TRIANGLE INEQUALITIES IN PATH METRIC SPACES

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Abstract. We study side-lengths of triangles in path metric spaces. We prove that unless such a space $X$ is bounded, or quasi-isometric to $\mathbb{R}_+$ or to $\mathbb{R}$, every triple of real numbers satisfying the strict triangle inequalities, is realized by the side-lengths of a triangle in $X$. We construct an example of a complete path metric space quasi-isometric to $\mathbb{R}^2$ for which every degenerate triangle has one side which is shorter than a certain uniform constant.

Given a metric space $X$ define
\[
K_3(X) := \{(a, b, c) \in \mathbb{R}^3_+ : \exists \text{ points } x, y, z \text{ so that } d(x, y) = a, d(y, z) = b, d(z, x) = c\}.
\]

Note that $K_3(\mathbb{R}^2)$ is the closed convex cone $K$ in $\mathbb{R}^3_+$ given by the usual triangle inequalities. On the other hand, if $X = \mathbb{R}$ then $K_3(X)$ is the boundary of $K$ since all triangles in $X$ are degenerate. If $X$ has finite diameter, $K_3(X)$ is a bounded set.

In [3, Page 18] Gromov raised the following

Question 1. Find reasonable conditions on path metric spaces $X$, under which $K_3(X) = K$.

It is not so difficult to see that for a path metric space $X$ quasi-isometric to $\mathbb{R}_+$ or $\mathbb{R}$, the set $K_3(X)$ does not contain the interior of $K$, see Section 3. Moreover, every triangle in such $X$ is $D$-degenerate for some $D < \infty$ and therefore $K_3(X)$ is contained in the $D$-neighborhood of $\partial K$.

Our main result is essentially the converse to the above observation:

Theorem 2. Suppose that $X$ is an unbounded path metric space not quasi-isometric to $\mathbb{R}_+$ or $\mathbb{R}$. Then:
1. $K_3(X)$ contains the interior of the cone $K$.
2. If, in addition, $X$ contains arbitrary long geodesic segments, then $K_3(X) = K$.

In particular, we obtain a complete answer to Gromov’s question for geodesic metric spaces, since an unbounded geodesic metric space clearly contains arbitrarily long geodesic segments. In Section 5 we give an example of a (complete) path metric space $X$ quasi-isometric to $\mathbb{R}^2$, for which
\[
K_3(X) \neq K.
\]

Therefore, Theorem 2 is the optimal result.

The proof of Theorem 2 is easier under the assumption that $X$ is a proper metric space: In this case $X$ is necessarily complete, geodesic metric space. Moreover, every unbounded sequence of geodesic segments $\overline{ox}$ in $X$ yields a geodesic ray. The reader
who does not care about the general path metric spaces can therefore assume that $X$

is proper. The arguments using the ultralimits are then replaced by the Arcela–Ascoli

theorem.

Below is a sketch of the proof of Theorem 2 under the extra assumption that $X$

is proper. Since the second assertion of Theorem 2 is clear, we have to prove only the

first statement. We define $R$-tripods $T \subset X$, as unions $\gamma \cup \mu$ of two geodesic segments

$\gamma, \mu \subset X$, having the lengths $\geq R$ and $\geq 2R$ respectively, so that:

1. $\gamma \cap \mu = o$ is the end-point of $\gamma$.
2. $o$ is distance $\geq R$ from the ends of $\mu$.
3. $o$ is a nearest-point projection of $\gamma$ to $\mu$.

The space $X$ is called $R$-compressed if it contains no $R$-tripods. The space $X$ is
called uncompressed if it is not $R$-compressed for any $R < \infty$.

We break the proof of Theorem 2 in two parts:

Part 1.

**Theorem 3.** If $X$ is uncompressed then $K_3(X)$ contains the interior of $K_3(\mathbb{R}^2)$.

The proof of this theorem is mostly topological. The side-lengths of triangles in $X$
determine a continuous map

$L: X^3 \to K$

Then $K_3(X) = L(X^3)$. Given a point $\kappa$ in the interior of $K$, we consider an $R$-tripod

$T \subset X$ for sufficiently large $R$. We then restrict to triangles in $X$ with vertices

in $T$. We construct a 2-cycle $\Sigma \in Z_2(T, \mathbb{Z}_2)$ whose image under $L_*$ determines a

nontrivial element of $H_2(K \setminus \kappa, \mathbb{Z}_2)$. Since $T^3$ is contractible, there exists a 3-chain

$\Gamma \in C_3(T^3, \mathbb{Z}_2)$ with the boundary $\Sigma$. Therefore the support of $L_*(\Gamma)$ contains the

point $\kappa$, which implies that $\kappa$ belongs to the image of $L$.

**Remark 4.** Gromov observed in [3] that uniformly contractible metric spaces $X$ have

large $K_3(X)$. Although uniform contractibility is not relevant to our proof, the key

argument here indeed has the coarse topology flavor.

Part 2.

**Theorem 5.** If $X$ is a compressed unbounded path metric space, then $X$ is quasi-
isometric to $\mathbb{R}$ or $\mathbb{R}_+$. 

Assuming that $X$ is compressed, unbounded and is not quasi-isometric to $\mathbb{R}$ and
to $\mathbb{R}_+$, we construct three diverging geodesic rays $\rho_i$ in $X$, $i = 1, 2, 3$. Define $\mu_i \subset X$
to be the geodesic segment connecting $\rho_1(i)$ and $\rho_2(i)$. Take $\gamma_i$ to be the shortest
segment connecting $\rho_3(i)$ to $\mu_i$. Then $\gamma_i \cup \mu_i$ is an $R_i$-tripod with $\lim_i R_i = \infty$, which
contradicts the assumption that $X$ is compressed.

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1. Preliminaries

Convention 6. All homology will be taken with the $\mathbb{Z}_2$-coefficients.

In the paper we will talk about ends of a metric space $X$. Instead of looking at the noncompact complementary components of relatively compact open subsets of $X$ as it is usually done for topological spaces, we will define ends of $X$ by considering unbounded complementary components of bounded subsets of $X$. With this modification, the usual definition goes through.

If $x$, $y$ are points in a topological space $X$, we let $P(x, y)$ denote the set of continuous paths in $X$ connecting $x$ to $y$. For $\alpha \in P(x, y)$, $\beta \in P(y, z)$ we let $\alpha \ast \beta \in P(x, z)$ denote the concatenation of $\alpha$ and $\beta$. Given a path $\alpha : [0, a] \to X$ we let $\bar{\alpha}$ denote the reverse path $\bar{\alpha}(t) = \alpha(a - t)$.

1.1. Triangles and their side-lengths. We set $K := K_3(\mathbb{R}^2)$; it is the cone in $\mathbb{R}^3$ given by

$$\{ (a, b, c) : a \leq b + c, b \leq a + c, c \leq a + b \}.$$ We metrize $K$ by using the maximum-norm on $\mathbb{R}^3$.

By a triangle in a metric space $X$ we will mean an ordered triple $\Delta = (x, y, z) \in X^3$. We will refer to the numbers $d(x, y), d(y, z), d(z, x)$ as the side-lengths of $\Delta$, even though these points are not necessarily connected by geodesic segments. The sum of the side-lengths of $\Delta$ will be called the perimeter of $\Delta$.

We have the continuous map

$$L : X^3 \to K$$

which sends the triple $(x, y, z)$ of points in $X$ to the triple of side-lengths

$$(d(x, y), d(y, z), d(z, x)).$$

Then $K_3(X)$ is the image of $L$.

