SOME NEW WELL-POSEDNESS RESULTS FOR THE KLEIN - GORDON - SCHRÖDINGER SYSTEM

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Abstract. We consider the Cauchy problem for the 2D and 3D Klein-Gordon-Schrödinger system. In 2D we show local well-posedness for Schrödinger data in $H^s$ and wave data in $H^\sigma \times H^{\sigma - 1}$ for $s = -1/4 + \text{ and } \sigma = -1/2$, whereas ill-posedness holds for $s < -1/4$ or $\sigma < -1/2$, and global well-posedness for $s \geq 0$ and $s - 1/4 \leq \sigma < s + 1/4$. In 3D we show global well-posedness for $s = 0$, $s - 1/2 < \sigma \leq s + 1$. Fundamental for our results are the studies by Bejenaru, Herr, Holmer and Tataru \[2\], and Bejenaru and Herr \[3\] for the Zakharov system, and also the global well-posedness results for the Zakharov and Klein-Gordon-Schrödinger system by Colliander, Holmer and Tzirakis \[5\].

1. Introduction and main results

We consider the Cauchy problem for the Klein-Gordon-Schrödinger system with Yukawa coupling

\begin{align*}
    i\partial_t u + \Delta u &= nu \\
    \partial_t^2 n + (1 - \Delta) n &= |u|^2
\end{align*}

with initial data

\begin{equation}
    u(0) = u_0, \quad n(0) = n_0, \quad \partial_t n(0) = n_1,
\end{equation}

where $u$ is a complex-valued and $n$ a real-valued function defined for $(x,t) \in \mathbb{R}^D \times [0,T]$, $D = 2$ or $D = 3$. This is a classical model which describes a system of scalar nucleons interacting with neutral scalar mesons. The nucleons are described by the complex scalar field $u$ and the mesons by the real scalar field $n$. The mass of the meson is normalized to be 1.

Our results do not use the energy conservation law but only charge conservation $\|u(t)\|_{L^2(\mathbb{R}^D)} \equiv \text{const}$ (for the global existence result), so they are equally true if one replaces $nu$ and $|u|^2$ by $-nu$ and/or $-|u|^2$, respectively.

We are interested in local and global solutions for data

\begin{equation}
    u_0 \in H^s(\mathbb{R}^D), \quad n_0 \in H^\sigma (\mathbb{R}^D), \quad n_1 \in H^{\sigma - 1}(\mathbb{R}^D).
\end{equation}

In the case $D = 3$ local well-posedness in Bourgain type spaces was proven by the author \[10\] under the assumptions

\begin{equation}
    s > -\frac{1}{4}, \quad \sigma > -\frac{1}{2}, \quad -2s < \frac{3}{2}, \quad \sigma - 2 < s < \sigma + 1.
\end{equation}

Moreover it was shown that up to the endpoints these conditions are sharp.

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Global well-posedness in $D = 3$ in spaces of Strichartz type was shown by Colliander, Holmer and Tzirakis [3] in the case $s = \sigma \geq 0$. This is also true in $D = 2$ by similar arguments. Unconditional uniqueness in the natural solution spaces in this case also holds [10].

In the case $D = 2$ we are now able to show that local well-posedness in Bourgain type spaces holds under the same assumptions as in $D = 3$ including the case $\sigma = -\frac{1}{2}$ (Theorem 1.3). The ill-posedness statement also carries over to the case $D = 2$ (Theorem 1.2).

We also show global well-posedness in $D = 2$ for $u_0 \in L^2$, $v_0 \in H^\sigma$, $n_1 \in H^{\sigma-1}$, if $-\frac{1}{2} \leq \sigma < \frac{1}{2}$, and more generally for $u_0 \in H^s$, $v_0 \in H^\sigma$, $n_1 \in H^{\sigma-1}$, if $s \geq 0$, $s - \frac{1}{2} \leq \sigma < s + \frac{1}{2}$ (Theorem 1.3).

In the cases $D = 2$ and $D = 3$ we show global well-posedness in the case $0 \leq s \leq \sigma \leq s + 1$ in spaces of Strichartz type (Theorem 1.4), and in the case $s \geq 0$, $s - \frac{1}{2} < \sigma < s$ in spaces of Bourgain type (Theorem 1.5).

In any of these cases unconditional uniqueness holds if $s, \sigma \geq 0$ in $D = 2$ and $D = 3$ (cf. Theorem 1.3 and Theorem 1.6, respectively).

The results in this paper are based on the $(3+1)$-dimensional estimates by Bejenaru and Herr [2] which they recently used to show a sharp well-posedness result for the Zakharov system. We also use the corresponding sharp $(2+1)$-dimensional local well-posedness results for the Zakharov system by Bejenaru, Herr, Holmer and Tataru [2].

Concerning the closely related wave Schrödinger system local well-posedness in $D = 3$ was shown for $s > -\frac{1}{4}$ and $\sigma > -\frac{1}{38}$ and also global well-posedness for certain $s, \sigma < 0$ by T. Akahori [1].

We use the standard Bourgain spaces $X^{m,b}$ for the Schrödinger equation, which are defined as the completion of $S(\mathbb{R}^D \times \mathbb{R})$ with respect to

$$
\|f\|_{X^{m,b}} := \|\langle \xi \rangle^m (\tau + |\xi|^2)^b \hat{f}(\xi,\tau)\|_{L^2_{\tau \xi}}.
$$

Similarly $X^{m,b}_\pm$ for the equation $i\partial_t n_{\pm} + A^{1/2}n_{\pm} = 0$ is the completion of $\mathcal{S}(\mathbb{R}^D \times \mathbb{R})$ with respect to

$$
\|f\|_{X^{m,b}_\pm} := \|\langle \xi \rangle^m (\tau \pm |\xi|^2)^b \hat{f}(\xi,\tau)\|_{L^2_{\tau \xi}}.
$$

For a given time interval $I$ we define $\|f\|_{X^{m,b}(I)} := \inf_{\tilde{f} \in f} \|\tilde{f}\|_{X^{m,b}}$ and similarly $\|f\|_{X^{m,b}_\pm(I)}$. We often skip $I$ from the notation.

In the following we mean by a solution of a system of differential equations always a solution of the corresponding system of integral equations.

Before formulating the main results of our paper we recall that the KGS system can be transformed into a first order (in $t$) system as follows: if

$$(u, n, \partial_t n) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^{\sigma-1})$$

is a solution of (1.1), (2.4), (3.8) with data $(u_0, n_0, n_1) \in H^s \times H^\sigma \times H^{\sigma-1}$, then defining $A := -\Delta + 1$ and

$$
n_{\pm} := n \pm iA^{-\frac{1}{2}} \partial_t n
$$

and

$$
n_{\pm 0} := n_0 \pm iA^{-\frac{1}{2}} n_1 \in H^\sigma,
$$

we get that

$$(u, n_+ , n_-) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^\sigma)$$

and

$$(u, n_+ , n_-) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^\sigma)$$
is a solution of the following problem:
\begin{equation}
i \partial_t u + \Delta u = \frac{1}{2} (n_+ - n_-) u
\end{equation}
\begin{equation}
i \partial_t n_\pm + A^{1/2} n_\pm = \pm A^{-1/2} |u|^2
\end{equation}
\begin{equation}
u(0) = u_0 \quad , \quad n_\pm (0) = n_{\pm 0} := n_0 \pm i A^{-1/2} n_1 .
\end{equation}
The corresponding system of integral equations reads as follows:
\begin{equation}
u(t) = e^{it \Delta} u_0 + \frac{1}{2} \int_0^t e^{i(t-\tau) \Delta} (n_+(\tau) + n_-(\tau)) u(\tau) d\tau
\end{equation}
\begin{equation}n_\pm (t) = e^{it A^{1/2}} n_{\pm 0} \pm i \int_0^t e^{i(t-\tau) A^{1/2}} A^{-1/2} |u(\tau)|^2 d\tau .
\end{equation}

Conversely, if
\[(u, n_+, n_-) \in X^{s,b}[0, T] \times X^{\sigma,b}_+[0, T] \times X^{\sigma,b}_-[0, T]
\]
is a solution of \([1, 5]\) with data \(u(0) = u_0 \in H^s\) and \(n_\pm (0) = n_{\pm 0} \in H^\sigma\), then we define \(n := \frac{1}{2} (n_+ + n_-)\), \(2i A^{-\frac{1}{2}} \partial_t n := n_+ - n_-\) and conclude that
\[(u, n, \partial_t n) \in X^{s,b}[0, T] \times (X^{\sigma,b}_+[0, T] + X^{\sigma,b}_-[0, T]) \times (X^{-1,b}_+[0, T] + X^{-1,b}_-[0, T])
\]
is a solution of \([1, 2]\) with data \(u(0) = u_0 \in H^s\) and
\[n(0) = n_0 = \frac{1}{2} (n_+(0) + n_-(0)) \in H^\sigma, \partial_t n(0) = \frac{1}{2i} A^{\frac{1}{2}} (n_+(0) - n_-(0)) \in H^{\sigma-1} .
\]
If \((u, n_+, n_-) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^s)\), then we also have \((u, n, \partial_t n) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^{\sigma-1})\).

