Global existence of solutions to a parabolic-elliptic chemotaxis system with critical degenerate diffusion

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Abstract This paper is devoted to the analysis of non-negative solutions for a degenerate parabolic-elliptic Patlak-Keller-Segel system with critical nonlinear diffusion in a bounded domain with homogeneous Neumann boundary conditions. Our aim is to prove the existence of a global weak solution under a smallness condition on the mass of the initial data, thereby completing previous results on finite blow-up for large masses. Under some higher regularity condition on solutions, the uniqueness of solutions is proved by using a classical duality technique.

Keywords: Chemotaxis; Keller-Segel model; Parabolic equation; Elliptic equation; Global existence; Uniqueness.

1 Introduction

Chemotaxis is the movement of biological organisms oriented towards the gradient of some substance, called the chemoattractant. The Patlak-Keller-Segel (PKS) model (see [13], [12] and [17]) has been introduced in order to explain chemotaxis cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density $u$, and a reaction diffusion equation for the chemoattractant concentration $\varphi$. It reads

\begin{equation}
\begin{aligned}
\partial_t u &= \text{div}(\nabla u^m - u \cdot \nabla \varphi) & x \in \Omega, t > 0, \\
-\Delta \varphi &= u - <u> & x \in \Omega, t > 0, \\
<\varphi(t)> &= 0 & t > 0, \\
\partial_{\nu} u &= \partial_{\nu} \varphi = 0 & x \in \partial\Omega, t > 0, \\
u(0,x) &= u_0(x) & x \in \Omega,
\end{aligned}
\end{equation}

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $\nu$ the outward unit normal vector to the boundary $\partial\Omega$ and $m \geq 1$. An important parameter in this model is the total mass $M$ of cells, which is formally conserved through the evolution:

\begin{equation}
M = <u> = \frac{1}{|\Omega|} \int_{\Omega} u(t,x) \, dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx.
\end{equation}

Several studies have revealed that the dynamics of (1) depend sensitively on the parameters $N$, $m$ and $M$. More precisely, if $N = 2$ and $m = 1$, it is well-known that the solutions of (1) may blow up in finite time if $M$ is sufficiently large (see [17, 16]) while solutions are global in time for $M$ sufficiently small [17], see also the survey articles [4, 10].

The situation is very different when $m = 1$ and $N \neq 2$. In fact, if $N = 1$, there is global existence of solutions of (1) whatever the value of the mass of initial data $M$, see [8] and the references therein. If $N \geq 3$, for all $M > 0$, there are initial data $u_0$ with mass $M$ for which the corresponding solutions of (1) explode in finite time (see [16]). Thus,
in dimension $N \geq 3$ and $m = 1$, the threshold phenomenon does not take place as in dimension 2, but we expect the same phenomenon when $N \geq 3$ and $m$ is equal to the critical value $m = m_c = \frac{2(N-1)}{N}$. More precisely, we consider a more general version of (1) where the first equation of (1) is replaced by

$$\partial_t u = \text{div}(\phi(u) \nabla u - u \nabla \phi), \quad t > 0, \quad x \in \Omega,$$

and the diffusitivity $\phi$ is a positive function in $C^1([0, \infty[)$ which does not grow to fast at infinity. In [8], the authors proved that there is a critical exponent such that, if the diffusion has a faster growth than the one given by this exponent, solutions to (1) (with $\phi(u)$ instead of $mu^{m-1}$) exist globally and are uniformly bounded, see also [6, 14] for $N = 2$. More precisely, the main results in [8] read as follows:

- If $\phi(u) \geq c(1 + u)^p$ for all $u \geq 0$ and some $c > 0$ and $p > 1 - \frac{2}{N}$ then all solutions of (1) are global and bounded.
- If $\phi(u) \leq c(1 + u)^p$ for all $u \geq 0$ and some $c > 0$ and $p < 1 - \frac{2}{N}$ then there exist initial data $u_0$ such that

$$\lim_{t \to T} ||u(., t)||_{\infty} = \infty,$$

for some finite $T > 0$.

Except for $N = 2$, the critical case $m = \frac{2(N-1)}{N}$ is not covered by the analysis of [8]. Recently, Cieślak and Laurençot in [7] show that if $\phi(u) \leq c(1 + u)^{1-\frac{2}{N}}$ and $N \geq 3$, there are solutions of (1) blowing up in finite time when $M$ exceeds an explicit threshold. In order to prove that, when $N \geq 3$ and $m = \frac{2(N-1)}{N}$, we have a threshold phenomenon similar to dimension $N = 2$ with $m = 1$, it remains to show that solutions of (1) are global when $M$ is small enough. The goal of this paper is to show that this is indeed true, see Theorem 2.2 below.

By combining Theorem 2.2 with the blow-up result obtained in [7], we conclude that, for $N \geq 3$ and $m = \frac{2(N-1)}{N}$, there exists $0 < M_1 \leq M_2 < \infty$ such that the solutions of (1) are global if the mass $M$ of the initial data $u_0$ is in $[0, M_1)$, and may explode in finite time if $M > M_2$. An important open question is whether $M_1 = M_2$ when $\Omega$ is a ball in $\mathbb{R}^N$ and $u_0$ is a radially symmetric function. Notice that, in the radial case, this result is true when $N = 2$ and $m = 1$, and the threshold value of the mass for blow-up is $M_1 = M_2 = 8\pi$, see [6, 16, 15, 18]. Again, for $N = 2$ and $m = 1$, but for regular, connected and bounded domain, it has been shown that $M_1 = 4\pi = \frac{M_2}{2}$ (see [15, 16] and the references therein). Such a result does not seem to be known for $N \geq 3$ and $m = \frac{2(N-1)}{N}$.

Still, in the whole space $\Omega = \mathbb{R}^N$ when the equation for $\varphi$ in (1) is replaced by the Poisson equation $\varphi = E_N * u$, with $E_N$ being the Poisson kernel, it has been shown in [9, 5, 2, 20, 21, 3] that:

- When $N \geq 3$ and $1 \leq m < 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution if $M = ||u_0||_1$ is sufficiently small, while finite time blow-up occurs for some initial data with sufficiently large mass.
- When $N \geq 2$ and $m > 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution whatever the value of $M$. 

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• When $N \geq 2$ and $m = 2 - \frac{2}{N}$, there is a threshold mass $M_c > 0$ such that solutions to this modified version of (1) exist globally if $M = \|u_0\|_1 \leq M_c$, and might blow up in finite time if $M > M_c$.

From now on, we assume that

$$N \geq 3 \quad \text{and} \quad m = \frac{2(N - 1)}{N}.$$

\section{Main Theorem}
Throughout this paper, we deal with weak solutions of (1). Our definition of weak solutions now reads:

\textbf{Definition 2.1.} Let $T \in (0; \infty]$. A pair $(u, \varphi)$ of functions $u : \Omega \times [0, T) \rightarrow [0, \infty)$, $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$ is called a weak solution of (1) in $\Omega \times [0, T)$ if

• $u \in L^\infty((0, T); L^\infty(\Omega))$; $u^m \in L^2((0, T); H^1(\Omega))$ and $<u> = M$.

• $\varphi \in L^2((0, T); H^1(\Omega))$ and $<\varphi> = 0$.

• $(u, \varphi)$ satisfies the equation in the sense of distributions; i.e,

\[ -\int_0^T \int_\Omega (\nabla u^m \cdot \nabla \psi - u \nabla \varphi \cdot \nabla \psi - \nabla u \cdot \nabla \varphi \cdot \nabla \psi - u \partial_t \psi) \, dx \, dt = \int_\Omega u_0(x) \, \psi(0, x) \, dx, \]

\[ \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega (u - M) \, \psi \, dx \, dt, \]

for any continuously differentiable function $\psi \in C^1([0, T] \times \overline{\Omega})$ with $\psi(T) = 0$ and $T > 0$.

