TRACER PARTICLES COUPLED TO AN INTERACTING BOSON GAS

ESTEBAN CÁRDENAS

Abstract. In this work, we investigate the mean-field limit of a model consisting of $m \geq 1$ tracer particles, coupled to an interacting boson field. We assume the mass of the tracer particles and the expected number of bosons to be of the same order of magnitude $N \geq 1$ and we investigate the $N \to \infty$ limit. In particular, we show that the limiting system can be effectively described by a pair of variables $(X_t, \varphi_t) \in \mathbb{R}^{3m} \times H^1(\mathbb{R}^3)$ that solve a mean-field equation. Our methods are based on proving estimates for the number of bosonic particles in a suitable fluctuation state $\Omega_{N,t}$. The main difficulty of the problem comes from the fact that the interaction with the tracer particles can create or destroy bosons for states close to the vacuum.

Contents

1. Introduction 1
   1.1. The Hamiltonian 2
   1.2. The Mean-Field Equations 3
2. Main Results 4
   2.1. Discussion of Main Results 4
3. Fluctuation States 8
   3.1. Regularization of the Problem 10
   3.2. The Truncated Dynamics 12
   3.3. Proof of Theorem 3.1 14
4. Proof of our Main Results 15
5. Particle Number Estimates of the Truncated Dynamics 17
   5.1. Dynamical Estimates 17
   5.2. Proof of Lemmata 19
6. Well-Posedness of the Mean-Field Equations 25
7. Well-posedness of the Regularized, Truncated Dynamics 29
Appendix A. The Regularization Lemmata 32
Appendix B. The Interpolation Lemma 34
References 35

1. Introduction

In this article, we consider $m \geq 1$ heavy quantum particles (tracer particles) that are coupled to a weakly interacting gas of non-relativistic scalar bosons. We are interested in the regime for which the average number of bosons and the mass of the tracer particles are, in appropriate physical units, large and of the same order of magnitude $N$. The microscopic
states of this system are described by wave functions that belong to the Hilbert space
\[
\mathcal{H} := L^2_X \otimes \mathcal{F}_b
\]
where \( L^2_X = L^2(\mathbb{R}^{3m}, dX) \) is the state space for the tracer particles, with positions labeled by \( X = (X^{(1)}, \ldots, X^{(m)}) \) \( \in \mathbb{R}^{3m} \), and \( \mathcal{F}_b \) is the spinless bosonic Fock space
\[
\mathcal{F}_b := \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{F}_n \quad \text{where} \quad \mathcal{F}_n := L^2(\mathbb{R}^3, dx)^{\otimes \text{sym}^n}.
\]
We will denote by \( \Omega \) := (1, 0, \ldots) the vacuum vector in \( \mathcal{F}_b \) and make extensive use of the operator-valued distributions \( a_x \) and \( a_x^* \) satisfying the canonical commutation relations (CCR)
\[
[a_x, a_y^*] = \delta(x - y), \quad [a_x, a_y] = [a_x^*, a_y^*] = 0, \quad x, y \in \mathbb{R}^3.
\]
The reader is referred to [33, Section 2] for a nice account on creation and annihilation operators, and to [32, Section X.7] for a thorough mathematical treatment.

If \( \Psi_{N,t} \in \mathcal{H} \) denotes the wave function of the above system at time \( t \geq 0 \), the main goal of the present article is to derive an effective description of this object, in the large \( N \) limit. From a heuristic point of view, one expects that as the mass of the tracer particles gets larger, classical behaviour should be dominant. On the other hand, for low enough temperatures, one expects a boson gas to experience condensation–most of the particles will be in the same quantum state. This leads us to introduce a pair of interacting mean-field variables
\[
t \in \mathbb{R} \longmapsto (X_t, \varphi_t) \in \mathbb{R}^{3m} \times L^2(\mathbb{R}^3)
\]
where \( X_t \) describes the classical trajectories followed by the tracer particles and \( \varphi_t \) is a self-interacting wave function describing the spatial probability distribution of the condensate.

### 1.1. The Hamiltonian
Throughout this article, we assume that the boson-tracer and boson-boson interactions are mediated by even, real-valued functions \( w : \mathbb{R}^3 \to \mathbb{R} \) and \( v : \mathbb{R}^3 \to \mathbb{R} \). We will always assume enough regularity so that the \((n+m)\)-particle Hamiltonian
\[
H_{N,n} := -\frac{\Delta_X}{2N} \otimes 1 - \sum_{i=1}^n 1 \otimes \Delta x_i + 1 \otimes \frac{1}{N} \sum_{i,j} v(x_i - x_j) + \sum_{i=1}^n \sum_{\ell=1}^m w(x_i - X^{(\ell)}) \quad n \geq 1
\]
is self-adjoint in its natural domain, with the obvious modification for \( n = 0 \). The dynamics of our combined system will then be described by the second-quantized Hamiltonian in \( \mathcal{H} \)
\[
\mathcal{H}_N := -\frac{\Delta_X}{2N} \otimes 1 + 1 \otimes T_b + 1 \otimes \frac{1}{2N} \int_{\mathbb{R}^3} v(x - y) a_x^* a_y^* a_x a_y \, dy + \int_{\mathbb{R}^3} w(x, X) \otimes a_x^* a_x \, dx,
\]
equivalently defined as the direct sum \( \bigoplus_{n \geq 0} H_{N,n} \). Here and in the sequel, \( T_b \) denotes the kinetic energy of the bosons
\[
T_b := \int_{\mathbb{R}^3} a_x^* (-\Delta_x) a_x \, dx = (0) \oplus \bigoplus_{n \geq 1} \sum_{i=1}^n (-\Delta_{x_i})
\]
and we will be using the following short-hand notation for the total boson-tracer interaction
\[
w(x, X) := \sum_{\ell=1}^m w(x - X^{(\ell)}), \quad x \in \mathbb{R}^3, \quad X = (X^{(1)}, \ldots, X^{(m)}) \in \mathbb{R}^{3m}.
\]

The Spectral Theorem then gives meaning to the evolution associated to the Schrödinger equation–from now on referred to as the *microscopic dynamics* of the system–
\[
\begin{cases}
i\hbar\partial_t \Psi_{N,t} = \mathcal{H}_N \Psi_{N,t} \\ \Psi_{N,t} |_{t=0} = \Psi_{N,0}
\end{cases}
\]
by means of exponentiation \( \Psi_{N,t} = \exp(-it\mathcal{H}_N)\Psi_{N,0} \). A few comments are in order.
Remark 1.1. Since the Hamiltonian $\mathcal{H}_N$ is diagonal with respect to Fock space $\mathcal{F}_b$, it commutes with the particle number operator

$$N_b := \int_{\mathbb{R}^3} a_x^* a_x dx = \bigoplus_{n \geq 0} n. \quad (1.10)$$

Therefore, if $\Psi_{N,0}$ contains in quantum-mechanical average $N$ bosons, so will $\Psi_{N,t}$ for all later times. In other words, it holds that

$$\langle \Psi_{N,t}, 1 \otimes N_b \Psi_{N,t} \rangle_{\mathbb{H}} = \langle \Psi_{N,0}, 1 \otimes N_b \Psi_{N,0} \rangle_{\mathbb{H}} = N, \quad \forall t \geq 0. \quad (1.11)$$

The initial data that we work with will always satisfy this condition.

Remark 1.2. Equation (1.9) is equivalent to an infinite system of equations: one for each component of $\Psi_{N,t}$ in its direct sum decomposition. Thus, an effective description of the microscopic dynamics in terms of the mean-field variables drastically reduces the number of unknowns one should solve for. Describing such an approximation is the content of the present paper.

Remark 1.3. On a formal level, each term present in the definition of $\mathcal{H}_N$ is of order $N$. Indeed, for the tracer particles, we expect position and velocities to be of order $1$, so that their kinetic energy is of order $N$ due to their heavy mass. For the boson field, we expect its kinetic term to be of order $N$ since there should be in average $N$ of them, each of which has order $1$ kinetic energy. Similarly, the interaction terms are scaled so that they remain of the same order of magnitude in $N$; one can directly count the powers on creation and annihilation operators present in each term. This is the so-called weak coupling scaling, or mean-field regime.

Remark 1.4. If the potentials $v$ and $w$ satisfy Condition 2—stated below—it is well-known that the finite particle Hamiltonian $H_{N,n}$ is self-adjoint in $H^2_{\mathbb{X}} \otimes H^2(\mathbb{R}^3)^{\otimes \text{sym} n}$. Further, for smooth $\Phi \in \mathcal{F}_b$ with finitely many non-zero entries, one may verify that the boson-boson interaction satisfies the estimate

$$\frac{1}{N} \left\| \int_{\mathbb{R}^6} v(x-y) a_x^* a_y^* a_x a_y dy \Phi \right\|_{\mathcal{F}_b} \lesssim \| T_b \Phi \|_{\mathcal{F}_b} + \frac{1}{N^2} \| (1 + N_b)^3 \Phi \|_{\mathcal{F}_b}. \quad (1.12)$$

Similarly, the boson-tracer interaction is controlled by $1 \otimes N_b$. Consequently, we have the following characterization for the domain of $\mathcal{H}_N$:

$$\mathcal{D}(\mathcal{H}_N) \cap \mathcal{D}(1 \otimes N_b^3) = H^2_{\mathbb{X}} \otimes \mathcal{D}(T_b) \cap \mathcal{D}(1 \otimes N_b^3), \quad \forall N \geq 1. \quad (1.13)$$

1.2. The Mean-Field Equations. Let us now introduce the equations satisfied by the mean-field variables. Heuristically, the classical trajectory of a tracer particle should obey Newton’s equation, with a time-dependent force depending on the location of the bosons. Similarly, the quantum boson condensate should feel a time-dependent potential, related to the positions of the tracer particles, plus its usual self-interactions as is well-known from the realm of Hartree’s equation. Thus, we introduce the following system of equations, from now on referred to as the mean-field equations

$$\begin{cases}
\dot{X}_t = -\int_{\mathbb{R}^3} \nabla x w(x, X_t|\varphi_t)^2 dx \\
i \dot{\varphi}_t = -\Delta \varphi_t + w(x, X_t) \varphi_t + (v * |\varphi_t|^2) \varphi_t \\
(X_t, \dot{X}_t, \varphi_t)|_{t=0} = (X_0, V_0, \varphi_0)
\end{cases} \quad (1.14)$$

provided some initial data $(X_0, V_0, \varphi_0)$ has been specified at time zero. Global well-posedness in $\mathbb{R}^{6m} \times H^1(\mathbb{R}^3)$, for the class of potentials we are going to be working with, is proven in
Section 6. In particular, the \( L^2 \) norm of the bosons and the energy of the system are conserved quantities.

2. Main Results

2.1. Discussion of Main Results. In this section we state our main theorems, which connect the microscopic dynamics (1.9) with the mean-field equations (1.14). First, let us introduce the following objects; we will make extensive use of them throughout this article.

- We denote by \( \mathbf{X} \) and \( \mathbf{P} = -i\nabla \mathbf{X} \) the tracer particle’s position and momentum observables in \( L^2_{\mathbf{X}} \), with domains \( \mathcal{D}(\mathbf{X}) \) and \( \mathcal{D}(\mathbf{P}) \), respectively.
- For \( f \in L^2(\mathbb{R}^3) \) we introduce the Weyl operator on \( \mathcal{F}_b \), defined as the unitary transformation
  \[
  \mathcal{W}(f) := \exp(a^*(f) - a(f)),
  \]
  where
  \[
  a(f) := \int_{\mathbb{R}^3} f(x)a_x dx \quad \text{and} \quad a^*(f) := \int_{\mathbb{R}^3} \overline{f(x)}a_x^* dx.
  \]

See [33, Lemma 2.2] for a list of the properties satisfied by Weyl operators.

The following Condition contains the mathematical properties satisfied by the initial data \( \Psi_{N,0} \) that we work with.

**Condition 1 (Initial Data).** We assume that the initial state of the combined system is a tensor product of the form

\[
\Psi_{N,0} = u_{N,0} \otimes \Phi_{N,0}, \quad N \geq 1
\]

where \( u_{N,0} \in L^2_{\mathbf{X}} \) and \( \Phi_{N,0} \in \mathcal{F}_b \) satisfy the additional requirements:

1. \( u_{N,0} \in \mathcal{D}(\mathbf{P}^3) \cap \mathcal{D}(\mathbf{X}^3) \) has unit \( L^2 \)-norm. Further, we assume that there exists \( (\mathbf{X}_0, \mathbf{V}_0) \in \mathbb{R}^{3m} \times \mathbb{R}^{3m} \) and \( C_0 > 0 \) such that for any \( 1 \leq p \leq 3 \) and all \( N \geq 1 \) it holds
   \[
   \| \mathbf{X} - \mathbf{X}_0 \|_{L^p_{\mathbf{X}}} + \| N^{-1} \mathbf{P} - \mathbf{V}_0 \|_{L^p_{\mathbf{P}}} \leq \frac{C_0}{N^{p/2}}.
   \]

2. There exists \( \varphi_0 \in H^1(\mathbb{R}^3) \), of unit \( L^2 \)-norm, such that
   \[
   \Phi_{N,0} = \mathcal{W}(\sqrt{N} \varphi_0)\Omega.
   \]

**Remark 2.1.** Given \( u \in \mathcal{D}(\mathbf{P}^3) \cap \mathcal{D}(\mathbf{X}^3) \), we may construct an initial condition that satisfies Condition 1 by appropriate re-scaling, that is, by setting

\[
u_{N}(\mathbf{X}) = N^{3m/4} \exp(i N \mathbf{X} \cdot \mathbf{V}_0) u(N^{1/2}(\mathbf{X} - \mathbf{X}_0)), \quad \mathbf{X} \in \mathbb{R}^{3m}, \quad N \geq 1.
\]

In particular, \( \| u_N \|_{L^2} = \| u \|_{L^2} \) for each \( N \).

In (1)–besides regularity requirements—we ask the tracer particle’s wave function to be such that its position and velocity distribution converge to a Dirac delta, centered at the phase space point \( (\mathbf{X}_0, \mathbf{V}_0) \in \mathbb{R}^{3m} \times \mathbb{R}^{3m} \). In particular, we choose the fastest rate of convergence that is in agreement with Heisenberg’s uncertainty principle: \( \Delta \mathbf{X} \Delta \mathbf{V} \geq N^{-1} \). Note that convergence is rather strong, since it includes up to six moments in the position and momentum observables. Our aim is to understand how these two conditions are propagated in time, provided one replaces \( (\mathbf{X}_0, \mathbf{V}_0) \) with \( (\mathbf{X}_t, \dot{\mathbf{X}}_t) \).
As for the bosons, in (2) we ask the initial datum to be a coherent state; each n-particle entry may be described as a symmetrization of the boson field \( \varphi_0 \), in the sense that

\[
\mathcal{W}(\sqrt{N}\varphi_0)\Omega = \exp(-N/2) \sum_{n=0}^{\infty} \frac{N^{n/2}}{\sqrt{n!}} \varphi_0^\otimes n.
\]

(2.4)

The time evolution of coherent states has been widely studied in the literature, and the following approach is now standard. We introduce the one-particle density

\[
\Gamma_{N,t}(x,y) := \frac{1}{N} \langle \Psi_{N,t}, \mathbb{1} \otimes a_x^\dagger a_y \Psi_{N,t} \rangle_{\mathscr{H}}, \quad (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3
\]

(2.5)

and we let \( \Gamma_{N,t} \) be the linear operator in \( L^2(\mathbb{R}^3) \) whose integral kernel is \( \Gamma_{N,t}(x,y) \). In particular, this operator is positive, self-adjoint and trace-class. Similarly, for the solution of the mean-field equations \( \varphi_t \in L^2(\mathbb{R}^3) \), we let \( |\varphi_t\rangle \langle \varphi_t| \) be the orthogonal projection onto the subspace spanned by \( \varphi_t \). One is then interested in the convergence of \( \Gamma_{N,t} \) towards \( |\varphi_t\rangle \langle \varphi_t| \).

The class of potentials that we work with is described in the following Condition. Most notably, we are able to include Coulomb potentials in the boson-boson interaction. On the other hand, requiring three derivatives in the boson-tracer interaction is, almost certainly, not optimal.

**Condition 2** (Potentials). We assume \( w : \mathbb{R}^3 \to \mathbb{R} \) and \( v : \mathbb{R}^3 \to \mathbb{R} \) are even, real-valued functions that satisfy

1. \( w \in C^3_b(\mathbb{R}^3) \).
2. \( v = v_1 + v_2 \) where \( v_1 = \lambda|x|^{-1} \) for some \( \lambda \in \mathbb{R} \) and \( v_2 \in L^\infty(\mathbb{R}^3) \).

Our main results are the following two theorems.

**Theorem 2.1.** Let \( (\Psi_{N,t})_{N \geq 1} \) satisfy Condition 1 with respect to \( (X_0, V_0, \varphi_0) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3) \) and assume the potentials \( v \) and \( w \) satisfy Condition 2. Let \( (X_t, \varphi_t) \) solve the mean-field equations (1.14), with the same initial data, and let \( \Psi_{N,t} = \exp(-itH_N)\Psi_{N,0} \) solve the microscopic dynamics (1.9). Then, there exists \( C > 0 \) such that for all \( t \geq 0 \) and \( N \geq 1 \) it holds that

\[
|\langle \Psi_{N,t}, X \otimes \mathbb{1} \Psi_{N,t} \rangle_{\mathscr{H}} - X_t| \leq \frac{Ce^{Ct}}{N^{1/4}}.
\]

(2.6)

\[
|N^{-1} \langle \Psi_{N,t}, P \otimes \mathbb{1} \Psi_{N,t} \rangle_{\mathscr{H}} - \dot{X}_t| \leq \frac{Ce^{Ct}}{N^{1/4}}.
\]

(2.7)

**Theorem 2.2.** Under the same conditions of Theorem 2.1, there exists \( C > 0 \) such that for all \( t \geq 0 \) and \( N \geq 1 \) it holds that

\[
\text{Tr} |\Gamma_{N,t} - |\varphi_t\rangle \langle \varphi_t| | \leq \frac{Ce^{Ct}}{N^{1/4}}.
\]

(2.8)

**Remark 2.2.** The conclusion of Theorem 2.1 characterizes the limits of the first moments of the probability measures

\[
\mu_{N,t}^X(E) := \| \mathbb{1}(X \in E)\Psi_{N,t} \|_{\mathscr{H}}^2 \quad \text{and} \quad \mu_{N,t}^V(E) := \| \mathbb{1}(N^{-1}P \in E)\Psi_{N,t} \|_{\mathscr{H}}^2
\]

(2.9)

where \( E \in \mathcal{B}(\mathbb{R}^{3m}) \) is a Borel set. Since we are actually able to control their variance in Lemma 4.1, it follows easily that there is weak-* convergence as \( N \to \infty \):

\[
\mu_{N,t}^X \rightharpoonup^* \delta_{X_t} \quad \text{and} \quad \mu_{N,t}^V \rightharpoonup^* \delta_{\dot{X}_t}.
\]

(2.10)
Remark 2.3. In the particular situation of Theorem 2.2, the Hilbert-Schmidt norm controls the trace norm. Namely, for the linear operator $S := \Gamma_{N,\ell} - |\varphi_l\rangle\langle\varphi_l|$ it holds that
\[
\text{Tr}|S| \leq 2|S|_{\text{HS}}. \tag{2.11}
\]
Therefore, we will actually prove convergence in Hilbert-Schmidt norm and then apply (2.11).