Our local well-posedness result in 2D reads as follow:

**Theorem 1.1.** The Klein - Gordon - Schrödinger system \([1, 2, 3]\) in 2D is locally well-posed for data
\[u_0 \in H^s(\mathbb{R}^2), n_0 \in H^\sigma(\mathbb{R}^2), n_1 \in H^{\sigma-1}(\mathbb{R}^2)
\]
under the assumptions
\[s > -\frac{1}{4}, \sigma \geq -\frac{1}{2}, \sigma - 2s < \frac{3}{2}, \sigma - 2 < s < \sigma + 1 .
\]
More precisely, there exists \(T > 0\), \(T = T(||u_0||_{H^s}, ||n_0||_{H^\sigma}, ||n_1||_{H^{\sigma-1}})\) and a unique solution
\[u \in X^{s,\frac{1}{2}+}[0, T],
\]
\[n \in X^{s,\frac{1}{2}+}_+[0, T] + X^{s,\frac{1}{2}+}_-[0, T], \partial_t n \in X^{s-1,\frac{1}{2}+}_+[0, T] + X^{s-1,\frac{1}{2}+}_-[0, T] .
\]
This solution has the property
\[u \in C^0([0, T], H^s(\mathbb{R}^2)), n \in C^0([0, T], H^\sigma(\mathbb{R}^2)), \partial_t n \in C^0([0, T], H^{\sigma-1}(\mathbb{R}^2)) .
\]
Under the additional assumption \(s, \sigma \geq 0\) we also have (unconditional) uniqueness in these latter spaces.

These conditions are sharp up to the endpoints. We namely have the following result, which can be proven exactly as in the case \(D = 3\).

**Theorem 1.2.** Let \(u_0 \in H^s(\mathbb{R}^2), n_0 \in H^\sigma(\mathbb{R}^2), n_1 \in H^{\sigma-1}(\mathbb{R}^2)\). Then the flow map \((u_0, n_0, n_1) \mapsto (u(t), n(t), \partial_t n(t))\), \(t \in [0, T]\), does not belong to \(C^2\) for any \(T > 0\), provided \(\sigma - 2s - \frac{3}{2} > 0\) or \(s < -\frac{1}{4}\) or \(\sigma < -\frac{1}{2}\).

The global well-posedness result for \(D = 2\) in the case of \(L^2\)-Schrödinger data is the following.
Theorem 1.3. The Klein - Gordon - Schrödinger system \((1), (3), (3)\) in 2D is globally well-posed for data

\[ u_0 \in H^s(\mathbb{R}^2), \quad v_0 \in H^s(\mathbb{R}^2), \quad n_1 \in H^{\sigma-1}(\mathbb{R}^2) \]

under the assumptions

\[ s \geq 0, \quad s - \frac{1}{2} \leq \sigma < s + \frac{3}{2}, \]

i.e. for any \(T > 0\) there exists a unique solution

\[ u \in X^{s+\frac{1}{2},+}[0,T], \]

\[ n \in X^{s,\frac{1}{2}+}[0,T] + X^{s-1,\frac{1}{2}+}[0,T], \quad \partial_t n \in X^{s-1,\frac{1}{2}+}[0,T] + X^{\sigma,\frac{1}{2}+}[0,T]. \]

This solution has the property

\[ u \in C^0([0,T], H^s(\mathbb{R}^2)), \quad n \in C^0([0,T], H^s(\mathbb{R}^2)), \quad \partial_t n \in C^0([0,T], H^{\sigma-1}(\mathbb{R}^2)). \]

Under the additional assumption \(\sigma \geq 0\) we also have (unconditional) uniqueness in these latter spaces, especially there exists a unique global classical solution for smooth data.

A global well-posedness result in 2D and also in 3D in the range \(0 \leq s \leq \sigma \leq s + 1\) can be proven without using Bourgain type spaces but only Strichartz type estimates.

Theorem 1.4. Let the space dimension \(D\) be 2 or 3. Assume \(0 \leq s \leq \sigma \leq s + 1\) and \(u_0 \in H^s(\mathbb{R}^2), \quad v_0 \in H^s(\mathbb{R}^2), \quad n_1 \in H^{\sigma-1}(\mathbb{R}^2)\). Then the Klein - Gordon - Schrödinger system \((1), (3), (3)\) is globally well-posed, i.e. for any \(T > 0\) there exists a unique solution

\[ u \in C^0([0,T], H^s) \cap \bigcap_{2 \leq r < \infty, 2 \leq q \leq \infty} L^q([0,T], H^{s,r}), \]

\[ n \in C^0([0,T], H^s), \quad \partial_t n \in C^0([0,T], H^{\sigma-1}), \]

where \(\frac{2}{r} + \frac{D}{r} = \frac{D}{2}\).

For negative \(\sigma\) we have to use Bourgain type spaces again.

Theorem 1.5. The Klein - Gordon - Schrödinger system \((1), (3), (3)\) in 3D is globally well-posed for data

\[ u_0 \in H^s(\mathbb{R}^3), \quad v_0 \in H^s(\mathbb{R}^3), \quad n_1 \in H^{\sigma-1}(\mathbb{R}^3) \]

under the assumptions

\[ s \geq 0, \quad s - \frac{1}{2} < \sigma < s, \]

i.e. for any \(T > 0\) there exists a unique solution

\[ u \in X^{s+\frac{1}{2},+}[0,T], \]

\[ n \in X^{s,\frac{1}{2}+}[0,T] + X^{s-1,\frac{1}{2}+}[0,T], \quad \partial_t n \in X^{s-1,\frac{1}{2}+}[0,T] + X^{\sigma,\frac{1}{2}+}[0,T]. \]

This solution has the property

\[ u \in C^0([0,T], H^s(\mathbb{R}^3)), \quad n \in C^0([0,T], H^s(\mathbb{R}^3)), \quad \partial_t n \in C^0([0,T], H^{\sigma-1}(\mathbb{R}^3)). \]

Remark: It would be desirable to have a similar result in the case \(s = 0, \quad 1 < \sigma < \frac{3}{2}\) as in the case \(D = 2\), but our estimates given in spaces of Bourgain type seem to be not quite strong enough to prove this.

Combining this with the unconditional uniqueness result [10] in the case \(s = \sigma = 0\) we also get
Theorem 1.6. Assume s, σ ≥ 0 and s − \frac{1}{2} < σ ≤ s + 1 and u_0 ∈ H^s, n_0 ∈ H^σ, n_1 ∈ H^{σ−1}. Then the Klein-Gordon-Schrödinger system (1), (2), (3) in 3D has a unique global solution

\[ u ∈ C^0([0, T], H^s), \quad n ∈ C^0([0, T], H^σ), \quad \partial_t n ∈ C^0([0, T]; H^{σ−1}). \]

Concerning the standard facts for the linear Cauchy problem (which are independent of the specific phase function) in spaces of Bourgain type we refer to [7, Section 2] or [8]. We also use the following well-known fact [8, Lemma 1.10], which we prove for the sake of completeness.

Lemma 1.1. If s ∈ \mathbb{R}, T ≤ 1, 0 ≤ b' < b < \frac{1}{2} or 0 ≥ b > b' > −\frac{1}{2}, the following estimate holds:

\[ \|u\|_{X^{s,b}[0,T]} \lesssim T^{b-b'}\|u\|_{X^{s,b'}[0,T]}, \]

Proof. Let ψ be a smooth time-cutoff function, ψ_T(t) = ψ(\frac{t}{T}), and 0 ≤ b' < b < \frac{1}{2}. By the well-known Sobolev multiplication law in 1D we get for 0 ≤ s ≤ s_1, s_2 and s ≤ s_1 + s_2 − \frac{1}{2}:

\[ \|fg\|_{H^{s'}} \lesssim \|f\|_{H^s}\|g\|_{H^{s_2}}. \]

Thus

\[ \|\psi T u\|_{H^{s'}} \lesssim \|\psi T\|_{H^{\frac{1}{2}−b'}}\|u\|_{H^{s'}} \lesssim T^{b-b'}\|u\|_{H^{s'}}, \]

so that

\[ \|\psi T u\|_{X^{s,b}} \lesssim \|e^{-i\Delta} \psi T u\|_{H^{s'} H^s} \lesssim T^{b-b'}\|e^{-i\Delta} u\|_{H^{s'} H^s} = T^{b-b'}\|u\|_{X^{s,b}}, \]

which is enough to prove the claimed estimate. The case 0 ≥ b > b' > −\frac{1}{2} follows by duality.

This obviously also holds for the spaces \( X^{s,a,b} \).

The wellknown Strichartz estimates are collected in

Proposition 1.1. (Schrödinger equation)

Let 2 ≤ q, \bar{q} ≤ ∞, 2 ≤ r, \bar{r} ≤ ∞ (excluding r, \bar{r} = ∞ in the case D = 2, \frac{2}{q} + \frac{2}{\bar{q}} = \frac{2}{r}, \frac{2}{q} + \frac{2}{\bar{r}} = \frac{2}{\bar{r}}, \frac{1}{\bar{r}} + \frac{1}{\bar{r}} = 1 = \frac{1}{q} + \frac{1}{q}). Then for any interval I = (0,T):

\[ \|e^{±it\Delta} u_0\|_{L^r_t(L^q)} \lesssim \|u_0\|_{L^\bar{r}}. \]

(9)

\[ \|\int_0^t e^{±i(t−s)\Delta} u(s)ds\|_{L^r_t(L^q)} \lesssim \|u\|_{L^\bar{r}_t(X^s_t,L^q)} \].

(10)

(Klein-Gordon equation) for D ≥ 2 :

Let 2 ≤ q, \bar{q} ≤ ∞, 2 ≤ r, \bar{r} < ∞, \frac{2}{q} + \frac{2}{\bar{q}} = \frac{2}{r}, \frac{2}{q} + \frac{2}{\bar{r}} = \frac{2}{\bar{r}}, \frac{1}{\bar{r}} + \frac{1}{\bar{r}} = 1 = \frac{1}{q} + \frac{1}{q}, \mu = D(\frac{1}{q} − \frac{1}{r}), \mu = 1 + \rho − D(\frac{1}{q} − \frac{1}{r}), \mu = 1 + \rho − D(\frac{1}{q} − \frac{1}{r}). Then for any interval I = (0,T):

\[ \|e^{±it(−\Delta)\frac{1}{2}} u_0\|_{L^r_t(L^q)} \lesssim \|u_0\|_{H^s}, \]

(11)

\[ \|\int_0^t e^{±i(t−s)(−\Delta)^{\frac{1}{2}}} (1−\Delta)^{-\frac{1}{2}} u(s)ds\|_{L^r_t(L^q)} \lesssim \|u\|_{L^\bar{r}_t(X^s_t,H^{s'}_r)}, \]

(12)

where the implicit constants are independent of I.

