Continuous monitoring of $\ell_p$ norms in data streams

Jarosław Blasiok∗ Jian Ding† Jelani Nelson‡

May 18, 2017

Abstract

In insertion-only streaming, one sees a sequence of indices $a_1, a_2, \ldots, a_m \in [n]$. The stream defines a sequence of $m$ frequency vectors $x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^n$ with $(x^{(t)})_i \overset{\text{def}}{=} |\{j : j \in [t], a_j = i\}|$. That is, $x^{(t)}$ is the frequency vector after seeing the first $t$ items in the stream. Much work in the streaming literature focuses on estimating some function $f(x^{(m)})$. Many applications though require obtaining estimates at time $t$ of $f(x^{(t)})$, for every $t \in [m]$. Naively this guarantee is obtained by devising an algorithm with failure probability $\ll 1/m$, then performing a union bound over all stream updates to guarantee that all $m$ estimates are simultaneously accurate with good probability. When $f(x)$ is some $\ell_p$ norm of $x$, recent works have shown that this union bound is wasteful and better space complexity is possible for the continuous monitoring problem, with the strongest known results being for $p = 2$ [HTY14, BCIW16, BCI+17]. In this work, we improve the state of the art for all $0 < p < 2$, which we obtain via a novel analysis of Indyk’s $p$-stable sketch [Ind06].

1 Introduction

Estimating statistics of frequency vectors implicitly defined by insertion-only update streams, as defined in the abstract, was first studied by Flajolet and Martin in [FM85]. They studied the so-called distinct elements problem, in which $f(x)$ is the support size of $x$. In the insertion-only model, the support size of $x$ is equivalent to the number of distinct $a_i$ appearing in the stream. One goal in such streaming algorithms, both for this particular distinct elements problem as well as for many other function estimation problems studied in subsequent works, is to minimize the space consumption of the stream-processing algorithm, ideally using $o(n)$ words of memory (note there is always a trivial $n$ space algorithm by storing $x$ explicitly in memory).

For over two decades, work on estimating statistics of frequency vectors of streams remained dormant, until the work of [AMS99] on estimating the $p$-norm $\|x\|_p = (\sum_i x_i^p)^{1/p}$ in streams for integer $p \geq 1$. Since then several works have studied these and several other problems, from the perspective of both upper and lower bounds, including estimating $\|x\|_p$ for all $0 < p \leq 2$ (not necessarily integral) [AMS99, BCIW16, AO11, AO13, BKSV14, Gan15], empirical entropy [CBM06, CCM10, BG06, HNO08] and other information-theoretic quantities [IM08, GIM08, BO10a], cascaded norms [CM05, JW09, Jay13], and several others. There have also been general theorems classifying which statistics of frequency vectors admit space-efficient streaming estimation algorithms [BO10b, BC15, BOR15, BCWY16, BCKY17].

Taking a dynamic data structural viewpoint, “streaming algorithms” is simply a synonym for “dynamic data structures” but with an implied focus on minimizing memory consumption (typically striving for an algorithm using sublinear memory). Elements in the stream can be viewed as updates to the frequency
vector $x$ (seeing $a \in [n]$ in the stream can be seen as update$(a, 1)$, causing the change $x_a \to x_a + 1$), and the request for an estimate of some statistic of $x$ is a query. In this data structural language, all the works cited in the previous paragraph provide Monte-Carlo guarantees of the following form for queries: starting from any fixed frequency vector and after executing any fixed sequence of updates, the probability that the output of a subsequent query then fails is at most $\delta$. Here we say a query fails if, say, the output is not a good approximation to some particular $f(x)$ (this will be made more formal later). In many applications however, one does not simply want the answer to one query at the end of some large number of updates, but rather one wants to continuously monitor the data stream. That is, the sequence of data structural operations is an intermingling of updates and queries. For example, one may have a threshold $T$ in mind, and if $f(x)$ ever increases beyond $T$ some data analyst should be alerted. Such a goal could be achieved (approximately) by querying after every update to determine whether the updated frequency vector satisfies this property. Indeed, the importance of supporting continuous queries in append-only databases (analogous to the insertion-only model of streaming) was recognized 25 years ago in [TGN92], with several later works focused on continuous stream monitoring with application areas in mind such as trend detection, anomaly detection, financial data analysis, and (bio)sensor data analysis [BW01, CCC02, OJW03].

If one assumes that a query is being issued after every update, then in a stream of $m$ updates the failure probability should be set to $\delta \ll 1/m$ so that, by a union bound, all queries succeed. Most Monte-Carlo streaming algorithms achieve some space $S$ to achieve failure probability $1/3$, at which point one can achieve failure probability $\delta$ by running $\Theta(\lg(1/\delta))$ instantiations of the algorithm in parallel and returning the median estimate (see for example [AMS99]). This method increases the space from $S$ to $\Theta(S\lg(1/\delta))$, and for many problems (such as $\ell_p$-norm estimation) it is known that at least in the so-called strict turnstile model (i.e. update$(a, \Delta)$ is allowed for both positive and negative $\Delta$, but we are promised $\Delta_i \geq 0$ for all $i$ at all times) this form of space blow-up is necessary [JW13]. Nevertheless, although improved space lower bounds have been given when desiring that the answer to a single query fails with probability at most $\delta$, now such blow-up has been shown necessary for the continuous monitoring problem in which one wants, with failure probability $1/3$, to provide simultaneously correct answers for $m$ queries intermingled with $m$ updates. In fact to the contrary, in certain scenarios such as estimating distinct elements or the $\ell_2$-norm in insertion-only streams, improved upper bounds have been given!

**Definition 1.** We say a Monte-Carlo randomized streaming algorithm $A$ provides strong tracking for $f$ in a stream of length $m$ with failure probability $\eta$ if at each time $t \in [m]$, $A$ outputs an estimate $\tilde{f}_t$ such that

$$\mathbb{P}(\exists t \in [m] : |\tilde{f}_t - f(x(t))| > \epsilon f(x(t))) < \eta.$$  

We say that $A$ provides weak tracking for $f$ if

$$\mathbb{P}(\exists t \in [m] : |\tilde{f}_t - f(x(t))| > \epsilon \sup_{t' \in [m]} f(x(t'))) < \eta.$$  

Note if $f$ is monotonically increasing, then for insertion-only streams $\sup_{t' \in [m]} f(x(t'))$ is simply $f(x(m))$. The first non-trivial tracking result we are aware of which outperformed the median trick for insertion-only streaming was the ROUGHESTIMATOR algorithm given in [KNW10] for estimating the number of distinct elements in a stream. ROUGHESTIMATOR provided a strong tracking guarantee for $f(x) = |support(x)|$ (the distinct elements problem) for constant $\epsilon, \eta$, using the same space as what is what is required to answer only a single query. This strong tracking algorithm was used as a subroutine in the main non-tracking algorithm of that work for approximating the number of distinct elements in a data stream up to $1 + \epsilon$.

