MULTIPlicITY ALONG POINTS OF A RADICAL COVERING OF A REGULAR VARIETY

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ABSTRACT. We study the maximal multiplicity locus of a variety $X$ over a field of characteristic $p > 0$ that is provided with a finite surjective radical morphism $\delta : X \to V$, where $V$ is regular, for example, when $X \subset \mathbb{A}^{n+1}_k$ is a hypersurface defined by an equation of the form $T^n - f(x_1, \ldots, x_n) = 0$ and $\delta$ is the projection onto $V := \text{Spec}(k[x_1, \ldots, x_n])$. The multiplicity along points of $X$ is bounded by the degree, say $d$, of the field extension $K(V) \subset K(X)$. We denote by $F_d(X) \subset X$ the set of points of multiplicity $d$. Our guiding line is the search for invariants of singularities $x \in F_d(X)$ with a good behavior property under blowups $X' \to X$ along regular centers included in $F_d(X)$, which we call invariants with the pointwise inequality property.

A finite radical morphism $\delta : X \to V$ as above will be expressed in terms of an $\mathcal{O}_V$-submodule $\mathcal{M} \subseteq \mathcal{O}_X$. A blowup $X' \to X$ along a regular equimultiple center included in $F_d(X)$ induces a blowup $V' \to V$ along a regular center and a finite morphism $\delta' : X' \to V'$. A notion of transform of the $\mathcal{O}_V$-module $\mathcal{M} \subseteq \mathcal{O}_V$ to an $\mathcal{O}_V'$-module $\mathcal{M}' \subseteq \mathcal{O}_{V'}$ will be defined in such a way that $\delta' : X' \to V'$ is the radical morphism defined by $\mathcal{M}'$. Our search for invariants relies on techniques involving differential operators on regular varieties and also on logarithmic differential operators. Indeed, the different invariants we introduce and the stratification they define will be expressed in terms of ideals obtained by evaluating differential operators of $V$ on $\mathcal{O}_V$-submodules $\mathcal{M} \subseteq \mathcal{O}_V$.

§ 1 Introduction

Let $k$ be a field of characteristic $p > 0$, and fix $q = p^r$, a power of $p$. Let $X \subset \mathbb{A}^{n+1}_k$ be a hypersurface defined by a polynomial equation of the form $T^n - f(x_1, \ldots, x_n) = 0$. The maximal multiplicity along points of $X$ is $\leq q$, and we denote by $F_q(X) \subset X$ the set of points where the multiplicity is $q$. This subset is closed, as is indicated above, and when it is nonempty, it can be thought of as the set of the worst singularities. Our guideline is the search for invariants of singularities $x \in F_q(X)$. We focus our attention on the projection $X \to V := \mathbb{A}^n_k$, which is a finite morphism of generic rank $q$, rather than on the immersion $X \subset \mathbb{A}^{n+1}_k$. The following theorem, which collects results taken from [26, Sections 5 and 6], serves as starting point in our discussion.

Theorem 1.1. Let $\delta : X \to V$ be a finite and surjective morphism of Noetherian integral schemes, where $V$ is excellent and regular, and $\dim \mathcal{O}_{V,x}$ is constant along closed points. We set $d := [K(X) : K(V)]$, the generic rank. Then the multiplicity along points of $X$ is at most $d$. Let $F_d(X)$ denote the set of points where the multiplicity is $d$ and assume that it is nonempty. Then the following holds.

1. $F_d(X) \subset X$ is closed and homeomorphic to its image $\delta(F_q(X)) \subset V$ (via $\delta$); moreover, $F_d(X) = \delta^{-1}(\delta(F_d(X)))$.

2. An integral subscheme $Y \subset X$ included in $F_d(X)$ is regular if and only if its schematic image $\delta(Y) \subset V$ is also regular. In that case the blowup of $X$ along $Y$ and the blowup of $V$ along $\delta(Y)$ fit into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & X_1 \\
\downarrow \delta & & \downarrow \delta_1 \\
V & \xleftarrow{\delta_1} & V_1
\end{array}
\]
for a unique morphism \( \delta_1 : X_1 \to V_1 \). This morphism is again finite and surjective and has generic rank \( d \). In particular, the maximal multiplicity of \( X_1 \) is \( \leq d \).

**Remark 1.2.**

(i) The closeness of \( F_\delta(X) \) under these general hypothesis was proved by Dade \[8\]; see (3.16).

(ii) The approach of viewing a variety, or a scheme, as a finite cover of a regular one, has been shown to be efficient when studying the multiplicity along singular points. For instance, Lipman \[21\] uses it to discuss the multiplicity of complex analytic and algebraic varieties and the effect of blowing up along equimultiple centers. This idea was taken further in \[20\], by the second author, to give an alternative proof of resolution of singularities in characteristic zero by using the multiplicity as the main invariant.

(iii) Assume that \( V = \text{Spec}(S) \), where \( S \) is of finite type over a field \( k \), and that \( X \subset V \times \mathbb{A}^1_k \) is defined by a polynomial of the form \( T^d + f_1 T^{d-1} + \cdots + f_d \), where \( f_i \in S \), and \( d \) is not divisible by the characteristic of \( k \). By using elimination theory we can produce an \( \mathcal{O}_V \)-ideal \( J \) and a positive integer \( b \) such that \( \delta(F_\delta(X)) = \text{Sing}(J, b) := \{ x \in V : \nu_x(J) \geq b \} \) and such that this description holds under any sequence of blowups that arises by applying Theorem \[1.1(2)\] successively. Here \( \nu_x(J) \) denotes the order of \( J \) at the regular local ring \( \mathcal{O}_{V, x} \). We refer to \[20\] Section 3 for a detailed discussion (see also \[25\]). We only stress the fact that \( (J, b) \) is obtained quite explicitly: it is determined by the weighted homogeneous polynomials on the coefficients \( f_1, \ldots, f_d \) that are invariant under the change of variables \( T \to T + \lambda, \lambda \in S \), such as the discriminant \[26\] Theorem 3.5.

We return to our original situation, where \( X \) is a hypersurface defined by a purely inseparable equation \( T^q - f(x_1, \ldots, x_n) = 0 \), and where \( \delta : X \to V \) is the projection. When applying the theorem to this morphism, the resulting \( \delta_1 : X_1 \to V_1 \) is locally in the same situation as \( X \to V \), that is, defined by a purely inseparable equation of degree \( q \). This process can be repeated as long as the maximal multiplicity along points of \( X_1 \) is \( q \). The intention in this first part of the introduction is to illustrate how the study of the singularities along points \( x \in F_\delta(X) \) and of blowups along regular centers included in \( F_\delta(X) \) naturally leads to the consideration of \( \mathcal{O}_V \)-submodules \( \mathcal{M} \subset \mathcal{O}_V \) and transformations of \( \mathcal{O}_V \)-submodules under suitable blowups of the regular scheme \( V \).

In contrast to the situation described in Remark \[1.2(iii)\], with \( S = k[x_1, \ldots, x_n] \), here the only nonzero coefficient of the equation (apart from the principal one) is the constant coefficient \( f := f(x_1, \ldots, x_n) \), and the change of variables \( T' := T + g \) with \( g \in S \) produces the equation \( T'^q - (f + g^q) = 0 \), that is, \( f \) is changed by \( f + g^q \). This tells us that we should not consider the ideal \( (f) \subset S \) if we attempt to find a substitute for \( (J, b) \), since this ideal has nothing to do with \( (f + g^q) \). Instead, we may consider the element \( f \) only up to equivalence, where \( f \sim f' \) if \( f - f' \in S^q \). In the case that \( S \) is a more general regular ring, we can also perform a change of variable of the form \( T' := u^{-1}T \) with \( u \in S^* \), and the equation will take the form \( T'^q - f u^q \). This suggests that it is a better idea to take the full \( S^q \)-submodule \( M \subset S \) generated by \( f \) and to consider \( S^q \)-submodules only up to equivalence, where \( M \sim M' \) if \( M + S^q = M' + S^q \). Finally, if we are willing to completely forget the immersion \( X \subset \mathbb{A}^{n+1}_k \) and regard \( X \) simply as a \( S \)-scheme, then we have to look at the whole \( S^q \)-subalgebra of \( S \) generated by \( f \), denoted by \( S^q[f] \).

In general, we will attach to any \( S^q \)-submodule \( M \subset S \) a finite morphism \( X \to V \) so that \( M \) and \( M' \) define the same morphism if and only if \( S^q[M] = S^q[M'] \). One important step in our search for invariants is the introduction of assignments of \( S \)-ideals \( M \to I(M) \) so that \( I(M) = I(S^q[M]) \). Our main device will be higher-order differential operators that are \( S^q \)-linear, such as \( \partial^\alpha / \partial x^\alpha \), with \( \alpha = (\alpha_1, \ldots, \alpha_n) \neq 0 \) and \( \sum_{i=1}^n \alpha_i < q \). To be more precise, for \( i = 1, \ldots, q - 1 \), we denote by \( \text{Diff}^i_{S^q} \) the \( S \)-module of differential operators \( D : S \to S \) of order \( \leq i \) such that \( D(1) = 0 \) (see \[3.1\]); for example, \( \text{Diff}^1_{S^q} = \text{Der}_S \), the set of derivations on \( S \). All these operators are \( S^q \)-linear, and we have \( \text{Diff}^i_{S^q}(S^q[M]) = \text{Diff}^i_{S^q}(M) \) (Proposition \[3.10\]).

We return to our hypersurface \( X \subset \mathbb{A}^{n+1}_k \) defined by the polynomial \( \Theta(T) := T^q - f(x_1, \ldots, x_n) \). Recall that \( \delta : X \to V = \mathbb{A}^n_k \) denotes the projection. The following proposition (reformulated and generalized in Proposition \[4.9\]) offers a first example of the role played by differential operators.

**Proposition 1.3.** \( \delta(F_\delta(X)) \subset V \) is the closed subset defined by the \( S \)-ideal \( \text{Diff}^{q-1}_{S^q}(f) = \text{Diff}^{q-1}_{S^q}(S^q[f]) \).
We remark that this is a direct consequence of the classical fact that \( F_q(X) \subset \mathbb{A}^{n+1}_k \) (the set of points where the order of \( \Theta(T) \) is \( \geq q \)) is the closed subset of \( \mathbb{A}^{n+1}_k \) defined by the ideal \( \text{Diff}^q_{S[T]}(\Theta(T)) \). Indeed, this ideal is generated by \( \Theta(T), \text{Diff}^q_{S[T]}(\Theta(T)), (\partial^i/\partial T^i)(\Theta(T)), i = 1, \ldots, q-1 \), where differential operators of \( S \) are extended to \( S[T] \) by acting as 0 on \( T \). Note that \( \text{Diff}^q_{S_+}(\Theta(T)) = \text{Diff}^q_{S_+}(f) \) and that \( (\partial^i/\partial T^i)(\Theta(T)) = 0 \) for \( i = 1, \ldots, q-1 \). The proposition follows from these observations.

The above proposition describes \( \delta(F_q(X)) \) with an ideal of \( S \), which is intrinsically attached to the \( V \)-scheme \( X \). The next natural step would be stratifying this closed set. This will be done by defining upper-semicontinuous functions, which in general are defined in terms of the order of suitable ideals along points of the regular scheme \( V \). Recall that a function \( s : Y \to \mathbb{N} \), from a topological space \( Y \), is upper-semicontinuous if the sets \( \{ x \in Y : s(x) \geq m \} \), \( m \in \mathbb{N} \), are closed.

According to our previous discussion, the order of \( f \in \mathcal{O}_{V,x} \) along points \( x \in \delta(F_q(X)) \) is not a good function to look at. Instead, we might consider

\[
(1.2) \quad \nu_p^q(f) := \sup \{ \nu_p(f + g^q) : g \in S_p \}.
\]

By using this function we obtain the following description.

**Proposition 1.4.** \( \delta(F_q(X)) = \{ x \in V : \nu_p^q(f) \geq q \} \).

See Proposition 2.18 for an equivalent formulation. A drawback of the above function along points \( x \in \delta(F(X)) \) is that it is not upper-semicontinuous, as it is well known, and we recall it in the following:

**Example 1.5.** Let \( f := x_1^q x_2 \in k[x_1, x_2] \) with algebraically closed \( k \). We obtain \( \text{Diff}^q_{S_+}(f) = (x_1^q) \), so that \( Z := \delta(F_q(X)) \subset V \) is the \( x_2 \)-axis. At \( \xi \in Z \), the generic point, we have \( \nu_x^q(f) = q \). However, at any closed point \( x = (x, \lambda) \) of \( Z \), we have \( x_1^q x_2 = x_1^q(x_2 - \lambda) + x_1^q \lambda \sim x_1^q(x_2 - \lambda) \) since \( \lambda \) is a \( q \)-power; hence \( \nu_x^q(f) \geq q+1 \) (in fact, it is an equality). Were \( x \mapsto \nu_x^q(f) \) upper-semicontinuous on \( Z \), the set \( \{ x \in Z : \nu_x^q(f) = q \} \) would be open in \( Z \), which is not the case.

We will use differential operators on the regular ring \( S \) to define an upper-semicontinuous function that coincides “almost always” with \( \nu_x^q(f) \). To motivate its definition, we recall that if \( D \) is a differential operator of order \( i \) and \( f \) has order \( m \) at a point \( x \in V \), then \( D(f) \) has order at least \( m - i \) at \( x \). In characteristic zero, we can always select \( D \) so that \( D(f) \) has order exactly \( m-i \). This is not longer true in positive characteristic. If we apply differential operators of order \( < q \) to the polynomial \( x_1^q x_2 \), we cannot get rid of the factor \( x_1^q \). However, \( \partial/\partial x_2 \), which has degree 1, lowers the order of \( x_1^q x_2 \) at the origin in exactly one unit. Similarly, \( \partial^i/\partial x_2^i \) has degree \( p \) and lowers the order of \( x_1^q x_2 \) at the origin in exactly \( p \) units.

**Definition 1.6.** For \( x \in V \), we define

\[
\eta_x(f) = \inf \{ \nu_x(\text{Diff}^i_{S_+}(f)) + i : i = 1, \ldots, q-1 \}.
\]

This defines an upper-semicontinuous function on \( V \) with values in \( \mathbb{N} \): \( \eta_x(f) \geq m \) if and only if \( x \) belongs to the intersection of the closed subsets \( \{ x \in V : \nu_x(\text{Diff}^i_{S_+}(f)) \geq m-i \} \). We stress the fact that this function is intrinsically attached to the \( V \)-scheme \( X \), that is, if \( S^q[f] = S^q[g] \), then \( \eta_x(f) = \eta_x(g) \). The fact that \( \eta_x(f) \) coincides with \( \nu_x^q(f) \) almost always on \( \delta(F_q(X)) \) will be made precise in Lemma 3.17. The next proposition, reformulated and generalized in Proposition 4.18 shows that \( \delta(F_q(X)) \) can be described by this function.

**Proposition 1.7.** For \( x \in V, x \in \delta(F_q(X)) \) if and only if \( \eta_x(f) \geq q \).

**Example 1.8.** Set \( f = x_1^q x_2 \), as in the previous example. We obtain \( \text{Diff}^i_{S_+}(f) = (x_1^q) \) for all \( i = 1, \ldots, q-1 \). Therefore \( \eta_x(f) \) is constant and equal to \( q+1 \) along \( \delta(F_q(X)) \), even at the generic point.

Our definition of \( \eta_x(f) \) and the above proposition, which shows that \( \eta_x(f) \) can be used to stratify \( \delta(F_q(X)) \), offer a first glimpse of the role played by the collection of ideals

\[
(1.3) \quad (\text{Diff}^1_{S_+}(f), \ldots, \text{Diff}^{q-1}_{S_+}(f)).
\]

We will return to this point later, where collections like this will be studied under the name of \( q \)-differential collections. This is the subject of Section 5. We now turn the discussion to blowups.
Regular hypersurfaces $H \subset V$ included in $\delta(F_q(X))$ will play a privileged role in our discussion. They will arise naturally when blowing up along a regular center. For such a hypersurface, the blowup of $X$ along $\delta^{-1}(H) \subset X$ can be interpreted in terms of the $S^q$-module generated by $f$ as follows.

**Proposition 1.9.** Let $H \subset V$ be a regular hypersurface, say with defining equation $h = 0$. Then the following are equivalent:

1. $H$ is included in $\delta(F_q(X))$.
2. $f = h^q f_1 + g^q$ for some $f_1, g \in S$.

If these conditions are satisfied, then the blowup of $X$ along $\delta^{-1}(H)$ is defined by the equation $T^q - f_1 = 0$.

This will be reformulated in a more general form in Proposition [22]. The outcome is that blowing up $X$ along a regular 1-codimensional center included in $F_q(X)$ corresponds to a notion of factorization of a $q$-power of the $S^q$-module generated by $f$ or, more precisely, of the equivalence class of this module. We will illustrate this in the following example.

**Example 1.10.** Let $X \subset \mathbb{A}^3_\mathbb{K}$ be defined by the equation $T^p - x_1^{3p-2}x_2(x_2 - 2x_1 + x_1^2) + x_2^p$ ($q = p > 2$), and set $S = k[x_1, x_2]$. We obtain the $S^p$-submodule of $S$ generated by $x_1^{3p-2}x_2(x_2 - 2x_1 + x_1^2) + x_2^p$, which is equivalent to that generated by $x_1^{3p-2}x_2(x_2 - 2x_1 + x_1^2)$. We apply $\text{Diff}_{S^p}^{-1}$ to any of these modules and obtain an ideal divisible by $x_1^{2p}$, which tells us that $\delta(F_p(X))$ includes the hypersurface $x_1 = 0$. Changing $x_1^{3p-2}x_2(x_2 - 2x_1 + x_1^2)$ by $f_1 := x_1^{2p}x_2(x_2 - 2x_1 + x_1^2)$ corresponds to a blowup $X_1 \to X$ along a regular one-codimensional center. Changing again this polynomial by $f_2 := x_1^{p-2}x_2(x_2 - 2x_1 + x_1^2)$ corresponds to a blowup $X_2 \to X_1$ of the same kind. Note that $\delta(F_p(X_2))$ has no one-dimensional component. The maximum of $\eta_2(f_2)$ along points of $V$ is $p$, and this maximum is only attained at the origin $(0, 0)$. By Proposition [7], $\delta(F_p(X_2)) = \{(0, 0)\}$.

We now show with an example that a blowup of $X$ along a more general regular center included in $F_q(X)$ leads to a notion of transformation of the $S^q$-module assigned to $X$ under the induced blowup of $V$ (Theorem [14]). (2)). We will give the details in the second part of Section 2 see Corollary 2.25 for a generalized version.

**Example 1.11.** Let us consider the hypersurface $X \subset \mathbb{A}^3_\mathbb{K}$ of equation $T^p - x_1^{p-2}x_2(x_2 - 2x_1 + x_1^2)$. This has attached the (equivalence class of the) $S^p$-submodule of $S$ generated by $f := x_1^{p-2}x_2(x_2 - 2x_1 + x_1^2)$. We obtain from our previous example that $\delta(F_p(X)) = \{(0, 0)\}$, and hence $F_p(X) = \{(0, 0, 0)\}$. Let $V_1 \to V$ be the blowup at $\{(0, 0)\}$, and let $X_1 \to X$ be the blowup at $\{(0, 0, 0)\}$. We set $S_1 := k[x_1, x_2, x_3]$ and $S_2 := k[x_2, x_3]$. Then $X_1$ is defined by $T_1^p - \frac{x_1 x_2 (x_2 - 2 + x_3)}{x_2} \in S_1[T_1]$ at the $x_1$-chart and by $T_2^p - \frac{x_2 x_3 (x_2 - 2 + x_3)}{x_2} \in S_2[T_2]$ at the $x_2$-chart. (The strict transform of $X$ is already included in the previous two charts.) We easily check that the $S^p_1$-submodule of $S_1$ and the $S^p_2$-submodule of $S_2$ generated respectively by

\[
\frac{x_2}{x_1} \left( \frac{x_2}{x_1} - 2 + x_3 \right) \quad \text{and} \quad \left( \frac{x_1}{x_2} \right)^{p-2} \left( 1 - 2 \frac{x_1}{x_2} + \left( \frac{x_1}{x_2} \right)^3 \right) \]

are equivalent $\mathcal{M} \subset \mathcal{O}_{\mathcal{V}_1}$. In addition, if $\mathcal{L} \subset \mathcal{O}_{\mathcal{V}_1}$ denotes the exceptional ideal of the blowup $V_1 \to V$, then we have the relation $F_{\mathcal{L}} \cdot \mathcal{M} = \mathcal{O}_{\mathcal{V}_1}^p \cdot f$, where $F$ stands for Frobenius.

So far we have illustrated how (or why) hypersurfaces defined by purely inseparable equations and the notion of blowups along suitable equimultiple regular centers relate to $S^q$-submodules and transforms of $S^q$-submodules. This approach appears already in [1] Theorem 5 and also in [11]. There is yet a natural question that arises when viewing (1.3): there is a factorization of the exceptional divisor to the power one, and we may ask if this power is optimal. For example, let us look at the first expression. Can we have an equivalent expression (in the sense that they both differ by an element in $S^p_1$ in which $\frac{x_1}{x_2}$ factors to a higher power? More generally, given a sequence of monoidal transformation, can we characterize an “optimal exceptional monomial” in the previous sense?

These questions are studied within the frame “jumping phenomenon”, concerning the behavior of singularities in positive characteristic. The reader is referred to [22] and [13] for more on this concept. The characterization of the optimal exceptional monomial can be achieved by using logarithmic differential operators with poles along such exceptional monomial. Unfortunately, our work does not enlighten these questions. Our results are related rather to the characterization of an optimal exceptional monomial in which the exponents are multiple of a fixed power $q$ of $p$; see, for instance, Theorem [18]. As from a technical point of view,
we do use logarithmic differential operators to search for invariants that do not get “worst” under suitable blowups; see Theorem 1.20

§1.1  Content of the Paper. We now turn to the more technical part of Introduction, where we present the main results. First, we set the notation. Given a ring \( B \) of characteristic \( p > 0 \), we denote by \( F : B \to B \) the Frobenius endomorphism. Given \( q = p^r \) and an ideal \( I \subset B \), we denote \( F^r I := F^r(I) = \{x^q : x \in I\} \) and \( B^q := F^q B = \{x^q : x \in B\} \), so that \( F^r I \) is an ideal of \( B^q \). For a domain \( B \), we set \( B^{1/q} := \{x \in L : x^q \in B\} \), where \( L \) is an algebraic closure of the fraction field of \( B \). This notation extends also to the setting of sheaves of rings and ideals on schemes of characteristic \( p \). Given a scheme \( V \), when we use the expressions \( \mathcal{O}_V \)-ideal, \( \mathcal{O}_V \)-module, and \( \mathcal{O}_V \)-algebra, we assume that they are quasi-coherent. If \( \mathcal{I} \) is an \( \mathcal{O}_V \)-ideal on \( V \), then \( \mathcal{V}(\mathcal{I}) \subset V \) denotes the support of \( \mathcal{O}_V/\mathcal{I} \).

Let \( V \) be a Noetherian irreducible regular scheme of characteristic \( p > 0 \), and assume that \( V \) is \( F \)-finite, which means that \( \mathcal{O}_V \) is a finite \( \mathcal{O}_V^p \)-module. This ensures that \( V \) is excellent and that the sheaves of (absolute) differential operators \( \text{Diff}^i_{V, q} \), \( i = 0, 1, 2, \ldots \), are locally free of finite rank. We fix \( q = p^r \), a power of \( p \). We consider \( \mathcal{O}_V^p \)-submodules of \( \mathcal{O}_V \). Such a module \( \mathcal{M} \subset \mathcal{O}_V \) has attached a finite surjective radicial morphism \( \delta : X \to V \), namely the one defined by the \( \mathcal{O}_V \)-algebra \( \langle \mathcal{O}_V^q, [\mathcal{M}] \rangle^{1/q} \). The generic rank \( d := [K(X) : K(V)] \) of this morphism is a power of \( p \), which is different from \( q \) in general. We denote by \( F_d(X) \subset X \) the set of points of multiplicity \( d \). Finally, we attach to \( \mathcal{M} \) the collection of \( \mathcal{O}_V \)-ideals

\[
\text{Diff}^i_{V, +}(\mathcal{M}) \sqsubseteq \cdots \sqsubseteq \text{Diff}^{i-1}_{V, +}(\mathcal{M}),
\]

where \( \text{Diff}^i_{V, +} \) denotes the sheaf of differential operators of order \( i \) that annihilates \( 1 \). The next theorem contains results already mentioned in the first part of the introduction in the particular case that \( X \) is a hypersurface defined by a purely inseparable equation.

**Theorem 1.12.** In the above setting, the following holds.

(1) There are equality of sets

\[
\delta(F_d(X)) = \{x \in V : \mathcal{M}_x \subseteq \mathcal{O}_{V, x}^q + m_{V, x}^q = \mathcal{V}(\text{Diff}^{i-1}_{V, +}(\mathcal{M}))\}.
\]

(2) Let \( Y \subset X \) be a closed irreducible subscheme, and let \( Z \subset V \) denote its image. The following statements are equivalent:

(a) \( Y \) is regular and included in \( F_d(X) \).

(b) \( Z \) is regular, and \( \mathcal{M} \subset \mathcal{O}_V^p + \mathcal{I}(Z)^q \).

(3) Assume that the equivalent conditions in (2) are satisfied, and consider the commutative diagram

\[
\begin{array}{c}
Y \subset X \\
\downarrow \delta
\end{array}
\]

obtained in Theorem 1.12(2). Then \( \delta_1 \) is the \( V_1 \)-scheme attached to the \( \mathcal{O}_{V_1}^q \)-module \( \mathcal{M}_1 := (\mathcal{M} \mathcal{O}_{V_1}^q + \mathcal{O}_{V_1}^p : F^q \mathcal{L}_1) \subset \mathcal{O}_{V_1}^q \). Here \( \mathcal{M} \mathcal{O}_{V_1}^q \) denotes the \( \mathcal{O}_{V_1}^q \)-submodule of \( \mathcal{O}_{V_1} \) generated by the sections of \( \mathcal{M} \) when these are viewed as sections on the blowup \( V_1 \), and \( \mathcal{L}_1 \subset \mathcal{O}_{V_1} \) denotes the exceptional ideal of the blowup \( V_1 \)

**Definition 1.13.** Given an \( \mathcal{O}_{V_1}^q \)-submodule \( \mathcal{M} \subset \mathcal{O}_V \), we define \( \text{Sing}(\mathcal{M}, 1) := \mathcal{V}(\text{Diff}^{i-1}_{V_1, +}(\mathcal{M})) \), which is a closed subset of \( V \). A regular closed subscheme \( Z \subset V \) included in \( \text{Sing}(\mathcal{M}, 1) \) is called a permissible center for \( (\mathcal{M}, 1) \). If \( V_1 \) is the blowup of \( V \) along \( Z \) and \( \mathcal{L}_1 \) denotes the exceptional ideal, then we call the \( \mathcal{O}_{V_1}^q \)-module \( (\mathcal{M} \mathcal{O}_{V_1}^q + \mathcal{O}_{V_1}^p : F^q \mathcal{L}_1) \) the 1-transform of \( \mathcal{M} \) by the blowup.

**Remark 1.14.** Roughly speaking, the theorem enables us to replace diagrams like (1.5) by diagrams of the form

\[
\begin{array}{c}
V \leftarrow V_1 \\
\mathcal{M} \quad \mathcal{M}_1
\end{array}
\]
where \( V \leftarrow V_1 \) is a blowup along a permissible center for \((\mathcal{M}, 1)\), and \( \mathcal{M}_1 \) is the 1-transform of \( \mathcal{M} \). Similarly, an iteration of diagrams like (1.5) corresponds to a sequence

\[
\begin{array}{cccc}
V & \leftarrow \pi_1 & V_1 & \leftarrow \pi_2 & \ldots & \leftarrow \pi_r & V_r \\
\mathcal{M} & \leftarrow \mathcal{M}_1 & \ldots & \leftarrow \mathcal{M}_r
\end{array}
\]

where \( V_i \leftarrow V_{i+1} \) is the blowup along a permissible center for \((\mathcal{M}_i, 1)\), and \( \mathcal{M}_{i+1} \) is the 1-transform of \( \mathcal{M}_i \).

We saw already the role played by the \( O_V \)-ideal \( \text{Diff}_{V+}^1(\mathcal{M}) \). We now consider the full collection

\[
\mathcal{G}(\mathcal{M}) := (\text{Diff}_{V+}^1(\mathcal{M}), \ldots, \text{Diff}_{V+}^{q-1}(\mathcal{M})).
\]

This is an example of the following more general concept.

