ON THE RADIUS OF ANALYTICITY FOR THE SOLUTION OF THE FIFTH ORDER KDV-BBM MODEL

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Abstract. We consider the initial value problem (IVP) associated to a fifth order KdV-BBM type model that describes the propagation of the unidirectional water waves. We prove the local well-posedness in the space of the analytic functions, so called Gevrey class. We also discuss the evolution of radius of analyticity in such class by providing explicit formulas for upper and lower bounds.

1. Introduction

Our interest in this work is to study the well-posedness in the spaces of analytic functions, the so called Gevrey class of functions, and the evolution of radius of analyticity of the solution to the following initial value problem (IVP)

\[
\begin{align*}
\eta_t + \eta_x - \gamma_1 \eta_{xxt} + \gamma_2 \eta_{xxx} + \delta_1 \eta_{xxxx} + \delta_2 \eta_{xxxxx} + \frac{3}{2} \eta \eta_x + \gamma (\eta^2)_{xxx} - \frac{7}{24} (\eta_x^2)_x - \frac{1}{8} (\gamma^2)_x &= 0, \\
\eta(x, 0) &= \eta_0(x),
\end{align*}
\]

where

\[
\begin{align*}
\gamma_1 &= \frac{1}{2} (b + d - \rho), \quad \gamma_2 = \frac{1}{2} (a + c + \rho), \\
\delta_1 &= \frac{1}{4} [2(b_1 + d_1) - (b - d + \rho)(\frac{1}{6} - a - d) - d(c - a + \rho)], \\
\delta_2 &= \frac{1}{4} [2(a_1 + c_1) - (c - a + \rho)(\frac{1}{6} - a) + \frac{1}{3} \rho], \quad \gamma = \frac{1}{24} [5 - 9(b + d) + 9 \rho].
\end{align*}
\]

The parameters appearing in (1.2) and (1.3) satisfy \( a + b + c + d = \frac{1}{3}, \ \gamma_1 + \gamma_2 = \frac{1}{6}, \ \gamma = \frac{1}{24} (5 - 18 \gamma_1) \) and \( \delta_2 - \delta_1 = -\frac{19}{360} - \frac{1}{6} \gamma_1 \) with \( \delta_1 > 0 \) and \( \gamma_1 > 0 \).

The higher order water wave model (1.1) describing the unidirectional propagation of water waves was recently introduced by Bona et al. [2] by using the second order approximation in the two-way model, the so-called abcd—system derived in [6] [7]. In the literature, this model is also known as the fifth order KdV-BBM type equation. The IVP (1.1) posed on

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the spatial domain $\mathbb{R}$ was studied by the authors in [2] considering initial data in $H^s(\mathbb{R})$ and proving local well-posedness for $s \geq 1$. When the parameter $\gamma$ satisfies $\gamma = \frac{9}{48}$, the model (1.1) posed on $\mathbb{R}$ possesses hamiltonian structure and the flow satisfies

$$E(\eta(\cdot, t)) := \int_{\mathbb{R}} \eta^2 + \gamma_1(\eta_x)^2 + \delta_1(\eta_{xx})^2 \, dx = E(\eta_0).$$ (1.4)

We note that, this conservation law holds in the periodic case as well.

The energy conservation (1.4) was used in [2] to prove the global well-posedness for data in $H^s(\mathbb{R})$, $s \geq 2$. While, for data with regularity $\frac{3}{2} \leq s < 2$, splitting to high-low frequency parts technique was used by the authors in [2] to get the global well-posedness result. This global well-posedness result was further improved in [15] for initial data with Sobolev regularity $s \geq 1$. Furthermore, the authors in [15] showed that the well-posedness result is sharp by proving that the mapping data-solution fails to be continuous at the origin whenever $s < 1$. For similar results in the periodic case we refer to [27].

As mentioned earlier, the main interest of this work is to find solutions $\eta(x, t)$ of the IVP (1.1) with real-analytic initial data $\eta_0$ which admit extension as an analytic function to a complex strip $S_{\sigma_0} := \{x + iy : |y| < \sigma_0\}$, for some $\sigma_0 > 0$ at least for a short time. Analytic Gevrey class introduced by Foias and Temam [17] is a suitable function space for this purpose. After getting this result, a natural question one may ask is whether this property holds globally in time, but with a possibly smaller radius of analyticity $\sigma(t) > 0$. In other words, is the solution $\eta(x, t)$ of the IVP (1.1) with real-analytic initial data $\eta_0$ is analytic in $S_{\sigma(t)}$ for all $t$ and what is the lower-bound of $\sigma(t)$? This question will also be addressed in this work.

