CLASSIFICATION OF ENDMORPHISMS WITH AN ANNIHILATING POLYNOMIAL ON ARBITRARY VECTOR SPACES

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Abstract. The aim of this work is to offer a solution to the problem of the classification of endomorphisms with an annihilating polynomial on arbitrary vector spaces. For these endomorphisms we provide a family of invariants that allows us to classify them when the group of automorphisms acts by conjugation.

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1. Introduction

The classification of mathematical objects is a classical problem: try to determine the structure of a quotient set up to some equivalence. The classification of endomorphisms on finite-dimensional vector spaces, where the group of automorphisms acts by conjugation, is well-known.

In this work we generalize this classification to endomorphisms admitting an annihilating polynomial on arbitrary vector spaces. This classification also generalizes the solution of the classification problem for finite potent endomorphisms that has been recently provided by the author in [3]. As far as we know, a solution of the classification problem for the set of all endomorphisms with an annihilating polynomial is not stated explicitly in the literature.

The paper is organized as follows. In section 2 we briefly recall the basic definitions of this work: the statement of the classification problem for endomorphisms, and the well-known theory of the classification of endomorphisms on finite-dimensional vector spaces, including the description of a method to construct Jordan bases in this case.

Section 3 is devoted to giving the main result of this work. Indeed, we offer invariants to classify endomorphisms with an annihilating polynomial on
arbitrary vector spaces (Theorem 5.11). As an example we offer the explicit description of the quotient set obtained from the classification of these endomorphisms on a countable dimensional vector space (Example 1).

2. Preliminaries

2.A. The Classification Problem. Let $V$ be an arbitrary $k$-vector space, and let $\text{End}_k(V)$ be the $k$-vector space of endomorphisms of $V$.

We have an action of the group of automorphisms of $V$, $\text{Aut}_k(V)$, on $\text{End}_k(V)$ by conjugation:

$$\text{Aut}_k(V) \times \text{End}_k(V) \rightarrow \text{End}_k(V)$$

$$(\tau, f) \mapsto \tau f \tau^{-1}.$$ 

Let us consider a subset $X \subset \text{End}_k(V)$ that is invariant under the action of $\text{Aut}_k(V)$ by conjugation.

The classification problem on $X$ refers to the possible answer to the question: which is the characterization of the quotient set $X/\text{Aut}_k(V)$.?

Henceforth, if $f \in \text{End}_k(V)$ and $H$ is a $k$-subspace of $V$ invariant by $f$, to simplify we shall again write $f: H \rightarrow H$ and $f: V/H \rightarrow V/H$ to refer to the induced linear operators.

2.B. Classification of Endomorphisms on Finite-Dimensional Vector Spaces. The solution of the classification problem for endomorphisms on vector spaces of finite dimension is well-known.

Let $E$ be a finite-dimensional vector space over a field $k$, and let $T \in \text{End}_k(E)$ be an endomorphism of $E$. We have that $T$ induces a structure of $k[x]$-module from the action

$$k[x] \times E \rightarrow E$$

$$(p(x), e) \mapsto p(T)e.$$ 

We shall write $E_T$ to denote the vector space $E$ with this $k[x]$-module structure.

It is known that two endomorphisms $T, \tilde{T} \in \text{End}_k(E)$ are equivalent, i.e. there exists an automorphism $\tau \in \text{Aut}_k(V)$ such that $T = \tau \tilde{T} \tau^{-1}$, if and only if the $k[x]$-modules $E_T$ and $E_{\tilde{T}}$ are isomorphic.

The above action, which determines de $k[x]$-module structure of $E$ (via $T$), is equivalent to a morphism of rings

$$\phi_T: k[x] \rightarrow \text{End}_k(E)$$

$$p(x) \mapsto \phi_T[p(x)],$$

where $\phi_T[p(x)](e) = p(T)(e)$.

Since $\text{Ker} \phi_T \neq \{0\}$, there exists a unique monic polynomial $a_T(x)$ such that $\text{Ker} \phi_T = (a_T(x))$, $a_T(x)$ being the annihilator polynomial of $T$.

