On the existence of deformed Lie-Poisson structures for quantized groups

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Abstract

The geometrical description of deformation quantization based on quantum duality principle makes it possible to introduce deformed Lie-Poisson structure. It serves as a natural analogue of classical Lie bialgebra for the case when the initial object is a quantized group. The explicit realization of the deformed Lie-Poisson structure is a difficult problem. We study the special class of such constructions characterized by quite a simple form of tangent vector fields. It is proved that in such a case it is sufficient to find four Lie compositions that form two deformations of the first order and four Lie bialgebras. This guarantees the existence of two families of deformed Lie-Poisson structures due to the intrinsic symmetry of the initial compositions. The explicit example is presented.

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1 Introduction

Quantum duality principle \[1,2\] asserts that quantization of a Lie bialgebra \((A, A^*)\) gives rise to a dual pair of Hopf algebras \((U_p(A), U_p(A^*))\) or in dual terms – \((\text{Fun}_p(G), \text{Fun}_p(G^*))\). Here \(G\) and \(G^*\) are the universal covering groups for \(A\) and \(A^*\) respectively. Every quantum algebra of this type can be interpreted as a quantum group (in quantum formal series Hopf terms),

\[ U_p(A) \approx (\text{Fun}_p(G^*)), \]

and vice versa. This leads to the natural dualization of classical limits for a given quantized object.

Thus the canonical form of deformation quantization of Lie bialgebra \((A, A^*)\) must be a 2-parametric family of Hopf algebras with two dual classical limits. Within certain assumptions this family forms an analytic variety \(\mathcal{D}\) and the classical limits – its boundary. The existence of a variety \(\mathcal{D}\) with such properties is equivalent to attributing its members the quantum duality. The Lie bialgebra appears here in the form of two vector fields tangent to \(\mathcal{D}\).

From the point of view of Lie-Poisson structures and their symmetries it can be shown natural to consider the varieties of this type and their boundaries entirely placed in the domain of noncommutative and noncocommutative Hopf algebras. In this case the main property of quantum duality is preserved and the obtained deformed vector fields can be treated as a deformed Lie-Poisson construction with respect to a quantized group.

The paper is organized as follows. In section 2 we describe the main features of canonical dual limits and varieties of the type \(\mathcal{D}\). The deformed (“lifted”) \(\mathcal{D}_2\) versions of \(\mathcal{D}\) are defined in subsection 2.2 as analogs with noncommutative and noncocommutative boundaries. In section 3 the general properties of multiplication and comultiplication deforming functions in \(\mathcal{D}_2\) are discussed. The sufficient conditions for the existence of the deformation \(\mathcal{D} \rightarrow \mathcal{D}_2\) are established in section 4. All the constructions involved are explicitly illustrated by the example considered in Section 5.

2 Deformed Poisson structures
2.1 Dual limits. Analytic variety \( \mathcal{D} \)

Consider the variety \( \mathcal{H} \) of Hopf algebras with fixed number of generators. Its points \( H \in \mathcal{H} \) are parameterized by the corresponding structure constants. We must find a (smooth) curve in \( \mathcal{H} \) containing \( U_p(A) \) and intersecting with the orbit \( \text{Orb}(\text{Fun}(G^*)) \). In the limit to be obtained the multiplication \( m \) in \( U_p(A) \) must become Abelian. For the universal enveloping algebra \( U(A) \) such a procedure is trivially described by a contraction. The corresponding transformation of basis \( \{a_i\} \),

\[
B(t) : a_i \rightarrow a_i/t,
\]
leads to new structure constants \( C'_{ij} \)

\[
C'_{ij}(t) = tC_{ij}.
\]

Algebras \( U(A_t) \) form a line in \( \text{Orb}U(A) \) with the limit point \( U(A_0) \equiv \text{Abelian} \). Applying operators \( B(t) \) to \( U_p(A) \) we obtain a smooth one-parametric curve

\[
B(t)U_p(A) \equiv U_p(A_t)
\]

in the orbit \( \text{Orb}(U_p(A)) \) and in \( \mathcal{H} \) – the 2-parametric subset

\[
\{U_p(A_t)\}_{p>0,t>0} \equiv \mathcal{D}(A, A^*)
\]

formed by the dense set of smooth curves.

\( \mathcal{D}(A, A^*) \) is an analytic variety with the coordinates \( p \) and \( t \) but the coproduct structure constants may tend to infinity when \( t \rightarrow 0 \). According to the results of [7] and [8] there always exists such a reparameterization of \( \mathcal{D} \) that in new coordinates both limits exist. In [9] it was demonstrated that for a certain class \( \mathcal{F} \) of quantizations the required reparameterization is very simple.

