THE NONCOMMUTATIVE SPECIAL RELATIVITY

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Abstract: We adapt the axioms of the quantum mechanics to the quantum Minkowski space-time coordinates and their transformations under the quantum Lorentz group to show how we can formulate the noncommutative special relativity and its quantum physical observables. We establish in this formalism the quantum analog of the lifetime dilatation formula and the relativistic relations between the energy-momentum four-vector and the mass and the velocity. From the explicit construction of states, we establish the causality principle in the noncommutative special relativity and show that for a free particle moving in the quantum Minkowski space-time, only the length of the velocity and one of its components can be measured exactly and simultaneously. In addition these observables present discret spectrums which imply quantized lifetimes of moving unstable particles.

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1 Introduction

It is well known that a given physical theory describes well only a limited class of phenomena measured in certain range of precision and disagrees with other phenomena beyond this range which need a more general theory for their description. In general, the latter involves new fundamental constant which can be viewed as a kind of deformation parameter. The old theory can be recovered in the limit where the new fundamental constant vanishes (which in general means that we do not reach enough precision to detect the effects tied to this fundamental constant). The special relativity and the quantum mechanics provide most famous examples of such a theory generalization. The special relativity may be regarded as a deformed theory of Galilei’s relativity where the deformation parameter is the inverse of the velocity of light and the quantum mechanics is a deformed theory of the classical mechanics with the Planck constant as deformation parameter.

On the other hand, in view of the enormous and unsuccessful effort to overcome the divergence difficulties in the relativistic quantum field theories, it is believed that the framework of special relativity has to be changed.

In the past few years, attention has been paid to formulate the particle evolution in quantum Minkowski space through the construction of the q-analogs of the relativistic plane waves [1] or the Hilbert space representation of a q-deformed Minkowski space [2]. Despite all the theoretical interests, the relevance of the noncommutative Minkowski space-time and its transformations under the quantum Lorentz group to measurable effects in particle physics has not been discussed very much.

The above considerations make especially interesting the study of the noncommutative special relativity in the frame of the quantum Minkowski space-time and its transformations under the quantum Lorentz group to derive measurable observables describing the evolution of particles in the noncommutative space-time.

In this paper, we adapt the quantum mechanics axioms to the quantum coordinates of Minkowski space-time and their transformations under quantum Lorentz group to show how one can derive the quantum physical observables corresponding to those of the usual special relativity.

This paper is organized as follows: In section 2, we recall the basic properties derived from the correspondence between quantum $SL(2, C)$ and Lorentz groups developed in [3]. In section 3, we consider Hilbert spaces on which the generators of the Hopf algebras corresponding to the quantum Lorentz symmetries and the quantum Minkowski space-time act. We establish the transformation rules of states according to the properties of coordinate transformations which are given in terms of tensorial products. The structure of the commutation rules between the generators of the Lorentz symmetries and the coordinates of the Minkowski space-time permits us to introduce the velocity operator and to study the boost from which we establish the quantum analog of the lifetime dilatation formula of unstable particles. We end this section by nothing that this construction is also valid if we replace the coordinates $X_N$ of the quantum Minkowski space-time by the energy-momentum four-vector $P_N$. In this case we obtain the analog of the usual relativistic formulas giving the energy and the vector-momentum in terms of the mass and the velocity. In section 4, we carry on the investigation of the Hilbert space states to describe the evolution of particles in the quantum Minkowski space-time. From this
state investigation, we establish the principle of the causality in the noncommutative special relativity and show that for a free particle moving in the noncommutative space-time, only the length of the velocity and one of its components can be measured exactly. In addition these observables present discreet spectrums which lead to a quantized lifetime of unstable paricles. We conclude this paper in section 5 by discussing the case of particles moving in the light-cone. We also derive directly from this formalism construction of the quantum Lorentz subgroup of the three dimensional rotation of the space coordinates leaving the time coordinate invariant. This permits us to deduce the properties of quantum spheres.

2 The quantum Lorentz group

To explore the different properties of the quantum Lorentz group and the quantum Minkowski space time, it is very convenient to represent the generators $\Lambda^M_N (N, M = 0, 1, 2, 3)$ of the quantum Lorentz group in terms of generators $M_{\alpha}^{\beta}$ ($\alpha, \beta = 1, 2$) and $M_{\alpha}^{\beta}$ of the quantum $SL(2, C)$ group which is well known. Let us start by recalling in this section, some basic properties derived from the correspondence between quantum $SL(2, C)$ and Lorentz groups developped in [3].

The unimodularity of $M_{\alpha}^{\beta}$ is expressed by $\varepsilon_{\alpha\beta} M^\alpha_{\gamma} M^\beta_{\delta} = \varepsilon_{\gamma\delta} I_{A\gamma}$, $\varepsilon_{\alpha\beta} M^\alpha_{\gamma} M^\beta_{\delta} = \varepsilon_{\alpha\beta} I_{A\alpha}$, and $\varepsilon_{\alpha\beta} M^\alpha_{\gamma} M^\beta_{\delta} = \varepsilon_{\gamma\delta} I_{A\alpha}$ where $I_{A\alpha}$ is the unity of the $\ast$ algebra $A_{\alpha}$ generated by $M^\beta_{\alpha}$ and the spinor metric $\varepsilon_{\alpha\beta}$ and its inverse $\varepsilon^{\alpha\beta}$ ($\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta} \varepsilon_{\delta\alpha}$) satisfy the conditions $\varepsilon_{\alpha\beta} \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \varepsilon_{\beta\alpha} = -\varepsilon_{\beta\alpha} \varepsilon_{\beta\alpha} = -\varepsilon_{\beta\alpha} \varepsilon_{\beta\alpha}$.

The commutation rules are given by $M_{\alpha}^{\gamma} (f_{\pm\gamma} \ast a) = (a \ast f_{\pm\gamma}) M_{\alpha}^{\gamma}$ and $M_{\alpha}^{\gamma} (f_{\pm\gamma} \ast a) = (a \ast f_{\pm\gamma}) M_{\alpha}^{\gamma}$, for any $a \in A$, where $f_{\pm\alpha}$ and $f_{\pm\alpha} \in A$, dual to the Hopf algebra $A_{\alpha}$, are linear functionals : $A \rightarrow C$. The convolution product is defined as $f \ast a = (id \otimes f) \Delta(a)$ and $a \ast f = (f \otimes id) \Delta(a)$. The subalgebra $A_{\alpha} \subset A$ generated by $f_{\pm\alpha}$ and $f_{\pm\alpha}$ is a Hopf algebra acting on the generators of $A$ as $f_{\pm\alpha} (M_{\alpha}^{\beta}) = a_{\pm\alpha} R_{\alpha\beta}$ and $f_{\pm\alpha} (M_{\alpha}^{\beta}) = a_{\pm\alpha} R_{\alpha\beta}$ [4] where $a \neq 0$ is a real number satisfying $\varepsilon_{\alpha\beta} a_{\alpha\beta} = -(a + a^{-1}) = -Q$. The R-matrices $R_{\alpha\gamma} = \delta_{\alpha\gamma} \delta_{\alpha\beta} + a_{\pm\alpha} \varepsilon_{\alpha\beta} \varepsilon_{\alpha\gamma}$ satisfy the Hecke conditions $(R_{\pm\alpha} + a_{\pm\alpha}) (R_{\pm\alpha} - 1) = 0$ and the Yang-Baxter equations [5].

