AN UNDECIDABILITY RESULT FOR THE ASYMPTOTIC
THEORY OF $p$-ADIC FIELDS

KONSTANTINOS KARTAS

ABSTRACT. Fix a prime $p$. We prove that the set of sentences true in all but finitely many finite extensions of $\mathbb{Q}_p$ is undecidable in the language of valued fields with a cross-section. The proof goes via reduction to positive characteristic, ultimately adapting Pheidas’ proof of the undecidability of $\mathbb{F}_p((t))$ with a cross-section. This answers a variant of a question of Derakhshan-Macintyre.

CONTENTS

Introduction 1
1. Preliminaries 3
2. Local field approximation 6
3. Truncations of $(\mathbb{N}; 0, 1, +, |_p)$ 8
4. Proof of the main Theorem 12
5. Final remarks 13
Acknowledgements 14
References 14

INTRODUCTION

Let $L_{\text{val},x}$ be the language of valued fields with a cross-section (see §1.3.3). Ax-Kochen [AK65] and independently Ershov [Ers65] showed that $\mathbb{Q}_p$, equipped with the normalized cross section $s : n \mapsto p^n$, is decidable in $L_{\text{val},x}$. Every finite extension of $\mathbb{Q}_p$ is also decidable in $L_{\text{val},x}$, for a suitable choice of a cross-section (see §2.4 [Kar22]). Combining the results of [AK65] with the theory of pseudofinite fields, Ax showed in Theorem 17 [Ax68] that the (asymptotic) theory of $\{\mathbb{Q}_p : p \in \mathbb{P}\}$ is decidable in $L_{\text{val},x}$ and also that the (asymptotic) theory of $\{\mathbb{Q}_p(\zeta_n) : p \nmid n\}$, namely the collection of all finite unramified extensions of $\mathbb{Q}_p$, is decidable in $L_{\text{val},x}$. Recall that the asymptotic theory of a class $\mathcal{C}$ of $L$-structures is defined to be the set of sentences $\phi \in L$ which are true in all but finitely many $M \in \mathcal{C}$.

During this research, the author was funded by EPSRC grant EP/20998761 and was also supported by the Onassis Foundation - Scholarship ID: FZP 020-1/2019-2020.
In sharp contrast, we have that $\mathbb{F}_p((t))$ is undecidable in $L_{\text{val},x}$. This was already known to Ax (unpublished) and an elementary proof was given later by Becker-Denef-Lipshitz \cite{BDL80}, which was also reworked by Cherlin in §4 \cite{Che82}. Pheidas \cite{Phe87} generalized the result for $k((t))$, where $k$ is an arbitrary field of characteristic $p$, and showed that already the existential $L_{\text{val},x}$-theory is undecidable. The decidability problem for $\mathbb{F}_p((t))$ in the language of rings $L_{\text{rings}}$ is still an open problem.

An important related open question in mixed characteristic is whether $\text{Th}([K : [K : \mathbb{Q}_p] < \infty])$, i.e. the theory of all finite $p$-adic extensions, is decidable. This question was first raised (in print) by Derakhshan-Macintyre in §9 \cite{DM22} for the language of rings $L_{\text{rings}}$. In Theorem 9.1 \cite{DM22}, they showed that the theory of adele rings of all number fields in $L_{\text{rings}}$ is decidable if and only if $\text{Th}([K : [K : \mathbb{Q}_p] < \infty])$ is decidable in $L_{\text{rings}}$, for each prime $p$. It is also natural to consider $\text{Th}([K : [K : \mathbb{Q}_p] < \infty])$ in any of the standard expansions $L$ of $L_{\text{rings}}$ (see e.g., pg. 20-22 \cite{Der20}), and ask whether it is decidable (Problem 6.2 \cite{Der20}).

In the present paper, we resolve negatively the asymptotic version of this problem in the presence of a cross-section. While each individual finite $p$-adic extension is decidable in $L_{\text{val},x}$, the asymptotic theory of all of them is less well-behaved:

**Theorem A.** The asymptotic $L_{\text{val},x}$-theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable.

We note that for each $K$ with $[K : \mathbb{Q}_p] < \infty$ there are many choices of a cross-section and any such choice will lead to an undecidability result (see Convention 2.2.6). By carefully keeping track of quantifiers, we will prove in §1.3 that already the asymptotic $\exists\forall$-theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable in $L_{\text{val},x}$.

The key idea of the proof will be to use highly ramified $p$-adic fields to approximate $\mathbb{F}_p((t))$ à la Krasner-Kazhdan-Deligne (see Section 2) and then adapt Pheidas’ proof of the undecidability of $\mathbb{F}_p((t))$ in $L_{\text{val},x}$ (see \cite{Phe87}). In more detail, the proof of Theorem A consists of the following steps:

1. Encode the asymptotic theory of totally ramified $p$-adic fields, i.e., the asymptotic theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ (see the proof of Theorem A).

2. Observe that $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^e)$, for $K/\mathbb{Q}_p$ totally ramified of degree $e$, and thereby encode the asymptotic theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ in $L_t$ with a predicate for powers of $t$ (see Corollary 2.2.7).

3. Encode the asymptotic theory of truncated fragments of $(\mathbb{N}; 0, 1, +, |_p)$ (see §3.2.5), where $m \mid_p n$ if and only if $n = p^s \cdot m$ for some $s \in \mathbb{N}$.

4. Show that the latter is undecidable by encoding the Diophantine problem of $(\mathbb{N}; 0, 1, +, |_p)$ (see §3.3).

We note that Derakhshan-Macintyre had already suggested that the difficulty in understanding the (asymptotic) theory of $p$-adic fields must lie in unbounded
ramification (see the last paragraph of [DM22]). This is precisely the ingredient that makes the transition to positive characteristic work and is the reason why this asymptotic class exhibits a different behavior from the two asymptotic theories mentioned in the first paragraph of the introduction.

1. Preliminaries

1.1. Interpretability. Our formalism follows closely §5.3 [Hod93], where details and proofs may be found.

1.1.1. Interpretations of structures. Given a language \( L \), an unnested atomic \( L \)-formula is one of the form \( x = y \) or \( x = c \) or \( F(x) = y \) or \( R(x) \), where \( x, y \) are variables, \( c \) is a constant symbol, \( F \) is a function symbol and \( R \) is a relation symbol of the language \( L \).