For $\varphi \in H^1(\Omega)$ satisfying $<\varphi> = 0$, we denote by $C_s$ the Sobolev constant where

$$\|\nabla \varphi\|_2 \geq C_s \|\varphi\|_{2*}, \quad \text{where} \quad 2^* = \frac{2N}{N - 2}. \quad (3)$$

The main theorem gives the existence and uniqueness of a time global weak solution to (1) which corresponds to a degenerate version of the “Nagai model” for the semi-linear Keller-Segel system, when $u_0 \in L^\infty(\Omega)$ and the initial data is assumed to be small.

\textbf{Theorem 2.2.} Define

$$M_* := \left(\frac{2 \, C_s^2}{(m - 1) \, |\Omega|^{\frac{1}{2}}}\right)^{\frac{N}{2}}, \quad (4)$$

where $C_s$ is the Sobolev constant in (3).

Assume that $u_0$ is nonnegative function in $L^\infty(\Omega)$, which satisfies

$$\|u_0\|_1 < M_*.$$ \quad (5)

Then the equation (1) has a global weak solution $(u, \varphi)$ in the sense of Definition 2.1. Moreover, if we assume that

$$\varphi \in L^\infty((0, T); W^{2, \infty}(\Omega))$$ \quad (6)

for all $T > 0$ then this solution is unique.
In order to prove the previous theorem, we introduce the following approximated equations

$$(KS)_\delta \left\{ \begin{array}{l} 
\partial_t u_\delta = \text{div} (\nabla (u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) \quad x \in \Omega, t > 0, \\
-\Delta \varphi_\delta = u_\delta - <u_\delta> \quad x \in \Omega, t > 0, \\
\partial_\nu u_\delta = \partial_\nu \varphi_\delta = 0 \quad x \in \partial \Omega, t > 0, \\
u_\delta(0, x) = u_0(x) \quad x \in \Omega,
\end{array} \right.$$  

where $\delta \in (0, 1)$, and we show that under a smallness condition on the mass of initial data, the Liapunov function

$$L_\delta(u, \varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi_\delta|^2 - u_\delta \varphi_\delta) \, dx,$$

yields the $L^m$ bound of $u_\delta(t)$ independent of $\delta$. Then using Gagliardo-Nirenberg and Poincaré inequalities, we obtain for $p > m$, the $L^p$ bound for $u_\delta(t)$ independent of $\delta$. As a consequence of Sobolev embedding theorem, we improve the regularity of $\varphi_\delta$. And thus, under the same assumptions on the initial data, Moser’s iteration technique yields the uniform bound of $u_\delta$. Then, thanks to the local well-posedness result [8, Theorem 3.1] we obtain the existence of a global solution of $(KS)_\delta$. The existence of solutions stated in Theorem 2.2 is then proved using a compactness method; for that purpose we show an additional estimate on $\partial_t u_\delta^m$ which, together with the already derived estimates, guarantees the compactness in space and time of the family $(u_\delta)_{\delta \in (0, 1)}$. Finally, in the presence of nonlinear diffusion and under some additional regularity assumption on $\varphi_\delta$, we prove the uniqueness using a classical duality technique.

3 Approximated Equations

The first equation of (1) is a quasilinear parabolic equation of degenerate type. Therefore, we cannot expect the system (1) to have a classical solution at the point where $u$ vanishes. In order to prove Theorem 2.2, we use a compactness method and introduce the following approximated equations of (KS):

$$(KS)_\delta \left\{ \begin{array}{l} 
\partial_t u_\delta = \text{div} (\nabla (u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) \quad x \in \Omega, t > 0, \\
-\Delta \varphi_\delta = u_\delta - <u_\delta> \quad x \in \Omega, t > 0, \\
\partial_\nu u_\delta = \partial_\nu \varphi_\delta = 0 \quad x \in \partial \Omega, t > 0, \\
u_\delta(0, x) = u_0(x) \quad x \in \Omega,
\end{array} \right.$$  

where $\delta \in (0, 1)$.

The main purpose of this section is to construct the time global strong solution of (7).

3.1 Existence of global strong solution of $(KS)_\delta$

**Theorem 3.1.** For $\delta \in (0, 1)$ and $T > 0$, we consider an initial condition $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and such that $\|u_0\|_1 < M_\ast$ where $M_\ast$ is defined in (4). Then $(KS)_\delta$ has a global strong solution $(u_\delta, \varphi_\delta)$ which is bounded in $L^\infty((0, T) \times \Omega)$ for all $T > 0$ uniformly with respect to $\delta \in (0, 1)$.

The starting point of the proof of Theorem 3.1 is the following local well-posedness result [8, Theorem 1.3]:

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Lemma 3.2. Let the same assumptions as that in Theorem 3.1 hold. There exists a maximal existence time $T_{max}^δ < \infty$ and a unique solution $(u_δ, ϕ_δ)$ of (KS)$_δ$ in $[0, T_{max}^δ) \times \Omega$. Moreover, if $T_{max}^δ < \infty$ then $\lim_{t \to T_{max}^δ} ||u_δ(t, .)||_{∞} = \infty$.

In addition $u_δ(t) > u_0$ for all $t \in [0, T_{max}^δ)$.

To prove Theorem 3.1 we need to prove some lemmas which control $L^m$ norm, $L^p$ norm and $L^∞$ norm of the solution $u_δ$ of (7).

3.2 $L^p$-estimates, $1 \leq p \leq ∞$.

Our goal is to show that if $||u_0||_1$ is small enough then all solutions are global in time and uniformly bounded.

Let us first prove the $L^m$ bound for $u_δ$.

Lemma 3.3. Let the same assumptions as that in Theorem 3.1 hold and $(u_δ, ϕ_δ)$ be the nonnegative maximal solution of (KS)$_δ$. Then, $u_δ$ satisfies the following estimate

$||u_δ(t)||_m \leq C_0$, for all $t \in [0, T_{max}^δ)$

and $||u_δ(t)||_1 = ||u_0||_1$ where $C_0$ is a constant independent of $T_{max}^δ$ and $δ$.

Proof. In this proof, the solution to equation (7) should be denoted by $(u_δ, ϕ_δ)$ but for simplicity we drop the index.

Let us define the functional $L_δ$ by

$L_δ(u, ϕ) = \int_Ω (b_δ(u) + \frac{1}{2}||∇ϕ||^2 - u ϕ) dx$,

where

$b_δ(u) := \int_1^u \int_1^2 \frac{m(σ + δ)^{m-1}}{σ} dσ dσ$,

such that $b_δ(1) = b_δ'(1) = 0$ and $b(u) ≥ 0$. According to [11] it is a Liapunov functional for (KS)$_δ$. Indeed,

$$\frac{d}{dt} L_δ(u(t), ϕ(t)) = \int_Ω b_δ(u) ∂_t u dx - \int_Ω Δϕ ∂_t ϕ dx - \int_Ω ∂_t u ϕ dx - \int_Ω u ∂_t ϕ dx$$

$$= \int_Ω ∂_t u (b_δ(u) - ϕ) dx - \int_Ω (∆ϕ + u) ∂_t ϕ dx$$

$$= \int_Ω \text{div} (m (u + δ)^{m-1} \nabla u - u \nabla ϕ) (b_δ(u) - ϕ) dx - \int_Ω < u(t) > ∂_t ϕ dx$$

$$= - \int_Ω (m (u + δ)^{m-1} \nabla u - u \nabla ϕ) (b_δ''(u) \nabla u - ∇ϕ) dx - M \frac{d}{dt} \int_Ω ϕ dx$$

$$\leq 0.$$  

Then, we can conclude that for all $t \in [0, T_{max}^δ)$ we have $L_δ(u(t), ϕ(t)) ≤ L_δ(u_0, ϕ_0)$. Using Sobolev inequality (3), Hölder inequality, and Young inequality we obtain

$$\int_Ω u ϕ dx ≤ ||ϕ||_2^2 ||u||_{2N} \leq C_s^{-1}||∇ϕ||_2 ||u||_{2N} \leq \frac{1}{2}||∇ϕ||_2^2 + C_s^{-1} ||u||_{2N}^2.$$
Since \( \frac{2N}{N+2} < m \), and using interpolation inequality we get,

\[
\|u\|_\frac{2N}{N+2} \leq \|u\|_1^{\frac{1}{N}} \|u\|_m^{\frac{N-1}{N}} \leq M^{\frac{2}{N}} \|\Omega\|^{\frac{2}{N}} \|u\|_m^m.
\]

