Quantum Brownian motion model for stock markets

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We investigate the relevance between quantum open systems and stock markets. A Quantum Brownian motion model is proposed for studying the interaction between the Brownian system and the reservoir, i.e., the stock index and the entire stock market. Based on the model, we investigate the Shanghai Stock Exchange of China from perspective of quantum statistics, and thereby examine the behaviors of the stock index violating the efficient market hypothesis, such as fat-tail phenomena and non-Markovian features. Our interdisciplinary works thus help to discover the underlying quantum characteristics of stock markets and develop new research fields of econophysics.

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I. INTRODUCTION

Since the Brownian motion in statistic physics was introduced into economics successfully, econophysics has tremendously developed and generated much more powerful tools, especially thanks to the pioneering works of Stanley and his coauthors in the 1990s.[1,2] Recently, the integration of quantum mechanics and finance has become a new interdisciplinary field of econophysics. A number of relevant researches were done in order to solve financial problems.[3–16] As one of the significant basic financial instruments, the stock market is critically concerned at all time. In 1970, Fama established the efficient market hypothesis (EMH), in which a Brownian motion model is put forward to describe the fluctuation of the stock price.[17] The perfect correspondence between the physical model and the financial concepts hence provides a compelling rationale. However, it has been discovered that the stock price does not fully fulfill the assumption of classical Brownian motion (CBM), since negative evidence is found such as non-Markovian memory features[18–20] and fat-tail phenomena.[21–26] For now, researches are mainly concentrated on the chaos theory such as the fractal Hurst model.[27] However, due to difficulty of mathematics, it is not simple to achieve a breakthrough in practical. Besides, the abandonment of statistic physics also seems like a retrogression.

To fix the flaws in the EMH, it is pivotal to explain the fluctuation of stock price in a correct way. In 1933, Frisch presented a damping oscillator model[28] which assumes the stock price swings and dissipates as a harmonic oscillator impelled by the initial information entering the market. Nevertheless, this model seems to have a big flaw that it cannot explain why the price always remains a small scale of persistent fluctuation. Recently in Ref. [7] it is suggested that one can use a quantum damped harmonic oscillator model instead of above to describe the fluctuation of the price of a single stock, and the probability distribution can be described by a quantum wave function. Hence, to investigate a single stock, we should not use a classic model of particle motion, but quantum methods of wave packet motion instead.

In this paper, we propose a quantum Brownian motion (QBM) model—a basic model of quantum open systems—to describe the fluctuation of the entire stock market and the features of the stock index. Within quantum finance methods, we are able to examine the irrationality in transaction, fat-tail phenomena and non-Markovian features in stock markets. With the help of the QBM model, we can study the underlying quantum characteristics of stock markets and develop new areas of econophysics.

II. BROWNIAN MOTION OF THE STOCK MARKET

In the CBM model, a free one-dimensional Brownian particle forced by the fluctuation-dissipation characteristics of environment leads to a random walk process after a sufficiently long time \( t \), with its coordinate \( x(t) \) following a Gaussian distribution with a variance of \( \sigma^2 t \), where \( \sigma^2 = kT/M\gamma \) with \( M \) the mass of the particle, \( \gamma \) the dissipation coefficient and \( kT \) the temperature of environment. Generally, the probability distribution of particles in phase space is described by the Markovian Klein-Kramers equation[29,30]:

\[
\frac{\partial}{\partial t} P(x,p,t) = -\frac{p}{M} \frac{\partial}{\partial x} P + 2\gamma \frac{\partial}{\partial p} (pP) + 2\gamma M k T \frac{\partial^2}{\partial p^2} P, \tag{1}
\]

of which the second-order partial differentials terms contribute to a Gaussian distribution solution[31]. In addition, \( p(t)/M = dx/dt \) is a white noise process with its autocorrelation function \( R(\tau) = \langle p(t + \tau) p(t) \rangle / M^2 = \sigma^2 \delta(\tau) \), which implies to be a Markovian process. In financial studies, the EMH assumes that the motion of
a stock market index $S(t)$ is a stochastic process, i.e.,
\[ d\ln S = \left(\mu - \sigma^2/2\right) dt + \sigma dW, \]
where $\mu$ and $\sigma$ represent the drift rate and volatility of the stock index, and $\sigma dW$ a free CBM process.

