AN EXAMPLE OF CIRCLE ACTIONS ON SYMPLECTIC CALABI-YAU MANIFOLD WITH NON-EMPTY FIXED POINTS.

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Abstract. Let \((X, \sigma, J)\) be a compact Kähler Calabi-Yau manifold equipped with a symplectic circle action. By Frankel’s theorem \([\text{F}]\), the action on \(X\) is non-Hamiltonian and \(X\) does not have any fixed point. In this paper, we will show that a symplectic circle action on a compact non-Kähler symplectic Calabi-Yau manifold may have a fixed point. More precisely, we will show that the symplectic \(S^1\)-manifold constructed by D. McDuff \([\text{McD}]\) has the vanishing first Chern class. This manifold has the Betti numbers \(b_1 = 3\), \(b_2 = 8\), and \(b_3 = 12\). In particular, it does not admit any Kähler structure.

1. Introduction

Let \((X, \omega)\) be a symplectic manifold and let \(G\) be a Lie group acting on \(X\). We say that the \(G\)-action is symplectic if \(g^*\omega = \omega\) for every \(g \in G\). Equivalently, the action is symplectic if and only if \(i_{\xi}\omega\) is a closed 1-form for every Lie algebra element \(\xi \in \mathfrak{g}\), where \(i_{\xi}\omega = \omega(\xi, \cdot)\) is an interior product with the fundamental vector field \(\xi\) of \(\xi\). When \(i_{\xi}\omega\) is exact for every \(\xi \in \mathfrak{g}\), then we say that the \(G\)-action is Hamiltonian. If the action is Hamiltonian, there exists a map \(\mu : X \to \mathfrak{g}^*\) defined by

\[
\mu(p)(\xi) := \mu_{\xi}(p), \forall p \in X, \forall \xi \in \mathfrak{g},
\]

where \(\mu_{\xi} : X \to \mathbb{R}\) is a \(C^\infty\)-function on \(X\) such that \(d\mu_{\xi} = i_{\xi}\omega\). We call \(\mu\) a moment map for the \(G\)-action. Therefore, it is natural to ask the followings.

**Question 1.** Let \((X, \omega)\) be a compact symplectic manifold. Then what conditions on \((X, \omega)\) make the symplectic action to be Hamiltonian? (non-Hamiltonian, respectively)

The following results are related to the question [11].

**Theorem 2.** \([\text{CKS}, \text{F}, \text{LO}, \text{McD}, \text{O}]\) Let \((X, \omega)\) be a compact symplectic manifold.

1. If \(X\) is simply-connected, then any symplectic action is Hamiltonian.
2. If \((X, \omega, J)\) is a Kähler manifold and if a given symplectic circle action preserves \(J\), then the action is Hamiltonian if and only if the fixed point set \(X^{S^1}\) is non-empty. (T. Frankel 1959 \([\text{F}]\),)
3. If \(\dim(X) = 4\), then any symplectic circle action is Hamiltonian if and only if the fixed point set \(X^{S^1}\) is non-empty. (D. McDuff 1988 \([\text{McD}]\),)
4. If \((X, \omega)\) satisfies the weak Lefschetz property, then any symplectic circle action is Hamiltonian if and only if the fixed point set \(X^{S^1}\) is non-empty. (K. Ono 1988 \([\text{O}]\),)
5. If \((X, \omega)\) is a monotone symplectic manifold, then any symplectic circle action is Hamiltonian. (G. Lupton, J. Oprea 1995 \([\text{LO}]\),)

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(6) If \((X, \omega)\) is symplectic Calabi-Yau, i.e. \(c_1(X) = 0\) in \(H^2(X; \mathbb{R})\), then any symplectic circle action is non-Hamiltonian. (Y. Cho, M. K. Kim, D. Y. Suh 2012 [CKS].)

In particular, if we combine the results (2) and (6) of Theorem 2 then we have the following corollary.

**Corollary 3.** Let \((X, \omega, J)\) be a compact Kähler Calabi-Yau manifold. Assume that there is a symplectic circle action preserving \(J\). Then \(X\) has no fixed point. In particular, the automorphism group of any simply connected Kähler Calabi-Yau manifold is discrete.

The main purpose of this paper is to announce that Corollary 3 does not hold on symplectic Calabi-Yau manifolds in general. Here, we state our main theorem.

**Theorem 4.** There exists a compact symplectic Calabi-Yau manifold \((X, \omega)\) equipped with a symplectic circle action such that the fixed point set \(X^{S^1}\) is non-empty.

In fact, there exists a 6-dimensional compact symplectic manifold equipped with a symplectic non-Hamiltonian circle action with non-empty fixed point set, which is constructed by D. McDuff in [McD]. As far as the authors know, McDuff’s example is the only one well-known example of symplectic non-Hamiltonian \(S^1\)-manifold with non-empty fixed point set. From now on, we denote the McDuff’s manifold by \((W, \tilde{\omega})\). As we will see in Section 2, \((W, \tilde{\omega})\) can be obtained by the quotient of some Hamiltonian \(S^1\)-manifold \((X, \omega)\) with the moment map \(\mu : X \rightarrow [0, 7] \subset \mathbb{R}\) with two boundaries \(\mu^{-1}(0)\) and \(\mu^{-1}(7)\), where the quotient map is given by some \(S^1\)-equivariant diffeomorphism between \(\mu^{-1}(0)\) and \(\mu^{-1}(7)\). From Section 2 to Section 3 we give an explicit computation of \(H_4(W; \mathbb{R})\) by following steps below.

**Step 1.** According to [McD], the critical values of the moment map \(\mu : X \rightarrow [0, 7]\) are 1, 2, 5, and 6. Hence the set of regular values of \(\mu\) is

\[\{0, 1\} \cup \{1, 2\} \cup \{2, 5\} \cup \{5, 6\} \cup \{6, 7\}.\]

Choose one value for each connected open regular interval. (In this paper, we will choose \(t_1 = 0 \in [0, 1]\), \(t_2 = 1.5 \in (1, 2)\), \(t_3 = 3.5 \in (2, 5)\), and \(t_4 = 5.5 \in (5, 6)\).

And then, we compute \(H_4(\mu^{-1}(t); \mathbb{R})\) and \(H_2(\mu^{-1}(t); \mathbb{R})\) for each regular value \(t = t_1, \ldots, t_4\), and describe the generators of \(H_4(\mu^{-1}(t); \mathbb{R})\) and \(H_2(\mu^{-1}(t); \mathbb{R})\) by some submanifolds of \(\mu^{-1}(t)\). Step 1 will be discussed in Section 3 and we use the result of Step 1 in the remaining steps.

**Step 2.** Note that the moment map \(\mu\) is a Morse-Bott function, so we can express \(X\) as the union of “elementary cobordisms” as follows.

\[X = X_1 \cup X_2 \cup X_3 \cup X_4,\]

where \(X_1 := \mu^{-1}([0, 1.5])\), \(X_2 := \mu^{-1}([1.5, 3.5])\), \(X_3 := \mu^{-1}([3.5, 5.5])\), and \(X_4 := \mu^{-1}([5.5, 7])\). In Section 3 we will study the topology of each elementary cobordism \(X_i\) to compute \(H_4(X_i; \mathbb{R})\) and \(H_2(X_i; \mathbb{R})\) for \(i = 1, \ldots, 4\).

**Step 3.** For each \(i = 1, \ldots, 4\), we will compute \(H_4(X_i; \mathbb{R})\) and describe the generators of \(H_4(X_i; \mathbb{R})\) in Section 3. In Section 3 we will compute \(H_2(X_i; \mathbb{R})\) and describe the generators of \(H_2(X_i; \mathbb{R})\) by some 2-dimensional symplectic submanifolds of \(X\) for \(i = 1, \ldots, 4\). After then, we compute \(H_4(X_1 \cup X_2; \mathbb{R})\) and \(H_4(X_3 \cup X_4; \mathbb{R})\) for \(j = 1\) or 2 in Section 4. Finally, we will show the followings in Section 5:

- \(H_4(W; \mathbb{R}) \cong \mathbb{R}^3\),
- \(H_2(W; \mathbb{R}) \cong \mathbb{R}^8\),
- \(\langle c_1(W), A \rangle = 0\) for every generator \(A \in H^2(W; \mathbb{R})\).
Remark 1. Using the fixed point formula for Hirzebruch genus, the Euler characteristic of $W$ equals to the sum of the Euler characteristic of the fixed components. So one can easily deduce that the Euler characteristic $\chi(W) = 0$ which is the alternating sum of the Betti numbers. Hence we have $H_3(W; \mathbb{R}) = \mathbb{R}^{12}$.