**Definition 1.15.** A \( q \)-differential collection of ideals is a sequence of \( O_V \)-ideals \( \mathcal{G} = (I_1, \ldots, I_{q-1}) \) such that \( \text{Diff}_{V+}^i(I_j) \subseteq I_{i+j} \) whenever \( i + j < q \). Given such a sequence and a point \( x \in V \), we set

\[
\eta_x(\mathcal{G}) := \min\{\nu_x(I_i) + i : 1 \leq i \leq q - 1\},
\]

where \( \nu_x(I_i) \) denotes, as usual, the order of \( I_i \) at \( x \).

In the case where \( \mathcal{G} = \mathcal{G}(\mathcal{M}) \) for an \( O_V \)-submodule \( \mathcal{M} \subseteq O_V \), we simply write \( \eta_x(\mathcal{M}) \) instead of \( \eta_x(\mathcal{G}(\mathcal{M})) \). This number was already introduced in the first part of Introduction in the case that \( \mathcal{M} \) is principal and \( V \) is an affine space. The same argument given there shows that \( \nu_x(\mathcal{G}) \) defines an upper-semicontinuous function on \( V \) with values on \( \mathbb{N} \). It rests on the fact that for an \( O_V \)-ideal \( J \), the function \( x \mapsto \nu_x(J) \) is upper-semicontinuous (see, e.g., [9 Chapter 2] or [10 Chapter 3] for an affine space \( V \)). We include in these notes a self-contained proof within our setting (see Proposition 4.7). The following theorem establishes that, under an appropriate definition of transformation of \( q \)-differential collections, the function \( x \mapsto \eta_x(\mathcal{G}) \) satisfies the so-called **fundamental pointwise inequality**.

**Theorem 1.16.** Let \( \mathcal{G} \) be a \( q \)-differential collection on \( V \); let \( Z \subseteq V \) be a regular center included in the maximum locus of \( \eta(\mathcal{G}) \), say \( \eta_x(\mathcal{G}) = aq + b \) for all \( x \in Z \) (with \( a, b \in \mathbb{N}_0 \) and \( 0 \leq b < q \)), let \( V \leftarrow V_1 \) be the blowup along \( Z \), and let \( \mathcal{L} \) denote the exceptional ideal. Then the collection of \( O_{V_1} \)-ideals

\[
(1.9) \quad \mathcal{G}^{(a)}_1 := (I_1O_{V_1} : \mathcal{L}^{qa}), \ldots, (I_{q-1}O_{V_1} : \mathcal{L}^{qa})
\]

is also \( q \)-differential, and the following pointwise inequality holds:

\[
(1.10) \quad \eta_{x_1}(\mathcal{G}^{(a)}_1) \geq \eta_{x_1}(\mathcal{G}^{(a)}_1), \quad \forall x_1 \in V_1.
\]

The collection of \( O_{V_1} \)-ideals \( \mathcal{G}^{(a)}_1 \) will be called the \( a \)-transform of \( \mathcal{G} \) by the blowup. The exponent \( qa \) of \( \mathcal{L} \) appearing in its definition is somehow related with a natural factorization in the definition of transformation of modules, as we will further see. To get a flavor of this theorem, it is worth bearing in mind the following classical example. Let \( J \) be a nonzero \( O_V \)-ideal, and let \( Z \subseteq V \) be a closed regular subscheme included in the maximum locus of the function \( x \mapsto \nu_x(J) \), say \( \nu_x(J) = b \), for all \( x \in Z \). Let \( V \leftarrow V_1 \) be the blowup of \( V \) along \( Z \), and let \( H_1 \subseteq V_1 \) denote the exceptional hypersurface. Then there is a factorization \( JO_{V_1} = I(H_1)^bJ_1 \) for some \( O_{V_1} \)-ideal \( J_1 \) that does not vanish along \( H_1 \). Then we have the following pointwise inequality.

**Theorem 1.17.** \( \nu_{x_1}(J) \geq \nu_{x_1}(J_1) \) for all \( x_1 \in V_1 \).

We include a self-contained proof of this theorem in Proposition 4.15.

We return to the consideration of \( O_V \)-submodules \( \mathcal{M} \subseteq O_V \). Part (1) of the following theorem was already mentioned in Proposition 1.7 in a particular case. Part (2) is somehow related with the outcome of Proposition 1.9.

**Theorem 1.18.** For an \( O_V \)-module \( \mathcal{M} \subseteq O_V \), the following holds.

(1) \( \text{Sing}(\mathcal{M}, 1) = \{ x \in V : \eta_x(\mathcal{M}) \geq q \} \).
(2) Assume that \( \operatorname{Sing}(\mathcal{M}, 1) \) is nonempty, and write the maximum possible value of \( \eta_x(\mathcal{M}) \) in the form \( aq + b \) with \( 0 \leq b < q \). Then a regular center \( Z \subset V \) included in the maximum locus of \( x \mapsto \eta_x(\mathcal{M}) \) defines a sequence

\[
(1.11) \quad V \stackrel{\pi}{\longrightarrow} V_1 \quad \overset{=}{{\sim}} \ldots \overset{=}{{\sim}} V_1
\]

where
(a) \( V \leftarrow V_1 \) is the blowup along \( Z \), and \( \mathcal{M}_1(1) \) is the 1-transform of \( \mathcal{M} \) by \( \pi \).
(b) For each index \( 1 \leq i < a \), the exceptional divisor of \( \pi \), say \( H_1 \subset V_1 \), is included in \( \operatorname{Sing}(\mathcal{M}_1, 1) \), the isomorphism \( V_i \leftarrow V_{i+1} \) is the blowup of \( V_i \) along \( H_i \), and \( \mathcal{M}_1^{(i+1)} \) is the 1-transform of \( \mathcal{M}_1(i) \).
(c) \( H_1 \nsubseteq \operatorname{Sing}(V_1, \mathcal{M}_1(1)) \).

(3) If \( (\mathcal{G}(\mathcal{M}))_1^{(a)} \) denotes the a-transform of \( \mathcal{G}(\mathcal{M}) \), then there is a componentwise inclusion

\[
(1.12) \quad \mathcal{G}(\mathcal{M}_1^{(a)}) \subseteq (\mathcal{G}(\mathcal{M}))_1^{(a)}.
\]

In particular, \( \eta_{x_1}(\mathcal{M}_1^{(a)}) \geq \eta_{x_1}(\mathcal{G}(\mathcal{M}))_1^{(a)} \) for all \( x_1 \in V_1 \), whence

\[
(1.13) \quad \{ x_1 \in V_1 : \eta_{x_1}(\mathcal{G}(\mathcal{M}))_1^{(a)} \geq q \} \subseteq \operatorname{Sing}(\mathcal{M}_1^{(a)}, 1).
\]

We call \( \mathcal{M}_1^{(a)} \) the a-transform of \( \mathcal{M} \) by the blowup.

Remark 1.19. The function \( x \mapsto \eta_x(\mathcal{M}) \) does not satisfy a pointwise inequality in general: The inclusion \((1.12)\) is in general strict, and therefore we cannot make use of Theorem \((1.10)\) to deduce a pointwise inequality \( \eta_{x_1}(\mathcal{M}) \geq \eta_{x_1}(\mathcal{G}(\mathcal{M}))_1^{(a)} \) for \( x_1 \in V_1 \), which would be desirable. For example, take \( p = q = 3 \), \( V = \operatorname{Spec}(k[x_1, x_2, x_3, x_4]) \), and \( \mathcal{M} = \mathcal{O}_V^1 : x_1^2 x_2^3 x_3 x_4 \). The maximum of \( \eta_x(\mathcal{M}) \) is 5 = 3 \cdot 1 + 2, whence \( a = 1 \), and this maximum is attained only at the origin. However, after blowing up the origin, the restriction to the \( x_1 \)-chart of the 1-transform \( \mathcal{M}_1^{(1)} \) of \( \mathcal{M} \) is generated by \( x_1^2 x_2^3 x_3 x_4^2 \), where \( x_1 = \frac{x_2}{x_3} \). Thus the maximum of \( \eta_x(\mathcal{M}_1^{(1)}) \) is \( \geq 6 = 3 \cdot 2 \). This example also shows that even \( \left[ \frac{\eta_x(\mathcal{M})}{q} \right] \) does not satisfy the pointwise inequality. Nevertheless, \((1.13)\) implies that a sequence of permissible transformations of \( q \)-differential collections starting from \( \mathcal{G}(\mathcal{M}), \) in a way that we will specify, induces a sequence of permissible transformations of \( \mathcal{M} \). This will be discussed in Remark \((5.11)\).

We end this presentation by illustrating how \( q \)-differential collections define invariants of singularities satisfying the pointwise inequality in a different manner. Given a sequence of monoidal transformations, it is natural to define invariants that take into account the hypersurfaces introduced in the previous blowups. Following this line, we consider 3-tuples \((\mathcal{M}, \Lambda, \mathcal{L})\), where \( \mathcal{M} \subset \mathcal{O}_V \) is an \( \mathcal{O}_V^1 \)-module on an \( F \)-finite regular scheme \( V \), \( \Lambda \) is a finite collection of hypersurfaces with only normal crossings, and \( \mathcal{L} \) is an invertible \( \mathcal{O}_V \)-ideal included in the ideal \( \mathcal{I}(H) \) for each \( H \in \Lambda \). For example, we could take for \( \Lambda \) the empty collection and \( \mathcal{L} = \mathcal{O}_V \). We attach to such 3-tuple the collection of \( \mathcal{O}_V \)-ideals

\[
\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) := ((\operatorname{Diff}^{1}_{V, \Lambda+, +}(\mathcal{M}) : \mathcal{L}^1), (\operatorname{Diff}^{2}_{V, \Lambda+, +}(\mathcal{M}) : \mathcal{L}^2), \ldots, (\operatorname{Diff}^{q-1}_{V, \Lambda+, +}(\mathcal{M}) : \mathcal{L}^{q-1})),
\]

where \( \operatorname{Diff}^{k}_{V, \Lambda+, +} \subseteq \operatorname{Diff}^{k}_{V, +} \) denotes the subsheaf consisting of those differential operators that are logarithmic with respect to \( \mathcal{I}(H) \) for all \( H \in \Lambda \); see Section \( \S \).

Theorem 1.20. Within the previous setting, the following properties hold:
(1) \( \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) \) is a \( q \)-differential collection, and there is a componentwise inclusion \( \mathcal{G}(\mathcal{M}) \subseteq \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) \).

In particular, \( \eta_x(\mathcal{M}) \geq \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \) for \( x \in V \), whence

\[
\{ x \in V : \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq q \} \subseteq \operatorname{Sing}(\mathcal{M}, 1).
\]

(2) Let \( Z \subset V \) be a regular center included in the maximum locus of the function \( x \mapsto \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \), say \( \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) = aq + b \) for all \( x \in Z \) \( (a, b \in \mathbb{N}_0, 0 \leq b < q) \), and suppose that \( \Lambda \) has only normal crossings with \( Z \). Let \( V \leftarrow V_1 \) be the blowup along \( Z \), and let \( H_1 \subset V_1 \) be the exceptional divisor. We set:
(a) \( \mathcal{M}_1^{(a)} := \) the a-transform of \( \mathcal{M} \) by the blowup.
(b) $\Lambda_1 := \text{the collection of the strict transforms of each } H \in \Lambda$, plus the exceptional divisor $H_1$.
(c) $L_1 := (\mathcal{O}_V)(H_1)$.

Then $\Lambda_1$ is a collection of hypersurfaces with only normal crossings, $L_1$ is included in $\mathcal{I}(H)$ for each $H \in \Lambda_1$, so that $\mathcal{G}(\mathcal{M}_1, \Lambda_1, L_1)$ is a $q$-differential collection on $V_1$, and the following pointwise inequality holds:

$$\eta_{\tau(x)}(\mathcal{G}(\mathcal{M}, \Lambda, L)) \geq \eta_{\tau_1}(\mathcal{G}(\mathcal{M}_1, \Lambda_1, L_1)), \quad x_1 \in V_1.$$ 

We refer to Remark 5.29 and Example 5.27 for a discussion on how this theorem can be used to define sequences of permissible transformations over $(V, \mathcal{M})$.

§1.2 On the Organization of the Paper. In the first part of Section 2 we introduce definitions and some basic constructions related to $\mathcal{O}_V^q$-modules and their transformations under blowups. In the second part, we discuss the interplay between $\mathcal{O}_V^q$-modules and their associated $V$-schemes. In particular, we will prove Theorem 1.12 except for the second equality in (1).

In Section 3 we revise some properties about $p$-basis and differential operators of $F$-finite regular local rings and apply them to the introduction of local invariants associated with $\mathcal{O}_V^q$-modules, such as the function $x \rightarrow \eta_q(\mathcal{M})$, which is related to the notion of Hironaka’s slope discussed in [7]. The application of this function to the study of $\mathcal{O}_V^q$-modules and transformations of $\mathcal{O}_V^q$-modules will be treated in Section 4. At this point, we can prove the second equality in Theorem 1.12 (1), and also part (1) and (2) of Theorem 1.18.

Finally, in Section 5, we study $q$-differential collections (Definition 1.15). In the first part, we prove Theorem 1.16 and part (3) of Theorem 1.18. In the final part, we delve into the study of invariants of singularities by using logarithmic differential operators. The main objective of the section is the proof of Theorem 1.20.

The presentation of the paper is not limited to the proof of the results presented in this introduction. Instead, we choose a detailed exposition that, we hope, will serve as a reference for future works. We also tried to keep the paper as self-contained as possible. In particular, we include proofs of the results stated in Theorem 1.14 adapted to our setting.

§1.3 Historical Remarks. Our discussion makes systematic use of differential operators and logarithmic differential operators. So it relates strongly with some published works. Let us first mention [11], where Giraud studies Jung conditions for finite radical coverings of regular varieties in positive characteristic. The outcome of [5] is a first breakthrough in the resolution of singularities of radical extensions in positive characteristic. Cossart and Piltant [6] proved resolution of singularities for arbitrary three-dimensional schemes of positive characteristic, and more recently for arithmetical threefolds [7]. These are proofs that introduce suitable invariants satisfying the pointwise inequality. Other invariants in positive characteristic arise in the work of Kawanoue and Matsuki in [17] and [18]. Also in this line, and related with our exposition, is the joint work of the second author with Benito [2] (see also [3]). All these cited papers also make use of either of differential operators or of logarithmic differential operators.

§2 $\mathcal{O}_V^q$-Modules and Transformations by Blowups

Throughout this section, $V$ denotes a fixed irreducible $F$-finite regular scheme of characteristic $p > 0$, and $q = p^e$ denotes a fixed power of $p$. In Section 2.1 we present an exposition on $\mathcal{O}_V^q$-submodules of $\mathcal{O}_V$ with a view toward their associated finite radical morphisms. We will introduce two notions of equivalence of $\mathcal{O}_V^q$-modules and discuss different notions of transformations of modules by blowups. In Section 2.2 we discuss the interplay between $\mathcal{O}_V^q$-submodules of $\mathcal{O}_V$ and finite radical morphisms $\delta : X \rightarrow V$, in connection with blowups. In particular, we prove Theorem 1.12 except for the second equality in (1).

§2.1 Basic Constructions. An $\mathcal{O}_V^q$-submodule $\mathcal{M} \subset \mathcal{O}_V$ has attached a finite radical morphism $\delta : X \rightarrow V$. This is the $V$-scheme defined by the $\mathcal{O}_V$-algebra $(\mathcal{O}_V^{\frac{1}{q}}[\mathcal{M}])^{1/q} \subset \mathcal{O}_V^{1/q}$, where $\mathcal{O}_V^{\frac{1}{q}}[\mathcal{M}] \subset \mathcal{O}_V$ denotes the $\mathcal{O}_V^q$-subalgebra of $\mathcal{O}_V$ spanned by $\mathcal{M}$. Two $\mathcal{O}_V^q$-submodules that span the same $\mathcal{O}_V^q$-subalgebra of $\mathcal{O}_V$ have attached the same finite radical morphism, so they should be considered as equivalent. We call this relation of submodules weak equivalence. We also introduce a stronger relation, which is more suitable for most of the constructions that we will introduce.

Definition 2.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathcal{O}_V^q$-submodules of $\mathcal{O}_V$.
(1) $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be equivalent if $\mathcal{M}_1 + \mathcal{O}_V^q = \mathcal{M}_2 + \mathcal{O}_V^q$. In notation, $\mathcal{M}_1 \sim \mathcal{M}_2$. 

(2) \(M_1\) and \(M_2\) are said to be weakly equivalent if \(O_V^q[M_1] = O_V^q[M_2]\).

(3) \(M_1\) is said to be trivial if \(M \sim 0\).

Here the sum of two submodules of \(O_V\) is the smallest submodule containing both. The relations introduced in (1) and (2) are clearly equivalence relations. Note that \(O_V^q + \mathcal{M}\) is the largest \(O_V^q\)-submodule in its equivalence class; in particular, each equivalence class has a canonical representative. Since \(O_V^q[M] = O_V^q[M + O_V^q]\), equivalent modules are weakly equivalent. The converse is not true: take \(V = \text{Spec}(\mathbb{F}_p[t])\) and \(M\) the \(O_V^q\)-module spanned by \(t\). Note that \(t^2 \in \Gamma(V, O_V^q[M])\); however, it is not a global section of \(M\). Finally, \(M \sim 0\) if and only if \(M \subseteq O_V^q\).

2.2. One important operation on \(O_V^q\)-submodules of \(O_V\) is the extension defined by taking conductors with respect to a hypersurface in a sense we specify now. Let \(L\) be an invertible ideal of \(O_V\). Then \(F^eL\) is an invertible ideal of \(O_V^q\), and we have \(L^q = F^eL \cdot O_V\). We deduce from this equality that all the \(O_V^q\)-submodules of \(L^q = F^eL \cdot O_V\) are of the form \((F^eL)N\) for some \(O_V^q\)-module \(N \subseteq O_V\).

Given an \(O_V^q\)-module \(\mathcal{M} \subseteq O_V\), the conductor \((\mathcal{M} : F^eL) \subseteq O_V\) (defined in the class of \(O_V^q\)-submodules of \(O_V\)) is, by definition, the largest \(O_V^q\)-submodule \(N \subseteq O_V\) such that \((F^eL)N \subseteq \mathcal{M}\). Since \((F^eL)N \subseteq L^q\), it is clear that \((\mathcal{M} : F^eL) = (\mathcal{M} \cap L^q : F^eL)\). As \(\mathcal{M} \cap L^q\) is a submodule of \(L^q\), the discussion in the previous paragraph shows that \(\mathcal{M} \cap L^q = (F^eL)N\) for an \(O_V^q\)-submodule \(N\), and this submodule has to be the conductor \((\mathcal{M} : F^eL)\). Thus

\[
\mathcal{M} \cap L^q = (F^eL)(\mathcal{M} : F^eL).
\]

We remark that even though the assignment \(\mathcal{M} \mapsto (\mathcal{M} : F^eL)\) is well-defined in the class of \(O_V^q\)-submodules of \(O_V\), it is not compatible with any of our notions of equivalence. For instance, take \(V = \text{Spec}(\mathbb{F}_p[s, t])\), \(\mathcal{M} = O_V^q \cdot st^q\), \(\mathcal{M}' = O_V^q \cdot (st^q + 1)\) and \(L = O_V \cdot t\). Note that \(\mathcal{M} \sim \mathcal{M}'\), yet \((\mathcal{M} : F^eL) = O_V^q\cdot s\) and \((\mathcal{M} : F^eL) = O_V^q\cdot (st^q + 1)\) are neither equivalent nor weakly equivalent. This presents a difficulty as we will consider \(O_V^q\)-modules only up to equivalence (and up to weak equivalence). One way to overcome this difficulty is by making use of the canonical representative in each equivalence class.

**Definition 2.3.** Let \(\mathcal{M} \subseteq O_V\) be an \(O_V^q\)-submodule, and let \(L \subseteq O_V\) be an invertible ideal. The \(L\)-conductor of \(\mathcal{M}\) is the \(O_V^q\)-module

\[
\mathcal{M}_L := ((O_V^q + \mathcal{M}) : F^eL) \subseteq O_V.
\]

It follows from the definition that \(N_L = \mathcal{M}_L\) if \(N \sim \mathcal{M}\), that is, the definition of \(L\)-conductor is compatible with the notion of equivalence. We gather more properties in the next lemma.

**Lemma 2.4.** Let \(\mathcal{M} \subseteq O_V\) be an \(O_V^q\)-module, and let \(L \subseteq O_V\) be an invertible ideal. Then the following statements hold:

1. \(\mathcal{M}_L = (\mathcal{M} : F^eL)\mathcal{M}_L\).
2. \(\mathcal{M}_L = \mathcal{M}_L + O_V^q\).
3. If \(L' \subseteq O_V\) is a second invertible ideal, then \(\mathcal{M}_{L'} = (\mathcal{M}_L : L')\).

**Proof.** (1) follows from \((2.1)\) when applied to the submodule \(O_V^q + \mathcal{M}\). As for (2), we have \((F^eL)O_V^q \subseteq O_V^q \subseteq O_V^q + \mathcal{M}\), so that \(O_V^q \subseteq \mathcal{M}_L\), and hence \(\mathcal{M}_L = O_V^q + \mathcal{M}_L\). Note finally that

\[
\mathcal{M}_{L_{L'}} = (O_V^q + \mathcal{M} : (F^eL')(F^eL')) = ((O_V^q + \mathcal{M} : F^eL') : F^eL') = (\mathcal{M}_L : F^eL') = (O_V^q + \mathcal{M} : F^eL') = (\mathcal{M}_L : L'),
\]

which proves (3). \(\square\)

Our definition of \(L\)-conductor is not compatible with the notion of weak equivalence in general. For instance, in the following example, we have a strict inclusion \(O_V^q[\mathcal{M}_L] \subset (O_V^q[M])_L\), which implies that the weakly equivalent modules \(\mathcal{M}\) and \(O_V^q[\mathcal{M}]\) have \(L\)-conductors that are not weakly equivalent. Take \(V = \text{Spec}(\mathbb{F}_p[[t]])\), \(\mathcal{M} = O_V^q \cdot t^{q-1}\) and \(L = O_V \cdot t\). Note that \(\mathcal{M}_L = \mathcal{M}\) and \(\Gamma(V, O_V^q[\mathcal{M}]) = \mathbb{F}_p[t^{q-1}, t^q]\). On the other hand, \(\Gamma(V, O_V^q[M])_L\) contains \(t^{q-2}\) since \(t^q t^{q-2} = (t^{q-1})^2\).

In Lemma 2.6, we introduce conditions under which the compatibility with the weak equivalence holds. We first study, in Lemma 2.5, the condition \(\mathcal{M} \subseteq O_V^q + I\) for a general \(O_V\)-ideal \(I\). Note that \(O_V^q + I\) is an \(O_V^q\)-subalgebra, namely, \(O_V^q + I = O_V^q[I]\).
Lemma 2.5. Given an $O^q_V$-module $M \subseteq O$ and an $O_V$-ideal $I$, the following conditions are equivalent.

1. $M \subseteq O^q_V + I$.
2. $O^q_V[M] \subseteq O^q_V + I$.
3. $M \sim ((O^q_V + M) \cap I)$.
4. $M \sim M'$ for some $O^q_V$-module $M'$ included in $I$.

Proof. The equivalence (1) $\iff$ (2) is trivial. For the implication (1) $\Rightarrow$ (3), observe that (1) implies that $O^q_V + M \subseteq O^q_V + I$, and hence

$$M \sim O^q_V + M = (O^q_V + M) \cap (O^q_V + I) = O^q_V + ((O^q_V + M) \cap I) \sim ((O^q_V + M) \cap I).$$

Finally, the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are clear.

Lemma 2.6. Let $M \subseteq O$ be an $O^q_V$-submodule, and let $L \subseteq O_V$ be an invertible ideal. Assume that $M \subseteq O^q_V + L^q$. Then

1. $M \sim (F^q L).M_L$.
2. Let $M \subseteq O_V$ be another $O^q_V$-module, and assume that $M$ and $M'$ are weakly equivalent. Then $M' \subseteq O^q_V + L^q$, and the $O^q_V$-modules $M_L$ and $M'_L$ are weakly equivalent.

Proof. By Lemma 2.5, the condition $M \subseteq O^q_V + L^q$ implies that $M \sim ((O^q_V + M) \cap L^q)$, and the latter is $(F^q L).M_L$ according to Lemma 2.4(1). This proves (1).

Before proving (2), as a preliminary step, we show that the condition $M \subseteq O^q_V + L^q$ implies that

$$O^q_V[M_L] = O^q_V[(O^q_V[M])_L],$$

and then we will show that (2) easily follows from this equality.

Clearly, $M_L \subseteq (O^q_V[M])_L$, and hence $O^q_V[M_L] \subseteq O^q_V[(O^q_V[M])_L]$. As for the reverse inclusion, we only need to prove that

$$(O^q_V[M])_L \subseteq O^q_V[M_L].$$

Since $F^q L \subseteq O^q_V$ is an invertible ideal, the previous inclusion is equivalent to

$$(F^q L)(O^q_V[M])_L \subseteq (F^q L)O^q_V[M_L].$$

To prove this last inclusion, note that, on the other hand, Lemma 2.4(1), applied to $O^q_V[M] = O^q_V + O^q_V[M]$ and $L$, says that

$$O^q_V[M] \cap L^q = (F^q L)(O^q_V[M])_L.$$  \hspace{1cm} (2.3)

On the other hand, by statement (1), $M \sim (F^q L).M_L$, whence they are also weakly equivalent, and

$$O^q_V[M] = O^q_V[(F^q L).M_L]$$

$$= O^q_V + \sum_{n=1}^{\infty} (F^q L).M_L)^n$$

$$\subseteq O^q_V + \sum_{n=1}^{\infty} (F^q L).M_L)^n.$$  \hspace{1cm} (2.4)

We finally claim that

$$(F^q L)(O^q_V[M])_L = O^q_V[M] \cap L^q$$

$$\subseteq \left( O^q_V + \sum_{n=1}^{\infty} (F^q L).M_L)^n \right) \cap L^q$$

$$= (O^q_V \cap L^q) + \sum_{n=1}^{\infty} (F^q L).M_L)^n$$

$$= (F^q L)O^q_V[M_L].$$

In fact, the first equality is (2.3), and the first inclusion follows from (2.4). As for the second equality, we just need to note that $\sum_{n=1}^{\infty} (F^q L).M_L)^n$ is already included in $L^q$. Finally, the last equality is clear since
\(\mathcal{O}_V^q \cap \mathcal{L} = F^* \mathcal{L}\). This completes the proof of (2.2). We now turn to the proof of (2). Notice first that 
\[ \mathcal{M} \subseteq \mathcal{O}_V^q[\mathcal{M}] \subseteq \mathcal{O}_V^q + L^q, \]
whence equality \(2.2\) holds for \(\mathcal{M}\) as well. Therefore
\[ \mathcal{O}_V^q[\mathcal{M}] = \mathcal{O}_V^q[(\mathcal{O}_V^q[\mathcal{M}])^\ell] = \mathcal{O}_V^q[(\mathcal{O}_V^q[\mathcal{M}])^\ell] = \mathcal{O}_V^q[\mathcal{M}], \]
which completes the proof of (2). \(\square\)

2.7. We now discuss the notion of pull-back of \(\mathcal{O}_V^q\)-submodules of \(\mathcal{O}_V\) by morphisms. Let \(V_1\) be a second irreducible \(F\)-finite regular scheme, and let \(\pi : V_1 \to V\) be a morphism. As usual, we write \(\pi^q : \pi^{-1}\mathcal{O}_V \to \mathcal{O}_{V_1}\) for the underlying morphism of sheaves. Clearly,
\[ \pi^{-1}\mathcal{O}_V^q = (\pi^{-1}\mathcal{O}_V)^q, \]
and hence by restriction of \(\pi^q\) we obtain a commutative diagram

\[
\begin{array}{ccc}
\pi^{-1}\mathcal{O}_V & \xrightarrow{\pi^q} & \mathcal{O}_{V_1} \\
\downarrow & & \downarrow \\
\pi^{-1}\mathcal{O}_V^q & \xrightarrow{\pi^q} & \mathcal{O}_{V_1}^q
\end{array}
\]

where the vertical arrows are inclusions. This commutative diagram yields a morphism of \(\mathcal{O}_V^q\)-algebras
\[
(2.5) \quad \Phi : \pi^{-1}\mathcal{O}_V \otimes_{\pi^{-1}\mathcal{O}_V^q} \mathcal{O}_{V_1}^q \to \mathcal{O}_{V_1}.
\]

**Definition 2.8.** Within the previous setting, if \(\mathcal{M}\) is an \(\mathcal{O}_V^q\)-submodule of \(\mathcal{O}_V\), then we denote by \(\mathcal{M}\mathcal{O}_{V_1}^q\) the image of the composition
\[
\pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_V^q} \mathcal{O}_{V_1}^q \quad \overset{t \otimes \text{id}}{\longrightarrow} \quad \pi^{-1}\mathcal{O}_V \otimes_{\pi^{-1}\mathcal{O}_V^q} \mathcal{O}_{V_1}^q \quad \overset{\Phi}{\longrightarrow} \quad \mathcal{O}_{V_1},
\]
where \(t\) denotes the inclusion \(\pi^{-1}\mathcal{M} \subseteq \pi^{-1}\mathcal{O}_V\).

