A note on $U_q(D^{(3)}_4)$ - Demazure crystals

Alyssa M. Armstrong and Kailash C. Misra

Abstract. We show that there exists a suitable sequence $\{w^{(k)}\}_{k \geq 0}$ of Weyl group elements for the perfect crystal $B = B^{1,3l}$ such that the path realizations of the Demazure crystals $B_{w^{(k)}}(\Lambda_2)$ for the quantum affine algebra $U_q(D^{(3)}_4)$ have tensor product-like structure with mixing index $\kappa = 1$.

1. Introduction

Consider the quantum affine algebra $U_q(g)$ (cf. [13]) associated with the affine Lie algebra $g$ (cf. [2]). For a dominant integral weight $\lambda$ of level $l > 0$, let $V(\lambda)$ denote the integrable highest weight $U_q(g)$-module with highest weight $\lambda$ and the pair $(L(\lambda), B(\lambda))$ denote its crystal base ([6], [8], [12]). The crystal $B(\lambda)$ has many interesting combinatorial properties and can be realized as elements (called paths) in the semi-infinite tensor product $\cdots \otimes B \otimes B \otimes B$ where $B$ is a perfect crystal of level $l$ for $U_q(g)$ [3]. A perfect crystal $B$ of level $l$ for the quantum affine algebra $U_q(g)$ can be thought of as a crystal for a level zero representation of the derived subalgebra $U'_q(g)$ [3].

Let $W$ denote the Weyl group for $g$ and $U_q^+(g)$ denote the positive part of $U_q(g)$. For each $w \in W$, the $U_q^+(g)$-submodule $V_w(\lambda)$ generated by the one-dimensional extremal weight space $V(\lambda)_w$ of $V(\lambda)$ is called a Demazure module. In [9], Kashiwara showed that the Demazure crystal $B_w(\lambda)$ for the Demazure module $V_w(\lambda)$ is a subset of the crystal $B(\lambda)$ for $V(\lambda)$ satisfying a certain recursive property. In [10], it has been shown that under certain conditions, the Demazure crystal $B_w(\Lambda)$ as a subset of $\cdots \otimes B \otimes B \otimes B$ has tensor product-like structure. In this realization a certain parameter $\kappa$, called the mixing index enters the picture. It is conjectured that for all quantum affine algebras $\kappa \leq 2$. It is known that for $\lambda = l\Lambda$, where $\Lambda$ is a dominant weight of level 1 the mixing index $\kappa = 1$ for $g = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}, D_4^{(3)}$ [1], $D_4^{(3)}$ [11], $D_4^{(3)}$ [14] and $G_2^{(1)}$ [1].

In this paper using the perfect crystal $B = B^{1,3l}$ of level $3l$ constructed in [7] for $g = D_4^{(3)}$ we show that there exists a suitable sequence $\{w^{(k)}\}_{k \geq 0}$ of Weyl group elements in $W$ such that the Demazure crystals $B_{w^{(k)}}(\Lambda_2)$ has tensor product like structures with $\kappa = 1$. We observe that the dominant weight $\Lambda_2$ has level 3.
2. Quantum affine algebras and the perfect crystals

In this section we recall necessary facts in crystal base theory for quantum affine algebras. Our basic references for this section are [2], [3], [8], [3], and [5].

Let \( I = \{0, 1, \ldots, n\} \) be the index set and let \( A = (a_{ij})_{i,j \in I} \) be an affine Cartan matrix and \( D = \text{diag}(s_0, s_1, \ldots, s_n) \) be a diagonal matrix with all \( s_i \in \mathbb{Z}_{>0} \) such that \( DA \) is symmetric. The dual weight lattice \( \mathcal{P}^\vee \) is defined to be the free abelian group \( \mathcal{P}^\vee = \mathbb{Z} h_0 \oplus \mathbb{Z} h_1 \oplus \cdots \oplus \mathbb{Z} h_n \oplus \mathbb{Z} d \) of rank \( n + 2 \), whose complexification \( \mathfrak{h} = \mathbb{C} \otimes \mathcal{P}^\vee \) is called the Cartan subalgebra. We define the linear functionals \( \alpha_i \) and \( \Lambda_i \) \( i \in I \) on \( \mathfrak{h} \) by

\[
\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{i0}, \quad \Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (i, j \in I).
\]

