Mixed norm estimates for Hermite multipliers

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Abstract. In this article mixed norm estimates are obtained for some integral operators, from which those for the Hermite semigroup and the Bochner-Riesz means associated with the Hermite expansions are deduced. Also, mixed norm estimates for the Littlewood-Paley $g$—functions and $g^*$—functions for the Hermite expansions are obtained, which lead to those for Hermite multipliers.

1. Introduction

The aim of this article is to study mixed norm estimates for multiplier operators associated to Hermite expansions. Given a function $\varphi$ defined on the set of all positive integers one can define the operator

$$\varphi(H) = \sum_{k=0}^{\infty} \varphi(2k+n) P_k$$

where $P_k$ are the spectral projections associated to the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. In [13] sufficient conditions on $\varphi$ have been given so that $\varphi(H)$ defines a bounded linear operator on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. In this paper we are interested in mixed norm estimates for $\varphi(H)$. Let $L^{p,2}(\mathbb{R}^n)$ stand for the space of functions $f(r\omega)$ on $\mathbb{R}^+ \times S^{n-1}$ for which

$$\|f\|_{L^{p,2}(\mathbb{R}^n)} = \left( \int_0^\infty \left( \int_{S^{n-1}} |f(r\omega)|^2 d\omega \right)^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{1}{p}}$$
are finite. We show that under some conditions on the multiplier \( \varphi \), the mixed norm estimates
\[
\| \varphi(H)f \|_{L^{p,2}(\mathbb{R}^n)} \leq C \| f \|_{L^{p,2}(\mathbb{R}^n)}
\]
hold for all \( 1 < p < \infty \).

This work is motivated by the recent paper \([2]\) by Ciaurri and Roncal where they have studied the boundedness of Riesz transforms associated to the Hermite operator written in polar coordinates. The Hermite operator \( H \) admits a family of eigenfunctions given by the multidimensional Hermite functions \( \Phi_{\alpha}(x) \). Written in polar coordinates the same operator admits another family of eigenfunctions which are of the form \( g(r)Y(\omega), \ x = r\omega \in \mathbb{R}^n \) where \( Y \) are spherical harmonics and \( g \) are suitable Laguerre functions. In terms of these eigenfunctions one can define Riesz transforms for the Hermite operator. In \([2]\) the authors have proved the boundedness of these Riesz transforms. It turns out that their result is equivalent to mixed norm estimates for the standard Riesz transforms \( R_j = A_jH^{-\frac{1}{2}}, \ j = 1,2,\ldots,n \), associated to Hermite expansions. Here \( A_j = \frac{\partial}{\partial x_j} + x_j \) are the annihilation operators of quantum mechanics.

In view of the above observation it is natural to ask for mixed norm estimates for certain operators associated to Hermite expansions. In this paper we consider the Hermite semigroup, Bochner-Riesz means and more general multiplier transformations for the Hermite expansions. The connection between the two kinds of expansions viz., the one in terms of the standard Hermite functions and the other in terms of spherical harmonics times Laguerre functions is provided by the Hecke-Bochner formula for the Hermite expansion, see Theorem 3.4.1 in \([13]\). However, our investigation of mixed norm estimates for Hermite multipliers originates from the observation that the kernel of such operators are of the form \( K_0(|x-y|,|x+y|) \) and hence one can make use of Funk-Hecke formula to study such operators. We will see that mixed norm estimates for Hermite multipliers reduces to a vector valued inequality for a sequence of operators, all of them related to the original operator via Funk-Hecke formula. These component operators can also be viewed as Laguerre multipliers in view of the Hecke-Bochner formula.

Though in this paper we have only treated the Hermite operator, all the results can be proved for the cases of Dunkl Harmonic Oscillator and Special Hermite operator. The main tools used in this paper viz., the Funk-Hecke formula and Hecke-Bochner identity are also available for \( h \)-harmonics and bigraded spherical harmonics. Hence the proofs of the main results can be suitably modified to cover these more general operators. We plan to return to some of these problems elsewhere.
The plan of the paper is as follows. In the next section we study mixed norm estimates of integral operators with kernels of the form $K_0(|x - y|, |x + y|)$. The results proved in Section 2 will be applied in Section 3 to prove mixed norm estimates for the Hermite semigroup and Bochner-Riesz means associated to Hermite expansions. In Section 4 we study mixed norm estimates for $g$–functions for the Hermite semigroup which will be used in Section 5 to prove our main result on Hermite multipliers.

2. Mixed norm estimates for some integral operators

In this section we study mixed norm estimates for certain operators given by (singular) kernels having some special properties. More precisely we consider operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy$$

where the kernel $K(x, y) = K_0(|x - y|, |x + y|)$ for some $K_0$. We are interested in estimates of the form

$$\|Tf\|_{L^{p,2}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,2}(\mathbb{R}^n)}.$$

We prove that under some mild assumptions on $T$ the above mixed norm estimates hold. Examples of such operators are provided by the Hermite semigroup $e^{-tH}$, Bochner-Riesz means $S^\delta_R$ associated to Hermite expansions and more generally Hermite multipliers. We study these operators in later sections using the main result proved in this section.

We need several results from the theory of spherical harmonics on $S^{n-1}$. For each $m = 0, 1, 2, \cdots$, let $\mathcal{H}_m$ be the space of spherical harmonics of degree $m$. Let $Y_{m,j}$, $j = 1, 2, \cdots, d(m)$ be an orthonormal basis for $\mathcal{H}_m$, where $d(m)$ is the dimension of $\mathcal{H}_m$. We can take these spherical harmonics to be real valued. It is well known that the reproducing kernel

$$\sum_{j=1}^{d(m)} Y_{m,j}(\omega) \ Y_{m,j}(\omega')$$

depends only on $\omega \cdot \omega'$ and hence there is a function, denoted by $C_m^{n-1}(u)$, defined on $[-1, 1]$ such that

$$\sum_{j=1}^{d(m)} Y_{m,j}(\omega) \ Y_{m,j}(\omega') = C_m^{n-1}(\omega \cdot \omega').$$
It can be shown that $C_{m}^{n-1}(u)$ are polynomials, called ultra spherical polynomials of type $(n^2-1)$ satisfying the differential equation

$$(1 - u^2) \varphi''(u) - (n - 1) u \varphi'(u) + m(m + n - 2) \varphi(u) = 0.$$ 

Moreover, it is known that

$$C_{m}^{n-1}(1) = \sum_{j=1}^{d(m)} Y_{m,j}(\omega)^2 = d(m) \omega_{n-1}^{-1},$$

$\omega_{n-1}$ being the surface measure of $S^{n-1}$. We let

$$P_{m}^{n-1}(u) = C_{m}^{n-1}(1) - 1 C_{m}^{n-1}(u)$$

to stand for the normalised ultra spherical polynomials. We note that $|P_{m}^{n-1}(u)| \leq 1$ for all $u \in [-1,1]$.