Let $a_T(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r}$ be the annihilator polynomial of $T$, where $p_i(x)$ are irreducible polynomials on $k[x]$. The decomposition of $k$-vector spaces

$$E = \text{Ker} p_1(T)^{n_1} \oplus \cdots \oplus \text{Ker} p_r(T)^{n_r},$$

where the subspaces $\text{Ker} p_i(T)^{n_i} \subset E$ are invariant by $T$, is compatible with the respective $k[x]$-module structures.

Indeed, the classification of endomorphisms on finite-dimensional vector spaces is reduced to studying the $k[x]$-module structure of $\text{Ker} p(T)^n$, $p(x)$ being an irreducible polynomial on $k[x]$. This structure is determined by a decomposition of $k[x]$-modules:

$$\text{Ker} p(T)^n \simeq [k[x]/p(x)]^{\nu_p(E,p(T))} \oplus \cdots \oplus [k[x]/p(x)]^{\nu_{p_i}(E,p(T))},$$
where $\nu_h(E, p(T)) \neq 0$ and

$$\nu_i(E, p(T)) = \dim_K \left( \text{Ker} p(T)^i / \text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1} \right),$$

with $K = k[x]/p(x)$.

Again writing the annihilator polynomial of $T$ as $s_T(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r}$, the invariant factors $\{\nu_i(E, p(T))\}_{1 \leq j \leq r, 1 \leq i \leq n_j}$ determine the $k[x]$-module structure of $E_T$ and, therefore, they classify the endomorphism $T$.

Furthermore, if $K_j = k[x]/p_j(x)$, then $p_j(x)$ is a polynomial of degree $d_j = \dim K_j$.

We shall now describe a method for constructing Jordan bases of $E$ for $T$.

Let us first assume that $s_T(x) = p(x)^m$, with $p(x)$ an irreducible polynomial on $k[x]$, and let again $K = k[x]/p(x)$, with $d = \dim_K K = \text{gr}(p(x))$.

For each $1 \leq i \leq n$, let $\{e^i_h\}_{1 \leq h \leq \nu_i(E, p(T))}$ be a basis of

$$\text{Ker} p(T)^i / \left( \text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1} \right)$$

as a $K$-vector space.

If we write

$$\pi_i : \text{Ker} p(T)^i \rightarrow \text{Ker} p(T)^i / \left( \text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1} \right)$$

to denote the natural projection, let $\{e^i_h\}_{1 \leq h \leq \nu_i(E, p(T))}$ be a family of vectors of $\text{Ker} p(T)^i$ such that $\pi_i(e^i_h) = e^i_h$ for all $1 \leq h \leq \nu_i(E, p(T))$.

**Remark 2.1.** Since both $\text{Ker} p(T)^i$ and $\text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1}$ are $k$-vector subspaces of $E$ invariant by $T$, then $\text{Ker} p(T)^i + p(T) \text{Ker} p(T)^{i+1}$ is also a $k[x]$-submodule of $(\text{Ker} p(T)^i)_T$. We should emphasize that the definition of the quotient set $\text{Ker} p(T)^i / [\text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1}]$ is independent of its structure as a $k$-vector space, $K$-vector space or $k[x]$-module (via $T$).

**Lemma 2.2.** If $\dim_K K = d$, then $\bigcup_{1 \leq h \leq \nu_i(E, p(T))} \{e^i_h, T(e^i_h), T^2(e^i_h), \ldots, T^{d-1}(e^i_h)\}$ is a linearly independent family of vectors of the $k$-vector space $\text{Ker} p(T)^i$.

**Proof.** Let us consider a linear combination

$$\sum_{0 \leq m \leq d-1} \lambda_m T^m(e^i_h) = 0,$$

$$\sum_{0 \leq m \leq d-1} \lambda_m x^m e^i_h = 0$$

with $\lambda_m \in k$.

