point function and the static potential for the renormalization of these equal-time correlation functions. To this end, we make use of the explicit expressions obtained for Eqs. (62)–(67) in Ref. [12] in dimensional regularization. Thus, by integrating the result for (62)–(64) according to Eq. (61), we obtain for the equal-time correlation function

\[ \langle A_i^a(p_1)A_j^b(p_2) \rangle = \left[ \frac{1}{2|p_1|^2} + \frac{N_c g^2}{(4\pi)^2} \frac{1}{2|p_1|^2} \left( \frac{1}{\epsilon} - \ln \frac{p_1^2}{\mu^2} + C_A \right) \right] \delta^{ab} t_{ij}(p_1)(2\pi)^3\delta(p_1 + p_2) \]  

(68)

in the limit \( \epsilon \to 0 \), where \( d = 3 - 2\epsilon \) is the dimension of space and \( \mu \) an arbitrary mass scale. The value of the constant \( C_A \) is not relevant to our purposes, but being an integral over an explicitly known function of \( p_{1,2}^2/|p_1|^2 \), we have carefully checked that it is finite.

From the explicit expressions for Eqs. (65)–(67) in dimensional regularization [12], we find directly

\[ \langle c_i^a(p_1)c_j^b(p_2) \rangle = \left[ \frac{1}{2|p_1|^2} + \frac{N_c g^2}{(4\pi)^2} \frac{1}{2|p_1|^2} \left( \frac{1}{3\epsilon} - \ln \frac{p_1^2}{\mu^2} + C_c \right) \right] \delta^{ab} (\pi)^3\delta(p_1 + p_2) , \]

(69)

\[ V(p_1) = \left[ \frac{1}{|p_1|^2} + \frac{N_c g^2}{(4\pi)^2} \frac{1}{|p_1|^2} \left( \frac{11}{3\epsilon} - \ln \frac{p_1^2}{\mu^2} + C_V \right) \right] . \]

(70)

This procedure to regularize the equal-time correlation functions finds further support in the cases where the equal-time functions in the form (42)–(45) and (58)–(59) can be evaluated directly in dimensional regularization (in \( d = 3 - 2\epsilon \) dimensions). For Eqs. (44) and (45), identical results are obtained in both ways [17] [also trivially for Eq. (42) and the loop integral in Eq. (58) which is just three times the one of Eq. (15)].

The results (68) and (69) for the equal-time two-point correlation functions can be renormalized in analogy to the procedures developed for covariant theories: we introduce renormalized correlation functions (or correlation functions of the renormalized fields)

\[ \langle A_i^a_R(p_1)A_j^b_R(p_2) \rangle = \frac{1}{Z_A} \langle A_i^a(p_1)A_j^b(p_2) \rangle , \]

\[ \langle c_i^a_R(p_1)c_j^b_R(p_2) \rangle = \frac{1}{Z_c} \langle c_i^a(p_1)c_j^b(p_2) \rangle . \]

(71)

The simplest choice of the normalization conditions is

\[ \langle A_i^a_R(p_1)A_j^b_R(p_2) \rangle |_{p_i^2 = \kappa^2} = \frac{1}{2|p_1|^2} \delta^{ab} t_{ij}(p_1)(2\pi)^3\delta(p_1 + p_2) , \]

\[ \langle c_i^a_R(p_1)c_j^b_R(p_2) \rangle |_{p_i^2 = \kappa^2} = \frac{1}{|p_1|^2} \delta^{ab} (\pi)^3\delta(p_1 + p_2) , \]

(72)

at the renormalization scale \( \kappa \). With these normalization conditions and the results (68),
(69), we obtain

\[
Z_A(\kappa) = 1 + \frac{N_c g^2}{(4\pi)^2} \left( \frac{1}{\epsilon} - \ln \frac{\kappa^2}{\mu^2} + C_A \right),
\]

\[
Z_c(\kappa) = 1 + \frac{N_c g^2}{(4\pi)^2} \frac{4}{3} \left( \frac{1}{\epsilon} - \ln \frac{\kappa^2}{\mu^2} + C_c \right),
\] (73)

to order \( g^2 \).

