NON-ARCHIMEDEAN PSEUDO-DIFFERENTIAL OPERATORS WITH BESSEL POTENTIALS

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ABSTRACT. In this article, we study a class of non-archimedean pseudo-differential operators associated via Fourier transform to the Bessel potentials. These operators (which we will denote as $J^\alpha$, $\alpha > n$) are of the form

$$(J^\alpha \varphi)(x) = \mathcal{F}_{\xi \to x}^{-1} \left( [\max\{1, ||\xi||_p\}]^{-\alpha} \hat{\varphi}(\xi) \right), \varphi \in \mathcal{D}'(\mathbb{Q}_p^n), \ x \in \mathbb{Q}_p^n.$$ 

We show that the fundamental solution $Z(x,t)$ of the $p$-adic heat equation naturally associated to these operators satisfies $Z(x,t) \leq 0$, $x \in \mathbb{Q}_p^n$, $t > 0$. So this equation describes the cooling (or loss of heat) in a given region over time.

Unlike the archimedean classical theory, although the operator symbol $-J^\alpha$ is not a function negative definite, we show that the operator $-J^\alpha$ satisfies the positive maximum principle on $C_0(\mathbb{Q}_p^n)$. Moreover, we will show that the closure $\overline{-J^\alpha}$ of the operator $-J^\alpha$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{T(t)\}$ on $C_0(\mathbb{Q}_p^n)$.

On the other hand, we will show that the operator $-J^\alpha$ is $m-$dissipative and is the infinitesimal generator of a $C_0-$semigroup of contractions $T(t)$, $t \geq 0$, on $L^2(\mathbb{Q}_p^n)$. The latter will allow us to show that for $f \in L^1((0,T) : L^2(\mathbb{Q}_p^n))$, the function

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds,$$ 

is the mild solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -J^\alpha u(x,t) + f(t) & t > 0, \ x \in \mathbb{Q}_p^n \\ u(x,0) = u_0 \in L^2(\mathbb{Q}_p^n). \end{cases}$$

1. Introduction

In this article, we study a class of non-archimedean pseudo-differential operators associated via Fourier transform to the Bessel potentials. If $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ (here $\mathbb{Q}_p^n$ denotes the $p$-adic numbers and $\mathcal{D}'(\mathbb{Q}_p^n)$ is called the space of distributions in $\mathbb{Q}_p^n$), $\alpha \in \mathbb{C}$ we define the $n-$dimensional $p-$adic Bessel potential of order $\alpha$ of $f$ by

$$(J^\alpha f)^\wedge = (\max\{1, ||x||_p\})^{-\alpha} \hat{f}.$$ 

Suppose $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. Defining on $\mathbb{Q}_p^n$ and with values in $\mathbb{R}_+$ := \{ $r \in \mathbb{R} : r \geq 0$ \} the function $K_\alpha$ as follows

$$K_\alpha(x) = \begin{cases} \frac{1-p^{-\alpha}}{1-p^{-\alpha}} \left( ||x||_p^{-n} - p^{\alpha-n} \right) \Omega(||x||_p) & \text{if } \alpha \neq n \\ (1-p^{-\alpha}) \log_p(\frac{1}{||x||_p}) \Omega(||x||_p) & \text{if } \alpha = n \end{cases}$$

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we have that $K_\alpha \in L^1(\mathbb{Q}_p^n)$ and $\tilde{K}_\alpha(\xi) = (\max\{1, ||\xi||_p\})^{-\alpha}$, see Remark 2.

For our purposes, in this article, we will consider $\alpha \in \mathbb{R}$ with $\alpha > n$. The condition $\alpha > n$ is completely necessary to obtain the inequality (3.3), which will be crucial in this article.

For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and taking inverse Fourier transform on both sides of (1.1), we have that

$$(J^\alpha \varphi)(x) = \mathcal{F}^{-1}_{\xi \to x} \left[ \tilde{K}_\alpha(\xi) \hat{\varphi}(\xi) \right] = \mathcal{F}^{-1}_{\xi \to x} \left[ (\max\{1, ||\xi||_p\})^{-\alpha} \hat{\varphi}(\xi) \right], \quad x \in \mathbb{Q}_p^n,$$

is a pseudo-differential operator with symbol $\tilde{K}_\alpha(\xi) = (\max\{1, ||\xi||_p\})^{-\alpha}$.

In this paper we study the fundamental solution (denoted by $Z(x, t) := Z_t(x)$, $x \in \mathbb{Q}_p^n$, $t > 0$) of the $p$-adic pseudodifferential equations of the form

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = J^\alpha u(x, t), & t \in [0, \infty), \quad x \in \mathbb{Q}_p^n \\
u(x, 0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n),
\end{cases}
$$

which is the $p$-adic counterparts of the archimedean heat equations.

Unlike the fundamental solution studied at [3], [5], [11], [12], [15], [16], [19], [23], [24], [25], [27], [28], et al., we obtain that $Z(x, t) \leq 0$, $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = e^{-t}$, $x \in \mathbb{Q}_p^n$, $t > 0$, among other properties, see Theorem 1. Since the heat kernel contains a large amount of redundant information, in our case, the $p$-adic heat equation describes the cooling (or loss of heat) in a given region over time.

The connections between pseudodifferential operators whose symbol is a negative definite function and that satisfies the positive maximum principle have been studied intensively in the archimedean setting, since, a sufficient and necessary condition for that a pseudodifferential operator satisfies the positive maximum principle is that its symbol be a negative definite function, see e.g. [6], [10], [13], [20], et al. In our case, the pseudodifferential operator $-J^\alpha$ satisfies the positive maximum principle on $C_0(\mathbb{Q}_p^n)$, however, its symbol $(\max\{1, ||\xi||_p\})^{-\alpha}$ is not a negative definite function, see Theorem 2 and Remark 1 respectively.

