Italian Domination of Cartesian Products of Directed Cycles

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Abstract

An Italian dominating function on a (di)graph $G$ with vertex set $V(G)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that for every $v \in V(G)$ with $f(v) = 0$, it is adjacent to a vertex $w$ with $f(w) = 2$. The Roman domination number $\gamma(G)$ is the minimum weight, i.e. $\sum_{v \in V(G)} f(v)$, of a Roman dominating function on $G$. A weak Roman dominating function of a graph $G$, introduced by Domke et al. \cite{4}, is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that for every $v \in V(G)$ with $f(v) = 0$, it is adjacent to a vertex $w$ with $f(w) \neq 0$, and the derived function $g : V(G) \rightarrow \{0, 1, 2\}$ with $g(v) = 1$, $g(w) = f(w) - 1$, and $g(x) = f(x)$ for all $x \neq v, w$ is such that $g$ is a dominating function. The weak Roman domination number $\gamma_w(G)$ is the minimum weight of a weak Roman dominating function.

A 2-rainbow dominating function of a graph $G$, introduced by Brešar et al. \cite{1}, is a function $f : V(G) \rightarrow \mathcal{P}\{1, 2\}$ such that for every $v \in V(G)$ with $f(v) = \emptyset$, it is adjacent to a vertex $w$ with $1 \in f(w)$ and it is adjacent to a vertex $x$ with $2 \in f(x)$. The 2-rainbow domination number $\gamma_{r2}(G)$ is the minimum weight, i.e. $\sum_{v \in V(G)} |f(v)|$, of a 2-rainbow dominating function.

A $\{2\}$-dominating function of a graph $G$, introduced by Domke et al. \cite{4}, is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that for every $v \in V(G)$, its closed neighbourhood has weight at least two. The $\{2\}$-domination number $\gamma_{\{2\}}(G)$ is the minimum weight of a $\{2\}$-dominating function.

An Italian dominating function, or Roman $\{2\}$-dominating function of a graph $G$, introduced by Chellali et al. \cite{2} is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that for every $v \in V(G)$ with $f(v) = 0$, it is adjacent to a vertex $w$ with $f(w) = 2$ or two vertices $x_1, x_2$ such that $f(x_1) = f(x_2) = 1$. The Italian domination number $\gamma_I(G)$ is the minimum weight of a Italian dominating function.

Chellali et al. \cite{2} derived the following chains of inequalities relating these domination parameters.

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Theorem 1.1. [2] For every graph $G$,
\[ \gamma(G) \leq \gamma_r(G) \leq \gamma_I(G) \leq \gamma_R(G) \leq 2\gamma(G), \quad \gamma_I(G) \leq \gamma_2(G). \]

Moreover, Chellali et al. [2] demonstrated the following sharpness results for certain classes of graphs. A graph is a cactus if it is connected and any two of its cycles have at most one vertex in common.

Theorem 1.2. [2] For every tree $T$, $\gamma_I(T) = \gamma_r^2(T)$.

Theorem 1.3. [2] If $G$ is a cactus graph with no even cycle, then $\gamma_I(G) = \gamma_r^2(G) \leq 2\gamma(G)$.

Finally, Chellali et al. [2] determined that the decision problem for Italian domination is NP-complete, even when restricted to bipartite graphs.

Klostermeyer and MacGillivray [10] demonstrated the following lower bound on the Italian domination number of trees.

Theorem 1.4. [10] For a tree $T$ with at least two vertices, then $\gamma_I(T) \geq \gamma(T) + 1$.

Henning and Klostermeyer [8] characterized the trees for which $\gamma_I(T) = \gamma(T) + 1$ and for which $\gamma_I(T) = 2\gamma(T)$.

The Italian domination number has been studied for Cartesian products of cycles by multiple authors. Li et al. [11] determined the following results based on the weak $\{2\}$-domination number.

Theorem 1.5. [11] For $n \geq 3$,
\[ \gamma_I(C_n \square C_3) = \begin{cases} n, & n \equiv 0 \pmod{3}; \\ n+1, & n \not\equiv 0 \pmod{3}. \end{cases} \]

Theorem 1.6. [11] For $n \geq 4$,
\[ \gamma(C_n \square C_4) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil, & n \equiv 0, 1, 3, 4, 5 \pmod{8}; \\ \left\lceil \frac{3n}{2} \right\rceil + 1, & n \equiv 2, 6, 7 \pmod{8}. \end{cases} \]

Stepień et al. [12] obtained the following based on the 2-rainbow domination number.

