A characterization of the Arf property for quadratic quotients of the Rees algebra

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Abstract

We provide a characterization of the Arf property in both the numerical duplication of a numerical semigroup and in a member of a family of quotients of the Rees algebra studied in [2].

Introduction

Let $R$ be a Noetherian one-dimensional local domain, $I$ an ideal of $R$ and $t$ and indeterminate. Let $R[It] = \bigoplus_{n \in \mathbb{N}} I^nt^n$ be the Rees algebra associated with $R$ and $I$. In [2] the authors, looking for a unified approach to Nagata’s idealization and the amalgamated duplication of a ring (see [3]), studied the following family of quotients of the Rees algebra

$$R(I)_{a,b} = \frac{R[It]}{(t^2 + at + b) \cap R[It]}.$$ 

showing that Nagata’s idealization is obtained for $a = b = 0$ and amalgamated duplication for $a = 1$, $b = 0$. A remarkable fact about this family of ring is that we can always find domains between its members, if the original ring $R$ is itself a domain. In particular, it was shown in [2] that this ring construction can be connected to a semigroup construction called numerical duplication (see [4]). More precisely let $S$ be a numerical semigroup, let $E$ be a semigroup ideal, $m \in S$ an odd integer. For any set of integers $A \subseteq \mathbb{Z}$ we set $2 \cdot A = \{2a : a \in A\}$. Then we define the numerical duplication $S \bowtie m E$ of $S$ with respect to $E$ and $m$ as the numerical semigroup

$$S \bowtie m E = 2 \cdot S \cup (2 \cdot E + m).$$

Now, if we start with an algebroid branch $R$ and $b \in R$ with $v(b)$ odd, the member of the family of the type $R(I)_{0,-b}$ has its value semigroup equal to
the numerical duplication of \( v(R) \) with respect to \( v(I) \) and \( v(b) \). In this paper we show that this is true in general for every Noetherian, one-dimensional, analytically irreducible, local domain \( R \).

In [1] Arf solved the classification problem of singular branches, using their multiplicity sequence. Later, inspired by the work of Arf, Lipman in [9] introduced the notions of Arf ring and Arf closure of a ring. These rings share the same multiplicity sequence. Hence the idea is to calculate the Arf closure of the coordinate ring of a curve and then its value semigroup, which is an Arf numerical semigroup, in order to obtain its multiplicity sequence.

In this paper we provide a characterization of the Arf property both in the numerical duplication and the rings \( R = R(I)_0^- b \).

More precisely, in Section 1 we recall all the basic notions on numerical semigroups and Arf rings.

In Section 2, we prove the characterization of the Arf property for the numerical duplication (Theorem 2.4).

In Section 3, we show that \( R \) is a Noetherian one-dimensional analytically irreducible local domain and its value semigroup is \( v(R) \cong v(b) v(I) \) (Theorem 3.1), then we prove a series of technical lemmas for the purpose of proving Theorem 3.12 that is the extension to \( R \) of the previous numerical characterization.

Several computations are performed by using the GAP system [10] and, in particular, the NumericalSgps package [6].

1 Preliminaries

A numerical semigroup \( S \) is an additive submonoid of \( \mathbb{N} \) with finite complement in \( \mathbb{N} \). The multiplicity of \( S \) is \( \mu(S) = \min(S \setminus \{0\}) \). The Frobenius number of \( S \) is \( F(S) = \max(\mathbb{N} \setminus S) \) and the conductor of \( S \) is \( c(S) = F(S) + 1 \).

A semigroup ideal of \( S \) is a subset \( E \subseteq S \) such that \( E + S \subseteq E \). We call \( \hat{e} = \min E \); then the integral closure of \( E \) in \( S \) is the semigroup ideal \( \overline{E} = \{ s \in S : s \geq \hat{e} \} \), if \( E = \overline{E} \) then \( E \) is integrally closed. We say that \( E \) is stable if \( E + E = E + \hat{e} \).

An Arf numerical semigroup is a numerical semigroup \( S \) in which for every \( x, y, z \in S \), such that \( x \geq y \geq z \), it results \( x + y - z \in S \); equivalently \( S \) is Arf if and only if every integrally closed semigroup ideal is stable (see [9] Theorem 2.2).

Given an Arf numerical semigroup \( S = \{ 0 = s_0 < s_1 < s_2 < \ldots \} \) the sequence \( (e_0, e_1, e_2, \ldots) \), with \( e_i = s_{i+1} - s_i \), is the multiplicity sequence of \( S \).
Note that $e_0$ corresponds to the multiplicity of $S$.

We call an **Arf sequence** a non increasing sequence of positive integers $(e_0, e_1, e_2, \ldots)$ such that

1. exists $n \in \mathbb{N}$ such that $e_k = 1$ for all $k \geq n$,

2. for every $i \in \mathbb{N}$ exists $k \geq 1$ such that $e_i = \sum_{j=1}^{k} e_{i+j}$.

A sequence of positive integers $(e_0, e_1, e_2, \ldots)$ is an Arf sequence if and only if it is a multiplicity sequence of an Arf numerical semigroup, that is $S = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \ldots\}$, see for instance [7, Proposition 1].

From the Arf numerical semigroup $S$ we can construct a chain of Arf numerical semigroups $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ with $S_n = S_0$ and $S_{i+1} = (S_i \setminus \{0\}) - e_i$, namely the blow up of $S_i$. The multiplicity of $S_i$ is $e_i$ and $S_n = \mathbb{N}$ for $n >> 0$.

