Quasi-conformal Rigidity of Negatively Curved Three Manifolds

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Abstract

In this paper we study the rigidity of infinite volume 3-manifolds with sectional curvature $-b^2 \leq K \leq -1$ and finitely generated fundamental group. In particular, we generalize the Sullivan’s quasi-conformal rigidity for finitely generated fundamental group with empty dissipative set to negative variable curvature 3-manifolds. We also generalize the rigidity of Hamenstädt or more recently Besson-Courtois-Gallot, to 3-manifolds with infinite volume and geometrically infinite fundamental group.

1 Introduction

Let $\tilde{M}$ be a simply connected complete Riemannian manifold with sectional curvature $-b^2 \leq K \leq -1$. Let $\text{ISO}(\tilde{M})$ denote the group of isometries of $\tilde{M}$. Let $\Gamma$ be a non-elementary, torsion-free, discrete subgroup of $\text{ISO}(\tilde{M})$, and set $M := \tilde{M}/\Gamma$.

First we recall some terminologies that is required for the statement of the theorem. Let $S_\infty$ denote the boundary of $\tilde{M}$. On $S_\infty$ one can define a metric in the following way. Let $v$ be a vector in the unit tangent bundle $S\tilde{M}$. The geodesic $v(t)$ defines two points on $S_\infty$ given by $v(\infty)$ and $v(-\infty)$. Let $\pi_t$ be the projection of $S_\infty \setminus v(-\infty)$ along the geodesics which are asymptotic to $v(-\infty)$ to the horosphere which is tangent to $v(-\infty)$ and passing through $v(t)$. Let $\text{dist}_{v,t}$ be the distance on the horosphere induced by restriction of the Riemannian distance, dist. On $S_\infty \setminus v(-\infty) \times S_\infty \setminus v(-\infty)$ define a function $\eta_v$ as $\eta_v(\xi, \zeta) := e^{-l_v(\xi, \zeta)}$ with $l_v(\xi, \zeta) := \sup\{ t | \text{dist}_{v,t}(\pi_t(\xi), \pi_t(\zeta)) \leq 1 \}$. By our curvature assumption $-b^2 \leq K \leq -1$, the function $\eta_v$ is a distance on $S_\infty \setminus v(-\infty)$, see [25].
Every element of $\gamma \in \Gamma$ has either exactly one or two fixed points in $S_\infty$, and $\gamma$ is called loxodromic if it has two fixed points $^4$. The group $\Gamma$ is called purely loxodromic if all $\gamma \in \Gamma$ are loxodromic. The limit set of $\Gamma$ denoted by $\Lambda_\Gamma$ is the unique minimal closed $\Gamma$-invariant subset of $S_\infty$ $^22$. If $\Gamma$ is purely loxodromic and $\Lambda_\Gamma = S_\infty$, then it can be either cocompact or $\tilde{M}/\Gamma$ is geometrically infinite, hence $\Gamma$ has infinite co-volume. The convex hull $CH_\Gamma$ is the smallest convex set in $\tilde{M} \cup S_\infty$ containing $\Lambda_\Gamma$. The group $\Gamma$ is called convex-cocompact if $CH_\Gamma/\Gamma$ is compact.

The critical exponent of $\Gamma$ is the unique positive number $D_\Gamma$ such that the Poincaré series of $\Gamma$ given by $\sum_{\gamma \in \Gamma} e^{-s\text{dist}(x,\gamma x)}$ is divergent for $s < D_\Gamma$ and convergent for $s > D_\Gamma$. If the Poincaré series diverges at $s = D_\Gamma$ then $\Gamma$ is called divergent.

Let $f : (X, \rho_X) \rightarrow (Y, \rho_Y)$ be an embedding between two topological metric spaces. Then $f$ is called quasi-conformal embedding $^47$ if there exists a constant $\kappa > 0$ such that, for any $x \in X$ and $r > 0$ there is $r_f(x, r) > 0$ with

$$f(X) \cap B'(f(x), r_f(x, r)) \subset f(B(x, r)) \subset B'(f(x), \kappa r_f(x, r)).$$

where $B$ and $B'$ denotes a ball in $X$ and $Y$ respectively. When $f(X) = Y$ then $f$ is a quasi-conformal homeomorphism.

A torsion-free discrete subgroup $\Gamma$ of $\text{ISO}(\tilde{M})$ is called topologically tame if $\tilde{M}/\Gamma$ is homeomorphic to the interior of a compact manifold-with-boundary.

**Theorem 1.1.** Let $\Gamma' \subset \text{PSL}(2, \mathbb{C})$ be a topologically tame discrete group with $\Lambda_{\Gamma'} = S^2$, and isomorphic $\chi : \Gamma' \rightarrow \Gamma$ to a convex-cocompact discrete subgroup $\Gamma$ of $\text{ISO}(\tilde{M})$ (here $\tilde{M}$ is $n$-dimensional). Let $f : S^2 \rightarrow S_\infty$ be a quasi-conformal embedding which conjugate $\Gamma'$ to $\Gamma$, i.e. $f \circ \gamma = \chi(\gamma) \circ f$, for $\gamma \in \Gamma'$. Then $D_\Gamma \geq D_{\Gamma'}$, and equality if and only if $\mathbb{H}^3$ embeds isometrically into $\tilde{M}$ and the action of $\Gamma$ stabilizes the image.

To state our next theorem we need to introduce one additional terminology. We take $\tilde{M}$ to be a 3-manifold in the following.

Let $\mathcal{M}_\eta^\lambda$ denote the $\lambda$-dimensional hausdorff measure on $(S_\infty \setminus v(-\infty), \eta_v)$. We say $\Gamma$ is hausdorff-conservative if there exists a constant $\alpha(v) > 0$ such that $\alpha^{-1} r^{D_\Gamma} \leq \mathcal{M}_\eta^\lambda (B(\xi, r) \cap \Lambda_\Gamma) \leq \alpha r^{D_\Gamma}$ for any ball $B(\xi, r)$ of radius $r$ about $\xi \in \Lambda_\Gamma$ in $(S_\infty \setminus v(-\infty), \eta_v)$. From this definition, we note that if $\Gamma$ is a finitely generated torsion-free discrete subgroup of $\text{PSL}(2, \mathbb{C})$ with $D_\Gamma = 2$, then hausdorff-conservative implies conservative (classical definition,
§5). Conversely, if $\Gamma$ is a topologically tame, conservative, discrete subgroup of $\text{PSL}(2, \mathbb{C})$, then $\Gamma$ is hausdorff-conservative, see Proposition 5.2. We believe all finitely generated conservative discrete subgroup of $\text{PSL}(2, \mathbb{C})$ are hausdorff-conservative, see Remark 5.3. For a convex-cocompact $\tilde{M}/\Gamma$ with $-b^2 \leq K \leq -1$, it follows from [12], $\Gamma$ is hausdorff-conservative. Now we are ready to state the theorem which generalizes Sullivan’s quasi-conformal rigidity theorem.

**Theorem 1.2 (Main).** Let $\Gamma$ be a topologically tame, purely loxodromic discrete subgroup of $\text{ISO}(\tilde{M})$ with $\Lambda_\Gamma = S_\infty$. Let $\Gamma'$ be a topologically tame discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Suppose $f : S_\infty \to S^2$ is a quasi-conformal homeomorphism conjugate $\Gamma$ to $\Gamma'$. Then $D_\Gamma \geq D_{\Gamma'}$, and $\Gamma = \gamma \Gamma' \gamma^{-1}$ with $\gamma \in \text{PSL}(2, \mathbb{C})$ if and only if $D_\Gamma = D_{\Gamma'}$ and $\Gamma$ is hausdorff-conservative.

**Corollary 1.3.** Let $M = \tilde{M}/\Gamma$ be a complete topologically tame 3-manifold with $-b^2 \leq K \leq -1$, $\Gamma$ purely loxodromic, and $\Lambda_\Gamma = S_\infty$. Let $h : M \to N$ be a quasi-isometric homeomorphism to a hyperbolic manifold $N$. Then $M$ is isometric to $N$ if and only if $D_\Gamma = 2$ and $\Gamma$ is hausdorff-conservative.

Let us point out that Theorem 1.2 generalizes known rigidity theorems in two directions for three dimensional manifolds.

First assume $M$ is hyperbolic ($b = 1$) but not necessarily geometrically finite. Since $M$ is topologically tame and $\Lambda_\Gamma = S^2$ we have $D_\Gamma = 2$ by analytical tameness (see Proposition 3.3). Hence by Theorem 1.2, $M$ is quasi-conformal stable. This is a case of the Sullivan rigidity theorem for topologically tame $\Gamma$ with empty dissipative set. Next let us assume $M$ is compact with $-b^2 \leq K \leq -1$. Then the critical exponent $D_\Gamma$ is equal to $h_M$ the topological entropy of $M$, and by [10], any homotopy equivalence between $M$ and a compact hyperbolic 3-manifold is induced by a homeomorphism. Therefore it follows from Corollary 1.3 we have: $M$ is isometric to a compact hyperbolic 3-manifold if and only if they are homotopically equivalent and $h_M = 2$. This is the Hamenstädt’s rigidity or more recently Besson-Courtois-Gallot theorem for 3-manifolds.