The class \( \mathcal{F} \) is fixed by the following conditions (see [4]):

(a) equations

\[
(m(\text{id} \otimes S)\Delta)_{\downarrow V(A)} = (m(S \otimes \text{id})\Delta)_{\downarrow V(A)} = (\eta\epsilon)_{\downarrow V(A)}
\]

define uniquely the antipode \( S \) of \( U_p(A) \).
(b) 

$((S^\uparrow \otimes \text{id})\Delta)_{\downarrow V(A)} = ((S \otimes \text{id})\Delta)_{\downarrow V(A)},$

$((\text{id} \otimes S^\uparrow)\Delta)_{\downarrow V(A)} = ((\text{id} \otimes S)\Delta)_{\downarrow V(A)}$

(here $S^\uparrow$ is a linear operator : $H \to H$ that coincides with the antipode $S$ on $V(A)$ and is homomorphically extended to $H$ ; $S^\uparrow(1) = 1$).

If the Hopf algebra $U_p(A)$ belongs to $\mathcal{F}$ the second classical limit can be visualized by a simple change the parameters:

$$(p, t) \Rightarrow (h, t), \quad h = p/t. \quad (5)$$

Then the coproduct in $U_h(A_t)$ becomes well defined in the limit $t \to 0$ (with $h$ fixed) while the multiplication structure constants in these new coordinates preserve their finite limit values. The limit points correspond to Hopf algebras with Abelian multiplication. They form an analytic curve $\mathcal{P}_h$. Each point of this curve can be interpreted as an algebra of exponential coordinate functions on the group $G^*_h$:

$$\lim_{t \to 0} U_h(A_t) \equiv U_h(A_0) \approx \text{Fun}(G^*_h) \quad (6)$$

The limit $h \to 0$ describes the trivial contraction of $G^*_h$ into the Abelian group $AB$.

Thus every deformation quantization of the type $\mathcal{F}$ can be written in the form $U_h(A_t)$ (respectively $\text{Fun}_t(G^*_h)$) that reveals two canonical dual classical limits:

$$U_h(A_t) \xrightarrow{h \to 0} U(A_t) \quad \text{Fun}_t(G_t) \quad \text{Fun}_t(G^*_t) \quad \text{Fun}_h(\mathcal{A}B) \quad \approx U_h(\mathcal{A}b) \quad (7)$$

All the reasoning is invariant with respect to interchange $A \leftrightarrow A^*$.

### 2.2 Deformed variety $\mathcal{D}_{\uparrow}$

It was also demonstrated that for a given analytic variety $\mathcal{D}_{h,t}$ one can easily (and, up to equivalence, uniquely) reconstruct the corresponding Lie bialgebra $(A, A^*)$. Thus the Lie bialgebra and the corresponding Lie-Poisson structure can be identified with the analytic subvariety $\mathcal{D}_{h,t}$ of $\mathcal{H}$ whose boundaries
are formed by the contraction curves of $U(A_t)$ and $\text{Fun}(G_h^*)$ intersecting in the common limit – $U(\text{Ab}) \approx \text{Fun}(\mathcal{A}\mathcal{B})$.

To analyze the corresponding construction for quantum algebras and quantum groups one must study the possibility to ”lift” the whole picture (that is the variety $\mathcal{D}_{h,t}$ with its boundaries) in the domain of noncommutative and noncocommutative Hopf algebras in $\mathcal{H}$. Suppose such lift exists, in this case the corresponding quasiclassical limits form the contraction curves of quantized algebra and quantized group. They still have a common limit, but now one finds in the intersection point nontrivial Hopf algebra instead of Abelian and co-Abelian one.

According to the quantization ideology we must be able to treat the obtained picture as a deformation of the initial $\mathcal{D}_{h,t}$. One is to attribute an extra deformation parameter $z$ to the obtained varieties $\mathcal{D}_{h,t,z}$, $\mathcal{D}_{h,t,0} \approx \mathcal{D}_{h,t}$, Thus to describe the deformed Lie-Poisson construction for a given Lie bialgebra $(A, A^*)$ one must find in $\mathcal{H}$ a tree-dimensional analytic subvariety $\mathcal{D}_{h,t,z}$. From beneath it is delimited by the initial $\mathcal{D}_{h,t,0}$. The other two facets – $\mathcal{D}_{0,t,z}$ and $\mathcal{D}_{h,0,z}$ – can not be interpreted as canonical deformation quantizations. Parameter $z$ describes here the simultaneous change of both the multiplication and comultiplication in $\mathcal{D}_{0,t,0}$ and $\mathcal{D}_{h,0,0}$ respectively – the situation similar to that appearing in quantum analogs of cotangent bundle [3]. Among the intersections of facets the lower two – $\mathcal{D}_{0,t,0}$ and $\mathcal{D}_{h,0,0}$ – are the initial dual classical limits while the third one – $\mathcal{D}_{0,0,z}$ – describes the trivial contraction of the Hopf algebra that plays the role of commutative and cocommutative one in the deformed case.

