ON THE R-BOUNDEDNESS OF SOLUTION OPERATOR FAMILIES FOR TWO-PHASE STOKES RESOLVENT EQUATIONS

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ABSTRACT. The aim of this paper is to show the existence of \( R \)-bounded solution operator families for two-phase Stokes resolvent equations in \( \Omega = \Omega_+ \cup \Omega_- \), where \( \Omega_\pm \) are uniform \( W^{2-1/r}_r \) domains of \( N \)-dimensional Euclidean space \( \mathbb{R}^N \) \( (N \geq 2, N < r < \infty) \). More precisely, given a uniform \( W^{2-1/r}_r \) domain \( \Omega \) with two boundaries \( \Gamma_\pm \) satisfying \( \Gamma_+ \cap \Gamma_- = \emptyset \), we suppose that some hypersurface \( \Gamma \) divides \( \Omega \) into two sub-domains, that is, there exist domains \( \Omega_{\pm} \subseteq \Omega \) such that \( \Omega_+ \cap \Omega_- = \emptyset \) and \( \Omega \setminus \Gamma = \Omega_+ \cup \Omega_- \), where \( \Gamma \setminus \Gamma_+ = \emptyset \), \( \Gamma \setminus \Gamma_- = \emptyset \), and the boundaries of \( \Omega_{\pm} \) consist of two parts \( \Gamma_+ \) and \( \Gamma_- \), respectively. The domains \( \Omega_{\pm} \) are filled with viscous, incompressible, and immiscible fluids with density \( \rho_\pm \) and viscosity \( \mu_\pm \), respectively. Here \( \rho_\pm \) are positive constants, while \( \mu_\pm = \mu_\pm (x) \) are functions of \( x \in \mathbb{R}^N \). On the boundaries \( \Gamma_+, \Gamma_- \), and \( \Gamma = \emptyset \), we consider an interface condition, a free boundary condition, and the Dirichlet boundary condition, respectively.

We also show, by using the \( R \)-bounded solution operator families, some \( L_{p} \) regularity as well as generation of analytic semigroup for a time-dependent problem associated with the two-phase Stokes resolvent equations. This kind of problems arises in the mathematical study of the motion of two viscous, incompressible, and immiscible fluids with free surfaces. The essential assumption of this paper is the unique solvability of a weak elliptic transmission problem for \( f \in L_{q}(\Omega)^{N} \), that is, it is assumed that the unique existence of solutions \( \theta \in W^{1}_{q}(\Omega) \) of the variational problem: \( (\rho^{-1} \nabla \theta, \nabla \phi)_{\Omega} = (f, \nabla \phi)_{\Omega} \) for any \( \varphi \in W^{1}_{q}(\Omega) \) with \( 1 \leq q < \infty \) and \( q' = q/(q - 1) \), where \( \rho \) is defined by \( \rho = \rho_{\pm} \) \( (x \in \Omega_{\pm}) \), \( \rho = \rho_{-} \) \( (x \in \Omega_-) \) and \( W^{1}_{q}(\Omega) \) is a suitable Banach space endowed with norm \( \| \cdot \|_{W^{1}_{q}(\Omega)} := \| \nabla \cdot \|_{L_{q}(\Omega)} \). Our assumption covers e.g. the following domains as \( \Omega: \mathbb{R}^N, \mathbb{R}_+^N, \mathbb{R}_-^N \), perturbed \( \mathbb{R}_+^N, \mathbb{R}_-^N \) layers, perturbed layers, and bounded domains, where \( \mathbb{R}_+^N \) and \( \mathbb{R}_-^N \) are the open upper and lower half spaces, respectively.

1. INTRODUCTION

1.1. Problem. Let \( \Omega \) be a domain of \( \mathbb{R}^N \), \( N \geq 2 \), with two boundaries \( \Gamma_\pm \) satisfying \( \Gamma_+ \cap \Gamma_- = \emptyset \). Suppose that some hypersurface \( \Gamma \) divides \( \Omega \) into two sub-domains, that is, there exist domains \( \Omega_{\pm} \subseteq \Omega \) such that \( \Omega_+ \cap \Omega_- = \emptyset \) and \( \Omega \setminus \Gamma = \Omega_+ \cup \Omega_- \), where \( \Gamma_+ \cap \Gamma_- = \emptyset \), \( \Gamma_+ \cap \Gamma_- = \emptyset \), and the boundaries of \( \Omega_{\pm} \) consist of two parts \( \Gamma_+ \) and \( \Gamma_- \), respectively. Set \( \Omega = \Omega_+ \cup \Omega_- \) and \( \Sigma_{\epsilon,\lambda_0} = \{ \lambda \in \mathbb{C} \mid | \arg \lambda | \leq \pi - \epsilon, | \lambda | \geq \lambda_0 \} \) for \( 0 < \epsilon < \pi/2 \) and \( \lambda_0 > 0 \). In this paper, we show the existence of \( R \)-bounded solution operator families for the following two-phase Stokes resolvent equations with resolvent parameter \( \lambda \) varying in \( \Sigma_{\epsilon,\lambda_0} \):

\[
\begin{align*}
\lambda \mathbf{u} - \rho^{-1} \nabla \mathbf{v}(\mathbf{u}, \theta) &= \mathbf{f}, & \text{div } \mathbf{u} &= g & \text{in } \hat{\Omega}, \\
\mathbf{T}(\mathbf{u}, \theta) \mathbf{n} &= [h], & [\mathbf{u}] &= 0 & \text{on } \Gamma, \\
\mathbf{T}(\mathbf{u}, \theta) \mathbf{n}_+ &= k & \text{on } \Gamma_+, \\
\mathbf{u} &= 0 & \text{on } \Gamma_-.
\end{align*}
\]

(1.1)

Here the unknowns \( \mathbf{u} = (u_1(x), \ldots, u_N(x))^{T} \) and \( \theta = \theta(x) \) are an \( N \)-vector function and a scalar function, respectively, while the right members \( f = (f_1(x), \ldots, f_N(x))^{T} \), \( g = g(x) \), \( h = (h_1(x), \ldots, h_N(x))^{T} \), and \( k = (k_1(x), \ldots, k_N(x))^{T} \) are given functions. Let \( \rho_\pm \) be positive constants and \( \mu_\pm = \mu_\pm (x) \) scalar functions defined on \( \mathbb{R}^N \), and let \( \varphi_\theta \) be the indicator function of \( D \subseteq \mathbb{R}^N \). Then \( \rho = \rho_+ \chi_{\Omega_+} + \rho_- \chi_{\Omega_-} \), \( \mu = \mu_+ \chi_{\Omega_+} + \mu_- \chi_{\Omega_-} \), and \( \mathbf{T}(\mathbf{u}, \theta) = \mu \mathbf{D}(\mathbf{u}) - \mathbf{I} \), where \( \mathbf{I} \) is \( \mathbb{N} \times \mathbb{N} \) identity matrix and \( \mathbf{D}(\mathbf{u}) \) is the doubled deformation tensor, that is, the \((i,j)\)-entry \( D_{ij}(\mathbf{u}) \) of \( \mathbf{D}(\mathbf{u}) \) is given by \( D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i \) for \( i, j = 1, \ldots, N \) and \( \partial_i = \partial/\partial x_i \). In

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\(^{\dagger}\)\( M^T \) denotes the transposed \( M \).
addition, $\mathbf{n}$ denotes on $\Gamma$ a unit normal vector, pointing from $\Omega_+$ to $\Omega_-$, while $\mathbf{n}_+$ the unit outward normal vector on $\Gamma_+$. For any function $f$ defined on $\Omega$, $[f]$ denotes a jump of $f$ across the interface $\Gamma$ as follows:

$$[f] = [f](x) = \lim_{y \to x, y \in \Omega_+} f(y) - \lim_{y \to x, y \in \Omega_-} f(y) \quad (x \in \Gamma).$$

Here and subsequently, we use the following symbols for differentiations:

Let $f = f(x)$, $g = (g_1(x), \ldots, g_N(x))^T$, and $\mathbf{M} = (M_{ij}(x))$ be a scalar, an $N$-vector, and an $N \times N$-matrix function defined on a domain of $\mathbb{R}^N$, respectively, and then

$$\nabla f = (\partial_1 f(x), \ldots, \partial_N f(x))^T, \quad \Delta f = \sum_{j=1}^N \partial_j^2 f(x), \quad \Delta g = (\Delta g_1(x), \ldots, \Delta g_N(x))^T,$$

$$\operatorname{div} g = \sum_{j=1}^N \partial_j g_j(x), \quad \nabla^2 g = \{ \partial_i \partial_j g_k(x) \mid i, j, k = 1, \ldots, N \},$$

$$\nabla \mathbf{g} = \begin{pmatrix}
\partial_1 g_1(x) & \cdots & \partial_N g_1(x) \\
\vdots & \ddots & \vdots \\
\partial_1 g_N(x) & \cdots & \partial_N g_N(x)
\end{pmatrix}, \quad \operatorname{Div} \mathbf{M} = \begin{pmatrix}
\sum_{j=1}^N \partial_j M_{1j}(x), \ldots, \sum_{j=1}^N \partial_j M_{Nj}(x)
\end{pmatrix}^T.$$

Problem (1.1) arises from a linearized system of some two-phase problem of the Navier-Stokes equations for viscous, incompressible, and immiscible fluids without taking surface tension into account. There are a lot of studies of two-phase problems for the Navier-Stokes equations. To see the history of study briefly, we restrict ourselves to the case where the two fluids are both viscous, incompressible, and immiscible in the following. Such a situation was treated in several function spaces as follows:

$L_2$-in-time and $L_2$-in-space setting. Denisova [2, 4] treated the motion of a drop $\Omega_{+t}$, which is the region occupied by the drop at time $t > 0$, in another liquid $\Omega_{-t} = \mathbb{R}^3 \setminus \overline{\Omega_{+t}}$. More precisely, [2] showed some estimates of solutions for linearized problems and [4] an unique existence theorem local in time for the two-phase problem describing the aforesaid situation with or without surface tension. In addition, Denisova [2] proved the unique existence of global-in-time solutions for small initial data and its exponential stability in the case where $\Omega_{-t}$ is bounded and surface tension does not work. Concerning non-homogeneous incompressible fluids, Tanaka [30] showed the unique existence of global-in-time solutions for small initial data when $\Omega_{-t}$ is bounded, but surface tension is taken into account.

Hölder function spaces. A series of papers Denisova-Solonnikov [9, 10] and Denisova [3] treated the same motion as in [2, 4] mentioned above. Especially, [9] and [3] established estimates of solutions for some linearized problems, and [10] proved an unique existence theorem local in time for the two-phase problem with surface tension. On the other hand, the unique existence of global-in-time solutions for small initial data was proved by Denisova [6] without surface tension and by Denisova-Solonnikov [11] with surface tension in the case where $\Omega_{-t}$ is bounded. Furthermore, there are other topics Denisova [5] and Denisova-Nečasová [8], which consider thermocapillary convection and Oberbeck-Boussinesq approximation, respectively.

$L_p$-in-time and $L_p$-in-space setting. Prüss and Simonett [20, 21, 22] treated a situation that two fluids occupy $\Omega_{\pm t} = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1}, \pm (x_N - h(x', t)) > 0\}$, respectively, where $h(x', t)$ is an unknown scalar function describing the interface $\Gamma = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1}, x_N = h(x', t)\}$ of the fluids. [21] and [22] proved the local solvability of the two-phase problem with surface tension and with both surface tension and gravity, respectively, for small initial data. On the other hand, [20] pointed out that the Rayleigh-Taylor instability occurs if gravity works and if the fluid occupying $\Omega_{+t}$ is heavier than the other one. Furthermore, Hieber and Saito [15] extended the results of the Newtonian case of [21, 22] to a generalized Newtonian one. Köhne, Prüss, and Wilke [16] showed the local solvability and the global solvability in the case where $\Omega_{\pm t}$ are bounded and surface tension is taken into account.

$L_p$-in-time and $L_q$-in-space setting. Shibata-Shimizu [28] showed a maximal $L_p$-$L_q$ regularity theorem for a linearized system of the two-phase problem considered in [20, 22] mentioned above.

This paper is a continuation of Shibata-Shimizu [28]. Our aim is in the present paper to prove the existence of $\mathcal{R}$-bounded solution operator families of (1.1) for $\hat{\Omega} = \Omega_+ \cup \Omega_-$ with uniform $W^{2-1/r}_p$ domains $\Omega_{\pm}$ ($N < r < \infty$), which is introduced in Definition 1.1 below. In addition, the $\mathcal{R}$-bounded solution operator
families enable us to show generation of analytic semigroup and some maximal $L^p-L^q$ regularity theorem for a time-dependent problem associated with (1.1), which are provided in Subsection 2.4 and Subsection 2.5, respectively. We want to emphasize that the maximal $L^p-L^q$ regularity theorem extends [28] to uniform $W^{-2-r}_r$ domains and to variable viscosities.

The strategy of this paper follows Shibata [26]. We extend his method for one-phase problem to one for two-phase problem. For example, a two-phase version of the weak Dirichlet-Neumann problem (it is called a weak elliptic transmission problem in the present paper) introduced in Definition 1.1 below, which plays an important role in this paper, and especially in derivation of reduced Stokes resolvent equations (cf. Subsection 2.1 below) and in Lemma 5.7 below. One of the main advantage of the reduced equations is that we can eliminate the divergence equation: $\text{div } u = q$ in $\bar{\Omega}$, which is difficult to treat in localized problems, from the problem (1.1). On the other hand, Lemma 5.7 enable us to control localized pressure term. There however is a remark on Shibata’s paper [26]. It seems to be difficult to obtain [26, Theorem 3.8] and to obtain [26, Theorem 3.10], because the $R$-boundedness of $\lambda_{0D}(\lambda), \lambda_{0N}(\lambda)$ was not proved in his paper (cf. [26, Proof of Theorem 3.1,Proof of Theorem 3.4]). We essentially need the $R$-boundedness of such operators since the right members $f$ for (3.7), (3.5) of [26] contain $\lambda V_F(g), \lambda V_D(g)$, respectively. Natural spaces for ranges of the operators $\lambda_{0D}(\lambda), \lambda_{0N}(\lambda)$ are given by negative Sobolev spaces, which is main difficulty to modify this. To overcome this difficulty, we introduce in this paper Proposition 3.10 which allows us to avoid such negative spaces. Following the strategy of Proposition 3.10 we can also complete his results.

1.2. Notation and main results. We first state notation used throughout this paper.

Let $N$ be a norm of all natural numbers and $N_0 = N \cup \{0\}$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in N_0^N$, we set $D^\alpha f = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} f$. Let $G$ be an open set of $\mathbb{R}^N$. Then $L^q(G)$ and $W^m_q(G)$ with $m \in N$ denote the usual $K$-valued Lebesgue space and Sobolev space on $G$ for $K = \mathbb{R}$ or $K = \mathbb{C}$, while $\| \cdot \|_{L^q(G)}$ and $\| \cdot \|_{W^m_q(G)}$ are defined, respectively. We essentially need the $R$-boundedness of such operators since the right members $f$ for (3.7), (3.5) of [26] contain $\lambda V_F(g), \lambda V_D(g)$, respectively. Natural spaces for ranges of the operators $\lambda_{0D}(\lambda), \lambda_{0N}(\lambda)$ are given by negative Sobolev spaces, which is main difficulty to modify this. To overcome this difficulty, we introduce in this paper Proposition 3.10 which allows us to avoid such negative spaces. Following the strategy of Proposition 3.10 we can also complete his results.

Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}_0^N$, we set $D^\alpha f = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} f$. Let $G$ be an open set of $\mathbb{R}^N$. Then $L^q(G)$ and $W^m_q(G)$ with $m \in \mathbf{N}$ denote the usual $K$-valued Lebesgue space and Sobolev space on $G$ for $K = \mathbb{R}$ or $K = \mathbb{C}$, while $\| \cdot \|_{L^q(G)}$ and $\| \cdot \|_{W^m_q(G)}$ are defined, respectively. Here we set $W^0_q(G) = L^q(G)$. In addition, $W^s_q(G)$ with $s \in (0, \infty) \setminus \mathbf{N}$ denotes the $K$-valued Sobolev-Slobodzki space endowed with norm $\| \cdot \|_{W^s_q(G)}$, and also $C^\infty_0(G)$ the function space of all $C^\infty$ functions $f : G \to K$ such that $\text{supp } f$ is compact and $\text{supp } f \subset G$.

For two Banach spaces $X$ and $Y$, $\mathcal{L}(X,Y)$ is the set of all bounded linear operators from $X$ to $Y$, and $\mathcal{L}(X)$ the abbreviation of $\mathcal{L}(X,X)$. Let $U$ be a domain of $C$, and then $\text{Hol}(U,\mathcal{L}(X,Y))$ stands for the set of all $\mathcal{L}(X,Y)$-valued holomorphic functions defined on $U$.

For $d \in \mathbf{N}$ with $d \geq 2$, $X^d$ denotes the $d$-product space of $X$. Let $\| \cdot \|_X$ be a norm of $X$, while $\| \cdot \|_X$ also denotes the norm of the product space $X^d$ for short, that is, $\| \mathbf{f} \|_X = \sum_{j=1}^d \| f_j \|_X$ for $\mathbf{f} = (f_1, \ldots, f_d) \in X^d$.

Let $a = (a_1, \ldots, a_N)^T$ and $b = (b_1, \ldots, b_N)^T$, and then we write $a \cdot b = \sum_{j=1}^N a_j b_j$ and $a \otimes b = (a_i b_j)$ that is an $N \times N$ matrix with the $(i,j)$-entry $a_i b_j$. On the other hand, for any vector functions $u, v$ on $G$, we set $(u,v)_G = \int_G u \cdot v \, dx$ and $(u,v)_{\partial G} = \int_{\partial G} u \cdot v \, d\sigma$, where $\partial G$ is the boundary of $G$ and $d\sigma$ the surface element of $\partial G$.

Given $1 < q < \infty$, we set $q' = q/(q-1)$. Let $L^{q,\text{loc}}(\overline{G})$ be the vector space of all measurable functions $f : G \to K$ such that $f \in L^q(G \cap B)$ for any ball $B$ of $\mathbb{R}^N$. We define a homogeneous space $W^{q,\text{loc}}(G)$ by $W^{q,\text{loc}}_0(G) = \{ f \in L^{q,\text{loc}}(G) \mid \nabla f \in L^q(G)^N \}$ with norm $\| \cdot \|_{W^{q,\text{loc}}_0(G)} := \| \nabla \cdot \|_{L^q(G)}$, where we have to identify two elements differing by a constant. In addition, let $W^{q,\text{loc}}_{1,0}(G)$ and $W^{1,0}_{q,0}(G)$ be Banach spaces defined by $W^{q,\text{loc}}_{1,0}(G) = \{ f \in W_{1,0}^{q,\text{loc}}(G) \mid f = 0 \text{ on } \partial G \} \ (X \in \widehat{\{W,W\}})$ with norms $\| \cdot \|_{W^{q,\text{loc}}_{1,0}(G)} := \| \nabla \cdot \|_{L^q(G)}$ and $\| \cdot \|_{W^{1,0}_{q,0}(G)} := \| \cdot \|_{W^{1,0}_{q,0}(G)}$, respectively.

Throughout this paper, the letter $C$ denotes generic constants and $C_{a,b,c,...}$, means that the constant depends on the quantities $a, b, c, \ldots$. The values of constants $C$ and $C_{a,b,c,...}$ may change from line to line.

Secondly, we show some definitions. Uniform $W^{2-1/r}_r$ domains are defined as follows:

**Definition 1.1.** Let $1 < r < \infty$ and $D$ be a domain of $\mathbb{R}^N$ with boundary $\partial D$. We say that $D$ is a uniform $W^{2-1/r}_r$ domain, if there exist positive constants $\alpha, \beta$, and $K$ such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial D$ there are a coordinate number $j$ and a $W^{2-1/r}_r$ function $h(x') (x' = (x_1, \ldots, \hat{x}_j, \ldots, x_N))$ defined on $B'_\alpha(x_0')$,
with \( x'_0 = (x_{01}, \ldots, \hat{x}_{0j}, \ldots, x_{0N}) \) and \( \|h\|_{W^{-1/r}(B_\alpha'(x'_0))} \leq K \), such that
\[
D \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x'), x' \in B_\alpha'(x'_0) \} \cap B_\beta(x_0),
\]
\[
\partial D \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x'), x' \in B_\alpha'(x'_0) \} \cap B_\beta(x_0).
\]

Here \((x_1, \ldots, \hat{x}_j, \ldots, x_N) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)\), \( B_\alpha'(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\} \), and \( B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\} \).

We next introduce the definition of the \( \mathcal{R} \)-boundedness of operator families.

**Definition 1.2.** Let \( X \) and \( Y \) be two Banach spaces. A family of operators \( \mathcal{T} \subset \mathcal{L}(X, Y) \) is called \( \mathcal{R} \)-bounded on \( \mathcal{L}(X, Y) \), if there exist constants \( C > 0 \) and \( p \in [1, \infty) \) such that the following assertion holds: For each natural number \( n \), \( \{T_j\}_{j=1}^n \subset \mathcal{T} \), \( \{f_j\}_{j=1}^n \subset X \) and for all sequences \( \{r_j(u)\}_{j=1}^n \) of independent, symmetric, \((-1, 1)\)-valued random variables on \([0, 1]\), there holds the inequality:
\[
\left( \int_0^1 \left\| \sum_{j=1}^n r_j(u)T_jf_j \right\|_Y^p \, du \right)^{1/p} \leq C \left( \int_0^1 \left\| \sum_{j=1}^n r_j(u)f_j \right\|_X^p \, du \right)^{1/p}.
\]

The smallest such \( C \) is called \( \mathcal{R} \)-bound of \( \mathcal{T} \) on \( \mathcal{L}(X, Y) \), which is denoted by \( \mathcal{R}_{\mathcal{L}(X, Y)} \).

**Remark 1.3.** The constant \( C \) in Definition 1.2 depends on \( p \). On the other hand, it is well-known that \( \mathcal{T} \) is \( \mathcal{R} \)-bounded for any \( p \in [1, \infty) \), provided that \( \mathcal{T} \) is \( \mathcal{R} \)-bounded for some \( p \in [1, \infty) \). This fact follows from Kahane’s inequality (cf. [13] Theorem 2.4)).

Furthermore, we introduce a weak elliptic transmission problem. In the present paper, \( \Gamma_+ = \emptyset \) or \( \Gamma_- = \emptyset \) are admissible, but note that \( \Gamma \neq \emptyset \). Let \( W^1_{q, \Gamma_+}(\Omega) \) and \( \tilde{W}^1_{q, \Gamma_+}(\Omega) \) be Banach spaces defined by
\[
X^1_{q, \Gamma_+}(\Omega) = \begin{cases} 
\{ f \in W^1_q(\Omega) \mid f = 0 \text{ on } \Gamma_+ \} & \text{if } \Gamma_+ \neq \emptyset, \\
\{ f \in \tilde{W}^1_q(\Omega) \} & \text{if } \Gamma_+ = \emptyset
\end{cases}
\]
for \( \Omega \in \{W, \tilde{W}\} \), and their norms are given by \( \| \cdot \|_{W^1_{q, \Gamma_+}(\Omega)} = \| \cdot \|_{W^1_q(\Omega)} \) and \( \| \cdot \|_{\tilde{W}^1_{q, \Gamma_+}(\Omega)} = \| \nabla \cdot \|_{L_q(\Omega)} \), respectively. The unique solvability of the weak elliptic transmission problem is defined in the following.

**Definition 1.4.** Let \( 1 < q < \infty \) and \( q' = q/(q - 1) \). Let \( W^1_q(\Omega) \) be a closed subspace of \( \tilde{W}^1_{q, \Gamma_+}(\Omega) \), and suppose that \( W^1_{q, \Gamma_+}(\Omega) \) is dense in \( W^1_q(\Omega) \). Set \( \rho = \rho_+ \chi_{\Omega_+} + \rho_- \chi_{\Omega_-} \) for positive constants \( \rho_\pm \). Then we say that the weak elliptic transmission problem is uniquely solvable on \( W^1_q(\Omega) \) for \( \rho_\pm \) if the following assertion holds: For any \( f \in L_q(\Omega)^N \), there is a unique \( \theta \in W^1_q(\Omega) \) satisfying the variational equation:
\[
(\rho^{-1} \nabla \theta, \nabla \varphi)_0 = (f, \nabla \varphi)_0 \quad \text{for all } \varphi \in \tilde{W}^1_q(\Omega),
\]
which possesses the estimate: \( \| \nabla \theta \|_{L_q(\Omega)} \leq C \| f \|_{L_q(\Omega)} \) with a positive constant \( C \) independent of \( \theta, \varphi, \) and \( f \).

**Remark 1.5.** (1) Let \( 1 < q < \infty \), \( q' = q/(q - 1) \), and let the weak elliptic transmission problem be uniquely solvable on \( W^1_q(\Omega) \) for \( \rho_+ = \rho_- = 1 \). We define \( J_q(\Omega) \) and \( G_q(\Omega) \) by
\[
J_q(\Omega) = \{ f \in L_q(\Omega)^N \mid (f, \nabla \varphi)_0 = 0 \quad \text{for all } \varphi \in \tilde{W}^1_q(\Omega) \},
\]
\[
G_q(\Omega) = \{ f \in L_q(\Omega)^N \mid \nabla \theta = \text{div} \varphi \quad \text{for some } \theta \in W^1_q(\Omega) \}.
\]

Then, by the standard proof, the so-called Helmholtz decomposition: \( L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega) \) holds.

(2) In applications, we choose \( W^1_q(\Omega) \) in such a way that the weak elliptic transmission problem is uniquely solvable for \( \rho_\pm \). Typical examples are as follows: \( W^1_q(\mathbb{R}^N) = \tilde{W}^1_q(\mathbb{R}^N) \); \( W^1_q(\mathbb{R}_+^N) = \tilde{W}^1_q(\mathbb{R}_+^N) \) with \( \Gamma_+ = \emptyset \) and \( \Gamma_- = \mathbb{R}_0^N \); \( \{ (x', x_N) \mid x' \in \mathbb{R}^{N-1}, x_N = 0 \} \); \( W^1_q(\mathbb{R}_-^N) = \tilde{W}^1_q(\mathbb{R}_-^N) \) with \( \Gamma_- = \mathbb{R}_0^N \) and \( \Gamma_+ = \emptyset \); \( \mathbb{R}^N \); \( \mathbb{R}_0^N \); \( \mathbb{R}_+^N \) when \( \Omega \) is a bounded domain, a layer, or a perturbed layer. We refer e.g. to [16 Appendix A.1] for the treatment of weak elliptic transmission problems.
(3) We set $W^1_q(\hat{\Omega}) + W^1_q(\Omega) = \{ \theta_1 + \theta_2 \mid \theta_1 \in W^1_q(\hat{\Omega}), \theta_2 \in W_q^1(\Omega) \}$. Suppose that the weak elliptic transmission problem is uniquely solvable on $W^1_q(\Omega)$ for $\rho_\pm$. Then, for any $\alpha \in L^q(\hat{\Omega})^N$, $\beta \in W_q^{1-1/q}(\Gamma)$, \( \gamma \in W_q^{1-1/q}(\Gamma_+) \), there exists a unique $\theta \in W_q^1(\hat{\Omega}) + W_q^1(\Omega)$ satisfying the weak problem:

$$
\begin{aligned}
(\rho^{-1}\nabla \theta, \nabla \varphi)_{\hat{\Omega}} &= (\alpha, \nabla \varphi)_{\hat{\Omega}} \quad \text{for all } \varphi \in W^1_q(\Omega), \quad [\theta] = \beta \quad \text{on } \Gamma, \quad \theta = \gamma \quad \text{on } \Gamma_+,
\end{aligned}
$$

which possesses the estimate:

$$
\| \nabla \theta \|_{L^q(\Omega)} \leq C \left( \| \alpha \|_{L^q(\hat{\Omega})} + \| \beta \|_{W_q^{1-1/q}(\Gamma)} + \| \gamma \|_{W_q^{1-1/q}(\Gamma_+)} \right)
$$

with some positive constant $C$ independent of $\alpha$, $\beta$, $\gamma$, $\theta$, and $\varphi$. Thus, it is possible to define a linear operator $K : L^q(\hat{\Omega})^N \times W_q^{1-1/q}(\Gamma) \times W_q^{1-1/q}(\Gamma_+) \to W_q^1(\hat{\Omega}) + W_q^1(\Omega)$ by $K(\alpha, \beta, \gamma) = \theta$ satisfying (1.3).

If $\Gamma_+ = \emptyset$, then we denote $K(\alpha, \beta, \gamma)$ by $K(\alpha, \beta, \emptyset)$ when $\Gamma_+ \neq \emptyset$ and by $K(\alpha, \beta)$ when $\Gamma_+ = \emptyset$.

We now state our main results. To this end, we introduce a data space for the divergence equation: $\text{div} \ u = g$ on $\hat{\Omega}$ with boundary conditions: $[u] \cdot n = 0$ on $\Gamma$ and $u \cdot n_- = 0$ on $\Gamma_-$, where $n_-$ is the unit outward normal vector on $\Gamma_-$. Let $W_q^{-1}(\Omega)$ be the dual space of $W_q^1(\Omega)$ for $1 < q < \infty$ and $q' = q/(q-1)$, and let $\| \cdot \|_{W_q^{-1}(\Omega)}$ and $\langle \cdot, \cdot \rangle_{\Omega}$ be its norm and the duality pairing between $W_q^{-1}(\Omega)$ and $W_q^1(\Omega)$, respectively. Then we set

$$
L_q(\Omega) \cap W_q^{-1}(\Omega) = \left\{ g \in L_q(\Omega) \mid \exists \ M > 0 \text{ s.t. } \langle g, \varphi \rangle_{\Omega} \leq M \| \nabla \varphi \|_{L^q(\Omega)} \right\}
$$

Let $g \in L_q(\Omega) \cap W_q^{-1}(\Omega)$, and thus $g$ can be extended uniquely to an element of $W_q^{-1}(\Omega)$. Such an extended $g$ is again denoted by $g$ for short. We can see $g$ as a functional on $\{ \nabla \theta \mid \theta \in W_q^1(\Omega) \} \subset L_q(\Omega)^N$, which, combined with Hahn-Banach’s theorem, furnishes that there is a $G \in L_q(\Omega)^N$ such that $\|g\|_{W_q^{-1}(\Omega)} = \|G\|_{L^q(\Omega)}$ and $\langle g, \varphi \rangle_{\Omega} = \langle G, \nabla \varphi \rangle_{\Omega}$ for all $\varphi \in W_q^1(\Omega)$. In what follows, $G$ is restricted to the functional on $\{ \nabla \theta \mid \theta \in W_q^1(\Omega) \}$. Let $\tilde{L}_q(\Omega) = L_q(\Omega)^N/J_q(\Omega)$, and let $[G] = \{ G + f \mid f \in J_q(\Omega) \} \subset \tilde{L}_q(\Omega)$. Then $g \mapsto [G]$ is well-defined, so that we denote $[G]$ by $G(g)$. Especially, we have, for $g \in L_q(\Omega) \cap W_q^{-1}(\Omega)$ and for any representative $g \in L_q(\Omega)^N$ of $G(g)$,

$$
(g, \varphi)_{\Omega} = -\langle g, \nabla \varphi \rangle_{\Omega} \quad \text{for all } \varphi \in W_q^1(\Gamma, \Gamma_+)\cap W_q^{-1}(\Omega).
$$

Here we set $W_q^{-1}(\Omega) = L_q(\Omega) \cap W_q^{-1}(\Omega)$. Then $W_q^1(\hat{\Omega}) \cap W_q^{-1}(\Omega)$ is a Banach space endowed with norm $\| \cdot \|_{W_q^1(\hat{\Omega}) \cap W_q^{-1}(\Omega)} := \| \cdot \|_{W_q^1(\hat{\Omega})} + \| \cdot \|_{W_q^{-1}(\Omega)}$, and the function space is characterized as the data space for the divergence equation above. The following theorem presents the main result of this paper.

**Theorem 1.6.** Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $N = r < \infty$, and $\max(q, q') \leq r$ with $q' = q/(q-1)$. Let $\rho_\pm$ be positive constants. Suppose that the following three conditions holds:

(a) $\Omega_\pm$ are uniform $W^{2-1/r}_p$ domains;

(b) The weak elliptic transmission problem is uniquely solvable on $W^1_q(\Omega)$ and $W^1_q(\Omega)$ for $\rho_\pm$;

(c) $\mu_\pm$ are real valued uniformly continuous functions defined on $R^N$ and there exist positive constants $\mu_{\pm 1}$, $\mu_{\pm 2}$ such that

$$
\mu_{\pm 1} \leq \mu_+(x) \leq \mu_{\pm 2}, \quad \mu_{\pm 1} \leq \mu_-(x) \leq \mu_{\pm 2} \quad \text{for any } x \in R^N.
$$

In addition, $\mu_\pm \in W^1_{r, \text{loc}}(R^N)$ and $\| \nabla \mu_\pm \|_{L^r(B)} \leq K_{r, \tau}$ with some positive constant $K_{r, \tau}$ for any ball $B \subset R^N$ with radius $\tau$.

(1) **Existence.** Set

$$
\begin{aligned}
X_q &= \{(f, g, h, k) \mid f \in L_q(\hat{\Omega})^N, g \in W^1_q(\hat{\Omega}) \cap W^{-1}_q(\Omega), h, k \in W^1_q(\Omega) \},
\end{aligned}
$$

$$
\begin{aligned}
X_q &= \{(F_1, \ldots, F_{11}) \mid F_1, F_2, F_3, F_7 \in L_q(\hat{\Omega})^N, F_5, F_6, F_7 \in L_q(\hat{\Omega}), F_5 \in L_q(\hat{\Omega}) \},
\end{aligned}
$$

$$
\begin{aligned}
F_6 \in L_q(\hat{\Omega})^{N^2}, F_7 \in W^1_q(\hat{\Omega})^N, F_{10} \in L_q(\Omega_\pm)^N, F_{11} \in W^1_q(\Omega_\pm)^N \}. \end{aligned}
$$

Then there exists a constant $\lambda_0 \geq 1$ and operator families:

$$
\text{A}(\lambda) \in \mathrm{Hol}(\Sigma_{c, \lambda_0}, L(X_q, W^2_q(\hat{\Omega})^N)), \quad \text{P}(\lambda) \in \mathrm{Hol}(\Sigma_{c, \lambda_0}, L(X_q, W^1_q(\hat{\Omega}) + W^1_q(\Omega)))).
$$
such that, for any $\lambda \in \Sigma_{\varepsilon, \lambda_0}$ and for any $(f, g, h, k) \in X_q$ and $q \in G(g)$, $u = A(\lambda)F_\lambda(f, g, h, k)$ and $\theta = P(\lambda)F_\lambda(f, g, h, k)$ are solutions to \[ (1.1) \] and furthermore,
\[ \mathcal{R}_{L(\mathcal{S}, L_q(\Omega))} \left( \left\{ \left( \lambda \frac{d}{dx} \right)^l (R_\lambda A(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \right\} \right) \leq \gamma_0 \quad (l = 0, 1), \]
and
\[ \mathcal{R}_{L(\mathcal{S}, L_q(\Omega))} \left( \left\{ \left( \lambda \frac{d}{dx} \right)^l \nabla P(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \right\} \right) \leq \gamma_0 \quad (l = 0, 1) \]

for some positive constant $\gamma_0$. Here we have set $\tilde{N} = N^3 + N^2 + N$, $R_\lambda u = (\nabla^2 u, \lambda^{1/2} \nabla u, \lambda u)$, and

$F_\lambda(f, g, h, k) = (f, \nabla g, \lambda^{1/2} g, g, \nabla h, \lambda^{1/2} h, h, \nabla k, \lambda^{1/2} k, k)$.

