Left orderability for surgeries on the $[1, 1, 2, 2j]$ two-bridge knots

Khanh Le
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Abstract

Let $M$ be a $\mathbb{Q}$-homology solid torus. In this paper, we give a cohomological criterion for the existence of an interval of left-orderable Dehn surgeries on $M$. We apply this criterion to prove that the two-bridge knot that corresponds to the continued fraction $[1, 1, 2, 2j]$ for $j \geq 1$ admits an interval of left-orderable Dehn surgeries. This family of two-bridge knots gives some positive evidence for a question of Xinghua Gao.

1 Introduction

A group $G$ is left orderable if it admits a strict total ordering on the group elements such that $g < h$ implies that $fg < fg$ for all elements $f, g, h \in G$. Left-orderability arises naturally in the study of low-dimensional topology, foliation theory, and group theory. Well-known examples of left-orderable groups include torsion-free abelian groups, free (non-abelian) groups, surface groups and the group of orientation preserving homeomorphisms of the real line. In 3-manifold topology, left-orderability is an important concept due to its role in the L-space conjecture.

Conjecture 1.1 (The L-space conjecture). For an irreducible $\mathbb{Q}$-homology 3-sphere $M$, the following are equivalent

1. $\pi_1(M)$ is left-orderable.
2. $M$ is not an L-space.
3. $M$ admits a coorientable taut foliation.

An L-space is a $\mathbb{Q}$-homology 3-sphere with $\dim \hat{HF}(M) = |H_1(M; \mathbb{Z})|$ where $\hat{HF}(M)$ is the Heegaard Floer homology of $M$ [OS05, Definition 1.1]. There has been a substantial amount of evidence in favor of this conjecture. For example, the L-space conjecture holds for all graph manifolds [BC17] and [Han+20].

In view of Conjecture 1.1, there have been a lot of ideas developed to study left-orderability of 3-manifold groups. It is a well-known fact that a countable group is left-orderable if and only if it embeds in the group of orientation-preserving homeomorphisms of the real line [Ghy01, Theorem 6.8]. In the case of an irreducible compact 3-manifold, its fundamental group is left-orderable if and only if it admits a non-trivial homomorphism onto a left-orderable group [BRW05, Theorem 1.1]. In particular, all manifolds with positive first Betti number are left-orderable. Therefore, it is interesting to construct left orders on $\mathbb{Q}$-homology spheres, for example those coming from doing Dehn filling and from taking cyclic branched covering of $\mathbb{Q}$-homology solid torus.

A fruitful way to build left-orderings on $\mathbb{Q}$-homology spheres is by lifting $\text{PSL}_2(\mathbb{R})$ representations to $\tilde{\text{PSL}}_2(\mathbb{R})$. This strategy has been employed with a lot of success, for example see [Tra15a, Tra15b, Hu15]. Recently, Dunfield, Culler and independently Gao have introduced the idea of using the extension locus of a compact 3-manifold with torus boundary $M$ to order families of $\mathbb{Q}$-homology spheres arising by doing Dehn filling on $M$ [CD18, Gao19]. Furthermore, they gave several criteria implying the existence of intervals of left-orderable Dehn fillings on $M$. To state their results, we need the following definition:
### Definition 1.2.
A compact 3-manifold $Y$ has few characters if each positive-dimensional component of the $\text{PSL}_2(\mathbb{C})$-character variety $X(Y)$ consists entirely of characters of reducible representations. An irreducible $\mathbb{Q}$-homology solid torus $M$ is called longitudinally rigid when $M(0)$ has few characters where $M(0)$ is the closed manifold obtained from $M$ by doing Dehn filling along the homological longitude.

We summarize their results in the following:

**Theorem 1.3** ([CD18 Theorem 7.1] and [Gao19 Theorem 5.1]). Suppose that $M$ is longitudinally rigid irreducible $\mathbb{Z}$-homology solid torus. Then the following are true:

1. If the Alexander polynomial of $M$ has a simple root $\xi \neq 1$ on the unit circle, then there exists $a > 0$ such that for every rational $r \in (-a,0) \cup (0,a)$ the Dehn filling $M(r)$ is orderable.

2. If the Alexander polynomial of $M$ has a simple positive real root $\xi \neq 1$, then there exists a nonempty interval $(-a,0]$ or $[0,a)$ such that for every rational $r$ in the interval, the Dehn filling $M(r)$ is orderable.

**Remark 1.4.** In fact, their techniques also apply to the case where $M$ is a $\mathbb{Q}$-homology solid torus with some further hypothesis on $\xi$ in the first statement. The full version of the second statement is stated below in *Theorem 2.11*.

Culler and Dunfield also proved the following criterion for left-orderability:

**Theorem 1.5.** Suppose that $M$ is a hyperbolic $\mathbb{Z}$-homology solid torus, whose trace field has a real embedding, then there exists $a > 0$ such that for every rational $r \in (-a,0) \cup (0,a)$ the Dehn filling $M(r)$ is orderable.

In view of *Theorem 1.3* it is natural to ask when a $\mathbb{Q}$-homology solid torus is longitudinally rigid. Since the character variety is notoriously hard to compute, see for example [BP13; Che20], longitudinal rigidity is difficult to study in a general setting. Culler and Dunfield gave a topological condition which implies longitudinal rigidity. In particular, they introduced the following concept:

**Definition 1.6.** Let $M$ be a knot exterior. We say that $M$ is lean if the longitudinal Dehn filling $M(0)$ is prime and every closed essential surface in $M(0)$ is a fiber in a fibration over $S^1$.

For example, the $(-2,3,2s+1)$-pretzel knots were shown to be lean for $s \geq 3$, so there is an interval of about 0 of left-orderable Dehn surgeries on these knot complements [Nie19 Theorem 4]. However as remarked in [CD18 Section 1.6], this leanness condition is rather restrictive. In particular, for a knot complement $K$ in $S^3$ being lean implies that $K$ fibers. Nevertheless, the first statement of *Theorem 1.3* was proved to be true without the condition of longitudinal rigidity by Herald and Zhang [HZ19 Theorem 1]. Motivated by this result, Xinghua Gao asked:

**Question 1.7.** [Gao19 Section 7] Can the longitudinal rigidity condition be dropped from the second statement of *Theorem 1.3*? Is it possible to prove $H^1(\pi_1(M(0)); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$ where $\rho$ is the non-abelian reducible representation of $\pi_1(M)$ coming from the root of the Alexander polynomial?

Following the suggestion in this question, we say that $\mathbb{Q}$-homology solid torus is locally longitudinal rigid at a root of the Alexander polynomial if $H^1(\pi_1(M(0)); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$ where $\rho$ is a non-abelian reducible representation coming from this root, see *Definition 2.15* for a precise definition. In fact, we prove that the second item of *Theorem 1.3* still holds true under this weakened hypothesis.

**Theorem 1.8.** Suppose that $M$ is an irreducible $\mathbb{Q}$-homology solid torus and that the Alexander polynomial of $M$ has a simple positive real root $\xi \neq 1$. Furthermore, suppose that $M$ locally longitudinally rigid at $\xi$. Then there exists a nonempty interval $(-a,0]$ or $[0,a)$ such that for every rational $r$ in the interval, the Dehn filling $M(r)$ is orderable.

As an application, we apply this result to produce an interval of left-orderable Dehn surgeries on an infinite family of two-bridge knots complement.

