CONVEXITY IN THE FIGURE EIGHT SOLUTION TO
THE THREE-BODY PROBLEM

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The figure eight is a remarkable solution to the Newtonian three-body problem in which the three equal masses chase each around a planar curve having the qualitative shape and symmetries of a figure eight. Here we prove that each lobe of this eight is convex.

1. Introduction

The figure eight is a recently discovered periodic solution to the Newtonian three-body problem in which three equal masses traverse a single closed planar curve which has the form of a figure eight (figure 1). See [Moore], and [CM]. In particular, it has one point of self-intersection, the origin, which divides the eight into two symmetric parts, its two lobes. In [CM] it was proved that each lobe is star-shaped. Here we prove convexity of the lobes.

Theorem 1. Each lobe of the eight solution is a convex curve.

In the final section we describe how the theorem generalizes to prove the convexity of eights for many three body potentials besides Newton’s.

A computer proof based on interval arithmetic appears in [KZ].

2. Preliminaries.

We present a number of properties of the eight established in [CM] and three assertions relating mechanics and plane geometry. The convexity proof relies on these properties and assertions.

2.1. Center of Mass. Write \((q_1(t), q_2(t), q_3(t))\) for the location of the three masses at time \(t\). The \(q_i(t)\) are points in the plane. At each time \(t\) we have that \(q_1(t) + q_2(t) + q_3(t) = 0\).

2.2. Symmetry. Write \(R_y(x, y) = (-x, y)\) for the reflection about the \(y\) axis. Then the eight solution enjoys the following symmetries:

\[
(q_1(t), q_2(t), q_3(t)) = (R_y(q_3(t - T/6)), R_y(q_1(t - T/6)), R_y(q_2(t - T/6)))
\]

\[
(q_1(t), q_2(t), q_3(t)) = (-q_1(-t), -q_3(-t), -q_2(-t)).
\]
The right-hand side of these equations define transformations $s$ and $\sigma$ on the space of all $T$-periodic loops. These transformations generate an action of the dihedral group

$$D_6 = \{s, \sigma|s^6 = 1, \sigma^2 = 1, s\sigma = \sigma s^{-1}\},$$

the symmetry group of a regular hexagon, which is consequently a symmetry group of the eight.

Invariance under $s^2 \in D_6$ implies that $(s^2(q_1, q_2))(t) = (q_1(t), q_2(t), q_3(t))$. Setting $q = q_1$ this last equation reads

$$q_1(t) = q(t), q_2(t) = q(t + T/3), q_3(t) = q(t + 2T/3).$$

We call three-body solutions satisfying (1) *choreographies*. The curve $q(t)$ is the curve of the eight, whose lobes are the subject of theorem 1.

$D_6$ invariance of the figure eight implies that it is completely determined by the three arcs $q_1([-T/12, 0]), q_2([-T/12, 0]), q_3([-T/12, 0])$ swept out by the three masses over the time interval $[-T/12, 0]$. In order to prove theorem 1 it is enough to prove that the curvatures of these three arcs are never zero (with the exception of the point $q_1(0)$ which is taken to be the origin – the self-intersection point of the eight).

A configuration $(q_1, q_2, q_3)$ satisfying $q_1 + q_2 + q_3 = 0$ is called an *Euler configuration* if one of the $q_i = 0$. Then necessarily the other two masses $q_j, q_k$ are of the form $\zeta, -\zeta$ so that the entire configuration $(q_1, q_2, q_3)$ is collinear with mass $i$ at the origin located at the midpoint of the segment defined by the other two masses $j$ and $k$. Upon translating time if necessary, and relabeling mass labels, we can insist that at the time $0$ the configuration is an Euler configuration with $1$ at the origin and $3$ in the first quadrant as indicated in figure 1. And at the initial time $t = -T/12$ the three masses form an isosceles triangle with mass $2$ at the vertex and lying on the negative $x$-axis.

