BRACE BLOCKS FROM BILINEAR MAPS
AND LIFTINGS OF ENDOMORPHISMS

A. CARANTI AND L. STEFANELLO

Abstract. We extend two constructions of Alan Koch, exhibiting methods to construct brace blocks, that is, families of group operations on a set $G$ such that any two of them induce a skew brace structure on $G$.

We construct these operations by using bilinear maps and liftings of endomorphisms of quotient groups with respect to a central subgroup.

We provide several examples of the construction, showing that there are brace blocks which consist of distinct operations of any given cardinality.

One of the examples we give yields an answer to a question of Cornelius Greither. This example exhibits a sequence of distinct operations on the $p$-adic Heisenberg group $(G, \cdot)$ such that any two operations give a skew brace structure on $G$ and the sequence of operations converges to the original operation “$\cdot$”.

1. Introduction

Alan Koch gave in [Koc21] the following construction. Let $(G, \cdot)$ be a group, and let $\psi$ be an endomorphism of $G$ with an abelian image. Define an operation on $G$ via

$$ g \circ h = g \cdot (\psi g)^{-1} \cdot h \cdot \psi g, \quad (1.1) $$

where we denote by $\psi g$ the value of the map $\psi$ on $g$. Koch proved that $(G, \cdot, \circ)$ is a bi-skew brace.

In [CS21] we characterised the endomorphisms $\psi$ of $G$ such that the operations

$$ g \circ h = g \cdot (\psi g)^{\varepsilon} \cdot h \cdot (\psi g)^{-\varepsilon} \quad (1.2) $$

yield (bi-)skew braces $(G, \cdot, \circ)$, for $\varepsilon = \pm 1$. In particular, when

$$ [\psi G, \psi G] \leq Z(G), \quad (1.3) $$

all operations (1.2) yield bi-skew braces $(G, \cdot, \circ)$, for $\varepsilon = \pm 1$. 

Date: 25 July 2022, 10:11 CEST — Version 4.03.

2010 Mathematics Subject Classification. 20B05 20D45 20N99 08A35 16T25.

Key words and phrases. endomorphisms, regular subgroups, skew braces, brace blocks, normalising graphs, Yang–Baxter equation.

Both authors are members of INdAM—GNSAGA. The first author gratefully acknowledges support from the Department of Mathematics of the University of Trento. The second author gratefully acknowledges support from the Department of Mathematics of the University of Pisa.
In [Koc22] Koch showed that if \((G, \cdot)\) is a group and \(\psi\) is an endomorphism of \(G\) with an abelian image, then \(\psi\) is also an endomorphism of the group \((G, \circ)\), for \(\circ\) as in (1.1). Iterating this procedure, Koch was able to construct a countable family \((\circ_n : n \in \mathbb{N})\) of group operations on the set \(G\), starting with the original operation “\(\cdot\)” as \(\circ_0\), such that \((G, \circ_n, \circ_m)\) is a skew brace for all \(n, m \geq 0\). Koch named such a family a brace block.

In this paper we extend both of Koch’s constructions. An immediate generalisation could be obtained by replacing the assumption that \(\psi\) has an abelian image with assumption (1.3). We work, however, in the following slightly more general context. Let \((G, \cdot)\) be a group, and let \(K \leq A \leq G\) be subgroups such that \(K \leq Z(G)\) and \(A/K\) is abelian. Consider the set
\[
A = \{ \psi \in \text{End}(G/K) : \psi(G/K) \leq A/K \},
\]
which is a ring under the usual operations. Given \(\psi \in A\), let \(\psi^{\uparrow}\) be a lifting of \(\psi\) to \(G\), that is, a map on \(G\) such that
\[
(\psi^{\uparrow} g)K = \psi(gK)
\]
for \(g \in G\). Such a \(\psi^{\uparrow}\) need not be unique, nor an endomorphism of \(G\).

We could try and extend Koch’s iterative construction by using the \(\psi\) and \(\psi^{\uparrow}\), possibly employing different elements of \(A\) at each step. However, in doing so certain bilinear maps occur naturally, in the form of products of suitable central commutators. We thus incorporate these bilinear maps in our construction, by introducing, on the model of a construction in [Car18], the set
\[
B = \{ \alpha : G \times G \to K : \alpha \text{ is bilinear, and } \alpha(K, G) = \alpha(G, K) = 1 \}.
\]
Given \((\psi, \alpha) \in A \times B\), we can now define on \(G\) the operation
\[
g \circ_{\psi, \alpha} h = g \cdot \psi^{\uparrow} g \cdot h \cdot (\psi^{\uparrow} g)^{-1} \cdot \alpha(g, h),
\]
which does not depend on the choice of the particular lifting \(\psi^{\uparrow}\) of \(\psi\).

In Theorem 3.2 we show that \((G, \circ_{\psi, \alpha}, \circ_{\phi, \beta})\) is a skew brace for all \((\psi, \alpha), (\phi, \beta) \in A \times B\). In other words, the operations “\(\circ_{\psi, \alpha}\)” form a brace block, for \((\psi, \alpha) \in A \times B\). Koch’s construction in [Koc21] occurs here by specialising \(K = 1\) (so that both \(\alpha\) and \(\beta\) are the trivial bilinear map), replacing \(\varphi\) with \(g \mapsto \varphi g^{-1}\), and taking \(\psi = 0\) to be the endomorphism of \(G\) mapping every element \(g \in G\) to 1.

In Theorem 4.3 we also show that, given a sequence \((\psi_n, \alpha_n) \in A \times B\), the operations “\(\circ_{\psi_n, \alpha_n}\)” can be iterated, yielding operations of the same form, thereby extending Koch’s construction of brace blocks in [Koc22]. As mentioned above, it is in this step that our introduction of bilinear
maps proves to be necessary, as such maps occur naturally in the iterative steps as products of central commutators (see Theorem 4.3 and the following Remark 4.4).

Perhaps it could be noted that, given \((\psi, \alpha), (\varphi, \beta) \in A \times B\), when doing our construction first with \((\psi, \alpha)\) and then with \((\varphi, \beta)\), we obtain an operation \(o_{(\xi, \gamma)}\), for a certain \((\xi, \gamma) \in A \times B\), where \(\xi = \psi + \varphi + \psi \varphi\) is the Jacobson circle of \(\psi\) and \(\varphi\) (see Theorem 4.3).

We provide several examples of our construction at work.

In particular, Example 5.10 shows that we can obtain brace blocks of any given cardinality consisting of distinct operations, and answers a question posed to us by Cornelius Greither at the Conference on Hopf algebras and Galois module theory held (online) in Omaha in May 2021. Namely, we exhibit a sequence of distinct (topological) group operations on the \(p\)-adic Heisenberg group \((G, \cdot)\) such that \(G\) is a skew brace with respect to any two of them, and the sequence converges to the original operation \(\cdot\).

The interplay between set-theoretic solutions of the Yang–Baxter equation of Mathematical Physics, (skew) braces, regular subgroups of the holomorph, and Hopf–Galois structures has spawned a considerable body of literature in recent years (see, for example, [Rum07, GV17, CJLO14, GP87, Byo96, SV18]).

We discuss some preliminaries about skew braces in Section 2. In Section 3 we describe our construction, which we then apply in Section 4 to the extension of Koch’s iterative approach mentioned above. Section 5 contains examples. We discuss set-theoretic solutions of the Yang–Baxter equation in Section 6. In Section 7 we rephrase brace blocks in terms of the **normalising graph**, exploiting the connections between regular subgroups and skew braces.

We are very grateful to Cornelius Greither for the suggestion that led to Example 5.10, and to the referee for a number of useful suggestions.

