Renormalization group and critical behaviour in gravitational collapse

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We present a general framework for understanding and analyzing critical behaviour in gravitational collapse. We adopt the method of renormalization group, which has the following advantages. (1) It provides a natural explanation for various types of universality and scaling observed in numerical studies. In particular, universality in initial data space and universality for different models are understood in a unified way. (2) It enables us to perform a detailed analysis of time evolution beyond linear perturbation, by providing rigorous controls on nonlinear terms. Under physically reasonable assumptions we prove: (1) Uniqueness of the relevant mode around a fixed point implies universality in initial data space. (2) The critical exponent $\beta_{\text{BH}}$ and the unique positive eigenvalue $\kappa$ of the relevant mode is exactly related by $\beta_{\text{BH}} = \beta/\kappa$, where $\beta$ is a scaling exponent used in calculating the eigenvalue. (3) The above (1) and (2) hold also for discretely self-similar case (replacing “fixed point” with “limit cycle”). (4) Universality for different models holds under a certain condition. [These are summarized as Theorems 1, 2 and 3 of Sec. II.]

According to the framework, we carry out a rather complete (though not mathematically rigorous) analysis for perfect fluids with pressure proportional to density, in a wide range of the adiabatic index $\gamma$. The uniqueness of the relevant mode around a fixed point is established by Lyapunov analyses. This shows that the critical phenomena occurs not only for the radiation fluid but also for perfect fluids with $1 < \gamma \lesssim 1.88$. The accurate values of critical exponents are calculated for the models. In particular, the exponent for the radiation fluid $\beta_{\text{BH}} \simeq 0.35580192$ is also in agreement with that obtained in numerical simulation.

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1. INTRODUCTION

Gravitational collapse with formation of black holes is one of the main problems of classical general relativity. A very important question here is how formation of a black hole depends on initial data, for example, which initial data evolve into black holes and which do not. It is concerned with many important problems such as cosmic censorship $^{[1]}$

Choptuik $^{[2]}$ rigorously analyzed such a problem in a relatively simple system, namely, spherically symmetric scalar field collapse. He proved that the field disperses at later times if the initial field is sufficiently weak, and also found a sufficient condition for black hole formation. However, the behaviour of the field and the space-time for initial data sets around the threshold of the black hole formation remained open, and he posed a question whether the phase transition between spacetimes with and without black holes is of the first or of the second order, i.e., whether the mass of the black hole formed is a continuous function at the threshold.

Choptuik $^{[3]}$ numerically analyzed the system and obtained a result which strongly suggests the latter. Furthermore, he discovered a “critical behaviour” of the gravitational collapse of the system. His result can be summarized as follows: Let the initial distribution of the scalar field be parametrized smoothly by a parameter $p$, such that the solutions with the initial data $p > p_c$ contain a black hole while those with $p < p_c$ do not. For several one-parameter families investigated, near-critical solutions ($p \approx p_c$) satisfy the following: (1) the critical solution (i.e. $p = p_c$) is universal in the sense that it approaches the identical space-time for all families, (2) the critical solution has a discrete self-similarity, and (3) for supercritical solutions ($p > p_c$) the black hole mass satisfies $M_{\text{BH}} \propto (p - p_c)^{\beta_{\text{BH}}}$ and the critical exponent $\beta_{\text{BH}}$, which is about 0.37, is universal for all families. Abrahams and Evans $^{[4]}$ found similar phenomena in axisymmetric collapse of gravitational wave and obtained $\beta_{\text{BH}} = 0.38$. Evans and Coleman $^{[5]}$ found similar phenomena with $\beta_{\text{BH}} \simeq 0.36$ in spherically symmetric collapse of radiation fluid, in which case the self-similarity is not discrete but continuous. Employing a self-similar ansatz they also found a numerical solution which fits the inner region of the near-critical solutions very well. Since the above values of $\beta_{\text{BH}}$ were close to each other, some people considered that there is a universality over many systems.

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For a deeper understanding of these phenomena, an important direction is to pursue the analogy not only phenomenologically but also theoretically between the “critical phenomena” in gravitational collapse and those in phase transitions in statistical mechanics. Argyres [9] reviewed the above numerical results in terms of renormalization group (RG). Independently, the present authors [10] showed that both qualitative and quantitative understanding is possible by introducing renormalization group ideas. They showed that uniqueness of the relevant (unstable) eigenmode of the linear perturbations around the critical self-similar space-time is essential for the universality to be observed and that the critical exponent $\beta_{BH}$ is given by $\beta_{BH} = 1/\kappa$, where $\kappa$ is the only eigenvalue with positive real part. Emphasized there was that the standard renormalization group argument implies that the linear perturbation analysis is sufficient to obtain the exact value of $\beta_{BH}$. They confirmed that their scenario holds in the case of radiation fluid and obtained a very accurate value of the exponent, $\beta_{BH} = 0.35580192$. The difference from those obtained for scalar field collapse (0.37) and gravitational wave collapse (0.38) was beyond the possible numerical errors in their simulations. This showed that there is no universality between radiation fluid collapse and other systems. Maison [11] applied the method to the systems of perfect fluids with pressure proportional to density and showed how $\beta_{BH}$ depends on the equation of state, under the assumption that the same phenomena occur and the same scenario holds as the radiation fluid. Gundlach [12] confirmed $\beta_{BH} = 1/\kappa$ in scalar field collapse.

Some new and intriguing phenomena in the critical behaviour have been found such as bifurcation [13,14] and coexistence of first and second order phase transitions [15]. However, in this paper we will pursue a slightly different direction, namely concentrate on providing deeper understanding of basic aspects of the critical behaviour.

This paper is the fully expanded version of Ref. [10]. (Some of the materials here have already been treated in [10].) The aim of the paper is twofold. First, we present a general, mathematically rigorous, framework in which critical behaviour observed in various models so far can be analyzed and understood in a unified way, based on a renormalization group picture. Now that there seems to be some consensus on the qualitative picture on the simplest case, we in this paper focus our attention to two problems which remain to be more clarified. (1) We give a sufficient condition under which the critical exponent is given exactly by $\beta_{BH} = \beta/\kappa$. In the above, $\beta$ is a scaling exponent which appears in calculating the Lyapunov exponent (see Sec. II for details). Note that in [10] we only considered the radiation fluid collapse, where one can take $\beta = 1$. (2) We also consider the problem of universality, in the wider sense of the word that details of the definition of the model is irrelevant for the critical behaviour, and we present a sufficient condition under which the universality of this kind occurs. Both conditions are sufficiently weak that they are expected to be applicable to most systems for which the critical behaviour has been observed.

Second, we present a more thorough analysis of the perfect fluid collapse, with various adiabatic index $\gamma$. In addition to presenting the details of our analysis on the radiation fluid collapse reported in [10], we here report a result of a new method of analysis, the so called Lyapunov analysis, which was performed to further confirm the uniqueness of the relevant mode around the critical solution. A Lyapunov analysis and the shooting method for an ordinary differential equation adopted in [10] are complementary to each other in the following sense: (1) The former extracts eigenmodes in the descending order of its real part, whereas the latter affords information only of a finite region of complex $\kappa$ plane. (2) The Lyapunov analysis is useful even if the eigenmodes do not form a complete set. (3) It is easier in the latter than in the former to numerically obtain accurate values of the eigenvalues $\kappa$ hence the critical exponent $\beta_{BH}$. Our analysis in the second part is sufficiently convincing for physicists, we believe. But mathematically it is unsatisfactory in the sense that it does not provide rigorous theorems. Rigorous full analysis of the system is a subject of future studies.

This paper is organized as follows. First, we present our general scenario in Sec. II. We define renormalization group transformations acting on the phase space (Sec. II A), and see how all the aspects of critical phenomena for radiation fluid collapse, etc. are deduced from the behaviour of the RG flow near a fixed point (Sec. II B). We present a sufficient condition for the exactness of the relation $\beta_{BH} = \beta/\kappa$ in Sec. II C. We then consider more general cases, i.e. discrete self-similarity (Sec. II D) and problem of universality for different systems (Sec. II E).

Then we turn our attention to detailed study of perfect fluid collapse. After reviewing the equations of motion in Sec. II F, we first study self-similar solution in Sec. II G. Then we study perturbations around the self-similar solution and confirm our picture to an extent in Sec. II H by numerical study. In Sec. II I we establish uniqueness of the relevant mode by the Lyapunov analysis. The uniqueness of the relevant mode implies that that self-similar solution is responsible for the critical behaviour. So we list in Sec. VII the value of the critical exponent $\beta$ for various $\gamma$. Sec. VIII is for conclusions and discussions.

II. SCENARIO BASED ON RENORMALIZATION GROUP IDEAS

In this section we present a scenario of critical behaviour based on renormalization group ideas, which gives clear understanding of every aspect of the phenomena. We first introduce renormalization group transformations acting on the phase space of partial differential equations (PDEs) in Sec. II A. In Sec. II B we give a (non-rigorous) scenario for the simplest case where the system
has scale invariance and a (continuously) self-similar solution plays an important role. Even in this simplest case we can argue most of the essential features of the critical behaviour. This case covers a perfect fluid collapse treated in detail in the subsequent sections. We then proceed in Sec. IIC to make our scenario into a rigorous mathematical framework which starts from basic assumptions, based on renormalization group ideas. The case where discrete self-similar solution becomes relevant such as scalar field collapse is understood essentially in the same way, as is explained in Sec. IID. In general cases where the system is not scale invariant, the renormalization group transformation drives the equations of motion to a fixed point. Renormalization group method then tells us which systems can exhibit the same critical behaviour, as will be explained in Sec. IIE. As examples, we consider perfect fluid with a modified equation of state, and also show that scalar fields with any potential term (which depends on the scalar field only) should show the same critical behaviour.

Currently, there seem to be two kinds of renormalization group type analysis of PDEs, [17] and [18]. Although these two are certainly philosophically related, they differ in details. We here adapt the approach of [18], and extend their work to cases where the critical behaviour is observed. It would also be very interesting to consider applications of the method of [17].

A. Renormalization group transformation

We give a general formalism which deals with time evolution of initial data considered as a flow of a renormalization group transformation. The argument is general but the notations are so chosen in order to suit the case considered in the subsequent sections. In particular, we explicitly consider only the case of spherical symmetry, although our framework can in principle be applied to more general cases in n-space dimensions with suitable changes: The independent space variable r is increased to r1, r2, . . . , rn, and the differential operators R and T become partial differential operators on a space of functions of n variables.

We are interested in the time evolution of n unknown functions u = (u1, u2, . . . , un), which are real-valued functions of time t and spatial coordinate r, and which satisfy a system of partial differential equations (PDEs):

\[ L \left( u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r} ; t, r \right) = 0. \] (2.1)

This real formulation does not restrict applicable systems in any way, because any PDEs for complex unknown functions can be rewritten in the real form by doubling the number of components. We employ this real form, because this makes the dimensional counting (e.g. number of relevant modes) simpler and clearer. See a Remark after Claim 1 of Sec. IIB1. (However, in practical problems, e.g. complex scalar fields, one could proceed without employing the real formulation, as long as some caution is taken in counting dimensions.)

The time evolution of this system is determined once one specifies the values of u(t, ·) at the initial time t = −1. We call the space of functions u(t, ·) (at fixed time t) the phase space \( \Gamma \), which is considered as a real vector space. We leave the definition of \( \Gamma \) (e.g. how smooth functions in \( \Gamma \) should be) rather vague, until we develop more detailed theory of time evolution in Sec. IIE.

To introduce a renormalization group transformation, we first define a scaling transformation \( S(s, \alpha, \beta) \) which depends on real parameters \( s, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), and \( \beta \):

\[ S(s, \alpha, \beta) : u_i(t, r) \mapsto u_i^{(s, \alpha, \beta)}(t, r) \equiv e^{\alpha s} u_i(e^{-s} t, e^{-\beta s} r) \] (2.2)

(As will be clear \( \beta \) here is related with, but not equal to, \( \beta_{\text{BH}} \).

Now we make the fundamental

Assumption S (Invariance under scaling) The system of PDEs (2.1) are invariant under the scaling transformation (2.2) in the following sense: If \( u \) is a solution of (2.1), then for suitably chosen constants \( \alpha, \beta \) and for an arbitrary scale parameter \( s \) (close to 0), the new function \( u^{(s, \alpha, \beta)} \) is also a solution of the system (2.1).

In the following we denote by \( \alpha, \beta \) the constants which make the above assumption hold for the system of PDEs (2.1), and denote \( u^{(s, \alpha, \beta)} \) simply by \( u^{(s)} \) unless otherwise stated. We consider only those systems for which the above Assumption S holds in Sec. IIA through Sec. IID. In Sec. IIE, we consider systems for which Assumption S does not hold, and address the problem of universality in its wider sense.

We now define a renormalization group transformation (RGT) acting on the phase space \( \Gamma \) as:

\[ \mathcal{R}_s : U_i(\xi) = u_i(−1, \xi) \mapsto U_i^{(s)}(\xi) = e^{\alpha s} u_i(−e^{-s}, e^{-\beta s} \xi) \] (2.3)

where \( \alpha, \beta \) are constants of Assumption S, \( s > 0 \), and \( u \) is a solution of the PDE system. That is, \( U^{(s)} \) is given by developing the initial data \( u(−1, r) \equiv U(r) \) from \( t = −1 \) to \( t = −e^{-s} \) by the PDE, and rescaling the spatial coordinate \( r \) and the unknown functions \( u_i \). We have written the argument of \( U \) as \( \xi \) rather than \( r \), in order to draw attention to the fact that the physical (or geometrical) length is given by

\[ r = e^{-\beta s} \xi \] (2.4)

rather than by \( \xi \) itself.

Because of our assumption of the invariance of the PDEs under scaling, \( \mathcal{R}_s \) forms a (semi) group:
\[ \mathcal{R}_{s_1 + s_2} = \mathcal{R}_{s_2} \circ \mathcal{R}_{s_1}. \] (2.5)

From this it follows immediately that the s-derivative of \( \mathcal{R}_s \) at \( s = 0 \) is in fact an infinitesimal generator of \( \mathcal{R}_s \):

\[ \hat{\mathcal{R}} \equiv \lim_{s \to 0} \frac{\mathcal{R}_s - 1}{s}, \quad \mathcal{R}_s = \exp \left( s\hat{\mathcal{R}} \right). \] (2.6)

A fixed point of the renormalization group \( \{ \mathcal{R}_s \mid s \in \mathbb{R} \} \) is a point \( U^* \) in \( \Gamma \) satisfying \( \mathcal{R}_s(U^*) = U^* \) for all \( s > 0 \), and can also be characterized by \( \hat{\mathcal{R}}(U^*) = 0 \). A function \( u \) is called self-similar with parameters \((\alpha, \beta)\) if it satisfies \( u(t, r) = u(s, \alpha, \beta)(t, r) \). Each self-similar solution \( u_{ss} \) of the PDEs with parameters \((\alpha, \beta)\) of Assumption S is related to a fixed point \( U^* \) of \( \mathcal{R}_s \) by \( u_{ss}(t, r) = (-t)^\alpha U^*(r(-t)^{-\beta}) \).

The tangent map (or Fréchet derivative) \( T_{s,U}(F) \) of \( \mathcal{R}_s \) at \( U \) is defined by

\[ T_{s,U}(F) \equiv \lim_{\epsilon \to 0} \frac{\mathcal{R}_s(U + \epsilon F) - \mathcal{R}_s U}{\epsilon}. \] (2.7)

Note that \( T_{s,U} \) is a linear operator for fixed \( s \) and \( U \), in contrast with \( \mathcal{R}_s \). The above definition suggests the following formal relation:

\[ \mathcal{R}_s(U + F) = \mathcal{R}_s(U) + T_{s,U}(F) + O(F)^2 \] (2.8)

where \( O(F)^2 \) denotes a term whose norm is of order \( \| F \|^2 \) (with some suitable norm \( \| \cdot \| \)).