The vector fields $V_{0,t,z}$ and $W_{h,0,z}$ tangent to $\mathcal{D}_{h,t,z}$ and normal to $\mathcal{D}_{0,t,z}$ and $\mathcal{D}_{h,0,z}$ respectively play here the role of deformed Poisson compositions for $\text{Fun}_z(G)$ and $\text{Fun}_z(G^*)$.

### 3 Analytic properties of $\mathcal{D}_{h,t,z}$

Let $m(t, h, z)$ and $\Delta(t, h, z)$ be the projections of multiplication and comultiplication in $H_{h,t,z} \in \mathcal{D}_{h,t,z}$ on the space $A \wedge A$ and $A$ respectively. Then their Taylor expansions are

$$m(t, h, z) = \sum_{i,j,k=0; (i,j,k) \neq (0,1,0)} t^i h^j z^k m_{i,j,k},$$

$$\Delta(t, h, z) = \sum_{i,j,k=0; (i,j,k) \neq (0,1,0)} t^i h^j z^k \Delta_{i,j,k}.$$
\[\Delta(t, h, z) = \sum_{i, j, k} t^i h^j z^k \Delta_{ijk}. \quad (9)\]

Missing terms in the expansion correspond to the property of the deformation quantization \(D_{h,t,0}\) where the multiplication in \(U_h(A)\) and the comultiplication in \(\text{Fun}_t(G)\) are untouched in the first order. Here \(m_{000}\) and \(\Delta_{000}\) refer to the commutative multiplication and primitive comultiplication respectively; in combersome expressions we shall denote them simply by \(m_0\) and \(\Delta_0\).

Let \(\mu\) and \(\delta\) be the antisymmetrized compositions for \(m\) and \(\Delta\),

\[
\mu(t, h, z) = \sum_{i, j, k = 0} t^i h^j z^k \mu_{ijk}, \quad (10)
\]

\[
\delta(t, h, z) = \sum_{i, j, k = 0} t^i h^j z^k \delta_{ijk}, \quad (11)
\]

The compositions \(\mu_{100}\) and \(\delta_{010}\) are just the \(A\) and \(A^*\) Lie multiplication and comultiplication.

Consider now the neighborhood of \(H_{0,0,0} \in D_{h,t,z}\) and write the bialgebra properties of \(H_{t,h,z}\) in terms of \(\mu\) and \(\delta\):

\[
\delta \circ \mu = (m \otimes^\wedge m) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes^\wedge \Delta), \quad (12)
\]

Here

\[
m \otimes^\wedge m \equiv (m \otimes m) \circ (\text{id} - \tau \otimes \text{id}), \quad (13)
\]

\[
\Delta \otimes^\wedge \Delta \equiv (\text{id} - \tau \otimes \tau) \circ (\Delta \otimes \Delta). \quad (14)
\]

Inserting the expansions \((10, 11)\) in the first nontrivial order one finds:

\[
z^2 \delta_{001} \circ \mu_{001} + th \delta_{010} \circ \mu_{100} + tz \delta_{001} \circ \mu_{100} + h z \delta_{010} \circ \mu_{001} = \\
z^2 (m_0 \otimes \mu_{001} + \mu_{001} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{001} + \delta_{001} \otimes \Delta_0) + \\
th (m_0 \otimes \mu_{100} + \mu_{100} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{010} + \delta_{010} \otimes \Delta_0) + \\
tz (m_0 \otimes \mu_{100} + \mu_{100} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{001} + \delta_{001} \otimes \Delta_0) + \\
hz (m_0 \otimes \mu_{001} + \mu_{001} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{010} + \delta_{010} \otimes \Delta_0). \quad (15)
\]