**Theorem 1.9.** For every two-bridge knot $K_j$ corresponding to the continued fraction $[1,1,2,2,2j]$ where $j \geq 1$, there exists a nonempty interval $(-a,0]$ or $[0,a)$ such that for every rational $r$ in the interval, the Dehn filling $M(r)$ is left-orderable.
Remark 1.10. As we will see in Lemma 3.2, the Alexander polynomial of \( K_j \) has all simple positive real roots that are not 1, and is not monic for \( j \geq 2 \). In particular, the complement of \( K_j \) is not lean for \( j \geq 2 \). Furthermore, the trace field of \( K_j \) for \( 1 \leq j \leq 30 \) has no real places, and it is most likely that the trace fields of all knots in this family share this property. Therefore, Theorem 1.9 is not a direct consequence of Theorem 1.3 or Theorem 1.5. The family of two-bridge knots \( [1,1,2,2,2] \) is a genuinely new family of knots with an interval left-orderable Dehn surgeries which cannot be obtained from prior techniques.

1.1 Outline

In Section 2, we review some background materials on group cohomology, PSL\(_2\)-representation variety, formal deformation of representation and holonomy extension locus. At the end of this section, we will give a proof of Theorem 1.8. In Section 3, we will carry out the group cohomology calculation and prove that the complement of \( K_j \) is locally longitudinally rigid at all roots of the Alexander polynomial. As a result, Theorem 1.9 will follow from Theorem 1.8.

2 Preliminaries

2.1 Group cohomology and PSL\(_2\)-representation variety

Following the notation in [CD18] and [Gao19], we set \( G = \text{PSL}_2(\mathbb{R}) \) and \( G_\mathbb{C} = \text{PSL}_2(\mathbb{C}) \) throughout the paper. For a compact manifold \( M \) and a group \( H \), we let \( R_H(M) = \text{Hom}(\pi_1(M), H) \) be the representation variety. When \( H = G_\mathbb{C} \), we denote \( R(M) := R_{G_\mathbb{C}}(M) \). Since \( \pi_1(M) \) is finitely generated, \( R(M) \) can be identified with an algebraic subset in some affine space \( \mathbb{C}^N \). The group \( G \) acts on \( R(M) \) by conjugation. Let us consider the minimal Hausdorff quotient \( X(M) := R(M)//G_\mathbb{C} \) and the quotient map \( \pi : R(M) \to X(M) \).

Given a representation \( \rho \in R(M) \), a character of \( \rho \) is the map \( \chi_{\rho} : \pi(M) \to \mathbb{C} \) defined by \( \chi_{\rho} (\gamma) = \text{tr}^\rho(\gamma) \). By [HP04] Theorem 1.3, there exists a bijection between the points of \( X(M) \) and the characters of representations in \( R(M) \) such that the point \( t(\rho) = [\rho] \) corresponds to \( \chi_{\rho} \). Therefore, we refer to \( X(M) \) as the \( \text{PSL}_2(\mathbb{C}) \)-character variety of \( M \).

Let \( \Gamma \) be a group and \( \rho : \Gamma \to G_\mathbb{C} \) be a representation. The Lie algebra of \( G_\mathbb{C} \) can be identified with the set of trace-less 2-by-2 matrices over \( \mathbb{C} \). Using the adjoint representation, the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) becomes a \( \Gamma \)-module by

\[
\gamma \cdot v = \rho(\gamma)v\rho(\gamma)^{-1}.
\]

We denote this \( \Gamma \)-module by \( \mathfrak{sl}_2(\mathbb{C})_\rho \). The space of 1-cocycles is

\[
Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = \{ z : \Gamma \to \mathfrak{sl}_2(\mathbb{C}) | z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma') \ \forall \gamma, \gamma' \in \Gamma \}.
\]

Alternatively when \( \Gamma \) is a finitely presented group, we can also describe the space of cocycles as maps \( \Gamma \to \mathfrak{sl}_2(\mathbb{C}) \) satisfying the group relations of \( \Gamma \). In particular, suppose that \( \Gamma = \langle \gamma_1, \ldots, \gamma_n | w_1(\gamma_1), \ldots, w_k(\gamma_k) \rangle \) is a finite presentation and that \( z(\gamma_i) = v_i \in \mathfrak{sl}_2(\mathbb{C}) \). Given any element \( w \in \Gamma \), we can express \( w \) as a word \( w(\gamma_i) \) in the generators \( \gamma_i \)'s of \( \Gamma \). The equation

\[z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma') \ \forall \gamma, \gamma' \in \Gamma \]  

(1)

determines the image of \( z(w) \). This gives us a well-defined cocycle on \( \Gamma \) if and only if \( z(\gamma_j) = 0 \) for all relations \( w_j \) of \( \Gamma \), see [Wei64] Equation 4. The space of 1-coboundaries is

\[
B^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = \{ b : \Gamma \to \mathfrak{sl}_2(\mathbb{C}) | \exists v \in \mathfrak{sl}_2(\mathbb{C}), b(\gamma) = (\gamma - 1_\Gamma) \cdot v \}.
\]

(2)

Finally, the group cohomology is defined by

\[
H^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho)/B^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho).
\]

Definition 2.1. Suppose that \( V \) is an affine algebraic variety in \( \mathbb{C}^n \). Let

\[
I(V) = \{ f \in \mathbb{C}[x_1, \ldots, x_n] | f(x) = 0 \ \forall x \in V \}
\]

3
be the vanishing ideal of \( V \). Define the Zariski tangent space to \( V \) at \( p \) to be the vector space of derivatives of polynomials.

\[
T^\text{Zar}_p(V) = \left\{ \frac{d\gamma}{dt} \bigg|_{t=0} \in \mathbb{C}^n \mid \gamma \in (\mathbb{C}[t])^n, \gamma(0) = p \text{ and } f \circ \gamma \in t^2\mathbb{C}[t] \ \forall f \in I(V) \right\}.
\]

It was observed by Weil in [Wei64] that for any Lie group \( H \) and \( \rho \in R_H(M) \) the Zariski tangent space embeds in the space of 1-cocycles \( Z^1(\pi_1(M); \mathfrak{h}) \) where \( \mathfrak{h} \) is the Lie algebra of \( H \). In particular, we have the following inequalities

\[
\dim Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) \geq \dim T^\text{Zar}_p (R(\Gamma)).
\]

### 2.2 Formal deformation of representation

We will review some background materials on formal deformations of representations and integrability of cocycles. The concept of integrable cocycles will be important to building a certain path of representations required in the proof of Theorem 1.8, see also Lemma 2.17. For this discussion, let \( \Gamma \) be a finitely presented group, \( A_k := \mathbb{R}[t]/(t^{k+1}) \) for \( k \in \mathbb{N} \) and \( A_\infty := \mathbb{R}[t] \). Consider the following groups \( G_k := \text{PSL}_2(A_k) \) and \( G_\infty := \text{PSL}_2(A_\infty) \).

**Definition 2.2.** Let \( \rho : \Gamma \to G \) be a representation. A formal deformation of \( \rho \) is a representation \( \rho_\infty : \Gamma \to G_\infty \) such that \( \rho = p_0 \circ \rho_\infty \) where \( p_0 : G_\infty \to G \) is the homomorphism induced by evaluating the formal power series at \( t = 0 \).

For any formal deformation \( \rho_\infty : \Gamma \to G_\infty \) of \( \rho \), we can write

\[
\rho_\infty(\gamma) = \exp \left( \sum_{i=1}^{\infty} t^i u_i(\gamma) \right) \rho(\gamma)
\]

where \( u_i : \Gamma \to \mathfrak{sl}_2(\mathbb{R}) \) is a cochain. Since \( \rho_\infty \) is a homomorphism, a calculation using the Taylor series for the exponential map implies that \( u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \). Conversely, we have the following definition:

**Definition 2.3.** A cochain \( u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \) is integrable if there exists a formal deformation \( \rho_\infty \) of \( \rho \) given by Equation (3). In this case, we say that \( \rho_\infty \) is a formal deformation of \( \rho \) with leading term \( u_1 \).

