Exact Volume of Zonotopes Generated by a Matrix Pair

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Abstract
In this article, we define a class of special zonotopes generated by a matrix pair with finite-interval parameters. We discuss the relationship between the volume of these zonotopes and the controllability of one aspect (the volume of the controllable region) of the dynamic systems. We present a corollary and develop an effective recursive method to compute the volume of the special zonotopes. Furthermore, we develop two recursive and analytical volume-computation methods for the finite- and infinite-time controllable regions with real eigenvalues. We conduct numerical experiments to demonstrate the effectiveness of these new volume-computation methods for zonotopes and regions.

Keywords: volume computation, zonotope, algorithm, computational complexity, discrete-time systems, controllable region, controllability

1. Introduction
In control theory and engineering, linear dynamic systems in the discrete-time case can be formulated as follows:

\[ x_{k+1} = Ax_k + Bu_k, \quad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^r, \quad (1) \]

where \( x_k \) and \( u_k \) are the state variable and input variable, respectively, and matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times r} \) are the state matrix and input matrix, respectively, in the system models [10], [11]. To investigate the controllability
of the linear dynamic systems (1), the input variables $u_k$ are needed to be bounded and normalized for the following reasons.

1) The many practical controlled plants are with the bounded input variables or the input saturation elements, that is, the input variables $u_k$ are bounded;

2) To compare properly the state control ability of the input variables between the different systems or in the one system with the different parameters in system models \{A, B\}, the state variables and the input variables of these systems are with the matching scale and normalization, respectively.

Therefore, in this paper, the state variables between the different systems are with the matching scale, and the input variables $u_k$ are bounded and normalized as $\|u_k\|_{\infty} \leq 1$. Then, the $N$-steps controllable region $R_{c,N}$ and reachable region $R_{r,N}$ of the systems (1) can be defined as

$$R_{c,N} = \{x_0|x_0 = -(A^{-N}P_N)U_N, \|U_N\|_{\infty} \leq 1\}$$

$$= \{x_0|x_0 = (A^{-N}P_N)U_N, \|U_N\|_{\infty} \leq 1\} \quad (2)$$

$$R_{r,N} = \{x_N|x_N = P_NU_N, \|U_N\|_{\infty} \leq 1\} \quad (3)$$

where $x_0$ and $x_N$ are respectively the initial and terminal states of the dynamic systems in the control process, $U_N^T = [u_{N-1}^T, u_{N-2}^T, \ldots, u_0^T]$ is the control input sequence, and $P_N = [B, AB, \ldots, A^{N-1}B]$ is the controllable matrix [14] [6] [15] [12] [13] [8] [9] [7] [11]. Because the controllable region and reachable region defined as above can be transformed each other, without loss of generality, only the reachable region $R_{r,N}$ and the reach ability are discussed later and the obtained conclusions can be generalizd conveniently to the controllable region $R_{c,N}$ and the control ability.

Based on the definition of the reachable region by Eq. (3), we know,

1) The larger the size of the reachable region $R_{r,N}$ is (e.g., $R_{r,N}^{(1)} \subset R_{r,N}^{(2)}$), the more the reachable states of the systems in $N$-steps are, and the larger the reachable range in the state space is;

2) For the reaching control problem that the state is controlled from the origin in the state space to the given same state $x_1$, if the size of the reachable region $R_{r,N}$ is larger,

2.1) there exists a control strategy with the less control time and the faster response speed;

2.2) there exist more control strategies, that is, the larger the size of the solution space of the input sequence for the reaching control problem and then
the easier to design and implement for the reaching control systems.

Therefore, it follows that the size of the reachable region can reflect well the state-reaching control ability of the input variables of the linear time-invariant discrete-time systems (1). From the perspective of geometric analysis, in fact, the reachable region in control theory and engineering field can be regarded as a geometry in $n$-dimension state space and can be characterized by its surface, shape and volume. When the geometry shapes are same or approximate, the larger the geometry volume are, the larger the geometry size.

To accurately measure the controllability of the systems, the volumes of the regions $R_{c,N}$ and $R_{r,N}$ must be computed. Based on volume computing, the controllability can be optimized and then the control performance of the closed-loop control systems for the open-loop systems (1) can be prompted.

In fact, the regions defined in eqs. (2) and (3) can be considered a class of zonotopes spanned by a vector set with a parameter set in the finite interval. These zonotopes can be defined as follows [16] [7] [3].

**Definition 1.** The zonotopes spanned by the $n$-dimensional ($n$-D) vectors of matrix $Z_m = \begin{bmatrix} z_1, z_2, \ldots, z_m \end{bmatrix} \in R^{n \times m}$ and the parameter set with a finite interval are defined as

$$C_q(Z_m) = \left\{ \sum_{i=1}^{m} c_i z_i \bigg | \forall c_i \in [0, 1], i = 1, m \right\} \quad (4)$$

where $q = \text{rank}(A_m), c_i (i = 1, m)$ are the parameters representing the zonotope, and vectors $z_i (i = 1, m)$ are called as the generators of the zonotopes. The zonotopes are the $q$-D parallel polytopes in the $n$-D space, and they are convex.

Similar to the above definition of zonotope $C_q(Z_m)$, as eqs. (2) and (3) describe the controllability region $R_{c,N}$ and reachability region $R_{r,N}$, we can define a new type of zonotope generated by the matrix pair $\{A, B\}$ as follows.

