THE NUMBER OF EDGES OF THE EDGE POLYTOPE OF A FINITE SIMPLE GRAPH

TAKAYUKI HIBI, AKI MORI, HIDEFUMI OHSUGI AND AKIHIRO SHIKAMA

Abstract. Let \([d] = \{1, \ldots, d\}\) be the vertex set and \(\Omega_d\) the set of finite simple graphs on \([d]\), where \(d \geq 3\). We write \(\varepsilon(G)\) for the number of edges of the edge polytope \(P_G \subset \mathbb{R}^d\) of \(G \in \Omega_d\). For example, \(\varepsilon(K_d) = \frac{(d-1)(d-2)}{2}\), where \(K_d\) is the complete graph on \([d]\). Let \(\mu_d = \max\{\varepsilon(G) : G \in \Omega_d\}\). We wish to find \(G \in \Omega_d\) with \(\mu_d = \varepsilon(G)\) and to compute \(\mu_d\). One may expect \(\mu_d = \varepsilon(K_d)\). In fact, it turns out to be true that \(\mu_d = \varepsilon(K_d)\) for each \(3 \leq d \leq 14\). However, it will be proved that, for each \(d \geq 15\), one has \(\mu_d \geq \varepsilon(K_d) + 50(d-14)\). It remains unsolved to find \(G \in \Omega_d\) with \(\mu_d = \varepsilon(G)\) and to compute \(\mu_d\) if \(d \geq 15\).

1. Introduction

The study on edge polytopes of finite graphs has been achieved by many authors from viewpoints of commutative algebra on toric ideals and combinatorics of convex polytopes. We refer the reader to [5] and [6] for foundations of edge polytopes.

Recall that a finite simple graph is a finite graph with no loop and no multiple edge. Let \([d] = \{1, \ldots, d\}\) be the vertex set and \(\Omega_d\) the set of finite simple graphs on \([d]\), where \(d \geq 3\). Let \(e_i\) denote the \(i\)th unit coordinate vector of the Euclidean space \(\mathbb{R}^d\). Let \(G \in \Omega_d\) and \(E(G)\) the set of edges of \(G\). If \(e = \{i, j\} \in E(G)\), then we set \(\rho(e) = e_i + e_j \in \mathbb{R}^d\). The edge polytope \(P_G\) of \(G \in \Omega_d\) is the convex hull of the finite set \(\{\rho(e) : e \in E(G)\}\) in \(\mathbb{R}^d\). Let \(\varepsilon(G)\) denote the number of edges of \(P_G\).

For example,

\[
\varepsilon(K_d) = \frac{(d-1)(d-2)}{2},
\]

where \(K_d\) is the complete graph on \([d]\), and

\[
\varepsilon(K_{m,n}) = \frac{mn(m+n-2)}{2},
\]

where \(K_{m,n}\) is the complete bipartite graph on the vertex set \([m] \cup \{m+1, \ldots, m+n\}\) for which \(m \geq 1, n \geq 1\) and \(m+n \geq 3\) ([7, Theorem 2.5]).

Given \(d \geq 3\), we are interested in the maximum of possible numbers of edges of edge polytopes arising from finite simple graphs on \([d]\). Let

\[\mu_d = \max\{\varepsilon(G) : G \in \Omega_d\}\].

Following [3, Question 1.3], for each \(d \geq 3\), we wish to find a finite graph \(G \in \Omega_d\) with \(\mu_d = \varepsilon(G)\) and to compute \(\mu_d\).

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Theorem 0.1. Work with the same notation as above.

(a) If $3 \leq d \leq 13$ and $G \in \Omega_d$ with $G \neq K_d$, then $\varepsilon(G) < \varepsilon(K_d)$.
(b) Let $d = 14$ and $G \in \Omega_d$ with $G \neq K_d$. Then $\varepsilon(G) \leq \varepsilon(K_{14})$. Moreover, $\varepsilon(G) = \varepsilon(K_{14})$ if and only if either $G = K_{14} - K_{4,5}$ or $G = K_{14} - K_{5,5}$.
(c) If $d \geq 15$, then there exists $G \in \Omega_d$ with $\varepsilon(G) - \varepsilon(K_d) = 50(d - 14)$.

Corollary 0.2. If $3 \leq d \leq 14$, then $\mu_d = d(d - 1)(d - 2)/2$. If $d \geq 15$, then

$$\mu_d \geq \frac{d(d - 1)(d - 2)}{2} + 50(d - 14).$$

We devote Section 1 to giving a proof of Theorem 0.1. At present, for $d \geq 15$, it remains unsolved to find $G \in \Omega_d$ with $\mu_d = \varepsilon(G)$ and to compute $\mu_d$. However, taking into consideration the process of achieving our proof of Theorem 0.1, we cannot escape from the temptation of presenting the following Conjecture 0.3. A complete bipartite graph of the form $K_{s,s}$ or $K_{s,s+1}$ in $\Omega_r$ is known as a Turán graph $T(r, 2)$.

Conjecture 0.3. Let $G \in \Omega_d$ with $\mu_d = \varepsilon(G)$. Then the complementary graph $\overline{G}$ of $G$ consists of complete bipartite Turán graphs and isolated vertices.

Now, in Section 2, we study various phenomena which support Conjecture 0.3. Apart from routine work of making computational observations on $\varepsilon(G) - \varepsilon(K_d) = d(\varepsilon(G) - \varepsilon(K_d))$ with $G \in \Omega_d$, we focus our attention on the crucial fact, which is obtained in the previous section, that, with fixing $r \geq 2$ and a finite simple graph $H$ on $[r]$, the function $\varepsilon(K_d - H) - \varepsilon(K_d)$, where $d \geq r$, is a linear polynomial in $d$ and its coefficient of $d$ is $\psi(H) - 2|E(H)|$, where $\psi(H)$ is the number of induced paths of length 2 appearing in $H$. We discuss the question which simple graph $H \in \Omega_r$ maximizes $\psi(H) - 2|E(H)|$ and find that a simple graph $H \in \Omega_r$ maximizes $\psi(H) - 2|E(H)|$ only if $H$ is either empty or a complete bipartite graph (Proposition 2.5). This result together with the formula

$$\varepsilon(K_d) - \varepsilon(G) = \sum_{i=1}^m (\varepsilon(K_d) - \varepsilon(K_d - H_i)),$$

where $G \in \Omega_d$ and where $H_1, \ldots, H_m$ are nonempty connected components of $\overline{G}$ (Proposition 2.2), enables us to believe Conjecture 0.3. Finally, if Conjecture 0.3 would be true, then one has $\mu_{15} = 1415$ (Example 2.9).

