Boundedness of Lusin-area and \(g^\lambda\) Functions on Localized BMO Spaces over Doubling Metric Measure Spaces

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Dedicated to Professor Kôzô Yabuta in celebration of his 70th birthday

Abstract. Let \(\mathcal{X}\) be a doubling metric measure space. If \(\mathcal{X}\) has the \(\delta\)-annular decay property for some \(\delta \in (0, 1]\), the authors then establish the boundedness of the Lusin-area function, which is defined via kernels modeled on the semigroup generated by the Schrödinger operator, from localized spaces \(\text{BMO}_\rho(\mathcal{X})\) to \(\text{BLO}_\rho(\mathcal{X})\) without invoking any regularity of considered kernels. The same is true for the \(g^\lambda\) function and unlike the Lusin-area function, in this case, \(\mathcal{X}\) is not necessary to have the \(\delta\)-annular decay property. Moreover, for any metric space, the authors introduce the weak geodesic property and the monotone geodesic property, which are proved to be respectively equivalent to the chain ball property of Buckley. Recall that Buckley proved that any length space has the chain ball property and, for any metric space equipped with a doubling measure, the chain ball property implies the \(\delta\)-annular decay property for some \(\delta \in (0, 1]\). Moreover, using some results on pointwise multipliers of \(\text{bmo}(\mathbb{R})\), the authors construct a counterexample to show that there exists a nonnegative function which is in \(\text{bmo}(\mathbb{R})\), but not in \(\text{blo}(\mathbb{R})\); this further indicates that the above boundedness of the Lusin-area and \(g^\lambda\) functions even in \(\mathbb{R}^d\) with the Lebesgue measure or the Heisenberg group also improves the existing results.

1 Introduction

Since the space \(\text{BMO}(\mathbb{R}^d)\) of functions with bounded mean oscillation on \(\mathbb{R}^d\) was introduced by John and Nirenberg [21], it then plays an important role in harmonic analysis and partial differential equations. It is well-known that \(\text{BMO}(\mathbb{R}^d)\) is the dual space of the Hardy space \(H^1(\mathbb{R}^d)\) (see, for example, [33, 14]), and also a good substitute of \(L^\infty(\mathbb{R}^d)\) in the study of boundedness of operators. However, the space \(\text{BMO}(\mathbb{R}^d)\) is essentially related to the Laplacian \(\Delta\), where \(\Delta \equiv \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}\).

On the other hand, there exists an increasing interest on the study of Schrödinger operators on \(\mathbb{R}^d\) and the sub-Laplace Schrödinger operators on connected and simply connected nilpotent Lie groups with nonnegative potentials satisfying the reverse Hölder

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inequality; see, for example, [11, 40, 32, 22, 8, 10, 9, 23, 38, 18, 19]. Let \( L \equiv -\Delta + V \) be the Schrödinger operator on \( \mathbb{R}^d \), where the potential \( V \) is a nonnegative locally integrable function. Denote by \( B_q(\mathbb{R}^d) \) the class of nonnegative functions satisfying the reverse Hölder inequality of order \( q \). For \( V \in B_{d/2}(\mathbb{R}^d) \) with \( d \geq 3 \), Dziubański et al [8, 10, 9] studied the \( \text{BMO} \)-type space \( \text{BMO}_L(\mathbb{R}^d) \) and the Hardy space \( H^1_L(\mathbb{R}^d) \) and, especially, proved that the dual space of \( H^1_L(\mathbb{R}^d) \) is \( \text{BMO}_L(\mathbb{R}^d) \). Moreover, they obtained the boundedness on these spaces of the Littlewood-Paley \( g \)-function associated to \( L \). Let \( \mathcal{X} \) be an RD-space in [16], which means that \( \mathcal{X} \) is a space of homogeneous type in the sense of Coifman and Weiss [4, 5] with the additional property that a reverse doubling condition holds. Let \( \rho \) be a given admissible function modeled on the known auxiliary function determined by \( V \in B_{d/2}(\mathbb{R}^d) \) (see [38] or (2.3) below). The localized Hardy space \( H^1_\rho(\mathcal{X}) \), the \( \text{BMO} \)-type space \( \text{BMO}_\rho(\mathcal{X}) \) and the \( \text{BLO} \)-type space \( \text{BLO}_\rho(\mathcal{X}) \) associated with \( \rho \) were introduced and studied in [38, 37]. Moreover, the boundedness from \( \text{BMO}_\rho(\mathcal{X}) \) to \( \text{BLO}_\rho(\mathcal{X}) \) of several maximal operators and the Littlewood-Paley \( g \)-function, which are defined via kernels modeled on the semigroup generated by the Schrödinger operator, was obtained in [37].

Let \( \mathcal{X} \) be a doubling metric measure space. The main purpose of this paper is to investigate behaviors of the Lusin-area and \( g^*_\lambda \) functions on localized \( \text{BMO} \) spaces over \( \mathcal{X} \), which is not necessary to be an RD-space. So far, it is still not clear whether the doubling property of \( \mathcal{X} \) is sufficient to guarantee the boundedness of the Lusin-area function on these localized \( \text{BMO} \) spaces over \( \mathcal{X} \). However, in this paper, when \( \mathcal{X} \) has the \( \delta \)-annular decay property for some \( \delta \in (0, 1] \) which was introduced by Buckley in [1], we establish the boundedness of the Lusin-area function, which is defined via kernels modeled on the semigroup generated by the Schrödinger operator, from localized spaces \( \text{BMO}_\rho(\mathcal{X}) \) to \( \text{BLO}_\rho(\mathcal{X}) \) without invoking any regularity of considered kernels. The corresponding boundedness of the \( g^*_\lambda \) function from \( \text{BMO}_\rho(\mathcal{X}) \) to \( \text{BLO}_\rho(\mathcal{X}) \) is also obtained in this paper. Moreover, an interesting phenomena is that unlike the Lusin-area function, the boundedness of the \( g^*_\lambda \) function needs neither the regularity of the kernels nor the \( \delta \)-annular decay property of \( \mathcal{X} \), which reflects the difference between the Lusin-area function and the \( g^*_\lambda \) function. These results are new even on \( \mathbb{R}^d \) with the Lebesgue measure and the Heisenberg group, and apply in a wide range of settings, for instance, to the Schrödinger operator or the degenerate Schrödinger operator on \( \mathbb{R}^d \), or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups. Moreover, via some results on the pointwise multiplier of \( \text{bmo}(\mathbb{R}) \) from [31], we construct a counterexample to show that there exists a nonnegative function which is in \( \text{bmo}(\mathbb{R}) \) of Goldberg [13], but not in \( \text{blo}(\mathbb{R}) \) of [17]. Thus, \( \text{blo}(\mathbb{R}) \cap \{ f \geq 0 \} \) is a proper subspace of \( \text{bmo}(\mathbb{R}) \), which further indicates that our above results on the boundedness of the Lusin-area and \( g^*_\lambda \) functions even in \( \mathbb{R}^d \) with the Lebesgue measure or the Heisenberg group also improve the existing results.

Moreover, motivated by Tessera [35], we introduce two properties, for any metric space, the weak geodesic property and the monotone geodesic property, which are slightly stronger variants of the corresponding ones of Tessera [35] (see Remark 4.1 below) and are then proved to be respectively equivalent to the chain ball property introduced by Buckley [1]. It was proved by Buckley [1] that any length space, namely, the metric space in which the distance between any pair of points equals the infimum of the lengths of rectifiable
paths joining them, has the chain ball property and, for any metric space equipped with a doubling measure, the chain ball property implies the $\delta$-annular decay property for some $\delta \in (0,1]$. As an application, we prove that any length space equipped with a doubling measure has the weak geodesic property and hence the $\delta$-annular decay property for some $\delta \in (0,1]$ without using the property of John domains as in [1].

This paper is organized as follows. Let $\mathcal{X}$ be a doubling metric measure space and $\rho$ an admissible function on $\mathcal{X}$. In Section 2, we first recall the notions of the spaces $\text{BMO}_\rho(\mathcal{X})$ and $\text{BLO}_\rho(\mathcal{X})$. When $\mathcal{X} = \mathbb{R}$, we construct a counterexample to show that there exists a nonnegative function $f \in \text{bmo}(\mathbb{R})$, but $f \not\in \text{blo}(\mathbb{R})$; see Proposition 2.1 below.

In Section 3, if $\mathcal{X}$ has the $\delta$-annular decay property for some $\delta \in (0,1]$ and the Littlewood-Paley $g$-function $g(f)$ is bounded on $L^2(\mathcal{X})$, we prove that if $f \in \text{BMO}_\rho(\mathcal{X})$, then $[S(f)]^2 \in \text{BLO}_\rho(\mathcal{X})$ with norm no more than $C\|f\|^2_{\text{BMO}_\rho(\mathcal{X})}$, where $C$ is a positive constant independent of $f$; see Theorem 3.1 below. As a corollary, we obtain the boundedness of the Lusin-area function from $\text{BMO}_\rho(\mathcal{X})$ to $\text{BLO}_\rho(\mathcal{X})$; see Corollary 3.1 below. The corresponding results for the $g^*_\delta$ function $g^*_\delta(f)$ are established in Theorem 3.2 and Corollary 3.2 below, where $\mathcal{X}$ is not necessary to have the $\delta$-annular decay property. Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 are true for the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups. Moreover, for these specific examples, it is known that the corresponding Littlewood-Paley $g$-function is bounded on $L^2(\mathcal{X})$; see [37] for the detailed explanations.

We remark that the results obtained in Section 3 are also new even on $\mathbb{R}^d$ with the Lebesgue measure and the Heisenberg group, since we do not need any regularity of involved kernels. However, to establish the boundedness of the Lusin-area function on a doubling metric measure space $\mathcal{X}$, we need certain regularity of $\mathcal{X}$, namely, the $\delta$-annular decay property of $\mathcal{X}$, which reflects the speciality of the Lusin-area function, comparing with the corresponding results of the $g^*_\delta$ function. Moreover, $\mathbb{R}^d$ with the Lebesgue measure and the Heisenberg group have the $\delta$-annular decay property.

In Section 4, for any metric space, we introduce the notions of the weak geodesic property and the monotone geodesic property in Definition 4.1 below, which are proved respectively equivalent to the chain ball property of Buckley in Theorem 4.1 below. As an application of this result and [1, Theorem 2.1], we obtain in Corollary 4.1 below that for any metric space equipped with a doubling measure, either the weak geodesic property or the monotone geodesic property guarantees its $\delta$-annular decay property for some $\delta \in (0,1]$. As an application of Corollary 4.1, we prove that any length space equipped with a doubling measure has the $\delta$-annular decay property for some $\delta \in (0,1]$; see Proposition 4.1 below.

Finally, we make some conventions. Throughout this paper, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_1$, do not change in different occurrences. If $f \leq Cg$, we then write $f \preceq g$ or $g \succeq f$; and if $f \preceq g \preceq f$, we then write $f \sim g$. We also use $B$ to denote a ball of $\mathcal{X}$, and for $\lambda > 0$, $\lambda B$ denotes the ball with the same center as $B$, but radius $\lambda$ times the radius of $B$. Moreover, set $B^c \equiv \mathcal{X} \setminus B$. Also, for any set $E \subset \mathcal{X}$, $\chi_E$ denotes its characteristic function. For all $f \in L^1_{\text{loc}}(\mathcal{X})$ and balls $B$, we always set $f_B \equiv \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y)$.
2 The spaces \( \text{BMO}_\rho(\mathcal{X}) \) and \( \text{BLO}_\rho(\mathcal{X}) \)

In this section, we first recall the notions of localized BMO spaces over doubling metric measure spaces. Moreover, via some results on pointwise multipliers of \( \text{bmo}(\mathbb{R}) \), an example is constructed to show that there exists a nonnegative function which is in \( \text{bmo}(\mathbb{R}) \), but not in \( \text{blo}(\mathbb{R}) \).

We begin with the notions of doubling metric measure spaces \([4, 5]\) and admissible functions \([38]\).

**Definition 2.1.** Let \( (\mathcal{X}, d) \) be a metric space endowed with a regular Borel measure \( \mu \) such that all balls defined by \( d \) have finite and positive measure. For any \( x \in \mathcal{X} \) and \( r \in (0, \infty) \), set the ball \( B(x, r) \equiv \{ y \in \mathcal{X} : d(x, y) < r \} \). The triple \( (\mathcal{X}, d, \mu) \) is called a doubling metric measure space if there exists a constant \( C_1 \in [1, \infty) \) such that for all \( x \in \mathcal{X} \) and \( r \in (0, \infty) \), \( \mu(B(x, 2r)) \leq C_1 \mu(B(x, r)) \) (doubling property).

