The total irregularity strength of caterpillars with odd number of internal vertices of degree three

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Abstract. Given a graph $G$ consisting of vertex set $V$ and edget set $E$, respectively. Assume $G$ is simple, connected, and the edges do not have direction. A function $f$ that maps $V \cup E$ into a set of $k$-integers is named a totally irregular total $k$-labelling if no vertices have the same weight and also the edges of $G$ get distinct weights. We call the minimum number $k$ for which $G$ has totally irregular total $k$-labelling as total irregularity strength of $G$, $ts(G)$. In this article, we construct labels of vertices and edges of caterpillar graphs which have $q$ internal vertices of degree 3 where $q$ is 5, 7, and 9. We obtain the exact values of $ts$ in the following: $n + 2$ if the caterpillars have $q=5$ internal vertices, $n + 3$ for $q=7$, and $n + 4$ for $q=9$.

1. Introduction
The notion of graph labelling was introduced in Alexander Rosa’s paper in 1967. The definition is as follows: “graph labelling is a function that has elements of $G(V, E)$ as its domain and a set of positive integers as its co-domain. If the function assigns $V(G)$ to the co-domain, then it is called a vertex labelling. Meanwhile, if the function maps $E(G)$ to the set of positive integers, then it is named an edge labelling. Moreover, the function is mentioned as a total labelling when it has $V \cup E$ as its domain” [1].

Further, the definition of edge irregular total labelling was given in Baca et al. as follows: “a total $k$-labelling $f: V \cup E \rightarrow \{1, 2, \cdots, k\}$ is named as an edge irregular total $k$-labelling when $w(e_1) \neq w(e_2)$ for each different pair of edges $e_1, e_2$ in $E(G)$. If $e = xy$, then the weight $w(e) = f(x) + f(y) + f(xy)$. The minimum number $k$ in such a way $G$ has such labelling as a total edge irregularity strength of $G$, indicated by $tes(G)$” [2]. Whereas, the tes of a tree $T(V, E)$ with a maximum degree $\Delta(T)$ was proved in [3]:

$$tes(T) = \max \left\{ \left\lfloor \frac{\Delta(T) + 1}{2} \right\rfloor, \left\lfloor \frac{\Delta(T) + 2}{3} \right\rfloor \right\}$$

(1)

Note that the symbol $[x]$ means the least integer greater than or equal to $x$.

The definition of vertex irregular total $k$-labelling was also given in Baca et al. as follows: “a total $k$-labelling $f: V \cup E \rightarrow \{1, 2, \cdots, k\}$ is named as an vertex irregular total $k$-labelling if the weights $w(v_1) \neq w(v_2)$ whenever $v_1 \neq v_2$ in $V(G)$, where $w(v) = f(v) + \sum_{v \in E} f(vy)$. We call the minimum positive integer $k$ for which $G$ has such labelling as a total vertex irregularity strength of $G$, symbolized
by \(tv(G)\)” [1]. The \(tv(G)\) of a tree \(T(V,E)\) that consists of \(n\) pendant vertices and it does not have vertices of degree 2 was proved in [4]:

\[
\text{tv}(T) = \left\lfloor \frac{n+1}{2} \right\rfloor + (2^n - 1)
\]

Furthermore, the notion of totally irregular total labelling was initiated in [5]: “the function \(f\) is called out as a totally irregular total \(k\)-labelling if both of the vertex-weights and the edge-weights are distinct. The minimum integer \(k\) such that \(G\) has totally irregular total \(k\)-labelling is named as total irregularity strength of \(G\), denoted by \(ts(G)\)”. The lower bound for \(ts\) of any graph was also given in [5]:

\[
\text{ts}(G) \geq \max\{tes(G), tvs(G)\}
\]

Some researchers have found \(tes\) and \(tvs\) of any graph classes as in [6], [7], [8-13], and [14]. Whereas, several exact values of \(ts\) of any graph have also been found, such as in [14-18]. A caterpillar graph is a tree such that removing its pendant edges will form a path [19]. The exact values of \(ts\) of some classes of caterpillars are still unknown [15]. Therefore, we investigate \(ts\) of caterpillar graphs where the number of internal vertices of degree 3 is odd.

