EXOTIC DEFORMATION QUANTIZATION

VALENTIN OVSIENTKO

1 Introduction

Let $\mathcal{A}$ be one of the following commutative associative algebras: the algebra of all smooth functions on the plane: $\mathcal{A} = C^\infty(\mathbb{R}^2)$, or the algebra of polynomials $\mathcal{A} = \mathbb{C}[p, q]$ over $\mathbb{R}$ or $\mathbb{C}$. There exists a non-trivial formal associative deformation of $\mathcal{A}$ called the Moyal $\star$-product (or the standard $\star$-product). It is defined as an associative operation $\mathcal{A}^\otimes \to \mathcal{A}[[\hbar]]$ where $\hbar$ is a formal variable. The explicit formula is:

$$F \star \hbar G = FG + \sum_{k \geq 1} \frac{(i\hbar)^k}{2^kk!} \{F, G\}_k,$$

where $\{F, G\}_1 = \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial G}{\partial p}$ is the standard Poisson bracket, the higher order terms are:

$$\{F, G\}_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k F}{\partial p^{k-i} \partial q^i} \frac{\partial^k G}{\partial p^i \partial q^{k-i}}.$$  (2)

The Moyal product is the unique (modulo equivalence) non-trivial formal deformation of the associative algebra $\mathcal{A}$ (see [13]).

**Definition 1.** A formal associative deformation of $\mathcal{A}$ given by the formula (1) is called a $\star$-product if:
1) the first order term coincides with the Poisson bracket: $\{F, G\}_1 = \{F, G\}$;
2) the higher order terms $\{F, G\}_k$ are given by differential operators vanishing on constants: $\{1, G\}_k = \{F, 1\}_k = 0$;
3) $\{F, G\}_k = (-1)^k \{G, F\}_k$. 


Definition 2. Two $\star$-products $\star_h$ and $\star'_h$ on $\mathcal{A}$ are called equivalent if there exists a linear mapping $A_h : \mathcal{A} \to \mathcal{A}[[\hbar]]$:

$$A_h(F) = F + \sum_{k=1}^{\infty} A_k(F)\hbar^k$$

interwining the operations $\star_h$ and $\star'_h$: $A_h(F) \star'_h A_h(G) = A_h(F \star_h G)$.

Consider now $\mathcal{A}$ as a Lie algebra: the commutator is given by the Poisson bracket. The Lie algebra $\mathcal{A}$ has a unique (modulo equivalence) non-trivial formal deformation called the Moyal bracket or the Moyal $\star$-commutator: $\{F, G\}_t = \frac{1}{\hbar}(F \star_h G - G \star_h F)$, where $t = -\hbar^2/2$.

The well-known De Wilde–Lecomte theorem [4] states the existence of a non-trivial $\star$-product for an arbitrary symplectic manifold. The theory of $\star$-products is a subject of deformation quantization. The geometrical proof of the existence theorem was given by B. Fedosov [9] (see [8] and [20] for clear explanation and survey of recent progress).

The main idea of this paper is to consider the algebra $\mathcal{F}(M)$ of functions (with singularities) on the cotangent bundle $T^*M$ which are Laurent polynomials on the fibers. In contrast to above algebra $\mathcal{A}$ it turns out that for such algebras the standard $\star$-product is no more unique at least if $M$ is one-dimensional: $\dim M = 1$.

We consider $M = S^1, \mathbb{R}$ in the real case, and $M = \mathcal{H}$ (the upper half-plane) in the holomorphic case. The main result of this paper is an explicit construction of a new $\star$-product on the algebra $\mathcal{F}(M)$ non-equivalent to the standard Moyal product. This $\star$-product is equivariant with respect to the Möbius transformations. The construction is based on the bilinear $SL_2$-equivariant operations on tensor-densities on $M$, known as Gordan transvectants and Rankin-Cohen brackets.

We study the relations between the new $\star$-product and extensions of the Lie algebra $\text{Vect}(S^1)$.

The results of this paper are closely related to those of the recent work of P. Cohen, Yu. Manin and D. Zagier [3] where a one-parameter family of associative products on the space of classical modular forms is constructed using the same $SL_2$-equivariant bilinear operations.

2 Definition of the exotic $\star$-product

2.1 Algebras of Laurent polynomials. Let $\mathcal{F}$ be one of the following
associative algebras of functions:

\[ \mathcal{F} = C[p, 1/p] \otimes C^\infty(\mathbb{R}) \quad \text{or} \quad \mathcal{F} = C[p, 1/p] \otimes \text{Hol}(\mathcal{H}) \]

This means, it consists of functions of the type:

\[ F(p, q) = \sum_{i=-N}^{N} p^i f_i(q) \]  

where \( f_i(q) \in C^\infty(\mathbb{R}) \) in the real case, or \( f_i(q) \) are holomorphic functions on the upper half-plane \( \mathcal{H} \) or \( f_i \in C[q] \) (respectively).