## 2. Preliminaries

In what follows, if \((G, \cdot)\) is a group, we write \(\text{End}(G, \cdot)\) for the monoid of endomorphisms of \((G, \cdot)\). We denote the action of \(\psi \in \text{End}(G, \cdot)\) on \(g \in G\) by a left exponent \(\psi g\), and thus we compose such endomorphisms right-to-left. If \(\varepsilon \in \mathbb{Z}\), then we have clearly \(\psi(g^\varepsilon) = (\psi g)^\varepsilon\); thus we write simply \(\psi g^\varepsilon\) for such an element.

We recall here the definitions of skew braces, bi-skew braces, and brace blocks, given in [GV17, Chi19, and Koc22] respectively. (We slightly generalise the last definition, in order to possibly admit brace blocks with arbitrary cardinality.)

Let \((G, \cdot)\) be a group with identity 1. We denote by \(g^{-1}\) the inverse of an element \(g \in G\) with respect to \(\cdot\).
A. CARANTI AND L. STEFANELLO

Definition 2.1. A skew (left) brace is a triple \((G, \cdot, \circ)\), where \((G, \cdot)\) and \((G, \circ)\) are groups, and for all \(g, h, k \in G\),
\[
g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k).
\]

It is immediate to check that \((G, \cdot)\) and \((G, \circ)\) share the same identity 1; see [GV17, Lemma 1.7].

Definition 2.2. A bi-skew brace is a triple \((G, \cdot, \circ)\), where both \((G, \cdot, \circ)\) and \((G, \circ, \cdot)\) are skew braces.

Definition 2.3. Let \(G\) be a non-empty set. A brace block on \(G\) is a family \(F\) of group operations on \(G\) such that \((G, \circ, \cdot)\) is a (bi-)skew brace for all \(\circ, \cdot \in F\).

Also, we write \(\iota\) for the homomorphism mapping \(g \in G\) to conjugation by \(g\):
\[
\iota: (G, \cdot) \to \text{Aut}(G, \cdot) \\
g \mapsto (x \mapsto g \cdot x \cdot g^{-1}).
\]

3. The main construction

Let \((G, \cdot)\) be a group, let \(K\) be a subgroup of \(G\) contained in the centre \(Z(G)\) of \(G\), and let \(A\) be a subgroup of \(G\) such that \(A/K\) is abelian.

Define
\[
\mathcal{A} = \{ \psi \in \text{End}(G/K) : \psi(G/K) \leq A/K \}.
\]
The set \(\mathcal{A}\) is a ring under the usual operations, with unity only when \(A = G\).

Given \(\psi \in \mathcal{A}\), define \(\psi^\uparrow\) to be a lifting of \(\psi\), that is, a set-theoretic map \(\psi^\uparrow: G \to A\) such that the following diagram is commutative:
\[
\begin{array}{ccc}
G & \xrightarrow{\psi^\uparrow} & A \\
\downarrow & & \downarrow \\
G/K & \xrightarrow{\psi} & A/K
\end{array}
\]

This is equivalent to saying that for all \(g \in G\) we have
\[
(\psi^\uparrow g)K = \psi(gK).
\]
Liftings are clearly not unique if \(K \neq 1\), and need not be endomorphisms of \(G\).

Given a lifting \(\psi^\uparrow\) of \(\psi \in \mathcal{A}\), we have for \(g, h \in G\) that
\[
\psi^\uparrow(g \cdot h)K = \psi((g \cdot h)K) = \psi(gK) \cdot \psi(hK) \\
= (\psi^\uparrow g)K \cdot (\psi^\uparrow h)K = (\psi^\uparrow g \cdot \psi^\uparrow h)K,
\]
that is,
\[
\psi^\uparrow(g \cdot h) \equiv \psi^\uparrow g \cdot \psi^\uparrow h \pmod{K}.
\]
Similarly, as $A/K$ is abelian, we obtain
\[ \psi^g \cdot \psi^h \equiv \psi^h \cdot \psi^g \pmod{K}; \]
since $K \leq Z(G, \cdot)$, this yields
\[ \iota((\psi^g \cdot h)) = \iota(\psi^h \cdot \psi^g) = \iota(\psi^g \cdot \psi^h). \tag{3.1} \]

**Remark 3.1.** Let $\psi, \varphi \in A$ have liftings $\psi^\uparrow$ and $\varphi^\uparrow$, respectively. Then the map $\psi^\uparrow + \varphi^\uparrow : G \to A$ defined by
\[ \psi^\uparrow + \varphi^\uparrow g = \psi^\uparrow g \cdot \varphi^\uparrow g \]
is a lifting of $\psi + \varphi$, and the map $\psi^\uparrow \varphi^\uparrow : G \to A$ defined by
\[ \psi^\uparrow \varphi^\uparrow g = \psi^\uparrow (\varphi^\uparrow g) \]
is a lifting of $\psi \varphi$.

Now consider
\[ B = \{ \alpha : G \times G \to K : \alpha \text{ is bilinear, and } \alpha(K, G) = \alpha(G, K) = 1 \}. \]
(Similar maps were used in a related context in [Car18].) If $(\psi, \alpha) \in A \times B$ and $\psi^\uparrow$ is a lifting of $\psi$, then we define an operation on $G$ via
\[ g \circ_{\psi, \alpha} h = g \cdot \psi^\uparrow g \cdot h \cdot (\psi^\uparrow g)^{-1} \cdot \alpha(g, h) = g \cdot \iota(\psi^\uparrow g) h \cdot \alpha(g, h). \]
Note that the operation $\circ_{\psi, \alpha}$ is indeed independent of the particular choice of the lifting $\psi^\uparrow$ of $\psi$, as if $\psi^\uparrow$ is another lifting of $\psi$, we have $\psi^\uparrow g \equiv \psi^\uparrow g \pmod{K}$ for $g \in G$, so that $\iota(\psi^\uparrow g) = \iota(\psi^\uparrow g)$, as $K \leq Z(G, \cdot)$.

We claim that $(G, \cdot, \circ_{\psi, \alpha})$ is a bi-skew brace. A sharper statement actually holds.

**Theorem 3.2.** For $(\varphi, \beta), (\psi, \alpha) \in A \times B$ we have that
\[ (G, \circ_{\varphi, \beta}, \circ_{\psi, \alpha}) \]
is a bi-skew brace.

In other words, the family $(\circ_{\varphi, \beta} : (\varphi, \beta) \in A \times B)$ is a brace block.

Before going into the proof, we record the following useful fact, which is a slight generalisation of (3.1):
\[ \iota((\varphi^\uparrow h \cdot \psi^\uparrow g)) = \iota(\psi^\uparrow g \cdot \varphi^\uparrow h). \]

In fact we have
\[ (\varphi^\uparrow h \cdot \psi^\uparrow g)K = (\varphi^\uparrow h)K \cdot (\psi^\uparrow g)K = (\psi^\uparrow g)K \cdot (\varphi^\uparrow h)K = (\varphi^\uparrow h \cdot \psi^\uparrow g)K, \]
as $A/K$ is abelian.

