Entangling two unequal atoms through a common bath

F. Benatti\textsuperscript{a,b}, R. Floreanini\textsuperscript{b} and U. Marzolino\textsuperscript{a,b}

\textsuperscript{a}Dipartimento di Fisica Teorica, Universit\`a di Trieste, 34014 Trieste, Italy
\textsuperscript{b}Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, 34014 Trieste, Italy

Abstract

The evolution of two, non-interacting two-level atoms immersed in a weakly coupled bath can be described by a refined, time coarse grained Markovian evolution, still preserving complete positivity. We find that this improved reduced dynamics is able to entangle the two atoms even when their internal frequencies are unequal, an effect which appears impossible in the standard \textit{weak coupling limit} approach. We study in detail this phenomenon for an environment made of quantum fields.

1 Introduction

Independent, non-interacting atoms immersed in a common bath, represent an example of an open quantum system, \textit{i.e.} a subsystem in interaction with a large environment \cite{[1]-[6]}. Their time evolution can be given as a reduced dynamics, obtained by eliminating the degrees of freedom of the environment and by subsequently performing a Markovian, memoryless approximation, justified by the very rapid decay of correlations in the bath. The resulting atom evolution is irreversible and incorporates dissipative and noisy effects induced by the environment.

In many physical instances, the atoms can be treated in a non-relativistic approximation, as independent finite-level systems, with negligibly small size. On the other hand, the environment can be modeled by a set of weakly coupled quantum fields, typically the electromagnetic field, in a given temperature state, interacting with the atoms through a dipole type coupling \cite{[7]}. Although ignoring the internal atom dynamics and the full vectorial structure of the electromagnetic field, this simplified setting is nevertheless perfectly adequate for studying the behaviour of physical systems like ions in traps, atoms in optical cavities and fibers, impurities in phonon fields \cite{[7]-[9]}.

The derivation of an acceptable subdynamics for the atoms is notoriously tricky \cite{[10], [1]-[3]} and time evolutions that are not even positive have been adopted in the literature in order to
describe their physical properties [5, 11, 12]. Instead, a physically consistent time evolution for the atom subsystem can be obtained through a suitable coarse grained procedure, within the weak coupling approximation [1, 3, 13, 14, 15]: the resulting subdynamics is described by a one parameter (≡ time) family of completely positive maps that form a quantum dynamical semigroup.

We shall explicitly discuss below such a derivation for a subsystem composed by two, unequal, mutually non-interacting atoms. For simplicity we shall restrict the attention to two-level atoms weakly coupled to a collection of independent, free, massless scalar fields in $3 + 1$ space-time dimensions, assumed to be in a state at temperature $T \equiv 1/\beta$.

As well known, the interaction with an environment usually leads to decoherence and noise, typical mixing enhancing phenomena. Therefore, one generally expects that quantum correlations that might have been present at the beginning between the two atoms be destroyed when they are immersed in the bath.

However, an external environment can also mediate indirect interactions between otherwise totally decoupled subsystems and therefore a mean to correlate them. This phenomenon is generic on a short, microscopic time-scale where the dynamics is unitary and reversible; however, its persistence is not expected in general on longer time-scales where irreversible, dissipative and decohering effects described by Markovian master equations appear.

Nevertheless, there are instances where purely dissipative, non-Hamiltonian contributions to the master equation can lead to entanglement generation [10]-[23, 8]. This is due to the particular form of the (Kraus) operators appearing there, which couple indirectly (i.e. not dynamically) the two subsystems. This phenomenon has been established in the case of subsystems formed by two identical two-level systems [24]-[27] or harmonic oscillators [28, 29] evolving with a reduced dynamics obtained via the so-called weak coupling limit [30]. This technique is applicable when the time-scale over which the dissipative effects become visible is so large that the free dynamics of the subsystems can be effectively averaged out, thus eliminating very rapid oscillations; this typically occurs for environments with very fast decaying correlations [2, 3, 5]. Nevertheless, it turns out that, in the case of atoms with unequal frequencies, this procedure of averaging out fast oscillations prevents generation of entanglement; in other terms, the environments for which the weak coupling limit procedure is justified are unable to correlate two atoms with different internal frequencies, while they do when the atoms are identical.

In the following, we shall study in detail the conditions that allow the two otherwise independent, unequal atoms to become initially entangled through the action of the environment, when the weak coupling limit procedure is not applicable. Following [15], we shall instead derive a time evolution for the two atoms that allows a finite coarse-graining time interval: remarkably, it results still expressible in terms of a completely positive quantum dynamical semigroup, which reduces to the standard one obtained through the weak coupling limit in the limit of an infinitely large coarse-graining time interval. We shall see that in this refined

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1 A preliminary investigation on these topics in the case of two qubits weakly coupled to a Ohmic bath has been reported in [31].
framework the sharp dependence on the atom frequencies of the entanglement capability of the environment mentioned above looks more like a mathematical artifact than a real physical effect.

2 Two-atom reduced dynamics

As explained above, we shall study the behaviour of a system composed by two, unequal two-level atoms, that start interacting at time $t = 0$ with an environment made of a collection of independent, massless, scalar quantum fields at temperature $1/\beta$. We are interested in the evolution of the atoms as open quantum systems and not in the details of their internal dynamics; therefore, we shall model them, in a nonrelativistic way, as simple qubits, described in terms of a two-dimensional Hilbert space.

