Topological Kirchhoff Law
and Bulk-Edge Correspondence for Valley-Chern and Spin-Valley-Chern Numbers

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The valley-Chern and spin-valley-Chern numbers are the key concepts in valleytronics. They are topological numbers in the Dirac theory but not in the tight-binding model. We analyze the bulk-edge correspondence between the two phases which have the same Chern and spin-Chern numbers but different valley-Chern and spin-valley-Chern numbers. The edge state between them is topologically trivial in the tight-binding model but is shown to be as robust as the topological edge. We construct Y-junctions made of topological edges. They satisfy the topological Kirchhoff law, where the topological charges are conserved at the junction. We may interpret a Y-junction as a scattering process of particles which have four topological numbers. It would be a milestone of future topological electronics.

Topological insulator is one of the most fascinating concepts found in this decade. It is characterized by topological numbers such as the Chern ($C$) number and the $Z_2$ index. When the spin $s_z$ is a good quantum number, the spin-Chern ($C_s$) number replaces the role of the $Z_2$ index. We consider honeycomb lattice systems. Electrons resides either in the $K$ or $K'$ valley in the low-energy Dirac theory. Accordingly we can define the valley-Chern ($C_v$) number[18,20] and the spin-valley-Chern ($C_{sv}$) number[18] in the Dirac theory. This valley degree of freedom leads to valleytronics.[21-27]. However, the $C_v$ and $C_{sv}$ numbers are ill-defined in the tight-binding model because the topological numbers are defined by the summation of Berry curvatures over the entire Brillouin zone. Namely, a state is indexed by the two topological numbers in the tight-binding model, while it is indexed by the four topological numbers in the Dirac theory.

The are four independent spin-valley dependent Chern numbers in the Dirac theory of honeycomb systems. Each Chern number can be controlled independently by changing the sign of spin-valley dependent Dirac masses. There are 16 types of topological insulators, as shown in the table. They are quantum anomalous Hall (QAH) insulator, four types of spin-polarized QAH (SQAΗ) insulators, quantum spin Hall (QSH) insulator and the band insulator with charge-density-wave (CDW) or antiferromagnetic (AF) order. The CDW and AF insulators are regarded trivial in the tight-binding model.

In this paper, we study the bulk-edge correspondence with respect to the $C_v$ and $C_{sv}$ numbers by examining the boundary of two insulators which have the same $C$ and $C_v$ numbers but different $C_s$ and $C_{sv}$ numbers. First we show that gapless edge states appear though they are trivial in the tight-binding model. Furthermore, we show that they are as robust as the topologically protected edges.

We propose a topological electronics based on the edge states in the Dirac theory. We are able to assign four topological numbers to each edge states. By joining three different topological insulators at one point, we can construct a Y-junction made of topological edge states. The edge states at the junction satisfies the conservation of four topological numbers, which we call the topological Kirchhoff law. We can change the connectivity of edge states by changing the topological property of bulk insulators, for instance, by applying electric field. The process may be interpreted as a pair annihilation of two Y-junctions.

Hamiltonian: The honeycomb lattice consists of two sublattices made of $A$ and $B$ sites. We consider a buckled system with the layer separation $2\ell$ between these two sublattices. The states near the Fermi energy are $\pi$ orbitals residing near the $K$ and $K'$ points at opposite corners of the hexagonal Brillouin zone. The low-energy dynamics in the $K$ and $K'$ valleys is described by the Dirac theory. In what follows we use notations $s_z=\uparrow, \downarrow$, $t_z= A, B$, $\eta=K,K'$ in indices while $s_z' = \pm 1$ for $\alpha =\uparrow, \downarrow$, $t_z' = \pm 1$ for $i= A, B$, and $\eta_i = \pm 1$ for $i= K, K'$ in equations. We also use the Pauli matrices $\sigma_\alpha$ and $\tau_i$ for the spin and the sublattice pseudospin, respectively.

