Free energy and $\theta$ dependence of SU($N$) gauge theories

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We study the dependence of the free energy on the CP violating angle $\theta$, in four-dimensional SU($N$) gauge theories with $N \geq 3$, and in the large-$N$ limit, using the Wilson lattice formulation for numerical simulations, we compute the first few terms of the expansion of the ground-state energy $A$ in substantial agreement with the Witten-Veneziano formulæ which relates $A$ to the topological susceptibility. Our results are consistent with the sine potential, or from a strong-coupling limit. Using the Wilson lattice formulation for numerical simulations, we verify that the topological susceptibility has a nonzero large-$N$ limit, related to sources of supersymmetric SU($N$) gauge theories in various realizations of supersymmetry. The partition function reads:

$$Z(\theta) = \int [dA] \exp(-\int d^4 x L_\theta) \equiv \exp[-V F(\theta)]$$

where $F(\theta)$ is the free (ground state) energy. Witten has argued that in the large-$N$ limit $F(\theta)$ is a multibranch function of the type

$$F(\theta) \equiv N^2 \min_k H[(\theta + 2\pi k)/N]$$

The conjecture was refined and simplified:

$$\Delta F(\theta) = A \min_k (\theta + 2\pi k)^2 + O(1/N).$$

This conjecture has been supported by gauge/string duality arguments.

1. INTRODUCTION

Interest in SU($N$) gauge theories has recently been revived by studies in the context of M-theory, AdS/CFT correspondence, and $\mathcal{N} = 2$ SUSY broken to $\mathcal{N} = 1$. A number of predictions have been put forth, and we have set out to check these predictions by numerical simulation.

One aspect of these studies regards the spectrum of confining strings in four-dimensional SU($N$) gauge theories. A considerable amount of numerical work has been done on this subject (see [4] and references therein). In particular, in Ref. [3] we have presented computations of the string tensions $\sigma_k$, related to sources with $Z_N$ charge $k$, for SU(4) and SU(6) gauge theories. Our results are consistent with the sine formula $\sigma_k/\sigma_1 = \sin(k \pi/N)/\sin(\pi/N)$, with an accuracy of approximately 1% [2%] for SU(4) [SU(6)]; this formula emerges in various realizations of supersymmetric SU($N$) gauge theories, and in the context of M-theory. On the other hand, our results show deviations from Casimir scaling $\sigma_k/\sigma = k(N-k)/(N-1)$, as might be expected from the short distance behaviour of the potential, or from a strong-coupling limit.

In the present work, we study the $\theta$ dependence of SU($N$) gauge theories, for $N \geq 3$ and in the large-$N$ limit, by numerical simulation; for a longer write-up of our work, see [3]. The angle $\theta$ appears in the Euclidean Lagrangian as:

$$L_\theta = \frac{1}{4} F_{\mu\nu}^a(x) F^{a\mu\nu}(x) - i\theta q(x)$$

where $q(x)$ is the topological charge density

$$q(x) = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F^{a\rho\sigma}(x).$$

The conjecture was refined and simplified:

$$\Delta F(\theta) = A \min_k (\theta + 2\pi k)^2 + O(1/N).$$

This conjecture has been supported by gauge/string duality arguments.

*Talk presented by H. Panagopoulos
Monroe Carlo studies of the \( \theta \)-dependence are made very difficult by the complex nature of the \( \theta \) term. In fact \( \theta \neq 0 \) cannot be directly simulated on the lattice. Here we restrict ourselves to relatively small \( \theta \) values, where one may expand:

\[
\Delta F(\theta) = A_2 \theta^2 + A_4 \theta^4 + \ldots
\]  

(7)

\( A_2 \) gives the topological susceptibility, i.e.

\[
A_2 = \chi/2, \quad \chi = \langle Q^2 \rangle_{\theta=0}/V;
\]

where \( Q \) is the topological charge \( Q = \int d^4x q(x) \).

Higher order coefficients in Eq. (7) can be related to higher moments of the probability distribution \( p(Q) \) of the topological charge. For instance,

\[
A_4 = \frac{1}{24V} \left[ (Q^4) - 3\langle Q^2 \rangle^2 \right]_{\theta=0}.
\]

(9)

A convenient parameterization of the free energy using dimensionless scaling coefficients is:

\[
\sigma^{-2} \Delta F(\theta) = \frac{1}{2} C \theta^2 (1 + b_2 \theta^2 + b_4 \theta^4 + \ldots)
\]  

(10)

\( \sigma: \) string tension, \( C = \chi/\sigma^2, \ b_2 = A_4/A_2 \).

Witten’s conjecture implies that \( C \) has a finite nonzero large-\( N \) limit \( C_\infty \) and \( b_{2i} = O(1/N^2) \). In particular one expects that \( b_2 \) is small, and that it should rapidly decrease with increasing \( N \). A nonzero value of \( \chi_\infty \) is essential to resolve the \( U(1) \) problem, and is related to the \( \eta' \) mass \( \xi_\infty \).

\[
\chi_\infty = 2A = \frac{f^2 n_f^2}{4N_f} + O(1/N).
\]

(11)

2. CALCULATION AND RESULTS

We present results for four-dimensional SU(\( N \)) gauge theories with \( N = 3, 4, 6 \). In a nutshell, we: Determine the ratio \( C = \chi/\sigma^2 \); verify the expected behavior \( C \sim A + B/N^2 \), determining the large-\( N \) limit by a fit; obtain a rather accurate estimate of \( C_\infty \); estimate the values of \( b_2 \), which are found to be very small and rapidly decreasing with \( N \), in support of Witten’s conjecture.

