Expansive actions of automorphisms of locally compact groups $G$ on $\text{Sub}_G$

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Abstract
For a locally compact metrizable group $G$, we consider the action of $\text{Aut}(G)$ on $\text{Sub}_G$, the space of all closed subgroups of $G$ endowed with the Chabauty topology. We study the structure of groups $G$ admitting automorphisms $T$ which act expansively on $\text{Sub}_G$. We show that such a group $G$ is necessarily totally disconnected, $T$ is expansive and that the contraction groups of $T$ and $T^{-1}$ are closed and their product is open in $G$; moreover, if $G$ is compact, then $G$ is finite. We also obtain the structure of the contraction group of such $T$. For the class of groups $G$ which are finite direct products of $\mathbb{Q}_p$ for distinct primes $p$, we show that $T \in \text{Aut}(G)$ acts expansively on $\text{Sub}_G$ if and only if $T$ is expansive. However, any higher dimensional $p$-adic vector space $\mathbb{Q}_p^n$, ($n \geq 2$), does not admit any automorphism which acts expansively on $\text{Sub}_G$.

Keywords Expansive automorphisms · Space of closed subgroups · Chabauty topology · Contraction subgroups of automorphisms · Connected Lie groups · Totally disconnected groups · $p$-adic vector spaces

Mathematics Subject Classification Primary 37B05 · 37F15; Secondary 22.20 · 22.50 · 22E35 · 54H20

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1 Introduction

Let $(X, d)$ be a metric space and let Homeo$(X)$ denote the space of homeomorphisms of $X$. Let $\Gamma$ be a group which acts on $X$ by homeomorphisms; i.e. there is a group homomorphism from $\Gamma$ to Homeo$(X)$. The $\Gamma$-action on $X$ is said to be expansive if there exists $\delta > 0$ which satisfies the following: If $x, y \in X$ with $x \neq y$, then $d(\gamma(x), \gamma(y)) > \delta$ for some $\gamma \in \Gamma$. A homeomorphism $\varphi$ of $X$ is said to be expansive if the group $\{\varphi^n\}_{n \in \mathbb{Z}}$ acts expansively on $X$ (equivalently, we say that the $\varphi$-action on $X$ is expansive).

Let $G$ be a locally compact Hausdorff group with the identity $e$. An automorphism $T$ of $G$ is said to be expansive if $\cap_{n \in \mathbb{Z}} T^n(U) = \{e\}$ for some neighbourhood $U$ of $e$. If $T$ is expansive on $G$, then $G$ is metrizable and the above definition is equivalent to the one given in terms of the left invariant metric $d$ on $G$.

The notion of expansivity was introduced by Utz in [24] and studied by many in different contexts (see Bryant [7], Schmidt [19], Glöckner and Raja [11], Shah [22] and references cited therein).

We now recall the definition of $K$-contraction groups of $T$. Let $T \in \text{Aut}(G)$ and let $K$ be a compact $T$-invariant subgroup of $G$. The group $C_K(T) = \{x \in G \mid T^n(x)K \rightarrow K$ in $G/K$ as $n \rightarrow \infty\}$ is called the $K$-contraction group of $T$. It is $T$-invariant and the group $C(T) := C_{e_1}(T)$ is called the contraction group of $T$. If $T$ is expansive and if $G$ is not discrete, by Theorem 2.9 of [22], it follows that $T$ cannot be distal and either $C(T)$ or $C(T^{-1})$ is nontrivial.

We know that an automorphism of the $n$-dimensional torus $\mathbb{T}^n$ (which belongs to $GL(n, \mathbb{Z})$) is expansive if and only if all its eigenvalues are of absolute value other than 1 (see e.g. [14] or [19]). However, the unit circle does not admit any expansive homeomorphism [1, Theorem 2.2.26]. Here we are going to show that a large class of homeomorphisms on certain compact spaces are not expansive; namely, the homomorphisms of Sub$_G$ arising from the automorphisms of almost connected locally compact infinite groups $G$.

Let $G$ be a locally compact (Hausdorff) topological group. Let Sub$_G$ denote the set of all closed subgroups of $G$ equipped with the Chabauty topology [8]. Then Sub$_G$ is compact and Hausdorff. It is metrizable if $G$ is so (see [10] and Chapter E of [4] for more details). Let Aut$(G)$ denote the group of automorphisms of $G$. There is a natural action of Aut$(G)$ on Sub$_G$; namely, $(T, H) \mapsto T(H), T \in \text{Aut}(G), H \in \text{Sub}_G$. For each $T \in \text{Aut}(G)$, the map $H \mapsto T(H)$ defines a homeomorphism of Sub$_G$ [13, Proposition 2.1], and the corresponding map from Aut$(G)$ \rightarrow Homeo(Sub$_G$) is a group homomorphism. Here, we would like to study the action of an automorphism of a locally compact first countable (metrizable) group $G$ on Sub$_G$ in terms of expansivity.

Let Sub$_G^a$ denote the space of all closed abelian subgroups of $G$. Note that Sub$_G^a$ is closed in Sub$_G$, and hence compact, and it is invariant under the action of Aut$(G)$. Moreover, if $T \in \text{Aut}(G)$ acts expansively on Sub$_G$, then it acts expansively on Sub$_G^a$. We show that a nontrivial connected Lie group $G$ does not have any automorphism which acts expansively on Sub$_G^a$ (see Theorem 3.1). For a compact group $G$ and $T \in \text{Aut}(G)$, we show that $T$ acts expansively on Sub$_G$ if and only if $G$ is finite (see Theorem 3.2). As a consequence of these results, we get that any automorphism of a
nontrivial connected locally compact group $G$ does not act expansively on $\text{Sub}_G$ (see Corollary 3.1).

For a general locally compact group $G$ and $T \in \text{Aut}(G)$, we show that if $T$ acts expansively on $\text{Sub}_G$, then $G$ is totally disconnected and $T$ acts expansively on $G$ (more generally, see Theorem 4.1). Let $\mathbb{Q}_p$ denote the $p$-adic field for a prime $p$. It is locally compact and totally disconnected and, it is a topological group with addition as the group operation. For $G = \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_n}$, a finite direct product, where $p_1, \ldots, p_n$ are distinct primes, we show that $T$ acts expansively on $\text{Sub}_G$ if and only if $T$ is expansive (see Corollary 4.2). However, we show that any higher dimensional $p$-adic vector space $\mathbb{Q}_p^n$, $(n \geq 2)$, does not admit any automorphism which acts expansively on $\text{Sub}_\mathbb{Q}_p$ (see Proposition 4.1). Using these results and a structure theorem for totally disconnected locally compact contraction groups obtained by Glöckner and Willis in [12], we prove that if an automorphism of $G$ acts expansively on $\text{Sub}_G$, then its contraction group is either trivial or a finite direct product of $\mathbb{Q}_p$ for distinct primes $p$ (see Theorem 4.2 and Remark 4.2).