For any function $F$ on $[-1,1]$ integrable with respect to the measure $(1 - u^2)^{\frac{n-3}{2}} du$, the Funk-Hecke formula says that

$$\int_{S^{n-1}} F(\omega \cdot \omega') Y_{m,j}(\omega')\ d\omega' = Y_{m,j}(\omega) \int_{-1}^{1} F(t) P_{m}^{n-1}(t) (1 - t^2)^{\frac{n-3}{2}} dt.$$ 

We make use of this formula in the proof of our main result in this section. Given the kernel $K(x,y) = K_0(|x-y|, |x+y|)$ we define a sequence of kernels $K_m$ by setting

$$K_m(x,y) = K_0(|x-y|, |x+y|) P_m^{n-1}(x' \cdot y')$$

where $x' = \frac{x}{|x|}, \ y' = \frac{y}{|y|}$. Consider the operators

$$T_m f(x) = \int_{\mathbb{R}^n} K_m(x,y) f(y) \ dy.$$ 

We observe that $T_m f$ is radial whenever $f$ is radial. Indeed, when $f(y) = g(|y|)$ we have

$$T_m f(x) = \int_{0}^{\infty} g(s) \left( \int_{S^{n-1}} K_m(x,sy') dy' \right) s^{n-1} \ ds.$$ 

But then

$$\int_{S^{n-1}} K_m(x,sy') dy' = \int_{S^{n-1}} K_0(|x-sy'|, |x + sy'|) P_m^{n-1}(x' \cdot y') \ dy'$$

is clearly radial in $x$ as the measure $dy'$ is rotation invariant. Thus we can also consider $T_m$ as an operator on $L^p(\mathbb{R}^+, r^{n-1}dr)$. We also let $\tilde{T}$ stand for the integral operator with kernel $|K(x,y)|$. 
**Theorem 2.1.** Let $T$ be an integral operator with kernel $K(x, y) = K_0(|x-y|, |x+y|)$. If $\widetilde{T}$ is bounded on $L^p(\mathbb{R}^n)$ then $T$ satisfies the mixed norm estimate

$$
\|Tf\|_{L^p,2(\mathbb{R}^n)} \leq C\|f\|_{L^p,2(\mathbb{R}^n)}.
$$

**Proof.** For a fixed $r$, let us calculate the spherical harmonic coefficients of $Tf(r\omega)$.

$$
\int_{S^{n-1}} Tf(r\omega) Y_{m,j}(\omega) \ d\omega = \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} K(r\omega, y) Y_{m,j}(\omega) \ d\omega \right) f(y) \ dy.
$$

If $y = s\omega'$, the kernel, $K(x, y) = K_0(|r\omega - s\omega'|, |r\omega + s\omega'|)$ is a function of $\omega \cdot \omega'$ and hence by Funk-Hecke formula we have

$$
\int_{S^{n-1}} K(r\omega, s\omega') Y_{m,j}(\omega) \ d\omega
$$

$$
= Y_{m,j}(\omega') \int_{-1}^{1} K_0(q_+(r, s; u)^{\frac{1}{2}}, q_-(r, s; u)^{\frac{1}{2}}) P_{m-1}^{n} (u) (1 - u^2)^{\frac{n-3}{2}} \ du
$$

where $q_\pm(r, s, u) = r^2 + s^2 \pm 2rsu$. Therefore,

$$
\int_{S^{n-1}} Tf(r\omega) Y_{m,j}(\omega) \ d\omega
$$

$$
= \int_{0}^{\infty} f_{m,j}(s) \left( \int_{-1}^{1} K_0(q_+(r, s; u)^{\frac{1}{2}}, q_-(r, s; u)^{\frac{1}{2}}) P_{m}^{n-1} (u) (1 - u^2)^{\frac{n-3}{2}} \ du \right) s^{n-1} \ ds
$$

where

$$
f_{m,j}(s) = \int_{S^{n-1}} f(s\omega') Y_{m,j}(\omega') \ d\omega'.
$$

By the same Funk-Hecke formula applied to $K(x, y) P_{m}^{n-1}(x' \cdot y')$ we get

$$
\int_{S^{n-1}} K(x, y) P_{m}^{n-1}(x' \cdot y') \ dy'
$$

$$
= \int_{-1}^{1} K_0(q_+(r, s; u)^{\frac{1}{2}}, q_-(r, s; u)^{\frac{1}{2}}) P_{m}^{n-1} (u) (1 - u^2)^{\frac{n-3}{2}} \ du.
$$
Consequently

\[
\int_{S^{n-1}} Tf(r\omega) Y_{m,j}(\omega) \, d\omega = \int_0^\infty \left( \int_{S^{n-1}} f_{m,j}(s) K_m(x,y) \, dy' \right) s^{n-1} \, ds = T_m f_{m,j}(r)
\]

Thus we see that

\[
Tf(r\omega) = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} T_m f_{m,j}(r) Y_{m,j}(\omega).
\]

This leads us to

\[
\int_{S^{n-1}} |Tf(r\omega)|^2 \, d\omega = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |T_m f_{m,j}(r)|^2.
\]

By considering \(T_m f_{m,j}\) as \(T_m\) applied to the radial function \(f_{m,j}(|y|)\) we see that

\[
T_m f_{m,j}(x) = \int_{\mathbb{R}^n} K(x,y) P_m^{n-1}(x' \cdot y') f_{m,j}(|y|) \, dy
\]

and hence \(|T_m f_{m,j}(x)| \leq \tilde{T}(|f_{m,j}|)(x)\) since \(P_m^{n-1}\) is bounded by 1. Thus the mixed norm inequality will follow from

\[
\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} \tilde{T}(|f_{m,j}|)(r)^2 \right)^{\frac{p}{2}} r^{n-1} \, dr \leq C \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} r^{n-1} \, dr.
\]

If \(\tilde{T}\) is bounded on \(L^p(\mathbb{R}^n)\), the above is an immediate consequence of a theorem of Marcinkiewicz and Zygmund \[8\]. \(\square\)

In the next theorem we obtain a sufficient condition on the operators \(T_m\) so that the original operator satisfies a mixed weighted norm inequality. On \(\mathbb{R}^+\) consider the measure \(d\mu_\alpha = r^{2\alpha+1} \, dr\) with respect to which it becomes a homogeneous space. We define Muckenhoupt’s \(A_p^\alpha\) weights adapted to the measure \(d\mu_\alpha\) as follows. A positive weight
function \( w \) is said to belong to \( A_\alpha^p(\mathbb{R}^+) \) if it satisfies
\[
\left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r) \, d\mu_\alpha \right) \left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r)^{-1/p-1} \, d\mu_\alpha \right)^{p-1} \leq C
\]
for all intervals \( Q \subset \mathbb{R}^+ \).

