On the nonexistence of stationary weak solutions to the compressible fluid equations

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Abstract

In this paper we prove that under some integrability conditions for the density and the velocity fields the only stationary weak solutions to the compressible fluid equations on \( \mathbb{R}^N \) correspond to the zero density. In the case of compressible magnetohydrodynamics equations similar integrability conditions for density, velocity and the magnetic fields lead to the zero density and the zero magnetic field.

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1 Introduction

1.1 The compressible Navier-Stokes(Euler) equations

We are concerned here on the compressible Navier-Stokes(Euler for \( \mu = \lambda = 0 \)) equations on \( \mathbb{R}^N \), \( N \geq 1 \).

\[
(\text{NS}) \begin{cases}
\partial_t \rho + \text{div}(\rho v) = 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) = -\nabla p + \mu \Delta v + (\mu + \lambda) \nabla \text{div} v + f,
\end{cases}
\]

\( p = p(\rho, S) \geq 0, \quad p = 0 \) only if \( \rho = 0 \).
The system (NS) describes compressible gas flows with the adiabatic exponent $\gamma$, and $\rho, v, S, p$ and $f$ denote the density, velocity, specific entropy, pressure and the external force respectively. Since the results below does not depend on the particular form of the entropy equation, nor the specific form of $p(\rho, S)$, we omit specifications of them. We treat the viscous case $\mu > 0$ (compressible Navier-Stokes equations) and the inviscid case $\mu = \lambda = 0$ (compressible Euler equations) simultaneously. For surveys of the known mathematical theories of the equations we refer to [3, 4, 5, 6, 7]. Our aim here is to prove nonexistence of nontrivial stationary weak solutions to the system (NS) under suitable integrability conditions. In the inviscid case and viscous case with $2\mu + \lambda = 0$ our integrability condition covers the finite energy condition, while for the viscous case with $2\mu + \lambda \neq 0$ we need extra integrability for velocity $v \in L^{\infty}((\mathbb{R}^N))$ besides the finite energy condition. These could be regarded as Liouville theorem for the stationary compressible fluid equations. The Liouville type of theorems for the nonstationary incompressible Euler equations are recently studied by the author of this paper in [1, 2], where we need to impose extra condition for the sign of the integral of pressure as well as the integrability conditions for the velocity. In the case of compressible fluid equations, however, we do not need such extra sign condition for pressure integral, since the sign of pressure is automatically nonnegative pointwise. Similar nonexistence results hold for the compressible magnetohydrodynamic equations for $N \geq 2$, which will be treated in the next section. A stationary weak solutions of (NS) are defined as follows.

**Definition 1.1** We say that a triple $(v, \rho, S) \in [L^2_{\text{loc}}(\mathbb{R}^N)]^N \times L^\infty_{\text{loc}}(\mathbb{R}^N) \times L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a stationary weak solution of (NS) if

$$\int_{\mathbb{R}^N} \rho v \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in C^\infty_0(\mathbb{R}^N), \quad (1.1)$$

$$\int_{\mathbb{R}^N} \rho v \otimes v : \nabla \phi \, dx = - \int_{\mathbb{R}^N} p \, \text{div} \phi \, dx - \mu \int_{\mathbb{R}^N} v \cdot \Delta \phi \, dx$$

$$- (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \text{div} \phi \, dx - \int_{\mathbb{R}^N} f \cdot \phi \, dx \quad \forall \phi \in [C^\infty_0(\mathbb{R}^N)]^N,$$

$$p = p(\rho, S) \geq 0, \quad p = 0 \text{ only if } \rho = 0. \quad (1.2)$$

The following is our main nonexistence theorem for (NS).

**Theorem 1.1** Let $N \geq 1$, and let the external force $f \in [L^1_{\text{loc}}(\mathbb{R}^N)]^N$ satisfy $\text{div} f = 0$ in the sense of distribution. Suppose $(\rho, v, S)$ is a stationary weak
solution to (NS) satisfying one of the following conditions depending on $\mu$ and $\lambda$.