Let $\epsilon \geq 0$. We say that a triple $(a, b, c) \in K$ is $\epsilon$-degenerate if, after reordering if necessary the coordinates $a, b, c$, we obtain

$$a + \epsilon \geq b + c.$$ Therefore every $\epsilon$-degenerate triple is within distance $\leq \epsilon$ from the boundary of $K$. A triple which is not $\epsilon$-degenerate is called $\epsilon$-nondegenerate. A triangle in a metric space $X$ whose side-lengths form an $\epsilon$-degenerate triple, is called $\epsilon$-degenerate. A 0-degenerate triangle is called degenerate.

1.2. Basic notions of metric geometry. For a subset $E$ in a metric space $X$ and $R < \infty$ we let $N_R(E)$ denote the metric $R$-neighborhood of $E$ in $X$:

$$N_R(E) = \{ x \in X : d(x, E) \leq R \}.$$ 

Definition 7. Given a subset $E$ in a metric space $X$ and $\epsilon > 0$, we define the $\epsilon$-nearest-point projection $p = p_{E, \epsilon}$ as the map which sends $X$ to the set $2^E$ of subsets in $E$:

$$y \in p(x) \iff d(x, y) \leq d(x, z) + \epsilon, \quad \forall z \in E.$$ If $\epsilon = 0$, we will abbreviate $p_{E,0}$ to $p_E$. 

3
Quasi-isometries. Let $X, Y$ be metric spaces. A map $f : X \to Y$ is called an $(L, A)$-quasi-isometric embedding (for $L \geq 1$ and $A \in \mathbb{R}$) if for every pair of points $x_1, x_2 \in X$ we have
\[
L^{-1}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A.
\]

A map $f$ is called an $(L, A)$-quasi-isometry if it is an $(L, A)$-quasi-isometric embedding so that $N_A(f(X)) = Y$. Given an $(L, A)$-quasi-isometry, we have the quasi-inverse map $\bar{f} : Y \to X$ which is defined by choosing for each $y \in Y$ a point $x \in X$ so that $d(f(x), y) \leq A$. The quasi-inverse map $\bar{f}$ is an $(L, 3A)$-quasi-isometry. An $(L, A)$-quasi-isometric embedding $f$ of an interval $I \subset \mathbb{R}$ into a metric space $X$ is called an $(L, A)$-quasi-geodesic in $X$. If $I = \mathbb{R}$, then $f$ is called a complete quasi-geodesic.

Every quasi-isometric embedding $\mathbb{R}^n \to \mathbb{R}^n$ is a quasi-isometry, see for instance [5].

Geodesics and path metric spaces.

A geodesic in a metric space is an isometric embedding of an interval into $X$. By abusing the notation, we will identify geodesics and their images. A metric space is called geodesic if any two points in $X$ can be connected by a geodesic. By abusing the notation we let $xy$ denote a geodesic connecting $x$ to $y$, even though this geodesic is not necessarily unique.

The length of a continuous curve $\gamma : [a, b] \to X$ in a metric space, is defined as
\[
\text{length}(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < ... < t_n = b \right\}.
\]

A path $\gamma$ is called rectifiable if $\text{length}(\gamma) < \infty$.

A metric space $X$ is called a path metric space if for every pair of points $x, y \in X$ we have
\[
d(x, y) = \inf \{ \text{length}(\gamma) : \gamma \in P(x, y) \}.
\]

We say that a curve $\gamma : [a, b] \to X$ is $\epsilon$-geodesic if
\[
\text{length}(\gamma) \leq d(\gamma(a), \gamma(b)) + \epsilon.
\]

It follows that every $\epsilon$-geodesic is $(1, \epsilon)$-quasi-geodesic. We refer the reader to [3, 2] for the further details on path metric spaces.

1.3. Ultralimits. Our discussion of ultralimits of sequences of metric space will be somewhat brief, we refer the reader to [5, 3, 4, 2, 6] for the detailed definitions and discussion.

Choose a nonprincipal ultrafilter $\omega$ on $\mathbb{N}$. Suppose that we are given a sequence of pointed metric spaces $(X_i, o_i)$, where $o_i \in X_i$. The ultralimit

\[
(X_\omega, o_\omega) = \omega\text{-lim}(X_i, o_i)
\]
is a pointed metric space whose elements are equivalence classes of sequences $x_i \in X_i$. The distance in $X_\omega$ is the $\omega$-limit:

$$\omega\text{-}\lim d(x_i, y_i).$$

One of the key properties of ultralimits which we will use repeatedly is the following. Suppose that $(Y_i, p_i)$ is a sequence of pointed metric spaces. Assume that we are given a sequence of $(L_i, A_i)$-quasi-isometric embeddings

$$f_i : X_i \to Y_i$$

so that $\omega\text{-}\lim d(f(o_i), p_i)) < \infty$ and

$$\omega\text{-}\lim L_i = L < \infty, \quad \omega\text{-}\lim A_i = 0.$$ 

Then there exists the ultralimit $f_\omega$ of the maps $f_i$, which is an $(L, 0)$-quasi-isometric embedding

$$f_\omega : X_\omega \to Y_\omega.$$ 

In particular, if $L = 1$, then $f_\omega$ is an isometric embedding.

**Ultralimits of constant sequences of metric spaces.** Suppose that $X$ is a path metric space. Consider the constant sequence $X_i = X$ for all $i$. If $X$ is a proper metric space and $o_i$ is a bounded sequence, the ultralimit $X_\omega$ is nothing but $X$ itself. In general, however, it could be much larger. The point of taking the ultralimit is that some properties of $X$ improve after passing to $X_\omega$.

**Lemma 8.** $X_\omega$ is a geodesic metric space.

**Proof.** Points $x_\omega, y_\omega$ in $X_\omega$ are represented by sequences $(x_i), (y_i)$ in $X$. For each $i$ choose a $1/i$-geodesic curve $\gamma_i$ in $X$ connecting $x_i$ to $y_i$. Then

$$\gamma_\omega := \omega\text{-}\lim \gamma_i$$

is a geodesic connecting $x_\omega$ to $y_\omega$. \hfill \Box

Similarly, every sequence of $1/i$-geodesic segments $\overline{yx_i}$ in $X$ satisfying

$$\omega\text{-}\lim d(y, x_i) = \infty,$$

yields a geodesic ray $\gamma_\omega$ in $X_\omega$ emanating from $y_\omega = (y)$.

If $o_i \in X$ is a bounded sequence, then we have a natural (diagonal) isometric embedding $X \to X_\omega$, given by the map which sends $x \in X$ to the constant sequence $(x)$.

