On the Decomposition Theorem for Gluons

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Abstract—Recently, the problem of spin and orbital angular momentum (AM) separation has widely been discussed. Nowadays, all discussions about the possibility to separate the spin AM from the orbital AM in the gauge invariant manner are based on the ansatz that the gluon field can be presented in form of the decomposition where the physical gluon components are additive to the pure gauge gluon components, i.e. \( A_\mu = A_\mu^{\text{phys}} + A_\mu^{\text{pure}} \). In the present paper, we show that in the non-Abelian gauge theory this gluon decomposition has a strong mathematical evidence in the frame of the contour gauge conception. In other words, we reformulate the gluon decomposition ansatz as a theorem on decomposition and, then, we use the contour gauge to prove this theorem. In the first time, we also demonstrate that the contour gauge possesses the special kind of residual gauge related to the boundary field configurations and expressed in terms of the pure gauge fields. As a result, the trivial boundary conditions lead to the inference that the decomposition includes the physical gluon configurations only provided by the contour gauge condition.

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1. INTRODUCTION

One of interesting subjects of modern disputes in both the theoretical and experimental communities is the possible separation of nucleon spin into the intrinsic spin and orbital angular momentum (AM) of partons [1]. Nowadays two concurrent decompositions, as known as Jaffe–Manohar’s decomposition (JM decomposition) [2] and Ji’s decomposition (J decomposition) [3], have widely been discussed. The JM decomposition refers to a complete decomposition of the nucleon spin into the spin and orbital parts of quarks and gluons individually. While, the J decomposition possesses the gauge invariance by construction but, at the same time, it does not lead to the separable quark and gluon contributions of spin and orbital AMs to the whole nucleon spin.

In [4, 5], the gauge invariant analogue of JM decomposition has been proposed. Considering the Coulomb gauge condition, they have advocated that the gluon field can formally be presented as

\[ A_\mu(x) = A_\mu^{\text{phys}}(x) + A_\mu^{\text{pure}}(x). \]

It is worth to notice that this decomposition has being assumed as the first-step ansatz in all existing discussions on the gauge-invariant separation of spin AM from the orbital AM (see, for example, [6–17]).

In the Abelian \( U(1) \) gauge theory the physical components \( A_\mu^{\text{phys}} \) of Eq. (1) correspond to the transverse components \( A_\mu^t \) which are gauge invariant in contrast to the longitudinal components \( A_\mu^L \) associated with \( A_\mu^{\text{pure}} \) which are gauge-transforming and they should be eliminated by the gauge condition used in the Lagrangian approach. As a result, in the Abelian theory the decomposition of Eq. (1) is absolutely natural and there is no doubt of its validity at all.

In the non-Abelian \( SU(3) \) gauge theory, both the transverse and longitudinal components are gauge-transforming. Hence, the mentioned decomposition is actually questioned regarding the definition of the physical components. In particular, the use of covariant-type gauge conditions should inevitably lead to the inability to separate the spin and orbital AMs in the gauge-invariant manner because the coordinate dependence of gluon configurations cannot be independently determined for every of components, see for example [19].

Meanwhile, the decomposition plays a role of key-stone in many discussions devoted to the gauge-invariant separation of spin AM from orbital AM.

In the paper, we consider the decomposition of Eq. (1) as a statement which must be proven, if it is possible, within the non-Abelian theory. It turns out...
that the proof can be implemented and elucidated with
the help of a contour gauge which extends the standard
local axial-type gauges and is free from the Gribov
ambiguities [20, 21]. Contrasting with the local gauge,
in the contour gauge (which refers to the nonlocal type
of gauges) we should first fix the gauge orbit represen-
tative and then we search the local gauge condition
which is suitable for a given gauge orbit representative.
The useful features of the contour gauge can be readily
understood in the frame of the Hamiltonian formal-
ism where the contour gauge condition defines the
manifold surface crossing over a group orbit of the
fiber uniquely (see [22] for further details).

As a new observation, we demonstrate that even in
the contour gauge we can deal with a special kind of
the residual gauge freedom. However, this residual
gauge is located in the nontrivial boundary pure gauge
configurations defined at infinity.