Note that it also follows from Theorem 1.2, the quasi-conformal version of the Hamenstädt’s theorem for compact 3-manifold $M$ can be stated as:

**Corollary 1.4.** Let $\Gamma$ be a cocompact discrete subgroup of $\text{ISO}(\tilde{M})$. Let $\Gamma' \subset \text{PSL}(2, \mathbb{C})$ be a discrete group. Suppose $f : S_\infty \to S^2$ is a quasi-conformal homeomorphism conjugate $\Gamma$ to $\Gamma'$. Then $D_\Gamma \geq D_{\Gamma'}$, and equality if and only if $\tilde{M}/\Gamma$ is isometric to $\mathbb{H}^3/\Gamma'$. 

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The proves of these theorems relies on our next result,

**Theorem 1.5.** Let $\tilde{M}/\Gamma$ be a topologically tame 3-manifold with $-b^2 \leq \mathcal{K} \leq -1$. Suppose that $\Gamma$ is purely loxodromic and that $\Lambda(\Gamma) = S_\infty$. Then $2 \leq D$ and $\Gamma$ is harmonically ergodic. If $D = 2$ then $\Gamma$ is also divergent.

In section 2, we state some of the topological properties of negatively pinched 3-manifolds. In particular, we define geometrically infinite ends for negatively pinched 3-manifolds, and then state our theorem which describe the geometrical properties of this type of end, it is a crucial step in the proof of Theorem 1.3. Section 3 discusses measures on $S_\infty$ and the ergodicity of $\Gamma$ with respect to these measures. In section 4, we give proofs of part I of the theorems. And section 5 is used to complete the proofs.

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## 2 Topological Ends

Every isometry of $\tilde{M}$ can be extend to a Lipschitz map on $S_\infty := \partial \tilde{M}$. For a torsion-free $\Gamma$, every element $\gamma \in \Gamma$ is one of the following types: (1) *parabolic* if it has exactly one fixed point in $\tilde{M} \cup S_\infty$ which lies in $S_\infty$; (2) *loxodromic* if it has exactly two distinct fixed points in $\tilde{M} \cup S_\infty$, both lying in $S_\infty$.

Denote by $\Lambda(\Gamma) \subset \partial \tilde{M}$ the limit set of $\Gamma$, which is the unique minimal closed $\Gamma$–invariant subset of $S_\infty$. Most of the important properties of the limit set in the constant curvature space continue to hold in the variable curvature space \[13\]. In particular: (i) $\Lambda(\Gamma) = \overline{\Omega(\Gamma) \cap S_\infty}$; (ii) $\Lambda(\Gamma)$ is the closure of the set of fixed points of loxodromic elements of $\Gamma$; and (iii) $\Lambda(\Gamma)$ is a perfect subset of $\Gamma$. The set $\Omega(\Gamma) := S_\infty \setminus \Lambda(\Gamma)$ is the region of discontinuity. The action of $\Gamma$ on $\tilde{M} \cup \Omega(\Gamma)$ is proper and discontinuous, see \[15\]. The manifold $M_T := \tilde{M} \cup \Omega(\Gamma)/\Gamma$ with possibly nonempty boundary is traditionally called the Kleinian manifold. We also let $\Lambda_c(\Gamma)$ denote the conical limit set of $\Gamma$, i.e. $\xi \in \Lambda_c(\Gamma)$ if for some $x \in \tilde{M}$ (and hence for every $x$) there exist a sequence $(\gamma_n)$ of elements in $\Gamma$, a sequence $(t_n)$ of real numbers, and a real number $C > 0$, such that $\gamma_n x \rightarrow \xi$ and $\text{dist}(c_x(t_n), \gamma_n x) < C$ where $c_x$ is the geodesic ray connecting $x$ and $\xi$. Equivalently, a point belongs to $\Lambda_c(\Gamma)$ if it belongs to infinitely many shadows cast by balls of some fixed radius.
centered at points of a fixed orbit of $\Gamma$. Note that $\Lambda_c(\Gamma)$ is a $\Gamma$–invariant subset of $\Lambda(\Gamma)$, hence a dense subset.

**Proposition 2.1 (Margulis Lemma).** There exists a number $\epsilon_b$ which only depend on the pinching constant $b$ of $M$, such that the group $\Gamma_\epsilon$ generated by elements in $\Gamma$ of length at most $\epsilon_b$ with respect to a fixed point in $M$ is almost nilpotent of rank at most 2. Then the number, $2\epsilon_b$ is called the Margulis constant.

Note that, if $M$ is orientable and $\Gamma$ is torsion-free, then Margulis Lemma implies $\Gamma_{\epsilon_b}$ is abelian.

Let $\epsilon \leq \epsilon_b$ be given. Then $M$ may be written as the union of a thin part $M_{[0,\epsilon)}$ consisting of all points at which there is based a homotopically nontrivial loop of length $\leq \epsilon$ and a thick part $M_{[\epsilon,\infty)} = M - M_{[0,\epsilon)}$. Note that $M_{[\epsilon,\infty)}$ is compact if $M$ is of finite volume. Also the thin part of $M$ is completely classified by the next proposition.

**Proposition 2.2.** Each connected component of $M_{[0,\epsilon)}$ is diffeomorphic to one of the following:

- **parabolic rank-1 cusp**: $S^1 \times \mathbb{R} \times [0,\infty)$.
- **parabolic rank-2 cusp**: $T^2 \times [0,\infty)$.
- **solid torus about the axis of a loxodromic** $\gamma$: $D^2 \times S^1$.

For simplicity we restrict to the case where $M$ has no cusps. It follows from the existence of a compact core $C(M)$ for $M$ 14, that $M$ has only finitely many ends 14. In fact, each component of $\partial C(M)$ is the boundary of a neighborhood of an end of $M$, and this gives a bijective correspondence between ends of $M$ and components of $\partial C(M)$.

We define the simplicial ruled surfaces as follows. Let $S$ be a surface of positive genus and let $T_P$ be a triangulation defined with respect to a finite collection $P$ of points of $S$. This means that $T_P$ is a maximal collection of nonisotopic essential arcs with end points in $P$; these arcs are the edges of the triangulation, and the components of the complement in $S$ of the union of the edges are the faces. Let $f: S \to M$ be a map which takes edges to geodesic arcs and faces to nondegenerate geodesic ruled triangles in $M$. The map $f$ induces a singular metric on $S$. If the total angle about each vertex of $S$ with respect to this metric is at least $2\pi$, then the pair $(S, f)$ is
called a simplicial ruled surface. It follows from the definition of the induced metric on $S$ that $f$ preserves lengths of paths and is therefore distance non-increasing. Any geodesic ruled triangle in $M$ has Gaussian curvature at most $-a^2$. This means that each 2-simplex of $S$ inherits a Riemannian metric of curvature at most $-a^2$. Since we have required the total angle at each vertex to be at least $2\pi$, by Gauss-Bonnet theorem the curvature of $S$ is negative in the induced metric.

**Definition 2.3.** An end $E$ is said to be a geometrically infinite if there exists a divergent sequence of geodesics, i.e: there exists a sequence of closed geodesics $\alpha_k \subset M_E$, such that for any neighborhood $U$ of $E$, there exists some positive integer $N$ such that $\alpha_k \subset U$ for all $k > N$. If in addition for some surface $S_E$ we have that $U$ is homeomorphic to $S_E \times [0, \infty)$, and there exists a sequence of simplicial ruled surfaces $S_E \xrightarrow{f_l} U$ such that $f_l(S_E)$ is homotopic to $S_E \times 0$ in $U$ and leaves every compact subset of $M$, then $E$ is said to be simply degenerate. The sequence $(S_E \xrightarrow{f_l} U)$ is called an exiting sequence. A end which is not geometrically infinite will be called geometrically finite.

**Theorem 2.4 (Hou).** Let $M = \tilde{M}/\Gamma$ be a topologically tame negatively pinched 3-manifold with $\Gamma$ purely loxodromic. Then all geometrically infinite ends of $M$ are simply degenerate. And if $\Lambda(\Gamma) = S_\infty$, then there are no nonconstant positive superharmonic functions, or nonconstant subharmonic functions bounded above, on $M$.

### 3 $\Gamma$-action

In this section we will study the action of $\Gamma$ on $S_\infty$ and prove ergodicity of $\Gamma$ for topologically tame 3-manifolds with $\Lambda(\Gamma) = S_\infty$. We will prove that for such a manifold, the Green series is divergent, and that the Poincaré series is also divergent if $D = 2$. Theorem [1.3] will also be proved in this section.