Remark 2.4 (Convergence Rates). If one compares Condition 1 for $p = 1$, with the result of Theorem 2.1, it is evident that there is a loss of a factor $N^{-1/4}$ in the rate of convergence. For $v = 0$, it follows easily from our methods that optimal convergence rate is achieved and we only need $u_{N,0} \in \mathcal{D}(X) \cap \mathcal{D}(P)$ with (1) in Condition 1 valid only up to $p = 1$. For $v \neq 0$, we can achieve optimal convergence rate provided we take the following stronger assumptions: (i) $u_{N,0} \in \mathcal{D}(X^4) \cap \mathcal{D}(P^4)$ with (1) in Condition 1 valid for $p = 4$ and (ii) $w \in C^4_b(\mathbb{R}^3)$. This last statement, however, requires more work as one needs to deal with higher powers of the observables $X$ and $P$.

The author did not investigate the optimality of the rate of convergence in Theorem 2.2. We believe, however, the optimal rate to be $N^{-1}$ provided one assumes $p$ to be large enough in Condition 1, and $w$ to be regular enough in Condition 2.

Remark 2.5 (Tracer-tracer Interactions). With our methods, it would be possible to include interactions between the tracer particles $V(X^{(i)} - X^{(\ell)})$, provided the potential function is regular enough. We decide to omit such a term in order to highlight the more interesting boson-tracer interaction $w(x, X)$ and the more technical boson-boson term $v(x - y)$.

Our contribution. There have been several studies in the literature that consider derivation of effective equations for many-particle systems of interacting bosons. The classical references are [22, 34, 16, 17]. More recently, [9, 10, 11, 12] together with [26, 33] have motivated much of the activity in the field; for a non-exhaustive list we refer to [29, 30, 7, 2, 5, 6, 3, 4, 18, 21, 20, 19, 25, 31].

On the other hand, there have been only few studies of systems that include interactions between bosons and one or more tracer particle. Most remarkably:

(1) In [8], Deckert, Fröhlich, Pizzo and Pickl consider the $(N + 1)$-particle Hilbert space $L^2(\mathbb{R}^{3(N+1)})$ with the $(N + 1)$-particle Hamiltonian $H_{N,N}$ for $v = 0$, i.e. an ideal (non-interacting) Bose gas. They study the case for which $\varphi_0$ is localized in a region that expands as $N \to \infty$ and derive a slightly different system of mean-field equations. We note that they consider only $C^4_b$ potentials and initial data, but include a smooth tracer-tracer interaction term.

(2) In [7], Chen and Soffer consider our same model for the case in which there is only one tracer particle, i.e. $m = 1$. The main advantage of this model comes from translation invariance of the system: one can decompose the Hilbert space into blocks of constant total momentum $P \in \mathbb{R}^3$, i.e. there is a direct integral representation
\[
\mathcal{H} = \int_{\mathbb{R}^3} \mathcal{H}(P) dP
\]
that allows to mod out the tracer particle dependence in the diagonal term $\int_{\mathbb{R}^3} w(x - X)a_x^*a_x dx$, see their equation (1.11). Notably, they are able to give a mean-field description that converges in Fock space norm once the intertwining between the bosons and the tracer particle’s momenta has been understood.
(3) Very recently, Lampart and Pickl in [27] studied the mean-field limit of a boson gas coupled to a tracer particle, whose mass remains finite. They are able to derive a norm approximation of the microscopic dynamics via Bogoliubov theory of the excitations of the condensate.

(4) In the context of quantum friction/Cherenkov radiation, interactions between tracer particles and ideal gases of bosons have been studied by Fröhlich and Gang in [15, 13, 14]. Their starting point are the effective equations derived in [8].

Our main contribution in this paper can be summarized as follows. Once translational symmetry is broken, the argument of [7] breaks down and a new approach towards finding a mean-field description is needed; we develop this approach while including, at the same time, the possibility for the Bose gas to be non-ideal, including the case of Coulomb interactions.

2.1.1. Idea of the proof. The proof of our main results is heavily inspired by [8] (to control expectations of tracer particle’s observables) and by [33] (to control vacuum fluctuations) and the idea is as follows. First, we introduce fluctuation states \( \Omega_{N,t} \in \mathcal{H} \) that show up naturally when trying to replace \( a_x \) with \( \sqrt{N} \varphi_t(x) \). However, a major difficulty arises when one writes down its infinitesimal generator. Namely, there is a (non-diagonal) term of the form

\[
\sqrt{N} \int_{\mathbb{R}^3} \left( w(x, X) - w(x, X_t) \right) \otimes (a_x \varphi_t(x) + a_x^* \varphi_t(x)) \, dx
\]

which has an apparent growth of \( \sqrt{N} \). In other words, the presence of tracer particles can potentially create or destroy a lot of bosons for states close to the vacuum. In order to control this term, we study simultaneously the difference \( X - X_t \) and apply a Taylor estimate on this interaction \( w(x - X) \). Note that, at the same time, this requires control of the difference \( N^{-1} P - X_t \) thanks to Ehrenfest’s Theorem. Subsequently, once that control of \( \Omega_{N,t} \) has been established, important remainder terms can be estimated and the proof of our main results follows in a few pages.

2.1.2. Organization of the paper. In Section 3 we introduce the fundamental notion of a fluctuation state, together with its regularized version. In Section 4 we prove our main results, based on particle number estimates satisfied by fluctuation states. In Section 5 we complete the proof of these particle number estimates. In Sections 6 and 7 we prove some required well-posedness results.

2.1.3. Notation.

- \( C^k_b(\mathbb{R}^3) \) is the class of \( k \in \mathbb{N} \) times continuously differentiable functions, with all derivatives bounded. We equip it with the norm \( \| f \|_{C^k_b} := m \sum_{0 \leq |\alpha| \leq k} \| \partial^\alpha f \|_{L^\infty} \).

- Given two quantities \( A(N, t) \) and \( B(N, t) \), we say that \( A \lesssim B \) if there exists a constant \( C > 0 \), depending only on \( w, v, \varphi_0 \) and \( C_0 \), such that

\[
A(N, t) \leq C B(N, t), \quad \forall N \geq 1, \quad \forall t \geq 0.
\]

(2.14)

- We drop the tensor product symbol \( \otimes \) for products of operators.
- We drop the subscript \( \mathcal{H} \) associated to the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), whenever there is no risk of confusion.
- \( \chi(A) \) stands for the characteristic function of the measurable set \( A \subset \mathbb{R} \).
- Given a Banach space \( E \), we denote by \( B(E) \) the Banach space of bounded, linear maps \( E \to E \), equipped with its usual norm \( \| \cdot \|_{B(E)} \).
2.1.4. Fundamenta l estimates. Let us give a list of estimates that will be used throughout this work, together with close variations of them; since they are well-known, we omit their proof. In the sequel, we shall refer to them in italics, i.e. as Particle Number Estimates, Potential Estimates, etc. 

**Taylor Estimates.** For all \( x, y \in \mathbb{R}^3 \) it holds that
\[
|w(x) - w(y)| + |\nabla w(x) - \nabla w(y)| \lesssim |x - y|. \tag{2.15}
\]

**Particle Number Estimates.** For all \( \Phi \in \mathcal{H} \) and \( f = f(x, X) \) it holds
\[
\left\| \int_{\mathbb{R}^3} f(x, X) a_x^* a_x \Phi \right\| \leq \sup_{X \in \mathbb{R}^{3m}} \|f(\cdot, X)\|_{L^2} ((N_0 + 1)^{1/2} \Phi). \tag{2.16}
\]
\[
\left\| \int_{\mathbb{R}^3} f(x, X) a_x^* a_x \Phi \right\| \leq \|f\|_{L^2} \|N_0 \Phi\|. \tag{2.17}
\]

In addition, for all \( \Phi \in \mathcal{H} \) and \( \phi_1 = \phi_1(x) \) and \( \phi_2 = \phi_2(y) \) it holds
\[
\left\| \int_{\mathbb{R}^6} v(x - y) \phi_1(x) \phi_2(y) a_x^* a_x dx dy \Phi \right\| \lesssim \left( \int_{\mathbb{R}^6} v(x - y)^2 |\phi_1(x)|^2 |\phi_2(y)|^2 dx dy \right)^{1/2} (N_0 + 1) \Phi, \tag{2.18}
\]
\[
\left\| \int_{\mathbb{R}^6} v(x - y) \phi_1(x) a_x^* a_x dx dy \Phi \right\| \lesssim \left( \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} v(x - y)^2 |\phi_1(x)|^2 dy \right)^{1/2} (N_0 + 1)^{3/2} \Phi. \tag{2.20}
\]

**Potential Estimates.** Let \( v \) satisfy Condition 2. Then, the following operator inequality holds
\[
\left\langle \varphi, v^2 \varphi \right\rangle \leq |v|^2 \|\varphi\|_{H^1}^2, \quad \varphi \in H^1(\mathbb{R}^3), \tag{2.19}
\]
for some \(|v| > 0\). Consequently, for any \( \varphi \in H^1(\mathbb{R}^3) \) it holds
\[
\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} v(x - y)^2 |\varphi(y)|^2 dy \leq |v|^2 \|\varphi\|_{H^1}^2. \tag{2.20}
\]
\[
\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} v(x - y) |\varphi(y)|^2 dy \leq |v| \|\varphi\|_{L^2} \|\varphi\|_{H^1}. \tag{2.21}
\]
\[
\int_{\mathbb{R}^6} v(x - y)^2 |\varphi(x)|^2 |\varphi(y)|^2 dy dx \leq |v|^2 \|\varphi\|_{L^2}^2 \|\varphi\|_{H^1}^2. \tag{2.22}
\]

**Local Lipschitz Property.** Let \( v \) satisfy Condition 2 and define the map \( J(\varphi) := (v \ast |\varphi|^2) \varphi \) for \( \varphi \in H^1(\mathbb{R}^3) \). Then, the Leibniz rule and (2.21) imply that for all \( \varphi, \psi \in H^1(\mathbb{R}^3) \)
\[
\|J(\varphi) - J(\psi)\|_{H^1} \leq L(v) \left( \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 \right) \|\varphi - \psi\|_{H^1}. \tag{2.23}
\]
where \( L(v) = C|v| \). For a detailed account on the Lipschitz property for the case \( v = \lambda|x|^{-\gamma} \) with \( \gamma \in (0, 3/2) \), we refer the reader to [23, Lemma A.2], whose analysis follows [28, Lemma 1].

3. Fluctuation States

Let \( \Psi_{N, t} = \exp(-it\mathcal{H}_N)\Psi_{N, 0} \) and \((X_t, \varphi_t)\) be the solutions of (1.9) and (1.14), respectively. In this section, we study the number of particles present in the fluctuation state
\[
\Omega_{N, t} := e^{-iS(t)} \mathcal{W}(\sqrt{N}\varphi_t) \exp(-it\mathcal{H}) \mathcal{W}(\sqrt{N}\varphi_0)(u_{N, 0} \otimes \Omega), \quad t \in \mathbb{R}, \ N \geq 1. \tag{3.1}
\]
Here, \( S(t) \) is a scalar term that modifies \( \Omega_{N, t} \) only by an overall phase; it is not a physically measurable quantity. We choose it to be
\[
S(t) := N \int_0^t \int_{\mathbb{R}^3} \left( w(x, X_s) |\varphi_s(x)|^2 + \frac{1}{2} (\varphi_s^*(x) \varphi_s(x) |\varphi_s(x)|^2) dx ds \right) \tag{3.2}
\]
The dynamics that drive states similar to $\Omega_{N,t}$ has been studied extensively, with the first rigorous study being carried out in [16] and more recently re-activated in [33]. When no tracer particle is present, the usual infinitesimal generator consists of kinetic terms, plus second quantized operators that are polynomials of order bigger than or equal to two in the creation and annihilation operators $a_x$ and $a_x^*$; the scalar term $S(t)$ and the boson field $\varphi_t$ can be chosen so that no zero or first order terms are present. However, when interacting with tracer particles, the linear terms that arise from $\int \bar{w}(x,X)a_x^*a_x dx \dagger$ cannot be removed unless one is willing to introduce a $X$-dependent boson field $\varphi_t(x,X)$. Note that doing so would have the effect of making $-D_X$ not commute with the Weyl operator $\mathcal{W}(\sqrt{N}\varphi_t(X))$. We believe that the consequences outweigh the benefits and shall not take this approach.

Our main result concerning particle number estimates for fluctuation states is the following theorem.

**Theorem 3.1.** Let $\Omega_{N,t}$ be the fluctuation state defined in (3.1), with initial data satisfying Condition 1. Then, there is a constant $C > 0$ such that for all $t \geq 0$ and $N \geq 1$ the following estimates hold

\[
\|N_b^{1/2}\Omega_{N,t}\|^2 \leq Ce^{Ct}N^{1/2}, \tag{3.3}
\]
\[
\|N_b\Omega_{N,t}\|^2 \leq Ce^{Ct}N^{3/2}. \tag{3.4}
\]

**Remark 3.1.** Even though we are not able to show that the fluctuation state $\Omega_{N,t}$ has particle number moments that are uniformly bounded in $N \geq 1$, we are able to reduce by a factor $N^{1/2}$ the a priori growth of these moments. This in turn is enough to prove our main theorems, the price to pay being having a worse convergence rate with respect to the initial data.

The proof of Theorem 3.1 consists in a detailed study of the linear generator that drives the time evolution of $\Omega_{N,t}$; let us then describe it in more detail. The reader should be aware that we will regularize the problem in a way that the following formal calculations are properly justified— for the moment we focus on the algebraic aspects of the proof.

First, note that $\Omega_{N,t} = \mathcal{U}_N(t,0)(\psi_{N,0} \otimes \Omega)$, where we introduce on $\mathcal{H}$ the following two-parameter family of unitary transformations

\[
\mathcal{U}_N(t,s) := e^{-iS(t)}\mathcal{W}^*\sqrt{N}\varphi_t \exp(-i(t-s)\mathcal{H}_N)e^{iS(s)}\mathcal{W}(\sqrt{N}\varphi_s), \quad t, s \in \mathbb{R}, \quad N \geq 1. \tag{3.5}
\]

In particular, straightforward differentiation with respect to time $t$ gives

\[
i\partial_t\mathcal{U}_N(t,s) = \left( \mathcal{W}_{N,t}^* \mathcal{H}_N \mathcal{W}_{N,t} + i\partial_t \mathcal{W}_{N,t}^* \mathcal{W}_{N,t} - i\dot{S}(t) \right) \mathcal{U}_N(t,s) =: \mathcal{L}(t) \mathcal{U}_N(t,s) \tag{3.6}
\]

where we have introduced the short-hand notation $\mathcal{W}_{N,t} := \mathcal{W}(\sqrt{N}\varphi_t)$. In order to get a more explicit representation of the linear generator $\mathcal{L}(t)$— defined in (3.6)—we proceed as follows. First, since $(X_t, \varphi_t)$ satisfies the mean field equations (1.14), the time derivative of the Weyl operator may be computed as

\[
i\partial_t \mathcal{W}_{N,t} = [\mathcal{H}_{X,\varphi}(t), \mathcal{W}_{N,t}] \tag{3.7}
\]
\[
\mathcal{H}_{X,\varphi}(t) = T_b + \int_{\mathbb{R}^3} w(x, X_t) a_x^*a_x dx + \int_{\mathbb{R}^3} (v*|\varphi_t|^2)(x) a_x^*a_x dx, \tag{3.8}
\]

\[\text{1In [7], the } X \text{ dependence is cleverly eliminated from the very beginning, thanks to the translational symmetry of the model.}\]
see for reference [16, Lemma 3.1]. Moreover, a straightforward, although tedious, application of the translation properties

\[ W_{N,t}^* a_x W_{N,t} = a_x + \sqrt{N} \varphi_t(x) \quad \text{and} \quad W_{N,t}^* a_x^* W_{N,t} = a_x^* + \sqrt{N} \varphi_t(x) \]  

in the second-quantized operators of (3.6), let us cast the infinitesimal generator of the dynamics in the form

\[ \mathcal{L}(t) = -\frac{\Delta x}{2N} + T_b + \mathcal{I}(t) + \mathcal{I}_4. \]  

Here, the first two terms correspond to the kinetic energy of the system. The third term corresponds to the time-dependent operator

\[ \mathcal{I}(t) := \mathcal{L}^{(d)}(t) + \mathcal{L}^{(od)}(t) \]  

that contains the diagonal and off-diagonal interaction terms, given by

\[ \mathcal{L}^{(d)}(t) := \int_{\mathbb{R}^3} \left( (w(x, x) - w(x, X_t)) (N|\varphi_t(x)|^2 - a_x^*a_x) \right) \, dx \]

\[ + \int_{\mathbb{R}^3} (v * |\varphi_t|^2)(x) a_x^*a_x \, dx \]

\[ + \int_{\mathbb{R}^6} v(x-y)\varphi_t(x)\varphi_t(y) a_y^*a_x^*a_ya_x \, dy \]

\[ \mathcal{L}^{(od)}(t) := \sqrt{N} \int_{\mathbb{R}^3} \left( (w(x, x) - w(x, X_t)) (a_x^*\varphi_t(x) + a_x\varphi_t^*(x)) \right) \, dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^6} v(x-y)(\varphi_t(x)\varphi_t(y)a_y^*a_y^* + \varphi_t(x)\varphi_t(y)a_x^*a_y) \, dy \]

\[ + \frac{1}{\sqrt{N}} \int_{\mathbb{R}^6} v(x-y) a_x (\varphi_t(y)a_y^* + \varphi_t(y)a_y) a_x \, dy. \]

The fourth term in (3.10) is the time-independent, diagonal quartic term

\[ \mathcal{I}_4 := \frac{1}{2N} \int_{\mathbb{R}^6} v(x-y)a_x^*a_y^*a_xa_y \, dy. \]

Note that both the kinetic terms, \( \mathcal{L}^{(d)}(t) \) and \( \mathcal{I}_4 \) commute with the particle number operator \( N_b \), but \( \mathcal{L}^{(od)}(t) \) does not. In other words, \( [\mathcal{L}(t), N_b] = [\mathcal{L}^{(od)}(t), N_b] \).

The main difficulty of the present article is understanding the first term of \( \mathcal{L}^{(od)}(t) \) as it involves a \( \sqrt{N} \) factor. To control the other two terms, we proceed very similarly as in [33].

Namely, we introduce a truncated dynamics whose generator is \( \mathcal{L}(t) \) but with a cut-off in the interaction term; this is described in Subsection 3.2. First, we regularize the problem at hand.

3.1. \textbf{Regularization of the Problem.} The purpose of regularizing the problem is twofold: (1) all of the upcoming calculations are rigorously justified, and (2) the proof of well-posedness for the (auxiliary) truncated dynamics is short. In this subsection, we show how to regularize both the microscopic dynamics (1.9) and the mean-field equations (1.14). This in turn is enough to regularize the fluctuation state \( \Omega_{N,t} \).

Let us be more precise. Consider some initial datum \( \Psi_{N,0} = u_{N,0} \otimes W(\sqrt{N}\varphi_0) \) satisfying Condition 1, and let \( v = \lambda|x|^{-1} + v_2 \) and \( w \) be potentials satisfying Condition 2. We will regularize these objects according to the following definition.

\textbf{Definition 3.2. In what follows, we always assume that} \( \varepsilon \in (0,1) \).