An early work in this direction is due to Kato and Masuda [25]. They considered a large class of evolution equations and developed a general method to obtain spatial analyticity of the solution. In particular, the class considered in [25] contains the KdV equation. Further development in this field can be found in the works of Hayashi [20], Hayashi and Ozawa [21], de Bouard, Hayashi and Kato [14], Kato and Ozawa [26], Bona and Grubić [13], Bona, Grujić and Kalish [11, 12], Grujić and Kalish [18, 19] and references there in. In recent literature, many authors have devoted much effort to get analytic solutions to several evolution equations, see for example [1, 22, 23, 24, 28, 29, 30, 31] and references therein.

Now, we introduce some notations and define function spaces in which this work will be developed. Throughout this work we use $C$ to denote a constant that may vary from one line to the next.
The Fourier transform of a function $f$ is defined by
\[
\hat{f}(\xi) = \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx,
\]
whose inverse transform is given by
\[
f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \, d\xi.
\]
We define a Fourier multiplier operator $J$ by
\[
\hat{J}^{s} f(\xi) = \langle \xi \rangle^{s} \hat{f}(\xi).
\]
For given $s \in \mathbb{R}$, we define the usual $L^2$-based Sobolev space $H^s$ denotes of order $s$ with norm
\[
\|f\|_{H^s}^2 = \int_{\mathbb{R}} (\xi)^{2s} |\hat{f}(\xi)|^2 \, d\xi = \|J^s f\|_{L^2(\mathbb{R})}^2 =: \|J^s f\|^2,
\]
where $\langle \cdot \rangle = 1 + |\cdot|.$

For $\sigma > 0$ and $s \in \mathbb{R}$, the analytic Gevrey class $G^{\sigma,s}$ is defined as the subspace of $L^2(\mathbb{R})$ with norm,
\[
\|f\|_{G^{\sigma,s}}^2 = \int_{\mathbb{R}} (\xi)^{2s} e^{2\sigma(\xi)} |\hat{f}(\xi)|^2 \, d\xi.
\]
The Gevrey norm of order $(\sigma, s)$ can be written in terms of the operator $J^s$ as
\[
\|f\|_{G^{\sigma,s}} = \|J^s e^{\sigma J} f\|_{L^2(\mathbb{R})} := \|J^s e^{\sigma J} f\|.
\]
To make the notation more compact we define $J^{s,\sigma} \eta := J^s e^{\sigma J} \eta$ so that the Gevrey norm can be expressed as
\[
\|\eta\|_{G^{\sigma,s}} = \|J^{s,\sigma} \eta\|_{L^2(\mathbb{R})} := \|J^{s,\sigma} \eta\|.
\]

Note that, a function in the Gevrey class $G^{\sigma,s}$ is a restriction to the real axis of a function analytic on a symmetric strip of width $2\sigma$. Hence, our interest is to prove the well-posedness result for the IVP (1.1) for given data in $G^{\sigma,s}$ for appropriate $s$ and analyse how $\sigma = \sigma(t)$ evolves in time.

Now we are in position to state the main results of this work. The first result deals with the local existence of the IVP (1.1) for given data in the usual Gevrey space $G^{\sigma,s}(\mathbb{R})$ and reads as follows.

**Theorem 1.1.** Let $s \geq 1$, $\sigma > 0$ and $\eta_0 \in G^{\sigma,s}(\mathbb{R})$ be given. Then there exist a time $T = T(\|\eta_0\|_{G^{\sigma,s}} > 0$ and $\eta \in C([0,T];G^{\sigma,s})$ satisfying the IVP (1.1).

The second main result deals with the evolution of the radius of analyticity in time. More precisely, we have the following global well-posedness result.
**Theorem 1.2.** Let \( \sigma := \sigma(t) > 0 \) and \( \eta_0 \in G^{\sigma,2}(\mathbb{R}) \). Then, the solution \( \eta \) of the IVP (1.1) with initial value \( \eta_0 \) remains analytic for all positive times. More precisely \( u \in C(0,T;G^{\sigma,2}(\mathbb{R})) \) for all \( T > 0 \) where a lower bound of the radius of analyticity \( \sigma(t) \) is given by (see (3.17))

\[
\sigma_0 \exp\{-(\|\eta_0\|_{G^{\sigma,2}} + 2\|\eta_0\|_{G^{\sigma,2}}^2)t - \frac{3}{2}t^{5/2}(\|\eta_0\|_{H^2}^{3/2} + \|\eta_0\|_{H^2}^2) - t^2(\|\eta_0\|_{H^2}^{3/2} + \|\eta_0\|_{H^2}^2)^2\}.
\]

and an upper bound is given by

\[
C\sigma_0 \exp\{-\|\eta_0\|_{H^2t}^2\}.
\]

Moreover

\[
\|\eta(t)\|_{G^{\sigma,2}} \leq \|\eta_0\|_{G^{\sigma,2}} + C t^{1/2}(\|\eta_0\|_{H^2}^{3/2} + \|\eta_0\|_{H^2}^2), \quad (1.10)
\]

The local well-posedness will be established using multilinear estimates combined with a contraction mapping argument. The global well-posedness in the spaces \( H^s \) with \( s \geq 2 \) will be obtained via energy-type arguments together with the local theory.

