On Explicit Evaluation of Ratio’s of Theta Function Which is Analogous to Ramanujan’s
Function $a_{m,n}$

S. Vasanth Kumar

ABSTRACT: In this article, Ramanujan defined $a_{m,n}$ [3], B. N. Dharmendra and S. Vasanth Kumar defined $E_{m,n}$ [5] for any positive real numbers $m$ and $n$ involving Ramanujan’s product of theta-functions. We established new relation between $a_{m,n}$ and $E_{m,n}$ and explicit evaluations of $E_{m,n}$.

Key Words: Modular equation, Theta-function.

Contents

1 Introduction 1

2 Preliminary Results 2

3 Modular relation between $a_{m,n}$ and $E_{m,n}$ 3

4 Explicit evaluation of $E_{m,n}$ 4

1. Introduction

The Ramanujan’s general theta function [11] is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

$$= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}$$

where,

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

Three special cases of $f(a, b)$ are defined as follows:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q; q)_{\infty}.$$
On page 338 in his first notebook [11], Ramanujan defines
\[
a_{m,n} = \frac{ne^{-(n-1)\pi/4} \sqrt{mn} \psi^2(e^{-\pi \sqrt{mn}})}{\psi^2(e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})},
\]
where \(m\) and \(n\) are positive real numbers.

In [3], on pages 337 - 338, Ramanujan has listed eighteen particular values. Berndt, Chan and Zhang [4] have been established all these values. For some general theorems and explicit evaluation on \(a_{m,n}\) one can refer [6,7,8,10].

Following the above definition [9], Mahadeva Naika et al. defined a new function \(b_{m,n}\) and in [5], B. N. Dharmendra and S. Vasanth Kumar defined the Ramanujan theta function \(E_{m,n}\). They established new properties of \(b_{m,n}\) and \(E_{m,n}\) and find its explicit values.

In [9], defined the theta function
\[
b_{m,n} = \frac{ne^{-(n-1)\pi/4} \sqrt{mn} \psi^2(e^{-\pi \sqrt{mn}})}{\psi^2(e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}.
\]

In [5], B. N. Dharmendra and S. Vasanth Kumar defined the Ramanujan theta function
\[
E_{m,n} = \frac{f(e^{-\pi \sqrt{mn}}) \psi(e^{-\pi \sqrt{mn}})}{e^{-\pi/(1-m) \sqrt{mn}} f(e^{-\pi \sqrt{mn}}) \psi(e^{-\pi \sqrt{mn}})}.
\]

The main purpose of this paper to be establish new relation between \(a_{m,n}\) and \(E_{m,n}\) and explicit evaluation of \(E_{m,n}\).

2. Preliminary Results

In this section, we tend to collect many identities that square measure helpful in proving our main results.

Lemma 2.1. [6] If \(m\) is any positive rational,
\[
a_{m,3} = \frac{3q^{1/2} \psi^2(-q^3) \varphi^2(q^3)}{\psi^2(-q) \varphi^2(q)},
\]
then we have,
\[
a_{m,3}^2 = \frac{9(1 + P^4)}{P^4(9 + P^4)} = \frac{9(1 - Q^4)}{Q^4(Q^4 - 9)}, \quad Q^4 \neq 9.
\]

Lemma 2.2. [5] If \(n\) is any positive rational,
\[
E_{3,n} = \frac{f(q) \psi(-q^3)}{q^{-1/6} f(q^3) \psi(-q)}; \quad q := e^{-\pi \sqrt{n}}.
\]
then we have,
\[
E_{3,n}^6 = \frac{P^4 + 9}{P^4(1 + P^4)}.
\]
Lemma 2.3. [6] If \( m \) is any positive rational,

\[ a_{m,5} = \frac{5q\psi^2(-q^5)\varphi^2(q^5)}{\psi^4(-q)\varphi^2(q)}, \quad (2.7) \]

\[ P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad Q = \frac{\varphi(q)}{\varphi(q^5)}, \quad (2.8) \]

then we have,

\[ a_{m,5} = \frac{5(1 + P^2)}{P^2(5 + P^2)} = 5\left(1 - \frac{Q^2}{Q^2 - 5}\right), \quad Q \neq \sqrt{5}. \quad (2.9) \]

Lemma 2.4. [5] If \( n \) is any positive rational,

\[ E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}; \quad q := e^{-\pi\sqrt{n}} \quad (2.10) \]

\[ P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/6}f(q^5)}, \quad (2.11) \]

then we have,

\[ E_{5,n}^5 = \frac{P^2 + 5}{P^2(P^2 + 1)}. \quad (2.12) \]