For $\ell_p$-estimation for $p \in (0, 2]$, without tracking, it is known that $O(\epsilon^{-2}\lg(1/\delta))$ words of memory is achievable to return a $(1 + \epsilon)$-approximate value of $f(x) = \|x\|_p$ with failure probability $\delta$ [AMS99]. This upper bound thus implies a strong tracking algorithm with space complexity $O(\epsilon^{-2}\lg m)$ for tracking failure probability $\eta = 1/3$, by setting $\delta < 1/(3m)$ and performing a union bound. The work [HTY14] considered the strong tracking variant of $\ell_p$-estimation in insertion-only streams for for any $p$ in the more restricted interval $[1, 2]$. They showed that the same algorithms of [AMS99], unchanged, provide

\footnote{For constant $\delta$ and $p = 2$, [AMS99] shows that space $O(\epsilon^{-2}(\lg n + \lg \lg m))$ bits is achievable in insertion-only streams.}
strong tracking with $\eta = 1/3$ with space $O(\varepsilon^{-2}(\lg n + \lg \lg m + \lg(1/\varepsilon)))$ words$^4$. This is an improvement over the standard median trick and union bound when the stream length is very long ($m > n^{o(1)}$) and $\varepsilon$ is not too small ($\varepsilon > 1/m^{o(1)}$). They also showed that in an update model which allows deletions of items (“turnstile streaming”), any algorithm which only maintains a linear sketch $P\times x$ of $x$ must use $\Omega(\lg m)$ words of memory for constant $\varepsilon$, showing that the median trick is optimal for this restricted class of algorithms.

A different algorithm was given in [BCIW16] for strong tracking for $\ell_2$ using space $O(\varepsilon^{-2}(\lg(1/\varepsilon) + \lg \lg m))$. It was then most recently shown in [BCI+17] that the AMS sketch itself of [AMS99] (though with 8-wise independent hash functions instead of the original 4-wise independence proposed in [AMS99]) provides strong tracking in space $O(\varepsilon^{-2} \lg \lg m)$, and weak tracking in space $O(1/\varepsilon^2)$. That is, the AMS sketch provides weak tracking without any asymptotic increase in space complexity over the requirement to correctly answer only a single query.

Despite the progress in upper bounds for tracking $\ell_2$, the only non-trivial improvement for tracking $\ell_p$ is the $O(\varepsilon^{-2}(\lg n + \lg \lg m + \lg(1/\varepsilon)))$ upper bound of [HTY14]. Although this bound provides an improvement for very long streams ($m$ super-polynomial in $n$), it does not provide any improvement over the standard median trick for the case most commonly studied case in the literature of $m, n$ being polynomially related.

Our contribution. We show that Indyk’s $p$-stable sketch [Ind06] for $0 < p \leq 2$, derandomized using bounded independence as in [KNW10a], provides weak tracking while using $O(\lg(1/\varepsilon)/\varepsilon^2)$ words of space. It also provides strong tracking using $O(\varepsilon^{-2}(\lg \lg m + \lg(1/\varepsilon)))$ words of space. Our bounds thus both improve the space complexity achieved in [HTY14] for $\ell_p$-tracking, and well as the range of $p$ supported from $p \in (1, 2]$ to all $p \in (0, 2]$ (note for $p > 2$, it is known that any algorithm requires polynomial space even to obtain a 2-approximation for a single query, i.e. the non-tracking variant of the problem [BYJKS04]).

2 Notation

We use $[n]$ for integer $n$ to denote $\{1, \ldots, n\}$. We measure space in words unless stated otherwise, where a single word is at least $\lg(nm)$ bits. For $p \in (0, 2]$, we let $D_p$ denote the symmetric $p$-stable distribution, scaled so that for $Z \sim D_p$, $\mathbb{P}(|Z| > 1) = \frac{1}{2}$. The distribution $D_p$ has the property that it is supported on the reals, and for any fixed vector $v \in \mathbb{R}^n$ and $Z_1, \ldots, Z_n, Z$ i.i.d. from $D_p$, $\sum_{i=1}^n Z_i x_i$ is equal in distribution to $\|x\|_p \cdot Z$. See [Nol17] for further reading on these distributions.

For two vectors $u, v \in \mathbb{R}^n$ we write $u \preceq v$ to denote coordinatewise comparison, i.e. $u \preceq v$ iff $\forall i u_i \leq v_i$. For a finite set $S$, we write $\#S$ to denote cardinality of this set.

3 Preliminaries

The following lemma is standard. A proof with explicit constants can be found in [Nel11] Theorem 42.

Lemma 2. If $Z \sim D_p$, then $\mathbb{P}(Z > \lambda) \leq \frac{C_p}{\lambda^p}$ for some explicit constant $C_p$ depending only on $p$.

We also state some other results we will need.

Lemma 3 (Paley-Zygmund). If $Z \geq 0$ is a random variable with finite variance, then

$$\mathbb{P}(Z > \theta \mathbb{E} Z) \geq (1 - \theta^2)^{\frac{(\mathbb{E} Z)^2}{\mathbb{E}(Z^2)}}.$$

Corollary 4. For fixed vector $v \in \mathbb{R}^n$, if $\sigma \in \{\pm 1\}^n$ is a vector of 4-wise independent random signs, then

$$\mathbb{P}((\sigma, v)^2 \geq \frac{2}{3} \|v\|_2^2) \geq \frac{1}{27}.$$ 

$^2$ For $p = 2$ their space is as written including the space required to store all hash functions, but for $1 < p < 2$ this space bound assumes that the storage of hash functions is for free.
Proof. This follows from $E(\sigma, v)^4 < 3(E(\sigma, v)^2)^2$ and the Paley-Zygmund inequality.

**Theorem 5.** ([BCIW16] Theorem 15) Let $v^{(1)}, v^{(2)}, \ldots, v^{(m)} \in \mathbb{R}^n$, be a sequence of vectors such that $0 \leq v^{(1)} \leq v^{(2)} \leq \cdots \leq v^{(m)}$. Let $\sigma \in \{\pm\}^n$ be a vector of $4$-wise independent random signs. Then

$$
\Pr \left( \sup_{i \leq m} |\langle \sigma, v^{(i)} \rangle| > \lambda \|v^{(n)}\|_2 \right) < \frac{C}{\lambda^2}
$$

for some universal constant $C$.

