INTERPLAY OF RANDOM INPUTS AND ADAPTIVE COUPLINGS IN THE WINFREE MODEL

SEUNG-YEAL HA
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826 and
Korea Institute for Advanced Study, Hoegiro 85, Seoul 02455, Republic of Korea

DOHEON KIM
School of Mathematics, Korea Institute for Advanced Study
Hoegiro 85, Seoul 02455, Republic of Korea

BORA MOON*
Department of Mathematics, Research Institute for Natural Sciences
Hanyang University, Seoul 04763, Republic of Korea

(Communicated by Feng-Yu Wang)

Abstract. We study a structural robustness of the complete oscillator death state in the Winfree model with random inputs and adaptive couplings. For this, we present a sufficient framework formulated in terms of initial data, natural frequencies and adaptive coupling strengths. In our proposed framework, we derive propagation of infinitesimal variations in random space and asymptotic disappearance of random effects which exhibits the robustness of the complete oscillator death state for the random Winfree model.

1. Introduction. Emergence of collective behaviors is often observed in natural and man-made many-body systems, e.g., aggregation of bacteria, flocking of birds, flashing of fireflies and herding of sheep, etc [1, 4]. Among such collective behaviors, our main interest in this paper lies on the synchronization such as collective firing of pacemaker cells, fireflies and neurons, etc. Synchronization was first mathematically modeled by Arthur Winfree in a half century ago in his bachelor thesis [29] and it motivated the work of Yoshiki Kuramoto in [19, 20]. In this paper, we continue the earlier studies begun in a series of recent papers [14, 15, 16, 17] on the emergence of asymptotic patterns such as the complete oscillator death (COD) states, complete synchronization, partial oscillator death, incoherent states and chimera states for the Winfree model. Although the Winfree model was the first mathematical model for synchronization [29], there are few works in literature on the emergence

2020 Mathematics Subject Classification. Primary: 70F99, 92B25.
Key words and phrases. Complete oscillator death, local sensitivity analysis, synchronization, Winfree model.

The work of S. Y. Ha was supported by the NRF grant (2020R1A2C3A01003881), the work of D. Kim was supported by a KIAS Individual Grant (MG073901) at Korea Institute for Advanced Study, and the work of B. Moon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2019R1I1A1A01059585) and the Ministry of Science and ICT (NRF-2020R1A4A3079066).

* Corresponding author.
of asymptotic patterns of the continuous equation [3, 21, 22, 23, 24] and the ODE
[14, 15, 16, 17] compared to the extensive research on the Kuramoto model, to name a few [5, 8, 9, 10, 19, 20]. Recently, uncertainty quantification(UQ) analysis has
been studied in the context of flocking and synchronization for the collective models
in [2, 6, 11, 12, 13]. We also refer to a recent book [18] on UQ for kinetic and hyperbolic equations. Our main focus lies in the robustness of the COD states under
the random effects via the local sensitivity analysis. To fix the idea, let \( \theta_i = \theta_i(t) \)
be the phase of the \( i \)-th limit-cycle oscillator with natural frequency \( \nu_i \), and let \( \kappa_{ij} \)
be the mutual coupling strength between the \( i \)-th and \( j \)-th oscillators. Under this
setting, we assume that the phase \( \theta_i \) is governed by the Winfree model:
\[
\dot{\theta}_i = \nu_i - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \sin \theta_i (1 + \cos \theta_j), \quad t > 0, \quad 0 \leq i \leq N. \tag{1.1}
\]
In most previous literature, the Winfree model (1.1) has been studied only on a
static network for deterministic data such that the pairwise coupling strength \( \kappa_{ij} \)
is a constant in time. However, in reality, mutual coupling strengths can depend on
several factors such as time, relative phase \( \theta_i - \theta_j \) and random inputs, etc. To deal
with more natural modeling, we adopt an evolutionary wiring coupling scheme with
Hebbian-like adaptive coupling, which is a dynamic law for the pairwise coupling
strength that adapts to the phase configuration discussed in [25, 27], and introduce
random parameters \( z \in \Omega \), where a nonempty open set \( \Omega \), whose probability density
function is given by \( \pi = \pi(z) \). In fact, the explicit form of probability density
function \( \pi \) does not play any role in later analysis. Then, the random Winfree phase
process \( \theta_i = \theta_i(t, z) \) is governed by the Cauchy problem to the random Winfree
model with adaptive coupling: for \( i, j = 1, \cdots, N \),
\[
\begin{align*}
\partial_t \theta_i(t, z) &= \nu_i - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \sin \theta_i (1 + \cos \theta_j), \quad t > 0, \quad z \in \Omega, \\
\partial_t \kappa_{ij}(t, z) &= -\gamma(z) \kappa_{ij} + \mu(z) \cos(\theta_j - \theta_i), \\
\theta_i(0, z) &= \theta_i^0(z), \quad \kappa_{ij}(0, z) = \kappa_{ij}^0(z).
\end{align*}
\tag{1.2}
\]
Here, \( \gamma, \mu \in C^\infty_b(\Omega) \) are nonnegative and infinitely differentiable bounded random
friction coefficient and learning enhancement rate, respectively, and the function
space \( C^\infty_b(\Omega) \) denotes the set of all infinitely differentiable function with bounded
derivatives. We assume that the randomness in the initial data, \( \gamma \) and \( \mu \) follow from
the same random source \( z \). If randomness in \( \gamma, \mu \) and the initial data are quenched
and satisfy
\[
\gamma = 0, \quad \mu = 0 \quad \text{and} \quad \kappa_{ij}^0 = \kappa_{ij}, \quad 1 \leq i, j \leq N,
\]
system (1.2) simply reduces to the deterministic Winfree model (1.1). For the
proposed random dynamical system (1.2), we are mainly interested in the following
questions:

(Q1) : Under adaptive coupling strength, does the random system (1.2)
admit a stable complete oscillator death state(or equilibrium)?

(Q2) : If so, are the complete oscillator death states for (1.2) robust in
the presence of randomness?

Note that the first question concerns about the existence of random equilibrium
state, whereas the second question deals with about the robustness of equilibrium
state. In the sequel, for notational simplicity, we set
\[
z \in \Omega \subset \mathbb{R}, \quad \Theta := (\theta_1, \cdots, \theta_N), \quad K := (\kappa_{ij})_{1 \leq i, j \leq N}.
\]
To see the random effect in (1.2), we expand the phase and coupling strengths via Taylor’s expansion in \( z \)-variable:
\[
\begin{align*}
\theta_i(t, z + dz) &= \theta_i(t, z) + \partial_z \theta_i(t, x) dz + \frac{1}{2} \partial^2_{zz} \theta_i(t, z)(dz)^2 + \cdots , \\
\kappa_{ij}(t, z + dz) &= \kappa_{ij}(t, z) + \partial_z \kappa_{ij}(t, x) dz + \frac{1}{2} \partial^2_{zz} \kappa_{ij}(t, z)(dz)^2 + \cdots .
\end{align*}
\]
(1.3)

Thus, the local sensitivity estimates [26, 28] deal with the dynamic behavior of the sensitivity vectors \( \partial^2_z \Theta \) and matrices \( \partial^2_z K \) consisting of coefficients in the R.H.S. of (1.3).

Next, we briefly discuss our two main results: First, we present a sufficient framework leading to COD states for system (1.1):
\[
\mu(z) > 0, \quad \gamma(z) > 0, \quad \forall \ z \in \Omega, \quad \sup_{z \in \Omega} \left| \frac{\gamma(z)}{\mu(z)} \right| < 1.
\]

In this framework, we show that there exists a unique complete oscillator death \( \Theta^\infty (z) := (\theta^\infty_1(z), \ldots, \theta^\infty_N(z)) \) satisfying
\[
\frac{\gamma(z)}{\mu(z)} \nu_i = -\frac{1}{N} \sum_{1 \leq j \leq N} \cos(\theta^\infty_j - \theta^\infty_i) \sin \theta^\infty_i (1 + \cos \theta^\infty_i), \quad 1 \leq i \leq N,
\]
and it is asymptotically stable in the sense that for each \( z \in \Omega \), there exists a random positive constant \( C = C(z) \) such that
\[
\sum_{1 \leq i \leq N} |\theta_i(t, z) - \theta^\infty_i(z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\kappa_{ij}(t, z) - \kappa^\infty_{ij}(z)| \leq e^{-C(z) t}, \quad \text{as } t \to \infty.
\]

See Theorem 3.2 and Theorem 3.3 for details.

Second, we provide the local sensitivity estimates for the \( z \)-variations \( \partial^2_z \Theta \) and \( \partial^2_z K \) (Theorem 4.1 and Theorem 5.1):

- (Uniform bound for \( \partial^2_z \Theta \) and \( \partial^2_z K \)): For \( \ell \geq 1 \), there exist positive random functions \( C(z), C_\ell(z) \) such that
\[
\sum_{1 \leq i \leq N} |\partial^2_z \theta_i(t, z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\partial^2_z \kappa_{ij}(t, z)| \leq \left( \sum_{1 \leq i \leq N} |\partial^2_z \theta^0_i(z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\partial^2_z \kappa^0_{ij}(z)| \right) e^{-C(z) \ell} + C_\ell(z)(1 - e^{-C(z) \ell}),
\]
where the random functions \( C(z), C_\ell(z) \) depend only on given random initial data and parameters \( \mu(z), \gamma(z) \) (See Theorem 4.1).

- (\( \ell \)-stability): For a solution \((\Theta, K)\) and an equilibrium solution \((\Theta^\infty, K^\infty)\) to (1.1), we have the following exponential decay of \( \ell \)-difference between \( \partial^2_z \Theta, \partial^2_z K \) and \((\partial^2_z \Theta^\infty, \partial^2_z K^\infty)\): For \( \ell \geq 1 \), there exists positive random function \( \tilde{E}_\ell(z) \) such that
\[
\sum_{1 \leq i \leq N} |\partial^2_z(\theta_i - \theta^\infty_i)(t, z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\partial^2_z(\kappa_{ij} - \kappa^\infty_{ij})(t, z)| \leq \tilde{E}_\ell(z) \sum_{0 \leq p \leq \ell} \left( \sum_{1 \leq i \leq N} |\partial^2_z(\theta^0_i - \theta^\infty_i)(z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\partial^2_z(\kappa^0_{ij} - \kappa^\infty_{ij})(z)| \right) e^{-C(z) \ell}.
\]
(See Theorem 5.1 for details.)
Note that the above estimate for $\ell_1$-stability implies $z$-variations of all orders of $(\Theta, K)$ converges exponentially to those of an equilibrium state. Our stability analysis is based on a robust $\ell_1$-metric functional as a Lyapunov functional which was introduced in [5] rather than a standard linearization technique for a given equilibrium state.

The rest of this paper is organized as follows. In Section 2, we briefly summarize previous results on the deterministic Winfree model with constant coupling strengths, and present several a priori estimates. In Section 3, we provide a sufficient framework leading to the existence and asymptotic stability of complete oscillator death state. In Section 4 and Section 5, we present a sufficient framework for local sensitivity analysis, and study a uniform bound and $\ell_1$-stability of $z$-variations $(\partial_m^z \Theta, \partial_m^z K)$. Finally, Section 6 is devoted to a brief summary of our main results and some open questions to be pursued in future work.

**Notation:** Throughout the paper, we use the following handy notation:

- $R(\theta_i(t, z)) := \max_{1 \leq i \leq N} |\theta_i(t, z)|, \quad R(\theta_i^0(z)) := \max_{1 \leq i \leq N} |\theta_i^0(z)|,$
- $\kappa^0_m(z) := \min_{1 \leq i, j \leq N} \kappa_{ij}^0(z), \quad \kappa^0_M(z) := \max_{1 \leq i, j \leq N} \kappa_{ij}^0(z), \quad ||\nu||_{\infty} := \max_{1 \leq i \leq N} |\nu_i|,$
- $||\gamma|| := \max_{1 \leq r < \infty} |\partial_r^z \gamma(z)|, \quad ||\mu|| := \max_{1 \leq r \leq \infty} |\partial_r^z \mu(z)|.$

2. Preliminaries. In this section, we briefly introduce the Winfree model with uniform coupling strength and discuss several basic asymptotic states which emerge from the initial configuration, and then provide elementary estimates to be used crucially in later analysis.

2.1. The Winfree model. Consider the Winfree model with a uniform coupling strength:

$$\dot{\theta}_i = \nu_i - \frac{\kappa}{N} \sum_{1 \leq j \leq N} \sin \theta_i(1 + \cos \theta_j), \quad t > 0, \quad 0 \leq i \leq N. \quad (2.1)$$

Note that when the coupling is turned off, i.e., $\kappa \equiv 0$, the phase dynamics is given by the simple motion:

$$\dot{\theta}_i = \nu_i, \quad \text{i.e.,} \quad \theta_i(t) = \theta_i^0 + \nu_i t, \quad i = 1, \cdots, N. \quad (2.2)$$

As noted in [14, 15, 17, 22], for the description of asymptotic states, it is more convenient to use the “rotation number” $\rho_i$ defined as an asymptotic phase velocity:

$$\rho_i := \lim_{t \to \infty} \frac{\theta_i(t)}{t}, \quad \text{if R.H.S. exists.}$$

For zero coupling case, the relation (2.2) yields

$$\rho_i - \rho_j = \lim_{t \to \infty} \frac{\theta_i(t) - \theta_j(t)}{t} = \nu_i - \nu_j.$$ 

Thus, if two oscillators have different natural frequencies, mutual entrainment will not emerge. Thus, sufficiently large coupling strength is needed to guarantee the mutual entrainment between oscillators.

Next, we discuss one distinguished asymptotic state, namely “complete oscillator death (COD) state” which roughly says that all oscillators eventually stops rotating. Precise definition for the COD state will be given as follows.
**Definition 2.1.** [14, 15] Let $\Theta := (\theta_1, \cdots, \theta_N)$ be an ensemble of Winfree oscillators. Then, the phase configuration $\Theta$ tends to “complete oscillator death (COD) state” if and only if the rotation numbers of all oscillators are zero:

$$|\{i : \rho_i = 0\}| = N,$$

where $|A|$ is the size of set $A$.

**Remark 1.** It is easy to see that for the equilibrium, their rotation numbers are all zero as well:

$$\rho_i = 0, \quad i = 1, \cdots, N.$$ 

Thus, equilibria are also COD states.

In the following theorem, we quote the result [17] on the emergence of the COD state without a proof.