In some situations we will take the dimension of $M$ to be 3, otherwise we will assume $M$ is $n$-dimensional in general.

Set the following notations throughout the paper. Let $\Gamma' \subset \text{PSL}(2, \mathbb{C})$ be a discrete torsion-free subgroup. Denote $S^2 := \partial \mathbb{H}^3$, and $S_\infty := \partial \tilde{M}$.

There are many equivalent ways of equipping $S_\infty$ with a metric which is compatible with $\Gamma$-action. Fix a point $x \in \tilde{M}$. Let $\xi, \zeta$ in $S_\infty$ be given. Set $c_y(t)$ as the geodesic ray connecting $y$ and $\zeta$. 


In [22], Gromov defined a metric on $S_\infty$ as follows. For $y, z \in \tilde{M}$, let us consider arbitrary continuous curve $c(t)$ in $\tilde{M}$ with initial point and end point denoted by $c(t_0) = y$ and $c(t_1) = z$ respectively. Define a nonnegative real-valued function $G_x$ on $\tilde{M} \times \tilde{M}$ by

$$G_x(y, z) := \inf_{c} \left( \int_{[t_0, t_1]} e^{-\text{dist}(x, c(t))} \, dt \right).$$

In particular, Gromov showed the function $G_x$ extends continuously to $S_\infty \times S_\infty$. Every element of $\Gamma$ extends to $S_\infty$ as a Lipschitz map with respect to $G_x$.

In [31] the following metrics are shown to be equivalent to the Gromov’s metric.

$K_x$ metric: Let $B_\zeta$ denote the Busemann function based at $x_0$. Set $B_\zeta(x, y) = B_\zeta(x) - B_\zeta(y)$, for $x, y \in \tilde{M}$, the function $B_\zeta(x, y)$ is called the Busemann cocycle. Define $\beta_x : S_\infty \times S_\infty \to \mathbb{R}$ by $\beta_x(\xi, \zeta) := B_\xi(x, y) + B_\zeta(x, y)$ where $y$ is a point on the geodesic connecting $\xi$ and $\zeta$. The $K_x$ metric is then defined by

$$K_x(\xi, \zeta) := e^{-\frac{1}{2} \beta_x(\xi, \zeta)}.$$

$L_x$ metric: Let $\alpha_x(\xi, \zeta)$ denote the distance between $x$ and the geodesic connecting $\xi$ and $\zeta$. The function $L_x : S_\infty \times S_\infty \to \mathbb{R}$ is then defined by

$$L_x(\xi, \zeta) := e^{-\alpha_x(\xi, \zeta)}.$$

$d_x$ metric: Define a function $l_x : S_\infty \times S_\infty \to \mathbb{R}$ by $l_x(\xi, \zeta) := \sup\{\tau \mid \text{dist}(c_\xi^x(\tau), c_\zeta^x(\tau)) = 1\}$. Geometrically, a neighborhood about $\xi$ in $S_\infty$ with respect to the topology induced by $l_x$ is the shadow cast by the intersection of 1-ball about $c_\xi^x(\tau)$ and $\tau$-sphere about $x$. The $d_x$ metric is then defined by

$$d_x(\xi, \zeta) := e^{-l_x(\xi, \zeta)}.$$

It was originally observed for symmetric spaces by Mostow [36] that the boundary map is quasi-conformal. This property continue to hold in negatively curved spaces, see [24] and [40]. Here we give a proof of this fact with respect to the above metrics.
Proposition 3.1. Let $h$ be a quasimorphism between two negatively pinched curved spaces. The boundary extension map $\tilde{h}$ is quasi-conformal on the boundary with respect to $d_x, L_x, K_x, \eta_0$-metrics.

Proof. For the proof of $\eta_0$-metric See Proposition 3.1 in [11]. Fix $x \in \tilde{M}$. Let us take $d_x$-metric. Set $\lambda \geq L$. Denote by $S(x; y, R)$ the shadow cased from $x$ of the metric sphere $S(y, R)$ with center located at $y$ and radius $R$, i.e. $S(x; y, R) = \{ x \in S_\infty | c^2_x \cap S(y, R) \neq \emptyset \}$. Let $B(\xi, r)$ be a ball of radius $r$ in $S_\infty$. Using triangle comparison we can show there exists a constant $\alpha_b \geq 1$ depends on pinching constant $b$ such that

$$S(x; c^2_x(t_r), \lambda) \subset B(\xi, r) \subset S(x; c^2_x(t_r), \alpha_b \lambda)$$

for some $t_r > 0$ which depends only on $r$. The images $\tilde{\phi}(S(x; c^2_x(t_r), \lambda))$ and $\tilde{\phi}(S(x; c^2_x(t_r), \alpha_b \lambda))$ are quasi-spheres, i.e. there exists a constant $\beta_\phi > 0$ depends on $\phi$ such that $S(\tilde{\phi}(x); \tilde{\phi}(c^2_x(t_r)), \beta_\phi^{-1} \lambda) \subset \tilde{\phi}(S(x; c^2_x(t_r), \lambda))$ and $\tilde{\phi}(S(x; c^2_x(t_r), \alpha_b \lambda)) \subset S(\tilde{\phi}(x); \tilde{\phi}(c^2_x(t_r)), \beta_\phi \alpha_b \lambda)$. On the other hand, by estimates in [11] there exists positive numbers $A_1(\beta_\phi, \lambda)$ and $A_2(\alpha_b, \beta_\phi, \lambda)$ such that

$$B(\tilde{\phi}(\xi), A_1 e^{-R}) \subset S(\tilde{\phi}(x); \tilde{\phi}(c^2_x(t_r)), \beta_\phi^{-1} \lambda),$$

$$S(\tilde{\phi}(x); \tilde{\phi}(c^2_x(t_r)), \beta_\phi \alpha_b \lambda) \subset B(\tilde{\phi}(\xi), A_2 e^{-R})$$

where $R = \text{dist}(\tilde{\phi}(x), \tilde{\phi}(c^2_x(t_r)))$. Hence the result follows by setting $r_\phi(\xi, r) = A_1 e^{-R}$ and $\kappa = A_2 / A_1$. \qed

Proposition 3.2. Let $f : \partial \tilde{N} \longrightarrow S_\infty$ be an embedding conjugate $\Gamma_1$ to $\Gamma_2$ under isomorphism $\chi : \Gamma_1 \longrightarrow \Gamma_2$ $(f \circ \gamma = \chi(\gamma) \circ f)$. Then $f(\Lambda_{\Gamma_1}) = \Lambda_{\Gamma_2}$.

Proof. Let $\gamma \in \Gamma_1$. Since $\gamma f^{-1}(\Lambda_{\Gamma_2}) = f^{-1}(\chi(\gamma)\Lambda_{\Gamma_2})$, and by $\Gamma_2$-invariance of $\Lambda_{\Gamma_2}$, we have $f^{-1}(\Lambda_{\Gamma_2})$ is $\Gamma_1$-invariant closed set. Note that $f^{-1}(\Lambda_2)$ is nonempty, since fixed points of elements of $\Gamma_1$ are also fixed points of elements of $\Gamma_2$, hence $f(\Lambda_{\Gamma_1}) \subseteq \Lambda_{\Gamma_2}$. Similarly we also have $f(\Lambda_{\Gamma_1}) \supseteq \Lambda_{\Gamma_2}$, and result follows. \qed

Proposition 3.3. Let $\Gamma$ be a topologically tame, torsion-free, discrete subgroup of ISO($\tilde{M}$) with $\Lambda_{\Gamma} = S_\infty$. Let $\Gamma'$ be a topologically tame, discrete subgroup of PSL(2, $\mathbb{C}$). Suppose $f : S_\infty \longrightarrow S^2$ is a homeomorphism conjugate $\Gamma$ to $\Gamma'$. Then $D_{\Gamma'} = 2$ and $\Gamma'$ is divergent.
Proof. By Proposition 3.2 the hyperbolic manifold $N = \mathbb{H}^3 / \Gamma'$ is topologically tame and $\Lambda_{\Gamma'} = S^2$. It follows from analytical tameness and Theorem 9.1 of [10], there exists no non-trivial positive superharmonic function on $N$ with respect to the hyperbolic Laplacian $\Delta$. Let $P(y, \xi)$ denote the Poisson kernel on $\mathbb{H}^3$. The $D_{\Gamma'}$-dimensional conformal measure (Patterson-Sullivan measure, see end of §3) $\sigma_y$ has Radon-Nikodym derivative of $P(y, \xi)^{D_{\Gamma'}}$, i.e. $\frac{d\sigma_y}{d\gamma^*\sigma_y}(\xi) = P(\gamma^{-1}y, \xi)^{D_{\Gamma'}}$. The $\Gamma'$-invariant function $h(y) := \sigma_y(S^2)$ satisfies $\Delta h = D_{\Gamma'}(D_{\Gamma'} - 2)h$, which implies $h$ is non-trivial superharmonic if $D_{\Gamma'} \neq 2$. And it follows that $\Gamma'$ must also be divergent.

