Quantitative $W_1$ Convergence of Langevin-Like Stochastic Processes with Non-Convex Potential State-Dependent Noise

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Abstract

We prove quantitative convergence rates at which discrete Langevin-like processes converge to the invariant distribution of a related stochastic differential equation. We study the setup where the additive noise can be non-Gaussian and state-dependent and the potential function can be non-convex. We show that the key properties of these processes depend on the potential function and the second moment of the additive noise. We apply our theoretical findings to studying the convergence of Stochastic Gradient Descent (SGD) for non-convex problems and corroborate them with experiments using SGD to train deep neural networks on the CIFAR-10 dataset.

1 Introduction

Stochastic Gradient Descent (SGD) is one of the workhorses of modern day machine learning. In many nonconvex optimization problems, such as training deep neural networks, SGD is able to produce solutions with good generalization error. Further, there is evidence that the generalization error of an SGD solution can be significantly better than Gradient Descent (GD) [12]. This suggests that, to understand the behavior of SGD, it is not enough to consider the limiting cases (such as small step-size or large batch-size), when it degenerates to GD. We take an alternate view of SGD as a sampling algorithm, and aim to understand its convergence to an appropriate stationary distribution.

There has been rapid recent progress in understanding the finite-time behavior of MCMC methods, by comparing them to stochastic differential equations (SDEs), such as the Langevin diffusion. It is natural in this context to think of SGD as a discrete time approximation of an SDE. But there are two significant barriers to extending previous analyses to the case of SGD, because those analyses are mostly restricted to isotropic Gaussian noise. First, the noise in SGD can be far from Gaussian. For instance, sampling from a minibatch leads to a discrete distribution. Second, the noise depends significantly on the current state (the optimization variable). For instance, if the objective is an average over a training sample of a non-negative loss, as the objective approaches zero, the noise variance of minibatch SGD goes to zero. Any attempt to cast SGD as an SDE must thus be able to handle this kind of noise.

This motivates the study of Langevin MCMC-like methods that have a state-dependent noise term:

\[ w_{(k+1)\delta} = w_{k\delta} - \delta \nabla U(w_{k\delta}) + \sqrt{\delta} \xi(w_{k\delta}, \eta_k), \]

where \( w_t \) is the state variable at time \( t \), \( \delta \) is the step-size, \( U : \mathbb{R}^d \to \mathbb{R} \) is a potential, \( \xi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) is the noise function, and \( \eta_k \) are sampled iid from some set \( \Omega \) (for example, in minibatch SGD, \( \Omega \) is the set of subsets of indices in the training sample. We discuss the SGD example in more detail in Section 6).

Throughout this paper, we assume that \( \mathbb{E}_\eta [\xi(x, \eta)] = 0 \) for all \( x \). We define a matrix-valued function \( M(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) to be the square root of the covariance matrix of \( \xi \), i.e. for all \( x \),

\[ M(x) := \sqrt{\mathbb{E}_\eta [\xi(x, \eta)\xi(x, \eta)^T]}. \]

Where for a symmetric positive semidefinite matrix \( G \), \( A = \sqrt{G} \) is the unique symmetric positive semidefinite matrix such that \( A^2 = G \).

Given the above definition of \( M \), it is natural to consider the following stochastic process:

\[ dy_t = -\nabla U(y_{k\delta})dt + \sqrt{\delta} M(y_{k\delta})dB_t \quad t \in [k\delta, (k+1)\delta] \]
It can be verified that at discrete time intervals, (2) is equivalent to $y_{k+1} = y_k - \delta \nabla U(y_k) + \sqrt{\delta} M(y_k) \theta_k$ Where $\theta_1, \ldots, \theta_k \sim \mathcal{N}(0, I)$. Thus (2) is essentially (1) with $\xi(x, \eta_k)$ replaced by the simpler $M(x) \theta_k$.

Finally, we can consider the continuous-time limit of (2):

$$dx_t = -\nabla U(x_t) dt + M(x_t) dB_t$$

(3)

We will let $p^*$ denote the invariant distribution of (3). In Theorem 1, we establish quantitative rates at which that (2) converges to $p^*$. In Theorem 2, we establish quantitative rates at which (1) also converges to $p^*$. Notice in particular that the only property of the SDE (3) that corresponds to the noise function in (1) is the covariance function $M$.

Our contributions are as follows:

1. We prove a quantitative convergence rate for (2) to $p^*$ of (3) in Theorem 2. This is the first quantitative rate for overdamped Langevin MCMC when the diffusion matrix is state-dependent. Our rate is comparable to earlier work with similar assumptions of nonconvexity, but assuming constant diffusion matrix [2, 4, 15].

2. We prove a quantitative convergence rate for (1) to $p^*$ of (3) in Theorem 2. Prior to this work, convergence of processes of the form has only been established in [3] under much more restrictive convexity assumptions.

3. Based on our theory, we describe a "large-noise" version of SGD and empirically evaluate its generalization performance. See Section 6.

2 Related work

There has been a long line of work in the study of convergence properties of stochastic processes. We review the ones most relevant to our work here.

Recent work on non-asymptotic convergence rates of Langevin MCMC algorithms began with [5] and [6], which established quantitative rates for Unadjusted Langevin MCMC under log-concavity assumptions (i.e., (2) with convex $U(x)$ and $M(x) = c \cdot I$ for some constant $c$). Another line of work [7, 8, 2, 4, 15] analyzed the convergence of MCMC algorithms under nonconvexity assumptions. In particular, [4] and [15] considered the Overdamped Langevin MCMC algorithm under similar assumptions as Assumption A, but still assuming $M(x) = c \cdot I$ for some constant $c$. Finally, [3] analyzed the convergence of (1) to (3) under much more restrictive assumptions of convexity of $U(x)$. In addition, [3] requires that (1) be a contractive process. In this paper, we show that convergence is possible as long as $M(x)$ is sufficiently regular and is lower-bounded globally by some constant. Following our presentation of Theorem 1 and 2, we compare our result with some of the above mentioned results.

On the other hand, authors of [16, 11] have drawn connections between SGD and a SDE of the form (3). Furthermore, [1] proved quantitative rates at which iterates of (rescaled) SGD approaches a normal distribution, assuming strong convexity around a local minima. In the specific setting of deep learning, authors of [12, 11, 13] studied the generalization properties of SGD in deep learning. In particular noted that the generalization error of SGD in deep learning seems strongly correlated with the magnitude of the noise covariance, and suggested that this may be explained by considering the underlying SDE.

3 An illustration of the importance of considering inhomogeneity

To motivate further discussion, we present a simple example to illustrate how $M(x)$ can significantly skew the invariant distribution of (3) away from $e^{-U(x)}$.

Figure 1: 1-dimensional example exhibiting the importance of inhomogeneity: A simple construction showing how $M(x)$ can affect the shape of the invariant distribution. While $U(x)$ has two local minima, $-\log(p^*(x))$ only has the smaller minima at $x = -2$. 1(d) represents samples obtained from simulating using the process (2).
We will define the potential $U(x)$ and the diffusion function $M(x)$ as

$$
U(x) := \begin{cases}
\frac{1}{2}x^2, & \text{for } x \in [-1, 4] \\
\frac{1}{4}(x + 2)^2 - 1, & \text{for } x \leq -1 \\
\frac{1}{2}(x - 8)^2 - 16, & \text{for } x \geq 4
\end{cases}
M(x) = \begin{cases}
\frac{1}{2}(x + 2), & \text{for } x \in [-2, 8] \\
1, & \text{for } x \in [-2] \\
6, & \text{for } x \geq 8
\end{cases}
$$

A plot of $U(x)$ can be found in Figure 1a. We highlight the fact that $U(x)$ is constructed to have two local minima: one shallow minimum at $x = -2$, and one deeper minimum at $x = 8$. A plot of $M(x)$ can be found in Figure 1b. $M(x)$ is constructed to have increasing magnitude at larger values of $x$.

Remarkably, $-\log p^*(x)$ is has only one local minima at $x = -2$. The larger minima of $U(x)$ at $x = 8$ was completely smoothed over by the effect of the large diffusion $M(x)$. This is very different from when the noise is homogeneous (e.g. $M(x) = I$), in which case $p^*(x) \propto e^{-U(x)}$. We also simulate (3) (using (2)) for the given $U(x)$ and $M(x)$ for 1000 samples (each simulated for 1000 steps), and plot the histogram in Figure 1d.

4 Assumptions and definitions

In this section, we state the assumptions and definitions that would be required for our main theoretical results in Theorem 1 and Theorem 2.

4.1 Assumptions

**Assumption A** We assume that $U(x)$ satisfies

1. The function $U(x)$ is continuously-differentiable on $\mathbb{R}^d$ and has Lipschitz continuous gradients; that is, there exists a positive constant $L > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$
\|\nabla U(x) - \nabla U(y)\|_2 \leq L\|x - y\|_2.
$$

2. The function $U(x)$ has a stationary point at zero: $\nabla U(0) = 0$.

3. There exist constants $m, R > 0$ such that for all $x, y \in \mathbb{R}^d$ with $\|x - y\|_2 > R$, we have:

$$
\langle \nabla U(x) - \nabla U(y), x - y \rangle \geq m\|x - y\|_2^2.
$$

in addition, for all $x, y$ with $\|x - y\|_2 \leq R$, we have:

$$
\|\nabla U(x) - \nabla U(y)\|_2 \leq L_R\|x - y\|_2.
$$

**Assumption B** We make the following assumptions on $\xi$ and $M$:

1. For all $x$, $\xi(x, \eta)$ has zero mean, i.e. $E_{\eta \sim q(\cdot)}[\xi(x, \eta)] = 0$

2. For almost sure $\eta$, and for all $x$, $\|\xi(x, \eta)\|_2 \leq \beta$

3. For almost sure $\eta$, and for all $x, y$, $\|\xi(x, \eta) - \xi(y, \eta)\|_2 \leq L_\xi\|x - y\|_2$

4. There is a positive constant $c_M$ such that for all $x$, $2c_M < M(x)$

**Remark 1** We discuss these assumptions in a specific setting at the end of Section 6.1.

In the rest of this paper, we will define a matrix valued function $N(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$:

$$
N(x) := \sqrt{M(x)^2 - c_M^2 I}
$$

From Assumption A, we can prove that $N(x)$ and $M(x)$ are bounded and Lipschitz. See Lemma 3 and 4 in Appendix C for details. These properties will be crucial in ensuring convergence.
4.2 Definitions and notation

For a $k^{th}$-order tensor $G \in \mathbb{R}^d$ and a vector $v \in \mathbb{R}^d$, we define the product $A = Gv$ such that $[A]_{i_1 \ldots i_k} = \sum_{j=1}^d [M]_{i_1 \ldots i_{k-1}j} \cdot v_j$.

For a tensor, we use $\|G\|_2$ to denote the operator norm:

$$\|G\|_2 = \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \|Mv\|_2.$$  

It can be verified that for all $k$, $\| \cdot \|_2$ is a norm over $\mathbb{R}^{dk}$. Furthermore, when $G \in \mathbb{R}^d$, this is the Euclidean norm, and when $G \in \mathbb{R}^{d \times d}$, this is the largest singular value.

We use the notation $\langle \cdot \rangle$ to denote both tensor and matrix inner products.

1. For vectors $u, v \in \mathbb{R}^d$, $\langle u, v \rangle = \sum_{i=1}^d u_i v_i$ (the dot product).

2. For matrices $A, B \in \mathbb{R}^{d \times d}$, $\langle A, B \rangle := \sum_{i=1}^d \sum_{j=1}^d A_{ij} B_{ji}$ (the trace inner product).

We will use $\otimes$ to denote outer product. For two vectors $u, v$, $A = u \otimes v$ means that $A_{ij} = u_j v_i$. We extend this notation to matrix vector outer products: $A = M \otimes v$ if $A_{ij} = G_{ij} v_i$, and similarly $A = v \otimes G$ if $A_{ij} = v_i M_{ij}$.

We will use $\otimes^2$ to denote $u \otimes u$ and $\otimes^3$ to denote $u \otimes u \otimes u$.

Given any distribution $p \in \mathcal{P}(\Omega)$ and $q \in \mathcal{Q}(\Omega)$, a joint distribution distribution $\zeta \in \mathcal{P}(\Omega \times \Omega)$ is a coupling between $p$ and $q$ if its marginals are equal to $p$ and $q$ respectively. Let $\Gamma(p, q)$ denote the space of all couplings between $p$ and $q$. Then the 1-Wasserstein distance is defined as

$$W_1(p, q) = \min_{\zeta \in \Gamma} \mathbb{E}_{(x,y) \sim \zeta} \|x - y\|_2$$

(5)

Our proof centers around a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$, defined in Appendix 2. Intuitively, $f(x-y)$ plays the role of the metric $\|z\|_2$. We will define a $f$-Wasserstein distance as follows:

$$W_f(p, q) = \min_{\zeta \in \Gamma} \mathbb{E}_{(x,y) \sim \zeta} [f(x - y)].$$

(6)

We overload the notation and sometimes use $W_f(x, y)$ for random variables $x$ and $y$ to denote the $W_f$ distance between their distributions. We will prove our convergence results in $W_f$, which then implies a convergence in $W_1$ by using Lemma 15.4.

Given a random variable $x$, we use $\text{Law}(x)$ to denote its distribution.

5 Main results

In this section, we present our main convergence results beginning with convergence under Gaussian noise in Section 5.1 and proceeding to the non-Gaussian case in Section 5.2.

5.1 Convergence under Gaussian noise

In this section, we study the convergence rate of (2) to $p^*$ (invariant distribution of (3)). We will assume that $U$, $\xi$ and $M$ satisfy Assumptions A and B.

Theorem 1 Let $\hat{\epsilon}$ be some target accuracy satisfying $\epsilon \leq R$. Let us define

$$\alpha_\delta := \frac{L_R + L^2_\xi}{2L^3} \quad R_\xi := \max \left\{ R, \frac{16\beta^2 L N}{c_\delta} \right\} \quad \lambda := \min \left\{ \frac{m}{2}, \frac{2\lambda_0^2}{28 L^2}, \epsilon + \frac{\lambda_0^2 R_\xi^2}{\lambda_0^2} \right\} \quad \delta := \min \left\{ \frac{m}{8L^2}, \frac{2\epsilon}{32 L^2}, \frac{\epsilon^2}{8\delta^2} \right\} \quad k := \frac{1}{\lambda^3} \log \left( \frac{64A \sqrt{R^2 + \beta^2}}{\beta L_N} \right)$$

Let $\hat{y}_0 \sim N(\bar{y}_0, \beta^2/m)$, for any $\|\hat{y}_0\|_2 \leq 2R$. Let $\hat{y}_k$ be as defined in (2), initialized at $\hat{y}_0$, and let $\hat{p}_k^\beta = \text{Law}(\hat{y}_k)$. Then, we have,

$$W_1(p^*, \hat{p}_k^\beta) \leq \hat{\epsilon},$$

where $p^*$ is the invariant distribution to (3).

Remark 2 For ease of reference: $m, L, R$ are from Assumption A, $L_N = \frac{4L^3}{c_M}$ is from Lemma 4, $c_M, \beta, L_\xi$ are from Assumption B.
Remark 3 Finding a suitable $y_0$ can be done very quickly using gradient descent wrt $U(\cdot)$. The convergence rate of $c$ to the ball of radius $R$ is very fast, due to Assumption A.

Remark 4 In a nontrivial instance, $\epsilon = \hat{c} \cdot \frac{\lambda}{14(L + L_{\xi}^2) e^{3\alpha_q R_q^2}}$.

Theorem 1 is saying that one can get $\epsilon$ error in $W_1$ between $y_{k\delta}$ from (2) and $p^*$ after $k$ steps, i.e. the convergence is given by $k$.

Substituting in the parameters, and after some algebra, we see that for a sufficiently small $\epsilon, \delta = \epsilon^2/\beta^2$, and

$$k = \tilde{O}\left(\max\{L^2/m, \beta^2\} \cdot \exp\left(9 \cdot \frac{L_R}{2c_M} + \frac{8\beta^2 L_{\xi}^2}{c_M^4} \right) \cdot \max\left\{R^2, \frac{2\epsilon^2 \beta^2 L_{\xi}^2}{m^2 c_M^2}\right\}\right)$$

Remark 5 As illustrated in Section 6.2, the $m$ from Assumption B.3 should be thought of as a regularization term which is can be set arbitrarily large. In the following discussion, we will assume that $\epsilon^{12} \beta^2 R_{\xi}^2 / m^2 c_M^2$ is dominated by the $R^2$ term.

To gain intuition about this term, let’s consider what it looks like under a sequence of increasingly weaker assumptions:

a. Strongly convex, Gaussian noise: $U(x)$ $m$-strongly convex, $\xi(x, \eta) \sim \mathcal{N}(0, I)$ for all $x$. In (reality we need to consider a truncated Gaussian so as not to violate Assumption B.2, but this is a minor issue). In this case, $L_{\xi} = 0, c_M = 1, R = 0, \beta = \tilde{O}(\sqrt{d})$, so $k = \tilde{O}(\frac{1}{\epsilon^2})$. This is the same rate as obtained in [3]. However, [3] gets a $W_2$ bound which is stronger than $W_1$ bound.

b. Non-convex, Gaussian noise: $U(x)$ not strongly convex but satisfies Assumption A [2] and $\xi(x, \eta) \sim \mathcal{N}(0, I)$. In this case, $L_{\xi} = 0, c_M = 1, \beta = \tilde{O}(\sqrt{d})$ This is the setting studied by [4] and [15]. The rate we recover is $k = \tilde{O}\left(\frac{d}{\epsilon^2} \cdot \exp\left(\frac{9}{2} LR^2\right)\right)$, which is in line with [4], and is the best $W_1$ rate obtainable from [15].

c. Non-convex, Inhomogenous noise: $U(x)$ satisfies Assumption A and $\xi$ satisfies Assumption B. To simplify matters, suppose the problem is rescaled so that $c_M = 1$. Then the main additional term compared to setting 2. above is $\exp\left(\frac{72\beta^2 L_{\xi}^2 R_{\xi}^2}{c_M^3}\right)$. This seems to suggest that the effect of a $L_{\xi}$-Lipschitz noise can play a similar role in hindering mixing as a $L_{\xi}$-Lipschitz nonconvex drift.

In the case when dimension is high, computing $M(y)$ could be difficult, but if for each $x$, one has access to samples whose covariance is $M(x)$, then one can approximate $M(y)\theta_k$ via the Central Limit Theorem (e.g. [3]) by drawing a sufficiently large number of samples. The proof of Theorem 1 can be modified relatively easily to accommodate this. We discuss this in further detail in Appendix A.