The recent paper Tuan–Ziegler [10] studies extremal problems on the number of faces of edge polytopes.

1. Proof of Theorem 0.1

In this section, we give a proof of Theorem 0.1. First, the following lemma is studied in [8, Lemma 1.4].

Lemma 1.1. Let $e$ and $f$ $(e \neq f)$ be edges of a graph $G \in \Omega_d$. Then, the convex hull of $\{\rho(e), \rho(f)\}$ is an edge of the edge polytope $\mathcal{P}_G$ if and only if one of the following conditions is satisfied.
(i) $e$ and $f$ have a common vertex in $[d]$.
(ii) $e = \{i, j\}$ and $f = \{k, l\}$ have no common vertex, and the induced subgraph of $G$ on the vertex set $\{i, j, k, l\}$ has no cycle of length 4.

The complementary graph $\overline{G}$ of a graph $G \in \Omega_d$ is the graph whose vertex set is $[d]$ and whose edges are the non-edges of $G$. For a vertex $i$ of a graph $G$, let $\deg_G(i)$ denote the degree of $i$ in $G$. See [2] to learn basics of graph theory. We translate Lemma 1.1 in terms of the complement $\overline{G}$ of $G$.

Lemma 1.2. Let $H$ be the complement of a graph $G \in \Omega_d$ Then, we have

$$\varepsilon(G) = \sum_{i=1}^{d} \left( \frac{d - 1 - \deg_H(i)}{2} \right) + a(H) + b(H) + c(H)$$

$$= \varepsilon(K_d) + \frac{1}{2} \sum_{i=1}^{d} \deg_H(i)^2 - (2d - 3)|E(H)| + a(H) + b(H) + c(H),$$

where $a(H)$, $b(H)$ and $c(H)$ are the number of induced subgraphs of $H$ on 4 vertices of the form

(a) a path of length 3;
(b) a cycle of length 4;
(c) a path of length 2 and one isolated vertex,

respectively.

Proof. First, the number of pairs of edges satisfying Lemma 1.1 (i) is equal to

$$\sum_{i=1}^{d} \left( \frac{d - 1 - \deg_H(i)}{2} \right) = \sum_{i=1}^{d} \frac{(d - 1 - \deg_H(i))(d - 2 - \deg_H(i))}{2}$$

$$= \sum_{i=1}^{d} \frac{(d - 1)(d - 2) - (2d - 3)\deg_H(i) + \deg_H(i)^2}{2}$$

$$= \varepsilon(K_d) + \frac{1}{2} \sum_{i=1}^{d} \deg_H(i)^2 - (2d - 3)|E(H)|.$$

Second, the number of pairs of edges satisfying Lemma 1.1 (ii) is equal to the number of the induced subgraphs of $G$ satisfying one of the following:

(a) a path of length 3;
(b) two disjoint edges;
(c) a cycle of length 3 and one edge.

Note that each induced subgraph has exactly one such pair of edges. The complement of each (a), (b), and (c) is

(a) a path of length 3;
(b) a cycle of length 4;
(c) a path of length 2 and one isolated vertex,

respectively. $\square$
For a graph $H \in \Omega_r$ with $r \leq d$, let $K_d - H$ denote the graph $G \in \Omega_d$ such that $E(G) = E(K_d) \setminus E(H)$. By Lemma 1.2, we have the following:

**Proposition 1.3.** Let $H \in \Omega_r$ and let $\psi(H)$ denote the number of induced paths in $H$ of length 2. Then, the function $\varphi(d) = \varepsilon(K_d - H) - \varepsilon(K_d)$ for $d = r, r+1, r+2, \ldots$ is a linear polynomial of $d$ whose initial coefficient is $\psi(H) - 2|E(H)|$.

*Proof.* It is sufficient to show that $\varphi(d + 1) - \varphi(d) = \psi(H) - 2|E(H)|$. Let $H_1 = K_d - H$ and $H_2 = K_{d+1} - H$. Then, $H_2$ is obtained by adding one isolated vertex $d + 1$ to $H_1$. Hence, it follows that $a(H_1) = a(H_2)$, $b(H_1) = b(H_2)$, $c(H_1) + \psi(H) = c(H_2)$ and $\deg_{H_1}(i) = \deg_{H_2}(i)$ for all $1 \leq i \leq d$. Thus, by Lemma 1.2 we have

$$
\varphi(d + 1) - \varphi(d) = 
\varepsilon(K_{d+1} - H) - \varepsilon(K_d) - \varepsilon(K_d - H) + \varepsilon(K_d)
\sum_{i=1}^{d+1} \left( d - \deg_{H_2}(i) \right) - \sum_{i=1}^{d} \left( d - 1 - \deg_{H_2}(i) \right) + \psi(H)
\frac{d(d-1)(d-2)}{2} - \frac{(d+1)d(d-1)}{2}
\left( \frac{d}{2} + \sum_{i=1}^{d} \left( \left( d - \deg_{H_2}(i) \right) - \left( d - 1 - \deg_{H_1}(i) \right) \right) \right) + \psi(H) - \frac{3d(d-1)}{2}
\left( \frac{d}{2} + \sum_{i=1}^{d} (d - 1 - \deg_{H_1}(i)) \right) + \psi(H) - \frac{3d(d-1)}{2}
\psi(H) - \sum_{i=1}^{d} \deg_{H_1}(i)
\psi(H) - 2|E(H)|,
$$
as desired. \qed

A graph $G \in \Omega_d$ is called *bipartite* if $|d|$ admits a partition into two sets of vertices $V_1$ and $V_2$ such that, for every edge $\{i, j\}$ of $G$, either $i \in V_1, j \in V_2$ or $j \in V_1, i \in V_2$ is satisfied. A *complete bipartite* graph is a bipartite graph such that every pair of vertices $i, j$ with $i \in V_1$ and $j \in V_2$ is adjacent. Let $K_{m,n}$ denote the complete bipartite graph with $|V_1| = m$ and $|V_2| = n$.