Definition 1.1.2. An \( n \)-dimensional interpretation of an \( L \)-structure \( M \) in the \( L' \)-structure \( N \) is a triple consisting of:

1. An \( L' \)-formula \( \partial \Gamma(x_1, ..., x_n) \).
2. A map \( \phi \mapsto \phi \Gamma \), that takes an unnested atomic \( L \)-formula \( \phi(x_1, ..., x_m) \) and sends it to an \( L' \)-formula \( \phi \Gamma(y_1, ..., y_m) \), where each \( y_i \) is an \( n \)-tuple of variables.
3. A surjective map \( f \Gamma : \partial \Gamma(N^n) \rightarrow M \) such that for all unnested atomic \( L \)-formulas \( \phi(x_1, ..., x_m) \) and all \( \overline{a}_i \in \partial \Gamma(N^n) \), we have

\[
M \models \phi(f \Gamma(\overline{a}_1), ..., f \Gamma(\overline{a}_m)) \iff N \models \phi \Gamma(\overline{a}_1, ..., \overline{a}_m)
\]

An interpretation of an \( L \)-structure \( M \) in the \( L' \)-structure \( N \) is an \( n \)-dimensional interpretation, for some \( n \in \mathbb{N} \). In that case, we also say that \( M \) is interpretable in \( N \). The formulas \( \partial \Gamma \) and \( \phi \Gamma \) (for all unnested atomic \( \phi \)) are the defining formulas of \( \Gamma \).

Proposition 1.1.3 (Reduction Theorem 5.3.2 [Hod93]). Let \( \Gamma \) be an \( n \)-dimensional interpretation of an \( L \)-structure \( M \) in the \( L' \)-structure \( N \). There exists a map \( \phi \mapsto \phi \Gamma \), extending the map of Definition 1.1.2(2), such that for every \( L \)-formula \( \phi(x_1, ..., x_m) \) and all \( \overline{a}_i \in \partial \Gamma(N^n) \), we have that

\[
M \models \phi(f \Gamma(\overline{a}_1), ..., f \Gamma(\overline{a}_m)) \iff N \models \phi \Gamma(\overline{a}_1, ..., \overline{a}_m)
\]

Proof. We describe how \( \phi \mapsto \phi \Gamma \) is built, for completeness (omitting details). By Corollary 2.6.2 [Hod93], every \( L \)-formula is equivalent to one in which all atomic subformulas are unnested. One can then construct \( \phi \mapsto \phi \Gamma \) by induction on the complexity of formulas. The base case is handled by Definition 1.1.2(2). This definition extends inductively according to the following rules:

1. \( (\neg \phi) \Gamma = \neg(\phi) \Gamma \).
2. \( (\bigwedge_{i=1}^n \phi_i) \Gamma = \bigwedge (\phi_i) \Gamma \).
(3) $(\forall \phi)_\Gamma = \forall x_1, ..., x_n (\partial_T (x_1, ..., x_n) \to \phi_T)$
(4) $(\exists \phi)_\Gamma = \exists x_1, ..., x_n (\partial_T (x_1, ..., x_n) \land \phi_T)$

The resulting map satisfies the desired conditions of the Proposition. \hfill \Box

**Definition 1.1.4.** The map $\text{Form}_L \to \text{Form}_{L'} : \phi \mapsto \phi_\Gamma$ constructed in the proof of Proposition 1.1.3 is called the reduction map of the interpretation $\Gamma$.

1.1.5. *Complexity of interpretations.* The complexity of the defining formulas of an interpretation defines a measure of complexity of the interpretation itself:

**Definition 1.1.6** (§5.4 (a) [Hod93]). An interpretation $\Gamma$ of an $L$-structure $M$ in an $L'$-structure $N$ is quantifier-free if the defining formulas of $\Gamma$ are quantifier-free. Other syntactic variants are defined analogously (e.g., existential interpretation).

**Remark 1.1.7.** Note that the reduction map of an existential interpretation sends positive existential formulas to existential formulas but does not necessarily send existential formulas to existential formulas.

1.1.8. *Recursive interpretations.*

**Definition 1.1.9** (Remark 4, pg. 215 [Hod93]). Suppose $L$ is a recursive language. Let $\Gamma$ be an interpretation of an $L$-structure $M$ in the $L'$-structure $N$. We say that the interpretation $\Gamma$ is recursive if the the map $\phi \mapsto \phi_\Gamma$ on unnested atomic formulas is recursive.

**Remark 1.1.10** (Remark 4, pg. 215 [Hod93]). If $\Gamma$ is a recursive interpretation of an $L$-structure $M$ in the $L'$-structure $N$, then the reduction map of $\Gamma$ is also recursive.

1.2. *Decidability.*

1.2.1. **Definition.** Fix a countable language $L$ and let $\text{Sent}_L$ be the set of well-formed $L$-sentences, identified with $\mathbb{N}$ via some Gödel numbering. Let $T$ be an $L$-theory, not necessarily complete. Recall that $T$ is decidable if we have an algorithm to decide whether $T \models \phi$, for any given $\phi \in \text{Sent}_L$. More formally, let $\chi_T : \text{Sent}_L \to \{0, 1\}$ be the partial characteristic function of $T \subseteq \text{Sent}_L$. We say that $T$ is decidable if $\chi_T$ is a partial recursive function. Let $M$ be an $L$-structure. We say that $M$ is decidable if $\text{Th}(M)$ is decidable.

1.2.2. **Turing reducibility of theories.** See Definition 14.3 [Pap94] for the formal definition of a Turing machine with an oracle.

**Definition 1.2.3.** A theory $T$ is Turing reducible to a theory $T'$ if there is a Turing machine which decides membership in $T$ using an oracle for $T'$.

**Remark 1.2.4.** In particular, if $T$ is Turing reducible to $T'$ and $T'$ is decidable, then so is $T$. 

Example 1.2.5. If $\Gamma$ is a recursive interpretation of an $L$-structure $M$ in the $L'$-structure $N$, then $\text{Th}(M)$ is Turing reducible to $\text{Th}(N)$. Indeed, for any given $\phi \in \text{Sent}_L$, we have that $M \models \phi \iff N \models \phi_\Gamma$, where $\phi \mapsto \phi_\Gamma$ is the reduction map of $\Gamma$. This furnish us with an algorithm to decide whether $M \models \phi$ using an oracle for $\text{Th}(N)$.

1.3. Languages. We write $L_{\text{oag}} = \{0, +, <\}$ for the language of ordered abelian groups and $L_{\text{rings}} = \{0, 1, +, \cdot\}$ for the language of rings.