As in the continuous case, with some restriction on the coefficients of the equation, we can also prove the global well-posedness result in the periodic case too, i.e., for given data \( \eta_0 \in G^{\sigma,2}(\mathbb{T}) \).

Before finishing this section we record the following estimates that will be used in sequel.

**Remark 1.3.** Observe that if \( \eta(x,t) \) is a solution of the IVP (1.1), \( c_1 = \min\{\gamma_1, \delta_1\} \) and \( C_1 = \max\{\gamma_1, \delta_1\} \), then

\[
c_1 \|\eta(\cdot,t)\|_{H^2} \leq E(\eta(\cdot,t)) = E(\eta_0) \leq C_1 \|\eta(\cdot,t)\|_{H^2}^2. \tag{1.11}
\]

Also, it is clear that \( c_1 \|\eta(\cdot,t)\|_{H^2}^2 \leq C_1 \|\eta_0\|_{H^2}^2 \) and \( c_1 \|\eta_0\|_{H^2}^2 \leq C_1 \|\eta(\cdot,t)\|_{H^2}^2 \). Therefore

\[
\frac{c_1}{C_1} \|\eta_0\|_{H^2}^2 \leq \|\eta(\cdot,t)\|_{H^2}^2 \leq \frac{C_1}{c_1} \|\eta_0\|_{H^2}^2. \tag{1.12}
\]

2. **Local Well-posedness Theory in** \( G^{\sigma,s}, s \geq 1 \)

In this section we focus upon the local well-posedness issues for the IVP (1.1) for given data \( \eta_0 \in G^{\sigma,s}, s \geq 1 \). We start writing the IVP (1.1) in an equivalent integral equation format. Taking the Fourier transform in the first equation in (1.1) with respect to the spatial variable and organizing the terms, we get

\[
\left(1 + \gamma_1 \xi^2 + \delta_1 \xi^4\right)\hat{\eta}_t = \xi(1 - \gamma_2 \xi^2 + \delta_2 \xi^4)\hat{\eta} + \frac{1}{4}(3\xi - 4\gamma_3 \xi^3)\hat{\eta}^2 - \frac{1}{8}\xi\hat{\eta}^3 - \frac{7}{48}\xi^2\hat{\eta}_t. \tag{2.1}
\]
The fourth-order polynomial
\[ \varphi(\xi) := 1 + \gamma_1 \xi^2 + \delta_1 \xi^4 > 0, \]
is strictly positive because \( \gamma_1 \), and \( \delta_1 \) are taken to be positive.

Now, we define the Fourier multiplier operators \( \phi(\partial_x) \), \( \psi(\partial_x) \) and \( \tau(\partial_x) \) as follows
\[
\hat{\phi}(\partial_x)f(\xi) := \phi(\xi) \hat{f}(\xi), \quad \hat{\psi}(\partial_x)f(\xi) := \psi(\xi) \hat{f}(\xi) \quad \text{and} \quad \hat{\tau}(\partial_x)f(\xi) := \tau(\xi) \hat{f}(\xi),
\]
where the symbols are given by
\[
\phi(\xi) = \frac{\xi(1 - \gamma_2 \xi^2 + \delta_2 \xi^4)}{\varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)} \quad \text{and} \quad \tau(\xi) = \frac{3\xi - 4\gamma_3 \xi^3}{4\varphi(\xi)}.
\]

With this notation, the IVP (1.1) can be written in the form
\[
\begin{aligned}
& i \eta_t = \phi(\partial_x)\eta + \tau(\partial_x)\eta^2 - \frac{1}{8} \psi(\partial_x)\eta^3 - \frac{7}{48} \psi(\partial_x)\eta_x^2, \\
& \eta(x, 0) = \eta_0(x).
\end{aligned}
\]
(2.3)

Consider first the following linear IVP associated to (2.3)
\[
\begin{aligned}
& i \eta_t = \phi(\partial_x)\eta, \\
& \eta(x, 0) = \eta_0(x),
\end{aligned}
\]
(2.4)
whose solution is given by \( \eta(t) = S(t) \eta_0 \), where \( S(t) \eta_0 = \hat{e}^{-i\phi(\xi)t} \hat{\eta}_0 \) is defined via its Fourier transform. Clearly, \( S(t) \) is a unitary operator on \( H^s \) and \( G^{\sigma,s} \) for any \( s \in \mathbb{R} \), so that
\[
\| S(t) \eta_0 \|_{H^s} = \| \eta_0 \|_{H^s}, \quad \text{and} \quad \| S(t) \eta_0 \|_{G^{\sigma,s}} = \| \eta_0 \|_{G^{\sigma,s}}
\]
(2.5)
for all \( t > 0 \). Duhamel’s formula allows us to write the IVP (2.3) in the equivalent integral equation form,
\[
\eta(x, t) = S(t) \eta_0 - i \int_0^t S(t - t') \left( \tau(\partial_x)\eta^2 - \frac{1}{8} \psi(\partial_x)\eta^3 - \frac{7}{48} \psi(\partial_x)\eta_x^2 \right)(x, t') dt'.
\]
(2.6)

