Continuous-variable qubit on an optical transverse mode

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Continuous-variable (CV) quantum computing is a potentially useful method for performing several types of task more efficiently than is possible with standard (discrete-variable) quantum computing [1]. To perform a system to generate and detect CV qubits on an optical transverse mode. As a proof-of-principle experiment, we generate six CV qubit states and observe their probability distributions in position and momentum space. This enabled us to prepare a non-Gaussian initial state for CV quantum computing. Other potential applications of the CV qubit include adiabatic control of a beam profile, phase shift keying on transverse modes, and quantum cryptography using CV qubit states.

I. INTRODUCTION

Continuous-variable (CV) quantum computing is a potentially useful method for performing several types of task more efficiently than is possible with standard (discrete-variable) quantum computing [1]. To perform CV quantum computing, it must be possible to apply an arbitrary time evolution exp(−iG) to the continuous variables (X and P) at will. Although in practice it is difficult to apply interactions G that are represented by greater than second-order polynomials of X and P, if a non-Gaussian state can somehow be prepared, the quadratic interaction G2 will be suitable for performing CV quantum computing. Photon subtraction from a squeezed vacuum state is a good way to prepare non-Gaussian states in optical longitudinal modes [3–5], and it has been demonstrated that, by combining the photon subtraction and displacement operations, a class of non-Gaussian states can be prepared within a given system [6]. Because its states can ideally be represented using the qubits C1|ω⟩ + C1|ω⟩, where C1 and C1 are complex numbers, and |ω⟩ and |ω⟩ are even and odd Schrödinger’s cat states, respectively, on the longitudinal mode, this class is called the “continuous-variable qubit.” The even and odd cats represent superpositions of two macroscopic states, where even and odd denote, respectively, whether the interference is constructive or destructive. In the case of optical longitudinal modes, the macroscopic state represents coherent (classical) light. In practice, however, CV qubits generated by photon subtraction will not be ideal; rather, they will be probabilistically generated states that are mixed to a certain extent with vacuum fluctuation. As a result, the purity around the |ω⟩ state will be far from unity (as shown Figure 11 in Ref. [7]). In addition, as the success rate in generating the even cat state is not very high, it must be substituted for with a squeezed vacuum. Thus, more innovations will be necessary in order to achieve CV quantum computing on optical longitudinal modes.

In addition to longitudinal modes, light also has transverse modes and, as analogies can be found between paraxial wave optics and quantum mechanics, CV quantum computing on optical transverse modes may be possible [8–9]. In this work, we will assume that light is always in a coherent state in its longitudinal mode, either in free space or in linear optical elements such as lenses and beam splitters. Based on the formulation in Ref. [10], we can quantize the transverse mode as

\[
\begin{align*}
\hat{a}_x & = \frac{\sqrt{2} \hat{x}}{w_0} + \frac{w_0}{\sqrt{2\hbar}} \hat{p}_x, \\
\hat{a}_y & = \frac{\sqrt{2} \hat{y}}{w_0} + \frac{w_0}{\sqrt{2\hbar}} \hat{p}_y,
\end{align*}
\]

where \(w_0\) is the intensity-1/e2 radius of the vacuum state |vac⟩ along the transverse mode. \(\hat{x}, \hat{y}\) are operatorized coordinates, and the momentum operators \(\hat{p}_x, \hat{p}_y\) are defined so as to satisfy the canonical commutation relation \([\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar\). As with the longitudinal mode, the vacuum state is defined to be \(\hat{a}_x, \hat{a}_y |\text{vac}\rangle = 0\). The Hermite-Gaussian (HG) mode of the principal numbers \(n_x, n_y\) can be written as |HG\(n_x, n_y\rangle \rangle = \{|\hat{a}_x\rangle^n_x |\hat{a}_y\rangle^n_y \sqrt{n_x! n_y!} |\text{vac}\rangle\rangle\rangle, while the Laguerre-Gaussian (LG) mode of the azimuthal quantum number \(l\) and the magnetic quantum number \(m\) is written as |LG\(l, m\rangle\rangle = \{|\hat{a}_\pm\rangle^{l+m} |\hat{a}_\pm\rangle^{l-m} \sqrt{(l+m)! (l-m)!} |\text{vac}\rangle\rangle\rangle, where the \(\hat{a}_\pm\) are defined as \(\hat{a}_\pm \equiv (\hat{a}_x \mp i \hat{a}_y) \sqrt{2}\), and the quantum numbers take the values of \(n_x, n_y = 0, 1, 2, \cdots, l = 0, 1, 2, \cdots, m = -l, \cdots, +l\). Thus, the HG and LG modes represent the number of states that can be occupied by the transverse mode. The orbital angular momentum (OAM) operator \(\hat{l}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x\) may be more familiar than annihilation operators given in Eq. \(1\); because the OAM can be written using annihilation operators as \(\hat{l}_z = (\hat{a}_+ \hat{a} - \hat{a}_- \hat{a}_-) \hbar /2\), the oscillators in Eq. \(1\) can be regarded as Schwinger’s model oscillators for OAM \(11\).

