A Rule-based Operational Semantics of Graph Query Languages *

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Abstract. We consider a core language of graph queries. These queries are seen as formulas to be solved with respect to graph-oriented databases. For this purpose, we first define a graph query algebra where some operations over graphs and sets of graph homomorphisms are specified. Then, the notion of pattern is introduced to represent a kind of recursively defined formula over graphs. The syntax and formal semantics of patterns are provided. Afterwards, we propose a new sound and complete calculus to solve patterns. This calculus, which is based on a rewriting system, develops only one derivation per pattern to be solved. Our calculus is generic in the sense that it can be adapted to different kinds of graph databases provided that the notions of graph and graph homomorphism (match) are well defined.

Keywords: Operational semantics, Rewrite systems, Graph query languages

1 Introduction

Rewriting techniques have been widely used in different areas such as operational semantics of declarative languages or automated theorem proving. In this paper, our main aim is to propose to use such techniques in the case of graph-oriented database languages.

Current developments in database theory show a clear shift from relational to graph-oriented databases. Relational databases are now well mastered and have been largely investigated in the literature with an ISO standard language SQL [89]. On the other side, the wide use of graphs as a flexible data model for numerous database applications [16] as well as the emergence of various languages such as SPARQL [17], Cypher [11] or G-CORE [2] to quote a few. An ongoing ISO project of a standard language, called GQL, has emerged recently for graph-oriented databases

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3 https://www.gqlstandards.org/
Representing data graphically is quite legible. However, there is always a dilemma in choosing the right notion of graphs when modeling applications. This issue is already present in some well investigated domains such as modeling languages [6] or graph transformation [15]. Graph-oriented data representation does not escape from such dilemma. We can quote for example RDF graphs [18] on which SPARQL is based or Property Graphs [11] currently used in several languages such as Cypher, G-CORE or the forthcoming GQL language.

In addition to the possibility of using different graph representations for data, graph database languages feature new kinds of queries such as graph-to-graph queries, cf. CONSTRUCT queries in SPARQL or G-CORE, besides the classical graph-to-relation (table) queries such as SELECT or MATCH queries in SPARQL or Cypher. The former constitute a class of queries which transforms a graph database to another graph database. The later transforms a graph to a multiset of solutions represented in general by means of a table just as in the classical relational framework.

In general, graph querying processing integrates features shared with graph transformation techniques (database transformation) and goal solving (variable assignments). Our main aim in this paper is to define an operational semantics, based on rewriting techniques, for graph-oriented queries. We propose a generic rule-based calculus, called \textit{gql-narrowing} which is parameterized by the actual interpretations of graphs and their matches (homomorphisms). That is to say, the obtained calculus can be adapted to different definitions of graph and the corresponding notion of match. The proposed calculus consists on a dedicated rewriting system and a narrowing-like procedure which follows closely the formal semantics of patterns or queries, the same way as (SLD-)Resolution calculus is related to formal models underlying Horn or Datalog clauses. The use of rewriting techniques in defining the proposed operational semantics paves the way to syntactic analysis and automated verification techniques for the proposed core language.

In order to define a sound and complete calculus, we first propose a uniform formal semantics for queries. Actually, we do consider graph-to-graph queries and graph-to-table queries as two facets of one same syntactic object that we call \textit{pattern}. The semantics of a pattern is a set of matches, that is to say, a set of graph homomorphisms and not only a set of variable assignments as proposed in [3,11]. From such set of matches, one can easily display either the tables by considering the images of the variables as defined by the matches or the graph target of the matches or even both tables and graphs. Our semantics for patterns allows us to write nested patterns in a natural way, that is, new data graphs can be constructed on the fly before being queried.

The paper is organized as follows: next section introduces a graph query algebra featuring some key operations needed to express the proposed calculus. Section 3 defines the syntax of patterns and queries as well as their formal semantics. In Section 4, a sound and complete calculus is given. First we introduce a rewriting system describing how query results are found. Then, we define gql-
narrowing, which is associated with the proposed rules. Concluding remarks and related work are given in Section 5.

2 Graph Query Algebra

During a query answering process, different intermediate results can be computed and composed. In this section, we introduce a Graph Query Algebra \( GQ \) which consists of a family of operations over graphs, matches (graph homomorphisms) and expressions. These operations are used later on to define the semantics of queries, see Sections 3 and 4.

2.1 Signature for the Graph Query Algebra

The algebra \( GQ \) is defined over a signature. The main sorts of this signature are \( Gr \), \( Som \), \( Exp \) and \( Var \) to be interpreted as graphs, sets of matches, expressions and variables, respectively, as explained in Sections 2.2, 2.3, 2.4 and 2.5. The sort \( Var \) is a subsort of \( Exp \). The main operators of the signature are:

- \( Match : Gr, Gr \rightarrow Som \)
- \( Join : Som, Som \rightarrow Som \)
- \( Bind : Som, Exp, Var \rightarrow Som \)
- \( Filter : Som, Exp \rightarrow Som \)
- \( Build : Som, Gr \rightarrow Som \)
- \( Union : Som, Som \rightarrow Som \)

The above sorts and operations are given as an indication while being inspired by concrete languages. They may be modified or adapted according to actual graph-oriented query languages.

2.2 An Actual Interpretation of Graphs

Various interpretations of sorts \( Gr \) and \( Som \) can be given. In order to provide concrete examples, we have to fix an actual interpretation of these sorts. For all the examples given in the paper, we have chosen to interpret the sort \( Gr \) as generalized RDF graphs. We could of course have chosen other notions of graphs such as property graphs. Our choice here is motivated by the simplicity of RDF graph definition (set of triples).

Below, we define generalized RDF graphs. They are the usual RDF graphs but they may contain isolated nodes. Let \( \mathcal{L} \) be a set, called the set of \emph{labels}, made of the union of two disjoint sets \( \mathcal{C} \) and \( \mathcal{V} \), called respectively the set of \emph{constants} and the set of \emph{variables}.

**Definition 1 (graph).** Every element \( t = (s, p, o) \) of \( \mathcal{L}^3 \) is called a triple and its members \( s, p \) and \( o \) are called respectively the subject, the predicate and the object of \( t \). A graph \( G \) is a pair \( G = (G_N, G_T) \) made of a subset \( G_N \) of \( \mathcal{L} \) called the set of nodes of \( G \) and a subset \( G_T \) of \( \mathcal{L}^3 \) called the set of triples of \( G \), such
that the subject and the object of each triple of \( G \) are nodes of \( G \). The nodes of \( G \) which are neither a subject nor an object are called the isolated nodes of \( G \). The set of labels of a graph \( G \) is the subset \( \mathcal{L}(G) \) of \( \mathcal{L} \) made of the nodes and predicates of \( G \), then \( \mathcal{C}(G) = \mathcal{C} \cap \mathcal{L}(G) \) and \( \mathcal{V}(G) = \mathcal{V} \cap \mathcal{L}(G) \). The graph with an empty set of nodes and an empty set of triples is called the empty graph and is denoted by \( \emptyset \). Given two graphs \( G_1 \) and \( G_2 \), the graph \( G_1 \) is a subgraph of \( G_2 \), written \( G_1 \subseteq G_2 \), if \( (G_1)_N \subseteq (G_2)_N \) and \( (G_1)_T \subseteq (G_2)_T \), then \( \mathcal{L}(G_1) \subseteq \mathcal{L}(G_2) \). The union \( G_1 \cup G_2 \) is the graph defined by \( (G_1 \cup G_2)_N = (G_1)_N \cup (G_2)_N \) and \( (G_1 \cup G_2)_T = (G_1)_T \cup (G_2)_T \), then \( \mathcal{L}(G_1 \cup G_2) = \mathcal{L}(G_1) \cup \mathcal{L}(G_2) \).

In the rest of the paper we write graphs as a couple made of a set of triples and a set of nodes: for example the graph \( G \) which are neither a subject nor an object are called the isolated nodes of \( G \).

**Example 1.** We define a toy database which is used as a running example throughout the paper. The database consists of persons who are either professors or students, with topics such that each professor teaches some topics and each student studies some topics.

\[
G_{ex} = \{(\text{Alice, is, Professor}), \ (\text{Alice, teaches, Mathematics}), \\
(\text{Bob, is, Professor}), \ (\text{Bob, teaches, Informatics}), \\
(\text{Charlie, is, Student}), \ (\text{Charlie, studies, Mathematics}), \\
(\text{David, is, Student}), \ (\text{David, studies, Mathematics}), \\
(\text{Eric, is, Student}), \ (\text{Eric, studies, Informatics}) \}
\]

Below, we define the notion of *match* which will be used, notably, to represent results of queries.

**Definition 2 (match).** A graph homomorphism from a graph \( L \) to a graph \( G \), denoted \( m : L \rightarrow G \), is a function from \( \mathcal{L}(L) \) to \( \mathcal{L}(G) \) which preserves nodes and preserves triples, in the sense that \( m(L_N) \subseteq G_N \) and \( m^3(L_T) \subseteq G_T \). A match is a graph homomorphism \( m : L \rightarrow G \) which fixes \( \mathcal{C} \), in the sense that \( m(c) = c \) for each \( c \) in \( \mathcal{C}(L) \).

When \( n \) is an isolated node of \( L \) then the node \( m(n) \) does not have to be isolated in \( G \). A match \( m : L \rightarrow G \) determines two functions \( m_N : L_N \rightarrow G_N \) and \( m_T : L_T \rightarrow G_T \), restrictions of \( m \) and \( m^3 \) respectively. A match \( m : L \rightarrow G \) is invertible if and only if both functions \( m_N \) and \( m_T \) are bijections. This means that a function \( m \) from \( \mathcal{L}(L) \) to \( \mathcal{L}(G) \) is an invertible match if and only if \( \mathcal{C}(L) = \mathcal{C}(G) \) with \( m(c) = c \) for each \( c \) in \( \mathcal{C}(L) \) and \( m \) is a bijection from \( \mathcal{V}(L) \) to \( \mathcal{V}(G) \): thus, \( L \) is the same as \( G \) up to variable renaming. It follows that the symbol used for naming a variable does not matter as long as graphs are considered only up to invertible matches.

Notice that RDF graphs are graphs according to Definition 1 but without isolated nodes, and where constants are either IRIs (Internationalized Resource Identifiers) or literals and where all predicates are IRIs and only objects can be literals. Blank nodes in RDF graphs are the same as variable nodes in our graphs. An isomorphism of RDF graphs, as defined in [13], is an invertible match, isomorphism of graphs as in Definition 2.
2.3 More Definitions on Matches

Below we introduce some useful definitions on matches. Notice that we do not consider a match \( m \) as a simple variable assignment but rather as a graph homomorphism with a clear source and target graphs. This nuance in the definition of matches is important in the rest of the paper.

**Definition 3 (compatible matches).** Two matches \( m_1 : L_1 \to G_1 \) and \( m_2 : L_2 \to G_2 \) are compatible, written as \( m_1 \sim m_2 \), if \( m_1(x) = m_2(x) \) for each \( x \in V(L_1) \cap V(L_2) \). Given two compatible matches \( m_1 : L_1 \to G_1 \) and \( m_2 : L_2 \to G_2 \), let \( m_1 \triangleright m_2 : L_1 \cup L_2 \to G_1 \cup G_2 \) denote the unique match such that \( m_1 \triangleright m_2 \sim m_1 \) and \( m_1 \triangleright m_2 \sim m_2 \) (which means that \( m_1 \triangleright m_2 \) coincides with \( m_1 \) on \( L_1 \) and with \( m_2 \) on \( L_2 \)).