To verify the presumption of EMH, we analyze the 5, 10, 15,..., 100 min lines data of the 1999-2013 Shanghai Composite Index (SCI) of China. It is obvious that $\sigma$ and $\mu$ are proportional to $\sqrt{t}$ and $t$, respectively, which implies the stock market index obeys the presumed stochastic process (see Fig. 1(a)). However, it is found that the probability distribution of the stock index increment does not fit the Gaussian distribution but deviates with a fat tail, while this difference will disappear as $\tau$ increases (see Fig. 1(b)). Hence, it is implied that the presumption of Gaussian distribution is only valid in the long-term limit. In addition, Fig. 1(c) shows that the autocorrelation function $R(\tau)$ of the stock market of an arbitrary period has not only a Dirac-peak at $\tau = 0$ but also non-zero correlations even when $|\tau| > 20$ min. Such phenomena indicate that the stock market is non-Markovian and thus imply again that the CBM model is imperfect.

**FIG. 1:** (Color online) The features of the 1999-2013 Shanghai Composite Index of China: (a) The drift rate (blue line) and volatility (red line) derived from 5, 10, 15,..., 100 min lines of the index. (b) The distributions of the stock index increment $\Delta S(\tau) = \ln S(t + \tau) - \ln S(t)/\tau$ with $\tau = 5, 10, 20, 40, 80$ min (the drift term is already eliminated) and the distribution of a Gaussian one from top to bottom. (c) The autocorrelation features of the stock index increment of six arbitrary time periods in 1999-2013.

III. QUANTUM BROWNIAN MOTION MODEL

Based on quantum finance [7], one single stock $i$ is described as a wave function $|\psi_i\rangle$. The operators $X$ and $P$ are corresponded to the stock price and the trend of it, related with the classical coordinate and momentum $x_i = \ln S_i$ and $p_i = m_i d(\ln S_i)/dt$, respectively, where the parameter $m_i$ reflects the inertia of the stock. The probability distribution $\langle \psi_i | \psi_i \rangle = |\psi_i(x,t)|^2$ reflects the probability density of measuring and finding the stock price at $x$. Due to the uncertainty relation $[X,P] = i\hbar$, the more we learn of the stock price the less we can provide an estimate of the trend of it (and vice versa). In the actual transaction, we can only acquire the knowledge of the distribution of the trading volume in a certain price range, and thus cannot measure the value of the stock exactly. As a compensation, we can estimate of the trend to some extent. In a more general sense, a single stock price is treated as a quantum harmonic oscillator [7] with Hamiltonian $H_i = P^2/2m_i + m_i\omega_i^2 X^2/2$ as a closed system. The probability distribution of the ground state $|0\rangle$ is not a Dirac-peak distribution, which indicates that the stock price will still remain a small fluctuating uncertainty even if there is no excitations such as macroscopic outside information from economics, politics, etc. The phenomena cannot be explained by a classical oscillator, and are, in fact, related with the uncertainty of irrational transaction: If the transaction is perfectly rational, the stock price should be invariable, while the irrationality of the transaction will introduce extra fluctuation of the price and thus lead to a small uncertainty.

To investigate the behavior of the stock index, we regard it as a particle influenced by a large number of stocks of the stock market—thermal reservoir. The stock index is thus regarded as a system coupled with an environment consisting tremendous amount of harmonic oscillators, with linear interactions between the stock index and these single stocks. The effect of the macroscopic information is regarded as a potential well $V(x)$. Because of the huge degrees of freedom of the entire stock market,
it is essential for the stock index to be considered as an open system. Hence, we introduce a density operator \( \rho \) to describe the stock index by counting the average of the interaction between the system and environment by partial trace, and finally derive the evolution of the open system. This is the so-called quantum master equation [33]. Regarding the Brownian particle as an open system and the thermal reservoir as an environment consisting of physical significance: On the one hand, the characteristic time of \( \tau = \frac{kT}{\gamma} \) is, ad hoc though, trivial to be considered in Eq. (3) since the Markovian approximation is already used, which is valid only if the characteristic time of \( E \) is less than the relaxation time, i.e., \( \text{Max}\{\Omega_{\text{cut}}^{-1}, \frac{\hbar}{4\pi kT}\} \ll \gamma^{-1} \).

The stock market can thus be seen from the perspective of physical significance: On the one hand, the external information \( V(x) \) influences the stock index, and is transmitted by the stock index to the environment consisting of large numbers of single stocks; on the other hand, the linear response to the information of the detailed transactions is fed back to the stock index. Such physical mechanism is based on the fluctuation-dissipation theorem and reflects the exchange of energy between the quantum open system and environment [34].

In summary, the correspondence between quantum systems and stock markets is shown in Tables I and II.