We will also consider the algebra of polynomials: \( C[p, 1/p, q] \) (Laurent polynomials in \( p \)).

2.2 Transvectants. Consider the following bilinear operators on functions of one variable:

\[ J_{k}^{m,n}(f, g) = \sum_{i+j=k} (-1)^i \binom{k}{i} \frac{(2m-1)! (2n-j)!}{(2m-k)! (2n-k)!} f^{(i)} g^{(j)} \]  

where \( f = f(z), g = g(z), f^{(i)}(z) = \frac{d^i f(z)}{dz^i} \).

These operators satisfy a remarkable property: they are equivariant under Möbius (linear-fractional) transformations. Namely, suppose that the transformation \( z \mapsto \frac{az+b}{cz+d} \) (with \( ad - bc = 1 \)) acts on the arguments as follows:

\[ f(z) \mapsto f(\frac{az+b}{cz+d}) (cz+d)^m, \quad g(z) \mapsto g(\frac{az+b}{cz+d}) (cz+d)^n, \]

then \( J_{k}^{m,n}(f, g) \) transforms as: \( J_{k}^{m,n}(f, g)(z) \mapsto J_{k}^{m,n}(f, g)(\frac{az+b}{cz+d}) (cz+d)^{m+n-k} \).

In other words, the operations (4) are bilinear \( SL_2 \)-equivariant mappings on tensor-densities:

\[ J_{k}^{m,n} : \mathcal{F}_m \otimes \mathcal{F}_n \to \mathcal{F}_{m+n-k} \]

where \( \mathcal{F}_l \) is the space of tensor-densities of degree \(-l\): \( \phi = \phi(z)(dz)^{-l} \).

The operations (4) were discovered more than one hundred years ago by Gordan [11] who called them the transvectants. They have been rediscovered many times: in the theory of modular functions by Rankin [18] and by Cohen [2] (so-called Rankin-Cohen brackets), in differential projective geometry by Janson and Peetre [12]. The “multi-dimensional transvectants” were defined in [14] in the context of the the Virasoro algebra and symplectic and contact geometry.
2.3 Main definition. Define the following bilinear mapping $\mathcal{F}^\otimes 2 \rightarrow \mathcal{F}[[\hbar]]$, for $F = p^m f(q), G = p^n g(q)$, where $m, n \in \mathbb{Z}$, put:

$$F^\star \hbar G = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{2^{2k}} J_{k}^{m+n-k} f^{m,n}(f, g)$$

(5)

Note, that the first order term coincides with the Poisson bracket.

This operation will be the main subject of this paper. We call it the exotic $\star$-product.

2.4 Remark. Another one-parameter family of operations on modular forms: $f \star^\kappa g = \sum_{n=0}^{\infty} t^{\kappa}_{n}(k, l) J_{n}^{k,l}(f, g)$, where $t^{\kappa}_{n}(k, l)$ are very interesting and complicated coefficients, is defined in [3].

3 Main theorems

We formulate here the main results of this paper. All the proofs will be given in Sections 4-7.

3.1 Non-equivalence. The Moyal $\star$-product (4) defines a non-trivial formal deformation of $\mathcal{F}$. We will show that the formula (5) defines a $\star$-product non-equivalent to the standard Moyal product.

Theorem 1. The operation (5) is associative, it defines a formal deformation of the algebra $\mathcal{F}$ which is not equivalent to the Moyal product.

The associativity of the product (5) is a trivial corollary of Proposition 1 below. To prove the non-equivalence, we will use the relations with extensions of the Lie algebra of vector fields on $S^1$: Vect($S^1$) $\subset$ $\mathcal{F}$ (cf. Sec.5).

It is interesting to note, that the constructed $\star$-product is equivalent to the standard Moyal product if we consider it on the algebra $C^\infty(T^* M \setminus M)$ of all smooth functions (not only Laurent polynomials on fibers), cf. Corollary 1 below.

3.2 $sl_2$-equivariance. The Lie algebra $sl_2(\mathbb{R})$ has two natural embeddings into the Poisson Lie algebra on $\mathbb{R}^2$: the symplectic Lie algebra $sp_2(\mathbb{R}) \cong sl_2(\mathbb{R})$ generated by quadratic polynomials ($p^2, pq, q^2$) and another one with generators: ($p, pq, pq^2$) which is called the Möbius algebra.