In what follows, a short-time solution of (2.6) will be obtained via the contraction mapping principle in the space \( C([0, T]; G^{\sigma,s}) \). This will provide a proof of Theorem 1.1.

2.0.1. Multilinear Estimates. Various multilinear estimates are now established that will be useful in the proof of the local well-posedness result. First, we record the following \( G^{\sigma,s} \) version of the “sharp” bilinear estimate obtained in [10].
Lemma 2.1. For \( s \geq 0 \), there is a constant \( C = C_s \) for which
\[
\| \omega(\partial_x)(uv) \|_{G^{s,s}} \leq C \| u \|_{G^{s,s}} \| v \|_{G^{s,s}}
\]  
(2.7)
where \( \omega(\partial_x) \) is the Fourier multiplier operator with symbol
\[
\omega(\xi) = \frac{|\xi|}{1 + \xi^2}.
\]

Proof. Using the definition of the \( G^{s,s} \)-norm from (1.8), one can obtain
\[
\| \omega(\partial_x)(uv) \|_{G^{s,s}}^2 = \| (\xi)^s e^{\sigma(\xi)} \omega(\xi) \hat{u} \ast \hat{v}(\xi) \|_{L^2}^2
= \int_{\mathbb{R}} (\xi)^{2s} e^{2\sigma(\xi)} \left( \int_{\mathbb{R}} \hat{u}(\xi - \xi_1) \hat{v}(\xi_1) d\xi_1 \right)^2 d\xi.
\]  
(2.8)

Note that, for \( s \geq 0 \) one has \( (\xi)^s \leq (\xi - \xi_1)^s (\xi_1)^s \) and also \( e^{\sigma(\xi)} \leq e^{\sigma(\xi - \xi_1)} e^{\sigma(\xi_1)} \). Using these inequalities, the estimate (2.8) yields
\[
\| \omega(\partial_x)(uv) \|_{G^{s,s}}^2 \leq \int_{\mathbb{R}} \frac{\xi^2}{(1 + \xi^2)^2} \left( \int_{\mathbb{R}} (\xi - \xi_1)^{2s} e^{2\sigma(\xi - \xi_1)} \hat{u}(\xi - \xi_1) \hat{v}(\xi_1) d\xi_1 \right)^2 d\xi.
\]  
(2.9)

Now, using \( \frac{\xi^2}{(1 + \xi^2)^2} \leq \frac{1}{1 + \xi^2} \), the Cauchy-Schwartz inequality and the definition of the \( G^{s,s} \)-norm, we obtain from (2.9) that
\[
\| \omega(\partial_x)(uv) \|_{G^{s,s}}^2 \leq \int_{\mathbb{R}} \frac{1}{1 + \xi^2} d\xi \| u \|_{G^{s,s}}^2 \| v \|_{G^{s,s}}^2
\leq C \| u \|_{G^{s,s}}^2 \| v \|_{G^{s,s}}^2,
\]  
(2.10)
and this completes the proof.

It is worth noting that the counterexample in [10] can be adapted to show that the inequality (2.7) fails for \( s < 0 \).

Lemma 2.2. For any \( s \geq 0 \) and \( \sigma > 0 \), there is a constant \( C = C_s \) such that the inequality
\[
\| \tau(\partial_x)\eta^2 \|_{G^{s,s}} \leq C \| \eta \|_{G^{s,s}}^2
\]  
(2.11)
holds, where the operator \( \tau(\partial_x) \) is defined in (2.2).

Proof. Since \( \delta_1 > 0 \), one can easily verify that \( |\tau(\xi)| \leq C \omega(\xi) \) for some constant \( C > 0 \). Using this fact and the definition of the \( G^{s,s} \)-norm and the estimate (2.7) from Lemma 2.1, one can obtain
\[
\| \tau(\partial_x)\eta^2 \|_{G^{s,s}} \leq \| (\xi)^s e^{\sigma(\xi)} \tau(\xi) \hat{\eta} \ast \hat{\eta}(\xi) \|_{L^2}
\leq \| (\xi)^s e^{\sigma(\xi)} \omega(\xi) \hat{\eta} \ast \hat{\eta}(\xi) \|_{L^2}^2
\leq C \| \eta \|_{G^{s,s}}^2,
\]
Lemma 2.3. For $s \geq \frac{1}{6}$, there is a constant $C = C_s$ such that

$$\|\psi(\partial_x)\eta^3\|_{G^{\sigma,s}} \leq C\|\eta^3\|_{G^{\sigma,s}}. \quad (2.12)$$