(2) **Uniqueness.** There exists a $\lambda_0 \geq 1$ such that if $u \in W_q^2(\tilde{\Omega}) \cap J_q(\Omega)$ and $\theta \in W_q^1(\tilde{\Omega}) + W_q^1(\Omega)$ satisfies the homogeneous equations:

\[ \lambda u - \rho^{-1} \text{Div} T(u, \theta) = 0 \quad \text{in } \tilde{\Omega}, \quad [T(u, \theta)n] = 0, \quad [\lambda u] = 0 \quad \text{on } \Gamma, \]
\[ T(u, \theta)n_+ = 0 \quad \text{on } \Gamma_+, \quad u = 0 \quad \text{on } \Gamma_- \]

with $\lambda \in \Sigma_{\varepsilon, \lambda_0}$, then $u = 0$ in $\tilde{\Omega}$.

**Remark 1.7.** The symbols $F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}$, and $F_{11}$ are variables corresponding to $f, \nabla g, \lambda^{1/2} g, g, \nabla h, \lambda^{1/2} h, h, \nabla k, \lambda^{1/2} k$, and $k$, respectively. The norm of space $X_q$ is given by

\[ \|(F_1, \ldots, F_{11})\|_{X_q} = \|(F_1, F_2, F_3, F_4, F_5, F_6, F_7)\|_{L_q(\tilde{\Omega})} + \|(F_8, F_9)\|_{W_q^1(\Omega)} + \|F_{10}\|_{W_q^1(\Omega)}. \]

This paper is organized as follows: The next section first tells us some equivalence of \[ (1.1) \] and two-phase reduced Stokes resolvent equations, which are obtained by elimination of pressure term from \[ (1.1) \], in Subsection 2.1 and Subsection 2.2. Secondly, we state our main result for the two-phase reduced Stokes resolvent equations in Subsection 2.3 and Subsection 2.4. We state generation of analytic semigroup and some maximal $L_p-L_q$ regularity theorem for two-phase problems of time-dependent Stokes equations in Subsection 2.3 and Subsection 2.4, respectively, with help of Theorem 1.6 and the main result stated in Subsection 2.3. Section 3 proves our main result for the two-phase reduced Stokes resolvent equations in the case where $\Omega = \tilde{\Omega} = R^N \cup R^N_+$, $R^N_+ = \{(x', x_N) \mid x' \in R^{N-1}, x_N > 0\}$, with constant viscosity coefficients. Section 4 proves our main result for the two-phase reduced Stokes resolvent equations with variable viscosity coefficients when $\tilde{\Omega}$ is a perturbed $R^N$ by using results obtained in Section 3. Section 5 shows the main result stated in Subsection 2.3 by using results obtained in Section 4 together with some localization technique.

2. **GENERATION OF ANALYTIC SEMIGROUP AND MAXIMAL REGULARITY**

In this section, after introducing the Stokes operator in \[ (2.1) \] below, we consider the following initial-boundary value problem:

\[ \begin{cases}
\partial_t v - \rho^{-1} \text{Div} T(v, \pi) = f, & \text{in } \tilde{\Omega} \times (0, \infty), \\
[T(v, \pi)n] = [h], & [\pi] = 0 \text{ on } \Gamma \times (0, \infty), \\
T(v, \pi)n_+ = k & \text{on } \Gamma_+ \times (0, \infty), \\
v = 0 & \text{on } \Gamma_- \times (0, \infty), \\
v|_{t=0} = v_0 & \text{in } \Omega,
\end{cases} \]

which is called the two-phase Stokes equations in this paper. We discuss the generation of analytic semigroup associated with \[ (2.1) \] and some maximal $L_p-L_q$ regularity property for \[ (2.1) \]. To consider the generation of analytic semigroup, we have to formulate \[ (2.1) \] in the semigroup setting, that is, we have to eliminate the pressure term from \[ (2.1) \]. Throughout this section, for some $1 < q < \infty$ and positive constants $\rho_\pm$, we assume that the weak elliptic transmission problem is uniquely solvable on $W_q^1(\Omega)$ for $\rho_\pm$. The assumption plays an essential role to eliminate the pressure term from \[ (2.1) \].
2.1. Two-phase reduced Stokes resolvent equations. Let \( 1 < q < \infty \), \( q' = q/(q-1) \), and \( u \in W^2_q(\hat{\Omega}) \). Set \( K(u) = K(\alpha, \beta, \gamma) \in W^1_q(\hat{\Omega}) + W^1_q(\Omega) \), defined in Remark 1.3, with
\[
(2.2) \quad \alpha = \rho^{-1} \text{Div}(\mu D(u)) - \nabla \text{div} u, \quad \beta = \langle [\mu D(u)n], n > - [\text{div} u], \quad \gamma = \langle \mu D(u)n_+, n_+ > - \text{div} u.
\]
Then \( u \to \nabla K(u) \) is a bounded linear operator from \( W^2_q(\hat{\Omega}) \) to \( L_q(\hat{\Omega}) \) with \( \|\nabla K(u)\|_{L_q(\hat{\Omega})} \leq C\|u\|_{W^2_q(\hat{\Omega})} \) for some positive constant \( C \) independent of \( u \). We consider the equations as follows:
\[
\begin{align*}
\lambda u - \rho^{-1} \text{Div} T(u, K(u)) &= f \quad \text{in } \hat{\Omega}, \\
T(u, K(u))n &= [h] \quad \text{on } \Gamma, \\
[u] &= 0 \quad \text{on } \Gamma, \\
T(u, K(u))n_+ &= k \quad \text{on } \Gamma_+, \\
u &= 0 \quad \text{on } \Gamma_-.
\end{align*}
\]
which is called the two-phase reduced Stokes resolvent equations. In this subsection, we construct a solution to (2.3) on the assumption that (1.1) is solvable. To this end, we treat the following auxiliary problem:
\[
(2.4) \quad (\lambda u, \varphi)_{\hat{\Omega}} + (\nabla u, \nabla \varphi)_{\hat{\Omega}} = (f, \nabla \varphi)_{\hat{\Omega}} \quad \text{for all } \varphi \in W^1_q,_{r_+}(\Omega), \quad [u] = g \quad \text{on } \Gamma, \quad u = h \quad \text{on } \Gamma_+,
\]
which is the weak elliptic transmission problem with resolvent parameter \( \lambda \). Employing the same argument as in the proof of our main result in the present paper, we can show the following proposition.

**Proposition 2.1.** Let \( 0 < \varepsilon < \pi/2, 1 < q < \infty, N < r < \infty, \text{and } \max(q, q') \leq r \) with \( q' = q/(q-1) \). Suppose that \( \Omega_\pm \) are uniform \( W^{2-1/r}_\infty \) domains. Then there is a positive number \( \lambda_0 \geq 1 \) such that, for any \( \lambda \in \Sigma_{r, \lambda_0} \), \( f \in L_q(\hat{\Omega}) \), \( g \in W^{1-1/q}(\Gamma) \), and \( h \in W^{1-1/q}(\Gamma_+) \), (2.4) admit a unique solution \( u \in W^1_q(\hat{\Omega}) \cap W^{1-1/q}_q(\Omega) \).

We solve (2.3) by means of solutions to (2.4). Given \( f \in L_q(\hat{\Omega}) \), \( h \in W^1_q(\hat{\Omega}) \), and \( k \in W^1_q(\Omega_+) \), we choose by Proposition 2.1 some \( g \) in such a way that \( g \) solves the weak problem:
\[
(2.5) \quad (\lambda g, \varphi)_{\hat{\Omega}} + (\nabla g, \nabla \varphi)_{\hat{\Omega}} = -(f, \nabla \varphi)_{\hat{\Omega}} \quad \text{for all } \varphi \in W^1_q,_{r_+}(\Omega),
\]
\[
(2.6) \quad [g] = \langle [h], n > \quad \text{on } \Gamma, \quad g = [k, n_+] \quad \text{on } \Gamma_+.
\]
Let \( u \in W^2_q(\hat{\Omega}) \) and \( \theta \in W^1_q(\hat{\Omega}) + W^1_q(\Omega) \) be solutions to (2.1) with \( f, g, h, \) and \( k \) as above. Then, by the definition of \( K(u) \) and Gauss’s divergence theorem together with \( [u] = 0 \) on \( \Gamma, u = 0 \) on \( \Gamma_- \), we see that
\[
(f, \nabla \varphi)_{\hat{\Omega}} = (\lambda u - \rho^{-1} \nabla K(u) + \rho^{-1} \nabla \theta, \nabla \varphi)_{\hat{\Omega}}
\]
\[
= - (\lambda g, \varphi)_{\hat{\Omega}} - (\nabla g, \nabla \varphi)_{\hat{\Omega}} + (\rho^{-1} \nabla \theta - K(u), \nabla \varphi)_{\hat{\Omega}}
\]
for any \( \varphi \in W^1_q,_{r_+}(\Omega) \). This combined with (2.5) and the denseness of \( W^1_q,_{r_+}(\Omega) \) in \( W^1_q(\Omega) \) furnishes that
\[
(\rho^{-1} \nabla \theta - K(u), \nabla \varphi)_{\hat{\Omega}} = 0 \quad \text{for all } \varphi \in W^1_q(\Omega).
\]
In addition, it holds that \([K(u) - \theta] = 0 \) on \( \Gamma \) and \( K(u) - \theta = 0 \) on \( \Gamma_+ \), since \( g \) satisfies (2.6) and
\[
\langle [h], n > = \langle [\mu D(u)n], n > - [\theta] + [K(u) - \theta] + [\text{div} u] = [K(u) - \theta] + [g] \quad \text{on } \Gamma,
\]
\[
\langle [k, n_+] > = \langle [\mu D(u)n_+], n_+ > - \theta = K(u) - \theta + \text{div} u = K(u) - \theta + g \quad \text{on } \Gamma_+.
\]
Thus the unique solvability of the weak elliptic transmission problem implies \( K(u) = \theta \), which means that the solution \( u \in W^2_q(\hat{\Omega}) \) of (1.1) solves (2.3) for \( f \in L_q(\hat{\Omega}) \), \( h \in W^1_q(\hat{\Omega}) \), \( k \in W^1_q(\Omega_+) \), and \( g \) of (2.5) - (2.6).

2.2. Reduced Stokes implies Stokes. In this subsection, we solve (1.1) on the assumption that (2.3) is solvable. Let \( 1 < q < \infty \) and \( q' = q/(q-1) \). Given \( f \in L_q(\hat{\Omega}) \), \( h \in W^1_q(\hat{\Omega}) \), and \( k \in W^1_q(\Omega_+) \), let \( \kappa \in W^1_q(\hat{\Omega}) + W^1_q(\Omega) \) be a solution to the weak problem:
\[
(\rho^{-1} \nabla \kappa, \nabla \varphi)_{\hat{\Omega}} = (f, \nabla \varphi)_{\hat{\Omega}} \quad \text{for all } \varphi \in W^1_q(\Omega),
\]
\[
[k] = - [\mu D(u)n], n > \quad \text{on } \Gamma, \quad \kappa = - [k, n_+] \quad \text{on } \Gamma_+.
\]
Then the system (1.1) is reduced to

$$
\begin{align*}
\lambda u - \rho^{-1} \text{Div}(u, \theta - \kappa) &= f - \rho^{-1} \nabla \kappa, & \text{div } u = g & \text{ in } \hat{\Omega}, \\
[T(u, \theta - \kappa)] n &= [h] - [h], & n > n, & [u] = 0 \text{ on } \Gamma, \\
T(u, \theta - \kappa)n_+ &= k^v - k, & n_+ > n_+ \text{ on } \Gamma_+, \\
u &= 0 \text{ on } \Gamma_-. 
\end{align*}
$$

It thus suffices to consider (1.1) under the condition that

$$
(2.7) \quad (f, \nabla \varphi)_\Omega = 0 \text{ for all } \varphi \in W^1_2(\Omega), \quad <[h], n> = 0 \text{ on } \Gamma, \quad <k, n_+> = 0 \text{ on } \Gamma_+.
$$

For $G = (G_1, G_2) \in L_q(\hat{\Omega})^N \times W^1_q(\hat{\Omega})$, we set $L(G) = L(G_1, G_2) = K(G_1 - \nabla G_2, -[G_2], -G_2)$ by $K$ of Remark 1.4 [4]. Then $G \mapsto \nabla L(G)$ is a bounded linear operator from $L_q(\hat{\Omega})^N \times W^1_q(\hat{\Omega})$ to $L_q(\hat{\Omega})^N$.

Given $g \in W^1_q(\hat{\Omega}) \cap W^2_q(\hat{\Omega})$, we choose a representative $g$ of $\mathcal{G}(g)$. For these $g$, $\mathcal{G}$ and $f$, $h$, $k$ satisfying (2.7), let $u \in W^2_2(\Omega)^N$ be a solution to the two-phase reduced Stokes resolvent equations as follows:

$$
\begin{align*}
\lambda u - \rho^{-1} \text{Div}(u, K(u)) &= f + \rho^{-1} \nabla L(\lambda g, g) \text{ in } \hat{\Omega}, \\
[T(u, K(u))]n &= [h] + [g]n \text{ on } \Gamma, \\
[u] &= 0 \text{ on } \Gamma, \\
T(u, K(u))n_+ &= k + gn_+ \text{ on } \Gamma_+, \\
u &= 0 \text{ on } \Gamma_-. 
\end{align*}
$$

Then, by (1.4), (2.7) and by the definition of $K(u)$, $L(\lambda g, g)$, we have

$$
0 = (f, \nabla \varphi)_\Omega = (\lambda u - \rho^{-1} \text{Div}(\mu D(u))) + \rho^{-1} \nabla K(u) - \rho^{-1} \nabla L(\lambda g, g, \nabla \varphi)_\Omega = (\lambda u, \nabla \varphi)_\Omega - (\nabla \text{Div}(u, \nabla \varphi)_\Omega + (\lambda g, \varphi)_\Omega + (\nabla g, \nabla \varphi)_\Omega
$$

for any $\varphi \in W^1_2(\Omega)$, which, combined with Gauss’s divergence theorem together with $[u] = 0$ on $\Gamma$, $u = 0$ on $\Gamma_-$, furnishes that $(\lambda \text{Div}(u - g, \varphi)_\Omega + (\nabla \text{Div}(u - g, \nabla \varphi)_\Omega = 0$ for all $\varphi \in W^1_2(\Omega)$. In addition, we see, by (2.7) and the definition of $K(u)$, that $[g] = [\mu D(u)n], n > -[K(u)] = [\text{Div}(u)]$ on $\Gamma$, $g = \mu D(u)n_+, n_+ > -K(u) = \text{Div}(u)$ on $\Gamma_+$, which implies that $[\text{Div}(u - g)] = 0$ on $\Gamma$, $\text{Div}(u - g) = 0$ on $\Gamma_+$. Thus, by Proposition 2.1, $\text{Div}(u = g$ in $\hat{\Omega}$, which means that $u$ and $\theta = K(u) - L(\lambda g, g)$ solves (1.1).

2.3. $\mathcal{R}$-bounded solution operator families of reduced Stokes. According to what was pointed out in Subsection 2.1 and Subsection 2.2, we consider the two-phase reduced Stokes resolvent equations (2.8) instead of (1.1) through Section 4. More precisely, we prove the following theorem.

**Theorem 2.2.** Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $N < r < \infty$, and $\max(q, q') \leq r$ with $q' = q/(q - 1)$. Let $\rho_\pm$ be positive constants. Suppose that (a), (b), and (c) stated in Theorem 1.1 hold. For any open set $G$ of $\mathbb{R}^N$, let $X_{\mathcal{R}, q}(G)$ and $X_{\mathcal{R}, q}(G)$ be a pair of $\mathcal{R}$-bounded solution families of reduced Stokes.

Then there exist a positive number $\lambda_0 \geq 1$ and an operator family $B(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_0}, L(\mathcal{X}_{\mathcal{R}, q}(\hat{\Omega}), W^2_q(\hat{\Omega})^N))$ such that, for any $\lambda \in \Sigma_{\varepsilon, \lambda_0}$ and $(f, h, k) \in X_{\mathcal{R}, q}(\hat{\Omega})$, $u = B(\lambda)F_{\mathcal{R}, \lambda}(f, h, k)$ is a unique solution to (2.8), and furthermore,

$$
(2.8) \quad \mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\hat{\Omega}), L_q(\hat{\Omega}))^N} \left\{ \left( \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda B(\lambda) \right) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \right) \right\} \leq \gamma_0 \quad (l = 0, 1)
$$

for some positive constant $\gamma_0$. Here we have set $\tilde{N} = N^3 + N^2 + N$, $R_\lambda u = (\text{Div} u, \lambda^{1/2} \text{Div} u, \lambda u)$, and

$$
F_{\mathcal{R}, \lambda}(f, h, k) = (f, \nabla h, \lambda^{1/2} h, h, \nabla k, \lambda^{1/2} k).
$$
Remark 2.3. (1) The symbols $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, and $H_7$ are variables corresponding to $f$, $\nabla h$, $\lambda^{1/2}h$, $\nabla k$, $\lambda^{1/2}k$, and $k$, respectively. The norm of space $X_{R,q}(\hat{\Omega})$ is given by \[ \|X_{R,q}(\hat{\Omega})\|_{L_q(\hat{\Omega})} = \|(H_1, \ldots, H_7)\|_{L_q(\hat{\Omega})} + \|H_4\|_{W^{1,2}_q(\hat{\Omega})} + \|(H_5, H_6)\|_{L_q(\hat{\Omega})} + \|H_7\|_{W^{1,2}_q(\hat{\Omega})}. \]

(2) If $u$ satisfies (2.3) with $\mathbf{f} \in J_q(\Omega)$, then $\langle [\mathbf{h}], u \rangle_0 = 0$ on $\Gamma$, and $\langle \mathbf{k}, u \rangle_+ = 0$ on $\Gamma_+$. This fact can be obtained in the same manner as in Subsection 2.2 with $q = 0$. It then holds that $u$ belongs to $J_q(\Omega)$ by Gauss’s divergence theorem together with $\|u\| = 0$ on $\Gamma$, $u = 0$ on $\Gamma_+$. Here and subsequently, we can see $J_q(\Omega)$ as a closed subspace of $L_q(\hat{\Omega})^N$, that is, $J_q(\Omega)$ are regarded as Banach spaces endowed with $\|\cdot\|_{L_q(\hat{\Omega})}$.

At this point, we introduce several propositions used throughout this paper. The following two propositions are fundamental properties of the $R$-boundedness (cf. [12], Proposition 3.4, [12], Remark 3.2. (4)).

Proposition 2.4. (1) Let $X$ and $Y$ be Banach spaces, and let $T$ and $S$ be $R$-bounded families in $L(X,Y)$. Then $T + S = \{ T + S \mid T \in T, S \in S \}$ is also $R$-bounded in $L(X,Y)$ and $R_{L(X,Y)}(T + S) \leq R_{L(X,Y)}(T) + R_{L(X,Y)}(S).

(2) Let $X$, $Y$, and $Z$ be Banach spaces, and let $T$ and $S$ be $R$-bounded families in $L(X,Y)$ and $L(Y,Z)$, respectively. Then $ST = \{ ST \mid S \in S, T \in T \}$ is also $R$-bounded in $L(X,Z)$ and $R_{L(X,Z)}(ST) \leq R_{L(X,Y)}(T)R_{L(Y,Z)}(S).

Proposition 2.5. Let $1 \leq q < \infty$. Let $m(\lambda)$ be a bounded function defined on a subset $\Lambda$ of the complex plane $C$, and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(\Omega)$ with an open set $G$ of $\mathbb{R}^N$. Then \[ R_{L_q(\Omega)}(M_m(\lambda) \mid \lambda \in \Lambda) \leq K_q^2 \|m\|_{L_\infty(\Lambda)}, \] where $K_q$ is a positive constant in Khintchine’s inequality (cf. also [18], Theorem 2.4).

The next one is used to estimate terms arising from uniform $W^{2-1/r}_r$ domains, for example unit normal vectors $\mathbf{n}$, $\mathbf{n}_+$.\footnote{The book [1] Proposition 5.22 only considered bounded boundary, but we can extend the result to uniform $W^{2-1/r}_r$ domains as mentioned in [1], Remark 5.23 (1).}

Proposition 2.6. Let $1 \leq q \leq r < \infty$ and $N < r < \infty$. Suppose that $\Omega_\pm$ are uniform $W^{2-1/r}_r$ domains. Then there exists a positive constant $C_{N,q,r}$ such that, for any $\sigma > 0$, $a \in L_q(\hat{\Omega})$, and $b \in W^{1,2}_q(\hat{\Omega})$, it holds the estimate:

\[ \|ab\|_{L_q(\hat{\Omega})} \leq \sigma \|b\|_{L_q(\hat{\Omega})} + C_{N,q,r} \left( \sigma^{N/q} \|a\|_{L_q(\hat{\Omega})} + \|a\|_{L_q(\hat{\Omega})} \right) \|b\|_{L_q(\hat{\Omega})}. \]

Proof. We first show the following inequality: For $q \leq 1 + 1/s$ and $N(1/q - 1/s) < 1$,

\[ \|u\|_{L_q(\hat{\Omega})} \leq C_{N,q,r,s} \left( \|\nabla u\|_{L_q(\hat{\Omega})}^{N(\frac{1}{4} - \frac{1}{2})} + \|u\|_{L_q(\hat{\Omega})}^{N(\frac{1}{2} - \frac{1}{2})} \right) \] for any $u \in W_q^l(\hat{\Omega})$ with some positive constant $C_{N,q,r,s}$ independent of $u$. To this end, let $E_{\pm}$ be extension operators for $\Omega_\pm$, introduced in [1], Proposition 5.22\footnote{The book [1] Proposition 5.22 only considered bounded boundary, but we can extend the result to uniform $W^{2-1/r}_r$ domains as mentioned in [1], Remark 5.23 (1).} that is, $\|E_{\pm}u_\pm\|_{L_q(\hat{\Omega})} \leq C_{N,q,r,s}\|u_\pm\|_{L_q(\hat{\Omega})}$ for $l = 0, 1$ and for any $u_\pm \in W_q^l(\Omega_\pm)$, respectively. These inequalities combined with Sobolev embedding inequality:

\[ \|f\|_{L_q(\hat{\Omega})} \leq C_{N,q,s}\|\nabla f\|_{L_q(\hat{\Omega})}^{\frac{N(\frac{1}{4} - \frac{1}{2})}{N(\frac{1}{2} - \frac{1}{2})}} \] with $q \leq 1 + 1/s$ and $N(1/q - 1/s) < 1$ yield that

\[ \|u_\pm\|_{L_q(\hat{\Omega})} \leq \|E_{\pm}u_\pm\|_{L_q(\hat{\Omega})} \leq C_{N,q,r,s}\|\nabla E_{\pm}u_\pm\|_{L_q(\hat{\Omega})}^{N(\frac{1}{4} - \frac{1}{2})} \|E_{\pm}u_\pm\|_{L_q(\hat{\Omega})}^{N(\frac{1}{2} - \frac{1}{2})} \leq C_{N,q,r,s}\|u_\pm\|_{W_q^1(\Omega_\pm)}^{N(\frac{1}{4} - \frac{1}{2})} \|u_\pm\|_{L_q(\hat{\Omega})}^{1-N(\frac{1}{4} - \frac{1}{2})}, \]

respectively. Let $u_\pm = u(\mathbf{R}_\pm)$ for $u \in W_q^1(\hat{\Omega})$. Then we have

\[ \|u\|_{L_q(\hat{\Omega})}^q \leq 2^q \left( \|u_+\|_{L_q(\hat{\Omega})}^q + \|u_-\|_{L_q(\hat{\Omega})}^q \right) \leq C_{N,q,r,s}\|u_+\|_{W_q^1(\hat{\Omega})}^{q/p_1}\|u_+\|_{L_q(\hat{\Omega})}^{q/p_2} + \|u_-\|_{W_q^1(\hat{\Omega})}^{q/p_1}\|u_-\|_{L_q(\hat{\Omega})}^{q/p_2}, \]
where we have set \(1/p_1 = N(1/q - 1/s)\) and \(1/p_2 = 1 - N(1/q - 1/s)\). We combine the last inequality with Hölder’s inequality: 
\[a_+ b_+ + a_- b_- \leq (a_+^p + a_-^p)^{1/p_1}(b_+^{p_1} + b_-^{p_1})^{1/p_2} \]
for \(a_\pm = \|u_\pm\|^{q/p_1}_{L^q(\Omega_\pm)}\) and \(b_\pm = \|u_\pm\|^{q/p_2}_{L^q(\Omega_\pm)}\), respectively, in order to obtain

\[\|u\|_{L^q(\Omega)} \leq C_{N,q,r,s}\|u\|^{N(\frac{1}{q} - \frac{1}{s})}_{W^1_q(\Omega)}\|u\|^{1-N(\frac{1}{q} - \frac{1}{s})}_{L^q(\Omega)},\]

which implies (2.9). The required estimate of Proposition 2.9 follows from (2.9) in the same manner as in the proof of Lemma 2.4.

We devote the last part of this subsection to the proof of Theorem 1.6.

**Proof of Theorem 1.6.** We prove Theorem 1.6 under the assumption that Theorem 2.2 holds.

**Step 1:** Proof of (1.5). We will be shown in Remark 1.7 below that the unit normals \(n, n_+\) can be regarded as vector functions defined on \(R^N\) and that, for any \(f \in L_q(\Omega_\pm)\) and \(g \in W^1_q(\Omega_\pm)\),

\[(2.10) \quad \|f\nu\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}, \quad \|g\nabla \nu\|_{W^1_q(\Omega)} \leq C\|g\|_{W^1_q(\Omega)}, \quad \|g\nu\|_{W^1_q(\Omega)} \leq C\|g\|_{W^1_q(\Omega)}

with \(\nu \in \{n, n_+\}\) and some positive constant \(C\).

Let \((f,g,h,k) \in X_q\) and \(g \in \mathcal{G}(g)\). Suppose that \((f,h,k)\) satisfy (2.7). Then, in view of Subsection 2.2 and Theorem 2.2, we set

\[\mathbf{u} = \mathbf{F}(\lambda)F_{\mathbb{R},\lambda}(f + \rho^{-1}\nabla L(\lambda g, g), h + gn, k + gn_+), \quad \theta = K(\mathbf{u}) - L(\lambda g, g)\]
to see that \((\mathbf{u}, \theta)\) solves the problem (1.1).

From now on, we show the estimates (1.5), (1.6). By Theorem 2.2 and Proposition 2.4, we easily have (1.5). To prove (1.6), we check the definition of \(R\)-boundedness. Let \(n \in \mathbb{N}_1\), \((\lambda_j)_{j=1}^n \subset \Sigma_{e,\lambda_0}\), and \(\{\mathbf{F}_j\}_{j=1}^n \subset \mathcal{X}_q\) with \(\mathbf{F}_j = (F_{1j}, \ldots, F_{11j})\). Since \(\{\lambda(\kappa/d(\lambda))\}^q \nabla K(\mathbf{A}(\lambda)\mathbf{F}) = \nabla K(\{(\kappa/d(\lambda))\}^q \mathbf{A}(\lambda)\mathbf{F})\) (\(l = 0, 1\)), we have, by Proposition 2.5 and (1.5),

\[
\int_0^1 \left\| \sum_{j=1}^n r_j(u) \left[ \left( \frac{d}{d\lambda} \right) \right]^{l} \mathbf{F}(\lambda) \right\|^q_{L^q(\Omega)} du \\
\leq C_{\gamma_0} \left( \int_0^1 \left\| \sum_{j=1}^n r_j(u) \left[ \left( \frac{d}{d\lambda} \right) \right]^{l} \mathbf{A}(\lambda) \right\|^q_{W^1_q(\Omega)} du + \int_0^1 \left\| \sum_{j=1}^n r_j(u) (F_{4j}, F_{5j}) \right\|^q_{L^q(\Omega)^n \times W^1_q(\Omega)} du \right) \\
\leq C_{\gamma_0} \left\{ \left( \lambda^{-q} + \lambda^{-q/2} + 1 \right) \int_0^1 \left\| \sum_{j=1}^n r_j(u) \mathbf{F}_j \right\|^q_{\mathcal{X}_q} du + \int_0^1 \left\| \sum_{j=1}^n r_j(u) \mathbf{F}_j \right\|^q_{\mathcal{X}_q} du \right\} \\
\leq C_{\gamma_0, \lambda_0} \int_0^1 \left\| \sum_{j=1}^n r_j(u) \mathbf{F}_j \right\|^q_{\mathcal{X}_q} du,
\]

which furnishes (1.6).

\(^1\)As was discussed in Subsection 2.2, it suffices to consider \((f,h,k)\) satisfying (2.7). In fact, we can extend it to any \((f,h,k) \in X_{\mathbb{R},e}(\Omega)\), similarly to the proof of Step 1, with the help of \(\kappa\) used in Subsection 2.2.
Step 2: Uniqueness. Let \( u \in W^2_q(\hat{\Omega})^N \cap J_q(\Omega) \) and \( \theta = \theta_1 + \theta_2 \in W^1_q(\hat{\Omega}) + W^1_q(\Omega) \) satisfy

\[
\begin{align*}
\lambda u - \rho^{-1} \text{Div}(u, \theta) &= 0 \quad \text{in} \ \hat{\Omega}, \\
[T(u, \theta)n] &= 0 \quad \text{on} \ \Gamma, \\
[u] &= 0 \quad \text{on} \ \Gamma, \\
T(u, \theta)n_+ &= 0 \quad \text{on} \ \Gamma_+, \\
u &= 0 \quad \text{on} \ \Gamma_-.
\end{align*}
\]

(2.11)

We prove that \( u = 0 \) in \( \hat{\Omega} \), which leads to the uniqueness of \( \text{ii.} \) To this end, it suffices to show that

\[
(\rho u, \psi)_{\hat{\Omega}} = 0 \quad \text{for any} \ \psi \in C^\infty_0(\hat{\Omega})^N
\]

(2.12)
in what follows. In fact, it holds that \( u = 0 \) in \( \Omega_\pm \) if we choose \( \psi \in C^\infty_0(\Omega_\pm)^N \) in (2.12), respectively.

The assumption (b), stated in Theorem 1.6, allows us to choose a \( \kappa \in W^1_q(\Omega) \) satisfying

\[
(\rho^{-1} \nabla \kappa, \nabla \varphi)_{\hat{\Omega}} = (\psi, \nabla \varphi)_{\hat{\Omega}} \quad \text{for any} \ \varphi \in W^1_q(\Omega).
\]

In addition, since the two-phase reduced Stokes resolvent equations is solvable for \( q' = q/(q-1) \), we have a solution \( v \in W^2_q(\hat{\Omega})^N \) to the equations:

\[
\begin{align*}
\lambda v - \rho^{-1} \text{Div}(v, K(v)) &= \psi - \rho^{-1} \nabla \kappa \quad \text{in} \ \hat{\Omega}, \\
[T(v, K(v))n] &= 0 \quad \text{on} \ \Gamma, \\
[v] &= 0 \quad \text{on} \ \Gamma, \\
T(v, K(v))n_+ &= 0 \quad \text{on} \ \Gamma_+, \\
v &= 0 \quad \text{on} \ \Gamma_-.
\end{align*}
\]

(2.13)

Then \( \psi - \rho^{-1} \nabla \kappa \in J'_q(\Omega) \) implies that \( v \in J'_q(\Omega) \) as was discussed in Remark 2.3. Setting \( K(v) = w_1 + w_2 \in W^1_q(\hat{\Omega}) + W^1_q(\Omega) \) we have, by Gauss’s divergence theorem, \( (u, \nabla \kappa)_{\hat{\Omega}} = 0 \), and \( (u, \nabla w_2)_{\hat{\Omega}} = 0 \),

\[
(\rho u, \psi)_{\hat{\Omega}} = (\rho u, \lambda v - \rho^{-1} \text{Div}(v, w_1 + w_2 + \kappa))_{\hat{\Omega}}
\]

\[
= \lambda(\rho u, v)_{\hat{\Omega}} - (u, \text{Div}(\mu D(v)))_{\hat{\Omega}} + (u, \nabla w_1)_{\hat{\Omega}}
\]

\[
= \lambda(\rho u, v)_{\hat{\Omega}} + (\mu D(v))_{\hat{\Omega}} - (u, [\mu D(v)n])_{\Gamma} - (u, \mu D(v)n_+)_{\Gamma_+}
\]

\[
- (\text{div} u, w_2)_{\Gamma} + (u, w_1 n_+)_{\Gamma} + (u, w_1 n_+)_{\Gamma_+}.
\]

Noting that \( [w_2] = 0 \) on \( \Gamma \) and \( w_2 = 0 \) on \( \Gamma_+ \), we see that \( [\mu D(v)n - w_1 n] = [\mu D(v)n - K(v)n] = 0 \) on \( \Gamma \) and \( \mu D(v)n - w_1 n = \mu D(v) - K(v)n = 0 \) on \( \Gamma_+ \). In addition, it holds that \( \text{div} u = 0 \) in \( \hat{\Omega} \), since

\[
0 = - (u, \nabla \varphi)_{\hat{\Omega}} = (\text{div} u, \varphi)_{\hat{\Omega}} \quad \text{for any} \ \varphi \in C^\infty_0(\hat{\Omega}),
\]

where we have used \( u \in J_q(\Omega) \) and the relation \( C^\infty_0(\hat{\Omega}) \subset W^1_q(\Omega) \). Hence, (2.13) implies that

\[
(\rho u, \psi)_{\hat{\Omega}} = \lambda(\rho u, v)_{\hat{\Omega}} + (\mu D(v))_{\hat{\Omega}}.
\]

On the other hand, it holds by the first equation of (2.11) that \( \lambda \rho u - \text{Div}(u, \theta) = 0 \) in \( \hat{\Omega} \), which, combined with Gauss’s divergence theorem, furnishes that

\[
0 = (\lambda \rho u - \text{Div}(u, \theta), v)_{\hat{\Omega}}
\]

\[
= \lambda(\rho u, v)_{\hat{\Omega}} + (\mu D(u))_{\hat{\Omega}} - ([\mu D(u)n], v)_{\Gamma} - ([\mu D(u)n_+], v)_{\Gamma_+}
\]

\[
- (\theta_1, \text{div} v)_{\hat{\Omega}} + ([\theta_1 n], v)_{\Gamma_+} + (\theta_1 n_+), v)_{\Gamma_+}
\]

since \( (\nabla \theta_2, v)_{\hat{\Omega}} = 0 \) by \( v \in J'_q(\Omega) \). We thus obtain \( \lambda(\rho u, v)_{\hat{\Omega}} + (\mu D(u), D(v))_{\hat{\Omega}} = 0 \) in the same manner as we have obtained (2.14) from (2.13). The last identity combined with (2.13) implies (2.12), which completes the proof of the uniqueness.
2.4. Generation of analytic semigroup. In this and the next subsection, we discuss time-dependent problems. We now consider the following initial-boundary value problem:

\[
\begin{aligned}
\partial_t u - \rho^{-1} \text{Div} T(u, K(u)) &= 0 \quad \text{in } \hat{\Omega} \times (0, \infty), \\
\text{[}T(u, K(u))u] &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
[u] &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
T(u, K(u))n_+ &= 0 \quad \text{on } \Gamma_+ \times (0, \infty), \\
u &= 0 \quad \text{on } \Gamma_- \times (0, \infty), \\
[\partial_t u - \rho^{-1} \text{Div} T(u, K(u)) - Au]_{t=0} &= u_0 \quad \text{in } \hat{\Omega}.
\end{aligned}
\] (2.15)

To discuss the generation of analytic semigroup associated with (2.15), we formulate (2.15) in the semigroup setting. For this purpose, we introduce the Stokes operator \(A\) and its domain \(D_q(A)\) as follows:

\[
D_q(A) = \{u \in W_0^2(\hat{\Omega}) \cap J_q(\Omega) \mid [\mathcal{T}_n(\mu D(u)n)] = 0 \quad \text{on } \Gamma, \\
[u] = 0 \quad \text{on } \Gamma, \quad [\mathcal{T}_n(\mu D(u)n_+)] = 0 \quad \text{on } \Gamma_+ , \quad u = 0 \quad \text{on } \Gamma_-, \}
\]

where we have set \(\mathcal{T}_n f = f - f \cdot n = n \) and \(\mathcal{T}_n f = f - f \cdot n_+ = n_+\) that are the tangential parts of \(N\)-vector \(f\) with respect to \(n\) and \(n_+\), respectively. Then it is possible to rewrite (2.15) as follows:

\[
\partial_t u - Au = 0 \quad (t > 0), \quad [\partial_t u - \rho^{-1} \text{Div} T(u, K(u)) - Au]_{t=0} = u_0.
\]

By Theorem 2.2 the resolvent set \(\rho(A)\) of \(A\) contains \(\Sigma_{\epsilon, \lambda_0}\). In addition, denoting the resolvent operator of \(A\) by \((\lambda - A)^{-1}\) and noting Remark 2.3 (2), we see that, for any \(\lambda \in \Sigma_{\epsilon, \lambda_0}\) and \(f \in J_q(\Omega)\), \((\lambda - A)^{-1}f = B(\lambda)f, 0, 0, 0, 0 ) \in J_q(\Omega)\). Since the \(R\)-boundedness of \(B(\lambda)\) implies the usual boundedness, it holds that

\[
\|((\lambda - A)^{-1}) \|_{L(J_q(\Omega))} \leq \frac{M_{\epsilon, \lambda_0}}{|\lambda|} \quad (\lambda \in \Sigma_{\epsilon, \lambda_0})
\]

with some positive constant \(M_{\epsilon, \lambda_0}\). By this resolvent estimate, we have the following theorem.

