Grassmannian Topological
Kazama-Suzuki
Models and Cohomology

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We investigate in detail the topological gauged Wess-Zumino-Witten models describing topological Kazama-Suzuki models based on complex Grassmannians. We show that there is a topological sector in which the ring of observables (constructed from the Grassmann odd scalars of the theory) coincides with the classical cohomology ring of the Grassmannian for all values of the level $k$. We perform a detailed analysis of the non-trivial topological sectors arising from the adjoint gauging, and investigate the general ring structure of bosonic correlation functions, uncovering a whole hierarchy of level-rank relations (including the standard level-rank duality) among models based on different Grassmannians. Using the previously established localization of the topological Kazama-Suzuki model to an Abelian topological field theory, we reduce the correlators to finite-dimensional purely algebraic expressions. As an application, these are evaluated explicitly for the $\mathbb{CP}(2)$ model at level $k$ and shown for all $k$ to coincide with the cohomological intersection numbers of the two-plane Grassmannian $G(2, k+2)$, thus realizing the level-rank duality between this model and the $G(2, k+2)$ model at level one.
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1 Introduction

In [1], Kazama and Suzuki discovered a large new class of \(N = 2\) superconformal field theories (SCFTs) by investigating under which conditions the \(N = 1\) super-GKO construction [2] for a \(G/H\) coset conformal field theory actually gives rise to an \(N = 2\) SCFT. Some aspects of these Kazama-Suzuki models, among which are the particularly interesting cosets with \(\text{rk } G = \text{rk } H\) and \(G/H\) Kähler, have been investigated both in the context of \(N = 2\) SCFT [3, 4, 5, 6, 7, 8] and in the context of string model building [9]. These investigations have unravelled the rich and beautiful structure of the Kazama-Suzuki models arising from the interplay between the geometry and topology of coset spaces on the one hand and the \(N = 2\) superconformal algebra on the other.

Certain global aspects of (two-dimensional) field theories are, however, frequently easier to extract from an action-based path integral formulation of the theory. Such a Lagrangian realization of Kazama-Suzuki models was provided in [10, 11]. There it was shown that, under the conditions for \(N = 2\) superconformal symmetry determined in [1], also the Lagrangian realization of the \(N = 1\) coset models as supersymmetric gauged Wess-Zumino-Witten (WZW) models [12] actually possesses the expected \(N = 2\) symmetry.

This Lagrangian realization was investigated further by Nakatsu and Sugawara [13] who, in particular, showed how with a suitable change of variables a direct link could be established between the gauge-theoretic and the more standard conformal field theory realization of these models. The Lagrangian
realization of Kazama-Suzuki models was also used by Henningson in [14, 15] to discuss other aspects of Kazama-Suzuki models which are more transparent in this framework, namely the elliptic genus and mirror symmetry.

Our starting point for this and the companion paper [16] is the work of Witten [10] who analyzed in detail the topologically twisted \( G/H = \mathbb{CP}(1) \) models (a.k.a. \( N = 2 \) minimal models) and their coupling to topological gravity. In [16], a large part of Witten’s analysis was generalized to arbitrary topological \( G/H \) Kazama-Suzuki models.

One of the new features of these models is the presence of topologically non-trivial sectors emerging from the fact that the gauge group of the model is not \( H \) but rather, because of the adjoint action, \( H/Z \) where \( Z = Z(G) \cap H \). It was pointed out by Hori [17] (in the case of bosonic coset models) that taking into account the twisted sectors should be tantamount to a resolution of what is known as the ‘fixed point problem’ [18] in the conformal field theory context. Thus, in [16], building on work by Gawedzki [19] and Hori [17], we presented a generalization of some of the standard properties and constructions of coset models (Polyakov-Wiegmann identities, wave functions, holomorphic factorization, . . . ) to the topologically non-trivial situations one encounters in Kazama-Suzuki models. Using these results, it was then possible to show that the techniques of localization [10] and diagonalization [20] are still applicable in the present context.

In particular, it was shown that the path integral of the topological \( G/H \) model can be localized to that of a bosonic \( H/H \) model (plus quantum corrections coming from the chiral anomaly of the \( G/H \) model). More precisely, one should perhaps refer to these as \( G/\text{Ad}H \) or \( G/(H/Z) \) and \( H/(H/Z) \) models. But in any case, by the results of [20], the latter can in turn be reduced to an Abelian topological field theory, namely a \( T/T \) model, where \( T \) is the maximal torus of \( H \).

At this stage one has simplified the original non-linear and non-Abelian supersymmetric model to such an extent that explicit calculations of correlation functions become not only feasible but even, in some cases, straightforward. The purpose of the present paper is to gain a better understanding of the (topological) Kazama-Suzuki models by performing some explicit calculations in the models based on complex Grassmannians \( G(m,m+n) \).

It is, more or less, known from the conformal field theory literature that the chiral \((c,c)\) ring [3] of these models at level \( k = 1 \) (which becomes the observable ring of the topological model) is just the classical (Dolbeault)
cohomology ring $[3]$.

In the present context it is straightforward to establish that there is indeed a topological sector of the model (the trivial sector) in which the correlation functions of suitable observables (constructed from the Grassmann odd scalars of the twisted model) calculate the cohomology ring (intersection numbers) of the Grassmannian $G(m, m + n)$. This can be established without making use of the simplifying features (localizability, diagonalizability) of the theory mentioned above. Remarkably, with a suitable $k$-dependent normalization of the correlation functions, the fact that these correlation functions reproduce the classical cohomology relations is independent of the level $k$. In this case, we also find an a priori geometrical explanation for the emergence of cohomology from the Kazama-Suzuki model by relating the observables to the classical geometry of vector bundles on $G(m, m + n)$.

Kazama-Suzuki models are also known to possess a level-rank duality symmetry which, in the case at hand, amounts to the statement that the models one obtains by permuting $k$ and $m$ and $n$ are all related. Analyzing the ring structure in general $G(m, m + n)$ level $k$ models, we find here that this is but a particular case of a whole hierarchy of relations between correlation functions in models based on projective spaces and more general Grassmannians, see (7.17). These relations take the form of morphisms between subrings of the chiral rings of different Grassmannian models.

Now, level-rank duality (see e.g. [21, 22, 23]) is something that is usually not particularly transparent in the path integral framework. However, calculations at higher level (which tend to be difficult in conformal field theory) are usually not more difficult than low-level calculations in this framework and thus provide a useful check on level-rank duality. In particular, having reduced a general $G(m, m + n)$ correlation function to a finite dimensional algebraic expression, we find (and prove in detail for $m = 2$) that the chiral ring of the $\mathbb{CP}(m)$ model at level $k$ (in the bosonic sector in which there are no fermionic zero modes) is the cohomology ring of the Grassmannian $G(m, m + k)$, thus realizing the expected equivalence $G(m, m + 1)_k \equiv G(m, m + k)_1$. That the path integral approach is, in a sense, ‘level-rank dual’ to the conformal field theory approach is also epitomized by the fact that the emergence of the classical cohomology ring directly from the level $k = 1$ Grassmannian models in the bosonic sector is slightly less transparent here.

This paper is organized as follows: In section 2 we introduce the topological
Kazama-Suzuki models and recall the relevant results from [10], in particular as regards the localization and diagonalization of these models. We also make some general comments on fermionic zero modes and ghost number anomalies (selection rules) for hermitian symmetric models.

In order to learn how to take into account the topologically non-trivial sectors, we investigate in some detail the global properties of the gauge group $H/Z = S(U(m) \times U(n))/\mathbb{Z}_{m+n}$ in the appendix. Based on that, in section 3 we derive explicit expressions for the Chern classes of $H/Z$-bundles and the corresponding Dolbeault-index.

In section 4, we work out in detail the general structure of the (abelianized) Grassmannian Kazama-Suzuki models. In section 5 we review the classical description of the cohomology of Grassmannians and show that in the topologically trivial sector (with zero modes of the Grassmann odd scalars) the correlation functions reproduce this cohomology. In sections 6 and 7 we consider bosonic correlation functions, show how they can in general be reduced to purely algebraic finite-dimensional expressions, discuss level-rank dualities and work out in detail the equivalence of the correlation functions of the $\mathbb{CP}(2)_k$ model with the cohomology of $G(2,k+2)$.

There are, of course, a large number of things that still remain to be understood, e.g. the non-hermitian-symmetric models and their cohomological (or other) interpretation. One would also like to find, within the present setting, an a priori explanation for the appearance of the classical cohomology of the right coset $G/H$ from the adjoint gauging of $H$ in the bosonic sector of Kazama-Suzuki models (and hence for the emergence of quantum cohomology in suitable perturbations thereof). Here we are only able to provide such an explanation in the fermionic sector (section 5.4).

As regards the topologically non-trivial sectors, in particular the torsion sectors arising from the action of $Z$ on $H$, our results lend further support to the suggestion [14] that taking these into account should be tantamount to resolving the fixed point (field identification) problem [17, 18]. In particular, we find that torsion sectors only arise when $m$ and $n$ are not coprime, and that only one topological sector will ever contribute to a given correlation function unless $k$, $m$ and $n$ all have a common factor (sections 4.4 and 7.1), which is precisely when one encounters problems in the algebraic GKO approach to coset models.

As a resolution of these problems appears to be somewhat more accessible [18] in the $N = 2$ models than in the bosonic models considered in [17], our
detailed analysis of the global aspects of Kazama-Suzuki models in [16] and the present paper should provide a useful basis for testing these ideas and finding a geometrical interpretation of the procedure suggested in [18].

Other open issues are an understanding of level-rank duality at the path integral level (see [21] for some relevant considerations), and questions related to mirror symmetry (and the corresponding topological B-models) in (tensor products of) Kazama-Suzuki models. The results presented here and in [16] are therefore only a preliminary step towards a full understanding of the global properties of Kazama-Suzuki models.

2 The Topological Kazama-Suzuki Model

In this section we will briefly recall the relevant results from [16]. For simplicity of exposition, these will only be presented for $H/Z$-bundles which lift to $H$-bundles. In subsequent sections, these will then be worked out explicitly for the topological Kazama-Suzuki models based on complex Grassmannians, taking into account the additional non-trivial sectors.

2.1 Lagrangian Realization of the Topological Kazama-Suzuki Model

Let $G$ be a compact semi-simple Lie group (which we will also throughout assume to be simply laced), and $H$ a closed subgroup of $G$. A Lagrangian realization for the $N = 1$ super-GKO construction [2] is provided by a gauged supersymmetric WZW model [12] with action

\[
S_{G/H}(g, A, \psi, \bar{\psi}) = S_{G/H}(g, A) + \frac{i}{4\pi} \int_{\Sigma} \bar{\psi} D_{\bar{z}} \psi + D_{\bar{z}} \bar{\psi} \psi
\]

\[
S_{G/H}(g, A) = -\frac{1}{8\pi} \int_{\Sigma} g^{-1} dAg * g^{-1} dAg - i\Gamma(g, A)
\]

\[
\Gamma(g, A) = \frac{1}{12\pi} \int_{M; \partial M = \Sigma} (g^{-1} dg)^3
\]

\[
-\frac{1}{4\pi} \int_{\Sigma} Adg g^{-1} + Ag^{-1} dg + Ag^{-1} Ag.
\]

Here $g$ is a map from the two-dimensional closed surface $\Sigma$ to the group $G$, $A$ is a $\mathfrak{h} \equiv \text{Lie}H$ valued gauge field for the (anomaly free) adjoint subgroup $H$ of $G_L \times G_R$. $\psi$ and $\bar{\psi}$ are Weyl fermions taking values in the complexification.
of \( \mathfrak{t} \), the orthogonal complement to \( \mathfrak{h} \) in \( \mathfrak{g} \equiv \text{Lie} \, G \),

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t} \ , \quad \psi, \bar{\psi} \in \mathfrak{t}^\mathbb{C} \ ,
\]

so that

\[
D_z \psi = \partial_z \psi + [A_z, \psi] \\
D_z \bar{\psi} = \partial_z \bar{\psi} + [A_z, \bar{\psi}] .
\]

Also, a trace (pairing via a non-degenerate bilinear invariant form) will always be understood in integrals of Lie algebra valued fields.

This action has the local gauge symmetry

\[
(g, A, \psi, \bar{\psi}) \rightarrow (h^{-1}gh, h^{-1}Ah + h^{-1}dh, h^{-1}\psi h, h^{-1}\bar{\psi}h) .
\]

and an \( N = 1 \) (actually \( (1,1) \)) supersymmetry (whose precise form we will not need to know).

We now consider (as the most interesting class of Kazama-Suzuki models), pairs of compact Lie groups \((G, H)\) of equal rank, with \( G \) semi-simple and such that the coset space \( G/H \) is Kähler. This is equivalent to the statement that \( H \) is the centralizer of a torus of \( G \), so that it is always of the form \( H = H' \times U(1) \), where \( H' \) is a product of simple factors and possibly further \( U(1) \)'s. In terms of Lie algebras this means that, with respect to a choice of Weyl chamber of \( \mathfrak{g}^\mathbb{C} \), one has a direct sum decomposition

\[
\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{t}^+ \oplus \mathfrak{t}^- \\
[\mathfrak{t}^\pm, \mathfrak{t}^\pm] \subset \mathfrak{t}^\pm \\
[\mathfrak{h}, \mathfrak{t}^\pm] \subset \mathfrak{t}^\pm
\]

such that \( \mathfrak{t}^+ \) is spanned by the root vectors corresponding to the positive roots \( \alpha \in \Delta^+(G/H) = \Delta^+(G) \setminus \Delta^+(H) \), and such that \( \mathfrak{t}^- = \mathfrak{t}^- \). (We hope that temporarily denoting the roots by \( \alpha \) should not give rise to any confusion with the fermionic field \( \alpha \) that will appear below.) It follows, that \( \mathfrak{t}^\pm \) are isotropic with respect to the non-degenerate invariant form on \( \mathfrak{g} \) which we will simply denote by \( \text{Tr} \), i.e. such that \( \text{Tr} ab = 0 \) for \( a, b \in \mathfrak{t}^+ \) or \( a, b \in \mathfrak{t}^- \), and that the decomposition of \( \mathfrak{t}^\mathbb{C} \), the complexified tangent space of \( G/H \) at the identity element, equips \( G/H \) with a Kähler structure. In this case, the action, which in terms of the fields

\[
(\alpha, \beta) = \Pi^\pm \psi \ , \quad (\bar{\alpha}, \bar{\beta}) = \Pi^\pm \bar{\psi} \ ,
\]

7
\[ S_{KS}(g, A, \alpha, \beta) = S_{G/H}(g, A) + \frac{i}{2\pi} \int_{\Sigma} \beta D_z \alpha + \bar{\beta} D_z \bar{\alpha} , \]  

(2.7)

actually has an \( N = 2 \) supersymmetry \[11, 10, 14\] and provides a Lagrangian realization of the \( G/H \) Kazama-Suzuki model. The action at level \( k \) is \( S_{KS}^k = k S_{KS} \).

If \( g \) and \( h \) are such that
\[ [k, k] \subset h , \]  

(2.8)

which implies that the algebras \( k \) are Abelian,

\[ [t^\pm, t^\pm] = 0 \]
\[ [t^+, t^-] \subset h , \]  

(2.9)

then \( G/H \) is what is known as a hermitian-symmetric space. In this case, \( H \) is of the form \( H' \times U(1) \) with \( H' \) containing no further \( U(1) \)-factors. It is Kazama-Suzuki models based on these spaces which are in a sense the easiest to understand and which have received the most attention in the literature (see e.g. \[3, 4, 5\]). Among these are the complex Grassmannians which we will study in detail in later sections.

The action of the topological Kazama-Suzuki model is now obtained from the above by what is commonly referred to as ‘twisting’. In the present case this is tantamount to regarding the fields \( \alpha \) and \( \bar{\alpha} \) as Grassmann odd scalars, and \( \beta = \bar{\beta} = \bar{\beta} \) as anti-commuting \((1, 0)\) and \((0, 1)\) forms respectively,

\[ S_{TKS}(g, A, \alpha, \beta) = S_{G/H}(g, A) + \frac{1}{2\pi} \int_{\Sigma} \beta D_z \alpha + \bar{\beta} D_z \bar{\alpha} , \]  

(2.10)

(here we have absorbed a factor of \( i \) into \( \alpha \) and \( \bar{\alpha} \)).

As a remnant of the \( N = 2 \) symmetry, the topological action has a BRST-like scalar supersymmetry \( \delta \) (this is actually the sum \( \delta = Q + \bar{Q} \) of two nilpotent supersymmetries \( Q \) and \( \bar{Q} \), which are separate invariances of the action), namely

\[ \delta g = g \alpha + \bar{\alpha} g \]
\[ \delta \alpha = -\frac{1}{2} [\alpha, \alpha] \]
\[ \delta \bar{\alpha} = \frac{1}{2} [\bar{\alpha}, \bar{\alpha}] \]
\[ \delta \beta_z = \Pi_-(g^{-1}D_z g - [\alpha, \beta_z]) \]
\[ \delta \bar{\beta}_z = \Pi_+(D_z gg^{-1} + [\bar{\alpha}, \bar{\beta}_z]) \]  

(2.11)
Here $[,]$ denotes a graded commutator so that e.g. $[\alpha, \alpha] = 2\alpha^2$. The terms quadratic in the Grassmann-odd fields are absent if $G/H$ is hermitian symmetric, and this is the case we will henceforth consider.

As this is, admittedly, a rather ham-handed way of introducing the topologically twisted model, and the topological nature of this theory is by no means obvious, let us make the following remarks, aimed at justifying calling this the topologically twisted Kazama-Suzuki model.

1. The twisted action differs from the supersymmetric action (2.1) by a term of the form (spin-connection) $\times R$-current, where the $U(1)_R$ charges of the fields taking values in $\mathfrak{t}^\pm$ are $\pm 1$ respectively. This is analogous to the relation between $N=2$ sigma model and its topological twist, the topological sigma model [24] or $A$-model [25], and simply amounts to shifting the gauge field by half the spin connection.