Notice that the physical quantities do not depend
on the choice of gauges, as it must be. The axial-type
gauges are related to the certain fixed direction in a
space. In this case, the gauge independency should be
treated as an independency with respect to the chosen
direction which is ensured by additional requirements
[23]. Moreover, the use of contour gauge implies that
in the Hamiltonian formalism the gauge condition, as
an additional condition, can be completely resolved
regarding the gauge function excluding the gauge
transforms in the finite region of space. Accordingly,
in this meaning the physical observables considered in
the contour gauge are gauge invariant ones by con-
struction.

2. LOCAL AND NONLOCAL GAUGE
TRANSFORM CONVENTIONS

Before going further, it is important to remind the
convention system regarding the gauge transforma-
tions which match the corresponding Wilson path
functional (see [24] for more details). In what follows,
for the sake of shortness, we say simply the Wilson line
dependently on the form of a path unless it leads to
misunderstanding. For the non-Abelian gauge theory,
let us now assume that the fermion and gauge fields are
transformed as

\[ \psi^\theta(x) = e^{-i\theta x^a T^a} \psi(x) \equiv U(x) \psi(x), \]  
\[ A^\theta_\mu(x) = U(x) A_\mu(x) U^{-1}(x) + i \frac{1}{g} U(x) \partial_\mu U^{-1}(x), \]

where \( \theta = \theta^a T^a \) with \( T^a \) being the generators of cor-
responding representations. With the local transfor-
mations fixed as in Eqs. (2) and (3), we can readily see
that the covariant derivative and the gauge-invariant
fermion string operator take the following forms:

\[ \partial^\theta_\mu = i \partial_\mu + g A_\mu(x), \]
\[ \psi^{g,-inv}(x, y) = \psi(y) \langle y; x \rangle \psi(x). \]

with the Wilson line defined as

\[ [x; x_0]_A = \mathcal{P} \exp \left\{ ig \int_{P(x_0, x)} d\omega_A(\omega) \right\} \]
\[ = g(x|A) \equiv g(P), \]

where \( P(x_0, x) \) stands for a path connecting the starting
\( x_0 \) and destination \( x \) points in the Minkowski space.

Inserting the point \( x_0 \) in the Wilson line of the
gauge-invariant string operator, see Eq. (4), we get that

\[ \psi^{g,-inv}(x, y) = \psi(x) | y; x_0 \rangle \langle x_0; x | \psi(x). \]  

Equation (6) hints that the path-dependent nonlo-
cal gauge transformation of fermions can be intro-
duced in the form of

\[ \psi^\theta(x) = g^{-1}(x|A) \psi(x), \]

where \( \psi^\theta(x) \) is nothing but the Mandelstam gauge-
invariant fermion field \( \Psi(x|A) \), modulo the global
gauge transforms [25, 26]. This transformation leads,
in turn, to (cf. Eq. (3))

\[ A^\theta_\mu(x) = g^{-1}(x|A) A_\mu(x) g(x|A) \]
\[ + i g^{-1}(x|A) \partial^\theta_\mu g(x|A). \]

Therefore, we have the following correspondence
between local and nonlocal gauge transformation

\[ U(x) \leftrightarrow g^{-1}(x|A) \]

which is extremely important for the further discus-
sions because the wrong correspondence results in the
substantially wrong conclusions (see for example
[27]).

3. CONTOUR GAUGE CONCEPTION

In the paper, we make a reexamination of Eq. (1)
for clarification of the conditions which provide the
decomposition validity, unless the decomposition
does not take place at all. We intend to consider Eq. (1)
as a statement which must be proven at least within the
gauge condition that is more suitable for a demonstra-
tion of Eq. (1). To this end, we adhere the contour
gauge conception.

At the beginning, we remind that, within the Ham-
iltonian formulation of gauge theory [28], the
extended functional integration measure over the gen-
eralized momenta, \( E_{\mu} \) and coordinates, \( A_\mu \), includes
two kinds of the functional delta-functions. The first
kind of delta-functions reflects the primary (second-
ary etc.) constraints on \( E_{\mu} \) and \( A_\mu \), while the second
delta-function refers to the so-called addi-
tional constraints (or gauge conditions) the exact
forms of which have been dictated by the gauge free-
dom. If the primary (secondary etc.) constraints are
needed to exclude the unphysical gauge field compo-
nents, the gauge conditions would allow, in the most ideal case, to fix the corresponding Lagrange factor related to the gauge orbit. Focusing on the Lagrangian formulation [29], since the infinite volume of gauge orbit is factorized out in the functional measure over the gauge field components, the gauge conditions work for the elimination of unphysical gluon components.

