Identities of symmetry for generalized Euler polynomials

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Abstract. In this paper, we derive eight basic identities of symmetry in three variables related to generalized Euler polynomials and alternating generalized power sums. All of these are new, since there have been results only about identities of symmetry in two variables. The derivations of identities are based on the $p$-adic fermionic integral expression of the generating function for the generalized Euler polynomials and the quotient of integrals that can be expressed as the exponential generating function for the alternating generalized power sums.

1. Introduction and preliminaries

Let $p$ be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $d$ be a fixed odd positive integer. Then we let

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z},$$

and let $\pi : X \to \mathbb{Z}_p$ be the map given by the inverse limit of the natural maps

$$\mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}.$$ 

If $g$ is a function on $\mathbb{Z}_p$, we will use the same notation to denote the function $g \circ \pi$. Let $\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{Q}_p^*$ be a (primitive) Dirichlet character of conductor $d$. Then it will be pulled back to $X$ via the natural map $X \to \mathbb{Z}/d\mathbb{Z}$. Here we fix, once and for all, an imbedding $\mathbb{Q} \to \mathbb{C}_p$, so that $\chi$ is regarded as a map of $X$ to $\mathbb{C}_p$ (cf. [3]).

For a continuous function $f : X \to \mathbb{C}_p$, the $p$-adic fermionic integral of $f$ is defined by

$$\int_X f(z)d\mu_{-1}(z) = \lim_{N \to \infty} \sum_{j=0}^{dp^N-1} f(j)(-1)^j.$$

Then it is easy to see that

$$\int_X f(z+1)d\mu_{-1}(z) + \int_X f(z)d\mu_{-1}(z) = 2f(0). \quad (1.1)$$

More generally, we deduce from (1.1) that, for any odd positive integer $n,$

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\( \int_X f(z + n)d\mu_{-1}(z) + \int_X f(z)d\mu_{-1}(z) = 2 \sum_{a=0}^{n-1} (-1)^a f(a) \),

and that, for any even positive integer \( n \),
\[
\int_X f(z + n)d\mu_{-1}(z) - \int_X f(z)d\mu_{-1}(z) = 2 \sum_{a=0}^{n-1} (-1)^{a-1} f(a).
\]

Let \( | \cdot |_p \) be the normalized absolute value of \( \mathbb{C}_p \), such that \( |p|_p = \frac{1}{p} \), and let
\[
E = \left\{ t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}} \right\}.
\]

Then, for each fixed \( t \in E \), the function \( e^{zt} \) is analytic on \( \mathbb{Z}_p \) and hence considered as a function on \( X \), and, by applying (1.2) to \( f \) with \( f(z) = \chi(z)e^{zt} \), we get the \( p \)-adic integral expression of the generating function for the generalized Euler numbers \( E_{n,\chi} \) attached to \( \chi \):

\[
\int_X \chi(z)e^{zt}d\mu_{-1}(z) = \frac{2}{e^{zt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a)e^{at} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!} \quad (t \in E).
\]

So we have the following \( p \)-adic integral expression of the generating function for the generalized Euler polynomials \( E_{n,\chi}(x) \) attached to \( \chi \):

\[
\int_X \chi(z)e^{(x+z)t}d\mu_{-1}(z) = \frac{2e^{xt}}{e^{zt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a)e^{at} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p).
\]

Also, from (1.1) we have:

\[
\int_X e^{zt}d\mu_{-1}(z) = \frac{2}{e^{zt} + 1} \quad (t \in E).
\]

Let \( T_k(n, \chi) \) denote the \( k \)th alternating generalized power sum of the first \( n + 1 \) nonnegative integers attached to \( \chi \), namely

\[
T_k(n, \chi) = \sum_{a=0}^{n} (-1)^a \chi(a)a^k = (-1)^0\chi(0)0^k + (-1)^1\chi(1)1^k + \cdots + (-1)^n\chi(n)n^k.
\]

From (1.2), (1.6), and (1.7), one easily derives the following identities: for any odd positive integer \( w \),
In what follows, we will always assume that the $p$-adic integrals of the various (twisted) exponential functions on $X$ are defined for $t \in E$ (cf. (1.3)), and therefore it will not be mentioned.