We define the s-derivative of \( T_{s,U} \) at \( U \) as

\[ \hat{T}_U \equiv \lim_{s \to 0} \frac{T_{s,U} - 1}{s}, \] (2.9)

which is formally related with \( \hat{\mathcal{R}} \) by

\[ \hat{\mathcal{R}}(U + F) = \hat{\mathcal{R}}(U) + \hat{T}_U(F) + O(F)^2. \] (2.10)

At a fixed point \( U^* \) of \( \mathcal{R}_s \), (2.9) simplifies to

\[ T_s(F) \equiv \lim_{\epsilon \to 0} \frac{\mathcal{R}_s(U^* + \epsilon F) - U^*}{\epsilon}, \] (2.11)

and we denote the tangent map at \( U^* \) simply by \( \hat{T} \). In the following, \( \hat{T} \) (at \( U^* \)) forms a (semi)group like \( \mathcal{R}_s \), and can be represented in terms of its infinitesimal generator \( \hat{T} \) as

\[ \hat{T} \equiv \hat{T}_{U^*} \equiv \lim_{s \to 0} \frac{T_s - 1}{s}, \quad T_s = \exp \left( s\hat{T} \right). \] (2.12)

An eigenmode \( F \) of \( \hat{T} \) with eigenvalue \( \kappa \) is a (complex-valued) function satisfying \( (\kappa \in \mathbb{C}) \)

\[ \hat{T} F = \kappa F. \] (2.13)

Note that we here consider eigenmode (and spectral) problem on complexification of \( \Gamma \), i.e. we allow complex eigenvalues and eigenvectors. This is necessary for our later analysis. Although complex eigenvectors are not elements of \( \Gamma \), we can express vectors in \( \Gamma \) by taking the real part of a linear combination of complex eigenvectors.

These modes determine the flow of the RGT near the fixed point \( U^* \). A mode with \( \Re \kappa > 0 \), a relevant mode, is tangent to a flow diverging from \( U^* \), and one with \( \Re \kappa < 0 \), an irrelevant mode, to a flow converging to it. A \( \Re \kappa = 0 \) mode is called marginal.

**Remark.** Many equations of gravitational systems, including those of the perfect fluid, are not exactly of the form of (2.3). More precisely, we have two kinds of unknowns \( u = (u_1, u_2, \ldots, u_n) \) and \( c = (c_1, c_2, \ldots, c_m) \), which satisfy the following PDEs of two different classes: (1) the evolution equations:

\[ L_E \left( u, c, \frac{\partial u}{\partial r}, \frac{\partial c}{\partial r}, r \right) = 0, \] (2.14)

and (2) the constraint equations:

\[ L_C \left( u, c, \frac{\partial c}{\partial r} \right) = 0. \] (2.15)

In this case, the well-posed Cauchy problem is to specify the initial values of \( u_i(-1, r) \) for \( i = 1, 2, \ldots, n \) (and appropriate boundary conditions for \( c_i \)’s). In this sense, the phase space variables are \( u_i \)’s, while \( c_i \)’s should be considered as constraint variables.

Solving (2.13) for constraint variables \( c_i \)’s in terms of integrals of \( u_i \)’s and substituting them into (2.14) yields evolution equations for \( u_i \)’s of the type of (2.1), with the difference that \( u \) in the argument of \( L \) now represents some complicated functionals (integrals) of \( u_i \)’s. In this form, we can apply our formalism of the renormalization group transformation.

**Remark.** In the above, we have presented the formalism in \((t, r)\) coordinate with unknowns \( u_i \), in order to directly reflect our physical interpretation. However, mathematical treatment can in principle be made much simpler by introducing new variables

\[ x \equiv \ln r - \beta \ln(-t), \quad s \equiv -\ln(-t), \]

\[ u_i(s, x) \equiv c^{\alpha_i} u_i(t, r). \] (2.16)

In terms of these, the scaling transformation \( S(s', \alpha, \beta) \) becomes a simple translation in \( s \):

\[ u_i(s, x) \rightarrow u_i^1(s', x) \equiv v_i(s + s', x), \] (2.17)

\(^1\)Mathematically, in order to determine the behaviour of the flow around \( U^* \), we have to know not only the point (discrete) spectrum but also the continuous spectrum of \( T \). See Assumption L1 of Sec. IIIC for details.
Historically, because of closeness of numerical values of the critical exponents of a scalar field $\beta_s$, a radiation fluid $\beta_r$, and gravitational waves $\beta_{BH}$, the following universality in still wider sense, though stated less precisely, was speculated.

(U3b) The value of $\beta_{BH}$ is the same over many systems, including a scalar field, a radiation fluid, and gravitational waves.

The precise value $0.35580192...$ in Ref. [10] and the values for scalar field collapse (0.37) and gravitational wave collapse (0.38) shows that this is not true.

We now present our scenario through two claims, and their derivations. We do this on two different levels. First, in Sec. II B 1 (resp. Sec. II B 2) we present claims concerning qualitative (resp. quantitative) pictures on the critical behaviour, and present their (nongrigorous) ‘derivation’ based on naive linear perturbation ideas. Presentation in this Sec. II B is intended to provide basic ideas, so the following Claims and their ‘proofs’ are not intended to be precise mathematical statements. In particular, naive perturbation is not sufficient to deal with long time asymptotics rigorously, and this is where the “renormalization group” idea comes in. We thus present in Sec. II C more detailed, rigorous derivation of Theorem [10], a rigorous version of Claims 1 and 2, based on renormalization group philosophy.

A mathematical remark about the phase space is in order. In order to formulate the time evolution, we consider our phase space $\Gamma$ to be a real Banach space of functions $U(\xi)$’s which satisfy certain decay and smoothness properties, with a norm $\| \cdot \|$ under which the phase space is complete. Concrete choice of the decay and smoothness properties should be determined on physical grounds for each models, together with the requirement that $\Gamma$ is invariant under the time evolution. The term “approach” and “deviate” in the statement (S1) should be interpreted in terms of the norm of the phase space, although other notion of convergence, e.g. pointwise convergence, could equally work. Precise final definition should be given based on rigorous mathematical analysis, which is beyond the scope of the present paper.

In the concrete cases discussed later, the fixed point $U^*$ does not represent an asymptotically flat space-time. Though one is usually interested in asymptotically flat space-time in gravitational collapse, we define our phase space $\Gamma$ to be sufficiently large to include $U^*$. Asymptotically flat data are included in the phase space (if one

Note the term “universality” is used in two ways. In (U1) and (U2), it refers to the independence on the initial data (or a family of initial data), while in (U3a), it refers to the independence on the system’s EOM (i.e. details in the definition of the model considered). In statistical mechanics, where the term was used for the first time in a similar context, “universality” refers to the latter (although there is no “initial data” in equilibrium statistical mechanics and thus it is impossible to have a counterpart for the former). It is one of our main purpose in this paper to give sufficient conditions under which the two kinds of universality hold.

2If there are some gauge degrees of freedom, which is often the case for general relativistic systems, we should consider $\Gamma$ as a space of gauge equivalence classes of functions. Or we could fix the gauge and consider $\Gamma$ to be a usual function space. In the latter case, we should be careful in gauge-fixing so as not to count “gauge modes” as physically meaningful eigenmodes in linear perturbation analysis.
chooses the functions $U_i$ appropriately) and form a subspace of $\Gamma$.

1. Critical space-time

We first consider the qualitative behaviour of (near-) critical solutions.

Claim 1 Suppose there is a fixed point $U^*$, with no marginal modes. If (S1) and (U1) hold with this $U^*$, then there should be a unique relevant mode with a real eigenvalue for this $U^*$. Conversely, if $U^*$ has a unique relevant mode, then (S1) and (U1) hold with this $U^*$, at least for all the initial data close to $U^*$. (Except, of course, for the exactly critical initial data.)

Remark. Because we are considering real PDEs, the ODEs for eigenmodes are also real, and thus all eigenmodes with complex eigenvalues (if any) appear in complex conjugate pairs. This means that if such a system has a unique relevant mode, the eigenvalue must be real.

Remark. (Reason for employing the real formulation.) One could proceed without employing the real formulation. However, in this case, dimension counting becomes more complicated; because the desired critical behaviour is observed only when the relevant subspace is (real) one-dimensional, one has to see whether a given eigenvector spans a subspace of (real) dimension 1 or (real) dimension 2. By reducing the system of PDEs in its real form, we can avoid such subtleties. We emphasize again that this does not restrict the applicability of our claims in any way. For example, a system of PDEs (for complex unknowns) which has exactly one relevant mode with a complex eigenvalue is easily seen to have a relevant invariant subspace which is (real) two-dimensional, thus leading to the same conclusion (i.e. this does not show usual critical behaviour) as is obtained from the real formulation. Based on this observation, we in this paper drop “Re” in the exponent relation $\beta_{BH} = \beta/\kappa$.

Proof. Suppose $U^*$ has a unique relevant mode. Then the renormalization group flow around the fixed point $U^*$ is contracting except for the direction of the relevant mode (Fig. 1), at least in some neighbourhood $N \subset \Gamma$ around $U^*$. Thus there will be a critical surface or a stable manifold $W^s(U^*)$ of the fixed point $U^*$, of codimension one, whose points will all be driven towards $U^*$. And there will be an unstable manifold $W^u(U^*)$ of dimension one, whose points are all driven away from $U^*$. A one-parameter family of initial data $I \subset N$ will in general intersect with the critical surface, and the intersecting point $U_c$ will be driven to $U^*$ under the RGT:

$$\lim_{s \to \infty} ||U^*_c(s) - U^*|| = 0.$$  \hspace{1cm} (2.18)

Thus, $U_c$ is the initial data with critical parameter $p_c$. An initial data $U_{init}$ in the one-parameter family $I$, which is close to $U_c$ will first be driven towards $U^*$ along the stable manifold, but eventually be driven away along the unstable manifold. Therefore, claims (S1) and (U1) hold.

To show the converse, first suppose $U^*$ had no relevant mode. Then any initial data (sufficiently close to $U^*$) would all be driven towards $U^*$ until finally be absorbed into $U^*$, which contradicts (S1). Next suppose $U^*$ had more than one relevant modes. Then the stable manifold of $U^*$ would have codimension more than one, and thus a generic one-parameter family of initial data would not intersect with the stable manifold of $U^*$. This means that a generic critical or near-critical solution would not approach the self-similar solution $U^*$; instead, they would be driven towards a different self-similar solution with a unique relevant mode. Thus claims (S1) and (U1) do not hold.

Remark. Eq. (2.18) states that the fixed point $U^*$ is on the closure of the subspace of $\Gamma$ which consists of asymptotically flat data. This explains, in rather precise terms, that the self-similar solution plays such an important role that its perturbation gives an exact critical exponent even when one is interested only in asymptotically flat collapses. Asymptotically flat data can converge to the self-similar fixed point with infinite mass in the above sense, even though the ADM mass conserves. It should be emphasized that there is no contradiction involved here. That is, the mass measured in the unit of the scaled coordinate becomes infinite as the space-time approaches the self-similar solution, while the physical mass measured in the unit of the original coordinate remains finite.

3PDEs, when one is considering in more than two dimensions, as remarked at the beginning of Sec. IA.
Remark. Some reader might doubt the validity of our claim (2.18), because often the critical space-time satisfies \( \lim_{r \to \infty} a(r) = a_\infty > 1 \). However, it is not difficult to find norms \( \| \cdot \| \), for which (2.18) holds; e.g. take
\[
\|a\| \equiv \sup_{r>0} (1+r)^{-1} |a(r)|, \quad \text{or} \quad \|a\| \equiv \sup_{r>0} e^{-r} |a(r)|.
\]
(2.19)

The remaining, highly nontrivial, question is whether the Assumptions in Sec. II C are satisfied in this norm, because the spectrum of perturbations can depend on the choice of the norm (and thus on the function space). In other words, the norm and the function space \( \Gamma \) should be so chosen that the Assumptions hold (and that \( \Gamma \) includes \( U^* \)): if we can find such a norm and \( \Gamma \), the critical behaviour should be observed; this is what is proved in Sec. II C. The question of the right choice of the norm must be answered for each models under proper mathematical consideration. We emphasize that our goal is to present a general framework (physically reasonable sufficient conditions) under which the critical behaviour is bound to happen, which is independent of the details of specific models.

Remark. Fixed points with more than one relevant modes are responsible for the so-called multicritical behaviour in statistical mechanics. We give examples of self-similar solutions with more than one relevant modes (and thus are not responsible for generic critical behaviour) in section V H. Another example is provided by the self-similar solutions of complex scalar fields \( |20\rangle \). In this model the continuous self-similar solution has three relevant modes, and thus is irrelevant for a generic critical behaviour. In other words, we have to adjust three parameters (instead of one) in the initial condition, to observe a critical behaviour governed by this self-similar solution.

2. Critical exponent

We now turn our attention to the quantitative aspect of the critical behaviour, i.e. the critical exponent \( \beta_{BH} \).

Claim 2 If the relevant mode is unique with a real eigenvalue \( \kappa \), the black hole mass \( M_{BH} \) satisfies
\[
M_{BH} \propto (p - p_c)^{\beta_{BH}}
\]
(2.20)
with
\[ \beta_{BH} = \frac{\beta}{\kappa} \]  
(2.21)
for slightly supercritical solutions, where \( \beta \) is the scaling exponent of the scaling transformation (2.2). Here \( f(p) \propto (p - p_c)^{\beta_{BH}} \) means there exist \( p \)-independent positive constants \( C_1, C_2 \) such that \( C_1 < f(p)(p - p_c)^{-\beta_{BH}} < C_2 \) holds for \( p \) sufficiently close to \( p_c \).

‘Proof.’ The claim, in particular the exactness of the relation (2.21), is one of the main conclusions of this paper. The derivation of this claim is rather lengthy, although it is a variation on rather standard treatment of similar problems in (rigorous) renormalization group analysis of critical behaviour in statistical mechanics. We first explain the rough idea, and then explain each step in detail.

We first consider the fate of an initial data \( U_{\text{init}} \) in the one-parameter family, which is close to \( U_c (\epsilon = p - p_c) \):
\[
U_{\text{init}} = U_c + \epsilon F, \quad \|F\| = 1.
\]
(2.22)
The data will first be driven towards \( U^* \) along the critical surface, but eventually be driven away along the unstable manifold, until it finally blows up and forms a black hole, or leads to a flat space-time.

To trace the time evolution, we divide the whole evolution into two stages: I (\( s \leq s_1 \)) and II (\( s > s_1 \)). See Fig. 4 the stage I corresponds to \( U_{\text{init}} \to Q \), while stage

\[ ^5 \text{More general situation, where the initial data is not close to } U^* \text{ but is close to } W^*(U^*), \text{ can be treated in a similar way} \]

FIG. 2. Schematic view of the flow. Each dot denotes approximate location of the flow under iterations of \( R_\sigma \), starting from several different initial data. Note that the points are most densely distributed around the fixed point \( U^* \). The decomposition into two stages I and II is shown by vertical dotted lines. The set \( A_\pm \) will be used in our detailed analysis in Sec. II C.
II to $Q \to$ (black hole). The stage I is the portion of the journey where $U^{(s_1)}_{\text{init}}$ is close to $U^*$ and we can use the perturbation. Stage II is the region where perturbation breaks down, and $s_1$ is so chosen to make this separation possible. Concretely, we choose $s_1$ as the smallest $s$ such that $\|U^{(s_1)}_{\text{init}} - U^*\|$ becomes of order 1 (after $U^{(s_1)}_{\text{init}}$ begins to deviate from $U^*$). As will be explained in detail, near-critical solution $U^{(s_1)}_{\text{init}}$ spends most of its time in stage I, and the time of stage II is relatively short.

Based on the analysis of the flow, we then derive the exponent by interpreting the result in our original coordinate. As will be explained, what matters in determining the exponent is the behaviour of the flow in stage I.

**Step 1. Flows in stage I:**
The analysis of this stage is essential, because the flow spends most of its time at this stage, and gives the value of the exponent.

