Sliding Surface Charges on AdS$_3$

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ABSTRACT: We consider Einstein gravity on a patch of AdS$_3$ spacetime between two radii $r_1, r_2$. We compute surface charges and their algebra at an arbitrary radius $r$ such that it reduces to a given set of surface charges at $r_1, r_2$. The $r$-dependent charges become integrable upon addition of an appropriate boundary $Y$-term. We observe that soft excitations at each boundary are independent of those at the other boundary. We explicitly construct solution geometries which interpolate between these radii with specified surface charges. The interpolation is smooth provided that the mass and angular momentum measured at the two boundaries are equal.
1 Introduction

As we learn in the undergraduate electromagnetism course, formulating Maxwell’s theory in a spacetime with boundaries requires imposing appropriate boundary conditions on the bulk fields and that these boundary conditions are outcome of degrees of freedom which reside at the boundary. The idea of “asymptotic symmetries” or “soft charges” provides us with a way to enumerate/label the boundary degrees of freedom through the boundary conditions we impose, without knowing about the details of the boundary theory which has effectively given rise to the boundary conditions, see e.g. [1–3] for further discussions.

In theories of gravity, motivated by quantifying the information carried to asymptotic null infinity through gravity waves, Bondi, van der Burg, Metzner and Sachs (BMS) [4, 5] have studied asymptotic symmetries and algebras for asymptotically flat metrics with certain falloff behavior. Similar analysis of asymptotic symmetries on asymptotically AdS3 spacetimes leads to two Virasoro algebras at the Brown-Henneaux central charge [6]. As the electromagnetic examples alluded above indicate, in a gravitational theory one may also consider imposing appropriate boundary conditions elsewhere and not just at the asymptotic region of spacetime. In this case, putting a material boundary (which has energy momentum tensor) would backreact on the bulk geometry as well. However, structure of the metric in special loci in spacetime may naturally lead to specific set of boundary conditions. Horizons are examples of such special loci. Presence of new “near horizon degrees of freedom” which are labeled by the near horizon symmetries, has been the main idea behind the soft hair proposal [7] to describe black hole
microstates and to address the information paradox. In stationary black holes we typically deal with a Killing horizon, a null surface of constant surface gravity generated by a Killing vector field. It is then natural to require that boundary conditions on metric fluctuations near a (Killing) horizon should respect the properties mentioned. This has been carried out in [8–10] yielding a specific near horizon symmetry algebra. One may then try to analyze near horizon symmetries of more general class of horizons with less restrictive near horizon boundary conditions, as e.g. done in [11–17].

In addressing questions regarding black holes, we hence typically deal with two boundaries, one at the horizon and the other in the asymptotic region. Each of these boundary conditions keep different physical properties fixed at each boundary: Asymptotic boundary conditions like those of Brown-Henneaux [6] or BMS [4, 5, 18] generically keep the ADM charges (mass and angular momentum) fixed while the near horizon ones keep the temperature (and horizon angular velocity) fixed [9, 10, 19–21]. Each of these yield different set of charges and algebras at each boundary. The question of imposing two different boundary conditions at two different loci was recently addressed in [22, 23], both of which are in the context of AdS3 gravity and employ Chern-Simons description of the theory.

In this paper we address a similar problem of gravity on a patch of AdS3 spacetime restricted between two arbitrary radii, as depicted in Fig. 2. In this sense our work is a continuation and extension of [22, 23], but we put more emphasis on the metric formulation and the geometry. We impose boundary conditions which yield two chiral U(1) current algebras at each boundary, much like the near horizon case analyzed in [8, 9]. We analyze surface charges at an arbitrary radius between the two boundaries. We show smoothness of the geometry requires that the zero mode charges of these U(1) current algebras associated with each boundary should be equal two each other. Moreover, while the charge algebra at the two boundaries are the same, the set of charges associated with one boundary, except for the zero modes, is not at all correlated with the charges at the other boundary. We demonstrate this by explicitly constructing the family of solutions to AdS3 gravity which smoothly interpolate between the two boundaries of arbitrary charges. In the following sections we establish these results and in the last section discuss the physical meaning and implications of these for the soft hair proposal.

2 Review of AdS3 gravity and associated surface charges

The AdS3 gravity is described by the action

\[ S = \int_{\mathcal{M}} L[g] d^3x + S_{\text{bdy}}, \]  

(2.1)

where

\[ L[g] = \frac{\sqrt{-g}}{16\pi G} (R + 2\Lambda), \]  

(2.2)

is the Lagrangian in which \( R \) is Ricci scalar and \( \Lambda = -1/l^2 \) is cosmological constant. \( S_{\text{bdy}} \) is boundary term which ensures a well-posed action principle, i.e.

\[ S_{\text{bdy}} = \int_{\mathcal{B}} L^\mu_{\text{bdy}}[g] d^2x_\mu, \]  

(2.3)

with

\[ L^\mu_{\text{bdy}}[g] = \frac{\sqrt{-g}}{16\pi G} K n^\mu \]

where \( \mathcal{B} \) is the boundary of \( \mathcal{M} \). Here we are assuming that topologically \( \mathcal{M} = \mathbb{R} \times \Sigma \) where \( \Sigma \) is a two-dimensional spacelike manifold. The boundary of this manifold \( \partial \Sigma \) may consist of
an $S^1$ or more circles. In this work we focus on formulating the boundary value problem on a patch of AdS$_3$ where the boundary consists of two coaxial cylinders of radii $r_1, r_2$, as depicted in Fig. 2. However, we develop the formulation of computing surface charges at a surface of arbitrary radius $r$, depicted in Fig. 1. $n^\mu$ is normal vector to the timelike boundary $B = \mathbb{R} \times \partial \Sigma$ and $K = \nabla_\mu n^\mu$ is trace of extrinsic curvature of $B$. We note that the boundary term (2.3) is one-half the Gibbons–Hawking–York boundary term [24]. The factor of one-half is needed to make the AdS$_3$ gravity action (2.1) regular [25].