Proof. Consider first when $\frac{1}{6} \leq s < \frac{5}{2}$. In this case, it appears that

$$\left| (1 + |\xi|)^s \psi(\xi) \right| = \frac{(1 + |\xi|)^s \xi}{(1 + \gamma_1 \xi^2 + \delta_1 \xi^4)} \leq C \frac{1}{(1 + |\xi|)^{3-s}}.$$ 

The last inequality implies that

$$\|\psi(\partial_x)\eta^3\|_{G^{\sigma,s}} = \|(1 + |\xi|)^s \psi(\xi)e^{\sigma(\xi)}\hat{\eta}^3(\xi)\|_{L^2} \leq C \left\| \frac{1}{(1 + |\xi|)^{3-s}} e^{\sigma(\xi)}\hat{\eta}^3(\xi) \right\|_{L^2} \leq C \left\| \frac{1}{(1 + |\xi|)^{3-s}} \right\|_{L^2} \|e^{\sigma(\xi)}\hat{\eta}^3(\xi)\|_{L^\infty}. \quad (2.13)$$

Let $\hat{f}(\xi) := e^{\sigma(\xi)}\hat{\eta}(\xi)$. Then using $e^{\sigma(\xi)} \leq e^{\sigma(\xi - \xi_1 - \xi_2)}$, we get

$$e^{\sigma(\xi)}\hat{\eta}^3(\xi) \leq \int_{\mathbb{R}} e^{\sigma(\xi - \xi_1 - \xi_2)}\hat{\eta}(\xi - \xi_1 - \xi_2)e^{\sigma(\xi_1)}\hat{\eta}(\xi_1)e^{\sigma(\xi_2)}\hat{\eta}(\xi_2)d\xi_1d\xi_2 = \hat{f}^3(\xi). \quad (2.14)$$

Using (2.14) and the fact that $\left\| \frac{1}{(1 + |\xi|)^{3-s}} \right\|_{L^2}$ is bounded for $s < \frac{5}{2}$, we obtain from (2.13) that

$$\|\psi(\partial_x)\eta^3\|_{G^{\sigma,s}} \leq \|\hat{f}^3(\xi)\|_{L^\infty} \leq C\|f\|_{L^3}^3. \quad (2.15)$$

From one dimensional Sobolev embedding, we have

$$\|f\|_{L^3} \leq C\|f\|_{H^{\frac{5}{2}}} = C\|\eta\|_{G^{\sigma,s}}. \quad (2.16)$$

Therefore, for $\frac{1}{6} \leq s < \frac{5}{2}$, we obtain from (2.15) and (2.16) that

$$\|\psi(\partial_x)\eta^3\|_{G^{\sigma,s}} \leq C\|\eta\|_{G^{\sigma,s}}^3. \quad (2.17)$$

For $s \geq \frac{5}{2}$, we observe that $G^{\sigma,s}$ is a Banach algebra. Also, note that $|\psi(\xi)| \leq C\frac{|\xi|}{1 + |\xi|^2}$. So, using the same procedure as in Lemma 2.2 we obtain

$$\|\psi(\partial_x)(\eta^2)\|_{G^{\sigma,s}} \leq C\|\eta\|_{G^{\sigma,s}}\|\eta^2\|_{G^{\sigma,s}} \leq C\|\eta\|_{G^{\sigma,s}}^3,$$

as desired. □
Lemma 2.4. For $s \geq 1$, the inequality
\[ \| \psi(\partial_x) \eta_x^2 \|_{G^{s, s}} \leq C \| \eta \|_{G^{s, s}}^2 \] (2.18)
holds.

Proof. Observe that
\[ \psi(\xi) \leq C \omega(\xi) \frac{1}{1 + |\xi|}. \]
The inequality (2.7) then allows the conclusion
\[ \| \psi(\partial_x) \eta_x^2 \|_{G^{s, s}} \leq C \| \omega(\partial_x) \eta_x^2 \|_{G^{s, s-1}} \leq C \| \eta_x \|_{G^{s, s-1}} \| \eta \|_{G^{s, s-1}} \leq C \| \eta \|_{G^{s, s}}, \]
since $s - 1 \geq 0$. □

In what follows, we use the estimates derived above to provide a proof of the local well-posedness result in the $G^{s, s}(\mathbb{R})$ space whenever $s \geq 1$.