On the other hand, the study of the $m$-dissipative operators self-adjoint on the Hilbert spaces is of great importance, since these are exactly the generators of contraction semigroups and $C_0$-semigroups, see [4], [18]. Motivated by it, we are interested in knowing if the pseudo-differential operator $-J^\alpha$ is an $m$-dissipative operators and self-adjoint on $L^2(\mathbb{Q}_p^n)$.

The article is organized as follows: In Section 2 we will collect some basic results on the $p$-adic analysis and fix the notation that we will use through the article. In Section 3 we study a class of non-archimedean pseudo-differential operators associated via Fourier transform to the Bessel potentials, those operators we denote by $J^\alpha$, $\alpha \in \mathbb{C}$. For our purposes, we will consider the case when $\alpha > n$. In addition, we will study certain properties corresponding to the fundamental solution $Z(x, t)$, $x \in \mathbb{Q}_p^n$, $t > 0$ of the $p$-adic heat equation naturally associated to these operators.

In Section 4 we will show that the operator $-J^\alpha$ satisfies the positive maximum principle on $C_0(\mathbb{Q}_p^n)$. Moreover, as for all $\lambda > 0$ we have that $\text{Ran}(\lambda + J^\alpha)$ is dense in $C_0(\mathbb{Q}_p^n)$, we have that the closure $-\overline{J^\alpha}$ of the operator $-J^\alpha$ on $C_0(\mathbb{Q}_p^n)$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{T(t)\}$ on $C_0(\mathbb{Q}_p^n)$, see Theorem 3. In the section 5 we will show that the operator $-J^\alpha : L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n)$ is $m$-dissipative and self-adjoint, see Theorem 4 and Lemma 5 respectively. We can get that the linear operator $-J^\alpha$ is the infinitesimal
rational numbers $\mathbb{Q}$ is the mild solution of the initial value problem on $[0 \, \frac{p}{b}]$. The field of $\mathbb{F}$ with generator of a $C_0-$semigroup of contraction $T(t), t \geq 0$, on $L^2(\mathbb{Q}_p^n)$. Moreover, when considering the problem the inhomogeneous initial problem
\begin{equation*}
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = -J^a u(x, t) + f(t) & t > 0, \, x \in \mathbb{Q}_p^n \\
u(x, 0) = u_0 \in L^2(\mathbb{Q}_p^n),
\end{cases}
\end{equation*}
with $f : [0, T] \to L^2(\mathbb{Q}_p^n), \, T > 0$, we have that the function \( u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T, \) is the mild solution of the initial value problem on $[0, T]$, see Remark \( \Box \).

2. Fourier Analysis on $\mathbb{Q}_p^n$: Essential Ideas

2.1. The field of $p$-adic numbers. Along this article $p$ will denote a prime number. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm $| \cdot |_p$, which is defined as
\[ |x|_p = \begin{cases} 0, & \text{if } x = 0 \\ p^{-\gamma}, & \text{if } x = p^\gamma a_b \end{cases}, \]
where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$-adic order of $x$.

Any $p$-adic number $x \neq 0$ has a unique expansion of the form
\[ x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j, \]
where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the fractional part of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number
\[ \{x\}_p = \begin{cases} 0, & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=0}^{\text{ord}(x)-1} x_j p^j, & \text{if } \text{ord}(x) < 0. \end{cases} \]
We extend the $p$-adic norm to $\mathbb{Q}_p^n$ by taking
\[ ||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n. \]

For $r \in \mathbb{Z}$, denote by $B_r^a(a) = \{x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^r\}$ the ball of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_0^a(0) =: B_0^a$. Note that $B_r^a(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p^n : |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius $p^r$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^a$ equals the product of $n$ copies of $B_0 = \mathbb{Z}_p$, the ring of $p$-adic integers of $\mathbb{Q}_p$. We also denote by $S_r^a(a) = \{x \in \mathbb{Q}_p^n : ||x - a||_p = p^r\}$ the sphere of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_0^a(0) =: S_0^a$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$.

As a topological space $(\mathbb{Q}_p^n, || \cdot ||_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [25] Section 1.3], or [11] Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, || \cdot ||_p)$ is a locally compact topological space.
We will use $\Omega(p^{-r}|x-a|_p)$ to denote the characteristic function of the ball $B^n_p(a)$. We will use the notation $1_A$ for the characteristic function of a set $A$. Along the article $d^n x$ will denote a Haar measure on $Q^n_p$ normalized so that $\int_{Q^n_p} d^n x = 1$.

2.2. Some function spaces. A complex-valued function $\varphi$ defined on $Q^n_p$ is called \textit{locally constant} if for any $x \in Q^n_p$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B^n_{l(x)}.$$ 

A function $\varphi : Q^n_p \to \mathbb{C}$ is called a \textit{Bruhat-Schwartz function} (or a \textit{test function}) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $D(Q^n_p) =: D$. Let $D'(Q^n_p) =: D'$ denote the set of all continuous functional (distributions) on $D$. The natural pairing $D'(Q^n_p) \times D(Q^n_p) \to \mathbb{C}$ is denoted as $(T, \varphi)$ for $T \in D'(Q^n_p)$ and $\varphi \in D(Q^n_p)$, see e.g. \cite{1} Section 4.4.

Every $f \in L^1_{loc}$ defines a distribution $f \in D'(Q^n_p)$ by the formula

$$(f, \varphi) = \int_{Q^n_p} f(x) \varphi(x) d^n x.$$ 

Such distributions are called \textit{regular distributions}.