For Stage I, we can rely on linear perturbations to get

$$U^{(s_1)}_{\text{init}} = \mathcal{R}_{s_1} U_{\text{init}} = \mathcal{R}_{s_1} (U_e + \epsilon F) = U^{(s_1)}_e + \epsilon \mathcal{S}_{s_1} F + O(\epsilon^2) = U^{(s_1)}_e + \epsilon \epsilon^{s_1} F_{\text{rel}} + O(\epsilon^2),$$

where $F_{\text{rel}}$ is roughly the component of the relevant mode in $F$. The derivation of (2.23) requires some calculations, together with suitable assumptions on invariant manifolds of $\mathcal{T}$. For the ease of reading, we postpone this derivation to section II C 3 and proceed assuming (2.23) is correct.

Due to (2.18), we have (for large $s_1$)

$$U^{(s_1)}_{\text{init}} \approx U^*(\xi) + \epsilon \epsilon^{s_1} F_{\text{rel}}.$$  

Our choice of $s_1$ implies

$$|\epsilon| \epsilon^{s_1} \approx 1.$$  

(Here 1 means a positive small which is independent of $\epsilon$.) Note that $s_1$ diverges logarithmically in $\epsilon$ as $\epsilon \to 0$.

**Step 2. Flows in stage II:**

We further evolve $U^{(s_1)}_{\text{init}}$ until it finally diverges or leads to a flat space-time. At this point, we assume that the phase space $\Gamma$ has a rather simple structure, i.e. all points (at least in a neighbourhood of $U^*$) on one side of $W^s(U^*)$ blow up and lead to a black hole, while those on the other side disperse and lead to a flat space-time. Once one admits this nontrivial assumption on the global structure of $\Gamma$, stage II becomes relatively trivial, because of the large second term of (2.24). That is, the data $U^{(s_1)}_{\text{init}}$ differs from $U^*$ so much that one can tell the fate of this data depending on the sign of $\epsilon$. In particular, if a black hole is formed, the solution will blow up in a finite, rather short, time (uniform in $\epsilon$). This means the radius of its apparent horizon, and thus its mass, will be $O(1)$ measured in $\xi$.

**Step 3. Going back to the original scale:**

Finally, we translate the above result back into our original coordinate $(t, r)$. The relation $r = \xi e^{-\beta s}$ implies that the radius of the apparent horizon, which is $O(1)$ measured in $\xi$, is in fact $O(e^{-\beta s})$ measured in $r$. So we have from (2.23)

$$M_{\text{BH}} = O(e^{-\beta s}) = O(|\epsilon|^{\beta/\kappa}).$$

Therefore the critical exponent is given exactly by

$$\beta_{\text{BH}} = \frac{\beta}{\kappa}.$$  

This concludes our ‘derivation.’

**Remark.** As can be seen from the above ‘proof,’ an intuitive understanding of the exactness of $\beta_{\text{BH}}$ is as follows. In the limit $\epsilon \to 0$, the time spent in a finite region around $U^*$ diverges whereas the time spent outside is usually finite before a black hole is formed, if there are no other fixed points, etc near the flow. So the change of the profiles of the fields are dominated by the linear stage so that the exact critical exponent is given in terms of the eigenvalue of linear perturbation around $U^*$. The rigorous argument is presented in Sec. II C.

It follows immediately from the above claim that conditions (S2) and (U2) hold and the critical exponent $\beta_{\text{BH}}$ is exactly given by (2.27). The above ‘proofs’ show that our claims are true not only in asymptotically flat cases but also in cases with any definition of mass proportional to the typical scale of the functions $U(\xi)$.

If one admits Claim 2, it would in fact be a big surprise if the condition (U3b) did hold, since the critical solution of other systems, such as that of a scalar field system and that of a gravitational wave, are so different that one cannot expect the eigenvalues of the perturbations around them to be the same. An explicit example of nonuniversality is given by the critical exponent $\beta_{\text{BH}}$ for various parameter values of $\gamma$ for the perfect fluid: see section VII.

C. Detailed analysis of renormalization group flows

Simple linear perturbations are not sufficient for rigorous study of the long time asymptotics of the solution. In this section, we describe the “renormalization group” type analysis in detail, which goes beyond the linear perturbation and provides firm grounds on our claims. Our goal is to derive Theorem 1, which is a rigorous version of Claims 1 and 2, from a few “assumptions.” The “assumptions” from which we start are grouped into two: (1) assumptions on the linear time evolution of the system, defined by $\mathcal{T}$ (or equivalently $\mathcal{T}_{s,U^*} = e^{s\mathcal{T}}$), and (2)
certain regularity (smoothness) properties of $\mathcal{R}_\sigma$ and $\mathcal{T}_\sigma$ which physically look innocent.

In the following $\sigma$ is a positive constant suitably chosen for both of the following assumptions to hold. The first assumption specifies the properties of the linear part of the time evolution, generated by $\mathcal{T}_{\sigma,U^*}$:

**Assumption L1 (Invariant subspaces of $\mathcal{T}_{\sigma,U^*}$)** For a suitably chosen $\sigma$, the whole tangent space $T_U\Gamma$ is a direct sum of invariant subspaces of $\mathcal{T}_{\sigma,U^*}$:

$$T_U\Gamma = E^s(U^*) \oplus E^s(U^*).$$  \hfill (2.28)

Here $E^s(U^*)$ and $E^s(U^*)$ are invariant subspaces of $\mathcal{T}_{\sigma,U^*}$, of dimension $N$ and of codimension $N$, respectively. There exists $\bar{\kappa} > 0$ such that the restrictions of $\mathcal{T}_{\sigma,U^*}$ on these invariant subspaces satisfy:

$$\|\mathcal{T}_{\sigma,U^*}(F)\| \geq e^{\bar{\kappa}\sigma}\|F\| \quad (F \in E^s(U^*))$$

$$\|\mathcal{T}_{\sigma,U^*}(F)\| \leq e^{-\bar{\kappa}\sigma}\|F\| \quad (F \in E^s(U^*))$$  \hfill (2.29)

The estimates (2.29) in Assumption L1 could be further deduced from suitable assumptions on the spectrum of $\mathcal{T}_{\sigma,U^*}$, because $\mathcal{T}_{\sigma,U^*} = \exp(s\mathcal{T}_{\sigma,U^*})$. One possibility of such an assumption would be:

The spectrum of $\mathcal{T} \equiv \mathcal{T}_{\sigma,U^*}$ is a union $\Sigma_- \cup \Sigma_+$. Hence (i) $\Sigma_+$ is a finite set with total multiplicity $N$, (ii) $\Sigma_- \subseteq \{ z \in \mathbb{C} | \text{Re} z \leq -\bar{\kappa} \}$ and $\Sigma_+ \subseteq \{ z \in \mathbb{C} | \text{Re} z \geq \bar{\kappa} \}$ for some $\bar{\kappa} > 0$, and (iii) $\mathcal{T}$ is a sectorial operator.

For a related estimate, see e.g. Henry [21, Theorem 1.5.3].

Before stating the second assumption, we set up the following convention on the choice of the norm $\|\cdot\|$.

**Convention on the choice of the norm $\|\cdot\|$.** For a vector $F$, we denote by $F^s$ (resp. $F^u$) the $E^s(U^*)$ (resp. $E^u(U^*)$) component of $F$:

$$F = F^s + F^u$$  \hfill (2.30)

with $F^s \in E^s(U^*)$, $F^u \in E^u(U^*)$. Assumption L1 guarantees the possibility of the decomposition. We define the "box norm"

$$\|F\|_{\text{box}} \equiv \sup \{\|F^s\|, \|F^u\|\}.$$  \hfill (2.31)

Note that this box norm is equivalent to the original $\|\cdot\|$, because

$$K_0\|F^s\|, K_0\|F^u\| \leq \|F\| \leq \|F^s\| + \|F^u\|$$  \hfill (2.32)

6Here we of course mean both continuous and discrete spectrum.

holds with some ($F$-independent) $K_0 > 0$. The first inequality of (2.33) is due to completeness, while the second is nothing but the triangle inequality. Because this box norm simplifies our presentation (e.g. conditions on various constants in Propositions), we extensively use this box norm in the following, writing it simply as $\|\cdot\|$.

Sometimes, we write $\|F\| = \|F^s\|, \|F\| = \|F^u\|$.

**Notation** We extensively use following abbreviations throughout Sec. 11 to Sec. 11:

- We write $a \odot b \equiv \max\{a, b\}$.
- $\tilde{O}(a)$ denotes a scalar or vector depending on the context, whose magnitude (measured in absolute value $|\cdot|$ for a scalar, in the box norm $\|\cdot\|$ for a vector) is bounded by $|a|$.
- We also write $f \leq g \pm h$ to simultaneously represent $f < g + h$ and $f > g - h$. That is, $f \leq g \pm h$ is equivalent to $f = g + \tilde{O}(h)$ when $h \geq 0$.

The second assumption refers to regularity properties of the flow, and essentially requires that the map $\mathcal{R}_\sigma$ has a Lipschitz continuous Fréchet derivative. Although it is sufficient to require only Hölder continuity (of the derivative) with a positive exponent in the sequel, we here require Lipschitz continuity, to make the presentation as simple as possible.

**Assumption L2 (Smoothness of $\mathcal{R}_\sigma$)** There exist positive constants $\delta_0, K_1, K_2$ such that:

(i) the formal relation (2.8) holds for $\|U - U^*\|, \|F\| \leq \delta_0$:

$$\mathcal{R}_\sigma(U + F) = \mathcal{R}_\sigma(U) + \mathcal{T}_{\sigma,U}(F) + \tilde{O}(K_1\|F\|^2)$$  \hfill (2.33)

(ii) $\mathcal{T}_{\sigma,U}$ is uniformly bounded for $\|U - U^*\| \leq \delta_0$:

$$\|\mathcal{T}_{\sigma,U}(F)\| \leq K_2\|F\|.$$  \hfill (2.34)

We take $\delta_0$ sufficiently small so that $\delta_0K_1 \leq K_2$.

Note that Assumption L2 implies the following Lipschitz-type continuity for $\mathcal{R}_\sigma$ and $\mathcal{T}_{\sigma,U}$ ($\|U_i - U^*\| \leq \delta_0 (i = 1, 2)$):

$$\mathcal{R}_\sigma(U_1) - \mathcal{R}_\sigma(U_2) = \tilde{O}(2K_2\|U_1 - U_2\|),$$  \hfill (2.35)

$$(\mathcal{T}_{\sigma,U_1} - \mathcal{T}_{\sigma,U_2}) F = \tilde{O}(6K_1 \cdot \|U_1 - U_2\| \cdot \|F\|).$$  \hfill (2.36)

[To prove (2.33) from Assumption L2, first note it suffices to consider the case of $\|F\| = \|U_1 - U_2\|/2$, because $\mathcal{T}_{\sigma,U}$ is a linear operator. Then evaluate $\Delta \mathcal{R} = \mathcal{R}_\sigma(U_2 + F) - \mathcal{R}_\sigma(U_2) - \mathcal{R}_\sigma(U_1 + F) + \mathcal{R}_\sigma(U_1)$ for $\|F\| = \|U_1 - U_2\|/2$ in two ways using (2.33); (1) express it in terms of $\mathcal{T}_{\sigma,U_1}$, (2) expand each $\mathcal{R}_\sigma(\cdot)$ at $(U_1 + U_2)/2$ and show that $\Delta \mathcal{R}$ is $\tilde{O}(10K_1\|F\|^2)$.)

Although Assumption L2 physically looks rather innocent, its verification is a nontrivial mathematical problem depending on individual cases. We emphasize that $K_1$
term in Assumption L2 is not necessarily bounded uniformly in $\sigma$, and can diverge for large $\sigma$. For example, they will certainly diverge (as $\sigma \to \infty$) for non-critical initial data which will lead to a black hole. What is claimed in the Assumption is that one can find finite nonzero $\delta_0$ and $\delta_1$ for which the above claims hold.

We follow the time evolution by considering a discrete dynamical system generated by $R_{n\sigma}$ $(n \in \mathbb{Z}, n \geq 0)$. The reason for considering the discrete system is twofold. First, it is naturally generalized to the case with discrete self-similarity, as discussed in Sec. II D. Second, the operator $R_{\sigma}$ could be more regular than $R$, thanks to a smoothing effect of the time evolution over a nonzero period $\sigma$. (I.e. the linearization $T$ may not be bounded, while $T$ can be).

1. Local behaviour of the flow: Existence of the critical surface (stable manifold)

Based on the above assumptions, we now start our renormalization group study of flows. We here consider the qualitative aspects of the flow in a vicinity of the fixed point $U^*$.

We remark that there are considerable number of mathematical literature concerning the local structure of invariant manifolds. Proposition 1 is a celebrated “invariant manifold theorem”; see e.g. Shub 22, Theorem 5.2, Theorem II.4 or 22, Theorem III.8, for precise statements and a proof by sophisticated methods under slightly different assumptions. Although the cited theorems are sufficient for our needs in this section, they do not cover the case considered in Sec. II E, where we consider the universality over different models. We therefore present our version of an elementary proof, in order to make the presentation more self-contained, and also to pave a natural way to our later analysis in Sec. II E.

We consider the time evolution of an initial data $U$ (close to $U^*$) under $R_{n\sigma}$, by expressing it as $(n \geq 0)$

$$U^{(n\sigma)} \equiv R_{n\sigma}(U) = U^* + F_n \tag{2.37}$$

where the second equality defines the difference $F_n$ between $R_{n\sigma}(U)$ and $U^*$. See Fig. 3(a). Assumption L1 guarantees that we can always decompose this way.

We begin with the following lemma, which claims that an initial data with $\|F_n^u\| < \|F_n^u\|$ is driven away from $U^*$ (at least until $\|F_n^u\|$ becomes somewhat large).

**Lemma 1** Define $\delta_1 > 0$ by

$$\delta_1 = \min \left\{ \delta_0, \frac{1 - e^{-2\sigma}}{2K_1} \right\}, \tag{2.38}$$

and consider the time evolution of an initial data $U = U^* + F_0$ with $\|F_0\| < \|F_0\| < \delta_1$. Then as long as $\|F_n\| \leq \delta_1$, we have

$$\|F_n\| \leq \delta_1, \tag{2.39}$$

$$e^{2\sigma/2} \|F_n\| < \|F_{n+1}\| < 2K_2 \|F_n\| \tag{2.40}$$

**Proof.** We proceed by induction in $n$. We start by obtaining recursion relations for $F_n^u$ and $F_n^s$. Assumption L2 reads

$$R_{\sigma}(U^* + F_n) = R_{\sigma}(U^*) + T_{\sigma,U^*}(F_n) + \mathcal{O}(K_1 \|F_n\|^2) \tag{2.41}$$

which implies (note: $R_{\sigma}(U^*) = U^*$)

$$F_{n+1} = T_{\sigma,U^*}(F_n) + \mathcal{O}(K_1 \|F_n\|^2)$$

$$= T_{\sigma,U^*}(F_n^u) + T_{\sigma,U^*}(F_n^s) + \mathcal{O}(K_1 \|F\|^2), \tag{2.42}$$

where in the second step we used the linearity of $T_{\sigma,U^*}$ and the fact that $F_n^s$ and $F_n^u$ belong to invariant subspaces of $T_{\sigma,U^*}$. Decomposing as in (2.37), we have recursion relations for $F_i (i = s$ or $u)$:

$$F_{i+1} = T_{\sigma,U^*}(F_i^u) + \mathcal{O}(K_1 \|F_i\|^2). \tag{2.43}$$

Our inductive assumption $\|F_i^u\| < \|F_i^u\|$ implies $\|F\| = \|F_i^u\|$ by the definition of the box norm (2.33). So taking the norm of (2.43) and utilizing Assumption L1 and Assumption L2 to bound $\|T_{\sigma,U^*}(F_i^u)\|$, we have recurrences:

$$\|F_{n+1}^u\| \leq \|F_n^u\| + K_1 \|F_n^u\|^2 \tag{2.44}$$

$$\|F_{n+1}^u\| \geq e^{2\sigma} \|F_n^u\| - K_1 \|F_n^u\|^2 \tag{2.45}$$

$$\|F_{n+1}^u\| \leq e^{-2\sigma} \|F_n^u\| + K_1 \|F_n^u\|^2. \tag{2.46}$$

Now the recursion (2.44) immediately implies (for $\|F_n^u\| \leq \delta_1 \leq \delta_0 \leq K_2$)

$$\|F_{n+1}^u\| \leq (K_2 + K_1 \|F_n^u\|) \|F_n^u\| \leq 2K_2 \|F_n^u\|. \tag{2.47}$$

On the other hand, (2.43) implies (for $\|F_n^u\| \leq \delta_1$; use also the definition of $\delta_1$)
\[ \|F_{n+1}^u\| \geq (e^{\bar{\kappa} \sigma} - K_{1}\|F_{n}^u\|)\|F_{n}^u\| \geq \left( \frac{e^{\bar{\kappa} \sigma} + 1}{2} \right) \|F_{n}^u\|. \]  

(2.48)

These two prove (2.4). [Note: \((1 + a)/2 \geq \sqrt{a}\) for positive \(a\).]