From Definition 2.1, it is easy to see that there exists positive constants \( C_2 \) and \( n \) such that for all \( x \in \mathcal{X}, r \in (0, \infty) \) and \( \lambda \in [1, \infty) \),

\[
\mu(B(x, \lambda r)) \leq C_2 \lambda^n \mu(B(x, r)).
\]

In what follows, we always let \( B(x, r) \equiv \{ y \in \mathcal{X} : d(x, y) \leq r \} \), \( V_r(x) \equiv \mu(B(x, r)) \) and \( V(x, y) \equiv \mu(B(x, d(x, y))) \) for all \( x, y \in \mathcal{X} \) and \( r \in (0, \infty) \).

**Definition 2.2** ([38]). A positive function \( \rho \) on \( \mathcal{X} \) is called admissible if there exist positive constants \( C_0 \) and \( k_0 \) such that for all \( x, y \in \mathcal{X} \),

\[
\frac{1}{\rho(x)} \leq C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x, y)}{\rho(y)}\right)^{k_0}.
\]

Obviously, if \( \rho \) is a constant function, then \( \rho \) is admissible. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class \( \mathcal{B}_q(\mathcal{X}, d, \mu) \) (see, for example \([15, 29, 32]\) for its definition on \( \mathbb{R}^n \), and \([34]\) for its definition on spaces of homogenous type).

Recall that a nonnegative potential \( V \) is said to be in \( \mathcal{B}_q(\mathcal{X}, d, \mu) \) (for short, \( \mathcal{B}_q(\mathcal{X}) \)) with \( q \in (1, \infty] \) if there exists a positive constant \( C \) such that for all balls \( B \) of \( \mathcal{X} \),

\[
\left\{ \frac{1}{|B|} \int_B [V(y)]^q \, dy \right\}^{1/q} \leq C \int_B V(y) \, dy
\]

with the usual modification made when \( q = \infty \). It was proved in \([34, \text{pp.8-9}]\) that if \( V \in \mathcal{B}_q(\mathcal{X}) \) for some \( q \in (1, \infty] \) and the measure \( V(z) d\mu(z) \) has the doubling property, then \( V \) is an \( \mathcal{A}_q(\mathcal{X}, d, \mu) \)-weight for some \( p \in [1, \infty) \) in the sense of Muckenhoupt, and also \( V \in \mathcal{B}_{q+\epsilon}(\mathcal{X}) \) for some \( \epsilon > 0 \). Here it should be pointed out that, generally speaking, \( V \in \mathcal{B}_q(\mathcal{X}) \) cannot guarantee the doubling property of \( V(z) d\mu(z) \), but when \( \mu(B(x, r)) \) is continuous respect to \( r \) for all \( x \in \mathcal{X} \) or \( \mathcal{X} \) has the \( \delta \)-annular decay property (see Definition 3.1 below), \( V \in \mathcal{B}_q(\mathcal{X}) \) does imply the doubling property of \( V(z) d\mu(z) \) by \([34, \text{Theorem 17}]\) or \([26, \text{Proposition 3.7}]\), respectively. Following \([32]\), for all \( x \in \mathcal{X} \), set

\[
\rho(x) \equiv \sup \left\{ r > 0 : \frac{r^2}{\mu(B(x, r))} \int_{B(x, r)} V(y) \, dy \leq 1 \right\}.
\]
see also [38]. It was proved in [38, Proposition 2.1] that if the measure $V(z) \mu(z)$ has the doubling property, then $\rho$ in (2.3) is an admissible function when $n \geq 1$, $q > \max\{1, n/2\}$ and $V \in B_q(\mathcal{X})$.

Now we recall the notions of the spaces $BMO_\rho(\mathcal{X})$ and $BLO_\rho(\mathcal{X})$ (see [37]).

**Definition 2.3 ([37]).** Let $\rho$ be an admissible function on $\mathcal{X}$, $\mathcal{D} \equiv \{B(x, r) \subset \mathcal{X} : x \in \mathcal{X}, \ r \geq \rho(x)\}$ and $q \in [1, \infty)$. A function $f \in L^q_{\text{loc}}(\mathcal{X})$ is said to be in the space $BMO_\rho^q(\mathcal{X})$ if

$$
\|f\|_{BMO_\rho^q(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y) - f_B|^q \mu(y) \right\}^{1/q} + \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^q \mu(y) \right\}^{1/q} < \infty.
$$

**Remark 2.1.** We denote $BMO_\rho^1(\mathcal{X})$ simply by $BMO_\rho(\mathcal{X})$. The space $BMO_\rho(\mathbb{R}^d)$ when $\rho \equiv 1$ was first introduced by Goldberg [13]. If $q > \frac{d}{2}$, $V \in B_q(\mathbb{R}^d)$ and $\rho$ is as in (2.3), then $BMO_\rho(\mathbb{R}^d)$ is just the space $BMO_L(\mathbb{R}^d)$ introduced by Dziubanski et al in [9]. For all $q \in [1, \infty)$, $BMO_\rho^q(\mathcal{X}) \subseteq BMO(\mathcal{X})$.

The following technical lemma is just Lemma 3.1 in [37].

**Lemma 2.1.** Let $\rho$ be an admissible function on $\mathcal{X}$ and $q \in [1, \infty)$. Then $BMO_\rho(\mathcal{X}) = BMO_\rho^q(\mathcal{X})$ with equivalent norms.

**Definition 2.4 ([37]).** Let $\rho$ and $\mathcal{D}$ be as in Definition 2.3 and $q \in [1, \infty)$. A function $f \in L^q_{\text{loc}}(\mathcal{X})$ is said to be in the space $BLO_\rho(\mathcal{X})$ if

$$
\|f\|_{BLO_\rho^q(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B \left[ f(y) - \text{essinf}_B f \right]^q d\mu(y) \right\}^{1/q} + \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^q d\mu(y) \right\}^{1/q} < \infty.
$$

**Remark 2.2.** (i) The space $BLO(\mathbb{R}^d)$ with the Lebesgue measure was introduced by Coifman and Rochberg [3], and extended by Jiang [20] to the setting of $\mathbb{R}^d$ with a non-doubling measure. The localized $BLO$ space was first introduced in [17] in the setting of $\mathbb{R}^d$ with a non-doubling measure.

(ii) For all $q \in [1, \infty)$, $BLO_\rho^q(\mathcal{X}) \subset BMO_\rho^q(\mathcal{X})$. We denote $BLO_\rho(\mathcal{X})$ simply by $BLO_\rho(\mathcal{X})$.

Even when $\rho \equiv 1$, it is not so difficult to show that for all $q \in [1, \infty)$, $BLO_\rho^q(\mathbb{R}^d)$ is a proper subspace of $BMO_\rho^q(\mathbb{R}^d)$. For example, if we set $f(x) \equiv (\log |x|) \chi_{\{|x| \leq 1\}}(x)$ for all $x \in \mathbb{R}$, then it is easy to show that $f \in BMO_1(\mathbb{R})$, but $f \not\in BLO_1(\mathbb{R})$. Notice that the above function is non-positive. However, it is not so easy to show that there exists a nonnegative function which is in $BMO_1^q(\mathbb{R}^d)$, but not in $BLO_1^q(\mathbb{R}^d)$.

Let $\mathcal{X} = (\mathbb{R}, \cdot, dx)$. Denote $BMO_\rho(\mathbb{R})$ and $BLO_\rho(\mathbb{R})$ with $\rho \equiv 1$, respectively, by $\text{bmo}(\mathbb{R})$ and $\text{blo}(\mathbb{R})$. In the rest of this section, we construct the following interesting counterexample.
Proposition 2.1. There exists a nonnegative function $f \in \text{bmo}(\mathbb{R})$, but $f \notin \text{blo}(\mathbb{R})$.

We first recall some notation and notions. Let $\phi$ be a positive non-decreasing function on $(0, \infty)$. Define

\[
\text{BMO}^\phi(\mathbb{R}) \equiv \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \sup_{\text{balls } B \subset \mathbb{R}} \frac{\text{MO}(f, B)}{\phi(r_B)} < \infty \right\}
\]

and

\[
\widetilde{\text{BMO}}^\phi(\mathbb{R}) \equiv \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : |f_{B(0,1)}| + \sup_{\text{balls } B \subset \mathbb{R}} \frac{\text{MO}(f, B)}{\phi(r_B)} < \infty \right\},
\]

where $\text{MO}(f, B) = \frac{1}{|B|} \int_B |f(x) - f_B| \, dx$ and $r_B$ denotes the radius of ball $B$. Recall that $f_B = \frac{1}{|B|} \int_B f(y) \, dy$. Then $\text{BMO}^\phi(\mathbb{R})$ modulo constants is a Banach space, but $\widetilde{\text{BMO}}^\phi(\mathbb{R})$ is itself a Banach space modulo null-functions; see [31].

The following conclusion is just Lemma 2.2 in [31].

Lemma 2.2. If $|F(x) - F(y)| \leq C|x - y|$, then $\text{MO}(F(f), B) \leq 2C \text{MO}(f, B)$.

For a positive non-decreasing function $\phi$ on $(0, \infty)$, we define strictly positive functions $\Phi^*(r)$ and $\Phi_*(r)$ by setting

\[
\Phi^*(r) \equiv \begin{cases} \int_1^r \frac{\phi(t)}{t} \, dt, & \text{if } 2 \leq r; \\ \int_1^2 \frac{\phi(t)}{t} \, dt, & \text{if } 0 < r \leq 2, \end{cases}
\]

and

\[
\Phi_*(r) \equiv \begin{cases} \int_r^2 \frac{\phi(t)}{t} \, dt, & \text{if } 0 < r \leq 1; \\ \int_1^2 \frac{\phi(t)}{t} \, dt, & \text{if } 1 < r. \end{cases}
\]

The following result is just Lemma 2.4 in [31].

Lemma 2.3. Assume that $\frac{\phi(t)}{t}$ is almost decreasing. Then $\Phi^*(|x|), \Phi_*(|x|) \in \widetilde{\text{BMO}}^\phi(\mathbb{R})$.

Recall that a function $g$ on $\mathbb{R}$ is called a pointwise multiplier on $\text{bmo}(\mathbb{R})$, if the pointwise multiplication $fg$ belongs to $\text{bmo}(\mathbb{R})$ for all $f \in \text{bmo}(\mathbb{R})$.

Set

\[
(2.4) \quad \psi(r) = \left[ \int_{\min(1, r)}^2 \frac{1}{t} \, dt \right]^{-1} \quad \text{for } r \in (0, \infty).
\]

Then $\psi$ is increasing and $\psi(t)$ is almost decreasing. The following Lemma 2.4 is a special case of Theorem 3 in [31].

Lemma 2.4. A function $g$ on $\mathbb{R}$ is a pointwise multiplier on $\text{bmo}(\mathbb{R})$ if and only if $g \in \widetilde{\text{BMO}}^\psi(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where $\psi$ is as in (2.4).

Then we have the following conclusion.
Proposition 2.2. Let $\psi$ be as in (2.4). Set

$$\Psi^*_r(r) \equiv \begin{cases} \int_r^2 \frac{\psi(t)}{t} \, dt, & \text{if } 0 < r \leq 1; \\ \int_1^2 \frac{\psi(t)}{t} \, dt, & \text{if } 1 < r, \end{cases}$$

and

$$g(x) \equiv \sin \Psi^*_r(|x|) \quad \text{for } x \in \mathbb{R}. \quad (2.5)$$

Then $g$ is a pointwise multiplier on $\text{bmo}(\mathbb{R})$.

Proof. By Lemma 2.4, we only need to prove that $\sin \Psi^*_r(|x|) \in \widehat{\text{BMO}}^\psi(\mathbb{R}) \cap L^\infty(\mathbb{R})$. From Lemma 2.3, it follows that $\Psi^*_r(|x|) \in \widehat{\text{BMO}}^\psi(\mathbb{R})$, which via Lemma 2.2 shows that $\sin \Psi^*_r(|x|) \in \widehat{\text{BMO}}^\psi(\mathbb{R})$. Obviously, $\sin \Psi^*_r(|x|) \in L^\infty(\mathbb{R})$, which completes the proof of Proposition 2.2.

Now we prove Proposition 2.1.

Proof of Proposition 2.1. Let $g$ be as in (2.5). For $x \in \mathbb{R}$, set

$$f(x) \equiv \begin{cases} \log(2/|x|), & \text{if } |x| \leq 2; \\ 0, & \text{if } |x| > 2. \end{cases}$$

Then we shall show $|fg| \in \text{bmo}(\mathbb{R})$, but $|fg| \notin \text{blo}(\mathbb{R})$.

Since $g$ is a pointwise multiplier on $\text{bmo}(\mathbb{R})$, $fg \in \text{bmo}(\mathbb{R})$, and so $|fg| \in \text{bmo}(\mathbb{R})$.