In the Klein-Gordon case the proof of (11) can be found in [9]. The proof of (12) follows by the well-known TT∗-method, as described in [6], in combination with the Christ-Kiselev lemma [4]. In the Schrödinger case (10) follows in the same way from the standard estimate [9].

We use the following notation. The Fourier transform is denoted by \( \hat{\cdot} \) or \( \mathcal{F} \), where it should be clear from the context, whether it is taken with respect to the space and time variables simultaneously or only with respect to the space variables. A ≤ B and A ≥ B is shorthand for A ≤ cB and A ≥ cB, respectively, with
a positive constant $c$, and $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$. For real numbers $a$ we denote by $a^+$ and $a^-$ a number sufficiently close to $a$, but larger and smaller than $a$, respectively.

2. Local well-posedness for $D = 2$.

We now formulate and prove the decisive bilinear estimates. We follow closely the arguments and notation from [3].

**Proposition 2.1.** The following estimate holds

$$
\|un\|_{X^{0, -\frac{\pi}{2}}} \lesssim \|u\|_{X^{0, \frac{\pi}{2}}} \|n\|_{X^{0, \frac{\pi}{2}} +}.
$$

Because we are going to use dyadic decompositions of $\hat{u}$ and $\hat{v}$ we take the notation from [3] and start by choosing a function $\psi \in C_0^\infty((-2, 2))$, which is even and nonnegative with $\psi(r) = 1$ for $|r| \leq 1$. Defining $\psi_N(r) = \psi(\frac{r}{N}) - \psi(\frac{r}{N+1})$ for dyadic numbers $N = 2^n \geq 2$ and $\psi_1 = \psi$ we have $1 = \sum_{N \geq 1} \psi_N$. Thus $\text{supp} \psi_1 \subset [-2, 2]$ and $\text{supp} \psi_N \subset [-2N, -N/2] \cup [N/2, 2N]$ for $N \geq 2$. For $f : \mathbb{R}^2 \to \mathbb{C}$ we define the dyadic frequency localization operators $P_N$ by

$$
\mathcal{F}_x(P_Nf)(\xi) = \psi_N(|\xi|)\mathcal{F}_x f(\xi).
$$

For $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ we define the modulation localization operators

$$
\mathcal{F}(S_Lu)(\xi, \tau) = \psi_L(|\xi|^2)u(\xi, \tau)
$$

$$
\mathcal{F}(W_K^u)(\xi, \tau) = \psi_L(|\xi| + |\tau|)u(\xi, \tau)
$$

in the Schrödinger case and the wave case.

We also define an equidistant partition of unity in $\mathbb{R}$,

$$
1 = \sum_{j \in \mathbb{Z}} \beta_j, \beta_j(s) = \psi(s - j)(\sum_{k \in \mathbb{Z}} \psi(s - k))^{-1}.
$$

Finally, for $A \in \mathbb{N}$ we define an equidistant partition of unity on the unit circle

$$
1 = \sum_{j=0}^{A-1} \beta_j^A, \beta_j^A(\theta) = \beta_j(\frac{A\theta}{\pi}) + \beta_{j-A}(\frac{A\theta}{\pi}).
$$

Then $\text{supp}(\beta_j^A) \subset \Theta_j^A$, where

$$
\Theta_j^A := \left[\frac{\pi}{A}(j-2), \frac{\pi}{A}(j+2)\right] \cup \left[-\pi + \frac{\pi}{A}(j-2), -\pi + \frac{\pi}{A}(j-2)\right].
$$

Now we introduce the angular frequency localization operators $Q_j^A$ by

$$
\mathcal{F}(Q_j^Af)(\xi) = \beta_j^A(\theta)Ff(\xi),
$$

where $\xi = |\xi|(\cos \theta, \sin \theta)$. For $A \in \mathbb{N}$ we can now decompose $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ as

$$
u = \sum_{j=0}^{A-1} Q_j^Au.
$$

**Proof of Proposition 2.1** Defining

$$
I(f, g_1, g_2) = \int \int f(\xi_3, \tau_3)g_1(\xi_1, \tau_1)g_2(\xi_2, \tau_2)d\xi_1d\xi_2d\xi_3d\tau_1d\tau_2d\tau_3,
$$

where * denotes the region $\{\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \tau_i = 0\}$ we have to show

$$
|I(\hat{u}_1, \hat{u}_1, \hat{u}_2)| \lesssim \|u_1\|_{X^{0, \frac{\pi}{2}+}}\|u_2\|_{X^{0, \frac{\pi}{2}+}}\|n\|_{X^{0, \frac{\pi}{2}+}}.
$$
We use dyadic decompositions

\[ u_k = \sum_{N, L \geq 1} S_{L, N} P_{N, L} u_k, \quad n = \sum_{N, L \geq 1} W_{L, N} P_{N, L}. \]

Defining

\[ g_k^{L, N} = \mathcal{F} S_{L, N} P_{N, L} u_k, \quad f^{L, N} = \mathcal{F} W_{L, N} P_{N, L} \]

we have

\[ I(\tilde{u}, \tilde{u}_1, \tilde{u}_2) = \sum_{N, N_1, N_2 \geq 1, L, L_1, L_2 \geq 1} I(f^{L, N}, g_1^{L, N_1}, g_2^{L, N_2}). \]

**Case 1:** \( N_1 \sim N_2 \geq N \geq 2^{10}. \)

Fix \( M = 2^{-4} N_1 \) and decompose

\[ I(f^{L, N}, g_1^{L, N_1}, g_2^{L, N_2}) = \sum_{0 \leq j_1, j_2 \leq M^{-1}, |j_1 - j_2| \leq 16} I(f^{L, N}, g_1^{L, N_1, M, j_1}, g_2^{L, N_2, M, j_2}), \]

\[ + \sum_{64 \leq A \leq M} \sum_{0 \leq j_1, j_2 \leq A^{-1}, |j_1 - j_2| \leq 32} I(f^{L, N}, g_1^{L, N_1, A, j_1}, g_2^{L, N_2, A, j_2}). \]

The first sum is estimated using [2] Prop. 4.7 by

\[ L_{L_1}^{\frac{1}{2}} L_{L_2}^{\frac{1}{2}} L_{N}^{\frac{1}{2}} \frac{N}{N_1} \| f^{L, N} \| \| g_1^{L, N, M} \| \| g_2^{L, N, M} \|. \]

The second sum is treated using [2] Prop. 4.4 and Prop. 4.6 and \( A \leq M \ll N_1 \).

We distinguish two cases.

**a.** \( L_1, L_2 \leq N_1^2. \)

We define \( \alpha := 2^{-4} \min(\frac{L_1}{N_1}, \frac{L_2}{N_1}, \max(L_1, L_2, L)^{-\frac{1}{2}}, N_1). \) The part where \( A \leq \alpha \) can be estimated for fixed \( A \) using [2] Prop. 4.4 by

\[ N_1^{-\frac{1}{2}} \left( \frac{N}{N_1} \right)^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]

Summing over \( 64 \leq A \leq \alpha \) and \( j_1, j_2 \) and using \( \sum_{64 \leq A \leq \alpha} A^{-\frac{1}{2}} \lesssim \alpha^{-\frac{1}{2}} \) we get the bound

\[ N_1^{-\frac{1}{2}} \left( \frac{N}{N_1} \right)^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]

Next we consider the part \( A \geq \alpha \). It is estimated using [2] Prop. 4.6 by

\[ N^{-\frac{1}{2}} \left( \frac{N}{A} \right)^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \max(L_1, L_2, L)^{-\frac{1}{2}} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]

Summing over \( \alpha \leq A \leq N_1 \) and \( j_1, j_2 \) and using \( \sum_{A \geq \alpha} A^{-\frac{1}{2}} \lesssim \alpha^{-\frac{1}{2}} \) we get the bound

\[ (L_1 L_2)^{\frac{1}{2}} \max(L_1, L_2, L)^{-\frac{1}{2}} N^{-\frac{1}{2}} N_1^{\frac{1}{2}} \left( \frac{N}{N_1} \right)^{\frac{1}{2}} N_1^{-\frac{1}{2}} \max(L_1, L_2, L)^{\frac{1}{2}}, \]

\[ \sum_{L_1, L_2} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]

**b.** \( \max(L_1, L_2, L) \gtrsim N_1^2. \)

[2] Prop. 4.6 gives the following bound for fixed \( A \):

\[ (L_1 L_2)^{\frac{1}{2}} \max(L_1, L_2, L)^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \frac{N}{A} \right)^{\frac{1}{2}} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]

\[ \lesssim N^{-\frac{1}{2}} N_1^{-\frac{1}{2}} \left( \frac{N}{A} \right)^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| f^{L, N} \| \| g_1^{L, N_1, A, j_1} \| \| g_2^{L, N_2, A, j_2} \|. \]
Summation over $64 \leq A \leq N_1$ and $j_1, j_2$ using $\sum A^{-\frac{1}{2}} \lesssim 1$ gives the bound

$$N^{-\frac{1}{2}} N_1^0 - (L_1L_2L) \lesssim \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$ 

**Case 2:** $N_1 \ll N_2$ or $N_2 \ll N_1$.

Using [2] Prop. 4.8 we get the bound

$$N^{-\frac{1}{2}} (L_1L_2L) \lesssim \min(\frac{N_1}{N_2}, \frac{N_2}{N_1}) \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$ 

**Case 3:** $N \lesssim 1$ ($\Rightarrow N_1 \sim N_2$ or $N_1, N_2 \lesssim 1$).

[2] Prop. 4.9 gives the bound

$$(L_1L_2L) \lesssim \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$ 

In any of these cases dyadic summation over $L_1, L_2, L$ and $N_1, N_2, N$ gives the desired bound. \qed

**Proposition 2.2.** Assume $s > -\frac{1}{2}$, $\sigma \geq -\frac{1}{2}$, $s < \sigma + 1$. Then the following estimate holds:

$$\|un\|_{X^{s-\frac{1}{2},2}} \lesssim \|u\|_{X^{s-\frac{1}{2},2}} \|n\|_{X^{s-\frac{1}{2},2}}.$$ 

**Proof.** We have to show

$$|I(\tilde{a}, \tilde{a}_1, \tilde{a}_2)| \lesssim \|u_1\|_{X^{s-\frac{1}{2},2}} \|u_2\|_{X^{s-\frac{1}{2},2}} \|n\|_{X^{s-\frac{1}{2},2}}.$$ 

Using dyadic decompositions as in the proof of Proposition 2.1 we consider different cases.