Then,

\[
\int_\Omega u \varphi \, dx \leq \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{C_s^2}{2} M^{\frac{2}{N}} \|\Omega\|^{\frac{2}{N}} \|u\|_m^m.
\]

Substituting this into the Liapunov functional, we find:

\[
L_\delta(u, \varphi) \geq \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi|^2) \, dx - \frac{1}{2} \|\nabla \varphi\|_2^2 - \frac{C_s^2}{2} M^{\frac{2}{N}} \|\Omega\|^{\frac{2}{N}} \|u\|_m^m
\]

\[
\geq \int_\Omega b_\delta(u) \, dx - \frac{C_s^2}{2} M^{\frac{2}{N}} \|\Omega\|^{\frac{2}{N}} \|u\|_m^m.
\]

We next observe that:

\[
b_\delta(u) = m \int_1^u \int_1^s \frac{(\delta + s)^{m-1}}{s} \, ds \, dz \geq m \int_1^u \int_1^s \frac{1}{s} \, ds \, dz \geq \frac{u^m}{m-1} - \frac{m}{m-1} u + 1 \geq \frac{u^m}{m-1} - \frac{m}{m-1} u.
\]

Then:

\[
L_\delta(u, \varphi) \geq \frac{1}{m-1} \|u\|_m^m - \frac{C_s^2}{2} \|\Omega\|^{\frac{2}{N}} M^{\frac{2}{N}} \|u\|_m^m - \frac{m}{m-1} M \|\Omega\|.
\]

Let us define \( \omega_M \) by

\[
\omega_M := \frac{1}{m-1} - \frac{C_s^2}{2} M^{\frac{2}{N}} \|\Omega\|^{\frac{2}{N}} = \frac{\|\Omega\|^{\frac{2}{N}}}{2 C_s^2} (M_s^{\frac{2}{N}} - M^{\frac{2}{N}}).
\]

Since \( M = \|u_0\|_1 < M_s \), then \( \omega_M \) is positive. Finally we get,

\[
L_\delta(u_0, \varphi_0) + \frac{m}{m-1} M \|\Omega\| \geq L_\delta(u(t), \varphi(t)) + \frac{m}{m-1} M \|\Omega\| \geq \omega_M \|u(t)\|_m^m \quad \text{for } t \in [0, T_{\text{max}}^\delta).
\]

In addition, we can see that \( L_\delta(u_0, \varphi_0) \leq C \) where \( C \) is independent of \( \delta \in (0, 1) \). In fact,

\[
L_\delta(u_0, \varphi_0) = \int_\Omega (b_\delta(u_0) + \frac{1}{2} |\nabla \varphi_0|^2 - u_0 \varphi_0) \, dx,
\]

and, since \((\delta + s)^{m-1} \leq s^{m-1} + s^{m-1} \leq 1 + s^{m-1}\) we obtain

\[
b_\delta(u_0) = m \int_1^{u_0} \int_1^s \frac{(\delta + s)^{m-1}}{s} \, ds \, dz \leq m \int_1^{u_0} \int_1^{s^{m-1}} \frac{1}{s} \, ds \, dz \leq m(u_0 \ln u_0 - u_0 + 1) + \frac{m}{m-1} \left( \frac{u_0^m}{m} - u_0 + 1 \right).
\]

Using Young inequality we get

\[
L_\delta(u_0, \varphi_0) \leq m \|u_0\|_2^2 + m \|\Omega\| + \frac{\|u_0\|_m^m}{m-1} + \frac{\|u_0\|_1 \|\Omega\|}{m-1} + \frac{1}{2} \|\nabla \varphi_0\|_2^2 + \frac{1}{2} \|u_0\|_2^2 + \frac{1}{2} \|\varphi_0\|_2^2.
\]

since \( u_0 \in L^\infty(\Omega) \) and \( \varphi_0 \in H^1(\Omega) \) we get \( L_\delta(u_0, \varphi_0) \leq C \) where \( C \) is independent of \( \delta \) and the proof of the lemma is complete. \( \square \)
Thanks to Lemma 3.3, let us now show that for all \( p > m \) the \( L^p \) bound for \( u_\delta \).

**Lemma 3.4.** Let the same assumptions as that in Theorem 3.1 hold. Then for all \( T > 0 \) and all \( p \in (1, \infty) \) there exists \( C(p,T) \) independent on \( \delta \) such that, for all \( t \in [0,T^{\delta}_{\max}) \cap [0,T] \), the solution \((u_\delta, \varphi_\delta)\) to \((KS)_\delta\) satisfies
\[
\|u_\delta(t)\|_p \leq C(p,T), \quad (8)
\]
and
\[
\int_0^t \int_\Omega (\delta + u_\delta)^{m-1} u_\delta^{p-2} |\nabla u_\delta|^2 \, dx ds \leq C(p,T). \quad (9)
\]

To prove the previous lemma we need the following preliminary result [20].

**Lemma 3.5.** Consider \( 0 < q_1 < q_2 \leq 2^* \). There is \( C_1 \) depending only on \( N \) such that
\[
\|u\|_{q_2} \leq C_1 \|u\|_{H^1(\Omega)} \|u\|_{q_1}^{1-\theta}, \quad \text{for } u \in H^1(\Omega), \quad (10)
\]
with
\[
\theta = \frac{2N (q_2 - q_1)}{q_2[(N+2)q_1 + 2N(1-q_1)]} \in [0,1].
\]

**Proof.** For \( u \in H^1(\Omega) \) we have by Sobolev inequality
\[
\|u\|_{2^*} \leq C_N \|u\|_{H^1}. \quad (11)
\]
By interpolation inequality we have for \( 0 < q_1 < q_2 \leq 2^* \)
\[
\|u\|_{q_2} \leq \|u\|_{2^*} \|u\|_{q_1}^{1-\theta}, \quad (12)
\]
where \( \frac{1}{q_2} = \frac{\theta(N-2)}{2N} + \frac{1-\theta}{q_1} \). Hence, substitute (11) into (12) and the lemma is proved. \( \square \)

Now, we recall the following generalized Poincaré inequality.

**Lemma 3.6.** For \( u \in H^1(\Omega) \) we have for \( 0 < q_1 \leq 1 \) the following inequality
\[
\|u\|_{q_1}^2 \leq C_2(q_1) (\|\nabla u\|_{2^*}^2 + \|u\|_{q_1}^2),
\]
where \( C_2 \) depends only on \( \Omega \) and \( q_1 \).

Now using the last two lemmas, let us prove Lemma 3.4.

**Proof.** In this proof, the solution to equation (7) should be denoted by \((u_\delta, \varphi_\delta)\) but for simplicity we drop the index.

