Interpolation of functional by integral continued C-fractions

Volodymyr L. Makarov
Institute of Mathematics of the NAS of Ukraine, Kiev, Ukraine
E-mail: makarovich@gmail.com

and

Mykhaylo M. Pahirya
State University of Mukachevo, Ukraine
E-mail: pahirya@gmail.com

January 22, 2018

Abstract. The functional interpolation problem on a continual set of nodes by an integral continued C-fraction is studied. The necessary and sufficient conditions for its solvability are found. As a particular case, the considered integral continued fraction contains a standard interpolation continued C-fraction which is used to approximate the functions of one variable.

Keywords: continuity nodes, integral continued C–fraction, interpolation of functional

AMS subject classification: 30B70, 41A20, 65D05, 65D15

1 Introduction

Recently, the considerable amount of scientific studies were devoted to the generalization of interpolation theory of functions of real (complex) variable to the case of functionals and operators in abstract spaces, see the monographs [1, 2] for example. Continued fractions [3] and branched continued fractions introduced by Skorobohat’ko V.Ya. [4] were generalized by integral continued fractions which proposed by Syvavko M.S. [5]. The problem of interpolation with integral continued fractions was first considered in the article [6], further expansions and generalizations of this work are contained in the paper [7].

Another class of interpolation integral continued fractions has been investigated in the paper [8]. This class differs from the previously studied integral continued fractions by the fact that $n–s$ floor of fraction containing $n$–tuple integral. Interpolation integral operator continued fractions in Banach spaces were investigated in the article [9]. The natural generalization of the classical Thiele continued fraction to the interpolation integral Thiele–type continued fractions were proposed in [10, 11, 12].

The purpose of this work is the study of interpolation of a functional given on the continual set of nodes by the integral C–type continued fractions. Such integral continued fractions contains the interpolation continued C–fraction as a particular case, so it is a generalization of one of the types of continued fraction used for interpolation of functions [13].

2 Statement of the problem

Assume that $x(z), x_i(z) \in C[0,1], i = 0, n$, are some given functions, $x_i(z) \neq x_j(z)$. Let $F(x(·))$ be a certain functional defined in the space of piecewise continuous functions $Q[0,1]$. First we define continual nodes

$$x^0(z) = x_0(z), \quad x^i(z, \xi) = x_0(z) + H(z - \xi)(x_i(z) - x_0(z)), \quad i = 1, n, \quad (1)$$

where $0 \leq \xi \leq 1$, and $H(·)$ is the Heaviside function.
Let \( b_0, a_i, b_i, i = 1, n \), be numbers, functions, operators, functionals etc. We denote a finite continued fraction as

\[
D_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n}}},
\]

as

\[
D_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n}}}.
\]

Let \( \mathcal{R} \subset \mathbb{R} \) be a compact, \( \mathcal{R} = \{ x_i : x_i \in \mathcal{R}, x_i \neq x_j, i, j = 0, n \} \) and the function \( f \in C(\mathcal{R}) \) is defined by its values at the points of the set \( \mathcal{R}, y_i = f(x_i), i = 0, n \). The function \( f \) can be interpolated by the set of values \( \{ y_i \} \) in different ways, for example, polynomials, splines and continued fractions. A problem of function interpolation by interpolation continued C–fraction (C–ICF)

\[
D_n^{(c)}(x) = a_0^{(c)} + \frac{a_1^{(c)}(x - x_0)}{1} + \frac{a_2^{(c)}(x - x_1)}{1} + \cdots + \frac{a_n^{(c)}(x - x_{n-1})}{1}
\]

investigated in monograph [13]. The coefficients of C–ICF are determined through the interpolation nodes \( \mathcal{R} \) and the set of values of the function \( \{ y_i \} \) by means of a finite-continued fraction recurrence

\[
da_k^{(c)} = \frac{1}{x_k - x_{k-1}} \left( -1 + \frac{a_{k-1}^{(c)}(x_k - x_{k-2})}{-1} + \cdots + \frac{a_2^{(c)}(x_k - x_1)}{-1} + \frac{a_1^{(c)}(x_k - x_0)}{y_k - y_0} \right), \quad k = 2, n,
\]

\[
da_0^{(c)} = y_0, \quad a_1^{(c)} = \frac{y_1 - y_0}{x_1 - x_0}.
\]

If all the interpolation nodes \( x_i, i = 0, n \), tend to the same value \( x_\ast \in \mathcal{R} \), then C–ICF transforms into a regular continued C–fraction that will correspond to the power series for \( f \) around the point \( x_\ast \).

