Smarandache Curves In Terms of Sabban Frame of Fixed Pole Curve

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Abstract: In this paper, we study the special Smarandache curve in terms of Sabban frame of Fixed Pole curve and we give some characterization of Smarandache curves. Besides, we illustrate examples of our results.

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1 Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [11]. Special Smarandache curves have been studied by some authors. Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case [1]. M. Çetin, Y. Tunçer and K. Karacan investigated special smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential goemetric properties of Smarandache curves [5]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache

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curves, [4]. Ö. Bektaş and S. Yüce studied some special Smarandache curves according to Darboux Frame in $E^3$ [2]. M. Turgut and S. Yılmaz studied a special case of such curves and called it Smarandache $TB_2$ curves in the space $E^4_1$ [11]. N. Bayrak, Ö. Bektaş and S. Yüce studied some special Smarandache curves in $E^3_1$ [3]. K. Taşköprü, M. Tosun studied special Smarandache curves according to Sabban frame on $S^2$ [10].

In this paper, the special Smarandache curves such as $CT_C, T_C(C \wedge T_C), CT_C(C \wedge T_C)$ created by Sabban frame, $\{C, T_C, C \wedge T_C\}$, that belongs to fixed pole of a $\alpha$ curve are defined. Besides, we have found some results.

2 Preliminaries

The Euclidean 3-space $E^3$ be inner product given by

$$\langle \cdot, \cdot \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Let $\alpha: I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, $\kappa$ and $\tau$ respectively, the Frenet formulae is given by [6]

$$\begin{cases}
    T' = \kappa N \\
    N' = -\kappa T + \tau B \\
    B' = -\tau N.
\end{cases} \quad (2.1)$$

Accordingly, the spherical indicatrix curves of Frenet vectors are $(T), (N)$ and $(B)$ respectively. These equations of curves are given by [7]

$$\begin{cases}
    \alpha_T(s) = T(s) \\
    \alpha_N(s) = N(s) \\
    \alpha_B(s) = B(s)
\end{cases} \quad (2.2)$$

Let $\gamma: I \to S^2$ be a unit speed spherical curve. We denote $s$ as the arc-length parameter of $\gamma$. Let us denote by

$$\begin{cases}
    \gamma(s) = \gamma(s) \\
    t(s) = \gamma'(s) \\
    d(s) = \gamma(s) \wedge t(s).
\end{cases} \quad (2.3)$$
We call $t(s)$ a unit tangent vector of $\gamma$. $\{\gamma, t, d\}$ frame is called the Sabban frame of $\gamma$ on $S^2$. Then we have the following spherical Frenet formulae of $\gamma$:

$$\begin{align*}
\gamma' &= t \\
t' &= -\gamma + \kappa_g d \\
d' &= -\kappa_g t
\end{align*}$$

(2.4)

where is called the geodesic curvature of $\kappa_g$ on $S^2$ and

$$\kappa_g = \langle t', d \rangle \ [8]$$

(2.5)

3 Smarandache Curves According to Sabban Frame of Fixed Pole Curve

In this section, we investigate Smarandache curves according to the Sabban frame of fixed pole curve ($C$). Let $\alpha_C(s) = C(s)$ be a unit speed regular spherical curves on $S^2$. We denote $s_C$ as the arc-length parameter of fixed pole curve ($C$)

$$\alpha_C(s) = C(s) \quad (3.1)$$

Differentiating (3.1), we have

$$\frac{d\alpha_C}{ds_C} \frac{ds_C}{ds} = C'(s)$$

and

$$T_C \frac{ds_C}{ds} = \varphi' \cos \varphi T - \varphi' \sin \varphi B \quad (3.2)$$

From the equation (3.2)

$$T_C = \cos \varphi T - \sin \varphi B$$

and

$$C \land T_C = N$$

From the equation (2.3)

$$\begin{align*}
C(s) &= C(s) \\
T_C(s) &= \cos \varphi T - \sin \varphi B \\
(C \land T_C)(s) &= N(s)
\end{align*}$$

is called the Sabban frame of fixed pole curve ($C$). From the equation (2.5)
\[ \kappa_g = \langle T_C', C \wedge T_C \rangle \implies \kappa_g = \frac{\|W\|}{\varphi'} \]

Then from the equation (2.4) we have the following spherical Frenet formulae of \((C)\):

\[
\begin{cases}
    C'' = T_C \\
    T_C' = -C + \frac{\|W\|}{\varphi'} (C \wedge T_C) \\
    (C \wedge T_C)' = -\frac{\|W\|}{\varphi'} T_C
\end{cases}
\tag{3.3}
\]