The \( \alpha_i \)'s are called the simple roots and the \( \Lambda_i \)'s are called the fundamental weights. We denote by \( \Pi = \{ \alpha_i \mid i \in I \} \) the set of simple roots. We also define the affine weight lattice to be \( P = \{ \lambda \in \mathfrak{h}^* \mid \lambda(\mathcal{P}^\vee) \subset \mathbb{Z} \} \). The quadruple \((A, \mathcal{P}^\vee, \Pi, \mathcal{P})\) is called an affine Cartan datum. We denote by \( \mathfrak{g} \) the affine Kac-Moody algebra corresponding to the affine Cartan datum \((A, \mathcal{P}^\vee, \Pi, \mathcal{P})\) (see [2]). Let \( \delta \) denote the null root and \( c \) denote the canonical central element for \( \mathfrak{g} \) (see [2] Ch. 4). Now the affine weight lattice can be written as \( P = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \cdots \oplus \mathbb{Z} \Lambda_n \oplus \mathbb{Z} d \). Let \( \mathcal{P}^+ = \{ \lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \} \). The elements of \( P \) are called the affine weights and the elements of \( \mathcal{P}^+ \) are called the affine dominant integral weights.

Let \( \mathcal{P}^\vee = \mathbb{Z} h_0 \oplus \cdots \oplus \mathbb{Z} h_n, \mathfrak{h} = \mathbb{C} \otimes \mathcal{P}^\vee, \mathcal{P} = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \cdots \oplus \mathbb{Z} \Lambda_n \) and \( \mathcal{P}^+ = \{ \lambda \in \mathcal{P} \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \} \). The elements of \( \mathcal{P} \) are called the classical weights and the elements of \( \mathcal{P}^+ \) are called the classical dominant integral weights. The level of a (classical) dominant integral weight \( \lambda \) is defined to be \( l = \lambda(c) \). We call the quadruple \((A, \mathcal{P}^\vee, \Pi, \mathcal{P})\) the classical Cartan datum.

For the convenience of notation, we define \( [k]_x = \frac{x^k - x^{-k}}{x - x^{-1}} \), where \( k \) is an integer and \( x \) is a symbol. We also define \( \left[ \frac{m}{k} \right]_x = \frac{[m]_x!}{[k]_x! [m-k]_x!} \), where \( m \) and \( k \) are nonnegative integers, \( m \geq k \geq 0 \). \( [k]_x! = [k]_x [k-1]_x \cdots [1]_x \) and \( [0]_x! = 1 \).

The quantum affine algebra \( U_q(\mathfrak{g}) \) is the quantum group associated with the affine Cartan datum \((A, \mathcal{P}^\vee, \Pi, \mathcal{P})\). That is, it is the associative algebra over \( \mathbb{C}(q) \) with unity generated by \( e_i, f_i (i \in I) \) and \( q^h (h \in \mathcal{P}^\vee) \) satisfying the following defining relations:

(i) \( q^0 = 1, q^h q^{h'} = q^{h+h'} \) for all \( h, h' \in \mathcal{P}^\vee \),
(ii) \( q^he_i q^{-h} = q^\alpha_i(h) e_i, q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \) for \( h \in \mathcal{P}^\vee \),
(iii) \( e_if_j - f_je_i = \delta_{ij} \frac{K_i - K_j^{-1}}{q_i - q_j} \) for \( i, j \in I \), where \( q_i = q^{s_i} \) and \( K_i = q^{\rho_i} \),
(iv) \( \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j^{(k)} e_i^{(k)} = 0 \) for \( i \neq j \),
(v) \( \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j^{(k)} = 0 \) for \( i \neq j \),

where \( e_i^{(k)} = \frac{e_i}{[k]_{q_i}^{1-a_{ij}}}, \) and \( f_i^{(k)} = \frac{f_i}{[k]_{q_i}^{a_{ij}}} \). We denote by \( U'_q(\mathfrak{g}) \) the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, K_i^{\pm 1} (i \in I) \). The algebra \( U'_q(\mathfrak{g}) \) can be regarded as the quantum group associated with the classical Cartan datum \((A, \mathcal{P}^\vee, \Pi, \mathcal{P})\).
Definition 2.1. An affine crystal (respectively, a classical crystal) is a set $B$ together with the maps $\wt: B \to \mathbb{P}$ (respectively, $\wt: B \to \tilde{\mathbb{P}}$), $\varepsilon_i, \phi_i: B \to B \cup \{0\}$ and $\varepsilon_i, \phi_i: B \to \mathbb{Z} \cup \{-\infty\} \ (i \in I)$ satisfying the following conditions:

(i) $\varepsilon_i(b) = \varepsilon_i(b) + \langle h_i, \wt(b) \rangle$ for all $i \in I$,
(ii) $\wt(\varepsilon_i(b)) = \wt(b) + \alpha_i$ if $\varepsilon_i(b) \in B$,
(iii) $\wt(\phi_i(b)) = \wt(b) - \alpha_i$ if $\phi_i(b) \in B$,
(iv) $\varepsilon_i(\varepsilon_i(b)) = \varepsilon_i(b) - 1$, $\phi_i(\varepsilon_i(b)) = \varepsilon_i(b) + 1$ if $\varepsilon_i(b) \in B$,
(v) $\varepsilon_i(\phi_i(b)) = \varepsilon_i(b) + 1$, $\phi_i(\phi_i(b)) = \phi_i(b) - 1$ if $\phi_i(b) \in B$,
(vi) $\phi_i(b) = b'$ if and only if $b = \varepsilon_i b'$ for $b, b' \in B, i \in I$.
(vii) If $\phi_i(b) = -\infty$ for $b \in B$, then $\varepsilon_i b = \phi_i b = 0$.