**Figure 1.** Figure eight. Solid circles labeled $js$ and $je$ ($j = 1, 2, 3$) represent places at $t = -T/12$ and $t = 0$ of $j$th mass.
The eight minimizes the usual action of mechanics (integral of the kinetic minus potential energy) among all $T$-periodic loops enjoying $D_6$ symmetry. Equivalently (see [CM]) the path $(q_1(t), q_2(t), q_3(t))$ of the eight over the fundamental time interval $[-T/12, 0]$ minimizes the action among all paths starting at time $-T/12$ in an isosceles configuration with 2 being the vertex and ending at time 0 in an Euler configuration with 1 being the origin. An important consequence of minimization proved in [CM], p.896-897 is that there are no times in the fundamental domain besides the endpoints at which the configuration is either collinear or isosceles. It follows that for all $t \in (-T/12, 0)$ we have

\begin{align*}
(2) \quad & r_{13} < r_{12} < r_{23}, \\
(3) \quad & q_1 \land q_2 = q_2 \land q_3 = q_3 \land q_1 < 0
\end{align*}

where $r_{ij} = |q_i - q_j|$ denotes the distance between mass $i$ and mass $j$. Here, we write $(x, y) \land (u, v) = xv - yu$ for planar vectors $(x, y), (u, v)$. We call equation (2) the distance ordering inequality.

### 2.3. Initial and Final Velocities.

At the Euler time, $t = 0$, the velocities of 2 and 3 are antiparallel to the velocity of 1, and half its size. See figure 1. This fact follows from the action minimization of the eight. At the isosceles time $t = -T/12$, 2’s velocity is vertical, pointing down, and the velocities of 1 and 3 are such that their tangent lines pass through 2. This fact follows from the three tangents theorem [FFO], and the angular momentum properties, described below.

### 2.4. Angular momentum and star-shapedness.

Write

$$\ell_j = q_j \land \dot{q}_j$$

for the angular momentum of the $j$th particle. Action minimization of the eight implies that its total angular momentum is zero:

$$\ell_1 + \ell_2 + \ell_3 = 0$$

of the eight. Newton’s equations (see [CM], p.896) imply

$$\dot{\ell}_3 = \left( \frac{1}{r_{13}^2} - \frac{1}{r_{23}^2} \right) (q_1 \land q_2)$$

valid for all time. Upon taking account the distance inequality (2) and (3) we find that $\dot{\ell}_3 < 0$ on the arc 3. Similarly, we get

$$\dot{\ell}_1 > 0, \dot{\ell}_2 > 0, \dot{\ell}_3 < 0.$$

By the symmetry $\ell_{1s} = \ell_{3s} = -2\ell_{2s} < 0$. (The inequalities $\ell_{1s} < 0$ and $\ell_{1e} = 0$ are consistent with $\dot{\ell}_1 > 0$.) Also $\ell_{2s} > 0$ and $\dot{\ell}_2 > 0$ imply $\ell_{2e} = -\ell_{3e} > 0$. (See figure 2.) Therefore over the interior $(-T/12, 0)$ of our fundamental domain we have

$$\ell_1 < 0, \ell_2 > 0, \ell_3 < 0$$
More generally, set 
\[ \ell = q \wedge \dot{q} \]
as \( q \) varies over the eight. It follows that on the right lobe \((x > 0)\) we have 
\[ \ell < 0 \text{ for } x > 0. \]
(See figure 2).

\[ \text{Figure 2. } \ell(t) \text{ vs. } t. \]

A curve in the plane is called ‘star-shaped’ with respect to the origin if every ray starting at the origin intersects the curve at most once. For a smooth curve, this is equivalent to the assertion that, when written in polar coordinates as \((r(t), \theta(t))\), the function \(\theta(t)\) is strictly monotone and does not vary by more than \(2\pi\). Since \(\ell = r^2 \dot{\theta}\) the star-shapedness of a curve (such as one lobe of the eight) which lies in the half-plane \(x > 0\) is thus equivalent to \(\ell \neq 0\).

2.5. Three tangents theorem. The following theorem can be found in [FFO] where it was used to find and establish existence of a choreographic three-body lemniscate for a non-Newtonian potential.