**Definition 2.** The zonotopes generated by the matrix pair $\{A, B\}$ and the parameter set with a finite interval are defined as

$$E_q(P_N) = \left\{ \sum_{i=1}^{rN} c_i p_i \bigg | \forall c_i \in [0, 1], i = 1, rN \right\} \quad (5)$$
where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$, $P_N = [B, AB, \ldots, A^{N-1}B] = [p_1, p_2, \ldots, p_N]$, $q = \text{rank}(P_N)$, $c_i (i = 1, r^N)$ are the parameters representing the zonotope, and the matrix pair $\{A, B\}$ is called the generator pair of the zonotopes.

It follows from the definition of the zonotopes $E_q(P_N)$ that the regions $R_{c,N}$ and $R_{r,N}$ can be gotten from the zonotope $E_q(P_N)$ by some linear transformations \cite{1} \cite{2}. Since the controllability of the dynamic systems is related to these regions, and the geometric volume is a key index for investigating these regions, to investigate the controllability can in some respects be carried out by investigating the volumes of these zonotopes.

In fact, the exact volume of the zonotope $C_q(Z_m)$ generated by $m$ vectors $z_i(i = 1, m)$ can be computed as the sum of the determinants of any $n$ vectors from the vectors $z_i(i = 1, m)$. These relevant results can be summarized in the following theorem \cite{5} \cite{7}.

**Theorem 1.** For any full row rank matrix $Z_m \in \mathbb{R}^{n \times m}$, the volume of the $n$-D zonotope $C_n(Z_m)$ spanned by the vectors of $Z_m$ can be computed as

$$V_n(C_n(Z_m)) = \sum_{(i_1, i_2, \ldots, i_n) \in \Omega^m_n} |\det \Lambda_{i_1 i_2 \ldots i_n}|$$

where $\Lambda_{i_1 i_2 \ldots i_n} = [z_{i_1}, z_{i_2}, \ldots, z_{i_n}]$, and the column-label $n$-tuple set $\Omega^m_n$ consists of all possible $n$-tuples $(i_1, i_2, \ldots, i_n)$ whose elements are picked from the set $\{1, 2, \ldots, m\}$ and are sorted by their values. The computational complexity of the volume-computation method, i.e., the times computing the $n \times n$ determinant values, is

$$\frac{m!}{(m-n)!n!}$$

times, noted as the polynomial time $\mathcal{O}(m^n)$ on the vector number $m$.

The volume of the zonotope $E_q(P_N)$ generated by the matrix pair $\{A, B\}$, as computed by eq. (6), will have complexity $\mathcal{O}((rN)^n)$, i.e., the complexity will be $\mathcal{O}(N^n)$ on the time variable $N$. For many practical problems in control theory and engineering, the dimensions $n$ and $r$ in the matrix pair $\{A, B\}$ are finite, but the sampling-step number $N$ is a time variable that will gradually increase. Considering that $N$ is gradually increasing and even will approach infinity, the focus of the computational complexity for the zonotope $E_q(P_N)$ is on the time variable $N$ but not the finite dimension variables $n$ and $r$. Therefore, we focus on the following two methods to compute the volume.
Problem 1. The exact volume computation of the finite-time zonotope \( E_q(P_N) \) generated by the matrix pair \( \{A, B\} \) with the lower complexity on time variable \( N \).

Problem 2. The analytically exact volume computation of the infinite-time zonotope \( E_q(P_\infty) \) generated by the matrix pair \( \{A, B\} \) with complexity \( O(1) \).

In this paper, first, for Problem 1 the recursive computation volume of the finite-time zonotope \( E_q(P_N) \) with the general matrix pair \( \{A, B\} \) will be discussed in section 2, and a new computation method with complexity \( O(N^{n-1}) \) is obtained. In section 3, the same problem for the matrix \( A \) with \( n \) real eigenvalues will be discussed, and a new computation method with complexity \( O(N) \) is proposed and proven. For Problem 2 the analytic computation method for infinite \( N \) and \( n \) real eigenvalues will be given and proven with complexity \( O(1) \) in section 4. Finally, the numerical experiments for the computation methods proposed in this paper will be carried out in section 5. The effective computation methods for the zonotope \( E_q(P_N) \) when the matrix \( A \) has a more complex eigenvalue distribution than \( n \) real eigenvalues will be investigated in future work.

2. Volume Computation of Zonotope Generated by Matrix Pair

As mentioned earlier, the regions \( R_{c,N} \) and \( R_{r,N} \) can be represented essentially by the zonotopes \( E_n(A^{-N}P_N) \) and \( E_n(P_N) \), respectively. Therefore, based on Theorem 1 the volumes of these regions can be computed conveniently by computation of these zonotope volumes. So, we have

\[
V_n(R_{c,N}) = |\det A|^{-N} V_n(R_{r,N})
\]

\[
V_n(R_{r,N}) = 2^n V_n(E_n(P_N))
\]

Hence, only the volume computation of the zonotope \( E_n(P_N) \) is studied in detail.

