Equivariant gluing theory on regular instanton moduli spaces

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Abstract

We follow the idea of gluing theory in instanton moduli spaces and discuss the case when there is a finite group $\Gamma$ acting on the 4-manifolds $X_1, X_2$ with $x_1, x_2$ as isolated fixed points, how to glue two $\Gamma$-invariant ASD connections over $X_1, X_2$ together to get a $\Gamma$-invariant ASD connection on the connected sum $X_1 \# X_2$.

Keywords: instanton moduli space, equivariant gluing theory

MSC Classification: 57R18, 57R57, 81T13

1 Introduction

From the late 1980s, gauge theoretic techniques were applied in the area of finite group actions on 4-manifolds. [1] showed that on $S^4$, there is no smooth finite group action with exactly 1 fixed point by arguing that instanton-one invariant connections form a 1-manifold whose boundary can be identified as fixed points of the group action. In [2], gauge theoretic techniques were used in studying fixed points of a finite group action on 3-manifolds. [3] studied pseudofree orbifolds using ASD moduli spaces. A 4-dimensional pseudofree orbifold is a special kind of orbifold which can be expressed as $M^5/S^1$, a quotient of a pseudofree $S^1$-action on a 5-manifold $M^5$. In [4] Austin studied the orbifold $S^4/\mathbb{Z}_\alpha$, which is a compactification of $L(\alpha, \beta) \times \mathbb{R}$ where $L(\alpha, \beta)$ is a Lens space. He also gave a criterion for the existence of instantons on $S^4/\mathbb{Z}_\alpha$ and calculated the dimension of the instanton moduli space. A more general kind of orbifold, orbifold with isolated fixed points, was discussed in [5], especially when the group-action around each singular point is a cyclic group.

In the study of instanton moduli spaces, gluing theory tells us that given two anti-self-dual connections $A_1, A_2$ on 4-manifolds $X_1, X_2$ respectively, we can glue them together to get a new ASD connection on the space $X_1 \# X_2$. It plays an important role in the process of compactifying moduli spaces. This paper follows the idea of gluing theory (cf. Chapter 7 of [6]) and discusses the case when there is a finite group $\Gamma$ acting on the 4-manifolds $X_1, X_2$ with $x_1, x_2$ as isolated fixed points, how to glue two
ASD Γ-invariant ASD connections over \( X_1, X_2 \) together to get an ASD Γ-invariant connection on \( X_1 \neq X_2 \).

The main differences between the original gluing theory and the Γ-equivariant case are the following. Firstly, over the fixed points \( x_1 \) and \( x_2 \), the Γ-actions induce two isotropy representations, which are required to be equivalent. Secondly, the gluing parameter depends on the isotropy representations. Finally, we need to deal with the regularity of \( A_i \) in the Γ-invariant spaces.

2 Set-up

Suppose \( X_1, X_2 \) are smooth, oriented, compact, Riemannian 4-manifolds, and \( P_1, P_2 \) are principal \( G \)-bundles over \( X_1, X_2 \) respectively where \( G = SU(2) \). Let \( \Gamma \) be a finite group acting on \( P_i, X_i \) from the left which is smooth and orientation preserving and such that the action on \( P_i \) cover the action on \( X_i \).

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\[
\begin{array}{c}
r \subset P_i \supset G \\
\downarrow \\
X_i \ni x_i
\end{array}
\]

\( x_1 \in X_1^\Gamma, x_2 \in X_2^\Gamma \) are two isolated fixed points with equivalent isotropy representations. i.e., there exists \( h \in G \) such that

\[
\rho_2(\gamma) = h \rho_1(\gamma) h^{-1} \quad \forall \gamma \in \Gamma
\]

(1)

where \( \rho_1, \rho_2 \) are isotropy representations of \( \Gamma \) at \( x_1, x_2 \) respectively.

Now we fix two metrics \( g_1, g_2 \) on \( X_1, X_2 \) such that the Γ-action preserves the metrics. This can be achieved by the following lemma.