Let \( L_{p,2}(w) \) stand for the space of functions \( f(\omega) \) on \( \mathbb{R}^+ \times S^{n-1} \) for which
\[
\| f \|_{L_{p,2}(w)} = \left( \int_0^\infty \left( \int_{S^{n-1}} |f(\omega)|^2 d\omega \right)^{\frac{p}{2}} \frac{1}{w(r)} r^{n-1} dr \right)^{\frac{1}{p}}
\]
are finite.

**Theorem 2.2.** Let \( T \) be as in the previous theorem and let \( T_m \) be defined as before. If \( T_m \) are uniformly bounded on \( L_{p,0}(\mathbb{R}^+, w d\mu_{\frac{n}{2}-1}) \) for some \( p_0 \), \( 1 < p_0 < \infty \) and all \( w \in A_\alpha^{n-1}(\mathbb{R}^+) \), then the mixed weighted norm inequality
\[
\| Tf \|_{L_{p,2}(w)} \leq C \| f \|_{L_{p,2}(w)}
\]
holds for all \( p, 1 < p < \infty \) and all \( w \in A_\alpha^{n-1}(\mathbb{R}^+) \).

As in the proof of Theorem 2.1 the mixed norm weighted inequality reduces to
\[
\int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |T_m f_{m,j}(r)|^2 \right) \frac{1}{w(r)} r^{n-1} dr \leq C \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right) \frac{1}{w(r)} r^{n-1} dr.
\]

We can easily deduce this inequality from the following extrapolation theorem (Theorem 3.1 in [5], Proposition 3.3 in [2]).

**Theorem 2.3.** Suppose that for some pair of nonnegative functions \( f, g \), for some fixed \( 1 < p_0 < \infty \) and for all \( w \in A_\alpha^{p_0}(\mathbb{R}^+) \) we have
\[
\int_0^\infty g(r)^{p_0} w(r) \, d\mu_\alpha \leq C(w) \int_0^\infty f(r)^{p_0} w(r) \, d\mu_\alpha.
\]
Then for all $1 < q < \infty$ and all $w \in A_q^\alpha(\mathbb{R}^n)$ we have
\[
\int_0^\infty g(r)^q w(r) \, d\mu_\alpha \leq C \int_0^\infty f(r)^q w(r) \, d\mu_\alpha.
\]

We can now easily deduce our theorem. The uniform boundedness of $T_m$ on $L^p_0(\mathbb{R}^n, w \, d\mu_\frac{n-1}{2})$ can be used along with the above theorem to give
\[
\int_0^\infty |T_m f(r)|^q w(r) \, d\mu_\frac{n-1}{2} \leq C(w) \int_0^\infty |f(r)|^q w(r) \, d\mu_\frac{n-1}{2}
\]
for any $1 < q < \infty$. If we take
\[
g = \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |T_m f_{m,j}|^2 \right)^{\frac{1}{2}}, \quad f = \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |f_{m,j}|^2 \right)^{\frac{1}{2}},
\]
then with $p = 2$
\[
\int_0^\infty g(r)^2 w(r) \, d\mu_\frac{n-1}{2} \leq C \int_0^\infty f(r)^2 w(r) \, d\mu_\frac{n-1}{2}
\]
holds. By Theorem \[2.3\] we get
\[
\int_0^\infty g(r)^p w(r) \, d\mu_\frac{n-1}{2} \leq C \int_0^\infty f(r)^p w(r) \, d\mu_\frac{n-1}{2}
\]
for any $1 < p < \infty$, $w \in A_p^\frac{n-1}{2}(\mathbb{R}^n)$. This completes the proof.

3. Hermite and Laguerre semigroups

In this section we apply the results of the previous section to study mixed weighted norm inequalities for the Hermite semigroup. The Hermite semigroup $e^{-tH}$, $t > 0$, generated by the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ is an integral operator whose kernel $K_t(x, y)$ is explicitly known. Since the normalised Hermite functions $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^n$ are eigenfunctions of $H$ with eigenvalues $(2|\alpha| + n)$ it follows that
\[
K_t(x, y) = \sum_{\alpha \in \mathbb{N}^n} e^{-(2|\alpha|+n)t} \Phi_\alpha(x)\Phi_\alpha(y).
\]
In view of the Mehler’s formula (Lemma 1.1.1, \[13\]) we see that
\[
K_t(x, y) = \pi^{-\frac{n}{2}} (\sinh 2t)^{-\frac{n}{2}} e^{-\frac{1}{2}(\coth 2t)(|x|^2+|y|^2)} + (\cosech 2t) x \cdot y.
\]
For each \( t > 0 \), \( e^{-tH} \) defines a bounded operator on \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \). The kernel of the operator \( e^{-tH} \) can be factorised as

\[
K_t(x, y) = \pi^{-n/2} (\sinh 2t)^{-n/2} e^{-\frac{1}{2}(\coth t)|x-y|^2} e^{-\frac{1}{4}(\tanh t)|x+y|^2}
\]

and hence we can make use of the results of the previous section to prove mixed norm estimates. If \( T = e^{-tH} \) then the operator \( T_m \) defined in the previous section with kernel \( K_t(x, y) P_m^{2m-1}(x' \cdot y') \) turns out to be a Laguerre semigroup.

For each \( \alpha > -1 \), let \( L_k^\alpha(r), r > 0 \) be the Laguerre polynomials of type \( \alpha \) and define the normalised Laguerre functions \( \psi_k^\alpha(r) \) by

\[
\psi_k^\alpha(r) = \left( \frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} L_k^\alpha(r^2) e^{-1/2r^2}.
\]

Then \( \psi_k^\alpha, k = 0, 1, 2, \cdots \) forms an orthonormal basis for \( L^2(\mathbb{R}^+, d\mu_\alpha) \) where \( d\mu_\alpha(r) = r^{2\alpha+1} dr \). Moreover, \( \psi_k^\alpha \) are eigenfunctions of the Laguerre operator

\[
L_\alpha = -\frac{d^2}{dr^2} + r^2 - \frac{2\alpha + 1}{r} \frac{d}{dr}
\]

with eigenvalues \( (4k + 2\alpha + 2) \). The Laguerre semigroup generated by \( L_\alpha \) is again an integral operator whose kernel is given by

\[
K_t^\alpha(r, s) = \sum_{k=0}^{\infty} e^{-(4k+2\alpha+2)} \psi_k^\alpha(r) \psi_k^\alpha(s).
\]