2.1 Surface charge analysis, Lee-Wald symplectic form

Variation of the total Lagrangian $L_T[g] = L[g] + \partial_\mu L^\mu_{\text{bdy}}[g]$ is

$$\delta L[g] = -\frac{\sqrt{-g}}{16\pi G} \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \partial_\mu \Theta^\mu[g; \delta g],$$

(2.4)

where

$$\mathcal{E}^{\mu\nu} = G^{\mu\nu} - \frac{1}{l^2} g^{\mu\nu}. \tag{2.5}$$

Equations of motion are $\mathcal{E}^{\mu\nu} = 0$. Eq.(2.4) defines the ‘surface term’ $\Theta^\mu$ only up to a total divergence,

$$\Theta^\mu[g; \delta g] = \Theta^\mu_{\text{lw}}[g; \delta g] + \delta L^\mu_{\text{bdy}}[g] + \partial_\nu Y^{\mu\nu}[g; \delta g],$$

(2.6)

where

$$\Theta^\mu_{\text{lw}}[g; \delta g] = \frac{\sqrt{-g}}{8\pi G} \nabla^\nu (g^{\mu\nu} \delta g_{\alpha\beta}),$$

(2.7)

is the “Lee-Wald symplectic potential” [26]. As we see the above analysis does not specify $Y$-term (usually called boundary term). We will discuss in the next subsection that $Y$ may be fixed (restricted) through other physical requirements.

The symplectic current

$$\omega^\mu[g; \delta_1 g, \delta_2 g] = \delta_1 \Theta^\mu[g; \delta_2 g] - \delta_2 \Theta^\mu[g; \delta_1 g],$$

(2.8)

Figure 1: The horizontal shaded surfaces $\Sigma_{t_1}, \Sigma_{t_2}$ show two constant time slices with radii less than $r$. $B_{t_2}$ is the gray cylinder at radius $r$ between $\Sigma_{t_1}, \Sigma_{t_2}$.
is a skew-symmetric function of metric variations and is conserved on-shell, i.e.

$$
\partial_\mu \omega^{\mu}[g; \delta_1 g, \delta_2 g] \approx 0,
$$

where $$\approx$$ denotes that the equality holds on-shell. That is, when $$g_{\mu\nu}$$ satisfies equation of motion $$\mathcal{E}^{\mu\nu} = 0$$ and $$\delta g_{\mu\nu}$$'s satisfy linearized equations of motion $$\delta \mathcal{E}^{\mu\nu} \approx 0$$. Being a total variation, the $$\delta L_{\text{bry}}^{\mu}[g]$$ term in (2.6) does not contribute to $$\omega$$ (2.8).

Pre-symplectic form can be defined through integrating symplectic current over a spacelike co-dimension one surface $$\Sigma$$ with boundary $$\partial \Sigma$$, \(^1\)

$$
\Omega[g; \delta_1 g, \delta_2 g] = \int_{\Sigma} \omega^{\mu}[g; \delta_1 g, \delta_2 g] d^2 x_\mu.
$$

The surface charge variation associated with the symmetry generator (diffeomorphism) $$\xi$$ is defined through pre-symplectic form as

$$
\delta Q_\xi := \Omega[g; \delta g, \delta_1 g]
= \oint_{\partial \Sigma} \left( Q^{\mu\nu}_\xi[g; \delta g] + \mathcal{Y}^{\mu\nu}_\xi[g; \delta g] \right) dx_{\mu\nu}
$$

with $$\delta_1 g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$. The first term is the contribution of the Lee-Wald part,

$$
Q^{\mu\nu}_\xi = \frac{\sqrt{-g}}{8\pi G} \left( h^{\lambda[\mu} \nabla_{\lambda} \xi_{\nu]} - \xi^{\lambda} \nabla_{[\mu} h_{\nu]} - \frac{1}{2} h \nabla^{[\mu} \xi_{\nu]} + \xi^{[\mu} \nabla_{\lambda} h_{\nu]} \right),
$$

where $$h_{\mu\nu} = \delta g_{\mu\nu}$$ is a metric perturbation and $$h = g^{\mu\nu} h_{\mu\nu}$$, and the second term is

$$
\mathcal{Y}^{\mu\nu}_\xi \equiv \delta Y^{\mu\nu}[g; \delta_1 g] - \delta_1 \delta Y^{\mu\nu}[g; \delta g] - Y^{\mu\nu}[g; \delta_1 \delta g].
$$

Here we are assuming a general field dependent variation, that is when $$\xi = \xi[g]$$ for which $$\delta_1 \xi \neq 0, \delta_2 \xi \neq 0$$.