Remark 2. Note that $b_1(W) = 3$. Therefore, $W$ does not admit any Kähler structure (since $b_1$ is odd).

2. McDuff’s example

In this section, we review the 6-dimensional semifree symplectic non-Hamiltonian $S^1$-action $(W, \omega)$ constructed by McDuff [McD], and then define some submanifolds of $W$ by which we will describe $H_1(W; \mathbb{R})$ and $H_2(W; \mathbb{R})$ in the below. Henceforward, the coefficient of homology and cohomology groups is $\mathbb{R}$, and we use integral indices $1 \leq i, j \leq 4$, and if they are used simultaneously, then $i \neq j$.

As we introduced in Section 1 $(W, \bar{\omega})$ can be obtained by the quotient of some Hamiltonian $S^1$-manifold $(X, \omega)$ with the moment map $\mu : X \to [0, 7] \subset \mathbb{R}$ with two boundary components $\mu^{-1}(0)$ and $\mu^{-1}(7)$ such that

$$W = X/\sim, \quad x \sim y \Leftrightarrow \{x \in \mu^{-1}(0), y \in \mu^{-1}(7), \tau(x) = y\}$$

for some $S^1$-equivariant diffeomorphism $\tau : \mu^{-1}(0) \to \mu^{-1}(7)$. We summarize properties of $(X, \omega)$ as follows:

Properties of $(X, \omega)$. [McD] $(X, \omega)$ satisfies the followings.

1. $X$ has four critical levels at $\lambda = 1, 2, 5, 6$, and the critical point set is the union of four two-tori $Z^\lambda$ for $\lambda = 1, 2, 5, 6$ such that $\mu(Z^\lambda) = \lambda$.

2. Let $B = (S^1)^4$ be a 4-dimensional torus with coordinates $(x^1, x^2, x^3, x^4)$. By [McD] Lemma 5.(i)], there exists an $S^1$-invariant smooth map $\pi : X \to B$ satisfying the followings.

   a. For each regular value $s$, the restriction map $\pi|_{\mu^{-1}(s)} : \mu^{-1}(s) \to B$ induces a diffeomorphism $\mu^{-1}(s)/S^1 \cong B$.

   b. Let $\xi$ be the fundamental vector field of the $S^1$-action and let $J$ be an $S^1$-invariant $\omega$-compatible almost complex structure. If two points $x, x' \in X$ are connected by a flow $J\xi$, then $\pi(x) = \pi(x')$.

   c. Let $L_{ij} \subset B$ a two-torus on which the two coordinates other than $x^i$ and $x^j$ are constant. Then

$$\pi(Z^\lambda) = \begin{cases} L_{13} & \text{for } s = 1, 5, \\ L_{24} & \text{for } s = 2, 6. \end{cases}$$

To avoid confusion, we denote $Z^\lambda$ by $Z^\lambda_{ij}$ when $\pi(Z^\lambda) = L_{ij}$.

3. Let $\sigma_{ij} = dx^i \wedge dx^j$ and $\sigma_i = dx^i$ on $B$. For any regular value $s$, the Chern classes of the principal $S^1$-bundle $\pi|_{\mu^{-1}(s)} : \mu^{-1}(s) \to B$ is as follows.

$$c_1(\pi|_{\mu^{-1}(s)}) = \begin{cases} 0 & \text{for } s \in [0, 1), \\ -[\sigma_{42}] & \text{for } s \in (1, 2), \\ -[\sigma_{21} + \sigma_{42}] & \text{for } s \in (2, 5), \\ -[\sigma_{31}] & \text{for } s \in (5, 6), \\ 0 & \text{for } s \in (6, 7). \end{cases}$$

4. For $s = 0, 7$, we may consider $\mu^{-1}(s)$ as $T^5$ (since $c_1(\pi|_{\mu^{-1}(s)}) = 0$) with coordinates $x^1, x^2, x^3, x^4, x^5$ so that $\pi|_{\mu^{-1}(s)}$ is expressed as

$$\pi|_{\mu^{-1}(s)} : \mu^{-1}(s) \to B = T^4, \quad (x^1, x^2, x^3, x^4, x^5) \mapsto (x^1, x^2, x^3, x^4),$$

Obviously, the circle action on $\mu^{-1}(s)$ is given by

$$t \cdot (x^1, x^2, x^3, x^4, x^5) = (x^1, x^2, x^3, x^4, tx^5)$$
for every $t \in S^1$.

Our manifold $(W, \tilde{\omega})$ is obtained by gluing $\mu^{-1}(0)$ to $\mu^{-1}(7)$, where the gluing map $\tau : \mu^{-1}(0) \to \mu^{-1}(7)$ is given by an involution

$$\tau : \mu^{-1}(0) \to \mu^{-1}(7), \quad (x^1, x^2, x^3, x^4, x^5) \mapsto (x^3, x^4, x^1, x^2, x^5).$$

Let $\tilde{\pi} : X \to W$ be the quotient map induced by the gluing. Then, $W$ carries the generalized moment map

$$\psi : W \to S^1, \quad \tilde{\pi}(x) \mapsto \exp\left(\frac{2\pi i \mu(x)}{7}\right), \quad \forall x \in X.$$

To describe the generators of $H_1(W)$ and $H_2(W)$ explicitly, we define some submanifolds of $W$. First, we define loops in $W$. For $s = 0$ or 7, denote by $L^s_i \subset \mu^{-1}(s) = T^5$ a circle on which coordinates other than $x^i$ are constant $1 \leq i \leq 4$, and denote by $L^s_5$ a circle in $\mu^{-1}(s) = T^5$ on which coordinates other than $x^5$ are constant. Here, $f$ is the initial alphabet of “fiber”.

**Remark 3.** Sometimes, we will describe the generators of $H_*(W)$ by submanifolds of $X$, i.e. the reader should regard any submanifold $K \subset X$ as an image $\tilde{\pi}(K) \subset W$. For example, the gluing map $\tau$ identifies $L^0_0$ with $L^7_0$ and identifies $L^0_5$ with $L^7_5$. Hence we should keep in mind that $L^0_5 = L^7_0$ and $L^0_0 = L^7_5$ as generators of $H_1(W)$.

We define one more loop in $W$. Let $\text{Crit} \psi$ be the set of all critical points of $\psi$, i.e., $\text{Crit} \psi = \bigcup_{\lambda=1,2,5} Z^\lambda$. Fix two points $y_0, y_1 \in W$ in the level set $\psi^{-1}(1)$, $\psi^{-1}(i)$, respectively. And, choose two paths $\gamma_1, \gamma_2 : [0, 1] \to W - \text{Crit} \psi$ such that

$$\gamma_1(0) = \gamma_2(1) = y_0, \quad \gamma_1(1) = \gamma_2(0) = y_1,$$

$$\psi(\gamma_1(t)) = \exp(\pi i t), \quad \psi(\gamma_2(t)) = \exp(\pi i (t + 1)).$$

Let $\gamma : [0, 1] \to W$ be the loop $\gamma_1 \gamma_2$ so that $|\gamma| : \gamma \to S^1$ is a bijection. In Section 8 we will prove

**Proposition 1.** $H_1(W) = \langle L^0_0, L^0_2, \gamma \rangle \cong \mathbb{R}^3$.

Second, we define some tori and a sphere in $W$. For $s = 0$ or 7, denote by $L^s_{ij} \subset \mu^{-1}(s) = T^5$ a two-torus on which coordinates other than $x^i$ and $x^j$ are constant, and denote by $L^5_{ij}$ a two-torus in $\mu^{-1}(s) = T^5$ on which the two coordinates other than $x^i$ and $x^j$ are constant for $1 \leq i, j \leq 4$. Also, let $G_{61} \subset W$ be an $S^1$-invariant sphere which connects $Z^0_{1+3}$ and $Z^1_{13}$ and whose image through $\tilde{\mu}$ is the counterclockwise arc from $\exp(\frac{2\pi i 5}{4})$ to $\exp(\frac{2\pi i 3}{4})$.

**Remark 4.** Note that $G_{61}$ can be chosen as follows. Let $\gamma : [0, 1] \to W$ be a path such that

- $\gamma(0) \in Z^0_{1+3}$,
- $\gamma(1) \in Z^1_{13}$, and
- $\langle \text{grad} \gamma, J \xi \rangle > 0$, where $\langle , \rangle$ is a Riemannian metric induced by $\omega$ and $J$.