### 3.1 \(CT_C\)-Smarandache Curves

**Definition 3.1** Let \(S^2\) be a unit sphere in \(E^3\) and suppose that the unit speed regular curve \(\alpha_C(s) = C(s)\) lying fully on \(S^2\). In this case, \(CT_C\)-Smarandache curve can be defined by

\[ \psi(s^*) = \frac{1}{\sqrt{2}}(C + T_C). \tag{3.4} \]

Now we can compute Sabban invariants of \(CT_C\)-Smarandache curves. Differentiating (3.4), we have

\[ T_{\psi} \frac{ds^*}{ds} = \frac{1}{\sqrt{2}} \left( (\cos \varphi - \sin \varphi)T + \frac{\|W\|}{\varphi'} N - (\cos \varphi + \sin \varphi)B \right), \]

where

\[ \frac{ds^*}{ds} = \sqrt{2 + \left( \frac{\|W\|}{\varphi'} \right)^2}. \tag{3.5} \]

Thus, the tangent vector of curve \(\psi\) is to be

\[ T_{\psi} = \frac{1}{\sqrt{2 + \left( \frac{\|W\|}{\varphi'} \right)^2}} \left( (\cos \varphi - \sin \varphi)T + \frac{\|W\|}{\varphi'} N - (\cos \varphi + \sin \varphi)B \right). \tag{3.6} \]

Differentiating (3.6), we get

\[ T_{\psi} \frac{ds^*}{ds} = \frac{1}{(2 + \left( \frac{\|W\|}{\varphi'} \right)^2)^{\frac{3}{2}}} (\lambda_1 T + \lambda_2 N + \lambda_3 B) \tag{3.7} \]
where

\[ \lambda_1 = -2\varphi'(\sin \varphi + \cos \varphi) - \kappa \left( 2 \frac{\|W\|}{\varphi'} + \left( \frac{\|W\|}{\varphi'} \right)^2 \right) - \frac{\|W\|^2}{\varphi'} (\sin \varphi + \cos \varphi) - \frac{\|W\|}{\varphi'} \left( \frac{\|W\|}{\varphi'} \right)' (\cos \varphi - \sin \varphi) \]

\[ \lambda_2 = (2 + \left( \frac{\|W\|}{\varphi'} \right)^2) \left( \kappa (\cos \varphi - \sin \varphi) + \tau (\cos \varphi + \sin \varphi) \right) + 2 \left( \frac{\|W\|}{\varphi'} \right)' \]

\[ \lambda_3 = \tau \frac{\|W\|}{\varphi'} \left( 2 + \left( \frac{\|W\|}{\varphi'} \right)^2 \right) - 2\varphi' (\cos \varphi - \sin \varphi) - \frac{\|W\|^2}{\varphi'} (\cos \varphi - \sin \varphi) + \frac{\|W\|}{\varphi'} \left( \frac{\|W\|}{\varphi'} \right)' (\cos \varphi + \sin \varphi). \]

Substituting the equation (3.5) into equation (3.7), we reach

\[ T'_\psi = \frac{\sqrt{2}}{(2 + \left( \frac{\|W\|}{\varphi'} \right)^2)^2} (\lambda_1 T + \lambda_2 N + \lambda_3 B). \quad (3.8) \]

Considering the equations (3.4) and (3.6), it easily seen that

\[ (C \land T_C)_\psi = \frac{1}{\sqrt{4 + 2\left( \frac{\|W\|}{\varphi'} \right)^2}} \left( - \frac{\|W\|}{\varphi'} (\cos \varphi - \sin \varphi) T + 2 N + \frac{\|W\|}{\varphi'} (\cos \varphi + \sin \varphi) B \right) \quad (3.9) \]

From the equation (3.8) and (3.9), the geodesic curvature of \( \psi(s^*) \) is

\[ \kappa^g_\psi = \langle T'_\psi, (C \land T_C)_\psi \rangle \]

\[ = \frac{1}{(2 + \left( \frac{\|W\|}{\varphi'} \right)^2)^2} \left( - \frac{\|W\|}{\varphi'} (\cos \varphi - \sin \varphi) \lambda_1 + 2 \lambda_2 + \frac{\|W\|}{\varphi'} (\cos \varphi + \sin \varphi) \lambda_3 \right). \]