Definition 2.2. Let $B_1$ and $B_2$ be affine or classical crystals. A crystal morphism (or morphism of crystals) $\Psi: B_1 \to B_2$ is a map $\Psi: B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that

(i) $\Psi(0) = 0$,
(ii) if $b \in B_1$ and $\Psi(b) \in B_2$, then $\wt(\Psi(b)) = \wt(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\phi_i(\Psi(b)) = \phi_i(b)$ for all $i \in I$,
(iii) if $b, b' \in B_1$, $\Psi(b), \Psi(b') \in B_2$ and $\phi_i b = b'$, then $\phi_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \varepsilon_i \Psi(b')$ for all $i \in I$.

A crystal morphism $\Psi: B_1 \to B_2$ is called an isomorphism if it is a bijection from $B_1 \cup \{0\}$ to $B_2 \cup \{0\}$.

For crystals $B_1$ and $B_2$, we define the tensor product $B_1 \otimes B_2$ to be the set $B_1 \times B_2$ whose crystal structure is given as follows:

$$
\varepsilon_i(b_1 \otimes b_2) = \begin{cases} 
\varepsilon_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varepsilon_i(b_2), \\
\varepsilon_i(b_1) \otimes \varepsilon_i(b_2) & \text{if } \varepsilon_i(b_1) < \varepsilon_i(b_2),
\end{cases}
$$

$$
\phi_i(b_1 \otimes b_2) = \begin{cases} 
\phi_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varepsilon_i(b_2), \\
\phi_i(b_1) \otimes \phi_i(b_2) & \text{if } \varepsilon_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}
$$

$$
\wt(b_1 \otimes b_2) = \wt(b_1) + \wt(b_2),
$$

$$
\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \wt(b_1) \rangle),
$$

$$
\phi_i(b_1 \otimes b_2) = \max(\phi_i(b_1), \phi_i(b_2) + \langle h_i, \wt(b_2) \rangle).
$$

Let $B$ be a classical crystal. For an element $b \in B$, we define

$$
\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i, \quad \phi(b) = \sum_{i \in I} \phi_i(b) \Lambda_i.
$$

Definition 2.3. Let $l$ be a positive integer. A classical crystal $B$ is called a perfect crystal of level $l$ if

1. there exists a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $B$,
2. $B \otimes B$ is connected,
3. there exists a classical weight $\lambda_0 \in \tilde{\mathbb{P}}$ such that $\wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i$,
4. for any $b \in B$, $\langle c, \varepsilon(b) \rangle \geq l$, 

$$
\#(B_{\lambda_0}) = 1, \quad \text{where } B_{\lambda_0} = \{b \in B \mid \wt(b) = \lambda_0\}.
$$
(5) for any $\lambda \in \tilde{P}^+$ with $\lambda(c) = l$, there exist unique $b^\lambda, b_\lambda \in B$ such that

$$\varepsilon(b^\lambda) = \lambda = \varphi(b_\lambda).$$

The following crystal isomorphism theorem plays a fundamental role in the theory of perfect crystals.

**Theorem 2.4.** [3] Let $B$ be a perfect crystal of level $l$ ($l \in \mathbb{Z}_{\geq 0}$). For any $\lambda \in \tilde{P}^+$ with $\lambda(c) = l$, there exists a unique classical crystal isomorphism

$$\Psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes B$$

given by $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$,

where $u_\lambda$ is the highest weight vector in $B(\lambda)$ and $b_\lambda$ is the unique vector in $B$ such that $\varphi(b_\lambda) = \lambda$.

Set $\lambda_0 = \lambda, \lambda_{k+1} = \varepsilon(b_{\lambda_k}), b_0 = b_{\lambda_0}, b_{k+1} = b_{\lambda_{k+1}}$. Applying the above crystal isomorphism repeatedly, we get a sequence of crystal isomorphisms

$$B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes B \xrightarrow{\sim} B(\lambda_2) \otimes B \otimes B \xrightarrow{\sim} \cdots$$

In this process, we get an infinite sequence $p_\lambda = (b_k)_{k=0}^\infty \in B^{\otimes \infty}$, which is called the ground-state path of weight $\lambda$. Let $\mathcal{P}(\lambda) := \{p = (p(k))_{k=0}^\infty \in B^{\otimes \infty} | p(k) \in B, p(k) = b_k \text{ for all } k \gg 0\}$. The elements of $\mathcal{P}(\lambda)$ are called the $\lambda$-paths. The following result gives the path realization of $B(\lambda)$.