Given a representation \( \rho : \Gamma \to G \) and a cochain \( u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \), the existence of a formal deformation \( \rho \) with leading term \( u_1 \) is equivalent to the vanishing of a series of obstruction classes in \( H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \) [HPS01] Proposition 3.1 and Corollary 3.2]. In particular, we have the following proposition from [HPS01].

**Proposition 2.4.** Let \( \rho \in R_G(\Gamma) \) and \( u_i \in C^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \) for \( 1 \leq i \leq k \) be given. Suppose that we have constructed a representation \( \rho_k := \rho_k^{(u_1, \ldots, u_k)} : \Gamma \to G_k \) given by

\[
\rho_k(\gamma) = \exp \left( \sum_{i=1}^{k} t^i u_i(\gamma) \right) \rho(\gamma) \mod t^{k+1}.
\]

There exists an obstruction class \( \zeta_{k+1} := \zeta_{k+1}^{(u_1, \ldots, u_k)} \in H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \) with the following properties:

1. There is a cochain \( u_{k+1} \) such that \( \rho_{k+1}^{(u_1, \ldots, u_{k+1})} : \Gamma \to G_{k+1} \) given by

\[
\rho_{k+1}^{(u_1, \ldots, u_{k+1})}(\gamma) = \exp \left( \sum_{i=1}^{k+1} t^i u_i(\gamma) \right) \rho(\gamma) \mod t^{k+2}
\]

is a homomorphism if and only if \( \zeta_{k+1} = 0 \).

2. The obstruction \( \zeta_{k+1} \) is natural in the following sense: if \( f : \Gamma' \to \Gamma \) is a homomorphism then

\[
f^* \rho_k^{(u_1, \ldots, u_k)} = \rho_k^{(f^* u_1, \ldots, f^* u_k)}
\]

is a homomorphism and \( f^* \zeta_{k+1}^{(u_1, \ldots, u_k)} = \zeta_{k+1}^{(f^* u_1, \ldots, f^* u_k)} \).
Consequently, an infinite sequence \( \{u_i\}_{i=1}^{\infty} \subset C^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})) \) defines a formal deformation of \( \rho, \rho_{\infty} : \Gamma \to G_{\infty} \) via Equation (3) if and only if \( u_1 \) is a cocycle and \( \zeta^{(u_1, \ldots, u_k)}_{k+1} = 0 \) for all \( k \geq 1 \).

**Remark 2.5.** Proposition 2.4 was stated over \( \mathbb{C} \) in [HPS01]. Since the construction of the obstruction \( \zeta_{k+1} \), see [HPS01] Definition 3.4, and the proof of Proposition 2.4 is purely homological, it remains true over \( \mathbb{R} \).

### 2.3 Holonomy extension locus

Now we recall some definitions and results about the holonomy extension locus from [Gao19]. The group \( G \) acts on \( P^1_\mathbb{C} \) by Mobius transformation leaving \( P^1_\mathbb{R} \) invariant. Any nontrivial abelian subgroup of \( G \) either contains only parabolic elements and has one fixed point in \( P^1_\mathbb{C} \) or contains only hyperbolic or elliptic elements and has two fixed points in \( P^1_\mathbb{C} \). Let \( \tilde{G} = \text{PSL}_2(\mathbb{R}) \) be the universal covering group of \( G \). The group \( \tilde{G} \) also acts on \( P^1_\mathbb{C} \) by pulling back the action of \( G \). We say that an element \( \tilde{g} \in \tilde{G} \) is hyperbolic, parabolic, elliptic, or trivial, respectively.

We denote by \( M \) a compact 3-manifold with a single torus boundary component and define the augmented representation \( R^\text{aug}_G(M) \). Since abelian subgroups of \( G \) act with global fixed points, we define the augmented representation variety \( R^\text{aug}(M) \) to be the subvariety of \( R_G(M) \times P^1_\mathbb{C} \) consisting of pairs \( (\rho, z) \) where \( z \) is a fixed point of \( \rho(\pi_1(\partial M)) \). Since the action of \( \tilde{G} \) on \( P^1_\mathbb{C} \) comes from pulling back the action of \( G \), we can also define \( R^\text{aug}_G(M) \) to be the real analytic subvariety of \( R_G(M) \times P^1_\mathbb{C} \) consisting of pairs \( (\rho, z) \) where \( z \) is a fixed point of \( \rho(\pi_1(\partial M)) \). Similarly, we define \( R^\text{aug}_G(\partial M) \) to be the real analytic subvariety of \( R_G(\partial M) \times P^1_\mathbb{C} \) consisting of pairs \( (\rho, z) \) where \( z \) is a fixed point of \( \rho(\pi_1(\partial M)) \).

Given a hyperbolic, parabolic or central element \( \tilde{g} \in \tilde{G} \) with a fixed point \( v \in P^1_\mathbb{C} \), let \( g \in G \) be the image of \( \tilde{g} \) and \( a \) be a square root of the derivative of \( g \) at \( v \). We define

\[
ev(\tilde{g}, v) := (\ln(|a|), \text{trans}(\tilde{g}))
\]

where \( \text{trans} : \tilde{G} \to \mathbb{R} \) is the translation number given by

\[
\text{trans}(\tilde{g}) = \lim_{n \to \infty} \frac{\tilde{g}^n(0)}{n}.
\]

for some \( x \in \mathbb{R} \). This limit exists for all \( \tilde{g} \in \tilde{G} \), see [Ghy01] Section 5.1. It is shown in [Gao19] Lemma 3.1 that \( \ev(-, v) \) is a homomorphism when restricted to hyperbolic or parabolic abelian subgroups of \( \tilde{G} \) fixing \( v \). We get a group homomorphism

\[
\ev(\tilde{\rho}(-), v) : \pi_1(\partial M) \to \mathbb{R} \times \mathbb{Z}
\]

for \( \tilde{\rho} \in R^\text{aug}_G(\partial M) \) whose image in \( \tilde{G} \) is hyperbolic, parabolic or central. In other words, we can view \( \ev(\tilde{\rho}(-), v) \) as an element of \( \text{Hom}(\pi_1(\partial M), \mathbb{R} \times \mathbb{Z}) \). We are now ready to define the holonomy extension locus.

**Definition 2.6.** Let \( PH_G(M) \) be the subset of \( R^\text{aug}_G(M) \) whose restriction to \( \pi_1(\partial M) \) is either hyperbolic, parabolic or central. Consider the restriction map \( i^* : R^\text{aug}_G(M) \to R^\text{aug}_G(\partial M) \) induced by the inclusion \( i : \partial M \to M \). Define \( \text{EV} : i^*(PH_G(M)) \to H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z}) \) by

\[
(\tilde{\rho}, v) \mapsto \ev((\tilde{\rho}(-), v)).
\]

**Definition 2.7.** Consider the composition

\[
PH_G(M) \subset R^\text{aug}_G(M) \overset{i^*}{\longrightarrow} R^\text{aug}_G(\partial M) \overset{\text{EV}}{\longrightarrow} H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z})
\]

The closure of \( \text{EV} \circ i^*(PH_G(M)) \) in \( H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z}) \) is called the holonomy extension locus of \( M \) and denoted \( H\text{L}_G(M) \).