Let $(R, \mathfrak{m})$ be a Noetherian one-dimensional local domain and $\overline{R}$ its integral closure in its field of fractions $Q(R)$. We assume that $R$ is **analytically irreducible**, that is its completion $\hat{R}_\mathfrak{m}$ is a domain, or, equivalently, $\overline{R}$ is a DVR and a finitely generated $R$-module. Since the integral closure $\overline{R}$ is a DVR, every non zero element of $R$ has a value as an element of $\overline{R}$. The set of values $v(R) = S$ is a numerical semigroup. The multiplicity of $R$ is equal to the multiplicity of its value semigroup $\mu(R) = \mu(v(R))$.

For any two $R$-submodules $E, F$ of $\overline{R}$ set

$$(E : F) = \{x \in \overline{R}: xF \subseteq E\}.$$  

The blow up of $R$ is $L(R) = \bigcup_{n \in \mathbb{N}} (\mathfrak{m}^n : \mathfrak{m}^n)$. If we fix $R = R_0$ and $R_{i+1} = L(R_i)$ then the multiplicity sequence of $R$ is the sequence $(\mu(R_0), \mu(R_1), \ldots)$.

We will also assume that $R$ is **residually rational**, namely its residue field $k = R/\mathfrak{m}$ is isomorphic to the residue field of $\overline{R}$. With this assumption, for any $x, y \in R$ such that $v(x) = v(y)$ there exists $u \in R$ invertible such that $y = ux$. Furthermore, for any fractional ideals $I, J$ of $R$ such that $J \subseteq I$, it results $\lambda(I/J) = |v(I) \setminus v(J)|$.

An element $x \in R$ is said to be integral over the ideal $I$ if $x$ satisfies a relation

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

with $a_j \in I^j$ for $j = 1, 2, \ldots, n$. The set $\overline{I}$ of all elements of $R$ which are integral over $I$ is an ideal of $R$ (see [29] or [33]), called the integral closure of $I$ in $R$.

In our setting, the integral closure of an ideal $I$ is equal to

$$\overline{I} = I\overline{R} \cap R = \{x \in R : v(x) \geq i_1\},$$
with $i_1 = \min v(I)$ (see Proposition 1.6.1, Proposition 6.8.1]). If $I = \mathcal{T}$ then $I$ is integrally closed. It follows that the ideal $I$ is integrally closed if and only if the semigroup ideal $v(I)$ is integrally closed. The necessity easily follows from the definitions. For the sufficiency we have $\lambda(\mathcal{T}/I) = |v(\mathcal{T}) \setminus v(I)| = 0$, then $I = \mathcal{T}$. Notice that here the assumption that $R$ is residually rational is needed.

Let $x \in I$ be such that $v(x) = i_1$. The ideal $I$ is stable if $I^2 = xI$, or, equivalently, if $(I : I) = x^{-1}I$. The ring $R$ is an Arf ring if every integrally closed ideal is stable. If $R$ is Arf then its value semigroup $S = v(R)$ is an Arf numerical semigroup and the multiplicity sequence of $R$ coincides with the multiplicity sequence of $S$.

2 Arf property in the numerical duplication

In this section $S = \{0 = s_0 < s_1 < s_2 < \ldots\}$ will be a numerical semigroup, $E$ a semigroup ideal of $S$ and $m \in S$ an odd integer. Recall that the quotient of $S$ by a positive integer $d$ is

$$\frac{S}{d} = \{x \in \mathbb{N} : dx \in S\}.$$

**Proposition 2.1.** For every $d > 0$, if $S$ is Arf so is $\frac{S}{d}$.

**Proof.** Let $x, y, z \in \frac{S}{d}$ with $x \geq y \geq z$, then we have $dx, dy, dz \in S$ with $dx \geq dy \geq dz$ and since $S$ is Arf it follows that

$$d(x + y - z) = dx + dy - dz \in S,$$

hence $x + y - z \in \frac{S}{d}$. \hfill \Box$

By definition of numerical duplication it is clear that $(S \bowtie m E)/2 = S$, hence we immediately get the following

**Corollary 2.2.** If $S \bowtie m E$ is Arf so is $S$.

**Lemma 2.3.** If $S \bowtie m E$ is Arf then $E$ is integrally closed.

**Proof.** Suppose by contradiction that $E$ is not integrally closed. Then there exists $i \in \mathbb{N}$ such that $s_i \in E$ and $s_{i+1} \notin E$. Consider $2s_{i+1}, 2s_i + m, 2s_i \in S \bowtie m E$, since $2s_{i+1} \geq 2s_i, 2s_i + m \geq 2s_i$ and $S \bowtie m E$ is Arf, then

$$2s_{i+1} + 2s_i + m - 2s_i = 2s_{i+1} + m \in S \bowtie m E,$$

which means $s_{i+1} \in E$, contradiction. \hfill \Box
Let $S$ be an Arf numerical semigroup, and let $(e_0, e_1, e_2, \ldots)$ be its multiplicity sequence. Fix $n \in \mathbb{N}$ to be the smallest integer such that $e_k = 1$ for every $k \geq n$. We recall that $e_0 = \min S \setminus \{0\} = s_1$, and that $s_{i+1} = e_0 + \ldots + e_i$, in particular $s_{n+1} = s_n + 1$ and $s_n$ is the conductor of $S$.

In the proof of the following result we will use the fact that for an Arf numerical semigroup $S$, if $x, x + 1 \in S$ then $x + \mathbb{N} \subseteq S$ (see for instance [11, Lemma 11]).

**Theorem 2.4.** The numerical duplication $D = S \rtimes m E$ is Arf if and only if $S$ is Arf, $E$ is integrally closed and, if $\min(E) < s_n$, $e_0$ is odd and $m = e_0 = e_1 = \ldots = e_{n-1}$.