1.3.1. Valued fields language. Let $L_{\text{val}}$ be the three-sorted language of valued fields, with sorts for the field, the value group and the residue field.

- The field sort $K$ is equipped with the language of rings $L_{\text{rings}}$.
- The value group sort $\Gamma$ is equipped with $L_{\text{oag}}$, together with a constant symbol for $\infty$.
- The residue field sort $k$ is equipped the language of rings $L_{\text{rings}}$.

We also have function symbols for the valuation map $v : K \to \Gamma$ and a residue map $\text{res} : K \to k$ (where $\text{res}(x) = 0$ when $vx < 0$ by convention).

Convention 1.3.2. We shall write $x \in O$ as an abbreviation of the formula $x \in K \land vx \geq 0$.

1.3.3. Ax-Kochen/Ershov language. Historically, the Ax-Kochen/Ershov formalism also included a function symbol for a cross-section, i.e. group homomorphism $s : \Gamma \to K^\times$ satisfying $v \circ s = \text{id}_\Gamma$ (see [AK65] and [Koc74]). One can also extend $s$ by defining $s(\infty) = 0$. We write $L_{\text{val}, \times}$ for the language $L_{\text{val}}$ enriched with such a cross-section symbol $s : \Gamma \to K$.

1.4. Ax-Kochen/Ershov Theorem. Among other results, Ax-Kochen [AK65] and independently Ershov [Ers65] obtained the following:

Fact 1.4.1 (Ax-Kochen/Ershov). The field $\mathbb{Q}_p$, equipped with the normalized cross-section $s : n \mapsto p^n$, is decidable in $L_{\text{val}, \times}$.

Remark 1.4.2. (a) More generally, by Theorem 4 [Koc74], any unramified henselian field $(K, v)$, equipped with a normalized cross-section (viz., $s(1) = p$) and with perfect residue field $k$, is decidable in $L_{\text{val}, \times}$.

(b) Finite extensions of $\mathbb{Q}_p$ are also decidable in $L_{\text{val}, \times}$, for a suitable choice of a cross-section (see §2.4 [Kar22]).

Theorem 12 [AK66] in fact shows that $\mathbb{Q}_p$ admits quantifier elimination in $L_{\text{val}, \times}$ relative to the value group. The definable sets of the latter are perfectly understood by classical quantifier elimination results for Presburger arithmetic. Nevertheless, the following is worth noting:

Remark 1.4.3. The definable sets in the Ax-Kochen/Ershov language are complicated and are generally not definable in the valued field language. For instance,
the image of the cross-section is not definable without the cross-section (see Example, pg. 609 [Mac76]). For this reason, at least for the purpose of studying definable subsets of the $p$-adics, the Ax-Kochen/Ershov formalism was superseded by Macintyre’s language (see pg. 606 [Mac76]).

Despite Remark 1.4.3, the Ax-Kochen/Ershov formalism has remained relevant. The fact that $p$-adic fields are decidable in such an expressive language is a strong result. As was mentioned in the introduction, this is in stark contrast with the fact that positive characteristic local fields are undecidable in $L_{\text{val}, x}$.

2. Local field approximation

2.1. Motivation. A powerful philosophy, often referred to as the Krasner-Kazhdan-Deligne principle (due to [Kra56], [Kaz86] and [Del84]), says that a highly ramified $p$-adic field $K$ (e.g., $K = \mathbb{Q}_p(p^{1/n})$ with $n$ large) is in many respects "close" to a positive characteristic valued field. Although this is reflected in many aspects of $K$, perhaps the most elementary one is that the residue ring $\mathcal{O}_K/(\pi)$ "approximates" a positive characteristic valuation ring (see Lemma 2.2.1, Remark 2.2.2). Our goal in this section will be to prove Corollary 2.2.7 which will serve as a bridge between Phiedas’ work in positive characteristic and our problem in mixed characteristic (see §3.1).

2.2. Computations modulo $p$. Let $K/\mathbb{Q}_p$ be a finite totally ramified extension of degree $n$, with value group $\Gamma$ and residue field $k$. For completeness, we record here the following computation:

Lemma 2.2.1. For each uniformizer $\pi$ of $\mathcal{O}_K$ we have an isomorphism $\mathcal{O}_K/(\pi) \cong \mathbb{F}_p[t]/(t^n)$, which maps the image of $\pi$ in $\mathcal{O}_K/(\pi)$ to the image of $t$ in $\mathbb{F}_p[t]/(t^n)$.

Proof. Write $\mathcal{O}_K = \mathbb{Z}_p[\pi]$, where $\pi$ is a root of an Eisenstein polynomial $E(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \in \mathbb{Z}_p[t]$ (see Proposition 11, pg.52 [Lan94]). In particular, we have that $a_i \equiv 0 \mod p\mathbb{Z}_p$ and the reduction of $E(t)$ modulo $p\mathbb{Z}_p$ is equal to $E(t) = t^n \in \mathbb{F}_p[t]$. We now compute

$$\mathcal{O}_K/(\pi) = \mathbb{Z}_p[t]/(\pi, E(t)) \cong \mathbb{F}_p[t]/(E(t)) = \mathbb{F}_p[t]/(t^n)$$

and observe that the above isomorphism sends $\pi + (p)$ to $t + (t^n)$. \hfill \Box

Remark 2.2.2. If $K, \pi$ are as above, then $\mathcal{O}_K/((\pi^n)) = \mathcal{O}_K/(\pi) \cong \mathbb{F}_p[t]/(t^n)$ and Kazhdan says that $K$ is $n$-close to $\mathbb{F}_p[[t]]$ (see §0 [Kaz86]). In what follows, it will indeed be useful to think of $\mathcal{O}_K/(p)$ as being very close to $\mathbb{F}_p[[t]]$ for large $n$ (see §3.1.2).
2.2.3. **Residue rings.** For each \(n \in \mathbb{N}\), we view \(\mathbb{F}_p[t]/(t^n)\) as an \(L_t \cup P\)-structure, where \(L_t\) is the language of rings \(L_{\text{rings}}\) together with a constant symbol for \(t\) and \(P\) is a unary predicate, whose interpretation is the set \(\{0, 1, t, \ldots, t^{n-1}\}\). Note that we have tacitly replaced the equivalence class \(t^k + (t^n)\) with \(t^k\), which is harmless and common when dealing with truncated/modular arithmetic.