Using the generating function identity (1.1.47) in [13] we can easily see that

\[
K_t^\alpha(r, s) = (\sinh 2t)^{-1} e^{-\frac{1}{2}(\coth 2t)(r^2+s^2)} (rs)^{-\alpha} I_\alpha(rs \csc 2t),
\]

where \( I_\alpha \) are the modified Bessel functions, \( I_\alpha(z) = e^{-iz^2} J_\alpha(iz) \). We note that for \( \alpha > -\frac{1}{2} \),

\[
J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\frac{1}{2})\Gamma(\alpha+1)} \int_{-1}^{1} e^{itz} (1 - t^2)^{\alpha-\frac{1}{2}} dt
\]

and consequently,

\[
I_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\frac{1}{2})\Gamma(\alpha+1)} \int_{-1}^{1} e^{itz} (1 - t^2)^{\alpha-\frac{1}{2}} dt.
\]

We will make use of these formulas in the sequel. We denote the Laguerre semigroup \( e^{-tL_\alpha} \) by \( T_t^\alpha \).
The Hermite and Laguerre semigroups are intimately connected. Consider $T_m$ applied to a radial function $f \in L^p(\mathbb{R}, r^{n-1}dr)$. Then

$$T_m f(r) = \int_0^\infty f(s) \left( \int_{S^{n-1}} K_t(rx', sy') P_m^{n-1}(x' \cdot y') \, dy' \right) s^{n-1} \, ds.$$

We claim that $T_m f(r) = C_n r^m T^{n+m-1}_t \tilde{f}(r)$ where $\tilde{f}(r) = r^{-m}f(r)$. This follows immediately once we show that

$$\int_{S^{n-1}} K_t(rx', sy') P_m^{n-1}(x' \cdot y') \, dy' = C_n r^m s^m K_t^{n+m-1}(r, s)$$

By Funk-Hecke formula the above kernel is given by

$$\int_{-1}^1 e^{zu} P_m^{n-1}(u) (1-u^2)^{n-3} \frac{u}{2} \, du = \Gamma \left( n - \frac{1}{2} \right) \Gamma \left( \frac{n-1}{2} \right) \left( \frac{z}{2} \right)^{-\frac{n}{2}+1} I_{\frac{n}{2}+m-1}(z).$$

**Proof.** The function $P_m^{\lambda}(t) = C_m^{\lambda}(t) C^{-1}_m(1)$ is given by the Rodrigues type formula

$$(1-t^2)^{n-3} P_m^{n-1}(t) = \frac{(-1)^m}{2^m (n-1)_m} \left( \frac{d}{dt} \right)^m (1-t^2)^{n-3+m}.$$

In view of this formula

$$\int_{-1}^1 e^{zu} P_m^{n-1}(u) (1-u^2)^{n-3} \frac{u}{2} \, du = \frac{(-1)^m}{2^m (n-1)_m} \int_{-1}^1 e^{zu} \left( \frac{d}{du} \right)^m (1-u^2)^{n-3+m} \, du.$$
Integrating by parts we see that
\[
\int_{-1}^{1} e^{zu} \left( \frac{d}{du} \right)^m (1 - u^2)^{\frac{n-3}{2}+m} du
\]
\[
= (-1)^m z^m \int_{-1}^{1} e^{zu} (1 - u^2)^{\frac{n-3}{2}+m} du
\]
\[
= (-1)^m 2^{m+\frac{n}{2}-1} z^{-\frac{n}{2}+1} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n-1}{2} + m \right) I_{\frac{n}{2}+1}(z).
\]
Since \(\frac{\Gamma(\frac{n+1}{2}+m)}{(\frac{n+1}{2})^m} = \Gamma(\frac{n-1}{2})\) we get the lemma. \(\square\)

If we recall the definition of the kernels \(K_{\frac{n}{2}+m-1}(r, s)\) and combine it with the result of the above lemma we immediately obtain the following:

**Proposition 3.2.** When \(f(x) = f_{m,j}(r)Y_{m,j}(\omega)\), we have
\[
e^{-tH} f(r\omega) = c_n \ r^m Y_{m,j}(\omega) T_{\frac{n}{2}+m-1} f_{m,j}(r)
\]
where \(c_n = \Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})2^{\frac{n}{2}-1}\) and \(\tilde{f}_{m,j}(s) = s^{-m} f_{m,j}(s)\).

**Remark 3.1.** The above proposition can also be proved directly using Hecke-Bochner formula for the Hermite projections (see Theorem 3.4.1 in [13]). However, the above approach comes in handy when we try to prove uniform estimates for the family of operators taking \(f_{m,j}\) into \(r^m T_{\frac{n}{2}+m-1} \tilde{f}_{m,j}\).

We are now ready to prove the following mixed weighted norm inequality for the Hermite semigroup. Since \(P_{\frac{n}{2}}^{-1}(u)\) is bounded it follows that \(|T_m f(r)| \leq T_0 |f|(r) = T_{\frac{n}{2}}^{-1} |f|(r)\). From the work [7] we know that the Laguerre semigroup \(T_t^\alpha\) satisfies the weighted norm inequality
\[
\int_0^\infty |T_t^\alpha f(r)|^p w(r) \ d\mu_\alpha \leq C \int_0^\infty |f(r)|^p w(r) \ d\mu_\alpha
\]
for all \(w \in A_{p,loc}^\alpha(\mathbb{R}^+), 1 \leq p < \infty\). Here \(A_{p,loc}^\alpha(\mathbb{R}^+)\) is defined as the set of all positive weight functions satisfying
\[
\left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r) \ d\mu_\alpha \right)^{\frac{1}{p}} \left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r)^{-\frac{1}{p^*}} \ d\mu_\alpha \right)^{\frac{1}{p^*}} \leq C
\]
for all $Q \subset \mathbb{R}^+$ of length less than or equal to one. Combining this with the classical theorem of Marcinkiewicz and Zygmund [8] mentioned already we get the following.

**Theorem 3.3.** The mixed norm inequalities
\[ \|e^{-tH}f\|_{L^{p,2}(w)} \leq C\|f\|_{L^{p,2}(w)} \]
hold for all $w \in A^{n-1}_{p,\text{loc}}, 1 \leq p < \infty$.