In this paper, we show how to generate the transverse modes for the odd cat |⇑⟩ \(\equiv \{|\text{vac}\rangle + |\text{coh}\rangle\rangle \sqrt{2\hbar}\rangle\rangle, the even cat |⇓⟩ \(\equiv \{|\text{vac}\rangle - |\text{coh}\rangle\rangle \sqrt{2\hbar}\rangle\rangle, and for any arbitrary superposition of these:

\[ |	heta, \phi\rangle = \cos(\theta/2)|x\rangle + e^{i\phi} \sin(\theta/2)|x\rangle, \]

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where $|\text{coh}\rangle$ is a coherent state for the transverse mode with small amplitude ($|\langle \hat{a}_x | \rangle | \sim 1$), $N_{t,1}$ are normalization factors, and $|x, \pm \rangle \equiv \{ \pm | \rangle \} \sqrt{2}$ is the pole state of a CV qubit. As a proof-of-principle experiment, we generated six CV qubit states and observed their probability distributions in $X$ and $P$ space; these measurements corresponded to quadrature amplitude homodyne detection. As the measured distributions agreed closely with theoretical expectations under which perfect purity is assumed, and their generation was deterministic, we can expect that CV quantum computing would function more ideally on the transverse mode than on the longitudinal mode.

The rest of this paper is organized as follows. In Sec. II, we briefly review how quantum dynamical behavior can be simulated with optical transverse modes. In Sec. III we show how to create CV qubits and perform homodyne detection on optical transverse modes. In Sec. IV we develop Wigner functions and probability distributions for eight typical CV qubit states. In Sec. V we report on the results of a proof-of-principle experiment. In Sec. VI we propose three applications, namely “adiabatic control of beam profile,” “phase shift keying on the transverse mode,” and “quantum cryptography using CV qubit states.” In Sec. VII we comment on related research, and, finally, In Sec. VIII we provide a summary for this paper.

II. SIMULATING QUANTUM DYNAMICS ON OPTICAL TRANSVERSE MODES

We will derive the formula of the beam radius of a Gaussian beam as an example of the analogies existing between quantum mechanics and paraxial optics. In the case of coherent light in a longitudinal mode, the component of the electric field $E_x$ at position $(x, y, z)$ and time $t$ can be written as

$$E_x(x, y, z, t) = \Psi(x, y, z)E_0 \exp[i(k(z - ct))],$$

where $k$ is the wave number, $c$ is the speed of light, and $E_0$ is the field amplitude. Using Dirac notation, we can represent the envelope $\Psi(x, y, z)$ as

$$\Psi(x, y, ct') = \langle x, y | \hat{U}(t') | \psi \rangle,$$

where $t' \equiv z/c$, $\hat{U}(t')$ is defined as the time evolution operator and $|x, y \rangle$ is the simultaneous eigenstate of $\hat{x}$ and $\hat{y}$. A unitary operator $\hat{U}(t')$ can be represented using a Hamiltonian $\hat{H}$ as $\hat{U}(t') = \exp[-i\hat{H}t'/\hbar]$. In order to obey the paraxial Helmholtz equation, $\hat{H}$ should be

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m'},$$

where $m' \equiv \hbar k/c$. Thus, it is seen that the time evolution of the envelope $\Psi(x, y, z)$ looks like the wavefunction of a free particle of mass $m'$. The propagator for a free particle $K(x, x', t') \equiv \langle x, y' | \hat{U}(t') | x', y' \rangle$ is given by [11]

$$K(x, x', t') = \sqrt{\frac{m'}{2\pi i\hbar}} \exp \left[ \frac{i\hbar (x - x')^2}{2\pi i\hbar} \right].$$

Because the vacuum state is defined as $\hat{a}_x |\text{vac}\rangle = 0$, its wavefunction $\psi_{\text{vac}}(x, y) \equiv \langle x, y | \text{vac} \rangle$ is Gaussian:

$$\psi_{\text{vac}}(x, y) = \sqrt{N_0} \exp \left[ -\frac{x^2 + y^2}{w_0^2} \right],$$

where $N_0$ is a normalization factor. By integration of the product of the wavefunction and the propagator for the vacuum state, $\Psi(x, y, ct') = \iint K(x, x', t')K(y, y', t')\psi(x', y')dx'dy'$, the general form of the Gaussian beam can be obtained:

$$\Psi_{\text{vac}}(x, y, z) = \sqrt{N_e} \exp \left[ iP(z) + \frac{i\hbar k(x^2 + y^2)}{2g(z)} \right],$$

where $N_e$ is a normalization factor, $P(z) = -\arctan(z/z_0)$, and $z_0 = k w_0^2/2$. In paraxial wave optics, $q(z) \equiv 1/R(z) + 2t/(k w_0^2 z)$ is called the beam parameter, where $R(z) = z \sqrt{1 + t(z/R(z))^2}$ is the concave radius and

$$w(z) = w_0 \sqrt{1 + (z/z_0)^2}$$

is the beam radius. The derivation above is performed in detail in Appendix A.