We readily check that for any \(x_1 \in V\),
\[
(2.6) \quad (\mathcal{M}\mathcal{O}_{V_1}^q)_{x_1} = \mathcal{M}_{\pi(x_1)}\mathcal{O}_{V_1, x_1}^q,
\]
where the term on the right is the \(\mathcal{O}_{V_1, x_1}^q\)-submodule of \(\mathcal{O}_{V_1, x_1}\) spanned by the image of the composition map \(\mathcal{M}_{\pi(x_1)} \hookrightarrow \mathcal{O}_{V, \pi(x_1)} \xrightarrow{\pi_{V, x_1}^q} \mathcal{O}_{V_1, x_1}\). The following proposition is a direct consequence of this equality.

**Proposition 2.9.** Fix a morphism \(\pi : V_1 \to V\) of \(F\)-finite regular schemes.

1. If \(\mathcal{B} \subseteq \mathcal{O}_V\) is an \(\mathcal{O}_V^q\)-subalgebra, then \(\mathcal{B}\mathcal{O}_{V_1}^q \subseteq \mathcal{O}_{V_1}\) is an \(\mathcal{O}_{V_1}^q\)-subalgebra.
2. If \(\mathcal{M}, \mathcal{M}' \subseteq \mathcal{O}_V\) are equivalent (resp., weakly equivalent) \(\mathcal{O}_V^q\)-modules, then \(\mathcal{M}\mathcal{O}_{V_1}^q\) and \(\mathcal{M}'\mathcal{O}_{V_1}^q\) are equivalent (resp., weakly equivalent) \(\mathcal{O}_{V_1}^q\)-modules.

We now formulate different notions of transformation of modules when we blow up along a regular center.

**Definition 2.10.** Let \(Z \subseteq V\) be an irreducible regular subscheme, and let \(V \leftarrow Z \to V_1\) be the blowup of \(V\) along \(Z\), where \(H_1\) denotes the exceptional hypersurface. Let \(\mathcal{M} \subseteq \mathcal{O}_V\) be an \(\mathcal{O}_V^q\)-submodule.

1. The **total transform** of \(\mathcal{M}\) by the blowup is the \(\mathcal{O}_{V_1}^q\)-submodule \(\mathcal{M}\mathcal{O}_{V_1}^q \subseteq \mathcal{O}_{V_1}\).
2. For each positive integer \(a\), the **a-transform** of \(\mathcal{M}\) by the blowup is the \(I(H_1)^a\)-conductor of \(\mathcal{M}\mathcal{O}_{V_1}^q \subseteq \mathcal{O}_{V_1}\) (see Definition 2.3(1)). We denote it \(\mathcal{M}^{(a)}_1\).

According to Proposition 2.9, if \(\mathcal{M}\) and \(\mathcal{N}\) are equivalent \(\mathcal{O}_V^q\)-modules, then \(\mathcal{M}\mathcal{O}_{V_1}^q\) and \(\mathcal{N}\mathcal{O}_{V_1}^q\) are equivalent \(\mathcal{O}_{V_1}^q\)-modules; therefore \(\mathcal{M}\) and \(\mathcal{N}\) have equivalent a-transforms for all \(a \geq 1\).

The notion of a-transform of an \(\mathcal{O}_V^q\)-module \(\mathcal{M} \subseteq \mathcal{O}_V\) will be of particular interest when \(\mathcal{M}\) is equivalent to an \(\mathcal{O}_V^q\)-module included in \(I(Z)^a\). In that case, if \(X \to V\) denotes the \(V\)-scheme attached to \(\mathcal{M}\) (i.e., the one defined by \(\mathcal{O}_V \cap (\mathcal{O}_V^q[\mathcal{M}])^{1/q}\)), then we will further see that the \(V_1\)-scheme attached to the a-transform of \(\mathcal{M}\) will be obtained from \(\delta\) by an iteration of diagrams like 1.5.

**Definition 2.11.** For an \(\mathcal{O}_V^q\)-module \(\mathcal{M} \subseteq \mathcal{O}_V\) and a positive integer \(a\), we set
\[
\text{Sing}(\mathcal{M}, a) := \{x \in V : \mathcal{M}_x \subseteq \mathcal{O}_{V_1,x}^q + m_{V_1,x}^a\}.
\]
An irreducible regular closed subscheme \(Z \subseteq V\) is called a permissible center for \((\mathcal{M}, a)\) if the following equivalent conditions hold:
Therefore, given \( \delta \), the equivalence of (1) and (2) follows from Lemma 2.6 and they clearly imply (3). The converse will be proved in Proposition 4.11 using \( p \)-basis.

Remark 2.12. We will see in Proposition 4.19 that each \( \text{Sing}(\mathcal{M}, a) \subset V \) is a closed subset.

Proposition 2.13. Assume that \( Z \subset V \) is permissible for \((\mathcal{M}, a)\), and let \( V \leftarrow V_1 \supset H_1 \) be the blowup along \( Z \), where \( H_1 \) denotes the exceptional hypersurface. Then \( H_1 \) is permissible for \((\mathcal{M}O_{V_1}^{q}, a)\), and we have \( \mathcal{M}O_{V_1}^{q} \sim F'((\mathcal{I}(H_1))^{a})M_1^{(a)}, \) where \( M_1^{(a)} \) is the \( a \)-transform of \( \mathcal{M} \) (Definition 2.11(2)).

Let \( \mathcal{N} \subset O_V \) be another \( O_V^{q} \)-submodule of \( O_V \) that is weakly equivalent to \( \mathcal{M} \) (i.e., \( O_{V}^{q}[\mathcal{M}] = O_{V}^{q}[\mathcal{N}] \)). Then \( Z \) is also permissible for \((\mathcal{N}, a)\), and we have \( O_{V_1}^{q}[\mathcal{M}^{(a)}] = O_{V_1}^{q}[\mathcal{N}^{(a)}] \).

Proof. By the definition of \( Z \) to be permissible for \((\mathcal{M}, a)\), we have \( \mathcal{M} \sim \mathcal{M}' \) for some \( \mathcal{M}' \subset O_V^{q} \). The last inclusion implies that \( \mathcal{M}'O_{V_1}^{q} \subset \mathcal{M}O_{V_1}^{q} \). Since \( \mathcal{M}O_{V_1}^{q} \sim \mathcal{M}'O_{V_1}^{q} \) (Proposition 2.19), we deduce that \( H_1 \) is permissible for \((\mathcal{M}O_{V_1}^{q}, a)\). This enables us to apply Lemma 2.9 to \( \mathcal{M}O_{V_1}^{q} \) and \( \mathcal{I}(H_1)^{a} \) and conclude that \( \mathcal{M}O_{V_1}^{q} \sim F'((\mathcal{I}(H_1))^{a})M_1^{(a)} \). This proves the first part of the proposition.

Assume now that \( O_{V_1}^{q}[\mathcal{M}] = O_{V_1}^{q}[\mathcal{N}] \) for a second \( O_V^{q} \)-module \( \mathcal{N}' \subset O_V \). Then \( \mathcal{N}' \subset O_{V_1}^{q}[\mathcal{M}'] \subset O_{V_1}^{q} + \mathcal{I}(Z)^{qg} \), which implies that \( Z \) is also permissible for \((\mathcal{N}', a)\). On the other hand, \( O_{V_1}^{q}[\mathcal{M}'O_{V_1}^{q}] = O_{V_1}^{q}[\mathcal{N}'O_{V_1}^{q}] \) by Proposition 2.15. Now the first part tells us that \( H_1 \) is permissible for \((\mathcal{M}'O_{V_1}^{q}, a)\), which means that \( \mathcal{M}'O_{V_1}^{q} \subset O_{V_1}^{q} + \mathcal{I}(H_1)^{qg} \). Therefore we can apply Lemma 2.9 to \( \mathcal{M} := \mathcal{M}O_{V_1}^{q}, \quad \mathcal{L} := \mathcal{I}(H_1)^{a}, \quad \mathcal{M} := \mathcal{M}O_{V_1}^{q} \). The conclusion is that \( O_{V_1}^{q}[\mathcal{M}^{(a)}] = O_{V_1}^{q}[\mathcal{N}^{(a)}] \). This completes the proof of the proposition.

§2.2 Relation with Finite Radicial Morphisms. In this second part of the section, we discuss the interplay between \( O_V^{q} \)-submodules \( \mathcal{M} \subset O_V \) and certain class of finite radicial morphisms \( \delta : X \to V \). We begin with some remarks about this class of morphisms.

2.14. We denote by \( \mathcal{C}_q(V) \) the class of all finite surjective morphisms \( \delta : X \to V \), where \( X \) is an integral scheme, and there is an inclusion \( \delta_*O_X^{q} \subset O_V \). We can view \( \mathcal{C}_q(V) \) as a full subcategory of the category of \( V \)-schemes. In this regard, the assignment \( (X, \delta) \to (\delta_*O_X) \) defines an antiequivalence between \( \mathcal{C}_q(V) \) and the category of \( O_V \)-subalgebras of \( O_{V}^{q} \). Here \( O_V \) is seen as embedded in the function field \( K(V) \) of \( V \), and \( O_V^{q} \) is the \( O_V \)-subalgebra of \( K(V) \) (a fixed algebraic closure of \( K(V) \)) whose sections on an open subset \( U \subset V \) are given by \( \Gamma(U, O_{V}^{q}) = \{ s \in K(V) : s^q \in \Gamma(U, O_V) \} \). The following lemma expresses a characteristic property of radial morphisms.

Lemma 2.15. Given \( \delta : X \to V \) in \( \mathcal{C}_q(V) \) and a reduced \( V \)-scheme \( Y \), there is at most one \( V \)-morphism \( f : Y \to X \).

Proof. We may assume that both \( V \) and \( X \) are affine, say \( X = \text{Spec}(B) \) and \( V = \text{Spec}(S) \), so \( B^q \subset S \subset B \). We have to prove that there is at most one \( S \)-homomorphism \( \varphi : B \to \Gamma(Y, O_V) \). Now for any two such homomorphisms, say \( \varphi \) and \( \varphi' \), we have that for any \( b \in B \), \( (\varphi(b))^q = \varphi'(b^q) = \varphi'(b^q) \), whence \( \varphi(b) = \varphi'(b) \) since \( \Gamma(Y, O_V) \) is reduced.

Corollary 2.16. The assignment \( (X, \delta) \to \delta_*O_X^{q} \) defines an antiequivalence between \( \mathcal{C}_q(V) \) and the category of \( O_V^{q} \)-subalgebras of \( O_V \). The only possible morphisms in the latter category are inclusions. Therefore, given \( \delta : X \to V \) with associated \( O_V^{q} \)-subalgebra \( \mathcal{B} \subset O_V \) and \( \delta' : X' \to V \) with \( O_V^{q} \)-subalgebra \( \mathcal{B}' \subset O_V \), there exists a \( V \)-morphism \( X' \to X \) if and only if \( \mathcal{B} \subset \mathcal{B}' \), and in that case the \( V \)-morphism is unique. In particular, \( X \) and \( X' \) are \( V \)-isomorphic if and only if \( \mathcal{B} = \mathcal{B}' \).

Proof. The first statement (on the antiequivalence of categories) follows from the discussion in 2.13 and from the fact that the category of \( O_V \)-subalgebras of \( O_{V}^{q} \) is equivalent, via Frobenius, to the category of \( O_V^{q} \)-subalgebras of \( O_V \). Next, given two \( O_V^{q} \)-subalgebras \( \mathcal{B}, \mathcal{B}' \subset O_V \), assume that there exists a morphism of \( O_V^{q} \)-algebras \( \mathcal{B} \to \mathcal{B}' \). It follows from Lemma 2.15 that the inclusion \( \mathcal{B} \subset O_V \) and the composition \( \mathcal{B} \to \mathcal{B}' \subset O_V \) coincide, whence \( \mathcal{B} \to \mathcal{B}' \) has to be an inclusion. The last two conclusions in the corollary are now immediate.
Before continuing with our discussion, we need to review some properties of integral closure of ideals.

**2.17.** Given a Noetherian ring $R$ and an ideal $I \subset R$, we denote by $\overline{I}$ the integral closure of $I$. This is an ideal of $R$ and consists of all $r \in R$ satisfying a relation of the form $r^n + a_1r^{n-1} + \cdots + a_nr^0 = 0$ for some $n \geq 1$ and $a_i \in I$, $i = 1, \ldots, n$. An ideal $I$ is said to be integrally closed if $\overline{I} = I$. Given an inclusion of ideals $J \supseteq I$, $J$ is said to be integral over $I$ or that $I$ is a reduction of $J$ if $J \subseteq \overline{I}$. In this case, $\overline{I} = J$. We list some properties that will be used along this section. Properties (b), (c), and (d) can be found in the table of basic properties of $\mathbb{A}$, and (g) can be deduced from (23) of that table. In what follows, $I \subset J$ denotes ideals in the Noetherian ring $R$.

(a) $J$ is integral over $I$ if and only if the Rees ring $R_J := R \oplus I \cdot J^2 \oplus \cdots$ is finite over $R_I := R \oplus I \cdot I^2 \oplus \cdots$.

This follows from [16] Proposition 5.2.1

(b) If $J$ is integral over $I$, then for any $R$-algebra $S$, $JS$ is integral over $IS$ (as $S$-ideals).

(c) Given an integral extension $R \subset S$, we have $\overline{I} = R \cap IS$.

(d) Given a multiplicative subset $T \subset R$, we have $\overline{I} \cdot T^{-1} = T^{-1}J$ (in $T^{-1}R$).

(e) If $J = (f) \subset R$ for a nonzero divisor $f \in R$, then $J$ has no proper reductions, that is, if $I \subset J$ is a reduction of $J$, then $I = J$.

**Proof.** Assume that $I$ is a reduction of $J$. This means that there is a relation of the form $f^n + a_1f^{n-1} + \cdots + a_n = 0$ with $a_i \in I^n$, $n > 0$. Assume that this relation is of minimal degree. We obtain that $a_n = a'_n \cdot f$ for some $a'_n \in I$. If $n > 1$, then the relation of integral dependence becomes $f(f^{n-1} + a_1f^{n-2} + \cdots + (a_{n-1} + a'_n)) = 0$, which implies that $f^{n-1} + a_1f^{n-2} + \cdots + (a_{n-1} + a'_n) = 0$ since $f$ is not a zero-divisor. This contradicts the minimality of our initial relation. Therefore $n = 1$ and $f = -a_n \in I$.

(f) If $(R, m)$ is a regular local ring, then $m$ has no proper reductions (see [28] Section 6, Cor. 2).

(g) If $R$ is a regular local ring, and $P \subset R$ is a prime ideal such that $R/P$ is regular, then $P^n = \overline{P^n}$, where $P^{(n)}$ is the ideal generated by $\{x^n : x \in P\}$; in particular, $P^n$ is integrally closed.

(h) Corollary of (c) and (g): If $R$ is an $F$-finite regular local ring, and if $O \subset R$ is an $R^d$-subalgebra ($q = p^d$), then for any prime $P \subset R$ such that $R/P$ is regular, the contraction $O \cap P^n \subset O$ is the integral closure of $F^d \cdot O \cdot P \subset O$, the ideal of $O$ generated by $\{x^q : x \in P\}$; in particular, $O \cap P^n$ is an integrally closed ideal of $O$.

**Proof.** By (c) $\overline{F^d \cdot O} = O \cap F^d \cdot (\overline{R}) = O \cap P^{(n)}$, and by (g) this is equal to $O \cap P^n$.

Property (d) enables us to extend the notion of integral closure to coherent ideals on schemes. Namely, given a scheme $W$ and an $\mathcal{O}_W$-ideal $I$, there is an $\mathcal{O}_W$-ideal $\overline{I}$, called the integral closure of $I$, such that if $W \subset W$ is an affine open subset, then $\overline{I}(U)$ is the integral closure of $I(U) \subset O_V(U)$. Given an inclusion of $\mathcal{O}_W$-ideals $I \subset J$, $J$ is said to be integral over $I$ if $\overline{J} = \overline{I}$. In this case, property (b) implies that $J \cdot \mathcal{O}_W$ is integrally closed over $I \cdot \mathcal{O}_W$ for any morphism of schemes $W' \to W$. Property (e) implies that if $L$ is an invertible ideal of $\mathcal{O}_W$, then $L$ is not integral over any $\mathcal{O}_W$-ideal properly included in $L$. Finally, if $V$ is an irreducible $F$-finite regular scheme and $Z \subset V$ is an irreducible regular closed subscheme, then $(g)$ implies that $I(Z)^{\#} \subset O_V$ is integrally closed. In addition, if $B \subset O_V$ is an $\mathcal{O}_V^d$-subalgebra, then (h) implies that the contraction $B \cap I(Z)^{\#} \subset B$ is an integrally closed $B$-ideal; indeed, it is the integral closure of the $B$-ideal $F^d(I(Z)) \cdot B$.

We return to our general discussion and delve into the proof of the properties stated in Theorem [11,2]. We also include proofs of the statements given in Theorem [11,2] (in our particular setting) to make the presentation self-contained.

**Proposition 2.18.** Given $\delta : X \to V$ in $\mathcal{O}_q(V)$, the maximal multiplicity along points of $X$ is $d := [K(X) : K(V)]$, the generic rank. Let $F_d(X) \subset X$ denote the set of points where the multiplicity is $d$, and let $\mathcal{M} \subset \mathcal{O}_V$ be any $\mathcal{O}_V^d$-submodule of $\mathcal{O}_V$ that defines $\delta$. Then

$$\delta(F_d(X)) = \text{Sing} (\mathcal{M}, 1).$$

(By Proposition 4.9 this implies that $F_d(X) \subset X$ is a closed subset.)

**Proof.** Since $\text{Sing} (\mathcal{M}, 1) = \text{Sing}(\mathcal{O}_V^d \cdot [\mathcal{M}], 1)$, the proposition follows from the following local version. □

**Lemma 2.19.** Let $(S, \mathfrak{m}) \subset (B, M)$ be a finite extension of local domains of characteristic $p$ such that $(S, \mathfrak{m})$ is regular and $F$-finite, and assume that $B^q \subset S$. Then the multiplicity of $B$ is at most $d := \text{Frac}(B) : \text{Frac}(S)$. In addition, the following are equivalent.
(1) The multiplicity of $B$ is $d$.
(2) $B = S + M$ and $M = \overline{mB}(\subset B)$.
(3) $B^q \subseteq S^q + \mathfrak{m}^q(\subset S)$; in particular, there is an inclusion $F^* M \subseteq \mathfrak{m}^q$ (as subsets of $S$).

Proof. The first part of the lemma and the equivalence (1) $\iff$ (2) follows from Zariski’s formula [27, VIII, Cor. 1 of Thm. 24] and a result by Rees relating multiplicity with integral closure of ideals ([24, Thm. 3.2]). We refer to [28, Section 4] for details. Since we are dealing only with local domains, and the extension $S \subset B$ is purely inseparable of finite exponent, all the required hypothesis in the references are satisfied.

As for the equivalence (2) $\iff$ (3), clearly (2) is equivalent to the equality $B^q = S^q + \overline{mB} \cdot B^q$ (the integral closure being taken inside $B^q$). By [2.17]h the latter is translated as $B^q = S^q + (\mathfrak{m}^q \cap B^q)$, which is clearly equivalent to the inclusion $B^q \subseteq S^q + \mathfrak{m}^q$. □

Part (2) and formula (2.7) of the next proposition are of technical nature, and they are there for later use.

Proposition 2.20. Given $\delta : X \to V$ in $\mathcal{C}_q(V)$, let $d := [K(X) : K(V)]$ and assume that $F_d(X) \neq \emptyset$. Let $Y \subset X$ be an integral subscheme of $X$ and set $Z := (\delta(Y))_{\text{red}} \subset V$. If $\mathcal{M} \subset \mathcal{O}_V$ is any $\mathcal{O}_V$-submodule of $\mathcal{O}_V$ defining $\delta$, then the following conditions are equivalent.
(1) $Y$ is regular and included in $F_d(X)$.
(2) $Z$ is regular, $\delta$ induces an isomorphism $Y \cong Z$, and $\mathcal{I}(Y) = \mathcal{I}(Z) \cdot \mathcal{O}_X$.
(3) $Z \subset V$ is regular, and $\mathcal{M} \subset \mathcal{O}_V + \mathcal{I}(Z)^q$, or say $Z$ is a permissible center for $(\mathcal{M}, 1)$ (Definition 2.17).

In addition, if these equivalent conditions hold, then
(2.7) $F^* (\mathcal{I}(Y)) = \mathcal{I}(Z)^q \cap \mathcal{O}_V \setminus \mathcal{M}$.

Proof. Note that we can replace $\mathcal{M}$ by $\mathcal{O}_V \setminus \mathcal{M}$, and the outcome of the proposition does not change. It is clear therefore that everything follows from the following local version.

Lemma 2.21. Let the hypothesis and notation be as in Lemma 2.19 and assume that the multiplicity of $B$ is $d := [\text{Frac}(B) : \text{Frac}(S)]$. Let $Q \subset B$ be a prime ideal and $q := S \cap Q \subset S$. Then the following conditions are equivalent.
(1) $B/Q$ is regular, and $B_P$ has multiplicity $d$ for every prime ideal $P \subset B$ that includes $Q$.
(2) $B/Q$ is regular, and $B_Q$ has also multiplicity $d$.
(3) $S/q$ is regular, $B = S + Q$, and $Q = \overline{qB}$.
(4) $S/q$ is regular, and $B^q \subset S^q + q^q$.

If these equivalent conditions hold, then $F^* Q = q^q \cap B^q$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. We now prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

By Lemma 2.19 we have $B = S + M$ and $M = \mathfrak{mB}$. By [2.17]h the latter equality implies that $M/Q$ is integral over $(mB + Q)/Q$ (as an ideal of $B/Q$). Assume (2). As $B/Q$ is regular, its maximal ideal $M/Q$ has no proper reductions (we are using [2.17]f); therefore $M/Q = (mB + Q)/Q$, which means that $M = mB + Q$. It follows that $B = S + M = S + Q + mB$, and hence $B = S + Q$ by Nakayama’s lemma. This implies that $S/q = B/Q$, and hence $S/q$ is regular. We now prove that $Q = \overline{qB}$. By applying the implication (1) $\Rightarrow$ (3) of Lemma 1 to the inclusion $S_q \subseteq B_Q$ we obtain that for all $x \in Q$, $x^q \in q^q S_q \cap S = q^q$, where the last equality holds since usual and symbolic powers of a regular prime in a regular ring coincide. We have shown that any $x \in Q$ is integral over $qB$ since such $x$ satisfies the relation of integral dependence $T^q - x^q = 0$; thus $Q = \overline{qB}$. This completes the proof of (2) $\Rightarrow$ (3).

Assume (3). It follows that $B^q = S^q + F^* Q$ and that $F^* Q = F^* q \cdot B^q$, and the latter is equal to $q^q \cap B^q$ by [2.17]h when applied to $O := B^q$ and $P := q$. We have shown that (3) implies (4) and the last sentence of the lemma.

Assume finally (4). On the one hand, we have that $B^q = S^q + q^q \cap B^q$, and by applying [2.17]h, as we did before, we obtain that $B^q = S^q + F^* q \cdot B^q$. By applying the inverse of Frobenius we obtain $B = S + qB$. This equality implies that $B/\overline{qB} \cong S/q$; in particular, $\overline{qB}$ is prime. Since $S/q \subset B/Q$ is finite, by dimension reasons we have that $Q = \overline{qB}$. In particular, $B/Q \cong S/q$ is regular. On the other hand, let $P \subset B$ be any prime ideal including $Q$, and set $p := P \cap S$. The inclusion $B^q \subseteq S^q + q^q$ implies clearly that $(B_P)^q \subseteq S_P^q + (pS_P)^q$. According to Lemma 2.19 the latter inclusion implies that $B_P$ has multiplicity $d$. This completes the proof of (4) $\Rightarrow$ (1) and hence also the proof of the lemma. □
Propositions 2.18 and 2.20 express $F_d(X)$ and the condition of being a regular center included in $F_d(X)$ completely in terms of the $\mathcal{O}_V^q$-module $\mathcal{M} \subset \mathcal{O}_V$ which defines $\delta : X \to V$. We now express certain transformations of $X$ in terms of transformations of the module $\mathcal{M}$. For instance, the next proposition shows that the definition of pull-back of $\mathcal{O}_V^q$-modules given in Definition 2.28 corresponds to a notion of base change of finite morphisms.

**Proposition 2.22.** Let $\mathcal{M} \subset \mathcal{O}_V$ be an $\mathcal{O}_V^q$-submodule, and let $\delta : X \to V$ be the associated morphism in $\mathcal{E}_q(V)$. Given a morphism $V \leftarrow V_1$ of irreducible $F$-finite regular schemes, the reduced fiber product $(X \times_V V_1)_{\text{red}}$ is integral, the projection $\delta_\pi : (X \times_V V_1)_{\text{red}} \to V_1$ belongs to $\mathcal{E}_q(V_1)$, and this is the $V_1$-scheme defined by the $\mathcal{O}_{V_1}^q$-submodule $\mathcal{M}_{\mathcal{O}_{V_1}} \subset \mathcal{O}_{V_1}$. Diagrammatically,

$$
\begin{array}{c}
X \\
\downarrow \pi \delta \\
V \\
\downarrow \pi \\
\mathcal{M} \\
\mathcal{M} \mathcal{O}_{V_1}^q
\end{array}
\quad
\begin{array}{c}
(X \times_V V_1)_{\text{red}} \\
\downarrow \delta_\pi \\
V_1
\end{array}
$$

**Proof.** Since $\delta : X \to V$ is finite and $\mathcal{O}_X^q \subset \mathcal{O}_V$, the base change $X \times_V V_1 \to V_1$ is also finite and satisfies $\mathcal{O}_{X \times_V V_1}^q \subset \mathcal{O}_{V_1}$. Therefore $\mathcal{O}_{(X \times_V V_1)_{\text{red}}} = (\mathcal{O}_{X \times_V V_1})_{\text{red}}$ is the natural image of $\mathcal{O}_{X \times_V V_1} = \pi^{-1} \mathcal{O}_X \otimes_{\pi^{-1} \mathcal{O}_V} \mathcal{O}_{V_1}$ in $\mathcal{O}_{V_1}^q$; in particular, $(X \times_V V_1)_{\text{red}}$ is integral. Equivalently, by applying $F^e$ we obtain that $(\mathcal{O}_{(X \times_V V_1)_{\text{red}}}^q)^\eta$ is the natural image of $\pi^{-1} \mathcal{O}_X^q \otimes_{\pi^{-1} \mathcal{O}_V^q} \mathcal{O}_{V_1}^q$ in $\mathcal{O}_{V_1}^q$, which is $\mathcal{O}_{X}^q \mathcal{O}_{V_1}^q = \mathcal{O}_{V_1}^q [\mathcal{M}] \mathcal{O}_{V_1}^q$; see Definition 2.28. The latter is equal to $\mathcal{O}_{V_1} \mathcal{M}_{\mathcal{O}_{V_1}}$ by Proposition 2.28(2). This shows that the projection $\delta_\pi : (X \times_V V_1)_{\text{red}} \to V_1$ is the $V_1$-scheme in $\mathcal{E}_q(V_1)$ defined by $\mathcal{M} \mathcal{O}_{V_1}^q$. 

For completeness, we include the proof of Theorem 1.1(2). We formulate it in our particular setting of purely inseparable morphisms, though the same proof we give here applies to the general case.

**Proposition 2.23.** Let the setting be as in ??Proposition 0310, and assume that the equivalent conditions given there are satisfied. Let $f : X_1 \to X$ denote the blowup of $X$ along $Y$, and let $\pi : V_1 \to V$ denote the blowup of $V$ along $Z$.