**Theorem 2.7.** Let \(1 < q < \infty, N < r < \infty, \) and \(\max(q', q) \leq r\) with \(q' = q/(q - 1)\). Let \(\rho_{\pm}\) be positive constants. Suppose that the conditions (a), (b), and (c) stated in Theorem 1.6 hold. Then the Stokes operator \(A\) generates a \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) on \(J_q(\Omega)\), which is analytic.

2.5. Maximal \(L_p - L_q\) regularity. Since the system (2.1) is linear, we consider the following two problems:

\[
\begin{aligned}
\partial_t u - \rho^{-1} \text{Div} T(u, \theta) &= 0, \quad \text{div} u = 0 \quad \text{in } \hat{\Omega} \times (0, \infty), \\
[T(u, \theta)u] &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
T(u, \theta)n_+ &= 0 \quad \text{on } \Gamma_+ \times (0, \infty), \\
u &= 0 \quad \text{on } \Gamma_- \times (0, \infty), \\
[\partial_t u - \rho^{-1} \text{Div} T(u, \theta)]_{t=0} &= u_0 \quad \text{in } \hat{\Omega},
\end{aligned}
\] (2.17)

\[
\begin{aligned}
\partial_t u - \rho^{-1} \text{Div} T(u, \theta) &= f, \quad \text{div} u = g \quad \text{in } \hat{\Omega} \times (0, \infty), \\
[T(u, \theta)u] &= [h], \quad [u] = 0 \quad \text{on } \Gamma \times (0, \infty), \\
T(u, \theta)n_+ &= k \quad \text{on } \Gamma_+ \times (0, \infty), \\
u &= 0 \quad \text{on } \Gamma_- \times (0, \infty), \\
[\partial_t u - \rho^{-1} \text{Div} T(u, \theta)]_{t=0} &= 0 \quad \text{in } \hat{\Omega}.
\end{aligned}
\] (2.18)

To state maximal regularity theorems for (2.17) and (2.18), we introduce several function spaces. For a Banach space \(X\), we denote the usual Lebesgue space and Sobolev space of \(X\)-valued functions defined on time interval \(I\) by \(L_p(I, X)\) and \(W^m_p(I, X)\) with \(m \in \mathbb{N}\), and their associated norms by \(\| \cdot \|_{L_p(I, X)}\) and \(\| \cdot \|_{W^m_p(I, X)}\), respectively. We set for \(\gamma > 0\)

\[
L_{p, \gamma}(I, X) = \{ f : I \rightarrow X \mid e^{-\gamma t}f \in L_p(I, X) \}, \quad L_{p, 0, \gamma}(R, X) = \{ f \in L_{p, \gamma}(R, X) \mid f(t) = 0 \text{ for } t < 0 \},
\]

where \(\gamma > 0\).
\[ W^m_{p,\gamma}(I, X) = \{ f \in L^p_{p,\gamma}(I, X) \mid e^{-\gamma t}D^j f(t) \in L^p(I, X) \ (j = 1, \ldots, m) \}, \]
\[ W^m_{p,0,\gamma}(R, X) = W^m_{p,\gamma}(R, X) \cap L^p_{p,0,\gamma}(R, X). \]

Let \( L, L^{-1}, F, \) and \( F^{-1} \) denote the Laplace transform, the Laplace inverse transform, the Fourier transform, and the Fourier inverse transform, which are denoted by

\[ L[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) \, dt, \quad L^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\lambda) \, d\lambda \ (\lambda = \gamma + i\tau), \]
\[ F[f](\tau) = \int_{-\infty}^{\infty} e^{-i\tau t} f(t) \, dt, \quad F^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau t} g(\tau) \, d\tau. \]

Note that \( L[f](\lambda) = L[F(e^{-\gamma t}f(t))](\tau) \) and \( L^{-1}[g](t) = e^{\gamma t}F^{-1}[g(\gamma + it)](t) \). For any real number \( s \geq 0 \), let \( H^s_{p,\gamma}(R, X) \) be the Bessel potential space of order \( s \) defined by

\[ H^s_{p,\gamma}(R, X) = \{ f \in L^p_{p,\gamma}(R, X) \mid e^{-\gamma t}(\Lambda^s_{\gamma} f)(t) \in L^p(R, X), \quad (\Lambda^s_{\gamma} f)(t) = L^{-1}[\lambda^s L[f]](t). \]

We also set \( H^0_{p,\gamma}(R, X) = \{ f \in H^0_{p,\gamma}(R, X) \mid f(t) = 0 \text{ for } t < 0 \} \). For solutions of problems (2.17) and (2.18), \( W^{2,1}_{q,p,\gamma}(\Omega \times (0, \infty)) \) and \( W^{2,1}_{q,p,0,\gamma}(\Omega \times R) \) are defined by

\[ W^{2,1}_{q,p,\gamma}(\Omega \times (0, \infty)) = W^{1}_{p,\gamma}((0, \infty), L^q(\Omega)^N) \cap L^p_{p,\gamma}((0, \infty), W^2_q(\Omega^N)), \]
\[ W^{2,1}_{q,p,0,\gamma}(\Omega \times R) = W^{1}_{p,0,\gamma}(R, L^q(\Omega^N)) \cap L^p_{p,0,\gamma}(R, W^2_q(\Omega^N)). \]

First we discuss a maximal \( L_p-L_q \) regularity theorem for (2.17). Setting \( u(t) = T(t)u_0 \) and \( \theta(t) = K(u(t)) \), we see that \( \text{div} \ u(t) = 0 \) in \( \Omega \) for \( t > 0 \) by \( u(t) \in J_q(\Omega) \), and thus \( u(t) \) and \( \theta(t) \) satisfy (2.17). Since \( \{ T(t) \}_{t \geq 0} \) is analytic, we have, for some \( \lambda_0 \geq 1 \) and for any \( t > 0 \),

\[ \| T(t)u_0 \|_{J_q(\Omega)} \leq C_{p,0,\lambda_0} e^{\lambda_0 t} \| u_0 \|_{J_q(\Omega)} \quad \text{for } u_0 \in J_q(\Omega), \]
\[ \| \partial_t T(t)u_0 \|_{J_q(\Omega)} \leq C_{p,0,\lambda_0} e^{\lambda_0 t} \| u_0 \|_{J_q(\Omega)} \quad \text{for } u_0 \in J_q(\Omega), \]
\[ \| \partial_t^2 T(t)u_0 \|_{J_q(\Omega)} \leq C_{p,0,\lambda_0} e^{\lambda_0 t} \| u_0 \|_{\mathcal{D}_q(A)} \quad \text{for } u_0 \in \mathcal{D}_q(A) \]

with some positive constant \( C_{p,\lambda_0} \). We then obtain in the same manner as in [27] Theorem 3.9

\[ \| e^{-2\lambda_0 t}(\partial_t u, \nabla u, \nabla^2 u) \|_{L^p((0, \infty), L^q(\Omega))} \leq C_{p,q,\lambda_0} \| u_0 \|_{L^{2(1-1/p)}_q(\Omega)} \]

for some positive constant \( C_{p,q,\lambda_0} \) with \( 1 < p, q < \infty, N > r > \infty, \) and \( \max(q, q') \leq r \) with \( q' = q/(q-1) \). Let \( \rho_{p,\lambda} \) be positive constants. Suppose that the conditions (a), (b), and (c) stated in Theorem 1.6 hold. Then we have the following two assertions:

1. There exists a positive constant \( \gamma_0 \geq 1 \) such that, for any \( u_0 \in D^{2(1-1/p)}_{q,p}(\Omega) \), the problem (2.17) admits a unique solution \( (u, \theta) \in W^{2,1}_{q,p,\gamma}(\Omega \times (0, \infty)) \times L^p_{p,\gamma}(0, \infty), W^1_q(\Omega) + W^1_q(\Omega) \), which satisfies

\[ \| e^{-\gamma_0 t}(\partial_t u, \nabla u, \nabla^2 u) \|_{L^p(0, \infty), L^q(\Omega)} + \| e^{-\gamma_0 t}\nabla \theta \|_{L^p((0, \infty), L^q(\Omega))} \leq C_{p,q,\gamma_0} \| u_0 \|_{D^{2(1-1/p)}_{q,p}(\Omega)} \]

with some positive constant \( C_{p,q,\gamma_0} \).

2. There exists a positive constant \( \gamma_0 \geq 1 \) such that, for any

\[ f \in L_{p,0,\gamma_0}(R, L^q_1(\Omega)^N), \quad g \in H^{1/2}_{p,0,\gamma_0}(R, L^q_1(\Omega)^N) \cap L_{p,0,\gamma_0}(R, W^1_q(\Omega) \cap W^1_q(\Omega)), \]
\[ h \in H^{1/2}_{p,0,\gamma_0}(R, L^q_1(\Omega)^N) \cap L_{p,0,\gamma_0}(R, W^1_q(\Omega)^N), \]
\[ k \in H^{1/2}_{p,0,\gamma_0}(R, L^q_1(\Omega^+)^N) \cap W^1_{p,0,\gamma_0}(R, L^q_1(\Omega^+)^N) \]

and for any representative \( g \in W^1_{p,0,\gamma_0}(R, L^q_1(\Omega)^N) \) of \( G(g) \), the problem (2.18) a unique solution

\[ (u, \theta) \in W^{2,1}_{q,p,\gamma_0}(\Omega \times R) \times L_{p,0,\gamma_0}(R, W^1_q(\Omega) + W^1_q(\Omega)), \]

such that
which possesses the estimate:

\begin{equation}
\|e^{-\gamma t}(\partial_t u, u, \Lambda^{1/2}_0\nabla u, \nabla^2 u)\|_{L^p(R, L^2(\Omega)))} + \|e^{-\gamma t}\nabla \theta\|_{L^p(R, L^q(\Omega)))} \leq C_{p,q,\gamma_0}N_{p,q,\gamma_0}(f, g, h, k)
\end{equation}

for some positive constant \(C_{p,q,\gamma_0}\) with

\[
N_{p,q,\gamma_0}(f, g, h, k) = \|e^{-\gamma t}(f, \nabla g, \Lambda^{1/2}_0\nabla h, \Lambda^{1/2}_0 h)\|_{L^p(R, L^2(\Omega)))} + \|e^{-\gamma t}(g, h)\|_{L^p(R, W^2_q(\Omega)))}
+ \|e^{-\gamma t}(\nabla k, \Lambda^{1/2}_0 k)\|_{L^p(R, L^2(\Omega)))} + \|e^{-\gamma t}k\|_{L^p(R, W^2_q(\Omega)))}.
\]

In addition, if \(g = 0, h = 0, \) and \(k = 0,\) then

\begin{equation}
\gamma \|e^{-\gamma t}u\|_{L^p(R, L^2(\Omega)))} \leq C_{p,q,\gamma_0}e^{-\gamma t}f\|L^p(R, L^2(\Omega))) \quad \text{for any } \gamma \geq \gamma_0.
\end{equation}

Proof. We prove the assertion (2). Smooth functions having compact supports with respect to time variable are dense in the spaces for \(f, g, h, \) and \(k,\) so that we may assume that \(f, g, h, \) and \(k,\) are smooth and supported compactly with respect to time variable. Applying the Laplace transform with respect to variable \(t \in \mathbb{R}\) to \((2.18),\) we have

\[
\left\{ \begin{array}{l}
\lambda v - \rho^{-1} \text{Div} \mathbf{T}(v, \pi) = L[f](\lambda), \quad \text{div} v = L[g](\lambda) \quad \text{in } \hat{\Omega}, \\
\mathbf{T}(v, \pi) = L[h](\lambda) \quad \text{on } \Gamma, \\
\mathbf{T}(v, \pi) n_+ = L[k](\lambda) \quad \text{on } \Gamma_+, \\
v = 0 \quad \text{on } \Gamma_-
\end{array} \right.
\]

On the other hand, we observe that \((L[g](\lambda), \varphi)_{\Omega} = -(L[g](\lambda), \nabla \varphi)_{\Omega}\) for all \(\varphi \in W^1_{q, \Gamma_+}(\Omega),\) because \((g(t), \varphi)_{\Omega} = -(g(t), \nabla \varphi)_{\Omega}\) for \(t \in \mathbb{R}\) by \((1.3).\) This implies that \(L[g](\lambda) \in W^1_q(\Omega)\) and \(L[g](\lambda) \in G(L[g](\lambda)\), so that we define, in view of Theorem 1.6 \(u \) and \(\theta\) by

\[
u = L^{-1}\{A(\lambda)F(\lambda) (L[f], L[g], L[h], L[k])\}, \quad \theta = L^{-1}\{P(\lambda)F(\lambda) (L[f], L[g], L[h], L[k])\}.
\]

Since we assume that \(f, g, h, \) and \(k,\) are supported compactly, it holds that \(L[f], L[g], L[h], \) and \(L[k]\) are holomorphic functions with respect to \(\lambda.\) Thus \(u \) and \(\theta\) are defined independently of \(\gamma \geq \gamma_0\) for \(\lambda = \gamma + i\tau,\) where \(\gamma_0\) is a positive number greater than \(\lambda_0\) stated in Theorem 1.6. Then,

\begin{equation}
negthinspace e^{-\gamma t} \left(\partial_t u, \Lambda^{1/2}_0\nabla u, \nabla^2 u\right) = \mathcal{F}^{-1} \left[ R_{\mu_0}(\lambda) \mathcal{F}[e^{-\gamma t}f] \right], \quad
\end{equation}

\begin{equation}
e^{-\gamma t}u = \mathcal{F}^{-1} \left[ (\mu_0^{-1}(\lambda) \mathcal{F}[e^{-\gamma t}f] \right] , \quad e^{-\gamma t}\nabla \theta = \mathcal{F}^{-1} \left[ \nabla \mathcal{F}[\mu_0^{-1}(\lambda) \mathcal{F}[e^{-\gamma t}f]] \right],
\end{equation}

with \(\mu_0 = \gamma_0 + i\tau\) and \(\mathbf{F} = (f, \nabla g, \Lambda^{1/2}_0 g, \partial_t g, g, \nabla h, \Lambda^{1/2}_0 h, h, \nabla k, \Lambda^{1/2}_0 k, k),\) which, combined with Weis's operator valued Fourier multiplier theorem (cf. \([31]\) Theorem 3.4) together with Theorem 1.6 and Proposition 2.3 allows us to conclude that the estimate \((2.19)\) holds.

Analogously, we can obtain the estimate \((2.20)\) if \(\mu_0\) is replaced by \(\lambda = \gamma + i\tau (\gamma \geq \gamma_0)\) in the second formula of \((2.21).\) Finally, \((2.21)\) combined with the argumentation used in \([24]\) Section 7 furnishes that \(u(t) = 0, \theta(t) = 0\) for \(t < 0\) and the uniqueness holds. This completes the proof of Theorem 2.8.

3. Two-phase reduced Stokes resolvent equations in \(\mathbb{R}^N\)

In this section, we discuss \(R\)-bounded solution operator families to the two-phase reduced Stokes resolvent equations with an interface condition in \(\mathbb{R}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_-;\) that is, we consider the following resolvent problem with resolvent parameter \(\lambda\) varying in \(\Sigma_\varepsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \varepsilon\}:

\begin{equation}
\left\{ \begin{array}{l}
\lambda u - \rho^{-1} \text{Div} \mathbf{T}(u, K_f(u)) = f \quad \text{in } \mathbb{R}^N, \\
[\mathbf{T}(u, K_f(u)) n_0] = [h] \quad \text{on } \partial \mathbb{R}^N, \\
[u] = 0 \quad \text{on } \partial \mathbb{R}^N,
\end{array} \right.
\end{equation}

where \(n_0 = (0, \ldots, 0, -1)^T\) and \(\mathbf{T}(u, K_f(u)) = \mu \mathbf{D}(u) - K_f(u)\mathbf{I}\). Here \(\rho = \rho_+ \chi_{\mathbb{R}^N_+} + \rho_- \chi_{\mathbb{R}^N_-}\) for positive constants \(\rho_{\pm}\) and suppose that

(d) viscosity coefficient \(\mu\) is given by \(\mu = \mu_+ \chi_{\mathbb{R}^N_+} + \mu_- \chi_{\mathbb{R}^N_-}\) for positive constants \(\mu_{\pm}\) satisfying \(\mu_{\pm_1} \leq \mu_{\pm} \leq \mu_{\pm_2}\), respectively, where \(\mu_{\pm_1}\) and \(\mu_{\pm_2}\) are the same constants as in Theorem 1.6.
Furthermore, for $1 < q < \infty$ and $q' = q/(q - 1)$, let $K_f(u)$ be defined by $K_f(u) = K(\alpha, \beta)$ with 
\[
\alpha = \rho^{-1} \text{Div}(\mu D(u)) - \nabla \text{div} u, \quad \beta = \langle [\mu D(u)] n_0 \rangle, n_0 > -[\text{div} u] \quad \text{for } u \in W^2_q(\mathbb{R}^N)^N,
\]
where $K(\alpha, \beta)$ is given in Remark 3.1 with $\Omega = \mathbb{R}^N$, that is, $K_f(u)$ is the unique solution to 
\[
(\rho^{-1} \nabla K_f(u), \nabla \varphi)_{\mathbb{R}^N} = (\rho^{-1} \text{Div}(\mu D(u)) - \nabla \text{div} u, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for all } \varphi \in \tilde{W}^1_{q'}(\mathbb{R}^N),
\]
\[
[K_f(u)] = \langle [\mu D(u)] n_0 \rangle, n_0 > -[\text{div} u] \quad \text{on } R^N_0.
\]
Especially, we know that $\|\nabla K_f(u)\|_{L_q(\mathbb{R}^N)} \leq \gamma_0 \|\nabla u\|_{W^1_{q'}(\mathbb{R}^N)}$. Here and hereafter, $\gamma_0$ denotes a generic constant depending solely on $N, q, \rho_+, \rho_-, \mu_+ \mu_2, \mu_{-1}$, and $\mu_{-2}$.

We will prove the following theorem in this section.

**Theorem 3.1.** Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and $\rho_{\pm}$ be positive constants. Suppose that the condition (d) holds. For any open set $G$ of $\mathbb{R}^N$, let $Y_{\mathbb{R}, q}(G)$ and $Y'_{\mathbb{R}, q}(G)$ be defined as 
\[
Y_{\mathbb{R}, q}(G) = \{(f, h) \mid f \in L_q(G)^N, h \in W^1_q(G)^N\},
\]
\[
Y'_{\mathbb{R}, q}(G) = \{(H_1, H_2, H_3) \mid H_1, H_3 \in L_q(G)^N, H_2 \in L_q(G)^{N^2}\}.
\]
Then there exists an operator family $S_f(\lambda) \in \text{Hol}(\Sigma_\varepsilon, L(\mathcal{Y}_{\mathbb{R}, q}(\mathbb{R}^N), W^2_q(\mathbb{R}^N)^N))$ such that, for any $\lambda \in \Sigma_\varepsilon$ and $(f, h) \in Y_{\mathbb{R}, q}(\mathbb{R}^N)$, $u = S_f(\lambda) G_{\mathbb{R}, \lambda}(f, h)$ is a unique solution to the problem (3.1), and furthermore, 
\[
\mathcal{R}_{L(\mathcal{Y}_{\mathbb{R}, q}(\mathbb{R}^N), L_q(\mathbb{R}^N)^N)}(\left\{ \left( \frac{\partial}{\partial \lambda} \right)^i (R_s S_f(\lambda)) \mid \lambda \in \Sigma_\varepsilon \right\}) \leq \gamma_1 \quad (l = 0, 1).
\]

Here and subsequently, we set $\tilde{N} = N^3 + N^2 + N$, $R_s u = (\nabla^2 u, \lambda^{1/2} \nabla u, \lambda u)$, $G_{\mathbb{R}, \lambda}(f, h) = (f, \nabla h, \lambda^{1/2} h)$ and $\gamma_1$ denotes a constant depending solely on $N, q, \varepsilon, \rho_+, \rho_-, \mu_+ \mu_2, \mu_{-1}$, and $\mu_{-2}$.

In view of Subsection 2.1 it is sufficient to consider the two-phase Stokes resolvent equations in $\tilde{\mathbb{R}}^N$:

\[
\begin{cases}
\lambda \rho u - \text{Div}(\mu D(u)) + \nabla \theta = \rho f & \text{in } \mathbb{R}^N, \\
\text{div } u = g & \text{in } \mathbb{R}^N, \\
([\mu D(u) - \theta] n_0) = [h] & \text{on } \mathbb{R}^N_0, \\
[u] = 0 & \text{on } \mathbb{R}^N_0.
\end{cases}
\]

Here, the Fourier transform $\mathcal{F}$ and its inverse formula $\mathcal{F}^{-1}$ are defined by

\[
\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-i x \xi} f(x) \, dx, \quad \mathcal{F}^{-1}[g](\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i x \xi} g(\xi) \, d\xi,
\]
respectively. We first consider the divergence equation: $\text{div } u = g$ in $\mathbb{R}^N$.

**Lemma 3.2.** Let $1 < q < \infty$. For $g \in W^1_{q'}(\mathbb{R}^N) \cap W^2_q(\mathbb{R}^N)$, we set 
\[
V(g) = (V_1(g), \ldots, V_N(g))^T, \quad V_j(g) = -\mathcal{F}^{-1} \left[ \frac{i \xi_j}{|\xi|^2} \mathcal{F}[g](\xi) \right](x) \quad (j = 1, \ldots, N).
\]

Then $V(g) \in W^1_q(\mathbb{R}^N)^N \cap W^2_q(\mathbb{R}^N)^N$ and $u = V(g)$ solves the divergence equation: $\text{div } u = g$ in $\mathbb{R}^N$. In addition, there are operators
\[
V^1 \in \mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N)^{N^3}), \quad V^2 \in \mathcal{L}(L_q(\mathbb{R}^N), L_q(\mathbb{R}^N)^{N^2}), \quad V^3 \in \mathcal{L}(\tilde{W}^{-1}_q(\mathbb{R}^N), L_q(\mathbb{R}^N)^N)
\]
such that $R_s V(g) = (V^1(\nabla g), V^2(\lambda^{1/2} g), V^3(\lambda g))$, where the dual space of $\tilde{W}^{-1}_q(\mathbb{R}^N)$ with $q' = q/(q - 1)$ is written by $\tilde{W}^{-1}_q(\mathbb{R}^N)$ endowed with norm $\| \cdot \|_{\tilde{W}^{-1}_q(\mathbb{R}^N)}$.

**Proof.** It is clear that $u = V(g)$ solves the divergence equation: $\text{div } u = g$ in $\tilde{\mathbb{R}}^N$ and that by the Fourier multiplier theorem of Mikhlin (cf. [19] Appendix, Theorem 2))
\[
\|\nabla V(g)\|_{L_q(\mathbb{R}^N)} \leq \gamma_0 \|g\|_{L_q(\mathbb{R}^N)}, \quad \|\partial_k \nabla V(g)\|_{L_q(\mathbb{R}^N)} \leq \gamma_0 \|\partial_k g\|_{L_q(\mathbb{R}^N)} \quad (k = 1, \ldots, N - 1).
\]
Since $\text{div} \, V(g) = g$ in $\mathbb{R}^N$, it holds that $\partial^2_{ij} V(g) = \partial_N g - \partial_N \sum_{k=1}^{N-1} \partial_k V(g)$ in $\mathbb{R}^N$, which, combined with the last inequalities, furnishes that $\|\partial^2_{ij} V(g)\|_{L_q(\mathbb{R}^N)} \leq \gamma_0 \| \nabla g \|_{L_q(\mathbb{R}^N)}$.

Next we estimate $V(g)$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, and then $(V(g), \varphi)_{\mathbb{R}^N} = -(g, F[|\xi|^2 \cdot \mathcal{F}^{-1} [\varphi]](\xi))_{\mathbb{R}^N}$. The Fourier multiplier theorem again yields that

$$\|(V(g), \varphi)_{\mathbb{R}^N}\| \leq \|g\|_{\mathcal{W}^{-1,q}(\mathbb{R}^N)} \left\| \nabla \mathcal{F} \left[ \frac{\xi \cdot \mathcal{F}^{-1} [\varphi](\xi)}{|\xi|^2} \right] \right\|_{L_q(\mathbb{R}^N)},$$

which implies that $\|V(g)\|_{L_q(\mathbb{R}^N)} \leq \gamma_0 \|g\|_{\mathcal{W}^{-1,q}(\mathbb{R}^N)}$. We thus see that $V(g) \in W^1_q(\mathbb{R}^N)^N \cap W^2_q(\mathbb{R}^N)^N$ and the existence of operators $V^i (i = 1, 2, 3)$. This completes the proof of the lemma.

Note that $\|V(g)\| = 0$ on $\mathbb{R}^N_0$ since $V(g) \in W^1_q(\mathbb{R}^N)^N$ by Lemma 3.2. Setting $u = V(g) + v$ in (3.3) and noting $\text{Div}(\mu D(v)) = \mu \Delta v$ by the condition (d) and by $\text{div} \, v = 0$ in $\mathbb{R}^N$, we have

$$(3.7) \begin{cases} \rho \lambda v - \mu \Delta v + \nabla \theta = \tilde{f} & \text{in } \mathbb{R}^N, \\ \text{div} \, v = 0 & \text{in } \mathbb{R}^N, \\ [\mu D(v) - \theta I] n_0 = [\tilde{h}] & \text{on } \mathbb{R}^N_0, \\ [v] = 0 & \text{on } \mathbb{R}^N_0, \end{cases}$$

where $\tilde{f} = \rho f - \rho \lambda V(g) + \text{Div}(\mu D(V(g)))$ and $\tilde{h} = h - \mu D(V(g)) n_0$.

The following theorem was essentially proved in [28, Theorem 1.1, Theorem 1.2], but we again show them here from viewpoint of the existence of $\mathcal{R}$-bounded solution operator families of (3.7).

**Theorem 3.3.** Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\rho_{\pm}$ be positive constants. Suppose that the condition (d) holds. Then there exists an operator family $S_l(\lambda) \in \text{Hol}(\Sigma_\epsilon, L(Y_{\mathcal{R},q}(\mathbb{R}^N), W^2_q(\mathbb{R}^N)^N))$ such that, for any $\lambda \in \Sigma_\epsilon$ and $(\tilde{f}, \tilde{h}) \in Y_{\mathcal{R},q}(\mathbb{R}^N)$, $v = S_l(\lambda)G_{\mathcal{R},l}(\tilde{f}, \tilde{h})$ is a unique solution to the problem (3.7) with some pressure term $\theta$.

In addition,

$$\mathcal{R}_{\mathcal{L}(Y_{\mathcal{R},q}(\mathbb{R}^N), L_q(\mathbb{R}^N)^N)} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda S_l(\lambda) \right) \mid \lambda \in \Sigma_\epsilon \right\} \right) \leq \gamma_1 \quad (l = 0, 1).$$

**Proof.** **Step 1:** Reduction to $\tilde{f} = 0$. We first reduce (3.7) to the case $\tilde{f} = 0$. To this end, we consider problems in $\mathbb{R}^N$ as follows:

$$\begin{cases} \rho_+ \lambda \psi_+ - \mu_+ \Delta \psi_+ + \nabla \varphi_+ = \tilde{f}, & \text{in } \mathbb{R}^N, \\ \rho_- \lambda \psi_- - \mu_- \Delta \psi_- + \nabla \varphi_- = \tilde{f}, & \text{in } \mathbb{R}^N, \\ \text{div} \, \psi_+ = 0, \quad \text{div} \, \psi_- = 0. \end{cases}$$

Then we have the following solution formulas (cf. [28, Section 2]):

$$\psi_\pm = A_\pm(\lambda) \tilde{f} := \mathcal{F}^{-1} \left[ \frac{\mathcal{F}[\tilde{f}](\xi) - |\xi|^2 \varphi_\pm \chi_{\mathbb{R}^N}}{\rho_\pm \lambda + \mu_\pm |\xi|^2} > 0 \right](x), \quad \varphi_\pm = -\mathcal{F}^{-1} \left[ \frac{<i\xi, \mathcal{F}[\tilde{f}](\xi)>}{|\xi|^2} \right](x).$$

By [14] Theorem 3.3, proof of Theorem 3.2,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N)^N)} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda A_\pm(\lambda) \right) \mid \lambda \in \Sigma_\epsilon \right\} \right) \leq \gamma_1 \quad (l = 0, 1).$$

Here, we set

$$\psi = A(\lambda) \tilde{f} := (A_+(\lambda) \tilde{f}) \chi_{\mathbb{R}^N} + (A_-(\lambda) \tilde{f}) \chi_{\mathbb{R}^N}, \quad \varphi = \varphi_+ \chi_{\mathbb{R}^N} + \varphi_- \chi_{\mathbb{R}^N}.$$
and also setting $v = \mathcal{A}(\lambda)\tilde{f} + w$ and $\theta = \varphi + \kappa$ in (3.17) yields that
\[
\begin{aligned}
\rho\lambda w - \mu \Delta w + \nabla \kappa &= 0 \quad \text{in } \hat{\mathbf{R}}^N, \\
\text{div } w &= 0 \quad \text{in } \hat{\mathbf{R}}^N, \\
[\mu D(w) - \kappa I]n_0 &= [\tilde{h}] - [\mu D(\mathcal{A}(\lambda)\tilde{f})]n_0 \quad \text{on } \mathbf{R}^N_0, \\
[w] &= -[\mathcal{A}(\lambda)\tilde{f}] \quad \text{on } \mathbf{R}^N_0.
\end{aligned}
\]

To analyze this system, it is enough to consider the equations:
\[
\begin{aligned}
\rho\lambda u - \mu \Delta u + \nabla \theta &= 0 \quad \text{in } \hat{\mathbf{R}}^N, \\
\text{div } u &= 0 \quad \text{in } \hat{\mathbf{R}}^N, \\
[\mu D(u) - \theta I]n_0 &= [h] \quad \text{on } \mathbf{R}^N_0, \\
[u] &= [k] \quad \text{on } \mathbf{R}^N_0
\end{aligned}
\tag{3.11}
\]
for given $h = (h_1, \ldots, h_N)^T \in W^1_q(\hat{\mathbf{R}}^N_0)^N$ and $k = (k_1, \ldots, k_N)^T \in W^2_q(\hat{\mathbf{R}}^N)^N$ with $k_N = -\psi_N$, where $\psi_N$ is the $N$th component of $\psi$ defined as (3.3).

**Step 2: Solution formulas of (3.11).** We rewrite (3.11) as follows:
\[
\begin{aligned}
\rho\pm \lambda u_\pm - \mu \pm \Delta u_\pm + \nabla \theta_\pm &= 0 \quad \text{in } \mathbf{R}^N_\pm, \\
\text{div } u_\pm &= 0 \quad \text{in } \mathbf{R}^N_\pm, \\
[\mu \partial_N u_j + \partial_j u_N] &= -[h_j] \quad \text{on } \mathbf{R}^N_0, \\
[2\mu \partial_N u_N - \theta] &= -[h_N] \quad \text{on } \mathbf{R}^N_0, \\
[u_j] &= [k_j] \quad \text{on } \mathbf{R}^N_0
\end{aligned}
\tag{3.12}
\]
where $u_\pm = (u_1, \ldots, u_N)^T$, $u_\pm = u_\pm |_{\mathbf{R}_\pm^N}$, and $\theta_\pm = \theta |_{\mathbf{R}_\pm^N}$. Here and subsequently, $j$ and $J$ run from 1 to $N - 1$ and 1 to $N$, respectively, and we set $y^j = (y_1, \ldots, y_{N-1})$ for $y = (y_1, \ldots, y_N)$.