**Lemma 9.** For every geodesic segment $\gamma_\omega = \overline{x_\omega y_\omega}$ in $X_\omega$ there exists a sequence of $1/i$-geodesics $\gamma_i \subset X_\omega$, so that

$$\omega\text{-}\lim \gamma_i = \gamma_\omega.$$ 

**Proof.** Subdivide the segment $\gamma_\omega$ into $n$ equal subsegments

$$\overline{z_{\omega,j} z_{\omega,j+1}}, \quad j = 1, ..., n,$$

where $x_\omega = z_{\omega,1}, y_\omega = z_{\omega,n+1}$. Then the points $z_{\omega,j}$ are represented by sequences $(z_{k,j}) \in X$. It follows that for $\omega$-all $k$, we have

$$\left| \sum_{j=1}^{n} d(z_{k,j}, z_{k,j+1}) - d(x_k, y_k) \right| < \frac{1}{2i}.$$
Connect the points $z_{k,j}, z_{k,j+1}$ by $\frac{1}{2^n}$-geodesic segments $\alpha_{k,j}$. Then the concatenation

$$\alpha_n = \alpha_{k,1} \ast \ldots \ast \alpha_{k,n}$$

is an $\frac{1}{i}$-geodesic connecting $x_k$ and $y_k$, where

$$x_\omega = (x_k), \quad y_\omega = (y_k).$$

It is clear from the construction, that, if given $i$ we choose sufficiently large $n = n(i)$, then

$$\omega \text{-lim } \alpha_{n(i)} = \gamma.$$

Therefore we take $\gamma_i := \alpha_{n(i)}$. \hfill \Box

1.4. **Tripods.** Our next goal is to define *tripods* in $X$, which will be our main technical tool. Suppose that $x, y, z, o$ are points in $X$ and $\mu$ is an $\epsilon$-geodesic segment connecting $x$ to $y$, so that $o \in \mu$ and

$$o \in p_{\mu, \epsilon}(z).$$

Then the path $\mu$ is the concatenation $\alpha \cup \beta$, where $\alpha, \beta$ are $\epsilon$-geodesics connecting $x, y$ to $o$. Let $\gamma$ be an $\epsilon$-geodesic connecting $z$ to $o$.

**Definition 10.** 1. We refer to $\alpha \cup \beta \cup \gamma$ as a *tripod* $T$ with the vertices $x, y, z$, legs $\alpha, \beta, \gamma$, and the center $o$.

2. Suppose that the length of $\alpha, \beta, \gamma$ is at least $R$. Then we refer to the tripod $T$ as $(R, \epsilon)$-tripod. An $(R, 0)$-tripod will be called simply an $R$-tripod.

The reader who prefers to work with proper geodesic metric spaces can safely assume that $\epsilon = 0$ in the above definition and thus $T$ is a geodesic tripod.

![Figure 1. A tripod.](image)

**Definition 11.** Let $R \in [0, \infty), \epsilon \in [0, \infty)$. We say that a space $X$ is $(R, \epsilon)$-compressed if it contains no $(R, \epsilon)$-tripods. We will refer to $(R, 0)$-compressed spaces as $R$-compressed. A space $X$ which is not $(R, \epsilon)$-compressed for any $R < \infty$, $\epsilon > 0$ is called uncompressed.

Therefore $X$ is uncompressed if and only if there exists a sequence of $(R_i, \epsilon_i)$-tripods in $X$ with

$$\lim_i R_i = \infty, \quad \lim_i \epsilon_i = 0.$$
1.5. **Tripods and ultralimits.** Suppose that $X$ is uncompressed and thus there exists a sequence of $(R_i, \epsilon_i)$-tripods $T_i$ in $X$ with

$$\lim_i R_i = \infty, \quad \lim_i \epsilon_i = 0,$$

so that the center of $T_i$ is $o_i$ and the legs are $\alpha_i, \beta_i, \gamma_i$. Then the tripods $T_i$ clearly yield a geodesic $(\infty, 0)$-tripod $T_\omega$ in $(X_\omega, o_\omega) = \omega\text{-lim}(X, o_i)$. The tripod $T_\omega$ is the union of three geodesic rays $\alpha_\omega, \beta_\omega, \gamma_\omega$ emanating from $o_\omega$, so that

$$o_\omega = p_{\mu_\omega}(\gamma_\omega).$$

Here $\mu_\omega = \alpha_\omega \cup \beta_\omega$. In particular, $X_\omega$ is uncompressed.

Conversely, in view of Lemma 9, we have:

**Lemma 12.** If $X$ is $(R, \epsilon)$-compressed for $\epsilon > 0$ and $R < \infty$, then $X_\omega$ is $R'$-compressed for every $R' > R$.

**Proof.** Suppose that $X_\omega$ contains an $R'$-tripod $T_\omega$. Then $T_\omega$ appears as the ultra-limit of $(R' - \frac{1}{i}, \frac{1}{i})$-tripods in $X$. This contradicts the assumption that $X$ is $(R, \epsilon)$-compressed. \(\Box\)

Let $\sigma : [a, b] \to X$ be a rectifiable curve in $X$ parameterized by its arc-length. We let $d_\sigma$ denote the path metric on $[a, b]$ which is the pull-back of the path metric on $[a, b]$. By abusing the notation we denote by $d$ the restriction to $\sigma$ of the metric $d$. Note that, in general, $d$ is only a pseudo-metric on $[a, b]$ since $\sigma$ need not be injective. However, if $\sigma$ is injective then $d$ is a metric.

We repeat this construction with respect to the tripods: Given a tripod $T \subset X$, define an abstract tripod $T_{mod}$ whose legs have the same length as the legs of $T$. We have a natural map

$$\tau : T_{mod} \to X$$

which sends the legs of $T_{mod}$ to the respective legs of $T$, parameterizing them by the arc-length. Then $T_{mod}$ has the path metric $d_{mod}$ obtained by pull-back of the path metric from $X$ via $\tau$. We also have the restriction pseudo-metric $d$ on $T_{mod}$:

$$d(A, B) = d(\tau(A), \tau(B)).$$

Observe that if $\epsilon = 0$ and $X$ is a tree then the metrics $d_{mod}$ and $d$ on $T$ agree.

**Lemma 13.**

$$d \leq d_{mod} \leq 3d + 6\epsilon.$$

**Proof.** The inequality $d \leq d_{mod}$ is clear. We will prove the second inequality. If $A, B \in \alpha \cup \beta$ or $A, B \in \gamma$ then, clearly,

$$d_{mod}(A, B) \leq d(A, B) + \epsilon,$$

since these curves are $\epsilon$-geodesics.

Therefore consider the case when $A \in \gamma$ and $B \in \beta$. Then

$$D := d_{mod}(A, B) = t + s,$$

where $t = d_o(A, o), s = d_B(o, B)$.

**Case 1:** $t \geq D/3$. Then, since $o \in \alpha \cup \beta$ is $\epsilon$-nearest to $A$, it follows that

$$D/3 \leq t \leq d(A, o) + \epsilon \leq d(A, B) + 2\epsilon.$$
Hence
\[ d_{\text{mod}}(A, B) = \frac{3D}{3} \leq 3(d(A, B) + 2\epsilon) = 3d(A, B) + 6\epsilon, \]
and the assertion follows in this case.

**Case 2:** \( t < D/3 \). By the triangle inequality,
\[ D - t = s \leq d(o, B) + \epsilon \leq d(o, A) + d(A, B) + \epsilon \leq t + 2\epsilon + d(A, B). \]
Hence
\[ \frac{D}{3} = D - \frac{2}{3}D \leq D - 2t \leq 2\epsilon + d(A, B), \]
and
\[ d_{\text{mod}}(A, B) = \frac{3D}{3} \leq 3d(A, B) + 6\epsilon. \]
\[ \square \]

2. **Topology of configuration spaces of tripods**

We begin with the model tripod \( T \) with the legs \( \alpha_i, i = 1, 2, 3 \), and the center \( o \). Consider the configuration space \( Z := T^3 \setminus \text{diag} \), where \( \text{diag} \) is the small diagonal
\[ \{(x_1, x_2, x_3) \in T^3 : x_1 = x_2 = x_3\}. \]
We recall that the homology is taken with the \( \mathbb{Z}_2 \)-coefficients.