2. Methods
According to the method in [2],[15-18], we summarize the steps used in this paper as follows:

a. Defining caterpillars which have \(q\) internal vertices of degree 3 where \(q = 5,7,9\). The caterpillar is denoted by \(S_{n,3,\ldots,3,n-q}\).

b. Calculating \(tes\) of the caterpillars based on eq.(1).

c. Calculating \(tvs\) of the caterpillars based on eq.(2).

d. Determining lower bound of \(ts\) of \(G = S_{n,3,\ldots,3,n}\) according to inequality (3).

e. Setting the lower bound \(k = \max\{tes(G), tvs(G)\}\) where \(tes(G)\) is given in (1) and \(tvs(G)\) is provided in (2), respectively.

f. Proving that \(ts\) of the caterpillars is least than or equal to \(k\) by constructing a totally irregular total \(k\)-labelling (trial and error process) on the caterpillars and the process is continued until we obtain a fixed pattern of the labelling;

g. Formulating vertex and edge-labels and formulating the \(ts\) of the caterpillars.

h. Proving the formula of the weights and showing that the weights are different.

i. Obtaining the \(ts = k\).

3. Results and Discussion
We present the results of \(ts\) of caterpillar graphs having odd number of internal vertices of degree 3.

3.1. Caterpillar graphs with 5 internal vertices of degree 3
We present the concept of the caterpillars in Definition 3.1.1.

**Definition 3.1.1.** Caterpillar \(S_{n,3,\ldots,3,n}\) is tree which is formed from double star \(S_{n,n}\) by inserting five vertices \((v^2, v^3, v^4, v^5, v^6)\) on the bridge that connects the centrals of the stars \((v^1\) and \(v^7)\) and the five vertices are incident to pendant edges \((v^1_j\) \(2 \leq j \leq 6)\). Whereas, the vertices of the double stars are \(v^1_i: 1 \leq i \leq n - 1\) and \(v^7_i: 1 \leq i \leq n - 1\). The caterpillar has \(2n+10\) vertices, \(2n+9\) edges, and \(2n+3\) pendant vertices. It has maximum degree is \(\Delta = n\).

An illustration of caterpillar caterpillar with 5 internal vertices of degree 3 is presented in Figure 1.
The $ts$ of the caterpillars as in Definition 3.1.1 is presented in Theorem 3.1.1.

**Theorem 3.1.1.** If $S_{n,3,3,...,3}^5, n \geq 5$ is the caterpillar as in Definition 3.1.1, then

$$ts(S_{n,3,3,...,3}^5) = \left\lfloor \frac{2n + 4}{2} \right\rfloor = n + 2.$$  

**Proof.** Based on eq. (1):

$$tes(S_{n,3,3,...,3}^5) = \max \left\{ \left\lfloor \frac{|E| + 2}{3} \right\rfloor, \left\lfloor \frac{|V| + 1}{2} \right\rfloor \right\} = \max \left\{ \left\lfloor \frac{2n + 9 + 2}{3} \right\rfloor, \left\lfloor \frac{n + 11}{2} \right\rfloor \right\} = \left\lfloor \frac{2n + 11}{3} \right\rfloor$$

Further, based on eq. (2): $tvs(S_{n,3,3,...,3}^5) = \left\lfloor \frac{n + 1}{2} \right\rfloor = \left\lfloor \frac{2n + 3 + 1}{2} \right\rfloor = \left\lfloor \frac{2n + 4}{2} \right\rfloor$.

The lower bound of $ts$ is obtained from eq. (3):

$$ts \left( S_{n,3,3,...,3}^5 \right) \geq \max \left\{ tes \left( S_{n,3,3,...,3}^5 \right), tvs \left( S_{n,3,3,...,3}^5 \right) \right\} = \max \left\{ \left\lfloor \frac{2n + 11}{3} \right\rfloor, \left\lfloor \frac{2n + 4}{2} \right\rfloor \right\} = \left\lfloor \frac{2n + 4}{2} \right\rfloor = n + 2, n \geq 5.$$  