Given $\rho \in [0, \infty)$, we denote by $L^\rho(Q^n_p, d^n x) = L^\rho(Q^n_p) := L^\rho$, the $\mathbb{C}$-vector space of all the complex valued functions $g$ satisfying $\int_{Q^n_p} |g(x)|^\rho d^n x < \infty$. $L^\infty := L^\infty(Q^n_p) = L^\infty(Q^n_p, d^n x)$ denotes the $\mathbb{C}$-vector space of all the complex valued functions $g$ such that the essential supremum of $|g|$ is bounded.

Let denote by $C(Q^n_p, \mathbb{C}) =: C_C$ the $\mathbb{C}$-vector space of all the continuous functions which are continuous, by $C(Q^n_p, \mathbb{R}) =: C_R$ the $\mathbb{R}$-vector space of continuous functions. Set

$$C_0(Q^n_p, \mathbb{C}) := \left\{ f : Q^n_p \to \mathbb{C}; \ f \text{ is continuous and } \lim_{x \to \infty} f(x) = 0 \right\},$$

where $\lim_{x \to \infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in Q^n_p \setminus B(\epsilon)$. We recall that $(C_0(Q^n_p), || \cdot ||_{L^\infty})$ is a Banach space.

2.3. Fourier transform. Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in Q_p$. The map $\chi_p(\cdot)$ is an additive character on $Q_p$, i.e. a continuous map from $(Q_p, +)$ into $S$ (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0) \chi_p(x_1)$, $x_0, x_1 \in Q_p$. The additive characters of $Q_p$ form an Abelian group which is isomorphic to $(Q_p, +)$, the isomorphism is given by $\xi \mapsto \chi_p(\xi)$, see e.g. \cite{1} Section 2.3.

Given $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in Q^n_p$, we set $x \cdot \xi := \sum_{j=1}^n x_j \xi_j$. If $f \in L^1$ its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \int_{Q^n_p} \chi_p(\xi \cdot x) f(x) d^n x, \text{ for } \xi \in Q^n_p.$$ 

We will also use the notation $\mathcal{F}_x \rightarrow \xi f$ and $\hat{f}$ for the Fourier transform of $f$. The Fourier transform is a linear isomorphism from $D(Q^n_p)$ onto itself satisfying

$$(2.1) \quad (\mathcal{F}(\mathcal{F}f))(\xi) = f(-\xi),$$
for every $f \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [1]. Section 4.8. If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k \to \infty} \int_{||x|| \leq p^k} \chi_p(\xi \cdot x)f(x)d^n x, \quad \text{for} \ \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in $L^2$. We recall that the Fourier transform is unitary on $L^2$, i.e. $||f||_{L^2} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2$ and that (2.1) is also valid in $L^2$, see e.g. [22, Chapter III, Section 2].

3. Pseudodifferential Operators and Heat Kernel Associated with Bessel Potentials

In this section, we study a class of non-archimedean pseudo-differential operators associated via Fourier transform to the Bessel potentials. We will also study some aspects associated with the fundamental solution of the heat equation associated with these operators.

**Definition 1.** [22, Definition-p. 137] If $f \in \mathcal{D}'(\mathbb{Q}_p^n)$, $\alpha \in \mathbb{C}$ we define the $n-$dimensional $p-$adic Bessel potential of order $\alpha$ of $f$ by

$$(J^\alpha f)^\wedge = (\max\{1, ||x||_p\})^{-\alpha} \hat{f}. \quad (3.1)$$

We will define the distribution with compact support $G^\alpha$ as

$$\tilde{G}^\alpha(x) = (\max\{1, ||x||_p\})^{-\alpha}. \quad (3.2)$$

**Remark 1.** [22, Proposition 5.1-p. 137] For $\alpha, \beta \in \mathbb{C}$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$, we have that $J^\alpha f = G^\alpha * f \in \mathcal{D}'(\mathbb{Q}_p^n)$ and $J^\alpha(J^\beta f) = J^{\alpha+\beta} f$. The map $\varphi \to J^\alpha \varphi$ is a homeomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto $\mathcal{D}(\mathbb{Q}_p^n)$. Furthermore $J^\alpha$ is continuous in $\alpha$ in the sense that whenever $\{\alpha_k\} \to \alpha$ in $\mathbb{C}$ then $J^{\alpha_k} \varphi \to J^\alpha \varphi$, when $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

The $n$-dimensional $p-$adic gamma function $\Gamma^\alpha_p$ is defined as

$$\Gamma^\alpha_p(\alpha) = \frac{1 - p^{-n}}{1 - p^{-\alpha}} \quad \text{for} \ \alpha \in \mathbb{C} \setminus \{0\}. \quad (3.3)$$

Suppose $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. Define on $\mathbb{Q}_p^n$ and with values in $\mathbb{R}_+$ the function $K_\alpha$ as follows:

$$K_\alpha(x) = \begin{cases} 
\frac{1}{\Gamma^\alpha_p(\alpha)} (||x||_p^{\alpha-n} - p^{-\alpha} n) \Omega(||x||_p) & \text{if} \ \alpha \neq n \\
(1 - p^{-n}) \log_p(\frac{p}{||x||_p}) \Omega(||x||_p) & \text{if} \ \alpha = n 
\end{cases} \quad (3.3)$$

**Remark 2.** (i) Note that $K_\alpha(x)$, $x \in \mathbb{Q}_p^n$, is a non-negative radial function. We have that $G^\alpha = K_\alpha$, $K_\alpha \in L^1(\mathbb{Q}_p^n)$ and $K_\alpha(\xi) = (\max\{1, ||\xi||_p\})^{-\alpha}$, see [22] Lemma 5.2-p. 138. Moreover, can verify that $\int_{\mathbb{Q}_p^n} K_\alpha(x)d^n x = 1$, if $\text{Re}(\alpha) > 0$, see [22] Remarks-p. 138 and (5.5)-p. 139.