**Theorem 6.** ([KNW10a] DKN10) If $Z_i \sim \mathcal{D}_p$ for $i \in [n]$ are $k$-wise independent random variables, then for every vector $x \in \mathbb{R}^n$ and every pair $a, b \in \mathbb{R} \cup \{\pm \infty\}$ we have

$$
\Pr \left( \langle Z, x \rangle \in (a, b) \right) = \Pr \left( \|x\|_p Z_1 \in (a, b) \right) \pm O(k^{-1/p})
$$

**Theorem 7.** ([BR94] Lemma 2.3) Let $X_1, \ldots, X_n \in \{0, 1\}$ be a sequence of vector $\{0, 1\}$-wise independent random variables, and let $\mu = \sum x_i$. Then

$$
\forall \lambda > 0, \quad \Pr \left( \sum x_i \geq (1 + \lambda)\mu \right) \leq \exp(-\Omega(\min\{\lambda, \lambda^2\} \mu)) + \exp(-\Omega(k))
$$

4 Overview of approach

Indyk’s $p$-stable sketch picks a random matrix $\Pi \in \mathbb{R}^d \times n$ such that each entry is drawn according to the distribution $\mathcal{D}_p$. It then maintains the sketch $\Pi x^{(t)}$ of the current frequency vector. This sketch can be easily updated as the frequency vector changes, i.e. after observing an index $a_j \in [n]$ we update the sketch by $\Pi x^{(t+1)} := \Pi x^{(t)} + \Pi a_j$. An $\|x\|_p$-estimate query is answered by returning the median of $|\langle \sigma, v^{(i)} \rangle|$ over $j \in [d]$. Since storing $\Pi$ in memory explicitly is prohibitively expensive, we generate it so that the entries in each row are $k$-wise independent for $k = O(1/\epsilon^2)$ (as done in [KNW10a]), and the $d$ seeds used to generate the rows of $\Pi$ are $O(\lg(1/\epsilon^2))$-wise independent. We also work with discretized $p$-stable random variables to take bounded memory. All together, the bounded independence and discretization, also performed in [KNW10a], allow us to store $\Pi$ using low memory.

We then show that instantiating Indyk’s algorithm with $d = O(\epsilon^{-2} \lg(1/\epsilon))$ provides the weak tracking guarantee with failure probability $\delta$. The analysis of the correctness of this algorithm is as follows. Let $\pi_i$ denote the $i$th row of $\Pi$. We first show a result resembling the Doob’s martingale inequality — namely, in Section 4.1 we show that for a fixed $i$, if we look at the evolution of $\langle \pi_i, x^{(t)} \rangle$ as $t$ increases, the largest attained value (sup $t \leq m \langle \pi_i, x^{(t)} \rangle$) is with good probability not much larger than the median of the distribution $|\langle \pi_i, x^{(m)} \rangle|$, which is the typical magnitude of the counter at the end of the stream. This fact resembles similar facts shown in [BCIW16] [BCI+17] for when the $\pi_i$ have independent Rademachers as entries, though our situation is complicated by the fact that $p$-stable random variables have much heavier tails.

We then, discussed in Section 5.1 show how the previous paragraph implies a weak tracking algorithm with $d = O(\epsilon^{-2} \lg(1/\epsilon))$: we split the sequence of updates into $poly(1/\epsilon)$ intervals such that the $\ell_p$-norm of the frequency vector of updates in each of those intervals, i.e. $\|x^{(t+1)} - x^{(t)}\|_p$, is of the order $\epsilon\Theta(1)\|x^{(m)}\|_p$. We then union bound over the $poly(1/\epsilon)$ intervals to argue that the algorithm’s estimate is good at each of the interval endpoints. This is the source of the extra factor of $\log(1/\epsilon)$ in our space bound: to obtain $\epsilon^{-O(1)}$ failure probability to union bound over these intervals. On the other hand, within each of the intervals most of the counters do not change too rapidly by the argument developed in Section 5.

Finally, in Section 5.2 we show how given an algorithm satisfying a weak tracking guarantee, one can use it to get a strong-tracking algorithm with slightly larger space complexity. This argument was already present in [BCI+17]. One first identifies $q$ points in the input stream at which the $\ell_p$ norm roughly doubles when compared to the previously marked point. There are only $O(\log m)$ such intervals. It is then enough to ensure that our algorithm satisfies weak tracking for all those $O(\log m)$ prefixes simultaneously, in order to deduce that the algorithm in fact satisfies strong tracking. This is done by union bound over $O(\log m)$ bad events (as opposed to standard union bound over $O(m)$ bad events), which introduces an extra $\log \log m$ factor in the space complexity as when compared to weak tracking.

4
5 Analysis

We first show two lemmas that play a crucial role in our weak tracking analysis.

**Lemma 8.** Let $x \in \mathbb{R}^n$ be a fixed vector, and $Z \in \mathbb{R}^n$ be a random vector with $k$-wise independent entries drawn according to $D_p$. Then

$$
P\left(\sum x_i^2 Z_i^2 \geq \lambda^2 \|x\|_p^2\right) \leq \frac{C}{\lambda^p} + O(k^{-1/p})$$

for some universal constant $C$.

**Proof.** Let $E_0$ be the event $\sum x_i^2 Z_i^2 \geq \lambda^2 \|x\|_p^2$. Note that $E_0$ depends only on $|Z_i|$, and does not depend on the signs of the $Z_i$. We write $Z_i = |Z_i|\sigma_i$, where $\sigma_i$ are $k$-wise independent random signs. Conditioning on $|Z_i|$, 

$$E_{\sigma}\left(\left(\sum x_i |Z_i|\sigma_i\right)^2 |Z_1|, \ldots, |Z_n|\right) = \sum x_i^2 Z_i^2$$

and therefore for any $|Z_1|, \ldots, |Z_n|$ for which $E_0$ holds, by Corollary 4

$$\mathbb{P}\left(\left(\sum x_i |Z_i|\sigma_i\right)^2 \geq \frac{2}{3} \lambda^2 \|x\|_p^2 |Z_1|, \ldots, |Z_n|\right) \geq \frac{1}{27},$$

and thus

$$\mathbb{P}_{\sigma, Z}\left(\left(\sum x_i |Z_i|\sigma_i\right)^2 \geq \frac{2}{3} \lambda^2 \|x\|_p^2 |Z_1|, \ldots, |Z_n|\right) \geq \frac{1}{27} \mathbb{P}(E_0).$$