1. There exists a unique morphism $\delta_1 : X_1 \to V_1$ making the diagram

$$
\begin{array}{c}
X \\
\downarrow \delta \\
V \\
\downarrow \pi \\
\mathcal{M} \\
\mathcal{M} \mathcal{O}_{V_1}^q
\end{array}
\quad
\begin{array}{c}
X_1 \\
\downarrow \delta_1 \\
V_1
\end{array}
$$

commutative. In addition, $\delta_1$ belongs to $\mathcal{E}_q(V_1)$, and $I(Y) \cdot \mathcal{O}_{X_1} = I(H_1) \cdot \mathcal{O}_{X_1}$, where $H_1 \subset V_1$ is the exceptional divisor of $\pi$.

2. We now consider the commutative diagram

$$
\begin{array}{c}
X \\
\downarrow \delta \\
V \\
\downarrow \pi \\
\mathcal{M} \\
\mathcal{M} \mathcal{O}_{V_1}^q
\end{array}
\quad
\begin{array}{c}
X \times_V V_1 \underset{f_1}{\leftarrow} X_1 \\
\downarrow \delta_\pi \\
V_1
\end{array}
$$

deduced from the previous one and the universal property of fiber products. Then $\delta_\pi$ is in $\mathcal{E}_q(V_1)$, has generic rank $d$, and is defined by the $\mathcal{O}_{V_1}^q$-submodule $\mathcal{M}_{\mathcal{O}_{V_1}} \subset \mathcal{O}_{V_1}$. In addition, the equivalent conditions of ??Proposition 0310 hold for $\delta_\pi$ and the inverse image subscheme $Y_1 := \pi_\delta^{-1}(Y) \subset X \times_V V_1$, which in turn satisfies $(\delta_\pi(Y_1))_{\text{red}} = H_1$. Finally, $f_1$ is the blowup of $X \times_V V_1$ along $Y_1$.

**Proof.** (1) Let $T$ denote a variable. In the following chain of extensions of sheaves over $V$:

$$
\mathcal{O}_V[(I(Z))T] \subset \mathcal{O}_X[(I(Z))T] = \mathcal{O}_X[(I(Z) \cdot \mathcal{O}_X)T] \subset \mathcal{O}_X[(I(Y))T]
$$

the first one is finite since $\mathcal{O}_V \subset \mathcal{O}_X$ is finite, and the last one is also finite by 2.17(a). Indeed, the $\mathcal{O}_X$-ideal $I(Y)$ is integral over $I(Z) \cdot \mathcal{O}_X$ by Proposition 2.20. It follows that the extension of sheaves of
graded rings $\mathcal{O}_Y[(I(Z))T] \subset \mathcal{O}_X[(I(Y))T]$ over $V$ is finite, and hence this extension induces a morphism $X_1 := \text{Proj}(\mathcal{O}_X[(I(Y))T]) \xrightarrow{\delta_1} V_1 := \text{Proj}(\mathcal{O}_Y[(I(Z))T])$, which is compatible with the blowups $X \overset{\pi}{\leftarrow} X_1$ and $V \overset{\pi}{\leftarrow} V_1$. Note that $V_1$ is irreducible, F-finite, and regular, and by construction $\delta_1$ belongs to $\mathcal{C}_q(V_1)$. The uniqueness of $\delta_1$ making the diagram commutative follows from the universal property of the blow-up $V_1 \rightarrow V$. Finally, $(I(Y) \cdot \mathcal{O}_X)/(I(Z) \cdot \mathcal{O}_X) = I(H_1) \cdot \mathcal{O}_{X_1}$ by 2.17(b), whence $(I(Y) \cdot \mathcal{O}_X) = (I(H_1) \cdot \mathcal{O}_{X_1})$; see 2.17(d).

(2) Note that $X \times V$ is the blowup of $X$ along $I(Z)\mathcal{O}_X$, and in particular it is integral. It clearly follows that $X \times V \to V_1$ belongs to $\mathcal{C}_q(V_1)$ and has generic rank $d$. Proposition 2.22 tells us that $\delta_1$ is defined by the $\mathcal{O}_{V_1}^q$-submodule $\mathcal{M} \mathcal{O}_{V_1}^q \subset \mathcal{O}_Y$. Next, $\pi^{-1}_1(Y) \subset X \times V_1$ is naturally isomorphic to $Y \times V_1 \cong Y \times H_1$, which in turn is isomorphic to $H_1$ via the projection $\pi$ since $Y \cong Z$ (recall that we assume the equivalent conditions of Proposition 2.20). Therefore the equivalent conditions of Proposition 2.20 hold for $\delta_1$ and $Y_1 \subset X \times V_1$.

We finally check that $f_1$ is the blowup of $X \times V_1$ along $Y_1$. Let $f_1^1 : X_1^1 \rightarrow X \times V_1$ be the blowup along $Y_1$. We aim to prove that $X_1$ and $X_1^1$ are $(X \times V_1)$-isomorphic. This will be done by proving that there exist a unique $(X \times V_1)$-morphism $h : X_1 \rightarrow X_1^1$ and a unique $(X \times V_1)$-morphism $u : X_1^1 \rightarrow X$. On the one hand, $(I(Y) \cdot \mathcal{O}_{X \times V_1}) \cdot \mathcal{O}_{X_1} = (I(Y) \cdot \mathcal{O}_{X_1})$ is an invertible $\mathcal{O}_{X_1}$-ideal since $f = \pi_1 h_1$ is the blowup along $Y$; thus by the universal property of $f_1^1$ there exists a unique $(X \times V_1)$-morphism $X_1 \xrightarrow{\delta_1} X_1^1$. On the other hand, $(I(Y) \cdot \mathcal{O}_{X \times V_1}) \cdot \mathcal{O}_{X_1} = (I(Y) \cdot \mathcal{O}_{X_1})$ is invertible since $f_1^1$ is the blowup along $Y_1$; thus by the universal property of $f$ there exists a unique $(X \times V_1)$-morphism $X_1 \xrightarrow{\delta_1 = \delta} X_1^1$. It is only left to prove that $u$ is an $(X \times V_1)$-morphism, that is, $f_1 = f_1^1$. Since $u$ is an $X$-morphism, we have that $\pi_1 u = f_1 = f_1^1$, but $\pi_1 u = \pi_1 f_1 = \pi_1 f_1^1$. Looking at the extremal sheaves of this equality and the universal property of $\delta$ and the fact that $X_1^1$ is integral, we conclude that $\delta_1 u = f_1^1 = \pi_1 f_1$, that is, $u$ is also a $V_1$-morphism. Therefore $u$ is an $(X \times V_1)$-morphism, as was aimed to prove. This completes the proof of (2).

□

**Proposition 2.24.** Let the setting be as in Proposition 2.20 and assume that the equivalent conditions given there are satisfied. Assume, in addition, that $H := Z \subset V$ is one-codimensional and set $\mathcal{M}_1 := \mathcal{M}_{Z(H)}$. Then the $V$-morphism $X_1 \rightarrow X$ defined by the inclusion $\mathcal{O}_{X_1}^q[\mathcal{M}_1] \subseteq \mathcal{O}_X^q[\mathcal{M}]$ is the blowup of $X$ along $Y$.

**Proof.** We first check that $X \overset{\pi}{\leftarrow} X_1$ factorizes as $X \overset{\pi}{\leftarrow} X_1^1 \overset{\pi}{\leftarrow} X_1$, where $X_1 \overset{\pi}{\leftarrow} X_1^1$ is the blowup of $X$ along $Y$, by showing that $I(Y)\mathcal{O}_{X_1} = I(H)\mathcal{O}_{X_1}$, an invertible $\mathcal{O}_{X_1}$-ideal. Since one inclusion is clear, this is equivalent to showing that $F^e(I(Y)) \cdot \mathcal{O}_{X_1} \subseteq F^e(I(H)) \cdot \mathcal{O}_{X_1}$, or simply $F^e(I(Y)) \subseteq F^e(I(H)) \cdot \mathcal{O}_{X_1}$.

We now prove this inclusion of subsheaves of $\mathcal{O}_Y$. By Proposition 2.20 $F^e(I(Y)) = \mathcal{O}_X^q[\mathcal{M}] \cap \mathcal{H}(H)^q$, and this is equal to $\mathcal{O}_Y^q[\mathcal{M}] \cap \mathcal{H}(H)^q$ by Lemma 2.3 when applied to $L = H$. Finally, this intersection is included in $\mathcal{O}_Y^q \cap \mathcal{H}(H)^q = F^e(I(H)) \cdot \mathcal{O}_X^q[\mathcal{M}] = F^e(I(H)) \cdot \mathcal{O}_X^q[\mathcal{M}] = F^e(I(H))$.

We now show that $u$ is an isomorphism. By Proposition 2.20 $\delta f_1 = \delta_1 u$, and this is only left to show the reverse inclusion.

Now from $I(Y) \cdot \mathcal{O}_{X_1} = I(H) \cdot \mathcal{O}_{X_1}$ we obtain $F^e(I(Y)) \mathcal{O}_{X_1} = F^e(I(H)) \cdot \mathcal{O}_{X_1}$. On the other hand, $F^e(I(Y)) = \mathcal{O}_Y^q[\mathcal{M}] \cap \mathcal{H}(H)^q$, and this clearly includes $F^e(I(H)) \cdot \mathcal{M}_1$ by the definition of $\mathcal{M}_1$. We deduce that $F^e(I(H)) \cdot \mathcal{M}_1 \subseteq F^e(I(H)) \cdot \mathcal{O}_{X_1}$, whence $\mathcal{M}_1 \subseteq \mathcal{O}_{X_1}^q$. Thus $\mathcal{O}_X^q[\mathcal{M}_1] \subseteq \mathcal{O}_{X_1}^q$, as was to be proved. □

**Corollary 2.25.** Let the setting be as in Proposition 2.20 and assume that the equivalent conditions given there are satisfied. Assume in addition that $d > 1$ (equivalently, $\mathcal{M} \not\subseteq \mathcal{O}_Y^q$). Let $f : X_1 \rightarrow X$ and $\pi : V_1 \rightarrow V$ be the blowups of $X$ along $Y$ and of $V$ along $Z$, respectively, and let $\delta_1 : X_1 \rightarrow V_1$ be the morphism in $\mathcal{C}_q(V_1)$ given in Proposition 2.20. Let $H_1 \subset V_1$ be the exceptional divisor of $\pi$. Then the following holds.

1. $\delta_1 : X_1 \rightarrow V_1$ is defined by the 1-transform $\mathcal{M}_1^{(1)} \subseteq \mathcal{O}_V$ of $\mathcal{M}$ under $\pi$ (Definition 2.10).
2. More generally, let $a$ denote the largest integer such that $H_1 \subset V_1$ is permissible for $(\mathcal{M} \mathcal{O}_{V_1}, a)$, and for each $1 \leq i \leq a$, we consider the i-transform $\mathcal{M}_1^{(i)} \subseteq \mathcal{O}_V$ of $\mathcal{M}$ under $\pi$ and its associated morphism
\[ \delta_i : X_i \to V_1 \text{ in } \mathcal{C}_q(V_1). \text{ Then } a \geq 1, \text{ and there is a commutative diagram} \]

\[
\begin{array}{ccccccc}
X & \xleftarrow{\pi} & X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_a} & X_a \\
\downarrow{\delta} & & \downarrow{\delta_0} & & \downarrow{\delta_1} & & \cdots & & \downarrow{\delta_a} \\
V & \xleftarrow{\pi} & V_1 & = & V_1 & = & \cdots & = & V_a \\
\end{array}
\]

where the first square is Cartesian (so that \( \delta_0 \) is defined by the \( \mathcal{O}_{V_1} \)-module \( \mathcal{M} \mathcal{O}_{V_1} \subseteq \mathcal{O}_{V_1} \) by Proposition 2.24), and \( f_i \) is the blowup of \( X_{i-1} \) along \( (\delta_i^{-1}(H_1))_{\text{red}} \subseteq X_{i-1} \). In addition, \( H_1 \) is permissible for \( (\mathcal{M}^{(a)}, 1), i = 1, \ldots, a - 1, \) and for \( (\mathcal{M}^{(a)}, 1) \).

Proof. By Proposition 2.24 \( \delta_1 \) can be expressed as \( \delta_0 f_1 \), where \( f_1 \) is the blowup along \( (\delta_0^{-1}(H_1))_{\text{red}} \subseteq X_0 \). Applying Proposition 2.24 to \( \delta_0 \) and \( H_1 \), we see that \( \delta_1 \) is defined by \( (\mathcal{M} \mathcal{O}_{V_1}) (I(H_1)) \), which is the definition of \( \mathcal{M}^{(a)} \). This proves (1).

The proof of (2) follows by iteration of (1) or of Proposition 2.24. We only need to check that for \( 1 \leq i < a \),

(a) \( H_1 \) is permissible for \( (\mathcal{M}^{(i)}, 1) \),
(b) \( \mathcal{M}^{(i+1)} = (\mathcal{M}^{(i)}) (I(H_1)) \), and
(c) \( H_1 \) is not permissible for \( (\mathcal{M}^{(a)}, 1) \).

(b) follows from Lemma 2.23(1) when applied to \( \mathcal{M} \mathcal{O}_{V_1}^{q} \subset \mathcal{O}_{V_1} \), \( \mathcal{L} = I(H_1)^i \), and \( \mathcal{L}' = I(H_1) \). By the definition of \( a \) and Lemma 2.23(1), \( \mathcal{M} \mathcal{O}_{V_1}^{q} \subset \mathcal{O}_{V_1}^{q} + F^e(I(H_1)^{a}) \mathcal{M}^{(a)} \), but the same does not hold with \( a \) replaced by \( a + 1 \). This implies (c), and also that for \( i < a \), \( \mathcal{M}^{(i)} = (\mathcal{O}_{V_1}^{q} + \mathcal{M} \mathcal{O}_{V_1}^{q} : F^e(I(H_1)^{i})) = \mathcal{O}_{V_1}^{q} + F^e(I(H_1)^{a-1}) \mathcal{M}^{(a)} \). Note that (a) follows from this. \( \square \)

Remark 2.26. Proposition 1.16 shows that the number \( a \) in the above corollary coincides with the largest integer such that \( Z \) is permissible for \( (\mathcal{M}, a) \).

Remark 2.27. On the proof of Theorem 1.1 (for \( \delta \in \mathcal{C}_q(V) \)) and Theorem 1.12. Proposition 2.13 covers the proof of the first part and (1) of Theorem 1.1 except for the closeness of \( F_q(X) \), whose proof will be postponed until Proposition 4.9. It also covers the proof of the first equality in (1) of Theorem 1.12, the second one being also included in Proposition 4.9. Propositions 2.23 and 2.24 cover the proof of (2) of Theorem 1.1 and of (2) of Theorem 1.12. Finally, Corollary 2.25(1) covers the proof of (3) of Theorem 1.12.

§3 Differential Operators and the Definition of Local Invariants for Points \( x \in F_q(X) \)

Let \( V \) denote an irreducible \( F \)-finite regular scheme, let \( \mathcal{M} \subset \mathcal{O}_V \) be an \( \mathcal{O}_V^{q} \)-module, and let \( \delta : X \to V \) be the finite morphism attached to \( \mathcal{M} \). We set \( d := [K(X) : K(V)] \). One of the main purposes of this paper is defining invariants of singularities for points \( x \in F_q(X) \). Proposition 2.13 provides the description \( \delta(F_q(X)) = \{ x \in V : \mathcal{M}_x \subseteq \mathcal{O}_{V,x}^{q} + m_{V,x}^{n} \} \). Our definition of invariants will arise of course in studying the inclusion \( \mathcal{M}_x \subseteq \mathcal{O}_{V,x}^{q} + m_{V,x}^{n} \) and similar conditions. Differential operators will play a relevant role in this vein.

We formulate our discussion in this section at the local level. We fix an \( F \)-finite regular local ring \( (R, m) \) and an \( R^q \)-submodule \( M \subset R \). We shall assign to \( M \) two different numerical invariants. Both invariants are compatible with our notions of equivalence of Definition 2.1. The first one, denoted by \( \nu_{m}^{(q)}(M) \), is the highest integer \( n \) such that \( M \subset R^q + m^n \subset R \). This will be, to some extend, an analog to the notion of order of an ideal but applied here to \( R^q \)-submodules of \( R \). The second invariant, denoted by \( \eta_{m}(M) \), will be defined in terms of differential operators; see 3.15.

We will make use of \( p \)-basis to show that these invariants are closely related. For instance, \( \nu_{m}^{(q)}(M) = \eta_{m}^{(q)}(M) \) if \( R/m \) is perfect. A \( p \)-basis of \( R \) provides, for each element \( f \in R \) and each power \( q = p^r \), a natural notion of \( q \)-expansion, which, in some way to be clarified, can be interpreted as a Taylor expansion of \( f \). This approach will lead us, for instance, to a simple proof of a classical result expressing the order of an element at a local regular ring in terms of differential operators (see Corollary 3.15).
Given \( V \) and an \( O_V \)-module \( \mathcal{M} \subset O_V \) as above, we get for each point \( x \in V \) an \( F \)-finite regular local ring \( O_{V,x} \) and an \( O_{V,x} \)-module \( \mathcal{M}_x \subset O_{V,x} \). Hence we obtain two functions \( x \mapsto \nu_x^q(\mathcal{M}) := \nu_{n_V,x}(\mathcal{M}_x) \) and \( x \mapsto \eta_x(\mathcal{M}) := \eta_{n_{V,x}}(\mathcal{M}_x) \).

Both functions appear as an attempt to extend to modules properties known for ideals. If we fix an ideal \( \mathcal{J} \) over \( V \), the order function: \( x \mapsto \nu_x(\mathcal{J}) \) is upper-semicontinuous. However, of the two functions defined by \( O_V^q \)-modules, only the second one has this property. In fact, we introduce the function \( x \mapsto \eta_x(\mathcal{M}) \) to remedy the fact that \( \nu_x^q(\mathcal{M}) \) is not upper-semicontinuous. We will discuss this in the next section (see [3.5]).

### §3.1 Differential Operators

We summarize some well-known facts about differential operators. We refer the reader to [10, Chapter 3] and [12, 16.8] for details of the definitions and facts that will be stated here without proofs.

3.1. Given a ring \( R \), we denote \( T_R := \langle 1 \otimes r - r \otimes 1 : r \in R \rangle \subset R \otimes R \). For an integer \( i \geq 0 \), we denote by \( \text{Diff}_i^R \) the (left) \( R \)-submodule of \( \text{End}_R(R, +) \) consisting of those \( Z \)-linear maps \( D : R \to R \) whose (left) \( R \)-linear extension \( \varphi_D : R \otimes R \to R \) (\( \varphi_D(r_1 \otimes r_2) = r_1D(r_2) \)) annihilates the ideal \( T_R^{i+1} \subset R \otimes R \) and hence factors through the quotient \( P^+_R := (R \otimes R)/T^{i+1}_R \). This quotient is called the \( R \)-module of principal parts of order \( i \).

There is a chain of \( R \)-modules \( \text{Diff}^0_R \subset \text{Diff}^1_R \subset \text{Diff}^2_R \subset \cdots \). An element \( D : R \to R \) in any of these modules is called an absolute differential operator or simply a differential operator of the ring \( R \). The order of \( D \) is the first integer \( i \) such that \( D \in \text{Diff}^i_R \), and it will be denoted \( |D| \). So \( \text{Diff}_i^R \) consists of the differential operators of order at most \( i \).

We denote by \( \text{Diff}^i_{R,+} \) the \( R \)-submodule of \( \text{Diff}^i_R \) consisting of the operators \( D \) such that \( D(1) = 0 \). For \( i = 0 \), we have \( \text{Diff}^0_R = \text{End}_R(R, +) \equiv R \). This identification leads to a decomposition

\[
\text{Diff}_i^R = R \oplus \text{Diff}^i_{R,+}.
\]

The composite of two differential operators is again a differential operator. More precisely, if \( D \in \text{Diff}^i_R \) and \( D' \in \text{Diff}^j_R \), then \( D \circ D' \in \text{Diff}^{i+j}_R \). In addition, if \( D' \in \text{Diff}^j_{R,+} \), then \( D \circ D' \in \text{Diff}^{i+j}_{R,+} \).

Given an \( R \)-submodule \( \mathcal{D} \subset \text{Diff}^i_R \) and a nonempty subset \( \Phi \subset R \), we denote by \( \mathcal{D}(\Phi) \) the abelian subgroup of \( R \) generated by the evaluations \( D(r) \) with \( D \in \mathcal{D} \) and \( r \in \Phi \), which is clearly an ideal of \( R \). Note that for two nonempty subsets \( \Phi, \Phi' \subset R \), we have

\[
\mathcal{D}(\Phi \cup \Phi') = \mathcal{D}(\Phi) + \mathcal{D}(\Phi').
\]

Given an ideal \( I \subset R \) and \( D \in \text{Diff}^i_R \), we will often use the following property:

\[
D(I^n) \subseteq I^{n-i} \quad \text{for all integers } i : 0 \leq i \leq n.
\]

3.2. If the ring \( R \) has characteristic \( p > 0 \), and if \( q \) denotes a power of \( p \), then any \( D \in \text{Diff}^{i-1}_R \) is \( R^q \)-linear. In fact, if \( \varphi : R \otimes R \to R \) denotes the morphism of left \( R \)-modules extending \( D \), then by definition \( \varphi(T^q_R) = 0 \).

Therefore \( D(x^q) = \varphi(1 \otimes x^q) = (\varphi(1) \otimes r) + (1 \otimes x^q - x^q \otimes r) = x^q \varphi(1 \otimes r) + (1 \otimes x^q - x^q \otimes 1)(1 \otimes r) = x^qD(r) + (1 \otimes x - x \otimes 1)(1 \otimes r) = x^qD(r) \) in particular,

\[
\text{Diff}^{i-1}_{R,+}(R^q) = 0.
\]

3.3. Logarithmic Differential Operators. Given an ideal \( I \subset R \), a differential operator \( D : R \to R \) is said to be \( I \)-logarithmic if \( D(I^k) \subseteq I^k \) for all \( k \geq 0 \). We denote by \( \text{Diff}^i_{R,I} \) the \( R \)-submodule of \( \text{Diff}^i_R \) consisting of those differential operators that are \( I \)-logarithmic. We also denote \( \text{Diff}^i_{R,I,+} := \text{Diff}^i_{R,I} \cap \text{Diff}^i_{R,+} \). According to (3.3), there are inclusions

\[
I^i \text{Diff}^i_R \subset \text{Diff}^i_{R,I} \quad \text{and} \quad I^i \text{Diff}^i_{R,+} \subset \text{Diff}^i_{R,I,+}.
\]

More generally, if \( \Lambda = \{I_1, \ldots, I_j\} \) is a finite collection of ideals of \( R \), then we denote \( \text{Diff}^i_{R,\Lambda} := \text{Diff}^i_{R,I_1} \cap \cdots \cap \text{Diff}^i_{R,I_j} \). The elements of this \( R \)-module are called \( \Lambda \)-logarithmic differential operators. We also define \( \text{Diff}^i_{R,\Lambda,+} := \text{Diff}^i_{R,\Lambda} \cap \text{Diff}^i_{R,+} \).

If \( D \in \text{Diff}^i_{R,\Lambda} \) and \( D' \in \text{Diff}^j_{R,\Lambda} \), then, as we already mentioned, \( D \circ D' \) is a differential operator of order \( \leq i + j \), and clearly \( D \circ D' \in \text{Diff}^{i+j}_{R,\Lambda} \).
As a consequence of (3.3), if \( L \subset R \) is an ideal included in \( I_1 \cap \cdots \cap I_r \), then
\[
(3.6) \quad L^i \text{Diff}^i_R \subseteq \text{Diff}^i_{R,A} \quad \text{and} \quad L^i \text{Diff}^i_{R,+} \subseteq \text{Diff}^i_{R,A,+}.
\]

Note that \( \text{Diff}^i_{R,A} = \text{Diff}^i_R \) if \( A \) is either the empty family or if \( A = \{ R \} \), so everything we say in the logarithmic setting applies also to the original case.

### 3.4. Compatibility with Localizations.

If \( S \) is a multiplicative subset of \( R \) such that \( R \to S^{-1}R \) is injective, then a differential operator \( D : R \to R \) can be extended uniquely to a differential operator \( D_S : S^{-1}R \to S^{-1}R \) (of the same order as \( D \)). This follows easily from the good localization property of the module of principal parts, namely, \( S^{-1} P^i_R = P^i_{S^{-1}R} \). In the case that \( R \) is a ring of characteristic \( p > 0 \) (as we will always assume in these notes), there is a handy description of the action of \( D_S \) on fractions. Let \( q \) be a power of \( p \) strictly greater than the order of \( D \). By (3.2) \( D \) is \( R^q \)-linear, and similarly \( D_S \) is \( (S^{-1}R)^q \)-linear. Therefore, given \( a/f \in S^{-1}R \) with \( f \in S \), we have
\[
(3.7) \quad D_S(a/f) = D_S(a f^{q-1}/f^q) = (1/f^q) D_S(a f^{q-1})/f^q.
\]

If \( A \) is a finite collection of ideals of \( R \), then we denote \( S^{-1}A = \{ S^{-1}I : I \in A \} \). The above description of \( D_S \) shows easily that if \( D \) is \( A \)-logarithmic, then \( D_S \) is \( S^{-1}A \)-logarithmic.

The assignment \( D \mapsto D_S \) extends naturally to a morphism of \( S^{-1}R \)-modules
\[
(3.8) \quad S^{-1}(\text{Diff}^i_{R,A}) \to \text{Diff}^i_{S^{-1}R,S^{-1}A}.
\]

When \( R \) is a domain, then this map is clearly injective. We now show that if \( R \) is an \( F \)-finite Noetherian domain, then the map (3.8) is bijective, giving a natural isomorphism
\[
(3.9) \quad S^{-1}(\text{Diff}^i_{R,A}) \cong \text{Diff}^i_{S^{-1}R,S^{-1}A}.
\]

In fact, it is only left to prove that given \( \tilde{D} \in \text{Diff}^i_{S^{-1}R,S^{-1}A} \), there exist \( D \in \text{Diff}^i_{R,A} \) and \( f \in S \) such that \( \tilde{D} = 1/f D_S \). Let \( q \) be a power of \( p \) such that \( q > i \). Then the restriction of \( \tilde{D} \) to \( R \) is \( R^q \)-linear, and we have \( \tilde{D}(I^k) \subseteq S^{-1}I^k \) for all \( k \geq 0 \) and \( f \in A \). Now, since \( R \) is an \( F \)-finite Noetherian domain, \( R \) and each of its ideals are finite \( R^q \)-modules. This observation and the fact that \( A \) is finite shows that there is an element \( f \in S \) such that \( f \tilde{D}(R) \subseteq R \), which implies that \( f \tilde{D} \in \text{Diff}^i_R \), and such that \( f \tilde{D}(I^k) \subseteq I^k \) for all \( I \in A \) and \( k \leq i \). It is easy to show that this already implies that \( f \tilde{D}(I^k) \subseteq I^k \) for all \( k \geq 0 \). Therefore \( f \tilde{D} \in \text{Diff}^i_{R,A} \). Note finally that \( \tilde{D} = 1/f D_S \). This completes the proof of our isomorphism.

An analogous discussion also holds for the module \( \text{Diff}^i_{R,A,+} \).

Finally, let \( R \) be an \( F \)-finite Noetherian domain, let \( A \) be a finite collection of ideals, and let \( q = p^e \) be a power of \( p \). Let \( M \subset R \) be an \( R^q \)-submodule, and let \( J \subset R \) be an ideal. We deduce from (3.7) and from the explicit description of this isomorphism given in (3.8) that the following equalities hold:
\[
(3.10) \quad S^{-1}(\text{Diff}^i_{R,A}(J)) = \text{Diff}^i_{S^{-1}R,S^{-1}A}(S^{-1}J), \quad \forall i \geq 0,
\]
\[
S^{-1}(\text{Diff}^i_{R,A,+}(M)) = \text{Diff}^i_{S^{-1}R,S^{-1}A,+}((F^e(S))^{-1}M), \quad \forall i = 1, \ldots, q - 1.
\]

### §3.2 \( p \)-Basis and Taylor Operators.

We recall the general notion of \( p \)-basis and the associated Taylor operators for \( F \)-finite regular local rings. Later we will focus only on those \( p \)-basis that extend a regular system of parameters. These are the \( p \)-basis that we will use to study the invariants that will be assigned to \( R^q \)-modules.