**Theorem 2.2.** [17] Suppose that a constant parameter $\alpha$ and initial configuration $\Theta_0$ satisfy

$$0 < \alpha < \pi, \quad \Theta_0^i \in (-\alpha, \alpha), \quad i = 1, \cdots, N.$$ 

Then, there exists a coupling strength $\kappa(\nu_i, \alpha) > 0$ depending on $\nu_i$ and $\alpha$, and a unique equilibrium $\Theta^\infty$ such that for any solution $\Theta$ to (2.1), we have

$$\lim_{t \to \infty} \Theta(t) = \Theta^\infty.$$ 

**Remark 2.** In [14, 15], existence of some chimera states have been studied on a locally coupled networks and the robustness of COD state has been investigated in [16] under the interplay of general network and time-delayed interactions. In particular, the latter work confirms that the COD state is independent of time-delay, as long as it is sufficiently small.

### 2.2. Elementary estimates.

In this subsection, we provide two elementary estimates to be crucially used in later analysis.

**Lemma 2.3.** Suppose that for a given $\alpha \in (0, \frac{\pi}{4})$, system parameters $\gamma, \mu, \|\nu\|_\infty$ and the initial data satisfy the following conditions: For $z \in \Omega$,

$$\|\nu\|_\infty < \frac{\mu(z)}{\gamma(z)} \cos 2\alpha(1 + \cos \alpha) \sin \alpha, \quad \kappa_m^0(z) \geq \frac{\mu(z)}{\gamma(z)} \cos 2\alpha, \quad R(\theta_0^i(z)) < \alpha,$$

and let $(\Theta, K)$ be a solution to (1.2). Then, for $z \in \Omega$, we have

1. $\sup_{0 \leq t < \infty} \kappa_{ij}(t, z) \leq \max \left\{ \kappa_M^0(z), \frac{\mu(z)}{\gamma(z)} \right\}$,
2. $\inf_{0 \leq t < \infty} \kappa_{ij}(t, z) \geq \frac{\mu(z)}{\gamma(z)} \cos 2\alpha, \quad \sup_{0 \leq t < \infty} R(\theta_i(t, z)) \leq \alpha.$

**Proof.** The proof of the first inequality is similar to the second one, so we only show the estimates in (ii). For $z \in \Omega$, we choose a time-dependent index $M$ such that

$$M = M(t, z) := \arg \max_{1 \leq i \leq N}|\theta_i(t, z)|, \quad t \in (0, \infty),$$

and define the set $T(z)$ and its supremum as

$$T(z) := \{T \in [0, \infty) : R(\theta_i(t, z)) < \alpha, \quad t \in [0, T]\}, \quad T^\infty(z) := \sup T(z).$$

Since $R(\theta_0^i(z)) < \alpha$ and $R(\theta_i(t, z))$ is a Lipschitz continuous function of $t$, there exists $\delta > 0$ such that

$$R(\theta_i(t, z)) < \alpha, \quad \text{for all} \quad t \in [0, \delta).$$
Thus, the set \( \mathcal{T}(z) \) is nonempty, and \( T^\infty(z) \) is well-defined.

Suppose that \( T^\infty(z) < \infty \). Then, we have
\[
|\theta_M(t, z)| < \alpha, \quad \text{for} \quad t \in [0, T^\infty),
\]
and by the continuity of \( R(\theta_i(\cdot, z)) \), one has
\[
\lim_{t \to T^\infty} R(\theta_i(t, z)) = \alpha, \quad |\theta_M(T^\infty(z))| = \alpha.
\]

Now, we consider the temporal evolution of \( \kappa_{ij} \):
\[
\partial_t \kappa_{ij} = -\gamma \kappa_{ij} + \mu \cos(\theta_j - \theta_i) \geq -\gamma \kappa_{ij} + \mu \cos 2\alpha, \quad t \in [0, T^\infty).
\]

This yields
\[
\kappa_{ij}(t, z) \geq \left( \kappa_{ij}^0(z) - \frac{\mu(z)}{\gamma(z)} \cos 2\alpha \right) e^{-\gamma(z) t} + \frac{\mu(z)}{\gamma(z)} \cos 2\alpha \geq \frac{\mu(z)}{\gamma(z)} \cos 2\alpha, \quad t \in [0, T^\infty).
\]

On the other hand,
\[
\partial_t |\theta_M(T^\infty(z))| = \nu_M \text{sgn}(\theta_M(T^\infty(z))) - \frac{1}{N} \sum_j \kappa_M j(T^\infty(z)) \sin |\theta_M(T^\infty(z))|(1 + \cos |\theta_M(T^\infty(z))|)
\]
\[
\leq ||\nu||_\infty - \frac{\mu(z)}{\gamma(z)} \cos 2\alpha \sin |\theta_M(T^\infty(z))|(1 + \cos |\theta_M(T^\infty(z))|)
\]
\[
= ||\nu||_\infty - \frac{\mu(z)}{\gamma(z)} \cos 2\alpha \sin(1 + \cos \alpha) < 0.
\]

This shows that \( |\theta_M| \) is strictly decreasing at \( t = T^\infty(z) \) so that there is \( \delta_0 > 0 \) such that
\[
|\theta_M(T^\infty(z) - \delta_0)| > \alpha.
\]

This contradicts to (2.3). Therefore, for \( z \in \Omega \), we have
\[
T^\infty(z) = \infty \quad \text{and} \quad R(\theta_i(t, z)) < \alpha, \quad \text{for all} \quad t \geq 0.
\]

Next, we introduce an index set: for a given positive integer \( n \),
\[
\Lambda(n) := \{(k_1, \cdots, k_n) \in (\mathbb{N} \cup \{0\})^n : k_1 + 2k_2 + \cdots + nk_n = n\},
\]
\[
\Lambda(n)' := \{(k_1, \cdots, k_n) \in (\mathbb{N} \cup \{0\})^{n-1} \times \{0\} : k_1 + 2k_2 + \cdots + nk_n = n\}.
\]

Note that \( (0, \cdots, 0, 1) \) is an element of \( \Lambda(n) \) but is not an element of \( \Lambda(n)' \). Then, the \( n \)-th derivative of the composite function \( f(g(x)) \) is given by the following formula:
\[
\frac{d^n}{dx^n} f(g(x)) = \sum_{(k_1, \cdots, k_n) \in \Lambda(n)} \frac{n!}{k_1! \cdots k_n!} f^{(k)}(g(x)) \prod_{p=1}^{n} \left( \frac{g(p)(x)}{p!} \right)^{k_p},
\]
where \( k := k_1 + \cdots + k_n \).

Now we state a Grönwall type lemma without a proof.
Lemma 2.4. Let \( y : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\} \) be a differential function satisfying
\[
y' \leq -\alpha y + C e^{-\beta t}, \quad t > 0, \quad y(0) = y_0,
\]
where \( \alpha \neq \beta \) and \( C \) are nonnegative constants. Then, \( y \) satisfies
\[
y(t) \leq y_0 e^{-\alpha t} + \frac{C}{\alpha - \beta} (e^{-\beta t} - e^{-\alpha t}), \quad t \geq 0.
\]

3. Asymptotic stability of COD state. In this section, we present an existence of COD state and its exponential stability using \( \ell_1 \)-type Lyapunov functional approach.

3.1. Existence of COD state. Note that the equilibrium solution is trivially COD state. Thus, we will show that system (3.1) exhibits equilibrium state using the inverse function theorem. Note that the equilibrium \((\Theta^\infty(z), K^\infty(z))\) satisfies
\[
0 = \nu_i - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \sin \theta_i^\infty(z)(1 + \cos \theta_j^\infty(z)), \quad z \in \Omega, \quad 1 \leq i, j \leq N,
\]
\[
0 = -\gamma(z) \kappa_{ii}^\infty(z) + \mu(z) \cos(\theta_i^\infty(z) - \theta_i^\infty(z)),
\]
or equivalently
\[
\frac{\gamma(z)}{\mu(z)} \nu_i = \frac{1}{N} \sum_{1 \leq j \leq N} \cos(\theta_j^\infty(z) - \theta_i^\infty(z)) \sin \theta_i^\infty(z)(1 + \cos \theta_j^\infty(z)). \tag{3.1}
\]

Next, we will look for a solution to (3.1) via the inverse function theorem, i.e., the existence of solutions to (3.1) reduces to the construction of a suitable map and its range where the map is surjective. For this, we define \( \mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) by
\[
\mathcal{F}(x) = (f_1(x), \cdots, f_N(x)),
\]
\[
f_i(x) := \frac{1}{N} \sum_{1 \leq j \leq N} \cos(x_j - x_i) \sin(x_i)(1 + \cos(x_j)), \quad i = 1, \cdots, N.
\]

We recall some basic properties on the operator norm of matrices to be used in the proof of the following lemma. Let \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) be an \( m \times n \) real matrix. We defined the operator norm of \( A \) as
\[
\|A\|_{op} := \sup_{x \in \mathbb{R}^n, \|x\| \leq 1} \|Ax\|,
\]
where \( \|\cdot\|\) is the \( \ell_2 \)-norm. Then, the operator norm satisfies the following properties:
\[
\|A\|_{op}^2 \leq \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2, \quad \|A_1 A_2 \cdots A_k\|_{op} \leq \|A_1\|_{op} \|A_2\|_{op} \cdots \|A_k\|_{op},
\]
and for the \( m \times 1 \) matrix, its operator norm coincides with its \( \ell_2 \)-norm. In the following lemma, we will show that system (3.1) admits a solution provided that \( \sup_{z \in \Omega} \frac{1}{N} \|\nu_i\| \) is sufficiently small following the arguments in [16].

Lemma 3.1. Let \( 0 < r \leq \left( \frac{1}{100N+25} \right)^{\frac{1}{2}} \) be given. Then for any \( y \in \mathbb{R}^N \) satisfying \( \|y\| \leq r \), there exists \( x_0 \in \mathbb{R}^N \) such that \( \|x_0\| \leq r \) and \( \mathcal{F}(x_0) = y \).
Proof. We split its proof into two steps.

- Step A: (The Jacobian matrix $D\mathcal{F}(0)$ is a diagonal matrix): By direct calculation, we have

$$
(D_j f_i)(x) = \begin{cases} 
\frac{2}{N} \sin x_i \sin \left( \frac{3}{2} x_j - x_i \right) \cos \left( \frac{1}{2} x_j \right), & \text{if } j \neq i, \\
\frac{1}{N} \sum_{1 \leq k \leq N} \cos(x_k - 2x_i)(1 + \cos x_k) - \frac{1}{N} \sin^2 x_i, & \text{if } j = i.
\end{cases}
$$

Let $P := D\mathcal{F}(0)$ be the Jacobian matrix of $\mathcal{F}$ at $x = 0$. Then $P$ is a diagonal matrix with $2$'s on the diagonal. We choose $y \in \mathbb{R}^N$ to satisfy $\|y\| \leq r$, and set

$$
\varphi(x) := x + P^{-1}(y - \mathcal{F}(x)).
$$

Then, it is easy to see that the fixed point of $\varphi$ corresponds to the desired equilibrium $x_0$:

$$
\varphi(x_0) = x_0 \iff x_0 + P^{-1}(y - \mathcal{F}(x_0)) = x_0 \iff y = \mathcal{F}(x_0).
$$

- Step B: (Existence of fixed points for the map $\varphi$): It follows from (3.2) that

$$
D\varphi(x) = I - P^{-1}D\mathcal{F}(x) = P^{-1}(P - D\mathcal{F}(x)).
$$

Next, we are going to show that $\varphi$ is a contraction mapping from $B_r(0) := \{x \in \mathbb{R}^N : \|x\| \leq r\}$ into itself.

- Step B.1 (The map $\varphi$ is a map from $B_r(0)$ into $B_r(0)$). Since $P^{-1}$ is a diagonal matrix with $\frac{1}{2}$'s on its diagonal, we have

$$
\|P^{-1}\|_{op} = \frac{1}{2}. \tag{3.3}
$$

On the other hand, for $\|x\| \leq r$, we obtain

$$
\|D\mathcal{F}(x) - P\|_{op}^2 
\leq \sum_{i \neq j, 1 \leq i, j \leq N} \left( -\frac{2}{N} \sin x_i \sin \left( \frac{3}{2} x_j - x_i \right) \cos \left( \frac{1}{2} x_j \right) \right)^2 
+ \sum_{1 \leq i \leq N} \left( \frac{1}{N} \sum_{1 \leq k \leq N} \cos(x_k - 2x_i)(1 + \cos x_k) - \frac{1}{N} \sin^2 x_i - 2 \right)^2 
\leq \frac{4}{N^2} \sum_{i \neq j, 1 \leq i, j \leq N} \left| \frac{3}{2} x_i - x_j \right|^2 
+ \frac{1}{N^2} \sum_{1 \leq i \leq N} \left[ \sum_{1 \leq k \leq N} \left( 1 - \cos x_k + (1 - \cos(x_k - 2x_i))(1 + \cos x_k) \right) + \sin^2 x_i \right]^2 
\leq \frac{4}{N^2} \sum_{i \neq j} x_i^2 \left( \frac{3}{2} |x_j| + |x_i| \right)^2 + \frac{1}{N^2} \sum_{1 \leq i \leq N} \left[ \sum_{1 \leq k \leq N} \left( \frac{1}{2} x_k^2 + (|x_k| + 2|x_i|)^2 \right) + x_i^2 \right]^2 
\leq \frac{25(N - 1)}{N} r^4 + \frac{r^4}{N} \left( \frac{19N}{2} + 1 \right)^2 r^4 \leq (100N + 25)r^4 \leq 1. \tag{3.4}
$$

Hence, for $\|x\| \leq r$, we use (3.3) and (3.4) to obtain

$$
\|\varphi(x)\| \leq \|\varphi(x) - \varphi(0)\| + \|\varphi(0)\| 
\leq \sup_{0 < t < 1} \left\| \frac{d}{dt} \varphi(tx) \right\| + \|P^{-1}y\| 
= \sup_{0 < t < 1} \left\| D\varphi(tx)x \right\| + \|P^{-1}y\|
$$
\[ \leq \sup_{0 < t < 1} \| P^{-1} \parallel \| P - DP(tx)) \parallel \| x \parallel + \| P^{-1} \parallel \| y \parallel \leq \frac{1}{2} \cdot 1 \cdot r + \frac{1}{2} \cdot r = r. \]

Therefore, \( \varphi \) maps \( B_r(0) \) into itself.