Finally, (2.46) implies
\[ \|F_{n+1}^u\| \leq e^{-\bar{\kappa} \sigma}\|F_{n}^u\| + K_{1}\|F_{n}^u\|^2 = (e^{-\bar{\kappa} \sigma} + K_{1}\|F_{n}^u\|)\|F_{n}^u\|. \]

(2.49)

This, combined with \(\|F_{n}^u\| < \|F_{n+1}^u\|\) just proven, implies (2.39), because \(e^{-\bar{\kappa} \sigma} + K_{1}\|F_{n}^u\| \leq e^{-\bar{\kappa} \sigma} + K_{1}\delta_{1} < 1\). \(\square\)

**Proposition 1** Under Assumptions 1 and 2, we have (i) [Existence of critical solutions.] For any \(F^u \in E^u(U^*)\) with \(\|F^u\| \leq \delta_1\), there exists \(F^u_{\bar{c}}(F^u) \in E^u(U^*)\) such that
\[ U^* + F^u + F^u_{\bar{c}}(F^u) \rightarrow U^* \text{ as } n \rightarrow \infty. \]  

Moreover,
\[ \|F^u_{\bar{c}}(F^u)\| \leq C_{1}\|F^u\|^2, \text{ with } C_{1} = \frac{2K_{1}}{1 - e^{-\bar{\kappa} \sigma}}. \]  

(2.51)

We define the “local stable manifold” as
\[ W_{\text{loc}}(U^*) \equiv \{ U^* + F^u + F^u_{\bar{c}}(F^u) \mid \|F^u\| \leq \delta_1 \}. \]  

(2.52)

(ii) [Motion on \(W_{\text{loc}}(U^*)\).] There exists \(\bar{\kappa} > 0\) such that
\[ \|R_{n,\sigma}(U) - U^*\| \leq e^{-n\bar{\kappa}}\|U - U^*\| \]  

(2.53)

and
\[ \|R_{(n+1),\sigma}(U) - U^*\| \leq e^{-\bar{\kappa}}\|R_{n,\sigma}(U) - U^*\| \]  

(2.54)

for any vector \(U \in W_{\text{loc}}(U^*)\) which satisfies \(\|U - U^*\| \leq \delta_1\). We can in fact take \(\bar{\kappa} \approx \bar{\kappa}\) by taking \(\|U - U^*\|\) small.

(iii) [Invariant manifolds.] Above \(W_{\text{loc}}(U^*)\) in fact forms a manifold of codimension \(N\).

We only present a proof for (i), (ii) of the Proposition. Detailed information presented in (iii) is not necessary for our later analysis, so we only make brief comments on their proof at the end of the proof of (i) and (ii). Our proof is modeled after the argument by Bleher and Sinai \(\cite{23}\), which first appeared in a similar context in statistical mechanics.

**Proof of Proposition 1.** (i) and (ii)

We mainly consider the case of \(N = 1\), which is of our main interest. For the other cases, see the end of this proof. We denote the normalized relevant mode of \(T_{\sigma,U}\) by \(F_{\text{rel}}\). \(T_{\sigma,U} = F_{\text{rel}}\) for fixed \(F_{0}\) and various \(a_0\). Our goal is to show that there is at least one value of \(a_0\) (called \(a_c\), as a function of \(F_0\)) for which \(\lim_{n \rightarrow \infty} R_{n,\sigma}(U(0)) = U^*\).

**Step 2.** Recursion relations for \(a_n\) and \(F_n^u\). We obtained recursions for \(F_n^u\) and \(F_n^u = a_n F_{\text{rel}}\) in the proof of Lemma 2.43. Utilizing Assumption L1, and using \(\|F\| \leq \max\{\|a_n\|, \|F_n^u\|\}\) which follows from the definition of the box norm (2.31), we can convert (2.43) into recursion relations for \(a_n\) and \(h_n \equiv \|F_n^u\|\). The result is (recall: \(a \lor b = \max\{a, b\}\))
\[ a_{n+1} = e^{\kappa \sigma} a_n + \bar{O}(K_1(a_n \lor h_n)^2) \]  

(2.55)

\[ h_{n+1} \leq e^{-\kappa \sigma} h_n + \bar{O}(K_1(a_n \lor h_n)^2). \]  

(2.56)

Roughly speaking, the recursion shows that the map for \(a\) is expanding, while that for \(h\) is contracting, with “shifts” of \(\bar{O}(K_1(a_n \lor h_n)^2)\). If we neglect the second term, \(a_n\) would blow up. Thus the stable set \(W_{\text{loc}}(U^*)\) is located by carefully balancing the inhomogeneous \(\bar{O}(K_1(a_n \lor h_n)^2)\) term with the first expanding term, so that \(a_n\) will remain bounded (in fact goes to zero) as we iterate the recursion. That we can in fact find at least one such \(a_0 (= a_c)\) for given \(h_0\) can be shown by using the argument due to Bleher and Sinai \(\cite{23}\), as explained in Steps 3–5.

**Step 3.** Continuity of \(a_n\) and \(h_n\) as functions of \(a_0\). from (2.33) \(R_{\sigma}\) is a continuous function of its arguments. This implies in particular that \(a_{n+1}\) and \(F_{n+1}^u\) are continuous functions of \(a_n\) and \(F_n^u\). Iterating this for \(n\) times, we see that \(a_n\) and \(F_n^u\) are continuous functions of the initial data, \(a_0\), for fixed \(F_0\). This implies \(h_n \equiv \|F_n^u\|\) is also continuous in \(a_0\).

**Step 4.** Solving the recursions. We now solve the recursion, following the technique of Bleher and Sinai. We will show that there are positive sequences \(r_n, t_n\) such that
\[ (a) \quad r_n, t_n \rightarrow 0 \text{ as } n \rightarrow \infty \]
\[ (b) \quad 0 \leq h_{n+1} < r_{n+1} \text{ holds, as long as } |a_n| < t_n \quad \text{and} \quad 0 \leq h_n < r_n. \]
\[ (c) \quad \text{As long as } 0 \leq h_n < r_n, \text{ the set of all } a_n \text{ when } a_n \text{ sweeps the interval } (-t_n, t_n) \text{ contains the interval } (-t_{n+1}, t_{n+1}) \text{ inside}. \]

The proof of (a)—(c) is postponed to “Step 5” for the ease of reading.

Now given (a)—(c), we can show the existence of the critical \(a_c\) as follows. Let \(I_n\) be the interval \(I_n := \{ z \in \mathbb{R} : |z| < t_n \}\). Because \(a_n\) is a continuous function of \(a_0\), we can consider the inverse image \(\text{Inv}_{n}(I_n)\) of the interval \(I_n\) under the \(n\)-iterations of the recursion. But (a)—(c) show that \(\text{Inv}_{n}(I_n)\) are nonincreasing (in \(n\)) sets, with nonempty intersections. So, taking an initial condition \(a_c \in \bigcap_n \text{Inv}_{n}(I_n)\), we can guarantee for all \(n \geq 0\)
\[ |a_n| < t_n \quad \text{and} \quad h_n < r_n. \]  

(2.57)

Because \(r_n, t_n \rightarrow 0 \text{ as } n \rightarrow \infty\), (2.57) shows \(U^{(n)} \rightarrow U^*\) for this special \(a_c\), i.e. \(U^* + F^u + F_n^u \in W^*(U^*)\), with \(F^u = a_c F_{\text{rel}}\): existence of a critical solution for given \(F^u\).
Step 5. Proof of (a)–(c). Now we give an example of sequences $r_n, t_n$ which satisfy (a)–(c) of Step 4. We define $r_n, t_n$ according to (for $0 < r_0 < \delta_1$)

$$ r_n \equiv (e^{-\kappa \sigma} + K_1 r_0)^n r_0 $$

$$ t_n \equiv C_1 r_n^2 \quad \text{with} \quad C_1 \equiv \frac{2K_1}{1 - e^{-\kappa \sigma}}. $$

(2.58)

(2.59)

Then (a) is easily seen to be satisfied, because $e^{-\kappa \sigma} + K_1 r_0 \leq (1 + e^{-\kappa \sigma})/2 < 1$ by the choice of $\delta_1$. To prove (b) and (c), first note the following sufficient condition:

$$ r_{n+1} \geq e^{-\kappa \sigma} r_n + K_1 (r_n \vee t_n)^2 $$

$$ t_{n+1} \leq e^{\kappa \sigma} t_n - K_1 (r_n \vee t_n)^2. $$

(2.60)

(2.61)

The choice of $\delta_1, C_1$ guarantees $C_1 r_n \leq 1$, and thus $t_n \leq r_n$. This, together with $\kappa \geq \bar{\kappa}$, immediately shows that (2.60) and (2.61) are satisfied. Eq. (2.59) for $n = 0$ proves (2.54).

Step 6. Motion of the critical solution. Now that we have proven the existence of a critical solution, we concentrate on its time evolution. We first note $|a_n| \leq h_n$ holds for the critical solution for all $n \geq 0$, because otherwise the solution is shown to be driven away from $U^*$ by Lemma 5. But then, the recursion (2.55) is reduced to

$$ h_{n+1} \leq [e^{-\kappa \sigma} + K_1 h_n] h_n $$

(2.62)

which, in view of $h_n \leq r_n \leq r_0$, implies

$$ \| R_{(n+1)\sigma}(U) - U^* \| \leq e^{-\kappa \sigma} \| R_{n \sigma}(U) - U^* \| $$

with

$$ e^{-\kappa \sigma} = e^{-\bar{\kappa} \sigma} + K_1 r_0 \left( 1 + \frac{1 - e^{-\bar{\kappa} \sigma}}{2} < 1 \right). $$

(2.63)

Cases of $N \neq 1$ and a comment on Proposition 4. (iii) All the above proves the existence of $W_{\text{loc}}(U^*)$ and the estimate (2.54) for the case of $N = 1$. For $N \geq 2$ we can proceed similarly and find the critical surface (this time, we have to adjust $N$ parameters). For $N = 0$, similar argument shows that the map $R_{\sigma}$ is a contraction in a neighbourhood of $U^*$, thus proving that all the initial data close to $U^*$ is attracted to $U^*$.

Proof of Proposition 1. (iii) can be performed, e.g., as in [23, Theorem II.4].

2. Local behaviour of the flow: Unstable manifold

In the previous section, we established the existence of the critical surface (stable manifold). We also showed that initial data which is sufficiently away from the critical surface (characterized by $\| U - U^* \|_u \leq \| U - U^* \|_u$) is driven away from $U^*$. In this and the following section, we concentrate on the fate of near (but not exactly) critical solutions. We in this section define an “unstable manifold.” Then in Sec. II.C.3, we show that all the near-critical initial data (starting from an neighbourhood of $U^*$) are at first driven towards $U^*$ along the critical surface, but are eventually driven away along an unstable direction. We also give quantitative estimates on time intervals until the deviation of the data from $U^*$ becomes large.

We concentrate on the case where there is a unique relevant mode ($N = 1$ in Assumption L1). More precisely we assume, in addition to Assumptions L1 and L2 of Sec. II.C.1.

Assumption L1A (Uniqueness of the relevant mode) The relevant mode of $T_{\sigma, U^*}$ is unique, i.e. $N = 1$ in Assumption L1. We denote the relevant mode by $F_{\text{rel}}$ and the relevant real eigenvalue by $\kappa \geq \bar{\kappa}$:

$$ T_{\sigma, U^*} F_{\text{rel}} = e^{\kappa \sigma} F_{\text{rel}}. $$

(2.64)

Throughout this section, we use following notation.

- $\epsilon_2, \delta_2$ are positive constants chosen so that

$$ \delta_2, \epsilon_2 \leq \min \left\{ \delta_1, \frac{1 - e^{-\kappa \sigma}}{12K_1} \right\}, $$

(2.65)

$$ \delta_2 + \epsilon_2 \leq \frac{e^{\kappa \sigma} - 1}{12K_1}. $$

(2.66)

- Given $U \in \Gamma$, define $n_1(U; \epsilon)$ to be the largest integer such that $\| R_{k \sigma}(U) - U^* \| \leq \epsilon$ holds for all $0 \leq k \leq n_1(U; \epsilon)$. I.e.

$$ n_1(U; \epsilon) \equiv \max \{ n || R_{k \sigma}(U) - U^* || \leq \epsilon \text{ for } k \leq n \} $$

(2.67)

To trace the time evolution, we have to specify the unstable manifold. If $R_{\sigma}$ is invertible, we can define the unstable manifold as

$$ W^n(U^*) = \{ U \mid R_{n \sigma}(U) \rightarrow U^* \text{ as } n \rightarrow -\infty \}, $$

(2.68)

interpreting $R_{n \sigma} = (R_{-n \sigma})^{-1}$ for negative $n$. However, we here do not assume the invertibility of $R_{\sigma}$. So the definition of $W^n(U^*)$ requires some additional work. Although our definition is somewhat artificial, it suffices our need.

We define several sets, including the local unstable manifold $W^n_{\text{loc}}(U^*)$. Although all of these are defined with respect to the fixed point $U^*$, we do not explicitly write $U^*$ except for $W^n_{\text{loc}}(U^*)$, to simplify notation. We first introduce a line segment $D_r$ for $r \in \mathbb{R}$ as

$$ D_r = \{ (1 - t)(U^* + rF_{\text{rel}}) + tR_{\sigma}(U^* + rF_{\text{rel}}) \mid 0 \leq t \leq 1 \} $$

(2.69)
and define $\tilde{W}_r$ as $[n_1]$ is defined in (2.67)

$$\tilde{W}_r = \{ R_{n\sigma}(U) | U \in D_r, n \leq n_1(U^* + r F_{\text{rel}}; \epsilon_2) \} .$$

(2.70)

Then we define the local unstable manifold at $U^*$ as

$$W^u_{\text{loc}}(U^*) = \lim_{r \to 0} [ \tilde{W}_r \cup \tilde{W} - r ] .$$

(2.71)

Lemma 3 below guarantees that $W^u_{\text{loc}}(U^*)$ is well defined, and all flows close to $U^*$ are squeezed around this $W^u_{\text{loc}}(U^*)$.