Hence, since

$$\sum_{0 \leq m \leq d-1} \lambda_m x^m e^i_h = 0$$

as a $k[x]$-module, if $\lambda_m \neq 0$ in some case then $\{e^i_h\}_{1 \leq h \leq \nu_i(E, p(T))}$ will be a family of linearly dependent vectors of $\text{Ker} p(T)^i / [\text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1}]$ as a $K$-vector space, which is impossible. Therefore, the statement is deduced. \qed

**Proposition 2.3.** One has that

$$H^p_{\text{cl}}(T) = \bigoplus_{1 \leq h \leq \nu_i(E, p(T))} \langle e^i_h, T(e^i_h), T^2(e^i_h), \ldots, T^{d-1}(e^i_h) \rangle$$

is a supplementary subspace of $\text{Ker} p(T)^{i-1} + p(T) \text{Ker} p(T)^{i+1}$ on the $k$-vector space $\text{Ker} p(T)^i$. 
Proof. It follows from Lemma 2.2 that the dimension of $H_i^{p(T)}$ as a $k$-vector space is $d \cdot \nu_i(E, p(T))$, which coincides with

$$\dim_k \left( \ker p(T)^i / \ker p(T)^{i-1} + p(T) \ker p(T)^{i+1} \right) = \dim_k [\ker p(T)^i] - \dim_k [\ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}].$$

Thus, to prove the claim it is sufficient to check that

$$[\ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}] \cap H_i^{p(T)} = \{0\}.$$

Let us now consider $e \in [\ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}] \cap H_i^{p(T)}$. One has that

$$e = \sum_{0 \leq m \leq d - 1} \gamma_{ml} T^m (e_1^l) \in [\ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}],$$

and, bearing in mind that $\pi_i(e) = 0$, we conclude that $\gamma_{ml} = 0$ (for all $l, m$) and, thereby, $e = 0$ because, otherwise, $\{e_1^l\}_{1 \leq l \leq \nu_i(E, p(T))}$ will be a family of linearly dependent vectors of $\ker p(T)^i / [\ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}]$ as a $k$-vector space, which is impossible. \hfill \square

Accordingly, bearing in mind that

$$\ker T^{i-1} \oplus p(T)^{n-i} H_i^{p(T)} \oplus \cdots \oplus p(T) H_{i+1}^{p(T)} = \ker p(T)^{i-1} + p(T) \ker p(T)^{i+1}$$

for all $1 \leq i \leq n$, we have constructed a family of $k$ subspaces of $E = \ker p(T)^n$, $\{H_i^{p(T)}\}_{1 \leq i \leq n}$, such that

$$E = \ker p(T)^{n-1} \oplus H_n^{p(T)}$$

$$\ker p(T)^{n-1} = \ker p(T)^{n-2} \oplus p(T) H_n^{p(T)} \oplus H_{n-1}^{p(T)}$$

$$\vdots$$

$$(2.1) \quad \ker p(T)^i = \ker p(T)^{i-1} \oplus p(T)^{n-i} H_n^{p(T)} \oplus \cdots \oplus p(T) H_{i+1}^{p(T)} \oplus H_i^{p(T)}$$

$$\vdots$$

$$\ker p(T)^2 = \ker p(T)^1 \oplus p(T)^{n-2} H_n^{p(T)} \oplus \cdots \oplus p(T) H_3^{p(T)} \oplus H_2^{p(T)}$$

$$\ker p(T) = p(T)^{n-1} H_n^{p(T)} \oplus p(T)^{n-2} H_{n-1}^{p(T)} \oplus \cdots \oplus p(T) H_2^{p(T)} \oplus H_1^{p(T)}.$$

Thus, if we now write

$$< e_h^i > T = \bigoplus_{0 \leq s \leq i-1} < p(T)^s [e_h^i], p(T)^s [T(e_h^i)], \ldots, p(T)^s [T^{d-1}(e_h^i)] >,$$

then

$$E = \bigoplus_{1 \leq i \leq n} < e_h^i > T.$$

Therefore, $1 \leq h \leq \nu_i(E, p(T))$

In general, if $T \in \text{End}_k(E)$ with annihilator polynomial

$$a_T(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r},$$

we have shown that there exist families of vectors $\{e_h^i\}_{1 \leq h \leq \nu_i(E, p(T))}$ with

$$e_h^i \in \ker p_j(T)^i$$

$$e_h^i \notin \ker p_j(T)^{i-1} + p_j(T) \ker p_j(T)^{i+1},$$
for all $1 \leq j \leq r$ and $1 \leq i \leq n_j$, such that if we set
\[
< e_h^{ij} > := \bigoplus_{0 \leq s \leq i-1} < p_j(T)^s[e_h^{ij}], p_j(T)^s[T(e_h^{ij})], \ldots, p_j(T)^s[T^{d_j-1}(e_h^{ij})] > ,
\]
then
\[
E = \bigoplus_{1 \leq j \leq r} \bigoplus_{1 \leq i \leq n_j} < e_h^{ij} > .
\]
Indeed, each family of vectors
\[
\bigcup_{1 \leq j \leq r} \left\{ e_h^{ij} \right\}_{1 \leq i \leq n_j, 1 \leq h \leq \nu_t(E, p_j(T))}
\]
generates a Jordan basis of $E$ for $T$.

