1. Introduction

Blueprints are a common generalization of commutative (semi)rings and monoids. The associated geometric objects, blue schemes, are therefore a common generalization of usual scheme theory and $\mathbb{F}_1$-geometry (as considered by Kato [5], Deitmar [3] and Connes–Consani [2]). The possibility of forming semiring schemes allows us to talk about idempotent schemes and tropical schemes (cf. [11]). All this is worked out in [9].

It is known, though not covered in literature yet, that the Proj-construction from usual algebraic geometry has an analogue in $\mathbb{F}_1$-geometry (after Kato, Deitmar and Connes–Consani). In this note we describe a generalization of this to blueprints. Privately, Koen Thas has announced a treatment of Proj for monoidal schemes (see [13]).

We follow the notations and conventions of [10]. Namely, all blueprints that appear in this note are proper and with a zero. We remark that the following constructions can be carried out for the more general notion of a blueprint as considered in [9]; the reason that we restrict to proper blueprints with a zero is that this allows us to adopt a notation that is common in $\mathbb{F}_1$-geometry.

Namely, we denote by $A^n_B$ the (blue) affine $n$-space $\text{Spec}(B[T_1,...,T_n])$ over a blueprint $B$. In case of a ring, this does not equal the usual affine $n$-space since $B[T_1,...,T_n]$ is not closed under addition. Therefore, we denote the usual affine
n-space over a ring $B$ by $\mathbb{A}^n_B = \text{Spec}(B[T_1, \ldots, T_n]^+)$. Similarly, we use a superscript “+” for the usual projective space $\mathbb{P}^n_B$ and the usual Grassmannian $\text{Gr}(k, n)_B^+$ over a ring $B$.

2. Graded blueprints and Proj

Let $B$ be a blueprint and $M$ a subset of $B$. We say that $M$ is additively closed in $B$ if for all additive relations $b = \sum a_i$ with $a_i \in M$ also $b$ is an element of $M$. Note that, in particular, 0 is an element of $M$. A graded blueprint is a blueprint $B$ together with additively closed subsets $B_i$ for $i \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ and $a \in B_i, b \in B_j$, the product $ab$ is an element of $B_{i+j}$ and such that for every $b \in B$, there is a unique finite subset $I$ of $\mathbb{N}$ and unique non-zero elements $a_i \in B_i$ for every $i \in I$ such that $b = \sum a_i$. An element of $\bigcup_{i \geq 0} B_i$ is called homogeneous. If $a \in B_i$ is non-zero, then we say, more specifically, that $a$ is homogeneous of degree $i$.

We collect some immediate facts for the graded blueprint $B$ as above. The subset $B_0$ is multiplicatively closed, i.e. $B_0$ can be seen as a subblueprint of $B$. The subblueprint $B_0$ equals $B$ if and only if for all $i > 0$, $B_i = \{0\}$. In this case we say that $B$ is trivially graded. By the uniqueness of the decomposition into homogeneous elements, we have $B_i \cap B_j = \{0\}$ for $i \neq j$.

This means that the union $\bigcup_{i \geq 0} B_i$ has the structure of a wedge product $\bigvee_{i \geq 0} B_i$. Since $\bigvee_{i \geq 0} B_i$ is multiplicatively closed, it can be seen as a subblueprint of $B$. We define $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$ and call the subblueprint $B_{\text{hom}}$ the homogeneous part of $B$.

Let $S$ be a multiplicitive subset of $B$. If $b/s$ is an element of the localization $S^{-1}B$ where $f$ is homogeneous of degree $i$ and $s$ is homogeneous of degree $j$, then we say that $b/s$ is a homogeneous element of degree $i - j$. We define $S^{-1}B_0$ as the subet of homogeneous elements of degree 0. It is multiplicatively closed, and inherits thus a subblueprint structure from $S^{-1}B$. If $S$ is the complement of a prime ideal $p$, then we write $B_{(p)}$ for the subblueprint $(B_{(p)})_0$ of homogeneous elements of degree 0 in $B_p$.

An ideal $I$ of a graded blueprint $B$ is called homogeneous if it is generated by homogeneous elements, i.e. if for every $c \in I$, there are homogeneous elements $p_i, q_j \in I$ and elements $a_i, b_j \in B$ and an additive relation $\sum a_ip_i + c = \sum b_jq_j$ in $B$.