In this connection, the contour gauge implies that in order to fix completely the gauge function (orbit representative) or to eliminate the unphysical gluons, one can demand the Wilson path functional between the starting point \(x_0\) and the final destination point \(x\), \(P(x_0, x)\), to be equal to unity, i.e.

\[
g(x|A) = [x; x_0]_A = 1, \tag{10}
\]

where the path \(P(x_0, x)\) is now fixed and \(x_0\) is a very special starting point that might depend on the destination point \(x\), see also [30].

In fact, the well-known axial gauge, like \(A^+ = 0\), is a particular case of the most general nonlocal contour gauge determined by the condition of Eq. (10) if the fixed path is the straightforward line connecting \(\pm \infty\) with \(x\).

In the past, the contour gauge had been a subject of intense studies (see, for example, [20, 21]). The obvious preponderance of the contour gauge use is that the quantum gauge theory is free from the Gribov ambiguities. By construction, the contour gauge does not suffer from the residual gauge freedom and gives, from the technical point of view, the simplest way to fix totally the gauge in the finite space. Briefly, within the contour gauge conception we first fix an arbitrary point \((x_0, g(x_0))\) in the fiber \(\mathcal{P}(\mathcal{X}, \pi G)\) [31]. We define afterwards two directions: one direction is determined in the base \(\mathcal{X}\) (where the direction is nothing but the tangent vector of a curve which goes through the given point \(x_0\)), the other direction - in the fiber where the direction can be uniquely determined as the tangent subspace related to the parallel transport. These two directions form the horizontal vector (or direction)

\[
H_\mu = \partial_\mu - igA^\mu_\nu(x)D^\nu \tag{11}
\]

which is invariant under the structure group on the fiber by construction [31]. In Eq. (11), \(D^\nu\) denotes the corresponding shift generator along the group fiber and can be presented as \(g\partial/\partial g\). In \(\mathcal{P}(\mathcal{X}, \pi G)\), the functional \(g(x|A)\) of Eq. (5) is a solution of the parallel transport equation given by

\[
\frac{dg_\mu(s)}{ds} H_\mu(A)g(x(s)|A) = 0 \tag{12}
\]

provided that \(p(s) = (x(s), g(x(s)))\) is defined with the curve \(x(s) \in \mathcal{X}\) parameterized by \(s\). If we impose the condition of Eq. (10), to fulfill Eq. (12) we need to suppose either \(A^\mu_\nu(x) = 0\) (this is a trivial case) or \(D^\nu = 0\) (that is a natural requirement if \(D^\nu = g\partial/\partial g\) as above).

We can thus uniquely define the point in the fiber bundle, \(\mathcal{P}(\mathcal{X}, \pi G)\), which has the unique horizontal vector corresponding to the given tangent vector at \(x \in \mathcal{X}\). We remind that the tangent vector at the point \(x\) is uniquely determined by the given path passing through \(x\). That is, within the Hamilton formalism based on the geometry of gluons the condition of Eq. (10) corresponds to the determining of the surface on \(\mathcal{P}(\mathcal{X}, \pi G)\). This surface is parallel to the base plane with the path and singles out the identity element, \(g = 1\), in every fiber of \(\mathcal{P}(\mathcal{X}, \pi G)\) [22].

The contour gauge refers to the nonlocal class of gauges and naturally generalizes the familiar local axial-type of gauges. It is also worth to notice that two different contour gauges can correspond to the same local axial gauge where the residual gauge left unfixed [32, 33]. This statement reflects the fact that, in contrast to the local axial gauge, the contour gauge does not possess the residual gauge freedom in the finite region of a space. However, as shown below, the boundary gluon configurations can generate the special class of the residual gauges.

