Hamiltonian Formulation of Mimetic Gravity

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Abstract

The Hamiltonian formulation of Mimetic Gravity is formulated. Although there are two more equations than those of general relativity, these are proved to be the constraint equation and the conservation of energy-momentum tensor. The closure of the Poisson brackets is proved and comparison with the Hamiltonian dust is done.
1 Introduction

Not long ago, Chamseddine and Mukhanov [1] proposed a theory of mimetic dark matter by modifying Einstein’s general relativity. This theory was shown to be a conformal extension of Einstein’s theory of gravity. They defined a physical metric in terms of an auxiliary metric and first derivatives of a scalar field. Interestingly, the equations of motion differ from Einstein equations by the appearance of an extra mode of the gravitational field which can reproduce Dark Matter. An equivalent formulation of the model was given in [2] where the action is written without the auxiliary metric. Then the ghost free models of this theory were discussed in [3]. It was found that the theory is free of ghost instabilities for a positive energy density. In a recent paper [4], the mimetic dark matter theory was extended by adding an arbitrary potential and then cosmological solutions were studied. It was shown how various cosmological solutions can be found by choosing an appropriate potential.

In this paper we are going to consider the canonical formulation of this theory. There are advantages of setting up the Hamiltonian dynamics for this new model such as quantization. Arnowitt, Deser and Misner constructed a canonical formulation of gravity ([5]). The tetrad form of this ADM formalism was derived in [6]. Schwinger constructed an invariant action operator under both local Lorentz and general coordinate transformations [7]. Ahtekar [8], [9] introduced new variables which lead to simplifications in the gravitational constraints, and his theory lead to loop quantum gravity. Later, the canonical formulations of matter couplings to gravity were considered (see for example ([10] - [15])).

The aim of this paper is to analyze the hamiltonian equations of motion and to check that the degrees of freedom are those of general relativity. Also, comparison with the canonical formulation of dust will be done.

2 Hamiltonian

2.1 ADM at a glance

To put the action in its canonical form, spacetime should be split into 3-dimensional hypersurfaces with constant time coordinate t. The full metric,
\[ g_{\mu\nu} \text{ is } [5] \]
\[
\begin{bmatrix}
N_i N^i - N^2 & N_k \\
N_i & h_{ij}
\end{bmatrix}
\]
(1)

where \( N \) and \( N_i \) are the lapse and shift functions. \( g_{ij} = h_{ij} \) is the induced metric on the 3-dimensional hypersurfaces, with \( h^{ij} h_{jk} = \delta^i_k \).

After this splitting of spacetime, the gravitational action is given by [5]
\[
L_{ADM} = \int d^4x \left( \dot{h}^{ij} \pi_{ij} - NR^0 - N_i R_i \right)
\]
(2)

where \( \pi_{ij} \) is the canonical momentum given in terms of the extrinsic curvature
\[
\pi_{ij} = \sqrt{h} \left( h_{ij} K - K_{ij} \right)
\]
(3)

and the intrinsic curvatures are
\[
R^0 \equiv -\sqrt{h} \left[ 3R + h^{-1} \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} \right) \right]
\]
\[
R_i \equiv -2\pi^{ij}_i
\]
(4)

\( N \) and \( N_i \) enter as Lagrange multipliers; variation with respect to them give constraint equations.

### 2.2 Extended Theory

The action of the extended mimetic dark matter theory, where an arbitrary potential is added, can be written as the following [2], [4]
\[
S = -\int d^4x (-g)^{1/2} \left( \frac{1}{2} \dot{R} + \frac{1}{2} \lambda \left( 1 - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) + L_m \right)
\]
(5)

where \( L_m \) is \( V(\phi) \). To construct the canonical formalism of this theory we rewrite this action in a 3 + 1 dimensional form. The momentum conjugate to \( \phi \) is given by
\[
p = \frac{\partial L}{\partial \dot{\phi}} = N \sqrt{h} \lambda \left( g^{00} \partial_0 \phi + g^{0i} \partial_i \phi \right)
\]
(6)

where \( g^{00} = -1/N^2 \) and \( g^{0i} = N_i/N^2 \) and \( g^{ij} = h^{ij} - N^i N^j / N^2 \).