We choose \( p > 1, \, K \geq 0 \) and we multiply the first equation in (7) by \((u - K)^{p-1}_+\) and
integrate by parts using the boundary conditions for \( u \) and \( \varphi \) to see that

\[
\frac{1}{p} \frac{d}{dt} \|(u - K)_+\|_p^p = -m(p - 1) \int_\Omega (\delta + u)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx \\
+ \ (p - 1) \int_\Omega u \nabla \varphi \cdot (u - K)_+^{p-2} \cdot \nabla u \, dx \\
= -m(p - 1) \int_\Omega (\delta + u - K + K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx \\
+ \ (p - 1) \int_\Omega (u - K) \nabla \varphi \cdot (u - K)_+^{p-2} \nabla u \, dx \\
\leq -m(p - 1) \int_\Omega (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx \\
+ \ (p - 1) \int_\Omega (u - K)_+^{p-1} \nabla \varphi \cdot u \, dx + (p - 1) K \int_\Omega \nabla \varphi \cdot (u - K)_+^{p-2} \cdot \nabla u \, dx \\
\leq -m(p - 1) \int_\Omega (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx \\
- \frac{p - 1}{p} \int_\Omega (u - K)_+^{p-1} \Delta \varphi \, dx - K \int_\Omega (u - K)_+^{p-1} \Delta \varphi \, dx \\
\leq -m(p - 1) \int_\Omega (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx + (I),
\]

where, thanks to the second equation in (7),

\[
(I) \ = \ \frac{p - 1}{p} \int_\Omega (u - K)_+^p (u - M) \, dx + K \int_\Omega (u - K)_+^{p-1} (u - M) \, dx \\
= \frac{p - 1}{p} \|(u - K)_+\|_p^{p+1} + \frac{p - 1}{p} (K - M))\|(u - K)_+\|_p^p \\
+ \ K\|(u - K)_+\|_p^p + K(K - M))\|(u - K)_+\|_p^{p-1} \\
\leq \ K^2 \|(u - K)_+\|_p^{p-1} + 2K \|(u - K)_+\|_p^p + \|(u - K)_+\|_p^{p+1},
\]

Since for \( a > 0 \) and \( b > 0 \) we have \( a^{p-1}b \leq a^{p+1} + b^{p+1} \) and \( a^p \leq a^{p+1} + b^{p+1} \) then,

\[
(I) \ \leq \ 3\|(u - K)_+\|_p^{p+1} + (2K)^{p+1} + K^{p+1}, \quad (13)
\]

and we get

\[
\frac{d}{dt}\|(u - K)_+\|_p^p \leq -m(p - 1) \int_\Omega (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 \, dx \\
+ \ 3p\|(u - K)_+\|_p^{p+1} + C_p K^{p+1}, \quad (14)
\]

for all \( t \in [0, T_{\max}]. \)

The term \( \|(u - K)_+\|_p^{p+1} \) can be estimated with the help of Lemma 3.5 and Lemma 3.6.

Assuming now that \( p > 2 \) we remark that \( 0 < \frac{2}{p+m-1} \leq 1 \) and \( 1 < \frac{2(p+1)}{p+m-1} = \frac{2}{N-2} \frac{1+p}{1+N-2} \leq \frac{2N}{N-2} \), then thanks to Lemma 3.5 and Lemma 3.6 we obtain

\[
\|(u - K)_+\|_p^{p+m-1} \leq C(p) \left( \frac{2(p+1)}{2(p+m-1)} \right)^{\frac{1+p}{2}} \|(u - K)_+\|_p^{\frac{2(p+1)}{2(p+m-1)}} \frac{2}{p+m-1} \\
+ \ \|(u - K)_+\|_p^{\frac{2(p+1)}{2(p+m-1)}} \frac{2(p+1)}{2(p+m-1)} \right), \quad (15)
\]

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where
\[ \theta = \frac{p + m - 1}{p + 1} \in (0, 1). \] (16)

Since
\[ \| (u - K)_{+}^{\frac{p + m - 1}{2}} \|^{2 \frac{p + m - 1}{p + 1}} = \int_{\Omega} (u - K)_{+}^{p + 1} \, dx = \| (u - K)_{+} \|_{p+1}^{p+1}, \] (17)

\[ \| (u - K)_{+} \|^{2 \frac{p + m - 1}{p + 1}} = \left( \int_{\Omega} (u - K)_{+} \, dx \right)^{p + 1} \leq \| (u - K)_{+} \|_{1}^{2}, \] (18)

and by Lemma 3.3
\[ \| (u - K)_{+} \|_{1} = \int_{u \geq K} (u - K) \, dx \leq \frac{1}{K^{m-1}} \int_{u \geq K} K^{m-1} \, u \, dx \leq \frac{\| u \|_{m}}{K^{m-1}} \leq \frac{C_{0}}{m}, \] (19)

we substitute (17), (18) and (19) into (15) and obtain
\[ \| (u - K)_{+} \|_{p+1}^{p+1} \leq C_{3}(p) \left\{ \| \nabla (u - K)_{+} \|_{m+1}^{m+1} \| K^{\frac{2(m-1)}{p}} \|^{\frac{p}{2}} + K^{-(m-1)(p+1)} \right\}. \] (20)

We may choose \( K = K_{*} \) large enough such that
\[ 3 \, p \, C_{3}(p) \, K_{*}^{\frac{2(m-1)}{m}} \leq \frac{2 \, p \, (p - 1) \, m}{(m + p - 1)^{2}}, \]

Hence
\[ \frac{d}{dt} \| (u - K_{*})_{+} \|_{p} \leq C(p) \, K_{*}^{p+1}, \]

so that
\[ \| (u(t) - K_{*})_{+} \|_{p} \leq C(p) \, t + \| u_{0} \|_{p}, \text{ for } t \in [0, T_{\max}]. \]

As
\[ \int_{\Omega} \| u \|^{p} \, dx \leq \int_{u < 2K_{*}} (2 \, K_{*})^{p-1} \| u \| \, dx + \int_{u \geq 2K_{*}} \| u - K_{*} + K_{*} \|^{p} \, dx \]
\[ \leq (2K_{*})^{p-1} \, M + \int_{u \geq 2K_{*}} (2 \, |u - K_{*}|)^{p} \, dx, \]
\[ \leq (2K_{*})^{p-1} \, M + 2^{p} \| (u - K_{*})_{+} \|_{p}, \]

the previous inequality warrants that
\[ \| u(t) \|_{p} \leq C(p, T), \text{ for } t \in [0, T_{\max}) \cap [0, T], \] (21)

where \( C(p, T) \) is a constant independent of \( \delta \).

We next take \( K = 0 \) in (14), integrate with respect to time and use (8) to obtain (9).

Thanks to Lemma 3.4, we can improve the regularity of \( \varphi_{\delta} \).

**Lemma 3.7.** Let the same assumptions as that in Theorem 3.1 hold, the solution \( \varphi_{\delta} \) satisfies
\[ \| \nabla \varphi_{\delta}(t) \|_{\infty} \leq L(T), \text{ for } t \in [0, T_{\max}) \cap [0, T], \]

where \( T > 0 \) and \( L \) is a positive constant independent of \( \delta \).
Lemma 3.8. Let \( N \geq 3 \), \( r \geq 4 \), \( u \in L_r^\infty(\Omega) \), and \( u^{\frac{r+m-1}{2}} \in H^1(\Omega) \). Then it holds that

\[
||u||_r \leq C_1^{\frac{r}{r+m-1}} ||u||_r^{1-\theta} \left( ||u||_r^{\frac{r+m-1}{2}} \right)^{\frac{r}{r+m-1}},
\]

(22)

with

\[
\theta = \frac{3 N (r + m - 1)}{(3N + 2) r + 4N (m - 1)} \in (0, 1).
\]

Proof. For \( r \geq 4 \), we can see that

\[
||u||_r = \left( \int_\Omega |u|^{\frac{r+m-1}{2}} \right)^{\frac{r}{r+m-1}} = ||u^{\frac{r+m-1}{2}}||_r^{\frac{r}{r+m-1}},
\]

and

\[
\frac{r}{2(r+m-1)} < 1 < \frac{2r}{r+m-1} < 2 < \frac{2N}{N-2}.
\]