(i) In the inviscid case ($\mu = \lambda = 0$); there exists $w \in L^1_{loc}([0, \infty))$, which is positive almost everywhere on $[0, \infty)$ such that

$$\int_{\mathbb{R}^N} \frac{(\rho|v|^2 + p)}{1 + |x|} \, dx < \infty.$$

(ii) In the viscous case ($\mu > 0$);

(a) if $2\mu + \lambda = 0$,

$$\int_{\mathbb{R}^N} (\rho|v|^2 + p) \, dx < \infty.$$  \hfill (1.5)

(b) if $2\mu + \lambda \neq 0$,

$$\int_{\mathbb{R}^N} (\rho|v|^2 + |v|^\frac{N}{N-1} + p) \, dx < \infty.$$  \hfill (1.6)

Then, $\rho(x) = 0$ for almost every $x \in \mathbb{R}^N$.

Remark 1.1 We note that in the special case of $N \geq 3$, $\mu > 0, \mu + \lambda > 0$, $f = \rho \nabla \Phi$, where the connected component of $\{\Phi(x) > -c\}$ is unbounded, P.L. Lions showed nonexistence of stationary solutions under appropriate integrability condition for $\rho, v$ (see Section 6.7 of [6]). Even when $\mu > 0$, $\mu + \lambda > 0$ and $N \geq 3$, the above theorem does not have mutual implication relation with this.

1.2 The compressible MHD equations

Here we are concerned on the compressible magnetohydrodynamic equations on $\mathbb{R}^N$,

(MHD) \[
\begin{cases}
\partial_t \rho + \text{div}(\rho v) = 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v - H \otimes H) = -\nabla (p + \frac{1}{2}|H|^2) + \mu \Delta v + (\mu + \lambda) \nabla \text{div} v + f, \\
\partial_t H - \text{curl}(v \times H) = 0, \\
\text{div} H = 0, \\
p = p(\rho, S) \geq 0, \quad p = 0 \text{ only if } \rho = 0.
\end{cases}
\]
The system (MHD) describes compressible charged gas flows (plasma gas) with the adiabatic exponent $\gamma$, and $\rho, v, H, S, p$ and $f$ denote the density, velocity, magnetic field, specific entropy, pressure and the external force respectively. A stationary weak solution of (MHD) is defined as follows.

**Definition 1.2** We say that a triple $(\rho, v, H, S) \in L_\infty^\infty(\mathbb{R}^N) \times [L_\text{loc}^2(\mathbb{R}^N)]^N \times W^{1,2}_\text{loc}(\mathbb{R}^N)$ is a stationary weak solution of (MHD) if

\[
\int_{\mathbb{R}^N} \rho v \cdot \nabla \psi(x) \, dx = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^N),
\]

\[
\int_{\mathbb{R}^N} (\rho v \otimes v - H \otimes H) : \nabla \phi(x) \, dx = -\int_{\mathbb{R}^N} (p + \frac{1}{2}|H|^2) \text{div} \phi(x) \, dx - \mu \int_{\mathbb{R}^N} v \cdot \Delta \phi \, dx
\]

\[
-(\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \phi \, dx - \int_{\mathbb{R}^N} f \cdot \phi \, dx \quad \forall \phi \in [C_0^\infty(\mathbb{R}^N)]^N,
\]

\[
\int_{\mathbb{R}^N} (v \times H) \cdot \nabla \phi(x) \, dx = 0 \quad \forall \phi \in [C_0^\infty(\mathbb{R}^N)]^N,
\]

\[
\int_{\mathbb{R}^N} H \cdot \nabla \eta(x) \, dx = 0 \quad \forall \eta \in C_0^\infty(\mathbb{R}^N)
\]

\[
p = p(\rho, S) \geq 0, \quad p = 0 \text{ only if } \rho = 0.
\]

Similarly to Theorem 1.1 we have the following theorem for (MHD).

**Theorem 1.2** Let the external force $f \in [L_\text{loc}^1(\mathbb{R}^N)]^N$ satisfy $\text{div} f = 0$ in the sense of distribution. Suppose $(\rho, v, H, S)$ is a stationary weak solution to (MHD) satisfying the following conditions depending on $\mu$ and $\lambda$.