Let $B$ be a graded blueprint. Then we define Proj $B$ as the set of all homogeneous prime ideals $p$ of $B$ that do not contain $B_0^+ = \bigvee_{i \geq 0} B_i$. The set $X = \text{Proj } B$ comes together with the topology that is defined by the basis

$$U_h = \{p \in X | h \notin p\}$$

where $h$ ranges through $B_{\text{hom}}$ and with a subset sheaf $O_X$ that is the sheafification of the association $U_h \mapsto B[h^{-1}]_0$ where $B[h^{-1}]$ is the localization of $B$ at $S = \{h^i | i \geq 0\}$.

Note that if $B$ is a ring, the above definitions yield the usual construction of Proj $B$ for graded rings. In complete analogy to the case of graded rings, one proves the following theorem:

**Theorem 1.** The space $X = \text{Proj } B$ together with $O_X$ is a blue scheme. The stalk at a point $p \in \text{Proj } B$ is $O_{X,p} = B_{(p)}$. If $h \in B_{\text{hom}}^+$, then $U_h \cong \text{Spec } B[h^{-1}]_0$. The inclusions $B_0 \hookrightarrow B[h^{-1}]_0$ yield morphisms $\text{Spec } B[h^{-1}]_0 \to \text{Spec } B_0$, which glue to a structural morphism $\text{Proj } B \to \text{Spec } B_0$. \hfill \Box

If $B$ is a graded blueprint, then the associated semiring $B^+$ inherits a grading. Namely, let $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$ the homogeneous part of $B$. Then we can define $B^+_0$ as the additive closure of $B_0$ in $B^+$, i.e. as the set of all $b \in B$ such that there is an additive relation of the form $b = \sum a_i$ in $B$ with $a_i \in B_i$. Then $\bigvee B^+_0$ defines a grading of $B^+$. Similarly, the grading of $B$ induces a grading on a tensor product $B \otimes_D C$ with respect to blueprint morphisms $C \to B$ and $C \to D$ under the assumption that the image of $C$ is contained in $B_0$. Consequently, a grading of $B$ implies a grading of $B_{\text{inv}} = B \otimes_{\mathbb{F}_1} [\mathbb{F}_1^+]$ (see [9, Lemma 1.4] and [10, p. 11]) and of the ring $B^+_\mathbb{Z} = B_{\text{inv}}^+$. Analogously, if both $B$ and $D$ are graded and the image of $C$ lies in both $B_0$ and $D_0$, then $B \otimes_D C$ inherits a grading from the gradings of $B$ and $D$.

3. Projective space

The functor $\text{Proj}$ allows the definition of the projective space $\mathbb{P}^n_B$ over a blueprint $B$. Namely, the free blueprint $C = B[T_0, \ldots, T_n]$ over $B$ comes together with a natural grading (cf. [9, Section 1.12] for the definition of free blueprints). Namely, $C_i$ consists of all monomials $bT_0^{e_0} \cdots T_n^{e_n}$ such that $e_0 + \cdots + e_n = i$ where $b \in B$. Note that $C_0 = B$ and $C_0 = C$.

The projective space $\mathbb{P}^n_B$ is defined as $\text{Proj } B[T_0, \ldots, T_n]$. It comes together with a structure morphism $\mathbb{P}^n_B \to \text{Spec } B$.

In case of $B = \mathbb{F}_1^+$, the projective space $\mathbb{P}^n_{\mathbb{F}_1^+}$ is the monoidal scheme that is known from $\mathbb{F}_1$-geometry (see [4], [1, Section 3.1.4]) and [10, Ex. 1.6]). The topological space of $\mathbb{P}^n_{\mathbb{F}_1^+}$ is finite. Its points correspond to the homogeneous prime ideals $(S_i)_{i \leq n} \subset \mathbb{F}_1^+[S_0, \ldots, S_n]$ where $I$ ranges through all proper subsets of $\{0, \ldots, n\}$.

In case of a ring $B$, the projective space $\mathbb{P}^n_B$ does not coincide with the usual projective space since the free blueprint $B[S_0, \ldots, S_n]$ is not a ring, but merely the blueprint of all monomials of the form $bS_0^{e_0} \cdots S_n^{e_n}$ with $b \in B$. However, the associated scheme $\mathbb{P}^n_B = (\mathbb{P}^n_B)^+$ coincides with the usual projective space over $B$, which equals $\text{Proj } B[S_0, \ldots, S_n]^+$.

4. Closed subschemes

Let $X$ be a scheme of finite type. By an $\mathbb{F}_1$-model of $X$ we mean a blue scheme $X$ of finite type such that $X^+_\mathbb{Z}$ is isomorphic to $X$. Since a finitely generated $\mathbb{Z}$-algebra is, by definition, generated by a finitely generated multiplicative
subset as a \( \mathbb{Z} \)-module, every scheme of finite type has an \( F_1 \)-model. It is, on the contrary, true that a scheme of finite type possesses a large number of \( F_1 \)-models.

