b-articulation points and b-bridges in strongly biconnected directed graphs

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Abstract

A directed graph $G = (V, E)$ is called strongly biconnected if $G$ is strongly connected and the underlying graph of $G$ is biconnected. This class of directed graphs was first introduced by Wu and Grumbach. Let $G = (V, E)$ be a strongly biconnected directed graph. An edge $e \in E$ is a b-bridge if the subgraph $G \setminus \{e\} = (V, E \setminus \{e\})$ is not strongly biconnected. A vertex $w \in V$ is a b-articulation point if $G \setminus \{w\}$ is not strongly biconnected, where $G \setminus \{w\}$ is the subgraph obtained from $G$ by removing $w$. In this paper we study b-articulation points and b-bridges.

Keywords: Directed graphs, Graph algorithms, Strongly biconnected directed graphs

1. Introduction

A directed graph $G = (V, E)$ is called strongly biconnected if $G$ is strongly connected and the underlying graph of $G$ is biconnected. This class of directed graphs was first introduced by Wu and Grumbach. Let $G = (V, E)$ be a strongly biconnected directed graph. In this paper we study edges and vertices whose removal destroys strongly biconnectivity in $G$. An edge $e \in E$ is a b-bridge if the subgraph $G \setminus \{e\} = (V, E \setminus \{e\})$ is not strongly biconnected. A vertex $w \in V$ is a b-articulation point if $G \setminus \{w\}$ is not strongly biconnected, where $G \setminus \{w\}$ is the subgraph obtained from $G$ by removing $w$. $G$ is 2-edge-strongly-biconnected (respectively, 2-vertex-strongly-biconnected) if the number of vertices in $G$ is at least 3 and $G$ has no b-bridges (respectively, b-articulation points). Note that each 2-edge-biconnected directed graph is 2-edge-connected, but the converse is not necessarily true (see Figure 1). Moreover, 2-vertex-connected directed graphs are not necessarily 2-vertex-biconnected, as illustrated in figure 2.

Articulation points and bridges of an undirected graph can be identified in linear time [7, 8, 6]. In 2010, Georgiadis [3] gave a linear time algorithm to
Figure 1: A strongly biconnected graph $G$. This graph is 2-edge-connected since it has no strong bridges. But $G$ contains a b-bridge $(5, 6)$. Thus, $G$ is not 2-edge-strongly biconnected.

Figure 2: A strongly biconnected graph $G = (V, E)$. $G$ is 2-vertex-connected. Since vertex 6 is a $b$-articulation point, $G$ is not 2-vertex-strongly biconnected.
test whether a directed graph is 2-vertex-connected. Italiano et al. [5] gave linear time algorithms for calculating strong articulation points and strong bridges in directed graphs. Wu and Grumbach [4] introduced a class of directed graphs, called strongly biconnected graphs. In this paper we study b-articulation points and b-bridges.

2. Computing b-bridges

This section illustrates how to compute b-bridges in strongly biconnected graphs.

Lemma 2.1. Let $G = (V, E)$ be a strongly biconnected directed graph and let $e$ be a strong bridge in $G$. Then $e$ is a b-bridge.

Proof. Since $e$ is a strong bridge in $G$ and $G$ is strongly connected, the subgraph $(V, E \setminus \{e\})$ is not strongly connected. Therefore, $e$ is a b-bridge.

Note that b-bridges are not necessarily strong bridges, as shown in Figure 3.

Figure 3: A strongly biconnected graph $G$. Edge $(2, 7)$ is both a strong bridge and a b-bridge. Note that the underlying graph of $G \setminus \{(5, 6)\}$ has an articulation point. Thus, $(5, 6)$ is a b-bridge. If we remove $(5, 6)$ from $G$, the remaining subgraph is still strongly connected. Therefore, $(5, 6)$ does not form a strong bridge in $G$. 
Wu and Grumbach [4] introduced strongly biconnected components. A strongly biconnected component of a strongly connected graph $G = (V,E)$ is a maximal vertex subset $U \subseteq V$ such that the induced subgraph on $U$ is strongly biconnected.

Let $G$ be a strongly biconnected directed graph. The following lemma shows how to decrease the number of strongly biconnected components in a strongly connected subgraph of $G$.