2.2.4. **Interpreting \(\mathbb{F}_p[t]/(t^n)\).** As a consequence of Lemma 2.2.1 we obtain:

**Proposition 2.2.5.** Let \(K/\mathbb{Q}_p\) be totally ramified of degree \(n \in \mathbb{N}\) and \(s: \Gamma \to K^\times\) be a cross-section. The structure \(\mathbb{F}_p[t]/(t^n)\) in \(L_t \cup P\) is \(\exists\forall\)-interpretable in \(K\) in the language \(L_{\text{val}, x}\). Moreover, the reduction map \(\phi \mapsto \phi_{\Delta_K}\) does not depend on \(K\) or the choice of \(s\).

**Proof.** Let \(\gamma\) be the minimal positive element in \(\Gamma\). It is definable by the \(\forall\)-formula in the free variable \(\gamma \in \Gamma\) written below

\[
\gamma > 0 \land \forall \delta \in \Gamma^{>0}(\gamma \leq \delta)
\]

henceforth abbreviated by \((\gamma\text{ minimal positive})\).

We now define a 1-dimensional interpretation \(\Delta_K\) of \(\mathbb{F}_p[t]/(t^n)\) in \(K\). Take \(\partial_{\Delta_K}(x)\) to be the formula \(x \in \emptyset\). The reduction map on unnested atomic formulas is described as follows:

1. If \(\phi(x)\) is the formula \(x = 0\) (resp. \(x = 1\) and \(x = t\)), we take \(\phi_{\Delta_K}(x)\) to be the formula \(\exists y \in \mathcal{O}(x = py)\) (resp. \(\exists y \in \mathcal{O}(x = 1 + py)\) and \(\exists \gamma \in \Gamma[(\gamma\text{ minimal positive}) \land (x = s(\gamma))]\)).
2. If \(\phi(x, y)\) is the formula \(x = y\), we take \(\phi_{\Delta_K}(x, y)\) to be the formula \(\exists z \in \mathcal{O}(x = y + pz)\).
3. If \(\phi(x, y, z)\) is \(x \circ y = z\), then we take \(\phi_{\Delta_K}\) to be the formula \(\exists w \in \mathcal{O}(x \circ y = z + pw)\), where \(\circ\) is either \(\cdot\) or \(+\).
4. If \(\phi(x)\) is the formula \(x \in P\), we take \(\phi_{\Delta_K}(x)\) to be the formula \(\exists \gamma \in \Gamma^{\geq 0}(x = s(\gamma))\)

The coordinate map \(f_{\Delta_K}: \mathcal{O}_K \to \mathbb{F}_p[t]/(t^n)\) is the projection modulo \(p\), given by Lemma 2.2.1. The isomorphism \(\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n)\) identifies \(\pi\) with \(t\) and the image of \(s(\Gamma^{\geq 0})\) in \(\mathcal{O}_K/(p)\) is equal to \(\{0, 1, t, \ldots, t^{n-1}\}\), via the above identification. One readily checks that the above data defines an \(\exists\forall\)-interpretation of the \(L_t \cup P\)-structure \(\mathbb{F}_p[t]/(t^n)\) in the \(L_{\text{val}, x}\)-structure \(K\).

Finally, the reduction map \(\phi \mapsto \phi_{\Delta_K}\) does not depend on \(K\), because of the inductive construction of the reduction map (see Proposition 1.1.3) and the fact that \(\phi \mapsto \phi_{\Delta_K}\) does not depend on \(K\) when \(\phi\) is any of the unnested atomic formulas listed above.

**Convention 2.2.6.** For the rest of the paper, fix once and for all a choice of a cross-section \(s_K: \Gamma \to K^\times\), for each finite extension \(K\) of \(\mathbb{Q}_p\). For each such \(K\), a cross-section does indeed exist and corresponds to a choice of a uniformizer for \(\mathcal{O}_K\).
Corollary 2.2.7. The asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is Turing reducible to the asymptotic existential-universal \( L_{\text{val},x} \)-theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \).

Proof. For \( K/\mathbb{Q}_p \) totally ramified of degree \( n \), let \( \Delta_K \) be the interpretation of the \( L_t \cup P \)-structure \( \mathbb{F}_p[t]/(t^n) \) in the \( L_{\text{val},x} \)-structure \( K \), provided by Proposition 2.2.5. The reduction map \( \Delta_K \) does not depend on \( K \) and will simply be denoted by \( \phi \mapsto \phi_\Delta \).

Claim: For any existential \( \phi \in \text{Sent}_{L_t \cup P} \), the sentence \( \phi_\Delta \) is equivalent to an \( \exists \forall \)-sentence.

Proof. For sentences \( \phi \) of the form \( \exists x \psi(x) \), where \( \psi(x) \) is a quantifier-free formula without negations, this follows from the fact that \( \Delta_K \) is an \( \exists \forall \)-interpretation. We may therefore focus on formulas \( \psi(x) \) of the form \( f(x_1, \ldots, x_m, t) \neq 0 \) (resp. \( f(x_1, \ldots, x_m, t) \in P \)). Such a formula is logically equivalent to \( \neg f(x_1, \ldots, x_m, y) = 0 \wedge y = t \) (resp. \( \neg f(x_1, \ldots, x_m, y) \in P \wedge y = t \)). Now \( \neg f(x_1, \ldots, x_m, y) = 0 \Delta \) (resp. \( \neg f(x_1, \ldots, x_m, y) \in P \Delta \)) is universal and \( y = t \) is existential-universal. The conclusion follows. \( \square \)

For any \( \phi \in \text{Sent}_{L_t \cup P} \) and any totally ramified extension \( K/\mathbb{Q}_p \) of degree \( n \), we have that \( \mathbb{F}_p[t]/(t^n) \models \phi \iff K \models \phi_\Delta \). For any given \( n \in \mathbb{N} \), there are finitely many totally ramified extensions \( K/\mathbb{Q}_p \) of degree \( n \) (Proposition 14 [Lan94]). It follows that

\[ \mathbb{F}_p[t]/(t^n) \models \phi \text{ for almost all } n \in \mathbb{N} \iff K \models \phi_\Delta \text{ for almost all } K \text{ with } [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \]

The conclusion follows from the Claim. \( \square \)

3. Truncations of \( (\mathbb{N}; 0, 1, +, |_p) \)

3.1. Motivation.

3.1.1. Pheidas’ work. In Theorem 1 [Phe87], Pheidas showed that the Diophantine problem for \( (\mathbb{N}; 0, 1, +, |_p) \) is undecidable. The proof goes by defining multiplication via a positive existential formula and using Matiyasevich’s negative solution to Hilbert’s tenth problem [Mat70]. In Lemma 1(c) [Phe87], it is shown that the Diophantine problem for \( (\mathbb{N}; 0, 1, +, |_p) \) can be encoded in the existential theory of \( \mathbb{F}_p((t)) \) in \( L_t \) with a predicate \( P \) for \( \{0, 1, t, t^2, \ldots \} \). It follows that the latter is also undecidable.