We have previously proposed a generic Hamiltonian for honeycomb systems, which contains eight interaction terms mutually commutative in the Dirac limit. Among them four contribute to the Dirac mass. The other four contribute to the shift of the energy spectrum. We are able to make a full control of the Dirac mass and the energy shift independently at each spin and valley by varying these parameters, and materialize various topological phases. [22,23]
\[ H = -t \sum_{\langle i,j \rangle, \alpha} c_{i,\alpha}^\dagger c_{j,\alpha} + i \frac{\lambda_{SO}}{\sqrt{3}} \sum_{\langle i,j \rangle, \alpha, \beta} \nu_{ij} c_{i,\alpha}^\dagger \sigma_{\alpha,\beta} c_{j,\beta} \\
- \lambda_V \sum_{i, \alpha} c_{i,\alpha}^\dagger c_{i,\alpha} + i \frac{\lambda_\Omega}{\sqrt{3}} \sum_{\langle i,j \rangle, \alpha} \nu_{ij} c_{j,\alpha} c_{i,\beta} \\
+ \lambda_{SX} \sum_{i, \alpha} c_{i,\alpha}^\dagger \sigma_{\alpha,\beta} c_{i,\beta}, \quad (1) \]

where \( c_{i,\alpha}^\dagger \) creates an electron with spin polarization \( \alpha \) at site \( i \), and \( \langle i, j \rangle / \langle j, i \rangle \) run over all the nearest/next-nearest neighbor hopping sites. We explain each term. The first term represents the nearest-neighbor hopping with the transfer energy \( t \). The second term represents the SO coupling with \( \lambda_{SO} \). The third term is the staggered sublattice potential term with \( \lambda_V \) and \( \lambda_\Omega \) defined by the photo-irradiation, where \( \lambda_{SO} = h v_F^2 A^2 \Omega^{-1} \) with \( \Omega \) the frequency and \( A \) the dimensionless intensity. The antiferromagnetic exchange magnetization might be induced by crossing proximity effects. The second candidate is proskrite transition-metal oxide grown on \([111]\)-direction, which has antiferromagnetic order incoherently. This material has also the buckled structure as in the case of silicene. Parameters are \( t \approx 0.2eV \), \( \lambda_{SO} = 3.9meV \), \( \lambda_V = 0.23A^2 \). The Haldane term is induced by the photoirradiation, where \( \lambda_{SO} = 3.9meV \), \( \lambda_V = 0.23A \). The forth term is the Haldane term with \( \lambda_\Omega \). The fifth term represents the antiferromagnetic exchange magnetization with \( \lambda_{SX} \).

We give typical sample parameters though we treat them as free parameters. Silicene is a good candidate, where \( t = 1.6eV, \lambda_{SO} = 3.9meV \) and \( \ell = 0.23A \). The Haldane term might be induced by the photo-irradiation, where \( \lambda_{SO} = h v_F^2 A^2 \Omega^{-1} \) with \( \Omega \) the frequency and \( A \) the dimensionless intensity. The antiferromagnetic exchange magnetization might be induced by crossing proximity effects. The second candidate is proskrite transition-metal oxide grown on \([111]\)-direction, which has antiferromagnetic order incoherently.

The low-energy Hamiltonian is described by \( H_\eta = h v_F (\eta k_x \tau_x + k_y \tau_y) + \lambda_{SO} \tau_z n_{\tau_z} \\
- \lambda_V \tau_z + \lambda_\Omega n_{\tau_z}, \quad (2) \]

where \( v_F = \sqrt{2m_0 t} \) is the Fermi velocity. The coefficient of \( \tau_z \) is the mass of Dirac fermions in the Hamiltonian,

\[ \Delta_{s_z} = \eta s_z \lambda_{SO} - \lambda_V + \eta \lambda_\Omega + s_z \lambda_{SX}. \quad (3) \]

The band gap is given by \( 2|\Delta_{s_z}| \).