From equation (6) the Lagrangian can be written as
\[
L = -\frac{1}{2} Nh^{1/2} \lambda \left( 1 + g^{00} \dot{\phi}^2 - h^{ij} \partial_i \phi \partial_j \phi + \frac{N^i N^j}{N^2} \partial_i \phi \partial_j \phi \right) + p \dot{\phi} - N \sqrt{h} V(\phi)
\]
(7)
Then the Hamiltonian is
\[ H = p\dot{\phi} - L = \frac{1}{2} N h^{1/2} \lambda \left( 1 + g^{00} \dot{\phi}^2 - h^{ij} \partial_i \phi \partial_j \phi + \frac{N^i N^j}{N^2} \partial_i \phi \partial_j \phi \right) + N\sqrt{h}V(\phi) \] (8)

Plugging for \( \dot{\phi} \) in terms of \( p \) from equation (6), we get
\[ H = -\frac{Np^2}{2\sqrt{h}\lambda} + \frac{1}{2} N\sqrt{h} \lambda \left[ 1 - h^{ij} \partial_i \phi \partial_j \phi \right] + pN^i \partial_i \phi + N\sqrt{h}V(\phi) \] (9)

Solving the equation \( \frac{\partial H}{\partial \lambda} = 0 \) gives for \( \lambda \),
\[ \lambda = \frac{p}{\sqrt{h} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi - 1} \] (10)

Plugging back we get the total action
\[ \int d^4x \left( L_{\text{ADM}} + p\dot{\phi} - N p \sqrt{h} \lambda \left[ 1 - h^{ij} \partial_i \phi \partial_j \phi \right] + pN^i \partial_i \phi - N\sqrt{h}V(\phi) \right) \] (11)

where \[ L_{\text{ADM}} = \dot{h}^{ij} \pi_{ij} - NR^0 - N^i R_i \] (12)

### 3 Equations of Motion

The total action is given by
\[ S = \int d^4x \left( h^{ij} \pi_{ij} + p\dot{\phi} - N \left( R^0 + p \sqrt{h} \partial_i \phi \partial_j \phi - 1 \right) - N^i \left( R_i + p\partial_i \phi \right) - N\sqrt{h}V(\phi) \right) \] (13)

The equations of motion are

- Varying with respect to \( \pi_{ij} \)
\[ \dot{h}^{ij} = \{ h^{ij}, H \} = 2Nh^{-1/2} \left( \pi^{ij} - \frac{1}{2} h^{ij} \pi \right) + N^{i|j} + N^{j|i} \] (14)

This equation is independent of \( \phi \) since the part of the action depending on the scalar field is independent of \( \pi_{ij} \).
\[ \dot{\pi}_{ij} = \{\pi_{ij}, H\} = -N\sqrt{h} \left( 3R_{ij} - \frac{1}{2} h_{ij}^3 R \right) + \frac{1}{2} Nh^{-1/2} h_{ij} \left( \pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2 \right) \\
- 2Nh^{-1/2} \left( \pi_{im} \pi^m_j - \frac{1}{2} \pi_{ij} \right) + \sqrt{h} \left( N|_{ij} - h_{ij}^l N^l_{|m} \right) + (\pi_{ij} N^m)_{|m} \\
- N^{|m}_{im} \pi_{mj} - N^{|m}_{jm} \pi_{mi} + \frac{Np\partial_i \phi \partial_j \phi}{2\sqrt{h^k_i \partial_k \phi \partial_l \phi - 1}} + \frac{1}{2} Nh^V(\phi) h_{ij} \tag{15} \]

- w.r.t. \( N \)
\[ R^0 + p\sqrt{h^i_j \partial_i \phi \partial_j \phi - 1} + \sqrt{h} V(\phi) = H_{grav} + H_\phi = 0 \tag{16} \]

- w.r.t. \( N^i \)
\[ R_i + p\partial_i \phi = H_{i grav} + H_{i \phi} = 0 \tag{17} \]

