2-positive almost order zero maps and decomposition rank

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Abstract

We consider 2-positive almost order zero (disjointness preserving) maps on C*-algebras. Generalizing the argument of M. Choi for multiplicative domains, we give an internal characterization of almost order zero for 2-positive maps. It is also shown that complete positivity can be reduced to 2-positivity in the definition of decomposition rank for unital separable C*-algebras.

1 Introduction

In [33] W. Winter and J. Zacharias gave the structure theorem for completely positive order zero maps based on the work of M. Wolff on disjointness preserving linear maps [32]. Recall that a positive linear map \( \varphi : A \to B \) between two C*-algebras is said to have order zero, if \( \varphi(a)\varphi(b) = 0 \) for any positive elements \( a, b \in A \) with \( ab = 0 \). This notion of order zero maps led to a geometric dimension, known as decomposition rank or nuclear dimension, in the context of the classification theorem of nuclear C*-algebras [19, 34].

The purpose of this paper is to explore the condition of 2-positivity in connection with order zero maps.

In the first part of the paper we show the following one variable characterization of 2-positive almost order zero maps.

**Theorem 1.1.** For \( \varepsilon > 0 \) there exists \( \delta > 0 \) satisfying the following condition: for two C*-algebras \( A \) and \( B \), the canonical approximate unit \( h_\lambda, \lambda \in \Lambda \) of \( A \), and a 2-positive contraction \( \varphi \) from \( A \) to \( B \), if a positive contraction \( a \in A \) satisfies

\[
\limsup_{\lambda} \left\| \varphi(a)^2 - \varphi(a^2)\varphi(h_\lambda) \right\| < \delta,
\]

then

\[
\sup_{b \in A, \|b\| \leq 1} \left( \limsup_{\lambda} \left\| \varphi(a)\varphi(b) - \varphi(h_\lambda)\varphi(ab) \right\| \right) < \varepsilon.
\]

Particularly, a 2-positive map \( \varphi \) from a unital C*-algebra \( A \) to a C*-algebra \( B \) has order zero if \( \varphi(a)^2 = \varphi(a^2)\varphi(1_\Lambda) \) for any positive element \( a \in A \).

In the second part of the paper, we study the relation between 2-positivity and decomposition rank. The notion of decomposition rank (Definition 6.1) was introduced by E. Kirchberg and W. Winter in their work [19], in which they showed that finiteness of decomposition rank implies quasidiagonality for C*-algebras. In [30] W. Winter showed...
that finiteness of decomposition rank (for separable C*-algebras, see [11] for non-separable cases) also implies the absorption of the Jiang-Su algebra which plays a central role in the classification theorem of nuclear C*-algebras [9]. For unital separable simple nuclear monotracial C*-algebras, we showed the converse, i.e., quasidiagonality and Jiang-Su absorption imply finiteness of decomposition rank [22, 23]. Our second main result characterizes finiteness of decomposition rank by 2-positive maps instead of completely positive maps.

**Theorem 1.2.** Let $A$ be a unital separable C*-algebra. Then the decomposition rank of $A$ is at most $d$ if and only if for a finite subset $F$ of contractions in $A$ and $\varepsilon > 0$, there exist finite dimensional C*-algebras $F_i$, $i = 0, 1, \ldots, d$, a 2-positive contraction $\psi : A \to \bigoplus_{i=0}^{d} F_i$, and 2-positive order zero contractions $\varphi_i : F_i \to A$, $i = 0, 1, \ldots, d$ such that $\sum_{i=0}^{d} \varphi_i : \bigoplus_{i=0}^{d} F_i \to A$ is contractive and

$$\left\| \left( \sum_{i=0}^{d} \varphi_i \right) \circ \psi(x) - x \right\| < \varepsilon, \quad \text{for all } x \in F.$$

Here we simply write $\sum_{i=0}^{d} \varphi_i \left( \bigoplus_{i=0}^{d} x_i \right) = \sum_{i=0}^{d} \varphi_i(x_i)$ for $x_i \in F_i$.

Before closing this section, let us collect some notations and terminologies. For a C*-algebra $A$, we let $A_{sa}$ and $A_+$ denote the set of self-adjoint elements and the cone of positive elements in $A$. For a subset $S \subset A$, $S^1$ denotes the set of contractions in $S$. If $A$ is a unital C*-algebra, $1_A$ denotes the unit of $A$.

For any two elements $a$ and $b$ in a C*-algebra $A$ we let $[a, b]$ denote the commutator $ab - ba \in A$, and by $a \approx \varepsilon b$ for $\varepsilon > 0$ we mean that $\|a - b\| < \varepsilon$.

Unless stated otherwise we consider two C*-algebras $A$ and $B$, and by a “map” $\varphi : A \to B$ we mean a “linear map” from $A$ to $B$. We let $\text{id}_A$ denote the identity map on $A$, i.e., $\text{id}_A(a) = a$ for any $a \in A$. For $n \in \mathbb{N}$, $M_n$ denotes the C*-algebra of complex $n \times n$ matrices. A map $\varphi$ from $A$ to $B$ is called positive if $\varphi(A_+) \subset B_+$. For a natural number $k$, a map $\varphi$ is called $k$-positive if $\varphi \otimes \text{id}_{M_k} : A \otimes M_k \to B \otimes M_k$ is positive. If a map $\varphi : A \to B$ is $k$-positive for any $k \in \mathbb{N}$, $\varphi$ is called completely positive.

## 2 Orthogonality domains for 2-positive maps

**Definition 2.1.** Let $A$ and $B$ be two C*-algebras, and let $h_\lambda \in A_+^1$, $\lambda \in \Lambda$ be an approximate unit. For a bounded linear map $\varphi$ from $A$ to $B$, we define a subspace $\text{OD}(\varphi)$ of $A$ by

$$\text{OD}(\varphi) = \{ a \in A : \varphi(a)\varphi(b) = \lim_{\lambda} \varphi(h_\lambda)\varphi(ab), \quad \varphi(b)\varphi(a) = \lim_{\lambda} \varphi(ba)\varphi(h_\lambda) \quad \text{for any } b \in A \}.$$

It follows from the definition that $\lim_{\lambda} \| [\varphi(h_\lambda), \varphi(a)] \| = 0$ for any $a \in \text{OD}(\varphi)$.
In this section we mainly deal with 2-positive maps for Kadison’s inequality in the following form, which makes OD(ϕ) into a C*-algebra. For two (not necessarily unital) C*-algebras A and B , if a map φ : A → B is contractive and 2-positive, then the original Kadison’s inequality tells us that

$$\varphi \otimes \text{id}_{M_2} \left( \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix} \right)^2 \leq \varphi \otimes \text{id}_{M_2} \left( \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix} \right)^2$$

for any \( a \in A \), see [14, p.770], for example. Then we have \( \varphi(a)^* \varphi(a) \leq \varphi(a^*a) \) for any \( a \in A \), [3, Corollary 2.8]. By using this inequality, we can see that OD(ϕ) is a C*-algebra.

**Proposition 2.2.** If a map \( \varphi : A \rightarrow B \) is 2-positive, then the following statements hold.

(i) OD(ϕ) is a C*-algebra which contains the multiplicative domain of \( \varphi \).

(ii) OD(ϕ) is independent of the choice of the approximate unit.