It is well-known that the Moyal product (4) is the unique non-trivial formal deformation of the associative algebra of functions on $\mathbb{R}^2$ equivariant under the action of the symplectic algebra. This means, (4) satisfies the Leibnitz property:

$$\{F, G \star \hbar H\} = \{F, G\} \star \hbar H + G \star \hbar \{F, H\}$$

(6)
where $F$ is a quadratic polynomial (note that $\{F, G\}_s = \{F, G\}$ if $F$ is a quadratic polynomial).

**Theorem 2.** The product (3) is the unique formal deformation of the associative algebra $F$ equivariant under the action of the Möbius algebra.

The product (3) is the unique non-trivial formal deformation of $F$ satisfying (3) for $F$ from the Möbius $sl_2$ algebra.

### 3.3 Symplectomorphism $\Phi$

The relation between the Moyal product and the product (3) is as follows. Consider the symplectic mapping

$$\Phi(p, q) = \left(\frac{p^2}{2}, \frac{q}{p}\right).$$

defined on $\mathbb{R}^2 \setminus \mathbb{R}$ in the real case and on $\mathcal{H}$ in the complex case.

**Proposition 1.** The product (3) is the $\Phi$-conjugation of the Moyal product:

$$F \hat{\star}_h G = F \hat{\star}_h^\Phi G := (F \circ \Phi \hat{\star}_h G \circ \Phi) \circ \Phi^{-1} \quad (8)$$

**Remark.** The mapping (7) (in the complex case) can be interpreted as follows. It transforms the space of holomorphic tensor-densities of degree $-k$ on $\mathbb{CP}^1$ to the space $C^k[p, q]$ of polynomials of degree $k$. Indeed, there exists a natural isomorphism $z^n(dz)^{-m} \mapsto p^m q^n$ (where $m \geq 2n$) and $(p^m q^n) \circ \Phi = p^{2m-n} q^n$.

### 3.4 Operator formalism

The Moyal product is related to the following Weil quantization procedure. Define the following differential operators:

$$\hat{\tilde{p}} = i\hbar \frac{\partial}{\partial q}, \quad \hat{\tilde{q}} = q \quad (9)$$

satisfying the canonical relation: $[\hat{\tilde{p}}, \hat{\tilde{q}}] = i\hbar \mathbf{1}$. Associate to each polynomial $F = F(p, q)$ the differential operator $\hat{F} = \text{Sym} \hat{\tilde{p}} \hat{\tilde{q}}$ symmetric in $\hat{\tilde{p}}$ and $\hat{\tilde{q}}$.

The Moyal product on the algebra of polynomials coincides with the product of differential operators: $F \hat{\star}_h G = \hat{F} \hat{G}$.

We will show that the $\star$-product (4) leads to the operators:

$$\hat{\tilde{p}}^\Phi = \left(\frac{i\hbar}{2}\right)^2 \Delta, \quad \hat{\tilde{q}}^\Phi = \frac{1}{4\hbar}(\Delta^{-1} \circ A + A \circ \Delta^{-1}) \quad (10)$$

(where $\Delta = \frac{\partial^2}{\partial q^2}$ and $A = 2q \frac{\partial}{\partial q} + 1$ is the dilation operator) also satisfying the canonical relation.
Remark that \( \hat{p}^\Phi \) and \( \hat{q}^\Phi \) given by (10) on the Hilbert space \( L^2(\mathbb{R}) \), are not equivalent to the operators (9) since \( \hat{q}^\Phi \) is symmetric but not self-adjoint (see [6] on this subject).

### 3.5 “Symplectomorphic” deformations

Let us consider the general situation.

**Proposition 2.** Given a symplectic manifold \( V \) endowed with a \( \star \)-product \( \star \bar{\hbar} \) and a symplectomorphism \( \Psi \) of \( V \), if there exists a hamiltonian isotopy of \( \Psi \) to the identity, then the \( \Psi \)-conjugate product \( \star^\Psi \bar{\hbar} \) defined according to the formula (8) is equivalent to \( \star \bar{\hbar} \).

**Corollary 1.** The \( \star \)-product (5) considered on the algebra of all smooth functions \( C^\infty(T^*\mathbb{R} \setminus \mathbb{R}) \) is equivalent to the Moyal product.

### 4 Möbius-invariance

In this section we prove Theorem 2. We show that the operations of transvectant (9) are \( \Phi \)-conjugate of the terms of the Moyal product.

