Complete coalescent diagram of the Painlevé equations

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Abstract
We will revise Garnier-Okamoto’s coalescent diagram of isomonodromic deformations and give a complete coalescent diagram. In our viewpoint, we have ten types of isomonodromic deformations and two of them give the same type of the Painlevé equation. We can naturally put the thirty-fourth Painlevé equation in our diagram, which corresponds to the Flaschka-Newell form of the second Painlevé equation.

1 Introduction
In this paper, we will revise Garnier-Okamoto’s coalescent diagram of isomonodromic deformations and will show a complete coalescent diagram. In the original form, the Painlevé equations are classified in six types. But in our picture, there exist ten different types of isomonodromic deformations. Since two of them give the same type of the Painlevé equation, the Painlevé equations are classified in eight different types.

We will also show that the Painlevé equations are classified in five types as a nonlinear single equation. Especially we show a unified equation of the fourth Painlevé equation and the thirty-fourth Painlevé equation. These five types are classified into fourteen types by scaling transformations. We exclude four types of them since they are quadrature. The remaining ten types of equations correspond to the different singularity types of isomonodromic deformations. In our form, it is easy to understand the relation between the type of the Painlevé equations and the singularity type of isomonodromic deformations.

It is known that different forms of isomonodromic deformations

$$\frac{\partial Y}{\partial x} = A(x, t)Y, \quad \frac{\partial Y}{\partial t} = B(x, t)Y$$

exist for some types of the Painlevé equations. One of the most famous example is the Flaschka-Newell form and the Miwa-Jimbo form for the second Painlevé equation $P2(\alpha)$

$$y'' = 2y^3 + ty + \alpha.$$ (1)
The Flaschka-Newell form (FN) is

\[ A^{FN}(x, t) = -4 \left( \begin{array}{c} x^2 \\ yx \\ -ax^2 \end{array} \right) + \left( \begin{array}{c} t + 2y^2 \\ 2z \\ -t - 2y^2 \end{array} \right) - \left( \begin{array}{c} 0 \\ \alpha \\ 0 \end{array} \right) \frac{1}{x}, \]

\[ B^{FN}(x, t) = \left( \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right) x + \left( \begin{array}{c} y \\ 0 \\ 0 \end{array} \right). \] (2)

The Miwa-Jimbo form (MJ) is

\[ A^{MJ}(x, t) = \left( \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right) x^2 + \left( \begin{array}{c} u \\ 0 \\ -u \end{array} \right) x + \left( \begin{array}{c} z + \frac{t}{2} \\ \frac{2}{u}(\theta + yz) \\ -z - \frac{t}{2} \end{array} \right), \]

\[ B^{MJ}(x, t) = \frac{x}{2} \left( \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} 0 \\ u \\ -\frac{2}{u} \end{array} \right). \] (3)

We take a slightly different form from the original Flaschka-Newell form. Our form is a ‘real’ form in a sense. \( A^{FN}(x, t) \) has an irregular singularity of the Poincaré rank three at \( x = \infty \) and a regular singularity at \( x = 0 \). \( A^{MJ}(x, t) \) has an irregular singularity of the Poincaré rank three but has no other singularities. They are not connected by any rational transform of the independent variable.

In this paper, we show that both (MJ) and (FN) comes from different degeneration from the sixth Painlevé equations. Moreover, we will show that it is natural to consider (FN) is a deformation for the thirty-fourth Painlevé equation \( P34(\alpha) \) in Gambier’s classification [3],

\[ y'' = \frac{y'^2}{2y} + 2y^2 - ty - \frac{\alpha}{2y}, \]

which is equivalent to the second Painlevé equation. Instead of the original \( P34 \), we change the sign \( t \to -t \). We call this equation as \( P34' \).

It is known by Garnier and Okamoto that all types of the Painlevé equations are represented as isomonodromic deformations of a single linear equation with order two [8]. (MJ) is essentially equivalent to the Garnier-Okamoto form. From the viewpoint of Garnier-Okamoto form, we obtain a well-known coalescent diagram of the Painlevé equations:

\[ (1 + 1 + 1 + 1) \rightarrow (1 + 1 + 2) \rightarrow (4) \rightarrow (7/2) \]

Here \( j \) is a pole order of the connection \( A(x, t) \). This diagram is easy to understand and explains coalescence of the Painlevé equations [12]. But it seems that (FN) is out of the coalescent diagram since the type of singularities of (FN) is \( (1 + 4) \). Later we will show a complete coalescent diagram of the Painlevé equations from the sixth Painlevé equations, which contains (FN) as the type \( (1 + 5/2) \).

Before we show the complete coalescent diagram, we will review the third Painlevé equation \( P3 \)

\[ y'' = \frac{1}{y} y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \delta. \] (4)

\( P3 \) is divided into four type

- (P3-A) \( \gamma \neq 0, \delta \neq 0 \)
- (P3-B) \( \gamma \neq 0, \delta = 0 \) or \( \gamma = 0, \delta \neq 0 \)
- (P3-C) \( \gamma = 0, \delta = 0 \)
- (P3-D) \( \alpha = 0, \gamma = 0 \) or \( \beta = 0, \delta = 0 \).
Since the case (P3-D) is quadrature, we exclude the case (P3-D). The cases (P3-A), (P3-B) and (P3-C) are called the type $D^{(1)}_6$, the type $D^{(1)}_7$, the type $D^{(1)}_8$, respectively. The meaning of type is the Dynkin diagram of the intersection form of boundary divisors of the Okamoto initial value spaces \[13\]. In [9] we show that the corresponding linear equations for $D^{(1)}_7$ and $D^{(1)}_8$ type has singularities of type $(1)(1/2)$ and $(1/2)^2$. These three different types of the third equations are noticed by Painlevé \[11\].

In the same way, the fifth Painlevé equation
\[
y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y'^2 - \frac{1}{t} y' + \frac{(y-1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1},
\]
has three type
\[
\begin{align*}
(P5-A) & \quad \delta \neq 0 \\
(P5-B) & \quad \delta = 0, \gamma \neq 0 \\
(P5-C) & \quad \delta = 0, \gamma = 0
\end{align*}
\]

(P5-A) is a generic case. In the case (P5-B), the fifth Painlevé equation is equivalent to the third Painlevé equation of type $D^{(1)}_6$. In the case (P5-C), the fifth Painlevé equation is quadrature and we exclude the case (P5-C). We denote the case (P5-B) as deg-P5.

In this paper, we study a complete coalescent diagram of singularity type:

The next diagram is the type of the Painlevé equation corresponding to the singularity diagram:

In both diagrams, we have two boxes and four ovals. We will show that the Painlevé equations in a box are equivalent (Theorem 1). The Painlevé equations and their isomonodromic deformations in an oval can be unified in one equation (Theorem 2).

We add four new types to old diagrams. All of them have a singularity whose order is a half integer. The types $(1)^2(3/2)$, $(1)(5/2)$, $(1)(3/2)$ and $(3/2)^2$ correspond to deg-P5, P34, $P3(D^{(1)}_7)$ and $P3(D^{(1)}_8)$, respectively. The third Painlevé equation of $D^{(1)}_6$ type and the second Painlevé equation have two different types of isomonodromic deformations. The type $(0)^2(1/2)$ is corresponding to the fifth Painlevé equation in the case $\delta = 0$. A transformation between (FN) and type $(1)(5/2)$ is also pointed out in [7]. We also add eight new arrows. We have two types of coalescence. One is confluence of two singularities $(r_1)(r_2) \rightarrow (r_1 + r_2)$. The second is decrease in the Poincaré rank $(r) \rightarrow (r - 1/2)$ when $r = 2, 3, 4$. In the old diagram, the second type appeared only in the case $P2 \rightarrow P1$. 
Theorem 1 The coalescent diagram which starts a linear differential equation with four regular singularities consists of ten types of singularities. We obtain eight different types of the Painlevé equations from this diagram. The third Painlevé equation of $D_6^{(1)}$ type and the second Painlevé equation have two types of isomonodromic deformations.

The first Painlevé equation P1

$$y'' = 6y^2 + t.$$ 

can be considered as deg-P2. Painlevé showed that a unified equation of P1 and P2 [10]:

$$y'' = \alpha(2y^3 + ty) + \beta(6y^2 + t) \quad (6)$$

In [10], Painlevé took $\beta = 1$. If $\alpha = 0$, (6) is nothing but P1. We will show (6) is equivalent to P2 if $\alpha \neq 0$ in the section 2.

deg-P5 is also a special case of P5, and P3($D_7^{(1)}$) and P3($D_8^{(1)}$) are also special cases of P3. In the section 2 we show the equation $P_{4,34}'(\alpha, \beta, \gamma)$

$$y'' = \frac{y^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + t) + \gamma y(y + t)(3y + t) \quad (7)$$

is a unified equation of P4 and P34'. If $\gamma = 0$, $P_{4,34}'(\alpha, \beta, \gamma)$ is equivalent to P34''. If $\gamma \neq 0$, $P_{4,34}'(\alpha, \beta, \gamma)$ is equivalent to P4. The authors cannot find the unified equation (7) in literature.

Thus we obtain the following observation.

