The Matrix Model Curve Near the Singularities

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Abstract

In $\mathcal{N} = 1$ supersymmetric $SO(N)/USp(2N)$ gauge theories with the tree-level superpotential $W(\Phi)$ that is an arbitrary polynomial of the adjoint matter $\Phi$, the massless fluctuations about each quantum vacuum are described by $U(1)^n$ gauge theory. By turning on the parameters of $W(\Phi)$ to the special values, the singular vacua where the additional fields become massless can be reached. Using the matrix model prescription, we study the intersections of $n = 0$ and $n = 1$ branches. The general formula for the matrix model curve at the singularity which is valid for arbitrary $N$ is obtained and this generalizes the previous results for small values of $N$ from strong-coupling approach. Applying the analysis to the degenerated case, we also obtain a general matrix model curve which is not only valid at a special point but also on the whole branch.
1 Introduction

A new recipe for the computation of the exact quantum effective superpotential for the glueball field was proposed by Dijkgraaf and Vafa [1, 2, 3] using a zero-dimensional matrix model. Extremization of the effective glueball superpotential has led to the quantum vacua of the supersymmetric gauge theory. For $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with the adjoint matter $\Phi$, the gauge group $U(N)$ breaks into $\prod_{i=1}^{n} U(N_i)$ for some $n$. At low energies, the effective theory becomes $\mathcal{N} = 1$ gauge theory with gauge group $U(1)^n$. The low energy dynamics have been studied in [4, 5, 6, 7, 8, 9, 10].

Recently in [11], the matrix model curve for $U(N)$ gauge theory was obtained through the glueball approach. For example, the intersections of $n = 1$ and $n = 2$ branches for cubic superpotential occur when a vacuum with gauge group $U(N_1) \times U(N_2)$ meets a vacuum with gauge group $U(N)$ where $N = N_1 + N_2$. The general formulas for the parameters of tree-level superpotential are functions of $N_1$ and $N_2$. The locations of the $n = 1$ and $n = 2$ singularities and the expectation value of glueball field at the singularities were obtained in [5] from the $\mathcal{N} = 2$ factorization problem using the strong-coupling approach [12, 13]: for small values of $N$ it was possible to solve explicitly. However, this general factorization problem will meet some difficulty as the $\mathcal{N}$ increases. The outcome of [11] allows us to write down the quantum vacua at the singularity for general $(N_1, N_2)$ which will generalize [5].

On the other hand, the $\mathcal{N} = 1$ matrix model curve for $SO(N)/USp(2N)$ gauge theories with the tree-level superpotential $W(\Phi)$ is characterized by

$$g_m^2 = F_{2(n+1)}(x) = W'_{2n+1}(x)^2 + \mathcal{O}(x^{2n})$$

where the tree-level superpotential is given by

$$W_{2(n+1)}(\Phi) = \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} \text{Tr} \Phi^{2r}. \quad (1.1)$$

In [14, 15, 16], the explicit constructions for the matrix model curve using the factorization problem were obtained for small values of $N$, when we consider the quartic superpotential ($n = 1$), in the strong-coupling approach. Here the matrix model curve possesses an arbitrary parameter. We expect there is a chance to have an extra double root by restricting ourselves to the particular value for superpotential parameter. The singularity arises from an additional monopole becoming massless in the strong-coupling description. We apply the method of [11] to the $SO(N)/USp(2N)$ gauge theories.

In this paper, we study how the generic picture can be changed at the strong-coupling singularities where the additional fields become massless and the presence of extra massless fields will lead to an interacting superconformal field theory. These singularities can be obtained by
turning on the parameters of $W(\Phi)$ to the particular values. By solving the glueball equations of motion at the $n = 0$ and $n = 1$ singularity, one gets a general formula for the parameter and fluctuating fields at these singularities. Our general formula extends the results of [14, 15, 16] to provide an information on the $\mathcal{N} = 1$ matrix model curve for arbitrary $(N_0, N_1)$.

In section 2.1, in order to find out the glueball equations of motion for given effective superpotential, we compute the derivatives of dual periods with respect to the fluctuating fields explicitly. This will lead to the solutions for the two kinds of parametrization of the matrix model curve (2.14) in terms of $N_0$, $N_1$, and the scale $\Lambda$ of $SO(N)$ gauge theory. Based on this general formula, we compare our results with the matrix model curve from previous results by further restricting some parameter of superpotential and we find an exact agreement. In section 2.2, the coupling constant at the singularity goes to vanish as we compute the derivative of dual period with respect to the glueball field.

In section 3.1, based on the general formula for the matrix model curve of $USp(2N)$ gauge theory (3.2), we compare our results with the matrix model curve from strong-coupling approach and we find an exact agreement. In section 3.2, the coupling constant corresponding to the gauge coupling constant of the nontrivial $U(1)$ at the singularity goes to vanish.

In section 4, we apply the method of section 2.1 to the degenerated case for $SO(N)$ gauge theory in which the matrix model curve (4.6) is parametrized by two variables with one constraint. In this case, we also find an exact agreement from strong-coupling approach.

In Appendices A and B, we present some detailed computations which are necessary to sections 2 and 3.

2 The $n = 0$ and $n = 1$ singularity: $SO(N)$ gauge theory

2.1 The glueball equations of motion

Let us consider the intersections of the $n = 0$ and $n = 1$ branches. These occur at special values of the tree-level superpotential parameter and a vacuum with unbroken gauge group $^{1} SO(N_0) \times U(N_1)$ intersects a vacuum with unbroken gauge group $SO(N)$ with

$$N = N_0 + 2N_1.$$ 

$^{1}$In several examples we present below, the unbroken gauge group contains $U(1)$ factor. We also consider the glueball superfield for the $U(1)$ gauge group and extremize the corresponding glueball field. Recently the issue when glueball superfields should be included and extremized or set to zero has been studied in [17]. According to this general prescription for how string theory deals with low rank gauge groups including $U(1)$ group in the geometric dual description, the generalized dual Coxeter number for $U(1)$ is 1 which is positive and one should include the corresponding glueball superfield and extremize the glueball superpotential with respect to it. Therefore, the theory has a dual confining description since the string theory computes not for the standard gauge theory but the associated higher rank gauge theory [17].
Although the structure of these singularities has been discussed in [14] implicitly by applying the strong-coupling approach, in this paper we study these intersection singularities in detail using the glueball description [11]. We will describe the approach to the singularity from the \( n = 1 \) branch since the approach from the \( n = 0 \) branch generally behaves without any singularity: the matrix model curve on the \( n = 0 \) branch is regular as we go through the intersection with the \( n = 1 \) branch.

Let us take the tree-level superpotential to be quartic (1.1). We expect that the general feature of the analysis for this particular superpotential holds for the general superpotential of arbitrary degree \( 2(n + 1) \). As we approach the \( n = 0 \) and \( n = 1 \) singularity, both the matrix model curve and the SW curve possess an extra double root. Using the matrix model curve, one can compute the effective glueball superpotential and \( U(1) \) gauge coupling near the singularity.

The matrix model curve is given by [18, 19]

\[
y_m^2 = W'_3(x)^2 + f_2 x^2 + f_0 = \left( x^2 + x_0^2 \right) \left( x^2 + x_1^2 \right) \left( x^2 + x_2^2 \right).
\]  

(2.1)

Here we assume that all three branch cuts \([-ix_2, -ix_1], [-ix_0, ix_0], \) and \([ix_1, ix_2]\) are along the imaginary axis and the contour of noncompact cycle \( B_0 \) as from the origin to the cut-off \( \Lambda_0 \) is along the real axis. The compact cycles \( A_i \) have to intersect the noncompact cycles \( B_i \) as \((A_i, B_j) = \delta_{ij} \) where \( i = 0, 1 \).

If we parametrize the tree-level superpotential as

\[
W'_3(x) = x^3 + mx,
\]

we obtain the following relation,

\[
m = \frac{1}{2} \left( x_0^2 + x_1^2 + x_2^2 \right).
\]  

(2.2)

The first parametrization of the matrix model curve (2.1) implies that \( m \) is a parameter and \( f_2 \) and \( f_0 \) are fluctuating fields that are related to the two glueball fields \( S_1 \) and \( S_0 \) respectively. The second parametrization in terms of the roots \( \pm ix_0, \pm ix_1 \) and \( \pm ix_2 \) will be more convenient and these fields are subject to the constraint (2.2). One can always interchange from one parametrization to the other through the matrix model curve (2.1).

We evaluate the derivatives of the dual periods of the matrix model curve on the \( n = 1 \) branch, with quartic tree-level superpotential. The periods \( S_i \) of holomorphic 3-form for the deformed geometry over compact \( A_i \) cycles and dual periods \( \Pi_i \) of holomorphic 3-form over noncompact \( B_i \) cycles are written in terms of the integrals over \( x \)-plane \((i = 0, 1)\). The periods
are the glueball fields $S_i$ [20, 19]:

\[ 2\pi i S_0 = \int_{-i\Delta_0}^{i\Delta_0} y_m \, dx = \int_{-i\Delta_0}^{i\Delta_0} \sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)} \, dx, \]

\[ 2\pi i S_1 = \int_{i\Delta_1}^{i\Delta_2} y_m \, dx = \int_{i\Delta_1}^{i\Delta_2} \sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)} \, dx, \]

and their conjugate periods:

\[ 2\pi i \Pi_0 = \int_0^{\Lambda_0} y_m \, dx = \int_0^{\Lambda_0} \sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)} \, dx, \]

\[ 2\pi i \Pi_1 = \int_{i\Lambda_1}^{i\Lambda_2} y_m \, dx = \int_{i\Lambda_1}^{i\Lambda_2} \sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)} \, dx. \]

These periods provide the effective glueball superpotential [18] (See also [19, 20, 21, 22, 23, 24]):

\[ W_{\text{eff}} = 2\pi i [(N_0 - 2) \Pi_0 + 2N_1 \Pi_1] - 2(N - 2) S \log \left( \frac{\Lambda}{\Lambda_0} \right). \]  

(2.3)

One can generalize this to add the $b_1 S_1$ term but as in [11] after the trivial calculation $b_1$ must be zero at the singularity (according to the computation of Appendix B, $\partial S_1/\partial f_0$ and $\partial S_1/\partial f_2$ are divergent at the singularity and therefore in order to have consistent equations of motion $b_1$ should vanish) and $S = S_0 + 2S_1$. Over a cycle $A_i$ surrounding the $i$-th cut, the periods of $T(x)$ are [14]

\[ N_0 = \frac{1}{2\pi i} \oint_{A_0} T(x) dx, \quad N_1 = \frac{1}{2\pi i} \oint_{A_1} T(x) dx \]

(2.4)

where

\[ T(x) = \frac{d}{dx} \log \left( P_N(x) + \sqrt{P_N^2(x) - 4\epsilon x^{2(1+\epsilon)}\Lambda^{2N-2(1+\epsilon)}} \right), \quad P_N(x) = \det (x - \Phi) \]

with $\epsilon = 0$ for $N$ odd, and $\epsilon = 1$ for $N$ even. Let us stress that $N_0$ and $N_1$ (they are always integers) are defined as the on-shell periods of the one-form $T(x)$ at the $n = 0$ and $n = 1$ singularity. As we will see below, the precise values of $(N_0, N_1)$ can be determined through (2.4) with an appropriate choice of cycles.