**Definition 2.8.** We call a point in \( H\text{L}_G(M) \) a hyperbolic/parabolic/central point if it comes from a representation \( \tilde{\rho} \in PH_G(M) \) such that \( i^*(\tilde{\rho}) \) is hyperbolic/parabolic/central. We call points in \( H\text{L}_G(M) \) but not in \( \text{EV} \circ i^*(PH_G(M)) \) ideal points.
To get concrete coordinates on the holonomy extension locus as well as the Dehn surgery space, let us pick a basis \((\mu, \lambda)\) for \(H_1(\partial M; \mathbb{R})\) where \(\lambda\) is the homological longitude of \(M\). We identify \(H^1(\partial M; \mathbb{R})\) with \(\mathbb{R}^2\) using the dual basis \((\mu^*, \lambda^*)\). Let \(L_r\) be the line through the origin in \(\mathbb{R}^2\) of slope \(-r\) where \(r \in \mathbb{Q} \cup \{\infty\}\). In terms of the dual basis \((\mu^*, \lambda^*)\), the line \(L_r\) consists of linear functions that vanish on the primitive element \(\gamma\) representing the slope \(r\) in \(\pi_1(\partial M)\) with respect to the basis \((\mu, \lambda)\). The structure of the holonomy extension locus is summarized as follows:

**Theorem 2.9.** [Gao19, Theorem 3.1] The holonomy extension locus

\[
 HL_G(M) = \bigcup_{i,j \in \mathbb{Z}} H_{i,j}(M)
\]

is a locally finite union of analytic arcs and isolated points. Each component \(H_{i,j}(M)\) contains at most one parabolic point and has finitely many ideal points locally. The locus \(H_{0,0}\) contains the horizontal axis \(L_0\), which comes from representations to \(G\) with abelian image.

The holonomy extension locus gives a tool to detect left-orderable Dehn surgeries. We have the following lemma:

**Lemma 2.10.** [Gao19, Lemma 3.8] If \(L_r\) intersects the component \(H_{0,0}(M)\) of \(HL_G(M)\) at non-parabolic and non-ideal points, and assume that \(M(r)\) is irreducible, then \(M(r)\) is left-orderable.

Using the previous lemma, Xinghua Gao gives a criterion in terms of the \(\text{PSL}_2(\mathbb{R})\)-character variety to produce an interval of left-orderable Dehn surgery around the 0-filling.

**Theorem 2.11.** [Gao19, Theorem 5.1] Suppose that \(M\) is a longitudinally rigid irreducible \(\mathbb{Q}\)-homology solid torus and that the Alexander polynomial of \(M\) has a simple positive real root \(\xi \neq 1\). Then there exists a nonempty interval \((-a, 0]\) or \([0, a)\) such that for every rational \(r\) in the interval, the Dehn filling \(M(r)\) is orderable.

For completeness, we include the proof of this theorem. The key to the proof of **Theorem 2.11** is to produce an arc in \(H_{0,0}(M)\) transverse to the horizontal axis. By construction, this arc does not contain any parabolic or ideal points. **Theorem 2.11** then follows from **Lemma 2.10**. To construct an arc in \(H_{0,0}(M)\), we start by deforming abelian representations coming from the roots of the Alexander polynomial into irreducible representations. In particular, let \(\xi\) be a simple positive real root of the Alexander polynomial and \(\alpha : \pi_1(M) \to \mathbb{R}_+\), the multiplicative group of the real numbers, such that \(\alpha\) factors through \(H_1(M; \mathbb{Z})_{\text{free}} \cong \mathbb{Z}\) and takes a generator of \(H_1(M; \mathbb{Z})_{\text{free}}\) to \(\xi\). We let \(\rho_\alpha : \pi_1(M) \to G_{\mathbb{C}}\) be the associated diagonal representation given by

\[
 \rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha(\gamma)^{1/2} & 0 \\ 0 & \alpha(\gamma)^{-1/2} \end{pmatrix}
\]

where \(\alpha(\gamma)^{1/2}\) is either square root. The condition on the root of the Alexander polynomial allows one to deform \(\rho_\alpha =: \rho_0\) into an analytic path of representations \(\rho_t : \pi_1(M) \to G\) where \(t \in [-1, 1]\), see **Gao19, Lemma 5.1**. Furthermore this path of representations has the following properties.

**Lemma 2.12.** [Gao19, Lemma 5.1] The path \(\rho_t : [-1, 1] \to R_G(M)\) constructed above satisfies:

1. The representations \(\rho_t\) are irreducible over \(G_{\mathbb{C}}\) for \(t \neq 0\).
2. The corresponding path \([\rho_t]\) of characters in \(X_G(M)\) is also a non-constant analytic path.
3. The function \(\text{tr}^2(\gamma)\) is nonconstant in \(t\) for some \(\gamma \in \pi_1(\partial M)\).

**Proof of Theorem 2.11** Let \(\rho_t\) be the path of representations from **Lemma 2.12**. Using this path, we can produce an arc in \(H_{0,0}(M)\) as follows. Since \(\rho_0\) factors through \(H_1(M; \mathbb{Z})_{\text{free}} \cong \mathbb{Z}\), we can lift this representation to \(\tilde{\rho}_0 : \pi_1(M) \to \tilde{G}\). As the obstruction of lifting a representation from \(G\) to \(\tilde{G}\) has discrete values and is continuous on \(R_G(M)\), we can lift the path \(\rho_t\) to a path \(\tilde{\rho}_t\) in \(R_{\tilde{G}}(M)\). Adjusting \(\tilde{\rho}_0\) by the appropriate central element of \(\tilde{G}\), we can assume that \(\text{trans}(\tilde{\rho}_0(\mu)) = 0\). The image \(\rho_0(\lambda)\) is trivial implies...
Remark 2.13. The condition that $M$ is longitudinally rigid ensures that the representation $\rho_t$ obtained by deforming the abelian representation $\rho_0$ does not factor through the longitudinal filling. We can weaken this hypothesis by a local condition at the non-abelian reducible representation $\rho_\xi^+$ that corresponds to a root $\xi$ of the Alexander polynomial.

Recall that, we have the following theorem of Burde and de Rham:

**Theorem 2.14** ([Bur67] and [Rha67]). Let $\alpha : \pi_1(M) \to \mathbb{C}^*$ be a representation and define $\rho_\alpha$ as in Equation (4). Then there exists a reducible, non-abelian representation $\rho_\xi^+ : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ such that $|\rho_\xi^+| = |\rho_\alpha|$ in $X(M)$ if and only if $\alpha$ factors through $H_1(M; \mathbb{Z})_{\text{free}} \cong \mathbb{Z}$ sending a generator to the root $\xi$ of the Alexander polynomial of $M$.

**Definition 2.15.** Suppose that $M$ be an irreducible $\mathbb{Q}$-homology solid torus. Let $\xi$ be a root of the Alexander polynomial of $M$ and $\rho_\xi^+$ be a non-abelian reducible representation associated to $\xi$. We say that $M$ is locally longitudinally rigid at $\xi$ if

$$H^1(M(0); \mathfrak{s}_2(\mathbb{C})_{\rho_\xi^+}) = 0.$$ 

Before proving [Theorem 1.8] we need the following lemmas from [HP05] in the real setting. We include the proof of these lemmas for completeness.

**Lemma 2.16.** Let $\xi$ be a simple positive real root of the Alexander polynomial that is not 1 and

$$\phi := \rho_\xi^+ : \pi_1(M) \to \text{PSL}_2(\mathbb{R})$$

be a non-abelian reducible representation that corresponds to $\xi$. Then the map

$$H^2(\pi_1(M); \mathfrak{s}_2(\mathbb{R})_\phi) \to H^2(\pi_1(\partial M); \mathfrak{s}_2(\mathbb{R})_\phi)$$

induced by the inclusion $\pi_1(\partial M) \hookrightarrow \pi_1(M)$ is injective.