Lemma 1. For any Riemannian metric \( g \) on \( X \),

\[
\tilde{g} := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* g
\]

defines a Γ-invariant metric.

The proof is straightforward. We omit it here.

3 Glue bundles and get an approximate ASD connection \( A' \)

The first step is to glue manifolds \( X_1 \) and \( X_2 \) by connecting sum.

Fix a large enough constant \( T \) and a small enough constant \( \delta \). Let \( \lambda > 0 \) be a constant satisfying \( \lambda e^\delta \leq \frac{1}{2} b \) where \( b := \lambda e^T \). We first glue \( X_1' := X_1 \setminus B_{x_1}(\lambda e^{-\delta}) \) and \( X_2' := X_2 \setminus B_{x_2}(\lambda e^{-\delta}) \) together as shown in Figure 1, where \( e_\pm \) are defined in polar coordinates by

\[
e_\pm : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R} \times S^3
\]
and

$$f : \Omega_1 = (-\delta, \delta) \times S^3 \to \Omega_2 = (-\delta, \delta) \times S^3$$

(2)
is defined to be a $\Gamma$-equivariant conformal map that fixes the first component. Denote the connected sum by $X_1 \#_\lambda X_2$ or $X$. On the new manifold $X$, we define the metric $g_\lambda$ to be a weighted average of $g_1$, $g_2$ on $X_1$ and $X_2$, compared by the diffeomorphism $f$. If $g_\lambda = \sum m_i g_i$ on $X'_i$, we can arrange $1 \leq m_i \leq 2$. This means points are further away from each other on the gluing area.

We now turn to the bundles $P_i$. Suppose $A_i$ are ASD $\Gamma$-invariant connections on $P_i$. We want to glue $P_i|_{X'_i}$ together so that $A_1$ and $A_2$ match on the overlapping part.

The first step is to replace $A_i$ by two $\Gamma$-invariant connections which are flat on the annuli $\Omega_i$. Define the cut-off function $\eta_i$ on $X_i$ such that

$$\eta_1(x) = \begin{cases} 0 & x \in [-\delta, +\infty) \times S^3 \\ 1 & x \in X_1 \setminus B_{x_1}(b) \end{cases}, \quad \eta_2(x) = \begin{cases} 0 & x \in [-\infty, \delta) \times S^3 \\ 1 & x \in X_2 \setminus B_{x_2}(b) \end{cases},$$

(3)and $\eta_i(x)$ depends only on $|x - x_i|$ when $x \in B_{x_i}(b)$. Therefore $\eta_i$ are $\Gamma$-invariant.

Recall from [7] that in the Euclidean ball $B^4$, an exponential gauge with respect to a connection $A$ is a gauge under which $A(0) = 0$ and $A(\partial_r) = 0$. Choose an exponential
Lemma 2. There exists a canonical and $\rho$-equivariant gauge on $B$.

Proposition 3. A $A'$ is almost ASD.

The next step is to glue $P_1|_{X_1}$ and $P_2|_{X_2}$ together to get a principal $G$-bundle $P$ over $X$ and glue $A'_1$ and $A'_2$ together to get a $\Gamma$-invariant connection $A'$ on $P$.

**Lemma 2.** There exists a canonical $(\Gamma, G)$-equivariant map $\varphi$:

$$G \cong P_1|_{x_1} \xrightarrow{\varphi} P_2|_{x_2} \cong G$$

where $h$ is defined in (1) and $(\Gamma, G)$-equivariant means $\varphi$ is $\Gamma$-equivariant and $G$-equivariant.

**Proof.** The $G$-equivariance is obvious and the $\Gamma$-equivariance follows from:

$$P_1|_{x_1} \xrightarrow{\varphi} P_2|_{x_2}, \quad P_1|_{x_1} \xrightarrow{\varphi} P_2|_{x_2}$$

$$g \mapsto hg \mapsto \rho_2(\gamma)hg \quad g \mapsto \rho_1(\gamma)g \mapsto h\rho_1(\gamma)g$$

and $\rho_2(\gamma)hg = h\rho_1(\gamma)h^{-1}hg = h\rho_1(\gamma)g$ for any $\gamma \in \Gamma$.