One can represent a general formula for the derivatives of the effective superpotential, which holds for the general $(N_0, N_1)$ with the help of Appendix A. As we mentioned before, $x_i (i = 0, 1, 2)$ lie on the imaginary line and $x_0 < x_1 < x_2$ which allows us to compute the elliptic integrals without any ambiguities. In principle, this formula provides the solutions for the $x_i$

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2In [20], the period $S_0$ was an integral over $x$-plane from $-\Delta_0$ to $\Delta_0$. In our notation here if we replace $x_0 \to -i\Delta_0$, the $S_i$'s agree with those in [20]. For the dual period $\Pi_0$, since there is no singularity on the Riemann surface $y_m(x)$, we can change the lower limit smoothly on the branch cut as $0 \to \Delta_0$. Taking into account this we can see the agreement with [20].

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in terms of the parameters $N_0, N_1$, and $m$. Moreover, according to the matching conditions (2.1) between the two parametrizations, this will lead to the expectation values of the fields $f_2$ and $f_0$ eventually. Although our general formula is valid not just near the $n = 0$ and $n = 1$ singularity, we are interested in the solutions near the $n = 0$ and $n = 1$ singularity 3.

To compute the expectation values on the $n = 0$ and $n = 1$ singularity, we can simply take a limit $x_1 \to x_0$. Or one can compute those by starting from the general formula and then substituting the condition $x_1 = x_0$ at the final stage. We will elaborate this in Appendix A.

- The equation of motion for a field $f_0$

The derivative of $\Pi_0$ with respect to $f_0$ is given, by recognizing $f_0 = x_0^4x_2^2$, the $x$-independent part inside of the square root in the dual periods, as

$$4\pi i \frac{\partial \Pi_0}{\partial f_0} = \int_0^{\Lambda_0^2} \frac{dt}{2(t + x_0^2)\sqrt{t(t + x_0^2)}}$$

$$= \frac{1}{2x_0\sqrt{x_0^2 - x_0^2}} \left[ \sin^{-1}\left( \frac{(x_0^2 - 2x_0^2)(t + x_0^2) + 2x_0^2(x_0^2 - x_0^2)}{x_0^2(t + x_0^2)} \right) \right]_0^{\Lambda_0^2}$$

$$\approx \frac{1}{2x_0\sqrt{x_0^2 - x_0^2}} \left[ \sin^{-1}\left( \frac{x_0^2 - 2x_0^2}{x_0^2} \right) + \frac{\pi}{2} \right].$$

(2.5)

where we change the integration variable as $t = x^2$ and in the last expression we take $\Lambda_0$ very large.

Similarly, one can execute the integral (we change a variable $t = -x^2$)

$$4\pi i \frac{\partial \Pi_1}{\partial f_0} = -\int_{x_2^2}^{\Lambda_0^2} \frac{dt}{2(t - x_0^2)\sqrt{t(x_0^2 - x_0^2)}}$$

$$= -\frac{1}{2x_0\sqrt{x_0^2 - x_0^2}} \left[ \sin^{-1}\left( \frac{(-x_0^2 + 2x_0^2)(t - x_0^2) + 2x_0^2(x_0^2 - x_0^2)}{x_0^2(t - x_0^2)} \right) \right]_{x_2^2}^{\Lambda_0^2}$$

$$\approx -\frac{1}{2x_0\sqrt{x_0^2 - x_0^2}} \left[ \sin^{-1}\left( \frac{-x_0^2 + 2x_0^2}{x_0^2} \right) + \frac{\pi}{2} \right].$$

(2.6)

We also drop the irrelevant terms in the last equation as we take the large limit of $\Lambda_0$.

By using these two results (2.5) and (2.6) and taking $\partial W_{\text{eff}}/\partial f_0 = 0$ ($W_{\text{eff}}$ is given by (2.3)), we obtain one equation of motion for $f_0$ after manipulating the trigonometric functions:

$$(N_0 - 2) \frac{\partial \Pi_0}{\partial f_0} + 2N_1 \frac{\partial \Pi_1}{\partial f_0} = 0 \iff -\left( \frac{x_0^2 - 2x_0^2}{x_0^2} \right) = \cos \left( \frac{2\pi N_1}{N - 2} \right).$$

(2.7)

- The equation of motion for a field $f_2$

3Other kind of singularities in different context may arise. For example, $\mathcal{N} = 1$ Argyres-Douglas (AD) points. See the paper [25]. It would be interesting to study the effective superpotential in both the glueball and the strong-coupling approach [26].
The derivative of $\Pi_0$ with respect to $f_2$ is given, by recognizing the $x^2$ part inside of the square root in the dual periods, as (we change a variable $t = x^2$)

$$4\pi i \frac{\partial \Pi_0}{\partial f_2} = \int_0^\Lambda_0^2 \frac{\sqrt{t}}{2(t + x_0^2) \sqrt{t + x_2^2}} \, dt = \left[ \frac{1}{2} \log \left( 2t + x_0^2 + 2\sqrt{t(t + x_2^2)} \right) \right]_{x_0^2}^{\Lambda_0^2} - 4\pi i x_0^2 \frac{\partial \Pi_0}{\partial f_2}$$

$$\simeq \frac{1}{2} \log \left| \frac{4\Lambda_0^2}{x_2^2} \right| - 4\pi i x_0^2 \frac{\partial \Pi_0}{\partial f_0}. \quad (2.8)$$

Note that the second term was given by (2.5) multiplied by $x_0^2$. Moreover one obtains by using a change of variable $t = -x^2$

$$4\pi i \frac{\partial \Pi_1}{\partial f_2} = \int_{x_2^2}^{\Lambda_0^2} \frac{t}{2(t + x_0^2) \sqrt{t - x_2^2}} \, dt = \left[ \frac{1}{2} \log \left( 2t - x_0^2 + 2\sqrt{t(t - x_2^2)} \right) \right]_{x_2^2}^{\Lambda_0^2} - 4\pi i x_0^2 \frac{\partial \Pi_1}{\partial f_0}$$

$$\simeq \frac{1}{2} \log \left| \frac{4\Lambda_0^2}{x_2^2} \right| - 4\pi i x_0^2 \frac{\partial \Pi_1}{\partial f_0} \quad (2.9)$$

where the second term is given by (2.6) multiplied by $x_0^2$.

By using these two results (2.8) and (2.9) and taking $\partial W_{\text{eff}}/\partial f_2 = 0$, we obtain one equation of motion for $f_2$ by cooperating with the equation of motion for $f_0$ (2.7):

$$(N_0 - 2) \frac{\partial \Pi_0}{\partial f_2} + 2N_1 \frac{\partial \Pi_1}{\partial f_2} + (N - 2) \log \left( \frac{\Lambda}{\Lambda_0} \right) = 0 \iff \frac{1}{2} (N - 2) \log \left| \frac{x_2^2}{4\Lambda^2} \right| = 0 \quad (2.10)$$

which implies the $(N - 2)$ branches labeled by $\eta$. That is, $\eta$ is the $(N - 2)$-th root of unity for $N$ even and the $(N - 2)$-th root of minus unity. That is,

$$\eta^{N-2} = 1, \text{ for } N \text{ even, } \eta^{N-2} = -1 \text{ for } N \text{ odd.}$$

From the two solutions (2.7) and (2.10), we obtain the following results together with (2.2),

$$x_2^2 = 4\eta \Lambda^2, \quad x_0^2 = 2\eta \Lambda^2 \left( 1 + \cos \frac{2\pi N_1}{N - 2} \right), \quad m = 2\eta \Lambda^2 \left( 2 + \cos \frac{2\pi N_1}{N - 2} \right). \quad (2.11)$$

Matching the two parametrizations in (2.1), we obtain the expectation values of the fields $f_2$ and $f_0$ at the double root singularity:

$$\langle f_2 \rangle = 4\eta^2 \Lambda^4 \left( 1 + 2 \cos \frac{2\pi N_1}{N - 2} \right),$$

$$\langle f_0 \rangle = 16\eta^3 \Lambda^6 \left( 1 + \cos \frac{2\pi N_1}{N - 2} \right)^2. \quad (2.12)$$

Since the total glueball field is related to $f_2$ through $S = -\frac{f_2}{4}$, we also obtain the expectation value of the glueball field at the $n = 0$ and $n = 1$ singularity:

$$\langle S \rangle = -\eta^2 \Lambda^4 \left( 1 + 2 \cos \frac{2\pi N_1}{N - 2} \right). \quad (2.13)$$
The general formulas (2.11), (2.12) and (2.13) for the matrix model curve, the parameter of the tree level superpotential and the expectation value of the glueball field are new. Previously, the locations of \( n = 0 \) and \( n = 1 \) singularities and the expectation value of the glueball field can be obtained only for small numbers of \( N \) where the factorization problem could be solved explicitly [14] by restricting the superpotential parameter further. The difficulty of the solving the general factorization problem when \( N \) is large can be avoided by looking at both the glueball equations of motion at the \( n = 0 \) and \( n = 1 \) singularity [11] and the quantum vacua at the singularity for general \((N_0,N_1)\). Now the matrix model curve can be summarized as

\[
y_m^2 = x^2 \left[ x^2 + 2\eta\Lambda^2(2 + c) \right]^2 + 4\eta^2\Lambda^4 (1 + 2c) x^2 + 16\eta^3\Lambda^6 (1 + c)^2
\]

\[
= \left[ x^2 + 2\eta\Lambda^2(1 + c) \right]^2 \left( x^2 + 4\eta\Lambda^2 \right), \quad c \equiv \cos \left( \frac{2\pi N_1}{N - 2} \right) \tag{2.14}
\]

where \( N_0 \) and \( N_1 \) are given in (2.4) together with \( N = N_0 + 2N_1 \).