#### 3.5. \( p \)-Basis.

Let \( R \) be an \( F \)-finite domain of characteristic \( p \). An ordered set \( \{ z_1, \ldots, z_n \} \subset R \) such that the monomials \( z_{\alpha} := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \), with \( 0 \leq \alpha_i < p \), form a basis for \( R \) over \( R^q \) is called a \( p \)-basis. Note that if \( q \) is a power of \( p \) and
\[
A_q := \{ \alpha \in \mathbb{N}_0^n : 0 \leq \alpha_i < q \},
\]
then \( R \) is an \( R^q \)-free module with basis \( \{ z_{\alpha} : \alpha \in A_q \} \). Therefore any \( f \in R \) has a unique expression of the form
\[
f = \sum_{\alpha \in A_q} c_{\alpha} z_{\alpha} \quad \text{with} \quad c_{\alpha} = c_{\alpha}(f) \in R.
\]
This expression will be referred to as the q-expansion of f with respect to the p-basis \((z_1, \ldots, z_n)\) or simply as the q-expansion of f if there is no risk of confusion.

A well-known result by Kunz states that if a local ring admits a p-basis, then it must be regular. Conversely, Proposition 3.8 recalls the fact that F-finite regular local rings admit p-basis.

### 3.6. Taylor Differential Operators

A p-basis \(\{z_1, \ldots, z_n\}\) is also a differential basis for \(R\), that is, \(\Omega_R^1\) is free with basis \(\{dz_1, \ldots, dz_n\}\). In fact, in the context of F-finite rings, these two concepts are the same; see [20]. Thus \(R\) is differentially smooth over the prime field in the sense of Grothendieck [12, 16.10], and for each \(\gamma \in \mathbb{N}_0^n\), there exists a unique differential operator \(D_{\gamma} : R \to R\) such that

\[
D_{\gamma}(z^\alpha) = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} z^{\alpha - \gamma}, \quad \forall \alpha \in \mathbb{N}_0^n.
\]

(Here \(\binom{\alpha}{\gamma}\) denotes \(\prod_{k=1}^n \binom{a_k}{\gamma_k}\).) In addition, \(D_{\gamma}\) has order \(|\gamma|\), and for each \(i \geq 0\), \(\text{Diff}_R^i\) is a free \(R\)-module with basis \(\{D_{\gamma} : |\gamma| \leq i\}\), and \(\text{Diff}_{R_\gamma}^i\) is free with basis \(\{D_{\gamma} : 0 < |\gamma| \leq i\}\) (observe that \(D_0\) is the identity).

The operators \(D_{\gamma}\) are called Taylor operators (with respect to the given p-basis). Note that from (3.11) we deduce that

\[
D_{\gamma} \circ D_{\gamma'} = \binom{\gamma + \gamma'}{\gamma} D_{\gamma + \gamma'}, \quad \gamma, \gamma' \in \mathbb{N}_0^n.
\]

In particular, a Taylor operator decomposes as

\[
D_{\gamma} = D_{\gamma_1 e_1} \circ \cdots \circ D_{\gamma_n e_n},
\]

where \(e_1, \ldots, e_n \in \mathbb{N}_0^n\) are the canonical vectors.

Since the characteristic is \(p > 0\), it is worth mentioning that a binomial coefficient of integers might be zero when viewed as an element of \(R\). In fact, \(\binom{a}{b}\) is zero in \(R\) if and only if \(\binom{a}{b} \equiv 0 \mod p\). Lucas’s theorem states that if \(a = \sum a_i p^i\) and \(b = \sum b_i p^i\) are the p-adic expansions of nonnegative integers \(a\) and \(b\), then \(\binom{a}{b} \equiv \prod \binom{a_i}{b_i} \mod p\); in particular,

\[
\binom{a}{b} \neq 0 \mod p \quad \text{if and only if} \quad b_i \leq a_i, \quad \forall i.
\]

### 3.7. Compatibility of the action of Taylor operators with the q-expansions

Let \(q\) be a power of \(p\). There is a compatibility of the action of Taylor operators \(D_{\gamma}\), when \(\gamma \in \mathcal{A}_q\), with q-expansions of elements of \(R\). To express this, note first that \(D_{\gamma}\) is \(R^q\)-linear, for it is the composition of differential operators of order \(\leq q - 1\) by (3.12), and these are \(R^q\)-linear by (3.12). Therefore, if \(f = \sum_{\mathcal{A}_q} c_\alpha z^{\alpha}\) is the q-expansion of an element \(f \in R\), then we have

\[
D_{\gamma}(f) = \sum_{\mathcal{A}_q} c_\alpha \binom{\alpha}{\gamma} z^{\alpha - \gamma}.
\]

The important observation is that this is the q-expansion of \(D_{\gamma}(f)\) with respect to the given p-basis. Indeed, \(\binom{\alpha}{\gamma} = \binom{\alpha}{\gamma} q\) in \(R\), and if \(\binom{\alpha}{\gamma} z^{\alpha - \gamma} = \binom{\alpha}{\gamma} z^{\alpha' - \gamma} \neq 0\), then we have \(\alpha = \alpha'\).

The following proposition recalls the fact that F-finite regular local rings admit p-basis.

**Proposition 3.8.** Let \((R, m)\) be an F-finite regular local ring, and let \(\{x_1, \ldots, x_r\}\) be a regular system of parameters. Then \(\{x_1, \ldots, x_r\}\) can be extended to a p-basis for \(R\). More precisely, given a subset \(\{y_1, \ldots, y_s\} \subset R\), the following conditions are equivalent.

1. The classes of \(y_1, \ldots, y_s\) modulo \(m\) form a p-basis for the residue field \(R/m\).
2. \(\{x_1, \ldots, x_r, y_1, \ldots, y_s\}\) is a p-basis for \(R\).

**Proof.** The existence of a p-basis for \((R, m)\) follows from the main result of [19]. The result of [20] ensures that a p-basis for \(R\) is the same thing as a differential basis. Thus the proposition follows from the exact sequence \(0 \to m/m^2 \to \Omega_R^1 \otimes R/m \to \Omega^1_{R/m} \to 0\) proved in [18, Satz 1, (b)].

**Definition 3.9.** An ordered p-basis \(\{x_1, \ldots, x_r, y_1, \ldots, y_s\}\) for \((R, m)\) as in Proposition 3.8 is called an adapted p-basis.
We denote by $D_{\alpha, \beta}$, $(\alpha, \beta) \in \mathbb{N}_0^{r+s}$, the Taylor operators associated with an adapted $p$-basis $(x_1, \ldots, x_r, y_1, \ldots, y_s)$. The set $A_q$ takes the form

$$A_q = \{(\alpha, \beta) \in \mathbb{N}_0^{r+s} : 0 \leq \alpha, \beta_j < q, 1 \leq i \leq r, 1 \leq j \leq s\}.$$ 

We also denote $A_q^+ := A_q \setminus \{(0,0)\}$. Given $f \in R$, its $q$-expansion has the form

$$f = \sum_{A_q} c_{\alpha, \beta}^q x^\alpha y^\beta = c_{0,0}^q + \sum_{A_q^+} c_{\alpha, \beta}^q x^\alpha y^\beta.$$ 

The first part of the following proposition expresses the order of an element $f$ in $(R, m)$ in terms of the $q$-expansion with respect to an adapted $p$-basis. The second part concerns the condition $f \in R^q + m^n$, which will be particularly useful in the study of $R^q$ submodules of $R$.

**Proposition 3.10.** Let $(R, m)$ be an $F$-finite regular local ring, and let $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ be an adapted $p$-basis. Given $f \in R$ with $q$-expansion $f = \sum_{(\alpha, \beta) \in A_q} c_{\alpha, \beta}^q x^\alpha y^\beta$, for any integer $n$, the following holds.

1. $f \in m^n$ if and only if $c_{\alpha, \beta}^q x^\alpha y^\beta \in m^n$ for all $(\alpha, \beta) \in A_q$.
2. $f \in R^q + m^n$ if and only if $c_{\alpha, \beta}^q x^\alpha y^\beta \in m^n$ for all $(\alpha, \beta) \in A_q^+$.

**Proof.** (1) For each integer $t \geq 0$ we denote by $M_t$ the Abelian subgroup of $R$ generated by the elements of $m^t$ that are of the form $c x^\alpha y^\beta$ with $c \in R$ and $(\alpha, \beta) \in A_q$. By definition we have that $M_t \subseteq m^t$ for all $t \geq 0$.

We claim that these are all equalities. This will imply (1) by the uniqueness of the $q$-expansion. To show the claim, we first observe that $M_t M_u \subseteq M_{t+u} \subseteq m^{t+u}$. In fact, given $c x^\alpha y^\beta \in m^t$ and $c' x^{\alpha'} y^{\beta'} \in m^u$, with $(\alpha, \beta), (\alpha', \beta') \in A_q$, the product $(cc') x^{\alpha+\alpha'} y^{\beta+\beta'}$ belongs to $m^{t+u}$, and it can be clearly expressed in the form $c x^{\alpha''} y^{\beta''}$ with $(\alpha'', \beta'') \in A_q$. In view of this observation, our claim will follow if we show that $M_1 = m$, or say $m \subseteq M_1$. Fix $g \in m$, say with $x$-expansion $g = \sum_{A_q} d_{\alpha, \beta}^q x^\alpha y^\beta$. If we reduce this equality modulo $m$, we get $0 = \sum_{(\alpha, \beta) \in A_q} d_{\alpha, \beta}^q \bar{y_1}^{\beta_1} \cdots \bar{y_s}^{\beta_s}$, where the bar stands for reduction modulo $m$. Given that $\bar{y_1}, \ldots, \bar{y_s}$ is a $p$-basis of $R/m$, the coefficients $d_{\alpha, \beta}^q$ have to be zero, that is, $d_{\alpha, \beta} \in m$. Therefore $g \in M_1$. This proves the inclusion $m \subseteq M_1$, and hence the proof of our claim is complete.

(2) For the nontrivial direction, given $f \in R^q + m^n$, we have $f = \lambda^q + g$ with $g \in m^m$. We write $g$ in its $q$-expansion, say $g = \sum_{A_q} d_{\alpha, \beta}^q x^\alpha y^\beta$. By (1) $d_{\alpha, \beta}^q x^\alpha y^\beta \in m^n$ for all $(\alpha, \beta) \in A_q$. The conclusion follows as clearly $c_{\alpha, \beta} = d_{\alpha, \beta}$ for all $(\alpha, \beta) \in A_q^+$. \hspace{1cm} $\square$

For the following corollary, we recall that if $R$ is a regular local ring, and if $p \subset R$ is a regular prime, that is, if $R/p$ is regular, then the symbolic powers and the usual powers of $p$ coincide. In other words, $p^n = (p^n R_p) \cap R$.

**Corollary 3.11.** Let $(R, m)$ be an $F$-finite regular local ring, and let $p \subset R$ be a regular prime. Then $R^q + p^n = (R^q + p^n R_p) \cap R$ for all $n \geq 0$.

**Proof.** The inclusion $\subseteq$ is clear. We now prove the reverse inclusion. Choose an adapted $p$-basis $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ for $R$ so that the first $x_1, \ldots, x_r$ (say) generate $p$. Fix $f \in (R^q + p^n R_p) \cap R$, say with $x$-expansion $f = \sum_{A_q} c_{\alpha, \beta}^q x^\alpha y^\beta$. It follows from the definition of $p$-basis that $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ is also a $p$-basis for $R_p$, and it is in fact an adapted $p$-basis for $(R_p, p R_p)$. The $q$-expansion of $f$ (viewed in $R_p$) remains the same. Now by Proposition 3.10 (2) $c_{\alpha, \beta}^q x^\alpha y^\beta \in p^n R_p$ for all $(\alpha, \beta) \in A_q^+$. Thus $c_{\alpha, \beta}^q x^\alpha y^\beta \in (p^n R_p) \cap R = p^n$ if $(\alpha, \beta) \in A_q^+$, and therefore $f \in R^q + p^n$. This proves the inclusion $\supseteq$ and ends the proof of the corollary. \hspace{1cm} $\square$

### §3.3 On the Definition of Invariants for Points $x \in \delta(F_d(X))$. We now consider the two numerical values $v_m^{(q)}(f)$ and $v_m(f)$ mentioned at the introduction of the paper (and of this section). These numerical values can be interpreted, to some extent, as analogues of the order of an ideal, but applied to modules. Lemma 3.17 will establish the connection between these two invariants. The proof of this lemma relies strongly on adapted $p$-basis and Taylor differential operators.

**3.12. The $q$-Order of an $R^q$-Module.** Let $(R, m)$ be an $F$-finite regular local ring and fix $q = p^e$, a power of $p$. For $f \in R$, we define the $q$-order of $f$ as

$$v_m^{(q)}(f) := \sup \{n \in \mathbb{N}_0 : f \in R^q + m^n \}. $$
Note that $\nu_m^{(q)}(f) = \sup \{ \nu_m(g) : g - f \in R^q \}$. In particular, $\nu_m(f) \leq \nu_m^{(q)}(f)$. As a matter of fact, the equality holds if $q \nmid \nu_m(f)$: Suppose that $q \nmid \nu_m(f)$ and $\nu_m(f) < \nu_m^{(q)}(f)$; then $f = \lambda^q + g$ for some $\lambda, g \in R$ with $\nu_m(g) > \nu_m(f)$, so $\nu_m(f) = \nu_m(\lambda^q) = q\nu_m(\lambda)$, which is a contradiction.

We extend the above definition to any subset $S \subset R$ by setting $\nu_m^{(q)}(S) := \sup \{ n \in \mathbb{N}_0 : S \subseteq R^q + m^n \}$. Note that $\nu_m^{(q)}(S) = \min \{ \nu_m(f) : f \in S \}$. Since $R^q + m^n$ is an $R^q$-subalgebra of $R$, we obtain the following:

**Lemma 3.13.** $\nu_m^{(q)}(S) = \nu_m^{(q)}(R^q[S])$, where $R^q[S]$ is the $R^q$-subalgebra of $R$ generated by $S$.

3.14. Fix now an adapted $p$-basis $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ for $(R, m)$. Given $f \in R$, we consider its $q$-expansion, say $f = \sum A_q \epsilon_{\alpha, \beta}^q x^\alpha y^\beta$. Note that by Proposition 3.10 (2)

$$\nu_m^{(q)}(f) = \min \{ \nu_m(\epsilon_{\alpha, \beta}^q x^\alpha y^\beta) : (\alpha, \beta) \in A_q^+ \}.$$  

This implies that $\nu_m^{(q)}(f)$ is infinite if and only if $f \in R^q$. It also shows that

$$f^+ := \sum_{A_q^+} \epsilon_{\alpha, \beta}^q x^\alpha y^\beta,$$

which is an element whose definition depends on the choice of the adapted $p$-basis, has intrinsically defined order. In fact, by Proposition 3.10 (1)

$$\nu_m(f^+) = \min \{ \nu_m(\epsilon_{\alpha, \beta}^q x^\alpha y^\beta) : (\alpha, \beta) \in A_q^+ \} = \nu_m^{(q)}(f).$$

3.15. **Definition of $\eta_m(M)$**. We have introduced the $q$-order $\nu_m^{(q)}(f)$ of an element $f \in R$. This definition has some similarities with the usual order of $f$ at the local ring, although both notions are different. For example, as noted previously, $\nu_m^{(q)}(f) = \nu_m(f)$ whenever $q \nmid \nu_m(f)$. However, $\nu_m^{(q)}(1) = \infty$.

The notion of $q$-order was extended to $R^q$-submodules $M \subset R$. Lemma 3.13 expresses the compatibility of this definition with the two notions of equivalence for $R^q$-modules introduced in Definition 2.1. As was indicated in Example 1.5, the drawback of the $q$-order is that it does not define an upper-semicontinuous function in the latter fact in the next section.

Then, for any $i = 1, \ldots, q - 1$, and for any nonempty subset $S \subset R$,

$$\text{Diff}^i_{R,\Lambda,+}(S) = \text{Diff}^i_{R,\Lambda,+}(R^q[S]).$$

**Proposition 3.16.** Fix a ring $R$ of characteristic $p$, a finite collection of ideals $\Lambda$, and $q$, a power of $p$. Then for any $i = 1, \ldots, q - 1$, and for any nonempty subset $S \subset R$,

$$\text{Diff}^i_{R,\Lambda,+}(S) = \text{Diff}^i_{R,\Lambda,+}(R^q[S]).$$

**Proof.** We only need to show that $\text{Diff}^i_{R,\Lambda,+}(R^q[S]) \subseteq \text{Diff}^i_{R,\Lambda,+}(S)$. This follows from the following argument. Given $r \in R$ and $D \in \text{Diff}_{R,\Lambda}$, we denote by $Dr$ the element in $\text{Diff}_{R,\Lambda}$ obtained from the structure of right $R$-module, whereas $D(r)(e) \in R$ denotes the evaluation. Given now elements $f, g \in R$ and $D \in \text{Diff}^i_{R,\Lambda,+}$, we obtain $D(fg) = ((D(\lambda) - D(f_1)\lambda)(g) + gD(f)$. Note that $D(f - D(f)1_R)$ and $gD$ are clearly elements of $\text{Diff}^i_{R,\Lambda,+}$; therefore $\text{Diff}^i_{R,\Lambda,+}(fg) \subseteq \text{Diff}^i_{R,\Lambda,+}(f) + \text{Diff}^i_{R,\Lambda,+}(g) =: \text{Diff}^i_{R,\Lambda,+}\{(f, g)\}$. An inductive argument then shows that $\text{Diff}^i_{R,\Lambda,+}(f_1 \cdots f_r) \subseteq \text{Diff}^i_{R,\Lambda,+}\{(f_1, \ldots, f_r)\}$. Since the elements of $\text{Diff}^i_{R,\Lambda,+}$ are $R^q$-linear, we conclude that $\text{Diff}^i_{R,\Lambda,+}(R^q[S]) \subseteq \text{Diff}^i_{R,\Lambda,+}(S)$.}

The following lemma shows the connection between the two numerical values that we have introduced.

**Lemma 3.17.** Fix $f \in R$, and fix an adapted $p$-basis $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ for $(R, m)$. We consider the $q$-expansion of $f$, say $f = \sum_{A_q} \epsilon_{\alpha, \beta}^q x^\alpha y^\beta$. Then we have the inequality $\nu_m^{(q)}(f) \leq \eta_m(f)$. 


Assume now that \( \nu_m^{(q)}(f) < \infty \), say
\[
\nu_m^{(q)}(f) = qa + b,
\]
where \( 0 \leq b < q \).

Then the conclusions in each of the following three cases hold:

Case 1: \( b > 0 \). In this case, \( \nu_m(\text{Diff}^b_{R^b+}(f)) = qa \), and hence we have the equality \( \nu_m^{(q)}(f) = \eta_m(f) \).

Case 2: \( b = 0 \), and there exists a term \( t_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta \) of order \( \nu_m^{(q)}(f) \) with \( \alpha \neq 0 \). In this case, \( \nu_m(\text{Diff}^{q-1}_{R^{q-1}+}(f)) < qa \), and we also have the equality \( \nu_m^{(q)}(f) = \eta_m(f) \).

Case 3: \( b = 0 \), and the only terms of the \( q \)-expansion of \( f \) different from \( t_{\alpha, \beta} \) and of order \( \nu_m^{(q)}(f) \) are all of the form \( t_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta \). In this case, \( \nu_m(\text{Diff}^{q-1}_{R^{q-1}+}(f)) \leq qa \), and \( qa := \nu_m^{(q)}(f) < \eta_m(f) < q(a+1) \).

In particular, \( qa + b := \nu_m^{(q)}(f) \leq \eta_m(f) < q(a+1) \), and the inequality in the middle is an equality if \( q \nmid \nu_m^{(q)}(f) \).

Proof. If \( f \in R^q + m^a \), then for any \( i = 1, \ldots, q-1 \), \( \text{Diff}^i_{R^b+}(f) \subseteq \text{Diff}^i_{R^b+}(m^a) \subseteq m^{a-i} \) by (3.4) and (3.3), whence \( n \leq \nu_m(\text{Diff}^i_{R^b+}(f)) + i \). This shows that \( n \leq \eta_m(f) \) by the definition of \( \eta_m(f) \). We conclude that \( \nu_m^{(q)}(f) \leq \eta_m(f) \) by the definition of \( \nu_m^{(q)}(f) \).

Suppose now that \( \nu_m^{(q)}(f) < \infty \), that is, \( f \in R^q + m \) and \( f \notin R^q \).

Assume that we are in Case 1, that is, \( b > 0 \). Then by (3.15) there exists a term \( t_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta \) with \( (\alpha, \beta) \in \mathbb{A}_q^+ \) such that \( qa + b = \nu_m(c_{\alpha, \beta} x^\alpha y^\beta) = q\nu_m(c_{\alpha, \beta} + |\alpha|) \). Since \( |\alpha| = b \) mod \( q \), not all entries of \( \alpha \) are zero. Let \( p' < q \) be the highest power of \( p \) that divides every \( \alpha_i \). Note that \( p' \) divides \( b \) since \( \sum \alpha_i = b \) mod \( q \), whence \( p' \leq b < q \). Note also that by the maximal property of \( p' \), there is some index \( i_0 \) such that \( p' \) is the highest power of \( p \) dividing \( \alpha_{i_0} \). Let \( \gamma(1) \in \mathbb{N}_0^b \) be the multinomial that has \( p' \) at its \( i_0 \)-entry and zeros at the other entries. The term \( D_{\gamma(1),0}(c_{\alpha, \beta} x^\alpha y^\beta) = (\alpha_{i_0}) c_{\alpha, \beta} x^{\alpha_{i_0}-\gamma(1)} y^\gamma \) is a nonzero term in the \( q \)-expansion of \( D_{\gamma(1),0}(f) \) by (3.7) and since \( (\alpha_{i_0}) \neq 0 \) mod \( p \) (see (3.14)). The order of this term is \( qa + b - p' \). Therefore \( D_{\gamma(1),0}(f) \) has the order \( qa + b - p' \) by (3.3) and Proposition \( \ref{prop:order} \). If \( p' = b \), then we are done. Otherwise, \( 0 < p' < b \), and we apply the same reasoning with \( f \) replaced by \( D_{\gamma(1),0}(f) \), \( c_{\alpha, \beta} x^\alpha y^\beta \) by \( (\alpha_{i_0}) c_{\alpha, \beta} x^{\alpha_{i_0}-\gamma(1)} y^\gamma \), and \( b \) by \( b - p' \), and we continue in this way. At the end, we will obtain a differential operator of the form \( D_{\gamma(1),0}(f) \) such that \( \nu_m(D_{\gamma(1),0}(\cdots D_{\gamma(1),0}(f))) = qa \) and \( |\gamma(1)| + \cdots |\gamma(1)| = b \). Since this operator has order \( b \), by (3.3) we obtain \( \nu_m(\text{Diff}^{b-1}_{R^{b-1}+}(f)) = qa \), whence \( \eta_m(f) \leq \nu_m(\text{Diff}^{b-1}_{R^{b-1}+}(f)) + b = \nu_m^{(q)}(f) \). Combining this with the conclusion of the first paragraph, we finally obtain that \( \eta_m(f) = \nu_m^{(q)}(f) \).

Assume now that we are in Case 2, so \( \nu_m^{(q)}(f) = qa \), and there exists a term \( c_{\alpha, \beta} x^\alpha y^\beta \) with \( (\alpha, \beta) \in \mathbb{A}_q^+ \) such that \( \nu_m(c_{\alpha, \beta} x^\alpha y^\beta) = \nu_m^{(q)}(f) \) and \( \alpha \neq 0 \). We may assume that \( \alpha \neq 0 \). Set \( \alpha' := (\alpha_1, 0, \ldots, 0) \in \mathbb{N}_0^b \). Note that \( D_{\alpha',0} \) has order \( 0 \) lower than \( \alpha \), and hence \( \alpha' \) has the nonzero term \( c_{\alpha',0} x^\alpha y^\beta \) in its \( q \)-expansion by (3.7). Since this term has the order \( \nu_m^{(q)}(f) - \alpha_1 \), we obtain that \( \nu_m(D_{\alpha',0}(f)) = \nu_m^{(q)}(f) - |\alpha'| \) by (3.3) and Proposition \( \ref{prop:order} \). It follows that \( \nu_m(\text{Diff}^{[\alpha]}_{R^{[\alpha]}+}(f)) + |\alpha'| = qa = \nu_m^{(q)}(f) \), hence \( \eta_m(f) \leq \nu_m^{(q)}(f) \). This has to be an equality by the conclusion of the first paragraph.

We finally assume that we are in Case 3, so again \( \nu_m^{(q)}(f) = qa \), but now we can decomposes \( f^+ := f - c_{0,0}^q \) as \( f_m + f_h \), where \( f_m \neq 0 \) is the sum of those terms \( c_{0,0}^q \) of order \( \nu_m^{(q)}(f) = \nu_m(f^+) \), and \( f_h \) is the sum of higher-order terms. Any \( D \in \text{Diff}^{q-1}_{R^{q-1}} \) is \( R^q \)-linear, and hence \( \nu_m(D(f_m)) \geq \nu_m^{(q)}(f) \), as we easily check. We conclude that for any \( i = 1, \ldots, q-1 \) and for any \( D \in \text{Diff}^{i}_{R^{i}+} \), \( D(f) \) is the sum of \( D(f_m) \), which is an element of order \( \geq \nu_m^{(q)}(f) \), and \( D(f_h) \), which is an element of order \( \geq \nu_m(f_h) - i > \nu_m^{(q)}(f) - i \) by (3.3). Therefore \( D(f) \) has the order \( > \nu_m^{(q)}(f) - i \). This proves that \( \eta_m(f) > \nu_m^{(q)}(f) \). Finally, there exists at least one nonzero term \( c_{0,0}^q x^0 y^0 \) of \( f_m \), and we may assume that \( \beta_1 \neq 0 \). We define \( \beta' := (\beta_1, 0, \ldots, 0) \in \mathbb{N}_0^b \). Since \( D_{0,0}^{[\beta]}(f) \) has the term \( D_{0,0}^{[\beta]}(c_{0,0}^q x^0 y^0) = c_{0,0}^{[\beta]} x^{\beta_1} y^{\beta_2} \cdots y^{\beta_q} \) in its \( q \)-expansion (again by (3.7), we conclude by Proposition \( \ref{prop:order} \) that \( D_{0,0}^{[\beta]}(f) \) has the order \( \leq \nu_m^{(q)}(f) \). Therefore \( \nu_m(D_{0,0}^{[\beta]+}(f)) \leq qa \), and so \( \eta_m(f) \leq qa + (q-1) < q(a+1) \). This proves the conclusion in Case 3. \( \square \)

**§3.4 Applications of Lemma 3.17** We present some corollaries of Lemma 3.17. The first corollary states a particular case of a more general and classical result of commutative algebra, namely, which expresses
the order of an element at a regular local ring in terms of differential operators. In the second corollary, we characterize the subring \( R^i \) in terms of differential operators. The last two corollaries are technical, and they will be used in Section 4.2 to give a characterization of permissible centers for a pair \((\mathcal{M}, a)\). This characterization will be useful for the study of transformation of \( \mathcal{O}_V^n \)-modules.

**Corollary 3.18.** Let \((R, m)\) be an \( F\)-finite regular local ring, and let \( f \in R \). Then for \( n \geq 1 \), \( f \in m^n \) if and only if \( \text{Diff}^{-1}_{R^i}(f) \subseteq m \).

Proof. If \( f \in m^n \), then \( \text{Diff}^{-1}_{R^i}(f) \subseteq m \) by (3.3). Suppose now that \( f \notin m^n \). If \( f \notin m \), then \( \text{Diff}^{-1}_{R^i}(f) \notin m \) since \( \text{Diff}^{-1}_{R^i} \) contains the identity. If \( f \in m \), then we choose \( q \), a power of \( p \), which is greater than \( n \). Note that \( 0 < \nu_m(f) < q \), whence \( \nu_{m}^{(q)}(f) = \nu_m(f) \), as was noted in (3.12). Note that we are in Case 1 of Lemma 3.17 with \( a = 0 \) and \( b := \nu_{m}^{(q)}(f) > 0 \); therefore \( \nu_m(\text{Diff}^{b}_{R^i}(f)) = 0 \). Since \( b \leq n - 1 \), we conclude that \( \text{Diff}^{-1}_{R^i}(f) \notin m \). \( \square \)

**Corollary 3.19.** Let \( R \) be an \( F\)-finite regular local ring. Then

\[ R^q = \{ f \in R : \text{Diff}^{-1}_{R^i}(f) = 0 \}. \]

Proof. The inclusion \( \subseteq \) follows from (3.3). Conversely, if \( \text{Diff}^{-1}_{R^i}(f) = 0 \), then \( \text{Diff}^{i}_{R^i}(f) = 0 \) for all \( i = 1, \ldots, q - 1 \), whence \( \nu_m(f) = \infty \). Therefore by Lemma 3.17, \( \nu_{m}^{(a)}(f) = \infty \). This implies that \( f \in R^q \), as was noted in (3.14). \( \square \)

**Corollary 3.20.** Let \((R, m)\) be an \( F\)-finite regular local ring. Given an integer \( a \geq 1 \), \( f \in R^q + m^{qa} \) if and only if \( \text{Diff}^{-1}_{R^i}(f) \subseteq m^{q(a-1)+1} \).