In absence of any interaction with the external fields, the single atom internal dynamics can thus be taken to be driven by a generic $2 \times 2$ Hamiltonian matrix. As a result, the total atom Hamiltonian $H_S$ can be expressed as:

$$H_S = H_S^{(1)} + H_S^{(2)}$$

where

$$H_S^{(a)} = \frac{\omega_a}{2} \vec{n} \cdot \vec{\sigma}^{(a)} \equiv \frac{\omega_a}{2} \sum_{i=1}^{3} n_i \sigma_i^{(a)}, \quad \alpha = 1, 2,$$

(1)

As we shall see in the following, the choice of modelling the environment in terms of relativistic quantum fields (at finite temperature) allows an analytic treatment of the reduced two-atom dynamics, without additional approximations besides the Born and Markov ones (see below).
and position the first atom at the origin of the reference frame, so that \( f^{(1)}(\vec{x}) \equiv f(\vec{x}) \), while the second is displaced by an amount \( \vec{\ell} \) with respect to it, \( f^{(2)}(\vec{x}) = f(\vec{x} + \vec{\ell}) \). Since the atom-field interaction takes place on the whole region occupied by the atoms, the field operators entering the interaction Hamiltonian above are smeared over the atom spatial extension:

\[
\Phi_i[f^{(\alpha)}] = \int d^3x f^{(\alpha)}(\vec{x}) \Phi_i(0, \vec{x}) , \quad \alpha = 1, 2 .
\]

(4)

The total Hamiltonian \( H \) describing the complete system, the two atoms together with the external fields \( \Phi_i \), can thus be written as

\[
H = H_S + H_\Phi + \lambda H' \equiv H_0 + \lambda H' ,
\]

(5)

with \( \lambda \) a small coupling constant. Through the standard Liouville-von Neumann equation, \( \partial_t \rho_{\text{tot}}(t) = -i[H, \rho_{\text{tot}}(t)] \), it generates the evolution in time of the state of the total system, described in general by a density matrix \( \rho_{\text{tot}} \), starting at \( t = 0 \) from the initial configuration: \( \rho_{\text{tot}}(0) \).

We shall assume the atom and the fields to be initially prepared in an uncorrelated state, with the fields in the temperature state \( \rho_\beta \) and the atoms in a generic initial state \( \rho(0) \), so that \( \rho_{\text{tot}}(0) = \rho(0) \otimes \rho_\beta \). The reduced time evolution of the two atoms is then obtained by integrating over the unobserved field degrees of freedom and is formally given by the transformation map: \( \rho(0) \mapsto \rho(t) \equiv \text{Tr}_\Phi[\rho_{\text{tot}}(t)] \). This map is in general very complicated, because of nonlinearities and memory effects; nevertheless, it can be approximated by a linear, memoryless map when the coupling with the environment is small and its own internal dynamics is sufficiently fast [1]-[5]. Indeed, in such cases the details of the internal environment dynamics result irrelevant, being the time scale of the subsystem evolution typically very long compared with the decay time of the correlations in the bath.

In order to derive the equation obeyed by the reduced density matrix \( \rho(t) \) in the case at hand, it is convenient to work in the interaction representation

\[
\tilde{\rho}_{\text{tot}}(t) = e^{itH_0} \rho_{\text{tot}}(t) e^{-itH_0} ,
\]

(6)

so that

\[
\frac{\partial \tilde{\rho}_{\text{tot}}(t)}{\partial t} = -i\lambda \left[H'(t), \tilde{\rho}_{\text{tot}}(t) \right] , \quad H'(t) = e^{itH_0} H' e^{-itH_0} .
\]

(7)

For simplicity, we have assumed that the environment couples with the same strength to the two atoms: this makes the following analytic derivation of the master equation more transparent, without compromising its generality.

A discussion on the validity of this so-called Markovian approximation is reported in [13]. There, a non-Markovian weak coupling approximation of the reduced dynamics is also introduced; it leads to a two-parameter family of dynamical maps, with a time-dependent generator [3]. We stress that this approach is completely different from the one discussed below, which instead describes the reduced two-atom dynamics in terms of a Markovian, one parameter semigroup. In particular, while in [13] the standard weak coupling limit can be reached only in the asymptotic, long-time regime, in the treatment presented below it can always be obtained for any time by letting the coarse-graining parameter become large.
One then focuses on the changes of the reduced state \( \tilde{\rho}(t) \equiv \text{Tr}_\Phi[\tilde{\rho}_{\text{tot}}(t)] \) over a time interval \( \Delta t \); by taking the trace over the field variables of the integrated version of the equation (7) one gets (to lowest order in \( \lambda \)):

\[
\frac{\tilde{\rho}(t + \Delta t) - \tilde{\rho}(t)}{\Delta t} = \frac{1}{\Delta t} \int_t^{t + \Delta t} ds \frac{\partial \tilde{\rho}(s)}{\partial s}
\]

\[
= -\frac{\lambda^2}{\Delta t} \int_t^{t + \Delta t} dt_1 \int_t^{t_1} dt_2 \text{Tr}_\Phi \left( \left[ H'(t_1), \left[ H'(t_2), \tilde{\rho}_{\text{tot}}(t) \right] \right] \right) + O(\lambda^4) .
\]

(8)

One notices that the variation of \( \tilde{\rho}(t) \) starts to become relevant at order \( \lambda^2 \), i.e. on time scales of order \( \tau = \lambda^2 t \). Then, one can equivalently write:

\[
\frac{\tilde{\rho}(t + \Delta t) - \tilde{\rho}(t)}{\Delta t} = \frac{1}{\Delta t} \int_{\tau}^{\tau + \lambda^2 \Delta t} ds \frac{\partial \tilde{\rho}(s/\lambda^2)}{\partial s} ,
\]

(9)

so that in the limit of small \( \lambda \) (and finite \( \Delta t \)) one can readily approximate the r.h.s. of (9) with \( \partial_t \tilde{\rho}(t) \). At this point, one further observes that the environment, containing an infinite number of degrees of freedom, is much larger than the subsystem immersed in it, so that its dynamics is hardly affected by its presence. It is therefore justified to replace in the double integral of (8) the evolved total state \( \tilde{\rho}_{\text{tot}}(t) \) with the product state \( \tilde{\rho}(t) \otimes \rho_\beta \), taking the initial state \( \rho_\beta \) as a reference state for the bath [1]-[3].