Consider the set of integral continued fractions (ICF) of the form

\[
Q_n(x(\cdot), \xi) = a_0 + \frac{\int_{0}^{1} a_1(z_1)[x(z_1) - x_0(z_1)]dz_1}{1} + \frac{\int_{0}^{1} a_2(z_2)[x(z_2) - x_1(z_2, \xi)]dz_2}{1} + \cdots + \frac{\int_{0}^{1} a_{n-1}(z_{n-1})[x(z_{n-1}) - x^{n-2}(z_{n-1}, \xi)]dz_{n-1}}{1} + \frac{\int_{0}^{1} a_n(z_n)[x(z_n) - x^{n-1}(z_n, \xi)]dz_n}{1},
\]

where \( a_0, a_1(z), a_2(z), \ldots, a_n(z) \) are certain kernels.

We formulate an interpolation problem as follows: Inside the set of ICF defined by [14] find an integral continued fraction which satisfies the interpolation conditions at continual nodes [1]

\[
F(x_0(\cdot)) = Q_n(x_0(\cdot), \xi), \quad F(x^i(\cdot, \xi)) = Q_n(x^i(\cdot, \xi), \xi), \quad i = 1, n, \quad \forall \xi \in [0, 1].
\]
3 Interpolation of functional by integral interpolation
continued C–fraction

Define the kernels \(a_0, a_1(z), \ldots, a_n(z)\) from the condition that C–IICFL \((\text{4})\) satisfies \((\text{5})\). Note that from \((\text{4}), (\text{5})\) you can directly get the expression

\[
Q_n(x^k(\cdot), \xi) = a_0 + \frac{1}{1} \int \frac{a_1(z_1)[x^k(z_1, \xi) - x_0(z_1)]dz_1}{1} + \frac{1}{1} \int \frac{a_2(z_2)[x^k(z_2, \xi) - x^1(z_2, \xi)]dz_2}{1} + \cdots + \\
\frac{1}{1} \int \frac{a_k(z_k)[x^k(z_k, \xi) - x^{k-1}(z_k, \xi)]dz_k}{1}, \quad k = 0, n.
\]

\[\text{(6)}\]

**Theorem 1.** Let the functional \(F(x(\cdot))\) be \((n-1)\)–times Gateaux differentiable and the following formulas are valid. In order that C–IICF \((\text{4})\) to satisfy the continual interpolation conditions \((\text{5})\) it is necessary that its kernels are determined by the formulas

\[a_k(\xi) = \frac{-1}{x_k(\xi) - x_{k-1}(\xi)} d \left( \frac{1}{0} \int \frac{a_{k-1}(z_{k-1})[x^k(z_{k-1}, \xi) - x^{k-2}(z_{k-1}, \xi)]dz_{k-1}}{1} - 1 \right) + \]

\[\frac{1}{0} \int \frac{a_{k-2}(z_{k-2})[x^k(z_{k-2}, \xi) - x^{k-3}(z_{k-2}, \xi)]dz_{k-2}}{1} + \frac{1}{0} \int \frac{a_2(z_2)[x^k(z_2, \xi) - x^1(z_2, \xi)]dz_2}{1} + \cdots + \]

\[\frac{1}{0} \int \frac{a_1(z_1)[x^k(z_1, \xi) - x_0(z_1)]dz_1}{1} + F(x^k(\cdot, \xi)) - F(x_0(\cdot)) \right), \quad k = 2, n, \]

\[a_0 = F(x_0(\cdot)), \quad a_1(\xi) = \frac{-1}{x_1(\xi) - x_0(\xi)} d F(x^1(\cdot, \xi)). \]

**Proof.** For the case \(k = 0, 1\) the formulas are obvious. When \(k = m\), from \((\text{5}), (\text{6})\) we get

\[F(x^m(\cdot, \xi)) = a_0 + \frac{1}{1} \int \frac{a_1(z_1)[x^m(z_1, \xi) - x_0(z_1)]dz_1}{1} + \frac{1}{1} \int \frac{a_2(z_2)[x^m(z_2, \xi) - x^1(z_2, \xi)]dz_2}{1} + \cdots + \\
\frac{1}{1} \int \frac{a_m(z_m)[x^m(z_m, \xi) - x^{m-1}(z_m, \xi)]dz_m}{1}.
\]