**Proposition 1.4.** Let $G = K_d - K_{m,n}$ such that $m + n \leq d$ and $m, n \geq 1$. Then,

$$
\varepsilon(G) - \varepsilon(K_d) = \frac{1}{2}mn(m + n - 6)d - \frac{1}{4}mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13).
$$

*Proof.* Let $H = K_{m,n}$. Then,

$$
\psi(H) - 2|E(H)| = \left( m \binom{n}{2} + n \binom{m}{2} \right) - 2mn = \frac{1}{2}mn(m + n - 6).
$$
Moreover, since $K_{m+n} - K_{m,n}$ is the disjoint union of $K_m$ and $K_n$, we have

$$\varphi(m+n) = \left( \frac{m(m-1)(m-2)}{2} + \frac{n(n-1)(n-2)}{2} + \binom{m}{2} \binom{n}{2} \right)$$

$$- \frac{(m+n)(m+n-1)(m+n-2)}{2} = \frac{1}{4} mn(mn - 7m - 7n + 13)$$

by Lemma 1.1. Hence, by Proposition 1.3,

$$\varepsilon(G) - \varepsilon(K_d) = \frac{1}{2} mn(m + n - 6)(d - (m + n)) + \frac{1}{4} mn(mn - 7m - 7n + 13)$$

$$= \frac{1}{2} mn(m + n - 6)d - \frac{1}{4} mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13),$$

as desired. \qed

By Proposition 1.4, we prove Theorem 0.1 (c).

**Proposition 1.5.** If $d \geq 15$, then there exists a graph $G \in \Omega_d$ with $\varepsilon(G) - \varepsilon(K_d) = 50(d - 14)$.

**Proof.** By Proposition 1.4, if $G = K_d - K_{m,n} \in \Omega_d$, then

$$\varepsilon(G) - \varepsilon(K_d) = \frac{1}{2} mn(m + n - 6)d - \frac{1}{4} mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13).$$

When $m = n = 5$, we obtain $\varepsilon(G) - \varepsilon(K_d) = 50(d - 14)$ as desired. \qed

Next, we prove that the complete graph $K_d$ with $d \leq 14$ vertices maximize $\varepsilon(G)$. Let $k_3(H)$ denote the number of triangles (cycles of length 3) of $H$. The following lemma is important.

**Lemma 1.6.** Let $H$ be the complement graph of $G \in \Omega_d$. Then, we have

$$\varepsilon(G) \leq \varepsilon(K_d) + \frac{d^2 - 16d + 29}{7} |E(H)| - \frac{3}{7} (d - 8) k_3(H).$$

**Proof.** The number of pairs of edges satisfying Lemma 1.1 (i) is, by Lemma 1.2

$$\varepsilon(K_d) - (2d - 3)|E(H)| + \frac{1}{2} \sum_{i=1}^{d} \deg_H(i)^2.$$

For an edge $\{j, k\}$ of $H$, let $k_3(j, k)$ be the number of triangles including $\{j, k\}$. We define three subsets of $[d] \setminus \{j, k\}$:

$$X_{j,k} = \{ \ell \in [d] \setminus \{j, k\} : \{j, \ell\} \in E(H), \{k, \ell\} \notin E(H) \},$$

$$Y_{j,k} = \{ \ell \in [d] \setminus \{j, k\} : \{k, \ell\} \in E(H), \{j, \ell\} \notin E(H) \},$$

$$Z_{j,k} = \{ \ell \in [d] \setminus \{j, k\} : \{j, \ell\} \notin E(H), \{k, \ell\} \notin E(H) \}.$$
It then follows that, \(|X_{j,k}| + |Y_{j,k}| + |Z_{j,k}| + k_3(j, k) = d - 2\), and

\[
\frac{1}{2} \sum_{i=1}^{d} \deg_H(i)^2 = \frac{1}{2} \sum_{\{j,k\} \in E(H)} (\deg_H(j) + \deg_H(k)) = \frac{1}{2} \sum_{\{j,k\} \in E(H)} (|X_{j,k}| + |Y_{j,k}| + 2k_3(j, k) + 2) = |E(H)| + 3k_3(\chi) + \frac{1}{2} \sum_{\{j,k\} \in E(H)} (|X_{j,k}| + |Y_{j,k}|). \]

Second, we count the number of pairs satisfying Lemma 1.1 (ii). By Lemma 1.2, this number is equal to \(a(\chi) + b(\chi) + c(\chi)\) where \(a(H), b(H)\) and \(c(H)\) are the number of induced subgraphs of \(H\) on 4 vertices of the form

(a) a path of length 3;
(b) a cycle of length 4;
(c) a path of length 2 and one isolated vertex,

respectively. Here, we fix an edge \(e = \{j, k\}\) of \(H\), and count the number of the above induced subgraphs including the edge \(e\). If \(e\) is included in the above induced subgraph, then the rest two vertices \(\ell\) and \(m\) of the induced subgraph satisfy exactly one of the following conditions:

(i) \(\ell \in X_{j,k}, m \in Y_{j,k}\);
(ii) \(\ell \in Y_{j,k}, m \in Z_{j,k}\);
(iii) \(\ell \in Z_{j,k}, m \in X_{j,k}\).