#### 4.1 Lie algebra \( \text{Vect}(\mathbb{R}) \) and modules of tensor-densities

Let \( \text{Vect}(\mathbb{R}) \) be the Lie algebra of smooth (or polynomial) vector fields on \( \mathbb{R} \):

\[
X = X(x) \frac{d}{dx}
\]

with the commutator

\[
[X(x) \frac{d}{dx}, Y(x) \frac{d}{dx}] = (X(x)Y'(x) - X'(x)Y(x)) \frac{d}{dx}.
\]

The natural embedding of the Lie algebra \( sl_2 \subset \text{Vect}(\mathbb{R}) \) is generated by the vector fields \( d/dx, xd/dx, x^2d/dx \).

Define a 1-parameter family of \( \text{Vect}(\mathbb{R}) \)-actions on \( C^\infty(\mathbb{R}) \) given by

\[
L^{(\lambda)}_X f = X(x)f'(x) - \lambda X'(x)f(x)
\]

where \( \lambda \in \mathbb{R} \). Geometrically, \( L^{(\lambda)}_X \) is the operator of Lie derivative on tensor-densities of degree \(-\lambda\):

\[
f = f(x)(dx)^{-\lambda}.
\]

Denote \( \mathcal{F}_\lambda \) the \( \text{Vect}(\mathbb{R}) \)-module structure on \( C^\infty(\mathbb{R}) \) given by (11).

#### 4.2 Transvectant as a bilinear \( sl_2 \)-equivariant operator

The operations (9) can be defined as bilinear mappings on \( C^\infty(\mathbb{R}) \) which are \( sl_2 \)-equivariant:
Statement 4.1. For each $k = 0, 1, 2, \ldots$ there exists a unique (up to a constant) bilinear $sl_2$-equivariant mapping

$$F_\mu \otimes F_\nu \to F_{\mu + \nu - k}.$$ 

It is given by $f \otimes g \mapsto J_k^{\mu,\nu}(f, g)$.

Proof: straightforward (cf. [11], [12]).

4.3 Algebra $F$ as a module over $\text{Vect}(R)$. The Lie algebra $\text{Vect}(R)$ can be considered as a Lie subalgebra of $F$. The embedding $\text{Vect}(R) \subset F$ is given by:

$$X(x) \frac{d}{dx} \mapsto pX(q).$$

The algebra $F$ is therefore, a $\text{Vect}(R)$-module.

Lemma 4.2. The algebra $F$ is decomposed to a direct sum of $\text{Vect}(R)$-modules:

$$F = \oplus_{m \in \mathbb{Z}} F_m.$$ 

Proof. Consider the subspace of $F$ consisting of functions homogeneous of degree $m$ in $p$: $F = p^m f(q)$. This subspace is a $\text{Vect}(R)$-module isomorphic to $F_m$. Indeed, $\{pX(q), p^m f(q)\} = p^m (X f' - mX f) = p^m L_X^{(m)} f$.

4.4 Projective property of the diffeomorphism $\Phi$. The transvectants (4) coincide with the $\Phi$-conjugate operators (2) from the Moyal product:

Proposition 4.3. Let $F = p^m f(q), G = p^n g(q)$, then

$$\Phi^{-1}\{\Phi^* F, \Phi^* G\}_k = \frac{k!}{2^k} p^{m+n-k} J_k^{m,n}(f, g).$$

(12)

Proof. The symplectomorphism $\Phi$ of $R^2$ intertwines the symplectic algebra $sp_2 \cong sl_2$ and the Möbius algebra: $\Phi^*(p, pq, pq^2) = (\frac{1}{2}p^2, \frac{1}{2}pq, \frac{1}{2}q^2)$. Therefore, the operation $\Phi^{-1}\{\Phi^* F, \Phi^* G\}_k$ is Möbius-equivariant.

On the other hand, one has: $\Phi^* F = \frac{1}{2^m} p^{2m} f(q)$ and $\Phi^* G = \frac{1}{2^n} p^{2n} g(q)$. Since $\Phi^* F$ and $\Phi^* G$ are homogeneous of degree $2m$ and $2n$ (respectively), the function $\{\Phi^* F, \Phi^* G\}_k$ is also homogeneous of degree $2(m + n - k)$. Thus, the operation $\{F, G\}_k^\Phi = \Phi^{-1}\{\Phi^* F, \Phi^* G\}_k$ defines a bilinear mapping on the space of tensor-densities $F_m \otimes F_n \to F_{m+n-k}$ which is $sl_2$-equivariant.

Statement 4.1 implies that it is proportional to $J_k^{m,n}$. One easily verifies the coefficient of proportionality for $F = p^m, G = p^n q^k$, to obtain the formula (12).