Returning to the Schrödinger representation, one finally gets the following linear, Markovian master equation for the two-atom state \( \rho(t) \):

\[
\frac{\partial \rho(t)}{\partial t} = -i[H_S, \rho] + D[\rho(t)] ,
\]

(10)

where the bath-dependent contribution \( D[\rho(t)] \) contains both an Hamiltonian and a dissipative term

\[
D[\rho(t)] = -i[H_{12}, \rho(t)] + \mathcal{L}[\rho(t)] ,
\]

(11)

with

\[
H_{12} = \frac{i\lambda^2}{2\Delta t} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 \theta(s_1 - s_2) \text{Tr}_\Phi \left( \rho_\beta \left[ H'(s_1), H'(s_2) \right] \right) ,
\]

(12)

\[
\mathcal{L}[\rho(t)] = \frac{\lambda^2}{\Delta t} \text{Tr}_\Phi \left[ L (\rho(t) \otimes \rho_\beta) L - \frac{1}{2} \left\{ L^2, \rho(t) \otimes \rho_\beta \right\} \right], \quad L = \int_0^{\Delta t} ds H'(s) ,
\]

(13)

the curly brackets representing the anticommutator, while \( \theta(s) \) is the step function.

It is important to observe that, for any interval \( \Delta t \), the master equation (10)-(13) generates a quantum dynamical semigroup of completely positive maps. Indeed, the generator in the r.h.s. of (11), besides the Hamiltonian piece, contains a dissipative term which turns out to be itself completely positive, being the composition of two completely positive maps, the trace over the environment degrees of freedom and a linear operator on the total system, written in
canonical Stinespring form \[33, 34, 6\]. Notice that, on the contrary, in the usual weak coupling limit approach to the derivation of a Markovian master equation, complete positivity is ensured by an ergodic average prescription, that, as mentioned in the Introduction, eliminates fast oscillating terms \[10, 30\]; in the present formalism, this corresponds to letting the time coarse-graining parameter $\Delta t$ going to infinity: it is thus applicable only to environments with sharp decaying correlations. In the following, we shall instead keep $\Delta t$ finite and consider therefore more general situations.

### 3 Master equation

For the case at hand, a more explicit expression for the generator in (11) can be obtained by recalling (2) and (4). Indeed, after straightforward manipulations, the master equation driving the dissipative dynamics of the two atoms state takes the following Kossakowski-Lindblad form \[35, 36\]

$$\frac{\partial \rho(t)}{\partial t} = -i [H_{\text{eff}}, \rho(t)] + \mathcal{L}[\rho(t)] ,$$

with

$$H_{\text{eff}} = H_S - \frac{i}{2} \sum_{\alpha, \beta = 1}^{2} \sum_{i,j=1}^{3} H^{(\alpha\beta)}_{ij} \sigma_i^{(\alpha)} \sigma_j^{(\beta)} ,$$

and

$$\mathcal{L}[\rho] = \sum_{\alpha, \beta = 1}^{2} \sum_{i,j=1}^{3} C^{(\alpha\beta)}_{ij} \left[ \sigma_j^{(\beta)} \rho \sigma_i^{(\alpha)} - \frac{1}{2} \{ \sigma_i^{(\alpha)} \sigma_j^{(\beta)}, \rho \} \right] .$$

The coefficients of the Kossakowski matrix $C^{(\alpha\beta)}_{ij}$ and of the effective Hamiltonian $H_{\text{eff}}$ are determined by the field correlation functions in the thermal state $\rho_\beta$:

$$G^{(\alpha\beta)}_{ij}(t-t') = \int d^3x \, d^3y \, f^{(\alpha)}(\vec{x}) \, f^{(\beta)}(\vec{y}) \langle \Phi_i(t, \vec{x}) \Phi_j(t', \vec{y}) \rangle ,$$

through their Fourier,

$$G^{(\alpha\beta)}_{ij}(z) = \int_{-\infty}^{\infty} dt \, e^{itz} G^{(\alpha\beta)}_{ij}(t) ,$$

and Hilbert transform,

$$K^{(\alpha\beta)}_{ij}(z) = \int_{-\infty}^{\infty} dt \, \text{sign}(t) \, e^{itz} G^{(\alpha\beta)}_{ij}(t) = \frac{P}{\pi i} \int_{-\infty}^{\infty} dw \, \frac{G^{(\alpha\beta)}_{ij}(w)}{w-z} ,$$

respectively ($P$ indicates principle value).
More specifically, one finds that the Kossakowski matrix reads:

\[
C^{(\alpha \beta)}_{ij} = \lambda^2 \sum_{\xi, \xi' = \{+, -, 0\}} e^{i(\xi \omega_\alpha + \xi' \omega_\beta) \Delta t/2} \psi^{(\xi)}_{ki} \psi^{(\xi')}_{lj} 
\times \frac{\Delta t}{2\pi} \int_{-\infty}^{\infty} d\omega \ G^{(\alpha \beta)}_{kl}(\omega) \sin \left[ \frac{(\xi \omega - \omega_\alpha) \Delta t}{2} \right] \sin \left[ \frac{(\xi' \omega + \omega_\beta) \Delta t}{2} \right],
\]

(20)

where

\[
\psi^{(0)}_{ij} = n_i n_j, \quad \psi^{(\pm)}_{ij} = \frac{1}{2} (\delta_{ij} - n_i n_j \pm i \epsilon_{ijk} n_k),
\]

(21)

are the components of auxiliary three-dimensional tensors, giving the free evolution of the atom operators:

\[
\sigma^{(\alpha)}_i(t) = e^{itH_S} \sigma^{(\alpha)}_i e^{-itH_S} = \sum_{\xi = \{+, -, 0\}} \sum_{j=1}^{3} e^{i\xi \omega_\alpha t} \psi^{(\xi)}_{ij} \sigma^{(\alpha)}_j.
\]

The \(6 \times 6\) matrix \(C^{(\alpha \beta)}_{ij}\) turns out to be non-negative, since, as already mentioned, the evolution generated by (9) is completely positive.\(^3\)

An expression similar to the one in (20) holds also for \(H^{(\alpha \beta)}_{ij}\) in (15), with \(G^{(\alpha \beta)}_{kl}(\omega)\) replaced by \(K^{(\alpha \beta)}_{kl}(\omega)\).