5.2 Convergence under non-Gaussian noise

In this section, we prove the convergence of (1) to the invariant distribution $p^*$ of (3). We will assume that $U, \xi$ and $M$ satisfy Assumptions A and B.

Theorem 2 Let $\epsilon$ be some target accuracy satisfying $\epsilon \leq R$. Let us define

$$\alpha_q := \frac{L_R + L_{\xi}}{2c_M} \quad \epsilon_q := \epsilon \cdot \frac{\lambda}{14(L + L_{\xi}^2) e^{3\alpha_q R_q^2}} \quad \mathcal{R}_q := \max\{R, \frac{16\beta^2 L_{\xi}^2}{m^2 c_M}\} \quad \lambda := \min\{\frac{m^2}{2}, \frac{2\epsilon^2}{3\alpha_q R_q^2}\} \quad \delta := \frac{1}{\epsilon^2} (e^8 \log \frac{1}{\epsilon})$$

Where $C_3 = \text{poly}(L, R, 1/m, \beta, L_{\xi})$ is some universal constant specified in the proof.

Let $\tilde{w}_0 \sim \mathcal{N}(\tilde{w}_0, \beta^2/m)$, for any $\|\tilde{w}_0\|_2 \leq 2R$. Let $\tilde{w}_1$ have dynamic as defined in (2) and let $\tilde{p}^{\infty}_{\delta}$ denote its distribution. Then for

$$\tilde{n} := \frac{1}{\lambda \delta} \log \left(\frac{16\lambda \sqrt{R^2 + \beta^2/m}}{6(L + L_{\xi}^2) \epsilon}\right),$$

we get $W_1(p^*, \tilde{p}^{\infty}_{\delta}) \leq \epsilon$

Remark 6 For a desired accuracy $\epsilon$, the number of steps needed is of order $\tilde{n} = \tilde{O}\left(\frac{1}{\epsilon^2} \cdot e^{27\alpha_q R_q^2}\right)$. The $\epsilon$ dependence is considerably worse than in Theorem 1. This is because we need to take many steps of (1) in order to approximate a single step of (2). For details, see the coupling construction in Equations (28) - (31).

In [3], the authors proved a convergence result of similar flavor, i.e. a sequence of the form (1) converges to $p^*$ of (3). The $\epsilon$ dependence in their paper is $1/\epsilon$. This is faster than our rate, but their proof made a number of much stronger assumptions. In particular, they assumed that $U$ is strongly convex, and (1) is contractive.
6 Application to stochastic gradient descent

6.1 SGD as SDE

In this section, we will try to cast SGD in the form of \( \eta \). We will consider an objective of the form

\[
U(w) = \frac{1}{n} \sum_{i=1}^{n} U_i(w)
\]

(8)

. We reserve the letter \( \eta \) to denote a random batch from \{1..n\}, sampled with replacement (will specify the batch size as needed). We will define \( \zeta(w, \eta) \) as follows

\[
\zeta(w, \eta) := \frac{1}{|\eta|} \sum_{i \in \eta} \nabla U_i(w) - \nabla U(w)
\]

For a single sample, i.e. \(|\eta| = 1\), we define

\[
H(w) := \mathbb{E} \left[ \zeta(w, \eta) \zeta(w, \eta)^T \right]
\]

(9)

I.e. \( H(x) \) is the covariance matrix of a single sampled gradient minus the true gradient.

A standard run of SGD, with minibatch size \( b \), has the following form:

\[
w_{k+1} = w_k - \delta \frac{1}{|\eta_k|} \sum_{i \in \eta_k} \nabla U_i(w_k)
\]

\[
= w_k - \delta \nabla U(w_k) + \sqrt{\delta} \left( \sqrt{\delta} \zeta(w, \eta_k) \right).
\]

(10)

Notice that (10) is in the form of (1), with \( \xi(w, \eta) = \sqrt{\delta} \zeta(w, \eta) \). The covariance matrix of the noise term is \( \mathbb{E} \left[ \xi(w, \eta) \xi(w, \eta)^T \right] = \frac{\delta}{|\eta_k|} \cdot H(w)^2 \).

Because the magnitude of the noise covariance scales with \( \sqrt{\delta} \), it follows that as \( \delta \to 0 \), (10) converges to the deterministic gradient flow ODE.

However, the loss of randomness as \( \delta \to 0 \) might not be desirable. It has been observed in certain cases that as SGD approaches GD, through either small step-size or large batch-size, the generalization error goes up [11]. In Section 6.3.1, we present a set of empirical results to support this claim.

We argue that the right way to take the limit of SGD is to fix the \( \sqrt{\delta} \) term in (10). Specifically we define the continuous limit of (10) as

\[
dx_t = -\nabla U(x_t)dt + \sqrt{\frac{\delta}{b}} H(x_t)dB_t
\]

(11)

(Recall that \( b := |\eta_k| \) in (10)). Notice that the above is similar to (3), with \( M(x) = \sqrt{\frac{\delta}{b}} H(x) \), which matches the covariance matrix of (10). Our definition is thus motivated by Theorem 2, which states that (1) converges to (3).

Let \( s \) be some stepsize, and let \( \sigma \) be an arbitrary constant. Consider the following stochastic sequence:

\[
\hat{w}_{k+1}^s = \hat{w}_k^s - \delta \frac{1}{|\eta_k|} \sum_{i \in \hat{\eta}_k} \nabla U_i(x) + \sigma \sqrt{s} \left( \frac{1}{|\eta_k|} \sum_{i \in \hat{\eta}_k} \nabla U_i(x) - \frac{1}{|\hat{\eta}_k|} \sum_{i \in \hat{\eta}_k} \nabla U_i(x) \right)
\]

\[
= \hat{w}_k^s - \delta \nabla U(\hat{w}_k^s) + s \zeta(\hat{w}_k^s, \hat{\eta}_k) + \sigma \sqrt{s} \left( \zeta(\hat{w}_k^s, \hat{\eta}_k) - \zeta(\hat{w}_k^s, \hat{\eta}_k) \right)
\]

(12)

Where \( \hat{\eta}_k, \hat{\eta}_k^2, \hat{\eta}_k^3 \) mini-batches, sampled iid and with replacement, and \(|\hat{\eta}_k| = |\hat{\eta}_k^2| \). Intuitively, in addition to the SGD noise, we inject additional noise by adding the difference between two independently sampled mini-batches.

We first note that (12) is in the form of (1), with

\[
\xi(w, \eta) = \sqrt{s} \zeta(\hat{w}_k^s, \hat{\eta}_k) + \sigma \left( \zeta(\hat{w}_k^s, \hat{\eta}_k) - \zeta(\hat{w}_k^s, \hat{\eta}_k) \right)
\]

(13)

The noise covariance matrix is \( \mathbb{E} \left[ \xi(w, \eta) \xi(w, \eta)^T \right] = \left( \frac{s}{|\hat{\eta}_k|} + \frac{2 \sigma^2}{|\hat{\eta}_k^2|} \right) H(w)^2 \).

If we pick

\[
\sigma = \sqrt{\frac{\delta}{b} - \frac{s}{|\hat{\eta}_k^2|}} \cdot \frac{|\hat{\eta}_k^2|}{2},
\]

(14)
then we guarantee that $E[\xi(w, \eta)\xi(w, \eta)^T] = \frac{s}{m}H(w)^2$, which matches (11). By Theorem 2 for sufficiently small $s$ and sufficiently large $k$, $\hat{w}_k$ of (12) converges to the invariant distribution of (11).

We stress that we are not proposing (12) as a practical algorithm. The reason that (12) is interesting is that it gives us a family of $(\hat{w}_k)_{k \in \mathbb{R}_+}$ which converges to (11), and is implementable in practice. In section 6.3.2 we implement and (12) evaluate its performance. From the experiments, it appears that (12) has similar test accuracy to vanilla (10) with step-size $\delta$. We thus hypothesize that the test accuracy depends largely on the shape and scale of the noise covariance matrix, which implies that the generalization properties of (10) for large $\delta$ should extend to its limit (11).

We remark that (10) proposed a different way of injecting noise, multiplying the sampled gradient with a suitably scaled Gaussian noise.

### 6.2 Satisfying Assumptions in Section 4.1

Finally, we remark on how the function $U(w)$ defined in (8), along with the stochastic sequence $\hat{w}_k$ defined in (12) can satisfy the assumptions in Section 4.1.

First, let us additionally assume that for each $i$, $U_i(w)$ has the form

$$U_i(w) = U'_i(w) + V(w)$$

Where $V(w) := m(\|x\|_2 - R/2)^2$ is a $m$-strongly convex regularizer outside a ball of radius $R$. $U'_i(w)$ has a minima at 0 and has $L_R$-Lipschitz gradients. Suppose further that $m \geq 4 \cdot L_R$. These additional assumptions make sense when we are only interested in $U(w)$ over $B_R(0)$, so $V(w)$ plays the role of a function that keeps us within $B_R(0)$.

It can immediately be verified that $U(w)$ satisfies Assumption A with $L = m + L_R$.

The noise term $\xi$ in (13) satisfies Assumption B.1 by definition, and satisfies Assumption B.3 with $L_\xi = (\sqrt{s} + 2\sigma)L$. Assumption B.2 is bounded if $\xi(w, \eta)$ is bounded for all $w$, i.e. the sampled gradient does not deviate from the true gradient by more than a constant. We will need to assume directly Assumption B.4, as it is a property of the distribution of $\nabla U_i(w)$ for $i = 1...n$.

### 6.3 Experiments

In this section, we present experimental results. In all experiments, we use two different neural network architectures on the CIFAR-10 dataset (14). The first architecture is a simple convolutional neural network, which we call CNN in the following, and the other is the VGG19 network (17). To make our experiments consistent with the setting of SGD, we do not use batch normalization or dropout. In all of our experiments, we run SGD algorithm 1000 epochs such that the algorithm converges sufficiently.

Let $H(w)^2$ be the covariance matrix of a single sample as defined in (9). For all SGD variants studied in this section, the covariance matrix will be some scaling of $H(w)^2$. We define the relative variance of a sequence $(w_k)_{k=1}$ as the scaling of $H(w)^2$ in its continuous limit. For a SGD sequence $w_k$ with $\delta$ stepsize and $b$ batchsize, one can verify that the relative scaling of $w_k$ is $\frac{s}{b}$. The authors of (11) have also observed this ratio is correlated with the quality of SGD solutions.

Out of pragmatic graph plotting considerations, we actually define relative variance to be the scaling wrt the noise when learning rate=0.01 and batch size=128.

#### 6.3.1 Accuracy vs relative Variance

In our first experiment, we show that there is a positive correlation between the relative variance of SGD (with respect to a particular baseline) and the final test accuracy of the trained model. We choose constant learning rate from

$$\{0.002, 0.004, 0.006, 0.008, 0.01, 0.02, 0.04, 0.06, 0.08, 0.1\}$$

and batch size from

$$\{32, 64, 128, 256, 512\}.$$
In this section, we implement and examine the performance of the Algorithm proposed in (12). In the Figure 3, each × denotes a baseline SGD run, with learning-rate specified in the legend and batch-size specified by plot title. For example, in the first plot of Figure 3, the red × denotes a SGD run with learning rate $\delta^x = 0.002$ and batch-size $b^x = 256$. For each ×, we have a corresponding ⋄, of the same color. The ⋄ corresponds to a run of (12), with $s = \delta^x$, $|\hat{\eta}_k^1| = |\hat{\eta}_k^2| = |\hat{\eta}_k^3| = b^x$, and $\sigma$ chosen so that the noise term $\xi$ as defined in (13) has covariance $\mathbb{E} [\xi(w, \eta)\xi(w, \eta)^T] = \frac{105}{128}H(w)^2$. In addition to × and ⋄, we also plot in small teal marker all the other runs from Section 6.3.1. This helps highlight the linear trend between log(relative variance) and test accuracy that we observed in Section 6.3.1.

As can be seen, the (test error, relative variance) values for the ⋄ runs fall close to the linear trend. (Though there are some outliers). Specifically, a run of (12) produces similar test accuracy to vanilla SGD runs with the same relative variance (e.g. SGD runs with the same minibatch size and 10 times the learning rate). We highlight two potential implications: First, just like in Section 6.3.1, we observe that the test accuracy is strongly correlated with relative variance, even for noise of the form (13), which can have rather different higher moments than $\zeta$. Second, since the ⋄ points fall close to the linear trend, we hypothesize that for all $s \rightarrow 0$, and for all $\sigma$ chosen so as in (14), the test accuracy of (12) will be similar to the test accuracy of (10). Then by our convergence result, (11) should also have similar test error. If true, then this implies that we only need to study $U(x)$ and $M(x)$ to explain the generalization properties of SGD.

Finally, Figure 4 presents a similar experiment to 3. This time, for each ×, we have a ⋄ run with $s = \delta^x$, $|\hat{\eta}_k^1| = |\hat{\eta}_k^2| = |\hat{\eta}_k^3| = 128$, and $\sigma$ chosen so that the noise term $\xi$ as defined in (13) has covariance $\mathbb{E} [\xi(w, \eta)\xi(w, \eta)^T] = \frac{105}{128}H(w)^2$. We see that once again, the ⋄ runs fall close to the linear trend.

Figure 2: Relationship between final test accuracy and the relative variance of the SGD algorithm.

Figure 3: Injecting noise with minibatch $b^x$.
7 Acknowledgements

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A Proofs for Convergence under Gaussian Noise (Section 5.1)

A.1 Proof of Theorem 1

In this section, we state our main Theorem. Our proof proceeds by recursively applying Lemma 1 over many steps.

Proof of Theorem 1

Let \( \alpha_q, R_q, \lambda, \epsilon, \delta, k \) be as defined in the Theorem statement.

For the rest of this proof, consider \( f \) defined as in Lemma 15 using the parameters \((\alpha_q, R_q, \epsilon)\).

Using Lemma 15, we know that

\[
\frac{1}{2} \exp(-3\alpha_q R_q^2)(\|z\|_2 - 2\epsilon) \leq f(z)
\]

As a consequence, for any two distributions \( p, p' \),

\[
\frac{1}{2} \exp(-3\alpha_q R_q^2)(W_1(p, p') - 2\epsilon) \leq W_f(p, p')
\]

\[
\Rightarrow \quad W_1(p, p') \leq 2 \exp(3\alpha_q R_q^2) W_f(p, p') + 2\epsilon
\]

Suppose that we have the guarantee that

\[
W_f(p^*, \hat{p}_{i\delta}^y) \leq \frac{7(L + L_N^2)\epsilon}{\lambda}
\]  

(17)

Then,

\[
W_1(p^*, \hat{p}_{i\delta}^y) \leq 2 \exp(3\alpha_q R_q^2) W_f(p, p') + 2\epsilon
\]

\[
\leq 2 \exp(3\alpha_q R_q^2) \left( \frac{7(L + L_N^2)\epsilon}{\lambda} \right) + 2\epsilon
\]

\[
= \left( 2 \exp(3\alpha_q R_q^2) \left( \frac{7(L + L_N^2)\epsilon}{\lambda} \right) + 2 \right) \epsilon
\]

\[
\leq \epsilon
\]

Where the last inequality is by our choice of \( \epsilon \), and we have concluded our proof. The rest of this proof will be dedicated to showing (17).

Let \( \hat{p}_y^y := \text{Law}(\hat{y}_t) \). Let \( \hat{x}_0 \sim p^* \) and \( \hat{x}_t \) be defined as in (3). Let \( \hat{p}_t^y := \text{Law}(x_t) \), then \( \hat{p}_t^y = p^* \) for all \( t \), by definition of \( p^* \).

First, by our choice of the initial \( \hat{p}_0^y \),

\[
\mathbb{E} \left[ \|\hat{y}_0\|^2 \right] \leq 2(R^2 + \beta^2/m)
\]

Combined with our choice of \( \delta \leq \frac{m}{3\beta^2} \), we can apply Lemma 8 with \( \xi_k(y) = M(y)\theta_k, \theta_k \sim \mathcal{N}(0, I) \), to get that for all \( j \),

\[
\mathbb{E} \left[ \|\hat{y}_{j\delta}\|^2 \right] \leq 4(R^2 + \beta^2/m)
\]  

(18)

Now, consider an arbitrary integer \( i \). For \( t \in [i\delta, (i+1)\delta) \), (2) and (3) evolve as

\[
dx_t = -\nabla U(x_t)dt + M(x_t)dS_t
\]

\[
d\hat{y}_t = -\nabla U(y_{i\delta})dt + M(y_{i\delta})dS_t
\]

The above is the same process as (20) and (21), for which (22) is a coupling.

We can thus apply Lemma 1 with the given \( \alpha_q, R_q, \epsilon, p_0^y := \hat{p}_{i\delta}^y, p_0 := \hat{p}_{i\delta}^y, \) and \( T = \delta \). Then

\[
W_f \left( \hat{p}_{(i+1)\delta}^y, \hat{p}_{(i+1)\delta}^y \right) \leq e^{-\lambda \delta} W_f (\hat{p}_{i\delta}^y, \hat{p}_{i\delta}^y) + 3\delta(L + L_N^2)\epsilon
\]

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Applying the above recursively,
\[
W_f(\tilde{p}_k^T, \tilde{p}_k) \leq e^{-\lambda \delta k} W_f(p^*, \tilde{p}_0) + \sum_{i=1}^{k} e^{-\lambda \delta (k-i)} (3\delta (L + L_N^{2}) \epsilon)
\]
\[
\leq e^{-\lambda \delta k} W_f(p^*, \tilde{p}_0) + \frac{6(L + L_N^{2}) \epsilon}{\lambda}
\]  \quad (19)

Where we use the fact that \( \tilde{p}_i^T = p^* \) for all \( t \). Using Lemma 9 and the definition of \( \tilde{p}_0 \) in Theorem 1, we can upper bound the initial error as
\[
W_f(p^*, \tilde{p}_0) \leq W_1(p^*, \tilde{p}_0) \leq \sqrt{\mathbb{E}_p^* \left[ \|x\|_2^2 \right]} + \sqrt{\mathbb{E}_p \left[ \|y_0\|_2^2 \right]} \leq 64 \sqrt{R^2 + \beta^2 / m}
\]

By our definition of \( k \) in the Theorem statement and by our definition of \( \epsilon \) at the start of the proof,
\[
k = \frac{1}{\lambda \delta} \log \left( \frac{64 \sqrt{R^2 + \beta^2 / m}}{(L + L_N^{2}) \epsilon} \right)
\]

which implies
\[
e^{-\lambda \delta k} W_f(p^*, \tilde{p}_0) \leq e^{-\lambda \delta k} 64 \sqrt{R^2 + \beta^2 / m} \leq \frac{(L + L_N^{2}) \epsilon}{\lambda}
\]

Then (19) gives
\[
W_1(p^*, \tilde{p}_0) \leq \frac{7(L + L_N^{2}) \epsilon}{\lambda}
\]

thus proving (17) and concluding our proof. ■

A.2 A coupling construction

In this subsection, we will study the evolution of (3) and (2) over a small time interval. Specifically, we will study
\[
dx_t = -\nabla U(x_t) dt + M(x_t) dS_t
\]
\[
dy_t = -\nabla U(y_t) dt + M(y_t) dS_t
\]

One can verify that (20) is equivalent to (3), and (21) is equivalent to a single step of (2) (i.e. over an interval \( t \leq \delta \)). We first give the explicit coupling between (20) and (21):

Define \((x_t, y_t)\) using the following coupled SDE:
\[
x_t = x_0 + \int_0^t -\nabla U(x_s) ds + \int_0^t e_M (I - 2\gamma_s \gamma_s^T) dB_s + \int_0^t N(x_s) dW_s
\]
\[
y_t = y_0 + \int_0^t -\nabla U(y_t) dt + \int_0^t e_M dB_s + \int_0^t N(y_t) dW_s
\]  \quad (22)

Where \( dB_t \) and \( dW_t \) are two independent standard Brownian motion, and \( \gamma_t := \frac{x_t - y_t}{\|x_t - y_t\|_2} \cdot \{ \|x_t - y_t\|_2 \in [2\epsilon, R \gamma] \} \).