Proof. **Necessity.** From Corollary 2.2 and Lemma 2.3 $S$ is Arf and $E$ is integrally closed. Now if $\min(E) < s_n$ then $s_{n-1} \in E$. Suppose that $m \geq 2e_{n-1} = 2(s_n - s_{n-1})$, then

$$2s_{n-1} + m \geq 2s_n.$$ 

Since $2s_{n-1} + m + 1$ is even and $s_n$ is the conductor of $S$ we obtain $2s_{n-1} + m + 1 = 2s_k$ for some $k \in \mathbb{N}$. Setting $x = 2s_{n-1} + m$ we have $x, x + 1 \in D$ which is Arf, hence $x + \mathbb{N} \subseteq D$, and so $x + 2 \in D$; this means

$$2s_{n-1} + m + 2 = 2s_n + m$$ 

$$2(s_n - s_{n-1}) = 2$$ 

$$e_{n-1} = 1,$$

which is a contradiction. Therefore $m < 2e_{n-1}$, hence $2s_{n-1} + m < 2s_n$, since $D$ is Arf, this implies

$$2s_n + 2s_n - (2s_{n-1} + m) = 2s_n + 2e_{n-1} - m \in D.$$ 

Furthermore $2s_n + 2e_{n-1} - m$ is odd and

$$2s_n + 2e_{n-1} - m \geq 2s_n > 2s_{n-1} + m.$$ 

It follows that

$$2s_n + 2e_{n-1} - m \geq 2s_n + m$$ 

$$\Rightarrow m \leq e_{n-1} \leq e_0.$$ 

Since $m \in S$ and it is odd then we must have $m = e_0 = e_1 = \ldots = e_{n-1}$.

**Sufficiency.** If $\min(E) \geq s_n$ then $E = x + \mathbb{N}$ with $x \geq s_n$ so it results $D = 2S \cup ((2x + m) + \mathbb{N})$ and it is easy to check that $D$ is Arf.
Otherwise if \( \min(E) < s_n \) and \( m = e_0 = e_1 = \ldots = e_{n-1} \) then \( S = e_0 \mathbb{N} \cup (ne_0 + \mathbb{N}) \) and \( E = \{ie_0, (i+1)e_0, \ldots, (n-1)e_0 \} \cup (ne_0 + \mathbb{N}) \) for some \( i \leq n \).

Hence, if \( D = \{ 0 = d_0 < d_1 < \ldots < d_k < \ldots \} \) then after some easy calculations it results

\[
(d_{k+1} - d_k : k \in \mathbb{N}) = (2e_0, 2e_0, \ldots, 2e_0, 2, 2, \ldots, 2, 1, \ldots),
\]

which is an Arf sequence, so \( D \) is Arf.

**Example 2.5.** Let \( S = \langle 3, 7, 8 \rangle = \{ 0, 3, 6, \rightarrow \} \), \( S \) is Arf and its multiplicity sequence is \( (3, 3, 1, \ldots) \), so \( n = 2 \). Let \( E = S \setminus \{ 0 \} \) and \( m = 3 \), \( E \) is integrally closed, \( \min(E) = 3 < 6 = s_n \) and \( m = e_0 = e_1 \). The numerical duplication is

\[
S \times^m E = \langle 6, 9, 14, 16, 17, 19 \rangle = \{ 0, 6, 9, 12, 14, \rightarrow \},
\]

and it is an Arf numerical semigroup.

**Remark 2.6.** In the case \( \min(E) < s_n \) of Theorem 2.4, the elements of \( D = S \times^m E \) less than the conductor of \( D \) are multiples of \( m = e_0 \), so they are of the form \( ke_0 \) for some \( k \in \mathbb{N} \).

Recall that the Arf closure \( \text{Arf}(S) \) of a numerical semigroup \( S \) is the smallest Arf numerical semigroup that contains \( S \) (see \[11\]). Let \( \bar{E} \) be the integral closure in \( \text{Arf}(S) \) of the ideal generated by \( E \) in \( \text{Arf}(S) \). More explicitly, if \( \bar{e} = \min E \), then \( \bar{E} = \{ s \in \text{Arf}(S) : s \geq \bar{e} \} \).

**Proposition 2.7.** With the notation introduced above we have

\[
\text{Arf}(S) \times^m \bar{E} \subseteq \text{Arf}(S \times^m E).
\]

**Proof.** Since \( S = (S \times^m E)/2 \subseteq \text{Arf}(S \times^m E)/2 \) and, by Proposition 2.1, \( \text{Arf}(S \times^m E)/2 \) is Arf, we got

\[
\text{Arf}(S) \subseteq \frac{\text{Arf}(S \times^m E)}{2};
\]

it follows that \( 2 \cdot \text{Arf}(S) \subseteq \text{Arf}(S \times^m E) \). Now with the same argument used in the proof of Lemma 2.3 we have that \( 2 \cdot \bar{E} + m \subseteq \text{Arf}(S \times^m E) \). In fact let \( e \in \bar{E} \), then \( 2e \geq 2\bar{e} \) and \( 2\bar{e} + m \geq 2\bar{e} \). Since \( \text{Arf}(S \times^m E) \) is Arf and \( 2e, 2\bar{e}, 2\bar{e} + m \in \text{Arf}(S \times^m E) \) we have

\[
2e + 2\bar{e} + m - 2\bar{e} = 2e + m \in \text{Arf}(S \times^m E).
\]

Therefore \( \text{Arf}(S) \times^m \bar{E} \subseteq \text{Arf}(S \times^m E) \). \( \square \)
Theorem 2.4 gives us sufficient conditions so that the inclusion of Proposition 2.7 is an equality. Recall that \( E \) is the integral closure in \( S \) of the semigroup ideal \( E \).