**Proof.** Since OD(ϕ) = OD(ϕ/∥ϕ∥), we may assume ∥ϕ∥ ≤ 1 in both (i) and (ii).

(i) Since \( \varphi \) is a bounded self-adjoint map, it is straightforward to check that OD(ϕ) is a self-adjoint Banach space which contains the multiplicative domain of \( \varphi \). It remains to show that OD(ϕ) is closed under multiplication. Let \( a, b \) be contractions in OD(ϕ), \( c \) a contraction in \( A \), and \( \varepsilon \in (0, 1) \). Taking a large \( k \in \mathbb{N} \) we have \( (1 - t^{1/k})t < \varepsilon^2/8 \) for any \( t \in [0, 1] \). Because of Kadison’s inequality, for any \( \lambda \in \Lambda \) with \( \| h_\lambda^{1/2} a^* a_h^{1/2} - a a \| < \varepsilon^2/8 \) we have

$$\|(1 - \varphi(h_\lambda)^{1/k})\varphi(ab)\varphi(ab)^*(1 - \varphi(h_\lambda)^{1/k})\| \leq \|(1 - \varphi(h_\lambda)^{1/k})\varphi(aa^*)\varphi(h_\lambda)^{1/k}\|$$

$$< \|(1 - \varphi(h_\lambda)^{1/k})\varphi(h_\lambda)\| + \varepsilon^2/8 < \varepsilon^2/4.$$  

Since \( \varphi(h_\lambda) \), \( \lambda \in \Lambda \) almost commutes with \( \varphi(a) \), it follows that

$$\lim_{\lambda} \| \varphi(h_\lambda)\varphi(ab)\varphi(c) - \varphi(h_\lambda)^2\varphi(abc) \| = 0,$$

which implies \( \lim_{\lambda} \| \varphi(h_\lambda)^n(\varphi(ab)\varphi(c) - \varphi(h_\lambda)\varphi(abc)) \| = 0 \) for any \( n \in \mathbb{N} \). Then we have

$$\lim_{\lambda} \| \varphi(h_\lambda)^{1/k}(\varphi(ab)\varphi(c) - \varphi(h_\lambda)\varphi(abc)) \| = 0.$$  

Thus, there exists \( \lambda_0 \in \Lambda \) such that for any \( \lambda \geq \lambda_0 \),

$$\| \varphi(ab)\varphi(c) - \varphi(h_\lambda)\varphi(abc) \| < \varepsilon.$$

Since \( \varepsilon > 0 \) is arbitrary, we have \( \varphi(ab)\varphi(c) = \lim_{\lambda} \varphi(h_\lambda)\varphi(abc) \). By OD(ϕ)* = OD(ϕ), we also have \( \varphi(c)\varphi(ab) = \lim_{\lambda} \varphi(cab)\varphi(h_\lambda) \) for any \( a, b \in \text{OD}(\varphi) \) and \( c \in A \).

(ii) Let \( k_\mu \in A_+^1 \), \( \mu \in I \) be another approximate unit of \( A \) and let OD(ϕ, k) be the subspace in Definition [2.1] determined by \( \{k_\mu\}_{\mu \in I} \). Since OD(ϕ) and OD(ϕ, k) are C*-algebras, it suffices to show OD(ϕ+|k) ⊂ OD(ϕ, k).

Let \( a \in \text{OD}(\varphi)^1_+ \), and let \( \lambda_0 \in \Lambda \) and \( \mu_0 \in I \) be such that \( \| \varphi((h_\lambda - k_\mu)a) \| < \varepsilon \) for any \( \lambda \geq \lambda_0 \) and \( \mu \geq \mu_0 \). Then it follows that

$$\| \varphi(h_\lambda - k_\mu)\varphi(a) \| = \lim_{\mu} \| \varphi((h_\lambda - k_\mu)a)\varphi(h_\mu) \| < \varepsilon.$$  

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By Kadison’s inequality, for any $b \in A^1$ we have
\[ \| \varphi(h_\lambda - k_\mu)\varphi(ab) \|^2 \leq \| \varphi(h_\lambda - k_\mu)\varphi(a)\varphi(h_\lambda - k_\mu) \| < 2\varepsilon, \]
for any $\lambda \geq \lambda_0$ and $\mu \geq \mu_0$. Then it follows that $\lim_{\mu} \varphi(k_\mu)\varphi(ab) = \varphi(a)\varphi(b)$. Since $a$ is self-adjoint, we also see that $\lim_{\mu} \varphi(ba)\varphi(k_\mu) = \varphi(b)\varphi(a)$ for any $b \in A$.

To prepare for the Schwartz inequality (Proposition 2.5) and the next section, we need the following calculation of non-invertible positive elements. This argument is a slight variation of [25] Lemma 1.4.4.

**Lemma 2.3.** Let $A$ be a C$^*$-algebra. For two positive elements $a$ and $b$ in the second dual $A^{**}$ with $a \leq b$, there exists a unique contraction $x$ in $A^{**}$ such that $b^{1/2}x = a^{1/2}$ and $p(b)x = x$, where $p(b)$ is the strong limit of $(\frac{1}{n}1_{A^{**}} + b)^{-1}b \in A^{**}$. If furthermore $[a, b] = 0$, then there exists a unique contraction $y$ in $A^{**}$ such that $b = a$ and $p(b)y = y$.

We write $b^{-1/2}a^{1/2} = x$ and $b^{-1}a = y$.

**Proof.** For $n \in \mathbb{N}$, we set $x_n = (\frac{1}{n}1_{A^{**}} + b)^{-1/2}a^{1/2} \in A^{**}$. Then it follows that $x_n^*x_n = a^{1/2}(\frac{1}{n}1_{A^{**}} + b)^{-1}a^{1/2} \leq a^{1/2}(\frac{1}{n}1_{A^{**}} + a)^{-1}a^{1/2} \leq 1_{A^{**}}$ for any $n \in \mathbb{N}$. Since the unit ball of $A^{**}$ is compact in the $\sigma$-weak (ultraweak) topology, there exists a subnet $x_{n_\lambda}$, $\lambda \in \Lambda$ of $\{x_n\}_{n \in \mathbb{N}}$ which converges to a contraction $x \in A^{**}$. Thus we have that
\[ b^{1/2}x = \sigma\text{-weak}\lim_{\lambda} b^{1/2}(\frac{1}{n_\lambda}1_{A^{**}} + b)^{-1/2}a^{1/2} = p(b)a^{1/2} = a^{1/2}. \]

If $x' \in A^{**}$ satisfies $b^{1/2}x' = a^{1/2}$ and $p(b)x' = x'$, then we have $x - x' = p(b)(x - x') = \sigma\text{-weak}\lim_{n \to \infty} (\frac{1}{n}1_{A^{**}} + b)^{-1}b(x - x') = 0$.

In the case of $[a, b] = 0$, by a similar argument, we can define a positive contraction $y$ in $A^{**}$ as the strong limit of $a^{1/2}(\frac{1}{n}1_{A^{**}} + b)^{-1}a^{1/2}$, $n \in \mathbb{N}$. This $y$ also satisfies the desired conditions.

**Corollary 2.4.** Let $A$ and $B$ be C$^*$-algebras.

(i) Suppose that $\varphi : A \to B$ is a 2-positive map and $a$ and $b$ are two elements in $A$. Then there exists a unique element $\varphi(b^{*}b)^{-1/2}\varphi(b^{*}a) \in B^{**}$ satisfying
\[ \varphi(b^{*}b)^{1/2}(\varphi(b^{*}b)^{-1/2}\varphi(b^{*}a)) = \varphi(b^{*}a) \]
and $(1_{B^{**}} - p(\varphi(b^{*}b)))\varphi(b^{*}b)^{-1/2}\varphi(b^{*}a) = 0$.

(ii) Suppose that $\varphi : A \to B$ is a positive map, $x$ is a normal element in $A$, and $y$ is a positive element in $A$ satisfying $xx^* \leq \|x\|^2y$. Then there exists a unique element $\varphi(y)^{-1/2}\varphi(x) \in B^{**}$ such that
\[ \varphi(y)^{1/2}((\varphi(y)^{-1/2}\varphi(x)) = \varphi(x) \quad \text{and} \quad (1_{B^{**}} - p(\varphi(y)))\varphi^{-1/2}(y)\varphi(x) = 0. \]
Proof. In both cases we may assume that $\varphi$ is contractive. We may further assume that $a$ and $x$ are contractions in $A$.

(i) By Kadison’s inequality we have $\varphi(a^*b)^*\varphi(a^*b) \leq \varphi(b^*b)$. From Lemma 2.3 we obtain the contraction $\varphi(b^*b)^{-1/2} \varphi(a^*b) \in B**$. By the polar decomposition of $\varphi(b^*a)$ in $B**$, there exists a contraction $\varphi(b^*b)^{-1/2} \varphi(b^*a) \in B**$ satisfying the desired conditions. The uniqueness of $\varphi(b^*b)^{-1/2} \varphi(b^*a) \in B**$ follows from these conditions automatically.

(ii) Since $x$ is normal, Kadison’s inequality implies that

$$\varphi(x)\varphi(x)^* \leq \varphi(xx^*) \leq \varphi(y),$$

see [14, p.770]. By the same argument as in the proof of (i) we obtain a unique element $\varphi(y)^{-1/2} \varphi(x) \in B**$ satisfying the desired conditions. \hfill \Box

The following Schwartz inequality was given by M. Choi in [14, Proposition 4.1] for strictly positive maps and invertible elements. Regarding $\varphi(a^*b)\varphi(b^*b)^{-1/2} \varphi(b^*a)$ as $(\varphi(b^*b)^{-1/2} \varphi(b^*a))^* \varphi(b^*b)^{-1/2} \varphi(b^*a)$ obtained in Corollary 2.4 we extend his result to the case of non-invertible elements.