By Lemma 3 (20) has the same distribution as \( x_t \) in (22), and (21) has the same distribution as \( y_t \) in (22). Thus, for any \( t \), the joint distribution \( \text{Law}(x_t, y_t) \) defined by (22) is a valid coupling for (20) and (21).

A.3 One Step Contraction

**Lemma 1** Given some initial distribution \( p_0^v, p_0^y \), satisfying \( \mathbb{E}_{y \sim p_0^y} \left[ \|y\|_2^2 \right] \leq 4(R^2 + \beta^2 / m) \), let us define
\[
\alpha_q := \frac{L_N + L_N^{2}}{2 \epsilon q_r}
\]

1. \( \alpha_q := \frac{L_N + L_N^{2}}{2 \epsilon q_r} \)
2. $\mathcal{R}_q := \max \left\{ R, \frac{16\beta^2 L_N}{m c_M} \right\}$

3. $\lambda := \min \left\{ \frac{m}{2}, \frac{2\gamma L_N}{3L_R^2} \right\} e^{-3\alpha_q R_q^2}$

Let $\epsilon$ be a target accuracy satisfying $\epsilon \leq \min \left\{ R, \frac{R}{m c_M + 1} \right\}$.

Let $f$ be as defined in Lemma 13 with parameters $(\alpha_q, \mathcal{R}_q, \epsilon)$. Let $p^n_t$ and $p^n_0$ be distributions of (20) and (21) respectively (initialized at $p^n_0$ and $p^n_0$).

Then for any $T \leq \min \left\{ \frac{m}{16L^2}, \frac{8}{16L\sqrt{R^2 + \beta^2/m}} \right\}$

$$W_f(p^n_T, p^n_0) \leq e^{-\lambda T} W_f(p^n_0, p^n_0) + 3T(L + L_N^2)\epsilon$$

**Remark 7** For ease of reference: $m, L, L_R, R$ are from Assumption 4, $L_N = \frac{4\beta L_N}{c_M}$ is from Lemma 4, $c_M, \beta$ are from Assumption 3.

**Proof of Lemma 1**

Let $\zeta^*_0$ be the optimal coupling between $p^n_0$ and $p^n_0$ wrt $W_f$, and let $(x_0, y_0) \sim \zeta^*_0$. Thus

$$\mathbb{E} [f(x_0 - y_0)] = W_f(p^n_0, p^n_0)$$

Now let $(x_t, y_t)$ evolve as (22). Since (22) is a coupling between (20) and (21), we can verify that the distribution $(x_t, y_t)$ is a valid coupling between $p^n_t$ and $p^n_t$. The claim in the Lemma thus reduces to showing that

$$\mathbb{E} [f(x_T - y_T)] \leq e^{-\lambda T} W_f(p^n_0, p^n_0) + 3T(L + L_N^2)\epsilon \quad (23)$$

The rest of the proof is dedicated to showing (23).

For convenience, let us define $z_t = x_t - y_t$.

For the rest of this proof, we will seek to bound $\mathbb{E} [f(z_T)] = \mathbb{E} [f(x_T - y_T)]$.

For convenience, let $\nabla := \nabla U(x_t) - \nabla U(y_t)$, $N_t := N(x_t) - N(y_t)$. It follows from (22) that

$$dz_t = -\nabla_t dt + 2c_M^2\gamma_t \gamma_t^T dB_t + (N_t + N(y_t) - N(y_0))dW_t \quad (24)$$

Using Ito's Lemma, the dynamics of $f(z_t)$ is given by

$$df(z_t) = \begin{cases} 
\langle \nabla f(z_t), dz_t \rangle + 2c_M^2\gamma_t (\nabla^2 f(z_t) (\gamma_t^T)) dt + \frac{1}{2} \text{tr} \left( \nabla^2 f(z_t)(N_t + N(y_t) - N(y_0))^2 \right) dt 
+ \langle \nabla f(z_t), -\nabla_t \rangle dt + 2c_M^2\gamma_t (\nabla^2 f(z_t) (\gamma_t^T)) dt + \frac{1}{2} \text{tr} \left( \nabla^2 f(z_t)(N_t + N(y_t) - N(y_0))^2 \right) dt 
+ \langle \nabla f(z_t), 2c_M^2\gamma_t dB_t + (N_t + N(y_t) - N(y_0))dW_t \rangle 
\end{cases} \quad (1) \quad (2) \quad (3) \quad (4)$$

(4) goes to 0 when we take expectation, so we will focus on (1), (2), (3). We will consider 3 cases

**Case 1:** $\|z_t\|_2 \leq 2\epsilon$

In this case,

$$\|\nabla f(z_t)\|_2 = q'(g(z_t))\|\nabla g(z_2)\|_2 \leq 1$$

(See bounds given in Lemma 13). Therefore, by Assumption 13 and the fact that $\epsilon \leq R$,

$$1 \leq \|\nabla_t\|_2 \leq L_R \|z_t\|_2 \leq L_R \epsilon$$

Since $\gamma_t = 0$ in this case by definition, (2) = 0.
Using Lemma 15.2.c.
\[ \|\nabla^2 f(z_t)\|_2 \leq \frac{2}{\epsilon} \]

So
\[ (3) \leq \frac{1}{2\epsilon} \left( \text{tr}(N_t^2) + \text{tr}\left((N(y_t) - N(y_0))^2\right) \right) \]
\[ \leq \frac{L_N^2}{\epsilon} \left( \|z_t\|_2^2 + \|y_t - y_0\|_2^2 \right) \]
\[ \leq 2L_N^2 \epsilon + \frac{L_N^2 \|y_t - y_0\|_2^2}{\epsilon} \]

Where the first inequality is by definition of \( L_N \) in Lemma 1.

Summing these,
\[ (1) + (2) + (3) \leq LR \epsilon + L_N^2 \epsilon + \frac{L_N^2 \|y_t - y_0\|_2^2}{\epsilon} \]

Case 2: \( \|z_t\|_2 \in (2\epsilon, R_q) \)
In this case, \( \gamma_t = \frac{z_t}{\|z_t\|_2^2} \). By Lemma 16 and Lemma 17,
\[ \nabla f(z_t) = q'(g(z_t))\nabla g(z_t) \]
\[ = q'(g(z_t)) \frac{z_t}{\|z_t\|_2^2} \]
\[ \nabla^2 f(z_t) = q''(g(z_t))\nabla g(z_t) \nabla^2 g(z_t) + q'(g(z_t))\nabla^2 g(z_t) \]
\[ = q''(g(z_t)) \frac{z_t z_T}{\|z_t\|_2^2} + q'(g(z_t)) \frac{1}{\|z_t\|_2} \left( I - \frac{z_t z_T}{\|z_t\|_2^2} \right) \]

Once again, by Assumption A.3,
\[ (1) \leq q'(g(z_t))\|\nabla \epsilon\|_2 \leq q'(g(z_t)) \cdot LR \cdot \|z_t\|_2 \leq q'(g(z_t)) \cdot LR \cdot g(z_t) + 2\epsilon LR \]

Where the last inequality uses Lemma 17.4.

Using the expression for \( \nabla^2 f(z_t) \),
\[ (2) = 2c_M^2 \text{tr}(\nabla^2 f(z_t)\gamma_T^T) = 2c_M^2 \cdot q''(g(z_t)) \]

Finally,
\[ (3) = \frac{1}{2} \text{tr}\left(\nabla^2 f(z_t)(N_t + N(y_t) - N(y_0))^2\right) \]
\[ = \frac{1}{2} \text{tr}\left( q''(g(z_t)) \frac{z_t z_T}{\|z_t\|_2^2} + q'(g(z_t)) \frac{1}{\|z_t\|_2} \left( I - \frac{z_t z_T}{\|z_t\|_2^2} \right) \right) \]
\[ \leq \frac{1}{2} \text{tr}\left( q'(g(z_t)) \frac{1}{\|z_t\|_2} \left( I - \frac{z_t z_T}{\|z_t\|_2^2} \right) \right) \]
\[ \leq q'(g(z_t)) \cdot \frac{L_N^2}{\|z_t\|_2} + \frac{L_N^2 \|y_t - y_0\|_2^2}{\epsilon} \]
\[ \leq q'(g(z_t)) \cdot L_N^2 g(z_t) + \frac{L_N^2 \|y_t - y_0\|_2^2}{\epsilon} \]

Where the first inequality is by Lemma 18.4, the second inequality is by the fact that \( N \) is always positive definite, the third inequality is by Lemma 4, the fourth inequality uses Lemma 17.4.
Summing these,

\[ \sum_{s=1}^{3} \leq q'(g(z_t)) \cdot g(z_t) \cdot (L_R + L_N^2) + 2c_3 q''(g(z_t)) + \frac{L_N^2 \|y_t - y_0\|^2}{\epsilon} + 2cL_R \]

Where the last inequality follows from Lemma 18.1, and our choice of \(\alpha_q, R_q\) in the Lemma statement.

**Case 3:** \(\|z_t\|_2 \geq R_q\)

In this case, \(\gamma_t = 0\). Similar to case 2,

\[ \nabla f(z_t) = q'(g(z_t)) \frac{z_t}{\|z_t\|_2} \]

Thus by Assumption A

\[ \langle q'(g(z_t)) \frac{z_t}{\|z_t\|_2}, -\nabla f \rangle \leq -mq' q'(g(z_t)) \|z_t\|_2 \]

Where the inequality is by Assumption A

Like in Case 1, \(\mathcal{R}_2 = 0\). Finally,

\[ \sum_{s=1}^{3} \leq q'(g(z_t)) \cdot \left( \frac{8\beta^2 L_N^2}{c_M} + \frac{L_N^2 \|y_t - y_0\|^2}{\epsilon} \right) \]

Where the second inequality is by our definition of \(R_q\) in the Lemma statement, which ensures that \(\frac{8\beta^2 L_N^2}{c_M} \leq \frac{m}{T} R_q \leq \frac{m}{T} \|z_t\|_2\).

Thus

\[ \langle q'(g(z_t)) \frac{z_t}{\|z_t\|_2}, -\nabla f \rangle \leq -mq' q'(g(z_t)) \|z_t\|_2 + \left( \frac{L_N^2 \|y_t - y_0\|^2}{\epsilon} \right) \]

The second inequality uses Lemma 17.1, the third inequality uses Lemma 18.3, the third inequality uses Lemma 18.3 and Lemma 15.4.
Combining the three cases gives, for all $t \leq T$,
\[
\begin{align*}
    d\mathbb{E}[f(z_t)] &\leq \mathbb{E}[(1 + 2 + 3)dt] + \mathbb{E}[4] \\
    &\leq -(\min \left\{ \frac{m}{2} \cdot \frac{2\epsilon^2}{32R_q^2} \right\} e^{-3\alpha_q R_q^2} - \frac{L_3}{\epsilon}) \mathbb{E}[|y_t - y_0|^2] dt + 2(L_R + L_N^2)\epsilon dt \\
    &\leq -\left( \min \left\{ \frac{m}{2} \cdot \frac{2\epsilon^2}{32R_q^2} \right\} e^{-3\alpha_q R_q^2} - \frac{2L_3}{\epsilon} \left( t^2 L^2 \left( \mathbb{E} \left[ |y_0|^2 \right] \right) + t^2 \right) \right) dt + 2(L_R + L_N^2)\epsilon dt \\
    &\leq -\left( \min \left\{ \frac{m}{2} \cdot \frac{2\epsilon^2}{32R_q^2} \right\} e^{-3\alpha_q R_q^2} - \frac{L_3}{\epsilon} \right) \mathbb{E}[|f(z_t)|] dt + 3(L_R + L_N^2)\epsilon dt \\
    &:= -\lambda \mathbb{E}[f(z_t)] dt + 3(L + L_N^2)\epsilon dt
\end{align*}
\] (25)

The second inequality uses Lemma [7]. The third inequality uses our assumed upper bounds on $\mathbb{E} \left[ |y_0|^2 \right]$ and $T$ in the Lemma statement. The last equality is by definition of $\lambda$ in the Theorem statement, and the fact that $L_R \leq L$.

Then by Gronwall’s Lemma,
\[
\begin{align*}
    \mathbb{E}[f(z_T)] &\leq e^{-\lambda T} \mathbb{E}[f(z_0)] + \left( 1 - e^{-\lambda T} \right) \frac{3(L + L_N^2)\epsilon}{\lambda} \\
    &\leq e^{-\lambda T} \mathbb{E}[f(z_0)] + 3T(L + L_N^2)\epsilon \\
    &= e^{-\lambda T} W_f(p_0^*, p_0^*) + 3T(L + L_N^2)\epsilon
\end{align*}
\] (26)

\section*{A.4 Simulating the SDE}

One can verify that the SDE in [4] can be simulated (at discrete time intervals) as follows:
\[
y_{k+1} = y_k + \delta \nabla U(y_k) + \sqrt{\delta} M(y_k) \theta_k
\]

Where $\theta_k \sim \mathcal{N}(0, I)$. This however requires access to $M(y_k, \delta)$, which may be difficult to compute.

If for any $y$, one is able to draw samples from some $p_y$ such that

1. $\mathbb{E}_{x \sim p_y} [x] = 0$ 
2. $\mathbb{E}_{x \sim p_y} [xx^T] = M(y)$ 
3. $\|x\|_2 \leq \beta$ almost surely, for some $\beta$.

then one might sample a noise that is $\delta$ close to $M(y_k, \delta) \theta_k$ through Theorem [3].

Specifically, if one draws $n$ samples $\xi_1, \ldots, \xi_n \overset{iid}{\sim} p_y$, and let $S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i$, we guarantee that $W_2(S_n, M(y) \theta) \leq \frac{8\delta \beta}{\sqrt{n}}$. We remark that the proof of Theorem [1] can be modified to accomodate for this sampling error. The number of samples needed per step will be on the order of $n \approx (\delta \epsilon)^{-2}$.
In this section, we demonstrate that (1) also converges to $p^*$ of (3). The idea is similar to (A.4): We will use Central Limit Theorem to bound the error between (1) and (2). We can then rely on Theorem 1 to show convergence to $p^*$.

The notable difference between this section and A.4 is that we do not have to draw lots of samples each iteration at the same location. Thus (1) is closer to algorithms commonly seen in practice. For an example, see Section 6.

B.1 Proof of Theorem 2

Let $\alpha_q, R_q, \lambda, \epsilon, \hat{\epsilon}, \hat{n}$ be as defined in the Theorem statement.

Let $C_1, C_2 = \text{poly}(L, R, 1/m, \beta, L\xi)$ be the universal constants defined in Lemma 2. Let $T := \frac{\epsilon^8}{C_1}, \delta := \frac{1}{C_2}(T^{3/2}/\epsilon^2 \log \frac{1}{T\epsilon})$.

It can be verified that for some $C_3 = \text{poly}(C_1, C_2)$, $\delta = \frac{1}{C_3}(\epsilon^8 \log \frac{1}{\epsilon})$.

Which aligns with our definition of $\delta$ in the Theorem statement.

For the rest of this proof, consider $f$ defined as in Lemma 15 using the parameters $(\alpha_q, R_q, \epsilon)$. Using Lemma 15, we know that

$$\exp(-3\alpha_q R_q^2)(\|z\|_2^2 - 2\epsilon) \leq f(z)$$

As a consequence, for any two distributions $p, p'$,

$$\exp(-3\alpha_q R_q^2)(W_1(p, p') - 2\epsilon) \leq W_f(p, p')$$

In order to get $W_1(p^*, \hat{p}_w^\delta) \leq \hat{\epsilon}$, it then suffices (based on our definition of $\epsilon$) to guarantee that

$$W_f(p^*, \hat{p}_w^\delta) \leq \frac{7(L + L\xi^2)\epsilon}{\lambda}$$

(27)

(The above steps are the same as in the proof of Theorem 1 up to (17)).

The rest of the proof will be dedicated to proving (27). First, by our choice of the initial $\hat{w}_0$,

$$\mathbb{E} \left[\|\hat{w}_0\|_2^4\right] \leq 128 \left(R^4 + \frac{\beta^4}{m^2}\right)$$

Thus we can apply Lemma 14 (assume wlog that $\delta$ is small enough) to get that for all $k$,

$$\mathbb{E} \left[\|w_{k\delta}\|_2^2\right] \leq 1024 \left(R^4 + \frac{\beta^4}{m^2}\right)$$

Similarly, by Lemma 9 for all $t$,

$$\mathbb{E} \left[\|\hat{x}_t\|_2^2\right] \leq 32R^4 + \frac{512\beta^4}{m^2}$$

Consider some arbitrary integer $i$. We will apply Lemma 2 with $\hat{p}_0^\delta = \hat{p}_{i\delta}, \hat{p}_0^w = \hat{p}_{i\delta}, \epsilon, \alpha_q, R_q, \lambda, T = \delta$ as defined in the Theorem statement. Lemma 2 thus gives

$$W_f(\hat{p}_{(i+K)\delta}, \hat{p}_{(i+K)\delta}) \leq e^{-\lambda(K\delta)} W_f(\hat{p}_{i\delta}, \hat{p}_{i\delta}) + 6K\delta(L + L\xi^2)\epsilon$$

(recall that $K := T/\delta$).