**Definition 4 (building a match).** Let \( m : L \to G \) be a match and \( R \) a graph. The match \( \text{Build}(m,R) : R \to G \cup H_{m,R} \) is the unique match (up to variable renaming) such that for each variable \( x \) in \( R \):

\[
\text{Build}(m,R)(x) = \begin{cases} 
    m(x) & \text{when } x \in V(R) \cap V(L), \\
    \text{some fresh variable } \text{var}(m,x) & \text{when } x \in V(R) - V(L).
\end{cases}
\]

and \( H_{m,R} \) is the image of \( R \) by \( \text{Build}(m,R) \).

**Definition 5 (set of matches, assignment table).** Let \( L \) and \( G \) be graphs. A set \( \mathcal{m} \) of matches, all of them from \( L \) to \( G \), is denoted \( m : L \Rightarrow G \) and called a homogeneous set of matches, or simply a set of matches, with source \( L \) and target \( G \). The image of \( L \) by \( \mathcal{m} \) is the subgraph \( m(L) = \cup_{m \in \mathcal{m}} (m(L)) \) of \( G \). We denote \( \text{Match}(L,G) : L \Rightarrow G \) the set of all matches from \( L \) to \( G \). When \( L \) is the empty graph this set has one unique element which is the inclusion of \( \emptyset \) into \( G \), then we denote \( \mathcal{L}_G = \text{Match}(\emptyset,G) : \emptyset \Rightarrow G \) this one-element set and \( \emptyset_G : \emptyset \Rightarrow G \) its empty subset. The assignment table \( \text{Tab}(\mathcal{m}) \) of \( \mathcal{m} \) is the two-dimensional table with the elements of \( V(L) \) in its first row, then one row for each \( m \) in \( \mathcal{m} \), and the entry in row \( m \) and column \( x \) equals to \( m(x) \).

Thus, the assignment table \( \text{Tab}(\mathcal{m}) \) describes the set of functions \( m|_{V(L)} : V(L) \Rightarrow L \), made of the functions \( m|_{V(L)} : V(L) \to L \) for all \( m \in \mathcal{m} \). A set of matches \( \mathcal{m} : L \Rightarrow G \) is determined by the graphs \( L \) and \( G \) and the assignment table \( \text{Tab}(\mathcal{m}) \).

**Example 2.** In order to determine when professor \( ?p \) teaches topic \( ?t \) which is studied by student \( ?s \) we may consider the following graph \( L_{ex} \), where \( ?p, ?t \) and \( ?s \) are variables. In all examples, variables are preceded by a “\( ? \)”.

\[
L_{ex} = \{ (\ ?p, \ \text{teaches}, \ ?t ), (\ ?s, \ \text{studies}, \ ?t ) \}
\]

There are 3 matches from \( L_{ex} \) to \( G_{ex} \). The set \( \mathcal{m}_{ex} \) of all these matches is:

\[
\mathcal{m}_{ex} : L_{ex} \Rightarrow G_{ex} \quad \text{with} \quad \text{Tab}(\mathcal{m}_{ex}) =
\begin{array}{|c|c|c|}
\hline
\text{?p} & \text{?t} & \text{?s} \\
\hline
\text{Alice} & \text{Mathematics} & \text{Charlie} \\
\text{Alice} & \text{Mathematics} & \text{David} \\
\text{Bob} & \text{Informatics} & \text{Eric} \\
\hline
\end{array}
\]
2.4 Expressions

Query languages usually feature a term algebra dedicated to express operations over integers, booleans and so forth. We do not care here about the way basic operations are chosen but we want to deal with aggregation operations as in most database query languages. Thus, one can think of any kind of term algebra with operators which are classified as either basic operators (unary or binary) and aggregation operators (always unary). We consider that all expressions are well typed. Typically, and not exclusively, the sets $Op_1$, $Op_2$ and $Agg$ of basic unary operators, basic binary operators and aggregation operators can be:

$$Op_1 = \{-, \not=\}, \quad Op_2 = \{+, -, \times, /, =, >, <, \text{AND}, \text{OR}\}, \quad Agg = Agg_{\text{elem}} \cup \{\text{agg} \ \text{DISTINCT} \mid \text{agg} \in Agg_{\text{elem}}\}$$

where $Agg_{\text{elem}} = \{\text{MAX}, \text{MIN}, \text{SUM}, \text{AVG}, \text{COUNT}\}$.

A group of expressions is a non-empty finite list of expressions.

**Definition 6 (syntax of expressions).** Expressions $e$ and their sets of in-scope variables $V(e)$ are defined recursively as follows, with $c \in C$, $x \in V$, $op_1 \in Op_1$, $op_2 \in Op_2$, $agg \in Agg$, $gp$ is a group of expressions:

$$e ::= c \mid x \mid op_1 \ e \mid op_2 \ e \mid agg(e_1) \ \text{BY} \ gp$$

$$V(c) = \emptyset, \quad V(x) = \{x\}, \quad V(op_1 \ e) = V(e), \quad V(op_2 \ e_1 \ e_2) = V(e_1) \cup V(e_2), \quad V(agg(e)) = V(e)$$

$$V(agg(e \ \text{BY} \ gp)) = V(e)$$ (the variables in $gp$ must be distinct from those in $e$).

The value of an expression with respect to a set of matches $m$ (Definition 7) is a family of constants $ev(m, e) = (ev(m, e)_m)_{m \in m}$ indexed by the set $m$. When the expression $e$ is free from any aggregation operator then $ev(m, e)_m$ is simply $m(e)$. But in general $ev(m, e)_m$ depends on $e$ and $m$ and it may also depend on other matches in $m$ when $e$ involves aggregation operators. The value of a group of expressions $gp = (e_1, \ldots, e_k)$ with respect to $m$ is the list $ev(m, gp) = (ev(m, e_1)_m, \ldots, ev(m, e_k)_m)_{m \in m}$. To each basic operator $op$ is associated a function $[[op]]$ (or simply $op$) from constants to constants if $op$ is unary and from pairs of constants to constants if $op$ is binary. To each aggregation operator $agg$ in $Agg$ is associated a function $[[agg]]$ (or simply $agg$) from multisets of constants to constants. Note that each family of constants determines a multiset of constants: for instance a family $c = (c_m)_{m \in m}$ of constants indexed by the elements of a set of matches $m$ determines the multiset of constants $[[c_m \mid m \in m]]$, which is also denoted $c$ when there is no ambiguity. Some aggregation operators $agg$ in $Agg_{\text{elem}}$ are such that $[[agg]](c)$ depends only on the set underlying the multiset $c$, which means that $[[agg]](c)$ does not depend on the multiplicities in the multiset $c$: this is the case for MAX and MIN but not for SUM, AVG and COUNT. When $agg = agg_{\text{elem}}$ DISTINCT with $agg_{\text{elem}}$ in $Agg_{\text{elem}}$ then $[[agg]](c)$ is $[[agg_{\text{elem}}]]$ applied to the underlying set of $c$. For instance, $COUNT(c)$ counts the number of elements of the multiset $c$ with their multiplicities, while $COUNT \ \text{DISTINCT}(c)$ counts the number of distinct elements in $c$. 


Definition 7 (evaluation of expressions). Let \( L \) be a graph, \( e \) an expression over \( L \) and \( m : L \Rightarrow G \) a set of matches. The value of \( e \) with respect to \( m \) is the family \( \text{ev}(m, e) = (\text{ev}(m, e)_m)_{m \in m} \) defined recursively as follows. It is assumed that each \( \text{ev}(m, e)_m \) in this definition is a constant.

- \( \text{ev}(m, c)_m = c \),
- \( \text{ev}(m, x)_m = m(x) \),
- \( \text{ev}(m, op e_1)_m = [(op)] \text{ev}(m, e_1)_m \),
- \( \text{ev}(m, e_1 op e_2)_m = \text{ev}(m, e_1)_m [(op)] \text{ev}(m, e_2)_m \),
- \( \text{ev}(m, agg(e_1))_m = [(agg)](\text{ev}(m, e_1)) \),
- \( \text{ev}(m, agg(e_1) BY gp)_m = [(agg)](\text{ev}(m, gp, e_1)) \) where \( m, gp \) is the subset of \( m \) made of the matches \( m' \) in \( m \) such that \( \text{ev}(m, gp)_m = \text{ev}(m, gp)_m' \).

Note that \( \text{ev}(m, agg(e_1))_m \) is the same for all \( m \) in \( m \) while \( \text{ev}(m, agg(e_1) BY gp)_m \) is the same for all \( m \) and \( m' \) in \( m \) such that \( \text{ev}(m, gp)_m = \text{ev}(m, gp)_m' \).

2.5 Operations

The sorts \( \text{Gr} \), \( \text{Som} \), \( \text{Exp} \) and \( \text{Var} \) of the signature in Section 2.1 are interpreted in the algebra \( \mathcal{GQ} \) respectively as the set of graphs (Definition 1), the set of homogeneous sets of matches (Definition 4), the set of expressions (Definition 6) and its subset of variables. Then the operators of the signature are interpreted in the algebra \( \mathcal{GQ} \) by the operations with the same name in Definition 8. Whenever needed, we extend the target of matches: for every graph \( H \) and every match \( m : L \Rightarrow G \) where \( G \) is a subgraph of \( H \) we denote \( m : L \Rightarrow H \) when \( m \) is considered as a match from \( L \) to \( H \).

Definition 8 (\( \mathcal{GQ} \) operations).

- For all graphs \( L \) and \( G \):
  Match\((L, G) : L \Rightarrow G \) is the set of all matches from \( L \) to \( G \).
- For all sets of matches \( m : L \Rightarrow G \) and \( p : R \Rightarrow H \):
  Join\((m, p) = \{m \bowtie p \mid m \in m \land p \in p \land m \sim p \} : L \cup R \Rightarrow G \cup H \).
- For every set of matches \( m : L \Rightarrow G \), every expression \( e \) and every variable \( x \), let \( p_m(x) = \text{ev}(m, e)_m \) for each \( m \in m \). Then:
  Bind\((m, e, x) = \{m \bowtie p_m \mid m \in m \land m(x) = p_m(x) \} : L \cup \{x \Rightarrow G \cup \{p_m(x) \mid m \in m \} \} \) EQUIVALENTLY, this can be expressed as follows:
  if \( x \in V(L) \) then \( \text{Bind}(m, e, x) = \{m \mid m \in m \land m(x) = p_m(x) \} : L \Rightarrow G \),
  otherwise \( \text{Bind}(m, e, x) = \{m \bowtie p_m \mid m \in m \} : L \cup \{x \Rightarrow G \cup \{p_m(x) \mid m \in m \} \} \).
- For every set of matches \( m : L \Rightarrow G \) and every expression \( e \):
  Filter\((m, e) = \{m \mid m \in m \land \text{ev}(m, e)_m = \text{true} \} : L \Rightarrow G \).
- For every set of matches \( m : L \Rightarrow G \) and every graph \( R \):
  Build\((m, R) = \{\text{Build}(m, R) \mid m \in m \} : R \Rightarrow G \cup \text{Build}(m, R)(R)
  where \( \text{Build}(m, R)(R) = \bigcup_{m \in m} \text{Build}(m, R)(R) \).
- For all sets of matches \( m : L \Rightarrow G \) and \( p : L \Rightarrow H \):
  Union\((m, p) = (m : L \Rightarrow G \cup H) \cup (p : L \Rightarrow G \cup H) : L \Rightarrow G \cup H \).
3 Patterns and Queries

Syntax of graph-oriented databases is still evolving. We do not consider all technical syntactic details of a real-world language nor all possible constraints on matches. We focus on a core language. Its syntax reflects significant aspects of graph-oriented queries. Conditions on graph paths, which can be seen as constraints on matches, are omitted in this paper in order not to make the syntax too cumbersome. We consider mainly two syntactic categories: patterns and queries, in addition to expressions already mentioned in Section 2.4. Queries are either SELECT queries, as in most query languages, CONSTRUCT queries, as in SPARQL and G-CORE, or the new CONSELECT queries introduced in this paper. A SELECT query applied to a graph returns a table which describes a multiset of solutions or variable bindings, while a CONSTRUCT query applied to a graph returns a graph. A CONSELECT query applied to a graph returns both a graph and a table. On the other hand, a pattern applied to a graph returns a set of matches. Patterns are the basic blocks for building queries. They are defined in Section 3.1 together with their semantics. Queries are defined in Section 3.2 and their semantics is easily derived from the semantics of patterns. In this Section, as in Section 2 the set of labels $\mathcal{L}$ is the union of the disjoint sets $\mathcal{C}$ and $\mathcal{V}$, of constants and variables respectively. We assume that the set $\mathcal{C}$ of constants contains the numbers and strings and the boolean values true and false.