Lemma 2.5. [5] We have,

\[ a_{m,n} = a_{n,m} \]

and

\[ E_{m,n} = E_{n,m}. \]

3. Modular relation between \( a_{m,n} \) and \( E_{m,n} \)

Theorem 3.1. If \( x := E_{m,3} \) and \( y := a_{m,3} \) then

\[ x^3 - \frac{1}{x^3} = 3\left(y - \frac{1}{y}\right). \quad (3.1) \]

Proof. From Lemma (2.1), we obtain

\[ P^4 := \frac{9 - 9y + 3\sqrt{9y^2 - 14y + 9}}{2y}, \quad (3.2) \]

where,

\[ y := a_{m,3}^2. \]

Employing the above equation (3.2) in Lemma (2.2), we obtain

\[ (x^3(yx^3 - 3 + 3y^2) - y)(x^3(yx^3 + 3 - 3y^2) - y) = 0 \]

(3.3)

By examining the behavior of the above factors near \( q = 0 \), we can find a neighborhood about the origin, where the second factor is zero; whereas another factor is not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. \( \square \)
Theorem 3.2. If \( x := E_{m,5} \) and \( y := a_{m,5} \) then
\[
\left( x^3 + \frac{1}{x^3} \right) + 8 = 5 \left( y + \frac{1}{y} \right).
\] (3.4)

Proof. From Lemma (2.3), we obtain
\[
p^2 := \frac{5 - 5y + \sqrt{25y^2 - 30y + 25}}{2y}.
\] (3.5)

Employing the above equation (3.5) in Lemma (2.4), we get
\[
x^3(5 - x^3y - 8y + 5y^2) - y = 0
\] (3.6)

By examining the behavior of the above term near \( q = 0 \). This completes the proof. \( \square \)

4. Explicit evaluation of \( E_{m,n} \)

Corollary 4.1. Explicit values of \( E_{3,n} \)

| Sr. No | \( a_{3,n} \) | \( E_{3,n} \) |
|--------|----------------|----------------|
| 1      | \( a_{3,2} = \sqrt[3]{\frac{x}{4} - 1} \) | \( E_{3,2} = \frac{(-4\sqrt{3} + 12 - 12\sqrt{3} + 16\sqrt{2})\sqrt{3} - 1}{2} \) |
| 2      | \( a_{3,3} = \frac{1}{2} \) | \( E_{3,3} = \frac{(2 - \sqrt{3})^\frac{3}{2}}{2} \) |
| 3      | \( a_{3,4} = \frac{2}{\sqrt{3}} \) | \( E_{3,4} = \frac{(28 - 12\sqrt{3})^\frac{3}{2}}{2} \) |
| 4      | \( a_{3,5} = \frac{2}{\sqrt{3}} \) | \( E_{3,5} = \frac{(28 - 12\sqrt{3})^\frac{3}{2}}{2} \) |
| 5      | \( a_{3,6} = \frac{1}{(3\sqrt{3} + 1)} \) | \( E_{3,6} = \frac{1}{3} \) |
| 6      | \( a_{3,7} = 2\sqrt{3} - \sqrt{11} \) | \( E_{3,7} = \frac{1}{3} \) |
| 7      | \( a_{3,8} = \sqrt{2} - \sqrt{3} \) | \( E_{3,8} = \frac{1}{3} \) |
| 8      | \( a_{3,9} = \sqrt{2} - \sqrt{3} \) | \( E_{3,9} = \frac{1}{3} \) |
| 9      | \( a_{3,10} = \sqrt{2} - \sqrt{3} \) | \( E_{3,10} = \frac{1}{3} \) |
| 10     | \( a_{3,11} = \sqrt{2} - \sqrt{3} \) | \( E_{3,11} = \frac{1}{3} \) |
| 11     | \( a_{3,12} = \sqrt{2} - \sqrt{3} \) | \( E_{3,12} = \frac{1}{3} \) |
| 12     | \( a_{3,13} = \sqrt{2} - \sqrt{3} \) | \( E_{3,13} = \frac{1}{3} \) |

Proof. In Ramanujan notebook Part V \([3]\) he recorded many values of \( a_{3,n} \). In particularly, he recorded for \( n = 3,5,7,9,11,15,19,31,59 \).

Then, M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankar \([7]\) also evaluated the values of \( a_{3,n} \) for \( n = 2,35,55 \).

Noting all these values of \( n \), we have established the values for \( E_{3,n} \).