**Proposition 14.** \( H_1(Z) = 0 \).

**Proof.** \( T^3 \) is the union of cubes
\[ Q_{ijk} = \alpha_i \times \alpha_j \times \alpha_k, \]
where \( i, j, k \in \{1, 2, 3\} \). Identify each cube \( Q_{ijk} \) with the unit cube in the positive octant in \( \mathbb{R}^3 \). Then in the cube \( Q_{ijk} \) we choose the equilateral triangle \( \sigma_{ijk} \) given by the intersection of \( Q_{ijk} \) with the hyperplane
\[ x + y + z = 1 \]
in \( \mathbb{R}^3 \). Define the 2-dimensional complex
\[ S := \bigcup_{ijk} \sigma_{ijk}. \]
This complex is homeomorphic to the link of \((o, o, o)\) in \( T^3 \). Therefore \( Z \) is homotopy-equivalent to
\[ W := S \setminus (\sigma_{111} \cup \sigma_{222} \cup \sigma_{333}). \]
Consider the loops \( \gamma_i := \partial \sigma_{iii}, i = 1, 2, 3 \).

**Lemma 15.** 1. The homology classes \([\gamma_i], i = 1, 2, 3 \) generate \( H_1(W) \).

2. \([\gamma_1] = [\gamma_2] = [\gamma_3] \) in \( H_1(W) \).

**Proof.** 1. We first observe that \( S \) is a 2-dimensional spherical building. Therefore \( L \) is homotopy-equivalent to a bouquet of 2-spheres (see \[\text{[1, Theorem 2, page 93]}\]), which implies that \( H_1(S) = 0 \). Now the first assertion follows from the long exact sequence of the pair \((S, W)\).

2. Let us verify that \([\gamma_1] = [\gamma_2] \). The subcomplex
\[ S_{12} = S \cap (\alpha_1 \cup \alpha_2)^3 \]
is homeomorphic to the 2-sphere. Therefore \( S_{12} \cap W \) is the annulus bounded by the circles \( \gamma_1 \) and \( \gamma_2 \). Hence \([\gamma_1] = [\gamma_2]\). □

Lemma 16. \[ [\gamma_1] + [\gamma_2] + [\gamma_3] = 0 \]
in \( H_1(W) \).

Proof. Let \( B' \) denote the chain
\[
\sum_{\{ijk\} \in A} \sigma_{ijk},
\]
where \( A \) is the set of triples of distinct indices. Let
\[
B'' := \sum_{i=1}^{3} (\sigma_{ii(i+1)} + \sigma_{i(i+1)i} + \sigma_{(i+1)ii})
\]
where we set \( 3+1 := 1 \). We leave it to the reader to verify that
\[
\partial(B' + B'') = \gamma_1 + \gamma_2 + \gamma_3
\]
in \( C_1(W) \). □

By combining these lemmata we obtain the assertion of the proposition. □

Application to tripods in metric spaces. Consider an \((R, \varepsilon)\)-tripod \( T \) in a metric space \( X \) and its standard parametrization \( \tau : T_{mod} \to T \).

There is an obvious scaling operation
\[
u \mapsto r \cdot u
\]
on the space \((T_{mod}, d_{mod})\) which sends each leg to itself and scales all distances by \( r \in \mathbb{R} \). It induces the map \( T_{mod}^3 \to T_{mod}^3 \), denoted \( t \mapsto r \cdot t \), \( t \in T_{mod}^3 \).

We have the functions
\[
L_{mod} : T_{mod}^3 \to K, \quad L : T_{mod}^3 \to K,
\]
\[
L_{mod}(x, y, z) = (d_{mod}(x, y), d_{mod}(y, z), d_{mod}(z, x)),
\]
\[
L(x, y, z) = (d(x, y), d(y, z), d(z, x))
\]
computing side-lengths of triangles with respect to the metrics \( d_{mod} \) and \( d \).

For \( \rho \geq 0 \) set
\[
K_\rho := \{(a, b, c) \in K : a + b + c > \rho \}.
\]

Define
\[
T^3(\rho) := L^{-1}(K_\rho), \quad T_{mod}^3(\rho) := L_{mod}^{-1}(K_\rho).
\]

Thus
\[
T_{mod}^3(0) = T^3(0) = T^3 \setminus \text{diag}.
\]

Lemma 17. For every \( \rho \geq 0 \), the space \( T_{mod}^3(\rho) \) is homeomorphic to \( T_{mod}^3(0) \).

Proof. Recall that \( S \) is the link of \((o, o, o)\) in \( T^3 \). Then scaling determines homeomorphisms
\[
T_{mod}(\rho) \to S \times \mathbb{R} \to T_{mod}(0). \quad \square
\]

Corollary 18. For every \( \rho \geq 0 \), \( H_1(T_{mod}(\rho), \mathbb{Z}_2) = 0 \).
Corollary 19. The map induced by inclusion
\[ H_1(T^3(3\rho + 18\epsilon)) \to H_1(T^3(\rho)) \]
is zero.

Proof. Recall that
\[ d \leq d_{\text{mod}} \leq 3d + 6\epsilon. \]
Therefore
\[ T^3(3\rho + 18\epsilon) \subset T^3_{\text{mod}}(\rho) \subset T^3(\rho). \]
Now the assertion follows from the previous corollary. \(\square\)

3. Proof of Theorem 5

Suppose that \(X\) is uncompressed. Then for every \(R < \infty, \epsilon > 0\) there exists an \((R, \epsilon)\)-tripod \(T\) with the legs \(\alpha, \beta, \gamma\). Without loss of generality we may assume that the legs of \(T\) have length \(R\). Let \(\tau : T_{\text{mod}} \to T\) denote the standard map from the model tripod onto \(T\). We will continue with the notation of the previous section.

Given a map \(f : E \to T^3_{\text{mod}}\) (or a chain \(\sigma \in C_*(T^3_{\text{mod}})\)) let \(\hat{f}\) (resp. \(\hat{\sigma}\)) denote the map \(L \circ f\) from \(E\) to \(K\) (resp. the chain \(L_*(\sigma) \in C_*(K)\)). Similarly, we define \(\hat{f}_{\text{mod}}\) and \(\hat{\sigma}_{\text{mod}}\) using the map \(L_{\text{mod}}\) instead of \(L\).

Every loop \(\lambda : S^1 \to T^3_{\text{mod}}\), determines the map of the 2-disk
\[ \Lambda : D^2 \to T^3_{\text{mod}}, \]
given by
\[ \Lambda(r, \theta) = r \cdot \lambda(\theta) \]
where we are using the polar coordinates \((r, \theta)\) on the unit disk \(D^2\). Triangulating both \(S^1\) and \(D^2\) and assigning the coefficient \(1 \in \mathbb{Z}_2\) to each simplex, we regard both \(\lambda\) and \(\Lambda\) as singular chains in \(C_*(T^3_{\text{mod}})\).

We let \(a, b, c\) denote the coordinates on the space \(\mathbb{R}^3\) containing the cone \(K\). Let \(\kappa = (a_0, b_0, c_0)\) be a \(\delta\)-nondegenerate point in the interior of \(K\) for some \(\delta > 0\); set \(r := a_0 + b_0 + c_0\).

Suppose that there exists a loop \(\lambda\) in \(T^3_{\text{mod}}\) such that:
1. \(\hat{\lambda}(\theta)\) is \(\epsilon\)-degenerate for each \(\theta\). Moreover, each triangle \(\lambda(\theta)\) is either contained in \(\alpha_{\text{mod}} \cup \beta_{\text{mod}}\) or has only two distinct vertices.