**Topological numbers:** We consider the systems where the spin \( s_z \) is a good quantum number. The summation of the Berry curvature over all occupied states of electrons with spin \( s_z \) in the Dirac valley \( K_\eta \) yields \( C_n \).

\[ C_n = \frac{\eta}{2} \text{sgn}(\Delta_{s_z}). \quad (4) \]

There are four independent spin-valley dependent Dirac masses determined by the four parameters \( \lambda_{SO}, \lambda_V, \lambda_\Omega \) and \( \lambda_{SX} \). Accordingly, we can define

\[ C_v = C^K - C^K - C^K + C^K, \quad (5) \]

\[ C_{sv} = \frac{1}{2} (C^K - C^K - C^K + C^K). \quad (6) \]

It is to be emphasized that \( C_v \) and \( C_{sv} \) are not defined in the tight-binding model.

The possible sets of topological numbers are \((0, 0), (2, 0), (0, 1), (1, \frac{1}{2})\) up to the overall sign \( \pm \) in the tight-binding model. They are the trivial, QAH, QSH and two types of SQAH insulators, respectively. They are further classified into subsets according to the valley degree of freedom in the Dirac theory. Trivial insulators are divided into two; one with CDW order and the other with AF orders. Each type of SQAH insulators are divided into two: There are four types in all, which we denote by SQAH, with \( i = 1, 2, 3, 4 \). All of them are summarized in Table I.

**Bulk-edge correspondence:** The most convenient way to determine the topological charges is to employ the bulk-edge correspondence. When there are two topological distinct phases, a topological phase transition may occur between them. It is generally accepted that the band gap must close at the topological phase transition point since the topological number cannot change its quantized value without gap closing.

The answer to this problem is that the band gap may close at the topological phase transition point since the topological number cannot change its quantized value without gap closing. Note that the topological number is only defined in the gapped system and remains unchanged for any adiabatic process. Alternatively, we may consider a junction separating two different topological phases in a single honeycomb system. Gapless edge modes must appear along the boundary. We may as well analyze the energy spectrum of a nanoribbon in a topological phase, because the boundary of the nanoribbon separates a topological state and the vacuum whose topological numbers are zero: See Fig (a).

**CDW-AF junction:** We first investigate the trivial insulator in the tight-binding model, which consists of two subsets (CDW and AF) in the Dirac theory. It is well known that a nanoribbon made of either the CDW insulator or the AF insulator has no gapless edge modes, as is regarded to be a demonstration of their triviality: See Fig (b) and (c).

One may wonder how they can be topological in the Dirac theory without gapless edge modes in view of the bulk-edge correspondence. The answer to this problem is that the \( C_v \) and \( C_{sv} \) numbers are not defined in the vacuum. Indeed, only the charge and the spin are well defined to be zero in the Dirac theory. We ask how these two properties are compatible.

We investigate the junction made of the CDW and AF insulators, whose topological numbers are \((C_v, C_{sv}) = (0, 0, 2, 0)\) and \((0, 0, 0, 2)\), respectively. On one hand, we expect no gapless edge modes in the tight-binding model. On the other hand, there should be gapless edge modes in the CDW phase. We ask how these two properties are compatible.

To answer this problem we study a hybrid nanoribbon by separating a nanoribbon into two parts, one in the CDW phase and the other in the AF phase: See Fig (d). Only the \( C_v \) and
On the other hand, when we concentrate on the vicinity of the
manifestation of the fact that the
are shown in magenta (cyan). We have taken
of (e) CDW-AF, and (f) AF-AF
regions. We have calculated the band structure of such a hy-
and the other
and SQAH-SQAH junction: We next investigate the junction
in SQAH phase. We calculate the band structure of a nanotube geometry since no gapless edge states appear for a simple nanoribbon in the SQAH phase. We calculate the band structure of a nanotube geometry since no gapless edge states appear even for a simple nanotube in the SQAH phase owing to the lack of the edge itself. We take a hybrid nanotube where one half of the nanotube is SQAH and the other half is SQAH2, as illustrated in Fig. 2(a). We show the result in Fig. 2(b), where we see clearly a gapless edge mode highly enhanced at the Γ point. The extremal energy $\Delta_{\Gamma}$ of the gapless edge mode is given by (9), while the bulk gap is given by (10) also in this case. The edge is very robust.