**Proposition 2.5.** Let $A$ and $B$ be C*-algebras.

(i) Suppose that $\varphi$ is a 2-positive map from $A$ to $B$. Then for any $a, b \in A$ it follows that

$$\varphi(a^*b)\varphi(b^*b)^{-1} \varphi(b^*a) \leq \varphi(a^*a).$$

(ii) Suppose that $\varphi$ is a positive map from $A$ to $B$. Then for a self-adjoint element $x \in A$ and a positive element $y \in A$ with $yx = x$, it follows that

$$\varphi(x)\varphi(y)^{-1} \varphi(x) \leq \varphi(x^2).$$

**Proof.** (i) Since the $2 \times 2$ matrix

$$\begin{bmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{bmatrix} \in B \otimes M_2$$

is positive for any $n \in \mathbb{N}$. From $\|\varphi(b^*b)^{-1/2} \varphi(b^*a)\| \leq \|\varphi\|^{1/2}\|a\|$ for any $n \in \mathbb{N}$, we obtain an accumulation point $X \in B^{**}$ of $(\frac{1}{n}1_{B^{**}} + \varphi(b^*b)^{-1/2} \varphi(b^*a))_{n \in \mathbb{N}}$ in the sense of $\sigma$-weak topology. It is straightforward to see that $\varphi(b^*b)^{-1/2} \varphi(b^*a)$ and $(1_{B^{**}} - p(\varphi(b^*b)))X = 0$. By Corollary 2.4 we have $X = \varphi(b^*b)^{-1/2} \varphi(b^*a)$. Then it follows that the $2 \times 2$ matrix

$$\begin{bmatrix} \varphi(a^*a) & (\varphi(b^*b)^{-1/2} \varphi(b^*a))^* \\ \varphi(b^*b)^{-1/2} \varphi(b^*a) & p(\varphi(b^*b)) \end{bmatrix} \in B^{**} \otimes M_2$$

is also positive. Because of

$$0 \leq \begin{bmatrix} 1_{B^{**}} & 0 \\ -\varphi(b^*b)^{-1/2} \varphi(b^*a) & 1_{B^{**}} \end{bmatrix}^* \begin{bmatrix} \varphi(a^*a) & (\varphi(b^*b)^{-1/2} \varphi(b^*a))^* \\ \varphi(b^*b)^{-1/2} \varphi(b^*a) & p(\varphi(b^*b)) \end{bmatrix} \begin{bmatrix} 1_{B^{**}} & 0 \\ -\varphi(b^*b)^{-1/2} \varphi(b^*a) & 1_{B^{**}} \end{bmatrix} = \begin{bmatrix} \varphi(a^*a) - \varphi(a^*b)\varphi(b^*b)^{-1} \varphi(b^*a) & 0 \\ 0 & p(\varphi(b^*b)) \end{bmatrix},$$

$$\begin{bmatrix} 1_{B^{**}} & 0 \\ -\varphi(b^*b)^{-1/2} \varphi(b^*a) & 1_{B^{**}} \end{bmatrix} \begin{bmatrix} \varphi(a^*a) & (\varphi(b^*b)^{-1/2} \varphi(b^*a))^* \\ \varphi(b^*b)^{-1/2} \varphi(b^*a) & p(\varphi(b^*b)) \end{bmatrix} \begin{bmatrix} 1_{B^{**}} & 0 \\ -\varphi(b^*b)^{-1/2} \varphi(b^*a) & 1_{B^{**}} \end{bmatrix}^* = \begin{bmatrix} \varphi(a^*a) & 0 \\ 0 & p(\varphi(b^*b)) \end{bmatrix}.$$
we conclude that 
\[ \varphi(a^{*}a) \geq \varphi(a^{*}b)\varphi(b^{*}b)^{-1}\varphi(b^{*}a). \]

(ii) When \(yx = x\), the \(2 \times 2\) matrix \( \begin{bmatrix} x^2 & x \\ x & y \end{bmatrix} \in A \otimes M_2 \) is positive. By [4] Corollary 4.4], we can see that \( \begin{bmatrix} \varphi(x^2) & \varphi(x) \\ \varphi(x) & \varphi(y) \end{bmatrix} \in B \otimes M_2 \) is also positive, even for a positive map \( \varphi \). By the same argument as the proof of (i), we conclude that \( \varphi(x^2) \geq \varphi(x)^2 \varphi(y)^{-1}\varphi(x). \)  

\[ \square \]

\section{Proof of Theorem 1.1 and applications}

In the following lemma, for a unital C*-algebra \(A\) we denote by \(\mathcal{H}_A\) the separable Hilbert \(A\)-module \(A \otimes \ell^2(\mathbb{N})\) and by \((\cdot, \cdot)_{\mathcal{H}_A} : \mathcal{H}_A \times \mathcal{H}_A \rightarrow A\) the inner product on \(\mathcal{H}_A\), which is defined by 
\[ (x \mid y)_{\mathcal{H}_A} = \sum_{i=1}^{\infty} x_i^* y_i \in A, \quad \text{for } x = (x_i)_{i \in \mathbb{N}} \text{ and } y = (y_i)_{i \in \mathbb{N}} \in \mathcal{H}_A, \]

(see [15, 20] for detail). Let \(B(\mathcal{H}_A)\) denote the set of adjointable operators on \(\mathcal{H}_A\). We let \(\{e_i\}_{i \in \mathbb{N}}\) denote the canonical orthonormal basis of \(\ell^2(\mathbb{N})\), and regard \(a \in B(\mathcal{H}_A)\) as an \(\infty\)-matrix whose \((i, j)\)-entry is \(a_{i,j} := (1_A \otimes e_i \mid a1_A \otimes e_j)_{\mathcal{H}_A} \in A\) for \(i, j \in \mathbb{N}\).

\textbf{Lemma 3.1.} Let \(A\) be a unital C*-algebra. For \(\varepsilon > 0\) the following statements hold.

(i) If a positive contraction \(a \in B(\mathcal{H}_A)\) satisfies \(\|a_{1,1}\| < \varepsilon\), then \(\sum_{i=1}^{\infty} a_{i,1}^* a_{i,1} < \varepsilon\).

(ii) If a unitary \(u \in B(\mathcal{H}_A)\) satisfies \(\|u_{1,1}^* u_{1,1} - 1_A\| < \varepsilon\), then \(\sum_{i=2}^{\infty} u_{i,1}^* u_{i,1} < \varepsilon\).

\textbf{Proof.} (i) For a positive contraction \(a \in B(\mathcal{H}_A)\), we have an element \(b \in B(\mathcal{H}_A)\) with \(b^* b = a\), which implies that \(a_{1,1} = \sum_{i=1}^{\infty} b_{i,1}^* b_{i,1}\), where the right hand side is in the operator norm topology on \(A\). Then we have that
\[
\left\| \sum_{i=1}^{\infty} a_{i,1}^* a_{i,1} \right\| = \left\| (1_A \otimes e_1 \mid a^* a 1_A \otimes e_1)_{\mathcal{H}_A} \right\| \\
\leq \|b\|^2 \left\| (1_A \otimes e_1 \mid b^* b 1_A \otimes e_1)_{\mathcal{H}_A} \right\| \leq \left\| \sum_{i=1}^{\infty} b_{i,1}^* b_{i,1} \right\| < \varepsilon.
\]

(ii) From \(u^* u = 1_{B(\mathcal{H}_A)}\), it follows that \(\sum_{i=1}^{\infty} u_{i,1}^* u_{i,1} = 1_A\) in the operator norm topology. Then we have that
\[
\left\| \sum_{i=2}^{\infty} u_{i,1}^* u_{i,1} \right\| = \left\| 1_A - \sum_{i=1}^{\infty} u_{i,1}^* u_{i,1} \right\| < \varepsilon.
\]

\[ \square \]
Proof of Theorem 1.3. When a positive contraction $a \in A$ satisfies
\[
\limsup_{\lambda} \| \varphi(a^2) - \varphi(a^2) \varphi(h_\lambda) \| < \delta,
\]
there exist positive contractions $a_+, a_- \in A$ such that $a_+ a_- = a_-, \| a - a_- \| < \varepsilon$, and
\[
\limsup_{\lambda} \| \varphi(a_-)^2 - \varphi(a_-^2) \varphi(h_\lambda) \| < \delta.\]
Once we show that
\[
\sup_{b \in A^1} \left( \limsup_{\lambda} \| \varphi(a_-)(b) - \varphi(h_\lambda) \varphi(a_- b) \| \right) < \varepsilon,
\]
this implies $\sup_{b \in A^1} \left( \limsup_{\lambda} \| \varphi(a)(b) - \varphi(h_\lambda) \varphi(ab) \| \right) < 3\varepsilon$. Then we may assume that $h_\lambda a = a$ for a large $\lambda \in \Lambda$.