Proof of Theorem 1.1. Taking into account of the Duhamel’s formula (2.6), we define a mapping
\[ \Psi \eta(x, t) = S(t) \eta_0 - i \int_0^t S(t - t') \left( \tau(D_x) \eta^2 - \frac{1}{4} \psi(\partial_x) \eta^3 - \frac{7}{48} \psi(\partial_x) \eta_x^2 \right) (x, t') dt'. \] (2.19)
We show that the mapping $\Psi$ is a contraction on a closed ball $B_r$ with radius $r > 0$ and center at the origin in $C([0, T]; G^{s, s})$.

From (2.5), we know that $S(t)$ is a unitary group in $G^{s, s}(\mathbb{R})$. Using this fact, we obtain
\[ \| \Psi \eta \|_{G^{s, s}} \leq \| \eta_0 \|_{G^{s, s}} + CT \left[ \| \tau(\partial_x) \eta^2 \|_{C([0, T]; G^{s, s})} - \frac{1}{4} \psi(\partial_x) \eta^3 \|_{C([0, T]; G^{s, s})} - \frac{7}{48} \psi(\partial_x) \eta_x^2 \|_{C([0, T]; G^{s, s})} \right]. \] (2.20)
In view of the inequalities (2.11), (2.12) and (2.18), we obtain from (2.20) that
\[ \| \Psi \eta \|_{G^{s, s}} \leq \| \eta_0 \|_{G^{s, s}} + CT \left[ \| \eta \|_{C([0, T]; G^{s, s})}^2 + \| \eta \|_{C([0, T]; G^{s, s})}^3 + \| \eta \|_{C([0, T]; G^{s, s})}^2 \right]. \] (2.21)
Now, consider $\eta \in B_r$, then (2.21) yields
\[ \| \Psi \eta \|_{G^{s, s}} \leq \| \eta_0 \|_{G^{s, s}} + CT \left[ 2r + r^2 \right] r. \]
If we choose $r = 2\| \eta_0 \|_{H^s}$ and $T = \frac{1}{2r(2+r)}$, then $\| \Psi \eta \|_{G^{s, s}} \leq r$, showing that $\Psi$ maps the closed ball $B_r$ in $C([0, T]; G^{s, s})$ onto itself. With the same choice of $r$ and $T$ and the same sort of estimates, one can easily show that $\Psi$ is a contraction on $B_r$ with contraction constant equal to $\frac{1}{2}$ as it happens. The rest of the proof is standard so we omit the details. □

Remark 2.5. The following points follow immediately from the proof of the Theorem 1.1:

\[ \text{□} \]
(1) The maximal existence time $T_s$ of the solution satisfies

$$T_s \geq \bar{T} = \frac{1}{8C_s \| \eta_0 \|_{G^{\sigma, s}} (1 + \| \eta_0 \|_{G^{\sigma, s}})},$$

(2.22)

where the constant $C_s$ depends only on $s$.

(2) The solution cannot grow too much on the interval $[0, \bar{T}]$ since

$$\| \eta(\cdot, t) \|_{G^{\sigma, s}} \leq r = 2 \| \eta_0 \|_{G^{\sigma, s}}$$

(2.23)

for $t$ in this interval, where $\bar{T}$ is as above in (2.22).

3. Evolution of Radius of Analyticity

In this section we study the evolution of the radius of analyticity $\sigma(t)$ as $t$ grows.

Lemma 3.1. Let $r, s$ and $\sigma$ be non-negative numbers. Then there are absolute constants $c_1$ and $c_2$ such that for $u \in D(J^{r+s}e^{\sigma}J)$,

$$\| J^{s, \sigma} u \| \leq c_1 \| J^s u \| + c_2 \sigma^r \| J^{s+r, \sigma} u \|.$$

Proof. See Lemma 9 in [13].

Lemma 3.2. Let $s_1, s_2, s$ be such that $s_1 \leq s \leq s_2$ and $\sigma$ be non-negative number. Then

$$\| J^{s, \sigma} u \| \leq \| J^{s_1, \sigma} u \|^\theta \| J^{s_2, \sigma} u \|^{1-\theta},$$

where $s = \theta s_1 + (1-\theta)s_2$.

Proof. This inequality is a consequence of the Holder’s inequality. In fact

$$\| J^{s, \sigma} u \|^2 = \int_{\mathbb{R}} \langle \xi \rangle^{2s} e^{2\sigma |\xi|} |\hat{f}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}} \left( \langle \xi \rangle^{2s_1 \theta} e^{2\sigma \theta |\xi|} |\hat{f}(\xi)|^2 \right) \left( \langle \xi \rangle^{2(1-\theta)s_2} e^{2\sigma (1-\theta) |\xi|} |\hat{f}(\xi)|^2 \right) d\xi$$

$$\leq \| J^{s_1, \sigma} u \|^{2\theta} \| J^{s_2, \sigma} u \|^{2(1-\theta)}.$$