Theorem 2 In the coalescent diagram, equations in an oval can be represented as one unified equation. P5 and deg-P5 are unified as the standard fifth Painlevé equation. $P(D_6^{(1)})$, $P(D_7^{(1)})$ and $P(D_8^{(1)})$ are unified as the standard third Painlevé equation. $P_4$ and $P_{34}$ are unified as (7). $P_1$ and $P_2$ are unified as (6). In an oval, coalescence reduces the Poincaré rank of a singularity by 1/2. The corresponding linear equations are also unified in one unified equation.

As a single nonlinear equations, the Painlevé equations are classified into five types. Each type has a scaling transformation $t \to c_1 t, y \to c_2 y$ except P6. We obtain eight types of the Painlevé equations after we classify again each type by the scaling transformation.

In the section 2 we review the Painlevé equations. We will show that the Painlevé equations in the same box is equivalent. In the section 3 we show that (FN) comes from an isomonodromic deformation of the type (1)(5/2). We will give two types of isomonodromic deformations of the Painlevé equations. One is the canonical type in the section 4. This form is easy to study when we consider ten types of the Painlevé equations. And the most of the Hamiltonians are polynomials. In the section 4.2 we give a degeneration of the extended linear equation

$$\frac{d^2 u}{dx^2} + p(x, t) \frac{du}{dx} + q(x, t)u = 0, \quad \frac{\partial u}{\partial t} = a(x, t) \frac{\partial u}{\partial x} + b(x, t)u$$

of the Painlevé equations. The second is $SL$-type in the section 5. In this form the extended linear equations of the Painlevé equations in the same oval are also unified in one linear equations. But the Hamiltonians are not polynomials in this form. Most of equations and degenerations are already listed in [8], but we correct misprints in [8].

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2 List of the Painlevé equations

In this section we list up the Painlevé equations in unusual way. This classification is essential for our coalescent diagram. We also give some equivalence between different types of the Painlevé equations. We will give a proof of the second part of the Theorem 1, although this is well-known.

We list five types of the Painlevé equations:

\[ P_{1.2} \quad y'' = \alpha(2y^3 + ty) + \beta(6y^2 + t), \]
\[ P_{4.34'} \quad y'' = \frac{y^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + t) + \gamma y(y + t)(3y + t), \]
\[ P_{3} \quad y'' = \frac{1}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \delta, \]
\[ P_{5} \quad y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) y^2 - \frac{1}{t} y' + \frac{(y - 1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \frac{\gamma y}{t} + \frac{\delta y(y + 1)}{y - 1}, \]
\[ P_{6} \quad y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right) y^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right) y' \]
\[ + \frac{y(y - 1)(y - t)}{t^2(t - 1)^2} \left[ \alpha + \frac{\beta t}{y^2} + \frac{\gamma t - 1}{(y - 1)^2} + \frac{\delta t(t - 1)}{(y - t)^2} \right]. \]

Here \( \alpha, \beta, \gamma, \delta \) are complex parameters. \( P_{1.2}, P_{4.34'}, P_{3} \) and \( P_{5} \) have a scaling transformation. We will classify five types to fourteen types by scaling transformations.

2.1 Unified equation of P1 and P2

By the scaling transformation \( y \rightarrow cy, \ t \rightarrow c^2t, \ P_{1.2}(\alpha, \beta) \) is changed to \( P_{1.2}(c^6\alpha, c^5\beta) \). \( P_{1.2}(\alpha, \beta) \) is divided into three types

(P1-A) \( \alpha \neq 0 \),
(P1-B) \( \alpha = 0, \beta \neq 0 \),
(P1-C) \( \alpha = 0, \beta = 0 \).

Lemma 3 The case (P1-A) is equivalent to P2 and the case (P1-B) is equivalent to P1:

\[ P_{1} \quad y'' = 6y^2 + t, \]
\[ P_{2} \quad y'' = 2y^3 + ty + \alpha. \]

The case (P1-C) is trivial.

Proof. In the case (P1-B), we can set \( \beta = 1 \) by a scaling transformation and \( P_{1.2}(0, 1) \) is nothing but P1. In the case (P1-A), we set \( \alpha = \varepsilon^6 \) and change the variables

\[ y \rightarrow y\varepsilon^{-1} - \beta \varepsilon^{-6}, \ t \rightarrow t\varepsilon^{-2} + 6\beta^2 \varepsilon^{-12}. \]

Then we obtain P2

\[ y'' = 2y^3 + ty + \frac{4\beta^3}{\varepsilon^{15}}. \]

Therefore \( P_{1.2}(\varepsilon^6, \beta) \) is equivalent to \( P_{2}(4\beta^3 \varepsilon^{-15}) \). \qed
2.2 Unified equation of P34 and P4

By the scaling transformation \( y \rightarrow cy, \ t \rightarrow ct \), \( P_{34}'(\alpha, \beta, \gamma) \) is changed to \( P_{34}'(\alpha, c^3\beta, c^4\gamma) \). \( P_{34}'(\alpha, \beta, \gamma) \) is divided into three types

- (P4-A) \( \gamma \neq 0 \),
- (P4-B) \( \beta \neq 0, \gamma = 0 \),
- (P4-C) \( \beta = 0, \gamma = 0 \).

**Lemma 4** The case (P4-A) is equivalent to \( P_4 \) and the case (P4-B) is equivalent to \( P_{34} \):

\[
P_{34}' \quad y'' = \frac{y'^2}{2y} + 2y^2 + ty - \frac{\alpha}{2y},
\]

\[
P_4 \quad y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}.
\]

\( P_2 \) and \( P_{34}' \) are equivalent. The case (P4-C) is quadrature.

**Proof.** In the case (P4-C), \( P_{34}'(\alpha, 0, 0) \) has a solution

\[
y = C_1t^2 + C_2 + \frac{C_2^2 - \alpha}{4C_1}.
\]

In the case (P4-B), we can set \( \beta = 1 \) by a scaling transformation and \( P_{34}'(\alpha, 1, 0) \) is nothing but \( P_{34}'(\alpha) \). In the case (P4-A), we set \( \beta = d^3, \gamma = 2\varepsilon^4 \) and change the variables

\[
y \rightarrow \frac{y}{2\varepsilon}, \ t \rightarrow \varepsilon^{-1}t - \frac{d^3}{4\varepsilon^4}; \ \alpha \rightarrow -\beta/2.
\]

Then we obtain \( P_4(d^6\varepsilon^{-6}/16, \beta) \)

\[
y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2t^2y - \frac{d^6y}{8\varepsilon^6} + \frac{\beta}{y}.
\]

We will show the equivalence between \( P_2 \) and \( P_{34}' \). The second Painlevé equation (1) is represented by a Hamiltonian form:

\[
\mathcal{H}_{II} : \begin{cases}
q' = -q^2 + p - \frac{t}{2}, \\
p' = 2pq + a,
\end{cases} \tag{8}
\]

with the Hamiltonian

\[
\mathcal{H}_{II} = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - aq.
\]

If we remove \( p \) from (8), we obtain \( P_2(a - 1/2) \). If we remove \( q \) from (8), we obtain \( P_{34}(a^2) \). Therefore \( P_2 \) and \( P_{34} \) are equivalent by \( t \rightarrow -t \).

More precisely, if \( y \) satisfies the second Painlevé equation \( P_2(\alpha) \), the function \( p = y^2 + y' + t/2 \) satisfies \( P_{34}((\alpha + 1/2)^2) \). Conversely, if \( p \) satisfies \( P_{34}(\alpha) \), \( q = \frac{1}{2p}(p' - \sqrt{\alpha}) \) satisfies \( P_2(\sqrt{\alpha} - 1/2) \).

**Remark.** If we choose \(-\sqrt{\alpha}\) instead of \(\sqrt{\alpha}\), we obtain \( P_2(-\sqrt{\alpha} - 1/2) \) which is equivalent to \( P_2(\sqrt{\alpha} - 1/2) \) by a Bäcklund transformation. The equivalence of \( P_2 \) and \( P_{34} \) are known by [3].

We will use \( P_{34}' \) instead of \( P_{34} \). If we change the sign of \( t \), we obtain a canonical transformation \((p, q, H, t) \rightarrow (q, p, H, -t)\):

\[
dp \wedge dq - dH \wedge dt = -(dq \wedge dp - dH \wedge d(-t)).
\]
In the followings, we may use \( \sigma = \pm 1 \) to express the both of P3 and P3':

\[
y'' = \frac{y'^2}{2y} + 2y^2 + \sigma ty - \frac{\alpha}{2y}.
\]

Similarly, we express the both of P4, P3′ and P3′:

\[
y'' = \frac{y'^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + \sigma t) + \gamma y(y + \sigma t)(3y + \sigma t).
\]

### 2.3 P3

By the scaling transformation \( y \to c_1 y, \ t \to c_2 t \), P3(\( \alpha, \beta, \gamma, \delta \)) is changed to P3(\( c_1 c_2 \alpha, c_2/c_1 \beta, c_1^2/c_1 \gamma, c_2^2/c_1^2 \delta \)). P3(\( \alpha, \beta, \gamma, \delta \)) is divided into four types

- (P3-A) \( \gamma \neq 0, \delta \neq 0 \)
- (P3-B) \( \gamma \neq 0, \delta = 0 \) or \( \gamma = 0, \delta \neq 0 \)
- (P3-C) \( \gamma = 0, \delta = 0 \)
- (P3-D) \( \alpha = 0, \gamma = 0 \) or \( \beta = 0, \delta = 0 \).