Proposition 4.3 is proven.

Remark. Proposition 4.3 was proven in [11]. We do not know whether this elementary fact has been mentioned by classics.
4.5 Proof of Theorem 2. Proposition 4.3 implies that the formula (5) is a $\Phi$-conjugation of the Moyal product and is given by the formula (8). Proposition 1 is proven.

It follows that (5) is a $\star$-product on $\mathcal{F}$ equivariant under the action of the Möbius $sl_2$ algebra. Moreover, it is the unique $\star$-product with this property since the Moyal product is the unique $\star$-product equivariant under the action of the symplectic algebra. Theorem 2 is proven.

5 Relation with extensions of the Lie algebra $Vect(S^1)$

We prove here that the $\star$-product (5) is not equivalent to the Moyal product.

Let $Vect(S^1)$ be the Lie algebra of vector fields on the circle. Consider the embedding $Vect(S^1) \subset \mathcal{F}$ given by functions on $\mathbb{R}^2$ of the type: $X = pX(q)$ where $X(q)$ is periodical: $X(q + 1) = X(q)$.

5.1 An idea of the proof of Theorem 1. Consider the formal deformations of the Lie algebra $\mathcal{F}$ associated to the $\star$-products (4) and (5). The restriction of the Moyal bracket to $Vect(S^1)$ is identically zero. We show that the restriction of the $\star$-commutator

$$\{F, G\}_t = \frac{1}{i\hbar}(F\bar{\star}_t H - G\bar{\star}_t H), \quad t = -\frac{\hbar^2}{2}$$

associated to the $\star$-product (5) defines a series of non-trivial extensions of the Lie algebra $Vect(S^1)$ by the modules $\mathcal{F}_k(S^1)$ of tensor-densities on $S^1$ of degree $-k$.

5.2 Extensions and the cohomology group $H^2(Vect(S^1); \mathcal{F}_\lambda)$. Recall that an extension of a Lie algebra by its module is defined by a 2-cocycle on it with values in this module. To define an extension of $Vect(S^1)$ by the module $\mathcal{F}_\lambda$ one needs therefore, a bilinear mapping $c : Vect(S^1)^{\otimes 2} \to \mathcal{F}_\lambda$ which verify the identity $\delta c = 0$:

$$c(X, [Y, Z]) + L_X^{(\lambda)} c(Y, Z) + (\text{cycle}_{X, Y, Z}) = 0.$$  

(see [10]).

The cohomology group $H^2(Vect(S^1); \mathcal{F}_\lambda)$ were calculated in [19] (see [10]). This group is trivial for each value of $\lambda$ except $\lambda = 0, -1, -2, -5, -7$. The explicit formulæ for the corresponding non-trivial cocycles are given in [17]. If $\lambda = -5, -7$, then $\dim H^2(Vect(S^1); \mathcal{F}_\lambda) = 1$, the cohomology group is
generated by the unique (up to equivalence) non-trivial cocycle. We will obtain these cocycles from the $\star$-commutator.

### 5.3 Non-trivial cocycles on $\text{Vect}(S^1)$

Consider the restriction of the $\star$-commutator $\{ , \}_t$ (corresponding to the $\star$-product (3)) to $\text{Vect}(S^1) \subset \mathcal{F}$: let

\[ X = pX(q), \quad Y = pY(q), \]

then from (3):

\[ \{X, Y\}_t = \{X, Y\} + \sum_{k=1}^{\infty} \frac{i^k}{2^{2k+1}} \frac{1}{p^{2k-1}} J_{2k+1}^{1,1}(X, Y) \]

It follows from the Jacobi identity that the first non-zero term of the series $\{X, Y\}_t$ is a 2-cocycle on $\text{Vect}(S^1)$ with values in one of the $\text{Vect}(S^1)$-modules $\mathcal{F}_k(S^1)$.

Denote for simplicity $J_{2k+1}^{1,1}$ by $J_{2k+1}$.

From the general formula (4) one obtains:

**Lemma 5.1.** First two terms of $\{X, Y\}_t$ are identically zero: $J_3(X, Y) = 0$, $J_5(X, Y) = 0$, the next two terms are proportional to:

\[ \begin{align*}
J_7(X, Y) &= X'''Y^{(IV)} - X^{(IV)}Y''' \\
J_9(X, Y) &= 2(X'''Y^{(V)} - X^{(V)}Y''') - 9(X^{(IV)}Y^{(V)} - X^{(V)}Y^{(IV)})
\end{align*} \]

(13)

The transvectant $J_7$ defines therefore a 2-cocycle. It is a remarkable fact that the same fact is true for $J_9$:

**Lemma 5.2.** (see [17]). The mappings

\[ J_7 : \text{Vect}(S^1)^{\otimes 2} \rightarrow \mathcal{F}_{-5} \quad \text{and} \quad J_9 : \text{Vect}(S^1)^{\otimes 2} \rightarrow \mathcal{F}_{-7} \]

are 2-cocycles on $\text{Vect}(S^1)$ representing the unique non-trivial classes of the cohomology groups $H^2(\text{Vect}(S^1); \mathcal{F}_{-5})$ and $H^2(\text{Vect}(S^1); \mathcal{F}_{-7})$ (respectively).