Let $C$ be a subset of $S_\infty$. Let $\lambda$-dimensional Hausdorff measure of $C$ on the metric space $(S_\infty, \rho_x)$ be denoted by $\mathcal{M}_x^\lambda(C)$. Observe that for any $x \in \tilde{M}$ and any $\gamma \in \Gamma$, we have $\gamma^*\mathcal{M}_x^\lambda = \mathcal{M}_{\gamma^{-1}x}^\lambda$; this follows from the straightforward identity.

A family of finite Borel measures $[\nu_y]_{y \in \tilde{M}}$ will be called a $\lambda$-conformal density under the action of $\Gamma$ if for every $x \in \tilde{M}$ and every $\gamma \in \Gamma$ we have $\gamma^*\nu_y = \nu_{\gamma^{-1}y}$, and the Radon-Nikodym derivative $\frac{d\nu_y}{d\gamma^*\nu_y}(\zeta)$ at any point $\zeta \in S_\infty$ is equal to $e^{-\lambda B_\gamma(\gamma^{-1}y, \xi)}$. (This is to be interpreted as being vacuously true if, for example, the measures in the family are all identically zero). Although there can not be any $\Gamma$-invariant non-trivial finite Borel measure on $\Lambda_{\Gamma}$ for non-elementary $\Gamma$, we can always define a $\Gamma$-invariant non-trivial locally finite measure $\Pi_{\nu_x}$ on $\Lambda_{\Gamma} \times \Lambda_{\Gamma}$ by setting $d\Pi_{\nu_x}(\xi, \zeta) = e^{\lambda B_{\gamma}(\xi, \zeta)}d\nu_x(\xi)d\nu_x(\zeta)$. The measure $\Pi_{\nu_x}$ corresponds to the Bowen-Margulis measure on the unit tangent bundle $SM$ see [31].

Let us recall a fundamental fact about conformal density, which was originally proved by Sullivan for $\Gamma \subset SO(n, 1)$ and generalized to the pinched negatively curved spaces in [10]. It relates the divergence of $\Gamma$ at the critical exponent $D_{\Gamma}$ with ergodicity of the $D_{\Gamma}$-conformal density under the action of $\Gamma$.

We will say that two Borel measures on $S_\infty$ are in the same $\Gamma$-class if the Radon-Nikodym derivative of $\gamma^*\nu_1$ with respect to $\nu_1$ is equal to the Radon-Nikodym derivative of $\gamma^*\nu_2$ with respect to $\nu_2$.

**Proposition 3.4 (see [10]).** Let $\Gamma$ be a nonelementary, discrete, torsion-free and divergent at $D_{\Gamma}$. Suppose $[\nu]$ is a $D_{\Gamma}$-conformal density under the action of $\Gamma$, then $\Gamma$ act ergodically on $\Lambda_{\Gamma}$ and $\Lambda_{\Gamma} \times \Lambda_{\Gamma}$ with respect to $[\nu_x]$ and $[\Pi_{\nu_x}]$ respectively.
Proposition 3.5 (see [37]). Let \( \Gamma \) be nonelementary and discrete. Suppose that \( \Gamma \) acts ergodically on \( S_\infty \) with respect to a measure \( \nu \) defined on \( S_\infty \). Then every measure of \( S_\infty \) in the same measure class as \( \nu \) is a constant multiple of \( \nu \).

Proposition 3.6. Let \( \Gamma \) be a non-elementary discrete subgroup of the isometry group of \( \tilde{M} \). If \([\nu_y]^D_x \) is a non-trivial \( \Gamma \)-invariant \( D \)-conformal density, then \( D \neq 0 \).

Proof. Suppose \( D = 0 \). Then \( \nu_y \) is a \( \Gamma \)-invariant non-trivial finite Borel measure. Since \( \Gamma \) is non-elementary, there exists a loxodromic element \( \gamma \) in \( \Gamma \). Let \( \xi, \zeta \in S_\infty \) be the two distinct fixed points of \( \gamma \). Let \( < \gamma > \) be the group generated by \( \gamma \). Then \( \nu_y \) is clearly \( < \gamma > \)-invariant. But \( \gamma \) is loxodromic, so we must have \( \text{supp}(\nu_y) \subset \{ \xi, \zeta \} \). Then, by the fact that \( \Lambda(\Gamma) \) is infinite, we have \( \nu_y \) is an infinite measure, which is a contradiction. \( \square \)

Proposition 3.7. Let \( \Gamma \) be a discrete subgroup of \( \text{ISO}(\tilde{M}) \). Suppose \( M^\lambda \mathbb{K}_x \) is a finite measure. Then \( M^\lambda \mathbb{K}_x \) is a \( \lambda \)-conformal density under the action of \( \Gamma \).

There is a canonical way of constructing \( D \)-dimensional conformal density which is due to Patterson-Sullivan as follows; By applying a adjusting function we can always assume the Poincaré series diverges at \( D \). The measures

\[
\mu_{x,s} := \frac{\sum_{\gamma \in \Gamma} e^{-s \dist(x,\gamma x)} \delta_{\gamma x}}{\sum_{\gamma \in \Gamma} e^{-s \dist(x,\gamma x)}} \quad ; \quad s > D
\]

converges weakly to a limiting measure \( \mu_x \) as \( s_n \to D \) through a subsequence. It is trivial to see that \( \mu_x \) is supported on \( \Lambda_\Gamma \). The measure \([\mu_x]\) is called Patterson-Sullivan measure which is \( D \)-conformal under \( \Gamma \) see [39], [42].

From now on for \( \Gamma' \subset \text{PSL}(2, \mathbb{C}) \), we will denote the Patterson-Sullivan measure on \( \Lambda_{\Gamma'} \) by \( \sigma_y \).

Let \( \Lambda_\Gamma^c \) denote the set of conical limit points in \( \Lambda_\Gamma \). Recall a point \( \xi \in \Lambda_\Gamma \) is in \( \Lambda_\Gamma^c \) if and only if there exists \( \{ \gamma_n \} \subset \Gamma \) such that \( \dist(\gamma_n \xi, x) < c \) for some \( c > 0 \) and sequence of \( t_n \). Obviously \( \Lambda_\Gamma^c \) is \( \Gamma \)-invariant, and non-empty (a loxodromic fixed points are in \( \Lambda_\Gamma^c \)), hence it is a dense \( \Gamma \)-invariant subset of \( \Lambda_\Gamma \). A equivalent definition for the conical limit point \( \xi \) is that it must be contained in infinitely many shadows \( S(x; \gamma_n x, c) \). Hence \( \Lambda_\Gamma^c = \bigcup_{\lambda > 0} \cap_{m \geq 1} \cup_{n > m} S(x, \gamma_n x, \lambda) \). It is an easy fact from the construction of \( \mu_x \), no points in \( \Lambda_\Gamma^c \) can be a atom for \( \mu_x \), and if \( \text{supp}(\mu_x) \subset \Lambda_\Gamma^c \) then \( \Gamma \) is
divergent. In fact it is a deep result of Sullivan that $\Gamma$ is divergent if and only if $\text{supp}(\mu_x) \subseteq \Lambda_c^\Gamma$.

**Lemma 3.8 (see [12]; Sullivan’s Shadow Lemma).** Let $\mu_x$ be a $D_\Gamma$-conformal density with respect to $\Gamma$, which is not a single atom. Then there exists constants $\alpha > 0$ and $\lambda_0 \geq 0$, such that,

$$\alpha^{-1} e^{-D_\Gamma \text{dist}(\gamma^{-1}x)} \leq \mu_x(S(x; \gamma x, \lambda)) \leq \alpha e^{-D_\Gamma \text{dist}(\gamma^{-1}x)+2D\lambda},$$

for all $\gamma \in \Gamma$ and $\lambda \geq \lambda_0$.

**Proposition 3.9.** Let $\Gamma \subset \text{ISO}(\tilde{M})$ be a discrete subgroup. Suppose either $\Lambda^\Gamma = \Lambda_c^\Gamma$ or $\Lambda^\Gamma = S_\infty$ and $\Gamma$ is divergent. Then $\mu_x$ is positive on all non-empty relative open subsets of $\Lambda^\Gamma$.

**Proof.** Suppose $\Lambda^\Gamma = S_\infty$. It suffices to show $\mu_x$ is positive for any non-empty open ball $B(\xi, r)$ with respect to the $d_x$-metric. Fix $\lambda > \lambda_0$. Let $\zeta \in \Lambda_c^\Gamma \cap B(\xi, r)$ (note that $\Lambda_c^\Gamma$ is dense in $\Lambda^\Gamma$ so the intersection is nonempty). Then we can choose $\gamma \in \Gamma$ such that $S(x; \gamma x, \lambda) \subset B(\xi, r)$. By assumption $\Gamma$ is divergent, we have $\text{supp}(\mu_x) \subset \Lambda_c^\Gamma$. Since no points of $\Lambda_c^\Gamma$ can be a atom for conformal density, the result follows from Lemma 3.8. Same argument works if $\Lambda^\Gamma = \Lambda_c^\Gamma$. \hfill\Box

Let us define a function $\Theta : \tilde{M} \times \tilde{M} \times S_\infty \to \mathbb{R}^+$ by $\Theta(x, y, \zeta) := \exp(-B_\zeta(x, y))$.

