Random membrane model for lattice gluodynamics

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252143 Kiev, Ukraine. 10.12.2005

Abstract

Model for studying coupling dependence on lattice spacing \( a \) in gluodynamics is suggested. The model predicts \( g \to g_0 > 0 \) with \( a \to 0 \). Free energy density in the model does not depend on temperature.

1 Introduction

As it is known, in lattice gluodynamics only the closed surfaces formed by plaquettes

\[
\Box_{x\mu\nu} = U_\mu(x) U_\nu(x + \mu) U_\mu^\dagger(x + \nu) U_\nu^\dagger(x); \quad U_\mu(x) \in SU(N)
\]

(1)

contribute to the expansion in powers of \( \beta \) of the partition function

\[
Z = \int \exp \left\{ \beta \sum_x \sum_{\mu,\nu=0}^3 \Box_{x\mu\nu} \right\} \prod_{x\omega} dU_{\omega}(x); \quad \beta \equiv 2N/g^2
\]

(2)

where \( g \) is the coupling constant. A surface may be self-intersecting, but it should not intersect another, since in this case those two surfaces must be treated as one. Since no additional restrictions are imposed, each surface may be treated as a random membrane.

There is some space in and out of each surface which is unavailable for another surfaces. We prescribe to any surface \( S_k \) some effective volume \( V_k = V(S_k) \), that depends both on area and shape of \( S_k \). In fact it is a volume of a minimal imaginary external shell, that films over each plaquette of the considered surface with the layer of effective thickness \( b \) no less than \( 1/2 \) link. For instance, surface of a cube with \( n \) links on edge, has the volume of \( 6 \times (n + 1)^2 \times 1/2 \). One must also take into account some effective redundant volume

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\[ V(S) - bS, \] that appears because a gap between the surfaces can’t be reduced to one link distance because their shapes are not rectangular in general.

Since the states, which differ by of surface shifts give the same contribution, the number of states contributed by a surface must be proportional to the volume available for such shifts. Therefore, the pattern of exactly solvable model [1] may be adjusted to develop a model intended to compute mentioned closed surface contributions in lattice gluodynamics. Since within model framework all orders in \( \beta \) are taken into account, model application region may be extended to the weak coupling area \( g < 1 \).

2 Model

Let the four dimensional volume \( V_{tot} = N^3 N_r \) contain \( r \) surfaces with volumes \( V_k = V(S_k), k = 1, \ldots, r \). The available volume for any surface is \( V_{tot} - \sum_k V_k \).

Let \( R_S \) be the number of configurations, taken by an isolated shape \( S \) without shifts, then total number of configurations may be computed 'in the spirit’ of a van der Vaals approximation

\[
Z = \sum_{r=0}^{\infty} \sum_{\{S_j\}} \frac{(V_{tot} - \sum_k V_k)^r}{r!} \exp \left\{ t \sum_{k=1}^{r} S_k \right\} \prod_k R_{S_k}
\]

(3)

where

\[
x^r_+ = \begin{cases} x^r; & x > 0 \\ 0; & x < 0 \end{cases} = \frac{r!}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xp} p^{-1-r} dp
\]

(4)

and

\[
t = \begin{cases} \ln \left( g^{-2}/N \right) & \text{for } SU(N); \quad N \geq 2 \\ \ln g^{-2} & \text{for } SU(2) \end{cases}
\]

(5)

Further we shall change the order of summation and integration. To make it safe we shift the integration path

\[
\frac{(V_{tot} - \sum_k V_k)^r}{r!} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p(V_{tot} - \sum_k V_k)} p^{-1-r} dp
\]

(6)

in a such a way that \( c > \text{Re} \ p_a \) for singularity \( p = p_a \) with the largest \( \text{Re} \ p \) and for the partition function we may write

\[
Z = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pV_{tot}} \sum_{r=0}^{\infty} \sum_{\{S_j\}}^{\infty} \prod_{k} \exp \{ tS_k - pV_k \} R_{S_k} dp
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pV_{tot}} \sum_{r=0}^{\infty} p^{-1-r} \mathcal{R}^r
\]

(7)

where

\[
\mathcal{R} = \sum_{S=S_m}^{\infty} \exp \{ tS - pV(S) \} R_S dS
\]

(8)
Further we replace summation in (8) by integration

\[ \Re = \sum_{S=S_m}^{\infty} \exp \{ tS - pV(S) \} \Re S dS \rightarrow \int_{S_m}^{\infty} \exp \{ tS - pV(S) \} \Re S dS \quad (9) \]

Possible consequences of such approximation are discussed in Appendix I.