(P3-A) is P3(\( D_6^{(1)} \)), (P3-B) is P3(\( D_7^{(1)} \)), (P3-C) is P3(\( D_8^{(1)} \)) and (P3-D) is quadrature.

In usual we fix \( \gamma = 4, \delta = -4 \) for P3(\( D_6^{(1)} \)), \( \alpha = 2, \gamma = 0, \delta = -4 \) for P3(\( D_7^{(1)} \)) and \( \alpha = 4, \beta = -4, \gamma = 0, \delta = 0 \) for P3(\( D_8^{(1)} \)). See [4].

We will use another form of the third Painlevé equation P3′(\( \alpha, \beta, \gamma, \delta \))

\[
q'' = \frac{1}{q} q'^2 - \frac{q'}{x} + \frac{\alpha q^2 + \gamma q^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4q},
\]

since P3′ is more sympathetic to isomonodromic deformations than P3. We can change P3 to P3′ by \( x = t^2, ty = q \).

### 2.4 P5

By the scaling transformation \( t \to ct \), P5(\( \alpha, \beta, \gamma, \delta \)) is changed to P5(\( \alpha, \beta, c_\gamma, c^2 \delta \)). P5(\( \alpha, \beta, \gamma, \delta \)) is divided into three types

- (P5-A) \( \delta \neq 0 \)
- (P5-B) \( \gamma \neq 0, \delta = 0 \)
- (P5-C) \( \gamma = 0, \delta = 0 \).

The case (P5-A) is a generic P5 and we call (P5-B) as deg-P5. In usual we fix \( \delta = -1/2 \) for (P5-A) and \( \gamma = -2, \delta = 0 \) for (P5-B).

**Lemma 5** (P5-B) is equivalent to P3(\( D_6^{(1)} \)) and (P5-C) is quadrature.

**Proof.** P3′(\( D_6^{(1)} \)) is represented by a Hamiltonian form:

\[
\mathcal{H}'_{D_6} : \begin{cases} 
    tq' = 2pq^2 - q^2 + (\alpha_1 + \beta_1)q + t, \\
    tp' = -2p^2q + 2pq - (\alpha_1 + \beta_1)p + \alpha_1.
\end{cases}
\]

with the Hamiltonian

\[
t\mathcal{H}'_{D_6} = q^2 p^2 - (q^2 - (\alpha_1 + \beta_1)q - t)p - \alpha_1 q.
\]

If we eliminate \( p \) from (9), \( q \) satisfies P3′(\( 4(\alpha_1 - \beta_1), -4(\alpha_1 + \beta_1 - 1), 4, -4 \)). If we eliminate \( q \) from (9) and set \( y = 1 - 1/p \), \( y \) satisfies deg-P5(\( \alpha_1^2/2, -\beta_1^2/2, -2, 0 \)). We can write down \( y \) directly by \( q \):

\[
y = \frac{tq' - q^2 - (\alpha_1 + \beta_1)q - t}{tq' + q^2 - (\alpha_1 + \beta_1)q - t},
\]

Therefore deg-P5 is equivalent to P3(\( D_6^{(1)} \)). This is known by [4].
2.5 Summary

If we classify the five types of the Painlevé equation by scaling transformations, we obtain fourteen types of equations. Four of them are quadrature. Thus we have ten types of the Painlevé equations:

(P1-A), (P1-B), (P4-A), (P4-B), (P3-A), (P3-B), (P3-C), (P5-A), (P5-B), (P6).

(P1-A) and (P4-B) are equivalent and (P3-A) and (P5-B) are equivalent.

3 The Flaschka-Newell form and P34

In this section we will prove that (FN) comes from isomonodromic deformations of type (1)(5/2) and show that it it natural to consider the Flaschka-Newell form as an isomonodromic deformation of P3 not of P2. This proves the rest part of the Theorem[11]. The relation between the Flaschka-Newell form and P34 are noticed by Kapaev and Hubert[6][7].

At first we will review the Poincaré rank of irregular singularities. We consider a linear equation

\[ \frac{d^2u}{dx^2} + p_1(x) \frac{du}{dx} + p_2(x)u = 0. \]  

Assume that

\[ p_1(x) = c_0x^k + c_1x^{k-1} + \cdots, \quad p_2(x) = d_0x^l + d_1x^{l-1} + \cdots, \]

around \( x = \infty \) and \( c_0, d_0 \) are not zero. If

\[ r = \max (k + 1, (l + 2)/2) \]

is positive, \( x = \infty \) is an irregular singularity of (10). We call \( r \) as the Poincaré rank of (10) at \( x = \infty \). The Poincaré rank \( r \) may be a half integer. If \( x = \infty \) is an irregular singularity with the Poincaré rank \( r \), (10) has solutions with an asymptotics

\[ u_j \sim \exp (\kappa_j x^r). \]

Proposition 6 The Flaschka-Newell form of P2 is a double cover of a linear equation of the singularity type (1)(5/2). If we write the equation of the type (1)(5/2) as a single equation, the apparent singularity satisfies P34.

Proof. We consider the following deformation equation.

\[ \frac{dZ}{dw} = \begin{bmatrix} 0 & 2w \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2y & -y^2 - z - t/2 \\ 2y & 2y \end{bmatrix} + \begin{bmatrix} -\alpha + 1/2 & 0 \\ -2y^2 + 2z - t & \alpha - 1/2 \end{bmatrix} \frac{1}{2w} \]

\[ \frac{\partial Z}{\partial t} = \begin{bmatrix} y & -w \\ -1 & -y \end{bmatrix} Z. \]  

(11)

By the compatibility condition, we obtain P2(\( \alpha \))

\[ y' = z, \quad z' = 2y^3 + ty + \alpha. \]

If we change \( w = x^2 \) and \( Z = RY \) where

\[ R = \begin{bmatrix} \sqrt{x} & \sqrt{x} \\ -1/\sqrt{x} & 1/\sqrt{x} \end{bmatrix}, \]
we obtain the FN form (2). Since the exponents of (11) at \( w = \infty \) coincide, the Poincaré rank at \( w = \infty \) in (11) is \((3/2)\). We will rewrite (11) as a single equation of the second order.

We change the variables

\[
  w \to \frac{w}{2}, \quad z \to p^2 + q - \frac{t}{2}, \quad y \to -p.
\]

Then (11) is changed to

\[
  \frac{dZ}{dw} = \left[ \begin{pmatrix} p & x/2 - q/2 \\ 1 & -p \end{pmatrix} + \begin{pmatrix} -\alpha/2 + 1/4 & 0 \\ q - 2p^2 - t & \alpha/2 - 1/4 \end{pmatrix} \frac{1}{w} \right] Z, \\
  \frac{\partial Z}{\partial t} = \begin{pmatrix} -p & -x/2 \\ -1 & p \end{pmatrix} Z.
\]

The compatibility condition is

\[
  \begin{cases}
    q' = -2pq + \left( \alpha + \frac{1}{2} \right), \\
    p' = p^2 - q + \frac{t}{2}.
  \end{cases}
\]

We set

\[
  Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 = w^{1/4 - \alpha/2} u.
\]

Eliminating \( u_2 \) from (12), we get a single equation for \( u = u_1 \):

\[
  \frac{d^2 u}{dw^2} + p_1(w, t) \frac{du}{dw} + p_2(w, t) u = 0,
\]

\[
  \frac{\partial u}{\partial t} = a(w, t) \frac{\partial u}{\partial w} + b(w, t) u,
\]

where

\[
  p_1(w, t) = -\frac{1}{w - q} + \frac{1/2 - \alpha}{w}, \quad p_2(w, t) = -\frac{w}{2} + \frac{t}{2} + \frac{\mathcal{H}_{34}}{w} + \frac{pq}{w(w - q)},
\]

\[
  a(w, t) = -\frac{w}{w - q}, \quad b(w, t) = \frac{pq}{w - q},
\]

\[
  \mathcal{H}_{34} = -pq^2 + \left( \alpha + \frac{1}{2} \right) p + \frac{q^2}{2} - \frac{1}{2} t q.
\]

The isomonodromic deformation is equivalent to the Hamiltonian system (13) with the Hamiltonian \( \mathcal{H}_{34} \). If we eliminate \( p \) from (13), we obtain \( P34((\alpha + 1/2)^2) \) for \( q \).