Besides commutation rules between undotted or dotted generators only, we need to specify commutation rules between undotted and dotted generators which are controlled by R-matrices $f_{\pm\alpha} (M_{\alpha}^{\gamma}) = R_{\alpha\gamma}^{\pm\alpha\beta}$ which we assume they take the form of the R matrix of the quantum $SU(2)$ group. More precisely, the generators $M_{\alpha}^{\beta}$ must satisfy the uniterity condition, $M_{\alpha}^{\beta} = S(M_{\alpha}^{\beta})$ and $M_{\alpha}^{\beta} = S^{-1}(M_{\alpha}^{\beta})$, when they are considered as belonging to a range of functionals $f_{\pm\alpha}^{\beta}$, $f_{\pm\alpha}^{\beta} (M_{\alpha}^{\gamma}) = R_{\alpha\gamma}^{\pm\alpha\beta} = f_{\pm\alpha}^{\beta} (S(M_{\alpha}^{\gamma})) = a_{\pm\alpha} R_{\alpha\gamma}^{\pm\alpha\beta}$ and $f_{\pm\alpha}^{\beta} (S(M_{\alpha}^{\beta})) = R_{\alpha\gamma}^{\beta\alpha} = \varepsilon_{\alpha\beta} f_{\pm\alpha}^{\beta} (M_{\alpha}^{\beta}) \varepsilon_{\alpha\gamma}$. In this case, the spinor metrics must satisfy an additional condition, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ and $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, required by consistency condition between the unitarity and the unimodularity of the quantum $SU(2)$ group generators. The functionals $f_{\pm\alpha}^{\beta}$ and $f_{\pm\alpha}^{\beta}$ satisfy the conditions $f_{\pm\alpha}^{\beta} = f_{\pm\alpha}^{\beta}$ and $f_{\pm\alpha}^{\beta} = f_{\pm\alpha}^{\beta} \circ S = \varepsilon_{\alpha\beta} f_{\pm\alpha}^{\beta} \varepsilon_{\gamma\beta}$ and $f_{\pm\alpha}^{\beta} = f_{\pm\alpha}^{\beta} \circ S^{-1} = \varepsilon_{\alpha\gamma} f_{\pm\alpha}^{\beta} \varepsilon_{\gamma\beta}$.  

3
The completeness relations are given by
$$G_{\pm} = \frac{1}{Q} \varepsilon^{\alpha\nu} \sigma^I_{\alpha} \sigma^\gamma_{\pm} \varepsilon_{\nu\gamma} = \frac{1}{Q} \varepsilon^{\alpha\nu} \sigma^{I^\gamma}_{\alpha} \sigma^J_{\alpha\beta} \varepsilon_{\nu\beta}$$

where $\sigma^I_{\alpha} (n = 1, 2, 3)$ are the usual Pauli matrices, $\sigma^0_{\alpha\beta}$ is the identity matrix and $\bar{\sigma}^{I^\alpha}_{\alpha\beta} = \varepsilon^{\alpha\beta} R^{\alpha\beta}_{\lambda\nu} \varepsilon^{\nu\beta} \sigma^{I^\lambda}_{\alpha\beta}$. The inverse of the Minkowskian metric may be written under the form $G_{\pmIJ} = \frac{1}{Q} Tr(\sigma_J \sigma_{\pm}) = \frac{1}{Q} \varepsilon^{\alpha\nu} \sigma^I_{\alpha} \sigma^\gamma_{\pm} \varepsilon_{\nu\gamma}$ where $\delta_{\pm\alpha\beta} = G_{\pmIJ} \sigma^I_{\alpha\beta}$. The form of the antipode of $\Lambda^M_{\alpha}$ guarantees the orthogonality condition on the generators of quantum Lorentz group as:

$$G_{\pmNM} \Lambda^N_{\alpha} \Lambda^K_{\beta} = G_{\pmKL} I_{KL} \quad \text{and} \quad G_{\pm}^{LK} \Lambda^N_{\alpha} \Lambda^K_{\beta} = G_{\pmNM} I_{\alpha}$$

where $I_{\alpha}$ is the unity of the algebra $L$.

The completeness relations are given by $\sigma^I_{\alpha} \hat{\sigma}^I_{\alpha} \sigma^J_{\alpha\beta} = Q \delta^\alpha_{\beta} \delta^I_{\alpha\beta}$ and $\sigma^I_{\alpha} \sigma^J_{\alpha\beta} = Q \delta^\alpha_{\beta} \delta^I_{\alpha\beta}$ where the undotted and dotted spinorial indices are raised and lowered as $\sigma^I_{\alpha} = \sigma^I_{\rho\delta} \varepsilon^{\rho\alpha}$ and $\sigma^I_{\beta} = \varepsilon^{\beta\rho} \sigma^I_{\alpha\rho}$. These completeness relations may be used to convert a vector to a bispinor and vice versa

$$X_{\alpha\beta} = X^I \sigma^I_{\alpha\beta} \Leftrightarrow X^I = \frac{1}{Q} \varepsilon^{\alpha\nu} X_{\alpha\beta} \sigma^I_{\alpha\beta} \varepsilon_{\nu\gamma} G_{\pmIJ} \quad \text{or} \quad X^I = \frac{1}{Q} \varepsilon^{\beta\nu} \sigma^I_{\alpha\beta} X_{\alpha\beta} \varepsilon_{\nu\gamma}$$

where $X_I$ are real elements of the right invariant basis of a bimodule $M$ over the algebra $L$ which transform under the left coaction $\Delta_L : M \rightarrow L \otimes M$ as

$$\Delta_L(X_I) = \Lambda^K_{\alpha} \otimes X_K.$$ 

The functional $f^I_{\alpha}$ of $SL(2, C)$ induce that of Lorentz group $F^L_{\pm} : L \rightarrow C$ given by $F^L_{\pm} = \frac{1}{Q}(f^I_{\pm} \hat{\sigma} \sigma^I_{\rho\delta} \varepsilon^{\rho\alpha})$. They controle the noncommutativity between elements of $L$ and $X^I$ as

$$\Lambda_I^L (F^L_{\pm} \delta^M_{\alpha}) = (a \ast F^L_{\pm} \delta^M_{\alpha}) \Lambda_I^K,$$

$$X_{\pm} = (a \ast F^L_{\pm} \delta^M_{\alpha}) X_{\pm} \quad \text{and} \quad X(a)_L X(b)_K = F^L_{\pm} (S(\delta^M_{\alpha})) X(a)_L X(b)_K$$

where $a \in L$ and the indices $a, b = \pm$. These functionals satisfy $F^L_{\pm} (\delta^M_{\alpha}) = \delta^L_{\alpha} \delta^M_{\alpha}$, $(F^L_{\pm} (\delta^M_{\alpha}))^* = F^L_{\pm} (S(\delta^M_{\alpha}))$ for any $a \in L$ and the relations $G^{MN}_{\pm} F^L_{\pm} \ast F^K_{\pm} (a) = G^{LK}_{\pmM} \varepsilon(a)$ and $G^{KL}_{\pmN} F^L_{\pm} \ast F^K_{\pm} (a) = G^{MN}_{\pmL} \varepsilon(a)$ which imply that the length of the quantum four-vector, $G^{LK}_{\pmL} X_L X_K$, is bi-invariant, central and real. Since it commutes with everything, it is of the form $G^{LK}_{\pmL} X_L X_K =$
\(-\tau^2 I_L\) where \(\tau^2\) is a real number.