This gives us a contraction over a single "epoch" of $K$ steps of (1).
Applying the above recursively,
\[
W_f(p_{n,Kδ}^e, p_{n,Kδ}^w) \leq e^{-\lambda nKδ}W_f(p_0^e, p_0^w) + \sum_{j=1}^{n} e^{-\lambda(n-j)Kδ}6K\delta((L + L_δ^2)\epsilon)
\]
\[
\leq e^{-\lambda nKδ}W_f(p_0^e, p_0^w) + \frac{6((L + L_δ^2)\epsilon)}{\lambda}
\]
By our assumption on \(p_0^e\) and \(p_0^w\), we can upper bound the initial error as
\[
W_f(p^e, p_0^w) \leq W_t(p^e, p_0^w) \leq 8\sqrt{R^2 + \frac{4\beta^2}{m}}
\]
Thus, by our definition of \(\hat{n}\) in [7] we ensure that
\[
W_f(p^e, p_{nδ}^w) = W_f(p_{nδ}^e, p_{nδ}^w) \leq \frac{7((L + L_δ^2)\epsilon)}{\lambda}
\]
This proves (27) and hence the Theorem.

**B.2 Constructing a Coupling**

In this subsection, we construct a coupling between [1] and [3], given initial distributions \(x_0 \sim p_0^e\) and \(w_0 \sim p_0^w\). We will consider a finite time \(T = nδ\).

1. Let \((x_0, w_0)\) be jointly distributed according to the optimal coupling in \(W_f\). I.e. \(\mathbb{E}[f(x_0 - w_0)] = W_f(p_0^e, p_0^w)\).
2. Let \(B_t\) and \(W_t\) be two independent Brownian motion.
3. Using \(B_t\) and \(W_t\), define
\[
y_t = w_0 + \int_0^t -\nabla U(w(s))ds + \int_0^t c_M dB_s + \int_0^t N(w(s))dW_s
\]  
(28)
It can be verified that (28) is equivalent in distribution to (22), initialized at \(w_0\).
4. Having defined \(y_t\), we will define \(x_t\) as
\[
x_t = x_0 + \int_0^t -\nabla U(x(s))ds + \int_0^t c_M (I - 2\gamma T) dB_s + \int_0^T N(x(s))dW_s
\]  
(29)
Where \(\gamma_t := \frac{|x_t - y_t|}{\|x_t - y_t\|} \cdot \mathbb{I}\{|x_t - y_t| \leq 2\epsilon, R_q\}\). The coupling between (29) and (28) is identical to that in (22).
5. We now define a process \(v\):
\[
v_{kδ} = w_0 + \sum_{i=1}^{k} -\delta \nabla U(w_0) + \sqrt{\delta} \sum_{i=0}^{k} \xi(w_0, \eta_i)
\]  
(30)
One can verify that \(v_{kδ} = w_0 - (kδ)\nabla U(w_0) + \sqrt{\delta} \sum_{i=0}^{k} \xi(w_0, \eta_i)\). On the other hand, using Lemma 4 \(y_t \sim N(w_0 - t\nabla U(w_0), M(w_0)^2)\).
We will couple (30) and (28) so that their joint distribution at time \(T\) is the optimal coupling. By Corollary 21
\[
W_2(y_T, v_T)
\leq W_2(y_T - (w_0 - T\nabla U(y_0), v_T - (w_0 - T\nabla U(w_0)))
\leq 6\sqrt{\delta \beta \log n}
\]
(recall our definition of \(T = nδ\)). For convenience, define \(v_t := v_{[t/δ]}\)
6. Given the sequence \(v_{kδ}\), we can define
\[
w_{kδ} = w_0 + \delta \sum_{i=1}^{k} -\nabla U(w_{kδ}) + \sqrt{\delta} \sum_{k=0}^{n} \xi(w_{kδ}, \eta_k)
\]  
(31)
specifically, \(w_{kδ}\) in (31) and \(v_{kδ}\) in (30) are coupled through the shared \(\eta_k\) variables. For convenience, define \(w_t := w_{[t/δ]}\)

For convenience, let us also define \(z_t = x_t - y_t\), and \(z'_t = x_t - w_t\).
B.3 One epoch contraction

Lemma 2 Let us define

1. \( \alpha_q := \frac{L_N + L_N^2}{2c_M} \)
2. \( \mathcal{R}_q := \max \left\{ R, \frac{16\beta^2 L_N}{m \cdot c_M} \right\} \)
3. \( \lambda := \min \left\{ \frac{m}{2}, \frac{2c_M}{3\mathcal{R}_q} \right\} \) \( e^{-3\alpha_q \mathcal{R}_q^2} \)

Let \( \epsilon \) be a target accuracy satisfying \( \epsilon \leq \min \left\{ \frac{R}{\alpha_q e^{\epsilon R/q}}, 1 \right\} \).

Let \( f \) be as defined in Lemma 13 with parameters \( (\alpha_q, \mathcal{R}_q, \epsilon) \)

Then there exists universal constants \( C_1, C_2 = \text{poly}(L, R, 1/m, \beta, L_N) \), such that the following holds for all \( T \leq \frac{C_1}{\epsilon^2} \), and for all \( \delta \leq \frac{1}{C_2} \left( T^{3/2} \epsilon^2 \log \frac{1}{T} \right) \):

For any \( p_0^\alpha \) and \( p_0^w \) satisfying \( \mathbb{E}_{p_0^\alpha} \left[ \|x\|_2^4 \right] \leq 1024 \left( R^4 + \frac{4^4}{m^2} \right) \) and \( \mathbb{E}_{p_0^w} \left[ \|w\|_2^2 \right] \leq 32R^4 + \frac{512\alpha^4}{m^2} \), let \( p_t^\alpha \) denote the distributions of (3) initialized at \( p_0^\alpha \). Let \( p_t^w \) denote the distribution of (1) initialized at \( p_0^w \) with stepsize \( \delta \), then

\[
W_f(p_t^\alpha, p_t^w) \leq e^{-\lambda T} W_f(p_0^\alpha, p_0^w) + 6T(L + L_N^2) \epsilon
\]

Proof

Let \( x_t, y_t, v_t, w_t \) be defined according to the coupling described in Section B.2. We highlight the following two facts:

1. (\( x_t, y_t \)) has the same dynamics as (22).
2. As a consequence of the point 1. above, Law(\( x_t \)) = \( p_t^\alpha \) for all \( t \).
3. (31) has the same dynamics as (1), so Law(\( w_t \)) = \( p_t^w \)

Due to 1. above, the proof of Lemma 1 can be repeated here exactly to show that (23) holds. Specifically,

\[
\mathbb{E} \left[ f(x_T - y_T) \right] \leq e^{-\lambda T} W_f(p_0^\alpha, p_0^w) + 3T(L + L_N^2) \epsilon
\]

(32)

Where we use the fact that \( y_0 = w_0 \) in (28).

Suppose that we can show the following

\[
\mathbb{E} \left[ f(x_T - w_T) \right] - \mathbb{E} \left[ f(x_T - y_T) \right] \leq 3T L \epsilon
\]

(33)

Summing the (32) and (33) gives our conclusion.

The rest of this proof will be dedicated to showing (33).

Let \( z_T := x_T - y_T \), and let \( z_T' := x_T - w_T \).

By Taylor expansion,

\[
\mathbb{E} \left[ f(z_T) \right] = \mathbb{E} \left[ f(z_0) + \langle \nabla f(z_0), z_T - z_0 \rangle + \int_0^T \int_0^t \langle \nabla^2 f((1-s)z_0 + sz_T), (z_T - z_0) \rangle ds dt \right]
\]

\[
\mathbb{E} \left[ f(z_T') \right] = \mathbb{E} \left[ f(z_0) + \langle \nabla f(z_0), z_T' - z_0 \rangle + \int_0^T \int_0^t \langle \nabla^2 f((1-s)z_0 + sz_T'), (z_T' - z_0) \rangle ds dt \right]
\]

Taking the difference between these two gives

\[
\mathbb{E} \left[ f(z_T') - f(z_T) \right] = \mathbb{E} \left[ \langle \nabla f(z_0), z_T' - z_T \rangle \right]
\]

\[+ \mathbb{E} \left[ \int_0^T \int_0^t \left( \langle \nabla^2 f((1-s)z_0 + sz_T), (z_T' - z_0) \rangle - \langle \nabla^2 f((1-s)z_0 + sz_T'), (z_T - z_0) \rangle \right) ds dt \right]
\]

(1)

(2)
We will bound the $1$ and $2$ separately, starting with $1$. Recalling the dynamics in (31) and (28),

$$z_T' - z_T = x_T - x_T + w_T - x_T$$

$$= w_T - y_T$$

$$= \delta \sum_{k=1}^{n} (\nabla U(w_0) - \nabla U(w_{k\delta})) + \int_0^T c_M dB_t + \int_0^T N(y_0) dW_t + \sqrt{\delta} \sum_{k=0}^{n} \xi(w_{k\delta}, \eta_k)$$

(1) $= \mathbb{E}[(\nabla f(z_0), z_T' - z_T)]$

$= \mathbb{E} \left[ \left( \nabla f(z_0), \delta \sum_{k=1}^{n} (\nabla U(w_0) - \nabla U(w_{k\delta})) + \int_0^T c_M dB_t + \int_0^T N(w_0) dW_t + \sqrt{\delta} \sum_{k=0}^{n} \xi(w_{k\delta}, \eta_k) \right) \right]$ 

$= \mathbb{E} \left[ \nabla f(z_0), \delta \sum_{k=1}^{n} (\nabla U(w_0) - \nabla U(w_{k\delta})) \right]$ 

$\leq \Vert \nabla f(z_0) \Vert_2 \cdot \delta \sum_{k=0}^{n} \mathbb{E} \Vert \nabla U(w_0) - \nabla U(w_{k\delta}) \Vert_2$

$\leq n \delta L \sqrt{\mathbb{E} \left[ \Vert w_{k\delta} - w_0 \Vert_2^2 \right]}$

$\leq TL \sqrt{72T^2 L^2 \mathbb{E} \left[ \Vert w_0 \Vert_2^2 \right] + 12T \beta^2}$

$\leq TL \sqrt{72T^2 L^2 \mathbb{E} \left[ \Vert w_0 \Vert_2^2 \right] + 12T \beta^2}$

$\leq 9L \left( T^2 L \sqrt{\mathbb{E} \left[ \Vert w_0 \Vert_2^2 \right]} + T^{3/2} \beta \right)$

(34)

Where the third line uses the fact that $\int_0^T c_M dB_t + \int_0^T N(w_0) dW_t + \sqrt{\delta} \sum_{k=0}^{n} \xi(w_{k\delta}, \eta_k)$, is 0-mean conditioned on $z_0$. The second inequality is by Assumption A2 Lemma [15]. The third inequality is by Lemma [11]. The last inequality is by our assumption on $\mathbb{E} \left[ \Vert w_0 \Vert_2^2 \right]$ and the fact that there exists a $C_1 = \text{poly}(L, R, 1/m, \beta, L_\xi)$ such that (34) holds for any $T \leq \frac{\epsilon^4}{L_\xi}$.

We next analyze the term

$$2 = \mathbb{E} \left[ \int_0^1 \int_0^t \left( \langle \nabla^2 f((1-s)z_0 + sz_T'), (z_T' - z_0)^\otimes 2 \rangle - \langle \nabla^2 f((1-s)z_0 + sz_T), (z_T - z_0)^\otimes 2 \rangle \right) ds dt \right]$$

We decompose the difference as follows:

$$= \mathbb{E} \left[ \langle \nabla^2 f((1-s)z_0 + sz_T'), (z_T' - z_0)^\otimes 2 \rangle - \langle \nabla^2 f((1-s)z_0 + sz_T), (z_T - z_0)^\otimes 2 \rangle \right]$$

(3)

$$+ \mathbb{E} \left[ \langle \nabla^2 f((1-s)z_0 + sz_T') - \nabla^2 f((1-s)z_0 + sz_T), (z_T - z_0)^\otimes 2 \rangle \right]$$

(4)
We first study (4):

\[
\mathbb{E} \left[ \int_0^t \int_0^s \left\langle \nabla^2 f((1-t)x_0 + tz_T), (z_T - z_0)^{\otimes 2} \right\rangle dsdt \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \int_0^s \| \nabla^2 f((1-t)x_0 + tz_T) - \nabla^2 f((1-t)x_0 + tz_T) \|_2 \| z_T - z_0 \|_2^2 dsdt \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \int_0^s \frac{9}{\epsilon^2} \| z_T - z_T' \|_2 \| z_T - z_0 \|_2^2 dsdt \right]
\]

\[
= \frac{9}{2\epsilon^2} \mathbb{E} \left[ \| z_T - z_T' \|_2 \| z_T - z_0 \|_2 \right]
\]

\[
\leq \frac{18}{\epsilon^2} \sqrt{\mathbb{E}[\| y_T - v_T \|_2^2] + \mathbb{E}[\| v_T - w_T \|_2^2]} \sqrt{\mathbb{E}[\| x_T - x_0 \|_2^2] + \mathbb{E}[\| w_T - w_0 \|_2^2]}
\]

Where the first inequality is by Cauchy Schwarz, the second inequality is by Lemma 15, the fourth inequality is by Cauchy Schwarz, the fifth inequality is by Young’s inequality.

We can bound each of the four terms above as follows:

\[
\mathbb{E}[\| y_T - v_T \|_2^2] \leq 32d\delta \beta^2 \log \left( \frac{T}{\delta} \right) \tag{35}
\]

Where the first inequality uses the definition of the coupling between \( y \) and \( v \) in (30) and Corollary 21

\[
\mathbb{E}[\| v_T - w_T \|_2^2] \leq 256(T^2L^2 + TL^4 \epsilon^2 \mathbb{E} \| w_0 \|_2^2 + T\beta^2)
\]

\[
\leq 512(T^4L^4 + T^2L^4 \epsilon^2 + T^4L^4 \mathbb{E} \| w_0 \|_2^2 + T^2\beta^4)
\]

\[
\leq 512 \left( T^4L^4 + T^2L^4 \epsilon^2 + 128T^4L^4 \left( R^4 + \frac{\beta^4}{m^2} \right) + T^2\beta^4 \right) \tag{36}
\]

Where the first inequality uses Lemma 13, the second inequality uses Young’s inequality, the third inequality is by our assumption on \( \mathbb{E} \| w_0 \|_2^2 \), the fourth inequality is by our assumption on \( T \) in the Lemma statement.

\[
\mathbb{E}[\| x_T - x_0 \|_2^2] \leq 128T^2\beta^4 + 2^{14}T^4L^4 \mathbb{E} \| w_0 \|_2^2
\]

\[
\leq 128T^2\beta^4 + 2^{14}T^4L^4 \left( 32R^4 + \frac{512\beta^4}{m^2} \right) \tag{37}
\]

Where the first inequality is by Lemma 10 and the second inequality is by our assumption on \( \mathbb{E} \| w_0 \|_2^2 \).

Finally,

\[
\mathbb{E}[\| w_T - w_0 \|_2^2] \leq 10368T^4L^4 \mathbb{E} \| w_0 \|_2^2 + 128T^2\beta^4
\]

\[
\leq 10368T^4L^4 \left( 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \right) \tag{38}
\]

Where the first inequality is by Lemma 11, the second inequality is by Lemma 14 and our assumption on \( \mathbb{E} \| w_0 \|_2^2 \).

It can be verified that there exists a \( C_1 = \text{poly}(L, R, 1/m, \beta, L_\xi) \) such that for any \( T \leq \frac{\epsilon}{c_1^2} \)

\[
\frac{36}{36} + \frac{37}{36} + \frac{38}{36} \leq \frac{1}{36} TLe^3
\]

It can also be verified that there exists a \( C_2 = \text{poly}(L, R, 1/m, \beta, L_\xi) \) such that for any \( \delta \leq \frac{1}{c_2}(Te^3 \log \frac{1}{T}) \), (essentially apply Lemma 22 with \( x = \frac{1}{3} \) and \( c = Te^3 \))

\[
\frac{35}{36} \leq \frac{1}{36} TLe^3
\]
Combining the the upper bounds above,
\[
\frac{4}{\epsilon^2} = 18\left(\frac{35}{\epsilon^2} + \frac{36}{\epsilon^2} + \frac{37}{\epsilon^2} + \frac{38}{\epsilon^2}\right) \leq TLe
\] (39)

Next, we bound (3). For convenience, let \(A_s := \nabla^2 f((1-s)z_0 + s z'T), \nu = wT - yT\). Then
\[
\mathbb{E}\left[\int_0^1 \int_0^t \left(\nabla^2 f((1-s)z_0 + s z'T), (z'T - z_0) \right) ds dt \right]
\]
\[
= \mathbb{E}\left[\int_0^1 \int_0^t \left((zT + \nu - z_0)^T A_s (zT + \nu - z_0) - ((zT + z_0)^T A_s (zT + z_0) ds dt \right)
\]
\[
= \mathbb{E}\left[\int_0^1 \int_0^t \nu^T A_s (zT + \nu - z_0) + (zT + \nu - z_0)^T A_s \nu + \nu^T A_s \nu ds dt \right]
\]
\[
\leq \mathbb{E}\left[\int_0^1 \int_0^t 2\|\nu\|_2\|A_s\|_2 zT + \nu - z_0 + \|A_s\|_2\|\nu\|_2^2 ds dt \right]
\]
\[
= \mathbb{E}\left[\int_0^1 \int_0^t 2\|wT - yT\|_2^2 f((1-s)z_0 + s z'T), \|z'T - z_0\| + \|\nabla^2 f((1-s)z_0 + s z'T)\|_2 \|wT - yT\|_2^2 ds dt \right]
\]
\[
\leq \frac{1}{\epsilon^2} \mathbb{E}\left[\|wT - yT\|_2^2 \|z'T - z_0\|_2 + \|wT - yT\|_2^2 \right]
\]
\[
\leq \frac{4}{\epsilon^2} \left(\sqrt{\mathbb{E}\left[\|wT - yT\|_2^2\right]} + \mathbb{E}\left[\|vT - yT\|_2^2\right] \right) + \mathbb{E}\left[\|wT - yT\|_2^2\right] + \mathbb{E}\left[\|vT - yT\|_2^2\right]
\]
\[
\leq \frac{4}{\epsilon^2} \left(\mathbb{E}\left[\|wT - vT\|_2^2\right] + \mathbb{E}\left[\|vT - yT\|_2^2\right] \right) + \mathbb{E}\left[\|wT - vT\|_2^2\right] + \mathbb{E}\left[\|vT - yT\|_2^2\right]
\]
Where the second inequality uses Lemma 15.2.c. to bound \(|\nabla^2 f((1-s)z_0 + s z'T)| \leq \frac{1}{\epsilon^2}\), the second last inequality is by Young’s inequality, and the last inequality is holds for \(T \leq 1\).

