Ricci linear Weyl/Maxwell mutual sourcing:
Electric current from spacetime curvature

Aharon Davidson and Tomer Ygael

Physics Department, Ben-Gurion University of the Negev, Beer-Shea 84105, Israel

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Abstract

We elevate the field theoretical similarities between Maxwell and Weyl vector fields into a full local scale/gauge invariant Weyl/Maxwell mutual sourcing theory. In its simplest form, and exclusively in 4-dimensions, the associated Lagrangian is scalar field free, hosts no fermion matter fields, and Holdom kinetic mixing can be switched off. The theory is then characterized by the following distinctive features: (i) The Weyl/Maxwell mutual sourcing term is necessarily spacetime curvature (not just metric) dependent, implying that (ii) A non-vanishing spacetime curvature can induce an electric current. (iii) In line with Weyl-Dirac (and Einstein-Hilbert) action, the mutual sourcing term is inevitably Ricci linear, and comes thus with the bonus that (iv) The co-divergence of the Maxwell vector field plays the role of a dilaton.

*Email: davidson@bgu.ac.il †Email: toberyg@post.bgu.ac.il
Within the framework of Riemann geometry, with tensor fields serving as the fundamental objects, ordinary derivatives are consistently replaced by covariant derivatives to assure diffeomorphism invariance. Going one step further into the territory of Weyl geometry [1], the tensor fields are traded for so-called co-tensor fields, and the covariant derivatives are generalized into co-covariant (also known as starred \( \ast \)) derivatives, respectively. Resembling a \( U(1) \) local gauge theory, the star derivation procedure mandatorily introduces a new player into the game, the Weyl vector field \( a_\mu(x) \). The prototype theory in this category is Weyl-Dirac gravity [2], a local scale symmetric generalization Brans-Dicke theory [3]. A counter example is provided by \( C^2 \) conformal gravity [4] which, owing to the Weyl tensor \( C^\mu_{\nu\lambda\sigma} \) being an in-tensor (co-tensor of weight zero), solely in 4-dimensions, does not require the presence of \( a_\mu(x) \). Other theoretical directions include two scalar gravity-anti-gravity theories [5], and Kaluza-Klein reduced higher dimensional local scale symmetric theories [6]. However, while treated on equal canonical footing in the Lagrangian formalism, Maxwell vector field \( A_\mu(x) \) and Weyl vector field \( a_\mu(x) \) play completely different roles in theoretical physics.

From the geometric point of view, the differences between these two vector fields sharpens. While \( A_\mu(x) \) constitutes an in-vector, \( a_\mu(x) \) does not constitute a co-vector at all. However, in spite of their physical and geometrical differences, these two vector fields do share a similar transformation law under their corresponding local symmetries. To be specific,

\[
\text{Maxwell} : \quad A_\mu(x) \to A_\mu(x) - \partial_\mu \Phi(x) ,
\]

\[
\text{Weyl} : \quad a_\mu(x) \to a_\mu(x) - \partial_\mu \varphi(x) .
\]

In turn, both their kinetic terms, namely \( F_{\mu\nu} \) and \( f_{\mu\nu} \) respectively, transform alike as Weyl in-scalars, and a Holdom-style kinetic mixing [7] becomes then field theoretically permissible. In this essay, however, with or without invoking the kinetic mixing term, we elevate the apparent field theoretical similarities between Maxwell and Weyl vector fields into a full local scale/gauge invariant mutual sourcing theory. Inline with Einstein-Hilbert and especially with Weyl-Dirac actions, and solely in a 4-dimensional spacetime, we show that the scale symmetric non-kinetic Weyl/Maxwell mutual source mixing is necessarily spacetime curvature dependent (not just metric dependent), and inevitably Ricci linear. This way, a non-vanishing spacetime curvature becomes an unconventional source of the electromagnetic current.