3. Classification of endomorphisms with an annihilating polynomial

Let $V$ again be an arbitrary vector space over a ground field $k$, and let $B = \{ v_i \}_{i \in I}$ be a basis of $V$. It is known that $\dim(V) = \#B$ is independent of the basis chosen, $\#B$ being the cardinality of the set $B$.

Let $X_{V^{anh}}$ be the subset of $\text{End}_k(E)$ consisting of all endomorphisms of $V$ with an annihilating polynomial, that is:
\[
X_{V^{anh}} = \{ f \in \text{End}_k(V) \text{ such that } p(f) = 0 \text{ for a certain } p(x) \in k[x] \} .
\]

We should note that $X_{V^{anh}}$ is not a vector subspace of $\text{End}_k(E)$ (in general the sum of two endomorphisms with an annihilating polynomial does not admit an annihilating polynomial).

It is clear that the group of automorphisms of $V$, $\text{Aut}_k(V)$, acts on $X_{V^{anh}}$ by conjugation, because if $p(f) = 0$, then $p(\tau f \tau^{-1}) = 0$ for all $\tau \in \text{Aut}_k(V)$.

Let $\phi : k[x] \rightarrow \text{End}_k(V)$ be the morphism of rings that induces the $k[x]$-module structure in $V$. If $\varphi$ admits an annihilating polynomial, then $\text{Ker} \phi \varphi \neq \{0\}$ and there exists a unique monic polynomial $a_{\varphi}(x)$ such that $\text{Ker} \phi \varphi = (a_{\varphi}(x))$. The polynomial $a_{\varphi}(x)$ is the annihilator of $\varphi$.

The aim of this section is to offer the main result of this work: i.e. to provide the characterization of the quotient set $X_{V^{anh}}/\text{Aut}_k(V)$.

Remark 3.1. An endomorphism $\varphi \in \text{End}_k(V)$ is "finite potent" if $\varphi^n V$ is finite dimensional for some $n$. It is clear that a finite potent endomorphism admits an annihilating polynomial, and if $X_{V^{fp}}$ is the set consisting of all finite potent endomorphisms of $V$, then $X_{V^{fp}}$ is a subset of $X_{V^{anh}}$ that is invariant under the action of $\text{Aut}_k(V)$ by conjugation. We should note that the classification problem for finite potent endomorphisms has recently been solved by the author in [3].

To start, we shall construct a Jordan Basis of $V$ for an endomorphism with an annihilating polynomial by generalizing the method described in Subsection 2.1.

Let us consider $f \in X_{V^{anh}}$, and we first assume that $a_f(x) = p(x)^n$, $p(x)$ being an irreducible polynomial on $k[x]$.

As in Subsection 2.1, we consider
\[
\nu_t(V, p(f)) = \dim_K \left( \text{Ker} p(f)^t/\text{Ker} p(f)^{t-1} + p(f) \text{Ker} p(f)^{t+1} \right),
\]
with $K = k[x]/p(x)$.
Then also a well-defined expression in Ker \( p \), with \( \text{Ker} p \) a \( \nu \)-module, if \( \text{Ker} p \) is a \( \nu \)-vector space, one has that

\[
\sum_{l \in L} \left[ \sum_{0 \leq m \leq d-1} \lambda_{ml} f^m(v_l^i) \right] = 0,
\]

with \( \lambda_{ml} \in k \). Summing up the \( \nu \)-module, if \( \lambda_{ml} \neq 0 \) in some case, then \( \{v_l^i\}_{h \in S_{\nu i}(V)} \) will be a family of linearly dependent vectors of \( \text{Ker} p(f)^i \) as a \( \nu \)-vector space, which is impossible. Therefore, the statement is deduced. 