We can also show that the quantum symmetrization of the Minkowski metric is given by 
\(R_{KL}^{\pm NM} G_{KL} = G_{\pm}^{NM}\) and \(R_{KL}^{\pm NM} G_{\pm KL} = G_{\pm KL}\) where \(R_{KL}^{\pm NM} = F_{KL}^{\pm MN}(\Lambda_{LM})\) satisfy the Yang-Baxter equations and the cubic Hecke conditions 
\((R_{KL}^{\pm} + a^{\pm 2})(R_{KL}^{\pm} + a^{\mp 2})(R_{KL}^{\pm} - 1) = 0\).

In following we shall consider the right invariant basis \(X_I = X_{aI}\) of the bimodule algebra \(M\) as a quantum coordinate system of the Minkowski space-time \(M\) provided with the metric 
\(G^{IJ} = G_{+}^{IJ}\). \(X_0\) represents the time operator and \(X_i\) \((i = 1, 2, 3)\) represent the space coordinate operators.

To make the explicit calculation of the different commutation rules, we take an adequate choice of Pauli hermitian matrices of the form 

\[
\sigma_{\alpha\beta}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\alpha\beta}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\alpha\beta}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\alpha\beta}^3 = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}
\]

and the spinorial metrics \(\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = \epsilon_{\beta\dot{\alpha}} = -\epsilon_{\dot{\alpha}\alpha} = \left( \begin{array}{cc} 0 & -q^{-\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{array} \right)\) which presents an advantage arising from the fact, in this representation, we have \(\sigma_0^{\dot{\alpha}\beta} = -\sigma_{\alpha\beta}^0 = -\delta_\alpha^\beta, \sigma_{N\dot{\alpha}\alpha} = \sigma_{NI1} + \sigma_{N22} = -(q + q^{-1})\delta_N^0 = -Q\delta_N^0\) and \(\sigma_{N\dot{\alpha}\dot{\alpha}} = Q\delta_N^0\). We shall see in the next section that these properties lead directly to the restriction of the quantum lorentz group to the quantum subgroup of the three dimensional space rotations by restricting the quantum \(SL(2, C)\) group to the \(SU(2)\) group. These representations also give a form of the quantum metric \(G^{NM}\) into two independent blocks, one for the time component \(X_0\) and the other for space components \(X_k\) \((k = 1, 2, 3)\). The non vanishing elements of the metric are \(G^{00} = -q^{-\frac{1}{2}}, G^{11} = G^{22} = G^{33} = q^{\frac{1}{2}}, G^{12} = -G^{21} = -iq^{\frac{1}{2}}(q^{-\frac{1}{2}} - \frac{1}{q})\) and the non vanishing elements of its inverse are \(G^{00} = -q^{\frac{3}{2}}, G^{11} = G^{22} = q^{-\frac{1}{2}}q^{\frac{3}{2}}, G^{33} = q^{\frac{1}{2}}, G^{12} = -G^{21} = iq^{\frac{1}{2}}(q^{-\frac{1}{2}} - \frac{1}{q})\). In the classical limit \(q = 1\), this metric reduces to the classical Minkowski metric with signature \((- , + , + , + )\).

### 3 The lifetime dilatation in the quantum Minkowski space-time

Before to show how to derive \(q\)-analog physical properties of the usual special relativity, let us investigate the properties of the \(R\) matrix and its consequences on the different commutation rules. Let us recall that as a range of the functionals \(f^{\beta}_{\alpha}\) the generators \(M^{\alpha\beta}\) of the Hopf algebra \(A\) satisfy unitarity condition. Hence as a range of the functionals \(F_{NM}^M\), the generators of \(L\) have the following properties

\[
\Lambda_N^0 = \frac{1}{Q} \sigma_{N\dot{\gamma}}^0 M_{\alpha} \sigma_{\alpha}^0 S(M_{\rho}^{\beta}) \epsilon_{\dot{\gamma}\dot{\alpha}} = \frac{1}{Q} \sigma_{N\dot{\gamma}}^0 \epsilon_{\dot{\gamma}\dot{\alpha}} = -\frac{1}{Q} \sigma_{N\dot{\gamma}\dot{\gamma}}^0 = \delta_N^0
\]

(5)
\[ \Lambda_0^M = \frac{1}{Q} \varepsilon_\gamma \delta^0_{\sigma} \sigma^M_{\alpha} S(M^\beta_\rho) \varepsilon_\gamma \delta^0_{\sigma} = \frac{1}{Q} \varepsilon^\alpha_\gamma M^\alpha_\sigma \sigma^M_{\rho} \varepsilon_\rho \delta \]

from which, we get

\[ F^M_N(\Lambda^0_0 P) = \delta^M_N \delta^0_P = R_{NP}^0, \text{ and } F^M_N(\Lambda^0_0 Q) = \delta^M_N \delta^Q_0 = R_{NP}^Q. \] (7)

The quantum symmetrization of the coordinates may be written, from the right relation of (4), as

\[ X_L X_K = R_{LK}^NM X_N X_M \] (8)

leading for \( K = 0 \) to

\[ X_L X_0 - R_{L0}^NM X_N X_M = X_0 X_L \] (9)

which shows that the time coordinate operator commutes with the space coordinate operator. After a straightforward computation (8) gives the commutation relations between space coordinates as:

\[ X_3 Z - q^2 Z X_3 = (q - q^{-1}) X_0 Z, \]
\[ X_3 \overline{Z} - q^{-2} \overline{Z} X_3 = -q^{-2} (q - q^{-1}) X_0 \overline{Z}, \]
\[ Z \overline{Z} - \overline{Z} Z = (q^2 - q^{-2}) X^2_3 + q^{-1} (q^2 - q^{-2}) X_0 X_3 \] (12)

where \( Z = X_1 + iX_2 \) and \( \overline{Z} = X_1 - iX_2 \). The four-vector length \( G^{IJ} X_I X_J = -\tau^2 I \) may be written of the form

\[ q^{-\frac{1}{2}} X^2_0 - \frac{q^2}{Q} Z \overline{Z} - \frac{q^{-\frac{1}{2}}}{Q} \overline{Z} Z - q^{\frac{1}{2}} X^2_3 = \tau^2. \] (13)

Note that by redefining the Minkowski space-time coordinates as \( C = qX_0 - X_3, D = q^{-1}X_0 + X_3, A = Z \) and \( B = \overline{Z} \) we recover the commutation relations and the four-vector length given in Ref.[2,6,7].