We now bound each of the terms separately.
\[
\mathbb{E}\left[\|wT - vT\|_2^2\right] \leq 512\left(T^{7/2}L^4 + T^{3/2}L^4 + 128T^{7/2}L^4 \left(R^4 + \frac{\beta^4}{m^2}\right) + T^{3/2}L^4\right)
\] (40)

Where we use the same reasoning as 30).
\[
\mathbb{E}\left[\|vT - yT\|_2^2\right] \leq 32T^{-1/2}\delta d^2 \log n
\] (41)

Where the first inequality uses the definition of the coupling between \(y\) and \(v\) in 30) and Corollary 21.

Finally,
\[
\sqrt{T}\mathbb{E}\left[\|vT - x_0\|_2^2\] \leq \left(8\beta^2 T^{3/2} + 64T^{5/2}L^2 \left(R^2 + \frac{4\beta^2}{m^2}\right) \right)
\] (42)

Where we use Lemma 10 and our assumption on \(\mathbb{E}\left[\|x_0\|_2^2\]\).

It can be verified that there exists a \(C_1 = \text{poly}(L, R, 1/m, \beta, L)\) such that for any \(T \leq \frac{\epsilon^2}{cT}\),
\[
(40) + (42) \leq \frac{1}{16} TLe^2
\]

It can also be verified that there exists a \(C_2 = \text{poly}(L, R, 1/m, \beta, L)\) such that for any \(\delta \leq \frac{1}{T^2} (T^{3/2}\epsilon^2 \log \frac{1}{T^2})\), (Apply Lemma 22 with \(x = \frac{1}{2}\) and \(c = T^{3/2}\epsilon^2\))
\[
(41) \leq \frac{1}{16} TLe^2
\]
Thus
\[ \delta = \frac{8}{\epsilon} \left( \frac{40}{41} + \frac{41}{42} \right) \leq TL\epsilon \] (43)

We have thus shown that there exists a choice of \( C_1, C_2 = \text{poly}(L, R, 1/m, \beta, L\xi) \), such that for all \( T \leq \frac{\epsilon}{\xi} \) and for all 
\[
\delta \leq \min \left\{ \frac{1}{C_2} \left( T^3 \log \frac{1}{T\epsilon} \right), \frac{1}{C_2} \left( T^{3/2} \epsilon^2 \log \frac{1}{T\epsilon} \right) \right\},
\]
\[
1 + 3 + 1 \leq \frac{43}{44} + \frac{43}{44} + \frac{39}{43} \leq 3T\epsilon
\]

Since \( T \) is of order \( \epsilon^4 \), the second term \( \frac{1}{C_2} \left( T^{3/2} \epsilon^2 \log \frac{1}{T\epsilon} \right) \) is smaller (for sufficiently small \( \epsilon \)), it thus suffices that
\[
\delta \leq \frac{1}{C_2} \left( T^{3/2} \epsilon^2 \log \frac{1}{T\epsilon} \right).
\]
We have thus proved (33). \( \blacksquare \)

C Regularity of \( M \) and \( N \)

**Lemma 3**

1. \( \text{tr}(M(x)^2) \leq \beta^2 \)
2. \( \text{tr}(M(x)^2 - M(y)^2)^2 \) \( \leq \) \( 16\beta^2 L\xi^2 \|x - y\|_2^2 \)
3. \( \text{tr}(M(x)^2 - M(y)^2)^2 \) \( \leq \) \( 32\beta^3 L\xi \|x - y\|_2 \)

**Proof**

In this proof, we will use the fact that \( \xi(\cdot, \eta) \) is \( L\xi \)-Lipschitz from Assumption [3]

The first property is easy to see:

\[
\text{tr}(M(x)^2) = \text{tr} \left( \mathbb{E}_\eta [\xi(x, \eta)\xi(x, \eta)^T] \right) = \mathbb{E}_\eta \left[ \text{tr} (\xi(x, \eta)\xi(x, \eta)^T) \right] = \mathbb{E}_\eta \left[ \|\xi(x, \eta)\|_2^2 \right] \leq \beta^2
\]

We now prove the second and third claims. Consider a fixed \( x \) and fixed \( y \), let \( u_\eta := \xi(x, \eta), v_\eta := \xi(y, \eta) \). Then

\[
\text{tr} \left( (M(x)^2 - M(y)^2)^2 \right) = \text{tr} \left( \mathbb{E}_{\eta, \eta'} \left[ (u_{\eta} u_{\eta}^T - v_{\eta} v_{\eta}^T) (u_{\eta'} u_{\eta'}^T - v_{\eta'} v_{\eta'}^T) \right] \right)
\]

For any fixed \( \eta \) and \( \eta' \), let’s further simplify notation by letting \( u, u', v, v' \) denote \( u_\eta, u_{\eta'}, v_\eta, v_{\eta'} \). Thus

\[
\text{tr} \left( (uu^T - vv^T)(u'u'^T - v'v'^T) \right)
\]

\[
= \text{tr} \left( (u - v)u^Tv + u(u - v)^T + (u - v)(u - v)^T \right) (u' - v')v'^T + v'(u' - v')^T + (u' - v')(u' - v')^T) \]

\[
= \text{tr} \left( (u - v)u^Tv(u' - v')v'^T + (u - v)(u - v)^T (u' - v')(u' - v')^T \right)
\]

\[
+ \text{tr} \left( (u - v)u^Tv(u' - v')v'^T \right) + \text{tr} \left( (u - v)(u - v)^T (u' - v')(u' - v')^T \right) + \text{tr} \left( (u - v)(u - v)^T (u' - v')(u' - v')^T \right)
\]

\[
+ \text{tr} \left( (u - v)(u - v)^T (u' - v')(u' - v')^T \right)
\]

\[
\leq \min \left\{ 16\beta^2 L\xi^2 \|x - y\|_2^2, 32\beta^3 L\xi \|x - y\|_2 \right\}
\]
Where the last inequality uses Assumption B.2 and B.3; in particular, \(\|v\|_2 \leq \beta\) and \(\|u - v\|_2 \leq \min\{2\beta, L_y\|x - y\|_2\}\). This proves 2. and 3. of the Lemma statement.

**Lemma 4** Let \(N(x)\) be as defined in (1). Then

1. \(\text{tr}(N(x)^2) \leq \beta^2\)
2. \(\text{tr}\left((N(x) - N(y))^2\right) \leq L_N^2\|x - y\|_2^2\)
3. \(\text{tr}\left((N(x) - N(y))^2\right) \leq \frac{8\beta^2}{c_M} \cdot L_N\|x - y\|_2\)

for \(L_N = \frac{4\beta L_t}{c_M}\)

**Proof of Lemma 4**

The first inequality holds because \(N(x)^2 := M(x)^2 - c_M^2 I\), and then applying Lemma 3.1, and the fact that \(\text{tr}(M(x)^2 - c_M^2 I) \leq \text{tr}(M(x)^2)\) by Assumption B.4.

The second inequality is a immediate consequence of Lemma 5. Lemma 3.2, and the fact that \(\lambda_{\min}(N(x)^2) = \lambda_{\min}(M(x)^2 - c_M^2 I) \geq c_M^2\) by Assumption B.4.

The proof for the third inequality is similar to the second inequality, and follows from Lemma 3 and Lemma 5.

**Lemma 5** (Simplified version of Lemma 1 from [9]) Let \(A, B\) be positive definite matrices. Then

\[
\text{tr}\left((\sqrt{A} - \sqrt{B})^2\right) \leq \text{tr}(A - B)^2 A^{-1}
\]

### D Coupling Properties

**Lemma 6** Consider the coupled \((x_t, y_t)\) in (22). Let \(p_t\) denote the distribution of \(x_t\), and \(q_t\) denote the distribution of \(y_t\). Let \(p'_t\) and \(q'_t\) denote the distributions of \((20)\) and \((21)\).

If \(p_0 = p'_0\) and \(q_0 = q'_0\), then \(p_t = p'_t\) and \(q_t = q'_t\) for all \(t\).

**Proof of Lemma 6**

Consider the dynamics for \(x_t\) in (22). By reflection principle, \((I - 2\gamma \gamma^T)dB_s\) is also a standard Brownian motion.

Let \(d\hat{B}_t := \left[(I - 2\gamma \gamma^T)dB_t\right]_{\text{d}W_t}\), so that \(\hat{B}_t\) is a 2d dimensional standard Brownian motion. Let \(H(x) := \left[cI \quad N(x)\right]\). Then the dynamics of \(x_t\) in (22) can be written as

\[
dx_t = -\nabla U(x_t)dt + H(x_t)^T d\hat{B}_t
\]

Let \(p_t\) denote the density of \(x_t\) in (22). For all \(x\),

\[
\frac{d}{dt} p_t(x) = \text{div}(p_t(x) \nabla U(x)) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p_t(x) H(x)^T H(x) \right]_i
\]

\[
= \text{div}(p_t(x) \nabla U(x)) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p_t(x) \left(c_M^2 I + N(x)^2\right) \right]_i
\]

\[
= \text{div}(p_t(x) \nabla U(x)) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p_t(x) \left(M(x)^2\right) \right]_i
\]

(44)

On the other hand, let \(p'_t\) denote the density of \(x_t\) in (20). By Fokker Planck, for all \(x\),

\[
\frac{d}{dt} p'_t(x) = \text{div}(p'_t(x) \nabla U(x)) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p'_t(x) \left(M(x)^2\right) \right]_i
\]

which is equivalent to (44). Thus \(p_t = p'_t\) for all \(t\).
It can be verified that \( y_t \) from \( (21) \) has the distribution \( y_t \sim \mathcal{N}(-t\nabla U(y_0), M(y_0)^2) \). Next, consider \( y_t \) in \( (22) \), \( y_t \sim \mathcal{N}(-t\nabla U(y_0), c_{4t}I + N(y_0)^2) = \mathcal{N}(-t\nabla U(y_0), M(y_0)^2) \). Thus we show that \( q_t = q_t' \) for all \( t \). (The proofs for \( q_t = q_t' \) simple because the dynamics of \( y_t \) in \( (21) \) and \( (22) \) depend only on \( y_0 \).)

**Lemma 7** Consider the four processes defined in \( (22) \). For all \( t \), the processes satisfy

\[
E \left[ \|y_t - y_0\|_2^2 \right] \leq 2 \left( s^2 L^2 \|y_0\|_2^2 + s \beta^2 \right)
\]

**Proof**

By Ito’s Lemma,

\[
\frac{d}{dt} E \left[ \|y_t - y_0\|_2^2 \right] = 2E \left[ \langle y_t - y_0, -\nabla U(y_0) \rangle + \frac{1}{2} \text{tr} (M(y_0)^2) \right] \\
\leq \frac{1}{s} E \left[ \|y_t - y_0\|_2^2 \right] + s \|\nabla U(y_0)\|_2^2 + \beta^2 \\
\leq \frac{1}{s} E \left[ \|y_t - y_0\|_2^2 \right] + s L^2 \|y_0\|_2^2 + \beta^2
\]

Where the first inequality is by Lemma \( \ref{lemma3} \)

Rearranging terms,

\[
\frac{d}{dt} \left( E \left[ \|y_t - y_0\|_2^2 \right] + s^2 L^2 \|y_0\|_2^2 + s \beta^2 \right) \\
\leq \frac{1}{s} \left( E \left[ \|y_t - y_0\|_2^2 \right] + s^2 L^2 \|y_0\|_2^2 + s \beta^2 \right)
\]

By Gronwall’s inequality, for any \( t \leq s \),

\[
E \left[ \|y_t - y_0\|_2^2 \right] \leq \left( e^{t/s} - 1 \right) \left( s^2 L^2 \|y_0\|_2^2 + s \beta^2 \right) \\
\leq 2 \left( s^2 L^2 \|y_0\|_2^2 + s \beta^2 \right)
\]

**Lemma 8** Consider a sequence

\[
y_{k+1} = y_k - \delta \nabla U(y_k) + \sqrt{\delta} \xi_k(y_k)
\]

Where

1. \( \delta \leq m/(4L^2) \)
2. For all \( y, k \), \( \xi_k(y) \) satisfies \( E \left[ \|\xi_k(y)\|_2^2 \right] \leq \beta^2 \)
3. \( E \left[ \|\xi_k\|_2^2 \right] \leq 2 \left( R^2 + \frac{s^2}{m} \right) \)

Then for all \( k \),

\[
E \left[ \|y_k\|_2^2 \right] \leq 4 \left( R^2 + \beta^2 m \right)
\]

**Proof**

Let \( a(y) := (\|y\|_2 - R)_+ \). We can verify that

\[
\nabla a(y) = (\|y\|_2 - R) + \frac{y}{\|y\|_2} \\
\nabla^2 a(y) = \mathbb{1} \left\{ \|y\|_2 \geq R \right\} \frac{yy^T}{\|y\|_2^2} + (\|y\|_2 - R) + \frac{1}{\|y\|_2} \left( I - \frac{yy^T}{\|y\|_2^2} \right)
\]

Observe that

1. \( \|\nabla^2 a(y)\|_2 \leq \mathbb{1} \left\{ \|y\|_2 \geq R \right\} \leq 2 \)
2. \( \langle \nabla a(y), -\nabla U(y) \rangle \leq -ma(y) \). This can be verified by considering 2 cases. If \( \|y\|_2 \leq R \), then \( \nabla a(y) = 0 \) and \( a(y) = 0 \). If \( \|y\|_2 \geq R \), then by Assumption \[A\]

\[
\langle \nabla a(y), -\nabla U(y) \rangle \leq -m(\|y\|_2 - R)_+ \|y\|_2 \leq -m(\|y\|_2 - R)^+ = -m \cdot a(y)
\]

3. \( a(y) \geq \frac{1}{2} \|y\|_2^2 - 2R^2 \). This can be verified by considering two cases. If \( \|y\|_2 \leq 2R \), then \( a(y) \geq 0 \geq \frac{1}{2} \|y\|_2^2 - 2R^2 \). If \( \|y\|_2 \geq 2R \), then \( a(y) \geq \frac{1}{2} \|y\|_2^2 \geq \frac{1}{2} \|y\|_2^2 - 2R^2 \).

Using Taylor’s Theorem, and taking expectation of \( y_{k+1} \) conditioned on \( y_k \),

\[
E[a(y_{k+1})] = a(y_k) + E[(\nabla a(y_k), y_{k+1} - y_k)] + E \left[ \int_0^1 \langle \nabla^2 a(y_k + s(y_{k+1} - y_k), (y_{k+1} - y_k) \otimes 2 \rangle \, ds \right]
\]

\[
\leq a(y_k) + E[(\nabla a(y_k), y_{k+1} - y_k)] + E \left[ \int_0^1 \|(y_{k+1} - y_k)\|_2^2 \, ds \right]
\]

\[
\leq a(y_k) + E[(\nabla a(y_k), -\delta \nabla U(y_k))] + 2\delta^2 \|\nabla U(y_k)\|_2^2 + 2\delta E \left[ \|\xi_k(y_k)\|_2^2 \right]
\]

\[
\leq a(y_k) - m\delta a(y_k) + 2\delta^2 \|\nabla U(y_k)\|_2^2 + 2\delta E \left[ \|\xi_k(y_k)\|_2^2 \right]
\]

\[
\leq a(y_k) - m\delta a(y_k) + 2\delta^2 L_2^2 \|y_k\|_2^2 + 2\delta \beta^2
\]

\[
\leq a(y_k) - m\delta a(y_k) + 2\delta^2 L_2^2 a(y_k) + 2\delta L^2 R^2 + 2\delta \beta^2
\]

\[
\leq (1 - m\delta/2) a(y_k) + m\delta R^2 + 2\delta \beta^2
\]

Where the first inequality uses the upper bound on \( \|\nabla^2 a(y)\|_2 \) above, the second inequality uses the fact that \( E[\xi_k(y_k)] = 0 \) conditional on \( y_k \), the second inequality uses Young’s inequality, the third inequality uses claim 2. above, the fourth inequality uses claim 3. above, the fifth inequality uses our assumption that \( \delta \leq \frac{m}{4L^2} \).

Taking expectation wrt \( y_k \),

\[
E[a(y_{k+1})] \leq E[a(y_k)] - m\delta \left( E[a(y_k)] - 2R^2 + 2\beta^2/m \right)
\]

\[
\Rightarrow \quad E[a(y_{k+1})] - (2R^2/2 + 2\beta^2/m) \leq (1 - m\delta) \left( E[a(y_k)] - (2R^2 + 2\beta^2/m) \right)
\]

Thus \( E[a(y_{k+1})] - (2R^2 + 2\beta^2/m) \) is non-increasing. As a result, if \( E[\|y_0\|_2^2] \leq 2R^2 + 2\beta^2/m \), then \( E[a(y_0)] - (2R^2 + 2\beta^2/m) \leq 0 \), then \( E[a(y_k)] - (2R^2 + 2\beta^2/m) \leq 0 \) for all \( k \), which implies that

\[
E[\|y_k\|_2^2] \leq 2E[a(y_k)] + 2R^2 \leq 4(R^2 + \beta^2/m)
\]

**Lemma 9** Consider \( x_t \) as defined in (5). If \( x_0 \) satisfies

1. \( E[\|x_0\|_2^2] \leq \frac{2\beta^2}{m} \)
2. \( E[\|x_0\|_2^2] \leq 8R^4 + \frac{256\beta^4}{m^2} \)

Then for all \( t \),

1. \( E[\|x_t\|_2^2] \leq R^2 + \frac{4\beta^2}{m} \)
2. \( E[\|x_t\|_2^4] \leq 32R^4 + \frac{512\beta^4}{m^2} \)

In particular, the above bounds hold for \( p^* \).