Given a scheme \( X \) with an \( F_1 \)-model \( X \), we can associate to every closed subscheme \( Y \) of \( X \) the following closed subscheme \( Y \) of \( X \), which is an \( F_1 \)-model of \( Y \). In case that \( X = \text{Spec } B \) is the spectrum of a blueprint \( B = A//R \), and thus \( X \sim \text{Spec } B^+ \) is an affine scheme, we can define \( Y \) as \( \text{Spec } C \) for \( C = A//R(Y) \) where \( R(Y) \) is the pre-addition that contains \( \sum a_i = \sum b_j \) whenever \( \sum a_i = \sum b_j \) holds in the coordinate ring \( \Gamma(Y) \) of \( Y \). This is a process that we used already in [10, Section 3].

Since localizations commute with additive closures, i.e. \( (S^{-1}B)^+ = S^{-1}(B^+) \) where \( S \) is a multiplicative subset of \( B \), the above process is compatible with the restriction to affine opens \( U \subset X \). This means that given \( U = \text{Spec } (S^{-1}B) \), which is an \( F_1 \)-model for \( X \), \( U \) is an \( F_1 \)-model of \( Y \) that is associated to the closed subscheme \( Y' = X' \times_X Y \) of \( X' \) by the above process is the spectrum of the blueprint \( S^{-1}C \). Consequently, we can associate with every closed subscheme \( Y \) of a scheme \( X \) with an \( F_1 \)-model \( X \) a closed subscheme \( Y \) of \( X \), which is an \( F_1 \)-model of \( Y \); namely, we apply the above process to all affine open subschemes of \( X \) and glue them together, which is possible since additive closures commute with localizations.

In case of a projective variety, i.e. a closed subscheme \( Y \) of a projective space \( +\mathbb{P}_2^n \), we derive the following description of the associated \( F_1 \)-model \( Y \) in \( \mathbb{F}_1 \) by homogeneous coordinate rings. Let \( C \) be the homogeneous coordinate ring of \( Y \), which is a quotient of \( \mathbb{Z}[S_0, \ldots, S_n]^{+} \) by a homogeneous ideal \( I \). Let \( R \) be the pre-addition on \( \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]] \) that consists of all relations \( \sum a_i = \sum b_j \) such that \( \sum a_i = \sum b_j \) in \( C \). Then \( B = \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]]//R \) inherits a grading from \( \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]] \) by defining \( B_i \) as the image of \( \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]]_i \) in \( B \). Note that \( B \subset C \) and that the sets \( B_i \) equal the intersections \( B_i = C_i \cap B \) for \( i \geq 0 \) where \( C_i \) is the homogeneous part of degree \( i \) of \( C \). Then the \( F_1 \)-model \( Y \) of \( Y \) equals \( \text{Proj } B \).

### 5. \( F_1 \)-models for Grassmannians

One of the simplest examples of projective varieties that is not toric is the Grassmannian \( \text{Gr}(2, 4) \). The problem of finding \( F_1 \)-models for Grassmannians was originally posed by Soulé in [12], and solved by the authors by obtaining a torification from the Schubert cell decomposition (cf. [8,7]). In this note, we present Section 3.

Let \( \text{Gr}(2, 4) \) be the homogeneous coordinate ring of \( \text{Gr}(2, 4) \), and let \( \text{Proj } \mathbb{F}_1[\text{Gr}(2, 4)] \) be the image of \( \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]]_i \) in \( \mathbb{F}_1[\mathbb{Z}[S_0, \ldots, S_n]]//R \).

Define the blueprint \( \mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) = \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]/R \) where the congruence \( R \) is generated by the Plücker relation \( x_{12}x_{34} + x_{14}x_{23} = x_{13}x_{24} \) (the signs have been picked to ensure that the totally positive part of the Grassmannian is preserved, cf. [6]). Since \( R \) is generated by a homogeneous relation, \( \mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) \) inherits a grading from the canonical morphism

\[
\pi : \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] \longrightarrow \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]/R.
\]

Let \( \text{Gr}(2, 4)_{\mathbb{F}_1} = \text{Proj } \mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) \). The base extension \( \text{Gr}(2, 4)^+ \) is the usual Grassmannian, and \( \pi \) defines a closed embedding of \( \text{Gr}(2, 4)_{\mathbb{F}_1} \) into \( \mathbb{P}_{2, 4}^+ \), which extends to the classical Plücker embedding \( \text{Gr}(2, 4)_C^+ \subset \mathbb{P}_{2, 4}^+ \).