Having collected everything we may finally write

\[ Z = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp \{ pV_{tot} \} dp}{p - \Re} \quad (10) \]

With infinitely increasing lattice volume \( V_{tot} = N_a^3 N_T \) partition function may be with good accuracy written as

\[ Z \simeq \exp \{ p_a(t) V_{tot} \} \quad (11) \]

where \( p_a \) is the singularity of integrand in (10) with the largest \( \Re p \).

There two possible regimes. The first one is realized when the solution of

\[ p - \Re = 0, \quad (12) \]

do\n
ominates giving the rightmost singularity in a complex \( p \)-plane. Another regime is realized when singularity of \( \Re \) proper dominates.

Note, that in fact we must include the self volume of \( S_j \) into the available one, i.e. replace \( (V_{tot} - \sum_k V_k)^T \) by \( \prod_{j=1}^{r} (V_{tot} - \sum_k V_k + V_j) \). A simple but bulky computation allows us to write for the partition function

\[ Z = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp \{ pV_{tot} \} \exp \{ -\partial \Re / \partial p \} dp}{p - \Re} \quad (13) \]

However, this correction doesn’t change the results appreciably, since the position of singularity of \( \partial \Re / \partial p \) is the same as that of \( \Re \).

Domination and possible interchanges of regimes are defined not only by specific form of \( \Re \), but are also determined by the choice of the coupling constant (and consequently \( t \)) dependence on lattice spacing \( a \). Indeed, free energy density is

\[ F = -\frac{T}{V} \ln Z \simeq -\frac{T}{V} V_{tot} p_a(t) = -a^{-4} p_a(t) \quad (14) \]

where \( V = a^3 N_a^3 \) is a veritable volume of the system and \( T = (N_c a)^{-1} \) is the temperature. For any \( p_a \) which does not vanish with \( a \rightarrow 0 \) as does \( a^4 \times \text{const} \), free energy density becomes infinite, so such state becomes unavailable for the system. Therefore, we claim

\[ p_a(t) = a^4 P_a(t) \quad (15) \]
where \( P_a(t) \) is assumed to be finite. Note that when dependence \( g \) on \( a \) is specified and \( a \to 0 \), free energy density become constant \( F = -\lim_{a\to0} P_a(t) \equiv -P_0 \). In particular, it doesn’t depend on temperature.

Another two basic assumptions of the model are

\[
\Re_S \simeq C \exp \{\mu S\} S^{\gamma - 1} \quad (16)
\]

(see Appendix II) and

\[
V(S) \simeq bS + (S/\sigma)^\alpha \quad (17)
\]

Some reasons for above assumption\(^1\) are given in Appendix III.

Therefore we may finally write

\[
\Re \equiv \Re(\tau, p) \simeq C \int_0^{\infty} \exp \{-\tau S - p (S/\sigma)^\alpha\} S^{\gamma - 1} dS \quad (18)
\]

with

\[
\tau \equiv pb - \mu - t \quad (19)
\]

Integral in \((18)\) diverges for all \( \tau < 0 \) and converges for all \( \tau > 0 \). In a singular point \( \tau = 0 \) integral in \((18)\) converges to finite value

\[
\Re(0, p) \simeq \frac{C}{\alpha} p^{-\frac{\gamma}{\alpha}} \Gamma \left( \frac{\gamma}{\alpha}, p (S_m/\sigma)^\alpha \right) \quad (20)
\]

for any \( \Re p > 0 \). If singularity \( \tau = 0 \) of \( \Re(\tau, p) \) is leading in a limit \( a \to 0 \), it specifies the dependence \( t \) on \( a \). Indeed, from \( p_a(t) b = a^4 P_a(t) b = \mu + t_a \) we get \( g^2 = \frac{1}{a^2} e^{\mu - a^4 t_a} \) where \( P_0 = \lim_{a \to 0} P_a(t) \), hence \( g^2 \to g_0^2 = e^{\mu} > 0 \) with \( a \to 0 \).