In particular, the image of \(\hat{\lambda}\) is contained in
\[ K \setminus \mathbb{R}_+ \cdot \kappa. \]

2. The image of \(\hat{\lambda}\) is contained in \(K_\rho\), where \(\rho = 3r + 18\epsilon\).

3. The homology class \([\hat{\lambda}]\) is nontrivial in \(H_1(K \setminus \mathbb{R}_+ \cdot \kappa)\).

Lemma 20. If there exists a loop \(\lambda\) satisfying the assumptions 1—3, and \(\epsilon < \delta/2\), then \(\kappa\) belongs to \(K_3(X)\).
Figure 2. Chains $\hat{\Lambda}$ and $\hat{B}$.

**Proof.** We have the 2-chains

$$\hat{\Lambda}, \hat{\Lambda}_{\text{mod}} \in C_2(K \setminus \kappa),$$

with

$$\hat{\lambda} = \partial \hat{\Lambda}, \hat{\lambda}_{\text{mod}} = \partial \hat{\Lambda}_{\text{mod}} \in C_1(K_\rho).$$

Note that the support of $\hat{\lambda}_{\text{mod}}$ is contained in $\partial K$ and the 2-chain $\hat{\Lambda}_{\text{mod}}$ is obtained by coning off $\hat{\lambda}_{\text{mod}}$ from the origin. Then, by Assumption 1, for every $w \in D^2$:

i. Either $d(\hat{\Lambda}(w), \hat{\Lambda}_{\text{mod}}(w)) \leq \epsilon$.

ii. Or $\hat{\Lambda}(w), \hat{\Lambda}_{\text{mod}}(w)$ belong to the common ray in $\partial K$.

Since $d(\kappa, \partial K) > \delta \geq 2\epsilon$, it follows that the straight-line homotopy $H_t$ between the maps

$$\hat{\Lambda}, \hat{\Lambda}_{\text{mod}} : D^2 \to K$$

misses $\kappa$. Since $K_\rho$ is convex, $H_t(S^1) \subset K_\rho$ for each $t \in [0, 1]$, and we obtain

$$[\hat{\Lambda}_{\text{mod}}] = [\hat{\Lambda}] \in H_2(K \setminus \kappa, K_\rho).$$

Assumptions 2 and 3 imply that the relative homology class

$$[\hat{\Lambda}_{\text{mod}}] \in H_2(K \setminus \kappa, K_\rho)$$

is nontrivial. Hence

$$[\hat{\Lambda}] \in H_2(K \setminus \kappa, K_\rho)$$

is nontrivial as well. Since $\rho = 3r + 18\epsilon$, according to Corollary 19, $\lambda$ bounds a 2-chain

$$B \in C_2(T^3(r)).$$
Set $\Sigma := B + \Lambda$. Then the absolute class
\[
[\hat{\Sigma}] = [\hat{\Lambda} + \hat{B}] \in H_2(K \setminus \kappa)
\]
is also nontrivial. Since $T^3_{\text{mod}}$ is contractible, there exists a 3-chain $\Gamma \in C_3(T^3_{\text{mod}})$ such that
\[
\partial \Gamma = \Sigma.
\]
Therefore the support of $\hat{\Gamma}$ contains the point $\kappa$. Since the map
\[
L : T^3 \rightarrow K
\]
is the composition of the continuous map $\tau^3 : T^3 \rightarrow X^3$ with the continuous map $L : X^3 \rightarrow K$, it follows that $\kappa$ belongs to the image of the map $L : X^3 \rightarrow K$ and hence $\kappa \in K_3(X)$. \hfill \Box

Our goal therefore is to construct a loop $\lambda$, satisfying Assumptions 1—3.

Let $T \subset X$ be an $(R, \epsilon)$-tripod with the legs $\alpha, \beta, \gamma$ of the length $R$, where $\epsilon \leq \delta/2$. We let $\tau : T_{\text{mod}} \rightarrow T$ denote the standard parametrization of $T$. Let $x, y, z, o$ denote the vertices and the center of $T_{\text{mod}}$. We let $\alpha_{\text{mod}}(s), \beta_{\text{mod}}(s), \gamma_{\text{mod}}(s) : [0, R] \rightarrow T_{\text{mod}}$ denote the arc-length parameterizations of the legs of $T_{\text{mod}}$, so that $\alpha(R) = \beta(R) = \gamma(R) = o$.

We will describe the loop $\lambda$ as the concatenation of seven paths
\[
p_i(s) = (x_1(s), x_2(s), x_3(s)), i = 1, \ldots, 7.
\]
We let $a = d(x_2, x_3), b = d(x_3, x_1), c = d(x_1, x_2)$.

1. $p_1(s)$ is the path starting at $(x, x, o)$ and ending at $(o, x, o)$, given by
\[
p_1(s) = (\alpha_{\text{mod}}(s), x, o).
\]

Note that for $p_1(0)$ and $p_1(R)$ we have $c = 0$ and $b = 0$ respectively.

2. $p_2(s)$ is the path starting at $(o, x, o)$ and ending at $(y, x, o)$, given by
\[
p_2(s) = (\beta_{\text{mod}}(s), x, o).
\]

3. $p_3(s)$ is the path starting at $(y, x, o)$ and ending at $(y, o, o)$, given by
\[
p_3(s) = (y, \alpha_{\text{mod}}(s), o).
\]

Note that for $p_3(R)$ we have $a = 0$.

4. $p_4(s)$ is the path starting at $(y, o, o)$ and ending at $(y, y, o)$, given by
\[
p_4(s) = (y, \beta_{\text{mod}}(s), o).
\]

Note that for $p_4(R)$ we have $c = 0$. Moreover, if $\alpha * \beta$ is a geodesic, then
\[
d(\tau(x), \tau(o)) = d(\tau(y), \tau(o)) \Rightarrow \hat{p}_4(R) = \hat{p}_1(0)
\]
and therefore $\hat{p}_1 * \ldots * \hat{p}_4$ is a loop.

5. $p_5(s)$ is the path starting at $(y, y, o)$ and ending at $(y, y, z)$ given by
\[
(y, y, \gamma_{\text{mod}}(s)).
\]

6. $p_6(s)$ is the path starting at $(y, y, z)$ and ending at $(x, x, z)$ given by
\[
(\beta_{\text{mod}} * \alpha_{\text{mod}}, \beta_{\text{mod}} * \alpha_{\text{mod}}, z).
\]
7. \( p_\ell(s) \) is the path starting at \((x, x, z)\) and ending at \((x, x, o)\) given by 
\[ (x, x, \gamma_{mod}(s)). \]

Thus 
\[ \lambda := p_1 \ast \ldots \ast p_\ell \]
is a loop.

Since \( \alpha \ast \beta \) and \( \gamma \) are \( \epsilon \)-geodesics in \( X \), each path \( p_i(s) \) determines a family of \( \epsilon \)-degenerate triangles in \((T_{mod}, d)\). It is clear that Assumption 1 is satisfied.

The class \([\hat{\lambda}_{mod}]\) is clearly nontrivial in \( H_1(\partial K \setminus 0)\). See Figure 3. Therefore, since \( \epsilon \leq \delta/2 \), 
\[ [\hat{\lambda}] = [\hat{\lambda}_{mod}] \in H_1(K \setminus \mathbb{R}^+ \cdot \kappa) \setminus \{0\}, \]
see the proof of Lemma 20. Thus Assumption 2 holds.