Indeed, let $R$ be the radialisation operator defined by
\[ Rf(r) = \int_{S^{n-1}} f(r\omega) \, d\omega. \]
Defining $Sf(x) = T^{n-1}_{s}(Rf \cdot w^{-1/p})(x) w(|x|)^{\frac{1}{p}}$ we see that
\[
\int_{\mathbb{R}^n} |Sf(x)|^p \, dx = C \int_0^{\infty} |T^{n-1}_{s}(Rf \cdot w^{-1/p})(r)|^p \, w(r) \, d\mu_{\frac{n}{2}-1}
\leq C \int_0^{\infty} |Rf(r)|^p \, r^{n-1} \, dr.
\]
Since
\[ |Rf(r)|^p \leq \left( \int_{S^{n-1}} d\omega \right)^{\frac{p}{n}} \int_{S^{n-1}} |f(r\omega)|^p \, d\omega \]
it follows that
\[
\int_{\mathbb{R}^n} |Sf(x)|^p \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, dx.
\]
By the theorem of Marcinkiewicz and Zygmund we have
\[
\int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} |Sf_j(x)|^2 \right)^{\frac{p}{2}} \, dx \leq C \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} |f_j(x)|^2 \right)^{\frac{p}{2}} \, dx
\]
for any sequence $f_j \in L^p(\mathbb{R}^n)$. Applying this to the radial functions $g_{m,j}(x) = f_{m,j}(|x|)w(|x|)^{\frac{1}{p}}$ where $f_{m,j} \in L^p(\mathbb{R}^+, w(r) \, d\mu_{\frac{n}{2}-1})$ we obtain
\[
\int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |T^{n-1}_{s}f_{m,j}(r)|^2 \right)^{\frac{p}{2}} \, w(r) \, d\mu_{\frac{n}{2}-1}
\leq C \int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} \, w(r) \, d\mu_{\frac{n}{2}-1}.
\]
This proves the theorem.

**Remark 3.2.** In view of Proposition 3.2 the main result can be restated as the following vector valued inequality for the Laguerre semi-groups $T_t^{\frac{m}{2} + m - 1}$: for $1 \leq p \leq \infty$, $w \in A^{\frac{m}{2} - 1}_{p, \text{loc}}$ we have

$$
\int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} r^{2m} |T_t^{\frac{m}{2} + m - 1} f_{m,j}(r)|^2 \right)^{\frac{p}{2}} \frac{w(r)}{r} \, d\mu_{\frac{m}{2} - 1}(r)
\leq C_{n,t} \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} \frac{w(r)}{r} \, d\mu_{\frac{m}{2} - 1}(r).
$$

Using Theorem 2.1 we can also prove mixed norm estimates for Riesz means associated to Hermite expansions on $\mathbb{R}^n$, $n \geq 2$. For $\delta \geq 0$ we define the Riesz means of order $\delta$ by

$$
S_\delta f(x) = \sum \left( 1 - \frac{2k + n}{R} \right)^\delta P_k f(x)
$$

where $P_k$ are the Hermite projection operators defined by

$$
P_k f(x) = \int_{\mathbb{R}^n} \Phi_k(x,y) f(y) \, dy
$$

with $\Phi_k(x,y) = \sum_{|\alpha| = k} \Phi_\alpha(x) \Phi_\alpha(y)$, $\Phi_\alpha$ being the normalised Hermite functions on $\mathbb{R}^n$. The basic result for the Riesz means is the following: if $\delta > \frac{n-1}{2}$, the operators $S_\delta^R$ are all uniformly bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. (Theorem 3.3.2 in [13]). This follows from the fact that (see Theorem 3.3.1 [13])

$$
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |s_\delta^R(x,y)| \, dy \leq C
$$

where $C$ is independent of $R$. Here $s_\delta^R(x,y)$ is the kernel associated to $S_\delta^R$, viz.,

$$
s_\delta^R(x,y) = \sum_k \left( 1 - \frac{2k + n}{R} \right)^\delta \Phi_k(x,y).
$$

It follows that the operator $\tilde{S}_\delta^R$ given by

$$
\tilde{S}_\delta^R f(x) = \int_{\mathbb{R}^n} |s_\delta^R(x,y)| f(y) \, dy
$$
is also bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $\delta > \frac{n-1}{2}$. Using this fact we can easily obtain the following mixed norm estimates for $S^\delta_R$.

**Theorem 3.4.** Let $n \geq 2$ and $\delta > \frac{n-1}{2}$. Then we have the uniform estimates

$$
\|S^\delta_R f\|_{L^p,2(\mathbb{R}^n)} \leq C \|f\|_{L^p,2(\mathbb{R}^n)}
$$

for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. (Here $C$ is independent of $R$).

**Proof.** To prove this theorem, all we have to do is to observe that $s^\delta_R(x,y) = K_0(|x-y|,|x+y|)$. To see this it is enough to check that $\Phi^\delta_k(x,y)$ have the same property. The heat kernel $K_t(x,y)$ can be defined even for complex $t$ from the open unit disc and we have

$$
t^k \Phi^\delta_k(x,y) = \frac{1}{2\pi} \int_0^{2\pi} K_{te^{i\theta}}(x,y) e^{-ik\theta} d\theta.
$$

Since $K_{te^{i\theta}}(x,y)$ depends only on $x \cdot y$ the same is true for $\Phi^\delta_k(x,y)$. And this proves the theorem.

**Remark 3.3.** The above theorem does not say anything for $S^\delta_R$ when $\delta$ is below the critical index $\frac{n-1}{2}$. We conjecture that for $0 \leq \delta \leq \frac{n-1}{2}$, the mixed norm estimates of the theorem holds for all $p$ satisfying $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. This is weaker than the Bochner-Riesz conjecture for Hermite expansions and hence stands a better chance of getting proved.

**4. $g-$functions associated to the Hermite semigroups**

For each positive integer $k$, the $g-$function $g_k$ associated to the Hermite semigroup $T_t = e^{-tH}$ is defined by

$$
g_k(f,x)^2 = \int_0^\infty |\partial^k_t T_t f(x)|^2 t^{2k-1} dt.
$$

These $g-$functions have been studied in [13] and the following estimates are known: for $1 < p < \infty$,

$$
C_1 \|f\|_p \leq \|g_k(f)\|_p \leq C_2 \|f\|_p
$$

and when $p = 2$ we have

$$
\|g_k(f)\|_2 = C_k \|f\|_2.
$$

In this section we are interested in obtaining mixed norm estimates for $g_k-$functions. The $g_k-$functions are singular integral operators with kernels taking values in the Hilbert space $L^2(\mathbb{R}^+, t^{2k-1} dt)$. More
precisely the kernel is given by $\partial_t^k K_t(x, y)$ and hence the results in Section 1 can be applied to study these functions.

Let us consider the $L^2$-norm of $g_k(f, x)$ in the angular variable first.

$$\int_{S^{n-1}} g_k(f, r\omega)^2 \, d\omega = \int_{S^{n-1}} \left( \int_0^\infty t^{2k-1} |\partial_t^k T_t f(r\omega)|^2 \, dt \right) \, d\omega.$$ 

Interchanging the order of integration we see that

$$\int_{S^{n-1}} |\partial_t^k T_t f(r\omega)|^2 \, d\omega = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |\int_{S^{n-1}} \partial_t^k T_t f(r\omega) Y_{m,j}(\omega) \, d\omega|^2.$$ 

It follows that

$$\int_{S^{n-1}} \partial_t^k T_t f(r\omega) Y_{m,j}(\omega) \, d\omega = \partial_t^k T_{t,m} f_{m,j}(r)$$

where the kernel of $T_{t,m}$ is $K_t(x, y) P_{\frac{n}{2}+m-1}^\frac{1}{2} (x' \cdot y')$. We also know that

$$T_{t,m} f_{m,j}(r) = r^m T_{t}^{\frac{n}{2}+m-1} \tilde{f}_{m,j}(r)$$

and consequently,

$$\int_{S^{n-1}} \left( \int_0^\infty t^{2k-1} |\partial_t^k T_t f(r\omega)|^2 \, dt \right) \, d\omega = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} \left( \int_0^\infty t^{2k-1} |\partial_t^k T_{t}^{\frac{n}{2}+m-1} \tilde{f}_{m,j}(r)|^2 \, dt \right) r^{2m}.$$ 

Thus we are led to study $g_k$-functions defined for the Laguerre semigroups.