**Proof.** Let us prove that $J_9$ is a 2-cocycle on $\text{Vect}(S^1)$. The Jacobi identity for the bracket $\{ , \}_t$ implies:

\[ \{X, J_9(Y, Z)\} + J_9(X, \{Y, Z\}) + J_3(X, J_7(Y, Z)) + (\text{cycle}_{X,Y,Z}) = 0 \]

for any $X = pX(q), Y = pY(q), Z = pZ(q)$. One checks that the expression $J_3(X, J_7(Y, Z))$ is proportional to $X'''(Y'''Z^{(IV)} - Y^{(IV)}Z''')$, and so one gets:

\[ J_3(X, J_7(Y, Z)) + (\text{cycle}_{X,Y,Z}) = 0. \]
We obtain the following relation:

$$\{X, J_0(Y, Z)\} + J_0(X, \{Y, Z\}) + (\text{cycle}_{X,Y,Z}) = 0$$

which means that $J_0$ is a 2-cocycle. Indeed, recall that for any tensor density $a$, $\{pX, p^ma\} = p^m L^{(m)}_{X/dx}(a)$. Thus, the last relation coincides with the relation $\delta J_0 = 0$.

Let us show now that the cocycle $J_7$ on $\text{Vect}(S^1)$ is not trivial. Consider a linear differential operator $A : \text{Vect}(S^1) \to \mathcal{F}$. It is given by: $A(X(q)d/dq) = (\sum_{i=0}^{K} a_i X^{(i)}(q))(dq)^5$, then $\delta A(X, Y) = L_Y^{(5)} A(Y) - L_Y^{(5)} A(X) - A([X, Y])$. The higher order part of this expression has a non-zero term $(5-K)a_K X' Y^{(K)}$ and therefore $J_7 \neq \delta A$.

In the same way one proves that the cocycle $J_9$ on $\text{Vect}(S^1)$ is non-trivial.

Lemma 5.2 is proven.

It follows that the $*$-product (5) on the algebra $\mathcal{F}$ is not equivalent to the Moyal product.

Theorem 1 is proven.

6 Operator representation

We are looking for an linear mapping (depending on $\hbar$) $F \mapsto \hat{F}^\Phi$ of the associative algebra of Laurent polynomials $\mathcal{F} = C[p, 1/p, q]$ into the algebra of formal pseudodifferential operators on $\mathbb{R}$ such that

$$\hat{F}^\Phi \hat{\ast}_h \hat{G}^\Phi = \hat{\Phi}^* F \hat{\ast} \hat{G}.$$  

Recall that the algebra of Laurent polynomials $C[p, 1/p, q]$ with the Moyal product is isomorphic to the associative algebra of pseudodifferential operators on $\mathbb{R}$ with polynomial coefficients (see [1]). This isomorphism is defined on the generators $p \mapsto \hat{p}, q \mapsto \hat{q}$ by the operators (3) and $p^{-1} \mapsto \hat{p}^{-1}$:

$$\hat{p}^{-1} = \frac{1}{i\hbar} (\partial/\partial q)^{-1}.$$

6.1 Definition. Put:

$$\hat{F}^\Phi = \Phi^* F$$  \hspace{1cm} (14)

then $\hat{F}^\Phi \hat{G}^\Phi = \Phi^* F \hat{\ast} \Phi^* G = \Phi^* F \hat{\ast}_h \Phi^* G = \Phi^* (F \hat{\ast}_h G) = \hat{F}^\Phi \hat{\ast}_h \hat{G}^\Phi$.