3.1.2. Plan of action. Our intuition that \( \mathbb{F}_p[t]/(t^n) \) approximates \( \mathbb{F}_p[t] \) when \( n \) is large, suggests that we can adapt Pheidas’ strategy and show that the asymptotic existential theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is also undecidable in \( L_t \cup P \). This will be established in Proposition 3.1.1 and in combination with Corollary 2.2.7 will pave the way for proving Theorem A in Section 4.3.
3.2. $p$-divisibility. The notion of $p$-divisibility was introduced by Denef in [Den79], in order to show that the Diophantine problem of a polynomial ring of positive characteristic is undecidable. Given $n, m \in \mathbb{N}$ and $p \in \mathbb{P}$, we write $n \mid_p m$ if $m = p^s n$, for some $s \in \mathbb{N}$. Let us also write $L_{p-\text{div}} = \{0, 1, +, \mid_p\}$ for the language of addition and $p$-divisibility.

3.2.1. Encoding $|_p$ in $F_p[t]/(t^n)$. The following result is due to Pheidas:

**Lemma 3.2.2** (Lemma 1(a) [Phe87]). Let $n, m \in \mathbb{N}$ with $0 < n \leq m$. Then $n \mid_p m$ if and only if there exists $a \in F_p[t]$ such that $t^{-m} - t^{-n} = a^{-p} - a^{-1}$.

**Remark 3.2.3.** If $n \mid_p m$, then the proof of Lemma 1(a) [Phe87] provides $a = \left( t^{-n+p^{-1}} + t^{-n+p^{-2}} + \ldots + t^{-n} \right)^{-1}$. Note that we have slightly rephrased the original formulation of Lemma 1(a) [Phe87], so that the witness $a$ has positive valuation.

We shall use a truncated version of Lemma 3.2.2, whose proof is identical, modulo some additional bookkeeping:

**Lemma 3.2.4.** Let $n, m, N \in \mathbb{N}$ with $0 < n \leq m < N/3$. Then $n \mid_p m$ if and only if there exists $\alpha \in F_p[t]/(t^N)$ such that $\alpha^p(t^n - t^m) = t^m t^n (1 - \alpha^{p-1})$ and $\alpha^{3p} \neq 0$ in $F_p[t]/(t^N)$.

**Proof.** $\Rightarrow$: Let $a = \left( t^{-n+p^{-1}} + t^{-n+p^{-2}} + \ldots + t^{-n} \right)^{-1} \in F_p[t]$. After clearing denominators in Lemma 3.2.2, we get that $a^p(t^n - t^m) = t^m t^n (1 - a^{p-1})$. Reducing the equation modulo $t^N$, yields $\alpha^p(t^n - t^m) = t^m t^n (1 - \alpha^{p-1})$, where $\alpha$ is the image of $a$ in $F_p[t]/(t^N) \cong F_p[t]/(t^N)$. Note that $\nu_t a^{3p} = 3m < N$ and thus $\alpha^{3p} \neq 0$ in $F_p[t]/(t^N)$.

$\Leftarrow$: Let $n = p^s k$ and $m = p^i k$ with $p \mid i, k$. Let $\alpha$ be as in our assumption and $a \in F_p[t]$ be a lift of $\alpha$. Since $\alpha^{3p} \neq 0$, we get that $\nu_t a^p < N/3$. We will have by assumption that $t^{-m} - t^{-n} = a^{-p} - a^{-1} + t^{N-m-n} a^{-p}$, for some $z \in F_p[t]$. Set $\varepsilon = t^{N-m-n} a^{-p}$ and note that $\nu_t \varepsilon > 0$ because $m, n, \nu_t a^p < N/3$ and $\nu_t \varepsilon > 0$. By Lemma 1 [Phe87], we may find $a_1, a_2 \in F_p[t]$ such that $t^{-m} - t^{-k} = a_1^{-p} - a_1^{-1}$ and $t^{-m} - t^{-i} = a_2^{-p} - a_2^{-1}$. We compute that

$$t^{-i} - t^{-k} = b^p - b + \varepsilon$$

where $b = a^{-1} + a_1^{-1} - a_2^{-1}$.

We claim that $i = k$. Otherwise, the left hand side must have negative valuation. Since $\nu_t \varepsilon > 0$, this forces the right hand side to have $p$-divisible valuation. This is contrary to the fact that $p \mid i, k$. It follows that $i = k$ and thus $n \mid_p m$. \qed

3.2.5. Interpreting $I_n$ in $F_p[t]/(t^n)$. Motivated by Lemma 3.2.4, we introduce for each $n \in \mathbb{N}$ the $L_{p-\text{div}} \cup \{\infty\}$-structure $I_n = (\{0, 1, \ldots, n-1, \infty\} ; 0, 1, \infty, \oplus, \mid_p)$, where:

- For $x, y \in I_n$, we have $x \mid_p y$ if $y = p^s x$ for some $s \in \mathbb{N}$ and $1 \leq x, y < n/3$. 

• The operation $\oplus : I_n \times I_n \to I_n$ stands for truncated addition, i.e. given $x, y \in \{0, 1, ..., n-1\}$ we have that $x \oplus y = x + y$ if $x + y < n$ and $\infty$ otherwise. Moreover, $\infty \oplus x = x \oplus \infty = \infty$ for all $x \in \{0, 1, ..., n-1, \infty\}$.

**Proposition 3.2.6.** For each $n \in \mathbb{N}$, there is an $\exists$-interpretation $\Gamma_n$ of the $L_p$-language in the $I_n$-structure $I_n$ in the $L = P$-structure $\mathbb{F}_p[t]/(t^n)$. Moreover, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n$.