### 2.2 Surface charge analysis, fixing the boundary $$Y$$-terms

One may verify conservation of the variations $$\delta Q_\xi$$ and use this requirement to fix the $$Y$$-term. To this end, we compute charge variation at an arbitrary radius $$r$$ at two times, by integrating over two different constant time slices $$\Sigma_{t_1}, \Sigma_{t_2}$$, respectively $$\delta Q_\xi^{(1)}$$, $$\delta Q_\xi^{(2)}$$; see Fig. 1. Conservation means $$\delta Q_\xi^{(1)} = \delta Q_\xi^{(2)}$$. Starting from (2.11) and using (2.9), we find

$$
\delta Q_\xi^{(1)} - \delta Q_\xi^{(2)} \approx \int_{B_{12}} \omega^{\mu}[g; \delta g, \delta_2 g] d^2 x_\mu
$$

\begin{equation}
\approx \int_{B_{12}} \left( \delta \Theta^{\mu}[g; \delta_1 g] - \delta_1 \delta \Theta^{\mu}[g; \delta g] - \Theta^{\mu}[g; \delta_1 \delta g] \right) d^2 x_\mu.
\end{equation}

where $$\Sigma_{t_1} \cup \Sigma_{t_2} \cup B_{12}$$ are boundaries of a region of spacetime bounded between $$\Sigma_{t_1}, \Sigma_{t_2}$$ and $$B_{12}$$ is time-like (or null) boundary of that region, e.g. as depicted in Fig. 1. In the ‘asymptotic symmetries’ analysis as in the seminal Brown and Henneaux work [6], $$B_{12}$$ is part of boundary of AdS$_3$ between the two constant time slices. In our current analysis $$B_{12}$$ is at a generic $$r$$, \(^1\)The volume element is defined as $$d^{(D-p)} x_{\mu_1 \cdots \mu_p} = \frac{1}{p!(D-p)!} \varepsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D} d x^{\mu_{p+1}} \cdots d x^{\mu_D}$$ where $$D$$ and $$p$$ are dimensions of spacetime and integration surface respectively.
as shown in Fig. 1, or may have two disconnected regions, as depicted in Fig. 2. A sufficient condition for (2.14) to vanish, is

\[ \Theta \cdot n \big|_{B_{12}} \approx 0, \tag{2.15} \]

where \( n_\mu \) is the normal to \( B_{12} \). Since we know the expression for the Lee-Wald contribution, (2.15) may be used to specify the \( Y \)-term. We will use this in the next section in our charge analysis.

Now let us examine the integrability of charges. The necessary and sufficient condition for integrability of the charge variation (2.11) is that the exterior derivative of the charge variation in the phase space is zero. By taking second variation of (2.11), one finds [27]

\[ \delta \hat{\phi}_2 Q_\xi - \delta \hat{\phi}_1 Q_\xi \approx - \oint_{\partial \Sigma} 2\xi^\mu \omega^\nu [g; \delta_1 g, \delta_2 g] dx_{\mu \nu} + \delta_2 \phi_1 \delta_1 \zeta - \delta_1 \phi_2 \zeta. \tag{2.16} \]

Therefore the integrability condition implies that the right-hand side of the above equation should vanish. For the field independent cases the two last terms in (2.16) drop out, recovering the Lee-Wald or Iyer-Wald integrability condition [26, 28].

3 A general family of interpolating solutions

We adapt radial coordinate \( r \) so that \( n_\mu dx^\mu = \frac{r}{l} dr \). Let \( t \) and \( \phi \sim \phi + 2\pi \) denote time and angular coordinates respectively. All solutions to the field equations (2.5) are locally AdS\(_3\) and we consider those which can be described through the following ansatz [21],

\[ ds^2 = \frac{l^2}{r^2} dr^2 - \left( r A^+ - \frac{l^2 A^-}{r} \right) \left( r A^- - \frac{l^2 A^+}{r} \right), \tag{3.1} \]

where \( A^\pm = \lambda^\pm dr + \zeta^\pm dt + \eta^\pm d\phi \) are two one-forms. The Einstein field equations then yield,

\[ dA^\pm = 0, \tag{3.2} \]

A simple choice of the coordinate system is when \( \partial_r \) is hypersurface orthogonal, i.e. when \( g_{rt} = g_{r\phi} = 0 \). For this choice, (3.2) may be solved as, \( \eta^\pm = \eta^\pm (t, \phi), \, \zeta^\pm = \zeta^\pm (t, \phi) \) where,

\[ \partial_t \eta^\pm = \partial_\phi \zeta^\pm \quad \text{with} \quad \lambda^\pm = 0. \tag{3.3} \]

The above equations have infinitely many solutions, and the solution may be specified by other physical requirements. One such solution discussed in [8, 9] is \( \zeta^- \) constant and fixed, and \( \eta^\pm = J^\pm (\phi) \), with \( J(\phi + 2\pi) = J(\phi) \). This solution is appropriate for describing geometries with horizons at constant left and right temperatures \( \zeta^\pm \). Another solution is

\[ A^\pm = \eta^\pm (x^\pm) dx^\pm, \quad \eta^\pm (x^\pm + 2\pi) = \eta^\pm (x^\pm), \tag{3.4} \]

where \( x^\pm = t/l \pm \phi \). With this choice and for the special case of \( \eta^\pm = J_0^\pm \), (3.1) describes a Bañados, Teitelboim and Zanelli (BTZ) black hole [29] of outer horizon radii \( r_+ = l \).\(^2\)

\(^2\)To be more precise, vanishing of the right-hand side of (2.14) implies that projection of surface term on timelike boundary should be a total variation on the phase space, that is \( \Theta [g; \delta g] \cdot n = \delta (B[g] \cdot n) \). One can then replace the boundary term \( L^\mu_{\text{bdy}} [g] \) by \( L^\mu_{\text{bdy}} [g] - B^\mu [g] \) to have a well-defined variation principle. With this new boundary term, we again recover (2.15) as the conservation condition.