**Proposition 3.3.** If \( H_{l}^{p(f)} \) is the \( k \)-subspace of \( \text{Ker} p(f)^i \) generated by the family of vectors

\[
\bigcup_{h \in S_{\nu i}(V)} \{v_h^i, f(v_h^i), f^2(v_h^i), \ldots, f^{d-1}(v_h^i)\},
\]

then \( H_{l}^{p(f)} \) is a supplementary subspace of \( \text{Ker} p(f)^i p(f) \text{Ker} p(f)^{i+1} \) on the \( k \)-vector space \( \text{Ker} p(f)^i \).

**Proof.** Given a vector \( v \in \text{Ker} p(f)^i \), considering the structure of

\[
\text{Ker} p(f)^i / [\text{Ker} p(f)^{i-1} p(f) \text{Ker} p(f)^{i+1}]
\]
as a \( K \)-vector space, one has that

\[
\pi_i(v) = \sum_{l \in L} \mu_l q_l(x) v_l^i,
\]

with \( q_l(x) = a_0^l + a_1^l x + \cdots + a_{d-1}^l x^{d-1} \in k[x]/p(x) = K \), \( L \subseteq S_{\nu i}(V) \), \( \sum_{l \in L} \) being a well-defined expression in \( \text{Ker} p(f)^i / [\text{Ker} p(f)^{i-1} p(f) \text{Ker} p(f)^{i+1}] \).

Since from the definition of the sum on the quotient set we deduce that \( \sum_{l \in L} \) is also a well-defined expression in \( \text{Ker} p(f)^i \), bearing in mind that

\[
\pi_i\left(v - \sum_{l \in L} \left[ \sum_{0 \leq m \leq d-1} \mu_l a_m^l f^m(v_l^i) \right] \right) = 0,
\]
it clear that \( \text{Ker} p(f)^i = [\text{Ker} p(f)^{i-1} p(f) \text{Ker} p(f)^{i+1}] + H_{l}^{p(f)} \).
Moreover, analogously to Proposition \(2.3\) it is easy to check that

\[
\{\text{Ker}(p(f))^{i-1} + p(f)\} \cap H^p(f) = \{0\},
\]

and the claim is proved. \(\square\)

Thus, recurrently, as in expressions \(2.1\), we can consider \(k\)-vector subspaces of \(V\), \(\{H^p(f)\}_{1 \leq r \leq n}\), such that:

\[
\text{Ker} f^i = \text{Ker} f^{i-1} \oplus p(f)^n \oplus \cdots \oplus p(f)H^p(f) \oplus H^p(f),
\]

for every \(1 \leq i < n\).

**Remark 3.4.** Recall from \([3]\) that there is no relationship of order between the invariants \(\nu_i(V, p(f))\).

For example, if \(V\) is a \(k\)-vector space of countable dimension with a basis \(\{e_1, e_2, \ldots, e_n, \ldots\}\), and we consider \(f, g \in X^a_{V}\) defined by:

\[
f(e_i) = \begin{cases} 
e 1, 2 & \text{if } i = 1, 2 \\ 0 & \text{if } i \geq 3 \end{cases},
\]

\[
g(e_j) = \begin{cases} e_{j+1} & \text{if } j \geq 3 \text{ odd} \\ 0 & \text{if } j \geq 4 \text{ even} \end{cases},
\]

then \(a_f(x) = a_g(x) = x^2\), and one has that:

- \(H^f_1 = \langle e_1 \rangle \), \(f(H^f_1) = \langle e_2 \rangle \), and \(H^f_1 = \langle e_1 - e_2, e_4, \ldots \rangle\);
- \(\nu_1(V, f) = \mathbb{N}_0\), and \(\nu_2(V, f) = 1\);
- \(H^g_1 = \langle e_1 \rangle \), \(g(H^g_1) = \langle e_2 \rangle \), and \(H^g_1 = \langle e_1, e_2 \rangle\);
- \(\nu_1(V, 2) = 2\), and \(\nu_2(V, f) = \mathbb{N}_0\);

\(\mathbb{N}_0\) being the cardinal of the set of all natural numbers.