We can explicitly demonstrate this general result by comparing them with the results obtained in [14]. Let us consider \( SO(N) \) case where \( N = 4, 5, 6, 7, \) and 8.

- **\( SO(4) \)**

For \( SO(4) \) case, we only consider the breaking pattern \( SO(4) \to SO(2) \times U(1) \). In this case from the solutions (2.11), (2.12) and (2.13), one predicts the matrix model curve, by putting \((N, N_1) = (4, 1)\) (note that \( N = N_0 + 2N_1 \), in terms of two parametrizations

\[
y_m^2 = x^2 \left( x^2 + m \right)^2 - 4\Lambda^4 x^2 = x^4 \left( x^2 + 4\eta\Lambda^2 \right)
\]

with \( m = 2\eta\Lambda^2 \) where \( \eta \) is 2-nd root of unity and also we find \( \langle S \rangle = \Lambda^4 \). In [14], the factorization problem resulted in the matrix model curve \( \tilde{y}_m^2 = x^2 (x^2 - v^2)^2 - 4\Lambda^4 x^2 \) and one can easily check that the intersections with the \( n = 0 \) branch occur at \( v^2 = -2\eta\Lambda^2 \) where \( \tilde{y}_m^2 \) has an additional double root because the \( x^2 \) term in \( \tilde{y}_m^2 \) vanishes and there exists an overall factor \( x^4 \). Therefore, we have \( y_m^2 = \tilde{y}_m^2 \). These subspaces are also on the unbroken \( SO(4) \) branch corresponding to \( n = 0 \). At these points, the characteristic function \( P_4(x) = x^2 (x^2 + 2\eta\Lambda^2) \) is equal to \( 2\rho^2 x^2 \Lambda^2 T_2 \left( \frac{\rho}{2\rho} \right) \) with \( \rho^4 = 1 \), by identifying \( \eta = -\rho^2 \) (In other words, these are the vacua that survive when the \( N = 2 \) theory is perturbed by a quadratic superpotential \((n = 0)\) and the \( SO(4) \) gauge theory becomes massive at low energies) \(^4\). Therefore, this is the agreement with the glueball approach here exactly.

- **\( SO(5) \)**

\(^4\)We explicitly write some of the first Chebyshev polynomials \( T_l(x) \) where \( l = 1, 2, \ldots, 6 \) as follows:

\[
T_1(x) = x, \quad T_2(x) = 2x^2 - 1,
\]

\[
T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1,
\]

\[
T_5(x) = 16x^5 - 20x^3 + 5x, \quad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.
\]
In $SO(5)$ case, there is a breaking pattern $(N_0, N_1) = (3, 1)$. In this case, one predicts the matrix model curve, by putting $(N, N_1) = (5, 1)$ (note that $N = N_0 + 2N_1$),

$$y_m^2 = x^2 \left( x^2 + m \right)^2 - 4\Lambda^6 = \left( x^2 + \eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)$$

with $m = 3\eta \Lambda^2$ where $\eta^3 = -1$. For $SO(2M + 1)$ case, we take a minus sign inside the log (2.10). That is, $\eta^{N-2} = -1$ for $N$ odd. We find $\langle S \rangle = 0$. In [14], the factorization problem resulted in the matrix model curve $\tilde{y}_m^2 = x^2 \left( x^2 - l^2 \right)^2 - 4\Lambda^6$ and one can easily check that the intersections with the $n = 0$ branch occur at $l^2 = -3\eta \Lambda^2$ where the additional double root appears in $\tilde{y}_m^2$. At these points, the characteristic function $P_4(x) = x^2 \left( x^2 + 3\eta \Lambda^2 \right)$ can be written as $2\rho^3 x \Lambda^3 T_3 \left( \frac{\rho}{2\rho^2} \right)$ with $\rho^6 = 1$, by identifying $\eta = -\rho^2$. Therefore, this is the agreement with the glueball approach here. The $\tilde{f}_0$ is equal to $\langle f_0 \rangle$.

- $SO(6)$

For $SO(6)$ case, we consider two breaking patterns characterized by $(N_0, N_1) = (2, 2)$ and $(4, 1)$. In this case from the relations (2.11), (2.12) and (2.13), one predicts the matrix model curve, by putting $(N_0, N_1) = (2, 2)$,

$$y_m^2 = x^2 \left( x^2 + m \right)^2 - 4\eta^2 \Lambda^4 x^2 = x^4 \left( x^2 + 4\eta \Lambda^2 \right)$$

with $m = 2\eta \Lambda^2$ and $\langle S \rangle = \eta^2 \Lambda^4$, where $\eta$ is 4-th root of unity. From the results in [14], the value of glueball is given as $\tilde{S} = -\epsilon^2 \Lambda^4$, where $\epsilon$ is 4-th root of unity. This is the agreement with the glueball approach here by identifying $\epsilon^2 = -\eta^2$ precisely. The factorization problem [14] resulted in the matrix model curve $\tilde{y}_m^2 = x^2 \left( x^2 - a^2 + \epsilon \sqrt{\Lambda}^2 \right)^2 + 4\epsilon^2 \Lambda^4 \right)$. There were some typos in [14]. The intersections with the $n = 0$ branch occur at $a^2 = \epsilon \sqrt{\Lambda}^2 - 2\eta \Lambda^2$ where $\tilde{y}_m^2$ has two double roots at $x = 0$. At these points, the characteristic function $P_6(x) = x^2 \left( x^2 - \epsilon \sqrt{\Lambda}^2 + 2\eta \Lambda^2 \right) \left( x^2 + \epsilon \sqrt{\Lambda}^2 + 2\eta \Lambda^2 \right)$ is equal to $2\epsilon^2 x^2 \Lambda^4 T_2 \left( \frac{P_4(x)}{2x^2 + \Lambda} \right)$ with $\epsilon^4 = 1$ where $P_4(x) = x^2 \left( x^2 + 2\eta \Lambda^2 \right)$. This branch was constructed by multiplication map by $K = 2$ of $P_4(x)$. How do we know this is the solution of $n = 0$ branch? One can write down $P_6(x) - 2\eta \Lambda^4 x^2 = x^2 \left( x^2 + 2\eta \Lambda^2 \right)^2$ and $P_6(x) + 2\eta \Lambda^4 x^2 = x^2 \left( x^2 + 2\eta \Lambda^2 \right)^2 - 4\eta \Lambda^4 x^2$. Then the first branch has a single double root and the second branch can be written as $x^4 \left( x^2 + 4\eta \Lambda^2 \right)$ which has an extra double root. Therefore, these points are on the branch with $n = 0$ and unbroken $SO(6)$. These are the vacua that survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic superpotential $(n = 0)$ and the $SO(6)$ gauge theory becomes massive at low energies.

On the other hand, for other breaking pattern, one predicts the matrix model curve, by putting $(N_0, N_1) = (4, 1)$,

$$y_m^2 = x^2 \left( x^2 + m \right)^2 + 4\eta^2 \Lambda^4 x^2 + 16\eta^3 \Lambda^6 = \left( x^2 + 2\eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)$$

with $m = 4\eta \Lambda^2$ and $\langle S \rangle = -\eta \Lambda^4$, where $\eta$ is 4-th root of unity. From the results in [14], the value of glueball is given as $\tilde{S} = -\epsilon \Lambda^4$, where $\epsilon$ is 2-nd root of unity. This is the agreement with
the glueball approach here by identifying with $\epsilon = \eta^2$. The factorization problem [14] resulted in the matrix model curve $\widetilde{y}_m^2 = x^2 (x^2 + A)^2 + 4\epsilon \Lambda^4 (x^2 + A)$. The intersections with the $n = 0$ branch occur at $A = 4\eta \Lambda^2$ where $\widetilde{y}_m^2$ has an additional double root because $\widetilde{y}_m^2$ contains $(x^2 + 2\eta \Lambda^2)^2$. At these points, the characteristic function $P_6(x) = x^2 (x^4 + 4\eta \Lambda^2 x^2 + 2\eta^2 \Lambda^4)$ is equal to $2\rho^4 x^2 \Lambda^4 \mathcal{I}_4 \left( \frac{x}{2\rho \Lambda} \right)$ with $\rho^8 = 1$, by identifying $\eta = -\rho^2$. These are the vacua that survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic superpotential ($n = 0$) and the $SO(6)$ gauge theory becomes massive at low energies.

• $SO(7)$

As in [14], we consider two breaking patterns described by $(N_0, N_1) = (3, 2)$ and $(5, 1)$. For $(N_0, N_1) = (3, 2)$, the matrix model curve, by putting $(N_0, N_1) = (3, 2)$, implies

$$y_m^2 = x^2 \left( x^2 + m \right)^2 + 2\eta^2 \left( 1 - \frac{\sqrt{5}}{2} \right) \Lambda^4 x^2 + 2\eta^3 \left( 3 - 3\sqrt{5} \right) \Lambda^6$$

with $m = \frac{(7 - \sqrt{5}) \eta \Lambda^2}{2}$ and $\eta^5 = -1$. We also find $\langle S \rangle = -\frac{1}{2} \eta^2 \left( 1 - \frac{\sqrt{5}}{2} \right) \Lambda^4$. In [14], the factorization problem turned out the matrix model curve $\widetilde{y}_m^2 = x^2 \left( x^2 - A + \frac{\epsilon \Lambda^2}{2A} \right)^2 - \frac{4\epsilon^2 \Lambda^4}{A^2} x^2 - \frac{4\Lambda^{10}}{\Lambda^2}$ with $\epsilon^4 = 1$. There were some typos in [14]. The intersections with the $n = 0$ branch occur at $A^2 = \frac{74 + 3\sqrt{5}}{2} \Lambda^4 \eta^2$ where the additional extra double root appears in $\widetilde{y}_m^2$ with $\epsilon^{4/5} = -\eta$. At these points, the characteristic function $P_6(x) = x^2 (x^2 - a) (x^2 - a - b)$ given in [14] is written as $2\rho^5 x \Lambda^5 \mathcal{I}_5 \left( \frac{x}{2\rho \Lambda} \right)$ with $\rho^{10} = 1$, by identifying $\eta = -\rho^2$. These are the vacua that survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic superpotential ($n = 0$).