By the same argument, we may replace the condition
\[
\sup_{\lambda} \left( \limsup_{\lambda} \| \varphi(a)(b) - \varphi(h_\lambda) \varphi(ab) \| \right) < \varepsilon,
\]
in Theorem 1.1 by $\limsup_{\lambda} \| \varphi(a)(b) - \varphi(h_\lambda) \varphi(ab) \| < \varepsilon$, for any positive contraction $b \in A$ with a large $\lambda \in \Lambda$ such that $h_\lambda b = b$.

For $\alpha \in (0,1)$ and $t \in [0,1]$, we set $f_\alpha(t) = \min \{ \max \{ 0, \alpha^{-1} t - 1 \}, 1 \}$. Since $f_\alpha \in C_0([0,1]^1)$, there exists $g_\alpha \in C([0,1])$ such that $g_\alpha \cdot \text{id}_{[0,1]} = f_\alpha$. Here $\text{id}_{[0,1]} \in C([0,1])$ means the continuous function defined by $\text{id}_{[0,1]}(t) = t$ for $t \in [0,1]$.

For $\varepsilon \in (0,1)$ we let $\alpha_1 \in (0,1)$ be such that $\| \text{id}_{[0,1]} \cdot (1 - f_{\alpha_1}) \| < \varepsilon^2/16$. Set $\varepsilon_1 = (\varepsilon/(8\|g_{\alpha_1}\|))^2 > 0$. Let $\alpha_2 \in (0,1/4)$ be such that $\| \text{id}_{[0,1]} \cdot (1 - f_{\alpha_2}) \| < \varepsilon_1/4$, and let $\delta_1 > 0$ be such that $\delta_1 < \varepsilon_1/4\|g_{\alpha_2}\|$. By approximating $\text{id}_{[0,1]}^{1/2}$ with polynomials, we let $\delta \in (0,\delta_1)$ be such that for any positive contractions $x, y$ in a C*-algebra, the condition $\| [x,y] \| < 6\delta$ implies $\| [x^{1/2}, y] \| < \delta_1$.

Let $A, B, h_\lambda \in A^1$, and $\varphi$ be as in the theorem. Suppose that a positive contraction $a \in A$ satisfies $\limsup_{\lambda} \| \varphi(a^2) - \varphi(a^2) \varphi(h_\lambda) \| < \delta$ and $h_\lambda a = a$ for a large $\lambda \in \Lambda$. From $\limsup_{\lambda} \| [\varphi(a^2), \varphi(h_\lambda)] \| < 2\delta$, it follows that $\limsup_{\lambda} \| [\varphi(a), \varphi(h_\lambda)] \| < \delta_1$. Let $b$ be a positive contraction in $A$ with $h_\lambda b = b$ for a large $\lambda \in \Lambda$. Set self-adjoint elements
\[
x = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix}, \quad y = \begin{bmatrix} h_\lambda & 0 \\ 0 & h_\lambda \end{bmatrix} \in A \otimes M_2.
\]

By (ii) of Proposition 2.5, we have
\[
\varphi \otimes \text{id}_{M_2}(x) \varphi \otimes \text{id}_{M_2}(y) \alpha^{-1} \varphi \otimes \text{id}_{M_2}(x) \leq \varphi \otimes \text{id}_{M_2}(x^2).
\]
This inequality implies that
\[
\begin{bmatrix} \varphi(a^2) & \varphi(ab) \\ \varphi(ba) & \varphi(a^2 + b^2) \end{bmatrix} \geq \begin{bmatrix} \varphi(h_\lambda) & 0 \\ 0 & \varphi(h_\lambda) \end{bmatrix}^{-1/2} \begin{bmatrix} 0 & \varphi(a) \\ \varphi(a) & \varphi(b) \end{bmatrix} \begin{bmatrix} \varphi(h_\lambda) & 0 \\ 0 & \varphi(h_\lambda) \end{bmatrix}^{-1/2} \begin{bmatrix} 0 & \varphi(a) \\ \varphi(a) & \varphi(b) \end{bmatrix}.
\]

\[
\begin{bmatrix} \varphi(a)g_{\alpha_2}(\varphi(h_\lambda)) \varphi(a) & \varphi(a)g_{\alpha_2}(\varphi(h_\lambda)) \varphi(b) \\ \varphi(b)g_{\alpha_2}(\varphi(h_\lambda)) \varphi(a) & \varphi(a)g_{\alpha_2}(\varphi(h_\lambda)) \varphi(a) + \varphi(b)g_{\alpha_2}(\varphi(h_\lambda)) \varphi(b) \end{bmatrix}.
\]
Set $y = \varphi(a^2 + b^2) - \varphi(a)g_{\alpha_2}(\varphi(h_\lambda))\varphi(a) + \varphi(b)g_{\alpha_2}(\varphi(h_\lambda))\varphi(b)$. Then the following matrix $X \in B \otimes M_2$ is a positive element,

$$X = \begin{bmatrix}
\varphi(h_\lambda)(\varphi(a^2) - \varphi(a)g_{\alpha_2}(\varphi(h_\lambda))\varphi(a))\varphi(h_\lambda) & \varphi(h_\lambda)(\varphi(ab) - \varphi(a)g_{\alpha_2}(\varphi(h_\lambda))\varphi(b))\varphi(h_\lambda) \\
\varphi(h_\lambda)(\varphi(ba) - \varphi(b)g_{\alpha_2}(\varphi(h_\lambda))\varphi(a))\varphi(h_\lambda) & \varphi(h_\lambda) \cdot y \cdot \varphi(h_\lambda)
\end{bmatrix}.$$ 

By the choice of $g_{\alpha_2}, f_{\alpha_2}, \delta_1 > \delta$, and $\alpha_2 \in (0, 1/4)$ we have that

$$\limsup_{\lambda} \|\varphi(h_\lambda)(\varphi(a^2) - \varphi(a)g_{\alpha_2}(\varphi(h_\lambda))\varphi(a))\varphi(h_\lambda)\| \leq \limsup_{\lambda} \|\varphi(h_\lambda)\varphi(a^2)\varphi(h_\lambda) - \varphi(a)f_{\alpha_2}(\varphi(h_\lambda))\varphi(h_\lambda)\varphi(a)\| + 2\delta_1 \|g_{\alpha_2}\|
$$

$$\leq \limsup_{\lambda} \|\varphi(h_\lambda)\varphi(a^2)\varphi(h_\lambda) - \varphi(a)\varphi(h_\lambda)\varphi(a)\| + \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2}
$$

$$\leq \limsup_{\lambda} \|\varphi(a^2)\varphi(h_\lambda) - \varphi(a)^2\| + \delta_1 + \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2}
$$

$$< 2\delta_1 + \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2} < \varepsilon.$$

Applying (i) of Lemma 3.1 to $X/\|X\| \in B_\sim \otimes M_2$, we have that

$$\limsup_{\lambda} \|\varphi(h_\lambda)(\varphi(ba) - \varphi(b)g_{\alpha_2}(\varphi(h_\lambda))\varphi(a))\varphi(h_\lambda)\|^2 < \|X\|\varepsilon_1 < 5\varepsilon_1.$$

Then it follows that

$$\limsup_{\lambda} \|\varphi(h_\lambda)(\varphi(ba)\varphi(h_\lambda) - \varphi(b)\varphi(a))\| < \sqrt{\varepsilon_1} \left(\sqrt{\delta_1} + \frac{1}{2}\right) + \frac{\varepsilon_1}{4},$$

$$\limsup_{\lambda} \|f_{\alpha_1}(\varphi(h_\lambda))(\varphi(ba)\varphi(h_\lambda) - \varphi(b)\varphi(a))\| < \|g_{\alpha_1}\| \left(\sqrt{\varepsilon_1} \left(\sqrt{\delta_1} + \frac{1}{2}\right) + \frac{\varepsilon_1}{4}\right) < \frac{\varepsilon}{2}.$$ 