4. CONTORU GAUGE AND THE GLUONFIELD DECOMPOSITION

We are going over to the discussion of the contour gauge defined by the condition of Eq. (10). Having used the path–dependent gauge transformations for gluons (see Eq. (8)), and having calculated the derivation of the Wilson line [34], we readily derive that in the gauge \([x; -\infty]_A = 1\) the gluon field can be presented as the following decomposition

\[
A^\nu_\mu(x) = \int_{-\infty}^{x} d\omega \omega g_{\mu
u}(\omega A^\nu_\mu + A^\nu_\mu(x - n^{\infty}), \tag{13}
\]

where \(g_{\mu\nu}\) is the gluon strength tensor; the starting point is now equal to \(\infty\) and the path parametrization is given by

\[
\omega x^{\infty} = x - n \lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon x^{\infty}}}{\epsilon} . \tag{14}
\]

This path parametrization includes the vector \(n\) defined a given fixed direction. As usual, the vector \(n\) becomes a minus light–cone basis vector, \(n = (0^+, n^-, 0_3)\), within the approaches where the light–cone quantization formalism has been applied.

Notice that the decomposition of Eq. (13) differs substantially from [35] by the absence of \(\epsilon\)-function. Indeed, the given contour gauge chooses either one \(0\)-function or the other, see [33] for details.
From Eq. (13), we can see that the contour gauge allows the gluon field to be naturally separated on the $G$-dependent and $G$-independent components. That is, instead of Eq. (13) it is instructive to write the separation as [cf. (17)]

$$A^{b,c}_i(x) = A^c_i(x|G) + A^{b,c}_i(-\infty),$$

(15)

where $A^c_i(x|G)$ is nothing but the first term of Eq. (13) and the boundary gluon configuration defined as $A^{b,c}_i(-\infty) \equiv A^{b,c}_i(x - n\infty)$. It is worth to notice that (a) the $G$-dependent configuration $A^c_i(x|G)$ stems from the nontrivial deformation of a path [34]; (b) the gluon separation presented by Eq. (15) resembles the equation of [17] but differs slightly by meaning.

In the contour gauge, see Eq. (10), the boundary gluon configurations have to fulfill the condition as

$$\mathbb{P} \exp \left\{ i g A^{b,c}_i (-\infty) \int_{-\infty}^{x} d\omega_i \right\} = \mathbb{I}.$$  

(16)

Therefore, since the integral over $d\omega_i$ in Eq. (16) is divergent as $1/\epsilon$ at $\epsilon$ goes to zero, the combination $n_i A^{b,c}_i (-\infty)$ should behave as $\epsilon^2$. Indeed, the exponential function of Eq. (16) reads (here, we deal with the space where the dimension is $D = 4$)

$$A^{b,c}_i(-\infty) \int_{-\infty}^{x} d\omega_i \equiv A^{b,c}_i(x - n\infty) \int_{-\infty}^{x} d\omega_i$$

$$\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon}\right)^{-1} A^{b,c}_i(n\epsilon)\frac{1}{\epsilon} = 0.$$  

(17)

Hence, the boundary gluon configurations obey the transversity condition as

$$n_i(\theta, \varphi) A^{b,c}_i(n(\theta, \varphi)) = 0.$$  

(18)

Here, since the vector $n$ defines the fixed direction it is more convenient to use the spherical co-ordinates in the Euclidean space (or the pseudo-spherical system in the Minkowski space) where the vector $n$ depends on the angle co-ordinates $(\theta, \varphi)$ only, see below. If the space dimension is $D > 4$, the transversity condition of Eq. (18) is not necessary to fulfill the contour gauge condition.

We are in position to show that in the contour gauge the boundary gluon configurations can be presented in the form of the pure gauge configurations. First of all, the starting point $x_0$ plays the special role in the considered formalism because all the paths originate from this point and the base $X$ touches the principle fiber bundle $\mathcal{P}$ only at this point in the general path-dependent gauge by construction.