The first equation of (14) has a regular singularity \( w = 0 \) and an irregular singularity of the Poincaré rank \( 3/2 \) at \( w = \infty \). It also has an apparent singularity \( w = q \). When we write the Painlevé equations as isomonodromic deformations of linear equations of the second order, they have an apparent singularity. And the apparent singularity is the Painlevé function. Moreover

\[
  p = \text{Res}_{w=q} p_2(w, t)
\]

is a canonical coordinate \( \Psi \). In the Flaschka-Newell case, the apparent singularity \( q \) satisfies \( P34 \) but not \( P2 \).
4 Isomonodromic deformations of canonical type

This section is the revision of the section 4.3 in [8]. In this part, we list up isomonodromic deformations of the canonical type $L_J$:

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + p(x,t)\frac{\partial u}{\partial x} + q(x,t)u &= 0, \\
\frac{\partial u}{\partial t} &= a(x,t)\frac{\partial u}{\partial x} + b(x,t)u.
\end{align*}
\]

The extended linear equation $L_J$ is called the canonical type if it is obtained from the canonical type equation $L_{VI}$ by the process by step-by-step confluence. And the Fuchsian equation $L_{VI}$ is called the canonical type if either of the local exponents at any singular point is zero. The compatibility condition of (15) is

\[
\begin{align*}
p_t(x,t) - (p(x,t)a(x,t))_x + a_{xx}(x,t) + 2b_x(x,t) &= 0, \\
q_t(x,t) - 2q(x,t)a_x(x,t) - q_x(x,t)a(x,t) + p(x,t)b_x(x,t) + b_{xx}(x,t) &= 0.
\end{align*}
\]

The second equation is an essential deformation equation and is a Hamiltonian system with the Hamiltonian $\mathcal{H}_J$. $b(x,t)$ is determined from the first equation by integration and if we change $b(x,t) \to b(x,t) + s(t)$, the compatibility condition is also satisfied. We can eliminate $s(t)$ by the transformation $u \to u \exp \int s(t) dt$. In the following list, we may change $b(x,t)$ up to an additive term $s(t)$.

4.1 List of canonical type

We have ten types of isomonodromic deformations: P1, P2, P34, P3'(D_{6}^{(1)}), P3'(D_{7}^{(1)}), P3'(D_{8}^{(1)}), P4, P5, deg-P5 and P6. We will show isomonodromic deformation not only for P3' but also for the original P3. We need two types of P3'(D_{7}^{(1)}) for degeneration from deg-P5. One is the case $\gamma = 0$ and the other is the case $\delta = 0$. We also show the isomonodromic deformations for P1,2 and P4,34. But these unified equations are not necessary for degenerations.

We will list up seventeen types, but they are classified in ten types up to algebraic transformations. We will show the degeneration diagram:

Here double lines mean algebraic transformations and equations in a box are equivalent to each other.
Painlevé 1.2:
\[ p(x, t) = -2\eta x^2 - \eta t - \frac{1}{x - y}, \]
\[ q(x, t) = -4\beta x^3 - (\eta + 2\beta t)x - 2\mathcal{H}_{1.2} + \frac{z}{x - y}, \]
\[ a(x, t) = \frac{1}{2(x - y)}, \]
\[ b(x, t) = \frac{\beta\eta^{-1} - \eta y}{2} - \frac{z}{2(x - y)}, \]
\[ \mathcal{H}_{1.2} = \frac{1}{2}z^2 - \left(\eta y^2 + \frac{1}{2}\right)z - 2\beta y^3 - t\beta y - \frac{1}{2}\eta y. \]
\[ \alpha = \eta^2. \]

Painlevé I:
\[ p(x, t) = -\frac{1}{x - y}, \]
\[ q(x, t) = -4x^3 - 2tx - 2\mathcal{H}_I + \frac{z}{x - y}, \]
\[ a(x, t) = \frac{1}{2(x - y)}, \]
\[ b(x, t) = -\frac{z}{2(x - y)}, \]
\[ \mathcal{H}_I = \frac{1}{2}z^2 - 2y^3 - ty. \]

Painlevé II:
\[ p(x, t) = -2x^2 - t - \frac{1}{x - y}, \]
\[ q(x, t) = -2(\alpha + 1)x - 2\mathcal{H}_{II} + \frac{z}{x - y}, \]
\[ a(x, t) = \frac{1}{2(x - y)}, \]
\[ b(x, t) = -\frac{y}{2} - \frac{z}{2(x - y)}, \]
\[ \mathcal{H}_{II} = \frac{1}{2}z^2 - \left(\eta y^2 + \frac{1}{2}\right)z - \left(\alpha + \frac{1}{2}\right)y. \]

Painlevé 4.34: If \( \sigma = +1 \), this gives P4.34'. If \( \sigma = -1 \), this gives P4.34.
\[ p(x, t) = \theta - \sigma \eta t + \frac{1 - \kappa_0}{x} - \eta x - \frac{1}{x - y}, \]
\[ q(x, t) = \frac{\theta^2}{4} + \frac{(\kappa_0 - 1)\eta}{2} - \sigma \mathcal{H}_{4.34} - \frac{yz}{x(x - y)}, \]
\[ a(x, t) = \frac{\sigma x}{x - y}, \]
\[ b(x, t) = -\frac{\sigma yz}{x - y}. \]
\[ \mathcal{H}_{4.34} = \sigma yz^2 - \sigma \left(\eta y^2 - \theta y + \kappa_0\right)z + \sigma \left(\frac{\theta^2}{4} + \frac{(\kappa_0 - 1)\eta}{2}\right)y - \eta t y z. \]
\[
\alpha = \kappa_0^2, \; \beta = -\eta \theta, \; \gamma = \frac{1}{2} \eta^2. \\
\]

Painlevé 34: If \( \sigma = +1 \), this gives \( P34' \). If \( \sigma = -1 \), this gives \( P34 \).

\[
p(x, t) = -\frac{1}{x - y} + \frac{1 - \kappa_0}{x}, \\
q(x, t) = -\frac{x - \sigma t}{2} - \frac{\sigma \mathcal{H}_{34}}{2x} + \frac{yz}{x(x - y)}, \\
a(x, t) = \frac{\sigma x}{x - y}, \\
b(x, t) = -\frac{\sigma y z}{x - y}, \\
\mathcal{H}_{34} = \sigma \left( y z^2 - \kappa_0 z - \frac{\eta^2}{2} \right) - \frac{1}{2} t y. \\
\alpha = \kappa_0^2.
\]

Painlevé IV:

\[
p(x, t) = \frac{1 - \kappa_0}{x} - \frac{x + 2t}{2} - \frac{1}{x - y}, \\
q(x, t) = \frac{1}{2} \theta_{\infty} - \frac{\mathcal{H}_{IV}}{2x} + \frac{yz}{x(x - y)}, \\
a(x, t) = \frac{2x}{x - y}, \\
b(x, t) = -\frac{1}{2} (y + 2t) - \frac{2yz}{x - y}, \\
\mathcal{H}_{IV} = 2yz^2 - (y^2 + 2ty + 2\kappa_0)z + \theta_{\infty} y, \\
\alpha = -\kappa_0 + 2\theta_{\infty} + 1, \; \beta = -2\kappa_0^2.
\]

Painlevé III(\(D_6^{(1)}\)):

\[
p(x, t) = \frac{\eta_0 t}{x^2} + \frac{1 - \theta_0}{x} - \eta_{\infty} t - \frac{1}{x - y}, \\
q(x, t) = \frac{\eta_{\infty}(\theta_0 + \theta_{\infty}) t}{2x} - \frac{t \mathcal{H}_{III} + y z}{2x^2} + \frac{yz}{x(x - y)}, \\
a(x, t) = \frac{2yx}{t(x - y)} + \frac{x}{t}, \\
b(x, t) = -\eta_{\infty} y - \frac{2y^2 z}{t(x - y)}. \\
t \mathcal{H}_{III} = 2y^2 z^2 - \{ 2\eta_{\infty} t y^2 + (2\theta_0 + 1)y - 2\eta_0 t \} z + \eta_{\infty}(\theta_0 + \theta_{\infty}) t y. \\
\alpha = -4\eta_{\infty} \theta_{\infty}, \; \beta = 4\eta_0 (1 + \theta_0), \; \gamma = 4\eta_{\infty}^2, \; \delta = -4\eta_0^2.
Painlevé III\((D_7^{(1)})\): The case \(\gamma = 0\).
\[
\begin{align*}
p(x, t) &= \frac{\eta_0 t}{x^2} + \frac{1 - \theta_0}{x} - \frac{1}{x - y}, \\
q(x, t) &= \frac{\theta_\infty t}{2x} - \frac{t\mathcal{H}_{D_7} + yz}{2x^2} + \frac{yz}{x(x - y)}, \\
a(x, t) &= \frac{2yx}{t(x - y)} + \frac{x}{t}, \\
b(x, t) &= -\frac{2y^2 z}{t(x - y)}.
\end{align*}
\]
\[
t\mathcal{H}_{D_7} = 2y^2 z^2 - \{(2\theta_0 + 1)y - 2\eta_0 t\} z + \theta_\infty ty.
\]
\[
\alpha = -4\theta_\infty, \quad \beta = 4(\theta_0 + 1)\eta_0, \quad \gamma = 0, \quad \delta = -4\eta_0^2.
\]

Painlevé III\((D_7^{(1)})-2\): The case \(\delta = 0\).
\[
\begin{align*}
p(x, t) &= \eta_\infty t + \frac{1}{x} - \frac{1}{x - y}, \\
q(x, t) &= \frac{\theta_0 t}{2x^3} - \frac{t\mathcal{H}_{D_7-2} + yz}{2x^2} + \frac{\theta_\infty \eta_\infty t}{2x} + \frac{yz}{x(x - y)}, \\
a(x, t) &= \frac{2yx}{t(x - y)} + \frac{x}{t}, \\
b(x, t) &= -\frac{2y^2 z}{t(x - y)}.
\end{align*}
\]
\[
t\mathcal{H}_{D_7-2} = 2y^2 z^2 - \{2\eta_\infty y^2 + y\} z + \theta_\infty \eta_\infty ty + \frac{\theta_0 t}{y}.
\]
\[
\alpha = -4\theta_\infty \eta_\infty, \quad \beta = 4\theta_0, \quad \gamma = 4\eta_\infty^2, \quad \delta = 0.
\]