We need to care that by counting among all edges \(e = \{j, k\}\) of \(H\), an induced subgraph of type (a) appears once as the form (i), twice as the form (ii) or (iii), an induced subgraph of type (b) appears four times as the form (i), an induced subgraph of type (c) appears twice as the form (ii) or (iii). Thus, we set the weight of the form (i), (ii), (iii) for \(1/4, 1/2, 1/2\). The total number of induced subgraphs of \(H\) in the above statement is at most

\[
\sum_{\{j,k\} \in E(H)} \left( \frac{1}{4} |X_{j,k}| |Y_{j,k}| + \frac{1}{2} |Y_{j,k}| |Z_{j,k}| + \frac{1}{2} |Z_{j,k}| |X_{j,k}| \right). \]

Subject to \(|X_{j,k}| + |Y_{j,k}| + |Z_{j,k}| = d - 2 - k_3(j, k)|, we study the maximum number of

\[
\alpha = \sum_{\{j,k\} \in E(H)} \left( \frac{|X_{j,k}| + |Y_{j,k}|}{2} + \frac{1}{4} |X_{j,k}| |Y_{j,k}| + \frac{1}{2} |Y_{j,k}| |Z_{j,k}| + \frac{1}{2} |Z_{j,k}| |X_{j,k}| \right). \]
Each summand of $\alpha$ satisfies
\[
\frac{|X_{j,k}| + |Y_{j,k}|}{2} + \frac{1}{4} |X_{j,k}| |Y_{j,k}| + \frac{1}{2} |Y_{j,k}| |Z_{j,k}| + \frac{1}{2} |Z_{j,k}| |X_{j,k}|
\]
\[
= \frac{1}{4} |X_{j,k}| |Y_{j,k}| + \frac{1}{2} (|X_{j,k}| + |Y_{j,k}|) (|Z_{j,k}| + 1)
\]
\[
= \frac{1}{4} |X_{j,k}| |Y_{j,k}| + \frac{1}{2} (|X_{j,k}| + |Y_{j,k}|) (d - 1 - k_3(j, k) - (|X_{j,k}| + |Y_{j,k}|))
\]
\[
\leq \frac{1}{4} \left( \frac{|X_{j,k}| + |Y_{j,k}|}{2} \right)^2 + \frac{1}{2} (|X_{j,k}| + |Y_{j,k}|) (d - 1 - k_3(j, k) - (|X_{j,k}| + |Y_{j,k}|))
\]
\[
= -\frac{7}{16} (|X_{j,k}| + |Y_{j,k}|)^2 + \frac{d - 1 - k_3(j, k)}{2} (|X_{j,k}| + |Y_{j,k}|).
\]

The last function has the maximum number $\frac{1}{7} (d - 1 - k_3(j, k))^2$ when $|X_{j,k}| + |Y_{j,k}| = \frac{4}{7} (d - 1 - k_3(j, k))$. Hence,
\[
\sum_{\{j,k\} \in E(H)} \frac{1}{7} (d - 1 - k_3(j, k))^2 \leq \sum_{\{j,k\} \in E(H)} \frac{1}{7} (d - 1)(d - 1 - k_3(j, k))
\]
\[
= \frac{1}{7} \sum_{\{j,k\} \in E(H)} (d - 1)^2 - \frac{1}{7} \sum_{\{j,k\} \in E(H)} (d - 1)k_3(j, k)
\]
\[
= \frac{1}{7} (d - 1)^2 |E(H)| - \frac{3}{7} (d - 1)k_3(H)
\]
is an upper bound of $\alpha$. Thus,
\[
\varepsilon(K_d) - (2d - 3)|E(H)| + |E(H)| + 3k_3(H) + \frac{1}{7} (d - 1)^2 |E(H)| - \frac{3}{7} (d - 1)k_3(H)
\]
is an upper bound of $\varepsilon(G)$ as desired. \[\square\]

By Lemma 1.6, we prove Theorem 1.1 (a).

**Proposition 1.7.** If $3 \leq d \leq 13$ and $G \in \Omega_d$ with $G \neq K_d$, then $\varepsilon(G) < \varepsilon(K_d)$.

**Proof.** If $d = 3$, then it is trivial. If $d = 4$, then $\varepsilon(K_4) = 12$. Since the number of edges of $G$ is less than or equal to 5, we have $\varepsilon(G) \leq \frac{5}{2} = 10 < \varepsilon(K_4)$. Let $d \geq 5$.

Let $H$ be the complement graph of $G$. By Lemma 1.6,
\[
\varepsilon(G) - \varepsilon(K_d) \leq \frac{d^2 - 16d + 29}{7} |E(H)| - \frac{3}{7} (d - 8)k_3(H).
\]

If $5 \leq d \leq 13$, then $\frac{d^2 - 16d + 29}{7} < 0$. Since $|E(H)| > 0$ and $k_3(H) \geq 0$,
\[
\varepsilon(G) - \varepsilon(K_d) \leq \frac{d^2 - 16d + 29}{7} |E(H)| - \frac{3}{7} (d - 8)k_3(H) < 0
\]
if $8 \leq d \leq 13$.

Let $5 \leq d \leq 7$. Then,
\[
\varepsilon(G) - \varepsilon(K_d) \leq \begin{cases} 
-\frac{26}{7} |E(H)| + \frac{9}{7} k_3(H) & \text{if } d = 5, \\
-\frac{31}{7} |E(H)| + \frac{9}{7} k_3(H) & \text{if } d = 6, \\
-\frac{34}{7} |E(H)| + \frac{9}{7} k_3(H) & \text{if } d = 7.
\end{cases}
\]
Hence, if $k_3(H) \leq 2$, then $\varepsilon(G) - \varepsilon(K_d)$ is negative. On the other hand, if $k_3(H) \geq 3$, then $|E(H)| \geq 5$. Since $k_3(H) \leq \binom{d}{3}$, it follows that $\varepsilon(G) - \varepsilon(K_d)$ is negative. \hfill $\square$

If $d = 14$, then there exists a graph $G (\neq K_{14})$ such that $\varepsilon(G) = \varepsilon(K_{14})$.

**Proposition 1.8.** Let $d = 14$ and $G \in \Omega_d$ with $G \neq K_d$. Then $\varepsilon(G) \leq \varepsilon(K_{14})$. Moreover, $\varepsilon(G) = \varepsilon(K_{14})$ if and only if either $G = K_{14} - K_{4,5}$ or $G = K_{14} - K_{5,5}$.

