Conformal Gravity from AdS/CFT

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Abstract. We explicitly calculate the induced gravity theory at the boundary of an asymptotically Anti-de Sitter five dimensional Einstein gravity. We also display the action that encodes the dynamics of radial diffeomorphisms. It is found that the induced theory is a four dimensional conformal gravity plus a scalar field. This calculation confirms some previous results found by a different approach.

1. Introduction

After the original AdS/CFT conjecture many, and very important, results has been found. The original conjecture proposes a connection between string theory on $\text{AdS}_5 \times S^5$ and super Yang Mills theory in four dimensions [1]. Since gravity emerge as the lower energy limit of String theory the conjecture was later extended to a connection between Gravity on AdS$_5$ and a Yang Mills theory four dimensions. Furthermore, the Yang Mills theory has been replaced by a generic conformal field theory shaping up the current version of the conjecture which one can identified with a realization of the Holographic proposal by Susskind [2] and 't Hooft [3].

Conformal theories are very interesting because many of their properties emerge generically as consequence of the conformal symmetry and not from the details of the theory considered. For instance the classical thermodynamics of a conformal theory can be determined, with the exception of parameters as the Stephan-Boltzmann constant, by the conformal symmetry. Using this result one can confirm that the thermodynamics of asymptotically AdS black holes can be matched by a conformal field theory [4].

The conjecture opens up possibilities to be explored in realm of gravitation. The conjecture is a relation of duality, thus one can expect that classical solution on the bulk can be connected with highly quantum corrected solutions at the AdS boundary, or viceversa. In this way the conjecture allows to unveil quantum effects computed using a classical theory. Indeed the analysis of anomalies of CFT theories using this approach has been very fruitful [5].

On the other hand, asymptotically (locally) AdS spaces require a careful treatment of the boundary conditions and their associated boundary terms otherwise divergent actions, charges, variations arise. In fact several different approaches to deal with those divergences has been developed [6, 7, 8, 9]. In particular for the context of this work one must mention the Holographic renormalization proposed in [5].

It is well known that gravity in higher dimensions can induce an effective theories in lower dimensions. This within the context of the AdS/CFT conjecture offers a very interesting issue to
address. In [10] is shown that classical bulk solutions can be reinterpreted at the AdS boundary as highly quantum corrected solutions.

Following the ideas proposed in [11], in this work the conjecture is used to show that Einstein gravity in five dimensions, adequately regularized, yields an effective theory of gravity in four dimensions. The effective theory coincides with the bosonic part of the super conformal gravity discussed in [12].

Let us consider the regularized action

\[
I_{grav} = \frac{1}{16\pi G} \int_M d^4x d\rho \sqrt{g}(R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} K - \frac{3}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} - \frac{1}{16\pi G} \int_{\partial M} d^4x \sqrt{\gamma} R[\gamma].
\]

It is worth to stress that in this action every term is divergent but their addition happens to be finite and well behaved. Using this action principle (1) as the theory in the bulk five dimensional bulk in this letter is shown that an effective conformal gravity theory arises at the boundary.

2. Computations

The intension of this work is to rewrite the action above (1) and showing that it actually can be understood as 4 dimensional theory for the diffeomorphisms that preserve the asymptotical AdS scaling on the metric. To do that we begin with the Fefferman-Graham-type line element for a general five dimensional asymptotical AdS,

\[
ds^2 = l^2d\rho^2 + g_{ij}(x, \rho)dx^i dx^j
\]

where \( \rho = \infty \) defines the asymptotical (locally) AdS region and \( g_{ij}(x, \rho) \) has the expansion

\[g_{ij}(x, \rho) = e^{2\rho} g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x) + e^{-2\rho} g_{ij}^{(4)}(x) - 2e^{-2\rho}\rho h_{ij}(x) + \ldots\]

In the following we will set \( l = 1 \), thus \( \Lambda = -6 \). With this expansion the Einstein equations can be solved iteratively. This yields [5]:

\[
Tr(g^{(4)}) = \frac{Tr(g^{(2,2)})}{4}, g_{ij}^{(2)} = -\frac{1}{2}(R_{ij}^{(0)} R^{(0)0j} - \frac{1}{6} R^{(0)} g_{ij}^{(0)}), Tr(h) = 0
\]

where traces are defined with \( g_{ij}^{(0)} \).

Now, we consider a coordinate transformation that leaves the asymptotic form of the metric (2) invariant. Using the prescription in [11], that transformation reads

\[
\rho \rightarrow \rho + \frac{1}{2} \varphi(x) + e^{-2\rho} f^{(2)}(x) + \ldots
\]

\[x^i \rightarrow x^i + e^{-2\rho} h^{(2)}(x) + \ldots\]

Note that in the new coordinate system a fixed \( \rho \) in fact corresponds to a function of \( x \). In fact the boundary is defined by

\[\rho = \bar{\rho} + \frac{1}{2} \varphi(x) + O(e^{-n\bar{\rho}}) = F(x),\]

with \( \bar{\rho} \rightarrow \infty \).
The induced metric at the boundary and the unit normal are respectively:

$$\gamma_{ij} = g_{ij} + \partial_i F \partial_j F$$

$$n^a = \frac{1}{\sqrt{1 + g^{ij} \partial_i F \partial_j F}} (-1, g^{ij} \partial_j F)$$