Theorem 1.7. [12] For $n \geq 5$, $\gamma_I(C_n \square C_5) = 2n$.

Gao et al. [6] determined the following general bounds for Cartesian products of cycles.

Theorem 1.8. [6] For $m \equiv n \equiv 0 \pmod{3}$, $\gamma_I(C_n \square C_m) = \frac{mn}{3}$.

Theorem 1.9. [6] For $m \not\equiv 0 \pmod{3}$ or $n \not\equiv 0 \pmod{3}$,
\[ \left\lfloor \frac{nm}{3} \right\rfloor \leq \gamma_I(C_n \square C_m) \leq \left\lceil \frac{2mn + n + 2m + 1}{6} \right\rceil. \]

Volkmann [13] initiated the study of the Italian domination number in digraphs. For a digraph $D$, we require that for every $v \in V(D)$ with $f(v) = 0$, it has an inneighbour $w$ with $f(w) = 2$ or two inneighbours $x_1, x_2$ such that $f(x_1) = f(x_2) = 1$. We let the maximum out-degree of a digraph be denoted by $\Delta^+(D)$ and the maximum in-degree of a digraph be denoted by $\Delta^-(D)$. Volkmann [13] provided the following preliminary results.

Proposition 1.10. [13] Let $D$ be a digraph of order $n$. Then $\gamma_I(D) \geq \left\lceil \frac{2n}{2 + \Delta^+(D)} \right\rceil$.

Proposition 1.11. [13] Let $D$ be a digraph of order $n$. Then $\gamma_I(D) \leq n$, and $\gamma_I(D) = n$ if and only if $\Delta^+(D), \Delta^-(D) \leq 1$. 

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Proposition 1.12. [13] If $D$ is a directed path or a directed cycle of order $n$, then $\gamma_I(D) = n$.

Kim [9] considered the Italian domination number of the Cartesian and strong products of directed cycles, and proved the following.

Theorem 1.13. [9] If $m = 2r$ and $n = 2s$ for some positive integers $r, s$, then $\gamma_I(C_m \square C_n) = \frac{mn}{2}$.

Theorem 1.14. [9] For an odd integer $n \geq 3$, $\gamma_I(C_2 \square C_n) = n + 1$.

Theorem 1.15. [9] For an integer $n \geq 3$, $\gamma_I(C_3 \square C_n) = 2n$.

Theorem 1.16. [9] For positive integers $m, n \geq 2$, $\gamma_I(C_m \otimes C_n) = \left\lceil \frac{mn}{2} \right\rceil$.

Kim [9] also stated the following conjecture.

Conjecture 1.17. [9] For an odd integer $n$, $\gamma_I(C_4 \square C_n) = 2n + 2$.

In this work, we verify this conjecture and completely settle the question of the Italian dominating number of the Cartesian product of cycles.

2 Main Result

Throughout this section, we consider the vertex set of the directed graph $C_m \square C_n$ to be $\mathbb{Z}_m \times \mathbb{Z}_n$, with arcs of the form $(i, j) \rightarrow (i + 1, j)$ and $(i, j) \rightarrow (i, j + 1)$. We note that $C_m \square C_n$ and $C_n \square C_m$ are isomorphic and will use this fact implicitly in deriving our results. Our approach is to establish properties regarding the set of vertices assigned zero that hold in at least one $\gamma_I(C_m \square C_n)$-function. We first show we can avoid having a line of three vertices assigned zero.

Lemma 2.1. There exists a $\gamma_I(C_m \square C_n)$-function such that no three consecutive vertices (horizontally or vertically) are assigned 0.