2. It has been shown in [13] that at the conformal field theory level this modification of the action amounts to twisting the energy-momentum tensor of the theory by the $U(1)$-current $J^{N=2}$ of the $N=2$ algebra in the way defining the $A$-model of any $N=2$ superconformal field theory [26], i.e. one has

$$
\begin{align*}
\mathcal{T}^{\text{KS}}_{zz} &= \mathcal{T}^{\text{KS}}_{zz} + \frac{1}{2} \partial_z J^{N=2}_z \\
\mathcal{T}^{\text{KS}}_{\bar{z}\bar{z}} &= \mathcal{T}^{\text{KS}}_{\bar{z}\bar{z}} + \frac{1}{2} \partial_{\bar{z}} J^{N=2}_{\bar{z}}.
\end{align*}
$$

This is not obvious a priori, as $J^{N=2}$, unlike the $R$-current of the gauge theory, is not only bilinear in the fermions but also contains a purely bosonic part.

3. In the Lagrangian realization we have chosen, the energy-momentum tensor is BRST exact modulo the equations of motion of the gauge field [16]. Hence correlation functions of gauge and BRST invariant and metric independent operators are topological provided that they do not depend on the gauge field $A$.

2.2 Localization and Diagonalization of the Topological Kazama-Suzuki Model

In [16] it was shown that the BRST symmetry of this theory can be used to localize the path integral to that of a bosonic $H/H$- ($H/(H/Z)$-) like model.
Heuristically speaking, the BRST symmetry permits one to linearize the $G/H$-part of the bosonic action, and up to a chiral anomaly the resulting determinant cancels against that arising from the integration over the non-zero-modes of the Grassmann-odd fields (taking values in $(\mathfrak{g}/\mathfrak{h})^C$).

For notational simplicity we assume that the gauge group is (locally) of the form $H = H' \times U(1)$ with $H'$ simple (the generalization to several simple factors of $H'$ or additional $U(1)$’s is immediate). We have a corresponding decomposition of the group valued field $h$ and the gauge field $A$ as $h = (h', \exp i\varphi)$, $A = (A', A^0)$. We denote by $c_G$ and $c_H = c_{H'}$ the dual Coxeter numbers of $G$ and $H$, respectively. We also define the Weyl vector of $G/H$ to be

$$\rho_{G/H} = \frac{1}{2} \sum_{\alpha \in \Delta^+(G/H)} \alpha .$$

As

$$\text{Tr} \rho_{G/H} \alpha = 0 \quad \forall \alpha \in \Delta^+(H'),$$

it can be regarded as the generator of the $U(1)$-factor of $H$. We therefore expand

$$A^0 = \frac{2}{c_G} \rho_{G/H} a ,$$

where $(2/c_G)\rho_{G/H}$ is the fundamental weight of $G$ dual to the single simple root of $G$ which is not a simple root of $H'$, since

$$\text{Tr} \rho_{G/H} \alpha = \frac{c_G}{2} \quad \forall \alpha \in \Delta^+(G/H) .$$

Finally, we denote by $F_a = da$ the $U(1)$-part of the curvature $F_A$ and by $R$ the scalar curvature of the metric implicit in the action (2.10), normalized such that

$$\frac{1}{2\pi} \int_{\Sigma_g} R = \chi(\Sigma_g) = 2 - 2g$$

for a genus $g$ surface $\Sigma_g$. Then the result of [16] is that the topological Kazama-Suzuki model at level $k$ defined by the action (2.10) localizes to the bosonic theory with action

$$S^k_{\text{loc}}(h, A) = (k + c_G - c_{H'})S_{H'/H'}(h', A') + (k + c_G)S_{U(1)/U(1)}(\exp i\varphi, a)$$

$$+ \frac{1}{2\pi} \int \text{Tr} \rho_{G/H} \varphi R$$

$$= (k + c_G - c_{H'})S_{H'/H'}(h', A')$$

$$+ \frac{1}{2\pi} \int \text{Tr} \varphi((k + c_G)F_a + \rho_{G/H} R)$$

(2.18)
(and the integral over the Grassmann odd zero modes, if any, still to be done).

In \[20\], it was shown that the \(H'\) gauge symmetry of the first term \(S_{H'/H'}(h', A')\) can be used to abelianize the theory, i.e. to reduce it to a \(T'/T'\) model (plus quantum corrections) where \(T'\) is the maximal torus of \(H'\) (and hence \(T = T' \times U(1)\) a maximal torus of \(H\)). These quantum corrections are of three kinds. The first one is a shift of the level of the action by \(c_{H'}\). Denoting the \(T'\) and \(T\) valued fields by \(t'\) and \(t = (t', \exp i \varphi) = \exp i \phi\), and letting \(A'\) and \(A = (A', a)\) from now on stand for the torus components of the gauge fields, one thus obtains

\[
S_{\text{eff}}^k(t, A) = (k + c_G)S_{T'/T'}(t', A') + (k + c_G)S_{U(1)/U(1)}(\exp i \varphi, a)
\]

\[
+ \frac{1}{2\pi} \int \text{Tr} \rho_G/H' \varphi R = \frac{1}{2\pi} \int \text{Tr} \rho_G/H' \varphi R
\]

\[
= \frac{1}{2\pi} \int \text{Tr} \phi((k + c_G)F_A + \rho_G/H R) .
\] \hspace{1cm} (2.19)

In both \((2.18)\) and \((2.19)\) we have used the fact that, in a \(U(1)/U(1)\) model the kinetic term \(\phi d^\ast d \phi\) for the compact scalar field \(\phi\) can be absorbed into the \(\phi da\) term by a shift \(a \to a + \ast d \phi\) of the \(U(1)\) gauge field. Alternatively, it suffices to note that the \(a\)-integral imposes \(d \phi = 0\), so that the kinetic term does not contribute anyway.

The second correction is a finite-dimensional determinant arising from the ratio of the functional determinant from the \(H'/T'\) components of the gauge field and the Jacobian (Faddeev-Popov determinant) from the change of variables (choice of gauge) \(h' \to t'\) involved in going from \((2.18)\) to \((2.19)\). This is the Weyl determinant \(\Delta_{H'}^{H'}(t')\) appearing in the finite-dimensional Weyl integral formula relating integrals of class functions on \(H'\) to integrals over \(T'\). It arises as a term \(\sim \int R \log \Delta_{W'}^{H'}(t')\), but because of topological invariance we may treat the fields \(t'\) as position independent. Hence we can say that the localized and abelianized theory is defined by the action \((2.19)\) with the modified \(t'\)-measure

\[
D[t'] \to D[t'](\Delta_{W'}^{H'}(t'))^{\chi(\Sigma_g)/2} .
\] \hspace{1cm} (2.20)

In addition there is a contribution to the action arising from the phase of the determinant one obtains upon Abelianization which is trivial for simply-connected gauge groups but will play a role in the present context. It is a
gauge anomaly term proportional to $\text{Tr} \rho_H F_A$ and, taking this into account, the total abelianized action is

$$S = \frac{1}{2\pi} \int \text{Tr} \left[ \phi(k + c_G) F_A + \rho_{G/H} R \right] + 2\pi \rho_H F_A.$$ \hspace{1cm} (2.21)

We have thus managed to reduce the original non-Abelian supersymmetric topological Kazama-Suzuki model to a much more tractable bosonic Abelian topological field theory. This localized and abelianized action will be the starting point of our investigations in the subsequent sections, the topological sectors being accounted for by suitable constraints on $\int F_A$.

### 2.3 Further Considerations

In performing actual calculations in the model described above, there are some further things that are good to keep in mind, related to chiral anomalies, fermionic zero modes and the range of integration over the torus gauge fields $A = (A', a)$.

Let us start by considering the latter. The functional integral of the Kazama-Suzuki model (2.7) (and its topological partner (2.10)) includes by definition a sum over all the topological sectors of $H/Z$, i.e. over all the isomorphism classes of principal $H/Z$ bundles on $\Sigma$. If $H = H' \times U(1)$ with $H'$ simply connected, then this means that one has to sum over all the Chern classes of the $U(1)$ connection $a$ (with an appropriate normalization determined by the $Z$-action on the $U(1)$-factor, and with the obvious generalization of this statement when there are several $U(1)$ factors as e.g. in the $G/T$ models). However, even if $H'$ is simply connected, so that there are no non-trivial $H'$-bundles, there will in general be a further summation over (torsion) topological sectors coming from the action of $Z$ on $H'$. How this is done will be explained in detail in the appendix and in section 3.

In addition to this summation over topological sectors, arising from the possibility of having non-trivial $H/Z$-bundles, there is a further source of non-trivial topological sectors (in the abelianized theory). Namely, it turns out \cite{20, 27} that global obstructions to diagonalization of $H'$-valued maps $h' \rightarrow t'$ translate themselves into the requirement, that in the Abelianized theory one has to also sum over all the topological sectors (Chern classes) of $T'$-bundles on $\Sigma$. Thus, in the final action (2.19) all the $T$-gauge fields seem to appear on a more or less equal footing.

Nevertheless, the gauge field(s) associated with the explicit $U(1)$ factor(s)
of \( H \) continue to play a special role. To see this, we go back to the original action (2.10) (and to \( H/Z \)-bundles that lift to \( H \) bundles). Correlation functions in this theory are constrained by certain selection rules arising from the chiral anomalies of the bosonic WZW and fermionic parts of the action. These chiral anomalies are associated with the explicit \( U(1) \)-factor(s) of \( H \). Let us consider, for example, the chiral transformation generated by \( \rho_{G/H} \) on the left-moving fermionic fields,

\[
(\alpha, \beta) \rightarrow (e^{i\theta f} \alpha, e^{-i\theta f} \beta) .
\]  

(2.22)

Classically, this is an invariance of the action. However, because of fermionic zero modes, the fermionic measure is not invariant, and one finds a chiral anomaly proportional to the number of \( \alpha \) minus the number of \( \beta \) zero modes. By Riemann-Roch, this is determined by \( \chi(\Sigma) \) and the Chern class \( c_1(a) \) of the \( U(1) \)-bundle. This chiral anomaly has to be compensated for by introducing operators into the correlation functions soaking up these zero modes.

Likewise, one can consider a chiral transformation on the bosonic part of the action. In terms of the variables \( \phi \) of the action (2.19), this amounts to the chiral shift

\[
\phi \rightarrow \phi + 2\theta_b \rho_{G/H} ,
\]

(2.23)

under which the action transforms as

\[
S_{eff}^k(t, A) \rightarrow S_{eff}^k(t, A) + \theta_b \Delta S_{eff}^k
\]

\[
\Delta S_{eff}^k = 2 \text{Tr}(\rho_{G/H}) \left[ \frac{2(k + c_G)}{c_G} c_1(a) + \chi(\Sigma) \right] .
\]

(2.24)

By the Freudenthal - de Vries formula and (2.14) one has

\[
12 \text{Tr}(\rho_{G/H})^2 = c_G d_G - c_{H'} d_{H'} ,
\]

(2.25)

where \( d_G \) denotes the dimension of \( G \). Using this identity and the fact that for hermitian symmetric spaces one has \[7\]

\[
c_G d_G - c_{H'} d_{H'} = 3c_G \text{dim}_\mathbb{C}(G/H) ,
\]

(2.26)

one can rewrite (2.24) as

\[
\Delta S_{eff}^k = -\frac{1}{2}k \chi(\Sigma) \text{dim}_\mathbb{C}(G/H) + (k + c_G) \text{dim}_\mathbb{C}(G/H) \left[ \frac{1}{2} \chi(\Sigma) + c_1(a) \right] .
\]

(2.27)
(see [4.11] for the general expression for Grassmannian $H/Z$ bundles). The first term represents the constant background charge of the (twisted) Kazama-Suzuki models, due to the anomalous transformation behaviour of the $N = 2$ $U(1)$ current $J^{N=2}$ in the twisted theory. The second term is a shift of this background charge due to fermionic zero modes.

Once $c_1(a)$ has been fixed in terms of $\chi(\Sigma)$ by the number of fermionic zero modes in the correlation functions, this gives a $c_1(a)$-independent chiral anomaly which has to be compensated by the chiral charges (weights) of the $g$-dependent observables of the theory.

Consider for example the sector $c_1(a) = -\chi(\Sigma)/2$. In that case, there are generically no fermionic zero modes, and the chiral anomaly of the bosonic part of the action is just

$$\Delta S_{\text{eff}}^k = -\frac{1}{2}k\chi(\Sigma)\dim_{\mathbb{C}}(G/H)$$

in that case. This means that genus zero level one correlation functions can possibly be interpreted as integrals over $G/H$ provided that one identifies an observable with chiral charge $q$ (in units of $(k + c_G)$) with a $2q$-form on $G/H$. This suggests the possibility to understand correlation functions in these models (and those related to them via level-rank duality) in terms of the cohomology of $G/H$, as has indeed been argued in [3, 7]. This also (strongly) suggests that such an interpretation is harder to come by in the non-hermitian symmetric case.

### 3 Global Aspects of Grassmannian Kazama-Suzuki Models

The complex Grassmannian manifold $G(m, m+n)$ of complex $m$-planes in $\mathbb{C}^{m+n}$ can be described as the hermitian symmetric coset space

$$G(m, m+n) = U(m+n)/U(m) \times U(n) \approx SU(m+n)/S(U(m) \times U(n))$$

so that, in the notation of the previous section, $G = SU(m+n)$ and $H = S(U(m) \times U(n))$. As mentioned above, the true gauge group of this model is, as one is gauging the adjoint action and the fermions also live in the adjoint, the group $H/Z$ where $Z = Z(G) \cap H = \mathbb{Z}_{m+n}$. Topological sectors (isomorphism classes of principal bundles of the gauge group) on a two-dimensional closed surface $\Sigma$ are classified by the fundamental group of the
gauge group. Thus, as a first step towards analyzing the Grassmannian Kazama-Suzuki models based on this coset, one needs to understand the global properties of $H/Z$ and, in particular, determine a set of generators for $\pi_1(H/Z)$. Referring for the detailed analysis to the appendix, we quote here the relevant results.

The fundamental group of $H/Z = S(U(m) \times U(n))/\mathbb{Z}_{m+n}$ is found to be

$$\pi_1(H/Z) = \mathbb{Z} \times \mathbb{Z}_{(m|n)} ,$$

(3.2)

where $(m|n)$ denotes the greatest common divisor of $m$ and $n$.

For an element $x$ of the Lie algebra of $H$ we will denote by $\gamma_x$ the path $\gamma_x(t) = \exp 2\pi i xt$ in $H$. Loops in $H/Z$ can be represented by (possibly open) paths in $H$ projecting down to closed loops in $H/Z$. We also regard $H = S(U(m) \times U(n))$ as a subgroup of $SU(m+n)$ and denote by $\alpha^l$ and $\lambda_l$ the fundamental roots and weights of $SU(m+n)$.

The elements of the fundamental group of $S(U(m) \times U(n))/\mathbb{Z}_{m+n}$ can be represented by paths $\gamma_x$ in $S(U(m) \times U(n))$ with

$$x = px_{\text{free}} + qx_{\text{tor}} \quad (p, q) \in \mathbb{Z} \times \mathbb{Z}_{(m|n)} .$$

(3.3)

Here $x_{\text{free}}$, the generator of the free part $\mathbb{Z}$, is given by

$$x_{\text{free}} = a\lambda_{m+n-1} + b\lambda_1 ,$$

(3.4)

with $a, b$ some solution to

$$am + bn = (m|n) ,$$

(3.5)

while the generator of the torsion part $\mathbb{Z}_{(m|n)}$ is

$$x_{\text{tor}} = \frac{1}{(m|n)}(n\lambda_{m+n-1} - m\lambda_1) .$$

(3.6)

Given the above results, it is now straightforward to work out what the allowed Chern classes are in the various twisted sectors of the $G/H$ model. Knowledge of these will allow us to determine two of the ingredients entering into the calculation of correlation functions, namely the index for the fermionic zero modes and the phase of a determinant one obtains upon diagonalization.
3.1 Chern Classes

The topological sectors (isomorphism classes of \( H' = H/Z \) bundles) on a compact closed surface \( \Sigma \) are labelled by pairs

\[ (p, q) \in \mathbb{Z} \times \mathbb{Z}_{(m|n)} \quad , \tag{3.7} \]

the \((p, q)\)'th topological sector corresponding to a transition function

\[ (p, q) \Leftrightarrow (\gamma_{x_{\text{free}}})^p(\gamma_{x_{\text{tor}}})^q \quad , \tag{3.8} \]

along the boundary of some disc \( D \subset \Sigma \). In this sector the Chern classes will be characterized by

\[ \frac{1}{2\pi} \int_{\Sigma} F_A = px_{\text{free}} + qx_{\text{tor}} \quad , \tag{3.9} \]

After diagonalization, the transition functions are given by a product of \( \mathbb{Z} \) with single valued functions taking values in the torus of \( SU(m) \times SU(n) \). The topological information is contained in the winding modes of these maps so that for the non-trivial torus bundles one generates via diagonalization \( [27] \) one can restrict one’s attention to those of the form \( \gamma_{x} \) with \( x \in \Gamma^r(SU(m)) \oplus \Gamma^r((SU(n)) \). (Note that the winding modes around \( \alpha^m \) are already contained in \( \gamma_{x_{\text{free}}} \)).

Thus after diagonalization one has the Chern classes

\[ \frac{1}{2\pi} \int_{\Sigma} F_A = px_{\text{free}} + qx_{\text{tor}} + \sum_{l \neq m} n_l \alpha^l \quad , \tag{3.10} \]

with \( n_l \in \mathbb{Z} \). Rewriting \( x_{\text{free}} \) and \( x_{\text{tor}} \) in terms of roots, this can be written more explicitly as

\[ \frac{1}{2\pi} \int_{\Sigma} F_A = \sum_{l} \left( n_l + \frac{(m|n) + (a-b)(l-m)}{m+n} + q \frac{l-m}{(m|n)} \right) \alpha^l \quad , \tag{3.11} \]

(with the understanding that \( n_m = 0 \)). This shows quite clearly which torus bundles arise in a given topological sector \((p, q)\). It also makes manifest the invariance of the parametrization of the topological sectors under \( q \rightarrow q + c(m|n) \), \( c \in \mathbb{Z} \).