By Kadison’s inequality and $b^2 \leq h_\lambda$ for a large $\lambda \in \Lambda$, we see that

$$\|f_{\alpha_1}(\varphi(h_\lambda))\varphi(ba) - \varphi(ba)\|^2 \leq \|(1 - f_{\alpha_1}(\varphi(h_\lambda)))\varphi(h_\lambda)\|^2 < \frac{\varepsilon^2}{16},$$

$$\|f_{\alpha_1}(\varphi(h_\lambda))\varphi(b)\varphi(a) - \varphi(b)\varphi(a)\|^2 \leq \|(1 - f_{\alpha_1}(\varphi(h_\lambda)))\varphi(h_\lambda)\|^2 < \frac{\varepsilon^2}{16}.$$ 

Therefore we conclude that

$$\limsup_{\lambda} \|\varphi(ba)\varphi(h_\lambda) - \varphi(b)\varphi(a)\| < \varepsilon.$$ 

Theorem 1.1 can be used to give an alternative proof of the structure theorem for completely positive order zero maps [32], [33], [10]. Our approach is effective even for 2-positive maps.

**Corollary 3.2.** Let $A$, $B$ be two C*-algebras, and $h_\lambda \in A_\Lambda^+ \subseteq B_\sim$, $\lambda \in \Lambda$ be the canonical approximate unit of $A$. Suppose that $\varphi$ is a 2-positive map from $A$ to $B$ such that

$$\varphi(a)^2 = \lim_{\lambda} \varphi(a^2)\varphi(h_\lambda),$$

8
for any positive element $a \in A$. Then there exist a $*$-homomorphism $\pi$ from $A$ to $B^{**}$ and a positive element $h$ in the multiplier algebra $M(C^*(\varphi(A)))$ of $C^*(\varphi(A))$ such that

$$\pi(a) \in M(C^*(\varphi(A))) \cap \{h\}^\prime$$

and $\varphi(a) = h\pi(a)$, for any $a \in A$. In particular, $\varphi$ is completely positive.

Proof. We may assume that $\varphi$ is contractive.

We let $h$ be the strong limit of $\varphi(h_\lambda)$, $\lambda \in \Lambda$ in $B^{**+1}$. Since $h\varphi(a) = \varphi(a^{1/2})^2 = \varphi(a)h$ for any $a \in A^1_+$, it follows that $h \in M(C^*(\varphi(A))) \cap (C^*(\varphi(A)))'$. By Lemma 2.3 and by $h \geq \varphi(a)$ for any $a \in A^1_+$, we can define a positive element $\pi(a) = h^{-1}\varphi(a) \in B^{**}$ for any $a \in A^1_+$. Set $f_n(h) = (\frac{1}{n}1_{B^{**}} + h)^{-1} \in M(C^*(\varphi(A))) \subset B^{**}$ for $n \in \mathbb{N}$. Note that for $a, b \in A^1_+$ and $m, n \in \mathbb{N}$

$$\|\varphi(a)(f_n(h) - f_m(h))\varphi(b)\|^2 \leq \left\|h^4(f_n(h) - f_m(h))^2\right\|,$$

then, by Dini’s theorem, $\varphi(a)f_n(h)\varphi(b) \in C^*(\varphi(A))$ converges to $\varphi(a)h^{-1}\varphi(b)$ in the operator norm topology. Thus we have $h^{-1}\varphi(a) \in M(C^*(\varphi(A)))$ for any $a \in A^1_+$.

By the uniqueness of $h^{-1}\varphi\left(\frac{a+b}{|a|+|b|}\right)$ for $a, b \in A_+$ in Lemma 2.3, it follows that

$$\pi\left(\frac{a+b}{|a|+|b|}\right) = \pi\left(\frac{a}{|a|+|b|}\right) + \pi\left(\frac{b}{|a|+|b|}\right).$$

Considering the linear span of $A^1_+$, we obtain a self-adjoint linear map $\pi : A \to M(C^*(\varphi(A)))$. Applying Theorem 11 to $\varphi / \|\varphi\|$, for $a, b \in A_+$ we have $\varphi(a)\varphi(b) = h\varphi(ab)$, which implies that $\pi(a)\pi(b) = \pi(ab)$. $\square$

Corollary 3.3. Every 2-positive order zero map is completely positive. More generally, a 2-positive map is completely positive if its restriction to any commutative $C^*$-subalgebra is order zero.

Proof. Let $\varphi : A \to B$ be a 2-positive map between two $C^*$-algebras, $h_\lambda \in A^1_+, \lambda \in \Lambda$ the canonical approximate unit of $A$, and $a$ a positive contraction in $A$. In order to show that $\varphi(a)^2 = \lim_{\lambda} \varphi(a^2)\varphi(h_\lambda)$, we may assume that $ah_\lambda = a$ for a large $\lambda_0 \in \Lambda$. Let $C$ be the commutative $C^*$-subalgebra of $A$ generated by $a$ and $h_{\lambda_0}$. By the assumption, $\varphi|_C$ is an order zero completely positive map. Then it follows that $\varphi(a)^2 = \varphi(a^2)\varphi(h_\lambda)$ for $\lambda \geq \lambda_0$ (see the proof of 32 Lemma 3.1 for a direct argument). By Corollary 3.2 we conclude that $\varphi$ is completely positive on $A$. $\square$

Combining the proof above with Corollary 3.2, we see the following structure theorem.

Corollary 3.4. Let $A$ and $B$ be two $C^*$-algebras. For a 2-positive order zero map $\varphi : A \to B$, there exist a representation $\pi$ of $A$ on $B^{**}$, and a positive contraction $h \in B^{**}$ satisfying the same condition in Corollary 3.2.

The next result is motivated by the question in 13 Section 5] for general $C^*$-algebras.

Corollary 3.5. Let $A$ and $B$ be $C^*$-algebras, and let $h_\lambda \in A^1_+, \lambda \in \Lambda$ be the canonical approximate unit of $A$. For a 2-positive linear map $\varphi$ from $A$ to $B$, the following holds.

$$\text{OD}(\varphi) = \text{span}\{a \in A^1_+ : \varphi(a)^2 = \lim_{\lambda} \varphi(a^2)\varphi(h_\lambda)\}.$$ 

Proof. From Theorem 11 the right hand side is contained in $\text{OD}(\varphi)$. Since the orthogonality domain $\text{OD}(\varphi)$ is a $C^*$-algebra, it can be decomposed into the span of $\text{OD}(\varphi)^1_+$. By the definition of $\text{OD}(\varphi)$, we see that $a \in \text{OD}(\varphi)^1_+$ implies $\varphi(a)^2 = \lim_{\lambda} \varphi(a^2)\varphi(h_\lambda)$. $\square$
4 Examples of $k$-positive order $\varepsilon$ maps

In the previous section we have seen that the class of order zero maps is explicitly divided into the two cases, positive but not completely positive and completely positive (Corollary 3.3). A well-known example of positive order zero map, but not 2-positive, is the transposition on a matrix algebra. This section studies the possibility of constructing $k$-positive maps of almost order zero but not $k+1$-positive.

From now on we denote by $\{e^{(m)}_{i,j}\}_{i,j=1}^n$ the canonical matrix units of $M_n$ and $\tr_n$ the normalized trace on $M_n$. The following construction of $k$-positive almost order zero maps relies on Tomiyama’s work in [29].

Example 4.1. Fix a natural number $k$ and $\varepsilon > 0$. Let $n$ be a natural number such that $k < n$. For $\lambda \in (0, \infty)$, we let $\psi_\lambda$ be the linear map from $M_n$ to $M_n$ defined by

$$\psi_\lambda(a) = \lambda \tr_n(a) 1_{M_n} + (1 - \lambda) a \quad \text{for} \ a \in M_n.$$  

Because of [29] Theorem2], we can see that $\psi_\lambda$ is $k$-positive if and only if $\lambda \leq 1 + \frac{1}{nk-1}$. We let $\lambda \in (0, \infty)$ be such that $\frac{1}{n(k+1)-1} < \lambda - 1 \leq \frac{1}{nk-1}$.