\(^3\)Note that in our coordinate system we only cover the outer horizon in \( r^2 > 0 \) region. The inner horizon is then located at \( r^2 = -l^2 \). Note also that the radial coordinate \( r \) in (3.1) and the usual BTZ radial coordinate are related as \( r^2_{\text{BTZ}} = \frac{r^2}{(l^2 + J_0^2)} = \frac{r^2 (J_0^2)}{J_0^2 + J_0^2} \), see [30, 31]. Therefore, this solution corresponds to a BTZ black hole with surface gravity and horizon angular velocity, \( \kappa = \frac{2J_0 J_\phi}{l (J_0^2 + J_\phi^2)} \), \( \Omega = \frac{J_\phi}{l (J_0^2 + J_\phi^2)} \).
In this work, we are interested in solutions for which \( A_\pm \) reduces to (3.4) at two specific radii, say \( r_1 \) and \( r_2 \). To this end, we introduce

\[
A_\pm = \mathcal{I}_\pm^{(f)} dx^\pm + A_\pm^r dr
\]

where

\[
\mathcal{I}_\pm^{(f)} := \mathcal{J}_\pm(x^\pm) f + \mathcal{K}_\pm(x^\pm) (1 - f)
\]

and \( f = f(r) \) is a smooth function of \( r \) such that,

\[
f|_{r=r_1} = 1, \quad f|_{r=r_2} = 0, \quad f'|_{r=r_1, r_2} = 0.
\]

Here \( \mathcal{J}_\pm \) and \( \mathcal{K}_\pm \) are four periodic functions,

\[
\mathcal{J}_\pm(x^\pm + 2\pi) = \mathcal{J}_\pm(x^\pm), \quad \mathcal{K}_\pm(x^\pm + 2\pi) = \mathcal{K}_\pm(x^\pm),
\]

which satisfy

\[
\frac{1}{2\pi} \int_0^{2\pi} \mathcal{J}_\pm dx^\pm = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{K}_\pm dx^\pm := J_0^\pm.
\]

The above and (3.6) yield,

\[
\frac{1}{2\pi} \int_0^{2\pi} \mathcal{I}_\pm(r) dx^\pm = J_0^\pm,
\]

which is independent of \( f(r) \) and the radial coordinate \( r \).

Equations of motion, or equivalently (3.2), imply that the \( r \)-component of \( A_\pm \) should be of the form

\[
A_\pm^r = Z_\pm(x^\pm) f' \quad \text{with} \quad \partial_\pm Z_\pm := \mathcal{J}_\pm - \mathcal{K}_\pm,
\]

where \( f' = df/dr \). Therefore, \( \mathcal{I}_\pm^{(f)} \) can be viewed as the \textit{interpolating} function. In the following, for the ease of notation we drop superscript \( (f) \) on interpolating function. As one can see, for the special case of \( f = \text{const.} \), (3.5) reduces to the solution (3.4). Therefore, the set of geometries described by (3.5) interpolates between two solutions of the form (3.4) with two different values of \( \eta^\pm, \eta^\pm = \mathcal{J}^\pm \) at \( r_1 \) and \( \eta^\pm = \mathcal{K}^\pm \) at \( r_2 \). The first equality in (3.9), that the zero mode of \( \mathcal{J}, \mathcal{K} \) functions should be equal, is in fact the condition for having a smooth interpolating solution. This may also be seen from the definition of \( Z_\pm \) (3.11). As \( Z_\pm \) should also be periodic in \( x^\pm \) and hence \( \int_0^{2\pi} \partial_\pm Z_\pm dx^\pm = 0 \).

The metric components correspond to the solution (3.5) with (3.11) are

\[
\begin{align*}
g_{rr} &= \frac{l^2}{r^2} + (f')^2 \left[ l^2 \left( Z_+^2 + Z_-^2 \right) + 2 Z_+ Z_- \mathcal{R}(r) \right], \\
g_{r\pm} &= f' \left[ l^2 Z_\pm + Z_+ \mathcal{R}(r) \right] \mathcal{I}_\pm, \\
g_{\pm\pm} &= l^2 \mathcal{I}_\pm^2, \\
g_{+-} &= \mathcal{R}(r) \mathcal{I}_- \mathcal{I}_+,
\end{align*}
\]

where

\[
\mathcal{R}(r) = -\frac{1}{2} \left( r^2 + \frac{l^4}{r^2} \right).
\]
Figure 2: Penrose diagram showing a patch of AdS$_3$ space between the two radii $r_1, r_2$. The vertical axis is the direction of time. We are computing the charges on a constant time slice at arbitrary $r$, $r_1 < r < r_2$ assume having the charge $J^\pm, K^\pm$ respectively at $r_1, r_2$.