Let us consider the point $x_0$ where, say, two different paths are started (see Fig. 1). This starting point has two tangent vectors associated with $P(x_0, x_1)$ and $P(x_0, x_2)$. In its turn, every of tangent vectors has the unique horizontal vector $H^{(i)}_\mu$ defined in the fiber. Then, making use of Eq. (10) we can obtain that

$$\mathbb{P} \exp \left\{ i g \int_{L(x_0)} d\omega_i A_i(\omega) \right\} = \mathbb{I} \quad \text{and} \quad \mathbb{P} \exp \left\{ i g \int_{L(x_0)} d\omega_i \wedge d\omega_i G_{\mu\nu}(\omega) \right\} = \mathbb{I},$$  

(19)

(20)

where $L(x_0)$ implies the loop with a basepoint $x_0 = \text{--}\infty$ and $\Omega$ is the corresponding surface related to the loop $L(x_0)$, see Fig. 1. Notice that the exact form of $\Omega$ is irrelevant for our study. Moreover, the fixed form of $\Omega$ may result in the additional ambiguity [18] which, however, does not influence on our conclusions.

Further, the simplest solution of Eq. (19) takes the form of

$$A_i(\omega) = \frac{1}{g} \int U(\omega) \partial_i U^{-1}(\omega),$$  

(21)

which leads to $G_{\mu\nu}(\omega) = 0$ and, therefore, to Eq. (20) after the use of the Stocks theorem (modulo the question on the surface $\Omega$).\footnote{The comprehensive analysis of the non-Abelian Stocks theorem can be found in [18].} Notice that the nontrivial (nonzero) solution of Eq. (20) may a priori exist. However, this solution (which ensures the nullification of the whole integration) cannot be presented in
the form of pure gauges anyway. In the path group theory it states that any loop as an element of the loop subgroup can homotopically be transformed to the “null element” which is, in our case, the basepoint $x_0 = -\infty$.

As a result, the pure gauge representation of Eq. (21) can be associated only with the boundary configurations, i.e.

$$A^b_c(x_0) = \frac{i}{g} U(x_0) \partial_i U^{-1}(x_0), \quad (22)$$

because the configuration $A_\mu(x|G)$ of Eq. (15) nullifies the integration of Eq. (19) by construction.

Finally, combining Eqs. (15) and (22), we have proved that in the contour gauge the gluon field can indeed be presented as the following decomposition

$$A^b_c(x) = A_p(x|G) + \frac{i}{g} U(x_0) \partial_i U^{-1}(x_0)|_{x \to x_0}, \quad (23)$$

where both terms are perpendicular to the chosen direction vector $n_\mu$.

Equation (23) shows that the residual gauge of contour gauge is entirely located at the boundary. Indeed, in order to understand the nature of the residual gauge associated with the boundary gluon configurations within the contour gauge, we consider the simplest and illustrative example of $\mathbb{R}^2$ where $A$ and $B$ have the same starting point $O$, see Fig. 2. It is more convenient to work with the spherical system, i.e. $A(R_\gamma, \phi_{\gamma}) \equiv (R_\gamma \cos \phi_{\gamma}, R_\gamma \sin \phi_{\gamma})$ etc. If the radius vectors of both $A$ and $B$ differ from zero even infinitesimally, we can distinguish these two vectors. However, if $R_A = R_B = 0$, the starting point $O$ loses information on the vectors $A$ and $B$ because of $O = (0 \cdot \cos \phi_A, 0 \cdot \sin \phi_A) = (0 \cdot \cos \phi_B, 0 \cdot \sin \phi_B)$. Notice that in general case the angles can be arbitrary ones. In this sense, we say that the starting point $O$ is the angle independent point.

Since $x_0 = \lim_{R \to \infty} \varphi(R_0, \theta_0, \varphi)$, we have

$$A^b_c(\varphi(\theta, \varphi), \varphi) = \frac{i}{g} U(\varphi(\theta, \varphi)) \partial_i U^{-1}(\varphi(\theta, \varphi)), \quad (24)$$

where $\varphi \to -\infty$ and $\theta$ and $\varphi$ are not fixed ensuring the residual gauge freedom in the similar way as demonstrated in Fig. 2.

The most convenient way to fix the residual gauge freedom is to assume that (see for instance [32, 33])

$$A^b_c(\varphi(\theta, \varphi)) \equiv A^b_c(\varphi(\theta, \varphi)) = 0. \quad (25)$$

In this case, the decomposition presented by Eq. (1) becomes a trivial one.