For simplicity, the fields giving rise to the environment are taken to be independent and further assumed to obey a free evolution; in this case, one finds:

\[
\langle \Phi_i(x) \Phi_j(y) \rangle = \text{Tr} \left[ \Phi_i(x) \Phi_j(y) \rho_\beta \right] = \delta_{ij} \ G(x - y),
\]

(22)

where \(G(x - y)\) is the standard four-dimensional Wightmann function for a single relativistic scalar field in a state at inverse temperature \(\beta\) \[32\], that, with the usual \(i\epsilon\) prescription, can be written as:

\[
G(x) = \int \frac{d^4k}{(2\pi)^4} \theta(k^0) \delta(k^2) \left[ (1 + \mathcal{N}(k^0)) e^{-ik \cdot x} + \mathcal{N}(k^0) e^{ik \cdot x} \right] e^{-\epsilon k^0},
\]

(23)

where

\[
\mathcal{N}(k^0) = \frac{1}{e^{\beta k^0} - 1}.
\]

(24)

Although the \(i\epsilon\) prescription, assuring the convergence of the integral in (23), originates from causality requirements, in the present setting it can be related to the finite size of the two atoms. Indeed, the correlations in (17) actually involve the Fourier transform \(\hat{f}(\vec{k}) = \int d^3x e^{i\vec{k} \cdot \vec{x}} f(\vec{x})\) of the shape function \(f(\vec{x})\) in (3); it can be easily computed to be \(\hat{f}(\vec{k}) = e^{-|\vec{k}|^2/2} \). Inserting it back in (17), this contribution can be conveniently attached to the

\(^5\)On the other hand, let us remark that direct use of the standard second order perturbative approximation (e.g. see [8, 9]) often leads to physically inconsistent results [5, 11, 12], giving a finite time evolution for \(\rho(t)\) that in general does not preserve the positivity of probabilities.
definition of the Wightmann function $G(x)$, so that the integrand in (23) gets an extra $e^{-\varepsilon k^0}$ overall factor.

Using (23) and (24), the Fourier transform in (18) can now be explicitly evaluated; taking for simplicity the limit of pointlike atoms, (the size $\varepsilon$ can be taken to vanish since it does not play any more the role of a regularization parameter), one gets:

$$G^{(\alpha \beta)}_{ij}(\omega) = \delta_{ij} G^{(\alpha \beta)}(\omega) \ ,$$

with:

$$G^{(11)}(\omega) = G^{(22)}(\omega) = \frac{\omega}{2\pi \left(1 - e^{-\beta \omega}\right)} \ ,$$

$$G^{(12)}(\omega) = G^{(21)}(\omega) = \frac{\omega}{2\pi \left(1 - e^{-\beta \omega}\right)} \frac{\sin(\omega \ell)}{\ell \omega} \ ,$$

where $\ell$ denotes the modulus of the displacement vector $\vec{\ell}$; then, recalling (19), for the Hilbert transform one similarly finds:

$$K^{(\alpha \beta)}_{ij}(z) = \delta_{ij} K^{(\alpha \beta)}(z) \ , \quad K^{(\alpha \beta)}(z) = \frac{P}{\pi i} \int_{-\infty}^{\infty} dw \frac{G^{(\alpha \beta)}(w)}{w - z} \ .$$

With these results and taking into account that $\sum_k \psi^\dagger_k \psi^\dagger_{kj} = \psi^\dagger \delta(\xi + \xi')$, the Kosakowski matrix takes the more explicit form:

$$C^{(\alpha \beta)}_{ij} = \epsilon^{(\alpha \beta)} C_{ij} - i \epsilon^{(\alpha \beta)} \sum_{k=0}^{\lambda} \epsilon_{ijk} n_k + \left[C^{(\alpha \beta)}_0 - C^{(\alpha \beta)}_+\right] n_i n_j \ ,$$

where

$$C^{(\alpha \beta)}_{\pm} = I^{(\alpha \beta)}_\pm \cos \left(\omega_{\alpha \beta} \Delta t / 2\right) + i I^{(\alpha \beta)}_\pm \sin \left(\omega_{\alpha \beta} \Delta t / 2\right) \ , \quad \omega_{\alpha \beta} \equiv \omega_\alpha - \omega_\beta \ ,$$

with

$$I^{(\alpha \beta)}_\pm = \frac{\Delta t}{4\pi} \int_{-\infty}^{\infty} d\omega \left[G^{(\alpha \beta)}(\omega) \pm G^{(\alpha \beta)}(-\omega)\right] \frac{\sin \left[(\omega - \omega_\alpha)\Delta t / 2\right]}{(\omega - \omega_\alpha)\Delta t / 2} \frac{\sin \left[(\omega - \omega_\beta)\Delta t / 2\right]}{(\omega - \omega_\beta)\Delta t / 2} \ ,$$

while

$$C^{(\alpha \beta)}_0 \equiv I^{(\alpha \beta)}_0 = \frac{\Delta t}{4\pi} \int_{-\infty}^{\infty} d\omega \left[G^{(\alpha \beta)}(\omega) + G^{(\alpha \beta)}(-\omega)\right] \left[\frac{\sin (\omega \Delta t / 2)}{\omega \Delta t / 2}\right]^2 \ .$$

Only the following combinations $G^{(\alpha \beta)}_{\pm}(\omega) = G^{(\alpha \beta)}(\omega) \pm G^{(\alpha \beta)}(-\omega)$ actually occur in the previous integrals, and from the explicit expressions in (26) one obtains:

$$G^{(11)}_+ = G^{(22)}_+ = \frac{\omega}{2\pi} \left[1 + e^{-\beta \omega}\right] \ , \quad G^{(12)}_+ = G^{(21)}_+ = \frac{\omega}{2\pi} \left[1 + e^{-\beta \omega}\right] \frac{\sin(\omega \ell)}{\omega \ell} \ ,$$

$$G^{(11)}_- = G^{(22)}_- = \frac{\omega}{2\pi} \ , \quad G^{(12)}_- = G^{(21)}_- = \frac{\omega}{2\pi} \frac{\sin(\omega \ell)}{\omega \ell} \ ;$$