Painlevé III\((D_8^{(1)})\):
\[
\begin{align*}
p(x, t) &= \frac{2}{x} - \frac{1}{x - y}, \\
q(x, t) &= \frac{t}{2x^3} - \frac{t\mathcal{H}_{D_8} + yz}{2x^2} + \frac{t}{2x} + \frac{yz}{x(x - y)}, \\
a(x, t) &= \frac{2yx}{t(x - y)} + \frac{x}{t}, \\
b(x, t) &= -\frac{2y^2 z}{t(x - y)},
\end{align*}
\]
\[
t\mathcal{H}_{D_8} = 2y^2 z^2 + yz + ty + \frac{t}{y},
\]
\[
\alpha = -4, \quad \beta = 4, \quad \gamma = 0, \quad \delta = 0.
\]

Painlevé III\'(\(D_6^{(1)})\):
\[
\begin{align*}
p(x, t) &= \frac{\eta_0 t}{x^2} + \frac{1 - \theta_0}{x} - \eta_\infty - \frac{1}{x - y}, \\
q(x, t) &= \frac{\eta_\infty (\theta_0 + \theta_\infty)}{2x} - \frac{t\mathcal{H}'_{D_6}}{x^2} + \frac{yz}{x(x - y)}, \\
a(x, t) &= \frac{yx}{t(x - y)}, \\
b(x, t) &= -\frac{y^2 z}{t(x - y)}.
\end{align*}
\]
\[ t\mathcal{H}_{D_6}' = y^2z^2 - \{\eta_\infty y^2 + \theta_0 y - \eta_0 t\} z + \frac{1}{2}\eta_\infty(\theta_0 + \theta_\infty)y. \]

\[ \alpha = -4\eta_\infty\theta_\infty, \quad \beta = 4\eta_0(1 + \theta_0), \quad \gamma = 4\eta_\infty^2, \quad \delta = -4\eta_0^2. \]

Painlevé III\((D_7^{(1)})\): The case \(\gamma = 0\)

\[
\begin{align*}
p(x, t) &= \frac{\eta_0 t}{x^2} + \frac{1 - \theta_0}{x} - \frac{1}{x-y}, \\
q(x, t) &= -\frac{t\mathcal{H}_{D_7}'}{x^2} + \frac{\theta_\infty}{2x} + \frac{yz}{x(x-y)}, \\
a(x, t) &= \frac{yx}{t(x-y)}, \\
b(x, t) &= -\frac{y^2z}{t(x-y)}, \\
t\mathcal{H}_{D_7}' &= y^2z^2 + (-\theta_0 y + \eta_0 t)z + \frac{\theta_\infty}{2}y. 
\end{align*}
\]

\[
\alpha = -4\theta_\infty, \quad \beta = 4(\theta_0 + 1)\eta_0, \quad \gamma = 0, \quad \delta = -4\eta_0^2.
\]

Painlevé III\((D_7^{(1)})\)-2: The case \(\delta = 0\)

\[
\begin{align*}
p(x, t) &= -\eta_\infty + \frac{1}{x} - \frac{1}{x-y}, \\
q(x, t) &= \frac{\theta_0 t}{2x^3} + \frac{\theta_\infty\eta_\infty}{2x^2} - \frac{t\mathcal{H}_{D_7,-2}'}{x^2} + \frac{yz}{x(x-y)}, \\
a(x, t) &= \frac{yx}{t(x-y)}, \\
b(x, t) &= -\frac{y^2z}{t(x-y)}, \\
t\mathcal{H}_{D_7,-2}' &= y^2z^2 - \eta_\infty y^2z + \frac{\theta_\infty\eta_\infty}{2}y + \frac{\theta_0 t}{2y}. 
\end{align*}
\]

\[
\alpha = -4\theta_\infty\eta_\infty, \quad \beta = 4\theta_0, \quad \gamma = 4\eta_\infty^2, \quad \delta = 0.
\]

Painlevé III\((D_8^{(1)})\):

\[
\begin{align*}
p(x, t) &= \frac{2}{x} - \frac{1}{x-y}, \\
q(x, t) &= \frac{t}{2x^3} - \frac{t\mathcal{H}_{D_8}'}{x^2} + \frac{1}{2x} + \frac{yz}{x(x-y)}, \\
a(x, t) &= \frac{yx}{t(x-y)}, \\
b(x, t) &= -\frac{y^2z}{t(x-y)}, \\
t\mathcal{H}_{D_8}' &= y^2z^2 + yz + \frac{y}{2} + \frac{t}{2y}. 
\end{align*}
\]

\[
\alpha = -4, \quad \beta = 4, \quad \gamma = 0, \quad \delta = 0.
\]
Painlevé V:

\[
\begin{align*}
\begin{aligned}
p(x,t) &= \frac{1 - \kappa_0}{x} + \frac{\eta t}{(x-1)^2} + \frac{1-\theta - \frac{1}{x-1}}{x-y}, \\
q(x,t) &= \frac{\kappa}{x(x-1)} - \frac{t\mathcal{H}_V}{x(x-1)^2} + \frac{y(y-1)z}{x(x-1)(x-y)}, \\
a(x,t) &= \frac{y-1}{x-1}, \\
b(x,t) &= -\frac{y(y-1)^2z}{t(x-y)}.
\end{aligned}
\end{align*}
\]

\[
\mathcal{H}_V = y(y-1)^2z^2 - \{\kappa_0(y-1)^2 + \theta y(y-1) - \eta ty\}z + \kappa(y-1).
\]

\[
\alpha = \frac{1}{2}\kappa_0^2, \quad \beta = -\frac{1}{2}\kappa_0^2, \quad \gamma = -(1+\theta)\eta, \quad \delta = -\frac{1}{2}\eta^2, \\
\kappa = \frac{1}{4}(\kappa_0 + \theta)^2 - \frac{1}{4}\kappa_\infty^2.
\]

Painlevé deg-V:

\[
\begin{align*}
\begin{aligned}
p(x,t) &= \frac{1}{x-1} + \frac{1-\kappa_0}{x} - \frac{1}{x-y}, \\
q(x,t) &= \frac{\gamma t}{2(x-1)^2} + \frac{\kappa}{x(x-1)} - \frac{t\mathcal{H}_{Vd}}{x(x-1)^2} + \frac{y(y-1)z}{x(x-1)(x-y)}, \\
a(x,t) &= \frac{y-1}{x-1}, \\
b(x,t) &= -\frac{y(y-1)^2z}{t(x-y)}.
\end{aligned}
\end{align*}
\]

\[
\mathcal{H}_{Vd} = y(y-1)^2z^2 - \kappa_0(y-1)^2z + \kappa(y-1) + \frac{\gamma ty}{2(y-1)}.
\]

\[
\alpha = \frac{1}{2}\kappa_0^2, \quad \beta = -\frac{1}{2}\kappa_0^2, \quad \delta = 0, \quad \kappa = -\frac{1}{2}(\alpha + \beta) = \frac{1}{4}(\kappa_0^2 - \kappa_\infty^2).
\]

Painlevé VI:

\[
\begin{align*}
\begin{aligned}
p(x,t) &= \frac{1 - \kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta - \frac{1}{x-1}}{x-t - \frac{1}{x-y}}, \\
q(x,t) &= \frac{\kappa}{x(x-1)} - \frac{t(t-1)\mathcal{H}_{VI}}{x(x-1)(x-t)} + \frac{y(y-1)z}{x(x-1)(x-y)}, \\
a(x,t) &= \frac{y-t}{t(t-1)}, \\
b(x,t) &= -\frac{y(y-1)(y-t)z}{t(t-1)(x-y)}.
\end{aligned}
\end{align*}
\]

\[
t(t-1)\mathcal{H}_{VI} = y(y-1)(y-t)z^2 - \{\kappa_0(y-1)(y-t) + \kappa_1 y(y-t) + (\theta-1)y(y-1)\}z + \kappa(y-t).
\]

\[
\alpha = \frac{1}{2}\kappa_\infty^2, \quad \beta = -\frac{1}{2}\kappa_0^2, \quad \gamma = \frac{1}{2}\kappa_1^2, \quad \delta = \frac{1}{2}(1-\theta)^2, \\
\kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_\infty^2.
\]
4.2 Degeneration

In this section we list up all of degeneration of the Painlevé equations and extended linear equations $L_J$. Here we consider degeneration of the extended linear system, which includes a deformation equation. In some cases, we should take a change of the dependent variable $u \to f(x, t, \varepsilon)u$. In [8] Okamoto did not treat the extended linear equations. If $b(x, t)$ is changed up to a function $r(t)$ in the limit, we denote $b \to b + r$.