**Theorem 2** (Three tangents). Let \((q_1(t), q_2(t), q_3(t))\) be three planar curves whose total linear and total angular momentum are zero. Then the three instantaneous tangent lines to these three curves are coincident – they all three intersect in the same (time-dependent) point or are parallel.

**Proof.** Fix the time \(t\). Because \(\dot{q}_1 + \dot{q}_2 + \dot{q}_3 = 0\), translating all the \(q_i\) in the same fixed direction does not change the condition of having zero angular momentum. So, without loss of generality, we can choose the origin to be the point of intersection of the tangent lines to \(q_1\) and \(q_2\) at time \(t\). Because the point \(q_1(t)\) lies along the line through the origin in the direction \(\dot{q}_1\) we have that \(q_1(t) \wedge \dot{q}_1(t) = 0\). Similarly \(q_2(t) \wedge \dot{q}_2(t) = 0\). But the total
angular momentum is zero so we must have that \( q_3(t) \wedge \dot{q}_3(t) = 0 \) which asserts that the line tangent to the curve of \( q_3 \) at \( t \) also passes through the origin. QED

Remark: the proof also works for unequal masses \( m_1, m_2, m_3 \). Simply use the correct mass-weighted formulae for linear and angular momentum.

2.6. Splitting Lemma. We will use the following ‘Splitting Lemma’ in several places in the proof. A line in the plane divides the plane into three pieces – two open half-planes and the line itself. We say that a point lies strictly on one side of the line if it lies in one of the open half-planes. We say that this line splits the points \( A \) and \( B \) of the plane if the two points lie in opposite open half planes.

**Lemma 1.** Let \((q_1(t), q_2(t), q_3(t))\) be a planar solution to Newton’s three-body equation with attractive \( 1/r \) potential. Suppose that at time \( t_* \) the arc \( q_i(t) \) of mass \( i \) has an inflection point and nonzero speed. Then the tangent line \( \ell \) to this arc at time \( t_* \) must either (A) split the other two masses \( q_j(t_*) \) and \( q_k(t_*) \) or (B) all three masses must lie on this tangent line.

**Proof.** Suppose, to the contrary, that either both \( q_j(t_*) \) and \( q_k(t_*) \) lie strictly on one side of \( \ell \), or that one lies on \( \ell \) while the other lies strictly on one side. According to Newton’s equations the acceleration \( \ddot{q}_i(t_) \) is a linear combination of \( q_j(t_*) - q_i(t_*) \) and \( q_k(t_*) - q_i(t_*) \) and the coefficients of this linear combination are positive. Thus, translating \( \ell \) and the configuration of masses back to the origin by subtracting \( q_i(t_*) \) we see that this acceleration lies strictly on one side of the line through 0 spanned by the velocity \( \dot{q}_i(t_*) \). Consequently, the acceleration and velocity of \( q_i(t) \) are linearly independent at \( t_* \). But the condition of being an inflection point is precisely that the acceleration and velocity be linearly dependent. QED

Remark. The same proof works if the Newtonian potential \(-\sum_{i<j} m_i m_j / r_{ij}\) is replaced by any potential \( V = \sum_{i<j} f(r_{ij}) \) where \( df/dr > 0 \).

2.7. A Convexity Proposition. A parameterization \( t \) of a curve \( C \) is called *nondengenerate* if under this parameterization the derivative \( dC(t)/dt \) is never zero. A smooth, possibly self-intersecting curve is called *locally convex* if its curvature never vanishes.

**Proposition 1.** Let \( C \) be a smooth locally convex planar curve parameterized by a nondegenerate parameter \( t \). Let \( \ell(t) \) be the the tangent line to \( C \) at \( C(t) \). Let \( m \) be a line not intersecting \( C \). Let \( P(t) \) be the point of intersection of \( \ell(t) \) and \( m \). Then \( dP/dt \neq 0 \) for all values of \( t \) (*).