Denote the subgroup of $(\Gamma, G)$-equivariant gluing parameters by

$$Gl^\Gamma := \text{Hom}_{(\Gamma, G)}(P_1|_{x_1}, P_2|_{x_2}). \quad (4)$$

**Proposition 3.** The subgroup of $(\Gamma, G)$-equivariant gluing parameters $Gl^\Gamma$ takes three forms:

$$Gl^\Gamma \cong \begin{cases} G & \text{if } \rho_1(\Gamma), \rho_2(\Gamma) \subset C(G), \\ U(1) & \text{if } \rho_1(\Gamma), \rho_2(\Gamma) \not\subset C(G) \text{ and are contained in some } U(1) \subset G, \\ C(G) & \text{if } \rho_1(\Gamma), \rho_2(\Gamma) \text{ are not contained in any } U(1) \text{ subgroup in } G, \end{cases} \quad (5)$$

where $C(G)$ is the center of $G$.

**Proof.** By formula (1), $\rho_1(\Gamma)$ and $\rho_2(\Gamma)$ are isomorphic and have isomorphic centralisers. For any element $h'$ in the centraliser of $\rho_1(\Gamma)$, $\varphi' : g \mapsto hh'g$ is also a $(\Gamma, G)$-equivariant map between $P_1|_{x_1}$ and $P_2|_{x_2}$ since for all $\gamma \in \Gamma$

$$hh'\rho_1(\gamma)g = h\rho_1(\gamma)h'g = \rho_2(\gamma)hh'g.$$
\[ \Rightarrow \rho_1(\gamma)h^{-1}h'g = h^{-1}h'\rho_1(\gamma)g, \]

which implies \( h^{-1}h' \) commutes with \( \rho_1(\gamma) \).

Therefore \( GL^F \) is isomorphic to the centraliser of \( \rho_1(\Gamma) \) in \( G \). The three cases in (5) are the only three groups that are centraliser of some subgroup in \( G \) when \( G = SU(2) \).

Recall that annuli \( \Omega_i \) are identified by \( f : \Omega_1 \rightarrow \Omega_2 \) defined in (2). Take \( \phi \in GL^\Gamma \), we glue \( P_1|_{\Omega_1} \) and \( P_2|_{\Omega_2} \) together to get \( P := (P_1|_{\Omega_1})#_\phi(P_2|_{\Omega_2}) \) with a \( \Gamma \)-action; glue \( A'_1 \) and \( A'_2 \) together to get a \( \Gamma \)-invariant \( A' \). For different gluing parameter \( \phi_1, \phi_2 \), \( A'_1(\phi_1) \) and \( A'_2(\phi_2) \) are gauge equivalent if and only if the parameters \( \phi_1, \phi_2 \) are in the same orbit of the action of \( \Gamma A_1 \times \Gamma A_2 \) on \( GL^\Gamma \). We denote \( A'_1(\phi) \) by \( A'_1 \) when the gluing parameter is contextually clear.

4 Constructing an ASD connection from \( A' \)

The general idea is to find a solution \( a \in \Omega^1(X, adP)^\Gamma \) so that \( A := A' + a \) is anti-self-dual, i.e.,

\[ F_A^+ = d_A^+ a + (a \wedge a)^+ = 0. \]  (6)

To do so, we wish to find a right inverse \( R^\Gamma \) of \( d_A^+ \) and an element \( \xi \in \Omega^{2,+}(X, adP)^\Gamma \) satisfying

\[ F_{A'}^+ + \xi + (R^\Gamma \xi \wedge R^\Gamma \xi)^+ = 0. \]  (7)

Then \( a = R^\Gamma \xi \) is a solution of equation (6).