(1.8) \[
\begin{align*}
\int_X \chi(x)e^{xt}d\mu_{-1}(x) &= \frac{e^{wdt} + 1}{e^{dt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a)e^{at} \\
&= \sum_{a=0}^{wd-1} (-1)^a \chi(a)e^{at} \\
&= \sum_{k=0}^{\infty} T_k(wd - 1, \chi) \frac{t^k}{k!} \quad (t \in E).
\end{align*}
\]

In what follows, we will always assume that the $p$-adic integrals of the various (twisted) exponential functions on $X$ are defined for $t \in E$ (cf. (1.3)), and therefore it will not be mentioned.

\[\text{ IDENTITIES OF SYMMETRY FOR GENERALIZED EULER POLYNOMIALS}\]

(1.9) \[
\frac{\int_X \chi(x)e^{xt}d\mu_{-1}(x)}{\int_X e^{wdt}d\mu_{-1}(y)} = \frac{e^{wdt} + 1}{e^{dt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a)e^{at}
\]

(1.10) \[
= \sum_{a=0}^{wd-1} (-1)^a \chi(a)e^{at}
\]

In this paper, we will be able to produce 8 identities of symmetry in three variables regarding generalized Euler polynomials and alternating generalized power sums. The case of two variables was treated in [7].

The following is stated as Theorem 4.2 and an example of the full six symmetries in $w_1, w_2, w_3$.

\[
\begin{align*}
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_1 y_1) E_{l, \chi}(w_2 y_2) T_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_1 y_1) E_{l, \chi}(w_3 y_2) T_m(w_2 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_2 y_1) E_{l, \chi}(w_1 y_2) T_m(w_3 d - 1, \chi) w_2^{l+m} w_1^{k+m} w_3^{k+l} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_3 y_1) E_{l, \chi}(w_2 y_2) T_m(w_1 d - 1, \chi) w_3^{l+m} w_2^{k+m} w_1^{k+l} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_2 y_1) E_{l, \chi}(w_3 y_2) T_m(w_1 d - 1, \chi) w_2^{l+m} w_3^{k+m} w_1^{k+l} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_3 y_1) E_{l, \chi}(w_1 y_2) T_m(w_2 d - 1, \chi) w_3^{l+m} w_1^{k+m} w_2^{k+l}.
\end{align*}
\]

The derivations of identities are based on the $p$-adic integral expression of the generating function for the generalized Euler polynomials in (1.4) and the quotient of integrals in (1.8)-(1.10) that can be expressed as the exponential generating function for the alternating generalized power sums. These abundance of symmetries would not be unearthed if such $p$-adic integral representations had not been available. We indebted this idea to the paper [7].
2. Several types of quotients of $p$-adic fermionic integrals

Here we will introduce several types of quotients of $p$-adic fermionic integrals on $X$ or $X^3$ from which some interesting identities follow owing to the built-in symmetries in $w_1, w_2, w_3$. In the following, $w_1, w_2, w_3$ are all positive integers and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identities in (2.1) and (2.0). To ease notations, from now on we will suppress $\mu_{-1}$ and denote, for example, $d\mu_{-1}(x)$ simply by $dx$.

(a) Type $\Lambda_1^i$ (for $i = 0, 1, 2, 3$)

\[
I(\Lambda_{23}^i) = \frac{\int_{X^3} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 w_2 x_1 + w_1 w_2 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-1} y_j)) t} dx_1 dx_2 dx_3}{(\int_X e^{d(w_1 w_2 x_1 + w_2 x_2 + x_3) t} dx_4)^t}
\]

\[
= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-1} y_j) t} (e^{d w_1 w_2 t} + 1)^i}{(e^{d w_1 t} + 1)(e^{d w_2 t} + 1)(e^{d w_3 t} + 1)}
\]

\[
I(\Lambda_{13}^i) = \frac{\int_{X^3} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-1} y_j)) t} dx_1 dx_2 dx_3}{(\int_X e^{d(w_1 w_2 x_1 + w_2 x_2 + x_3) t} dx_4)^t}
\]

\[
= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-1} y_j) t} (e^{d w_1 w_2 t} + 1)^i}{(e^{d w_1 t} + 1)(e^{d w_2 t} + 1)(e^{d w_3 t} + 1)}
\]

\[
\times \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_2 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_3 t} \right).
\]