3.1 Patterns

In Definition 9 patterns are built from graphs by using six operators: BASIC, JOIN, BIND, FILTER, BUILD and UNION. Then, in Definition 10 the formal semantics of patterns is given by an evaluation function.

Definition 9 (syntax of patterns). Patterns $P$ and their scope graph $[P]$ are defined recursively as follows.

- The symbol $\Box$ is a pattern, called the empty pattern, and $[\Box]$ is the empty graph $\emptyset$.
- If $L$ is a graph then $P = \text{BASIC}(L)$ is a pattern, called a basic pattern, and $[P] = L$.
- If $P_1$ and $P_2$ are patterns then $P = P_1 \text{ JOIN } P_2$ is a pattern and $[P] = [P_1] \cup [P_2]$.
- If $P_1$ is a pattern, $e$ an expression such that $\mathcal{V}(e) \subseteq \mathcal{V}([P_1])$ and $x$ a variable then $P = P_1 \text{ BIND } e \text{ AS } x$ is a pattern and $[P] = [P_1] \cup \{x\}$.
- If $P_1$ is a pattern and $e$ an expression such that $\mathcal{V}(e) \subseteq \mathcal{V}([P_1])$ then $P = P_1 \text{ FILTER } e$ is a pattern and $[P] = [P_1]$.
- If $P_1$ is a pattern and $R$ a graph then $P = P_1 \text{ BUILD } R$ is a pattern and $[P] = R$.
- If $P_1$ and $P_2$ are patterns such that $[P_1] = [P_2]$ then $P = P_1 \text{ UNION } P_2$ is a pattern with $[P] = [P_1] = [P_2]$. 


The value of a pattern over a graph is a set of matches, as defined now.

**Definition 10 (evaluation of patterns, set of solutions).** The set of solutions or the value of a pattern $P$ over a graph $G$ is a set of matches $[[P]]_G : [P] \Rightarrow G^{(P)}$ from the scope graph $[P]$ of $P$ to a graph $G^{(P)}$ that contains $G$. This value $[[P]]_G : [P] \Rightarrow G^{(P)}$ is defined inductively as follows:

- $[[\square]]_G = \emptyset_G : \emptyset \Rightarrow G$.
- $[[\text{BASIC}(L)]]_G = \text{Match}(L, G) : L \Rightarrow G$.
- $[[P_1 \text{ JOIN } P_2]]_G = \text{Join}([[P_1]]_G, [[P_2]]_{G^{(P_1)}}) : [P_1] \cup [P_2] \Rightarrow G^{(P_1)(P_2)}$.
- $[[P_1 \text{ BIND } e \text{ AS } x]]_G = \text{Bind}([[P_1]]_G, e, x) : [P_1] \cup \{x\} \Rightarrow G^{(P_1)} \cup [[P_1]]_G(e)$.
- $[[P_1 \text{ FILTER } e]]_G = \text{Filter}([[P_1]]_G, e) : [P_1] \Rightarrow G^{(P_1)}$.
- $[[P_1 \text{ BUILD } R]]_G = \text{Build}([[P_1]]_G, R) : R \Rightarrow G^{(P_1)} \cup [[P_1]]_G(R)$.
- $[[P_1 \text{ UNION } P_2]]_G = \text{Union}([[P_1]]_G, [[P_2]]_{G^{(P_2)}}) : [P_1] \Rightarrow G^{(P_1)(P_2)}$.

**Remark 1.** In all cases, the graph $G^{(P)}$ is built by adding to $G$ “whatever is required” for the evaluation. When $P$ is the empty pattern, the value of $P$ over $G$ is the empty subset of Match($\emptyset, G$). Syntactically, each operator OP builds a pattern $P$ from a pattern $P_1$ and a parameter param, which is either a pattern $P_2$ (for JOIN and UNION), a pair $(e, x)$ made of an expression and a variable (for BIND), an expression $e$ (for FILTER) or a graph $R$ (for BUILD). Semantically, for every pattern $P = P_1$ OP param, let us denote $m_1 : X_1 \Rightarrow G_1$ for $[[P_1]]_G : [P_1] \Rightarrow G^{(P_1)}$ and $m : X \Rightarrow G'$ for $[[P]]_G : [P] \Rightarrow G^{(P)}$. In every case it is necessary to evaluate $m_1$ before evaluating param: for JOIN and UNION this is because pattern $P_2$ is evaluated on $G_1$, for BIND and FILTER because expression $e$ is evaluated with respect to $m_1$, and for BUILD because of the definition of Build. Note that the semantics of $P_1$ JOIN $P_2$ and $P_1$ UNION $P_2$ is not symmetric in $P_1$ and $P_2$ in general, unless $G^{(P_1)} = G$ and $G^{(P_2)} = G$, which occurs when $P_1$ and $P_2$ are basic patterns. Given a pattern $P = P_1$ OP param, the pattern $P_1$ is a subpattern of $P$, as well as $P_2$ when $P = P_1$ JOIN $P_2$ or $P = P_1$ UNION $P_2$. The semantics of patterns is defined in terms of the semantics of its subpatterns (and the semantics of its other arguments, if any). Thus, for instance, BUILD patterns can be nested at any depth.

**Definition 11.** For every pattern $P$, the set $\mathcal{V}(P)$ of in-scope variables of $P$ is the set $\mathcal{V}([[P]])$ of variables of the scope graph $[P]$. An expression $e$ is over a pattern $P$ if $\mathcal{V}(e) \subseteq \mathcal{V}(P)$.

**Example 3.** Let $R_{ex}$ be the following graph, where $?p$, $?z$ and $?s$ are variables.

$$R_{ex} = \{ (?p, \text{teaches, } ?z), (?s, \text{studies, } ?z) \}$$

Note that $R_{ex}$ is the same as $L_{ex}$, except for the name of one variable. In order to determine when professor $?p$ teaches some topic which is studied by student $?s$, whatever the topic, we consider the following pattern $P_{ex}$.

$$P_{ex} = \text{BASIC (} L_{ex} \text{) BUILD } R_{ex}$$
$$= \text{BASIC (} \{ (?p, \text{teaches, } ?t), (?s, \text{studies, } ?t) \} \text{) BUILD } \{ (?p, \text{teaches, } ?z), (?s, \text{studies, } ?z) \}$$
Note that the variable \(?z\) in \(R_{\text{ex}}\) does not appear in \(L_{\text{ex}}\). Since there are 3 matches from \(L_{\text{ex}}\) to \(G_{\text{ex}}\) (Example 2), the value of \(P_{\text{ex}}\) over \(G_{\text{ex}}\) is:

\[
p_{\text{ex}} : R_{\text{ex}} \Rightarrow G'_{\text{ex}} \text{ with } \text{Tab}(p_{\text{ex}}) = \begin{array}{ccc}
?p & ?z & ?s \\
\text{Alice} & ?z_1 & \text{Charlie} \\
\text{Alice} & ?z_2 & \text{David} \\
\text{Bob} & ?z_3 & \text{Eric}
\end{array}
\]

where \(?z_1\), \(?z_2\) and \(?z_3\) are 3 fresh variables and:

\[
G'_{\text{ex}} = G_{\text{ex}} \cup \{ (\text{Alice, teaches, } ?z_1), (\text{Charlie, studies, } ?z_1), (\text{Alice, teaches, } ?z_2), (\text{David, studies, } ?z_2), (\text{Bob, teaches, } ?z_3), (\text{Eric, studies, } ?z_3) \}
\]

### 3.2 Queries

We consider three kinds of queries: CONSTRUCT queries, SELECT queries and CONSELECT queries. We define the semantics of queries from the semantics of patterns. According to Definition 10, all patterns have a graph-to-set-of-matches semantics. In contrast, CONSTRUCT queries have a graph-to-graph semantics and SELECT queries have a graph-to-multiset-of-solutions or graph-to-table semantics while CONSELECT have a graph-to-graph-and-table semantics.

**Definition 12 (syntax of queries).** Let \(S\) be a set of variables, \(R\) a graph and \(P\) a pattern. A query \(Q\) has one of the following three shapes:

1. CONSTRUCT \(R\) WHERE \(P\)
2. SELECT \(S\) WHERE \(P\)
3. CONSELECT \(S, R\) WHERE \(P\)

**Definition 13 (result of CONSTRUCT queries).** Given a pattern \(P_1\) and a graph \(R\) consider the query \(Q = \text{CONSTRUCT } R \text{ WHERE } P_1\) and the pattern \(P = P_1 \text{ BUILD } R\). The result of the query \(Q\) over a graph \(G\), denoted \(\text{Result}_C(Q, G)\), is the subgraph of \(G^{(P)}\) image of \(R\) by the set of matches \([P]_G\).

Thus, the result of a CONSTRUCT query \(Q\) over a graph \(G\) is the graph \(\text{Result}_C(Q, G) = [P]_G(R)\) built by “gluing” the graphs \(m(R)\) for all matches \(m\) in \([P]_G\), where \(m(R)\) is a copy of \(R\) with each variable \(x \in \mathcal{V}(R) - \mathcal{V}(P)\) replaced by a fresh variable (which means, fresh for each \(m\) and each \(x\)).