For $(N_0, N_1) = (5, 1)$, the matrix model curve, by putting $(N_0, N_1) = (5, 1)$, implies

$$y_m^2 = x^2 \left( x^2 + m \right)^2 + 2\eta^2 \left( 1 + \frac{\sqrt{5}}{2} \right) \Lambda^4 x^2 + 2\eta^3 \left( 7 + 3\sqrt{5} \right) \Lambda^6$$

with $m = \frac{(7 + \sqrt{5}) \eta \Lambda^2}{2}$. The intersections with the $n = 0$ branch occur at $A^2 = \frac{74 - 3\sqrt{5}}{2} \Lambda^4 \eta^2$ where the additional extra double root appears in $\widetilde{y}_m^2$. At these points, the characteristic function $P_6(x)$ is equal to $2\rho^5 x \Lambda^5 \mathcal{I}_5 \left( \frac{x}{2\rho \Lambda} \right)$ with $\rho^{10} = 1$, by identifying $\eta = -\rho^2$.

• $SO(8)$

Next we move to $SO(8)$ case. As in [14], we consider two breaking patterns described by $(N_0, N_1) = (4, 2)$ and $(6, 1)$. For $(N_0, N_1) = (4, 2)$, the matrix model curve, by putting $(N_0, N_1) = (2, 2)$, implies

$$y_m^2 = x^2 \left( x^2 + m \right)^2 + 4\eta^3 \Lambda^6 = \left( x^2 + \eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)$$

9
with \( m = 3\eta \Lambda^2 \) and \( \langle S \rangle = 0 \), where \( \eta \) is 6-th root of unity. On the confining branch, the value of \( \tilde{S} \) is zero and \( \tilde{f}_0 = 4\epsilon \Lambda^6 \) where \( \epsilon = \pm 1 \). This is the agreement with the glueball approach here by putting \( \epsilon = \eta^3 \). The factorization problem \([14]\) resulted in the matrix model curve \( \tilde{y}_m^2 = x^2(x^2 - a^2)^2 + 4\epsilon \Lambda^6 \). The intersections with the \( n = 0 \) branch occur at \( a^2 = -3\eta \Lambda^2 \) where \( \tilde{y}_m^2 \) has an additional double root because \( \tilde{y}_m^2 \) contains \( \left( x^2 + \epsilon^{1/3} \Lambda^2 \right)^2 \). At these points, the characteristic function \( P_8(x) = x^4(x^2 + 3\eta \Lambda^2)^2 + 2\eta^6 \Lambda^6 x^2 \) can be written as \( 2\rho^6 x^2 \Lambda^6 \mathcal{T}_0 \left( \frac{x}{2\rho^6} \right) \) with \( \rho^{12} = 1 \), by identifying \( \eta = -\rho^2 \). These are the vacua that survive when the \( N = 2 \) theory is perturbed by a quadratic superpotential \( (n = 0) \) and the \( SO(8) \) gauge theory becomes massive at low energies.

On the other hand, on the Coulomb branch \( \tilde{S} \) is parametrized by \( a \). Let us consider the breaking pattern \( SO(8) \to SO(6) \times U(1) \) where we have \( (N, N_1) = (8, 1) \). One can write down the matrix model curve as

\[
y_m^2 = x^2 \left( x^2 + m \right)^2 + 8\eta^2 \Lambda^4 x^2 + 36\eta^6 \Lambda^6 = \left( x^2 + 3\eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)
\]

with \( m = 5\eta \Lambda^2 \) where \( \eta^6 = 1 \) and \( \langle S \rangle = -2\eta^2 \Lambda^4 \). The factorization problem \([14]\) turned out the matrix model curve \( \tilde{y}_m^2 = x^2 \left( x^2 + \frac{4\epsilon \Lambda^6}{a^6} - a^2 \right)^2 - \frac{8\epsilon \Lambda^6}{a^6} x^2 + \frac{4\epsilon \Lambda^6}{a^6} \left( a^2 - \frac{8\epsilon \Lambda^6}{a^6} \right) \) with \( \epsilon^2 = 1 \) and the glueball \( \tilde{S} \) becomes \( \frac{2\epsilon \Lambda^6}{a^6} \) implying \( a^2 = -\frac{4}{\Lambda^2} \). Since we are interested in the special point \( n = 0 \) and \( n = 1 \) singularity we should constrain one more double root for the matrix model curve. The intersections with the \( n = 0 \) branch occur at \( \frac{4\epsilon \Lambda^6}{a^6} - a^2 = 5\eta \Lambda^2 \). Therefore, it leads to the value \( a^2 = -\eta \Lambda^2 \) by identifying \( \epsilon = \eta^3 \). At these points, the characteristic function \( P_8(x) = x^2(x^2 - a^2)^2 \left( x^2 + \frac{4\epsilon \Lambda^6}{a^6} \right) - 2\epsilon \Lambda^6 x^2 = x^2(x^2 + \eta \Lambda^2)^2 \left( x^2 + 4\eta \Lambda^2 \right) - 2\eta^3 \Lambda^6 x^2 \) can be written as \( 2\rho^6 x^2 \Lambda^6 \mathcal{T}_0 \left( \frac{x}{2\rho^6} \right) \) with \( \rho^{12} = 1 \), by identifying \( \eta = -\rho^2 \).

It was noticed in \([14]\) that there exists also a Coulomb branch where the \( SO(8) \) breaks into \( SO(4) \times U(2) \). That is, \( (N_0, N_1) = (4, 2) \). We want to show that for given matrix model curve, \( (N_0, N_1) \) are determined uniquely as follows.\(^5\) In order to see the precise values \( (N_0, N_1) \) on the Coulomb branch we calculate them by putting \( \eta = -1, \Lambda = 1 \), and \( a = 1 \), that are consistent with the previous paragraph, into the expressions given in \([14]\). The results can be rewritten as

\[
\tilde{y}_m^2 = \left( x^2 - 4 \right)^2 \left( x^2 - 3 \right)^2, \quad P_8(x) = x^2 \left( x^2 - 1 \right)^2 \left( x^2 - 4 \right) + 2x^2.
\]

By using these relations, the function \( T(x) = \text{Tr} \frac{1}{x - \Phi} \) is given as \([14]\)

\[
T(x) = \frac{P_8'(x) - 2P_8(x)}{\sqrt{P_8(x)^2 - 4x^4 \Lambda^{12}}} + \frac{2}{x} = \frac{6}{\sqrt{x^2 - 4}} + \frac{2}{x}.
\]

As we can see from the first equation of (2.15), there exist three branch cuts on the \( x \)-plane \([-2, -\sqrt{3}], [-\sqrt{3}, \sqrt{3}] \), and \([\sqrt{3}, 2] \). Since we are assuming \( n = 0 \) and \( n = 1 \) singular case, these

\(^5\) We are grateful to D. Shih \([11]\) for relevant discussion on the \( U(4) \) gauge theory in the Coulomb branch where although there exist two classical limits, \( U(4) \to U(2) \times U(2) \) and \( U(4) \to U(3) \times U(1) \), only the latter can be applied to the matrix model curve at \( n = 1 \) and \( n = 2 \) singularity through the computations on \( (N_1, N_2) \).
three branch cuts before taking the limit are joined at the locations of \( x = \pm \sqrt{3} \) after taking the limit and they become a single branch cut \([-2,2]\). Therefore, we can explicitly calculate \((N_0, N_1)\) as follows through (2.4):

\[
N_0 = \frac{1}{2\pi i} \oint_{A_0} T(x)dx = \frac{2}{2\pi i} \int_{-\sqrt{3}}^{\sqrt{3}} \left( \frac{6}{\sqrt{x^2 - 4}} + \frac{2}{x} \right) dx = \left( \frac{12}{\pi} \int_{0}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} \right) + 2 = 6,
\]

\[
N_1 = \frac{1}{2\pi i} \oint_{A_1} T(x)dx = \frac{2}{2\pi i} \int_{\sqrt{3}}^{2} \left( \frac{6}{\sqrt{x^2 - 4}} + \frac{2}{x} \right) dx = \frac{6}{\pi} \int_{\sqrt{3}}^{2} \frac{dx}{\sqrt{4-x^2}} = 1.
\]

where we used the theorem of residue around the origin: \( \frac{1}{2\pi i} \oint_{A_0} \frac{1}{x} dx = 1 \).

Although there exist two classical limits, \( SO(8) \rightarrow SO(4) \times U(2) \) and \( SO(8) \rightarrow SO(6) \times U(1) \), at the \( n = 0 \) and \( n = 1 \) singularity, one must use one of them, \( SO(8) \rightarrow SO(6) \times U(1) \). Let us emphasize that \( N_0 \) and \( N_1 \) are defined to be the on-shell periods of the one-form \( T(x) \) at the \( n = 0 \) and \( n = 1 \) singularity.