2.1. recursive computation method

From Theorem 1 we have the following corollary on the volume computation for the special zonotope \( E_n(P_N) \) generated by the matrix pair \( \{A, B\} \).
Corollary 1. For any matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$, the volume of the zonotope $E_n(P_N)$ generated by pair $\{A, B\}$ can be computed recursively by the following equation with computational complexity $\mathcal{O}(N^{n-1})$ on time variable $N$:

$$V_n(E_n(P_N)) = (1 + |\det A|)V_n(E_n(P_{N-1})) - |\det A|V_n(E_n(P_{N-2}))$$

$$+ \sum_{j=1}^{r} \sum_{k=1}^{r} \left( \sum_{(i_1,i_2,\ldots,i_n)\in \Theta^j_{0,0} \times \Theta^{n-j-k}_{1,N-2} \times \Theta^k_{N-1,N-1}} |\det \Psi_{i_1i_2\ldots i_n}| \right) \quad (10)$$

where $\Psi_{i_1i_2\ldots i_n} = [p_{i_1}, p_{i_2}, \ldots, p_{i_n}]$, $p_i(i = 1, rN)$ is the $i$-th vector of matrix $P_N$, and the $j$-tuple set $\Theta_{N,M}^j$ consists of all possible $j$-tuples $(i_1, i_2, \ldots, i_j)$ whose elements are picked from the set \{rN + 1, rN + 2, \ldots, r(M + 1)\} and sorted by the values

$$\Theta^j_i \times \Theta^{m-j}_i = \{(i_1, i_2, \ldots, i_m) | \forall (i_1, \ldots, i_j) \in \Theta^j_i, \forall (i_{j+1}, \ldots, i_m) \in \Theta^{m-j}_i \}$$
Proof. (1) By the volume-computation equation (6), we have

\[ V_n(E_n(P_N)) = \sum_{(i_1, i_2, \ldots, i_n) \in \Theta_{0,N-1}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ = \left\{ \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-2}^n} + \sum_{(i_1, \ldots, i_n) \in \Theta_{1,N-1}^n} - \sum_{(i_1, \ldots, i_n) \in \Theta_{1,N-2}^n} \right\} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ + \sum_{j=1}^r \sum_{k=1}^r \sum_{(i_1, \ldots, i_n) \in \Theta_{0,0}^j \times \Theta_{1,N-2}^{n-j} \times \Theta_{N-1,1}^k} \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-3}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ = V_n(E_n(P_{N-1})) + |\det A| \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-2}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ - |\det A| \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-3}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ + \sum_{j=1}^r \sum_{k=1}^r \sum_{(i_1, i_2, \ldots, i_n) \in \Theta_{0,0}^j \times \Theta_{1,N-2}^{n-j} \times \Theta_{N-1,1}^k} \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-3}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

\[ = (1 + |\det A|) V_n(E_n(P_{N-1})) - |\det A| V_n(E_n(P_{N-2})) \]

\[ + \sum_{j=1}^r \sum_{k=1}^r \sum_{(i_1, i_2, \ldots, i_n) \in \Theta_{0,0}^j \times \Theta_{1,N-2}^{n-j} \times \Theta_{N-1,1}^k} \sum_{(i_1, \ldots, i_n) \in \Theta_{0,N-3}^n} |\det \Psi_{i_1 i_2 \cdots i_n}| \]

(2) Considering the combination computation in eq. (10) and the recursive time length \(N\), the computational complexity of eq. (10) in the \(N\)-th recursive computation stage is less than or equal to

\[ (r \times r^{r/2})^2 \times \frac{(rN - 2)!}{(rN - n)! (n - 2)!} + 1 \] (11)

times computing the \(n \times n\) determinant. Then the complexity for the full recursive computation can be noted as the polynomial time \(O(N^{n-1}r^{n+r})\), \(i.e., O(N^{n-1})\) on time variable \(N\).

When \(r = 1\), \(i.e., B\) in matrix pair \(\{A, B\}\) is an \(n \times 1\) vector, eq. (10)
can be simplified to
\[
V_n(E_n(P_N)) = (1 + |\det A|) V_n(E_n(P_{N-1})) - |\det A| V_n(E_n(P_{N-2})) + \sum_{(i_2, i_3, \ldots, i_{n-1}) \in \Theta_{n-2}^{i_1 N-2}} |\det [B, A^{i_2}B, \ldots, A^{i_{n-1}}B, A^{N-1}B]| \tag{12}
\]

By eq. (10) or (12), the volume of zonotope \(E_n(P_N)\) can be computed recursively with the time variable \(N\), and the corresponding complexity will be reduced from \(O(N^n)\) to \(O(N^{n-1})\).

3. Volume Computation for Matrix \(A\) with \(n\) real eigenvalues

3.1. a lemma on the determinant of quasi-Vandermonde matrices

By Corollary 1, the volume computation of the zonotope \(E_n(P_N)\) spanned by the general matrix pair \(\{A, B\}\) can be made, and the more effective computation methods of the zonotope volume for a matrix \(A\) with \(n\) real eigenvalues will be studied here.

First, for that matrix pair, the following lemma about the sign of a class of quasi-Vandermonde matrices is proposed and proven.