**Proof**

We consider the potential function \( a(x) = (\|x\|_2 - R)_+ \). We verify that

\[
\nabla a(x) = (\|x\|_2 - R)_+ \frac{x}{\|x\|_2}
\]

\[
\nabla^2 a(x) = \mathbb{1} \{\|x\|_2 \geq R\} \frac{xx^T}{\|x\|_2^2} + \frac{(\|x\|_2 - R)_+}{\|x\|_2} \left( I - \frac{xx^T}{\|x\|_2^2} \right)
\]

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Therefore,
\[
\frac{d}{dt} \mathbb{E}[a(x_t)] = \mathbb{E}\left[\langle \nabla a(x_t), -\nabla U(x_t) dt \rangle\right] + \mathbb{E}\left[\text{tr}(M(x_t) \nabla^2 a(x_t))\right] \\
\leq -m \mathbb{E}[a(x_t)] + 2\beta^2
\]
Thus if \( \mathbb{E}[a(x_0)] \leq \frac{2\beta^2}{m} \), then for all \( t \),
\[
\mathbb{E}[a(x_t)] \leq \frac{2\beta^2}{m}
\]
This implies that, for all \( t \),
\[
\mathbb{E}\left[\|x_t\|^2\right] \leq \mathbb{E}[2a(x_t) + R^2] \leq R^2 + \frac{4\beta^2}{m}
\]
Where the inequality uses the fact that \( 0 < a(x) < 1 \) and \( \text{tr}(M(x)) \leq \beta^2 \).
Thus if \( \mathbb{E}[a(x_0)] \leq \frac{2\beta}{m} \), then for all \( t \),
\[
\mathbb{E}[a(x_t)] \leq \frac{2\beta}{m}
\]
To bound the fourth moment, Let \( a(x) = (\|x\|^2 - R^2)^\frac{3}{2} \). Then
\[
\nabla a(x) = 2\left(\|x\|^2 - R^2\right)^{\frac{1}{2}} \\
\nabla^2 a(x) = 4\left(\|x\|^2 - R^2\right)x x^T + 2\left(\|x\|^2 - R^2\right)I
\]
We can further verify that
\[
a(x) = (\|x\|^2 - R^2)^2 \geq \frac{1}{2}\|x\|^4 - 8R^4
\]
The proof considers two cases: when \( \|x\|^2 \geq 2R^2 \), \( (\|x\|^2 - R^2)^\frac{3}{2} \geq 0 \) \( \geq \frac{1}{2}\|x\|^4 - 8R^4 \). On the other hand, when \( \|x\|^2 \leq 2R^2 \), \( (\|x\|^2 - R^2)^\frac{3}{2} \geq \|x\|^2 \geq \frac{1}{2}\|x\|^4 - R^4 \).
By Ito’s Lemma,
\[
\frac{d}{dt} \mathbb{E}[a(x_t)] = \mathbb{E}\left[\langle \nabla a(x_t), -\nabla U(x_t) dt \rangle\right] + \mathbb{E}\left[\text{tr}(M(x_t) \nabla^2 a(x_t))\right] \\
= \mathbb{E}\left[2\left(\|x_t\|^2 - R^2\right)^\frac{3}{2}, \nabla U(x_t)\right] + \mathbb{E}\left[\text{tr}(M(x_t) \left(4\left(\|x_t\|^2 - R^2\right)x_t x_t^T + 2\left(\|x_t\|^2 - R^2\right)I\right))\right] \\
\leq -2m \mathbb{E}\left[\left(\|x_t\|^2 - R^2\right)^\frac{3}{2}\|x_t\|^2\right] + 6\beta^2 \mathbb{E}\left[\|x_t\|^2\right] \\
\leq -2m \mathbb{E}[a(x_t)] + \frac{256\beta^4}{m} + \frac{m}{2} \mathbb{E}[\|x_t\|^2] \\
\leq -2m \mathbb{E}[a(x_t)] + \frac{256\beta^4}{m} + m \mathbb{E}[a(x_t)] + 8mR^4 \\
\leq -m \mathbb{E}[a(x_t)] + 8mR^4 + \frac{256\beta^4}{m}
\]
The above is negative if
\[
\mathbb{E}[a(x_t)] \geq 8R^4 + \frac{256\beta^4}{m^2}
\]
Therefore, if \( \mathbb{E}[a(x_0)] \leq 8R^4 + \frac{256\beta^4}{m^2} \), then \( \mathbb{E}[a(x_t)] \leq 8R^4 + \frac{256\beta^4}{m^2} \) for all \( t \).
Finally,
\[
\mathbb{E}[a(x_t)] \leq 8R^4 + \frac{256\beta^4}{m^2} \\
\Rightarrow \mathbb{E}[\|x_t\|^2] \leq 2\mathbb{E}[a(x_t)] + 16R^4 \leq 32R^4 + \frac{512\beta^4}{m^2}
\]
\[\blacksquare\]
Lemma 10 Let \( dx_t = -\nabla U(x_t)dt + M(x_t)dB_t \) initialized at \( x_0 \). Then for any \( T \leq \frac{1}{16T} \)

1. \( P\left( \|x_T - x_0\|_2 \geq 2\sqrt{T}\beta^2 + 8TL\|x_0\|_2 + c \right) \leq \exp \left( -\frac{c^2}{8T}\right) \)

2. For any desired probability \( \rho \), let \( c_\rho := \sqrt{8T\beta^2 \log \frac{1}{\rho}} \), then

\[
P\left( \|x_T - x_0\|_2 - \left( 2\sqrt{T}\beta^2 + 8TL\|x_0\|_2 \right) \geq c_\rho \right) \leq \rho
\]

3. \( E\left[ \left\|x_T - x_0\right\|_2 \right] \leq 8\beta^2 T + 64T^2 L^2 \|x_0\|_2^2 \)

4. \( E\left[ \left\|x_T - x_0\right\|_2^4 \right] \leq 128T^2 \beta^4 + 214T^4 L^4 \|x_0\|_2^2 \)

**Proof**

Let’s define \( y_t := x_t - x_0 \).

Consider any \( s, t \in \mathbb{R}^+ \) satisfying \( t \leq T \) and \( s \leq \frac{1}{8T}\beta^2 \). Let us define

\[
g(t, y) := \exp \left( s(1 - t(1/(4T) + 2s\beta^2))\|y\|_2^2 - st(2\beta^2 + 32TL^2\|x_0\|_2^2) \right)
\]

One can verify that

\[
\frac{\partial}{\partial t} g(t, y) = g(t, y) \left( -s/(4T) - 2s^2\beta^2 \right)\|y\|_2^2 - s(2\beta^2 + 32TL^2\|x_0\|_2^2) + \nabla_U(x_0 + y_0) \cdot \nabla g(t, y)
\]

Applying Ito’s Lemma,

\[
\frac{d}{dt} E[g(t, y)] = E \left[ g(t, y) \left( -s/(4T) - 2s^2\beta^2 \right)\|y\|_2^2 - s(2\beta^2 + 32TL^2\|x_0\|_2^2) \right] + E \left[ \nabla g(t, y) \cdot \nabla_U(x_0 + y_0) \right] dt
\]

\[
\leq - s/(4T) \cdot 2s^2\beta^2 E \left[ g(y)\|y\|_2^2 \right] - s(2\beta^2 + 32TL^2\|x_0\|_2^2) E \left[ g(t, y) \right] + 2s(1 - t(1/(4T) + 2s\beta^2)) E \left[ g(t, y) \right] + 2s^2(1 - t(1/(4T) + 2s\beta^2)) E \left[ g(y)\|y\|_2^2 \right] + 2s(1 - t(1/(4T) + 2s\beta^2)) E \left[ \|y\|_2^2 \right]
\]

\[
\leq - s/(4T) \cdot 2s^2\beta^2 E \left[ g(y)\|y\|_2^2 \right] - s(2\beta^2 + 32TL^2\|x_0\|_2^2) E \left[ g(t, y) \right] + 2s\beta^2 E \left[ g(y)\|y\|_2^2 \right] + 2s^2\beta^2 E \left[ \|y\|_2^2 \right] + 2s^2\beta^2 E \left[ g(t, y) \right]
\]

\[
\leq 0
\]

Thus,

\[
E[g(t, y)] \leq E[g(0, y_0)] = 1
\]

Set \( s := \frac{1}{8T}\beta^2 \). Then by our assumptions on \( s \) and \( t \leq T \), \( (1 - t(1/(4T) + 2s\beta^2))\|y\|_2^2 \geq \frac{1}{2}\|y\|_2^2 \), so for any \( c \),

\[
P\left( \|y\|_2^2 - 2t(2\beta^2 + 32TL^2\|x_0\|_2^2) \geq c \right)
\]

\[
= P\left( \|y\|_2^2/2 - t(2\beta^2 + 32TL^2\|x_0\|_2^2) \geq c/2 \right)
\]

\[
\leq P\left( (1 - t(1/(4T) + 2s\beta^2))\|y\|_2^2/2 - t(2\beta^2 + 32TL^2\|x_0\|_2^2) \geq c/2 \right)
\]

\[
= P\left( g(t, y) \geq \exp \left( \frac{s c}{2} \right) \right)
\]
Where the last line is by definition of \(g(t, y_t)\).

Using Markov’s inequality, we get

\[
P \left( g(t, y_t) \geq \frac{sc}{2} \right) \leq \frac{\mathbb{E}[g(t, y_t)]}{\exp \left( \frac{sc}{2} \right)} \leq \exp \left( -\frac{sc}{2} \right) = \exp \left( -\frac{c}{8\beta^2 T} \right)
\]

Put together,

\[
P \left( \| y_t \|_2^2 \geq 2T(2\beta^2 + 32TL^2\|x_0\|_2^2) + c \right) \leq \exp \left( -\frac{c}{8\beta^2 T} \right)
\]

As an implication,

\[
P \left( \| y_t \|_2 \geq 2\sqrt{T\beta^2 + 8TL\|x_0\|_2} + c \right) \leq \exp \left( -\frac{c^2}{8\beta^2 T} \right)
\]

To bound the second moment,

\[
\begin{aligned}
&\mathbb{E} \left[ \| y_t \|_2^2 - 4T\beta^2 - 64T^2L^2\|x_0\|_2^2 \right] \\
\leq &\int_0^\infty \exp \left( -\frac{s}{8\beta^2 T} \right) ds \\
\leq &8\beta^2 T
\end{aligned}
\]

Thus

\[
\begin{aligned}
\mathbb{E} \left[ \| y_t \|_2^2 \right] \leq &8\beta^2 T + 4\beta^2 T + 64T^2L^2\|x_0\|_2^2 \\
\leq &8\beta^2 T + 64T^2L^2\|x_0\|_2^2
\end{aligned}
\]

To bound the 4\textsuperscript{th} moment,

\[
\begin{aligned}
&\mathbb{E} \left[ (\| y_t \|_2^2 - 4T\beta^2 - 64T^2L^2\|x_0\|_2^2)^2 \right] \\
= &\int_0^\infty P \left( (\| y_t \|_2^2 - 4T\beta^2 - 64T^2L^2\|x_0\|_2^2)^2 \geq s \right) ds \leq \int_0^\infty \exp \left( \frac{\sqrt{s}}{8\beta^2 T} \right) ds = 16\beta^4T^2
\end{aligned}
\]

Thus

\[
\begin{aligned}
\mathbb{E} \left[ \| y_t \|_4^4 \right] \leq &2\mathbb{E} \left[ (\| y_t \|_2^2 - 4T\beta^2 - 64T^2L^2\|x_0\|_2^2)^2 \right] + 2(4T\beta^2 + 64T^2L^2\|x_0\|_2^2)^2 \\
\leq &32\beta^2T^2 + 64T^2\beta^4 + 2^{14}T^4L^4\|x_0\|_2^4 \\
\leq &128T^2\beta^4 + 2^{14}T^4L^4\|x_0\|_2^4
\end{aligned}
\]

**Lemma 11** Let \(w_{(k+1),a} = -\delta \nabla U(w_{k,a}) + \sqrt{\delta}(w_{k,a}, \eta_k)\) be as defined in [31]. Then for any \(\delta, n\) such that \(n\delta \leq 1/(16L)\), and for any desired accuracy \(\rho\),

1. \(P \left( \| w_{n\delta} - w_0 \|_2 \geq 9n\delta L\|w_0\|_2 + 4\sqrt{n\delta \beta^2} + c \right) \leq \exp \left( -\frac{c^2}{4n\delta \beta^2} \right)\)

2. For any desired probability \(\rho\), let \(c_\rho := 2\sqrt{n\delta \beta^2 \log(1/\rho)}\), then

\[
P \left( \| w_{n\delta} - w_0 \|_2 \geq 9n\delta L\|w_0\|_2 + 4\sqrt{n\delta \beta^2} + c_\rho \right) \leq \rho
\]

3. \(\mathbb{E} \left[ \| w_{n\delta} - w_0 \|_2^2 \right] \leq 72n^2\delta^2L^2\|w_0\|_2^2 + 12n\delta \beta^2\)
4. \( \mathbb{E} \left[ \|w_{n \delta} - w_0\|_2^4 \right] \leq 10368n^4 \delta^4 L^4 \|w_0\|_2^4 + 128n^2 \delta^2 \beta^4 \)

**Proof**

Let us define the constant \( s := \frac{1}{4n \delta \beta^2} \). Condition on \( \eta_1...\eta_k \), and taking expectation with respect to \( \eta_{k+1} \) gives

\[
\mathbb{E} \left[ \exp \left( s\|w_{k+1} - w_0\|_2^2 \right) \right] = \mathbb{E} \left[ \exp \left( s\|w_k - w_0\|_2^2 + 2s \delta \langle w_k - w_0, \nabla U(w_k) \rangle + 2s \langle w_k - w_0 - \delta \nabla U(w_k), \sqrt{\theta}(w_k, \eta_k) \rangle + s\delta \|\xi(w_k, \eta_k)\|_2^2 \right) \right]
\]

\[
\leq \exp \left( s\|w_k - w_0\|_2^2 + 2s \delta \langle w_k - w_0, \nabla U(w_k) \rangle + s\delta \beta^2 \right) \cdot \mathbb{E} \left[ \exp \left( 2s \langle w_k - w_0 - \delta \nabla U(w_k), \sqrt{\theta}(w_k, \eta_k) \rangle \right) \right]
\]

\[
\leq \exp \left( s\|w_k - w_0\|_2^2 + 2s \delta \langle w_k - w_0, \nabla U(w_k) \rangle + s\delta \beta^2 \right) \cdot \exp \left( 2s^2 \|w_k - \delta \nabla U(w_k) - w_0\|_2^2 \cdot \delta \beta^2 \right)
\]

\[
\leq \exp \left( s \left( 1 + 2s \delta \beta^2 + \frac{1}{2n} \right) \|w_k - w_0\|_2^2 + 2s \delta^2 \|\nabla U(w_k)\|_2^2 + s\delta \beta^2 + 2s^2 \delta^3 \beta^2 L^2 \|w_0\|_2^2 \right)
\]

\[
\leq \exp \left( s \left( 1 + 2s \delta \beta^2 + \frac{1}{2n} \right) \|w_k - w_0\|_2^2 + \frac{8}{n} s \delta^2 L^2 \|w_k - w_0\|_2^2 + 8sn \delta^2 L^2 \|w_0\|_2^2 + s\delta \beta^2 + 2s^2 \delta^3 \beta^2 L^2 \|w_0\|_2^2 \right)
\]

\[
\leq \exp \left( s \left( 1 + \frac{2}{n} \right) \|w_k - w_0\|_2^2 + 9sn \delta^2 L^2 \|w_0\|_2^2 + s\delta \beta^2 \right)
\]

Where we use multiple times the assumption that \( n \delta \beta^2 \leq \frac{1}{8} \). The last line is by the fact that \( s = \frac{1}{4n \delta \beta^2} \). The second inequality crucially uses Lemma 12.

Applying the above recursively,

\[
\mathbb{E} \left[ \exp \left( s\|w_n - w_0\|_2^2 \right) \right] \leq \mathbb{E} \left[ \exp \left( \sum_{k=1}^{n} \exp \left( \left( 1 + \frac{2}{n} \right)^{n-k} \left( 9sn \delta^2 L^2 \|w_0\|_2^2 + s\delta \beta^2 \right) \right) \right) \right]
\]

\[
\leq \mathbb{E} \left[ \exp \left( 8sn \left( 9sn \delta^2 L^2 \|w_0\|_2^2 + s\delta \beta^2 \right) \right) \right]
\]

Using Markov’s inequality, and recalling the definition of \( s \), we immediately get

\[
P \left( \|w_{n \delta} - w_0\|_2^2 - 72n \delta^2 L^2 \|w_0\|_2^2 - 8n \delta \beta^2 \geq c \right) \leq \exp \left( -c \right)
\]

\[
= \exp \left( -\frac{c}{4n \delta \beta^2} \right)
\]

From here, each of the claims of this Lemma can be proven identically as the proof of Lemma 10 and so will be skipped.

\[\Box\]

**Lemma 12** For any 0-mean random vector \( X \) which satisfies \( \|X\|_2 \leq \beta \) almost surely, and for any fixed vector \( v \),

\[
\mathbb{E} \left[ e^{c\langle v, X \rangle} \right] \leq e^{2\|v\|_2^2 \beta^2}
\]

**Proof**
Let \( \epsilon \) be a Rademacher random variable, and let \( \lambda \) be a fixed scalar. Then
\[
\mathbb{E}[e^{\epsilon \lambda}] = \frac{1}{2} \{ e^{-\lambda} + e^\lambda \} = \frac{1}{2} \left( \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} \leq 1 + \sum_{k=1}^{\infty} \frac{\lambda^{2k}}{2^k k!} = e^{\lambda^2/2}
\]

Using Jensen’s inequality and the convexity of \( f(x) = e^{(v \cdot x)} \),
\[
e^{(v \cdot \mathbb{E}[X])} \leq \mathbb{E}[e^{(v \cdot X)}]
\]

Thus for any \( v \),
\[
\mathbb{E}_X \left[ e^{(v \cdot X)} \right] = \mathbb{E}_X \left[ e^{(v \cdot X - \mathbb{E}[X])} \right] \leq \mathbb{E}_{X,X'} \left[ e^{(v \cdot X - X')} \right] = \mathbb{E}_{X,X'} \left[ \mathbb{E}_v \left[ e^{(v \cdot (X - X'))} \right] \right] \leq \mathbb{E}_{X,X'} \left[ e^{(v \cdot (X - X'))^2/2} \right] \leq e^{2\|v\|^2\delta^2}
\]

Where the second last inequality uses our earlier result on Rademacher variable.