Gapless edge mode in Dirac theory: We proceed to construct the Dirac theory of the gapless edge states [20]. They emerge along a curve where the Dirac mass vanishes, $\Delta_n^\eta (x, y) = 0$. Let us take the edge along the $x$ axis. The zero modes emerge along the line determined by $\Delta_n^\eta (y) = 0$, when $\Delta_n^\eta (y)$ changes the sign. We may set $k_x = \xi = \pm 1$. Here, $\psi_A$ is a two-component amplitude with the up spin and down spin, $\psi_A = (\psi_A^\uparrow, \psi_A^\downarrow)$. Setting $\psi_A (x, y) = e^{i k_x x} \phi_A (y)$, we obtain $H_\eta \psi_A (x, y) = E_\eta \psi_A (x, y)$, together with a linear dispersion relation $E_{\eta \xi} = \eta \xi \hbar v_F k_x$. We can explicitly solve this as

$$\phi_A^\eta (y) = C \exp \left[ \frac{\xi}{\hbar v_F} \int_y^\eta \Delta_n^\eta (y') dy' \right].$$

where $C$ is the normalization constant. The sign $\xi$ is determined so as to make the wave function finite in the limit $|y| \to ...
The number of topological edge states is determined by the process of these particles. In this scattering process, the topological charges is zero.

Topological Kirchhoff law: We consider a configuration where three different topological insulators meet at one point: See Fig.3. In this configuration there are three edges forming a Y-junction. It is convenient to assign the topological numbers to each edge which are the difference between those of the two adjacent topological insulators. Namely, when the topological insulator with $(C^L, C^R, C^L, C^R)$ is on the left-hand side of the one with $(C^R, C^L, C^R, C^L)$, we assign the numbers $[c^L - c^R, C^L, C^L - c^R, C^L]$ to the boundary, as illustrated in Fig.4. The condition which edges can make a Y-junction is the conservation of these topological numbers at the junction. This law is a reminiscence of the Kirchhoff law, which dictates the conservation of currents at the junction of electronic circuits. We call it the topological Kirchhoff law.

The number of Y-junctions is given by the combination of selecting 3 from 16 topological insulators, i.e., $16C_3 = 560$. The number of topological edge states is determined by the combination of selecting 2 from 16 topological insulators. We have $16C_2 = 120$ types of topological edge states. We show typical examples of Y-junctions in Fig.3.

We present an interesting interpretation of the topological Kirchhoff law. We may regard each topological edge state as a world line of a particle carrying the four topological charges. The Y-junction may be interpreted as a scattering process of these particles. In this scattering process, the topological charges conserve.

Topological electronic circuits: We can construct electronic circuits made of edge states by joining Y-junctions. Each topological edge state carries conductance $\frac{e^2}{h}$, whose magnitude is given by the Chern number $c$ in unit of $e^2/\hbar$. In general, the edge states carry charge $c$, spin $s$, valley-charge $v$, and spin-valley-charge $sv$. The present-day electronic circuits only use the charge degree of freedom. In our circuits of topological edges we can make a full use of four types of charges. This would greatly enhance the ability of information processing.

We can control the position of edge state by controlling the parameters of the bulk states. The easiest way is to apply electric field $E_z$ locally. Let us review the topological phase transition taking place as $E_z$ changes by taking $\lambda_1 = \lambda_2 = 0$, where the Dirac mass is given by $\Delta_0 = \eta_s \lambda_0 - E_z$. The condition $\Delta_0 = 0$ implies $E_z = \pm E_{cr}$ with $E_{cr} = \lambda_0/\ell$. It follows that $(c, s) = (0, 0)$ for $|E_z| < E_{cr}$ and $(0, \frac{1}{2})$ for $|E_z| > E_{cr}$. For instance, the two CDW domains are made in this way in Fig.4(a). Applying $E_z$ only to a part in the QSH domain near the SQAH domain, we can turn this part into the CDW domain as in Fig.4(b).

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