P6→P5: We change the variables

$$ t \to 1 + \varepsilon t, \quad \kappa_1 \to \varepsilon^{-1} \eta + \theta + 1, \quad \theta \to -\varepsilon^{-1} \eta, $$

$$(\alpha \to \alpha, \beta \to \beta, \gamma \to -\delta \varepsilon^{-2} + \gamma \varepsilon^{-1}, \quad \delta \to \delta \varepsilon^{-2}).$$

In the limit $\varepsilon \to 0$, $L_{VI}$ goes to $L_V$ and

$$ \mathcal{H}_{VI} \to \varepsilon^{-1} \mathcal{H}_V + O(\varepsilon^0), \quad (\varepsilon \to 0). $$

P5→deg-P5: We change the variables

$$ z \to z + \frac{\gamma}{2\varepsilon(y-1)}, \quad \eta \to \varepsilon, \quad \theta \to \gamma \varepsilon^{-1}. $$

Then

$$ \mathcal{H}_V + \frac{\theta^2}{4t} \to \mathcal{H}_{Vd} + O(\varepsilon^1), \quad (\varepsilon \to 0). $$

For $L_V$ we change

$$ u \to (x - 1) \theta^{\gamma/2} u $$

and $b \to b - \theta(y - 1)/(2t)$ at first. Then $L_V$ goes to $L_{Vd}$ in the limit $\varepsilon \to 0$.

P5→P4: We change the variables

$$ t \to 1 + \sqrt{2} \varepsilon t, \quad y \to \frac{\varepsilon}{\sqrt{2}} y, \quad z \to \sqrt{2} \varepsilon^{-1} z, \quad x \to \frac{\varepsilon}{\sqrt{2}} x, $$

$$ \kappa_0 \to \varepsilon^{-2}, \quad \theta \to \varepsilon^{-2} + 2\theta_\infty - \kappa_0, \quad \eta \to -\varepsilon^{-2}, $$

$$(\alpha \to \varepsilon^{-4}/2, \beta \to \beta/4, \gamma \to -\varepsilon^{-4}, \quad \delta \to -\varepsilon^{-4}/2 + \alpha \varepsilon^{-2}).$$

After changing variables, we set $u \to \exp(\varepsilon^{-1} t/\sqrt{2}) u$. Then in the limit $\varepsilon \to 0$, $L_V$ goes to $L_{IV}$ with $b \to b + t + y/2$ and

$$ \sqrt{2} \left( \mathcal{H}_V + \frac{(\kappa_0 + \theta)^2 - \kappa_\infty^2}{4 t} \right) \to \varepsilon^{-1} (\mathcal{H}_{IV} + 2\theta_\infty t) + O(\varepsilon^0), \quad (\varepsilon \to 0). $$

P5→P′(D_6^{(1)}): We change the variables

$$ y \to 1 + \varepsilon y, \quad z \to z/\varepsilon, \quad x \to 1 + \varepsilon x, $$

$$ \kappa_0 \to \varepsilon^{-1} \eta_\infty, \quad \kappa_\infty \to \varepsilon^{-1} \eta_\infty - \theta_\infty, \quad \theta \to \theta_0, \quad \eta \to \varepsilon \eta_0, $$

$$(\alpha \to \frac{1}{8} \varepsilon^{-2} \gamma + \frac{1}{4} \varepsilon^{-1} \alpha, \beta \to -\frac{\varepsilon^{-2} \gamma}{8}, \gamma \to \frac{\varepsilon \beta}{4}, \delta \to \frac{\varepsilon^2 \delta}{8}).$$

In the limit $\varepsilon \to 0$, $L_V$ goes to $L_{D_6}$ and

$$ \mathcal{H}_V \to \mathcal{H}_{D_6} + O(\varepsilon^1), \quad (\varepsilon \to 0). $
deg-P5→P3′(D_{7}^{(1)})-2: We change the variables
\[ y \to 1 + \varepsilon y, \quad z \to \varepsilon^{-1}z, \quad x \to 1 + \varepsilon x, \]
\[ \kappa_\infty \to \varepsilon^{-1}\eta_\infty, \quad \kappa_0 \to \varepsilon^{-1}\eta_\infty + \theta_\infty, \quad \gamma \to \theta_0\varepsilon/4, \]
\[ (\alpha \to -\varepsilon^{-2}\gamma/8, \quad \beta \to \varepsilon^{-1}\alpha/4, \quad \gamma \to \beta\varepsilon/4). \]
In the limit \( \varepsilon \to 0 \), \( L_{Vd} \) goes to \( L_{D_{7}^{-2}} \) and
\[ \mathcal{H}_{Vd} \to \mathcal{H}_{D_{7}^{-2}} + O(\varepsilon), \quad (\varepsilon \to 0). \]

deg-P5→P34: We change the variables
\[ t \to 1 + \sigma\varepsilon^2t, \quad y \to \varepsilon^2 y, \quad z \to z\varepsilon^{-2}, \quad x \to \varepsilon^2 x, \]
\[ \kappa_\infty \to \sigma\sqrt{-2\varepsilon^{-3}}, \quad \gamma \to \varepsilon^{-6}, \]
\[ (\alpha \to -\varepsilon^{-6}, \quad \beta \to -\alpha/2, \quad \gamma \to \varepsilon^{-6}). \]
In the limit \( \varepsilon \to 0 \), \( L_{Vd} \) goes to \( L_{34} \) and
\[ \mathcal{H}_{Vd} = \varepsilon^{-2} \left( \sigma\mathcal{H}_{34} - t^2/2 \right) + \sigma t\varepsilon^{-4}/2 - \varepsilon^{-6}/2 + O(\varepsilon^{-1}), \quad (\varepsilon \to 0). \]

P3′→P3: This transformation is algebraic and we do not take any limit. If we change the variables
\[ t \to t^2, \quad y \to ty, \quad z \to z/t, \quad x \to tx, \]
\( L'_J \) is changed to \( L_J \) with \( b \to b + \eta_\infty y \) if \( J = D_{6}^{(1)} \) and
\[ \mathcal{H}'_J = \frac{1}{2t}\mathcal{H}_J + \frac{yz}{2t^2}, \]
for \( J = D_{6}^{(1)}, D_{7}^{(1)}, D_{7}^{(1)}-2, D_{8}^{(1)}. \)

P3′(D_{6}^{(1)})→P3′(D_{7}^{(1)}): We change the parameters
\[ \eta_\infty \to \varepsilon, \quad \theta_\infty \to \theta_\infty\varepsilon^{-1}. \]
In the limit \( \varepsilon \to 0 \), \( L'_{D_6} \) goes to \( L'_{D_7} \) and
\[ \mathcal{H}'_{D_6} = \mathcal{H}'_{D_7} + O(\varepsilon), \quad (\varepsilon \to 0). \]

P3(D_{6}^{(1)})→P3(D_{7}^{(1)}) is as the same.

P3′(D_{6}^{(1)})→P3′(D_{7}^{(1)})-2: We change the parameters
\[ \eta_0 \to \varepsilon, \quad \theta_0 \to \theta_0\varepsilon^{-1}, \quad z \to z + \frac{\theta_0}{2\varepsilon y}. \]
Then we have
\[ \mathcal{H}'_{D_6} + \frac{\theta_0^2}{4t} \to \mathcal{H}'_{D_7-2} + O(\varepsilon), \quad (\varepsilon \to 0). \]
For \( L'_{D_6} \), we change
\[ u \to x^{\theta_0/2}u \]
at first. Then \( L'_{D_6} \) goes to \( L'_{D_7-2} \). P3′(D_{6}^{(1)})→P3′(D_{7}^{(1)})-2 is as the same.
$P_3'(D_7^{(1)}) \to P_3'(D_7^{(1)})^{-2}$: This transformation is algebraic and we do not take any limit. If we change the variables

\[
y \to t/y, \ z \to (\theta_\infty y/2 - y^2z)/t,
\]

\[
\theta_0 \to \theta_\infty - 1, \ \theta_\infty \to \theta_0, \ \eta_0 \to \eta_\infty,
\]

\[
(\alpha \to -\beta, \ \beta \to -\alpha, \ \delta \to -\gamma).
\]

we have

\[
\mathcal{H}'_{D_7} = \mathcal{H}'_{D_7^{-2}} - \frac{yz}{t} - \frac{\theta_\infty(\theta_\infty - 2)}{4t}.
\]

For $L'_D$, we change

\[
u \to u x(\theta_0 + 1)/2
\]
at first. Changing the variable $x \to t/x$, $L'_D$ is changed to $L'_D^{-2}$ with $b \to b - yz/t$.

$P_3'(D_7^{(1)}) \to P_3'(D_8^{(1)})$: We change the variables

\[
t \to 2t, \ y \to 2y, \ z \to z/2 + 1/(4\varepsilon y),
\]

\[
\eta_0 \to \varepsilon, \ \theta_0 \to -1 + \varepsilon^{-1}, \ \theta_\infty \to 1/2.
\]

Then

\[
\mathcal{H}'_{D_7} = \frac{1}{2} \mathcal{H}'_{D_8} - \frac{1}{8\varepsilon^2 t} + \frac{1}{4\varepsilon t} + O(\varepsilon), \quad (\varepsilon \to 0).
\]

For $L'_D$, we change

\[
u \to u x(1+\theta_0)/2 u
\]
at first. Changing the variable $x \to 2x$, $L'_D$ goes to $L'_D$ in the limit $\varepsilon \to 0$. $P_3(D_7^{(1)}) \to P_3(D_8^{(1)})$ is as the same.