The expression (70) for the static potential needs to be renormalized, too. This is most naturally achieved by a renormalization of the coupling constant as was first suggested in Ref. [20], through

\[
g^2 V(p) \bigg|_{p^2 = \kappa^2} = \frac{\tilde{g}^2_R(\kappa)}{p^2},
\] (74)

see Eq. (57). Hence, from Eq. (70),

\[
\tilde{g}^2_R(\kappa) = g^2 \left[ 1 + \frac{N_c g^2}{(4\pi)^2} \frac{11}{3} \left( \frac{1}{\epsilon} - \ln \frac{\kappa^2}{\mu^2} + C_V \right) \right],
\] (75)

which implies for the corresponding beta function to one-loop order,

\[
\kappa^2 \frac{\partial}{\partial \kappa^2} \tilde{g}^2_R(\kappa) = \frac{\bar{\beta}_0}{(4\pi)^2} \tilde{g}^4_R(\kappa),
\] (76)

that

\[
\bar{\beta}_0 = -\frac{11}{3} N_c.
\] (77)

This is the well-known result from covariant perturbation theory (for Yang-Mills theory in covariant gauges), and has also been found in Ref. [12].

For the rest of this section, we will pursue a more conventional way of renormalizing the coupling constant (which, however, leads to the same result). To this end, we consider the equal-time ghost-gluon three-point correlation function \( \langle c^a(p_1) \bar{c}^b(-p_4) A^c_i(p_3) \rangle \) to order \( g^3 \) (one loop). The calculation of this correlation function is performed in analogy with the determination of the equal-time two-point correlation functions in Section 3, using the result for the vacuum wave functional obtained in Section 2. In this particularly simple case (and to the order considered), the external “propagators” (equal-time two-point functions) can be factorized to define the equal-time proper three-point vertex \( \Gamma_{i}^{abc}(p_1, p_2, p_3) \) as

\[
\langle c^a(p_1) \bar{c}^b(p_2) A^c_i(p_3) \rangle = -\int \frac{d^3 p_4}{(2\pi)^3} \frac{d^3 p_5}{(2\pi)^3} \frac{d^3 p_6}{(2\pi)^3} \langle c^a(p_1) \bar{c}^d(-p_4) \rangle \times \Gamma_{j}^{def}(p_4, p_5, p_6) \langle c^e(-p_5) \bar{c}^b(p_2) \rangle \langle A^f_i(-p_6) A^c_i(p_3) \rangle.
\] (78)
\[ \Gamma = - \quad \text{diagram} \quad - \quad \text{diagram} \]

Figure 7: The proper ghost-gluon vertex to one-loop order.

The explicit perturbative result is

\[ \Gamma_{abc}(p_1, p_2, p_3) = -igf^{abc} \left( p_{1,k} - \frac{N_c g^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{[p_1 \cdot p_2 - (p_1 \cdot \hat{q})(p_2 \cdot \hat{q})]}{2|q| (p_1 - q)^2 (p_2 + q)^2} \right) \] (79)

\[ + \frac{2N_c g^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{p_{l,m} t_{lm}(p_1 - q) t_{mn}(p_2 + q)}{q^2 2|p_1 - q| 2|p_2 + q|} \times \frac{\delta_{km}(p_{1,r} - p_{3,r} - q_r) - \delta_{mr}(p_{1,k} - p_{2,k} - 2q_k) - \delta_{rk}(p_{2,m} - p_{3,m} + q_m)}{|q| + |p_1 - q| + |p_2 + q|} \] (80)

\[ \times t_{jk}(p_3)(2\pi)^3 \delta(p_1 + p_2 + p_3). \]

It is represented diagrammatically in Fig. 7. Note that due to the transversality of the gauge, two powers of the external momenta can be factorized from the loop integrals [cf. Eq. (75)] and, as a result, the integrals are UV finite. This phenomenon is well-known in another transverse gauge, the Landau gauge [21]. For future use, we note that by very lengthy algebra the tensor structure in Eq. (80) can be simplified as follows:

\[ p_{l,m} t_{lm}(p_1 - q) t_{mn}(p_2 + q) \left[ \delta_{km}(p_{1,r} - p_{3,r} - q_r) \right. \]

\[ - \delta_{mr}(p_{1,k} - p_{2,k} - 2q_k) - \delta_{rk}(p_{2,m} - p_{3,m} + q_m) \] \]

\[ = 2 \left( q^2 p_{1,k} + (p_1 \cdot p_2) q_k - \frac{(p_1 - q) \cdot (p_2 + q)}{(p_1 - q)^2 (p_2 + q)^2} \left[ \frac{q \cdot (p_1 - q)}{q^2} \right] \right) t_{jk}(p_3)(2\pi)^3 \delta(p_1 + p_2 + p_3) \] (81)