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they contain the dependence on the bath temperature $1/\beta$ and on the separation $\ell$ between the two atoms. Because of the presence of the Boltzmann factors, the integrals $I^{(\alpha\beta)}_{\pm,0}$ in (30), (31) cannot in general be expressed in terms of elementary functions. However, in the case of a bath at high temperature (i.e. for small $\beta$), the square bracket in (32) behaves as $2/\beta \omega$ and the above integrals can be explicitly evaluated (see the Appendix). In the physical situation for which $\ell \leq \Delta t$ one finds:

$$I^{(\alpha\beta)}_+ = \frac{1}{\pi \beta \omega_{\alpha\beta} \Delta t} \left\{ \sin \left( \frac{\ell \omega_{\alpha}/2}{\ell \omega_{\alpha}/2} \right) \sin \left[ \frac{\omega_{\alpha} \left( 1 - \ell / \Delta t \right) - \omega_{\beta}}{2} \right] \right.$$ 

$$\left. + \sin \left( \frac{\ell \omega_{\beta}/2}{\ell \omega_{\beta}/2} \right) \sin \left[ \frac{\omega_{\alpha} - \omega_{\beta} \left( 1 - \ell / \Delta t \right)}{2} \right] \right\}, \quad (34)$$

$$I^{(\alpha\beta)}_- = \frac{1}{\pi \ell \omega_{\alpha\beta} \Delta t} \sin \left[ \frac{\omega_{\alpha\beta} \left( \Delta t - \ell \right)}{2} \right] \sin \left[ \frac{\ell \left( \omega_{\alpha} + \omega_{\beta} \right)}{2} \right], \quad (35)$$

$$I^{(\alpha\beta)}_0 = \frac{1}{4\pi \beta} \left( 2 - \frac{\ell}{\Delta t} \right). \quad (36)$$

Inserting these results back in (29) and (31), one finally obtains the explicit expression for the Kossakowski matrix $C^{(\alpha\beta)}_{ij}$, in the large temperature limit.

Coming now to the Hamiltonian contribution to the master equation, one sees that the effective Hamiltonian $H_{\text{eff}}$ in (15) can be split into two parts, $H_{\text{eff}} = \bar{H}_S + H_{\text{eff}}^{(12)}$, the first is just a renormalization of the starting system Hamiltonian, while the second one represents an environment induced direct coupling term for the two atoms. The term $\bar{H}_S$ has the same form as the Hamiltonian in (1) but with redefined frequencies

$$\bar{\omega}_{\alpha} = \omega_{\alpha} - i \frac{\Delta t}{2\pi} \int_{-\infty}^{\infty} d\omega \left[ K^{(\alpha\alpha)}(\omega) - K^{(\alpha\alpha)}(-\omega) \right] \frac{\sin \left[ (\omega - \omega_{\alpha}) \Delta t / 2 \right]}{(\omega - \omega_{\alpha}) \Delta t / 2} \right)^2. \quad (37)$$

Recalling the definition of $K^{(\alpha\alpha)}(\omega)$ in (27), one sees that it can be split as:

$$K^{(\alpha\alpha)}(\omega) = \frac{1}{2\pi i} \left[ P \int_{0}^{\infty} dz \frac{z}{z - \omega} \right.$$ 

$$\left. + P \int_{0}^{\infty} dz \frac{1 - e^{\beta z}}{1 - e^{\beta z}} \left( \frac{1}{z + \omega} - \frac{1}{z - \omega} \right) \right]. \quad (38)$$

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6This condition assures that the two atoms actually feel the presence of the quantum fields; indeed, due to relativistic causality [32], the fields would not be able to interact with the atoms in the time interval $\Delta t$ if they were to far apart.

7Since this is an approximated result, positivity of the matrix is not a priori guaranteed and should be formally imposed in order to preserve the properties of the exact expression (20). In particular, positivity of the two diagonal submatrices $C^{(\alpha\alpha)}_{ij}$, requires $\beta \omega_{\alpha}/2 \leq 1$, which are satisfied by the requirement of $\beta$ small.
into a vacuum and a temperature-dependent piece. Although not expressible in terms of simple functions, the temperature dependent second term is a finite, odd function of $\omega$; on the contrary, the remaining, vacuum contribution in (38) results divergent, and therefore so are the shifted frequencies $\tilde{\omega}_\alpha$. As a consequence, the definition of effective Hamiltonian $H_{\text{eff}}$ requires the introduction of a suitable cutoff and a renormalization procedure. This is not a surprise: the appearance of the divergences is due to the non-relativistic treatment of the two-level atoms, while any sensible calculation of energy shifts would have required the use of quantum field theory techniques [7]. In order to make $H_{\text{eff}}$ well defined we follow a simple prescription: perform a suitable temperature independent subtraction, so that the expressions in (37) reproduce the correct quantum field theory result, obtained by considering the external fields in the vacuum state.

The induced two-atom interaction term $H_{\text{eff}}^{(12)}$ can instead be expressed as

$$H_{\text{eff}}^{(12)} = \sum_{i,j=1}^{3} H_{ij}^{(12)} \sigma_i^{(1)} \otimes \sigma_j^{(2)},$$

where

$$H_{ij}^{(12)} = \left( \cos \left[ \frac{\omega_{12} \Delta t}{2} \right] \delta_{ij} + \sin \left[ \frac{\omega_{12} \Delta t}{2} \right] \right) \sum_{k=0}^{3} \epsilon_{ijk} n_k J_{+} + \left( J_{0} - \cos \left[ \frac{\omega_{12} \Delta t}{2} \right] J_{+} \right) n_i n_j,$$  (40)

with

$$J_{+} = -i \frac{\Delta t}{4 \pi} \int_{-\infty}^{\infty} d\omega \left[ K^{(12)}(\omega) + K^{(12)}(-\omega) \right] \sin \left[ (\omega - \omega_1) \Delta t / 2 \right] \sin \left[ (\omega - \omega_2) \Delta t / 2 \right],$$  (41)

$$J_{0} = -i \frac{\Delta t}{8 \pi} \int_{-\infty}^{\infty} d\omega \left[ K^{(12)}(\omega) + K^{(12)}(-\omega) \right] \left[ \frac{\sin \left( \omega \Delta t / 2 \right)}{\omega \Delta t / 2} \right]^2.$$  (42)