By sequentially inverting the continued fraction, we get

\[\frac{1}{1} \int \frac{a_m(z_m)[x^m(z_m, \xi) - x^{m-1}(z_m, \xi)]dz_m}{1} + \frac{1}{1} \int \frac{a_{m-1}(z_{m-1})[x^m(z_{m-1}, \xi) - x^{m-2}(z_{m-1}, \xi)]dz_{m-1}}{1} - 1 \right) + \]

\[\frac{1}{0} \int \frac{a_2(z_2)[x^m(z_2, \xi) - x^1(z_2, \xi)]dz_2}{1} + \frac{1}{0} \int \frac{a_1(z_1)[x^m(z_1, \xi) - x_0(z_1)]dz_1}{1} + \cdots + F(x^m(\cdot, \xi)) - F(x_0(\cdot)).\]
where the interpolation conditions\(^{(5)}\) are determined by the formulas\(^{(7)}\). The subsequent differentiation of both parts of this relation with respect to the variable \(x\) and kernels C–IICF\(^{(4)}\) yields

By applying the L’Hospital rule and considering\(^{(7)},\ (8)\), we get

\[
Q_n(x_0, \xi) = \lim_{z_1 \to 0} \int_0^1 a_1(z_1)[x^2(z_1, \xi) - x_0(z_1)]dz_1
\]

Next, we substitute the result into\(^{(9)}\). Consequently, we obtain

\[
P_2 = 1 + \int_0^1 a_2(z_2)[x^2(z_2, \xi) - x^1(z_2, \xi)]dz_2 = 1 - \int_\xi^1 \frac{d}{dz_2} \left( \frac{\int_{z_2}^1 a_1(z_1)[x_2(z_1) - x_0(z_1)]dz_1}{F(x^2(\cdot, \xi)) - F(x_0(\cdot))} \right) dz_2 =
\]

\[
= 1 - \lim_{z_2 \to 1} K_2[z_2, x_2] + \frac{1}{F(x^2(\cdot, \xi)) - F(x_0(\cdot))} \int_0^1 a_1(z_1)[x_2(z_1) - x_0(z_1)]dz_1
\]

where

\[
K_2[z_2, x_2] = \frac{1}{z_2} \int_0^1 a_1(z_1)[x_2(z_1) - x_0(z_1)]dz_1
\]

By applying the L’Hospital rule and considering\(^{(7)},\ (8)\), we get

\[
\lim_{z_2 \to 1} K_2[z_2, x_2] = \lim_{z_2 \to 1} a_1(z_2)[x_2(z_2) - x_0(z_2)] = \lim_{z_2 \to 1} a_1(z_2)|_{x_1(z_2) \to x_2(z_2)} =
\]

\[
= \lim_{z_2 \to 1} x_2(z_2) - x_0(z_2) \frac{f'(\int_0^1 x^1(t, z_2)dt)(x_0(z_2) - x_1(z_2))}{f'(\int_0^1 x^2(t, z_2)dt)(x_0(z_2) - x_2(z_2))} = 1, \quad \forall x_2(z_2).
\]

We substitute the calculated value of the limit \(K_2[z_2, x_2]\) into the expression for \(P_2\), and then substitute the result into\(^{(9)}\). Consequently, we obtain

\[
(Q_n(x^2(\cdot, \xi)), F(x^2(\cdot, \xi)) = F(x^2(\cdot, \xi)).
\]
Let \( k = 3 \). From (6) we get

\[
Q_n(x^3(\cdot, \xi), \xi) = F(x_0(\cdot)) + \frac{1}{1} \int_0^1 a_1(z_1) [x^3(z_1, \xi) - x_0(z_1)] dz_1 + \\
\int_0^1 a_2(z_2) [x^3(z_2, \xi) - x^1(z_2, \xi)] dz_2 + \int_0^1 a_3(z_3) [x^3(z_3, \xi) - x^2(z_3, \xi)] dz_3
\]