4 Symmetries of the interpolating solutions

In the previous section we introduced a four-function family of AdS$_3$ gravity solutions which interpolate between two, two-function family of solutions between radii $r_1, r_2$. In this section we show that there are two set of charges associated with this family. Consider diffeomorphisms,

$$\xi^r = 0, \quad \xi^\pm = \pm \frac{T^\pm}{I^\pm},$$

with $I^\pm$ given in (3.6) and

$$T^\pm = \epsilon_\pm f + \psi_\pm (1 - f),$$

where $\epsilon_\pm$ and $\psi_\pm$ are arbitrary periodic functions of $x^\pm$. These diffeomorphisms rotate us in the family of interpolating solutions (3.12), explicitly,

$$\delta_\xi g_{\mu\nu} = g_{\mu\nu} (\mathcal{J}_\pm + \delta_\xi \mathcal{J}_\pm, \mathcal{K}_\pm + \delta_\xi \mathcal{K}_\pm) - g_{\mu\nu} (\mathcal{J}_\pm, \mathcal{K}_\pm)$$

with

$$\delta_\xi \mathcal{J}_\pm = \pm \partial_\pm \epsilon_\pm, \quad \delta_\xi \mathcal{K}_\pm = \pm \partial_\pm \psi_\pm.$$  (4.4)

Therefore,

$$\delta_\xi \mathcal{I}_\pm = \pm \partial_\pm T^\pm,$$  (4.5a)
$$\delta_\xi \mathcal{Z}_\pm = \pm (\epsilon_\pm - \psi_\pm).$$  (4.5b)

That is, (4.1) generate symmetries in our interpolating family of solutions.

4.1 Algebra of symmetry generators

The generators (4.1) are field dependent, they depend on background fields $\mathcal{J}^\pm, \mathcal{K}^\pm$ through $\mathcal{I}_\pm$. Therefore, the algebra of the symmetry generators is given by the “adjusted bracket”[27]:

$$[\xi_1, \xi_2]_* = [\xi_1, \xi_2]_{1_{ic}} - \delta^{(g)}_{\xi_1} \xi_2 + \delta^{(g)}_{\xi_2} \xi_1,$$  (4.6)
where $[,]_{Lie}$ is Lie bracket and $\delta^{(g)}_{\xi_2}$ denotes the change induced in $\xi_2$ due to the variation of metric $\delta_{\xi_2}g_{\mu\nu} = \mathcal{L}_{\xi_2}g_{\mu\nu}$. For the residual symmetry generators (4.1), one can simply show that

$$[\xi_1, \xi_2]_* = 0.$$  \hspace{1cm} (4.7)

By expanding the vector field $\xi = \xi(\epsilon^\pm, \psi^\pm)$ in modes $X^\pm_m = \xi(e^{im\pi\pm}, 0)$ and $Y^\pm_m = \xi(0, e^{im\pi\pm})$, one obtains four copies of the $U(1)$ algebra

$$[X^\pm_m, X^\pm_n]_* = 0, \quad [Y^\pm_m, Y^\pm_n]_* = 0,$$

$$[X^\pm_m, Y^\pm_n]_* = 0, \quad [X^\pm_m, Y^\mp_n]_* = 0, \quad [Y^\mp_m, Y^\pm_n]_* = 0, \quad [Y^\mp_m, Y^\mp_n]_* = 0.$$  \hspace{1cm} (4.8a, 4.8b, 4.8c)

### 4.2 Isometries of the interpolating family

The Killing vectors of the interpolating family $\zeta$ are a subset of (4.1) which keep the metric (3.12) intact, $\mathcal{L}_{\zeta}g_{\mu\nu} = 0$. That is, $\delta_{\zeta}\mathcal{J}_\pm = \delta_{\zeta}K_\pm = 0$. Moreover, since metric (3.12) depends on $\mathcal{Z}_\pm$, we should also have $\delta_{\zeta}\mathcal{Z}_\pm = 0$. Therefore,

$$\zeta^r = 0, \quad \zeta^\pm = \frac{\alpha^\pm}{I^\pm},$$  \hspace{1cm} (4.9)

where $\alpha^\pm$ are arbitrary constants.

In general our family of solutions (3.12) have two periodic (and hence globally defined) Killing vectors.$^4$ At generic values of radius $r$, one of these Killing vector fields is spacelike and the other is timelike. Requiring that for constant $\mathcal{J}_\pm, K_\pm$ these Killing vectors reduce to the canonically normalized Killing vectors of a BTZ black hole, fixes $\alpha^\pm = \kappa/2$ where $\kappa$ is the surface gravity. With this normalization, $\zeta \cdot \zeta = -\frac{\kappa^2}{4\pi^2} \left(\chi^2 - l^2\right)^2$. Therefore, we have a Killing horizon at $r_h = l$ (cf. footnote 3).