We define the renormalized coupling constant in analogy with the covariant case as

\[ \Gamma_{abc}^{R,\gamma}(p_1, p_2, p_3) \bigg|_{p_1^2 = p_2^2 = p_3^2 = \kappa^2} = Z_c(\kappa) Z_A^{1/2}(\kappa) \Gamma_{abc}^{\gamma}(p_1, p_2, p_3) \bigg|_{p_1^2 = p_2^2 = p_3^2 = \kappa^2} \]

\[ = -ig_{\kappa}(\kappa) f^{abc} p_{1,k} t_{jk}(p_3)(2\pi)^3 \delta(p_1 + p_2 + p_3) \] (82)

at the symmetric point. As a consequence, using Eq. (79) and the UV finiteness of the loop integrals (79), (80),

\[ g_R(\kappa) = g \left[ 1 + \frac{N_c g^2}{(4\pi)^2} \frac{11}{6} \left( \frac{1}{\epsilon} - \ln \frac{\kappa^2}{\mu^2} + C \right) \right] \],

(83)
with a finite constant $C$ given by $(11/6)C = (4/3)C_c + (1/2)C_A + C_v$, where $C_v$ is obtained from the finite loop integrals in Eqs. (12)–(80).

For the beta function defined in analogy with Eq. (76), we obtain from Eq. (83),

$$\beta_0 = -\frac{11}{3} N_c.$$  \hfill (84)

This beta function coincides with the one obtained in Eq. (77) before with the renormalized coupling constant defined through the static potential. The integration of the renormalization group equation (76) gives the well-known (one-loop) result

$$g^2_R(\kappa) = \frac{(4\pi)^2}{11/3 N_c \ln \left( \frac{\kappa^2}{\Lambda^2_{QCD}} \right)}$$ \hfill (85)

[and the same for $\bar{g}^2_R(\kappa)$ (75)]. It must be noted that for renormalization group improvements like Eq. (85) to be sensible we have to suppose that the three-dimensional formulation presented here is multiplicatively renormalizable to all orders in the same way as the usual formulation of a renormalizable covariant quantum field theory, which is not known at present (even the renormalizability of the Lagrangian functional integral approach to Coulomb gauge Yang-Mills theory has not yet been shown). Equation (85) and the developments to follow are hence to some degree speculative, but it seemed of some interest to us to explore the consequences of the natural assumption of multiplicative renormalizability.

With these qualifications, we go on to use a standard renormalization group argument to extract the asymptotic UV behavior of the equal-time two-point correlation functions. To this end, differentiate Eq. (71) with respect to $\kappa^2$ using the $\kappa$-independence of the “bare” two-point functions. It is then seen that the $\kappa$-dependence of the renormalized two-point functions is determined by the anomalous dimensions $(\kappa^2 \partial \ln Z_{A,c}/\partial \kappa^2)$. Evaluating the latter from Eq. (73) and replacing $g^2$ in the results with $g^2_R(\kappa)$, we obtain the desired renormalization group equations for the equal-time two-point functions, explicitly

$$\kappa^2 \frac{\partial}{\partial \kappa^2} \langle A_{R,i}(P_1) A_{R,j}(P_2) \rangle = \frac{N_c g^2_R(\kappa)}{(4\pi)^2} \langle A_{R,i}(P_1) A_{R,j}(P_2) \rangle,$$

$$\kappa^2 \frac{\partial}{\partial \kappa^2} \langle c_{R}^{a}(P_1) c_{R}^{b}(P_2) \rangle = \frac{4}{3} \frac{N_c g^2_R(\kappa)}{(4\pi)^2} \langle c_{R}^{a}(P_1) c_{R}^{b}(P_2) \rangle. \hfill (86)$$

In these equations, we substitute from Eq. (85) for $g^2_R(\kappa)$ and integrate. Using the normalization conditions (72) for the determination of the integration constants, one obtains the
momentum dependence of the equal-time two-point functions:

\[
\langle A^a_{R,i}(p_1) A^b_{R,j}(p_2) \rangle = \frac{1}{2|p_1|} \left( \ln \left( \frac{\kappa^2}{\Lambda^2_{QCD}} \right) \right)^{3/11} \delta^{ab} t_{ij}(p_1)(2\pi)^3 \delta(p_1 + p_2) .
\]  

\[
\langle c^a_R(p_1) c^b_R(p_2) \rangle = \frac{1}{p_1^2} \left( \ln \left( \frac{\kappa^2}{\Lambda^2_{QCD}} \right) \right)^{4/11} \delta^{ab} \delta(p_1 + p_2) .
\]  