TABLE I: The correspondence between quantum closed systems and single stock features.

| Quantum closed systems | Single stock features |
|------------------------|-----------------------|
| Coordinate representation \( X_i \) | Stock price \( \ln S_i \) |
| Momentum representation \( P_i \) | Trend of stock price \( m_\text{i} \text{d} (\ln S_i) / \text{d}t \) |
| Harmonic oscillator Hamiltonian \( H_i \) | Stock price oscillating energy |
| Angular frequency \( \omega_i \) | Characteristic oscillating frequency |
| Wave function \( \psi_i(x,t) \) | Probability density distribution of stock price \( P(x) \) |
| Uncertainty principle \( \{X, P\} = i\hbar \) | Uncertainty of irrational transaction |

TABLE II: The correspondence between quantum open systems and stock index features.

| Quantum open systems | Stock index features |
|----------------------|----------------------|
| System density operator \( \rho_A(x,x',t) \) | Probability distribution of stock index |
| System potential well \( V(x) \) | Macroscopic information influence |
| Thermal reservoir of harmonic oscillators \( \rho_E \) | Environment with large amount of single stocks |
| Temperature \( kT \) | Fluctuation of stock market |
| Thermal reservoir dissipation coefficient \( \gamma \) | Dissipation of stock market |
| Thermal reservoir spectral density \( J(\omega) \) | Autocorrelation features of stock market |

IV. MOMENTS OF QUANTUM BROWNIAN MOTION

From Eq. (3), it is simple to derive the moments of \( X \) and \( P \), i.e., variance, kurtosis, etc. For convenience, we choose the first-order moments to be zero so that \( \langle X \rangle = \langle P \rangle = 0 \) and all physical parameters to be dimensionless. We thus have \( \sigma^2_x = \langle X^2 \rangle \), which reads

\[
\sigma^2_x(t) = \sigma^2_x(0) + \frac{1}{2M\gamma^2} \left( 1 - e^{-2\gamma t} \right) \sigma^2_p(0),
\]

where \( \sigma^2_p = \langle P^2 \rangle \) and \( \sigma_{px} = \langle XP + PX \rangle \) [32]. For the QBM model, the volatility of the stock index is influenced by \( kT \) and \( \gamma \) in distinct ways; while for the CBM model, there is only one effective parameter \( kT/M\gamma \). The difference of \( \sigma^2_x \) between QBM and CBM is furthermore compared in Figs. (2a) and (2b).

In high temperature \( (kT > 10^{-2}) \), the \( \sigma^2_x(t) \) of QBM and CBM are equal. While in low temperature, due to
the uncertainty principle, the second term in right side of Eq. (4) should satisfy \( \sigma_x^2(0) \geq \hbar^2/4\sigma_x^2(0) \) and cannot be ignored then (see Fig. 2(a)). Hence, it is necessary to assume that the fluctuation of the stock index has an additional minimum variance caused by irrationality, which is dominant provided \( kT \ll \hbar \gamma \). The influence of \( \gamma \) is shown in Fig. 2(b). When \( 10^2 < \gamma < 10^6 \), the dissipation can be ignored, and \( \sigma_x^2(t) \rightarrow (kT/M\gamma t) \). However, if \( \gamma > 10^6 \), it is found that \( \sigma_x^2(t) \rightarrow \sigma_x^2(0) \). A non-zero \( \sigma_x^2(0) \) indeed reflects the fluctuation of quantum measurement, therefore provides a valid explanation of the non-Gaussian distribution of the stock index. For example, with a non-zero positive \( \gamma \) the evolution of the QBM model is derived that the volatility of the stock index strongly depends on the initial states of the system. For example, with a non-zero positive \( \gamma \) at \( t = 0 \), \( \kappa_x^4(t) \) evolves like the actual kurtosis of SCI with an exponential decrease (see Fig. 2(c)). The dependence comes from the imprecision of measurement of the stock index \[33\], and the involved initial \( \langle X^4 \rangle \) only matters during a short time after the measurement. The dependence of the initial states of \( \rho_A \) caused by imprecision of measurement therefore provides a valid explanation of the non-Gaussian distribution of the stock index.

However, even if the \( \sigma_x^2(t) \) of QBM has different behaviors from that of CBM, the good fit of the stock index volatility with \( \sqrt{t} \) (see Fig. 2(a)) implies the difference is negligible, unless in some extreme cases of \( kT \) or \( \gamma \) (extreme downturn or resistance in stock markets).