\( \circ \) Step B.2 (The map \( \varphi \mid_{B_r(0)} \) is a contraction): Let \( x_1, x_2 \in B_r(0) \). Then, we have

\[ \| \varphi(x_1) - \varphi(x_2) \| \leq \sup_{0 < t < 1} \left\| \frac{d}{dt} \varphi(tx_1 + (1 - t)x_2) \right\| = \sup_{0 < t < 1} \left\| D \varphi(tx_1 + (1 - t)x_2)(x_1 - x_2) \right\| \]

\[ \leq \sup_{0 < t < 1} \| P^{-1} \parallel \| P - DF(tx_1 + (1 - t)x_2)) \parallel \| x_1 - x_2 \| \leq \frac{1}{2} \cdot 1 \cdot \| x_1 - x_2 \| \]

Finally, by the contraction mapping principle, \( \varphi \mid_{B_r(0)} \) has the unique fixed point \( x_0 \). Then, the relation \( \varphi(x_0) = x_0 \) clearly implies \( F(x_0) = y \).

As a direct application of Lemma 3.1, we obtain the existence of COD state for (1.2).

**Theorem 3.2.** Suppose that the random parameters \( \gamma, \mu \) and natural frequencies satisfy

\[ \sup_{z \in \Omega} \left( \frac{\gamma(z)}{\mu(z)} \| \nu \|_\infty \right) \leq \left( \frac{1}{100N + 25} \right)^{\frac{1}{2}}. \]

Then, system (3.1) admits a solution satisfying

\[ \sup_{z \in \Omega} R(\theta_i^\infty(z)) \leq \sup_{z \in \Omega} \left( \frac{\gamma(z)}{\mu(z)} \| \nu \|_\infty \right). \]

**Remark 3.** If we assume that the nature of the coupling is anti-Hebbian-like, say \( |\sin \theta| \), then the COD state may not exist, no matter how small \( \sup_{z \in \Omega} \frac{\gamma(z)}{\mu(z)} \| \nu \|_\infty \) is. For instance, consider the case:

\[ N = 2, \nu_1 = \nu_2 = \nu \neq 0. \]

For a fixed \( z \in \Omega \), suppose that the following system admits a solution \( (\theta_1^\infty, \theta_2^\infty) \):

\[ \frac{\gamma}{\mu} \nu_i = \frac{1}{N} \sum_{1 \leq j \leq N} \sin |\theta_j^\infty - \theta_i^\infty| \sin \theta_i^\infty (1 + \cos \theta_j^\infty), \quad 1 \leq i \leq N. \]

Note that the above system is equivalent to the following:

\[ 0 \neq \frac{\gamma}{\mu} \nu = \frac{1}{2} \sin |\theta_2^\infty - \theta_1^\infty| \sin \theta_1^\infty (1 + \cos \theta_2^\infty) = \frac{1}{2} \sin |\theta_2^\infty - \theta_1^\infty| \sin \theta_2^\infty (1 + \cos \theta_1^\infty). \]

We define:

\[ f(\theta) := \frac{\sin \theta}{1 + \cos \theta}. \]

Then, the relation (3.6) implies \( f(\theta_1^\infty) = f(\theta_2^\infty) \). Since \( f \) is one-to-one, we get \( \theta_1^\infty = \theta_2^\infty \). We substitute this to (3.6) to get \( \nu = 0 \), which is contradictory to (3.5).
3.2. Exponential relaxation toward the COD state. In this subsection, we present an exponential convergence of the perturbed solution toward the COD state whose existence is guaranteed by Theorem 3.2. Let \((\Theta^\infty(z), K^\infty(z))\) be an COD state of \((1.2)\):

\[
0 = \nu_i - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa^\infty_{ij} \sin \theta^\infty_i (1 + \cos \theta^\infty_j),
\]

\[
0 = -\gamma(z) \kappa^\infty_{ij} + \mu(z) \cos(\theta^\infty_j - \theta^\infty_i).
\]

For a solution \((\Theta, K)\) to \((1.2)\), we consider its fluctuations around the given COD state:

\[
\hat{\theta}_i(t, z) := \theta_i(t, z) - \theta^\infty_i(z), \quad \hat{\kappa}_{ij}(t, z) := \kappa_{ij}(t, z) - \kappa^\infty_{ij}(z).
\]

Then, the fluctuations \((\hat{\theta}_i, \hat{\kappa}_{ij})\) satisfy

\[
\partial_t \hat{\theta}_i = -\frac{1}{N} \sum_{1 \leq j \leq N} \hat{\kappa}_{ij} \sin \theta_i (1 + \cos \theta_j) - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa^\infty_{ij} (\sin \theta_i - \sin \theta^\infty_j)(1 + \cos \theta_j)
\]

\[
- \frac{1}{N} \sum_{1 \leq j \leq N} \kappa^\infty_{ij} \sin \theta^\infty_i (\cos \theta_j - \cos \theta^\infty_j),
\]

\[
\partial_t \hat{\kappa}_{ij} = -\gamma(z) \hat{\kappa}_{ij} + \mu(z) \left( \cos(\theta_j - \theta_i) - \cos(\theta^\infty_j - \theta^\infty_i) \right).
\]

(3.7)

**Theorem 3.3.** Suppose that for \(z \in \Omega\), phase bound \(\alpha\), random system parameters \(\gamma(z), \mu(z), ||\nu||_\infty\) and the initial data satisfy the following conditions:

(i) \(\alpha \in (0, \frac{\pi}{6})\), \(\cos 2\alpha \cos \alpha(1 + \cos \alpha) \geq \frac{\gamma(z)}{\mu(z)} \max \left\{ \kappa^0_M(z), \frac{\mu(z)}{\gamma(z)} \right\} \sin^2 \alpha + 2 \sin 2\alpha,\)

(ii) \(\kappa^0_m(z) \geq \frac{\mu(z)}{\gamma(z)} \cos 2\alpha, \quad R(\theta^0_i(z)) < \alpha,\)

(iii) \(||\nu||_\infty < \frac{\mu(z)}{\gamma(z)} \cos 2\alpha(1 + \cos \alpha) \sin \alpha, \quad \gamma(z) \frac{\mu(z)}{\gamma(z)} ||\nu||_\infty \leq \left( \frac{1}{100N^2} \right) < \alpha.\)

Then, for \(z \in \Omega\) and \(t \in (0, \infty)\), we have

\[
\sum_{1 \leq i \leq N} |\hat{\theta}_i(t, z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}(t, z)| \leq e^{-C(z)t} \left( \sum_{1 \leq i \leq N} |\hat{\theta}^0_i(z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\kappa^0_{ij}(z)| \right),
\]

where the random variables \(C(z) := \min\{A(z), B(z)\}\) and

\[
A(z) := \frac{\mu(z)}{\gamma(z)} \left( \cos 2\alpha \cos \alpha(1 + \cos \alpha) - \frac{\gamma(z)}{\mu(z)} \max \left\{ \kappa^0_M(z), \frac{\mu(z)}{\gamma(z)} \right\} \sin^2 \alpha - 2 \sin 2\alpha \right),
\]

\[
B(z) := \gamma(z)(1 - 2 \sin \alpha).
\]

**Proof.** In (3.7), we multiply \(\text{sgn}(\hat{\theta}_i)\) and \(\text{sgn}(\hat{\kappa}_{ij})\) to get

\[
\partial_t |\hat{\theta}_i| = -\frac{1}{N} \sum_{1 \leq j \leq N} \hat{\kappa}_{ij} \sin \theta_i (1 + \cos \theta_j) \text{sgn}(\hat{\theta}_i) - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa^\infty_{ij} |\hat{\theta}_i| \cos \theta_i (1 + \cos \theta_j)
\]

\[
+ \frac{1}{N} \sum_{1 \leq j \leq N} \kappa^\infty_{ij} \sin \theta^\infty_i \sin \theta_i \hat{\theta}_j \text{sgn}(\hat{\theta}_i),
\]

(3.8)

\[
\partial_t |\hat{\kappa}_{ij}| = -\gamma |\hat{\kappa}_{ij}| - \mu \sin \hat{\theta}_j |\hat{\theta}_j| - \hat{\theta}_i |\text{sgn}(\hat{\kappa}_{ij})|,
\]

(3.9)
where the intermediate values \( \hat{\theta}_i \in (-\alpha, \alpha) \) and \( \hat{\theta}_{ij} \in (-2\alpha, 2\alpha) \) follow from the mean value theorem. Next, we sum (3.8) over \( i \) and \( j \) to obtain

\[
\frac{\partial}{\partial t} \sum_{1 \leq i \leq N} |\hat{\theta}_i| \leq \frac{2}{N} \sin \alpha \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| - \frac{\mu}{\gamma} \cos 2\alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\hat{\theta}_i| \\
+ \max \left\{ \frac{\mu}{\gamma}, \kappa_0 \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} |\hat{\theta}_j|,
\]

\[(3.10)\]

\[
\frac{\partial}{\partial t} \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| \leq -\gamma \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| + \mu \sin 2\alpha \sum_{1 \leq i, j \leq N} |\hat{\theta}_j - \hat{\theta}_i| \\
\leq -\gamma \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| + 2\mu N \sin 2\alpha \sum_{1 \leq i \leq N} |\hat{\theta}_i|.
\]

\[(3.11)\]

The relation (3.10) + \( \frac{1}{\gamma N} (3.11) \) yields the following Grönwall’s inequality:

\[
\frac{\partial}{\partial t} \left( \sum_{1 \leq i \leq N} |\hat{\theta}_i| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| \right) \leq - \left( A(z) \sum_{1 \leq i \leq N} |\hat{\theta}_i| + \frac{B(z)}{\gamma N} \sum_{1 \leq i, j \leq N} |\hat{\kappa}_{ij}| \right),
\]

where the condition (i) for \( \alpha \) guarantees the positivity of \( A(z) \) and \( B(z) \). This yields the desired estimate.

\( \square \)

4. Uniform bound for \( z \)-variations. In this section, we present a uniform bound for \( z \)-variations. First, we will discuss a framework and then study lower-order estimate and higher-order estimates, respectively.

4.1. A framework for a uniform bound. In this subsection, we present a framework for a uniform bound of \( z \)-variations. For \( \ell \geq 1 \) and given solution \((\Theta, K)\) to (1.2), we introduce \( \ell \)-equivalent functional \( \mathcal{L} \):

\[
\mathcal{L}(\partial^\ell_{\xi} \Theta, \partial^\ell_{\xi} K) := \sum_{1 \leq i \leq N} |\partial^\ell_{\xi} \theta_i(t, z)| + \frac{1}{\gamma N} \sum_{1 \leq i, j \leq N} |\partial^\ell_{\xi} \kappa_{ij}(t, z)|.
\]

Next, we state our framework (A) as follows.

- (A1): The phase bound \( \alpha \), the random parameters \( \gamma, \mu \) and natural frequencies satisfy the following relations: For \( z \in \Omega \),

\[
\alpha \in (0, \frac{\pi}{6}), \quad \cos 2\alpha \cos \alpha (1 + \cos \alpha) > \frac{\gamma(z)}{\mu(z)} \max \left\{ \kappa_0, \frac{\mu(z)}{\gamma(z)} \right\} \sin^2 \alpha + 2 \sin 2\alpha,
\]

\[
||\nu||_\infty < \frac{\mu(z)}{\gamma(z)} \cos 2\alpha (1 + \cos \alpha) \sin \alpha, \quad \frac{\gamma(z)}{\mu(z)} ||\nu||_\infty \leq \left( \frac{1}{100N + 25} \right)^{\frac{1}{2}} < \alpha,
\]

\[
||\gamma||_\infty, ||\mu||_\infty < \infty.
\]

- (A2): Initial data \((\Theta^0, K^0)\) satisfy

\[
\kappa_m^0(z) \geq \frac{\mu(z)}{\gamma(z)} \cos 2\alpha, \quad R(\theta_i^0(z)) < \alpha, \quad \text{for} \ z \in \Omega.
\]

In the sequel, as long as there is no confusion, we suppress \( z \)-dependence in \( \gamma, \mu, \theta_i \) and \( \kappa_{ij} \):

\[
\gamma(z) = \gamma, \quad \mu(z) = \mu, \quad \theta_i(t) = \theta_i(t, z), \quad \kappa_{ij}(t) = \kappa_{ij}(t, z), \quad 1 \leq i, j \leq N.
\]
Before we begin our uniform bound estimate, we introduce some random functions:

\[ A(z) := \frac{\mu}{\gamma} \left( \cos 2\alpha \cos \alpha (1 + \cos \alpha) - \frac{\gamma}{\mu} \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} \sin^2 \alpha - 2 \sin 2\alpha \right), \]

\[ B(z) := \gamma (1 - 2 \sin \alpha), \quad C(z) := \min\{A, B\}, \quad C_1(z) := \left( \frac{|\gamma'|}{\gamma} \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} + |\mu'| \right) \frac{N}{C}, \]

where \( |\gamma'| \) and \( |\mu'| \) denote differentiation with respect to \( z \).

Now, we are ready to state our third main results as follows.