Now we turn our attention to prove that $|fg| \notin \text{blo}(\mathbb{R})$. Notice that

$$\Psi^*_r(r) \equiv \begin{cases} 1 + \int_0^1 \frac{dt}{t \log(2/t)}, & \text{if } 0 < r \leq 1; \\ 1, & \text{if } 1 < r. \end{cases}$$

So

$$g(x) = \sin \left( 1 + \int_{|x|}^1 \frac{dt}{t \log(2/t)} \right), \quad \text{if } |x| \leq 1.$$

For $k = 2, 3, 4, \cdots$, choose $r_k > 0$ such that

$$\Psi^*_r(r_k) = 1 + \int_{r_k}^1 \frac{dt}{t \log(2/t)} = \frac{\pi}{4} k.$$

Then $1 > r_2 > r_3 > r_4 > \cdots$, and $r_k \to 0$ as $k \to \infty$. Let $m \in \mathbb{N}$. For $x \in [r_{8m+3}, r_{8m+4})$, we have $\Psi^*_r(x) \in [(2m + \frac{1}{4})\pi, (2m + 1)\pi]$, which implies that $\sin \Psi^*_r(x) \geq 0$, $\cos \Psi^*_r(x) < 0$.
and \( \sin \Psi_*(x) + \cos \Psi_*(x) < 0 \). Then we have the following:

| \( x \) | \( r_{8m+4} \) | \( r_{8m+3} \) |
|---|---|---|
| \( (fg)'(x) \) | + | 0 |
| \( (fg)''(x) \) | - | - |
| \( fg(x) \) | 0 | \( \sqrt{2 \log(2/r_{8m+4})} \) |

In fact, for \( x \in (r_{8m+4}, r_{8m+3}) \),

\[
(fg)'(x) = \left( -\frac{1}{x} \right) \sin \Psi_*(x) + \lfloor \log(2/x) \rfloor \cos \Psi_*(x) \left( -\frac{1}{x \log(2/x)} \right),
\]

and

\[
(fg)''(x) = \frac{1}{x^2} (\sin \Psi_*(x) + \cos \Psi_*(x)) - \frac{1}{x} (\cos \Psi_*(x) - \sin \Psi_*(x)) [\Psi_*(x)]' < 0.
\]

Hence \( fg \) is nonnegative, increasing and strictly concave on \([r_{8m+4}, r_{8m+3})\), and so

\[
\frac{1}{r_{8m+3} - r_{8m+4}} \int_{r_{8m+4}}^{r_{8m+3}} \left[ |fg|(x) - \text{essinf}(|fg|) \right] dx
\]

\[
= \frac{1}{r_{8m+3} - r_{8m+4}} \int_{r_{8m+4}}^{r_{8m+3}} fg(x) dx
\]

\[
\geq \frac{1}{2} \frac{\sqrt{2 \log(2/r_{8m+3})}}{r_{8m+3}} \to \infty \quad \text{as} \quad m \to \infty,
\]

which implies that \( |fg| \notin \text{blo} (\mathbb{R}) \). This finishes the proof of Proposition 2.1.

\[\square\]

3 Boundedness of Lusin-area and \( g^*_\lambda \) functions

Let \( \rho \) be an admissible function and \( \mathcal{X} \) a doubling metric measure space. In this section, we consider the boundedness of certain variant of Lusin-area and \( g^*_\lambda \) functions from \( \text{BMO}_\rho (\mathcal{X}) \) to \( \text{BLO}_\rho (\mathcal{X}) \). We remark that unlike the boundedness of the \( g^*_\lambda \) function, to obtain the boundedness of the Lusin-area function, we need to assume that \( \mathcal{X} \) has the \( \delta \)-annular decay property. Several remarks on this property are given in Section 4.

Definition 3.1. For \( \delta \in (0, 1] \) and a doubling metric measure space \((\mathcal{X}, d, \mu), (\mathcal{X}, d, \mu)\) is said to have the \( \delta \)-annular decay property if there exists a constant \( K \in [1, \infty) \) such that for all \( x \in \mathcal{X}, s \in (0, \infty) \) and \( r \in (s, \infty) \),

\[
\mu(B(x, r + s)) - \mu(B(x, r)) \leq K \left( \frac{s}{r} \right)^{\delta} \mu(B(x, r)).
\]
Observe that if \( r \in (0, s] \), then (3.1) is a simple conclusion of the doubling property (2.1) of \( \mu \).

Let \( p \) be an admissible function on \( X \) and \( \{Q_t\}_{t>0} \) a family of operators bounded on \( L^2(X) \) with integral kernels \( \{Q_t(x, y)\}_{t>0} \) satisfying that there exist constants \( C, \delta_1 \in (0, \infty), \delta_2 \in (0, 1) \) and \( \gamma \in (0, \infty) \) such that for all \( t \in (0, \infty) \) and \( x, y \in X \),

\[
(Q_1): |Q_t(x, y)| \leq C \frac{1}{(x, y)} \gamma \left( \frac{|x-y|}{t + d(x, y)} \right)^{\delta_1};
\]

\[
(Q_2): |\int_X Q_t(x, z) \, d\mu(z)| \leq C \frac{1}{(x, y)}^{\delta_2}.
\]

For all \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \), define the Littlewood-Paley \( g \)-function by setting

\[
g(f)(x) \equiv \left\{ \int_0^\infty |Q_t(x)|^2 \frac{dt}{t} \right\}^{1/2}, \tag{3.2}
\]

and Lusin-area and \( g_\lambda^* \) functions, respectively, by setting

\[
S(f)(x) \equiv \left\{ \int_0^\infty \int_{d(x, y) < t} |Q_t(x, y)|^2 \frac{dt}{V_t(y)} \, d\mu(y) \right\}^{1/2}, \tag{3.3}
\]

and

\[
g_\lambda^*(f)(x) \equiv \left\{ \int_{X \times (0, \infty)} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(x, y)|^2 \frac{dt}{V_t(y)} \, d\mu(y) \right\}^{1/2}, \tag{3.4}
\]

where \( \lambda \in (0, \infty) \).

We first have the following technical lemma.

**Lemma 3.1.** Assume that the Littlewood-Paley \( g \)-function \( g(f) \) in (3.2) is bounded on \( L^2(X) \). Then the Lusin-area function \( S(f) \) in (3.3) and the \( g_\lambda^* \) function \( g_\lambda^*(f) \) in (3.4) with \( \lambda \in (n, \infty) \) are bounded on \( L^2(X) \), where \( n \) is as in (2.1).

**Proof.** Since for all \( x \in X \), \( S(f)(x) \leq g_\lambda^*(f)(x) \). We only need to prove the \( L^2(X) \)-boundedness of \( g_\lambda^*(f) \).

To this end, we have

\[
\int_X \left[ g_\lambda^*(f)(x) \right]^2 \, d\mu(x) = \int_X \int_{X \times (0, \infty)} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(x, y)|^2 \frac{dt}{V_t(y)} \, d\mu(y) \, d\mu(x)
\]

\[
\leq \int_X \int_0^\infty |Q_t(x)|^2 \frac{dt}{t} \sup_{t>0} \left[ \int_X \left( \frac{t}{t + d(x, y)} \right)^\lambda \frac{1}{V_t(y)} \, d\mu(x) \right] \, d\mu(y)
\]

\[
= \int_X \left[ g(f)(y) \right]^2 \sup_{t>0} \left[ \int_X \left( \frac{t}{t + d(x, y)} \right)^\lambda \frac{1}{V_t(y)} \, d\mu(x) \right] \, d\mu(y).
\]

Moreover, for all \( y \in X \) and \( t > 0 \), we obtain

\[
\int_X \left( \frac{t}{t + d(x, y)} \right)^\lambda \frac{1}{V_t(y)} \, d\mu(x)
\]
From (3.7) and (3.6), \( \lambda \) completes the proof of Lemma 3.1.

**Theorem 3.1.** Let \( \mathcal{X} \) be a doubling metric measure space having the \( \delta \)-annular decay property for some \( \delta \in (0, 1) \). Let \( \rho \) be an admissible function on \( \mathcal{X} \) and the Lusin-area function \( S(f) \) as in (3.3). Assume that the Littlewood-Paley \( g \)-function in (3.2) is bounded on \( L^2(\mathcal{X}) \). Then there exists a positive constant \( C \) such that for all \( f \in \text{BMO}_\rho(\mathcal{X}) \), \([S(f)]^2 \in \text{BLO}_\rho(\mathcal{X})\) and \([\|S(f)\|^2_{\text{BLO}_\rho(\mathcal{X})}] \leq C\|f\|^2_{\text{BMO}_\rho(\mathcal{X})}\).

**Proof.** By the homogeneity of \( \| \cdot \|_{\text{BMO}_\rho(\mathcal{X})} \) and \( \| \cdot \|_{\text{BLO}_\rho(\mathcal{X})} \), we may assume that \( f \in \text{BMO}_\rho(\mathcal{X}) \) and \( \|f\|_{\text{BMO}_\rho(\mathcal{X})} = 1 \). Let \( B \equiv B(x_0, r) \). We prove Theorem 3.1 by considering the following two cases. First, we notice that the \( L^2(\mathcal{X}) \)-boundedness of \( g \) via Lemma 3.1 implies that \( S(f) \) is bounded on \( L^2(\mathcal{X}) \).

**Case 1.** \( r \geq \rho(x_0) \). In this case, we prove that

\[
(3.5) \quad \frac{1}{\mu(B)} \int_B [S(f)(x)]^2 \, d\mu(x) \lesssim 1.
\]

For any \( x \in B \), write

\[
[S(f)(x)]^2 = \int_0^{\rho(x)} \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} + \int_{\rho(x)}^{\infty} \int_{d(x, y) < t} \cdots \equiv [S_1(f)(x)]^2 + [S_2(f)(x)]^2.
\]

By the \( L^2(\mathcal{X}) \)-boundedness of \( S(f) \), (2.1) and Lemma 2.1, we have

\[
(3.6) \quad \frac{1}{\mu(B)} \int_B [S_1(f\chi_{2B})(x)]^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_{2B} |f(x)|^2 \, d\mu(x) \lesssim 1.
\]

Fix \( x \in B \). Notice that if \( d(x, y) < t \), then

\[
(3.7) \quad t + d(y, z) \sim t + d(x, z) \quad \text{and} \quad V_t(y) + V(y, z) \sim V_t(x) + V(x, z).
\]

From (3.7) and \((Q)_i\), it follows that for all \( y \in \mathcal{X} \) with \( d(x, y) < t \), we have

\[
(3.8) \quad |Q_t(f\chi_{2B})(y)| \lesssim \int_{2B} \frac{1}{V_t(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right) |f(z)| \, d\mu(z) \lesssim \int_{2B} \frac{1}{V_t(x) + V(x, z)} \left( \frac{t}{t + d(x, z)} \right) |f(z)| \, d\mu(z) \lesssim \left( \frac{t}{r} \right)^\gamma \sum_{j=1}^{\infty} \frac{2^{-j\gamma}}{V_{2^{-j-1}r}(x)} \int_{d(x, z) < 2^{-j}r} |f(z)| \, d\mu(z) \lesssim \left( \frac{t}{r} \right)^\gamma.
\]
Observe that by (2.2), for any \( a \in (0, \infty) \), there exists a constant \( \bar{C}_a \in [1, \infty) \) such that for all \( x, y \in X \) with \( d(x, y) \leq a \rho(x) \),

\[
\rho(y)/\bar{C}_a \leq \rho(x) \leq \bar{C}_a \rho(y).
\]

By this and \( r \geq \rho(x_0) \), we obtain that for all \( x \in B \), \( \rho(x) \lesssim r \). Notice that for all \( x, y \in X \) satisfying \( d(x, y) < t \), we have

\[
V_t(x) \sim V_t(y).
\]

It then follows from (3.8) and (3.10) together with (3.12) that

\[
\frac{1}{\mu(B)} \int_B |S_1(f\chi_{(2B)^c})(x)|^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_B \int_0^{8\rho(x)} \left( \frac{t}{r} \right)^{2\gamma} \frac{dt}{t} \, d\mu(x) \lesssim 1,
\]

which together with (3.6) tells us that

\[
\frac{1}{\mu(B)} \int_B |S_1(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

Fix \( x \in B \). Notice that for all \( y \in X \) with \( d(x, y) < t \) and \( t \geq 8\rho(x) \), by (2.2), we have

\[
\frac{\rho(y)}{t + \rho(y)} \lesssim \left( \frac{\rho(x)}{t} \right)^{1+\kappa_0}.
\]

From (3.7), (3.13) and (Q)_i, it follows that

\[
|Q_t(f)(y)| \leq \int_X \frac{1}{V_t(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^{\gamma} \left( \frac{\rho(y)}{t + \rho(y)} \right)^{\delta_1} |f(z)| \, d\mu(z)
\]
\[
\lesssim \int_X \frac{1}{V_t(x) + V(x, z)} \left( \frac{t}{t + d(x, z)} \right)^{\gamma} \left( \frac{\rho(x)}{t} \right)^{1+\kappa_0} |f(z)| \, d\mu(z)
\]
\[
\lesssim \left( \frac{\rho(x)}{t} \right)^{\delta_1} \sum_{j=0}^{\infty} \frac{2^{-j\gamma}}{V_{2^{j-1}t}(x)} \int_{d(x, z) < 2^{j-1}t} |f(z)| \, d\mu(z) \lesssim \left( \frac{\rho(x)}{t} \right)^{\delta_1}.
\]

Thus,

\[
\frac{1}{\mu(B)} \int_B |S_2(f)(x)|^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_B \int_{8\rho(x)}^{\infty} \left( \frac{\rho(x)}{t} \right)^{2\delta_1} \frac{dt}{t} \, d\mu(x) \lesssim 1,
\]

which along with (3.12) yields (3.5). Moreover, the fact that (3.5) holds for all balls \( B(x_0, r) \) with \( r \geq \rho(x_0) \) tells us that \( S(f)(x) < \infty \) for almost every \( x \in X \).