**Proof.** Take $\partial_{\Gamma_n}(x)$ to be the formula $x \in P$. The reduction map of $\Gamma_n$ on unnested atomic formulas is described as follows:

1. If $\phi$ is the formula $x = 0$ (resp. $x = 1, x = \infty$ and $x = y$), we take $\phi_{\Gamma_n}$ to be the formula $x = 1$ (resp. $x = t, x = 0$ and $x = y$).
2. If $\phi(x, y, z)$ is $x + y = z$, then we take $\phi_{\Gamma_n}(x, y, z)$ to be the formula $x \cdot y = z$.
3. If $\phi(x, y)$ is the formula $x \mid_p y$, we take $\phi_{\Gamma_n}(x, y)$ to be the formula $\exists z(z^p(x - y) = x \cdot y(1 - z^{-p-1}) \land z^p \neq 0)$.

The coordinate map $f_{\Gamma_n} : \{0, 1, t, ..., t^{n-1}\} \to I_n$ is equal to $v|_P$, i.e., the valuation map $v$ restricted on $P$. Condition (3) of Definition 3.1.2 is readily verified for unnested atomic formulas of type (1). For the formula described in (2), one has to use the isomorphism of monoids $(P, \cdot, 1) \cong (\{0, 1, ..., n-1, \infty\}, \oplus, 0)$. Using Lemma 3.2.4, one also verifies it for the unnested atomic formula of type (3). We deduce that $\Gamma_n$ is an interpretation. It is clear from the description of the reduction map on unnested formulas that $\Gamma_n$ is existential.

Finally, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n \in \mathbb{N}$, because of the inductive construction of the reduction map (see Proposition 3.1.3) and the fact that $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n$ when $\phi$ is any of the unnested atomic formulas listed above.

\[\square\]

**Corollary 3.2.7.** The (resp. asymptotic) $L_p$-theory of $\{I_n : n \in \mathbb{N}\}$ is Turing reducible to the (resp. asymptotic) $L = P$-theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$.

**Proof.** Immediate from Proposition 3.2.6. \[\square\]

### 3.3. Undecidability of fragments.

Using the undecidability of the Diophantine problem over $(\mathbb{N}; 0, 1, +, |_p)$ (Theorem 1 [Phe87]) as a black box, we shall prove that the asymptotic existential theory of $\{I_n : n \in \mathbb{N}\}$ is also undecidable.

The proof becomes more transparent by using matrix norms. Recall the maximum absolute row sum norm $\| \cdot \|_\infty$ on the set of all matrices over $\mathbb{R}$, which is defined as $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, for $A \in M_{m \times n}(\mathbb{R})$. One readily checks that $\| \cdot \|_\infty$ is both sub-additive and sub-multiplicative, meaning that $\|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$ and $\|A \cdot B\|_\infty \leq \|A\|_\infty \cdot \|B\|_\infty$, whenever the operations are defined. Note also that when $A = (a_1, ..., a_n)^t \in M_{n \times 1}(\mathbb{N})$, we have $\|A\|_\infty = \max \{a_i : i = 1, ..., n\}$. 


For $A = (a_{ij})$ and $B = (b_{ij})$ in $M_{m \times n}(\mathbb{N})$, it will be convenient to use the notation $A \mid_p B$, which means that $a_{ij} \mid_p b_{ij}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

**Proposition 3.3.1.** The asymptotic existential $L_{p-\text{div}} \cup \{\infty\}$-theory of $\{I_n : n \in \mathbb{N}\}$ is undecidable. Moreover, let $T$ be the subtheory consisting of sentences $\phi$ of the form $\exists x \psi(x)$, where $x = (x_1, ..., x_n)$ and $\psi(x)$ is a conjunction of a quantifier-free formula without negations with a formula of the form $\wedge_{i=1}^n N x_i \neq \infty$, for some $N \in \mathbb{N}$. Then $T$ is undecidable.

**Proof.** We shall encode the Diophantine problem of $(\mathbb{N}; 0, 1, +, \mid_p)$ in $T$. Let $\Sigma$ be an arbitrary system in variables $x = (x_1, ..., x_n) \in M_{n \times 1}(\mathbb{N})$ of the form

$$\begin{cases} A_1 x + b_1 = A_2 x + b_2 \\ A_3 x + b_3 \mid_p A_4 x + b_4 \end{cases}$$

where $A_i \in M_{m \times n}(\mathbb{N})$ and $b_i \in M_{m \times 1}(\mathbb{N})$. Consider also the formula $\Sigma(x) \in \text{Form}_{p-\text{div}}$ associated with the system $\Sigma$.

**Claim:** We have that

$$\quad (\mathbb{N}; 0, 1, +, \mid_p) \models \exists x \Sigma(x) \iff T \models \exists x (\Sigma(x) \bigwedge_{i=1}^n 3 M x_i \neq \infty)$$

where $M = \max_{1 \leq i \leq 4}\{\|A_i\|_{\infty} + \|b_i\|_{\infty}\}$.

**Proof.** "$\Rightarrow":$ Consider a witnessing tuple $c = (c_1, ..., c_n)^T \in M_{n \times 1}(\mathbb{N})$. If $c = 0$, then the conclusion is clear. Otherwise, consider $m = \|c\|_{\infty} = \max\{c_i : i = 1, ..., n\} \geq 1$ and choose $N \in \mathbb{N}$ such that $N > 3 \cdot M \cdot m$. Using the sub-additive and sub-multiplicative properties of $\|\cdot\|_{\infty}$, we see that

$$\|A_i c + b_i\|_{\infty} \leq \|A_i\|_{\infty} \|c\|_{\infty} + \|b_i\|_{\infty} \leq m \cdot (\|A_i\|_{\infty} + \|b_i\|_{\infty}) \leq M \cdot m < N/3$$

for $i = 1, ..., 4$. In particular, we get that both $\oplus$ and $\mid_p$ specialize to their ordinary counterparts in $\mathbb{N}$ and that $\Sigma(c)$ holds true in $I_N$, viewing $c$ as a tuple in $I^n_N$. Each conjunct $3 M x_i \neq \infty$ also holds true for $c_i$ because $3 M c_i \leq 3 M \cdot m < N$.