(i) In the inviscid case ($\mu = \lambda = 0$);

(i-a) The case $N \geq 3$ : There exists $w \in L_\text{loc}^1([0, \infty))$, which is non-increasing, positive almost everywhere on $[0, \infty)$ such that

\[
\int_{\mathbb{R}^N} \rho |v|^2 + |H|^2 + p \frac{1 + |x|}{1 + |x|} \, dx < \infty
\]

(i-b) The case $N = 2$ :

\[
\int_{\mathbb{R}^N} (\rho |v|^2 + |H|^2 + p) \, dx < \infty.
\]
(ii) In the viscous case ($\mu > 0$) for all $N \geq 2$:

(ii-a) if $2\mu + \lambda = 0$,

$$\int_{\mathbb{R}^N} \left( \rho |v|^2 + |H|^2 + p \right) dx < \infty. \quad (1.14)$$

(ii-b) if $2\mu + \lambda \neq 0$,

$$\int_{\mathbb{R}^N} \left( \rho |v|^2 + |v|^\frac{N}{N-1} + |H|^2 + p \right) dx < \infty. \quad (1.15)$$

Then, $\rho = 0$ and $H = 0$ (and $v = 0$ in the case (ii-b)) almost everywhere on $\mathbb{R}^N$ if $N \geq 3$. In the case $N = 2$ we just conclude that $\rho = 0$ almost everywhere on $\mathbb{R}^N$.

Remark 2.1 Contrary to the case of previous section, our argument of the proof of the above theorem does not work for $N = 1$, and we do not yet know if similar nonexistence results hold or not in those cases.

## 2 Proof of the Main Theorems

In order to prove Theorem 1.1 we introduce a class of weight functions as follows.

**Definition 2.1** We say that a function $w(\cdot) \in C^3([0, \infty))$ is admissible if it satisfies the following conditions:

(i)

$$w(r), w'(r), w''(r) \geq 0 \quad \text{and} \quad w'''(r) \leq 0 \quad \forall r > 0. \quad (2.1)$$

(ii) There exists a constant $C$ such that

$$w''(r) + \frac{1}{r} w'(r) + \frac{1}{r^2} w(r) \leq \frac{C}{1 + r} \quad \forall r \geq 0. \quad (2.2)$$

The class of all admissible function will be denoted by $\mathcal{W}$. 

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As examples we find that \( w_1(r), w_2(r) \in \mathcal{W} \), where
\[
  w_1(r) = \log(\cosh r), \quad w_2(r) = \int_0^r \arctan s \, ds.
\]

**Proof of Theorem 1.1**

(i) The case \( \mu = \lambda = 0 \): Let us consider a radial cut-off function \( \sigma \in C_0^\infty(\mathbb{R}^N) \) such that
\[
  \sigma(|x|) = \begin{cases} 
    1 & \text{if } |x| < 1 \\
    0 & \text{if } |x| > 2,
  \end{cases}
\]  
and \( 0 \leq \sigma(x) \leq 1 \) for \( 1 < |x| < 2 \). Let us choose a weight function \( w \in \mathcal{W} \). Then, for each \( R > 0 \), we define
\[
  \varphi_R(x) = w(|x|)\sigma\left(\frac{|x|}{R}\right) = w(|x|)\sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\]

We choose the vector test function \( \phi \) in (1.2) as
\[
  \phi = \nabla \varphi_R(x).
\]