**Proof.** By Lemma 1.6, the maximum value of (3) is $24 - \frac{3}{7}$ more accurately by focus on Proposition 1.8. Hence, if $\varepsilon(G) - \varepsilon(K_d) \leq \frac{1}{7}|E(H)| - \frac{18}{7}k_3(H)$.

However, this is not enough. We need to evaluate the function appears in Lemma 1.6 more accurately by focus on $d = 14$. Recall that

\begin{equation}
\frac{|X_{j,k}| + |Y_{j,k}|}{2} + \frac{1}{4}|X_{j,k}||Y_{j,k}| + \frac{1}{2}|Y_{j,k}||Z_{j,k}| + \frac{1}{2}|Z_{j,k}||X_{j,k}|
\end{equation}

\begin{equation}
\leq -\frac{7}{16}(|X_{j,k}| + |Y_{j,k}|)^2 + \frac{13 - k_3(j,k)}{2}(|X_{j,k}| + |Y_{j,k}|),
\end{equation}

and the function (4) has the maximum value $\frac{1}{7}(13 - k_3(j,k))^2$ when $|X_{j,k}| + |Y_{j,k}| = \frac{1}{7}(13 - k_3(j,k))$. If $1 \leq k_3(j,k) \leq 12$, then the maximum value of (4) satisfies

\begin{equation}
\frac{1}{7}(13 - k_3(j,k))^2 = 24 - \frac{13}{7}k_3(j,k) - \frac{11}{7} + \frac{1}{7}(k_3(j,k) - 1)(k_3(j,k) - 12) < 24 - \frac{13}{7}k_3(j,k).
\end{equation}

If $k_3(j,k) = 0$, the maximum value of (4) is $24 + 1/7$, however, since

\begin{equation}
4 \left( \frac{|X_{j,k}| + |Y_{j,k}|}{2} + \frac{1}{4}|X_{j,k}||Y_{j,k}| + \frac{1}{2}|Y_{j,k}||Z_{j,k}| + \frac{1}{2}|Z_{j,k}||X_{j,k}| \right)
\end{equation}

is an integer, the maximum value of (3) is $24$ or less. Thus, for $k_3(j,k) = 0, 1, 2, \ldots, 12$, the maximum value of (3) is $24 - \frac{13}{7}k_3(j,k)$ or less. Thus, by the same argument in Lemma 1.6, $\varepsilon(G) - \varepsilon(K_{14})$ is at most

\begin{equation}
-25|E(H)| + |E(H)| + 3k_3(H) + 24|E(H)| - \frac{3 \cdot 13}{7}k_3(H) = -\frac{18}{7}k_3(H) \leq 0.
\end{equation}

Therefore, $\varepsilon(G) \leq \varepsilon(K_{14})$.

Suppose that $\varepsilon(G) = \varepsilon(K_{14})$. Then, we have $k_3(H) = 0$. Moreover,

\begin{equation}
\frac{|X_{j,k}| + |Y_{j,k}|}{2} + \frac{1}{4}|X_{j,k}||Y_{j,k}| + \frac{1}{2}|Y_{j,k}||Z_{j,k}| + \frac{1}{2}|Z_{j,k}||X_{j,k}| = 24
\end{equation}

and $|X_{j,k}| + |Y_{j,k}| + |Z_{j,k}| = 12$ for an arbitrary edge $\{j, k\}$ of $H$. It is easy to see that $|X_{j,k}| + |Y_{j,k}| = 7, 8$. It then follows that, for an arbitrary edge $\{j, k\}$, $(|X_{j,k}|, |Y_{j,k}|, |Z_{j,k}|) \in \{(3, 4, 5), (4, 3, 5), (4, 4, 4)\}$. In particular, the degree of each vertex is either 0, 4 or 5. Here, for each induced subgraphs (a), (b) or (c), the sum of the weight is, 5/4, 1 or 1. If we have a induced subgraph of type (a), the above value is more than the number of induced subgraphs (a), (b) and (c). Thus, $H$ cannot have any path of length 3 as an induced subgraph.

Suppose that an edge $\{j, k\}$ of $H$ satisfies $(|X_{j,k}|, |Y_{j,k}|, |Z_{j,k}|) = (4, 4, 4)$. Let $X_{j,k} = \{j_1, j_2, j_3, j_4\}$ and $Y_{j,k} = \{k_1, k_2, k_3, k_4\}$. Since $H$ has no triangles, both $\{k, j_1, j_2, j_3, j_4\}$ and $\{j, k_1, k_2, k_3, k_4\}$ are independent sets. Moreover, since $H$ has no
induced path of length 3, the induced subgraph of $H$ with the vertex set \{\(j, j_1, j_2, j_3, j_4, k, k_1, k_2, k_3, k_4\)\} is \(K_{5,5}\). Since the degree of any vertex is either, 0, 4 or 5, the degree of the rest vertices are 0. Therefore, in this case, we have \(G = K_{14} - K_{5,5}\).

We concern about the case for each edge \(\{j, k\}\), \(|\{X_{j,k}, |Y_{j,k}|, |Z_{j,k}|\} \neq (4,4,4)\) is satisfied. For an edge \(\{j, k\}\), let \(X_{j,k} = \{j_1, j_2, j_3\}\) and \(Y_{j,k} = \{k_1, k_2, k_3, k_4\}\). Since \(H\) does not have triangles, \(\{k, j_1, j_2, j_3\}\) and \(\{j, k_1, k_2, k_3, k_4\}\) are both independent sets. Moreover, since \(H\) has no induced path of length 3, the induced subgraph of \(H\) with vertex set \(\{j, j_1, j_2, j_3, k, k_1, k_2, k_3, k_4\}\) is \(K_{4,5}\). Since \(|\{X_{j,k}, |Y_{j,k}|, |Z_{j,k}|\} \neq (4,4,4)\) for each edge \(\{j, k\}\), the degree of every vertex in \(\{k, j_1, j_2, j_3\}\) is 4. In this case, \(K_{4,5}\) is a connected component of \(H\). Since the degree of the rest five vertices is 4 or less, it follows that the rest vertices are isolated vertices. Therefore, \(G = K_{14} - K_{4,5}\). □

2. A conjecture on graphs \(G\) with \(\mu_d = \varepsilon(G)\)

For \(d \geq 15\), it remains unsolved to find \(G \in \Omega_d\) with \(\mu_d = \varepsilon(G)\). The purpose of this section is to give necessary conditions for such a graph \(G\) and a conjecture. First, by Lemma 1.2 we can compute \(\varepsilon(G) - \varepsilon(K_d)\) for typical graphs \(G\).