**Corollary 2.8.** If one of the following conditions holds

1. \( \min(E) \geq s_n \),
2. \( S \) is Arf and \( m = e_0 = e_1 = \ldots = e_{n-1} \),

then \( \overline{E} = \bar{E} \) and \( \text{Arf}(S) \ltimes^m \overline{E} = \text{Arf}(S \ltimes^m E) \).

It is easy to see that the previous equality is not true in the general case. In particular, the following example shows that neither the equality \( \text{Arf}(S) = \frac{\text{Arf}(S \ltimes^m E)}{2} \) holds true in general.

**Example 2.9.** Let \( S = \langle 5, 8, 11, 12, 14 \rangle = \{0, 5, 8, 10, \rightarrow\} \), \( E = S \setminus \{0\} \) and \( m = 5 \). Note that \( S \) is Arf, so \( S = \text{Arf}(S) \). The numerical duplication of \( S \) with respect to \( E \) and \( m \) is

\[
S \ltimes^m E = \{0, 10, 15, 16, 20, 21, 22, 24, \rightarrow\}.
\]

Since \( 15, 16 \in S \ltimes^m E \), its Arf closure is \( \text{Arf}(S \ltimes^m E) = \{0, 10, 15 \rightarrow\} \); moreover \( 9 \in \text{Arf}(S \ltimes^m E)/2 \), but \( 9 \notin \text{Arf}(S) = S \).

A couple of questions naturally arise.

**Question 2.10.** Are there any sufficient and necessary conditions so that the inclusion of Proposition 2.7 is an equality? Is there a way to express \( \text{Arf}(S \ltimes^m E) \) in terms of \( S, E \) and \( m \) ?

### 3 Arf property in \( \mathcal{R} \)

In this section \( (R, \mathfrak{m}) \) will be a Noetherian, analytically irreducible, residually rational, one-dimensional, local domain with \( \text{char}(R) \neq 2 \); \( v : Q(R) \to \mathbb{Z} \) will denote the valuation on \( Q(R) \) associated to \( \overline{R} \). Let \( I \) be an ideal of \( R \) and let \( t \) be an indeterminate; the Rees algebra (also called Blow-up algebra) associated with \( R \) and \( I \) is the graded subring of \( R[t] \) defined as

\[
R[It] = \bigoplus_{n \in \mathbb{N}} I^n t^n.
\]
Let $b \in R$ such that that $v(b) = m$ is odd, we define

$$R = \frac{R[t]}{(t^2 - b) \cap R[t]}.$$ 

Then $R$ is a subring of $R[\alpha]$ with $\alpha = t + (t^2 - b)$. Furthermore $R$ and $R[\alpha]$ have the same integral closure $\overline{R}$ and the same field of fractions $Q(R)[\alpha]$ (see [2 Corollary 1.8]). Since $\alpha$ is integral over $R$, the integral closure $\overline{R}^{Q(R)[\alpha]}$ of $R$ in $Q(R)[\alpha]$ is the same as the integral closure of $R$. The extension field $Q(R) \subseteq Q(R)[\alpha]$ is finite and since $\text{char}(R) \neq 2$, it is also separable.

**Theorem 3.1.** The ring $R$ is a Noetherian one-dimensional local domain analytically irreducible and residually rational. If $v' : Q(R)[\alpha] \to \mathbb{Z}$ is the extension on $Q(R)[\alpha]$ of the valuation of $\overline{R}$, then $v'_{|Q(R)} = 2v$ and

$$v'(R) = v(R) \ltimes v(b) v(I) = S \ltimes v^m v(I).$$

**Proof.** From [2] we know that if $R$ is Noetherian, one-dimensional and local so is $R$. Moreover since $v(b)$ is odd, the polynomial $t^2 - b$ is irreducible in $Q(R)[t]$, then, from [5 Corollary 1.3], $R$ is a domain.

Now we prove that $R$ is analytically irreducible. It is enough to prove that $\overline{R}$ is local and a finitely generated $R$-module. Let $x \in \overline{R}$ be an element of valuation 1, $k = \frac{m-1}{2}$ and $\beta = \frac{\alpha}{x^k} \in Q(R)[\alpha]$; then $\beta^2 = \frac{b}{x^{2k}} \in \overline{R}$ since $v\left(\frac{b}{x^{2k}}\right) = 1 > 0$, so $\beta$ is integral over $\overline{R}$. We prove that $\overline{R} = \overline{R} + \mathbb{Z}\beta$.

The inclusion $\overline{R} + \mathbb{Z}\beta \subseteq \overline{R}$ follows from the fact that $\overline{R} \subseteq \overline{R}$ and that $\beta$ is integral over $\overline{R}$. Viceversa let $p + q\alpha \in \overline{R} = \overline{R}^{Q(R)[\alpha]}$, where $p, q \in Q(R)$, $q \neq 0$. Then, since $Q(R) \subseteq Q(R)[\alpha]$ is algebraic, from [3 Theorem 2.1.17] the coefficients $2p$ and $p^2 - q^2b$ of the minimal polynomial of $p + q\alpha$ over $Q(R)$ are in $\overline{R}$. In addition, $v(p^2) = 2v(p)$ is even and $v(q^2b) = 2v(q) + m$ is odd, then $v(p^2) \neq v(q^2b)$, therefore

$$0 \leq v(p^2 - q^2b) = \min\{2v(p), 2v(q) + m\}$$

$$\Rightarrow \begin{cases} v(p) \geq 0 \Rightarrow p \in \overline{R} \\ 2v(q) + m \geq 0 \Rightarrow v(q) \geq -k \Rightarrow qx^k \in \overline{R}. \end{cases}$$