Contrary to the situation with the ordinary Lie bialgebra quantization (that is typical to \(D_{h,t,0}\)) the four relations obtained from \((15)\) refer only to the
facets of $\mathcal{D}_{h,t,z}$. To describe the necessary deformation into the nontrivial three-dimensional domain in $\mathcal{H}$ one must consider the higher orders with much more complicated relations. Thus one of the 3-d order equations (the $thz$-coefficients) looks like

$$
\delta_{001} \circ \mu_{110} + \delta_{010} \circ \mu_{101} + \delta_{110} \circ \mu_{101} + \delta_{101} \circ \mu_{101} = \\
(m_0 \otimes \mu_{110} + \mu_{110} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{001} + \delta_{001} \otimes \Delta_0) + \\
\left( m_0 \otimes \mu_{101} + \mu_{101} \otimes m_0 + \\
m_0^s \otimes \mu_{100} + \mu_{100} \otimes m_0^s + \\
m_0^s \otimes \mu_{001} + \mu_{100} \otimes m_0^s \right) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \delta_{101} + \delta_{101} \otimes \Delta_0) + \\
(m_0 \otimes \mu_{100} + \mu_{100} \otimes m_0) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ \left( \frac{\Delta_0 \otimes \delta_{011} + \delta_{011} \otimes \Delta_0}{\Delta_{010} \otimes \delta_{001} + \delta_{001} \otimes \Delta_{010}} + \frac{\Delta_{010} \otimes \delta_{010} + \delta_{010} \otimes \Delta_{010}}{\Delta_{010} \otimes \delta_{001} + \delta_{001} \otimes \Delta_{010}} \right).
$$

Note that symmetric parts denoted by $m^s_{ijk}$ and $\Delta^s_{ijk}$ also appear in this relation. To be able to obtain the explicit realizations of deformed Lie-Poisson construction one must investigate the possibility to simplify the deformation equations imposing some restrictions on the bialgebras involved.

### 4 Twice-first-order class of deformations

Here we shall formulate the conditions that will guarantee the existence of the nonempty $\mathcal{D}_{h,t,z}$. In some sense this will also show the way how to realize it explicitly.

**Theorem.** Let

1. $A$ be a Lie algebra and $A^*$ a Lie coalgebra fixed by the compositions $\mu_{100} : V \wedge V \to V$ and $\delta_{010} : V \to V \wedge V$,

2. $\mu_{001}$ and $\delta_{001}$ be the deforming functions defining first order deformations of $A$ and $A^*$ respectively,

3. the following four pairs of compositions be Lie bialgebras: $(\mu_{100}, \delta_{010})$, $(\mu_{001}, \delta_{010})$, $(\mu_{100}, \delta_{001})$, $(\mu_{001}, \delta_{001})$. 

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Then there exists in $\mathcal{H}$ the three-dimensional analytic subvariety $D_{h,t,z}$ ($h, t, z \in K$) such that

1. for every fixed $\tilde{z} \neq 0$ the corresponding two-dimensional subvariety $D_{h,t,\tilde{z}}$ has the boundaries consisting of one-dimensional $D_{0,t,\tilde{z}}$ and $D_{h,0,\tilde{z}}$ and zero-dimensional $D_{0,0,\tilde{z}}$ subvarieties that can be identified with the contractions of (in general noncommutative and noncocommutative) Hopf algebras $H_{0,t,\tilde{z}}$ and $H_{h,0,\tilde{z}}$ with the common limit $H_{0,0,\tilde{z}}$ (in general noncommutative and noncocommutative),

2. in the limit
   \[ \lim_{\tilde{z} \to 0} D_{h,t,z} = D_{h,t,0} \]
   the subvariety $D_{h,t,0}$ describes the deformation quantization of the Lie bialgebra $(\tilde{\mu}_{101}, \tilde{\delta}_{011})$ with the curves $D_{0,t,0}$ and $D_{h,0,0}$ presenting the canonical dual classical limits for $H_{h,t,0}$, $D_{0,0,0}$ coincides with the common contraction limit of $H_{0,t,0}$ and $H_{h,t,0}$.

**Proof.** According to the condition 2 for every $h, t, z, \in K$ the compositions

\[ \tilde{\mu}_{101} \equiv z\mu_{001} + t\mu_{100}, \]

and

\[ \tilde{\delta}_{011} \equiv z\delta_{001} + h\delta_{010} \]

define on $V$ Lie algebra and Lie coalgebra correspondingly. It is easy to verify that $(\tilde{\mu}_{101}, \tilde{\delta}_{011})$ form a Lie bialgebra. Inserting (17,18) into the equation

\[ \tilde{\delta}_{011} \circ \tilde{\mu}_{101} = (m_{000} \otimes \tilde{\mu}_{101} + \tilde{\mu}_{101} \otimes m_{000}) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_{000} \otimes \tilde{\delta}_{011} + \tilde{\delta}_{011} \otimes \Delta_{000}) \]

one gets the relation that splits into four equations. Each of them describes the bialgebraic property of the corresponding pair $(\mu_{100}, \delta_{010})$, $(\mu_{001}, \delta_{010})$, $(\mu_{100}, \delta_{001})$, and $(\mu_{001}, \delta_{001})$. The condition 3 guarantees these properties.