### 5 Charge analysis and $Y$-term

Having specified the symmetry generators, we now compute the charges and their algebra for the set of solutions (3.12). The charge expression (2.11) on $\partial\Sigma$ can be written as

$$\delta Q_\xi = \int_0^{2\pi} \left( Q_\xi^+ + Y_\xi^- \right) dx^+ - \int_0^{2\pi} \left( Q_\xi^- + Y_\xi^+ \right) dx^-.$$  \hspace{1cm} (5.1)

The above charge expressions are computed at a fixed, but arbitrary value of the radial coordinate $r$. Therefore only the two $Q_\xi^\pm$ components contribute to the charge. A straightforward computation yields the following explicit form expressions for the interpolating solutions,

$$Q_\xi^{\pm r} = \mp \frac{l}{4\pi G} T^{\pm r} \delta I_{\mp r} - \frac{1}{16\pi G} \left[ \delta \sqrt{-g} \delta g^{\pm r} - \delta \sqrt{-g} \delta g^{\mp r} \right]$$

$$+ \partial_\tau \left\{ \frac{r^2 R'/f'}{32\pi G} \left[ \pm T^{\mp} \delta \left( \frac{Z^{\mp}}{I_{\mp}} \right) \mp T^{\mp} \delta \left( \frac{Z^{\mp}}{I_{\pm}} \right) \right] \right\}.$$

$^4$Note that all the geometries we consider here are locally AdS$_3$ and hence there are six local Killing vectors, generating $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ algebra. However, generically only two of these are also periodic in $x^\pm$ and are hence globally defined [32]. There is a similar situation for the class of Bañados geometries [33] which is studied in detail in [30, 31].
The second line in (5.2) is a total derivative and hence do not contribute to the charge (5.1). The first term in (5.2) yields an integrable charge, while the second term bars integrability. This second term is, however, of the form of a $Y$-term, cf. (2.13). So, fixing the $Y$-term as

$$Y^{\mu\nu}[g; \delta g] = \frac{-g}{8\pi G} \delta \epsilon^{\mu\nu}$$

(5.3)

where $\epsilon^{\mu\nu} = n[^\nu m^\nu]$ with $n_\nu dx^\nu = \frac{L}{r} dr, m = \frac{r}{r} \partial_r$, renders the charge integrable. One may immediately verify that with the $Y$-term (5.3), the charge conservation condition (2.15) (vanishing flux through boundary) is also satisfied: A direct computation leads to,

$$\Theta_{\text{LW}}[g; \delta g] = -\frac{1}{16\pi G} \delta (\sqrt{-g} K^r) + \frac{1}{16\pi G} \left[ \partial_+ (\sqrt{-g} \delta g^+) + \partial_- (\sqrt{-g} \delta g^-) \right],$$

(5.4a)

$$\partial_\mu Y^{r\mu} = -\frac{1}{16\pi G} \left[ \partial_+ (\sqrt{-g} \delta g^+) + \partial_- (\sqrt{-g} \delta g^-) \right],$$

(5.4b)

and hence $\Theta \cdot n = \frac{L}{r} \left( \Theta_{\text{LW}} + \delta L \mid_{\text{bdy}} + \partial_\mu Y^{r\mu} \right) = 0$. That is, the integrability (2.16) and charge conservation equations are coming about through the same choice of the boundary $Y$-term.

A physically motivated requirement is that the $Y$-term (5.3) should vanish at the boundaries. Since $Y$ is proportional to $f'$ this latter can be satisfied for generic values of $r_1, r_2$ if $f'(r_1) = f'(r_2) = 0$, as we have already required in (3.7). By this assumption $\int_{\Sigma_t} \omega_{\text{LW}}$ is conserved where $\partial_\Sigma_t$ is the circle at $r = r_1$ or $r = r_2$ (see Fig. 2). In other words, the conserved and integrable charges of geometries (3.4) at $r = r_1$ and $r = r_2$ can be computed from $\int_{\Sigma_t} \omega_{\text{LW}}$ and there is no need for the $Y$ term.

Thus, charges associated with the symmetry generators (4.1) are

$$Q_\xi = Q_\xi^+ + Q_\xi^-,$$

(5.5)

with

$$Q_\xi^\pm(r) = \frac{l}{4\pi G} \int_0^{2\pi} T^\pm \mathcal{I}_\pm dx^\pm.$$  

(5.6)

These charges are integrable and conserved. We stress that the integrals (5.1) are computed at a fixed but arbitrary $r$ and hence the charges, as (5.6) explicitly indicates, are $r$ dependent. In particular,

$$Q_\xi^\pm(r_1) = \frac{l}{4\pi G} \int_0^{2\pi} \epsilon^\pm J_\pm dx^\pm, \quad Q_\xi^\pm(r_2) = \frac{l}{4\pi G} \int_0^{2\pi} \psi^\pm K_\pm dx^\pm.$$  

(5.7)

The above expressions are precisely the charge expressions for geometries given by (3.4) [21]. That is, the charges (5.6) are also interpolating between the charges associated with $f = 0, f = 1$ geometries.

6 Charge algebra and redundancies of the phase space

The algebra among charges can be obtained by the fundamental theorem of the covariant phase space method (see e.g. [34, 35]) which states that

$$\{Q_{\xi_1}, Q_{\xi_2}\} = Q_{[\xi_1, \xi_2]} + C(\xi_1, \xi_2)$$

(6.1)

where the bracket is defined by $\delta_{\xi_1} Q_{\xi_2} = \{Q_{\xi_1}, Q_{\xi_2}\}$ and $C(\xi_1, \xi_2)$ is a possible central extension. That is, the algebra among charge modes is isomorphic to the algebra of residual symmetry.
generators up to central extension terms. In our case the symmetry generators commute (4.7) and hence we only remain with the central extension, which is
\[ \{Q^\pm_{\xi_1}, Q^\pm_{\xi_2}\} = \pm \frac{l}{4\pi G} \int_0^{2\pi} T_1^\pm \partial_x T_2^\pm \, dx^\pm, \] (6.2)
\[ \{Q^\pm_{\xi_1}, Q^\mp_{\xi_2}\} = 0. \]