\( 1 \) We let \( w^\varepsilon \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}) \) be a Schwartz class function such that

\[ \lim_{\varepsilon \downarrow 0} \|w - w^\varepsilon\|_{C^3_b} = 0. \]
Remark 3.1. If the potential estimates imply that the \(\varepsilon\)-regular\(\varepsilon\)-regularized mean-field equations are of rather technical character, we postpone it to Appendix \(\varepsilon\).

**Lemma 3.1.** Let \(\Psi_0^\varepsilon \in H^2(\mathbb{R}^3)\) be an initial datum for the boson field that satisfies

\[
\lim_{\varepsilon \to 0} \| \Psi_0^\varepsilon - \Psi_0^\varepsilon \|_{H^1} = 0.
\]

(4) For each \(N \geq 1\), we let \(u_{N,0}^\varepsilon \in \mathcal{S}(\mathbb{R}^3)\) be an initial datum that satisfies

\[
\lim_{\varepsilon \to 0} \| (|X|^3 + |P|^3)(u_{N,0} - u_{N,0}^\varepsilon) \|_{L^2} = 0.
\]

**Remark 3.2.** For the boson-boson interaction, the pointwise bound \((\varepsilon^2 + x^2)^{-1/2} \leq |x|^{-1}\) implies that the Potential Estimates and the Local Lipschitz Property are both satisfied by \(v^\varepsilon\), uniformly in \(\varepsilon\). On the other hand, since giving explicit formulas for the other regularized objects is not enlightening, we refrain to do so.

### 3.1.1. Regularization of the Microscopic Dynamics

In the sequel, we refer to the solution \(\Psi_{N,t}^\varepsilon = \exp(-it\mathcal{H}_{N}^\varepsilon)\Psi_{N,0}^\varepsilon\) of the Schrödinger equation

\[
\begin{aligned}
&i\partial_t \Psi_{N,t}^\varepsilon = \mathcal{H}_{N}^\varepsilon \Psi_{N,t}^\varepsilon, \\
&\Psi_{N,t}^\varepsilon|_{t=0} = u_{N,0}^\varepsilon \otimes \mathcal{W}(\sqrt{N}) \Omega
\end{aligned}
\]

as the regularized microscopic dynamics, where the Hamiltonian \(\mathcal{H}_{N}^\varepsilon\) is given by (1.6) but with \((v^\varepsilon, w^\varepsilon)\) replacing the original potentials \((v, w)\). In particular, \(\Psi_{N,t}^\varepsilon \in \mathcal{D}(\mathcal{H}_{N}^\varepsilon)\) for all \(t \geq 0\) and (3.19) holds in the strong sense.

The next result establishes the validity of the approximation when \(\varepsilon \downarrow 0\). Since the proof is of rather technical character, we postpone it to Appendix A.

**Lemma 3.1.** Let \(\Psi_{N,t}^\varepsilon\) and \(\Psi_{N,t}^\varepsilon\) be the microscopic dynamics and the regularized microscopic dynamics, respectively. Then, for all \(N \geq 1\) and \(t \geq 0\) it holds that

\[
\lim_{\varepsilon \to 0} \| \Psi_{N,t}^\varepsilon - \Psi_{N,t}^\varepsilon \|_{\mathcal{H}} = 0.
\]

### 3.1.2. Regularization of the Mean-field Equations

In the sequel, we consider a regularized pair of mean-field variables \((X_t^\varepsilon, \varphi_t^\varepsilon)\) satisfying the coupled equations

\[
\begin{aligned}
&\dot{X}_t^\varepsilon = -\int_{\mathbb{R}^3} \nabla x w^\varepsilon(x, X_t^\varepsilon) |\varphi_t^\varepsilon(x)|^2 dx, \\
&i\partial_t \varphi_t^\varepsilon = -\Delta \varphi_t^\varepsilon + w^\varepsilon(x, X_t^\varepsilon) \varphi_t^\varepsilon + (v^\varepsilon |\varphi_t^\varepsilon|^2) \varphi_t^\varepsilon, \\
&(X_{t=0}^\varepsilon, X_{t=0}^\varepsilon, \varphi_0^\varepsilon) = (X_0, V_0, \varphi_0^\varepsilon),
\end{aligned}
\]

from now on referred to as the regularized mean-field equations. In particular, as proven in Section 6, the regularized boson field \(\varphi_t^\varepsilon\) remains in \(H^2(\mathbb{R}^3)\) for all later times and (3.21) holds in the strong sense.

The following result is the analogous of Lemma 3.1. For the same reasons, its proof gets postponed to Appendix A.

**Lemma 3.2.** Let \((X_t^\varepsilon, \varphi_t^\varepsilon)\) and \((X_t^\varepsilon, \varphi_t^\varepsilon)\) be the mean-field variables and regularized mean-field variables, respectively. Then, it holds that for all \(t \geq 0\)

\[
\lim_{\varepsilon \to 0} \left( |X_t - X_t^\varepsilon| + |\dot{X}_t - \dot{X}_t^\varepsilon| + \| \varphi_t - \varphi_t^\varepsilon \|_{H^1} \right) = 0.
\]
3.1.3. Regularization of the Fluctuation State. In terms of the regularized microscopic dynamics \( \Psi_{N,t}^\varepsilon \) and the regularized mean-field variables \((X_t^{\varepsilon}, \varphi_t^{\varepsilon})\), we define the regularized fluctuation state as

\[ \Omega_{N,t} := e^{-iS^\varepsilon(t)} \mathcal{W}(\sqrt{N} \varphi_t^{\varepsilon}) \exp(-it \mathcal{H}_N^\varepsilon) \mathcal{W}(\sqrt{N} \varphi_0^{\varepsilon})(u_{N,0}^{\varepsilon} \otimes \Omega), \quad N \geq 1, \ t \in \mathbb{R}. \quad (3.23) \]

Here, \( S^\varepsilon(t) \) is the regularized version of the scalar term \( S(t) \) defined in (3.2). The following result establishes the validity of the approximation \( \varepsilon \downarrow 0 \)

**Lemma 3.3.** Let \( t \geq 0, \ N \geq 1 \) and \( k \in \mathbb{N} \). Then, the following holds:

1. \( \lim_{\varepsilon \downarrow 0} \| \Omega_{N,t} - \Omega_{N,t}^\varepsilon \|_\mathcal{H} = 0. \)
2. \( \|N_b^k \Omega_{N,t} - \Omega_{N,t}^\varepsilon\| + \|N_b^k \Omega_{N,t} - \Omega_{N,t}^\varepsilon\| \leq N^k \) uniformly in \( t \in \mathbb{R} \) and \( \varepsilon \in (0,1). \)
3. \( \lim_{\varepsilon \downarrow 0} \|N_b^{k/2}(\Omega_{N,t} - \Omega_{N,t}^\varepsilon)\|_\mathcal{H} = 0. \)

**Proof.** (1) Since \( \|u_{N,0}\|_{L^2_X} \) and \( \|\mathcal{W}(\sqrt{N} \varphi_0^{\varepsilon})\|_{\mathcal{F}_b} \) are uniformly bounded in \( N \geq 1 \) and \( \varepsilon \in (0,1) \), one obtains that

\[ \|\Omega_{N,t} - \Omega_{N,t}^\varepsilon\|_\mathcal{H} \leq \left\| \left( \mathcal{W}(\sqrt{N} \varphi_t) - \mathcal{W}(\sqrt{N} \varphi_t^{\varepsilon}) \right) \varphi_{N,t}^\varepsilon \right\|_\mathcal{H} + \|\varphi_{N,t} - \varphi_{N,t}^{\varepsilon}\|_\mathcal{H} + |S(t) - S^\varepsilon(t)|. \quad (3.24) \]

It is known [16, Lemma 3.1] that the map \( f \in L^2 \mapsto \mathcal{W}(f)\Phi \) is continuous for every \( \Phi \in \mathcal{F}_b \).

Since \( \varphi_t \to \varphi_t^{\varepsilon} \) in \( L^2 \), this implies that the first term vanishes in the limit. The \( \varepsilon \downarrow 0 \) limit of the second term is zero, thanks to Lemma 3.1. Finally, since \( \varphi_t \to \varphi_t^{\varepsilon} \) in \( H^1 \), one may easily adapt the proof of Lemma 3.2 to show that \( S(t) = \lim_{\varepsilon \downarrow 0} S^\varepsilon(t) \). Thus, \( \Omega_{N,t} = \mathcal{H} - \lim_{\varepsilon \downarrow 0} \Omega_{N,t}^{\varepsilon}. \)

(2) The proof may be adapted from the weak bounds presented in [33, Lemma 3.6]; one only needs to use the translation property (3.9) of the Weyl operator \( \mathcal{W}(\sqrt{N} \varphi_t) \), combined with the uniform-in-time growth \( \|N_b^k \varphi_{N,t}^\varepsilon\| \leq N^k. \)

(3) The Cauchy-Schwarz inequality implies

\[ \|N_b^{k/2}(\Omega_{N,t} - \Omega_{N,t}^\varepsilon)\|_\mathcal{H}^2 \leq \left( \|N_b^k \Omega_{N,t}^\varepsilon\|_\mathcal{H} + \|N_b^k \Omega_{N,t}^\varepsilon\|_\mathcal{H} \right) \|\Omega_{N,t} - \Omega_{N,t}^\varepsilon\|_\mathcal{H}. \quad (3.25) \]

It suffices to apply the first two parts of the lemma. \( \square \)

3.2. The Truncated Dynamics. Let \( M \geq 1 \), from now on referred to as the *cut-off parameter*. Given a solution \((X_t^{\varepsilon}, \varphi_t^{\varepsilon})\) of the mean-field equations, we introduce on \( \mathcal{H} \) the auxiliary time-dependent operator

\[ L_M^\varepsilon(t) := -\frac{\Delta_X}{2N} + T_b + \chi(N_b \leq M) I_M^\varepsilon(t) + I_M^\varepsilon. \quad (3.26) \]

Here, the new \( \varepsilon \)-dependent operators are defined according to the formulae (3.10)–(3.14), but with the regularized potentials and mean-field variables replacing the old ones, i.e. we change

\[ (v,w,X_t,\varphi_t) \to (v^{\varepsilon},w^{\varepsilon},X_t^{\varepsilon},\varphi_t^{\varepsilon}). \]

We shall refer to \( I_M^\varepsilon(t) := \chi(N_b \leq M) I_M^\varepsilon(t) \) as the truncated interaction term; we also refer to \( L_M^\varepsilon(t) \) as the truncated generator. As will become clear later on, the particle number cut-off will let us “close equations” when trying to run a Grönwall-type argument. In particular, estimates will be given in terms of both \( M \) and \( N \) and we will choose \( M = N \) at the end of the calculations. Let us note that besides \( L_M^\varepsilon(t) \) one could have chosen different auxiliary operators. For instance, one could have considered introducing more or less terms with a cut-off in particle number. In particular, we decide not to introduce a cut-off in the quartic term because we do not have good operator bounds on \( I_M \) (\( v^{\varepsilon} \) is not bounded uniformly in \( \varepsilon \))—it is, however, diagonal and time independent.
In Section 7, we prove that to the truncated generator we can associate a unique unitary propagator\(^2\) that satisfies
\[
i\partial_t U_N^{(\varepsilon,M)}(t,s) = \mathcal{L}_M^{\varepsilon}(t) U_N^{(\varepsilon,M)}(t,s) \quad \text{and} \quad U_N^{(\varepsilon,M)}(t,t) = \mathbb{1}, \quad \forall t,s \in \mathbb{R}. \quad (3.27)
\]
We show that this equation is well-posed and establish basic results related to propagation of regularity—see Proposition 7.1 and 7.2. Our main ingredients are the uniform boundedness of the interaction term, together with the map \(t \in \mathbb{R} \mapsto \mathcal{I}_M^{\varepsilon}(t) \in \mathcal{B} (\mathcal{H}) \) being locally Lipschitz; see Lemma 7.1.

For the rest of the section, we focus on the case where the initial condition that is being evolved satisfies (up to regularization) Condition 1, i.e. we introduce
\[
\Omega_{N,t}^{(\varepsilon,M)} := U_N^{(\varepsilon,M)}(t,0)(u_{N,0}^{\varepsilon} \otimes \Omega), \quad t \geq 0, \; N, M \geq 1, \; \varepsilon \in (0,1). \quad (3.28)
\]
One of our most important estimates is contained in the following proposition; it is concerned with particle number estimates for \(\Omega_{N,t}^{(\varepsilon,M)}\). Note that, if one chooses \(M = N\), we obtain estimates for the third particle number moment that are uniform in \(N \geq 1\).

**Proposition 3.1.** Let \(\Omega_{N,t}^{(\varepsilon,M)}\) be as above, with \((u_{N,0}^{\varepsilon})_{N \geq 1}\) being regularized initial data, satisfying Condition 1. Then, there is \(C > 0\) such that for all \(t \geq 0\) and \(M, N \geq 1\) it holds that
\[
\limsup_{\varepsilon \downarrow 0} \| N_b^{3/2} \Omega_{N,t}^{(\varepsilon,M)} \| \leq C \exp \left( C t \left(1 + M/N\right)^2 \right), \quad (3.29)
\]
\[
\limsup_{\varepsilon \downarrow 0} \| N_b^2 \Omega_{N,t}^{(\varepsilon,M)} \| \leq C \sqrt{N} \exp \left( C t \left(1 + M/N\right)^2 \right). \quad (3.30)
\]

**Remark 3.3.** The proof of Proposition 3.1 contains the most intricate technical work of this article. This is due to the fact that the time evolution of powers of the particle number operator \(N_b\) get intertwined with powers of the tracer particle observables \(X\) and \(P\); we postpone its proof and dedicate Section 5 to it.

One of the main consequences of Proposition 3.1 is that it allows to control the difference between the original and the truncated evolution. The following result is the first step in this direction.

**Corollary 1.** Let \(\Omega_{N,t}^{(\varepsilon,M)}\) be as in Proposition 3.1. Then, there is \(C > 0\) such that for all \(t \geq 0\), and \(N = M \geq 1\) it holds that
\[
\limsup_{\varepsilon \downarrow 0} \| \chi(N_b > N) \mathcal{I}_t^{\varepsilon}(t) \Omega_{N,t}^{(\varepsilon,N)} \| \leq \frac{C e^{Ct}}{\sqrt{N}}. \quad (3.31)
\]

**Proof.** For simplicity, let us introduce the following notation \(\chi_N \equiv \chi(N_b > N)\) for the projection operator. Without loss of generality we take \(N \geq 2\) and \(t \geq 0\); they shall remain fixed. Moreover, since the upcoming estimates are uniform in \(\varepsilon \in (0,1)\), we shall not write it explicitly. Finally, we let \(\Phi\) denote a generic element of \(L^2_X \otimes \mathcal{D}(N_b^2)\) and we let \(C > 0\) denote a constant, independent of these quantities, whose value may change from line to line.

---

\(^2\)A strongly continuous, two-parameter family of unitary transformations \((t,s) \in \mathbb{R}^2 \mapsto U(t,s) \in \mathcal{B} (\mathcal{H})\) is said to be a unitary propagator, if \(U(t,s) = U(t,r)U(r,s)\) and \(U(t,t) = \mathbb{1}\).
First, we recall that $\mathcal{I}(t) = \mathcal{L}^{(d)}(t) + \mathcal{L}^{(od)}(t)$. For the diagonal part, we easily find thanks to our Particle Number Estimates and the Potential Estimates that
\[
\|\chi_N \mathcal{L}^{(d)}(t) \Phi\| = \|\mathcal{L}^{(d)}(t) \chi_N \Phi\| \\
\leq C \left( \|\chi_N \Phi\| + \|(N_b + 1) \chi_N \Phi\| \right).
\]
(3.32)
Next, for the non-diagonal term we use the relations $\chi_N a_z a_x = a_x \chi_N - 1$ and $\chi_N a_x^* a_z = a_x^* \chi_N - 1$ to find, thanks again to our Particle Number Estimates and the Potential Estimates, that
\[
\|\chi_N \mathcal{L}^{(od)}(t) \Phi\| \leq C \sqrt{N} \left( \|\chi_N (N_b + 1)^{1/2} \chi_N - 1 \Phi\| + \|\chi_N (N_b + 1)^{1/2} \chi_N + 1 \Phi\| \right) \\
+ C \left( \|\chi_N (N_b + 1)^{1/2} \chi_N - 2 \Phi\| + \|\chi_N (N_b + 1)^{1/2} \chi_N + 2 \Phi\| \right) \\
+ \frac{C}{\sqrt{N}} \left( \|\chi_N (N_b + 1)^{3/2} \chi_N - 1 \Phi\| + \|\chi_N (N_b + 1)^{3/2} \chi_N + 1 \Phi\| \right).
\]
(3.33)
In order to control the intermediate terms involving $N_b^j$ for $0 \leq j \leq 3/2$ we use the following inequalities:
\[
\|(N_b + 1)^j \chi(N_b > M) \Phi\| \leq \frac{1}{(M + 1)^j} \|(N_b + 1)^j \Phi\|, \quad 0 \leq j \leq J
\]
(3.34)
for $M = N - 2, \cdots, N + 2$ and $J = 3/2$. Gathering estimates we find that
\[
\|\chi(N_b > N) \mathcal{I}(t) \Phi\| \leq C \sqrt{N} \|(N_b + 1)^{3/2} \Phi\|.
\]
(3.35)
The last inequality for $\Phi = \Omega^{(N)}_{N, t}$, combined with Proposition 3.1, finishes the proof. \qed

3.3. Proof of Theorem 3.1. Starting from Proposition 3.1 (particle number estimates for the truncated dynamics), we can sacrifice growth in $N$ and prove Theorem 3.1 (particle number estimates for the original dynamics).

Proof of Theorem 3.1. Let $N \geq 1$, $\varepsilon \in (0, 1)$ and $t \geq 0$. The following proof is heavily inspired by that of [33, Lemma 3.7]; the idea is as follows: pick $k \in [0, 2]$ and write
\[
\|N_b^{k/2} \Omega_{N, t}^{(\varepsilon, N)}\|^2 = \|N_b^{k/2} (\Omega_{N, t}^{(\varepsilon, N)})\|^2 + \langle N_b \Omega_{N, t}^{(\varepsilon, N)} (\Omega_{N, t}^{(\varepsilon, N)}) \rangle + \langle N_b \Omega_{N, t}^{(\varepsilon, N)} (\Omega_{N, t}^{(\varepsilon, N)}) \rangle, \\
\leq \|N_b^{k/2} \Omega_{N, t}^{(\varepsilon, N)}\|^2 + \|N_b \Omega_{N, t}^{(\varepsilon, N)}\| \|\Omega_{N, t}^{(\varepsilon, N)}\|.
\]
(3.36)
We control each term separately. Since $k \leq 2$, the first one may be controlled using Proposition 3.1. For the second one, we use Lemma 3.3 and Proposition 3.1 to obtain
\[
\|N_b^{k/2} \Omega_{N, t}^{(\varepsilon, N)}\| + \|N_b \Omega_{N, t}^{(\varepsilon, N)}\| \leq \exp(Ct)(N_b^k + N^{1/2}), \quad t \in \mathbb{R}, \ N \geq 1, \ \varepsilon \in (0, 1).
\]
(3.37)
In addition, we may use Duhamel’s formula to find that
\[
\|\Omega_{N, t}^{(\varepsilon, N)} - \Omega_{N, t}^{(\varepsilon, N)}\| \leq \int_0^t \left( \|L^{(\varepsilon)}(s) - L^{(\varepsilon)}(s)\| \Omega_{N, s}^{(\varepsilon, N)}\| ds \leq \int_0^t \chi(N_b > N) L^{(\varepsilon)}(s) \Omega_{N, s}^{(\varepsilon, N)}\| ds,
\]
(3.38)
which, in combination with Corollary 1, controls the last term in (3.36). Thus, we gather estimates and find that
\[
\limsup_{\varepsilon \downarrow 0} \|N_b^{k/2} \Omega_{N, t}^{(\varepsilon, N)}\|^2 \leq \exp(Ct) \left(1 + N_b^{k - 1/2}\right).
\]
(3.39)
Thus, the proof of the theorem is complete after we use Lemma 3.3 and select $k = 1$ and $k = 2$ in the last estimate. \qed
4. Proof of our Main Results

Throughout this section, the assumptions made in Theorem 2.1 hold and we keep on using the same notation from the previous section. In particular, $\Omega_{N,t}$ denotes the fluctuation state defined in (3.1) and, thanks to Theorem 3.1, the following particle number estimates hold

\[
\frac{\|N_b \Omega_{N,t}\|}{N} + \frac{\|N_b^{1/2} \Omega_{N,t}\|}{N^{1/2}} \leq Ce^{Ct} N^{1/4}
\]

for some constant $C > 0$, independent of $N \geq 1$ and $t \geq 0$.