Due to the result proved by P.Etingof and D.Kazhdan [7] the Lie bialgebra $(\tilde{\mu}_{101}, \tilde{\delta}_{011})$ can be quantized. The reasoning presented by N.Reshetikhin [8] shows that the corresponding Hopf algebras can be treated as deformation quantizations of $(\tilde{\mu}_{101}, \tilde{\delta}_{011})$. In [10] it was proved that these quantized objects form in $\mathcal{H}$ an analytic subvariety $D_{x,y}$ ($x, y \in K$) where parameters $x, y$
correspond to the trivial contractions of compositions $\tilde{\mu}_{101}$ and $\tilde{\delta}_{011}$. This means that (at least when conditions exposed in subsection 2.2 are valid) we can pass to the set of bialgebras $\{(x\tilde{\mu}_{101}, y\tilde{\delta}_{011})\}$ that can be written as

$$x\tilde{\mu}_{101} \Rightarrow \tilde{\mu}'_{101} \equiv z'\mu_{001} + t\mu_{100},$$

$$y\tilde{\delta}_{011} \Rightarrow \tilde{\delta}'_{011} \equiv z''\delta_{001} + h\delta_{010}. \quad (21)$$

The corresponding quantizations $\mathcal{D}_{x,y} = \mathcal{D}_{h,t,z,z''}$ form in $\mathcal{H}$ the four-dimensional subvariety with analytic coordinates $h, t, z', z'' \in \mathbb{K}$. Its boundaries contain the two-dimensional domains of classical objects - $\mathcal{D}_{0,t,z,0} \ni U(\tilde{\mu}'_{101})$ and $\mathcal{D}_{h,0,0,z''} \ni \text{Fun}(G(\tilde{\delta}'_{011}))$, where $G(\tilde{\delta}'_{011})$ is the universal covering Lie group corresponding to the Lie coalgebra $\tilde{\delta}'_{011}$.

Now let us equalize the parameters $z'$ and $z''$

$$\mathcal{D}_{h,t,z,z} \equiv \mathcal{D}_{h,t,z}.$$ 

Contrary to the four-dimensional picture in the three-dimensional domain $\mathcal{D}_{h,t,z}$ thus obtained the two-dimensional facets $\mathcal{D}_{0,t,z}$ and $\mathcal{D}_{h,0,z}$ contain (in general) noncommutative and noncocommutative objects. The lower facet $\mathcal{D}_{h,t,0}$ corresponds obviously to the quantizations of initial Lie bialgebra $(A, A^*) = (t\mu_{100}, h\delta_{010})$. Thus in the boundary of $\mathcal{D}_{h,t,z}$ only two edge curves are classical:

$$\mathcal{D}_{0,t,0} \ni U(t\mu_{100}),$$

$$\mathcal{D}_{h,0,0} \ni \text{Fun}(G(h\delta_{010})).$$

The third edge - $\mathcal{D}_{0,0,z}$- is the "diagonal" of the bialgebra $(\mu_{001}, \delta_{001})$ quantizations. There always exists in $\mathcal{D}_{h,t,z}$ such a neighborhood of $\mathcal{D}_{h,t,0}$ where Hopf algebras $H_{0,t,z}, H_{h,0,z}$ and $H_{0,0,z}$ are (in general) nonequivalent to each other. Thus the curves $\mathcal{D}_{0,t,z}$ and $\mathcal{D}_{h,0,z}$ can be treated as nontrivial contractions of $H_{0,t,z}$ and $H_{h,0,z}$ ( $H_{0,0,z}$ being their common noncommutative and noncocommutative limit).

We have proved that given two $\mu$- and two $\delta$- compositions (Lie and co-Lie respectively) that form four Lie bialgebras and two first order deformations one can always construct the parameterized (by $z$ ) set $\mathcal{D}_{h,t,z}$ with the properties described in subsection 2.2.

**Note.** The theorem holds also true after the interchange

$$\mu_{100} \Leftrightarrow \mu_{001},$$

$$\delta_{010} \Leftrightarrow \delta_{001}.$$
This means that one really obtain two different three-dimensional pictures from the four-dimensional one. The two subvarieties of ordinary deformation quantizations $D_{h,t,0,0}^0$ and $D_{0,0,z',z''}^0$ can be treated on equal footing – one can use the "diagonal" from any of them.

5 Example

The Theorem proved above can be used to construct the explicit examples of the analytic variety $D_{h,t,z}$ and the vector fields corresponding to the deformed Lie-Poisson structure.

Consider the 6-dimensional vector space $V$ over $\mathbb{C}$ with the basis $\{p_x, p_y, p_z, l_x, l_y, l_z\}$.

