VALIDATION OF A MODEL OF THE DOMINO EFFECT?

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ABSTRACT. A recent paper proposing a model of the limiting speed of the domino effect is discussed with reference to its need and the need of models in general for validation against experimental data. It is shown that the proposed model diverges significantly from experimentally derived speed estimates over a significant range of domino spacing using data from the existing literature and this author’s own measurements, hence if its use had had economic importance its use outside its range of validity could have led to loses of one sort or another to its users.

1. INTRODUCTION

The purpose of mathematical modelling is to capture some aspect of the behaviour of reality in a set of equations, or some other mathematical structure, to allow us to examine its behaviour in the comfort of our offices. But any mathematical model introduces some level of simplification so before we place our confidence in a model it is a good idea to compare the behaviour that the model predicts with reality at least for some subset of cases of interest. This is validation, and before we use the predictions of a model we should have shown that at the very least it reproduce the known behaviour of the modelled system. Any student of mathematical modelling must learn the importance of validation if their work is to be applied with any confidence.

In this note I investigate the validity of a model of domino wave speed propose by Efthimiou and Johnson [1]. The reason that I am focusing on this model is that in their paper the authors present their model and derive the speed in terms of a domino spacing parameter $(d/H)$ the ratio of the gap between adjacent dominoes and their height, see figure [1], but do not quote any experimental results in to support the model despite such data being easily acquired.

2. BACKGROUND

A number of people have previously constructed models to represent the propagation of a wave of domino collapse. Some of these are the models of van Leeuwen [4], McLachlan et al. [5] and Banks [6] which are references in [1], and Stronge [2] and Stronge and Shu [3]. Of these
McLachlan et al’s model is for "thin" dominoes (or for geometrical similar dominoes that is the height to thickness ratio is constant), and Bank’s models assume that only pairs of adjacent dominoes are involved at any one time in the collapse wave. Efthimiou and Johnson’s model makes both of these assumptions. The others model “thick” dominoes (that is of non-negligible thickness) and also van Leeuwen and Stronge and Shu model the effects of multiple dominoes being involved in the wave (it seems that this is also implicit in McLachlan et al)) they also model the interactions between the dominoes in more detail. (McLachlan et al’s use a dimensional argument which is independent of any assumption about the number of dominoes involved in the interactions giving rise to the collapse wave) All of these quote some experimental results though [4] and [6] quote the experimental results of [5]. So there is no shortage of data against which a model may be validated.

My first reaction when I read [1], before I had chased up references, was that it must be easy to measure the wave speed. As a result I set about collecting equipment to make my own measurements, which I will compare with [1]’s theoretical calculations and other experimental data below.

3. Theory

McLachlan et al. [5] conclude from dimensional analysis that the limiting wave speed \( V \) for thin dominoes satisfies:

\[
\frac{v}{\sqrt{gh}} = G(L/H)
\]

for some function \( G \). Which for thin dominoes is the same as:

\[
\frac{v}{\sqrt{gh}} = G_1(d/H)
\]

Where \( H \) is the height of the dominoes, \( d \) the gap between adjacent dominoes, and \( L \) the distance between equivalent points on neighbouring dominoes (that is the pitch of the domino array) The proposed model of Efthimiou and Johnson aims to determine what the form of the function \( G_1 \) must be (see figure 1 for the significance of the variables).

Notes: There are additional invisible arguments in functions \( G \) and \( G_1 \) as the normalised speed may also depend on the dimensionless constants relevant to the system, in this case these include the coefficient of friction between dominoes, and the coefficient of restitution for inter domino impacts. The coefficient of friction between the surface and the dominoes is of lesser relevance as in domino experiments it is usual to arrange things so that there is no slipping between the dominoes and the surface. Efthimiou and Johnson make particular assumptions
which mean that none of these dimensionless parameter appear explicitly in their model.

Figure 1. Geometry of Domino Array

4. Generation of Validation Data

Initially I had toyed with the idea of videoing domino waves, then extracting the speed from an analysis of the video’s frames. I abandoned this approach when I realised that audio recording would be more convenient. The way that I decided to measure the domino wave speed was to use the sound recorder and microphone on an old laptop to record the sound of a domino array collapse. (This is far less demanding in terms of cost of equipment than the high speed photography reported in [2] and [3]). Then to analyse the recording to extract the frequency of dominoes hitting their adjacent domino. This signal is encoded in the envelope of the recording so analysis techniques analogous to the processing in a crystal AM radio receiver, or a simple form of DEMON (Detection of Envelope Modulation On Noise) analysis similar to that used in passive Sonar processing is required. The initial sections of each recording were progressively discarded to identify and eliminate any start up transients. For most of the recordings the transients were at most slight and easily eliminated but four (the two with the closest and the two with the widest relatively spacing of the dominoes) must be regarded with caution as the results were irregular for these (they should probably be repeated more carefully with a better surface and more regular dominoes than the high street ones that I used).