On the other hand, in terms of the alternative parametrization \( [A.78] \) one has

\[ \frac{1}{2\pi} \int_{\Sigma} F_A = r\alpha^m + s\lambda_{m+n-1} \quad . \tag{3.12} \]
The claim is that, modulo the root lattice \( \Gamma_r^{r,(m)} \oplus \Gamma_r^{r,(n)} \), (3.9) and (3.12) are equivalent. In order to compare them, it is helpful to split both expressions into their \( SU(m) \times SU(n) \times U(1) \)-parts. Using various Lie algebra identities, one obtains (modulo the root lattice \( \Gamma_r^{r,(m)} \oplus \Gamma_r^{r,(n)} \))

\[
(3.9) = p\frac{(m|n)}{mn}\lambda_m \\
+ (pb - \frac{qm}{(m|n)})\lambda_1^{(m)} + (pa + \frac{qn}{(m|n)})\lambda_n^{(n)} .
\]

and

\[
(3.12) = r\frac{(m+n)+sm}{mn}\lambda_m \\
+ r\lambda_1^{(m)} + (r+s)\lambda_n^{(n)} .
\]

This now permits an explicit comparison of the two parametrizations. In establishing their equivalence, the crucial observation is that there is an \((m|n)\)-fold degeneracy in the parametrisation of the \(U(1)\)-term of (3.14), i.e. for each value of \(r(m+n)+sm\) there are \((m|n)\) distinct pairs \((r,s)\) with \(r \in \mathbb{Z}, s \in \mathbb{Z}_{m+n}\) giving that value. Thus, in particular, a fixed value of the \(U(1)\) Chern class (we will be working with later on in evaluating correlation functions, as this fixes the number of fermionic zero modes) corresponds to a fixed value of \(p\), with \(q\) varying freely, while in terms of \((r,s)\) one is dealing with an \((m|n)\)'s worth of distinct values of \(r\) and \(s\).

### 3.2 The Phase of the Determinant

In calculating the ratio of determinants arising from diagonalization, one obtains a phase factor \(e^{i\Gamma}\),

\[
i\Gamma = \frac{1}{2\pi} \int \text{Tr} 2\rho_H F_A \log(-1) ,
\]

where \(\rho_H = \rho^{(m)} + \rho^{(n)}\) is the Weyl vector (half the sum of the positive roots) of \(SU(m) \times SU(n)\). Writing

\[
\log(-1) = i\pi + 2\pi is \quad s \in \mathbb{Z} ,
\]

and using the representation (3.10) for \(\int F_A\), this can be written more explicitly as

\[
i\Gamma = 2\pi i(2s+1) \text{Tr} \rho_H(px_{free} + qx_{tor} + \sum_{l \neq m} n_l \alpha^l) .
\]
As the Weyl vector $\rho^{(m)}$ of $SU(m)$ can alternatively be written as the sum
of the fundamental weights of $SU(m)$,

$$\rho^{(m)} = \sum_{i=1}^{m-1} \lambda_i^{(m)}, \quad (3.18)$$

(and likewise for $SU(n)$), the third term will always give an integral multiple
of $2\pi i$ and hence not contribute to the phase. The contribution from the
first two terms can be readily determined using (3.11) and

$$\text{Tr} \rho_H \alpha^l = 1 \quad l \neq m$$

$$\text{Tr} \rho_H \alpha^m = 1 - \frac{2}{m+n}, \quad (3.19)$$

and one finds

$$i\Gamma = 2\pi i (2s+1) \left[ \frac{p}{2}(a(n-1) + b(m-1)) + \frac{q}{2(m|n)}(n(n-1) - m(m-1)) \right]. \quad (3.20)$$

As the term in square brackets is always in $\frac{1}{2}\mathbb{Z}$, the ambiguity in the branch
of the logarithm is irrelevant and one actually has

$$i\Gamma = 2\pi i \left[ \frac{p}{2}(a(n-1) + b(m-1)) + \frac{q}{2(m|n)}(n(n-1) - m(m-1)) \right]. \quad (3.21)$$

Thus the whole effect of the phase $\Gamma$ can be incorporated in the Abelianised
action as a gauge anomaly term $\sim \text{Tr} \rho_H F_A$,

$$S = \frac{1}{2\pi} \int \text{Tr} \left[ \phi \left( (k + c_G)F_A + \rho_G/H R \right) + 2\pi \rho_H F_A \right], \quad (3.22)$$

as anticipated in (2.21).

### 3.3 Index Theorem and Fermionic Zero Modes

We now consider a fixed principal $H' = H/Z$ bundle $P'$ labelled by $(p, q) \in \mathbb{Z} \times \mathbb{Z}_{(m|n)}$. We consider the $\bar{\partial}$-part of the fermionic action given by $\int \bar{\partial}_\alpha \alpha^+ + \beta_\alpha \bar{\partial}_A \alpha^+$. The index of this fermionic system is given by

$$\text{Index}(\bar{\partial}_A) = \frac{1}{2} \dim_C \mathfrak{t}^+ \chi(\Sigma) + c_1(V^+) \quad (3.23)$$

where $V^+$ is the vector bundle

$$V^+ = P' \times_{H'} \mathfrak{t}^+ \quad (3.24)$$
associated to $P'$ via the adjoint action of $H'$ on $\mathfrak{t}^+$. As the complex dimension of $\mathfrak{t}^+$, arising as the trace of the identity matrix of $\text{End}_C \mathfrak{t}^+$, is $mn$, the first term of (3.23) is simply $mn(1 - g)$. The second term can be calculated as the trace of the adjoint action of $\int F_A/2\pi$ on $\mathfrak{t}^+$. As $\int F_A$ in (3.9) is in the torus, this action is diagonal. Since on a root vector $E_\alpha \in \mathfrak{t}^+$ corresponding to $\alpha \in \Delta^+(G/H)$ one has
\[ [x, E_\alpha] = (\text{Tr} x \alpha) E_\alpha \text{ , (3.25)} \]
the first Chern class can be calculated to be
\[ c_1(V^+) = \text{Tr} \text{ad}_{\mathfrak{t}^+}(px_{\text{free}} + qx_{\text{tor}}) \]
\[ = \sum_{\alpha \in \Delta^+(G/H)} \text{Tr} \alpha(px_{\text{free}} + qx_{\text{tor}}) \]
\[ = \text{Tr} 2\rho_{G/H}(px_{\text{free}} + qx_{\text{tor}}) \]
\[ = \text{Tr} 2\rho_{G/H}px_{\text{free}} \]
\[ = \text{Tr}(m + n)p\lambda_m x_{\text{free}} \]
\[ = p(m + n)(m/n) = p(m|n) \text{ . (3.26)} \]
This also follows readily from the parametrisation (3.13), in terms of which only the $\lambda_m$-term contributes. In any case the index is
\[ \text{Index}(\bar{\partial}_A) = mn(1 - g) + p(m|n) \text{ . (3.27)} \]
As expected, the index does not depend on the torsion class $q$ but only on the winding number $p$. Combining this with a vanishing theorem, one concludes that generically (in genus zero) there are no fermionic zero modes in the topological sector $p = (g - 1)mn/(m|n)$ ($p = -mn/(m|n)$), with $q$ arbitrary. Modulo the root lattice of $SU(m) \times SU(n)$, the corresponding generator $px_{\text{free}}$ is equal to $(-\lambda_m)$. In fact, it is easy to see directly that they have the same winding number $\nu$ (A.54),
\[ \nu(\frac{mn}{(m|n)}x_{\text{free}}) = \frac{mn}{(m|n)} \frac{(m|n)}{m + n} = \frac{mn}{m + n} \]
\[ = \text{Tr} \lambda_m^2 = \nu(\lambda_m) \text{ . (3.28)} \]
Again, this also follows immediately from the parametrisation (3.13). Thus, in this sector the $U(1)$ gauge field can be represented as $a\lambda_m$ with $\int da = -2\pi$.  

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Thus there are an \((m|n)\)'s worth of torsion sectors contributing to scalar correlation functions (meaning, correlation functions of the purely bosonic operators to be discussed below). In this sector with no zero modes the Chern classes can be written as

\[
\frac{1}{2\pi} \int F_A = -\lambda_m + qx_{\text{tor}} \\
= -\lambda_m + \frac{q}{(m|n)}(n\lambda_{m+n-1} - m\lambda_1) \\
= -\lambda_m + \frac{q}{(m|n)}(n\lambda_{n-1}^{(m)} - m\lambda_1^{(m)})
\]

(3.29)

Finally, in this sector only the torsion part of \(\Gamma\) contributes to the phase,

\[
e^{i\Gamma} = (-1)^{\frac{q}{2m|n}((n-1)-m(m-1))}.
\]

(3.30)

4 The Topological Grassmannian Kazama-Suzuki Model

4.1 The Abelianized Action

It follows from (2.19), that the Abelianized action of the Grassmannian topological Kazama-Suzuki model (without the phase factor) is (we now denote \(S_{\text{eff}}^{k}\) simply by \(S\))

\[
S = \frac{1}{2\pi} \int (k + m + n) \text{Tr} \phi F_A + \text{Tr} \rho_{G/H} \phi R,
\]

(4.1)

where the torus valued field \(t\) is parametrized as \(t = \exp i\phi\), \(\phi\) a compact scalar field defined modulo \(2\pi\Gamma\), the root (and integral) lattice of \(SU(m+n)\). Thus, if we expand \(\phi\) in terms of the simple roots of \(SU(m+n)\),

\[
\phi = \sum_{l=1}^{m+n-1} \phi_l \alpha^l,
\]

(4.2)

the components \(\phi_l\) have period \(2\pi\). In order to disentangle the \(U(1)\) from the \(SU(m) \times SU(n)\) parts of the action, it will turn out to be convenient to decompose \(\phi\) orthogonally (with respect to the trace Tr) as

\[
\phi = \phi' + \frac{m+n}{mn} \lambda_m \phi_m \\
\phi' = \phi^{(m)} + \phi^{(n)},
\]

(4.3)
where $\phi^{(m)}$ ($\phi^{(n)}$) has an expansion in terms of simple roots of $SU(m)$ ($SU(n)$). This will lead to an almost complete decoupling of the $SU(m)$ and $SU(n)$ sectors and thus greatly simplifies the analysis.

For example, for the path integral measure we will need the Weyl determinants of $SU(m)$ and $SU(n)$. The former is (including a factor $1/m!$, $m!$ the order of the Weyl group of $SU(m)$)

$$
\Delta^{(m)}_W(t) = \frac{1}{m!} \prod_{\alpha \in \Delta^+(SU(m))} 4 \sin^2 \frac{1}{2} \text{Tr} \alpha \phi^{(m)}
$$

and thus depends only on $\phi^{(m)}$ (and likewise for $SU(n)$). The normalization is such that

$$
\int_{U(1)^{m-1}} dt \Delta^{(m)}_W(t) = 1 .
$$

Also, this decomposition is such that under the chiral shift (chiral $U(1)$ transformation)

$$
\phi \rightarrow \phi + 2\theta \rho_{G/H} ,
$$

which we will consider later on in the context of selection rules, the field $\phi'$ is inert while $\phi_m$ transforms as

$$
\phi_m \rightarrow \phi_m + mn \theta .
$$

By topological invariance (or by integration over the one-form modes of the gauge field $A$ which imposes $d\phi = 0$) we can assume that $\phi$ is constant. Hence we can use the condition (3.10) on the Chern classes to rewrite the first part of the action as

$$
\frac{1}{2\pi} \int \text{Tr} \phi F_A = \text{Tr} \phi (px_{\text{free}} + qx_{\text{tor}} + \sum_{l \neq m} n_l \alpha^l) .
$$

By the same token, the scalar curvature term, to which only $\phi_m$ will contribute as $\rho_{G/H}$ is proportional to $\lambda_m$ (A.13), becomes

$$
\frac{1}{2\pi} \int \text{Tr} \rho_{G/H} \phi R = \frac{m+n}{4\pi} \int \phi_m R
$$

$$
= (m+n)(1-g)\phi_m ,
$$

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so that the total action can be written as

\[ S = (k + m + n) \text{Tr} \phi(px_{\text{free}} + qx_{\text{tor}} + \sum_{l \neq m} n_l \alpha_l') + (m + n)(1 - g) \phi_m . \]  

(4.10)

Under the chiral shift (1.6) this action changes as

\[ S \rightarrow S + [kp(m|n) + (m + n)((1 - g)mn + p(m|n))] \theta . \]  

(4.11)

In the sector with generically no zero modes \((px_{\text{free}} \sim (g - 1)\lambda_m)\), one sees that the shift \(k \rightarrow k + cG\) cancels against the scalar curvature term and one obtains

\[ S = k(g - 1) \phi_m + (k + m + n) \text{Tr} \phi'( qx_{\text{tor}} + \sum_{l \neq m} n_l \alpha_l') . \]  

(4.12)

Thus, under the chiral shift (4.6) the action transforms as

\[ S \rightarrow S + kmn(g - 1) \theta , \]  

(4.13)

in agreement with the general result (2.24).

4.2 Bosonic Observables

In [16], we also determined the bosonic observables of the Grassmannian Kahama-Suzuki models. These are functionals of the group valued field \(g\) which are invariant under the BRST transformation

\[ \delta g = g\alpha + \bar{\alpha}g \]  

(4.14)

and \(H\) gauge transformations. It follows readily from (A.23) that the upper left-hand \(U(m)\) block \(g^{(m)}\) of \(g\) and the lower right-hand \(U(n)\) block \((g^{-1})^{(n)}\) of \(g^{-1}\) are \(\delta\)-invariant,

\[ \delta g^{(m)} = \delta(g^{-1})^{(n)} = 0 . \]  

(4.15)

As the remaining components of \(g\) are paired exactly with the fields \(\alpha\) and \(\bar{\alpha}\) under \(\delta\), there are no further BRST invariants one can construct from \(g\). It remains to impose gauge invariance. \(H\) gauge transformations act on \(g^{(m)}\) and \((g^{-1})^{(n)}\) by conjugation with \(h^{(m)}\) and \(h^{(n)}\) respectively. Hence, a complete set of gauge and BRST invariant operators can be obtained as
traces of $g^{(m)}$ and $(g^{-1})^{(n)}$. For the cohomological interpretation of the operators it turns out to be convenient to consider as the basic set of operators the traces of $g^{(m)}$ and $(g^{-1})^{(n)}$ in the exterior powers of the fundamental representations of $U(m)$ and $U(n)$ respectively. We thus define

$$O_a(g) := \text{Tr} \wedge^a g^{(m)}, \quad a = 1, \ldots, m,$$

$$\bar{O}_b(g) := \text{Tr} \wedge^b (g^{-1})^{(n)}, \quad b = 1, \ldots, n.$$  (4.16)

Since $\det g = 1$, there is one relation between these operators, namely

$$\det g^{(m)} \equiv O_m = \det((g^{-1})^{(n)}) \equiv \bar{O}_n.$$  (4.18)

Altogether, one therefore has $\text{rk}(G) = m + n - 1$ independent basic gauge and BRST invariant operators generating the ring of observables of the topological Kazama-Suzuki model.

In the localized and abelianized theory, these observables reduce to the elementary symmetric functions of the diagonal entries of $t$. Thus, in terms of

$$t = \exp i\phi = \text{diag}(t_1, \ldots, t_{n+m})$$

$$t_1 = e^{i\phi_1}$$

$$t_k = e^{i(\phi_k - \phi_{k-1})}, \quad k = 2, \ldots, m + n - 1$$

$$t_{m+n} = e^{-i\phi_{m+n-1}},$$  (4.19)

one has

$$O_a(\phi) = \sum_{1 \leq i_1 < \cdots < i_a \leq m} t_{i_1} \cdots t_{i_a},$$

$$\bar{O}_b(\phi) = \sum_{m+1 \leq j_1 < \cdots < j_b \leq m+n} (t_{j_1} \cdots t_{j_b})^{-1}.$$  (4.20)

Let us consider some examples. In the $\mathbb{C}P(1)$ model there is one and only one scalar operator, namely

$$O_1 = g_{11}.$$  (4.22)

In the localized theory this becomes

$$O_1 = e^{i\phi_1}.$$  (4.23)
In the \( \mathbb{C}P(2) \) model, there are two scalar operators, namely

\[
\begin{align*}
\mathcal{O}_1 &= g_{11} + g_{22} \to e^{i\phi_1} + e^{i(\phi_2 - \phi_1)} \\
\mathcal{O}_2 &= g_{11}g_{22} - g_{12}g_{21} \to e^{i\phi_2}.
\end{align*}
\]

(4.24)

And quite generally one finds that in the \( G(m, m+n) \) models the operator \( \mathcal{O}_m = \det g^{(m)} \) reduces to

\[
\mathcal{O}_m = \mathcal{O}_n \to e^{i\phi_m}.
\]

(4.25)

As a final example, we consider the simplest Grassmannian which is not a projective space, namely \( G(2,4) \). In that case one has three operators. In the localized and abelianized theory they are

\[
\begin{align*}
\mathcal{O}_1 &= e^{i\phi_1} + e^{i(\phi_2 - \phi_1)} \\
\mathcal{O}_2 &= \mathcal{O}_2 = e^{i\phi_2} \\
\mathcal{O}_1 &= e^{i\phi_3} + e^{i(\phi_2 - \phi_3)}.
\end{align*}
\]

(4.26)

We now need to determine the weight of these operators under the chiral shift (4.6). It follows from (4.7) and the explicit expressions for the operators given above, that

\[
\begin{align*}
\mathcal{O}_a(\phi + 2\theta\rho_{G/H}) &= e^{ian\theta} \mathcal{O}_a(\phi) \\
\bar{\mathcal{O}}_b(\phi + 2\theta\rho_{G/H}) &= e^{ibm\theta} \bar{\mathcal{O}}_b(\phi),
\end{align*}
\]

(4.27)

so that the weights (chiral charges) of the operators are

\[
\begin{align*}
w(\mathcal{O}_a) &= an & a &= 1, \ldots, m \\
w(\bar{\mathcal{O}}_b) &= bm & b &= 1, \ldots, n.
\end{align*}
\]

(4.28)

Hence, separating the \( \phi_m \)-dependence of these operators from that on \( \phi' \), we can write

\[
\begin{align*}
\mathcal{O}_a(\phi) &= e^{i\frac{a}{m}\phi_m} \mathcal{O}_a(\phi^{(m)}) \\
\bar{\mathcal{O}}_b(\phi) &= e^{i\frac{b}{m}\phi_m} \bar{\mathcal{O}}_b(\phi^{(n)}),
\end{align*}
\]

(4.29)

with \( \mathcal{O}_m = \bar{\mathcal{O}}_n = 1. \)
We are now in a position to discuss the selection rules for a general (bosonic) correlator in the topological $G(m, m+n)$ Kazama-Suzuki model at level $k$ which we will denote by

$$\langle O_{1}^{r_{1}} \ldots O_{m}^{r_{m}} \bar{O}_{1}^{s_{1}} \ldots \bar{O}_{n}^{s_{n}} \rangle_{G(m, m+n)_{k}} , \quad (4.30)$$

or simply by $\langle \ldots \rangle_{k}$ if there can be no confusion about which Grassmannian we are discussing. Here we take all the exponents $r_{a}$ and $s_{b}$ to be positive, as the operators can take on the value zero prior to abelianization. Also, as the correlation functions are, by standard arguments, independent of the positions of the operator insertions, there is no need to indicate these insertion points explicitly in (4.30).