Let $k = \left\lfloor \frac{2n + 4}{2} \right\rfloor$. We will prove that $ts(S_{n,3,3,...,3}^5) \leq k$ by constructing a totally irregular total $k$-labelling $h : V \cup E \rightarrow \{ 1, 2, \ldots, k \}$ with $k = \left\lfloor \frac{2n + 4}{2} \right\rfloor = n + 2$. Meanwhile, vertex and edges-labels are given in Table 1.

| Table 1. Labels of elements of the caterpillars with $5$ internal vertices |
|-----------------------------|-----------------------------|
| **Vertex-labels**           | **Edge-labels**             |
| $h(v_1^i)$, $i = 1$         | $h(v_1^i v_1^j)$, $i = 1$  |
| $i - 1, \forall 2 \leq i \leq n - 1$ | $2, \forall 2 \leq i \leq n - 1$ |
| $h(v_1^i)$, $i + 2, \forall n - 3 \leq i \leq n - 1$ | $i + 1, \forall 1 \leq i \leq n - 4$ |
| $h(v_1^j)$, $j = 1, 7$     | $h(v_1^j v_1^j + 1)$, $\forall 1 \leq j \leq 5$ |
| $\left[ \frac{2n + 4}{2} \right] + (j - 6), \forall 2 \leq j \leq 6$ | $\left[ \frac{2n + 4}{2} \right] - 3, j = 6$ |
| $h(v_1^j)$, $\forall 2 \leq j \leq 6$ | $h(v_1^j v_1^j)$, $\forall 2 \leq j \leq 6$ |

We calculate the weights of elements of the caterpillars in Table 2.
Table 2. Weights of elements of the caterpillars with 5 internal vertices

| Weights of vertices | Weights of edges |
|---------------------|------------------|
| $\forall 1 \leq i \leq n - 1$ | $\forall 1 \leq i \leq n - 1$ |
| $w(v^j)$ | $w(v^j v^{j+1})$ |
| $3n$, for $j = 1$ | $2n + 1$, for $j = 1$ |
| $4n + (2j - 4), \forall 2 \leq j \leq 5$ | $3n + (2j - 5), \forall 2 \leq j \leq 5$ |
| $4n + (j - 1), \text{ for } j = 6$ | $2n + 2$, for $j = 6$ |
| $\frac{n(n+1)}{2} - 4$, for $j = 7$ | $3n + (2j - 6), \forall 2 \leq j \leq 6$ |
| $w(v^j v^i)$ | $2 + i$, for $j = 1$ |
| $1 + i$, $j = 1$ | $w(v^j v^i)$ |
| $n + i$, $j = 7$ | $1 + n + i$, for $j = 7$ |
| $w(v^i)$ | $2n + (j - 2), \forall 2 \leq j \leq 6$ |

It is shown that the labels of elements of the caterpillars are nor more than $k = \left\lfloor \frac{2n+4}{2} \right\rfloor$. Further, no vertices have a same weight and also all edges have distinct weights. It proves $ts(S_{n,3,\ldots,3,n}) = n + 2$ for $n \geq 5$.

3.2. Caterpillar graphs with 7 internal vertices of degree 3

The notion of a caterpillar with 7 internal vertices of degree 3 is presented in Definition 3.2.1.

Definition 3.2.1. Caterpillar graphs $(S_{n,3,\ldots,3,n})$ are obtained from double star $S_{n,n}$ by inserting seven vertices $(v^2, v^3, v^4, v^5, v^6, v^7, v^8)$ on the bridge between the two centrals $(v^1$ and $v^9)$ and the seven vertices are incident to pendant edges $(v^j_1 | 2 \leq j \leq 8)$. Whereas, the vertices of the double stars are $\{v^j_1: 1 \leq i \leq n - 1\}$ and $\{v^i_1: 1 \leq i \leq n - 1\}$. The caterpillar has $2n + 14$ vertices, $2n + 13$ edges, and $2n + 5$ pendant vertices. It has maximum degree is $\Delta = n$.

The $ts$ of the caterpillars as in Definition 3.2.1 is provided in Theorem 3.2.1.