Using the same Hamiltonian [15] as above, Eq. 9 can also be derived from the same initial state [7] using Heisenberg matrices instead of the Schrödinger equation. The Heisenberg equation $dA/dt' = [A, H]/(i\hbar)$ gives the slope of the ray $\hat{v}_x = d\hat{x}/dz = \hat{p}_x/\hbar k$. Because the operator $\hat{A}$ evolves as $\hat{A}(t') \equiv \hat{U}^d(t') \hat{A} \hat{U}(t')$, the time evolution of $\hat{x}$ and $\hat{v}_x$ can be summarized using the matrix

$$\begin{bmatrix} \hat{x}(t') \\ \hat{v}_x(t') \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{v}_x \end{bmatrix}. $$

This matrix is identical to the ray matrix in geometrical optics [13]. $\hat{x}(t') = \hat{x}_{0} + z (\hat{x}_{0} + \hat{v}_x) + z^2 \hat{v}_x^{(2)}$ obtained from Eq. 9. The vacuum state [7] has the distributions $\langle \hat{x}^2 \rangle = w_0^2/4$ and $\langle \hat{v}_x^2 \rangle = 1/(k w_0^2)$, according to Eq. 8. Because the beam radius is defined as $w(z) = 2\sqrt{\langle \hat{x}^2(t') \rangle}$, therefore, we can obtain the same answer as in Eq. 8. Note that the vacuum state satisfies the diffraction limit $\Delta x \Delta v_x = 1/(2k)$, where we set $\Delta x \equiv \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$ and $\Delta v_x \equiv \sqrt{\langle \hat{v}_x^2 \rangle - \langle \hat{v}_x \rangle^2}$. As shown above, the wavefunction of the vacuum state $\psi_{\text{vac}}$ is Gaussian in both the longitudinal and transverse modes. On the other hand, the Hamiltonians of the respective modes differ; whereas the Hamiltonian for free propagation along the longitudinal mode is similar to that of a harmonic oscillator, the Hamiltonian for free propagation along the transverse modes is similar to that of a free particle and can be written using Eq. 9. However, this difference is not critical for CV quantum computing, as the Hamiltonian of a harmonic oscillator in transverse mode can be constructed using a single-lens system, as shown in Sec. III.
III. METHOD FOR GENERATING AND OBSERVING A CV QUBIT

Fig. 1 shows the process by which a CV qubit is generated on an optical transverse mode. A laser beam is passed through a single mode fiber and then split by half wave plate (HWP) 1 and polarizing beam splitter (PBS) 1 with transmittance $T$. Displacement $d$ and relative phase $\varphi$ are added between the two beams using mirrors (M) 1 and M2. By passing the beams through PBS2, HWP2, and PBS3, they are caused to interfere. Lens (L) 1 is set so that both beams are focused onto the $z = 0$ plane. L2, with focal length $f$, is placed at $z = L_1$, and a CCD camera placed at $z = L_1 + L_2$ acquires the position or momentum distributions.

To observe the momentum distribution, L2 can be inserted at $z = L_1$ and a CCD camera set at $z = L_1 + L_2$ to measure the intensity distribution, as shown schematically in Fig. 1. We assume that the spacings are constant, as $L_1 = L_2 = f(1 - \cos \theta_d)$, where $f$ is the focal length of L2, which is the definition of the rotation angle in the phase space $\theta_L$. The position $\hat{x}$ and the slope of the ray $\hat{v}_x$ on the CCD plane become

$$\left[ \begin{array}{c} \hat{x}' \\ \hat{v}_x' \end{array} \right] = \left[ \begin{array}{cc} \cos \theta_L & f_0 \sin \theta_L \\ -\sin \theta_L/f_0 & \cos \theta_L \end{array} \right] \left[ \begin{array}{c} \hat{x} \\ \hat{v}_x \end{array} \right],$$

where $f_0 = f \sin \theta_L$ is the conversion factor between $v_x$ and $x$. The CCD camera acquires the intensity distribution on the $z = L_1 + L_2$ plane. To simplify the calculation of the distribution, we reduce the $y$ dependence using $I(x') = \int |\Psi(x',y,L_1 + L_2)|^2 dy$. When the lengths are set to $L_1 = L_2 = f_1$, the rotation angle becomes $\theta_L = \pi/2$ and the resulting intensity distribution, $I(x') = \int |\tilde{\psi}(p_x,p_y)|^2 dp_y$, reflects the momentum distribution, where $x' = p_x f/(\hbar k)$ and $\tilde{\psi}(p_x,p_y)$ is the wavefunction in momentum space. When $L_1 = L_2 = 0$, the intensity distribution reflects the position distribution as $I(x) = \int |\psi(x,y)|^2 dy$. 

where $N_{arb}$ is the normalization factor.