**Theorem 4.1.** Suppose that the framework \((A1)-(A2)\) holds. Then, for \( z \in \Omega \) and \( \ell \geq 1 \), there exists random functions \( C_\ell(z) \) such that

\[ \mathcal{L}(\partial^j_t \Theta(t, z), \partial^{j'}_t K(t, z)) \leq \mathcal{L}(\partial^j_t \Theta^0(z), \partial^{j'}_t K^0(z)) e^{-C_\ell} + C_\ell(z) (1 - e^{-C_\ell}), \quad t \geq 0, \quad (4.2) \]

where the coefficients \( C_\ell \) with \( \ell \geq 2 \) are defined inductively by the following relation:

\[ C_\ell(z) := \frac{\widetilde{C}_{\ell-1}(z)}{C(z)} \]

and

\[ \widetilde{C}_{\ell-1}(z) := 2\gamma N^2 \times \sum_{r_1, r_2, r_3 \neq \ell, r_1 + r_2 + r_3 = \ell} \frac{\ell! C_{r_1}(z)}{r_1! k_1! \cdots k_1'! k_2'! \cdots k_3'!} \prod_{p=1}^{r_2} \left( \frac{C_p(z)}{p!} \right)^{k_p} \prod_{p=1}^{r_3} \left( \frac{2C_p(z)}{p!} \right)^{k_p} \]

\[ + 3N \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} \sum_{(k_1, \ldots, k_3) \in A(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \prod_{p=1}^{\ell-1} \left( \frac{C_p(z)}{p!} \right)^{k_p} \]

\[ + \|\gamma\| N^2 \sum_{0 \leq r \leq \ell - 1} \frac{\ell!}{r!(\ell - r)!} \overline{C}_r(z) \]

\[ + \frac{\mu N}{\gamma} \sum_{(k_1, \ldots, k_\ell) \in A(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \prod_{p=1}^{\ell-1} \left( \frac{2C_p(z)}{p!} \right)^{k_p} \]

\[ + \|\mu\| N \sum_{0 \leq r \leq \ell - 1} \frac{\ell!}{(\ell - r)! k_1! \cdots k_r!} \prod_{p=1}^{r} \left( \frac{2C_p(z)}{p!} \right)^{k_p}, \]

\[ \overline{C}_r(z) := \max \left\{ \mathcal{L}(\partial^j_t \Theta^0(z), \partial^{j'}_t K^0(z), C_p(z) \right\}. \]

**Proof.** We present a proof of Theorem 4.1 for \( \ell = 1 \) and \( \ell \geq 2 \) in the following two subsections. Before we close this proof, we briefly comment on how to get the desired estimate \((4.2)\). For this, we need to estimate the terms

\[ |\partial^j_t \theta_i(t, z)|, \quad |\partial^j_k \kappa_{ij}(t, z)|, \]

which are at most Lipschitz differentiable, not differentiable. For the estimate of \( |\partial^j_t \theta_i(t, z)| \), we multiply the differentiable equation for \( \partial^j_t \theta_i(t, z) \) by \( \text{sgn}(\partial^j_t \theta_i(t, z)) \), and then integration by parts to derive the differentiable equation for \( |\partial^j_t \theta_i(t, z)| \). However, this is not a rigorous argument, because one need to differentiate discontinuous function \( \text{sgn}(\cdot) \). To remedy this difficulty, we first approximate the sign function \( \text{sgn}(\cdot) \) by a sequence of regularized functions which are parametrized by a
small positive parameter \( \varepsilon \). Although the detailed argument for this kind of regularization can be found in [7], we present a simple case here for reader’s convenience. In what follows, we only consider the case of \( \ell = 1 \), and the other cases can be treated in the same way. Of course, the following arguments are also applicable in the case of \( \partial_i |\partial_i^2 \partial_i|, \partial_j |\partial_j^2 \partial_i| \) in Section 5 as well.

Now, we consider a regularization of \( \text{sgn}(x) \) via a sequence of smooth functions \( \eta_t'(x) \) defined by

\[
\eta_t'(x) = \begin{cases} 
1, & x \geq \varepsilon, \\
-1, & x \leq -\varepsilon, \\
-\eta_t' < 1, & |x| < \varepsilon,
\end{cases}
\]

satisfying the following properties:

\[
\eta_t(0) = 0, \quad |\eta_t'(x) x - \eta_t(x)| \leq \mathcal{O}(\varepsilon), \quad ||x| - |\eta_t(x)|| \leq \mathcal{O}(\varepsilon). \tag{4.3}
\]

Note that \( \eta_t(x) \) is a regularization of \( |x| \). By multiplying the equation for \( \varepsilon \)-variation \( \partial_t \theta_i(t, z) \) by \( \eta_t'(\partial_t \theta_i) \), we have

\[
\partial_t \eta_t(\partial_t \theta_i) = -\frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \cos \theta_i(1 + \cos \theta_j) \eta_t'(\partial_t \theta_i) \partial_t \theta_i \\
- \frac{\eta_t'(\partial_t \theta_i)}{N} \sum_{1 \leq j \leq N} (\partial_t \kappa_{ij} \sin \theta_i(1 + \cos \theta_j) - \kappa_{ij} \sin \theta_i \sin \theta_j \partial_t \theta_j)
\]

\[
:= \mathcal{I}_1 + \mathcal{I}_2. \tag{4.4}
\]

Below, we estimate the terms \( \mathcal{I}_i, \ i = 1, 2 \) one by one.

- (Estimate of \( \mathcal{I}_1 \)): For this, we use (4.3)\(_2\) to obtain

\[
\mathcal{I}_1 = -\frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \cos \theta_i(1 + \cos \theta_j) \eta_t(\partial_t \theta_j) \\
- \frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \cos \theta_i(1 + \cos \theta_j)(\eta_t'(\partial_t \theta_i) \partial_t \theta_i - \eta_t(\partial_t \theta_i)) \leq -\frac{\mu}{\gamma} \cos 2\alpha \cos(1 + \cos \alpha) \eta_t(\partial_t \theta_i) + \mathcal{O}(\varepsilon). \tag{4.5}
\]

- (Estimate of \( \mathcal{I}_2 \)): Again, we use (4.3)\(_3\) to treat the bad term \( \mathcal{I}_2 \) as follows.

\[
\mathcal{I}_2 \leq \frac{2 \sin \alpha}{N} \sum_{1 \leq j \leq N} \eta_t(\partial_z \kappa_{ij}) + \frac{1}{N} \max \left\{ \frac{\alpha}{\kappa M}, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} \eta_t(\partial_t \theta_j) + \mathcal{O}(\varepsilon). \tag{4.6}
\]

Similarly, we multiply the equation for \( \partial_z \kappa_{ij}(t, z) \) by \( \eta_t'(\partial_z \kappa_{ij}) \) and use (4.3) to get

\[
\partial_t \eta_t(\partial_z \kappa_{ij}) \leq -\gamma \eta_t(\partial_z \kappa_{ij}) + \mu \sin 2\alpha (\eta_t(\partial_t \theta_j) + \eta_t(\partial_t \theta_i)) + |\gamma| \max \left\{ \frac{\alpha}{\kappa M}, \frac{\mu}{\gamma} \right\} + |\mu'| + \mathcal{O}(\varepsilon) N. \tag{4.7}
\]

Next, we combine all the estimates (4.4), (4.5), (4.6) and (4.7) to obtain

\[
\partial_t \sum_{1 \leq i \leq N} \eta_t(\partial_t \theta_i) \\
\leq \frac{2}{N} \sin \alpha \sum_{1 \leq i, j \leq N} \eta_t'(\partial_z \kappa_{ij}) - \frac{\mu}{\gamma} \cos 2\alpha \cos(1 + \cos \alpha) \sum_{1 \leq i \leq N} \eta_t'(\partial_t \theta_i) \\
+ \max \left\{ \frac{\alpha}{\kappa M}, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} \eta_t'(\partial_t \theta_j) + \mathcal{O}(\varepsilon) N, \tag{4.8}
\]

\[
+ \max \left\{ \frac{\alpha}{\kappa M}, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} \eta_t'(\partial_t \theta_j) + \mathcal{O}(\varepsilon) N, \tag{4.8}
\]

R. W. B. and M. T. H.
Finally, we let stay on the track of formal argument for easy presentation without bothering with the following two subsections using a formal method, of course, the aforementioned 4.2.

These relations yield
\[ \left| \gamma \right| \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} + \left| \mu' \right| + \mathcal{O}(\varepsilon) \right) N^2. \] (4.9)

Then, the linear combination \( (4.8) + \frac{1}{\gamma N} (4.9) \) and Grönwall’s lemma yield
\[ \sum_{1 \leq i,j \leq N} \eta^r(\partial_i^r \kappa_{ij} (t, z)) + \frac{1}{\gamma N} \sum_{1 \leq i,j \leq N} \eta^r(\partial_i^r \kappa_{ij} (t, z)) \leq \sum_{1 \leq i,j \leq N} \eta^r(\partial_i^r \kappa_{ij}^0 (z)) + \frac{1}{\gamma N} \sum_{1 \leq i,j \leq N} \eta^r(\partial_i^r \kappa_{ij}^0 (z)) \varepsilon - C(z)t + (C_1(z) + \mathcal{O}(\varepsilon))(1 - e^{-C(z)t}). \]

Finally, we let \( \varepsilon \to 0 \) to yield the desired estimate. The rest of proof will be given in the following two subsections using a formal method, of course, the aforementioned rigorous regularization arguments can be employed to be more precise, but we stay on the track of formal argument for easy presentation without bothering with cumbersome calculations.

4.2. Lower-order estimate. In this subsection, we will provide the estimate (4.2) for \( \ell = 1 \), i.e., for \( z \in \Omega \),
\[ \mathcal{L}(\partial_i \Theta(t, z), \partial_i K(t, z)) \leq \mathcal{L}(\partial_i^0 \Theta^0(z), \partial_i K^0(z)) e^{-Ct} + C_1(1 - e^{-Ct}), \quad \text{for all} \quad t \geq 0. \] (4.10)

Derivation of (4.10): Note that \( z \)-variations \( \partial_i \partial_i \kappa_{ij} (t, z), \partial_i \kappa_{ij} (t, z) \) satisfy
\[
\partial_t \partial_i \partial_i \kappa_{ij} = -\gamma \partial_i \kappa_{ij} - \gamma' \kappa_{ij} + \mu \sin(\theta_j - \theta_i)(\partial_i \theta_j - \partial_j \theta_i) + \mu' \cos(\theta_j - \theta_i).
\]

These relations yield
\[
\partial_t \sum_{1 \leq i \leq N} |\partial_i \partial_i \kappa_{ij}| \leq \frac{2}{\gamma} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_i \kappa_{ij}| \mu \cos 2\alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_i \theta_i| \\
+ \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq i \leq N} |\partial_i \theta_i|, \quad (4.11)
\]
\[
\partial_t \sum_{1 \leq i,j \leq N} |\partial_i \kappa_{ij}| \leq -\gamma \sum_{1 \leq i,j \leq N} |\partial_i \kappa_{ij}| + 2N \mu \sin 2\alpha \sum_{1 \leq i \leq N} |\partial_i \theta_i| \\
+ \left( |\gamma| \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} + |\mu'| \right) N^2. \quad (4.12)
\]

Finally, the combination (4.11) + \( \frac{1}{\gamma N} (4.12) \) implies
\[
\partial_t \left( \sum_{1 \leq i \leq N} |\partial_i \theta_i| + \frac{1}{\gamma N} \sum_{1 \leq i,j \leq N} |\partial_i \kappa_{ij}| \right)
\]
where the condition (A1) guarantees the positivity of $A$ and $B$. By applying Grönwall’s lemma to the above inequality, we have the desired result (4.10).

4.3. Higher-order estimates. In this subsection, we present uniform bound estimates for $\{ (\partial^{\ell}_{z} \Theta, \partial^\ell_{z} K) \}_{\ell \geq 2}$.

First, note that $z$-variation $(\partial^\ell_{z} \Theta, \partial^\ell_{z} K)$ satisfy

\[ \partial_{t}(\partial^\ell_{z} \Theta) = -\frac{1}{N} \sum_{1 \leq \ell \leq N} \frac{\ell!}{r_{1}! r_{2}! r_{3}!} \partial^{\ell}_{z} \kappa_{i j} \partial^{\ell}_{z} \gamma \cos(\Theta_{i} - \Theta_{j}) + \left( \partial^{\ell}_{z} \kappa_{i j} \right) + \left( \partial^{\ell}_{z} \kappa_{i j} \right) N,\]

where the condition (4.3) guarantees the positivity of $A$ and $B$. By applying Grönwall’s lemma to the above inequality, we have the desired result (4.10).

Next, we claim: for $\ell \geq 1$,

\[ L(\partial^\ell_{z} \Theta(t, z), \partial^\ell_{z} K(t, z)) \leq L(\partial^\ell_{z} \Theta_{0}, \partial^\ell_{z} K_{0}) e^{-Ct} + C_{t}(1 - e^{-Ct}), \quad t \geq 0. \tag{4.14} \]

**Derivation of (4.14):** We use induction argument on $\ell$ to derive the estimate (4.14). Let $m$ be a positive integer.

- Initial step ($\ell = 1$): This case is exactly done in Section 4.2.
- Inductive step ($\ell \geq 2$): Suppose that the estimate (4.14) holds for $1 \leq m \leq \ell - 1$:

\[ L(\partial^{m}_{z} \Theta(t, z), \partial^{m}_{z} K(t, z)) \leq L(\partial^{m}_{z} \Theta_{0}, \partial^{m}_{z} K_{0}) e^{-Ct} + C_{t}(1 - e^{-Ct}), \quad t \geq 0. \tag{4.15} \]

Then, we claim that the estimate (4.14) holds for $m = \ell$:

\[ L(\partial^{\ell}_{z} \Theta(t, z), \partial^{\ell}_{z} K(t, z)) \leq L(\partial^{\ell}_{z} \Theta_{0}, \partial^{\ell}_{z} K_{0}) e^{-Ct} + C_{t}(1 - e^{-Ct}), \quad t \geq 0. \]

By induction hypothesis (4.15), for $1 \leq m \leq \ell - 1$ and $z \in \Omega$,

\[ L(\partial^{m}_{z} \Theta(t, z), \partial^{m}_{z} K(t, z)) \leq \max \left\{ L(\partial^{m}_{z} \Theta_{0}(z), \partial^{m}_{z} K_{0}(z)), C_{m}(z) \right\} =: \overline{C_{m}(z)}. \]

Next, we set $S^{(k)}(\cdot), I^{(k)}(\cdot)$ and $\cos^{(k)}(\cdot)$:

\[ S^{(k)}(\cdot) := \frac{d^{k}}{dx^{k}} \sin x \bigg|_{x=\theta}, \quad I^{(k)}(\cdot) := \frac{d^{k}}{dx^{k}} \bigg( 1 + \cos x \bigg) \bigg|_{x=\theta}, \]

\[ \cos^{(k)}(\cdot) := \frac{d^{k}}{dx^{k}} \cos x \bigg|_{x=\theta}. \]

For simplicity, we introduce some handy notation: for $r \in \mathbb{N}$ and $(k_{1}, \ldots, k_{r}) \in (\mathbb{N} \cup \{0\})^{r}$ with $k := k_{1} + \cdots + k_{r}$, we define

\[ S(r, k_{1}, \ldots, k_{r}, \theta_{i}) := \frac{r!}{k_{1}! \cdots k_{r}!} S^{(k)}(\theta_{i}) \prod_{p=1}^{r} \left( \frac{\partial^{p} \theta_{i}}{p!} \right)^{k_{p}}, \]

\[ I(r, k_{1}, \ldots, k_{r}, \theta_{j}) := \frac{r!}{k_{1}! \cdots k_{r}!} I^{(k)}(\theta_{j}) \prod_{p=1}^{r} \left( \frac{\partial^{p} \theta_{j}}{p!} \right)^{k_{p}}, \]

\[ C(r, k_{1}, \ldots, k_{r}, \theta_{j} - \theta_{i}) := \frac{r!}{k_{1}! \cdots k_{r}!} \cos^{(k)}(\theta_{j} - \theta_{i}) \prod_{p=1}^{r} \left( \frac{\partial^{p} (\theta_{j} - \theta_{i})}{p!} \right)^{k_{p}}, \]
We multiply the above equations by \( \text{sgn} \) and \( \Sigma \). As long as there is no confusion for \( \{k_1, \cdots, k_r\} \) when we use the above notation in \( \Sigma \), we omit it as \( \{k_1, \cdots, k_r\} \).