(ii) Since the function $f(x) = 1$, $x \in \mathbb{Q}_p^n$, is constant we have in particular that $f$ is locally constant. Moreover, the function $|| \cdot ||_p$ is also locally constant, see [23, Example 1-p. 79]. Therefore the function $(\max\{1, ||\xi||_p\})^{-\alpha}$ is locally constant.

For our purposes from now on we consider fix $\alpha > n$. Therefore,

$$\frac{1 - p^{-\alpha}}{1 - p^{-n}} < 0. \quad (3.4)$$
Following the notation given in [22] and taking into account Remark 1 and Remark 2 for \( \varphi \in D(Q^n_p) \) we define the pseudo-differential operator \( J^\alpha \) by

\[
(3.5) \quad (J^\alpha \varphi)(x) := \mathcal{F}_{\xi \rightarrow x}^{-1} \left[ \hat{K}_\alpha(\xi)\hat{\varphi}(\xi) \right] = \mathcal{F}_{\xi \rightarrow x}^{-1} \left[ (\max\{1, ||\xi||_p\})^{-\alpha}\hat{\varphi}(\xi) \right], \ x \in \mathbb{Q}_p^n,
\]

with symbol \( \hat{K}_\alpha(\xi) = (\max\{1, ||\xi||_p\})^{-\alpha} \).

Note that if \( \varphi \in D(Q^n_p) \) then \( \hat{\varphi} \in D(Q^n_p) \), see [25] Lemma 4.8.1. Moreover, \( supp(\hat{K}_\alpha \hat{\varphi}) = supp(\hat{\varphi}) \), so that by Remark 2(ii) we have that \( \hat{K}_\alpha \hat{\varphi} \in D(Q^n_p) \). Therefore the operator \( J^\alpha : D(Q^n_p) \rightarrow D(Q^n_p) \) is well defined, and by Remark 1 and Remark 2 we have that

\[
(J^\alpha \varphi)(x) = (K_\alpha * \varphi)(x), \ \text{for} \ \varphi \in D(Q^n_p).
\]

**Lemma 1.** For \( t > 0 \) we have that

\[
(3.6) \quad \int_{Q^n_p} e^{-t\hat{K}_\alpha(\xi)} \leq e^{-t} - 1,
\]

i.e. \( e^{-t\hat{K}_\alpha(\xi)} = e^{-t(\max\{1, ||\xi||_p\})^{-\alpha}} \in L^1(Q^n_p) \).

**Proof.** Since \( e^{-t p^{-j\alpha}} \leq 1 \) for \( j \geq 1 \), we have that

\[
\int_{Q^n_p} e^{-t\hat{K}_\alpha(\xi)} d^n \xi = \int_{Q^n_p} e^{-t(\max\{1, ||\xi||_p\})^{-\alpha}} d^n \xi
= e^{-t} + (1 - p^{-n}) \sum_{j=1}^{\infty} e^{-t p^{-j\alpha}} p^{nj}
\leq e^{-t} + \sum_{j=1}^{\infty} (p^{nj} - p^{n(j-1)}) = e^{-t} - 1.
\]

The proof of the following Lemma is similar to the one given in [23] Proposition 1.

**Lemma 2.** Consider the Cauchy problem

\[
(3.7) \quad \begin{cases}
\frac{\partial u}{\partial \tau}(x, t) = J^\alpha u(x, t), & t > 0, x \in \mathbb{Q}_p^n \\
u(x, 0) = u_0(x) \in D(Q^n_p).
\end{cases}
\]

Then

\[
u(x, t) = \int_{Q^n_p} \chi_p \left( -\xi \cdot x \right) e^{-t(\max\{1, ||\xi||_p\})^{-\alpha}} \hat{u}_0(\xi) d^n \xi
\]
is a classical solution of [3.7]. In addition, \( u(\cdot, t) \) is a continuous function for any \( t \geq 0 \).

We define the heat Kernel attached to operator \( J^\alpha \) as

\[
Z(x, t) := \mathcal{F}_{\xi \rightarrow x}^{-1} \left( e^{-t(\max\{1, ||\xi||_p\})^{-\alpha}} \right).
\]

When considering \( Z(x, t) \) as a function of \( x \) for \( t \) fixed, we will write \( Z_t(x) \). On the other hand, by Lemma 1 and [22] Chapter III-Theorem 1.1-(b) we have that \( Z_t(x), t > 0 \), is uniformly continuous.
Theorem 1. For any $t > 0$, the heat kernel has the following properties:

(i) $Z(x, t) = \left\{ \begin{array}{ll}
\sum_{i=0}^{\gamma} p^{i\alpha} (e^{-tp^{-\alpha}} - e^{-tp^{-(i+1)\alpha}}) < 0 & \text{if } ||x||_p = p^{-\gamma}, \, \gamma \geq 0, \\
0, & \text{if } ||x||_p = p^\gamma, \, \gamma > 1,
\end{array} \right.
\]

i.e., $Z(x, t) \leq 0$ and $\text{supp}(Z(x, t)) = \mathbb{Z}_p^n$, for any $t > 0$ and $x \in \mathbb{Q}_p^n$.

(ii) $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = e^{-t}$ with $x \in \mathbb{Q}_p^n$.

(iii) $Z_t(x) * Z_{t_0}(x) = Z_{t+t_0}(x)$, for any $t_0 > 0$.

(iv) $\lim_{t \to 0^+} Z(x, t) = \delta(x)$.