Hence $p + q\alpha = p + qx^k\beta \in \overline{R} + \mathbb{Z}\beta$. Now if we denote by $\overline{m}$ the maximal ideal of $\overline{R}$, the ring $\overline{R} + \mathbb{Z}\beta$ is local with maximal ideal $\overline{m} = \overline{R} + \mathbb{Z}\beta$; indeed the inverse of $p + q\beta \in \overline{R} + \mathbb{Z}\beta$ with $p \in \overline{R} \setminus \overline{m}$, is $\frac{p - q\beta}{p^2 - q^2b}$, in fact $p^2 - q^2b$ is invertible since $v(p^2) \neq v(q^2b)$ and

$$v\left(p^2 - q^2b\right) = \min\left\{v(p^2), v\left(q^2b\right) \right\} = v(p^2) = 0.$$
It follows that $\overline{R} = R + R\beta$ is local.

Now we prove that $\overline{R}$ is a finitely generated $R$-module. The field extension $Q(R) \subseteq Q[R][\alpha]$ is finite and separable, then, by [3, Theorem 3.1.3], the integral closure of $\overline{R}$ in $Q(R)[\alpha]$, which is equal to $\overline{R}^{Q(R)[\alpha]} = \overline{R}$, is a finite module over $R$. Now $\overline{R}$ and $\overline{R} \simeq R + I\alpha$ are finite modules over $R$, so $\overline{R}$ is a finite module over $R$.

Now let $v'(Q(R)) = d\mathbb{Z}$ for some $d \in \mathbb{N}$. It results $v'(x) = d$ and $b \in (x^m) \setminus (x^{m+1})$, namely $b = ux^m$ with $u \in \overline{R}$ invertible. It follows that $v'(b) = mv'(x)$; in addition from $\alpha^2 = b$ we obtain $2v'(\alpha) = v'(b) = mv'(x) = md$.

Since $v'(Q(R)[\alpha]) = v'(Q(R) + Q(R)\alpha) = v'(Q(R)) \cup [v'(Q(R)) + v'(\alpha)] = d\mathbb{Z} \cup (d\mathbb{Z} + v(\alpha)) = \mathbb{Z}$, it must be $d = 2$, so $v'(Q(R)) = 2v$ and also $v'(\alpha) = m$. It easily follows that $v'(\overline{R}) = v(R) \otimes v(I) = S \otimes v(I)$.

Finally we show that $\overline{R}$ is residually rational. Recall that $\overline{m}$ is the maximal ideal of $\overline{R}$ and $\overline{M} = \overline{m} + \overline{R}\beta$ is the maximal ideal of $\overline{R}$. From [2, Proposition 2.1] the maximal ideal of $\overline{R}$ is $M = m + I\alpha$. Thus

$$\overline{R}/\overline{M} = \frac{\overline{R} + \overline{R}\beta}{\overline{m} + \overline{R}\beta} \simeq \overline{R}/\overline{m} \simeq \overline{R}/m \simeq \frac{R + I\alpha}{m + I\alpha} = \overline{R}/M.$$ 

Remark 3.2. Considering the valuation $v' : Q(R)[\alpha] \to \mathbb{Z}$, we have the following diagram

\[
\begin{array}{ccc}
R & \longrightarrow & \mathcal{R} \\
\downarrow & & \downarrow \\
\overline{R} & \longrightarrow & \overline{\mathcal{R}} \\
\downarrow & & \downarrow \\
Q(R) & \longrightarrow & Q(R)[\alpha] \\
\end{array}
\quad
\begin{array}{ccc}
2 \cdot S & \longrightarrow & S \otimes v(I) \\
\downarrow & & \downarrow \\
2 \cdot N & \longrightarrow & N \\
\downarrow & & \downarrow \\
2 \cdot \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\end{array}
\]

Proposition 3.3. If $\mathcal{R}$ is Arf, so is $R$.

Proof. Let $J$ be an integrally closed ideal of $R$ and let $x \in J$ such that $v(x) = \min v(J)$. Fix $\tilde{J} = \{y \in \mathcal{R} : v'(y) \geq v'(x)\}$, $\tilde{J}$ is an integrally
closed ideal of \( R \), so it is stable, namely \((\tilde{J} : J) = x^{-1}\tilde{J}\). Furthermore, since \( \overline{R} \cap R = R \) then \( J = \tilde{J} \cap R = \tilde{J} \cap \overline{R} \). It suffices to prove that \((J : J) = x^{-1}J\).

It is clear that \((J : J) \subseteq x^{-1}J\), in fact, if \( j \in (J : J) \), then by definition \( xj \in J \), so \( j \in x^{-1}J \).

Viceversa let \( j \in x^{-1}J \) and \( j' \in J \), we have

\[
j \in x^{-1}J \subseteq x^{-1}\tilde{J} = (\tilde{J} : J) \quad \implies \quad jj' \in \tilde{J}.
\]

Further

\[
j \in x^{-1}J \subseteq x^{-1}\overline{R} \quad \implies \quad jj' \in \overline{R},
\]

it follows that \( jj' \in \tilde{J} \cap \overline{R} = J \), so \( j \in (J : J) \). \( \square \)

We recall that the conductor of \( R \) is \( C = (R : \overline{R}) \). The conductor is an ideal both of \( R \) and \( \overline{R} \).