**Proof.** Let $\varphi^\uparrow$ and $\psi^\uparrow$ be liftings of $\varphi$ and $\psi$, respectively.

It is not difficult to verify that $(G, \circ_{\varphi, \beta})$ is indeed a group. Associativity follows from the fact that
\[(g \circ_{\varphi, \beta} h) \circ_{\varphi, \beta} k \quad \text{and} \quad g \circ_{\varphi, \beta} (h \circ_{\varphi, \beta} k)\]
both expand to

\[ g \circ \varphi g \cdot h \cdot \varphi h \cdot k \cdot (\varphi^{-1} h)^{-1} \cdot (\varphi g)^{-1} \cdot \beta(g, h) \cdot \beta(g, k) \cdot \beta(h, k). \]

The identity is 1 and the inverse of \( g \) is

\[ \overline{g} = (\varphi g)^{-1} \cdot g^{-1} \cdot \varphi g \cdot \beta(g, g). \]

To show that \((G, \circ_{\varphi, \beta}, \circ_{\psi, \alpha})\) is a skew brace, we need to verify that for all \( g, h, k \in G \) we have

\[ (g \circ_{\psi, \alpha} h) \circ_{\varphi, \beta} g \circ_{\varphi, \beta} (g \circ_{\psi, \alpha} k) \]
equals

\[ g \circ_{\psi, \alpha} (h \circ_{\varphi, \beta} k). \]

We have

\[
\overline{g} \circ_{\varphi, \beta} (g \circ_{\psi, \alpha} k) = \overline{g} \circ_{\varphi, \beta} (g \cdot (\varphi g)k \cdot \alpha(g, k)) \\
= (\varphi^{-1} g \cdot (\varphi g) \cdot \beta(g, g) \cdot (\varphi^{-1} g \cdot (\varphi g)k \cdot \alpha(g, k)) \\
\cdot \beta(g^{-1} g, k) \cdot \beta(g^{-1}, k) \\
= (\varphi^{-1} g \cdot (\varphi g) \cdot \beta(g^{-1}, k) \cdot \alpha(g, k) \\
= (\varphi^{-1} g \cdot \beta(g^{-1}, k) \cdot \alpha(g, k)
\]

and

\[
(g \circ_{\psi, \alpha} h) \circ_{\varphi, \beta} g \circ_{\varphi, \beta} (g \circ_{\psi, \alpha} k) \\
= g \cdot \varphi^{-1} g \cdot h \cdot (\varphi^{-1} g) \cdot k \cdot (\varphi g)^{-1} \cdot \varphi^{-1} g \cdot \beta(g^{-1}, k) \cdot \alpha(g, k) \\
\cdot \beta(h, k) \cdot \alpha(g, h) \cdot \alpha(g, k) \\
= g \cdot \varphi^{-1} g \cdot h \cdot (\varphi^{-1} g) \cdot k \cdot (\varphi g)^{-1} \cdot \varphi^{-1} g \cdot \beta(h, k) \cdot \alpha(g, h) \cdot \alpha(g, k) \\
= g \cdot \varphi^{-1} g \cdot h \cdot (\varphi^{-1} g) \cdot k \cdot (\varphi g)^{-1} \cdot \beta(h, k) \cdot \alpha(g, h) \cdot \alpha(g, k) \\
= g \cdot \varphi^{-1} g \cdot h \cdot (\varphi^{-1} g) \cdot k \cdot (\varphi g)^{-1} \cdot \beta(h, k) \cdot (\varphi g)^{-1} \cdot \alpha(g, h) \cdot \alpha(g, k) \\
= g \circ_{\psi, \alpha} (h \circ_{\varphi, \beta} k),
\]
as claimed. \( \square \)
Remark 3.3. Specialising $\varphi(xK) = K$ and $\beta(x, y) = 1$ for all $x, y \in G$ in Theorem 3.2, so that $g \circ_{x, \beta} h = g \cdot h$, we obtain that $(G, \cdot, \circ_{\psi, \alpha})$ is a bi-skew brace.

4. ITERATING THE CONSTRUCTION

Let $(\psi, \alpha) \in \mathcal{A} \times \mathcal{B}$, and let $\psi^\uparrow$ be a lifting of $\psi$. Write “$\circ$” for “$\circ_{\psi, \alpha}$”. Then $(G, \circ)$ is a group and $(K, \circ)$ and $(A, \circ)$ are both subgroups of $(G, \circ)$. (Note in particular that “$\circ$” and “$\cdot$” coincide on $K$, as $(K, \cdot) \leq Z(G, \cdot)$.) Moreover, $\psi^\uparrow K \leq K$, as

$$\psi^\uparrow k K = \psi(kK) = \psi K = K.$$ 

We obtain that $K \leq Z(G, \circ)$, and since $(A/K, \circ)$ remains abelian, we can consider the ring

$$A_\circ = \{ \varphi \in \text{End}(G/K, \circ) : \varphi G/K \leq A/K \}.$$ 

Note that the bilinear map $\alpha$ does not play a role in the group operation of $G/K$, as the image of $\alpha$ is contained in $K$. We claim that $A \subseteq A_\circ$. We prove this result in a more general setting.

Lemma 4.1. Let $B$ be a group, and let $C \leq B$ be an abelian subgroup. Consider the ring $R$ of endomorphisms of $B$ with image in $C$. If $\psi, \varphi \in R$, then $\varphi$ is also an endomorphism of $(B, \circ_{\psi})$, where

$$x \circ_{\psi} y = x \cdot \psi x \cdot y \cdot (\psi x)^{-1}.$$ 

Proof. For all $x, y \in B$ we have, as $C$ is abelian,

$$\varphi(x \circ_{\psi} y) = \varphi(x \cdot \psi x \cdot y \cdot (\psi x)^{-1}) = \varphi x \cdot \varphi(\psi x) \cdot \varphi y \cdot \varphi(\psi x)^{-1} = \varphi x \cdot \varphi y.$$

and

$$\varphi x \circ_{\psi} \varphi y = \varphi x \cdot \varphi(\psi x) \cdot \varphi y \cdot \varphi(\psi x)^{-1} = \varphi x \cdot \varphi y.$$ 

Now consider

$$B_\circ = \{ \beta : G \times G \to K : \beta \text{ is bilinear with respect to } \circ, \text{ and } \beta(K, G) = \beta(G, K) = 1 \}.$$ 

Lemma 4.2. The inclusion $B \subseteq B_\circ$ holds.

Proof. Let $\beta \in B$, and let $g, h, k \in G$. Then

$$\beta(g \circ h, k) = \beta(g \cdot \psi^\uparrow g \cdot h \cdot (\psi^\uparrow g)^{-1} \cdot \alpha(g, h), k)$$

$$= \beta(g, k) \cdot \beta(h, k) = \beta(g, k) \circ \beta(h, k).$$

Similarly,

$$\beta(g, h \circ k) = \beta(g, k) \circ \beta(h, k).$$
This result implies, very much as in [Koc22], that given \((\varphi, \beta) \in A \times B \subseteq A \circ B\), if \(\varphi^\uparrow\) is a lifting of \(\varphi\), then we may iterate our main construction with respect to \((G, \circ)\) and \((\varphi, \beta)\). Therefore \((G, \circ, \ast)\) is a bi-skew brace, where
\[
g \ast h = g \circ \varphi^\uparrow g \circ h \circ \overline{\varphi} g \circ \beta(g, h).
\]
(Here an overline denotes the inverse with respect to \(\circ\).

Two questions arise naturally. Is \((G, \cdot, \ast)\) a bi-skew brace? And if this is the case, can we obtain it directly from our construction applied to \((G, \cdot)\)?

The next result provides an affirmative answer to both questions.