**Lemma 1.** For any \(n > 0\), if \(0 < k_1 < k_2 < \cdots < k_n\) and \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n\), we have
\[
F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{k_1, k_2 \cdots k_n} = \det \begin{bmatrix}
\lambda_1^{k_1} & \lambda_2^{k_1} & \cdots & \lambda_n^{k_1} \\
\lambda_1^{k_2} & \lambda_2^{k_2} & \cdots & \lambda_n^{k_2} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{k_n} & \lambda_2^{k_n} & \cdots & \lambda_n^{k_n}
\end{bmatrix} > 0 \tag{13}
\]

**Proof.** Let \(\alpha_i = [\lambda_1^{k_i} \lambda_2^{k_i} \cdots \lambda_n^{k_i}]^T, i = 1, n\). In fact, \(F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{k_1, k_2 \cdots k_n}\) defined above is the oriented volume of the polytope spanned by the vectors \(\alpha_i (i = \overline{1, n})\) in \(n\)-D space. From the representation of vectors \(\alpha_1\) and \(\alpha_2\), we know that vector \(\alpha_2\) can be regarded as the linear transformation result from vector \(\alpha_1\) via the following transformation matrix:
\[
\Lambda_1 = [\text{diag} \{\lambda_1, \lambda_2, \cdots, \lambda_n\}]^{k_2 - k_1}
\]
where \(\text{diag}\{\bullet\}\) denotes the diagonal matrix. Similarly, vector \(\alpha_i (i = 3, 4, \ldots, n)\) can be obtained from vector \(\alpha_{i-1}\).
When \( \lambda_i(i = 1, n) \) satisfy \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \), it can be proved that after the linear transformation via matrix \( \Lambda_1 \), vector \( \alpha_1 \) to vector \( \alpha_2 \) constitute a right-handed system and satisfy the right-hand rule. Analogously, we know that vector \( \alpha_i(i = 3, 4, \ldots, n) \) is in the right-handed systems spanned by vectors \( \{\alpha_1, \alpha_2, \cdots, \alpha_{i-1}\} \). Therefore, considering that vectors \( \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \) satisfy the right-handed system and are in the first quadrant of the \( n \)-D space, according to the basic theory of linear algebra, the oriented volume of the polytope spanned by these vectors must satisfy

\[
\det [\alpha_1, \alpha_2, \cdots, \alpha_n] > 0
\]
i.e., eq. (13) holds.

3.2. Recursive Computation Method with Linear Time Complexity

3.2.1. Algorithm

When the \( n \) eigenvalues of the matrix \( A \) are different and real, there must exist a transformation matrix \( W \) such that

\[
\Lambda = WA W^{-1} = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}
\]
\[
\Gamma = WB
\]
\[
\mathcal{P}_N = [\Gamma, \Lambda \Gamma, \cdots \Lambda^{N-1} \Gamma] = WP_N
\]

and then it is proven easily that for the reversible transformation matrix \( W \), the volume of the zonotope \( E_n(P_N) \) generated by the matrix pair \( (\Lambda, \Gamma) \) satisfies

\[
V_n(E_n(P_N)) = |\det W|^{-1} V_n(E_n(\mathcal{P}_N))
\]

Therefore, \( V_n(E_n(P_N)) \) with \( n \) differentially real eigenvalues can be gotten by computing \( V_n(E_n(\mathcal{P}_N)) \). Later we will discuss in detail how to effectively compute \( V_n(E_n(\mathcal{P}_N)) \).

As we know, the most practical discrete-time systems are the sampling systems from the continuous-time systems, and the eigenvalues of continuous-time systems and the corresponding sampling systems satisfy

\[
\lambda_i = \exp(\mu_i T) \quad i = 1, n
\]

where \( T \) is the sampling period, and \( \mu_i \) and \( \lambda_i(i = 1, n) \) are the eigenvalues of the continuous-time systems and sampling systems, respectively. Therefore,
the relationships between \( \lambda_i \) and \( \mu_i \) are

\[
\lambda_i > 0 \iff \text{Im}(\mu_i) = 0
\]
\[
\text{Im}(\lambda_i) \neq 0 \iff \text{Im}(\mu_i) \neq 0
\]

and there exists no \( \lambda_i \leq 0 \) for finite eigenvalues \( \mu_i \), where \( \text{Im}(z) \) is the imaginary part of the complex number \( z \). Hence, if the eigenvalues of the sampling systems are real, they must be positive.

When matrices \( B \) and \( \Gamma \) are only vectors, \( i.e. \) the linear discrete systems (1) are with a single input, then the volumes of the zonotopes \( E_n(P_N) \) and \( E_n(\mathcal{P}_N) \) can be computed recursively with complexity \( O(N) \), and the corresponding result can be determined by the following theorem.

**Theorem 2.** If \( \Lambda \) is a diagonal matrix and \( \Gamma \) is only a vector, the volume of the zonotope \( E_n(\mathcal{P}_N) \) generated by matrix pair \( \{\Lambda, \Gamma\} \) can be computed with computational complexity \( O(N) \) by the following equation:

\[
V_n(E_n(\mathcal{P}_N)) = \left| \prod_{i=1}^{n} \beta_i \right| V_N^{\lambda_1 \lambda_2 \cdots \lambda_n}
\]

(19)

where

\[
\begin{align*}
\Lambda &= \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \\
\Gamma &= [\beta_1, \beta_2, \ldots, \beta_n]^T \\
V_N^{\lambda_1 \lambda_2 \cdots \lambda_n} &= \sum_{(i_1, i_2, \ldots, i_n) \in \Omega_{N-1}^n} F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{i_1 i_2 \cdots i_n} \\
&= V_{N-1}^{\lambda_1 \lambda_2 \cdots \lambda_n} + \sum_{j=1}^{n} (-1)^{n+j} \lambda_j^{N-1} V_{N-1}^{\lambda_1 \lambda_2 \cdots \lambda_n \setminus \lambda_j}
\end{align*}
\]

(20)

where \( \lambda_1 \lambda_2 \cdots \lambda_n \setminus \lambda_j \) means that \( \lambda_j \) is deleted from sequence \( \lambda_1 \lambda_2 \cdots \lambda_n \).