**Case II.** \( r < \rho(x_0) \). In this case, if \( r \geq \rho(x_0)/8 \), then by (2.1) and (3.5), we have

\[
\frac{1}{\mu(B)} \int_B \left\{ |S(f)(x)|^2 - \inf_B |S(f)(y)|^2 \right\} \, d\mu(x) \lesssim \frac{1}{\mu(8B)} \int_{8B} |S(f)(x)|^2 \, d\mu(x) \lesssim 1,
\]
which is desired. If \( r < \rho(x_0)/8 \), it suffices to prove that for \( \mu \)-almost every \( y \in B \),

\[
\frac{1}{\mu(B)} \int_B \{ |S(f)(x)|^2 - |S(f)(y)|^2 \} \, d\mu(x) \lesssim 1.
\]

For all \( x \in B \), write

\[
|S(f)(x)|^2 = \int_0^{8r} \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y) \, dt}{V_t(y) \, t} + \int_{8r}^{8\rho(x_0)} \cdots + \int_0^{\infty} \cdots
\]
\[
\equiv |S_r(f)(x)|^2 + |S_{r,x_0}(f)(x)|^2 + |S_{\infty}(f)(x)|^2.
\]

Observe that for \( \mu \)-almost every \( y \in B \),

\[
\frac{1}{\mu(B)} \int_B \{ |S(f)(x)|^2 - |S(f)(y)|^2 \} \, d\mu(x)
\]
\[
\leq \frac{1}{\mu(B)} \int_B \{ |S_r(f)(x)|^2 + |S_{\infty}(f)(x)|^2 + |S_{r,x_0}(f)(x)|^2 - |S_{r,x_0}(f)(y)|^2 \} \, d\mu(x).
\]

We first prove that

\[
(3.15) \quad \frac{1}{\mu(B)} \int_B |S_r(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

Write \( f \equiv f_1 + f_2 + f_B \), where \( f_1 \equiv (f - f_B)\chi_{2B} \) and \( f_2 \equiv (f - f_B)\chi_{(2B)^c} \). By the \( L^2(\mathcal{X}) \)-boundedness of \( S(f) \), (2.1) and Lemma 2.1, we have

\[
(3.16) \quad \frac{1}{\mu(B)} \int_B |S_r(f_1)(x)|^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_{2B} |f - f_B|^2 \, d\mu(x) \lesssim 1.
\]

Fix \( x \in B \). Then for all \( y \in \mathcal{X} \) with \( d(x, y) < t \), by \( (Q)_i \), (3.7), (2.1) and the fact that \( |f_{2j+1}B - f_B| \lesssim j \) for all \( j \in \mathbb{N} \), we have

\[
|Q_t(f_2)(y)| \leq \int_{(2B)^c} \frac{1}{V_t(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_B| \, d\mu(z)
\]
\[
\lesssim \int_{(2B)^c} \frac{1}{V_t(x) + V(x, z)} \left( \frac{t}{t + d(x, z)} \right)^\gamma |f(z) - f_B| \, d\mu(z)
\]
\[
\lesssim \sum_{j=1}^{\infty} \left( \frac{t}{2j+1} \right)^\gamma \left[ \frac{1}{V_{2j-1}(x)} \int_{2j+1B} |f(z) - f_{2j+1}B| \, d\mu(z) \right] + \int_{2j+1B} |f(z) - f_{2j+1}B - f_B| \, d\mu(z)
\]
\[
\lesssim \left( \frac{t}{r} \right)^\gamma \sum_{j=1}^{\infty} j^{2-\gamma} \lesssim \left( \frac{t}{r} \right)^\gamma,
\]

which together with (3.10) leads to that

\[
\frac{1}{\mu(B)} \int_B |S_r(f_2)(x)|^2 \, d\mu(x) \lesssim \int_0^{8r} \left( \frac{t}{r} \right)^{2\gamma} \frac{dt}{t} \lesssim 1.
\]
Let \( k \) be the smallest positive integer satisfying \( 2^k r \geq \rho(x_0) \). Then,

\[
|f_B| \leq |f_B - f_{2B}| + |f_{2B} - f_{2^2B}| + \cdots + |f_{2^{k-1}B} - f_{2^kB}| + |f_{2^kB}| \lesssim \log \frac{\rho(x_0)}{r}.
\]

On the other hand, fix \( x \in B(x_0, r) \) with \( r < \rho(x_0)/8 \). Then for all \( y \in \mathcal{X} \) satisfying \( d(x, y) < t \) with \( t \in (0, 8r) \), by (3.9), we have \( \rho(y) \sim \rho(x_0) \). Hence, by \((Q)_{ii}\) and (3.18), we have

\[
|Q_t(f_B)(y)| \lesssim \left( \frac{t}{\rho(y)} \right)^{\delta_2} |f_B| \lesssim \left( \frac{t}{\rho(x_0)} \right)^{\delta_2} \log \frac{\rho(x_0)}{r},
\]

which via \( t \leq 8r < \rho(x_0) \) further yields (3.17).

Now we turn our attention to prove that

\[
\frac{1}{\mu(B)} \int_B |S_{\infty}(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

Fix \( x \in B(x_0, r) \). Let \( a \in [1/8, \infty) \) and \( \overline{C}_a \) be as in (3.9). We first prove that for all \( f \in \text{BMO}_\rho(\mathcal{X}) \) with \( \|f\|_{\text{BMO}_\rho(\mathcal{X})} = 1 \), \( y \in \mathcal{X} \) with \( d(x, y) < t \) and \( t \leq 8\overline{C}_a \rho(x_0) \),

\[
|Q_t(f)(y)| \lesssim 1.
\]

In fact, by \((Q)_{i}\) and (3.7), we obtain

\[
|Q_t(f - f_{B(x,t)})(y)| \lesssim \int_{\mathcal{X}} \frac{1}{V_t(y)} + \frac{1}{V_t(x)} \left( \frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_{B(x,t)}| \, d\mu(z)
\]
\[
\lesssim \sum_{j=0}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x, z) < 2^j t} |f(z) - f_{B(x,t)}| \, d\mu(z)
\]
\[
\lesssim \sum_{j=0}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x, z) < 2^j t} |f(z) - f_{B(x,t)}| \, d\mu(z) \lesssim 1.
\]

It follows from (3.9) that for all \( y \in \mathcal{X} \) with \( d(x, y) < t \leq 8\overline{C}_a \rho(x_0) \), \( \rho(y) \sim \rho(x_0) \sim \rho(x) \), which together with the fact that for all \( x \in \mathcal{X} \), \( |f_B(x, t)| \leq |f_{B(x, t)} - f_{B(x, \rho(x))}| + |f_{B(x, \rho(x))}| \lesssim 1 + \log \frac{\rho(x)}{t} \) (by (3.18)), and \((Q)_{ii}\) shows that

\[
|Q_t(f_{B(x,t)})(y)| \lesssim \left( \frac{t}{\rho(y)} \right)^{\delta_2} \left( 1 + \log \frac{\rho(x)}{t} \right) \lesssim \left( \frac{t}{\rho(x)} \right)^{\delta_2} \left( 1 + \log \frac{\rho(x)}{t} \right) \lesssim 1.
\]

Combining this and (3.21) proves (3.20).

Using (3.20), (3.9), (3.10) and (3.14), we have that for all \( x \in B \),

\[
\int_{8\rho(x_0)}^\infty \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \lesssim 1.
\]
By symmetry, we also have
\[ \leq \int_{8\rho(x_0)}^{8\tilde{C}_a\rho(x_0)} \int_{d(x,y)<t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} + \int_{8\rho(x_0)}^{\infty} \cdots \]
\[ \lesssim 1 + \int_{8\tilde{C}_a\rho(x_0)}^{\infty} \left( \frac{\rho(x)}{t} \right)^{\frac{2d}{d+k_0}} \frac{dt}{t} \lesssim 1, \]
which yields (3.19).

By (3.15) and (3.19), we reduce the proof of Theorem 3.1 to show that for \( \mu \)-almost every \( x' \in B \),
\[ \frac{1}{\mu(B)} \int_{B} \{ [S_{r,x_0}(f)(x)]^2 - [S_{r,x_0}(f)(x')]^2 \} \ d\mu(x) \lesssim 1. \] (3.22)

For any \( x, x' \in B \) such that \( S_{r,x_0}(f)(x) \) and \( S_{r,x_0}(f)(x') \) are finite, write
\[ [S_{r,x_0}(f)(x)]^2 - [S_{r,x_0}(f)(x')]^2 \]
\[ = \int_{8r} \int_{\delta(x,y)<t} \left| Q_t(f)(y) \right|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} - \int_{8r} \int_{\delta(x',y)<t} \cdots \]
\[ \leq \int_{8r} \int_{B(x,t) \backslash B(x',t)} \left| Q_t(f - f_B)(y) \right|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \]
\[ + \int_{8r} \int_{B(x,t) \cap B(x',t)} \left| Q_t(f_B)(y) \right|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \equiv J_1 + J_2, \]
where \( B(x,t) \backslash B(x',t) \equiv [B(x,t) \backslash B(x',t)] \cup [B(x',t) \backslash B(x,t)] \).

By the facts that \( x, x' \in B \) and \( t \geq 8r \), we have \( B(x,t-2r) \subseteq [B(x,t) \cap B(x',t)] \). Since \( \mathcal{X} \) has the \( \delta \)-annular decay property for some \( \delta \in (0,1) \), we obtain
\[ \mu(B(x,t) \backslash B(x',t)) \leq \mu(B(x,t)) - \mu(B(x,t-2r)) \lesssim \left( \frac{r}{t} \right)^{\delta} \mu(B(x,t)). \]

By symmetry, we also have \( \mu(B(x',t) \backslash B(x,t)) \lesssim \left( \frac{r}{t} \right)^{\delta} \mu(B(x',t)) \), which together with (2.1) implies that
\[ \mu(B(x,t) \backslash B(x',t)) \lesssim \left( \frac{r}{t} \right)^{\delta} \mu(B(x,t)). \]

(3.23)

By \((Q_i), (3.7), (3.23), (3.10)\) and (2.1), we obtain
\[ J_1 \lesssim \int_{8r} \left( \frac{r}{t} \right)^{\delta} \left[ \int_{\mathcal{X}} \frac{1}{V_t(x) + V(x,z)} \left( \frac{t}{t + d(x,z)} \right)^{\gamma} |f(z) - f_B| \mu(z) \right] \frac{2}{t} \]
\[ \lesssim \int_{8r} \left( \frac{r}{t} \right)^{\delta} \left[ \frac{1}{\mu(2B)} \int_{2B} |f(z) - f_B| \mu(z) \right] \]
\[ + \sum_{j=1}^{\infty} \frac{t^{\gamma}}{(t + 2^{j-1}r)^{\gamma}} \left( \frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(z) - f_B| \mu(z) \right) \frac{2}{t}. \]
\[
\left(\frac{r}{t}\right) - 2\sum_{j=0}^{\infty} \frac{t^\gamma}{(t+2^{-j}r)^\gamma} \right] \frac{dt}{t}.
\]

Moreover, if \(2\gamma < \delta\), we then have
\[
J_1 \lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \frac{dt}{t} + r^{\delta - 2\gamma} \int_{8r}^{\infty} \frac{dt}{t^{\delta - 2\gamma + 1}} \lesssim 1;
\]
if \(2\gamma \geq \delta\), letting \(\epsilon \in (0, \delta/2)\) yields that
\[
J_1 \lesssim 1 + \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[ \sum_{j=0}^{\infty} \frac{t^\gamma}{(t+2^{-j}r)^\gamma} \right] \frac{dt}{t} \lesssim 1 + r^{\delta - 2\epsilon} \int_{8r}^{\infty} \frac{dt}{t^{\delta - 2\epsilon + 1}} \lesssim 1.
\]
Thus, \(J_1 \lesssim 1\).