Upon expanding in modes \( F_m^\pm := Q^\pm(e^\pm = e^{imx^\pm}, \psi^\pm = 0), \) \( H_m^\pm := Q^\pm(e^\pm = 0, \psi^\pm = e^{imx^\pm}), \) we find
\[ F_m^\pm = \frac{l}{2G} \int \mathcal{I}_m^\pm, \quad H_m^\pm = \frac{l}{2G} (1 - f) \mathcal{I}_m^\pm, \] (6.3)
where \( \mathcal{I}_m^\pm = f \mathcal{J}_m^\pm + (1 - f) \mathcal{K}_m^\pm \) and
\[ \mathcal{J}_m^\pm = \frac{1}{2\pi} \int_0^{2\pi} e^{imx^\pm} \mathcal{J}^\pm \, dx^\pm, \quad \mathcal{K}_m^\pm = \frac{1}{2\pi} \int_0^{2\pi} e^{imx^\pm} \mathcal{K}^\pm \, dx^\pm. \] (6.4)

The charge algebra (6.2) then reduces to
\[ \{F_m^\pm, F_n^\pm\} = \mp \left( \frac{ilm}{2G} \right) f^2 \delta_{m+n,0} \] (6.5a)
\[ \{H_m^\pm, H_n^\pm\} = \mp \left( \frac{ilm}{2G} \right) (1 - f)^2 \delta_{m+n,0} \] (6.5b)
\[ \{F_m^\pm, H_n^\pm\} = \mp \left( \frac{ilm}{2G} \right) f(1 - f) \delta_{m+n,0} \] (6.5c)
and the brackets not displayed vanish. By replacing the above brackets with commutators, \( i\{,\} \rightarrow [\, ,\] , we obtain four \( U(1) \) current algebras
\[ [F_m^\pm, F_n^\pm] = \pm mkf^2 \delta_{m+n,0} \] (6.6a)
\[ [H_m^\pm, H_n^\pm] = \pm mk(1 - f)^2 \delta_{m+n,0} \] (6.6b)
\[ [F_m^\pm, H_n^\pm] = \pm mkf(1 - f) \delta_{m+n,0} \] (6.6c)
where \( k = \frac{l}{2G} = \frac{\xi}{3} \) and \( c \) is the Brown-Henneaux central charge.\(^5\)

**Redundancy in phase space.** The family of interpolating solutions are specified by four functions \( \mathcal{J}_\pm, \mathcal{K}_\pm \). At first sight this matches with the number of charges, \( F^\pm, H^\pm \). However, we note that in metric (3.1), and also in the charge (5.6), at any given \( r \) only a certain combination of these four functions, namely \( \mathcal{I}_\pm \), appear. This suggests that there is a redundancy in our description of the phase space and also the charges. To establish this, let us define,
\[ I_m^\pm := F_m^\pm + H_m^\pm, \] (6.7a)
\[ D_m^\pm := (1 - f) F_m^\pm - f H_m^\pm. \] (6.7b)

Then, (6.3) yields,
\[ I_m^\pm = \frac{l}{2G} \mathcal{I}_m^\pm, \quad D_m^\pm = 0. \] (6.8)

As we see the \( D^\pm \) charges vanish on the phase space and the phase space points are governed by just two \( r \) dependent functions \( \mathcal{I}_\pm \).

\(^5\)Note that the normalization of our charges is different than those used in [8] by a factor of 1/2, yielding a factor of 1/4 difference in commutators, in factor \( k \).
Black hole entropy. We discussed that our family of interpolating geometries in general correspond to a BTZ black hole with different kinds of soft hair excitations. These soft hair excitations can change as we move along the radial $r$ direction, nonetheless the value of the zero modes remain the same. In our coordinates, the horizon is at $r = l$ and line element at the bifurcation circle (a constant $t$ slice at $r = l$) is $ds_h^2 = l^2 (\mathcal{I}_+ + \mathcal{I}_-) d\phi^2$. Therefore, the Bekenstein-Hawking entropy is

$$S_{\text{BH}} = \frac{l}{4G} \int_0^{2\pi} d\phi \left( \mathcal{I}_+ + \mathcal{I}_- \right) = \frac{2\pi l}{4G} \left( J_0^+ + J_0^- \right) = \pi (I_0^+ + I_0^-). \quad (6.9)$$

The mass and angular momentum of the BTZ black hole are

$$M = \frac{l}{4G} \left( (J_0^+)^2 + (J_0^-)^2 \right), \quad J = \frac{l}{4G} \left( (J_0^+)^2 - (J_0^-)^2 \right). \quad (6.10)$$

These correspond to exact symmetries (Killing vector) charges and hence their values are independent of the radius $[36]$, as expected.

7 Discussion and outlook

In this paper we continued the analysis of $[22, 23]$ and studied AdS$_3$ gravity on a portion of AdS$_3$ space between two arbitrary constant $r$ slices $r_1, r_2$, as depicted in Fig. 2. In our family of solutions (3.1) with (3.5), constant $J^\pm, K^\pm$ case describes a BTZ black hole whose horizon is at $r = l$. For $J^\pm = K^\pm$ case our analysis reduces to those of $[8, 9]$ where the near horizon symmetries of BTZ black holes were studied. To see the connection to the near horizon symmetries, we note that one could have taken our inner boundary to be at the horizon, $r_1 = l$, where our boundary conditions reduce to those in $[8, 9]$.