$P_3'(D_7^{(1)})^{-2} \to P_3'(D_8^{(1)})$: We change the variables

\[
t \to -2t, \ z \to z + 1/(2y),
\]

\[
\eta_\infty \to \varepsilon, \ \theta_0 \to -1/2, \ \theta_\infty \to \varepsilon^{-1}.
\]

Then

\[
\mathcal{H}'_{D_7^{-2}} = -\frac{1}{2} \mathcal{H}'_{D_8} - \frac{1}{8t} + O(\varepsilon), \quad (\varepsilon \to 0).
\]

For $L'_D^{-2}$ we change

\[
u \to x^{1/2} u
\]
at first. Changing the variables, $L'_D^{-2}$ goes to $L'_D$ in the limit $\varepsilon \to 0$. $P_3(D_7^{(1)})^{-2} \to P_3(D_8^{(1)})$ is as the same.

$P_3(D_6^{(1)}) \to P_2$: We change the variables

\[
t \to 1 + \varepsilon^2 t, \ y \to 1 + 2\varepsilon y, \ z \to 1 + \frac{z}{2\varepsilon}, \ x \to 1 + 2\varepsilon x,
\]

\[
\eta_0 \to -\varepsilon^{-3}/4, \ \eta_\infty \to \varepsilon^{-3}/4, \ \theta_0 \to -\varepsilon^{-3}/2 - 2\alpha - 1, \ \theta_\infty \to \varepsilon^{-3}/2,
\]

\[
(\alpha \to -\frac{\varepsilon^6}{2}, \ \beta \to \frac{1}{2}\varepsilon^{-6}(1 + 4\alpha\varepsilon^3), \ \gamma \to \frac{\varepsilon}{4}, \ \delta \to -\frac{\varepsilon^{-6}}{4}).
\]
I. After changing variables, we set \( u \to \exp(-\varepsilon^{-1}t/4)u \). Then in the limit \( \varepsilon \to 0 \), \( L_{D_6} \) goes to \( L_{II} \) and

\[
\mathcal{H}_{D_6} + \eta_0(\theta_0 + \theta_\infty) \to \varepsilon^{-2}\mathcal{H}_{II} + O(\varepsilon^{-1}), \quad (\varepsilon \to 0).
\]

P4→P4,34: This transformation is algebraic and we do not take any limit. We change the variables

\[
t \to \sigma \varepsilon t - \varepsilon^{-1}\theta/2, \ y \to 2\varepsilon y, \ z \to \varepsilon^{-1}z/2,
\]

\[
x \to 2\varepsilon x, \ \theta_\infty \to (\kappa_0 - 1)/2 + \varepsilon^{-2}\theta^2/8,
\]

\[
(\alpha \to \varepsilon^{-6}\beta^2/16, \ \beta \to -2\alpha, \ 2\varepsilon^4 \to \gamma).
\]

Then \( L_{IV} \) is changed to \( L_{4,34} \) with \( b \to b - \sigma\varepsilon^2 y - \varepsilon^2 t + \sigma\theta/2 \) and

\[
\mathcal{H}_{IV} = \sigma\varepsilon^{-1}\mathcal{H}_{4,34}
\]

for \( \eta = 2\varepsilon^2 \).

P4→P2: We change the variables

\[
t \to -\varepsilon^{-3}(1 - 2^{-2/3}\varepsilon^4 t), \ y \to \varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 y), \ z \to 2^{-2/3}\varepsilon z,
\]

\[
x \to \varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 x), \ u \to \exp(2^{-5/3}\varepsilon^{-2}t)u,
\]

\[
\kappa_0 \to \varepsilon^{-6}/2, \ \theta_\infty \to -\alpha - 1/2, \quad \left( \alpha \to -2\alpha - \frac{1}{2\varepsilon^6}, \ \beta \to -\frac{1}{2\varepsilon^{12}} \right).
\]

In the limit \( \varepsilon \to 0 \), \( L_{IV} \) goes to \( L_{II} \) and

\[
\mathcal{H}_{IV} \to 2^{2/3}\varepsilon^{-1}\mathcal{H}_{II} - \varepsilon^{-3}(\alpha + 1/2) + O(\varepsilon^0), \quad (\varepsilon \to 0).
\]

P4→P34: We change the variables

\[
t \to \varepsilon t + \sigma\varepsilon^{-3}/4, \ y \to 2\sigma\varepsilon y, \ z \to \sigma\varepsilon^{-1}z/2 + \sigma\varepsilon^{-3}/8,
\]

\[
x \to \sigma\varepsilon x, \ u \to \exp \frac{\sigma x}{8\varepsilon^3} u,
\]

\[
\theta_\infty \to \varepsilon^{-6}/32, \quad (\alpha \to \varepsilon^{-6}/16, \ \beta \to -2\alpha).
\]

In the limit \( \varepsilon \to 0 \), \( L_{IV} \) goes to \( L_{34} \) and

\[
\mathcal{H}_{IV} \to -\frac{1}{\varepsilon}\mathcal{H}_{34} - \frac{\sigma\kappa_0}{4\varepsilon^3} + O(\varepsilon^0), \quad (\varepsilon \to 0).
\]

P3\((D_7^{(1)})\)→P1: We change the variables

\[
t \to (\varepsilon^{-10} + \varepsilon^{-6}t)/2, \ y \to 1 + 2\varepsilon^2 y, \ z \to \varepsilon^{-2}z/2 - \frac{\varepsilon^{-5}}{4\sqrt{-2}} - \frac{\varepsilon^{-3}y}{2\sqrt{-2}},
\]

\[
\theta_0 \to -1 + 3\sqrt{-2}\varepsilon^{-5}/4, \ \theta_\infty \to -1/2, \ \eta_0 \to \sqrt{-2}\varepsilon^5,
\]

\[
(\alpha \to 2, \ \beta \to -6, \ \delta \to 8\varepsilon^{10}).
\]

Then

\[
\mathcal{H}_{D_7} \to \frac{5}{8}\eta_0^2 - \frac{\eta_0}{4} - \frac{1}{4} \to 2\varepsilon^6\mathcal{H}_I + O(\varepsilon^7), \quad (\varepsilon \to 0).
\]
For $L_{D_7}$, we change
\[ u \to x^{(\theta_0 - 1)^2} \exp \left( \frac{\eta_0 t}{2x} - \frac{3\eta_0 t}{4} \right) u \]
at first. Changing the variable
\[ x \to 1 + 2\varepsilon^2 x, \]
$L_{D_7}$ goes to $L_I$ in the limit $\varepsilon \to 0$.

P3($D_7^{(1)})$-2 → P1: We change the variables
\[
\begin{align*}
t &\to (\varepsilon^{-10} + \varepsilon^{-6} t)/2, \\
y &\to 1 - 2\varepsilon^2 y, \\
z &\to -\varepsilon^{-2} z/2 - \frac{\varepsilon^{-5}}{2\sqrt{-2}}, \\
\theta_0 &\to -1/2, \\
\theta_\infty &\to 1/2 - 3\varepsilon^{-5}/(2\sqrt{-2}), \\
\eta_\infty &\to \sqrt{-2}\varepsilon^5,
\end{align*}
\]
Then
\[ \mathcal{H}_{D_7-2} + \frac{\eta_\infty^2 t}{2} \to 2\varepsilon^6 (\mathcal{H}_I) - 2 + O(\varepsilon^7), \quad (\varepsilon \to 0). \]

For $L_{D_7-2}$, we change
\[ u \to x^{(\theta_0 - 1)^2} \exp \left( \frac{2\theta_\infty x}{3} - \frac{3\eta_0 t}{2} \right) u \]
at first. Changing the variable
\[ x \to 1 - 2\varepsilon^2 x, \]
$L_{D_7-2}$ goes to $L_I$ in the limit $\varepsilon \to 0$.

P34 → P1: We change the variables
\[
\begin{align*}
t &\to -\sigma\varepsilon^2 t + 6\sigma\varepsilon^{-10}, \\
y &\to 2\varepsilon^{-4} y - 2\varepsilon^{-10}, \\
z &\to \varepsilon^4 z/2 + \varepsilon y + \varepsilon^{-5}, \\
\kappa_0 &\to -4\varepsilon^{-15},
\end{align*}
\]
Then
\[ \mathcal{H}_{34} \to -\sigma\varepsilon^{-2}\mathcal{H}_I + 6\sigma\varepsilon^{-20} - \sigma\varepsilon^{-8} t + O(\varepsilon^{-1}), \quad (\varepsilon \to 0). \]

For $L_{34}$, we change
\[ u \to x^{\kappa_0/2} u \]
at first. Changing the variable
\[ x \to 2\varepsilon^{-4} x - 2\varepsilon^{-10}, \]
Then $L_{34}$ goes to $L_I$ in the limit $\varepsilon \to 0$.

P1 → P2: This transformation is algebraic and we do not take any limit. We change the variables
\[
\begin{align*}
t &\to \varepsilon^{-2} t + 6\varepsilon^{-2} \theta^2, \\
y &\to \varepsilon^{-1} y - \varepsilon^{-1} \theta, \\
z &\to \varepsilon z - 2\varepsilon \theta y + 4\varepsilon \theta^2, \\
\eta &\to \varepsilon^3, \\
\beta &\to \varepsilon^5 \theta,
\end{align*}
\]
Then
\[ \mathcal{H}_{1,2} = \varepsilon^2 \left( \mathcal{H}_{II} + \frac{\theta}{2} - \theta^2 t \right) \]
for $\alpha = 4\theta^3$ in P2. For $L_{1,2}$, we change
\[ u \to \exp \left( -\frac{\beta x^2}{\eta} + \frac{2\beta^2 x}{\eta^3} \right) u \]
at first. Changing the variable
\[ x \rightarrow \varepsilon^{-1}x - \varepsilon^{-1}\theta, \]

\( L_{1,2} \) is changed to \( L_I \) for \( \alpha = 4\theta^3 \).