Proof. Suppose by way of contradiction that every $\gamma_I(C_m \square C_n)$-function has three consecutive vertices assigned 0. Let $f$ be a $\gamma_I(C_m \square C_n)$-function such that the number of sets of three consecutive vertices assigned 0 is minimized. Then there exist consecutive vertices $(i, k)$, $(i + 1, k)$, $(i + 2, k)$ such that $f((i, k)) = f((i + 1, k)) = f((i + 2, k)) = 0$. Now, since $f$ is a $\gamma_I(C_m \square C_n)$-function, we require that $f((i + 1, k - 1)) = f((i + 2, k - 1)) = 2$. Consider the function $g$ given by

$$g((x, y)) = \begin{cases} 
0, & (x, y) = (i + 2, k - 1); \\
1, & (x, y) = (i + 2, k); \\
\max\{1, f((x, y))\}, & (x, y) = (i + 3, k - 1); \\
f((x, y)), & \text{otherwise}.
\end{cases}$$

If $f((i + 3, k - 1)) \neq 0$, then $g$ is an Italian dominating function with smaller weight than $f$, a contradiction. If either $f((i + 2, k - 2)) \neq 0$ or $f((i + 2, k - 3)) \neq 0$, then $g$ is a $\gamma_I(C_m \square C_n)$-function with fewer sets of three consecutive vertices assigned 0, contradicting the definition of $f$. Otherwise, we have $f((i + 3, k - 1)) = f((i + 2, k - 2)) = f((i + 2, k - 3)) = 0$, and since $f$ is a $\gamma_I(C_m \square C_n)$-function, we require that $f((i + 1, k - 2)) = 2$. Now consider the function $h$ given by

$$h((x, y)) = \begin{cases} 
0, & (x, y) = (i + 2, k - 1); \\
1, & (x, y) \in \{(i + 1, k - 2), (i + 2, k - 2), (i + 2, k), (i + 3, k - 1)\}; \\
f((x, y)), & \text{otherwise}.
\end{cases}$$

Then $h$ is a $\gamma_I(C_m \square C_n)$-function with fewer sets of three consecutive vertices assigned 0, contradicting the definition of $f$. The result follows. \qed
We now show we can further avoid having adjacent vertices assigned zero.

Lemma 2.2. There exists a $\gamma_I(C_m \square C_n)$-function such that no pair of adjacent vertices is assigned 0.

Proof. Suppose by way of contradiction that every $\gamma_I(C_m \square C_n)$-function has a pair of adjacent vertices assigned 0. Let $f$ be a $\gamma_I(C_m \square C_n)$-function with no set of three consecutive vertices assigned 0 (whose existence is guaranteed by the previous lemma) such that the number of pairs of adjacent vertices assigned 0 is minimized. Then without loss of generality, there exist adjacent vertices $(i, k), (i + 1, k)$ such that $f((i, k)) = f((i + 1, k)) = 0$. Then we have $f((i - 1, k)) \neq 0$ and $f((i + 1, k - 1)) = 2$.

Consider the function $g$ given by

$$g((x, y)) = \begin{cases} 1, & (x, y) \in \{(i + 1, k - 1), (i + 1, k)\}; \\ f((x, y)), & \text{otherwise.} \end{cases}$$

If either $f((i + 2, k - 1)) \neq 0$ or $f((i + 2, k - 2)) \neq 0$, then $g$ is a $\gamma_I(C_m \square C_n)$-function with fewer pairs of adjacent vertices assigned 0, contradicting the definition of $f$. Otherwise, we have $f((i + 2, k - 1)) = f((i + 2, k - 2)) = 0$.

Now, consider the function $h$ given by

$$h((x, y)) = \begin{cases} 0, & (x, y) = (i + 1, k - 1); \\ 1, & (x, y) \in \{(i + 1, k), (i + 2, k - 1)\}; \\ f((x, y)), & \text{otherwise.} \end{cases}$$

If $f((i + 1, k - 2)) \neq 0$, then $h$ is a $\gamma_I(C_m \square C_n)$-function with fewer pairs of adjacent vertices assigned 0, contradicting the definition of $f$.

Finally, we may assume $f((i + 2, k - 1)) = f((i + 1, k - 2)) = f((i + 2, k - 2)) = 0$. Then $f((i, k - 2)) \neq 0$ and $f((i + 2, k - 3)) = 2$. By the arguments above, we may assume that $f((i + j, k - 1 - 2j)) = 2$ for all $j \geq 0$. Let $S$ be the set of vertices of the form $(i + 1 + j, k - 1 - 2j)$.