If we now write

\[
<v^i_h>_f = \bigcup_{0 \leq s \leq i-1} \{p(f)^s[v^i_h], p(f)^s[f(v^i_h)], \ldots, p(f)^s[f^{d-1}(v^i_h)]\},
\]

then

\[
\bigcup_{1 \leq i \leq n} \bigcup_{h \in \mathcal{S}_n(V, p(f))} <v^i_h>_f
\]

is a Jordan basis of \(V\) for \(f\).

In general, if \(f \in X^a_{V}\) with annihilator polynomial

\[
a_f(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r},
\]

we have shown that there exist families of vectors \(\{v^{ij}_h\}_{h \in \mathcal{S}_n(V, p_j(f))}\) with

\[
v^{ij}_h \in \text{Ker} p_j(f)^i
\]

\[
v^{ij}_h \notin \text{Ker} p_j(f)^{i-1} + p_j(f)\text{Ker} p_j(f)^{i+1},
\]

for all \(1 \leq j \leq r\) and \(1 \leq i \leq n_j\), such that if we set

\[
<v^{ij}_h>_f = \bigcup_{0 \leq s \leq i-1} \{p_j(f)^s[v^{ij}_h], p_j(f)^s[f(v^{ij}_h)], \ldots, p_j(f)^s[f^{d-1}(v^{ij}_h)]\},
\]

then

\[
\bigcup_{1 \leq j \leq r} \bigcup_{1 \leq i \leq n_j} \bigcup_{h \in \mathcal{S}_n(V, p_j(f))} <v^{ij}_h>_f
\]

is a Jordan basis of \(V\) for \(f\).
then
\[
\bigcup_{1 \leq j \leq r} \bigcup_{1 \leq i \leq n_j} <v_{ij}^h> \\
\text{is a Jordan basis of } V \text{ for } f.
\]
Indeed, each family of vectors

\[(3.2) \bigcup_{1 \leq j \leq r} \bigcup_{1 \leq i \leq n_j} \{v_{ij}^h\}_{h \in S_{\nu_i(V,p_j(f))}}\]

generates a Jordan basis of \( V \) for \( f \).

Note that

\[\#\left[ \bigcup_{1 \leq j \leq r} (S_{\nu_i(V,p_j(f)))} \cup \cdots \cup S_{\nu_i(V,p_j(f)))} \right] = \dim(V).\]

Remark 3.5. Recall from [2] that for every \( \phi \in \text{End}_k(V) \) possessing an annihilating polynomial of an arbitrary infinite-dimensional vector space \( V \) there exists a Jordan basis of \( V \) associated with \( \phi \). We should note that the above construction of Jordan bases for endomorphisms with an annihilating polynomial is compatible with the results of [2]. However, from the proof of the existence of Jordan bases given in [2] a Classification Theorem for these endomorphisms can not be obtained, because from the statements of this paper it is not possible to deduce that the dimensions of the vector subspaces that determine a Jordan basis are independent of the choices made.

We shall now use the existence of Jordan bases for endomorphisms with an annihilating polynomial to characterize the quotient set \( X^{\text{anh}}_V / \text{Aut}_k(V) \).

Let \( \tau \) be an automorphism of \( V \).

Lemma 3.6. If \( v \in V \), then

\[p(f)^s(v) = 0 \iff p(\bar{f})^s(\tau(v)) = 0\]

with \( \bar{f} = \tau f \tau^{-1} \).

Proof. One has that:

\[p(\bar{f})^s(\tau(v)) \iff \tau[p(f)^s(v)] = 0 \iff p(f)^s(v) = 0.\]

\[\square\]

Corollary 3.7. If \( \bar{f} = \tau f \tau^{-1} \), then

- \( \tau[\text{Ker}(f)^r] = \text{Ker}(\bar{f})^r \) for all \( r \geq 1 \) and \( p(x) \in k[x] \).
- \( \tau[\text{Ker}(f)^{i-1} + p(f) \text{ Ker}(f)^{i+1}] = \text{Ker}(\bar{f})^{i-1} + p(\bar{f}) \text{ Ker}(\bar{f})^{i+1} \) for all \( i \geq 1 \) and \( p(x) \in k[x] \).

