AFFINE TRANSFORMATIONS OF CIRCLE AND SPHERE

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Abstract. A non-degenerate two-dimensional linear operator $\varphi$ transforms the unit circle into ellipse. Let $p$ be the length of its bigger axis and $q$ — the smaller. We can define the deformation coefficient $k(\varphi)$ as $q/p$. Analogously, if $\varphi$ is a non-degenerate three-dimensional operator, then it transforms the unit sphere into ellipsoid. If $p > q > r$ are lengths of its axes, then deformation coefficient $k(\varphi)$ will be defined as $r/p$. In this work we compute the mean value of deformation coefficient in two-dimensional case and give an estimation of the mean value in three-dimensional case.

1. Introduction

This work is a continuation of the work [1], where we study the deformation of angles under the action of a linear operator in $\mathbb{R}^2$. Here we study the deformation of the unit circle and also made some comments about three-dimensional case.

Let $\varphi$ be a non-degenerate linear operator in $\mathbb{R}^2$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0$$

be its matrix in standard base. Operator $\varphi$ transforms unit circle $C$ into ellipse with canonical equation

$$\frac{x'^2}{p^2} + \frac{y'^2}{q^2} = 1, \quad p \geq q$$

in appropriate coordinate system $(x', y')$. The number $q/p \leq 1$ will be called the deformation coefficient $k(\varphi)$ of operator $\varphi$.

In Section 2 we compute the deformation coefficient $k(\varphi)$.

Theorem 2.1.

$$k(\varphi) = \sqrt{\frac{a^2 + b^2 + c^2 + d^2 - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}{a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}}$$

In Section 3 compute the mean value $\overline{k}_2$ of coefficients $k(\varphi)$.

Theorem 3.1. $\overline{k}_2 = 3 - 4 \ln(2)$.

In Section 4 we demonstrate how to obtain a coarse upper bound for $\overline{k}_2$: $\overline{k}_2 < \frac{1}{2}$ (Theorem 4.1) with the aim to generalize this result to three dimensional case: $\overline{k}_3 < \frac{1}{3}$ (Theorem 5.1).
2. Deformation coefficient in two-dimensional case

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-degenerate linear operator and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be its matrix in the standard base. Operator $\varphi$ transforms unit circle into ellipse with axes $p$ and $q$, $p > q$.

**Theorem 2.1.**

$$k(\varphi) = \frac{q}{p} = \sqrt{a^2 + b^2 + c^2 + d^2 - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}.$$ 

**Proof.** If $\varphi^*$ is the conjugate operator, then $A^t$ is its matrix in the standard base. $p^2$ and $q^2$ are eigenvalues of operator $\varphi^* \varphi$ with matrix

$$A^t A = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}.$$ 

Thus, $p^2$ and $q^2$ are roots of $A^t A$ characteristic polynomial

$$s(x) = x^2 - (a^2 + b^2 + c^2 + d^2) x + (ad - bc)^2 :$$

$$p = \sqrt{\frac{a^2 + b^2 + c^2 + d^2 - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}{2}};$$

$$q = \sqrt{\frac{a^2 + b^2 + c^2 + d^2 - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}{2}}.$$

The change of variables simplifies these formulas. Put

$$a := x + y, \quad b := x - y, \quad c := z + t, \quad d := z - t.$$

In new variables

$$k(\varphi) = \sqrt{\frac{x^2 + y^2 + z^2 + t^2 - \sqrt{(x^2 + y^2 + z^2 + t^2)^2 - (x^2 + z^2 - y^2 - t^2)^2}}{x^2 + y^2 + z^2 + t^2 - \sqrt{(x^2 + y^2 + z^2 + t^2)^2 - (x^2 + z^2 - y^2 - t^2)^2}}}.$$ 

The next change of variables

$$x := r \sin(\alpha), \quad z := r \cos(\alpha), \quad y := \rho \sin(\beta), \quad t := \rho \cos(\beta)$$

allows us the further simplification:

$$k(\varphi) = \frac{|r - \rho|}{r + \rho}.$$
3. Computation of the mean value

Theorem 3.1. \( k_2 = 3 - 4 \ln(2) \).