**Harmonic Density**

Let $\lambda_1$ and $\tilde{\lambda}_1$ denote the first of the spectrum of $\Delta$ on $M = \tilde{M}/\Gamma$, and of $\tilde{\Delta}$ on $\tilde{M}$, respectively. Recall that for a noncompact open manifold, the first of the spectrum is defined as

$$\lambda_1 := \inf_{f \in C^\infty_o \setminus \{0\}} \left( \frac{\int |\nabla f|^2}{\int f^2} \right),$$

where $C^\infty_o$ is the space of smooth functions on $M$ with compact support. Note that we always have $\lambda_1 \leq \tilde{\lambda}_1$.

The $\lambda_1$-harmonic functions has been studied by Ancona in [2] and [3].

**Proposition 3.10 (Ancona).** For each $s < \lambda_1$, the elliptic operator $\tilde{\Delta} + sI$ has a Green function $G_s(x, y)$, and there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{\gamma \in \Gamma} \hat{G}_s(x, \gamma y)$ converges for $s < \lambda_1$ and diverges for $s \geq \lambda_1$, where $\hat{G}_s(x, \gamma y) := \exp(f(d(y, \gamma y)))G_s(x, \gamma y)$. Furthermore, $\mathcal{P}_s(x, y, \zeta) := \lim_{z \to \zeta} \frac{G_s(x, z)}{G_s(y, z)}$ defines the Poisson kernel of $\tilde{\Delta} + sI$ at $\zeta \in S_\infty$.
Similarly to the construction of $\mu_x$ from $Z_\Gamma$, one can also construct a family of Borel measures from $\sum_{\gamma \in \Gamma} G_s(x, \gamma y)$.

**Proposition 3.11.** Let $x$ be any point of $\tilde{M}$. There exists a family of Borel measures $[\omega^1_y]_{y \in \tilde{M}}$ on $S_\infty$ such that (i) for all $x, y \in \tilde{M}$, Radon-Nikodym derivative $d\omega^1_y/d\omega^1_x$ at any point $\zeta \in S_\infty$ is equal to $\mathcal{P}_{\lambda_1}(x, y, \zeta)$ and (ii) $\omega^1_x$ is of mass 1.

Let us denote the harmonic density of $\tilde{\Delta}$ by $[\omega_y]_{y \in \tilde{M}}$ with $\omega_x$ normalized of mass 1. By definition this means that every harmonic function $f$ on $M$ with boundary values $f_\infty$ is given by

$$f(x) = \int_{S_\infty} f_\infty(\xi) d\omega_x(\xi).$$

The existence and uniqueness of harmonic density follows from the solvability of the Dirichlet problem on $\tilde{M} \cup S_\infty$ see [1] and the Riesz Representation Theorem. The Radon-Nikodym derivative of $\omega_x$ at $\xi \in S_\infty$ is given by the Poisson kernel $\mathcal{P}(x, y, \xi)$ of $\tilde{\Delta}$, i.e. $\frac{d\omega_y}{d\omega_x}(\xi) = \mathcal{P}(y, x, \xi)$. For any $\Gamma$-invariant subset $C \subset S_\infty$, the function $h_C$ on $\tilde{M}$ defined by $h_C(y) := \int_{S_\infty} \chi_C \mathcal{P}(y, x, \xi) d\omega_x(\xi)$ is $\Gamma$-invariant, hence defines a harmonic function on $M$.

**Proposition 3.12.** Let $M = \tilde{M}/\Gamma$ be a negatively pinched topologically tame 3-manifold with $\Lambda(\Gamma) = S_\infty$. Then $\Gamma$ is ergodic with respect to harmonic density $[\omega_y]_{y \in \tilde{M}}$.

**Proof.** Suppose not, and let $C \subset S_\infty$ be a $\Gamma$-invariant subset with $\omega_x(C) > 0$ and $\omega_x(C^c) > 0$. By Fatou’s conical convergence theorem, we have $\chi_C(\xi) = \lim_{t \to \infty} h_C(c_\xi^t(t))$ for $\xi \in S_\infty$. Hence, $h_C$ defines a positive nonconstant $\Gamma$-invariant harmonic function, which contradicts Theorem 2.4. Therefore $\Gamma$ must be ergodic.

**Proposition 3.13.** Let $M$ be noncompact and satisfy the hypothesis of Proposition 3.12. Then $\omega^1_x = \omega_x$.

**Proof.** Let us note that $\lambda_1 = 0$. This follows from the fact that for a noncompact, complete Riemannian manifold $M$, if $\lambda_1(M) > 0$ then there exists a positive Green’s function $G$ on $M$. If such a $G$ exists, then $1 - \exp(-G)$ defines a positive superharmonic function, which is a contradiction to Theorem 2.4. Hence we must have $\lambda_1 = 0$. Therefore $\mathcal{P}_{\lambda_1} = \mathcal{P}$, i.e $\frac{d\omega^1_y}{d\omega^1_x} = \frac{d\omega_y}{d\omega_x}$. Hence, by Proposition 3.12 and uniqueness, we have the desired result.
Superharmonic Functions

Let $\xi \in S\infty$ be given. Let $E$ be a continuous unit vector field on $\tilde{M}$ with $E(x) = \Phi_x(\xi)$. Then by using the first length variation formula, one can show that $B_\xi$ is $C^1$ and that $-\text{grad}B_\xi = \mathcal{E}$. In fact, the Busemann function is $C^{2,\alpha}$, see [27].

Let $[\mu_y]^D$ denote $D$-conformal density. Let us define a nonnegative function $u$ on $\tilde{M}$ by

$$u(y) := \int_{S\infty} \Theta^D(y,x,\xi) d\mu_x(\xi).$$

**Proposition 3.14.** The function $u$ is a $\Gamma$-invariant and positive. It is superharmonic if $D \leq (n-1)a$, and subharmonic if $(n-1)a \leq D \leq (n-1)b$.

**Proof.** We can write $u(y)$ as $\mu_y(S\infty)$. Since $u(\gamma y) = \gamma^* \mu_y(S\infty) = \mu_y(S\infty) = u(y)$ for $\gamma \in \Gamma$, we have that $u$ is $\Gamma$-invariant.

Let $x \in M$ be fixed. It follows from, $|\nabla B_\xi(y,x)| = |\mathcal{E}| = 1$ and Rauch's theorem that we have $\exp(-DB_\xi(y,x))D(D-(n-1)b) \leq \Delta\Theta^D \leq \exp(-DB_\xi(y,x))D(D-(n-1)a)$. This implies the result.

**Proposition 3.15.** Suppose $\Gamma$ is nonelementary, i.e. has no abelian subgroup of finite index. Suppose that there are no nontrivial $\Gamma$-invariant positive-valued superharmonic function on $\tilde{M}$. Then $(n-1)a \leq D \leq (n-1)b$.

**Proof.** Let $\Gamma x$ be the orbit of $x$ under $\Gamma$. Then the growth rate of the number of points of $\Gamma x$ in $\text{ball}(x,r)$ as $r$ increases is bounded by $\text{vol}(\text{ball}(x,r))$. By the volume comparison theorem we have $C_n \text{exp}((n-1)b r) \geq \text{vol}(\text{ball}(x,r))$, for some constant $C_n$ which depends only on dimension $n$. Therefore, when $s > (n-1)b$ we get $Z_\Gamma(x,s) < \infty$, which implies that $D \leq (n-1)b$.

Next suppose that we have $D \leq (n-1)a$. Then by Proposition 3.14, $u(x)$ is a $\Gamma$-invariant positive superharmonic and $\Delta u \leq D(D-(n-1)a)u$. It now follows from the hypothesis that $u$ is constant and that either $D = 0$ or $D = (n-1)a$. However, since $\Gamma$ is nonelementary and $[\mu_y]$ is $\Gamma$-invariant, Proposition 3.6 implies that $D \neq 0$. Hence, $D = (n-1)a$, and the result follows.

The next proposition was originally proved by Sullivan [11] using a Borel-Cantelli type of argument. The proof is purely measure theoretic (see [37], [46]). The proposition relates the ergodicity of $\Gamma$ with the divergence of the Poincaré series at $D$. 

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Proposition 3.16 (Sullivan). Suppose that $\Gamma$ is nonelementary, discrete and torsion-free, and is divergent at $D$. Then $\Gamma$ is ergodic with respect to $[\mu]^D$.