For $t \geq 0$, let us introduce the following operators to take into account the classical-quantum error associated to the tracer particle variables

\[
\Delta X(t) := X - X_t \quad \text{and} \quad \Delta V(t) = N^{-1} P - \hat{X}_t.
\]

We note that propagation of regularity for the microscopic state $\Psi_{N,t} = e^{-i t \mathcal{H}_N} \Psi_{N,0}$ follows from standard techniques. In particular, $\Psi_{N,t} \in \mathcal{D}(X^3) \cap \mathcal{D}(P^3)$ for all $t \geq 0$.

Let us start with the following estimate, from which Theorem 2.1 follows easily after elementary manipulations.

**Lemma 4.1.** There exists $C > 0$ such that for all $t \geq 0$ and $N \geq 1$ the following inequality holds

\[
\|\Delta X(t) \Psi_{N,t}\|^2 + \|\Delta V(t) \Psi_{N,t}\|^2 \leq C \exp(Ct) N^{-1/2}.
\]

**Proof.** Let us first assume that $(t \rightarrow \Psi_{N,t}) \in C^1(\mathbb{R}, \mathcal{H}) \cap C(\mathbb{R}, \mathcal{D}(\mathcal{H}_N))$. We start with the difference in position; straightforward differentiation and the Cauchy-Schwarz inequality gives

\[
\frac{d}{dt} \|\Delta X(t) \Psi_{N,t}\|^2 = 2 \langle \Psi_{N,t}, \Delta X(t) \cdot \Delta V(t) \Psi_{N,t} \rangle \leq \|\Delta X(t) \Psi_{N,t}\|^2 + \|\Delta V(t) \Psi_{N,t}\|^2.
\]

The difference in velocity is a little more intricate. First, differentiation with respect to time and the Cauchy-Schwarz inequality gives

\[
\frac{d}{dt} \|\Delta V(t) \Psi_{N,t}\|^2 \leq 2 \langle \Psi_{N,t}, \Delta V(t) \cdot \Delta V(t) \Psi_{N,t} \rangle \leq \|\Delta V(t) \Psi_{N,t}\|^2 + \|\Delta V(t) \Psi_{N,t}\|^2.
\]

Here, we have used the commutation relation $i [P, \mathcal{H}_N] = -i \int \nabla_X w(x, X) a_x^* a_x \, dx$ together with the equation of motion $\dot{X}_t = -\int \nabla_X w(x, X) |\varphi_t(x)|^2 \, dx$. Next, we control the second term in the right hand side; the triangle inequality gives

\[
\|\left( \int_{\mathbb{R}^3} \nabla_X w(x, X) a_x^* a_x \frac{dx}{N} - \int_{\mathbb{R}^3} \nabla_X w(x, X_t) |\varphi_t(x)| \, dx \right) \Psi_{N,t}\| \leq \|\int_{\mathbb{R}^3} \nabla_X w(x, X) \left( \frac{a_x^* a_x}{N} - |\varphi_t(x)|^2 \right) \, dx \Psi_{N,t}\| + \|\int_{\mathbb{R}^3} \left( \nabla_X w(x, X) - \nabla_X w(x, X_t) \right) |\varphi_t(x)|^2 \, dx \Psi_{N,t}\|.
\]
The first term on the right hand side can be estimated using the translation property (3.9) and the Particle Number Estimates. We obtain
\[
\left\| \int_{\mathbb{R}^3} \nabla \mathcal{W}(x, X) \left( \frac{a_x^* a_y}{N} - |\varphi_t(x)|^2 \right) dx \psi_{N,t} \right\|
\]
\[
= \int_{\mathbb{R}^3} \nabla \mathcal{W}(x, X) \left( \frac{a_x^* a_y}{N} + \frac{1}{\sqrt{N}} (a_x^* \varphi_t(x) + \text{h.c}) \right) dx \Omega_{N,t}
\]
\[
\leq 2 \| \nabla \mathcal{W} \|_{L^\infty} \| \varphi_0 \|_{L^2}
\]
\[
\times \left( \frac{1}{N} \| \Omega_{N} \| + \frac{1}{N^{1/2}} \| \Omega_{N,t} \| \right). \quad (4.7)
\]

Similarly, we use a first order Taylor estimate to find that
\[
\left\| \int_{\mathbb{R}^3} \left( \nabla \mathcal{W}(x, X) - \nabla \mathcal{W}(x, X_t) \right) |\varphi_t(x)|^2 dx \psi_{N,t} \right\| \leq \| \mathcal{W} \|_{C^2} \| \varphi_0 \|_{L^2} \| \Delta X(t) \psi_{N,t} \|. \quad (4.8)
\]

We put the last four estimates together to find that for some \( C > 0 \) it holds that
\[
\frac{d}{dt} \| \Delta V(t) \psi_{N,t} \|^2 \lesssim \| \Delta V(t) \psi_{N,t} \|^2 + \| \Delta X(t) \psi_{N,t} \|^2 + \exp \left( \frac{Ct}{N^{1/2}} \right), \quad t \geq 0, N \geq 1, \quad (4.9)
\]
where we have used the number estimates (4.1). We combine (4.4) and (4.9) to find that, thanks to the Grönwall inequality, the following estimate holds for some \( C > 0 \)
\[
\| \Delta X(t) \psi_{N,t} \|^2 + \| \Delta V(t) \psi_{N,t} \|^2 \lesssim e^{Ct} \left( \| \Delta X(0) \psi_{N,t} \|^2 + \| \Delta V(0) \psi_{N,t} \|^2 + \frac{1}{N^{1/2}} \right). \quad (4.10)
\]

Since \( u_{N,0} \) satisfies Condition 1, the right hand side is of order \( \exp(Ct)N^{-1/2} \). Thus, the proof of the lemma is complete under our time differentiability assumption. To remove it, one may re-do all the calculations for \( \psi_{N,t} \) and \( X^f_t \), introduced in Section 3. Moreover, thanks to Lemma 3.1 and 3.2, we may adapt the argument presented in Lemma 3.3 to show that
\[
\| \Delta X(t) \psi_{N,t} \|^2 + \| \Delta V(t) \psi_{N,t} \|^2 \leq \limsup_{\epsilon \to 0} \left( \| \Delta X^f(t) \psi_{N,t} \|^2 + \| \Delta V^f(t) \psi_{N,t} \|^2 \right). \quad (4.11)
\]
Further, all we used in the proof of (4.10) were standard bounds, and the particle number estimates (4.1). Thus, our claim follows in view of (3.39). \( \square \)

The proof of our first main theorem now becomes a straightforward corollary of the last result.

**Proof of Theorem 2.1.** Let us fix \( t \geq 0 \) and \( N \geq 1 \). Since \( \psi_{N,t} \) has unit norm, one easily finds thanks to the Cauchy-Schwarz inequality that
\[
| \langle \psi_{N,t}, X \psi_{N,t} \rangle - X_t | \leq \langle \psi_{N,t}, |X - X_t| \psi_{N,t} \rangle \leq \| \Delta X(t) \psi_{N,t} \|. \quad (4.12)
\]
Thus, it suffices to apply Lemma 4.1. The argument for the difference in velocity remains unchanged. \( \square \)

**Proof of Theorem 2.2.** Let \( \varphi_t \) be the boson field solving the system (1.14). Then, the translation property (3.9) satisfied by the Weyl operator implies that
\[
\frac{1}{N} \langle \psi_{N,t}, a_x^* a_y \psi_{N,t} \rangle = \mathcal{W}(x) \varphi_t(y) + \frac{1}{N} \langle \Omega_{N,t}, \left( a_x^* a_y + \sqrt{N} \varphi_t(y) a_x^* + \sqrt{N} \varphi_t(x) a_y \right) \psi_{N,t} \rangle. \quad (4.13)
\]
In particular, we have the following expression for the difference of the integral kernels:

\[
\Gamma_{N,t}(x, y) - \overline{\varphi_t}(x)\varphi_t(y) = \frac{1}{N} \langle a_x \Omega_{N,t}, a_y \Omega_{N,t} \rangle + \frac{\varphi_t(y)}{\sqrt{N}} \langle a_x \Omega_{N,t}, \Omega_{N,t} \rangle + \frac{\overline{\varphi_t}(x)}{\sqrt{N}} \langle \Omega_{N,t}, a_y \Omega_{N,t} \rangle.
\] (4.14)

Next, let \( K(x, y) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \) be the integral kernel of the Hilbert-Schmidt operator \( \mathcal{K} \). We integrate both sides of the last identity against \( K(x, y) \) and use the Cauchy-Schwarz inequality to find

\[
|\text{Tr} \, \mathcal{K}(\Gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|)| \leq \frac{\|K\|_{L^2}}{N} \left( \int_{\mathbb{R}^3} |a_x \Omega_{N,t}|^2 \right)^{1/2} + \frac{2\|K\|_{L^2}}{\sqrt{N}} \left( \int_{\mathbb{R}^3} |\varphi_t(y)|^2 |a_x \Omega_{N,t}|^2 \right)^{1/2}
\leq \frac{\|K\|_{HS}}{N} \int_{\mathbb{R}^3} |a_x \Omega_{N,t}|^2 dx + \frac{2\|K\|_{HS}}{\sqrt{N}} \left( \int_{\mathbb{R}^3} |a_x \Omega_{N,t}|^2 dx \right)^{1/2}
= \frac{\|K\|_{HS}}{N} \|\Omega_{N,t}\|_{L^2}^2 + \frac{2\|K\|_{HS}}{\sqrt{N}} \|\Omega_{N,t}\|_{L^2},
\] (4.15)

where we have also used the fact that \( \Omega_{N,t} \) and \( \varphi_t \) have unit norms. It follows now from the number estimates (4.1) that there is \( C > 0 \) such that for all \( t \geq 0 \) and \( N \geq 1 \)

\[
|\text{Tr} \, \mathcal{K}(\Gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|)| \leq C\|\mathcal{K}\|_{HS} \left( \frac{e^{Ct}}{N^{1/2}} + \frac{e^{Ct}}{N^{1/4}} \right).
\] (4.16)

Therefore, from the previous estimates we conclude that

\[
\|\Gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|\|_{HS} = \sup_{\|\mathcal{K}\|_{HS} \leq 1} |\text{Tr} \, \mathcal{K}(\Gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|)| \leq \frac{C e^{Ct}}{N^{1/4}},
\] (4.17)

for all \( t \geq 0 \) and \( N \geq 1 \). In view of (2.11), this completes the proof. \( \square \)

5. Particle Number Estimates of the Truncated Dynamics

The main goal of this section is proving Proposition 3.1, which contains the key estimate used to prove Theorem 3.1. Let us then fix \( N, M \geq 1 \).

5.1. Dynamical Estimates. We will be working with the following space of smooth functions

\[
\mathcal{D} := \mathcal{D}_\infty \cap \mathcal{D}(T_b), \quad \text{where} \quad \mathcal{D}_\infty := \bigcap_{k=1}^\infty \mathcal{D}(|X|^k) \cap \mathcal{D}(|P|^k) \cap \mathcal{D}(N^k).
\] (5.1)

In particular, it follows from our results in Section 7 that the regularized, truncated dynamics propagates smoothness in the sense that

\[
\mathcal{U}^{(M,\varepsilon)}_N(t,s) \mathcal{D} \subset \mathcal{D}, \quad \forall t, s \in \mathbb{R}.
\] (5.2)

All of the upcoming calculations are justified in this space.

We will be mostly concerned in proving estimates for \( \|N^{3/2} \mathcal{U}^{(M,\varepsilon)}_N(t,0) \Phi\| \), when \( \Phi \in \mathcal{D} \) and \( t \geq 0 \). Let us note that thanks to the non-diagonal term

\[
\sqrt{N} \int_{\mathbb{R}^3} \left( w(x, X) - w(x, X^f) \right) (a_x \overline{\varphi_t}(x) + a_x^* \varphi_t(x)) dx
\] (5.3)
present in the generator of the (regularized) truncated dynamics, there is a link between the production of particles and the interaction between bosons and tracer particles. In order to account for this phenomenon, we introduce the time-dependent operators

\[
\Delta X(t, \varepsilon) := X - X^\varepsilon_t \\
\Delta V(t, \varepsilon) := N^{-1} P - \dot{X}^\varepsilon_t.
\]

where \( X \) and \( P \) are the tracer particle’s position and momentum observables. It will be convenient to work with their Euclidean components. Namely, if \( i \in \{1, \ldots, m\} \) labels the different tracer particles and \( k \in \{1, 2, 3\} \) labels their components, we write

\[
\Delta X(t, \varepsilon) = (\Delta X_k^{(i)}(t, \varepsilon))_{i=1,k=1}^{m,3} \quad \text{and} \quad \Delta V(t, \varepsilon) = (\Delta V_k^{(i)}(t, \varepsilon))_{i=1,k=1}^{m,3}.
\]

We will be using the following quadratic form to quantify the production of bosonic particles associated to the interaction with the tracer particles. Here and in the sequel, we let \( \mathcal{U}^{(M,\varepsilon)}_{N,t} \equiv \mathcal{U}^{(M,\varepsilon)}_{N}(t,0) \) unless stated otherwise.

**Definition 5.1.** For \( \Phi \in \mathbb{D} \), \( t \geq 0 \) and \( \varepsilon \in (0, 1) \) we introduce the quantity

\[
\mathcal{G}_\Phi(t, \varepsilon) := \sum_{i=1}^m \sum_{k=1}^3 \left( \| \Delta X_k^{(i)}(t, \varepsilon) \|^3 \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 + \| \Delta V_k^{(i)}(t, \varepsilon) \|^3 \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \right)
\]

\[
+ \frac{1}{N^2} \sum_{i=1}^m \sum_{k=1}^3 \left( \| \Delta X_k^{(i)}(t, \varepsilon) \| \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 + \| \Delta V_k^{(i)}(t, \varepsilon) \| \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \right)
\]

\[
+ \frac{1}{N^2} \left( \|(N_b + 1)^{3/2} \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 + \|(N_b + 1)^{1/2} \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \right).
\]

**Remark 5.1.** When we evaluate at our (regularized) initial condition \( \Phi_N = \Phi_N^0 \otimes \Omega \), the following order-of-magnitude estimate holds at time \( t = 0 \):

\[
\limsup_{\epsilon \downarrow 0} \mathcal{G}_\Phi_N(0, \varepsilon) \lesssim N^{-3}.
\]

Thus, our goal will be to show that there is \( C \), independent of \( N, M, t, \varepsilon \) and \( \Phi \), such that

\[
\partial_t \mathcal{G}_\Phi(t, \varepsilon) \leq C(1 + M/N)^2 \mathcal{G}_\Phi(t, \varepsilon)
\]

holds. We then apply the Grönwall inequality.

The proof of (5.9) will be split into three parts, contained in the following three lemmas.

**Lemma 5.1.** There is \( C > 0 \), independent of \( N \) and \( M \), such that for all \( \Phi \in \mathbb{D} \) the following holds

(a) For any \( t \geq 0 \) and \( \varepsilon \in (0, 1) \)

\[
dt \|(N_b + 1)^{3/2} \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \leq C(1 + \sqrt{M/N}) N^3 \mathcal{G}_\Phi(t, \varepsilon).
\]

(b) For any \( t \geq 0 \) and \( \varepsilon \in (0, 1) \)

\[
dt \|(N_b + 1)^{1/2} \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \leq C(1 + \sqrt{M/N}) N^3 \mathcal{G}_\Phi(t, \varepsilon).
\]

(c) For any \( t \geq 0 \) and \( \varepsilon \in (0, 1) \)

\[
\|(N_b + 1)^2 \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2 \leq C \exp \left( C t(1 + \sqrt{M/N}) \right) N \|(N_b + 1)^{3/2} \mathcal{U}^{(M,\varepsilon)}_{N,t} \Phi \|^2.
\]
Lemma 5.2. Let $i \in \{1, \ldots, m\}$ and $k \in \{1, 2, 3\}$. There is $C > 0$, independent of $N$ and $M$, such that for all $\Phi \in \mathcal{D}$ the following holds

(b1) For any $t \geq 0$ and $\varepsilon \in (0, 1)$
\[
\frac{d}{dt} \| \Delta X_{k}^{(i)}(t, \varepsilon) U_{N,t}^{(M,\varepsilon)} \Phi \|^2 \leq CN^2 \mathcal{G}(t, \varepsilon). 
\]

(b2) For any $t \geq 0$ and $\varepsilon \in (0, 1)$
\[
\frac{d}{dt} \| \Delta V_{k}^{(i)}(t, \varepsilon) U_{N,t}^{(M,\varepsilon)} \Phi \|^2 \leq C(1 + M/N)N^2 \mathcal{G}(t, \varepsilon). 
\]

Lemma 5.3. Let $i \in \{1, \ldots, m\}$ and $k \in \{1, 2, 3\}$. There is $C > 0$, independent of $N$ and $M$, such that for all $\Phi \in \mathcal{D}$ the following holds

(c1) For any $t \geq 0$ and $\varepsilon \in (0, 1)$
\[
\frac{d}{dt} \| \Delta X_{k}^{(i)}(t, \varepsilon)^3 U_{N,t}^{(M,\varepsilon)} \Phi \|^2 \leq C \mathcal{G}(t, \varepsilon). 
\]

(c2) For any $t \geq 0$ and $\varepsilon \in (0, 1)$
\[
\frac{d}{dt} \| \Delta V_{k}^{(i)}(t, \varepsilon)^3 U_{N,t}^{(M,\varepsilon)} \Phi \|^2 \leq C(1 + M/N)^2 \mathcal{G}(t, \varepsilon). 
\]

We postpone the proof of Lemma 5.1, 5.2 and 5.3 to the next subsection; let us first prove the very important Proposition 3.1.