Thus, we have demonstrated that the contour gauge use gives the most natural decomposition of gluon fields on the $G$-dependent gluon component, which can be called as the physical one, and the unphysical gluon component related to the pure gauge configuration. Moreover, in the finite domain of space the contour gauge does not suffer from the residual gauge, while the remaining (possible) residual gauge has entirely been isolated on the infinite boundary of a given space.

5. LOCAL AND NONLOCAL GAUGE MATCHING

Since the contour gauge as a nonlocal kind of gauges generalizes the standard local gauge of axial type, it is worth to discuss shortly the correspondence between the local and nonlocal gauge transforms. As mentioned, the local axial-type gauge suffers from the residual gauge transformations. While the nonlocal contour gauge fixes all the gauge freedom in the finite space provided the infinite starting point $x_0 = -\infty$. Indeed, if we consider the local axial gauge, $A^+0(x) = 0$, as an equation on the gauge function $\theta(x)$, see Eq. (3), we can readily derive that the solution of this equation takes the form of

$$\mathcal{U}_0(x^-, \bar{x}) = C(\bar{x}) \mathcal{U}(x^-, \bar{x}),$$

$$\mathcal{U}(x^-, \bar{x}) = \exp \left\{-ig \int_{x_0}^x d\omega A^+(\omega^-, \bar{x}) \right\}, \quad (26)$$

where $\bar{x} = (x^+, 0^-, x_L)$, $x_0$ is fixed and $C(\bar{x})$ is an arbitrary function which does not depend on $x^-$ and is given by

$$\mathcal{U}_0(x^- = x_0^-, \bar{x}) = C(\bar{x}). \quad (27)$$

The arbitrariness of $C$-function also reflects the fact we here deal with an arbitrary fixed starting point $x_0$. We then come to the residual gauge freedom requiring both $A^+0(x) = 0$ and $A^+(x) = 0$, we have

$$\mathcal{U}_0(x^- = x_0^-, \bar{x})|_{\theta=1} = \mathcal{U}_0(x^- = x_0^-, \bar{x}) = C(\bar{x}) = e^{i\bar{\theta}(\bar{x})}. \quad (28)$$
One can see that the function \(C\) determines the residual gauge transform.

The nonlocal contour gauge extends the local axial-type gauge and demands that the full integral in the exponential of Eq. (10) has to go to zero.\(^2\) Within the Hamiltonian formalism, with the help of contour gauge, the residual gauge function \(\tilde{\theta}(\vec{x})\) can be related to the configurations \(A^-\) and \(A^c\) which are also disappeared eliminating the whole gauge freedom (see \([24, 36]\) for details). That is, if we restore the full path in the Wilson line for a given process, we can derive that

\[
C(\vec{x}) = \tilde{C}(x_0^+, x_0^-, x_0^-) \\
\times \mathbb{P} \exp \left\{ ig \int_{x_0^+}^{x_0^-} d\omega A_i^c(\omega, \vec{x}_0^-) \right\} \\
\times \mathbb{P} \exp \left\{ -ig \int_{x_0^+}^{x_0^-} d\omega A^c(\omega, \vec{x}_0^-, x_0^-) \right\}
\]

and, therefore, we have

\[
G^{\mu}(k^+; \vec{x}) = \int_{-\infty}^{\infty} d\omega G^{\mu}(\omega, \vec{x}; A^c)^e
\]

Here, we emphasize that Eqs. (31), as well as Eq. (15), has been derived by direct solution of the contour gauge requirement, see Eq. (10). As mentioned, the important finding of the present paper is that despite the contour gauge fixes the whole gauge freedom in the finite domain of space, it is still possible to deal with the residual gauge which is, however, located at the boundary field configurations only. The nontrivial topological effects due to the boundary field configurations are forthcoming in the further our studies.

In [17], the representation that is similar to our Eq. (31) has rather been guessed in the local axial gauge, \(A^+ = 0\), where the corresponding residual gauge freedom is incorporated into the inhomogeneous term with \(\tilde{\delta}(k^+)\). In turn, the gauge \(A^+ = 0\) with the fixed residual gauge freedom in the finite domain of space is actually identical to the unique contour gauge [33].