Proof. If \( f \in R^q + m^{qa} \), then \( qa \leq \nu_{m}^{(a)}(f) \leq \nu_m(f) \leq q - 1 + \nu_m(\text{Diff}^{-1}_{R^i}(f)) \) by Lemma 3.17; therefore \( \nu_m(\text{Diff}^{-1}_{R^i}(f)) \geq qa - (q - 1) = qa(a - 1) + 1 \), that is, \( \text{Diff}^{-1}_{R^i}(f) \subseteq m^{q(a-1)+1} \). If \( f \notin R^q + m^{qa} \), then \( \nu_{m}^{(a)}(f) < qa \), say \( \nu_{m}^{(a)}(f) = qa' + b \) with \( a' < a \) and \( 0 \leq b < q \). Thus by Lemma 3.17, we have the inequality \( \nu_m(\text{Diff}^{-1}_{R^i}(f)) \leq qa' \), whence \( \nu_m(\text{Diff}^{-1}_{R^i}(f)) + q - 1 < qa \), that is, \( \text{Diff}^{-1}_{R^i}(f) \subseteq m^{q(a-1)+1} \). This completes the proof. \( \square \)

**Corollary 3.21.** Let \((R, m)\) be an \( F\)-finite regular local ring, and let \( x \in R \) be a nonzero element such that \( R/xR \) is regular. Given \( f \in R \setminus R^q \), let \( n \) denote the largest integer such that \( \text{Diff}^{-1}_{R^i}(f) \subseteq x^n R \), and let \( a \) denote the largest integer such that \( f \in R^q + x^qa R \). Then \( n = qa \).

Proof. We can write \( f = \lambda^q + x^qa g \) for some \( \lambda, g \in R \). Then \( \text{Diff}^{-1}_{R^i}(f) = \text{Diff}^{-1}_{R^i}(x^qa g) = x^qa \text{Diff}^{-1}_{R^i}(g) \subseteq x^qa R \). This shows that \( qa \leq n \). If \( qa < n \), then \( \text{Diff}^{-1}_{R^i}(f) \subseteq x^qa+1 R \), and by Corollary 3.20 using (3.3) we obtain \( f \in R^q + x^{qa(q+1)+1} R \), which contradicts the definition of \( a \). We conclude that \( qa = n \). \( \square \)

§4 Differential Operator Techniques Applied to the Study of \( \mathcal{O}_V^n \)-Modules and Morphisms

Let \((R, m)\) be an \( F\)-finite regular local ring, and let \( \text{Spec}(R) \leftarrow V_1 \supset H_1 \) be the blowup at the closed point, where \( H_1 \) denotes the exceptional hypersurface. Here \( V_1 \) is regular, and if \( \xi \in H_1 \) denotes the generic point of \( H_1 \), then \( \mathcal{O}_{V_1, \xi} \) is a valuation ring. Given an ideal \( J \) in \( R \), \( J \subseteq m^n \) for a positive integer \( n \) if and only if \( J : \mathcal{O}_{V_1} = \mathcal{I}(H_1)^n J_1 \) for some \( \mathcal{O}_{V_1} \)-ideal \( J_1 \). In particular, \( J \) has order \( n \) if \( n \) is the valuation of \( J \) at \( \mathcal{O}_{V_1, \xi} \). So the order of an ideal \( J \) can be characterized either in terms of differential operators at \((R, m)\) (see Corollary 3.18) or by the order of vanishing of the total transform \( J \cdot \mathcal{O}_{V_1} \) along the exceptional hypersurface \( H_1 \).

Here \( V \) is an irreducible \( F\)-finite regular scheme, and we aim to study the \( \mathcal{O}_V^n \)-submodules of \( \mathcal{O}_V \), a task initiated in Section 2 always intending to draw parallels with the realm of ideals on \( \mathcal{O}_V \). The notion of order of an ideal at a local regular ring \( \mathcal{O}_V, x \) is replaced here by that of \( \eta_x(\mathcal{M}) \) for a given \( \mathcal{O}_V^n \)-submodule \( \mathcal{M} \) (see (4.17)).

Once we fix a regular center \( Z \subset V \) and an \( \mathcal{O}_V^n \)-submodule \( \mathcal{M} \) of \( \mathcal{O}_V \), we study an analog to the order of vanishing for the total transform \( \mathcal{M} \mathcal{O}_{V_1}^n \) (Definition 2.10) along the exceptional hypersurface \( H_1 \) at the blowup, say \( V \leftarrow V_1 \subset H_1 \). The function \( \eta_x(\mathcal{M}) \) will ultimately enable us, at the end of this section, to
characterize $a$-transforms and the notion of permissible center for $(\mathcal{M}, a)$ in Definition 3.11 (see Proposition 4.18).

The section begins by reviewing some results concerning the behavior of differential operators when blowing up along suitable centers, results which are essential in our forthcoming discussions.

### §4.1 Behavior of Differential Operators with Morphisms

We recall some facts about the behavior of differential operators with respect to homomorphisms of rings, which include those that arise when taking monoidal transformations along regular centers.

**Lemma 4.1.** Let $R$ be an $F$-finite regular local ring, and let $\varphi : R \rightarrow R'$ be any ring homomorphism from $R$. Given $D' \in \text{Diff}_{R'}^i$ (resp., $\text{Diff}_{R'}^{i,+}$), there are differential operators $D_1, \ldots, D_n \in \text{Diff}_R$ (resp., $\text{Diff}_R^{i,+}$) and scalars $r_1', \ldots, r_n' \in R'$ such that

$$D' \circ \varphi = r_1'(\varphi \circ D_1) + \cdots + r_n'(\varphi \circ D_n).$$

**Proof.** We view differential operators of order $\leq i$ from $R$ to an $R$-module $M$ as elements of $\text{Hom}_R(P_R^i, M)$, where $P_R^i$ denotes the $R$-module of principal parts of order $\leq i$. In this regard, $D' \circ \varphi \in \text{Hom}_{R'}(P_{R'}^i, R')$. The existence of a finite $p$-basis for $R$ implies that $P_R^i$ is free of finite rank; therefore $D' \circ \varphi$ is a linear combination, say $D' \circ \varphi = \sum D'((\varphi(u_k)))(\varphi \circ D_k)$, where $\{u_1, \ldots, u_n\}$ is a basis for $P_R^i$, and $D_1, \ldots, D_n$ is the dual basis. In the case that $D'(1) = 0$, we have $0 = D'(1) = \sum D'((\varphi(u_k)))(\varphi(D_k(1)))$, whence $D' \circ \varphi = \sum D'((\varphi(u_k)))(\varphi \circ D_k) = \sum D'((\varphi(u_k)))(\varphi \circ D_k(1)) \varphi = \sum D'((\varphi(u_k)))(\varphi \circ (D_k - D_k(1) \text{id}_R)).$

Notice that $D_k - D_k(1) \text{id}_R \in \text{Diff}_R^{i,+}$. This completes the proof of the lemma. □

Given a morphism $\varphi : R \rightarrow R'$ of rings of characteristic $p$, and given an ideal $I \subset R$, we denote by $IR'$, as usual, the ideal of $R'$ generated by $\varphi(I)$. If $M$ is an $R'$-submodule of $R$, then $MR'$ denotes the $R'$-submodule generated by $\varphi(M)$.

**Corollary 4.2.** Let $R$ be an $F$-finite regular local ring, and let $\varphi : R \rightarrow R'$ be a homomorphism of rings. Let $M \subset R$ be an $R'$-submodule, and let $I \subset R$ be an ideal. Then the following inclusions hold.

1. $\text{Diff}_R^{i}(MR'^q) \subset \text{Diff}_R^{i}(M)R'$ for $i = 1, \ldots, q - 1$.
2. $\text{Diff}_R^{i}(IR') \subset \text{Diff}_R^{i}(I)R'$, $\forall i \geq 0$.

**Proof.** (1) Since $i < q$, the elements of $\text{Diff}_R^{i,+}$ are $R'$-linear, and hence $\text{Diff}_R^{i}(MR'^q) = \text{Diff}_R^{i,+}(\varphi(M))$, and by the previous lemma this set is included in the $R'$-module generated by $\varphi(\text{Diff}_R^{i,+}(M))$.

(2) Using that $\text{Diff}_R^{i}(I)R' = \text{Diff}_R^{i}(\varphi(I))$, which follows, for instance, from the statement right below (4.1), the inclusions in (2) follow immediately from Lemma 4.1. □

### §4.3 Let $R$ be a domain, and let $P$ be any prime ideal, say generated by $x_1, \ldots, x_n$. Then the blowup of Spec$(R)$ along Spec$(R/P) \subset$ Spec$(R)$ is obtained by gluing, in the natural way, the affine schemes Spec$(R_i)$, $i = 1, \ldots, n$, where $R_i$ denotes $\{z \in \mathbb{C}_2^i : z \in P^t, t \geq 0\}$, which is an $R$-subalgebra of $R_{x_i}$. Note that for any integer $k \geq 1$, we have the following equalities:

$$P^k R_i = x_i^k R_i = \left\{ \frac{z}{x_i^t} \in R_{x_i} : z \in P^{k+t}, t \geq 0 \right\} \subset R_i.$$

As was mentioned in 3.1 a differential operator $D : R \rightarrow R$ can be extended uniquely to a differential operator of $R_{x_i}$ (which we still denote $D$); however, this extension might not restrict to a differential operator of $R_i$, that is, $D(R_i)$ might not be included in $R_i$. For example, take $R = k[x_1, x_2]$ and $P = (x_1, x_2)$. Then $\frac{\partial}{\partial x_1} = \frac{2}{x_1}$, which is not in $R_i := k[x_1, \frac{2}{x_1}]$. The following lemma shows that this does not happen when $D$ is $P$-logarithmic (in our example, we would take $x_1^{1/2}$).

**Lemma 4.4.** Within the setting of 4.3, given a $P$-logarithmic differential operator $D : R \rightarrow R$, its extension to a differential operator of $R_{x_i}$ restricts to $R_i$, that is, $D(R_i) \subset R_i$. Moreover, $D$ is $x_i$,$R_i$-logarithmic.

**Proof.** We may assume that $i = 1$. Fix $\frac{z}{x_1} \in x_1^k R_i$ with $z \in P^{t+k}$ (4.1). Let $q$ be a power of $p$ that is strictly greater than the order of $D$, so that $D$ is $R_{x_1}^q$-linear. Then $D(\frac{z}{x_1}) = D(\frac{2^{q-1}(z/x_1)}{x_1^{q-1}}) = \frac{D(z/x_1)}{x_1^{q-1}}$. Notice that $z x_1^{q-1} \in P^{k+q}$, and since $D$ is $P$-logarithmic, we have that $D(z x_1^{q-1}) \in P^{k+q}$. In conclusion, $D(\frac{z}{x_1}) \in x_1^{k+q} R_1$ by (4.1).
We have proved that $D(x_1^k R_1) \subseteq x_1^k R_1$ for all $k \in \mathbb{N}_0$; in particular, $D(R_1) \subset R_1$. Hence $D$ is a differential operator of $R_1$ that is $x_1 R_1$-logarithmic (and of the same order as $D$).

**Corollary 4.5.** Within the setting of 4.3, if we identify differential operators of a domain with their extension to the fraction field, then for each $i \geq 0$, there are inclusions
\begin{equation}
(4.2)
\begin{align*}
x_1^i \text{ Diff}_R & \subseteq \text{ Diff}_i R_1, \quad \text{and} \quad x_1^i \text{ Diff}_{R,+} \subseteq \text{ Diff}_{i R,+}.
\end{align*}
\end{equation}
In particular, if $M \subseteq R$ is an $R^i$-submodule, and $I \subset R$ is an ideal, then there are inclusions
\begin{align*}
x_1^i \text{ Diff}_{R,+}(M) R_1 & \subseteq \text{ Diff}_{iR,+}(M R_1^q), \quad \forall i = 1, \ldots, q - 1, \\
x_1^i \text{ Diff}_R(I) R_1 & \subseteq \text{ Diff}_R(I R_1), \quad \forall i \geq 0.
\end{align*}

**Proof.** Lemma 4.4 shows that $\text{Diff}_{R,P} \subseteq \text{Diff}_{R,R,P}$, so the inclusions in (4.2) are consequences of the inclusion $x_1^i \text{ Diff}_R \subseteq \text{ Diff}_{R,P}$ observed in 3.5.

The last inclusions in the corollary follow from (4.2). We only need to take into account that $\text{Diff}_{R,+}(M R_1^q) = \text{Diff}_{R,+}(M)$ if $i < q$ and $\text{Diff}_R(I R_1) = \text{Diff}_R(I)$ for any $i \geq 0$. □

**§ 4.2 Applications to $\mathcal{O}_V^q$-Submodules of $\mathcal{O}_V$.** In what follows, $V$ denotes a connected $F$-finite regular scheme, and $q = p^e$ is a fixed power of $p$. We make use of the properties of localization of differential operators established in 3.9 to translate the results from 3.9 and 4.1 formulated there at a local level, into the global setting of schemes and sheaves of modules. We then discuss the role played by the function $x \mapsto \eta_x(\mathcal{M})$ in the characterization of permissible centers and in the definition of the “right” notion of transformation of modules.

**4.6.** Let $\mathcal{I}$ be an $\mathcal{O}_V$-ideal, and let $\mathcal{M}$ be an $\mathcal{O}_V^q$-submodule of $\mathcal{O}_V$. Then by (3.10) there are $\mathcal{O}_V$-ideals $\text{Diff}_V^i(\mathcal{I})$ for all $i \geq 0$ and $\text{Diff}_V^i(\mathcal{M})$ for all $i = 1, \ldots, q - 1$ such that for any $x \in V$,
\begin{equation}
(4.3)
\begin{align*}
(\text{Diff}_V^i(\mathcal{I}))_x & = \text{Diff}_{\mathcal{O}_V,x}(\mathcal{I}_x), \quad \forall i \geq 0, \\
(\text{Diff}_V^i(\mathcal{M}))_x & = \text{Diff}_{\mathcal{O}_V,x}(\mathcal{M}_x), \quad \forall i = 1, \ldots, q - 1.
\end{align*}
\end{equation}

Propositions 4.11, 4.12 below are just translations of the results of 3.4 into the language of sheaves. This translation is possible thanks to (1.3). For instance, the following proposition, which we will call the absolute Jacobian criterion, is a consequence of its local version stated in Corollary 3.18. It is at the core of most arguments in our discussion.

**Proposition 4.7.** Let $\mathcal{J}$ be an $\mathcal{O}_V$-ideal. Then for any integer $n \geq 0$,
\begin{equation}
\{x \in V : \nu_x(\mathcal{J}) \geq n\} = \{x \in V : \nu_x(\text{Diff}_V^{n-1}(\mathcal{J})) \geq 1\}.
\end{equation}
In particular, the assignment $x \mapsto \nu_x(\mathcal{J})$ defines an upper semicontinuous function $V \rightarrow \mathbb{N}_0$.

We now characterize the $\mathcal{O}_V^q$-submodules of $\subseteq \mathcal{O}_V^q$.

**Proposition 4.8.** For an $\mathcal{O}_V^q$-module $\mathcal{M} \subseteq \mathcal{O}_V$, the following conditions are equivalent.
\begin{enumerate}
\item $\text{Diff}_V^{q-1}(\mathcal{M}) = 0$.
\item $\text{Diff}_{\mathcal{O}_V,x}^{q-1}(\mathcal{M}_x) = 0$ for some $x \in V$.
\item $\mathcal{M}_x \subseteq \mathcal{O}_V^q$ for some point $x \in V$.
\item $\mathcal{M} \subseteq \mathcal{O}_V^q$.
\end{enumerate}

**Proof.** The implication (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) follows from Corollary 3.19. If (3) holds, then certainly $\mathcal{M}_x \subseteq \mathcal{O}_V^q$, where $\xi \in V$ is the generic point, and $K$ denotes the function field of $V$. Next, given $y \in V$, clearly $\mathcal{M}_y \subseteq \mathcal{M}_x$, and hence $\mathcal{M}_y \subseteq K^q \cap \mathcal{O}_{V,y} = \mathcal{O}_V^q$, where the last equality holds since $\mathcal{O}_{V,y}$ is regular and hence proper. This proves (3) $\Rightarrow$ (4). Finally, (4) $\Rightarrow$ (1) follows from (3.4). □

We now describe $\text{Sing}(\mathcal{M},a) = \{x \in V : \mathcal{M}_x \subseteq \mathcal{O}_V^q + n_{V,x}^{qa}\}$ (Definition 2.11) as a closed subset of $V$.

**Proposition 4.9.** For an $\mathcal{O}_V^q$-module $\mathcal{M} \subseteq \mathcal{O}_V$ and an integer $a \geq 1$, we have
\begin{equation}
\text{Sing}(\mathcal{M},a) = \{x \in V : \nu_x(\text{Diff}_V^{q-1}(\mathcal{M})) \geq qa - 1 + 1\}.
\end{equation}
In particular, $\text{Sing}(\mathcal{M},1) \subset V$ is the closed subset defined by $\text{Diff}_V^{q-1}(\mathcal{M})$, and each $\text{Sing}(\mathcal{M},a)$ is also closed.
Proposition 4.13. The equality follows from Corollary 3.20 and the last statement follows from Proposition 4.7 when applied to \( J = \text{Diff}^{q-1}_V(\mathcal{M}) \) and \( n = q(a - 1) + 1 \).

\[ \square \]

Remark 4.10. In view of Remark 2.24, Proposition 4.9 completes the proof of theorems 1.14 and 1.12.

In Definition 2.24, Proposition 4.9 completes the proof of theorems 1.14 and 1.12.

Proposition 4.11. Let \( \mathcal{M} \subseteq O_V \) be an \( O_V^q \)-module, and let \( a \) be a positive integer. Given an irreducible regular subscheme \( Z \subset V \), the following conditions on \( Z \) are equivalent.

1. \( \mathcal{M} \subseteq O^q_V + \mathcal{I}(Z)^a \).
2. \( Z \) is included in \( \text{Sing}(\mathcal{M}, a) \).
3. \( \xi \in \text{Sing}(\mathcal{M}, a) \), where \( \xi \) denotes the generic point of \( Z \).
4. \( \nu_k(\text{Diff}^{q-1}_V(\mathcal{M})) \geq q(a - 1) + 1 \).
5. \( \text{Diff}^{q-1}_V(\mathcal{M}) \subseteq \mathcal{I}(Z)^{q(a - 1) + 1} \).

Proof. The implications (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are clear, and (3) \( \Rightarrow \) (1) is a consequence of Corollary 3.11 with \( n = qa \). The equivalence of (3) and (4) follows from Proposition 4.9, and the equivalence of (4) and (5) holds because \( Z \) is a regular subscheme in a regular scheme.

The following proposition expresses a peculiarity of regular centers of codimension one. It is a consequence of its local version stated in Corollary 3.24.

Proposition 4.12. Let \( \mathcal{M} \) be a nontrivial \( O_V^q \)-module, and let \( H \subset V \) be an irreducible regular hypersurface. Let \( n \) be the largest integer such that \( \text{Diff}^{q-1}_V(\mathcal{M}) \subseteq \mathcal{I}(H)^n \), and let \( a \) be the largest integer such that \( H \) is permissible for \( (\mathcal{M}, a) \). Then \( n = qa \). In particular, \( n \) is a multiple of \( q \).

The next two propositions present general properties of differential operators to be used in our study of transformations of modules and of ideals under morphisms. They are just consequences of Corollary 4.15 and Corollary 4.16.

Proposition 4.13. Let \( \pi : V_1 \rightarrow V \) be a morphism between \( F \)-finite regular schemes, let \( \mathcal{I} \) be an \( O_V \)-ideal, and let \( \mathcal{M} \subseteq O_V \) be an \( O_V^q \)-submodule. Then we have the following inclusions:

1. \( \text{Diff}^i_{V_1}(\mathcal{M}O_{V_1}) \subseteq \text{Diff}^i_{V_1}(\mathcal{M})O_{V_1} \) for \( i = 1, \ldots, q - 1 \).
2. \( \text{Diff}^i_{V_1}(\mathcal{I}O_{V_1}) \subseteq \text{Diff}^i_{V_1}(\mathcal{I})O_{V_1} \), \( \forall i \geq 0 \).

Proposition 4.14. Let \( V \overset{\pi}{\leftarrow} V_1 \supset H_1 \) be the blowup of \( V \) along an irreducible regular center \( Z \subset V \). Given an \( O_{V_1}^q \)-module \( \mathcal{M} \) and an \( O_V \)-ideal \( J \), we have the following inclusions:

1. \( \mathcal{I}(H_1)^i(\text{Diff}^{q-1}_V(\mathcal{M}O_{V_1})) \subseteq \text{Diff}^i_{V_1}(\mathcal{M}O_{V_1}) \) for \( i = 1, \ldots, q - 1 \).
2. \( \mathcal{I}(H_1)^i(\text{Diff}^b_{V_1}(\mathcal{J}O_{V_1})) \subseteq \text{Diff}^i_{V_1}(\mathcal{J}O_{V_1}) \), \( \forall i \geq 0 \).

As a first application of the previous result, we show how they lead to a proof of Theorem 1.17 concerning the behavior of the order of an ideal by blowups along regular centers.

Proposition 4.15. Let \( V \overset{\pi}{\leftarrow} V_1 \supset H_1 \) be the blowup of \( V \) along an irreducible regular center \( Z \subset V \). Assume that the mapping \( x \mapsto \nu_x(J) \) is constant along \( Z \), say \( \nu_x(J) = b \) for \( z \in Z \). Let \( J_1 \) be the \( O_{V_1} \)-ideal such that \( \mathcal{J}O_{V_1} = \mathcal{I}(H_1)^bJ_1 \). Then for any \( x_1 \in V_1 \),

\[ \nu_x(J_1) \leq \nu_x(J) \]

Proof. We only need to prove the claim for points \( x_1 \in H_1 \). In this case, we have to prove that \( \nu_x(J_1) \leq b \), or, equivalently, \( \nu_x(\mathcal{J}O_{V_1}) \leq 2b \). To this end, by Proposition 4.7, it suffices to show that \( \nu_x(\text{Diff}^{2b}_{V_1}(\mathcal{J}O_{V_1})) = 0 \). Observe that

\[ \text{Diff}^{2b}_{V_1}(\mathcal{J}O_{V_1}) \supseteq \text{Diff}^b_{V_1}(\mathcal{J}O_{V_1}) \overset{4.13}{\supseteq} \text{Diff}^b_{V_1}(\mathcal{I}(H_1)^bJ_1). \]

Given that \( \nu_{\pi(x_1)}(J) = b \), Proposition 4.7 ensures that \( (\text{Diff}^b_{V_1}(\mathcal{J}))_{\pi(x_1)} = O_{V_1, \pi(x_1)} \), whence

\[ \text{Diff}^b_{V_1}(\mathcal{I}(H_1)^bJ_1)_{x_1} = \text{Diff}^b_{V_1}(\mathcal{I}(H_1)^bJ_1)_{x_1} \overset{4.13}{\subseteq} O_{V_1, x_1}. \]

We conclude that \( \text{Diff}^{2b}_{V_1}(\mathcal{J}O_{V_1})(x_1) = O_{V_1, x_1}, \) that is, \( \nu_{x_1}(\text{Diff}^{2b}_{V_1}(\mathcal{J}O_{V_1})) = 0 \), as was to be proved. \( \square \)
We add another application, which is of interest for the interpretation of the transformations of modules as blowups stated in Corollary 2.25(b).

**Proposition 4.16.** Let \( \mathcal{M} \subseteq O_V \) be an \( \mathcal{O}_V^n \)-module, and let \( a \) be a positive integer. Let \( Z \subset V \) be a closed irreducible and regular subscheme, and consider the blowup \( V' \leftarrow V_1 \supset H_1 \) of \( V \) along \( Z \), where \( H_1 \) denotes the exceptional divisor. Then \( Z \) is permissible for \((\mathcal{M}, a)\) if and only if \( H_1 \) is permissible for \((\mathcal{M} \mathcal{O}_{V_1}^q, a)\).

**Proof.** If \( Z \) is permissible for \((\mathcal{M}, a)\), then \( H_1 \) is permissible for \((\mathcal{M} \mathcal{O}_{V_1}^q, a)\) by Proposition 2.13. Conversely, if \( H_1 \) is permissible for \((\mathcal{M} \mathcal{O}_{V_1}^q, a)\), then by Proposition 4.12, \( \operatorname{Diff}_{V_1}^{q-1}(\mathcal{M} \mathcal{O}_{V_1}^q) \subseteq \mathcal{I}(H_1)^{qa} \). This implies by Proposition 4.14 that \( \mathcal{I}(H_1)^{q-1}(\operatorname{Diff}_{V_1}^{q-1}(\mathcal{M})\mathcal{O}_{V_1}) \subseteq \mathcal{I}(H_1)^{qa} \), that is, \( \operatorname{Diff}_{V_1}^{q-1}(\mathcal{M})\mathcal{O}_{V_1} \subseteq \mathcal{I}(H_1)^{(q(a-1)+1)} \). Therefore \( \operatorname{Diff}_{V_1}^{q-1}(\mathcal{M}) \subseteq \mathcal{I}(Z)^{(q(a-1)+1)} \), and hence by Proposition 4.11 \( Z \) is permissible for \((\mathcal{M}, a)\). \( \square \)

**4.17.** Fix an \( \mathcal{O}_V^n \)-submodule \( \mathcal{M} \subseteq O_V \). Then for each \( x \in V \), we get a local ring \( O_{V,x} \) and an \( \mathcal{O}_{V,x} \)-submodule \( \mathcal{M}_x \subset O_{V,x} \). Therefore we have two numerical functions \( \nu_x(q) \) of \( \mathcal{M}_x \) and \( \eta_x(\mathcal{M}) := \eta_{m_{V,x}}(\mathcal{M}_x) \); see Section 3.3. We get two numerical functions \( V \rightarrow \mathbb{N}_0 \), namely \( x \mapsto \nu_x(q) \) and \( x \mapsto \eta_x(\mathcal{M}) \). These functions are intrinsic to the \( \mathcal{O}_V^n \)-subalgebra generated by \( \mathcal{M} \), as was observed in Lemma 3.13 and Proposition 3.16. Note that

\[
\eta_x(\mathcal{M}) = \min \{ \nu_x(q) + i : i = 1, \ldots, q-1 \},
\]

and from this expression we see that the function \( x \mapsto \eta_x(\mathcal{M}) \) is upper-semicontinuous. Indeed, by Proposition 4.17 it is the minimum of upper semicontinuous functions.

Lemma 3.17 shows that \( \nu_x(q) \leq \eta_x(\mathcal{M}) \) for all \( x \in V \), and the equality holds, for instance, when \( \nu_x(q) \) is not divisible by \( q \).

The next proposition provides another description for \( \operatorname{Sing}(\mathcal{M}, a) \) (see Proposition 4.9), this time by using \( \eta_x(\mathcal{M}) \).

**Proposition 4.18.** Let \( \mathcal{M} \subset O_V \) be an \( \mathcal{O}_V^n \)-module. Then for any integer \( a \geq 0 \), we have that \( \{ x \in V : \eta_x(\mathcal{M}) \geq qa \} = \operatorname{Sing}(\mathcal{M}, a) \) (Definition 2.7). In particular, an irreducible regular subscheme \( Z \subset V \) is permissible for \((\mathcal{M}, a)\) if and only if \( \eta_x(\mathcal{M}) \geq qa \) for all \( x \in Z \) (see Definition 2.7).

**Proof.** It follows from Lemma 3.17 that \( \{ x \in V : \eta_x(\mathcal{M}) \geq qa \} = \{ x \in V : \nu_x(q) \geq qa \} \), and the latter is clearly \( \operatorname{Sing}(\mathcal{M}, a) \) by the definition of \( \eta_x(\mathcal{M}) \). The last conclusion is clear from the definition. \( \square \)

**Remark 4.19.** This proposition, together with Proposition 4.16 and Corollary 2.25, covers the proof of (1) and (2) of Theorem 1.18.