"$\Leftarrow":$ Let $c = (c_1, ..., c_n)^T \in I^n_N$ be a witness of the sentence $\exists x (\Sigma(x) \bigwedge_{i=1}^n 3 M x_i \neq \infty)$, for some $N > 3 \cdot M$. Since $3 M c_i \neq \infty$, we get that $c_i < N/3M$ for $i = 1, ..., n$. We therefore get that

$$\|A_i c + b_i\|_{\infty} \leq \|A_i\|_{\infty} \|c\|_{\infty} + \|b_i\|_{\infty} < \frac{N}{3M} \cdot (\|A_i\|_{\infty} + \|b_i\|_{\infty}) \leq \frac{N}{3M} \cdot M = N/3$$

for $i = 1, ..., 4$. In particular, we get that both $\oplus$ and $\mid_p$ specialize to their ordinary counterparts in $\mathbb{N}$. The corresponding tuple in $\mathbb{N}^n$ is the desired witness. $\square$

The conclusion follows from the fact that the Diophantine problem over $(\mathbb{N}; 0, 1, +, \mid_p)$ is undecidable (Theorem I [Phe87]). $\square$
4. Proof of the main theorem

4.1. Truncated polynomial rings.

**Proposition 4.1.1.** The asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is undecidable.

*Proof.* Let \( T \) be as in Proposition 3.2.6. We shall argue that the asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is Turing reducible to \( T \). The conclusion will then follow from Proposition 3.3.1. Let \( \Gamma_n \) be the \( \exists \)-interpretation of the \( L_{p \text{-div}} \cup \{ \infty \} \)-structure \( I_n \) in the \( L_t \cup P \)-structure \( \mathbb{F}_p[t]/(t^n) \), provided by Proposition 3.2.6. The reduction map of \( \Gamma_n \) does not depend on \( n \in \mathbb{N} \) and will simply be denoted by \( \phi \mapsto \phi_T \). Since \( \Gamma_n \) is existential, whenever \( \psi(x) \) is a quantifier-free \( L_{p \text{-div}} \cup \{ \infty \} \)-formula, the formula \( \psi_T(x) \) is an existential formula. Moreover, from the description of the reduction map on unnested atomic formulas (see the proof of Proposition 3.2.6), we have \( (\bigwedge_{i=1}^n Nx_i \neq \infty)_T = \bigwedge_{i=1}^n (Nx_i \neq \infty) \neq 0 \). It follows that \( T \) is Turing reducible to the asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \). \( \square \)

4.2. Totally ramified extensions.

**Proposition 4.2.1.** The asymptotic \( \exists \forall \)-theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) is undecidable in \( L_{\text{val},x} \).

*Proof.* This follows from Corollary 2.2.7 and Proposition 4.1.1. \( \square \)

**Remark 4.2.2.** (a) The same proof applies verbatim to any infinite collection of totally ramified extensions of \( \mathbb{Q}_p \), e.g., \( \{ \mathbb{Q}_p(p^{\alpha}) : n \in \mathbb{N} \} \) or \( \{ \mathbb{Q}_p(p^{1/n}) : n \in \mathbb{N} \} \).

(b) We do not know if the asymptotic existential theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) is decidable or not in \( L_{\text{val},x} \).

4.3. Finite extensions.

**Theorem A.** The asymptotic \( \exists \forall \)-theory of \( \{ K : [K : \mathbb{Q}_p] < \infty \} \) is undecidable in \( L_{\text{val},x} \).

*Proof.* Let \( T \) be the theory in question and \( T_{\text{tot}} \) be the asymptotic theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) in \( L_{\text{val}} \) with a cross-section. We shall encode \( T_{\text{tot}} \) in \( T \). Given an \( \exists \forall \)-sentence \( \phi \), we see that

\[ T_{\text{tot}} \models \phi \iff T \models (k = \mathbb{F}_p) \rightarrow \phi \]

The formal counterpart of \( (k = \mathbb{F}_p) \rightarrow \phi \) is logically equivalent to an \( \exists \forall \)-sentence. It follows that the asymptotic existential-universal \( L_{\text{val},x} \)-theory of \( \{ K : [K : \mathbb{Q}_p] < \infty \} \) is Turing reducible to the asymptotic existential-universal \( L_{\text{val},x} \)-theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \). The conclusion follows from Proposition 4.2.1. \( \square \)
5. Final remarks

5.1. $L_{\text{val}}$ vs $L_{\text{rings}}$. In view of [CDLM13] and by possibly increasing the complexity, one may replace $L_{\text{val}, x}$ with the 1-sorted language of rings $L_{\text{rings}}$ together with a unary predicate $P$ for the image of the cross-section in $K$. We need to use the following:

**Fact 5.1.1** (Theorem 2 [CDLM13]). There is an $\exists\forall$-formula $\phi(x)$ in $L_r$ such that $\mathcal{O}_K = \phi(K)$ for any henselian valued field $K$ with finite or pseudo-finite residue field.

**Remark 5.1.2.** In fact, Theorem 2 [CDLM13] is about the existence of an existential formula in $L_{\text{rings}} \cup P_2^{AS}$, where $P_2^{AS}(x) = \exists y(x = y^2 + y)$. However, any existential sentence in $L_{\text{rings}} \cup P_2^{AS}$ is equivalent to an $\exists\forall$-sentence in $L_{\text{rings}}$.

By Fact 5.1.1, the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in $L_{\text{val}}$ with a cross-section can be encoded in the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in $L_{\text{rings}} \cup P$. By Theorem A, the latter is undecidable. Our use of the cross-section/predicate formalism is essential and we do not know whether the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in $L_{\text{rings}}$.

5.2. **Residue rings and $\mathbb{F}_p((t))$.** If the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in $L_{\text{rings}}$, then by Corollary [2.2.7] this would also yield a positive answer to the following question:

**Question 5.2.1.** Is the (asymptotic) theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ decidable in $L_{\text{rings}}$?

**Observation 5.2.2.** The asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is decidable in $L_{\text{rings}}$.

**Proof.** Let $\mathbb{F}_p[t^{1/\infty}]$ be the direct limit of the injective system $\lim_{\longrightarrow} \mathbb{F}_p[t^{1/n}]$, where $\phi_{nm} : \mathbb{F}_p[t^{1/n}] \hookrightarrow \mathbb{F}_p[t^{1/m}]$ is the natural inclusion map for $n \mid m$. We write $\mathbb{F}_p[t^{1/n}]/(t)$ and $\mathbb{F}_p[t^{1/\infty}]/(t)$ for the quotients modulo $t$. Note that $\phi_{nm}(t \cdot \mathbb{F}_p[t^{1/n}]) \subseteq t \cdot \mathbb{F}_p[t^{1/m}]$ and therefore the induced maps $\phi_{nm} : \mathbb{F}_p[t^{1/n}] \hookrightarrow \mathbb{F}_p[t^{1/m}]$ and $\phi_n : \mathbb{F}_p[t^{1/n}]/(t) \hookrightarrow \mathbb{F}_p[t^{1/\infty}]/(t)$ are also injective.

**Claim 1:** The asymptotic existential $L_{\text{rings}}$-theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is equal to $\text{Th}_3(\mathbb{F}_p[t^{1/\infty}]/(t))$.