Fix the initial Lie bialgebra $(A, A^*)$ by the compositions

$$\mu_{100} = \left\{ C_{l_y l_x}^{l_x} = 1, C_{l_z l_x}^{l_x} = 1, C_{p_y l_x}^{p_x} = -1, C_{p_z l_x}^{p_x} = -1 \right\},$$

$$\delta_{010} = \left\{ D_{l_x}^{l_y} = i \right\}.$$ (22)

Chose the following first order deforming functions for algebras $A$ and $A^*$ respectively:

$$\mu_{001} = \left\{ C_{p_y}^{p_x} = i \right\},$$

$$\delta_{001} = \left\{ D_{p_y}^{p_x} = -1/2, D_{p_z}^{p_x} = -1/2 \right\}.$$ (24)

(25)

It is easy to check that $\tilde{\mu}_{101}$ and $\tilde{\delta}_{011}$ (see (20,21)) are Lie and co-Lie compositions for arbitrary complex parameters $(z', t)$ and $(z'', h)$. Combined into four pairs $-(\mu_{100}, \delta_{010})$, $(\mu_{001}, \delta_{010})$, $(\mu_{100}, \delta_{001})$ and $(\mu_{001}, \delta_{001})$. – they form the four Lie bialgebras mentioned in the Theorem.

The composition $\tilde{\delta}_{011}$ can be treated as the direct sum of Heizenberg and $\bar{e}(2)$ Lie coalgebras. This simplifies considerably the construction of quantum deformations of the Lie bialgebra $(\tilde{\mu}_{101}, \tilde{\delta}_{011})$. The direct sum structure and the solvability of the coalgebra $\tilde{\delta}_{011}$ makes it possible to use here the algorithm developed in [9]. The result is the four-dimensional variety $D_{h,t,z',z''}$ of Hopf
algebras with the following multiplication and comultiplication:

\[
[p_x, p_y] = 0, \\
[p_z, p_x] = iz'p_y, \\
[p_z, p_y] = -ih^2t^2 \sinh (z''p_x), \\
[l_z, l_x] = \frac{4}{z''^2p_x} \sinh (z''h_l x), \\
[l_z, l_y] = iz''p_x \sinh (z''h_l z) - 1, \\
[l_y, l_x] = tly, \\
[l_y, l_z] = \frac{4}{z''} (\cosh (z''h_l z) - 1), \\
[p_x, l_x] = -\frac{1}{2} (1 + \cosh (z''h_l z)) p_y - \frac{i \hbar^2 t^2}{2} \sinh (z''h_l z) \exp \left( \frac{z''}{2} p_x \right), \\
[p_z, l_x] = -\frac{1}{2} (1 + \cosh (z''h_l z)) p_z + \hbar^2 t \left( il_y + \frac{z''}{2} \sinh (z''h_l z) \right) \exp \left( -\frac{z''}{2} p_x \right), \\
[p_x, l_y] = i t \sinh (z''h_l z), \\
[p_y, l_x] = \hbar^2 t^2 \left( \cosh (z''h_l z) \exp \left( \frac{z''}{2} p_x \right) - \exp \left( -\frac{z''}{2} p_x \right) \right) - i \frac{z''}{2} p_y \sinh (z''h_l z), \\
[p_z, l_y] = \frac{1}{2} i z''^2 \hbar^2 t^2 \left( \cosh (z''h_l z) - 1 \right) \exp \left( -\frac{z''}{2} p_x \right) - i \frac{z''}{2} p_z \sinh (z''h_l z), \\
[p_x, l_z] = 0, \\
[p_y, l_z] = 0, \\
[p_z, l_z] = 2 \hbar t \sinh \left( \frac{z''}{2} p_x \right); \\
\]

\[
\Delta p_x = 1 \otimes p_x + p_x \otimes 1, \\
\Delta p_y = \exp \left( -\frac{z''}{2} p_x \right) \otimes p_y + p_y \otimes \exp \left( \frac{z''}{2} p_x \right), \\
\Delta p_z = \exp \left( -\frac{z''}{2} p_x \right) \otimes p_z + p_z \otimes \exp \left( \frac{z''}{2} p_x \right), \\
\Delta l_x = l_x \otimes \cosh (z''h_l z) + 1 \otimes l_x - i \frac{1}{2} l_y \otimes \sinh (z''h_l z), \\
\Delta l_y = l_y \otimes \cosh (z''h_l z) + 1 \otimes l_y + i \frac{z''}{2} l_x \otimes \sinh (z''h_l z), \\
\Delta l_z = l_z \otimes 1 + 1 \otimes l_z.
\]

(26)

Note that the $z'$ parameter appears only in one of the defining relations (in the commutator $[p_z, p_x]$).