Let our starting point be the familiar 4-dimensional action involving a linear electromag-
netic coupling term
\[ I_{EM} = \int \left( \mathcal{L}_G - \frac{1}{4} F^2 - J^\mu A_\mu \right) \sqrt{-g} \, d^4x , \tag{3} \]
with \( J^\mu \) serving as the external electromagnetic source current, and \( \mathcal{L}_G \) denoting the yet unspecified gravitational part of the Lagrangian. To keep gauge invariance manifest already at the Lagrangian level, one may invoke a Lagrange multiplier \( \Lambda \), and simply replace \( A_\mu \) by \( A_\mu - \Lambda_{;\mu} \). As dictated by the self consistency of the associated Maxwell equations \( F^\mu{}_{\nu} = J^\mu \), and directly by the variation with respect to \( \Lambda \), the theory maintains gauge invariance only provided \( J^\mu \) is locally conserved \( J^\mu_{;\mu} = 0 \). The action Eq.(3) is furthermore local scale invariant if \( \mathcal{L}_G \) is such, and if \( J_\mu \) happens to be a co-covariant vector of power
\[ [J_\mu] = -2 \iff [J^\mu] = -4 , \tag{4} \]
where in our Weyl-Dirac notations,
\[ [g_{\mu\nu}] = 2 , \quad [g^{\mu\nu}] = -2 \implies [\sqrt{-g}] = 4 , \tag{5} \]
\[ [A_\mu] = 0 , \quad [A^\mu] = -2 \implies [F_\mu{}^\nu] = 0 . \tag{6} \]

The last formula deserves some attention. Consider a co-covariant vector \( V_\mu \) of power \( [V_\mu] = n \), and recall that its covariant derivative \( V_{\mu;\nu} \) does not form a co-tensor. Alternatively, one invokes a co-covariant starred derivative, and show that the corresponding co-tensor role is then taken by
\[ V_{\mu*\nu} = V_{\mu;\nu} - (n - 1)a_\nu V_\mu + a_\mu V_\nu - g_{\mu\nu}a_\lambda V_\lambda . \tag{7} \]
In particular, notice the antisymmetric combination
\[ V_{\mu*\nu} - V_{\nu*\mu} = V_{\mu;\nu} - V_{\nu;\mu} + n(a_\mu V_\nu - a_\nu V_\mu) , \tag{8} \]
telling us that antisymmetric \( F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu*\nu} - A_{\nu*\mu} \) is in fact an in-tensor simply because \( A_\mu \) is an in-vector (meaning \( n = 0 \)) to start with. By the same token, if \( U^\mu \) is a co-contravariant vector of power \( [U^\mu] = n \), its star derivative is given by
\[ U^\mu_{*\nu} = U^\mu_{;\nu} - (n + 1)a_\nu U^\mu + a^\mu U_\nu - g^{\mu\nu}a_\lambda U_\lambda . \tag{9} \]
In particular, its co-divergence is given by
\[ U^\mu_{*\mu} = U^\mu_{;\mu} - (n + 4)a_\mu U^\mu . \tag{10} \]
It is only for the special case of \( n = -4 \), that we face the advantage of \( J^\mu_{\star\mu} = J^\mu_{;\mu} \).

The fact that \( A_\mu \) and \( a_\mu \) share similar transformation laws under their corresponding local symmetries, and exhibit kinetic terms of the one and the same structure, may prematurely suggest, in analogy with Eq.\( (3) \), an action à la

\[
\int \left( \mathcal{L}_G - \frac{1}{4} f^{\mu\nu} - j^\mu a_\mu \right) \sqrt{-g} \, d^4x .
\]

(11)

The trouble is that, while \( f_{\mu\nu} = a_{\mu;\nu} - a_{\nu;\mu} \) turns out to be a legitimate in-tensor, the Weyl vector \( a_\mu \) itself does not transform like a co-vector at all. This is to say that the action eq.\( (11) \) is not invariant under arbitrary local scale transformations.