**Proof.** We have $\phi|_{\pi_1(\partial M)}$ is non-trivial since $\text{tr}^2(\phi(\mu)) = \xi^k + 2 + \xi^{-k} > 4$ where $k$ is the index of $[\mu]$ in $H_1(M; \mathbb{Z})_{\text{free}}$. Since $\partial M$ is aspherical, we have $H^*(\partial M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong H^*(\pi_1(\partial M); \mathfrak{s}_2(\mathbb{R})_\phi)$. Since $\phi|_{\pi_1(\partial M)}$ is non-trivial, we have

$$H^0(\partial M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong \mathfrak{s}_2(\mathbb{R})^{\phi(\pi_1(\partial M))} \cong \mathbb{R}.$$ 

By duality and Euler characteristic, we have

$$H^2(\partial M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong \mathbb{R} \quad \text{and} \quad H^3(\partial M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong \mathbb{R}^2.$$ 

Since $\xi$ is a simple root of the Alexander polynomial, [HP05] Corollary 5.4 gives that

$$H^1(M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong H^1(\pi_1(M); \mathfrak{s}_2(\mathbb{R})_\phi) \cong \mathbb{R}.$$ 

By duality, we have

$$H^2(M, \partial M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong H^1(M; \mathfrak{s}_2(\mathbb{R})_\phi) \cong \mathbb{R}.$$ 

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Therefore, the following segment of the long exact sequence of pair for \((M, \partial M)\)
\[ H^1(M; \mathfrak{sl}_2(\mathbb{R})_\phi) \rightarrow H^1(\partial M; \mathfrak{sl}_2(\mathbb{R})_\phi) \rightarrow H^2(M, \partial M; \mathfrak{sl}_2(\mathbb{R})_\phi) \]
is short exact. Therefore from the long exact sequence of pair for \((M, \partial M)\) we see that the map
\[ H^2(M; \mathfrak{sl}_2(\mathbb{R})_\phi) \rightarrow H^2(\partial M; \mathfrak{sl}_2(\mathbb{R})_\phi) \]
is injective. The conclusion of the lemma follows from the following commutative diagram
\[
\begin{array}{ccc}
H^2(M; \mathfrak{sl}_2(\mathbb{R})_\phi) & \rightarrow & H^2(\partial M; \mathfrak{sl}_2(\mathbb{R})_\phi) \\
\uparrow & & \uparrow \\
H^2(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi) & \rightarrow & H^2(\pi_1(\partial M); \mathfrak{sl}_2(\mathbb{R})_\phi)
\end{array}
\]
and the fact that \(H^2(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi) \rightarrow H^2(M; \mathfrak{sl}_2(\mathbb{R})_\phi)\) is injective, see [HP05, Lemma 3.1].

**Lemma 2.17.** Let \(\xi\) be a simple positive real root of the Alexander polynomial that is not 1 and
\[ \phi := \rho^+_{\xi} : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{R}) \]
be a non-abelian reducible representation that corresponds to \(\xi\). All cocycles in \(Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)\) are integrable.

**Proof.** As noted in the proof of [Lemma 2.16] \(\phi_{|_\pi(\partial M)}\) is non-trivial. Since \(\phi(\pi_1(\partial M)) \subset \text{PSL}_2(\mathbb{R})\), the image \(\phi(\pi_1(\partial M))\) cannot be the Klein 4-group. By [HP05, Lemma 7.4], \(\phi_{|_\pi(\partial M)}\) is a smooth point of an irreducible component of \(R_G(\mathbb{Z}^2)\) with local dimension four.

Let \(i : \pi_1(\partial M) \rightarrow \pi_1(M)\) be an inclusion map and \(u_1 : \pi_1(M) \rightarrow \mathfrak{sl}_2(\mathbb{R})\) be a cocycle. Suppose we have cocycles \(u_2, \ldots, u_k : \pi_1(M) \rightarrow \mathfrak{sl}_2(\mathbb{R})\) such that
\[ \phi_k(\gamma) = \exp \left( \sum_{i=1}^{k} t^i u_i(\gamma) \right) \phi(\gamma) \]
is a homomorphism modulo \(t^{k+1}\). From [Proposition 2.4], we get an obstruction class
\[ \zeta^{(u_1, \ldots, u_k)}_{k+1} \in H^2(\pi_1(M); \mathfrak{sl}(\mathbb{R})_\phi), \]
which vanishes if and only if \(\phi_k\) can be extended to a homomorphism modulo \(t^{k+2}\).

The restriction \(\phi_k \circ i\) is a homomorphism modulo \(t^{k+1}\). Since \(\phi \circ i\) is a smooth point of \(R_G(\mathbb{Z}^2)\), \(\phi_k \circ i\) extends to a homomorphism modulo \(t^{k+2}\), see [HPS01, Lemma 3.7]. Therefore, the order \(k+1\) obstruction vanishes on the boundary:
\[ i^* \zeta^{(u_1, \ldots, u_k)}_{k+1} = \zeta^{(\iota^* u_1, \ldots, \iota^* u_k)}_{k+1} = 0. \]

By [Lemma 2.16], \(\iota^*\) is injective, and so the obstruction \(\zeta^{(u_1, \ldots, u_k)}_{k+1}\) vanishes for \(\pi_1(M)\) as well. Iterating this process starting with \(u_1\), we get an infinite sequence of cocycles \(\{u_i\}_{i=1}^{\infty}\) such that \(u_1\) is a cocycle and the obstruction
\[ \zeta^{(u_1, \ldots, u_k)}_{k+1} = 0 \]
for all \(k \geq 1\). By [Proposition 2.4], we get a representation \(\phi_\infty : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{R}[t])\)
\[ \phi_\infty(\gamma) = \exp \left( \sum_{i=1}^{\infty} t^i u_i(\gamma) \right) \phi(\gamma) \]
for all cocycle \(u_1\). Therefore, all cocycles of \(Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R}))\) are integrable. \(\square\)
Remark 2.18. This strategy of proving that all cocycles are integrable was carried out over \(\mathbb{C}\) in [HP05, Lemma 7.5]. The key tool is [HP05, Lemma 3.7] which uses the formal implicit function theorem. Since the formal implicit function theorem holds over \(\mathbb{R}\), we can also carry out this strategy over \(\mathbb{R}\). The same strategy to prove that certain cocycles are integrable over \(\mathbb{R}\) was also carried out in the proof of [HP05, Proposition 10.2].

Proof of Theorem 1.8. Following the proof of [Theorem 2.11], it suffices to prove that the arc \(A\) constructed in the proof of [Theorem 2.11] is not contained in \(L_0\). Arguing by contradiction, suppose this arc is contained in \(L_0\). As in the proof of [Theorem 2.11], this would imply that the path of representation \(\rho_t\) factors through \(M(0)\). Since \(\rho_t\) is irreducible for all \(t \neq 0\), we obtain an arc in \(X(M(0))\) that contains \([\rho_0] = [\rho^+_\xi]\).

On the other hand, we claim that there exists a path \(\phi_t : [-1, 1] \to R_G(M)\) such that \(\phi_0 = \rho^+_\xi\), the non-abelian reducible representation that corresponds to \(\xi\). For convenience, we let \(\phi := \rho^+_\xi\). We have the following isomorphism of cohomology groups

\[
H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\phi) = H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi) \otimes \mathbb{R} \mathbb{C}.
\]

Since \(\xi\) is a simple root of the Alexander polynomial, [HP05, Corollary 5.4] gives that \(H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)\) is one-dimensional. Therefore, \(Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)\) is four-dimensional. By Lemma 2.17, all cocycles in \(Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)\) are integrable. Therefore, \(\phi\) is a smooth point of \(R_G(M)\) with local dimension 4. Integrating a cocycle that generates \(H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)\), we obtain a path \(\phi_t : [-1, 1] \to R_G(M)\) which has \(\phi_0 = \phi\) and is transverse to the orbit of \(\phi\) at \(t = 0\).

We note that since \(\phi\) is a smooth point of \(R_G(M)\), it is contained in a unique irreducible component of \(R_G(M)\). Since the abelian representations of \(\pi_1(M)\) form an irreducible component of dimension 3, \(\phi\) is locally four-dimensional. In the proof of Theorem 2.11, it suffices to prove that the arc \(\phi_t\) is contained in \(X_G(M)\) which contains the character \([\rho_0] = [\rho^+_\xi]\) coming from an abelian representation. By [HP05, Proposition 10.2], \([\rho_0] = [\rho^+_\xi]\) is contained in precisely two real curves of characters. One of the curves is associated with abelian representations, and the other one with irreducible representations. Since \(\phi_t\) is non-abelian representation for all \(t\), the path \([\phi_t]\) is contained in the curve of irreducible characters.