The two "classical" facets $\mathcal{D}_{0, t, z', 0}$ and $\mathcal{D}_{h, t, 0, z''}$ of the variety $\mathcal{D}_{h, t, z', z''}$ contain $H_{0, t, z', 0}$ - the universal enveloping algebra of a Lie algebra (from now on we write down only the nonzero commutators and nontrivial cocommutators):

\[
[p_z, p_x] = iz'p_y, \\
[l_z, l_x] = tl_z, \\
[p_y, l_x] = -tp_y, \\
[l_y, l_x] = tl_y, \\
[p_z, l_x] = -tp_z.
\]

(28)
and an algebra \( H_{h,0,0,z''} \) of exponential coordinate functions of the group \( \tilde{E}(2) \times E(2) \). In the latter case the restriction \( \Delta_{IV} \) of the coproduct in \( H_{h,0,0,z''} \) coincides with that of the general case and is presented by the relations (27).

The "lower" – \( D_{h,t,0,0} \) and the "upper" – \( D_{0,0,z',z''} \) – facets of \( D_{h,t,z',z''} \) look like the ordinary deformation quantizations. The points \( H_{h,t,0,0} \) refer to the quantizations of \( (\mu_{100}, \delta_{010}) \)-bialgebra with the defining relations

\[
[l_z, l_x] = tl_z, \quad [p_y, l_x] = -tp_y - ih^3 t^2 l_z,
\]

\[
[p_z, l_x] = -tp_z + ih^2 t l_y;
\]

\[
\Delta l_x = l_x \otimes 1 + 1 \otimes l_x - i h l_y \otimes l_z.
\]

While the points \( H_{0,0,z',z''} \) describe the quantized Heisenberg algebra

\[
[p_z, p_x] = iz' p_y,
\]

\[
\Delta p_y = \exp \left( -\frac{z''}{2} p_x \right) \otimes p_y + p_y \otimes \exp \left( \frac{z''}{2} p_x \right),
\]

\[
\Delta p_z = \exp \left( -\frac{z''}{2} p_x \right) \otimes p_z + p_z \otimes \exp \left( \frac{z''}{2} p_x \right).
\]

Now let us put \( z' = z'' = z \) and consider the three-dimensional variety \( D_{h,t,z} \). This means that we chose only the "diagonal" points in \( H_{0,0,z',z''} \). The lower facet \( H_{h,t,0,0} \) rests unchanged and fixes the initial (in our interpretation the undeformed) Lie-Poisson structure – the Heisenberg Poisson algebra for the group \( G(\mu_{100}) \) or the \( \mu_{100} \) Poisson algebra for the Heisenberg group.

The two-dimensional facets \( D_{0,t,z} \) and \( D_{h,0,z} \) can no more be considered as ordinary quantizations. Each of them have only one classical edge \( D_{0,t,0} \in D_{0,t,z} \) and \( D_{h,0,0} \in D_{h,0,z} \), their common edge \( D_{0,0,z} \) refers to the quantized Heisenberg algebra \( H_{0,0,z} \) (defined by (22),28) with \( z' = z'' = z \). Thus considering the parameter \( z \) in the variety \( D_{h,0,z} \) as the deformation parameter we see that in the Hopf algebras \( H_{h,0,z} \),

\[
[p_z, p_x] = iz p_y,
\]

\[
\Delta p_y = \exp \left( -\frac{z}{2} p_x \right) \otimes p_y + p_y \otimes \exp \left( \frac{z}{2} p_x \right),
\]

\[
\Delta p_z = \exp \left( -\frac{z}{2} p_x \right) \otimes p_z + p_z \otimes \exp \left( \frac{z}{2} p_x \right),
\]

\[
\Delta l_x = l_x \otimes \cosh (z h l_z) + 1 \otimes l_x - i \frac{l_y}{z} l_y \otimes \sinh (z h l_z),
\]

\[
\Delta l_y = l_y \otimes \cosh (z h l_z) + 1 \otimes l_y + iz l_y \otimes \sinh (z h l_z),
\]

\[
\Delta l_z = l_z \otimes \cosh (z h l_z) + 1 \otimes l_z - i \frac{l_y}{z} l_y \otimes \sinh (z h l_z).
\]
not only the multiplication but also the comultiplication is deformed. The same is true also for $H_{0,t,z}$:

\[
[p_z, p_x] = izp_y, \\
[l_z, l_x] = tl_z, \\
[p_y, l_z] = -tp_z, \\
[p_z, l_z] = -tp_z;
\]

\[
\Delta p_y = \exp \left(-\frac{i}{2}p_x \right) \otimes p_y + p_y \otimes \exp \left(\frac{i}{2}p_x \right),
\]

\[
\Delta p_z = \exp \left(-\frac{i}{2}p_x \right) \otimes p_z + p_z \otimes \exp \left(\frac{i}{2}p_x \right);
\]

(34)

This property of $D_{h,0,z}$ and $D_{0,t,z}$ resembles that of the quantized version of the cotangent bundle $[3]$ where the multiplication on $\operatorname{Fun}(T^*G)$ is deformed simultaneously with the group $G \triangleright T^*G$ itself. Note that in our case the Hopf structure is preserved.