In search of a tenable coupling term to replace the problematic \( j^\mu a_\mu \), we first recall that co-covariant starred derivatives are generically linear in \( a_\mu \). For example, let \( S \) be a co-scalar of power \( n \), then

\[
S_{\star\mu} = S_{;\mu} - n a_\mu S ,
\]

(12)

with the bonus of having \([S_{\star\mu}] = n\) as well. Thus, a coupling term of the form

\[
\mathcal{L}_{\text{int}} = j^\mu S_{\star\mu} \propto j^\mu a_\mu + ...
\]

(13)

can certainly do, but only provided (i) \( n \neq 0 \) on self consistency grounds, and (ii) The source current \( j^\mu \) must constitutes a co-vector of the exact power

\[
[j^\mu] = -(n + 4) \Rightarrow [j_\mu] = -(n + 2) .
\]

(14)

Now, aiming towards Weyl/Maxwell mutual sourcing, one would like to identify \( j_\mu \) with \( A_\mu \). This is our goal, but for this to be the case, recalling that \([A_\mu] = 0\), we must first find a suitable candidate for \( S \), such that

\[
[A_\mu] = 0 \implies [S] = -2 .
\]

(15)

What are the options?

At this stage, fundamental scalar fields are yet (and if at all) to be introduced. In fact, the option of not introducing fundamental scalar fields into the theory is exclusively viable in four spacetime dimensions. So, in the absence of scalar fields, the answer to the above question must come from the geometry of the underlying 4-dim curved spacetime. The simplest curvature scalar to think of is no doubt the Ricci scalar \( R \). But unfortunately, \( R \)
cannot enter the game as is, but must be traded for its $\tilde{R}$ scale symmetric co-scalar variant. In 4-dimensions, it is given by

$$\tilde{R} = R + 6a_\mu^\mu - 6a^\mu a_\mu .$$

(16)

Note that we prefer the notation $\tilde{R}$, instead of the original $R^*$ or $^*R$, leaving the star symbol solely for denoting co-derivation. The crucial observation now is that $[\tilde{R}] = -2$, and the same is true for its co-derivative

$$\tilde{R}_{*\mu} = \tilde{R}_{;\mu} + 2a_\mu \tilde{R} .$$

(17)

In turn, the master requirement eq.(15) can now be satisfied by naturally choosing $S = \tilde{R}$. It is straightforward to verify that other powers of $\tilde{R}$, as well as higher order curvature co-scalars, such as $\tilde{R}^{\mu\nu} \tilde{R}_{\mu\nu}$ and $\tilde{R}^{\mu\nu\lambda\sigma} \tilde{R}_{\mu\nu\lambda\sigma}$, will not do.

We can now close the circle. Rather than assigning external non-dynamical source currents $J^\mu$ and $j^\mu$, we let the Maxwell vector field $A_\mu$ and the Weyl vector field $a_\mu$ source each other. The result is the simplest scalar-free local gauge/scale invariant Weyl/Maxwell mixing theory described by the action

$$I = -\int \left( \frac{1}{4} F^2 + \frac{1}{4} f^2 + \frac{1}{2} e A^\mu \tilde{R}_{*\mu} \right) \sqrt{-g} \, d^4x$$

(18)

where $e$ is a universal dimensionless constant. Note that, in analogy with Eq.(3), $A_\mu$ can always be supplemented by some $\Lambda_{*\mu}$ when needed. It is crucial to notice that $A^\mu \tilde{R}$ happens to be a co-contravariant vector of the special power $[A^\mu \tilde{R}] = -4$. Hence, by recalling Eq.(10) twice, we find

$$A^\mu \tilde{R}_{*\mu} = -A_{*\mu} \tilde{R} + (A^\mu \tilde{R})_{*\mu} = -(A^\mu - 2A_\mu a_\mu) \tilde{R} + (A^\mu \tilde{R})_{;\mu} .$$

(19)

Up to a total divergence, and by no coincidence, also up to a total co-divergence, Eq.(18) can be now re-written in the attractive $\tilde{R}$-linear form

$$I = -\int \left( \frac{1}{4} F^2 + \frac{1}{4} f^2 - \frac{1}{2} e A^\mu \tilde{R} \right) \sqrt{-g} \, d^4x$$

(20)

We aimed towards Weyl/Maxwell mutual sourcing, and have automatically been driven into its unified Weyl/Dirac/Maxwell embedding. Gravity just cannot stay out of the game.