If
\[
n = 3
\]

then, we find in \([3]\), \( a_{3,3} = \frac{1}{\sqrt{3}} \), substituting this value in (3.3) we obtain an equation
\[
-2x^3 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} x^6
\]
and solving for \( x \) we get the desired result.

i.e.,
\[
E_{3,3} = (2 - \sqrt{3})^\frac{3}{2}.
\]

Similarly we can obtain for remaining values of \( n \) which is mentioned in the above table 1. \( \square \)
Corollary 4.2. Explicit evaluation of $E_{5,n}$

| Sr.No | $a_{5,n}$ | $E_{5,n}$ |
|-------|-----------|-----------|
| 1     | $a_{5,2} = (\sqrt{2} + 1)(\sqrt{5} - 2)$ | $E_{5,2} = \frac{(\sqrt{5}+1)(\sqrt{2} - 1)}{2}$ |
| 2     | $a_{5,5} = \frac{1}{5}$ | $E_{5,5} = \frac{3 - \sqrt{5}}{2}$ |
| 3     | $a_{5,9} = (2 - \sqrt{3})^2$ | $E_{5,9} = (31 - 8\sqrt{15})^\frac{1}{4}$ |
| 4     | $a_{5,11} = \left(\frac{\sqrt{7} + \sqrt{5} - \sqrt{3} - 1}{8}\right)^8$ | $E_{5,11} = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{12 + 2\sqrt{15} + 2\sqrt{5} - 2\sqrt{2}}{16}\right)$ |
| 5     | $a_{5,13} = \left(\frac{\sqrt{9 + 2\sqrt{15}} - \sqrt{7 + 6\sqrt{2}}}{2}\right)^2$ | $E_{5,13} = \frac{(\sqrt{5} - 1)(\sqrt{13} - 1)}{4}$ |
| 6     | $a_{5,21} = 32 + 3\sqrt{105} - 4\sqrt{123 + 12\sqrt{105}}$ | $E_{5,21} = \left(\left(\sqrt{35} - 6\right)(15\sqrt{3} - 26)\right)^\frac{1}{4}$ |
| 7     | $a_{5,29} = \left(4 + 4\sqrt{145} - 4\sqrt{48 + 4\sqrt{145}}\right)^2$ | $E_{5,29} = \frac{13 + \sqrt{145}(7 - \sqrt{145})\sqrt{12 + \sqrt{145}}}{4}$ |
| 8     | $a_{5,33} = (2 - \sqrt{3})^2 \left(2\sqrt{3} - \sqrt{11}\right)^2$ | $E_{5,33} = \left((9 - 4\sqrt{5})(89 - 12\sqrt{55})\right)^\frac{1}{4}$ |
| 9     | $a_{5,69} = \frac{(5-\sqrt{23})(4\sqrt{5} - \sqrt{11})^2}{4}$ | $E_{5,69} = \left((1126 - 105\sqrt{115})(26 - 15\sqrt{3})\right)^\frac{1}{4}$ |
| 10    | $a_{5,77} = 11303 + 576\sqrt{385} - 1524\sqrt{55} - 4272\sqrt{7}$ | $E_{5,77} = \frac{1126 - 105\sqrt{115} \left(26 - 15\sqrt{3}\right)}{4}$ |

Proof. In [3] Ramanujan has recorded many values of $a_{5,n}$ for $n = 9, 11, 13, 29$. Then [7], M. S. Mahadeva Naika et al. also evaluated the values of $a_{5,n}$ for $n = 2, 5, 9, 33, 69, 77$. Noting all these values of $n$ we have established the values for $E_{5,n}$.

If $n = 5$

then, $a_{5,5} = \frac{1}{5}$, [3] substituting this value in (3.6) we obtain an equation

$$\frac{18}{5} x^3 - \frac{1}{5} x^6 - \frac{1}{5} = 0$$

and solving for $x$ we get the desired result.

i.e.,

$$E_{5,5} = \frac{3 - \sqrt{5}}{2}.$$ 

Similarly we can obtain for remaining values of $n$ which is mentioned in the above table 2.

Conclusion: Finally in this article we established new relation between $a_{m,n}$ and $E_{m,n}$ and explicit evaluations of $E_{3,n}$ and $E_{5,n}$ by setting particular values to $n$, similarly we can also obtain for other values of $m$.

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Vasanth Kumar,
Research Scholar, Department of Mathematics,
Bharathiar University, Coimbatore-641046,
India.

Assistant Professor, Department of Mathematics,
Mysuru Royal Institute of Technology, Mandya, India.
E-mail address: svmaths.174@gmail.com