To check the equivalence between the ket representation in Eq. (2) and the field envelope written as Eq. (11), we derive the wavefunction for the coherent state $|\psi_{coh}(x,y)\rangle \equiv (x,y)|\text{coh}\rangle$. As with the longitudinal mode, a coherent state on transverse mode is generated by using a displacement operator $\hat{D}_{\alpha}(\phi) \equiv \exp[\alpha \hat{a} - \alpha^* \hat{a}^*]$ on the vacuum state $|\text{vac}\rangle$, i.e., $|\text{coh}\rangle = \hat{D}_{\alpha}(\phi)|\text{vac}\rangle$. To simplify this formulation, we assume that the complex amplitude $\alpha$ is a real number $\alpha = d_0/\sqrt{2\omega_0}$ and that the direction of the displacement is along the $x$-axis; using these assumptions, the displacement operator becomes $\hat{D}_{\alpha} = \exp(-i\hat{p}_x/\hbar)$. From the relation $\hat{D}_{\alpha}^* \hat{D}_{\alpha} = \hat{D}_{\alpha}^2 = \hat{D}_{\alpha}$, we obtain $\langle x,y|\hat{D}_{\alpha}^* = \langle x-d,y|$. In addition, we find that $\hat{D}_{\alpha}^2 = \hat{D}_{\alpha}$. From these, we obtain $|\psi_{coh}(x,y)\rangle = (x,y)|\hat{D}_{\alpha}^*|\text{coh}\rangle = |\text{coh}\rangle$, as the wavefunction of the coherent state (the same conclusion is derived in Ref. [10]). The ket representation in Eq. (11) then becomes

$$|\text{arb}\rangle = \frac{1}{\sqrt{N_{arb}}} \left\{ \sqrt{T}|\text{vac}\rangle + e^{i\varphi} \sqrt{1-T}|\text{coh}\rangle \right\},$$

which is mathematically equivalent to a CV qubit as given in Eq. (2). The relation between $(T, \varphi)$ and $(\theta, \phi)$ are discussed further in Sec. IV.
IV. DETAILED DESCRIPTION OF CV QUBIT STATE

We can define a reduced Wigner function of the \( x \)-mode as

\[
W(x, p_x) = N_w \int_{-\infty}^{\infty} \psi(x + x'/2, y) \psi^*(x - x'/2, y) \times \exp \left[-i \frac{p_x x'}{\hbar}\right] dx' dy,
\]

where \( N_w \) is a normalization factor that satisfies \( \iint W(x, p_x) dx dp_x = 1 \) \( [10] \). The Wigner function for the arbitrary superposition of coherent states given by Eq. \( (11) \) then becomes

\[
W_{arb}(x, p_x) = \frac{1}{N_{arb}} \left\{ TW_{vac}(x, p_x) + (1 - T)W_{coh}(x, p_x) \right\} 
+ 2 \sqrt{T(1 - T)} W_{half}(x, p_x) \cos(\varphi - dp_x/\hbar),
\]

where \( W_{vac}(x, p_x) = \exp[-2x^2/(\theta_0^2 - 2p_x^2/(2\hbar^2)]/(\pi \hbar) \), \( W_{coh}(x, p_x) = W_{vac}(x - d, p_x) \), and \( W_{half}(x, p_x) = W_{vac}(x - d/2, p_x) \) are the Wigner functions for for the vacuum state, the coherent state with displacement \( d \), and the coherent state with displacement \( d/2 \), respectively.

The condition \( T = 1/2 \) and \( \varphi = 0, \pi \) provide the even and odd cats \( |\uparrow\rangle \) and \( |\downarrow\rangle \), respectively. The condition \( T = (1 \pm \sin \theta_d)/2 \) and \( \varphi = \mp \theta_d \) provides the \( |x\pm\rangle \) state. A detailed derivation of these states is provided in Appendix \( [13] \). The Wigner functions at the above conditions are plotted in Fig. 2, which introduces the non-dimensional position \( X = \sqrt{2}(x - d/2)/\theta_0 \) and the non-dimensional momentum \( P = w_0 p_x/\sqrt{2} \hbar \). According to the quantization in Eq. \( (1) \), these variables can be rewritten as \( X = (\hat{x}_a + \hat{x}_d)/2 \) and \( P = (\hat{x}_a - \hat{x}_d)/(2i) \), and their commutation relation becomes \([X, P] = i/2\). From Fig. 2(c) it is seen that \( |\downarrow\rangle \) is similar to a one-number state, and from Figs. 2(d)-(h) it is seen that the \( |x\pm\rangle \) are position squeezed states and that \( |\uparrow\rangle \) and \( |\downarrow\rangle \) are momentum squeezed states. The exact position and momentum distributions \( I(x) \) and \( \hat{I}(p_x) \), respectively, can be calculated from the Wigner function as \( I(x) = \iint W(x, p_x) dp_x dx \), yielding

\[
I_{arb}(x) = \frac{1}{N_{arb}} \left\{ TI_{vac}(x) + (1 - T)I_{coh}(x) \right\} 
+ 2 \sqrt{T(1 - T)} \sqrt{\cos \theta_d \cos \varphi - dp_x/\hbar} \right\},
\]

\[
\hat{I}_{arb}(p_x) = \frac{\hat{I}_{vac}(p_x)}{N_{arb}} \times \left\{ \cos \left( \varphi - \frac{dp_x}{\hbar} \right) \right\},
\]

where we set \( I_{vac}(x) = \iint W_{vac}(x, p_x) dp_x dx \), \( I_{coh}(x) = I_{vac}(x - d) \), \( I_{half}(x) = I_{vac}(x - d/2) \), and \( I_{vac}(p_x) = \iint W_{vac}(x, p_x) dx \). To derive the distributions in Eq. \( (14) \), we assume that the purity of the CV qubit is unity; if this is not so, then \( \psi(x + x'/2, y) \psi^*(x - x'/2, y) \) in Eq. \( (14) \) should be replaced by the matrix element of the density.
operator $\hat{\rho}$ as $(x + x'/2, y \mid \rho \mid x - x'/2, y)$. Nevertheless, as we will show in Sec. V, Eqs. (16) and (17) agree closely with the experimental results.