Suppose $S + (0, 1) = S$. It follows that $|S| \geq mn/2$, so $f$ has weight at least $mn$. If $f$ has weight greater than $mn$, then $1$ is an Italian dominating function of smaller weight, a contradiction. Otherwise, $1$ is a $\gamma_I(C_m \square C_n)$-function with no zeros, contradicting our assumption on $f$.

Otherwise, $S + (0, 1) \cap S = \emptyset$. Then by construction Consider the function $p$ given by

$$p((x, y)) = \begin{cases} 1, & (x, y) \in S \cup S + (0, 1); \\ f((x, y)), & \text{otherwise.} \end{cases}$$

Then $p$ is a $\gamma_I(C_m \square C_n)$-function with fewer pairs of adjacent vertices assigned 0, contradicting the definition of $f$. The result follows.

It therefore follows that there exists a $\gamma_I(C_m \square C_n)$-function where the vertices assigned zero form an independent set. We hence establish the independence number $\alpha$ for Cartesian products of cycles.

Lemma 2.3.

$$\alpha(C_m \square C_n) = \begin{cases} \frac{mn}{2}, & m \equiv n \equiv 0 \pmod{2}; \\ \frac{m(n-1)}{2}, & m \equiv 0, n \equiv 1 \pmod{2}; \\ \frac{m(n-1)}{2}, & m \equiv n \equiv 1 \pmod{2}, m \geq n. \end{cases}$$

Proof. $m \equiv n \equiv 0 \pmod{2}$: Suppose $I$ is an independent set of size greater than $mn/2$. Then there exists a row containing more than $m/2$ vertices of $I$. Then there exists a pair of adjacent vertices of $I$, a contradiction. Since there exists an independent set of size $mn/2$ by taking all vertices $(i, j)$ such that $i + j \equiv 0 \pmod{2}$, the result follows.

$m \equiv 0, n \equiv 1 \pmod{2}$: Suppose $I$ is an independent set of size greater than $m(n-1)/2$. Then there exists a column containing more than $(n-1)/2$ vertices of $I$. Then there exists a pair of adjacent vertices of $I$, a contradiction. Since there exists an independent set of size $m(n-1)/2$ by taking all vertices $(i, j)$ such that $i + j \equiv 0 \pmod{2}$ and $j \neq n$, the result follows.
Suppose $I$ is an independent set of size greater than $m(n - 1)/2$. Then there exists a column containing more than $(n - 1)/2$ vertices of $I$. Then there exists a pair of adjacent vertices of $I$, a contradiction. Since there exists an independent set of size $m(n - 1)/2$ by taking all vertices $(i, j) = (x + 2y, x)$, $1 \leq x \leq m$, $1 \leq y \leq (n - 1)/2$, the result follows.

We now have the tools to establish our main result.

**Theorem 2.4.**

\[
\gamma_I(C_m \square C_n) = \begin{cases} 
\frac{mn}{2}, & m \equiv n \equiv 0 \pmod{2}; \\
\frac{m(n+1)}{2}, & m \equiv 0, n \equiv 1 \pmod{2}; \\
\frac{m(n+1)}{2}, & m \equiv n \equiv 1 \pmod{2}, m \geq n.
\end{cases}
\]

**Proof.** Let $f$ be a $\gamma_I(C_m \square C_n)$-function. By Lemma 2.2, there exists a $\gamma_I(C_m \square C_n)$-function with no pair of adjacent vertices assigned zero. Hence, $|f| \geq mn - \alpha(C_m \square C_n)$, which by Lemma 2.4 are precisely the values given in the theorem statement. It remains to establish the existence of an Italian dominating function of this size. Let $I$ be an independent set of size $mn - \gamma_I(C_m \square C_n)$ which is shown to exist by Lemma 2.4. Define $f$ as follows:

\[
f((x, y)) = \begin{cases} 
1, & (x, y) \notin I; \\
0, & (x, y) \in I.
\end{cases}
\]

It is clear that $f$ is an Italian dominating function, as every vertex in $I$ has its two inneighbours not in $I$ by definition, and hence have value 1. The result follows.

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