As a consequence of (7), the relation (3) gives, for \( a = \Lambda^0_N \),

\[ \Lambda_L^I \Lambda_N^P F_{I}^{K}(\Lambda_P^0) = \Lambda_L^K \Lambda_N^0 = F_{L}^{I}(\Lambda_N^P) \Lambda_P^0 \Lambda_I^K = R_{LN}^{PI} \Lambda_P^0 \Lambda_I^K \] (14)

which reduces to

\[ \Lambda_L^0 \Lambda_N^0 = R_{LN}^{PI} \Lambda_P^0 \Lambda_I^K \] (15)

for \( K = 0 \) and to

\[ \Lambda_L^K \Lambda_0^0 = \Lambda_0^0 \Lambda_L^K \] (16)
for \( N = 0 \). The commutation relations between the Minkowski space-time coordinates \( X_L \) and the generators of the quantum Lorentz group \( \Lambda^M_N \) are given by the left relation of (4) which gives for \( a = \Lambda^0_N \)

\[
X_L \Lambda^0_N = F^*_L (\Lambda^F_N) \Lambda_P^0 X_K = \mathcal{R}^P_K \Lambda_P^0 X_K. 
\]

(17)

This relation reduces to

\[
X_L \Lambda^0_0 = F^*_L (\Lambda^0_0) \Lambda_P^0 X_K = \Lambda^0_0 X_L 
\]

(18)

for \( N = 0 \).

We are now ready to show that the above commutation relations suffice to construct the physical states and the different \( q \)-deformed observables derived from the quantum boost of particles at rest. Since the coordinates and their transformations are operators, we consider, for the usual quantum mechanics, that the evolution of a free particle in the noncommutative special relativity is described by Hilbert space states which are common eigenstates of a set of commuting elements of the quantum algebras \( \mathcal{M} \) and \( \mathcal{L} \). The corresponding eigenvalues give the measurable quantities tied to the evolution of this particle.

First we see from the relations (16) and (18) that \( \Lambda^0_0 \) commutes with the algebras \( \mathcal{L} \) and \( \mathcal{M} \), then it is a real c-number. Therefore, as in textbooks of special relativity we set \( \Lambda^0_0 = \gamma I_L \) and \( \Lambda^0_I = \gamma V_i \) where \( V_0 = I_L \) and \( V_i \) are the components of the velocity operator. Due to the fact that \( \gamma \) is a real c-number and \( \Lambda^0_I \) are real operators, the components \( V_i \) of the velocity are real operators. From (15) we see that \( \Lambda^0_I \) fulfil the same commutation relations (8) of the coordinates \( X_I \), then \( V_i \) satisfy the commutations rules \( V_L V_K = R^{NM}_{LN} V_N V_M \) which give the same relations (10-12) by replacing \( X_i \) by \( V_i \) and \( X_0 \) by 1. With these notations, the orthogonality relations (1) give

\[
G^{IJ} \Lambda^0_I \Lambda^0_J = \gamma^2 (G^{00} + G^{ij} V_i V_j) = G^{00} 
\]

leading to

\[
\gamma = (1 - |v|^2)^{-\frac{1}{2}}. 
\]

(19)

(20)

Since \( \gamma \) is a real c-number, the length of the velocity, \(-\frac{G^{00}}{G^{00}} V_i V_j = |v|^2 \), is also a real c-number. Then both \( \gamma \) and \( |V|^2 \) can be measured exactly. \( \Lambda^0_0 = (\varepsilon^{\alpha\beta} M^\sigma_\alpha) (\varepsilon^{\alpha\beta} M^\sigma_\alpha)^* \) is a real and positive operator, then \( \gamma > 0 \). From (20) we see that the upper limit of the velocity is 1 (the light velocity). This point will be discussed in more details in the next section.

Now, we show how the states transform under a Lorentz group. To do that, we observe that we are in presence of two quantum algebras, \( \mathcal{L} \) corresponds to the quantum symmetries and \( \mathcal{M} \) is generated by the quantum coordinates of the Minkowski space-time. Since the coordinates transform through the left coaction (2) as a tensor product \( \mathcal{L} \otimes \mathcal{M} \), we consider the space states of positive norm as a tensor product \( \mathcal{H}_L \otimes \mathcal{H}_M \) of two Hilbert spaces in which the algebras \( \mathcal{L} \) and \( \mathcal{M} \) act respectively (space of representations of the coordinate transformations). Then in
this formalism we have to construct bases for $\mathcal{H}_M$ and $\mathcal{H}_L$ which are eingenstate of the set of commuting elements of algebras $\mathcal{L}$ and $\mathcal{M}$ separately.

From the commutation rules of the coordinates (9-12), we see that the set of commuting operators are the proper time $\tau^2$, the time component $X_0$ and one of the space components, for example $X_3$. Then in a fixed reference system, a particle is described by a state belonging to the Hilbert space $\mathcal{H}_M$ labeled by the eigenvalues $\tau^2$, $t$ and $x_3$, $|\mathcal{P}\rangle = |t, x_3, \tau^2\rangle$ eingenstate of $X_0$, $X_3$ and $\tau^2$ respectively

$$X_0|\mathcal{P}\rangle = t|\mathcal{P}\rangle, \quad X_3|\mathcal{P}\rangle = x_3|\mathcal{P}\rangle, \quad \tau^2|\mathcal{P}\rangle = \tau^2|\mathcal{P}\rangle. \quad (21)$$

This state describes the evolution of a particle seen by an observer $O$ in a coordinate system $X_N$. For a second observer $O'$ connected with the observer $O$ by a quantum Lorentz transformation, the coordinate system tied to $O'$ is given by $X'_{N} = \Delta_L(X_N) = \Lambda_N^M \otimes X_M$ which fulfil the same commutation rules (9-12) as $X_N$. Then the state $|\mathcal{P}'\rangle$, describing the same particle seen by the observer $O'$ is also labeled by the eigenvalues of $\tau^2$, $X'_0$ and $X'_3$ as $|\mathcal{P}'\rangle = |t', x'_3, \tau^2\rangle \in \mathcal{H}_M$ satisfying:

$$X'_0|\mathcal{P}'\rangle = t'|\mathcal{P}'\rangle, \quad X'_3|\mathcal{P}'\rangle = x'_3|\mathcal{P}'\rangle, \quad \tau^2|\mathcal{P}'\rangle = \tau^2|\mathcal{P}'\rangle. \quad (22)$$

Since the coordinates transform with tensorial products as $X'_N = \Lambda_N^M \otimes X_M$ it is natural to suppose that the Hilbert state $|\mathcal{P}\rangle$ transforms into $|\mathcal{P}'\rangle$ as

$$|\mathcal{P}'\rangle = |\text{sym}_q\rangle \otimes |\mathcal{P}\rangle \quad (23)$$

where $|\text{sym}_q\rangle$ is a basis of the Hilbert space $\mathcal{H}_L$ where the generators $\Lambda_N^M$ of the quantum Lorentz group are represented. Note that:

- The state transformation (23) are different from those of the usual quantum mechanics where the states transform with the unitary transformations $U(G)$ of the classical group $G$ as $|\Psi'\rangle = U(G)|\Psi\rangle$.