To prepare a desired CV qubit state $|\psi\rangle$ by arbitrary superposition of coherent states $|\xi\rangle$, $T$ and $\varphi$ must be set as

$$T = \frac{1}{2} \left( 1 + \frac{z_q \sin \theta_d}{1 - x_q \cos \theta_d} \right),$$

$$\cos \varphi = \frac{x_q - \cos \theta_d}{\sqrt{(x_q - \cos \theta_d)^2 + (y_q \sin \theta_d)^2}},$$

$$\sin \varphi = \frac{y_q \sin \theta_d}{\sqrt{(x_q - \cos \theta_d)^2 + (y_q \sin \theta_d)^2}},$$

where $(x_q, y_q, z_q) \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ are the components of the Bloch vector of the qubit. Fig. 3 shows the Bloch vectors for the eight typical qubit states; a detailed derivation of these is given in Appendix C. Although the Wigner functions for these states take many forms, they can each be classified as a type of Bloch components of the Bloch vector of the qubit. Fig. 3 clearly shows the Bloch vectors for the eight typical CV qubit states, namely $|\psi\rangle$, $|\theta\rangle$, $|\varphi\rangle$, $|\phi\rangle$, $|\theta\rangle$, $|\varphi\rangle$, $|\phi\rangle$, and $|\phi\rangle$. At the limit $\theta_d \to \pi/2$, $|\psi\rangle$ and $|\phi\rangle$ are orthogonal. At the limit $\theta_d \to \pi/2$, $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

V. EXPERIMENTAL RESULTS FROM GENERATION TO OBSERVATION OF CV QUBITS

As a proof-of-principle experiment, we generated six typical CV qubit states, namely $|\psi\rangle$, $|\theta\rangle$, and $|\phi\rangle$, and four states on the equator of the Bloch sphere. We used a commercial external-cavity diode laser (toptica DL100) with wavelength $\lambda = 2\pi/k = 780$ nm to generate a beam that was transmitted through a single-mode fiber (780HP) in order to shape its spatial distribution. The power in front of PBS1 was 50 $\mu$W and the visibility of the interferometer was measured to be 0.97 by setting $T = 1/2$ and $d = 0$. A $f = 145$ mm lens was used as L2. The CCD camera for measuring the results outputted a monochromatic 720 pixel × 480 pixel video signal with an eight-bit analog-to-digital (A/D) resolution and a pixel size of 6.5 $\mu$m. To reduce the background light, an iris and a neutral-density (ND) filter of optical density (OD) 2 was inserted in front of the CCD sensor. To avoid saturating the video signal, an ND filter of OD3 and a combination of a HWP and a PBS were inserted into the optical path, although such attenuation would be unnecessary if the exposure time of the CCD camera could be shortened. The video signal output was acquired by a computer in order to analyze the distribution with y-dependence reduction, background removal and normalization. Instead of measuring the position and momentum distributions with the same CCD camera, we inserted a non-polarizing beam splitter at $z < 0$ in order to split each CV qubit beam into two beams for which we could view the respective distributions using two CCD cameras simultaneously.

The above fittings produced...
values of $\varphi$ required; then, by taking measurements at a number of positive phases $\varphi |\uparrow\rangle$ estimate the respective values of momentum distributions with Eq. (17), we were able to determine the momentum distributions of Figs. 5 (b-2), (c-2), and (d-2) are narrower; accordingly, these represent a state. At the relative phases obtained in Figs. 5 (b-1,a-2), the position and momentum distribution are both broader than the SQL, while all of the momentum distributions have been successfully observed.

Fig. 5 shows the observed position and momentum distributions for the four CV qubit states, which have been generated by fixing $T = 1/2$ and setting four arbitrary values of $\varphi$. The theoretical curves for $I_{\text{half}}(x)$ or $\tilde{I}_{\text{half}}(x)$ are also plotted as a reference for the standard quantum limit (SQL). At the relative phase obtained in Figs. 5 (a-1,a-2), the position and momentum distribution are both split into two peaks, which are characteristics of the odd cat state. At the relative phases obtained in Figs. 5 (b-1,b-2)(c-1,c-2)(d-1,d-2), all of the position distributions are broader than the SQL, while all of the momentum distributions are narrower; accordingly, these represent momentum squeezed states. The peaks of the momentum distributions of Figs. 5 (b-2), (c-2), and (d-2) are close to the center, the negative side, and the positive side, respectively, of the distributions. By fitting these momentum distributions with Eq. (17), we were able to estimate the respective values of $\varphi$: The fitted curves are shown as blue lines in Figs. 5 (a-2), (b-2), (c-2), and (d-2). Based on these estimated values of $\varphi$ and Eq. (16), we plotted the theoretical curves of the position distributions in Figs. 5 (a-1), (b-1), (c-1), and (d-1); all of these agree closely with theoretical predictions. The shapes of the Wigner functions will be similar to those of the $|\downarrow\rangle$, $|\uparrow\rangle$ and $|p_x+\rangle$ distributions. To directly measure the relative phases $\varphi$ in Eq. (12) as well as the transmittance $T$, another beam not displaced by mirror M2 would be required: then, by taking measurements at a number of values of $\theta_L$ in addition to $\theta_L = 0, \pi/2$, tomographic reconstruction of the Wigner function, as well as homodyne tomography on the quadrature amplitude, would be possible. Although the image of the Wigner function $\tilde{W}(x,p_x)$ can be constructed by optical means [17], it is necessary to use the tomographic method to observe the negative portion, $\tilde{W}(x,p_x) < 0$, which is needed to verify whether or not the state is non-Gaussian [18].