Notice that \(r < \rho(x_0)/8\) and \(t \in (8r, 8\rho(x_0))\). By (3.9), we have that for any \(x \in B\) and \(y \in \mathcal{X}\) with \(d(x, y) < t\), \(\rho(x_0) \sim \rho(x) \sim \rho(y)\). Choosing \(\eta \in (0, 1)\) such that \(\eta \delta_2 < \delta\), then by (3.18), (Q)\(n\) and (3.10), we have
\[
J_2 \lesssim \int_{8r}^{8\rho(x_0)} \left[ \log \frac{\rho(x_0)}{r} \right]^2 \left(\frac{r}{t}\right)^\delta \left(\frac{t}{\rho(x_0)}\right)^{\eta \delta_2} \frac{dt}{t} \lesssim \int_{8r}^{\infty} \left(\frac{r}{t}\right)^{\delta - \eta \delta_2} \frac{dt}{t} \lesssim 1.
\]
Combining the estimates for \(J_1\) and \(J_2\) yields (3.22), which completes the proof of Theorem 3.1.

As a consequence of Theorem 3.1, we have the following conclusion, which can be proved by an argument similar to the proof of [37, Corollary 6.1]. We omit the details.

**Corollary 3.1.** With the assumptions same as in Theorem 3.1, then there exists a positive constant \(C\) such that for all \(f \in \text{BMO}_\rho(\mathcal{X})\), \(S(f) \in \text{BLO}_\rho(\mathcal{X})\) and \(\|S(f)\|_{\text{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\text{BMO}_\rho(\mathcal{X})}\).

**Remark 3.1.** In Theorem 3.1 and Corollary 3.1, if we replace the assumption that the Littlewood-Paley \(g\)-function in (3.2) is bounded on \(L^2(\mathcal{X})\) by that the Lusin-area function \(S(f)\) in (3.3) is bounded on \(L^2(\mathcal{X})\), then Theorem 3.1 and Corollary 3.1 still hold.

Now we study the boundedness of \(g^*_\lambda\) function. In this case, \(\mathcal{X}\) is not necessary to have the \(\delta\)-annular decay property.

**Theorem 3.2.** Let \(\mathcal{X}\) be a doubling metric measure space. Let \(\rho\) be an admissible function on \(\mathcal{X}\) and the \(g^*_\lambda\) function \(g^*_\lambda(f)\) as in (3.4) with \(\lambda \in (3n, \infty)\). Assume that the Littlewood-Paley \(g\)-function in (3.2) is bounded on \(L^2(\mathcal{X})\). Then there exists a positive constant \(C\) such that for all \(f \in \text{BMO}_\rho(\mathcal{X})\), \([g^*_\lambda(f)]^2 \in \text{BLO}_\rho(\mathcal{X})\) and \(\|[g^*_\lambda(f)]^2\|_{\text{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\text{BMO}_\rho(\mathcal{X})}^2\).

**Proof.** Again, by the homogeneity of \(\|\cdot\|_{\text{BMO}_\rho(\mathcal{X})}\) and \(\|\cdot\|_{\text{BLO}_\rho(\mathcal{X})}\), we may assume that \(f \in \text{BMO}_\rho(\mathcal{X})\) and \(\|f\|_{\text{BMO}_\rho(\mathcal{X})} = 1\).

Let \(B \equiv B(x_0, r)\). For any nonnegative integer \(k\), let
\[
J(k) \equiv \{(y, t) \in \mathcal{X} \times (0, \infty) : d(y, x_0) < 2^{k+1}r \text{ and } 0 < t < 2^{k+1}r\}.
\]
For any \( f \in \text{BMO}_\rho(X) \) and \( x \in X \), write
\[
[g^*_\lambda(f)(x)]^2 = \int_{J(0)} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} t + \int_{\mathbb{R}^n} \int_{J(0)} \int_{|x| < t} \int_{|y| < t} \int_{|z| < t} f(y) f(z) d\mu(y) d\mu(z) dt.
\]

We now consider the following two cases. Notice that the \( L^2(X) \)-boundedness of \( g \) via Lemma 1.1 implies that \( g^*_\lambda(f) \) is bounded on \( L^2(X) \).

**Case I.** \( r \geq \rho(x_0) \). In this case, we first prove that
\[
\frac{1}{\mu(B)} \int_B [g^*_{\lambda, 0}(f)(x)]^2 d\mu(x) \lesssim 1.
\]

For any \( x \in B \), write
\[
[g^*_{\lambda, 0}(f)(x)]^2 \leq \int_{J(0)} \int_{d(x, y) < t} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} t
\]
\[
+ \int_{J(0)} \int_{d(x, y) < t} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(f\chi_{8B})(y)|^2 \frac{d\mu(y)}{V_t(y)} t
\]
\[
+ \int_{J(0)} \int_{d(x, y) < t} \left( \frac{t}{t + d(x, y)} \right)^\lambda |Q_t(f\chi_{8B}^c)(y)|^2 \frac{d\mu(y)}{V_t(y)} t
\]
\[
\equiv I_1(x) + I_2(x) + I_3(x).
\]

Notice that for all \( x \in B \), \( I_1(x) \leq |S(f)(x)|^2 \). It then follows from (3.5) that
\[
\frac{1}{\mu(B)} \int_B I_1(x) d\mu(x) \lesssim 1.
\]

We remark that in the proof of (3.5), we do not need the \( \delta \)-annular decay property of \( X \).

As for \( I_2(x) \), by the \( L^2(X) \)-boundedness of \( g^*_\lambda(f) \), (2.1) and Lemma 2.1, we have
\[
\frac{1}{\mu(B)} \int_B I_2(x) d\mu(x) \lesssim 1.
\]

To deal with \( I_3(x) \), we notice that for all \( z \in (8B)^c \) and \( y \in X \) with \( d(y, x_0) < 2r \), \( d(y, z) < d(x_0, z) \) and \( V(y, z) < V(x_0, z) \). Hence,
\[
I_3(x) \lesssim \int_0^{2r} \int_{d(x, y) < t} \left( \frac{t}{t + d(x, y)} \right)^\lambda
\]
\[
\times \left[ \int_{(8B)^c} \frac{1}{V_t(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^\gamma |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} t
\]
\[
\lesssim \int_0^{2r} \int_{d(x, y) < t} \left( \frac{t}{t + d(x, y)} \right)^\lambda \left[ \int_{(8B)^c} \frac{1}{V(x_0, z)} \left( \frac{t}{d(x_0, z)} \right)^\gamma |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} t
\]
we have
\[ y, t \]
where in the last inequality we used the fact that \( \lambda > n \). Furthermore, we obtain
\[
\frac{1}{\mu(B)} \int_B I_3(x) \, d\mu(x) \lesssim 1,
\]
which together with (3.25) and (3.26) proves (3.24).

Now we prove that
\[
(3.27) \quad \frac{1}{\mu(B)} \int_B [g_{\lambda, \infty}^*(f)(x)]^2 \, d\mu(x) \lesssim 1.
\]

Notice that for \( (y, t) \in J(k) \setminus J(k-1) \) with \( k \in \mathbb{N} \) and \( x \in B, t + d(x, y) \sim 2^k r \). Thus,
\[
[g_{\lambda, \infty}^*(f)(x)]^2 \lesssim \sum_{k=1}^{\infty} \int_{J(k) \setminus J(k-1)} \left( \frac{t}{2^k r} \right)^{\lambda} \int_{2^k + B} \frac{1}{V_t(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^{\gamma} \left( \frac{\rho(y)}{t + \rho(y)} \right)^{\delta_1} |f(z)| \, d\mu(z) \, dt \, d\mu(y) \, dt \frac{V_t(y)}{t}
\]
\[ + \sum_{k=1}^{\infty} \int_{J(k) \setminus J(k-1)} \left( \frac{t}{2^k r} \right)^{\lambda} \left[ \int_{(2^k + B)^c} \cdots d\mu(z) \right] \, dt \, d\mu(y) \, dt \equiv E_1(x) + E_2(x).
\]

The fact that \( r \geq \rho(x_0) \) and (2.2) imply that for all \( y \in \mathcal{X} \) with \( d(y, x_0) < 2^{k+1} r \),
\[
(3.28) \quad \rho(y) \lesssim [\rho(x_0)]^{\frac{1}{1+k_0}} \left( 2^k r \right)^{\frac{k_0}{1+k_0}}.
\]

By the assumption that \( \lambda \in (3n, \infty) \), we choose \( \eta_1 \in (0, \delta_1) \) such that \( \lambda - 2\eta_1 - 3n > 0 \).

By (3.28), we obtain
\[
E_1(x) \lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1} r} \int_{d(y, x_0) < 2^{k+1} r} \left( \frac{t}{2^k r} \right)^{\lambda} \left( 2^k r \right)^{2n} \left[ \rho(x_0) \right]^{\frac{1}{1+k_0}} \left( 2^k r \right)^{\frac{k_0}{1+k_0}} \, dt \, d\mu(y) \, dt \frac{V_t(y)}{t}
\]
\[ \lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1} r} \left( \frac{t}{2^k r} \right)^{\lambda} \left( 2^k r \right)^{3n} \left[ \rho(x_0) \right]^{\frac{1}{1+k_0}} \left( 2^k r \right)^{\frac{k_0}{1+k_0}} \, dt \, d\mu(y) \, dt \lesssim \sum_{k=1}^{\infty} \left[ \rho(x_0) \right]^{\frac{2\eta_1}{2^k r}} \lesssim 1.
\]

Choose \( \eta_2 \in (0, \delta_1) \) such that \( \lambda + 2\gamma - 2\eta_2 - n > 0 \), then by (3.28) and the fact that for \( z \in (2^k + B)^c \) and \( y \in \mathcal{X} \) with \( d(y, x_0) < 2^{k+1} r, d(y, z) \sim d(x_0, z) \) and \( V(y, z) \sim V(x_0, z) \), we have
\[
E_2(x) \lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1} r} \int_{d(y, x_0) < 2^{k+1} r} \left( \frac{t}{2^k r} \right)^{\lambda}
\]


\[
\begin{align*}
&\times \left[ \int_{(2^{k+4}B)_{\overline{2}}} \frac{1}{V(x_0, z)} \left( \frac{t}{d(x_0, z)} \right)^{\gamma} \left( \frac{\rho(y)}{t + \rho(y)} \right)^{\eta_2} |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_1(y)} \frac{dt}{t} \\
&\leq \sum_{k=1}^{\infty} \int_{0}^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \left( \frac{t}{2^k r} \right)^{\lambda} \left( \frac{\rho(x_0)}{t} \right)^{2\gamma_0} \left( \frac{\rho(y)}{t} \right)^{\eta_2} \frac{d\mu(y)}{V_1(y)} \frac{dt}{t} \\
&\leq \sum_{k=1}^{\infty} \int_{0}^{2^{k+1}r} \left( \frac{t}{2^k r} \right)^{\lambda + 2\gamma - n} \left( \frac{\rho(x_0)}{t} \right)^{k_0} \left( \frac{\rho(y)}{t} \right)^{k_0} \frac{d\mu(y)}{V_1(y)} \frac{dt}{t} \\
&\lesssim \sum_{k=1}^{\infty} \left[ \frac{\rho(x_0)}{2^k r} \right]^{2\eta_2} \lesssim 1,
\end{align*}
\]

which together with the estimate of \( E_1(x) \) yields (3.27).

Combining (3.24) and (3.27) yields that

\[
(3.29) \quad \frac{1}{\mu(B)} \int_B |g_\lambda^*(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

Moreover, from the fact that (3.29) holds for all balls \( B(x_0, r) \) with \( r \geq \rho(x_0) \), it follows that \( g_\lambda^*(f)(x) < \infty \) for almost every \( x \in X \).