(*) Some care is in order regarding those instants where the line \( \ell(t) \) does NOT intersect \( m \) by virtue of being parallel to it. The notion of convex and locally convex is a *projective* notion. (Local convexity is the assertion of linear independence of the first and the second derivatives of the curve, and
this linear independence is invariant under projective transformations.) So we should view the curve $C$ and the line $m$ as lying in the real projective plane $RP_2$. From this new perspective, the point $P(t)$ exists and is uniquely defined for all values of $t$. The theorem asserts that $dP/dt$ never vanishes. In the proof below we ignore these instants of parallelism i.e of $P(t)$ being at infinity relative to the affine plane within which we are working. Use a projective transformation to bring the point at infinity on $m$ to a finite point on the affine plane, and repeat the computation below to arrive at a proof valid for all values of $t$, including the instants of parallelism.

**Proof.** By a translation and rotation we can take $m$ to be the $y$-axis. If $(x(t), y(t))$ parameterize $C$, then the line $\ell(t)$ is given by \{(x(t), y(t)) + \lambda(\dot{x}(t), \dot{y}(t)) : \lambda \in \mathbb{R}\}$. It follows that the point of intersection of $\ell(t)$ with $m$ occurs at the point $P(t) = (0, p(t))$ where

$$p = -\frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{\dot{x}(t)}$$

A routine differentiation combined with the definition of the curvature $\kappa$ yields that

$$\frac{dp}{dt} = -\frac{v^3 x}{\dot{x}^2 \kappa}$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the curve’s speed. Every factor on the right hand side is nonzero. The coordinate $x$ is nonzero since the curve $C$ never intersects the line $m$. The only times at which $\dot{x} = 0$ are the excluded instants, those at which $\ell(t)$ is parallel to $m$. And $\kappa \neq 0$ by convexity of $C$. It follows that $dp/dt \neq 0$ everywhere except at the excluded instants. QED

3. To each mass its own quadrant.

A crucial ingredient of the proof of theorem 1 is that each mass “stays in its own quadrant” during the time interval $(-T/12, 0)$. Initially 3 is in quadrant 1, 1 is in quadrant 4 and 2 is on the x-axis between quadrants 2 and 3, but moving into quadrant 3. Hence, for a short time interval $(-T/12, -T/12 + \epsilon)$ mass 3 lies in quadrant 1, 1 in quadrant 4, and 2 in quadrant 3.

**Lemma 2.** Over the entire time interval $(-T/12, 0)$ body 1 lies in the 4th quadrant, body 2 lies in the 3rd quadrant, and body 3 lies in the 1st quadrant.

**Proof of Lemma.**

By way of contradiction, suppose one of the masses leaves its initial quadrant before the allotted time $T/12$. It must exit along the boundary of this quadrant. It cannot exit through the origin, as this would imply an Euler configuration and the only Euler configuration occurs at the endpoint of the interval.
We argue individually that each mass cannot be the first exiting mass. Suppose that 2 exits first. It cannot leave crossing the x-axis as this would contradict star-shapedness of the lobe it lies on. Neither can it exit through the y-axis. For if it exited through the y-axis, its x-coordinate would be zero. Now the x-coordinates of the other two masses cannot both be zero, otherwise this instant would be a syzygy instant. Thus at least one of the other two masses lie in their quadrants, which means their x-coordinate is positive. Thus the sum of the x-coordinates of the masses is positive, contradicting that the center of mass is at the origin.

Mass 1 cannot leave first. For it cannot leave through the x-axis, as this would again contradict star-shapedness. It cannot leave through the y-axis as this would violate the distance ordering (2).

\[ r_{13} < r_{12} < r_{23}. \]

(See figure 1.) To see this violation, write the exit point for mass 1 as \((0, y_1)\) with \(y_1 < 0\). Then the other masses must be at 
\((-x, y_2)\) and \((x, y_3)\) with \(x > 0\) (since the configuration cannot be a syzygy) and \(y_2 < 0, y_3 > 0\). We have

\[ r_{13}^2 = x^2 + (y_3 - y_1)^2, \quad r_{12}^2 = x^2 + (y_2 - y_1)^2. \]

But \(y_3 > 0, 0 > y_1, y_2\) and \(y_1 + y_2 + y_3 = 0\) so that
\(y_3 - y_1 = -2y_1 - y_2 = 2|y_1| + |y_2|\) while
\(|y_2 - y_1| < |y_2| + |y_1|\) so that \((y_3 - y_1)^2 > (y_2 - y_1)^2\) and \(r_{13} > r_{12}\), contradicting the distance ordering.