**Theorem 4.3.** Set \(\circ_0\) to be "·". Let \(((\psi_n, \alpha_n) : n \geq 1)\) be a sequence of elements of \(A \times B\), and for all \(n \geq 1\), define
\[
g \circ_n h = g \circ_{n-1} \psi^\uparrow_n g \circ_{n-1} h \circ_{n-1} \overline{\psi}_n g \circ_{n-1} \alpha_n(g, h),
\]
where \(\psi^\uparrow_n\) is a lifting of \(\psi_n\) and an overline denotes the inverse with respect to \(\circ_{n-1}\). Then for all \(n \geq 1\) we have
\[
g \circ_n h = g \cdot q^\ast g \cdot h \cdot (q^\ast g)^{-1} \cdot \beta_n(g, h),
\]
where
\[
q_n = \sum_{A \subseteq \{1, \ldots, n\}} \left( \prod_{j \in A} \psi_j^\uparrow \right) = \left( \sum_{A \subseteq \{1, \ldots, n\}} \left( \prod_{j \in A} \psi_j \right) \right)^\uparrow
\]
and \(\beta_n \in B\). In particular, \((G, \cdot, \circ_n)\) is a bi-skew brace.

**Remark 4.4.** The previous result is the main reason why we have introduced bilinear maps in our construction in Section 3. In the following proof one sees indeed that bilinear maps naturally occur in the form of products of suitable central commutators in the iterating steps, even if one starts the construction with an element of \(A\) only.

Before the proof, we introduce a technical lemma, in which we compute conjugates with respect to our new operations.

**Lemma 4.5.** Let \((\psi, \alpha) \in A \times B\). Then for all \(x, y \in G\) we have
\[
x \circ_{\psi, \alpha} y \circ_{\psi, \alpha} \overline{\psi} = x \cdot \psi^\uparrow x \cdot y \cdot (\psi^\uparrow x)^{-1} \cdot \psi^\uparrow y \cdot x^{-1} \cdot (\psi^\uparrow y)^{-1} \cdot \alpha(x, y) \cdot \alpha(y, x^{-1}),
\]
where an overline denotes the inverse with respect to \(\circ_{\psi, \alpha}\).

**Proof.** Write "\(\circ\)" for "\(\circ_{\psi, \alpha}\)". First, we compute \(y \circ \overline{\psi}\):
\[
y \circ \overline{\psi} = y \cdot \psi^\uparrow y \cdot (\psi^\uparrow x)^{-1} \cdot x^{-1} \cdot \psi^\uparrow x \cdot \alpha(x, x) \cdot (\psi^\uparrow y)^{-1} \cdot \alpha(y, x^{-1}),
\]
Therefore, employing (3.1), we deduce that
\[
x \circ y \circ x = x \cdot \psi^t x \cdot (y \cdot \psi^t y \cdot (\psi^t x)^{-1} \cdot x^{-1} \cdot \psi^t \cdot \alpha(x, x) \cdot (\psi^t y)^{-1} \cdot \alpha(y, x^{-1}))
\]
\[
\cdot (\psi^t x)^{-1} \cdot \alpha(x, y) \cdot \alpha(x, x^{-1})
\]
\[
= x \cdot \psi^t x \cdot y \cdot (\psi^t x)^{-1} \cdot \psi^t y \cdot x^{-1} \cdot (\psi^t y)^{-1} \cdot \psi^t x \cdot (\psi^t x)^{-1}
\]
\[
\cdot \alpha(x, y) \cdot \alpha(y, x^{-1})
\]
\[
= x \cdot \psi^t x \cdot y \cdot (\psi^t x)^{-1} \cdot \psi^t y \cdot x^{-1} \cdot (\psi^t y)^{-1} \cdot \alpha(x, y) \cdot \alpha(y, x^{-1}).
\]
as claimed.

\(\square\)

**Proof of Theorem 4.3.** We prove the result by induction on \(n\). For \(n = 1\), the result follows from Theorem 3.2 with \(q_1 = \psi_1^t\) and \(\beta_1 = \alpha_1\).

Suppose that the assertion holds for \(n - 1\), where \(n \geq 2\), so that we have
\[
g \circ_{n-1} h = g \cdot q^{n-1} g \cdot h \cdot (q^{n-1} g)^{-1} \cdot \beta_{n-1}(g, h).
\]
Thus \((G, \cdot, \circ_{n-1})\) is a bi-skew brace, again by Theorem 3.2.

Write \(\psi^t\) for \(\psi_1^t\), \(\alpha\) for \(\alpha_n\), \(q\) for \(q_{n-1}\), and \(\beta\) for \(\beta_{n-1}\). Then we have
\[
g \circ_n h = g \circ_{n-1} \psi^t g \circ_{n-1} h \circ_{n-1} \psi^t g \circ_{n-1} \alpha(g, h),
\]
where \(\psi^t g\) denotes the inverse with respect to \(\circ_{n-1}\). By Lemma 4.5 we have
\[
\psi^t g \circ_{n-1} h \circ_{n-1} \psi^t g
\]
\[
= \psi^t g \cdot q \psi^t g \cdot h \cdot (q \psi^t g)^{-1} \cdot \psi g \cdot (\psi^t g)^{-1} \cdot (\psi g) \cdot \beta(g, h) \cdot \beta(h, (\psi g)^{-1})
\]
\[
= \psi^t g \cdot q \psi^t g \cdot h \cdot (\psi^t g)^{-1} \cdot (\psi^t g)^{-1} \cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1})
\]
\[
= \psi^t + \psi g \cdot h \cdot (\psi^t + \psi g)^{-1} \cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1}).
\]
We obtain that
\[
g \circ_{n-1} (\psi^t + \psi g) \cdot h \cdot (\psi^t + \psi g)^{-1} \cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1})
\]
\[
= g \cdot q \cdot \psi^t + \psi^t g \cdot h \cdot (q \psi^t + \psi^t g)^{-1} \cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1})
\]
\[
\cdot (q \psi^t)^{-1} \cdot \beta(g, h)
\]
\[
= g \cdot q \psi^t + \psi^t g \cdot h \cdot (q \psi^t + \psi^t g)^{-1}
\]
\[
\cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1}) \cdot \beta(g, h).
\]
(Here we have used the fact that \([\psi^t g, \psi h] \in K \leq Z(G), as\. A/K is abelian.) Finally, note that if \(k \in K\) and \(x \in G\), then we have \(x \circ_{n-1} k = x \cdot k\); thus, as \(\alpha(g, h) \in K\), we find that \(g \circ_n h\) equals
\[
g \cdot q \psi^t + \psi^t g \cdot h \cdot (q \psi^t + \psi^t g)^{-1}
\]
\[
\cdot \psi g \cdot \alpha_1(h) \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1}) \cdot \beta(g, h) \cdot \alpha(g, h).
\]
Therefore we may set
\[
\beta_n(g, h) = [\psi^t g, \psi h] \cdot \beta(g, h) \cdot \beta(h, (\psi^t g)^{-1}) \cdot \beta(g, h) \cdot \alpha(g, h),
\]
which can be easily seen to satisfy the required properties, and we note that

\[
q + \psi^\uparrow + q\psi^\uparrow = q_{n-1} + \psi^\uparrow_n + q_{n-1}\psi^\uparrow_n
\]

\[
= \sum_{A \subseteq \{1, \ldots, n-1\}} \left( \prod_{j \in A} \psi^\uparrow_j \right) + \psi^\uparrow_n + \sum_{A \subseteq \{1, \ldots, n-1\}} \left( \prod_{j \in A} \psi^\uparrow_j \right) \psi^\uparrow_n
\]

\[
= \sum_{A \subseteq \{1, \ldots, n\} \setminus \emptyset} \left( \prod_{j \in A} \psi^\uparrow_j \right) = q_n.
\]

We conclude, appealing once more to Theorem 3.2, that \((G, \cdot, \circ_n)\) is a bi-skew brace. □

**Corollary 4.6.** In the setting of Theorem 4.3, \((G, \circ_n, \circ_m)\) is a bi-skew brace, for all \(n, m \geq 0\).

**Proof.** Assuming without loss of generality that \(n \leq m\), we can apply Theorem 4.3 starting with \((G, \circ_n)\), as \((\psi^{i+n}, \alpha_{i+n}) : i \geq 1\) is a sequence of elements of \(A_{\circ_n} \times B_{\circ_n}\) by Lemmas 4.1 and 4.2. □

## 5. Examples

### 5.1. Koch’s construction revisited.

We begin by recovering Koch’s construction of [Koc22].