As emphasized smoothness of geometries in the family of our solutions requires $J^\pm$ and $K^\pm$ to have equal zero modes (3.9), but are otherwise arbitrary and independent. In the Chern-Simons analysis of $[22, 23]$ this condition arises from matching of holonomies around noncontractable circles in the (Euclidean) interpolating geometry. Algebraically, each geometry in our family may be viewed as an element in a coadjoint orbit of our $U(1)$ currents symmetry algebra, where each orbit is specified by the $J_0^\pm$ values. (Note that in the algebra (6.6) $F_0^\pm, H_0^\pm$ commute with all elements of the algebra and hence these zero modes can be used to label the coadjoint orbits.) Therefore, our analysis reveals that as we move in the radial direction we may move in a given coadjoint orbit but it is not possible to smoothly traverse the orbits.

If we take $r_1 = l$ while taking the other boundary to the AdS boundary, $r_2 \to \infty$, we get a solution which covers the region outside the outer horizon of a BTZ black hole. In this case one may observe that we end up with a metric which does not satisfy the Brown-Henneaux boundary conditions, see $[21]$ for more discussions. In this solution the near horizon soft hair are given by $J^\pm$ functions while the asymptotic charges are given by $K^\pm$, which are not constrained at all by the near horizon charges. The fact that near horizon and asymptotic soft charges are not correlated with each other was also observed and briefly discussed in $[22]$. Here we find a similar phenomenon for an example with different boundary conditions at the two boundaries.

Given the line of analysis and arguments yielding the independence of near horizon and asymptotic charges, we expect that this feature is not limited to the cases in AdS$_3$ gravity and should hold more generically. In particular, we expect this to be true for asymptotic flat 4d Kerr
black hole, the near horizon symmetries of which was analyzed in \cite{10, 11, 17} and its asymptotic symmetries are BMS$_4$ \cite{18}. This has implications for the soft hair proposal \cite{7}, according which the black hole microstates, the soft hair, are labeled by the near horizon soft charges, while the information carried by the Hawking radiation may be labeled by the asymptotic BMS$_4$ charges.

At a technical level in our charge computations we found that the charge variation computed from the Lee-Wald contribution is not integrable and conserved. This was a result of the fact that the Lee-Wald flux does not vanish at the two boundaries at arbitrary $r_1, r_2$. This was remedied by the addition of a boundary $Y$-term (5.3) which vanishes at two radii. For $r_1 = l, r_2 \to \infty$ this matches with the earlier familiar results in the literature.

Finally, we would like to comment on the possibility of extension of asymptotic (boundary) symmetries into the bulk. To explain the point we start with an example. Let us consider the Virasoro excitations at the causal boundary of AdS$_3$ which are defined by the Brown-Henneaux boundary conditions \cite{6}. One may then ask if it is possible to extend the Brown-Henneaux excitations to the bulk of AdS$_3$. The answer is of course affirmative and Bañados geometries \cite{33} achieve this. These are the family of AdS$_3$ solutions uniquely specified by their Virasoro charges. Moreover, as discussed in \cite{27}, given the set of Bañados geometries one may compute the associated Virasoro charge at an arbitrary radius, i.e. we are dealing with symplectic symmetries/charges (and not just asymptotic symmetries/charges). The near horizon symmetries of \cite{8, 9} are also symplectic in the same sense. Similar extension of the asymptotic symmetries to the bulk has also been studied in 4d examples \cite{37, 38} and for near horizon extremal geometries \cite{39, 40}. Given our analysis here and results of \cite{22, 23}, one would expect that this should be a very general feature: (i) All the near horizon and/or asymptotic symmetries can be extended into the bulk and be made symplectic. (ii) This extension into the bulk is not unique at all. One can arbitrarily rotate in a given coadjoint orbit as we move in the radial direction. There is, of course, a specific extension, associated with the symplectic symmetries, in which one does not move in the orbit as we move in the radial direction. Explicitly, if we denote our radius dependent charges by $Q(r)$, then $Q(r_1), Q(r_2)$ can generically be different elements in the same coadjoint orbit of the charge algebra. (iii) In the example we studied here, while the charges $I^+_n$ (6.7) are $r$ dependent, the algebra of charges are $r$ independent. One can envisage examples where the algebra of charges are also $r$-dependent. The change in the algebra should, however, be such that it does not force us traverse across the orbits of the algebra at either of the boundaries. A first example of the latter was given in \cite{22}. It would of course be desirable to provide a rigorous proof for these statements in a general setting, beyond the examples so far studied in the literature.

**Acknowledgement**

We are grateful to Hamid Afshar, Daniel Grumiller, Marc Henneaux, Raphaela Wutte and Hossein Yavartanoo for useful discussions and Ali Seraj for this contribution at the early stages of this work. MMShJ would like to thank the hospitality of ICTP HECAP and ICTP EAIFR where this research carried out. We acknowledge the support by INSF grant No 950124 and Saramadan grant No. ISEF/M/98204. MMShJ acknowledge the Iran-Austria IMPULSE project grant, supported and run by Khawrizmi University.
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