Proposition 3.17. Suppose $D = (n-1)a$ and there are no nontrivial positive superharmonic functions on $M$. Then $\Gamma$ is divergent.

Proof. Fix a point $y \in \hat{M}$. Let us assume the Poincaré series converges at $D$ (i.e. $\sum_{\gamma \in \Gamma} \exp(-D \text{dist}(x, \gamma y)) < \infty$). Then this series defines a nontrivial $\Gamma$-invariant function on $\hat{M}$. Let us denote this function by $h(x)$.

By Rauch’s theorem and $|\nabla \text{dist}_{\gamma y}(x)|^2 = 1$ we get

$$\Delta h(x) \leq \sum_{\gamma \in \Gamma} \exp(-D \text{dist}(x, \gamma y))D(D - (n-1)a),$$

which implies $ \Delta h \leq 0$. We consider the series $\sum_{\gamma \in \Gamma} \log \tanh\left(\frac{(n-1)a \text{dist}_{\gamma y}(x)}{2}\right)$. It is easy to see that the convergence of this series on the set of points bounded away from $\Gamma y$ follows from the convergence of the Poincaré series at $D = (n-1)a$. Then by direct computation and Rauch’s theorem we have $\Delta f(x) \leq 0$ for $x \in \hat{M}\setminus\Gamma y$. Hence $1 - \exp(-f(x))$ defines a nontrivial positive $\Gamma$-invariant superharmonic function on $\hat{M}$.

Therefore, the convergence of the Poincaré series at $D = (n-1)a$ give raise to contradictions to our hypothesis, and the result follows.

Corollary 3.18. Suppose $D = (n-1)a$ and there are no nontrivial positive superharmonic functions on $M$. Then, $\Gamma$ is ergodic with respect to $[\mu]^D$.

Proof. The corollary follows from Proposition 3.16 and Proposition 3.17. \qed
Corollary 3.19. Let $M = \tilde{M}/\Gamma$ be a topologically tame 3-manifold with $-b^2 \leq K \leq -1$ and $\Lambda(\Gamma) = S_\infty$. If $D = 2$, then $\Gamma$ is divergent, hence ergodic with respect to $[\mu]^D$.

Proof. The corollary follows from Theorem 2.4, Proposition 3.17 and Corollary 3.18.

Proof of Theorem 1.5. Under the hypothesis of Theorem 1.5, it follows from Proposition 3.15 that $D \in [2, 2b]$. That $\Gamma$ is harmonically ergodic follows from Proposition 3.12. If $D = 2$, then by Corollary 3.19 we have $\Gamma$ is divergent. □

4 Part I of Theorems 1.1 and 1.2

Let $\Gamma$ be a torsion-free discrete subgroup of $\text{ISO}(\tilde{M})$ with $D_\Gamma = 2$. We assume $\Gamma$ is either convex-cocompact or $\Lambda_\Gamma = S_\infty$, hausdorff-conservative and divergent.

Proposition 4.1. The measure $\mathcal{M}_{K_x}^2$ is finite and positive on all non-empty relative open subsets of $\Lambda_\Gamma$, and $\mathcal{M}_{K_x}^2(A) = 0$ if and only if $\mathcal{M}_{\eta_v}^2(A) = 0$ for $A \subset \Lambda_\Gamma \setminus v(-\infty)$.

Proof. First note that if we replace $d_{v,t}$ with $d$ in the definition of $\eta_v$, we get an equivalent metric by Lemma 4 in [26].

Let $x \in \tilde{M}$ be any point. Denote $H_{v,x}$ the horosphere tangent to $v(\infty)$ and passing through $x$. Take two vectors $U^\zeta, U^\xi$ in $S\tilde{M}$ that are asymptotic to $\zeta$ and $\xi$ respectively, we have $\text{dist}(g_t U^\zeta, g_t U^\xi) \leq \alpha e^{-t}$ and $\text{dist}(g_t U^\zeta, g_t U^\xi) \leq \alpha e^{-t}$, where $g_t$ is the flow. This gives $\text{dist}(g_t U^\zeta, g_t U^\xi) \leq 2\alpha e^{-\tau} + 1$ with $\tau = l_x(\zeta, \xi)$. On the other hand we also have $\beta^{-1} e^t \leq \text{dist}(g_t U^\zeta, g_t U^\xi) \leq \beta e^{bt}$ for some positive constant $\beta$, which gives $e^{-s} \geq \beta^{-1}$ and $e^{-s} \leq \beta^{1/b}$ when $\text{dist}(g_t U^\zeta, g_t U^\xi) = 1$. Hence $\frac{\beta^{-1} e^{bt}}{2e+1} e^{-s} \leq e^{-\tau} \leq \beta e^{-s}$. Therefore $\eta_v$ and $K_x$ are equivalent on all points in $S_{v,x}$, where $S_{v,x}$ is the shadow of $H_{v,x}$ casted from $v(-\infty)$. By compactness of $S_\infty$ there are $\{v_1, \ldots, v_n\} \subset SM$ such that $\cup_{i=1}^n S_{v_i} = S_\infty$. Since $0 < \mathcal{M}_{\eta_v}^2(S_{v,x} \cap \Lambda_\Gamma) < \infty$, we have $\mathcal{M}_{K_x}^2$ is positive and finite on $\Lambda_\Gamma$. It follows from Propositions 3.17, 3.3 and 3.9 the measure $\mathcal{M}_{K_x}^2$ is positive on all relative open subsets. Let $A \subset S_\infty \setminus v(-\infty)$ be a $\mathcal{M}_{\eta_v}^2$-null set. Let $\delta > 0.$
Note that\( \bigcup_{x \in M} S_{v,x} = S_{\infty} \setminus v(-\infty) \). Hence there is \( B \subset A \) with \( B \subset S_{v,z} \) such that \( M_{K_z}^2(A \setminus B) < \delta \). But \( M_{K_z}^2(B) \leq c M_{\eta}^2(B) \) for some \( c > 0 \). By finiteness we have \( M_{K_z}^2(A) < \delta \). Same argument holds for the rest of the proposition. \( \square \)

**Corollary 4.2.** The measures \( \mu_x \) and \( M_{\eta}^2 \) are absolutely continuous with respect to each other. In particular \( M_{\eta}^2 \) is supported on \( \Lambda_\Gamma \).

**Proof.** The result follows from Proposition 3.4, 3.5 and Proposition 4.1. \( \square \)

We use Mostow and Gehring’s original idea to show the regularity of quasiconformal map [36], [20]. This method was extended in [25]. We will follow their presentations, but with necessary generalizations that will allow us to prove our theorems using results from previous sections.

Take the unite ball model of \( \mathbb{H}^3 \). Let \( u \) be a unit tangent vector at the origin. Let \( O_u \) be the unit circle on \( \partial \mathbb{H}^3 = S^2 \) which is contained in the unique totally geodesic plane perpendicular to \( u \) and passing through the origin. Also denote the point \( u(\infty) \) on \( S^2 \) by \( \varsigma \). Then for any pair \((p, \varsigma) \in \mathbb{B}_u := O_u \times \varsigma \) there is a unique semi-circle connecting them. The bundle of all these semi-circles is the upper hemisphere \( \Omega_u \) of \( S^2 \). We denote this bundle space by \((\Omega_u, \pi_u, \mathbb{B}_u)\) where \( \pi_u \) is the projection.

Let \( \phi : S^2 \rightarrow S_{\infty} \) be a quasi-conformal embedding conjugate \( \Gamma' \) to \( \Gamma \) under isomorphism \( \chi : \Gamma' \rightarrow \Gamma \), here \( \Gamma' \) is a topologically tame, torsion-free, discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \) with \( \Lambda_{\Gamma'} = S^2 \). And let \( \psi \) be the inverse of \( \phi \) when it is a quasi-conformal homeomorphism.

Let \( \rho_u \) be the metric on \( S^2 \setminus u(-\infty) \) which is defined same as \( \eta_v \) with \( v(-\infty) = \phi(u(-\infty)) \). The hausdorff measure \( M_{\rho_u}^2 \) on \( S^2 \setminus u(-\infty) \) with respect to \( \rho_u \)-metric is the usual Lebesgue measure. Hence there exists a constant \( \omega > 0 \) such that for all \( \theta \in S^2 \setminus u(-\infty) \), we have \( M_{\rho_u}^2(B_{\rho_u}(\theta, r)) = \omega r^2 \).

**Proposition 4.3 (see [36], [25]).** The measure \( \phi^* \mathcal{M}^1_{\eta} \) is absolutely continuous with respect to measure \( \mathcal{M}_{\rho_u}^1 \) on semi-circles. Here \( \mathcal{M}^1_{\eta} \) and \( \mathcal{M}_{\rho_u}^1 \) are 1-dimensional hausdorff measures with respect to the \( \eta \)-metric and \( \rho_u \)-metric respectively.