Next, we use the chain rule to rewrite (4.13) as follows:

\[
\frac{\partial_i}{\partial \ell} (\partial^\ell \dot{\theta}_i) = - \frac{1}{N} \sum_{1 \leq j \leq N} \left[ \theta_i (1 + \cos \theta_j) + \kappa_{ij} \right] \frac{\partial \theta_i}{\partial \ell} \theta_i (1 + \cos \theta_j) - \kappa_{ij} \sin \theta_i \sin \theta_j \frac{\partial^\ell \dot{\theta}_j}{\partial \ell}
\]

and

\[
\frac{\partial_i}{\partial \ell} (\partial^\ell \kappa_{ij}) = - \gamma \frac{\partial^\ell \kappa_{ij}}{\partial \ell} - \mu \sin(\theta_j - \theta_i) \left( \frac{\partial^\ell \dot{\theta}_j}{\partial \ell} - \frac{\partial^\ell \dot{\theta}_i}{\partial \ell} \right) - \sum_{0 \leq r \leq \ell - 1} \frac{\ell!}{r!(\ell - r)!} \frac{\partial^{\ell-r} \gamma \frac{\partial^\ell \kappa_{ij}}{\partial \ell}}{\partial \ell}
\]

We multiply the above equations by \( \text{sgn} (\partial^\ell \dot{\theta}_i) \) and \( \text{sgn} (\partial^\ell \kappa_{ij}) \), respectively, to have

\[
\frac{\partial_i}{\partial \ell} | \partial^\ell \dot{\theta}_i | = - \frac{1}{N} \sum_{1 \leq j \leq N} \left[ \text{sgn} (\partial^\ell \dot{\theta}_i) \frac{\partial^\ell \kappa_{ij}}{\partial \ell} \sin \theta_i (1 + \cos \theta_j) + \kappa_{ij} \cos \theta_i | \partial^\ell \dot{\theta}_i | (1 + \cos \theta_j) \right]
\]

and

\[
\frac{\partial_i}{\partial \ell} | \partial^\ell \kappa_{ij} | = \frac{\text{sgn}(\partial^\ell \dot{\theta}_i)}{N} \sum_{1 \leq j \leq N} \left[ \frac{\ell!}{r!(\ell - r)!} \frac{\partial^{\ell-r} \gamma \frac{\partial^\ell \kappa_{ij}}{\partial \ell}}{\partial \ell} \sin \theta_i \sin \theta_j \frac{\partial^\ell \dot{\theta}_j}{\partial \ell} \right]
\]
× \sum_{1 \leq j \leq N \atop (k_1, \cdots, k_j) \in \Lambda(\ell)'} \kappa_{ij} \left[ S(0, \theta_i) I(\ell, k_1, \cdots, k_\ell, \theta_j) + S(\ell, k_1, \cdots, k_\ell, \theta_j) I(0, \theta_j) \right]

and

\partial_\ell |\partial_\ell^t \kappa_{ij}| \quad (4.17)

\begin{align*}
&= -\gamma |\partial_\ell^t \kappa_{ij}| - \mu \text{sgn}(\partial_\ell^t \kappa_{ij}) \sin(\theta_j - \theta_i)(\partial_\ell^t \theta_j - \partial_\ell^t \theta_i) - \sum_{0 \leq r \leq \ell - 1} \frac{\text{sgn}(\partial_\ell^t \kappa_{ij}) \ell!}{r!(\ell - r)!} \partial_\ell^{\ell - r} \gamma \partial_\ell^r \kappa_{ij} \\
&\quad + \mu \text{sgn}(\partial_\ell^t \kappa_{ij}) \sum_{(k_1, \cdots, k_j) \in \Lambda(\ell)'} C(\ell, k_1, \cdots, k_\ell, \theta_j - \theta_i) \\
&\quad + \sum_{0 \leq r \leq \ell - 1} \frac{\text{sgn}(\partial_\ell^t \kappa_{ij}) \ell!}{r!(\ell - r)!} \partial_\ell^{\ell - r} \mu C(r, k_1, \cdots, k_r, \theta_j - \theta_i).
\end{align*}

○ Case A. Estimate for \( \partial_\ell (\partial_\ell^t \theta_i) \): In this case, we use (4.16) to get

\begin{align*}
\frac{1}{N} &\sum_{1 \leq i \leq N} |\partial_\ell^t \theta_i| \\
&\leq \frac{1}{N} \sum_{1 \leq i,j \leq N} \left[ |\partial_\ell^t \kappa_{ij}| \sin |\theta_i|(1 + \cos |\theta_j|) - \kappa_{ij} \cos |\theta_i| |\partial_\ell^t \theta_i|(1 + \cos |\theta_j|) \right] \\
&\quad + \kappa_{ij} \sin |\theta_i| |\theta_j| |\partial_\ell^t \theta_j| \\
&\quad + \frac{1}{N} \sum_{1 \leq i,j \leq N} \sum_{r_1, r_2, r_3 \neq \ell \atop r_1 + r_2 + r_3 = \ell} \frac{\ell!}{r_1!r_2!r_3!} |\partial_\ell^{\ell - 1} \kappa_{ij}| |S(r_2, k_1, \cdots, k_{r_2}, \theta_i)| |I(r_3, k_1', \cdots, k_{r_3}', \theta_j)| \\
&\quad + \frac{1}{N} \sum_{1 \leq i,j \leq N} \kappa_{ij} \left[ S(0, |\theta_i|)|I(\ell, k_1, \cdots, k_\ell, \theta_j)| + |S(\ell, k_1, \cdots, k_\ell, \theta_j)| |I(0, \theta_j)| \right]
\end{align*}

\begin{align*}
&\quad := \sum_{1 \leq k \leq 3} \mathcal{H}_{1k}.
\end{align*}

• (Estimate of \( \mathcal{H}_{11} \)): By the same calculation in Section 4.2, we deduce

\begin{align*}
\mathcal{H}_{11} &\leq \frac{2}{N} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_\ell^t \kappa_{ij}| - \frac{\mu}{\gamma} \cos 2 \alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_\ell^t \theta_i| \\
&\quad + \max \left\{ \kappa_0^0, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq i \leq N} |\partial_\ell^t \theta_i|.
\end{align*}

• (Estimate of \( \mathcal{H}_{12} \) and \( \mathcal{H}_{13} \)): For \( 0 \leq r_2, r_3 \leq \ell - 1 \), we use induction hypothesis to obtain

\begin{align*}
|S(r_2, k_1, \cdots, k_{r_2}, \theta_i)| &\leq \frac{r_2!}{k_1! \cdots k_{r_2}!} \prod_{p=1}^{r_2} \left( \frac{C_p(z)}{p!} \right)^{k_p}, \\
|I(r_3, k_1', \cdots, k_{r_3}', \theta_j)| &\leq \frac{2r_3!}{k_1'! \cdots k_{r_3}'!} \prod_{p=1}^{r_3} \left( \frac{C_p(z)}{p!} \right)^{k_p'}.
\end{align*}
for each $z \in \Omega$, with the convention
$$\prod_{p=1}^{0} \left( \frac{C_p(z)}{p!} \right)^{k_p} = 1.$$ 

Thus, we have
$$\mathcal{H}_{12} \leq 2\gamma N^2 \sum_{r_1, r_2, r_3 \neq \ell} \left( \frac{\ell!}{r_1!k_1! \cdots r_2!k_2'! \cdots r_3!k_3'} \prod_{p=1}^{r_1} \left( \frac{C_p(z)}{p!} \right)^{k_p} \prod_{p=1}^{r_2} \left( \frac{C_p(z)}{p!} \right)^{k_p'} \prod_{p=1}^{r_3} \left( \frac{C_p(z)}{p!} \right)^{k_p''} \right).$$

Thus, we have
$$\mathcal{H}_{13} \leq 3N \max \left\{ \kappa_{M}^0, \mu, \frac{\mu}{\gamma} \right\} \sum_{(k_1, \ldots, k_i) \in \Lambda(\ell)} \left( \ell! \prod_{p=1}^{\ell-1} \left( \frac{C_p(z)}{p!} \right)^{k_p} \right).$$

\(\diamond\) Case B. Estimate for $\partial_t (\partial_z^{\ell} \kappa_{ij})$: By direct estimate using (4.17), one has
$$\partial_t \sum_{1 \leq i, j \leq N} |\partial_z^t \kappa_{ij}|$$
$$\leq -\gamma \sum_{1 \leq i, j \leq N} |\partial_z^t \kappa_{ij}| + \mu \sum_{1 \leq i, j \leq N} \sin |\theta_j - \theta_i||\partial_z^t \theta_j - \partial_z^t \theta_i|$$
$$+ \sum_{1 \leq i, j \leq N} \frac{\ell!}{r!(\ell - r)!} |\partial_z^{t-r} \gamma||\partial_z^t \kappa_{ij}| + \mu \sum_{1 \leq i, j \leq N} |\mathcal{C}(\ell, k_1, \ldots, k_{r}, \theta_j - \theta_i)|$$
$$+ \sum_{1 \leq i, j \leq N} \frac{\ell!}{r!(\ell - r)!} |\partial_z^{t-r} \mu||\mathcal{C}(\ell, k_1, \ldots, k_{r}, \theta_j - \theta_i)|$$
$$\leq -\gamma \sum_{1 \leq i, j \leq N} |\partial_z^t \kappa_{ij}| + 2N \mu \sin 2\alpha \sum_{1 \leq i \leq N} |\partial_z^t \theta_i|$$
$$+ \sum_{1 \leq i, j \leq N} \frac{\ell!}{r!(\ell - r)!} |\partial_z^{t-r} \gamma||\partial_z^t \kappa_{ij}| + \mu \sum_{1 \leq i, j \leq N} |\mathcal{C}(\ell, k_1, \ldots, k_{r}, \theta_j - \theta_i)|$$
$$+ \sum_{1 \leq i, j \leq N} \frac{\ell!}{r!(\ell - r)!} |\partial_z^{t-r} \mu||\mathcal{C}(\ell, k_1, \ldots, k_{r}, \theta_j - \theta_i)|.$$
Now, we combine all the estimates in Case A and Case B to obtain
\[ \partial_t \mathcal{L}(\partial^\ell_z \Theta(t, z), \partial^\ell_z K(t, z)) \leq -C \mathcal{L}(\partial^\ell_z \Theta(t, z), \partial^\ell_z K(t, z)) + \widetilde{C}_{\ell-1}(z), \]
where the term \( \widetilde{C}_{\ell-1} \) can be estimated as follows.

\[
\begin{align*}
\widetilde{C}_{\ell-1}(z) := &\ 2N^2 \sum_{r_1, r_2, r_3 \neq \ell, r_1 + r_2 + r_3 = \ell} \frac{\ell!}{r_1 k_1! \cdots r_2 k_2! \cdots r_3 k_3!} \prod_{p=1}^{r_2} \left( \frac{C_p(z)}{p!} \right)^{k_p} \prod_{p=1}^{r_2} \left( \frac{C_p(z)}{p!} \right)^{k_p'} \\
&+ 3N \max \left\{ \kappa_M^0, \frac{\mu}{\gamma} \right\} \sum_{(k_1, \ldots, k_\ell) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \prod_{p=1}^{\ell-1} \left( \frac{C_p(z)}{p!} \right)^{k_p} \\
&+ \frac{\gamma}{N} \sum_{0 \leq r \leq \ell-1} \frac{\ell!}{r!(\ell-r)!} C_r(z) + \frac{\mu N}{\gamma} \sum_{(k_1, \ldots, k_\ell) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \prod_{p=1}^{\ell-1} \left( \frac{2C_p(z)}{p!} \right)^{k_p} \\
&+ \frac{\ell!}{r!(\ell-r)!} k_1! \cdots k_r! \prod_{p=1}^{r} \left( \frac{2C_p(z)}{p!} \right)^{k_p}.
\end{align*}
\]

Finally, we use Grönwall’s lemma to obtain
\[
\mathcal{L}(\partial^\ell_z \Theta(t, z), \partial^\ell_z K(t, z)) \leq \mathcal{L}(\partial^\ell_z \Theta(z), \partial^\ell_z K(z)) e^{-Ct} + \frac{\widetilde{C}_{\ell-1}(z)}{C(z)} (1 - e^{-Ct}).
\]

5. **Exponential stability of \( z \)-variations in \( \ell_1 \)-topology.** In this section, we provide a uniform \( \ell_1 \)-stability for the \( z \)-variations of fluctuations \( \tilde{\theta}_i \) and \( \tilde{\kappa}_{ij} \) which extend the exponential stability result in Theorem 3.3 for fluctuations. We will show that the \( z \)-variations of fluctuations are exponentially stable in \( \ell_1 \)-topology.