**Example 5.1.** Let \(\psi : G \to G\) be an abelian map, that is, an endomorphism of \(G\) such that \(\psi G = A\) is abelian.

Take \(K = [G, G] = 1 \leq Z(G)\). Then \(\psi \in A\), and we can consider \(\varphi = -\psi \in A\), where

\[
\varphi g = \psi(g^{-1}).
\]

Now take \((\varphi_n : n \geq 1),\) where \(\varphi_n = \varphi\) for all \(n \geq 1\). Then Theorems 3.2 and 4.3 yield a family of operations \((\circ_n : n \geq 1)\) defined by

\[
g \circ_n h = g \cdot q^n g \cdot h \cdot (q^n g)^{-1}
\]

such that \((G, \circ_n, \circ_m)\) is a bi-skew brace for all \(n, m \geq 1\). Here

\[
q_n = \sum_{A \subseteq \{1, \ldots, n\} \setminus \emptyset} \varphi^{\lvert A \rvert} = \sum_{i=1}^{n} \binom{n}{i} \varphi^i = \sum_{i=1}^{n} \binom{n}{i} (-\psi)^i.
\]

Note that all the central bilinear maps are trivial, as they have image in \(K = 1\).

**Example 5.2.** In a slight variation on Example 5.1, and with the same setting, we may take \((\psi_n : n \geq 1),\) where \(\psi_n = \psi\) for all \(n \geq 1\). Then we obtain a family of operations \((\circ_n : n \geq 1)\) such that \((G, \circ_n, \circ_m)\) is a bi-skew brace for all \(n, m \geq 1\), where

\[
g \circ_n h = g \cdot q^n g \cdot h \cdot (q^n g)^{-1}
\]
and
\[ q_n = \sum_{A \subseteq \{1, \ldots, n\}} \psi^{|A|} = \sum_{i=1}^{n} \binom{n}{i} \psi^i. \]

We can also give the following slight generalisation of Koch’s construction, whose proof follows as an immediate application of Theorem 3.2.

**Proposition 5.3.** Let \( A \) be an abelian subgroup of \( G \), and let
\[ \mathcal{A} = \{ \psi \in \text{End}(G) \mid \psi G \leq A \}. \]
For all \( \psi \in \mathcal{A} \), write
\[ g \circ_\psi h = g \cdot \psi g \cdot h \cdot \psi g^{-1}. \]
Then \((\circ_\psi \mid \psi \in \mathcal{A})\) is a brace block on \( G \).

### 5.2. Endomorphisms, liftings, and central bilinear maps.

The next example shows that the operations obtained with liftings are not covered by endomorphisms and central bilinear maps alone.

If \( G \) is a group, then for all \( i \geq 0 \) we write \( \gamma_k(G) \) for the \( i \)-th term of the lower central series, and \( Z_k(G) \) for the \( i \)-th term of the upper central series.

**Example 5.4.** Let \( p \geq 5 \) be a prime, and let \( H \) the group of upper unitriangular matrices in \( \text{GL}(5, p) \).

Write as usual \( e_{ij} \) for a matrix whose \((i, j)\)-entry is 1 and all other entries are zero, and let \( t_{ij} = 1 + e_{ij} \in H \), for \( i < j \).

The following elementary facts hold:

1. \( H \) has exponent \( p \).
2. \( \gamma_k(H) = \langle t_{ij} : j - i \geq k \rangle \).
3. \( H \) has nilpotence class four.
4. The upper and lower central series of \( H \) coincide.

Moreover we have
\[ [t_{12}, t_{23}] = t_{13}, \quad [t_{34}, t_{45}] = t_{35}, \quad [t_{13}, t_{35}] = t_{15} \neq 1. \]

Let \( G = \langle x, y \rangle \times H \), where \( \langle x, y \rangle \) is elementary abelian of order \( p^2 \).
Let \( K = \gamma_4(H) = \langle t_{15} \rangle \leq G \), and let \( A = \gamma_2(H) \leq G \).
We have the following facts:

1. \( K \leq Z(G) \).
2. \( A/K \) is abelian, while \( A \) is not.
3. \( \langle t_{13}, t_{35} \rangle K/K \) is elementary abelian of order \( p^2 \).
4. The assignment
\[ x \mapsto t_{13}, \quad y \mapsto t_{35}, \quad t_{ij} \mapsto 1 \]
uniquely defines an endomorphism \( \psi \) of \( G/K \).
(5) The endomorphism \( \psi \) does not extend to an endomorphism of \( G \), because if \( g \in t_{13}K \) and \( h \in t_{35}K \), then \([g, h] = [t_{13}, t_{35}]\), as \( K \leq Z(G) \), so that \([g, h] = t_{15} \neq 1\).

We claim that there is no \( \varphi \in \text{End}(G) \) and central bilinear map \( \alpha : G \times G \to K \) such that, for all \( g, h \in G \), one has
\[
g \circ_{\psi, \text{triv}} h = g \cdot \varphi g \cdot h \cdot \varphi g^{-1} \cdot \alpha(g, h),
\]
where \( \text{triv} : G \times G \to K \) extends to an endomorphism \( \alpha : G \times G \to K \) such that, for all \( g, h \in G \), one has
\[
g \circ_{\psi, \text{triv}} h = g \cdot \varphi g \cdot h \cdot \varphi g^{-1} \cdot \alpha(g, h),
\]
where \( \text{triv} : G \times G \to K \) is the trivial bilinear map, mapping everything to 1. In other words, endomorphisms and central bilinear maps alone cannot replace liftings.

If (5.1) holds, set \( g = x \) to get for all \( h \in H \) that
\[
x \circ_{\psi, \text{triv}} h = x \cdot \psi^i x \cdot h \cdot \psi^i x^{-1} = x \cdot h \cdot [h^{-1}, t_{13}]
= x \cdot h \cdot [h^{-1}, \varphi x] \cdot \alpha(x, h).
\]
Thus for all \( h \in G \) we have
\[
[h^{-1}, \varphi x^{-1} \cdot t_{13}]=[h^{-1}, \varphi x^{-1}] \cdot [\varphi x^{-1}, [h^{-1}, t_{13}]] \cdot [h^{-1}, t_{13}]
\equiv [h^{-1}, \varphi x^{-1}] \cdot [h^{-1}, t_{13}] \equiv 1 \pmod{K}.
\]
Since \( K = \gamma_4(H) \leq Z(G) \), we obtain that \( \varphi x^{-1} \cdot t_{13} = a \in Z_2(G) \).

A similar argument yields \( \varphi y^{-1} \cdot t_{35} = b \in Z_3(G) \). Now we have \([\gamma_2(G), Z_2(G)] = 1\). Moreover \( Z_2(G) = \langle x, y \rangle \times \gamma_3(H) \) is abelian.

Since \( t_{13}, t_{35} \in \gamma_2(G) \), we obtain
\[
1 = \varphi[x, y] = [t_{13} \cdot a^{-1}, t_{35} \cdot b^{-1}] = [t_{13}, t_{35}] = t_{35} \neq 1,
\]
a contradiction.