Therefore, for \(t \neq 0\), the character \([\phi_t]\) is the character of some irreducible representation. Up to shrinking either \(\rho_t\) or \(\phi_t\), we may assume that \([\phi_t] = [\rho_t]\) for all \(t\). Since \(\phi_t\) has the same character as an irreducible representation \(\rho_t\) for \(t \neq 0\), the representation \(\phi_t\) is conjugate to \(\rho_t\) for all \(t \neq 0\) by [CS83, Proposition 1.5.2]. Since \(\rho_t\) factors through \(M(0)\) for all \(t\) we also get that \(\phi_t\) factors through \(M(0)\) for all \(t\). We obtain a path \([\phi_t]\) in \(R_G(M(0))\) going through \(\phi = \rho^+_\xi\) that is transverse to the orbit of \(\rho^+_\xi\). The existence of this path implies that

\[
\dim \mathbb{R} Z^1(G(0), \mathfrak{sl}_2(\mathbb{R})_{\rho^+_\xi}) \geq \dim \mathbb{R} Z^1_{\text{zar}}(R(G(0))) \geq 1 + \dim \mathbb{R} B^1(G(0), \mathfrak{sl}_2(\mathbb{R})_{\rho^+_\xi}) = 4.
\]

This would imply that \(\dim \mathbb{R} H^1(G(0), \mathfrak{sl}_2(\mathbb{R})_{\rho^+_\xi}) \geq 1\). Since \(\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C}\), we have

\[
H^1(G(0), \mathfrak{sl}_2(\mathbb{C})_{\rho^+_\xi}) = H^1(G(0), \mathfrak{sl}_2(\mathbb{R})_{\rho^+_\xi}) \otimes \mathbb{C}.
\]

Therefore, the dimension of \(H^1(G(0), \mathfrak{sl}_2(\mathbb{C})_{\rho^+_\xi})\) is at least 1. This gives a desired contradiction to the condition that \(M\) is locally longitudinally rigid at \(\xi\).

3 The \([1,1,2,2,2,2]\) two-bridge knots

In this section, we apply [Theorem 1.8] to study left-orderability on the family of two-bridge knots \(K_j\) associated to the continued fraction \([1, 1, 2, 2, 2, 2]\) for \(j \geq 1\) and prove [Theorem 1.9] We first make some remarks about this family of two-bridge knot complements.

These knot complements are obtained by doing \(1/j\) Dehn filling on the unknot component of the link \(L_{25}^j\), see Figure 1. The first two members of the family are the knots \(8_{12}\) and \(10_{13}\) in Rolfsen’s table. As we will see in Lemma 3.2, the Alexander polynomial of \(K_j\) has all simple positive real roots, that are not 1, and is not monic for \(j \geq 2\). In particular, the complement of \(K_j\) is not lean for \(j \geq 2\). Furthermore, the trace field of \(K_j\) for \(1 \leq j \leq 30\) has no real places, and it is most likely that the trace fields of all knots in this family share this property. Therefore, [Theorem 1.9] is not a direct consequence of [Theorem 1.3] or [Theorem 1.5].
The family of two-bridge knots \([1, 1, 2, 2, 2j]\) is a new family of knots with an interval left-orderable Dehn surgeries which cannot be obtained from prior techniques.

### 3.1 Group presentation

We will denote by \( \Gamma \) the fundamental group of the complement of the knot \( K_j \). The knot corresponding to the continued fraction \([1, 1, 2, 2, 2j]\) has the associated fraction

\[
[1, 1, 2, 2, 2j] = \frac{1}{1 + \frac{1}{2 + \frac{1}{x + \frac{1}{y + \frac{1}{\cdots}}}}} = \frac{24j + 5}{14j + 3}.
\]

By [Ril72 Proposition 1], the knot group \( \Gamma \) has the presentation \( \Gamma = \langle x, y | tw = wt \rangle \). The word \( w \) is given by

\[
w = y^{e_1}x^{e_2}\ldots y^{e_{24j+3}}x^{e_{24j+4}}
\]

where \( e_i = (-1)^{i(14j+3)/(24j+5)} \). Also by [Ril72 Proposition 1], the homological longitude of \( K_j \) that commutes with \( x \) is given by \( \ell = wv \) where

\[
v = x^{e_{24j+4}}y^{e_{24j+3}}\ldots x^{e_2}y^{e_1}.
\]

We first give an explicit description of \( w \) in terms of \( x \) and \( y \) by giving a formula for the right-hand sides of Equation (5) and Equation (6). We have the following lemma.

**Lemma 3.1.** In the terms of the generators \( x, y \) of \( \Gamma \), the word \( w \) has the form

\[
w = (yx^{-1}y^{-1}x)u^{j}\quad \text{and} \quad v = s^{j}(xy^{-1}x^{-1}y)
\]

where

\[
u = (yx^{-1}yx)(y^{-1}x^{-1}yx)^{-1}(y^{-1}yx)^{-1}(y^{-1}x^{-1}y^{-1})(y^{-1}y^{-1}x)(yx^{-1}y^{-1}x)
\]

and \( s \) is \( u \) spelled backwards.

**Proof.** Since \( v \) is \( w \) spelled backwards, it suffices to prove the lemma for \( w \). Let us consider

\[
k_{i,j} = \frac{i(14j+3)}{24j+5}
\]

for \( 1 \leq i \leq 24j + 4 \). We first claim that

\[
[k_{i,j}] = [k_{i,m}] = \left\lfloor \frac{7i}{12} \right\rfloor
\]

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for all \( j \geq m \) and \( \varepsilon_m \leq i \leq 24m + 4 \) where \( \varepsilon_m = \max\{1, 24(m - 1) + 5\} \). Fixing \( i \), we can view \( k_{i,j} \) as a continuous function in the variable \( j \). Since \( i \geq 1 \), the derivative of \( k_{i,j} \) with respect to \( j \) is

\[
\frac{dk_{i,j}}{dj} = -\frac{2i}{(24j + 5)^2} < 0.
\]

The function \( k_{i,j} \) is strictly decreasing with and has a horizontal asymptote at \( 7i/12 \) as \( j \to +\infty \). Therefore, we have the following chain of inequalities

\[
\frac{7i}{12} < k_{i,j} < k_{i,m} = \frac{(14m + 3)i}{24m + 5}
\]

for all \( j \geq m \) and \( \varepsilon_m \leq i \leq 24m + 4 \). We have

\[
0 < k_{i,m} - \left\lfloor \frac{7i}{12} \right\rfloor \leq k_{i,m} - \frac{7i}{12} + \frac{11}{12} = \frac{264m + 55 + i}{288m + 60} < 1
\]

for all \( \varepsilon_m \leq i \leq 24m + 4 \). It follows that \( k_{i,j} \) is contained in the interval \( (\lfloor \frac{7i}{12} \rfloor, \lfloor \frac{7i}{12} \rfloor + 1) \) for all \( j \geq m \) and \( \varepsilon_m \leq i \leq 24m + 4 \). To verify \([\text{Equation } (8)]\) it remains to show that \( k_{i,m} \) is not an integer for all \( \varepsilon_m \leq i \leq 24m + 4 \). Since \( 14m + 3 \) and \( 24m + 5 \) are relatively prime, \( k_{i,m} \) is an integer if and only if \( 24m + 5 \) divides \( i \). But this is not possible since \( i \leq 24m + 5 \).