**Lemma 2** (i) [Structure of $W^u_{\text{loc}}(U^*)$] $W^u_{\text{loc}}(U^*)$ of (2.74) is a well-defined set, and is “one-dimensional” in the sense that its intersection with a plane $U^* + E^s(U^*) + a F_{\text{rel}}$ exists as a unique point for $|a| \leq \epsilon_2$. Moreover, any $U, U' \in W^u_{\text{loc}}(U^*)$ satisfy

$$\| U - U' \| < C_4 \| U - U' \|_u$$

with

$$C_4 \equiv \frac{7K_1\epsilon_2}{e^{\epsilon_2} - e^{-\epsilon_2} - 7K_1\epsilon_2} \quad (< 1)$$

(2.73)

(ii) [Contraction around $W^u_{\text{loc}}(U^*)$ in $E^s$-plane] Consider the time evolution of a vector $U$ under $R_{n\sigma}$, with the initial data satisfying $\| U - U^* \| \leq \epsilon_2$. We can find $\tilde{r}^m > 0$ (depending on $\epsilon_2$) such that

$$\text{dist}(R_{n\sigma}(U), W^u_{\text{loc}}(U^*)) \leq e^{-nK^m} \| U - U^* \|$$

holds as long as $\| R_{n\sigma}(U) - U^* \| \leq \epsilon_2$.

For a proof, see Appendix A.

Finally we define sets of vectors

$$A_\pm(\epsilon, \delta) \equiv \left\{ U_\pm \in \Gamma \mid e^{-2\epsilon_2} \leq \| U_\pm - U^* \|_u \leq \epsilon, \text{ dist}(U_\pm, W^u_{\text{loc}}(U^*)) \leq \delta \right\} ,$$

(2.75)

where for a given set $A \subset \Gamma$ and a vector $U \in \Gamma$, we denote their “distance” as

$$\text{dist}(U, A) \equiv \inf_{U' \in A} \| U - U' \|.$$

(2.76)

The set $A_\pm$ is schematically shown in Fig. 3 of Sec. II B 2.

3. Local behaviour of the flow: Near-critical solution

We now trace the time evolution of a slightly supercritical initial data $U$. Our final goal is to trace it until it finally blows up and forms a black hole, but in this section we trace it until its deviation from $W^u_{\text{loc}}(U^*)$ becomes of order one (Stage I of Sec. II B 2) as in (2.24), so that we can tell that it is not critical. As shown, this part plays the essential role in determining the critical exponents. Our presentation is modeled after a standard approach in applications of rigorous renormalization group techniques in statistical mechanics, see e.g. (24, 25). The rest of the flow (Stage II) is considered in Sec. II C 3.

Time evolution of an initial data sensitively depends on its location relative to $W^u_{\text{loc}}(U^*)$. So we decompose the initial data $U$ as

$$U = b F_{\text{rel}} + U_c,$$

(2.77)

where $U_c \in W^u_{\text{loc}}(U^*)$ [i.e. $R_{n\sigma}(U_c) \to U^*$ as $n \to \infty$], $F_{\text{rel}}$ is the normalized relevant eigenmode defined in (2.64), and $b \in \mathbb{R}$ is its coefficient. See Fig. 3(b) in Sec. II C 3. Proposition 3(i) guarantees that we can find $U_c$ and decompose this way for $\| U - U^* \| \leq \delta_1$. Generically $b = O(p - p_c)$, because $b$ measures the component of $F_{\text{rel}}$ in $U - U_c$.

Now, for $n > 0$, we define $F_n$ in terms of the decomposition $\mathcal{R}_{n\sigma}(U) = R_{n\sigma}(U_c) + F_n$,

$$F_n = b_n F_{\text{rel}} + F_n^*,$$

(2.78)

and further decompose $F_n$ in the spirit of (2.80):

$$F_n = b_n F_{\text{rel}} + F_n^*$$

(2.79)

See Fig. 3(b) in Sec. II C 3. We do emphasize that the decomposition (2.78) differs from that of (2.33) in Sec. II C 1, in that $F$ there measures the difference between $U$ and the fixed point $U^*$, while $F$ here measures the difference between $U$ and the (time-evolved) critical solution $U^{(n\sigma)}$. Also note that $F_n$ ($n > 0$) is not necessarily parallel to $F_{\text{rel}}$.

The initial condition implies $b_0 = b, F_0 = 0$.

After these preparations, concerning the flow in Stage I, we have:

**Proposition 2** (Local behaviour of non-critical flows) Consider the time evolution of an initial data $U_{\text{init}} = b_0 F_{\text{rel}} + U_c$, with $U_c \in W^u_{\text{loc}}(U^*)$, $\| U_c - U^* \| < \delta_2$, $0 < |b_0| \leq \epsilon_2$, and at each $n \sigma$ we decompose as in (2.78): $\mathcal{R}_{n\sigma}(U_{\text{init}}) = \mathcal{R}_{n\sigma}(U_c) + b_n F_{\text{rel}} + F_n^*$. Let $n_2 \equiv \max\{ n \in \mathbb{Z} \mid |b_k| \leq \epsilon_2 \quad 0 \leq k \leq n \}$. Then under Assumptions L1, L2, and L1A, we can find $C_2 > 1$ (independent of $b_0$ and $\epsilon_2$) such that the following holds:

(i) $n_2$ is finite and satisfies $(n \leq n_2)$

$$\frac{2}{3e^{\epsilon_2} - 1} \epsilon_2 \leq |b_{n_2}| \leq \epsilon_2,$$

(2.80)

$$\| F_n^* \| \leq |b_n|$$

(2.81)

We could trace the time evolution by employing the decomposition (2.77) also for $n > 0$, i.e. $\mathcal{R}_{n\sigma}(U) = U_{n\sigma} + b_n F_{\text{rel}}$ with $U_{n\sigma} \in W^u_{\text{loc}}(U^*)$. However, we here employ (2.78), because it also allows us to deal with the case of Sec. II B 2 in essentially the same way.
and (2.88) are of the form of (B3) and (B4) there, with an elementary exercise in calculus, and is summarized as to simplify the notation. In fact we can take \( \delta \searrow 0 \) as \(|b_0|\searrow 0\). In the above Proposition gives an estimate on the time interval spent by the flow in Stage I. It also provides information on the exit of Stage I, by showing that the exit is essentially squeezed into the tiny tube \( A_\pm \) around \( W_{loc}^s(U^*) \) (see Fig. 3).

Proof of Proposition 3. (i) We first derive recursion equations for \( b_n \) and \( F_n \).

Using (2.33) with \( U = U_c \), we have

\[
\mathcal{R}_{\sigma}(U) = \mathcal{R}_\sigma(U_c) + \mathcal{O}(K_1 ||F_n||^2). \tag{2.85}
\]

This implies, according to our decomposition (2.78),

\[
F_{n+1} = T_{\sigma,U_c}(F_n) + \mathcal{O}(K_1 ||F_n||^2)
\]

where in the second step we used (2.30). This corresponds to the first line of (2.42); the difference is we here have \( \mathcal{O}(6K_1 ||U_c|| F_n) \). Arguing as we did in deriving (2.43)\(\rightarrow\) (2.46), i.e. decomposing \( F \) into \( F^\sigma \) and \( F^u \) as in (2.79), we this time obtain

\[
\begin{align*}
\lambda_{n+1} &\leq e^{-\kappa \sigma} \lambda_n + \tilde{K}_1(b_n \vee f_n) \left[ \tilde{\mathcal{O}}(6\nu_n) + \mathcal{O}(b_n \vee f_n) \right] \tag{2.87} \\
\lambda_{n+1} &\leq e^{\kappa \sigma} \lambda_n + \tilde{K}_1(b_n \vee f_n) \left[ \tilde{\mathcal{O}}(6\nu_n) + \mathcal{O}(b_n \vee f_n) \right]. \tag{2.88}
\end{align*}
\]

where we have introduced \( \lambda_n \equiv \|U_c^{(n\sigma)} - U^*\|, f_n \equiv \|F^n\| \) to simplify the notation.

Our remaining task is to solve the recursions. This is an elementary exercise in calculus, and is summarized as Lemma 3 in Appendix B and the recursive inequalities (2.87) and (2.88) are of the form of (B3) and (B4) there, with \( \Lambda \equiv e^{\kappa \sigma}, \lambda \equiv e^{-\kappa \sigma}, \) and \( C_3 = 6K_1 \). And that \( \lambda_n \) satisfies the assumption of the Lemma follows from Proposition 3. (2.53). Part (i) of Proposition 2 thus follows immediately from Lemma 3.

(ii) We apply Lemma 3 to the time evolution of the vector \( U_2 \equiv U^{(n_2 \sigma/2)} \). Here \( n_2/2 \) should be interpreted as its integer part. We have, by Lemma 3 and Proposition 1,

\[
\|U^{(n_\sigma)} - U^*\| \leq |b_0| + \|U_c^{(n\sigma)} - U^*\| \leq \varepsilon_2 \left( 1 + e^{\kappa \sigma} \right)^{-(n_2-n)} + \delta_2 e^{-n_\sigma}. \tag{2.89}
\]

For \( n \in \left\lceil n_2/2, n_2 \right\rceil \), the above is bounded uniformly by \( \delta_2 + \varepsilon_2 \). This also shows that the initial data \( U_2 \) for this part of time evolution satisfies \( \|U_2 - U^*\| = \|U^{(n_2 \sigma/2)} - U^*\| \searrow 0 \) as \( n_2 \searrow \infty \). So taking \( \delta_2 + \varepsilon_2 \) sufficiently small, we can apply Lemma 3 to conclude that \( \text{dist}(U^{(n_\sigma)}, W_{loc}^s(U^*)) \) goes to zero as \( n_2 \) goes to infinity (i.e. \( b_0 \) goes to zero).

4. Global behaviour and physical consequences

We are now at the stage of presenting the rigorous version of Claims 1 and 2, together with necessary assumptions. After the analysis in the previous section, that is, the analysis of the flow at Stage I, we in this section consider Stage II of the flow. We want to trace the flow further until it blows up, and then calculate the mass of the Black hole. This stage involves nontrivial physical and mathematical processes whose detailed analysis is beyond the scope of this paper. In this sense, this section is essentially a repetition of Sec. 1B3, with a difference that we here explicitly list up all the assumptions (which are physically reasonable in view of numerical simulations) on the global behaviour of the flow, under which we can present the rigorous version of Claims 1 and 2, i.e. Theorem 1.

Assumption G1 (Flows in Stage II) For sufficiently small \( \delta_3 > 0 \) and for \( A_\pm(\varepsilon_2, \delta_3) \) defined in (2.77), we have:

- \( \text{U}_- \equiv \mathcal{R}_\sigma(U_-) \) tends to a flat space-time as \( s \to \infty \), for all \( U_- \in A_-(\varepsilon_2, \delta_3) \).
- \( \text{U}_+ \equiv \mathcal{R}_\sigma(U_+) \) blows up at some finite time \( s = s_{AH} > 0 \); \( \xi = \xi_{AH} \); \( s_{AH} \) and \( \xi_{AH} \) are uniformly bounded above and below for all \( U_+ \in A_+(\varepsilon_2, \delta_3) \). (Recall that \( \xi \) is related to \( r \) by \( r = \xi e^{-\beta s} \) as in (2.4).)

\[9\text{Precisely speaking, } |b_{n_2}| < \varepsilon_2 \text{ does not guarantee } ||U^{(n_2\sigma)} - U^*|| < \varepsilon_2, \text{ because } b_{n_2} \text{ measures } ||U^{(n_2\sigma)} - U^*||, \text{ and in general } ||U^{(n_2\sigma)} - U^*|| > 0. \] However, because \( ||U_c^{(n_2\sigma)} - U^*|| \) goes to zero as \( n_2 \) goes to infinity, the difference can be neglected as \( b_0 \) goes to zero.
Assumption G2 (Mass inside the apparent horizon) The mass \( M_{AH} \) inside the apparent horizon at its formation is related with its radius \( r_{AH} \) by \( C^{-1}r_{AH} \leq M_{AH} \leq Cr_{AH} \), where \( C \) is a finite positive constant independent of \( r_{AH} \).

Assumption G3 (Relation between masses) The final mass of the black hole \( M_{BH} \), and the mass at the formation of the apparent horizon \( M_{AH} \), are of the same order. More precisely, there exists a positive constant \( C' \) such that \( C'^{-1}M_{AH} \leq M_{BH} \leq C'M_{AH} \) holds for all the near-critical solutions.

Remarks on the meaning of these assumptions are in order.

Remark.

1. Assumption G1 specifies the behaviour of the flow in Stage II. Thanks to our analysis in Stage I, we have only to know the fate of the flow which is emerging from a very small neighbourhood of \( W^u(U^*) \) (recall Fig. 2 of Sec. II B). This means that physically it is sufficient to trace the time evolution of two vectors which is located on \( W^u(U^*) \).

2. Assumption G2 is valid for a spherically symmetric gravitational collapse (in fact, \( M_{AH} = r_{AH}/2 \)), which is the main topic in this paper. However we termed it as an Assumption, in order to make our presentation more general.

3. Assumption G3 involves a subtle physical process after the formation of an apparent horizon. Strictly speaking, the radius of the apparent horizon is not exactly related with the final Bondi–Sachs mass which gives the mass of the black hole. The final mass \( M_{BH} \) differs from (greater than, when the dominant energy condition is satisfied,) the mass at the apparent horizon \( M_{AH} \) by the matter absorbed into the black hole after the formation of the apparent horizon. However, in our case, it is expected from the form of the relevant mode and the blow-up profile in numerical simulations that this effect can only make the black hole slightly heavier, say by a factor of two or so. (It should be noted that usual numerical simulation assumes Assumption G3, and identifies the radius of apparent horizon with the final mass of the black hole.)

Under these preparations, we can now state our rigorous version of Claim 1 and Claim 2, which is the main result of this section.

**Theorem 1** Under Assumptions L1, L2, L1A and Assumptions G1, G2, G3, the universality (U1, U2) and scaling (S1, S2) of Sec. II B hold. More precisely, (i) All the initial data sufficiently close to \( U^* \) (but not exactly on \( W^c_{loc}(U^*) \)) once approach, but finally deviate from, \( U^* \). The final fate of these data are either a flat space-time or a black hole, depending on which side of \( W^c_{loc}(U^*) \) lies the initial data. (ii) Consider the time evolution of the initial data

\[
U_{init} = b^{prel} + U_c
\]

with \( \|U_c - U^*\| \leq \delta_2, 0 < |b| \leq \delta_2 \), which is on the side of black hole of \( W^c_{loc}(U^*) \). Then as \( b \) goes to zero, the mass of the black hole formed satisfies

\[
C_1|b|^\beta/\kappa \leq \frac{M_{BH}}{M_0} \leq C_2|b|^\beta/\kappa
\]

with \( \beta \)-independent positive constants \( C_1, C_2 \). Here \( M_0 \) is the initial (say, ADM or Bondi-Sachs) mass.

**Remark.** By refining the above argument leading to Theorem 1, especially refining the proof of Lemma 1, we can see the following: if there is a marginal mode, but we still have a stable manifold of \( U^* \) of codimension one, it can happen that the scaling of mass has a logarithmic correction term, like

\[
M_{BH} \propto |p^* - p|^\beta_{BH} (\ln |p^* - p|)^\beta_m
\]

with some exponents \( \beta_{BH} \) and \( \beta_m \).

**Proof.** Proposition 1 is a rigorous version of Step 1 of the ‘proof’ of Claim 2 of Sec. II B. Assumptions G1, G2 and G3 enable us to perform Step 2, proving (i) of the Theorem. This also shows

\[
r_{AH}/C \leq M_{BH} \leq C r_{AH}, \quad \epsilon_2/C \leq |b|e^{\epsilon_2} \leq C \epsilon_2,
\]

where \( C \) is a positive constant, and \( \tau \) is the total time until the formation of the apparent horizon. Since \( r_{AH}/M_0 = O(e^{-\tau}) \) by the definition of scaling transformation (2.2), we get (ii) of the Theorem.