Since \( \partial \Re / \partial \tau < 0 \) for \( \tau > 0 \), function \( \Re(\tau, p) \) is monotonously decreasing in this area. Therefore, if the solution \( p_a(t) = \Re[p_a(t) b - \mu - t, p_a(t)] \) of \( (12) \) exists, it is unique. There is some range of values of \( t \) and \( a \) where such solution exists and varying those parameters we may move \( p_a(t) \) in this area. When dependence \( t \) on \( a \) is specified, i.e. \( t = t_a \) this area shrinks into line \( p_a(t_a) \) defined by the single parameter \( a \), and according to \((15)\) \( p_a(t_a) \to 0 \) with \( a \to 0 \).

If \( t_a \) is chosen in a such way that \( \lim_{a \to 0} t_a \equiv t_0 = \text{const} \), then \( g^2 \to g_0^2 = N^{-1} e^{\mu - t_0} \). The case of \( t_a \to t_0 \to -\infty \) is unacceptable, since it means \( g \to \infty \) with \( a \to 0 \). It is easy to check that we cannot choose \( g \to 0 \) \((t_a \to t_0 \to +\infty) \) with \( a \to 0 \) because there is no solution \( p = \Re(\tau, p) \to 0 \) for such choice of \( t_a \). Indeed, in this case from \((18)\) we obtain

\[
\Re(\tau_a, p_a) \simeq C \sum_{n=0}^{\infty} \frac{(-p_a \sigma - \alpha)^n}{n!} \int_{S_m}^{\infty} \exp \{-\tau_a S\} S^{\gamma + \alpha n - 1} dS \quad (21)
\]

or

\[
\Re(\tau_a, p_a) \simeq C \tau_a^{-\gamma} \sum_{n=0}^{\infty} \frac{\Gamma (\gamma + \alpha n, \tau_a S_m)}{n!} \sigma^{-\alpha n} (-\tau_a^{-\alpha} p)^n \quad (22)
\]

\(^1\)As it follows from \((8)\), role of specific form of \( V(s) \) with \( a \to 0 \) is essentially diminished due condition \((18)\).
that with \( \omega \)

\[
\Gamma (\lambda, x) = x^{\lambda - 1} e^{-x} \sum_{n=0}^{N-1} \frac{1}{\Gamma(1-\lambda)} \left( \frac{1}{(-x)^n} + O(|x|^{-N}) \right)
\]  

(23)

leads to

\[
\Re(\tau_a, p) \simeq -CSm(\mu + t_a)^{-1} \exp ((\mu + t_a) S_m - n)
\]  

(24)

so

\( p_a = \Re(\tau_a, p_a) \to -\infty \) with \( a \to 0 \), but we claim \( p_a \to 0 \) in such a limit. Hence, there is no asymptotic freedom in suggested model.

The possibility for QCD (and gluodynamics in particular) to be a non-asymptotically free theory have been discussed for years. There are reasons to believe, that QCD is not perturbative at \( a \sim 0 \). On the basis of today’s numeric computations, it is difficult to anticipate the behavior of \( g \) in the limit of \( a \to 0 \), taking perturbative calculations as a guidance. Moreover, numerical studies showed deviations of the Callan-Symanzik \( \beta \)-function \( \beta_{CS}(g) \) from perturbative result when the correlation length begins to grow. These deviations are of such a pattern, as if the theory approaches the fixed point \( g_0 \) at which \( \beta_{CS} = 0 \) and consequently the theory is not asymptotically free. Solid arguments in favor of such behavior of \( \beta_{CS}(g_0) \) were given in [6]. Data on deep inelastic scattering does not eliminate the fixed point [7]. Phenomenological analysis of available monte-carlo lattice data in the \( SU(2) \)-gluodynamics shows no contradiction with the fixed point of \( \beta_{CS}(g) \) located at \( g_0 \simeq 0.563 
\]

Analytical estimations [9] also favor \( g_0 \neq 0 \).

3 Conclusions

The pattern of exactly solvable model [11] is adjusted to develop the random membrane model for lattice gluodynamics. We make use of the fact that only closed surfaces formed by plaquettes contribute to the expansion of the lattice gluodynamics partition function in \( \beta \) powers. Since within model framework all orders in \( \beta \) are taken into account, model application region may be extended to the weak coupling area \( g < 1 \). Arguments for the main assumptions, \( 16 \) and \( 17 \) of the model are given in Appendix II and Appendix III.