(c-0) Type $\Lambda_0^0$ (for $i = 0, 1, 2, 3$)

\[
I(\Lambda_{12}^0) = \int_{X^3} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 y + w_1 w_3 y + w_1 w_2 y) t} dx_1 dx_2 dx_3
\]

\[
= \frac{8 e^{(w_2 w_3 + w_1 w_3 + w_1 w_2) y t}}{(e^{d w_1 t} + 1)(e^{d w_2 t} + 1)(e^{d w_3 t} + 1)}
\]

\[
\times \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_2 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_3 t} \right).
\]

(c-1) Type $\Lambda_1^0$ (for $i = 0, 1, 2, 3$)

\[
I(\Lambda_{12}^1) = \frac{\int_{X^3} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3) t} dx_1 dx_2 dx_3}{(\int_{X^3} e^{d(w_1 w_2 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3) t} dx_2 dx_3)}
\]

\[
= \frac{8 e^{(w_2 w_3 + w_1 w_3 + w_1 w_2) y t}}{(e^{d w_1 t} + 1)(e^{d w_2 t} + 1)(e^{d w_3 t} + 1)}
\]

\[
\times \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_2 t} \right) \left( \prod_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_3 t} \right).
\]
where the inner sum is over all nonnegative integers 
(3.1)
\[ (c-0), \text{where they are any positive integers.} \]
First, let’s consider Type $\Lambda$
(2.8).
(2.1), (2.3), (2.5), and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6),
(a-0)

In the following,
All of the above
identities for generalized Euler polynomials
Identities for generalized Euler polynomials
In two different ways:
(2.8)
I
I
$\Lambda$
$\Lambda$
1
1
2
2
3
3

3. IDENTITIES FOR GENERALIZED EULER POLYNOMIALS

In the following, $w_1$, $w_2$, $w_3$ are all odd positive integers except for (a-0) and (c-0), where they are any positive integers. First, let’s consider Type $\Lambda_{23}$, for each $i = 0, 1, 2, 3$. The following results can be easily obtained from (1.6) and (1.8)-(1.10),
(a-0)

\[ I(\Lambda_{23}) = \int_x \chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1 \int_x \chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2 \]
\[ \times \int_x \chi(x_3)e^{w_1w_2(x_3+w_3y_3)t}dx_3 \]
(3.1)
\[ = \sum_{k=0}^{\infty} \frac{E_{k,\chi}(w_1y_1)}{k!}(w_2w_3t)^k \sum_{l=0}^{\infty} \frac{E_{l,\chi}(w_2y_2)}{l!}(w_1w_3t)^l \]
\[ \times \sum_{m=0}^{\infty} \frac{E_{m,\chi}(w_3y_3)}{m!}(w_1w_2t)^m \]

where the inner sum is over all nonnegative integers $k$, $l$, $m$ with $k + l + m = n$, and

\[ \binom{n}{k, l, m} = \frac{n!}{k!l!m!} \]

(a-1) Here we write $I(\Lambda_{23})$ in two different ways:

\[ I(\Lambda_{23}) = \int_x \chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1 \int_x \chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2 \]
\[ \times \int_x \chi(x_3)e^{w_1w_2(x_3+w_3y_3)t}dx_3 \]
(3.3)
\[ = \sum_{k=0}^{\infty} \frac{E_{k,\chi}(w_1y_1)}{k!}(w_2w_3t)^k \sum_{l=0}^{\infty} \frac{E_{l,\chi}(w_2y_2)}{l!}(w_1w_3t)^l \]
\[ \times \sum_{m=0}^{\infty} \frac{T_m(w_3d - 1, \chi)(w_1w_2t)^m}{m!} \]
(3.4) \[ I = \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_1 y_1) E_{l,\chi}(w_2 y_2) \right) \times T_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} \frac{t^n}{n!} \]

(2) Invoking (1.9), \[ I \] can also be written as

\[ I(A_{23}^2) = \sum_{a=0}^{w_3 d - 1} (-1)^a \chi(a) \int_X \chi(x_1) e^{w_2 w_3 (x_1 + w_1 y_1) t} \, dx_1 \]

\[ \times \int_X \chi(x_2) e^{w_1 w_3 (x_2 + w_2 y_2 + \frac{w_2}{w_3} a) t} \, dx_2 \]

\[ = \sum_{a=0}^{w_3 d - 1} (-1)^a \chi(a) \left( \sum_{k=0}^{\infty} E_{k,\chi}(w_1 y_1) \frac{(w_2 y_3 t)^k}{k!} \right) \]

\[ \times \left( \sum_{l=0}^{\infty} E_{l,\chi}(w_2 y_2 + \frac{w_2}{w_3} a) \frac{(w_1 y_3 t)^l}{l!} \right) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} E_{k,\chi}(w_1 y_1) \sum_{a=0}^{w_3 d - 1} (-1)^a \chi(a) \right) \]

\[ \times E_{n-k,\chi}(w_2 y_2 + \frac{w_2}{w_3} a) w_1^{n-k} w_3^{k} \frac{t^n}{n!} \]