One selection rule follows from considering the chiral shift (4.6) whose effect on the action we have determined before (2.27, 4.13). Thus, taking into account the weights (4.28) of the operators, the condition for the correlation function (4.30) to be non-zero is

$$\sum_{a=1}^{m} ar_{a} + \sum_{b=1}^{n} bms_{b} = (1 - g)kmn . \quad (4.31)$$

Since the left-hand side is manifestly positive, we see that this condition can only be satisfied if $g = 0$ (and possibly for the partition function in genus one, but that one vanishes because of fermionic zero modes), so we will henceforth work in genus $g = 0$.

For the $\mathbb{CP}(m) = G(m, m+1)$ models, (4.31) reduces to

$$\sum_{a=1}^{m} ar_{a} = km . \quad (4.32)$$

Let us make some observations here concerning this selection rule.

1. The right-hand side is the complex dimension of the Grassmannian $G(m, m+k)$. As we will recall below, the cohomology of $G(m, m+n)$ is generated by $m$ Chern classes $c_{a}$ of form-degree $2a$. With the (tentative) identification $O_{a} \sim c_{a}$ the selection rule (4.32) can thus be regarded as the condition that the correlator represent a top-form on $G(m, m+k)$. 

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2. We also note that (4.31) reduces to (4.32) if all the $s_\beta = 0$, i.e. if one considers chiral correlation functions in the $G(m, m+n)_k$ model of only the $O_a$ and not the $\bar{O}_b$. This is a special case of a more general relation between correlation functions in $G(m, m+n)$ and $\mathbb{CP}(m)$ models which we will derive below.

4.4 $\mathbb{Z}_m \times \mathbb{Z}_n$ Selection Rules

A further selection rule arises from a discrete symmetry of the Kazama-Suzuki model. The original Kazama-Suzuki action naively has a discrete $\mathbb{Z}_m \times \mathbb{Z}_n$-symmetry (acting in the obvious way via right multiplication on the group element $g$). Less naively, this symmetry could be broken by non-trivial bundles interfering with the $\mathbb{Z}_m \times \mathbb{Z}_n$ directions. In the present case, these discrete transformations do not see the winding number sector $(-\lambda_m)$, as that one lives entirely in the $U(1)$-direction which does not interfere with the $Z(SU(m)) \times Z(SU(n))$-transformations. However, the torsion sector could break this symmetry, and this is in fact what we will find below. Even in the absence of torsion, however, these considerations lead to a refinement of the selection rule arising from the invariance of the measure and the action under the transformation $g \to gz$.

As the generator of $\mathbb{Z}_m$ is the fundamental weight $\lambda_{m-1}^{(m)}$ of $SU(m)$, one needs to consider the behaviour of the theory under

$$\phi_{(m)} \to \phi^{(m)} + 2\pi \lambda_{m-1}^{(m)}.$$

(4.33)

There are three contributions to consider: the action, the measure, and the observables.

As $Tr \lambda_m \lambda_{m-1}^{(m)} = 0$, $Tr \lambda_{m-1}^{(m)} \alpha^l \in \mathbb{Z}$ for $l \neq m$, it is obvious that the only possible source of non-invariance of the action (4.10) is the torsion term. Thus, under this shift the action changes by

$$2\pi(k + m + n) Tr \lambda_{m-1}^{(m)} qx_{tor} = -\frac{2\pi qk}{(m|n)} \mod 2\pi \mathbb{Z}.$$  

(4.34)

On the other hand, the observables transform as

$$O_a \to e^{2\pi i \frac{a}{m}} O_a.$$  

(4.35)

The barred observables are invariant. Finally, the Weyl determinant (and hence the measure) is also invariant. Thus, putting everything together one
finds the condition
\[ \sum_{a=1}^{m} \frac{ar_a}{m} - \frac{qk}{(m|n)} \in \mathbb{Z}, \]
(4.36)
or
\[ \sum_{a=1}^{m} ar_a - \frac{q}{(m|n)} km = cm \quad c \in \mathbb{Z}. \]
(4.37)
Completely analogously, one finds that the condition arising from \( \mathbb{Z}_n \)-transformations is
\[ \sum_{b=1}^{n} bs_b + \frac{q}{(m|n)} kn = dn \quad c \in \mathbb{Z}. \]
(4.38)
Combining this with the selection rule
\[ \sum nar_a + \sum mbs_b = kmn, \]
(4.39)
one obtains the relation
\[ c + d = k. \]
(4.40)
This refinement of the selection rule has some interesting consequences. For instance, it implies that in the case when there is no torsion, \( (m|n) = 1 \), the barred and unbarred operators decouple completely at level \( k = 1 \), as the only allowed possibilities are then \( c = 0, 1 \).

At this point one sees something special happening when \( k \) and \( m \) and \( n \) all have a common divisor. Namely, one can ask the question if a given correlator can receive contributions from more than one torsion sector. For that, one needs to be able to solve (4.36) simultaneously for two different values \( q \) and \( q' \). Equivalently, one is trying to solve
\[ \frac{q - q'}{(m|n)} k \in \mathbb{Z}. \]
(4.41)
If \( (m|n)|k) = 1 \), this has no solution for \( q \neq q' \) in the allowed range. In fact, if \( (m|n) \) and \( k \) are coprime, then
\[ (q - q')k = c(m|n) \quad c \in \mathbb{Z} \]
(4.42)
implies that \( (q - q') \) is a multiple of \( (m|n) \) (and hence zero). We will see below that in many cases of interest only \( q = 0 \) is allowed.

The situation is clearly more complicated when \( k \) and \( (m|n) \) are not coprime, a manifestation of similar problems encountered in the conformal field theory approach in this case [18].
Finally, one could also consider $\mathbb{Z}_{m+n} = Z(G)$ transformations, but unsurprisingly it turns out that the resulting selection rules are automatically satisfied as a consequence of the relations obtained from considering $\mathbb{Z}_m \times \mathbb{Z}_n$ plus the usual $U(1)$ chiral selection rule.

4.5 Example: The $\mathbb{C}P(1)_k$ Model and the Cohomology of $\mathbb{C}P(k)$

As a warm-up exercise, we now discuss the (almost trivial) relation between the $\mathbb{C}P(1)_k$ model and the cohomology of $\mathbb{C}P(k)$. Of course, hardly any of the gyrations of the previous sections are needed for this example.

As there is only one operator $O_1$ in this model, the bosonic genus zero correlation functions $\langle O_1^r \rangle_k$ are completely determined by the selection rule (4.32) which in this case reads $r = k$. Thus, with an appropriate normalization, we obtain [10]

$$\langle O_1^r \rangle_{\mathbb{C}P(1)_k} = \delta_{r,k} \ .$$

(4.43)

This means that the observable ring of the $\mathbb{C}P(1)_k$ model is described by the single relation

$$O_1^{k+1} = 0 \ .$$

(4.44)

This is precisely the characterization of the classical cohomology ring of $\mathbb{C}P(k)$, with the identification of $O_1$ with the first Chern class $c_1$ of the (dual of the) tautological line bundle on $\mathbb{C}P(k)$ (see section 3.1). In this case, it is also readily seen that perturbing the action by the two-form descendant of $O_1$ [13, 14] deforms the classical cohomology of $\mathbb{C}P(k)$ to its quantum cohomology ring with defining relation $c_1^{k+1} = q$ (with $q$ the deformation parameter). This and its counterpart for other Grassmannians will be discussed in more detail elsewhere.

5 Cohomology and Chiral Rings I: $\alpha$ Zero Modes

In this section we will establish the first, and very direct, relationship between the observable ring of the topological Grassmannian Kazama-Suzuki models and the cohomology ring of the corresponding Grassmannians. We will first recall the classical description of this cohomology ring in terms of the Chern classes of the tautological vector bundles. We show that one can construct operators from the Grassmann odd scalars $\alpha$ and $\bar{\alpha}$ which are BRST and gauge invariant (hence observables) and which satisfy the cohomology relations purely algebraically, i.e. even outside of correlation
functions. We then show that these relations continue to be satisfied inside genus zero correlation functions, the only contribution coming from the topological sector \( p = q = 0 \) in which these \( \alpha \)-operators decouple completely from the scalar operators discussed above. This result can be established directly from the original action, i.e. without referring to the localization and abelianization of the theory. Finally, we provide a geometrical explanation of these results by identifying the matrix \( \alpha \bar{\alpha} \) with the curvature form of the tautological vector bundle. Although we will only spell this out in detail for the Grassmannian models, it will be evident that much of what follows is valid for all the hermitian symmetric Kazama-Suzuki models.

5.1 Cohomology of Grassmannians

We briefly recall the defining relations for the cohomology of a complex Grassmannian \( G(m, m + n) \) (see e.g. [29]). Given a complex rank \( m \) vector bundle \( E \) over some manifold \( M \), we denote by

\[
c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_m(E)
\]

its total Chern class. \( c(E) \) satisfies the Whitney formula

\[
c(E \oplus F) = c(E)c(F)
\]

(5.2)

Assume now that this vector bundle can actually be regarded as a sum of line bundles (splitting principle). Then the Whitney formula implies that \( c(E) \) factorizes as

\[
c(E) = (1 + x_1)(1 + x_2)\ldots(1 + x_m)
\]

(5.3)

where the \( x_i \in H^2(M) \) represent (perhaps symbolically) the first Chern classes of the individual line bundles. Alternatively, if one represents the Chern classes by their Weil representatives (in terms of curvatures of connections), the \( x_i \) can be regarded as the eigenvalues of the curvature matrix of \( E \). Clearly, therefore, the \( c_l(E) \) are just the \( l \)th elementary symmetric functions of the \( x_i \),

\[
c_l(E) = \sum_{1 \leq i_1 < \ldots < i_l \leq m} x_{i_1} \ldots x_{i_l}
\]

(5.4)

The Grassmannian \( G(m, m + n) \) is the space of all complex \( m \)-planes in \( \mathbb{C}^{m+n} \), i.e. every point of \( G(m, m + n) \) represents a particular \( m \)-plane. Attaching the corresponding \( m \)-plane to each point, one obtains a rank \( m \)
vector bundle $E$ over $G(n, n + k)$, the tautological vector bundle. Likewise, one can associate to each point the $n$-plane orthogonal to the $m$-plane it represents. In this way, one obtains an $n$-plane bundle $F$. The fact that at each point $x \in G(m, m + n)$ the fibres $E_x$ and $F_x$ are canonically related by $F_x = C^{m+n}/E_x$ can be rephrased as the statement that $E$ and $F$ fit into an exact sequence of vector bundles

$$O \to E \to G(m, m + n) \times C^{m+n} \to F \to 0 \quad . \quad (5.5)$$

As the bundle in the center is trivial, it follows from the Whitney formula that the total Chern classes of $E$ and $F$ are related by

$$c(E)c(F) = 1 \quad . \quad (5.6)$$

The same thing is true for the sequence of dual bundles,

$$0 \to F^* \to G(m, m + n) \times C^{m+n} \to E^* \to 0 \quad . \quad (5.7)$$

$$c(E^*)c(F^*) = 1 \quad . \quad (5.8)$$

Clearly, Chern classes of $E^*$ give rise to cohomology classes of $G(m, m + n)$. What is less evident (but nevertheless true) is that the cohomology is generated by these Chern classes and that the only relations they satisfy are those following from $(5.8)$ [29].

To be more specific, we denote by $c_i$, $i = 1, \ldots, m$ the Chern classes of $E^*$ and by $d_j$, $j = 1, \ldots, n$ those of $F^*$. Then the above relation reads more explicitly

$$(1 + c_1 + \ldots + c_m)(1 + d_1 + \ldots + d_n) = 1 \quad . \quad (5.9)$$

This gives rise to $m+n$ relations of form degree $2, 4, \ldots 2(m+n)$ respectively. The first $n$ of these are of the form $d_j = \ldots$ and can hence be used to express the $d_j$ entirely in terms of the Chern classes $c_i$ of $E^*$. The remaining $m$ relations are polynomial relations among the $c_m$ themselves. These will give rise to (and are equivalently described by) all the top-degree relations of form degree $2mn$ one can obtain from them. There will be one less top-degree relation than the number of top-form monomials one can build from the $c_i$ (meaning that the relations imply the obvious fact that the top-cohomology group is one-dimensional). The remaining overall scale is then fixed by the normalization

$$\int_{G(m, m+n)} c_m^n = 1 \quad . \quad (5.10)$$
E.g. for $\mathbb{C}P(m) = G(m, m+1)$ one has

$$\left(1 + c_1 + \ldots + c_m\right)(1 + d_1) = 1 \ , \quad (5.11)$$

implying first of all that $d_1 = -c_1$ and then that $c_l = c_1^l$. Thus in this case, all the $c_i$ with $i \geq 2$ can also be eliminated directly, leaving one with the one (obvious) relation $c_1^{m+1} = 0$ for $c_1$. This description of the cohomology is suitable for comparison with the $\mathbb{C}P(m)\,\mathbb{I}$ Kazama-Suzuki models, in which a priori we have $m$ operators $O_i$ corresponding to the Chern classes $c_i$. For comparison with the $\mathbb{C}P(1)\,\mathbb{I}_k$ models, on the other hand, it is more convenient (but completely equivalent, of course) to describe the cohomology of $\mathbb{C}P(k)$ using

$$\left(1 + c_1\right)(1 + d_1 + \ldots + d_k) = 1 \ , \quad (5.12)$$

leading to the elimination of the $d_j$ by $d_j = (-1)^j c_1^j$ and the remaining relation $c_1^{k+1} = 0$.

5.2 $\alpha$-Observables

We will now show that from the Grassmann odd scalars $\alpha$ and $\bar{\alpha}$ we can construct observables $C_l$ and $D_l$ satisfying the relation $(5.9)$. First of all, we recall that in the hermitian symmetric models $\alpha^+$ and $\bar{\alpha}^-$ are BRST invariant,

$$\delta \alpha = \delta \bar{\alpha} = 0 \ , \quad (5.13)$$

and that they have components $\alpha_{ij}$ and $\bar{\alpha}_{ji}$ with $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$. We can thus consider the $(m \times m)$ and $(n \times n)$ matrices

$$C = \alpha \bar{\alpha} \ , \quad D = \bar{\alpha} \alpha \ . \quad (5.14)$$

Under $H$ gauge transformations, these matrices transform in the adjoint representation so that e.g. the operators

$$C_a = \text{Tr}_{\Lambda^a} C \ , \quad 1 \leq a \leq m$$
$$D_b = \text{Tr}_{\Lambda^b} D \ , \quad 1 \leq b \leq n \quad (5.15)$$

(cf. $(4.17)$) are BRST and gauge invariant and hence qualify as observables of the topological Grassmannian Kazama-Suzuki model. We will now show that these operators satisfy $(5.9)$, i.e. that one has

$$\left(1 + C_1 + \ldots + C_m\right)(1 + D_1 + \ldots + D_n) = 1 \ . \quad (5.16)$$
To that end we recall that, for any matrix $M$, one has

$$\det(1 + M) = \sum_k \text{Tr}_{\wedge^k} M. \quad (5.17)$$

Moreover, the block-diagonal matrix $\text{diag}(C, D)$ can be written as the (graded) commutator of $\alpha$ and $\bar{\alpha}$,

$$\text{diag}(C, D) = \left[ \alpha, \bar{\alpha} \right]. \quad (5.18)$$

Hence, since $\text{Tr}$ in any representation vanishes on commutators, one has

$$1 = \sum_{p=0}^{m+n} \text{Tr}_{\wedge^p} [\alpha, \bar{\alpha}]$$

$$= \det(1 + [\alpha, \bar{\alpha}]) = \det(1 + \text{diag}(C, D))$$

$$= \det(1 + C) \det(1 + D)$$

$$= \sum_{a=0}^m \text{Tr}_{\wedge^a} C \sum_{b=0}^n \text{Tr}_{\wedge^b} D$$

$$= (1 + C_1 + \ldots + C_m)(1 + D_1 + \ldots + D_n). \quad (5.19)$$

We thus see that in the Grassmannian Kazama-Suzuki models one can construct operators which satisfy the cohomology relations of the corresponding Grassmannian purely classically and algebraically, highlighting the intimate relationship between cohomology and Kazama-Suzuki models.

### 5.3 Correlation Functions and Cohomology

We will now investigate to which extent and under which conditions the relation (5.16) continues to hold at the level of correlation functions. Let us therefore consider a general correlator involving the $\alpha$-operators $C_a$ and $D_b$ as well as the scalar operators $\mathcal{O}_a$ and $\bar{\mathcal{O}}_b$,

$$\langle \prod_{a=1}^m C_{\gamma_a} \prod_{b=1}^n D_{\delta_b} \prod_{a=1}^m \mathcal{O}_{\alpha_a} \prod_{b=1}^n \bar{\mathcal{O}}_{\beta_b} \rangle_{G(m,m+n)_k}. \quad (5.20)$$

This correlation function is constrained by conditions (selection rules) arising from the chiral anomalies of the bosonic and fermionic parts of the action respectively. As the operators $C_a$ and $D_b$ do not depend on the group-valued field $g$, the former simply reads (we are now using the unlocalized form of
the action - its transformation under the chiral shift is obtained from (4.11) by dropping the second term, the quantum correction)

\[ kp(m|n) + \sum_{a=1}^{m} anr_a + \sum_{b=1}^{n} bms_b = 0 \quad . \]  (5.21)

This implies that \( p \) is non-positive. In the sector \( p(m|n) = mn(g - 1) \) in which there are generically no fermionic zero modes this selection rules reduces to that of the purely bosonic correlation functions discussed above, namely (4.31). Here, on the other hand, we are interested in the situation in which there are \( \alpha \) zero modes (and hence generically no \( \beta \) zero modes).