Proof. (i) For $x \in \mathbb{Q}_p^n$ and $t > 0$, we have by (3.5) that

$$Z(x, t) = e^{-t} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-x \cdot \xi) d^n \xi + \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-x \cdot \xi) e^{-t||\xi||_p^{-\alpha}} d^n \xi.$$  

Consider the following cases:

If $||x||_p = p^\gamma$, with $\gamma \geq 1$, then by (3.8) and the $n$-dimensional version of [24] Example 6-p. 42], we have that

$$Z(x, t) = e^{-t} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-x \cdot \xi) e^{-t||\xi||_p^{-\alpha}} d^n \xi.$$

By using the formula

$$\int_{||w||_p=1} \chi_p \left(-p^{-j} x \cdot w \right) d^n w = \left\{ \begin{array}{ll}
1 - p^{-n}, & \text{if } j \leq -\gamma, \\
-p^{-n}, & \text{if } j = -\gamma + 1, \\
0, & \text{if } j \geq -\gamma + 2,
\end{array} \right.$$

we get that $Z(x, t) = 0$.

On the other hand, if $||x||_p = p^{-\gamma}$, $\gamma \geq 0$, then by (3.8) and the $n$-dimensional version of [24] Example 6-p. 42] we have that

$$Z(x, t) = e^{-t} + \sum_{j=1}^{\infty} e^{-tp^{-j\alpha}} \int_{||w||_p=1} \chi_p(-x \cdot \xi) d^n \xi$$

(3.9)$$= e^{-t} + \sum_{j=1}^{\infty} e^{-tp^{-j\alpha}} p^{nj} \int_{||w||_p=1} \chi_p(-p^{-j} x \cdot w) d^n w \quad (\text{taking } w = p^j \xi).$$

Now, we have that

$$\int_{||w||_p=1} \chi_p \left(-p^{-j} x \cdot w \right) d^n w = \left\{ \begin{array}{ll}
1 - p^{-n}, & \text{if } j \leq \gamma, \\
-p^{-n}, & \text{if } j = \gamma + 1, \\
0, & \text{if } j \geq \gamma + 2.
\end{array} \right.$$

We proceed by induction on $\gamma$. Note that if $\gamma = 0$, then by (3.9) and (3.10) we have that

$$Z(x, t) = e^{-t} - e^{-tp^{-\alpha}}.$$
If $\gamma = 1$, then by (3.9) and (3.10) we have that
\[
Z(x, t) = e^{-t} + (p^n - 1)e^{-tp^{-\alpha}} - p^n e^{-tp^{-2\alpha}}
\]
\[
= (e^{-t} - e^{-tp^{-\alpha}}) + p^n (e^{-tp^{-\alpha}} - e^{-tp^{-2\alpha}}).
\]
Suppose that
\[
Z(x, t) = \sum_{i=0}^{n} p^n (e^{-tp^{-\alpha}} - e^{-tp^{-(i+1)\alpha}})
\]
is satisfied for $\gamma = n$.

Let’s see if the hypothesis is met for $\gamma = n + 1$. By (3.9) and (3.10) we have that
\[
Z(x, t) = e^{-t} + (1 - p^{-n})e^{-tp^{-\alpha}}p^n + (1 - p^{-n})e^{-tp^{-2\alpha}}p^{2n} + \ldots
\]
\[
+ (1 - p^{-n})e^{-tp^{-\gamma(n+1)\alpha}}p^{n\alpha} + (1 - p^{-n})e^{-tp^{-(\gamma+1)\alpha}}p^{(\gamma+1)n}
\]
\[
- p^{-n}e^{-tp^{-(\gamma+2)\alpha}}p^{(\gamma+2)n}
\]
\[
= e^{-t} + (p^n - 1)e^{-tp^{-\alpha}} + (p^{2n} - p^n)e^{-tp^{-2\alpha}} + \ldots + (p^{n\alpha} - p^{(n-1)\alpha})e^{-tp^{-\gamma}}
\]
\[
+ (p^{(\gamma+1)n} - p^{n\gamma})e^{-tp^{-(\gamma+1)\alpha}} - p^{(\gamma+1)n}e^{-tp^{-(\gamma+2)\alpha}}.
\]
So by the hypothesis of induction we have that
\[
Z(x, t) = (e^{-t} - e^{-tp^{-\alpha}}) + p^n (e^{-tp^{-\alpha}} - e^{-tp^{-2\alpha}}) + p^{2n} (e^{-tp^{-2\alpha}} - e^{-tp^{-3\alpha}}) + \ldots
\]
\[
+ p^{n\alpha} (e^{-tp^{-\gamma}} - e^{-tp^{-(\gamma+1)\alpha}}) + p^{(\gamma+1)n} (e^{-tp^{-(\gamma+1)\alpha}} - e^{-tp^{-(\gamma+2)\alpha}}).
\]
Therefore, taking into account that the function $f(x) = e^{-tp^{-\alpha}}$ is an increasing function in the real variable $x$, we have that
\[
Z(x, t) \leq 0, \text{ for any } t > 0 \text{ and } x \in Q_p^n,
\]
\[(ii)\] Let $t > 0$. By the definition of $Z(x, t)$, (i) and (3.6), we have that $|Z(x, t)| \leq e^{-t} - 1$. Therefore,
\[
(3.11) \quad Z(x, t) \in L^1(Q_p^n).
\]

Since $\hat{Z}(x, t) = e^{-t((\max(1, ||x||_p))^{-\alpha}}$ we have that $\hat{Z}(0, t) = e^{-t}$. On the other hand, $\hat{Z}(\xi, t) = \int_{Q_p^n} \chi_p(\xi \cdot x) Z(x, t) d^n x$ and $\hat{Z}(0, t) = \int_{Q_p^n} Z(x, t) d^n x$. Therefore, $\int_{Q_p^n} Z(x, t) d^n x = e^{-t}$.

\[(iii)\] It is an immediate consequence of the definition of $Z(x, t)$ and (3.11).

\[(iv)\] For $\varphi \in D(Q_p^n)$ we have that
\[
\lim_{t \to 0^+} \langle Z_t(x), \varphi \rangle = \lim_{t \to 0^+} \bigg\langle e^{-t((\max(1, ||x||_p))^{-\alpha}}, \mathcal{F}^{-1}_{\xi \to x}(\varphi) \bigg\rangle = \langle 1, \mathcal{F}^{-1}_{\xi \to x}(\varphi) \rangle = \langle \delta, \varphi \rangle.
\]

\[\square\]

\textbf{Remark 3.} (i) By the previous theorem, we have that the family $(Z_t)_{t>0}$ it’s not a convolution semigroup on $Q_p^n$, see e.g. [2].