Then we get a 2-sphere $G_{61} := S^1 \cdot \gamma$.

Now, we define two more tori. Let $L^1_{1+3}, L^2_{2+4}$ be loops in $\mu^{-1}(0)$ defined by

\begin{align*}
\{ & (\exp(2\pi i t), 1, \exp(2\pi i t), 1, 1) \mid t \in \mathbb{R} \}, \\
\{ & (1, \exp(2\pi i t), 1, \exp(2\pi i t), 1) \mid t \in \mathbb{R} \},
\end{align*}

respectively. And, let $L^1_{1+3}, L^2_{2+4}$ be loops in $\mu^{-1}(7)$ defined by

\begin{align*}
\{ & (\exp(2\pi i t), 1, \exp(2\pi i t), 1, 1) \mid t \in \mathbb{R} \}, \\
\{ & (1, \exp(2\pi i t), 1, \exp(2\pi i t), 1) \mid t \in \mathbb{R} \},
\end{align*}

respectively. By varying constant coordinates of $L_{ij}$ other than $x_i, x_j$, we may assume that $\tilde{\pi}(L^1_{1+3}), \tilde{\pi}(L^2_{2+4})$ do not intersect any $L_{ij}$ in $B$. By (2)-(b) of “properties
of \((X, \omega)\), this implies that we may assume that \(L_{1+3}\) and \(L'_{1+3}\) (also \(L_{2+4}\) and \(L'_{2+4}\)) are connected by gradient flow of \(J_\xi\). And the trajectory of \(L_{1+3}\) along the gradient flow of \(J_\xi\) swept by the gradient flow beginning at \(L_{1+3}\) and \(L_{2+4}\) are diffeomorphic to \(L_{1+3} \times [0, 7]\) and \(L_{2+4} \times [0, 7]\) whose images \(\tilde{\pi}(L_{1+3} \times [0, 7])\) and \(\tilde{\pi}(L_{2+4} \times [0, 7])\) are called \(T^2_{1+3}\) and \(T^2_{2+4}\), respectively. In Section 8 we will prove the followings.

**Proposition 2.**

\[
H^2(W) = \langle L^0_{12}, L^0_{13}, L^0_{14}, L^0_{24}, Z^2_{13} \rangle = \mathbb{R}^5.
\]

**Proposition 3.** \(c_1(TW|_Q) = 0\) for each generator \(Q\) of \(H^2(W)\).

### 3. Homology groups of a regular level set \(\mu^{-1}(s)\)

In this section, we define \(L^s_i, L^s_{di}, L^s_{dij}\) in \(\mu^{-1}(s)\) for \(s = 1.5, 3.5, 5.5\) as we have defined for \(s = 0, 7\); and we compute the homology groups \(H_1(\mu^{-1}(s))\) for regular values \(s = 0, 1.5, 3.5\) for \(i = 1, 2\).

Let \(\pi : P \rightarrow B\) be an oriented smooth \(S^1\)-bundle. Then, the Gysin sequence is the following long exact sequence of de Rham cohomology groups:

\[
\begin{array}{ccccccc}
H^p(B) & \xrightarrow{\pi^*} & H^{p+2}(B) & \xrightarrow{\epsilon^*} & H^{p+2}(P) & \xrightarrow{\pi^*} & H^{p+1}(B) & \xrightarrow{\epsilon^*} & H^{p+3}(B) \\
\downarrow{\iota^*} & & \downarrow{\iota^*} & & \downarrow{\iota^*} & & \downarrow{\iota^*} & & \downarrow{\iota^*}
\end{array}
\]

where \(\pi^*\) is the pullback induced by \(\pi\), \(\epsilon^*\) is the wedge product of a differential form with the Euler class \(e\) of the bundle, and \(\iota^*\) is the integration along the fiber of differential forms. Applying \(\text{Hom}(\cdot, \mathbb{R})\) to this, we obtain the following exact sequence:

\[
\begin{array}{ccccccc}
H^p(B) & \xrightarrow{(\iota^*)^*} & H^{p+2}(B) & \xrightarrow{(\iota^*)^*} & H^{p+2}(P) & \xrightarrow{(\iota^*)^*} & H^{p+1}(B) & \xrightarrow{(\iota^*)^*} & H^{p+3}(B) \\
\end{array}
\]

where \(\iota^*\) means transpose. Here, \((\iota^*)^*\) is equal to the homology functor \(H_1(\pi)\). Let \(\iota : C \rightarrow B\) be a smooth inclusion of a circle \(C\). Since \(\pi_s\) is natural, we obtain the following commutative diagram:

\[
\begin{array}{ccccccc}
H^{p+2}(P) & \xrightarrow{(\iota^*)^*} & H^{p+1}(B) \\
\downarrow{\iota^*} & & \downarrow{\iota^*} \\
H^{p+1}(P|_C) & \xrightarrow{(\iota^*)^*} & H^{p+1}(C)
\end{array}
\]

where \(\iota^* : \iota^* P = P|_C \rightarrow A\) is the pullback bundle

\[
\begin{array}{ccccccc}
\pi^* P & \xrightarrow{\iota^*} & P \\
\iota^* & & \iota^* \\
C & \xrightarrow{\iota^*} & B
\end{array}
\]

Applying \(\text{Hom}(\cdot, \mathbb{R})\) to (3.3) for the case when \(p = 0\), we obtain

\[
\begin{array}{ccccccc}
H_2(P) & \xrightarrow{(\iota^*)^*} & H_1(B) \\
\downarrow{(\iota^*)^*} & & \downarrow{(\iota^*)^*} \\
H_2(P|_C) & \xrightarrow{(\iota^*)^*} & H_1(C)
\end{array}
\]

We will apply the Gysin sequence to the principal \(S^1\)-bundle \(\tilde{\pi}|_{\mu^{-1}(s)} : \mu^{-1}(s) \rightarrow B\) for regular values \(s = 0, 1.5, 3.5, 5.5, 7\). For simplicity, denote \(\mu^{-1}(s)\) and \(\tilde{\pi}|_{\mu^{-1}(s)}\) by \(P\) and \(\pi_s\), respectively. To express homology groups of level sets by their generators, we define some submanifolds of level sets. Pick a circle \(L^s_i\) in \(P\) such that

\[
\pi(L^s_i) = L_i
\]
for \( s = 1.5, 3.5, 5.5 \). This is always possible because the restricted bundle \( P|_{L_i} \) is a trivial bundle. Denote by \( L^i_0 \) for \( s = 1.5, 3.5, 5.5 \) a fiber of \( \mu^{-1}(s) \). If exists, pick a torus \( L^i_0 \) in \( P \) such that

\[
\pi(L^i_0) = L_{ij}
\]

for \( s = 1.5, 3.5, 5.5 \). This is possible only when \( P|_{L_{ij}} \) is trivial. Denote by \( L^i_0 \) the bundle \( P|_{L_i} = \pi^{-1}(L_i) \) for \( s = 1.5, 3.5, 5.5 \).

### 3.1. Homology of \( \mu^{-1}(s) \) for \( s = 0 \).

In this case, \( c_1(P) = 0 \), and \( \pi \) is

\[
(x^1, x^2, x^3, x^4) \mapsto (x^1, x^2, x^3, x^4).
\]

It is easy to see that

\[
\begin{align*}
H_1(\mu^{-1}(0)) &= (L^0_0 | 1 \leq i \leq 4) + (L^0_0 | 1 \leq i, j \leq 4) \\ H_2(\mu^{-1}(0)) &= (L^0_0 | 1 \leq i \leq 4) \\
\end{align*}
\]

\( \cong \mathbb{R}^5 \), \( \cong \mathbb{R}^{10} \).