Then, after routine computations, the equation (1.2) becomes
\[
  0 = \int_{\mathbb{R}^N} \rho(x) \left[ \frac{W''(|x|)(v \cdot x)^2}{|x|^2} + w'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \sigma_R(|x|) \, dx
  + \int_{\mathbb{R}^N} \rho(x) w'(|x|) \sigma' \left( \frac{|x|}{R} \right) \left( \frac{(v \cdot x)^2}{R|x|^2} \right) \, dx
  + \int_{\mathbb{R}^N} \frac{1}{R} \rho(x) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \sigma' \left( \frac{|x|}{R} \right) \, w(|x|) \, dx
  + \int_{\mathbb{R}^N} \rho(x) \frac{(v \cdot x)^2}{R^2|x|^2} \sigma'' \left( \frac{|x|}{R} \right) \, w(|x|) \, dx
  + \int_{\mathbb{R}^N} p(x) \left[ w''(|x|) + (N - 1) \frac{w'(|x|)}{|x|} \right] \sigma_R(|x|) \, dx
  + \frac{2}{R} \int_{\mathbb{R}^N} p(x) w'(|x|) \sigma' \left( \frac{|x|}{R} \right) \, dx
  + \frac{N - 1}{R} \int_{\mathbb{R}^N} p(x) \frac{1}{|x|} \sigma' \left( \frac{|x|}{R} \right) \, w(|x|) \, dx
  + \int_{\mathbb{R}^N} p(x) \frac{1}{R^2} \sigma'' \left( \frac{|x|}{R} \right) \, w(|x|) \, dx
:= I_1 + \cdots + I_8
\]
From the condition (2.2) we find that
\[
\int_{\mathbb{R}^N} \left( \rho(x)|v(x)|^2 + |p(x)| \right) \left[ w''(|x|) + \frac{1}{|x|} w'(|x|) + \frac{1}{|x|^2} w(|x|) \right] \, dx \\
\leq C \int_{\mathbb{R}^N} \frac{\rho(x)|v(x)|^2 + |p(x)|}{1 + |x|} \, dx < \infty. \tag{2.7}
\]

Since
\[
\int_{\mathbb{R}^N} \rho(x) \left[ w''(|x|) \frac{(v \cdot x)^2}{|x|^2} + w'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \\
\leq 2 \int_{\mathbb{R}^N} \rho(x)|v(x)|^2 \left[ w''(|x|) + \frac{w'(|x|)}{|x|} \right] \, dx < \infty,
\]

One can use the dominated convergence theorem to show that
\[
I_1 \to \int_{\mathbb{R}^N} \rho(x) \left[ w''(|x|) \frac{(v \cdot x)^2}{|x|^2} + w'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \tag{2.8}
\]
as \( R \to \infty \). Similarly,
\[
I_5 \to \int_{\mathbb{R}^N} p(x) \left[ w''(|x|) + (N - 1) \frac{w'(|x|)}{|x|} \right] \, dx \tag{2.9}
\]
as \( R \to \infty \). For \( I_2 \) we estimate
\[
|I_2| \leq \int_{R<|x|<2R} \rho(x)|v(x)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \frac{w'(|x|)}{|x|} \right| \, dx \\
\leq 2 \sup_{1<s<2} \left| \sigma'(s) \right| \int_{R<|x|<2R} \rho(x)|v(x)|^2 \frac{w'(|x|)}{|x|} \, dx \\
\to 0 \tag{2.10}
\]
as \( R \to \infty \) by the dominated convergence theorem. Similarly
\[
|I_3| \leq 2 \int_{R<|x|<2R} \frac{|x|}{R} \rho(x)|v(x)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \frac{w(|x|)}{|x|^2} \right| \, dx \\
\leq 4 \sup_{1<s<2} \left| \sigma'(s) \right| \int_{R<|x|<2R} \rho(x)|v(x)|^2 \frac{w'(|x|)}{|x|} \, dx \to 0, \tag{2.11}
\]