**Example 2.1.** Suppose that the complement of a graph \(G \in \Omega_d\) has exactly one nonempty connected component \(H\). If \(H\) is the complete graph \(K_n\), a path \(P_n\) of length \(n\), or a cycle \(C_n\) of length \(n\), then by Lemma 1.2 we can compute \(\varepsilon(G) - \varepsilon(K_d)\).

| \(H\) | \(\frac{1}{2} \sum \text{deg}_H(t)^2\) | \(-(2d - 3)|E(H)|\) | \(a(H)\) | \(b(H)\) | \(c(H)\) | \(\varepsilon(G) - \varepsilon(K_d)\) |
|-----|-----------------|-----------------|-------|-------|-------|-----------------|
| \(K_n\) | \(\frac{1}{2} n(n - 1)^2\) | \(\frac{-2d - 3}{2} n(n - 1)\) | 0 | 0 | 0 | \(-\frac{1}{2} n(n - 1)(2d - n - 2)\) |
| \(P_n(n \geq 2)\) | \(2n - 1\) | \(\frac{-2d - 3}{2} n\) | \(n - 2\) | 0 | \(n - 1)(d - 5) + 2\) | \(3 - (n + 1)(d - 1)\) |
| \(C_n(n \geq 5)\) | 8 | \(\frac{2d - 3}{2} n\) | 0 | 1 | 4(d - 4) | \(-4d + 6\) |

In particular, \(\varepsilon(G) - \varepsilon(K_d) < 0\) for all \(G\).

Next, we study graphs \(G\) such that the complement graph of \(G\) is not connected. The following proposition is important.

**Proposition 2.2.** Let \(G \in \Omega_d\) and let \(H_1, H_2, \ldots, H_m\) be nonempty connected components of \(\overline{G}\). Then,

\[
\varepsilon(K_d) - \varepsilon(G) = \sum_{i=1}^{m} (\varepsilon(K_d) - \varepsilon(K_d - H_i)).
\]

**Proof.** We remove \(H_1, H_2, \ldots, H_m\) one by one. Notice that we need to prove that removing order does not do any effects to how many edges of the edge polytope increase or decrease upon removing the other connected components. This is equivalent to, for any connected component \(H_k, H_l\) \((k \neq l)\) there are no vertex set \(V = \{i, j, k, l\}\) such that we gain a edge of the edge polytope on \(\{i, j, k, l\}\) upon removing both \(H_k\) and \(H_l\). We need to remove at least 2 of \(\{(i, j), (i, k), (i, l), (j, k), (j, l), (k, l)\}\) to exterminate all the 4-cycles on \(K_4\) with vertex set \(V\). We cannot choose 2 sets of 2 edges without contradicting that \(H_k\) and \(H_l\) are disconnected. □

**Example 2.3.** Let \(G\) be a graph with edge set \(E(G) = E(K_d) - \{e_1, e_2, \ldots, e_n\}\), where \(e_1, e_2, \ldots, e_n \in E(K_d)\) have no common vertex each other. In Example 2.1,
we have seen that \( \varepsilon(K_d) - \varepsilon(K_d - \{e_i\}) = 2(d - 2) \). Hence, by Proposition 2.2, we have \( \varepsilon(K_d) - \varepsilon(G) = \sum_{i=1}^n 2(d - 2) = 2n(d - 2) > 0 \).

We now give several necessary conditions for a graph \( G \in \Omega_d \) with \( \mu_d = \varepsilon(G) \).

**Proposition 2.4.** Let \( d \geq 15 \) and let \( G \in \Omega_d \) be a graph with \( \mu_d = \varepsilon(G) \). Then, \( G \) satisfies the following:

(i) \( G \) is connected;
(ii) \( G \) is not bipartite;
(iii) \( G \) is not a complete multipartite graph;
(iv) \( G \) has at least one cycle of length 4.

**Proof.** (i) Let \( G \) be a graph that is not connected and let \( G' \) be a graph such that \( G' = G \cup \{i, j\} \) where \( i \) and \( j \) belong to different connected components of \( G \). Then, \( \{i, j\} \) belongs to no cycle of \( G' \). Hence, by Lemma 1.1

- For any edge \( e, f \in E(G) \), \{\( \rho(e), \rho(f) \}\) is an edge of \( \mathcal{P}_{G'} \) if and only if \{\( \rho(e), \rho(f) \)\} is an edge of \( \mathcal{P}_G \);
- For any edge \( e \in E(G) \), \{\( \rho(e), \rho(\{i, j\}) \)\} is an edge of \( \mathcal{P}_{G'} \).

Thus, we have \( \varepsilon(G) + |E(G)| = \varepsilon(G') \).

(ii) Let \( G \) be a connected bipartite graph on the vertex set \( V_1 \cup V_2 \) with \( |V_1| \geq 2 \). Let \( G' \) be a nonbipartite graph such that \( E(G') = E(G) \cup \{i, j\} \) with \( i, j \in V_1 \). It is easy to see that there exists no even cycle of \( G' \) containing \( \{i, j\} \). Thus, by the same argument in (i), we have \( \varepsilon(G) + |E(G)| = \varepsilon(G') \).

(iii) By Proposition 1.8, \( G \neq K_d \). Let \( G (\neq K_d) \) be a complete multipartite graph. Then, every nonempty connected component of \( \overline{G} \) is a complete graph. Thus, applying Proposition 2.2 to Example 2.1, we have \( \varepsilon(K_d) - \varepsilon(G) \).