This drastically changes the implications of (5.21) in the present context. Namely, since one wants \( \alpha \) zero modes, i.e. the index \( mn(1 - g) + p(m|n) \) to be positive, one is led to consider the case \( p = g = 0 \) with \( mn \) zero modes, one of each species (component) \( \alpha_{ij} \) (there will be no \( \beta \) zero modes in that case). We will see below that they can be interpreted directly as tangents to \( G(m, m + n) \).

Then the only way (5.21) can be satisfied is when there are no scalar insertions whatsoever, \( r_a = s_b = 0 \). Thus the scalar operators \( O_a \) and \( \bar{O}_b \) decouple completely from correlation functions involving \( \alpha \) zero modes in this sector. Furthermore, the dependence on the level \( k \) has disappeared. As the fermions are inert under the action of \( \mathbb{Z}_m \times \mathbb{Z}_n \), the refined selection rules (4.36) now imply that the torsion \( q = 0 \) as well.

Each operator \( C_a (D_b) \) soaks up \( a \) (\( b \)) \( \alpha \) and \( \bar{\alpha} \) zero modes. Hence another condition required for the non-vanishing of (5.20) is

\[ \sum_{a=1}^{m} a\gamma_a + \sum_{b=1}^{n} b\delta_b = nm \quad . \]  (5.22)

With the interpretation of \( C_a \) as a \((a,a)\)-form on \( G(m, m + n) \), this is precisely the condition required to interpret the fermionic correlator as a top-form on \( G(m, m + n) \).

It now follows that if these conditions are satisfied and if one normalizes the correlation functions in agreement with (5.10), i.e. according to

\[ \langle C^m_{mn} \rangle_{G(m,m+n)_k} = 1 \quad , \]  (5.23)

that the non-vanishing correlation functions are exactly the intersection
numbers of $G(m, m + n)$,

$$\langle \prod_{a=1}^{m} C_{a}^{\gamma_{a}} \prod_{b=1}^{n} D_{b}^{\delta_{b}} \rangle_{G(m,m+n)} = \int_{G(m,m+n)} \prod_{a=1}^{m} c_{a}^{\gamma_{a}} \prod_{b=1}^{n} d_{b}^{\delta_{b}} .$$

(5.24)

Notice that this result follows directly from the classical properties of the observables and the selection rules and that, in particular, there was no need to invoke things like localization or diagonalization in order to evaluate the correlation functions.

We also want to draw attention to the fact that this result is independent of the level $k$. Thus for every $k$ there is a topological sector whose observable ring reproduces precisely the classical cohomology ring of $G(m, m + n)$. We will find a (slightly weaker) counterpart of this observation for bosonic correlation functions below. Here we just want to point out that this result is rather striking from the conformal field theory point of view, from which it is known that the level $k = 1$ chiral rings are the cohomology rings of the ‘target space’, while the higher level rings are rather different (unless they can be related to $k = 1$ rings via level-rank duality, as for the models based on the projective spaces $\mathbb{CP}(m)$). It would be nice to understand this result in terms of Gepner’s dihedrality and the spectral flow operation considered by Nakatsu and Sugawara.

5.4 A Geometrical Interpretation

While we have seen above that the $\alpha$-operators $C_a$ and $D_b$ can be identified (already classically) with the Chern classes $c_a$ and $d_b$, this does not yet explain why such an identification is possible. Here we will provide such an explanation by showing that $C$ and $D$ can be regarded as the curvature forms of the tautological vector bundles $E^*$ and $F^*$ respectively.

Let us start in complete generality by looking at a compact group $G$ and a compact reductive subgroup $H$. Thus, in terms of Lie algebras one has the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$$

(5.25)

with

$$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k} .$$

(5.26)

We will first calculate the curvature of a (particular) connection on the principal $H$-bundle $G \to G/H$ and then specialize to pairs $G$ and $H$ satisfying...
the Kazama-Suzuki conditions and then further such that $G/H$ is hermitian symmetric.

Let $\Theta = g^{-1}\delta_G g$ be the left-invariant Maurer-Cartan form on $G$, and denote by $\Theta^h$ and $\Theta^k$ the components in $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Then it is known that $\Theta^h$ is a connection on $G \to G/H$. Indeed it is easy to see that this is the case. There are two conditions an $\mathfrak{h}$-valued one-form on the total space $P$ of a principal $H$-bundle has to satisfy in order to qualify as a connection. The first is that evaluated on the vectors generating the $H$-action the connection should produce the corresponding Lie algebra element. This is true for $\Theta^h$ as the vertical part of the Maurer-Cartan form of $P = G$. The second is that it should transform in the adjoint representation as one moves up and down the fibres by the principal right action. Again this is clearly satisfied by $\Theta^h$, $\Theta^h_{gh} = h^{-1}\Theta^h g h$.

Now one can calculate the curvature of this connection using the Maurer-Cartan equation

$$\delta_G \Theta = -\frac{1}{2} [\Theta, \Theta] . \quad (5.27)$$

First of all, one has

$$\delta_G \Theta^h = -\frac{1}{2} [\Theta, \Theta]^h = -\frac{1}{2} [\Theta^h, \Theta^h] - \frac{1}{2} [\Theta^k, \Theta^k]^h . \quad (5.28)$$

The curvature of $\Theta^h$ is by definition the horizontal exterior derivative of $\Theta^h$. As $\Theta^h$ is vertical and $\Theta^k$ is horizontal, calculating the curvature is tantamount to dropping the first term on the right hand side of (5.28),

$$F(\Theta^h) := \delta_G^{\text{hor}} \Theta^h = \delta_G \Theta^h + \frac{1}{2} [\Theta^h, \Theta^h] = -\frac{1}{2} [\Theta^k, \Theta^k]^h . \quad (5.29)$$

Now let us make the further assumption (part of the definition of a Kazama-Suzuki model) that $\mathfrak{k}^c$ decomposes into two complex conjugate subalgebras $\mathfrak{t}^\pm$. In this case, the curvature can be written as

$$F(\Theta^h) = -[\Theta^+, \Theta^-]^h . \quad (5.30)$$

which begins to resemble the expression $[a, \bar{a}]$. To make this correspondence completely explicit, consider now the case $G' = U(m+n)$ and $H = U(m) \times U(n)$. Then $G/H = G(m,m+n)$ and what we are interested in is the
curvature of the tautological $m$-plane bundle over $G(m,m+n)$ or, what is the same, the curvature of the principal (and tautological) $U(m)$-bundle over $G(m,m+n)$. Consider (5.30) in this case. Let the indices $i, i_k, \ldots$ run from 1 to $m$, and indices $j, j_k, \ldots$ from $m+1$ to $m+n$. Then the Maurer-Cartan forms $\Theta^\pm$ have the index structure $\Theta^+_{ij}$ and $\Theta^-_{ji}$. Thus the $U(m)$ and $U(n)$ parts of the curvature read

$$F_{i_1i_2} = -\Theta^+_{i_1j} \Theta^-_{j_2},$$
$$F_{j_1j_2} = -\Theta^-_{j_1i} \Theta^+_{i_2},$$

(5.31)

or

$$F^{U(m)} = -\Theta^+ \Theta^-,$$
$$F^{U(n)} = -\Theta^- \Theta^+,$$

(5.32)

with

$$\text{Tr} F_{i_1i_2} = - \text{Tr} F_{j_1j_2}$$

(5.33)

representing the $U(1)$-part of the curvature.

If we now identify

$$\alpha \leftrightarrow i\Theta^+, \quad \bar{\alpha} \leftrightarrow i\Theta^-,$$

(5.34)

then we have the desired correspondence

$$C = \alpha \bar{\alpha} = F^{U(m)}$$
$$D = \bar{\alpha} \alpha = F^{U(n)},$$

(5.35)

explaining why the operators $C_a$ and $D_b$ represent the Chern classes $c_a$ and $d_b$.

For completeness' sake we also mention that in the hermitian symmetric case the BRST operator $\delta$ can be identified with the exterior covariant derivative $\delta^\text{hor}_G$. In fact, since

$$\delta_G \Theta^\pm = -[\Theta^b, \Theta^\pm],$$
$$\delta_G F^b = -[\Theta^b, F^b],$$

(5.36)

one has

$$\delta^\text{hor}_G \Theta^\pm = 0 \leftrightarrow \delta \alpha = \delta \bar{\alpha} = 0,$$
$$\delta^\text{hor}_G F^b = 0 \leftrightarrow \delta C = \delta D = 0.$$
6 General Structure of Bosonic Correlation Functions

We now reconsider the bosonic correlation functions (4.30). Their structure is much richer (and more intricate) than that of their fermionic counterparts discussed above, in particular because of the subtle level dependence of the correlation functions. As a consequence, in this case, we will have to work a bit harder to establish the correspondence between these correlation functions and intersection numbers of Grassmannians.

We thus consider a general correlation function

\[ \langle \prod_{a=1}^{m} O_{a}^{r_{a}} \prod_{b=1}^{n} \tilde{O}_{b}^{s_{b}} \rangle_{k,q} \]  

(6.1)

in the \( G(m, m+n) \) level \( k \) model in the topological sector determined by

\[ \frac{1}{2\pi} \int F_{A} = -\lambda_{m} + qx_{\text{tor}} + \sum_{l \neq m} n_{l} \alpha_{l} \]  

(6.2)

(actually, we will be summing over the \( n_{l} \), of course, and at some later stage also over the allowed values of \( q, q = 0, \ldots, (m|n) - 1 \)).

6.1 The \( U(1) \)-Part of the Action

As explained above, cf. (4.12), in genus zero and in the topological sector(s) of interest the action reduces to (treating \( \phi \) as a constant)

\[ S = -k \text{Tr} \lambda_{m} \phi + (k + m + n) \text{Tr} \phi'(qx_{\text{tor}} + \sum_{l \neq m} n_{l} \alpha_{l}) . \]  

(6.3)

As \( \phi_{m} \) appears only in the first term of (6.3), to which \( \phi' \) does not contribute, and the \( \phi_{m} \) dependence of the observable \( O_{a} \) can be extracted as a prefactor \( \exp i \frac{\pi}{k} \phi_{m} \) (likewise for the barred observables), the \( \phi_{m} \) integral can be done directly and its sole purpose in life is to impose the familiar selection rule

\[ \sum_{a=1}^{m} nar_{a} + \sum_{b=1}^{n} mbs_{b} = kmn . \]  

(6.4)

Furthermore, the \( SU(m) \) and \( SU(n) \) sectors of the theory now couple only through the selection rule (and possibly through \( x_{\text{tor}} \)), and in much of what follows we will only deal with the \( \phi^{(m)} \)-dependence explicitly, the \( \phi^{(n)} \)-dependence being analogous.
6.2 The \( SU(m) \times SU(n) \)-Part of the Action

Let us now take a look at the third term of (6.3), i.e. the term
\[
(k + m + n) \text{Tr} \phi' \sum_{l \neq m} n_l \alpha^l .
\]
(6.5)
The summation over the \( n_l \) can be regarded as a summation over the root lattice \( \Gamma^{r,(m)} \oplus \Gamma^{r,(n)} \) of \( SU(m) \times SU(n) \). It will thus impose the condition that \((k + m + n)\phi'\) be an element of \( 2\pi \) times the dual lattice, i.e. the weight lattice \( \Gamma^{w,(m)} \oplus \Gamma^{w,(n)} \).
\[
\sum_{n_l} \Rightarrow \quad \frac{1}{2\pi} \phi^{(m)} = \frac{1}{k + m + n} \lambda^{(m)} \text{ mod } \alpha^{(m)}
\]
\[
\frac{1}{2\pi} \phi^{(n)} = \frac{1}{k + m + n} \lambda^{(n)} \text{ mod } \alpha^{(n)} .
\]
(6.6)
Here and in the following we use \( \lambda^{(m)} (\alpha^{(m)}) \) to denote a generic element of the weight lattice \( \Gamma^{w,(m)} \) (root lattice \( \Gamma^{r,(m)} \)). Thus the integral over \( \phi' \) becomes a sum over a compact subset of the weight lattice of \( SU(m) \times SU(n) \) given by
\[
\frac{k + m + n}{2\pi} \phi' \in \left( \Gamma^{w,(m)}/(k + m + n)\Gamma^{r,(m)} \right) \times \left( \Gamma^{w,(n)}/(k + m + n)\Gamma^{r,(n)} \right) .
\]
(6.7)
In analogy with the observations in [20], this implies that correlation functions can be written in terms of sums over level \( k + c_G - c_H \) integrable representations of \( H \).

In a sense, our description of the observables and correlation functions is quite far removed from the description in the conformal field theory literature based on the cohomology of affine algebras, see e.g. [2, 8, 13]. However, it should be possible to establish a correspondence between the two as the integrable level \( k + c_G - c_H \) representations of \( H \) are also the fundamental building blocks of observables in the Lie algebraic approach to \( N = 2 \) coset models.

6.3 The Torsion-Part of the Action

We now take a brief look at the torsion part of the action, the second term of (6.3). Plugging \((k + m + n)\phi' = 2\pi(\lambda^{(m)} + \lambda^{(n)})\) into that term, one finds
\[
(k + m + n) \text{Tr} \phi' q x_{tor} = \frac{2\pi q}{(m|n)} \text{Tr}(n\lambda^{(n)} \lambda_{m+n-1} - m\lambda^{(m)}\lambda_1) .
\]
(6.8)
Expanding

\[ \lambda^{(m)} = \sum n^i \lambda_i^{(m)} \]
\[ \lambda^{(n)} = \sum n^{m+j} \lambda_j^{(m)} , \]  
and using

\[ \text{Tr} \lambda_i^{(m)} \lambda_1 = \frac{m - i}{m} \]
\[ \text{Tr} \lambda_j^{(n)} \lambda_{m+n-1} = \frac{j}{n} , \]

one finds that (modulo \(2\pi\mathbb{Z}\)) this term becomes

\[ (k + m + n) \text{Tr} \phi' q_{\text{tor}} = \frac{2\pi q}{(m|n)} (\sum i n_i + \sum j n_{m+j}) . \]  
Combining this with the phase factor, this can be written as

\[ \Gamma + (k + m + n) \text{Tr} \phi' q_{\text{tor}} = \frac{2\pi q}{(m|n)} (\sum i (n_i - 1) + \sum j (n_{m+j} - 1)) . \]  
If there were no further \(q\)-dependence, the sum over \(q\) would now impose the condition that the weight \(\lambda\) of \(SU(m) \times SU(n)\) determined by

\[ \lambda + \rho_H = \lambda^{(m)} + \lambda^{(n)} \]  
actually gives a representation of

\[ (SU(m) \times SU(n))/\mathbb{Z}(m|n) . \]  
This fact, that not all a priori possible integrable weights of \(SU(m) \times SU(n)\) do actually appear when \((m|n) \neq 1\), is one of the manifestations of the fixed point problem of conformal field theory.

### 6.4 Fourier Expansion

In order to simplify the calculations, one may try to reduce the \(\phi^{(m)}\)-sum over \(\Gamma^{w,(m)}/(k + m + n)\Gamma^{r,(m)}\) to a sum over \(\Gamma^{r,(m)}/(k + m + n)\Gamma^{r,(m)}\). The idea is to make use of the fact that the ‘difference’ between the root and the weight lattice is

\[ \Gamma^{w,(m)}/\Gamma^{r,(m)} = Z(SU(m)) = \mathbb{Z}_m \]  

\[ 39 \]
to write an element $\lambda^{(m)} \in \Gamma^{w,(m)}$ in some (natural) way as the sum of an element $\alpha^{(m)}$ of $\Gamma^{r,(m)}$ and a rest, the latter exponentiating to a non-trivial element of $\mathbb{Z}_m$. As the center of $SU(m)$ is generated by $\lambda^{(m)}_{m-1}$ (which can be thought of as an element of the Lie algebra of $SU(m+n)$ by embedding it as $(\lambda^{(m)}_{m-1},0^n)$), consider the decomposition

$$\lambda^{(m)} = \alpha^{(m)}_\lambda + p\lambda^{(m)}_{m-1}. \quad (6.16)$$

Here $p_\lambda \in \mathbb{Z}$. First of all, every element of the form of the right hand side of this equation is an element of $\Gamma^{w,(m)}$: the second term certainly is, and the first term is as well since every root is a weight. However, if $p_\lambda$ is a multiple of $m$, then $p_\lambda\lambda^{(m)}_{m-1}$ is actually an element of $\Gamma^{r,(m)}$. In fact,

$$m\lambda^{(m)}_{m-1} = \sum_{i=1}^{m-1} i\alpha^i. \quad (6.17)$$

Thus, to avoid an overcounting of elements of $\Gamma^{w,(m)}$ in (6.16), we impose the condition that $p_\lambda \in [0, \ldots, m-1]$. Finally, we need to establish that every element of $\Gamma^{w,(m)}$ can be written in the form (6.16) for some $\alpha^{(m)}_\lambda$ and $p_\lambda$. By what we know so far, this decomposition will, if it exists, then be unique.

Thus consider a fundamental root $\lambda^{(m)}_k$. This can be written in the above form as

$$\lambda^{(m)}_k = \sum_{i>k} (k-i)\alpha^i + (m-k)\lambda^{(m)}_{m-1}. \quad (6.18)$$

One way to verify this is to check that the right hand side satisfies the defining relations $\text{Tr} \alpha^i \lambda^{(m)}_k = \delta^i_k$ for $i = 1, \ldots, m-1$. Thus the coefficient $p_\lambda$ in this case is simply $(m-k)$ - which lies in the desired range. When using this result to obtain a decomposition of this type for a general element $\lambda^{(m)} \in \Gamma^{w,(m)}$, care needs to be taken to take the resulting $p_\lambda$, which at first takes the value $\sum n^k(m-k)$ for $\lambda^{(m)} = \sum n^k\lambda^{(m)}_k$, back into the range $[0, \ldots, m-1]$ using (6.17). But this can be done in a unique way, and this establishes the existence and uniqueness of the decomposition (6.16).