**Example 4.** Consider the query:

\[
Q_{C, \text{ex}} = \text{CONSTRUCT } R_{\text{ex}} \text{ WHERE } \text{BASIC } (L_{\text{ex}}) = \text{CONSTRUCT } \{ (?p, \text{teaches, } ?z), (?s, \text{studies, } ?z) \} \text{ WHERE } \text{BASIC } (\{ (?p, \text{teaches, } ?t), (?s, \text{studies, } ?t) \})
\]

The corresponding pattern \(P_{\text{ex}}\) and the value \(p_{\text{ex}} : R_{\text{ex}} \Rightarrow G'_{\text{ex}}\) of \(P_{\text{ex}}\) over \(G_{\text{ex}}\) are as in Example 3. It follows that the result of the query \(Q_{C, \text{ex}}\) over \(G_{\text{ex}}\) is the subgraph of \(G'_{\text{ex}}\) image of \(R_{\text{ex}}\) by \(p_{\text{ex}}\):
Result\(_C(Q_{C,ex}, G_{ex}) = \{ \text{(Alice, teaches, } ?z_1), \text{(Charlie, studies, } ?z_1), \text{(Alice, teaches, } ?z_2), \text{(David, studies, } ?z_2), \text{(Bob, teaches, } ?z_3), \text{(Eric, studies, } ?z_3) \}\)

Remark 2. CONSTRUCT queries in SPARQL are similar to CONSTRUCT queries considered in this paper: the variables in \(\mathcal{V}(R) - \mathcal{V}(P_1)\) play the same role as the blank nodes in SPARQL. By considering BUILD patterns, thanks to the functional orientation of the definition of patterns, our language allows BUILD subpatterns: this is new and specific to the present study.

For SELECT queries we proceed as for CONSTRUCT queries: we define a transformation from each SELECT query \(Q\) to a BUILD pattern \(P\) and a transformation from the result of pattern \(P\) to the result of query \(Q\). Definition 14 below would deserve more explanations. However this is not the subject of this paper, see [10] for details about how turning a table to a graph.

**Definition 14 (result of SELECT queries).** For every set of variables \(S = \{s_1, ..., s_n\}\), let \(Gr(S)\) denote the graph made of the triples \((r, c_j, s_j)\) for \(j \in \{1, ..., n\}\) where \(r\) is a fresh variable and \(c_j\) is a fresh constant string for each \(j\). Given a pattern \(P_1\) and a set of variables \(S = \{s_1, ..., s_n\}\) consider the query \(Q = \text{SELECT } S \text{ WHERE } P_1\) and the pattern \(P = P_1 \text{ BUILD } Gr(S)\). The value of \(P\) over a graph \(G\) is a set of matches \([P]_G\) which assignment table has \(n + 1\) columns, corresponding to the variables \(r, s_1, ..., s_n\). The result of the query \(Q\) over a graph \(G\), denoted \(\text{Result}_S(Q, G)\), is the multiset of solutions made of the rows of the assignment table of \([P]_G\) after dropping the column \(r\).

Example 5. Consider the query:

\[ Q_{S,ex} = \text{SELECT } \{?p, ?s\} \text{ WHERE } \text{BASIC } (L_{ex}) \]

Let \(R_{S,ex} = Gr(\{ ?p, ?s \}) = \{ (?r, A_p, ?p), (?r, A_s, ?s) \}\) where \(?r\) is a fresh variable and \(A_p, A_s\) are fresh distinct strings. Then the pattern corresponding to \(Q_{S,ex}\) is:

\[ P_{S,ex} = \text{BASIC } (L_{ex}) \text{ BUILD } R_{S,ex} \]

The value of \(P_{S,ex}\) over \(G_{ex}\) is:

\[ p_{S,ex} : R_{S,ex} \Rightarrow G'_{S,ex} \quad \text{with} \quad \text{Tab}(p_{S,ex}) = \begin{array}{ccc}
?r & ?p & ?s \\
?r_1 & Alice & ?p \\
?r_2 & Alice & David \\
?r_3 & Bob & Eric
\end{array} \]

where \(?r_1, ?r_2\) and \(?r_3\) are 3 fresh variables and:

\[ G'_{S,ex} = G_{ex} \cup \{ (?r_1, A_p, Alice), (?r_1, A_s, Charlie), (?r_2, A_p, Alice), (\text{no attribute}, A_s, David), (?r_3, A_p, Bob), (?r_3, A_s, Eric) \}\]

It follows that:

\[ \text{Result}_S(Q_{S,ex}, G_{ex}) = \begin{array}{ccc}
?p & ?s \\
Alice & Charlie \\
Alice & David \\
Bob & Eric
\end{array} \]
Definition 15 (result of CONSELECT queries). Given a pattern $P_1$ a graph $R$ and a set of variables $S = \{s_1, ..., s_n\}$. Let $Gr(S)$ be the graph as described in Definition 14, consider the query: $Q = \text{CONSELECT } S, R \text{ WHERE } P_1$ and the pattern: $P = P_1 \text{ BUILD } (Gr(S) \cup R)$. The result of the query $Q$ over a graph $G$, denoted $\text{Result}_{CS}(Q, G)$, is the pair consisting of the subgraph of $G^{(P)}$ image of $R$ by the set of matches $[[P]]G$ and the multiset of solutions made of the rows of the assignment table of $[[P]]G(Gr(S))$ after dropping the column $r$.

Example 6. We illustrate here the CONSELECT queries through a toy example. The idea is to have a query that both returns a graph and a table as result. Typically it may be helpful when one wants to query statistical facts about the generated graph. Let us consider the database defined in Example 1. We propose to ask the following query which generates a graph representing professors and their supervised students accompanied with simple statistics about the number of students supervised by each professor.

$$Q_{CS, ex} = \text{CONSELECT}_{CS, ex}, P_{CS, ex} \text{ WHERE } P'_{CS, ex}$$

$$S_{CS, ex} = \{(?p, ?\text{nbstudents})\}$$

$$P_{CS, ex} = \{(?s, \text{supervisedby}, ?p)\}$$

$$P'_{CS, ex} = \text{BASIC} \{\{(?p, \text{is}, \text{Professor}), (?p, \text{teaches}, ?c),
{(?s, \text{is}, \text{Student}), (?s, \text{studies}, ?c)}\}\}$$

$$\text{BIND (COUNT (} ?s \text{ BY } ?p \text{)) AS } ?\text{nbstudents}$$

The result $\text{Result}_{CS}(Q, G)$ of this query is the list of professors with the number of students they supervise (in our toy database, Alice has two students, and Bob has one student) together with the graph of students supervised by a professor. The expected graph and table are displayed below:

| Professor | nbstudents |
|-----------|------------|
| Alice     | 2          |
| Bob       | 1          |

4 A Sound and Complete Calculus

In this section we propose a calculus for solving patterns and queries based on a relation over patterns called qql-narrowing. It computes values (i.e., sets of solutions) of patterns (Definition 10) and results of queries (Definitions 13, 14 and 15) over any graph. This calculus is sound and complete with respect to the set-theoretic semantics given in Section 3.

In functional and logic programming languages, narrowing [4] or resolution [13] derivations are used to solve goals and may have the following shape where $g_0$ is the initial goal to solve (e.g., conjunction of atoms, equations or a (boolean) term) and $g_{n+1}$ is a “terminal” goal such as the empty clause, unifiable equations or the constant true:

$$g_0 \leadsto[\sigma_0] g_1 \leadsto[\sigma_1] g_2 \ldots g_n \leadsto[\sigma_n] g_{n+1}$$
From such a derivation, a solution is obtained by simple composition of local substitutions \( \sigma_n \circ \ldots \circ \sigma_1 \circ \sigma_0 \) with restriction to variables of the initial goal \( \sigma_0 \). In this paper, \( \sigma_0 \) is a pattern or a query and the underlying program is not a set of clauses or rewriting rules but a graph augmented by a set of rewriting rules defining the behavior of two functions \( \text{Solve} \) (for patterns) and \( \text{Solve}_Q \) (for queries).

An important difference between the setting developed in this paper and classical functional and logic languages comes from the use of functional composition \( \circ \) in \( \sigma_n \circ \ldots \circ \sigma_1 \circ \sigma_0 \). Depending on the shape of the considered patterns, solutions can be obtained by using additional composition operators such as \( \text{Join} \) (Definitions 8 and 10) which composes only compatible substitutions computed by different parts of a derivation (e.g., \( \text{Join}(\sigma_k \circ \ldots \circ \sigma_0, \sigma_n \circ \ldots \circ \sigma_{k+1}) \)). In order to have an easy way to handle such kinds of compositions when developing derivations starting from patterns, we introduce below the notion of configuration. We write \( \text{Pat} \) for the sort of patterns.

**Definition 16 (configuration).** Let \( \langle \_ \_ \_ \_ \_ \rangle : \text{Pat} \rightarrow \text{Configurations} \) be the unique constructor operator of the sort \( \text{Configurations} \). Let \( m : L \Rightarrow G \) be a set of matches from graph \( L \) to graph \( G \) and \( P \) a pattern. A configuration is denoted using a mixfix notation as a pair \( [P, m : L \Rightarrow G] \) or simply \( [P, m] \). An initial configuration is a configuration of the form \( [P, \emptyset] \) where \( \emptyset = \text{Match}(\emptyset, G) \) is the set with one unique element that is the inclusion of the empty graph into \( G \). A terminal configuration is a configuration of the form \( [\Box, m : L \Rightarrow G] \).

Roughly speaking, a configuration \( [P, m : L \Rightarrow G] \) represents a state where the considered pattern is \( P \), the current graph database is \( G \) which is the target of the current set of matches \( m : L \Rightarrow G \). Finding solutions of a pattern \( P \) over a graph \( G \) consists in starting from the term \( \text{Solve}([P, \emptyset : L \Rightarrow G]) \) which applies the function \( \text{Solve} \) to an initial configuration and then using appropriate rewriting rules to transform configurations until reaching a terminal configuration of the form \( [\Box, m : L \Rightarrow G'] \) where \( m : L \Rightarrow G' \) represents the expected set of matches (solutions) of \( P \) over \( G \) and where \( G' \) is the graph obtained after solving the pattern \( P \) over \( G \). Notice that graph \( G' \) contains \( G \) but is not necessarily equal to \( G \).

In Fig. 1 we provide a rewriting system, \( \mathcal{R}_{\text{gql}} \), which defines the function \( \text{Solve} \). This function is defined by structural induction on the first component of configurations, i.e., on the patterns. The second argument of configurations, i.e., the sets of matches, in the left-hand sides defining the function \( \text{Solve} \) are always variables of the form \( m : L \Rightarrow G \) or simply \( m \) and thus can be handled easily in the pattern-matching process of the left-hand sides of the proposed rules (no need to higher-order pattern-matching nor unification). In the rules of \( \mathcal{R}_{\text{gql}} \), the letters \( P, P_1 \) and \( P_2 \) are variables ranging over patterns (sort \( \text{Pat} \)) while variables \( L, G \) and \( R \) are ranging over graphs (sort \( \text{Gr} \)) and \( \emptyset \) is the constant denoting the empty graph. Symbol \( e \) is a variable of sort \( \text{Exp} \) and \( x \) is a variable of subsort \( \text{Var} \) while \( m, m' \) and \( p \) are variables of sort \( \text{Som} \). Some constraints of the rules use operations already introduced in Definition 8 such as \( \text{Match}, \text{Join}, \text{Bind}, \text{Filter}, \text{Build} \) and \( \text{Union} \).
The image contains a table with rewriting rules for patterns in a graph query language. The rules are as follows:

| Rule | Description |
|------|-------------|
| $r_0$ | $\text{Solve}([\square, m : L \Rightarrow G]) \rightarrow [\square, \emptyset_G : \emptyset \Rightarrow G]$ |
| $r_1$ | $\text{Solve}([\text{BASIC}(L), m : L \Rightarrow G]) \rightarrow [\square, p : L \Rightarrow G]$ where $p = \text{Match}(L, G)$ |
| $r_2$ | $\text{Solve}([P_1 \text{ JOIN } P_2, m]) \rightarrow \text{Solve}_{JL}(\text{Solve}([P_1, m]), P_2)$ |
| $r_3$ | $\text{Solve}_{JL}([\square, m], P) \rightarrow \text{Solve}_{JR}(m, \text{Solve}([P, m])$ |
| $r_4$ | $\text{Solve}_{JR}(m, [\square, m']) \rightarrow [\square, p]$ where $p = \text{Join}(m, m')$ |
| $r_5$ | $\text{Solve}([P \text{ BIND } e \text{ AS } x, m]) \rightarrow \text{Solve}_{BI}(\text{Solve}([P, m]), e, x)$ |
| $r_6$ | $\text{Solve}_{BI}([\square, m], e, x) \rightarrow [\square, p]$ where $p = \text{Bind}(m, e, x)$ |
| $r_7$ | $\text{Solve}([P \text{ FILTER } e, m]) \rightarrow \text{Solve}_{FR}(\text{Solve}([P, m]), e)$ |
| $r_8$ | $\text{Solve}_{FR}([\square, m], e) \rightarrow [\square, p]$ where $p = \text{Filter}(m, e)$ |
| $r_9$ | $\text{Solve}([P \text{ BUILD } R, m]) \rightarrow \text{Solve}_{BU}(\text{Solve}([P, m]), R)$ |
| $r_{10}$ | $\text{Solve}_{BU}([\square, m], R) \rightarrow [\square, p]$ where $p = \text{Build}(m, R)$ |
| $r_{11}$ | $\text{Solve}([P_1 \text{ UNION } P_2, m]) \rightarrow \text{Solve}_{UL}(\text{Solve}([P_1, m]), P_2)$ |
| $r_{12}$ | $\text{Solve}_{UL}([\square, m], P) \rightarrow \text{Solve}_{UR}(m, \text{Solve}([P, m])$ |
| $r_{13}$ | $\text{Solve}_{UR}(m, [\square, m']) \rightarrow [\square, p]$ where $p = \text{Union}(m, m')$ |

In the sequel, we write $P_{gql}(V)$ for the term algebra over the set of variables $V$ generated by the operations occurring in the rewriting system $\mathcal{R}_{gql}$.

Rule $r_0$ considers the degenerated case when one looks for solutions of the empty pattern $\square$. In this case there is no solution and the empty set of matches $\emptyset_G$ is computed.

Rule $r_1$ is key in this calculus because it considers basic patterns of the form $\text{BASIC}(L)$ where $L$ is a graph which may contain variables. In this case $\text{Solve}([\text{BASIC}(L), m])$ consists in finding all matches from $L$ to $G$. These matches can instantiate variables in $L$. Thus, the constraint $p = \text{Match}(L, G)$ of rule $r_1$ instantiates variables occurring in graph $L$. This variable instantiation process is close to the narrowing or the resolution-based calculi.
As said earlier the term \( \text{Solve}([P, L_G : \emptyset \Rightarrow G]) \) is intended to find both the solutions of pattern \( P \) over graph \( G \) and the graph \( G' \) obtained after transforming graph \( G \) along the evaluation of the subpatterns of \( P \). So, the aim of the gql-narrowing process is to infer all solutions (matches) of a pattern \( P \) over a graph \( G \) starting from the term \( \text{Solve}([P, L_G : \emptyset \Rightarrow G]) \).

In the context of functional-logic programming languages, several strategies of narrowing-based procedures have been developed to solve goals including even a needed strategy \([4]\). In this paper, we do not need all the power of narrowing procedures because manipulated data are mostly flat (mainly constants and variables). Thus the unification process used at every step in the narrowing relation is beyond our needs. On the other hand, the classical rewriting relation induced by the above rewriting system is not enough since variables in patterns \( P \) have to be instantiated and such an instantiation cannot be done by simply rewriting the initial term \( \text{Solve}([P, L_G : \emptyset \Rightarrow G]) \).

Consequently, we propose hereafter a new relation induced by the above rewriting system that we call gql-narrowing. Before the definition of this relation, we recall briefly some notations about first-order terms. Readers not familiar with such notations may consult, e.g., \([5]\).

**Definition 17 (position, subterm replacement, substitution, \( t \downarrow \text{gq} \)).** A position is a sequence of positive integers identifying a subterm in a term. For a term \( t \), the empty sequence, denoted \( \Lambda \), identifies \( t \) itself. When \( t \) is of the form \( g(t_1, \ldots, t_n) \), the position \( i.p \) of \( t \) with \( 1 \leq i \leq n \) and \( p \) is a position in \( t_i \), identifies the subterm of \( t_i \) at position \( p \). The subterm of \( t \) at position \( p \) is denoted \( t[p] \) and the result of replacing the subterm of \( t \) at position \( p \) with term \( s \) is written \( t[s]_p \). We write \( t \downarrow \text{gq} \) for the term obtained from \( t \) where all expressions of \( \mathcal{GQ} \)-algebra (i.e., operations such as Join, Bind, Filter, Match, etc.) have been evaluated. A substitution \( \sigma \) is a mapping from variables to terms. When \( \sigma(x) = u \) with \( u \neq x \), we say that \( x \) is in the domain of \( \sigma \). We write \( \sigma(t) \) to denote the extension of the application of \( \sigma \) to a term \( t \) which is defined inductively as \( \sigma(c) = c \) if \( c \) is a constant or \( c \) is a variable outside the domain of \( \sigma \). Otherwise \( \sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n)) \).

**Definition 18 (gql-narrowing \( \Rightarrow \)).** The rewriting system \( \mathcal{R}_{\text{gql}} \) defines a binary relation \( \Rightarrow \) over terms in \( \mathcal{P}_{\text{gql}}(\mathcal{V}) \) that we call gql-narrowing relation. We write \( t \Rightarrow [u, \operatorname{lhs} \rightarrow \operatorname{rhs}, \sigma] \) \( t' \) or simply \( t \Rightarrow t' \) and say that \( t \) is gql-narrowable to \( t' \) iff there exists a rule \( \operatorname{lhs} \rightarrow \operatorname{rhs} \) in the rewriting system \( \mathcal{R}_{\text{gql}} \), a position \( u \) in \( t \) and a substitution \( \sigma \) such that \( \sigma(\operatorname{lhs}) = t|_u \) and \( t' = t[\sigma(\operatorname{rhs})]_u \). Then \( \Rightarrow^* \) denotes the reflexive and transitive closure of the relation \( \Rightarrow \).

Notice that in the definition of term \( t' = t[\sigma(\operatorname{rhs})]_u \) above, the substitution \( \sigma \) is not applied to \( t \) as in narrowing \( (\sigma(t|\operatorname{rhs}|_u) \downarrow \text{gq}) \) but only to the right-hand side \( (\sigma(\operatorname{rhs})) \). This is mainly due to the possible use of additional function composition such as Join operation. If we consider again the rule \( r_1 \), \( t' \) would be of the following shape \( t' = t[\emptyset, (\text{Match}(\sigma(L), G) : \sigma(L) \Rightarrow G)]_u \). Notice that, in this case, the evaluation of \( \text{Match} \) operation instantiates possible variables occurring in the pattern \( \text{BASIC}(\sigma(L)) \) just like classical narrowing procedures.
Definition 19 (gql-narrowing derivations). Let $G$ be a graph, $P$ a pattern and $m$ a set of matches. The evaluation of $P$ over $G$ consists in computing gql-narrowing derivations of the form:

$$\text{Solve}([P, i : G]) \rightsquigarrow^* [\square, m]$$

Example 7. As in Example 3 we consider the pattern:

$$P_{ex} = \text{BASIC}(L_{ex}) \text{ BUILD } R_{ex}$$

$$= \text{BASIC} \{ (?p, \text{teaches}, ?t), (?s, \text{studies}, ?t) \} \text{ BUILD } \{ (?p, \text{teaches}, ?z), (?s, \text{studies}, ?z) \}$$

The expected gql-narrowing derivation is as follows:

$$\text{Solve}([P_{ex}, L_{G_{ex}}]) \rightsquigarrow_{r_0} \text{Solve}_{BU}(\text{Solve}([\text{BASIC}(L_{ex}), L_{Gex}], R_{ex}))$$

$$\rightsquigarrow_{r_1} \text{Solve}_{BU}([\square, \text{Match}(L_{ex}, G_{ex}], R_{ex})$$

$$\rightsquigarrow_{r_{10}} [\square, \text{Build}(\text{Match}(L_{ex}, G_{ex}], R_{ex})]$$

According to Example 3 this is the required result.

Example 8. We consider again Example 1 and enrich the database with a few triples stating membership to a lab for professors and fixing supervisors of some students.

$$G_A = \{ \text{(Alice, is, Professor), (Alice, teaches, Mathematics)},$$

$$\text{(Bob, is, Professor), (Bob, teaches, Informatics),}$$

$$\text{(Charlie, is, Student), (Charlie, studies, Mathematics),}$$

$$\text{(David, is, Student), (David, studies, Mathematics),}$$

$$\text{(Eric, is, Student), (Eric, studies, Informatics),}$$

$$\text{(Alice, member, Lab1), (Bob, member, Lab2),}$$

$$\text{(David, supervisedby, Alice), (Eric, supervisedby, Bob)} \}$$

We illustrate below a gql-narrowing derivation which solves a pattern $\pi_A$ over graph $G_A$. The goal from this pattern is to find students who are interns in some laboratory (set of matches) and add them to the database (new database). Let $\pi_A$ be the pattern $\pi_1 \text{JOIN } \pi_2$ where:

$$\pi_1 = (P_1 \text{ BUILD } R_1),$$

$$\pi_2 = (P_2 \text{ BUILD } R_2),$$

$$P_1 = \text{BASIC}(L_1), \text{with } L_1 = \{ (?x, \text{supervisedby, } ?p), (?p, \text{member, } ?l) \},$$

$$R_1 = \{ (?x, \text{member, } ?l) \},$$

$$P_2 = \text{BASIC}(L_2), \text{with } L_2 = \{ (?x, \text{member, } ?t), (?x, \text{is, Student}) \},$$

$$\text{and } R_2 = \{ (?x, \text{is, Intern}) \}.$$
of subpattern π₂ uses these triples and adds itself two new triples
\{ (David, is, Intern), (Eric, is, Intern) \}.

\[
\text{Solve}([\pi_A, \downarrow G_A : \emptyset \Rightarrow G_A])
\]

1) \(\leadsto_{r_3} \text{Solve}(_{L_1}([\pi_1, \downarrow G_A]), \pi_2)\)
2) \(\leadsto_{r_3} \text{Solve}(_{L_1}([\pi_2, \downarrow G_A]), R_1), \pi_2)\)
3) \(\leadsto_{r_1} \text{Solve}(_{L_1}([\sqcap, \downarrow \pi_3], R_1), \pi_2)\)
4) \(\leadsto_{r_1} \text{Solve}(_{L_1}([\sqcap, \downarrow \pi_4], \pi_2)\)
5) \(\leadsto_{r_3} \text{Solve}(_{R_1}(\pi_5, \text{Solve}([\pi_2, \downarrow \pi_4]))\)
6) \(\leadsto_{r_9} \text{Solve}(_{R_1}(\pi_5, \text{Solve}([\pi_2, \downarrow \pi_6], R_2))\)
7) \(\leadsto_{r_6} \text{Solve}(_{R_1}(\pi_5, \text{Solve}([\sqcap, \downarrow \pi_7], R_2))\)
8) \(\leadsto_{r_9} \text{Solve}(_{R_1}(\pi_5, [\sqcap, \downarrow \pi_8]))\)
9) \(\leadsto_{r_4} [\sqcap, \downarrow \pi_9] \)