One obtains the formulae (11). Indeed,

$$\hat{p}^\Phi = \hat{p}^2/2 = \frac{(i\hbar)^2}{2} \frac{\partial^2}{\partial q^2}. $$
Since \( q = \left( \frac{1}{2} \right) \tilde{\pi}_h p q + p q \tilde{\pi}_h \left( \frac{1}{2} \right) \), one gets:

\[
\hat{q}^\Phi = \frac{1}{4i\hbar} (\Delta \circ A + A \circ \Delta)
\]

### 6.2 \( sl_2 \)-equivariance

For the Möbius \( sl_2 \) algebra one has:

\[
\begin{align*}
\hat{p}^\Phi &= \frac{(i\hbar)^2}{2} \frac{\partial^2}{\partial q^2} \\
\hat{pq}^\Phi &= \frac{i\hbar}{4} + \frac{i\hbar}{2} q \frac{\partial}{\partial q} \\
\hat{pq^2}^\Phi &= \frac{q^2}{2}
\end{align*}
\]

**Lemma 6.1.** The mapping \( F \mapsto \hat{F}^\Phi \) satisfies the Möbius-equivariance condition:

\[
\{ \hat{X}, F \}^\Phi = [\hat{X}^\Phi, \hat{F}^\Phi]
\]

for \( X \in sl_2 \).

**Proof.** It follows immediately from Theorem 2. Indeed, the \( \star \)-product is \( sl_2 \)-equivariant (that is, satisfying the relation: \( \{ X, F \}_t = \{ X, F \} \) for \( X \in sl_2 \)).

**Remark.** Beautiful explicit formulae for \( sl_2 \)-equivariant mappings from the space of tensor-densities to the space of pseudodifferential operators are given in [3].

### 7 Hamiltonian isotopy

The simple calculations below are quite standard for the cohomological technique. We need them to prove Corollary 1 (from Sec. 3).

Given a symplectomorphism \( \Psi \) of a symplectic manifold \( V \) and a formal deformation \( \{ , \}_t \) of the Poisson bracket on \( V \), we prove that if \( \Psi \) is isotopic to the identity then the formal deformation \( \{ , \}_t^\Psi \) defined by:

\[
\{ F, G \}_t^\Psi = \Psi^{-1} \{ \Psi^* F, \Psi^* G \}_t
\]

is equivalent to \( \{ , \}_t \). The similar proof is valid in the case of \( \star \)-products.

Recall that two symplectomorphisms \( \Psi \) and \( \Psi' \) of a symplectic manifold \( V \) are isotopic if there exists a family of functions \( H(s) \) on \( V \) such that the symplectomorphism \( \Psi_1 \circ \Psi_2^{-1} \) is the flow of the Hamiltonian vector field with the Hamiltonian function \( H(s) \), \( 0 \leq s \leq 1 \).
Let $\Psi(s)$ be the flow of a family of functions $H = H(s)$. We will prove that the equivalence class of the formal deformation $\{ , \}_t^{\Psi(s)}$ does not depend on $s$.

### 7.1 Equivalence of homotopic cocycles

Let us first show that the cohomology class of the cocycle $C_{\Psi(s)}^3(F, G)$:

$$C_{\Psi(s)}^3(F, G) = \Psi_{\Psi(s)}^{-1}(\Psi_{\Psi(s)}^*F, \Psi_{\Psi(s)}^*G)$$

does not depend on $s$. To do this, it is sufficient to prove that the derivative $\dot{C}_3 = \frac{d}{ds}C_{\Psi(s)}^3|_{s=0}$ is a coboundary. One has

$$\dot{C}_3(F, G) = C_3(\{H, F\}, G) + C_3(F, \{H, G\}) - \{H, C_3(F, G)\}.$$

The relation $\delta C_3 = 0$ implies:

$$\dot{C}_3(F, G) = \{F, C_3(G, H)\} - \{G, C_3(F, H)\} - C_3(\{F, G\}, H).$$

This means, $\frac{d}{ds}C_{\Psi(s)}^3|_{s=0} = \delta B_H$, where $B_H(F) = C_3(F, H)$.

### 7.2 General case

Let us apply the same arguments to prove that the deformations $\{ , \}_t^{\Psi(s)}$ are equivalent to each other for all values of $s$. We must show that there exists a family of mappings $A(s)(F) = F + \sum_{k=1}^{\infty} A(s)_k(F)t^k$ such that $A_{\Psi(s)}^{-1}(\{A(s)(F), A(s)(G)\}_t) = \{F, G\}_t$.