Now, the $SU(m)$-part of the partition function involves a sum over the elements

$$\frac{1}{2\pi i} g^{(m)} = \frac{1}{k + m + n} \lambda^{(m)} \mod \Gamma^{r,(m)}$$

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\[
\frac{1}{k + m + n} (\alpha^{(m)} + p \lambda_{m-1}^{(m)}) \mod \Gamma^{r,(m)} \\
= \frac{1}{k + m + n} \alpha^{(m)} \mod \Gamma^{r,(m)} + \frac{1}{k + m + n} p \lambda_{m-1}^{(m)} \\
\in (\mathbb{Z}^{k+m+n})^{m-1} \times \mathbb{Z} \quad \text{as a set}. \quad (6.19)
\]

Thus, by ordering this sum in such a way that for each fixed \( \alpha^{(m)} \in \Gamma^{r,(m)} \)
the \( \lambda^{(m)} \) with \( p \lambda = 0, \ldots, m - 1 \) appear first, it should be possible to reduce
the sum to one over the first summand by first performing the sum over the
\( p \lambda \).

It turns out to be simpler, however, to proceed in the opposite order. Instead
of trying to reduce the sum over \( \Gamma^{w,(m)}/(k + m + n) \Gamma^{r,(m)} \) to a sum over
\( \Gamma^{r,(m)}/(k + m + n) \Gamma^{r,(m)} \) by summing over the \( p \lambda \), one may alternatively try
to sum over the latter for each fixed value of \( p \lambda \) and perform the sum over the
\( p \lambda \) at the end.

Using (6.17) and (6.19) one deduces that, in terms of the expansion
\[
\phi^{(m)} = \sum_{i=1}^{m-1} \phi^{(m)}_i \alpha^i \equiv \sum_{i=1}^{m-1} N_i \alpha^i , \quad (6.20)
\]
the allowed values of \( \phi^{(m)} \) can be parametrised as
\[
N_i = \frac{2\pi}{k + m + n} (n_i + \frac{i}{m} p \lambda) , \quad (6.21)
\]
where the \( n_i \) range from 0 to \( k + m + n - 1 \).

Next, denote by \( F(\phi^{(m)}) \) the expression
\[
F(\phi^{(m)}) = \Delta^{(m)}_{W}(\phi^{(m)}) \prod_{a=1}^{m-1} O^r_a(\phi^{(m)}) \quad (6.22)
\]
entering in the \( (m) \)-sector of the correlation function. The crucial property
of \( F \), following from the compactness of \( \phi^{(m)} \), is its periodicity,
\[
F(\phi^{(m)}) = F(\phi^{(m)} + 2\pi \alpha^{(m)} \quad \forall \alpha^{(m)} \in \Gamma^{r,(m)} . \quad (6.23)
\]

This implies that we can expand \( F \) in a finite Fourier series (finite, because
all the ingredients of \( F \) are polynomials),
\[
F(\phi^{(m)}) = \sum_{a \in \mathbb{Z}^{m-1}} f(a) e^{i \sum a^l N_l} \\
= \sum_{a \in \mathbb{Z}^{m-1}} f(a) e^{2\pi i \frac{a^l}{k+m+n} (n_l + \frac{l}{m} p \lambda)} . \quad (6.24)
\]
(a can be regarded as an element of $\Gamma^{w,(m)}$).

Now the sum over the (compact range of the) $n_l$ will imply that only the Fourier modes with $a^l = (k + m + n)b^l$ will contribute so that, after the sum over the $n_l$ one is left with

$$\left( \prod_{l=1}^{m-1} \sum_{n_l=0}^{k+m+n-1} \right) F(\phi^{(m)}) = \sum_{b \in (2)^{m-1}} f((k + m + n)b)e^{2\pi i (\sum l b_l) \frac{p_\lambda}{m}}. \quad (6.25)$$

If there is no torsion, then (6.25) is the only place where $p_\lambda$ appears. In that case, the sum over the $p_\lambda$ implies a further constraint on the Fourier modes that can contribute, namely

$$\sum_{l=1}^{m-1} l b_l = 0 \mod m. \quad (6.26)$$

Frequently, this just leaves the constant mode $f(0)$ as the only non-zero contribution to the ($m$)-sector of the correlation function, but we have not been able to determine when this will be the case in general.

At some point, one needs to combine this with the ($n$)-sector. In the absence of torsion, these are only coupled via the selection rules so the same considerations as above apply to that sector.

If there are torsion sectors then, not unexpectedly, things are slightly different. First of all, in that case $p_\lambda$ also appears in the action, as the only surviving contribution from the $\text{Tr} \phi^{(m)}q_{x_{tor}}$-term, the precise form being

$$- \frac{2\pi q}{(m|n)} p_\lambda^{(m)}.$$

Thus the summation over the $p_\lambda$ now requires

$$\sum_{l=1}^{m-1} l b_l - \frac{qm}{(m|n)} = 0 \mod m \quad (6.27)$$

(and likewise for the ($n$)-sector).

7 COHOMOLOGY AND CHIRAL RINGS II: BOSONIC CORRELATION FUNCTIONS

On the basis of what we have established so far, it is now possible to discuss more concretely some of the general and special properties of correlation functions in the bosonic sector. We will not attempt a complete
and completely explicit analysis of the correlation functions. However, we will analyze chiral correlators and their level-rank duality properties, and we will establish in detail the equality of the correlation functions of the \( \mathbb{C}P(2)_k \) model and the intersection numbers of the Grassmannian manifold \( G(2, k + 2) \) for arbitrary \( k \).

7.1 Generalities About Chiral Correlators

Recall that in the \( G(m, m + n) \)-models we have two families of bosonic operators, the \( O_p \) and the \( \bar{O}_q \). By ‘chiral correlators’ we mean correlation functions depending on operators from only one of these two families.

The three selection rules we have found are

\[
\begin{align*}
\sum_{a=1}^m nar_a + \sum_{b=1}^n mbs_b &= kmn \\
\sum_{a=1}^m ar_a - \frac{q}{(m|n)} km &= 0 \mod m \\
\sum_{b=1}^n bs_b + \frac{q}{(m|n)} kn &= 0 \mod n .
\end{align*}
\]

Compatibility of the first with the second and third requires that, if one has

\[
\begin{align*}
\sum_{a=1}^m ar_a - \frac{q}{(m|n)} km &= cm \\
\sum_{b=1}^n bs_b + \frac{q}{(m|n)} kn &= dn ,
\end{align*}
\]

that \( c \) and \( d \) be related by

\[
c + d = k .
\]

In particular, this implies that if there is no torsion one has a decoupling of the two chiral sectors at level \( k = 1 \),

\[
k = (m|n) = 1 \Rightarrow (c = 1, d = 0) \text{ or } (c = 0, d = 1) .
\]

In general, the chiral sectors need not decouple, but for now we will just consider chiral correlators depending only on the \( O_a \). For these, the selection
rules reduce to

\[
\begin{align*}
\sum_{a=1}^{m} a r_a &= km \\
\sum_{a=1}^{m} a r_a - \frac{q}{(m|n)} km &= cm \\
\frac{q}{(m|n)} k &= d = k - c .
\end{align*}
\] (7.5)

Here clearly the second equality in the third equation is a consequence of the first two. The first equation is just the old familiar selection rule. It coincides with the selection rule of the $\mathbb{CP}(m)_k$-model (for which $(m|n) = 1$, hence $q = 0$, so that the second and third equations are empty in that case).

The most elementary consequence of these equations is that $q = 0, c = k$ is always a solution, so that the trivial torsion sector will always contribute to chiral correlators.

The second most elementary consequence is the condition that $qk/(m|n)$ be an integer,

\[
\frac{q}{(m|n)} k \in \mathbb{Z} .
\] (7.6)

This immediately implies that, if $k$ and $(m|n)$ are coprime, $q$ is necessarily zero,

\[
(k|(m|n)) = 1 \Rightarrow q = 0 .
\] (7.7)

Hence, even when $(m|n) \neq 1$ and, in principle, there are non-trivial torsion sectors, these will not contribute to chiral correlators unless one is considering the exceptional cases in which all three parameters $k$ and $m$ and $n$ have a non-trivial common divisor. In particular, away from these exceptional points (which are precisely those for which the field identification problems arise in conformal field theory), all that remains of the selection rules is

\[
\sum_{a=1}^{m} a r_a = km .
\] (7.8)

If, on the other hand, $(k|(m|n)) \neq 1$, then non-trivial torsion sectors can contribute as well. Clearly, the strength of the present set-up is that there are no major obstacles in dealing with these models. Consider e.g. the case $k = m = n = 2$, so that $(m|n) = (k|(m|n)) = 2$ as well. Then one needs to
solve
\[ k = m = n = 2 \Rightarrow \quad r_1 + 2r_2 = 4 \]
\[ r_1 + 2r_2 - 2q = 2c \]
\[ q = 2 - c \quad . \quad (7.9) \]

Clearly, in this case both possible torsion sectors can contribute, \( q = 0 \) for \( c = 2 \) and \( q = 1 \) for \( c = 1 \). However, as it appears to be difficult to say anything of substance in general about the \( (k|(m|n)) \neq 1 \) correlators, we will henceforth focus on the case \( (k|(m|n)) = 1 \).

### 7.2 Chiral Correlators and Level-Rank Duality for \( (k|(m|n)) = 1 \)

We now consider a general chiral correlation function in the \( G(m, m + n)_k \) model when \( k \) and \( (m|n) \) are co-prime (and hence only \( q = 0 \) contributes). It follows from the (preliminary) results (6.25, 6.26, 7.7) of the previous sections that

\[ \langle \prod_{a=1}^{m} O_{a}^{r_{a}} \rangle_{G(m, m+n)_k} = \sum_{b \in (\mathbb{Z})^{m-1}} \sum_{l_{b}=0 \mod m} f_{m,\vec{r}}((k + m + n)b)\delta_{\sum_{a} r_{a}, km} \quad . \quad (7.10) \]

Here the \( f_{m,\vec{r}} \) are the Fourier coefficients of

\[ F(\phi^{(m)}) = \Delta_{W}^{(m)}(\phi^{(m)}) \prod_{a=1}^{m-1} O_{a}^{r_{a}}(\phi^{(m)}) \quad , \quad (7.11) \]

so that, in particular, \( \vec{r} = (r_1, \ldots, r_{m-1}) \) does not depend on \( r_{m} \) (which only enters via the selection rules). This will be important below.

Several immediate (and less immediate) conclusions can be drawn from this result:

First of all, \( k \) and \( n \) only enter in the symmetric combination \( k + m + n \) in the frequencies contributing to (7.10), and this is the only place where \( n \) appears. \( k \), on the other hand, also appears in the Kronecker delta (selection rule), but only in the combination \( k - r_{m} \). \( r_{m} \), in turn, only appears there.

Given this, one can read off some remarkable properties of these correlation functions. In order not to change the sum in (7.10), we do not want to change \( m \) and \( \vec{r} \), and we do not want to change \( k + m + n \), i.e. we want
to keep $k + n$ fixed. Let us consider first what happens when we exchange $k$ and $n$. The sole effect of this is to change the selection rule. But if we simultaneously replace $r_m$ by $r_m + n - k$, then we retain the selection rule we started off with,

$$0 = km - \sum_{a=1}^{m-1} ar_a - mr_m$$

$$= nm - \sum_{a=1}^{m-1} ar_a - m(r_m + n - k)$$

$$.$$ \hspace{1cm} (7.12)

Thus, simultaneous interchange of $k$ and $n$ plus an insertion of $\mathcal{O}_{m}^{n-k}$ does not change the correlation functions, and we obtain the level-rank duality

$$\langle \prod_{a=1}^{m} \mathcal{O}_{a}^{r_{a}} \rangle_{G(m,m+n)}^{k} = \langle \mathcal{O}_{m}^{n-k} \prod_{a=1}^{m} \mathcal{O}_{a}^{r_{a}} \rangle_{G(m,m+n)}^{n-k}.$$ \hspace{1cm} (7.13)

For $k = 1$, this relates the chiral ring of the $\mathbb{C}P(m)$ model to that of the level 1 Grassmannian model which is expected to describe its classical cohomology ring \hspace{1cm}.

To understand the necessity for the insertion of the operator $\mathcal{O}_m$, we observe the following: Let us compare the couplings to the scalar curvature in the $G(m, m + n)_k$ and $G(m, m + k)_n$ models. In the former this is

$$e^{i \frac{m+n}{4\pi} \int \phi_m R} \rightarrow e^{i \phi_m(m + n)} = \mathcal{O}_{m}^{(m+n)},$$ \hspace{1cm} (7.14)

while in the latter this is

$$e^{i \frac{m+k}{4\pi} \int \phi_m R} \rightarrow e^{i \phi_m(m + k)} = \mathcal{O}_{m}^{(m+k)},$$ \hspace{1cm} (7.15)

If one now inserts the operator $\mathcal{O}_{m}^{(n-k)}$ into the $G(m, m + k)_n$ correlation functions, then one sees that this just converts (7.13) into (7.14) and thus corrects for the difference between the gravitational background charges of the present topological model and the untwisted model (for which the fermion index does not involve the scalar curvature).

This is precisely analogous to the observation made by Witten in \hspace{1cm}, when discussing the relation between the quantum cohomology ring of Grassmannians and the Verlinde algebra, i.e. correlation functions of a $G/G$ model (see the discussion leading to eq. (4.63) of \hspace{1cm}).
There, the $G/G$ model in question is a $U(m)/U(m)$ model, which essentially differs from the sector of the $G(m, m + n)$ Kazama-Suzuki model we have been considering only by the absence of selection rules due to fermionic zero modes in the former. This difference is responsible for the drastically different ring structure of the classical and quantum cohomology rings.

Quantum cohomology rings have also been studied in the context of Kazama-Suzuki models [4, 5, 31], primarily in the language of integrable perturbations of the Landau-Ginzburg description of Kazama-Suzuki models (these are available e.g. for level one models of hermitian symmetric spaces based on simply-laced groups). We will explain elsewhere what the counterpart of this approach to quantum cohomology is in our setting.

Using the same reasoning as above we can also establish more general dualities among correlation functions. Namely, rather than just exchanging $k$ and $n$ we can also replace $k$ by $k + t$ and $n$ by $n - t$ and insert $O^t_m$ (the choice $t = n - k$ corresponding to what we did above). As this will also not change the selection rule,

$$0 = km - \sum_{a=1}^{m-1} ar_a - mr_m$$

$$= (k + t)m - \sum_{a=1}^{m-1} ar_a - m(r_m + t),$$

one obtains the **generalized level-rank duality** (which should be related to Gepner’s dihedrality [4])

$$\langle \prod_{a=1}^{m} O^{r_a}_a \rangle_{G(m, m+n)_k} = \langle O^t_m \prod_{a=1}^{m} O^{r_a}_a \rangle_{G(m, m+n-t)_{k+t}}.$$

In particular, this shows that one can always (formally) reduce everything to calculations at level $k = 0$, as previously observed by Gepner. The only caveat here is that $t$ should be chosen such that $k + t$, $m$ and $n - t$ are still co-prime as otherwise the correlation functions in the ‘dual’ model might also receive contributions from the torsion sectors $q \neq 0$ which we have not considered in the above.

While this presents one with a rather satisfactory overall picture of the chiral rings of Kazama-Suzuki models, there are a number of things that still remain to be understood or worked out in detail, in particular the role of the torsion sectors in models with $(k|(m|n)) \neq 1$. Ideally, one would also
like to establish a general result concerning the relation between chiral rings and cohomology rings. Lacking this, we will - in the remainder of this article - study in detail the $\mathbb{CP}(2)_k$ model to learn about the issues at stake.

7.3 The Cohomology Relations for $G(2, k + 2)$

We will now apply the results of the previous sections to the $\mathbb{CP}(2)_k$ model. The reason why we discuss this model in detail is that here we can prove from scratch and for all values of the level $k$ that the operator ring of the model is exactly the same as the cohomology ring of the Grassmannian $G(2, k + 2)$.

We start by taking a detailed look at the cohomology of the Grassmannian of two-planes $G(2, k + 2)$. We choose the defining relation for the cohomology of $G(2, k + 2)$ to be

\[(1 + c_1 + c_2)(1 + d_1 + \ldots + d_k) = 1 . \tag{7.18}\]

From this the $d_m$ can be recursively expressed in terms of the $c_l$ by the relation

\[d_m = -c_1d_{m-1} - c_2d_{m-2} . \tag{7.19}\]

Writing this out explicitly for some low values of $m$, one can guess a formula for $d_m(c_1, c_2)$ which can then be proved by induction. This formula is

\[d_m = (-1)^m \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l \binom{m-l}{l} c_1^{m-2l} c_2^l \tag{7.20}\]

The induction relies on the identity

\[\binom{m-l}{l} + \binom{m-l}{l-1} = \binom{m+1-l}{l} . \tag{7.21}\]

Thus the non-trivial relations among the generators $c_1$ and $c_2$ are

\[d_{k+1} = -c_1d_k - c_2d_{k-1} = 0 \tag{7.22}\]
\[d_{k+2} = -c_2d_k = 0 . \tag{7.23}\]

As the dimension of $G(2, k + 2)$ is $4k$, this means that the top-degree relations are

\[I(k, m) \equiv c_1^{k-1-2m} c_2^m d_{k+1} \]
\[
\begin{align*}
I(k, m) &= \sum_{l=0}^{[(k+1)/2]} c_1^{2k-2(l+m)} c_2^{l+m} {k+1-l \choose l} (-1)^l = 0 \\
II(k, m) &= c_1^{k-2-2m} c_2^m (c_2 d_k) \\
&= \sum_{l=0}^{[k/2]} c_1^{2k-2(l+m)} c_2^{l+m+1} {k-l \choose l} (-1)^l = 0. \\
&= c_2^m (c_2 d_k) \\
&= \sum_{l=0}^{m} c_2^{l+m+1} (-1)^l = 0.
\end{align*}
\]

Thus all in all there are \( k \) relations, the right number, as there are \( (k + 1) \)
possible top-form monomials, namely

\[
c_1^k, c_2^k, c_2^{k-2}, c_2^{k-4}, \ldots, c_2. \quad (7.26)
\]

The relations plus the normalization condition (5.10) thus determine the
integrals (intersection numbers)

\[
\int_{G(2,k+2)} c_1^{r_1} c_2^{r_2}, \quad r_1 + 2r_2 = 2k. \quad (7.27)
\]

These individual integrals are somewhat difficult to extract from the relations (7.24,7.25). Thus, what we will show instead below (section 8.4) is that these coincide with the correlation functions

\[
\langle O_1^{r_1} O_2^{r_2} \rangle_{\mathbb{C}P^2_k} \quad (7.28)
\]

of the \( \mathbb{C}P^2_k \) model by demonstrating that these correlators satisfy (7.24,7.25). These relations satisfy some recursion relations which will allow us to verify
that the predictions of the \( \mathbb{C}P^2_k \) model agree with the above relations by checking the single relation \( I(k, 0) \) for all \( k \). First of all, we observe that this
is the only relation which contains the top-degree monomial \( c_1^k \). Moreover,
all the other relations can be expressed as powers of \( c_2 \) times relations for
lower values of \( k \). In fact, it can easily be checked that one has

\[
I(k, m) = -c_2 (I(k-1, m-1) + II(k-1, m-1)) \quad \forall \, m > 0. \quad (7.29)
\]

\[
II(k, m) = c_2 I(k-1, m) \quad \forall \, m. \quad (7.30)
\]

Therefore, everything can be reduced to checking \( I(k, 0) \). To make this
slightly more explicit, say that for \( k = 3 \) one has the top-degree relations

\[
c_1^6 = x c_1^4 c_2 = y c_1^2 c_2^2 = z c_2^3. \quad (7.31)
\]
Then the above implies that at \( k = 4 \) one will have
\[
c_1^6 c_2 = x c_1^4 c_2^2 = y c_1^2 c_2^3 = z c_2^4,
\]
and the only top-form monomial whose coefficient is not determined is \( c_1^8 \).
As everything starts off at \( k = 1 \) with the relation \( c_1^2 = c_2 \), all relations are determined by the one relating \( c_1^{2k} \) to the other top-form monomials, i.e. by \( I(k, 0) \) (or, rather, for fixed \( k \), by all the \( I(l, 0) \) with \( l \leq k \)).