V. NON-MARKOVIAN FEATURES

Generally, the non-Markovian behaviors of QBM come from the non-Markovian characteristics of both the dissipation and fluctuation of the environment \[32\]. However, since in the long-term limit the effect of dissipation becomes small, we only consider the non-Markovian characteristics of fluctuation. From Eq. (2) we derive \[36\]

\[
\frac{d}{dt} \rho_A(t) = -i\hbar[H_A, \rho_A(t)] + \mathcal{K}(t)\rho_A(t) \tag{5}
\]

with

\[
\mathcal{K}(t)\rho_A(t) = -\frac{\gamma}{\hbar}\{X, \{P, \rho_A(t)\}\} - \frac{1}{2\hbar^2}\int_0^t d\tau D_1(\tau) [X, [X, \rho_A(\tau)]]
+ \frac{1}{2M\hbar^2}\int_0^t d\tau \{\tau D_1(\tau) |X, [P, \rho_A(\tau)]\},
\]

(6)

FIG. 2: (Color online) The dependence of the volatility of the quantum Brownian motion (QBM) (red solid line) and the classical one (CBM) (black dash line) on (a) the temperature \( kT \) (with \( \gamma = 10^3 \)) and (b) the dissipation coefficient \( \gamma \) (with \( kT = 0.1 \)) is compared. We set \( M = 10, \hbar = 0.01, t = 10, \) and \( \sigma_x^2(0) = 10^{-7} \) for both. (c) The kurtosis of QBM (red line) and the Shanghai Composite Index (SCI) from 1999 to 2013 (black line) is shown, with \( M = 20 \) and \( kT = \hbar = \gamma = 1 \). \( \langle X^2 \rangle = \langle P^2 \rangle = \hbar/2, \langle XP + PX \rangle = 0, \) and \( \langle X^4 \rangle = 200 \) when \( t = 0 \).
where $\gamma$ is time-independent since the dissipation term is supposed to be Markovian, whereas $D_1(\tau) = 2\hbar \int_0^\infty d\omega J(\omega) \coth(\hbar\omega/2kT) \cos \omega \tau$ is the autocorrelation function of fluctuation of the thermal reservoir [32]. Since $D_1(\tau)$ depends on the spectral density of environment and is not a Dirac function form in most cases, the memory kernel $K(t)$ is therefore time-dependent, from which the non-Markovian effect of QBM is introduced.

It is found that the stock market has non-Markovian features as shown in Fig. 3. In Fig. 3(b), $R(\tau)$ of three different periods are calculated with several rules found: the maximal critical points of the three lines all concentrate at $\tau = 120 \text{ min}$ and $\tau = 240 \text{ min}$ nearby, while the minimal critical points most concentrate at $\tau = 60 \text{ min}$ and $\tau = 180 \text{ min}$. The phenomena indicate a periodic time dependence of the autocorrelation similarity of the stock market. Besides, it is found that $R(\tau)$ decays exponentially with $\tau$ averagely. Hence we propose an autocorrelation function model to simulate the autocorrelation features of the stock market (see Fig. 3(b)):

$$R(\tau) = \left( \xi e^{-\eta \tau} \cos \Omega \tau \right)^2 = \frac{1}{2} \xi^2 e^{-\eta \tau} (\cos 2\Omega \tau + 1),$$

in which $\xi$ and $\eta$ represent the strength of the non-Markovian effect and the decay rate with $\tau$, respectively. $\Omega$ is an introduced frequency representing the periodicity of the entire stock market. $D_1(\tau)$ then yields

$$D_1(\tau) = 8M\gamma kT \delta(\tau) + 8M^2\gamma^2 R(\tau).$$

Substituting Eq. (8) into Eq. (9) and making a second-order time-convolutionless perturbation approximation[37] that $\rho_A(t) \rightarrow \rho_A(t)$, we eventually have

$$K(t)\rho_A(t) = -\frac{i\gamma}{\hbar}[X,\{P,\rho_A(t)\}] - \Delta(t)[X,[X,\rho_A(t)]] + \Lambda(t)[X,[P,\rho_A(t)]] ,$$

where

$$\Delta(t) = \frac{2M\gamma kT}{\hbar^2} + 2 \left( \frac{M\gamma \xi}{\hbar} \right)^2 \left\{ \frac{1}{\eta}(1 - e^{-\eta t}) + \frac{\eta}{\eta^2 + 4\Omega^2} \left[ 1 - e^{-\eta t} \left( \cos 2\Omega t - \frac{2\Omega}{\eta} \sin 2\Omega t \right) \right] \right\} ,$$