**Case 1:** $N_1 \sim N_2$.

This case can be treated by using Proposition 2.1 directly.

**Case 2.** $1 \leq N_1 \ll N_2$ ($\Rightarrow N \sim N_2$).

We have

$$L_{max} := \max(L, L_1, L_2) \gtrsim (\tau_1 + |\xi_1|^2 + \tau_2 + |\xi_2|^2 + \tau_3 + |\xi_3|) = \|\xi_1^2 + |\xi_2|^2 + |\xi_3| \gtrsim N_2^2.$$ 

Using the proof of [2] Prop. 4.8 we consider three cases.

**a.** $L = L_{max}$.

We get

$$|I(f_{L_1, N_1}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim L^{\frac{1}{2}} L_1^{\frac{1}{2}} (\frac{N_1}{N_2})^{\frac{1}{2}} \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \lesssim (L_1L_2L) \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$ 

**b.** $L_1 = L_{max}$.

Similarly we get

$$|I(f_{L_1, N_1}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim L^{\frac{1}{2}} L_1^{\frac{1}{2}} (\frac{N_1}{N_2})^{\frac{1}{2}} \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \lesssim (L_1L_2L) \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$ 

**c.** $L_2 = L_{max}$.

We get

$$|I(f_{L_1, N_1}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \lesssim (L_1L_2L) \|f\|_{L^1} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.$$
If \(-\frac{1}{2} < s \leq 0\) and \(\sigma \geq -\frac{1}{2}\) we get
\[
N_2^{-1+} \lesssim N_2^s N_2^{-\sigma} \sim N_2^s N_2^{-\sigma} \lesssim N_2^0 N_2^{-\sigma} \lesssim \frac{N_2^s}{N_2} N_2^{-\sigma},
\]
and in the case \(s > 0\) and \(\sigma > s - 1\) we get the same bound, because
\[
N_2^{-1+} \lesssim N_2^{-s+\sigma} \lesssim \frac{N_2^s}{N_2} N_2^{-\sigma} \lesssim \frac{N_2^s}{N_2} N_2^{-\sigma}.\]

In any case we thus get
\[
|I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim \frac{N_2^s}{N_2} N_2^{-\sigma} (L_1 L_2 L)^{\frac{s}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.
\]

**Case 3.** \(1 \leq N_2 \ll N_1\) (\(\Rightarrow N \sim N_1\)).

Similarly as in case 2 we get the bound
\[
|I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim (L_1 L_2 L)^{\frac{s}{2}} N_1^{-1+} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.
\]

If \(-\frac{1}{2} < s \leq 0\) and \(\sigma \geq -\frac{1}{2}\) we get
\[
N_1^{-1+} \lesssim N_1^s N_1^{-\sigma} \lesssim \frac{N_1^s}{N_1} N_1^{-\sigma},
\]
and if \(s > 0\) and \(\sigma > s - 1\) we get
\[
N_1^{-1+} \lesssim \frac{N_1^s}{N_1} N_1^{-\sigma} \lesssim \frac{N_1^s}{N_2} N_1^{-\sigma} \lesssim \frac{N_1^s}{N_2} N_1^{-\sigma},
\]
so that we get the same bound as in case 2.

Dyadic summation in all cases completes the proof of Prop. 2.2. \(\square\)

We also need the following bilinear estimate for our unconditional uniqueness result:

**Proposition 2.3.** For any \(\epsilon > 0\) the following estimate holds:
\[
\|uu\|_{X^{\epsilon,-\frac{1}{4}}} \lesssim \|u\|_{X^{\epsilon,-\frac{1}{4}}} \|u\|_{X^{\epsilon,-\frac{1}{4}}}.\]

**Proof.** We use dyadic decompositions as in the proof of Proposition 2.1.

**Case 1:** \(N_1 \sim N_2 \gtrsim N \gtrsim 2^{10}\).

We use (14). When estimating its first sum we consider different cases using the proof of [2] Prop. 4.7.

a. \(L = L_{max}\).

In this case we get the bound
\[
(L_1 L_2)^{\frac{s}{2}} N^{-\frac{s}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}
\]
and

- either \(N \sim N_1\) in which case we have
  \[
  (L_1 L_2)^{\frac{s}{2}} N^{-\frac{s}{2}} \lesssim (L_1 L_2)^{\frac{s}{2}} N_1^{-\frac{s}{2}} \sim (L_1 L_2)^{\frac{s}{2}} N_1^{-\frac{s}{2}} N_2^{0+} N^{0-},
  \]
- or \(NN_1 \lesssim L_{max}\) in which case we get
  \[
  (L_1 L_2)^{\frac{s}{2}} N^{-\frac{s}{2}} \lesssim (L_1 L_2)^{\frac{s}{2}} N_1^{-\frac{s}{2}} N_1^{-\frac{s}{2}} N_2^{0+}.
  \]

b. \(L_1 = L_{max}\).

In this case we get the bound
\[
(LL)^{\frac{s}{2}} N_1^{-\frac{s}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}
\]
and
\[
\lesssim (LL_1 L_2)^{\frac{s}{2}} N_1^{-\frac{s}{2}} N_2^{0+} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.
\]
Similarly as in the proof of Proposition 2.2 we have

**Case 4.**

\[ \sum_{A \leq N_1} A^{\frac{\sigma}{2}} \lesssim N_1^{\frac{\sigma}{2}}. \]

By [2, Prop. 4.6] for fixed \( b \),

\[ \sum_{A \leq N_1} A^{\frac{\sigma}{2}} \lesssim N_1^{\frac{\sigma}{2}}. \]

The second sum in (14) is estimated as follows.

This case is similar as case b.

By [2, Prop. 4.4] for fixed \( L \),

\[ \sum_{A \leq N_1} A^{\frac{\sigma}{2}} \lesssim N_1^{\frac{\sigma}{2}}. \]

Case 2. \( 1 \leq N_1 \ll N_2 \)

Similarly as in the proof of Proposition 2.2 we have

\[ |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim (L_1 L_2) \frac{\sigma}{2} N_2^{1+} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \]

Case 3. \( 1 \leq N_1 \ll N_2 \)

We have similarly as in case 2:

\[ |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim (L_1 L_2) L^{\frac{\sigma}{2}} N_1^{1+} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \]

Case 4. \( 1 \leq N \ll 1 \) (\( \Rightarrow N_1 \sim N_2 \) or \( 1 \leq N_1, N_2, N \ll 1 \))

By the bilinear Strichartz type estimate [2, Prop. 4.3] we get

\[ |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim \left( \min(N, N_2) \right) \frac{\sigma}{2} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \]

Dyadic summation in all cases completes the proof of Prop. 2.3.

**Proposition 2.4.** Assume \( s > -\frac{1}{2}, \sigma - 2s < \frac{1}{2}, \sigma < s + 2 \). Then the following estimate holds:

\[ \|u_1 u_2\|_{X^{-\frac{1}{2},-\frac{\sigma}{2}}} \lesssim \|u_1\|_{X^{s,\frac{\sigma}{2}}} \|u_2\|_{X^{s,\frac{\sigma}{2}}}. \]
Proof. With $I$ defined by (13) we have to show

$$|I(\hat{n}, \hat{u}_1, \hat{u}_2)| \lesssim \|u_1\|_{X^s_{1, \frac{1}{2}} - \frac{1}{2}} \|u_2\|_{X^s_{1, \frac{1}{2}} - \frac{1}{2}} \|n\|_{X^s_{1, \frac{1}{2}} - \frac{1}{2}}.$$  

Dyadically decomposing as in Proposition 2.2 we consider different cases.

Case 1. $N_1 \sim N_2 \gtrsim N \geq 2^{10}$. Fix $M = 2^{-4}N_1$ and use (14).

Case 1.1. $L_{\max} \lesssim N_1^7$.