VI. OTHER APPLICATIONS OF CV QUBIT ON TRANSVERSE MODE

As we explained in Sec. VI CV qubits on transverse modes are useful in achieving CV quantum computing. In this section, we will propose three further applications of CV qubits on transverse modes.

The first of these is “adiabatic control of the beam profile.” As shown in Fig. 2 many varieties of non-Gaussian state can be produced on the same focal plane by choosing values for the two experimental parameters $(T, \varphi)$. In
particular, we can examine the intensity distributions on the focal plane $\Delta x$ and the mean values of the slope of ray $\langle \hat{\phi}_x \rangle$. It is seen that $\Delta x$ of $|x\rangle$ is narrower than the SQL and that the $\langle \hat{\phi}_x \rangle$ of $|p_x\rangle$ has either a positive or a negative value. The intensity distribution of $|\downarrow\rangle$ has a local minimum at its center. Thus, by continuously varying $T$ and $\varphi$, the spatial distribution $\Delta x$ and the slope of ray $\langle \hat{\phi}_x \rangle$ can be continuously modified without replacing or mechanically tilting optical elements. This would be useful in the optical dipole trapping of cold atoms or as an optical tweezer for a microscope. Note that a similar phenomenon was predicted in Ref. [20] and demonstrated in Ref. [21]. In these methods, interference between the two higher order HG modes $|HG_{40}\rangle$ and $|HG_{50}\rangle$ were utilized. By contrast, our method requires the two lowest HG modes $|vac\rangle$ and $|coh\rangle$. More ideal beams for such applications can be generated experimentally.

A second application for transverse mode CV cubits is “phase shift keying (PSK) on transverse modes.” Multiplexing light with differing spatial modes by applying phase shift keying (PSK) in the quadrature amplitude, or mode division multiplexing (MDM), has gained interest as a means for increasing capacity in optical fiber communications [21]. Typically, four transverse mode states, such as $\{|HG_{00}\rangle, |HG_{10}\rangle, |HG_{01}\rangle, |HG_{11}\rangle\}$ or $\{|LG_{00}\rangle, |LG_{10}\rangle, |LG_{11}\rangle\}$, are used as the bases of a four-MDM system [22]. The cat states $\{|\uparrow\rangle, |\downarrow\rangle, |\uparrow_y\rangle, |\downarrow_y\rangle\}$ can be also adopted as the four bases of an MDM, where we define $|y\rangle \equiv \{D_y(-\alpha/2) + D_y(\alpha/2)\}/\sqrt{2N}$ and $|\downarrow_y\rangle \equiv \{D_y(-\alpha/2) - D_y(\alpha/2)\}/\sqrt{2N}$. Because these bases are mutually orthogonal, performance that is similar to that of a standard four-MDM system will be obtained. These bases can be also regarded as two sets of two transverse mode PSKs, i.e., $\{|\uparrow\rangle, |\downarrow\rangle\}$ and $\{|\uparrow_y\rangle, |\downarrow_y\rangle\}$. The number of bases can be increased by reducing the pitch of PSK for a transverse mode on the Bloch sphere; for example, $\{|\uparrow\rangle, |\downarrow\rangle, |p_x\rangle, |p_{-x}\rangle\}$ and $\{|\uparrow_y\rangle, |\downarrow_y\rangle, |p_y\rangle, |p_{-y}\rangle\}$ can be adopted as the twelve bases of an MDM, where we define $|p_y\rangle \equiv (|\uparrow_y\rangle \pm |\downarrow_y\rangle)/\sqrt{2}$ and $|p_{-y}\rangle \equiv (|\uparrow_y\rangle \pm i|\downarrow_y\rangle)/\sqrt{2}$. As a tradeoff for increasing the number of bases using this method, the bit error rate on decoding will increase; however, the resulting system could still represent a more efficient method of increasing communication capacity than the standard MDM. In standard MDM, 16 MDM is necessary to construct twelve bases, which means that $n_{x,y} \leq 3$ of HG mode or $l \leq 3$ of LG mode are required. Because of the diffraction limit, spatial modes of higher order require a larger cross section; as a result, transmission becomes more lossy as the order of the spatial modes increases [23]. Thus, it might be better overall to use PSK on two transverse mode than to use MDM.