### 3.2. Homology of \( \mu^{-1}(s) \) for \( s = 1.5 \).

In this case, \( c_1(P) = -[\sigma_{52}] \). So, \( L^i_0 \) does not exist, but other \( L^i_{1,5} \)'s exist. First, we calculate \( H_i(P) \). Substituting \( p = -1 \) into (3.1) and (3.2), we obtain the following:

\[
\begin{align*}
H^{-1}(B) &= 0 \xrightarrow{(e \wedge)} H^1(B) \cong \mathbb{R}^4 \xrightarrow{e^*} H^1(P) \cong \mathbb{R}^4 \xrightarrow{e^*} H_1(B) \cong \mathbb{R}^4 \cong \mathbb{R}^6, \\
H_{-1}(B) &= 0 \xrightarrow{(e \wedge)} H_{5}(B) \cong \mathbb{R}^4 \xrightarrow{e^*} H_5(B) \cong \mathbb{R}^4 \xrightarrow{e^*} H_2(B) \cong \mathbb{R}^4 \cong \mathbb{R}^6.
\end{align*}
\]

In (3.6), the first \( e \wedge \) is a zero map, and the second \( e \wedge \) is injective because \( e \) is nontrivial. By exactness,

\[
\ker \pi^* = 0 \quad \text{and} \quad \im \pi_* = 0.
\]

So,

\[
\im(\pi^*)_i = H_1(B) \quad \text{and} \quad \ker \pi_*^i = H_0(B).
\]

This means that \((\pi^*)_i \) is isomorphic. So,

\[
H_1(\mu^{-1}(1.5)) = (L^i_{1,5} | 1 \leq i \leq 4) \cong \mathbb{R}^4.
\]

Moreover, any fiber of \( P \) is trivial in \( H_i(P) \) because \((\pi^*)_i \) is equal to the homology functor \( H_i(\pi) \) and hence a fiber is sent to 0 by the isomorphism \((\pi^*)_i \).

Next, we calculate \( H_2(P) \). Substituting \( p = 0 \) into (3.1) and (3.2), we obtain the followings:

\[
\begin{align*}
H^0(B) &\cong \mathbb{R} \xrightarrow{\pi^*} H^1(B) \cong \mathbb{R}^6 \xrightarrow{\pi^*} H^2(B) \cong \mathbb{R}^4 \xrightarrow{\pi^*} H^3(B) \cong \mathbb{R}^4, \\
H_0(B) &\cong \mathbb{R} \xrightarrow{(e \wedge)} H_2(B) \cong \mathbb{R}^4 \xrightarrow{(e \wedge)} H_3(B) \cong \mathbb{R}^4.
\end{align*}
\]

In (3.7), the image of the first \( e \wedge \) is equal to \( \langle \sigma_{52} \rangle \), and the kernel of the second \( e \wedge \) is equal to \( \langle \sigma_2, \sigma_4 \rangle \). By exactness,

\[
\ker \pi^* = \langle \sigma_{42} \rangle \quad \text{and} \quad \im \pi_* = \langle \sigma_2, \sigma_4 \rangle.
\]

So,

\[
\im(\pi^*)_i = \langle \sigma_{42} \rangle^i = \langle L_{12}, L_{13}, L_{14}, L_{23}, L_{34} \rangle, \\
\ker \pi_*^i = \langle \sigma_2, \sigma_4 \rangle^i = \langle L_1, L_3 \rangle.
\]

From these, we obtain

\[
\begin{align*}
\im \pi_*^i &= \langle \pi_*^i(L_2), \pi_*^i(L_4) \rangle, \\
H_2(P) &= \langle L^i_{1,2}, L^i_{1,3}, L^i_{1,4}, L^i_{2,3}, L^i_{3,4}, \pi_*^i(L_2), \pi_*^i(L_4) \rangle \cong \mathbb{R}^7.
\end{align*}
\]

Substituting \( C = L_i \) into (3.4), we obtain

\[
\pi_*^i(L_2) = L^i_{2,1}, \quad \pi_*^i(L_4) = L^i_{4,1}, \quad L^i_{1,1} = L^i_{3,1} = 0
\]
in $H_2(P)$. That is,
\[
(3.8) \quad H_2(\mu^{-1}(1.5)) = \langle L_{11}^{1.5}, L_{15}^{1.5}, L_{21}^{1.5}, L_{24}^{1.5} \rangle + \langle L_{d1}^{1.5}, L_{d2}^{1.5} \rangle \cong \mathbb{R}^7
\]
where $L_{11}^{1.5} = L_{15}^{1.5} = 0$ in $H_2(\mu^{-1}(1.5))$.

3.3. Homology of $\mu^{-1}(s)$ for $s = 3.5$. In this case, $c_1(P) = -[\sigma_{31} + \sigma_4]$, and $L_{13}^{3.5}, L_{24}^{3.5}$ do not exist. Also, pick a class in $H_2(\pi^{-1}(L_{13} - L_{24}))$ and call it $(L_{13} - L_{24})^{3.5}$. This is possible because the pairing of $c_1(P)$ and $L_{13} - L_{24}$ is zero.

First, calculation of $H_1(P)$ is similar to $s = 1.5$. That is,
\[
H_1(\mu^{-1}(3.5)) = \langle L_i^{3.5} \mid 1 \leq i \leq 4 \rangle \cong \mathbb{R}^4,
\]
and any fiber of $P$ is trivial in $H_1(P)$.

Next, we calculate $H_2(P)$. Substituting $p = 0$ into (3.1) and (3.2), we obtain the followings:
\[
(3.9) \quad H^0(B) \cong \mathbb{R}^3, H^2(B) \cong \mathbb{R}^6, H^3(B) \cong \mathbb{R}^4, H^1(B) \cong \mathbb{R}^4
\]
\[
H_0(B) \cong \mathbb{R}^3, H_2(B) \cong \mathbb{R}^6, H_3(B) \cong \mathbb{R}^4, H_1(B) \cong \mathbb{R}^4
\]
In (3.10), the image of the first $e \wedge$ is equal to $\langle \sigma_{31} + \sigma_4 \rangle$, and the kernel of the second $e \wedge$ is trivial. By exactness,
\[
\ker \pi_* = \langle \sigma_{31} + \sigma_4 \rangle \quad \text{and} \quad \im \pi_* = \langle 0 \rangle.
\]
So,
\[
\im(\pi_*)' = \langle \sigma_{31} + \sigma_4 \rangle = \langle L_{12}, L_{13} - L_{24}, L_{14}, L_{23}, L_{34} \rangle,
\]
\[
\ker \pi_*' = \langle 0 \rangle = H_1(B).
\]
From these, we obtain $\im \pi_*' = \langle 0 \rangle$, and
\[
(3.10) \quad H_2(\mu^{-1}(3.5)) = \langle L_{12}^{3.5}, (L_{13} - L_{24})^{3.5}, L_{14}^{3.5}, L_{23}^{3.5}, L_{34}^{3.5} \rangle \cong \mathbb{R}^5.
\]
Substituting $C = L_i$ into (3.10), we obtain $L_i^{3.5} = 0$ in $H_2(\mu^{-1}(3.5))$ for $1 \leq i \leq 4$.

4. Elementary cobordism

In this section, we study an elementary cobordism $\mu^{-1}[a, b]$ for two regular values $a < b$ of $\mu$ such that $\lambda = \mu(Z^\lambda)$ is the unique critical value between them.

For each critical submanifold $Z^\lambda$ of $X$, the almost Kähler structure $J$ induces a complex structure on the normal bundle $\nu$ of $Z^\lambda$ in $X$ which splits as a sum $\nu = \nu^0 \oplus \nu^\pm$. By the equivariant symplectic neighborhood theorem, we may assume that a small neighborhood $N$ of $Z$ for a sufficiently small $\varepsilon$ is equivariantly symplectically diffeomorphic to the interior of $D_\varepsilon(\nu)$ of the zero section of $\nu$ where $D_\varepsilon(\cdot)$ is the disc bundle with radius $\varepsilon$. And, we may assume that $D_\varepsilon(\nu^-)$ and $D_\varepsilon(\nu^+)$ are contained in stable and unstable manifolds of $Z^\lambda$ with respect to the vector field $J_\varepsilon$, respectively. To calculate homology groups of $\mu^{-1}[a, b]$, we need calculate the first Chern classes of $\nu^\pm$. For this, we will show that
\[
(4.1) \quad c_1(\nu^-) = c_1((\pi|_{Z^\lambda})^* \mu^{-1}(a)), \quad c_1(\nu^+) = c_1((\pi|_{Z^\lambda})^* \mu^{-1}(b)),
\]
\[
(4.2) \quad c_1(\nu^-) = -c_1(\nu^+)
\]
where $\mu^{-1}(a), \mu^{-1}(b)$ are regarded as circle bundles over $B$. We can observe that $Z^\lambda$ are connected by $J_\varepsilon$ to $\mu^{-1}(a)|_{\pi(Z^\lambda)}$ and $\mu^{-1}(b)|_{\pi(Z^\lambda)}$. Since the flow of $J_\varepsilon$ is equivariant, this observation means that
\[
S_\varepsilon(\nu^-) \cong (\pi|_{Z^\lambda})^* \mu^{-1}(a) \quad \text{and} \quad S_\varepsilon(\nu^+) \cong (\pi|_{Z^\lambda})^* \mu^{-1}(b)
\]
as $S^1$-bundles where $S_\varepsilon(\cdot)$ is the sphere bundle with radius $\varepsilon$. So, we obtain (4.1). Also, we obtain (4.2) because the normal bundle of $\pi(Z^\lambda)$ in $B$ is isomorphic to
\( \nu^- \oplus \nu^+ \) by [MeM] p. 156] and is trivial. By using (4.1) and (4.2), we can calculate \( c_1(\nu^\pm) \) as follows:

\[
\begin{align*}
  c_1(\nu^-) &= -c_1(\nu^+) = 0 & \text{for } \lambda = 1, 6, \\
  c_1(\nu^-) &= -c_1(\nu^+) = \pm 1 & \text{for } \lambda = 2, 5,
\end{align*}
\]

up to orientation of \( Z^\lambda \). When \( \hat{\pi}(Z^\lambda) = L_{ij} \), put \( Z^\lambda_i = \hat{\pi}^{-1}(L_i) \cap Z^\lambda_j \). When \( S_c(\nu^\pm) \) is trivial, pick a section of \( S_t(\nu^\pm) \) and denote it by \( Z^\lambda_{ij} \). And, pick a section of \( S_c(\nu^\pm | Z^\lambda) \) and denote it by \( Z^\lambda_{i,j} \). For simplicity, we also denote

\[
S_c(\nu^\pm | Z^\lambda), S_c(\nu^\pm | Z^\lambda_j) \quad \text{by} \quad Z^\lambda_{i,j}, Z^\lambda_{j},
\]

respectively. And, let \( Z^\lambda_{ij} \) be a fiber of \( S_c(\nu^\pm) \). By using Gysin sequence as in Section 3 we can calculate homology groups of sphere bundles \( S_c(\nu^\pm) \) as follows:

\[
\begin{align*}
  H_1(S_c(\nu^\pm)) &= \begin{cases} 
    \langle Z^\lambda_{ij}, Z^\lambda_{ij} \rangle & \text{for } \lambda = 1, 6, \\
    \langle Z^\lambda_{ij}, Z^\lambda_{ij} \rangle & \text{for } \lambda = 2, 5,
  \end{cases} \\
  H_2(S_c(\nu^\pm)) &= \begin{cases} 
    \langle Z^\lambda_{ij}, Z^\lambda_{ij}, Z^\lambda_{ij} \rangle & \text{for } \lambda = 1, 6, \\
    \langle Z^\lambda_{ij}, Z^\lambda_{ij} \rangle & \text{for } \lambda = 2, 5.
  \end{cases}
\end{align*}
\]

**Remark 5.** In this review, \( Z^\lambda_{i,j} \) and \( Z^\lambda_{ij} \) are arbitrary sections by definition. However, in the next section, we will designate more specific sections to these.

Before we go further, we review Mayer-Vietoris sequence. For a pair of subspaces \( A, A' \) of a topological space \( Y \) such that \( Y \) is the union of the interiors of \( A, A' \), this exact sequence has the form:

\[
\cdots \longrightarrow H_{n+1}(Y) \xrightarrow{\partial^*} H_n(A \cap A') \xrightarrow{(\iota_* \circ \partial_*)} H_n(A) \oplus H_n(A') \xrightarrow{k_{2n-4}} H_n(Y) \longrightarrow \cdots
\]

The boundary maps \( \partial \), lowering the dimension may be made explicit as follows. An element \( y \) in \( H_n(Y) \) is the homology class of an \( n \)-cycle \( Y \) which, by barycentric subdivision for example, can be written as the sum of two \( n \)-chains \( u \) and \( v \) whose images lie wholly in \( A \) and \( A' \), respectively. Thus \( \partial y = \partial(u + v) = \partial u + \partial v \). This implies that the images of both these boundary \((n-1)\)-cycles are contained in the intersection \( A \cap A' \). Then, \( \partial_*([x]) \) is the class of \( \partial u \in H_{n-1}(A \cap A') \).

Since \( \mu \) is a Morse-Bott function, the elementary cobordism \( \mu^{-1}[a, b] \) is homeomorphic to the attaching space

\[
\mu^{-1}[a, a'] \bigcup_f D_c(\nu^-) \oplus D_c(\nu^+) \tag{4.7}
\]

for \( a < a' < \lambda \) with a attaching map \( f : S_c(\nu^-) \oplus D_c(\nu^+) \rightarrow \mu^{-1}(a') \) by [P] Section 11; [V]. Here, the restriction of \( f \) to \( S_c(\nu^-) \)

\[
f|_{S_c(\nu^-)} : S_c(\nu^-) \oplus 0 \rightarrow \mu^{-1}(a')|_{\hat{\pi}(\nu^-)}
\]

is defined by the flow of \( J\xi \). Similarly, \( \mu^{-1}[a, b] \) is homeomorphic to the attaching space

\[
D_c(\nu^-) \oplus D_c(\nu^+) \bigcup_f \mu^{-1}[b', b] \tag{4.8}
\]

for \( \lambda < b' < b \) with a attaching map \( f' : D_c(\nu^-) \oplus S_c(\nu^+) \rightarrow \mu^{-1}(b') \) whose restriction to \( S_c(\nu^+) \) is defined by the flow of \( J\xi \). Since \( D_c(\nu^+) \oplus S_c(\nu^+) \) is homotopically equivalent to \( S_c(\nu^+) \), we can obtain \( H_1 \) and \( H_2 \) of \( D_c(\nu^\pm) \oplus S_c(\nu^\pm) \) by (4.3), (4.4). Furthermore since we already know homology groups of level sets, we can calculate
homology of the elementary cobordism by using Mayer-Vietoris sequence. More precisely, we will calculate the first and second homology groups of two elementary cobordisms

\[(4.9) \quad \mu^{-1}[0, 1.5] \quad \text{and} \quad \mu^{-1}[1.5, 3.5].\]

in two ways of (4.7), (4.8). Then, we will calculate cohomology of

\[\mu^{-1}[0, 3.5] = \mu^{-1}[0, 1.5] \cup \mu^{-1}[1.5, 3.5],\]

and finally \(W\) again by Mayer-Vietoris sequence. When we apply Mayer-Vietoris sequence to attaching (4.7) or (4.8) in the below, we will use the following notations:

\[\text{Mayer-Vietoris sequence to (4.7), (4.8) for } n\]

\[\text{or}\]

\[\text{We attach } \lambda\]

\[\text{and the next sections.}\]

In this section, we calculate the first homology group of elementary cobordisms of (4.9). As stated in Remark 5, we will define \(Z_{\lambda}^{\pm}, Z_{\mu}^{\pm}, Z_{\nu}^{\pm}\) more precisely by specifying their images through \(i_{\lambda}\) in Mayer-Vietoris sequences appearing in this and the next sections.

Let \(\mu^{-1}[a, b]\) be one of (4.9), and let \(\nu\) be the unique critical value between \(a\) and \(b\). Let \(\nu^{\pm}\) be normal bundles over \(Z^{\lambda}\). To calculate \(H_1(\mu^{-1}[a, b])\), we apply Mayer-Vietoris sequence to (4.7), (4.8) for \(n = 1, \lambda = 1, 2\) as follows:

1. when we apply Mayer-Vietoris sequence to (4.7) for \(\lambda = 1,\)

\[
\begin{align*}
\text{2. when we apply Mayer-Vietoris sequence to (4.8) for } &\lambda = 1, \\
\text{3. when we apply Mayer-Vietoris sequence to (4.7) for } &\lambda = 2, \\
\text{4. when we apply Mayer-Vietoris sequence to (4.8) for } &\lambda = 2,
\end{align*}
\]

By observing these sequences, we can show the followings:

i. \(H_1(\mu^{-1}[a, b]) \cong \mathbb{R}^4\)

ii. \(H_1(\lambda) : H_1(\mu^{-1}[a, b]) \to H_1(\lambda)\) is isomorphic,
iii. For \( s = a \) or \( b \), inclusion induces
\[
H_1(\mu^{-1}(s)) \longrightarrow H_1(\mu^{-1}[a, b])
\]
\[
L^*_7 \longrightarrow L^*_7 \quad \text{L}^*_8 \longrightarrow L^*_8
\]
where \( L^*_i \) might be trivial in \( H_1(\mu^{-1}(s)) \).