On the other hand $|Z_i|\sigma_i$ has the same distribution as $Z_i$, and moreover

$$\mathbb{P}_Z\left(\left(\sum x_i Z_i\right)^2 \geq \frac{2}{3} \lambda^2 \|x\|_p^2\right) = \mathbb{P}_Z\left(|\langle x, Z \rangle| \geq \sqrt{\frac{2}{3}} \lambda \|v\|_p\right)$$

$$\leq \mathbb{P}_Z\left(\|x\|_p \hat{Z} \geq \sqrt{\frac{2}{3}} \lambda \|x\|_p\right) + O(k^{-1/p})$$

$$\leq \frac{C}{\lambda^p} + O(k^{-1/p})$$

(2)

where $\hat{Z} \sim D_p$. The inequalities are obtained via Theorem 6 and Lemma 2. Combining (1), (2) yields

$$\mathbb{P}_Z(E_0) \leq \frac{27C}{\lambda^p} + O(k^{-1/p}).$$

**Lemma 9.** Let $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in \mathbb{R}^n$ satisfy $0 \preceq x^{(1)} \preceq x^{(2)} \preceq \ldots \preceq x^{(m)}$. Let $Z \in \mathbb{R}^n$ have $k$-wise independent entries marginally distributed according to $D_p$. Then for some $C_p$ depending only on $p$,

$$\mathbb{P}\left(\sup_{k \leq m} |\langle Z, x^{(k)} \rangle| \geq \lambda \|x^{(m)}\|_p\right) \leq C_p \left(\frac{1}{\lambda^{2p/(2+p)}} + k^{-1/p}\right).$$


Proof. Observe that for any \( \beta \) we have

\[
\mathbb{P} \left( \sup_{k \leq m} |Z, x^{(k)}| \geq \lambda \| x^{(m)} \|_p \right) \leq \mathbb{P} \left( \sum_i Z_i^2 (x^{(m)})_i^2 \geq \beta^2 \| x^{(m)} \|_p^2 \right) + \mathbb{P} \left( \sup_{k \leq m} |Z, x^{(k)}| \geq \lambda \| x^{(m)} \|_p \right) \sum_i Z_i^2 (x^{(m)})_i^2 < \beta^2 \| x^{(m)} \|_p^2 \right).
\]

Lemma \( \mathbb{L} \) directly implies that

\[
\mathbb{P} \left( \sum_i Z_i^2 (x^{(m)})_i^2 \geq \beta^2 \| x^{(m)} \|_p^2 \right) \leq \frac{C}{\beta^p} + \frac{C}{\lambda^{1/p}}. \tag{3}
\]

On the other hand we can write \( Z_i = |Z_i| \sigma_i \), where \( \sigma_i \) are \( k \)-wise independent Rademacher random variables, independent from \( |Z_i| \). Let us define \( w^{(k)} \in \mathbb{R}^n \) for \( k \in [m] \) to be the vector with coordinates \( (w^{(k)})_i := (x^{(k)})_i |Z_i| \), so that \( \langle x^{(k)}, Z \rangle = \langle w^{(k)}, \sigma \rangle \), and in particular

\[
\sup_{k \leq m} |Z, x^{(k)}| = \sup_{k \leq m} |\langle \sigma, w^{(i)} \rangle|.
\]

Now, if we condition on \( |Z|, \ldots, |Z_n| \), then the sequence \( w^{(1)}, \ldots, w^{(k)} \) of vectors satisfies the assumptions of Theorem \( \mathbb{L} \) and we can conclude that

\[
\mathbb{P} \left( \sup_{k \leq m} |\langle \sigma, w^{(k)} \rangle| > \frac{\lambda}{\beta} \| w^{(m)} \|_2 \right) \leq \frac{C\beta^2}{\lambda^2}.
\]

Moreover if \( |Z_i| \) are such that \( \sum_i Z_i^2 (x^{(m)})^2_i \leq \beta^2 \| x^{(m)} \|_p^2 \), or equivalently \( \| w^{(m)} \|_2^2 \leq \beta^2 \| x^{(m)} \|_p^2 \), we have

\[
\mathbb{P} \left( \sup_{k \leq m} |\langle \sigma, w^{(k)} \rangle| > \lambda \| x^{(m)} \|_p \right) \leq \frac{C\beta^2}{\lambda^2},
\]

which implies

\[
\mathbb{P} \left( \sup_{k \leq m} |\langle Z, x^{(k)} \rangle| \geq \lambda \| x^{(m)} \|_p \right) \sum_i (Z_i x_i^{(m)})^2 < \beta^2 \| x^{(m)} \|_p^2 \right) \leq \frac{C\beta^2}{\lambda^2}.
\]

This together with Eq. \( \mathbb{L} \) yields

\[
\mathbb{P} \left( \sup_{k \leq m} |\langle Z, x^{(k)} \rangle| \geq \lambda \| x^{(m)} \|_p \right) \leq \frac{1}{\beta^p} + \frac{C\beta^2}{\lambda^2} + \frac{C}{\lambda^{1/p}}.
\]

We can take \( \beta := \Theta(\lambda^{-\frac{2}{p+2}}) \), to have \( \frac{1}{\beta^p} + \frac{C\beta^2}{\lambda^2} = \mathcal{O}(\lambda^{-\frac{2}{p+2}}) \).

\( \Box \)

### 5.1 Weak tracking of \( \| x \|_p \)

In this section we upper bound the number of rows needed in Indyk’s \( p \)-stable sketch with boundedly independent entries to achieve weak tracking.

**Lemma 10.** Let \( x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^n \) be any sequence satisfying \( 0 \leq x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(m)} \). Take \( \Pi \in \mathbb{R}^{d \times n} \) to be a random matrix with entries drawn according to \( \mathcal{D}_p \), and such that the rows are \( r \)-wise independent, and all entries within a row are \( s \)-wise independent.