### 2.2 The coupling constant near the singularity

To see the matrix of coupling constant near the singularity we consider the effective superpotential from which we can read off the matrix of coupling constant for \( U(1)^n \) gauge groups. As already discussed in [4, 21, 22, 23, 24], the effective superpotential is given as, by considering \( RP^2 \) contribution,

\[
W_{\text{eff}} = \int d^2\psi F_p + 4F_{RP^2}, \quad F_{RP^2} = -\frac{1}{2} \frac{\partial F_p}{\partial S_0}
\]

for some function \( F_p \). By performing the \( \psi \) integrals, expanding in powers of \( w_\alpha \), and collecting the terms proportional to \( w_\alpha w_\beta \), one obtains

\[
W_{\text{eff}} \sim \frac{1}{2} \sum_{i,j,k} \frac{\partial^2 F_p(S_k)}{\partial S_i \partial S_j} w_\alpha w_\beta - \frac{N_0 - 2}{2N_i} \frac{\partial^2 F_p}{\partial S_0 \partial S_l} w_\alpha w_\beta - \sum_{i,k} \frac{N_i}{2N_k} \frac{\partial^2 F_p}{\partial S_i \partial S_k} w_\alpha w_\beta
\]

where \( i, j, k = 1, 2, \ldots, n \). Since the \( SO(N_0) \) group does not have an \( U(1) \) factor, the corresponding \( U(1) \) gauge field does not exist. The \( w_\alpha \)'s come from the \( U(1) \) factor in \( U(N_i) = U(1) \times SU(N_i) \). The matrix of gauge couplings is given by the formula [4, 24, 21]

\[
\frac{1}{2\pi i} \tau_{ij} = \frac{\partial^2 F_p(S_k)}{\partial S_i \partial S_j} - \delta_{ij} \sum_{l=1}^{n} N_i \frac{\partial^2 F_p(S_k)}{\partial S_i \partial S_l} - \delta_{ij} \left( \frac{N_0 - 2}{N_i} \right) \frac{\partial^2 F_p(S_k)}{\partial S_0 \partial S_i}, \quad i, j = 1, 2, \ldots, n.
\]

In the quartic tree-level superpotential case \( (n = 1) \), there is only one coupling constant and it is given by, due to the cancellation of first two terms above,

\[
\frac{1}{2\pi i} \tau = -\left( \frac{N_0 - 2}{N_1} \right) \frac{\partial \Pi_1}{\partial S_0} = -\frac{i\pi}{16} \left( \frac{N_0 - 2}{N - 2} \right) \log \left( \frac{16}{1 - \frac{k^2}{4}} \right).
\]
Here $k'$ is defined as (B.3). Recall that the $\Pi_1$ is a derivative of the prepotential with respect to $S_1$: $\frac{\partial F_p(S_k)}{\partial S_1}$. The single derivative $\frac{\partial \Pi_1}{\partial S_0}$ was evaluated at the extremum of the effective superpotential. We used the result of Appendix A (B.8). So $\tau$ goes to zero as one approaches the $n = 0$ and $n = 1$ singularity. The $\tau$ is continuous as we move from the $n = 1$ branch to the $n = 0$ branch since the $\tau$ is zero on the $n = 0$ branch. The logarithmic behavior implies that the gauge coupling constant of the nontrivial $U(1)$ diverges as we approach the singularity. See also the similar behavior in [27]. This divergence comes from the additional monopole that becomes massless at the singularity.

3 The $n = 0$ and $n = 1$ singularity: $USp(2N)$ gauge theory

3.1 The glueball equations of motion

Now we discuss the intersections of the $n = 0$ and $n = 1$ branches for $USp(2N)$ gauge theory. A vacuum with unbroken gauge group $USp(2N_0) \times U(N_1)$ meets a vacuum with unbroken gauge group $USp(2N)$ with

$$2N = 2N_0 + 2N_1.$$ 

These intersections occur at the particular values of the tree-level superpotential parameter. One can proceed the method given in previous section similarly. The matrix model curve is given in (2.1) and the tree-level superpotential has a parameter $m$: $W'_3(x) = x^3 + mx$. The periods and their conjugate periods are written as those in $SO(N)$ case and they provide the effective superpotential [18]

$$W_{\text{eff}} = 2\pi i \left[ (N_0 + 2) \Pi_0 + 2N_1 \Pi_1 \right] - 2(2N + 2) S \log \left( \frac{\Lambda}{\Lambda_0} \right). \quad (3.1)$$

There is no $b_1$ term. Here $N_0$ and $N_1$ are defined by (2.4) and the corresponding operator $T(x)$ for $USp(2N)$ gauge theory is [14]

$$T(x) = \frac{d}{dx} \log \left[ B_{2N+2}(x) + \sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4} \log x^2} \right],$$

$$B_{2N+2}(x) = x^2P_{2N}(x) + 2\Lambda^{2N+2}, \quad P_{2N}(x) = \det (x - \Phi).$$

The derivatives of $\Pi_i$ with respect to $f_j$ are the same as those in $SO(N)$ case exactly and by using the equations of motion for $f_0$ and $f_2$ together with the effective superpotential (3.1), one obtains two relations

$$(N + 1) \log \left| \frac{x_2^2}{4\Lambda^2} \right| = 0, \quad -\left( \frac{x_2^2 - 2x_0^2}{x_2^2} \right) = \cos \left( \frac{2\pi N_1}{2N + 2} \right)$$

where $\eta$ is $(N + 1)$-th root of unity.
From these two equations we obtain the following results with (2.2),

\[ x_2^2 = 4\eta\Lambda^2, \quad x_0^2 = 2\eta\Lambda^2\left(1 + \cos\frac{2\pi N_1}{2N+2}\right), \quad m = 2\eta\Lambda^2\left(2 + \cos\frac{2\pi N_1}{2N+2}\right) \]

where \( \eta^{N+1} = 1 \). Note that compared with the \( SO(N) \) gauge theory, the \( N \) dependence appears in the denominator of cosine function differently and the property of the phase factor \( \eta \). By using the relation between the two parametrizations, one obtains the expectation values of the fields \( f_2, f_0 \) and \( S \) at the double root singularity

\[
\langle f_2 \rangle = 4\eta^2\Lambda^4 \left( 1 + 2\cos\frac{2\pi N_1}{2N+2} \right), \\
\langle f_0 \rangle = 16\eta^3\Lambda^6 \left( 1 + \cos\frac{2\pi N_1}{2N+2} \right)^2, \\
\langle S \rangle = -\eta^2\Lambda^4 \left( 1 + 2\cos\frac{2\pi N_1}{2N+2} \right).
\]

Also we write the matrix model curve as

\[
y_m^2 = x^2 \left[ x^2 + 2\eta\Lambda^2 (2 + c) \right]^2 + 4\eta^2\Lambda^4 (1 + 2c) x^2 + 16\eta^3\Lambda^6 (1 + c)^2 = \left[ x^2 + 2\eta\Lambda^2 (1 + c) \right]^2 \left( x^2 + 4\eta\Lambda^2 \right), \quad c \equiv \cos\left(\frac{2\pi N_1}{2N+2}\right). \tag{3.2}
\]

To demonstrate this general results one can compare the formula (3.2) with the explicit examples given [14] by imposing the additional condition for an extra double root. Let us consider \( USp(2), USp(4) \) and \( USp(6) \).

- \( USp(2) \)

One predicts the matrix model curve by inserting \( (N, N_1) = (1, 1) \)

\[
y_m^2 = x^2 \left( x^2 + m \right)^2 + 4\Lambda^4 x^2 + 16\eta\Lambda^6 = \left( x^2 + 2\eta\Lambda^2 \right)^2 \left( x^2 + 4\eta\Lambda^2 \right)
\]

with \( m = 4\eta\Lambda^2 \) and

\[
\langle f_2 \rangle = 4\Lambda^4, \quad \langle f_0 \rangle = 16\eta\Lambda^6, \quad \langle S \rangle = -\Lambda^4
\]

where \( \eta \) is 2-nd root of unity. The factorization problem [14] resulted in the matrix model curve \( y_m^2 = x^2 (x^2 - v^2)^2 + 4\Lambda^4 (x^2 - v^2) \). There were some typos in [14]. The intersections with the \( n = 0 \) branch occur at \( v^2 = -4\eta\Lambda^2 \) where \( y_m^2 \) has an additional double root. At these points, the characteristic function \( B_4(x) = x^2 (x^2 + 4\eta\Lambda^2) + 2\Lambda^4 \) is equal to \( 2\rho^2\Lambda^4 \left( \frac{x^2}{2\rho\Lambda} + 1 \right) \) with \( \rho^2 = 1 \) by identifying \( \rho = \eta \). Recall that the function \( B_4(x) \equiv x^2 P_2(x) + 2\Lambda^4 \). These are the vacua that survive when the \( N = 2 \) theory is perturbed by a quadratic superpotential \( (n = 0) \). These subspaces are on the unbroken \( USp(2) \) branch. This is an agreement with the glueball approach.
\* \* USp(4) \\
At first, we consider the breaking pattern USp(4) \( \rightarrow \) U(2), namely \( N = 2, N_1 = 2 \) and \( \eta^3 = 1 \). Putting these results, we obtain

\[
\langle f_2 \rangle = 0, \quad \langle f_0 \rangle = 4\Lambda^6, \quad \langle S \rangle = 0,
\]

where the matrix model curve becomes

\[
y_m^2 = x^2 \left( x^2 + m \right)^2 + 4\Lambda^6 = \left( x^2 + \eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)
\]

with \( m = 3\eta \Lambda^2 \). From the results of [14], the glueball field \( \tilde{S} \) vanishes. The factorization problem turned out \( \tilde{y}_m^2 = x^2 \left( x^2 - a^2 \right)^2 + 4\Lambda^6 \). Without any difficulty the intersections with the \( n = 0 \) branch occur at \( a^2 = -3\eta \Lambda^2 \) where the additional double root appears in \( \tilde{y}_m^2 \). In this case, the characteristic function \( B_6(x) = x^2 \left( x^2 + 3\eta \Lambda^2 \right)^2 + 2\Lambda^6 \) can be written as \( 2\rho^3 \Lambda^6 T_3 \left( \frac{x^2}{2\rho^2 \Lambda^2} + 1 \right) \) with \( \rho^3 = 1 \) by identifying \( \rho = \eta \). These are the vacua that survive when the \( \mathcal{N} = 2 \) theory is perturbed by a quadratic superpotential and the USp(4) gauge theory becomes massive at low energies.

Next breaking pattern is USp(4) \( \rightarrow \) USp(2) \( \times \) U(1). That is, \( N = 2, N_1 = 1 \), one obtains

\[
\langle f_2 \rangle = 8\eta^2 \Lambda^4, \quad \langle f_0 \rangle = 36\Lambda^6, \quad \langle S \rangle = -2\eta^2 \Lambda^4
\]

and the matrix model curve is

\[
y_m^2 = x^2 \left( x^2 + m \right)^2 + 8\eta^2 \Lambda^4 x^2 + 36\Lambda^6 = \left( x^2 + 3\eta \Lambda^2 \right)^2 \left( x^2 + 4\eta \Lambda^2 \right)
\]

with \( m = 5\eta \Lambda^2 \). The matrix model curve \( \tilde{y}_m^2 = \left( x^2 + \frac{4\Lambda^6}{ax^2} \right) \left[ (x^2 - a^2)^2 + \frac{4\Lambda^6}{a^2} (x^2 - 2a^2) \right] \) written in [14] has an extra double root when \( a^2 = -\eta \Lambda^2 \) and \( \tilde{S} = \frac{2\Lambda^6}{a^2} \). The intersections with the \( n = 0 \) branch occur at \( a^2 = -\eta \Lambda^2 \) and the characteristic function \( B_6(x) = (x^2 + \eta \Lambda^2)^2 (x^2 + 4\eta \Lambda^2) - 2\Lambda^6 \) becomes \( 2\rho^3 \Lambda^6 T_3 \left( \frac{x^2}{2\rho^2 \Lambda^2} + 1 \right) \) with \( \rho^3 = 1 \) by identifying \( \rho = \eta \).