Also $K^{(12)}(\omega)$ can be split as in (38) into a temperature dependent term, odd in $\omega$, and a vacuum piece. Clearly, only this second contribution enters the above integrals $J_{+0}$; it is finite (for non vanishing atom separation) and with the help of (32) can be explicitly computed:

$$K^{(12)}(\omega) + K^{(12)}(-\omega) = \frac{P}{2 \pi^2 i} \int_{-\infty}^{\infty} dz \frac{z}{z + \omega} \sin \frac{\ell z}{\ell z} = \frac{1}{2 \pi i} \frac{\cos \ell \omega}{\ell}.$$  (43)

Inserting this result in (41) and (42), one finally obtains, again for $\ell \leq \Delta t$ (see Appendix):

$$J_{+} = -\frac{1}{2 \pi \ell \omega_{12} \Delta t} \cos \left[ \frac{(\omega_1 + \omega_2) \ell}{2} \right] \sin \left[ \frac{\omega_{12} \Delta t}{2} \right], \quad J_{0} = \frac{1}{8 \pi} \left( \frac{1}{\Delta t} - \frac{1}{\ell} \right).$$  (44)

We are now ready to discuss the entanglement properties of the time evolution generated by the master equation (14)-(16).
4 Environment entanglement generation

In order to study the entanglement power of the thermal bath made of free quantum fields, we shall focus on the small $t$ behaviour of the dynamics generated by the equation (14); our aim is to investigate whether the two independent atoms can get entangled by the action of the environment in which they are immersed at the beginning of their dissipative evolution. Then, without loss of generality, one can limit the considerations to pure, separable initial states, and therefore take:

$$\rho(0) = |\varphi\rangle\langle \varphi | \otimes |\psi\rangle\langle \psi | ,$$

(45)

with $|\varphi\rangle$, $|\psi\rangle$ given single atom states; indeed, if the environment is unable to create entanglement out of pure states, it will certainly not correlate their mixtures.

Since we are dealing with a couple of two-level systems, one can use partial transposition as a criterion for entanglement creation [37, 38]. More precisely, the environment is able to create quantum correlations between the two atoms if and only if the operation of partial transposition spoils the positivity of the state $\rho(t)$.

The presence of negative eigenvalues in the partially transposed reduced density matrix $\hat{\rho}(t)$ can be ascertained by looking at the sign of the average

$$\mathcal{A}(t) = \langle \chi | \hat{\rho}(t) | \chi \rangle ,$$

(46)

with $|\chi\rangle$ a four-dimensional vector. Indeed, choose $|\chi\rangle$ to be orthogonal to $|\varphi\rangle \otimes |\psi\rangle$, so that the above average initially banishes, $\mathcal{A}(0) = 0$. Then, the two atoms, initially prepared in a state $\rho(0) \equiv \hat{\rho}(0)$ as in (45), will surely become entangled if $|\chi\rangle$ can be further chosen so that $\partial_t \mathcal{A}(0) < 0$. From this condition, a simple test for entanglement creation involving the elements of the Kossakowski matrix (28) and of the effective interaction Hamiltonian (39) can then be extracted [24, 39]. It explicitly reads:

$$\langle u|C(11)|u \rangle \langle v|C(22)^T|v \rangle < \langle u|\text{Re}(C(12) + iH^{(12)}_{\text{eff}})|v \rangle^2 ,$$

(47)

where $T$ means matrix transposition; the three-dimensional vectors $|u\rangle$ and $|v\rangle$ contain the information about the starting factorized state (45); in fact, their components can be expressed as:

$$u_i = \langle \varphi | \sigma_i | \varphi \rangle , \quad v_i = \langle \psi \perp | \sigma_i | \psi \rangle ,$$

(48)

where $|\varphi \perp\rangle$ and $|\psi \perp\rangle$ are the orthonormal complement to the initial atom states $|\varphi\rangle$ and $|\psi\rangle$, respectively. Therefore, the external quantum fields will be able to initially entangle the two atoms evolving with the Markovian dynamics generated by (14) if there exists an initial state of the form (15) for which the inequality (47) is satisfied.

In order to obtain a manageable expression for it, we first note that, without loss of generality, the unit vector $\vec{n}$ that defines the atom Hamiltonian in (11) can be oriented along the third axis. Further, as initial atom state we shall choose $\rho(0) = | -\rangle\langle - | \otimes | +\rangle\langle + |$, constructed out of the eigenstates of the single atom Hamiltonian, $\sigma_3 | \pm\rangle = \pm | \pm\rangle$. As a consequence, recalling (18), one finds that the three-dimensional vector $|u\rangle$ has components
\[ u_i = \{1, -i, 0\} \text{, and further } v_i = u_i. \] Then, using the explicit expressions for the elements of the Kossakowski matrix \( C^{(a_\beta)}_{ij} \) and of the induced interaction Hamiltonian \( H^{(12)}_{\text{eff}} \), the inequality (47) reduces to
\[
(1 - \frac{\beta \omega_1}{2}) (1 + \frac{\beta \omega_2}{2}) < \pi^2 \beta^2 \left[ (I^{(12)}_+)^2 + 4(J_+)^2 \right]. \tag{49}
\]
Notice that the l.h.s. of this expression is positive, since as discussed in the previous Section, complete positivity requires \( \beta \omega_\alpha / 2 \leq 1 \).

As remarked at the end of Section 2, the parameter \( \Delta t \) identifies the time scale over which the presence of the environment is felt by the system of the two atoms; clearly the weaker the coupling with the environment is, the longer one needs to wait for the bath induced effects to become apparent.