Since \( A_i \) are two ASD connections, we have the complex:

\[
0 \rightarrow \Omega^0(X_i, adP_i) \xrightarrow{d_{A_i}} \Omega^1(X_i, adP_i) \xrightarrow{d_{A_i}^+} \Omega^{2,+}(X_i, adP_i) \rightarrow 0.
\]

We assume that the second cohomology classes \( H^2_{A_1}, H^2_{A_2} \) are both zero. The \( \Gamma \)-action can be induced on this chain complex naturally. It is worth mentioning that the \( \Gamma \)-action preserves the metric, so the space \( \Omega^{2,+}(X_i, adP_i) \) is \( \Gamma \)-invariant. Define the following two averaging maps:

\[
ave : \Omega^1(X_i, adP_i) \rightarrow \Omega^1(X_i, adP_i)^\Gamma \\
a \mapsto \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* a
\]

\[
ave : \Omega^{2,+}(X_i, adP_i) \rightarrow \Omega^{2,+}(X_i, adP_i)^\Gamma \\
\xi \mapsto \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \xi.
\]

Note that these maps are surjective since any \( \Gamma \)-invariant element is mapped to itself.

**Proposition 4.** The following diagram
0 \longrightarrow \Omega^0(X_i, adP_i) \xrightarrow{d_{A_i}} \Omega^1(X_i, adP_i) \xrightarrow{d_{A_i}} \Omega^2+(X_i, adP_i) \longrightarrow 0

0 \longrightarrow \Omega^0(X_i, adP_i)^\Gamma \xrightarrow{d_{A_i}} \Omega^1(X_i, adP_i)^\Gamma \xrightarrow{d_{A_i}} \Omega^2+(X_i, adP_i)^\Gamma \longrightarrow 0

commutes.

Proof. It suffices to show that \( d_{A_i} : \Omega^1(X_i, adP_i) \to \Omega^2(X_i, adP_i) \) and \( \gamma \) commute for any \( \gamma \in \Gamma \). For any \( \eta \in \Omega^1(X_i, adP_i) \), we treat \( \eta \) as a Lie algebra valued 1-form on \( P_i \), then

\[
(d + A_i)(\gamma^* \eta) = \gamma^* d\eta + [A_i, \gamma^* \eta] = \gamma^* d\eta + [\gamma^* A_i, \gamma^* \eta] = \gamma^* ((d + A_i)\eta).
\]

By Proposition 4,

\[
\text{Im}(d_{A_i}^+ \circ \text{ave}) = \text{Im}(\text{ave} \circ d_{A_i}^+) = \Omega^2+(X_i, adP_i)^\Gamma.
\]

Therefore, \((H_{A_i}^2)^\Gamma = 0\) and there exists right inverses \( R_{\Gamma_i} : \Omega^2+(X_i, adP_i)^\Gamma \to \Omega^1(X_i, adP_i)^\Gamma\).

**Proposition 5.** \( R_{\Gamma_i} \) are bounded operators from \( \Omega^2+(X_i, adP_i)^\Gamma \) to \( \Omega^1(X_i, adP_i)^\Gamma \).

The proof of Proposition 5 follows from Proposition 2.13 of Chapter III of [8] and the fact that the \( X_i \) are compact.

By the Sobolev embedding theorem, we have

\[
||R_{\Gamma_i}^\Gamma \xi||_{L^3} \leq \text{const.} ||R_{\Gamma_i}^\Gamma \xi||_{L^4},
\]

and combined with Proposition 5, we have

\[
||R_{\Gamma_i}^\Gamma \xi||_{L^4} \leq \text{const.} ||\xi||_{L^2}.
\]

(8)

Define two operators \( Q_{\Gamma_i}^\Gamma : \Omega^2+(X_i, adP_i)^\Gamma \to \Omega^1(X_i, adP_i)^\Gamma \) by

\[
Q_{\Gamma_i}^\Gamma(\xi) := \beta_i R_{\Gamma_i}^\Gamma \gamma_i(\xi),
\]

where \( \beta_i, \gamma_i \) are cut-off functions defined in the Figure 2 where \( \beta_1 \) varies on \((1, \delta) \times S^3\), \( \beta_2 \) varies on \((-\delta, -1) \times S^3\) and \( \gamma_i \) varies on \((-1, 1) \times S^3\). We can choose \( \beta_i \) such that \( \frac{|\partial \beta_i|}{\partial t} < \frac{2}{\delta} \) pointwise, then

\[
||\nabla \beta_i||_{L^4} \leq 4\pi \left( \int_1^{\delta} \frac{2^4}{\delta^2} dt \right)^{1/4} \leq 64\pi \delta^{-3/4},
\]