Mass 3 cannot leave first. It cannot exit across the x-axis, for if it did then the center of mass of the system would have a negative x-coordinate, contradicting that the center of mass is at the origin. It cannot leave across the y-axis, for this would contradict star-shapedness.

Some further thought going back through these cases shows that we cannot have two or more masses exiting their respective quadrants simultaneously before the allotted time either. QED

4. Proof of theorem 1

Denote the arc swept out by mass \(j\) during the the time interval \([-T/12, 0]\) as arc \(j\). Write \(\kappa_j\) for the curvature of arc \(j\). We must show that \(\kappa_j \neq 0\) at each point of each arc, with the exception of the origin for arc 1. More precisely, with our orientation and labelling of the eight, we must show that \(\kappa_1 \leq 0\) with \(\kappa_1 < 0\) for \(t \neq 0\), that \(\kappa_2 > 0\) and that \(\kappa_3 < 0\).

4.1. Convexity of arc 1. We begin by showing that \(\ddot{y}_1 > 0\) along arc 1. Since each mass stays in its own quadrant, we have \((y_3 - y_1) > 0\). And by the distance ordering inequality (2) \(r_{13} < r_{12}\). It follows that
\[
\ddot{y}_1 = \frac{(y_3 - y_1)/r_{13}^3 + (y_2 - y_1)/r_{12}^3}{r_{13}^3 + r_{12}^3} > \frac{(y_3 - y_1)/r_{13}^3 + (y_2 - y_1)/r_{12}^3}{r_{12}^3} = -3y_1/r_{12}^3
\]
Next we show that $\dot{y}_1 > 0$ along the arc. From the fact that $\ddot{y}_1 > 0$, it suffices to show that $\dot{y}_1 > 0$ at the initial point of arc 1, the isosceles point. By the three tangents theorem and the fact that $\ell_1 < 0$, it follows that at the isosceles point $\dot{q}_1$ points from $q_1$ to the vertex $q_2$, so that $\dot{y}_1 > 0$.

We have seen above that $\ell_1 < 0$ while $\dot{\ell}_1 > 0$ along the arc. Combining these inequalities, we see that $\dot{\ell}_1 \dot{y}_1 - \ell_1 \ddot{y}_1 > 0$ holds along the arc. On the other hand, expanding the angular momentum, we get $\dot{\ell}_1 \dot{y}_1 - \ell_1 \ddot{y}_1 = (x_1 \dot{y}_1 - y_1 \dot{x}_1)\dot{y}_1 - (x_1 \dot{y}_1 - y_1 \dot{x}_1)\ddot{y}_1 = y_1(x_1 \ddot{y}_1 - \dot{y}_1 \dot{x}_1) = y_1 v_1^3 \kappa$. Thus $y_1 v_1^3 \kappa_1 > 0$. Since $y_1 < 0$, $v_1 > 0$ we have that $\kappa_1 < 0$.

4.2. Convexity of arc 2. Assume, by way of contradiction, that there exists an inflection point $\kappa_2 = 0$ on arc 2. Let $a$ be the last inflection point on arc 2 – the one whose time $t$ is closest to 0. From the initial conditions at $t = -T/12, 0$ described above we also know that $\kappa_2 > 0$ at the points $2s$ and $2e$. By continuity, $\kappa_2 > 0$ near both of these points. Then $\kappa_2 > 0$ on the arc $a \rightarrow 2e$.

We know by the previous subsection that arc 1 is convex ($\kappa_1 < 0$) and we also know that body 3 moves in ‘its own quadrant’ - the 1st quadrant. It follows that bodies 1 and 3 must lie within the shaded region in the figure.