Using [2, Prop. 4.4] for the second sum in (14) we get the bound

$$N_1^{-\frac{1}{4}} (L_1L_2L)^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \sum_{64 \leq A \leq M} \left(\frac{A}{N_1}\right)^{\frac{1}{4}}
\sum_{0 \leq j_1,j_2 \leq A - 1, 16 \leq |j_1 - j_2| \leq 32}
\|g_1^{L_1N_1,A,j_1}\|_{L^2} \|g_2^{L_2N_2,A,j_2}\|_{L^2}
\lesssim N_1^{-\frac{1}{4}} (L_1L_2L)^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1}\|_{L^2} \|g_2^{L_2N_2}\|_{L^2}.$$  

For the first sum in (14) we get the bound

- in the case $L = L_{\max}$
  - either $N \sim N_1$ and thus the bound
    $$I(f^{L,N},g_1^{L_1N_1,M,j_1},g_2^{L_2N_2,M,j_2})
    \lesssim \frac{N_1^7}{M^2} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}
    \lesssim \frac{(L_1L_2L)^{\frac{1}{2}}}{N_1^7} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}.$$  

- or $NN_1 \lesssim L_{\max}$ and thus
  $$I(f^{L,N},g_1^{L_1N_1,M,j_1},g_2^{L_2N_2,M,j_2})
  \lesssim \frac{N_1^7}{M^2} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}
  \lesssim \frac{(L_1L_2L)^{\frac{1}{2}}}{N_1^7} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}.$$  

b. In the case $L_1 = L_{\max}$ we get the bound

$$I(f^{L,N},g_1^{L_1N_1,M,j_1},g_2^{L_2N_2,M,j_2})
\lesssim \frac{(LL_2)^{\frac{1}{2}}}{M^2} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}
\lesssim \frac{(L_1L_2L)^{\frac{1}{2}}}{N_1^7} \|f^{L,N}\|_{L^2} \|g_1^{L_1N_1,M,j_1}\|_{L^2} \|g_2^{L_2N_2,M,j_2}\|_{L^2}.$$  

c. The case $L_2 = L_{\max}$ is similar.

Thus the first sum in (14) can be bounded like the second sum.

Case 1.2. $L_{\max} \gtrsim N_1^7$.

The first sum in (14) is treated exactly as before, whereas the second sum is
estimated using \cite{2} Prop. 4.6] by
\[
\sum_{0 \leq j_1, j_2 \leq A - 1, 16 \leq |j_1 - j_2| \leq 32} I(f^{L,N}, g_1^{L_1, N_1, A, j_1}, g_2^{L_2, N_2, A, j_2}) \lesssim \frac{(L_1 L_2 L)^{\frac{1}{2}} N^{-\frac{1}{2}} N_1^{\frac{1}{2}}}{N^2 N_1^2} \| f^{L,N} \|_{L^2} \| g_1^{L_1, N_1} \|_{L^2} \| g_2^{L_2, N_2} \|_{L^2},
\]
where we used the estimate
\[
\frac{1}{A} \sum_{A} A^{-\frac{1}{2}} \lesssim 1.
\]
Summarizing, we get
\[
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \lesssim (L_1 L_2 L)^{\frac{1}{2}} N^{-\frac{1}{2}} N_1^{\frac{1}{2}} \| f^{L,N} \|_{L^2} \| g_1^{L_1, N_1} \|_{L^2} \| g_2^{L_2, N_2} \|_{L^2},
\]
where we used \( s > -\frac{1}{4} \) and \( \sigma < 2s + \frac{3}{2} \) to get
\[
N_1^{-\frac{1}{2} +} \lesssim N_1^{-1} N_2^{-s} N_2^{-1 - 2s +} \lesssim N_1^{-1} N_2^{-s} N^{-\frac{1}{2} - 2s +} \lesssim N_1^{-1} N_2^{-s} N^{-1 - \sigma -}.
\]

Dyadic summation over \( N_1, N_2, N \) and \( L_1, L_2, L \) gives the claimed estimate. **Case 2.** \( N_1 \ll N_2 \sim N \) (or similarly \( N_2 \ll N_1 \sim N \)).

As in the proof of Prop. 2.2 we get the bound
\[
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \lesssim (L_1 L_2 L)^{\frac{1}{2}} N^{-1 +} \| f^{L,N} \|_{L^2} \| g_1^{L_1, N_1} \|_{L^2} \| g_2^{L_2, N_2} \|_{L^2},
\]
where we used \( \sigma < s + 2 \) to get in the case \( s > 0 \)
\[
N_2^{-1 +} \lesssim N_2^{-s +} \lesssim N_2^{-1 - s} N_1^{-1 - \sigma -}
\]
and \( \sigma < 2s + \frac{3}{2} \) to get in the case \( s \leq 0 \)
\[
N_2^{-1 +} \lesssim N_2^{-2s - 1 +} N_2^{-s} \lesssim N_2^{-s +} N_1^{-1 - s} N_2^{-1 - \sigma -} \lesssim N_2^{-s +} N_1^{-1 - \sigma -} N_2^{-1 - \sigma -},
\]
which is more than enough to get the claimed result after dyadic summation. **Case 3.** \( N \ll 1 \) (\( \Rightarrow N_1 \sim N_2 \) or \( N_1, N_2 \ll 1 \)).

Assuming without loss of generality \( L_1 \leq L_2 \) and using the bilinear Strichartz type estimate \cite{2} Prop. 4.3 we get
\[
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \lesssim \| f^{L,N} \|_{L^2} \| g_1^{L_1, N_1} \|_{L^2} \| g_2^{L_2, N_2} \|_{L^2},
\]
Furthermore we get by \cite{2} formula (4.22)
\[
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \leq \frac{L^2 \| f^{L,N} \|_{L^2} L^2 \| g_1^{L_1, N_1} \|_{L^2} L^2 \| g_2^{L_2, N_2} \|_{L^2},}
\]
so that by interpolation we arrive at

\[ |I(L^{s,N},g_1^{L,N_1},g_2^{L_2,N_2})| \lesssim N_1^{-\frac{1}{2}+\varepsilon} \|f\|_{L^2} \|g_1\|_{L^2} \|g_2\|_{L^2} \lesssim N_1^{-\frac{1}{2}+\varepsilon} \|f\|_{L^2} \|

using \( s > -\frac{1}{2} \). Dyadic summation again gives the claimed result. \( \square \)

**Proof of Theorem 1.1.** It is by now standard to use Proposition 2.2 and Proposition 2.4 to show the local well-posedness result (Theorem 1.1) for the system (4), (5), (6). Let \( (u, n, \partial_t n) \) with \( u \in X^{\sigma,0}t+0,T] \) and data \( u_0 \in H^s, n_0 \in H^s, n_1 \in H^s \) then \( n_\pm \) defined by (3) belongs to \( X^{\sigma,0}t+0,T] \) by Proposition 2.3 and thus \( n = \frac{1}{2}(n_+ + n_-) \) belongs to \( X^{\sigma,0}t+0,T] \) and \( \partial_t n = \frac{1}{2}A^*(n_+ - n_-) \) belongs to \( X^{-1-\frac{1}{2}+}[0,T] \) and one easily checks that \( (u, n_+, n_-) \) is a solution of the system (of integral equations belonging to) (4), (5), (6). But because this solution is uniquely determined the solution of the Klein-Gordon-Schrödinger system (1), (2), (3) with the required properties as explained before Theorem 1.1.

Moreover, if \( (u, n, \partial_t n) \) is a solution of (the system of integral equations belonging to) (1), (1), (1) with \( u \in X^{\sigma,0}t+0,T] \) and data \( u_0 \in H^s, n_0 \in H^s, n_1 \in H^s \), then \( n_\pm \) defined by (3) belongs to \( X^{\sigma,0}t+0,T] \) by Proposition 2.3 and thus \( n = \frac{1}{2}(n_+ + n_-) \) belongs to \( X^{\sigma,0}t+0,T] \) and \( \partial_t n = \frac{1}{2}A^*(n_+ - n_-) \) belongs to \( X^{-1-\frac{1}{2}+}[0,T] \) and one easily checks that \( (u, n_+, n_-) \) is a solution of the system (of integral equations belonging to) (4), (5), (6). But because this solution is uniquely determined the solution of the Klein-Gordon-Schrödinger system is also unique.

For the part concerning unconditional uniqueness we use an idea of Y. Zhou [11, 12], which we already applied in [10, Prop. 3.1]. Let

\( (u, n, \partial_t n) \in C^0([0,T], L^2(\mathbb{R}^2)) \times C^0([0,T], L^2(\mathbb{R}^2)) \times C^0([0,T], H^{-1}(\mathbb{R}^2)) \)

be any solution of the Klein-Gordon-Schrödinger system (1), (2), (3). This leads to a corresponding solution of the system (4), (5), (6) with

\( (u, n_+, n_-) \in C^0([0,T], L^2(\mathbb{R}^2)) \times C^0([0,T], L^2(\mathbb{R}^2)) \times C^0([0,T], L^2(\mathbb{R}^2)) \).

By Sobolev’s embedding theorem we get

\[ \|n_\pm u\|_{L^2([0,T], H^{-1+})} \lesssim \|n_\pm u\|_{L^2([0,T], L^2)} \lesssim T^{\frac{1}{2}} \|n_\pm\|_{L^\infty([0,T], L^2)} \|u\|_{L^\infty([0,T], L^2)} < \infty. \]

so that from (4) we have \( u \in X^{-1-\frac{1}{2}+}[0,T] \), because

\[ \|i\partial_t + \Delta\|_{L^2([0,T], H^{-1+})} + \|u\|_{L^2([0,T], H^{-1+})} \lesssim \|u\|_{X^{-1-\frac{1}{2}+}[0,T]} < \infty. \]

Interpolation with \( u \in X^{s,0}t+0,T] \) gives \( u \in X^{-1-\frac{1}{2}+}[0,T] \). Similarly we get

\[ \|u\|^2_{L^2([0,T], H^{-1+})} \lesssim T^{\frac{1}{2}} \|u\|_{L^\infty([0,T], L^2)}^2 < \infty \]

and from (5) we conclude \( n_+ \in X^{0,0}t+0,T] \). Proposition 2.3 shows that \( un_\pm \in X^{-1-\frac{1}{2}+}[0,T] \), thus \( u \in X^{-\frac{1}{2}+}[0,T] \) and \( n_\pm \in X^{0,0}t+0,T] \) for any \( \epsilon > 0 \). But in these spaces uniqueness holds by the first part of this proof, so that unconditional uniqueness is also proven. \( \square \)

3. **Global well-posedness results for the case \( D = 2 \)**

We first show a modified local well-posedness result in arbitrary space dimension \( D \).
Proposition 3.1. Let \( u_0 \in L^2(\mathbb{R}^D) \), \( n_0 \in H^\sigma(\mathbb{R}^D) \), \( n_1 \in H^{\sigma-1}(\mathbb{R}^D) \) and \( T \leq 1 \).