**D. Discrete self-similarity**

Significance of discretely self-similar solutions can be well understood with minor modifications to the scenario presented in Sec. II B.

\[\text{[10] We have included } M_0 \text{ in the expression (2.9) to avoid a nongeneric situation in which } M_0 \text{ behaves like power of } b_0 \text{ and } \beta_{BH} \text{ becomes different from } \beta/\kappa.\]
S

uation

for some \( \Delta \neq 0 \). With a slight abuse of notation, we use \( U_{lc} \) to denote the orbit itself, while \( U_{lc}^{(s)} \) denotes a point on it. Let us call a function \( u \) discretely self-similar with parameters \( (\alpha, \beta) \) if it satisfies \( u(t, r) = u^{(\Delta, \alpha, \beta)}(t, r) \) for some \( \Delta > 0 \). Each discretely self-similar solution \( u \) of the PDEs with parameters \( (\alpha, \beta) \) of the Assumption S in Sec. II B is related to a limit cycle \( U_{lc} \) by \( u(t, r) = (-t)^{\alpha} r^{\beta} \).

For a limit cycle \( U_{lc} \), it is convenient to consider eigenmodes of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \), where \( T_{s, U} \) is defined in (2.7). An eigen-

mode \( F \) of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \) is a function satisfying

\[
T_{\Delta, U_{lc}^{(0)}}(\cdot) F = e^{\kappa \Delta} F.
\]

The exponent \( \kappa \), which is equal to the Lyapunov exponent, gives the mean growth rate of the perturbation \( F \). Relevant, irrelevant, and marginal modes are defined according to the sign of \( \text{Re} \kappa \) as in Sec. I A. It is easily seen that if \( F \) is an eigenmode of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \) then \( T_{s, U_{lc}^{(0)}}(\cdot) F \) is an eigenmode of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \) with the same \( \kappa \). Note that for a limit cycle \( U_{lc} \) there is a trivial marginal mode \( F_0 \) of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \) which is the tangent vector of \( U_{lc}^{(0)} \) in \( \Gamma \):

\[
F_0 \equiv \lim_{s \to 0} \frac{U_{lc}^{(s)} - U_{lc}^{(0)}}{s} = \dot{R}(U_{lc}^{(0)}).
\]

To measure the ‘distance’ between a vector \( U \) and the limit cycle \( U_{lc} \), we introduce

\[
dist(U, U_{lc}) \equiv \inf_{s} \| U - U_{lc}^{(s)} \|.
\]

and we simply say “\( U \) is close to \( U_{lc} \)” when \( \text{dist}(U, U_{lc}) \) is small.

We can now show the discrete versions of the claims in Sec. II B.

Claim 1’ Suppose there is a limit cycle \( U_{lc} \) of periodicity \( \Delta \), with no nontrivial marginal modes. If (S1) and (U1) hold with this limit cycle \( U_{lc} \), then there should be a unique relevant mode for this \( U_{lc} \). Conversely, if \( U_{lc} \) has a unique relevant mode, then (S1) and (U1) hold with this \( U_{lc} \), at least for all the initial data sufficiently close to \( U_{lc} \). (Except, of course, for the exactly critical initial data.)

Claim 2’ If the relevant mode is unique, the black hole mass satisfies

\[
M_{\text{BH}} \propto (p - p^*)^{3/\kappa},
\]

for slightly supercritical solutions, where \( \kappa \) is the eigenvalue of the unique relevant mode of \( T_{\Delta, U_{lc}^{(0)}}(\cdot) \).

Here “self-similarity” in (S1) and (S2) should be understood as “discrete self-similarity.”

Rough idea of the ‘proof’ is as follows (see the following section for rigorous results). We reduce the problem of a limit cycle to that of a fixed point by considering the Poincaré section \( S, \) a codimension-one submanifold transverse to the limit cycle, and the Poincaré map \( P \) on it induced by the renormalization group flow \( \mathcal{R} \). (See Fig. 4.) \( (S, P) \) defines a discrete dynamical system, for which the limit cycle in \( \Gamma \) corresponds to a fixed point in \( S \). The behaviour of the map \( P \) on the Poincaré section \( S \) is the same as that of the map \( \mathcal{R}_\sigma \) on the whole phase space \( \Gamma \) in the continuously self-similar case. Thus it follows from Claims 1 and 2 that Claims 1’ and 2’ hold.

2. The Poincaré map and the local behaviour of the flow

In this and next sections, we give rigorous results under certain assumptions. In this section, we consider the local structure of the phase space and show how the Poincaré map reduces the problem of the limit cycle to that of the fixed point.

\[\text{Claim 1’} \quad \text{Claim 2’} \]

By reducing in this way, we are losing some information on detailed behaviour of the flow. For example, if one is interested in a ‘wiggle’ predicted in [22], one should study the continuous time evolution directly.

After the submission of this paper we learned that the “wiggle” has been studied and numerically confirmed by Hod and Piran [23].
Let us first list assumptions used in the derivation, which are similar to those in the case of continuous self-similarity. To simplify the notation, we often write $\mathcal{T}_\Delta$ for $\mathcal{T}_{\Delta, U_{lc}^{(0)}}$ in the following.

**Assumption L1’ (Invariant subspaces of $\mathcal{T}_\Delta$)** The tangent space $T_{U_{lc}^{(0)}} \Gamma$ of $\Gamma$ at $U_{lc}^{(0)}$ is a direct sum of invariant subspaces of $\mathcal{T}_\Delta$:

$$T_{U_{lc}^{(0)}} \Gamma = E^u(U_{lc}^{(0)}) \oplus E^c(U_{lc}^{(0)}) \oplus E^s(U_{lc}^{(0)}); \quad (2.98)$$

where $E^c(U_{lc}^{(0)})$ is a one-dimensional eigenspace spanned by $F_0$ satisfying

$$\mathcal{T}_\Delta(F_0) = F_0; \quad (2.99)$$

and $E^u(U_{lc}^{(0)})$ and $E^s(U_{lc}^{(0)})$ are of dimension $N$ and of codimension $N + 1$, respectively, and there exists $\bar{\kappa} > 0$ such that

$$\|\mathcal{T}_\Delta(F)\| \geq e^{\bar{\kappa}\Delta} \|F\|, \quad F \in E^u(U_{lc}^{(0)}),$$

$$\|\mathcal{T}_\Delta(F)\| \leq e^{-\bar{\kappa}\Delta} \|F\|, \quad F \in E^s(U_{lc}^{(0)}). \quad (2.100)$$

**Assumption L2’ (Smoothness of $\mathcal{R}$).** The renormalization group transformation $\mathcal{R}$ considered as a function $\mathcal{R} : \Gamma \to \mathbb{R} \times \Gamma \to \mathbb{R}$ of $U_{lc} \in \Gamma$ is of class $C^{1,1}$ in a tubular neighborhood of the limit cycle $U_{lc}$. Namely, for $|s' - s|, \|U' - U\| < \delta_0$ and $\sup_{0 \leq s' \leq s} \text{dist}(\mathcal{R}_{s'}(U), U_{lc}) < \delta'_0$,

$$\mathcal{R}_{s'}U - \mathcal{R}_sU = (s' - s)\bar{\mathcal{R}}(\mathcal{R}_sU) + O(|s' - s|^2), \quad (2.101)$$

$$\mathcal{R}_{s'}U' - \mathcal{R}_sU = \mathcal{T}_{s', U}(U' - U) + O(\|U' - U\|). \quad (2.102)$$

**Remark.** Assumption L1’ together with the existence of $\mathcal{R}$ in Assumption L2’ is a natural limit cycle version of Assumption L1 of Sec. [1]. Assumption L2’ is slightly stronger than Assumption L2. However, we consider it to be also physically reasonable and adopt it in order to rule out subtle and unexpected behaviour of the flow near the limit cycle $U_{lc}$. Assumption L2’ implies the following:

$$\|\mathcal{R}(U') - \mathcal{R}(U)\| = O(\|U' - U\|), \quad (2.103)$$

$$(\mathcal{T}_{s', U} - \mathcal{T}_{s, U}) F = O(\|s' - s\| \cdot \|F\|), \quad (2.104)$$

$$(\mathcal{T}_{s', U} - \mathcal{T}_{s, U}) F = O(\|U' - U\| \cdot \|F\|). \quad (2.105)$$

Let us define the Poincaré map and reduce the problem to that of a dynamical system on a submanifold of codimension one. Let $S$ be a submanifold of $\Gamma$ defined by

$$S \equiv U_{lc}^{(0)} + Y, \quad Y \equiv E^u(U_{lc}^{(0)}) \oplus E^s(U_{lc}^{(0)}). \quad (2.106)$$

A point moving along a flow curve $\mathcal{R}_sU$ starting from point $U$ on $S$ will return to $S$ for some $s \approx \Delta$, and we can define the Poincaré map $P : S \to S$ so that $P(U) \in S$. More precisely, we have the following lemma.

**Lemma 3 (The Poincaré map)**

Under Assumptions L1’ and L2’, the following (i) and (ii) hold.

(i) There is a $\delta$-neighborhood of $U_{lc}^{(0)}$ in $S$ and the Poincaré map $P$ which is a local $C^{1,1}$-diffeomorphism from $S$ to $S$ satisfying

$$P(U) = \mathcal{P}_p(U)(U), \quad (2.107)$$

where $p : S \to \mathbb{R}$ is $C^{1,1}$ and $p(U_{lc}^{(0)}) = \Delta$. In particular, the tangent map $T_{U_{lc}^{(0)}}^P \mathcal{P}$ at $U_{lc}^{(0)}$ is given by

$$\frac{T_{U_{lc}^{(0)}}^P \mathcal{P}}{U_{lc}^{(0)}} = \mathcal{T}_{\Delta, U_{lc}^{(0)}} \big|_{Y}. \quad (2.108)$$

(ii) Furthermore, a tubular $\delta$-neighbourhood of $U_{lc}$ is foliated by sections

$$S(s) \equiv U_{lc}^{(s)} + Y(s), \quad Y(s) \equiv \mathcal{T}_{s, U_{lc}^{(0)}}Y, \quad (s \in \mathbb{R}), \quad (2.109)$$

and on each section there exists the Poincaré map $P^{(s)}$ which satisfies all of (i) with $Y, \mathcal{P}, S$ and $U_{lc}^{(0)}$ replaced by $Y(s), \mathcal{P}^{(s)}, S(s)$ and $U_{lc}^{(s)}$. In particular, we have

$$\frac{T_{U_{lc}^{(s)}}^P \mathcal{P}}{U_{lc}^{(s)}} = \mathcal{T}_{\Delta, U_{lc}^{(s)}} \big|_{Y(s)}. \quad (2.110)$$

The Poincaré maps $P^{(s)}$ for various $s$ are related by

$$P^{(s+s')} \circ \tilde{P}_s = \tilde{P}_{s'} \circ P^{(s)}, \quad (2.111)$$

where $\tilde{P}_s : U \to \mathcal{R}_{\tilde{p}(U)}(U) \quad (s \in I \equiv [-\Delta, \Delta])$ is a foliation-preserving local diffeomorphism on $\Gamma$ which maps $S^{(s)}$ to $S^{(s+s')}$. Here $\tilde{p}$ is a $C^{1,1}$-function determined by $s'$ such that $\mathcal{R}_{\tilde{p}(U)}(U) \in S^{(s+s')}$, and satisfies $\tilde{p}(U_{lc}^{(s)}) = s'$. The tangent map $T_{U_{lc}^{(s)}}^P \tilde{P}_{s}$ at $U_{lc}^{(s)}$ satisfies

$$\frac{T_{U_{lc}^{(s)}}^P \tilde{P}_{s}}{U_{lc}^{(s)}} = \mathcal{T}_{s', U_{lc}^{(s)}}. \quad (2.112)$$

Moreover $\tilde{P}$ is $C^{1,1}$, as a function on $I \times (\delta$-neighbourhood of $U_{lc})$.

**Remark.** The choice of $S$ being a linear space, though in general it need not to be, makes the subsequent discussion much simpler. In particular, by reducing the flow on $\Gamma$ to maps on $S$, we can use the results in the fixed point case.

**Remark.** In fact, (i) is included in (ii). One could define $P_s$ first and then the Poincaré map by $P^{(s)} = P\big|_{S^{(s)}}$.}

**Proof.** (i) The proof here is based on the standard treatment of the Poincaré map (see, e.g. [3] Section 8). Assumption L1’ immediately implies that $S$ is transverse to
where the inverse derivative of \( f \), evaluated at \((U,p(U),\mathcal{P}(U))\), exists and is given by (2.114).

The Poincaré map on \( S \) is the restriction of \( \mathcal{P} \) to \( S \cap B(U_{lc}^{(0)},\delta_0^U,\Gamma) \), which we denote by \( \mathcal{P} \). The transversality of \( \mathcal{R}(U) \) and \( S \) implies that this restriction is a one-to-one map. Eq. (2.121) implies that
\[
T^\mathcal{P}_{U_{lc}^{(0)}} = \pi_{U_{lc}^{(0)}} \circ T_{\Delta, U_{lc}^{(0)}} \big|_Y = T_{\Delta, U_{lc}^{(0)}} \big|_Y ,
\]
where the second equality follows from invariance of \( Y \) under \( T_{\Delta, U_{lc}^{(0)}} \).

(ii) It follows from Assumption 1’ and \( T_{\Delta, U_{lc}^{(0)}} \circ T_{s, U_{lc}^{(0)}} = T_{s, U_{lc}^{(0)}} \circ T_{\Delta, U_{lc}^{(0)}} \) that \( T_{s, U_{lc}^{(0)}} E(U_{lc}^{(s)}) \) are invariant subspaces of \( T_{\Delta, U_{lc}^{(s)}} \) so that \( S^{(s)} \) and \( \mathcal{R}(U_{lc}^{(0)}) \) are transverse.

Then by the same argument as (i), one can define \( \mathcal{P}_s \) and the lemma is proved. (The existence of a tubular neighborhood on which \( \mathcal{P}_s \) is well defined follows from compactness of \( I \).) \( \square \)

Though we will consider qualitative behaviour mainly on the Poincaré section, we show here the qualitative behaviour in the whole phase space, which is the limit cycle version of Proposition 1(i).

**Proposition 2** (Invariant manifolds of \( \mathcal{R} \) at \( U_{lc} \))

Under Assumptions L1’ and L2’, we have the following: In a tubular neighbourhood of \( U_{lc} \), there exist a stable manifold \( W^s(U_{lc}) \) of codimension \( N \), and an unstable manifold \( W^u(U_{lc}) \) of dimension \( N + 1 \), whose tangent spaces at \( U_{lc}^{(s)} \) are \( T^s_{U_{lc}^{(s)}} (E^s(U_{lc}^{(0)}) \oplus E^r(U_{lc}^{(0)})) \) and \( T^u_{U_{lc}^{(s)}} (E^u(U_{lc}^{(0)}) \oplus (E^r(U_{lc}^{(0)}))) \), respectively.

In terms of the Poincaré sections, there exist a stable manifold \( W^s(U_{lc}^{(s)},S^{(s)}) \) and an unstable manifold \( W^u(U_{lc}^{(s)},S^{(s)}) \) of each \((S^{(s)},\mathcal{P}^{(s)})\) and are given by
\[
W^s(U_{lc}^{(s)},S^{(s)}) = \mathcal{P}_s W^s(U_{lc}^{(0)},S),
\]
\[
W^u(U_{lc}^{(s)},S^{(s)}) = \mathcal{P}_s W^u(U_{lc}^{(0)},S).
\]

The stable and unstable manifolds \( W^s(U_{lc}) \) and \( W^u(U_{lc}) \) of the whole flow are written as
\[
W^s(U_{lc}) = \big\{ \mathcal{R}_s W^s(U_{lc}^{(0)},S) \big\} \big|_{s \in \mathbb{R}},
\]
\[
W^u(U_{lc}) = \big\{ \mathcal{R}_s W^u(U_{lc}^{(0)},S) \big\} \big|_{s \in \mathbb{R}}.
\]

**Proof.** We now show that discrete dynamical system \((S,\mathcal{P})\) satisfies Assumptions L1 and L2 of Sec. II C, i.e., we show that the assumptions hold under the replacement
\[
(\Gamma,\mathcal{R}_\sigma,\mathcal{T}_{\sigma,U}) \longrightarrow (S,\mathcal{P},\mathcal{T}_U^\mathcal{P}).
\]
Assumption L2 holds for $(S, \mathcal{P})$ because $\mathcal{P}$ is of class $C^{1,1}$ (see (2.113)–(2.118)). Assumption L1 holds for $(S, \mathcal{P})$ because of (2.108) and Assumption L1'. Then Proposition 3 of Sec. 5.1C implies that in $S$ there exists a stable manifold of codimension $N$ in $S$ and an $N$-dimensional unstable manifold of $\Gamma^{l(0)}$. Again, more sophisticated argument guarantees their smoothness (Theorem 5.2, Theorem 5.11A); they are of class $C^{1,1}$.