(9)
We can choose \( \gamma_i \) such that \( \gamma_1 + \gamma_2 = 1 \) on \( \Omega^1 \# \Omega^2 \) where \( f \) is defined in (2).

Now we want to extend the operators \( Q^\Gamma_i \) to \( X = X_1 \# X_2 \). Firstly, extend \( \beta_i, \gamma_i \) to \( X \) in the obvious way. It is worth mentioning that after the extension \( \gamma_1 + \gamma_2 = 1 \) on \( X \). Secondly, for any \( \xi \in \Omega^{2+}(X, adP)^\Gamma \), \( \gamma_i \xi \) is supported on \( X_i \), thus \( R_i^\Gamma \gamma_i \xi \) makes sense. Finally, extend \( \beta_i R_i^\Gamma \gamma_i(\xi) \) to the whole \( X \). Therefore \( Q^\Gamma_i \) can be treated as an operator:

\[
Q^\Gamma_i : \Omega^{2+}(X, adP)^\Gamma \to \Omega^1(X, adP)^\Gamma.
\]

Define

\[
Q^\Gamma := Q^\Gamma_1 + Q^\Gamma_2 : \Omega^{2+}(X, adP)^\Gamma \to \Omega^1(X, adP)^\Gamma.
\]

**Lemma 6.** With definitions above, we have \( \forall \xi \in \Omega^{2+}(X, adP)^\Gamma \),

\[
||d^+_A Q^\Gamma(\xi) - \xi||_{L^2} \leq \text{const.}(b^2 + \delta^{-3/4})||\xi||_{L^2}.
\]

**Proof.**

\[
||d^+_A Q^\Gamma(\xi) - \xi||_{L^2} = ||d^+_A (Q^\Gamma_1(\xi) + Q^\Gamma_2(\xi)) - \gamma_1 \xi - \gamma_2 \xi||_{L^2} \\
= ||d^+_A Q^\Gamma_1(\xi) + d^+_A Q^\Gamma_2(\xi) - \gamma_1 \xi - \gamma_2 \xi||_{L^2} \\
\leq ||d^+_A Q^\Gamma_1(\xi) - \gamma_1 \xi||_{L^2} + ||d^+_A Q^\Gamma_2(\xi) - \gamma_2 \xi||_{L^2}.
\]

Suppose \( A'_i = A_i + a_i \), then

\[
d^+_A Q^\Gamma_i \xi = d^+_A \beta_i R_i^\Gamma \gamma_i \xi + [a_i, \beta_i R_i^\Gamma \gamma_i]^+ \\
= \beta_i d^+_A R_i^\Gamma \gamma_i \xi + \nabla \beta_i R_i^\Gamma \gamma_i \xi + [\beta_i a_i, R_i^\Gamma \gamma_i]^+.
\]
The three terms on the right hand side have the following estimates.

(i). $\beta d^+_A R^+_1 \gamma_i \xi = \beta_i \gamma_i \xi = \gamma_i \xi$.

(ii). $||\nabla \beta_i R^+_1 \gamma_i \xi||_{L^2} \leq ||\nabla \beta||_{L^4} ||R^+_1 \gamma_i \xi||_{L^4}$ by the Sobolev multiplication theorem and (8) and (9).

(iii). $||[\beta_i a_i, R^+_1 \gamma_i \xi]||_{L^2} \leq \text{const.} ||a_i||_{L^4} ||R^+_1 \gamma_i \xi||_{L^4} \leq \text{const.} \delta^{-3/4} ||\xi||_{L^2}$ by the Sobolev multiplication theorem and (??) and (8).