5.1. **A framework for exponential \( \ell_1 \)-stability.** In this subsection, we briefly describe a framework for the exponential stability of fluctuations. Let \((\Theta, K)\) and \((\Theta^\infty, K^\infty)\) be a solution to (1.2) and a unique complete oscillator death state whose existence is guaranteed by Theorem 3.2, respectively. Then, we have

\[
\begin{align*}
\partial_t \theta_i &= \nu_i - \frac{1}{N} \sum_{1 \leq j \leq N} \kappa_{ij} \sin \theta_i (1 + \cos \theta_j), \quad t > 0, \ z \in \Omega, \\
\partial_t \kappa_{ij} &= -\gamma(z) \kappa_{ij} + \mu(z) \cos(\theta_j - \theta_i),
\end{align*}
\]
Recall the fluctuations:
\[
\hat{\theta}_i := \theta_i - \theta^\infty_i, \quad \hat{\kappa}_{ij} := \kappa_{ij} - \kappa^\infty_{ij}, \quad 1 \leq i, j \leq N,
\]
and we set
\[
\hat{\Theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_N), \quad \hat{\kappa} := (\hat{\kappa}_{ij}),
\]
\[
\rho(z) := \max_{0 \leq i, j \leq N} \{ |\partial^\infty_i \theta_i(z)|, |\partial^\infty_i \kappa_{ij}(z)| \}, \quad \text{for } z \in \Omega.
\]

The random function \( \tilde{E}_\ell(z) \) is given by the following explicit relations:
\[
\tilde{E}_0(z) := 1, \quad \tilde{E}_1(z) := \max \left\{ 1, \frac{2E_1}{C} \right\}
\]
and
\[
\tilde{E}_\ell(z) := \max \left\{ 1, \frac{2E_\ell(z)}{C} \sum_{0 \leq p \leq \ell-1} \tilde{E}_p(z) \right\} \quad (\ell \geq 2),
\]
where
\[
E_1(z) := \rho \left( 2 + \sin^2 \alpha + \gamma (1 + \cos \alpha) + 5\rho \sin \alpha + 4 \frac{\mu'}{\gamma} \right) + 2 \left\lfloor \frac{\mu'}{\gamma} \right\rfloor \sin 2\alpha + |\gamma|,
\]
\[
E_\ell(z) := 3\rho + 5\rho^2 + 3 \frac{\rho}{N}
\]
\[
+ \frac{2}{N} \sum_{r_1, r_2, r_3 \neq \ell} \frac{\ell!}{r_1!k_1! \cdots r_2!k_2'! \cdots r_3!k_3'!} \prod_{p=1}^{r_2} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p} \prod_{p=1}^{r_1} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p'}
\]
\[
+ 2\rho \sum_{r_1, r_2, r_3 \neq \ell} \frac{\ell!}{r_1!k_1! \cdots r_2!k_2'! \cdots r_3!k_3'!} \mathcal{R}(r_2, k_1, \ldots, k_2) \prod_{p=1}^{r_3} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p'}
\]
\[
+ \rho \sum_{r_1, r_2, r_3 \neq \ell} \frac{\ell!}{r_1!k_1! \cdots r_2!k_2'! \cdots r_3!k_3'!} \prod_{p=1}^{r_2} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p} \mathcal{R}(r_3, k_1', \ldots, k_3')
\]
\[
+ \frac{3}{N} \sum_{(k_1, \ldots, k_\ell) \in \Lambda(\ell')} \frac{\ell!}{k_1! \cdots k_\ell!} \prod_{p=1}^{\ell-1} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p}
\]
\[
+ 3\rho \sum_{(k_1, \ldots, k_\ell) \in \Lambda(\ell')} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \left[ \mathcal{R}(\ell-1, k_1, \ldots, k_{\ell-1}) + \prod_{p=1}^{\ell-1} \left( \frac{\mathcal{C}_p}{p!} \right)^{k_p'} \right] + 4\rho \frac{\mu}{\gamma}
\]
Suppose that the framework of 4.1 and 5.2 holds. Lower-order estimate. Theorem 5.1. Suppose that the framework (A1) – (A2) holds, and let (Θ, K) and (Θ∞, K∞) be a solution to (1.2) and a unique complete oscillator death state, respectively. Then, for each z ∈ Ω and ℓ ≥ 1, we have

\[ L(\partial^2_\ell \hat{\Theta}(t, z), \partial^2_\ell \hat{K}(t, z)) \leq \hat{E}_\ell(z) \left( \sum_{0 \leq p \leq \ell} L(\partial^p \Theta^0(z), \partial^p \hat{K}^0(z)) \right) e^{-\frac{\ell + 1}{2} t}. \] (5.1)

Proof. The proof will be provided in next two subsections similar to the proof of Theorem 4.1.

5.2. Lower-order estimate. In this subsection, we study an exponential stability of z-variations of fluctuations. Note that ∂zθi, ∂zκij, ∂zθi∞ and ∂zκij∞ satisfy

\[
\begin{align*}
&\partial_t (\partial_z \theta_i) = -\frac{1}{N} \sum_{1 \leq j \leq N} \left[ \partial_z \kappa_{ij} \sin \theta_i (1 + \cos \theta_j) 
\right.
+ \kappa_{ij} \cos \theta_i \partial_z \theta_i (1 + \cos \theta_j) - \kappa_{ij} \sin \theta_i \sin \theta_j \partial_z \theta_j], \\
&\partial_t (\partial_z \kappa_{ij}) = -\gamma \partial_z \kappa_{ij} - \gamma' \kappa_{ij} - \mu \sin(\theta_j - \theta_i) (\partial_z \theta_j - \partial_z \theta_i) + \mu' \cos(\theta_j - \theta_i),
\end{align*}
\] (5.2)

and

\[
\begin{align*}
0 &= -\frac{1}{N} \sum_{1 \leq j \leq N} \left[ \partial_z \kappa_{ij}^\infty \sin \theta_i^\infty (1 + \cos \theta_j^\infty) + \kappa_{ij}^\infty \cos \theta_i^\infty \partial_z \theta_i^\infty (1 + \cos \theta_j^\infty) 
\right.
\left. - \kappa_{ij}^\infty \sin \theta_i^\infty \sin \theta_j^\infty \partial_z \theta_j^\infty \right], \\
0 &= -\gamma \partial_z \kappa_{ij}^\infty - \gamma' \kappa_{ij}^\infty - \mu \sin(\theta_j^\infty - \theta_i^\infty) (\partial_z \theta_j^\infty - \partial_z \theta_i^\infty) + \mu' \cos(\theta_j^\infty - \theta_i^\infty),
\end{align*}
\] (5.3)

where γ' and μ' denote the z-differentiations.
Then, by subtracting (5.3) from (5.2), multiplying \( \text{sgn}(\partial_z \hat{\theta}_i) = \text{sgn}(\partial_z \theta_i - \partial_z \theta_i^\infty) \) and summing up the resulting relation over \( i \) and \( j \), we have

\[
\partial_t \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i| = - \frac{1}{N} \sum_{1 \leq i,j \leq N} \left[ \partial_z \hat{k}_{ij} \sin \theta_i (1 + \cos \theta_j) \text{sgn}(\partial_z \hat{\theta}_i) + |\partial_z \hat{\theta}_i| \cos \theta_i (1 + \cos \theta_j) \right]
\]

\[
- \partial_z \hat{\theta}_j \kappa_{ij} \sin \theta_i \sin \theta_j \text{sgn}(\partial_z \hat{\theta}_i)
\]

\[
- \frac{1}{N} \sum_{1 \leq i,j \leq N} \partial_z \kappa_{ij}^\infty \left[ (\sin \theta_i - \sin \theta_i^\infty) (1 + \cos \theta_j) + \sin \theta_i^\infty (\cos \theta_j - \cos \theta_j^\infty) \right] \text{sgn}(\partial_z \hat{\theta}_i)
\]

\[
- \frac{1}{N} \sum_{1 \leq i,j \leq N} \partial_z \theta_i^\infty \kappa_{ij} \left[ \cos \theta_i (1 + \cos \theta_j) - \sin \theta_i \sin \theta_j \right] \text{sgn}(\partial_z \hat{\theta}_i)
\]

\[
- \frac{1}{N} \sum_{1 \leq i,j \leq N} \partial_z \theta_i^\infty \kappa_{ij}^\infty \left[ \cos \theta_i - \cos \theta_i^\infty \right] (1 + \cos \theta_j) \sin \theta_j + \sin \theta_i^\infty (\sin \theta_j - \sin \theta_j^\infty) \right] \text{sgn}(\partial_z \hat{\theta}_i)
\]

\[
= \sum_{1 \leq k \leq 5} H_{2k}, \quad (5.4)
\]

and

\[
\partial_t \sum_{1 \leq i,j \leq N} |\partial_z \hat{k}_{ij}| \quad (5.5)
\]

\[
= - \sum_{1 \leq i,j \leq N} \left[ \gamma |\partial_z \hat{k}_{ij}| + \mu \sin(\theta_j - \theta_i) (\partial_z \hat{\theta}_j - \partial_z \hat{\theta}_i) \text{sgn}(\partial_z \hat{k}_{ij}) \right]
\]

\[
- \sum_{1 \leq i,j \leq N} \mu \left[ \sin(\theta_j - \theta_i) - \sin(\theta_j^\infty - \theta_i^\infty) \right] (\partial_z \theta_j^\infty - \partial_z \theta_i^\infty) \text{sgn}(\partial_z \hat{k}_{ij})
\]

\[
- \sum_{1 \leq i,j \leq N} \gamma \kappa_{ij} \text{sgn}(\partial_z \hat{k}_{ij}) + \sum_{1 \leq i,j \leq N} \mu \left[ \cos(\theta_j - \theta_i) - \cos(\theta_j^\infty - \theta_i^\infty) \right] \text{sgn}(\partial_z \hat{k}_{ij}).
\]

**Lemma 5.2.** The functionals \( H_{2k} \) satisfy the following estimates:

\[
H_{21} \leq \frac{2}{N} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_z \hat{k}_{ij}| - \frac{\mu}{\gamma} \cos 2 \alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i|
\]

\[
+ \max \left\{ \kappa_{M,0}, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} |\partial_z \hat{\theta}_j|,
\]

\[
H_{22} \leq \rho (2 + \sin \alpha \gamma) \mathcal{L}(\hat{\theta}^0, \hat{K}^0) e^{-Ct}, \quad H_{23} \leq \rho \gamma (1 + \cos \alpha) \mathcal{L}(\hat{\theta}^0, \hat{K}^0) e^{-Ct},
\]

\[
H_{24} \leq 3 \rho^2 \sin \alpha \mathcal{L}(\hat{\theta}^0, \hat{K}^0) e^{-Ct}, \quad H_{25} \leq 2 \rho^2 (1 + \sin \alpha) \mathcal{L}(\hat{\theta}^0, \hat{K}^0) e^{-Ct}.
\]

**Proof.** In (5.4), we use similar arguments as in Theorem 3.3 to obtain the following estimates:

\[
H_{21} \leq \frac{2}{N} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_z \hat{k}_{ij}| - \frac{\mu}{\gamma} \cos 2 \alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i|
\]
Suppose the framework to get

\[
\begin{align*}
\mathcal{H}_{22} &\leq \frac{1}{N} \sum_{1 \leq i,j \leq N} |\partial_z \kappa_{ij}^\infty| \left[ \cos \hat{\theta}_i |\hat{\theta}_i| (1 + \cos |\theta_j|) + \sin |\theta_i^\infty| \sin |\hat{\theta}_j||\hat{\theta}_j| \right] \\
&\leq \rho(2 + \sin^2 \alpha) \sum_{1 \leq i \leq N} |\hat{\theta}_i| \leq \rho(2 + \sin^2 \alpha) \mathcal{L}(\hat{\Theta}^0, \hat{K}^0)e^{-Ct}, \\
\mathcal{H}_{23} &\leq \frac{1}{N} \sum_{1 \leq i,j \leq N} |\partial_z \theta_i^\infty| \left[ \cos \alpha(1 + \cos \alpha) + \sin^2 \alpha \right] |\bar{\kappa}_{ij}|
\end{align*}
\]

where the intermediate values \( \hat{\theta}_j \in (-\alpha, \alpha) \) follow from the mean value theorem. \( \square \)

Now, we are ready to provide exponential \( \ell_1 \)-stability estimate for the lower-order \( z \)-variations.

**Proposition 1.** Suppose the framework \((A1) - (A2)\) holds, and let \((\Theta, K)\) and \((\Theta^\infty, K^\infty)\) be a solution to \((1.2)\) and a unique complete oscillator death state, respectively. Then, for each \( z \in \Omega \) and \( t \geq 0 \), we have

\[
\mathcal{L}(\partial_z \hat{\Theta}(t, z), \partial_z \hat{K}(t, z)) \leq \max \left\{ 1, \frac{2\mathcal{E}_1}{C} \right\} \left( \sum_{0 \leq p \leq 1} \mathcal{L}(\hat{\Theta}^p, \hat{K}^p(z)) \right)e^{-\frac{C}{\rho}t}.
\]

**Proof.** First, we combine all the estimates in Lemma 5.2 to get

\[
\partial_t \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i| \leq \frac{2}{N} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_z \kappa_{ij}| - \frac{\mu}{\gamma} \cos 2\alpha \cos \alpha(1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i| + \max \left\{ \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} |\partial_z \hat{\theta}_j| + \rho \left( 2 + \sin^2 \alpha + \gamma(1 + \cos \alpha) + 5\rho \sin \alpha \right) \mathcal{L}(\hat{\Theta}^0(z), \hat{K}^0(z))e^{-Ct}.
\]

Second, it follows from \((5.5)\) that

\[
\partial_t \sum_{1 \leq i,j \leq N} |\partial_z \kappa_{ij}| \leq -\gamma \sum_{1 \leq i,j \leq N} |\partial_z \kappa_{ij}| + 2\mu N \sin 2\alpha \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i| + 2N(2\rho \mu + |\mu'| \sin 2\alpha) \sum_{1 \leq i \leq N} |\hat{\theta}_i| + |\gamma| \sum_{1 \leq i,j \leq N} |\bar{\kappa}_{ij}|
\]

\[
\leq -\gamma \sum_{1 \leq i,j \leq N} |\partial_z \kappa_{ij}| + 2\mu N \sin 2\alpha \sum_{1 \leq i \leq N} |\partial_z \hat{\theta}_i|
\]

\[
\mathcal{L}(\partial_z \hat{\Theta}(t, z), \partial_z \hat{K}(t, z)) \leq \max \left\{ 1, \frac{2\mathcal{E}_1}{C} \right\} \left( \sum_{0 \leq p \leq 1} \mathcal{L}(\hat{\Theta}^p, \hat{K}^p(z)) \right)e^{-\frac{C}{\rho}t}.
\]
5.3. Higher-order estimates. In this subsection, we provide higher-order \( \ell \)-stability estimates for \( \mathcal{L}(\partial^{\ell}_2 \hat{\Theta}, \partial^{\ell}_1 \hat{K}) \) \((\ell \geq 2)\).