By a direct computation, we can verify \([\text{Equation } (7)]\) when \( j = 1 \). From \([\text{Equation } (8)]\) we see that the right-hand side of \([\text{Equation } (5)]\) has prefix \( w_1 = yx^{-1}y^{-1}xu \) for all \( j \geq 1 \). We write \( w = w_1w'_j = (yx^{-1}y^{-1}x)uw'_j \). It remains to show that \( w'_j = w^{j-1} \). Using \([\text{Equation } (8)]\) we have

\[
|k_{i+24n,j}| = \left\lfloor \frac{7i}{12} + 14n \right\rfloor = \frac{7i}{12} + 14n = |k_{i,j}| + 14n
\]

for all \( 5 \leq i \leq 28 \) and \( 5 \leq i + 24n \leq 24j + 4 \). We have

\[
|k_{i,j}| \equiv |k_{i+24n,j}| \mod 2 \tag{9}
\]

for all \( 5 \leq i \leq 28 \) and \( 5 \leq i + 24n \leq 24j + 4 \). \([\text{Equation } (9)]\) implies that the parity of \( |k_{i,j}| \) repeats with period 24 when \( i \geq 5 \). Since the word for \( w \) in \( x \) and \( y \) only depends on this parity, the word \( w \) is given by \([\text{Equation } (7)]\) as claimed. This completes the proof of the lemma.

\[\square\]

### 3.2 The Alexander polynomial of \( K_j \)

Now we will compute the Alexander polynomial of \( K_j \) using non-abelian reducible representations. Let \( \rho : \Gamma \to \text{SL}_2(\mathbb{C}) \) be a non-abelian reducible representation of \( \Gamma \). Since \( \Gamma \) is generated by two conjugate meridians \( x \) and \( y \), the representation \( \rho \) can be conjugated to have the form

\[
x \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}
\tag{10}
\]

where \( t \neq \pm 1 \). By \([\text{Theorem } 2.14]\) for a knot group \( \Gamma \) the assignment in \([\text{Equation } (10)]\) defines a representation of \( \Gamma \) if and only if \( t^2 \) is a root of the Alexander polynomial \( \Delta(\tau) \in \mathbb{Z}[[\tau^{\pm 1}]] \). Consequently, we can use this fact to compute the Alexander polynomial of the knot \( K_j \) as follows.

Let \( F_2 \) be the free group on two letters \( X \) and \( Y \). Consider the representation \( P : F_2 \to \text{SL}_2(\mathbb{Z}[[\tau^{\pm 1}]]) \)

\[
X \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} \tau & 1 \\ 0 & \tau^{-1} \end{pmatrix}.
\]

Let \( W \) be the word in \( X \) and \( Y \) given by \([\text{Equation } (7)]\) A direct calculation shows that

\[
P(W) = \begin{pmatrix} 1 & -j\tau^3 + (5j + 1)\tau - (5j + 1)\tau^{-1} + j\tau^{-3} \\ 0 & 1 \end{pmatrix}
\]

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The representation $P$ factors through the natural projection $F_2 \to \Gamma$ if and only if $P(XW) = P(WY)$. Or equivalently, we have
\[ j\tau^4 - (6j + 1)\tau^2 + (10j + 3) - (6j + 1)\tau^{-2} + j\tau^{-4} = 0. \]
The expression above is the Alexander polynomial of $K_j$ evaluated at $\tau^2$. As a convention, we will normalize the Alexander polynomial so that the lowest term of $\Delta(\tau)$ is a non-zero constant term. We have the following lemma.

**Lemma 3.2.** The Alexander polynomial of $K_j$ has the form
\[ \Delta(\tau) = j\tau^4 - (6j + 1)\tau^3 + (10j + 3)\tau^2 - (6j + 1)\tau + j. \] Furthermore, $\Delta(\tau)$ has exactly 4 simple real roots.

**Proof.** The discussion prior to the lemma implies that
\[ \Delta(\tau^2) = j\tau^8 - (6j + 1)\tau^6 + (10j + 3)\tau^4 - (6j + 1)\tau^2 + j. \]
This gives us Equation (11) as claimed. For the claim about the roots of $\Delta$, we consider $\delta(\tau) = \Delta(\tau)/j$. We note that
\[ \delta_j(0) = 1, \quad \delta_j(1/2) = \frac{-3j + 2}{16j}, \quad \delta_j(1) = \frac{1}{j}, \quad \delta_j(2) = \frac{-3j + 2}{j}, \quad \delta_j(5) = \frac{96j - 55}{j}. \]
For all $j \geq 1$, we see that $\delta_j$ changes signs 4 times in the interval $[0, 5]$. By continuity, $\delta_j(\tau)$ has 4 distinct real roots in the interval $[0, 5]$. Therefore, $\Delta(\tau)$ has at least 4 positive real roots. Since $\Delta$ has degree 4, $\Delta$ has precisely 4 simple positive real roots for all $j \geq 1$.

### 3.3 The group cohomology $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C}))$

In this section, we will prove that the knots $K_j$ are locally longitudinal rigid by directly computing the group cohomology with coefficients in $\mathfrak{sl}_2(\mathbb{C})$. We first identify $\mathfrak{sl}_2(\mathbb{C})$ with $\mathbb{C}^3$ by choosing the following basis
\[ v_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad v_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]
With respect to this basis, the adjoint representation $\text{Ad} : \text{SL}_2(\mathbb{C}) \to \text{SL}_3(\mathbb{C})$ becomes
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad + bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}. \]

By Lemma 3.2, we can choose $t \in \mathbb{R}$ such that $t^2$ is a simple root of the Alexander polynomial $\Delta(\tau)$. Since the longitude $t$ belongs to the second commutator subgroup of $\Gamma$, any non-abelian reducible representation on $\Gamma$ factors through $\Gamma(0)$. We get a non-abelian reducible representation $\rho : \Gamma(0) \to \text{SL}_2(\mathbb{C})$ given by Equation (10). For convenience, we will write
\[ \rho(w) = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \]
where $f = -jt^3 + (5j + 1)t - (5j + 1)t^{-1} + jt^{-3}$. The action of $\Gamma(0)$ on $\mathfrak{sl}_2(\mathbb{C})$ is given by
\[ x \mapsto \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} t^2 & -2t & -1 \\ 0 & 1 & t^{-1} \\ 0 & 0 & t^{-2} \end{pmatrix}. \]
Using Equation (2), we see that the space of coboundaries can be parametrized by $d : \Gamma(0) \to \mathfrak{sl}_2(\mathbb{C})_\rho$ such that
\[ d(x) = \begin{pmatrix} (t^2 - 1)a & 0 \\ 0 & (t^2 - 1)c \end{pmatrix} \quad \text{and} \quad d(y) = \begin{pmatrix} (t^2 - 1)a - 2tb - c \\ t^{-1}c \\ (t^{-2} - 1)c \end{pmatrix}. \]
Proposition 3.3. Any cohomology class in $H^1(\Gamma(0); sl_2(\mathbb{C})_\rho)$ can be represented by a 1-cocycle $z \in Z^1(\Gamma(0); sl_2(\mathbb{C})_\rho)$ such that

$$z(x) = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad z(y) = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}.$$  \tag{13}

Proof. Let $z \in Z^1(\Gamma(0); sl_2(\mathbb{C})_\rho)$ be a 1-cocycle. Since $t^2 \neq 1$, by an appropriate choice of $a, b, c \in \mathbb{C}$ for a coboundary $d$ in Equation (12), we can assume that

$$z(x) = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad z(y) = \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}.$$  

The relation $z(wx) = z(xy)$ implies that

$$z(x) + (x-1) \cdot z(w) - w \cdot z(y) = 0.$$  

Or equivalently, we have

$$\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} t^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^{-2} - 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} - \begin{pmatrix} 1 & -2f & -f^2 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$  