Theorem 3.2.1. If $S_{n,3,\ldots,3,n}$, $n \geq 6$ is the caterpillar as in Definition 3.2.1, then

$$ts(S_{n,3,\ldots,3,n}) = \left\lfloor \frac{2n + 6}{2} \right\rfloor = n + 3.$$ 

Proof. The lower bound of $ts$ is as follows:

$$ts(S_{n,3,\ldots,3,n}) \geq \max\left\{ tes\left(S_{n,3,\ldots,3,n}\right), tvs\left(S_{n,3,\ldots,3,n}\right)\right\} = \max\left\{ \left\lfloor \frac{2n + 15}{3} \right\rfloor, \left\lfloor \frac{2n + 6}{2} \right\rfloor \right\} \geq \left\lfloor \frac{2n + 6}{2} \right\rfloor = n + 3, n \geq 6.$$ 

Let $k = \left\lfloor \frac{2n+6}{2} \right\rfloor$. We will prove $ts(S_{n,3,\ldots,3,n}) \leq k$ by constructing a totally irregular total $k$-labelling $g : V \cup E \to \{1, 2, \ldots, k\}$ with $k = \left\lfloor \frac{2n+6}{2} \right\rfloor = n + 3$. Labels of elements of the caterpillar are given in Table 3.

Table 3. Labels of elements of the caterpillars with 7 internal vertices

| Vertex-labels | Edge-labels |
|---------------|-------------|
| $g(v^i_1)$   | $g(v^j v^i)$ |
| $1$, $i = 1$ | $1$, $i = 1$ |
| $i - 1, \forall 2 \leq i \leq n - 1$ | $2$, $\forall 2 \leq i \leq n - 1$ |
It is obvious that the labels of elements of the caterpillar is not more than $k = \left\lfloor \frac{2n+6}{2} \right\rfloor$. Moreover, we evaluate the weights in Table 4.

| Weights of vertices | Weights of edges |
|---------------------|------------------|
| $\forall 1 \leq i \leq n - 1$ | $\forall 1 \leq i \leq n - 1$ |
| $3n + 1$, for $j = 1$ | $2n + 1$, for $j = 1$ |
| $4n + (2j - 4)$, $\forall 2 \leq j \leq 7$ | $w(v^j v^{j+1})$ |
| $4n + (j - 1)$, for $j = 8$ | $3n + (2j - 6)$, $\forall 2 \leq j \leq 7$ |
| $n(n+5) - 13$, $j = 9$ | $w(v^i v'_i)$ |
| $w(v^i v'_i)$, $\forall 2 \leq j \leq 8$ | $2 + i$, $j = 1$ |
| $2n + (j - 2)$, $\forall 2 \leq j \leq 8$ | $1 + n + i$, $j = 9$ |

Table 4. Weights of elements of the caterpillars with 7 internal vertices

Based on the above calculation, we can see that the vertices have different weights and no edges have a same weight. Therefore, the upper bound is obtained and $ts(S_{n,3,3,\ldots,3,n}) = \left\lfloor \frac{2n+6}{2} \right\rfloor = n + 3$ for $n \geq 6$.

An illustration of labelling on the caterpillar $S_{n,3,3,\ldots,3,n}$ is shown in Figure 2. The green colors show vertex-labels and the blue colors denote labels of edges.

**Figure 2.** The totally irregular total 9-labelling on the Caterpillar $S_{n,3,3,\ldots,3,n}$

### 3.3. Caterpillar graphs with 9 internal vertices of degree 3

**Definition 3.3.1.** Caterpillar $S_{n,3,3,\ldots,3,n}$ is a graph which is obtained from double star $S_{n,n}$ by inserting nine vertices $(v^2, v^3, v^4, v^5, v^6, v^7, v^8, v^9, v^{10})$ on the bridge connecting the two centres ($v^1$ and $v^{11}$) and the nine vertices are incident to pendant edges ($v^i | 2 \leq j \leq 10$). Meanwhile, the vertices of the double stars are $\{v^i | 1 \leq i \leq n - 1\}$ and $\{v^{11} | 1 \leq i \leq n - 1\}$. The caterpillar has $2n + 18$ vertices, $2n + 17$ edges, and $2n + 7$ pendant vertices. Its maximum degree is $\Delta = n$. 
The $ts$ of the caterpillars that contain nine internal vertices of degree 3 is proved in Theorem 3.3.1.