We substitute the value of the kernel \( a_3(z_3) \) and evaluate the last denominator to get

\[
P_3 = 1 + \int_0^1 a_3(z_3) [x^3(z_3, \xi) - x^2(z_3, \xi)] dz_3 = 1 - \int \frac{d}{dz_3} \left\{ \frac{1}{-1} \left[ \int_0^1 a_2(z_2) [x^3(z_2, z_3) - x^1(z_2, z_3)] dz_2 \right] + \right. \\
+ \left. \frac{1}{z_3} \int_0^1 a_1(z_1) [x^3(z_1, z_3) - x_0(z_1)] dz_1 \right\} dz_3 = 1 - \lim_{z_3 \to 1} K_3[z_3, x_3] + \int \frac{d}{dz_3} \left\{ \frac{1}{-1} \left[ \int_0^1 a_2(z_2) [x^3(z_2, z_3) - x^1(z_2, z_3)] dz_2 \right] + \right. \\
+ \left. \frac{1}{z_3} \int_0^1 a_1(z_1) [x^3(z_1, z_3) - x_0(z_1)] dz_1 \right\} dz_3
\]

where

\[
K_3[z_3, x_3] = \int z_3^1 a_2(z_2) [x^3(z_2) - x^1(z_2)] dz_2 + \int z_3^1 a_1(z_1) [x^3(z_1) - x_0(z_1)] dz_1 \\
+ \frac{1}{F(x^3(\cdot, z_3)) - F(x_0(\cdot))}
\]

By applying the L’Hospital rule, taking into account (10), we obtain

\[
\lim_{z_3 \to 1} K_3[z_3, x_3] = \lim_{z_3 \to 1} a_2(z_3) [x^3(z_3) - x^1(z_3)] + \int z_3^1 a_1(z_1) [x^3(z_1) - x_0(z_1)] dz_1 + \\
\frac{d}{dz_3} \left[ F(x^3(\cdot, z_3)) - F(x_0(\cdot)) \right] = \\
\lim_{z_3 \to 1} a_1(z_3) [x^3(z_3) - x_0(z_3)] + \frac{d}{dz_3} F(x^3(\cdot, z_3)) \frac{F(x^3(\cdot, z_3)) - F(x_0(\cdot))}{F(x^3(\cdot, z_3)) - F(x_0(\cdot))} = \\
\lim_{z_3 \to 1} a_1(z_3) [x^3(z_3) - x_0(z_3)] + \frac{d}{dz_3} F(x^3(\cdot, z_3)) \times \\
\frac{x_3(z_3) - x_1(z_3)}{x_2(z_3) - x_1(z_3)} \frac{d}{dz_3} F(x^2(\cdot, z_3))
\]
\[
[x_2(z_3) - x_0(z_3)] \frac{d}{dz_3} F(x^1(\cdot, z_3)) - [x_1(z_3) - x_0(z_3)] \frac{d}{dz_3} F(x^2(\cdot, z_3)) \\
[x_3(z_3) - x_0(z_3)] \frac{d}{dz_3} F(x^1(\cdot, z_3)) - [x_1(z_3) - x_0(z_3)] \frac{d}{dz_3} F(x^3(\cdot, z_3))
\]

Using equation (6) in the general case for \(k\) \(Q\) where

\[
\text{lim} \int_0^3 = 1
\]

\[
\int_0^3 (x^2(z_3)) ds
\]

\[
\int_0^3 (x^1(z_3)) ds
\]

\[
\int_0^3 (x^3(z_3)) ds
\]

\[
\frac{a_2(z_3)}{a_2(z_3)|_{x_2(z_3) \rightarrow x_3(z_3)} = 1, \quad \forall x_3(z_3),
\]

where

\[
\theta_1 = \int_0^1 x^1(s, z_3) ds + \tau_1 \left( \int_0^1 x^2(s, z_3) ds - \int_0^1 x^1(s, z_3) ds \right),
\]

\[
\theta_2 = \int_0^1 x^1(s, z_3) ds + \tau_2 \left( \int_0^1 x^3(s, z_3) ds - \int_0^1 x^1(s, z_3) ds \right), \quad \tau_1, \tau_2 \in (0, 1).
\]