Therefore $||d^+_A Q^F(\xi) - \gamma_i \xi||_{L^2} \leq \text{const.}(b^2 + \delta^{-3/4}) ||\xi||_{L^2}$ and the result follows.

The result of Lemma 6 means that $Q^F$ is almost a right inverse of $d^+_A$. Next we show there is a right inverse $R^F$ of $d^+_A$.

By Lemma 6, we can choose $\delta$ large enough and $b$ small enough so that $||d^+_A Q^F(\xi) - \xi||_{L^2} \leq 2/3 ||\xi||_{L^2}$ which implies

$$1/3||\xi||_{L^2} \leq ||d^+_A Q^F(\xi)||_{L^2} \leq 5/3 ||\xi||_{L^2}.$$ 

Then $d^+_A Q^F$ is invertible and

$$1/3||d^+_A Q^F(\xi)||_{L^2} \leq ||\xi||_{L^2}. \quad (10)$$

Define $R^F := Q^F(d^+_A Q^F)^{-1}$, then it is easy to see that $R^F$ is the right inverse of $d^+_A$. Note that $R^F$ depends on the gluing parameter $\varphi$, so we denote the operator by $R^F_{\varphi}$ when the gluing parameter is not contextually clear.

$R^F$ has the following good estimate:

$$||R^F \xi||_{L^4} = ||(Q^F_1 + Q^F_2)(d^+_A Q^F)^{-1}(\xi)||_{L^4}$$
$$\leq ||Q^F_1 (d^+_A Q^F)^{-1}(\xi)||_{L^4} + ||Q^F_2 (d^+_A Q^F)^{-1}(\xi)||_{L^4}$$
$$\leq ||R^+_1 \gamma_1 (d^+_A Q^F)^{-1}(\xi)||_{L^4} + ||R^+_2 \gamma_2 (d^+_A Q^F)^{-1}(\xi)||_{L^4}$$
(by (8)) \leq \text{const.} ||\gamma_1 (d^+_A Q^F)^{-1}(\xi)||_{L^4} + \text{const.} ||\gamma_2 (d^+_A Q^F)^{-1}(\xi)||_{L^4}$$
$$\leq \text{const.} ||(d^+_A Q^F)^{-1}(\xi)||_{L^2} \quad (11)$$

Then we have

$$||(R^F \xi_1 \wedge R^F \xi_2)^+ - (R^F \xi_2 \wedge R^F \xi_2)^+||_{L^2}$$
$$\leq ||R^F \xi_1 \wedge R^F \xi_1 - R^F \xi_2 \wedge R^F \xi_2||_{L^2}$$
$$= \frac{1}{2} ||(R^F \xi_1 + R^F \xi_2) \wedge (R^F \xi_1 - R^F \xi_2) + (R^F \xi_1 - R^F \xi_2) \wedge (R^F \xi_1 + R^F \xi_2)||_{L^2}$$
$$\leq \text{const.} ||R^F \xi_1 - R^F \xi_2||_{L^4} ||R^F \xi_1 + R^F \xi_2||_{L^4}$$
(by (11)) \leq \text{const.} ||\xi_1 - \xi_2||_{L^2} ||R^F \xi_1||_{L^4} + ||R^F \xi_2||_{L^4} \quad (12)$$

Define an operator $T : \xi \mapsto -F^+(A') - (R^F \xi \wedge R^F \xi)^+$, then solving equation (7) means finding a fixed point of the operator $T$. Here we apply the contraction mapping
1. There is an $r > 0$ such that for small enough $b$, $T$ is a map from the ball $B(r) \subset \Omega^2_{L^2} (X, adP)$ to itself. This follows from

$$||\kappa||_{L^2} < r \implies ||T\kappa||_{L^2} \leq ||F^+(A')||_{L^2} + ||R^T_\kappa \wedge R^T_\kappa||_{L^2} \leq \text{const.}b^2 + ||R^T_\kappa||^2_{L^2,1} \leq \text{const.}b^2 + \text{const.}||\kappa||^2_{L^2} \leq \text{const.}(b^2 + r^2) < r \text{ (for small } b, r \text{ with } b << r).$$

2. $T$ is a contraction for sufficiently small $r$, i.e., there exists $\lambda < 1$ such that

$$||T\xi_1 - T\xi_2|| \leq \lambda||\xi_1 - \xi_2|| \forall \xi_1, \xi_2.$$

This follows from (12).