Let $\iota : M_n \to (e^{(m)}_{1,1} \otimes 1_{M_n})M_m \otimes M_n (e^{(m)}_{1,1} \otimes 1_{M_n})$ be the canonical isomorphism. We define a linear map $\varphi^{(m)}_\lambda$ from $M_m \otimes M_n$ to $M_m \otimes M_n$ by

$$\varphi^{(m)}_\lambda(x) = (1 - \varepsilon)x + \varepsilon 1_{M_m} \otimes \psi_\lambda \circ \iota^{-1}(e^{(m)}_{1,1} \otimes 1_{M_n})x(e^{(m)}_{1,1} \otimes 1_{M_n}), \quad \text{for} \ x \in M_m \otimes M_n.$$  

Then for any $m \in \mathbb{N}$, this map $\varphi^{(m)}_\lambda$ is unital and $k$-positive, satisfying

$$\|\varphi^{(m)}_\lambda(x)^2 - \varphi^{(m)}_\lambda(x^2)\| < 6\varepsilon,$$

for any contraction $x$ in $M_m \otimes M_n$. By Theorem [1.1] we can regard $\varphi^{(m)}_\lambda$ as an almost order zero map.

For a large $m \in \mathbb{N}$, we have that $\varphi^{(m)}_\lambda$ is not $(k+1)$-positive. Actually, setting the unital completely positive map $\Phi_n : M_m \otimes M_n \to M_n$ by $\Phi_n(a \otimes b) = \tr_n(a)b$, and

$$\tilde{\lambda} = \frac{me\lambda}{(1-\varepsilon)m+me} > 0,$$

we see that

$$\Phi_n \circ \varphi^{(m)}_\lambda(\iota(a)) = \frac{1 - \varepsilon}{m} a + \varepsilon (\lambda \tr_n(a) 1_{M_n} + (1 - \lambda) a)$$  

$$= \frac{\varepsilon \lambda}{\tilde{\lambda}} (\lambda \tr_n(a) 1_{M_n} + (1 - \lambda) a), \quad \text{for} \ a \in M_n.$$  

Since

$$\lim_{m \to \infty} \tilde{\lambda} = \lambda \in \left(1 + \frac{1}{n(k+1)-1}, 1 + \frac{1}{nk-1}\right),$$

it follows that $\tilde{\lambda} > 1 + \frac{1}{n(k+1)-1}$ for a large $m \in \mathbb{N}$. Thus $\Phi_n \circ \varphi^{(m)}_\lambda|_{\iota(M_n)}$ is not $(k+1)$-positive, so $\varphi^{(m)}_\lambda$ is not.

In contrast to the above example, by fixing the size of the matrix algebras, the following proposition shows how close unital 2-positive almost order zero maps are to being completely positive.
We identify a $C^*$-algebra $B$ and $M_n$ as the $\ell^{-}$-direct sum of $\Lambda$ such that the canonical completely positive contraction $a \in M_n$. Then the linear map $M_n \ni a \mapsto \varphi(a) + n\varepsilon \mathrm{Tr}_n(a)1_B$ is completely positive, where $\mathrm{Tr}_n$ denotes the non-normalized trace on $M_n$.

**Proof.** We set $b_0 = \sum_{i=1}^{\infty} e_{i,i}^{(n)} \otimes e_{i,i}^{(n)} \in M_n \otimes M_n$ and $b = b_0 \otimes b_0 \in M_n \otimes M_n$. It is enough to show that the Choi matrix $(\varphi + \varepsilon n\mathrm{Tr}_n) \otimes \mathrm{id}_{M_n}(b)$ is a positive element in $B \otimes M_n$, (see [2, Proposition 1.5.12] for example). Since $\|\varphi(e_{i,i})\varphi(e_{i,i}) - \varphi(e_{i,i})\| < \varepsilon$, it follows that

$$
\left\| \varphi \otimes \mathrm{id}_{M_n}(b_0) \varphi \otimes \mathrm{id}_{M_n}(b_0) - \sum_{i,j=1}^{n} \varphi(e_{i,j}) \otimes e_{i,j} \right\| < n\varepsilon.
$$

Thus we have that

$$(\varphi + n\varepsilon \mathrm{Tr}_n) \otimes \mathrm{id}_{M_n}(b) = \sum_{i,j=1}^{n} \varphi(e_{i,j}) \otimes e_{i,j} + n\varepsilon \sum_{i=1}^{n} 1_B \otimes e_{i,i} \geq 0.$$

$$\square$$

### 5 One-way CPAP

In the rest of this paper, we focus on nuclear $\mathcal{C}^*$-algebras and aim to show the second main result Theorem 1.2. The following weaker characterization of nuclearity has implicitly appeared in Ozawa's survey [24], which was obtained in the context of [16] and [17]. Let us revisit this argument for our self-contained proof.

For a $\mathcal{C}^*$-algebra $B$ and a net $A_\lambda, \lambda \in \Lambda$ of $\mathcal{C}^*$-subalgebras of $B$, we denote by $\bigoplus_\lambda A_\lambda$ the $\ell_\infty$-direct sum of $\{A_\lambda\}_{\lambda \in \Lambda}$ (i.e., the set of nets $(a_\lambda)_{\lambda \in \Lambda}$ such that $a_\lambda \in A_\lambda$ and $\sup_{\lambda} \|a_\lambda\| < \infty$), and $\bigoplus_\lambda A_\lambda$ the $\ell_\infty$-direct sum (i.e., the set of nets $(a_\lambda)_{\lambda \in \Lambda}$ such that $\lim_{\lambda} \|a_\lambda\| = 0$). It is well-known that $\bigoplus_\lambda A_\lambda$ is a $\mathcal{C}^*$-algebra and $\bigoplus_\lambda A_\lambda$ is an ideal of $\bigoplus_\lambda A_\lambda$. When $A_\lambda = A$ for any $\lambda \in \Lambda$ we let

$$\ell_\infty(\Lambda, A) = \bigoplus_\lambda A_\lambda \quad \text{and} \quad c_0(\Lambda, A) = \bigoplus_\lambda A_\lambda.$$

We identify a $\mathcal{C}^*$-algebra $A$ with the $\mathcal{C}^*$-subalgebra of $\ell_\infty(\Lambda, A)$ consisting of equivalence classes of constant nets.

**Theorem 5.1.** A $\mathcal{C}^*$-algebra $A$ is nuclear if and only if there exists a net $\varphi_\lambda : M_{N_\lambda} \to A, \lambda \in \Lambda$ of completely positive contractions such that the canonical completely positive contraction

$$\Phi = (\varphi_\lambda)_\lambda : \bigoplus_\lambda M_{N_\lambda} \to \ell_\infty(\Lambda, A) \quad \text{satisfies} \quad \Phi\left(\left(\prod_\lambda M_{N_\lambda}\right)^1\right) \subseteq A^1.$$
The following lemma is essentially given in [16] Lemma 3.5 for completely positive maps. A generalization for 2-positive maps may be of independent interest.

For a given unital C*-algebra \( A \), we define

\[
\Lambda_A = \{ F \subset A^1 : \text{a finite subset of unitaries in } A \} \times \{ \varepsilon \in \mathbb{R} : \varepsilon > 0 \},
\]

and regard \( \Lambda_A \) as the (upward-filtering) ordered set by the inclusion order on \( 2^{A^1} \) and the standard order on \( \mathbb{R} \). For a C*-algebra \( A \), we let \( \text{dist}(x, F) \) denote \( \inf_{y \in F} \| x - y \| \) for \( x \in A \) and \( F \subset A \).