Proof of Theorem 1.2. Let $\sigma := \sigma(t) > 0$ and $\eta_0 \in G^{\sigma, 2}(\mathbb{R})$. Consider $v = J^s e^{\sigma}J =: J^{s, \sigma} \eta$, so that

$$v_t = J^s e^{\sigma}J \eta_t + \sigma' J^{s+1} e^{\sigma}J \eta.$$

(3.1)

Applying the operator $J^{s, \sigma}$ in (2.3), we obtain

$$v_t = \sigma' J v - i\phi(\partial_x) v - i\tau(\partial_x) J^{s, \sigma} (\eta^2) + \frac{i}{8} \psi(\partial_x) J^{s, \sigma} (\eta^3) + \frac{7i}{18} \psi(\partial_x) J^{s, \sigma} (\eta^2_x).$$

(3.2)
Multiply both sides of (3.2) by $v$ and then integrate in the space variable, to obtain
\[
\frac{1}{2} \partial_t \int v^2 = \int v \sigma' Jv - i \int v \Phi(\partial_x) v - i \int v J^s \sigma J(\partial_x)(\eta^2) + \frac{i}{8} \int v J^s \sigma J(\partial_x)(\eta^2)^2 + \frac{7i}{18} \int v J^s \sigma J(\partial_x)(\eta_x^2).
\] (3.3)

Observe that $i \Phi(\partial_x)v = \partial_x \mathcal{K} = \mathcal{K} \partial_x v$, where
\[
\mathcal{K} v(\xi) = \frac{1 - \gamma_2 \xi^2 + \delta_2 \xi^4}{\varphi(\xi)} \tilde{\mathcal{O}}(\xi).
\]

Note that
\[
i \int v \Phi(\partial_x)v = \int v \partial_x \mathcal{K} = - \int (\partial_x v)(\mathcal{K}v).
\] (3.4)

Using the commutativity of the operator $\mathcal{K}$ with $\partial_x$ and the fact that $\mathcal{K}$ is symmetric, one has
\[
i \int v \Phi(\partial_x)v = \int v \partial_x \mathcal{K}v = \int v \mathcal{K} \partial_x v = \int (\mathcal{K}v)(\partial_x v).
\] (3.5)

Now, combining (3.4) and (3.5) we conclude that
\[
i \int v \Phi(\partial_x)v = 0.
\]

Using the estimates from Lemmas 2.1, 2.2, 2.3, 2.4 and the Lemma 3.1, we get
\[
\frac{1}{2} \partial_t \| J^s e^{\sigma J} \eta \|^2 - \sigma' \| J^{s+1/2} e^{\sigma J} \eta \|^2 \lesssim \| J^s e^{\sigma J} \eta \|^3 + \| J^s e^{\sigma J} \eta \|^4
\]
\[
\lesssim \| J^s \eta \|^3 + \| J^s \eta \|^4 + \sigma \| J^{s+1/3} e^{\sigma J} \eta \|^3 + \sigma \| J^{s+1/4} e^{\sigma J} \eta \|^4. \tag{3.6}
\]

Considering the interpolation estimate in Lemma 3.2, it follows that
\[
\| J^{s+1/4} e^{\sigma J} \eta \| \lesssim \| J^{s+1/2} e^{\sigma J} \eta \|^{1/2} \| J^s e^{\sigma J} \eta \|^{1/2}, \tag{3.7}
\]
and
\[
\| J^{s+1/3} e^{\sigma J} \eta \| \lesssim \| J^{s+1/2} e^{\sigma J} \eta \|^{2/3} \| J^s e^{\sigma J} \eta \|^{1/3}. \tag{3.8}
\]

An use of the estimates (3.7) and (3.8) in (3.6), yields
\[
\frac{1}{2} \partial_t \| J^s e^{\sigma J} \eta \|^2 - \sigma' \| J^{s+1/2} e^{\sigma J} \eta \|^2 \lesssim \| J^s \eta \|^3 + \| J^s \eta \|^4
\]
\[
+ \sigma \| J^{s+1/2} e^{\sigma J} \eta \|^2 \| J^s e^{\sigma J} \eta \| + \sigma \| J^{s+1/2} e^{\sigma J} \eta \|^2 \| J^s e^{\sigma J} \eta \|^2 + \sigma \| J^{s+1/2} e^{\sigma J} \eta \|^2 \| J^s e^{\sigma J} \eta \|^2. \tag{3.9}
\]

Thus
\[
\frac{1}{2} \partial_t \| J^s e^{\sigma J} \eta \|^2 + C \left( -\sigma' - \sigma \| J^s e^{\sigma J} \eta \| - \sigma \| J^s e^{\sigma J} \eta \|^2 \right) \| J^{s+1/2} e^{\sigma J} \eta \|^2 \lesssim \| J^s \eta \|^3 + \| J^s \eta \|^4. \tag{3.10}
\]
Now, if

$$-\sigma' - \sigma \| J^2 e^{\sigma J} \eta \| - \sigma \| J^2 e^{\sigma J} \eta \|^2 = 0$$