Model predicts

\( g \to g_0 > 0 \) with \( a \to 0 \) and independence of free energy density on temperature.

4 Appendix I. Discrete Volume

For the discrete variable \( m \) one may define

\[
m'_+ = \begin{cases} 
  m^r & \text{for } m > 0 \\
  \delta_0^r & \text{for } m = 0 \\
  0 & \text{for } m < 0
\end{cases}
\]

(25)
and write a series
\[ \sum_{n=-\infty}^{\infty} n^r e^{-np} = \sum_{n=0}^{\infty} n^r e^{-np} \] (26)
which is a discrete version of Laplace transform, called \( Z \)-transformation (see e.g. [10]). The inverse transform
\[ m_n^r = \frac{1}{2\pi i} \int_{c-i\pi}^{c+i\pi} e^{-pm} \sum_{n=0}^{\infty} n^r e^{-np} dp \] (27)
may be used instead of (4), so for partition function we obtain
\[ Z = \sum_{r=0}^{\infty} \frac{1}{r!} \int_{c-i\pi}^{c+i\pi} e^{pV_{tot}} \sum_{n=0}^{\infty} n^r e^{-np} dp \sum_{\{S_j\}} \prod_k e^{tS_k - V_k(S_k)} \mathcal{R}[S_k] \] (28)
that leads to
\[ Z = \frac{1}{2\pi i} \int_{c-i\pi}^{c+i\pi} e^{pV_{tot}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{r!} n^r \mathcal{R}[r] e^{-np} dp \] (29)
where
\[ \mathcal{R} = \sum_S \exp \{ pV(S) + tS \} \mathcal{R}[S] \] (30)
and one may finally write
\[ Z = \frac{1}{2\pi i} \int_{c-i\pi}^{c+i\pi} \exp \{ pV_{tot} \} \frac{1 - \exp \{ \mathcal{R}(\tau, p) - p \}}{\mathcal{R}[\tau] - p} dp \] (31)
Since \( \exp \{ \mathcal{R} - p \} \) is entire function of \( \mathcal{R} - p \), singularities of the integrand in (31) and in (10) are located at the same position, and the difference between corresponding expressions for partition functions disappears with \( V_{tot} \to \infty \).

5 Appendix II

For simplicity we consider here the closed two-dimensional surface in the three-dimensional space, instead of four-dimensional space in suggested model. The slice between planes \( x = x_c \) and \( x = x_c + 1 \) is a two-dimensional surface closed in the \( x \)-direction. Its borders are located in the above planes. The borders are equal and are closed loops of length \( L_x \). The area of this slice is \( L_x \times 1 \). For \( N \) slices of common area \( S_{YZ} \) we get
\[ \mathcal{R}^{[N]}(S_{YZ}) = \sum_{(L_z)} \delta \left( S_{YZ} - \sum_x^N L_x \right) \prod_x^N R_{L_x} \]
\[ = \sum_{(L_z)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \{ pS_{YZ} \} \prod_x^N R_{L_x} e^{-pL_z} dp \] (32)
or

\[ R^{[N]}(SYZ) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp{\{pSYZ\}} R^N dp \]  \hspace{1cm} (33)

where

\[ R = \sum_L R_L e^{-pL} \simeq \int_{L_{min}}^{\infty} R_L e^{-pL} dL dp \]  \hspace{1cm} (34)

Summing over \( N \) we finally get

\[ \Re(\text{SYZ}) = \sum_{N=1}^{\infty} R^{[N]}(\text{SYZ}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp{\{pSYZ\}}}{1-R} dp \]  \hspace{1cm} (35)

It is known (see e.g. [11]) that

\[ R_L = C L^{\lambda-1} \exp{\{mL\}} \]  \hspace{1cm} (36)

where \( C, \lambda, m \) are constants \(^2\) and we get

\[
R \simeq C \int_{L_{min}}^{\infty} L^{\lambda-1} \exp\{- (p - m) L\} dL \\
= (p - m)^{-\lambda} CT \left( \lambda, (p - m) L_{min} \right) \simeq (p - m)^{-\lambda} CT \left( \lambda \right) . \]  \hspace{1cm} (37)