Let $\hat{f}(\xi', x_N)$ and $\mathcal{F}_\xi^{-1} [g(\xi', x_N)](x')$ be the partial Fourier transform with respect to $x'$ and its inverse formula defined by
\[
\hat{f}(\xi', x_N) = \int_{\mathbf{R}^{N-1}} e^{-ix' \cdot \xi'} f(x', x_N) \, dx', \quad \mathcal{F}_\xi^{-1} [g(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} e^{ix' \cdot \xi'} g(\xi', x_N) \, d\xi'.
\]

Apply the partial Fourier transform to (3.12), and we have
\[
\begin{aligned}
\rho\pm \lambda |\xi|^2 - \mu \pm \partial_N^2 |\xi|^2 u_\pm(\xi', x_N) + i\xi_j \theta_\pm(\xi', x_N) = 0, & \quad \pm x_N > 0, \\
\rho\pm \lambda |\xi|^2 - \mu \pm \partial_N^2 |\xi|^2 u_\pm(\xi', x_N) + \partial_N \theta_\pm(\xi', x_N) = 0, & \quad \pm x_N > 0, \\
\sum_{j=1}^{N-1} i\xi_j \tilde{u}_\pm(\xi', x_N) + \partial_N \tilde{u}_\pm(\xi', x_N) = 0, & \quad \pm x_N > 0,
\end{aligned}
\tag{3.13-14}
\]
\[
\begin{aligned}
[\mu(\partial_N u_N + \partial_j u_j)](\xi', x_N) &= -[h_j](\xi', x_N), \\
[2\mu \partial_N \tilde{u}_N - \theta](\xi', x_N) &= -[h_N](\xi', x_N),
\end{aligned}
\tag{3.15-16}
\]
\[
\begin{aligned}
[\tilde{u}_j](\xi', x_N) &= [k_j](\xi', x_N). 
\end{aligned}
\tag{3.17-18}
\]

Set $A = |\xi|$ and $B_\pm = \sqrt{(\rho_\pm / \mu_\pm) \lambda + |\xi'|^2}$. By (3.13)–(3.15), we have $(\partial_N^2 - A^2)\tilde{\theta}_\pm(\xi', x_N) = 0$ for $\pm x_N > 0$, and applying $\partial_N^2 - A^2$ to (3.13) and (3.14) yields that $(\rho_\pm \lambda + \mu_\pm |\xi'|^2 - \mu_\pm \partial_N^2) |\xi'|^2 - \mu_\pm \partial_N^2 |\xi'|^2 \tilde{u}_\pm(\xi', x_N) = 0$ for $\pm x_N > 0$. Thus, we will look for solutions to (3.13)–(3.15) of the forms:
\[
\tilde{u}_\pm(\xi', x_N) = \alpha_{\pm j} e^{\mp A x_N} + \beta_{\pm j} e^{\mp B_{\pm x_N}}, \quad \tilde{\theta}_\pm(\xi', x_N) = \gamma_{\pm} e^{\mp A x_N} (\pm x_N > 0).
\]

Inserting the above formulas into (3.13)–(3.18), we have the following relations:
\[
\begin{aligned}
\mu_\pm (B_\pm^2 - A^2) \alpha_{\pm j} + i\xi_j \gamma_{\pm} &= 0, \\
\mu_\pm (B_\pm^2 - A^2) \beta_{\pm j} + i\xi_j \gamma_{\pm} &= 0, \\
\mu_\pm (B_\pm^2 - A^2) \alpha_{\pm j} \gamma_{\pm} &= 0, \\
\mu_\pm (B_\pm^2 - A^2) \beta_{\pm j} \gamma_{\pm} &= 0.
\end{aligned}
\tag{3.19-20}
\]
(3.20) \( \mu_{\pm}(B_{\pm}^2 - A^2)\alpha_{\pm N} = A\gamma_{\pm} = 0 \)

(3.21) \( i\xi' \cdot \alpha'_{\pm} = A\alpha_{\pm N} = 0, \quad -i\xi' \cdot \alpha'_{\pm} + i\xi' \cdot \beta'_{\pm} \pm B_{\pm} \alpha_{\pm N} \mp B_{\pm} \beta_{\pm N} = 0 \)

(3.22) \( \mu_{+}\{i\xi_{j}\beta_{+ N} + (-A + B_{+})\alpha_{+ j} - B_{+} \beta_{+ j}\} - \mu_{-}\{i\xi_{j}\beta_{- N} + (A - B_{-})\alpha_{- j} - B_{-} \beta_{- j}\} = -[\hat{h}_j](\xi', 0) \)

(3.23) \( 2\mu_{+}\{(A-B_{+})\alpha_{- N} + B_{-} \beta_{- N}\} - 2\mu_{-}\{(A-B_{-})\alpha_{+ N} + B_{+} \beta_{+ N}\} - \gamma_{-} = -[\hat{h}_N](\xi', 0) \)

(3.24) \( \beta_{+ j} - \beta_{- j} = [\hat{k}_j](\xi', 0) \)

where we have set \( \alpha_{\pm} = (\alpha_{\pm 1}, \ldots, \alpha_{\pm N}) \) and \( \beta_{\pm} = (\beta_{\pm 1}, \ldots, \beta_{\pm N}) \).

From now on, we solve the equations (3.19) and (3.24). First, we write \( i\xi' \cdot \alpha'_{\pm}, \alpha_{\pm N} \), and \( \gamma_{\pm} \) by using \( i\xi' \cdot \beta'_{\pm} \) and \( \beta_{\pm N} \). By (3.21), we have

(3.25) \( \alpha_{\pm N} = \pm \frac{-i\xi' \cdot \beta'_{\pm} \pm B_{\pm} \beta_{\pm N}}{B_{\pm} - A} \), \( i\xi' \cdot \alpha'_{\pm} = \frac{A(-i\xi' \cdot \beta'_{\pm} \pm B_{\pm} \beta_{\pm N})}{B_{\pm} - A} \),

which, combined with (3.20), furnishes that

(3.26) \( \gamma_{\pm} = \frac{\mu_{\pm}(B_{\pm} + A)}{A}(-i\xi' \cdot \beta'_{\pm} \pm B_{\pm} \beta_{\pm N}) \).

Next, we give exact formulas of \( \alpha_{\pm j} \) and \( \beta_{\pm j} \). By (3.22) and (3.23),

(3.27) \( \mu_{+}\{(B_{+} - A)A\xi' \cdot \beta'_{+} + A(B_{+} - A)\beta_{+ N}\} - \mu_{-}\{(B_{-} + A)A\xi' \cdot \beta'_{-} + A(B_{-} + A)\beta_{- N}\} = -i\xi' \cdot [\hat{h}'](\xi', 0) \)

In addition, by (3.26), (3.25), (3.22), and (3.23),

(3.28) \( \mu_{+}\{(B_{+} + A)A\xi' \cdot \beta'_{+} - B_{+}(B_{+} + A)\beta_{+ N}\} - \mu_{-}\{(B_{-} - A)A\xi' \cdot \beta'_{-} + B_{-}(B_{-} + A)\beta_{- N}\} = -A[\hat{h}_N](\xi', 0) \)

It holds by (3.24) that

\( i\xi' \cdot \beta'_{-} = i\xi' \cdot \beta'_{+} \cdot [\hat{k}'](\xi', 0) \), \( \beta_{- N} = \beta_{+ N} = -[\hat{k}_N](\xi', 0) \),

which, inserted into (3.27) and (3.28), furnishes that

\( \{\mu_{+}(B_{+} + A) + \mu_{-}(B_{-} + A)\} i\xi' \cdot \beta'_{+} + \{-\mu_{+}A(B_{+} - A) + \mu_{-}A(B_{-} - A)\} \beta_{+ N} = P(h, k) \),

\( \{-\mu_{+}(B_{+} - A) + \mu_{-}(B_{-} - A)\} i\xi' \cdot \beta'_{-} + \{\mu_{+}B_{+}(B_{+} + A) + \mu_{-}B_{-}(B_{-} + A)\} \beta_{- N} = Q(h, k) \),

where

\( P(h, k) = i\xi' \cdot [\hat{h}'](\xi', 0) + \mu_{-}(B_{-} + A)i\xi' \cdot [\hat{k}'](\xi', 0) + \mu_{-}A(B_{-} - A)[\hat{k}_N](\xi', 0) \),

\( Q(h, k) = A\hat{h}_N](\xi', 0) + \mu_{-}(B_{-} - A)i\xi' \cdot [\hat{k}'](\xi', 0) + \mu_{-}B_{-}(B_{+} + A)[\hat{k}_N](\xi', 0) \).

We often denote \( P(h, k) \) and \( Q(h, k) \) by \( P \) and \( Q \) for short in the following. Let

\( L = \begin{pmatrix} \mu_{+}(B_{+} + A) + \mu_{-}(B_{-} + A) & -\mu_{+}A(B_{+} - A) + \mu_{-}A(B_{-} - A) \\ -\mu_{+}(B_{+} - A) + \mu_{-}(B_{-} - A) & \mu_{+}B_{+}(B_{+} + A) + \mu_{-}B_{-}(B_{-} + A) \end{pmatrix} \),

and then

\( det L = -(\mu_{+} - \mu_{-})^2A^2 + \{(3\mu_{+} - \mu_{-})\mu_{+}B_{+} + (3\mu_{-} - \mu_{+})\mu_{-}B_{-}\} A^2 + \{(\mu_{+}B_{+} + \mu_{-}B_{-})^2 + \mu_{+}\mu_{-}(B_{+} + B_{-})^2\} A + (\mu_{+}B_{+} + \mu_{-}B_{-})(\mu_{+}B_{+}^2 + \mu_{-}B_{-}^2) \).

The inverse matrix \( L^{-1} \) of \( L \) is given by

\( L^{-1} = \frac{1}{det L} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \)

with

(3.29) \( L_{11} = \mu_{+}B_{+}(B_{+} + A) + \mu_{-}B_{-}(B_{-} + A), \quad L_{12} = \mu_{+}A(B_{+} - A) - \mu_{-}A(B_{-} - A), \quad L_{21} = \mu_{+}(B_{+} - A) - \mu_{-}(B_{-} - A), \quad L_{22} = \mu_{+}(B_{+} + A) + \mu_{-}(B_{-} + A) \).
Thus we have
\begin{align*}
  i \xi' \cdot \beta_+ &= \frac{1}{\det \mathbf{L}} (L_{11} P + L_{12} Q), \quad \beta_{+N} = \frac{1}{\det \mathbf{L}} (L_{21} P + L_{22} Q), \\
  i \xi' \cdot \beta_- &= \frac{1}{\det \mathbf{L}} (L_{11} P + L_{12} Q) - i \xi' \cdot \hat{\mathbf{k}}' (\xi', 0), \quad \beta_{-N} = \frac{1}{\det \mathbf{L}} (L_{21} P + L_{22} Q) - \hat{\mathbf{k}}_N (\xi', 0).
\end{align*}

These relations yields that
\begin{align*}
  \mathcal{F}_+(\mathbf{h}, \mathbf{k}) &= -i \xi' \cdot \beta_+ + B_+ \beta_{+N} = - \frac{1}{\det \mathbf{L}} \{(L_{11} - B_+ L_{21}) P + (L_{12} - B_+ L_{22}) Q\}, \\
  \mathcal{F}_-(\mathbf{h}, \mathbf{k}) &= -i \xi' \cdot \beta_- - B_- \beta_{-N} \\
  &= - \frac{1}{\det \mathbf{L}} \{(L_{11} + B_- L_{21}) P + (L_{12} + B_- L_{22}) Q\} + i \xi' \cdot \hat{\mathbf{k}}' (\xi', 0) + B_- \hat{\mathbf{k}}_N (\xi', 0),
\end{align*}

which, inserted into (3.26) and (3.27), furnishes that
\begin{align*}
  \alpha_{+N} &= \frac{\mathcal{F}_+(\mathbf{h}, \mathbf{k})}{B_+ - A}, \quad \gamma_+ = \frac{\mu_- (B_+ + A) \mathcal{F}_+(\mathbf{h}, \mathbf{k})}{A},
\end{align*}

By (3.19) and (3.31), we have
\begin{align*}
  \alpha_{+j} &= - \frac{i \xi_j \mathcal{F}_+(\mathbf{h}, \mathbf{k})}{A (B_+ - A)},
\end{align*}

and furthermore, by (3.22) and (3.24),
\begin{align*}
  \mu_+ B_+ \beta_+ + \mu_- B_- \beta_- &= \left[ \hat{h}_{ij} \right] (\xi', 0) + \mu_- i \xi_j \hat{\mathbf{k}}_N (\xi', 0) \\
  + i \xi_j \{ (L_{21} P + L_{22} Q) - \frac{i \xi_j}{A} (\mu_+ \mathcal{F}_+(\mathbf{h}, \mathbf{k}) + \mu_- \mathcal{F}_-(\mathbf{h}, \mathbf{k})) \}
\end{align*}

The last relations imply that
\begin{align*}
  \beta_{+j} &= \frac{1}{\mu_+ B_+ + \mu_- B_-} \left[ \left[ \hat{h}_{ij} \right] (\xi', 0) + \mu_- i \xi_j \hat{\mathbf{k}}_N (\xi', 0) + \frac{i \xi_j}{\det \mathbf{L}} (L_{21} P + L_{22} Q) \right] \\
  - \frac{i \xi_j}{A} \left( \mu_+ \mathcal{F}_+(\mathbf{h}, \mathbf{k}) + \mu_- \mathcal{F}_-(\mathbf{h}, \mathbf{k}) \right) + \mu_+ B_+ \left[ \hat{k}_{ij} \right] (\xi', 0).
\end{align*}

By the symbols (3.30)-(3.32), we can give solution formulas of (3.11) as follows:
\begin{align*}
  u_{+1}(x) &= - \mathcal{F}_+^{-1} [\alpha_{+j} (B_+ - A) \mathcal{M}_+ (\pm \epsilon x)] (x') + \mathcal{F}_+^{-1} \left[ \beta_{+j} e^{\mp B_+ \pm B_-} \right] (x'), \\
  \theta_+ (x) &= \mathcal{F}_+^{-1} \left[ \gamma_{+} e^{\mp B_+ \pm B_-} \right] (x'), \quad \mathcal{M}_+ (a) = e^{-B_+ a} - e^{-A a}.
\end{align*}

**Step 3: Construction of solution operators for (3.31)**. Setting
\begin{align*}
  P'(\mathbf{h}, \mathbf{k}') &= i \xi' \cdot \left[ \hat{\mathbf{k}}' \right] (\xi', 0) + \mu_- (B_- + A) i \xi' \cdot \hat{\mathbf{k}}' (\xi', 0), \\
  Q'(\mathbf{h}, \mathbf{k}') &= A \left[ \hat{h}_{ij} \right] (\xi', 0) + \mu_- (B_- + A) i \xi' \cdot \hat{\mathbf{k}}' (\xi', 0), \\
  P_N (\mathbf{k}) &= \mu_- A (B_- - A) \left[ \hat{\mathbf{k}}_N \right] (\xi', 0), \quad Q_N (\mathbf{k}) = \mu_- B_- (B_- + A) \left[ \hat{\mathbf{k}}_N \right] (\xi', 0), \\
  \mathcal{F}_+ (\mathbf{h}, \mathbf{k}') &= - \frac{1}{\det \mathbf{L}} \{(L_{11} - B_+ L_{21}) P'(\mathbf{h}, \mathbf{k}') + (L_{12} - B_+ L_{22}) Q'(\mathbf{h}, \mathbf{k}') \}, \\
  \mathcal{F}_- (\mathbf{h}, \mathbf{k}') &= - \frac{1}{\det \mathbf{L}} \{(L_{11} + B_- L_{21}) P'(\mathbf{h}, \mathbf{k}') + (L_{12} + B_- L_{22}) Q'(\mathbf{h}, \mathbf{k}') \} + i \xi' \cdot \hat{\mathbf{k}}' (\xi', 0), \\
  \mathcal{F}_+(\mathbf{k}_N) &= - \frac{\mu_-}{\det \mathbf{L}} \left[ \hat{\mathbf{k}}_N \right] (\xi', 0) \{ A (B_- - A) (L_{11} - B_+ L_{21}) + B_- (B_- + A) (L_{12} - B_+ L_{22}) \}, \\
  \mathcal{F}_-(\mathbf{k}_N) &= - \frac{\mu_-}{\det \mathbf{L}} \left[ \hat{\mathbf{k}}_N \right] (\xi', 0) \{ A (B_- - A) (L_{11} + B_- L_{21}) + B_- (B_- + A) (L_{12} + B_- L_{22}) \} + B_- \left[ \hat{\mathbf{k}}_N \right] (\xi', 0),
\end{align*}
we see that
\[
P(h, k) = P'(h, k') + P_N(k_N), \quad Q(h, k) = Q'(h, k') + Q_N(k_N),
\]
\[
F_+(h, k) = F'_+(h, k') + F_+(k_N), \quad F_-(h, k) = F'_-(h, k') + F_-(k_N).
\]
We also define operators \(S_{\pm j}(\lambda)\) and \(T_{\pm j}(\lambda)\) by
\[
S_{\pm j}(\lambda)(h, k') = F_{\pm j}(h, k') - \mathcal{F}_{\pm j}(h, k') A \mathcal{M}_{\pm}(\pm x_N)(\xi') \frac{(\mu_{\pm B_x e^{\mp B_x x_N}})}{(\mu_{\mp B_x + \mp B_x} A)} e^{\mp B_x x_N}(\xi'),
\]
\[
T_{\pm j}(\lambda)k_N = F_{\pm j}(h, k') - \mathcal{F}_{\pm j}(h, k') A \mathcal{M}_{\pm}(\pm x_N)(\xi') \frac{(\mu_{\pm B_x e^{\mp B_x x_N}})}{(\mu_{\mp B_x + \mp B_x} A)} e^{\mp B_x x_N}(\xi'),
\]
Step 4: \(\mathcal{R}\)-boundedness of solution operator families \((3.35)\). We show the \(\mathcal{R}\)-boundedness of the operator families \((3.35)\). To this end, we introduce two classes of multipliers. Let \(0 < \varepsilon < \pi/2\) and \(\gamma_0 \geq 0\), and let \(m(\xi', \lambda)\) be a function defined on \(\mathbb{R}^{N-1} \setminus \{0\} \times \Sigma_{\varepsilon, \gamma_0}\), which is infinitely many times differentiable with respect to \(\xi' \in \mathbb{R}^{N-1} \setminus \{0\}\) and is holomorphic with respect to \(\lambda \in \Sigma_{\varepsilon, \gamma_0}\). Here we have \(\Sigma_{\varepsilon, 0} = \Sigma_{\varepsilon}\).

If there exists a real number \(s\) such that, for any multi-index \(\alpha' = (\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{N}_0^{N-1}\) and \((\xi', \lambda) \in (\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon, \gamma_0}\), there hold the estimates:
\[
|D_{\xi'}^s m(\xi', \lambda)| \leq C_{s, \alpha', \varepsilon, \gamma_0}(|\lambda|^{1/2} + A)^{s-|\alpha'|}, \quad \left|\frac{d}{d\lambda} m(\xi', \lambda)\right| \leq C_{s, \alpha', \varepsilon, \gamma_0}(|\lambda|^{1/2} + A)^{s-|\alpha'|}
\]
with some positive constant \(C_{s, \alpha', \varepsilon, \gamma_0}\), then \(m(\xi', \lambda)\) is called a multiplier of order \(s\) with type 1. If there exists a real number \(s\) such that, for any multi-index \(\alpha' = (\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{N}_0^{N-1}\) and \((\xi', \lambda) \in (\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon, \gamma_0}\), there holds the estimates:
\[
|D_{\xi'}^s m(\xi', \lambda)| \leq C_{s, \alpha', \varepsilon, \gamma_0}(|\lambda|^{1/2} + A)^{s-|\alpha'|}, \quad \left|\frac{d}{d\lambda} m(\xi', \lambda)\right| \leq C_{s, \alpha', \varepsilon, \gamma_0}(|\lambda|^{1/2} + A)^{s-|\alpha'|}
\]
with some positive constant \(C_{s, \alpha', \varepsilon, \gamma_0}\), then \(m(\xi', \lambda)\) is called a multiplier of order \(s\) with type 2. In what follows, we denote the set of all multipliers defined on \((\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon, \gamma_0}\) of order \(s\) with type \(l\) \((l = 1, 2)\) by \(M_{s, l, \varepsilon, \gamma_0}\). We here give typical examples of multiplies as follows: the Riesz kernel \(\xi_j/|\xi'|\) \((j = 1, \ldots, N - 1)\)
is a multiplier of order 0 with type 2. Functions $\xi$ and $\lambda^{1/2}$ are multipliers of order 1 with type 1. We also introduce the following two fundamental lemmas (cf. [28 Lemma 4.6, Lemma 4.8]).

**Lemma 3.4.** Let $s_1, s_2 \in \mathbb{R}$, $0 < \varepsilon < \pi/2$, and $\gamma_0 \geq 0$.

1. Given $m_i \in M_{s_1, 1, \varepsilon, \gamma_0}$ ($i = 1, 2$), we have $m_1 m_2 \in M_{s_1 + s_2, 1, \varepsilon, \gamma_0}$.
2. Given $l_i \in M_{s_1, \varepsilon, \gamma_0}$ ($i = 1, 2$), we have $l_1 l_2 \in M_{s_1 + s_2, \varepsilon, \gamma_0}$.
3. Given $n_i \in M_{s_1, 2, \varepsilon, \gamma_0}$ ($i = 1, 2$), we have $n_1 n_2 \in M_{s_1 + s_2, 2, \varepsilon, \gamma_0}$.

**Lemma 3.5.** Let $s \in \mathbb{R}$ and $0 < \varepsilon < \pi/2$. Then the following assertions hold:

1. $B^s_+ \in M_{s_1, \varepsilon, 0}$, $(A + B^s_+)^s \in M_{s_2, \varepsilon, 0}$, and $(\det L)^s \in M_{s_3, \varepsilon, 0}$.
2. $A^s \in M_{s_2, \varepsilon, 0}$, provided that $s \geq 0$.
3. For real numbers $a, b$ satisfying $a + b > 0$, we have $(aB_+ + bB_-)^s \in M_{s_4, \varepsilon, 0}$.
4. $L_{11}, L_{12}, L_{21},$ and $L_{22}$ defined as in [28] satisfy $L_{11}, L_{12} \in M_{2, 2, \varepsilon, 0}$ and $L_{21}, L_{22} \in M_{1, 2, \varepsilon, 0}$.

We start with the following lemma to show the $\mathcal{R}$-boundedness of the operators $S_{\pm j}(\lambda)$, $T_{\pm j}(\lambda)$.

**Lemma 3.6.** Let $0 < \varepsilon < \pi/2$, $\gamma_0 \geq 0$, and $1 < q < \infty$. Given multipliers

$$m_1 \in M_{-1, 1, \varepsilon, \gamma_0}, \quad m_2 \in M_{-2, 1, \varepsilon, \gamma_0}, \quad m_3 \in M_{-1, 2, \varepsilon, \gamma_0}, \quad m_4 \in M_{0, 1, \varepsilon, \gamma_0}, \quad m_5 \in M_{0, 2, \varepsilon, \gamma_0},$$

we define operators $K_{\pm i}(\lambda)$ on $W^1_q(\mathbb{R}^N)$ and $L_{\pm i}(\lambda)$ on $W^2_q(\mathbb{R}^N)$ ($i = 1, 2, 3$) by the formulas:

$$[K_{\pm 1}(\lambda)f](x) = F_{\xi}^{-1} \left[ m_1(\xi', \lambda)e^{\pm B_{\pm x} N}[f](\xi', 0) \right](x'),$$

$$[K_{\pm 2}(\lambda)f](x) = F_{\xi}^{-1} \left[ m_2(\xi', \lambda)Ae^{\pm B_{\pm x} N}[f](\xi', 0) \right](x'),$$

$$[K_{\pm 3}(\lambda)f](x) = F_{\xi}^{-1} \left[ m_3(\xi', \lambda)AM_{\pm}(\pm x_N)[f](\xi', 0) \right](x'),$$

$$[L_{\pm 1}(\lambda)g](x) = F_{\xi}^{-1} \left[ m_4(\xi', \lambda)e^{\pm B_{\pm x} N}[g](\xi', 0) \right](x'),$$

$$[L_{\pm 2}(\lambda)g](x) = F_{\xi}^{-1} \left[ m_5(\xi', \lambda)Ae^{\pm B_{\pm x} N}[g](\xi', 0) \right](x'),$$

$$[L_{\pm 3}(\lambda)g](x) = F_{\xi}^{-1} \left[ m_6(\xi', \lambda)AM_{\pm}(\pm x_N)[g](\xi', 0) \right](x')$$

for $\pm x_N > 0$ and $\lambda \in \Sigma_{\varepsilon, \gamma_0}$. Then there exist operator families $\tilde{K}_{\pm i}(\lambda)$, $\tilde{L}_{\pm i}(\lambda)$ with

$$\tilde{K}_{\pm i}(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \gamma_0}, \mathcal{L}(L_q(\mathbb{R}^N)^{N+1}, W^2_q(\mathbb{R}^N))), \quad \tilde{L}_{\pm i}(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \gamma_0}, \mathcal{L}(L_q(\mathbb{R}^N)^{N^2+N+1}, W^2_q(\mathbb{R}^N))),$$

such that $K_{\pm i}(\lambda)f = \tilde{K}_{\pm i}(\lambda)(\nabla^2 f, \lambda^{1/2} f)$, $L_{\pm i}(\lambda)g = \tilde{L}_{\pm i}(\lambda)(\nabla^2 g, \lambda^{1/2} \nabla g, \lambda g)$, and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^{N+1}, L_q(\mathbb{R}^N)^{N^2+N+1})}(\left\{ \left( \frac{d}{d\lambda} \right)^l (R_\lambda \tilde{K}_{\pm i}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0} \right\}) \leq \gamma_1,$n

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^{N^2+N+1}, L_q(\mathbb{R}^N)^{N^2+N+1})}(\left\{ \left( \frac{d}{d\lambda} \right)^l (R_\lambda \tilde{L}_{\pm i}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0} \right\}) \leq \gamma_1$$

for $l = 0, 1$ and $i = 1, 2, 3$.

**Proof.** It was essentially proved in [28] Lemma 5.1, 5.2, 5.3, 5.4].

**Lemma 3.6** enables us to obtain the following lemma.

**Lemma 3.7.** Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Given a multiplier $m_0 \in M_{0, 2, \varepsilon, 0}$, we define operators $K_{\pm i}(\lambda)$ on $W^1_q(\mathbb{R}^N) \times W^2_q(\mathbb{R}^N)^{N-1}$ ($i = 1, 2, 3$) by the formulas:

$$[K_{\pm 1}(\lambda)(h, k')](x) = F_{\xi}^{-1} \left[ m_0(\xi', \lambda) \frac{L_{21}P'(h, k') + L_{22}Q'(h, k')}{A \det L} Ae^{\pm B_{\pm x} N} \right](x'),$$

$$[K_{\pm 2}(\lambda)(h, k')](x) = F_{\xi}^{-1} \left[ m_0(\xi', \lambda) \frac{F_{\pm}(h, k')}{A} AM_{\pm}(\pm x_N) \right](x'),$$

$$[K_{\pm 3}(\lambda)(h, k')](x) = F_{\xi}^{-1} \left[ m_0(\xi', \lambda) \frac{F_{\pm}(h, k')}{(\mu_+ B_+ + \mu_- B_-)A} Ae^{\pm B_{\pm x} N} \right](x')$$
for \( \pm x_N > 0 \) and \( \lambda \in \Sigma_\varepsilon \). Then there exist operator families \( \widetilde{C}_{\pm 1}(\lambda) \) in \( \text{Hol}(\Sigma_\varepsilon, \mathcal{L}(L_q(\mathbb{R}^N)^N, W^2_q(\mathbb{R}^N_+))) \) such that \( \mathcal{K}_{\pm 1}(\lambda)(h, k') = \widetilde{C}_{\pm 1}(\lambda)(\nabla h, \lambda^{1/2} h, \nabla^2 k', \lambda^{1/2} \nabla k', \lambda k') \) and

\[
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N_+)^{N^2 + n + 1})} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda \widetilde{C}_{\pm 1}(\lambda) \right) \mid \lambda \in \Sigma_\varepsilon \right\} \right) \leq \gamma_1,
\]

for \( l = 0, 1 \) and \( i = 1, 2, 3 \), where \( N = N^2 + N + N^2(N - 1) + N(N - 1) + (N - 1) \).

**Proof.** We only show the case \( \mathcal{K}_{\pm 1}(\lambda) \). Note that

\[
\begin{align*}
\mathcal{K}_{\pm 1}(\lambda)(h, k')(x) &= F_{\xi'}^{-1} \left[ m_0(\xi', \lambda) \frac{L_{21}}{\det L} A e^{\pm B_{\pm} x_N} i \frac{\xi'}{A} \cdot [\widetilde{h}'](\xi', 0) \right](x') \\
&+ F_{\xi'}^{-1} \left[ m_0(\xi', \lambda) \frac{L_{22}}{\det L} A e^{\pm B_{\pm} x_N} [\widetilde{h}')(\xi', 0) \right](x') \\
&+ \mu F_{\xi'}^{-1} \left[ \frac{m_0(\xi', \lambda) (B_+ + A)L_{21} + (B_- - A)L_{22}}{\det L} A e^{\pm B_{\pm} x_N} i \frac{\xi'}{A} \cdot [\widetilde{k}'](\xi', 0) \right](x') \\
&= [\mathcal{K}_{\pm 1}^1(\lambda) h'](x) + [\mathcal{K}_{\pm 1}^2(\lambda) \lambda_N h](x) + [\mathcal{K}_{\pm 1}^3(\lambda) k'](x).
\end{align*}
\]

By Lemma 3.4 and Lemma 3.5,

\[
\begin{align*}
m_0 \frac{L_{21}}{\det L}, m_0 &\left( \frac{L_{22}}{\det L} \right), m_0 \left( \frac{B_+ + A}{\det L} \right) + (B_- - A) \frac{L_{22}}{\det L} &\in \mathbb{M}_{-2, 2, \varepsilon, 0},
\end{align*}
\]

which, combined with Lemma 3.6, furnishes that there exist operator families \( \widetilde{C}_{\pm 1}(\lambda) \) (i = 1, 2, 3) with

\[
\begin{align*}
\mathcal{K}_{\pm 1}^1(\lambda) &= \widetilde{C}_{\pm 1}^1(\lambda)(\nabla h, \lambda^{1/2} h'), \\
\mathcal{K}_{\pm 1}^2(\lambda) &= \widetilde{C}_{\pm 1}^2(\lambda)(\nabla h, \lambda^{1/2} h_N), \\
\mathcal{K}_{\pm 1}^3(\lambda) &= \widetilde{C}_{\pm 1}^3(\lambda)(\nabla^2 k', \lambda^{1/2} \nabla k', \lambda k'),
\end{align*}
\]

such that

\[
\begin{align*}
\mathcal{K}_{\pm 1}(\lambda) h' &= \widetilde{C}_{\pm 1}(\lambda)(\nabla h', \lambda^{1/2} h'), \\
\mathcal{K}_{\pm 1}(\lambda) h_N &= \widetilde{C}_{\pm 1}(\lambda)(\nabla h_N, \lambda^{1/2} h_N), \\
\mathcal{K}_{\pm 1}(\lambda) k' &= \widetilde{C}_{\pm 1}(\lambda)(\nabla^2 k', \lambda^{1/2} \nabla k', \lambda k').
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N_+)^{N^2 + n + 1})} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda \mathcal{K}_{\pm 1}(\lambda) \right) \mid \lambda \in \Sigma_\varepsilon \right\} \right) \leq \gamma_1,
\end{align*}
\]

for \( l = 0, 1 \). Setting

\[
\begin{align*}
\mathcal{K}_{\pm 1}(\lambda)(\nabla h, \lambda^{1/2} h, \nabla^2 k', \lambda^{1/2} \nabla k', \lambda k') &= \widetilde{C}_{\pm 1}(\lambda)(\nabla h', \lambda^{1/2} h') + \mathcal{K}_{\pm 1}^2(\lambda)(\nabla h_N, \lambda^{1/2} h_N) + \mathcal{K}_{\pm 1}(\lambda)(\nabla^2 k', \lambda^{1/2} \nabla k', \lambda k')
\end{align*}
\]

implies, by Proposition 2.3, that we have obtained the required operator \( \mathcal{K}_{\pm 1}(\lambda) \) of Lemma 3.7. \( \square \)

To treat \( T_{\pm, l}(\lambda) \), we use the following lemma.

**Lemma 3.8.** Let \( 0 < \varepsilon < \pi/2 \) and \( 1 < q < \infty \). Suppose that \( k_N \) is given by \( k_N = -\psi_N \), where \( \psi_N \) is the \( N \)th component of \( \psi = A(\lambda) \tilde{f} \) (\( \lambda \in \Sigma_\varepsilon \)) defined as (3.9). Given a multiplier \( m_0 \in \mathbb{M}_{0, 2, \varepsilon, 0} \), we define operators
\( \mathcal{K}_{\pm i}(\lambda) \) (\( i = 4, 5, 6 \)) by the formulas:

\[
[\mathcal{K}_{\pm 4}(\lambda)k_N](x) = F^{-1}_{\xi'} \left[ \frac{m_0(\xi', \lambda) L_2 P_{N}(k_N) + L_2 Q_{N}(k_N) A e^{\pm B_{\pm} x N}}{\det L} \right] (x'),
\]

\[
[\mathcal{K}_{\pm 5}(\lambda)k_N](x) = F^{-1}_{\xi'} \left[ \frac{m_0(\xi', \lambda) F_{\pm N}(k_N)}{A} A M_{\pm}(\pm x N) \right] (x'),
\]

\[
[\mathcal{K}_{\pm 6}(\lambda)k_N](x) = F^{-1}_{\xi'} \left[ \frac{m_0(\xi', \lambda) F_{\pm N}(k_N)}{(\mu_+ B_+ + \mu_- B_-) A} A e^{\pm B_{\pm} x N} \right] (x')
\]

for \( \pm x_N > 0 \) and \( \lambda \in \Sigma_\varepsilon \). Then there exist operator families \( \tilde{\mathcal{K}}_{\pm i}(\lambda) \in \text{Hol}(\Sigma_\varepsilon, \mathcal{L}(L_q(\mathbb{R}^N)^N, W^2_q(\mathbb{R}^N))) \) such that \( \mathcal{K}_{\pm i}(\lambda)k_N = \tilde{\mathcal{K}}_{\pm i}(\lambda)\tilde{f} \) and

\[
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N)^{N^2+N+1})} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_N \tilde{\mathcal{K}}_{\pm i}(\lambda) \right) \right| \lambda \in \Sigma_\varepsilon \right) \leq \gamma_1
\]

for \( l = 0, 1 \) and \( i = 4, 5, 6 \).