![Figure 3. The loop \( \hat{\lambda}_{mod} \).](image)

**Lemma 21.** The image of \( \hat{\lambda} \) is contained in the closure of \( K_{\rho'} \), where
\[ \rho' = \frac{2}{3}R - 4\epsilon. \]

*Proof.* We have to verify that for each \( i = 1, ..., 6 \) and every \( s \in [0, R] \), the perimeter (with respect to the metric \( d \)) of each triangle \( p_i(s) \in T_{mod}^3 \) is at least \( \rho' \). These inequalities follow directly from Lemma 13 and the description of the paths \( p_i \). \( \square \)
Therefore, if we take 
\[ R > \frac{9}{2}r - 33\epsilon \]
then the image of \( \hat{\lambda} \) is contained in 
\[ K_{3r+18\epsilon} \]
and Assumption 3 is satisfied. Theorem 3 follows. \( \square \)

4. Quasi-isometric characterization of compressed spaces

The goal of this section is to prove Theorem 5. Suppose that \( X \) is compressed. The proof is easier if \( X \) is a proper geodesic metric space, in which case there is no need considering the ultralimits. Therefore, we recommend the reader uncomfortable with this technique to assume that \( X \) is a proper geodesic metric space.

Pick a base-point \( o \in X \), a nonprincipal ultrafilter \( \omega \) and consider the ultralimit 
\[ X_\omega = \omega\text{-lim}(X, o) \]
of the constant sequence of pointed metric spaces. If \( X \) is a proper geodesic metric space then, of course, \( X_\omega = X \). In view of Lemma 12, the space \( X_\omega \) is \( R \)-compressed for some \( R \).

Assume that \( X \) is unbounded. Then \( X \) contains a sequence of \( 1/i \)-geodesic paths \( \gamma_i = \overline{ox_i} \) with 
\[ \omega\text{-lim} d(o, x_i) = \infty, \]
which yields a geodesic ray \( \rho_1 \) in \( X_\omega \) emanating from the point \( o_\omega \).

**Lemma 22.** Let \( \rho \) be a geodesic ray in \( X_\omega \) emanating from the point \( O \). Then the neighborhood \( E = N_R(\rho) \) is an end \( E(\rho) \) of \( X_\omega \).

**Proof.** Suppose that \( \alpha \) is a path in \( X_\omega \setminus B_{2R}(O) \) connecting a point \( y \in X_\omega \setminus E \) to a point \( x \in E \). Then there exists a point \( z \in \alpha \) such that \( d(z, \rho) = R \). Since \( X_\omega \) contains no \( R \)-tripods,
\[ d(p_{\rho}(z), O) < R. \]
Therefore \( d(z, O) < 2R \). Contradiction. \( \square \)

Set \( E_1 := E(\rho_1) \). If the image of the natural embedding \( \iota : X \to X_\omega \) is contained in a finite metric neighborhood of \( \rho_1 \), then we are done, as \( X \) is quasi-isometric to \( \mathbb{R}_+ \). Otherwise, there exists a sequence \( y_n \in X \) such that:
\[ \omega\text{-lim} d(\iota(y_n), \rho_1) = \infty. \]
Consider the \( \frac{1}{n} \)-geodesic paths \( \alpha_n \in P(o, y_n) \). The sequence \( (\alpha_n) \) determines a geodesic ray \( \rho_2 \subset X_\omega \) emanating from \( o_\omega \). Then there exists \( s \geq 4R \) such that
\[ d(\alpha_n(s), \gamma_i) \geq 2R \]
for \( \omega \)-all \( n \) and \( \omega \)-all \( i \). Therefore, for \( t \geq s \), \( \rho_2(t) \notin E(\rho_1) \). By applying Lemma 22 to \( \rho_2 \) we conclude that \( X_\omega \) has an end \( E_2 = E(\rho_2) = N_R(\rho_2) \). Since \( E_1, E_2 \) are distinct ends of \( X_\omega \), \( E_1 \cap E_2 \) is a bounded subset. Let \( D \) denote the diameter of this intersection.
Lemma 23. 1. For every pair of points \( x_i = \rho_i(t_i), \ i = 1, 2, \) we have

\[
\overline{x_1x_2} \subset N_{D/2+2R}(\rho_1 \cup \rho_2).
\]

2. \( \rho_1 \cup \rho_2 \) is a quasi-geodesic.

Proof. Consider the points \( x_i \) as in Part 1. Our goal is to get a lower bound on \( d(x_1, x_2) \). A geodesic segment \( \overline{x_1x_2} \) has to pass through the ball \( B(o_\omega, 2R), i = 1, 2, \) since this ball separates the ends \( E_1, E_2 \). Let \( y_i \in \overline{x_1x_2} \cap B(o_\omega, 2R) \) be such that

\[
\overline{x_iy_i} \subset E_i, \ i = 1, 2.
\]

Then

\[
d(y_1, y_2) \leq D + 4R,
\]

\[
d(x_i, y_i) \geq t_i - 2R,
\]

and

\[
\overline{x_iy_i} \subset N_R(\rho_i), \ i = 1, 2.
\]

This implies the first assertion of Lemma. Moreover,

\[
d(x_1, x_2) \geq d(x_1, y_1) + d(x_2, y_2) \geq t_1 + t_2 - 4R = d(x_1, x_2) - 4R.
\]

Therefore \( \rho_1 \cup \rho_2 \) is a \((1, 4R)\)-quasi-geodesic. \( \square \)

If \( \iota(X) \) is contained in a finite metric neighborhood of \( \rho_1 \cup \rho_2 \), then, by Lemma 23, \( X \) is quasi-isometric to \( \mathbb{R} \). Otherwise, there exists a sequence \( z_k \in X \) such that

\[
\omega\text{-lim } d(\iota(z_k), \rho_1 \cup \rho_2) = \infty.
\]

By repeating the construction of the ray \( \rho_2 \), we obtain a geodesic ray \( \rho_3 \subset X_\omega \) emanating from the point \( o_\omega \), so that \( \rho_3 \) is not contained in a finite metric neighborhood of \( \rho_1 \cup \rho_2 \). For every \( t_3 \), the nearest-point projection of \( \rho_3(t_3) \) to

\[
N_{D/2+2R}(\rho_1 \cup \rho_2)
\]

is contained in

\[
B_{2R}(o_\omega).
\]

Therefore, in view of Lemma 23 for every pair of points \( \rho_i(t_i) \) as in that lemma, the nearest-point projection of \( \rho_3(t_3) \) to \( \rho_1(t_1) \rho_2(t_2) \) is contained in

\[
B_{4R+D}(o_\omega).
\]

Hence, for sufficiently large \( t_1, t_2, t_3 \), the points \( \rho_i(t_i), \ i = 1, 2, 3 \) are vertices of an \( R \)-tripod in \( X \). This contradicts the assumption that \( X_\omega \) is \( R \)-compressed.

Therefore \( X \) is either bounded, or is quasi-isometric to a \( \mathbb{R}_+ \) or to \( \mathbb{R} \). \( \square \)
5. Examples

Theorem 24. There exist an (incomplete) 2-dimensional Riemannian manifold $M$ quasi-isometric to $\mathbb{R}$, so that:

a. $K_3(M)$ does not contain $\partial K_3(\mathbb{R}^2)$.

b. For the Riemannian product $M^2 = M \times M$, $K_3(M^2)$ does not contain $\partial K_3(\mathbb{R}^2)$ either.

Moreover, there exists $D < \infty$ such that for every degenerate triangle in $M$ and $M^2$, at least one side is $\leq D$.