Consider the Gauss map (hodograph) of arc 2. This is the map which assigns to a point of arc 2 the unit tangent to arc 2, $\dot{q}_2/|\dot{q}_2|$, at that point.

By Newton’s equation and the fact that $x_1 - x_2$ and $x_3 - x_2$ are positive we have that $\dot{x}_2 > 0$ on the entire arc 2. Since $\dot{x}_2 = 0$ at $2s$, this implies that $\dot{x}_2 > 0$ on the open arc of 2, from $2s$ to $2e$, and so in particular $\dot{x}_2 > 0$ at $a$. Since $\kappa_2 > 0$ on the arc $a \rightarrow 2e$, the vector $\dot{q}_2/|\dot{q}_2|$ must approach $2e$ from the point $a$ monotonically counterclockwise. Therefore the point $a$ lies on the arc between the points $2s$ and $2e$ on the right half of the circle as shown in the Gauss map (figure 4).
Figure 4. Gauss map of the unit tangent vector $\frac{\dot{q}_2}{|\dot{q}_2|}$.

But then the tangent line to arc 2 at $a$ cannot split the points 1 and 3, which, according to the splitting lemma (sec. 2.6), contradicts the assumption that $a$ is an inflection point.

Thus we have proved that there is no inflection point on the arc 2. In other words, $\kappa_2 > 0$ on the arc 2.

4.3. Convexity of arc 3. Assume, by way of contradiction, that there are one or more inflection points $\kappa_3 = 0$ on the arc 3. Let $b$ be the first such point, the one for which the time $t$ is closest to $-T/12$. Then, by the splitting lemma (sec. 2.6), the tangent line to arc 3 at $b$ must split bodies 1 and 2. In order to do that, the line must have passed earlier through either body 1 or body 2. We argue that both passings are impossible.

The tangent line to arc 3 cannot pass through body 1. For, by the three tangent theorem, at the instant this happened, the tangent line from the body 2 would also pass through the body 1. We have already proved that $\kappa_2 > 0$ on the arc 2. Thus the tangent line from the body 2 never pass through the body 1 in this interval. (See figures 3 and 4.) This is a contradiction.

The tangent line to arc 3 cannot pass through body 2. For if it did, by the three tangents theorem, the tangent line to 1's curve would also pass through body 2 at the same instant. To see that this latter passing is impossible, join the endpoints 2s and 2e of arc 2 by a straight line $m$. Arc 2 lies completely on one side of this line, by convexity.

In section 2.7 we proved the proposition that if $c(t)$ is a smooth convex curve parameterized at a nonzero speed, and if $m$ is a fixed line, then the intersection point $P(t)$ of $c$'s tangent line with $m$ moves monotonically:
Figure 5. Line $m$ and tangent lines to arc 1 at $t = -T/12$ and $t = 0$.

$dP/dt \neq 0$. We apply the proposition to our situation. At the final points e, the tangents to 1 and 2 are parallel, so that the intersection of $m$ with 1’s tangent lies in the massless quadrant $x < 0, y > 0$. At the initial point s the intersection point of $m$ and arc 1’s tangent is 2s. Consequently, in between $s$ and $e$ the intersection always lies in that part of $m$ lying in the massless quadrant. But in order for 1’s tangent to pass through 2, 1’s tangent would have to cross line $m$ between 2s and 2e, which is in the quadrant of arc 2, and hence it is impossible that this tangent passes through 2.

Therefore, we have proved that there is no inflection point on the arc 3. In other word, $\kappa_3 < 0$ on the arc 3.

4.4. Conclusion. Combining 4.1, 4.2, and 4.3 proves theorem 1.

5. Convexity for other potentials

Theorem 1 holds for the “eights” of other potentials. Indeed, our proof only depended on the properties and propositions of the eight listed in section 2 and a “monotonicity” property of the Newtonian potential discussed below.

To be precise, we need to define what we mean by an “eight” Let

$$V = V(r_{12}, r_{23}, r_{31})$$

be a three-body potential depending only on the interparticle distances $r_{ij}$ and invariant under interchange of the masses. Then the symmetry group
$D_6$ of the eight acts on solutions to the corresponding Newton equation, taking solutions to solutions, and so we can speak of $D_6$-invariant solutions.