(a-2) Here we write \[ I(A_{23}^2) \] in three different ways:

(1) \[ I(A_{23}^2) = \int_X \chi(x_1) e^{w_2 w_3 (x_1 + w_1 y_1) t} \, dx_1 \times \frac{\int_X \chi(x_2) e^{w_1 w_2 w_3 x_2 t} \, dx_2}{\int_X e^{w_1 w_2 w_3 x_2 t} \, dx_2} \]

\[ \times \frac{\int_X \chi(x_3) e^{w_1 w_2 x_3 t} \, dx_3}{\int_X e^{w_1 w_2 x_3 t} \, dx_3} \]

\[ = \left( \sum_{k=0}^{\infty} E_{k,\chi}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} T_l(w_2 d - 1, \chi) \frac{(w_1 y_3 t)^l}{l!} \right) \]

\[ \times \left( \sum_{m=0}^{\infty} T_m(w_3 d - 1, \chi) \frac{(w_1 w_2 t)^m}{m!} \right) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_1 y_1) T_l(w_2 d - 1, \chi) \right) \]

\[ \times T_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} \frac{t^n}{n!} \]

(2) Invoking (1.9), \[ I \] can also be written as
\begin{align*}
I(A_{23}^2) &= \sum_{a=0}^{w_2d-1} (-1)^a \chi(a) \int_X \chi(x_1) e^{w_2 w_3 (x_1 + w_1 y_1 + \frac{w_1}{w_2} a) t} dx_1 \\
&= \sum_{a=0}^{w_2d-1} (-1)^a \chi(a) \left( \sum_{k=0}^{\infty} E_{k, \chi}(w_1 y_1 + \frac{w_1}{w_2}) (\frac{w_2 w_3 t}{k!}) \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{w_2d-1} \binom{n}{k} \right) \left( \sum_{a=0}^{w_2d-1} (-1)^a \chi(a) E_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2}) \left( w_2 w_3 t \right)^{n-k} \frac{t^n}{n!} \right)
\end{align*}

(3.9) Invoking (1.9) once again, (3.8) can be written as

\begin{align*}
I(A_{23}^2) &= \sum_{a=0}^{w_2d-1} (-1)^a \chi(a) \sum_{b=0}^{w_2d-1} (-1)^b \chi(b) \int_X \chi(x_1) e^{w_2 w_3 (x_1 + w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) t} dx_1 \\
&= \sum_{a=0}^{w_2d-1} (-1)^a \chi(a) \sum_{b=0}^{w_2d-1} (-1)^b \chi(b) \sum_{n=0}^{\infty} E_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) \frac{(w_2 w_3 t)^n}{n!}
\end{align*}

(3.10) = \sum_{n=0}^{\infty} \left( w_2 w_3 \right)^n \sum_{a=0}^{w_2d-1} \sum_{b=0}^{w_2d-1} (-1)^{a+b} \chi(ab) E_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) \frac{t^n}{n!}

(3.11) \begin{align*}
I(A_{23}^2) &= \int_X \chi(x_1) e^{w_2 w_3 x_1 t} dx_1 \int_X e^{w_2 w_3 x_2 t} dx_2 \int_X e^{w_2 w_3 x_3 t} dx_3 \\
&= \sum_{k=0}^{\infty} T_k(w_1 y_1) \left( \frac{(w_2 w_3 t)^k}{k!} \right) \sum_{l=0}^{\infty} T_l(w_2 d - 1, \chi) \left( \frac{(w_1 w_2 t)^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} T_k(w_1 y_1) T_l(w_2 d - 1, \chi) T_m(w_3 d - 1, \chi) \right) \frac{t^n}{n!}
\end{align*}
(b) For Type $\Lambda_{13}^i$ ($i=0,1,2,3$), we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute $w_2w_3$, $w_1w_2$, $w_1w_3$ respectively for $w_1$, $w_2$, $w_3$ in (3.1), this amounts to replacing $t$ by $w_1w_2w_3t$ in (2.3). So, upon replacing $w_1$, $w_2$, $w_3$ respectively by $w_2w_3$, $w_1w_3$, and $w_1w_2$ and dividing by $(w_1w_2w_3)^n$, in each of the expressions of (3.1), (3.4), (3.5), (3.7), (3.9)-(3.11), we will get the corresponding symmetric identities for Type $\Lambda_{13}^i$ ($i=0,1,2,3$).