The final application is “quantum cryptography using CV qubit states.” In standard quantum cryptography with CV, four PSK states for the quadrature amplitude $\{|-\alpha_{\omega}\rangle, |+\alpha_{\omega}\rangle, |i\alpha_{\omega}\rangle, |-i\alpha_{\omega}\rangle\}$, where we define $|\beta_{\omega}\rangle \equiv D_{\omega}\langle\beta\rangle|0_{\omega}\rangle$ and $|0_{\omega}\rangle$ is the vacuum state of both the longitudinal and transverse modes, are sent randomly [24]. In quantum cryptography with CV qubit states, by contrast, the qubit states $\{|x\rangle, |x\rangle, |p_x\rangle, |p_{-x}\rangle\}$ are sent randomly. Because two pairs of these are mutually orthogonal, $(x - x') = (p_x - p_{-x}) = 0$, and the same level of security as in the Bennett-Brassard 1984 (BB84) protocol with weak coherent light can be obtained. In a Graded - Index (GI) fiber, propagation through a phase shifter can be described using $H = h_{G}\omega_{G}(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y)$ instead of Eq. [5]. The zigzag period of rays within a GI fiber, $cT' \equiv 2\pi/\omega_{G}$, is typically on the order of 1 mm, which is much larger than the wavelength $cT' > \lambda$. Thus, discriminating the four qubit states $\{|x\rangle, |x\rangle, |p_x\rangle, |p_{-x}\rangle\}$ is possible unless the optical path length fluctuates on the order of $cT'$. This limitation is very loose compared with that associated with standard quantum cryptography using CV. Discriminating the four longitudinal coherent states $\{|-\alpha_{\omega}\rangle, |\alpha_{\omega}\rangle, |\alpha_{\omega}\rangle, |+\alpha_{\omega}\rangle\}$ is possible unless the optical path length fluctuates on the order of $\lambda$. Note that, whereas qubit states in the longitudinal mode are easily decohered by transmission losses, those in transverse mode are conserved after transmission losses. Thus, quantum cryptography with CV qubit states is a good example of an application that is difficult in the longitudinal mode but easy in the transverse mode.

VII. COMMENTS ON RELEVANT RESEARCH

Much relevant work based on coherent light has been conducted by other researchers. Discrete-variable quantum computing with numbered states on two transverse modes, such as the LG and HG modes, is demonstrated in Ref. [25]. CV quantum computing using optical fiber is discussed in Ref. [8]. The even cat for a transverse mode was generated in the large displacement regime $|\alpha|^2 \gg 1$, and the absolute value of the Wigner function $|W(x,p_x)|$ was observed by optical means in Ref. [26]. Ref. [21] cautioned that the Wigner function for the field envelope simply mimics the form of the Wigner function in quantum mechanics. One reason for this is that no decoherence was naturally induced in their study; in our system, by contrast, there is a decoherence mechanism. According to Eq. [15], the mixed state $W_{mix} = (W_{vac} + W_{coh})/(2N_{arb})$ can be obtained by setting $T = 1/2$ and averaging $\varphi$.

The beam splitter is also one of the most important elements in CV quantum computing, and in such applications, beam splitter functionality similar to that for the longitudinal mode can be obtained for the transverse mode by using a Kerr nonlinear medium, as shown, for example, in [27, 28].

Another class of relevant research has focused on few-photon states and the additional spatial degrees of freedom that can be utilized in these. Two-photon states with a large transverse displacement regime, called the spatial qubit, have been generated by [29, 30], and en-
tangling the transverse modes of a few-photon state is considered as a resource for CV quantum computing in 31, 32.

VIII. SUMMARY

Based on the quantization of optical transverse modes defined in Eq. 11 and in Ref. 10, we experimentally generated continuous-variable (CV) qubits on an optical transverse mode. A CV qubit, which is a class of non-Gaussian state defined by the superposition of two coherent states, is useful as an initial state preparation in CV quantum computing. The Wigner functions and Bloch-vector representations of typical eight CV qubits were derived. Arbitrary CV qubit states can be generated continuous-variable (CV) qubits on an optical longitudinal mode. As further applications of CV qubits on transverse modes, we proposed “adiabatic control of transverse mode.”

Finally, we must stress that these results represent only the first step in achieving CV quantum computing with optical transverse modes. As long as the requirements of CV quantum computing are satisfied, we do not need to judge whether or not the system studied here is a quantum system. Needless to say, it is a macroscopic system consisting of coherent light in longitudinal mode. Because the probability distribution is squeezed from the standard quantum limit (SQL) and the Wigner function of the odd cat state can assume negative values, it would be natural to regard this as a quantum system in terms of the transverse mode. Nevertheless, even without regarding it as a quantum system, this system remains useful for CV quantum computing because it can produce a complete set of tools needed for CV computing. This example, therefore, may lead to new interpretations of the nature of quantum objects or quantum computing.