- w.r.t. \( p \)
\[ \dot{\phi} - N\sqrt{h^i_j \partial_i \phi \partial_j \phi - 1} - N^i \partial_i \phi = 0 \tag{18} \]

- w.r.t. \( \phi \)
\[ \dot{\phi} - \partial_k \left( \frac{Np\partial^k \phi}{\sqrt{h^i_j \partial_i \phi \partial_j \phi - 1}} + N^k p \right) + N\sqrt{h} dV(\phi) = 0 \tag{19} \]

The number of equations of motion are those of Einstein’s gravity plus two more equations coming from the variation of the \( \phi \) and \( p \). However, equation (18) is exactly the constraint equation, \( g^\mu\nu \partial_\mu \phi \partial_\nu \phi = 1 \); therefore, it gives no new info. While equation (19) is the Bianchi identity as explained below. Therefore, we end up with the same number of equations of motion as those of general relativity.

Considering only the part of the action depending on the scalar field \( \phi \), it is given by
\[ S_\phi = \int d^4x \left( \rho \dot{\phi} - Np\sqrt{h^i_j \partial_i \phi \partial_j \phi - 1} - N^i p\partial_i \phi - N\sqrt{h} V(\phi) \right). \tag{20} \]

The canonical components of stress energy tensor given by (16)
\[ T_{00} = -\frac{N}{\sqrt{h}} \left( \frac{\delta S_\phi}{\delta N} + 2N_i \frac{\delta S_\phi}{\delta N^i} + 2 \frac{NiN^j}{N^2} \frac{\delta S_\phi}{\delta h^{ij}} \right) \]
\[ T_{0i} = -\frac{N}{\sqrt{h}} \left( \frac{\delta S_\phi}{\delta N^i} + 2 \frac{N_j}{N^2} \frac{\delta S_\phi}{\delta h^{ij}} \right) \]
\[ T_{ij} = -\frac{2}{N\sqrt{h}} \left( \frac{\delta S_\phi}{\delta h^{ij}} \right). \tag{21} \]
These are given by
\[
T_{00} = \frac{N^2 p}{\sqrt{h}} \sqrt{h^{ij} \partial_i \phi \partial_j \phi} - 1 + \frac{2N}{\sqrt{h}} N^i p \partial_i \phi + \frac{N^i N^j}{\sqrt{h}} \frac{p \partial_i \phi \partial_j \phi}{\sqrt{h^{ij} \partial_i \phi \partial_j \phi} - 1} \\
T_{0i} = \frac{N}{\sqrt{h}} p \partial_i \phi + \frac{N^j}{\sqrt{h}} \frac{p \partial_i \phi \partial_j \phi}{\sqrt{h^{ij} \partial_i \phi \partial_j \phi} - 1} \\
T_{ij} = \frac{p \partial_i \phi \partial_j \phi}{\sqrt{h} \sqrt{h^{ij} \partial_i \phi \partial_j \phi} - 1}
\]  
which agrees with what is found in the Lagrangian formalism, \( T_{\mu\nu} = \lambda \partial_{\mu} \phi \partial_{\nu} \phi \), upon using equation (18). Starting from
\[
\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu_\nu) - \nabla_\mu \Gamma^\mu_\nu_\rho \sqrt{-g} T^\rho_\nu
\]
then it is easy to show that equation (19) is just the identity \( \nabla_\mu T^\mu_\nu = \nabla_0 T^0_\nu + \nabla_j T^j_\nu = 0 \).

To summarize, the equations of motion for mimetic gravity are those of Einstein’s gravity plus two more equations which we showed to be the constraint equation and the conservation of the energy-momentum tensor. However, the equations due to varying w.r.t \( h^{ij}, \pi_{ij}, N \) and \( N^i \) are those of pure Einstein’s gravity [5] but have extra terms as a function of the scalar field \( \phi \). This is how the mimetic dark matter enters the picture.