Homogeneous prime ideals in \( \mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) \) are described by their generators as the proper subsets \( I \subset \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\} \) such that \( I \) is either contained in one of the sets \( \{x_{12}, x_{34}\}, \{x_{14}, x_{23}\}, \{x_{13}, x_{24}\}, \) or otherwise \( I \) has a non-empty intersection with all three of them. In other words, \( I \) cannot contain elements in two of the above sets without also containing an element of the third one. \( \text{Gr}(2, 4)_{\mathbb{F}_1} \) is depicted in Fig. 1. It consists of \( 6 + 12 + 11 + 6 + 1 = 36 \) prime ideals of ranks

![Fig. 1](https://example.com/fig1.png)
0, 1, 2, 3 and 4, respectively (cf. [10, Def. 2.3] for the definition of rank), thus resulting in a model essentially different to the one presented in [8], which had 35 points corresponding to the coefficients of $N_{\text{Gr}(2,4)}(q) = 6 + 12(q - 1) + 11(q - 1)^2 + 5(q - 1)^3 + 1(q - 1)^4$. In spite of arising from different constructions, both $\mathbb{F}_1$-models for $\text{Gr}(2,4)$ have $6 = \binom{4}{2}$ closed points, supporting the naive combinatorial interpretation of $\text{Gr}(2,4)_{\mathbb{F}_1}$. These six points correspond to the $\mathbb{F}_1$-rational Tits points of $\text{Gr}(2,4)_{\mathbb{F}_1}$, which reflect the naive notion of $\mathbb{F}_1$-rational points of an $\mathbb{F}_1$-scheme (cf. [10, Section 2.2]).

As in the classical setting, the Grassmannian $\text{Gr}(2,4)_{\mathbb{F}_1}$ does admit a covering by six $\mathbb{F}_1$-models of affine 4-space, which correspond to the open subsets of $\text{Gr}(2,4)_{\mathbb{F}_1}$ where one of the generators is non-zero. However, these $\mathbb{F}_1$-models of affine 4-space are not the standard model $A^4_{\mathbb{F}_1} = \text{Spec}(\mathbb{F}_1[a, b, c, d])$, but the “$2 \times 2$-matrices” $M_{2,\mathbb{F}_1} = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]/\langle ad = bc + D \rangle)$ in case that one of $x_{12}$, $x_{34}$, $x_{14}$ or $x_{23}$ is non-zero, and the “twisted $2 \times 2$-matrices” $M^*_1,\mathbb{F}_1 = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]/\langle ad + bc = D \rangle)$ in case that one of $x_{13}$ or $x_{24}$ is non-zero.

References

[1] C. Chu, O. Lorscheid, R. Santhanam, Sheaves and $K$-theory for $\mathbb{F}_1$-schemes, Adv. Math. 229 (4) (2012) 2239–2286.
[2] A. Connes, C. Consani, Characteristic 1, entropy and the absolute point, preprint, arXiv:0911.3537v1, 2009.
[3] A. Deitmar, Schemes over $\mathbb{F}_1$, in: Number Fields and Function Fields—Two Parallel Worlds, in: Progr. Math., vol. 239, Birkhäuser Boston, Boston, MA, 2005, pp. 87–100.
[4] A. Deitmar, $\mathbb{F}_1$-schemes and toric varieties, Beiträge Algebra Geom. 49 (2) (2008) 517–525.
[5] K. Kato, Toric singularities, Amer. J. Math. 116 (5) (1994) 1073–1099.
[6] J. López Peña, $\mathbb{F}_1$-models for cluster algebras and total positivity, in preparation.
[7] J. López Peña, O. Lorscheid, Mapping $\mathbb{F}_1$-land an overview of geometries over the field with one element, in: Noncommutative Geometry, Arithmetic and Related Topics, Johns Hopkins University Press, 2011, pp. 241–265.
[8] J. López Peña, O. Lorscheid, Torified varieties and their geometries over $\mathbb{F}_1$, Math. Z. 267 (3–4) (2011) 605–643.
[9] O. Lorscheid, The geometry of blueprints. Part I: Algebraic background and scheme theory, Adv. Math. 229 (3) (2012) 1804–1846.
[10] O. Lorscheid, The geometry of blueprints. Part II: Tits–Weyl models of algebraic groups, preprint, arXiv:1201.1324, 2012.
[11] G. Mikhalkin, Tropical geometry, unpublished notes, 2010.
[12] C. Sou lé, Les variétés sur le corps à un élément, Mosc. Math. J. 4 (1) (2004) 217–244, 312.
[13] K. Thas, Notes on $\mathbb{F}_1$, I. Combinatorics of $D_2$-schemes and $\mathbb{F}_1$-geometry, in preparation.