**Case II. \( r < \rho(x_0) \).** In this case, if \( r \geq \rho(x_0)/16 \), then by (2.1) and (3.29), we obtain the desired estimate that

\[
\frac{1}{\mu(B)} \int_B \left\{ |g_\lambda^*(f)(x)|^2 - \operatorname{essinf}\frac{g_\lambda^*(f)(y)}{B} \right\} \, d\mu(x) \lesssim \frac{1}{\mu(8B)} \int_{8B} |g_\lambda^*(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

If \( r < \rho(x_0)/16 \), it is enough to show that for all \( x' \in B \) such that \( g_\lambda^*(f)(x') < \infty \),

\[
\frac{1}{\mu(B)} \int_B \left\{ |g_\lambda^*(f)(x)|^2 + |g_\lambda^*(f)(x')|^2 - |g_\lambda^*(f)(x')|^2 \right\} \, d\mu(x) \lesssim 1.
\]

We first prove that

\[
(3.30) \quad \frac{1}{\mu(B)} \int_B |g_\lambda^*(f)(x)|^2 \, d\mu(x) \lesssim 1.
\]

To this end, write \( f \equiv f_1 + f_2 + f_3 \), where \( f_1 \equiv (f - f_B)\chi_{8B} \) and \( f_2 \equiv (f - f_B)\chi_{(8B)^c} \). By the \( L^2(X) \)-boundedness of \( g_\lambda^*(f) \), (2.1) and Lemma 2.1, we have

\[
(3.31) \quad \frac{1}{\mu(B)} \int_B |g_\lambda^*(f_1)(x)|^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_{8B} |f - f_B|^2 \, d\mu(x) \lesssim 1.
\]

Notice that for \( z \in (8B)^c \) and \( y \in X \) with \( d(y, x_0) < 2r, d(y, z) \sim d(x_0, z) \) and \( V(y, z) \sim V(x_0, z) \). This together with (Q), (2.1) and the fact that \( |f_{2^{j+1}B} - f_B| \lesssim j \) for all \( j \in \mathbb{N} \) yields that

\[
|Q_{1}(f_2)(y)| \leq \int_{(8B)^c} \frac{1}{V_1(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^{\gamma} |f(z) - f_B| \, d\mu(z) \\
\lesssim \int_{(8B)^c} \frac{1}{V(x_0, z)} \left( \frac{t}{d(x_0, z)} \right)^{\gamma} |f(z) - f_B| \, d\mu(z) \lesssim \left( \frac{t}{r} \right)^{\gamma},
\]
where we omitted some routine computation. Hence, by an argument similar to the estimates of (3.11) and $I_3(x)$, we obtain

\[(3.32) \quad \frac{1}{\mu(B)} \int_B [g_{\lambda,0}^*(f_2)(x)]^2 \, d\mu(x) \leq \frac{1}{\mu(B)} \int_B \int_0^{2r} \int_{d(x,y) < 2r} \left( \frac{t}{t+d(x,y)} \right)^\lambda \left( \frac{t}{r} \right)^{2\gamma} \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} \]

\[\leq \frac{1}{\mu(B)} \int_B \int_0^{2r} \int_{d(x,y) < t} \left( \frac{t}{r} \right)^{2\gamma} \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} + \frac{1}{\mu(B)} \int_B \int_0^{2r} \int_{d(x,y) \geq t} \left( \frac{t}{t+d(x,y)} \right)^\lambda \left( \frac{t}{r} \right)^{2\gamma} \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} \lesssim 1. \]

For $y \in \mathcal{X}$ with $d(x_0, y) < 2r < \rho(x_0)/8$, by (3.9), we have $\rho(x_0) \sim \rho(y)$, which together with $(Q)_i$ and (3.18) leads to

\[|Q_t(f_B)(y)| \lesssim \left( \frac{t}{\rho(y)} \right)^{\delta_2} |f_B| \lesssim \left( \frac{t}{\rho(x_0)} \right)^{\delta_2} \log \frac{\rho(x_0)}{r} \lesssim \left( \frac{t}{r} \right)^{\delta_2}. \]

Then, similarly to the estimate of (3.32), we obtain

\[\frac{1}{\mu(B)} \int_B [g_{\lambda,0}^*(f_B)(x)]^2 \, d\mu(x) \lesssim 1, \]

which together with (3.31) and (3.32) yields (3.30).

The proof of Theorem 3.2 now is reduced to show that for all $x' \in B$ such that $g_{\lambda,\infty}^*(f)(x') < \infty$,

\[(3.33) \quad \frac{1}{\mu(B)} \int_B \{ [g_{\lambda,\infty}^*(f)(x)]^2 - [g_{\lambda,\infty}^*(f)(x')]^2 \} \, d\mu(x) \lesssim 1. \]

For $x, x' \in B$ such that $g_{\lambda,\infty}^*(x)$ and $g_{\lambda,\infty}^*(x')$ are finite, write

\[[g_{\lambda,\infty}^*(f)(x)]^2 - [g_{\lambda,\infty}^*(f)(x')]^2 \]

\[\leq \int_{\mathcal{X} \times (0, \infty) \setminus J(0)} \left| \left( \frac{t}{t+d(x, y)} \right)^\lambda - \left( \frac{t}{t+d(x', y)} \right)^\lambda \right| |Q_t(f)(y)|^2 \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} \]

\[\lesssim \sum_{k=1}^{\infty} \int_{J(k) \setminus J(k-1)} \frac{r t^\lambda}{(2k^2r)^{\lambda+1}} |Q_t(f - f_B)(y)|^2 \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} + \sum_{k=1}^{\infty} \int_{J(k) \setminus J(k-1)} \frac{r t^\lambda}{(2k^2r)^{\lambda+1}} |Q_t(f_B)(y)|^2 \frac{d\mu(y) \, dt}{V_t(y)} \, \frac{d\mu(x)}{t} \equiv G_1 + G_2. \]

Using the assumption that $\lambda \in (3n, \infty)$ and $(Q)_i$, we have

\[G_1 \lesssim \sum_{k=1}^{\infty} \int_{J(k) \setminus J(k-1)} \frac{r t^\lambda}{(2k^2r)^{\lambda+1}} \]
\[ \begin{align*}
\times \left[ \int_{2^{k+4}B} \frac{1}{V(y) + V(y, z)} \left( \frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_B| \frac{d\mu(z)}{V(y)} \right]^2 \frac{d\mu(y)}{V(y)} \frac{dt}{t} \\
+ \sum_{k=1}^\infty \int_{J(k) \setminus J(k-1)} \frac{rt^\lambda}{(2^{kR})^{\lambda+1}} \left[ \int_{(2^{k+4}B)^c} \cdots d\mu(z) \right]^2 \frac{d\mu(y)}{V(y)} \frac{dt}{t} \\
\lambda \sum_{k=1}^\infty \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^{kR})^{\lambda+1}} \left( \frac{2^{kR}}{t} \right)^{2n} k^2 d\mu(y) \frac{dt}{t} \\
+ \sum_{k=1}^\infty \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^{kR})^{\lambda+1}} \left( \frac{2^{kR}}{t} \right)^{2n} k^2 d\mu(y) \frac{dt}{t} \lesssim 1.
\end{align*} \]

Choose \( \eta_3 \in (0, 1) \) such that \( \eta_3(1 + k_0)\delta_2 < 1 \). It then follows from \((Q)_{ii}, (2.2), (3.18)\) and \( \lambda \in (n, \infty) \) that

\[ G_2 \lesssim \sum_{k=1}^\infty \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^{kR})^{\lambda+1}} \left[ \log \frac{\rho(x_0)}{r} \right]^2 \left( \frac{t}{\rho(y)} \right)^{\eta_3 \delta_2} \frac{d\mu(y)}{V(y)} \frac{dt}{t} \]

\[ \lesssim \sum_{k=1}^\infty \int_0^{2^{k+1}r} \frac{rt^\lambda}{(2^{kR})^{\lambda+1}} \left[ \log \frac{\rho(x_0)}{r} \right]^2 \left( \frac{2^{kR}}{\rho(x_0)} \right)^{\eta_3 \delta_2} + \left( \frac{2^{kR}}{\rho(x_0)} \right)^{\eta_3 (1 + k_0) \delta_2} \frac{2^{kR}}{t} \frac{dt}{t} \lesssim 1. \]

Combining the estimates for \( G_1 \) and \( G_2 \) yields (3.33), which completes the proof of Theorem 3.2. \( \square \)

As a consequence of Theorem 3.2, we have the following conclusion.

**Corollary 3.2.** With the assumptions same as in Theorem 3.2, then there exists a positive constant \( C \) such that for all \( f \in \text{BMO}_\rho(\mathcal{X}) \), \( g_\lambda^*(f) \in \text{BLO}_\rho(\mathcal{X}) \) and \( \|g_\lambda^*(f)\|_{\text{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\text{BMO}_\rho(\mathcal{X})}. \)

**Remark 3.2.** (i) In Theorem 3.2 and Corollary 3.2, if we replace the assumption that the Littlewood-Paley \( g \)-function in (3.2) is bounded on \( L^2(\mathcal{X}) \) by that the \( g_\lambda^* \) function \( g_\lambda^*(f) \) in (3.4) is bounded on \( L^2(\mathcal{X}) \), then Theorem 3.2 and Corollary 3.2 still hold.

(ii) Comparing with the classical known result in [28], it is still unclear if \( \lambda \in (n, \infty) \) is enough to guarantee Theorem 3.2 and Corollary 3.2. In the proof of Theorem 3.2, we need the assumption \( \lambda > 3n \) only in the estimates of \( E_1(x) \) and \( G_1 \). In [28], this can be reduced to \( \lambda > n \) via the fractional integral. However, in the current setting, corresponding result of the fractional integral is not available.

(iii) Let \( \mathcal{X} = (\mathbb{R}^d, |\cdot|, dx) \) and \( \{Q_t\}_{t>0} \) be the operators associated to the semigroups generated by the Schrödinger operator with nonnegative potential satisfying the reverse Hölder inequality on \( \mathbb{R}^d \); see Proposition 3.1 below. Then, Theorem 3.2 implies that the \( g_\lambda^* \) function \( g_\lambda^*(f) \) associate to the kernels \( \{Q_t\}_{t>0} \) is bounded from \( \text{BMO}_\rho(\mathbb{R}^d) \) to \( \text{BLO}_\rho(\mathbb{R}^d) \) for \( \lambda \in (3d, \infty) \), which improves the result in [19] that \( g_\lambda^*(f) \) is bounded on \( \text{BMO}_\rho(\mathbb{R}^d) \) for \( \lambda \in (3d + 4k_0, \infty) \), where \( k_0 \) is as in (2.2).

Notice that Buckley [1] showed that Heisenberg groups and connected and simply connected nilpotent Lie groups with a Carnot-Carathéodory (control) distance have the \( \delta \)-annular decay property (see also Example 4.1 below). By this fact, we have the following
simple corollary of Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2. We omit the details here; see [37, Section 7].

**Proposition 3.1.** Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2 are true if

\[ Q_t \equiv t^2 \frac{de^{-s\mathcal{L}}}{ds} \bigg|_{s=t^2}, \]

where \( \mathcal{L} = -\Delta + V \) is the Schrödinger operator or the degenerate Schrödinger operator on \( \mathbb{R}^d \), or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups, and \( V \) is a nonnegative function satisfying certain reverse Hölder inequality; see the details in [37, Section 7].

### 4 Several remarks on the \( \delta \)-annular decay property

To the best of our knowledge, the \( \delta \)-annular decay property in Definition 3.1 was introduced by Buckley [1] in 1999. However, if \((X,d,\mu)\) is a normal space of homogeneous type in the sense of Marcíaś and Segovia [24], the \( \delta \)-annular decay property was introduced by David, Journé and Semmes in 1985 in their celebrated paper on the \( T(b) \) theorem (see [7, p. 41]). A slight variant on manifolds also appeared in Colding and Minicozzi II [6] in 1998, which was called \( \epsilon \)-volume regularity property therein (see [6, p. 125]). Buckley [1] proved that for any metric space equipped with a doubling measure, the chain ball property implies the \( \delta \)-annular decay property for some \( \delta \in (0,1] \).

In this section, we first introduce two properties on any metric space, the weak geodesic property and the monotone geodesic property, which are proved to be respectively equivalent to the chain ball property introduced by Buckley [1]. As an application, we prove that any length space equipped with a doubling measure has the weak geodesic property and hence the \( \delta \)-annular decay property for some \( \delta \in (0,1] \). Finally, we give several examples of doubling metric measure spaces having the \( \delta \)-annular decay property.

We begin with the notions of the weak geodesic property, the monotone geodesic property, and the chain ball property.

**Definition 4.1.** Let \((X,d)\) be a metric space.

(I) \((X,d)\) is said to have the weak geodesic property (or called Property \((\tilde{M})\)) if there exists a positive constant \(C_3\) such that for all \(x \in X\), \(r, s \in (0, \infty)\) and \(y \in B(x, r+s)\),

\[ d(y, B(x, r)) \leq C_3 s. \]

(II) \((X,d)\) is said to have the monotone geodesic property if there exists a positive constant \(C_4\) such that for all \(s \geq 0\) and \(x, y \in X\) with \(d(x, y) \geq s\), there exists a finite chain \(x_0 \equiv y, x_1, \cdots, x_m \equiv x\) with \(m \in \mathbb{N}\) such that for \(0 \leq i < m\), \(d(x_i, x_{i+1}) \leq C_4 s\) and \(d(x_{i+1}, x) \leq d(x_i, x) - s\).

(III) Let \(\alpha, \beta \in (1, \infty)\). A ball \(B \equiv B(z, r) \subset X\) is said to be an \((\alpha, \beta)\)-chain ball, with respect to a “central” sub-ball \(B_0 \equiv B(z_0, r_0) \subset B\) if, for every \(x \in B\), there is an integer \(k \equiv k(x) \geq 0\) and a chain of balls, \(B_{x,i} \equiv B(z_{x,i}, r_{x,i})\), \(0 \leq i \leq k\), with the following properties:

(i) \(B_{x,0} = B_0\) and \(x \in B_{x,k}\),
Theorem 4.1. Let $B_{x,i} \cap B_{x,i+1}$ be non-empty, $0 \leq i < k$,

(iii) $x \in \alpha B_{x,i}$, $0 \leq i \leq k$,

(iv) $\beta r_{x,i} \leq r - d(z_{x,i}, z)$, $0 \leq i \leq k$.

The metric space $(X, d)$ is said to have the $(\alpha, \beta)$-chain ball property if every ball in $X$ is an $(\alpha, \beta)$-chain ball.