For many groups $G$, the space $\text{Sub}_G$ and $\text{Sub}^G_G$ have been identified (see e.g. Pourezza and Hubbard [17], Bridson, de la Harpe and Kleptsyn [6] and also Baik and Clavier [2]). Since the homeomorphisms of $\text{Sub}_G$ arising from the action of $\text{Aut}(G)$ form a large subclass of it, it is significant to study the expansivity of such homeomorphisms of $\text{Sub}_G$.

A homeomorphism $\varphi$ of a topological space $X$ is said to be distal if for every pair of distinct elements $x$, $y \in X$, the closure of the double orbit $\{(\varphi^n(x), \varphi^n(y)) \mid n \in \mathbb{Z}\}$ does not intersect the diagonal $\{(d, d) \mid d \in X\}$ in $X \times X$. In case $X$ is a compact metric space with a metric $d$, $\varphi$ is distal if $\inf_{n \in \mathbb{Z}} d(\varphi^n(x), \varphi^n(y)) > 0$, $x$, $y \in X$. Note that for homeomorphisms of compact infinite metric spaces, distality and expansivity are opposite phenomena [7, Theorem 2]. Shah and Yadav in [23] have discussed the distality of the actions of automorphisms on $\text{Sub}_G$ and $\text{Sub}^G_G$. The study of expansivity of these actions in the current paper contributes to and enhances the understanding of the dynamics of actions of automorphisms of $G$ on $\text{Sub}_G$.

Throughout, $G$ is a locally compact first countable (metrizable) topological group. For a closed subgroup $H$, let $H^0$ denote the connected component of the identity $e$ in $H$. For $T \in \text{Aut}(G)$, we say that a subset $A$ of $G$ is $T$-invariant if $T(A) = A$.

2 Basic results on expansive actions and $\text{Sub}_G$

Given a locally compact (metrizable) group, the Chabauty topology on $\text{Sub}_G$ was introduced by Claude Chabauty in [8]. A sub-basis of the Chabauty topology on $\text{Sub}_G$ is given by the sets of the following form $O_K = \{A \in \text{Sub}_G \mid A \cap K = \emptyset\}$, where $K$ is a compact subset of $G$, and $O_U = \{A \in \text{Sub}_G \mid A \cap U \neq \emptyset\}$, where $U$ is an open subset of $G$.

Any closed subgroup of $\mathbb{R}$ is either a discrete group generated by a real number or the whole group $\mathbb{R}$, and $\text{Sub}_\mathbb{R}$ is homeomorphic to $[0, \infty]$ with a compact topology. Any closed subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{N} \cup \{0\}$, and $\text{Sub}_{\mathbb{Z}}$ is homeomorphic to $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$. The space $\text{Sub}_{\mathbb{R}^2}$ is homeomorphic to $S^4$ (see [17]). Note that the space $\text{Sub}_{\mathbb{R}^n}$ is simply connected for all $n \in \mathbb{N}$ [15, Theorem 1.3].
We first recall some known results which will be useful.

**Lemma 2.1** [25, Corollary 5.22 & Theorem 5.26] Let \((X, d)\) be a compact metric space. Then the following hold for homeomorphisms of \(X\):

1. Expansivity is a topological conjugacy invariant.
2. Expansivity of a homeomorphism is independent of the metric chosen as long as the metric induces the topology of \(X\).

Moreover, the following hold for any homeomorphism \(\varphi\) of \(X\):

3. For every \(n \in \mathbb{Z}\setminus\{0\}\), if \(\varphi\) is expansive, then \(\varphi^n\) has only finitely many fixed points.
4. \(\varphi^n\) is expansive for some \(n \in \mathbb{Z}\setminus\{0\}\) if and only if \(\varphi^n\) is expansive for all \(n \in \mathbb{Z}\setminus\{0\}\).
5. If \(\varphi\) is expansive and \(Y\) is a closed \(\varphi\)-invariant subset of \(X\), then \(\varphi|_Y\) is also expansive.

**Remark 2.1** Note that (4) and (5) of Lemma 2.1 hold for any automorphism of a locally compact (not necessarily compact) group and (5) of Lemma 2.1 also holds for any (not necessarily compact) metric space.

Recall that there is a natural group action of \(\text{Aut}(G)\), the group of automorphisms of \(G\), on \(\text{Sub}_G\) defined as follows:

\[
\text{Aut}(G) \times \text{Sub}_G \to \text{Sub}_G; \quad (T, H) \mapsto T(H), T \in \text{Aut}(G), H \in \text{Sub}_G.
\]

The map \(H \mapsto T(H)\) is a homeomorphism of \(\text{Sub}_G\) for each \(T \in \text{Aut}(G)\), and the corresponding map from \(\text{Aut}(G)\) to \(\text{Homeo}(\text{Sub}_G)\) is a homomorphism.

The following elementary lemma will be useful and it is easy to prove. However we give a proof for the sake of completion.

**Lemma 2.2** If \(G\) is an infinite locally compact group, then \(\text{Sub}_{\text{a}}^G\) is also infinite. In particular, for an infinite closed subset \(\mathcal{H}\) of \(\text{Sub}_G\), the action of the trivial map \(\text{Id}\) on \(\mathcal{H}\) is not expansive.

**Proof** If \(G\) has infinitely many elements of finite order, then \(\text{Sub}_{\text{a}}^G\) is infinite. Suppose \(G\) has finitely many elements of finite order. Then \(G\) has an element say, \(x\) which does not have finite order. Then the cyclic subgroup \(G_x\) generated by \(x\) is either discrete and isomorphic to \(\mathbb{Z}\) or its closure \(\overline{G}_x\) is a compact group [16, Theorem 19]. In the first case, there are infinitely many closed subgroups; namely, \(G_{x^n}\) generated by \(x^n, n \in \mathbb{N}\). In the second case \(\overline{G}_x\) is a compact abelian infinite group. Then its Pontryagin dual is an infinite discrete abelian group, so has infinitely many subgroups. By Pontryagin duality, it follows that \(\overline{G}_x\) has infinitely many closed subgroups. Therefore \(\text{Sub}_{\overline{G}_x}\), and hence \(\text{Sub}_{\text{a}}^G\) is infinite.

As the \(T\)-action on \(\mathcal{H}\) has infinitely many fixed points, by Lemma 2.1 (3), the \(\text{Id}\)-action on \(\mathcal{H}\) is not expansive.

The following lemma shows that the expansivity carries over to the quotients modulo closed invariant subgroups.
Lemma 2.3 Let $G$ be a locally compact group, $T \in \text{Aut}(G)$ and let $H$ be a $T$-invariant closed normal subgroup of $G$. Let $\overline{T} \in \text{Aut}(G/H)$ be the corresponding map defined as $\overline{T}(gH) = T(g)H$, for all $g \in G$. If the $T$-action on $\text{Sub}_G$ is expansive, then the $\overline{T}$-action on $\text{Sub}_H$ as well as the $T$-action on $\text{Sub}_{G/H}$ are also expansive.