Using equation (6) in the general case for \(k = m\) we obtain

\[
Q_n(x^m(\cdot, \xi), \xi) = F(x_0(\cdot)) + \frac{1}{1} a_1(z_1)[x^m(z_1, \xi) - x_0(z_1)] dz_1 + \frac{1}{1} a_2(z_2)[x^m(z_2, \xi) - x^1(z_2, \xi)] dz_2
\]

\[
\int_0^1 a_{m-1}(z_{m-1}) [x^m(z_{m-1}, \xi) - x^{m-2}(z_{m-1}, \xi)] dz_{m-1}
\]

\[
+ \ldots + \int_0^1 a_m(z_m) [x^m(z_m, \xi) - x^{m-1}(z_m, \xi)] dz_m.
\]

Which permits us to find the value of the last denominator

\[
P_m = 1 + \int_0^1 a_m(z_m) [x^m(z_m, \xi) - x^{m-1}(z_m, \xi)] dz_m.
\]

To do that we use the values of \(a_m(z_m)\) from (7). Then

\[
P_m = 1 - \int_0^1 \frac{d}{dz_m} \left( \int_0^1 a_{m-1}(z_{m-1}) [x^m(z_{m-1}, z_m) - x^{m-2}(z_{m-1}, z_m)] dz_{m-1} \right)
\]

\[
-1 + \int_0^1 a_2(z_2)[x^m(z_2, z_m) - x^1(z_2, z_m)] dz_2 + \ldots + \int_0^1 a_1(z_1)[x^m(z_1, z_m) - x_0(z_1)] dz_1
\]

\[
+ \frac{1}{F(x^m(\cdot, z_m)) - F(x_0(\cdot))} dz_m =
\]
Theorem 3. The results of [14, 15] are employed. For brevity we omit the case \( k > 3 \).

From the interpolation condition (5) we get
\[
0 = \int \frac{1}{\xi} a_2(z_2)[x_m(z_2) - x_1(z_2)]dz_2 + \cdot \cdot + \int \frac{1}{\xi} a_1(z_1)[x_m(z_1) - x_0(z_1)]dz_1 + \cdot \cdot + \int \frac{1}{\xi} a_1(z_1)[x_m(z_1) - x_0(z_1)]dz_1 + \cdot \cdot + \int \frac{1}{\xi} a_1(z_1)[x_m(z_1) - x_0(z_1)]dz_1.
\]

Similarly, one can prove that
\[
\lim_{z_m \to 1} K_m[z_m, x_m] = \lim_{z_m \to 1} \frac{a_{m-1}(z_m)}{a_{m-1}(z_m)[x_{m-1}(z_m) - x_m(z_m)]} = 1, \quad \forall x_m(z_m).
\]

By substituting the obtained value of \( P_m \) into (11) we get the sought for result, namely
\[
Q_n(x^m, \xi) = F(x^m, \xi).
\]

For brevity we omit the case \( k > 3 \), which can be proved similarly.

**Remark.** Conditions (5) formulation of Theorem 2 can be changed to a more general, when the results of [7, 12] are employed.

**Theorem 3.** Assume the conditions of Theorem 2 are valid and \( \xi = 0, x(z) \equiv x, x_i(z) \equiv x_i, i = 0, n. \) Then C–ICF [4] will coincide with the C–ICF (2).

**Proof.** Taking into account the conditions of the theorem 3 we obtain the following form of a continued fraction (4)
\[
Q_n(x) = a_0 + \frac{1}{a_1(z_1)dz_1} + \frac{1}{a_2(z_2)dz_2} + \cdot \cdot + \frac{1}{a_n(z_n)dz_n}.
\]

From the interpolation condition (5) we get
\[
a_0 = f(x_0), \quad \tilde{a}_1 = \int a_1(z_1)dz_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad \tilde{a}_2 = \int a_2(z_2)dz_2 = \frac{1}{x_2 - x_1} \left( -1 + \frac{\tilde{a}_1(x_2 - x_0)}{f(x_2) - f(x_0)} \right).
\]

Using the mathematical induction we intend to show that the formula
\[
\tilde{a}_m = \int a_m(z_m)dz_m = \frac{1}{x_m - x_{m-1}} \left( -1 + \frac{\tilde{a}_{m-1}(x_m - x_{m-2})}{-1} + \frac{\tilde{a}_{m-2}(x_m - x_{m-3})}{-1} + \cdot \cdot \right)
\]
holds true for $m = \frac{3}{n}$.