**Proof.** (1) By Theorem 1 and eq. (10), we have

\[
V_n(E_n(\mathcal{P}_N)) = \sum_{(k_1, k_2, \ldots, k_n) \in \Omega_{N-1}^n} |\text{det} [A^{k_1} \Gamma, A^{k_2} \Gamma, \ldots, A^{k_n} \Gamma]| \\
= \left| \prod_{i=1}^{n} \beta_i \right| \sum_{(k_1, k_2, \ldots, k_n) \in \Omega_{N-1}^n} |F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{k_1 k_2 \cdots k_n}|
\]

(21)
For any \( \lambda_i \) and \( k_i \) that satisfy
\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \quad \text{and} \quad 0 \leq k_1 < k_2 < \cdots < k_n
\]
according to Lemma 1, eq. (21) can be rewritten as
\[
V_n(E_n(\overline{P}_N)) = \prod_{i=1}^{n} \beta_i \left| V_N^{\lambda_1 \lambda_2 \ldots \lambda_n} \right| \tag{22}
\]
where
\[
V_N^{\lambda_1 \lambda_2 \ldots \lambda_n} = \sum_{(i_1,i_2,\ldots,i_n) \in \Omega_{N-1}^n} F_{\lambda_1 \lambda_2 \ldots \lambda_n}^{i_1 \ldots i_n} \tag{23}
\]
Then the above \( V_N^{\lambda_1 \lambda_2 \ldots \lambda_n} \) can be computed recursively as follows:
\[
V_N^{\lambda_1 \lambda_2 \ldots \lambda_n} = V_{N-1}^{\lambda_1 \lambda_2 \ldots \lambda_n} + \sum_{(i_1,i_2,\ldots,i_{n-1}) \in \Omega_{N-2}^{n-1}} (-1)^{n+j} \lambda_{j}^{N-1} \sum_{j=1}^{n}(1) F_{\lambda_1 \lambda_2 \ldots \lambda_n \lambda_j}^{i_1 \ldots i_{n-1}}
\]
\[
= V_{N-1}^{\lambda_1 \lambda_2 \ldots \lambda_n} + \sum_{(i_1,i_2,\ldots,i_{n-1}) \in \Omega_{N-2}^{n-1}} \sum_{j=1}^{n} (-1)^{n+j} \lambda_{j}^{N-1} V_{N-1}^{\lambda_1 \lambda_2 \ldots \lambda_n \lambda_j}
\]
Therefore, eqs. (19) and (20) hold.

(2) The computational complexity of the recursive eq. (20) can be divided into two parts. One is \( n(N-2) \) for computing the power \( \lambda_j^i \), and the other is the rest of the complexity in the recursive process. The recursive process can be described by the following array:

\[
\begin{array}{ccccccc}
C_n^n & n & V_N^{\lambda_1 \lambda_2 \ldots \lambda_n} & \rightarrow & V_{N-1}^{\lambda_1 \lambda_2 \ldots \lambda_n} & \rightarrow & \cdots & V_1^{\lambda_1 \lambda_2 \ldots \lambda_n} \\
C_n^{n-1} & n-1 & V_{N-1}^{\lambda_1 \lambda_2 \ldots \lambda_n \lambda_j} & \rightarrow & V_{N-2}^{\lambda_1 \lambda_2 \ldots \lambda_n \lambda_j} & \rightarrow & \cdots & V_{n-1}^{\lambda_1 \lambda_2 \ldots \lambda_n \lambda_j} \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
C_n^1 & 1 & V_{N-n+1}^{\lambda_1} & \rightarrow & V_{N-n}^{\lambda_1} & \rightarrow & \cdots & V_1^{\lambda_1}
\end{array}
\]

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In the above array:

1) Each element $C_k^k$ of the first column is the variable number of the recursive variables $V^\lambda_1^{\lambda_2^{\cdots^{\lambda_k}}}$.

2) Each element of the second column is the increasing number of multiplications to compute the recursive variable $V^\lambda_1^{\lambda_2^{\cdots^{\lambda_k}}}$ by eq. (20).

3) Each element $V^\lambda_1^{\lambda_2^{\cdots^{\lambda_k}}}$ of the last column is essentially the value of each order Vandermonde determinant, and its complexity is $i(i-1)/2$.