**4.20.** The role played by \( \eta_x(\mathcal{M}) \) in the description of \( \delta(F_d(X)) \) and in the definition of the transformation of \( \mathcal{O}_V^n \)-modules. Let \( \mathcal{M} \subset O_V \) be an \( \mathcal{O}_V^n \)-module, let \( \delta : X \rightarrow V \) be the \( V \)-scheme attached to \( \mathcal{M} \), and set \( d := [K(X) : K(V)] \), the generic rank. Then \( \delta(F_d(X)) = \operatorname{Sing}(\mathcal{M}, 1) = \{ x \in V : \eta_x(\mathcal{M}) \geq q \} \), where the first equality was stated in Proposition 2.13 and the second one follows from Proposition 4.18 with \( a = 1 \). Therefore we can use the upper-semicontinuous function \( x \mapsto \eta_x(\mathcal{M}) \) to stratify \( F_d(X) \cong \delta(F_d(X)) \).

In this regard, it is natural to blow up \( V \) along a regular center, say \( Z \), included in the closed set where the semi-continuous function \( \eta_x(\mathcal{M}) \) attains its maximum value, say \( qa + b \) with \( 0 \leq b < q \). We are only interested in the case where \( a \geq 1 \) (which means that \( F_d(X) \neq \emptyset \)).

Let \( V \leftarrow V_1 \) be the blowup along such center \( Z \), and let \( H_1 \subset V_1 \) be the exceptional divisor. Proposition 4.18 implies that \( Z \) is permissible for \((\mathcal{M}, a)\) but not for \((\mathcal{M}, a+1)\). Then Proposition 4.16 implies that \( H_1 \) is permissible for \((\mathcal{M} \mathcal{O}_{V_1}^q, a)\) but not for \((\mathcal{M} \mathcal{O}_{V_1}^q, a + 1)\). Finally, Corollary 2.25 shows that \( \mathcal{M}_1(\mathcal{M}) := (\mathcal{O}_{V_1} + \mathcal{M} \mathcal{O}_{V_1} : F^n(\mathcal{I}(H_1)^{qa})) \) is the “right” notion of transform of \( \mathcal{M} \) in the sense that if \( \delta_1 : X_a \rightarrow V_1 \) is the associated \( V_1 \)-scheme in \( \mathcal{E}_a(V_1) \), then \( \delta_a(F_d(X_a)) \) does not include \( H_1 \). However, we should mention...
that with this notion of transformation, the function \( \eta \) does not always satisfy the pointwise inequality: 
\[
\eta_{\nu_x}(\mathcal{M}) \geq \eta_{\nu_x}(\mathcal{M}_a)
\]
for \( x_1 \in V_1 \), that is, there is no analog to Proposition 4.15; see the example discussed in Remark 1.19. This problem is due essentially to the fact that although we may factorize \( F^e(\mathcal{I}(H_1)^n) \) from \( \mathcal{M} \mathcal{O}_{V_1}^q \), we cannot factorize \( \mathcal{I}(H_1)^b \) in a natural way, as the latter is an ideal and \( \mathcal{M} \mathcal{O}_{V_1}^q \) is an \( \mathcal{O}_{V_1}^q \)-module. However, we seem to recover \( \theta \), as an invariant on the \( q \)-modules, later on, when we consider logarithmic differential operators.

§5 On \(-q\)-Differential Collections of Ideals

In this final section, we introduce in §5.1 the notion of \(-q\)-differential collection of ideals and their transformations by blowups. These collections have attached numerical invariants satisfying the pointwise inequality. In §5.2 we study \(-q\)-differential collections that arise from evaluating logarithmic differential operators with respect to a set of hypersurfaces with normal crossings on \( \mathcal{O}_{V}^q \)-submodules \( \mathcal{M} \subset \mathcal{O}_V \). As a result, we obtain invariants associated with \( \mathcal{O}_{V}^q \)-modules and hypersurfaces with normal crossings, and these invariants satisfy the pointwise inequality.

Along the section, \( V \) denotes, as usual in this paper, an irreducible \( F \)-finite regular scheme, and \( q = p^e \) a fixed power of \( p \).

§5.1 Definitions and Basic Properties. We introduce the main algebraic object of the section.

Definition 5.1. A collection \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \) of \( q - 1 \) \( \mathcal{O}_V \)-ideals is said to be \(-q\)-differential if

\[
\text{Diff}^q_1(\mathcal{I}_i) \subseteq \mathcal{I}_{i+j} \quad \text{whenever} \quad i + j \leq q - 1.
\]

In particular, \( \mathcal{I}_i = \text{Diff}^q_1(\mathcal{I}_i) \subseteq \text{Diff}^q_1(\mathcal{I}_i) \subseteq \mathcal{I}_{i+1} \) for all \( i = 1, \ldots, q - 2 \). We say that \( \mathcal{G} \) is nonzero and write \( \mathcal{G} \neq 0 \) if \( \mathcal{I}_{q-1} \neq 0 \). If \( \mathcal{G}' = (\mathcal{I}_1', \ldots, \mathcal{I}_{q-1}') \) is a second \(-q\)-differential collection of \( \mathcal{O}_V \)-ideals, then we write \( \mathcal{G} \subseteq \mathcal{G}' \) whenever \( \mathcal{I}_i \subseteq \mathcal{I}'_i \) for all \( 1 \leq i \leq q - 1 \).

As a main example (we will see others later, using logarithmic differential operators), given an \( \mathcal{O}_{V}^q \)-module \( \mathcal{M} \), we set

\[
\mathcal{G}(\mathcal{M}) := (\text{Diff}^q_1(\mathcal{M})_+, \text{Diff}^q_2(\mathcal{M})_+, \ldots, \text{Diff}^q_{q-1}(\mathcal{M})_+).
\]

This collection is \(-q\)-differential, as the composition of a differential operator of degree \( i \) with one of degree \( j \) is another one of degree \( i + j \). Note that by Proposition 4.18 \( \mathcal{G}(\mathcal{M}) \neq 0 \) if and only if \( \mathcal{M} \not\subseteq \mathcal{O}_{V}^q \). Note also that \( \mathcal{G}(\mathcal{M}) = \mathcal{G}(\mathcal{O}_{V}^q[\mathcal{M}]) \) by Proposition 5.16.

Example 5.2. Let \( V = \text{Spec}(\mathbb{F}_3[x_1, x_2, x_3, x_4, x_5]) \) and \( q = p = 3 \). We consider \( \mathcal{M} := \mathcal{O}_{V}^q \cdot x_1x_2x_3x_4x_5 \). Then

\[
\mathcal{G}(\mathcal{M}) = \left( \left\langle \frac{x_1x_2x_3x_4x_5}{x_j} : j = 1, \ldots, 5 \right\rangle, \left\langle \frac{x_1x_2x_3x_4x_5}{x_i^2x_j} : 1 \leq i < j \leq 5 \right\rangle \right)
\]

Definition 5.3. Given a \(-q\)-differential collection \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \), for each \( x \in V \), we define

\[
\eta_x(\mathcal{G}) := \min\{\nu_x(\mathcal{I}_i) + i : 1 \leq i \leq q - 1\} \in \mathbb{N}.
\]

Observe that if \( \mathcal{G} \neq 0 \), then \( \mathcal{I}_{q-1} \neq 0 \), and so \( \eta_x(\mathcal{G}) \leq \nu_x(\mathcal{I}_{q-1}) + q - 1 < \infty \) for all \( x \in V \). Note also that \( \eta_x(\mathcal{G}) \geq 1 \) for all \( x \in V \). Finally, the function \( x \mapsto \eta_x(\mathcal{G}) \) from \( V \) to \( \mathbb{N}_0 \) (with the usual order topology) is upper-semicontinuous since it is the minimum of the upper-semicontinuous functions \( x \mapsto \nu_x(\mathcal{I}_i) + i \), \( i = 1, \ldots, q - 1 \) (see Proposition 1.17).

In the case \( \mathcal{G} = \mathcal{G}(\mathcal{M}) \), we obtain \( \eta_x(\mathcal{G}(\mathcal{M})) = \eta_x(\mathcal{M}) \) (see 1.17).

Example 5.4. In the previous example the maximum value of \( \eta_x(\mathcal{G}(\mathcal{M})) = \eta_x(\mathcal{M}) \) is \( 5 \), and this maximum is attained only when \( x \) is the origin.

We will use the next technical lemma to show that the function \( \eta_x(\mathcal{G}) \) satisfies the pointwise inequality, with a suitable definition of transformation of \(-q\)-differential collection by blowups.

Lemma 5.5. Let \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \) be a nonzero \(-q\)-differential collection of ideals on \( V \). Given \( x \in V \), we write \( \eta_x(\mathcal{G}) = aq + b \) with \( 0 \leq b < q \). Then there is an index \( i \in \{1, \ldots, q - 1\} \) such that \( i \geq b \) and \( \nu_x(\mathcal{I}_i) = \eta_x(\mathcal{G}) - i \).
Proof. By definition \( \nu_x(\mathcal{I}_i) \geq \eta_x(\mathcal{G}) - i \) for all \( i = 1, \ldots, q - 1 \), and for some \( i \), the equality holds. Let \( i_0 \) be the largest \( i \) such that \( \nu_x(\mathcal{I}_i) = \eta_x(\mathcal{G}) - i \). The claim of the lemma is that \( i_0 \geq b \). This is obvious if \( b = 0 \), so we assume that \( b > 0 \). Suppose on the contrary that \( i_0 < b \), so that \( \nu_x(\mathcal{I}_{i_0}) = qa + (b - i_0) \) and \( 0 < b - i_0 < q \). We apply Lemma 5.5 with an element \( f \in (\mathcal{I}_{i_0})_x \) of order \( qa + (b - i_0) \). Note that we are in Case 1 of that lemma, and hence \( \nu_x(Diff_{\mathcal{L}^o}^{b-i_0}(\mathcal{I}_{i_0})) = qa \). Finally,

\[
qa = \eta_x(\mathcal{G}) - b \leq \nu_x(\mathcal{I}_b) \leq \nu_x(Diff_{\mathcal{L}^o}^{b-i_0}(\mathcal{I}_{i_0})) = qa,
\]

where the second inequality follows from the inclusion \( Diff_{\mathcal{L}^o}^{b-i_0}(\mathcal{I}_{i_0}) \subseteq \mathcal{I}_b \). This says that \( \nu_x(\mathcal{I}_b) = \eta_x(\mathcal{G}) - b \) if \( i_0 < b \), which is in contradiction with the definition of \( i_0 \). Thus \( i_0 \geq b \), and the proof is complete. \( \square \)

Corollary 5.6. Let \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \) be a \( q \)-differential collection of ideals on \( V \). Fix \( x \in V \) and write \( \eta_x(\mathcal{G}) = qa + b \) with \( 0 \leq b < q \). Then

\[
qa + b = \eta_x(\mathcal{G}) \leq \nu_x(\mathcal{I}_{q-1}) + q - 1 < q(a + 1).
\]

Proof. The first inequality follows from the definition of \( \eta_x(\mathcal{G}) \). We now show the second (strict) inequality. By Lemma 5.5 there exists an index \( i \geq b \) such that \( \nu_x(\mathcal{I}_i) = qa + b - i \leq qa \), whence \( \nu_x(\mathcal{I}_{q-1}) \leq qa \) since \( \mathcal{I}_i \subseteq \mathcal{I}_{q-1} \). This implies that \( \nu_x(\mathcal{I}_{q-1}) + q - 1 \leq qa + q - 1 < q(a + 1) \), as was to be shown. \( \square \)

The next lemma provides a tool to construct examples of \( q \)-differential collections of ideals from a given one. They will be used, for example, in Definition 5.8 to give different notions of transformation of a \( q \)-differential collection of ideals under a blowup.

Lemma-Definition 5.7. Fix a \( q \)-differential collection of ideals \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \) on \( V \).

1. If \( \mathcal{L} \subset \mathcal{O}_V \) is an invertible ideal, then the collection of ideals

\[
\mathcal{G}_{\mathcal{L}} := ((\mathcal{I}_1 : \mathcal{L}^i), \ldots, (\mathcal{I}_{q-1} : \mathcal{L}^q))
\]

is also \( q \)-differential.

2. Given a second \( F \)-finite regular scheme \( V_1 \), and given a morphism \( \pi : V_1 \to V \), the collection

\[
\mathcal{G}_{\pi V_1} := (\mathcal{I}_1 \mathcal{O}_{V_1}, \ldots, \mathcal{I}_{q-1} \mathcal{O}_{V_1})
\]

is also \( q \)-differential.

Proof. (1) Given integers \( i \geq 1 \) and \( j \geq 0 \) with \( i + j \leq q - 1 \), we have \( \mathcal{L}^i \mathcal{L}^j(\mathcal{I}_i : \mathcal{L}^i) = \mathcal{L}^j(\mathcal{I}_i : \mathcal{L}^i) \subseteq \mathcal{L}^i \mathcal{L}^j(\mathcal{I}_i) \subseteq \mathcal{L}^{i+j} \), and hence \( \mathcal{L}^i \mathcal{L}^j(\mathcal{I}_i : \mathcal{L}^i) \subseteq (\mathcal{I}_i : \mathcal{L}^{i+j}) \). This proves that \( \mathcal{G}_{\mathcal{L}} \) is a \( q \)-differential collection.

(2) For integers \( i \geq 1 \) and \( j \geq 0 \) with \( i + j \leq q - 1 \), Proposition 4.7.3(2) implies that \( \mathcal{L}^i \mathcal{L}^j(\mathcal{I}_i \mathcal{O}_{V_1}) \subseteq \mathcal{L}^{i+j} \mathcal{L}^j(\mathcal{I}_i \mathcal{O}_{V_1}) \). This proves that \( \mathcal{G}_{\pi V_1} \) is also a \( q \)-differential collection. \( \square \)

We now formulate notions of transformation by blowups.

Definition 5.8. Let \( \mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}) \) be a \( q \)-differential collection of \( \mathcal{O}_V \)-ideals, and let \( V \leftarrow^x V_1 \supset H_1 \) be the blowup of \( V \) along an irreducible regular center \( Z \).

1. We call \( \mathcal{G}_{\pi V_1} \) the total transform of \( \mathcal{G} \) by the blowup.

2. For a positive integer \( a \), we call \( (\mathcal{G}_{\pi V_1})_{\pi(V_1)^a} \) the \( a \)-transform of \( \mathcal{G} \).

The following theorem, stated in Introduction as Theorem 1.16, establishes the fundamental pointwise inequality for the invariant \( \eta \) attached to \( q \)-differential collections.

Theorem 5.9. Let \( V \leftarrow^x V_1 \supset H_1 \) be a blowup along an irreducible regular center \( Z \). Assume, in addition, that \( \eta_x(\mathcal{G}) \) is constant along points in \( Z \), say \( \eta_x(\mathcal{G}) = qa + b \) for all \( x \in Z \), where \( 0 \leq b < q \). If \( \mathcal{G}_x^{(a)} \) denotes the \( a \)-transform of \( \mathcal{G} \), then for any \( x_1 \in V_1 \),

\[
\eta_{\pi(x_1)}(\mathcal{G}) \geq \eta_{x_1}(\mathcal{G}_x^{(a)}).
\]

Proof. We only need to prove this inequality for points in \( H_1 \). Fix \( x_1 \in H_1 \) and set \( x := \pi(x_1) \in Z \), so that \( \eta_x(\mathcal{G}) = qa + b \). According to Lemma 5.5 there is an index \( i \) with \( b \leq i \leq q - 1 \) such that \( \nu_x(\mathcal{I}_i) = qa + b - i \). We fix such an index \( i \). Since by hypothesis \( \nu_x(\mathcal{G}) = qa + b \) for all \( z \in Z \), we have \( \nu_x(\mathcal{I}_i) \geq \nu_z(\mathcal{G}) - i = qa + b - i \) for all \( z \in Z \), and the equality holds for \( z = x \). As the order function is upper-semicontinuous (Proposition 1.7), after restricting to an open neighborhood of \( x \), we may assume that
\( \nu_z(\mathcal{I}_z) = qa + b - i \) for all \( z \in Z \). So, for our fixed index \( i \), \( \mathcal{I}_z \mathcal{O}_{V_i} = \mathcal{I}(H_1)^{aq + b - i} \mathcal{I}_1 \) for an \( \mathcal{O}_{V_1} \)-ideal \( \mathcal{I}_1 \). By Proposition 4.15 \( \nu_{x_1}(\mathcal{I}_1) \leq \nu_z(\mathcal{I}_z) = qa + b - i \). Finally,

\[
(\mathcal{I}_z \mathcal{O}_{V_1} : \mathcal{I}(H_1)^{aq + b - i} \mathcal{I}_1 : \mathcal{I}(H_1)^{aq}) \supseteq \mathcal{I}_1,
\]

where the last inclusion holds since \( i \geq b \). We conclude that \( \nu_{x_1}(\mathcal{I}_z \mathcal{O}_{V_1} : \mathcal{I}(H_1)^{aq}) \leq \nu_{x_1}(\mathcal{I}_1) \leq qa + b - i \), and by definition this implies that \( \eta_x(V_1(a_q)) \leq qa + b \), as was to be proved.

**Proposition 5.10.** Fix an \( \mathcal{O}^q_{V_1} \)-submodule \( \mathcal{M} \subseteq \mathcal{O}_{V_1} \), an integer \( a \geq 1 \), and an irreducible regular subscheme \( Z \subseteq V_1 \). Consider the blowup \( V_1 \leftarrow V_1 \supset H_1 \) of \( V_1 \) along \( Z \). Let \( \mathcal{M}^{(a_q)} \) and \( \mathcal{G}(\mathcal{M})^{(a_q)} \) denote, respectively, the \( a \)-transform of \( \mathcal{M} \) (Def. 2.10) and the \( a \)-transform of \( \mathcal{G}(\mathcal{M}) \) (Def. 5.8). Then

\[
\mathcal{G}(\mathcal{M}^{(a_q)}) \subseteq (\mathcal{G}(\mathcal{M}))^{(a_q)}.
\]

In particular, \( \eta_x(\mathcal{M}^{(a_q)}) \geq \eta_x((\mathcal{G}(\mathcal{M}))^{(a_q)}) \) for all \( x_1 \in V_1 \).

**Proof.** By the definition of the \( a \)-transform of a module there is an inclusion \( F^a(\mathcal{I}(H_1)^{aq}) \mathcal{M}_1 \subseteq \mathcal{M}^{(a_q)} \mathcal{O}_{V_1}^{\mathcal{O}_{V_1}} + \mathcal{O}_{V_1}^{a_q} \), and hence for each \( i = 1, \ldots, q - 1 \),

\[
\mathcal{I}(H_1)^{aq} \text{ Diff}_{V_1, +}(\mathcal{M}_1) = \text{ Diff}_{V_1, +}(\mathcal{I}(H_1)^{aq}) \mathcal{M}_1
\]

\[
\subseteq \text{ Diff}_{V_1, +}(\mathcal{M}^{(a_q)}) \subseteq \text{ Diff}_{V_1, +}(\mathcal{M}) \mathcal{O}_{V_1}^{a_q},
\]

where the last inclusion follows from Proposition 4.13. Therefore \( \text{ Diff}_{V_1, +}(\mathcal{M}_1) \subseteq (\text{ Diff}_{V_1, +}(\mathcal{M}) \mathcal{O}_{V_1} : \mathcal{I}(H_1)^{aq}) \). This yields the desired inclusion of \( q \)-differential collections.

**Remark 5.11.** We discuss an outcome of interest for the study of transformations of singularities that follows from the previous results. We fix an \( \mathcal{O}_{V_1}^q \)-submodule \( \mathcal{M} \subseteq \mathcal{O}_{V_1} \). We associate with \( \mathcal{M} \) the \( q \)-differential collection \( \mathcal{G}_{0} := \mathcal{G}(\mathcal{M}) \). Sequences of “permissible transformations” of this collection lead to “permissible transformations” of the module \( \mathcal{M} \) by blowups in a way we now describe. We consider the sequences of blowups

\[
\begin{align*}
V = V_0 & \overset{\pi_1}{\longrightarrow} V_1 \overset{\pi_2}{\longrightarrow} \cdots \overset{\pi_n}{\longrightarrow} V_n \\
\mathcal{G}_0 = \mathcal{G}(\mathcal{M}) & \quad \mathcal{G}_1 \quad \cdots \quad \mathcal{G}_n
\end{align*}
\]

that are constructed from the functions \( \eta \) as follows: for each \( i = 0, 1, \ldots, n - 1 \), \( V_i \leftarrow V_{i+1} \) is the blowup of \( V_i \) along an irreducible regular center \( Z_i \subset V_i \) included in the set of points \( x \in V_i \) at which \( \eta_x(Z_i) \) reaches its maximum value, say \( a_iq + b_i \) (\( 0 \leq b_i < q \)), and \( \mathcal{G}_{i+1} \) is the \( a_i \)-transform of \( \mathcal{G}_i \). By Theorem 5.9 we have

\[
a_iq + b_0 \geq a_1q + b_1 \geq \cdots \geq a_nq + b_n.
\]

Assume that \( a_{n-1} \geq 1 \). We now set \( \mathcal{M}_0 := \mathcal{M} \), and for \( i \geq 0 \), we define \( \mathcal{M}_{i+1} \) as the \( a_i \)-transform of \( \mathcal{M}_i \). It follows from Proposition 5.10 and induction on \( i \) that there are inclusions

\[
\mathcal{G}(\mathcal{M}_i) \subseteq \mathcal{G}_i, \quad i = 1, \ldots, n;
\]

in particular, \( \eta_x(\mathcal{M}_i) := \eta_x(\mathcal{G}(\mathcal{M}_i)) \geq \eta_x(Z_i) := a_iq + b_i \) for all \( x \in Z_i \). By Proposition 4.13 \( Z_i \) is permissible for \( (\mathcal{M}_i, a_{n-i}) \) for all \( i = 0, \ldots, n - 1 \), and hence the finite morphism \( X_{i+1} \to V_{i+1} \) attached to \( \mathcal{M}_{i+1} \) is obtained from the finite morphism \( X_i \to V_i \) attached to \( \mathcal{M}_i \) as \( \delta \) is obtained from \( \delta \) in Corollary 2.25.

**§5.2 Logarithmic Differential Invariants.** In this final part of the section, we explore new invariants of singularities that are constructed using logarithmic differential operators. These operators arise naturally when we fix a regular scheme \( V \) together with a sequence of blowups along regular centers \( V \leftarrow \cdots \leftarrow V_r \). In this setting, \( V_r \) is regular and contains exceptional hypersurfaces, say \( H_1, \ldots, H_r \), and it is useful to consider logarithmic differential operators with poles along these hypersurfaces. Roughly speaking, we will construct \( q \)-differential collections associated with modules and hypersurfaces with normal crossings and define invariants from the function \( \eta \) associated with \( q \)-differential collections (Definition 5.3). Our goal is to prove that these invariants satisfy the fundamental pointwise inequality.

**5.12.** Logarithmic differential operators were reviewed in 3.3. The discussion in 3.4 enables us to extend everything to the setting of sheaves on schemes. So we fix an irreducible \( F \)-finite regular scheme \( V \) and
Proposition 3.10 implies that $c = q$.

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and obtain numeral invariants with a good behavior under blowups.

By Proposition 3.16 this sequence of ideals is not affected if $\mathcal{M}$ is replaced by $\mathcal{O}_V$, so it is compatible with our notion of weak equivalence of $\mathcal{O}_V$-submodules. However, this sequence is not (in general) a $q$-differential collection in the sense of Definition 5.1. This is due to the fact that there is no inclusion $\text{Diff}_{V,\Lambda} \subset \text{Diff}_{V,\Lambda}^q$ in general. We will use the following lemmas to produce $q$-differential collections from the previous collection (Proposition 5.17).

We are interested only in the case where $\Lambda$ consists of the ideals of hypersurfaces with normal crossings. We review this definition.

Definition 5.13. A collection of hypersurfaces $\{H_1, \ldots, H_r\}$ on $V$ is said to have only normal crossings in $V$ if for each point $z \in V$, there exists a regular system of parameters $(x_1, \ldots, x_n)$ for $O_{V,z}$ such that for each $j = 1, \ldots, r$, the ideal $\mathcal{I}(H_j)_z \subseteq O_{V,z}$ is either $O_{V,z}$ or is $(x_{i_j})$ for some $i_j \in \{1, \ldots, n\}$. If in addition $Z \subset V$ is a closed regular subscheme, we say that $\{H_1, \ldots, H_r\}$ has only normal crossings with $Z$ (or that $Z$ has only normal crossings with $\{H_1, \ldots, H_r\}$) if for each $z \in Z$, there exists a regular system of parameters $(x_1, \ldots, x_n)$ for $O_{V,z}$ satisfying the above condition and such that $\mathcal{I}(Z)_z$ is generated by a subset of $\{x_1, \ldots, x_n\}$; see [14, Definition 2, p. 141].

We include the following proposition to show the role played by the ideal $\text{Diff}_{V,\Lambda}^q(\mathcal{M})$ in the characterization of the “optimal monomial” that can be extracted from the equivalence class of $\mathcal{M}$ in the following sense.

Proposition 5.14. Let $\{H_1, \ldots, H_r\}$ be a collection of hypersurfaces with only normal crossings, and let $\Lambda$ be the collection of their ideals. Then for nonnegative integers $m_1, \ldots, m_r$,

$$
\text{Diff}_{V,\Lambda}^{q-1}(\mathcal{M}) \subseteq \mathcal{I}(H_1)^{m_1} \cdots \mathcal{I}(H_r)^{m_r} \text{ if and only if } \mathcal{M} \subseteq \mathcal{O}_V \mathcal{I}(H_1)^{m_1} \cdots \mathcal{I}(H_r)^{m_r}.
$$

Proof. The “if” part is immediate from the definition of logarithmic differential operator. The other direction can be checked locally, and we only need to prove the following statement: Let $(R, \mathfrak{m})$ be an $F$-finite regular local ring, let $(x_1, \ldots, x_n)$ be a regular system of parameters, and let $\Lambda = \{x_1 R, \ldots, x_r R\}$ for some $r \leq n$. Given $f \in R$, assume that $\text{Diff}_{R,\Lambda}^{q-1}(f) = x_1^{m_1} \cdots x_r^{m_r} R$ for integers $m_1, \ldots, m_r \geq 0$. Then $f \in R^q + x_1^{m_1} \cdots x_r^{m_r} R$.

We extend $(x_1, \ldots, x_n)$ to a $p$-basis $(x_1, \ldots, x_n, y_1, \ldots, y_r)$ for $R$ and consider the $q$-expansion of $f$, say $f = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta$. We will show that given $(\alpha, \beta) \neq (0, 0)$ and $i \leq r$, the term $c_{\alpha,\beta} x^\alpha y^\beta$ is divisible by $x_1^{m_1}$. We can replace $R$ by its localization at the regular prime $(x_i)$ since the $p$-basis and the $q$-expansion remain unaltered. Thus we may assume that $i = r = n = 1$. Now $x_1^{m_1} \frac{\partial f}{\partial x_1} \in \text{Diff}_{R,\Lambda}^{q-1}$, and $x_1^{m_1} \frac{\partial f}{\partial x_1}(f)$ has $c_{\alpha,\beta} x^\alpha y^\beta$ as a term in its $q$-expansion. Since $x_1^{m_1} \frac{\partial f}{\partial x_1}(f)$ is divisible by $x_1^{m_1}$ (this is the assumption), Proposition 3.10 implies that $c_{\alpha,\beta} x^\alpha y^\beta$ is divisible by $x_1^{m_1}$.

We do not pursue this line in name, merely the factorization of an “optimal monomial” and the behavior under suitable blowups, and refer the reader to [22] and [13]. We instead try to produce a $q$-differential collection attached to $\mathcal{M}$ and $\Lambda$ so that we can apply the results of the first part of the section (mainly Theorem 5.9) and obtain numeral invariants with a good behavior under blowups.

A natural step would be to work with the collection of $\mathcal{O}_V$-ideals

$$(\text{Diff}_{V,\Lambda}^q(\mathcal{M}), \ldots, \text{Diff}_{V,\Lambda}^{q-1}(\mathcal{M})).$$

By Proposition 5.16 this sequence of ideals is not affected if $\mathcal{M}$ is replaced by $\mathcal{O}_V$-ideals, so it is compatible with our notion of weak equivalence of $\mathcal{O}_V$-submodules. However, this sequence is not (in general) a $q$-differential collection in the sense of Definition 5.1. This is due to the fact that there is no inclusion $\text{Diff}_{V,\Lambda} \subset \text{Diff}_{V,\Lambda}^q$ in general. We will use the following lemmas to produce $q$-differential collections from the previous collection (Proposition 5.17).
Lemma 5.15 ([15] Lemma 1.5]). Let \( R \) be a ring, and let \( x \in R \) be an element that is not a zero divisor. Given integers \( a, b \geq 0 \) and \( D \in \text{Diff}_R^a \), we have \( x^{a-b}Dx^b \in \sum_{j=0}^a x^j \text{Diff}_R^j \). In particular, \( x^{a-b}Dx^b \) is (\( x \))-logarithmic. (\( Dx^b \) denotes the operator \( Dx^b(r) = (Dx^b)_r, r \in R \).)