Lemma 13 Let \( v_{k,\delta} \) and \( w_{k,\delta} \) be as defined in (30) and (31). Then for any \( \delta, n \) satisfying \( n \leq \frac{1}{\delta^2} \),
\[
\mathbb{E} \left[ \|v_{n,\delta} - w_{n,\delta}\|^2 \right] \leq 256(n^2 \delta^2 L^2 + n\delta L^2)(n^2 \delta^2 L^2 \mathbb{E} \left[ \|w_0\|^2 \right] + n\delta^2)
\]

Proof

We first bound
\[
\mathbb{E} \left[ \|v_{(k+1),\delta} - v_0\|^2 \right] = \mathbb{E} \left[ \|v_{k,\delta} - \delta \nabla U(v_0) + \sqrt{\delta} \xi(v_0, \eta_0) - v_0\|^2 \right] = \mathbb{E} \left[ \|v_{k,\delta} - \delta \nabla U(v_0) - v_0\|^2 \right] + \mathbb{E} \left[ \|\sqrt{\delta} \xi(v_0, \eta_0)\|^2 \right] \\
\leq (1 + 1/n) \mathbb{E} \left[ \|v_{k,\delta} - v_0\|^2 \right] + (1 + n) \mathbb{E} \left[ \|\delta \nabla U(v_0)\|^2 \right] + \delta \beta^2 \\
\leq \exp(1/n) \mathbb{E} \left[ \|v_{k,\delta} - v_0\|^2 \right] + 2n \delta^2 L^2 \mathbb{E} \left[ \|v_0\|^2 \right] + \delta \beta^2 \\
= \exp(1/n) \mathbb{E} \left[ \|v_{k,\delta} - y_0\|^2 \right] + 2n \delta^2 L^2 \mathbb{E} \left[ \|w_0\|^2 \right] + \delta \beta^2
\]

Applying the above recursively, up to \( n \),
\[
\mathbb{E} \left[ \|v_{n,\delta} - y_0\|^2 \right] \leq \sum_{k=0}^{n} \exp(k/n) (2n \delta^2 L^2 \mathbb{E} \left[ \|v_0\|^2 \right] + \delta \beta^2) \\
\leq \exp(1) (2n^2 \delta^2 L^2 \mathbb{E} \left[ \|w_0\|^2 \right] + n\delta^2) \\
\leq 7(n^2 \delta^2 L^2 \mathbb{E} \left[ \|w_0\|^2 \right] + n\delta^2)
\]

(45)
For \( k \leq n \),

\[
\mathbb{E} \left[ \|v_{k+1} - w_{k+1}\|^2 \right] \\
\leq \mathbb{E} \left[ \|v_k - \delta \nabla U(w_0) - w_{k+1} + \delta \nabla U(w_k) + \sqrt{3} \delta \tilde{\xi}(w_0, \eta_k) - \sqrt{3} \delta \tilde{\xi}(w_k, \eta_k)\|^2 \right] \\
\leq \left(1 + \frac{1}{2n}\right)^2 \mathbb{E} \left[ \|v_k - \delta \nabla U(v_k) - w_{k+1} + \delta \nabla U(w_k)\|^2 \right] + 3n \delta^2 \mathbb{E} \left[ \|\nabla U(v_k) - \nabla U(w_0)\|^2 \right] + \delta L^2 \mathbb{E} \left[ \|w_k - w_0\|^2 \right] \\
\leq \left(1 + \frac{1}{2n}\right)^2 \mathbb{E} \left[ \|v_k - \delta \nabla U(v_k) - w_{k+1} + \delta \nabla U(w_k)\|^2 \right] + 3n \delta^2 L^2 \mathbb{E} \left[ \|v_k - v_0\|^2 \right] + \delta L^2 \mathbb{E} \left[ \|w_k - w_0\|^2 \right] \\
\leq \left(1 + \frac{1}{2n}\right)^2 \left(1 + \delta L\right)^2 \mathbb{E} \left[ \|v_k - w_{k+1}\|^2 \right] + 3n \delta^2 L^2 \left(7(n^2 \delta^2 L^2 \|w_0\|^2 + n \delta \beta^2)\right) + \delta L^2 \left(72n^2 \delta^2 L^2 \|w_0\|^2 + 12n \delta \beta^2\right)
\]

Where the second last inequality uses (45) and Lemma 11.

Applying the above recursively,

\[
\mathbb{E} \left[ \|v_n - w_n\|^2 \right] \\
\leq \sum_{k=0}^{n} \exp(k/n + 2k \delta L) \left(3n \delta^2 L^2 \left(7(n^2 \delta^2 L^2 \|w_0\|^2 + n \delta \beta^2)\right) + \delta L^2 \left(72n^2 \delta^2 L^2 \|w_0\|^2 + 12n \delta \beta^2\right)\right) \\
\leq 9n^2 \delta^2 L^2 \left(7(n^2 \delta^2 L^2 \|w_0\|^2 + n \delta \beta^2)\right) + 3n \delta L^2 \left(72n^2 \delta^2 L^2 \|w_0\|^2 + 12n \delta \beta^2\right) \\
\leq 256 \left(n^2 \delta^2 L^2 + n \delta L^2\right) \left(n^2 \delta^2 L^2 \|w_0\|^2 + n \delta \beta^2\right)
\]

Where the last inequality uses the fact that \( n \delta \leq \frac{1}{4} \).

Lemma 14 Consider the sequence \( w_{k,\delta} \) as defined in 11. If

1. \( \mathbb{E} \left[ \|w_0\|^2 \right] \leq 128 \left(R^4 + \frac{\beta^4}{m^2}\right) \)
2. \( \delta \leq \frac{m}{128R^2} \)

Then for all \( k \),

\[
\mathbb{E} \left[ \|w_{k,\delta}\|^2 \right] \leq 1024 \left(R^4 + \frac{\beta^4}{m^2}\right)
\]

Proof

To simplify notation, we use \( w_k \) instead of \( w_{k,\delta} \) in this proof.

Let us define

\[
a(x) = (\|x\|^2 - R)^+\)

We can verify that

\[
\nabla a(x) = 4(\|x\|^2 - R)^+ \frac{x}{\|x\|^2} \\
\nabla^2 a(x) = 12(\|x\|^2 - R)^+ \frac{x x^T}{\|x\|^4} + 3(\|x\|^2 - R)^+ \left(\frac{1}{\|x\|^2} - \frac{x x^T}{\|x\|^2}\right)
\]

We can also verify that

\[
\|\nabla^2 a(x)\|^2 \leq (\|x\|^2 - R)^+ \leq \|x\|^2 \leq R + \sqrt{\beta}
\]

Finally, notice that \( \|w_{k+1}\|^2 \leq (1 + \delta L)\|w_k\| + \sqrt{\beta} \)
Using Taylor’s Theorem,

\[ a(w_{k+1}) = a(w_k) - \delta \nabla U(w_k) + \sqrt{\delta} \xi(w_k, \eta_k) \]

\[ = a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \int_0^1 \int_s^1 \left( \nabla^2 a(w_k + s(coefficient)(\nabla U + \sqrt{\delta} \xi)(w_k, \eta_k)) \right) \left( w_{k+1} - w_k \right) \right) dsdt \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \int_0^1 \int_s^1 \left( \nabla^2 a(w_k + s(coefficient)(\nabla U + \sqrt{\delta} \xi)(w_k, \eta_k)) \right) \left( w_{k+1} - w_k \right)^2 \right) dsdt \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \int_0^1 \int_s^1 \left( \nabla^2 a(w_k + s(coefficient)(\nabla U + \sqrt{\delta} \xi)(w_k, \eta_k)) \right) \left( w_{k+1} - w_k \right)^2 \right) dsdt \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \left( 2(\|w_k\|_2 + R)\|w_{k+1} - w_k\|_2 \right) \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \left( 4(\|w_k\|_2^2 + R^2)(\delta^2 L^2\|w_k\|_2^2 + \delta^2) \right) \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \left( 4(\delta^2 L^2\|w_k\|_2^2 + \delta^2 L^2 R^2\|w_k\|_2^2 + \delta\|w_k\|_2^2 + \delta^2) \right) \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \left( 4(3\delta^2 L^2\|w_k\|_2^4 + 2\delta^2 L^2 R^4 + \frac{m\delta}{32}\|w_k\|_2^4 + \frac{64\delta}{m}\beta^4 + m\delta R^4) \right) \]

\[ \leq a(w_k) - \delta \left( \nabla a(w_k), \nabla U(w_k) \right) + \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \]

\[ + \left( \frac{m\delta}{4}\|w_k\|_2^4 + \frac{64\delta}{m}\beta^4 + 2m\delta R^4 \right) \]

Where the second inequality is by (46), the third inequality is by triangle inequality, the fourth inequality is by Assumptions A and B, the fifth inequality uses the assumption that \( \delta \leq R^2/\beta^2 \) and \( \delta \leq 1/L \). The last inequality is by Young’s inequality and the assumption that \( \delta \leq \frac{m}{128L^2} \).

We now note a few facts:

1. \( \left( \nabla a(w_k), -\nabla U(w_k) \right) \leq -m\delta a(w_k) \). This is true for \( \|w_k\|_2 \geq R \) by Assumption A and is true for \( \|w_k\|_2 \leq R \) as \( \nabla a(w_k) = 0 \) and \( a(w_k) = 0 \).

2. \( a(w_k) = (\|w_k\|_2 - R)^4 + \frac{3}{4}\|w_k\|_2^2 - 8R^4 \). This is true for \( \|w_k\|_2 \geq 2R \) as \( a(w_k) = 0 \) \( \geq \frac{1}{2}\|w_k\|_2^4 - 128R^4 \) and this is true for \( \|w_k\|_2 \geq 2R \) as \( a(w_k) = \frac{1}{2}\|w_k\|_2^2 \geq \frac{1}{8}\|w_k\|_2^2 - 8R^4 \).

3. \( E\left[ \sqrt{\delta} \left( \nabla a(w_k), \xi(w_k, \eta_k) \right) \right] = 0 \), where the expectation is conditional on \( w_k \).
Thus
\[
\mathbb{E} [a(w_{k+1})] \leq (1 - m\delta) \mathbb{E} [a(w_k)] + E \left[ \frac{m\delta}{4} \|w_k\|_2^4 + \frac{m\delta}{4} R^4 + \frac{64\delta}{m} \beta^4 + m\delta R^4 \right]
\]
\[
\leq (1 - m\delta) \mathbb{E} [a(w_k)] + E \left[ \frac{m\delta}{2} a(w_k) + 64m\delta R^4 + \frac{64\delta}{m} \beta^4 + 2m\delta R^4 \right]
\]
\[
\leq (1 - m\delta/2) \mathbb{E} [a(w_k)] + 128 \left( m\delta R^4 + \frac{\beta^4}{m} \right)
\]

Rearranging terms,
\[
\mathbb{E} [a(w_{k+1})] \leq 128 \left( R^4 + \frac{\beta^4}{m^2} \right)
\]
\[
\leq (1 - m\delta/2) \left( \mathbb{E} [a(w_k)] - 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \right)
\]
Thus \( \mathbb{E} [a(w_k)] - 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \) is non-increasing. If \( \mathbb{E} [a(w_0)] - 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \leq 0 \), then for all \( k \), \( \mathbb{E} [a(w_k)] - 128 \left( m\delta R^4 + \frac{\beta^4}{m} \right) \leq 0 \).

Using again 2. above,
\[
\mathbb{E} \left[ \|w_0\|_2^4 \right] \leq 128 \left( R^4 + \frac{\beta^4}{m^2} \right)
\]
\[
\Rightarrow \quad \mathbb{E} [a(w_0)] - 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \leq 0
\]
\[
\Rightarrow \quad \forall k, \quad \mathbb{E} [a(w_k)] - 128 \left( R^4 + \frac{\beta^4}{m^2} \right) \leq 0
\]
\[
\Rightarrow \quad \forall k, \quad \mathbb{E} \left[ \|w_k\|_2^2 \right] \leq 1024 \left( R^4 + \frac{\beta^4}{m^2} \right)
\]

\[\blacksquare\]

E  Defining \( q \) and related inequalities

Lemma 15 (Properties of \( f \)) Given parameters \((\alpha_q, R_q, \epsilon)\). Assume that \( \epsilon \leq \frac{R_q}{\alpha_q R_q + 1}\)

\( f(z) := q(g(z)) \)

Where \( q \) is as defined in Section E.1 (using \( \alpha_q, R_q \)), and \( g \) is as defined in Lemma 17 (using \( \epsilon \)). Then

1. (a) \( \nabla f(z) = q'(g(z)) \cdot \nabla g(z) \)
   
   (b) For \( r \geq 2\epsilon \), \( \nabla f(z) = q'(g(z)) \frac{ze^T}{\|z\|^2} \)

2. (a) \( \nabla^2 f(z) = q''(g(z)) \nabla g(z) \nabla g(z)^T + q'(g(z)) \nabla^2 g(z) \)
   
   (b) For \( r \geq 2\epsilon \), \( \nabla^2 f(z) = q''(g(z)) \frac{ze^T}{\|z\|^2} + q'(g(z)) \frac{1}{\|z\|^2} \left( I - \frac{ze^T}{\|z\|^2} \right) \)

3. For all \( z \), \( \|\nabla^2 f(z)\|_2 \leq \left( \frac{5\alpha_q R_q}{4} + \frac{1}{R_q} \right) + \frac{1}{\epsilon} \leq \frac{2}{\epsilon} \)

4. For all \( z \), \( \|\nabla^3 f(z)\|_2 \|z\| \leq \frac{R_q}{\alpha_q R_q + 1}\)

Proof of Lemma 15
1. (a) chain rule
   (b) Use definition of \( \nabla g(z) \) from Lemma 17

2. (a) chain rule
   (b) by Lemma 17
   (c) by Lemma 17 and Lemma 18.
   (d) by noting that \( q''(z) \) is always negative (Lemma 18).

3. It can be verified that
   \[
   \nabla^3 f(z) = q''(g(z)) \cdot \nabla g(z) \otimes 3 + q''(g(z)) \nabla g(z) \otimes \nabla^2 g(z) + q''(g(z)) \nabla^2 g(z) \otimes \nabla g(z)
   \]
   
   Thus
   \[
   \| \nabla^3 f(z) \|_2 \leq |q''(g(z))| \| \nabla g(z) \|_2^3 + 3q''(g(z)) \| \nabla^2 g(z) \|_2 \| \nabla^2 g(z) \|_2 + q''(g(z)) \| \nabla^3 g(z) \|
   \]
   
   \[
   \leq 5\left( \alpha_q + \frac{1}{R_q^2} \right) (\alpha_q R_q^2 + 1) + 3 \left( \frac{5\alpha_q R_q}{4} + \frac{4}{R_q} \right) \cdot \frac{1}{\epsilon} + \frac{1}{\epsilon^2}
   \]
   
   \[
   \leq \frac{9}{\epsilon^2}
   \]
   
   Where the first inequality uses Lemma 18 and Lemma 17 and the second inequality assumes that \( \epsilon \leq \frac{R_q}{\alpha_q R_q^2 + 1} \).

4. \( f(z) \in \left[ \frac{1}{2} \exp \left( -\frac{7\alpha_q R_q^2}{3} \right) g(||z||_2), g(||z||_2) \right] \in \left[ \frac{1}{2} \exp \left( -\frac{7\alpha_q R_q^2}{3} \right) (||z||_2 - 2\epsilon, ||z||_2) \right] \)

   The first containment is by Lemma 18. 2. 1/2 exp (−7αqRq^2 3) · g(z) ≤ q(g(z)) ≤ g(z). The second containment is by Lemma 17: g(||z||_2) ∈ [||z||_2 − 2\epsilon, ||z||_2].

---

**Lemma 16 (Properties of h)** Given a parameter \( \epsilon \),

\[
\begin{align*}
h(r) := & \begin{cases} 
\frac{r^3}{6\epsilon^2}, & \text{for } r \in [0, \epsilon] \\
\frac{\epsilon}{6} + \frac{(r-\epsilon)^2}{2\epsilon} - \frac{(r-\epsilon)^3}{6\epsilon^2}, & \text{for } r \in [\epsilon, 2\epsilon] \\
\epsilon + r, & \text{for } r \geq 2\epsilon 
\end{cases}
\end{align*}
\]

1. The derivatives of \( h \) are as follows:

   \[
   \begin{align*}
h'(r) := & \begin{cases} 
\frac{r^2}{2\epsilon^2}, & \text{for } r \in [0, \epsilon] \\
\frac{1}{2} + \frac{r-\epsilon}{\epsilon} - \frac{(r-\epsilon)^2}{2\epsilon^2}, & \text{for } r \in [\epsilon, 2\epsilon] \\
1, & \text{for } r \geq 2\epsilon 
\end{cases}
\end{align*}
\]

   \[
   \begin{align*}
h''(r) := & \begin{cases} 
\frac{r}{2\epsilon^2}, & \text{for } r \in [0, \epsilon] \\
\frac{1}{\epsilon}, & \text{for } r \in [\epsilon, 2\epsilon] \\
0, & \text{for } r \geq 2\epsilon 
\end{cases}
\end{align*}
\]

   \[
   \begin{align*}
h'''(r) := & \begin{cases} 
\frac{1}{2\epsilon^2}, & \text{for } r \in [0, \epsilon] \\
\frac{1}{\epsilon^2}, & \text{for } r \in [\epsilon, 2\epsilon] \\
0, & \text{for } r \geq 2\epsilon 
\end{cases}
\end{align*}
\]

2. (a) \( h' \) is positive, monotonically increasing.
   (b) \( h'(0) = 0 \), \( h'(r) = 1 \) for \( r \geq \epsilon \)
   (c) \( \frac{h'(r)}{r} \leq \min \{ \frac{1}{2} , \frac{1}{\epsilon} \} \) for all \( r \)

3. (a) \( h''(r) \) is positive
   (b) \( h''(r) = 0 \) for \( r = 0 \) and \( r \geq 2\epsilon \)
(c) \( h''(r) \leq \frac{1}{r} \)

(d) \( \frac{h''(r)}{r} \leq \frac{1}{2r} \)

4. \( |h'''(r)| \leq \frac{1}{r^2} \)

5. \( r - 2\epsilon \leq h(r) \leq r \)

**Proof of Lemma 16**
The claims can all be verified with simple algebra.

**Lemma 17 (Properties of \( g \))**  Given a parameter \( \epsilon \), let us define

\[
g(z) := h(\|z\|_2)
\]

Where \( h \) is as defined in Lemma 16 (using parameter \( \epsilon \)). Then

1. (a) \( \nabla g(z) = h'(\|z\|_2) \frac{z}{\|z\|_2} \)

   (b) For \( \|z\|_2 \geq 2\epsilon \), \( \nabla g(z) = \frac{\epsilon}{\|z\|_2} \).

   (c) For any \( \|z\|_2 \), \( \|\nabla g(z)\|_2 \leq 1 \)

2. (a) \( \nabla^2 g(z) = h''(\|z\|_2) \frac{zz^T}{\|z\|_2^2} + h'(\|z\|_2) \frac{1}{\|z\|_2} \left( I - \frac{zz^T}{\|z\|_2^2} \right) \)

   (b) For \( r \geq 2\epsilon \), \( \nabla^2 g(z) = \frac{1}{\|z\|_2} \left( I - \frac{zz^T}{\|z\|_2^2} \right) \).

   (c) For \( r \geq 2\epsilon \), \( \|\nabla^2 g(z)\|_2 = \frac{1}{\|z\|_2} \)

   (d) For all \( r \), \( \|\nabla^2 g(z)\|_2 \leq \frac{1}{\epsilon} \)

3. \( \|\nabla^3 g(z)\|_2 \leq \frac{5}{\epsilon^2} \)

4. \( \|z\|_2 - 2\epsilon \leq g(z) \leq \|z\|_2 \)

**Proof of Lemma 17**
All the properties can be verified with algebra. We provide a proof for 3. since it is a bit involved.

Let us define the functions \( \kappa^1(z) = \nabla(\|z\|_2), \kappa^2(z) = \nabla^2(\|z\|_2), \kappa^3(z) = \nabla^3(\|z\|_2) \).