Proposition 3.8. Let \( f \in X^{\text{anh}}_V \) with \( a_f(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r} \), \( p_i(x) \) being a irreducible polynomial in \( k[x] \). If \( \bar{f} = \tau f \tau^{-1} \). Then:

- \( a_{\bar{f}}(x) = a_f(x) \).
- \( \nu_i(V,p_j(f)) = \nu_i(V,p_j(\bar{f})) \) for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq n_j \).
Proof. Given \( q(x) \in k[x] \), it is clear that
\[
q(\bar{f}) = 0 \iff q(f) = 0,
\]
and hence \( a_f(x) = a_f(x) \). In particular, \( \bar{f} \in X^\text{anh}_V \).

On the other hand, for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq n_j \), bearing in mind Corollary 3.7 one has that
\[
0 \xrightarrow{\tau} \ker p_j(f)^{j-1} \oplus p_j(f)(\ker p_j(f)^{j+1}) \xrightarrow{\tau} \ker p_j(f)^i
\]
and hence \( \tau \) induces an isomorphism of \( k \)-vector spaces
\[
\ker p_j(\bar{f})^i / [\ker p_j(\bar{f})^{j-1} \oplus p_j(\bar{f})(\ker p_j(\bar{f})^{j+1})] \quad \sim \quad \ker p_j(f)^i / [\ker p_j(f)^{j-1} \oplus p_j(f)(\ker p_j(f)^{j+1})].
\]

Furthermore, for each family of vectors \( \{w_s\}_{s \in I} \subseteq \ker p_j(f)^i \), it is clear that
\[
\tau[\sum_{r, s \in I} f^{r,s}(w_s)] = \sum_{r, s \in I} f^{r,s}(\tau[w_s]).
\]

Thus, if we consider the structures of \( k[x] \)-module induced in \( \ker p_j(f)^i \) by \( f \) and in \( \ker p_j(\bar{f})^i \) by \( \bar{f} \) respectively, we have that
\[
\tau[\sum_{r, s \in I} x^{r,s}(w_s)] = \sum_{r, s \in I} x^{r,s}(\tau[w_s]),
\]
and, therefore, expression (3.3) is an isomorphism of \( K \)-vector spaces, from which the statement can be deduced. \( \square \)

Theorem 3.9 (Classification Theorem). Let \( f, g \in X^\text{anh}_V \) be two endomorphisms with an annihilating polynomial. Thus, \( f \sim g \) (mod. \( \text{Aut}_k(V) \)) if and only if:
- \( a_f(x) = a_f(x) = p_1(x)^{a_1} \cdots p_r(x)^{a_r}, p_i(x) \) being an irreducible polynomial in \( k[x] \).
- \( \nu_i(V, p_j(f)) = \nu_i(V, p_j(g)) \) for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq n_j \).

Proof. \( \Rightarrow \) If \([f] = [g] \in X^\text{anh}_V / \text{Aut}_k(V)\), there exists \( \tau \in \text{Aut}_k(V) \) such that:
- \( a_f(x) = a_f(x) = p_1(x)^{a_1} \cdots p_r(x)^{a_r}, p_i(x) \) being an irreducible polynomial in \( k[x] \).
- \( \nu_i(V, p_j(f)) = \nu_i(V, p_j(g)) \) for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq n_j \).

\( \Leftarrow \) If \( a_f(x) = a_f(x) = p_1(x)^{a_1} \cdots p_r(x)^{a_r} \) and \( \nu_i(V, p_j(f)) = \nu_i(V, p_j(g)) \) for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq n_j \), let us consider two families of vectors
\[
\{v^{ij}_h \}_{h \in S_{\nu_i(V, p_j(f))}} \quad \text{and} \quad \{w^{ij}_h \}_{h \in S_{\nu_i(V, p_j(g))}}
\]
determining Jordan bases of \( V \) for \( f \) and \( g \) respectively.

Let \( \tau \in \text{Aut}_k(V) \) be the automorphism defined by
\[
\tau[p_j(f)^m(f^a(v^{ij}_h))] = p_j(g)^m(g^a(w^{ij}_h)),
\]
for all \( 1 \leq j \leq r, 1 \leq i \leq n_j, h \in S_{\nu_i(V, p_j(g))}, 0 \leq a \leq d_j - 1, \) and \( 0 \leq m \).
An easy computation shows that
\[\tau[p_j(f)^m(f^a(v_h^{ij}))] = p_j(g)^m(g^a(v_h^{ij}))\]
for all \(b \geq 0\).