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and
\[
|I_4| \leq \int_{R < |x| < 2R} \frac{|x|^2}{R^2} \rho(x) |v(x)|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{w(|x|)}{|x|^2} \, dx
\]
\[
\leq 4 \sup_{1 < s < 2} |\sigma''(s)| \int_{R < |x| < 2R} \rho(x) |v(x)|^2 \frac{w(|x|)}{|x|^2} \, dx \to 0
\]
(2.12)
as \(R \to \infty\). The estimates for \(I_6, I_7\) and \(I_8\) are similar to the above, and we find
\[
|I_6| \leq 2 \int_{R < |x| < 2R} |p(x)| \frac{|x|}{R} \frac{w'(|x|)}{|x|} \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{w(|x|)}{|x|^2} \, dx
\]
\[
\leq 4 \sup_{1 < s < 2} |\sigma'(s)| \int_{R < |x| < 2R} |p(x)| \frac{w'(|x|)}{|x|} \frac{w(|x|)}{|x|^2} \, dx \to 0,
\]
(2.13)
\[
|I_7| \leq (N - 1) \int_{R < |x| < 2R} |p(x)| \frac{|x|}{R} \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{w(|x|)}{|x|^2} \, dx
\]
\[
\leq 2 \sup_{1 < s < 2} |\sigma'(s)| \int_{R < |x| < 2R} |p(x)| \frac{w(|x|)}{|x|^2} \, dx \to 0,
\]
(2.14)
and
\[
|I_8| \leq \int_{\mathbb{R}^N} |p(x)| \frac{|x|^2}{R^2} \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{w(|x|)}{|x|^2} \, dx
\]
\[
\leq 4 \sup_{1 < s < 2} |\sigma''(s)| \int_{R < |x| < 2R} |p(x)| \frac{w(|x|)}{|x|^2} \, dx \to 0
\]
(2.15)
as \(R \to \infty\) respectively. Thus passing \(R \to \infty\) in (2.6), we finally obtain
\[
\int_{\mathbb{R}^N} \rho(x) \left[ w''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx
\]
\[
+ \int_{\mathbb{R}^N} p(x) \left[ w''(|x|) + (N - 1) \frac{w'(|x|)}{|x|} \right] \, dx = 0.
\]
(2.16)
Since
\[
    w''(|x|)(v \cdot x)^2 \frac{w''(|x|)}{|x|^2} + \frac{w''(|x|)}{|x|}\left(|v|^2 - \frac{(v \cdot x)^2}{|x|^2}\right) \geq 0,
\]
and
\[
    w''(|x|) + (N - 1)\frac{w'(|x|)}{|x|} > 0,
\]
in (2.16), we need to have
\[
p(x) = p(\rho(x), S(x)) = 0 \quad \text{almost everywhere on } \mathbb{R}^N,
\]
and therefore \( \rho(x) = 0 \) for almost every \( x \in \mathbb{R}^N \).

(ii) The case of \( \mu > 0 \) and either \( 2\mu + \lambda = 0 \) or \( 2\mu + \lambda \neq 0 \): In this case we choose the vector test function
\[
    \phi = \nabla(|x|^2 \sigma_R(x))
\]
in (1.2), where \( \sigma_R \) is defined above. Then, each of the procedure of (i) can be repeated word by word with specific choice of function \( w(r) \equiv 1 \) on \( [0, \infty) \).

We just need to show
\[
    \mu \int_{\mathbb{R}^N} v \cdot \Delta \phi \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \text{div} \, v \, dx = o(1) \quad (2.17)
\]
as \( R \to \infty \).

If \( 2\mu + \lambda = 0 \), then
\[
    J := \mu \int_{\mathbb{R}^N} v \cdot \Delta \nabla(|x|^2 \sigma_R) \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla [\text{div} \nabla(|x|^2 \sigma_R)] \, dx
\]
\[
= 2(\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \Delta(|x|^2 \sigma) \left(\frac{|x|}{R}\right) \, dx = 0,
\]
and (2.17) holds true.

If \( 2\mu + \lambda \neq 0 \), then we compute and estimate
\[
|J| = 2|\mu + \lambda| \left| \int_{\mathbb{R}^N} v \cdot \nabla \Delta(|x|^2 \sigma) \left(\frac{|x|}{R}\right) \, dx \right|
\]
\[
\leq |2\mu + \lambda| \left| \int_{\mathbb{R}^N} (N + 5) \left[ \frac{(v \cdot x)}{R |x|^2} \sigma'(\frac{|x|}{R}) + \frac{(v \cdot x)^2}{R^2 \sigma''(\frac{|x|}{R})} \right] \, dx \right|
\]
\[
+ |2\mu + \lambda| \left| \int_{\mathbb{R}^N} \frac{|x|(v \cdot x)}{R^3} \sigma'''\left(\frac{|x|}{R}\right) \, dx \right|
\]
\[
\leq \frac{C}{R} \int_{R \leq |x| \leq 2R} |v(x)| \, dx \leq C \left( \int_{R \leq |x| \leq 2R} |v(x)|^{\frac{N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \to 0
\]
as $R \to \infty$, since $v \in L^N N^{-1}(\mathbb{R}^N)$ by the hypothesis in the viscous case. Thus (2.17) holds true. □