For each $m = 0, 1, 2, \cdots$, let us define

$$g_{k,m}(f, x)^2 = \int_0^\infty |\partial_t^k T_{t,m} f(x)|^2 t^{2k-1} \, dt,$$
where \( f \) is a radial function on \( \mathbb{R}^n \). The \( L^2 \)–theory of \( g_k \) functions for Laguerre semigroups immediately gives us

\[
\int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} g_{k,m}(f_{m,j}, r)^2 \right)^{\frac{p}{2}} r^{n-1} \, dr
\]

\[
= C_{k,n} \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} r^{n-1} \, dr
\]

for \( p = 2 \). We are interested in proving the inequality for general \( p, 1 < p < \infty \). By showing that \( g_{k,m} \) are singular integral operators uniformly bounded on \( L^p(\mathbb{R}^+, wd\mu_{\frac{n}{2}-1}) \) we will obtain the following result.

**Theorem 4.1.** For every \( k = 1, 2, \cdots \), the \( g_k \)–functions satisfy

\[
\|g_k(f)\|_{L^p(\mathbb{R}^+, w)} \leq C_k \|f\|_{L^p(\mathbb{R}^+)}
\]

for all \( w \in A_{\frac{n}{2}}(\mathbb{R}^+) \), \( 1 < p < \infty \).

In order to prove this theorem we need to recall Calderón-Zygmund theory of singular integral operators on the homogeneous space \((\mathbb{R}^+, d\mu_\alpha)\) developed by Calderón [11]. Let \( B(a, b) \) stand for the ball of radius \( b > 0 \) centred at \( a \in \mathbb{R}^+ \) and let \( \mu_\alpha(B(a, b)) \) stand for its measure. Note that \( g_{k,m} \) can be considered as a (singular) integral operator with \( L^2(\mathbb{R}^+, t^{2k-1} dt) \)–valued kernel given by

\[
\partial^k_t K_m(r, s; t) = \int_{\mathbb{S}^{n-1}} \partial^k_t K_1(x, y) P_{n-1}^{\frac{n}{2}-1}(x' \cdot y') \, dy',
\]

\( x = rx' \), \( y = sy' \). It is therefore, enough to show that \( K_m \) satisfy the following Calderón-Zygmund estimates uniformly in \( m \).

**Proposition 4.2.** The kernels \( K_m(r, s; t) \) satisfy the following uniform estimates: for \( r \neq s \)

\[
(i) \left( \int_0^\infty |\partial^k_t K_m(r, s; t)|^2 \, t^{2k-1} \, dt \right)^{\frac{1}{2}} \leq \frac{C_1}{\mu_{n/2-1}(B(r, |r-s|))}
\]

\[
(ii) \left( \int_0^\infty |\partial^k_t \partial_r K_m(r, s; t)|^2 \, t^{2k-1} \, dt \right)^{\frac{1}{2}} \leq \frac{C_2}{|r-s| \, \mu_{n/2-1}(B(r, |r-s|))}
\]

where \( C_1 \) and \( C_2 \) are independent of \( m \).

The measure \( \mu_{\frac{n}{2}-1}(B(r, |r-s|)) \) has been estimated in [9] (See Proposition 3.2) according to which

\[
\mu_{\frac{n}{2}-1}(B(r, |r-s|)) \approx |r-s|(r + s)^{n-1}.
\]
We will also make use of the following Lemma due to Nowak and Stem-
pak [9] (see Lemma 5.3 in [3]).

**Lemma 4.3.** Let \( c \geq \frac{1}{2}, 0 < B < A \) and \( \lambda > 0 \). Then

\[
\int_{0}^{1} \frac{(1-u)^{c-\frac{1}{2}}}{(A-Bu)^{c+\lambda+\frac{1}{2}}} du \leq \frac{C}{A^{c+\frac{1}{2}}(A-B)^{\lambda}}.
\]

We also need the following estimates on the kernel \( \partial^{k}_{t} K_{t}(x,y) \) proved in [13] (see inequalities (4.1.12) and (4.1.13)) and [12].

**Proposition 4.4.** We have the following estimates (for \( x \neq y \)):

(i) \[
\left( \int_{0}^{\infty} |\partial^{k}_{t} K_{t}(x,y)|^{2} t^{2k-1} dt \right)^{\frac{1}{2}} \leq C_{k} |x-y|^{-n}
\]

(ii) \[
\left( \int_{0}^{\infty} |\nabla_{x} \partial^{k}_{t} K_{t}(x,y)|^{2} t^{2k-1} dt \right)^{\frac{1}{2}} \leq C_{k} |x-y|^{-n-1}
\]

We can now easily prove the estimates in Proposition 4.2. In view of the above estimate (i) in Proposition 4.4 and the fact that \( P_{m}^{\frac{n}{2}-1}(u) \) is bounded we see that

\[
\int_{0}^{\infty} |\partial^{k}_{t} K_{m}(r,s,t)|^{2} t^{2k-1} dt \leq \int_{S^{n-1}} \left( \int_{0}^{\infty} |\partial^{k}_{t} K_{t}(rx',sy')|^{2} t^{2k-1} dt \right)^{\frac{1}{2}} dy' \leq C \int_{S^{n-1}} |rx' - sy'|^{-n} dy'.
\]

The last integral is equal to a constant multiple of

\[
\int_{0}^{\pi} (r^2 + s^2 - 2rs \cos \theta)^{-\frac{n}{2}} (\sin \theta)^{n-2} d\theta
\]

which is estimated by

\[
\int_{0}^{1} (r^2 + s^2 - 2rsu)^{-\frac{n}{2}} (1-u)^{\frac{n-3}{2}} du.
\]
By appealing to Lemma 4.3 with $A = r^2 + s^2$, $B = 2rs$, $c = \frac{n}{2} - 1$, $\lambda = \frac{1}{2}$ we obtain the estimate

$$
\left( \int_0^\infty |\partial_t^k K_m(r, s; t)|^2 \, t^{2k-1} \, dt \right)^{\frac{1}{2}} \leq C |r - s|^{-1} (r^2 + s^2)^{-\frac{n-1}{2}}.
$$

The desired estimate follows as $|r - s| (r^2 + s^2)^{\frac{n-1}{2}}$ is comparable to $\mu_{\frac{n}{2}-1}(B(r, |r - s|))$.