P2→P1: We change the variables
\begin{align*}
t &\rightarrow \varepsilon^2 t - 6\varepsilon^{-10}, \\
y &\rightarrow \varepsilon y + \varepsilon^{-5}, \\
z &\rightarrow \varepsilon^{-1} z + (\varepsilon y + \varepsilon^{-5})^2 + (\varepsilon^2 t - 6\varepsilon^{-10})/2, \\
\alpha &\rightarrow 4\varepsilon^{-15}.
\end{align*}

Then
\begin{align*}
\mathcal{H}_{II} &= \varepsilon^{-2}\mathcal{H}_I - 6\varepsilon^{-20} + \varepsilon^{-8}t - \varepsilon^{-5}/2 + O(\varepsilon), \quad (\varepsilon \to 0).
\end{align*}

For \( L_{II} \), we change
\[ u \rightarrow \exp \left( \frac{x^3}{3} + \frac{tx}{2} \right) u \]
at first. Changing the variable
\[ x \rightarrow \varepsilon x + \varepsilon^{-5}, \]

\( L_{II} \) goes to \( L_I \) in the limit \( \varepsilon \to 0 \).

**Remark.** In [8], there is a misprint in P3→P2.

## 5 Isomonodromic deformations of SL-type

We will list up five types of isomonodromic deformations of SL-type. This part is the revision of the section 4.4 in [8]. The isomonodromic deformation of SL-type is
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= p(x, t)u, \\
\frac{\partial u}{\partial t} &= A(x, t)\frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial A(x, t)}{\partial x} u.
\end{align*}

The compatibility condition is given by
\[ p_t(x, t) = 2p(x, t)A_x(x, t) + A(x, t)p_x(x, t) - \frac{1}{2} A_{xxx}(x, t). \]

In the following list, \( p(x, t) \) contains a Hamiltonian \( K \). The compatibility condition is the Hamiltonian system with the Hamiltonian \( K \). Eliminating \( z \), we obtain the Painlevé equation on \( y \), which is an apparent singularity of the linear equation. We remark that R. Fuchs studied the isomonodromic deformations of SL-type for P6 [2].

**Type (4), (7/2) : P1₂(\( \alpha, \beta \))**

\begin{align*}
p(x, t) &= \alpha x^4 + 4\beta x^3 + \alpha tx^2 + 2\beta tx + 2K_{1,2} + \frac{3}{4(x-y)^2} - \frac{z}{x-y}, \\
A(x, t) &= \frac{1}{2(x-y)}, \\
K_{1,2} &= \frac{z^2}{2} - \beta(2y^3 + ty) - \frac{\alpha}{2}(y^4 + ty^2).
\end{align*}
Type (1)(3), (1)(5/2) : $P_{4,34}'(\alpha, \beta, \gamma)$ for $\sigma = 1$, $P_{4,34}(\alpha, \beta, \gamma)$ for $\sigma = -1$.

$$p(x,t) = \frac{\gamma}{2}(x^2 + 2\sigma tx + t^2) + \frac{\beta}{2}(x + \sigma t) + \frac{\alpha - 1}{4x^2} + \frac{\sigma K_{4,34}}{x} + \frac{3}{4(x - y)^2} - \frac{yz}{x(x - y)},$$

$$A(x,t) = \frac{\sigma x}{x - y},$$

$$K_{4,34} = \sigma yz^2 - \sigma z + \frac{\sigma(1 - \alpha)}{4y} - \frac{\beta}{2}y(\sigma y + t) - \frac{\sigma \gamma}{2}y(y + \sigma t)^2.$$

Type (2)$^2$, (2)(3/2), (3/2)$^2$ : $P_{3}'(\alpha, \beta, \gamma, \delta)$

$$p(x,t) = \frac{a_0 t^2}{x^4} + \frac{a'_0 t}{x^3} + \frac{tk_{III}'}{x^2} + \frac{a''_0}{x} + a_\infty + \frac{3}{4(x - y)^2} - \frac{yz}{x(x - y)},$$

$$A(x) = \frac{yx}{t(x - y)},$$

$$tk_{III}' = y^2 z^2 - yz - \frac{a_0 t^2}{y^2} - \frac{a'_0 t}{y} - a''_0 y - a_\infty y^2,$$

$$a_0 = -\frac{\delta}{16}, \quad a'_0 = -\frac{\beta}{8}, \quad a_\infty = \frac{\gamma}{16}, \quad a''_0 = \frac{\alpha}{8}.$$

Type (1)$^2$(2), (1)$^2$(3/2) : $P_{5}(\alpha, \beta, \gamma, \delta)$

$$p(x,t) = \frac{a_1 t^2}{(x - 1)^4} + \frac{K_V t}{(x - 1)^2 x} + \frac{a_2 t^3}{(x - 1)^3} - \frac{z(y - 1)y}{x(x - 1)(x - y)} + \frac{a_\infty}{(x - 1)^2} + \frac{a_0}{x^2} + \frac{3}{4(x - y)^2},$$

$$A(x) = \frac{y - 1}{t} \cdot \frac{x(x - 1)}{x - y},$$

$$tk_{V} = y(y - 1)^2 \left[ -\frac{a_1 t^2}{(y - 1)^4} - \frac{a_2 t^3}{(y - 1)^3} + z^2 - \left( \frac{1}{y} + \frac{1}{y - 1} \right) z - \frac{a_\infty}{(y - 1)^2} - \frac{a_0}{y^2} \right],$$

$$a_0 = -\frac{\beta}{2} - \frac{1}{4}, \quad a_1 = -\frac{\delta}{2}, \quad a_2 = -\frac{\gamma}{2}, \quad a_\infty = \frac{1}{2}(\alpha + \beta) - \frac{3}{4}.$$ 

Type (1)$^4$ : $P_{6}(\alpha, \beta, \gamma, \delta)$

$$p(x,t) = \frac{a_0}{x^2} + \frac{a_1}{(x - 1)^2} + \frac{a_\infty}{x(x - 1)} + \frac{b_1}{(x - t)^2} + \frac{3}{4(x - y)^2} + \frac{t(t - 1)K_{VI}}{x(x - 1)(x - t)} - \frac{y(y - 1)z}{x(x - 1)(x - y)},$$

$$A(x) = \frac{y - t}{t(t - 1)} \cdot \frac{x(x - 1)}{x - y},$$

$$K_{VI} = \frac{y(y - 1)(y - t)}{t(t - 1)} \left[ z^2 - \left( \frac{1}{y} + \frac{1}{y - 1} \right) z - \frac{a_0}{y^2} - \frac{a_1}{(y - 1)^2} - \frac{a_\infty}{y(y - 1)} - \frac{b_1}{(y - t)^2} \right],$$

$$a_0 = -\frac{\beta}{2} - \frac{1}{4}, \quad a_1 = \frac{\gamma}{2} - \frac{1}{4}, \quad b_1 = -\frac{1}{2} \delta, \quad a_\infty = \frac{1}{2}(\alpha + \beta - \gamma + \delta - 1).$$

If we set $\alpha = 0$ in $P_{1,2}$, we obtain the standard isomonodromic deformations of $SL$-type for $P_1$. If we set $\gamma = 0$ in $P_{4,34}$, we obtain the $SL$-type for $P_{34}$. If we set $\gamma = 0$ in $P_{3}'$, we obtain the $SL$-type for $P_{3}'(D_{7}^{(1)})$. If we set $\gamma = 0$, $\delta = 0$ in $P_{3}'$, we obtain the $SL$-type for $P_{3}'(D_{5}^{(1)})$. If we set $\delta = 0$ in $P_{5}$, we obtain the $SL$-type for deg-$P_{5}$. 
We show the standard isomonodromic deformations of \( SL \)-type for P2 and P4.

P2(\( \alpha \)):

\[
p(x, t) = x^4 + tx^2 + 2\alpha x + 2K_{II} + \frac{3}{4(x-y)^2} = \frac{z}{x-y},
\]

\[
A(x, t) = \frac{1}{2} \cdot \frac{1}{x-y},
\]

\[
K_{II} = \frac{1}{2} z^2 - \frac{1}{2} y^4 - \frac{1}{2} ty^2 - \alpha y.
\]

P4(\( \alpha, \beta \)):

\[
p(x, t) = a_0 \frac{x^2}{x^2} + \frac{K_{IV}}{2x} + a_1 + \left( \frac{x + 2t}{4} \right)^2 + \frac{3}{4(x-y)^2} - \frac{yz}{x(x-y)},
\]

\[
A(x) = \frac{2x}{x-y},
\]

\[
K_{IV} = 2yz^2 - 2z - \frac{2a_0 y}{y} - 2a_1 y - 2y \left( \frac{y + 2t}{4} \right)^2,
\]

\[
a_0 = -\frac{\beta}{8} - \frac{1}{4}, \quad a_1 = -\frac{\alpha}{4}.
\]

Remark. In [8], there is a misprint in \( K_{IV} \).



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