**Proof** As $T(H) = H$, $\text{Sub}_H$ is $T$-invariant. Also, $\mathcal{H} = \{K \in \text{Sub}_G : H \subset K\}$ is a $T$-invariant closed subset of $\text{Sub}_G$. Therefore, by Lemma 2.1 (5), the $T$-action on both $\text{Sub}_H$ and $\mathcal{H}$ is expansive. Let $\pi : \mathcal{H} \to \text{Sub}_{G/H}$ be defined as $\pi(K) = K/H$. Then $\pi$ is a homeomorphism [20, Proposition 2]. By Lemma 2.1 (1), $\pi T \pi^{-1}$ acts expansively on $\text{Sub}_{G/H}$. As the $\pi T \pi^{-1}$-action on $\text{Sub}_{G/H}$ is the same as the $\overline{T}$-action on $\text{Sub}_{G/H}$, the $T$-action on $\text{Sub}_{G/H}$ is expansive. $\square$

The converse of the above lemma does not hold as illustrated by Example 4.1.

We say that a Hausdorff topological space $K$ acts continuously on a topological space $X$ if there exists a continuous map from $K \times X$ to $X$, $(k, x) \mapsto kx$, $k \in K$, $x \in X$. In particular, each $k \in K$ defines a continuous map on $X$.

The following lemma is well-known and easy to prove.

**Lemma 2.4** Let $(X, d)$ be a metric space and let $K$ be a compact space which acts continuously on $X$. Then this action of $K$ on $X$ is uniformly equicontinuous; (i.e. given $\epsilon > 0$, there exists $\delta > 0$, for any pair of elements $x, y \in X$ with $d(x, y) < \delta$, we have $d(kx, ky) < \epsilon$ for all $k \in K$).

Let $H$ be (Hausdorff) topological group. We say that $H$ acts continuously on a topological space $X$ by homeomorphisms if there exists a homomorphism $\rho : H \to \text{Homeo}(X)$ such that the corresponding map $H \times X \to X$ given by $(h, x) \mapsto \rho(h)(x)$ is continuous. The following lemma will be useful and it can be easily proven by using Lemma 2.4. The lemma will apply in particular to the action of $H = \text{Aut}(G)$ on $X = \text{Sub}_G$, when $G$ is a connected Lie group or a $p$-adic vector space.

**Lemma 2.5** Let $X$ be a metric space and let $H$ be a topological group acting continuously on $X$ by homeomorphisms. Let $\varphi, \psi \in H$ be such that $\varphi \psi = \psi \varphi$ and $\psi$ is contained in a compact subgroup of $H$. Then $\varphi$ is expansive if and only if $\varphi \psi$ is expansive.

Note that $\text{Aut}(G)$ with the modified compact-open topology is a topological group [21, 9.17], and by Lemma 2.4 of [23] it acts continuously on $\text{Sub}_G$ by homeomorphisms. Hence Lemma 2.5 holds for the case when $X = \text{Sub}_G$ and $H = \text{Aut}(G)$ with the modified compact-open topology, where $G$ is a locally compact metrizable group.

3 Actions of automorphisms of Lie groups $G$ on $\text{Sub}_G$

In this section we prove that a connected Lie group $G$ does not admit any automorphism which acts expansively on $\text{Sub}_G$. We also prove that a compact group $G$ admits an automorphism which acts expansively on $\text{Sub}_G$ if and only if $G$ is finite. As a consequence of these results, we show that any connected locally compact group $G$ does not admit an automorphism which acts expansively on $\text{Sub}_G$. 

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Theorem 3.1 Let $G$ be a nontrivial connected Lie group and let $T \in \text{Aut}(G)$. Then the $T$-action on $\text{Sub}_G^0$ is not expansive.

Proof Case I: Suppose $G \cong \mathbb{R}$. If $T = \pm \text{Id}$, then $T$ acts trivially on $\text{Sub}_\mathbb{R}$. It follows by Lemma 2.2 that the $T$-action on $\text{Sub}_\mathbb{R}$ is not expansive. Now suppose $T = \alpha \text{Id}$ for some $\alpha \in \mathbb{R}^*$. Replacing $T$ by $T^{-1}$ if necessary, we may assume that $0 < \alpha < 1$. Let $K = [1/2, 3/2]$. Then $K$ is compact and it acts continuously on $\text{Sub}_\mathbb{R}$ through the action given as follows: $(k, H) \mapsto kH$, $k \in K$, $H \in \text{Sub}_\mathbb{R}$. Let $d$ be a metric on $\text{Sub}_\mathbb{R}$.

Let $\epsilon > 0$ be fixed. By Lemma 2.4, there exists $\delta > 0$ such that if $H_1, H_2 \in \text{Sub}_\mathbb{R}$ with $d(H_1, H_2) < \delta$, then $d(kH_1, kH_2) < \epsilon/2$, $k \in K$.

We know that $T^n(\mathbb{Z}) = \alpha^n\mathbb{Z} \to \mathbb{R}$ and $T^{-n}(\mathbb{Z}) = \alpha^{-n}\mathbb{Z} \to \{0\}$ as $n \to \infty$. There exists $n_0 \in \mathbb{N}$ such that

$$\max\{d(\alpha^n\mathbb{Z}, \mathbb{R}), d(\alpha^{-n}\mathbb{Z}, \{0\})\} < \delta$$

for all $n \geq n_0$.

This implies that

$$\max\{d(k\alpha^n\mathbb{Z}, \mathbb{R}), d(k\alpha^{-n}\mathbb{Z}, \{0\})\} < \epsilon/2$$

for all $n \geq n_0$ and $k \in K$.

Now for each $k \in K$ and $n \in \mathbb{N}$ with $n \geq n_0$, we have

$$d(\alpha^n\mathbb{Z}, \alpha^n k\mathbb{Z}) \leq d(\alpha^n\mathbb{Z}, \mathbb{R}) + d(\alpha^n k\mathbb{Z}, \mathbb{R}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Similarly, we have $d(\alpha^{-n}\mathbb{Z}, \alpha^{-n} k\mathbb{Z}) < \epsilon$ for all $n \geq n_0$ and $k \in K$. Since the $T^n$-action on $\text{Sub}_\mathbb{R}$ is continuous for all $n \in \mathbb{N}$, there exists $\delta' > 0$ with $0 < \delta' < \delta$ such that for any $k \in \mathbb{N}$ if $d(k\mathbb{Z}, \mathbb{Z}) < \delta'$, then $d(\alpha^n\mathbb{Z}, \alpha^n k\mathbb{Z}) < \epsilon$ for all $|n| \leq n_0$. It is easy to see that for $k \in K$, if $k \to 1$, then $k\mathbb{Z} \to \mathbb{Z}$. This implies that there exists $k' \in K$, such that $d(k'\mathbb{Z}, \mathbb{Z}) < \delta'$. Therefore, $d(\alpha^n\mathbb{Z}, \alpha^n k'\mathbb{Z}) < \epsilon$ for all $n \in \mathbb{Z}$. Since $\epsilon > 0$ is arbitrary and $T = \alpha \text{Id}$, the $T$-action on $\text{Sub}_\mathbb{R}$ is not expansive.