Proof of Proposition 3.1. We will show in full detail the proof of (3.29) and omit that of (3.30); the proof of the latter follows directly from (3.29) and Lemma 5.1. Indeed, we put Lemma 5.1, 5.2 and 5.3 together and we conclude that there is $C > 0$ such that for all $t \geq 0$, $\varepsilon \in (0, 1)$, $N, M \geq 1$ and $\Phi \in \mathcal{D}$ it holds that
\[
\partial_t \mathcal{G}(t, \varepsilon) \leq C(1 + M/N)^2 \mathcal{G}(0, \varepsilon). \tag{5.10}
\]

Therefore, an application of the Grönwall inequality yields
\[
\mathcal{G}(t, \varepsilon) \leq \exp \left( C t (1 + M/N)^2 \right) \mathcal{G}(0, \varepsilon). \tag{5.11}
\]

In particular, it follows from the definition of $\mathcal{G}$ that
\[
\|(N_{b} + 1)^{3/2} U_{N,t}^{(M,\varepsilon)} \Phi \|^{2} \leq N^{3} \exp \left( C t (1 + M/N)^2 \right) \mathcal{G}(0, \varepsilon). \tag{5.12}
\]

Let us now take $\Phi \equiv \Phi_{N}^{\varepsilon} := u_{N,0}^{\varepsilon} \otimes \Omega \in \mathcal{D}$, where $u_{N,0}^{\varepsilon}$ is the regularized version of the initial data $u_{N,0}$ satisfying Condition 1. Thanks to (3.18), it follows that
\[
\limsup_{\varepsilon \downarrow 0} \mathcal{G}_{N}(0, \varepsilon) \lesssim N^{-3}. \tag{5.13}
\]

The proof is complete once we put the last two estimates together for $\Omega_{N,t}^{(\varepsilon,M)} = U_{N,t}^{(M,\varepsilon)} \Phi_{N}^{\varepsilon}$. □

5.2. Proof of Lemmata. Let us now turn to the proof of Lemma 5.1, 5.2 and 5.3. Since one should think about the following calculations as a regularized version of estimates done for the original dynamics, we will drop the $\varepsilon \in (0, 1)$ superscript and assume instead that all of the involved quantities are smooth; since our estimates are uniform in $\varepsilon$, there is no risk in doing so. We also denote $\mathcal{G}(t, \varepsilon) \equiv \mathcal{G}(t)$.

The following (formal) Interpolation Lemma will be helpful. Its use will only be put into use when all the manipulations find a rigorous meaning; for a proof see Appendix B.
Lemma 5.4 (Interpolation). Let $A$ and $B$ be self-adjoint operators such that $i[A, B] = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$. Then, there is $c > 0$ such that

$$\|A^2 B \Psi\|^2 + \|ABA \Psi\|^2 + \|BA^2 \Psi\|^2 \leq c\left(\|A^3 \Psi\|^2 + \|B^3 \Psi\|^2 + \lambda^2 \|A \Psi\|^2 + \lambda^2 \|B \Psi\|^2\right)$$

(5.14)

for all $\Psi$.

Proof of Lemma 5.1. Let us fix $N, M \geq 1$ and $\Phi \in \mathcal{M}$. We start by recalling the interaction terms to be $I_M(t) = \chi(N_b \leq M)(L^{(od)}(t) + L^{(d)}(t))$. In particular, we may write the truncated, off-diagonal term as

$$\chi(N_b \leq M)L^{(od)}(t) = L_1(t) + L_2(t) + L_3(t)$$

(5.15)

where we introduce the notation

$$L_1(t) = \chi(N_b \leq M)\sqrt{N} \int_{\mathbb{R}^3} (\overline{w}(x, X) - \overline{w}(x, X_t)) \left(a_x \varphi_t(x) + a_x^* \varphi_t(x)\right) dx$$

$$L_2(t) = \chi(N_b \leq M) \frac{1}{2} \int_{\mathbb{R}^6} v(x - y) \left(\varphi_t(x) \varphi_t(y) a_x^* a_y^* + \overline{\varphi_t(x) \varphi_t(y) a_x a_y}\right) dx dy$$

$$L_3(t) = \chi(N_b \leq M) \frac{1}{2\sqrt{N}} \int_{\mathbb{R}^6} v(x - y) a_x^* \left(\varphi_t(y) a_y^* + \overline{\varphi_t(y) a_y}\right) a_x dx dy.$$  

(5.16)

The labeling has been chosen so that each $L_i(t)$ depends on a $i$-th power of the creation and annihilation operators.

The terms $L_2(t)$ and $L_3(t)$ have already been analyzed in the literature; we will take some commutator estimates from [33]. More precisely, it is known that for all $j \in \mathbb{Z}$ there is $c(j) > 0$ such that for all smooth $\Psi \in \mathcal{M}$ and $t \geq 0$ it holds that

$$|\langle \Psi, [L_2(t) + L_3(t), (N_b + 1)^j] \Psi \rangle| \leq c(j) \left(1 + \sqrt{M/N}\right) \langle \Psi, (N_b + 1)^j \Psi \rangle.$$  

(5.17)

The constant $C(j)$ depends on the potentials only through $|v|$, as the above estimates are a consequence of Particle Number Estimates and the Potential Estimates. Therefore, it suffices to focus on deriving similar commutator estimates for $L_1(t)$ for the cases $j = 1, 2, 3$.

Proof of $(a_1)$. Let $\Psi \in \mathcal{M}$ be smooth but arbitrary. Starting from the definition of $L_1(t)$ one easily finds that

$$|\langle \Psi, [L_1(t), (N_b + 1)^3] \Psi \rangle|$$

$$\leq \sqrt{N} \langle \Psi, \int_{\mathbb{R}^3} (w(x, X) - \overline{w}(x, X_t)) [a_x \varphi_t(x) + a_x^* \varphi_t(x), (N_b + 1)^3] \Psi dx \rangle$$

$$\lesssim \sqrt{N} \langle \Psi, \int_{\mathbb{R}^3} (w(x, X) - \overline{w}(x, X_t)) \varphi_t(x) [a_x, (N_b + 1)^3] \Psi dx \rangle.$$  

The remaining commutator can be controlled as follows: by means of the relation $a_x N_b = (N_b + 1) a_x$ one finds that for any $\ell \in \mathbb{N}$ it holds that

$$[a_x, (N_b + 1)\ell] = \sum_{k=0}^{\ell-1} \binom{\ell}{k} (N_b + 1)^k a_x.$$  

(5.19)
We plug this relationship back in (5.18) and use the Cauchy-Schwarz inequality to find that
\[
\left| \langle \Psi, [\mathcal{L}_1(t), (N_b + 1)^3] \Psi \rangle \right| 
\leq \sqrt{N} \sum_{k=0}^{2} \left( \frac{3}{k} \right) \left| \langle \Psi, \int_{\mathbb{R}^3} (w(x, X) - w(x, X_t)) \varphi_t(x) (N_b + 1)^k a_x \Psi \, dx \rangle \right|
\leq \sqrt{N} \left\| (N_b + 1)^{3/2} \int_{\mathbb{R}^3} (w(x, X) - w(x, X_t)) \varphi_t(x) (N_b + 1)^{1/2} a_x \, dx \Psi \right\|
\leq \sqrt{N} \left\| (N_b + 1)^{3/2} \varphi_t(x) (N_b + 1)^{1/2} \Psi \right\|
\leq \left\| (N_b + 1)^{3/2} \Psi \right\|^2 + N \left\| \Delta X(t)(N_b + 1) \Psi \right\|^2
\] (5.20)
where, in the third line, we have used a Taylor estimate together with the Particle Number Estimates. In order to control the second term that appears in (5.20) we use the Interpolation Lemma for \( A = |\Delta X(t)|, B = N^{-1/2}(N_b + 1)^{1/2} \) and \( \lambda = 0 \) to find that
\[
N \left\| \Delta X(t)(N_b + 1) \Psi \right\|^2 \leq \left\| (N_b + 1)^{3/2} \Psi \right\|^2 + N^3 \left\| \Delta X(t) \right\|^3 \left\| \Psi \right\|^2.
\] (5.21)
By putting together estimates (5.17), (5.20) and (5.21) for \( \Psi = \mathcal{U}_{N,t}^{(M)} \Phi \) we find
\[
\frac{d}{dt} \left\| (N_b + 1)^{3/2} \mathcal{U}_{N,t}^{(M)} \Phi \right\|^2 \leq \left| \langle \mathcal{U}_{N,t}^{(M)} \Phi, [\mathcal{L}_1(t) + \mathcal{L}_2(t) + \mathcal{L}_3(t), (N_b + 1)^3] \mathcal{U}_{N,t}^{(M)} \Phi \rangle \right|
\leq (1 + \sqrt{M/N}) N^3 G_{\Phi}(t).
\] (5.22)

**Proof of (a2).** We repeat the argument presented above; the proof is actually shorter. In particular, the estimate
\[
\left| \langle \Psi, [\mathcal{L}_1(t), (N_b + 1)^2] \Psi \rangle \right| \leq \left\| (N_b + 1)^{1/2} \Psi \right\|^2 + N \left\| \Delta X(t) \Psi \right\|^2
\] (5.23)
is the analogous of (5.20) but the operators \( N_b \) and \( \Delta X(t) \) are already decoupled. For \( \Psi = \mathcal{U}_{N,t}^{(M)} \Phi \) the right hand side may be immediately bounded above by \( N^3 G_{\Phi}(t) \); an application of (5.17) for \( j = 1 \) then finishes the proof.

**Proof of (a3).** Similarly as in the proof of (a1), we estimate the following commutator as
\[
\left| \langle \Psi, [\mathcal{L}_1(t), (N_b + 1)^2] \Psi \rangle \right| \leq \sqrt{N} \left\| (N_b + 1)^{2} \Psi \right\| \left\| \int_{\mathbb{R}^3} (w(x, X) - w(x, X_t)) \varphi_t(x) (N_b + 1)^2 a_x \Psi \, dx \right\|
\leq \sqrt{N} \left\| (N_b + 1)^{2} \Psi \right\| \left\| (N_b + 1)^{3/2} \Psi \right\|
\leq \left\| (N_b + 1)^{2} \Psi \right\|^2 + N \left\| (N_b + 1)^{3/2} \Psi \right\|^2,
\] (5.24)
where we have used boundedness of \( w \) instead of a Taylor estimate to control the \( X \)-dependent term. We put together estimates (5.24) and (5.17) for \( j = 4 \), to find that for \( \Psi = \mathcal{U}_{N,t}^{(M)} \Phi \) we have
\[
\frac{d}{dt} \left\| (N_b + 1)^2 \mathcal{U}_{N,t}^{(M)} \Phi \right\|^2 \leq (1 + \sqrt{M/N}) \left\| (N_b + 1)^2 \mathcal{U}_{N,t}^{(M)} \Phi \right\|^2 + N \left\| (N_b + 1)^{3/2} \mathcal{U}_{N,t}^{(M)} \Phi \right\|^2.
\] (5.25)
The proof of the lemma is finished after we apply the Grönwall inequality. \( \square \)

**Proof of Lemma 5.2.** Let us introduce simplifications in the notation:
(i) Since the following proof is conceptually independent of the number of tracer particles $m$, we shall assume for simplicity that $m = 1$. In particular, we drop the boldface notation for $X$ and $V$; we write instead

$$X = X = (X_1, X_2, X_3) \in \mathbb{R}^3, \quad \text{and} \quad w(x - X) = \underline{w}(x, X)$$  \hspace{1cm} (5.26)$$

and similarly for velocity.

(ii) In order to avoid double subscripts, we write

$$X_t = X_t = (X_1(t), X_2(t), X_3(t))$$  \hspace{1cm} (5.27)$$

to denote time dependence of the components of the tracer’s particle position, and similarly for velocity.

(iii) Whenever there is no risk of confusion, we write

$$\Phi(t) \equiv \Phi_t := \mathcal{U}_{N,t}^{(M)} \Phi.$$  \hspace{1cm} (5.28)$$

Let us then fix $k \in \{1, 2, 3\}$, together with $N, M \geq 1$ and $\Phi \in \mathcal{D}$. 

**Proof of** (b1). Let us first calculate that

$$\frac{d}{dt} \|\Delta X_k(t) \Phi(t)\|^2 = \left\langle \Phi(t), \left(i \frac{\Delta X_k(t)}{2N}, \Delta X_k(t)^2\right) - 2 \Delta X_k(t) \dot{X}_k(t) \right\rangle \Phi(t).$$  \hspace{1cm} (5.29)$$

Moreover, the commutation relations $i[P_t, X_k] = \delta_{k,t}$ imply that the following identity holds

$$i[P^2, \Delta X_k(t)^2] = P_k \Delta X_k(t) + \Delta X_k(t) P_k.$$  \hspace{1cm} (5.30)$$

Thus, since $\Delta V_k(t) = N^{-1} P_k - \dot{X}_k(t)$, we find thanks to the Cauchy-Schwarz inequality that

$$\frac{d}{dt} \|\Delta X_k(t) \Phi(t)\|^2 = 2 \text{Re} \left\langle \Phi(t), \Delta V_k(t) \Delta X_k(t) \Phi(t) \right\rangle$$

$$\leq \|\Delta V_k(t) \Phi(t)\|^2 + \|\Delta X_k(t) \Phi(t)\|^2$$  \hspace{1cm} (5.31)$$

from which our claim follows easily.

**Proof of** (b2). First, let us identify the term contained in the generator $\mathcal{L}_M(t)$ that does not commute with $P_k = -i \hat{c}_k$; it is given by

$$\mathcal{F}_M(t) = \chi(N_k \leq M) \int_{\mathbb{R}^3} w(x - X) \left( N |\varphi_t(x)|^2 + \sqrt{N}(a_x \varphi_t(x) + h.c) - a_x^* a_x \right) dx.$$  \hspace{1cm} (5.32)$$

Moreover, we recall that the trajectory $X(t)$ satisfies the mean-field equations (1.14). Therefore, similarly as we calculated for (b1), we find

$$\frac{d}{dt} \|\Delta V_k(t) \Phi(t)\|^2 = 2 \text{Re} \left\langle \Phi(t), \left(i \left[\mathcal{F}_M(t), N^{-1} P_k\right] - \dot{X}_k(t)\right) \Delta V_k(t) \mathcal{U}_{N,t}^{(M)} \Phi(t) \right\rangle$$

$$\leq \|\Delta V_k(t) \Phi(t)\|^2 + \left\|\left(i \left[\mathcal{F}_M(t), N^{-1} P_k\right] - \dot{X}_k(t)\right) \Phi(t) \right\|^2.$$  \hspace{1cm} (5.33)$$

The first term that shows up in (5.33) is clearly bounded above by $N^2 \mathcal{G}_\Phi(t)$. Therefore, it remains to estimate the first one in terms of $N^2 \mathcal{G}_\Phi(t)$. To this end, we write

$$\Delta F_k(t) := i \left[\mathcal{F}_M(t), N^{-1} P_k\right] - \dot{X}_k(t)$$  \hspace{1cm} (5.34)$$
to denote the error in the force term exerted on the tracer particle. Further, we decompose it as $\Delta F_k(t) = \Delta F_k^{(1)}(t) + \Delta F_k^{(2)}(t)$, where

$$\Delta F_k^{(1)}(t) := \chi(N_b \leq M) \int_{\mathbb{R}^3} (\hat{c}_k w(x - X) - \hat{c}_k w(x - X_t))|\varphi_t(x)|^2 \, dx$$

(5.35)

$$\Delta F_k^{(2)}(t) := \chi(N_b \leq M) \int_{\mathbb{R}^3} \hat{c}_k w(x - X) \left( \frac{1}{\sqrt{N}} (a_x \varphi_t(x) + h.c) - \frac{1}{N} a_x^* a_x \right) \, dx .$$

(5.36)

Since $w \in C^2_c(\mathbb{R}^3, \mathbb{R})$, we may use a Taylor Estimate and the Particle Number Estimates to find that the following inequalities

$$\|\Delta F_k^{(1)}(t)\Psi\| \leq \|\nabla^2 w\|_{L^\infty} \|\Delta X(t)\Psi\|$$

(5.37)

$$\|\Delta F_k^{(2)}(t)\Psi\| \leq \frac{\|\nabla w\|_{L^\infty}}{\sqrt{N}} (1 + \sqrt{M/N}) \|(N_b + 1)^{1/2}\Psi\|. \quad (5.38)$$

hold, for all $\Psi \in \mathcal{D}$. By plugging these estimates back in (5.33) for $\Psi = \Phi(t)$ one easily finds that

$$\frac{d}{dt}\|\Delta V_k(t)\Phi(t)\|^2 \lesssim (1 + M/N) N^2 G_\Phi(t).$$

This finishes the proof of the lemma.

Proof of Lemma 5.2. We will keep on using the same simplifications and notation introduced at the beginning of the proof of lemma 5.2. Let us fix $k \in \{1, 2, 3\}$, together with $N, M \geq 1$ and $\Phi \in \mathcal{D}$.

The following general general calculation will turn out to be useful. Let $A(t)$ be a time-dependent, self-adjoint operator. As long as everything is well-defined, one formally has that

$$\frac{d}{dt}\|A(t)^3\Phi(t)\|^2 = \left\langle \Phi(t), \left[ i[\mathcal{L}_M(t), A(t)^6] - 6A(t)^5 \dot{A}(t) \right] \Phi(t) \right\rangle$$

$$\lesssim \|A(t)^3\Phi(t)\|^2 + \|A(t)^2\left[ i[\mathcal{L}_M(t), A] - \dot{A}(t) \right] A(t) \Phi(t)\|^2$$

$$+ \|A(t)\left[ i[\mathcal{L}_M(t), A] - \dot{A}(t) \right] A(t) \Phi(t)\|^2$$

$$+ \left\| \left[ i[\mathcal{L}_M(t), A] - \dot{A}(t) \right] A(t)^2 \Phi(t) \right\|^2. \quad (5.39)$$

Proof of (c1). First, we specialize to $A(t) = \Delta X_k(t)$ for which the above estimate turns into

$$\frac{d}{dt}\|\Delta X_k(t)^3\Phi_t\|^2 \lesssim \|\Delta X_k(t)^3\Phi_t\|^2 + \|\Delta X_k(t)^2\Delta V_k(t)\Phi_t\|^2$$

$$+ \|\Delta X_k(t)\Delta V_k(t)\Delta X_k(t)\Phi_t\|^2 + \|\Delta V_k(t)\Delta X_k(t)^2\Phi_t\|^2. \quad (5.40)$$

It suffices to show that every term contained in the right hand side above may be bounded in terms of $G_\Phi(t)$. The first one is trivial and follows from the definition, so we omit it. The other three may be controlled using the Interpolation Lemma with $A = \Delta X_k(t)$, $B = \Delta V_k(t)$ and $\lambda = -1$: we find that for all $t \geq 0$

$$\|\Delta X_k(t)^2\Delta V_k(t)\Phi_t\|^2 + \|\Delta X_k(t)\Delta V_k(t)\Delta X_k(t)\Phi_t\|^2 + \|\Delta V_k(t)\Delta X_k(t)^2\Phi_t\|^2 \lesssim G_\Phi(t). \quad (5.41)$$
Proof of \((c_2)\). Let us now specialize our general calculation to \(A(t) = \Delta V_k(t)\). We find that
\[
\frac{d}{dt}\|\Delta V_k(t)^3\Phi_t\|^2 \lesssim \|\Delta V_k(t)^3\Phi_t\|^2 + \|\Delta V_k(t)^2\Delta F_k(t)\Phi_t\|^2 \\
+ \|\Delta V_k(t)\Delta F_k(t)\Delta V_k(t)\Phi_t\|^2 + \|\Delta F_k(t)\Delta V_k(t)^2\Phi_t\|^2,
\]
where we recall that \(\Delta F_k(t)\) was defined in \((5.34)\). Let us introduce some notation for the different terms that we found above:
\[
T_1 := \|\Delta F_k(t)\Delta V_k(t)^2\Phi(t)\|^2 \\
T_2 := \|\Delta V_k(t)^2\Delta F_k(t)\Delta V_k(t)\Phi(t)\|^2 \\
T_3 := \|\Delta V_k(t)^2\Delta F_k(t)\Phi(t)\|^2.
\]
Our goal is to show that \(T_i \lesssim (1 + M/N)^2 \mathcal{G}_\Phi(t)\) for \(i \in \{1, 2, 3\}\).