- At this stage there is no relation between the eigenvalues of space coordinates and those of Lorentz group as velocities. So (23) may be regarded as the quantum analog of a point in the configuration space of the classical mechanics.

Under the quantum Lorentz transformation, the coordinates $X'_N$ act on $|\mathcal{P}'\rangle$ as:

$$X'_0|\mathcal{P}'\rangle = (\Lambda_0^0 \otimes X_0)|\mathcal{P}'\rangle + (\Lambda_0^k \otimes X_k)|\mathcal{P}'\rangle = \Lambda_0^0|\text{sym}_q\rangle \otimes X_0|\mathcal{P}\rangle + \Lambda_0^k|\text{sym}_q\rangle \otimes X_k|\mathcal{P}\rangle, \quad (24)$$
$$X'_i|\mathcal{P}'\rangle = (\Lambda_i^0 \otimes X_0)|\mathcal{P}'\rangle + (\Lambda_i^k \otimes X_k)|\mathcal{P}'\rangle = \Lambda_i^0|\text{sym}_q\rangle \otimes X_0|\mathcal{P}\rangle + \Lambda_i^k|\text{sym}_q\rangle \otimes X_k|\mathcal{P}\rangle. \quad (25)$$

In the following we consider the simplest case where a particle at rest is boosted with a velocity $v$. Let $|\mathcal{P}_0\rangle = |t, 0, \tau^2\rangle$ a state describing a particle at rest. This state satisfy beside $X_0|t, 0, \tau^2\rangle = |t, 0, \tau^2\rangle$, $\tau^2|t, 0, \tau^2\rangle = \tau^2|t, 0, \tau^2\rangle$ and $X_3|t, 0, \tau^2\rangle = 0|t, 0, \tau^2\rangle$ two additionnal conditions

$$X_1|\mathcal{P}\rangle = 0|\mathcal{P}\rangle, \quad X_2|\mathcal{P}\rangle = 0|\mathcal{P}\rangle. \quad (26)$$
The latter relations are possible because of the homogeneous feature of the commutation rules (10-12) and do not disagree with (13). In the next section, we shall show that this rest state is unique. By using the properties (26) of the rest state, the transformations (24-25) reduce to

\[ X_0'|P' = t'|P' = \Lambda_0^0|\text{sym}_q \otimes X_0|P_0 = \gamma|\text{sym}_q \otimes t|P_0 = \gamma t'|P', \tag{27} \]
\[ X_i'|P' = \Lambda_i^0|\text{sym}_q \otimes X_0|P_0 = V_i \gamma|\text{sym}_q \otimes t|P_0 = V_i \gamma t'|P' = V_i t'|P'. \tag{28} \]

These transformations require the knowledge of the commutation rules between \( \Lambda_N^0 = V_N \gamma \) only, which can be deduced from (9-12). Then as the coordinate system, the state describing the quantum boost are labeled by the length \( |v|^2 \), or \( \gamma \) and its component \( v_3 \). And, therefore the transformed Hilbert space state may be written under the form of tensor product of the rest state \( |P_0 \rangle \) and the boost state labelled by \( v_3 \) and the length of the velocity \( |v|^2 \) eigenvalues of the common eigenstate \( |v_3, |v|^2\rangle \) of \( V_3 \) and \( -\frac{G_{ij}}{G_{00}} V_i V_j \) as:

\[ |P'\rangle = |v_3, |v|^2\rangle \otimes |P_0 \rangle. \tag{29} \]

On this state, the transformation (27-28) lead to

\[ X_0'|P' = t'|P' = (\gamma \otimes t)|P' \text{ and } X_3'|P' = x_3'|P' = (v_3 \otimes \gamma t)|P' = v_3 t'|P' \tag{30} \]

which show the usual relation \( x_3' = v_3 t \) between the coordinate \( x_3 \) and the component \( v_3 \) of the velocity and the time for a free particle moving with the velocity \( |v| \). Note that we have assumed \( t \geq 0, \gamma \geq 0 \) then \( t' = \gamma t \geq 0 \). On the other hand the combination of the invariance of the proper time \( \tau^2 \) with (26-28) and their conjugate \( \langle P|X_i' = \langle P|V_i t \rangle \) gives

\[
\langle P'|G^{ij}X_i'X_j'|P\rangle = \langle P'|G^{00}(X_0'^2 + \frac{G^{ij}}{G^{00}} X_i'X_j')|P\rangle = \langle P'|G^{00}(t^2 + \frac{G^{ij}}{G^{00}} V_i V_j t^2)|P\rangle = \\
\langle P'|G^{00}(t^2 - |v|^2 t^2)|P\rangle = G^{00}(t^2 - |v|^2 t^2) = \\
\langle v_3, |v|^2|v_3, |v|^2\otimes \langle P_0|G^{ij}X_iX_j|P_0\rangle = \langle v_3, |v|^2|v_3, |v|^2\otimes \langle P_0|G^{00}t^2|P_0\rangle = \\
I \otimes \frac{G^{00}t^2}{G_{00}t^2} 
\]

leading to the \( q \)-deformed lifetime dilatation in the noncommutative special relativity

\[ t' = \frac{t}{1 - |v|^2 t^2}. \tag{31} \]

In the semi classical limit \( q \simeq 1 \pm \kappa + O(\kappa^2) \), where \( |v|_{q=0} = |v| \) denotes the classical velocity, the correction to the classical formula of the lifetime dilatation of unstable particles \( \Delta(t_{cl}) \) is given by

\[ \Delta(t_{q}' ) \simeq \Delta(t_{cl}) \pm \kappa \Delta(t_{cl}) \frac{|v|_{cl}^2}{1 - |v|_{cl}^2} + O(\kappa^2). \tag{32} \]