Proof. a. We start with the open concentric annulus $A \subset \mathbb{R}^2$, which has the inner radius $R_1 > 0$ and the outer radius $R_2 < \infty$. We give $A$ the flat Riemannian metric induced from $\mathbb{R}^2$. Let $M$ be the universal cover of $A$, with the pull-back Riemannian metric. Since $M$ admits a properly discontinuous isometric action of $\mathbb{Z}$ with the quotient of finite diameter, it follows that $M$ is quasi-isometric to $\mathbb{R}$. The metric completion $\bar{M}$ of $M$ is diffeomorphic to the closed bi-infinite flat strip. Let $\partial_1 M$ denote the component of the boundary of $\bar{M}$ which covers the inner boundary of $A$ under the map of metric completions $\bar{M} \to \bar{A}$.

As a metric space, $\bar{M}$ is $CAT(0)$, therefore it contains a unique geodesic between any pair of points. However, for any pair of points $x, y \in M$, the geodesic $\gamma = \overline{xy} \subset \bar{M}$ is the union of subsegments

$$\gamma_1 \cup \gamma_2 \cup \gamma_3$$

where $\gamma_1, \gamma_3 \subset M$, $\gamma_2 \subset \partial_1 M$, and the lengths of $\gamma_1, \gamma_3$ are at most $D_0 = \sqrt{R_2^2 - R_1^2}$.

Hence, for every degenerate triangle $(x, y, z)$ in $M$, at least one side is $\leq D_0$.

b. We observe that the metric completion of $M^2$ is $\bar{M} \times \bar{M}$; in particular, it is again a $CAT(0)$ space. Therefore it has a unique geodesic between any pair of points. Moreover, geodesics in $\bar{M} \times \bar{M}$ are of the form

$$\left(\gamma_1(t), \gamma_2(t)\right)$$

where $\gamma_i, i = 1, 2$ are geodesics in $\bar{M}$. Hence for every geodesic segment $\gamma \subset \bar{M} \times \bar{M}$, the complement $\gamma \setminus \partial M^2$ is the union of two subsegments of length $\leq \sqrt{2}D_0$ each. Therefore for every degenerate triangle in $M^2$, at least one side is $\leq \sqrt{2}D_0$. $\square$

Remark 25. The manifold $M^2$ is, of course, quasi-isometric to $\mathbb{R}^2$.

Our second example is a graph-theoretic analogue of the Riemannian manifold $M$.

Theorem 26. There exists a complete path metric space $X$ (a metric graph) quasi-isometric to $\mathbb{R}$ so that:

a. $K_3(X)$ does not contain $\partial K_3(\mathbb{R}^2)$.

b. $K_3(X^2)$ does not contain $\partial K_3(\mathbb{R}^2)$.

Moreover, there exists $D < \infty$ such that for every degenerate triangle in $X$ and $X^2$, at least one side is $\leq D$. 

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Proof. a. We start with the disjoint union of oriented circles $\alpha_i$ of the length $1 + \frac{1}{i}$, $i \in I = \mathbb{N} \setminus \{2\}$. We regard each $\alpha_i$ as a path metric space. For each $i$ pick a point $a_i \in \alpha_i$ and its antipodal point $b_i \in \alpha_i$. We let $\alpha_i^+$ be the positively oriented arc of $\alpha_i$ connecting $a_i$ to $b_i$. Let $\alpha_i^-$ be the complementary arc.

Consider the bouquet $Z$ of $\alpha_i$’s by gluing them all at the points $o_i$. Let $o \in Z$ be the image of the points $o_i$. Next, for every pair $i, j \in I$ attach to $Z$ the oriented arc $\beta_{ij}$ of the length

$$\frac{1}{2} + \frac{1}{4}(\frac{1}{i} + \frac{1}{j})$$

connecting $b_i$ and $b_j$ and oriented from $b_i$ to $b_j$ if $i < j$. Let $Y$ denote the resulting graph. We give $Y$ the path metric. Then $Y$ is a complete metric space, since it is a metric graph where the length of every edge is at least $1/2 > 0$. Note also that the length of every edge in $Y$ is at most 1.

The space $X$ is the infinite cyclic regular cover over $Y$ defined as follows. Take the maximal subtree

$$T = \bigcup_{i \in I} \alpha_i^+ \subset Y.$$ 

Every oriented edge of $Y \setminus T$ determines a free generator of $G = \pi_1(Y, o)$. Define the homomorphism $\rho : G \to \mathbb{Z}$ by sending every free generator to 1. Then the covering $X \to Y$ is associated with the kernel of $\rho$. (This covering exists since $Y$ is locally contractible.)
We lift the path metric from $Y$ to $X$, thereby making $X$ a complete metric graph. We label vertices and edges of $X$ as follows.

1. Vertices $a_n$ which project to $o$. The cyclic group $\mathbb{Z}$ acts simply transitively on the set of these vertices thereby giving them the indices $n \in \mathbb{Z}$.

2. The edges $\alpha_i^\pm$ lift to the edges $\alpha_i^+, \alpha_i^-$ incident to the vertices $a_n$ and $a_{n+1}$ respectively.

3. The intersection $\alpha_i^+ \cap \alpha_i^-(n+1)$ is the vertex $b_{in}$ which projects to the vertex $b_i \in \alpha_i$.

4. The edge $\beta_{ijn}$ connecting $b_{in}$ to $b_{j(n+1)}$ which projects to the edge $\beta_{ij} \subset Y$.

**Figure 5. The metric space $Y$.**

**Figure 6. The metric space $X$.**

**Lemma 27.** $X$ contains no degenerate triangles $(x,y,v)$, so that $v$ is a vertex,

$$d(x,v) + d(v,y) = d(x,y)$$
and \( \min(d(x, v), d(v, y)) > 2 \).

**Proof.** Suppose that such degenerate triangles exist.

**Case 1:** \( v = b_n \). Since the triangle \((x, y, v)\) is degenerate, for all sufficiently small \( \epsilon > 0 \) there exist \( \epsilon \)-geodesics \( \sigma \) connecting \( x \) to \( y \) and passing through \( v \).

Since \( d(x, v), d(v, y) > 2 \), it follows that for sufficiently small \( \epsilon > 0 \), \( \sigma = \sigma(\epsilon) \) also passes through \( b_{j(n-1)} \) and \( b_{k(n+1)} \) for some \( j, k \) depending on \( \sigma \). We will assume that as \( \epsilon \to 0 \), both \( j \) and \( k \) diverge to infinity, leaving the other cases to the reader.

Therefore
\[
\begin{align*}
d(x, v) &= \lim_{j \to \infty} (d(x, b_{j(n-1)}) + d(b_{j(n-1)}, v)), \\
d(v, y) &= \lim_{k \to \infty} (d(y, b_{k(n+1)}) + d(b_{k(n+1)}, v)).
\end{align*}
\]

Then
\[
\lim_{j \to \infty} d(b_{j(n-1)}, v) + \lim_{k \to \infty} d(b_{k(n+1)}, v) = 1 + \frac{1}{2i}.
\]

On the other hand, clearly,
\[
\lim_{j, k \to \infty} d(b_{j(n-1)}, b_{k(n+1)}) = 1.
\]

Hence
\[
d(x, y) = \lim_{j \to \infty} d(x, b_{j(n-1)}) + \lim_{k \to \infty} d(y, b_{k(n+1)}) + 1 < d(x, v) + d(v, y).
\]

Contradiction.