7.4 The CP(2)\( _k \) Model

We will now apply the result (7.10) to the correlation functions of the CP(2)\( _k \) model. Let us collect the ingredients that go into the calculation.

The observables are
\[
\begin{align*}
O_1 &= e^{i\phi_1} + e^{i\phi_2} - \phi_1 = e^{i\frac{1}{2}\phi_2} O_1 \\
O_2 &= e^{i\phi_2} ,
\end{align*}
\]
with
\[
O_1 = e^{i\frac{1}{2}(2\phi_1 - \phi_2)} + e^{-i\frac{1}{2}(2\phi_1 - \phi_2)} = e^{iN_1} + e^{-iN_1} .
\]

The Weyl determinant is
\[
\Delta_W^{(2)}(N_1) = 2\sin^2 \frac{1}{2}(2\phi_1 - \phi_2) = 2\sin^2 N_1 .
\]

In order to calculate the correlation function
\[
\langle O_1^{r_1} O_2^{r_2}\rangle_{\text{CP}(2)_k}
\]
(subject to the selection rule \( r_1 + 2r_2 = 2k \)), we have to expand \( O_1^{r_1} \Delta_W^{(2)} \) in a Fourier series in \( N_1 \). Actually, it can be easily read off from (7.10) that we can reduce (7.36) to a correlation function with only the \( O_1 \) insertion at level \( k - r_2 \). This is the counterpart of the observation made above, that at each higher \( k \) the only undetermined coefficient in the relations is that of \( c_1^{2k} \).

Using
\[
\Delta_W^{(2)}(N_1) = (1 - \frac{1}{2}e^{2iN_1} - \frac{1}{2}e^{-2iN_1})
\]

(7.37)
and
\[
O_{1}^{r_{1}} = \sum_{l=0}^{r_{1}} \binom{r_{1}}{l} e^{i(2l - r_{1})N_{1}} ,
\]
\[ (7.38) \]

one finds
\[
O_{1}^{r_{1}}(N_{1})\Delta_{W}^{(2)}(N_{1}) = \sum_{l=-1}^{r_{1}+1} \left[ \binom{r_{1}}{l} - \frac{1}{2} \left( \binom{r_{1}}{l+1} - \frac{1}{2} \binom{r_{1}}{l-1} \right) \right] e^{i(2l - r_{1})N_{1}} .
\]
\[ (7.39) \]

It readily follows that in this case only the constant mode \( r_{1} = 2l \) contributes to the correlation function and thus the correlation functions of the \( \mathbb{C}P(2)_{k} \) model are
\[
\langle O_{1}^{r_{1}}O_{2}^{r_{2}} \rangle_{\mathbb{C}P(2)_{k}} = \frac{1}{r_{1} + 1} \left( \frac{r_{1} + 1}{2r_{1}} \right) \delta_{r_{1}+2r_{2},2k} .
\]
\[ (7.40) \]

We see that these are already correctly normalized to
\[
\langle O_{2}^{k} \rangle_{\mathbb{C}P(2)_{k}} = 1 .
\]
\[ (7.41) \]

To check that the correlation functions (7.40) agree with the intersection numbers of \( G(2, k+2) \), we verify that they satisfy the relations (7.24,7.25), i.e. that the following identities hold:
\[
\sum_{l=0}^{[(k+1)/2]} \frac{1}{k+1 - (l + m)} \binom{2k - 2(l + m)}{k - (l + m)} \binom{k + 1 - l}{l} (-1)^{l} = 0 \quad (7.42)
\]
\[
\sum_{l=0}^{[k/2]} \frac{1}{k - (l + m)} \binom{2k - 2 - 2(l + m)}{k - 1 - (l + m)} \binom{k - l}{l} (-1)^{l} = 0 \quad (7.43)
\]

We also know that it is sufficient to check \( I(k,0) \). Let us do this for even \( k = 2j \). The argument for odd \( k \) is identical. Thus the identity we want to prove is
\[
\sum_{l=0}^{j} \frac{1}{2j + 1 - l} \binom{4j - 2l}{2j - l} \binom{2j + 1 - l}{l} (-1)^{l} = 0 .
\]
\[ (7.44) \]

Let us rewrite this as
\[
\sum_{l=0}^{j} \frac{1}{2j + 1 - 2l} \binom{4j - 2l}{2j - l} \binom{2j - l}{l} (-1)^{l} = 0 ,
\]
\[ (7.45) \]
and regard this as the value at 1 of the function
\[ F_{2j}(x) = \sum_{l=0}^{j} \frac{1}{2j + 1 - 2l} \binom{4j - 2l}{2j - l} \binom{2j - l}{l} (-1)^l x^{2j - 2l + 1} . \] (7.46)

Our aim is to show that \( F_{2j}(1) = 0 \). Consider first the function
\[ G_{2j}(x) = (1 - x^2)^{2j} = \sum_{l=0}^{2j} \binom{2j}{2j - l} (-1)^l x^{4j - 2l} . \] (7.47)

Using
\[ \frac{1}{q!} \left( \frac{d}{dx} \right)^q x^p = \binom{p}{q} x^{p-q} , \] (7.48)

one obtains
\[ \frac{1}{(2j-1)!} \left( \frac{d}{dx} \right)^{2j-1} G_{2j}(x) = \sum_{l=0}^{j} \binom{2j}{2j - l} \binom{4j - 2l}{2j - l} (-1)^l x^{2j - 2l + 1} . \] (7.49)

Now one can check that
\[ \binom{4j - 2l}{2j - 1} \binom{2j}{2j - l} = \frac{2j}{2j - 2l + 1} \binom{4j - 2l}{2j - l} \binom{2j - l}{l} , \] (7.50)

so that finally
\[ F_{2j}(x) = \frac{1}{(2j)!} \left( \frac{d}{dx} \right)^{2j-1} (1 - x^2)^{2j} . \] (7.51)

This makes it obvious that \( F_{2j}(1) = 0 \), as there are enough powers of \((1 - x^2)\).

In fact, for all (even and odd) values of \( k \) one has
\[ F_k(x) = \frac{1}{k!} \left( \frac{d}{dx} \right)^{k-1} (1 - x^2)^k \]
\[ F_k(1) = I(k,0) = 0 . \] (7.52)

This proves the identity \( I(k,0) \) and hence the claim that the \( \mathbb{C}P(2) \) level \( k \) model calculates the intersection numbers of the classical cohomology ring of \( G(2,k+2) \).

Clearly, in principle one can discuss other models along these lines (and we have verified the expected relation of the \( \mathbb{C}P(3)_k \) model to the cohomology of \( G(3,k+3) \) for certain low values of \( k \)); in practice, however, this is rather
cumbersome as both the explicit description of the cohomology relations and the evaluation of the correlation functions become (algebraically) more and more complex. Hence, at this point a more conceptual approach is called for, based perhaps on the spectral flow as discussed in [13] which would allow one to relate the bosonic correlation functions to the (more transparent) $\alpha \bar{\alpha}$ correlators discussed in section 5. We will leave this, as well as the other issues mentioned in the introduction, to future investigations.

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**A Global Aspects of the Gauge Group $H/Z$**

**A.1 Lie Algebra Conventions for $SU(m+n)$ and $S(U(m) \times U(n))$**

A standard choice for the simple roots $\alpha^l$ and fundamental weights $\lambda_l$, $l = 1, \ldots, m + n - 1$, of $SU(m+n)$ is

$$
\alpha^l = E_{l,l} - E_{l+1,l+1},
$$

$$
\lambda_l = \frac{1}{m+n} \text{diag}((m+n-l)I_l, -lI_{m+n-l}),
$$

$$
\text{Tr} \alpha^i \lambda_k = \delta^i_k,
$$

where $I_l$ denotes the $(l \times l)$ unit matrix. The root lattice

$$
\Gamma^r = \mathbb{Z}[\alpha^1, \ldots, \alpha^{m+n-1}]
$$

coincides with the integral lattice. The center of $SU(m+n)$,

$$
Z(SU(m+n)) = \mathbb{Z}_{m+n},
$$

is given in terms of $\Gamma^r$ and the weight lattice

$$
\Gamma^w = \mathbb{Z}[\lambda_1, \ldots, \lambda_{m+n-1}]
$$
by
\[ Z(SU(m+n)) = \Gamma^w / \Gamma^r \]  \hfill (A.5)
It is generated by
\[ e^{2\pi i \lambda_{m+n-1}} = e^{\frac{2\pi i}{m+n} \lambda_{m+n}} \equiv z_{m+n} \]  \hfill (A.6)
and
\[ e^{2\pi i p \lambda_{m+n-1}} = e^{2\pi i \lambda_{m+n-p}} = z_p \]  \hfill (A.7)
This implies that
\[ p\lambda_{m+n-1} - \lambda_{m+n-p} \in \Gamma^r \]  \hfill (A.8)
as can also be checked directly. Here we introduced the notation \( z_{m+n} \) for the generator of \( \mathbb{Z}_{m+n} \subset SU(m+n) \). Similarly, \( z_p \) and \( z_q \) will denote elements of \( \mathbb{Z}_m \subset SU(m) \) and \( \mathbb{Z}_n \subset SU(n) \) respectively.

It follows directly from the expression (A.1) for the \( \alpha^l \) that the Cartan matrix has components
\[ \text{Tr} \alpha^k \alpha^l = 2\delta_{k,l} - \delta_{k,l-1} - \delta_{k,l+1} \]  \hfill (A.9)
We will also need the inverse of that matrix, namely \( \text{Tr} \lambda_k \lambda_l \). From the definitions one finds, for \( k < l \),
\[ \text{Tr} \lambda_k \lambda_l = \frac{k(m + n - l)}{m + n} \]  \hfill (A.10)
In particular, one has
\[ \text{Tr} \lambda_i \lambda_m = \frac{in}{m+n} \quad i = 1, \ldots, m - 1 \]
\[ \text{Tr} \lambda_{m+j} \lambda_m = \frac{m(n-j)}{m+n} \quad j = 1, \ldots, n - 1 \]  \hfill (A.11)
As an application, one can use (A.10) to establish (A.8), by showing that the traces of the left hand side of (A.8) with all the \( \lambda_l \) are integral. In fact, one can even show slightly more than that, namely that
\[ p\lambda_{m+n-1} - \lambda_{m+n-p} \in \Gamma^r(SU(n)) \equiv \Gamma^{r,(n)} \]  \hfill (A.12)
for \( p \leq n \), and likewise that
\[ p\lambda_1 - \lambda_p \in \Gamma^r(SU(m)) \equiv \Gamma^{r,(m)} \]  \hfill (A.13)
for \( p \leq m \).

Finally, denoting as usual by \( \rho_G \) half the sum of the positive weights of \( G = SU(m + n) \), one has the standard result that

\[
\rho_G = \sum_{l=1}^{m+n-1} \lambda_l .
\]  
(A.14)

We regard \( H = S(U(m) \times U(n)) \) as a subgroup of \( G = SU(m + n) \) by embedding it in \( SU(m + n) \) in block diagonal form,

\[
H \ni h = \text{diag}(h_1^{(m)} e^{i\varphi/m}, h_2^{(n)} e^{-i\varphi/n}) ,
\]  
(A.15)

with \( h_1^{(m)} \in SU(m) \) and \( h_2^{(n)} \in SU(n) \). There is some redundancy in this description of \( h \), meaning that \( H \neq SU(m) \times SU(n) \times U(1) \), and we will come back to this in detail below.

The embedding is chosen such that the

\[
\begin{align*}
\alpha^i & \quad i = 1, \ldots, m - 1 \\
\alpha^{m+j} & \quad j = 1, \ldots, n - 1 ,
\end{align*}
\]  
(A.16)

are the simple roots of the \( SU(m) \) and \( SU(n) \) subgroups of \( SU(m + n) \) respectively. Furthermore, the \( U(1) \) factor is generated by the weight

\[
\lambda_m = \frac{1}{m+n} (n\mathbb{I}_m, -m\mathbb{I}_n) ,
\]  
(A.17)

as

\[
e^{i\frac{m+n}{mn} \lambda_m \varphi} = (e^{i\varphi/m \mathbb{I}_m}, e^{-i\varphi/n \mathbb{I}_n}) .
\]  
(A.18)

The Weyl vector \( \rho_{G/H} \) (half of the sum of the positive roots of \( G \) which are not positive roots of \( H \) - alternatively the sum over all the positive roots of \( G \) which contain \( \alpha^m \) as a summand) is proportional to \( \lambda_m \),

\[
\rho_{G/H} = \frac{1}{2} (n\mathbb{I}_m, -m\mathbb{I}_n) = \frac{n+m}{2} \lambda_m .
\]  
(A.19)

A useful expression for \( \lambda_m \) in terms of roots is

\[
(m + n) \lambda_m = \sum_{i=1}^{m-1} in\alpha^i + m\alpha^m + \sum_{j=1}^{n-1} (n - j) m\alpha^{m+j} .
\]  
(A.20)

One observation we will need is that

\[
e^{2\pi i \lambda_m} = z_{m+n} .
\]  
(A.21)

55
so that the subgroup of $\mathbb{Z}_{m+n}$ generated by $z_{m+n}^n$ acts only on the $U(1)$-part of $S(U(m) \times U(n))$. This observation will turn out to be crucial in the analysis of the fundamental group of $S(U(m) \times U(n))/\mathbb{Z}_{m+n}$.

Finally, in order to deal with the fermions of the model, we note that the above choice of embedding of $H$ into $G$ leads to the Lie algebra decomposition
\[ \mathfrak{g}_C = \mathfrak{h}_C \oplus \mathfrak{t}^+ \oplus \mathfrak{t}^- , \]
where $\mathfrak{t}^+$ corresponds to the upper right hand block. Thus, e.g. the Grassmann-odd scalars $\alpha$ and $\bar{\alpha}$ have components
\[ \alpha = (\alpha_{ij}) , \quad \bar{\alpha} = (\bar{\alpha}_{ji}) , \quad i = 1, \ldots, m , \quad j = m + 1, \ldots, m + n , \]
with $\alpha_{ij}$ living in the root space corresponding to the positive root
\[ \alpha^i + \ldots + \alpha^{-1} \in \Delta^+(G/H) . \]
As the roots $\alpha \in \Delta^+(G/H)$ are characterized by the fact that they contain the summand $\alpha^m$ exactly once, one has (cf. (2.16))
\[ \text{Tr} \alpha \rho_{G/H} = \frac{n + m}{2} = \frac{C_G}{2} \quad \forall \alpha \in \Delta^+(G/H) . \]

A.2 Global Aspects of $S(U(m) \times U(n))$

As mentioned above, there is some redundancy in the parametrization of $h \in S(U(m) \times U(n))$ in (A.15). This redundancy means that the relation between $S(U(m) \times U(n))$ and $SU(m) \times SU(n) \times U(1)$ is of the form
\[ S(U(m) \times U(n)) = (SU(m) \times SU(n) \times U(1))/Z(m,n) \]
for some discrete group $Z(m,n)$. It is the purpose of this section to determine $Z(m,n)$. Here for the first time the degree of ‘coprimeness’ of $m$ and $n$ will enter, as it will again in the analysis of the action of $\mathbb{Z}_{m+n}$ on $S(U(m) \times U(n))$.

Redundancy means that there are distinct triples
\[ (h_1^{(m)}, h_2^{(n)}, e^{i \frac{m+n}{mn} \lambda_m \varphi}) \neq (\tilde{h}_1^{(m)}, \tilde{h}_2^{(n)}, e^{i \frac{m+n}{mn} \lambda_m \tilde{\varphi}}) \]
representing the same element $h \in S(U(m) \times U(n))$. As a shift in $\varphi$ can only be compensated by an element of the center of $SU(m) \times SU(n)$, $\mathbb{Z}_m \times \mathbb{Z}_n$, we clearly have
\[ Z(m,n) \subset \mathbb{Z}_m \times \mathbb{Z}_n . \]
As we also have
\[ h_1^{(m)} e^{i\varphi/m} = h_1^{(m)} z_m e^{i(\varphi - 2\pi p)/m} \]  
(A.29)
and
\[ h_2^{(n)} e^{-i\varphi/n} = h_2^{(n)} z_n e^{-i(\varphi - 2\pi q)/n} , \]  
(A.30)
an element \((p, q) \in \mathbb{Z}_m \times \mathbb{Z}_n\) (thought of additively, i.e. as \((p \text{ mod } m, q \text{ mod } n)\) for \(p, q \in \mathbb{Z}\)) will be an element of \(Z(m, n)\) if there exist integers \(a = a_{p,q}\) and \(b = b_{p,q}\) such that
\[ \varphi - 2\pi p + 2\pi ma = \varphi - 2\pi q - 2\pi nb , \]  
(A.31)
i.e. such that
\[ p - q = ma + nb \]  
(A.32)
for some \(a, b \in \mathbb{Z}\). At this point some elementary number theory enters the game. Denote by \((m|n)\) the greatest common divisor of \(m\) and \(n\),
\[ (m|n) = \gcd(m, n) . \]  
(A.33)
The fact we will need is that for any \(a, b \in \mathbb{Z}\)
\[ am + bn = 0 \text{ mod } (m|n) \]  
(A.34)
and that for any integer \(c \in \mathbb{Z}\) one can find \(a_c, b_c \in \mathbb{Z}\) such that
\[ a_c m + b_c n = c(m|n) . \]  
(A.35)
This fact is usually stated in the form that for coprime integers \(m_0\) and \(n_0\) one can find \(a, b \in \mathbb{Z}\) such that
\[ am_0 + bn_0 = 1 \]  
(A.36)
Thus there is a solution to (A.32) for some \(a, b \in \mathbb{Z}\) iff \(p - q = 0 \text{ mod } (m|n)\) and one has
\[ Z(m, n) = \{(p, q) \in \mathbb{Z}_m \times \mathbb{Z}_n : p - q = 0 \text{ mod } (m|n)\} . \]  
(A.37)
It is relatively easy to see that this can be described more explicitly as
\[ Z(m, n) = \mathbb{Z}_{mn/(m|n)} . \]  
(A.38)
One way of doing this is to show that the element \((p, q) = (1, 1)\) is a generator of \(Z(m, n)\), its order obviously being the least common multiple of \(m\) and \(n\), i.e. \(mn/(m|n)\).