Specifically,

\[
\kappa^1(z) = \frac{z}{\|z\|_2}.
\]

\[
\kappa^2(z) = \frac{1}{\|z\|_2} \left( I - \frac{zz^T}{\|z\|_2^2} \right).
\]

\[
\kappa^3(z) = -\frac{1}{\|z\|_2^2 \|z\|_2} \bigotimes \left( I - \frac{zz^T}{\|z\|_2^2} \right) + \frac{1}{\|z\|_2} \left( \frac{z}{\|z\|_2} \bigotimes \kappa^2(z) + \kappa^2(z) \bigotimes \frac{z}{\|z\|_2} \right).
\]

It can be verified that

\[
\|\kappa^2(z)\|_2 = \frac{1}{\|z\|_2}
\]

\[
\|\kappa^3(z)\|_2 = \frac{1}{\|z\|_2^2}
\]

It can be verified that \( \nabla^2 g(z) \) has the following form:

\[
\nabla^3 g(z) = h'''(\|z\|_2) \bigotimes \kappa^3(z) + h''(\|z\|_2) \kappa^1(z) \bigotimes \kappa^2(z) + h'(\|z\|_2) \kappa^1(z) \bigotimes \kappa^1(z) + h'(\|z\|_2) \kappa^3(z) + h''(\|z\|_2) \kappa^1(z) \bigotimes \kappa^2(z)
\]

Thus

\[
\|\nabla^3 g(z)\|_2 \leq \left( h'''(\|z\|_2) + 3 \frac{h''(\|z\|_2)}{\|z\|_2} + \frac{h'(\|z\|_2)}{\|z\|_2^2} \right) \|z\|_2 \leq \frac{5}{\epsilon^2}
\]

Where we use properties of \( h \) from Lemma 16.

The last claim follows immediately from Lemma 16.4.
E.1 Defining \( q \)

In this section, we define the function \( q \) that is used in Lemma 15. We let \( \alpha_q > 0 \) and \( R_q > 0 \) be two arbitrary constants (these are parameters used in defining \( q \)). We begin by defining auxiliary functions \( \psi(r) \), \( \Psi(r) \) and \( \nu(r) \), all from \( \mathbb{R}^+ \) to \( \mathbb{R} \):

\[
\psi(r) := e^{-\alpha_q \tau(r)}, \quad \Psi(r) := \int_0^r \psi(s)ds, \quad \nu(r) := 1 - \frac{1}{2} \int_0^r \frac{\mu(s)\Psi(s)}{\psi(s)}ds, \quad (47)
\]

Where \( \tau(r) \) and \( \mu(r) \) are as defined in Lemma 19 and Lemma 20 with \( R = R_q \).

Finally we define \( q \) as

\[
q(r) := \int_0^r \psi(s)\nu(s)ds. \quad (48)
\]

We now state some useful properties of the distance function \( q \).

**Lemma 18** The function \( q \) defined in (48) (using parameters \( \alpha_q, R_q \)) has the following properties.

1. For all \( r \leq R_q \), \( q''(r) + \alpha_q r q'(r) \leq \frac{\exp\left(-\frac{7\alpha_q R_q^2}{2R_q^2}\right)}{32R_q^2} q(r) \)

2. For all \( r \), \( \frac{\exp\left(-\frac{7\alpha_q R_q^2}{2R_q^2}\right)}{2} \cdot r \leq q(r) \leq r \)

3. For all \( r \), \( \frac{\exp\left(-\frac{7\alpha_q R_q^2}{2R_q^2}\right)}{2} \leq q'(r) \leq 1 \)

4. For all \( r \), \( q''(r) \leq 0 \) and \( |q''(r)| \leq \left(\frac{2\alpha_q R_q}{R_q} + \frac{4}{R_q}\right) \)

5. For all \( r \), \( |q''(r)| \leq 5\alpha_q + 2\alpha_q R_q^2 + 1 + \frac{2(\alpha_q R_q^2 + 1)}{R_q^2} \)

**Proof of Lemma 18**

**Proof of 1.** It can be verified that

\[
\psi'(r) = \psi(r)(-\alpha_q \tau'(r))
\]

\[
\psi''(r) = \psi(r)\left((\alpha_q \tau'(r))^2 + \alpha_q \tau''(r)\right)
\]

\[
\nu'(r) = -\frac{1}{2} \int_0^r \frac{\mu(s)\Psi(s)}{\psi(s)}ds
\]

For \( r \in [0, R_q] \), \( \tau'(r) = r \), so that \( \psi'(r) = \psi(r)(-\alpha_q r) \). Thus

\[
q'(r) = \psi(r)\nu(r)
\]

\[
q''(r) = \psi'(r)\nu(r) = \psi(r)\nu(r)(-\alpha_q r) + \psi(r)\nu'(r)
\]

\[
= -\alpha_q r \nu'(r) + \psi(r)\nu'(r)
\]

\[
q''(r) + \alpha_q r q'(r) = \psi(r)\nu'(r)
\]

\[
= -\frac{1}{2} \int_0^r \frac{\mu(s)\Psi(s)}{\psi(s)}ds
\]

\[
= -\frac{1}{2} \int_0^r \frac{\Psi(s)}{\psi(s)}ds
\]

Where the last equality is by definition of \( \mu(r) \) in Lemma 20 and the fact that \( r \leq R_q \).

We can upper bound

\[
\int_0^{4R_q} \frac{\mu(s)\Psi(s)}{\psi(s)}ds \leq \int_0^{4R_q} \frac{\Psi(s)}{\psi(s)}ds \leq \int_0^{4R_q} \frac{sd\cdot\psi(4R_q)}{(4R_q)^2} = \frac{16R_q^2}{\psi(4R_q)} \leq 16R_q^2 \cdot \exp\left(\frac{7\alpha_q R_q^2}{3}\right)
\]

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Where the first inequality is by Lemma 20, the second inequality is by the fact that \( \psi(s) \) is monotonically decreasing, the third inequality is by Lemma 19.

Thus
\[
q''(r) + \alpha_q \tau'(r) \leq - \frac{1}{2} \left( \exp \left( -\frac{7\alpha_q R_q^2}{3} \right) \right) \Psi(r)
\]
\[
\leq - \frac{\exp \left( -\frac{7\alpha_q R_q^2}{3} \right)}{32R_q^2} q(r)
\]

Where the last inequality is by \( \Psi(r) \geq q(r) \).

**Proof of 2.** Notice first that \( \nu(r) \geq \frac{1}{2} \) for all \( r \). Thus
\[
q(r) := \int_0^r \psi(s)\nu(s)ds
\]
\[
\geq \frac{1}{2} \int_0^r \psi(s)ds
\]
\[
\geq \exp \left( -\frac{7\alpha_q R_q^2}{3} \right) \cdot r
\]

Where the last inequality is by Lemma 19.

**Proof of 3.** By definition of \( f \), \( q'(r) = \psi(r)\nu(r) \), and
\[
\exp \left( -\frac{7\alpha_q R_q^2}{3} \right) \leq \psi(r)\nu(r) \leq 1
\]

Where we use Lemma 19 and the fact that \( \nu(r) \in [1/2, 1] \)

**Proof of 4.** Recall that
\[
q''(r) = \psi'(r)\nu(r) + \psi(r)\nu'(r)
\]

That \( q'' \leq 0 \) can immediately be verified from the definitions of \( \psi \) and \( \nu \).

Thus
\[
|q''(r)| \leq |\psi'(r)\nu(r)| + |\psi(r)\nu'(r)|
\]
\[
\leq \alpha_q \tau'(r) + |\psi(r)\nu'(r)|
\]

From Lemma 19 we can upperbound \( \tau'(r) \leq \frac{5R_q}{2} \). In addition, \( \Psi(r) = \int_0^r \psi(s)ds \geq r\psi(r) \), so that
\[
\frac{\Psi(r)}{\psi(r)} \geq r
\]  
(49)

(Recall again that \( \psi(s) \) is monotonically decreasing). Thus \( \Psi(r)/r \geq r \) for all \( r \). In addition, using Lemma 19
\[
\Psi(r) = \int_0^r \psi(s)ds \leq 4R_q
\]  
(50)
Combining the previous expressions,

\[
|\psi(r)\nu'(r)| = \left| \frac{1}{2} \int_{0}^{4R_q} \frac{\mu(s)\Psi(s)}{\psi(s)} ds \right|
\]

\[
\leq \left| \frac{1}{2} \int_{0}^{4R_q} \frac{\Psi(r)}{\psi(s)} ds \right|
\]

\[
\leq \left| \frac{1}{2} \int_{0}^{4R_q} \frac{\Psi(r)}{\psi(s)} ds \right|
\]

\[
\leq \left| \frac{1}{2} \int_{0}^{4R_q} \frac{\Psi(r)}{\psi(s)} ds \right|
\]

\[
\leq \left| \frac{1}{2} \int_{0}^{4R_q} \sqrt{\frac{\psi(s)}{\nu'(r)}} ds \right|
\]

\[
\leq \frac{4}{R_q}
\]

Where the first inequality are by definition of \( \mu(r) \), and the second last inequality is by \([49]\). Combining with our bound on \( \psi'(r)\nu'(r) \) gives the desired bound.

**Proof of [5]**

\[ q'''(r) = \psi''(r)\nu(r) + 2\psi'(r)\nu'(r) + \psi(r)\nu''(r) \]

We first bound the middle term:

\[
|\psi'(r)\nu'(r)| = |\psi(r)(\alpha_q^2\tau'(r))^2 + \alpha_q\tau''(r))| \]

\[
\leq 5\alpha_q R_q \cdot \frac{4}{R_q}
\]

Where the second last line follows form Lemma \([19]\) and our proof of \([4]\).

Next,

\[ \psi''(r) = \psi(r)(\alpha_q^2\tau'(r) - \alpha_q\tau''(r)) \]

Thus applying Lemma \([19.1]\) and Lemma \([19.3]\),

\[
|\psi''(r)\nu(r)| \leq 2\alpha_q R_q^2 + \alpha_q
\]

Finally,

\[ \nu''(r) = \frac{1}{2} \int_{0}^{4R_q} \frac{\mu(s)\Psi(s)}{\psi(s)} ds \cdot \frac{d}{dr} \mu(r)\Psi(r)/\psi(r) \]

Expanding the numerator,

\[
\frac{d}{dr} \frac{\mu(r)\Psi(r)}{\psi(r)} = \mu'(r) \frac{\Psi(r)}{\psi(r)} + \mu(r) - \frac{\psi'(r)}{\psi(r)^2} \Psi(r)\nu'(r)
\]

\[
= \mu'(r) \frac{\Psi(r)}{\psi(r)} + \mu(r) + \mu(r)\Psi(r)\alpha_q\tau'(r)\]

Thus

\[ \psi(r)\nu''(r) = \frac{1}{2} \int_{0}^{4R_q} \frac{\mu(s)\Psi(s)}{\psi(s)} ds \cdot (\mu'(r)\Psi(r) + \mu(r)\psi(r) + \mu(r)\Psi(r)\alpha_q\tau'(r)) \]

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Using the same argument as from the proof of 4., we can bound
\[
\frac{1}{2} \int_0^{4\mathcal{R}_q} \frac{\mu(s)\Psi(s)}{\phi(s)} ds \leq \frac{1}{2} \int_0^{\mathcal{R}_q} sds \leq \frac{1}{\mathcal{R}_q^2}
\]

Finally, from Lemma 20, \(|\mu'(r)| \leq \frac{\pi}{3\mathcal{R}_q^2}\), so
\[
|\psi(r)\nu''(r)| \leq \frac{\pi/6 + 5\beta\mathcal{R}_q^2 / 4}{\mathcal{R}_q^2} \leq \frac{2(\alpha\mathcal{R}_q^2 + 1)}{\mathcal{R}_q^2}
\]

Lemma 19

Let \(\tau(r) : [0, \infty) \to \mathbb{R}\) be defined as
\[
\tau(r) = \begin{cases} 
\frac{r^2}{\mathcal{R}_q^2} + \mathcal{R}(r - \mathcal{R}) + \frac{(r - \mathcal{R})^2}{\mathcal{R}} - \frac{(r - \mathcal{R})^2}{2}, & \text{for } r \leq \mathcal{R} \\
\frac{5\mathcal{R}^2}{12} + \mathcal{R}(r - 2\mathcal{R}) - \frac{(r - 2\mathcal{R})^2}{2} + \frac{(r - 2\mathcal{R})^3}{12\mathcal{R}}, & \text{for } r \in [\mathcal{R}, 2\mathcal{R}] \\
0, & \text{for } r \geq 2\mathcal{R}
\end{cases}
\]

1. \(\tau'(r) \in [0, \frac{5\mathcal{R}}{2}], \text{ with maxima at } r = \frac{3\mathcal{R}}{2}\). \(\tau'(r) = 0 \text{ for } r \in \{0\} \cup [4\mathcal{R}, \infty)\)
2. As a consequence of 1, \(\tau(r)\) is monotonically increasing
3. \(\tau''(r) \in [-1, 1]\)

Proof of Lemma 19

We provide the derivatives of \(\tau\) below. The claims in the Lemma can then be immediately verified.
\[
\tau'(r) = \begin{cases} 
\tau, & \text{for } r \leq \mathcal{R} \\
\mathcal{R} + (r - \mathcal{R}) - \frac{(r - \mathcal{R})^2}{\mathcal{R}} - \frac{(r - \mathcal{R})^2}{2}, & \text{for } r \in [\mathcal{R}, 2\mathcal{R}] \\
\mathcal{R} - (r - 2\mathcal{R}) + \frac{(r - 2\mathcal{R})^2}{4\mathcal{R}}, & \text{for } r \in [2\mathcal{R}, 4\mathcal{R}] \\
0, & \text{for } r \geq 4\mathcal{R}
\end{cases}
\]

\[
\tau''(r) = \begin{cases} 
1, & \text{for } r \leq \mathcal{R} \\
1 - 2\frac{(r - \mathcal{R})}{\mathcal{R}}, & \text{for } r \in [\mathcal{R}, 2\mathcal{R}] \\
-1 + \frac{r - 2\mathcal{R}}{2\mathcal{R}}, & \text{for } r \in [2\mathcal{R}, 4\mathcal{R}] \\
0, & \text{for } r \geq 4\mathcal{R}
\end{cases}
\]

Lemma 20

Let
\[
\mu(r) := \begin{cases} 
1, & \text{for } r \leq \mathcal{R} \\
\frac{1}{2} + \frac{1}{2} \cos \left(\frac{\pi(r - \mathcal{R})}{3\mathcal{R}}\right), & \text{for } r \in [\mathcal{R}, 4\mathcal{R}] \\
0, & \text{for } r \geq 4\mathcal{R}
\end{cases}
\]

Then
\[
\mu'(r) := \begin{cases} 
0, & \text{for } r \leq \mathcal{R} \\
-\frac{\pi}{6\mathcal{R}} \sin \left(\frac{\pi(r - \mathcal{R})}{\mathcal{R}}\right), & \text{for } r \in [\mathcal{R}, 4\mathcal{R}] \\
0, & \text{for } r \geq 4\mathcal{R}
\end{cases}
\]

Furthermore, \(\mu'(r) \in [\frac{\pi}{6\mathcal{R}}, 0]\)

This Lemma can be easily verified by algebra.
F Miscellaneous

The following Theorem, taken from [9], establishes a quantitative CLT.

**Theorem 3** Let \( X_1, \ldots, X_n \) be random vectors with mean 0, covariance \( \Sigma \), and \( \|X_i\| \leq \beta \) almost surely for each \( i \). Let
\[
S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i,
\]
and let \( Z \) be a Gaussian with covariance \( \Sigma \), then
\[
W_2(S_n, Z) \leq \frac{6d\beta \sqrt{\log n}}{\sqrt{n}}.
\]

**Corollary 21** Let \( X_1, \ldots, X_n \) be random vectors with mean 0, covariance \( \Sigma \), and \( \|X_i\| \leq \beta \) almost surely for each \( i \). Let \( Y \) be a Gaussian with covariance \( n\Sigma \). Then
\[
W_2 \left( \sum_i X_i, Y \right) \leq \frac{6d\beta \sqrt{\log n}}{\sqrt{n}}.
\]

This is simply taking the result of Theorem 3 and scaling the inequality by \( \sqrt{n} \) on both sides.

The following Lemma is taken from [3] and included here for completeness.

**Lemma 22** For any \( c > 0 \), \( x > 3 \max \{ \frac{1}{c} \log \frac{1}{c}, 0 \} \), the inequality
\[
\frac{1}{c} \log(x) \leq x
\]
holds.

**Proof**
We will consider two cases:

**Case 1:** If \( c \geq \frac{1}{e} \), then the inequality
\[
\log(x) \leq cx
\]
is true for all \( x \).

**Case 2:** \( c \leq \frac{1}{e} \).

In this case, we consider the Lambert W function, defined as the inverse of \( f(x) = xe^x \). We will particularly pay attention to \( W_{-1} \) which is the lower branch of \( W \). (See Wikipedia for a description of \( W \) and \( W_{-1} \)).

We can lower bound \( W_{-1}(-c) \) using Theorem 1 from [??]:
\[
\forall u > 0, \quad W_{-1}(-e^{-u-1}) > -u - \sqrt{2u} - 1
\]
equivalently
\[
\forall c \in (0, 1/e), \quad -W_{-1}(-c) < \log \left( \frac{1}{c} \right) + 1 + \sqrt{2 \left( \log \left( \frac{1}{c} \right) - 1 \right)} - 1
\]
\[
= \log \left( \frac{1}{c} \right) + \sqrt{2 \left( \log \left( \frac{1}{c} \right) - 1 \right)}
\]
\[
\leq 3 \log \frac{1}{c}
\]

Thus by our assumption,
\[
x \geq 3 \cdot \frac{1}{c} \log \left( \frac{1}{c} \right)
\]
\[
\Rightarrow x \geq \frac{1}{c} (-W_{-1}(-c))
\]
then \( W_{-1}(-c) \) is defined, so
\[
x \geq \frac{1}{c} \max \{ -W_{-1}(-c), 1 \}
\]
\[
\Rightarrow (-cx)e^{-cx} \geq -c
\]
\[
\Rightarrow xe^{-cx} \leq 1
\]
\[
\Rightarrow \log(x) \leq cx
\]
The first implication is justified as follows: \( W_{-1}^{-1} : [-\frac{1}{e}, \infty) \rightarrow (-\infty, -1) \) is monotonically decreasing. Thus its inverse \( W_{-1}^{-1}(y) = ye^y \), defined over the domain \((-\infty, -1)\) is also monotonically decreasing. By our assumption, 
\[-cx \leq -3 \log \frac{1}{c} \leq -3, \text{ thus } -cx \in (-\infty, -1], \text{ thus applying } W_{-1}^{-1} \text{ to both sides gives us the first implication.} \]