**Case 2:** \( v = a_n \). Since the triangle \((x, y, v)\) is degenerate, for all sufficiently small \( \epsilon > 0 \) there exist \( \epsilon \)-geodesics \( \sigma \) connecting \( x \) to \( y \) and passing through \( v \). Then for sufficiently small \( \epsilon > 0 \), every \( \sigma \) also passes through \( b_{j(n-1)} \) and \( b_{kn} \) for some \( j, k \) depending on \( \sigma \). However, since \( j, k \geq 2 \),
\[
d(b_{j(n-1)}, b_{kn}) = \frac{1}{2} + \frac{1}{4j} + \frac{1}{4k} \leq \frac{3}{4} < 1 = \inf_{j,k} d(b_{j(n-1)}, v) + d(v, b_{kn}).
\]

Therefore \( d(x, y) < d(x, v) + d(v, y) \). Contradiction. \( \square \)

**Corollary 28.** \( X \) contains no degenerate triangles \((x, y, z)\), such that
\[
d(x, z) + d(z, y) = d(x, y)
\]
and \( \min(d(x, z), d(z, y)) \geq 3 \).

**Proof.** Suppose that such degenerate triangles exist. We can assume that \( z \) is not a vertex. The point \( z \) belongs to an edge \( e \subset X \). Since \( \text{length}(e) \leq 1 \), for one of the vertices \( v \) of \( e \)
\[
d(z, v) \leq 1/2.
\]

Since the triangle \((x, y, z)\) is degenerate, for all \( \epsilon \)-geodesics \( \sigma \in P(x, z) \), \( \eta \in P(z, y) \) we have:
\[
e \subset \sigma \cup \eta,
\]
provided that \( \epsilon > 0 \) is sufficiently small. Therefore the triangle \((x, y, v)\) is also degenerate. Clearly,
\[
\min(d(x, v), d(y, v)) \geq \min(d(x, z), d(y, z)) - 1/2 \geq 2.5.
\]

This contradicts Lemma \([27] \). \( \square \)
Hence part (a) of Theorem 26 follows.

b. We consider $X^2 = X \times X$ with the product metric

$$d^2((x_1, y_1), (x_2, y_2)) = d^2(x_1, x_2) + d^2(y_1, y_2).$$

Then $X^2$ is a complete path-metric space. Every degenerate triangle in $X^2$ projects to degenerate triangles in both factors. It therefore follows from part (a) that $X$ contains no degenerate triangles with all sides $\geq 18$. We leave the details to the reader. $\square$

6. Exceptional cases

**Theorem 29.** Suppose that $X$ is a path metric space quasi-isometric to a metric space $X'$, which is either $\mathbb{R}$ or $\mathbb{R}_+$. Then there exists a $(1, A)$-quasi-isometry $X' \to X$.

**Proof.** We first consider the case $X' = \mathbb{R}$. The proof is simpler if $X$ is proper, therefore we sketch it first under this assumption. Since $X$ is quasi-isometric to $\mathbb{R}$, it is 2-ended with the ends $E_+, E_-$. Pick two divergent sequences $x_i \in E_+, y_i \in E_-$. Then there exists a compact subset $C \subset X$ so that all geodesic segments $\gamma_i := \frac{x_i y_i}{d(x_i, y_i)}$ intersect $C$. It then follows from the Arcela-Ascoli theorem that the sequence of segments $\gamma_i$ subconverges to a complete geodesic $\gamma \subset X$. Since $X$ is quasi-isometric to $\mathbb{R}$, there exists $R < \infty$ such that $X = N_R(\gamma)$. We define the $(1, R)$-quasi-isometry $f: \gamma \to X$ to be the identity (isometric) embedding.

We now give a proof in the general case. Pick a non-principal ultrafilter $\omega$ on $\mathbb{N}$ and a base-point $o \in X$. Define $X_\omega$ as the $\omega$-limit of $(X, o)$. The quasi-isometry $f: \mathbb{R} \to X$ yields a quasi-isometry $f_\omega: \mathbb{R} = \mathbb{R}_\omega \to X_\omega$. Therefore $X_\omega$ is also quasi-isometric to $\mathbb{R}$.

We have the natural isometric embedding $\iota: X \to X_\omega$. As above, let $E_+, E_-$ denote the ends of $X$ and choose divergent sequences $x_i \in E_+, y_i \in E_-$. Let $\gamma_i$ denote an $\frac{1}{i}$-geodesic segment in $X$ connecting $x_i$ to $y_i$. Then each $\gamma_i$ intersects a bounded subset $B \subset X$. Therefore, by taking the ultralimit of $\gamma_i$’s, we obtain a complete geodesic $\gamma \subset X_\omega$. Since $X_\omega$ is quasi-isometric to $\mathbb{R}$, the embedding $\eta: \gamma \to X_\omega$ is a quasi-isometry. Hence $X_\omega = N_R(\gamma)$ for some $R < \infty$.

For the same reason,

$$X_\omega = N_D(\iota(X))$$

for some $D < \infty$. Therefore the isometric embeddings

$$\eta: \gamma \to X_\omega, \iota: X \to X_\omega$$

are $(1, R)$ and $(1, D)$-quasi-isometries respectively. By composing $\eta$ with the quasi-inverse to $\iota$, we obtain a $(1, R + 3D)$-quasi-isometry $\mathbb{R} \to X$.

The case when $X$ is quasi-isometric to $\mathbb{R}_+$ can be treated as follows. Pick a point $o \in X$ and glue two copies of $X$ at $o$. Let $Y$ be the resulting path metric space. It is easy to see that $Y$ is quasi-isometric to $\mathbb{R}$ and the inclusion $X \to Y$ is an isometric embedding. Therefore, there exists a $(1, A)$-quasi-isometry $h: Y \to \mathbb{R}$ and the restriction of $h$ to $X$ yields the $(1, A)$-quasi-isometry from $X$ to the half-line. $\square$

**Corollary 30.** Suppose that $X$ is a path metric space quasi-isometric to $\mathbb{R}$ or $\mathbb{R}_+$. Then $K_3(X)$ is contained in the $D$-neighborhood of $\partial K$ for some $D < \infty$. In particular, $K_3(X)$ does not contain the interior of $K = K_3(\mathbb{R}^2)$.
Proof. Suppose that $f: X \to X'$ is an $(L, A)$-quasi-isometry, where $X'$ is either $\mathbb{R}$ or $\mathbb{R}_+$. According to Theorem 29, we can assume that $L = 1$. For every triple of points $x, y, z \in X$, after relabelling, we obtain

$$d(x, y) + d(y, z) \leq d(x, z) + D,$$

where $D = 3A$. Then every triangle in $X$ is $D$-degenerate. Hence $K_3(X)$ is contained in the $D$-neighborhood of $\partial K$. 

Remark 31. One can construct a metric space $X$ quasi-isometric to $\mathbb{R}$ such that $K_3(X) = K$. Moreover, $X$ is isometric to a curve in $\mathbb{R}^2$ (with the metric obtained by the restriction of the metric on $\mathbb{R}^2$). Of course, the metric on $X$ is not a path metric.

**Corollary 32.** Suppose that $X$ is a path metric space. Then the following are equivalent:

1. $K_3(X)$ contains the interior of $K = K_3(\mathbb{R}^2)$.
2. $X$ is not quasi-isometric to the point, $\mathbb{R}_+$ and $\mathbb{R}$.
3. $X$ is uncompressed.

**Proof.** 1$\Rightarrow$2 by Corollary 30. 2$\Rightarrow$3 by Theorem 5. 3$\Rightarrow$1 by Theorem 3. 

Remark 33. The above corollary remains valid under the following assumption on the metric on $X$, which is weaker than being a path metric:

For every pair of points $x, y \in X$ and every $\epsilon > 0$, there exists a $(1, \epsilon)$-quasi-geodesic path $\alpha \in P(x, y)$.

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