**Lemma 2.2.** Let $G_s = (V, E_s)$ be a subgraph of a strongly biconnected directed graph $G = (V,E)$ such that $G_s$ is strongly connected and $G_s$ has $t$ strongly biconnected components. Let $(u,w)$ be an edge in $E \setminus E_s$ such that $u, w$ are in distinct strongly biconnected components $C^B_1, C^B_2$ of $G_s$ with $u, w \notin C^B_1 \cap C^B_2$. Then the graph $(V,E \cup \{(u,w)\})$ contains at most $t - 1$ strongly biconnected components.

**Proof.** Since $G_s$ is strongly connected, there exists a simple path $P$ from $w$ to $u$ in $G_s$. Path $p$ and edge $(u,w)$ form a simple cycle. Consequently, $u, w$ are in the same strongly biconnected component of $(V,E \cup \{(u,w)\})$. □

Algorithm 2.3 can compute all the b-bridges of a strongly biconnected directed graph.

**Lemma 2.4.** Let $e \in E \setminus E_y$. Then $e$ is not a b-bridge.

**Proof.** It is known that a strongly connected subgraph of a strongly connected graph can be obtained by finding a spanning tree in $G$ and in $G_r$ (see [2]). By lemma 2.2 the number of strongly biconnected components must decrease by at least one in each iteration of while loop. Therefore, the subgraph $(V,E_y)$ is strongly biconnected.

The correctness of Algorithm 2.3 follows from Lemma 2.1, Lemma 2.2, and Lemma 2.4.

**Theorem 2.5.** The Algorithm 2.3 runs in time $O(nm)$.

**Proof.** The set of strong bridges $U$ can be calculated in linear time using the algorithm of Italiano et al. [5]. A spanning tree of a strongly biconnected graph can be constructed in linear time using depth first search. Thus, lines 7–13 take linear time. By Lemma 2.2 the number of iterations of the while loop is at most $n - 1$. Moreover, the strongly biconnected components of a strongly connected graph can be found in linear time [4]. Therefore, the while loop of lines 14–18 takes time $O(nm)$. The for loop of lines 20–22 executes $n - |U|$ times. Thus, the total time taken by this loop is $O(nm)$. □
Algorithm 2.3.
Input: A strongly biconnected graph $G = (V, E)$.
Output: $B$, where $B$ is the set of b-bridges in $G$

1. $U \leftarrow \emptyset$
2. Calculate the set of strong bridges in $G$
3. for every strong bridge $(v, w)$ in $G$ do
   4. $U \leftarrow U \cup \{(v, w)\}$
5. select a vertex $y \in V$
6. $E_y \leftarrow \emptyset$
7. build a spanning tree $T$ rooted at $y$ in $G$
8. for each edge $(v, w)$ in $T$ do
   9. $E_y \leftarrow E_y \cup \{(v, w)\}$
10. build $G_r = (V, E_r)$, where $E_r = \{(v, w) \mid (w, v) \in E\}$
11. create a spanning tree $T_y$ rooted at $y$ in $G_r$
12. for each edge $(v, w)$ in $T_y$ do
   13. $E_y \leftarrow E_y \cup \{(w, v)\}$
14. while $G_y = (V, E_y)$ is not strongly biconnected do
   15. identify the strongly biconnected components of $G_y$
   16. find an edge $(u, w) \in E \setminus E_y$ such that $u, w$ are in distinct strongly
   17. biconnected components $C_1^B, C_2^B$ of $G_y$ with $u, w \notin C_1^B \cap C_2^B$
   18. $E_y \leftarrow E_y \cup \{(u, w)\}$.
19. $B \leftarrow U$
20. for every edge $(u, w) \in E_y \setminus B$ do
   21. if $G \setminus \{(u, w)\}$ is not strongly biconnected then
   22. $B \leftarrow B \cup \{e\}$.

3. Open problem

It is easy to see that the b-articulation points of a strongly biconnected directed graph $G$ can be computed in time $O((n - a)m)$, where $a$ is the number of strong articulation points in $G$. Strong articulation points can be computed in linear time [5, 1]. We leave as an open problem whether there is a linear time algorithm for calculating b-bridges and b-articulation points in directed graphs.

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