(iv) Let \( G \) be a connected graph having no cycle of length 4. It is known [9] that \( |E(G)| \leq d(1 + \sqrt{4d - 3})/4 \). It then follows that

\[
\varepsilon(G) \leq \frac{d(1 + \sqrt{4d - 3})/4}{2} - \frac{d(1 + \sqrt{4d})/4}{2} - \frac{d(1 + \sqrt{4d})/4}{2} - 1.
\]

Thus, for every \( d \geq 15 \), we have

\[
\varepsilon(K_d) - \varepsilon(G) > \frac{d(d - 1)(d - 2)}{2} - \frac{d(1 + \sqrt{4d})/4}{2} - 1
\]

\[
= \frac{d}{12} \left( 12d^2 - 4d\sqrt{d} - 49d + 8\sqrt{d} + 36 \right) > 0.
\]

Let \( H \in \Omega_r \). Recall that, by Proposition 1.3

\[
\varepsilon(K_d - H) - \varepsilon(K_d) = (\psi(H) - 2|E(H)|)(d - r) + \varepsilon(K_r - H) - \varepsilon(K_r).
\]

Hence, \( \varepsilon(K_d - H) - \varepsilon(K_d) \) is approximately \((\psi(H) - 2|E(H)|)d\) for \( d \gg r \). Thus, it is worth computing

\[
\nu_r = \max\{\psi(H) - 2|E(H)| : H \in \Omega_r \}.
\]
Proposition 2.5. Fix an integer $r \geq 2$. Then,

$$\nu_r = \max \left\{ 0, \frac{r - 6}{2} \left\lfloor \frac{r^2}{4} \right\rfloor \right\}.$$  

Moreover, $\psi(H) - 2|E(H)| = \nu_r$ if and only if

$$H = \begin{cases} 
\text{empty} & \text{if } r \leq 5, \\
\text{empty or complete bipartite} & \text{if } r = 6, \\
K_{s,s+1} & \text{if } r = 2s + 1 (3 \leq s \in \mathbb{Z}), \\
K_{s,s} & \text{if } r = 2s (4 \leq s \in \mathbb{Z}).
\end{cases}$$

Proof. If $H$ is empty, then $\psi(H) - 2|E(H)| = 0$. For complete bipartite graphs $H$, we computed $\psi(H) - 2|E(H)|$ in the proof of Proposition 1.4. For an edge $\{i, j\}$ of $H$, we set $k_3(i, j)$ be the number of triangles including $\{i, j\}$. Then,

$$\sum_{i=1}^{r} \text{deg}_H(i)^2 = \sum_{\{i,j\} \in E(H)} (\text{deg}_H(i) + \text{deg}_H(j)) \leq \sum_{\{i,j\} \in E(H)} (r + k_3(i, j)) = r|E(H)| + 3k_3(H).$$

Moreover, it is known [4] that $k_3(H) \geq \frac{4}{3r} |E(H)|^2 - \frac{r}{3} |E(H)|$. Hence, we have

$$\psi(H) - 2|E(H)| = \sum_{i=1}^{r} \left( \text{deg}_H(i) \right)^2 - 3k_3(H) - 2|E(H)|$$

$$= \frac{1}{2} \sum_{i=1}^{r} \text{deg}_H(i)^2 - 3k_3(H) - 3|E(H)|$$

$$\leq \frac{1}{2} (r - 6)|E(H)| - \frac{3}{2} k_3(H)$$

$$\leq \frac{1}{2} (r - 6)|E(H)| - \frac{2|E(H)|^2}{r} + \frac{r}{2} |E(H)|$$

$$= (r - 3)|E(H)| - \frac{2|E(H)|^2}{r}.$$

Case 1. Suppose that $r \leq 5$.

By (5), if $|E(H)| \neq 0$, then we have $\psi(H) - 2|E(H)| < 0$. Hence, $\nu_r = 0 = \psi(H) - 2|E(H)|$ where $H$ is an empty graph.

Case 2. Suppose that $r = 6$.

By (5), we have $\psi(H) - 2|E(H)| \leq 0$. Hence, $\nu_6 = 0$. Suppose that $\psi(H) - 2|E(H)| = 0$ and $H$ is not empty. Then, $k_3(H) = 0$ and $\sum_{i=1}^{6} \text{deg}_H(i)^2 = 6|E(H)|$.

By the above inequality on $\sum_{i=1}^{6} \text{deg}_H(i)^2$, we have $\text{deg}_H(i) + \text{deg}_H(j) = 6$ for an arbitrary edge $\{i, j\}$ of $H$. Suppose that $H$ is not bipartite. Then, $H$ has an odd cycle $C$ of length 5. Moreover, since $H$ has no triangle, $C$ has no chord. If $\{i, j\}$ is an edge of $C$, then there exists a vertex $k$ of $C$ such that neither $i$ nor $j$ is adjacent with $k$. It then follows that $\text{deg}_H(i) + \text{deg}_H(j) \leq 5$, which is a contradiction. Hence, $H$ is a bipartite graph. Since $\text{deg}_H(i) + \text{deg}_H(j) = 6$ holds for an arbitrary edge $\{i, j\}$ of $H$, it follows that $H$ is a complete bipartite graph.
Case 3. Suppose that \( r \geq 7 \).

Suppose that \( \frac{r^2}{4} + \frac{3}{r - 6}k_3(H) \leq |E(H)| \). Then, by (5),

\[
\left\lfloor \frac{r^2}{4} \right\rfloor + \frac{3}{r - 6}k_3(H) \leq |E(H)|.
\]

If \( |E(H)| \geq r^2/3 \), then we have

\[
(r - 3)|E(H)| - \frac{2|E(H)|^2}{r} \leq (r - 3)\frac{r^2}{3} - \frac{2r^3}{9} = \frac{r^2(r - 9)}{9} < \frac{r - 6}{2} \left\lfloor \frac{r^2}{4} \right\rfloor.
\]

This contradicts (6). Hence, we have

\[
\left\lfloor \frac{r^2}{4} \right\rfloor + \frac{3}{r - 6}k_3(H) \leq |E(H)| < \frac{r^2}{3}.
\]

Suppose that \( k_3(H) > 0 \). It follows that

\[
\left\lfloor \frac{r^2}{4} \right\rfloor + 1 \leq |E(H)| < \frac{r^2}{3}.
\]