**Proof.** For each $n \in \mathbb{N}$, we have a ring isomorphism

$$\mathbb{F}_p[t^{1/n}]/(t) \cong \mathbb{F}_p[t, X]/(X^n - t, t) \cong \mathbb{F}_p[X]/(X^n) \cong \mathbb{F}_p[t]/(t^n)$$

Now if $\phi \in L_{\text{rings}}$ is existential, then $\mathbb{F}_p[t^{1/\infty}]/(t) \models \phi$ if and only if $\mathbb{F}_p[t^{1/n}]/(t) \models \phi$ for all sufficiently large $n$. The conclusion follows. \(\square\) **Claim 1**

Finally, we prove:

**Claim 2:** $\text{Th}_3(\mathbb{F}_p[t^{1/\infty}]/(t))$ is decidable in $L_{\text{rings}}$. 

Proof. A straightforward adaptation of Proposition 6.2.1 [Kar20] shows that

\[ \mathbb{F}_p[[t]](t^{1/\infty}) \models \exists x \bigwedge_{1 \leq i,j \leq n} (v(f_i(x)) > v(g_j(x))) \]

where \( f_i(x), g_j(x) \in \mathbb{F}_p[x] \) are any multi-variable polynomials in \( x = (x_1, \ldots, x_m) \) for \( i, j = 1, \ldots, n \). Finally, the henselian valued field \( \mathbb{F}_p((t))(t^{1/\infty}) \) is existentially decidable in \( L_{\text{val}} \) by Corollary 7.5 [AF16]. \qed

\textbf{Claim 2}

Remark 5.2.3. (a) Observation 5.2.2 should be contrasted with Proposition 4.1.1, which shows that the asymptotic existential theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is undecidable in \( L_t \cup P \).

(b) The proof of Observation 5.2.2 does not go through for the language \( L_t \), as the ring isomorphisms in the proof of Claim 1 do not respect \( t \). We do not know if the asymptotic existential theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is decidable in \( L_t \).

(c) On the other hand, the asymptotic positive existential theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) in \( L_t \) is equal to \( \text{Th}_{\exists^+} \mathbb{F}_p[t] \) and is decidable by an effective version of Greenberg’s approximation theorem (see Theorem 3.1 and Theorem 6.1 [BDLvdD79]).

Acknowledgements

I would like to thank E. Hrushovski, who suggested the key idea of this paper, and J. Koenigsmann for careful readings and various suggestions. I also thank J. Derakhshan for helpful discussions and two anonymous referees for several fruitful comments.

References

[AF16] Sylvy Anscombe and Arno Fehm. The existential theory of equicharacteristic henselian valued fields. Algebra & Number Theory Volume 10, Number 3 (2016), 665-683., 2016.

[AK65] James Ax and Simon Kochen. Diophantine problems over local fields II. Amer. J. Math. 87, 1965.

[AK66] James Ax and Simon Kochen. Diophantine problems over local fields III. Annals of Mathematics, Second Series, Vol. 83, No. 3, pp. 437-456, 1966.

[Ax68] James Ax. The elementary theory of finite fields. Annals of Mathematics, Second Series, Vol. 88, No. 2 , pp. 239-271, 1968.

[BDL80] J. Becker, J. Denef, and L. Lipshitz. Further remarks on the elementary theory of formal power series rings. Model Theory of Algebra and Arithmetic, Pacholski et al. eds., LNM 834, Springer-Verlag NY, pp 1-9, 1980.

[BDLvdD79] J. Becker, J. Denef, L. Lipshitz, and L. van den Dries. Ultraproducts and approximation in local rings I. Inventiones math. 51,189-203, 1979.
Raf Cluckers, Jamshid Derakhshan, Eva Leenknegt, and Angus Macintyre. Uniformly defining valuation rings in henselian valued fields with finite or pseudo-finite residue fields. *Ann. Pure Appl. Logic* 164, 12, 1236-1246, 2013.

Gregory Cherlin. Undecidability of rational function fields in nonzero characteristic. *Logic Colloq., no. 82, North-Holland, Amsterdam.*, 1982.

Pierre Deligne. Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0. *Représentations des groupes réductifs sur un corps local*, - *Hermann Paris*, 1984.

Jan Denef. The diophantine problem for polynomial rings of positive characteristic. *Logic Colloq., no. 78, North-Holland, Amsterdam*, 1979.

Jamshid Derakhshan. Model theory of Adeles and number theory. *arXiv: 2007.09237 [math.LO]*, 2020.

Jamshid Derakhshan and Angus Macintyre. Model theory of adeles I. *Ann. Pure Appl. Logic*, 173(3):Paper No. 103074, 43, 2022.

Ju.L. Ershov. On elementary theories of local fields. *Algebra i Logika 4, No. 2, 5-30, 1965.*

Wilfrid Hodges. *Model theory*. Cambridge University Press, 1993.

Konstantinos Kartas. Decidability via the tilting correspondence. *arXiv:2001.04424[math.LO]*, (2020).

Konstantinos Kartas. *Contributions to the model theory of henselian fields*. PhD thesis, University of Oxford, 2022.

D. Kazhdan. Representations of groups over close local fields. *Journal d’ Analyse Mathématique* volume 47, pages 175-179, 1986.

Simon Kochen. The model theory of local fields. *ISILC Logic Conference (Proc. Internat. Summer Inst. and Logic Colloq., Kiel)*, pp. 384-425. *Lecture Notes in Math.*, Vol. 499, *Springer, Berlin.*, 1974.

Marc Krasner. Approximation des corps valués complets de caractéristique p par ceux de caractéristique 0. *Colloque d’Algebre Superleure Bruxelles*, 1956.

Serge Lang. *Algebraic Number Theory*. Springer, Graduate Texts in Mathematics book series (GTM, volume 110), 1994.

Angus J. Macintyre. On definable subsets of p-adic fields. *Journal of Symbolic Logic*, 41:605–10, 1976.

Yuri Matiyasevich. Enumerable sets are Diophantine. *Dokl. Akad. Nauk SSSR 191*, 279- 282, 1970.

Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, Inc., 1994.

Thanases Pheidas. An undecidability result for power series rings of positive characteristic. II. *Proceedings of the American Mathematical Society*, Vol. 100, pp. 526-530, 1987.

Mathematical Institute, Woodstock Road, Oxford OX2 6GG.

E-mail address: kartas@maths.ox.ac.uk