(ii) By (ii) in the previous theorem we have that the family of operators
\[
(\Theta(t)f)(x) := \int_{Q_p^n} Z(x - y, t)f(y) d^n y
\]
not preserve the function $f(x) \equiv 1$. Thus $\Theta(t)$ it’s not a Markov semigroup and moreover the fundamental solution $Z(x,t)$ it’s not a transition density of a Markov process. For more details, the reader can consult the theory of Markov processes, see e.g. [7], [10].

(iii) By Theorem 2(ii) and Fubini’s theorem, we have that the classical solution of (3.7) can be written as
$$u(x,t) = Z_t(x) * u_0(x), \quad t \geq 0, \quad x \in \mathbb{Q}_p^n.$$

4. The Positive Maximum Principle and Strongly Continuous, Positive, Contraction Semigroup On $C_0(\mathbb{Q}_p^n)$

In this section, we will show that the operator $-J^\alpha$ satisfies the positive maximum principle on $C_0(\mathbb{Q}_p^n)$ and that also the closure $\overline{-J^\alpha}$ of the operator $-J^\alpha$ on $C_0(\mathbb{Q}_p^n)$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{T(t)\}$ on $C_0(\mathbb{Q}_p^n)$. For more details, the reader can consult [2], [21].

**Definition 2.** An operator $(A, Dom(A))$ on $C_0(\mathbb{Q}_p^n)$ is said to satisfy the positive maximum principle if whenever $f \in Dom(A) \subseteq C_0(\mathbb{Q}_p^n, \mathbb{R})$, $x_0 \in \mathbb{Q}_p^n$, and $\sup_{x \in \mathbb{Q}_p^n} f(x) = f(x_0) \geq 0$ we have $Af(x_0) \leq 0$.

Let $x_0 \in \mathbb{Q}_p^n$ such that $\sup_{x \in \mathbb{Q}_p^n} \varphi(x) = \varphi(x_0) \geq 0$ with $\varphi \in D(\mathbb{Q}_p^n)$. By Remark 1, Remark 2 and (3.3) we have that

$$\begin{align*}
(J^\alpha \varphi)(x_0) &= (K_\alpha \ast \varphi)(x_0) \\
 &= \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{\mathbb{Q}_p^n} (||x_0||_p^{\alpha-n} - p^{\alpha-n}) \Omega(||x_0||_p, \varphi(x_0) - y) d^n y \\
(4.1) &\quad = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{\mathbb{Q}_p^n} (||x_0 - y||_p^{\alpha-n} - p^{\alpha-n}) \Omega(||x_0 - y||_p, \varphi(y)) d^n y.
\end{align*}$$

Consider the following cases:

**Case 1.** $\varphi(x_0) > 0$. In this case $x_0 \in supp(\varphi)$ and for all $x \in supp(\varphi)$ we have that $\varphi(x) > 0$.

If $supp(\varphi) \cap \mathbb{Z}_p^n = \phi$, then $||x_0||_p > p$. So that by (4.1) we have that

$$\begin{align*}
(J^\alpha \varphi)(x_0) &= 0.
\end{align*}$$

If $supp(\varphi) \subseteq \mathbb{Z}_p^n$, then $||x_0||_p = p^{-\beta}$, with $\beta \geq 0$. For the case where $||y||_p > 1$ we have that $||x_0 - y||_p = ||y||_p$ and consequently $x_0 - y \notin \mathbb{Z}_p^n$ and $\varphi(x_0 - y) = 0$. On the other hand, if $||y||_p \leq 1$ then $||x_0 - y||_p \leq 1$, and in this case $\varphi(x_0 - y) > 0$. So that by (4.1) and (3.3) we have that

$$\begin{align*}
(J^\alpha \varphi)(x_0) &= 1 - p^{-\alpha} \int_{\mathbb{Z}_p^n} (p^{-\beta(n-\alpha)} - p^{\alpha-n}) \varphi(x_0 - y) d^n y \geq 0.
\end{align*}$$

If $\mathbb{Z}_p^n \subseteq supp(\varphi)$, then there are two possibilities for $x_0$. For the case when $||x_0||_p \leq 1$ we have that $||x_0||_p^{\alpha-n} - p^{\alpha-n} \leq 0$ and $\varphi(x_0 - y) > 0$ for all $y \in supp(\varphi)$. So that by (4.1) and (3.3), we have that $(J^\alpha \varphi)(x_0) \geq 0$. For the case when $x_0 \in supp(\varphi) \setminus \mathbb{Z}_p^n$, by (4.1) we have that $(J^\alpha \varphi)(x_0) = 0$. 

Case 2. \( \varphi(x_0) = 0 \). In this case \( x_0 \notin \text{supp}(\varphi) \) and for all \( x \in \text{supp}(\varphi) \) we have that \( \varphi(x) < 0 \).

If \( \text{supp}(\varphi) \subseteq Z^n_p \), then \( ||x_0||_p > 1 \), or if \( \text{supp}(\varphi) \subseteq Q^n_p \setminus Z^n_p \) and \( x_0 \notin Z^n_p \), then by (4.4) we have that \( (J^\alpha \varphi)(x_0) = 0 \).