We now sketch a more complicated construction, which cannot be expressed as a direct product as the previous one.

**Example 5.5.** Let
\[
G = \langle x, y, t_1, t_2, t_3, t_4 : [y, x] = 1, \text{ and commutators of weight } > 4 \text{ vanish} \rangle.
\]
So \( G \) is the quotient group of the free group in the variety of nilpotent groups of class four in the generators \( x, y, t_1, t_2, t_3, t_4 \), by the normal subgroup generated by \( [y, x] \).

Let \( A = \gamma_2(G) \), and let \( K = \gamma_4(G) \). Note that \( A \) is non-abelian, as \([t_1, t_2], [t_3, t_4] \neq 1\), while \( A/K \) is abelian, as \([\gamma_2(G), \gamma_2(G)] \leq \gamma_4(G)\).

The assignment
\[
x \mapsto [t_1, t_2], \quad y \mapsto [t_3, t_4], \quad t_1 \mapsto 1
\]
extends to an endomorphism \( \psi \) of \( G/K \), as the quotient of \( G \) by the normal subgroup generated by the \( t_i \) (which contains \( K \)) is free abelian of rank two, and so is \( \langle [t_1, t_2], [t_3, t_4] \rangle K/K \). However, very much as in the previous example, \( \psi \) one sees does not extend to an endomorphism of \( G \).

One can check with the Nilpotent Quotient Algorithm [Nic96] of GAP [GAP10] that \( Z(G) = \gamma_4(G) \), and \( Z_2(G) = \gamma_3(G) \). Then an
argument entirely similar to the one in the previous example shows that an identity like (5.1) does not hold here.

5.3. **Groups of nilpotence class two.** Let $G$ be a group of nilpotence class two. Our construction applies to $G$ with $A = G$ and $K = [G,G]$. Here $A = \text{End}(A/K)$. In particular, we obtain the following two classes of examples.

**Proposition 5.6.** Let $G$ be a group of nilpotence class two. For all $n \in \mathbb{N}$ write

$$g \circ_n h = g \cdot g^n \cdot h \cdot g^{-n} = g \cdot h \cdot [h^{-1}, g]^n = g \cdot h \cdot [g, h]^n.$$

Then $(\circ_n | n \in \mathbb{Z})$ is a brace block on $G$.

**Proof.** As $G/K$ is abelian, $gK \mapsto g^nK$ is an endomorphism of $G/K$. Therefore we can apply Theorem 3.2 to derive the assertion. □

**Example 5.7.** In [Car16], the first author constructed examples of finite $p$-groups of class two, for $p > 2$, where $[G,G]$ coincides with the Frattini subgroup $\text{Frat}(G)$, such that $\text{End}(G)$ induces only the identity map $1$ and the trivial endomorphism $0$ on $G/[G,G]$ (there are many similar examples in the literature).

In particular, if $n \not\equiv 0, 1 \pmod{p}$, then $gK \mapsto g^nK$ is an endomorphism of the elementary abelian group $p$-group $G/Z(G)$ which cannot be lifted to an endomorphism of $G$. This means that the operation $\circ_n$ defined by

$$g \circ_n h = g \cdot g^n \cdot h \cdot g^{-n} = g \cdot h \cdot [g, h]^n$$

can be obtained either with a lifting or with a bilinear map, but not with an endomorphism alone.

**Remark 5.8.** See [CS22] for an application of Proposition 5.6 to skew braces and Rota–Baxter operators ([BG22]).

**Proposition 5.9.** Let $G$ be a group of nilpotence class two. For all $\psi \in \text{End}(G)$ write

$$g \circ_\psi h = g \cdot \psi g \cdot h \cdot \psi^{-1}.$$

Then $(\circ_\psi : \psi \in \text{End}(G))$ is a brace block on $G$.

**Proof.** As $K = [G,G]$ is fully invariant in $G$, every endomorphism of $G$ is a lifting of the endomorphism it induces on $G/K$. We conclude by applying Theorem 3.2 □

The following example allows us to obtain brace block of any given cardinality, consisting of distinct operations, because there are commutative rings $S$ with unity of any given cardinality. (The latter statement appears to be folklore, see for instance [Cla]: if the cardinality $n$ is finite, one may take $S = \mathbb{Z}/n\mathbb{Z}$. If the cardinality $n$ is infinite, one may take $S$ to be the polynomial ring in $n$ indeterminates with integer coefficients.)
Example 5.10. Let $R$ be a commutative ring with unity, and let $G$ be the Heisenberg group
\[ G = \{(a, b, c) \mid a, b, c \in R\}, \]
with group operation
\[ (a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + ab'). \]
Then $G$ is a group of nilpotence class two: an immediate computation shows that
\[ [(a, b, c), (a', b', c')] = [(0, 0, ab' - a'b)] \in \{(0, 0, c) \mid c \in R\} = Z(G). \]
For all $x \in R$, consider the map
\[ \psi_x : (a, b, c) \mapsto (xa, xb, x^2c). \]
It is clear that $\psi_x \in \text{End}(G)$, so $(\psi_x \mid x \in R)$ yields a brace block $(\circ_x \mid x \in R)$ on $G$. Explicitly, if $g = (a, b, c)$ and $h = (a', b', c')$, then
\[ g \circ_x h = g \cdot h \cdot (0, 0, x(ab' - a'b)). \]
We deduce that this brace block consists of all distinct operations, and has cardinality $|R|$. We consider two special cases.
- Suppose that $R$ has characteristic zero. Then $Z \subseteq R$, and it is immediate to see that the skew braces of the kind $(G, \cdot, \circ_n)$, $n \in Z$, are not isomorphic. We have found a brace block consisting of (at least) countably many non-isomorphic skew braces.
- Suppose that $R$ is a topological ring. Then $G$ is a topological group with the product topology, and if we take a sequence $(r_n \mid n \in \mathbb{N})$ of elements of $R$ converging to 0, we deduce that the corresponding operations $(\circ_{r_n} \mid n \in \mathbb{N})$ are distinct and converge to the original one, in the sense that for all $g, h \in G$,
\[ g \circ_{\infty} h = \lim_{n \to \infty} g \circ_{r_n} h = g \cdot h. \]
For example, if $R = \mathbb{Z}_p$, the ring of $p$-adic integers, then both the previous cases apply; for the latter, one can take $r_n = p^n$ for all $n \geq 0$. This addresses a question posed to us by Cornelius Greither at the Conference on Hopf algebras and Galois module theory held (online) in Omaha in May 2021, about finding a brace blocks with distinct operations converging to the original one.

5.4. More brace blocks of any given cardinality.

Example 5.11. Let $(X, \leq)$ be a well-ordered set with first element $x_0$. Let $(G, \cdot)$ be the free group of class two on $X$, and take $A = G$ and $K = G' = Z(G)$; the latter subgroup is free abelian in the commutators $[u, v]$, for $u > v$.
For all $y \in X$, consider the endomorphism $\psi_y$ of $G/K$ defined by
\[
\begin{cases}
\psi_y(x_0K) = yK, \\
\psi_y(xK) = K, & \text{for } x \in X \setminus \{x_0\}.
\end{cases}
\]
Then for $y \in X$ we have

$$x_0 \circ_{\psi_y, \text{triv}} x_0 = x_0 \cdot \psi^*_y(x_0) \cdot x_0 \cdot \psi^*_y(x_0)^{-1}$$

$$= x_0^2 \cdot x_0^{-1} \cdot y \cdot x_0 \cdot y^{-1}$$

$$= x_0^2 \cdot [x_0^{-1}, y] = x_0^2 \cdot [y, x_0],$$

so that all operations $\circ_{\psi_y, \text{triv}}$ are distinct, for $y \in X$.