It is sufficient to verify the existence of a mapping $a(F) = \sum_{k=1}^{\infty} a_k(F)t^k$ (the derivative: $a(F) = \frac{d}{ds}(A(s)(F))|_{s=s_0}$) such that

$$\frac{d}{ds}\{F, G\}_t^{\Psi(s)}|_{s=s_0} = \{a(F), G\}_t + \{F, a(G)\} - a(\{F, G\}_t)$$

One has:

$$\frac{d}{ds}\{F, G\}_t^{\Psi(s)}|_{s=s_0} = \{\{F, H\}_t, G\}_t + \{F, \{G, H\}_t\}_t - \{\{F, G\}_t, H\}_t$$

From the Jacobi identity:

$$\{\{F, H\}_t, G\}_t + \{F, \{G, H\}_t\}_t - \{\{F, G\}_t, H\}_t = 0$$

one obtains that the mapping $a(F)$ can be written in the form:

$$a(F) = \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} C_{2k+1}(F, H_{(s_0)})t^k$$
7.3 Proof of Corollary 2. Consider the ⋆-product (8) given by $F \star_{h}^{\Phi} G$, where $F \star_{h}^{\Phi} G$ is the Moyal product (1), $\Phi : (p, q) \mapsto (p^{2}/2, q/p)$. It is defined on $R^{2} \setminus R$.

The ⋆-product (8) on the algebra $C^{\infty}(R^{2} \setminus R)$ is equivalent to the Moyal product. Indeed, the symplectomorphism $\Phi$ is isotopic to the identity in the group of all smooth symplectomorphisms of $R^{2} \setminus R$. The isotopy is: $\Phi_{s} : (p, q) \mapsto (p^{1+s}, q/p)$, where $s \in [0, 1]$.

Recall, that the ⋆-product $F \star_{h}^{\Phi} G$ on the algebra $\mathcal{F}$ is not equivalent to the Moyal product since it coincides with the product (5).

The family $\Phi_{s}$ does not preserve the algebra $\mathcal{F}$. Theorem 1 implies that $F$ is not isotopic to the identity in the group of symplectomorphisms of $R^{2} \setminus R$ preserving the algebra $\mathcal{F}$.

8 Discussion

8.1 Difficulties in multi-dimensional case

There exist multi-dimensional analogues of transvectants [14] and [16].

Consider the projective space $RP^{2n+1}$ endowed with the standard contact structure (or an open domain of the complex projective space $CP^{2n+1}$). There exists an unique bilinear differential operator of order $k$ on tensor-densities equivariant with respect to the action of the group $Sp_{2n}$ (see [14], [17]):

$$J_{k}^{\lambda, \mu} : \mathcal{F}_{\lambda} \otimes \mathcal{F}_{\mu} \rightarrow \mathcal{F}_{\lambda+\mu-\frac{k}{n+1}}$$

where $\mathcal{F}_{\lambda} = \mathcal{F}_{\lambda}(P^{2n+1})$ is the space of tensor-densities on $P^{2n+1}$ of degree $-\lambda$:

$$f = f(x_{1}, \ldots, x_{2n+1})(dx_{1} \wedge \ldots dx_{2n+1})^{-\lambda}.$$

The space of tensor-densities $\mathcal{F}_{\lambda}(RP^{2n+1})$ is isomorphic as a module over the group of contact diffeomorphisms to the space of homogeneous functions on $R^{2n+2}$. The isomorphism is given by:

$$f \mapsto F(y_{1}, \ldots, y_{2n+2}) = y_{2n+2}^{-\lambda(n+1)} f\left(\frac{y_{1}}{y_{2n+2}}, \ldots, \frac{y_{2n+1}}{y_{2n+2}}\right)$$

Then, the operations (15) are defined as the restrictions of the terms of the standard ⋆-product on $R^{2n+2}$.

The same formula (15) defines a ⋆-product on the space of tensor-densities on $CP^{2n+1}$ (cf. [17]). However, there is no analogues of the symplectomorphism (1). I do not know if there exists a ⋆-product on the Poisson algebra $C[y_{2n+2}, y_{2n+2}^{-1}] \otimes C^{\infty}(RP^{2n+1})$ non-equivalent to the standard.
8.2 Classification problem. The classification (modulo equivalence) of ⋆-products on the Poisson algebra $\mathcal{F}$ is an interesting open problem. It is related to calculation of cohomology groups $H^2(\mathcal{F}; \mathcal{F})$ and $H^3(\mathcal{F}; \mathcal{F})$. The following result is announced in [7]: $\dim H^2(\mathcal{F}; \mathcal{F}) = 2$.

Let us formulate a conjecture in the compact case. Consider the Poisson algebra $\mathcal{F}(S^1)$ of functions on $T^*S^1 \setminus S^1$ which are Laurent polynomials on the fiber: $F(p, q) = \sum_{-N \leq i \leq N} p^i f_i(q)$ where $f_i(q + 1) = f_i(q)$.

**Conjecture.** Every ⋆-product on $\mathcal{F}(S^1)$ is equivalent to (1) or to (5).

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