**Proposition 2.5.** [3] There exists an isomorphism of classical crystals

$$\Psi_\lambda : B(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda)$$

given by $u_\lambda \mapsto p_\lambda$,

where $u_\lambda$ is the highest weight vector in $B(\lambda)$.

### 3. $U_q(D_4^{(3)})$-Perfect crystals

In this section we recall the perfect crystal $B^{1,l}$ for the quantum affine algebra $U_q(D_4^{(3)})$ of level $l > 0$ constructed in [7].

First we fix the data for $D_4^{(3)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}, \{h_0, h_1, h_2\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix},$$

and its Dynkin diagram is given by:

$$\alpha_0 - \alpha_1 \leq \alpha_2$$

The standard null root $\delta$ and the canonical central element $c$ are given by

$$\delta = \alpha_0 + 2\alpha_1 + \alpha_2 \quad \text{and} \quad c = h_0 + 2h_1 + 3h_2,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 - \delta, \alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2, \alpha_2 = -3\Lambda_1 + 2\Lambda_2$.

For positive integer $l$ define the set

$$B = B^{1,l} = \left\{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^6 \left| \begin{array}{c}
x_3 \equiv \bar{x}_3 \pmod{2}, \\
\sum_{i=1,2} (x_i + \bar{x}_i) + \frac{x_1 + \bar{x}_1}{2} \leq l
\end{array} \right. \right\}.$$
For $b = (x_1, x_2, x_3, x_3, \bar{x}_2, \bar{x}_1) \in B$ we denote

$$s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1, \
\text{and} \
\text{if } (E_1),$$

$$t(b) = x_2 + \frac{x_3 + \bar{x}_3}{2},$$

and

$$z_1 = \bar{x}_1 - x_1, \quad z_2 = \bar{x}_2 - x_3, \quad z_3 = x_3 - x_2, \quad z_4 = \frac{\bar{x}_3 - x_3}{2},$$

$$A = (0, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + 3z_4, 2z_1 + z_2 + 3z_4).$$

Now we define conditions $(E_1)$-$(E_6)$ and $(F_1)$-$(F_6)$ as follows.

$$E_i \quad (1 \leq i \leq 6) \quad \text{is defined from } (F_i) \quad \text{by replacing } > \text{ (resp. } \leq) \quad \text{with } \geq \text{ (resp. } <).$$

Then for $b = (x_1, x_2, x_3, x_3, \bar{x}_2, \bar{x}_1) \in B$, we define $\tilde{e}_i(b), \tilde{f}_i(b), \tilde{\varphi}_i(b), i = 0, 1, 2$ as follows. We use the convention: $(a)_+ = \max(a, 0)$.

$$\tilde{e}_0(b) = \begin{cases} 
(x_1 - 1, \ldots) & \text{if } (E_1), \\
(x_2 - 1, \bar{x}_3 - 1, \ldots, \bar{x}_1 + 1) & \text{if } (E_2), \\
(x_3 - 2, \ldots, \bar{x}_2 + 1, \ldots) & \text{if } (E_3), \\
(x_2 - 1, \ldots, \bar{x}_3 + 2, \ldots) & \text{if } (E_4), \\
(x_1 - 1, \ldots, x_3 + 1, \bar{x}_3 + 1, \ldots) & \text{if } (E_5), \\
(\ldots, \bar{x}_1 + 1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} 
(x_1 + 1, \ldots) & \text{if } (F_1), \\
(x_3 + 1, \bar{x}_3 + 1, \ldots, \bar{x}_1 - 1) & \text{if } (F_2), \\
(x_3 + 2, \ldots, \bar{x}_2 - 1, \ldots) & \text{if } (F_3), \\
(x_2 + 1, \ldots, \bar{x}_3 - 2, \ldots) & \text{if } (F_4), \\
(x_1 + 1, \ldots, x_3 - 1, \bar{x}_3 - 1, \ldots) & \text{if } (F_5), \\
(\ldots, \bar{x}_1 - 1) & \text{if } (F_6), \end{cases}$$
for dimensional subspaces of $V$

let $V$ be the subalgebra of $U$

for

the irreducible integrable highest weight $\lambda$, denote the length of $\lambda$. It is known that for $\bar{x}$, $\bar{x}$ for some $j$ becomes negative or $s(b)$ exceeds $l$, we understand it to be 0.