5.3.1. Pointwise estimate of \( \partial^{\ell}_i \hat{\theta}_i \). Note that \( \partial^{\ell}_i \hat{\theta}_i \) satisfies

\[
\partial_t \sum_{1 \leq i \leq N} |\partial^{\ell}_i \hat{\theta}_i| = \sum_{1 \leq k \leq 8} J_{1k}, \tag{5.8}
\]

where the terms in the R.H.S. of (5.8) are given as follows.

\[
J_{11} := - \frac{1}{N} \sum_{1 \leq i, j \leq N} [\partial^{\ell}_i \hat{k}_{ij} \sin \theta_i (1 + \cos \theta_j) \text{sgn}(\partial^{\ell}_i \hat{\theta}_i) \\
+ |\partial^{\ell}_i \hat{\theta}_i| \hat{k}_{ij} \cos \theta_i (1 + \cos \theta_j) - \partial^{\ell}_i \hat{\theta}_i \hat{k}_{ij} \sin \theta_i \sin \theta_j \text{sgn}(\partial^{\ell}_i \hat{\theta}_i)],
\]

\[
J_{12} := - \frac{1}{N} \sum_{1 \leq i, j \leq N} \partial^{\ell}_i \hat{k}_{ij}^{\infty} \left[ (\sin \theta_i - \sin \theta_j^{\infty}) (1 + \cos \theta_j) + \sin \theta_i^{\infty} (\cos \theta_j - \cos \theta_j^{\infty}) \right] \\
\times \text{sgn}(\partial^{\ell}_i \hat{\theta}_i)
\]

\[
- \frac{1}{N} \sum_{1 \leq i, j \leq N} \partial^{\ell}_i \theta^{\infty}_j \hat{k}_{ij} \left[ \cos \theta_i (1 + \cos \theta_j) - \sin \theta_i \sin \theta_j \right] \text{sgn}(\partial^{\ell}_i \hat{\theta}_i),
\]

\[
- \frac{1}{N} \sum_{1 \leq i, j \leq N} \partial^{\ell}_i \theta^{\infty}_j \hat{k}_{ij}^{\infty} \left[ (\cos \theta_i - \cos \theta_{j}^{\infty})(1 + \cos \theta_j) + \cos \theta_i^{\infty} (\cos \theta_j - \cos \theta_{j}^{\infty}) \right] \\
\times \text{sgn}(\partial^{\ell}_i \hat{\theta}_i)
\]

\[
+ \frac{1}{N} \sum_{1 \leq i, j \leq N} \partial^{\ell}_j \theta^{\infty}_j \hat{k}_{ij}^{\infty} \left[ (\sin \theta_i - \sin \theta_j^{\infty}) \sin \theta_j + \sin \theta_i^{\infty} (\sin \theta_j - \sin \theta_j^{\infty}) \right] \\
\times \text{sgn}(\partial^{\ell}_i \hat{\theta}_i),
\]

\[
J_{13} := - \frac{1}{N} \sum_{1 \leq i, j \leq N} \frac{\ell!}{r_1!r_2!r_3!} \partial^{\ell}_i \hat{k}_{ij} S(r_2, \theta_i) I(r_3, \theta_j) \text{sgn}(\partial^{\ell}_i \hat{\theta}_i),
\]

where the terms \( 1 \leq r_1, r_2, r_3 \neq \ell \) and \( k_1, \ldots, k_2, k_3 \in \Lambda(r_2) \) and \( k'_1, \ldots, k'_3 \in \Lambda(r_3) \).
Lemma 5.3. In the following lemma, we provide estimates for \( J_1 \) in the following lemma, we provide estimates for \( J_1 \):

\[
\begin{align*}
J_{14} := - \frac{1}{N} & \sum_{1 \leq i,j \leq N} \frac{\ell}{r_1 r_2 r_3} \partial_z^{r_1} \kappa_{ij} \left[ S(r_2, \theta_i) - S(r_2, \theta_i^\infty) \right] I(r_3, \theta_j) \operatorname{sgn}(\partial_z^r \hat{\theta}_i), \\
J_{15} := - \frac{1}{N} & \sum_{1 \leq i,j \leq N} \frac{\ell}{r_1 r_2 r_3} \partial_z^{r_1} \kappa_{ij} S(r_2, \theta_i^\infty) [I(r_3, \theta_j) - I(r_3, \theta_j^\infty)] \operatorname{sgn}(\partial_z^r \hat{\theta}_i), \\
J_{16} := - \frac{1}{N} & \sum_{1 \leq i,j \leq N} \kappa_{ij} S(r_2, \theta_i) I(r_3, \theta_j) \operatorname{sgn}(\partial_z^r \hat{\theta}_i), \\
J_{17} := - \frac{1}{N} & \sum_{1 \leq i,j \leq N} \kappa_{ij} [S(r_2, \theta_i) - S(r_2, \theta_i^\infty)] I(r_3, \theta_j) \operatorname{sgn}(\partial_z^r \hat{\theta}_i), \\
J_{18} := - \frac{1}{N} & \sum_{1 \leq i,j \leq N} \kappa_{ij}^\infty S(r_2, \theta_i^\infty) [I(r_3, \theta_j) - I(r_3, \theta_j^\infty)] \operatorname{sgn}(\partial_z^r \hat{\theta}_i).
\end{align*}
\]

In the following lemma, we provide estimates for \( J_{11} \).

**Lemma 5.3.** The terms \( J_{11} \) satisfy the following estimates:

(i) \( J_{11} \leq \frac{2}{N} \sin \alpha \sum_{1 \leq i,j \leq N} |\partial_z^r \hat{\kappa}_{ij}| - \frac{\mu}{\gamma} \cos 2\alpha \cos \alpha (1 + \cos \alpha) \sum_{1 \leq i \leq N} |\partial_z^r \hat{\theta}_i| + \max \left\{ \kappa_{0,M}, \frac{\mu}{\gamma} \right\} \sin^2 \alpha \sum_{1 \leq j \leq N} |\partial_z^r \hat{\theta}_j|, \)

(ii) \( J_{12} \leq \rho (3 + 5\rho) \sum_{1 \leq i \leq N} |\hat{\theta}_i| + 3 \frac{\rho}{N} \sum_{1 \leq i,j \leq N} |\hat{\kappa}_{ij}|, \)

(iii) \( J_{13} \leq \frac{2}{N} \sum_{r_1,r_2,r_3 \neq \ell} \frac{\ell!}{r_1! k_1! \cdots k_r! l_1' \cdots l_r'} \prod_{1 \leq i \leq r_2} \left( \frac{C_p}{p!} \right)^{k_i} \prod_{1 \leq p \leq r_3} \left( \frac{C_p}{p!} \right)^{k_p'} \times \sum_{1 \leq i,j \leq N} |\partial_z^r \hat{\kappa}_{ij}|, \)

(iv) \( J_{14} \leq 2\rho \sum_{r_1,r_2,r_3 \neq \ell} \frac{\ell!}{r_1! k_1! \cdots k_r! l_1' \cdots l_r'} R(r_2, k_1, \cdots, k_r; l_1', \cdots, l_r') \prod_{p=1}^{r_3} \left( \frac{C_p}{p!} \right)^{k_p'}. \)
Since the proof is rather lengthy, we leave it in Appendix A.

5.3.2. Pointwise estimate of $\partial_z^\ell \hat{\kappa}_{ij}$. Note that $\partial_z^\ell \hat{\kappa}_{ij}$ satisfies

$$\partial_i \sum_{1 \leq i, j \leq N} |\partial_z^\ell \hat{\kappa}_{ij}| = \sum_{1 \leq k \leq 4} J_{2k},$$

(5.9)

where the terms in the R.H.S. of (5.9) are given as follows.

$$J_{21} := -\gamma \sum_{1 \leq i, j \leq N} |\partial_z^\ell \hat{\kappa}_{ij}| - \mu \sum_{1 \leq i, j \leq N} \sin(\theta_j - \theta_i)(\partial_z^\ell \hat{\theta}_j - \partial_z^\ell \hat{\theta}_i)\text{sgn}(\partial_z^\ell \hat{\kappa}_{ij})$$

$$- \mu \sum_{1 \leq i, j \leq N} \left[ \sin(\theta_j - \theta_i) - \sin(\theta_j^\infty - \theta_i^\infty) \right] (\partial_z^\ell \theta_j^\infty - \partial_z^\ell \theta_i^\infty)\text{sgn}(\partial_z^\ell \hat{\kappa}_{ij}),$$

$$J_{22} := -\frac{\ell!}{r!} \rho \partial_z^{\ell-r} \gamma |\partial_z^\ell \hat{\kappa}_{ij}| \text{sgn}(\partial_z^\ell \hat{\kappa}_{ij}),$$

$$J_{23} := \mu \sum_{1 \leq i, j \leq N} \left[ C(\ell, k_1, \cdots, k_\ell, \theta_j - \theta_i) - C(\ell, k_1, \cdots, k_\ell, \theta_j^\infty - \theta_i^\infty) \right] \text{sgn}(\partial_z^\ell \hat{\kappa}_{ij}),$$

$$J_{24} := \frac{\ell!}{r!(\ell - r)!} \rho \partial_z^{\ell-r} \mu \left[ C(r, k_1, \cdots, k_r, \theta_j - \theta_i) - C(r, k_1, \cdots, k_r, \theta_j^\infty - \theta_i^\infty) \right].$$
× sgn(∂_{ij}^x \hat{R}_{ij}).

Lemma 5.4. The terms $J_{2i}$ satisfy the following estimates:

(i) $J_{21} \leq -\gamma \sum_{1 \leq i, j \leq N} |\partial_{ij}^x \hat{R}_{ij}| + 2\mu N \sin 2\alpha \sum_{1 \leq i \leq N} |\partial_{i}^x \hat{\theta}_i| + 4\rho \mu N \sum_{1 \leq i \leq N} |\hat{\theta}_i|,$

(ii) $J_{22} \leq \|\| \sum_{1 \leq i \leq N} \frac{\ell!}{r!(\ell - r)!} |\partial_{i}^x \hat{R}_{ij}|,$

(iii) $J_{23} \leq 2\mu N \sum_{(k_1, \ldots, k_\ell) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} P(\ell - 1, k_1, \cdots, k_{\ell-1}) \sum_{1 \leq i \leq N} |\partial_{i}^x \hat{\theta}_i|,$

(iv) $J_{24} \leq 2\|\|N \sum_{0 \leq r \leq \ell - 1} \frac{\ell!}{(r!)^\ell (\ell - r)!} P(\ell, k_1, \cdots, k_r) \sum_{1 \leq i \leq N} |\partial_{i}^x \hat{\theta}_i|.$

Proof. Since the proof is rather lengthy, we leave it in Appendix B.

Finally, we are ready to provide higher-order stability estimate using induction arguments.

Proof of Theorem 5.1: Below, we verify the assertion (5.1) by an induction argument.

• Initial step ($l = 1$): This follows from Proposition 1.
• Inductive step ($l \geq 2$): Suppose that for $1 \leq m \leq l - 1$, $L(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z)) \leq \hat{E}_m(z) \left( \sum_{0 \leq p \leq m} L(\partial_{p}^x \hat{\Theta}^{0}(z), \partial_{p}^x \hat{K}^{0}(z)) \right) e^{-\frac{L}{2}t}.$

Now, we claim that the estimate (5.1) holds for $m = \ell$. For this, we collect all estimates $J_{ij}$, and use induction hypothesis for $1 \leq m \leq \ell - 1$ to see that for $z \in \Omega$,

$\partial_{\ell}L(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z))$

$\leq -CL(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z)) + \hat{E}_\ell(z) \sum_{0 \leq p \leq \ell - 1} L(\partial_{p}^x \hat{\Theta}(t, z), \partial_{p}^x \hat{K}(t, z))$

$\leq -CL(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z)) + \hat{E}_\ell(z) e^{-\frac{L}{2}t} \sum_{0 \leq p \leq \ell - 1} \hat{E}_p(z) \sum_{0 \leq q \leq p} L(\partial_{q}^x \hat{\Theta}^{0}(z), \partial_{q}^x \hat{K}^{0}(z))$

$\leq C(L(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z))$

$+ \left( \hat{E}_\ell(z) \sum_{0 \leq p \leq \ell - 1} \hat{E}_p(z) \right) \sum_{0 \leq p \leq \ell - 1} L(\partial_{p}^x \hat{\Theta}^{0}(z), \partial_{p}^x \hat{K}^{0}(z)) e^{-\frac{L}{2}t}.$

Finally, we apply Grönwall’s lemma to yield

$L(\partial_{\ell}^x \hat{\Theta}(t, z), \partial_{\ell}^x \hat{K}(t, z))$

$\leq L(\partial_{\ell}^x \hat{\Theta}^{0}(z), \partial_{\ell}^x \hat{K}^{0}(z)) e^{-Ct}$

$+ \frac{2\hat{E}_\ell(z)}{C} \sum_{0 \leq p \leq \ell - 1} \hat{E}_p(z) \sum_{0 \leq p \leq \ell - 1} L(\partial_{p}^x \hat{\Theta}^{0}(z), \partial_{p}^x \hat{K}^{0}(z)) \left( e^{-\frac{L}{2}t} - e^{-Ct} \right)$

$\leq \hat{E}_\ell \sum_{0 \leq p \leq \ell} L(\partial_{p}^x \hat{\Theta}^{0}(z), \partial_{p}^x \hat{K}^{0}(z)) e^{-\frac{L}{2}t}.$
6. **Conclusion.** In this paper, we have provided the existence of complete oscillator death state for the globally coupled Winfree model with random inputs and adaptive couplings. As far as the authors know, the adaptive coupling has not been employed in the synchronization research for the Winfree model. Thus, we studied the interplay of adaptive couplings and random inputs. First, we showed that the Winfree model with Hebbian type adaptive couplings also allows an existence of complete oscillator death state using the inverse function theorem type argument, and provide pointwise bounds and stability estimate for the coupled Winfree model. Our findings show that the complete oscillator death state is structurally stable with respect to random inputs. There are lots of open questions that are not covered in this paper. For example, we have employed Hebbian type coupling as an adaptive coupling rule, but for non-Hebbian coupling such as sinusoidal function, it is not clear whether complete oscillator death state exists or not. These interesting issues will be addressed in a future work.

**Appendix A. Proof of Lemma 5.3.** In this section, we provide a proof of Lemma 5.3. Below, we will take care of only three terms $J_{i3}$, $i = 3, 4, 7$. The other terms can be treated similarly.