(87)

The momentum dependence of the “bare” two-point functions, obtained from Eq. (87) simply by multiplying with the corresponding wave function renormalization constants \(Z_{A,c}\), is obviously the same. By solving the renormalization group equations for \(Z_A\) and \(Z_c\) that involve the anomalous dimensions, it may be shown explicitly that the bare two-point functions are \(\kappa\)-independent, as they must be.

For the static potential, on the other hand, we immediately obtain from Eqs. (74) and (85) [for \(\bar{g}_R^2(\kappa)\)] the renormalization group improved result

\[
g^2 V(p) = \frac{(4\pi)^2}{113N_c p^2 \ln \left( \frac{p^2}{\Lambda^2_{QCD}} \right)} .
\]  

(88)

Note that the latter one-loop formula constitutes a very direct expression of asymptotic freedom.

The result (87) for the momentum dependence of the equal-time two-point functions has also been obtained in Ref. [22] from a Dyson-Schwinger equation for the equal-time ghost correlator, where the gauge-invariant one-loop running (85) of the renormalized coupling constant is used as an input. We briefly discuss that derivation here, adapted to the conventions of the present paper.

The renormalized equal-time two-point functions are parameterized as

\[
\langle A^a_{R,i}(p_1) A^b_{R,j}(p_2) \rangle = \frac{1}{2\omega(p_1)} \delta^{ab} t_{ij}(p_1)(2\pi)^3 \delta(p_1 + p_2) .
\]  

(89)

and

\[
\langle c^a_R(p_1) c^b_R(p_2) \rangle = \frac{d(p_1^2)}{p_1^2} \delta^{ab} (2\pi)^3 \delta(p_1 + p_2) .
\]  

(90)

and normalized according to the conditions (72). The renormalized coupling constant is defined as before in Eq. (82). Then the Dyson-Schwinger equation for the equal-time ghost
two-point function reads [3]

\[ d^{-1}(p^2) = Z_c - N_c g_R^2(\kappa) \int \frac{d^3q}{(2\pi)^3} \frac{1 - (\hat{p} \cdot \hat{q})^2}{2\omega(q^2)} \frac{d((p - q)^2)}{(p - q)^2}. \]  

(91)

Here we have approximated the full ghost-gluon vertex appearing in the exact equation by the tree-level vertex, as it is appropriate in order to obtain the (renormalization-group improved) one-loop expressions.

In order to solve Eq. (91), we make the following, properly normalized, ansätze for the two-point functions in the ultraviolet limit \( p^2 \gg \Lambda_{QCD}^2 \),

\[ \frac{|p|}{\omega(p^2)} = \left( \frac{\ln \left( \frac{\kappa^2}{\Lambda_{QCD}^2} \right)}{\ln \left( \frac{p^2}{\Lambda_{QCD}^2} \right)} \right)^\gamma, \quad d(p^2) = \left( \frac{\ln \left( \frac{\kappa^2}{\Lambda_{QCD}^2} \right)}{\ln \left( \frac{p^2}{\Lambda_{QCD}^2} \right)} \right)^\delta, \]  

(92)

with the exponents \( \gamma \) and \( \delta \) to be determined. The integral in Eq. (91) can then be calculated in the limit \( p^2 \gg \Lambda_{QCD}^2 \) and the Dyson-Schwinger equation yields the relation [22]

\[ \ln^{\delta} \left( \frac{\kappa^2}{\Lambda_{QCD}^2} \right) \ln^{\delta} \left( \frac{p^2}{\Lambda_{QCD}^2} \right) = N_c g_R^2(\kappa) \frac{4}{(4\pi)^2} \frac{4}{3\delta} \ln^{\gamma+\delta} \left( \frac{\kappa^2}{\Lambda_{QCD}^2} \right) \ln^{1-\gamma-\delta} \left( \frac{p^2}{\Lambda_{QCD}^2} \right) \],  

(93)

from which we infer the sum rule

\[ \gamma + 2\delta = 1 \]  

(94)

for the exponents as well as the identity

\[ g_R^2(\kappa) \frac{1}{(4\pi)^2} \frac{4}{3\delta} N_c \ln \left( \frac{\kappa^2}{\Lambda_{QCD}^2} \right) = 1 \]  

(95)

for the coefficients. Consistency of the latter relation with the well-known perturbative result [85] yields the exponents

\[ \gamma = \frac{3}{11}, \quad \delta = \frac{4}{11}, \]  

(96)

where we have used the sum rule (94) again. We have thus regained the result of Eq. (87).