In order to see the precise values \( (N_0, N_1) \) on the Coulomb branch we calculate them by putting \( \eta = 1, \Lambda = 1 \), and \( a = 1 \) that are also consistent with the previous paragraph, into the expressions given in [14]. The results can be rewritten as

\[
\tilde{y}_m^2 = \left( x^2 + 4 \right) \left( x^2 + 3 \right)^2, \quad B_6(x) = \left( x^2 + 1 \right)^2 \left( x^2 + 4 \right) - 2.
\]

(3.3)

By using these relations, the function \( T(x) \) is given as [14]

\[
T(x) = \frac{B_6'(x)}{\sqrt{B_6(x)^2 - 4\Lambda^{12}}} - \frac{2}{x} = \frac{6}{\sqrt{x^2 + 4}} - \frac{2}{x}.
\]

As we can see from the first equation of (3.3), there are three branch cuts on the \( x \) plane \([-2i, -\sqrt{3}i], [-\sqrt{3}i, \sqrt{3}i], \) and \([\sqrt{3}i, 2i]\). Since we are assuming \( n = 0 \) and \( n = 1 \) singular case,
these branch cuts are joined at $\pm \sqrt{3}i$. Therefore, we can explicitly calculate $(N_0, N_1)$ as follows:

$$
2N_0 = \frac{2}{2\pi i} \int_{-\sqrt{3}i}^{\sqrt{3}i} \left( \frac{6}{\sqrt{x^2 + 4}} - \frac{2}{x} \right) dx = \left( \frac{12}{\pi} \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}} \right) - 2 = 2,
$$

$$
N_1 = \frac{2}{2\pi i} \int_{\sqrt{3}i}^{2\sqrt{3}i} \left( \frac{6}{\sqrt{x^2 + 4}} - \frac{2}{x} \right) dx = \frac{6}{\pi} \int_{\sqrt{3}}^{2} \frac{dx}{\sqrt{4 - x^2}} = 1.
$$

where we used the residue theorem.

Although there exist two different classical limits corresponding to unbroken gauge group $USp(4) \to U(2)$ and $USp(4) \to USp(2) \times U(1)$, at the $n = 0$ and $n = 1$ singularity, one must use $USp(4) \to USp(2) \times U(1)$ since these are the values of $(N_0, N_1)$ at the $n = 0$ and $n = 1$ singularities of these branches.

- $USp(6)$

Let us consider the confining branch in which the gauge group breaks into $USp(6) \to USp(2) \times U(2)$. Then one predicts the matrix model curve for $(N, N_1) = (3, 2)$

$$
y_m^2 = x^2 (x^2 + m)^2 + 4\eta^2 \Lambda^4 x^2 + 16\eta^3 \Lambda^6 = \left( x^2 + 2\eta \Lambda \right)^2 (x^2 + 4\eta \Lambda^2)
$$

with $m = 4\eta \Lambda^2$ and

$$
\langle f_2 \rangle = 4\eta^2 \Lambda^4, \quad \langle f_0 \rangle = 16\eta^3 \Lambda^6, \quad \langle S \rangle = -\eta^2 \Lambda^4,
$$

where $\eta$ is 4-th root of unity. The factorization problem turned out $\tilde{y}_m^2 = x^2 \left( x^2 - a - \frac{2\eta \Lambda^4}{a} \right)^2 + 4\epsilon \Lambda^4 x^2 - 4\epsilon \Lambda^4 - \frac{8 \Delta^2}{a}$ with $\epsilon^2 = 1$ and the glueball field $\tilde{S} = -\epsilon \Lambda^4$ which is identical to $\langle S \rangle$ for $\epsilon = \eta^2$. The matrix model curve written in [14] has an extra double root when $a = (-2 \pm \sqrt{2}) \eta \Lambda^2$.

At these points, the characteristic function $B_8(x) = (x^2 - a)^2 \left( x^2 - \frac{2\eta \Lambda^4}{a} \right)^2 - 2\Lambda^8$ which is equal to $2\epsilon^2 \Lambda^8 \tilde{P}_2 \left( \frac{x^2 P(x)}{2\Lambda} + 1 \right)$ where $P_2(x) = x^2 - a - \frac{2\eta \Lambda^4}{a}$. Then how do we check these points are on the $n = 0$ branch with unbroken $USp(6)$? One can write down $B_8(x) - 2\Lambda^8 = x^2 (x^2 + 2 \eta \Lambda^2)^2 (x^2 + 4 \eta \Lambda^2)$ and $B_6(x) + 2\Lambda^8 = (x^2 + 4 \eta \Lambda^2)^2 (2\Lambda^4)^2$. Then the first branch has an extra double root and the second branch has two double roots. Therefore, these points are on the branch with $n = 0$ and unbroken $USp(6)$.

### 3.2 The coupling constant near the singularity

For $USp(2N)$ case, the matrix of gauge couplings is given by the formula [4, 24, 21], by taking the different contribution to the $F_{RP^2}$,

$$
\frac{1}{2\pi i} \tau_{ij} = \frac{\partial^2 F_p(S_k)}{\partial S_i \partial S_j} - \delta_{ij} \frac{1}{N_1} \sum_{l=1}^{n} N_l \frac{\partial^2 F_p(S_k)}{\partial S_l \partial S_i} - \delta_{ij} \left( \frac{2N_0 + 2}{N_i} \right) \frac{\partial^2 F_p(S_k)}{\partial S_0 \partial S_i}, \quad i, j = 1, 2, \ldots, n.
$$

The single gauge coupling for quartic superpotential is given similarly

$$
\frac{1}{2\pi i} \tau = \left( \frac{2N_0 + 2}{N_1} \right) \frac{\partial \Pi_1}{\partial S_0} = \frac{i\pi}{16} \left( \frac{2N_0 + 2}{2N + 2} \right)^2 \log \left( \frac{16}{1-x^2} \right)
$$
where we use a notation for $USp(2N) \rightarrow USp(2N_0) \times U(N_1)$. As we have seen in previous consideration for $SO(N)$ case, The $\tau$ is continuous as we move from the $n = 1$ branch to the $n = 0$ branch since the $\tau$ is zero on the $n = 0$ branch. The logarithmic behavior implies the gauge coupling constant of the nontrivial $U(1)$ diverges as we approach the singularity.

4 The matrix model curve for degenerated case: $SO(N)$ gauge theory

Contrary to non-degenerated case where every root of $W'(x)$ has D5-branes wrapping around it, in previous sections, there exists only one parameter denoted by $F$ for degenerated case. For the degenerated case [15, 16] where some roots of $W'(x)$ do not have wrapping D5-branes around them, the matrix model curve is described as [15, 16]

$$y_{m,d}^2 = \left( \frac{W'_3(x)}{x} \right)^2 + 4F = (x^2 + m)^2 + 4F \equiv (x^2 + a^2)(x^2 + b^2)$$

where $W'_3(x) = x^3 + mx$ as before. The first parametrization of this matrix model curve implies that $m$ is a parameter and $4F$ is a fluctuating field that is related to the glueball field. The second parametrization in terms of the roots $\pm ia$ and $\pm ib$ will be convenient and is subject to the constraint

$$m = \frac{1}{2} (a^2 + b^2).$$

At first sight, one can derive the results for degenerate case by using the results in section 2.1. The matrix model curve for the degenerated case can be represented by the matrix model curve $y_m$ (2.14) corresponding to the non-degenerated case,

$$y_{m}^2 = x^2 y_{m,d}^2 = x^4 \left( x^2 + 4\eta \Lambda^2 \right) \iff f_0 = 0. \quad (4.4)$$

where $\eta^{N-2} = 1$ for $N$ even and $\eta^{N-2} = -1$ for $N$ odd as before.

As we will see below, this naive consideration gives the precise results on the degenerated case. On the degenerated case, although the matrix model curve does not have singularity, the dynamical variables are less than those in non-degenerated case. Therefore, the glueball approach for this case is very powerful as the singular case and the equation of motions for the variables becomes drastically easy to solve.

The dual periods are given by the integrals for $x^2 y_{m,d}$ over $x$:

$$2\pi i \Pi_0 = \int_0^{\Lambda_0} \sqrt{x^2(x^2 + a^2)(x^2 + b^2)} \, dx,$$

$$2\pi i \Pi_1 = \int_{ib}^{i\Lambda_0} \sqrt{x^2(x^2 + a^2)(x^2 + b^2)} \, dx.$$
The corresponding effective glueball superpotential is given by (2.3). One can derive the matrix model curve by direct calculations from the effective superpotential for the degenerated case. Since there exists only one parameter, we have to consider the equation of motion of $F$ only. After differentiating the dual periods with respect to the field $F$, one obtains

$$
4\pi i \frac{\partial \Pi_0}{\partial F} = \int_0^{N_0} \frac{x^2 dx}{\sqrt{x^2(x^2 + a^2)(x^2 + b^2)}} \approx \frac{1}{2} \log \left| \frac{4\Lambda_0^2}{(a + b)^2} \right|,
$$

$$
4\pi i \frac{\partial \Pi_1}{\partial F} = \int_b^{i\Lambda_0} \frac{x^2 dx}{\sqrt{x^2(x^2 + a^2)(x^2 + b^2)}} \approx \frac{1}{2} \log \left| \frac{4\Lambda_0^2}{b^2 - a^2} \right|.
$$

By using these two results and taking $\partial W_{\text{eff}} / \partial F = 0$, we obtain the following equation,

$$
\left( \frac{4\Lambda^2}{(a + b)^2} \right)^{N_0 - 2} \times \left( \frac{4\Lambda^2}{b^2 - a^2} \right)^{2N_1} = \pm 1. \tag{4.5}
$$

It is noteworthy that by taking the special limit, we can reproduce the equation (4.4) because $b^2 = 4\eta \Lambda^2, a = 0$ and $y_{m,d}^2 = x^2(x^2 + 4\eta \Lambda^2)$. However, in general, the relation (4.5) leads to the more general result.