Remark 4.1. (i) Tessera in [35] introduced the following Property (M). A metric space $(X, d)$ is said to has Property (M) if there exists a positive constant $C$ such that the Hausdorff distance between any pair of balls with same center and any radii between $r$ and $r + 1$ is less than $C$. In other words, there exists a positive constant $C$ such that for all $x \in X$, $r > 0$ and $y \in B(x, r + 1)$, $d(y, B(x, r)) \leq C$; see [35, Definition 1]. Obviously, if $(X, d)$ has Property $(\tilde{M})$, then $(X, d)$ also has Property (M).

Conversely, let $Z$ be equipped with the usual Euclidean distance $| \cdot |$. Then $(Z, | \cdot |)$ has Property (M). Assume that $(Z, | \cdot |)$ has also Property $(\tilde{M})$. Then, by Definition 4.1(I), there exists a positive constant $C_3$ such that for all $r, s \in (0, \infty)$ and $y \in B(0, r + s)$, $d(y, B(0, r)) \leq C_3s$. If we choose $s < \min\{1, (C_3)^{-1}\}$ and $r \in (0, 1)$ with $r + s \geq 1$, then $C_3s < 1, B(0, r) = \{0\}$ and $B(0, r + s) = \{0, 1\}$; it then follows that $1 = d(1, B(0, r)) \leq C_3s < 1$, which is a contradiction. Thus, $(Z, | \cdot |)$ does not have Property $(\tilde{M})$. In this sense, we say that Property $(\tilde{M})$ is slightly stronger than Property (M).

(ii) Let $(X, \mu, d)$ be a doubling measure space having Property (M). Then using 3) of Proposition 2 in [35], by an argument same as in the proof of Theorem 4 of [35] (see also the proof of Lemma 3.3 of Colding and Minicozzi II [6]), we have that there exist positive constants $\delta$ and $C$ such that for all $x \in X$, $s \in [1, \infty)$ and $r \in (s, \infty)$,

$$\mu(B(x, r + s)) - \mu(B(x, r)) \leq C \left(\frac{s}{r}\right)^\delta \mu(B(x, r)).$$

Thus, when $\delta \in (0, 1]$, $(X, \mu, d)$ satisfies a slightly weaker property than the $\delta$-annular decay property.

Tessera in [35, pp. 51-52] also verified that the assumptions of Theorem 4 in [35] are optimal. Thus, in some sense, it is necessary to introduce the weak geodesic property to guarantee the $\delta$-annular decay property.

(iii) It is easy to check that $C_4 \geq 1$. In fact, if $m = 1$, that is, $x_0 \equiv y$ and $x_1 \equiv x$, then $s \leq d(x, y) = d(x_0, x_1) \leq C_4s$, which implies that $C_4 \geq 1$; if $m > 1$, that is, $x_0 \equiv y$, $x_1$, $\ldots$, $x_m \equiv x$, then $d(y, x_1) = d(x_0, x_1) \leq C_4s$ and $d(x_1, x) \leq d(x_0, x) - s = d(x, y) - s$, which also implies that $s = d(x, y) - (d(x, y) - s) \leq d(x, y) - d(x_1, x) \leq d(y, x_1) \leq C_4s$ and hence $C_4 \geq 1$.

(iv) The notion of $(\alpha, \beta)$-chain ball property in Definition 4.1(III) was first introduced by Buckley in [1]. Moreover, it is easy to see that in Definition 4.1(III), $B_{x,i} \subset B$ for all $x \in B$ and $i \in \{0, \cdots, k\}$. In fact, by (iv) of Definition 4.1(III) and the fact that $\beta \in (1, \infty)$, we have that for any $w \in B_{x,i}$, $d(w, z) \leq d(w, z_{x,i}) + d(z_{x,i}, z) < r_{x,i} + d(z_{x,i}, z) < \beta r_{x,i} + d(z_{x,i}, z) \leq r$, which implies that $B_{x,i} \subset B$.

The main result of this section is the following equivalences of the above three properties.

**Theorem 4.1.** Let $(X, d)$ be a metric space. Then the following are equivalent:
(I) \( (X, d) \) has the weak geodesic property;

(II) \( (X, d) \) has the monotone geodesic property;

(III) \( (X, d) \) has the \((\alpha, \beta)\)-chain ball property for some \( \alpha, \beta \in (1, \infty) \).

**Proof.** Similarly to the proof of [35, Proposition 2], we can show the equivalence of (I) and (II). We omit the details.

Now we prove that (II) implies (III). To this end, let \( (X, d) \) be a metric space having the monotone geodesic property with a positive constant \( C_4 \), and let \( B \equiv B(z, r) \) be any ball in \( X \). We show that \( B \) is a \((4C_4/3, 4/3)\)-chain ball with respect to the “central” sub-ball \( B_0 \equiv B(z, 3r/4) \subset B \).

For every \( x \in B \), let \( t_0 \equiv (r - d(x, z))/2 \). If \( x \in B_0 \), then \( k \equiv k(x) \equiv 0 \) and \( \{B_0\} \) is a desired chain.

Assume that \( x \notin B_0 \), then \( d(x, z) \geq 3r/4 \) and \( t_0 \leq r/8 \). Thus, \( d(x, z) \geq 6t_0 > t_0/C_4 \), since \( C_4 \geq 1 \) by Remark 4.1(iii). Since \( X \) has the monotone geodesic property, by Definition 4.1(II), there exists a finite chain \( x_{0,0} \equiv x, x_{0,1}, \ldots, x_{0,m_0} \equiv z \) with \( m_0 \in \mathbb{N} \) such that for \( 0 \leq i < m_0 \), \( d(x_{0,i}, x_{0,i+1}) \leq t_0 \) and \( d(x_{0,i+1}, z) \leq d(x_{0,i}, z) - t_0/C_4 \). In this case, \( B(x_{0,0}, 3t_0/2) = B(x, 3t_0/2) \ni x_{0,1} \) and \((4/3) \times 3t_0/2 = r - d(x, z)\). If \( x_{0,1} \in B_0 \), then \( k \equiv k(x) \equiv 1 \) and \( \{B_0, B(x, 3t_0/2)\} \) is a desired chain, since \( x \in (4C_4/3)B_0 \).

Assume that \( x_{0,1} \notin B_0 \) and let \( t_1 \equiv (r - d(x_{0,1}, z))/2 \), then \( d(x_{0,1}, z) \geq 3r/4 \) and \( t_1 \leq r/8 \). Thus, \( d(x_{0,1}, z) \geq 6t_1 > t_1/C_4 \), by \( C_4 \geq 1 \). By Definition 4.1(II), there exists a finite chain \( x_{1,0} \equiv x_{0,1}, x_{1,1}, \ldots, x_{1,m_1} \equiv z \) with \( m_1 \in \mathbb{N} \) such that for \( 0 \leq i < m_1 \), \( d(x_{1,i}, x_{1,i+1}) \leq t_1 \) and \( d(x_{1,i+1}, z) \leq d(x_{1,i}, z) - t_1/C_4 \). In this case, \( B(x_{0,1}, 3t_1/2) = B(x_{0,1}, 3t_1/2) \ni x_{1,1} \) and \((4/3) \times 3t_1/2 = r - d(x_{0,1}, z)\). Moreover, \( t_0 \leq t_1(2C_4)/(1 + 2C_4) \), since

\[
t_1 - t_0 = (r - d(x_{0,1}, z))/2 - (r - d(x, z))/2 = (d(x, z) - d(x_{0,1}, z))/2 \geq t_0/(2C_4).
\]

Then, \( d(x, x_{0,1}) \leq t_0 \leq t_1(2C_4)/(1 + 2C_4) < 2C_4 t_1 = (4C_4/3) \times (3t_1/2) \), that is,

\[
x \in (4C_4/3)B(x_{0,1}, 3t_1/2).
\]

If \( x_{1,1} \in B_0 \), then \( k \equiv k(x) \equiv 2 \) and \( \{B_0, B(x_{0,1}, 3t_1/2), B(x, 3t_0/2)\} \) is a desired chain.

Assume that \( x_{1,1} \notin B_0 \) and let \( t_{j+1} \equiv (r - d(x_{j,1}, z))/2 \), then \( d(x_{j,1}, z) \geq 3r/4 \) and \( t_{j+1} \leq r/8 \). Thus, \( d(x_{j,1}, z) \geq 6t_{j+1} > t_{j+1}/C_4 \), by \( C_4 \geq 1 \). By Definition 4.1(II), there exists a finite chain \( x_{j+1,0} \equiv x_{j,1}, x_{j+1,1}, \ldots, x_{j+1,m_{j+1}} \equiv z \) with \( m_{j+1} \in \mathbb{N} \) such that for \( 0 \leq i < m_{j+1} \), \( d(x_{j+1,i}, x_{j+1,i+1}) \leq t_{j+1} \) and \( d(x_{j+1,i+1}, z) \leq d(x_{j+1,i}, z) - t_{j+1}/C_4 \). In this case,

\[
B(x_{j,1}, 3t_{j+1}/2) = B(x_{j,1}, 3t_{j+1}/2) \ni x_{j+1,1} \quad \text{and} \quad (4/3) \times 3t_{j+1}/2 = r - d(x_{j,1}, z).
\]

Moreover, \( t_j \leq t_{j+1}(2C_4)/(1 + 2C_4) \), since \( x_{j-1,1} = x_{j,0} \) and

\[
t_{j+1} - t_j = (r - d(x_{j,1}, z))/2 - (r - d(x_{j-1,1}, z))/2 = (d(x_{j,0}, z) - d(x_{j,1}, z))/2 \geq t_j/(2C_4).
\]

Then,

\[
d(x, x_{j,1}) \leq d(x, x_{0,1}) + \sum_{\ell=1}^{j} d(x_{\ell-1,1}, x_{\ell,1})
\]
\[ = d(x_{0,0}, x_{0,1}) + \sum_{\ell=1}^{j} d(x_{\ell,0}, x_{\ell,1}) \]
\[ \leq \sum_{\ell=0}^{j} t_{\ell} \leq \sum_{\ell=1}^{j+1} t_{j+1}((2C_4)/(1 + 2C_4))^\ell \]
\[ < 2C_4 t_{j+1} = (4C_4/3) \times (3t_{j+1}/2), \]
that is, \( x \in (4C_4/3)B(x_{j,1}, 3t_{j+1}/2) \). If \( x_{j+1,1} \in B_0 \), then \( k \equiv k(x) \equiv j + 2 \) and
\[ \{B_0, B(x_{j,1}, 3t_{j+1}/2), \cdots, B(x_{0,1}, 3t_1/2), B(x, 3t_0/2)\} \]
is a desired chain.

To finish the proof that (II) implies (III), we must show \( x_{j_0,1} \in B_0 \) for some \( j_0 \in \mathbb{N} \cup \{0\} \). To this end, it is enough to show that (4.1) holds for \( j_0 \). Assume that (4.1) holds for \( j \in \mathbb{N} \) and we consider the case \( j+1 \). By the definitions of \( x_{j,1} \) and \( t_j \), we have
\[ t_{j+1} = \frac{1}{2}(r - d(x_{j,1}, z)) \]
\[ \geq \frac{1}{2} \left( r - \left( 1 + \frac{1}{2C_4} \right)^{j+1} [r - d(x, z)] \right) \],
\[ = \frac{1}{2} \left( 1 + \frac{1}{2C_4} \right)^{j+1} [r - d(x, z)] \]
and
\[ d(x_{j+1,1}, z) \leq d(x_{j+1,0}, z) - t_{j+1}/C_4 = d(x_{j,1}, z) - t_{j+1}/C_4 \]
\[ \leq r - \left( 1 + \frac{1}{2C_4} \right)^{j+1} [r - d(x, z)] - \frac{1}{2C_4} \left( 1 + \frac{1}{2C_4} \right)^{j+1} [r - d(x, z)] \]
\[ = r - \left( 1 + \frac{1}{2C_4} \right)^{j+2} [r - d(x, z)]. \]
Hence, by (iv),

$$\text{(iii), it follows that there exists } w \in \mathbb{N} \text{ such that } d(y_N, B(x_N, r_N)) > Ns_N. \text{ In this case, } y_N \notin B(x_N, r_N) \text{ and } Ns_N < d(y_N, x_N) \leq r_N + s_N. \text{ Thus,}$$

$$(N - 1)s_N < r_N.$$

We show that, for all $\alpha, \beta \in (1, \infty)$, there exists $N \in \mathbb{N}$ such that $B(x_N, r_N + 2s_N)$ is not an $(\alpha, \beta)$-chain ball. Otherwise, for some $\alpha, \beta \in (1, \infty)$ and for all $N \in \mathbb{N}$, if $B(x_N, r_N + 2s_N)$ is an $(\alpha, \beta)$-chain ball with respect to $B_{N,0} \equiv B(z_{N,0}, t_{N,0}) \subset B(x_N, r_N + 2s_N)$, then there exists an integer $k \equiv k(y_N) > 0$ and a chain of balls, $B_{N,i} \equiv B(z_{N,i}, t_{N,i}) \subset B(x_N, r_N + 2s_N)$, $0 \leq i \leq k$, satisfy that

(i) $y_N \in B_{N,k}$,
(ii) $B_{N,i} \cap B_{N,i+1}$ is non-empty, $0 \leq i < k$,
(iii) $y_N \in B(z_{N,i}, \alpha t_{N,i})$, $0 \leq i \leq k$,
(iv) $\beta_{N,i} \leq r_N + 2s_N - d(z_{N,i}, x_N)$, $0 \leq i \leq k$.