Having written

\[ 1 - R = 1 - CT \left( \lambda \right) (p - m)^{-\lambda} \]

\[ = x \sum_{n=0}^{\infty} \frac{(n+\lambda)!}{(n+1)! \Gamma(\lambda)} (-x)^n \left( CT(\lambda) \right)^{-(n+1)/\lambda} \]  \hspace{1cm} (38)

where \( x = p - m - \left( CT(\lambda) \right)^{1/\lambda} \), we come to an integrand in (35) having a simple pole in \( p = \mu \equiv m + \left( CT(\lambda) \right)^{1/\lambda} \). Since \( \lambda > 0 \) we get \( p = \mu > m \), so singular point of \( R \) at \( p = m \) is located leftward in complex \( p \)-plane. Therefore, the pole is a leading singularity and for \( SYZ \gg 1 \) we may write

\[ \Re(\text{SYZ}) \simeq \exp{\{\mu SYZ\}} \]  \hspace{1cm} (39)

If one assumes that

\[ \Re(\text{SZ}) \simeq \exp{\{\mu S_X\}} \]  \hspace{1cm} (40)

where \( S_X = S - SYZ \) part of surface that consists of \( x \)-planes plaquettes, one comes to

\[ \Re(S) \simeq \int_{S_{min}}^{S} \exp{\{\mu (S - S_X)\}} \exp{\{\mu S_X\}} dS_X \simeq S \exp{\{\mu S\}} \]  \hspace{1cm} (41)

Unfortunately such simple estimation allows to exclude corrections neither in a power type factor, nor in \( \mu \). Nonetheless, even in this case expression for \( \Re(S) \) will not contradict to the assumption\(^{16}\).

\(^2\)For closed loops without intersections \( C \simeq 6/5; \lambda \simeq 4/3; m \simeq 1 \) [11], but intersection doesn’t change those constants drastically.
6 Appendix III. Packing of surfaces.

The proper effective volume of the surface $S$ is very close to $bS$, but with increasing number of surfaces and their areas the packing problem appears. Despite four century history, this problem has been more or less solved only for objects of simple form (mainly for spherical ones) $[12, 13]$. In particular, for ‘random’ balls packing in a spherical bag of volume $v$, one can define the density $\rho = v_B/v$ where $v_B$ is the volume of the balls. The density depends on many conditions, but roughly it may be computed as $[12]$

$$\rho \simeq \frac{2}{3} - \frac{1}{3} N^{-\frac{1}{3}} \quad (42)$$

where $N$ is the number of balls. One may expect similar density behavior in a case of random volume of balls. Let $\overline{v} = v_B/N$ is an average volume of a ball, hence one may write

$$v \simeq \frac{v_B}{\frac{2}{3} - \frac{1}{3} (v_B/\overline{v})^{-\frac{1}{3}}} = \frac{3}{2} v_B + 3\overline{v}^{\frac{1}{3}} v_B^{\frac{2}{3}} + ... \quad (43)$$

If for fixed average volume $\overline{v}$ we increase volume of balls $v_B$, in accordance with $[13]$, by so doing we increase $v$. One may expect, that similar relation

$$V \simeq c_1 v_B + c_2 v_B^2 + ... \quad (44)$$

is true for a single ball volume $V_B$ and volume $V$ which this ball effectively occupied. Taking this relation as a pattern, we assume that there exist some constants $b > 0; \sigma > 0$ and $0 < \alpha < 1$, such that for objects of arbitrary form

$$V (S) \simeq bS + (S/\sigma)^\alpha \quad (45)$$

Although expression $[17]$ is regarded only as the model assumption, there is at least one more argument in favor $[17]$. For the convex body the volume of parallel shell with thickness $b$ is $V (S) = Sb + Mb^2 + \frac{4}{3} \pi b^3$ where $M$ is an average curvature integral $[14]$. Dimensional method allows to assume that $M \sim S^{1/2}$ that corresponds $\alpha = 1/2$ in $[17]$ and allows to expect that with varying $\alpha$ in $[15]$ a reasonable description may be found for a body of arbitrary form at least for $S \gg 1$.

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