Another example of this kind can be constructed in the realm of (absolutely) free groups.

**Example 5.12.** Let $X$ be a non-empty set, and $a \notin X$. Let $(G, \cdot)$ be the free group on $X \cup \{a\}$, and let $K = 1$, $A = \langle a \rangle$.

For all $y \in X$, consider the endomorphism $\psi_y$ of $G$ defined by

$$\begin{cases}
\psi_y(y) = a, \\
\psi_y(a) = 1, \\
\psi_y(x) = 1, & \text{for } x \in X \setminus \{y\}.
\end{cases}$$

Then we have $x \circ_{\psi_y, \text{triv}} x = x \cdot x$ for $x \neq y$, and $y \circ_{\psi_y, \text{triv}} y = y \cdot a \cdot y \cdot a^{-1} = y \cdot y \cdot [y^{-1}, a] \neq y \cdot y$, so that all operations $\circ_{\psi_y, \text{triv}}$ are distinct, for $y \in X$.

**6. Set-theoretic solutions of the Yang–Baxter equation**

We recall that a *set-theoretic solution of the Yang–Baxter equation*, defined in [Dri92], is a pair $(X, r)$, where $X \neq \emptyset$ is a set and

$$r: X \times X \to X \times X$$

$$(x, y) \mapsto (\sigma_x(y), \tau_y(x))$$

is a bijective map satisfying

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

We say that $(X, r)$ is

- *non-degenerate* if for all $x \in X$, $\sigma_x$ and $\tau_x$ are bijective;
- *involutive* if $r^2 = \text{id}_{X \times X}$.

In what follows, by a *solution* $(X, r)$ we mean a set-theoretic non-degenerate solution of the Yang–Baxter equation.

The connection between skew braces and set-theoretic non-degenerate solutions of the Yang–Baxter equation has been developed in various papers in the last few years; see, for example, [Rum07] and [GV17].

In the next statement, we summarise some of the results of [Rum07], [GV17], [Rum19], [KT20]; see [CS21] Section 5 for further details.

**Proposition 6.1.** Let $(G, \cdot, \circ)$ be a skew brace. Write $g^{-1}$ and $\overline{g}$ for the inverse of $g$ with respect to $\cdot$ and $\circ$, respectively. Then $(G, r)$ and $(G, r')$ are solutions, where

$$r(g, h) = (g^{-1} \cdot (g \circ h), \overline{g^{-1}} \cdot (g \circ h) \circ g \circ h),$$

$$r'(g, h) = ((g \circ h) \cdot g^{-1}, \overline{g \circ h} \cdot g^{-1} \circ g \circ h).$$
The solutions \((G, r_G)\) and \((G, r_{\varphi})\) are one the inverse of the other and coincide if and only if \((G, \cdot)\) is abelian.

Remark 6.2. In Proposition 6.1, if \((G, \cdot, \circ)\) is a bi-skew brace, then we can reverse the role of the two operations, obtaining other two solutions.

For the next result, assume the setting of Section 3.

**Theorem 6.3.** Let \((\psi, \alpha), (\varphi, \beta) \in A \times B\). Then \((G, r)\) and \((G, r')\) are solutions, where

\[
 r(g, h) = (\psi^\top - \varphi^\top) g \cdot h \cdot (\psi^\top - \varphi^\top) g^{-1} \cdot \beta(g^{-1}, h) \cdot \alpha(g, h),
\]
\[
 (\psi^\top h)^{-1} \cdot \psi^\top - \varphi^\top g \cdot h^{-1} \cdot (\psi^\top - \varphi^\top) g^{-1} \cdot g \cdot \psi^\top g \cdot h \cdot (\psi^\top g)^{-1} \cdot \psi^\top h \\
 \cdot \beta(g, h) \cdot \alpha(h^{-1}, g)),
\]
\[
 r'(g, h) = (g \cdot \psi^\top g \cdot h \cdot (\varphi^\top) g^{-1} \cdot \psi^\top h \cdot g^{-1} \cdot (\varphi^\top h)^{-1} \cdot \beta(h, g^{-1}) \cdot \alpha(g, h),
\]
\[
 \varphi^\top - \psi^\top h \cdot g \cdot (\varphi^\top - \psi^\top h)^{-1} \cdot \beta(h, g) \cdot \alpha(h^{-1}, g)).
\]

**Proof.** By Theorem 6.2, \((G, \circ_{\varphi, \beta}, \circ_{\psi, \alpha})\) is a bi-skew brace, so that we can apply Proposition 6.1. The (long but straightforward) computation to exhibit the solutions in this explicit formulation is similar to that of [CS21, Theorem 5.3]. \(\square\)

**Corollary 6.4.** Let \((\psi, \alpha) \in A \times B\). Then \((G, r), (G, r'), (G, \tilde{r}), (G, \tilde{r}')\) are solutions, where

\[
 r(g, h) = (\psi^\top g \cdot h \cdot (\psi^\top g)^{-1} \cdot \alpha(g, h),
\]
\[
 (\psi^\top h)^{-1} \cdot \psi^\top g \cdot h^{-1} \cdot (\psi^\top g)^{-1} \cdot g \cdot \psi^\top g \cdot h \cdot (\psi^\top g)^{-1} \cdot \psi^\top h \cdot \alpha(h^{-1}, g));
\]
\[
 r'(g, h) = (g \cdot \psi^\top g \cdot h \cdot (\varphi^\top) g^{-1} \cdot \psi^\top h \cdot g^{-1} \cdot (\varphi^\top h)^{-1} \cdot \alpha(g, h),
\]
\[
 (\psi^\top h)^{-1} \cdot g \cdot \psi^\top h \cdot \alpha(h^{-1}, g));
\]
\[
 \tilde{r}(g, h) = ((\psi^\top g)^{-1} \cdot h \cdot \psi^\top g \cdot \alpha(g^{-1}, h),
\]
\[
 (\psi^\top g)^{-1} \cdot h^{-1} \cdot \psi^\top g \cdot g \cdot h \cdot \alpha(g, h));
\]
\[
 \tilde{r}'(g, h) = (g \cdot h \cdot \psi^\top g \cdot h^{-1} \cdot (\varphi^\top h)^{-1} \cdot \alpha(h, g^{-1}),
\]
\[
 \psi^\top h \cdot g \cdot (\varphi^\top h)^{-1} \cdot \alpha(h, g)).
\]

**Proof.** It is enough to apply Theorem 6.3 to the bi-skew brace \((G, \cdot, \circ_{\psi, \alpha})\), as when both \(\varphi\) and \(\beta\) are trivial, the operation \(\circ_{\varphi, \beta}\) coincides with \(\cdot\). \(\square\)

Remark 6.5. Note that the solutions given in [KST20, Theorem 5.1], [Koc21, Corollary 5.4], and [Koc22, Theorem 4.15] are particular instances of the solutions exhibited in Theorem 6.3 and Corollary 6.4.
We briefly discuss here the concept of a brace block from the point of view of graphs.

Let \((G, 1)\) be a pointed set. Denote by \(\text{Perm}(G)\) the group of permutations on the set \(G\). For \(\eta \in \text{Perm}(G)\) and \(g \in G\), write \(\eta g\) for the image of \(g\) under \(\eta\).

**Definition 7.1.** A subgroup \(N\) of \(\text{Perm}(G)\) is **regular** if the map

\[
N \to G \\
\eta \mapsto \eta 1
\]

is bijective. We denote by \(\nu\) the inverse of this map, so that \(\nu(g)\) is the unique element of \(N\) such that \(\nu(g) 1 = g\), and \(N = \{\nu(g) : g \in G\}\).