Now we have proved that there exists a unique solution to equation (7).

**Theorem 7.** Suppose $A_1, A_2$ are $\Gamma$-invariant ASD connections on $X_1, X_2$ respectively with $H^2_{A_1} = 0$. Let $\lambda, T, \delta$ be positive real numbers such that $b := \lambda e^T > 2\lambda e^\delta$. Then we can make $\delta$ large enough and $b$ small enough so that for any $(\Gamma, G)$-equivariant gluing parameter $\varphi \in \text{Hom}_{(\Gamma, 1)}(P_1, P_2)$, there exists $a_\varphi \in \Omega^1(X, adP)^G$ with $||a_\varphi||_{L^4} \leq \text{const.}b^2$ such that $A'(\varphi) + a_\varphi$ is a $\Gamma$-invariant ASD connection on $X$. Moreover, if $\varphi_1, \varphi_2$ are in the same orbit of $\Gamma_{A_1} \times \Gamma_{A_2}$ on $Gl$, then $A'(\varphi_1) + a_\varphi_1, A'(\varphi_2) + a_\varphi_2$ are gauge equivalent.

**Proof.** We only need to prove the last statement.

If $\varphi_1, \varphi_2$ are in the same orbit of $\Gamma_{A_1} \times \Gamma_{A_2}$ on $Gl$, then $A'(\varphi_1), A'(\varphi_2)$ are gauge equivalent. For some gauge transformation $\sigma$ we have $\sigma^*A'(\varphi_1) = A'(\varphi_2)$. Applying $\sigma^*$ on both sides of the following formula

$$F^+_{A'(\varphi_1)} + \xi(\varphi_1) + (R^T_{\varphi_1} \xi(\varphi_1) \wedge R^T_{\varphi_1} \xi(\varphi_1))^+ = 0$$

gives

$$\sigma^*F^+_{A'(\varphi_1)} + \sigma^*\xi(\varphi_1) + \sigma^*(R^T_{\varphi_2} \xi(\varphi_1) \wedge R^T_{\varphi_2} \xi(\varphi_1))^+ = 0. \tag{13}$$

Since $\sigma^*$ and $d^+_{A'}$ commute and $R^T = Q^T(d^+_{A'} Q^T)^{-1}$, then $\sigma^*$ and $R^T$ commute. Then (13) becomes

$$F^+_{A'(\varphi_1)} + \sigma^*\xi(\varphi_1) + (R^T_{\varphi_2} \sigma^*\xi(\varphi_1) \wedge R^T_{\varphi_2} \sigma^*\xi(\varphi_1))^+ = 0$$

This means $\sigma^*\xi(\varphi_1), \xi(\varphi_2)$ are solutions to $F^+_{A'(\varphi_2)} + \xi + (R^T_{\varphi_2} \xi \wedge R^T_{\varphi_2} \xi)^+ = 0$, which implies $\sigma^*\xi(\varphi_1) = \xi(\varphi_2)$. The following deduction completes the proof.

$$\sigma^*\xi(\varphi_1) = \xi(\varphi_2) \implies \sigma^*R^T_{\varphi_1} \xi(\varphi_1) = R^T_{\varphi_2} \sigma^*\xi(\varphi_1) = R^T_{\varphi_2} \xi(\varphi_2)$$
\[ \Rightarrow \sigma^* a_{\varphi_1} = a_{\varphi_2} \Rightarrow \sigma^*(A'(\varphi_1) + a_{\varphi_1}) = A'(\varphi_2) + a_{\varphi_2}. \]

**Declarations**

The author declares that there is no conflict of interest.

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