**Lemma 5.2.** Let \( A \) be a unital C*-algebra and \( \mathcal{M} \) a unital C*-algebra which is closed under the polar decomposition by unitaries, i.e., for any \( x \in \mathcal{M} \), there exists a unitary \( u \in \mathcal{M} \) such that \( x = u|x| \). Suppose that for \( \lambda = (F, \varepsilon) \in \Lambda_A \), a 2-positive contraction \( \varphi : \mathcal{M} \to A \) satisfies \( \text{dist}(x, \varphi(\mathcal{M}^1)) < \varepsilon \) for all \( x \in F \). Then there exist unitaries \( U_x \in \mathcal{M} \), \( x \in F \) such that

\[
\| \varphi(U_x) - x \| < 3\sqrt{\varepsilon} \quad \text{for all } x \in F.
\]

**Proof.** Let \( y_x \in \mathcal{M}^1 \) be such that \( \| \varphi(y_x) - x \| < \varepsilon \) for \( x \in F \). For \( x \in F \), by the polar decomposition of \( y_x \), there exists a unitary \( U_x \in \mathcal{M} \) such that \( y_x = U_x|y_x| \). Since \( x \in F \) is a unitary, it follows that \( \| \varphi(y_x)^*\varphi(y_x) - 1_A \| < 2\varepsilon \). Then Kadison’s inequality implies that

\[
(1 - 2\varepsilon)1_A \leq \varphi(y_x)^*\varphi(y_x) \leq \varphi(y_x^*y_x) \leq \varphi(1_{\mathcal{M}}) \leq 1_A.
\]

By \( \varphi(1 - |y_x|) \leq \varphi(1 - y_x^*y_x) \leq 2\varepsilon 1_A \), we have that

\[
\| \varphi(U_x) - x \| < \| \varphi(U_x - y_x) \| + \varepsilon \leq \| \varphi((1 - |y_x|)^2) \|^{1/2} + \varepsilon \leq \sqrt{2\varepsilon} + \varepsilon < 3\sqrt{\varepsilon}.
\]

**Lemma 5.3** (Lemma 3.6 of [16], see also Lemma 4.1.4 of [10]).

For \( N \in \mathbb{N} \) and \( (F, \varepsilon) \in \Lambda_{MN} \), there exist unitaries \( v_i \in MN_i \), \( i = 1, 2, ..., M \) and permutations \( \sigma_x \), \( x \in F \) of \( \{1, 2, ..., M\} \) such that

\[
\max_{i=1,2,...,M} \| v_i \cdot x - v_{\sigma_x(i)} \| < \varepsilon \quad \text{for all } x \in F.
\]

**Proof of Theorem 5.1.** It is shown in [18, Theorem], [5, Theorem 3.1] that the nuclearity of \( A \) implies the completely positive approximation property (CPAP) which is stronger than the condition in Theorem 5.1. Then it is enough to show the converse direction.

It is well-known that \( A \) is nuclear if and only if the unitization \( A^\sim \) of \( A \) is nuclear. Thus we may assume that \( A \) is unital. Actually, for \( \tilde{\lambda} = (F^\sim, \varepsilon) \in \Lambda_{A^\sim} \), taking an approximate unit of \( A \) we have a positive contraction \( e \in A \) and \( \lambda_\varepsilon \in \mathbb{C} \) for \( x \in F^\sim \) such that \( (1_{A^\sim} - e)x \approx_\varepsilon \lambda_\varepsilon (1_{A^\sim} - e) \) and \( [x, e] \approx_\varepsilon 0 \) for all \( x \in F^\sim \). Let \( \tilde{e} \in A_1^\sim \) be such that \( e^{1/2}\tilde{e}^{1/2} \approx_\varepsilon e^{1/2} \). By the assumption of \( A \), we now obtain a completely positive contraction \( \varphi : MN \to A \) such that \( \text{dist}(y, \varphi(MN^1)) < \varepsilon \) for all \( y \in \{ \tilde{e} \} \cup \{ \tilde{e}^{1/2}x\tilde{e}^{1/2} : x \in F^\sim \} \subset A^1 \). Then we have \( e^{1/2}\varphi(1_{MN})e^{1/2} \approx_\varepsilon e \). Define a completely positive map \( \varphi^\sim : MN \oplus \mathbb{C} \to A \) by \( \varphi^\sim(x \oplus e) = e^{1/2}\varphi(x)e^{1/2} + c(1_{A^\sim} - e) \) for \( x \in MN \) and \( c \in \mathbb{C} \). Since \( \varphi^\sim(1_{MN} \oplus 1) \approx_\varepsilon 1_{A^\sim} \),
the canonical extension \( \varphi^\lambda : M_{N+1} \to A \) of \( \frac{1}{1+x} \hat{\varphi} \) is a completely positive contraction, which satisfies the condition in Theorem \textup{[5.1]} for \( A^\sim \).

Let \( \lambda = (F, \varepsilon) \in \Lambda_A \) be such that \( \varepsilon < 1 \). By the assumption, we now obtain a completely positive contraction \( \varphi : M_N \to A \) such that \( \text{dist}(x, \varphi(M_N^1)) < (\varepsilon/6)^4 \) for all \( x \in F \). By Lemma \textup{[5.2]} there are unitaries \( U_x \in M_N, x \in F \) such that \( \|\varphi(U_x) - x\| < \varepsilon^2/12 \) for \( x \in F \). By Lemma \textup{[5.3]}, for \( \{(U_x)_{x \in F}, \varepsilon/2\} \in \Lambda_{M_N} \), there exist unitaries \( v_i \in M_N, i = 1, 2, ..., M \) and permutations \( \sigma_x, x \in F \) of \( \{1, 2, ..., M\} \) such that

\[
\|v_i \cdot U_x - v_{\sigma_x(i)}\| < \varepsilon/2 \quad \text{for all } i = 1, 2, ..., M, \text{ and } x \in F.
\]

Due to the Kasparov-Stinespring dilatation theorem \textup{[15]}, (see also \textup{[20] Theorem 6.5}), there exists a \(*\)-homomorphism \( \pi : M_N \to B(\mathcal{H}_A) \) such that \( \varphi(a) = \pi(a)_{1,1} \in A \), where the notations of \( \mathcal{H}_A \) and \( a_{i,j} \in A \) for \( a \in B(\mathcal{H}_A) \) are same as in Lemma \textup{3.1}. We set \( a_j^{(i)} = \pi(v_i)_{j,1} \in A \) for \( i = 1, 2, ..., M \) and \( j \in \mathbb{N} \).

From (ii) of Lemma \textup{3.1} and \( \|\pi(U_x)^{1,1} \pi(U_x)_{1,1} - 1_A\| = \|\varphi(U_x)^* \varphi(U_x) - 1_A\| < \varepsilon/6 \), it follows that

\[
\left\| \sum_{j=2}^{\infty} \pi(U_x)^{j,1} \pi(U_x)_{j,1} \right\| < \varepsilon^2/6 \quad \text{for all } x \in F.
\]

Combining this with \( \|\pi(v_i) \cdot \pi(U_x) - \pi(v_{\sigma_x(i)})\| < \varepsilon/2 \), we have that for \( x \in F \)

\[
\left\| \sum_{j=1}^{\infty} (a_j^{(i)} x - a_j^{(\sigma_x(i))}) \right\|^{1/2} = \left\| (a_j^{(i)} x_j - (a_j^{(\sigma_x(i))})_j \right\|_{\mathcal{H}_A} < \varepsilon.
\]

Since \( v_i, i = 1, 2, ..., M \) are unitaries, we obtain \( L \in \mathbb{N} \) such that

\[
\left\| \sum_{j=1}^{L} a_j^{(i)} x_j - 1_A \right\| < \varepsilon.
\]

Let \( A^{**} \) be the second dual of \( A \) faithfully represented on a Hilbert space \( \mathcal{H} \) i.e., \( A \subset A^{**} \subset B(\mathcal{H}) \). For \( \lambda \in \Lambda_A \), we define a completely positive map \( \Phi_\lambda : B(\mathcal{H}) \to B(\mathcal{H}) \) by

\[
\Phi_\lambda(y) = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{L} a_j^{(i)*} y a_j^{(i)} \quad \text{for } y \in B(\mathcal{H}).
\]

Thus we have that for \( x \in F \) and \( y \in B(\mathcal{H})^1 \)

\[
\Phi_\lambda(y)x \approx_{\varepsilon} \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{L} a_j^{(i)*} y a_j^{(\sigma_x(i))} = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{L} a_j^{(\sigma_x^{-1}(i))*} y a_j^{(i)} \approx_{\varepsilon} \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{L} (a_j^{(i)*} x^*) y a_j^{(i)} = x \Phi_\lambda(y).
\]
From \( \left\| \sum_{j=1}^{L} a_{j}^{(i)} \lambda_{j} \lambda_{j}^{(i)} - I_{A} \right\| < \varepsilon \), for \( y \in B(\mathcal{H}) \cap A' \) it follows that \( \Phi_{\lambda}(y) \approx \varepsilon y \). So, \( \Phi_{\lambda} \) is close to a conditional expectation onto \( A' \). Let \( \omega \) be a (cofinal) ultrafilter on the ordered set \( \Lambda_{A} \). Then one can define a bounded map \( \Phi : B(\mathcal{H}) \to B(\mathcal{H}) \) by the weak* limit \( \Phi(y) = \text{weak*}- \lim_{\lambda \to \omega} \Phi_{\lambda}(y) \) in \( B(\mathcal{H}) \). By the above conditions of \( \Phi_{\lambda} \), it is straightforward to check that \( \Phi \) is a conditional expectation on \( B(\mathcal{H}) \cap A' \). Hence \( A' \) is an injective von Neumann algebra, and so is \( A'' = A^{**} \). Because of [7], we can see that \( A^{**} \) is AFD which implies the CPAP of \( A \).