Proof. Without loss of generality we can assume that \(|A| > 0\), i.e. \( ad - bc > 0 \). In new variables this condition can be rewritten as
\[
x^2 + z^2 - y^2 - t^2 > 0 \text{ or } r > \rho.
\]
We will compute the mean value of \( k(\phi) \) in the domain \( ad - bc > 0 \), i.e. in the domain
\[
r > \rho > 0, \quad 0 \leq \alpha \leq 2\pi, \quad 0 \leq \beta \leq 2\pi.
\]
We have
\[
\bar{k}_2 = \lim_{R \to \infty} \left( 4\pi^2 \int_0^R r \, dr \int_0^r \frac{\rho}{r + \rho} \, d\rho \right) = \lim_{R \to \infty} \left( \int_0^R \left( -\frac{1}{2} \rho^2 + 2\rho r - 2r^2 \ln(r + \rho) \right) \right) = 3 - 4 \ln 2 \approx 0.227411278.
\]

4. Upper bound for deformation coefficient

Let \( y \) and \( z \), \( y \geq z \), be lengths of vector-columns of matrix \( A \) and \( S \leq yz \) be the area of parallelogram, generated by these vectors. Characteristic polynomial of the matrix \( A'tA \) can be written in the following way:
\[
s(x) = x^2 - (y^2 + z^2) x + S^2.
\]
As
\[
p^2, q^2 = \frac{y^2 + z^2 \pm \sqrt{(y^2 + z^2)^2 - 4S^2}}{2},
\]
then
\[
q^2 \leq z^2 \leq y^2 \leq p^2, \text{ and } k(\phi) = \frac{q}{p} \leq \frac{z}{y}.
\]

Theorem 4.1. \( \bar{k}_2 < \frac{1}{2} \).

Proof. We have
\[
\bar{k}_2 < \int_0^1 dy \int_0^y \frac{z}{y} \, dz \int_0^1 dy \int_0^y dz = \frac{1}{2}.
\]
5. Three-dimensional case

Let $A$ be the matrix of linear operator $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $u, v, w, u \leq v \leq w$, be lengths vector-columns of this matrix, $S_1, S_2, S_3$ be areas of parallelograms, generated by pairs of vector-columns and $V$ be the volume of parallelepiped, generated by vector-columns. Then characteristic polynomial $s$ of the operator $\varphi^*\varphi$ is of form

$$s(x) = x^3 - (u^2 + v^2 + w^2)x^2 + (S_1^2 + S_2^2 + S_3^2)x - V^2.$$

**Proposition 5.1.** Let $x_1, x_2, x_3$, $x_1 \leq x_2 \leq x_3$, be (real) roots of $s$. Then $x_1 \leq u^2 \leq w^2 \leq x_3$.

**Proof.** Computer assisted check. □

Thus,

$$k(\varphi) = \frac{x_1}{x_3} \leq \frac{u}{w},$$

and we have the following estimation of the mean value $\overline{k}_3$ of three-dimensional deformation coefficient.

**Theorem 5.1.** $\overline{k}_3 < \frac{1}{3}$.

**Proof.**

$$\overline{k}_3 < \int_{0}^{1} dw \int_{0}^{w} dv \int_{0}^{v} \frac{u}{w} du \int_{0}^{1} dw \int_{0}^{w} dv \int_{0}^{v} du = \frac{1}{3}.$$ □

**Remark 5.1.** Actual value of $\overline{k}_3$ is quite difficult to compute. We have a rather coarse estimation: $\overline{k}_3 \approx 0.15$.

**References**

[1] Busjatskaja Irina, Kochetkov Yury, *Affine transformations of the plane and their geometrical properties*, arXiv 1603.02938v1.

[2] Halmos Paul, *Finite dimensional vector spaces*, Springer, 1974.

[3] Lang Serge, *Linear algebra*, Springer, 1987.

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