and

$$\Lambda(t) = \frac{2M^2\gamma^2 \xi^2}{\hbar^2 \eta^2} \left[ 1 - (1 + \eta t) e^{-\eta t} \right] + \frac{2M^2\gamma^2 \xi^2}{\hbar^4 (\eta^2 + 4\Omega^2)} \left\{ \frac{\eta^2 - 4\Omega^2}{\eta^2 + 4\Omega^2} - e^{-\eta t} \left[ \left( \frac{\eta^2 - 4\Omega^2}{\eta^2 + 4\Omega^2} + \eta t \right) \cos 2\Omega t \right. \right. $$

$$+ \left. \left( \frac{4\eta \Omega}{\eta^2 + 4\Omega^2} + 2\Omega t \right) \sin 2\Omega t \right\} .$$

The corresponding master equation with $K(t)$ is thus a non-Markovian equation which describes the Shanghai stock market appropriately. It is worth noting that the autocorrelation features of the stock market we observed cannot be explained by the CBM model from a microscopic view, whereas such features are actually fundamental phenomena of the QBM model, and such phenomena come directly from the spectral density $J(\omega)$, which actually introduces the weight of the contribution from a specific single stock oscillator $\omega$ to the stock index. For an Ohmic spectral density that $J(\omega) = 2M\gamma \omega/\pi$, the weight is exactly proportional to the energy ($\propto \hbar\omega$) of a single stock. Therefore, we simply derive $D_1(\tau) = 8M\gamma kT \delta(\tau)$ which implies the stock index is Markovian. While for a spectral density which satisfies Eq. (5), the non-Markovian features are introduced then. The result implies that one can understand the influence of single stocks to the stock index just through the non-Markovian
features of the index itself.

In addition, although the autocorrelation function \( R(\tau) \) introduced for fitting is monofrequency, because of the linearity of the interaction we can simply add higher-order terms with frequencies of \( 2\Omega, 3\Omega \ldots \) for more sophisticated calculations. We also note that, actually, the autocorrelation function should yield \( R(\tau)e^{-i\omega_0\tau} \) in general. Nevertheless, by assuming the phase information \( \phi \) of the stock market is completely time-uncorrelated, it is convenient to ignore the phase information and only concern on the amplitude \( R(\tau) \).

VI. DISCUSSION AND CONCLUSION

We have introduced the non-Markovian master equation of the stock market. However, the exact non-Markovian solution of Eq. [2] implies that the master equation contains not only the second-order products of \( X \) and \( P \) terms in Eq. [3], but also higher-orders products of \( X \) and \( P \). To investigate the influence of higher-order products by introducing the Wigner function \( W_\rho(x, p, t) = \int_{-\infty}^{+\infty} d\rho \rho_A(x - s/2, x + s/2, t)exp(ips/h), \) we can derive the Kolmogorov equation [33, 39]

\[
\frac{\partial}{\partial t}W_\rho(x, p, t) = -\frac{p}{M} \frac{\partial}{\partial x}W_\rho + 2\gamma \frac{\partial}{\partial p}(pW_\rho)
\]

\[+ h^2 A(t)(\frac{\partial^2}{\partial p^2}W_\rho - h^2 \Lambda(t) \frac{\partial^2}{\partial x \partial p}W_\rho) + G(W_\rho) \]

(12)

compared with Eq. [1], where

\[ G(W_\rho) = \sum_{k=3}^{\infty} \sum_{r=0}^{k} \frac{(-1)^k}{r!(k-r)!} \frac{\partial^k [A^{(k-r, r)} W_\rho]}{\partial x^{k-r} \partial p^r} \]

(13)

with time-dependent coefficients \( A^{(k-r, r)} (x, p, t) = \langle [x(t + dt) - x(t)]^{k-r} \rangle \langle [p(t + dt) - p(t)]^r \rangle dt \) \( (dt \to 0) \).

This term \( G(W_\rho) \) implies a deviation from the Gaussian distribution, which lasts even in the long-term limit, and it seems invalid, since the stock index always tends to return to a Gaussian one. Nevertheless, the Pawula theorem [40] proves that if any \( A^{(k-r, r)} (x, p, t) = 0 \), we have all \( A^{(k-r, r)} = 0 \) for \( r \geq 3 \). Therefore \( G(W_\rho) \) disappears in most conditions, which almost guarantees that the solution of \( W_\rho \) is a Gaussian distribution in the long-term limit.