6. THE SECOND HOMOLOGY GROUPS OF ELEMENTARY COBORDISMS

In this section, we calculate the second homology groups of elementary cobordisms of \((4.9)\). For this, we apply Mayer-Vietoris sequence to \((4.7), (4.8)\) for \( n = 2 \), \( \lambda = 1, 2 \). As we have showed three thing on the first homology in the previous section, we will describe the second homology groups by their generators, and deal with maps induced by \( \bar{\mu} \) and inclusions.

6.1. \( H_2 \) of \( \mu^{-1}[0, 1.5] \). Applying Mayer-Vietoris sequence to \((4.7)\) for \( n = 2 \), \( \lambda = 1 \), we obtain the followings:
\[
H_2(A \cap A') \cong \mathbb{R}^3 \xrightarrow{\langle i_*, j_* \rangle} H_2(A) \oplus H_2(A') \cong \mathbb{R}^{10} \oplus \mathbb{R} \xrightarrow{\kappa_* - \iota_*} H_2(Y) \xrightarrow{\partial_*}
\]
(6.1)
So, the rank of \( \text{im}(i_*, j_*) \) is rank 3. Since \( \langle i_*, j_* \rangle \) for \( n = 1 \) is injective, \( \partial_* \) is a zero-map. This implies that \( H_1(Y) \) is rank 3, and we can check that
\[
H_2(\mu^{-1}[0, 1.5]) \cong \langle L_0^0, L_0^0, L_1^0, L_1^0, \cdots \rangle \cong \mathbb{R}^5
\]
(6.2)
in which \( L_0^0 = Z_1 \) because \( L_0^0, Z_1 \) is contained in \( \text{im}(i_*, j_*) \). The map \( \bar{\mu} \) induces the following map:
\[
H_2(\mu^{-1}[0, 1.5]) \cong \mathbb{R}^5 \longrightarrow H_2(T^4) \cong \mathbb{R}^6
\]
(6.3)
\[
L_0^i \longrightarrow 0 \quad \text{for } i = 2, 4,
L_0^i \longrightarrow L_1^i \quad \text{for } 1 \leq i \neq j \leq 4.
\]
And, the inclusion induces the following surjection:
\[
H_2(\mu^{-1}(0)) \cong \mathbb{R}^{10} \longrightarrow H_2(\mu^{-1}[0, 1.5]) \cong \mathbb{R}^5
\]
(6.4)
\[
L_i^0 \longrightarrow 0 \quad \text{for } i = 1, 3,
L_i^0 \longrightarrow L_i^0 \quad \text{for } i = 2, 4,
L_i^0 \longrightarrow L_i^0 \quad \text{for } 1 \leq i \neq j \leq 4.
\]

Applying Mayer-Vietoris sequence to \((4.8)\) for \( n = 2 \), \( \lambda = 1 \), we obtain the followings:
\[
H_2(A \cap A') \cong \mathbb{R}^3 \xrightarrow{\langle i_*, j_* \rangle} H_2(A) \oplus H_2(A') \cong \mathbb{R}^{10} \oplus \mathbb{R}^7 \xrightarrow{\kappa_* - \iota_*} H_2(Y) \xrightarrow{\partial_*}
\]
(6.5)
By this,
\[
H_2(\mu^{-1}[0, 1.5]) \cong \langle L_1^1, L_2^1, L_1^2, L_2^2, \cdots \rangle \cong \mathbb{R}^5
\]
and \( Z_1^1 = L_1^1 \). Comparing this with (6.1), we have
\[
H_2(\mu^{-1}[0, 1.5]) \cong \langle L_1^1, L_2^1, L_1^2, L_2^2, \cdots \rangle \cong \langle L_1^1, L_2^1 \rangle \cong \mathbb{R}^8.
\]
(6.6)
The map \( \bar{\mu} \) induces the following surjection:
\[
H_2(\mu^{-1}[0, 1.5]) \cong \mathbb{R}^8 \longrightarrow H_2(T^4) \cong \mathbb{R}^6
\]
(6.7)
\[
L_i^0 \longrightarrow L_i^0 \quad \text{for } i = 1, 3,
L_i^0 \longrightarrow L_i^0 \quad \text{for } 1 \leq i \neq j \leq 4, \{i, j\} \neq \{2, 4\}.
\]
And, the inclusion induces the following injection:
\[
H_2(\mu^{-1}(1.5)) \cong \mathbb{R}^7 \longrightarrow H_2(\mu^{-1}[0, 1.5]) \cong \mathbb{R}^8
\]
(6.8)
\[
L_i^0 \longrightarrow L_i^0 \quad \text{for } i = 1, 3,
L_i^0 \longrightarrow L_i^0 \quad \text{for } 1 \leq i \neq j \leq 4, \{i, j\} \neq \{2, 4\}.
\]
and $L_{24}^0$ is not contained in its image.

Comparing (6.3) with (6.7), maps (6.4), (6.5) give us relations between $Z_{13}$ and all generators of $\mu^{-1}(0), \mu^{-1}(1.5)$ as follows:

$$\begin{align*}
L^0_i &= L^{1.5}_{4i} & \text{for } 1 \leq i \neq j \leq 4, \{i, j\} \neq \{2, 4\}.
L^0_i &= L^{1.5}_{4i} & \text{for } i = 2, 4.
L_{13}^{2j} &= 0 & \text{for } i = 1, 3.
Z_{13}^{1j} &= L_{13}^0 & = L_{13}^{1.5}.
\end{align*}$$

(6.9)

up to $(L_{24}^0, L_{4f}^0) = \langle L_{12}^{1.5}, L_{24}^{1.5} \rangle$ in $H_2(\mu^{-1}[0, 1.5])$ where $L_{24}^0 = L_{24}^0$ means that there exists no relation on $L_{24}^0$.

6.2 $H_2$ of $\mu^{-1}[1.5, 3.5]$. Applying Mayer-Vietoris sequence to (4.7) for $n = 2, \lambda = 2$, we obtain the followings:

$$\begin{align*}
H_2(A \cap A') &\cong \mathbb{R}^2 \xrightarrow{\{i, j\}} H_2(A') &\cong \mathbb{R}^2
\end{align*}$$

(6.10)

So, the rank of $im(i_*, j_*)$ is rank two. Since $(i_*, j_*)$ for $n = 1$ is injective, $\partial_*$ is a zero-map. This implies that $H_1(Y)$ is rank six, and we can check that

$$H_2(\mu^{-1}[1.5, 3.5]) = \langle L_{12}^{1.5}, L_{13}^{1.5}, L_{14}^{1.5}, L_{23}^{1.5}, L_{34}^{1.5} \rangle \oplus \langle Z_{24}^2 \rangle \cong \mathbb{R}^6.$$

(6.11)

The map $\hat{\mu}$ induces the following isomorphism:

$$\begin{align*}
H_2(\mu^{-1}[1.5, 3.5]) &\cong \mathbb{R}^6 \xrightarrow{\{i, j\}} H_2(T^4) &\cong \mathbb{R}^6
\end{align*}$$

(6.12)

$Z_{24}^2$ is not contained in its image.

Applying Mayer-Vietoris sequence to (4.8) for $n = 2, \lambda = 2$, we obtain the followings:

$$\begin{align*}
H_2(A \cap A') &\cong \mathbb{R}^2 \xrightarrow{\{i, j\}} H_2(A') &\cong \mathbb{R}^2
\end{align*}$$

(6.13)

and $Z_{24}^2$ is not contained in its image.

Comparing (6.12) with (6.16), maps (6.14), (6.17) give us relations between $Z_{24}^2$ and all generators of $\mu^{-1}(1.5), \mu^{-1}(3.5)$ as follows:

$$\begin{align*}
L_{13}^{2j} &= L_{13}^{3.5} & \text{for } 1 \leq i \neq j \leq 4, \{i, j\} \neq \{1, 3\}, \{2, 4\}.
L_{24}^{2j} &= L_{24}^{3.5} & \text{for } 1 \leq i \neq j \leq 4, \{i, j\} \neq \{1, 3\}, \{2, 4\}.
\end{align*}$$

(6.18)

in $H_2(\mu^{-1}[1.5, 3.5])$. 