Suppose $G \cong \mathbb{R}^n$ for some $n \geq 2$ and $T \in \text{GL}(n, \mathbb{R})$. Suppose $T$ has a real eigenvalue. Then we get a one dimensional $T$-invariant space $V \cong \mathbb{R}$ such that $T|_V = \alpha \text{Id}$, for some $\alpha \neq 0$. From above, the $T$-action on $\text{Sub}_V$, and hence on $\text{Sub}_{\mathbb{R}^n}$ is not expansive. Now suppose all the eigenvalues of $T$ are complex. There exists a two-dimensional subspace $W$ which is $T$-invariant. Then there exist $t > 0$ and a rotation map $S \in \text{GL}(2, \mathbb{R})$ such that $T|_W = t ASA^{-1}$ for some $A \in \text{GL}(2, \mathbb{R})$. Observe that $ASA^{-1}$ is contained in a compact subgroup of $\text{GL}(2, \mathbb{R})$. As $W \cong \mathbb{R}^2$, from the above argument, $t \text{Id}$-action on $\text{Sub}_W$ is not expansive. As $\text{Aut}(W) \cong \text{GL}(2, \mathbb{R})$, its topology coincides with the compact-open topology as well as with the modified compact-open topology, and it acts continuously on $\text{Sub}_W$ by homeomorphisms; (one can also directly show this by using the criteria for convergence of sequences in $\text{Sub}_G$ as in [4, Proposition E.1.2, pp 161]). By Lemma 2.5, the $T$-action on $\text{Sub}_W$, and hence on $\text{Sub}_{\mathbb{R}^n}$ is also not expansive. Note that $\text{Sub}_{\mathbb{R}^n} = \text{Sub}_{\mathbb{R}^n}^0$.

Case II: Suppose $G$ is not isomorphic to $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Let $R$ be the radical of $G$, i.e. $R$ is the largest connected closed solvable normal subgroup. Suppose $R \neq \{e\}$. Let $R_0 = R$, $R_1 = [R, R]$, $R_{m+1} = [R_m, R_m]$ for all $m \geq 1$. These are closed connected...
characteristic subgroups of $G$. As $R$ is solvable, there exists $k \in \mathbb{N} \cup \{0\}$ such that $R_k \neq \{e\}$ and $R_{k+1} = \{e\}$. As $R_k$ is a nontrivial $T$-invariant closed connected abelian Lie subgroup, $R_k \cong \mathbb{R}^n \times \mathbb{T}^m$ for some $m, n \in \mathbb{N} \cup \{0\}$ such that $m + n$ is nonzero. Suppose $m \neq 0$. Then $\mathbb{T}^m$ is characteristic in $G$, and hence it is $T$-invariant. Let $K$ be the set of all roots of unity in $\mathbb{T}^m$, i.e. $K = \cup_{i \in \mathbb{N}} K_i$, where $K_i = \{x \in \mathbb{T}^m \mid x^i = e\}$. Note that $K_i$ is a nontrivial proper closed $T$-invariant subgroup of $\mathbb{T}^m$, $i \in \mathbb{N}$, and $K_i \neq K_j$ if $i, j \in \mathbb{N}$ are distinct. By Lemma 2.1 (3), we get that the $T$-action on $\text{Sub}_{\mathbb{T}^m}$, and hence on $\text{Sub}_{G}^a$ is not expansive. If $m = 0$, then $R_k \cong \mathbb{R}^n$ and from Case I, the $T$-action on $R_k$, and hence on $\text{Sub}_{G}^a$ is not expansive.

Suppose $R = \{e\}$. Then $G$ is semisimple. Then the group Inn$(G)$ of all inner automorphisms of $G$ is a subgroup of finite index in Aut$(G)$ (see section 1 in [9]). There exist $x \in G$ and $n \in \mathbb{N}$ such that $T^n = \text{inn}(x)$; i.e. $T^n(g) = xgx^{-1}$, $g \in G$. Suppose $x^n \in Z(G)$ for some $m \in \mathbb{N}$. Then $T^{nm} = \text{Id}$ and by Lemma 2.2 and Lemma 2.1 (3), the $T$-action on $\text{Sub}_{G}^a$ is not expansive. Suppose $x^k \notin Z(G)$ for every $k \in \mathbb{N}$. Let $H_k$ be the closure of the cyclic subgroup generated by $x$. Then $H_k$ is an infinite closed abelian $T$-invariant subgroup. As $T$ acts trivially on $H_k$, by Lemma 2.2, the $T$-action on Sub$_{H_k}$, and hence on $\text{Sub}_{G}^a$ is not expansive. □

Let $X$ be a topological space and let $f$ be a continuous self map on $X$. A point $x \in X$ is said to be periodic if $f^n(x) = x$ for some $n \in \mathbb{N}$. The set Per$(f)$ denotes the set of all periodic points of $X$.

Let $G$ be a compact group and let $T \in \text{Aut}(G)$. We say that $(G, T)$ satisfies the descending chain condition if for every decreasing sequence $\{G_n\}$ of closed $T$-invariant subgroups in $G$, there exists $k \in \mathbb{N}$ such that $G_n = G_k$ for all $n \geq k$. Note that if $T$ acts expansively on $\text{Sub}_G$, then by Lemma 2.1 (3), $(G, T)$ satisfies the descending chain condition.

**Theorem 3.2** Let $G$ be a compact metrizable group and let $T \in \text{Aut}(G)$. Then the $T$-action on $\text{Sub}_G$ is expansive if and only if $G$ is finite.

**Proof** The ‘if’ statement is obvious. Now suppose $G$ is an infinite compact group. We show that the $T$-action on $\text{Sub}_G$ is not expansive. As noted above, if $(G, T)$ does not satisfy the descending chain condition, then by Lemma 2.1 (3), the $T$-action on $\text{Sub}_G$ is not expansive. Now suppose $(G, T)$ satisfies the descending chain condition. By Proposition 3.5 of [19], there exists a compact normal $T$-invariant subgroup $H$ of $G$ such that $G/H$ is Lie group and the $T$-action on $H$ is ergodic. Suppose $H$ is trivial. Then $G$ is a compact Lie group with finitely many connected components. As $G$ is infinite, $G^0$ is nontrivial and by Theorem 3.1, the $T$-action on $\text{Sub}_{G^0}$, and hence on $\text{Sub}_G$ is not expansive.

Now suppose $H$ is nontrivial. Then it is infinite. Let Per$(T)$ be the set of all periodic points of $T$ in $H$. As $T|_H$ is ergodic and $(H, T|_H)$ also satisfies the descending chain condition, by Theorem 7.5 of [14], Per$(T)$ is dense in $H$. Here Per$(T) = \cup_{n \geq 1} P_n$, where $P_n = \{x \in H \mid T^n(x) = x\}$ and each $P_n$ is a $T$-invariant closed subgroup of $H$, $n \in \mathbb{N}$. Since $T$ is ergodic, so is $T^n$, and hence $T^n \neq \text{Id}$ and $P_n \neq \text{Per}(T)$ for every $n \in \mathbb{N}$. By Lemma 2.1 (3), the $T$-action on $\text{Sub}_H$, and hence on $\text{Sub}_G$ is not expansive. □
Corollary 3.1 A nontrivial connected locally compact group $G$ does not admit any automorphism which acts expansively on $\text{Sub}_G$.