**Bounds on \(T_1\):**

Thanks to the force estimates \((5.37), (5.38)\) one easily finds that
\[
T_1 \lesssim \|\Delta X|\Delta V_k(t)^2\Phi(t)\|^2 + (1 + M/N)\|N^{-1/2}(N_b + 1)^{1/2}\Delta V_k(t)^2\Phi(t)\|^2 \\
\lesssim \sum_{\ell=1}^{3} \|\Delta X_\ell(t)\Delta V_k(t)^2\Phi(t)\|^2 + (1 + M/N)\|N^{-1/2}(N_b + 1)^{1/2}\Delta V_k(t)^2\Phi(t)\|^2
\]
Both the first and the second term that appear in the right hand side above can be controlled by \(\mathcal{G}_\Phi(t)\) using the Interpolation Lemma:

(i) For the the first term choose: \(A_\ell = \Delta X_\ell(t), B = \Delta V_k(t)\) with \(\ell \in \{1, 2, 3\}\) and \(\lambda_\ell \in \{-1, 0\}\).

(ii) For the second term choose: \(A = N^{-1/2}(N_b + 1)^{1/2}, B = \Delta V_k(t)\) and \(\lambda = 0\).

We conclude that \(T_1 \lesssim (1 + M/N)\mathcal{G}_\Phi(t)\).

**Bounds on \(T_2\):**

The idea will be to give an upper bound for \(T_2\) in terms on \(T_1\), plus a commutator term arising from \([\Delta F_k(t), \Delta V_k(t)] = \frac{1}{\sqrt{N}}[\Delta F_k(t), F_k]\). For the sake of the upcoming calculations, let us briefly denote by
\[
\mathcal{A}_{N,t}(x) := \frac{1}{\sqrt{N}}(a_x \varphi_t(x) + h.c) + \frac{1}{N} a_x^* a_x, \quad x \in \mathbb{R}^3
\]
the operator-valued distribution, found in the definition of the force term \(\Delta F_k(t)\). We find that
\[
T_2 \lesssim T_1 + \frac{1}{N^2} \|\chi(N_b \leq M)\int_{\mathbb{R}^3} e_i^2 w(x - X) \left( |\varphi_t(x)|^2 + \mathcal{A}_{N,t}(x) \right) dx \Delta V_k(t)\Phi(t)\|^2.
\]
Since \(w \in C_b^2(\mathbb{R}^3)\) and \(\|\varphi_t\|_{L^2} = \|\varphi_0\|_{L^2}\) we find, thanks to the cut-off term \(\chi(N_b \leq M)\) and the Particle Number Estimates, that
\[
T_2 \lesssim T_1 + \frac{1}{N^2}(1 + \sqrt{M/N} + M/N)^2\|\Delta V_k(t)\Phi(t)\|^2 \lesssim T_1 + (1 + M/N)^2 \mathcal{G}_\Phi(t)\.
\]
Thanks to the bound \(T_1 \lesssim (1 + M/N)^2 \mathcal{G}_\Phi(t)\) we conclude that \(T_2 \lesssim (1 + M/N)^2 \mathcal{G}_\Phi(t)\).

**Bounds on \(T_3\):**

Similarly as before, we give an upper bound for \(T_3\) in terms of \(T_2\), plus commutator terms. In particular, the following estimate is the only one in this article that requires three
derivatives of the boson-tracer interaction $w$. We find
\[ T_3 \lesssim T_2 + \frac{1}{N^2} \left\| \chi(N_b \leq M) \int_{\mathbb{R}^3} \delta^2_k w(x - X)(|\varphi_t(x)|^2 + A_{N,t}(x)) \, dx \right\|_{\Phi(t)}^2 \]
\[ \lesssim T_2 + \frac{1}{N^2} \left\| \chi(N_b \leq M) \int_{\mathbb{R}^3} \delta_k^2 w(x - X)(|\varphi_t(x)|^2 + A_{N,t}(x)) \, dx \right\|_{\Delta V_k(t) \Phi(t)}^2 \]
\[ + \frac{1}{N^2} \left\| \chi(N_b \leq M) \int_{\mathbb{R}^3} \delta_k^3 w(x - X)(|\varphi_t(x)|^2 + A_{N,t}(x)) \, dx \right\|_{\Phi(t)}^2 \]
\[ \lesssim T_2 + \frac{1}{N^2} (1 + \sqrt{M/N + M/N}) \left\| \Delta V_k(t) \Phi(t) \right\|_{\Phi(t)}^2 \]
\[ + \frac{1}{N^2} (1 + \sqrt{M/N + M/N}) \left\| \Phi(t) \right\|_{\Phi(t)}^2 \]
\[ \lesssim T_2 + (1 + M/N)^2 G_{\Phi}(t) + N^{-4}(1 + M/N)^2. \quad (5.50) \]

Thanks to the upper bound $T_2 \lesssim (1 + M/N)^2 G_{\Phi}(t)$ and $N^{-4} \leq G_{\Phi}(t)$ we conclude that $T_3 \lesssim (1 + M/N)^2 G_{\Phi}(t)$. This finishes the proof of the lemma. \hfill \square

6. WELL-POSEDNESS OF THE MEAN-FIELD EQUATIONS

In this section, we establish well-posedness of the mean-field equations, introduced in (1.14). The energy of the system will be described by the functional $E : \mathbb{R}^{6m} \times H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as
\[ E(X, V, \varphi) := \frac{1}{2} V^2 + \|\varphi\|^2_{H^1} + \frac{1}{2} \int_{\mathbb{R}^6} |\varphi(x)|^2 v(x-y)|\varphi(y)|^2 \, dx \, dy + \int_{\mathbb{R}^3} w(x, X)|\varphi(x)|^2 \, dx. \quad (6.1) \]

The fundamental estimate satisfied by the energy functional $E$ is the lower bound
\[ V^2 + \|\varphi\|^2_{H^1} \leq \mu \left( E(X, V, \varphi) + \|\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2} \right), \quad (X, V, \varphi) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3) \quad (6.2) \]
where $\mu = \mu(v, w) > 0$ is a constant depending only on the potentials. Its proof follows from the operator inequality (2.19) for $v$, and the boundedness of $w$.

For notational simplicity, given a solution $(X_t, \varphi_t)$ of the mean-field equations, we shall simply write $E(t) = E(X_t, \dot{X}_t, \varphi_t)$. In addition, we denote by $V_t = \dot{X}_t$ the velocity of the tracer particles. The main result of this section is

**Theorem 6.1.** Assume that $v$ and $w$ satisfy Condition 2. Then, for any initial condition $(X_0, V_0, \varphi_0) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3)$, there is a unique global solution
\[ (X, V, \varphi) \in C(\mathbb{R}; \mathbb{R}^{6m} \times H^1(\mathbb{R}^3)) \]

**to the mean-field equations** (1.14) **in mild form**; it preserves the $L^2$-norm of the boson field and the total energy of the system. In particular, for all $t \in \mathbb{R}$ it holds that
\[ V_t^2 + \|\varphi_t\|^2_{H^1} \leq \mu \left( E(0) + \|\varphi_0\|^2_{L^2} + \|\varphi_0\|^2_{L^2} \right). \quad (6.3) \]

Further, if $\varphi_0 \in H^2(\mathbb{R}^3)$, it holds that $\varphi \in C^1(\mathbb{R}; L^2(\mathbb{R}^3)) \cap C(\mathbb{R}; H^2(\mathbb{R}^3))$ and (1.14) holds in the strong sense.

Its proof is divided into three steps: local well-posedness, propagation of $H^2$ regularity, and conservation laws. These steps are given as the following three lemmas.
**Lemma 6.1 (Local Well-Posedness).** Let \((X_0, V_0, \varphi_0) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3)\) be initial conditions for which the system has energy \(E_0\). Then, there exists \(T = T(v, w, E_0, \|\varphi_0\|_{L^2}) > 0\) such that the system of equations

\[
\begin{aligned}
X_t &= X_0 + \int_0^t V_s ds \\
V_t &= V_0 - \int_0^t \int_{\mathbb{R}^3} \nabla x \, w(x, X_s) \, |\varphi_s(x)|^2 dx \\
\varphi_t &= e^{-it\Delta} \varphi_0 - i \int_0^t e^{-i(t-s)\Delta} \left( (v * |\varphi_s|^2) \varphi_s + w(x, X_s) \varphi_s \right) ds
\end{aligned}
\]  

has a unique solution \((X, V, \varphi) \in C([0, T]; \mathbb{R}^{6m} \times H^1(\mathbb{R}^3))\). In addition, the solution depends continuously on the initial data.

**Remark 6.1.** The map \((X_0, V_0, \varphi_0) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3) \mapsto T \in (0, \infty)\) is continuous; see (6.7) and (6.20).

**Proof.** Let us fix some initial conditions \((X_0, V_0, \varphi_0) \in \mathbb{R}^{6m} \times H^1(\mathbb{R}^3)\). In particular, it follows from (6.2) that the inequality

\[
M^2 := \mu(E_0 + \|\varphi_0\|_{L^2}^2 + \|\varphi_0\|^6_{L^2}) \geq \|\varphi_0\|_{H^1}^2 + |V_0|^2 \geq 0,
\]  

holds. For \(T = T(M, v, w) > 0\), we will set up a fixed-point argument in the Banach space

\[
Y_T := \left\{ (X, V, \varphi) \in C([0, T]; \mathbb{R}^{6m} \times H^1(\mathbb{R}^3)) \, | \sup_{|t| \leq T} \left( \|V(t) - V_0\| + \|\varphi(t) - e^{-it\Delta} \varphi_0\|_{H^1} \right) \leq M \right\},
\]

endowed with the norm \(\|(X, V, \varphi)\| := \sup_{0 \leq t \leq T} \left( |X(t)| + |V(t)| + \|\varphi(t)\|_{H^1} \right)\). We consider the map \(\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) on \(C([0, T]; \mathbb{R}^{6m} \times \mathbb{R}^{6m} \times H^1(\mathbb{R}^3))\) defined as

\[
\begin{aligned}
\mathcal{M}_1(X, V, \varphi)(t) &= X_0 + \int_0^t V_s ds \\
\mathcal{M}_2(X, V, \varphi)(t) &= V_0 - \int_0^t \int_{\mathbb{R}^3} \nabla x \, w(x, X_s) \, |\varphi_s(x)|^2 dx ds \\
\mathcal{M}_3(X, V, \varphi)(t) &= e^{-it\Delta} \varphi_0 - i \int_0^t e^{-i(t-s)\Delta} \left( (v * |\varphi_s|^2) \varphi_s + w(x, X_s) \varphi_s \right) ds.
\end{aligned}
\]

Thanks to Banach’s fixed point theorem, existence and uniqueness of solutions is established once we show that \(\mathcal{M}\) is a contraction from \(Y_T\) into itself, for \(T\) small enough. We will show that such \(T\) can be chosen to depend only on \(M, v\) and \(w\). The details of the proof of continuity with respect to initial data are left as an exercise to the reader.

**Proof of \(\mathcal{M}(Y_T) \subset Y_T\).** Let \((X, V, \varphi) \in Y_T\) and \(t \in [0, T]\). Then, we have the estimates

\[
\begin{aligned}
|\mathcal{M}_2(X, V, \varphi)(t) - V_0| &\leq T \|w\|_{C^1} \sup_{0 \leq \tau \leq T} \|\varphi_\tau\|_{L^2}^2 \\
\|\mathcal{M}_3(X, V, \varphi)(t) - e^{-it\Delta} \varphi_0\|_{H^1} &\leq T \sup_{0 \leq \tau \leq T} \left( \|v \ast |\varphi_\tau|^2\|_{H^1} + \|w(x, X_\tau) \varphi_\tau\|_{H^1} \right) \\
&\leq T (L(v) + 2 \|w\|_{C^1}) \sup_{0 \leq \tau \leq T} \left( \|\varphi_\tau\|_{H^1}^3 + \|\varphi_\tau\|_{H^1} \right)
\end{aligned}
\]  

(6.12) 

(6.13)
where, in deriving the last inequality, we have used the Lipschitz property (2.23) for $\psi = 0$. Thanks to the triangle inequality and the definition of $M$ and $Y_T$, it follows that
\[
\|\varphi\|_{H^1} + |V_\varphi| \leq \|e^{-i\Delta} \varphi_0\|_{H^1} + |V_0| + \|\varphi_t - e^{-i\Delta} \varphi_0\|_{H^1} + |V_\varphi - V_0| \leq 3M. \tag{6.14}
\]
Therefore, we may combine the last three estimates to conclude that there is $C^{(1)}_{v,w} > 0$ such that for all $t \in [0,T]$
\[
|M_2(X, V, \varphi)(t) - V_0| + \|M_3(X, V, \varphi)(t) - e^{-it\Delta} \varphi_0\|_{H^1} \leq C^{(1)}_{v,w} T (M + M^3). \tag{6.15}
\]
Thus, $\mathcal{M}$ maps $Y_T$ into itself provided $T \leq (C^{(1)}_{v,w}(1 + M^2))^{-1}$.

**Proof of Contractivity.** Consider $(X, V, \varphi)$ and $(\tilde{X}, \tilde{V}, \tilde{\varphi})$ in $Y_T$, and let $t \in [0,T]$. For the position variables, one simply has that
\[
|M_1(X, V, \varphi)(t) - M_1(\tilde{X}, \tilde{V}, \tilde{\varphi})(t)| \leq T \sup_{0 \leq \tau \leq T} |V_\varphi - \tilde{V}_\varphi|. \tag{6.16}
\]
For the velocity variables, we use a Taylor estimate and the triangle inequality to find that
\[
|M_2(X, V, \varphi)(t) - M_2(\tilde{X}, \tilde{V}, \tilde{\varphi})(t)| \leq T \sup_{0 \leq \tau \leq T} \left( \|w\|_{C^1_v} \left( \|\varphi\|_{L^2} + \|\tilde{\varphi}\|_{L^2} \right) \|\varphi_t - \tilde{\varphi}_t\|_{L^2} + \|w\|_{C^2_v} \|\varphi_t - \tilde{\varphi}_t\|^2_{L^2} |X_\tau - \tilde{X}_\tau| \right). \tag{6.17}
\]
For the boson fields, we use the Lipschitz property (2.23), a Taylor estimate and the triangle inequality to find that
\[
\|M_3(X, V, \varphi)(t) - M_3(\tilde{X}, \tilde{V}, \tilde{\varphi})(t)\|_{H^1} \leq T \sup_{0 \leq \tau \leq T} \|v\|_{C^3_v} \left( \|\varphi\|_{H^1} + \|\tilde{\varphi}\|_{H^1} \right) \|\varphi_t - \tilde{\varphi}_t\|_{H^1} + T \sup_{0 \leq \tau \leq T} 3 \|w\|_{C^2_v} \|\varphi_t\|_{H^1} |X_\tau - \tilde{X}_\tau| + \|\varphi_t - \tilde{\varphi}_t\|_{H^1}. \tag{6.18}
\]
Since the last three estimates are uniform in $t \in [0,T]$, we put them together to find that there is $C^{(2)}_{v,w} > 0$ such that
\[
\|\mathcal{M}(X, V, \varphi) - \mathcal{M}(\tilde{X}, \tilde{V}, \tilde{\varphi})\| \leq T C^{(2)}_{v,w} (M + M^2) \|(X, V, \varphi) - (\tilde{X}, \tilde{V}, \tilde{\varphi})\| \tag{6.19}
\]
where we have used (6.14) as well. Since $M + M^2 \leq 2(1 + M^2)$, it follows that by choosing
\[
T(M, v, w) := \min \left\{ \frac{1}{C^{(1)}_{v,w}}, \frac{1}{4C^{(2)}_{v,w}} \right\} \frac{1}{1 + M^2} \tag{6.20}
\]
the map $\mathcal{M}$ becomes Lipschitz continuous, with Lipschitz constant $1/2$. In particular, it becomes a contraction and the proof is complete. \qed

The following proof of propagation of regularity is heavily inspired by [1, Chapter 4].

**Lemma 6.2 (Propagation of $H^2$ Regularity).** Let $(X, V, \varphi) \in C([0,T]; \mathbb{R}^{6m} \times H^1(\mathbb{R}^3))$ be as in Lemma 6.1 with initial data $\varphi_0 \in H^2(\mathbb{R}^3)$. Then, we have that $\varphi \in C^1([0,T]; L^2(\mathbb{R}^3)) \cap C([0,T]; H^2(\mathbb{R}^3))$ and the mean-field equation (1.14) holds in the strong sense.

**Proof.** In terms of $X \in C^1([0,T], \mathbb{R}^{3m})$, as given by Lemma 6.1, we define the map
\[
F(t, \phi) := (v \ast |\phi|^2)\phi + w(x, X_t)\phi, \quad (t, \phi) \in [0,T] \times H^1(\mathbb{R}^3). \tag{6.21}
\]
In particular, a Taylor estimate combined with the Potential Estimate (2.21), imply that the following Lipschitz estimates are satisfied
\[
\|F(t, \phi) - F(s, \phi)\|_{L^2} \lesssim \sup_{0 \leq \tau \leq T} |\dot{X}_\tau| \|\phi\|_{L^2} |t - s| \tag{6.22}
\]
\[
\|F(t, \phi) - F(t, \psi)\|_{L^2} \lesssim (\|\phi\|^2_{H^1} + \|\psi\|^2_{H^1}) \|\phi - \psi\|_{L^2}. \tag{6.23}
\]

Let \( \varphi \in C([0, T], H^1) \) be the solution constructed in Lemma 6.1. Our goal will be to show that \( t \in [0, T] \mapsto \varphi_t \in L^2 \) is a Lipschitz map. Indeed, after a change of variables, we are able to write \( \varphi_t = e^{-it\Delta} \varphi_0 - i \int_0^t e^{-is\Delta} F(t - s, \varphi_{t-s}) \, dt \) for all \( t \in (0, T) \). Thus, for all \( h \) small enough we find thanks to (6.22) and (6.23)
\[
\|\varphi_{t+h} - \varphi_t\|_{L^2} \lesssim |h| \|\Delta \varphi_0\|_{L^2} + \int_t^{t+h} \|F(t + h - s, \varphi_{t+s})\|_{L^2} \, ds \\
+ h \|\Delta \varphi_0\|_{L^2} + \int_0^h (\nu^2 |h| + \nu^2 \|\varphi_{t+h} - \varphi_t\|_{L^2}) \, ds, \tag{6.24}
\]
where \( \nu := \sup_{0 \leq t \leq T} (\|\varphi_t\|_{H^1} + |\dot{X}_t|) < \infty \). A second change of variables in the last integral let us apply the Grönwall inequality and conclude that for some \( C > 0 \), independent of \( t \) and \( h \), it holds that
\[
\|\varphi_{t+h} - \varphi_t\|_{L^2} \leq Ce^{C\nu^2t}(\|\Delta \varphi_0\|_{L^2} + \nu^3 + \nu^2 t) |h|. \tag{6.25}
\]

Consequently, the map \( t \in [0, T] \mapsto f(t) := F(t, \varphi_t) \in L^2(\mathbb{R}^3) \) is Lipschitz continuous and, in particular, \( f \in W^{1,1}(0, T; L^2(\mathbb{R}^3)) \). One may then view the mean-field equation for \( \varphi_t \) as a semi-linear Schrödinger equation with source term \( f \). In particular \( W^{1,1} \) regularity of the source term implies that \( H^2 \) regularity of the solution is propagated in time; see [1, Proposition 4.1.16].