By Lemma 3.5,

\[
||u||_r = ||u^{\frac{r+m-1}{2}}||_r^{\frac{r}{r+m-1}} \leq \left( C_1^{\theta} ||u^{\frac{r+m-1}{2}}||_r^{\theta} ||u^{\frac{r+m-1}{2}}||_r^{1-\theta} \right)^{\frac{r}{r+m-1}},
\]

and

\[
\theta = \frac{2N \left( \frac{2r}{r+m-1} - \frac{r}{2(r+m-1)} \right)}{2r \left( \frac{r}{2(r+m-1)} \right)(2N(1 - \frac{r}{2(r+m-1)}) + (N + 2) \frac{r}{2(r+m-1)})}
\]

\[
= \frac{3N (r + m - 1)}{(3N + 2) r + 4N (m - 1)} \in (0, 1).
\]

In addition, we have

\[
||u^{\frac{r+m-1}{2}}||_r^{\frac{r}{2(r+m-1)}} = \left( \int_\Omega |u|^{\frac{r+m-1}{2}} \right)^{\frac{r}{2(r+m-1)}} = ||u||_r^{\frac{r+m-1}{2}},
\]

and we obtain (22). \( \square \)

Lemma 3.9. Let the same assumptions as that in Theorem 3.1 hold, and \((u_\delta, \varphi_\delta)\) be the nonnegative maximal solution of (7). For all \( T > 0 \), there is \( C_\infty(T) \) such that

\[
||u_\delta(t)||_\infty \leq C_\infty(T), \text{ for all } t \in [0, T_{\max}) \cap [0, T],
\]

where \( C_\infty(T) \) is a positive constant independent on \( \delta \).
Proof. In this proof we omit the index $\delta$, and we employ Moser’s iteration technique developed in \cite{1,21} to show the uniform norm bound for $u$. We multiply the first equation in (7) by $u^{r-1}$, where $r \geq 4$, and integrate it over $\Omega$. Then, we have
\[
\frac{d}{dt} \frac{\|u\|^r}{r} = - \int_{\Omega} (\nabla (u + \delta)^m - u \nabla \varphi) \cdot \nabla u^{r-1} \, dx \\
= -m(r - 1) \int_{\Omega} (u + \delta)^{m-1} u^{r-2} |\nabla u|^2 \, dx + (r - 1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u \, dx \\
\leq -m(r - 1) \int_{\Omega} u^{m+r-3} |\nabla u|^2 \, dx + (r - 1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u \, dx.
\]
By Young’s inequality and Lemma 3.7,
\[
\frac{1}{r} \frac{d}{dt} \|u\|^r_r \leq -\frac{4m(r - 1)}{(r + m - 1)^2} \int_{\Omega} |\nabla u|^{r+1-\frac{m}{2}} \, dx + \frac{2(r - 1)}{(r + m - 1)^2} \int_{\Omega} \nabla \varphi \cdot \nabla u^{r-1} \, dx \\
\leq -\frac{2m(r - 1)}{(r + m - 1)^2} \|\nabla u\|^{r+1-\frac{m}{2}} \|\nabla \varphi\|^{2(r-m)+1} \int_{\Omega} u^{r-1} \, dx + C(T) \int_{\Omega} u^{r-1} \, dx.
\]
Using Hölder and Young inequalities and Lemma 3.3 we obtain
\[
\frac{1}{r} \frac{d}{dt} \|u\|^r \leq \frac{2m(r - 1)}{r} \|\nabla u\|^{r+1-\frac{m}{2}} \|\nabla \varphi\|^{2(r-m)+1} \int_{\Omega} u^{r-1} \, dx + C(T) \int_{\Omega} u^{r-1} \, dx,
\]
where we have used that $r^{r-m} \leq r^2$ for $r \geq 4$. By Lemma 3.8, we have for $r \geq 4$
\[
\|u\|^r_r \leq C_1 \|u\|^{(1-\theta) \frac{r}{r+1}} \|\nabla u\|^{\frac{r+1-\frac{m}{2}}{r+1}} \int_{\Omega} u^{r-1} \, dx,
\]
where
\[
\theta = \frac{3N(r + m - 1)}{3N + 2r + 4N(m - 1)} < 1.
\]
Therefore, Young inequality and (25) yield that
\[
2 r^2 \|u\|^r_r \leq 2 r^2 C_1 \|u\|^{(1-\theta) \frac{r}{r+1}} \|\nabla u\|^{\frac{r+1-\frac{m}{2}}{r+1}} \\
\leq \frac{\theta r}{r + m - 1} \frac{m(r - 1)}{(r + m - 1)^2} \|u\|^{r+1-\frac{m}{2}} \int_{\Omega} (r + m - 1) \, dx + C_2(1) \int_{\Omega} \theta(r + m - 1) \, dx \\
\times \frac{2}{r} \|\nabla u\|^{(1-\theta) \frac{r}{r+1}} \|u\|^{(1-\theta) \frac{r}{r+1}} \int_{\Omega} u^{r-1} \, dx, \]
where $C_2(1)$ is the Poincaré constant defined in Lemma 3.6. Then we obtain
\[
2 r^2 \|u\|^r_r \leq \frac{m(r - 1)}{C_2(1) \int_{\Omega} \theta(r + m - 1) \, dx} \|u\|^{r+1-\frac{m}{2}} \int_{\Omega} u^{r-1} \, dx \\
+ \frac{\theta r}{r + m - 1} \frac{m(r - 1)}{(r + m - 1)^2} \|u\|^{r+1-\frac{m}{2}} \int_{\Omega} u^{r-1} \, dx.
\]
Now, since \( N > 2 \), which gives \( 4N \geq 3N + 2 \), we find the following upper bound for \( \theta \)

\[
\theta \leq \frac{3N}{3N + 2}
\]  

(26)

In addition,

\[
\frac{\theta r}{r(1 - \theta) + m - 1} \leq \frac{\theta}{1 - \theta} = -1 + \frac{1}{1 - \theta} \leq \frac{3N}{2},
\]

(27)

\[
\frac{r + m - 1}{r(1 - \theta) + m - 1} \leq \frac{r + m - 1}{(1 - \theta)(r + m - 1)} \leq \frac{1}{1 - \theta} \leq \frac{3N + 2}{2},
\]

(28)

and

\[
\frac{2(r + m - 1) + \theta r}{r(1 - \theta) + m - 1} \leq \frac{2 + \theta}{1 - \theta} \leq 9N + 4.
\]

(29)

As \( C_1 \geq 1 \) and \( r \geq 1 \), we get

\[
2 r^2 \|u\|^r_r \leq \frac{m}{C_2(1)} \frac{(r - 1)}{(r + m - 1)^2} \|u\|_{r + m - 1}^{r + m - 1} \left( \|\nabla u\|^2 + \|u\|^2 \right) + C \ r^{9N+4} \|u\|_2^{(1-\theta)(r+m-1)}.
\]

(30)

Using Lemma 3.6 we have

\[
\|u\|_{r + m - 1}^{r + m - 1} \leq C_2(1) \left( \|\nabla u\|^2 + \|u\|^2 \right).
\]

(31)

Using Hölder inequality, Young inequality and Lemma 3.3, we get

\[
\|u\|_{r + m - 1}^{r + m - 1} \leq \left( \|u\|_{1} \|u\|_{r + m - 1} \right) \leq \left( \|u\|_{r} \right) \|u\|_{1} \|u\|_{r + m - 1},
\]

then

\[
\frac{m}{(r + m - 1)^2} \|u\|_{r + m - 1}^{r + m - 1} \leq (r - 1) \frac{r + m - 1}{r - 1} \|u\|_{r} + \frac{2 - m}{r - 1} \left( \frac{m}{(r + m - 1)^2} \|u\|_{1} \right)^{\frac{r + m - 1}{r - 1}} \leq \frac{r^2}{2} \|u\|_{r} + \left( \frac{m}{(r + m - 1)^2} \|u\|_{1} \right)^{\frac{r + m - 1}{r - 1}} \leq \frac{r^2}{2} \|u\|_{r} + C r^4.
\]

(32)

Now substituting (32) and (31) into (30) we get

\[
2 r^2 \|u\|_r^r \leq \frac{m}{(r + m - 1)^2} \left( \|\nabla u\|^2 + \|u\|^2 \right) + C \ r^{9N+4} \|u\|_2^{(1-\theta)(r+m-1)}.
\]

hence

\[
\|u\|_r^r \leq C \ r^4 + C \ r^{9N+4} \|u\|_2^{(1-\theta)(r+m-1)}.
\]

We apply Young inequality again to the last term of the above inequality. It is easy to see that

\[
\frac{2}{3N + 2} \leq 1 - \theta \leq \frac{(1 - \theta)(r + m - 1)}{r(1 - \theta) + m - 1} = \frac{(1 - \theta)r + (1 - \theta)(m - 1)}{r(1 - \theta) + m - 1} < 1,
\]

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so that
\[ r^2 ||u||_r^r \leq \frac{m (r - 1)}{(r + m - 1)^2} \| \nabla u \|_{r - \frac{r + m - 1}{2}}^2 + C_4^r + 1 + (C r^{9N+4})^{3N+1} ||u||_r^r, \] (33)
for any \( r \in [4, \infty) \).