Proof Since $G$ is connected, it admits the largest compact normal subgroup $K$ such that $G/K$ is a Lie group. Let $T \in \text{Aut}(G)$. Here, $K$ is characteristic in $G$ and, in particular, it is $T$-invariant. If $K$ is infinite, then by Theorem 3.2, the $T|_K$-action on $\text{Sub}_K$ is not expansive. If $K$ is finite, then $G$ is a Lie group and by Theorem 3.1, the $T$-action on $\text{Sub}_G$, and hence on $\text{Sub}_G$ is not expansive.

4 Expansive actions on $\text{Sub}_G$ of automorphisms of locally compact groups $G$

In this section we prove certain results for the structure of locally compact groups $G$ which admits an automorphism $T$ that acts expansively on $\text{Sub}_G$. We show that such $G$ is necessarily totally disconnected, $T$ is expansive (on $G$), and the contraction group of $T$ is either trivial or a finite direct product of $\mathbb{Q}_p$, for some distinct primes $p$. We show that any automorphism of $G$, where $G$ is a finite direct product of $\mathbb{Q}_p$ for distinct primes $p$, is expansive if and only if it acts expansively on $\text{Sub}_G$. We also show that any higher dimensional $p$-adic vector space $\mathbb{Q}_p^n$, $(n \geq 2)$, does not admit any automorphism which acts expansively on $\text{Sub}_{\mathbb{Q}_p^n}$.

For $T \in \text{Aut}(G)$, let $M(T) = \{x \in G \mid \{T^n(x)\}_{n \in \mathbb{Z}}$ is relatively compact$\}$. Then $M(T)$ is a $T$-invariant subgroup of $G$. If $G$ is totally disconnected, then $M(T)$ is closed [27, Proposition 3] (see also Remark 3.1 in [3]).

Theorem 4.1 Let $G$ be a locally compact group $G$ and let $T \in \text{Aut}(G)$ be such that the $T$-action on $\text{Sub}_G$ is expansive. Then the following hold:

1. $G$ is totally disconnected.
2. $T$ is expansive.
3. $C(T)$ and $C(T^{-1})$ are closed, and $C(T)C(T^{-1})$ is open.
4. $M(T)$ is finite.

Proof (1) : As $T|_{G^0}$ is expansive, $G^0$ is trivial by Corollary 3.1. Hence $G$ is totally disconnected.

(2) : If $G$ is discrete, then $T$ is expansive. Suppose $G$ is not discrete and suppose $T$ is not expansive. There exists $\{U_n \mid n \in \mathbb{N}\}$, a neighbourhood basis of the identity $e$ consisting of compact open subgroups such that $K_n = \cap_{k \in \mathbb{Z}}T^k(U_n) \neq \{e\}$, $n \in \mathbb{N}$. Note that $K_n$ is a nontrivial closed $T$-invariant subgroup of $G$, $n \in \mathbb{N}$. By Lemma 2.1 (3), the $T$-action on $\text{Sub}_G$ is also not expansive.

(3) : By Corollaries 3.27 and 3.30 of [3], there exists a compact $T$-invariant subgroup $K = \overline{C(T)} \cap C(T^{-1})$ such that $C(T) = C_K(T) = KC(T)$. Since the $T$-action on $\text{Sub}_G$ is expansive and $T(K) = K$, $T|_K$ acts expansively on $\text{Sub}_K$. Then $K$ is finite by Theorem 3.2. By Lemma 3.31 (2) of [3], we get that $C(T) \cap M(T)$ is dense in $K$. Since $K$ is finite and $T$-invariant, and since $M(T)$ is closed and $T$-invariant, we have that $C(T|_{M(T)}) = C(T) \cap M(T) = C(T) \cap K = C(T|_K) = \{e\}$. Therefore $K = \{e\}$, and hence $C(T)$ is closed. Replacing $T$ by $T^{-1}$, we get that $C(T^{-1}) \cap M(T) = \{e\}$.
and $C(T^{-1})$ is also closed. By Lemma 1.1 (d) of [11], $C(T)C(T^{-1})$ is open in $G$ and (3) holds.

(4) : As observed above, $M(T)$ is a closed $T$-invariant subgroup. As shown in the proof of (3), $C(T|_{M(T)}) = M(T) \cap C(T) = \{ e \} = M(T) \cap C(T^{-1}) = C(T^{-1}|_{M(T)})$ (see also Theorem 3.32 in [3]). By (3), $\{ e \}$ is open in $M(T)$, and hence $M(T)$ is discrete. Now for every $x \in M(T)$, the $T$-orbit of $x$ is relatively compact, and hence finite. Therefore, $M(T) = \text{Per}(T) = \bigcup_{n \in \mathbb{N}} P_n^T$, where $P_n^T = \{ x \in M(T) \mid T^n(x) = x \}$. Each $P_n^T$ is a closed $T$-invariant subgroup and $P_n^T \subset P_{n+1}^T$, $n \in \mathbb{N}$. If possible, suppose $P_n^T \neq P_{n+1}^T$ for infinitely many $n$. By Lemma 2.1 (3), the $T$-action on $\text{Sub}_M(T)$ is not expansive, which leads to a contradiction. Therefore, $M(T) = P_n^T$ for some $n$, and hence $T^n = \text{Id}$. By Lemma 2.2, $M(T)$ is finite. 

\[ \square \]

**Corollary 4.1** Let $G$ be a locally compact group $G$ and let $T \in \text{Aut}(G)$. If the $T$-action on $G$ is distal and the $T$-action on $\text{Sub}_G$ is expansive, then $G$ is discrete.

**Proof** If the $T$-action on $\text{Sub}_G$ is expansive, then by Theorem 4.1 (2), we get that $T$ is expansive. Moreover, if the $T$-action on $G$ is also distal, then $C(T) = C(T^{-1}) = \{ e \}$, and by Theorem 4.1 (3), $G$ is discrete (see also [22, Theorem 2.9]). \[ \square \]

From Theorem 4.1, we know that if $G$ admits an automorphism which acts expansively on $\text{Sub}_G$, it is necessarily totally disconnected. Proposition 4.1 below shows that $\mathbb{Q}_p$ admits such an automorphism, while none of the $\mathbb{Q}_p^n$, $n \geq 2$, does.