Note that we can also consider the bicovariant bimodule \( M \) over \( L \) where, instead of space-time coordinates \( X_N \), we take as basis of vector space of all the right invariant elements of \( M \) the
coordinates $P_N$ of the energy-momentum four-vector. $P_0$ and $P_n$ are identified to the energy operator and the coordinates of the vector-momentum operator respectively. In this case, we have only to replace in this section $X_N$ by $P_N$, $\tau^2$ by $G_{00}m^2$, the rest state $|\mathcal{P}_0\rangle$ by $|m, 0, m^2\rangle$ and the $|P\rangle$ by $|E, p_3, m^2\rangle$ where $E$ and $p_3$ are real eigenvalues of $P_0$ and $P^3$ respectively and $m^2$ is the square of rest mass which is bi-invariant, real and central. The mass is given by $E^2 - |p|^2 = m^2$ where $|p|^2 = -\frac{G_{ij}}{c^2}P_iP_j$ is the length of the vector-momentum. As for the space-time coordinates we can measure exactly and simultaneously the mass, the energy, the length of the vector-momentum and only one of its components, for instance $p_3$. From momentum version of (30) we may see that the eigenvalues $E$ and $p_3$ are given in terms of the mass and the velocity by

$$E = \frac{m}{(1 - |v|^2)^{\frac{n}{2}}} = m\gamma, \quad p_3 = \frac{mv_3}{(1 - |v|^2)^{\frac{n}{2}}} = mv_3\gamma \quad \text{and} \quad |p|^2 = \frac{m|v|^2}{(1 - |v|^2)^{\frac{n}{2}}} \quad (33)$$

which are the quantum analog of the usual relativistic formulas of the energy and the vector-momentum of free particles moving in the Minkowski space with velocity $v$.

4 Evolution of particles in quantum Minkowski space-time

Having associated the quantum mechanics principles with the properties of the quantum co-ordinates of the Minkowski space-time and their transformations under the quantum Lorentz group, we investigated in this section the states describing the evolution of particles in the quantum the Minkowski space-time. Let $|t, x_3, \tau^2\rangle$ be a state satisfying (21). From (10-11), we see that the states $|n, t, x_3, \tau^2\rangle$ and $|-m, t, x_3, \tau^2\rangle$ given by

$$|n, t, x_3, \tau^2\rangle = \zeta^n\zeta t, x_3, \tau^2\rangle \quad \text{and} \quad |-m, t, x_3, \tau^2\rangle = \zeta^{-n}\zeta t, x_3, \tau^2\rangle \quad (34)$$

are eigenstates of $X_3$ with eigenvalues given respectively by

$$X_3|n, t, x_3, \tau^2\rangle = (q^{2n}x_3 + q^{-1}(q^{2n} - 1)t)|n, t, x_3, \tau^2\rangle = x_3^{(n)}|n, t, x_3, \tau^2\rangle, \quad (35)$$
$$X_3|-m, t, x_3, \tau^2\rangle = (q^{-2m}x_3 + q^{-1}(q^{-2m} - 1)t)|-m, t, x_3, \tau^2\rangle = x_3^{(-m)}|-m, t, x_3, \tau^2\rangle. \quad (36)$$

In following we shall assume that $q > 0$. Therefore we can see that these eigenvalues satisfy

$$x_3^{(n)} \leq x_3 \text{ and } x_3^{(-n)} \geq x_3 \quad \text{for } 0 < q < 1, \quad (37)$$
$$x_3^{(n)} \geq x_3 \text{ and } x_3^{(-n)} \leq x_3 \quad \text{for } q > 1 \quad (38)$$

then for $0 < q < 1 \ (q > 1)$, $\zeta$ and $\bar{\zeta}$ are lowering (raising) and raising (lowering) operators for the eigenvalues of $X_3$. To establish the conditions on the eigenvalues $x_3$, we consider the relations

(a) $\zeta \bar{\zeta} = q^{-2}(X_0 + qX_3)(X_0 - q^{-1}X_3) - q^{-\frac{1}{2}}\tau^2,
(b) \bar{\zeta} \zeta = q^{-2}(X_0 + qX_3)(X_0 - q^3X_3) - q^{-\frac{1}{2}}\tau^2 \quad (39)$
obtained from (13) and the commutation relation (12). Since the states must be of positive norm, the main value of (39) satisfy:

\begin{align*}
(a) & \quad q^{-2}(t + qx_3)(t - q^{-1}x_3) - q^{-2}a^2 \geq 0, \\
(b) & \quad q^{-2}(t + qx_3)(t - q'x_3) - q^{-2}a^2 \geq 0
\end{align*}

(40)

where \( \tau^2 = q^{-2}a^2 \). From (28), we can replace \( X_3 \) by \( V_{ij} \) into the norm of the vector \( X_3 \) to get \( q^{-\frac{3}{2}}t^2 - G^{ij}V_{ij}t^2 = q^{-\frac{3}{2}}a^2 \) or \( t^2(1 - |\gamma|^2) = a^2 \), hence \( a^2 = \frac{t^2}{\gamma^2} \). The conditions (40a-b) define the interval of the eigenvalues \( x_3 \) in which it is possible to construct states of positive norm belonging to Hilbert space. To define this interval, we begin to give the roots of (40a-b)

\begin{align*}
\alpha_{03}^{(a)} & = \frac{(q - q^{-1}) \pm (Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}}}{2} t \quad \text{where } x_{03}^{(a)} < x_{03}^{(a)+}, \\
\alpha_{03}^{(b)} & = -q^{-2}(q - q^{-1}) \pm (Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}} t \quad \text{where } x_{03}^{(b)} < x_{03}^{(b)+}.
\end{align*}

(41)

(42)

The reality of \( x_3 \) requires \( Q^2 - \frac{4}{\gamma^2} \geq 0 \), hence \( \gamma^2 \geq \frac{4}{Q^2} \). It is not very difficult to see that \( x_{03}^{(a)} < x_{03}^{(b)+} \) and \( x_{03}^{(a)+} < x_{03}^{(b)-} \) if \( 0 < q < 1 \) and \( x_{03}^{(b)+} < x_{03}^{(a)}- \) and \( x_{03}^{(b)} < x_{03}^{(a)+} \) if \( q > 1 \). So the conditions (40a-b) are both satisfied in the interval \( (x_{03}^{(a)-} < x_{03}^{(a)+}) \cap (x_{03}^{(b)+} < x_{03}^{(b)-}) \) which must be nonempty. This requires that

\begin{align*}
(a') & \quad x_{03}^{(b)+} < x_{03}^{(a)+} \quad \text{for } 0 < q < 1 \quad \text{and} \quad (b') & \quad x_{03}^{(a)-} < x_{03}^{(b)-} \quad \text{for } q > 1.
\end{align*}

(43)

The relation (43b) is satisfied if \( (q - q^{-1}) - (Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}} \leq -(q - q^{-1}) + Q^{-2}(Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}} \Rightarrow (1 - q^{-2})Q \leq (1 + q^{-2})(Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}} \) or \( 1 - q^{-2}Q^2 \leq (1 + q^{-2})^2(Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}} \) implying

\[ \gamma^2 \geq \frac{Q^2(1 + q^{-2})^2}{Q^2} = 1 \]

(44)

the relation (43a') gives the same condition (44) on \( \gamma \). This relation is necessary for the existence of Hilbert states of positive norm (physical states), hence the causality principle in noncommutative special relativity which states that there exist physical states only for real velocities in the range \( 0 \leq |v|_q \leq c = 1 \).