In order to obtain global well-posedness, we prove that both the energy of the system and the \( L^2 \) mass of the boson field are constant in time. Consequently, the time \( T > 0 \) introduced in Lemma 6.1—may be used to patch local solutions and cover the real line.

**Lemma 6.3 (Conservation Laws).** Let \( (X, V, \varphi) \in C([0, T]; \mathbb{R}^{6n} \times H^1(\mathbb{R}^3)) \) be as in Lemma 6.1. Then, it holds that for all \( t \in [0, T] \)
\[
\|\varphi_t\|_{L^2} = \|\varphi_0\|_{L^2} \quad \text{and} \quad E(t) = E(0), \tag{6.26}
\]
where \( E(t) \) is the energy of the system defined in (6.1).

**Proof.** Let us first consider the case \( \varphi_0 \in H^2(\mathbb{R}^3) \) and let \( (X, V, \varphi) \) be the local solution of the mean-field equations, as given by Lemma 6.1. Thanks to Lemma 6.2, the boson field remains in \( H^2 \) and satisfies \( \dot{\varphi}_t = -i h(t) \varphi_t \), where we introduce the self-adjoint operator
\[
h(t) := -\Delta + v * |\varphi_t|^2 + \overline{w}(x, X_t), \quad t \in [0, T]. \tag{6.27}
\]

**Conservation of \( L^2 \)-norm.** We differentiate at \( t \in (0, T) \) to find
\[
\frac{d}{dt} \|\varphi_t\|_{L^2}^2 = 2 \text{Re} \langle \varphi_t, i h(t) \varphi_t \rangle_{L^2} = 0 \tag{6.28}
\]
thanks to self-adjointness of \( h(t) \). It suffices to apply the Fundamental Theorem of Calculus.
Conservation of energy. Let \( DE_Z : \mathbb{R}^{6m} \times H^1(\mathbb{R}^3) \to \mathbb{R} \) be the Fréchet derivative of the energy functional \( E \), evaluated at \( Z \). Then, the chain rule gives at \( t \in (0, T) \)
\[
\frac{d}{dt} E(t) = DE_{X_t, V_t, \varphi_t}(\hat{X}_t, \hat{V}_t, \hat{\varphi}_t),
\]
\[
= \int_{\mathbb{R}^3} \nabla X w(x, X_t) |\varphi_t(x)|^2 dx \cdot \hat{X}_t + V_t \cdot \hat{V}_t + 2\text{Re} \langle h(t) \varphi_t, \hat{\varphi}_t \rangle_{L^2}. \tag{6.29}
\]
Thanks to the mean-field equations, the first and second term cancel out. The third term vanishes, thanks to self-adjointness of \( h(t) \).

Finally, for the case \( \varphi_0 \in H^1(\mathbb{R}^3) \) we take a sequence \( (\varphi_{0,n})_{n \geq 1} \subset H^2(\mathbb{R}^3) \) that converges to \( \varphi_0 \) in \( H^1(\mathbb{R}^3) \). In particular, thanks to continuity with respect to initial data, it holds that
\[
\lim_{n \to \infty} \| \varphi_{t,n} - \varphi_t \|_{H^1} + \| \dot{X}_{t,n} - \dot{X}_t \| + \| X_{t,n} - X_t \| = 0, \quad \forall t \in [0, T] \tag{6.30}
\]
where \((X_t, \varphi_t)\) solves \((1.14)\) with initial data \((X_0, V_0, \varphi_0)\) and \((X_{t,n}, \varphi_{t,n})\) solves \((1.14)\) with initial data \((X_0, V_0, \varphi_{0,n})\). Thus, it easily follows that for all \( t \in [0, T] \)
\[
E(t) = E(X_t, \dot{X}_t, \varphi_t) = \lim_{n \to \infty} E(X_{t,n}, \dot{X}_{t,n}, \varphi_{t,n}) = \lim_{n \to \infty} E(X_0, V_0, \varphi_{0,n}) = E_0, \tag{6.31}
\]
which concludes the proof. \( \Box \)

7. Well-posedness of the Regularized, Truncated Dynamics

Let \( N, M \geq 1 \) and \( \varepsilon \in (0, 1) \). Throughout this section, we shall drop the regularization parameter \( \varepsilon \in (0, 1) \) and assume instead that the potentials and the initial data that we work with satisfy the stronger assumptions:
\[
v \in L^\infty(\mathbb{R}^3), \quad w \in S(\mathbb{R}^3), \quad \text{and} \quad \varphi_0 \in H^2(\mathbb{R}^3). \tag{7.1}
\]
We will then only apply well-posedness results presented in this section, to the truncated dynamics. Indeed, we now turn our attention to the study of the Cauchy problem for solutions of the equation
\[
i \partial_t U_N^{(M)}(t, s) = \mathcal{L}_M(t) U_N^{(M)}(t, s), \quad \text{and} \quad U_N^{(M)}(t, t) = 1, \quad t, s \in \mathbb{R}. \tag{7.2}
\]
Here, \( \mathcal{L}_M(t) \) is the regularized, truncated generator introduced in \((3.26)\). We will decompose it in the form
\[
\mathcal{L}_M(t) = \mathbb{H} + \mathcal{I}_M(t), \quad \text{where} \quad \mathbb{H} := -\frac{1}{2N} \Delta X + T_b + \mathcal{I}_4. \tag{7.3}
\]
Remark 7.1. \( \mathbb{H} \) is a self-adjoint operator with a domain \( \mathcal{D}(\mathbb{H}) \) satisfying the analogous characterization \((1.13)\), but with \( N_b^2 \) instead of \( N_b^3 \); this thanks to boundedness of \( v \). In particular, its domain becomes a Banach space when endowed with the norm
\[
\| \Psi \|_{\mathbb{H}} := \| \Psi \| + \| \mathbb{H} \Psi \|, \quad \Psi \in \mathcal{D}(\mathbb{H}). \tag{7.4}
\]
Further, note that \( \mathbb{H} \) commutes with both the particle number operator \( N_b \) and the momentum operator \( P = -i\nabla X \).

Thanks to the particle number cut-off, the potentials being bounded, and the boson field being in \( C^1(\mathbb{R}, L^2_\alpha) \), the interaction term \( \mathcal{I}_M(t) \) enjoys the following two regularity properties.
Lemma 7.1. Let $T \geq 0$ and $N, M \geq 1$. Then, the following holds:

1. $\sup_{t \in \mathbb{R}} \| \mathcal{I}_M(t) \|_{B(\mathcal{H})} < \infty$.

2. There exists $C = C(v, w, T, N, M)$ such that

$$\| \mathcal{I}_M(t) - \mathcal{I}_M(s) \|_{B(\mathcal{H})} \leq C |t - s|, \quad \forall t, s \in [0, T].$$

(7.5)

Proof. (1) Using the Particle Number Estimates for $v, w \in L^\infty(\mathbb{R}^3)$ one finds, thanks to the particle number cut-off, that for all $t \in \mathbb{R}$

$$\| \mathcal{I}_M(t) \|_{B(\mathcal{H})} \lesssim C(N, M)(\|v\|_{L^\infty} + \|w\|_{L^\infty}) (1 + \|\varphi_t\|_{L^2}^2)$$

(7.6)

from which our claims follows easily.

(2) Let $t, s \in [0, T]$ and write the difference as

$$\mathcal{I}_M(t) - \mathcal{I}_M(s) = \chi(N_b \leq M)(N \Delta \mathcal{I}_0(t, s) + N^{1/2} \Delta \mathcal{I}_1(t, s) + \Delta \mathcal{I}_2(t, s) + N^{-1/2} \Delta \mathcal{I}_3(t, s))$$

(7.7)

where each term $\Delta \mathcal{I}_i(t)$ that has been introduced above, depends on a $i$-th power on creation and annihilation operator. Since the estimate for each one of these is very similar, we shall only show one in detail. Indeed, for the cubic term we have that

$$\| \chi(N_b \leq M) \Delta \mathcal{I}_3(t, s) \| \lesssim \| \chi(N_b \leq M) \|_{C^1(\mathbb{R}^6)} v(x - t) a_x^a (\varphi_t(y) - \varphi_s(y)) a_y^a a_x dx dy \|

\lesssim M^{3/2} \left( \sup_{x \in \mathbb{R}^3} v(x - y)^2 |(\varphi_t - \varphi_s)(y)|^2 dy \right)^{1/2}

\lesssim \|v\|_{L^\infty} M^{3/2} \|\varphi_t - \varphi_s\|_{L^2} \lesssim \|v\|_{L^\infty} M^{3/2} \sup_{\tau \in T} \|\partial_{\tau} \varphi_{\tau}\|_{L^2} |t - s|,$$

(7.8)

where, in the last line, we have used $\varphi_t - \varphi_s = \int_0^t \partial_\tau \varphi_{\tau} d\tau$. Since $(t \mapsto \varphi_t) \in C^1(\mathbb{R}, L^2_\mathcal{H})$, the Lipschitz constant of the right hand side of (7.8) is finite. \qed

We combine Lemma 7.1, together with [32, Theorem X.69] and [1, Proposition 4.1.16] to prove existence, uniqueness and basic propagation of regularity for the propagator of the dynamics defined by (7.2). Note that a similar result could have been obtained using the general theory of Kato [24].

Proposition 7.1. There exists a unique unitary propagator $(\mathcal{U}_N^{(M)}(t, s))_{t, s \in \mathbb{R}}$ such that for all $\Psi_0 \in \mathcal{D}(\mathcal{H})$ and $s \in \mathbb{R}$, the map $t \in \mathbb{R} \mapsto \Psi(t) := \mathcal{U}_N^{(M)}(t, s)\Psi_0 \in \mathcal{H}$ satisfies:

1. $\Psi \in C^1(\mathbb{R}, \mathcal{H}) \cap C(\mathbb{R}, \mathcal{D}(\mathcal{H}))$

2. $i\partial_t \Psi(t) = \mathcal{L}_M(t)\Psi(t)$ holds in $\mathcal{H}$.

Remark 7.2. This result can be extended to the following abstract setting. Consider time-dependent operators of the form

$$\mathcal{H}(t) = \mathcal{H}_0 + \mathcal{I}(t), \quad t \in \mathbb{R},$$

(7.9)

where $\mathcal{H}_0$ is self-adjoint with domain $\mathcal{D}(\mathcal{H}_0)$, and $\mathcal{I}(t)$ is bounded, self-adjoint and satisfies assumptions (1) and (2) of Lemma 7.1. Then, there is a unique unitary propagator $U(t, s)$ associated to the evolution of $\mathcal{H}(t)$, in the sense of Proposition 7.1. Note that we do not require $\mathcal{H}_0$ to be bounded from below, nor we require additional assumptions on the derivative of $\mathcal{I}(t)$ or on the commutator $[\mathcal{H}_0, \mathcal{I}(t)]$. 
Proof of Proposition 7.1. Let us first pass to the interaction picture. Namely, we define
\[ \tilde{\mathcal{I}}_M(t) := e^{it\mathcal{I}_M(t)}e^{-it\mathcal{I}_M}, \quad t \in \mathbb{R}. \] (7.10)
Then, Lemma 7.1 and [32, Theorem X.69] imply the existence of a unitary propagator \( \tilde{U}_N^M(t,s) \) – explicitly given by the absolutely convergent Dyson series – that satisfies the integral equation
\[ \tilde{U}_N^M(t,s) = 1 - i \int_s^t \tilde{\mathcal{I}}_M(r)\tilde{U}_N^M(r,s)dr, \quad t, s \in \mathbb{R}. \] (7.11)
Consequently, for \( t, s \in \mathbb{R} \) we let our original propagator be \( U_N^M(t,s) := e^{-iHt}\tilde{U}_N^M(t,s)e^{isH} \).

Let now \( \Psi(t) \) be as in the statement of the proposition. Then, it is straightforward to verify that for all \( t, s \in \mathbb{R} \) it holds that
\[ \Psi(t) = e^{-i(t-s)H}\Psi_0 - i \int_s^t e^{-i(t-r)H}\mathcal{I}_M(r)\Psi(r)dr. \] (7.12)
Since the interaction term \( \mathcal{I}_M(t) \) is locally Lipschitz continuous, we may adapt the argument presented in the proof of Lemma 6.2 to show that the map \( t \in \mathbb{R} \mapsto \Psi(t) \in \mathcal{H} \) is locally Lipschitz continuous, as well. In particular, the source term \( t \mapsto F(t) := \mathcal{I}_M(t)\Psi(t) \in \mathcal{H} \) becomes locally Lipschitz continuous and, therefore, belongs to \( W^{1,1}_{loc}(\mathbb{R}, \mathcal{H}) \). We may then apply [1, Proposition 4.1.6] to conclude propagation of regularity, in the sense that (1) and (2) hold true. Uniqueness follows from standard arguments using the symmetry of \( \mathcal{L}_M(t) \) and denseness of \( \mathcal{D}(\mathcal{H}) \subset \mathcal{H} \). \( \square \)

Next, we show that the regularized truncated dynamics propagates smoothness with respect to the tracer particle variables, together with boundedness with respect to the particle number operator and its powers. We remind the reader that the space \( \mathcal{D}_\infty \) has been introduced in (5.1)

Proposition 7.2. In the notation of Proposition 7.1, if \( \Psi_0 \in \mathcal{D}_\infty \), then \( \Psi(t) \in \mathcal{D}_\infty \) for all \( t \in \mathbb{R} \).

Proof. Let \( N, M \geq 1 \) be fixed and assume \( s = 0 \) for simplicity. We will be using a fixed-point argument. Indeed, let \( k \in \mathbb{N} \cup \{0\} \) and consider the following Banach space
\[ \mathcal{H}_k := \{ \Psi \in \mathcal{H} \mid \|\Psi\|_{\mathcal{H}_k} < \infty \} \] (7.13)
which we endow with the norm
\[ \|\Psi\|_{\mathcal{H}_k} := \|X|^k\Psi\|_{\mathcal{H}} + \|P|^k\Psi\|_{\mathcal{H}} + \|N_b^k\Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}. \] (7.14)

Step 1. First, we show that the interaction term \( \mathcal{I}_M(t) \) is continuous with respect to \( \mathcal{H}_k \), uniformly in \( t \in \mathbb{R} \). To this end, let \( \alpha \) be a multi-index with \( |\alpha| \leq k \). Then, the terms in \( \mathcal{I}_M(t) \) that depend on \( X \in \mathbb{R}^{3m} \) satisfy the following estimate
\[ \|C_X^\alpha \left( \int_{\mathbb{R}^m} \mathfrak{w}(x, X) \chi(N_b \leq M)(N_1 \varphi_1^2 + \sqrt{N}a_x\varphi_1(x) + a_x^*a_x)dx \right) \|_{\mathcal{H}} \leq C(k)\|\mathfrak{w}\|_{C_k^b} \|N + \sqrt{NM} + M\|\Psi\|_{\mathcal{H}_k} \] (7.15)
where \( C(k) > 0 \) depends only on \( k \). The other terms that show up in \( \mathcal{I}_M(t) \) do not depend on the tracer particle variables \( X \in \mathbb{R}^{3m} \) and can be controlled analogously, thanks to the particle number cut-off. We put all of the terms together and find that for all \( t \in \mathbb{R} \) it holds that
\[ \|P|^k\mathcal{I}_M(t)\Psi\|_{\mathcal{H}} \lesssim \left( N + \sqrt{NM} + M + \sqrt{M^3/N} \right) \|\Psi\|_{\mathcal{H}_k}, \quad \Psi \in \mathcal{H}_k. \] (7.16)
Next, for the particle number operator we can use the particle number cut-off to easily find that the following (rough) estimate holds
\[ \| N^k_b I_M(t) \Psi \|_{\mathcal{F}} \lesssim M^k \sup_{t \in \mathbb{R}} \| I_M(t) \|_{B(\mathcal{F})} \| \Psi \|_{\mathcal{F}} \lesssim M^k \sup_{t \in \mathbb{R}} \| I_M(t) \|_{B(\mathcal{F})} \| \Psi \|_{\mathcal{H}_k}. \]  
(7.17)

For the position variables, simply note that \( X \) commutes with \( I_M(t) \). Thus, we put these estimates together to find that for all \( t \in \mathbb{R} \)
\[ \| I_M(t) \Psi \|_{\mathcal{H}_k} \lesssim (N + M + \sqrt{M^3/K}) + M^k \sup_{t \in \mathbb{R}} \| I_M(t) \|_{B(\mathcal{F})} \| \Psi \|_{\mathcal{H}_k}. \]  
(7.18)

**Step 2.** Secondly, we show that the evolution group \( (e^{-iH_N})_{t \in \mathbb{R}} \) is continuous with respect to \( \mathcal{H}_k \), locally uniformly in time. To this end, we note that in one dimension the following estimate holds for the free Schrödinger propagator
\[ \| X^k e^{-itP^2} \psi \|_{L^2(\mathbb{R})} = \| (X + tP)^k \psi \|_{L^2(\mathbb{R})} \lesssim C(k)\| X^k \psi \|_{L^2(\mathbb{R})} + t^k \| P^k \psi \|_{L^2(\mathbb{R})} + \| \psi \|_{L^2(\mathbb{R})} \]  
(7.19)
as it can be verified by using the commutation relation \( XP = PX + i \) to control mixed powers. This in turn easily implies that
\[ \| e^{-iI_N^k} \|_{\mathcal{H}_k} = \| X^k e^{-itP^2} \psi \|_{\mathcal{H}_k} + \| P^k \|_{\mathcal{H}_k} + \| N^k_b \|_{\mathcal{H}_k} \lesssim C(1 + t^k) \]  
(7.20)
for some constant \( C = C(k) > 0 \) and all \( t \in \mathbb{R} \).

**Step 3.** Finally, we use the continuity estimates (7.18) and (7.20) to set up a fixed-point argument in \( C([0,T], \mathcal{H}_k) \) to show that the equation
\[ \hat{\Psi}(t) = e^{-itI_N^k} \Psi_0 - i \int_0^t e^{-i(t-r)I_N^k} I_M(r) \hat{\Psi}(r) \, dr. \]
has a unique solution. In particular, if \( \Psi(t) \) is the solution constructed in Proposition 7.1, it holds that \( \Psi(t) = \hat{\Psi}(t) \in \mathcal{H}_k \) for all \( t \in [0,T] \), thanks to uniqueness of solutions in \( C([0,T], \mathcal{H}) \) (i.e. the \( k = 0 \) case). Further, \( T < 1 \) can be chosen to depend only on the constants that show up in the continuity estimates. Thus, we can iterate the argument to cover \( \mathbb{R} \). Since \( k \in \mathbb{N} \) was arbitrary, this concludes the proof.