In order to get the estimate on the derivative we note that

$$
\partial_r \partial_t^k K_m(r, s; t) = \int_{S^{n-1}} \partial_r \partial_t^k K_t(x, y) \, P_m^{\frac{n}{2}-1}(x' \cdot y') \, dy'.
$$

Since $\partial_t^k K_t(x, y) = \sum_{j=1}^n \frac{\partial}{\partial x_j} \partial_t^k K_t(x, y)x_j'$ we can estimate

$$
\left( \int_0^\infty |\partial_r \partial_t^k K_m(r, s; t)|^2 \, t^{2k-1} \, dt \right)^{\frac{1}{2}}
$$

in terms of

$$
\int_{S^{n-1}} \left( \int_0^\infty |\nabla_x \partial_t^k K_t(x, y)|^2 \, t^{2k-1} \, dt \right)^{\frac{1}{2}} \, dy'.
$$

This, in view of (ii) of Proposition 4.4 leads to the integral

$$
\int_{S^{n-1}} |rx' - sy'|^{-n-1} \, dy'
$$

which can be estimated, as above, leading to $|r - s|^{-2}(r^2 + s^2)^{-\frac{(n-1)}{2}}$. This proves the second estimate in Proposition 4.2.

Thus the $g_k-$functions $g_{k,m}$ are all uniformly bounded on the space $L^p(\mathbb{R}^+, wd\mu_{\frac{n}{2}-1})$ for any $1 < p < \infty$, $w \in A_{\frac{n}{p}}^{\frac{n}{2}-1}(\mathbb{R}^+)$ and consequently, the $g_k-$functions satisfy mixed weighted norm inequalities. This proves Theorem 4.1.

Polarising the identity $\|g_k(f)\|_2 = C_k \|f\|_2$ and using the boundedness of $g_k-$functions we can prove the reverse inequality

$$
C_1 \|f\|_{L^p,2}(\mathbb{R}^n) \leq \|g_k(f)\|_{L^p,2}(\mathbb{R}^n).
$$
Indeed, polarising $\|g_k(f)\|_2 = C_k\|f\|_2$ and performing the integration over $S^{n-1}$ we get

$$C_k \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} f_{m,j}(r)h_{m,j}(r) \right) r^{n-1} dr$$

$$= \int_0^\infty \int_0^\infty \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} \partial^k_t T_{t,m} f_{m,j}(r) \partial^k_t T_{t,m} h_{m,j}(r) t^{2k-1} dt \ r^{n-1} dr.$$

The right hand side is dominated by

$$\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} g_{k,m}(f_{m,j},r) g_{k,m}(h_{m,j},r) \right) r^{n-1} dr.$$

Applying Hölder’s inequality for the vector valued functions $(g_{k,m}(f_{m,j}))$ and $(g_{k,m}(h_{m,j}))$, and using the boundedness of $g_{k,m}$—functions the above is dominated by $\|f\|_{L^p(R^n)} \|h\|_{L^{p'}(R^n)}$. By taking supremum over all $h \in L^{p'}(R^n)$ we get the required inequality.

5. Mixed norm estimates for Hermite multipliers

In this section we will prove mixed norm estimates for Hermite multipliers making use of our results on $g$—functions proved in Section 4. Given a bounded function $\varphi$ defined on the set of all positive integers we can define a bounded linear operator on $L^2(R^n)$ by means of spectral theorem:

$$\varphi(H)f = \sum_{k=0}^\infty \varphi(2k + n) \ P_kf.$$

This is clearly a bounded operator on $L^2(R^n)$ but without further assumptions on $\varphi$ it may not be possible to extend $\varphi(H)$ to $L^p(R^n)$, $p \neq 2$ as a bounded linear operator. Consider the finite difference operator $\Delta$ defined by

$$\Delta \varphi(k) = \varphi(k + 1) - \varphi(k)$$

and define $\Delta^{j+1} \varphi(k) = \Delta(\Delta^j \varphi)(k)$ for $j = 1, 2, \cdots$. The following theorem has been proved in [13], see Theorem 4.2.1.

**Theorem 5.1.** Assume that $k > \frac{n}{2}$ is an integer and the function $\varphi$ satisfies $|\Delta^j \varphi(k)| \leq C_j(2k + n)^{-j}$, $j = 0, 1, 2, \cdots, k$. Then $\varphi(H)$ is bounded on $L^p(R^n)$ for all $1 < p < \infty$. 
Actually the theorem is true for more general multipliers but we have stated it in the above form for the sake of simplicity. The proof relies on the $g-$function estimates, viz.,

$$C_1 \|f\|_p \leq \|g(f)\|_p \leq C_2 \|f\|_p$$

for the $g-$function defined for the semigroup $e^{-tH}$. Another ingredient is the boundedness of $g_k^*$ functions: when $k > \frac{n}{2}$ and $p > 2$ we have

$$\|g_k^*(f)\|_p \leq C \|f\|_p.$$

Here $g_k^*(f, x)$ is defined by

$$g_k^*(f, x)^2 = \int_0^\infty \int_{\mathbb{R}^n} t^{-n/2} (1 + t^{-1}|x - y|^2)^{-k} \left| \partial_t T_t f(y) \right|^2 dy \, t \, dt.$$

Under the hypothesis of Theorem 5.1 one proves that

$$g_{k+1}(\varphi(H)f, x) \leq C \, g_k^*(f, x)$$

and this can be used in conjunction with the boundedness of $g_k$ and $g_k^*$ functions to prove the multiplier theorem.

For each $m$, we introduce the following $g_{k,m}^*$ functions:

$$g_{k,m}^*(f, x)^2 = \int_0^\infty \int_{\mathbb{R}^n} t^{-\frac{n}{2}} (1 + t^{-1}|x - y|^2)^{-k} \left| \partial_t T_{t,m} f(y) \right|^2 dy \, t \, dt.$$

where $f$ is a radial function on $\mathbb{R}^n$. It is then clear that $g_{k,m}^*(f, x)$ is a radial function of $x$ and hence we consider $g_{k,m}^*(f)$ as defined on $L^p(\mathbb{R}^+, r^{n-1}dr)$. For any radial function $h$ on $\mathbb{R}^n$ look at

$$\int_{\mathbb{R}^n} g_{k,m}^*(f, x)^2 \, h(x) \, dx$$

$$= \int_0^\infty \left| \partial_t T_{t,m} f(y) \right|^2 \left( \int_{\mathbb{R}^n} t^{-\frac{n}{2}} (1 + t^{-1}|x - y|^2)^{-k} h(x) \, dx \right) \, t \, dt.$$