Now, suppose that \( x_3 \in (x_{03}^{(b)+}, x_{03}^{(a)+}) \), \( (0 < q < 1) \), is an eigenvalue of the eigenstate \(|t, x_3, \tau^2\rangle\) of \( X_3 \). If we consider the state \(| - m, t, x_3, \tau^2 \rangle\) eigenvector of \( X_3 \) with eigenvalue \( q^{-2m}x_3 + q^{-1}(q^{-2m-1}t)t > x_3 \) such that \( q^{-2(m+1)}x_3 + q^{-1}(q^{-2(m+1)-1}t)t > x_{03}^{(a)+} \), since the state norms are positive, then

\[ q^{-2m}x_3 + q^{-1}(q^{-2m-1}t)t = x_{03}^{(a)+} = \frac{(q - q^{-1})t}{2} + \frac{(Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}}}{2} t \]

implying

\[ x_3 = q^{2m}\left(\frac{Q}{2} + \frac{(Q^2 - \frac{4}{\gamma^2})^{\frac{1}{2}}}{2}\right) t - q^{-1}t. \]

(45)
Since the operator $Z$ decreases the value of $x_3$, it exists a number $r \in N^+$ such that $Z^{r} |t, x_3, \tau^2\rangle$ is eigenstate of $X_3$ with eigenvalue $x_{03}^{(b)+}$, hence

$$q^{2r}x_3 + q^{-1}(q^{2r} - 1)t = -q^{-2}\left(\frac{q - q^{-1}}{2} + \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right)t. \quad (46)$$

By setting (45) into (46), we get

$$q^{2(r+m)}\left(\frac{Q}{2} + \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right) = q^{-2}\left(\frac{Q}{2} - \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right) \quad (47)$$

from which we obtain

$$\frac{Q^2}{4}(q^{2(r+m+1)} - 1)^2 = \frac{Q^2 - \frac{4}{\tau^2}}{4}(q^{2(P+m+1)} + 1)^2$$

leading to

$$\gamma^2 = \frac{(q^{(r+m+1)} + q^{-(r+m+1)})^2}{Q^2}. \quad (48)$$

On the other hand, if we consider the state $|n, t, x_3, \tau^2\rangle$ eigenvector of $X_3$ with eigenvalue $q^{2n}x_3 + q^{-1}(q^{2n} - 1)t < x_3$ such that $q^{2(n+1)}x_3 + q^{-1}(q^{2(n+1)} - 1)t < x_{03}^{(b)-}$, we have necessary

$$q^{2n}x_3 + q^{-1}(q^{2n} - 1)t = x_{03}^{(b)-} = -q^{-2}\left(\frac{q - q^{-1}}{2} + \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right)t$$

implying

$$x_3 = q^{-2(n+1)}\left(\frac{Q}{2} - \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right)t - q^{-1}t. \quad (49)$$

Since the operator $\overline{Z}$ increases the value of $x_3$, there exists a number $s \in N^+$ such that $\overline{Z}^s |t, x_3, \tau^2\rangle$ is a eigenstate of $X_3$ with eigenvalue $x_{03}^{(a)+}$ yielding

$$q^{-2s}x_3 + q^{-1}(q^{-2s} - 1)t = \frac{(q - q^{-1})}{2}t + \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}t. \quad (50)$$

By setting (49) into (50), we get

$$q^{-2(s+n+1)}\left(\frac{Q}{2} - \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}\right) = \frac{Q}{2} + \frac{(Q^2 - \frac{4}{\tau^2})^{\frac{1}{2}}}{2}$$

which is equivalent to (47). Therefore $\gamma^2$ is quantized and given by

$$(\gamma^{(L)})^2 = \frac{(q^{(s+n+1)} + q^{-(s+n+1)})^2}{Q^2} = \frac{(q^{(r+m+1)} + q^{-(r+m+1)})^2}{Q^2} = \frac{(q^{(L+1)} + q^{-(L+1)})^2}{Q^2} \quad (51)$$
where $L = 0, 1, 2, \ldots, \infty$. By substituting (51) into (45), we get the states $|t, x_3^{(L,n)}(\tau^2)\rangle$ such that $X_3|t, x_3^{(L,n)}(\tau^2)\rangle = x_3^{(L,n)}|t, x_3^{(L,n)}(\tau^2)\rangle$ where

$$x_3^{(L,n)} = (Q \frac{q^{-(L+1-2n)}}{q^{L+1}} t - q^{-1})$$

(52)

and $n = 0, 1, \ldots, L$. Note that if we set $L = 2l$ and $m = l - n$ we can rewrite the states as $|t, x_3^{(l,m)}(\tau^2)\rangle$ eigenstates of $X_3$ with eigenvalues

$$x_3^{(l,m)} = q^{-1}(Q \frac{q^{2m}}{q^{2(l+1)}} - 1) t = q^{-1}(\frac{q^{2m}}{\gamma(l)} - 1) t$$

(53)

where $\gamma(l) = \frac{q^{2(l+1)} + q^{-(2l+1)}}{q}$, with $l = 0, \frac{1}{2}, 1, \ldots, \infty$ and $m$ runs by integer steps over the range $-l \leq m \leq l$. The substitution of (51) into (49) gives the same states with the same eigenvalues (52). Following the same analysis as above, we can show that the case $q > 1$ gives the same results for $\gamma(l)$ and for $x_3(l,m)$ (52). From (30) and (53) we deduce the quantization of the velocity as

$$|v|^2_q = 1 - \frac{1}{(\gamma(l))^2} \quad \text{and} \quad v_3^{(l,m)} = q^{-1}(\frac{q^{2m}}{\gamma(l)} - 1).$$

(54)

Note that in the case where we consider the energy-momentum four-vector $P_N$ instead of the coordinates $X_N$. The relation (33) combined with (51) and (54) show that the energy $E$ and the component $P_3$ of the vector-momentum present discrete spectrums given by $E(l) = m\gamma(l)$ and $P_3(l,m) = mv_3(\gamma(l))$. The basis of the Hilbert space states of free particle in the momentum representation is given by $|E(l), p_3(l,m), m\rangle$ normalized as $\langle E(l'), p_3(l',m), m|E(l'), p_3(l',m'), m\rangle = \delta_{l,l'}\delta_{n,n'}$, where $l, l' = 0, \frac{1}{2}, 1, \ldots, \infty$ and $-l \leq m, m' \leq l$.

### 5 Discussions and Conclusions

In this paper, we have seen that an adequate association of the quantum mechanics principles with the properties of the quantum Minkowski space-time and its quantum Lorentz transformations make possible the description of the evolution of particles in the noncommutative special relativity. In this formalism we have showed that only the length of the velocity and one of its components can be measured exactly and present discreet spectrums. From the quantum boost, we have established the principle of causality and the $q$-deformed analog of the lifetime dilatation formula for moving unstable particles from which we see that $\frac{\Delta(t_3)}{\Delta(t_2)} = \gamma(l)$, $(l = 0, \frac{1}{2}, 1, \ldots, \infty)$ is quantized. This quantization presents an interesting novelty because as opposite to the customary believe where we consider that the effects of noncommutativity of space may be observed only at very high energy, Planck scale $M_P$, which is beyond the reach of conceivable experiments, in this scenario, the effects of the evolution of free particles in the noncommutative special relativity are expressed in terms of quantized velocities (hence quantized currents for charged particles) which require not very energies but high precision on
Note that the case $L = 0$, $n = 0$ corresponds to $x_3 = 0$ and $\tau^2 = 1$ yielding $|v| = 0$ and corresponds to the unique state describing a particle at rest considered in the section 3.