**Appendix A. The Regularization Lemmata**

*Proof of Lemma 3.1.* Let us fix \( N \geq 1 \) and \( t \geq 0 \). Then, the triangle inequality and the unitarity of the evolution group \( (e^{-iH_N^\varepsilon})_{t \in \mathbb{R}} \) implies
\[ \| \Psi_{N,t} - \Psi_{N,0}^\varepsilon \| \lesssim \| \Psi_{N,0} - \Psi_{N,0}^\varepsilon \| + \| (e^{-iH_N^\varepsilon} - e^{-iH_N^\varepsilon}) \Psi_{N,0} \|, \quad \forall \varepsilon \in (0,1). \]  
(A.1)

We shall estimate these terms separately.

**The First Term.** The triangle inequality implies that
\[ \| \Psi_{N,0} - \Psi_{N,0}^\varepsilon \| \lesssim \| \mathcal{W}(\sqrt{N} \varphi_0) \Omega - \mathcal{W}(\sqrt{N} \varphi_0 ) \Omega \|_{\mathcal{F}_b} + \| u_{N,0} - u_{N,0}^\varepsilon \|_{L^2_\mathcal{X}} \]  
(A.2)
where we used the fact that the norms of \( u_{N,0}^\varepsilon \) and \( \mathcal{W}(\sqrt{N} \varphi_0 ) \Omega \) are uniformly bounded in \( N \) and \( \varepsilon \). It is known [16, Lemma 3.1] that the map \( f \in L^2 \rightarrow \mathcal{W}(f) \Phi \in \mathcal{F}_b \) is continuous for every \( \Phi \in \mathcal{F}_b \). Thus, we apply the definitions of the regularized quantities \( u_{N,0}^\varepsilon \) and \( \varphi_0^\varepsilon \) to conclude that the right hand side of (A.2) vanishes as \( \varepsilon \downarrow 0 \).

**The Second Term.** Since the Hamiltonians preserve particle number, for any \( n_0 \in \mathbb{N} \) we may write
\[ \| (e^{-iH_N^\varepsilon} - e^{-iH_N^\varepsilon}) \Psi_{N,0} \| \lesssim \| \chi(N_b \leq n_0)(e^{-iH_N^\varepsilon} - e^{-iH_N^\varepsilon}) \Psi_{N,0} \| + \| \chi(N_b > n_0) \Psi_{N,0} \|. \]  
(A.3)
Let us decompose $e^{-it\mathcal{H}_X^\varepsilon}\Psi_{N,0} = (\psi_n(t))_{n \geq 0}$ and $e^{-it\mathcal{H}_X}\Psi_{N,0} = (\psi_n^\varepsilon(t))_{n \geq 0}$ according to their direct sum representation in $\bigoplus_{n \geq 0} L^2_X \otimes F_n$. Then, we obtain the following standard estimates

$$\|\psi_n(t) - \psi_n^\varepsilon(t)\|^2_{L^2_X \otimes F_n}$$

(A.4)

$$\lesssim \int_0^t \|\langle \psi_n(s), (H_{N,n} - H_{N,n}^\varepsilon)\psi_n^\varepsilon(s)\rangle\|_{L^2_X \otimes F_n} \, ds \quad \text{ (in the sense of quad. forms)}$$

$$\lesssim \int_0^t \|\langle \psi_n(s), \left( \frac{n^2}{N}(v - v^\varepsilon)(x_1 - x_2) + n(w - w^\varepsilon)(x_1, X) \right)\psi_n^\varepsilon(s)\rangle\|_{L^2_X \otimes F_n} \, ds$$

$$\lesssim \int_0^t \left( \frac{n}{N} \|v - v^\varepsilon\|_{L^{3/2}} + n \|w - w^\varepsilon\| C_b \right) \|\psi_n(s)\|_{L^2_X \otimes H^1(\mathbb{R}^3)} \|\psi_n^\varepsilon(s)\|_{L^2_X \otimes H^1(\mathbb{R}^3)} \, ds.$$  

Here, we have used $|\langle \varphi_1, \psi_2 \rangle_{L^2(\mathbb{R}^3)}| \lesssim \|v\|_{L^{3/2}} \|\varphi_1\|_{L^6} \|\psi_2\|_{L^6} \lesssim \|v\|_{L^{3/2}} \|\varphi_1\|_{H^1} \|\psi_2\|_{H^1}$, which is a consequence of Hölder’s inequality and Sobolev’s Embedding Theorem for $d = 3$. It is also a standard exercise to check that, for each $n \in \mathbb{N}$, the $H^1$ norms of $\psi_n(t)$ and $\psi_n^\varepsilon(t)$ are uniformly bounded in $t \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. Therefore, for each fixed $n \in \mathbb{N}$, it holds that

$$\lim_{\varepsilon \downarrow 0} \|\psi_n(t) - \psi_n^\varepsilon(t)\|^2_{L^2_X \otimes F_n} = 0, \quad \forall t \in \mathbb{R}. \quad \text{(A.5)}$$

Thus, it easily follows now from (A.3) that for all $n_0 \in \mathbb{N}$

$$\limsup_{\varepsilon \downarrow 0} \| (e^{-it\mathcal{H}_X^\varepsilon} - e^{-it\mathcal{H}_X})\Psi \| \lesssim \|\chi(N_b > n_0)\Psi_{N,0}\|. \quad \text{(A.6)}$$

The proof is complete after we take the limit $n_0 \to \infty$.  

\[\square\]

**Proof of Lemma 3.2.** Throughout the proof, we will use the fact that the $H^1$ norms of both $\varphi_t$ and $\varphi_t^\varepsilon$ are uniformly bounded in $t$ and $\varepsilon$. They will be absorbed into universal constants.

**The Boson Field.** First, we note that difference between the boson fields may be written as

$$\varphi_t - \varphi_t^\varepsilon = e^{-it\Delta}(\varphi_0 - \varphi_0^\varepsilon) - i \int_0^t e^{-i(t-s)\Delta} \left( (v * |\varphi_s|^2)\varphi_s - (v^\varepsilon * |\varphi_s^\varepsilon|^2)\varphi_s + w(x, X_s)\varphi_s - w^\varepsilon(x, X_s)\varphi_s^\varepsilon \right) \, ds.$$  

Under our condition for the potential $v$, it is known that the map $\varphi \in H^1(\mathbb{R}^3) \mapsto (v * |\varphi|^2)\varphi \in H^1(\mathbb{R}^3)$ is locally Lipschitz continuous, i.e. it satisfies (2.23). Therefore, we apply the triangle inequality, a first order Taylor estimate and Lipschitz continuity to find that

$$\|\varphi_t - \varphi_t^\varepsilon\|_{H^1} \lesssim \|\varphi_0 - \varphi_0^\varepsilon\|_{H^1} + \int_0^t \|v - v^\varepsilon\| \|\varphi_t^\varepsilon\|^2_{H^1} \, ds.$$  

(A.8)

$$+ \int_0^t \left( \|w\|_{C^1_b} + \|\varphi_s\|^2_{H^1} + \|\varphi_s^\varepsilon\|^2_{H^1} \right) \|\varphi_s - \varphi_s^\varepsilon\|_{H^1} + \|X_s - X_s^\varepsilon\| + \|w - w^\varepsilon\|_{C^1_b} \, ds$$

For the term containing the difference $\delta v = v - v^\varepsilon$, we use the Leibniz rule to find that

$$\|(\delta v * |\varphi|^2)\varphi\|_{H^1} \lesssim \|(\delta v * |\varphi|^2)\varphi\|_{L^2} + \|(\delta v * \varphi^2)\varphi\|_{L^2} + \|(\delta v * |\varphi|^2)\nabla \varphi\|_{L^2}. \quad \text{(A.9)}$$

In addition, we note that an application of Hölder’s and Young’s inequality gives

$$\|(\delta v * \varphi^2)\varphi\|_{L^2} \lesssim \|\delta v\|_{L^{3/2}} \|\varphi\|_{L^6} \|\phi\|_{L^6} \|\psi\|_{L^6}, \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{5}{6}. \quad \text{(A.10)}$$
We apply the last estimate for \((p_1, p_2, p_3) = (6, 2, 6)\) and \((p_1, p_2, p_3) = (6, 6, 2)\), combined with the embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\), to find that
\[
\|\langle \nabla v \ast \varphi^\varepsilon_t \rangle \varphi^\varepsilon_t \rangle_{H^1} \lesssim \|\nabla v\|_{L^3(\mathbb{R}^3)} \|\varphi^\varepsilon_t \rangle_{H^1} \lesssim \varepsilon. \tag{A.11}
\]

**Tracer Particle Variables.** Similarly as before, we may use a Taylor estimate and the triangle inequality to find that
\[
|X_t - X^\varepsilon_t| \lesssim \int_0^t |\dot{X}_s - \dot{X}^\varepsilon_s| ds \tag{A.12}
\]
\[
|\dot{X}_t - \dot{X}^\varepsilon_t| \lesssim \int_0^t (\|w\|_{C^2} + \|\varphi_s\|_{L^2}^2 + \|\varphi^\varepsilon_s\|_{L^2}^2)(\|\varphi_s - \varphi^\varepsilon_s\|_{L^2} + |X_s - X^\varepsilon_s| + \|w - w^\varepsilon\|_{C^1}) ds.
\]

We combine the estimates for the boson field and the tracer particle position and apply the Grönwall inequality to the quantity \(\|\varphi_t - \varphi^\varepsilon_t\|_{H^1} + |X_t - X^\varepsilon_t| + |\dot{X}_t - \dot{X}^\varepsilon_t|\) to obtain
\[
\|\varphi_t - \varphi^\varepsilon_t\|_{H^1} + |X_t - X^\varepsilon_t| + |\dot{X}_t - \dot{X}^\varepsilon_t| \lesssim e^{Ct} (1 + t)(\varepsilon + \|\varphi_0 - \varphi^\varepsilon_0\|_{H^1} + \|w - w^\varepsilon\|_{C^1}), \tag{A.13}
\]
for some \(C > 0\). The proof of the lemma is finished once we apply the definitions of the regularized objects and take the limit \(\varepsilon \downarrow 0\). \(\square\)

**Appendix B. The Interpolation Lemma**

In this appendix we give a proof of the Interpolation Lemma, stated as Lemma 5.4.

**Proof.** Let \(\Psi\) be arbitrary and assume \(\lambda \in \mathbb{R}\) for simplicity. Then, using the commutation relations we find that
\[
\|ABA\Psi\|^2 = \langle \Psi, AB(A^2B)A\Psi \rangle = \langle \Psi, AB^2A^3\Psi \rangle - 2i\lambda \langle \Psi, ABA^2\Psi \rangle. \tag{B.1}
\]
Then, using the Cauchy-Schwarz inequality and Young’s inequality, we find that for all \(\varepsilon > 0\) there is \(C_\varepsilon\) (independent of \(\Psi\)) such that
\[
|\langle \Psi, AB^2A^3\Psi \rangle| \leq \varepsilon \|AB^2\Psi\|^2 + C_\varepsilon \|A^3\Psi\|^2,
\]
\[
|\langle \Psi, ABA^2\Psi \rangle| \leq \varepsilon \|ABA\Psi\|^2 + C_\varepsilon \lambda^2 \|A\Psi\|^2.
\]
If we let \(M(A, B) := \|A^3\Psi\|^2 + \|B^3\Psi\|^2 + \lambda^2 \|A\Psi\|^2 + \lambda^2 \|B\Psi\|^2\), we have found that
\[
\|ABA\Psi\|^2 \leq 2\varepsilon (\|AB^2\Psi\|^2 + \|ABA\Psi\|^2) + 2C_\varepsilon M(A, B). \tag{B.2}
\]
Using \(\|AB^2\Psi\|^2 \leq 2\|ABA\Psi\|^2 + 2\lambda^2 \|A\Psi\|^2\) and \(\|BA^2\Psi\|^2 \leq 2\|ABA\Psi\|^2 + 2\lambda^2 \|A\Psi\|^2\) we find (after re-updating \(C_\varepsilon\))
\[
\|ABA\Psi\|^2 + \|AB^2\Psi\|^2 + \|BA^2\Psi\|^2 \leq 10\varepsilon (\|AB^2\Psi\|^2 + \|ABA\Psi\|^2) + C_\varepsilon M(A, B). \tag{B.3}
\]
Similarly, we can interchange the roles of \(A\) and \(B\) to find that
\[
\|BAB\Psi\|^2 + \|B^2A\Psi\|^2 + \|AB^2\Psi\|^2 \leq 10\varepsilon (\|BA^2\Psi\|^2 + \|BABA\Psi\|^2) + C_\varepsilon M(A, B). \tag{B.4}
\]
A straightforward combination of the last two inequalities, for \(\varepsilon < 1/20\), finishes the proof of the lemma. \(\square\)

**Acknowledgments.** I am deeply grateful to Thomas Chen for mentoring, reading this manuscript, for holding several encouraging discussions, and for giving me the opportunity to work on the problem of tracer particles interacting with bosons. I would also like to thank Michael Hott for giving useful references and insight regarding singular Hartree-type potentials. I gratefully acknowledge support from the Provost’s Graduate Excellence Fellowship.
at The University of Texas at Austin, and by the NSF grants DMS-1716198 and DMS-2009800 through T. Chen, and by the NSF RTG Grant DMS-1840314 “Analysis of PDE”.

References

[1] T. Cazenave, A. Haraux, An introduction to semilinear evolution equations, Oxford Lecture Series in Mathematics and its Applications, 13. Oxford University Press, 1998.

[2] T. Chen, C. Hainzl, N. Pavlović, R. Seiringer, Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti, Commun. Pure Appl. Math., 68 (10), 1845–1884, 2015.

[3] X. Chen, J. Holmer, On the Klainerman-Machedon conjecture for the quantum BBGKY hierarchy with self-interaction, J. Eur. Math. Soc. (JEMS) 18 (6), 1161–1200 (2016).

[4] X. Chen, J. Holmer, Correlation structures, many-body scattering processes, and the derivation of the Gross-Pitaevskii hierarchy, Int. Math. Res. Not. IMRN 2016: 10, 3051–3110 (2016).

[5] T. Chen, N. Pavlović, On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies, Discr. Contin. Dyn. Syst., 27 (2), 715–739 (2010).

[6] T. Chen, N. Pavlović, Derivation of the cubic NLS and Gross-Pitaevskii hierarchy from many-body dynamics in $d = 3$ based on spacetime norms, Ann. Henri Poincaré, 15 (3), 543–588, (2014).

[7] T. Chen, A. Soffer, Mean field dynamics of a quantum tracer particle interacting with a boson gas, J. Funct. Anal., 276 (3), 971–1006, (2019).

[8] D.-A. Deckert, J. Fröhlich, P. Pickl, A. Pizzo, Effective dynamics of a tracer particle interacting with an ideal Bose gas, Comm. Math. Phys. 328 (2), 597-624 (2014).

[9] L. Erdős, B. Schlein, H.-T. Yau, Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate, Comm. Pure Appl. Math. 59 (12), 1659–1741 (2006).

[10] L. Erdős, B. Schlein, H.-T. Yau, Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems, Invent. Math. 167 (2007), 515–614.

[11] L. Erdős, B. Schlein, H.-T. Yau, Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential, J. Amer. Math. Soc. 22 (4), 1099–1156 (2009).

[12] L. Erdős, B. Schlein, H.-T. Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensates, Ann. of Math. (2), 172 (1), 291–370 (2010).

[13] J. Fröhlich, Z. Gang, Ballistic motion of a tracer particle coupled to a Bose gas, Adv. Math. 259, 252-268 (2014).

[14] J. Fröhlich, Z. Gang, Emission of Cherenkov radiation as a mechanism for Hamiltonian friction, Adv. Math. 264, 183-235 (2014).

[15] J. Fröhlich, Z. Gang, A. Soffer, Friction in a model of Hamiltonian dynamics, Comm. Math. Phys. 315 (2), 401-444 (2012).

[16] J. Ginibre, G. Velo, The classical field limit of scattering theory for nonrelativistic many-boson systems. I, Comm. Math. Phys. 66, 37–76 (1979).

[17] J. Ginibre, G. Velo, The classical field limit of scattering theory for non-relativistic many-boson systems. II, Comm. Math. Phys. 68, 45–68 (1979).

[18] P. Gressman, V. Sohinger, G. Staffilani, On the uniqueness of solutions to the periodic 3D Gross-Pitaevskii hierarchy, J. Funct. Anal. 266 (7), 4705–4764 (2014).

[19] M. Grillakis, M. Machedon, A. Margetis, Second-order corrections to mean field evolution for weakly interacting Bosons. I, Comm. Math. Phys. 294 (1), 273–301 (2010).

[20] M. Grillakis, M. Machedon, Pair excitations and the mean field approximation of interacting bosons, I, Comm. Math. Phys. 324 (2), 601–636 (2013).

[21] M. Grillakis, M. Machedon, Pair excitations and the mean field approximation of interacting bosons, II, Comm. Partial Differential Equations 42 (1), 24–67 (2017).

[22] K. Hepp, The classical limit for quantum mechanical correlation functions, Comm. Math. Phys. 35, 265–277 (1974).

[23] M. Hott. Convergence rate towards the fractional Hartree-equation with singular potentials in higher Sobolev trace norms, Reviews in Mathematical Physics, to appear.

[24] T. Kato Linear evolution equations of “hyperbolic” type, II, J. Math. Soc. Japan 25 (4): 648-666.

[25] K. Kirkpatrick, B. Schlein, G. Staffilani, Derivation of the two dimensional nonlinear Schrödinger equation from many body quantum dynamics, Amer. J. Math. 133 (1), 91–130 (2011).
[26] S. Klainerman, M. Machedon, *On the uniqueness of solutions to the Gross-Pitaevskii hierarchy*, Comm. Math. Phys. **279** (1), 169–185, (2008).

[27] J. Lampart, P. Pickl, *Dynamics of a tracer particle interacting with excitations of a Bose-Einstein condensate*, arXiv:2011.14428.

[28] E. Lenzmann, *Well-posedness for semi-relativistic Hartree equations of critical type*, Mathematical Physics, Analysis and Geometry, **10** (1), 43–64 (2007).

[29] M. Lewin, P.T. Nam, N. Rougerie, *Derivation of Hartree’s theory for generic mean-field Bose systems*, Adv. Math. **254**, 570-621 (2014).

[30] M. Lewin, P.T. Nam, B. Schlein, *Fluctuations around Hartree states in the mean-field regime*, Amer. J. Math. **137** (6), 1613–1650 (2015).

[31] P. Pickl, *A simple derivation of mean field limits for quantum systems*, Lett. Math. Phys., **97** (2), 151 – 164 (2011).

[32] M. Reed, B. Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Methods of Modern Mathematical Physics. Elsevier Science, 1975.

[33] I. Rodnianski, B. Schlein, *Quantum fluctuations and rate of convergence towards mean field dynamics*, Comm. Math. Phys. **291** (1), 31–61, (2009).

[34] H. Spohn, *Kinetic Equations from Hamiltonian Dynamics*, Rev. Mod. Phys. **52** (3), 569–615 (1980).

[35] H. Spohn, *On the Vlasov Hierarchy*, Math. Meth. in the Appl. Sci., **3**, 445–455 (1981).

(Esteban Cárdenas) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, 2515 SPEEDWAY, AUSTIN TX, 78712, USA

Email address: eacardenas@utexas.edu