3.2 \( T_C(C \land T_C) \)-Smarandache Curves

**Definition 3.2** Let \( S^2 \) be a unit sphere in \( E^3 \) and suppose that the unit speed regular curve \( \alpha_C(s) = C(s) \) lying fully on \( S^2 \). In this case, \( T_C(C \land T_C) \)-Smarandache curve can be defined by

\[ \psi(s^*) = \frac{1}{\sqrt{2}} (T_C + C \land T_C). \quad (3.10) \]
Now we can compute Sabban invariants of $T_C(C \wedge T_C)$ - Smarandache curves. Differentiating (3.10), we have

$$T_\psi \frac{ds^*}{ds} = \frac{1}{\sqrt{2}} ((-\sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi)T + \frac{\|W\|}{\varphi'} N + \left(\frac{\|W\|}{\varphi'} \sin \varphi - \cos \varphi\right)B)$$

where

$$\frac{ds^*}{ds} = \sqrt{1 + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \frac{2}{2}}.$$  \hspace{1cm} (3.11)

In that case, the tangent vector of curve $\psi$ is as follows

$$T_\psi = \frac{1}{\sqrt{1 + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \frac{2}{2}}} \left((-\sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi)T + \left(\frac{\|W\|}{\varphi'} \sin \varphi - \cos \varphi\right)B\right)$$ \hspace{1cm} (3.12)

Differentiating (3.12), it is obtained that

$$T_\psi' \frac{ds^*}{ds} = \frac{1}{(1 + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \frac{2}{2})^2} \left(\lambda_1 T + \lambda_2 N + \lambda_3 B\right)$$ \hspace{1cm} (3.13)

where

$$\lambda_1 = -\varphi' \cos \varphi + \|W\|(\sin \varphi + 2 \frac{\|W\|}{\varphi'} \cos \varphi - \frac{\varphi'}{\varphi'} + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \sin \varphi - 2 \kappa \frac{\|W\|^2}{\varphi'} - \left(\frac{\|W\|}{\varphi'}\right)'(\cos \varphi - 2 \frac{\|W\|}{\varphi'} \sin \varphi)$$

$$\lambda_2 = \kappa(-\sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 (-\sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi)) - \tau\left(\frac{\|W\|}{\varphi'}\right)'(\sin \varphi - \cos \varphi) - 2 \left(\frac{\|W\|}{\varphi'}\right)^3 (\sin \varphi - \cos \varphi) + \left(\frac{\|W\|}{\varphi'}\right)'$$

$$\lambda_3 = \varphi' \sin \varphi + \frac{\|W\|}{\varphi'}(\tau + \varphi' \cos \varphi + 2 \tau \left(\frac{\|W\|}{\varphi'}\right)^2 + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \varphi' \cos \varphi + 2 \|W\| \sin \varphi) + \left(\frac{\|W\|}{\varphi'}\right)'(\sin \varphi + 2 \frac{\|W\|}{\varphi'} \cos \varphi).$$

Substituting the equation (3.11) into equation (3.13), we get

$$T_\psi' = \frac{\sqrt{2}}{(1 + 2 \left(\frac{\|W\|}{\varphi'}\right)^2 \frac{2}{2})^2} (\lambda_1 T + \lambda_2 N + \lambda_3 B)$$ \hspace{1cm} (3.14)
Using the equations (3.10) and (3.12), we easily find

\[(C \land T_C)_{\psi} = \frac{1}{\sqrt{2 + 4(\|W\|_{\varphi'})^2}} \left(2\|W\|_{\varphi'} \sin \varphi - \cos \varphi\right)T + \right. \\
\left. + N + (2\|W\|_{\varphi'} \cos \varphi + \sin \varphi)B \right)

(3.15)

So, the geodesic curvature of \(\psi(s^*)\) is as follows

\[\kappa_{g} = \langle T_{\psi}, (C \land T_C)_{\psi} \rangle \]

\[= \frac{1}{\left(1 + 2(\|W\|_{\varphi'})^2\right)^{\frac{1}{2}}} \left(\left(2\|W\|_{\varphi'} \sin \varphi - \cos \varphi\right)\lambda_1 + \lambda_2 + (2\|W\|_{\varphi'} \cos \varphi + \sin \varphi)\lambda_3 \right).\]

### 3.3 \(CTC(C \land T_C)\)-Smarandache Curves

**Definition 3.3** Let \(S^2\) be a unit sphere in \(E^3\) and suppose that the unit speed regular curve \(\alpha_C(s) = C(s)\) lying fully on \(S^2\). In this case, \(CTC(C \land T_C)\)-Smarandache curve can be defined by

\[\psi(s^*) = \frac{1}{\sqrt{3}}(C + T_C + C \land T_C).\]

(3.16)