**Lemma 3.4.** Let \( J \) be an integrally closed ideal of \( R \). If \( J \subseteq C \) then \( J \) is stable.

**Proof.** Let \( x \in J \) such that \( v(x) = \min v(J) \), we show that \( J^2 = xJ \). Clearly \( xJ \subseteq J^2 \). Let \( i, j \in J \) then

\[
v(j) \geq v(x) \Rightarrow v(j) - v(x) = v(jx^{-1}) \geq 0 \Rightarrow jx^{-1} \in \overline{R},
\]

since \( i \in J \subseteq C \), it results \( ijx^{-1} \in R \), so

\[
v(ijx^{-1}) = v(i) + v(jx^{-1}) \geq v(i) \geq v(x) \Rightarrow ijx^{-1} \in J,
\]

therefore \( ij \in xJ \). \( \square \)

**Lemma 3.5.** Let \( J \) be an integrally closed ideal of \( R \). If \( C \subseteq J \) and \( x \in J \) is an element of minimum value in \( J \) then \( JC \subseteq xJ \).

**Proof.** Observe that \( x^{-1}J \subseteq \overline{R} \), hence

\[
JC = x(x^{-1}J)C \subseteq x\overline{RC} \subseteq xC \subseteq xJ.
\]

\( \square \)

Now, to prove Theorem 3.12 we need a series of technical lemmas. For this purpose we introduce some more notation. We fix an integrally closed ideal \( \tilde{J} \) of \( R \); set \( J = \tilde{J} \cap R \) and \( j_1 = \min v'(\tilde{J}) \), \( j_1 = \min v'(J) \). We denote the conductor of \( R \) with \( C_R = (R : \overline{R}) \). Note that the inclusion \( C_R \cap R \subseteq C \) may be strict. In addition, we will suppose that \( I \) is integrally closed and that \( C_R \nsubseteq \tilde{J} \).
Lemma 3.6. The ideal $J$ is integrally closed in $R$. Further

1. If $x + y\alpha \in \tilde{J}$ then $x \in J$ and $y\alpha \in \tilde{J}$.

2. If $\tilde{j}_1$ is even then $j_1 = \tilde{j}_1$ and there exists $x \in J$ such that $v'(x) = \tilde{j}_1$.

Proof. We have

$$\tilde{J} = \{x \in R : v'(x) \geq \tilde{j}_1\}, \quad J = \tilde{J} \cap R = \{x \in R : v'(x) \geq \tilde{j}_1\},$$

then $j_1 \geq \tilde{j}_1$; since for any $x, y \in R$ $v(x) \geq v(y)$ if and only if $v'(x) \geq v'(y)$, $\tilde{J} \subseteq J$, i.e. $J$ is integrally closed.

Let $z = x + y\alpha \in \tilde{J}$, with $x \in R$ and $y \in I$. Now $v'(x) = 2v(x)$ is even and $v'(y\alpha) = 2v(y) + m$ is odd, therefore $v'(x) \neq v'(y\alpha)$ and it results

$$v'(x) \geq \min\{v'(x), v'(y\alpha)\} = v'(z) \geq \tilde{j}_1.$$ 

It follows that $x \in \tilde{J} \cap R = J$ and consequently $y\alpha = z - x \in \tilde{J}$. Now, if $\tilde{j}_1$ is even, let $z = x + y\alpha \in \tilde{J}$ such that $v'(z) = \tilde{j}_1$. From the previous observations it must be $\tilde{j}_1 = v'(z) = v'(x) \geq j_1 \geq \tilde{j}_1$, hence $v'(x) = j_1 = \tilde{j}_1$ with $x \in J$.

Corollary 3.7. It results $v'(J) = v'(\tilde{J}) \cap v'(\overline{R}) = v'(\tilde{J}) \cap 2\mathbb{N}$.

Proof. Since $J \subseteq \tilde{J}$ and $J \subseteq \overline{R}$, then $v'(J) \subseteq v'(\tilde{J})$ and $v'(J) \subseteq v'(\overline{R})$, so $v'(J) \subseteq v'(\tilde{J}) \cap v'(\overline{R})$. Viceversa let $z = x + y\alpha \in R$ such that $v'(z) \in v'(\tilde{J}) \cap v'(\overline{R})$; $\tilde{J}$ integrally closed implies $z \in \tilde{J}$, hence $x \in J$, since $v'(z) \in v'(\overline{R}) = 2\mathbb{N}$, it is even, thus $v'(z) = v'(x) \in v'(J)$.

Lemma 3.8. If $I \subseteq C$ then $I\alpha \subseteq C_R$ and $\tilde{j}_1$ is even.

Proof. Recalling that $v'(R) = v(R) \times v(b) v(I)$, from [4, Proposition 2.1] we obtain

$$\min v'(C_R) = 2 \min v(I) + v(b) - 1 = \min v'(I) + v'(\alpha) - 1.$$ 

Now let $i \in I$, then

$$v'(i\alpha) = v'(i) + v'(\alpha) \geq \min v'(I) + v'(\alpha) > \min v'(C_R) \Rightarrow i\alpha \in C_R.$$ 

Hence $I\alpha \subseteq C_R$ and, in addition, $\min v'(I\alpha) > \min v'(C_R)$. Now, since $I\alpha \subseteq C_R \subseteq \tilde{J}$, it follows that $\min v'(I\alpha) \geq \min v'(C_R)$, contradicting $\tilde{j}_1 < \min v'(I\alpha)$. Assume by contradiction that $\tilde{j}_1$ is odd. Let $z = x + y\alpha \in \tilde{J}$ such that $v'(z) = \tilde{j}_1$, then $\tilde{j}_1 = v'(z) = v'(y\alpha) \in v'(I\alpha)$, contradicting $\tilde{j}_1 < \min v'(I\alpha)$. 

\[11\]
Lemma 3.9. If \( I \subseteq J \) then \( \tilde{j}_1 \) is even and \( I\alpha \subseteq \tilde{J} \). Further, let \( x \in J \) such that \( v'(x) = j_1 = \tilde{j}_1 \). If \( R \) is Arf then \( JI\alpha \subseteq x\tilde{J} \).