For $b \in B$ if $\bar{e}_i(b)$ or $\bar{f}_i(b)$ does not belong to $B$, namely, if $x_j$ or $\bar{x}_j$ for some $j$ becomes negative or $s(b)$ exceeds $l$, we understand it to be 0.

The following is one of the main results in [7]:

**Theorem 3.1.** [7] For the quantum affine algebra $U_q(D_4^{(3)})$ the set $B = B_1^{l,l}$ equipped with the maps $\bar{e}_i, \bar{f}_i, \bar{e}_i, \bar{f}_i, i = 0, 1, 2$ is a perfect crystal of level $l$.

As was shown in [7], the minimal elements are given by:

$$(B)_{\text{min}} = \{(\alpha, \beta, \beta, \beta, \beta, \alpha) | \alpha, \beta \in \mathbb{Z}_{\geq 0}, 2\alpha + 3\beta \leq l\}.$$

4. Demazure Modules and Demazure crystals

Let $W$ denote the Weyl group for the affine Lie algebra $\mathfrak{g}$ generated by the simple reflections $\{r_i | i \in I\}$, where $r_i(\mu) = \mu - \langle \mu, h_i \rangle e_i$ for all $\mu \in \mathfrak{h}^*$. For $w \in W$ let $l(w)$ denote the length of $w$ and $<$ denote the Bruhat order on $W$. Let $U_q^+(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the $e_i$’s. For $\lambda \in P^+$ with $\lambda(d) = 0$, consider the irreducible integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight $\lambda$ and highest weight vector $v_{\lambda}$. Let $(L(\lambda), B(\lambda))$ denote the crystal basis for $V(\lambda)$ ([6], [8]). It is known that for $w \in W$, the extremal weight space $V(\lambda)_{w,\lambda}$ is one dimensional. Let $V_w(\lambda)$ denote the $U_q^+(\mathfrak{g})$-module generated by $V(\lambda)_{w,\lambda}$. These modules $V_w(\lambda)$ ($w \in W$) are called the Demazure modules. They are finite dimensional subspaces of $V(\lambda)$ and satisfy the properties: $V(\lambda) = \bigcup_{w \in W} V_w(\lambda)$ and for $w, w' \in W$ with $w \leq w'$ we have $V_w(\lambda) \subseteq V_{w'}(\lambda)$.

In 1993, Kashiwara [9] showed that for each $w \in W$, there exists a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{V_w(\lambda) \cap L(\lambda)}{V_w(\lambda) \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}b.$$
The set $B_w(\lambda)$ is the crystal for the Demazure module $V_w(\lambda)$. The Demazure crystal $B_w(\lambda)$ has the following recursive property:

\[
\text{(4.1)} \quad \text{If } r_i w \succ w, \text{ then } B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \hat{f}_n^i B_w(\lambda) \setminus \{0\}.
\]

Suppose $\lambda(c) = l \geq 1$, and suppose $B$ be a perfect crystal of level $l$ for $U_q(\mathfrak{g})$. Then the crystal $B(\lambda)$ is isomorphic to the set of paths $\mathcal{P}(\lambda) = \mathcal{P}(\lambda, B)$ (see Proposition 2.5). Under this isomorphism the highest weight element $u_\lambda \in B(\lambda)$ is identified with the ground state path $p_\lambda = (\cdots \otimes b_3 \otimes b_2 \otimes b_1)$. We recall the path realizations of the Demazure crystals $B_w(\lambda)$ from [10]. Fix positive integers $d$ and $\kappa$. For a sequence of integers $\{i_j^{(j)}\} j \geq 1, 1 \leq a \leq d \in \{0, 1, \ldots, n\}$, we define the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq d\}$ of $B$ by

\[
B_0^{(j)} = \{b_j\}, B_a^{(j)} = \bigcup_{k \geq 0} \hat{f}_k^{i_j^{(j)}} B_{a-1}^{(j)} \setminus \{0\}.
\]

Next we define $B_{a}^{(j+1, j)}$, $j \geq 1, 0 \leq a \leq d$ by

\[
B_0^{(j+1, j)} = B_0^{(j+1)} \otimes B_{d}^{(j)}, B_{a}^{(j+1, j)} = \bigcup_{k \geq 0} \hat{f}_k^{i_j^{(j+1)}} B_{a-1}^{(j+1)} \setminus \{0\},
\]

and continue until we define

\[
\begin{align*}
B_0^{(j+\kappa-1, \ldots, j)} &= B_0^{(j+\kappa-1)} \otimes B_{d}^{(j+\kappa-2, \ldots, j)} \\
B_{a}^{(j+\kappa-1, \ldots, j)} &= \bigcup_{k \geq 0} \hat{f}_k^{i_j^{(j+\kappa-1, \ldots, j)}} B_{a-1}^{(j+\kappa-1, \ldots, j)} \setminus \{0\}.
\end{align*}
\]