AN EXAMPLE OF CIRCLE ACTIONS ON SYMPLECTIC CALABI-YAU MANIFOLDS 11

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7. Homology groups of the union of two elementary cobordisms

In this section, we calculate homology groups of $\mu^{-1}(0, 3.5]$. Put

\[ Y = \mu^{-1}(0, 3.5], \quad A = \mu^{-1}(0, 1.5], \quad A' = \mu^{-1}(1.5, 3.5]. \]

Applying Mayer-Vietoris sequence to (7.1) for $n = 1$, we obtain the followings:

\[
\begin{align*}
H_1(\Lambda \setminus A') &\cong \mathbb{R}^4 \quad \overset{\partial}{\longrightarrow} \quad H_1(\Lambda \setminus A) \oplus H_1(\Lambda) \cong \mathbb{R}^4 \oplus \mathbb{R}^4
\end{align*}
\]

Since two $(i_*, j_*)'$s are injective, $H_1(\mu^{-1}(0, 3.5])$ is rank 4, and

\[ H_1(\hat{\pi}) : H_1(\mu^{-1}(0, 3.5]) \longrightarrow H_1(B) \]

is isomorphic.

Applying Mayer-Vietoris sequence to (7.1) for $n = 2$, we obtain the followings:

\[
\begin{align*}
H_2(\Lambda \setminus A') &\cong \mathbb{R}^7 \quad \overset{(i_*, j_*)}{\longrightarrow} \quad H_2(\Lambda) \oplus H_2(A') \cong \mathbb{R}^8 \oplus \mathbb{R}^6 \quad \overset{\partial}{\longrightarrow} \quad H_2(\Lambda) \oplus H_2(A') \cong \mathbb{R}^8 \oplus \mathbb{R}^6
\end{align*}
\]

Since $(i_*, j_*)$ for $n = 1$ is injective as we have seen in (7.2), $\partial_*$ is zero-map and hence $k_* - l_*$ is surjective by exactness. Also since $(i_*, j_*)$ for $n = 2$ is injective, the rank of $\text{im} k_* - l_*$ is 7. So, $H_2(\mu^{-1}(0, 3.5])$ is rank 7. Next, we find generators of $H_2(\mu^{-1}(0, 3.5])$. By exactness, $\text{im}(i_*, j_*)$ for $n = 2$ gives relation in $H_2(\hat{\pi})$. More precisely, two elements

\[ (L_{12}^{1.5}, 0), \quad (L_{14}^{1.5}, 0) \in H_2(\Lambda) \oplus H_2(A') \]

give the relation $L_{12}^{1.5} = 0, L_{14}^{1.5} = 0$ in $H_2(\hat{\pi})$, respectively. By using (6.9), (6.18) in addition to these relations, we can check that $L_{12}^{1.5}, L_{13}^{1.5}, L_{14}^{1.5}, L_{15}^{1.5}, L_{23}^{1.5}, L_{14}^{1.5}, L_{24}^{1.5}, Z_{24}^{1.5}$ generate $H_2(\mu^{-1}(0, 3.5])$. And, this implies

\[ H_2(\mu^{-1}(0, 3.5]) = \langle L_{12}^{1.5}, L_{13}^{1.5}, L_{14}^{1.5}, L_{15}^{1.5}, L_{24}^{1.5} \rangle + \langle L_{24}^{1.5}, Z_{24}^{1.5} \rangle \cong \mathbb{R}^7 \]

because $H_2(\mu^{-1}(0, 3.5])$ is rank 7. In this way, we can also show that relations between $Z_{14}^{1.5}, Z_{24}^{1.5}$, and all generators of $\mu^{-1}(0, 1.5], \mu^{-1}(1.5), \mu^{-1}(3.5)$ are as follows:

\[
\begin{align*}
L_{ij}^{0} &= L_{ij}^{1.5} = L_{ij}^{3.5} = 0 \quad \text{for } 1 \leq i \neq j \leq 4, \quad \{i, j\} \neq \{1, 3\}, \quad \{2, 4\},
L_{ij}^{0} &= L_{ij}^{1.5} = L_{ij}^{3.5} = 0 \quad \text{for } i = 1, 2, 3, 4,
L_{ij}^{0} &= L_{ij}^{1.5} = Z_{13}. \quad \text{for } i = 1, 2, 3, 4,
L_{ij}^{0} &= L_{ij}^{1.5} = Z_{24}. \quad \text{for } i = 1, 2, 3, 4,
(L_{14} - L_{24})^{3.5} &= L_{14}^{1.5} - Z_{24}
\end{align*}
\]

in $H_2(\mu^{-1}(0, 3.5])$.

In the exactly same way with $H_n(\mu^{-1}(0, 3.5])$ for $n = 1, 2$, we can calculate $H_n(\mu^{-1}(3.5, 7])$ for $n = 1, 2$ as follows:

\[ H_2(\mu^{-1}(3.5, 7]) = \langle L_{12}^{1.5}, L_{13}^{1.5}, L_{14}^{1.5}, L_{23}^{1.5}, Z_{14}^{1.5} \rangle + \langle L_{13}, Z_{14} \rangle \cong \mathbb{R}^7. \]

And we can also show that relations between $Z_{13}^{1.5}, Z_{24}^{1.5}$, and all generators of $\mu^{-1}(3.5), \mu^{-1}(5.5), \mu^{-1}(7)$ are as follows:

\[
\begin{align*}
L_{ij}^{1.5} &= L_{ij}^{3.5} = L_{ij}^{7} \quad \text{for } 1 \leq i \neq j \leq 4, \quad \{i, j\} \neq \{1, 3\}, \quad \{2, 4\},
L_{ij}^{1.5} &= L_{ij}^{3.5} = L_{ij}^{7} = 0 \quad \text{for } i = 1, 2, 3, 4,
L_{ij}^{1.5} &= L_{ij}^{3.5} = L_{ij}^{7} \quad \text{for } i = 1, 2, 3, 4,
(L_{14} - L_{24})^{3.5} &= L_{12}^{1.5} - Z_{24}
\end{align*}
\]

in $H_2(\mu^{-1}(3.5, 7])$. 

Remark 6. We can observe that if we exchange $i, j, s, \lambda$ of generators of the homology group $H_n(\mu^{-1}[0, 3.5])$ for $n = 1, 2$ through

\[
i \mapsto 5-i, \quad j \mapsto 5-j, \quad s \mapsto 7-s, \quad \lambda \mapsto 7-\lambda,
\]

for example

\[
L_2^{1,5} \mapsto L_3^{5,5}, \quad L_{13}^0 \mapsto L_{42}^7, \quad Z_{13}^1 \mapsto Z_{42}^6,
\]

then we obtain $H_n(\mu^{-1}[3.5, 7])$ for $n = 1, 2$. This is also applicable to their relations. The reason for this relation between $H_n(\mu^{-1}[0, 3.5])$ and $H_n(\mu^{-1}[3.5, 7])$ can be found in [McD, p. 157].

8. Homology groups of $W$

In this section, we calculate homology groups of $W$. The manifold $W$ can be considered as the union of $\mu^{-1}[0, 3.5]$ and $\mu^{-1}[3.5, 7]$. Put

\[
Y = W, \quad A = \mu^{-1}[0, 3.5], \quad A' = \mu^{-1}[3.5, 7].
\]

Then, $A \cap A' = \mu^{-1}(0) \cup \mu^{-1}(3.5)$. And,

\[
\begin{align*}
H_2(A \cap A') &= H_2(\mu^{-1}(0)) \oplus H_2(\mu^{-1}(3.5)) \cong \mathbb{R}^{10} \oplus \mathbb{R}^5, \\
H_1(A \cap A') &= H_1(\mu^{-1}(0)) \oplus H_1(\mu^{-1}(3.5)) \cong \mathbb{R}^5 \oplus \mathbb{R}^4, \\
H_0(A \cap A') &= H_0(\mu^{-1}(0)) \oplus H_0(\mu^{-1}(3.5)) \cong \mathbb{R}^1 \oplus \mathbb{R}^1.
\end{align*}
\]

Henceforward, we fix orders of these summands. Applying Mayer-Vietoris sequence to this for $n = 0, 1$, we obtain the followings:

\[
\begin{align*}
H_1(A \cap A') &\cong \mathbb{R}^5 \oplus \mathbb{R}^4, \\
(0, L_7^3, L_3^5) &\mapsto (L_7^3, L_3^5), \\
(0, 0) &\mapsto (0, 0), \\
(L_2^0, 0), j = 1, 2 &\mapsto (L_2^0, L_2^7), \\
(L_2^0, 0), j = 3, 4 &\mapsto (L_2^0, L_2^7), \\
H_0(A \cap A') &\cong \mathbb{R}^2, \\
(0, 0) &\mapsto (0, 0), \\
(y_0, y_1) &\mapsto (y_0, y_1).
\end{align*}
\]