**Theorem 3.3.1.** If $S_{n,3,\ldots,3,n}$, $n \geq 7$ is the caterpillar as in Definition 3.3.1, then

$$ts(S_{n,3,\ldots,3,n}) = \left\lfloor \frac{2n+8}{2} \right\rfloor = n + 4.$$ 

**Proof.** It is similar to the proof of Theorem 3.1.1 and Theorem 3.2.1, we get a lower bound as follows:

$$ts(S_{n,3,\ldots,3,n}) \geq \max\left\{ tes(S_{n,3,\ldots,3,n}), tve(S_{n,3,\ldots,3,n}) \right\} = \max\left\{ \left\lfloor \frac{2n+19}{3} \right\rfloor, \left\lfloor \frac{2n+8}{2} \right\rfloor \right\} = \left\lfloor \frac{2n+8}{2} \right\rfloor = n + 4, n \geq 7.$$

Let $k = \left\lfloor \frac{2n+8}{2} \right\rfloor$. We should show $ts(S_{n,3,\ldots,3,n}) \leq k$ by constructing a totally irregular total $k$-labelling $p : V \cup E \rightarrow \{1, 2, \ldots, k\}$ with $k = \left\lfloor \frac{2n+8}{2} \right\rfloor = n + 4$. We define labels for elements of the caterpillar to Table 5.

| $p(v)$ For all $v \in V(G)$ | $p(e)$ For all $e \in E(G)$ |
|-------------------------------|-------------------------------|
| $p(v_i)$ for $i = 1$ | $p(v_i) 1, i = 1$ |
| $p(v_i)$ for $i = 1$ | $p(v_{i+1}) 2, \forall 2 \leq i \leq n - 1$ |
| $p(v_{i+1})$ for $i = 1$ | $p(v_{i+1}) 1 + (j - 10), \forall 2 \leq j \leq 10$ |
| $p(v_{i+1})$ for $i = 1$ | $p(v_{i+1}) \left\lfloor \frac{2n+8}{2} \right\rfloor + (j - 10), \forall 2 \leq j \leq 10$ |
| $p(v_{i+1})$ for $i = 1$ | $p(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |
| $p(v_{i+1})$ for $i = 1$ | $p(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |

It is shown above that the labels of elements of the caterpillar is less than or equal to $k = \left\lfloor \frac{2n+8}{2} \right\rfloor$.

Furthermore, we calculate the weights to Table 6.

| $w(v)$ For all $v \in V(G)$ | $w(e)$ For all $e \in E(G)$ |
|-------------------------------|-------------------------------|
| $w(v_i)$ for $i = 1$ | $w(v_i) 1 + i, j = 1$ |
| $w(v_i)$ for $i = 1$ | $w(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |
| $w(v_{i+1})$ for $i = 1$ | $w(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |
| $w(v_{i+1})$ for $i = 1$ | $w(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |
| $w(v_{i+1})$ for $i = 1$ | $w(v_{i+1}) 2, \forall 2 \leq j \leq 10$ |

We observe that all elements of the caterpillar do not have a same weight. Therefore, we get the upper bound and $ts(S_{n,3,\ldots,3,n}) = \left\lfloor \frac{2n+8}{2} \right\rfloor = n + 4$ for $n \geq 7$.

Figure 3 describes labelling on the caterpillar $S_{n,3,\ldots,3,n}$. The green colors indicate labels of vertices and the blue colors present edge-labels.
Figure 3. The totally irregular total 11-labelling on the Caterpillar $S_{n,3,3,...,3,n}$

4. Conclusion
In this research, we proved that $ts$ of $(S_{n,3,3,...,3,n})$ is equal to: $\left\lceil \frac{2n+4}{2} \right\rceil = n + 2$ for $q=5$, it is equal to $\left\lceil \frac{2n+6}{2} \right\rceil = n + 3$ for $q=6$, and it is equal to $\left\lceil \frac{2n+8}{2} \right\rceil = n + 4$ for $q=9$. In upcoming research, we will investigate $ts$ of the caterpillars which have $q$ internal vertices of degree 3 for odd $q > 9$.

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