Simulation Details: \bullet \ We use the standard Wilson gauge action and the Cabibbo-Marinari updating algorithm (one SU(\( N \)) update \( \rightarrow N(N-1)/2 \) SU(2) subgroup updates). \bullet \ Microcanonical over-relaxation and heat-bath updates are used in a 4:1 ratio. \bullet \ Values for \( \beta \) are chosen in the weak-coupling, (beyond the 1st order phase transition (\( N = 6 \)), beyond crossover (peak of specific heat \( N = 3, 4 \))). \bullet \ Topological quantities are computed via cooling. We measure \( Q \) typically every 100 sweeps. \bullet \ We have employed different lattice sizes, so that finite size effects are under control. Tables of our numerical results can be found in Ref. [3].

\[ \begin{array}{c}
N=6 \\
N=4 \\
N=3
\end{array} \]

\[ \begin{array}{c}
\text{N=6} \\
\text{N=4} \\
\text{N=3}
\end{array} \]

\( \begin{array}{c}
\text{FIGURE 1. Autocorrelation time } \tau_Q \text{ versus } \xi_\sigma \equiv \sigma^{-1/2}. \text{ Dotted lines show the linear fits.}
\end{array} \)

A severe form of critical slowing down is seen in the measurement of \( Q \), at large values of \( N \). Our results (Fig. 1) suggest an exponential behavior:

\[
\ln \tau_Q \approx N (\epsilon_N \xi_\sigma + \epsilon_N) \quad (12)
\]

(\( \xi_\sigma \equiv \sigma^{-1/2}, \text{ and } \epsilon_N, \epsilon_N \text{ are constants with only weak } N \text{-dependence}.\)) Thus, \( \ln \tau_Q \propto N \) (at fixed \( \xi_\sigma \)). This phenomenon is possibly due to the free-energy barriers present. There are no a priori arguments in favor of this behavior; it simply arises naturally from fits, unlike a power law. No similar effect is observed in plaquette or Polyakov line correlations; this suggests a decoupling between topological modes and nontopological ones, such as those determining confining properties.

In Fig. 2, the ratio \( C \) is plotted versus \( \sigma \), to evidentiﬁate possible scaling corrections (expected: \( O(a^2), \) logs). Corrections are indeed observed for \( N = 3 \) and \( N = 4 \), and we must allow for them in a fit, for a continuum extrapolation. Assuming
a linear behavior: \( C = C_{\text{cont}} + u \sigma \), to take into account the leading \( \mathcal{O}(\alpha^2) \) correction, we obtain:

\[
C_{\text{cont}} = 0.0282(12) \quad (N=3) \quad (13)
\]
\[
C_{\text{cont}} = 0.0257(10) \quad (N=4) \quad (14)
\]

For \( N = 6 \), there is no evidence of scaling corrections; we find: \( C_{\text{cont}} = 0.0236(10) \) for \( N = 6 \).

\[
\text{FIGURE 2.} \quad \text{The scaling ratio } C = \chi/\sigma^2 \text{ versus } \sigma. \quad \text{Dotted lines show the linear fits.}
\]

In Fig. 3, our results for \( C_{\text{cont}} \) are plotted versus \( 1/N^2 \) (the expected order of \( 1/N \) corrections). They are clearly consistent with a behavior of the type \( C_{\text{cont}}(N) = C_{\infty} + B/N^2 \). Assuming such a behavior, a fit to \( N = 3, 4, 6 \) results gives

\[
C_{\infty} = 0.0221(13), \quad B = 0.055(19), \quad (15)
\]

This is a rather accurate estimate of large-\( N \) limit of \( C \), and it is clearly nonzero.

To compare with literature, we set a standard value: \( \sqrt{\sigma} = 440 \, \text{MeV} \), for \( N \geq 3 \). We obtain: \( \chi_{\infty}^{1/4} = 170(3) \, \text{MeV} \), and \( \chi_{\infty}^{1/4} = 180(2) \, \text{MeV} \) for \( N = 3 \). From Eq. (14), with actual values of \( f_\pi, m_{\eta'} \), and \( N_f = 3 \), one has 191 MeV. Using Veneziano’s improvement: \( m_{\eta'}^2 \to m_{\eta'}^2 + m_\eta^2 - 2m_\pi^2 \), one obtains 180 MeV.

We now turn to \( b_2 \): The cancellations in its definition necessitate very high statistics. We have obtained sufficiently accurate results for \( N = 3, 4 \), showing scaling. A weighted average, with conservative errors, yields:

\[
b_2 = -0.023(7) \quad (N=3), \quad b_2 = -0.013(7) \quad (N=4), (16)
\]

Reducing statistical error on \( b_2 \) in the \( N = 6 \) case is extremely difficult. We can only give a rough estimate \( b_2 = -0.01(2) \) (\( \gamma = 0.348 \)) in this case, showing that \( b_2 \) is very small.

\[
\text{FIGURE 3.} \quad \text{The scaling ratio } C = \chi/\sigma^2 \text{ versus } 1/N^2. \quad \text{The line is the linear fit to } A + B/N^2.
\]

An alternative to cooling is based on the index theorem, using, e.g., the overlap formulation [7]. The rigorous properties of such an approach would make up for a useful consistency check.

In conclusion, we stress that further investigation of other observables in \( SU(N) \) theories is called for, to test predictions from: SUSY \( SU(N) \), M-theory and gauge/string duality.

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