Where:
\[
\begin{align*}
\pi_1 & = \text{Match}(L_1, G_A) \\
\pi_2 & = \text{Build}(\pi_1, R_1) : R_1 \Rightarrow G_B \text{ with } G_B = G_A \cup H_{\sqcap_1}, R_1 \\
\pi_3 & = \text{Match}(L_2, G_1) : L_2 \Rightarrow G_B \\
\pi_4 & = \text{Build}(\pi_3, R_2) : R_2 \Rightarrow G_C \text{ with } G_C = G_B \cup H_{\sqcap_2}, R_2 \\
\pi_5 & = \text{Join}(\pi_3, \pi_4) : R_1 \cup R_2 \Rightarrow G_B \cup G_C
\end{align*}
\]

There are two matches from \(L_1\) to \(G_A\). The set \(\pi_1\) of these matches is:

\[
\begin{array}{ccc}
?x & ?p & ?l \\
David & Alice & Lab1 \\
Eric & Bob & Lab2
\end{array}
\]

Let \(G_B\) be the intermediate graph, \(G_B = G_A \cup H_{\sqcap_1}, R_1\), as depicted below.

\[
G_B = \{ (Alice, is, Professor), (Alice, teaches, Mathematics), (Bob, is, Professor), (Bob, teaches, Informatics), (Charlie, is, Student), (Charlie, studies, Mathematics), (David, is, Student), (David, studies, Mathematics), (Eric, is, Student), (Eric, studies, Informatics), (Alice, member, Lab1), (Bob, member, Lab2), (David, supervisedby, Alice), (Eric, supervisedby, Bob), (David, member, Lab1), (Eric, member, Lab2) \}
\]

Then, the set of matches \(\pi_2 : R_1 \Rightarrow G_B\) consists of two matches.

\[
\begin{array}{ccc}
?x & ?l \\
David & Lab1 \\
Eric & Lab2
\end{array}
\]

There are two matches from \(L_2\) to \(G_B\). The set \(\pi_3\) of these matches is:

\[
\begin{array}{ccc}
?w & ?l \\
David & Lab1 \\
Eric & Lab2
\end{array}
\]
Let $G_C$ be the following graph $G_C = G_B \cup H_{p_2}$:

$$G_C = \{(Alice, is, Professor), (Alice, teaches, Mathematics),
(Bob, is, Professor), (Bob, teaches, Informatics),
(Charlie, is, Student), (Charlie, studies, Mathematics),
(David, is, Student), (David, studies, Mathematics),
(Eric, is, Student), (Eric, studies, Informatics),
(Alice, member, Lab1), (Bob, member, Lab2),
(David, supervisedby, Alice), (Eric, supervisedby, Bob),
(David, member, Lab1), (Eric, member, Lab2),
(David, is, Intern), (Eric, is, Intern)\}$$

The set of matches $p_3 : R_2 \Rightarrow G_C$ consists of two matches.

$$\text{Tab}(p_3) = \begin{array}{c}
?w \\
?x \\
\text{David} \\
\text{Eric} \\
\text{Lab1} \\
\text{Lab2}
\end{array}$$

Finally, $p_4 = \text{Join}(p_3, p_1) : R_1 \cup R_2 \Rightarrow G_B \cup G_C$ consists of two matches

$$\text{Tab}(p_4) = \begin{array}{c}
?w \\
?x \\
?l \\
\text{David} \\
\text{Lab1} \\
\text{Eric} \\
\text{Lab2}
\end{array}$$

**Theorem 1 (soundness).** Let $G$ be a graph, $P$ a pattern and $m$ a set of matches such that $\text{Solve}(\square, m) \Rightarrow^* [P]_G$. Then for all morphisms $m'$ in $m$, there exists a morphism $m'$ equal to $m$ up to renaming of variables such that $m'$ is in $[[P]]_G$.

**Proof.** The proof is done by induction on the length $n$ of derivation $\text{Solve}(\square, m) \Rightarrow^* [P]_G$.

**Base case.** $n = 1$. In this case only rules $r_0$ or $r_1$ are possible.

Case of rule $r_0$. In this case, the pattern $P$ is the empty pattern $\square$ and $m = \emptyset$.

The case vacuously holds.

Case of rule $r_1$. In this case, the pattern $P$ is of the form $\text{BASIC}(L)$ and the considered derivation is of the form $\text{Solve}(\square, m : L \Rightarrow G)$ where $m = \text{Match}(L, G)$. The claim obviously holds since $[[\text{BASIC}(L)]_G = \text{Match}(L, G) : L \Rightarrow G$ by Definition $10$.

**Induction Step.** $n > 1$. In this case the pattern $P$ can be of five different shapes as discussed below:

$P = \pi_1 \text{ JOIN } \pi_2$. Then, the gql-narrowing derivation $\text{Solve}(\square, m) \Rightarrow^* [P]_G$ has the following shape

$\text{Solve}(\pi_1 \text{ JOIN } \pi_2, m) \Rightarrow \text{Solve}_J L(\text{Solve}(\pi_1, m), \pi_2) \Rightarrow^{n_1}$

$\text{Solve}_J L([\square, m_1], \pi_2) \Rightarrow \text{Solve}_J R(m_1, \text{Solve}(\pi_2, m_1)) \Rightarrow^{n_2}$

$\text{Solve}_J R(m_1, \square, m_2 : Z \Rightarrow G') \Rightarrow [\square, m_3]$ with $m_3 = \text{Join}(m_1, m_2)$ and $Z$ a graph.
Notice that the length $n$ equals $n_1 + n_2 + 3$. By induction hypothesis, the set of matches $m_1$ and $m_2$ is sound. Then, the set $m_3$ is obtained by using the operation $\text{Join}$ over the sets $m_1$ and $m_2$ which ensures the soundness of the set $m_3$.

\[
P = \pi_1 \text{BIND } e_1 \text{ AS } x_1.
\]

Then, the gql-narrowing derivation $\text{Solve}([P, \xi_G]) \rightsquigarrow^n ([\square, m])$ has the following shape:

\[
\text{Solve}([\pi_1 \text{BIND } e_1 \text{ AS } x_1, \xi_G]) \rightsquigarrow \text{Solve}_{BD}(\text{Solve}([\pi_1, \xi_G]), e_1, x_1) \rightsquigarrow^n m_1
\]

$\text{Solve}_{BD}([\square, m_1 : Z \Rightarrow G_1]), e_1, x_1) \rightsquigarrow [\square, m_2]$ with $G' = G_1 \cup m(e_1)$ and $m_3 = \text{Bind}(m_1, e_1, x_1) : Z \Rightarrow G'$

Notice that the length $n$ equals $n_1 + 2$. Thus $n_1$ is less than $n$ and by induction hypothesis, the set of matches $m_1$ is sound. Then, the set $m_3$ is obtained as $\text{Bind}(m_1, ev(m_1, e_1), x_1)$ as expected by the semantics which ensures the soundness of the set $m_3$.

\[
P = \pi_1 \text{FILTER } e_1.
\]

Then, the gql-narrowing derivation $\text{Solve}([P, \xi_G]) \rightsquigarrow^n ([\square, m])$ has the following shape:

\[
\text{Solve}([\pi_1 \text{FILTER } e_1, \xi_G]) \rightsquigarrow \text{Solve}_{FR}(\text{Solve}([\pi_1, \xi_G]), e_1) \rightsquigarrow^n m_1
\]

$\text{Solve}_{FR}([\square, m_1 : Z \Rightarrow G_1]), e_1) \rightsquigarrow [\square, m_2]$ with $m_3 = \text{Filter}(m_1, e_1) : Z \Rightarrow G'$

Notice that the length $n$ equals $n_1 + 2$. Thus $n_1$ is less than $n$ and by induction hypothesis, the set of matches $m_1$ is sound. Then, the set $m_3$ is obtained as $\text{Filter}(m_1, ev(m_1, e_1))$ as expected by the semantics which ensures the soundness of the set $m_3$.

\[
P = \pi_1 \text{BUILD } R.
\]

Then, the gql-narrowing derivation $\text{Solve}([P, \xi_G]) \rightsquigarrow^n ([\square, m])$ has the following shape:

\[
\text{Solve}([\pi_1 \text{BUILD } R_1, \xi_G]) \rightsquigarrow \text{Solve}_{BR}(\text{Solve}([\pi_1, \xi_G]), R_1) \rightsquigarrow^n m_1
\]

$\text{Solve}_{BR}([\square, m_1 : Z \Rightarrow G_1]), R_1) \rightsquigarrow [\square, m_2]$ with $m_3 = \text{Build}(m_1, R_1) : Z \Rightarrow G'$

Notice that the length $n$ equals $n_1 + 2$. Thus $n_1$ is less than $n$ and by induction hypothesis, the set of matches $m_1$ is sound. Then, the set $m_3$ is obtained as $\text{Build}(m_1, R_1)$ as expected by the semantics which ensures the soundness of the set $m_3$.

\[
P = \pi_1 \text{UNION } \pi_2.
\]

Then, the gql-narrowing derivation $\text{Solve}([P, \xi_G]) \rightsquigarrow^n ([\square, m])$ has the following shape

\[
\text{Solve}([\pi_1 \text{UNION } \pi_2, \xi_G]) \rightsquigarrow \text{Solve}_{UL}(\text{Solve}([\pi_1, \xi_G]), \pi_2) \rightsquigarrow^n m_1
\]

$\text{Solve}_{UL}([\square, m_1]), \pi_2) \rightsquigarrow \text{Solve}_{UR}(m_1, \text{Solve}([\pi_2, m_1])) \rightsquigarrow^n m_3$

$\text{Solve}_{UR}(m_1, [\square, m_2 : Z \Rightarrow G']) \rightsquigarrow [\square, m_3]$ with $m_4 = \text{Union}(m_1, m_2) : Z \Rightarrow G'$ and $Z$ a graph.

Notice that the length $n$ equals $n_1 + n_2 + 3$. By induction hypothesis, the set of matches $m_1$ and $m_2$ is sound. Then, the set $m_4$ is obtained by using the operation $\text{Union}$ over the sets $m_1$ and $m_2$ as expected by the semantics which ensures the soundness of the set $m_4$.

**Theorem 2** (completeness). Let $G_1, G_2$ and $X$ be graphs, $P$ a pattern and $h : X \Rightarrow G_2$ a match in $[P]_{G_1}$. Then there exist graphs $G'_2$ and $X'$, a set of matches $m : X' \Rightarrow G'_2$, a derivation $\text{Solve}([P, \xi_G]) \rightsquigarrow^* [\square, m]$ and a match $m : X' \Rightarrow G'_2$ in $m$ such that $m$ and $h$ are equal up to variable renaming.
Proof. The proof is done by structural induction over patterns.

Base case. $P = \emptyset$. In this case, $[[P]]_G$ is empty and thus the statement vacuously holds.

Induction step. There are six cases to consider according to the shape of pattern $P$.

$P = \text{BASIC}(L)$. In this case the set of matches $[[P]]_G$ coincides with the set $m$ obtained after one step gql-narrowing $\text{Solve}([[\text{BASIC}(L), L]]_G) \leadsto_{r_1} \left[ \square, m : L \Rightarrow G_1 \right]$ where $m = \text{Match}(L, G_1)$. Obviously, $m = [[P]]_G$.