(c-0) \[ I(\Lambda_{12}^0) = \int_X \chi(x_1)e^{w_1(x_1+w_2y)t}dx_1 \int_X \chi(x_2)e^{w_2(x_2+w_3y)t}dx_2 \times \int_X \chi(x_3)e^{w_3(x_3+w_1y)t}dx_3 \]

\[ = \sum_{k=0}^{\infty} \frac{E_{k,\chi}(w_2y)}{k!}(w_1t)^k \sum_{l=0}^{\infty} \frac{E_{l,\chi}(w_3y)}{l!}(w_2t)^l \sum_{m=0}^{\infty} \frac{E_{m,\chi}(w_1y)}{m!}(w_3t)^m \]

\[ = \sum_{n=0}^{\infty} \sum_{k+l+m=n} \left( \frac{n}{k,l,m} \right) E_{k,\chi}(w_2y)E_{l,\chi}(w_3y)E_{m,\chi}(w_1y)w_1^{k}w_2^{l}w_3^{m} \frac{t^n}{n!}, \]

(c-1) \[ I(\Lambda_{12}^1) = \int_X \chi(x_1)e^{w_1x_1t}dx_1 \int_X e^{d_1w_1x_1t}dx_2 \int_X e^{d_2w_2x_2t}dx_3 \int_X e^{d_3w_3x_3t}dx_1 \]

\[ = \sum_{k=0}^{\infty} T_k(w_2d-1,\chi) \frac{(w_1t)^k}{k!} \sum_{l=0}^{\infty} T_l(w_3d-1,\chi) \frac{(w_2t)^l}{l!} \times \sum_{m=0}^{\infty} T_m(w_1d-1,\chi) \frac{(w_3t)^m}{m!} \]

\[ = \sum_{n=0}^{\infty} \sum_{k+l+m=n} \left( \frac{n}{k,l,m} \right) T_k(w_2d-1,\chi)T_l(w_3d-1,\chi) \times T_m(w_1d-1,\chi)w_1^{k}w_2^{l}w_3^{m} \frac{t^n}{n!}, \]

4. **Main Theorems**

As we noted earlier in the last paragraph of Section 2, the various types of quotients of $p$-adic fermionic integrals are invariant under any permutation of $w_1$, $w_2$, $w_3$. So the corresponding expressions in Section 3 are also invariant under any permutation of $w_1$, $w_2$, $w_3$. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting $w_1$, $w_2$, $w_3$ in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of $S_3$. In particular, the number of possible distinct expressions are 1, 2, 3, or 6 (a-0), (a-1(i)), (a-1(2)), and (a-2(2)) give the full six
identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.4 and 4.8, leaving the others as easy exercises for the reader. As for the case of Theorem 4.4, in addition to (4.11)-(4.13), we get the following three ones:

\[(4.1)\]
\[
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_1 y_1) T_i(w_3 d - 1, \chi) T_m(w_2 d - 1, \chi) w_1^{k+m} w_2^{k+l},
\]
\[(4.2)\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_2 y_1) T_i(w_1 d - 1, \chi) T_m(w_3 d - 1, \chi) w_2^{k+m} w_3^{k+l},
\]
\[(4.3)\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_3 y_1) T_i(w_2 d - 1, \chi) T_m(w_1 d - 1, \chi) w_3^{k+m} w_1^{k+l}.
\]