If $N$ satisfies $\beta - 4\alpha/(N - 1) > 1$, then

$$\bigcup_{0 \leq i \leq k} B_{N,i} \subset B(x_N, r_N),$$

that is, $y_N \notin B_{N,i}$ for all $0 \leq i \leq k$, which is contradict to (i).

In the following we show (4.3). By (iii) of Definition 4.1(III), we have that $x_N \in B(z_{N,0}, \alpha t_{N,0})$, which together with $y_N \notin B(x_N, r_N)$, (4.2) and (iii) leads to that

$$(N - 1)s_N \leq r_N \leq d(y_N, x_N) \leq d(y_N, z_{N,0}) + d(z_{N,0}, x_N) < 2\alpha t_{N,0}.$$

Hence, by (iv),

$$d(z_{N,0}, x_N) \leq r_N + 2s_N - \beta t_{N,0} \leq r_N + \left(\frac{4\alpha}{N - 1} - \beta\right) t_{N,0} < r_N - t_{N,0},$$

since $\beta - 4\alpha/(N - 1) > 1$. Thus, $B(z_{N,0}, t_{N,0}) \subset B(x_N, r_N)$, since, for $w \in B(z_{N,0}, t_{N,0})$,

$$d(w, x_N) \leq d(w, z_{N,0}) + d(z_{N,0}, x_N) < t_{N,0} + r_N - t_{N,0} = r_N.$$

Assume that $B_{N,i} \subset B(x_N, r_N)$ for $i \in \mathbb{N}$. We show $B_{N,i+1} \subset B(x_N, r_N)$. From (ii) and (iii), it follows that there exists $w \in (B_{N,i} \cap B_{N,i+1}) \subset B(x_N, r_N)$ and

$$Ns_N < d(w, y_N) \leq d(w, z_{N,i+1}) + d(z_{N,i+1}, y_N) \leq (1 + \alpha)t_{N,i+1}.$$

Hence, by (iv),

$$d(z_{N,i+1}, x_N) \leq r_N + 2s_N - \beta t_{N,i+1} \leq r_N + \left(\frac{(1 + \alpha)}{N} - \beta\right) t_{N,i+1} < r_N - t_{N,i+1},$$

since $\beta - 2(1 + \alpha)/N > \beta - 4\alpha/(N - 1) > 1$. Thus, $B(z_{N,i+1}, t_{N,i+1}) \subset B(x_N, r_N)$, which completes the proof of (4.3) and hence Theorem 4.1.
As an application of the chain ball property, Buckley in [1] proved the following useful result.

**Lemma 4.1** ([1]). Let $X = (X, d, \mu)$ be a doubling metric measure space with doubling constat $C_1$. Suppose that $(X, d)$ also has the $(\alpha, \beta)$-chain ball property for some $\alpha, \beta \in (1, \infty)$, then $\mu$ has the $\delta$-annular decay property for some $\delta \in (0, 1]$ dependent only on $\alpha, \beta$ and $C_1$.

As a consequence of Theorem 4.1 and Lemma 4.1, we have the following conclusion.

**Corollary 4.1.** Let $X = (X, d, \mu)$ be a doubling metric measure space. If $(X, d)$ has either the weak geodesic property or the monotone geodesic property, then $\mu$ has the $\delta$-annular decay property for some $\delta \in (0, 1]$.

**Remark 4.2.** By an argument similar to that used in the proof of [35, Theorem 4], we can also directly prove Corollary 4.1, without invoking Lemma 4.1. We omit the details.

As an application of Corollary 4.1, we show that any length space equipped with a doubling measure has the $\delta$-annular decay property, which is just [1, Corollary 2.2]. However, unlike the proof of [1, Corollary 2.2], we prove the following Proposition 4.1 without invoking the property of John domains. In what follows, for any rectifiable path $\gamma$, let $\ell(\gamma)$ denote its length.

**Proposition 4.1.** Any length space $(X, d)$ has the weak geodesic property. Moreover, if $\mu$ is a doubling measure on $(X, d)$ with doubling constant $C_1$, then $\mu$ has the $\delta$-annular decay property for some $\delta \in (0, 1]$ dependent only on $C_1$.

**Proof.** Let $x \in X$, $r, s \in (0, \infty)$ and $y \in \overline{B(x, r+s)}$. If $d(x, y) \leq r$, then $d(y, \overline{B(x, r)}) = 0 \leq s$. If $r < d(x, y) \leq r+s$, then for any given $\epsilon > 0$, there exists a rectifiable path $\gamma$ from $x$ to $y$ such that $\ell(\gamma) < d(x, y)+\epsilon$. Moreover, by the mean value theorem for the continuous function of $w \mapsto d(x, w)$ restricted to the path $\gamma$, there exists a $z \in \gamma$ such that $d(x, z) = r$. By splitting the path $\gamma$ into $\gamma_1$ from $x$ to $z$ and $\gamma_2$ from $z$ to $y$, we have by definition of the distance and choice of $\gamma$ that $d(x, z) + d(z, y) \leq \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma) < d(x, y) + \epsilon$. Thus, $d(y, \overline{B(x, r)}) \leq d(y, z) < d(x, y) + \epsilon - d(x, z) \leq s + \epsilon$. Letting $\epsilon \to 0$ yields that $d(y, \overline{B(x, r)}) \leq s$, which shows that $(X, d)$ has the weak geodesic property. This combined with Corollary 4.1 implies that $\mu$ has the $\delta$-annular decay property for some $\delta \in (0, 1]$ dependent only on $C_1$, which completes the proof of Proposition 4.1.

**Remark 4.3.** In the proof of Proposition 4.1, if $r < d(x, y) \leq r+s$, then by [2, p. 42, Exercise 2.4.13], we also have $d(y, \overline{B(x, r)}) \leq d(y, \overline{B(x, r)}) = d(x, y) - r \leq s$, which is another proof of this fact.

Now we give an equivalent characterization for the $\delta$-annular decay property. First, we introduce the following notion.

**Definition 4.2.** Let $\tau \in [1, \infty)$. A doubling metric measure space $(X, d, \mu)$ is said to have *Property $(P)_\tau$*, if there exist positive constants $\delta$ and $C_{(P)_\tau}$, such that for all $x \in X$, $s \in (0, \infty)$ and $r \in (\tau s, \infty)$,

\[
\mu(B(x, r+s)) - \mu(B(x, r)) \leq C_{(P)_\tau} \left( \frac{\delta}{\tau} \right) \mu(B(x, r)).
\]
Remark 4.4. (i) When \( \tau = 1 \), it was proved in [27, Remark 1.1] that if \( \mathcal{X} \) contains no less than two elements, then \( \delta \in (0, 1] \). Hence, if \( \mathcal{X} \) contains no less than two elements, Property \((P)_1\) is just the \( \delta \)-annular decay property and we denote it simply by Property \((P)\). Also, we denote the corresponding constant \( C_{(P)_1} \) in (4.4) by \( C_P \).

(ii) Observe that if \( r \in (0, \tau s] \), then (4.4) is a simple conclusion of the doubling property (2.1) of \( \mu \). Moreover, if \( r \in (0, s] \), then (4.4) is always true, which explains why we restrict that \( \tau \in [1, \infty) \) in Definition 4.2.

It is easy to show that Property \((P)_{\tau}\) with \( \tau \in (1, \infty) \) is equivalent to the \( \delta \)-annular decay property in the meaning as in the following Proposition 4.2. We omit the details. In what follows, for any \( a \in \mathbb{R} \), we denote by \( \lceil a \rceil \) the smallest integer no less than \( a \).

Proposition 4.2. Let \( \tau \in (1, \infty) \). Then

(i) Property \((P)\) implies Property \((P)_{\tau}\) with \( C_{(P)_{\tau}} \equiv C_P \).

(ii) Property \((P)_{\tau}\) implies Property \((P)\) with \( C_P \equiv ([\tau])^{1-\delta}C_{(P)_{\tau}}C_1 \), where \( C_1 \) is the same as in Definition 2.1.

Finally, we give several examples of doubling metric measure spaces having the \( \delta \)-annular decay property.

Example 4.1. (i) \((\mathbb{R}^d, |\cdot|, dx)\), the \( d \)-dimensional Euclidean space endowed with the Euclidean norm \(|\cdot|\) and the Lebesgue measure \( dx \). It is easy to show that \((\mathbb{R}^d, |\cdot|, dx)\) has the \( \delta \)-annular decay property for all \( \delta \in (0, 1] \).

(ii) \((\mathbb{R}^d, |\cdot|, w(x)dx)\), the \( d \)-dimensional Euclidean space endowed with the Euclidean norm \(|\cdot|\) and the measure \( w(x)dx \), where \( w \) is an \( A_{\infty}(\mathbb{R}^d) \) weight (see [12, p. 401] for its definition) and \( dx \) is the Lebesgue measure. Let \( w \) be an \( A_{\infty}(\mathbb{R}^d) \) weight and for any Lebesgue measurable set \( E \), let \( w(E) \equiv \int_E w(x)dx \). Then there exist positive constants \( C \) and \( \delta \in (0, 1] \) such that for all balls \( B \) and measurable subsets \( E \) of \( B \),

\[
\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{\delta}
\]

(see [12, p. 401, Theorem 2.9]). Clearly this inequality implies that \((\mathbb{R}^d, |\cdot|, w(x)dx)\) has the \( \delta \)-annular decay property.

(iii) Macias, Segovia and Torrea [25] introduced the condition \((H_{\alpha})\) with \( \alpha \in (0, 1] \) on a space of homogeneous type. Recall that a doubling metric measure space \((\mathcal{X}, d, \mu)\) is said to satisfy Condition \((H_{\alpha})\) with \( \alpha \in (0, 1] \), if there exists a positive constant \( C \) such that for all \( x \in \mathcal{X}, r \in (0, \infty) \) and \( s \in (0, r) \),

\[
\mu(B(x, r+s)) - \mu(B(x, r-s)) \leq C[\mu(B(x, r))]^{1-\alpha}[\mu(B(x, s))]^{\alpha}.
\]

If \( \mathcal{X} \) is an RD-space, namely, there exist constants \( 0 < \kappa \leq n \) and \( C \geq 1 \) such that for all \( x \in \mathcal{X} \) and \( 0 < r < 2 \text{diam}(\mathcal{X}) \) and \( 1 \leq \lambda < 2 \text{diam}(\mathcal{X})/r \),

\[
C^{-1}x^{\kappa}\mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^n\mu(B(x, r))
\]
(see [39]), where \(\text{diam} (X) \equiv \sup_{x, y \in X} d(x, y)\), then there exists a positive constant \(C\) such that for all \(x \in X\), \(s \in (0, \infty)\) and \(r \in (s, \infty)\),

\[
\mu(B(x, r + s)) - \mu(B(x, r)) \leq C \left(\frac{s}{r}\right)^{\kappa \alpha} \mu(B(x, r)).
\]

This shows that for an RD-space, Condition \((H_{\alpha})\) with \(\alpha \in (0, 1]\) implies the \(\delta\)-annular decay property.

(iv) \((\mathbb{H}^n, d, dx)\), the \((2n + 1)\)-dimensional Heisenberg group \(\mathbb{H}^n\) with a left-invariant metric \(d\) and the Lebesgue measure \(dx\). Buckley [1] showed that \((\mathbb{H}^n, d, dx)\) is a doubling metric measure space having the \(\delta\)-annular decay property for all \(\delta \in (0, 1]\).

(v) \((G, d, \mu)\), the nilpotent Lie group \(G\) with a Carnot-Carathéodory (control) distance \(d\) and a left invariant Haar measure \(\mu\). Fix a left invariant Haar measure \(\mu\) on \(G\). Then for all \(x \in G\), \(V_r(x) = V_r(e)\); moreover, there exist \(\kappa, D \in (0, \infty)\) with \(\kappa \leq D\) such that for all \(x \in G\), \(C^{-1} r^\kappa \leq V_r(x) \leq C r^\kappa\) when \(r \in (0, 1]\), and \(C^{-1} r^D \leq V_r(x) \leq C r^D\) when \(r \in (1, \infty)\); see [30] and [36]. By Proposition 4.1 and the fact that \((G, d)\) is a length space, we know that \((G, d, \mu)\) is a doubling metric measure space having the \(\delta\)-annular decay property for some \(\delta \in (0, 1]\).

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