\textbf{Proof.} We only consider the case \( \mathcal{K}_{\pm 4}(\lambda) \). First, we give some special formula of \( \tilde{k}_N = - (\tilde{\psi}_+(\xi, 0) - \tilde{\psi}_-(\xi, 0)) \). Let \( \tilde{f} = (f_1, \ldots, f_N)^T \). Since

\[
\tilde{\psi}_{\pm N}(x) = F^{-1}_{\xi} \left[ \frac{A^2}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} \right] \left( x' - \sum_{j=1}^{N-1} \frac{\xi_j \xi_j}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} \right) (x),
\]

it holds that

\[
\tilde{\psi}_{\pm N}(\xi', x_N) = \int_{-\infty}^{\infty} A^2 \tilde{f}_N(\xi', y_N) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x_N - y_N)\xi_N} \frac{1}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} d\xi_N \right) dy_N
\]

\[
- \sum_{j=1}^{N-1} \int_{-\infty}^{\infty} \xi_j \tilde{f}_j(\xi', y_N) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi_N e^{i(x_N - y_N)\xi_N}}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} d\xi_N \right) dy_N,
\]

On the other hand, we have, by the residue theorem,

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} d\xi_N = \frac{1}{2\rho_+ \lambda} \left( \frac{e^{-|a| A}}{A} - \frac{e^{-|a| B_\pm}}{B_\pm} \right),
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi_N e^{ia\xi_N}}{|\xi|^2(\rho_+ \lambda + \mu_\pm |\xi|^2)^2} d\xi_N = \text{sign}(a) \frac{i}{2\rho_+ \lambda} \left( e^{-|a| A} - e^{-|a| B_\pm} \right)
\]

for \( a \in \mathbb{R} \), where \( \text{sign}(a) = \pm 1 \) when \( \pm a > 0 \) and \( \text{sign}(a) = 0 \) when \( a = 0 \). Inserting these formulas into the above identity of \( \tilde{\psi}_{\pm N}(\xi', x_N) \) with \( x_N = 0 \) yields that

\[
\tilde{\psi}_{\pm N}(\xi', 0) = \int_{-\infty}^{\infty} \frac{A^2}{2\rho_+ \lambda} \left( \frac{e^{-|A|y_N}}{A} - \frac{e^{-B_\pm|y_N|}}{B_\pm} \right) \tilde{f}_N(\xi', y_N) dy_N
\]

\[
- \sum_{j=1}^{N-1} \int_{-\infty}^{\infty} \frac{i \xi_j \text{sign}(y_N)}{2\rho_+ \lambda} \left( e^{-|A|y_N} - e^{-B_\pm|y_N|} \right) \tilde{f}_j(\xi', y_N) dy_N.
\]

By \( \rho_+ \lambda = \mu_\pm (B_\pm^2 - A^2) \), we have

\[
\tilde{\psi}_{\pm N}(\xi', 0) = \frac{1}{2\mu_\pm} \int_{-\infty}^{\infty} \frac{A}{B_\pm(\pm x N) + A} e^{-B_\pm|y_N|} \tilde{f}_N(\xi', y_N) dy_N
\]

\[
- \frac{1}{2\mu_\pm} \int_{-\infty}^{\infty} \frac{A}{B_\pm + A} M_{\pm}(|y_N|) \tilde{f}_N(\xi', y_N) dy_N
\]

\[
- \frac{1}{2\mu_\pm} \sum_{j=1}^{N-1} \int_{-\infty}^{\infty} \frac{i \xi_j \text{sign}(y_N) M_{\pm}(|y_N|)}{B_\pm + A} \tilde{f}_j(\xi', y_N) dy_N.
\]
Thus, by Lemma 3.4 and Lemma 3.5 there exist \( m_{\pm} \in (M_{-2,2,\varepsilon,0})^N \) and \( n_{\pm} \in (M_{-1,2,\varepsilon,0})^N \) such that

\[
\| \tilde{k}_N \| (\xi, 0) = \sum_{s \in \{+, -\}} \int_{-\infty}^{\infty} A e^{-B_s|y_N|} m_s(\xi', \lambda) \cdot \tilde{f}(\xi', y_N) \, dy_N + \int_{-\infty}^{\infty} A \mathcal{M}_s(|y_N|) n_s(\xi', \lambda) \cdot \tilde{f}(\xi', y_N) \, dy_N,
\]

which, combined with the formula of \( \mathcal{K}_{\pm}(\lambda) \), furnishes that

\[
|\mathcal{K}_{\pm}(\lambda)k_N(x) = \sum_{s \in \{+, -\}} \int_{-\infty}^{\infty} F_{\xi'}^{-1} \left[ \frac{A e^{T_{\pm}B_{\pm}x_N} e^{-B_s|y_N|} l(\xi', \lambda) m_0(\xi', \lambda) m_s(\xi', \lambda) \tilde{f}(\xi', y_N)}{\det L} \right] (x') + \sum_{s \in \{+, -\}} \int_{-\infty}^{\infty} F_{\xi'}^{-1} \left[ \frac{A e^{T_{\pm}B_{\pm}x_N} \mathcal{M}_s(|y_N|) l(\xi', \lambda) m_0(\xi', \lambda) n_s(\xi', \lambda) \tilde{f}(\xi', y_N)}{\det L} \right] (x') =: |\tilde{K}_{\pm}(\lambda)\tilde{f}(x)|
\]

with \( l(\xi', \lambda) = \mu_s \{ L_{21}A(B_s - A) + L_{22}B_s(B_0 + A) \} \). By Lemma 3.4 and Lemma 3.5

\[
\frac{l(\xi', \lambda) m_0(\xi', \lambda) m_{\pm}(\xi', \lambda)}{\det L} \in (M_{-2,2,\varepsilon,0})^N,
\]

\[
\frac{l(\xi', \lambda) m_0(\xi', \lambda) n_{\pm}(\xi', \lambda)}{\det L} \in (M_{-1,2,\varepsilon,0})^N,
\]

which, combined with [29 Lemma 5.6] (cf. also [23 Lemma B.2]), shows that \( \tilde{K}_{\pm}(\lambda) \) is the required operator in Lemma 3.8. This completes the proof of the lemma. \( \square \)

We apply Lemma 3.14, Lemma 3.7, and Lemma 3.5 to 3.35, together with Proposition 2.4, Lemma 3.4, and Lemma 3.5, to see that there exist operator families \( \tilde{S}_{\pm}(\lambda), \tilde{T}_{\pm}(\lambda) \) with

\[
\tilde{S}_{\pm}(\lambda) \in \text{Hol} (\Sigma, \mathcal{L}(L_q(\hat{R}^N)^N, W_q^2(\hat{R}^N))) \quad \text{and} \quad \tilde{T}_{\pm}(\lambda) \in \text{Hol} (\Sigma, \mathcal{L}(L_q(\hat{R}^N)^N, W_q^2(\hat{R}^N)))
\]

such that

\[
\tilde{S}(\lambda)(\mathbf{h}, \mathbf{k'}) = \tilde{S}_{\pm}(\lambda)(\nabla \mathbf{h}, \lambda^{1/2} \mathbf{h}, \nabla^2 \mathbf{k'}, \nabla \mathbf{k'}, \lambda \mathbf{k'}), \quad \tilde{T}_{\pm}(\lambda)k_N = \tilde{T}_{\pm}(\lambda)\tilde{f}, \quad \text{and}
\]

\[
\mathcal{R}_{\tilde{L}(\hat{R}^N_\pm)}(L_q(\hat{R}^N)^N) \left( \left\{ \left( \frac{\lambda}{\mathbf{h}} \right)^l \left( \mathbf{R}_{\lambda} \tilde{S}_{\pm} \right)(\lambda) \right\} \mid \lambda \in \Sigma \right) \leq \gamma_1,
\]

\[
\mathcal{R}_{\tilde{L}(\hat{R}^N_\pm)}(L_q(\hat{R}^N)^N) \left( \left\{ \left( \frac{\lambda}{\mathbf{h}} \right)^l \left( \mathbf{R}_{\lambda} \tilde{T}_{\pm} \right)(\lambda) \right\} \mid \lambda \in \Sigma \right) \leq \gamma_1 \quad (l = 0, 1),
\]

where \( \Sigma \) is the same number as in Lemma 3.7. Thanks to these properties and Proposition 2.4, setting

\[
\tilde{S}(\lambda)(\mathbf{h}, \mathbf{k'}) = (S_{\pm}(\lambda)(\mathbf{f}, \mathbf{k'}), \ldots, S_{\pm}(\lambda)(\mathbf{h}, \mathbf{k'}))^T, \quad \tilde{T}_{\pm}(\lambda)k_N = (T_{\pm}(\lambda)k_N, \ldots, T_{\pm}(\lambda)k_N)^T,
\]

we can construct an operator family \( \mathcal{B}(\lambda) \in \text{Hol} (\Sigma, \mathcal{L}(L_q(\hat{R}^N)^N)^N, W_q^2(\hat{R}^N)^N) \) such that

\[
\mathcal{B}(\lambda)(\nabla \mathbf{h}, \lambda^{1/2} \mathbf{h}, \nabla^2 \mathbf{k'}, \nabla \mathbf{k'}, \lambda \mathbf{k'} \tilde{f}) = S(\lambda)(\mathbf{h}, \mathbf{k'}) + \mathcal{B}(\lambda)k_N,
\]

which solves the problem 3.11, and

\[
\mathcal{R}_{\tilde{L}(\hat{R}^N_\pm)}(\hat{R}^N)^N \left( \left\{ \left( \frac{\lambda}{\mathbf{h}} \right)^l \left( \mathbf{R}_{\lambda} \mathcal{B}(\lambda) \right) \right\} \mid \lambda \in \Sigma \right) \leq \gamma_1 \quad (l = 0, 1).
\]

Thus, we define an operator family \( \mathcal{S}(\lambda) \) as

\[
\mathcal{S}(\lambda)G_{\mathcal{R}_{\lambda} \mathcal{B}(\lambda)}(\mathbf{f}, \mathbf{h}) = \mathcal{A}(\lambda)\tilde{f} + \mathcal{B}(\lambda)(\nabla \mathbf{h}, \lambda^{1/2} \mathbf{h}, \nabla^2 \mathbf{k'}, \nabla \mathbf{k'}, \lambda \mathbf{k'} \tilde{f})
\]

with \( \mathbf{h} = \bar{\mathbf{h}} - \mu \mathbf{D}(\mathcal{A}(\lambda)\tilde{f})n_0 \) and \( \mathbf{k} = -\mathcal{A}(\lambda)\tilde{f} \), which, combined with [31,10] and Proposition 2.4 shows that \( \mathcal{S}(\lambda) \) is the required operator in Theorem 3.8. This completes the proof of Theorem 3.8. \( \square \)

Since \( R_N V(g) = (V^1(\nabla g), V^2(\lambda^{1/2}g), V^3(\lambda g)) \) as follows from Lemma 3.2, we have the following theorem by combining Theorem 3.8 with Lemma 3.2 and by setting

\[
Y_q = \left\{ (f, g, \mathbf{h}) \mid f \in L_q(\hat{R}^N)^N, g \in W_q^1(\hat{R}^N) \cap W_q^{-1}(\hat{R}^N), \mathbf{h} \in W_q^1(\hat{R}^N)^N \right\},
\]

\[
Y_q = \left\{ (F_1, \ldots, F_6) \mid F_1, F_2, F_6 \in L_q(\hat{R}^N)^N, F_2 \in L_q(\hat{R}^N)^N, F_3, F_5 \in L_q(\hat{R}^N)^N \right\},
\]

\[
G(\lambda, f, g, h) = (f, V^1(\nabla g), V^2(\lambda^{1/2}g), V^3(\lambda g), \nabla h, \lambda^{1/2}h) = (f, R_N V(g), \nabla h, \lambda^{1/2}h).
\]
**Theorem 3.9.** Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and $\rho_{\pm}$ be positive constants, and let $V$ be the same operator as in Lemma 3.2. Suppose that the condition (d) holds. Then there exists an operator family $T_\lambda(\lambda) \in H(\Sigma_c, L(\mathcal{X}_q, W^2_q(\mathbb{R}^N)^N))$ such that $u = V(\lambda) + T_\lambda(\lambda)G_\lambda(f, g, h)$ is a unique solution to the problem (3.4) with some pressure $\theta$ for $\lambda \in \Sigma_c$ and $(f, g, h) \in \mathcal{Y}_q$. In addition,

$$R_{\mathcal{Y}_q, L_q(\mathbb{R}^N)^N} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_2 T_\lambda(\lambda) \right) \mid \lambda \in \Sigma_c \right\} \right) \leq \gamma_1 \quad (l = 0, 1).$$

**Proof of Theorem 3.9.** Let $1 < q < \infty$ and $d' = q/(q - 1)$. According to what was pointed out in Subsection 2.1 we consider, as an auxiliary problem, the following weak problem:

$$(3.36) \quad \lambda(g, \varphi)_{R^N} + (\nabla g, \nabla \varphi)_{R^N} = -(f, \nabla \varphi)_{R^N} \quad \text{for all } \varphi \in W^1_q(\mathbb{R}^N), \quad [g] = [\tilde{h}], \quad n_0 > \text{ on } R^N_0.$$ Concerning this weak problem, we show the following proposition.

**Proposition 3.10.** Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Suppose that $V$ is the same operator as in Lemma 3.2. Then, for any $\lambda \in \Sigma_c$ and $(f, h) \in Y_{R,q}(\mathbb{R}^N)$, the problem (3.36) admits a unique solution $g \in W^1_q(\mathbb{R}^N) \cap W^{-1}_q(\mathbb{R}^N)$. In addition, there exists an operator family $V(\lambda) \in H(\Sigma_c, L(\mathcal{X}_q, W^2_q(\mathbb{R}^N)^N))$ such that

$$(3.37) \quad R_{\mathcal{Y}_q, L_q(\mathbb{R}^N)^N} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_2 V(\lambda) \right) \mid \lambda \in \Sigma_c \right\} \right) \leq \gamma_1 \quad (l = 0, 1)$$

and $V(\lambda)(f, \nabla h, \lambda^{1/2} h)$ for any $(f, h) \in Y_{R,q}(\mathbb{R}^N)$, where $g$ is the solution to (3.36).

**Proof.** We only show the existence of the $R$-bounded solution operator family $V(\lambda)$, since the unique solvability of the weak problem (3.36) was already mentioned in Proposition 2.1.

It suffices to consider the case $f \in C^0_{\infty}(\mathbb{R}^N)^N$ in what follows, since $C^0_{\infty}(\mathbb{R}^N)$ is dense in $L_q(\mathbb{R}^N)$. Then the $g$ satisfying (3.36) is given by $g = \varphi + \psi$ with

$$(\lambda - \Delta)\varphi = \text{div} f \quad \text{in } \mathbb{R}^N,$$

$$(\lambda - \Delta)\psi = 0 \quad \text{in } \mathbb{R}^N,$$

$$[\psi] = [\tilde{h}], \quad [\frac{\partial \psi}{\partial n_0}] = 0 \quad \text{on } R^N_0,$$

where $h = [h, n_0] >$ and $\partial \psi/\partial n_0 = n_0 \cdot \nabla \psi = -\partial \psi$. **Step 1: Solution formulas.** We give the exact solution formulas of $\varphi, \psi$. The $\varphi$ is given by

$$(3.38) \quad \varphi = F^{-1} \left[ \frac{\mathcal{F}[\text{div } f(\xi)]}{\lambda + |\xi|^2} \right]^N(x) = F^{-1} \left[ \frac{<i \xi, \mathcal{F}[f(\xi)]>}{\lambda + |\xi|^2} \right]^N(x).$$

On the other hand, we rewrite the system for $\psi$ as follows:

$$(3.39) \quad \left\{ \begin{array}{l}
(\lambda - \Delta)\psi_\pm = 0 \quad \text{in } \mathbb{R}^N_\pm, \\
\psi_+ - \psi_- = [h] \quad \text{on } R^N_0, \\
\partial_N \psi_+ - \partial_N \psi_- = 0 \quad \text{on } R^N_0,
\end{array} \right.$$
Step 2: Construction of $\mathcal{R}$-bounded solution operator families. Since $V(\varphi + \psi) = V(\varphi) + V(\psi)$, we consider $V(\varphi)$, $V(\psi)$ one by one. First we construct a $\mathcal{R}$-bounded solution operator family for $V(\varphi)$. By (3.39) and (3.38),

$$V(\varphi) = \mathcal{F}^{-1} \left[ \frac{\xi < \xi, \mathcal{F}[f](\xi)}{\|\xi\|^2(\lambda + \|\xi\|^2)} \right] (x) =: V^1(\lambda)f.$$

As was discussed in (3.8), we already know that

$$V^1(\lambda) \in \text{Hol}(\Sigma_{\varepsilon}, \mathcal{L}(L_q^2(\mathbb{R}^N)), \mathcal{R}_{\mathcal{L}(L_q^2(\mathbb{R}^N)^N)}),$$

$$\mathcal{R}_{\mathcal{L}(L_q^2(\mathbb{R}^N)^N)} \left( \left( \frac{d}{d \lambda} \right)^l \mathcal{A} V^1(\lambda) \left| \lambda \in \Sigma_{\varepsilon} \right) \right) \leq \gamma_1 \ (l = 0, 1).$$

Next, we consider the term $V(\psi)$. By (3.39), we have, for $j = 1, \ldots, N - 1$,

$$(3.41) \quad V_j(\psi)(\xi', x_N) = -\int_{-\infty}^{\infty} \hat{\psi}(\xi', y_N) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(x_N - y_N)\xi_N}}{\|\xi\|^2} d\xi_N \right) dy_N,$$

$$V_N(\psi)(\xi', x_N) = -\int_{-\infty}^{\infty} \hat{\psi}(\xi', y_N) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i\xi_N e^{i(x_N - y_N)\xi_N}}{\|\xi\|^2} d\xi_N \right) dy_N.$$ 

Since it holds, by the residue theorem, that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi_N}}{\|\xi\|^2} d\xi_N = 2A, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i\xi_N e^{i\alpha\xi_N}}{\|\xi\|^2} d\xi_N = -\text{sign}(a) \frac{e^{-|a|A}}{2} \ (a \in \mathbb{R} \setminus \{0\}),$$

we insert these formulas into (3.41) in order to obtain

$$V_j(\psi)(\xi', x_N) = -\frac{i\xi_j}{2A} \int_{-\infty}^{\infty} e^{-|x_N - y_N| A} \hat{\psi}(\xi', y_N) dy_N,$$

$$V_N(\psi)(\xi', x_N) = \frac{1}{2} \int_{-\infty}^{\infty} \text{sign}(x_N - y_N) e^{-|x_N - y_N| A} \hat{\psi}(\xi', y_N) dy_N.$$ 

This combined with (3.30) furnishes that

$$V_j(\psi) = -\mathcal{F}^{-1} \left[ \frac{i\xi_j}{4A} \hat{h}(\xi', 0) \int_{0}^{\infty} \left( e^{-|x_N - y_N| A} - e^{-|x_N + y_N| A} \right) e^{-By_N} dy_N \right] (x'),$$

$$V_N(\psi) = \mathcal{F}^{-1} \left[ \frac{\hat{h}(\xi', 0)}{4} \int_{0}^{\infty} \left( \text{sign}(x_N - y_N) e^{-|x_N - y_N| A} - \text{sign}(x_N + y_N) e^{-|x_N + y_N| A} \right) e^{-By_N} dy_N \right] (x').$$

By direct calculations, we have the following lemma.

Lemma 3.11. Let $0 < \varepsilon < \pi/2$ and $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$. We set

$$A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2}, \quad M(a) = \frac{e^{-Ba} - e^{-Aa}}{B - A} \ (\lambda \in \Sigma_{\varepsilon}, a > 0).$$

Then it holds that, for $\pm x_N > 0$,

$$\int_{0}^{\infty} \left( e^{-|x_N - y_N| A} - e^{-|x_N + y_N| A} \right) e^{-By_N} dy_N = \pm \frac{2A}{B + A} M(\pm x_N),$$

$$\int_{0}^{\infty} \left( \text{sign}(x_N - y_N) e^{-|x_N - y_N| A} - \text{sign}(x_N + y_N) e^{-|x_N + y_N| A} \right) e^{-By_N} dy_N$$

$$= -\frac{2A}{B + A} M(\pm x_N) - \frac{2}{B + A} e^{\mp Bx_N}. $$
This lemma yields that, for $\pm x_N > 0$ and $j = 1, \ldots, N - 1$,
\[
[V_J(\psi)](x', x_N) = \pm F_{\xi_j}^{-1} \left[ \left( \frac{i\xi_j}{2A(B + A)} \right) \left( \frac{e^{\mp B x_N}}{B + A} \right) \mu(\pm x_N) \left[ \hat{h} \right](\xi', 0) \right](x'),
\]
\[
[V_N(\psi)](x', x_N) = -\frac{1}{2} F_{\xi}^{-1} \left[ \frac{A}{B + A} \right] \mu(\pm x_N) \left[ \hat{h} \right](\xi', 0) (x') - \frac{1}{2} F_{\xi}^{-1} \left[ \frac{e^{\mp B x_N}}{B + A} \right] \left[ \hat{h} \right](\xi', 0) \right](x')
\]
\[= : [V_N(\psi)](x', x_N) + [V_N(\psi)](x', x_N).\]

By Lemma 3.6 and $h = \langle h, n_0 \rangle$, there exist $V_J(\lambda), V_N(\lambda) \in \text{Hol}(\Sigma_\varepsilon, L'(N^{N+1}_q, W_1^2(N^N_q)))$ such that $V_J(\psi) = V_J(\lambda)(\nabla h, \lambda^{1/2} h)$, $V_N(\psi) = V_N(\lambda)(\nabla h, \lambda^{1/2} h)$, and
\[
\mathcal{R}_{L'(N^{N+1}_q, L'(N^{N+1}_q))} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda V_J(\lambda), R_\lambda V_N(\lambda) \right) \mid \lambda \in \Sigma_\varepsilon \right\} \right) \leq \gamma_1 \quad (l = 0, 1).
\]

To treat $V_N(\psi)$, we show the following lemma.

Lemma 3.12. Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. We define an operator $K_{\pm}(\lambda)$ on $W_1^2(N^N_q)$ by the formulas:
\[
[K_{\pm}(\lambda) f](x) = F_{\xi}^{-1} \left[ \frac{e^{\mp B x_N}}{B + A} \left[ \hat{f} \right](\xi', 0) \right](x') \quad (\pm x_N > 0, \lambda \in \Sigma_\varepsilon).
\]

Then there exists an operator families $K_{\pm}(\lambda) \in \text{Hol}(\Sigma_\varepsilon, L'(N^{N+1}_q, W_1^2(N^N_q)))$ such that $K_{\pm}(\lambda)f = \hat{K}_{\pm}(\lambda)(\nabla f, \lambda^{1/2} f)$ and
\[
(3.42) \quad \mathcal{R}_{L'(N^{N+1}_q, L'(N^{N+1}_q))} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( R_\lambda K_{\pm}(\lambda) \right) \mid \lambda \in \Sigma_\varepsilon \right\} \right) \leq \gamma_1 \quad (l = 0, 1).
\]

Proof. By using the relation: $g(x_N) h(0) = - \int_0^\infty d y_N (g(x_N + y_N) h(y_N)) d y_N$ for functions $g, h$ satisfying $g(x_N + y_N) h(y_N) \to 0$ as $y_N \to \infty$, we rewrite $K_{\pm}(\lambda)f$ as
\[
K_{\pm}(\lambda)f = \pm \sum_{j=1}^{N-1} \int_0^\infty F_{\xi_j}^{-1} \left[ \frac{i\xi_j}{B(B + A)} e^{\mp B x_N} - B y_N \left( \lambda^{1/2} f(\xi', y_N) - \lambda^{1/2} f(\xi', - y_N) \right) \right](x') d y_N
\]
\[
\pm \sum_{j=1}^{N-1} \int_0^\infty F_{\xi_j}^{-1} \left[ \frac{i\xi_j}{B(B + A)} e^{\mp B x_N} \left( \hat{\partial_j f}(\xi', - y_N) - \hat{\partial_j f}(\xi', y_N) \right) \right](x') d y_N
\]
\[
\pm \int_0^\infty F_{\xi}^{-1} \left[ \frac{1}{B + A} e^{\mp B x_N} \left( \hat{\partial_N f}(\xi', y_N) + \hat{\partial_N f}(\xi', - y_N) \right) \right](x') d y_N
\]
for $\pm x_N > 0$, respectively, where we have used $B^2 = \lambda + A^2$. From now on, we show the estimate (3.42). Noting $\lambda = (B + A)(B - A)$ and $B/(B + A) = 1 - A/(B + A)$, we have, for $k, l = 1, \ldots, N - 1$ and $\pm x_N > 0$,
\[
\left( \partial_\xi \partial_{\xi'} \partial_\lambda \partial_{\lambda'} \partial_N \partial_{\partial_N} \lambda^{1/2} \partial_N \right) K_{\pm}(\lambda)(\nabla f, \lambda^{1/2} f)
\]
\[
= \pm \int_0^\infty F_{\xi}^{-1} \left[ \frac{\lambda^{1/2} m(\xi', \lambda)}{B} \right] A e^{\mp B x_N} \left( \lambda^{1/2} f(\xi', y_N) - \lambda^{1/2} f(\xi', - y_N) \right)(x') d y_N
\]
\[
\pm \sum_{j=1}^{N-1} \int_0^\infty F_{\xi_j}^{-1} \left[ \frac{i\xi_j m(\xi', \lambda)}{B} \right] A e^{\mp B x_N} \left( \hat{\partial_j f}(\xi', y_N) - \hat{\partial_j f}(\xi', - y_N) \right)(x') d y_N
\]
\[
\pm \int_0^\infty F_{\xi}^{-1} \left[ m(\xi', \lambda) A e^{\mp B x_N} \left( \hat{\partial_N f}(\xi', y_N) + \hat{\partial_N f}(\xi', - y_N) \right) \right](x') d y_N
\]
\[
\left( \partial_N^2, \lambda^{1/2} \partial_N \right) K_{\pm}(\lambda)(\nabla f, \lambda^{1/2} f)
\]
\[
= \pm \int_0^\infty F_{\xi}^{-1} \left[ \frac{\lambda^{1/2}}{B} \right] (n(\xi', \lambda) - \hat{A}(\xi', \lambda)) e^{\mp B x_N} \left( \lambda^{1/2} f(\xi', y_N) - \lambda^{1/2} f(\xi', - y_N) \right)(x') d y_N
\]
\[
\pm \sum_{j=1}^{N-1} \int_0^\infty F_{\xi_j}^{-1} \left[ \frac{i\xi_j}{B} \right] (n(\xi', \lambda) - \hat{A}(\xi', \lambda)) e^{\mp B x_N} \left( \hat{\partial_j f}(\xi', y_N) - \hat{\partial_j f}(\xi', - y_N) \right)(x') d y_N
\]

we assume that

\[ m(\xi', \lambda) = \left( -\frac{\xi_{k} \lambda}{A(B + A)}, \frac{\pm i \xi_{k} B}{A(B + A)}, \frac{\pm i \xi_{k} \lambda}{A(B + A)} \right), \]

\[ n(\xi', \lambda) = \left( B, \mp \lambda^{1/2} \right), \quad l(\xi', \lambda) = \left( \frac{B}{B + A}, \mp \lambda^{1/2} \right). \]

Since \( m(\xi', \lambda), l(\xi', \lambda) \in \mathbb{M}_{0,2,\xi,0} \) and \( n(\xi', \lambda) \in \mathbb{M}_{1,\xi,0} \), applying [20, Lemma 5.4] with Lemma 3.5 to the above formulas of \( K_{\pm}(\lambda)(\nabla f, \lambda^{1/2} f) \) furnishes (3.42). This completes the proof of the lemma. \( \square \)

By Lemma 3.12 and \( h = \langle h, n_{0} \rangle \), there exists \( V_{2}(\lambda) \in \text{Hol}(\Sigma_{c}, \mathcal{L}(L_{q}(\mathbb{R}^{N})^{2+N}, W_{2}^{2}(\mathbb{R}^{N}))) \) such that \( V_{2}(\psi) = V_{2}(\lambda)(\nabla h, \lambda^{1/2} h) \) and

\[ R_{\mathcal{L}(L_{q}(\mathbb{R}^{N})^{2+N}, L_{q}(\mathbb{R}^{N}))}^{\mathcal{L}(\mathbb{R}^{N})^{2+N}} \left( \left\{ \left( \frac{d}{d\lambda} \right)^{l} (R_{V}^{2}(\lambda)) \mid \lambda \in \Sigma_{c} \right\} \right) \leq \gamma_{1} \quad (l = 0, 1). \]

Recalling Remark 2.3 [1], we set, for \((H_{2}, H_{3}) \in L_{q}(\mathbb{R}^{N})^{2+N} \times L_{q}(\mathbb{R}^{N})^{2+N}\),

\[ V^{2}(\lambda)(H_{2}, H_{3}) = (V_{1}(\lambda)(H_{2}, H_{3}), \ldots, V_{N-1}(\lambda)(H_{2}, H_{3}), V_{N}(\lambda)(H_{2}, H_{3} + V_{N}(\lambda)(H_{2}, H_{3}))^{T}. \]

Then \( V(\lambda)H = V^{2}(\lambda)H_{1} + V^{2}(\lambda)(H_{2}, H_{3}) \) with \( H = (H_{1}, H_{2}, H_{3}) \in \mathcal{Y}_{R,q}(\mathbb{R}^{N}) \) satisfies (3.37). Moreover, for \((f, h) \in Y_{R,q}(\mathbb{R}^{N}), V(g) = V(\lambda)(f, \nabla h, \lambda^{1/2} h) \) with the solution \( g \) of (3.66). \( \square \)

We set \( S_{I}(\lambda)H = V(\lambda)H + T_{I}(\lambda)(H_{1}, R_{3}V(\lambda)H_{2}, H_{3}) \) for \( H = (H_{1}, H_{2}, H_{3}) \in \mathcal{Y}_{R,q}(\mathbb{R}^{N}) \). Then, Theorem 3.3 and Proposition 3.10 together with Proposition 2.3 shows that \( S_{I}(\lambda) \) is the required operator in Theorem 3.4. This completes the proof of Theorem 3.4. \( \square \)

4. Reduced Stokes resolvent equations on a bent space

Let \( \Phi: \mathbb{R}^{N}_{x} \to \mathbb{R}^{N}_{y} \) be a bijection of \( C^{1} \) class and let \( \Phi^{-1} \) be its inverse map, where subscripts \( x, y \) denote their variables, here and subsequently. Writing \((\nabla x \Phi)(x) = A + B(x)\) and \((\nabla y \Phi^{-1})(\Phi(x)) = A_{-1} + B_{-1}(x)\), we assume that \( A \) and \( A_{-1} \) are orthonormal matrices with constant coefficients and \( \text{det} A = \text{det} A_{-1} = 1 \), and also assume that \( B(x) \) and \( B_{-1}(x) \) are matrices of functions in \( W_{1}^{2}(\mathbb{R}^{N}) \) with \( N < r < \infty \) such that

\[ \| (B, B_{-1}) \|_{L_{\infty}(\mathbb{R}^{N})} \leq M_{1}, \quad \| \nabla x (B, B_{-1}) \|_{L_{r}(\mathbb{R}^{N})} \leq M_{2}. \]

We will choose \( M_{1} \) small enough eventually, so that we may assume that \( 0 < M_{1} \leq 1 \leq M_{2} \) in the following.

**Remark 4.1.** Since \( x = \Phi^{-1}(\Phi(x)) \), we have \( I = (\nabla y \Phi^{-1})(\nabla x \Phi) \). This implies that \((\nabla y \Phi^{-1})^{-1} = (\nabla x \Phi)\), which is equivalent to \((A_{-1} + B_{-1}(x))^{-1} = A + B(x)\).

Set \( \Omega_{+} = \Phi(\mathbb{R}^{N}_{x}) \) and \( \Gamma = \Phi(\mathbb{R}^{N}_{y}) \), and let \( \tilde{n} = \tilde{n}(y) \) be the unit normal vector on \( \Gamma \), which points from \( \Omega_{+} \) to \( \Omega_{-} \). In addition, setting \( \Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})^{T} \), we see that \( \Gamma \) is represented by \( \Phi_{-1}(y) = 0 \), since \( \Gamma = \Phi_{-1}(\{y \in \mathbb{R}^{N} \mid \Phi_{-1,N}(y) = 0\}) \) by \( x_{N} = \Phi_{-1,N}(y) \). This representation implies that

\[ \tilde{n}(\Phi(x)) = -\frac{\nabla y \Phi_{-1,N}}{\nabla y \Phi_{-1,N}} = \frac{(A_{N1} + B_{N1}(x), \ldots, A_{NN} + B_{NN}(x))}{(\sum_{i=1}^{N}(A_{Ni} + B_{Ni}(x))^{2})^{1/2}} = \frac{(A_{-1} + B_{-1}(x))^{T} n_{0}}{|(A_{-1} + B_{-1}(x))^{T} n_{0}|} \]
with \( \mathbf{n}_0 = (0, \ldots, 0, -1)^T \), where we have set \( A_{-1} = (A_i) \) and \( B_{-1}(x) = (B_i(x)) \). In particular, \( \bar{n} \) is defined on \( \mathbb{R}^N \) by (4.2). Since \( \sum_{i=1}^N (A_{N_i} + B_{N_i}(x))^2 = 1 + \sum_{i=1}^N (2A_{N_i}B_{N_i}(x) + B_{N_i}(x)^2) \) by the fact that \( A_{-1} \) is an orthonormal matrix, we see by (4.1) and (4.2) that \( \| \nabla_y \bar{n} \|_{L^\infty(\mathbb{R}^N)} \leq C_N M_2 \). Let \( \tilde{\mu}_\pm = \mu_\pm(y) \) be a viscosity coefficient that is defined on \( \mathbb{R}^N \) and satisfies conditions:

\[
\frac{1}{2} \mu_{\pm 1} \leq \tilde{\mu}_\pm(y) \leq \frac{3}{2} \mu_{\pm 2} \quad (y \in \mathbb{R}^N), \quad |\tilde{\mu}_\pm(y) - \mu_0| \leq M_1 \quad (y \in \mathbb{R}^N), \quad \| \nabla_y \tilde{\mu}_\pm \|_{L^\infty(\mathbb{R}^N)} \leq C_r,
\]

where \( \mu_{\pm 0} \) are some constant with \( \mu_{\pm 1} \leq \mu_{\pm 0} \leq \mu_{\pm 2} \), respectively, for the same constants \( \mu_{\pm 1}, \mu_{\pm 2} \) as in Theorem 1.6. In addition, we set

\[
(4.4) \quad \tilde{\mu}(y) = \tilde{\mu}_+(y)\chi_{\Omega_+}(y) + \tilde{\mu}_-(y)\chi_{\Omega_-}(y), \quad \bar{\rho}(y) = \rho_+\chi_{\Omega+}(y) + \rho_-\chi_{\Omega_-}(y) \quad (\rho_+: \text{positive constants}),
\]

and also set \( \mu_{\pm}(x) = \tilde{\mu}_{\pm}(\Phi(x)) \), \( \rho(x) = \bar{\rho}(\Phi(x)) \), and \( \mu_0(x) = \tilde{\mu}_0(\Phi(x)) \) with \( \tilde{\mu}_0(y) = \mu_{\pm 0}\chi_{\Omega_+}(y) + \mu_{\pm 0}\chi_{\Omega_-}(y) \). It then holds that

\[
(4.5) \quad \rho = \rho(x) = \rho_+\chi_{\Omega+}(x) + \rho_-\chi_{\Omega_-}(x), \quad \mu_0 = \mu_0(x) = \mu_{\pm 0}\chi_{\Omega+}(x) + \mu_{\pm 0}\chi_{\Omega_-}(x),
\]

\[
\mu(x) = \mu_{\pm}(x)\chi_{\Omega+}(x) + \mu_{\mp}(x)\chi_{\Omega_-}(x), \quad |\mu_\pm - \mu_0| \leq M_1 \quad (x \in \mathbb{R}^N), \quad \| \nabla_x \mu \|_{L^\infty(\mathbb{R}^N)} \leq C_r.
\]

First we consider the two-phase reduced Stokes equation with an interface condition:

\[
\begin{aligned}
\begin{cases}
\lambda \bar{u} - \bar{\rho}^{-1} \text{Div} \, \bar{T}(\bar{u}, \bar{K}_I(\bar{u})) = \bar{f} & \text{in } \bar{\Omega}, \\
[\bar{T}(\bar{u}, \bar{K}_I(\bar{u}))\bar{n}] = [\bar{h}] & \text{on } \Gamma, \\
[\bar{u}] = 0 & \text{on } \Gamma.
\end{cases}
\end{aligned}
\]

Here \( \bar{T}(\bar{u}, \bar{K}_I(\bar{u})) = \bar{\rho}(\bar{u}) - \bar{K}_I(\bar{u})\bar{I} \) and \( \bar{K}_I(\bar{u}) \) is a unique solution to the following weak problem:

\[
(4.7) \quad (\bar{\rho}^{-1}\nabla \bar{K}_I(\bar{u}), \nabla \bar{v})_{\bar{\Omega}} = (\bar{\rho}^{-1} \text{Div} \, \bar{\mu}(\bar{u}) - \nabla \text{div} \, \bar{u}, \nabla \bar{v})_{\bar{\Omega}} \quad \text{for all } \bar{v} \in \bar{W}^1_{\bar{\rho}}(\mathbb{R}^N),
\]

\[
(4.8) \quad [\bar{K}_I(\bar{u})] = [\bar{\mu}(\bar{u})\bar{n}], \quad \bar{n} > -[\text{div} \, \bar{u}] \quad \text{on } \Gamma.
\]

We then have the following theorem.