For $m = 1, 2$ formula (13) is obviously valid. Let’s assume that it is valid for $m = 1, k - 1$. Then for $m = k$ the continued fraction (12) can be written as

$$f(x_k) = a_0 + \left(\frac{x_k - x_0}{1} + \frac{a_1(x_m - x_0)}{f(x_m) - f(x_0)}\right) + \cdots + \left(\frac{x_k - x_0}{1} + \frac{a_1(x_m - x_0)}{f(x_m) - f(x_0)}\right)\cdots.$$

(14)

By inverting the continued fraction (14), we get

$$\tilde{a}_k = \int_0^1 a_k(z_k)dz_k = \frac{1}{x_k - x_{k-1}} - \frac{a_{k-1}(x_k - x_{k-3})}{1 - \frac{a_1(x_k - x_0)}{f(x_k) - f(x_0)}} + \cdots + \frac{a_2(x_k - x_1)}{1 - \frac{a_1(x_k - x_0)}{f(x_k) - f(x_0)}}.$$

(15)

By its form, the right-hand side of (15) coincides with the right-hand side of (3) albeit with different notation, in addition to that, the initial conditions for both formulas are the same. Consequently, formula (13) is the same as formula (3).

References

[1] Makarov, V.L. & Khlobistov, V.V. (1999) Fundamentals of the theory of polynomial operator interpolation. – Kyev: Inst. of Mathematics NAS of Ukraine (in Russian).

[2] Makarov, V.L., Khlobistov, V.V. & Yanovich, L.A. (2000) Interpolation of operators. – Kyev: Naukova dumka (in Russian).

[3] Jones, W. B. & Thron W. J. (1980) Continued Fractions. Analytic Theory and Applications. Encyclopedia of Mathematics and its Applications.

[4] Skorobohat’ko, V.Ya. (1983) Theory of Branching Continued Fractions and Its Application in Computational Mathematics, Moscow: Nauka (in Russian).

[5] Syvavko, M.S. (1994) Integral continued fractions. Kyev: Naukova dumka (in Ukrainian).

[6] Mykhal’chuk, B.R. (1999) Interpolation of nonlinear functionals by integral continued fractions. Ukr. Mat. Zh.Vol. 51, No 3. pp. 364-375 (in Ukrainian).

[7] Makarov, V.L., Khlobistov, V.V. & Mykhal’chuk, B.R. (2003) Interpolation integral continued fractions. Ukr. Mat.Zh. Vol. 55, No 4. 479-488 (in Ukrainian).

[8] Makarov, V.L. & Demkiv, I.I.(2008) A new class of interpolation integral continued fractions. Dopov. Nac. akad. nauk Ukr., No. 11, pp. 17–23 (in Ukrainian).

[9] Makarov, V.L., Khlobistov, V.V. & Demkiv, I.I. (2008) Interpolation integral operator fractions in a Banach space. Dopov. Nac. akad. nauk Ukr., No. 3, pp. 17–23 (in Ukrainian).
[10] Makarov, V. L. & Demkiv, I. I. (2014) Interpolating integral continued fraction of Theile type. Mat. Metody Fiz.–Mekh. Polya. Vol. 57, No. 4, pp. 44–50 (in Ukrainian).

[11] Makarov, V. L. & Demkiv, I. I. (2016) An integral interpolation chain fraction of Thiele type. Dopov. Nac. akad. nauk Ukr., No. 1, pp. 12–18 (in Ukrainian). doi: https://doi.org/10.15407/dopovidi2016.01.012

[12] Makarov, V. L. & Demkiv, I. I. (2016) Abstract interpolation Thiele-type fraction. Mat. Metody Fiz.–Mekh. Polya. Vol. 59, No. 2, pp. 50–57 (in Ukrainian).

[13] Pahirya, M.M. (2016) Approximation of functions by continued fractions. Uzhhorod: Grazda (in Ukrainian).

[14] Averbukh, V.I. & Smolyanov, O.G. (1967) The theory of differentiation in linear topological spaces. Uspehi Mat. Nauk. Vol. 22, No. 6, pp. 201-260 (in Russian).

[15] Makarov, V. L., Khlobystov, V. V., Kashpur, E. F. & Mikhal’chuk, B. R. (2003) Integral Newton-Type Polynomials with Continual Nodes. Ukr. Mat. Zh. Vol. 55, No. 6. pp. 779-789 (in Ukrainian).