In the frame of the path group formalism, we have the following path-dependent transformation, which generates the usual translation transformation,

\[
\text{U}^P(x, x+y)\psi](x) = [x + y; x]_L^{-1}\psi(x + y)
\]

where \(\psi(x)\) belongs to the spinor fundamental representation and has defined on the Minkowski space \(M = P/L\) (\(P\) denotes the corresponding path group, \(L\) stands for the loop subgroup of \(P\)) as an invariant function of the conjugacy classes, i.e. \(\psi(x) = g(p)\Psi(p)\) with \(p = (x, g) \in P\) [31]. Besides, in Eq. (34) the operator \(\text{U}^P\) which acts on the spinor manifold has the form of

\[
\text{U}^P(x, x+y)\psi](x) = \mathbb{P} \exp \left\{ -ig \int_{x}^{x+y} d\omega A^c(\omega, \vec{x}) \right\}.
\]

In the contour gauge where the Wilson line of Eq. (34) is fixed to be equal to unity, the transport operator \(\text{U}^P\) takes the trivial form of

\[
\text{U}^P(x, x+y)|_{c.g.} = \mathbb{P} \exp \left\{ -ig \int_{x}^{x+y} d\omega A^c(\omega, \vec{x}) \right\}.
\]

This operator does not include any information on the boundary configurations even if, say, \(y \to \pm \infty\) because the boundary field configurations obey Eq. (10) too. Moreover, in our case, the Wilson line of Eq. (34) is set to unity due to the nullified integrand, \(A^+ = A^- = 0\), and the nullified integral over \(A^c\).
Hence, if we introduce the quark-gluon operators, forming the spin and orbital AM, as the residual-gauge invariant operators, we have to use the covariant derivative as \( f^b c = i \theta^b_{\mu} + g A^b c_{\mu} (\infty) \). In this sense, our results and the results of [17] are not much at variance. For example, following [17], we readily obtain the structure function of the quark angular momentum operator. It reads

\[
f_{ij}(x) = N \int_\infty^{-\infty} dz e^{i p_{\perp} z} \int_\infty^{-\infty} d^2 y \phi_{ij}(y) \psi_{ij}(y) \psi(\infty) \psi(0) \phi(\infty) \phi(0),
\]

where the antisymmetric combination \( [i, j] \) has been introduced with \( i, j = 1, 2 \) and \( N \) is the normalization factor defined as in [17].

As above-mentioned, \( f_{ij}(x) \) as the physical quantity does not depend on the gauge choice. At the same time, the axlial-type (local or nonlocal) gauges are correlated with the fixed direction which is also necessary for the factorization procedure [23]. Therefore, we are able to treat the gauge independency in the meaning of an independency on the chosen direction as well. In the frame of the Hamiltonian formalism, we assume that the gauge condition (or an additional condition) can be completely resolved regarding the gauge function excluding the gauge transforms in the finite region. In a sense, the physical quark-gluon operators, considered in the contour gauge, are “gauge invariant” by construction because we do not deal with any gauge transforms in the finite region due to the fixed gauge function \( \Theta_{fi} \) (as above discussed, due to \( g = 1 \) in the fiber for the whole base \( \mathcal{F} \)), see [22] for details.

7. CONCLUSIONS

To conclude, we have expounded the useful correspondence between local and nonlocal gauges which is extremely important to avoid the substantially wrong conclusions appeared in the literature.

We have proposed the proof of the following statement which is valid in the non-Abelian theory: in the contour gauge the gluon field can be presented in the form of decomposition on the gluon configuration \( A_{\mu}(x|G) \) being the physical degree of freedom and the pure gauge gluon configuration \( A^\text{pure}_{\mu}(x) \) that is totally isolated on the boundary and includes the special type of residual gauge freedom. We have demonstrated that the contour gauge condition cannot finally eliminate this, new-found, special residual gauge the nature of which has been illustrated in detail.

In the case of the trivial boundary conditions, i.e. \( A^b c_{\mu} = 0 \), in the contour gauge the decomposition of Eq. (1) does not make a sense in the non-Abelian theory because only the boundary gluon configurations can be presented as the pure gauge gluon configurations. Moreover, if the boundary configurations have been nullified, there is no the gauge freedom at all and, therefore, we deal with the gauge invariant operators by construction modulo the global gauge transformations that are not essential for the bilinear forms.