Then, it is known [1] that \( k_3(H) \geq \frac{r}{9}(4|E(H)| - r^2) \). Thus,

\[
\left\lfloor \frac{r^2}{4} \right\rfloor + \frac{3}{r - 6} \cdot \frac{r}{9}(4|E(H)| - r^2) \leq |E(H)|.
\]

Therefore,

\[
|E(H)| \leq \frac{r^3 - 3(r - 6)\left\lfloor \frac{r^2}{4} \right\rfloor}{r + 18} = \left\lfloor \frac{r^2}{4} \right\rfloor + \frac{r}{r + 18} \left( r^2 - 4 \left\lfloor \frac{r^2}{4} \right\rfloor \right) < \left\lfloor \frac{r^2}{4} \right\rfloor + 1.
\]

This is a contradiction. Hence, \( k_3(H) = 0 \) and \( |E(H)| \geq \left\lfloor \frac{r^2}{4} \right\rfloor \). By Turán’s Theorem [11], \( H = K_{(r-1)/2,(r+1)/2} \) if \( r \) is odd and \( H = K_{r/2,r/2} \) if \( r \) is even.

**Corollary 2.6.** Let \( d \geq 15 \) and let \( G \in \Omega_d \) be a graph with \( \mu_d = \varepsilon(G) \). Then, any nonempty connected component \( H \) of \( G \) has at least \( 7 \) vertices (and hence at least \( 6 \) edges). In particular, \( |E(K_d)| - |E(G)| \geq 6 \).

**Proof.** Suppose that \( G \) has a nonempty connected component \( H \) having at most \( 6 \) vertices. Let \( r \) be the number of vertices of \( H \). By Proposition 1.3, we have

\[
\varepsilon(K_d - H) - \varepsilon(K_d) = (\psi(H) - 2|E(H)|)(d - r) + \varepsilon(K_r - H) - \varepsilon(K_r).
\]

Then, \( \psi(H) - 2|E(H)| \leq 0 \) by Proposition 2.5 and \( \varepsilon(K_r - H) - \varepsilon(K_r) < 0 \) by Proposition 1.7. Thus, \( \varepsilon(K_d - H) - \varepsilon(K_d) < 0 \). By Proposition 2.2, \( \varepsilon(G) < \varepsilon(G + H) \). This is a contradiction. \( \square \)

A complete bipartite graph of the form \( K_{s,s} \) or \( K_{s,s+1} \) in Proposition 2.5 is known as a Turán graph \( T(r, 2) \). Here, we present a remark on the function

\[
f(m, n) = \frac{1}{2}mn(m + n - 6)d - \frac{1}{4}mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13)
\]

where \( 1 \leq m, n \in \mathbb{Z} \) appearing in Proposition 1.4. The following proposition says that \( f(m, n) \) is maximum only if \( K_{m,n} \) is a Turán graph.
Proposition 2.7. Fix an integer $d \geq 15$. If $f(m_0, n_0) = \max \{ f(m, n) : 1 \leq m, n \in \mathbb{Z} \}$, then we have $|m_0 - n_0| \leq 1$.

Proof. Since $f(5, 5) = 50(d - 14) > 0$, $\max \{ f(m, n) : 1 \leq m, n \in \mathbb{Z} \} > 0$. Let $k = m_0 + n_0$. Then, $f(m_0, n_0) = \max \{ f(m, n) : 1 \leq m, n \in \mathbb{Z}, m + n = k \}$. If $m + n = k$, then

$$f(m, n) = \frac{1}{4}mn(2d(m + n) - 12d - 2(m + n)^2 + mn + 5(m + n) + 13)$$

$$= \frac{1}{4}\left(\frac{k^2}{4} - \left(\frac{m - k}{2}\right)^2\right)\left(2dk - 12d - \frac{7}{4}k^2 + 5k + 13 - \left(\frac{m - k}{2}\right)^2\right).$$

So, the integers $m_0$ and $n_0$ that maximize $f(m, n)$ are

$$(m_0, n_0) = \begin{cases} 
\left(\frac{k}{2}, \frac{k}{2}\right) & \text{if } k \text{ is even} \\
\left(\frac{k+1}{2}, \frac{k+1}{2}\right) & \text{if } k \text{ is odd}
\end{cases}$$

as desired. □

By Propositions 2.2 and 2.5, the following conjecture seems to be reasonable.

Conjecture 2.8. Let $G \in \Omega_d$ with $\mu_d = \varepsilon(G)$. Then the complementary graph $\overline{G}$ of $G$ consists of complete bipartite Turán graphs and isolated vertices.

Example 2.9. Let $\Omega'_{15}$ denote the set of all graphs on 15 vertices whose complement consists of complete bipartite graphs and isolated vertices. Let $G \in \Omega'_{15}$ be a graph such that $\varepsilon(G) = \max \{ \varepsilon(G') : G' \in \Omega'_{15} \}$. By Proposition 1.4, $\varepsilon(K_{15} - K_{m,n}) - \varepsilon(K_{15}) = \frac{15}{2}mn(m + n - 6) - \frac{1}{4}mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13)$. Then, $\varepsilon(K_{15} - K_{m,n}) - \varepsilon(K_{15}) \leq 0$ if $m + n < 8$. By Proposition 2.2, $\overline{G}$ has exactly one nonempty connected component. Moreover, $\varepsilon(K_{15} - K_{m,n}) - \varepsilon(K_{15})$ has the maximum value 50 when $m = n = 5$. Hence, $G = K_{15} - K_{5,5}$ and $\varepsilon(G) = 1415$.

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Takayuki Hibi, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: hibi@math.sci.osaka-u.ac.jp

Aki Mori, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: a-mori@cr.math.sci.osaka-u.ac.jp

Hidefumi Ohsugi, Department of Mathematics, College of Science, Rikkyo University, Toshima-ku, Tokyo 171-8501, Japan
E-mail address: ohsugi@rikkyo.ac.jp

Akihiro Shikama, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: a-shikama@cr.math.sci.osaka-u.ac.jp