The analysis of the acoustic recordings is fascinating problem in itself but I will not go into the detail of the experiment, it is the results and their relationship to the model predictions that I am interested in here.
A detailed description of the experimental method can be found in [7] and [8].

5. Results

All of the papers that report domino wave speed measurements report speeds \( \sim 0.5 \) to \( 1.7 \times \sqrt{gh} \). These are in broad agreement with my own measurements shown in figure 2.

Given the usual shapes of dominoes I would hope that the thin domino approximation would be not unreasonable down to values of \( d/H \sim 0.3 \).

As can also be seen in the figure the measured data are comparable to the predictions of [1] over a rather limited range of \( d/H \). This is in contradistinction to the models in the references (the predictions of Bank’s [6] is reproduced in the figure for reference) which in general give rather better agreement with experiment. The models which represent the effects of multiple dominoes being involved in the collapse wave being rather better than Bank’s model. Even so the reasonable agreement between the experimental data and the model predictions from Banks [6] is worth noting as it indicates that the single neighbour domino interaction assumption is not entirely misleading.

6. Discussion

The experimental data may be summarised as telling us that to a fair (hand-waving) approximation for common dominoes the normalised wave speed is independent of the normalised inter domino interval for all practical intervals. Also that the normalised wave speed is of the order 1, with variation about this value probably explained by the variation of sliding friction and possibly coefficient of restitution between different species of dominoes (see [3] and [4]).

It is difficult to see the variability in the experientially determined wave speeds in [2] due to the vertical scale required to show the model predictions. The experimental data shown in figure 2 together with that from [5] are shown in figure 3 with a vertical scale more suited to showing this data. In this plot the variation of wave speed with spacing parameter can be seen. For most of the data sets there appears to be a weak dependence of normalised wave speed on the spacing parameter \( h/H \). The data form McLachlan et al [5] shows what appears to be a stronger dependence on the spacing parameter apparently rising as the spacing parameter becomes small. Given the other data and McLachlan et al’s description of their experimental method (hand timing the collapse of an array of \( \sim 100 \) dominoes) this is possibly a misleading trend.

From figure 3 we can see that all the reliable data points give normalised wave speeds in the range \( \sim 1 - 1.6 \).
Figure 2. Domino Wave Speed Plot; the solid line is the model prediction from Efthimiou and Johnson [1], dashed line from Banks [6], + measurements by the current author with the dominoes vertical, ⭕ with them horizontal, × and □ measurements from Strong and Shu [3]

(Note the apparent systematic upward curve in McLachlan et al’s data as the domino spacing falls may be due to systematic effects in the experimental method that was employed.)

7. Summary

From the comparison of the model of Efthimiou and Johnson [1] and experiment we see that the area of agreement of experiment and model is rather limited. Had the model been part of a project with some economic impact we would have been at risk of being found to not have shown due diligence, which could result in unfavourable consequences for us and/or our employers in the event of a failure.
Validation of models is not a chore that we may do after the interesting parts of a study are completed but an essential activity if our work is not to be nugatory.

It is also worth while comparing the predictions in the literature with ones current models predictions, the differences may be important and in need of explanation.

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Appendix A. Experimental Data

For reference I include here the data from my experiments on domino wave speed.

Table 1. Experimental Results With Dominoes Vertical (italic script indicates less reliable data)

| $d/H$ | 0.04 | 0.14 | 0.23 | 0.33 | 0.43 | 0.53 | 0.62 | 0.72 | 0.82 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $V/\sqrt{gH}$ | 1.07 | 1.33 | 1.53 | 1.51 | 1.47 | 1.50 | 1.40 | 1.33 | 1.23 |

Table 2. Experimental Results With Dominoes Horizontal (italic script indicates less reliable data)

| $d/H$ | 0.28 | 0.47 | 0.67 | 0.87 |
|-------|-------|-------|-------|-------|
| $V/\sqrt{gH}$ | 1.15 | 1.19 | 1.15 | 0.68 |

Notes The last entry in table 2 has a spacing greater than the maximum for which one would expect the domino wave to propagate. At a value of $d/H > \sqrt{3}/2$ a domino strikes its neighbour below its’ mid
point, and under these conditions it may well not topple in the expected manner, this is van Leeuwen’s practical upper limit for the wave to propagate. So it is no surprise that the data for this point is unreliable and this was the largest spacing at which I could get the wave to propagate. Presumably it did propagate in this case as a result of the irregularities in the domino geometry and spacing, or some other unidentified reason.

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