$P = \pi_1 \text{JOIN} \pi_2$. By Definition $[[P]]_G = \text{Join}([[\pi_1]_G], [[\pi_2]_G])$.

Thus, $h$ being an element of $[[\pi_1 \text{JOIN} \pi_2]_G]$ there exist two matches $h_1$ in $[[\pi_1]_G$ and $h_2$ in $[[\pi_2]_G$ such that $h = h_1 \uplus h_2$. Let us consider the following gql-narrowing step $\text{Solve}([[\pi_1 \text{JOIN} \pi_2]_G]) \leadsto \text{Solve}_{JL}(\text{Solve}([[\pi_1]_G]), \pi_2)$. By induction hypothesis, there exists a derivation $\text{Solve}([[\pi_1]_G]) \leadsto^* \left[ \square, m_1 \right]$ and a match $m_1$ in $m_1$ such that $h_1$ and $m_1$ are equal up to variable renaming. Notice that $G'_1$ is isomorphic to $G'_{\pi_1}$. Now we can develop further the above derivation and get $\text{Solve}_{JL}(\text{Solve}([[\pi_1]_G]), \pi_2) \leadsto \text{Solve}_{JL}([\square, m_1], \pi_2) \leadsto \text{Solve}_{JL}(\text{Solve}_{JR}(m_1, \text{Solve}(\pi_2, m_1)))$. Again, by induction hypothesis, there exists a derivation $\text{Solve}_{JR}(m_1, \text{Solve}(\pi_2, m_1)) \leadsto^* \left[ \square, m_2 : Z \Rightarrow \left[ G_2 \right] \right]$ and a match $m_2$ in $m_2$ such that $h_2$ and $m_2$ are equal up to variable renaming.

Finally, we get the expected derivation $\text{Solve}_{JL}(\text{Solve}([[\pi_1]_G]), \pi_2) \leadsto^* \text{Solve}_{JL}(\text{Solve}(\pi_2, m_2)) \leadsto^* \text{Solve}_{JL}([\square, m_2 : Z \Rightarrow \left[ G_2 \right]])$.

$P = \pi_1 \text{BIND} e_1 AS x_1$. By Definition $[[\pi_1 \text{BIND} e_1 \text{AS} x_1]]_G = \text{Bind}([[P_1]_G, e_1, x_1])$.

Now, let us consider the following gql-narrowing step $\text{Solve}([[\pi_1 \text{BIND} e_1 \text{AS} x_1]_G]) \leadsto \text{Solve}_{BD}(\text{Solve}([[\pi_1]_G]), e_1, x_1)$. By induction hypothesis, there exists a derivation $\text{Solve}([[\pi_1]_G]) \leadsto^* \left[ \square, m_1 \right]$ such that for all matches $h_1$ in $[[P_1]_G$, there exists a match $m_1$ in $m_1$ equal to $h_1$ up to variable renaming. Therefore, for every element $e$ in $\text{Bind}([[P_1]_G, e_1, x_1])$ there exists a match $m$ in $\text{Bind}(m, e_1, x_1)$ with $m$ and $h$ equal up to variable renaming.

$P = \pi_1 \text{FILTER} e_1$. By Definition $[[\pi_1 \text{FILTER} e_1]]_G = \text{Filter}([[P_1]_G, e_1])$.

Now, let us consider the following gql-narrowing step $\text{Solve}([[\pi_1 \text{FILTER} e_1]_G]) \leadsto \text{Solve}_{FR}(\text{Solve}([[\pi_1]_G]), e_1)$. By induction hypothesis, there exists a derivation $\text{Solve}([[\pi_1]_G]) \leadsto^* \left[ \square, m_1 \right]$ such that for all matches $h_1$ in $[[P_1]_G$, there exists a match $m_1$ in $m_1$ equal to $h_1$ up to variable renaming. Therefore, for every element $e$ in $\text{Filter}([[P_1]_G, e_1])$ there exists a match $m$ in $\text{Filter}(m_1, e_1, x_1)$ with $m$ and $h$ equal up to variable renaming.

$P = \pi_1 \text{BUILD} R_1$. By Definition $[[\pi_1 \text{BUILD} R_1]]_G = \text{Build}([[P_1]_G, R_1])$.

Thus, $h$ is an element of $\text{Build}([[P_1]_G, R_1])$.
Now, let us consider the following gql-narrowing step
\( \text{Solve}([\pi_1 \text{ BUILD } R_1, \mathbf{L}_{G_1}]) \rightsquigarrow \text{Solve}_B(V)(\text{Solve}([\pi_1, \mathbf{L}_{G_1}]), R_1) \). By induction hypothesis, there exists a derivation \( \text{Solve}([\pi_1, \mathbf{L}_{G_1}]) \rightsquigarrow^* [\square, m_1] \) such that for all matches \( h \) in \([\pi_1]_{G_1}\), there exists a match \( m_1 \) in \( m_1 \) equal to \( h \) up to variable renaming. Therefore, for every element \( h \) in \( \text{Build}([\pi_1])_{G_1}, R_1 \) there exists a match \( m \) in \( \text{Build}(m_1, R_1) \) with \( m \) and \( h \) equal up to variable renaming.

\[ P = \pi_1 \text{ UNION } \pi_2. \] By Definition\[\text{[10]}[[\pi_1 \text{ UNION } \pi_2]]_{G_1} = \text{Union}([[\pi_1]]_{G_1}, [[\pi_2]]_{G_1, (\pi_1)}).\]

Thus, \( h \) being an element of \([\pi_1 \text{ UNION } \pi_2]]_{G_1}\), either \( h \) is an extension of a match \( h_1 \) in \([\pi_1]]_{G_1}\) or a match \( h_2 \) in \([\pi_2]]_{G_1, (\pi_1)}\).

Let us consider the following gql-narrowing step \( \text{Solve}([\pi_1 \text{ UNION } \pi_2, \mathbf{L}_{G_1}]) \rightsquigarrow \text{Solve}_L(V)(\text{Solve}([\pi_1, \mathbf{L}_{G_1}]), \pi_2) \). By induction hypothesis, there exists a derivation \( \text{Solve}([\pi_1, \mathbf{L}_{G_1}]) \rightsquigarrow^* [\square, m_1] \). If \( h \) is an extension of \( h_1 \) element of \([\pi_1]]_{G_1}\), then, by induction hypothesis, there exists a match \( m_1 \) in \( m_1 \) such that \( h_1 \) and \( m_1 \) are equal up to variable renaming and \( G_1 \) is isomorphic to \( G_1^{(\pi_1)} \). Now we can develop further the above derivation and get \( \text{Solve}([\pi_1 \text{ UNION } \pi_2, \mathbf{L}_{G_1}]) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_1, \mathbf{L}_{G_1}]), \pi_2) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_2, \mathbf{L}_{G_1}]), \pi_2) \). Again, by induction hypothesis, there exists a derivation \( \text{Solve}([\pi_2, m_1]) \rightsquigarrow^* [\square, m_2 : Z \Rightarrow G_2] \). If \( h \) is an extension of \( h_2 \) element of \([\pi_2]]_{G_2}\), then, by induction hypothesis, there exists a match \( m_2 \) in \( m_2 \) such that \( h_2 \) and \( m_2 \) are equal up to variable renaming and \( G_2 \) is isomorphic to \( G_2^{(\pi_2)} \). Finally, we get the expected derivation
\( \text{Solve}([\pi_1 \text{ UNION } \pi_2, \mathbf{L}_{G_1}]) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_1, \mathbf{L}_{G_1}]), \pi_2) \rightsquigarrow^* \text{Solve}_U(V)(\text{Solve}([\pi_2, m_1]), \pi_2) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_2, m_1]), \pi_2) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_2, m_1]), \pi_2) \rightsquigarrow \text{Solve}_U(V)(\text{Solve}([\pi_2, m_1]), \pi_2) \). Therefore, by definition of \( m_3 \), we have \( m \) can be either an extension of \( m_1 \) or \( m_2 \) in \( m_4 \) and the empty pattern \( \square \) is equal to \( h \) up to variable renaming.

All gql-narrowing derivation steps for solving patterns are needed since at each step only one position is candidate to a gql-narrowing step.

**Proposition 1 (determinism).** Let \( t_0 \rightsquigarrow t_1 \rightsquigarrow \ldots \rightsquigarrow t_n \) be a gql-narrowing derivation with \( t_0 = \text{Solve}(\mathbf{P}, \mathbf{L}_{G}) \). For all \( i \in [0..n] \), there exists at most one position \( u \) in \( t_i \) such that \( t_i \) can be gql-narrowed into \( t_{i+1} \).

**Proof.** The proof is by induction on \( i \).

Base case \((i = 0)\). Since the function \text{solve} is completely defined by structural induction on patterns, it follows that there exists one and only one rule that applies to the term \( t_0 = \text{Solve}(\mathbf{P}, \mathbf{L}_{G} : \emptyset \Rightarrow G) \) at position \( \Lambda \) according to the structure (constructor at the head) of pattern \( P \).

Induction step: Assume that \( t_i \) contains only one reducible subterm at position \( u_i = u.k \) via rule \( r \) with \( k \) being a natural number \((k > 0)\) and \( u \) a position in \( t_i \). Below, we discuss the different cases according to the considered rule \( r \).

\( r = r_0 \): Rule \( r_0 \) considers the degenerated case when one looks for solutions of the empty pattern \( \square \). In this case there is no solution and the empty set of
matches is computed $\emptyset_C$. The right-hand side of rule $r_0$ is in normal form and thus does not contain a possible reducible subterm. In addition, according to the shape of the other rules only the next upper position $u$ may become a potential reducible term via one of the rules $r_0, r_1, r_4, r_6, r_8, r_{10}, r_{12}, r_{13}$.

$r = r_1$: Rule $r_1$ considers the case when one looks for solutions when the pattern is of the form $BASIC(G_1)$ for some graph $G_1$. The right-hand side of the rule is in normal form and thus does not contain a possible reducible subterm. However, according to the shape of the other rules (height of the left-hand sides is equal to 1), only the next upper term at position $u$ may become a potential reducible term via one of the rules $r_0, r_3, r_4, r_6, r_8, r_{10}, r_{12}, r_{13}$.

$r = r_2$: In this case we have $t_i \sim_{[u,k,r_2,\sigma]} t_{i+1}$ with $t_{i+1} = [\sigma_i(SOLVE_{\text{JR}}(\text{Solve}(\{P_1, m\}), P_2))_{gq}]_{u,k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_2$ is the operation $SOLVE_{\text{JR}}$ which does not appear in the subterms of the left-hand sides of the rules in $R_{\text{gq}}$. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.1$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_3$ cannot be used (no possible pattern-matching). It remains position $u.k.1$ at which term $t_{i+1}$ can be reduced if term $\sigma_i(P_1)$ is headed by one of the constructors of patterns.

$r = r_3$: In this case we have $t_i \sim_{[u,k,r_3,\sigma]} t_{i+1}$ with $t_{i+1} = t_i[\sigma_i(SOLVE_{\text{JR}}(m, \text{Solve}(\{P, m\}))_{gq}]_{u,k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_3$ is the operation $SOLVE_{\text{JR}}$ which does not appear in the subterms of the left-hand sides of the rules. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.2$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_4$ cannot be used (no possible pattern-matching). It remains position $u.k.2$ at which term $t_{i+1}$ can be reduced if term $\sigma_i(P)$ is headed by one of the constructors of patterns.