4) Each element $V^\lambda_i$ of the last row equals $V^\lambda_i^{\lambda_{i-1}} + \lambda_i^2$, and the complexity of the power $\lambda_i^2$ has been computed above. In summary, the computational complexity $Q_N^{(n)}$ for $V^\lambda_1^{\lambda_2^{\cdots^{\lambda_n}}}$ is

$$Q_N^{(n)} = n(N-2) + \sum_{i=2}^{n} C_n^i \times i \times (N-n) + \sum_{i=2}^{n} \frac{i(i-1)}{2}$$

$$= n(N-2) + \sum_{i=2}^{n} \frac{n!}{(n-i)!i!} \times i \times (N-n) + \sum_{i=2}^{n} \frac{i(i-1)}{2}$$

$$= \mathcal{O}(n^{n/2+2}N) \quad (24)$$

i.e., the complexity for the volume of the zonotope $E_n(P_N)$ is linear complexity $O(N)$ on the time variable $N$. □

4. Analytic Computation Method for Infinite-time $E_n(P_\infty)$

For many analysis problems on the controllability of practical dynamic systems (1), our focus is on the infinite-time controllable region $R_{c,\infty}$ and reachable region $R_{r,\infty}$. The computational cost of these region volumes by Theorem 1 Corollary 1 and Theorem 2 will approach infinity. We now propose a theorem on an analytic computation method with complexity $O(1)$ that has nothing to do with the time variable $N (N \to \infty)$, and we prove it as follows.

**Theorem 3.** For the $n$ eigenvalues $\lambda_i (i = 1, n)$ of the matrix $A$ satisfying

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < 1,$$

the volume of the infinite-time $E_n(P_\infty)$ is as

$$V^\lambda_1^{\lambda_2^{\cdots^{\lambda_n}}} = \Phi_{\lambda_1^{\lambda_2^{\cdots^{\lambda_n}}}} \quad (25)$$
where
\[
\Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \left( \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right) \left( \prod_{i=1}^{n} \frac{1}{1 - \lambda_i} \right) \tag{26}
\]

**Proof.** The inductive method can be used to prove the theorem.
(1) When \( n = 1 \), we have
\[
V_{\infty}^{\lambda_1} = \sum_{i_1=0}^{\infty} F_{\lambda_1}^{i_1} = \frac{1}{1 - \lambda_1}
\]
and eq. (25) holds.
(2) Assume that for all \( n = s \leq m - 1 \), eq. (25) holds, i.e.,
\[
V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s} = \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_s} \tag{27}
\]
In addition, according to definition (23) of \( V_{N_{\infty}}^{\lambda_1, \lambda_2, \ldots, \lambda_s} \), we have
\[
V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s} = \sum_{(k_1, k_2, \ldots, k_s) \in \Omega_{0, \infty}^s} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s} + \sum_{(k_1, k_2, \ldots, k_s) \in \Omega_{1, \infty}^s} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s}
\]
\[
= \sum_{(k_2, \ldots, k_s) \in \Omega_{0, \infty}^{s-1}} \sum_{k_1=1}^{s} (-1)^{1+k} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s} + \sum_{(k_1, k_2, \ldots, k_s) \in \Omega_{0, \infty}^s} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s}
\]
\[
+ \Upsilon_s \sum_{(k_1, k_2, \ldots, k_s) \in \Omega_{1, \infty}^s} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s}
\]
\[
= \sum_{(k_2, \ldots, k_s) \in \Omega_{0, \infty}^{s-1}} \sum_{k_1=1}^{s} (-1)^{1+k} \Upsilon_s V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s} + \Upsilon_s \sum_{(k_1, \ldots, k_s) \in \Omega_{0, \infty}^s} F_{\lambda_1, \lambda_2, \ldots, \lambda_s}^{k_1 k_2 \ldots k_s}
\]
\[
= \sum_{k_1=1}^{s} (-1)^{1+k} \Upsilon_s V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s}^{\lambda_1 \lambda_2 \ldots \lambda_s}) + \Upsilon_s \sum_{k_1=1}^{s} (-1)^{1+k} \Upsilon_s V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s}^{\lambda_1 \lambda_2 \ldots \lambda_s} \tag{28}
\]
i.e.,
\[
(1 - \Upsilon_s) V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s} = \sum_{k_1=1}^{s} (-1)^{1+k} \Upsilon_s V_{\infty}^{\lambda_1, \lambda_2, \ldots, \lambda_s}^{\lambda_1 \lambda_2 \ldots \lambda_s} \tag{29}
\]
where
\[ \Upsilon_s = \prod_{i=1}^{s} \lambda_i, \quad \Upsilon_{s \setminus k} = \prod_{i=1, i \neq k}^{s} \lambda_i \]

(3) Next, it will be proved that for \( n = m \), eq. (25) holds.

Similar to the proof of eq. (29), considering the above assumption for \( V_{\lambda_1, \ldots, \lambda_s} \) with \( s \leq m - 1 \), we have

\[ (1 - \Upsilon_n) V_{\lambda_1, \ldots, \lambda_n} = \sum_{k=1}^{n} (-1)^{1+k} \Upsilon_{n \setminus k} \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} \]