Furthermore, we define a sequence $\{w^{(k)}\}$ of elements in the Weyl group $W$ by

\[
w^{(0)} = 1, w^{(k)} = r_{i_j^{(j)}} w^{(k-1)}, \text{ for } k > 0.
\]

Here, $j$ and $a$ are uniquely determined from $k$ by the relation $k = (j-1)d + a, j \geq 1, 1 \leq a \leq d$. Now for $k \geq 0$, we define subsets $\mathcal{P}^{(k)}(\lambda, B)$ of $\mathcal{P}(\lambda, B)$ as follows:

\[
\mathcal{P}^{(0)}(\lambda, B) = \{p_\lambda\},
\]

and for $k > 0$,

\[
\mathcal{P}^{(k)}(\lambda, B) = \begin{cases} \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_{a}^{(j, \ldots, 1)} & \text{if } j < \kappa \\ \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_{a}^{(j, \ldots, j-\kappa+1)} \otimes B \otimes (j-\kappa) & \text{if } j \geq \kappa. \end{cases}
\]

The following theorem shows that under certain conditions, the path realizations of the Demazure crystals $B_w(\kappa) (\lambda)$ is isomorphic to $\mathcal{P}^{(k)}(\lambda, B)$ and hence have tensor product-like structures.

**Theorem 4.1. [10]** Let $\lambda \in \tilde{P}^+$ with $\lambda(c) = l$ and $B$ be a perfect crystal of level $l$ for the quantum affine Lie algebra $U_q(\mathfrak{g})$. For fixed positive integers $d$ and $\kappa$, suppose we have a sequence of integers $\{i_j^{(j)}\} j \geq 1, 1 \leq a \leq d \in \{0, 1, 2, \ldots, n\}$ satisfying the conditions:

1. for any $j \geq 1$, $B_d^{(j+\kappa-1, \ldots, j)} = B_d^{(j+\kappa-1, \ldots, j+1)} \otimes B$,
2. for any $j \geq 1, 1 \leq a \leq d$, $\langle \lambda_j, h^{(j)} \rangle \leq \varepsilon_w^{(j)}(b), b \in B_a^{(j)}$,
3. the sequence of elements $\{w^{(k)}\}_{k \geq 0}$ is an increasing sequence with respect to the Bruhat order.
Then we have \( B_{\omega_6}(\lambda) \cong \mathcal{P}^{(k)}(\lambda, B) \).

The positive integer \( \kappa \) in Theorem 4.1 is called the *mixing index*. It is conjectured that for any affine Lie algebra \( \mathfrak{g} \), the mixing index \( \kappa \leq 2 \). It is known that the mixing index \( \kappa \) is dependent on the choice of the perfect crystal (see [15]). For \( \lambda = l\Lambda \) (where \( \Lambda \) is a dominant weight of level one) and the perfect crystal \( B = B^{1.\Lambda} \) it is known that there exists a suitable sequence of Weyl group elements \( \{ \tilde{w}^{(k)} \} \) which satisfy the conditions in Theorem 4.1 with \( \kappa = 1 \) for any classical quantum affine Lie algebra \( \mathcal{U}(B_3), \mathcal{U}(D_4) \), and \( \mathcal{U}(G_2^{(1)}) \).

For \( b \in B \), let \( \tilde{f}^{\text{max}}_i(b) \) denote \( \tilde{f}_i^{\omega^j(b)}(b) \). For \( j \geq 1 \), we set

\[
\begin{align*}
\tilde{b}^{(j)}_0 &= b_j, & \tilde{b}^{(j)}_a &= \tilde{f}^{\text{max}}_i(b_{a-1}) & (a = 1, 2, \ldots, d).
\end{align*}
\]

The following Proposition ([10], Proposition 2) will be useful to check the validity of condition (3) in Theorem 4.1.

**Proposition 4.2.** [10] For \( w \in W \), if \( \langle \mu^\prime, h_{\bar{j}} \rangle > 0 \) for some \( \mu^\prime \in \tilde{P}^+ \), then \( \tilde{r}_{\bar{j}}w > w \).

5. \( \mathcal{U}(D_4^{(3)}) \)-Demazure crystals

In this section we show that for the perfect crystal of level \( 3l \) for the quantum affine algebra \( \mathcal{U}(D_4^{(3)}) \) given in Section 3, there is a suitable sequence of Weyl group elements \( \{ \tilde{w}^{(k)} \} \) which satisfy the conditions (1), (2) and (3) for \( \lambda = l\Lambda_2 \) and hence Theorem 4.1 holds in this case with \( \kappa = 1 \). Thus we have path realizations of the corresponding Demazure crystals with tensor product-like structures.