Let us calculate Sabban invariants of \(CTC(C \land T_C)\)-Smarandache curves. Differentiating (3.16), we have

\[T_{\psi} \frac{ds^*}{ds} = \frac{1}{\sqrt{3}} \left((\cos \varphi - \sin \varphi - \|W\|_{\varphi'} \cos \varphi)T + \right. \\
\left. + \|W\|_{\varphi'} N + (- \sin \varphi - \cos \varphi + \|W\|_{\varphi'} \sin \varphi)B \right) \]

where

\[\frac{ds^*}{ds} = \sqrt{\frac{2(1 - \|W\|_{\varphi'}) + \left(\|W\|_{\varphi'}^2\right)^{\frac{1}{2}}}{3}}.\]

(3.17)

Thus, the tangent vector of curve \(\psi\) is

\[T_{\psi} = \frac{1}{\sqrt{2\left(1 - \|W\|_{\varphi'} + \left(\|W\|_{\varphi'}^2\right)^{\frac{1}{2}}\right)}} \left((\cos \varphi - \sin \varphi - \|W\|_{\varphi'} \cos \varphi)T + \right. \\
\left. + \|W\|_{\varphi'} N + (- \sin \varphi - \cos \varphi + \|W\|_{\varphi'} \sin \varphi)B \right) \]

(3.18)
Differentiating (3.18), it is obtained that

\[ T_\psi^* \frac{ds^*}{ds} = \frac{1}{2\sqrt{2}(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2)\frac{3}{2}}(\lambda_1 T + \lambda_2 N + \lambda_3 B) \]  

(3.19)

where

\[
\lambda_1 = (1 - \frac{\|W\|}{\varphi'}) + (\frac{\|W\|}{\varphi'})^2 \left( -2\varphi' (\sin \varphi + \cos \varphi) + 2\|W\| \sin \varphi - 2\kappa \frac{\|W\|}{\varphi'} \right) + \frac{\|W\|}{\varphi'} (\cos \varphi - \sin \varphi) + 2 (\frac{\|W\|}{\varphi'})' \left( -\cos \varphi + \frac{\|W\|}{\varphi'} \sin \varphi \right)
\]

\[
\lambda_2 = (1 - \frac{\|W\|}{\varphi'}) + (\frac{\|W\|}{\varphi'})^2 \left( 2\kappa (\cos \varphi - \sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi) + 2\varphi' (\sin \varphi - \cos \varphi) + 2\|W\| \cos \varphi \right) + (\frac{\|W\|}{\varphi'})^2 \sin \varphi + 2 (\frac{\|W\|}{\varphi'})' \left( \sin \varphi + \frac{\|W\|}{\varphi'} \cos \varphi \right) - \frac{\|W\|}{\varphi'} (\sin \varphi + \cos \varphi).
\]

Substituting the equation (3.17) into equation (3.19), we reach

\[ T_\psi = \frac{\sqrt{3}}{4 \left(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2\right)^2} (\lambda_1 T + \lambda_2 N + \lambda_3 B). \]  

(3.20)

Using the equations (3.16) and (3.18), we have

\[ (C \land T_C)_\psi = \frac{1}{\sqrt{6}} \sqrt{1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2} \left( \left(2 \frac{\|W\|}{\varphi'} \sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi \right) + (1 - \frac{\|W\|}{\varphi'})N + (\sin \varphi + 2 \frac{\|W\|}{\varphi'} \cos \varphi) \right. \]

\[ \left. + \frac{\|W\|}{\varphi'} \sin \varphi \right) B \]  

(3.21)

From the equation (3.20) and (3.21), the geodesic curvature of \( \psi(s^*) \) is

\[ \kappa_g^\psi = \langle T_\psi', (C \land T_C)_\psi \rangle = \frac{1}{4\sqrt{2}(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2)^{\frac{3}{2}}} \left[ \lambda_1 \left(2 \frac{\|W\|}{\varphi'} \sin \varphi - \frac{\|W\|}{\varphi'} \cos \varphi - \cos \varphi \right) \right. \]

\[ \left. + \lambda_2 (1 - \frac{\|W\|}{\varphi'}) + \lambda_3 (\sin \varphi + 2 \frac{\|W\|}{\varphi'} \cos \varphi + \frac{\|W\|}{\varphi'} \sin \varphi) \right]. \]
3.4 Example

Let us consider the unit speed spherical curve:

\[ \alpha(s) = \left\{ \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s, -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s, \frac{6}{65} \sin 10s \right\}. \]

It is rendered in Figure 1.

![Figure 1: Fixed Pole curve (T)](image)

In terms of definitions, we obtain Smarandache curves according to Sabban frame on \( S^2 \), see Figures 2 - 4.

![Figure 2: CTc - Smarandache Curve](image)
Figure 3: $T_C(C \wedge T_C)$ - Smarandache Curve

Figure 4: $CT_C(C \wedge T_C)$ - Smarandache Curve

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