**Theorem 4.2.** Let \( 0 < \varepsilon < \pi/2, 1 < q < \infty, N < r < \infty, \) and \( \text{max}(q, q') \leq r \) with \( q' = q/(q-1) \). Suppose that (4.1), (4.3), and (4.4) hold. Let \( Z_{G}(\mathbb{R}^N) \) and \( Z_{G}(\mathbb{R}^N) \), with an open set \( G \) of \( \mathbb{R}^N \), be defined as

\[
Z_{G}(\mathbb{R}^N) = L_q(G)^N \times W^1_q(G)^N,
\]

\[
Z_{G}(\mathbb{R}^N) = \{(H_1, \ldots, H_4) \mid H_1, H_3 \in L_q(G)^N, H_2 \in L_q(G)^{N^2}, H_4 \in W^1_q(G)^N\},
\]

while \( \mu^* := (1/2)\min(\mu_{\pm 1}, \mu_{\pm 2}, \mu_{\pm 2}) \). Then there exist \( 0 < M_1 < \min(1, \mu^*) \), \( \lambda_0 \geq 1 \), and \( \bar{S}_I(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_0}, \mathcal{C}(Z_{G}(\mathbb{R}^N), W^2_q(\mathbb{R}^N)^N)) \) such that, for any \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \) and \( (\bar{f}, \bar{h}) \in Z_{G}(\mathbb{R}^N) \), \( \bar{u} = \bar{S}_I(\lambda)H_{G, \lambda}(\bar{f}, \bar{h}) \) is a unique solution to the problem (4.9), and furthermore,

\[
R_{\mathcal{C}(Z_{G}(\mathbb{R}^N), L_q(\mathbb{R}^N))} \left\{ \left\{ \left( \frac{\lambda d}{d\lambda} \right)^l \left( R_{\lambda} \bar{S}_I(\lambda) \right) \right\} \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \right\} \leq \gamma_2 \quad (l = 0, 1, 2)
\]

with some positive constant \( \gamma_2 \). Here and subsequently, \( \bar{N} = N^3 + N^2 + N, R_{\lambda}u = (\nabla^2 u, \lambda^{1/2} \nabla u, \lambda u) \), and \( H_{G, \lambda}(\bar{f}, \bar{h}) = \left( \bar{f}, \nabla \bar{h}, \lambda^{1/2} \bar{h} \right) \); \( M_1 \) is a constant depending on \( N, q, r, \varepsilon, \rho_+, \rho_-; \mu_+, \mu_-, \mu_+; \mu_-, \mu_-, \lambda_0 \) is a constant depending on \( M_2, N, q, r, \varepsilon, \rho_+, \rho_-; \mu_+; \mu_-, \mu_+; \mu_-, \mu_+ \); \( \gamma_2 \) denotes a generic constant depending on \( M_2, \lambda_0, N, q, r, \varepsilon, \rho_+, \rho_-; \mu_+; \mu_-, \mu_+; \mu_-, \mu_+ \).

The remaining part of this section is mainly devoted to the proof of Theorem 4.2. We rewrite the problem (4.10) as follows:

\[
\begin{aligned}
\begin{cases}
\lambda \bar{u} - \bar{\rho}^{-1} \mu \text{Div} \, \bar{D}(\bar{u}) + \bar{\rho}^{-1} \nabla \bar{\theta} - \bar{\rho}^{-1} \nabla \bar{\mu} \bar{D}(\bar{u}) \nabla \bar{\mu} = \bar{f} & \text{in } \bar{\Omega}, \\
[\bar{\mu}(\bar{D}(\bar{u}) - \bar{D}(\bar{I}))\bar{n}] = [\bar{h}] & \text{on } \Gamma, \\
[\bar{u}] = 0 & \text{on } \Gamma.
\end{cases}
\end{aligned}
\]
with \( \tilde{\theta} = \tilde{K}_f(\tilde{u}) \). By the change of variable: \( y = \Phi(x) \), we transform the problem \( \tilde{\Omega} \) to some problem on \( \tilde{\mathbb{R}}^N \) with \( u(x) = \tilde{u}(y) \) and \( \theta(x) = \tilde{\theta}(y) \). Here we note the following fundamental relations:

\[
\frac{\partial}{\partial y_j} = \sum_{k=1}^{N} (A_{kj} + B_{kj}(x)) \frac{\partial}{\partial x_k}, \quad \nabla_y = (A_{-1} + B_{-1}(x))^T \nabla_x,
\]

\[
\frac{\partial^2}{\partial y_j \partial y_k} = \sum_{l,m=1}^{N} A_{lj} A_{mk} \frac{\partial^2}{\partial x_l \partial x_m} + \sum_{l,m=1}^{N} (A_{lj} B_{mk}(x) + A_{mk} B_{lj}(x) + B_{lj}(x) B_{mk}(x)) \frac{\partial^2}{\partial x_l \partial x_m}
\]

\[
+ \sum_{l,m=1}^{N} (A_{lj} + B_{lj}(x)) \left( \frac{\partial}{\partial x_l} B_{mk}(x) \right) \frac{\partial}{\partial x_m}.
\]

and furthermore,

\[
\Delta_y = \Delta_x + \sum_{k,l,m=1}^{N} (A_{lk} + B_{lk}(x)) \left( \frac{\partial}{\partial x_l} B_{mk}(x) \right) \frac{\partial}{\partial x_m}
\]

\[
+ \sum_{k,l,m=1}^{N} (A_{lk} B_{mk}(x) + A_{mk} B_{lk}(x) + B_{lk}(x) B_{mk}(x)) \frac{\partial^2}{\partial x_l \partial x_m},
\]

\[
\nabla_y \text{div}_y \tilde{u} = (A_{-1} + B_{-1}(x))^T \nabla_x \text{div}_x (A_{-1} u) + (A_{-1} + B_{-1}(x))^T \sum_{j,k=1}^{N} \nabla_x \left( B_{kj}(x) \frac{\partial}{\partial x_k} u_j \right),
\]

\[
D(\tilde{u}) = \nabla_x u (A_{-1} + B_{-1}(x)) + (A_{-1} + B_{-1}(x))^T (\nabla_x u)^T.
\]

Thus the first equation of \( (4.10) \) is reduced to

\[
\tilde{f} = \lambda \tilde{u} - \frac{\mu}{\rho} (\Delta_y \tilde{u} + \nabla_y \text{div}_y \tilde{u}) + \frac{1}{\rho} \nabla_y \tilde{\theta} - \frac{1}{\rho} D(\tilde{u}) \nabla_y \tilde{\theta}
\]

\[
= \lambda u - \frac{\mu}{\rho} (\Delta u + A_{-1}^T \text{div} (A_{-1} u)) + \frac{1}{\rho} A_{-1}^T \nabla \theta - \frac{\mu}{\rho} \left\{ \sum_{k,l,m=1}^{N} (A_{lk} + B_{lk}(x)) \left( \frac{\partial}{\partial x_l} B_{mk}(x) \right) \frac{\partial}{\partial x_m} u \right\}
\]

\[
+ B_{-1}(x)^T \nabla \text{div} (A_{-1} u) + (A_{-1} + B_{-1}(x))^T \sum_{j,k=1}^{N} \nabla \left( B_{kj}(x) \frac{\partial}{\partial x_k} u_j \right) + \frac{1}{\rho} B_{-1}(x)^T \nabla \theta
\]

\[
- \frac{1}{\rho} \{ (\nabla u) (A_{-1} + B_{-1}(x)) + (A_{-1} + B_{-1}(x))^T (\nabla u)^T \} (A_{-1} + B_{-1}(x))^T \nabla \mu,
\]

and we have, by setting \( v = A_{-1} u \) and \( f = A_{-1} \tilde{f} \) \( \Theta \),

\[
(4.11) \quad \lambda v - \rho^{-1} \mu \text{Div} D(v) + \rho^{-1} \nabla \theta + \rho^{-1} F^1(v) + \rho^{-1} P^1 \nabla \theta = f \quad \text{in} \quad \tilde{\mathbb{R}}^N.
\]

Here we have the following information for \( F^1(v) \) and \( P^1 \):

\[
(4.12) \quad F^1(v) = \mu (R^1 \nabla^2 v + S^1 \nabla v) + (T^1 \nabla v) \nabla \mu, \quad \| (P^1, R^1) \|_{L^\infty(\mathbb{R}^N)} \leq C_N M_1,
\]

\[
\| (\nabla P^1, \nabla R^1, S^1) \|_{L^\infty(\mathbb{R}^N)} \leq C_N M_2, \quad \| (T^1, \nabla T^1) \|_{L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)} \leq C_N M_2.
\]

Next we consider the interface condition of \( (4.10) \). By \( (4.2) \),

\[
[(A_{-1} + B_{-1}(x))^T n_0] \tilde{h}_0 = [(\tilde{C}(\tilde{u}) - \tilde{G})(A_{-1} + B_{-1}(x))^T n_0],
\]

which, multiplied by \( (A_{-1} + B_{-1}(x))^{-T} = ((A_{-1} + B_{-1}(x))^T)^{-1} \), furnishes that

\[
[(A_{-1} + B_{-1}(x))^T n_0] (A_{-1} + B_{-1}(x))^{-T} [\tilde{h}_0]
\]

is
\[ \begin{align*}
&= [\tilde{\mu}(A_{-1} + B_{-1}(x))^{-T}D(\bar{u})(A_{-1} + B_{-1}(x)Tn_0] - [\tilde{\mu}n_0] \\
&= [\mu(D(v) - \theta I)n_0] + [\mu((A_{-1} + B_{-1}(x))^{-T}A_{T1}^{-1} - I)](\nabla v)n_0] + [\mu(\nabla v)T A_{-1} B_{-1}(x)Tn_0] \\
&+ [\mu(A_{-1} + B_{-1}(x))^{-T}A_{T1}^{-1}(\nabla)(A_{-1} B_{-1}(x)T + B_{-1}(x)A_{T1}^{-1} + B_{-1}(x)B_{-1}(x)T)n_0].
\end{align*} \]

Since \((A_{-1} + B_{-1}(x))^{-T}A_{T1}^{-1} = (I + B_{-1}(x))^{-T}A_{T1}^{-1}\) and \((A_{-1} + B_{-1}(x))^{-1} = A + B(x)\) by Remark 4.1, it holds that, by (4.11),
\[
\begin{align*}
&\|(A_{-1} + B_{-1}(x))^{-T}A_{T1}^{-1} - I\|_{L_\infty(R^N)} \leq C_N M_1, \\
&\|(\nabla((A_{-1} + B_{-1}(x))^{-T}A_{T1}^{-1} - I))\|_{L_{r}(R^N)} \leq C_{N} M_2,
\end{align*}
\]

We thus see, by setting \(h = ((A_{-1} + B_{-1}(x))^{-T}n_0|A_{-1} + B_{-1}(x))^{-T}\tilde{h}\circ \Phi,\) that
\[
(4.13)
\]
\[
[\mu(D(v) - \theta I)n_0] + [F^2(v)n_0] = [h] \quad \text{on} \quad R^N_0,
\]
where \(F^2(v)\) satisfies the following properties:
\[
(4.14)
\]
\[
F^2(v) = \mu R^2 \nabla v, \quad \|R^2\|_{L_\infty(R^N)} \leq C_N M_1, \quad \|\nabla R^2\|_{L_r(R^N)} \leq C_N M_2.
\]

Finally, we consider the weak problem (4.17)-(4.18). Let (LHS) and (RHS) stand for the left-hand side and the right-hand side of (4.17), respectively. Then, for any \(\tilde{\varphi} \in \tilde{W}^1_q(R^N_0)\) and for \(\varphi(x) = \tilde{\varphi}(\Phi(x))\), we have
\[
(4.15)
\]
\[
(\text{LHS}) = (\rho^{-1}(A_{-1} + B_{-1})T)\nabla \varphi \cdot \det \nabla \Phi |(A_{-1} + B_{-1})T\nabla \varphi|_{R^N} = (\rho^{-1}\nabla \varphi)_{R^N} + (\rho^{-1}P^2 \nabla \varphi)_{R^N},
\]
\[
(\text{RHS}) = (\rho^{-1}\mu \text{Div } D(v) - \rho^{-1}F^1(v) - \nabla \text{div } v + F^3(v), \quad |\text{det } \nabla \Phi|(I + A_{-1}B_{-1}(x)T)\nabla \varphi|_{R^N} = (\rho^{-1}\mu \text{Div } D(v) - \rho^{-1}F^1(v) - \nabla \text{div } v + F^3(v), \quad \nabla \varphi)_{R^N} + (F^4(v), \quad \nabla \varphi)_{R^N}
\]
for some \(P^2, F^4(v),\) and \(F^4(v)\) satisfying
\[
(4.16)
\]
\[
F^3(v) = \nabla \text{div } v + \nabla \text{div } v + F^4(v) = \nabla \text{div } v - \nabla \text{div } v + F^3(v) + F^3(v), \quad \|P^2, P^4\|_{L_\infty(R^N)} \leq C_N M_1, \quad \|\nabla P^2, \nabla P^4\|_{L_r(R^N)} \leq C_N M_2.
\]

In addition,
\[
(4.17)
\]
\[
[\theta] = < [\mu(D(v)n_0], n_0 > - [\text{div } v] + [F^5(v)] \quad \text{on} \quad R^N_0,
\]
where \(F^5(v)\) is given by
\[
(4.18)
\]
\[
F^5(v) = \mu R^5 \nabla v + \nabla 5 \nabla v, \quad \|(R^5,R^5)\|_{L_\infty(R^N)} \leq C_N M_1, \quad \|(\nabla R^5, R^5)\|_{L_r(R^N)} \leq C_N M_2.
\]

Summing up (4.11), (4.13), (4.15), and (4.17), we have obtained the following system:
\[
(4.19)
\]
\[
\begin{align*}
\lambda v - \frac{1}{\rho} \text{Div } T(v, \theta) - \frac{\mu - \mu_0}{\rho} \text{Div } D(v) + \frac{1}{\rho} F^1(v) + \frac{1}{\rho} P^1 \nabla \theta &= f \quad \text{in } R^N, \\
T(v, \theta)n_0] + [((\mu - \mu_0)D(v)n_0] + [F^5(v)n_0] = [h] \quad \text{on} \quad R^N_0, \\
[v] &= 0 \quad \text{on} \quad R^N_0
\end{align*}
\]
with \(T(v, \theta) = \mu_0 D(v) - \theta I,\) and also for any \(\varphi \in \tilde{W}^1_q(R^N)\)
\[
(4.20)
\]
\[
(\rho^{-1}\nabla \theta, \nabla \varphi)_{R^N} + (\rho^{-1}P^2 \nabla \theta, \nabla \varphi)_{R^N} = (\rho^{-1}\text{Div } (\mu_0 D(v)) - \nabla \text{div } v + \rho^{-1}((\mu - \mu_0) \text{Div } D(v) - \rho^{-1}F^1(v) + F^3(v) + F^3(v), \quad \nabla \varphi)_{R^N,}
\]
\[
(4.21)
\]
\[
[\theta] = < [\mu_0 D(v)n_0], n_0 > - [\text{div } v] + < [((\mu - \mu_0)D(v)n_0], n_0 > + [F^5(v)] \quad \text{on} \quad R^N_0.
\]

From now on, we solve (4.19), (4.20), (4.21). Let \(\theta_1 = K_1(v)\) given by the solution to (5.2)-(5.3) with \(\mu = \mu_0.\) Setting \(\theta = K_1(v) + \theta_2(v)\) in (4.20)-(4.21), we have the weak problem for \(\theta_2 = \theta_2(v)\) as follows:
\[
(4.22)
\]
\[
(\rho^{-1}\nabla \theta_2, \nabla \varphi)_{R^N} + (\rho^{-1}P^2 \nabla \theta_2, \nabla \varphi)_{R^N} = (\rho^{-1}((\mu - \mu_0) \text{Div } D(v) - \rho^{-1}F^1(v) + F^3(v) + F^3(v) - \rho^{-1}P^2 \nabla K_1(v), \nabla \varphi)_{R^N,}
\]
\[
(4.23)
\]
\[
[\theta_2] = < [((\mu - \mu_0)D(v)n_0], n_0 > + [F^5(v)] \quad \text{on} \quad R^N_0.
\]
for any $\varphi \in \tilde{W}^1_q(\mathbb{R}^N)$. Substituting $\theta = K_I(v) + \theta_2(v)$ in the problem \eqref{eq:4.19}, we have

\begin{equation}
\begin{aligned}
\lambda v - \rho^{-1} \text{Div} T(v, K_I(v)) + \mathcal{U}^1(v) &= f \quad \text{in } \tilde{\mathbb{R}}^N, \\
[T(v, K_I(v))n_0] + \|\mathcal{U}^2(v)n_0\| &= |h| \quad \text{on } \mathbb{R}^N_0, \\
[v] &= 0 \quad \text{on } \mathbb{R}^N_0,
\end{aligned}
\end{equation}

where

$$
\mathcal{U}^1(v) = -\rho^{-1}(\mu - \mu_0) \text{Div} D(v) + \rho^{-1} F^1(v) + \rho^{-1} P^1 \nabla K_I(v) + \rho^{-1}(I + P^1) \nabla \theta_2(v),
$$

$$
\mathcal{U}^2(v) = (\mu - \mu_0) D(v) + F^2(v) - \theta_2(v) I = F^2(v) - \left\{ (\mu - \mu_0) D(v)n_0, n_0 > + F^5(v) \right\} I.
$$

At this point, we introduce a result about the unique solvability of the weak problem:

\begin{equation}
(\rho^{-1} \nabla \theta, \nabla \varphi)_{\mathbb{R}^N} + (\rho^{-1} \nabla^2 \nabla \theta, \nabla \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N}
\end{equation}

for all $\varphi \in \tilde{W}^1_q(\mathbb{R}^N)$, \eqref{eq:4.26}

$$
[h] = [g] \quad \text{on } \mathbb{R}^N_0.
$$

\textbf{Lemma 4.3.} Let $1 < q < \infty$. Then there exists a constant $M_3 \in (0, 1)$ and an operator

$$
\Psi \in \mathcal{L}(L_q(\mathbb{R}^N)^N \times W^1_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N) + \tilde{W}^1_q(\mathbb{R}^N))
$$

such that, for any $f \in L_q(\mathbb{R}^N)^N$ and $g \in W^1_q(\mathbb{R}^N)$, $\theta = \Psi(f, g)$ is a unique solution to \eqref{eq:4.20} - \eqref{eq:4.21}, which possesses the estimate: $\|\nabla \theta\|_{L_q(\mathbb{R}^N)} \leq C_{N,q}(\|f\|_{L_q(\mathbb{R}^N)} + \|g\|_{W^1_q(\mathbb{R}^N)})$ with a positive constant $C_{N,q}$ independent of $M_2$.

\textbf{Proof.} Since the weak problem \eqref{eq:4.22} - \eqref{eq:4.23} is uniquely solvable, we can prove Lemma 4.3 by the small perturbation method, so that we may omit the detailed proof. \hfill \square

By Lemma 4.3 we have $\theta_2(v) = \Psi(f, g)$ with

$$
f = \rho^{-1}(\mu - \mu_0) \text{Div} D(v) - \rho^{-1} F^1(v) + F^3(v) + F^4(v) - \rho^{-1} P^2 \nabla K_I(v),
$$

$$
g = <(\mu - \mu_0) D(v)n_0, n_0 > + F^5(v).
$$

We solve the problem \eqref{eq:4.21} by using Theorem 3.1. Substituting $v = S_I(\lambda) G_{R,\lambda}(f, h)$ in \eqref{eq:4.24} yields that

\begin{equation}
\begin{aligned}
\lambda v - \rho^{-1} \text{Div} T(v, K_I(v)) &= f - \mathcal{U}^1(S_I(\lambda) G_{R,\lambda}(f, h)) \quad \text{in } \tilde{\mathbb{R}}^N, \\
[T(v, K_I(v))n_0] &= |h - \mathcal{U}^2(S_I(\lambda) G_{R,\lambda}(f, h))n_0| \quad \text{on } \mathbb{R}^N_0, \\
[v] &= 0 \quad \text{on } \mathbb{R}^N_0.
\end{aligned}
\end{equation}

Set $\mathcal{V}(\lambda)(f, h) = (\mathcal{V}^1(\lambda)(f, h), \mathcal{V}^2(\lambda)(f, h))$ with $\mathcal{V}^i(\lambda)(f, h) = \mathcal{U}^i(S_I(\lambda) G_{R,\lambda}(f, h))$ $(i = 1, 2)$ and

$$
Y^i_{\lambda R,q}(\mathbb{R}^N) = \{(f, \nabla h, \lambda^{1/2} h) \mid (f, h) \in Y_{\lambda R,q}(\mathbb{R}^N)\}
$$

for each $\lambda \neq 0$. Then, for each $\lambda \neq 0$, $\varphi(\lambda)(f, h) := G_{R,\lambda}(f, h)$ is a bijection from $Y^i_{\lambda R,q}(\mathbb{R}^N)$ onto $Y^i_{\lambda R,q}(\tilde{\mathbb{R}}^N)$. Formally, if there is the inverse operator of $(I - \varphi(\lambda) V(\lambda) \varphi(\lambda)^{-1})$, then $v = S_I(\lambda) G_{R,\lambda} \varphi(\lambda)^{-1} (I - \varphi(\lambda) V(\lambda) \varphi(\lambda)^{-1})^{-1} \varphi(\lambda)(f, h)$ solves \eqref{eq:4.22} since $\varphi(\lambda)^{-1} (I - \varphi(\lambda) V(\lambda) \varphi(\lambda)^{-1})^{-1} \varphi(\lambda) = (I - V(\lambda))^{-1}$.

In what follows, we show the invertibility above and the $\mathcal{R}$-boundedness of the inverse operator. To this end, we estimate the remainder terms on the right-hand sides of \eqref{eq:4.27}. We combine Proposition 2.6 for $\Omega = \mathbb{R}^N$ with \eqref{eq:4.12}, \eqref{eq:4.13}, \eqref{eq:4.16}, and \eqref{eq:4.18} in order to obtain

\begin{equation}
\begin{aligned}
\|F^i(v)\|_{L_q(\mathbb{R}^N)} &\leq \gamma_3(M_1 + \sigma) \|\nabla^2 v\|_{L_q(\mathbb{R}^N)} + \gamma_{\sigma, M_2} \|\nabla v\|_{L_q(\mathbb{R}^N)} \quad (i = 1, 3, 4), \\
\|P^i v\|_{L_q(\mathbb{R}^N)} &\leq \gamma_3(M_1 + \sigma) \|\nabla^2 v\|_{L_q(\mathbb{R}^N)} + \gamma_{\sigma, M_2} \|\nabla v\|_{L_q(\mathbb{R}^N)} \quad (i = 2, 5), \\\n\|\nabla F^i(v)\|_{L_q(\mathbb{R}^N)} &\leq \gamma_3(M_1 + \sigma) \|\nabla^2 v\|_{L_q(\mathbb{R}^N)} + \gamma_{\sigma, M_2} \|\nabla v\|_{L_q(\mathbb{R}^N)} \quad (i = 2, 5), \\
\|P^i \nabla K_I(v)\|_{L_q(\mathbb{R}^N)} &\leq \gamma_3 M_1 \|\nabla v\|_{W^1_q(\mathbb{R}^N)} \quad (i = 1, 2).
\end{aligned}
\end{equation}
Here and subsequently, $\gamma_3$ is a generic constant depending, at most, on $N$, $q$, $r$, $\rho_+$, $\rho_-$, $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_2$; $\gamma_{\sigma,M_2}$ is a generic constant depending, at most, on $M_2$, $\sigma$, $N$, $q$, $r$, $\rho_+$, $\rho_-$, $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_2$. In addition, by Lemma 3.3, 4.28, and 4.35 together with Proposition 2.10, we have

$$
(4.29) \qquad \| (I + D^2) \nabla \theta (v) \|_{L^q (\mathbb{R}^N)} \leq \gamma_3 (M_1 + \sigma) \| \nabla v \|_{L^q (\mathbb{R}^N)} + \gamma_{\sigma,M_2} \| \nabla v \|_{L^q (\mathbb{R}^N)}.
$$

Define operators $V^i(\lambda)$, $i = 1, 2$, as $V^i(\lambda)H = U^i(S_f(\lambda)H)$ with $H = (H_1, H_2, H_3) \in \mathcal{Y}_{\sigma,q}(\mathbb{R}^N)$. Then we have $V^i(f, h) = V^i(\lambda)G_{\sigma,\lambda}(f, h)$ and have, by Proposition 2.10, 4.28, 4.29, and Theorem 3.3

$$
(4.30) \qquad \mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N), L^q(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l V^1(\lambda) | \lambda \in \Sigma_{e,\lambda_0} \right\} \right)^{1/q} \leq \gamma_1 \left( \gamma_3 (M_1 + \sigma) + \gamma_{\sigma,M_2} \lambda_0^{-1/2} \right),
$$

$$
\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N), L^q(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l \left( \nabla V^2(\lambda), \lambda^{1/2} V^2(\lambda) \right) | \lambda \in \Sigma_{e,\lambda_0} \right\} \right) \leq \gamma_1 \left( \gamma_3 (M_1 + \sigma) + \gamma_{\sigma,M_2} \lambda_0^{-1/2} \right)
$$

for $l = 0, 1$ and for any $\lambda_0 > 0$. In fact, since $\mathcal{F}^1$ is linear, we have, for any $\lambda_0 > 0$ and for any $n \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^n \subset \Sigma_{e,\lambda_0}$, and $\{H_j\}_{j=1}^n \subset \mathcal{Y}_{\sigma,q}(\mathbb{R}^N)$,

$$
\left( \int_0^1 \left\| \sum_{j=1}^n r_j(u) \mathcal{F}^1(S_f(\lambda_j)H_j) \right\|_{L^q(\mathbb{R}^N)}^q \, du \right)^{1/q} \leq \gamma_3 (M_1 + \sigma) \left( \int_0^1 \left\| \sum_{j=1}^n r_j(u) \nabla^2 S_f(\lambda_j)H_j \right\|_{L^q(\mathbb{R}^N)}^q \, du \right)^{1/q}
$$

$$
+ \gamma_{\sigma,M_2} \left( \int_0^1 \left\| \sum_{j=1}^n r_j(u) \nabla S_f(\lambda_j)H_j \right\|_{L^q(\mathbb{R}^N)}^q \, du \right)^{1/q}
$$

$$
\leq \gamma_1 \left( \gamma_3 (M_1 + \sigma) + \gamma_{\sigma,M_2} \lambda_0^{-1/2} \right) \left( \int_0^1 \left\| \sum_{j=1}^n r_j(u)H_j \right\|_{L^q(\mathbb{R}^N)}^q \, du \right)^{1/q}.
$$

It holds, by the linearity of $\mathcal{F}^1$, that

$$
\lambda \frac{d}{d\lambda} \mathcal{F}^1(S_f(\lambda)H) = \mathcal{F}^1 \left( \lambda \frac{d}{d\lambda} S_f(\lambda)H \right),
$$

so that we have in the same manner as above

$$
\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N), L^q(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l (S_f(\lambda)(\cdot)) | \lambda \in \Sigma_{e,\lambda_0} \right\} \right) \leq \gamma_1 \left( \gamma_3 (M_1 + \sigma) + \gamma_{\sigma,M_2} \lambda_0^{-1/2} \right).
$$

Analogously, we can obtain estimates for $\mathcal{R}$-bound of the other terms, and thus we have (4.30). Setting $V(\lambda)H = (V^1(\lambda)H, V^2(\lambda)H)$ for $H \in \mathcal{Y}_{\sigma,q}(\mathbb{R}^N)$ furnishes that

$$
(4.31) \quad V(\lambda)(f, h) = V(\lambda)G_{\sigma,\lambda}(f, h) \in \mathcal{Y}_{\sigma,q}(\mathbb{R}^N) \quad \text{for} \quad (f, h) \in \mathcal{Y}_{\sigma,q}(\mathbb{R}^N),
$$

$$
\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l (G_{\sigma,\lambda}V(\lambda)) | \lambda \in \Sigma_{e,\lambda_0} \right\} \right) \leq \gamma_1 \left( \gamma_3 (M_1 + \sigma) + \gamma_{\sigma,M_2} \lambda_0^{-1/2} \right) \quad (l = 0, 1).
$$

If we choose $\sigma$ and $M_1$ so small that $\gamma_1 \gamma_3 \sigma \leq 1/8$ and $\gamma_1 \gamma_3 M_1 \leq 1/8$ and if we choose $\lambda_0 \geq 1$ so large that $\gamma_{\sigma,M_2} \lambda_0^{-1/2} \leq 1/4$, then we have by (4.31)

$$
(4.32) \quad \mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l (G_{\sigma,\lambda}V(\lambda)) | \lambda \in \Sigma_{e,\lambda_0} \right\} \right) \leq \frac{1}{2} \quad (l = 0, 1).
$$

Since it holds by (4.31), (4.32) that

$$
\| \varphi_\lambda V(\lambda) \varphi_\lambda^{-1}(f, \nabla h, \lambda^{1/2} h) \|_{\mathcal{Y}_{\sigma,q}(\mathbb{R}^N)} = \| G_{\sigma,\lambda} V(\lambda)(f, h) \|_{\mathcal{Y}_{\sigma,q}(\mathbb{R}^N)}
$$

$$
= \| G_{\sigma,\lambda} V(\lambda) G_{\sigma,\lambda}(f, h) \|_{\mathcal{Y}_{\sigma,q}(\mathbb{R}^N)} \leq \frac{1}{2} \| (f, \nabla h, \lambda^{1/2} h) \|_{\mathcal{Y}_{\sigma,q}(\mathbb{R}^N)},
$$

there exists the inverse mapping $(I - \varphi_\lambda V(\lambda) \varphi_\lambda^{-1})^{-1} \in \mathcal{L}(\mathbb{R}^N)$ for any $\lambda \in \Sigma_{e,\lambda_0}$. In addition, $(I - G_{\sigma,\lambda} V(\lambda))^{-1} = \sum_{j=0}^{\infty} (G_{\sigma,\lambda} V(\lambda))^j$ exists by (4.32) and satisfies the estimate:

$$
(4.33) \quad \mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\sigma,q}(\mathbb{R}^N))} \left( \left\{ (\lambda \frac{d}{d\lambda})^l (I - G_{\sigma,\lambda} V(\lambda))^{-1} | \lambda \in \Sigma_{e,\lambda_0} \right\} \right) \leq 2 \quad (l = 0, 1).
If we set \(v = S_\ell(\lambda)G_{\mathcal{R},\lambda}\varphi^{-1}_\lambda(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi^{-1}_\lambda)\varphi(\mathbf{f}, \mathbf{h})\), then \(v\) is a solution to (4.24) as mentioned above. Noting that \(\varphi_\lambda\mathcal{V}(\lambda)\varphi^{-1}_\lambdaG_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}) = G_{\mathcal{R},\lambda}\mathcal{V}(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})\) by (4.31), we see that

\[
G_{\mathcal{R},\lambda}\varphi^{-1}_\lambda(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi^{-1}_\lambda)\varphi(\mathbf{f}, \mathbf{h}) = \sum_{j=0}^{\infty}(\varphi_\lambda\mathcal{V}(\lambda)\varphi^{-1}_\lambda)^jG_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}) = (I - G_{\mathcal{R},\lambda}\mathcal{V}(\lambda))^{-1}G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}).
\]

Set \(S_\ell(\lambda) = S_\ell(\lambda)(I - G_{\mathcal{R},\lambda}\mathcal{V}(\lambda))^{-1}\), and then \(v = S_\ell(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})\) is a solution to (4.24) for any \(\lambda \in \Sigma_{\varepsilon,\lambda_0}\) and \((\mathbf{f}, \mathbf{h}) \in \mathcal{Y}_{\mathcal{R},\lambda}(\mathcal{R}^N)\). Furthermore, by (4.33) and Theorem 3.4, we have

\[
(4.34) \quad S_\ell(\lambda) \in \mathcal{H}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_{\mathcal{R},\lambda}(\tilde{\mathcal{R}}^N), \mathcal{W}_q^2(\tilde{\mathcal{R}}^N)^N)),
\]

\[
\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},\lambda}(\mathcal{R}^N), \mathcal{L}(\mathcal{R}^N))} \left( \left\{ \left( \begin{array}{c} d \frac{d}{dx} \end{array} \right)^l (R_{\lambda}\mathcal{V}(\lambda)) \right| \lambda \in \Sigma_{\varepsilon,\lambda_0} \right) \right) \leq \gamma_2 \quad (l = 0, 1).
\]

The uniqueness of solutions to (4.24) can be proved in the same manner as in [20] Section 4.

Setting \(\bar{u} = A^T_{-1}v \circ \Phi^{-1} = [A^T_{-1}S_\ell(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})] \circ \Phi^{-1}\) and noting \(A^T_{-1} = (A_{-1})^{-1}\), we see that \(\bar{u}\) is a unique solution to (4.6). Recall that \(f = A_{-1} \circ \Phi\) and \(h = [A_{-1} + B_{-1}(x)]^{-T}\mathbf{n}_0[(A_{-1} + B_{-1}(x))^{-T} \circ \Phi, \Phi, \Phi]\mathbf{E}(x)[(\Phi h) \circ \Phi, \Phi, \Phi]\mathbf{E}(x)(\lambda^{1/2}\tilde{h}) \circ \Phi\),

we define, for \(H = (H_1, H_2, H_3, H_4) \in \mathcal{L}(\mathcal{Y}_{\mathcal{R},\lambda}(\hat{\Omega}))\), an operator \(\bar{S}_\ell(\lambda)\) by

\[
\bar{S}_\ell(\lambda)H = [A^T_{-1}S_\ell(\lambda)(A_{-1}H_1 \circ \Phi, (\Phi h) \circ \Phi, \Phi h)(H_2 \circ \Phi, \Phi h)(H_3 \circ \Phi) \circ \Phi^{-1}].
\]

Then we can show that \(\bar{S}_\ell(\lambda)\) satisfies (4.34) by (4.33) and Proposition 2.4, with \(\sigma = 1\), and also \(\bar{u} = \bar{S}_\ell(\lambda)H_{\mathcal{R},\lambda}(\bar{\mathbf{f}}, \bar{\mathbf{h}})\) solves (4.1) uniquely. This completes the proof of Theorem 4.2.

5. A proof of Theorem 2.2

As was discussed in Subsection 2.3 our main result Theorem 1.6 follows from Theorem 2.2 so that we prove Theorem 2.2 in this section.

5.1. Some preparations for the proof of Theorem 2.2

First we state several properties of uniform \(W^{2-1/r}_{r}\) domain (cf. [14] Proposition 6.1, [17]).

**Proposition 5.1.** Let \(N < r < \infty\) and let \(\Omega_{\pm}\) be uniform \(W^{2-1/r}_{r}\) domains in \(\mathcal{R}^N\). Let \(M_1\) the number given in Section 4. Then there exist constants \(M_2 > 0, 0 < d < 1\) \((i = 1, 2, 3)\), \(x^i_j \in \mathcal{R}^N\), \(j \in \mathcal{N}, i = 1, 2, 3\), \(x^i_j \in \Gamma, x^i_j \in \Gamma_\pm, x^i_j \in \Gamma_\mp, x^i_j \in \Omega, \) such that the following assertions hold:

1. The maps: \(\mathcal{R}^N \ni x \mapsto \Phi^i_j(x)(j \in \mathcal{N}, i = 1, 2, 3)\) are bijective such that \(\nabla \Phi^i_j = A^i_j + B^i_j(x)\) and \(\nabla(\Phi^i_j)^{-1} = A^i_{j-1} + B^i_{j-1}(x)\), where \(A^i_j, A^i_{j-1}\) are \(N \times N\) constant orthonormal matrices and \(B^i_j(x), B^i_{j-1}(x)\) are \(N \times N\) matrices of \(W^2_{r}(\mathcal{R}^N)\) functions which satisfy the conditions: \(\|B^i_j, B^i_{j-1}\|_{L_{\infty}(\mathcal{R}^N)} \leq M_1\) and \(\|B^i_j, B^i_{j-1}\|_{L_{\infty}(\mathcal{R}^N)} \leq M_2\).