Next, consider a one-parameter family of discrete dynamical systems $(S^{(x)}, \mathcal{P}^{(x)})$. Lemma 3(ii) states that $\mathcal{P}^{(x)}$ is a diffeomorphism from $S^{(x)}$ to $S^{(x)}$ which preserves the action of the Poincaré maps on them. This implies that $\lim_{n \to \infty} \mathcal{P}^{(x)n}(U) = U^{(x)}$ if and only if $\lim_{n \to \infty} \mathcal{P}^{(x)n}(\bar{U}^{(x)}) = \bar{U}^{(x)}$ so that $\mathcal{P}^{(x)}$ sends the stable manifold $W^s(U^{(0)}, \delta)$ to $W^s(U^{(x)}, \delta)$. It also implies that $W^s(U^{(x)}, \delta)$ is tangent to $T_{U^{(x)}} E^s(U^{(x)})$ so that $W^s(U^{(x)}, \delta)$ is tangent to $T_{U^{(x)}} (E^s(U^{(x)}) \oplus E^c(U^{(x)}))$. We have a similar result for the unstable manifold (consider $\mathcal{P}^{-1}$ in place of $\mathcal{P}$). Thus there exist the stable and unstable manifolds $W^s(U^{(0)})$ and $W^u(U^{(0)})$ of codimension $N$ and dimension $N + 1$ in $\Gamma$, which are given by (2.123) and (2.124), respectively.

We now consider quantitative behaviour of the renormalization group flow. We assume the following, which, because of Proposition 3, asserts that the stable manifold is of codimension 1 and the unstable manifold is of dimension 2.

**Assumption L1’ (Uniqueness of the relevant mode)** $E^u(U^{(0)})$ is one-dimensional, i.e., $N = 1$. Thus $E^u(U^{(0)})$ is an eigenspace spanned by $F^{\text{rel}}$ satisfying

$$ T_\Delta (F^{\text{rel}}) = e^{\kappa \sigma} F^{\text{rel}}, \quad (2.128) $$

where $\kappa$ is the unique positive eigenvalue of $T_\Delta$.

Consider the time evolution $U^{(x)} = R_s(U^{(\text{init})})$ of an initial data $U^{(\text{init})}$ in $S^{(x)}$. We are going to trace the evolution on the Poincaré section $S$. To measure how $U^{(\text{init})}$ is deviated from the critical surface, we introduce a decomposition of $U^{(\text{init})}$ in the tubular neighbourhood of $U^{(0)}$. Lemma 3(iii) implies that for any $U^{(\text{init})}$ there is a unique $S^{(x)} \ni U^{(\text{init})}$. We can decompose any $U^{(\text{init})}$ as $U^{(\text{init})} = U^{(\text{init})} + b F^{(s_0)}$ on $S^{(x)} \ni U^{(\text{init})}$.

$$ U^{(\text{init})} = U^{(\text{init})} + b \frac{F^{(s_0)}}{\|F^{(s_0)}\|}, \quad (2.129) $$

As in the continuously self-similar case, we generically have $b(0) = O(p - p_c)$. In the special case of $U^{(\text{init})} \in S$, Proposition 3 applies for the dynamical system $(S, \mathcal{P})$ and we immediately have the estimates in Proposition 3. In a general case where $U^{(\text{init})} \notin S$ (say, the point $A$ of Fig. 4), all we have to do is to estimate how $U^{(\text{init})}$ and $b$ behave before the flow intersects $S$ for the first time (at the point $B$ of Fig. 4); after that, we can directly apply Proposition 3. The estimate is given by the following lemma.

**Lemma 4** Under Assumptions L1’, L2 and L1’ there exist positive constants $\delta$ and $\epsilon$ which satisfy the following: Let $|b| \leq \epsilon$, $\|U - U^{(s_0)}\| \leq \delta$, $\delta' = \Delta - s_0$ and

$$ U = U_{\perp} + b F^{(s_0)}, \quad U_{\perp} \in W^s(U^{(0)}) \cap S^{(s_0)}, \quad (2.130) $$

$$ U = U_{\perp} + b F^{(s_0)}, \quad U_{\perp} \in W^s(U^{(0)}) \cap S. \quad (2.131) $$

Then we have

$$ \|U_{\perp} - U^{(s_0)}\| \leq C(\delta + \epsilon^2), $$

$$ b \leq \epsilon \leq \delta, \quad (2.132) $$

where $C$ is a positive constant.

**Proof.** Let $b' \equiv b/\|F^{(s_0)}\|$ and $\tilde{b}' \equiv \tilde{b}/\|F_1\|$. We have, for fixed $s_0$,

$$ \bar{U} = \bar{P}_{s'}(U_{\perp} + b' F^{(s_0)}) = \bar{P}_{s'}(U_{\perp} + b' F_{\perp}) + O(b'^2) $$

$$ = \bar{P}_{s'}(U_{\perp}) + b' F_{\perp} + O \left( b' \left( \|F^{(s_0)}\| \right) \right) $$

$$ = \bar{P}_{s'}(U_{\perp}) + b' F_{\perp} + O \left( b' \left( \|F^{(s_0)}\| \right) \right), \quad (2.134) $$

where we have used (2.112) at the fourth equality. $(O(x)$ above may depend on $s'$ or $s_0$.) Since $\bar{P}_{s'}(U_{\perp}) \in W^s(U^{(0)})$, we have

$$ \bar{U}_{\perp} = \bar{P}_{s'}(U_{\perp}) + O(\epsilon(\epsilon + \delta)), \quad (2.135) $$

$$ \tilde{b}' = b'(1 + O(\epsilon + \delta)). \quad (2.136) $$

The first equality gives

$$ \|\bar{U}_{\perp} - U^{(0)}\| \leq \|\bar{P}_{s'}(U_{\perp}) - \bar{P}_{s'}(U^{(s_0)})\| + O(\epsilon(\epsilon + \delta)) $$

$$ = O(\delta + \epsilon^2), \quad (2.137) $$

where we have used the smoothness of $\bar{P}$ in Lemma 3. Eq. (2.136) gives

$$ \tilde{b} = b' = b(1 + O(\epsilon + \delta)) = b(1 + O(1)). \quad (2.138) $$

Since $0 \leq s' \leq \Delta$, the $O(x)$ terms above are bounded by a constant independent of $s_0$. \qed
3. Global behaviour of the flow

Now that the local flow structures have been analyzed, we can state a rigorous theorem which corresponds to Theorem 13 under moderate assumptions on the global structures of the flow just as in the case of continuous self-similarity. One possibility is:

**Theorem 2** Suppose Assumptions L1, L2, L1'A, G2, and G3 hold, and Assumption G1 holds for the discrete dynamical system (S, P). Then the following hold.
(i) All the initial data sufficiently close to U0c (but not exactly on W*(U0c) once approach, but finally deviate from, U0c. The final fate of these data are either a flat spacetime or a black hole, depending on which side of W*(U0c) lies the initial data.

(ii) Consider the time evolution of the initial data \( (2.124) \) with \( |U_{\perp} - U_{\perp}^{(s)}| \leq \delta, 0 < |b| \leq \delta \), which is on the side of black hole of W*(U0c). Then as b goes to zero, the mass of the black hole formed satisfies

\[
C_1 |b|^{\beta/\kappa} \leq \frac{M_{BH}}{M_0} \leq C_2 |b|^{\beta/\kappa} \quad (2.139)
\]

with b-independent positive constants C1, C2. Here M0 is the initial (say, ADM or Bondi-Sah) mass.

Proof. Lemma 4 reduces the problem into the dynamical system (S, P) on the Poincaré section S, where Theorem 13 can be applied. □

E. Universality class

In this section, we focus on the problem of the universality in the original sense of the term in statistical mechanics, based on renormalization group philosophy. We have largely benefitted from the formulation of [43]. This is the rigorous version of the heuristic argument of universality class given in [43, 44].

Consider general cases in which the PDEs do not necessarily satisfy the fundamental assumption of scale invariance, Assumption S in Sec. 11.4, or self-similar solutions with the scale invariance is not relevant for a critical behaviour (e.g. having more than one relevant mode). Methods of renormalization group gives us a clear understanding on which systems can exhibit the same critical behaviour. The main point is that the renormalization group transformation drives the equations of motion to a fixed point, where they gain the scale invariance, and the problem reduce to the cases treated in previous sections. As a result, we understand why and to what extent condition (U3a) holds. Knowing the universality class (the class of models which exhibit the same critical behaviour) is very important from a physical point of view, because it guarantees that certain models exhibit the same critical behaviour with a model in the same class, whose analysis could be much simpler. (Imagine we want to deal with a realistic matter. It would be almost impossible to know or deal with the exact form of the microscopic interaction. Universality could make the problem easier and accessible, by reducing it to that for a simpler model, e.g. perfect fluid.)

Suppose unknowns \( u \) satisfy the PDEs of the following form, which is not necessarily scale invariant:

\[
L \left( u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r}, t, r \right) = 0. \quad (2.140)
\]

Let us define the scaling transformation \( S(s, \alpha, \beta) : u(t, r) \mapsto u^{(s)}(t, r) = e^{\beta s} u(e^{-\beta s} t, e^{-\beta s} r) \) as in (2.2). Then \( u^{(s)} \) satisfies equations

\[
L^{(s)} \left( u^{(s)}, \frac{\partial u^{(s)}}{\partial t}, \frac{\partial u^{(s)}}{\partial r}, t, r \right) = 0, \quad (2.141)
\]

where

\[
L^{(s)} \left( u^{(s)}, \frac{\partial u^{(s)}}{\partial t}, \frac{\partial u^{(s)}}{\partial r}, t, r \right) = e^{\gamma s} L \left( e^{-\alpha s} u^{(s)}, e^{(-\alpha+1)s} \frac{\partial u^{(s)}}{\partial t}, e^{(-\alpha+\beta s)} \frac{\partial u^{(s)}}{\partial r}, e^{-s t}, e^{-\beta s r} \right), \quad (2.142)
\]

and \( \gamma = (\gamma_1, \gamma_2, ..., \gamma_m) \) (where \( m \) is the number of component of \( L \)) is a set of constants which may depend on \( \alpha \) and \( \beta \) and are chosen so that \( L^{(s)} \) remains finite and nonzero when \( s \to \infty \). It should be noted that if \( L^{(s)} = L \) holds with some \( \gamma \) the system \( L = 0 \) is invariant under scaling transformation in the sense of Sec. 11.4.

A renormalization group transformation is now defined to be a pair of a transformation on the EOM: \( L \mapsto L^{(s)} \), and the following transformation on the \( \Gamma \):

\[
R_{s, L} : U_i(\xi) = u_i(-1, \xi) \quad \mapsto \quad U_i^{(s)}(\xi) = u_i^{(s)}(-1, \xi) = e^{\alpha s} u_i(-e^{-s}, e^{-\beta s} \xi), \quad (2.143)
\]

where we added subscript \( L \) to show explicitly the RGT’s dependence on \( L \). See Fig. 4. We emphasize that the renormalization group transformation is now to be considered as a pair of two transformations on different spaces: the space of EOM, and the space of initial data (phase space \( \Gamma \)). One can easily show the (semi)group property

\[
R_{s_1 + s_2, L} = R_{s_2, L^{(s_1)}} \circ R_{s_1, L}, \quad (2.144)
\]
We define
\[ W^c(U^*; V) \equiv \{ U^* + F^c + F^u(F^c) \} \] (2.152)
as we did in (2.52). Note that now the definition depends on \( V \).

**Sketch of the Proof of Proposition 3**. The proof is done by Bleher-Sinai argument as was done for Proposition 1.
We sketch the proof by mainly pointing out necessary modifications.

**Step 1. Decomposition.** Same as the proof of Proposition 3A. We employ the same decomposition (2.37), and for each \( F^n \), try to find \( a_c \) such that

\[
R_{na}(U^* + F^n + a_cF^\text{rel}) \to U^*.
\]

(2.153)

**Step 2. Recursion.** By (2.148), we have

\[
R^V_{a}(U^* + F) = R^V_{a}(U^* + F) + \tilde{a}(K_{3}||V(U^* + F)||).
\]

(2.154)

Calculating \( R^V_{\alpha}(U^* + F) \) just as in (2.42), we this time obtain [cf. (2.43), (2.55), (2.56)]

\[
a_{n+1} = e^{-\tilde{\alpha}}a_n + \tilde{O}(K_1||F_n||^2) + \tilde{O}(K_3v_n),
\]

(2.155)

\[
F^n_{a} = T_{a,U^*}(F^n) + \tilde{O}(K_1||F_n||^2) + \tilde{O}(K_3v_n),
\]

(2.156)

where \( v_n = \sup_{||U-U^*|| \leq \delta_0} ||V^{(\alpha)}(U)||. \) Note that the only difference from Sec. II C 1 is the presence of \( \tilde{O}(K_3v_n) \).

**Step 3. Continuity.** Same as the proof of Proposition 3A. Assumption 2C guarantees the continuity in \( a_0 \), even in the presence of \( V \).

**Step 4. Solving the recursion.** Same as the proof of Proposition 3A. Our goal is to find sequences \( r_n, t_n \) for which (a)—(c) of Step 4 of the proof of Proposition 3A hold.

**Step 5. Proof of (a)—(c).** This part significantly differs from the proof of Proposition 3A due to the presence of the term \( \tilde{O}(K_3\nu_n) \) in the recursion. This time, we define \( t_n = r_n \) as follows. First define \( \tilde{v}_n = \sup_{k \geq n} \nu_k \) and choose \( \delta'_1 \) and \( \delta''_1 \) as

\[
\delta'_1 \leq \frac{1 - e^{-\tilde{\alpha}}}{2K_1}, \quad \delta''_1 \leq \frac{1 - e^{-\tilde{\alpha}}}{2K_3} \delta'_1.
\]

(2.157)

Then define \( r_n = t_n \) recursively, starting from \( r_0 = \delta'_1 \) as

\[
r_{n+1} = (e^{-\tilde{\alpha}} + K_1\delta'_1) r_n + K_3\tilde{v}_n
\]

(2.158)

or \( \bar{A} = e^{-\tilde{\alpha}} + K_1\delta'_1 < 1 \)

\[
r_n = (\bar{A})^n \delta'_1 + K_3 \sum_{k=0}^{n-1} \tilde{v}_k (\bar{A})^{n-k-1} \]

(2.159)

We now show that (a)—(c) is satisfied by this choice of \( r_n \). First, dividing the sum over \( k \) in (2.159) according to \( k \leq n/2 \), (2.153) implies (\( \nu_k \) is non-increasing in \( k \) by definition)

\[
r_n \leq (\bar{A})^n \delta'_1 + K_3 \sum_{k=0}^{n/2} \tilde{v}_k (\bar{A})^{n-k-1} + K_3\tilde{v}_{n/2} \sum_{k=n/2}^{n-1} (\bar{A})^{n-k-1}.
\]

(2.160)

Three terms on the RHS goes to zero (as \( n \to \infty \)), either because \( e^{-\tilde{\alpha}} + K_1\delta'_1 < 1 \) or \( \tilde{v}_{n/2} \to 0 \). This proves (a). Similar reasoning also shows the following uniform bound:

\[
r_n \leq (\bar{A})^n \delta'_1 + K_3 \sum_{k=0}^{n-1} \delta''_1 (\bar{A})^{n-k-1} \leq 2\delta'_1.
\]

(2.161)

Now we turn to (b) and (c). First note that it is sufficient to prove [cf. (2.60), (2.61)]

\[
r_{n+1} \geq e^{-\tilde{\alpha}} r_n + K_1 (r_n + t_n)^2 + K_3 \nu_n
\]

(2.162)

\[
t_{n+1} \leq e^{\alpha} t_n - K_1 (r_n + t_n)^2 - K_3 \nu_n.
\]

(2.163)

Eq.(2.162) is satisfied, by our definition of \( r_n \). And (2.163) is satisfied, if

\[
(e^{\alpha} - e^{-\tilde{\alpha}} - 2K_1\delta'_1) r_n \geq 2K_3\tilde{v}_n.
\]

(2.164)

It is not difficult to see (by induction) that (2.164) is in fact satisfied under our choice of \( \delta'_1 \) and \( \delta''_1 \). This completes Step 5.