As $h$ is radial, the inner integral is given by

$$C_n \int_0^\infty h(r) \left( \int_{S^{n-1}} t^{-\frac{n}{2}} (1 + t^{-1}|r \cdot x' - s y'|^2)^{-k} dy' \right) \, dr.$$
where \( h(r) = h(x) \) with \( |x| = r \). For \( k > \frac{n}{2} \) the function \( t^{-\frac{n}{2}} (1 + t^{-1} r^2)^{-k} \) is integrable over \( \mathbb{R}^+ \) with respect to \( r^{n-1} \, dr \) and the integral
\[
\int_{S^{n-1}} t^{-\frac{n}{2}} (1 + t^{-1} |x'|^2)^{-k} \, dy'
\]
is nothing but the generalised Euclidean translation of \( t^{-\frac{n}{2}} (1 + t^{-1} r^2)^{-k} \), see Stempak [11], and [13]. Consequently the integral
\[
\int_{\mathbb{R}^n} t^{-\frac{n}{2}} (1 + t^{-1} |x - y|^2)^{-k} h(x) \, dx
\]
is dominated by the maximal function \( M_{\frac{n}{2} - 1} h(s) \). Thus we have obtained
\[
\int_0^\infty g_{k,m}^*(f, r)^2 h(r) \, r^{n-1} \, dr 
\]
\[
\leq C \int_0^\infty g_{k,m}(f, r)^2 M_{\frac{n}{2} - 1} h(r) \, r^{n-1} \, dr.
\]
Here the constant \( C \) is independent of \( m \). By taking \( h = 1 \), the boundedness of \( g_{k,m} \) on \( L^2(\mathbb{R}^+, d\mu_{\frac{n}{2} - 1}) \) leads to the same for \( g_{k,m}^* \). By standard arguments one can prove the uniform estimates
\[
\int_0^\infty (g_{k,m}^*(f, r))^p \, r^{n-1} \, dr \leq C \int_0^\infty |f(r)|^p \, r^{n-1} \, dr,
\]
for all \( p \geq 2 \). We will make use of these estimates in the following theorem.

**Theorem 5.2.** Let \( \varphi \) be as in Theorem 5.1. Then \( \varphi(H) \) satisfies the mixed norm estimates
\[
\|\varphi(H)f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}
\]
for \( 1 < p < \infty \).

In order to prove this theorem we proceed as follows. Let \( \varphi \) be as in the theorem and let \( F = \varphi(H)f \). As in the proof of Theorem 4.2.1 in [13] we have the estimate \( g_{k+1}(F, x) \leq C_k g_k^*(f, x) \) provided \( k > \frac{n}{2} \). The reverse inequality \( \|F\|_{L^p(\mathbb{R}^n)} \leq C \|g_{k+1}(F)\|_{L^p(\mathbb{R}^n)} \) together with the above estimate gives us
\[
\|F\|_{L^p(\mathbb{R}^n)} \leq C_k \|g_k^*(f)\|_{L^p(\mathbb{R}^n)}
\]
and we will show that \( \| g_k^*(f) \|_{L^p(R^n)} \leq C \| f \|_{L^p(R^n)} \) for \( p \geq 2 \) which will prove the theorem for \( p \geq 2 \). By duality we can take care of the case \( 1 < p < 2 \).

Note that

\[
\int_{S^{n-1}} g_k^*(f, rx')^2 \, dx' = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} g_{k,m}^*(f_{m,j}, r)^2
\]

which follows from the fact that

\[
\int_{S^{n-1}} t^{-\frac{n}{2}} (1 + t^{-1}|x - y|^2)^{-k} \, dx'
\]

is a radial function of \( y \) and that

\[
\int_{S^{n-1}} |\partial_t T_t f(y)|^2 \, dy' = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |\partial_t T_{t,m} f_{m,j}(s)|^2
\]

as observed earlier in the previous section.

Therefore, we are left with proving the inequality

\[
\int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} g_{k,m}^*(f_{m,j}, r)^2 \right)^{\frac{p}{2}} r^{n-1} \, dr \leq C \int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} r^{n-1} \, dr.
\]

The following argument is essentially taken from Rubio de Francia [10], see the proof of the main theorem. For \( p > 2 \) let \( q = \frac{p}{2} \) and take \( h \in L^{q'} \) with \( \| h \|_{q'} = 1 \) and

\[
\int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} g_{k,m}^*(f_{m,j}, r)^2 \right)^{\frac{p}{2}} r^{n-1} \, dr = \left( \int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} g_{k,m}^*(f_{m,j}, r)^2 \right) h(r) \, r^{n-1} \, dr \right)^{q}.
\]
The uniform inequality (5.1) gives us

\[
\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} g_{k,m}(f_{m,j}, r)^2 \right) h(r) \, r^{n-1} \, dr 
\leq C \int_0^\infty \sum_{m=0}^\infty \sum_{j=1}^{d(m)} g_{k,m}(f_{m,j}, r)^2 \, v(r) \, r^{n-1} \, dr
\]

where \( v(r) = M_{\frac{n}{2}-1} h(r) \in L^q(\mathbb{R}^+, r^{n-1}dr) \). As in [10] the last integral is dominated by

\[
\sum_{m=0}^\infty \sum_{j=1}^{d(m)} \int_0^\infty g_{k,m}(f_{m,j}, r)^2 \, u(r) \, r^{n-1} \, dr
\]

where \( u(r) = (M_{\frac{n}{2}-1} v^s)^{\frac{1}{s}} \in L^{q'}(\mathbb{R}^+, r^{n-1}dr) \) provided that \( 1 < s < q' \). Now it can be shown that the function \( u \in A_{\frac{n}{2}}^{-1}(\mathbb{R}^+) \subset A_{\frac{n}{2}+1}(\mathbb{R}^+) \), the weighted norm inequality for \( g_{k,m} \) gives

\[
\sum_{m=0}^\infty \sum_{j=1}^{d(m)} \int_0^\infty g_{k,m}(f_{m,j}, r)^2 \, u(r) \, r^{n-1} \, dr 
\leq C \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{q'}{2}} \left( \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{q}{2}} r^{n-1} \, dr \right)^{\frac{2}{q}}.
\]

Thus we have proved that for the operator \( T \) defined on \( L^p(\mathbb{R}^+, l^2) \) by

\[
Tf(r) = (g_{k,m}(f_{m,j})), \quad f = (f_{m,j})
\]

\( \|Tf\|_{L^p(\mathbb{R}^+, l^2)} < \infty \) for each \( f \in L^p(\mathbb{R}^+, l^2) \). By the uniform boundedness principle it follows that \( T \) is actually bounded on \( L^p(\mathbb{R}^+, l^2) \). This proves the inequality \( \|F\|_{L^{p,2}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,2}(\mathbb{R}^n)} \) for \( p \geq 2 \) and hence the theorem follows.

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