**Remark 5.4.** In [28] R. Smith showed that the complete positivity of contractive maps in the CPAP can be replaced by the complete contractivity. However, we cannot expect to replace completely positive contractions \( \varphi \) in Theorem 5.1 by completely contractive maps. In fact, there are many non-nuclear C*-algebras with the completely contractive approximation property (CCAP), although any C*-algebra \( A \) with the CCAP satisfies the following condition: there exists a net of complete contractions \( \varphi_{\lambda} : M_{\Lambda_{A}} \to A, \lambda \in \Lambda \) such that for \( a \in A^{1} \) there are \( x_{a,\lambda} \in M_{\Lambda_{A}} \), \( \lambda \in \Lambda \) satisfying \( \lim_{\lambda} \varphi_{\lambda}(x_{a,\lambda}) = a \).

### 6 Decomposition rank by 2-positive maps

Before proving Theorem 1.2 let us recall the definition of decomposition rank.

**Definition 6.1** (E. Kirchberg - W. Winter, [19]). For \( d \in \mathbb{N} \cup \{0\} \), a C*-algebra \( A \) is said to have decomposition rank at most \( d \), if for a finite subset \( F \) of contractions in \( A \) and \( \varepsilon > 0 \), there exist finite dimensional C*-algebras \( F_{i} \), \( i = 0, 1, \ldots, d \), a completely positive contraction \( \psi : A \to \bigoplus_{i=0}^{d} F_{i} \), and completely positive order zero contractions \( \varphi_{i} : F_{i} \to A, i = 0, 1, \ldots, d \) such that \( \sum_{i=0}^{d} \varphi_{i} : \bigoplus_{i=0}^{d} F_{i} \to A \) is contractive and

\[
\left\| \left( \sum_{i=0}^{d} \varphi_{i} \right) \circ \psi(x) - x \right\| < \varepsilon, \quad \text{for all } x \in F.
\]

**Theorem 6.2** (Theorem 1.2). Let \( A \) be a unital separable C*-algebra and \( d \in \mathbb{N} \cup \{0\} \). Then the following conditions are equivalent.

(i) The decomposition rank of \( A \) is at most \( d \).

(ii) For \( \lambda = (F, \varepsilon) \in \Lambda_{A} \), there are finite dimensional C*-algebras \( F_{i} \), \( i = 0, 1, \ldots, d \), a 2-positive contraction \( \psi : A \to \bigoplus_{i=0}^{d} F_{i} \), and 2-positive order zero contractions \( \varphi_{i} : F_{i} \to A, i = 0, 1, \ldots, d \) such that \( \sum_{i=0}^{d} \varphi_{i} : \bigoplus_{i=0}^{d} F_{i} \to A \) is contractive and

\[
\left\| \left( \sum_{i=0}^{d} \varphi_{i} \right) \circ \psi(x) - x \right\| < \varepsilon, \quad \text{for all } x \in F.
\]

(iii) There exist finite dimensional C*-algebras \( F_{i,\lambda} \), \( i = 0, 1, \ldots, d, \lambda \in \Lambda \) and nets \( \varphi_{i,\lambda} : F_{i,\lambda} \to A, i = 0, 1, \ldots, d, \lambda \in \Lambda \) of 2-positive order zero contractions such that
there are unitaries $U_i : F_i \to A$ is contractive for any $\lambda \in \Lambda$, where $F_\lambda = \bigoplus_{i=0}^d F_{i,\lambda}$, and the canonical contraction

$$\Phi = \left( \sum_{i=0}^d \varphi_{i,\lambda} \right)_\lambda : \bigoplus_{i=0}^d F_\lambda \to \frac{\ell^\infty(\Lambda, A)}{c_0(\Lambda, A)} \text{ satisfies } \Phi \left( \left( \bigoplus_{i=0}^d F_{i,\lambda} \right) \right) \supset A^1.$$

**Proof.** The implications (i) $\implies$ (ii) $\implies$ (iii) are trivial. We shall show (iii) $\implies$ (i). By Corollary 3.3, we see that $\sum_{i=0}^d \varphi_{i,\lambda}, \lambda \in \Lambda$ are completely positive contractions. Taking a conditional expectation from a matrix algebra onto $F_\lambda$, by Theorem 5.1 we know that $A$ is nuclear.

From the assumption of (iii), for $\mu = (F, \varepsilon) \in \Lambda_A$ we obtain finite dimensional $C^*$-algebras $F_{i,\mu}, i = 0, 1, \ldots, d$, and completely positive order zero contractions $\varphi_{i,\mu} : F_{i,\mu} \to A, i = 0, 1, \ldots, d$ such that

$$\text{dist} \left( x, \sum_{i=0}^d \varphi_{i,\mu} \left( \left( \bigoplus_{i=0}^d F_{i,\mu} \right) \right) \right) < \varepsilon, \text{ for all } x \in F.$$

Set $F_\mu = \bigoplus_{i=0}^d F_{i,\mu}$ and $\varphi_\mu = \sum_{i=0}^d \varphi_{i,\mu} : F_\mu \to A$ for $\mu \in \Lambda_A$. By Lemma 5.2 and $\|\varphi_\mu\| \leq 1,$ there are unitaries $U_{x,\mu} \in F_\mu, x \in F, \mu = (F, \varepsilon) \in \Lambda_A$, such that $\|\varphi_\mu(U_{x,\mu}) - x\| < 3\sqrt{\varepsilon}$ for all $x \in F$. For any unitary $x \in A$, we set $U_{x,\mu} = 1_{F_\mu}$ if $x \not\in F$ and $\mu = (F, \varepsilon)$, and set $U_x = (U_{x,\mu})_\mu \in \prod_\mu F_\mu$. We let $Q : \prod_\mu F_\mu \to \prod_\mu F_\mu$ be the quotient map, $\overline{U}_x = Q(U_x)$, and let $C$ be the $C^*$-subalgebra of $\prod_\mu F_\mu$ generated by $\{\overline{U}_x : x \text{ is a unitary in } A\}.$

Let $\varphi : \prod_\mu F_\mu \to \frac{\ell^\infty(\Lambda_A, A)}{c_0(\Lambda_A, A)}$ be the completely positive contraction defined by $\varphi \circ Q((x_{i,\mu})_\mu) = (\varphi_{i,\mu}(x_{i,\mu}))_\mu$ in $\ell^\infty(\Lambda_A, A)/c_0(\Lambda_A, A).$ By regarding $A$ as the $C^*$-algebra of $\ell^\infty(\Lambda_A, A)/c_0(\Lambda_A, A)$, it follows that $\varphi(\overline{U}_x) = x$ for any unitary $x \in A$, then $\varphi(C) = A$. Because of

$$\varphi(\overline{U}_x)^* \varphi(\overline{U}_x) = 1_A = \varphi(\overline{U}_x^* \overline{U}_x) \quad \text{and} \quad \varphi(\overline{U}_x) \varphi(\overline{U}_x)^* = 1_A = \varphi(\overline{U}_x \overline{U}_x^*),$$

we see that $\varphi|_C : C \to A$ is a unital $*$-homomorphism. Let $\widetilde{\varphi}$ be the $*$-isomorphism from $C/\ker(\varphi|_C)$ onto $A$ and $\widetilde{\psi} = \varphi|_C^{-1}$.

Applying the Choi-Effros lifting theorem [3] to $\widetilde{\psi}$, we obtain a unital completely positive map $\psi : A \to \prod_\mu F_\mu$ such that $\varphi \circ Q \circ \psi(a) = a$ for any $a \in A$. Note that $A$ is required to be nuclear and separable in order to apply [3] Theorem 3.10. Taking unital completely positive maps $\psi_\mu : A \to F_\mu, \mu \in \Lambda_A$ with $(\psi_\mu(a))_\mu = \psi(a)$ for $a \in A$, we conclude that $\psi_\mu$ and $\varphi_{i,\mu}$ satisfy the conditions in (i). \qed

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