Sticking to 4-dimensions, one is always free to add curvature quadratics terms, for example $\mathcal{L}_G = \tilde{R}^2$ or $\mathcal{L}_G = C^2$ without violating local scale invariance. Furthermore, when introducing fundamental scalar fields, the door gets open for the Weyl-Dirac $\phi^2 \tilde{R}$ term accompanied
by generalized (with co-covariant replacing covariant derivatives) Brans-Dicke kinetic $\omega_{BD}$-terms and a quadratic scalar potential. A more pretentious attempt would be adding Eq. (20) to the standard Einstein-Hilbert $\mathcal{L}_G = R$, which obviously does not respect Weyl scale symmetry. This would mean revising Einstein-Maxwell into into Einstein-Weyl/Maxwell theory, and modifying the well known Reissner-Nordstrom solution accordingly. On pedagogical and simplicity grounds, however, we hereby set $\mathcal{L}_G = 0$ and first study the action Eq. (20) on its own merits.

Here are some distinctive features of the simplest Weyl/Maxwell mutual sourcing theory:

- The highlight is, roughly speaking, the construction of the Maxwell conserved current $J_\mu$ from spacetime curvature (involving $a_\mu$ dependence). The variation with respect to $A_\mu$ is straightforward, giving rise to the field equation

$$F_{\ast \nu}^{\mu \nu} = \frac{1}{2}e g^{\mu \nu} \tilde{R}_{\ast \nu} ,$$

where one can make use of the identity $F_{\ast \nu}^{\mu \nu} = F_{\ast \nu}^{\mu \nu}$. Self consistency (and $g_{\ast \nu}^{\mu \nu} = 0$) then dictates the complementary co-scalar constraint

$$g^{\mu \nu} \tilde{R}_{\ast \mu \ast \nu} = 0$$

Here again, owing to $[\tilde{R}_{\ast \mu}] = -2$, one can take advantage of $g^{\mu \nu} \tilde{R}_{\ast \mu \ast \nu} = g^{\mu \nu} \tilde{R}_{\ast \mu \ast \nu}$, and recall Eq. (16) to further probe the structure of the Maxwell current

Maxwell current: \[ J_\mu = e(a_\mu \tilde{R} + \frac{1}{2} \tilde{R}_{;\mu}) . \]

- By the same token, the variation with respect to $a_\mu$ leads to the field equation $f_{\ast \nu}^{\mu \nu} = j_\mu$.

It takes some algebra though to establish the analogy with the Maxwell current, and verify that the Weyl current is given by

Weyl current: \[ j_\mu = e(A_\mu \tilde{R} + 3A_\nu^{\nu} a_\mu) . \]

- The co-divergence of the Maxwell vector field resembles a dilaton, with the formal definition being the coefficient of $\tilde{R}$ in the Lagrangian Eq. (20), namely

Dilaton: \[ \phi^2 = \frac{1}{2} eA_\mu^{\mu} = e(\frac{1}{2} A_\mu^{\mu} - a_\mu A_\mu) . \]

The fact that the roots of such a dilaton-like configuration are electromagnetic in origin is a natural consequence of the Weyl/Maxwell mutual sourcing.
Finally, imitating Holdom’s $U(1) \otimes U'(1)$ kinetic mixing, one may switch on the analogous scale/gauge symmetric Weyl/Maxwell kinetic mixing [8]

\[ \mathcal{L}_\epsilon = \frac{1}{2} \epsilon g^{\mu \lambda} g^{\nu \sigma} F_{\mu \nu} f_{\lambda \sigma}, \]  

parametrized by a dimensionless coefficient $\epsilon$. No dramatic effects are expected as long as minimally coupled charged scalar fields or fermion fields are not introduced. Once they do enter the theory, reflecting the opposite transformation laws of $A_\mu \rightarrow -A_\mu$ vs. $a_\mu \rightarrow +a_\mu$, CP symmetry gets explicitly violated.

* Electronic address: davidson@bgu.ac.il; URL: https://physics.bgu.ac.il/~davidson/
† Electronic address: tomeryg@post.bgu.ac.il

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