Recall that for a fixed prime $p$, and $a/b \in \mathbb{Q}\setminus\{0\}$, the $p$-adic absolute value of $a/b$ is defined as $|a/b|_p = p^{-n}$ if $a/b = p^n c/d$ for some $n \in \mathbb{Z}$, where $c$ and $d$ are co-prime to $p$, and $|0|_p = 0$. Observe that $|\cdot|_p$ defines a norm on $\mathbb{Q}$, and $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ with respect to the metric induced by this norm. Moreover, $|\cdot|_p$ extends canonically to $\mathbb{Q}_p$ such that $|xy|_p = |x|_p |y|_p$, $x, y \in \mathbb{Q}_p$. It is a locally compact totally disconnected field of characteristic zero, and $\mathbb{Q}_p^n$ is the $n$-dimensional $p$-adic vector space. Here, we consider $\mathbb{Q}_p^n$ as a group with addition as the operation. Any automorphism of $\mathbb{Q}_p^n$ belongs to $\text{GL}(n, \mathbb{Q}_p)$. Any automorphism $T$ of $\mathbb{Q}_p$ is of the form $T(x) = qx$, $x \in \mathbb{Q}_p$, where $q \in \mathbb{Q}_p^*$, i.e. $q = p^m z$, for some $m \in \mathbb{Z}$ and some $z \in \mathbb{Z}_p^* \subset \{ x \in \mathbb{Q}_p \mid |x|_p = 1 \}$. Note that $\mathbb{Z}_p^*$ is a compact multiplicative group. Observe that such a $T$ is expansive if and only if $q \notin \mathbb{Z}_p^*$, i.e. $m \neq 0$.

**Proposition 4.1** Let $T \in \text{GL}(n, \mathbb{Q}_p)$ for some $n \in \mathbb{N}$.

1. For $n = 1$, the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is expansive if and only if $T$ is expansive.
2. For $n \geq 2$, the $T$-action on $\text{Sub}_{\mathbb{Q}_p^n}$ is not expansive. That is, $\mathbb{Q}_p^n$, $(n \geq 2)$, does not admit any automorphism which acts expansively on $\text{Sub}_{\mathbb{Q}_p^n}$.

**Proof** (1) : The ‘only if’ statement follows from Theorem 4.1 (1). Now suppose $T$ acts expansively on $\mathbb{Q}_p$. Then $T = p^m \text{Id}$, for some $m \in \mathbb{Z}\setminus\{0\}$ for $a \in \mathbb{Z}_p^*$, i.e. $|a|_p = 1$. Note that any closed subgroup of $\mathbb{Q}_p$ is equal to $\{0\}$, $\mathbb{Q}_p$ or $p^n \mathbb{Z}_p$ for some $n \in \mathbb{Z}$, where $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$. Observe that the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is the same as the $bT$-action on $\text{Sub}_{\mathbb{Q}_p}$ for any $b \in \mathbb{Z}_p^*$. By Lemma 2.1 (4), without loss of any generality we may assume that $T = \text{Id}$.
If possible, suppose the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is not expansive. Let $d$ be a metric on $\text{Sub}_{\mathbb{Q}_p}$. Let $k_0$ be such that

$$\frac{1}{k_0} < d(H_1, H_2) \text{ for all } H_1, H_2 \in \{[0], \mathbb{Z}_p, \mathbb{Q}_p\} \text{ with } H_1 \neq H_2.$$ 

Let $k \in \mathbb{N}$ be such that $k > k_0$. Since the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is not expansive, for any such $k$, there exist $i_k, j_k \in \mathbb{Z}$ with $i_k < j_k$ such that for all $n \in \mathbb{Z}$,

$$d(T^n(p^{i_k}\mathbb{Z}_p), T^n(p^{j_k}\mathbb{Z}_p)) < \frac{1}{k}.$$ 

Putting $n = -i_k$, we get for $l_k = j_k - i_k$, $k \in \mathbb{N}$ with $k > k_0$ that

$$d(\mathbb{Z}_p, p^{l_k}\mathbb{Z}_p) < \frac{1}{k},$$

and hence, $p^{l_k}\mathbb{Z}_p \to \mathbb{Z}_p$. But as $l_k \in \mathbb{N}$, passing to a subsequence if necessary, we get that either $l_k \to \infty$ and $p^{l_k}\mathbb{Z}_p \to \{0\}$ in $\text{Sub}_{\mathbb{Q}_p}$, or $l_k = l \in \mathbb{N}$ and $p^{l_k}\mathbb{Z}_p = p^l\mathbb{Z}_p \neq \mathbb{Z}_p, k \in \mathbb{N}$. In either case, we get a contradiction. Hence the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is expansive.

(2) : Let $n \geq 2$ and let $T \in \text{GL}(n, \mathbb{Q}_p)$. We show that the $T$-action on $\text{Sub}_{\mathbb{Q}_p}$ is not expansive. By 3.3 of [26], there exists $m \in \mathbb{N}$ such that $T^m \sim \text{SAD}$, where $S$, $A$ and $D$ are semisimple, unipotent and diagonal matrix respectively, $A$, $S$ and $D$ commute with each other and $A$ as well as $S$ generate a relatively compact group. Take $U = SA$. Then $T^m = UD = DU$ where $U$ generates a compact group and $D(x_1, \ldots, x_n) = (p^{k_1}x_1, \ldots, p^{k_n}x_n)$ with $k_i \in \mathbb{Z}, 1 \leq i \leq n$. By Lemma 2.1 (4), it is enough to show that the $T^m$-action on $\text{Sub}_{\mathbb{Q}_p}$ is not expansive. Note that $\text{GL}(n, \mathbb{Q}_p)$ is a metrizable topological group. Using the criteria for convergence of sequences in $\text{Sub}_G$ for a metrizable group $G$ as in [4], it is easy to see that $\text{GL}(n, \mathbb{Q}_p)$ acts continuously on $\text{Sub}_{\mathbb{Q}_p}$ by homeomorphisms. By Lemma 2.5, it is enough to show that the $D$-action on $\text{Sub}_{\mathbb{Q}_p}$ is not expansive.

Let $H = \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ for all } i \geq 3\}$. $H \cong \mathbb{Q}_p^2$ is a $D$-invariant closed subgroup of $\mathbb{Q}_p^n$, and it enough to show that the $D|_H$-action on $\text{Sub}_H$ is not expansive. Hence we may assume that $n = 2$, i.e. $D(x_1, x_2) = (p^{k_1}x_1, p^{k_2}x_2)$ for some $k_1, k_2 \in \mathbb{Z}$.