## 5 Conclusions

In this work, we have accomplished a systematic perturbative solution of the Yang-Mills Schrödinger equation in Coulomb gauge for the vacuum wave functional following the e\( S \) method in many-body physics. This resulted in a novel functional integral representation for the calculation of equal-time correlation functions. We have derived a diagrammatical
representation of these functions, order by order in perturbation theory, where the vertices in the diagrams are determined from the perturbative calculation of the vacuum wave functional. The number of the vertices, which by themselves have a perturbative expansion, grows with the perturbative order. We have determined the equal-time gluon and ghost two-point correlation functions and the potential between static color charges to one-loop order in this way.

The results coincide with those of a straightforward calculation in Rayleigh-Schrödinger perturbation theory [17], and also with the values for equal times of the two-point space-time correlation functions from a Lagrangian functional integral representation [12]. We have emphasized that the latter coincidence is not trivial since the gauge fixing procedures in the Hamiltonian and the Lagrangian approach are profoundly different. We have also used the results of the Lagrangian approach to renormalize the equal-time two-point correlation functions and the static potential.

With the help of the non-renormalization of the ghost-gluon vertex which we also show, or, alternatively, from the static potential, we can extract the running of the correspondingly defined renormalized coupling constant. The result for the beta function is the one also found in covariant and other gauges, \( \beta_0 = -(11/3)N_c \) to one-loop order. We have used standard renormalization group arguments to determine the asymptotic ultraviolet behavior of the equal-time two-point functions and the static potential under the assumption of multiplicative renormalizability to all orders, with the result that

\[
\langle A_i^a(p_1)A_j^b(p_2) \rangle \propto \frac{(\ln(p_1^2/\Lambda_{QCD}^2))^{-3/11}}{|p_1|} \delta_{ij} t_{ij}(p_1)(2\pi)^3 \delta(p_1 + p_2),
\]

\[
\langle c^a(p_1)\bar{c}^b(p_2) \rangle \propto \frac{(\ln(p_1^2/\Lambda_{QCD}^2))^{-4/11}}{|p_1|^2} \delta_{ab} (2\pi)^3 \delta(p_1 + p_2)
\]

\[
g^2V(p_1) \propto \frac{(\ln(p_1^2/\Lambda_{QCD}^2))^{-1}}{p_1^2}
\]

(97)
to one-loop order in the perturbative (asymptotically free) regime.

It is clear from the presence of an infinite number of vertices in the functional integral representation of the equal-time correlation functions (to infinite perturbative order) that the corresponding Dyson-Schwinger equations contain an infinite number of terms, a very serious problem for the determination of an appropriate approximation scheme for nonperturbative solutions. The existence of simplified diagrammatic rules for the calculation of equal-time correlation functions via the \( \mathcal{E} \)-operator, to be appropriately extended to all perturbative orders, seems to point toward the possibility of formulating similar nonperturbative equations with a finite number of terms. It would indeed be very interesting to repeat the type of infrared analysis applied before to Yang-Mills theory in the Landau gauge [4, 8, 9] and to a variational ansatz in the Coulomb gauge [11, 14] for such a set of equations.
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A Gauge transformations in Lagrangian and Hamiltonian formalisms

While the Lagrangian approach to Yang–Mills theory offers some convenient features (such as manifestation of Lorentz invariance), the more cumbersome Hamiltonian approach yields equations of motion invariant under a larger set of gauge transformations. Prior to quantization, we discuss gauge invariance starting from the classical Lagrangian and Hamiltonian functions, respectively. Note that we will employ standard covariant notation in this appendix; in particular, spatial subindices refer to the covariant components of the corresponding 4-vector or tensor.

The Lagrangian function of the gauge sector,

\[ L = -\frac{1}{4} \int d^3x \, F_{\mu\nu}^a(x) F^{\mu\nu}_a(x), \]

is invariant under gauge transformations of the gauge field \( A_\mu(x) \equiv A_\mu^a(x) T^a \),

\[ A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{1}{g} U(x) \partial_\mu U^\dagger(x), \]

where \( U \in SU(N) \) and \([T^a, T^b] = f^{abc}T^c\).