In the case where $\tau^2 = 0$, corresponding to a particle on the light cone, the roots of (40a-b) reduce to $x_{03}^{(a)-} = -q^{-1}t < x_{03}^{(a)+} = qt$ and $x_{03}^{(b)+} = -q^{-1}t < x_{03}^{(b)-} = q^{-3}t$. Then the conditions (40a-b) are both satisfied in the interval $(x_{03}^{(a)-} = x_{03}^{(b)+}, x_{03}^{(a)+})$ if $0 < q < 1$ or in the interval $(x_{03}^{(a)-} = x_{03}^{(b)+}, x_{03}^{(b)-})$ if $q > 1$. The states of positive norm are given by $|t, x^{(n)}_3, \tau^2\rangle$ where the eigenvalues of $X_3$ are given by

$$x^{(n)}_3 = q^{2n}Qt - q^{-1}t, \quad \text{for } 0 < q \leq 1$$

$$x^{(n)}_3 = q^{-2(n+1)}Qt - q^{-1}t, \quad \text{for } q \geq 1$$

with $n = 0, 1, \ldots, \infty$. Note that the eigenstate $|t, -q^{-1}t, 0\rangle$ is stable under the action of $Z$ and $\bar{Z}$ in the sense that $\forall n, m \in N^+, \bar{Z}^n\bar{Z}^m|t, -q^{-1}t, 0\rangle$ is an eigenstate of $X_3$ with eigenvalue $-q^{-1}t$ hence $v_3 = -q^{-1}$. $\langle t, -q^{-1}t, 0|Z\bar{Z}\bar{Z}|t, -q^{-1}t, 0\rangle$ and $\langle t, -q^{-1}t, 0|Z\bar{Z}|t, -q^{-1}t, 0\rangle$ vanish, therefore, the light velocity reduce to the $|v|^2 = q^2v^2_3 = 1$ which is, by virtue of (31), the upper limit of velocities. We may retrieve this case in the limit $L \to \infty \Rightarrow \tau^2 \to 0$.

To boost again the state $|t, x_3, \tau^2\rangle$ we must know explicitly all the commutation relations of the sixteen generators $\Lambda^M_N$ to get the set of commuting generators and their eigenstate $|sym\rangle$ labelled by its different eigenvalues. This study permits to define the addition rules of velocities in the noncommutative special relativity [8].

When we restrict the generators of the quantum $SL(2, C)$ group to those of the $SU(2)$ by imposing unitarity condition, we get relations (5) and (6) which lead us to the restriction of the Minkowski space-time transformations to the orthogonal transformations of the three dimensional space $R_3$ equipped with the coordinate system $X_i$, $(i = 1, 2, 3)$. These transformations leave invariant the time coordinate $X_0$. In fact, as a consequence of (5) and (6) we have

$$\Delta_L(X_0) = \Lambda^0_0 \otimes X_0 = I \otimes X_0$$

$$\Delta_L(X_i) = \Lambda^j_i \otimes X_j$$

(57)

where the generators $\Lambda^j_i = \frac{1}{4}Q^\alpha_\gamma \sigma_\xi \sigma_\psi M_\alpha \sigma_\sigma \sigma_\beta M_\beta \varepsilon^{\xi\beta} = \frac{1}{4}Q^\alpha_\gamma \sigma_\xi \sigma_\psi M_\alpha \sigma_\sigma \sigma_\beta S(\sigma_\rho \beta)\varepsilon^{\xi\beta}$ generate a Hopf sub-algebra $SO_q(3)$ of $\mathcal{L}$ whose the axiomatic structure is derived from those of $\mathcal{L}$ as $\Delta(\Lambda^j_i) = \Lambda^k_i \otimes \Lambda^j_k \varepsilon(\Lambda^j_i) = \delta^j_i$ and $S(\Lambda^j_i) = G_{iK}^L \Lambda^j_L G^j_i = G_{ik} \Lambda^j_k G^j_i$ where $G^{ij}$ is the restriction of $G^{ij}$ to the quantum space $R_3$ satisfying $G^{ik}G_{kj} = \delta^i_j = G_{ij}G^{ki}$. The form of the antipode of $\Lambda^j_i$ implies the orthogonality properties

$$G^{ij} \Lambda^i_j \Lambda^k_k = G^{ik} \quad \text{and} \quad G_{ik} \Lambda^i_j \Lambda^k_k = G_{ij}.$$
Therefore, $\Lambda^j_i = \frac{1}{Q} \sigma^i \sigma^j \sigma^{\rho} \sigma^{\rho} S(M_{\rho}^{\alpha}) \varepsilon^{\gamma^{\beta}}$ establishes a correspondence between $SU_q(2)$ and $SO_q(3)$ group. In fact the two-dimensional representation of $SU_q(2)$ is given by $\left( \begin{array}{cc} \alpha & q \gamma^* \\ -\gamma & \alpha^* \end{array} \right)$ [9] and in the three dimensional space spanned by the basis $Qe_{-1} = Z, e_0 = X_3$ and $Qe_1 = \overline{Z}$, the generators $\Lambda^j_i = \frac{1}{Q} \sigma^i \sigma^j \sigma^{\rho} \sigma^{\rho} S(M_{\rho}^{\alpha}) \varepsilon^{\gamma^{\beta}}$ write

$$(d_{1,ij})_{i,j=-1,0,1} = \left( \begin{array}{ccc} \alpha^{*2} & -(1 + q^2) \alpha^* \gamma & -q \gamma^2 \\ \gamma^* \alpha^* & 1 - (1 + q^2) \gamma^* \gamma & \alpha \gamma \\ -q \gamma^{*2} & -(1 + q^2) \gamma^* \alpha & \alpha^2 \end{array} \right) \in M_3 \otimes C(SU_q(2))$$

which is an irreducible three-dimensional representation of $SU_q(2)$ considered in [9]. In the other hand if we replace $\overline{Z} = Qe_1, X_3 = e_0, Z = Qe_{-1}, \lambda = (q - q^{-1}) t = (q - q^{-1}) X_0$ and $\rho = q^{-2} t^2 - q^{-\frac{1}{2}} \tau^2$ into (19-22) we retrieve the algebra of the quantum spheres given by (2a-e) of [10]. In the framework developped above, the states $|t, x^{(n)}_3, \tau^2\rangle$ whose eigenvalues are given by (55-56) (case $\tau^2 = 0$) represent a basis of the Hilbert space where the quantum sphere algebra is represented. The fixed time $t$ represents the ray of the sphere.

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