For every \( k \in [m] \), define \( s_k \) to be median \( (|\langle \Pi x^{(k)} \rangle|_1, \ldots, |\langle \Pi x^{(k)} \rangle|_d) \). If \( d = \Omega(\varepsilon^{-2} (\log \frac{1}{\varepsilon} + \log \frac{1}{\beta})) \), \( r = \Omega(\log \frac{1}{\varepsilon} + \log \frac{1}{\beta}) \) and \( s = \Omega(\varepsilon^{-p}) \), then with probability at least \( 1 - \delta \) we have

\[
\forall k \in [m], \quad \| x^{(k)} \|_p - \varepsilon \| x^{(m)} \|_p \leq s_k \leq \| x^{(k)} \|_p + \varepsilon \| x^{(m)} \|_p
\]
Proof. Consider a sequence of indices $1 < t_1 < t_2 < \ldots < t_{q+1} = m$, constructed inductively in the following way. We take $t_1$ to be the smallest index with $\|x^{(t_1)}\|_p \geq \varepsilon^4\|x^{(m)}\|_p$. Given $t_k$, we take $t_{k+1}$ to be the smallest index such that $\|x^{(t_{k+1})} - x^{(t_k)}\|_p \geq \varepsilon^4\|x^{(m)}\|_p$ if there exists one, and $t_{k+1} = m$ otherwise.

Observe $q \leq \varepsilon^{-8}$. Indeed, for $p \geq 1$ we have

$$\|x^{(m)}\|_p = \|x^{(t_1)} + \sum_{1 \leq i < q} (x^{(t_{i+1})} - x^{(t_i)})\|_p \geq \|x^{(t_1)}\|_p + \sum_{1 \leq i < q} \|x^{(t_{i+1})} - x^{(t_i)}\|_p \geq q\varepsilon^4\|x^{(m)}\|_p$$

where the inequality $\|x^{(t_1)} + \sum_{i \geq 1} (x^{(t_{i+1})} - x^{(t_i)})\|_p \geq \|x^{(t_1)}\|_p + \sum \|x^{(t_{i+1})} - x^{(t_i)}\|_p$ holds because all vectors $x^{(t_1)}$ and $x^{(t_{i+1})} - x^{(t_i)}$ for every $i$ have non-negative entries — we can consider each coordinate separately, and use the fact that for $p \geq 1$ and non-negative numbers $a_i$ we have $(\sum a_i)^p \geq \sum a_i^p$ — or equivalently, $\|a\|_p^p \geq \|a\|_{p'}^p$. After rearranging this yields $q \leq \varepsilon^{-4p}$.

Similarly, for $p \leq 1$, we have that for non-negative numbers $a_i$, $(\sum_{i \leq q} a_i)^p \geq q^{p-1} \sum a_i^p$ (this is true because for fixed $\sum a_i$, the sum $\sum a_i^p$ is maximized when all $a_i$ are equal), and therefore

$$\|x^{(m)}\|_p = \|x^{(t_1)} + \sum_{1 \leq i < q} (x^{(t_{i+1})} - x^{(t_i)})\|_p \geq q^{p-1}\|x^{(t_1)}\|_p + \sum_{1 \leq i < q} \|x^{(t_{i+1})} - x^{(t_i)}\|_p \geq q^{p-1}\varepsilon^4\|x^{(m)}\|_p$$

which implies $q \leq \varepsilon^{-4}$.

For $j \in [m]$, let us define

$$l_j := \#\{i : \langle \pi_i, x^{(j)} \rangle < (1 - \varepsilon)\|x^{(j)}\|_p \}$$

$$u_j := \#\{i : \langle \pi_i, x^{(j)} \rangle > (1 + \varepsilon)\|x^{(j)}\|_p \}.$$

Let $\tilde{\pi}_i$ be a vector of i.i.d. random variables drawn according to $\mathcal{D}_p$. We know that $\langle \tilde{\pi}_i, x^{(j)} \rangle \sim \|x^{(j)}\|_p \mathcal{D}_p$. Hence $\mathbb{P}(\langle \tilde{\pi}_i, x^{(j)} \rangle > \|x^{(j)}\|_p) = \frac{1}{2}$, and $\mathbb{P}(\langle \tilde{\pi}_i, x^{(j)} \rangle > (1 + \varepsilon)\|x^{(j)}\|_p) \leq \frac{1}{2} - 2C\varepsilon$ for some universal constant $C$. Similarly $\mathbb{P}(\langle \tilde{\pi}_i, x^{(j)} \rangle < (1 - \varepsilon)\|x^{(j)}\|_p) \leq \frac{1}{2} - 2C\varepsilon$.

Entries of $\pi_i$ are $s$-wise independent, for $s \geq C_2\varepsilon^{-p}$ with some large constant $C_2$ depending on $C$. Thus by Theorem 4, $\mathbb{P}(\langle \pi_i, x^{(j)} \rangle > (1 - \varepsilon)\|x^{(j)}\|_p) \leq \mathbb{P}(\langle \tilde{\pi}_i, x^{(j)} \rangle > (1 - \varepsilon)\|x^{(j)}\|_p) + C\varepsilon \leq \frac{1}{2} - C\varepsilon$, and analogously for $\mathbb{P}(\langle \pi_i, x^{(j)} \rangle > (1 + \varepsilon)\|x^{(j)}\|_p) < \frac{1}{2} - C\varepsilon$.

Hence

$$\mathbb{E}l_j \leq d \left( \frac{1}{2} - C\varepsilon \right)$$

$$\mathbb{E}u_j \leq d \left( \frac{1}{2} - C\varepsilon \right).$$

For $j \in [q]$, let $S_j$ be the event

$$\left\{ t_i \leq d \frac{2}{\varepsilon} - \frac{C_d}{2} \right\} \land \left\{ u_i \leq d \frac{2}{\varepsilon} - \frac{C_d}{2} \right\}.$$

Note that for fixed $j$ and varying $i$, indicator random variables for the events $\langle \pi_i, x^{(j)} \rangle < (1 - \varepsilon)\|x^{(j)}\|_p$ are $r$-wise independent. Thus by Theorem 7, $\mathbb{P}(S_j) \geq 1 - C \exp(-\Omega(d\varepsilon^2)) - \exp(-\Omega(r))$. Taking $d = \Omega(\varepsilon^{-2}(\log \frac{1}{\varepsilon} + \log \frac{1}{\varepsilon}))$ and $r = \Omega(\log \frac{1}{\varepsilon})$ we obtain $\mathbb{P}(S_j) \geq 1 - \frac{\delta_s}{3}$, and hence by a union bound all $S_j$ hold simultaneously except with probability at most $\frac{1}{3}$ since the number of events $S_j$ is $q \leq \varepsilon^{-8}$.