**Proof.** Let \( \mathcal{L} \) be the Lebesgue measure on \( \mathbb{B}_u \). Then for all \( P \in \mathbb{B}_u \) we have the following derivative

\[
\lambda(P) := \lim_{r \to 0} \frac{M_{\eta}^2(\phi \circ \pi_u^{-1}(B_{\rho_u}(P,r) \cap \mathbb{B}_u))}{\mathcal{L}(B_{\rho_u}(P,r) \cap \mathbb{B}_u)}
\]
exists and finite for \( \mathcal{L} \)-almost everywhere, see \[18\].

Choose \( P \in \mathbb{B}_u \) with \( \lambda(P) < \infty \). For a semi-circle \( l := \pi_u^{-1}(P) \), let \( U_r(l) \) denote the \( r \)-neighborhood of \( l \), then \( \limsup_{r \to 0} \mathcal{M}_{\eta_u}^2(\bar{U}_r(l))/r < \infty \).

For any compact \( K \subset l \) with \( \mathcal{M}_{\rho_u}^1(K) = 0 \), choose a number \( C > 0 \) with \( \mathcal{M}_{\eta_u}^2(\bar{U}_r(l))/r < C \). Let \( \epsilon > 0 \) be given, by Besicovic’s covering theorem there exists \( \{\theta_1, \ldots, \theta_k\} \subset K \) such that \( kr < \epsilon \), \( K \subset \bigcup_k B_{\rho_k}(\theta_i, r) \) and any three of the balls \( B_{\rho_k}(\theta_i, r) \) with distinct centers are disjoint.

Let \( s_i := \inf\{s > 0|\bar{B}_{\rho_k}(\theta_i, r) \subset B_{\eta_k}(\bar{\phi}(\theta_i), s)\} \) and \( k > 0 \) (conformal constant) provided by Proposition \[3.1\] Then we have \( \bar{\phi}(K) \subset \bigcup_k B_{\eta_k}(\bar{\phi}(\theta_i), s_i) \), \( \bar{\phi}(S^2) \cap B_{\eta_k}(\bar{\phi}(\theta_i), s_i/\kappa) \subset \bar{\phi}(B_{\rho_k}(\theta_i, s_i)) \). Since \( \Gamma \) is hausdorff Conservative and by Proposition \[3.2\] Corollary \[4.2\], there exists \( \alpha > 0 \) such that

\[
\left( \sum_{i=1}^k s_i^2 \right)^2 \leq k \sum_{i=1}^k s_i^2 \leq k \kappa^2 \alpha \sum_{i=1}^k \mathcal{M}_{\eta_k}^2(\bar{\phi}(B_{\rho_k}(\theta_i, r)))
\]

\[
\leq 2k^2 \alpha k \mathcal{M}_{\eta_k}^2(\bar{\phi}(U_r(K))) \leq 2k^2 \alpha k \mathcal{M}_{\eta_k}^2(\bar{\phi}(U_r(l)))
\]

\[
\leq 2k^2 C \alpha (kr) \leq \text{const} \epsilon.
\]

Note the fact that any three of \( \bar{\phi}(B_{\rho_k}(\theta_i, r)) \) do not intersect is used to bound \( \sum_k \mathcal{M}_{\eta_k}^2(\bar{\phi}(B_{\rho_k}(\theta_i, r))) \) by \( 2\mathcal{M}_{\eta_k}^2(\bar{\phi}(U_r(K))) \).

Therefore the result follows from the last inequality.

The balls \( B_{\rho_k}(\theta, r), \theta \in S^2 \setminus u(-\infty), r > 0 \) form a Vitali relation for the Lebesgue measure \( \mathcal{M}_{\rho_u}^2 \). The following derivative

\[
J(\theta) := \lim_{r \to 0} \frac{\mathcal{M}_{\rho_u}^2(\bar{\phi}(B_{\rho_k}(\theta, r)))}{\mathcal{M}_{\rho_u}^2(B_{\rho_k}(\theta, r))}
\]

exists and finite for \( \mathcal{M}_{\rho_u}^2 \)-almost every \( \theta \in S^2 \setminus u(-\infty) \).

**Proposition 4.4.** Let \( \text{Lip}_u \) be defined by \( \text{Lip}_u : \theta \mapsto \limsup_{r \to 0} \rho_u(\theta, r)/r \).

Then \( \text{Lip}_u \in L_{\text{loc}}^2(S^2 \setminus u(-\infty), \mathcal{M}_{\rho_u}^2) \). In-fact there exists a constant \( k > 0 \) such that

\[
\sqrt{J(\theta)}/k \leq \liminf_{r \to 0} \rho_u(\theta, r)/r \leq \limsup_{r \to 0} \rho_u(\theta, r)/r \leq k\sqrt{J(\theta)}.
\]

**Proof.** Let \( \epsilon > 0 \). There is \( r_\epsilon > 0 \) such that for any \( r < r_\epsilon \) we have

\[
\omega f(\theta) r^2/2 \leq \mathcal{M}_{\eta_k}^2(\bar{\phi}(B_{\rho_k}(\theta, r))) \leq (2\omega f(\theta) + \epsilon)r^2.
\]
where the fact that $\mathcal{M}_{\rho_u}^2$ is Lebesgue measure, i.e. $\mathcal{M}_{\rho_u}^2(B_{\rho_u}(\theta, r)) = \omega r^2$ for some constant $\omega > 0$ has been used. Since $\Gamma$ is hausdorff-conservative and by Proposition 3.2, Corollary 4.2, there exists some constant $\alpha > 0$ such that

$$(r_\phi(\theta, r)/\beta)^2/\alpha \leq \mathcal{M}_{\eta}(\bar{\phi}(B_{\rho}(\theta, r))) \leq \alpha (r_\phi(\theta, r))^2.$$ 

Hence we have

$$\sqrt{\omega/2\alpha f(\theta)}r \leq r_\phi(\theta, r) \leq \sqrt{\alpha (2f(\theta)\omega + \epsilon))} \beta r$$

and the result follows by letting $\epsilon \to 0$. \hfill \Box

**Lemma 4.5.** The image under $\phi$ of almost every semi-circle has locally finite $\mathcal{M}_{\eta}$-measure.

**Proof.** Let $f : \Omega_u \to \mathbb{B}_u \times [0, 1]$ be a diffeomorphism which maps $\pi_u^{-1}(P)$ over $P$ onto $P \times [0, 1]$. For every compact subset $C \subset \Omega_u$ we can find a positive number $\alpha$ such that

- For all $x \in f(C)$ the Jacobian of $f^{-1}$ at $x$ are $< \alpha$,
- For all $P \in \mathbb{B}_u$ and $y \in \pi_u^{-1}(P) \cap C$ the local dilations at $y$ of $f|_{\pi_u^{-1}(P)}$ are $< \alpha$.

Since $\bar{\phi}$ is an embedding, $\bar{\phi}(\Omega_u)$ is relative compact subset of $S_\infty \setminus v(-\infty)$ and we have by Proposition 4.4, $\int_{\Omega_u} \text{Lip}_\phi^2 \text{d}\mathcal{M}_{\rho_u}^2 \leq k^2 \mathcal{M}_{\eta}(\bar{\phi}(\Omega_u)) < \infty$, and Hölder inequality gives $\int_{\Omega_u} \text{Lip}_\phi \text{d}\mathcal{M}_{\rho_u}^2 < \infty$. Hence

$$\int_{\mathbb{B}_u} \left( \int_{\pi_u^{-1}(P) \cap C} \text{Lip}_\phi \text{d}\mathcal{M}_{\eta}^1 \right) \text{d}\mathcal{L} \leq \alpha \int_{f(C)} \text{Lip}_\phi \circ f^{-1} \text{d}\mathcal{L} \text{d}t \leq \alpha^2 \int_C \text{Lip}_\phi \text{d}\mathcal{M}_{\rho_u}^2 < \infty$$

where $dt$ is Lebesgue measure on $[0, 1]$. Now by Proposition 4.3, $\bar{\phi}$ is absolutely continuous on $\pi_u^{-1}(P)$ therefore

$$\mathcal{M}_{\eta}(\bar{\phi}(\pi_u^{-1}(P) \cap C)) \leq \int_{\pi_u^{-1}(P) \cap C} \text{Lip}_\phi \text{d}\mathcal{M}_{\rho_u}^1 < \infty.$$ 

\hfill \Box
Next we adapt the idea in [25] to prove the inequality part of Theorems 1.1.

**Proof.** Part I of Theorems 1.1 and 1.2. For Theorem 1.2, the inequality follows from Theorem 1.3 and Proposition 3.3. Let \( \Gamma \) and \( \Gamma' \) be as in Theorem 1.1 and satisfies those conditions. Note that by Proposition 3.3, \( D_{\Gamma'} = 2 \).