If \( Z^n_p \subseteq \text{supp}(\varphi) \) and \( x_0 \notin Z^n_p \) then by (4.4) we have that \( (J^\alpha \varphi)(x_0) = 0 \), and if \( Z^n_p \subseteq \text{supp}(\varphi) \) and \( x_0 \in Z^n_p \) then by (4.2) we have that \( (J^\alpha \varphi)(x_0) = 0 \).

If \( \text{supp}(\varphi) \subseteq Q^n_p \setminus Z^n_p \) and \( x_0 \in Z^n_p \). Then, when \( y \in Q^n_p \setminus Z^n_p \), we have that \( ||x_0 - y||_p^{\alpha - n} = ||y||_p^{\alpha - n} \) and \( \Omega(||x_0 - y||_p) = 0 \). Moreover, if \( y \in Z^n_p \) then \( \varphi(y) = 0 \). Therefore, by (4.2) we have that \( (J^\alpha \varphi)(x_0) = 0 \).

From all the above we have shown the following theorem.

**Theorem 2.** The operator

\[
-J^\alpha \varphi(x) = -F_{\xi \to x}^{-1} \left( \left( \max \{1, ||\xi||_p \} \right)^{-\alpha} \varphi(\xi) \right), \quad x \in Q^n_p, \quad \varphi \in D(Q^n_p),
\]

satisfies the positive maximum principle on \( C_0(Q^n_p) \).

**Lemma 3.** For all \( \lambda > 0 \) we have that \( \text{Ran}(\lambda + J^\alpha) \) is dense in \( C_0(Q^n_p) \).

**Proof.** Let \( \lambda > 0 \) and \( \varphi \in D(Q^n_p) \). Considering the equation

\[
(\lambda + J^\alpha)u = \varphi,
\]

we have that \( u(\xi) = F_{\xi \to x}^{-1} \left( \frac{\varphi(\xi)}{\lambda + \varphi(\xi)} \right) \) is a solution of the equation \( 18 \). Since \( \varphi(\xi) \in D(Q^n_p) \), then by Remark 2 (ii) we have that \( \frac{\varphi(\xi)}{\lambda + \varphi(\xi)} \in D(Q^n_p) \). Therefore, \( u \in D(Q^n_p) \).

**Theorem 3.** The closure \( \overline{-J^\alpha} \) of the operator \(-J^\alpha \) on \( C_0(Q^n_p) \) is single-valued and generates a strongly continuous, positive, contraction semigroup \( \{T(t)\} \) on \( C_0(Q^n_p) \).

**Proof.** It follows from Theorem 2, Lemma 3 and 21, Chapter 4, Theorem 2.2], taking into account that \( D(Q^n_p) \) is dense in \( C_0(Q^n_p) \), see e.g. 22, Proposition 1.3).

**Remark 4.** A function \( f : Q^n_p \to \mathbb{C} \) is called negative definite, if

\[
\sum_{i,j=1}^m \left( f(x_i) + \overline{f(x_j)} - f(x_i - x_j) \right) \lambda_i \lambda_j \geq 0
\]

for all \( m \in \mathbb{N}, x_1, \ldots, x_m \in Q^n_p, \lambda_1, \ldots, \lambda_m \in \mathbb{C} \).

Note that for all \( \xi \in Q^n_p \) we have that \( \hat{K}_\alpha(\xi) = (\max \{1, ||\xi||_p \})^{-\alpha} \leq \hat{K}_\alpha(0) \), so that by 2 Chapter II we have that the function \( \hat{K}_\alpha(\xi) \) no is a function negative definite. Therefore, the operator \(-J^\alpha \) is a pseudo-differential operator that satisfies the positive maximum principle and whose symbol is not a negative definite function.

5. The Pseudo-differential Operator \(-J^\alpha \) on \( L^2(Q^n_p) \)

In this section, consider the operator \(-J^\alpha \) in \( L^2(Q^n_p) \). By 22, Remarks-p. 138, we have that if \( f \in L^2(Q^n_p) \) then \(-J^\alpha f \) is \( L^2(Q^n_p) \) and \( || -J^\alpha f ||_{L^2(Q^n_p)} \leq ||f||_{L^2(Q^n_p)} \). In this case \( D(-J^\alpha) = L^2(Q^n_p) \). The main objective of this section is to demonstrate that this operator is \( m \)-dissipative, which will be crucial to prove that \(-J^\alpha \) is the infinitesimal generator of a \( C_0 \)-semigroup of contraction \( T(t), t \geq 0 \), on \( L^2(Q^n_p) \). For more details, the reader can consult 4, 17, 18, 21.
Remark 5. The graph $G(-J^\alpha)$ of $-J^\alpha$ is defined by

$$G(-J^\alpha) = \{(u, f) \in L^2(Q^n_p) \times L^2(Q^n_p); \ u \in D(-J^\alpha) \and f = -J^\alpha u\}.$$ 

Therefore by [1] Remark 2.1.6 we have that $G(-J^\alpha)$ is closed in $L^2(Q^n_p)$.

Definition 3. [17] Definition 1.2] An operator $A$ with domain $D(A)$ in $L^2(Q^n_p)$ is called dissipative if

$$\text{re} \langle Af, f \rangle \leq 0, \ \text{for all} \ f \in D(A),$$

where $L^2(Q^n_p)$ is the Hilbert space with the scalar product

$$\langle f, g \rangle = \int_{Q^n_p} f(x)\overline{g}(x)d^n x, \ f, g \in L^2(Q^n_p).$$

Lemma 4. The operator $-J^\alpha$ is dissipative in $L^2(Q^n_p)$.