A planar solution to the Newton’s equation for $V$ will be called an eight solution if

i) it is invariant under the $D_6$ symmetries
ii) on the interior of each fundamental domain $((mT/12, (m + 1)T/12)$, $m = 0, ±1, ±2, ...)$ the configuration is never collinear and never isosceles
iii) the solution has no collisions

Such a solution will necessarily be a planar choreography (see (1) above), and so the three masses travel a single planar curve. Condition (i) implies that the center of mass is 0 and that the angular momentum is zero. If, in addition, our potential $V$ has the form

$$V = \sum_{i<j} f(r_{ij})$$

with

iv) $\frac{df}{dr} > 0$ (attractive two-body potential) and with
v) $g(r) = r^{-1}\frac{df}{dr}$ a strictly monotone decreasing function of $r$, then all properties and inequalities used in this paper hold.

Indeed, return to the starting point, the distance ordering inequality (2). At $t = -T/12$ and $t = 0$ we have $r_{233} = r_{12} + r_{31}$ and $r_{1e2} = r_{3e1} < r_{2e3}(= 2r_{1e2})$. By the property (ii), the possible distance orderings on the time interval $(-T/12, 0)$ are $r_{31} < r_{12} < r_{23}$ or $r_{12} < r_{31} < r_{23}$. Consider the equation for $\dot{\ell}_1$,

$$\dot{\ell}_1 = (g(r_{21}) - g(r_{31}))(q_2 \wedge q_3)$$

for a monotone decreasing function $g(r)$. We have $\dot{\ell}_1 > 0$ for the first ordering and $\dot{\ell}_1 < 0$ for the second ordering. But, since $\ell_{1s} < 0$ and $\ell_{1e} = 0$, $\dot{\ell}_1$ must be positive. So we must have the first ordering, namely, the equation (2). Then, all equalities and inequalities in this paper hold upon replacing $1/r^3$ with $g(r)$. Thus we have the following theorem.

**Theorem 3.** Let $V$ be a three-body potential of the form $V = \sum_{i<j} f(r_{ij})$ where $f$ satisfies (iv) and (v) immediately above, and admitting an eight solution as defined by (i)-(iii) above. Then each lobe of this eight for $V$ is convex.

The theorem begs the question, do eight solutions exist for any potentials besides Newton? Recall from [CM], p. 896-897 that if a solution which satisfies (i) and (ii) is known to minimize the action associated to $V$ among all paths satisfying (i), and if that solution is not identically collinear, then automatically the solution satisfies (ii). The power law potentials

$$V_a = (a)^{-1}(r_{12}^a + r_{23}^a + r_{31}^a),$$
for $a \leq -2$ admit such collision-free action minimizing solutions, and consequently they admit eight solutions. Moreover, the proof of [CM], specific to $a = -1$, is based on strict inequalities, and hence is valid for a range of exponents $-1 - \epsilon_1 < a < -1 + \epsilon_2$ for $\epsilon_1, \epsilon_2$ positive numbers.

As a corollary, we obtain

**Corollary 1.** For the power law potentials $V_a$ with $a < -2$ or with $a$ in some open interval about $-1$, there exist eight solutions and each lobe of these eight solutions is convex.

**References**

[CM] A. Chenciner, and R. Montgomery, *A remarkable periodic solution of the three-body problem in the case of equal masses*, Annals of Mathematics, 152 (2000), 881-901.

[FFO] T. Fujiwara, H. Fukuda, and H. Ozaki, *Choreographic three bodies on the lemniscate*, Journal of Physics A, 36 (2003), 2791-2800.

[KZ] T. Kapela and P. Zgliczynski, *The existence of simple choreographies for the $N$-body problem — a computer-assisted proof*, Nonlinearity, 16 (2003), 1899–1918.

[Moore] C. Moore, C. *Braids in Classical Gravity*, Physical Review Letters, 70 (1993), 3675–3679.

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