## 4 Poisson Brackets

In the presence of the scalar field, the combined constraints are
\[
H = H_{grav} + H_\phi; \quad H_i = H_{i,grav} + H_{i,\phi}.
\]  
This coupling is ‘non-derivative’ since the \( H_\phi \) does not depend on the momentum \( p \) [15]. For computing the poisson brackets, we consider smeared functions to have well-defined algebraic relationships. Then derivatives by fields are free of delta-distributions. Writing the constraints in smeared form [16] we have
\[
\mathbf{H}[N] = \int d^3x N(x) H(x) \\
\mathbf{D}[N^i] = \int d^3x N^i(x) H_i(x)
\]
Then we have
\[
\{H[N_1], H[N_2]\} = \{H_{\text{grav}}[N_1] + H_{\phi}[N_1], H_{\text{grav}}[N_2] + H_{\phi}[N_2]\}
\]
\[
= \{H_{\text{grav}}[N_1], H_{\text{grav}}[N_2]\} + \{H_{\text{grav}}[N_1], H_{\phi}[N_2]\}
\]
\[
+ \{H_{\phi}[N_1], H_{\text{grav}}[N_2]\} + \{H_{\phi}[N_1], H_{\phi}[N_2]\}
\]
(26)

Since the gravitational Hamiltonian constraint is free of spatial derivatives of the momenta, then we have
\[
\{H[N_1], H[N_2]\} = \{H_{\text{grav}}[N_1], H_{\text{grav}}[N_2]\} + \{H_{\phi}[N_1], H_{\phi}[N_2]\}
\]
(27)

Poisson brackets are defined as
\[
\{H_{\phi}[N_1], H_{\phi}[N_2]\} = \int d^3x \left( \frac{\delta H_{\phi}[N_1]}{\delta \varphi(x)} \frac{\delta H_{\phi}[N_2]}{\delta p(x)} - \frac{\delta H_{\phi}[N_1]}{\delta p(x)} \frac{\delta H_{\phi}[N_2]}{\delta \varphi(x)} \right)
\]
(28)

Functional derivatives are
\[
\frac{\delta H_{\phi}[N_1]}{\delta \varphi(x)} = N_1 \sqrt{h} \frac{dV}{dp} \quad \frac{\delta H_{\phi}[N_1]}{\delta p(x)} = N_1 \sqrt{h} \partial_k \partial_l \phi - 1
\]
(29)

which gives upon plugging in the integral
\[
\{H_{\phi}[N_1], H_{\phi}[N_2]\} = D_{\phi} \left[ h^{ij} \left( N_2 \partial_j N_1 - N_1 \partial_j N_2 \right) \right]
\]
(30)

Similarly we get
\[
\{D_{\phi}[N_1^i], D_{\phi}[N_2^i]\} = D_{\phi} \left[ N_1^i \partial_i N_2^j - N_2^j \partial_i N_1^j \right]
\]
(31)

where \( \frac{\delta D_{\phi}}{\delta \phi} = -\partial_i (p N^i) \) and \( \frac{\delta D_{\phi}}{\delta p} = N^i \partial_i \phi \). These equations are equivalent to the Dirac algebra [17] (see also [18]).

5 Canonical Formulation of Dust

The Hamiltonian formulation of dust was done by Brown and Kuchar [19]. They wrote the four-velocity \( u_\alpha \) of the dust as \( u_\alpha = -\partial_\alpha T + W_\alpha Z^\alpha \) where \( Z_i \) are the comoving coordinates of the dust particles, \( T \) is the proper time and \( W_\alpha \) are the 3-velocity components. The canonical action is given by

\[
S^D = \int d^4x \left( P \dot{T} + P_i \dot{Z}^i - N^D H^D - N^i H_i^D \right)
\]
(32)

where \( Z_i \) and \( T \) are the canonical coordinates,

\[
H^D = P \sqrt{h^{ij} u_i u_j - 1} \quad \text{and} \quad H_i^D = -Pu_i = P \partial_i T - PW_k \partial_i Z^k
\]
(33)

This agrees with our results when \( Z_i = 0 \). The scalar field \( \phi \) is then the proper time \( T \).
6 Conclusion

In this paper, we constructed the Hamiltonian of Mimetic Gravity. We showed that the two extra equations of motion are the constraint equation and the conservation of energy-momentum tensor. Then, the Poisson brackets are computed and the theory is shown to be closed. At last, our results are compared with the Hamiltonian action of dust.

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