Let $\mathcal{H}$ be the collection of all closed subgroups of $\mathbb{Q}_p^2$ of the form $(a, b)\mathbb{Q}_p$ with $(a, b) \in \mathbb{Q}_p^2 \setminus \{(0, 0)\}$ (Here, $\mathcal{H}$ is the set of $p$-adic lines in $\mathbb{Q}_p^2$). It is clear that $\mathcal{H}$ is a $D$-invariant closed subset of $\text{Sub}_{\mathbb{Q}_p^2}$. We will show that the $D$-action on $\mathcal{H}$ is not expansive. Since the $D$-action on $\mathcal{H}$ is the same as that of $p^{-k_2}D$, we may replace $D$ by $p^{-k_2}D$ and assume that $k_2 = 0$. Now if $k_1 = 0$, then $D = \text{Id}$ and by Lemma 2.2, the $D$-action on $\mathcal{H} \subset \text{Sub}_{\mathbb{Q}_p^2}$ is not expansive. Suppose $k_1 \neq 0$. Since $D = D_{11}^{k_1}$, where $D_1(x, y) = (px_1, x_2), x_1, x_2 \in \mathbb{Q}_p$, and $D_1$ also keeps $\mathcal{H}$ invariant, by Lemma 2.1 (4) we may replace $D$ by $D_1$ and assume that $D(x_1, x_2) = (px_1, x_2), x_1, x_2 \in \mathbb{Q}_p$. Observe that $D((0, 1)\mathbb{Q}_p) = (0, 1)\mathbb{Q}_p$ and that $D^n((a, 1)\mathbb{Q}_p) \to (0, 1)\mathbb{Q}_p$ as $n \to \infty$.
∞, for all a ∈ Q_p. Moreover, the set \{(a, 1)Q_p | a ∈ Q_p\} is uncountable as Q_p is so. By Theorem 1 of [18], the D-action on H is not expansive. Therefore, the D-action on Sub_{Q_p^2} is not expansive.

\[ \square \]

**Remark 4.1** Since the metric on \( \mathbb{Q}_p^m \) (naturally defined by the p-adic norm) is proper, Proposition 4.1 can also be proven using the explicit metric on Sub_{Q_p^2} given as in [10]. As mentioned in [10], this metric on Sub_G for a locally compact metrizable group G has been suggested by Biringer in [5].

The following example illustrates that the converse of Lemma 2.3 does not hold in general.

**Example 4.1** Let T = p Id in GL(2, Q_p). If H = Q_p x {0}, then H is T-invariant and by Proposition 4.1(1), the T-action on both Sub_H and Sub_{Q_p^2}/H are expansive, but the T-action on Sub_{Q_p^2} is not expansive by Proposition 4.1(2).

**Corollary 4.2** For a locally compact group G and T ∈ Aut(G), the following hold:

1. If \( G = \mathbb{Q}_{p_1}^{m_1} × ... × \mathbb{Q}_{p_n}^{m_n} \) for distinct primes \( p_1, ..., p_n \in \mathbb{N} \) and \( m_1, ..., m_n \in \mathbb{N} \), and if T is expansive, then T keeps \( \mathbb{Q}_{p_i}^{m_i} \) invariant for each i and G = C(T) × C(T^{-1}). Moreover if the T-action on Sub_G is expansive, then \( m_1 = ... = m_n = 1 \).
2. If \( G = \mathbb{Q}_{p_1}^{m_1} × ... × \mathbb{Q}_{p_n}^{m_n} \) for distinct primes \( p_1, ..., p_n \in \mathbb{N} \), then T is expansive if and only if the T-action on Sub_G is expansive.

**Proof** (1) : Suppose T is expansive on G as in (1). Then C(T)C(T^{-1}) is open in G [11, Lemma 1.1(d)]. Since G is not discrete, either C(T) or C(T^{-1}) is nontrivial. Suppose C(T) is nontrivial. Since G is torsion-free, by Theorem B of [12], C(T) is a direct product of T-invariant \( \mathbb{Q}_{p_i}^{l_i} \), for some prime \( p_i \) and \( l_i ≤ m_i \), and hence C(T) is closed. Similarly, either C(T^{-1}) is trivial, or it has a similar structure as C(T) described above and it is closed. In particular, C(T) ∩ C(T^{-1}) = \{e\} [3, Theorem 3.2(1⇔4)]. Therefore, C(T) × C(T^{-1}) is an open subgroup of G and it follows that G = C(T) × C(T^{-1}). Considering the structure of C(T) and C(T^{-1}) described above, it follows that each \( \mathbb{Q}_{p_i}^{m_i} \) is T-invariant. Now suppose the T-action on Sub_G is expansive. Then the T-action on Sub_{Q_{p_i}^{m_i}} is also expansive and by Proposition 4.1(2), \( m_i = 1 \) for each i.

(2) : Let G be as in (2). The one way implication ‘if’ follows from Theorem 4.1(2). Now suppose T is expansive. From (1), we get that \( \mathbb{Q}_{p_i} \) is T-invariant for each i, \( G = C(T) × C(T^{-1}) \) and any one of the following holds: C(T) = \{e\}, C(T) = G or C(T) (resp. C(T^{-1})) is a direct product of some \( \mathbb{Q}_{p_i} \). Now the T-action on \( \mathbb{Q}_{p_i} \) is expansive for each i.

Let \( \pi_i : G → \mathbb{Q}_{p_i} \) be the natural projection and let \( T_i : \mathbb{Q}_{p_i} → \mathbb{Q}_{p_i} \) be the map corresponding to T, such that \( \pi_i ∘ T = T_i ∘ \pi_i \), i = 1, ..., n. We show that Sub_G = Sub_{\mathbb{Q}_{p_1}} × ... × Sub_{\mathbb{Q}_{p_n}}. It is easy to see that Sub_{\mathbb{Q}_{p_i}} × ... × Sub_{\mathbb{Q}_{p_n}} ⊂ Sub_G.

Let H be a closed subgroup of G and let \( H_i = \pi_i(H) \) for each i. Then \( H_i \) is a subgroup of \( \mathbb{Q}_{p_i} \) for each i. We show that \( H = H_1 × ... × H_n \). This holds if H is trivial. Suppose H is nontrivial. Then not all \( H_i \) are trivial. Note that H ⊂ H_1 × ... × H_n. Let i ∈ {1, ..., n} be fixed such that \( H_i ≠ \{0\} \). It is enough to show that \( H_i ⊂ H \).
(through the canonical inclusion of $\mathbb{Q}_{p_i}$ in $G$). Let $x_i \in H_i$ be nonzero. There exists $y = (y_1, \ldots, y_n) \in H$ such that $y_i = \pi_i(y) = x_i$. Let $l = (p_1 \cdots p_n)/p_i$. Then $l \in \mathbb{N}$ and for all $j \neq i$, $l^k \to 0$ in $\mathbb{Q}_{p_j}$ as $k \to \infty$ and $l^k \in \mathbb{Z}^*$, which is a compact multiplicative group. It follows that there exists a sequence $\{ l_m \} \subset \mathbb{N}$ such that $l^{km} \to 1$ in $\mathbb{Q}_{p_i}$ as $m \to \infty$. Now $l^{km} y = (l^{km} y_1, \ldots, l^{km} y_n) \to a \in H$, where $a = (a_1, \ldots, a_n)$, $a_i = y_i = x_i$ and $a_j = 0$ for all $j \neq i$. Therefore $x_i \in H$, and hence $H_i \subset H$. Since this holds for all $i$ such that $H_i \neq \{0\}$ and $\pi_i(H) = H_i$, we get that $H = H_1 \times \cdots \times H_n$. Now $H_i = H \cap \mathbb{Q}_{p_i}$ is closed for each $i$. This implies that $\text{Sub}_G = \text{Sub}_{\mathbb{Q}_{p_1}} \times \cdots \times \text{Sub}_{\mathbb{Q}_{p_n}}$.