Proof. For every \( i \in I \) it follows that \( v'(i\alpha) = v'(i) + v'(\alpha) > v'(i) \), then \( \min v'(I\alpha) > \min v'(I) \). Moreover \( I \subseteq J \subseteq \tilde{J} \), hence

\[
\min v'(I\alpha) > \min v'(I) \geq \min v'(J) \geq \tilde{j}_1.
\]

It follows immediately that \( I\alpha \subseteq \tilde{J} \). Moreover, with a similar argument used in the previous proof it follows that \( \tilde{j}_1 \) is even. Now the choice of \( x \in J \) is allowed by Lemma 3.6. Let \( i \in I \) and \( j \in J \), then \( ij \in J\alpha \subseteq J^2 = xJ \) (\( R \) is Arf and \( J \) is integrally closed and \( x \) is of minimum value in \( J \)), so there exists \( j' \in J \) such that \( ij = xj' \), furthermore

\[
v'(j') = v'(ijx^{-1}) = v'(i) + v'(j) - v'(x) \geq v'(i),
\]

hence \( j' \in I \). Finally \( j\alpha = xj'\alpha \in xI\alpha \subseteq x\tilde{J} \), therefore \( JI\alpha \subseteq x\tilde{J} \). \( \square \)

Recall that, if \( R \) is Arf, then the multiplicity sequence of \( R \) coincides with the multiplicity sequence of \( S = v(R) \).

Lemma 3.10. If \( J \subseteq I \), then \( J\alpha \subseteq \tilde{J} \). Further, let \( x \in \tilde{J} \) be an element of minimum value in \( \tilde{J} \). If \( R \) is Arf and \( v(b) = e_0 = e_1 = \ldots = e_{n-1} \), then \( J^2 \subseteq x\tilde{J} \) and \( J^2\alpha \subseteq x\tilde{J} \).

Proof. If \( j \in J \) then \( j\alpha \in J\alpha \subseteq I\alpha \subseteq \mathcal{R} \), moreover \( v'(j\alpha) \geq v'(j) \geq \tilde{j}_1 \). It follows that \( j\alpha \in \tilde{J} \), therefore \( J\alpha \subseteq \tilde{J} \).

Now if \( \tilde{j}_1 \) is even, from Lemma 3.6 we can choose \( x \in J \); it follows that \( J^2 = xJ \subseteq x\tilde{J} \) and \( J^2\alpha = xJ\alpha \subseteq x\tilde{J} \).

On the other hand, if \( \tilde{j}_1 \) is odd we can choose \( y \in I \) such that \( x = ya \in \tilde{J} \). In this case it must be \( C \subseteq I \), otherwise if \( I \subseteq C \) (\( I \) is integrally closed), then from Lemma 3.8 \( \tilde{j}_1 \) can not be odd. Therefore, since \( C_\mathcal{R} \subseteq \tilde{J} \) the value of \( y\alpha \) is less than the conductor of \( v'(\mathcal{R}) \), then from Remark 2.6 we have \( v'(ya) = ke_0 \) for some \( k \in \mathbb{N} \) odd. From Corollary 3.7 it follows that the minimum of \( v'(J) \) is equal to \((k + 1)e_0 \), hence

\[
(k + 1)e_0 = ke_0 + e_0 = v'(ya) + v'(\alpha) = v'(yb) = j_1.
\]

Since \( yb \in R \), \( yb = xo \) is an element of minimum value in \( J \), therefore \( J^2 = x\alpha J \subseteq x\tilde{J} \) and \( J^2\alpha = x\alpha^2J = xbJ \subseteq xJ \subseteq x\tilde{J} \). \( \square \)

Lemma 3.11. Suppose that \( R \) is Arf and \( v(b) = e_0 = e_1 = \ldots = e_{n-1} \). If \( i\alpha \in \tilde{J} \setminus C_\mathcal{R} \) with \( v'(i\alpha) > \tilde{j}_1 \) then \( i \in J \).
Proof. Note that we can assume \( C \subseteq I \), otherwise if \( I \subseteq C \) from Lemma \(3.8\) \( I\alpha \subseteq C_R \), so \( I\alpha \cap (J \setminus C_R) = \emptyset \).

In view of Theorem \(2.4\) and Theorem \(3.1\), since \( i\alpha \notin C_R \) then \( v'(i\alpha) = ke_0 \) for some \( k \in \mathbb{N} \) odd. Moreover, \( v'(i\alpha) > j_1 \), so \( (k-1)e_0 \in v'(J) \), therefore

\[
v'(i) = v'(i\alpha) - v'\alpha = ke_0 - e_0 = (k-1)e_0 \in v'(J).
\]

Hence \( i \in I \subseteq R \) with \( v'(i) \geq j_1 \) so \( i \in \tilde{J} \cap R = J \).

Now we are ready to prove Theorem \(3.12\) so we no longer hold the assumptions made on \( I \) and \( \tilde{J} \).