2. \(\Omega = \left\{ \cup_{i=1,2,3} \cup_{j=1}^{\infty}(\Phi^i_j(H^i) \cap B^i_{d}(x^i_j)) \right\} \cup \left\{ \cup_{i=1,2,3} \cup_{j=1}^{\infty}(\Phi^i_j(H^i) \cap B^i_{d}(x^i_j)) \right\} \) with \(H^1 = \mathcal{R}^N, H^2 = \mathcal{R}^N, H^3 = \mathcal{R}^N,\) where \(\Phi^i_j(\mathcal{R}^N) \cap B^i_{d}(x^i_j) = \Omega_+ \cap B^i_{d}(x^i_j) = \Omega_- \cap B^i_{d}(x^i_j)\) \((i = 1, 2, 3, x^i_j \in \Omega_+, x^i_j \in \Omega_-, and B^i_j(\mathcal{R}^N) \cap B^i_{d}(x^i_j) = B^i_j(\mathcal{R}^N) \cap B^i_{d}(x^i_j))\) \((i = 1, 2, 3)\). Here and subsequently, we set \(\Gamma^1 = \Gamma, \Gamma^2 = \Gamma^+, \) and \(\Gamma^3 = \Gamma^-\) for the notational convenience.

3. There exist \(C^\infty\) functions \(\zeta^i_j, \tilde{\zeta}^i_j\) \((i = 1, 2, 3, j \in \mathcal{N})\) such that \(\|\zeta^i_j\|_{W^2_{r}(\mathcal{R}^N)} \leq c_0, 0 \leq \zeta^i_j, \tilde{\zeta}^i_j \leq 1,\) \(\text{supp} \zeta^i_j, \text{supp} \tilde{\zeta}^i_j \subset B^i_{d}(x^i_j)\) \((i = 1, 2, 3)\), \(\zeta^i_j = 1\) on \(\text{supp} \zeta^i_j, \sum_{i=1,2,3}^{\infty} \sum_{j=1}^{\infty} c^i_j = 1\) on \(\hat{\Omega}\), and \(\sum_{j=1}^{\infty} c^i_j = 1\) on \(\Gamma^i\) \((i = 1, 2, 3)\). Here \(c_0\) is a positive constant depending on \(M_2, \mathcal{N},\) and \(r\), but independent of \(j \in \mathcal{N}\).

4. There exists a natural number \(L \geq 2\) such that any \(L + 1\) distinct sets of \(\{B^i_{d}(x^i_j)\} \) \((i = 1, \ldots, 5, j \in \mathcal{N})\) have an empty intersection.
Since $\mu_\pm(x)$ is uniformly continuous in $\mathbb{R}^N$ as was assumed in the assumption (c), choosing $d^i > 0$ smaller, if necessary, allows us to assume that $|\mu_\pm(x) - \mu_\pm(x_j^i)| \leq M_1$ for any $x \in B_{d^i}(x_j^i)$ with $i = 1, \ldots, 5$ and $j \in \mathbb{N}$. Moreover, after choosing $M_2$ and $d^i$ according to $M_1$ in Proposition 5.1, we choose $M_2$ again so large that $\|\nabla \mu_\pm\|_{L_{r}(B_{d^i}(x_j^i))} \leq M_2$. Here and in the following, constants denoted by $C$ are independent of $j \in \mathbb{N}$. In view of (4.2), we may assume that unit normal vectors $n_j^i$ to $\Gamma_j^i = \Phi_j^i(B_{r}(\mathbb{R}^N))$ ($i = 1, 2, 3, j \in \mathbb{N}$) are defined on $\mathbb{R}^N$ together with $||n_j^i||_{L_\infty(\mathbb{R}^N)} = 1$, and also they satisfy, by Proposition 5.1(1), the conditions: $\|\nabla n_j^i\|_{L_\infty(\mathbb{R}^N)} \leq CM_2$. Note that $n = n_j^i$ on $B_{d^i}(x_j^i) \cap \Gamma$ and points from $\Omega_+ \to \Omega_-$, and besides, the unit outward normal $n_\pm$ to $\Gamma_\pm$ satisfy $n_\pm = n_j^2$ on $B_{d^i}(x_j^2) \cap \Gamma_\pm$ and $n_- = n_j^3$ on $B_{d^i}(x_j^3) \cap \Gamma_-$, respectively.

Summing up the above properties, we suppose in this section that

$$\mu_\pm \leq \mu_\pm(x_j^i) \leq \mu_\pm(2), \quad |\mu_\pm(x) - \mu_\pm(x_j^i)| \leq M_1 \quad (x \in B_{d^i}(x_j^i)), \quad \|\nabla \mu_\pm\|_{L_r(B_{d^i}(x_j^i))} \leq M_2.$$  

Let $B_j^i = B_{d^i}(x_j^i)$ with $i = 1, \ldots, 5$ and $j \in \mathbb{N}$ for short. Then, by the finite intersection property stated in Proposition 5.1(1), we see that, for any $s \in [1, \infty)$, there is a positive constant $C_{s, L}$ such that, for any $f \in L_s(G)$ with an open set $G$ of $\mathbb{R}^N$ and for $i = 1, \ldots, 5$,

$$\left(\sum_{j=1}^{\infty} \|f\|_{L_s(B_j^i)}^s\right)^{1/s} \leq C_{s, L} \|f\|_{L_s(G)}.$$  

In fact, when $1 \leq s < \infty$,

$$\sum_{j=1}^{\infty} \|f\|_{L_s(G \cap B_j^i)}^s = \int_G \left(\sum_{j=1}^{\infty} \chi_{B_j^i}(x)\right)^s |f(x)|^s \, dx \leq \left(\sum_{j=1}^{\infty} \chi_{B_j^i} \right)_{L_\infty(\mathbb{R}^N)} \|f\|_{L_s(G)}^s \leq L \|f\|_{L_s(G)}^s.$$  

Next we prepare two lemmas used to construct parametrices.

**Lemma 5.2.** Let $X$ be a Banach space and $X^*$ its dual space, while $\|\cdot\|_X$, $\|\cdot\|_{X^*}$, and $<\cdot, \cdot>$ be the norm of $X$, the norm of $X^*$, and the duality pairing between of $X$ and $X^*$, respectively. Let $n \in \mathbb{N}$, $l = 1, \ldots, n$, and $\{a_i\}_{i=1}^n \subset C$, and let $\{f_j^i\}_{j=1}^{\infty}$ be sequences in $X^*$ and $\{g_j^i\}_{j=1}^{\infty}$ be sequences of positive numbers. Assume that there exist maps $N_j^i : X \to [0, \infty)$ such that

$$\langle f_j^i, \varphi \rangle \leq M_3 g_j^i N_j^i(\varphi) \quad (l = 1, \ldots, n), \quad \left| \left(\sum_{i=1}^{n} a_i f_j^i, \varphi \right) \right| \leq M_3 h_j N_j^i(\varphi)$$  

for any $\varphi \in X$ with some positive constant $M_3$ independent of $j \in \mathbb{N}$ and $l = 1, \ldots, n$. If

$$\sum_{j=1}^{\infty} (g_j^i)^q < \infty, \quad \sum_{j=1}^{\infty} (h_j)^q < \infty, \quad \sum_{j=1}^{\infty} (N_j^i(\varphi))^q \leq (M_4 \|\varphi\|_X)^q$$  

with $1 < q < \infty$ and $q' = q/(q-1)$ for some positive constant $M_4$, then the infinite sum $f^l = \sum_{j=1}^{\infty} f_j^l$ exists in the strong topology of $X^*$ and

$$\|f^l\|_{X^*} \leq M_3 M_4 \left(\sum_{j=1}^{\infty} (g_j^i)^q\right)^{1/q}, \quad \left\|\sum_{i=1}^{n} a_i f^l\right\|_{X^*} \leq M_3 M_4 \left(\sum_{j=1}^{\infty} (h_j)^q\right)^{1/q}.$$  

**Proof.** Let $F_j^l = \sum_{j=1}^{\infty} f_j^l$. We can show that $\{F_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $X^*$, which implies the existence of $f^l$. Then the estimates follow immediately.

The following lemma follows from Lemma 5.2 and (5.2).

**Lemma 5.3.** Let $1 < q < \infty$, $q' = q/(q-1)$, $i = 1, \ldots, 5$, and $m \in \mathbb{N}_0$. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $W_q^m(\bar{\Omega})$ and let $\{g_j^i\}_{j=1}^{\infty}$ be sequences of positive numbers for $l = 0, 1, \ldots, m$. Assume that

$$\sum_{j=1}^{\infty} (g_j^i)^q < \infty, \quad \|\nabla^l f_j, \varphi\|_{L_q(\bar{\Omega} \cap \Omega_j^i)} \leq M_5 g_j^i \|\varphi\|_{L_q(\bar{\Omega} \cap \Omega_j^i)}$$  

for any $\varphi \in L_q(\bar{\Omega})$ with some positive constant $M_5$ independent of $j \in \mathbb{N}$ and $l = 0, 1, \ldots, m$. Then $f = \sum_{j=1}^{\infty} f_j$ exists in the strong topology of $W_q^m(\bar{\Omega})$ and $\|\nabla^l f\|_{L_q(\bar{\Omega})} \leq C_{q, L} M_5 \left(\sum_{j=1}^{\infty} (g_j^i)^q\right)^{1/q}$ with some positive constant $C_{q, L}$. 

Remark 5.4. At this point, we have a remark on unit normals \( \mathbf{n}, \mathbf{n}_+ \). We can see \( \mathbf{n}, \mathbf{n}_+ \) as vector functions defined on \( \mathbb{R}^N \) through the relations: \( \mathbf{n} = \sum_{j=1}^{\infty} \zeta_j^i \mathbf{n}_1^i, \mathbf{n}_+ = \sum_{j=1}^{\infty} \zeta_j^i \mathbf{n}_2^i \). Then it is clear that \( \mathbf{n} = \mathbf{n}_+ \) in \( B_j^1 \cap \Gamma \) and \( \mathbf{n}_+ = \mathbf{n}_2^j \) in \( B_j^2 \cap \Gamma_+ \). Moreover, we have \( \| \mathbf{f} \mathbf{n} \|_{L_q(\Gamma)} \leq C \| \mathbf{f} \|_{L_q(\Gamma)} \) for any function \( f \in L_q(\Omega) \). In fact, for \( f \in L_q(\Omega) \) and \( \varphi \in L_{q'}(\Omega) \),

\[
\|(f \mathbf{n}, \varphi)_{\mathcal{D}}\| \leq \sum_{j=1}^{\infty} \| f \|_{L_q(\Omega \cap B_j^1)} \| \varphi \|_{L_{q'}(\Omega \cap B_j^1)} \left( \sum_{j=1}^{\infty} \| f \|_{L_q(\Omega \cap B_j^1)} \right)^{1/q} \left( \sum_{j=1}^{\infty} \| \varphi \|_{L_{q'}(\Omega \cap B_j^1)} \right)^{1/q'},
\]

which, combined with (5.2), furnishes that \( \|(f \mathbf{n}, \varphi)_{\mathcal{D}}\| \leq C \| f \|_{L_q(\Omega)} \| \varphi \|_{L_{q'}(\Omega)} \). This inequality implies that the required estimate holds. Similarly, for \( g \in W^1_q(\Omega) \), we can prove \( \| g \nabla \mathbf{n} \|_{L_q(\Gamma)} \leq C \| g \|_{W^1_q(\Omega)} \), \( \| \mathbf{h} \|_{W^1_q(\Omega)} \leq C \| g \|_{W^1_q(\Omega)} \) with help of Lemma 2.6. It is clear that we can replace \( \mathbf{n} \) by \( \mathbf{n}_+ \) in the above inequalities.

5.2. Local solutions. In view of (5.1), we define local viscosity coefficients \( \nu_{ij}^{\pm}(x) \) by

\[
\nu_{ij}^{\pm}(x) = (\mu_\pm(x) - \mu_\pm(x_j^i)) \zeta_j^i + \mu_\pm(x_j^i).
\]

Note that \( M_1 \leq (1/2) \min(\mu_+, \mu_-, \mu_+, \mu_-) \) as was stated in Theorem 4.2. Then, using (5.1) and setting \( \mu_\pm = \mu_\pm(x_j^i) \), we have

\[
\frac{1}{2} \mu_\pm \leq \nu_{ij}^{\pm}(x) \leq \frac{3}{2} \mu_\pm, \quad |\nu_{ij}^{\pm}(x) - \mu_\pm| \leq M_1 \quad (x \in \mathbb{R}^N), \quad \| \nabla \nu_{ij}^{\pm} \|_{L_2(\mathbb{R}^N)} \leq C M_2, r
\]

with \( \mu_\pm \leq \mu_\pm \leq \mu_\pm \). In fact, \( \| \nabla \nu_{ij}^{\pm} \|_{L_2(\mathbb{R}^N)} \leq C (\| \mu_\pm - \mu_\pm \|_{L_2(\mathbb{R}^N)} + \| \nabla \mu_\pm \|_{L_2(\mathbb{R}^N)}) \), which, combined with the estimate:

\[
\| \mu_\pm - \mu_\pm \|_{L_2(\mathbb{R}^N)} = \int_{B_0^1(0)} \left| \mu_\pm(x) - \mu_\pm(x_j^i) \right|^2 \, dx
\]

\[
= \int_{B_0^1(0)} \int_0^1 \left| \nabla \mu_\pm(\theta x + x_j^i) \right|^2 \, d\theta \, dx \leq (d^4) \int_0^1 \left| \nabla \mu_\pm(\theta x + x_j^i) \right|^2 \, d\theta \leq (d^4) \| \nabla \mu_\pm \|_{L_2(\mathbb{R}^N)} \int_0^1 \frac{d\theta}{\theta^{N/2}}
\]

furnishes that \( \| \nabla \nu_{ij}^{\pm} \|_{L_2(\mathbb{R}^N)} \leq C M_2, r \). The condition (5.4) implies that \( \nu_{ij}^{\pm}(x) \) satisfy (4.3).

Set \( \mathcal{H}_j^5 = \Phi_j^1(\mathbb{R}^N), \mathcal{H}_j^5 = \mathcal{H}_j^5 \cup \mathcal{H}_j^{1, -} \cup \mathcal{H}_j^{1, +} \cup \mathcal{H}_j^3 \cup \mathcal{H}_j^4 \), \( \mathcal{H}_j^s = \Phi_j^2(\mathbb{R}^N), \mathcal{H}_j^s = \mathcal{H}_j^3 \cup \mathcal{H}_j^4 \cup \mathcal{H}_j^5, \mathcal{H}_j^s = \Phi_j^3(\mathbb{R}^N), \mathcal{H}_j^s = \mathcal{H}_j^3 \cup \mathcal{H}_j^4 \cup \mathcal{H}_j^5, \) and \( \mathcal{H}_j^s = \Phi_j^5(\mathbb{R}^N) \) in what follows. Let us define \( \nu_{ij}^{\pm}(x) \) and \( \rho_{ij}^{\pm}(x) \) by

\[
\nu_{ij}^{\pm}(x) = \begin{cases} \nu_{ij}^{+}(x), & i = 1, 2, 4, 5, \rho_{ij}^{+}(x) = \rho_+ + \chi_{\mathcal{H}_j^{1, +}}(x) \rho_- - \chi_{\mathcal{H}_j^{1, -}}(x), \quad i = 1, 2, 4, 5, \\
\nu_{ij}^{-}(x), & i = 3, 5, \rho_{ij}^{-}(x) = \rho_-, \quad i = 3, 5.
\end{cases}
\]

We see that, for \( i = 1, \ldots, 5 \) and \( j \in \mathbb{N} \),

\[
\nu_{ij}^{\pm}(x) = \mu(x) = \mu_+(x) \chi_{\mathcal{H}_j^{1, +}}(x) + \mu_-(x) \chi_{\mathcal{H}_j^{1, -}}(x), \quad \rho_{ij}^{\pm}(x) = \rho(x) = \rho_+ \chi_{\mathcal{H}_j^{1, +}}(x) + \rho_- \chi_{\mathcal{H}_j^{1, -}}(x), \quad x \in \text{supp } \zeta_j^i
\]

because \( \zeta_j^i = 1 \) on \( \text{supp } \zeta_j^i \). Moreover, we set \( T_j(u, \theta) = \nu_{ij}^{\pm}(x) \mathbf{D}(u) - \theta I \). Let \( (f, h, k) \in X_{R_2}(\bar{\Omega}) \). We consider the following problems:

\[
\begin{aligned}
\mathbf{u}_j^1 - (\rho_{ij}^{\pm})^{-1} \text{Div } T_j^1(u_j^1, K_j^1(u_j^1)) &= \zeta_j^i f \quad \text{in } \mathcal{H}_j^1, \\
[T_j^1(u_j^1, K_j^1(u_j^1)) n_j^1] &= \zeta_j^i h \quad \text{on } \Gamma_j^1, \\
[u_j^1] &= 0 \quad \text{on } \Gamma_j^1,
\end{aligned}
\]

and furthermore,

\[
\begin{aligned}
\mathbf{u}_j^2 - (\rho_{ij}^{\pm})^{-1} \text{Div } T_j^2(u_j^2, K_j^2(u_j^2)) &= \zeta_j^2 f \quad \text{in } \mathcal{H}_j^2, \\
T_j^2(u_j^2, K_j^2(u_j^2)) n_j^2 &= \zeta_j^2 k \quad \text{on } \Gamma_j^2, \\
\mathbf{u}_j^3 - (\rho_{ij}^{\pm})^{-1} \text{Div } T_j^3(u_j^3, K_j^3(u_j^3)) &= \zeta_j^3 f \quad \text{in } \mathcal{H}_j^3, \\
u_j^3 &= 0 \quad \text{on } \Gamma_j^3,
\end{aligned}
\]
\begin{align}
(5.9) \quad \lambda u^j_i - (\rho^j_i)^{-1} \text{Div} T^j_i(u^j_i, K^j_i(u^j_i)) &= \tilde{c}_i^j \mathbf{f} \quad \text{in} \ \mathcal{H}^j_i, \\
(5.10) \quad \lambda u^j_i - (\rho^j_i)^{-1} \text{Div} T^j_i(u^j_i, K^j_i(u^j_i)) &= \tilde{c}_i^j \mathbf{f} \quad \text{in} \ \mathcal{H}^j_i.
\end{align}

Here \( K^j_i(u^j_i) \) \((i = 1, \ldots, 5, j \in \mathbb{N})\) are given as follows: For \( u^j_i \in W^2_q(\mathcal{H}^j_i)^N, \ K^j_i(u^j_i) \in W^1_q(\mathcal{H}^j_i) + \tilde{W}^1_q(\mathcal{H}^j_i) \) denotes the unique solution to the weak problem:

\begin{align}
(5.11) \quad ((\rho^j_i)^{-1} \nabla K^j_i(u^j_i), \nabla \varphi)_{\mathcal{H}^j_i} &= ((\rho^j_i)^{-1} \text{Div}(\nu^j_i D(u^j_i))) - \nabla \text{div} u^j_i, \nabla \varphi)_{\mathcal{H}^j_i} \quad \text{for all} \ \varphi \in \tilde{W}^1_q(\mathcal{H}^j_i), \\
(5.12) \quad \| K^j_i(u^j_i) \| = \| \nu^j_i D(u^j_i) \|, \quad u^j_i > -\text{div} u^j_i \quad \text{on} \ \Gamma^j_i
\end{align}

with \( \| \nabla K^j_i(u^j_i) \| \leq C \| \nabla u^j_i \|_{L_4(\mathcal{H}^j_i)} \); For \( u^j_i \in W^2_q(\mathcal{H}^j_i)^N, \ K^j_i(u^j_i) \in W^1_q(\mathcal{H}^j_i) + \tilde{W}^1_q(\mathcal{H}^j_i) \) denotes the unique solution to the weak problem:

\begin{align}
((\rho^j_i)^{-1} \nabla K^j_i(u^j_i), \nabla \varphi)_{\mathcal{H}^j_i} &= ((\rho^j_i)^{-1} \text{Div}(\nu^j_i D(u^j_i))) - \nabla \text{div} u^j_i, \nabla \varphi)_{\mathcal{H}^j_i} \quad \text{for all} \ \varphi \in \tilde{W}^1_q(\mathcal{H}^j_i), \\
K^j_i(u^j_i) &= \nu^j_i D(u^j_i) n^j_i, \quad u^j_i > -\text{div} u^j_i \quad \text{on} \ \Gamma^j_i
\end{align}

with \( \| \nabla K^j_i(u^j_i) \| \leq C \| \nabla u^j_i \|_{L_4(\mathcal{H}^j_i)} \); For \( u^j_i \in W^2_q(\mathcal{H}^j_i)^N \) \((i = 3, 4, 5)\), \( K^j_i(u^j_i) \in \tilde{W}^1_q(\mathcal{H}^j_i) \) denotes the unique solution to the weak problem:

\begin{align}
((\rho^j_i)^{-1} \nabla K^j_i(u^j_i), \nabla \varphi)_{\mathcal{H}^j_i} &= ((\rho^j_i)^{-1} \text{Div}(\nu^j_i D(u^j_i))) - \nabla \text{div} u^j_i, \nabla \varphi)_{\mathcal{H}^j_i} \quad \text{for all} \ \varphi \in \tilde{W}^1_q(\mathcal{H}^j_i)
\end{align}

with \( \| \nabla K^j_i(u^j_i) \| \leq C \| \nabla u^j_i \|_{L_4(\mathcal{H}^j_i)} \).

We know that the following properties hold for the problems \((5.9)-(5.10)\). There exist a positive constant \( \lambda_0 \geq 1 \) and operator families \( S^j_i(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_0}, L(\mathcal{Z}^j_i(\mathcal{H}^j_i), W^2_q(\mathcal{H}^j_i)^N)) \) with
\[
\mathcal{Z}^j_i(\mathcal{H}^j_i) = \mathcal{Z}_{R,q}(\mathcal{H}^j_i) \quad (i = 1, 2), \quad \mathcal{Z}^j_i(\mathcal{H}^j_i) = L_q(\mathcal{H}^j_i)^N \quad (i = 3, 4, 5)
\]
such that, for any \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \),
\[
(5.13) \quad u^j_i = S^j_i(\lambda) H_{R,\lambda}(\tilde{c}_i^j f, \tilde{c}_i^j h), \quad u^j_i = S^j_i(\lambda) H_{R,\lambda}(\tilde{c}_i^j f, \tilde{c}_i^j k), \quad u^j_i = S^j_i(\lambda) \tilde{c}_i^j f \quad (i = 3, 4, 5)
\]
are unique solutions to \((5.9)-(5.10)\), respectively, where \( \mathcal{Z}_{R,q} \) and \( H_{R,\lambda} \) are given in Theorem 4.2. In addition,
\[
(5.14) \quad \mathcal{R}_{L(\mathcal{Z}^j_i(\mathcal{H}^j_i), L_q(\mathcal{H}^j_i)^N)} \left( \left\{ \left( \lambda \frac{d}{d\lambda} \right)^i \left( R_{\lambda} S^j_i(\lambda) \right) \right| \lambda \in \Sigma_{\varepsilon, \lambda_0} \right\} \right) \leq \gamma_4 \quad (l = 0, 1)
\]
with some positive constant \( \gamma_4 \) depending on \( \lambda_0 \), but independent of \( i = 1, \ldots, 5 \) and \( j \in \mathbb{N} \). Since the \( \mathcal{R} \)-boundedness implies the usual boundedness, we have, by \((5.13)\) and \((5.14)\) with \( l = 0, 1 \),
\[
(5.15) \quad \| R_{\lambda} u^j_i \|_{L_q(\mathcal{H}^j_i)} \leq \gamma_4 \left( \| (f, \nabla h, \lambda^{1/2} h) \|_{L_q(\Omega \cap B^j_i)} + \| h \|_{W^1_q(\Omega \cap B^j_i)} \right),
\]
\[
R_{\lambda} u^j_i \|_{L_q(\mathcal{H}^j_i)} \leq \gamma_4 \left( \| f \|_{L_q(\Omega \cap B^j_i)} + \| (\nabla k, \lambda^{1/2} k) \|_{L_q(\Omega \cap B^j_i)} + \| k \|_{W^1_q(\Omega \cap B^j_i)} \right),
\]
\[
R_{\lambda} u^j_i \|_{L_q(\mathcal{H}^j_i)} \leq \gamma_4 \| f \|_{L_q(\Omega \cap B^j_i)} \quad (i = 3, 4, 5)
\]
for any \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \), noting \( |\lambda|^{-1/2} \leq \lambda_0^{-1/2} \).

5.3. Construction of parametrices. For \( (f, h, k) \in X_{R,q}(\mathcal{O}) \), we consider the two-phase reduced Stokes equations \((2.3)\). By Lemma 5.3 together with \((5.2)-(5.13)\), we see that the infinite sum \( \sum_{i=1}^5 \sum_{j=1}^\infty \tilde{c}_i^j u^j_i \) exists in the strong topology of \( W^2_q(\mathcal{O})^N \), so that we define \( u \) by
\[
(5.16) \quad u = \sum_{i=1}^5 \sum_{j=1}^\infty \tilde{c}_i^j u^j_i \quad \text{in} \ W^2_q(\mathcal{O})^N.
\]

The existence of \( \mathcal{R} \)-bounded solution operator families \( S^j_i(\lambda), S^j_i(\lambda), S^j_i(\lambda) \) below follow from Theorem 4.2 and 20 Theorem 4.1, Theorem 4.4], respectively. In addition, concerning \( S^j_i(\lambda), S^j_i(\lambda), S^j_i(\lambda) \), we can construct such \( \mathcal{R} \)-bounded solution operator families with variable viscosities in \( \mathbb{R}^N \) under the same condition as \((4.3)\) by using Theorem 5.1 similarly to Section 4.
Then, noting (5.5), \( n = n_1^j \) on \( \text{supp } \zeta^j_1 \cap \Gamma \), and \( n_+ = n_2^j \) on \( \text{supp } \zeta^j_2 \cap \Gamma_+ \), we have

\[
\begin{cases}
\lambda u - \rho^{-1} \text{Div } T(u, K(u)) = f - U^0(\lambda)(f, h, k) \quad \text{in } \hat{\Omega}, \\
[T(u, K(u))n] = [h] - [\text{Div}(\lambda^j_1 D(u_j^i))] \quad \text{on } \Gamma, \\
u[0] = 0 \quad \text{on } \Gamma, \\
T(u, K(u))n_+ = k - U^2(\lambda)(f, h, k) \quad \text{on } \Gamma_+,
\end{cases}
\]

where we have set

\[
U^i(\lambda)(f, h, k) = V^i(\lambda)(f, h, k) + P^i(\lambda)(f, h, k) \quad (i = 0, 1, 2),
\]

\[
V^0(\lambda)(f, h, k) = \sum_{i=1}^{5} \sum_{j=1}^{\infty} (\rho_j^i)^{-1} \left\{ \zeta_j^i \text{Div}(\lambda^j_1 D(u_j^i)) - \text{Div}(\lambda^j_1 D(\zeta_j^i u_j^i)) \right\},
\]

\[
V^i(\lambda)(f, h, k) = \sum_{j=1}^{\infty} \left\{ \lambda^j_1 D(\zeta^j_1 u_j^i)n_j^i - \zeta^j_1 \rho_j^i D(u_j^i)n_j^i \right\} \quad (i = 1, 2),
\]

\[
P^0(\lambda)(f, h, k) = \sum_{i=1}^{5} \sum_{j=1}^{\infty} (\rho_j^i)^{-1} \left\{ \nabla K(\zeta_j^i u_j^i) - \zeta_j^i \nabla K_j(u_j^i) \right\},
\]

\[
P^i(\lambda)(f, h, k) = \sum_{j=1}^{\infty} \left\{ \zeta_j^i K_j(u_j^i)n_j^i - K(\zeta_j^i u_j^i)n_j^i \right\} \quad (i = 1, 2).
\]

Here we have used the fact that

\[
\nabla K(u) = \sum_{i=1}^{5} \sum_{j=1}^{\infty} \nabla K(\zeta^j_1 u_j^i) \quad \text{in } L_q(\hat{\Omega})^N, \quad K(u) = \sum_{j=1}^{\infty} K(\zeta^j_1 u_j^i) \quad \text{in } W_q^{-1/q}(\Gamma^i) \quad (i = 1, 2).
\]

In fact, we have the following observation: In view of Subsection 2.1 we see by (5.5) that

\[
K(\zeta^j_1 u_j^i) = K((\rho_j^i)^{-1} \text{Div}(\lambda^j_1 D(\zeta^j_1 u_j^i)) - \nabla \text{div}(\zeta^j_1 u_j^i), [g_j^i], h_j^i|_{\Gamma^i}) ,
\]

where \( \cdot |_{\Gamma^i} \) denotes the trace to \( \Gamma^i \) and

\[
(g_j^1, h_j^1) = (\langle \rho_j^i \text{Div}(\zeta_j^i u_j^i)n_j^1, n_j^2 > - \text{div}(\zeta_j^i u_j^i), 0),
\]

\[
(g_j^2, h_j^2) = (0, \langle \rho_j^i \text{Div}(\zeta_j^i u_j^i)n_j^2, n_j^2 > - \text{div}(\zeta_j^i u_j^i)), \quad g_j^i = h_j^i = 0 \quad (i = 3, 4, 5).
\]

On the other hand, by (5.5) and (5.10),

\[
\rho^{-1} \text{Div}(\lambda D(u)) - \nabla \text{div } u = \sum_{i=1}^{5} \sum_{j=1}^{\infty} (\rho_j^i)^{-1} \left\{ \zeta_j^i \text{Div}(\zeta_j^i u_j^i)) - \nabla \text{div}(\zeta^j_1 u_j^i) \right\} \quad \text{in } L_q(\hat{\Omega})^N,
\]

\[
< \mu D(u)n_j^i, n > - \text{[div } u] = \sum_{j=1}^{\infty} \left( < \rho_j^i \text{Div}(\zeta_j^i u_j^i)\n_j^3, n_j^3 > - \text{[div}(\zeta_j^i u_j^i) \right) \quad \text{in } W_q^{-1/q}(\Gamma),
\]

\[
< \mu D(u)n_+ n_+ > - \text{div } u = \sum_{j=1}^{\infty} \left( < \rho_j^i \text{Div}(\zeta_j^i u_j^i)\n_j^3, n_j^3 > - \text{div}(\zeta_j^i u_j^i) \right) \quad \text{in } W_q^{-1/q}(\Gamma_+).
\]

Thus the continuity of \( K \) implies (5.18) and \( K(u) = K(\alpha, \beta, \gamma) \) with \( (\alpha, \beta, \gamma) \) given by (5.2).

Now it holds that, by (5.13),

\[
u = \sum_{j=1}^{\infty} \zeta_j^1 S_{j}^1(\lambda) \left( \zeta_j^1 f, \zeta_j^1 (\nabla h) + \lambda^{-1/2}(\nabla \zeta_j^1) (\zeta_j^1)^1 h, \zeta_j^1 (\lambda^{1/2} h), \zeta_j^1 h \right)
\]

\[
+ \sum_{j=1}^{\infty} \zeta_j^2 S_{j}^2(\lambda) \left( \zeta_j^2 f, \zeta_j^2 (\nabla k) + \lambda^{-1/2}(\nabla \zeta_j^2) (\zeta_j^2)^1 k, \zeta_j^2 (\lambda^{1/2} k), \zeta_j^2 k \right) + \sum_{i=3}^{5} \sum_{j=1}^{\infty} \zeta_j^i S_{j}^i(\lambda) \left( \zeta_j^i f \right),
\]
so that we set, by \( H = (H_1, \ldots, H_7) \in \mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}) \),

\[
\begin{aligned}
S_j^1(\lambda)(\lambda) &= S_j^1(\lambda)(\tilde{C}_j^1 H_1, \tilde{C}_j^2 H_2 + \lambda^{-1/2}(\nabla \tilde{C}_j^1) H_3, \tilde{C}_j^2 H_3, \tilde{C}_j^2 H_4), \\
S_j^2(\lambda)(\lambda) &= S_j^2(\lambda)(\tilde{C}_j^2 H_1, \tilde{C}_j^2 H_5 + \lambda^{-1/2}(\nabla \tilde{C}_j^2) H_6, \tilde{C}_j^2 H_6, \tilde{C}_j^2 H_7), \\
S_j^3(\lambda)(\lambda) &= S_j^3(\lambda)(\tilde{C}_j^1 H_1) \quad (i = 3, 4, 5).
\end{aligned}
\]  

(5.19) 

It then clear that \( u = \sum_{j=1}^5 \zeta_j S_j(\lambda)(f, h, k) \). By (5.14) with Definition (1.2) it holds that

\[
\int_0^1 \left\| \sum_{j=1}^n r_j(u)R_{\lambda j}(\zeta_j S_j(\lambda)(\lambda)H_1) \right\|^q_{L_q(\Omega)} du \leq \gamma_4 \int_0^1 \left\| \sum_{j=1}^n r_j(u)H_j \right\|^q_{X_{\mathcal{R}, q}(\Omega \cap B_j)} du
\]

for any \( n \in \mathbb{N} \), \( \{\zeta_j\}_{j=1}^n \subset \Sigma_{\gamma, \lambda}, \) and \( \{H_j\}_{j=1}^n \subset \mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}) \). The inequality (5.20) with \( n = 1 \), together with Lemma (5.23) yields that the infinite sum \( \sum_{j=1}^\infty \zeta_j S_j(\lambda)(\lambda)H \) exists in the strong topology of \( W_q^2(\tilde{\Omega})^N \), so that we define \( T^i(\lambda)H = \sum_{j=1}^\infty \zeta_j S_j(\lambda)(\lambda)H \) for \( i = 1, \ldots, 5 \). In addition, by Lemma (5.2)

\[
\left\| \sum_{j=1}^\infty a_j R_{\lambda j} T^i(\lambda)(\lambda)H \right\|^q_{L_q(\Omega)} \leq \gamma_4 \sum_{j=1}^\infty \left\| \sum_{j=1}^n a_j R_{\lambda j} \left( \zeta_j S_j(\lambda)(\lambda)H \right) \right\|^q_{L_q(\Omega)} \quad (i = 1, \ldots, 5)
\]

for any \( n \in \mathbb{N}, \{a_j\}_{j=1}^n \subset \mathbb{C}, \{\zeta_j\}_{j=1}^n \subset \Sigma_{\gamma, \lambda}, \) and \( \{H_j\}_{j=1}^n \subset \mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}) \). The last inequality combined with (5.20), (5.23) furnishes that, by the monotone convergence theorem,

\[
\int_0^1 \left\| \sum_{j=1}^\infty r_j(u)R_{\lambda j} T^i(\lambda)(\lambda)H \right\|^q_{L_q(\Omega)} du \leq \gamma_4 \int_0^1 \left\| \sum_{j=1}^n r_j(u)H_j \right\|^q_{X_{\mathcal{R}, q}(\Omega \cap B_j)} du \leq \gamma_4 \int_0^1 \left\| \sum_{j=1}^n r_j(u)H_j \right\|^q_{X_{\mathcal{R}, q}(\tilde{\Omega})} du,
\]

which implies that, for \( i = 1, \ldots, 5 \),

\[
T^i(\lambda) \in \text{Hol}(\Sigma_{\gamma, \lambda}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), W_q^2(\tilde{\Omega})^N)) \quad \mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), L_q(\tilde{\Omega}))^N} \left\{ R_{\lambda j} T^i(\lambda) \right\} \leq \gamma_4.
\]

Analogously, we have the \( \mathcal{R} \)-boundedness of \( \{((\lambda d/da)(R_{\lambda j} T^i(\lambda))) \mid \lambda \in \Sigma_{\gamma, \lambda} \} \) on \( \mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), L_q(\tilde{\Omega})^N) \). Thus, setting \( S(\lambda)H = \sum_{j=1}^5 T^i(\lambda)H \) yields that, by Proposition (2.4)

\[
S(\lambda) \in \text{Hol}(\Sigma_{\gamma, \lambda}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), W_q^2(\tilde{\Omega})^N)), \quad u = S(\lambda)(f, h, k),
\]

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), L_q(\tilde{\Omega}))^N} \left\{ \left( \int \frac{d}{da} \right)^l R_{\lambda j} S(\lambda) \right\} \leq \gamma_4 \quad (l = 0, 1, 2).
\]

5.4. Estimates of the remainder terms \( \mathcal{U}^i(\lambda)(f, h, k) \). In this subsection, we prove the following lemma.