In summary, we first examine the limitations of the efficient market hypothesis, and propose a rigorous correspondence between quantum mechanics and the stock market with help of econophysics. Based on the quantum harmonic oscillator single stock model introduced in Ref. [7], we propose a quantum Brownian motion model applicable for the stock index and the entire stock market. With the Caldeira-Leggett equation used, the volatility of the quantum Brownian motion and the classical one is compared, while the calculation of the kurtosis implies that the fat-tail phenomena come from the imprecision of measurement of the stock index and reflect the basic postulate of quantum measurement in quantum physics. The non-Markovian master equation is then introduced to describe the autocorrelation feature of the stock market in detail. This feature is found to be fundamental and is related with the weight of the contribution from single stocks to the stock index. We remark that the quantum Brownian motion model is a practicable model fitting the actual stock market better and is more acceptable for application owing to its understandable physical characteristics.

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[1] R. N. Mantegna and H. E. Stanley, Introduction to Econophysics: Correlations and Complexity in Finance (Cambridge University Press, 1999).
[2] R. N. Mantegna and H. E. Stanley, Nature (London) 376, 46 (1995).
[3] B. E. Baaquie, Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates (Cambridge University Press, 2004).
[4] B. E. Baaquie, Phys. Rev. E 77, 036106 (2008).
[5] M. Schaden, Physica A 316, 511 (2002).
[6] B. E. Baaquie, Phys. Rev. E 80, 046119 (2009).
[7] C. Ye and J. Huang, Physica A 387, 1255 (2008).
[8] F. Bagarello, Physica A 388, 4397 (2009).
[9] C. Zhang and L. Huang, Physica A 389, 5769 (2010).
[10] P. Pedram, Physica A 391, 2100 (2012).
[11] E. W. Piotrowski, Physica A 324, 196 (2003).
[12] I. Pakula, E. W. Piotrowski, and J. Sladkowski, Physica A 385, 397 (2007).
[13] L. A. Cotfas, Physica A 392, 371 (2013).
[14] G. Barad, Metalurgia International 18, 66 (2013).
[15] B. E. Baaquie, Y. Cao, A. Lau, and P. Tang, Physica A 391, 1408 (2012).
[16] B. E. Baaquie and P. Tang, Physica A 391, 1287 (2012).
[17] E. F. Fama, Journal of Finance 46, 1575 (1991).
[18] W. Wan and J.-W. Zhang, Front. Phys. China 3, 489 (2008).
[19] L. K. C. Chan and J. Lakonishok, J. Financ. Econ. 33, 173 (1993).
[20] S. Picozzi and B. J. West, Phys. Rev. E 66, 046118 (2002).
[21] Y. Zhang, J.-W. Zhang, and Z.-X. Wang, Phys. China 33, 734 (2004).
[22] C. Yan, J.-W. Zhang, Y. Zhang, and Y. N. Tang, Physica A 353, 425 (2005).
[23] J.-W. Zhang, Y. Zhang, and H. Kleinert, Physica A 377, 166 (2007).
[24] K. Kiyono, Z. R. Struzik, and Y. Yamamoto, Phys. Rev. Lett. 96, 068701 (2006).
[25] R. Chicheportiche and J.-P. Bouchaud, Phys. Rev. E 86, 041115 (2012).
[26] B. Podobnik, P. C. Ivanov, K. Biljakovic, D. Horvatic, H. E. Stanley, and I. Grosse, Phys. Rev. E 72, 026121 (2005).
[27] Mandelbr.Bb and J. W. Vanness, SIAM Rev. 10, 422 (1968).
[28] R. Frisch and F. V. Waugh, Econometrica 1, 387 (1933).
[29] O. Klein, Ark. Mat. Astr. Fys. 16, 1 (1922).
[30] H. Kramers, Physica 7, 284 (1940).
[31] D. Zwillinger, Handbook of Differential Equations, 3rd ed. (Academic Press Inc, 1997).
[32] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, 2002).
[33] A. O. Caldeira and A. J. Leggett, Physica A 121, 587 (1983).
[34] P. Hanggi and G. L. Ingold, Chaos 15, 026105 (2005).
[35] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, Rev. Mod. Phys. 82, 1155 (2010).
[36] B. Hu, J. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992).
[37] F. Shibata and T. Arimitsu, J. Phys. Soc. Japan 49, 891 (1980).
[38] A. O. Bolivar, Ann. Phys. 326, 1354 (2011).
[39] A. Kolmogorov, Math. Ann. 104, 414 (1931).
[40] R. F. Pawula, Phys. Rev. 162, 186 (1967).