Assume

\[
\|un\|_{X^{0,-\frac{1}{2}}} \lesssim T \|u\|_{X^{0,\frac{1}{2}}} \|n\|_{X^{\frac{\sigma}{2},\frac{1}{2}}} \tag{15}
\]

and

\[
\\|u^2\|_{X^{\sigma-1,-\frac{1}{2}}} \lesssim T^k \|u\|^2_{X^{0,\frac{1}{2}}},
\]

where \( k, l > 0 \).

Then there exists \( 1 \geq T > 0 \) such that the system of integral equations [7], [8] has a unique solution \( u \in X^{0,\frac{1}{2}+}[0, T] \), \( n_\pm \in X^{\sigma,\frac{1}{2}+}[0, T] \).

\( n_\pm \) fulfills for \( 0 \leq t \leq T \):

\[
\|n_\pm(t)\|_{H^\sigma} \leq \|n_{\pm 0}\|_{H^\sigma} + cT^k\|u_0\|^2_{L^2}. \tag{17}
\]

T can be chosen such that

\[
T^l(\|n_{+ 0}\|_{H^\sigma} + \|n_{- 0}\|_{H^\sigma}) \lesssim 1 \tag{18}
\]

\[
T^q\|u_0\|_{L^2} \lesssim 1 \tag{19}
\]

\[
T^k\|u_0\|_{L^2} \lesssim 1 \tag{20}
\]

\[
T^k\|u_0\|^2_{L^2} \lesssim \|n_{+ 0}\|_{H^\sigma} + \|n_{- 0}\|_{H^\sigma}. \tag{21}
\]

Remark: No implicit constant appears on the right hand side of (17).

Proof. We construct a fixed point of \( S = (S_0, S_+, S_-) \) in

\[
M : = \{ u \in X^{0,\frac{1}{2}+}[0, T], n_\pm \in X^{\sigma,\frac{1}{2}+}[0, T] : \n\|u\|_{X^{0,\frac{1}{2}+}} \lesssim \|u_0\|_{L^2}, \|n_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_-\|_{X^{\sigma,\frac{1}{2}+}} \lesssim \|n_{+ 0}\|_{H^\sigma} + \|n_{- 0}\|_{H^\sigma}, \}
\]

where \( S_0u \) and \( S_\pm n_\pm \) denote the right hand sides of our integral equations (7) and (8). Then we get for \( u, n_\pm \in M \):

\[
\|S_0u\|_{X^{0,\frac{1}{2}+}} \lesssim \|u_0\|_{L^2} + T^l\|u\|_{X^{0,\frac{1}{2}+}}(\|n_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_-\|_{X^{\sigma,\frac{1}{2}+}}) \lesssim \|u_0\|_{L^2} + T^l\|u\|^2_{X^{0,\frac{1}{2}+}} \lesssim \|u_0\|_{L^2}
\]

by (18), and

\[
\|S_+ n_+\|_{X^{\sigma,\frac{1}{2}+}} + \|S_- n_-\|_{X^{\sigma,\frac{1}{2}+}} \lesssim \|n_+\|_{H^\sigma} + \|n_-\|_{H^\sigma} + T^k\|u\|^2_{X^{0,\frac{1}{2}+}} \lesssim \|n_+\|_{H^\sigma} + \|n_-\|_{H^\sigma} + T^k\|u_0\|^2_{L^2} \lesssim \|n_+\|_{H^\sigma} + \|n_-\|_{H^\sigma}
\]

by (21), such that \( S : M \rightarrow M \).

In order to show the contraction property we estimate as follows. For \( u, n_\pm \), \( (\tilde{u}, \tilde{n}_\pm) \in M \) we get

\[
\|S_0u - S_0\tilde{u}\|_{X^{0,\frac{1}{2}+}} \lesssim T^l(\|u - \tilde{u}\|_{X^{0,\frac{1}{2}+}}(\|n_+\|_{X^{\sigma,\frac{1}{2}+}} \|\tilde{n}_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_-\|_{X^{\sigma,\frac{1}{2}+}} \|\tilde{n}_-\|_{X^{\sigma,\frac{1}{2}+}}) + (\|u\|_{X^{0,\frac{1}{2}+}} + \|\tilde{u}\|_{X^{0,\frac{1}{2}+}}(\|n_+ - \tilde{n}_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_- - \tilde{n}_-\|_{X^{\sigma,\frac{1}{2}+}}) \lesssim T^l(\|u - \tilde{u}\|_{X^{0,\frac{1}{2}+}}(\|n_+\|_{H^\sigma} + \|n_-\|_{H^\sigma} + \|u_0\|_{L^2}(\|n_+ - \tilde{n}_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_- - \tilde{n}_-\|_{X^{\sigma,\frac{1}{2}+}}) \lesssim \frac{1}{2}(\|u - \tilde{u}\|_{X^{0,\frac{1}{2}+}} + \|n_+ - \tilde{n}_+\|_{X^{\sigma,\frac{1}{2}+}} + \|n_- - \tilde{n}_-\|_{X^{\sigma,\frac{1}{2}+}})\]

by (13) and (14). Similarly
\[
\|S_+ n_+ - S_+ \bar{n}_+\|_{X^0_{\frac14} +} + \|S_- n_- - S_- \bar{n}_-\|_{X^0_{\frac14} +} \\
\lesssim T^k \|u\|_{X^0_{\frac14} +} + \|\bar{u}\|_{X^0_{\frac14} +} \|u - \bar{u}\|_{X^0_{\frac14} +} \\
\lesssim T^k \|u_0\|_{L^2} \|u - \bar{u}\|_{X^0_{\frac14} +} \\
\leq \frac{1}{2} \|u - \bar{u}\|_{X^0_{\frac14} +}
\]
by (21). Thus the contraction mapping principle gives a unique solution in $[0, T]$. This solution fulfills $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Moreover we get from the integral equations (5) for $0 \leq t \leq T$:
\[
\|n_\pm(t)\|_{H^s} \leq \|n_{\pm 0}\|_{H^s} + cT^k \|u\|_{L^2}^2 \lesssim \|n_{\pm 0}\|_{H^s} + cT^k \|u_0\|_{L^2}^2,
\]
using that $e^{\mp i t A^\frac12}$ is unitary.

This version of the local well-posedness result will now be used to show the global well-posedness result Theorem 1.3. We first show

**Proposition 3.2.** In space dimension $D = 2$ assume $-\frac{1}{2} \leq \sigma < \frac{3}{2}$ and $T \leq 1$. Then the estimates (15) and (16) hold with $1 > k, l \geq \frac{1}{2} - \sigma$ and $k + l \geq \frac{1}{2} - \sigma$.

**Proof.** We estimate $I$ (defined by (13)) by H"older’s inequality and Sobolev’s embedding:
\[
|I(\bar{n}, \bar{u}_1, \bar{u}_2)| \lesssim \|n\|_{L^1 L^\infty} \|u_1\|_{L^2 L^2} \|u_2\|_{L^2 L^2} \lesssim \|n\|_{X^\frac{1}{4} +} \|u_1\|_{X^0_{\frac14}} \|u_2\|_{X^0_{\frac14}}.
\]
Thus
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim \|u\|_{X^0_{\frac14}} \|n\|_{X^\frac{1}{4} +}.
\]
This implies by Lemma 1.1
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim T^\frac{\theta}{2} \|u\|_{X^\frac{1}{4}} \|n\|_{X^\frac{1}{4} +} \lesssim T^{1 - \frac{\theta}{4}} \|u\|_{X^0_{-\frac{1}{4} +}} \|n\|_{X^\frac{1}{4} +}.
\]
Moreover by Proposition 2.1
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim T^\frac{\theta}{2} \|u\|_{X^0_{\frac{1}{4}}} \|n\|_{X^\frac{1}{4} +} \lesssim T^{\frac{1}{2} - \theta} \|u\|_{X^0_{\frac{1}{4}}} \|n\|_{X^\frac{1}{4} +}.
\]
Interpolation gives for $0 \leq \theta \leq 1$
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim T^{1 - \frac{\theta}{4}} \|u\|_{X^0_{\frac{1}{4}}} \|n||_{X^{\frac{1}{4} - \frac{\theta}{4}}}.
\]
By duality we also get
\[
\|u\|_{X^0_{\frac{1}{4}} - \frac{\theta}{4}} \lesssim T^{1 - \frac{\theta}{4}} \|u\|_{X^0_{\frac{1}{4}}}.
\]
- If $-\frac{1}{2} \leq \sigma \leq 0$ we choose $\theta = \frac{2}{3} (1 - \sigma)$ and get
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim T^\frac{\theta}{2} \|u\|_{X^0_{\frac{1}{4}}} \|n\|_{X^\frac{1}{4} +}.
\]
- If $0 \leq \sigma < \frac{1}{2}$ we choose $\theta = 0$ and get
\[
\|un\|_{X^0_{-\frac{1}{4} +}} \lesssim T^{1 - \frac{\theta}{4}} \|u\|_{X^0_{\frac{1}{4}}} \|n||_{X^{\frac{1}{4} - \frac{\theta}{4}}}.
\]
• If \(-\frac{1}{2} \leq \sigma \leq 0\) we choose \(\theta = 0\) and get
\[
i[|u|^2]_{X^{s,-\frac{1}{2}}} \lesssim |u|^2_{X^{s,-\frac{1}{2}}} \lesssim T^{-1} u^2_{X^{s,\frac{3}{4}}}.
\]
• If \(0 \leq \sigma < \frac{3}{2}\) we choose \(\theta = \frac{3}{2} \sigma\) and get
\[
i[|u|^2]_{X^{s,-\frac{1}{2}}} \lesssim T^{-\frac{3}{2}} u^2_{X^{s,\frac{3}{4}}}.
\]
Thus we conclude that (15), (16) hold:

\[
\text{if } -\frac{1}{2} \leq \sigma \leq 0 \text{ with } k = \frac{1}{2} + \frac{2}{7}, l = 1 - \Rightarrow k + l = \frac{3}{2} + \frac{5}{7} \geq \frac{5}{4},
\]
\[
\text{if } 0 \leq \sigma \leq 1 \text{ with } k = \frac{1}{2} + \frac{2}{7}, l = 1 - \Rightarrow k + l = \frac{3}{2},
\]
\[
\text{if } 1 \leq \sigma < \frac{3}{2} \text{ with } k = 1 - , l = 1 - \Rightarrow k + l = 2 - \frac{5}{4} + > \frac{5}{4}.
\]