The matrix model curve for the degenerated case can be represented as

$$
y_{m,d}^2 = (x^2 + m)^2 + 4F, \quad m = \frac{K^2 - 16F}{4K}, \quad K \equiv \left[ \frac{(-4\Lambda^2)^{N_0 - 2}}{(-16F)^{N_1}} \right]^{1/2} \tag{4.6}
$$

where $K$ becomes $(a + b)^2$ in the second parametrization. This general formula provides the matrix model curve for degenerated case and it depends on the parameter $F, N_0$ and $N_1$ where $N_0$ can be zero. Turning on the parameter $F$ to the special value will lead to the symmetry breaking $SO(N) \rightarrow SO(N)$.

- **SO(4)**

For the breaking pattern $SO(4) \rightarrow U(2)$, by plugging the values $N = 4, N_0 = 0$, and $N_1 = 2$ into the general formula (4.6), one gets $K = \frac{4\eta F}{\Lambda^2}$ and $m = \frac{\eta F}{\Lambda^2} - \eta \Lambda^2$. As studied in [15], the solutions for the factorization problem of degenerated case can be represented as $\tilde{y}_{m}^2 = (x^2 + D)^2 + 4G$ with $D = \frac{G}{4\Lambda^2} - \epsilon \Lambda^2$ where $\epsilon$ is 2-nd root of unity (Note that $D = b/2$ and $4G = c - b^2/4$ in the notation of [15]). Therefore by identifying $D, G, \epsilon$ with $m, F, \eta$ respectively, the two approaches, glueball approach and strong-coupling approach are equivalent to each other. The special point comes from the condition $G = -\Lambda^4$ in which the $SO(4)$ goes to $SO(4)$. If we define $\rho = -\eta$, we can rewrite the matrix model curve as $\tilde{y}_{m}^2 = (x^2 + 2\rho \Lambda^2)^2 - 4\Lambda^4$, which agrees with the general formula in the equation (4.4).

- **SO(5)**

For the breaking pattern $SO(5) \rightarrow SO(3) \times U(1)$ since the $K$ becomes $\frac{4\Lambda^6}{F}$, we can write the $m$ as $m = -\frac{F^2}{\Lambda^6} + \frac{4\Lambda^6}{F}$. By identifying $G$ in the result [15] with $F$, we find an exact agreement.
From the results in [15], the matrix model curve is given as \( \tilde{y}_m^2 = (x^2 - D)^2 + 4G \) where \( D = \frac{G^2}{\Lambda^6} - \frac{\Lambda^6}{G^6} \). Note that \( G = b/4 \) and \( D = a \) in the notation of [15]. The particular point comes from the condition \( G^3 = -\Lambda^{12} \) where \( SO(5) \to SO(5) \). If we define \( \rho = -\eta^2 \) where \( \eta^3 = -1 \) (therefore \( \rho^3 = -1 \)), we can write \( G = \eta \Lambda^4 = -\rho^2 \Lambda^4 \) and then \( \tilde{y}_m^2 = (x^2 + 2\rho \Lambda^2)^2 - 4\rho^2 \Lambda^4 \). This curve agrees with the general formula (4.4).

\( \bullet SO(6) \)

For the breaking pattern \( SO(6) \to SO(4) \times U(1) \), since the \( K \) becomes \( \frac{4\eta \Lambda^4}{\sqrt{\eta}} \), where \( \eta^4 = 1 \) we can write down \( m = \eta \left( \frac{\Lambda^4}{\sqrt{\eta}} - \frac{F_\eta \eta}{\eta \Lambda^4} \right) \). By identifying \( \epsilon \) and \( G^2 \) in [15] with \( -\eta \) and \( F \) respectively, we find an exact agreement between the glueball approach and strong-coupling approach. In [15] the special point is given by the condition \( G^4 = \Lambda^8 \) where there exists a breaking pattern \( SO(6) \to SO(6) \). Putting this value, we obtain the matrix model curve as \( \tilde{y}_m^2 = (x^2 + 2\rho \Lambda^2)^2 - 4\rho^2 \Lambda^4 \) where \( \rho \) is 4-th root of unity. This curve agrees with the one obtained from glueball approach exactly.

\( \bullet SO(7) \)

For the breaking pattern \( SO(7) \to SO(3) \times U(2) \) since the \( K \) becomes \( -\frac{4\Lambda^{10}}{F} \), we can write down \( m = -\frac{\Lambda^{10}}{F} + \frac{F_\eta^{3} \eta}{\Lambda^{20}} \). By identifying \( G \) in [15] with \( F \), we find an exact agreement. In [15], the particular point is given by the condition \( G^5 = -\Lambda^{20} \) where \( SO(7) \to SO(7) \). Putting this value, we obtain the matrix model curve \( \tilde{y}_m^2 = (x^2 + 2\rho \Lambda^2)^2 - 4\rho^2 \Lambda^4 \) where \( \rho \) satisfies \( \rho^5 = -1 \). This curve agrees with the one obtained from the glueball approach.

When we increase the \( N \) beyond 7, we do not have any results for the matrix model curve from the strong-coupling approach so far, we have to resort to the expression (4.6) only. In this formula, the matrix model curve is given by \( N_0, N_1 \), and \( F \). In general, the \( N_0 \) and \( N_1 \)’s are obtained from the relations (2.4). However, contrary to the nondegenerated case, the contour integrals do not lead to the final results for \( N_0 \) and \( N_1 \) easily, because the matrix model curve does not have double root, which will make the \( T(x) \) complicated.

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**Appendix A**  \textbf{The derivatives of dual periods on the } \textit{n} = 1 \textbf{branch} 

The derivative of \( \Pi_0 \) with respect to \( f_0 \) when \( x_1 \) is arbitrary (not near the singularity) can
be obtained from the definition of $\Pi_0$ in section 2 as follows:

\[
4\pi i \frac{\partial \Pi_0}{\partial f_0} = \int_0^{\Lambda_0} \frac{dx}{\sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)}} = \frac{1}{2} \int_0^{\Lambda_2} \frac{dt}{\sqrt{t(t + x_0^2)(t + x_1^2)(t + x_2^2)}}
\]

\[
= \frac{1}{x_1\sqrt{x_2^2 - x_0^2}} F(\phi|R)
\]

(A.1)

where we make a change of variable $t = x^2$ and in the final relation we used $t = \frac{-x^2}{1-y^2}$ and then $y^2 = \frac{x^2}{x_2^2 - x_0^2}z^2$ with $\Lambda_0$ large. Although the dual period $\Pi_0$ is different from the one in $U(N)$ case, the derivative of $\Pi_0$ with respect to $f_0$ in the $t$-integration is the same as $\partial \Pi_2/\partial f_0$ in (A.4) of [11] up to an overall constant. Also we have $x_0 < x_1 < x_2$ as before. Here $\phi$, $R$ and the first kind elliptic integral $F(\phi|R)$ are

\[
\phi = \sin^{-1}\left(\frac{x_2 - x_0}{x_2^2}\right), \quad R = \sqrt{\frac{x_2^2(x_1^2 - x_0^2)}{x_1^2(x_2^2 - x_0^2)}},
\]

\[
F(\phi|R) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - R^2 \sin^2 \theta}} = \int_0^{w} \frac{dz}{\sqrt{(1 - z^2)(1 - R^2 z^2)}} \equiv F\left(w = \sin^{-1} \phi|R\right). \quad (A.2)
\]

Similarly the derivative of $\Pi_1$ with respect to $f_0$ is given by

\[
4\pi i \frac{\partial \Pi_1}{\partial f_0} = \int_{ix_2}^{i\Lambda_2} \frac{dx}{\sqrt{(x^2 + x_0^2)(x^2 + x_1^2)(x^2 + x_2^2)}} = -\frac{1}{2} \int_{ix_2}^{\Lambda_2} \frac{dt}{\sqrt{t(t - x_0^2)(t - x_1^2)(t - x_2^2)}}
\]

\[
= -\frac{1}{x_1\sqrt{x_2^2 - x_0^2}} F(\psi|R)
\]

(A.3)

where we introduce a new angle $\psi$ as follows:

\[
\psi = \sin^{-1}\left(\frac{x_1}{x_2}\right).
\]

(A.4)

From these general formulas we can derive the equations (2.5) and (2.6) by using the properties of trigonometric functions. That is, $\phi$ can be represented as

\[
\phi = \frac{1}{2} \left[\sin^{-1}\left(\frac{x_2^2 - 2x_0^2}{x_2^2}\right) + \frac{\pi}{2}\right]. \quad (A.5)
\]

Under the limit, $x_1 \rightarrow x_0$ (at the singularity), the parameter $R$ goes to zero and the elliptic function behaves as $F(\phi|0) = \phi$ from (A.2). After some calculations, we obtain the following relations which are the same as (2.5) and (2.6) precisely,

\[
4\pi i \frac{\partial \Pi_0}{\partial f_0} = \frac{1}{2x_0\sqrt{x_2^2 - x_0^2}} \left[\sin^{-1}\left(\frac{x_2^2 - 2x_0^2}{x_2^2}\right) + \frac{\pi}{2}\right],
\]

\[
4\pi i \frac{\partial \Pi_1}{\partial f_0} = -\frac{1}{2x_0\sqrt{x_2^2 - x_0^2}} \left[-\sin^{-1}\left(\frac{x_2^2 - 2x_0^2}{x_2^2}\right) + \frac{\pi}{2}\right]. \quad (A.6)
\]
where we use the fact that \( \psi = \phi + \pi/2 \) under this limit.