If $T$ acts expansively on $G$, then $T_i$ acts expansively on $\mathbb{Q}_{p_i}$ for each $i$. By Proposition 4.1 (1), the $T_i$-action on $\text{Sub}_{\mathbb{Q}_{p_i}}$ is expansive for each $i$. Now the assertion holds by Theorem 2.2.5 of [1].

Glöckner and Willis have obtained a structure theorem for totally disconnected locally compact contraction groups in [12]. We use this theorem along with Proposition 4.1 and Corollary 4.2 and get the following structure theorem for groups $G$ which admit a contractive automorphism that acts expansively on $\text{Sub}_G$. Recall that $T$ is said to be contractive if $C(T) = G$. Recall also that we call a subgroup $H$ of $G$ $T$-invariant if $T(H) = H$; (in [12], such a group $H$ is called $T$-stable).

**Theorem 4.2** Let $G$ be a locally compact group which admits a contractive automorphism $T$. Then $T$ acts expansively on $\text{Sub}_G$ if and only if $G = \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_n}$ for distinct primes $p_1, \ldots, p_n$ such that each $\mathbb{Q}_{p_i}$ is $T$-invariant.

**Proof** If $T$ is contractive, then $T$ is expansive [11, Remark 1.10]. One way implication ‘if’ follows from Corollary 4.2 (2). (In fact we do not need $\mathbb{Q}_{p_i}$ to be $T$-invariant).

Now we assume that $T$ is contractive and acts expansively on $\text{Sub}_G$. By Proposition 4.1 (1), $G$ is totally disconnected. By Theorem B of [12], $G = T \times D$, where $T$ and $D$ are closed $T$-invariant groups, $T$ is the torsion group, $D$ is the group of divisible elements in $G$ and $D$ is a direct product of $T$-invariant, nilpotent $p$-adic Lie groups for certain primes $p$, i.e.

$$D = G_{p_1} \times \cdots \times G_{p_n}$$

where each $G_p$ is the group of $\mathbb{Q}_p$-rational points of a unipotent linear algebraic group defined over $\mathbb{Q}_p$, by Theorem 3.5 (ii) of [26], for $p = p_1, \ldots, p_n$. Suppose $n > 1$. (We will refer to $G_p$ itself as a unipotent $p$-adic algebraic group).

We first show that $T$ is trivial. If possible, suppose $T$ is nontrivial. Then $T$ admits a nontrivial closed $T$-invariant subgroup $T_1$ such that $T_1$ has no proper closed $T$-invariant subgroup and $T_1 \cong F(-\mathbb{N}) \times F_{\mathbb{N}_0}^\times$, a restricted direct product for a finite group $F$, with $T|_{T_1}$ as the right-shift [12, Theorem 3.3 and Theorem A (a)]. Recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z} = \mathbb{N}_0 \cup -\mathbb{N}$, $F$ is a finite group with the discrete topology, $F_{\mathbb{N}_0}^\times = \prod_{n \in \mathbb{N}_0} F$, endowed with the compact product topology. Recall also that $F(-\mathbb{N}) \subset F(-\mathbb{N})$ is a subgroup consisting of $(x_n)_{n \in -\mathbb{N}}$ with $x_n = 1$ for all but finitely many $n \in -\mathbb{N}$, where $1$ is the identity element of the finite group $F$. Since $T_1$ is nontrivial, so is $F$. Let $\mathcal{H} = \{ H \subset F_{\mathbb{N}_0}^\times \mid H$ is a compact group$\}$. Since the set of subsets of $\mathbb{N}_0$ is uncountable, it implies that $\mathcal{H}$ is uncountable. Moreover, as $T((x_n)) = ((x_{n-1}))$ for
By Theorem 1 of [18], $T^n(H) \to \{e\}$ in $\text{Sub}_{T^1}$ for all $H \in \mathcal{H}$ [26, Proposition 2.1]. By Theorem 1 of [18], $T|_{T^1}$ is not expansive. This leads to a contradiction. Therefore, $T$ is trivial and $G = D$.

We show that $D = \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_n}$. Let $p$ be a fixed prime in $\{p_1, \ldots, p_n\}$. Then $G_p$ is a $T$-invariant unipotent $p$-adic algebraic group, $T|_{G_p}$ is contractive and the $T$-action on $\text{Sub}_{G_p}$ is expansive. It is enough to show that $G_p = \mathbb{Q}_p$.

Let $Z$ be the center of $G_p$. Then $Z$ is $T$-invariant and $Z \cong \mathbb{Q}_p^m$ for some $m \in \mathbb{N}$. By Proposition 4.1 (2), we get that $Z \cong \mathbb{Q}_p$. If $G_p$ is abelian, then $G_p = \mathbb{Q}_p$. If possible, suppose $G_p$ is not abelian. Let $Z^{(1)}$ be the closed subgroup of $G_p$ such that $Z \subset Z^{(1)}$ and $Z^{(1)}/Z$ is the center of $G_p/Z$. Then $Z^{(1)}$ is $T$-invariant and it is also a unipotent $p$-adic algebraic subgroup of $G_p$, and $Z^{(1)}/Z$ is abelian and isomorphic to $\mathbb{Q}_p^{m_1}$, for some $m_1 \in \mathbb{N}$. By Lemma 2.3, the $T$-action on $\text{Sub}_{G_p}/Z$, and hence on $\text{Sub}_{Z^{(1)}/Z}$ is expansive. By Proposition 4.1 (2), $Z^{(1)}/Z$ is isomorphic to $\mathbb{Q}_p$. Let $x \in Z^{(1)}/Z$. Since $Z^{(1)}$ is unipotent, we have $\{x\}_t \subset Z^{(1)}$, the one-parameter subgroup with $x = x_1$. As $Z^{(1)}/Z$ is isomorphic to $\mathbb{Q}_p$, we get that $Z^{(1)} = \{x\}_t \subset \mathbb{Q}_p Z$, and it is abelian and isomorphic to $\mathbb{Q}_p^2$. Since $Z^{(1)}$ is $T$-invariant, by Proposition 4.1 (2), the $T$-action on $\text{Sub}_{Z^{(1)}}$ is not expansive, which leads to a contradiction. Hence $G_p$ is abelian and isomorphic to $\mathbb{Q}_p$.

Remark 4.2 If a locally compact group $G$ admits an automorphism $T$ which acts expansively on $\text{Sub}_G$, then we have that $G$ is totally disconnected, $C(T)C(T^{-1})$ is open in $G$ and $C(T)$ and $C(T^{-1})$ are closed, and hence locally compact. This implies in particular that such $G$ is either discrete, or at least one of $C(T)$ or $C(T^{-1})$ is nontrivial and such a nontrivial contraction group is a direct product of finitely many $\mathbb{Q}_{p_i}$ for distinct primes $p_i$ (cf. Theorem 4.2). It would be interesting to study in detail the structure of $C(T)C(T^{-1})$ for such $T$. It would also be interesting to study the structure of (infinite) discrete groups $G$ admitting automorphisms which act expansively on $\text{Sub}_G$.

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