Here, we use the fact that the level set $\mu^{-1}(0)$ is glued to $\mu^{-1}(7)$ by the involution $\tau$. The image and kernel of $(i_*, j_*)$ for $n = 1$ are as follows:

\[
\begin{align*}
\text{im}(i_*, j_*) &= \langle (L_7^3, L_3^5) \mid 1 \leq i \leq 4 \rangle + \langle (L_2^0, L_2^7), (L_2^0, L_2^7) \rangle \cong \mathbb{R}^6, \\
\text{ker}(i_*, j_*) &= \langle (L_7^0 + L_7^0, -L_7^{1,5} - L_7^{3,5}), (L_2^0 + L_2^0, -L_2^{3,5} - L_2^{4,5}), (L_2^0, 0) \rangle \cong \mathbb{R}^3.
\end{align*}
\]

And, the image and kernel of $(i_*, j_*)$ for $n = 0$ are as follows:

\[
\begin{align*}
\text{im}(i_*, j_*) &= \langle (y_0, y_0) = (y_1, y_1) \rangle \cong \mathbb{R}^1, \\
\text{ker}(i_*, j_*) &= \langle (-y_0, y_1) \rangle \cong \mathbb{R}^1.
\end{align*}
\]

We can show that $\partial_*(\gamma) = (-y_0, y_1)$ by definition of $\partial_*$. By these, we can conclude

\[
H_1(W) = \langle L_7^0, L_7^0, \gamma \rangle \cong \mathbb{R}^3
\]

where $L_1^0 = L_3^0, L_2^0 = L_4^0$. This is the proof of Proposition [1].
Applying Mayer-Vietoris sequence to this for \( n = 2 \), we obtain the followings:

\[
H_2(A(A') \cong \mathbb{R}^{10} \oplus \mathbb{R}^5 \\
\xrightarrow{(i \rightarrow j)} H_2(A) \oplus H_2(A') \xrightarrow{k - \iota_*} H_2(Y) \xrightarrow{\partial_*}
\]

for \( i = 1, 2, 3, 4 \) and \( 1 \leq j \neq k \leq 4 \), \( \{ j, k \} \neq \{ 1, 3 \}, \{ 2, 4 \} \). In the sequence, the author uses the dashed line \( \rightarrow \) to mean that an element of \( H_2(A) \oplus H_2(A') \) gives a relation in \( H_2(Y) \). For example, \((L_{13}^5, L_{31}^5)\) gives the relation \( L_{13}^5 = L_{31}^5 \). First, the rank of \( \langle i, j \rangle \) is 9. For this, it is easy that four \((L_{jk}, L_{jk}^5)\)'s, \((L_{34}^5, L_{34}^5), (L_{24}^5, L_{24}^5), (L_{13}^5 - Z_{24}^5, Z_{13}^5 - L_{24}^5)\) are independent in \( H_2(A) \oplus H_2(A') \) because we know basis of \( H_2(A), H_2(A') \). Let \( M' \) be the subspace of \( H_2(A) \oplus H_2(A') \) generated by these, and let \( M'' \) be its subspace generated by \((L_{jk}, L_{jk}^5)\)'s. The remaining are

\[
\langle L_{12}^5, L_{34}^5 \rangle, \ (L_{14}^5, L_{32}^5), \ (L_{13}^5, L_{54}^5), \ (L_{14}^5, L_{12}^5). \]

Since two sums

\[
(L_{12}^5, L_{34}^5) + (L_{34}^5, L_{12}^5) = (L_{12}^5 + L_{34}^5, L_{34}^5, L_{12}^5 + L_{34}^5), \\
(L_{14}^5, L_{32}^5) + (L_{32}^5, L_{14}^5) = (L_{14}^5 + L_{32}^5, L_{32}^5 + L_{14}^5)
\]

are contained in \( \langle L_{jk}, L_{jk}^5 \rangle \subset M\), we only have to consider

\[
(L_{13}^5, L_{54}^5), \ (L_{14}^5, L_{32}^5). \]

Let \( M'' \) be the subspace of \( H_2(A) \oplus H_2(A') \) generated by these two. Then, \( M'' \cap M' = M'' \cap M' = 0 \). The \( \langle i, j \rangle \) is 9. This implies that the rank of \( k_* - l_* \) is 5 by exactness of the sequence. Since the image of \( k_* - l_* \) is generated by

\[
\{ L_{12}^5, L_{34}^5, L_{13}^5, L_{24}^5, Z_{24}^5 \text{ or } Z_{13}^5 \}
\]

by relations \((7.3), (7.7)\), this is a basis of \( \text{im} k_* - l_* \) because its dimension is 5.

Recall that \( \ker (i_* - j_*) \) for \( n = 1 \) is equal to

\[
\langle (L_0^0 + L_{34}^0 - L_{13}^0 - L_{34}^3), \ (L_0^0 + L_{24}^0 - L_{24}^3 - L_{13}^3) \rangle \cong \mathbb{R}^3
\]

by \((8.3)\). We have

\[
\partial_* (T_{13}^2) = (L_0^0 + L_{13}^0 - L_{13}^3 - L_{13}^3), \\
\partial_* (T_{24}^2) = (L_0^0 + L_{24}^0 - L_{24}^3 - L_{13}^3), \\
\partial_* (G_{01}) = (L_0^0, 0).
\]

That is,

\[
\text{im} \partial_* = \langle \partial_* (T_{13}^2), \partial_* (T_{24}^2), \partial_* (G_{01}) \rangle.
\]

Since we already know that the rank of \( \ker \partial_* \) is equal to 3, we can conclude that

\[
H^2(W) = \langle L_{12}^0, L_{14}^0, L_{13}^0, L_{24}^0 \rangle + \langle Z_{24}^5 \text{ or } Z_{13}^5 \rangle + \langle T_{13}^2, T_{24}^2, G_{01} \rangle \cong \mathbb{R}^8
\]

by exactness. This is the proof of Proposition \( 2 \).}

Last, we prove Proposition 3 i.e. the main theorem.
Proof of Theorem 4. We only have to show that $c_1(TW|_{T^2_{1+3}}) = 0$ and $c_1(TW|_{T^2_{2+4}}) = 0$ because $c_1(TW|_Q) = 0$ for other generators of $H^2(W)$ is easy. The tangent spaces of $X$ restricted to $L_{1+3} \times [0, 7]$ is trivial. Then, $TW|_{T^2_{1+3}}$ is constructed by gluing of $(L_{1+3} \times [0, 7]) \times \mathbb{C}^3$ through the map

$$(L_{1+3} \times 0) \times \mathbb{C}^3 \to (L_{1+3} \times 7) \times \mathbb{C}^3,$$

$$(x, 0, (z_1, z_2, z_3)) \mapsto ((x, 7), (z_2, z_1, z_3)).$$

This gluing map is just a writing of $\tau$ by using complex coordinates. The bundle $TW|_{T^2_{1+3}}$ has three subbundles $\eta_i$ for $i = 1, 2, 3$ whose pullbacks $\tilde{\eta}_i = \tilde{\pi}^* \eta_i$ to $L_{1+3} \times [0, 7]$ are as follows:

$\eta_1 = \{(x, s, z, 0) \in (L_{1+3} \times [0, 7]) \times \mathbb{C}^3 \mid x \in L_{1+3}, s \in [0, 7], z \in \mathbb{C}\},$

$\eta_2 = \{(x, s, 0, 0, z) \in (L_{1+3} \times [0, 7]) \times \mathbb{C}^3 \mid x \in L_{1+3}, s \in [0, 7], z \in \mathbb{C}\},$

$\eta_3 = \{(x, s, z, -z, 0) \in (L_{1+3} \times [0, 7]) \times \mathbb{C}^3 \mid x \in L_{1+3}, s \in [0, 7], z \in \mathbb{C}\}.$

Then, $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are easily trivial. The subbundle $\eta_3$ has a nonvanishing section

$$(x, s) \mapsto \left(x, s, \exp(\pi s i/7), -\exp(\pi s i/7), 0\right).$$

And, this gives a nonvanishing section of $\tilde{\eta}_3$. So, $\tilde{\eta}_3$ is trivial, and we can conclude that $c_1(TW|_{T^2_{1+3}}) = 0$. Similarly, $c_1(TW|_{T^2_{2+4}}) = 0$. □

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