(3.11)

and $s = 2$, then from (3.10) and (1.12), one gets

$$\frac{1}{2} \partial_t \| J^2 e^{\sigma J} \eta \|^2 \lesssim \| J^2 \eta \|^3 + \| J^2 \eta \|^4 \sim \| J^2 \eta_0 \|^3 + \| J^2 \eta_0 \|^4.$$  

(3.12)

From (3.12) one can infer that

$$\| J^2 e^{\sigma J} \eta \| \leq \| J^2 e^{\sigma J} \eta_0 \| + C t^{1/2} (\| J^2 \eta_0 \|^3/2 + \| J^2 \eta_0 \|^2) =: X_0 + t^{1/2} y_0.$$  

(3.13)

The estimate (3.13) and a blow-up alternative imply that if $T_s$ is the maximal time of existence of solution $\eta$ to the IVP (1.1), then $T_s = \infty$. We prove this by using a contradiction argument. If possible suppose that $0 < T_s < \infty$. Let $L$ and $F$ be the linear and nonlinear parts of the IVP (1.1). Then for any $0 < T < T_s$, we have

$$\begin{cases}
L \eta + F \eta = 0, & 0 \leq t \leq T < T_s, \\
\eta(x,0) = \eta_0(x).
\end{cases}$$

(3.14)

We consider also the IVP

$$\begin{cases}
L u + F u = 0, & 0 \leq t \leq T_0, \\
u(x,0) = \eta(x,T).
\end{cases}$$

(3.15)

where $T_0$ is the local existence time given by (2.22), i.e.

$$T_0 = \frac{1}{8C_s \| \eta(\cdot,T) \|_{G^{\sigma,2}} (1 + \| \eta(\cdot,T) \|_{G^{\sigma,2}})}.$$  

Using (3.13) and (1.12), we get

$$\| \eta(\cdot,T) \|_{G^{\sigma,2}} \leq \| \eta_0 \|_{G^{\sigma,2}} + C T_s^{1/2} \left( \| \eta_0 \|_{H^{2}}^{3/2} + \| \eta_0 \|_{H^{2}}^{2} \right).$$

Thus

$$T_0 \geq \frac{1}{8C_s \left( 1 + \| \eta_0 \|_{G^{\sigma,2}} + C T_s^{1/2} \left( \| \eta_0 \|_{H^{2}}^{3/2} + \| \eta_0 \|_{H^{2}}^{2} \right) \right)^2} =: \mathcal{T}_0.$$  

Now, we choose $0 < T < T_s$ such that

$$T + \mathcal{T}_0 > T_s.$$  

If $v$ is such that $u(x,t) = v(x,T + t)$, $0 \leq t \leq T_0$, we obtain

$$w(x,t) = \begin{cases}
\eta(x,t), & 0 \leq t \leq T, \\
v(x,t), & T \leq t \leq T + T_0.
\end{cases}$$
is also a solution of the IVP \((1.1)\), with initial data \(\eta_0\) in the time interval \([0, T + T_0]\) with \(T + T_0 > T_s\), which contradicts the definition of the maximality of \(T_s\). Hence the solution is global.

Now, we move to find the lower and upper bounds for \(\sigma(t)\). Note that using \((3.13)\), the estimate \((3.11)\) is true if
\[
-\sigma' = \sigma \|J^2 e^{\sigma J} \eta\| + \sigma \|J^2 e^{\sigma J} \eta\|^2 \\
\leq \sigma (X_0 + t^{1/2} y_0 + 2X_0^2 + 2ty_0^2) =: \sigma A(t).
\]

On the other hand, the estimate \((3.16)\) is equivalent to
\[
\sigma(t) \geq \sigma_0 e^{-\int_0^t A(t') dt'}
\]
\[
= \sigma_0 e^{-(X_0 + 2X_0^2)t - \frac{1}{2}t^{1/2} y_0 - \frac{1}{2}t^2 Y_0^2},
\]
where \(\sigma_0 = \sigma(0)\). This provides the lower bound for \(\sigma(t)\).

Now, we proceed to find an upper bound \(\sigma(t)\). Considering \((3.16)\) and \((1.12)\), we have
\[
-\sigma' \geq \sigma \|J^2 e^{\sigma J} \eta\|^2 \\
\geq \sigma \|J^2 \eta\|^2 \\
\geq \sigma \|J^2 \eta_0\|^2.
\]

Consequently
\[
\sigma(t) \leq C \sigma_0 e^{-\|J^2 \eta_0\|^2 t}.
\]

We observe that the radius of analyticity \(\sigma(t)\) goes to zero when \(t\) goes to infinity
\[
\lim_{t \to \infty} \sigma(t) = 0.
\]

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