Substituting (33) into (24) we end up with
\[ \frac{d}{dt} ||u||_r^r \leq r C_4^r + r + r (C r^{9N+4})^{3N+1} ||u||_r^r \leq C_5^r + C r^\alpha ||u||_r^r, \] (34)
for any \( r \in [4, \infty) \), where \( \alpha = (9N + 4)(3N + 1) + 1 \). After integrating (34) from 0 to \( t \), we obtain the \( L^r \) estimate for \( u \) as follows:
\[ \sup_{0 < t < T} ||u(t)||_r^r \leq ||u_0||_r^r + T C_5^r + C r^\alpha T \sup_{0 < t < T} ||u(t)||_r^r. \] (35)

Since
\[ ||u_0||_r \leq ||u_0||_r^{\frac{r-1}{r}} ||u_0||_1^{\frac{1}{r}} \leq C_6, \]
then
\[ \sup_{0 < t < T} ||u(t)||_r \leq C_7(T)^{\frac{1}{r}} r^\alpha \max \left\{ C_6, \sup_{0 < t < T} ||u(t)||_r^r \right\}, \] (36)
and we obtain for \( r \geq 4 \)
\[ \sup_{0 < t < T} ||u(t)||_r \leq C_7(T)^{\frac{1}{r}} r^\alpha \max \left\{ C_6, \sup_{0 < t < T} ||u(t)||_r^r \right\}. \] (37)

We are now in a position to derive the claimed \( L^\infty \) estimate. To this end, we set
\[ \alpha_p := \max \left\{ C_6, \sup_{0 < t < T} ||u(t)||_4^p \right\} \]
for \( p \geq 0 \). Then we take \( r = 4^p \) with \( p \geq 0 \) in (37) which reads
\[ \alpha_p \leq 4^{\frac{p}{2^p}} C_7(T)^{\frac{1}{2^p}} \max \left\{ C_6, \sup_{0 < t < T} ||u(t)||_{4^{p-1}} \right\}, \]
\[ \leq 4^{\frac{p}{2^p}} C_7(T)^{\frac{1}{2^p}} \alpha_{p-1} \]
since \( p \leq 2^p \) for \( p \geq 1 \). Arguing by induction we conclude that
\[ \alpha_p \leq 4^\alpha \sum_{k=1}^p 2^{-k} C_7(T) \sum_{k=1}^p 4^{-k} \alpha_0. \]
Then by using Lemma 3.3 we get
\[ \sup_{0 < t < T} ||u(t)||_{4^p} \leq 4^\alpha C_7(T) \alpha_0 \leq C_8(T). \]

Consequently, by letting \( p \) tend to \( \infty \), we see that \( u \in L^\infty((0, T) \times \Omega) \) and
\[ \sup_{0 < t < T} ||u(t)||_\infty \leq C_8(T). \] (38)

Since the right hand side is independent of \( \delta \), we have proved the lemma. \( \square \)
Lemma 3.10. Let the same assumptions as that in Theorem 3.1 hold, and \((u_\delta, \varphi_\delta)\) be the solution to (T). Then for all \(T > 0\) there is \(C_0(T)\) such that the solution \(u_\delta\) satisfies the following derivation estimate

\[
\int_0^T \| \partial_t u_\delta^m \|_{(W^{1,N+1})'} \, dt \leq C_0(T).
\]

Proof. Consider \(\psi \in W^{1,N+1}(\Omega)\) and \(t \in (0, T)\), we have

\[
\left| \int_{\Omega} m u_\delta^{m-1}(t) \partial_t u_\delta(t) \psi \, dx \right|
= m \left| \int_{\Omega} \nabla (u_\delta^{m-1}) \cdot (\nabla u_\delta^m - u_\delta \nabla \varphi_\delta) \, dx \right|
\leq \int_{\Omega} \left[ u_\delta^{m-1} |\nabla u_\delta^m| |\nabla \psi| + u_\delta^m |\nabla \psi| |\nabla \varphi_\delta| \right.
+ |\psi| m(m - 1) u_\delta^{2m-3} |\nabla u_\delta|^2 + |\psi|(m - 1)u_\delta^{m-1} |\nabla u_\delta| |\nabla \varphi_\delta| \bigg] \, dx
\leq m \left[ \| u_\delta \|^m_{L^{\infty}} \| \nabla u_\delta^m \|_2 \| \nabla \psi \|_2 + \| \nabla \psi \|_2 \| u_\delta \|^m_{L^{\infty}} \| \nabla \varphi_\delta \|_{L^{\infty}} |\Omega| \right]^{\frac{1}{2}}
+ |\psi| \frac{4m(m - 1)}{(2m - 1)^2} \| \nabla u_\delta^{m-\frac{1}{2}} \|_2^2 + \| \psi \|_2 \frac{m - 1}{m} \| \nabla u_\delta^m \|_2 \| \nabla \varphi_\delta \|_{L^{\infty}} \bigg].
\]

Using Lemma 3.8, Lemma 3.9, and the embedding of \(W^{1,N+1}(\Omega)\) in \(L^{\infty}(\Omega)\), we end up with

\[
| \partial_t u_\delta^m(t), \psi | \leq C(T) \left( \| \nabla u_\delta^m(t) \|_2 + \| \nabla u_\delta^{m-\frac{1}{2}}(t) \|_2^2 + 1 \right) \| \psi \|_{W^{1,N+1}},
\]

and a duality argument gives

\[
\| \partial_t u_\delta^m(t) \|_{(W^{1,N+1})'} \leq C(T) \left( \| \nabla u_\delta^m(t) \|_2 + \| \nabla u_\delta^{m-\frac{1}{2}}(t) \|_2^2 + 1 \right).
\]

Integrating the above inequality over \((0,T)\) and using Lemma 3.4 with \(p = 2\) and \(p = m\), give Lemma 3.10. \(\square\)

4 Proof of Theorem 2.2

4.1 Existence

In this section, we assume that \(u_0\) is a nonnegative function in \(L^{\infty}(\Omega)\) satisfying (5). For \(\delta \in (0, 1)\), \((u_\delta, \varphi_\delta)\) denotes the solution to \((KS)_\delta\) constructed in Section 3. To prove existence of a weak solution, we use a compactness method. For that purpose, we first study the compactness properties of \((u_\delta, \varphi_\delta)_\delta\).