The varieties $D_{h,t,z}$ with fixed $z = \bar{z} \neq 0$ are especially interesting for us. Their boundaries lie in $D_{h,0,z}$ and $D_{0,t,\bar{z}}$. The internal points $H_{h,t,z}$ can be interpreted as quantizations of $H_{0,t,z}$ by $H_{h,0,\bar{z}}$ and $\bar{z}$ measures the difference between the canonical $D_{h,t,0}$ and deformed $D_{h,t,\bar{z}}$ pictures. It is evident that Hopf algebras $H_{h,t,z}$ (see (31,32) are inequivalent to the quantized Heisenberg algebra $H_{0,0,\bar{z}}$. The same is true for $H_{h,0,z}$ and $H_{h,0,\bar{z}}$. Thus the boundaries $D_{h,0,z}$ and $D_{0,t,\bar{z}}$ can be treated as the nontrivial contraction curves with the common contraction limit $D_{0,0,\bar{z}}$.

In the initial variety $D_{h,0,0}$ the Lie-Poisson structure is defined by the tangent fields $V_{0,t,0}$:

\[
\delta(l_x) = -il_y \wedge l_z,
\]

(35)

and $W_{h,0,0}$ :

\[
\mu(l_z, l_x) = l_z, \quad \mu(l_y, l_z) = l_y, \\
\mu(p_y, l_x) = -p_y, \quad \mu(p_z, l_x) = -p_z + i\hbar^2 l_y.
\]

(36)

In the deformed case the field $V_{0,t,\bar{z}}$ contains both multiplication and comultiplication components. To simplify the comparison with $V_{0,t,0}$ we shall present the first terms of its power series expansion in $\bar{z}$ and $t$ :

\[
\mu(l_z, l_y) = \frac{1}{2} t \bar{z}^2 l_z^2 + ..., \\
\mu(p_x, l_y) = it \bar{z} l_x + ..., \\
\mu(p_y, l_z) = -\frac{1}{2} t \bar{z}^2 l_z p_y + ..., \\
\mu(p_z, l_y) = -\frac{1}{2} t \bar{z}^2 l_z p_z + ..., \\
\mu(p_z, l_z) = t \bar{z} p_z + ..., \\
\delta(l_x) = -il_y \wedge l_z + ..., \\
\delta(l_y) = i\bar{z}^2 l_x \wedge l_z + ... .
\]

(37)
The field $V_{0,t,0}$ is reobtained in the limit $\lim_{\tilde{z} \to 0} V_{0,t,\tilde{z}} = V_{0,t,0}$. The co-Poisson structure defined by $V_{0,t,0}$ is deformed to describe the correlation between the possible additional quantization of $H_{0,t,\tilde{z}}$ and the deformation of its multiplication structure constants.

The field $W_{h,0,\tilde{z}}$ does not obtain the additional components in the coproduct sector:

$$
\mu(l_z, l_x) = l_z + ..., \quad \mu(l_y, l_x) = l_y, \quad \mu(l_z, l_y) = \frac{i}{2} \hbar \tilde{z}^2 l_z^2 + ..., \\
\mu(p_y, l_x) = -p_y + ..., \quad \mu(p_z, l_x) = -p_z + i\hbar \tilde{l}_z + ..., \\
\mu(p_x, l_x) = \frac{1}{2} \hbar^2 \tilde{z}^2 l_z^2 + ..., \quad \mu(p_x, l_y) = i\hbar \tilde{z} l_z + ..., \\
\mu(p_y, l_y) = -\frac{i}{2} \hbar \tilde{z}^2 l_z p_y + ..., \quad \mu(p_z, l_y) = -\frac{i}{2} \hbar \tilde{z}^2 l_z p_z + ..., \\
\mu(p_z, l_z) = \hbar \tilde{z} p_x + ... .
$$

Considered as function of $z$ the field $W_{h,0,z}$ describes the explicit dependence of the Poisson algebra on the quantization of the group $H_{0,t,z}$ where $z$ now plays the role of the deformation parameter.

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