**Proof.** We proceed by induction on \( ab \). If \( ab = 0 \), then either \( a = 0 \), in which case \( D \) is \( R \)-linear and \( x^{-b}Dx^b = D \in \text{Diff}_R^0 \), or else \( b = 0 \), in which case the result is trivial. Assume now that \( ab > 0 \), whence \( a > 0 \) and \( b > 0 \). Then

\[
x^{a-b}Dx^b = x^{a-b}(Dx)x^{b-1} = x^{a-b}(Dx - xD + xD)x^{b-1} = x^{(a-1)-(b-1)}(Dx - xD)x^{b-1} + x^{a-(b-1)}Dx^b - 1.
\]

The first term in the last expression is in \( \sum_{k=0}^{a-1} x^k \text{Diff}_R^k \) by the inductive hypothesis since \( (a-1)(b-1) < ab \) and \( Dx - xD \in \text{Diff}_R^{a-1} \). Similarly, the second term is in \( \sum_{k=0}^{a} x^k \text{Diff}_R^k \) since \( a(b-1) < ab \). Therefore \( x^{a-b}Dx^b \in \sum_{k=0}^{a-1} x^k \text{Diff}_R^k \). The induction is now complete. \( \square \)

Lemma 5.16. Fix \( \mathcal{O}_V \)-ideals \( \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{q-1} \) and an invertible ideal \( \mathcal{L} \), and assume that

\[
\mathcal{L}^i \text{Diff}^i_V((\mathcal{J}_i : \mathcal{L}^i)) \subseteq \mathcal{J}_{i+j} \quad \text{whenever} \quad i + j \leq q - 1, j \geq 0, i \geq 1.
\]

Then the collection of \( \mathcal{O}_V \)-ideals \( ((\mathcal{J}_1 : \mathcal{L}^1), \ldots, (\mathcal{J}_{q-1} : \mathcal{L}^{q-1})) \) is \( q \)-differential (Definition/qdiffcoll).

**Proof.** Observe first that

\[
\mathcal{L}^{i+j} \text{Diff}^i_V((\mathcal{J}_i : \mathcal{L}^i)) = \mathcal{L}^{i+j-q} \mathcal{L}^q \text{Diff}^i_V((\mathcal{J}_j : \mathcal{L}^j))
\]

\[
= \mathcal{L}^{i+j-q} \text{Diff}^i_V(\mathcal{L}^q(\mathcal{J}_j : \mathcal{L}^j))
\]

\[
\subseteq \mathcal{L}^{i+j-q} \text{Diff}^i_V(\mathcal{L}^{q-i} \mathcal{J}_j).
\]

In the second equality, we use that \( \mathcal{L}^q = F^q \mathcal{L} \cdot \mathcal{O}_V \) (since \( \mathcal{L} \) is invertible) and that differential operators of order \( \leq q - 1 \) are \( \mathcal{O}_V^q \)-linear. We now apply Lemma 5.15 with \( a = j \) and \( b = q - i \), and obtain that the expression on the right in (5.2) is included in \( \sum_{k=0}^{i} \mathcal{L}^k \text{Diff}^k_V(\mathcal{J}_i) \), which is included in \( \mathcal{J}_{i+j} \), by the hypothesis. Summarizing, \( \mathcal{L}^{i+j} \text{Diff}^i_V((\mathcal{J}_i : \mathcal{L}^i)) \subseteq \mathcal{J}_{i+j} \), and hence \( \text{Diff}^i_V((\mathcal{J}_i : \mathcal{L}^i)) \subseteq (\mathcal{J}_{i+j} : \mathcal{L}^{i+j}) \), as was to be proved. \( \square \)

Proposition 5.17. Let \( \Lambda = \{\mathcal{I}_1, \ldots, \mathcal{I}_r\} \) be a family of nonzero \( \mathcal{O}_V \)-ideals, and let \( \mathcal{L} \) be an invertible \( \mathcal{O}_V \)-ideal included in each \( \mathcal{I}_k \), \( k = 1, \ldots, r \). Then for any \( \mathcal{O}_V^q \)-module \( \mathcal{M} \), the collection of ideals

\[
\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) := ((\text{Diff}^i_{\mathcal{V}, \Lambda}^i(\mathcal{M}) : \mathcal{L}^1), (\text{Diff}^i_{\mathcal{V}, \Lambda}^i(\mathcal{M}) : \mathcal{L}^2), \ldots,
\]

\[
(\text{Diff}^{q-1}_{\mathcal{V}, \Lambda}^i(\mathcal{M}) : \mathcal{L}^{q-1}))
\]

is \( q \)-differential. In addition, there is a componentwise inclusion

\[
\mathcal{G}(\mathcal{M}) \subseteq \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})
\]

**Proof.** By (3.6) there are inclusions \( \mathcal{L}^i \text{Diff}^i_V \subseteq \text{Diff}^i_{\mathcal{V}, \Lambda} \) and \( \mathcal{L}^i \text{Diff}^i_{\mathcal{V}^+} \subseteq \text{Diff}^i_{\mathcal{V}, \Lambda^+} \). Thus

\[
\mathcal{L}^i \text{Diff}^i_V((\text{Diff}^i_{\mathcal{V}, \Lambda^+}(\mathcal{M})) \subseteq \text{Diff}^i_{\mathcal{V}, \Lambda}(\text{Diff}^i_{\mathcal{V}, \Lambda^+}(\mathcal{M})) \subseteq \text{Diff}^{i+j}_{\mathcal{V}, \Lambda^+}((\mathcal{M}).
\]

This shows that \( \mathcal{L} \) and the collection \( \text{Diff}^i_{\mathcal{V}, \Lambda^+}(\mathcal{M}) \subseteq \cdots \subseteq \text{Diff}^{q-1}_{\mathcal{V}, \Lambda^+}(\mathcal{M}) \) satisfy the hypothesis of Lemma 5.16 whence the collection \( \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) \) is \( q \)-differential. As for (5.4), we note that for \( i = 1, \ldots, q - 1 \),

\[
\text{Diff}^i_{\mathcal{V}, \Lambda^+}(\mathcal{M}) = (\mathcal{L}^i \text{Diff}^i_{\mathcal{V}, \Lambda}(\mathcal{M}) : \mathcal{L}^i) \subseteq (\text{Diff}^i_{\mathcal{V}, \Lambda^+}(\mathcal{M}) : \mathcal{L}^i).
\]

\( \square \)

Corollary 5.18. Let the setting and notation be as in Proposition 5.17. Then

\[
\eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \leq \eta_x(\mathcal{G}(\mathcal{M})), \quad \forall x \in V.
\]

In particular, if \( Z \subseteq V \) is a regular irreducible subscheme and \( a \) is a positive integer such that \( \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq q a \) for all \( x \in Z \), then \( Z \) is permissible for \( (\mathcal{M}, a) \).

**Proof.** The inequality follows from (5.3), and the last assertion follows from Proposition 4.18. \( \square \)
Remark 5.19. So far we have associated with a 3-uple \((\mathcal{M}, \Lambda, \mathcal{L})\) as above an upper-semicontinuous function \(V \rightarrow \mathbb{N}\) given by \(x \mapsto \eta_x(\mathcal{M}, \Lambda, \mathcal{L})\). This function coincides with \(\eta_x(\mathcal{M})\) outside the support of the ideals of \(\Lambda\) and \(\mathcal{L}\). Now we want to formulate a notion of transformation of triples \((\mathcal{M}, \Lambda, \mathcal{L})\) by blowups and to show that the fundamental pointwise inequality holds for the function \(x \mapsto \eta_x(\mathcal{M}, \Lambda, \mathcal{L})\). To do this, we will restrict ourselves to the case where \(\Lambda\) consists of ideals of hypersurfaces with only normal crossings. We also require a version of Corollary 4.5 with logarithmic differential operators, which is our main tool to compare invariants before and after a blowup. We begin with the formulation of the latter task.

5.20. Let \(R\) be an \(F\)-finite regular local ring, let \(P \subset R\) be a regular prime, and let \((x_1, \ldots, x_n)\) be a regular system of parameters for \(R\) such that \(P = (x_1, \ldots, x_s)\) for some index \(s \leq n\). We refer to the discussion of the blowup of \(\text{Spec}(R)\) along \(P\) in \([4,3]\) and consider

\[
R_1 := \left\{ \frac{z}{x_1^j} : z \in P^t, t \geq 0 \right\} = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_s}{x_1}\right] \subset R_{x_1}.
\]

For each index \(j = 1, \ldots, n\), we set

\[
x_j' = \begin{cases} \frac{x_j}{x_1^j} & \text{if } j \leq s, \\ x_j & \text{if } j > s. \end{cases}
\]

Notice that \(x_j' R_1\) is the restriction to \(\text{Spec}(R_1)\) of the strict transform of \(x_j R\) by the blowup along \(P\).

We fix a subset \(\Phi \subseteq \{1, \ldots, n\}\) and set \(\Lambda := \{x_j : j \in \Phi\}\), which is a collection of ideals of \(R\), and \(\Lambda_1 := \{x_j' R_1 : j \in \Phi\} \cup \{x_1 R_1\}\), which is a collection of ideals in \(R_1\). In the formulation of the following lemma, we identify differential operators of \(R\) or of \(R_1\) with their extensions to the fraction field.

Lemma 5.21. There are inclusions

\[
\text{Diff}^i_{R, \Lambda_1 \cup \{P\}} \subseteq \text{Diff}^i_{R_1, \Lambda_1} \quad \text{and} \quad \text{Diff}^i_{R, \Lambda_1 \cup \{P\}, +} \subseteq \text{Diff}^i_{R_1, \Lambda_1, +}
\]

for all \(i \geq 0\). In particular, given an \(R^g\)-submodule \(M \subseteq R\), there are inclusions

\[
x_1^j \text{Diff}^i_{R, \Lambda_1 \cup \{P\}}(M) R_1 \subseteq \text{Diff}^i_{R_1, \Lambda_1, +}(MR_1^{j})
\]

for all \(i = 1, \ldots, q-1\). (\(MR_1^j\) denotes the \(R_1\)-submodule of \(R_1\) generated by \(M\).)

Proof. We prove the first inclusion in \((5.5)\), the second being just a consequence. Fix \(D \in \text{Diff}^i_{R, \Lambda_1 \cup \{P\}}\). By Lemma \([4,3]\), \(D\) extends to a differential operator of \(R_1\), and this operator is \(x_1 R_1\)-logarithmic. To show that \(D \in \text{Diff}^i_{R_1, \Lambda_1}\), it is only left to show that \(D(x_j' R_1) \subseteq x_j^k R_1\) for all \(k \geq 0\) and \(j \in \Phi, j \neq 1\).

Fix \(j \in \Phi\). Assume first that \(j \leq s\), so that \(x_j' = \frac{x_j}{x_1^j}\) and the elements of \(x_j R_1\) are of the form \(\frac{x_j^k x_1^j}{x_1^l}\) where \(z \in P^t\). After multiplication and division by a power of \(x_1\), we may assume that \(k + t\) is a power of \(p\) that is greater than the order of \(D\). Hence \(D(x_j^k z/x_1^{k+t}) = D(x_j^k z)/x_1^{k+t}\). Since \(D\) is \(x_1 R_1\)-logarithmic and also \(P\)-logarithmic, and since \(x_j^k z \in x_j^k R \cap P^{k+t}\), the numerator of the latter fraction belongs to \(x_j^k R \cap P^{k+t} = x_j^k P^t\). Thus the fraction belongs to \(x_j^k R_1\). This proves that \(D(x_j' R_1) \subseteq x_j^k R_1\) for all \(k \geq 0\).

Assume now that \(j > s\), so that \(x_j' = x_j\), and the elements of \(x_j R_1\) are of the form \(x_j^k / x_1^l\) with \(z \in P^t\). As in the previous case, we may assume that \(t\) is a power of \(p\) that is greater than the order of \(D\). Then \(D(x_j^k z/x_1^l) = D(x_j^k z)/x_1^l\). Since \(x_j^k z \in x_j^k \cap P^t\), and since \(D\) is \(x_j R_1\)-logarithmic and \(P\)-logarithmic, the numerator of the latter fraction is in \(x_j^k R \cap P^t = x_j^k P^t\). Thus the fraction belongs to \(x_j^k R_1 = x_j^k R_1\). This proves that \(D(x_j' R_1) \subseteq x_j^k R_1\) for all \(k \geq 0\).

As for the inclusion in \((5.5)\), we just need to observe that \(x_1^j \text{Diff}^i_{R, \Lambda_1, +} \subseteq \text{Diff}^i_{R_1, \Lambda_1, +, +}\), which follows from \((3.5)\) and then use the second inclusion in \((5.5)\). \qed

Proposition 5.22. Let \(\Lambda\) be a finite collection of hypersurfaces on \(V\) with only normal crossings, and let \(Z \subset V\) be an irreducible regular closed subscheme such that \(\Lambda\) has only normal crossings with \(Z\) (Definition \([4,7]\)). Let \(V \leftarrow V_1 \supset H_1\) be the blowup along \(Z \subset V\), and let \(\Lambda_1\) be the collection of the strict transforms of each \(H \in \Lambda\) plus the exceptional hypersurface \(H_1 \subset V_1\). Then \(\Lambda_1\) has only normal crossings on \(V_1\), and for any \(\mathcal{O}_{V_1}\)-module \(\mathcal{M}\), we have

\[
\mathcal{I}(H_1) \left( \text{Diff}^i_{V_1, \Lambda_1, +}(-\mathcal{M}) \mathcal{O}_{V_1} \right) \subseteq \text{Diff}^i_{V_1, \Lambda_1, +}(\mathcal{M}) \mathcal{O}_{V_1} \quad \text{for } i = 1, \ldots, q - 1.
\]

(Here and further, logarithmic differential operators with respect to a collection of hypersurfaces mean logarithmic differential operators with respect to the ideals of these hypersurfaces.)
Proof. The verification that $\Lambda_1$ has normal crossings is straightforward. The above inclusion is a consequence of [5.6].

5.23. In view of Propositions 5.17 and 5.22 we now consider 3-tuples $(\mathcal{M}, \Lambda, \mathcal{L})$, where $\mathcal{M}$ is an $\mathcal{O}_V^a$-module on a connected $F$-finite regular scheme $V$, $\Lambda$ is a finite collection of hypersurfaces on $V$ with only normal crossings, and $\mathcal{L}$ is an invertible $\mathcal{O}_V$-ideal included in $\mathcal{I}(H)$ for all $H \in \Lambda$. We attach to such a triple $(\mathcal{M}, \Lambda, \mathcal{L})$ the $q$-differential collection $\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})$ obtained in Proposition 5.17. We set

\[ \text{Sing}(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) := \{ x \in V : \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq q \} \subset V. \]

A permissible center for $(\mathcal{M}, \Lambda, \mathcal{L})$ is a closed irreducible regular subscheme $Z \subset V$ included in $\text{Sing}(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}))$ such that $\Lambda$ has only normal crossings with $Z$. Let $V \overset{\pi}{\rightarrow} V_1 \supset H_1$ be the blow-up of $V$ along $Z$. Given a positive integer $a$, we define the a-transform of $(\mathcal{M}, \Lambda, \mathcal{L})$ as the 3-tuple $(\mathcal{M}_1^{(a)}, \Lambda_1, \mathcal{L}_1)$ on $V_1$, where

1. $\mathcal{M}_1^{(a)} := (\mathcal{M} \mathcal{O}_{V_1})_{\mathcal{I}(H_1)}$ is the a-transform of $\mathcal{M}$ (Definition 5.10),
2. $\Lambda_1$ is the collection of the strict transforms of the hypersurfaces in $\Lambda$ plus the exceptional hypersurface $H_1$, and
3. $\mathcal{L}_1 := (\mathcal{LO}_{V_1})_{\mathcal{I}(H_1)} \subset \mathcal{O}_{V_1}$.

Note that the family $\Lambda_1$ has only normal crossings and that $\mathcal{L}_1$ is included in $\mathcal{I}(H')$ for all $H' \in \Lambda_1$. Hence $(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1)$ is in the same situation as the original triple $(\mathcal{M}, \Lambda, \mathcal{L})$. See Example 5.22.

Proposition 5.24. Within the setting of 5.23 assume in addition that $\eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq q a$ for all $x \in Z$. Then $Z$ is permissible for $(\mathcal{M}, a)$, and there is an inclusion

\[ \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})_{\mathcal{O}_{V_1}} \subseteq \mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1), \]

where the term on the left is the a-transform of $\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})$ (Definition 5.3).

Proof. Corollary 5.13 implies that $\eta_z(\mathcal{M}) := \eta_x(\mathcal{G}(\mathcal{M})) \geq \eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq q a$ for all $x \in Z$. Hence $Z$ is permissible for $(\mathcal{M}, a)$ by Proposition 5.13. Next, we have to prove that

\[ ((\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \subseteq ((\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

for $i = 1, \ldots, q - 1$. Since $Z$ is permissible for $(\mathcal{M}, a)$, Proposition 2.13 shows that $\mathcal{M} \mathcal{O}_{V_1} \sim (\mathcal{I}(H_1)^{(a)})^a \mathcal{M}$; therefore $\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M} \mathcal{O}_{V_1}) = (\mathcal{I}(H_1)^{qa} \text{Diff}_{V_{\Lambda_1}+}(\mathcal{M})_{\mathcal{O}_{V_1}})$. We use this to prove the above inclusion:

\[ ((\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ \subseteq ((\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ = (\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ = (\mathcal{I}(H_1)^{qa} \text{Diff}_{V_{\Lambda_1}+}(\mathcal{M})_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ \subseteq (\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i)_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ = (\mathcal{I}(H_1)^{qa} \text{Diff}_{V_{\Lambda_1}+}(\mathcal{M})_{\mathcal{O}_{V_1}} : \mathcal{I}(H_1))^{qa} \]

\[ = (\text{Diff}_{V_{\Lambda_1}+}(\mathcal{M}) : \mathcal{L}_1^i). \]

5.25. Within the setting of 5.23 assumption in addition that $a \geq 1$ and that there exist $0 \leq b < q$ such that $\eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) = qa + b$ for all $x \in Z$. Then $Z$ is permissible for $(\mathcal{M}, a)$, and

\[ \eta_{x_1}(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq \eta_{x_1}(\mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1)), \quad \forall x_1 \in V_1. \]

Proof. The fact that $Z$ is permissible for $(\mathcal{M}, a)$ was already noted in Proposition 5.24. Next, by Theorem 5.9 we have that $\eta_{x_1}(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) \geq \eta_{x_1}(\mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1))$ for $x_1 \in V_1$, and from Proposition 5.24 we deduce the inequality $\eta_{x_1}((\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})_{\mathcal{O}_{V_1}})_{\mathcal{I}(H_1)})^{qa} \geq \eta_{x_1}(\mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1)), \forall x_1 \in V_1$. These two inequalities imply the corollary.

Remark 5.26. We use our previous results to define “permissible” sequences of transformations of a module by using functions satisfying the pointwise inequality:
Fix an $O^q_V$-submodule $\mathcal{M} \subseteq O_V$. We attach to $\mathcal{M}$ the 3-tuple $(\mathcal{M}_0, \Lambda_0, \mathcal{L}_0) := (\mathcal{M}, \emptyset, O_V)$ with its associated $q$-differential collection $\mathcal{G}(\mathcal{M}_0, \Lambda_0, \mathcal{L}_0) = \mathcal{G}(\mathcal{M})$. From this collection and its associated function $\eta$ we can define sequences of transformations of the triple $(\mathcal{M}_0, \Lambda_0, \mathcal{L}_0)$, say

$$\begin{align*}
V_0 &\leftarrow \pi_1 V_1 \leftarrow \pi_2 \cdots \leftarrow \pi_r V_r \\
(\mathcal{M}_0, \Lambda_0, \mathcal{L}_0) &\leftarrow (\mathcal{M}_1, \Lambda_1, \mathcal{L}_1) \cdots \leftarrow (\mathcal{M}_r, \Lambda_r, \mathcal{L}_r)
\end{align*}$$

constructed as follows. For $i = 0, \ldots, r - 1$, $V_i \leftarrow V_{i+1}$ is a blowup with a permissible center $Z_i$ for the triple $(\mathcal{M}_i, \Lambda_i, \mathcal{L}_i)$, along which the function $x \mapsto \eta_i(\mathcal{M}_i, \Lambda_i, \mathcal{L}_i)$ reaches its maximum value, say $a_i q + b_i$ (with $0 \leq b_i < q$ and $a_i \geq 1$). The triple $(\mathcal{M}_{i+1}, \Lambda_{i+1}, \mathcal{L}_{i+1})$ is the $a_i$-transform of $(\mathcal{M}_i, \Lambda_i, \mathcal{L}_i)$.

By Corollary 5.28 the functions $\eta$ defined from the $q$-differential collections $\mathcal{G}(\mathcal{M}_i, \Lambda_i, \mathcal{L}_i)$ satisfy the fundamental pointwise inequality; in particular,

$$a_1 q + b_1 \geq a_2 q + b_2 \geq a_3 q + b_3 \geq \cdots .$$

By the same corollary, $Z_i$ is permissible for $(\mathcal{M}_i, a_i)$.

**Example 5.27.** Let $V = \text{Spec}(\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5])$ and $q = p = 3$. We consider $\mathcal{M} := O^q_V \cdot x_1 x_2 x_3 x_4 x_5$, $\Lambda := \emptyset$, and $\mathcal{L} := O_V$. Then

$$\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) = \mathcal{G}(\mathcal{M}) = \left( \left\langle \frac{x_1 x_2 x_3 x_4 x_5}{x_j} : j = 1, \ldots, 5 \right\rangle, \left\langle \frac{x_1 x_2 x_3 x_4 x_5}{x_i x_j} : 1 \leq i < j \leq 5 \right\rangle \right).$$

The maximum value of $\eta_i(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) = \eta_i(\mathcal{M})$ is $5 = 3 \cdot 1 + 2$ (so that $a = 1$), and this maximum is attained only when $x$ is the origin.

We blowup $V$ at the origin and look at the affine chart $V_1 := \text{Spec}(\mathbb{F}_q[x_1, x_2', x_3', x_4', x_5'])$, where $x'_i = \frac{x_i}{x_1}$ for $i = 2, \ldots, 5$. The 1-transform of $\mathcal{M}$ (on $V_1$) is $\mathcal{M}_1 := O_{V_1} \cdot x_1^2 x_2' x_3' x_4' x_5'$. Note also that $\Lambda_1 = \{O_{V_1} \cdot x_1\}$ and $\mathcal{L}_1 = O_{V_1} x_1$. A straightforward computation yields

$$\mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1) = \left( \left\langle \frac{x_1^2 x_2' x_3' x_4' x_5'}{x_j} : j = 2, \ldots, 5 \right\rangle : O_{V_1} \cdot x_1 \right),$$

$$\left( \left\langle \frac{x_1^2 x_i' x_j' x_k'}{x_i' x_j'} : 2 \leq i < j \leq 5 \right\rangle : O_{V_1} \cdot x_1^2 \right)$$

$$= \left( \left\langle \frac{x_1 x_2' x_3' x_4' x_5'}{x_j} : j = 2, \ldots, 5 \right\rangle, \left\langle \frac{x_1^2 x_i' x_j' x_k'}{x_i' x_j'} : 2 \leq i < j \leq 5 \right\rangle \right).$$

The maximum value of $\eta_i(\mathcal{G}(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1))$ along points of $V_1$ is $4 = 3 \cdot 1 + 1$ and is attained along the regular subscheme $Z_1$ defined by $\langle x_2', x_3', x_4', x_5' \rangle$, whereas the maximum value of $\eta_i(\mathcal{G}(\mathcal{M}_1)) = \eta_i(\mathcal{M}_1)$ is $6 = 3 \cdot 2$, and it is only attained at the origin.

We blow up $V_1$ along $Z_1$ and look at the affine chart $V_2 := \text{Spec}(\mathbb{F}_q[x_1, x_2', x_3', x_4', x_5'])$, where $x''_i := \frac{x_i'}{x_2'}$ for $i = 3, 4, 5$. The 1-transform of $\mathcal{M}_1$ (on $V_2$) is $\mathcal{M}_2 := O_{V_2} \cdot x_1^2 x_2' x_3' x_4'' x_5''$. Note also that $\Lambda_2 = \{O_{V_2} \cdot x_1, O_{V_2} \cdot x_2\}$ and that $\mathcal{L}_2 = O_{V_2} \cdot x_1 x_2'$. A straightforward computation yields

$$\mathcal{G}(\mathcal{M}_2, \Lambda_2, \mathcal{L}_2) = \left( \left\langle \frac{x_1 x_2' x_3'' x_4'' x_5''}{x_j} : j = 3, 4, 5 \right\rangle : O_{V_2} \cdot x_1 x_2' \right),$$

$$\left( \langle x_1^2 x_2' x_3'' x_4'' x_5'' : j = 3, 4, 5 \rangle : O_{V_2} \cdot x_1^2 x_2' \right)$$

$$= \left( \left\langle \frac{x_1 x_2' x_3'' x_4'' x_5''}{x_j} : j = 3, 4, 5 \right\rangle, \langle x_3'', x_4'', x_5'' \rangle \right).$$

The maximum value of $\eta_i(\mathcal{G}(\mathcal{M}_2, \Lambda_2, \mathcal{L}_2))$ is $3$, and it is attained along the regular subscheme $Z_2$ defined by the ideal $\langle x_3'', x_4'', x_5'' \rangle$. On the other hand, the maximum of $\eta_i(\mathcal{G}(\mathcal{M}_2)) = \eta_i(\mathcal{M}_2)$ is $6 = 3 \cdot 2$, and it is only attained at the origin.

We blow up $V_2$ along $Z_2$ and look at the affine chart $V_3 := \text{Spec}(\mathbb{F}_q[x_1, x_2', x_3', x_4'', x_5''])$, where $x'''_i := \frac{x_i''}{x_3''}$ for $i = 4, 5$. The 1-transform of $\mathcal{M}_2$ is $\mathcal{M}_3 := O_{V_3} \cdot x_1^2 x_2' x_4'' x_5''$. Note also that $\Lambda_3 = \{O_{V_3} \cdot x_1, O_{V_3} \cdot x_2', O_{V_3} \cdot x_5''\}$
and that \( L_3 = \mathcal{O}_{V_3} \cdot x_1 x_2^2 x_3^m \). A straightforward computation yields

\[
\mathcal{G}(M_3, \Lambda_3, L_3) = \left( \langle x_1^2 x_2^2 x_3^m : \lambda = \{4, 5\} : \mathcal{O}_{V_2} \cdot x_1 x_2 x_3^m \rangle, \langle x_1^2 x_2^2 x_3^m : \mathcal{O}_{V_3} \rangle \right)
\]

The process stops here, at least over this chart, as now the maximum of \( \eta_x(M_3, \Lambda_3, L_3) \) is 2 < 3. Moreover, the maximum of \( \eta_x(M_3) \) is still 5, and hence the \( V_3 \)-scheme, say

\[ X_3 = \text{Spec}(\mathcal{O}_{V_3}^{3,1/3}) \]

still has points of multiplicity 3. In fact, the dimension of the closed set of points of multiplicity 3 of \( X_3 \) is higher than that of the closed set of points of multiplicity 3 of the original scheme, say \( X = \text{Spec}(\mathcal{O}_{V_1}^{3,1/3}) \) (recall here that \( X \) has highest multiplicity \( q = 3 \) and that \( X_3 \) is obtained form \( X \) by blowing up successively at regular equimultiple centers of multiplicity 3).

Nevertheless, at least in this example, points of multiplicity 3 of \( X_3 \) are easier to deal with than those of the original scheme \( X \). In fact, we can eliminate points of multiplicity 3 of \( X_3 \) by simply blowing up at the higher-dimensional components of the closed set of points of multiplicity 3, namely by blowing up first at the components of codimension 2, followed by a blowup at a component of codimension 3.

We hope that the invariants developed in this paper, defined in terms of differentials and of logarithmic differentials, will eventually lead to a simplification of the highest multiplicity locus and hence to a reduction of points of multiplicity \( q \).

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