Let us first discuss the standard weak coupling limit approximation; in this case, one actually let the coupling constant \( \lambda \) to approach zero, so that changes in the two-atom density matrix become visible only for infinitely large \( \Delta t \). In this limit however, the two integrals on the r.h.s. of (49) become vanishingly small.\(^8\) Thus, for atoms with unequal frequencies, the inequality can never be satisfied, and thus no entanglement is generated. On the contrary, when the two frequencies coincide, \( \omega_1 = \omega_2 = \omega \), the condition (49) becomes:
\[
1 - \left( \frac{\beta \omega}{2} \right)^2 < \left[ \frac{\sin (\omega \ell)}{\omega \ell} \right]^2 + \frac{\beta^2}{4} \left[ \frac{\cos (\omega \ell)}{\ell} \right]^2. \tag{50}
\]
This result generalizes the one discussed in [26], where the contribution of the environment induced interaction Hamiltonian (the second term in the r.h.s. of (49)) was neglected. In particular, one sees that, in this case, for any given (small) inverse temperature \( \beta \), there is always an atom separation \( \ell \) below which the inequality (50) is satisfied, and therefore entanglement created between the two atoms. This phenomenon is forbidden only for infinitely large separation or infinitely large temperature, in which case the environment induced decoherence and noisy effects dominate.

The sharp dependence of the entanglement capability of the environment on the atom frequencies in the weak coupling limit approach is however striking; it originates in the elimination of fast oscillating terms in the reduced two-atom dynamics through an ergodic average, a procedure that is justified only in the limit of a vanishing \( \lambda \) and very fast decay correlations in the environment.

Instead, if the coupling of the atoms to the bath is weak, but not infinitesimally small, environment induced changes in the atom density matrix \( \rho(t) \) can be seen on finite time intervals \( \Delta t \). In this case, it is the full condition (19) that regulates the entanglement capability of the thermal bath. One can check that indeed this inequality can be satisfied even for \( \omega_1 \neq \omega_2 \), and therefore that a bath made of thermal quantum fields can correlate two unequal atoms.

\(^8\)Indeed, in the limit \( \Delta t \to \infty \) both integrals vanish since, for \( \omega_1 \neq \omega_2 \), the two functions \( \sin((\omega - \omega_1)\Delta t/2)/(\omega - \omega_1)\Delta t/2 \) and \( \sin((\omega - \omega_2)\Delta t/2)/(\omega - \omega_2)\Delta t/2 \) have disjoint supports.
In order to show this, let us first note that the Hamiltonian contribution in (49), being positive, can only enhance entanglement generation; this is the result of the hermiticity of the induced coupling term $H_{\text{eff}}^{(12)}$ in (39). One can therefore limit the considerations to a simpler inequality, in which the term $(J_+)^2$ is neglected; when this reduced condition is satisfied, also the full one in (49) will clearly be. Recalling (34), and keeping for simplicity only first order terms in $\ell$, the condition for environment assisted entanglement generation reduces to:

$$\left(1 - \frac{\beta \omega_1}{2}\right) \left(1 + \frac{\beta \omega_2}{2}\right) < \left[\sin \left(\frac{\omega_{12} \Delta t/2}{2}\right)\right]^2 - \left(\frac{\ell}{\Delta t}\right) \sin \left(\frac{\omega_{12} \Delta t}{\omega_{12} \Delta t}\right).$$

(51)

For given bath temperature and atom frequencies, this condition is satisfied provided a sufficiently small time interval $\Delta t$ results from the coupling with the environment. Further, the smaller the atom separation is, the easier the condition of (51) will be met.

Finally, note that, in contrast to situation encountered in the weak coupling limit approximation, here there is no sharp change between the regime of entanglement generation and the one of solely decoherence; the transition is smoothly regulated by the coarse graining parameter $\Delta t$, i.e. ultimately by the strength of the coupling of the atoms to the environment.

5 Discussion

We have seen that two atoms, prepared initially in a separable state, can get entangled as a result of their independent interaction with a common bath made of thermal quantum fields even when their internal frequencies are unequal. This result is based on a novel Markovian approximation of the reduced atom dynamics, that allows an explicit dependence on the time scale $\Delta t$, measuring the interval over which the atoms feel the presence of the environment.

This conclusion contrasts with the one obtained through the usual weak coupling limit approach to the atom reduced dynamics; in that case, the entanglement power of the external environment is reduced to zero for atoms with unequal frequencies as a consequence of the procedure of taking the ergodic average. In the light of the results presented in the previous Section, this conclusion appears however a mathematical artifact, originating in letting $\lambda$ to zero and $\Delta t$ to infinity conditions hardly met in actual physical situations. Instead, for weakly coupled baths with finite $\Delta t$, environment assisted entanglement generation is always allowed, and can be controlled through the external parameters, the bath inverse temperature $\beta$ and the atom spatial separation $\ell$.

In the high temperature case (i.e. $\beta$ small) and arbitrary $\Delta t$, we have explicitly shown that this conclusion holds because of the condition (51). Similarly, in situations allowing a
large, but finite $\Delta t$, a different approximation of the full entanglement condition (47) can be given; it can be obtained using techniques and procedures analogous to the ones discussed in the previous Sections. Neglecting again the Hamiltonian contribution, one explicitly finds:

\[
(1 - R_1)(1 + R_2) < \frac{1}{4} \left[ \frac{\sin (\omega_{12} \Delta t/2)}{\omega_{12} \Delta t/2} \right]^2 \left[ \left( \frac{\omega_1 R_2}{\omega_2 R_1} \right)^{1/2} S_1 + \left( \frac{\omega_2 R_1}{\omega_1 R_2} \right)^{1/2} S_2 \right]^2,
\]

where

\[
R_\alpha = \frac{1 - e^{-\beta \omega_\alpha}}{1 + e^{-\beta \omega_\alpha}}, \quad S_\alpha = \frac{\sin (\omega_\alpha \ell)}{\omega_\alpha \ell}, \quad \alpha = 1, 2.
\]

It is a further generalization of the condition discussed in \[26\] in the case of identical atoms, to which it reduces for $\omega_1 = \omega_2$ and $\Delta t$ infinite. Although valid only for large (but finite) $\Delta t$, it can always be satisfied with suitably chosen $\beta$ and $\ell$. In particular, (52) is always true in the zero temperature case, i.e. in the limit $\beta \to \infty$; in other words, a bath made of quantum fields in the vacuum state is always able to generate entanglement, for any finite spatial separation of the two atoms.