Proof of Theorem 1.3. By persistence of higher regularity it suffices to consider the case \(s = 0\) and \(-\frac{1}{2} \leq \sigma < \frac{3}{2}\). We first use our local well-posedness result Theorem 4.1 which gives under our assumptions a local solution. Because \(\|u(t)\|_{L^2}\) is conserved this solution exists as long as \(\|n_+(t)\|_{H^s} + \|n_-(t)\|_{H^s}\) remains bounded. If this is the case for any \(t > 0\) we are done. Otherwise we can suppose that at some time \(t\) we have
\[
n_+(t)\|_{H^s} + \|n_-(t)\|_{H^s} > \|u(t)\|^2_{L^2} + 1 = \|u_0\|^2_{L^2} + 1.
\]
Take this time as initial time \(t = 0\) so that
\[
n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s} > \|u_0\|^2_{L^2} + 1.
\]
We want to apply now our modified local well-posedness result Proposition 3.1. The estimates (15), (16) are fulfilled by Proposition 5.2 with \(k + l > \frac{3}{4} > 1\) and \(1 > k, l \geq \frac{3}{4}\). Estimate (21) is also fulfilled. We now choose \(T\) such that (18) and (20) are satisfied, namely
\[
T \sim \min \left(\frac{1}{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}, \frac{1}{\|u_0\|_{L^2}^2}, 1\right).
\]
Then (19) is automatically satisfied, because (18) holds and \(\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s} \gtrsim \|u_0\|_{L^2}\). Using (17) we see that it is possible to use this local existence theorem \(m\) times with intervals of length \(T\), before \(\|n_+(t)\|_{H^s} + \|n_-(t)\|_{H^s}\) at most doubles. Here we have
\[
m \sim \frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{T^{k}\|u_0\|^2_{L^2}}.
\]
After \(m\) iterations we arrive at the time
\[
mT \sim \frac{T^{1-k}([\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s})}}{\|u_0\|^2_{L^2}}\min(\frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s} + \|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}})}\frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{\|u_0\|^2_{L^2}}
\]
\[
\sim \min(\frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{\|u_0\|^2_{L^2}}) \frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{\|u_0\|^2_{L^2}}\frac{\|n_+(0)\|_{H^s} + \|n_-(0)\|_{H^s}}{\|u_0\|^2_{L^2}}
\]
\[
\gtrsim \min(\frac{1}{\|u_0\|_{L^2}^2}, 1)
\]
using \( k + l > 1 \), \( k \geq \frac{1}{4} \), and \( ||n_{\pm 0}||_{H^s} + ||n_{0}||_{H^s} \gg ||u_0||_{L^2}^2 \). This is independent of \( ||n_{\pm 0}||_{H^s} + ||n_{0}||_{H^s} \). Using conservation of \( ||u(t)||_{L^2} \) again it is thus possible to repeat the whole procedure with time steps of equal length. This proves the global existence result.

In the range \(-\frac{1}{2} \leq \sigma < \frac{1}{4} \) we can give a much easier proof using Strichartz’ estimates for the Klein-Gordon equation as follows. In order to estimate the wave part we get from the integral equation (8):

\[
||n_{\pm}||_{L^\infty(0,T),H^{s}} \lesssim ||n_{\pm 0}||_{H^{s}} + ||u||_{L^\frac{4}{3}-(0,T),H^{s+r,3+s}}^2
\]

where we defined \( \tilde{q} = 4+ \), \( \tilde{r} = \infty - \) such that \( \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2} \), and moreover \( 0 = 1 + \rho - 2(\frac{2}{\tilde{q}} - \frac{1}{\tilde{r}}) + \frac{\rho}{\tilde{r}} \iff \rho = -\frac{1}{2} + \), so that \( \sigma + \rho < 0 \). Thus by Sobolev’s embedding and conservation of \( ||u(t)||_{L^2} \):

\[
||n_{\pm}||_{L^\infty(0,T),H^{s}} \lesssim ||n_{\pm 0}||_{H^{s}} + ||u||_{L^\frac{4}{3}-(0,T),L^1}^2 \lesssim ||n_{\pm 0}||_{H^{s}} + T^{\frac{2}{3}-}||u_0||_{L^2}^2
\]

which implies global existence.

\[\Box\]

**Proof of Theorem 1.2** for \( D = 2 \). Using persistence of regularity it suffices to consider the case \( s = 0 \), \( 0 \leq \sigma \leq 1 \). Let \( T \leq 1 \) and \( \frac{1}{q} + \frac{1}{r} = \frac{1}{2} \), \( r < \infty \). Using the notation from the proof of Proposition 3.1 we get by Strichartz’ estimates for the Schrödinger equation:

\[
||S_0 u||_{L^\infty((0,T),L^2) \cap L^4((0,T),L^r)} \lesssim ||u_0||_{L^2} + ||n u||_{L^4((0,T),L^r)} \lesssim ||u_0||_{L^2} + ||n||_{L^2((0,T),L^r)} ||u||_{L^\infty((0,T),L^\infty)}.
\]

Here \( \frac{1}{r_1} := \frac{1}{r} - \epsilon \) and \( \frac{1}{r_0} = \epsilon \) for a sufficiently small \( \epsilon > 0 \).

**a.** \( 0 \leq \sigma < 1 \).

Here \( \frac{1}{q} = \frac{1}{2} + \frac{\sigma}{q} + \epsilon \), \( \frac{1}{r} = \frac{2}{q} - \frac{\sigma}{q} - \epsilon \), \( \frac{1}{r_1} = 1 - \frac{\sigma}{q} - \epsilon \), \( \frac{1}{r_2} = \frac{q}{q} + \epsilon \), so that \( \frac{1}{q} + \frac{1}{r} = \frac{1}{q} \).

Choose \( \frac{1}{q} = \frac{1}{2} - \frac{\sigma}{q} + \epsilon \), \( \frac{1}{r} = \frac{1}{2} + \frac{q}{q} \), so that \( H^s_x \subset L^q_x \) and \( \frac{1}{q} = \frac{1}{2} + \frac{q}{q} \). Then \( \frac{1}{q} = \frac{1}{2} + \frac{q}{q} \) and \( \frac{1}{q} = \frac{1}{2} + \frac{q}{q} \). Thus we get the estimate

\[
||S_0 u||_{L^\infty((0,T),L^2) \cap L^4((0,T),L^r)} \lesssim ||u_0||_{L^2} + T^{\frac{1}{2}+\frac{\sigma}{2}} ||n||_{L^\infty((0,T),H^s)} ||u||_{L^\infty((0,T),L^\infty)}.
\]

because \( T \leq 1 \).

**b.** \( \sigma = 1 \).

We choose \( \frac{1}{q} = \epsilon \), \( \frac{1}{r_1} = \frac{1}{2} \), \( \frac{1}{r} = 0 \), \( \frac{1}{r_2} = 1 - \epsilon \), so that again \( \frac{1}{q} + \frac{1}{r} = \frac{1}{2} \), \( \frac{1}{r_1} = \frac{1}{2} + \frac{r_1}{r_1} \), \( \frac{1}{r_2} = \frac{1}{2} + \frac{r_1}{r_1} \) and \( H^\sigma \subset L^r_x \). Thus we get the estimate

\[
||S_0 u||_{L^\infty((0,T),L^2) \cap L^4((0,T),L^r)} \lesssim ||u_0||_{L^2} + T^{\frac{1}{2}-\epsilon} ||n||_{L^\infty((0,T),H^s)} ||u||_{L^\infty((0,T),L^\infty)}.
\]

Moreover

\[
||S_{\pm} n_{\pm}||_{L^\infty((0,T),H^s)} \leq ||n_{\pm 0}||_{H^s} + ||u||^2_{L^1((0,T),H^{s-1})} + ||n_{\pm 0}||_{H^s} + ||u||_{L^2((0,T),L^{2q})} + ||n_{\pm 0}||_{H^s} + ||u||_{L^3((0,T),L^q)} + ||n_{\pm 0}||_{H^s} + ||u||_{L^3((0,T),L^q)}.
\]

where \( \frac{1}{q} = \frac{1}{2} - \frac{q}{q} \), so that \( L^q_x \subset H^{s-1} \), and \( \frac{1}{r_1} = \frac{1}{2} + \frac{q}{q} \), \( \frac{1}{r_2} = \frac{q}{q} \), so that \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2} \). Giving similar estimates for the differences \( S_0 u - S_0 \tilde{u} \) and \( S_{\pm} n_{\pm} - S_{\pm} \tilde{n}_{\pm} \)
choosing $T$ subject to the conditions

$$T^\frac{1}{2} (\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s}) \lesssim 1 \quad (22)$$

$$T^\frac{1}{2} \|u_0\|_{L^2} \lesssim 1 \quad (23)$$

$$T^\frac{1}{2} \|u_0\|_{L^2}^2 \lesssim \|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s} \quad (24)$$

then Banach’s fixed point theorem shows that there exists a unique solution of our system of integral equations \((7), (8)\) on \([0, T]\) such that

$$\|u\|_{L^\infty((0, T), L^2) \cap L^\infty((0, T), L^r)} \lesssim \|u_0\|_{L^2}$$

and

$$\|n_{\pm}\|_{L^\infty((0, T), H^s)} \leq \|n_{\pm}\|_{H^s} + c T^\frac{1}{2} \|u_0\|_{L^2}^2. \quad (25)$$