The upshot of this is that the relation between \(S(U(m) \times U(n))\) and \(SU(m) \times SU(n) \times U(1)\) is

\[ S(U(m) \times U(n)) = (SU(m) \times SU(n) \times U(1))/Z_{mn/(m|n)} \quad \text{.} \tag{A.39} \]

As an example consider \((m, n = 1)\), i.e. \(\mathbb{CP}(m)\). In that case, we have \((m|n) = 1, mn/(m|n) = m\), and one recovers the familiar result

\[ S(U(m) \times U(1)) = (SU(m) \times U(1))/Z_m = U(m) \quad . \tag{A.40} \]

### A.3 The Fundamental Group of \(S(U(m) \times U(n))/Z_{m+n}\)

The relevant gauge group in the \(G/H\) gauged Wess-Zumino-Witten models is not \(H\) but rather

\[ H' = H/(Z(G) \cap H) \quad , \tag{A.41} \]

(not to be confused with the group we called \(H'\) in section 2) where \(Z(G)\) denotes the center of \(G\), as the gauge group has to be a subgroup of the adjoint group \(G/Z(G)\) of \(G\). The aim will now be to

1. study the action of \(Z(G) \cap H\) on \(H\);
2. determine the fundamental group \(\pi_1(H')\);
3. exhibit an explicit set of generating paths for \(\pi_1(H')\).

Note that for our purposes it is not enough to just determine \(\pi_1(H')\) by some means. It is (3) that we need in order to be able to explicitly introduce the twisted topological sectors into the path integral via transition functions (equivalently, via conditions on the allowed Chern classes).

For \(G = SU(m + n), Z(G) = Z_{m+n}\) is generated by

\[ z_{m+n} = e^{2\pi i \lambda_{m+n}} = e^{2\pi i \lambda_{m+n-1}} \quad . \tag{A.42} \]

But clearly \(z_{m+n}\) is an element of \(H = S(U(m) \times U(n)) \subset SU(m + n)\), as both blocks of \(z_{m+n}\) are unitary, and the determinant of \(z_{m+n}\) is equal to one. Hence we have

\[ H \cap Z(G) = Z(G) = Z_{m+n} \quad , \tag{A.43} \]
and

\[ H' = S(U(m) \times U(n))/\mathbb{Z}_{m+n} \]  \hspace{1cm} (A.44)

Now, let us recall the observation (A.21) that

\[ z^n_{m+n} = e^{2\pi i \lambda_m} \]  \hspace{1cm} (A.45)

This means that the subgroup of \( \mathbb{Z}_{m+n} \) generated by \( z^n_{m+n} \) acts only on the \( U(1) \)-part of \( S(U(m) \times U(n)) \) which is generated by \( \lambda_m \). The reason why it is necessary to disentangle this subgroup from the rest is that it essentially does not change the fundamental group. ‘Essentially’ means that \( \pi_1(U(1)) \) is isomorphic to \( \pi_1(U(1)/\mathbb{Z}_r) \), the isomorphism being given by

\[ \pi_1(U(1)) = \mathbb{Z} \ni 1 \rightarrow r \in \pi_1(U(1)/\mathbb{Z}_r) = \mathbb{Z} \]

\[ \pi_1(U(1)/\mathbb{Z}_r)/\pi_1(U(1)) = \mathbb{Z}/r\mathbb{Z} = \mathbb{Z}_r \]  \hspace{1cm} (A.46)

Thus the question is what is the order of the group generated by \( z^n_{m+n} \) or, in other words, what is the smallest positive integer \( p \) such that \( z^{np}_{m+n} = 1 \). Now

\[ z^{np}_{m+n} = 1 \Leftrightarrow \frac{pm}{m+n} \in \mathbb{Z} \]  \hspace{1cm} (A.47)

To determine the smallest integer for which this is the case, we once again write \( m = m_0(m|n) \), \( n = n_0(m|n) \), and try to find the smallest solution \( p \) to

\[ pm = a(n+m) \quad a \in \mathbb{Z} \]  \hspace{1cm} (A.48)

or

\[ pm_0 = a(m_0 + n_0) \]  \hspace{1cm} (A.49)

As \( n_0 \) and \( m_0 + n_0 \) have no common factor, the ‘minimal’ solution is \( a = n_0 \) and \( p = m_0 + n_0 \) or \( p = (m+n)/(m|n) \). Thus the subgroup of \( \mathbb{Z}_{m+n} \) acting only on the \( U(1) \) is

\[ \{ z^{np}_{m+n}, p \in \mathbb{Z} \} \approx \mathbb{Z}_{(m+n)/(m|n)} \]  \hspace{1cm} \hspace{1cm} (A.50)

the generator

\[ z^{np}_{m+n} = z_{m+n}^{(m|n)} \]  \hspace{1cm} (A.51)

of \( \mathbb{Z}_{(m+n)/(m|n)} \) being obtained for \( p = b - a \) where \( am + bn = (m|n) \).

In particular, in the coprime case \( (m|n) = 1 \), this is all of \( \mathbb{Z}_{m+n} \), in line with the conformal field theory contention that in this case no complicating features due to fixed points and/or topological sectors should arise.
In general, the result (A.50) implies that the group giving rise to new topological sectors in $H'$ is the quotient group

$$Z_{m+n}/Z_{(m+n)/(m|n)} = Z_{(m|n)}.$$

(A.52)

The above discussion suggests (and almost proves - we will not fill in the missing steps) that the fundamental group of $H'$ is

$$\pi_1(S(U(m) \times U(n))/Z_{m+n}) = \mathbb{Z} \times Z_{(m|n)},$$

(A.53)

and we will now proceed to exhibit explicitly a set of generators.

As principal $H'$-bundles on a surface $\Sigma$ are labelled by $\pi_1(H')$, what (A.53) shows is that

- new topological sectors occur precisely when $m$ and $n$ are not coprime;
- these are always torsion, so that there are a finite number of new topological sectors to be taken into account;
- the sum over the new topological sectors is a finite sum over the elements of $Z_{(m|n)}$.

A.4 Generating Loops of the Fundamental Group of $S(U(m) \times U(n))/Z_{m+n}$

As mentioned before, for our purposes it is not sufficient to just determine $\pi_1(H')$ in some way (as above). What we need in order to implement the topological sectors in the path integral and to read off the corresponding Chern classes is an explicit set of representatives of the generators of $\pi_1(H')$.

As we are working at the level of $H$, this amounts to a suitable set of paths in $H$ which are closed in $H$ up to the action of $H \cap Z(G) = Z_{m+n}$.

A remark on notation. For an element $x$ of the Lie algebra of $H$ we will denote by $\gamma_x$ the path $\gamma_x(t) = \exp 2\pi i xt$ in $H$. We will also call winding number of $\gamma_x$ the real number

$$\nu(\gamma_x) \equiv \nu(x) = \text{Tr} \lambda_m x.$$

(A.54)

This winding number is sensitive only to the free part of the fundamental group (living in the $U(1)$-factor) and not to the torsion part. Of course, this is only literally a winding number if $\gamma_x$ is a closed path in $H$ or $H'$ (and
need not be an integer in the latter case). Thus, single valued loops \( \gamma_x \) with \( x \in \Gamma^r \) give integer winding numbers, as in

\[
x = \sum_l n_l \alpha^l \Rightarrow \nu(\gamma_x) = n_m .
\]  

(A.55)

We will first determine a generator of the fundamental group of \( S(U(m) \times U(n)) \). With the parametrization

\[
h = (h_1^{(m)} e^{i \phi/m}, h_2^{(n)} e^{-i \phi/n}) ,
\]  

(A.56)

we had the identification

\[
S(U(m) \times U(n)) = (SU(m) \times SU(n) \times U(1))/\mathbb{Z}_{mn/(m|n)} .
\]  

(A.57)

This is reflected in the fact that the minimal closed loop living entirely in the \( \phi \)-direction,

\[
g(t) = e^{2\pi i \frac{m+n}{(m|n)} \lambda_m t} = (e^{2\pi i \frac{n}{(m|n)} t \mathbb{1}_m}, e^{-2\pi i \frac{m}{(m|n)} t \mathbb{1}_n})
\]  

(A.58)

has winding number

\[
\nu(x = \frac{m+n}{(m|n)} \lambda_m) = \frac{m+n}{(m|n)} \text{Tr} \lambda_m^2 = \frac{mn}{(m|n)} .
\]  

(A.59)

While this is thus the image of the generator of \( \pi_1(U(1)) \) under the identification \( (A.57) \), this is not the generator of \( \pi_1(H) \). To see this, consider the loop

\[
\gamma_{\alpha^m}(t) = e^{2\pi i \alpha^m t} .
\]  

(A.60)

Clearly, this loop has winding number one,

\[
\nu(\alpha^m) = 1 ,
\]  

(A.61)

and can be chosen to be the generator of \( \pi_1(S(U(m) \times U(n))) \). Moreover, using \( (A.20) \), one can verify that

\[
\frac{m+n}{(m|n)} \lambda_m = \frac{mn}{(m|n)} \alpha^m \mod \Gamma^r(SU(m)) \oplus \Gamma^r(SU(n)) ,
\]  

(A.62)

so that

\[
(\gamma_{\alpha^m})^{\frac{mn}{(m|n)}}(t) \sim g(t) ,
\]  

(A.63)
where \( \sim \) means 'homotopic to'. This holds because the terms depending on the roots of \( SU(m) \) and \( SU(n) \) will give contractible loops in \( S(U(m) \times U(n)) \). What the above means in terms of bundles is that a principal \( H \) bundle constructed with \( \gamma_p \) as a transition function will not lift to a principal \( SU(m) \times SU(n) \times U(1) \) bundle unless \( p \) is a multiple of \( mn/(m|n) \).

We now come to the thornier issue of generators for \( \pi_1(H') \). From the point of view of \( H \), these are certain paths in \( H \) which start at the identity element and end at an element of \( \mathbb{Z}^{m+n} \subset S(U(m) \times U(n)) \), so that they define non-trivial closed loops in \( H' = H/\mathbb{Z}^{m+n} \).

What we expect to find are a generator of the free part \( \mathbb{Z} \subset \pi_1(H') \) which is such that its \( (m+n)/(m|n) \)'th power is homotopic to the loop \( \gamma_{\alpha^m} \), the generator of \( \pi_1(H) \), and a generator of the torsion part \( \mathbb{Z}^{(m|n)} \subset \pi_1(H') \) which is such that its \( (m|n)' \)th power is contractible.

First of all, recall that \( \mathbb{Z}^{m+n} \) is generated by

\[
\gamma_{m+n} = e^{2\pi i (m+n-1)} \quad (A.64)
\]

with

\[
\gamma_{m+n} = e^{2\pi ip (m+n-1)} = e^{2\pi i (m+n-p)} \quad . \quad (A.65)
\]

It is thus natural to parametrize the loops of interest (ending at some point of \( \mathbb{Z}^{m+n} \)) as \( \gamma_x \) with \( x \in \Gamma^w \), the weight lattice.

We begin with the free part. Up to homotopy, the most general loop of the type \( \gamma_x \) with \( x \in \Gamma^w \) is of the form \( x = a\lambda_{m+n-1} + b\lambda_1 \) since, as noted before, any other weight is equal to a multiple of either \( \lambda_1 \) or \( \lambda_{m+n-1} \) modulo the root lattice of \( SU(m) \times SU(n) \). Thus, let us consider the loop \( \gamma_x \) with

\[
x = a\lambda_{m+n-1} + b\lambda_1 \quad . \quad (A.66)
\]

Its winding number is

\[
\nu(\gamma_x) = \frac{am + bn}{m + n} \quad . \quad (A.67)
\]

Thus, using again the general result that

\[
\min_{a,b} (am + bn) = (m|n) \quad , \quad (A.68)
\]

we see that the minimum winding number is

\[
\min_x \nu(\gamma_x) = \frac{(m|n)}{m + n} \quad , \quad (A.69)
\]
which is obtained for \( x = x_{\text{free}} \),

\[
x_{\text{free}} = a\lambda_{m+n-1} + b\lambda_1 , \quad am + bn = (m|n) .
\]  (A.70)

But, comparing with the generator of the fundamental group of \( H \), this is precisely what we expect as this means that we get an \( \frac{m+n}{(m|n)} \)’s worth of new generators for the free part of \( \pi_1(H') \), corresponding to the \( \mathbb{Z}_{(m+n)/(m|n)} \) subgroup of \( \mathbb{Z}_{m+n} \) acting on the \( U(1) \)-part of \( H \).

A solution to \( am + bn = (m|n) \) is only unique up to the shift \( a \rightarrow a + \frac{cn}{(m|n)} \), \( b \rightarrow b - \frac{cm}{(m|n)} \). Under this shift \( x_{\text{free}} \) transforms as

\[
x_{\text{free}} \rightarrow x_{\text{free}} + c\left( \frac{n}{(m|n)}\lambda_{m+n-1} - \frac{m}{(m|n)}\lambda_1 \right) .
\]  (A.71)

We will see below that the term in brackets represents the generator of the torsion subgroup \( \mathbb{Z}_{(m|n)} \) of \( \pi_1(H') \), so the ambiguity in the choice of \( a \) and \( b \) only amounts to a mixing of the generators.

We now come to the torsion part. We are still working with paths in \( H \) which are of the form \( \gamma_x \) with \( x \in \Gamma^w \), as these correspond to non-trivial loops in \( H' \). Also, as before, we can restrict our attention to \( x \) of the form \( x = a\lambda_{m+n-1} + b\lambda_1 \). A torsion loop should not be detectable by the \( U(1) \) winding number measured by \( \nu(\gamma_x) \), so that one is looking for non-trivial solutions to

\[
\nu(\gamma_x) = 0 \iff am + bn = 0 \iff am_0 + bn_0 = 0 .
\]  (A.72)

As \( m_0 \) and \( n_0 \) are coprime, i.e. have no common factors, this can only have a solution if \( a \) has a factor of \( n_0 \) and \( b \) has a factor of \( m_0 \). In fact, the general solution is

\[
(a, b) = (cn_0, -cm_0) \quad c \in \mathbb{Z} .
\]  (A.73)

The minimal solution \( c = 1 \) corresponds to the element

\[
x_{\text{tor}} = n_0\lambda_{m+n-1} - m_0\lambda_1 = \frac{n}{(m|n)}\lambda_{m+n-1} - \frac{m}{(m|n)}\lambda_1 \in \Gamma^w .
\]  (A.74)

It has the crucial property that

\[
(m|n)x_{\text{tor}} \in \Gamma^r(SU(m)) \oplus \Gamma^r(SU(n)) ,
\]  (A.75)

so that the loop \( \gamma_{(m|n)x_{\text{tor}}} \) is homotopically trivial,

\[
\gamma_{(m|n)x_{\text{tor}}} = (\gamma_{x_{\text{tor}}})^{(m|n)} \sim 0 \in \pi_1(H') ,
\]  (A.76)
precisely the property one would expect of the generator of a $\mathbb{Z}_{(m|n)}$ subgroup of $\pi_1(H')$. To establish (A.73), we recall that both $m\lambda_1$ and $n\lambda_{m+n-1}$ are equivalent to $\lambda_m$ modulo $\Gamma^r$. Hence their difference is zero modulo $\Gamma^r$ and defines a trivial loop in $H$ (and hence, in particular, in $H'$).

In summary, the elements of the fundamental group of $S(U(m)\times U(n))/\mathbb{Z}_{m+n}$ can be written as paths $\gamma_x$ in $S(U(m) \times U(n))$ with

$$x = px_{free} + qx_{tor} \quad (p, q) \in \mathbb{Z} \times \mathbb{Z}_{(m|n)} . \quad (A.77)$$

There is an alternative approach to determining the transition functions (and hence Chern classes) of $S(U(m) \times U(n))/\mathbb{Z}_{m+n}$-bundles which is simpler and more direct but makes less manifest the group structure of the fundamental group. One starts with $S(U(m) \times U(n))$-bundles and the simple result that for these the generator can be chosen to be $\alpha^m$. One then notes that the allowed transition functions for $S(U(m) \times U(n))/\mathbb{Z}_{m+n}$-bundles are then necessarily of the form

$$ra^m + s\lambda_{m+n-1} \quad , \quad (r, s) \in \mathbb{Z} \times \mathbb{Z}_{m+n} \quad (A.78)$$

where $\lambda_{m+n-1}$ is the generator of $\mathbb{Z}_{m+n}$ and hence $s = 0, \ldots, m + n - 1$. This should be compared with the (rather more complicated) expression

$$px_{free} + qx_{tor} \quad (p, q) \in \mathbb{Z} \times \mathbb{Z}_{(m|n)}$$

$$x_{free} = a\lambda_{m+n-1} + b\lambda_1 \quad am + bn = (m|n)$$

$$x_{tor} = \frac{1}{(m|n)}(n\lambda_{m+n-1} - m\lambda_1) \quad (A.79)$$

obtained above. It can be shown with some work that these two parametrizations are (despite appearance) equivalent and we will briefly indicate the relation between the two in section 3.1.

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