By the definition of \( V_{\lambda_1, \lambda_2, \ldots, \lambda_n} \), it can be proved that for any \( \lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_n} \), \( \lambda_{j_1} \) is a factor of \( V_{\lambda_1, \lambda_2, \ldots, \lambda_n} \). And then, by the definition of \( \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} \), \( V_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) can be represented as

\[ V_{\lambda_1, \lambda_2, \ldots, \lambda_n} = H_{\lambda_1, \lambda_2, \ldots, \lambda_n} \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} \] (31)

where \( H_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) is an undetermined polynomial function on \( \lambda_i (i = 1, n) \). By eqs. (30) and (31), it can be proved that the highest order of \( H_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) is \( n - 1 \), i.e., the polynomial function \( H_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) can be described as

\[ H_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \sum_{i=0}^{n-1} \sum_{\sum k=1}^{n} c^{(n,i)}_{j_1, j_2, \ldots, j_n} \lambda_1^{j_1} \lambda_2^{j_2} \ldots \lambda_n^{j_n} \] (32)

where \( c^{(n,i)}_{j_1, j_2, \ldots, j_n} \) is an undetermined coefficient. Because of the symmetry of \( H_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) on \( \lambda_i (i = 1, n) \), we have

\[ c^{(n,i)}_{j_1, j_2, \ldots, j_n} = c^{(n,i)}_{j_1 k_2, \ldots, j_n} \]

(33)

where \( \{k_1, k_2, \ldots, k_n\} \) is any other permutation of \( \{j_1, j_2, \ldots, j_n\} \).

Therefore, by eqs. (30) and (31), we know that to prove eq. (25) is true is equivalent to proving the following equation is true:

\[ H_{\lambda_1, \lambda_2, \ldots, \lambda_n} (1 - \Upsilon_n) \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \sum_{k=1}^{n} (-1)^{1+k} \Upsilon_{n \setminus k} \Phi_{\lambda_1, \lambda_2, \ldots, \lambda_n} \lambda_k \] (34)
Next, eq. (34) will be proved true for any real variables \( \lambda_i (i = \overline{1, n}) \) and then eq. (25) must be true for all \( \lambda_i \in (0, 1) \).

After playing by \( 1 - \lambda_n \) and then letting \( \lambda_n = 1 \), considering that \( \Phi_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}, 1} = \Phi_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}} \), the two sides in the above equation can be rewritten as follows.

left side = \( H_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}, 1} (1 - \Upsilon_{n-1}) \Phi_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}} \)
right side = \( \sum_{k=1}^{n-1} (-1)^{1+k} \Upsilon_{n-1}V_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}} \lambda_k \)

Therefore, by eq. (34), we have

\[ H_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}, 1} = 1 \]  (37)

i.e.,

\[ H_{\lambda_1 \lambda_2 \cdots \lambda_{n-1}, 1} = \sum_{i=0}^{n-1} \sum_{\sum_{k=1}^{n} j_k = i} c_{j_1 j_2 \cdots j_{n-1}, 0}^{(n,i)} \lambda_1^{j_1} \lambda_2^{j_2} \cdots \lambda_{n-1}^{j_{n-1}} = 1 \]  (38)

Then, when \( \lambda_i = 0 (i = \overline{1, n-1}) \), we can get

\[ c_{0,0,\ldots,0}^{(n,0)} = 1 \]  (39)

In addition, for some \( i \in [1, n-1] \) and \( \{j_1, j_2, \ldots, j_{n-1}\} \) in eq. (38), if

\[ c_{j_1 j_2 \cdots j_{n-1}, 0}^{(n,i)} \neq 0, \]  (40)

then the dimension of the real solution space of variables \( \lambda_i (i = \overline{1, n-1}) \) in eq. (38) is less than or equal to \( n - 2 \). But, because the equation holds for any \( \lambda_i (i = \overline{1, n-1}) \), the dimension of the real solution space must be \( n - 1 \). The above two conclusions contradict each other, and so it can be proved that

\[ c_{j_1 j_2 \cdots j_{n-1}, 0}^{(n,i)} = 0 \]  \( \exists j_i = 0 \)  (41)

Then, by (33), for all \( i \in [1, n-1] \) and \( j_k (k = \overline{1, n}) \) with \( \sum_{k=1}^{n} j_k = i \), we have

\[ c_{j_1 j_2 \cdots j_n}^{(n,i)} = 0 \]  (42)
Because $\sum_{k=1}^{n} j_k = i$ and $i < n$, it is sure that at least one element of $\{j_1, j_2, \cdots, j_n\}$ is zero. Therefore, for any $c_{j_1j_2\cdots j_n}^{(n,i)}$, we have

$$c_{j_1j_2\cdots j_n}^{(n,i)} = 0 \quad \sum_{k=1}^{n} j_k = i, \quad i \in [1, n - 1]$$

(43)

i.e.,

$$H_{\lambda_1\lambda_2\cdots\lambda_n} = 1$$

(44)

Thus, by eq. (31), we know that when $m = n$, eq. (25) is also true.

In summary, eq. (25) is proved to be true by the inductive method. □

For the volume computation of the infinite-time zonotope $E_n(P_{\infty})$, the computational complexity of eq. (25) is $O(n^2)$, and it has nothing to do with the time variable $N(N \to \infty)$.

5. Numerical Experiments

In this section, two numerical experiments for volume computation of the controllable and reachable regions are carried out.