The derivative of \( \Pi_0 \) with respect to \( f_2 \) (when we consider the type of integration, this corresponds to \( \partial \Pi_2/\partial f_1 \) of \( U(N) \) case) can be obtained from the definition of \( \Pi_0 \) by the same change of variables before

\[
4\pi i \frac{\partial \Pi_0}{\partial f_2} = \frac{1}{2} \int_0^{\Lambda_0^2} \frac{t \, dt}{\sqrt{t \left( t + x_0^2 \right) \left( t + x_1^2 \right) \left( t + x_2^2 \right)}}
\]

\[
= \frac{x_0^2}{x_1 \sqrt{x_2^2 - x_0^2}} \int_0^{\Lambda_0^2 \left( x_0^2 + x_0^2 \right)} \left( \frac{1}{1 - n^2 z^2} - 1 \right) \frac{dz}{\sqrt{(1 - z^2) \left( 1 - R^2 z^2 \right)}}
\]

\[
= \frac{x_0^2}{x_1 \sqrt{x_2^2 - x_0^2}} \left[ \Pi \left( n; \sin^{-1} \left[ \frac{1}{n} \sqrt{\frac{\Lambda_0^2}{\Lambda_0^2 + x_0^2}} \right] R \right) - F(\phi|R) \right]
\]

\[
= \frac{x_0^2}{x_1 \sqrt{x_2^2 - x_0^2}} \Pi \left( n; \sin^{-1} \left[ \frac{1}{n} \sqrt{\frac{\Lambda_0^2}{\Lambda_0^2 + x_0^2}} \right] R \right) - 4\pi i x_0^2 \frac{\partial \Pi_0}{\partial f_0}
\]

(A.7)

where we introduce a new notation for \( \phi \) and the third kind elliptic integral \( \Pi(c; \phi|R) \) is given by

\[
n^2 \equiv \frac{x_2^2}{x_2^2 - x_0^2} = \frac{1}{\sin \phi},
\]

\[
\Pi(c; \phi|R) = \int_0^\phi \frac{d\theta}{(1 - c^2 \sin^2 \theta) \sqrt{1 - R^2 \sin^2 \theta}}
\]

\[
= \int_0^w \frac{dz}{(1 - c^2 z^2) \sqrt{(1 - z^2) \left( 1 - R^2 z^2 \right)}} \equiv \Pi \left( c; w = \sin^{-1} \phi|R \right).
\]

(A.8)

Note that if we take a limit \( \Lambda_0^2 \to \infty \), the first term of (A.7) contains a divergence term. Therefore, we should keep \( \Lambda_0 \) term in the formula explicitly.

Next we evaluate this formula at the singular point with \( x_1^2 \to x_0^2 \). Under this limit, since the \( R \) goes to zero we have only to consider the following integral,

\[
\int_0^{\epsilon +} \frac{dz}{(1 - n^2 z^2) \sqrt{1 - z^2}} = \frac{1}{\sqrt{n^2 - 1}} \tanh^{-1} \left( \frac{\sqrt{n^2 - 1} z}{\sqrt{1 - z^2}} \right) \bigg|_0^{\epsilon +}
\]

\[
\approx \frac{-1}{2\sqrt{n^2 - 1}} \log \left| \frac{en^3}{2 (n^2 - 1)} \right|
\]

\[
= \frac{-1}{2\sqrt{n^2 - 1}} \log \left| \frac{x_0^2}{4\Lambda_0^2} \right|
\]

(A.9)

where we define

\[
\epsilon \equiv -\frac{x_0^2}{2n\Lambda_0^2}
\]

(A.10)
and drop out $O(\epsilon)$ in (A.9). We also used the fact that $\tanh^{-1} x = \frac{1}{2} \log \left| \frac{1+i x}{1-i x} \right|$. Taking into account an overall factor and $x_1^2 \to x_0^2$, the first equation of (A.7) becomes

$$-\frac{1}{2} \log \left| \frac{x_2^2}{4A_0^2} \right|.$$  

(A.11)

Therefore (A.7) reproduces to (2.8) at the singularity.

Similarly one can execute the integral

$$4\pi i \frac{\partial \Pi}{\partial f_2} = \frac{1}{2} \int_{x_2^2}^{\Lambda_0^2} \frac{t \, dt}{\sqrt{t(1-x_0^2)(t-x_1^2)(t-x_2^2)}}$$

$$= \frac{(x_2^2-x_1^2)}{x_1\sqrt{x_2^2-x_0^2}} \int_0^{\sqrt{(x_2^2-x_0^2)^2}} \left( \frac{1}{1-n^2z^2} + \frac{x_1^2}{x_2^2-x_1^2} \right) \frac{dz}{\sqrt{(1-z^2)(1-R^2z^2)}}$$

$$= \frac{(x_2^2-x_1^2)}{x_1\sqrt{x_2^2-x_0^2}} \pi \left( \frac{n}{\sin^{-1} \left( \frac{\Lambda_0^2-x_0^2}{\Lambda_0^2-x_1^2} \sin \psi \right)} \left| R \right\rangle + \frac{x_1^2}{(x_2^2-x_1^2)} F \left( \psi \left| R \right\rangle \right) \right) - 4\pi i x_1^2 \frac{\partial \Pi}{\partial f_0}$$

(A.12)

where

$$\tilde{n}^2 = \frac{x_2^2}{x_1^2} = \frac{1}{\sin^2 \psi}, \quad \tilde{e} = -\frac{1}{2} \frac{x_0(x_0^2-x_2^2)}{x_2^2(x_0^2-x_1^2)}.$$  

(A.13)

As in previous calculation, we can evaluate the first term of (A.12) at the singularity and obtain the same result (A.11). Therefore, we reproduced the relation (2.9).

Appendix B  The derivatives of periods on the $n = 1$ branch

Next let us consider the derivatives of $S_1$. By changing the variable $t$ to $z$,

$$t \equiv \frac{x_1^2-x_0^2y^2}{1-y^2}, \quad y^2 \equiv \frac{x_1^2-x_0^2}{x_1^2-x_0^2} z^2$$  

(B.1)

we can get the following formula:

$$4\pi i \frac{\partial S_1}{\partial f_0} = \int_{ix_1}^{ix_2} \frac{dx}{\sqrt{(x^2+x_0^2)(x^2+x_1^2)(x^2+x_2^2)}} = \int_{x_1^2}^{x_2^2} \frac{dt}{2\sqrt{(t-x_0^2)(t-x_1^2)(t-x_2^2)}}$$

$$= \frac{i}{x_1\sqrt{x_2^2-x_0^2}} F \left( \frac{\pi}{2} \right)$$

(B.2)

where we introduce

$$n' = \frac{x_2^2-x_1^2}{x_2^2-x_0^2}, \quad k'^2 = \frac{x_0^2(x_2^2-x_1^2)}{x_1^2(x_2^2-x_0^2)}.$$  

(B.3)
Although the form of $S_1$ is different from the one in $U(N)$ case, the above $t$-integration is the same as $\partial S_2/\partial f_0$ in (A.8) of [11]. Moreover one has
\[
4\pi i \frac{\partial S_1}{\partial f_2} = \int_{ix_1}^{ix_2} \frac{x^2 dx}{\sqrt{(x^2 + x_0^2) (x^2 + x_1^2)}} = \int_{x_1^2}^{x_2^2} \frac{t dt}{2 \sqrt{t (t - x_0^2) (t - x_1^2) (t - x_2^2)}}
\]
\[
= \frac{i (x_1^2 - x_0^2)}{x_1 \sqrt{x_2^2 - x_0^2}} \Pi \left( n' \mid \frac{\pi}{2} k' \right) + \frac{i x_0^2}{x_1 \sqrt{x_2^2 - x_0^2}} F \left( \frac{\pi}{2} k' \right). \tag{B.4}
\]
This corresponds to the $\partial S_2/\partial f_1$ for $U(N)$ case. We can evaluate the above formula under the limit $x_1^2 \to x_0^2$ and in this case $k' \to 1$:
\[
F \left( \frac{\pi}{2} k' \right) \to \frac{1}{2} \log \left( \frac{16}{1 - k'^2} \right) + \mathcal{O} \left( 1 - k'^2 \right),
\]
\[
\Pi \left( n' \mid \frac{\pi}{2} k' \right) \to \frac{(x_1^2 - x_0^2)}{(x_1^2 - x_2^2)} \sqrt{\frac{x_1^2}{x_2^2 - x_0^2}} \sin^{-1} \left( \frac{x_1^2 - x_0^2}{x_2^2} \right). \tag{B.5}
\]
These relations were given already in [11].

Let us compute the partial derivative $\frac{\partial \Pi_1}{\partial s_0}$ near the $n = 0$ and $n = 1$ singularity that is necessary to the coupling constant at the singularity. By using the chain rule one writes
\[
\frac{\partial \Pi_1}{\partial s_0} = \frac{\partial \Pi_1}{\partial f_2} \frac{\partial f_2}{\partial s_0} + \frac{\partial \Pi_1}{\partial f_0} \frac{\partial f_0}{\partial s_0} = \frac{1}{4} \left( \kappa \frac{\partial \Pi_1}{\partial f_0} - \frac{\partial \Pi_1}{\partial f_2} \right) \tag{B.6}
\]
where we use the relation $S_0 = S - 2S_1 = -4f_2 - 2S_1$ in order to eliminate $S_0$ and define the $\kappa$ as follows:
\[
\kappa \equiv \frac{\partial s_1}{\partial f_2} \frac{\partial f_2}{\partial s_0} = x_0^2 + 2 \frac{x_0^2 (x_2^2 - x_0^2)}{\log \left( \frac{16}{1 - k'^2} \right)} \sin^{-1} \left( \frac{x_1^2 - x_0^2}{x_2^2} \right). \tag{B.7}
\]
Inserting the results for $x_2$ and $x_0$ given in (2.11), we can rewrite as
\[
\frac{\partial \Pi_1}{\partial s_0} \simeq \frac{1}{4} \times 2 \frac{x_0^2 (x_2^2 - x_0^2)}{\log \left( \frac{16}{1 - k'^2} \right)} \sin^{-1} \left( \frac{x_1^2 - x_0^2}{x_2^2} \right) \times \frac{\partial \Pi_1}{\partial f_0}
\]
\[
= \frac{i \pi N_1 (N_0 - 2)}{16 (N - 2)^2} \frac{1}{\log \left( \frac{16}{1 - k'^2} \right)} \tag{B.8}
\]
where we used $\frac{\partial \Pi_1}{\partial f_0} = \frac{i}{8} \frac{1}{x_0 \sqrt{x_2^2 - x_0^2}} (1 - \frac{2N_1}{N - 2})$. For $USp(2N)$ gauge theory, the computation can be done similarly.

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