All the above considerations are based on the condition (47) for entanglement enhancement; when satisfied, it assures that quantum correlations among the two atoms are generated as soon as $t > 0$. It is however unable to determine the fate of this quantum correlations as time increases and in particular in the asymptotically long time regime. On general grounds, one expects that the effects of decoherence and dissipation that counteract entanglement production be dominant at large times, so that no entanglement is left in the end. There are however instances in which the entanglement generated at the beginning of the evolution persists also for asymptotically long times \[27, 28, 5\]. In order to fully clarify this situation, a complete study and classification of the set of the equilibrium states of the refined master equation (10)-(13) is necessary\[1\]. Work on this topic is presently in progress and will be reported elsewhere.

Appendix

We indicate here how to compute the integrals that appear in the expressions of the Kos-sakowski matrix $C_{ij}^{(\alpha \beta)}$, (30) and (31), and in that of the effective Hamiltonian interaction term $H_{\text{eff}}^{(12)}$, (41) and (42). In the high temperature case, the explicit evaluation of (30)

\[\text{Only partial results on the classification the equilibrium states of completely positive quantum dynamical semigroups have been so far obtained \[2, 40\].}\]
involves the computation of integrals of the following two types:

\[
I_1 = \int_{-\infty}^{+\infty} dx \sin(cx) \frac{\sin(x-a)}{x-a} \frac{\sin(x-b)}{x-b},
\]

\[
I_2 = \int_{-\infty}^{+\infty} dx \frac{\sin(cx) \sin(x-a)}{x-a} \frac{\sin(x-b)}{x-b},
\]

with \(a, b, c\) positive constants. By decomposing the products of trigonometric functions in terms of linear combinations of sines and cosines, one can split e.g. \(I_1\) into the sum of three simpler integrals:

\[
I_1 = I_0 + I(c) - I(-c), \tag{56}
\]

with

\[
I_0 = \frac{\cos(a-b)}{2} \int_{-\infty}^{+\infty} dx \frac{\sin(cx)}{(x-a)(x-b)}, \quad I(c) = \frac{1}{4} \int_{-\infty}^{+\infty} dx \frac{\sin[(2-c)x-a-b]}{(x-a)(x-b)}. \tag{57}
\]

By first changing the integration variable to \(y = (2-c)x - a - b\) in \(I(c)\), with \(c \leq 2\), and then reducing the denominators in partial fractions in both integrands, one can express \(I_0\) and \(I(c)\) as combinations of the following integral (e.g. see [41]):

\[
\int_{-\infty}^{+\infty} dx \frac{\sin(\alpha x)}{x+z} = \pi \cos(\alpha z), \quad \alpha > 0. \tag{58}
\]

Explicitly, one finds:

\[
I_0 = -\pi \frac{\cos(a-b)}{a-b} \sin [(a+b)c/2] \sin [(a-b)c/2], \tag{59}
\]

\[
I(c) = \frac{\pi}{4(a-b)} \left( \cos [a(c-1)+b] - \cos [a+b(c-1)] \right), \tag{60}
\]

so that, recalling (56), one finally obtains:

\[
I_1 = \pi \sin \left[ \frac{(a+b)c}{2} \right] \sin \left[ \frac{(a-b)(1-c/2)}{a-b} \right]. \tag{61}
\]

This result holds for \(c \leq 2\); when \(c > 2\), one is forced to use a different integration variable in the expression of \(I(c)\) in (57), \(y' = (c-2)x + a + b\), and as a result ends up with a vanishing value for \(I_1\). As a function of the parameter \(c\), the integral \(I_1\) is however continuous, since the expression in (61) also vanishes at the boundary point \(c = 2\).

From the result (61), one further obtains:

\[
\lim_{c \to 0} \left( \frac{I_1}{c} \right) = \int_{-\infty}^{+\infty} dx \frac{\sin(x-a)}{x-a} \frac{\sin(x-b)}{x-b} = \pi \left( \frac{a+b}{2} \right) \frac{\sin(a-b)}{a-b}. \tag{62}
\]
The integral $I_2$ in (55) can be evaluated using similar manipulations. When $c \leq 2$, one explicitly finds

$$I_2 = \frac{\pi}{a - b} \left( \frac{\sin(ac/2)}{a} \sin [a(1-c/2) - b] + \frac{\sin(bc/2)}{b} \sin [a - b(1-c/2)] \right),$$

(63)

while for $c > 2$, a simpler expression holds:

$$I_2 = \pi \frac{\sin a}{a} \frac{\sin b}{b}.$$  

(64)

Here again one sees that $I_2$ is a continuous function of $c$, since the expression in (63) reduces to the one in (64) at the boundary value $c = 2$. Further, from the expression in (63), one easily obtains the following limiting results:

$$\lim_{c \to 0} \left( \frac{I_2}{c} \right) = \int_{-\infty}^{+\infty} dx \frac{\sin(x - a)}{x - a} \frac{\sin(x - b)}{x - b} = \pi \frac{\sin(a - b)}{a - b}.$$ 

(65)

and similarly,

$$\lim_{a,b \to 0} I_2 = \int_{-\infty}^{+\infty} dx \frac{\sin(c x)}{x} \left( \frac{\sin x}{x} \right)^2 = \pi c \left( 1 - \frac{c}{4} \right).$$ 

(66)

The integrals appearing in the evaluation of the Hamiltonian contribution $H_{\text{eff}}^{(12)}$ can instead be all reduced to expressions of the form:

$$J = \int_{-\infty}^{+\infty} dx \cos(c x) \frac{\sin(x - a)}{x - a} \frac{\sin(x - b)}{x - b}.$$ 

(67)

With the help of manipulations similar to the one used above, $J$ can be reduced to combinations of the following integral [41]:

$$\int_{-\infty}^{+\infty} dx \frac{\cos(a x)}{x + z} = \pi \sin(\alpha z), \quad \alpha > 0.$$ 

(68)

When $c \leq 2$, the integral in (67) can be cast in the following form:

$$J = \pi \cos \left[ \frac{(a + b)c}{2} \right] \frac{\sin [(a - b)(1-c/2)]}{(a - b)},$$ 

(69)

while it vanishes for $c > 2$. In the limit of vanishing $a$ and $b$, it reduces to $J = \pi(1 - c/2)$. 

16
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