Lemma 4.1. There are functions \(u\) and \(\varphi\) and a sequence \((\delta_n)_{n \geq 1}\), \(\delta_n \rightarrow 0\), such that, for all \(T > 0\) and \(p \in (1, \infty)\),

\[
u_{\delta_n} \rightarrow u, \ \text{in} \ L^p((0,T) \times \Omega) \ \text{as} \ \delta_n \rightarrow 0, \quad (39)
\]

\[
\varphi_{\delta_n} \rightarrow \varphi, \ \text{in} \ L^p((0,T); W^{2,p}(\Omega)) \ \text{as} \ \delta_n \rightarrow 0. \quad (40)
\]

In addition, \(u \in L^{\infty}((0,T) \times \Omega)\) for all \(T > 0\) and is nonnegative.
Proof. Thanks to Lemma 3.4 and Lemma 3.9, $(u^n_\delta)_\delta$ is bounded in $L^2((0, T); H^1(\Omega))$ while $(\partial_t u^n_\delta)_\delta$ is bounded in $L^1((0, T); (W^{1,N+1})'(\Omega))$ by Lemma 3.10. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $(W^{1,N+1})'(\Omega)$, it follows from [19, corollary 4] that $(u^m_\delta)$ is compact in $L^2((0, T) \times \Omega)$ for all $T > 0$. Since $r \mapsto r^{\frac{m}{m-1}}$ is H"older continuous, it is easy to check that the previous compactness property implies that $(u_\delta)$ is compact in $L^{2m}((0, T) \times \Omega)$ for all $T > 0$. There are thus a function $u \in L^{2m}((0, T) \times \Omega)$ for all $T > 0$ and a sequence $(\delta_n)_{n \geq 1}$ such that

$$u_{\delta_n} \rightarrow u \text{ in } L^{2m}((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0,$$

(41)

for all $T > 0$, owing to Lemma 3.9, we may also assume that

$$u_{\delta_n} \xrightarrow{\ast} u \text{ in } L^\infty((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0.$$

(42)

for all $T > 0$. It readily follows from (41) and (42), and H"older inequality that (39) holds true. Since elliptic regularity ensure that

$$||\varphi_{\delta_k} - \varphi_{\delta_n}||_{W^{2,p}} \leq C(p) ||u_{\delta_k} - u_{\delta_n}||_p,$$

for all $k \geq 1$, $n \geq 1$, and $p \in (1, \infty)$, a straightforward consequence of (39) is that $(\varphi_{\delta_n})_{n \geq 1}$ is a Cauchy sequence in $L^p((0, T); W^{2,p}(\Omega))$ and thus converges to some function $\varphi$ in that space. Finally, the nonnegativity of $u$ follows easily from that of $u_{\delta_n}$ by (39). \hfill \Box

Proof of Theorem 2.2 (existence). It remains to identify the equations solved by the limit $(u, \varphi)$ of $(u_{\delta_n}, \varphi_{\delta_n})_{n \geq 1}$ constructed in Lemma 4.1. To this end we first note that, owing to (39) and the boundedness of $(u_{\delta_n}, n)$ and $u$ in $L^\infty((0, T) \times \Omega)$, we have

$$u^m_{\delta_n} \rightarrow u^m \text{ in } L^p((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0,$$

(43)

for all $T > 0$. Since $(\nabla(u_{\delta_n} + \delta_n \frac{m+1}{2}))_{n \geq 1} \text{ and } (\nabla u^m_{\delta_n})_{n \geq 1}$ are bounded in $L^2((0, T) \times \Omega)$ for all $T > 0$ by Lemma 3.4 with $p = 2$ and $p = m + 1$, we may extract a further subsequence (not relabeled) such that

$$\nabla(u_{\delta_n} + \delta_n \frac{m+1}{2}) \rightarrow \nabla u^m \frac{m+1}{2} \text{ in } L^2((0, T) \times \Omega),$$

(44)

$$\nabla u^m_{\delta_n} \rightarrow \nabla u^m \text{ in } L^2((0, T) \times \Omega),$$

(45)

for all $T > 0$. Then if $\psi \in L^4((0, T) \times \Omega; \mathbb{R}^N)$,

$$\left| \int_0^T \int_\Omega \psi \cdot \left[ \nabla(u_{\delta_n} + \delta_n)^{m} - \nabla u^m \right] \ dx ds \right|$$

$$= \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \left[ (u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - u^{\frac{m+1}{2}} \nabla u^{\frac{m+1}{2}} \right] \ dx ds \right|$$

$$\leq \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} ((u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} - u^{\frac{m-1}{2}}) \ dx ds \right|$$

$$+ \frac{2}{m+1} \left| \int_0^T \int_\Omega u^{\frac{m-1}{2}} \psi \cdot \left( \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \ dx ds \right|$$

$$\leq \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \left( \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \ dx ds \right|.$$
for all $T > 0$.

Now, we are going to show that $(u, \varphi)$ in Lemma 4.1 is the desired weak solution in Theorem 2.2. Let $T > 0$ and $\psi \in C^1([0, T] \times \overline{\Omega})$ with $\psi(0) = 0$. The solution of (7) satisfies

$$\int_0^T \int_\Omega [\nabla (u_{\delta_n} + \delta_n)^m \cdot \nabla \psi - u_{\delta_n} \nabla \varphi_{\delta_n} \cdot \nabla \psi - u_{\delta_n} \partial_t \psi] \, dx \, dt = \int_\Omega u_0 \psi(0, x) \, dx,$$  

(47) and,

$$\int_0^T \int_\Omega [\nabla \varphi_{\delta_n} \cdot \nabla \psi + M \psi - u_{\delta_n} \psi] \, dx \, dt = 0.$$  

(48)

From (46) we see that

$$\int_0^T \int_\Omega \nabla (u_{\delta_n} + \delta_n)^m \cdot \nabla \psi \, dx \, dt \rightarrow \int_0^T \int_\Omega \nabla u^m \cdot \nabla \psi \, dx \, dt \text{ as } \delta_n \rightarrow 0.$$

From (39) we get

$$\int_0^T \int_\Omega u_{\delta_n} \partial_t \psi \, dx \, dt \rightarrow \int_0^T \int_\Omega u \partial_t \psi \, dx \, dt \text{ as } \delta_n \rightarrow 0.$$

From (39) and (40) we get

$$\int_0^T \int_\Omega u_{\delta_n} \varphi_{\delta_n} \cdot \nabla \psi \, dx \, dt \rightarrow \int_0^T \int_\Omega u \varphi \cdot \nabla \psi \, dx \, dt \text{ as } \delta_n \rightarrow 0.$$

Thus we conclude that $u$ satisfies

$$\int_0^T \int_\Omega (\nabla u^m \cdot \nabla \psi - u \nabla \varphi \cdot \nabla \psi - u \cdot \partial_t \psi) \, dx \, dt = \int_\Omega u_0(x) \cdot \psi(0, x) \, dx.$$

Similarly, from (40) we see that

$$\int_0^T \int_\Omega \nabla \varphi_{\delta_n} \cdot \nabla \psi \, dx \, dt \rightarrow \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx \, dt \text{ as } \delta_n \rightarrow 0,$$

and from (39) we see that

$$\int_0^T \int_\Omega u_{\delta_n} \psi \, dx \, dt \rightarrow \int_0^T \int_\Omega u \psi \, dx \, dt \text{ as } \delta_n \rightarrow 0.$$

Thus, we have constructed a weak solution $(u, \varphi)$ of (KS).