For \( \lambda = l\Lambda_2, l \geq 1 \), the \( \Lambda_2 \)-minimal element in the perfect crystal \( B = B^{1.3l} \) is \( \tilde{b} = (0, l, l, l, 0) \) and in this case \( \lambda_j = \lambda = l\Lambda_2 \) for \( j \geq 1 \). Hence \( b_j = \tilde{b} \) for all \( j \geq 1 \). Set \( d = 6 \) and choose the sequence \( \{ i^{(j)}_a \mid j \geq 1, 1 \leq a \leq 6 \} \) defined by

\[
\begin{align*}
i^{(j)}_1 &= i^{(j)}_3 = 2, & i^{(j)}_2 &= i^{(j)}_4 = i^{(j)}_6 = 1, & i^{(j)}_5 &= 0. 
\end{align*}
\]

Hence, by the action of \( \tilde{f}_1 \) on \( B \) we have, for \( j \geq 1 \)

\[
\begin{align*}
b^{(j)}_0 &= (0, l, l, l, l, 0), & b^{(j)}_1 &= \tilde{f}^{\text{max}}_2(i^{(j)}_0) = (0, 0, 3l, l, l, 0),

b^{(j)}_2 &= \tilde{f}^{\text{max}}_1(b^{(j)}_1) = (0, 0, 0, 4l, l, 0), & b^{(j)}_3 &= \tilde{f}^{\text{max}}_2(b^{(j)}_2) = (0, 0, 0, 3l, l, 0),

b^{(j)}_4 &= \tilde{f}^{\text{max}}_1(b^{(j)}_3) = (0, 0, 0, 0, 4l, l), & b^{(j)}_5 &= \tilde{f}^{\text{max}}_2(b^{(j)}_4) = (3l, 0, 0, 0, 0, 0),

b^{(j)}_6 &= \tilde{f}^{\text{max}}_1(b^{(j)}_5) = (0, 3l, 0, 0, 0, 0).
\end{align*}
\]

We define conditions \( P \) and \( Q_j, 1 \leq j \leq 6 \) for \( b \in B \) as follows:
\begin{align*}
(P) : & \quad z_3 \geq 0; z_3 + 3z_4 \geq (-2z_2)_+; z_1 + z_2 + z_3 + 3z_4 \geq 0; t(b) < 2l; s(b) < 3l \\
(Q_1) : & \quad z_3 < 0; z_3 \geq 0; z_1 + z_2 + 3z_4 \geq 0; z_1 + 2z_2 + z_3 + 3z_4 \geq 0; t(b) \leq 2l; s(b) \leq 3l \\
(Q_2) : & \quad z_2 \geq 0; z_4 < 0; z_3 + 3z_4 < 0; z_1 + z_2 \geq 0; z_1 + 2z_2 + z_3 \geq 0; s(b) \leq 3l \\
(Q_3) : & \quad z_2 \geq 0; z_4 \geq 0; z_3 + 3z_4 \geq 0; z_1 + z_2 < 0; z_2 + z_3 \geq 0; s(b) \leq 3l \\
(Q_4) : & \quad z_2 \geq 0; z_3 \geq 0; z_3 + z_3 + 3z_4 < 0; z_3 + 3z_4 \geq 0; t(b) < 2l; s(b) \leq 3l \\
(Q_5) : & \quad x_1 > 0; z_3 \geq 0; z_3 + 3z_4 \geq 0; z_1 + z_2 + z_3 + 3z_4 \geq 0; \\
& \quad t(b) < 2l; s(b) \leq 3l \\
(Q_6) : & \quad x_1 > 0; z_3 \geq 0; z_4 \geq 0; z_1 + z_2 + 3z_4 < 0; z_2 + 3z_4 \geq 0; z_2 + z_3 \geq 0; t(b) < 2l; s(b) \leq 3l \\
\end{align*}

By direct calculations it can be seen that the subsets \( \{B^{(j)}_a \mid j \geq 1, 0 \leq a \leq 6 \} \)
of \( B \) are given as follows.