By construction, one has that
\[(\tau f \tau^{-1})[p_j(g)^m(g^a(v_h^{ij}))] = (\tau f)[p_j(f)^m(f^a(v_h^{ij}))] = \tau[p_j(f)^m(f^{a+1}(v_h^{ij}))] = p_j(g)^m(g^{a+1}(v_h^{ij})) = g[p_j(g)^m(g^a(v_h^{ij}))],\]
for all \(1 \leq j \leq r, 1 \leq i \leq n_j, \ h \in S_{\nu_i(V_p_j(g))}, 0 \leq a \leq d_j - 1, \) and \(0 \leq m < i\).

Accordingly, \(\tau f \tau^{-1} = g\) and \([f] = [g] \in X_{\text{det}}^h/\text{Aut}_k(V)\). \(\square\)

**Example 1.** Let \(V\) be a countable dimensional \(k\)-vector space. Let
\[Y = \{0\} \cup \mathbb{N} \cup \{\aleph_0\},\]
\(\aleph_0\) being the cardinal of the set of all natural numbers, and let \(H\) be the set consisting of all monic irreducible polynomials \(p(x) \in k[x]\).

If \(H_r \subset H \times \ldots \times H\) is the set such that
\[(p_1(x), \ldots, p_r(x)) \in H_r \iff p_j(x) \neq p_h(x) \text{ for all } j \neq h,\]
and
\[Y = \{(n, \nu_1, \ldots, \nu_n) \text{ with } n \in \mathbb{N} \text{ and } \nu_i \in Y\},\]
it follows from Theorem 3.9 that
\[X_{\text{det}}^h/\text{Aut}_k(V) = \bigcup_{r \in \mathbb{N}} (X_r \times \prod_r Y_r),\]
where
\[\prod_r Y_r = \left\{(n_1, \nu_{i_1}, \ldots, \nu_{n_1}, \ldots, n_i, \nu_{i_1}, \ldots, \nu_{n_i}, \ldots, n_r, \nu_{i_1}, \ldots, \nu_{n_r}) \mid \nu_{i_j} \in Y, \nu_{n_i} \neq 0, \text{ and } \nu_{j_i} = \aleph_0 \text{ for at least one } i_j \right\}.\]

**Remark 3.10.** Recently, in [1], D. Hernández Serrano and the author have offered an algebraic definition of infinite determinants \(\text{det}_V^h(1 + \varphi)\) on an arbitrary \(k\)-vector space \(V, \varphi\) being a finite potent endomorphism. It is known that \(\varphi\) is finite potent if and only if \(a_{1+\varphi}(x) = x^m \cdot p(x)\), with \((x, p(x)) = 1\) and \(\dim_k \text{Ker} p(\varphi) < \infty\). Accordingly, \(a_{1+\varphi}(x) = (x - 1)^m p(x - 1)\), and since
\[1 + \varphi \text{ is invertible } \iff \text{det}_V^h(1 + \varphi) \neq 0,\]
it follows from the above classification and the statements of [1] that for each finite potent endomorphism \(\varphi\) \(1 + \varphi\) is invertible if and only if \(\nu_i(V, 1 + \varphi) = 0\) for all \(i > 0\).

**Remark 3.11 (Final Consideration).** If \(E\) is a finite-dimensional \(k\) vector space, and \(f \in \text{End}_k(E)\) with \(a_f(x) = p_1(x)^{n_1} \cdot \ldots \cdot p_r(x)^{n_r}, p_i(x)\) being a irreducible polynomial in \(k[x]\), and invariants \(\nu_i(V, p_j(f))\) \(\forall 1 \leq i \leq n_j, \ 1 \leq j \leq r\), we should note that the structure of \(E\) as a \(k[x]\)-module induced by \(f\) is:
\[E_f \simeq \bigoplus_{1 \leq j \leq r} (k[x]/p_j(x))^{\nu_j(V, p_j(f))}.\]

Thus, Theorem 3.9 offers a classification of endomorphisms with an annihilator polynomial on arbitrary vector spaces that generalizes the well-known classification of endomorphisms on finite-dimensional vector spaces.
References

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