4.2 Uniqueness

In this section, we prove the uniqueness statement of Theorem 2.2 under the additional assumption (6) on $\varphi$. The proof relies on a classical duality technique, and on the method presented in [2]

Proof. The proof estimates the difference of weak solutions in dual space $H^1(\Omega)'$ of $H^1(\Omega)$, motivated by the fact that the nonlinear diffusion is monotone in this norm.
Assume that we have two different weak solutions \((u_1, \varphi_1)\) and \((u_2, \varphi_2)\) to equations (1) corresponding to the same initial conditions, and fix \(T > 0\). We put

\[
(u, \varphi) = (u_1 - u_2, \varphi_1 - \varphi_2) \text{ in } [0, T] \times \Omega.
\]

Then \(\varphi\) is the strong solution of

\[
- \Delta \varphi = u \quad \text{in } \Omega, \\
\partial_\nu \varphi = 0 \text{ on } \partial \Omega,
\]

and

\[
< \varphi > = 0.
\]

Since \(\partial_t u \in L^2((0, T); H^1(\Omega)')\), we have

\[
- \Delta \partial_t \varphi = \partial_t u_1 - \partial_t u_2 = \partial_t u \text{ in } H^1(\Omega)',
\]

and

\[
\frac{1}{2} \frac{d}{dt} ||\nabla \varphi||_2^2 = \int_\Omega \nabla \varphi \cdot \nabla \partial_t \varphi \, dx = - < \Delta \partial_t \varphi, \varphi >_{(H^1)' \times H^1} = < \partial_t u, \varphi >_{(H^1)' \times H^1}.
\]

Now it follows from (1) that \(u\) satisfies the equation

\[
\left\{ \begin{array}{l}
\partial_t u = \text{div}(\nabla (u_m^1 - u_m^2)) - \text{div}(u_1 \nabla \varphi + u_2 \nabla \varphi_2) \\
\partial_\nu u = 0 \\
u(0, x) = 0.
\end{array} \right.
\]

Substituting (51) in (50), we obtain

\[
\frac{1}{2} \frac{d}{dt} ||\nabla \varphi||_2^2 = \int_\Omega (u_m^1 - u_m^2) \Delta \varphi \, dx + \int_\Omega u_1 |\nabla \varphi|^2 \, dx + \int_\Omega u \nabla \varphi_2 \cdot \nabla \varphi \, dx.
\]

The first integral on the right-hand side of (52) is nonnegative due to the fact that \(z \mapsto z^m\) is an increasing function. The second integral on the right-hand side of (52) can be estimated by

\[
\left| \int_\Omega u_1 |\nabla \varphi|^2 \, dx \right| \leq ||u_1||_\infty \int_\Omega |\nabla \varphi|^2 \, dx.
\]

For the last integral, using an integration by parts we obtain

\[
\int_\Omega u \nabla \varphi_2 \cdot \nabla \varphi \, dx = - \int_\Omega \Delta \varphi \nabla \varphi_2 \cdot \nabla \varphi \, dx = \int_\Omega \nabla \varphi \cdot \nabla (\nabla \varphi_2 \cdot \nabla \varphi) \, dx = \sum_{i,j} \int_\Omega \partial_i \varphi \partial_j^2 \varphi_2 \partial_j \varphi + \sum_{i,j} \int_\Omega \partial_i \varphi \partial_j \varphi_2 \partial_i^2 \varphi \, dx.
\]

Integrating by parts the second integral on the right-hand side of (53),

\[
\sum_{i,j} \int_\Omega \partial_i \varphi \partial_j \varphi_2 \partial_i^2 \varphi \, dx = \sum_{i,j} \frac{1}{2} \int_\Omega \partial_j \varphi_2 \partial_j |\partial_i \varphi|^2 \, dx = \frac{1}{2} \int_\Omega \nabla \varphi_2 \cdot \nabla (|\nabla \varphi|^2) \, dx = - \frac{1}{2} \int_\Omega \Delta \varphi_2 |\nabla \varphi|^2 \, dx \leq C(T) ||\nabla \varphi||_2^2,
\]

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since $-\Delta \varphi_2 = u_2 - \frac{u_2}{2} \in L^\infty((0,T) \times \Omega)$. Together with \eqref{4.10} the previous inequality implies

\[
\left| \int_\Omega u \nabla \varphi_2 \cdot \nabla \varphi \, dx \right| \leq C(T) \int_\Omega (|\nabla^2 \varphi_2| + 1) |\nabla \varphi|^2 \, dx.
\]

\[
\leq C(T) \left( ||\varphi_2||_{L^\infty((0,T);W^{2,\infty}(\Omega))} + 1 \right) \int_\Omega |\nabla \varphi|^2 \, dx,
\]

provided that the $L^\infty((0,T);W^{2,\infty}(\Omega))$ norm of the function $\varphi_2$ is bounded. Thus, substituting the above estimates in \eqref{4.10}, one finally obtains

\[
\frac{d}{dt} \int_\Omega |\nabla \varphi|^2 \, dx \leq C(T) \int_\Omega |\nabla \varphi|^2 \, dx. \tag{54}
\]

Notice that $||\nabla \varphi(0)||_2 = 0$ which follows from \eqref{4.7} and the property $u(0) = 0$. Thus, inequality \eqref{54} implies

\[
||\nabla \varphi(t)||_2^2 \leq e^{C(T)} t ||\nabla \varphi(0)||_2^2 = 0.
\]

Consequently, $\nabla \varphi(t) = 0$ for all $t \in [0,T]$ and, since $< \varphi(t) >= 0$, we have $\varphi(t) = 0$ for all $t \in [0,T]$. Using \eqref{4.7}, we conclude that $u(t) = 0$ for all $t \in [0,T]$. Consequently $(u_1, \varphi_1) = (u_2, \varphi_2)$.

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\section*{References}

[1] N. D. Alikakos. $L^p$ bounds of solutions of reaction-diffusion equations. \textit{Communication in Partial Differential Equations} 4(1979), no. 8, 827-868.

[2] J. Bedrossian, N. Rodriguez and A. Bertozzi. Local global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion, \textit{Nonlinearity} 24 (2011)1683-1714.

[3] J. Bedrossian, I. C. Kim. Global Existence and Finite Time Blow-Up for Critical Patlak-Keller-Segel Models with Inhomogeneous Diffusion, preprint arXiv:1108.5301.

[4] A. Blanchet. On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher. \textit{To appear in Sémin. Équ. Dériv. Partielles}.

[5] A. Blanchet, J. A. Carrillo. Ph. Laurençot. Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. \textit{Calc. Var. Partial Differential Equations} 35 (2009), no. 2,133-168.

[6] V. Calvez, J. A. Carrillo. Volume effect in the Keller-Segel model: energy estimates preventing blow-up. \textit{J. Math. Pures. Appl.} 86 (2006)155-175.

[7] T. Cieślak, Ph. Laurençot. Finite time blow-up for radially symmetric solutions to a critical quasilinear Smoluchowski-Poisson system.\textit{C.R. Acad. Sci. Paris, ser. I} 347(2009) 237-242.
[8] T. Cieślak, M. Winkler. Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity* 21 (2008) 1057-1076.

[9] J. Dolbeault, B. Perthame. Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$. *C. R. Math. Acad. Sci. Paris* 339 (2004), no. 9, 611-616.

[10] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.* 105(3):103-165, 2003.

[11] D. Horstmann. Lyapunov functions and $L^p$-estimates for a class of reaction-diffusion systems. *Colloq. math.* 87 (2001) no. 1, 113-127.

[12] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* 329(2)(1992) 819-824.

[13] E. F. Keller and L. A. Segel. Initiation of slide mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.

[14] R. Kowalczyk. Preventing blow-up in a chemotaxis model. *J. Math. Anal. Appl.* 305 (2005)566-588.

[15] T. Nagai. Blow-up of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains. *J. Inequal. Appl.* 6 (2001) 37-55.

[16] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system, *Advances in Mathematical Sciences and Applications* 5 (1995), no. 2, 581-601.

[17] C.S. Patlak, Random walk with persistence and external bias, *Bull. Math. Biophys.* 15 (1953) 311-338.

[18] B. Perthame, PDE models for chemotactic movements. Parabolic, hyperbolic and kinetic, *Appl. Math.* 49 (6) (2005) 539-564.

[19] J. Simon, Compact sets in the space $L^p(0,T;B)$. 1987, *Annali di Mathematica Pura ed Applicata (IV)*, vol. CXLVI, 65-96.

[20] Y. Sugiyama. Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems. *Differential and Integral Equations* 19 (2006), no. 8, 841-876.

[21] Y. Sugiyama. Time global existence and asymptotic behavior for solutions to degenerate quasi-linear parabolic systems of chemotaxis. *Differential and Integral Equations* 20 (2007), no. 2, 133-180.