Note that if \( I \) is integrally closed, the condition \( \min(v(I)) < s_n \) (similar to the one of Theorem \(2.4\)) is equivalent to \( C \subseteq I \).

**Theorem 3.12.** \( \mathcal{R} \) is Arf if and only if \( R \) is Arf, \( I \) is integrally closed and if \( C \subseteq I \) then \( v(b) = e_0 = e_1 = \ldots = e_{n-1} \).

**Proof.** Necessity. From Proposition \(3.3\) \( R \) is Arf. Since \( \mathcal{R} \) is Arf, \( S \ast^v(b) v(I) \) is an Arf numerical semigroup, so from Theorem \(2.4\) \( v(I) \) is integrally closed, equivalently \( I \) is integrally closed. Furthermore, if \( C \subseteq I \), equivalently if \( \min(v(J)) < s_n \), then \( v(b) = e_0 = e_1 = \ldots = e_{n-1} \).

Sufficiency. Let \( \tilde{J} \) be an integrally closed ideal of \( \mathcal{R} \). From Lemma \(3.4\) applied to \( \tilde{J} \) and \( \mathcal{R} \), if \( \tilde{J} \subseteq C_R \), then \( \tilde{J} \) is stable, so suppose that \( C_R \subseteq \tilde{J} \).

We denote with \( J = \tilde{J} \cap R \), from Lemma \(3.6\) \( J \) is an integrally closed ideal of \( R \), so it is stable. Let \( x \in \tilde{J} \) be an element of minimum value, we want to prove that \( x\tilde{J} = \tilde{J}^2 \), the inclusion \( x\tilde{J} \subseteq \tilde{J}^2 \) is clear, so it is suffice to prove that \( \tilde{J}^2 \subseteq x\tilde{J} \). Now the two ideals \( I \) and \( C \) of \( R \) are both integrally closed, so one is contained in the other.

If \( I \subseteq C \) from Lemma \(3.3\) and Lemma \(3.8\) we can choose \( x \in J \) and it results \( J \subseteq J + C_R \). In the view of Lemma \(3.5\) applied to \( \tilde{J} \) and \( \mathcal{R} \) we obtain

\[
\tilde{J}^2 \subseteq (J + C_R)^2 = J^2 + JC_R + C_R^2 \subseteq xJ + JC_R + JC_R \subseteq x\tilde{J}.
\]

If \( C \subseteq I \) in this case we have \( v(b) = e_0 = e_1 = \ldots = e_{n-1} \). Again \( I \) and \( J \) are two integrally closed ideal of \( R \) and we distinguish two cases.

If \( I \subseteq J \) then from Lemma \(3.6\) and Lemma \(3.9\) we can choose \( x \in J \) and we have \( JI\alpha \subseteq x\tilde{J} \). It follows

\[
\tilde{J}^2 \subseteq (J + I\alpha)^2 = J^2 + JI\alpha + I^2\alpha \subseteq J^2 + x\tilde{J} + J^2 = xJ + x\tilde{J} \subseteq x\tilde{J}.
\]

If \( J \subseteq I \) then let \( x_1 + y_1\alpha, x_2 + y_2\alpha \in \tilde{J} \). For \( k = 1, 2 \) we distinguish three cases.
1. \( y_k\alpha \) is of minimum value in \( \tilde{J} \); then, since \( R \) is residually rational, there exists \( u_k \in R \) invertible such that \( u(y_k\alpha) = x \).

2. \( y_k\alpha \in \tilde{J}\setminus C_R \) with \( v'(y_k\alpha) > \tilde{j}_1 \). In this case, from Lemma 3.11, \( y_k \in J \).

3. \( y_k\alpha \in C_R \).

In view of Lemma 3.10 and Lemma 3.5 we have to verify six different cases.

(1,1) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) = x_1x_2 + x_1y_2\alpha + y_1\alpha(x_2 + y_2\alpha) \\
\in J^2 + y_2\alpha\tilde{J} + y_1\alpha\tilde{J} \subseteq x\tilde{J} + u_2(y_2\alpha)\tilde{J} + u_1(y_1\alpha)\tilde{J} = x\tilde{J}
\]

(1,2) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) = x_1(x_2 + y_2\alpha) + y_1\alpha(x_2 + y_2\alpha) \\
\in J(J + J\alpha) + y_1\alpha\tilde{J} \subseteq J^2 + J^2\alpha + u_1(y_1\alpha)\tilde{J} \subseteq x\tilde{J}.
\]

(1,3) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) = x_1(x_2 + y_2\alpha) + y_1\alpha(x_2 + y_2\alpha) \\
\in J(J + C_R) + y_1\alpha\tilde{J} \subseteq J^2 + JC_R + u_1(y_1\alpha)\tilde{J} \subseteq x\tilde{J} + JC_R + x\tilde{J} \subseteq x\tilde{J}.
\]

(2,2) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) \in (J + J\alpha)^2 = J^2 + J^2\alpha + J^2b \subseteq x\tilde{J} + x\tilde{J} + J^2 \subseteq x\tilde{J}.
\]

(2,3) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) \in (J + J\alpha)(J + C_R) = J^2 + JC_R + J^2\alpha + J\alpha C_R \subseteq x\tilde{J} + JC_R + x\tilde{J} + JC_R \subseteq x\tilde{J}.
\]

(3,3) \[
(x_1 + y_1\alpha)(x_2 + y_2\alpha) \in (J + C_R)^2 = J^2 + JC_R + C_R^2 \subseteq x\tilde{J} + JC_R + JC_R \subseteq x\tilde{J}.
\]

This proves that \( \tilde{J}^2 \subseteq x\tilde{J} \).

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