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Appendix A: Hamiltonian identical to paraxial optics

We derive the Hamiltonian for free propagation given by Eq. 6. The wave equation is written as

\[
\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right\} E_x = 0. \tag{A1}
\]

By using the envelope \( \Psi(x, y, z) \) defined in Eq. 5, the partial derivative with \( z \) can be written as

\[
\frac{\partial^2 E_x}{\partial z^2} = \left\{ -k^2 + \frac{2ik}{\hbar} \frac{\partial \Psi}{\partial t} + \left( \frac{\partial \Psi}{\partial z} \right)^2 + \frac{\partial^2 \Psi}{\partial z^2} \right\} E_x. \tag{A2}
\]

By applying the slowly varying envelope approximation, the wave equation becomes the paraxial Helmholtz equation

\[
\frac{\partial \Psi}{\partial z} = -\frac{1}{2ik} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \Psi. \tag{A3}
\]

According to Eq. 4, the envelope evolves as

\[
\frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar} \langle x, y | \hat{H} \hat{U}(t') | \psi \rangle, \tag{A4}
\]

which is otherwise known as the Schrödinger equation. The momentum operator \( \hat{p}_x \) acts on the position eigenstates as

\[
\langle x', y' | \hat{p}_x = \hbar \frac{\partial}{\partial x'} \rangle(x', y'). \tag{A5}
\]

Therefore, assuming \( \hat{H} = (\hat{p}_x^2 + \hat{p}_y^2)/(2m') + m' = \hbar k/c \), the paraxial Helmholtz equation (A3) and the Schrödinger equation (A4) become identical. The wavefunction of the vacuum state written as Eq. 7 is obtained from Eq. (A3). The Gaussian beam having the form of Eq. 8 is obtained by using the formula

\[
\int_{-\infty}^{\infty} e^{- (ax^2 + bx + c)} dx = \sqrt{\frac{\pi}{d}} \exp \left[ \frac{b^2 - 4ac}{4a} \right]. \tag{A6}
\]

Appendix B: Generation of typical states

The normalization factors of the even \(| \uparrow \rangle \) and odd cats \(| \downarrow \rangle \) introduced in Sec. 4 are written as \( N_{\uparrow} = 1 + \cos \theta_d \) and \( N_{\downarrow} = 1 - \cos \theta_d \), respectively. The typical states \(| x \mp \rangle \) and \(| p \mp \rangle \) are expanded by the vacuum state and coherent states as

\[
|x \mp \rangle = \frac{1}{\sin \theta_d} \left[ c_d \pm s_d \right] | \text{vac} \rangle + \frac{c_d \mp s_d}{\sqrt{2}} | \text{coh} \rangle, \tag{B1}
\]

\[
|p \mp \rangle = \frac{1}{\sqrt{2}} \left[ c_d \pm is_d \right] | \text{vac} \rangle + \frac{c_d \mp is_d}{\sin \theta_d} | \text{coh} \rangle, \tag{B2}
\]

where we set \( c_d \equiv \cos(\theta_d/2) = \sqrt{N_{\uparrow}/2} \) and \( s_d \equiv \sin(\theta_d/2) = \sqrt{N_{\downarrow}/2} \) for simplicity. Note that \( \sin \theta_d = 2c_d s_d = \sqrt{N_{\downarrow} N_{\uparrow}} \). From the relations of \( c_d \pm s_d = \sqrt{1 \pm \sin \theta_d} \) and \( (c_d \mp is_d)^2 = \exp(\pi i \theta_d) \), we obtain the condition for generating the typical states written in Sec. 4 as well.
Then, we obtain in Eq. (2) for the vacuum and coherent states $C_{\text{vac}} \equiv |\text{vac}\rangle \langle \theta, \phi|$ and $C_{\text{coh}} \equiv |\text{coh}\rangle \langle \theta, \phi|$, respectively, become as follows:

$$C_{\text{vac}} = \frac{c_d + s_d}{\sqrt{2 \sin \theta_d}} \cos \frac{\theta}{2} - \frac{c_d - s_d}{\sqrt{2 \sin \theta_d}} e^{i\phi} \sin \frac{\theta}{2},$$

$$C_{\text{coh}} = -\frac{c_d - s_d}{\sqrt{2 \sin \theta_d}} \cos \frac{\theta}{2} + \frac{c_d + s_d}{\sqrt{2 \sin \theta_d}} e^{i\phi} \sin \frac{\theta}{2}.$$  

(1)

(2)

Then, we obtain

$$|C_{\text{vac}}|^2 = \frac{1}{2 \sin^2 \theta_d} [1 - x_q \cos \theta_d + z_q \sin \theta_d],$$

$$|C_{\text{coh}}|^2 = \frac{1}{2 \sin^2 \theta_d} [1 - x_q \cos \theta_d - z_q \sin \theta_d],$$

(3)

(4)

The normalization factor $N_{\text{arb}}$ becomes

$$N_{\text{arb}} = \frac{1}{|C_{\text{vac}}|^2 + |C_{\text{coh}}|^2} = \frac{\sin^2 \theta_d}{1 - x_q \cos \theta_d}.$$  

(5)

(6)

From the relation $T = N_{\text{arb}}|C_{\text{vac}}|^2$ and $\varphi = \text{arg}[C_{\text{coh}}C_{\text{vac}}^*]$, Eqs. (18)-(19)-(20) are obtained. By comparing the condition for generating the typical states derived in Appendix B we can find a representation of the typical states with the Bloch vector $(x_q, y_q, z_q)$ shown in Fig. 3.

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