**Example 7.2** (Cayley’s theorem). Let “\(\circ\)" be an operation on \(G\) such that \((G, \circ)\) is a group with identity \(1\). Take

\[
\lambda_\circ: (G, \cdot) \to \text{Perm}(G) \\
g \mapsto (h \mapsto g \circ h),
\]

the **left regular representation** with respect to “\(\circ\)”. Then \(\lambda_\circ(G)\) is a regular subgroup of \(\text{Perm}(G)\).

Conversely, every regular subgroup \(N\) of \(\text{Perm}(G)\) occurs as \(\lambda_\circ(G)\), for a suitable group operation “\(\circ\)" on \(G\). Indeed, if \(N = \{\nu(g) : g \in G\}\), we can use transport of structure via the bijection \(\nu\) to obtain a group operation “\(\circ\)" on \(G\) such that \(\nu(g \circ h) = \nu(g) \nu(h)\). It is now immediate to see that \(\nu = \lambda_\circ\), so that \(N\) is precisely the image of the left regular representation with respect to “\(\circ\)”. We have obtained the following result.

**Proposition 7.3.** Let \((G, 1)\) be a pointed set. The following data are equivalent:

1. an operation \(\circ\) on \(G\) such that \((G, \circ)\) is group with identity \(1\);
2. a regular subgroup \(N = \{\nu(g) : g \in G\} \leq \text{Perm}(G)\).

The correspondence is given by

\[
g \circ h = \nu(g) h.
\]

In particular, the map

\[
\nu: (G, \circ) \to N
\]

is an isomorphism, and \(N = \lambda_\circ(G)\).

The following result is basically [GV17, Theorem 4.2], with a slight change of point of view.

**Theorem 7.4.** Let \((G, \circ_1)\) and \((G, \circ_2)\) be groups with identity \(1\). Then \((G, \circ_1, \circ_2)\) is a skew brace if and only if \(\lambda_{\circ_2}(G)\) normalises \(\lambda_{\circ_1}(G)\).

Now we can consider graphs.
Definition 7.5. Let $(G, 1)$ be a pointed set. The normalising graph of $G$ is the undirected graph $\mathfrak{N}$ whose vertices are the regular subgroups of $\text{Perm}(G)$, and where two vertices are joined by an edge if and only if the corresponding subgroups normalise each other.

By Theorem 7.4, we derive that having an edge in the normalising graph is equivalent to having a bi-skew brace structure on $G$. In particular, cliques (complete subgraphs) in the normalising graph of $(G, 1)$ correspond to brace blocks.

Similar ideas and correspondences were studied in [Koh22]. A particular case of the normalising graph is tackled in [Spa22].

References

[BG22] Valeriy G. Bardakov and Vsevolod Gubarev, *Rota-Baxter groups, skew left braces, and the Yang-Baxter equation*, J. Algebra 596 (2022), 328–351, https://arxiv.org/abs/2105.00428 MR 4370524

[Byo96] Nigel P. Byott, *Uniqueness of Hopf Galois structure for separable field extensions*, Comm. Algebra 24 (1996), no. 10, 3217–3228. MR 1402555

[Car16] Andrea Caranti, *A simple construction for a class of $p$-groups with all of their automorphisms central*, Rend. Semin. Mat. Univ. Padova 135 (2016), 251–258. https://arxiv.org/abs/1406.7651 MR 3506071

[Car18] A. Caranti, *Multiple holomorphs of finite $p$-groups of class two*, J. Algebra 516 (2018), 352–372, https://arxiv.org/abs/1801.10410 MR 3863482

[Chi19] Lindsay N. Childs, *Bi-skew braces and Hopf Galois structures*, New York J. Math. 25 (2019), 574–588. MR 3982254

[CJO14] Ferran Cedó, Eric Jespers, and Jan Okniński, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. 327 (2014), no. 1, 101–116. MR 3177933

[Cla] Pete L. Clark, *Some cardinality questions*, http://alpha.math.uga.edu/~pete/settheorypart4.pdf

[CS21] A. Caranti and L. Stefanello, *From endomorphisms to bi-skew braces, regular subgroups, the Yang–Baxter equation, and Hopf–Galois structures*, J. Algebra 587 (2021), 462–487, https://arxiv.org/abs/2104.01582 MR 4304796

[CS22] A. Caranti and L. Stefanello, *Skew braces from Rota–Baxter operators: A cohomological characterisation, and some examples*, https://arxiv.org/abs/arXiv:2201.03936, to appear, Ann. Mat. Pura Appl. (4), https://doi.org/10.1007/s10231-022-01230-x

[Dri92] V. G. Drinfel’d, On some unsolved problems in quantum group theory, Quantum groups (Leningrad, 1990), Lecture Notes in Math., vol. 1510, Springer, Berlin, 1992, pp. 1–8. MR 1183474

[GAP19] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.10.2*, 2019, https://www.gap-system.org

[GP87] Cornelius Greither and Bodo Pareigis, *Hopf Galois theory for separable field extensions*, J. Algebra 106 (1987), no. 1, 239–258. MR 878476

[GV17] L. Guarnieri and L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comp. 86 (2017), no. 307, 2519–2534. MR 3647970

[Koc21] Alan Koch, *Abelian maps, bi-skew braces, and opposite pairs of Hopf-Galois structures*, Proc. Amer. Math. Soc. Ser. B 8 (2021), 189–203, https://arxiv.org/abs/2007.08967 MR 4273165
[Koc22] ______, Abelian maps, brace blocks, and solutions to the Yang-Baxter equation, J. Pure Appl. Algebra 226 (2022), no. 9, Paper No. 107047, https://arxiv.org/abs/2102.06104 MR 4381676

[Koh22] Timothy Kohl, Mutually normalizing regular permutation groups and Zappa-Szép extensions of the holomorph, Rocky Mountain J. Math. 52 (2022), no. 2, 567–598, https://arxiv.org/abs/arXiv:2005.10989 MR 4423796

[KST20] Alan Koch, Laura Stordy, and Paul J. Truman, Abelian fixed point free endomorphisms and the Yang-Baxter equation, New York J. Math. 26 (2020), 1473–1492. MR 4184834

[KT20] Alan Koch and Paul J. Truman, Opposite skew left braces and applications, J. Algebra 546 (2020), 218–235. MR 4033084

[Nic96] Werner Nickel, Computing nilpotent quotients of finitely presented groups, Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 25, Amer. Math. Soc., Providence, RI, 1996, pp. 175–191. MR 1364184

[Rum07] Wolfgang Rump, Braces, radical rings, and the quantum Yang-Baxter equation, J. Algebra 307 (2007), no. 1, 153–170. MR 2278047

[Rum19] ______, A covering theory for non-involutive set-theoretic solutions to the Yang-Baxter equation, J. Algebra 520 (2019), 136–170. MR 3881192

[Spa22] Filippo Spaggiari, The mutually normalizing regular subgroups of the holomorph of a cyclic group of prime power order, https://arxiv.org/abs/arXiv:2203.09316

[SV18] Agata Smoktunowicz and Leandro Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), J. Comb. Algebra 2 (2018), no. 1, 47–86. MR 3763907

(A. Caranti) Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38123 Trento, Italy
Email address: andrea.caranti@unitn.it
URL: https://caranti.maths.unitn.it/

(L. Stefanello) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo, 5, I-56127 Pisa, Italy
Email address: lorenzo.stefanello@phd.unipi.it
URL: https://people.dm.unipi.it/stefanello