In the new system of coordinates (4) for a surface at constant $\rho$ one can obtain an expansion in powers of $\rho$ for $\sqrt{\gamma}$, extrinsic curvature $K$ and the Ricci scalar near the boundary. Those expansions read respectively,

$$\sqrt{\gamma} = e^{4\mu} \sqrt{g(0)} + \frac{1}{2} \sqrt{g(0)} e^{2\rho} (Tr(g(2)) + g^{(0)ij} \partial_i F \partial_j F) + \frac{1}{2} \sqrt{g(0)} \left( Tr(g(4)) + \frac{1}{4} Tr(g(2))^2 \right)$$

$$K = \frac{1}{2} Tr(\gamma \mathcal{L}_n g_{ij}) = -4 + e^{-2\rho} (g^{(0)ij} \partial_i F \partial_j F + g^{(0)ij} \nabla_i \nabla_j (0) F + Tr(g(2))) + e^{-4\rho} (2 Tr(g(4))$$

$$R[\gamma] = e^{-2\rho} (R(0) - 6 g^{(0)ij} \nabla_i \nabla_j (0) F - 6 g^{(0)ij} \partial_i F \partial_j F) + e^{-4\rho} (-g^{(2)ij} R_{ij} - R^{(0)ij} \partial_i F \partial_j F$$

with $\mu = 0 \ldots 4$ and $x^4 = \rho$, where

$$\partial_i x^\mu = \delta^\mu_i + \partial_i F \delta^\mu_4$$

We want to use the above expansions in order to get the finite part of the action (1). First we integrate the bulk term of the action (1):

$$\int_M d^4 x \sqrt{g} (R - 2\Lambda)_{on-shell} = -8 \int_{\partial M} d^4 x \int_{\rho = F}^{\rho = F} d\rho \sqrt{g} = -8 \int_{\partial M} d^4 x \int_{\rho = F}^{\rho = F} d\rho (e^{4\rho} \sqrt{g(0)}$$

$$+ \frac{1}{2} \sqrt{g(0)} e^{2\rho} (Tr(g(2))) + \frac{1}{2} \sqrt{g(0)} (Tr(g(4)) + \frac{1}{4} Tr(g(2))^2 - \frac{1}{2} Tr(g(2/2)) + \ldots)$$

$$= \int_{\partial M} d^4 x (2e^{4F} \sqrt{g(0)} + \frac{1}{3} \sqrt{g(0)} e^{2F} (R(0)) - \frac{1}{4} \sqrt{g(0)} (R^{(0)ij} R_{ij})$$

$$- \frac{1}{3} R(0)^2) F + \ldots)$$

$$= 0$$

The Ricci scalar is defined up to total derivatives of $O(e^{-4\rho})$. All indices are raised and lowered respect to $g(0)$.

Here we have defined

$$Tr(\gamma \mathcal{L}_n g_{ij}) = \gamma^{ij} \partial_i x^\mu \partial_j x^\nu (\mathcal{L}_n g)_{\mu\nu}$$

with $\mu = 0 \ldots 4$ and $x^4 = \rho$, where

$$\partial_i x^\mu = \delta^\mu_i + \partial_i F \delta^\mu_4$$

We want to use the above expansions in order to get the finite part of the action (1). First we integrate the bulk term of the action (1):
The term proportional to $F$ has a divergent part that must be eliminated with an additional counterterm as shown in [5]. Finally the action becomes

$$I_{\text{grav}} = \frac{1}{16\pi G} \int_{\partial M} \sqrt{g(0)} \left( -\frac{1}{16} (R(0)_{ij} R(0)^{ij} - \frac{1}{3} R(0)^2) + \frac{1}{64} (\partial_i \varphi \partial^i \varphi)^2 \right)$$

$$+ \frac{1}{16} \partial_i \varphi \partial^i \varphi \nabla_j (0) \nabla^j (0) \varphi - \frac{1}{8} (R(0)_{ij} R(0)^{ij} - \frac{1}{3} R(0)^2) \varphi$$

(14)

This action can be recognized as the action for 4-dimensional conformal gravity plus the anomalous part obtained in [13].

This action was obtained by Riegert [14] as the local form of the action which gives a trace anomaly proportional to $R(0)_{ij} R(0)^{ij} - \frac{1}{3} R(0)^2$ and corresponds to the local form of the anomalous part of the effective action associated with the Super Yang-Mills theory in $d = 4$ [13]. Also from [15] we know that this field encodes part of the degrees of freedom contained in the traceless part of $g^{(4)}$ which, along with $g^{(0)}$, contains all the degrees of freedom of the solutions for pure gravity in five dimensions.

3. conclusions

In this work we have proven that four dimensional conformal gravity can be obtained through a AdS/CFT mechanism from five dimensional Einstein gravity. We have demonstrated this explicitly using the Fefferman-Graham expansion and regularizing the action. As expected the radial diffeomorphisms induced Weyl transformations on the boundary which produced the anomaly. This calculation confirms the previous result obtained in [13] by other method. The idea is more direct than in [13], however the required calculations are far more complex.

Acknowledgments

R.A. would like to thank Abdus Salam International Centre for Theoretical Physics (ICTP) for its support. This work was partially funded by grants FONDECYT 1040202 and DI 06-04. (UNAB).

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