For $i \in [d]$ and $j \in [q]$, let $E_{i,j}$ be the event

$$\exists s \in [t_j, t_{j+1} - 1], \| (x^{(s)} - x^{(t_j)}, \pi_i) \| > \varepsilon\|x^{(m)}\|_p.$$
By construction of the sequence \( t_j \), all \( x(s) - x(t_j) \) above have \( \ell_p \) norm at most \( \varepsilon^4 \|x(m)\|_p \), we can invoke Lemma 9 to deduce that \( \mathbb{P}(E_{ij}) \leq C_3 \left( \frac{\varepsilon}{d} \right)^{2/3} + C_3 \varepsilon^{-1/p} \). Again if we pick \( s \geq C_4 \varepsilon^{-p} \) for sufficiently large \( C_4 \) and small enough \( \varepsilon \) we have \( \mathbb{P}(E_{ij}) \leq \frac{C_4}{4} \varepsilon \). Therefore for any fixed \( j \), we have
\[
\mathbb{E} \sum_i 1_{E_{ij}} \leq \frac{C_4}{4} \varepsilon
\]
And finally again by Theorem 7 for each \( j \)
\[
\mathbb{P}(\sum_i 1_{E_{ij}} \geq \frac{C_4}{2} d \varepsilon) \leq \exp(-C'd \varepsilon) + \exp(-C'\varepsilon)
\]
We have \( d \geq C_3 \varepsilon^{-2} \log \frac{1}{\varepsilon^2} \), and \( q \leq \varepsilon^{-8} \), hence for sufficiently small \( \varepsilon \), we have \( \exp(-C'd \varepsilon) \leq \frac{d}{2q} \). On the other hand if \( r = \Omega(\log \frac{1}{\varepsilon^2}) \) is sufficiently large, we have \( \exp(-C'\varepsilon) \leq \frac{d}{2q} \). We invoke the union bound over all \( j \) to deduce that with probability at least 1 - \( \frac{d}{2} \) the following event \( V \) holds:
\[
\forall j, \sum_i 1_{E_{ij}} \leq \frac{C_4}{2} d \varepsilon.
\]
We know that with probability at least 1 - \( \delta \) simultaneously \( V \) and all the events \( S_j \) hold. We will show now that, when these events all hold, then \( \forall k \|x^{(k)}\|_p - K \|x^{(m)}\|_p \leq s_k \leq \|x^{(k)}\|_p + K \|x^{(m)}\|_p \) for some universal constant \( K \). Indeed, consider some \( k \), and let us assume that \( t_j \leq k \leq t_{j+1} \). With event \( S_j \) satisfied, we know that \( \#\{i : \|\langle \pi_i, x^{(k)} \rangle \| \leq \|x^{(t_j)}\|_p + \varepsilon \|x^{(m)}\|_p \} \geq d \left( \frac{1}{2} + \frac{C_3}{4} \varepsilon \right) \) and with event \( V \) satisfied, we know that for all but \( \frac{C_3}{4} d \) of indices \( i \) we have \( \|\langle \pi_i, x^{(k)} - x^{(t_j)} \rangle \| \leq \varepsilon \|x^{(m)}\| \).

By the triangle inequality \( |\langle \pi_i, x^{(k)} \rangle| \leq |\langle \pi_i, x^{(t_j)} \rangle| + |\langle \pi_i, x^{(k)} - x^{(t_j)} \rangle| \), yielding
\[
\#\{i : |\langle \pi_i, v_k \rangle| \leq \|v_{t_j}\|_p + 2 \varepsilon \|v_m\|_p \} \geq \frac{d}{2}.
\]
With similar reasoning we can deduce that
\[
\#\{i : |\langle \pi_i, x^{(k)} \rangle| \geq \|x^{(t_j)}\|_p - 2 \varepsilon \|x^{(m)}\|_p \} \geq \frac{d}{2},
\]
which implies the median of \( |\langle \pi_i, x^{(k)} \rangle| \) over \( i \in [d] \) is in the range \( \|x^{(t_j)}\|_p \pm 2 \varepsilon \|x^{(m)}\|_p \). In other words
\[
\|x^{(t_j)}\|_p - 2 \varepsilon \|x^{(m)}\|_p \leq s_k \leq \|x^{(t_j)}\|_p + 2 \varepsilon \|x^{(m)}\|_p.
\]
Finally we also have \( \|x^{(k)}\|_p - \|x^{(t_j)}\|_p \leq \varepsilon \|x^{(m)}\|_p \) by construction of the sequence \( \{t_j\}_{j=1}^q \), so the claim follows up to rescaling \( \varepsilon \) by a constant factor.

Lemma 11. The above algorithm can be implemented using \( O(\varepsilon^{-2} \log(1/(\varepsilon \delta)) \log m) \) bits of memory to store fixed precision approximations of all counters \( (\Pi x^{(k)})_i \), and \( O(\varepsilon^{-p} \log(1/(\varepsilon \delta)) \log (nm)) \) bits to store \( \Pi \).

Proof. Consider a sketch matrix \( \Pi \) as in Lemma 10 — i.e. \( \Pi \in \mathbb{R}^{d \times n} \) with random \( D_p \) entries, such that all rows are \( r \)-wise independent and all entries within a row are \( s \)-wise independent. Moreover let us pick some \( \gamma = \Theta(\varepsilon m^{-1}) \) and consider discretization \( \tilde{\Pi} \) of \( \Pi \), namely each entry \( \tilde{\Pi}_{ij} \) is equal to \( \Pi_{ij} \) rounded to the nearest integer multiple of \( \gamma \). The analysis identical to the one in \( [KNW10a, A.6] \) shows that this discretization have no significant effect on the accuracy of the algorithm, and moreover that one can sample from a nearby distribution using only \( \tau = \Theta(\log m \varepsilon^{-1}) \) uniformly random bits. Therefore we can store such a matrix succinctly using \( O(rs(\log n + \tau) + r \log d) \) bits of memory, by storing a seed for a random \( r \)-wise independent hash function \( h : [d] \rightarrow \{0,1\}^{O(s(\log n + \tau))} \) and interpreting each \( h(i) \) as a seed for an \( s \)-wise
independent hash function describing the $i$-th row of $\Pi$ [Vad12 Corollary 3.34]. Hence the total space complexity of storing the sketch matrix $\Pi$ in a succinct manner is $O\left(\frac{\lg \delta^{-1} + \lg \varepsilon^{-1}}{\varepsilon^2}(\lg n + \lg m)\right)$ bits.