Let \( g \) be the Riemannian metric of \( \tilde{M} \). Set \( h = (D_{\Gamma}/2)g \) as the new metric of \( \tilde{M} \). The boundary space of \( (\tilde{M},g) \) and \( (\tilde{M},h) \) can be trivially identified, and \( \eta_{(2/D_{\Gamma})v} = \eta_{v}^{D_{\Gamma}/2} \). The critical exponent of \( \Gamma \) with respect to \( h \) is 2, hence by Lemma 4.5 there is a non-trivial curve in \( S_{\infty} \setminus v(\infty) \) with finite \( D_{\Gamma}/2 \)-dimensional hausdorff measure with respect to \( \eta_{v} \). However as noted before the curvature assumption \( -b^2 \leq K \leq -1 \) of \( g \) implies the \( \eta_{v} \)-metric is a distance on \( S_{\infty} \setminus v(\infty) \), but the distance-hausdorff dimension is \( \geq 1 \) for any non-trivial curves. Therefore we have \( D_{\Gamma}/2 \geq 1 \).

**Lemma 4.6.** Let \( \Gamma' \) be a divergent, torsion-free discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \) with \( \Lambda_{\Gamma'} = S^2 \) and \( D_{\Gamma'} = 2 \). Then the maps \( \phi \) and \( \psi \) are absolutely continuous with respect to \( \sigma_{y} \) and \( \mu_{x} \).

**Proof.** By ergodicity of \( \Gamma \), \( \Gamma' \) and equivariance of \( \bar{\phi}, \bar{\psi} \) and also Proposition 4.2, it suffices to show there exists a \( A \subset S^2 \setminus u(\infty) \) with \( \mathcal{M}_{\rho_{u}}^{2}(A) > 0 \) such that the Radon-Nikodym derivative of \( \bar{\phi} \) at every \( x \in A \) with respect to \( \mathcal{M}_{\rho_{u}}^{2} \) and \( \mathcal{M}_{\eta_{v}}^{2} \) is non-zero. Using the fact that \( \eta_{v} \) is a distance function, it follows from Proposition 1.3, for \( \mathcal{L} \)-almost all \( P \in \mathbb{B}_{u} \) the length of \( \bar{\phi}(\pi_{u}^{-1}(P)) > 0 \) is bounded by \( \int_{\pi_{u}^{-1}(P)} \text{Lip}_{\phi} \, d\mathcal{M}_{\rho_{u}}^{1} \). Hence if we set \( A := \{ x \in \Omega_{u} | \text{Lip}_{\phi}(x) > 0 \} \), then for \( \mathcal{L} \)-almost all \( P \in \mathbb{B}_{u} \), \( \mathcal{M}_{\rho_{u}}^{1}(\pi_{u}^{-1}(P) \cap A) > 0 \) which implies \( \mathcal{M}_{\rho_{u}}^{2}(A) > 0 \). Therefore the result follows from Proposition 4.4. \( \square \)

5 Part II of Theorems 1.1 and 1.2

Let \( \xi_{1}, \xi_{2}, \xi_{3}, \xi_{4} \in S_{\infty} \). The **cross-ratio** \( |\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}| \) of these four points is defined as

\[
|\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}| := \frac{e^{-\beta_{x}(\xi_{1}, \xi_{2})}e^{-\beta_{x}(\xi_{3}, \xi_{4})}}{e^{-\beta_{x}(\xi_{1}, \xi_{3})}e^{-\beta_{x}(\xi_{2}, \xi_{4})}}.
\]

This definition is consistent with the hyperbolic space cross-ratio.

If \( \Gamma_{1}, \Gamma_{2} \) are discrete subgroups of \( \tilde{M} \) such that both \( \Gamma_{1}, \Gamma_{2} \) are divergent, and there exists a equivariant (under some group morphism \( \chi \)), nonsingular
(with respect to $\mu_1, \mu_2$ Patterson-Sullivan measures on $\Lambda_{\Gamma_1}$ and $\Lambda_{\Gamma_2}$ respectively), measurable map $f : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$. Then

$$d(f \times f)^* \Pi_2(\xi, \zeta) = e^{-D_{\Gamma_2} \beta_2(f\xi, f\zeta)} g(\xi)g(\zeta)d\mu_1(\xi)d\mu_2(\zeta)$$

where $g := \frac{df^*(\mu_2)}{d\mu_1}$, and $\Pi_i$ is the measure defined in §3 through $\mu_i$. From the properties of $f$, $(f \times f)^* \Pi_2$ is a constant $a > 0$ multiple of $\Pi_1$. Hence $e^{D_{\Gamma_2} \beta_2(f\xi, f\zeta)} g(\xi)g(\zeta) = ae^{D_{\Gamma_1} \beta_1(\xi, \zeta)}$. Therefore for $\mu_1$-almost everywhere we have

$$|f(\xi_1), f(\xi_2), f(\xi_3), f(\xi_4)| = |\xi_1, \xi_2, \xi_3, \xi_4|^{D_{\Gamma_1}/D_{\Gamma_2}}.$$

This was the idea of Sullivan for the following lemma:

**Lemma 5.1.** Let $\Gamma_1, \Gamma_2$ be discrete subgroups of $\text{ISO}(\tilde{M})$ with $D_{\Gamma_1} = D_{\Gamma_2}$ and $\Gamma_1, \Gamma_2$ are divergent. Suppose there exists an equivariant nonsingular measurable map $f : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ with respect to Patterson-Sullivan measures space $(\Lambda_{\Gamma_1}, \mu_1)$ and $(\Lambda_{\Gamma_2}, \mu_2)$. Then $f$ preserves cross-ratio $\mu_1$-almost everywhere.

For a finitely generated discrete subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{C})$. The **conservative set** of $\Gamma$ on $S^2$ coincides with $\Lambda_{\Gamma}$ up-to Lebesgue measure zero. The group $\Gamma$ is called **conservative** if and only if $\Lambda_{\Gamma}$ has full Lebesgue measure. Since for a topologically tame $\Gamma$, the Hausdorff dimension of $\Lambda_{\Gamma}$ is equal to $D_{\Gamma}$, therefore we have the following:

**Proposition 5.2.** Let $\Gamma$ be a topologically tame, torsion-free discrete subgroup of $\text{PSL}(2, \mathbb{C})$ with conservative $\Gamma$, then $\Gamma$ is Hausdorff-conservative.

**Remark 5.3.** It is a conjecture that all finitely generated discrete subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{C})$ are topologically tame.

Next we recall the statement of Sullivan’s quasi-conformal stability for discrete subgroups of $\text{PSL}(2, \mathbb{C})$.

**Theorem 5.4 (Sullivan [13]).** Let $\Gamma$ be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Then $\Gamma$ is quasi-conformally stable (i.e. if $f$ is a quasi-conformal automorphism of $S^2$ with $f\Gamma f^{-1} \subset \text{PSL}(2, \mathbb{C})$, then $f$ is a Möbius transformation) if and only if $\Gamma$ is conservative.

**Corollary 5.5.** Let $N = \mathbb{H}^3/\Gamma$ be a complete hyperbolic 3-manifold for a conservative $\Gamma$. Then $N$ is quasi-isometrically stable, i.e. If there is a quasi-isometric homeomorphism $h : N \to M$ to a hyperbolic manifold $M$, then $N$ is isometric to $M$. 

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Proof. Theorem 1.3 part II. By Theorem 1.2, $\Gamma$ is divergent for $D_{\Gamma} = 2$. From Proposition 3.3, $\Gamma'$ is also divergent and $D_{\Gamma'} = 2$. Lemma 1.6 then implies $f$ is absolutely continuous with respect to $\sigma_y$ and $\mu_x$. Hence by Lemma 5.1, $f$ preserves cross ratio $\sigma_y$-everywhere. By Proposition 3.2, $\Lambda_{\Gamma'} = S^2$ and since $\sigma_y$ is non-zero constant multiple of Lebesgue measure, we can modify $f$ on the Lebesgue measure null subset of $S^2$ to a map which is cross ratio preserving on $S^2$. We denote the new map also by $f$. By Bourdon’s theorem 4.4, $f$ extends into the space as an isometry, i.e. $\mathbb{H}^3$ and $\tilde{M}$ are isometric. Hence the result follows from Theorem 5.4.

Proof. Theorems 1.1 part II. Here $f$ embeds $S^2$ into $S^\infty$. If we suppose $D_{\Gamma} = D_{\Gamma'} = 2$, then by using same argument as the proof of Theorem 1.2, $f$ extends to a isometric embedding of $\mathbb{H}^3$ into $\tilde{M}$ by 4.4. Since $f(S^2)$ is a $\Lambda_{\Gamma}$-invariant closed subset of $S^\infty$, by Proposition 3.2, $f(S^2) = \Lambda_{\Gamma}$. Hence the boundary space of the isometric embedded image of $\mathbb{H}^3$ coincides with $\Lambda_{\Gamma}$, therefore the result follows.

Proof. Corollary 1.3. This follows from Propositions 3.1, 3.3 and Theorem 1.2.

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