**Example 1.** Computing the volume of the finite-time reachable region of the following linear discrete-time system:

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.9596 & -2.9196 & 2.96 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k$$

(45)

The three eigenvalues of the matrix $A$ are $\{0.9517, 1.0000, 1.0083\}$, and the volume of the finite-time reachable region can be computed by **Theorem 1**, **Corollary 1** and **Theorem 2**. The numerical experiments are carried out with the Intel Core i7-7700 3.6GHz CPU and MATLAB R2012a. The computational results are shown in Table 1 where $N$ is the number of sampling steps, $v_r$ is the region volume, $n_d$ is the number of times computing the $n \times n$ determinants, $C_t$ is the computational time cost, and $n_p$ is the number of multiplications only in recursive equations (19) and (20). From the table, we can see that the volumes computed by the three methods are exactly the same, but the computation methods proposed in **Corollary 1** and **Theorem 2** can greatly reduce the computational complexity.
### Table 1: Numerical results for the reachable regions

| $N$ | $v_r$  | $n_d$ | $C_t(s)$ | $n_d$ | $C_t(s)$ | $n_p$ | $C_t(s)$ |
|-----|--------|-------|----------|-------|----------|-------|----------|
| 100 | 4.622E9 | 1.617E5 | 6.887E-1 | 4.852E3 | 2.953E-2 | 1.470E3 | 1.334E-2 |
| 200 | 1.162E11 | 1.313E6 | 5.427 | 1.970E4 | 1.086E-1 | 2.970E3 | 1.615E-2 |
| 300 | 8.015E11 | 4.55E6 | 1.857E1 | 4.455E4 | 2.395E-1 | 4.470E3 | 1.872E-2 |
| 400 | 3.553E12 | 1.059E7 | 7.940E4 | 4.219E-1 | 5.970E3 | 2.127E-2 |
| 500 | 1.274E13 | 2.071E7 | 8.519E1 | 1.243E5 | 6.586E-1 | 7.470E3 | 2.383E-2 |
| 600 | 4.057E13 | 3.582E7 | 3.857E2 | 1.791E5 | 9.508E-1 | 8.970E3 | 2.638E-2 |
| 700 | 1.199E14 | 5.692E7 | 2.370E2 | 1.791E5 | 9.508E-1 | 8.970E3 | 2.638E-2 |
| 800 | 3.373E14 | 8.501E7 | 3.585E2 | 3.188E5 | 1.299 | 1.047E4 | 3.166E-2 |

**Example 2.** Computing the volume of the finite- and infinite-time controllable region of the following linear discrete-time system

\[ x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.5629 & 5.6007 & -7.5179 & 4.48 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_k \] (46)

The four eigenvalues of matrix $A$ are $\{1.2049, 1.1589, 1.0755, 1.0407\}$, the volume of the finite- and infinite-time controllable regions can be computed by **Theorem 1** **Corollary 1** **Theorem 2** and **Theorem 3** respectively, and the experimental tools are as in Example 1. The numerical results shown in Table 2 are for the volume computation of the finite-time controllable region by **Theorem 1** **Corollary 1** and **Theorem 2** and the numerical results shown in Table 3 are for the volume computation of the infinite-time controllable region by **Theorem 3** where $n_{inf}$ is the number of multiplications only in eqs. (19) and (26). These results show the effectiveness of the computational methods proposed in this paper.

### 6. Conclusions

In this article, we define a class of special zonotopes generated by matrix pair $\{A, B\}$ with finite-interval parameters, and then some effective computation methods with low computational complexity for the matrix $A$ with three eigenvalue-distribution cases: any $n$ eigenvalues $\lambda_i$, $n$ differential eigenvalues...
Table 2: Numerical results for the finite-time controllable regions

| N  | $v_r$  | $n_d$ | $C_t(s)$ | $n_d$ | $C_t(s)$ | $n_p$ | $C_t(s)$ |
|----|-------|-------|----------|-------|----------|-------|----------|
| 50 | 2.388E8 | 2.303E5 | 8.134E-1 | 1.843E4 | 6.650E-2 | 1.696E3 | 1.323E-2 |
| 100| 7.495E8 | 3.921E6 | 1.376E1  | 1.569E5 | 5.336E-1 | 3.496E3 | 1.915E-2 |
| 150| 8.671E8 | 2.026E7 | 7.051E1  | 5.403E5 | 1.819     | 5.296E3 | 2.497E-2 |
| 200| 8.846E8 | 6.468E7 | 2.251E2  | 1.294E6 | 4.366     | 7.096E3 | 3.080E-2 |
| 250| 8.871E8 | 1.589E8 | 5.523E2  | 2.542E6 | 8.567     | 8.896E3 | 3.663E-2 |
| 300| 8.874E8 | 3.308E8 | 1.154E3  | 4.411E6 | 1.485E1  | 1.070E4 | 4.245E-2 |
| 350| 8.874E8 | 6.146E8 | 2.136E3  | 7.024E6 | 2.359E1  | 1.250E4 | 4.823E-2 |
| 400| 8.874E8 | 1.051E9 | 3.646E3  | 1.051E7 | 3.532E1  | 1.430E4 | 5.405E-2 |

Table 3: Numerical results for the infinite-time controllable regions by Theorem [3]

| $v_r$ | $n_{inf}$ | $C_t(s)$ |
|-------|------------|----------|
| 8.874E8 | 26         | 1.328E-3 |

$\lambda_i \geq 0$, and $n$ different eigenvalues $\lambda_i \in [0, 1)$. Effective computation methods for the zonotope $E_q(P_N)$, where the matrix $A$ has more complex eigenvalue-distribution cases, such as complex eigenvalues and repeated eigenvalues, will be investigated in our future work.

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