Lemma 5.5. Let \( \lambda_0 \) and \( \gamma_4 \) be the same numbers as in (5.14). Let \( \mathcal{U}^i(\lambda), \mathcal{V}^i(\lambda), \) and \( \mathcal{P}^i(\lambda) \) \( (i = 0, 1, 2) \) be the operators defined in (5.17) and set

\[
\begin{aligned}
\mathcal{U}(\lambda)(f, h, k) &= \mathcal{V}(\lambda)(f, h, k) + \mathcal{P}(\lambda)(f, h, k), \\
\mathcal{V}(\lambda)(f, h, k) &= (\mathcal{V}^0(\lambda)(f, h, k), \mathcal{V}^1(\lambda)(f, h, k), \mathcal{V}^2(\lambda)(f, h, k)), \\
\mathcal{P}(\lambda)(f, h, k) &= (\mathcal{P}^0(\lambda)(f, h, k), \mathcal{P}^1(\lambda)(f, h, k), \mathcal{P}^2(\lambda)(f, h, k)).
\end{aligned}
\]

Then there exists an operator family \( \mathcal{U}(\lambda) \in \text{Hol}(\Sigma_{\gamma, \lambda}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega})) \) such that

\[
\begin{aligned}
\mathcal{U}(\lambda)(f, h, k) &= \mathcal{U}(\lambda)(f, h, k) \quad \text{for } (f, h, k) \in \mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}), \\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, q}(\tilde{\Omega}))^N} \left\{ \left( \int \frac{d}{da} \right)^l R_{\lambda j} \mathcal{U}(\lambda) \right\} \leq \gamma_4 (\sigma_2 + \gamma_2 \sigma_1 + \gamma_1 \gamma_2 \lambda_1^{1/2}) \quad (l = 0, 1, 2)
\end{aligned}
\]

\( ^\dagger \) As was mentioned in Remark (2.28), the symbols \( H_1, H_2, H_3, H_4, H_5, H_6, \) and \( H_7 \) are variables corresponding to \( f, \nabla h, \lambda^{1/2}h, h, \nabla k, \lambda^{1/2}k, \) and \( k \), respectively.

\( ^\ddagger \) Holomorphic property can be proved in the same manner as in (25) Proposition 5.3 (ii).
for any $\sigma_1, \sigma_2 > 0$ and for any $\lambda_1 \geq \lambda_0$. Here and subsequently, $\gamma_{\sigma_1}, \gamma_{\sigma_2}$ are positive constants depending on $\sigma_1, \sigma_2$, respectively.

**Proof. Step 1: Case $V^i(\lambda)$.** First we consider $V^0(\lambda)(f, h, k)$. We write $\text{Div}(\mu D(\varphi u)) = \varphi \text{Div}(\mu D(u)) = C_1(\mu, \varphi)\nabla u + C_0(\mu, \varphi)u$ for any scalar functions $\mu, \varphi$ and for any $N$-vector function $u$, where we have set

$$C_0(\mu, \varphi)u = <\nabla u, \mu > - \nabla \varphi + <\nabla \mu, \varphi > u + \mu \{(\nabla^2 \varphi)u + (\Delta \varphi)u\},$$

$$C_1(\mu, \varphi)\nabla u = \mu \{D(u)\nabla \varphi + (\nabla \varphi)\text{div}u + (\nabla u)\nabla \varphi\}.$$ 

Using the above symbols $C_0, C_1$ and \((5.19)\), we write

$$\text{Div}(\nu_j^i D(\zeta_j^i u_j^i)) - \zeta_j^i \text{Div}(\nu_j^i D(u_j^i)) = C_1(\nu_j^i, \zeta_j^i)\nabla S_j^i(\lambda)F_{R, \lambda}(f, h, k) + C_0(\nu_j^i, \zeta_j^i)S_j^i(\lambda)F_{R, \lambda}(f, h, k)$$

for $i = 1, \ldots, 5$ and $j \in \mathbb{N}$. By \((5.4)\) and Proposition \((2.6)\) with $\sigma = 1$, we have, for $H \in H_{R, q}(\tilde{\Omega})$,

$$\|C_0(\nu_j^i, \zeta_j^i)S_j^i(\lambda)H\|_{L_q(\tilde{\Omega})} \leq C\|S_j^i(\lambda)H\|_{W^4_q(\tilde{\Omega})}, \quad \|C_1(\nu_j^i, \zeta_j^i)\nabla S_j^i(\lambda)H\|_{L_q(\tilde{\Omega})} \leq C\|\nabla S_j^i(\lambda)H\|_{L_q(\tilde{\Omega})},$$

which, combined with \((5.14)\) and Proposition \((2.5)\) furnishes that

$$\int_0^1 \left\| \sum_{i=1}^n r_i(u)C_1(\nu_j^i, \zeta_j^i)\nabla S_j^i(\lambda)H \right\|_{L_q(\tilde{\Omega})}^q du \leq \left(\gamma \lambda_1^{-1/2}\right)^q \int_0^1 \left\| \sum_{i=1}^n r_i(u)H \right\|_{X_{R, q}(\Omega \cap B_{1/2}^i)}^q du,$$

for any $\lambda_1 \geq \lambda_0$ and for any $n \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^n \subset \Sigma_{\epsilon, \lambda_1}$, and \(\{H_i\}_{i=1}^n \subset H_{R, q}(\tilde{\Omega})\). Define $V^0(\lambda)H$ as

$$V^0(\lambda)H = \sum_{i=1}^n \sum_{j=1}^5 \left\{ C_1(\nu_j^i, \zeta_j^i)\nabla S_j^i(\lambda)H + C_0(\nu_j^i, \zeta_j^i)S_j^i(\lambda)H \right\} \quad \text{with } H \in H_{R, q}(\tilde{\Omega}).$$

In the same manner as we have obtained \((5.21)\) from \((5.20)\), we can prove, by \((5.22)\), the following properties:

$$V^0(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_1}, L(\mathbb{R}^{Nq}(\tilde{\Omega}), L_q(\tilde{\Omega}))) \cap \text{Hol}(\Sigma_{\epsilon, \lambda_1}, L_q(\tilde{\Omega}))$$

for any $\lambda_1 \geq \lambda_0$. Here and therefore, $\lambda_1$ denotes any number satisfying $\lambda_1 \geq \lambda_0$. Analogously, we can construct operator families $V^i(\lambda)$ \((i = 1, 2)\) such that

$$V^i(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_1}, L(\mathbb{R}^{Nq}(\tilde{\Omega}), W^1_q(\tilde{\Omega}))) \cap \text{Hol}(\Sigma_{\epsilon, \lambda_1}, L_q(\tilde{\Omega}))$$

for any $\lambda_1 \geq \lambda_0$. Here and therefore, we set

$$\tilde{X}_{R, q}(\Omega) = L_q(\tilde{\Omega})^{N^2} \times L_q(\tilde{\Omega})^N \times W^1_q(\tilde{\Omega})^N, \quad \tilde{F}_{R, \lambda}u = (\nabla u, \chi^{1/2}u, u).$$

**Step 2: Case $P^i(\lambda)$.** We consider the term:

$$\rho_j^i \{\nabla K(\zeta_j^i u_j^i) - \zeta_j^i \nabla K(u_j^i)\} = \rho_j^i \nabla (K(\zeta_j^i u_j^i) - \zeta_j^i K_j^i(u_j^i)) + (\rho_j^i)^{-1}(\nabla \zeta_j^i)K_j^i(u_j^i).$$

We start with the following inequalities of Poincaré type with uniform constant, which are proved in the same manner as in the proof of \([25]\) Lemma 3.4, Lemma 3.5].

**Lemma 5.6.** Let $1 < q < \infty$. Then there exists a constant $c_1 > 0$, independent of $j \in \mathbb{N}$, such that

$$\|\varphi - c_1^j(\varphi)\|_{W^1_q(\partial \Omega \cap B_j^i)} \leq c_1 \|\nabla \varphi\|_{L_q(\partial \Omega \cap B_j^i)} \quad \text{for any } \varphi \in W^1_q(\partial \Omega),$$

$$\|\psi - c_1^j(\psi)\|_{W^1_q(\partial \Omega \cap B_j^i)} \leq c_1 \|\nabla \psi\|_{L_q(\partial \Omega \cap B_j^i)} \quad \text{for any } \psi \in W^1_q(\Omega),$$

$$\|\varphi\|_{W^1_q(\partial \Omega \cap B_j^i)} \leq c_1 \|\nabla \varphi\|_{L_q(\partial \Omega \cap B_j^i)} \quad \text{for any } \varphi \in W^1_q(\Omega),$$

$$\|\psi\|_{W^1_q(\partial \Omega \cap B_j^i)} \leq c_1 \|\nabla \psi\|_{L_q(\partial \Omega \cap B_j^i)} \quad \text{for any } \psi \in W^1_q(\Omega), \quad i = 3, 4, 5.$$

Here $c^j_1(\varphi)$ and $c^j_1(\psi)$ \((i = 1, 3, 4, 5)\) are suitable constants depending on $\varphi$ and $\psi$, respectively.
To handle \((\rho_j^i)^{-1}(\nabla \zeta_j^i)K_j^i(u_j^i)\), we use the following lemma.

**Lemma 5.7.** Let \(1 < q < \infty\). Then there exists a constant \(c_2\), independent of \(j \in \mathbb{N}\), such that

\[
\|K_j^i(u)\|_{L_q(\Omega_j^i \cap B_{\delta^i})} \leq c_2 \left( \|\nabla u\|_{L_q(\Omega_j^i)} + \|\nabla u\|_{L_q(\Omega_j^i \cap B_{\delta^i})} \right)
\]

for any \(u \in W^2_q(\Omega_j^i)^N\) and for any \(v \in W^2_q(\Omega_j^i)^N\) (\(i = 2, \ldots, 5\)), respectively, where \(\delta^i\) is symbols defined by \(\delta^i = 1\) (\(i = 2, 3\)) and \(\delta^i = 0\) (\(i = 4, 5\)).

**Remark 5.8.** Applying Young’s inequality to \((5.25)\), we have

\[
\|K_j^i(u)\|_{L_q(\Omega_j^i \cap B_{\delta^i})} \leq \sigma_1 \|\nabla u\|_{L_q(\Omega_j^i)} + \gamma_1 \|\nabla u\|_{L_q(\Omega_j^i)} \quad (i = 1, \ldots, 5)
\]

for any \(\sigma_1 > 0\) and for any \(u \in W^2_q(\Omega_j^i)^N\).

**Proof of Lemma 5.7.** We here show the case \(K_j^i(u)\). In the following, \(C\) stands for generic constants independent of \(j \in \mathbb{N}\), and recall that \(\mathcal{H}_j^0 = \Phi_j^1(R^N) = R^N\), \(\mathcal{H}_j^1 = \mathcal{H}_{1,j}^1 \cup \mathcal{H}_{1,j}^2\) (\(\mathcal{H}_{3,j}^1 = \Phi_j^1(R^N)\), respectively), and \(\Gamma_j^1 = \Phi_j^1(R^N)\).

Let \(\eta_j \in C_0^\infty(\mathcal{H}_j^0 \cap B_{\delta^i})\) in such a way that \(\int_{\Omega_j^i} \eta_j \, dx = 1\) and \(\eta_j \geq 0\). Fix \(u \in W^2_q(\mathcal{H}_j^1)^N\) in what follows. Since \(K_j^1(u)\) satisfies the weak problem \((5.10)-(5.12)\) for any constant \(c\), we may assume that \(\int_{\Omega_j^i} \eta_j K_j^1(u) \, dx = 0\).

Given \(\psi \in C_0^\infty(\mathcal{H}_j^0 \cap B_{\delta^i})\), we define a function by \(\tilde{\psi} = \psi - \eta_j \int_{\Omega_j^i} \psi \, dx\). Then,

\[
\tilde{\psi} \in C_0^\infty(\mathcal{H}_j^0 \cap B_{\delta^i}), \quad \|\tilde{\psi}\|_{L_q(\mathcal{H}_j^0)} \leq C\|\psi\|_{L_q(\mathcal{H}_j^0)}, \quad \int_{\Omega_j^i \cap B_{\delta^i}} \tilde{\psi} \, dx = 0
\]

for \(q' = q/(q - 1)\). These properties combined with Lemma 5.7 yields that

\[
|\tilde{\psi}(\varphi)|_{\mathcal{H}_j^0} = |\tilde{\psi}(\varphi - c_j^i(\varphi))|_{\mathcal{H}_j^0} \leq \|\tilde{\psi}\|_{L_q(\mathcal{H}_j^0)} \varphi - c_j^i(\varphi)\|_{L_q(\mathcal{H}_j^0 \cap B_{\delta^i})} \leq C\|\psi\|_{L_q(\mathcal{H}_j^0)} \|\varphi\|_{L_q(\mathcal{H}_j^0)}
\]

for any \(\varphi \in W^2_q(\mathcal{H}_j^0)^N\). Thus \(\|\tilde{\psi}\|_{W^{-1}_q(\mathcal{H}_j^0)} \leq C\|\psi\|_{L_q(\mathcal{H}_j^0)}\), where \(W^{-1}_q(\mathcal{H}_j^0)\) is the dual spaces of \(W^2_q(\mathcal{H}_j^0)^N\).

Let \(W^2_q(\mathcal{H}_{1,j}^1)^N\) be function spaces defined as \(W^2_q(\mathcal{H}_{1,j}^1)^N = \{\theta \in W^2_q(\mathcal{H}_{1,j}^1)^N \mid \nabla \theta \in W^2_q(\mathcal{H}_{1,j}^1)^N\}\), respectively. We choose a \(\Psi \in W^2_q(\mathcal{H}_{1,j}^1)^N \cap W^2_q(\mathcal{H}_{1,j}^1)^N\) satisfying the following equations:

\[
-\Delta \Psi = \tilde{\psi} \quad \text{in } \mathcal{H}_j^1, \quad \left[\frac{\partial \Psi}{\partial n_j}\right] = 0 \quad \text{on } \Gamma_j^1, \quad [\rho_j^1 \Psi] = 0 \quad \text{on } \Gamma_j^2
\]

and the estimate: \(\|\nabla \Psi\|_{W^2_q(\mathcal{H}_j^1)^N} \leq C(\|\tilde{\psi}\|_{L_q(\mathcal{H}_j^0)} + \|\tilde{\psi}\|_{W^{-1}_q(\mathcal{H}_j^0)})\). Then the estimates of \(\tilde{\psi}\) above yields that

\[
\|\nabla \Psi\|_{W^2_q(\mathcal{H}_j^1)^N} \leq C\|\psi\|_{L_q(\mathcal{H}_j^0)}\]

and furthermore, by Gauss’s divergence theorem,

\[
(\nabla \Psi, \nabla \theta)_{\mathcal{H}_j^1} - \left(\frac{\partial \Psi}{\partial n_j}, [\theta]\right)_{\Gamma_j^1} = (\tilde{\psi}, \theta)_{\mathcal{H}_j^1} \quad \text{for any } \theta \in W^1_q(\mathcal{H}_j^1) + W^{-1}_q(\mathcal{H}_j^1)^N.
\]

This identity allows us to see that

\[
(K_j^1(u), \psi)_{\mathcal{H}_j^0} = (K_j^1(u), \tilde{\psi})_{\mathcal{H}_j^0} = (K_j^1(u), \psi)_{\mathcal{H}_j^0} = (\nabla K_j^1(u), \nabla \Psi)_{\mathcal{H}_j^1} - \left(\frac{\partial \Psi}{\partial n_j}\right)_{\Gamma_j^1}
\]
Here and hereafter, for the sake of simplicity, we write $\sigma$, which can be proved by Proposition 5.1 and \[13, \text{Section 4, Proposition 16.2}\]. These inequalities combined

$$d\sigma$$

which, combined with

Thus, by Gauss’s divergence theorem, we have

$$(5.28) \quad (K_j^1(u), \psi)_{\mathcal{H}_j^0} = -\left(\nu_j^1 D(u), \nabla^2 \psi\right)_{\mathcal{H}_j^0} + \int_{\Gamma_j^1} \left(\nabla \cdot D(u)n_j^1\right) d\sigma - \left(\nabla \cdot D(u)n_j^1\right)_{\Gamma_j^1},$$

where $d\sigma$ denotes the surface element of $\Gamma_j^1$.

At this point, we introduce trace inequalities as follows: there exists a positive constant $c_3$, independent of $j \in \mathbb{N}$, such that

$$(5.29) \quad \|f \|_{L_q(\Omega)} \leq c_3 \|f \|_{L_q(\mathcal{H}_j^1)} \quad \text{for any } f \in W_q^1(\mathcal{H}_j^1),$$

which can be proved by Proposition 5.1 and \[13, \text{Section 4, Proposition 16.2}\]. These inequalities combined with (5.28) furnish that

$$|\langle K_j^1(u) \psi \rangle_{\mathcal{H}_j^0}| \leq C \left(\|\nabla u\|_{L_q(\mathcal{H}_j^1)} + \|\nabla u\|_{L_q(\mathcal{H}_j^1)}^{1/q} \|\nabla u\|_{L_q(\mathcal{H}_j^1)}^{1/q} \right) \|\psi\|_{L_q(\mathcal{H}_j^0)},$$

which implies that the required estimate (5.25) holds. This completes the proof of the lemma.

We consider $(\rho_j)^{-1}(\nabla \zeta_j^i)K_j^1(u_j^1)$. By Definition (5.25) and (5.14), we have, for any $n \in \mathbb{N}$, \{\lambda_l\}_{l=1}^n \subset \Sigma_{e,\lambda_1}$, and \{\mathcal{H}_j\}_{j=1}^n \subset \mathcal{X}_{\mathcal{R},q}(\hat{\Omega}),

$$(5.30) \quad \int_0^1 \left\| \sum_{i=1}^n r_l(u)\zeta_j^i \right\|_{L_q(\mathcal{H}_j^0)}^q \leq \{\gamma_4(\sigma_1 + \gamma_\sigma, \lambda_1^{-1/2})\} \int_0^1 \left\| \sum_{i=1}^n r_l(u)\right\|_{\mathcal{X}_{\mathcal{R},q}(\hat{\Omega} \cap B_j)}^q \quad \text{for any } \sigma_1 > 0 \text{ and any } \lambda_1 \geq \lambda_0.$$
In addition, for any \( \varphi \in W_q^1(\Omega) \), we have

\[
((\rho_j')^{-1}\nabla(K(\zeta_j'\mathbf{u}_j') - \zeta_j'K(\mathbf{u}_j')), \nabla\varphi)_\Omega = ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

\[
- ((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega - ((\rho_j')^{-1}\zeta_j'\nabla K_j(\mathbf{u}_j'), \nabla(\varphi - c_3(\varphi)))_\Omega
\]

\[
= ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega - ((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

\[
+ ((\rho_j')^{-1}\nabla K_j(\mathbf{u}_j'), (\nabla\zeta_j')(\varphi - c_3(\varphi)))_\Omega - ((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

\[
= ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega - ((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

\[
+ ((\rho_j')^{-1}\nabla K_j(\mathbf{u}_j'), (\nabla\zeta_j')(\varphi - c_3(\varphi)))_\Omega - ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

where \( c_3(\varphi) \) are constants given in Lemma 5.6 for \( i = 1, 3, 4, 5 \) and \( c_3(\varphi) = 0 \). Let \( W_q^{-1}(\Omega) \) be the dual space of \( W_q^1(\Omega) \) and \( < \cdot, \cdot > \) denote the duality pairing between \( W_q^{-1}(\Omega) \) and \( W_q^1(\Omega) \). Thus, if we define \( I_j' \in W_q^{-1}(\Omega) \) by

\[
< I_j', \varphi >_\Omega = ((\rho_j')^{-1}C_1(\rho_j', \zeta_j')\nabla\mathbf{u}_j' + C_0(\rho_j', \zeta_j')\nabla\varphi)_\Omega - (\nabla(\nabla\zeta_j') \cdot \mathbf{u}_j' + (\nabla\zeta_j') \nabla\mathbf{u}_j', \nabla\varphi)_\Omega
\]

\[
- 2((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega - ((\rho_j')^{-1}(\Delta\zeta_j')K_j(\mathbf{u}_j'), \varphi - c_3(\varphi))_\Omega
\]

\[
+ ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega - ((\rho_j')^{-1}(\nabla\zeta_j')K_j(\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

\[
+ ((\rho_j')^{-1}\nabla K_j(\mathbf{u}_j'), (\nabla\zeta_j')(\varphi - c_3(\varphi)))_\Omega - ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(\zeta_j'\mathbf{u}_j')) - \nabla \text{div}(\zeta_j'\mathbf{u}_j'), \nabla\varphi)_\Omega
\]

then we have, for \( i = 1, \ldots, 5 \),

\[
((\rho_j')^{-1}\nabla(K(\zeta_j'\mathbf{u}_j') - \zeta_j'K(\mathbf{u}_j')), \nabla\varphi)_\Omega = < I_j', \varphi >_\Omega \quad \text{for all} \quad \varphi \in W_q^1(\Omega).
\]

Let \( \mathbf{F} \) be an element of \( L(W_q^{-1}(\Omega), L_q(\Omega))^N \) such that, for \( \theta \in W_q^{-1}(\Omega) \),

\[
< \theta, \varphi >_\Omega = < \mathbf{F}(\theta), \nabla\varphi >_\Omega \quad \text{for all} \quad \varphi \in W_q^1(\Omega), \quad ||\mathbf{F}(\theta)||_{L_q(\Omega)} = ||\theta||_{W_q^{-1}(\Omega)}.
\]

Such a \( \mathbf{F} \) can be constructed by the Hahn-Banach theorem. Since it holds that \( ((\rho_j')^{-1}\nabla(K(\zeta_j'\mathbf{u}_j') - \zeta_j'K(\mathbf{u}_j')), \nabla\varphi)_\Omega = (\mathbf{F}(I_j'), \nabla\varphi)_\Omega \), we see that \( \nabla(K(\zeta_j'\mathbf{u}_j') - \zeta_j'K(\mathbf{u}_j')) \) is given by the following formula:

\[
(5.32) \quad \nabla(K(\zeta_j'\mathbf{u}_j') - \zeta_j'K(\mathbf{u}_j')) = \nabla \mathbf{K}(\mathbf{F}(I_j'), [\mathbf{g}_j'], [\mathbf{f}_j'], r^j).
\]

To see the \( \mathcal{R} \)-boundedness, for \( i = 1, \ldots, 5 \), we define operators \( I_j'(\lambda) \) by

\[
< I_j'(\lambda)\mathbf{H}, \varphi >_\Omega = ((\rho_j')^{-1}C_1(\rho_j', \zeta_j')\nabla(S_j'(\lambda)\mathbf{H}) + C_0(\rho_j', \zeta_j')S_j'(\lambda)\mathbf{H}, \nabla\varphi)_\Omega
\]

\[
- (\nabla(\nabla\zeta_j') \cdot S_j'(\lambda)\mathbf{H}) + (\nabla\zeta_j') \text{div}(S_j'(\lambda)\mathbf{H}), \nabla\varphi)_\Omega - 2((\rho_j')^{-1}(\nabla\zeta_j')K_j(S_j'(\lambda)\mathbf{H}), \nabla\varphi)_\Omega
\]

\[
- ((\rho_j')^{-1}(\Delta\zeta_j')K_j(S_j'(\lambda)\mathbf{H}), \varphi - c_3(\varphi))_\Omega + ((\rho_j')^{-1}\text{Div}(\rho_j'\mathbf{D}(S_j'(\lambda)\mathbf{H})), \nabla(\nabla\zeta_j')(\varphi - c_3(\varphi)))_\Omega
\]

\[
- (\text{div}(S_j'(\lambda)\mathbf{H}), \nabla(\nabla\zeta_j')(\varphi - c_3(\varphi)))_\Omega + [\mathbf{E}]_j(S_j'(\lambda)\mathbf{H}), \varphi
\]

for any \( \mathbf{H} \in \mathcal{X}_{\mathcal{R}_q}(\bar{\Omega}) \) and for any \( \varphi \in W_q^1(\Omega) \). In addition, we define operators \( J_j'(\lambda) \) by

\[
J_j'(\lambda)\mathbf{H} = < \nu_j'(\mathbf{D}(\nabla\zeta_j')(S_j'(\lambda)\mathbf{H}))\mathbf{n}_j', \mathbf{n}_j' > - \mathcal{E}(\nabla\zeta_j')(S_j'(\lambda)\mathbf{H}) \quad (i = 1, 2).
\]

By Lemma 5.6 and (5.29), we have

\[
||\mathbf{E}]_j(S_j'(\lambda)\mathbf{H}), \varphi || \leq \gamma_4\left(||K_j'(S_j'(\lambda)\mathbf{H})||_{L_q(\Omega)}^{1/q} ||\nabla K_j'(S_j'(\lambda)\mathbf{H})||_{L_q(\Omega)}^{1/q} \right)
\]

\[
+ ||K_j'(S_j'(\lambda)\mathbf{H})||_{L_q(\Omega)}^{1-1/q} ||\nabla K_j'(S_j'(\lambda)\mathbf{H})||_{L_q(\Omega)}^{1/q} + ||\nabla S_j'(\lambda)\mathbf{H}||_{L_q(\Omega)}^{1-1/q} ||\nabla^2 S_j'(\lambda)\mathbf{H}||_{L_q(\Omega)}^{1/q}
\]
\[
\|B_j^q(S_j^q(\lambda)H), \varphi\| \leq \gamma_4 \left( (\sigma_2 + \sigma_1 \gamma_{\sigma_2}) \|\nabla^2 S_j^q(\lambda)H\|_{L_q(\mathcal{H}_j)} + \gamma_{\sigma_1} \gamma_{\sigma_2} \|\nabla S_j^q(\lambda)H\|_{L_q(\mathcal{H}_j)} \right) \|\nabla \varphi\|_{L_{\infty}(\Omega \cap B_j^q)}
\]
for any \( \sigma_1, \sigma_2 > 0 \). Similarly to the last inequality, we can estimate \([B_j^q(u_j^i), \varphi]\). Since \([B_j^q(u_j^i), \varphi]\) are linear with respect to \(u_j^i\), the inequalities of \([B_j^q(S_j^q(\lambda)H), \varphi]\) (\(i = 1, 3\)) above yields that

\[
(5.33) \quad \left| \sum_{l=1}^{n} a_l T_j^q(\lambda_l)H_l, \varphi > \Omega \right| \leq \gamma_4 \left\{ (\sigma_2 + \sigma_1 \gamma_{\sigma_2}) \left| \sum_{l=1}^{n} a_l \nabla^2 S_j^q(\lambda_l)H_l \right|_{L_q(\mathcal{H}_j)} + \gamma_{\sigma_1} \gamma_{\sigma_2} \left| \sum_{l=1}^{n} a_l S_j^q(\lambda_l)H_l \right|_{W_q^1(\mathcal{H}_j)} \right\} \|\nabla \varphi\|_{L_{\infty}(\Omega \cap B_j^q)}
\]
with \(i = 1, \ldots, 5\) for any \( \varphi \in W_q^1(\Omega) \) and for any \( n \in \mathbb{N} \), \( \{a_l\}_{l=1}^{n} \subset C \), \( \{\lambda_l\}_{l=1}^{n} \subset \Sigma_{\varepsilon, \lambda_1} \), and \( \{H_l\}_{l=1}^{n} \subset \mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}) \). The estimate (5.33) with \( n = 1 \), together with (5.13) and (5.14), shows that

\[
| \sum_{l=1}^{n} T_j^q(\lambda_l)H_l, \varphi > \Omega | \leq \left( \frac{M}{\|H\|_{\mathcal{X}_{\mathcal{R}, \varphi}(\Omega \cap B_j^q)}} \right) \|\nabla \varphi\|_{L_{\infty}(\Omega \cap B_j^q)}
\]
for any \( \varphi \in W_q^1(\Omega) \) and \( H \in \mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}) \) with some positive constant \( M \) independent of \( j \in \mathbb{N} \), which, combined with Lemma 5.2, furnishes that the infinite sum \( I(\lambda)H = \sum_{j=1}^{\infty} T_j^q(\lambda_l)H_l \) exists in the strong topology of \( W_q^1(\Omega) \). In addition, by (5.30) with Hölder’s inequality and by Lemma 5.2 again, we have

\[
\| \sum_{l=1}^{n} a_l T_j^q(\lambda_l)H_l \|_{W_q^1(\Omega)} \leq 2\gamma_4 \left\{ (\sigma_2 + \sigma_1 \gamma_{\sigma_2}) \sum_{j=1}^{\infty} \left| \sum_{l=1}^{n} a_l \nabla^2 S_j^q(\lambda_l)H_l \right|_{L_q(\mathcal{H}_j)} + \sum_{j=1}^{\infty} \left| \sum_{l=1}^{n} a_l S_j^q(\lambda_l)H_l \right|_{W_q^1(\mathcal{H}_j)} \right\} \|\nabla \varphi\|_{W_q^1(\Omega)}
\]

This inequality combined with monotone convergence theorem, Proposition 2.5, and (5.14), together with the formulas (5.19), yields that, by Definition 1.2 and (5.2),

\[
\int_{0}^{1} \left| \sum_{l=1}^{n} r_l(u)T_j^q(\lambda_l)H_l \right|_{W_q^1(\Omega)} du \leq \gamma_4 \left( (\sigma_2 + \sigma_1 \gamma_{\sigma_2}) + \gamma_{\sigma_1} \gamma_{\sigma_2} \lambda_1^{-1/2} \right) \|\nabla \varphi\|_{W_q^1(\Omega)}
\]
for any \( \sigma_1, \sigma_2 > 0 \) and \( \lambda_1 \geq \lambda_0 \). Thus, setting \( I(\lambda)H = \sum_{i=1}^{5} I_i(\lambda)H \) and using Proposition 2.3, we have

\[
(5.34) \quad I(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}), W_q^1(\Omega))), \quad I(\lambda)F_{R, \lambda}(f, h, k) = \sum_{i=1}^{5} I_i^j(\lambda),
\]

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}), W_q^1(\Omega))} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l I(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \lambda_1} \right\} \right) \leq \gamma_4 (\sigma_2 + \sigma_1 \gamma_{\sigma_2} + \gamma_{\sigma_1} \gamma_{\sigma_2} \lambda_1^{-1/2} ) \quad (l = 0, 1).
\]

Analogously, we can prove the existence of operator families \( J^1(\lambda), J^2(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}), W_q^1(\Omega))) \) such that

\[
(5.35) \quad J^1(\lambda)F_{R, \lambda}(f, h, k) = \sum_{j=1}^{\infty} a_j^1, \quad J^2(\lambda)F_{R, \lambda}(f, h, k) = \sum_{j=1}^{\infty} b_j^2,
\]

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R}, \varphi}, \mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}))} \left( \left\{ \left( \frac{d}{d\lambda} \right)^l \left( \tilde{F}_{R, \lambda}(\lambda) \right) \mid \lambda \in \Sigma_{\varepsilon, \lambda_1} \right\} \right) \leq \gamma_4 (\sigma_2 + \sigma_1 \gamma_{\sigma_2} + \gamma_{\sigma_1} \gamma_{\sigma_2} \lambda_1^{-1/2} ) \quad (l = 1, 2)
\]
with \(i = 1, 2\) and \(l = 0, 1\) for any \( \sigma_1, \sigma_2 > 0 \) and for any \( \lambda_1 \geq \lambda_0 \).

In view of (5.32), we define \( L^0(\lambda)H \) as \( \mathcal{L}(\mathcal{F}(I(\lambda)H), [J^1(\lambda)H], J^2(\lambda)H|_{r^+}) \) for \( H \in \mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}) \). Then, by the continuity of \( \mathcal{K} \), (5.32), (5.33), and Proposition 2.3, we see that

\[
(5.36) \quad L^0(\lambda) \in \text{Hol}(\Sigma_{\varepsilon, \lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R}, \varphi}(\bar{\Omega}), L_q(\Omega)^N)), \quad L^0(\lambda)F_{R, \lambda}(f, h, k) = \sum_{i=1}^{5} \sum_{j=1}^{\infty} \nabla \left( K(\zeta^i_j u_j^i - \zeta^j_i K_j^i(u_j^i)) \right),
\]
for any $\sigma_1, \sigma_2 > 0$ and $\lambda_1 \geq 0$. Summing up (5.23), (5.24), (5.31), (5.35), and (5.36), we define $U(\lambda)H$ as
\[
U(\lambda)H = (V^0(\lambda)H + K^0(\lambda)H) + L^0(\lambda)H, V^1(\lambda)H + J^1(\lambda)H, V^2(\lambda)H + J^2(\lambda)H 
\]
for $H \in X_{R,q}(\Omega)$, and then $U(\lambda)$ is the required operator in Lemma 5.5. This completes the proof of the lemma.

5.5. **Proof of Theorem 2.2.** In Lemma 5.5 we choose $\sigma_1$, $\sigma_2$, and $\lambda_1$ in such a way that $\gamma_4 \sigma_2 < 1/8$, $\gamma_4 \gamma_2 \sigma_1 < 1/8$, and $\gamma_4 \gamma_2 \gamma_1^{-1/2} < 1/4$, successively, and thus
\[
\mathcal{R}_{L}(X_{R,q}(\Omega),L_{q}(\Omega)) \left( \left\{ \left( \frac{d}{d\lambda} \right)^l L^0(\lambda) | \lambda \in \Sigma_{\varepsilon,\lambda_1} \right\} \right) \leq \gamma_4(\sigma_2 + \sigma_1 \gamma_2 + \gamma_1 \gamma_2 \lambda_1^{-1/2}) \quad (l = 0, 1)
\]
These inequalities imply that
\[
\mathcal{R}_{L}(X_{R,q}(\Omega)) \left( \left\{ \left( \frac{d}{d\lambda} \right)^l (I - F_{R,q}U(\lambda))^{-1} | \lambda \in \Sigma_{\varepsilon,\lambda_1} \right\} \right) \leq 2 \quad (l = 0, 1).
\]
Similarly to Section 4, setting $B(\lambda) = S(\lambda)(I - F_{R,q}U(\lambda))^{-1}$ with (5.21) yields that $u = B(\lambda)F_{R,q}(f,h,k)$ solves the problem (2.3) and $B(\lambda)$ satisfies (2.8). The uniqueness of (2.8) follows from the solvability of the weak elliptic transmission problem on $W_0^s(\Omega)$ for $\rho_{\pm}$ and the solvability of (2.8) for $q^2$ in the same manner as in the proof of Theorem 1.6. This completes the proof of Theorem 2.2.

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