Proof of Theorem 1.2 Since the proof is similar to that of Theorem 1.1, we will be brief here. Let $w \in \mathcal{W}$. In the inviscid case we choose the vector test function $\phi$ in (1.8) as previously, namely

$$\phi = \nabla \varphi_R(x),$$

where

$$\varphi_R(x) = w(|x|)\sigma_R(|x|) = w(|x|)\sigma\left(\frac{|x|}{R}\right)$$

and $\sigma(\cdot)$ is the cutoff function given by (2.3). Then, the equation (1.8) become

$$\int_{\mathbb{R}^N} \rho(x) \left[ w''(|x|) \frac{(v \cdot x)^2}{|x|^2} + \frac{w'(|x|)}{|x|} \left( |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right) \right] \sigma_R(x) \, dx$$

$$- \int_{\mathbb{R}^N} \left[ w''(|x|) \frac{(H \cdot x)^2}{|x|^2} + \frac{w'(|x|)}{|x|} \left( |H|^2 - \frac{(H \cdot x)^2}{|x|^2} \right) \right] \sigma_R(x) \, dx$$

$$+ \int_{\mathbb{R}^N} \left( p(x) + \frac{1}{2}|H|^2 \right) \left[ w''(|x|) + (N - 1) \frac{w'(|x|)}{|x|} \right] \sigma_R(x) \, dx$$

$$= o(1)$$

as $R \to \infty$. Passing $R \to \infty$ in (2.19), and rearranging the remaining terms, we have

$$\int_{\mathbb{R}^N} \rho(x) \left[ w''(|x|) \frac{(v \cdot x)^2}{|x|^2} + \frac{w'(|x|)}{|x|} \left( |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right) \right] \, dx$$

$$+ \int_{\mathbb{R}^N} \left[ \frac{w'(|x|)}{|x|} - w''(|x|) \right] \frac{(H \cdot x)^2}{|x|^2} \, dx$$

$$+ \frac{N - 3}{2} \int_{\mathbb{R}^N} \frac{|H|^2 w'(|x|)}{|x|} \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |H|^2 w''(|x|) \, dx$$

$$+ \int_{\mathbb{R}^N} \left( p(x) \left[ w''(|x|) + (N - 1) \frac{w'(|x|)}{|x|} \right] \, dx = 0$$

(2.20)

Since $w''(r) \leq 0$, we find that $w''(r)$ is a nonnegative, non-increasing function on $[0, \infty)$. Hence,

$$\frac{w'(|x|)}{|x|} = \frac{w''(|x|)}{|x|} + \int_0^{|x|} w''(s) \, ds \geq \frac{\int_0^{|x|} w''(s) \, ds}{|x|} \geq w''(|x|),$$

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and
\[ \frac{w'(|x|)}{|x|} - w''(|x|) \geq 0 \]
for almost every \( x \in \mathbb{R}^N \). Therefore for \( N \geq 3 \), all of the integrals in (2.20) are nonnegative, and hence
\[ p = p(\rho, S) = 0, \quad H = 0, \]
and \( \rho = 0, H = 0 \) almost everywhere on \( \mathbb{R}^N \). For the case \( N = 2 \) we just set \( w(r) = r^2 \) on \([0, \infty)\). Then, (2.20) is reduced to
\[ \int_{\mathbb{R}^N} [\rho(x)|v(x)|^2 + 2p(x)]dx = 0, \]
which implies \( p(\rho, S) = 0 \), and hence \( \rho = 0 \) almost everywhere on \( \mathbb{R}^N \). The proof for the viscous case is the same as that of Theorem 1.1, and we omit it here. \( \square \)

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