$r = r_4$: In this case we have $t_i \sim_{[u,k,r_4,\sigma]} t_{i+1}$ with $t_{i+1} = [\sigma_i([\Box, Join(m, m'))]_{gq}]_{u,k}$. Notice that the subterm $Join(m, m')$ is part of a built-in constraint and is supposed to be evaluated (in normal form). Thus the subterm of $t_{i+1}$ at position $u.k$ cannot be reduced further. However $t_{i+1}$ can be reducible at position $u$ by using one of the rules $r_0, r_3, r_4, r_6, r_8, r_{10}, r_{12}, r_{13}$.

$r = r_5$: In this case we have $t_i \sim_{[u,k,r_5,\sigma]} t_{i+1}$ with $t_{i+1} = t_i[\sigma_i(SOLVE_{\text{BT}}(\text{Solve}(\{P, m\}, e, x))]_{gq}]_{u,k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_5$ is the operation $SOLVE_{\text{BT}}$ which does not appear in the subterms of the left-hand sides of the rules in $R_{\text{gq}}$. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.1$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_6$ cannot be used (no possible pattern-matching). It remains position $u.k.1$ at which term $t_{i+1}$ can be reduced if term $\sigma_i(P)$ is headed by one of the constructors of patterns.

$r = r_6$: In this case we have $t_i \sim_{[u,k,r_6,\sigma]} t_{i+1}$ with $t_{i+1} = t_i[\sigma_i([\Box, Bind(m, e, x)])]_{gq}]_{u,k}$. Notice that the subterm $Bind(m, e, x)$ is part of a built-in constraint and is supposed to be evaluated (in normal form). Thus, the subterm of $t_{i+1}$ at
position $u.k$ cannot be reducible further. However $t_{i+1}$ can be reducible at position $u$ by using one of the rules $r_0, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{12}, r_{13}$.

$r = r_7$ : In this case we have $t_i \leadsto_{\{u.k, r_7, \sigma_1\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1(Solve_{FR}(Solve(\{P, m\}, e)))]_{u.k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_7$ is the operation $Solve_{FR}$ which does not appear in the subterms of the left-hand sides of the rules in $R_{gql}$. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.1$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_8$ cannot be used (no possible pattern-matching). It remains position $u.k.1$ at which term $t_{i+1}$ can be reduced if term $\sigma_1(P)$ is headed by one of the constructors of patterns.

$r = r_8$ : In this case we have $t_i \leadsto_{\{u.k, r_8, \sigma_1\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1(\square, Filter(m, e))]_{u.k}$. Notice that the subterm $Filter(m, e)$ is part of a built-in constraint and is supposed to be evaluated (in normal form). Thus, the subterm of $t_{i+1}$ at position $u.k$ cannot be reducible further. However $t_{i+1}$ can be reducible at position $u$ by using one of the rules $r_0, r_3, r_4, r_5, r_6, r_7, r_{10}, r_{12}, r_{13}$.

$r = r_9$ : In this case we have $t_i \leadsto_{\{u.k, r_9, \sigma_1\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1(Solve_{BU}(Solve(\{P, m\}, R))]_{u.k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_9$ is the operation $Solve_{BU}$ which does not appear in the subterms of the left-hand sides of the rules in $R_{gql}$. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.1$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_{10}$ cannot be used (no possible pattern-matching). It remains position $u.k.1$ at which term $t_{i+1}$ can be reduced if term $\sigma_1(P)$ is headed by one of the constructors of patterns.

$r = r_{10}$ : In this case we have $t_i \leadsto_{\{u.k, r_{10, \sigma_1}\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1([\square, Build(m, R)]))]_{u.k}$. Notice that the subterm $Build(m, R)$ is part of a built-in constraint and is supposed to be evaluated (in normal form). Thus, the subterm of $t_{i+1}$ at position $u.k$ cannot be reducible further. However $t_{i+1}$ can be reducible at position $u$ by using one of the rules $r_0, r_3, r_4, r_5, r_6, r_7, r_{10}, r_{12}, r_{13}$.

$r = r_{11}$ : In this case we have $t_i \leadsto_{\{u.k, r_{11, \sigma_1}\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1(Solve_{UL}(Solve(\{P_1, m\}, P_2))]_{u.k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_2$ is the operation $Solve_{UL}$ which does not appear in the subterms of the left-hand sides of the rules in $R_{gql}$. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.1$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_{12}$ cannot be used (no possible pattern-matching). It remains position $u.k.1$ at which term $t_{i+1}$ can be reduced if term $\sigma_1(P_1)$ is headed by one of the constructors of patterns.

$r = r_{12}$ : In this case we have $t_i \leadsto_{\{u.k, r_{12, \sigma_1}\}} t_{i+1}$ with $t_{i+1} = t_i[\sigma_1(Solve_{UR}(m, Solve(\{P, m\}, P))]_{u.k}$. $t_{i+1}$ cannot be reduced at position $u$ because $t_i$ was not reducible at position $u$ and the head of the right-hand side of rule $r_{12}$ is the operation $Solve_{UR}$ which does not appear in the subterms of the left-hand sides of the rules. Therefore, $t_{i+1}$ can be reducible either at position $u.k$ or $u.k.2$ since $t_i$ was reducible at position $u.k$ only. $t_{i+1}$ is not reducible at position $u.k$ because rule $r_{13}$ cannot be used (no
possible pattern-matching). It remains position \( u.k.2 \) at which term \( t_{i+1} \) can be reduced if term \( \sigma_i(P) \) is headed by one of the constructors of patterns.

\[
r = r_{13}: \text{In this case we have } t_i \xrightarrow{[u.k.\tau_{13}.\sigma_i]} t_{i+1} \text{ with } t_{i+1} = t_i[\sigma_i(\bigcup, \text{Union}(m, m'))]_{gql}
\]

|u.k. Notice that the subterm \( \text{Union}(m, m') \) is part of a built-in constraint and is supposed to be evaluated (in normal form). Thus, the subterm of \( t_{i+1} \) at position \( u.k \) cannot be reducible further. However \( t_{i+1} \) can be reducible at position \( u \) by using one of the rules \( r_0, r_3, r_4, r_6, r_8, r_{10}, r_{12}, r_{13} \).

**Proposition 2 (termination).** The relation \( \sim \) is terminating.

**Proof.** The proof is quite direct. One possible ordering is the lexicographical ordering \( >_{\text{lex}} \) defined over terms of the rewriting system \( R_{\text{gql}} \) as follows: \( t_1 >_{\text{lex}} t_2 \) \( \iff (M(t_1), H(t_1)) >_{\text{lex}} (M(t_2), H(t_2)) \) where

\[ H(t) \text{ is the height of term } t \text{ and } M(t) \text{ is the multiset including all values } M(P) \text{ for every pattern } P \text{ occurring in } t. >_{\text{lex}} \text{ is we}

For rules \( r_1, r_2, r_3, r_5, r_7, r_9, r_{11}, r_{12} \), we have \((M(lhs), H(lhs)) >_{\text{lex}} (M(rhs), H(rhs))\) because \( M(lhs) > M(rhs) \). For rules \( r_0, r_4, r_6, r_8, r_{10}, r_{13} \) we have \((M(lhs), H(lhs)) >_{\text{lex}} (M(rhs), H(rhs))\) because \( M(lhs) = M(rhs) \) and \( H(lhs) > H(rhs) \).

In Fig. 2 we enrich the rewriting system \( R_{\text{gql}} \) by means of six additional rules which tackle queries as in Section 3.2. Remember that the result of a query \( Q \) over a graph \( G \) is a graph \( \text{Result}_{C}(Q, G) \) when \( Q \) is a CONSTRUCT query whereas it is a table \( \text{Result}_{S}(Q, G) \) when \( Q \) is a SELECT query, or both in the case of a CONSELECT query \( \text{Result}_{CS}(Q, G) \). The details of the display functions \( \text{Print}_{C}, \text{Print}_{S} \) and \( \text{Print}_{CS} \) used in rules \( r_{15}, r_{17} \) and \( r_{19} \) are omitted as they are out of the scope of the present paper.

For SELECT and CONSELECT queries we use the graph \( \text{Gr}(S) \) associated to the set of variables \( S \) (see, Definition 14 or 10 for details).

**Fig. 2.** \( R_{\text{gql}} \) (continued): Rewriting rules for queries

\[
\begin{align*}
r_{14} : & \quad \text{Solve}_{Q}(\text{CONSTRUCT } R \text{ WHERE } P, G) \rightarrow \text{Display}_{C}(R, \text{Solve}(\bigcup P \text{ BUILD } R, \text{Gr}(G))) \\
r_{15} : & \quad \text{Display}_{C}(R, [\bigcup, m]) \rightarrow \text{Print}_{C}(R, m) \\
r_{16} : & \quad \text{Solve}_{Q}(\text{SELECT } S \text{ WHERE } P, G) \rightarrow \text{Display}_{S}(S, \text{Solve}(\bigcup P \text{ BUILD } \text{Gr}(S), \text{Gr}(G))) \\
r_{17} : & \quad \text{Display}_{S}(S, [\bigcup, m]) \rightarrow \text{Print}_{S}(S, m) \\
r_{18} : & \quad \text{Solve}_{Q}(\text{CONSELECT } S, R \text{ WHERE } P, G) \rightarrow \text{Display}_{CS}(S, R, \text{Solve}(\bigcup P \text{ BUILD } \text{Gr}(S) \cup R, \text{Gr}(G))) \\
r_{19} : & \quad \text{Display}_{CS}(S, R, [\bigcup, m]) \rightarrow \text{Print}_{CS}(S, R, m)
\end{align*}
\]

The Soundness and completeness of the calculus with respect to queries are direct consequences of Theorems 1 and 2 and Definitions 13, 14 and 15.

5 Conclusion and Related Work

We propose a rule-based calculus for a core graph query language. The calculus is generic and could easily be adapted to different graph structures and extended
to actual graph query languages. For instance, graph path variables can be added to the syntax and matches could be constrained by positive, negative or path constraints just by performing matches with constraints $\text{Match}(L, G, \Phi)$ where $\Phi$ represents constraints in a given logic over items of graphs $L$ and $G$. To our knowledge, the proposed calculus is the first rule-based sound and complete calculus dedicated to graph query languages. This work opens new perspectives regarding the application of verification techniques to graph-oriented database query languages. Future work includes also an implementation of the proposed calculus.

Among related work, we quote first the use of declarative (functional and logic) languages in the context of relational databases (see, e.g. [7,12,1]). In these works, the considered databases follow the relational paradigm which differs from the graph-oriented one that we are tackling in this paper. Our aim is not to make connections between graph query languages and functional logic ones. We are rather interested in investigating formally graph query languages, and particularly in using dedicated rewriting techniques for such languages.

The notion of pattern present in this paper is close to the syntactic notions of clauses in [11] or graph patterns in [3]. For such syntactic notions, some authors associate as semantics sets of variables bindings (tables) as in [11,14] or simply graphs as in [2]. In our case, we associate both variable bindings and graphs since we associate sets of graph homomorphisms to patterns. This semantics is borrowed from a first work on formal semantics of graph queries based on category theory [10]. Our semantics allows composition of patterns in a natural way. Such composition of patterns is not easy to catch if the semantics is based only on variable bindings but can be recovered when queries have graph outcomes as in G-CORE [2].

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