But, by interchanging \(l\) and \(m\), we see that (4.1), (4.2), and (4.3) are respectively equal to (4.11), (4.12), and (4.13). As to Theorem 4.8, in addition to (4.17) and (4.18), we have:

\[(4.4)\]
\[
\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_2 d - 1, \chi) T_i(w_3 d - 1, \chi) T_m(w_1 d - 1, \chi) w_1^{k} w_2^{l} w_3^{m},
\]
\[(4.5)\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_3 d - 1, \chi) T_i(w_1 d - 1, \chi) T_m(w_2 d - 1, \chi) w_2^{k} w_3^{l} w_1^{m},
\]
\[(4.6)\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_3 d - 1, \chi) T_i(w_2 d - 1, \chi) T_m(w_1 d - 1, \chi) w_3^{k} w_2^{l} w_1^{m},
\]
\[(4.7)\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_2 d - 1, \chi) T_i(w_1 d - 1, \chi) T_m(w_3 d - 1, \chi) w_2^{k} w_1^{l} w_3^{m}.
\]

However, (4.4) and (4.5) are equal to (4.17), as we can see by applying the permutations \(k \rightarrow l, l \rightarrow m, m \rightarrow k\) for (4.4) and \(k \rightarrow m, l \rightarrow k, m \rightarrow l\) for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.18), by applying permutations \(k \rightarrow l, l \rightarrow m, m \rightarrow k\) for (4.6) and \(k \rightarrow m, l \rightarrow k, m \rightarrow l\) for (4.7).
Theorem 4.1. Let \( w_1, w_2, w_3 \) be any positive integers. Then we have:

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_1 y_1) E_{l, \chi}(w_2 y_2) E_{m, \chi}(w_3 y_3) w_1^{l+m} w_2^{k+l} w_3^{k+l} = \\
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_2 y_1) E_{l, \chi}(w_3 y_2) E_{m, \chi}(w_1 y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l} = \\
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_3 y_1) E_{l, \chi}(w_1 y_2) E_{m, \chi}(w_2 y_3) w_1^{l+m} w_2^{k+l} w_3^{k+m}.
\]

(4.8)

Theorem 4.2. Let \( w_1, w_2, w_3 \) be any odd positive integers. Then we have:

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_1 y_1) E_{l, \chi}(w_2 y_2) T_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} = \\
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_2 y_1) E_{l, \chi}(w_3 y_2) T_m(w_1 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l} = \\
\sum_{k+l+m=n} \binom{n}{k, l, m} E_{k, \chi}(w_3 y_1) E_{l, \chi}(w_1 y_2) T_m(w_2 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l}.
\]

(4.9)

Theorem 4.3. Let \( w_1, w_2, w_3 \) be any odd positive integers. Then we have:

\[
w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_1 d-1} (-1)^a \chi(a) E_{n-k, \chi}(w_2 y_2) + \frac{w_2}{w_1} a w_3^{n-k} w_2^k = \\
w_2^n \sum_{k=0}^{n} \binom{n}{k} E_{k, \chi}(w_2 y_1) \sum_{a=0}^{w_2 d-1} (-1)^a \chi(a) E_{n-k, \chi}(w_3 y_2) + \frac{w_3}{w_1} a w_1^{n-k} w_3^k = \\
w_3^n \sum_{k=0}^{n} \binom{n}{k} E_{k, \chi}(w_1 y_1) \sum_{a=0}^{w_3 d-1} (-1)^a \chi(a) E_{n-k, \chi}(w_1 y_2) + \frac{w_1}{w_2} a w_3^{n-k} w_1^k.
\]

(4.10)
Theorem 4.4. Let $w_1$, $w_2$, $w_3$ be any odd positive integers. Then we have the following three symmetries in $w_1$, $w_2$, $w_3$:

\[(4.11)\]

\[
\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_1 y_1) T_1(w_2 d - 1, \chi) T_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{k+l}
\]

\[(4.12)\]

\[
= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_2 y_1) T_1(w_3 d - 1, \chi) T_m(w_3 d - 1, \chi) w_2^{l+m} w_3^{k+m} w_1^{k+l}
\]

\[(4.13)\]

\[
= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_3 y_1) T_1(w_1 d - 1, \chi) T_m(w_2 d - 1, \chi) w_3^{l+m} w_1^{k+m} w_2^{k+l}.
\]