The second coordinate of the previous equation implies that $\delta = \alpha$. \qed

We will need the following lemma

Lemma 3.4. Let $z \in Z^1(\Gamma(0); sl_2(\mathbb{C})_\rho)$ be given by Equation (13). Suppose that

$$z(w) = \omega_1 v_+ + \omega_2 v_0 + \omega_3 v_- \quad \text{and} \quad z(v) = \nu_1 v_+ + \nu_2 v_0 + \nu_3 v_-.$$  

Then

$$\omega_1 = \alpha(-4jt^3 + (10j + 2)t^2 - 2jt^{-3}) + \beta h,$$

$$\omega_2 = \left(\frac{1}{2}j(j+1)t^7 - 5j(j+1)t^5 + \frac{1}{2}(35j^2 + 31j + 2)t^3 - \frac{1}{2}(52j^2 + 28j + 4)t \right. \right.$$  

$$+ \frac{1}{2}j(35j - 3)t^{-1} - \frac{1}{2}j(10j - 6)t^{-3} + \frac{1}{2}j(j - 1)t^{-5}\left.\beta \right)$$  

$$\nu_2 = \left(\frac{1}{2}j(j+1)t^7 - 5j(j+1)t^5 + \frac{1}{2}(35j^2 + 27j + 2)t^3 - \frac{1}{2}(52j^2 + 12j)t \right. \right.$$  

$$+ \frac{1}{2}j(35j - 7)t^{-1} - \frac{1}{2}j(10j - 6)t^{-3} + \frac{1}{2}j(j - 1)t^{-5}\left.\beta \right)$$  

$$\omega_3 = -\nu_3 = tf\beta$$  

where $h \in \mathbb{C}$.

Proof. The proof of this lemma is a direct calculation. By a repeat application of the cocycle relation in Equation (1) we have

$$z(w) = z(xy^{-1}y^{-1}x) + (y^{-1}y^{-1}x) \cdot \sum_{i=0}^{j-1} u^i \cdot z(u),$$

$$z(v) = s^j \cdot z(xy^{-1}y^{-1}y) + \sum_{i=0}^{j-1} s^i \cdot z(s).$$  

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We also have
\[ \sum_{i=0}^{j-1} (\mathrm{Ad} \circ \rho)(u^i) = \begin{pmatrix} j & (j^2 - j)(t^3 - 5t + 5t^{-1} - t^{-3}) & -\frac{1}{6}(2j^3 - 3j^2 + j)(t^3 - 5t + 5t^{-1} - t^{-3})^2 \\ 0 & j & -\frac{1}{2}(j^2 - j)(t^3 - 5t + 5t^{-1} - t^{-3}) \\ 0 & 0 & j \end{pmatrix}, \]
\[ \sum_{i=0}^{j-1} (\mathrm{Ad} \circ \rho)(s^i) = \begin{pmatrix} j & -(j^2 - j)(t^3 - 5t + 5t^{-1} - t^{-3}) & -\frac{1}{6}(2j^3 - 3j^2 + j)(t^3 - 5t + 5t^{-1} - t^{-3})^2 \\ 0 & j & -\frac{1}{2}(j^2 - j)(t^3 - 5t + 5t^{-1} - t^{-3}) \\ 0 & 0 & j \end{pmatrix}, \]
and
\[ z(u) = \begin{pmatrix} -(4t^3 + 10t - 2t^{-3})\alpha + h'\beta \\ (t^7 - 9t^5 + 27t^3 - 30t + 10t^{-1} - t^{-3})\beta \\ -(t^4 + 5t^2 - 5 + t^{-2})\beta \end{pmatrix} \quad \text{and} \quad z(s) = \begin{pmatrix} (4t^3 - 10t + 2t^{-3})\alpha + h''\beta \\ (t^7 - 9t^5 + 27t^3 - 22t + 8t^{-1} - t^{-3})\beta \\ (t^4 - 5t^2 + 5 - t^{-2})\beta \end{pmatrix}. \]
for some $h', h'' \in \mathbb{C}$. The lemma will follow once we note that
\[ z(yx^{-1}y^{-1}x) = \begin{pmatrix} 2t\alpha - (t^4 - 3t^2 + 1)\beta \\ (t^3 - 2t)\beta \\ (t^2 - 1)\beta \end{pmatrix} \quad \text{and} \quad z(xy^{-1}x^{-1}y) = \begin{pmatrix} (-6t + 2t^{-1})\alpha + t^4\beta \\ t^3\beta \\ (-t^2 + 1)\beta \end{pmatrix}. \]

Now we are ready to show that $H^1(\Gamma(0); \mathfrak{s}_{\mathbb{C}}(\rho)) = 0$.

**Proof of Theorem 1.9.** Now we will show that $K_j$ is locally longitudinally rigid for all $j \geq 1$ at any root of the Alexander polynomial. Let $[z] \in H^1(\Gamma(0); \mathfrak{s}_{\mathbb{C}}(\rho))$. By Proposition 3.3, we can assume that $z$ satisfies Equation (13). Since $z(\ell) = 0$, we must have
\[ z(w) + w \cdot z(v) = 0. \]

Let us write $z(w) = \omega_1v_+ + \omega_2v_0 + \omega_3v_-$ and $z(v) = \nu_1v_+ + \nu_2v_0 + \nu_3v_-$. The second coordinate of this equation implies that $\omega_2 + \nu_2 + f\nu_3 = 0$. Using Lemma 3.4, we have
\[ \frac{(t^4 - 1)(t^4 + j(t^4 - 4t^2 + 1)^2)}{t^5} \beta = 0. \]

Suppose that $\beta \neq 0$. Since $t \in \mathbb{R} \setminus \{ \pm 1 \}$, we have
\[ t^4 + j(t^4 - 4t^2 + 1)^2 = 0. \]

Since $t \in \mathbb{R}$ and $j \geq 1$, the above equation holds if and only if
\[ t = 0 \quad \text{and} \quad t^4 - 4t^2 + 1 = 0. \]

This is the desired contradiction. Therefore, we must have $\beta = 0$.

From the first coordinate of the relation $z(xw) = z(wy)$, we have $(t^2 - 1)\omega_1 + 2f\alpha = 0$. Using $\beta = 0$ and Lemma 3.4, this equation is equivalent to
\[ (t^4 - 1)(2jt^4 - (6j + 1)t^2 + 2j))\alpha = 0. \]
Similarly, if $\alpha \neq 0$, we must have $(2jt^4 - (6j + 1)t^2 + 2j) = 0$. It follows that $t^2$ is a root of both
\[ \Delta(\tau) \quad \text{and} \quad h(\tau) := (2jt^2 - (6j + 1)\tau + 2j). \]

Note that the roots of $h(\tau)$ are reciprocal of each other. Since $\Delta(\tau)$ is a reciprocal polynomial, the roots of $\Delta(\tau)$ come in reciprocal pairs. Therefore, $h(\tau)$ divides $\Delta(\tau)$. By Gauss’s lemma, we can write $\Delta(\tau) = h\kappa(\tau)$ for $\kappa(\tau) \in \mathbb{Z}[\tau]$. This implies that $\Delta(0) = h(0)k(0)$ or $j = 2k(0)$. This contradicts the fact that $j \geq 1$ and $k(0) \in \mathbb{Z}$. Therefore, $\alpha = 0$ and $z$ can only be the zero cocycle. Consequently, $H^1(\Gamma(0); \mathfrak{s}_{\mathbb{C}}(\rho)) = 0$ where $\rho$ is any non-abelian reducible representation of $\Gamma(0)$. In other words, the knot $K_j$ is locally longitudinally rigid at any root of the Alexander polynomial. By Theorem 1.8, there exists an interval of left-orderable Dehn surgeries near 0. 
\[ \square \]
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