Additionally we have to store the sketch of the current frequency vector itself, i.e. for all $i \in [d]$ we need to store $(\tilde{\pi}_i, x^{(k)})$; for every such counter we need $O(\lg m \varepsilon^{-1}) = O(\lg m)$ bits, and there are $d = O\left(\frac{\lg \varepsilon^{-1} + \lg \delta^{-1}}{\varepsilon^2}\right)$ counters.

We thus have the following main theorem of this section.

**Theorem 12.** For any $p \in (0, 2]$ there is an insertion-only streaming algorithm that provides the weak tracking guarantees for $f(x) = \|x\|_p$ with probability $1 - \delta$ using $O\left(\frac{\lg m + \lg n}{\varepsilon^2}(\lg \varepsilon^{-1} + \lg \delta^{-1})\right)$ bits of memory. □

### 5.2 Strong tracking of $\|x\|_p$

In this section we discuss achieving a strong tracking guarantee. The same argument for $\ell_2$-tracking appeared in [BCJ\textsuperscript{+}17]. The reduction is in fact general, and shows that for any monotone function $f$ the strong tracking problem for $f$ reduces to the weak tracking version of the same problem with smaller failure probability.

**Lemma 13.** Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be any monotone function of $\mathbb{R}^n$ (i.e. $x \leq y \implies f(x) \leq f(y)$), such that $\min f(e_i) = 1$ (where $e_i$ are standard basis vectors). Let $A$ be an insertion-only streaming algorithm satisfying weak tracking for any sequence of updates with probability $1 - \delta$ and accuracy $\varepsilon$. Then for a sequence of frequency vectors $0 \leq x^{(1)} \leq \ldots \leq x^{(m)}$ algorithm $A$ satisfies strong tracking with probability $1 - \delta \lg f(x^{(m)})$ and accuracy $2\varepsilon$.

**Proof.** Define $t_1 < t_2 < \cdots < t_q$ so that $t_i$ is the smallest index in $[m]$ larger than $t_{i-1}$ with $f(x^{(t_i)}) \geq 2^i$ (if no such index exists, define $q = i$ and $t_q = m$). Note that $q \leq \lg f(x^{(m)})$.

The algorithm will fail with probability at most $\delta$ to satisfy the conclusion of Theorem 12 for a particular sequence of vectors $x^{(1)}, x^{(2)}, \ldots, x^{(t_j)}$. That is, for every $j$, with probability $1 - \delta$, we have that

$$\forall i \leq t_j, f(x^{(i)}) - \varepsilon f(x^{(t_j)}) \leq \tilde{f}^i \leq f(x^{(i)}) + \varepsilon f(x^{(t_j)}),$$

where $\tilde{f}^i$ is the estimate output by the algorithm at time $t$.

We can union bound over all $j \in [q]$ to deduce that except with probability $q\delta \leq \delta \lg f(x^{(m)})$,

$$\forall i \leq t_j, f(x^{(i)}) - \varepsilon f(x^{(t_j)}) \leq \tilde{f}^i \leq f(x^{(i)}) + \varepsilon f(x^{(t_j)}).$$

By construction of the sequence of $t_j$, we know that for every $i$, if we take $t_j$ to be smallest such that $i \leq t_j$, then $f(x^{(t_j)}) \leq 2f(x^{(i)})$, and the claim follows. □

**Theorem 14.** For any $p \in (0, 2]$ there is an insertion-only streaming algorithm that provides strong tracking guarantees for estimating the $\ell_p$-norm of the frequency vector with probability $1 - \delta$ and multiplicative error $1 + \varepsilon$, with space usage bounded by $O\left(\frac{\lg m + \lg n}{\varepsilon^2}(\lg \varepsilon^{-1} + \lg \delta^{-1} + \lg \lg m)\right)$ bits.

**Proof.** This follows from Lemma 11 and Lemma 13 by observing that after a sequence of $m$ insertions, the $\ell_p$ norm of the frequency vector is bounded by $m^2$, i.e. $\lg(\|x^{(m)}\|_p) = O(\lg m)$. □

### References

[AKO11] Alexandr Andoni, Robert Krauthgamer, and Krzysztof Onak. Streaming algorithms via precision sampling. In *Proceedings of the 52nd IEEE Annual Symposium on Foundations of Computer Science (FOCS)*, pages 363–372, 2011.
[AMS99] Noga Alon, Yossi Matias, and Mario Szegedy. The space complexity of approximating the frequency moments. *J. Comput. Syst. Sci.*, 58(1):137–147, 1999.

[BC15] Vladimir Braverman and Stephen R. Chestnut. Universal sketches for the frequency negative moments and other decreasing streaming sums. In *Proceedings of the 18th International Workshop on Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques (APPROX)*, pages 591–605, 2015.

[BCI+17] Vladimir Braverman, Stephen R. Chestnut, Nikita Ivkin, Jelani Nelson, Zhengyu Wang, and David P. Woodruff. BPTree: an $\ell_2$ heavy hitters algorithm using constant memory. In *Proceedings of the 36th SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS)*, 2017.

[BCIW16] Vladimir Braverman, Stephen R. Chestnut, Nikita Ivkin, and David P. Woodruff. Beating countsketch for heavy hitters in insertion streams. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 740–753, 2016.

[BCKY17] Vladimir Braverman, Stephen R. Chestnut, Robert Krauthgamer, and Lin F. Yang. Streaming symmetric norms via measure concentration. In *Proceedings of the 49th Annual ACM Symposium on Theory of Computing (STOC), to appear*, 2017.

[BCWY16] Vladimir Braverman, Stephen R. Chestnut, David P. Woodruff, and Lin F. Yang. Streaming space complexity of nearly all functions of one variable on frequency vectors. In *Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS)*, pages 261–276, 2016.

[BG06] Lakshminath Bhuvanagiri and Sumit Ganguly. Estimating entropy over data streams. In *Proceedings of the 14th Annual European Symposium on Algorithms (ESA)*, pages 148–159, 2006.

[BGKS06] Lakshminath Bhuvanagiri, Sumit Ganguly, Deepanjan Kesh, and Chandan Saha. Simpler algorithm for estimating frequency moments of data streams. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 708–713, 2006.

[BKSV14] Vladimir Braverman, Jonathan Katzman, Charles Seidell, and Gregory Vorsanger. An optimal algorithm for large frequency moments using $o(n^{1-2/k})$ bits. In *Proceedings of the 17th International Workshop on Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques (APPROX)*, pages 531–544, 2014.

[BO10a] Vladimir Braverman and Rafail Ostrovsky. Measuring independence of datasets. In *Proceedings of the 42nd ACM Symposium on Theory of Computing (STOC)*, pages 271–280, 2010.