- **(Estimate of $J_{i3}$):** By direct estimates, it is easy to see that

\[
J_{i3} := -\frac{1}{N} \sum_{1 \leq i, j \leq N} \sum_{1 \leq r_1, r_2, r_3 \leq N} \frac{\ell !}{r_1!r_2!r_3!} \partial_{s_i}^{\ell, i} \kappa_{ij} S(r_2, k_1, \ldots, k_r, \theta_i) I(r_3, k'_1, \ldots, k'_r, \theta_j) \text{sgn}(\partial_{s_j} \hat{\theta}_i)
\]

\[
\leq \frac{2}{N} \sum_{1 \leq i, j \leq N} \sum_{1 \leq r_1, r_2, r_3 \leq N} \frac{\ell !}{r_1!r_2!r_3!} \kappa_{ij} \prod_{p=1}^{r_2} \left( C_p \right)^{k_1} \prod_{p=1}^{r_3} \left( C_p \right)^{k_1'} \sum_{1 \leq i, j \leq N} \left| \partial_{s_i} \kappa_{ij} \right|
\]

- **(Estimate of $J_{i4}$):** Recall that

\[
J_{i4} := -\frac{1}{N} \sum_{1 \leq i, j \leq N} \sum_{1 \leq r_1, r_2, r_3 \leq N} \frac{\ell !}{r_1!r_2!r_3!} \partial_{s_i}^{\ell, i} \kappa_{ij}^\infty [S(r_2, \theta_i) - S(r_2, \theta_i^\infty)] \times I(r_3, k'_1, \ldots, k'_r, \theta_j) \text{sgn}(\partial_{s_j} \hat{\theta}_i).
\]

Now, we need to estimate $S(r, \theta_i) - S(r, \theta_i^\infty)$: For $(k_1, \ldots, k_r) \in \Lambda(r)$,

\[
S(r, \theta_i) - S(r, \theta_i^\infty) := \frac{\ell !}{k_1! \cdots k_r!} \left( R_1(r, \theta_i, \theta_i^\infty) + R_2(r, \theta_i, \theta_i^\infty) \right),
\]

where

\[
R_1 := (S^{(k)}(\theta_i) - S^{(k)}(\theta_i^\infty)) \prod_{p=1}^{r} \left( \frac{\partial \theta_i}{\partial \theta_p} \right)^2.
\]
\[ R_2 := S^{(k)}(\theta_0^\infty) \sum_{q_1=0}^{p-1} \prod_{q=1}^{r} \left( \frac{\partial^q \theta}{q!} \right)^{k_q} \sum_{q_2=p+1}^{r+1} \left( \frac{\partial^q \theta}{q!} \right)^{k_2} \left[ \left( \frac{\partial^p \theta}{p!} \right)^{k_p} - \left( \frac{\partial^p \theta}{p!} \right)^{k_p} \right], \]

where we set \( k_0 = k_{r+1} = 0 \).

\( \diamond \) (Estimate of \( R_1 \)): We apply the mean value theorem to get
\[ |S^{(k)}(\theta_i) - S^{(k)}(\theta_i^\infty)| = |S^{(k+1)}(\hat{\theta}_i)||\hat{\theta}_i| \leq |\hat{\theta}_i|, \tag{A.2} \]
where \( \hat{\theta}_i \in (-\alpha, \alpha) \).

\( \diamond \) (Estimate of \( R_2 \)):
\[
\left| \left( \frac{\partial^p \theta}{p!} \right)^{k_p} - \left( \frac{\partial^p \theta}{p!} \right)^{k_p} \right| \leq \left| \frac{\partial^p \hat{\theta}_i}{p!} \sum_{p'=0}^{k_p-1} \left| \frac{\partial^p \theta}{p!} \right| \left| \frac{\partial^p \theta}{p!} \right|^{k_p-1-p'} \right| \\
\leq k_p \left| \frac{\partial^p \hat{\theta}_i}{p!} \right| \left( \max \\left\{ \frac{C_r}{q!} \right\} \right)^{k_p-1}
\]
for \( k_p > 0 \), and the difference is 0 if \( k_p = 0 \). So, we have
\[
R_2 \leq \sum_{p=1}^{r} k_p \left( \max \left\{ \frac{C_r}{q!} \right\} \right)^{k_p-1} \prod_{0 \leq q \leq r+1 \atop q \neq p} \left( \max \left\{ \frac{C_r}{q!} \right\} \right)^{k_q} \sum_{0 \leq q \leq r+1 \atop q \neq p} \left| \frac{\partial^q \hat{\theta}_i}{q!} \right| \tag{A.3}
\]
Recall that
\[
R(r_2, k_1, \ldots, k_{r_2}) := \prod_{p=1}^{r_2} \left( \frac{C_p}{p!} \right)^{k_p} + \sum_{p=1}^{r_2} k_p \left( \max \left\{ \frac{C_r}{q!} \right\} \right)^{k_p-1} \prod_{0 \leq q \leq r+1 \atop q \neq p} \left( \max \left\{ \frac{C_r}{q!} \right\} \right)^{k_q}
\]
In (A.1), we combine (A.2) and (A.3) to obtain
\[
\mathcal{J}_{14} \leq \frac{2\rho}{N} \sum_{1 \leq i \leq N \atop r_1, r_2, r_3 \neq \ell} \frac{\ell!}{r_1!k_1! \cdots r_2!k_2! \cdots k_{r_3}!} \times R(r_2, k_1, \ldots, k_{r_2}) \prod_{p=1}^{r_3} \left( \frac{C_p}{p!} \right)^{k_p} \sum_{0 \leq q \leq \ell-1} \left| \frac{\partial^q \hat{\theta}_i}{q!} \right| \\
\leq 2\rho \sum_{r_1, r_2, r_3 \neq \ell} \frac{\ell!}{r_1!k_1! \cdots r_2!k_2! \cdots k_{r_3}!} \times R(r_2, k_1, \ldots, k_{r_2}) \prod_{p=1}^{r_3} \left( \frac{C_p}{p!} \right)^{k_p} \sum_{0 \leq q \leq \ell-1} \left| \frac{\partial^q \hat{\theta}_i}{q!} \right|.
\]
Similar arguments can be applied to \( \mathcal{J}_{15} \) and \( \mathcal{J}_{16} \).
5.4

⋄

k

⋄

•

where

J

The term \( J_{18} \) can be treated similarly.

**Appendix B. Proof of Lemma 5.4.** In this section, we provide a proof of Lemma 5.4. In the sequel, we only take care of the the term \( J_{23} \) and \( J_{24} \).

• (Estimate of \( J_{23} \)): First, we need to estimate \( C(\ell, \theta_j - \theta_i) - C(\ell, \theta_j^\infty - \theta_i^\infty) \). For \((k_1, \ldots, k_\ell) \in \Lambda(\ell)'\),

\[
C(\ell, k_1, \ldots, k_\ell, \theta_j - \theta_i) - C(\ell, k_1, \ldots, k_\ell, \theta_j^\infty - \theta_i^\infty) = \frac{\ell!}{k_1! \cdots k_{\ell-1}!} (P_1 + P_2),
\]

where

\[
P_1 := (\cos(k) \theta_j - \theta_i) - \cos(k) (\theta_j^\infty - \theta_i^\infty) \prod_{p=1}^{\ell-1} \left( \frac{\partial^p \theta_j - \partial^p \theta_i}{p!} \right)^{k_p},
\]

\[
P_2 := \cos(k) (\theta_j^\infty - \theta_i^\infty) \sum_{q_1=0}^{\ell-1} \prod_{p=1}^{q_1} \left( \frac{\partial^p \theta_j - \partial^p \theta_i}{q_1!} \right)^{k_p} \prod_{q_2=p+1}^{\ell} \left( \frac{\partial^q \theta_j^\infty - \partial^q \theta_i^\infty}{q_2!} \right)^{k_{q_2}} \times \left\{ \left( \frac{\partial^p \theta_j - \partial^p \theta_i}{p!} \right)^{k_p} - \left( \frac{\partial^p \theta_j^\infty - \partial^p \theta_i^\infty}{p!} \right)^{k_p} \right\},
\]

where we set \( k_0 = k_\ell = 0 \).

⋄ (Estimate of \( P_1 \)): We apply the mean value theorem to get

\[
|\cos(k) \theta_j - \theta_i) - \cos(k) (\theta_j^\infty - \theta_i^\infty)| = |\cos(k+1)(\hat{\theta})||\hat{\theta}_j - \hat{\theta}_i| \leq |\hat{\theta}_j| + |\hat{\theta}_i|
\]

where \( \hat{\theta} \in (-2\alpha, 2\alpha) \). Thus, we have

\[
P_1 \leq \left( |\hat{\theta}_j| + |\hat{\theta}_i| \right) \prod_{p=1}^{\ell-1} \left( \frac{2C_p}{p!} \right)^{k_p}.
\]

⋄ (Estimate of \( P_2 \)): Note that

\[
\left( \frac{\partial^p \theta_j - \partial^p \theta_i}{p!} \right)^{k_p} \left( \frac{\partial^p \theta_j^\infty - \partial^p \theta_i^\infty}{p!} \right)^{k_{p-1}} \prod_{p'=0}^{k_p-1} \left( \frac{\partial^p \theta_j - \partial^p \theta_i}{p!} \right)^{p'} \left( \frac{\partial^p \theta_j^\infty - \partial^p \theta_i^\infty}{p!} \right)^{k_{p'-1}} \leq \frac{k_p}{p!} (|\partial^p \hat{\theta}_j| + |\partial^p \hat{\theta}_i|) \left( \frac{2\max(C_p, \rho)}{p!} \right)^{k_p-1},
\]

for \( k_p > 0 \), and the difference is 0 if \( k_p = 0 \). This yields

\[
P_2 \leq \sum_{p=1}^{\ell-1} \prod_{0 \leq q \leq \ell} \left( \frac{2\max(C_q, \rho)}{q!} \right)^{k_q} \frac{k_p}{p!} \left( \frac{2\max(C_p, \rho)}{p!} \right)^{k_p-1} (|\partial^p \hat{\theta}_j| + |\partial^p \hat{\theta}_i|)
\]

\[
\frac{\ell!}{k_1! \cdots k_{\ell-1}!} \left( R(\ell-1, k_1, \ldots, k_{\ell-1}) + \prod_{p=1}^{\ell} \left( \frac{C_p}{p!} \right)^{k_p} \right).\]
Recall that
\[ \mathcal{P}(\ell - 1, k_1, \ldots, k_{\ell-1}) := \prod_{p=1}^{\ell-1} \left( \frac{2 \max \{ C_p, \rho \} }{ p! } \right) \sum_{0 \leq p \leq \ell} \prod_{q \neq p}^{\ell-1} \left( \frac{2 \max \{ C_q, \rho \} }{ q! } \right)^{k_q} \sum_{p=1}^{\ell-1} \left( | \partial_y^p \hat{\theta}_i | + | \partial_x^p \hat{\theta}_i | \right). \]

We combine all these estimates to get

\[ J_{2a} \leq \mu \sum_{1 \leq j, l \leq N \atop (k_1, \ldots, k_l) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \mathcal{P}(\ell - 1, k_1, \ldots, k_{\ell-1}) \sum_{p=0}^{\ell-1} \left( | \partial_y^p \hat{\theta}_j | + | \partial_x^p \hat{\theta}_i | \right) \]

\[ \leq 2 \mu N \sum_{1 \leq j, l \leq N \atop (k_1, \ldots, k_l) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_{\ell-1}!} \mathcal{P}(\ell - 1, k_1, \ldots, k_{\ell-1}) \sum_{0 \leq p \leq \ell-1} \left( | \partial_y^p \hat{\theta}_j | + | \partial_x^p \hat{\theta}_i | \right). \]

\[ \bullet \text{(Estimate of } J_{2a} \text{)}: \text{ By direct estimate, we have} \]

\[ J_{2a} \leq ||u|| \sum_{1 \leq j, l \leq N \atop 0 \leq r \leq \ell-1 \atop (k_1, \ldots, k_r) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_r!} \mathcal{P}(r, k_1, \ldots, k_r) \sum_{p=0}^{r} \left( | \partial_y^p \hat{\theta}_j | + | \partial_x^p \hat{\theta}_i | \right) \]

\[ \leq 2 ||u|| N \sum_{1 \leq j, l \leq N \atop 0 \leq r \leq \ell-1 \atop (k_1, \ldots, k_r) \in \Lambda(\ell)} \frac{\ell!}{k_1! \cdots k_r!} \mathcal{P}(r, k_1, \ldots, k_r) \sum_{0 \leq p \leq \ell-1} \left( | \partial_y^p \hat{\theta}_j | + | \partial_x^p \hat{\theta}_i | \right). \]
[14] S. Y. Ha, D. Ko, J. Park and S. W. Ryoo, Emergence of partial locking states from the ensemble of Winfree oscillators, *Quart. Appl. Math.*, 75 (2017), 39–68.

[15] S. Y. Ha, D. Ko, J. Park and S. W. Ryoo, Emergent dynamics of Winfree oscillators on locally coupled networks, *J. Differ. Equ.*, 260 (2016), 4203–4236.

[16] S. Y. Ha and D. Kim, Robustness and asymptotic stability for the Winfree model on a general network under the effect of time-delay, *J. Math. Phys.*, 59 (2018), 112702.

[17] S. Y. Ha, J. Park and S. W. Ryoo, Emergence of phase-locked states for the Winfree model in a large coupling regime, *Discrete Contin. Dyn. Syst.*, 35 (2015), 3417–3436.

[18] S. Jin and L. Pareschi, *Uncertainty Quantification for Hyperbolic and Kinetic Equations*, SEMA SIMAI Springer Series Book 14, Springer, 2018.

[19] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*, Springer-Verlag, Berlin, 1984.

[20] Y. Kuramoto, International symposium on mathematical problems in mathematical physics, Lecture notes in theoretical physics, 30 (1975), 420.

[21] S. Louca and F. M. Atay, Spatially structured networks of pulse-coupled phase oscillators on metric spaces, *Discrete Contin. Dyn. Syst.*, 34 (2014), 3703–3745.

[22] W. Oukil, A. Kessi and Ph. Thieullen, Synchronization hypothesis in the Winfree model, *Dyn. Syst.*, 32 (2017), 326–339.

[23] D. D. Quinn, R. H. Rand and S. Strogatz, Singular unlocking transition in the Winfree model of coupled oscillators, *Physical Rev. E.*, 75 (2007), 036218.

[24] D. D. Quinn, R. H. Rand and S. Strogatz, *Synchronization in the Winfree Model of Coupled Nonlinear Interactions*, A. ENOC 2005 Conference, Eindhoven, Netherlands, August 7–12, 2005 (CD-ROM).

[25] Q. Ren and J. Zhao, Adaptive coupling and enhanced synchronization in coupled phase oscillators, *Phys. Rev. E.*, 76 (2007), 016207.

[26] A. Saltelli, M. Ratto, T. Andres, F. Campolongo, J. Cariboni, D. Gatelli, M. Saisana and S. Tarantola, Introduction to sensitivity analysis, *Global sensitivity analysis. The Primer*, (2008), 1–51.

[27] P. Seliger, S. C. Young and L. S. Tsimring, Plasticity and learning in a network of coupled phase oscillators, *Phys. Rev. E.*, 65 (2002), 041906.

[28] R. C. Smith, *Uncertainty quantification: Theory, Implementation, and Applications*, 2013.

[29] A. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theoret. Biol.*, 16 (1967), 15–42.

Received April 2021; 1st revision June 2021; 2st revision July 2021; early access August 2021.

E-mail address: syha@snu.ac.kr
E-mail address: doheonkim@kias.re.kr
E-mail address: boramoon@hanyang.ac.kr