Theorem 4.5. Let $w_1$, $w_2$, $w_3$ be any odd positive integers. Then we have:

\[(4.14)\]

\[
w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{w_3 d-1} (-1)^a \chi(a) E_{k,\chi}(w_2 y_1) + \frac{w_2}{w_1} a T_{n-k}(w_3 d - 1, \chi) w_2^{n-k} w_3^{k}
\]

\[
= w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{w_3 d-1} (-1)^a \chi(a) E_{k,\chi}(w_3 y_1) + \frac{w_3}{w_1} a T_{n-k}(w_2 d - 1, \chi) w_3^{n-k} w_1^{k}
\]

\[
= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{w_2 d-1} (-1)^a \chi(a) E_{k,\chi}(w_1 y_1) + \frac{w_1}{w_2} a T_{n-k}(w_3 d - 1, \chi) w_1^{n-k} w_3^{k}
\]

\[
= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{w_2 d-1} (-1)^a \chi(a) E_{k,\chi}(w_3 y_1) + \frac{w_3}{w_2} a T_{n-k}(w_1 d - 1, \chi) w_3^{n-k} w_1^{k}
\]

\[
= w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{w_1 d-1} (-1)^a \chi(a) E_{k,\chi}(w_2 y_1) + \frac{w_2}{w_3} a T_{n-k}(w_1 d - 1, \chi) w_2^{n-k} w_1^{k}.
\]
Theorem 4.6. Let $w_1$, $w_2$, $w_3$ be any odd positive integers. Then we have the following three symmetries in $w_1$, $w_2$, $w_3$:

$$
(w_1w_2)^n \sum_{a=0}^{w_1-1} \sum_{b=0}^{w_2-1} (-1)^{a+b} \chi(ab) E_{w_1,w_2}(w_3y_1 + \frac{w_3}{w_1}a + \frac{w_3}{w_2}b)
$$

(4.15)

$$
(w_2w_3)^n \sum_{a=0}^{w_2-1} \sum_{b=0}^{w_3-1} (-1)^{a+b} \chi(ab) E_{w_2,w_3}(w_1y_1 + \frac{w_1}{w_2}a + \frac{w_1}{w_3}b)
$$

$$
(w_3w_1)^n \sum_{a=0}^{w_3-1} \sum_{b=0}^{w_1-1} (-1)^{a+b} \chi(ab) E_{w_3,w_1}(w_2y_1 + \frac{w_2}{w_3}a + \frac{w_2}{w_1}b).
$$

Theorem 4.7. Let $w_1$, $w_2$, $w_3$ be any positive integers. Then we have the following two symmetries in $w_1$, $w_2$, $w_3$:

$$
\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,w_1}(w_1y_1) E_{l,w_2}(w_2y_2) E_{m,w_3}(w_3y_1) w_k^2 w_l^2 w_m^2
$$

(4.16)

$$
= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,w_1}(w_1y_1) E_{l,w_2}(w_2y_2) E_{m,w_3}(w_3y_1) w_k^2 w_l^2 w_m^2.
$$

Theorem 4.8. Let $w_1$, $w_2$, $w_3$ be any odd positive integers. Then we have the following two symmetries in $w_1$, $w_2$, $w_3$:

$$
\sum_{k+l+m=n} \binom{n}{k,l,m} T_k(w_1d-1,\chi) T_l(w_2d-1,\chi) T_m(w_3d-1,\chi) w_k^2 w_l^2 w_m^2
$$

(4.17)

$$
\sum_{k+l+m=n} \binom{n}{k,l,m} T_k(w_1d-1,\chi) T_l(w_2d-1,\chi) T_m(w_3d-1,\chi) w_k^2 w_l^2 w_m^2.
$$

References

[1] E. Deeba and D. Rodriguez, Stirling’s and Bernoulli numbers, Amer. Math. Monthly 98 (1991), 423–426.
[2] F. T. Howard, Applications of a recurrence for the Bernoulli numbers, J. Number Theory 52 (1995), 157–172.
[3] D. S. Kim, Identities of symmetry for generalized Bernoulli polynomials, Submitted.
[4] D. S. Kim and K. H. Park, Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under $S_3$, Submitted.
[5] D. S. Kim and K. H. Park, Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under $S_3$, Submitted.
[6] T. Kim, Symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, J. Difference Equ. Appl. 14 (2008), 1267–1277.
[7] T. Kim, A note on the generalized Euler numbers and polynomials, arXiv:0907.4889v1[math.NT].
[8] N. Koblitz, A new proof of certain formulas for $p$-adic $L$-functions, Duke Math. J. 46 (1979), 455–468.
[9] H. Tuenter, A symmetry of power sum polynomials and Bernoulli numbers, Amer. Math. Monthly 108 (2001), 258–261.
[10] S. Yang, An identity of symmetry for the Bernoulli polynomials, Discrete Math. 308 (2008), 550–554.