Proof. Let fixed $\varphi \in D(Q^n_p)$. Since $D(Q^n_p)$ is dense in $L^2(Q^n_p)$, see [22, Chapter III], and the Parseval-Steklov equality, see [1] Section 5.3, we have that

$$\langle -J^\alpha \varphi, \varphi \rangle = \langle -\mathcal{F}_{\xi \rightarrow x}^{-1} \left[ \max \{1, ||\xi||_p\} \right]^{-\alpha} \hat{\varphi}(\xi), \varphi \rangle = \langle -(\max \{1, ||\xi||_p\})^{-\alpha} \hat{\varphi}, \hat{\varphi} \rangle = -\int_{Q^n_p} (\max \{1, ||\xi||_p\})^{-\alpha} |\hat{\varphi}(\xi)|^2 d^n \xi \leq 0.$$

\qed

Remark 6. By [21] Chapter 4-Lemma 2.1, we have that the linear operator $-J^\alpha$ on $C_0(Q^n_p)$ is dissipative, in the sense that for all $f \in C_0(Q^n_p)$ and $\lambda > 0$ we have that $||\lambda f - Af||_{L^\infty(Q^n_p)} \geq \lambda ||f||_{L^\infty(Q^n_p)}$. This definition is valid for any Banach space with its corresponding norm, for more details see [1, 17].

Definition 4. [1] Definition 2.2.2] An operator $A$ in $L^2(Q^n_p)$ is $m-$dissipative if

(i) $A$ is dissipative;

(ii) for all $\lambda > 0$ and all $f \in L^2(Q^n_p)$, there exists $u \in D(A)$ such that $u - \lambda Au = f$.

Lemma 5. The operator $-J^\alpha$ is self-adjoint, i.e.

$$\langle -J^\alpha f, g \rangle = \langle f, -J^\alpha g \rangle, \ \text{for all} \ f, g \in L^2(Q^n_p).$$

Proof. For $f, g \in L^2(Q^n_p)$ and the Parseval-Steklov equality, we have that

$$\langle -J^\alpha f, g \rangle = \langle -\mathcal{F}_{\xi \rightarrow x}^{-1} \left[ \max \{1, ||\xi||_p\} \right]^{-\alpha} \hat{f}(\xi), g \rangle = -\int_{Q^n_p} (\max \{1, ||\xi||_p\})^{-\alpha} \hat{f}(\xi)\overline{g}(\xi)d^n \xi$$

$$= -\int_{Q^n_p} \left[ \max \{1, ||\xi||_p\} \right]^{-\alpha} \hat{f}(\xi) \left[ \int_{Q^n_p} \chi_p(x \cdot \xi)\overline{g}(x)d^n x \right] d^n \xi$$

$$= -\int_{Q^n_p} \hat{f}(\xi) \left[ \max \{1, ||\xi||_p\} \right]^{-\alpha} \overline{g}(\xi)d^n \xi$$

$$= \langle f, -\mathcal{F}_{\xi \rightarrow x}^{-1} \left[ \max \{1, ||\xi||_p\} \right]^{-\alpha} \hat{g}(\xi) \rangle = \langle f, -J^\alpha g \rangle.$$

\qed
**Theorem 4.** The operator \(-J^\alpha : L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n)\) is \(m\)-dissipative.

**Proof.** The result follow from Remark 5, Lemma 3, and Lemma 5 by well-known results in the theory of dissipative operators, see e.g. [1] Theorem 2.4.5. □

**Remark 7.** The (infinitesimal) generator of a semigroups \((T(t))_{t \geq 0}\) is the linear operator \(L\) defined by

\[
D(L) = \left\{ f \in L^2(\mathbb{Q}_p^n); \frac{T(t)x - x}{t} \text{ has a limit in } L^2(\mathbb{Q}_p^n) \text{ as } t \to 0^+ \right\} ,
\]

and

\[
Lf = \lim_{t \to 0^+} \frac{T(t)f - f}{t},
\]

for all \(f \in D(L)\).

The linear operator \(-J^\alpha\) is the generator of a contraction semigroups \((T(t))_{t \geq 0}\) in \(L^2(\mathbb{Q}_p^n)\), i.e. the family of semigroups \((T(t))_{t \geq 0}\) satisfies:

(i) \(|T(t)||_{L^2(\mathbb{Q}_p^n)} \leq 1\) for all \(t \geq 0\);
(ii) \(T(0) = I\);
(iii) \(T(t + s) = T(t)T(s)\) for all \(s, t \geq 0\),
(iv) for all \(f \in L^2(\mathbb{Q}_p^n)\), the function \(t \mapsto T(t)f\) belongs to \(C([0, \infty), L^2(\mathbb{Q}_p^n))\).

For more details, the reader can consult [3] Section 3.4.

**Remark 8.** By [1] Theorem 4.3, [1] Theorem 4.5-(a), [1] Definition 2.1 and Theorem 4, we have that the linear operator \(-J^\alpha\) is the infinitesimal generator of a \(C_0\)-semigroup of contractions \(T(t), t \geq 0\), on \(L^2(\mathbb{Q}_p^n)\).

Let's consider the problem the inhomogeneous initial value problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = -J^\alpha u(x, t) + f(t) & t > 0, x \in \mathbb{Q}_p^n \\
u(x, 0) = u_0 \in L^2(\mathbb{Q}_p^n),
\end{cases}
\]

(5.1)

where \(f : [0, T] \to L^2(\mathbb{Q}_p^n), T > 0\).

Then, for \(f \in L^1([0, T) : L^2(\mathbb{Q}_p^n))\) we have that the function

\[
u(t) = T(t)u_0 + \int_0^t T(t - s)f(s)ds, \quad 0 \leq t \leq T,
\]

is the mild solution of the initial value problem (5.1) on \([0, T]\).

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