Using conservation of mass we have $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, and thus get a global solution unless we have after a number of iterations

$$\|n_{+}(t)\|_{H^s} + \|n_{-}(t)\|_{H^s} \gg \|u_0\|_{L^2}^2 + 1,$$

which we thus may suppose. Take this time as initial time $t = 0$ so that

$$\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s} \gg \|u_0\|_{L^2}^2 + 1.$$

Then (24) is automatically satisfied. Using (22) we choose

$$T^\frac{1}{2} \sim \frac{1}{\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s}}. \quad (26)$$

Then (23) is also satisfied. By (25) we see that after $m$ iterations of size (26) the quantity $\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s}$ at most doubles, where

$$m \sim \frac{\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s}}{T^\frac{1}{2} \|u_0\|_{L^2}^2}.$$

The total time after $m$ iterations is

$$mT \sim T^\frac{1}{2} \frac{\|n_{+0}\|_{H^s} + \|n_{-0}\|_{H^s}}{\|u_0\|_{L^2}^2} \sim \frac{1}{\|u_0\|_{L^2}^2},$$

by (26), which is independent of $\|n_{\pm0}\|_{H^s}$. We can now repeat the whole procedure with time steps of equal length, thus leading to a global solution.

\[\square\]

4. Global well-posedness results for the case $D = 3$

We generalize the argument of Colliander-Holmer-Tzirakis \([5]\) for data $u_0 \in H^s$, $u_0 \in H^s$, $n_1 \in H^{s-1}$ from the case $\sigma = s \geq 0$ to the region $s \geq 0$, $s - \frac{1}{2} \leq \sigma \leq s + 1$.

**Proof of Theorem \([1, 4]\) for $D = 3$.** Using persistence of regularity it again suffices to consider the case $s = 0$, $0 \leq \sigma \leq 1$. Let $T \leq 1$ and $\frac{2}{q} + \frac{2}{r} = \frac{3}{2}$. Similarly as in the 2D case we estimate

$$\|S_0 u\|_{L^\infty((0, T), L^2) \cap L^\infty((0, T), L^r)} \lesssim \|u_0\|_{L^2} + \|nu\|_{L^r(0, T), L^r)}$$

$$\lesssim \|u_0\|_{L^2} + \|n\|_{L^6(0, T), L^r)} \|u\|_{L^4(0, T), L^3)}$$

$$\lesssim \|u_0\|_{L^2} + T^\frac{1}{2} \|n\|_{L^\infty((0, T), H^s)} \|u\|_{L^4(0, T), L^3)}.$$ 

Here $\frac{1}{q} = \frac{1}{p} + \frac{2}{r}$, $\frac{1}{r} = \frac{5}{6} - \frac{2}{r}$, $\frac{1}{r} = \frac{1}{4} - \frac{2}{r}$, $\frac{1}{r} = \frac{1}{6} + \frac{2}{r}$, and $H^{\sigma} \subset L^r$, so that $\frac{3}{2} = \frac{1}{r} + \frac{1}{r}$, thus Strichartz’ estimate applies. Furthermore we
define $\frac{1}{q} = \frac{1}{p} + \frac{\sigma}{T}$ so that $\frac{1}{r} = \frac{1}{p} + \frac{1}{T}$, and also $\frac{1}{p} = \frac{1}{q} + \frac{\sigma}{T}$ so that Hölder’s estimate applies. Moreover
\[
\| S_{\pm} n_{\pm} \|_{L^\infty((0,T),H^{s})} \leq \| n_{\pm 0} \|_{H^{s}} + c \| u \|^2_{L^2((0,T),H^{s-1})}
\leq \| n_{\pm 0} \|_{H^{s}} + c \| u \|^2_{L^2((0,T),L^{p})}
\leq \| n_{\pm 0} \|_{H^{s}} + c T^{\frac{1}{2} - \frac{\sigma}{T}} \| u \|^2_{L^2(0,T),L^{p}},
\]
where $\frac{1}{p} = \frac{2}{q} - \frac{\sigma}{T}$, so that $L^{p}_{x} \subset H^{\sigma-1}_{x}$, and $\frac{1}{p} = \frac{2}{q} - \frac{\sigma}{T}$, so that $\frac{2}{q} + \frac{\sigma}{T} = \frac{1}{4} + \frac{\sigma}{T} = \frac{1}{2}$. Giving similar estimates for the differences $S_{0}u - \bar{S}_{0}u$ and $\bar{S}_{\pm} n_{\pm} - S_{\pm} n_{\pm}$ and choosing $T$ subject to the conditions
\[
T^{\frac{1}{2} - \frac{\sigma}{T}} (\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}) \lesssim 1
\]
(27)
\[
T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}} \lesssim 1
\]
(28)
\[
T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}} \lesssim 1
\]
(29)
\[
T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}} \lesssim \| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}
\]
(30)
then Banach’s fixed point theorem shows that there exists a unique solution of our system of integral equations \([7, 5]\) on \([0, T]\) such that
\[
\| u \|_{L^\infty((0,T),L^{p})} \lesssim \| u_{0} \|_{L^{2}}
\]
and
\[
\| n_{\pm 0} \|_{L^\infty((0,T),H^{s})} \leq \| n_{\pm 0} \|_{H^{s}} + c T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}}^{3}.
\]
(31)
Using conservation of mass we have $\| u(t) \|_{L^{2}} = \| u_{0} \|_{L^{2}}$, and thus get a global solution unless we have after a number of iterations
\[
\| n_{+}(t) \|_{H^{s}} + \| n_{-}(t) \|_{H^{s}} \gg \| u_{0} \|_{L^{2}}^{3} + 1,
\]
which we thus may suppose. Take this time as initial time $t = 0$ so that
\[
\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}} \gg \| u_{0} \|_{L^{2}}^{3} + 1.
\]
Then (30) is automatically satisfied. Using (27) we choose
\[
T^{\frac{1}{2} - \frac{\sigma}{T}} \sim \frac{1}{\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}}
\]
(32)
Then (28) is also satisfied, because $\| u_{0} \|_{L^{2}} \ll \| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}$ and 
\[
(T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}})^{3} \leq (T^{\frac{1}{2}} \| u_{0} \|_{L^{2}})^{3} \ll T^{\frac{1}{2}} (\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}) \sim T^{\frac{1}{2} - (\frac{1}{2} + \frac{\sigma}{T})} \lesssim 1,
\]
so that (29) is also satisfied. By (31) we see that after $m$ iterations of size (32) the quantity $\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}$ at most doubles, where
\[
m \sim \frac{\| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}}}{T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}}^{2}}.
\]
The total time after $m$ iterations
\[
m T \sim T^{\frac{1}{2} - \frac{\sigma}{T}} \| u_{0} \|_{L^{2}}^{2} \| n_{+0} \|_{H^{s}} + \| n_{-0} \|_{H^{s}} \sim \frac{1}{\| u_{0} \|_{L^{2}}^{2}},
\]
by (32), which is independent of $\| n_{\pm 0} \|_{H^{s}}$. We can now repeat the whole procedure with time steps of equal length, thus leading to a global solution. \(\square\)

**Proof of Theorem 1.5.** Using persistence of regularity it suffices to consider the case $s = 0$, $\frac{1}{2} < \sigma < 0$. Using the local wellposedness theorem [10, Theorem 1.1] and conservation of mass we only have to give a bound for $\| n(t) \|_{H^{s}} + \| \partial n(t) \|_{H^{s-1}}$. We use Strichartz’ estimate for the Klein-Gordon equation and get
\[
\| n \|_{L^\infty((0,T),H^{s})} \lesssim \| n_{0} \|_{H^{s}} + \| n_{1} \|_{H^{s-1}} + \| u \|^2_{L^2((0,T),H^{s+r},r')}.
\]
where \( \tilde{r} = \infty - \), \( \tilde{q} = 2+ \), so that \( \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2} \). Moreover \( 0 = 1 + \rho - 3 \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) + \frac{1}{\tilde{q}} \)
\( \iff \rho = 0 \). Thus Strichartz’ estimate applies. By Sobolev’s embedding we have \( L^1(\mathbb{R}^3) \subset H^{\rho+\frac{1}{\tilde{q}}}(\mathbb{R}^3) = H^{\rho-1+}(\mathbb{R}^3) \). Thus we arrive at
\[
\|n\|_{L^\infty((0,T),H^\sigma)} \lesssim \|n_0\|_{H^\sigma} + \|n_1\|_{H^{\sigma-1}} + \|u\|^2_{L^4((0,T),L^2)}
\lesssim \|n_0\|_{H^\sigma} + \|n_1\|_{H^{\sigma-1}} + T \frac{1}{2} \|u_0\|^2_{L^2}.
\]
Similarly \( \|\partial_t n\|_{L^\infty((0,T),H^{\sigma-1})} \) can be estimated. \( \square \)

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