The Weyl gauge, \( A_0^a(x) = 0 \), can be found by choosing the time-ordered exponential

\[ U^\dagger(x) = T \exp \left\{ -g \int^t dt' A_0(x, t') \right\}. \]

To remain in the Weyl gauge, the transformation (100) may be followed by time-independent transformations \( U(x) \) only. We can therefore fix the Coulomb gauge, \( \partial_i A_i^a(x) = 0 \), at one instant of time but it is impossible to fix both gauges simultaneously for all times.

In the Hamiltonian formalism, on the other hand, gauge transformations are generated by (first-class) constraints in configuration space [23]. To see that, supplement the Hamiltonian function

\[ H = \frac{1}{2} \int d^3x \, (\Pi_0^a(x) + B_0^2(x)) - \int d^3x A_0^a(x) \hat{D}_i^a(x) \Pi_i^b(x) \]

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by the constraints
\[ \phi_1^a(x) = \Pi_0^a(x) \approx 0, \quad \phi_2^a(x) = \dot{D}_i^{ab}(x)\Pi_b^i(x) \approx 0 \]  
with some arbitrary Lagrange multiplier fields \( \{\lambda_k^a(x)\} \),

\[ H_E = H + \sum_{k=1,2} \int d^3x \lambda_k^a(x)\phi_k^a(x). \]  

We defined \( \Pi_a^a(x) = F_{\mu a}^a(x) \) and \( \dot{D}_i^{ab}(x) = \delta^{ab}\partial_i - gf^{abc}A_i^c(x) \). The extended Hamiltonian \( H_E \) in Eq. (103) is equivalent to the original Hamiltonian \( H \) since the constraints \( \{\phi_k^a(x)\} \) vanish weakly (in the Dirac sense [23]). The infinitesimal time evolution of the gauge field \( A_\mu^a(x, t) \) from \( t_0 \) to \( t = t_0 + \delta t \), generated by \( H_E \) through the Poisson brackets,

\[ A_\mu^a(x, t) = A_\mu^a(x, t_0) + \delta t \{A_\mu^a(x, t_0), H\} + \delta t \sum_{k=1,2} \int d^3y \lambda_k^b(y)\{A_\mu^a(x, t_0), \phi_k^b(y)\}, \]  
gives for two different sets of Lagrange multiplier functions \( \{\lambda_k^b(x)\} \) and \( \{\lambda_k^b(x)\} \) two different results \( A_\mu^a \) and \( A_\mu^{ma} \), respectively. These differ to \( \mathcal{O}(\delta t) \) by

\[ A_\mu^{ma}(x, t) - A_\mu^a(x, t) = \delta t \sum_{k=1,2} \int d^3y (\lambda_k^{mb}(y) - \lambda_k^{mb}(y))\{A_\mu^a(x, t), \phi_k^b(y)\} \]  
and are physically equivalent. Thus, the function

\[ G = \sum_{k=1,2} \int d^3y \tau_k^a(y)\phi_k^a(y) \]  
generates infinitesimal gauge transformations in the (extended) Hamiltonian formalism with arbitrary functions \( \tau_1^a(x) \) and \( \tau_2^a(x) \). Computing the Poisson brackets in Eq. (105) yields

\[ A_0^a(x) \rightarrow A_0^a(x) + \tau_1^a(x) \]  
\[ A_i^a(x) \rightarrow A_i^a(x) - \dot{D}_i^{ab}(x)\tau_2^b(x) \]  

The difference to the gauge transformations (99) in the Lagrangian formalism is that the time component and the spatial components of the gauge field transform independently. The two functions \( \tau_1^a(x) \) and \( \tau_2^a(x) \) allow for a larger set of gauge transformations than the single function \( U(x) \) in the Lagrangian formalism. The simultaneous fixing of Weyl and Coulomb gauges, which is impossible in the Lagrangian formalism, can be accomplished in the Hamiltonian formalism by appropriately choosing \( \tau_1^a(x) \) and \( \tau_2^a(x) \). See Ref. [24] for the abelian case. Subsequently, the non-abelian gauge-fixed theory can be canonically quantized with projection on the physical Hilbert space [14], or with Dirac brackets [22] enforcing all constraints strongly. Both quantization prescriptions produce the Hamiltonian operator given by Eq. (3).
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