[BO10b] Vladimir Braverman and Rafail Ostrovsky. Zero-one frequency laws. In *Proceedings of the 42nd ACM Symposium on Theory of Computing (STOC)*, pages 281–290, 2010.

[BO13] Vladimir Braverman and Rafail Ostrovsky. Approximating large frequency moments with pick-and-drop sampling. In *Proceedings of the 16th International Workshop on Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques (APPROX)*, pages 42–57, 2013.

[BOR15] Vladimir Braverman, Rafail Ostrovsky, and Alan Roytman. Zero-one laws for sliding windows and universal sketches. In *Proceedings of the 18th International Workshop on Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques (APPROX)*, pages 573–590, 2015.

[BR94] Mihir Bellare and John Rompel. Randomness-efficient oblivious sampling. In *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 276–287, 1994.
[BW01] Shivnath Babu and Jennifer Widom. Continuous queries over data streams. *SIGMOD Record*, 30(3):109–120, 2001.

[BYJKS04] Ziv Bar-Yossef, T. S. Jayram, Ravi Kumar, and D. Sivakumar. An information statistics approach to data stream and communication complexity. *Journal of Computer and System Sciences*, 68(4):702–732, 2004.

[CBM06] Amit Chakrabarti, Khanh Do Ba, and S. Muthukrishnan. Estimating entropy and entropy norm on data streams. *Internet Mathematics*, 3(1):63–78, 2006.

[CČĆ+02] Donald Carney, Ugur Çetintemel, Mitch Cherniack, Christian Convey, Sangdon Lee, Greg Seidman, Michael Stonebraker, Nesime Tatbul, and Stanley B. Zdonik. Monitoring streams - A new class of data management applications. In *Proceedings of the 28th International Conference on Very Large Data Bases (VLDB)*, pages 215–226, 2002.

[CCM10] Amit Chakrabarti, Graham Cormode, and Andrew McGregor. A near-optimal algorithm for estimating the entropy of a stream. *ACM Trans. Algorithms*, 6(3):51:1–51:21, 2010.

[CKS03] Amit Chakrabarti, Subhash Khot, and Xiaodong Sun. Near-optimal lower bounds on the multi-party communication complexity of set disjointness. In *Proceedings of the 18th Annual IEEE Conference on Computational Complexity (CCC)*, pages 107–117, 2003.

[CM05] Graham Cormode and S. Muthukrishnan. Space efficient mining of multigraph streams. In *Proceedings of the 24th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS)*, pages 271–282, 2005.

[DKN10] Ilias Diakonikolas, Daniel M. Kane, and Jelani Nelson. Bounded independence fools degree-2 threshold functions. In *Proceedings of the 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 11–20, 2010.

[FM85] Philippe Flajolet and G. Nigel Martin. Probabilistic counting algorithms for data base applications. *J. Comput. Syst. Sci.*, 31(2):182–209, 1985.

[Gan15] Sumit Ganguly. Taylor polynomial estimator for estimating frequency moments. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 542–553, 2015.

[GIM08] Sudipto Guha, Piotr Indyk, and Andrew McGregor. Sketching information divergences. *Machine Learning*, 72(1-2):5–19, 2008.

[Ind06] Piotr Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. *J. ACM*, 53(3):307–323, May 2006.
[IW03] Piotr Indyk and David P. Woodruff. Tight lower bounds for the distinct elements problem. In Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 283–, 2003.

[IW05] Piotr Indyk and David P. Woodruff. Optimal approximations of the frequency moments of data streams. In Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC), pages 202–208, 2005.

[Jay09] T. S. Jayram. Hellinger strikes back: A note on the multi-party information complexity of AND. In Proceedings of the 12th International Workshop on Randomization and Approximation Techniques (RANDOM), pages 562–573, 2009.

[Jay13] T. S. Jayram. On the information complexity of cascaded norms with small domains. In IEEE Information Theory Workshop (ITW), pages 1–5, 2013.

[JW09] T. S. Jayram and David P. Woodruff. The data stream space complexity of cascaded norms. In Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 765–774, 2009.

[JW13] T. S. Jayram and David P. Woodruff. Optimal bounds for johnson-lindenstrauss transforms and streaming problems with subconstant error. ACM Trans. Algorithms, 9(3):26:1–26:17, 2013.

[KNPW11] Daniel M. Kane, Jelani Nelson, Ely Porat, and David P. Woodruff. Fast moment estimation in data streams in optimal space. In Proceedings of the 43rd ACM Symposium on Theory of Computing (STOC), pages 745–754, 2011.

[KNW10a] Daniel M. Kane, Jelani Nelson, and David P. Woodruff. On the exact space complexity of sketching and streaming small norms. In Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1161–1178, 2010.

[KNW10b] Daniel M. Kane, Jelani Nelson, and David P. Woodruff. An optimal algorithm for the distinct elements problem. In Proceedings of the 29th SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS), pages 99–110, 2010.

[Li08] Ping Li. Estimators and tail bounds for dimension reduction in $\ell_\alpha$ ($0 < \alpha \leq 2$) using stable random projections. In Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 10–19, 2008.

[Li09] Ping Li. Compressed counting. In Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 412–421, 2009.

[Nel11] Jelani Nelson. Sketching and streaming high-dimensional vectors. PhD thesis, Massachusetts Institute of Technology, 2011.

[Nol17] J. P. Nolan. Stable Distributions - Models for Heavy Tailed Data. Birkhauser, Boston, 2017. In progress, Chapter 1 online at [http://fs2.american.edu/jpnolan/www/stable/stable.html](http://fs2.american.edu/jpnolan/www/stable/stable.html).

[NW10] Jelani Nelson and David P. Woodruff. Fast manhattan sketches in data streams. In Proceedings of the 29th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS), pages 99–110, 2010.

[OJW03] Chris Olston, Jing Jiang, and Jennifer Widom. Adaptive filters for continuous queries over distributed data streams. In Proceedings of the ACM SIGMOD International Conference on Management of Data (SIGMOD), pages 563–574, 2003.

[TGNO92] Douglas B. Terry, David Goldberg, David A. Nichols, and Brian M. Oki. Continuous queries over append-only databases. In Proceedings of the ACM SIGMOD International Conference on Management of Data (SIGMOD), pages 321–330, 1992.
[Vad12] Salil P. Vadhan. Pseudorandomness. *Foundations and Trends in Theoretical Computer Science*, 7(1-3):1–336, 2012.

[Woo04] David Woodruff. Optimal space lower bounds for all frequency moments. In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 167–175, 2004.