\begin{align*}
B^{(j)}_0 &= \{(0, l, l, l, 0, l, 0)\} \\
B^{(j)}_1 &= B^{(j)}_0 \cup \{(0, x_2, x_3, l, l, 0, 0) \mid z_3 > 0, s(b) = 3l\} \\
B^{(j)}_2 &= B^{(j)}_1 \cup \{(0, x_2, x_3, x_3, l, 0, 0) \mid z_2 < 0, z_3 \geq 0, s(b) = 3l\} \\
B^{(j)}_3 &= B^{(j)}_2 \cup \{(0, x_2, x_3, x_2, x_2, 0, 0) \mid z_3, z_3 + 3z_4 \geq 0, t(b) < 2l, s(b) = 3l\} \\
B^{(j)}_4 &= B^{(j)}_3 \cup \{(0, x_2, x_3, x_3, x_2, x_1, 0) \mid x_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq (2z_2)_+, t(b) < 2l, s(b) = 3l\} \\
B^{(j)}_5 &= B^{(j)}_4 \cup C \cup D_1 \cup D_2 \cup \ldots D_6 \\
B^{(j)}_6 &= B,
\end{align*}

where \( C = \{(0, x_2, x_3, x_2, x_2, x_2) \mid (P) \text{ holds}\} \) and for \( 1 \leq j \leq 6 \),
\( D_j = \{(x_1, x_2, x_3, x_3, x_2, x_1) \mid (Q_j) \text{ holds}\} \).

The following lemma is useful and follows by easy calculations.

**Lemma 5.1.** Let \( k \in \mathbb{Z}_{>0} \) and \( k = 6(j-1) + a, j \geq 1, 1 \leq a \leq 6 \). Then we have

\[ w^{(k)}_a \Lambda_2 = \Lambda_2 - m_0 a_0 - m_1 a_1 - m_2 a_2 \text{ where} \]

\[ m_0 = \begin{cases} 3j^2 + 3j & \text{if } a = 1, 2, 3, 4 \\ 3j^2 + 9j + 6 & \text{if } a = 5, 6 \end{cases} \]

\[ m_1 = \begin{cases} 6j^2 + 3j & \text{if } a = 1 \\ 6j^2 + 9j + 3 & \text{if } a = 2, 3 \end{cases} \]

\[ m_2 = \begin{cases} 6j^2 + 12j + 6 & \text{if } a = 4, 5 \\ 3j^2 + 15j + 9 & \text{if } a = 6 \end{cases} \]

\[ m_3 = \begin{cases} 3j^2 + 3j + 1 & \text{if } a = 1, 2 \\ 3j^2 + 6j + 3 & \text{if } a = 3, 4, 5, 6 \end{cases} \]

**Theorem 5.2.** For \( \lambda = \lambda \Lambda_2, l \geq 1 \) and the given perfect crystal \( B = B_1^{\Lambda_2} \) for the quantum affine algebra \( U_q(D_4^{(3)}) \) with \( d = 6 \) and the sequence \( \{s_a^{(j)}\} \) given in \( [5, 7] \), conditions (1), (2) and (3) in Theorem 4.1 hold with \( \kappa = 1 \). Hence we have path realizations of the corresponding Demazure crystals \( B_{w^{(k)}_a}(\Lambda_2) \) for \( U_q(D_4^{(3)}) \) with tensor product-like structures.

**Proof.** We have already shown above by explicit descriptions of the subsets \( B^{(j)}_a \) that \( B^{(j)}_6 = B \) which implies that condition (1) in Theorem 4.1 holds for \( \kappa = 1 \).
Observe that \( \langle l\Lambda_2, h_{i(a)}^{(j)} \rangle = 0 \) for all \( b \in B_{a-1}^{(j)}, a = 2, 4, 5, 6 \). Also \( \langle l\Lambda_2, h_{i(a)}^{(j)} \rangle = l \) for \( a = 1, 3 \). Observe that for all \( b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in B_0^{(j)} \) or \( B_0^{(j)} \) we have \( \bar{x}_2 = l \). Hence \( \varepsilon_{h_i^{(j)}}(b) = \bar{x}_2 + \frac{1}{2}(x_3 - \bar{x}_3)_+ = l + \frac{1}{2}(x_3 - \bar{x}_3)_+ \geq l \) for \( a = 1, 3 \) and condition (2) holds.

To prove condition (3), we use Lemma 5.1 to obtain:

\[
\langle w^{(k)} l\Lambda_2, h_{i(a)}^{(j)} \rangle = \begin{cases} 
3j + 2 & \text{if } a = 2 \\
3j + 3 & \text{if } a = 5, 6 \\
3j + 4 & \text{if } a = 6 \\
6j + 3 & \text{if } a = 1 \\
6j + 6 & \text{if } a = 4 \\
\end{cases}
\]

where \( k = 6(j - 1) + a, j \geq 1 \). Hence \( \langle w^{(k)} l\Lambda_2, h_{i(a)}^{(j)} \rangle > 0 \) for all \( j \geq 1 \). Therefore, by Proposition 4.2 \( w^{(k+1)} = r_{i(a)}^{(j)} w^{(k)} \), which implies that condition (3) holds.

\[ \square \]

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