As a last point, we want to mention that the gluon decomposition of [4], which is formally similar to Eq. (1), has a status of the ansatz rather then a strong inference formulated and proven in our studies. Moreover, it has a distinguished feature that the gluon fields are separated into the physical and pure gauge gluon configurations before the gauge condition has been fixed. Hence, in this case, in order to formulate the ansatz they should demand to impose an addition requirement to extract \( A^\text{pure}_{\mu}(x) \) which is finally defined by \( G^\text{pure}_{\mu}(x) = 0 \). In its turn, this requirement appears naturally working within contour gauge conception, see Eq. (23). Equation (37), is formally not at odds with [4, 17] but, in a sense, we are in contradiction with [3, 14] where the “dynamical” type of angular momentum decompositions has been presented.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. X. Ji, F. Yuan, and Y. Zhao, Nat. Rev. Phys. 3, 27 (2021).
2. R. L. Jaffe and A. Manohar, Nucl. Phys. B 337, 509 (1990).
3. X. D. Ji, Phys. Rev. Lett. 78, 610 (1997).
4. X. S. Chen et al., Phys. Rev. Lett. 100, 232002 (2008).
5. X. S. Chen et al., Phys. Rev. Lett. 103, 062001 (2009).
6. X. Ji, Phys. Rev. Lett. 106, 259810 (2011).
7. M. Wakamatsu, Int. J. Mod. Phys. A 29, 1430012 (2014).
8. M. Wakamatsu, Eur. Phys. J. A 51, 52 (2015).
9. M. Wakamatsu et al., Ann. Phys. 392, 287 (2018).
10. C. Lorce, Phys. Lett. B 719, 185 (2013).
11. C. Lorcé, Phys. Rev. D 88, 044037 (2013).
12. E. Leader and C. Lorcé, Phys. Rept. 541, 163 (2014).
13. M. Wakamatsu, Phys. Rev. D 84, 037501 (2011).
14. M. Wakamatsu, Phys. Rev. D 83, 014012 (2011).
15. M. Wakamatsu, Phys. Rev. D 81, 114010 (2010).
16. P. M. Zhang and D. G. Pak, Eur. Phys. J. A 48, 91 (2012).
17. S. Bashinsky and R. L. Jaffe, Nucl. Phys. B 536, 303 (1998).
18. Y. A. Simonov, Sov. J. Nucl. Phys. 50, 134 (1989).
19. A. V. Belitsky and A. V. Radyushkin, Phys. Rep. 418, 1 (2005).
20. S. V. Ivanov et al., Sov. J. Nucl. Phys. 44, 145 (1986).
21. S. V. Ivanov and G. P. Korchemsky, Phys. Lett. B 154, 197 (1985).
22. I. V. Anikin, arXiv: 2105.09430 [hep-ph].
23. I. V. Anikin et al., Nucl. Phys. B 828, 1 (2010).
24. I. V. Anikin et al., Phys. Rev. D 95, 034032 (2017).
25. S. Mandelstam, Ann. Phys. 19, 1 (1962).
26. B. S. DeWitt, Phys. Rev. 125, 2189 (1962).
27. C. Lorce, Phys. Rev. D 87, 034031 (2013).
28. L. D. Faddeev and A. A. Slavnov, Front. Phys. 50, 1 (1980).
29. L. D. Faddeev and V. N. Popov, Sov. Phys. Usp. 16, 777 (1973).
30. H. Weigert and U. W. Heinz, Z. Phys. C 56, 145 (1992).
31. M. B. Mensky, Theor. Math. Phys. 173, 1668 (2012).
32. I. V. Anikin and O. V. Teryaev, Phys. Lett. B 690, 519 (2010).
33. I. V. Anikin and O. V. Teryaev, Eur. Phys. J. C 75, 184 (2015).
34. L. Durand and E. Mendel, Phys. Lett. B 85, 241 (1979).
35. Y. Hatta, Phys. Rev. D 84, 041701 (2011).
36. A. V. Belitsky et al., Nucl. Phys. B 656, 165 (2003).
37. M. Burkardt, Phys. Rev. D 88, 014014 (2013).