\( \square \)

To complete the story, we have to make additional assumptions on the global behaviour of the flow in the presence of \( V \). Concretely we assume:

**Assumption GV (Global behaviour in the presence of \( V \))** Consider the time evolution of initial data \( U_1 \in A_{\pm}(\epsilon_2, \delta_3) \) in the presence of \( V \). Then there exist \( \epsilon_2, \delta_3, \delta_4 > 0 \) such that if \( |\|V(U_1)||| \leq \delta_4 \) then Assumptions G1, G2, G3 hold for suitably chosen \( C', \tilde{C}' \).

In the above, \( A_{\pm}(\epsilon_2, \delta_3) \) was defined in (2.73), in terms of dynamics without \( V \).

Under these preparations, we can now state our theorem.

**Theorem 3** Suppose there exist \( \alpha, \gamma \) and \( \beta > 0 \) satisfying \( L^s \to L^* (s \to \infty) \), where \( L^* \) is a fixed point of the scaling transformation; \( (L^*)^{s} = L^* \). Suppose there is a fixed point or limit cycle \( U^* \) of \( R_{s,L} \), with a unique relevant mode, i.e. Assumptions L1, L2, L1A of Sec. II C hold. Moreover, suppose Assumption 2C and Assumption GV hold. Then,

(i) if there exists \( \delta_5 > 0 \) such that for \( |\|U - U^*||| \leq \delta_0 \) we have (1) \( |\|V(s)(U)||| \to 0 \) as \( s \to \infty \) and (2) \( |\|V^{(s)}(U)||| < \delta_5 \) for sufficiently small \( \delta_5 \), then the system \( L = 0 \) exhibits qualitatively the same critical behaviour as the system \( L^* = 0 \). (i.e. there is a critical behaviour and near-critical solutions once approach and
for $U$ compose the initial data of a critical solution. We now trace the time evolution of logarithmic ratio of both hand sides.

The critical exponent (if we define it as the limit of this small $|V(s)|$) leads to qualitatively different global behaviour. (Because the solution is expected to blow up, we cannot expect nice regularity in $V$ in this region.)

Theorem 3 explains how and why claim (U3a) of Sec. 11B holds. The mere condition $L \to L^*$ is not sufficient to guarantee the universality. The following proof in fact suggests the deviation in the critical behaviour:

$$M_{BH} \propto |p - p_c|^{\beta / \kappa} f(p)$$

where

$$f(p) \approx \exp \left[ c_2 \int_{-1}^{c_1 \log |p - p_c|} V(s) ds \right],$$

with positive constants $c_1, c_2$. Concretely, $f(p)$ will be a power of logarithm in $|p - p_c|$ for $|V(s)| \propto 1/s$, and

$$f(p) \propto \exp \left( \text{const.} \log^{1-\alpha} |p - p_c| \right)$$

for $|V(s)| \propto s^{-\alpha} (0 < \alpha < 1)$. The deviation in (2.167) is greater than logarithmic, although it does not change the critical exponent (if we define it as the limit of the logarithmic ratio of both hand sides).

**Proof of Theorem 3** Proposition 3 shows the existence of a critical solution. We now trace the time evolution of off-critical solutions.

We employ the scheme of Sec. 1IC3. That is, we decompose the initial data $U$ as [cf. (2.77)]

$$U = U_c + b F^{rel}$$

and for $n > 0$ decompose [cf. (2.78)]

$$R_{n,s}^V(U) = R_{n,s}^V(U_c) + F_n$$

and further decompose $F_n$ as in (2.79):

$$F_n = b_n F^{rel} + F_n^{n}.$$  

Note that $U_c$ here means the critical solution in the presence of $V$ (and its time evolution). Then arguing as we did in deriving (2.85), we now obtain, using (2.147)

$$R_{n+1,s}^V(U) = R_{n,s}^V(U_c^{n+1}) + F_n$$

$$= R_{n,s}^V(U_c^{n+1}) + T_{\sigma, U_c^{n+1}}^V(F_n) + O(K_1 ||F||^2)$$

which implies

$$F_{n+1} = T_{\sigma, U_c^{n+1}}^V(F_n) + O(K_1 ||F||^2).$$

Now using (2.149) and (2.36),

$$T_{\sigma, U_c}^V(F) = [T_{\sigma, U}^V - T_{\sigma, U_c}^V](F) + [T_{\sigma, U}^V - T_{\sigma, U_c}^V](F)$$

$$= T_{\sigma, U_c}^V(F) + \bar{O}(K_3 ||V(U)|| \cdot ||F||)$$

$$+ \bar{O}(6K_1 \cdot ||U - U_c|| \cdot ||F||)$$

So decomposing $F_n$ into $F_n^{rel}$ and $F_n^{n}$ in (2.172), we have recursions for $b_n$ and $f_n = ||F_n^{rel}||, g_n = ||U_c^{n+1} - U^*||$

$$b_{n+1} = e^{n\sigma} b_n + K_3 (|b_n| \land f_n) \bar{O}(v_n)$$

$$+ K_1 (|b_n| \land f_n) \bar{O}(|b_n| \land f_n) + 6\bar{O}(g_n)$$

$$f_{n+1} \leq e^{-n\sigma} f_n$$

$$+ (|b_n| \land f_n) \bar{K}_1 (|b_n| \land f_n) + 6K_1 g_n + 3K_1 v_n$$

As seen, the only difference here from (B3) and (B4) is the appearance of $v_n$, and this proves the quantitative estimate which corresponds to Proposition 2. In particular, Lemma 3 guarantees that the presence of $O(v_n)$ does not affect the critical behaviour.

The above concludes the local analysis of the flow in the neighbourhood of $U^*$, which corresponds to that in Sec. 1IC3 and 1IC4. The analysis of global behaviour, corresponding to that of Sec. 1IC4, can be carried out in a similar manner under the Assumption GV.

**Remark.** By refining the the proof of Proposition 3, we can show in general (p.c(0) denotes the critical value without $V$)

$$|p_c(V) - p_c(0)| = O(||V(U^*)||).$$

This estimate gives an answer to the following question considered in [25]: fix the value of $p$ to the critical value when $V = 0$ (i.e. $p = p_c(0)$), and vary the magnitude of $V$. What kind of critical behaviour can we observe? The above estimate, together with Theorem 3 shows that we will observe the same critical exponent $\beta_{BH} = \beta / \kappa$, because

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14 Heuristically, we can go further. If $V$ is of definite sign, we can heuristically get a lower bound on the difference, and show

$$|p_c(V) - p_c(0)| = O(||V(U^*)||).$$

This estimate gives an answer to the following question considered in [25]: fix the value of $p$ to the critical value when $V = 0$ (i.e. $p = p_c(0)$), and vary the magnitude of $V$. What kind of critical behaviour can we observe? The above estimate, together with Theorem 3 shows that we will observe the same critical exponent $\beta_{BH} = \beta / \kappa$, because
\(|p_\circ(V) - p_\circ(0)| \leq O(|||V(U^*)|||).
\] (2.178)

We conclude this section by giving two examples to which the above analysis can be applied. Note that the following arguments are not rigorous in contrast to what we have been doing, in the sense that no rigorous verification of Assumptions are given.

1. Example: perfect fluid with a modified equation of state

The first example is a perfect fluid, with equation of state \(f\) is a given function
\[p = (\gamma - 1)\rho + f(\rho).
\] (2.179)

The EOM for this system is given by (3.3) of the next section, supplemented by the above equation of state. When \(f \equiv 0\), this is a usual perfect fluid, to be studied in detail beginning with the next section. It will be shown that the system with \(f \equiv 0\) exhibits a critical behaviour governed by a self-similar solution \(U^*\), which is a fixed point of a renormalization group transformation induced from a scaling transformation \(S\) whose effect on \(\rho\) and \(p\) is:

\[
\rho^*(x) \equiv e^{-2\beta s} \rho(e^{-\beta t}, e^{-\beta r})
\] (2.180)

\[
p^*(x) \equiv e^{-2\beta s} p(e^{-\beta t}, e^{-\beta r}),
\] (2.181)

where \(\beta > 0\) is arbitrary due to gauge invariance. The EOM (3.3) is invariant under the scale transformation, and so is the equation of state with \(f \equiv 0\):

\[L_6^{(s)} \equiv p - (\gamma - 1)\rho - f(\rho), \]

(2.182)

where

\[f^*(x) \equiv e^{-2\beta s} f(e^{2\beta s} x).\] (2.183)

At the fixed point in question, \(\rho^* = (\gamma - 1)\rho^*\) are finite smooth functions of their argument (Sec. IV). So we can expect Assumption 2C to hold in a neighbourhood of the fixed point, by simply taking \(|||V||| \equiv f(\rho)\). Then Theorem 3 implies that the model belongs to the same universality class as that of \(f \equiv 0\), as long as \((\gamma - 1)\rho^*\) is a given function)

\[\beta > 0\] is arbitrary due to gauge invariance. The case \((\gamma - 1)\rho^*\) is arbitrary due to gauge invariance. The

Thus the system with \(f \equiv 0\) is an example to which our scenario in Sec. II.B and Sec. II.C applies.

Our question is to determine the models (forms of \(f\)) which belong to the same universality class as the \(f \equiv 0\) model. Now under the scaling transformation the EOM in question, \(L_6 \equiv p - (\gamma - 1)\rho - f(\rho)\), is transformed into

\[L_6^{(s)} = p^{(s)} - (\gamma - 1)\rho^{(s)} - f^{(s)}(\rho^{(s)}),\] (2.182)

where

\[f^{(s)}(x) \equiv e^{-2\beta s} f(e^{2\beta s} x).\] (2.183)

At the fixed point in question, \(\rho^* = (\gamma - 1)\rho^*\) are finite smooth functions of their argument (Sec. IV). So we can expect Assumption 2C to hold in a neighbourhood

\[
M_{\text{BH}}(p_\circ(0); V) \propto |p_\circ(0) - p_\circ(V)|^{2 \beta H} \propto |||V(U^*)|||^{2 \beta H},
\] (2.177)

where we used Theorem 3 in the first step, and (2.176) in the second.

We fix the gauge by the condition \(g(\cdot, 0) = 1\). The equations of motion can be written in terms of retarded time \(u\), area radius \(r\) and field variable \(h(u, r)\) as \(L_V[h] = 0\), where functional \(L\) labeled by \(V\) is given by

\[L_V[h] \equiv D_V h - \frac{1}{2} (\rho - (\gamma - 1)\rho) \frac{\partial \tilde{g}_V[h]}{\partial r} + \frac{1}{2} \tilde{g}_V[h] V'(\langle h \rangle),
\] (2.187)

where \(\langle \cdot \rangle\), \(g\), \(\tilde{g}\), and \(D_V\) are functionals defined as

\[\langle f \rangle \equiv \frac{1}{r} \int_0^r dr f,
\]

\[g[f] \equiv \exp(-4\pi r^{-1}(1 - \langle f \rangle)^2)),
\]

\[\tilde{g}_V[f] \equiv \langle g[f] \rangle (1 - 8\pi r^2 V(\langle f \rangle))),
\]

\[D_V f = \frac{\partial f}{\partial u} - \frac{1}{2} \tilde{g}_V[f] \frac{\partial f}{\partial r}.\] (2.188)

In the equation of motion, \(L_V[h] = 0\) above, metric variables \(g\) and \(\tilde{g}\) have been solved as functionals \(g\) and \(\tilde{g}_V\), respectively, of \(h\), and the scalar field is given by \(\phi = \langle h \rangle\). Hereafter, for a functional \(F\), \(F[h]\) denotes a function determined by a function \(h\) and \(F[h](a, b)\) denotes the value of function \(F\) at \((a, b)\). For functionals \(F = \langle \cdot \rangle, g, g_V, D_V\), we rewrite the values

\[F[h](e^{-\beta u}, e^{-\beta r})\] in terms of \(h^{(s)}(u, r)\) using the relation

\[h^{(s)}(u, r) = e^{\alpha s} h(e^{-\beta u}, e^{-\beta r});\]
\[ \langle h \rangle (e^{-\beta r} u, e^{-\beta r}) = e^{-\Delta \varepsilon h(s)}(u, r), \quad (2.189) \]
\[ g[h] (e^{-\beta r} u, e^{-\beta r}) = g[h(s)] e^{2 \varepsilon s}(u, r), \quad (2.190) \]
\[ \bar{g}_V [h] (e^{-\beta r} u, e^{-\beta r}) = \left( g[h(s)] \right) e^{-2 \varepsilon s}(u, r), \quad (2.191) \]
\[ (D_V h)(e^{-\beta r} u, e^{-\beta r}) = \left( e^{-(\alpha + 1) s} \frac{\partial h(s)}{\partial u} \right) \cdot \left( g[h(s)] \right), \quad (2.192) \]

where
\[ V(s)(a) = e^{-2 \beta s} V(a). \quad (2.193) \]

We find the renormalized functional \((L_V)(s)\) with parameters \(\alpha\), \(\beta\) and \(\gamma\) by substituting \((2.189)-(2.192)\) into \(e^{\gamma L_V} h(s)\) and reinterpreting it as \((L_V)(s)(u, r)\), the value of function \((L_V)(s)(h(s))\) at \((u, r)\).

Let us find fixed points, i.e., \(L_V\) satisfying \((L_V)(s) = L_V\). It follows from \((2.192)-(2.192)\) that we must choose \(\alpha = 0, \beta = 1\). Then \((2.189)-(2.192)\) and \((2.193)\) read
\[ \langle h \rangle (e^{-\beta r} u, e^{-\beta r}) = \langle h(s) \rangle (u, r), \quad (2.194) \]
\[ g[h] (e^{-\beta r} u, e^{-\beta r}) = g[h(s)](u, r), \quad (2.195) \]
\[ \bar{g}_V [h] (e^{-\beta r} u, e^{-\beta r}) = \bar{g}_V [h(s)](u, r), \quad (2.196) \]
\[ (D_V h)(e^{-\beta r} u, e^{-\beta r}) = e^{-(\bar{D}_V, s)}(D_V, h(s))(u, r), \quad (2.197) \]

and
\[ V(s)(a) = e^{-2 \beta s} V(a). \quad (2.198) \]

Transformation of \(L\) is thus given by
\[ (L_V)(s)[h(s)](u, r) \equiv e^{\gamma L_V} h(s)(u, r) \]
\[ = e^{\gamma L_V} h(s)(u, r) \]
\[ = e^{\gamma L_V} h(s)(u, r). \quad (2.199) \]

Choosing \(\gamma = 1\), we have \((L_V)(s) = L_V(s)\), and from \((2.198)\) we find that \(L_0\) is the only fixed point, i.e., \(L_0 = 0\) is the only model having scale invariance, among the models considered here.

Choptuik’s original result \((1)\), which was further confirmed by a theoretical work by Gundlach \((2)\), was that the system \(L_0 = 0\) shows critical behaviour with this choice of parameters, \(\alpha = 0, \beta = 1\), for the RGT.

The system has a limit cycle which should have a unique relevant mode, according to our scenario in previous sections.

It follows directly from \((2.198)\) and \((2.199)\) that
\[ V(s) \rightarrow V^* = 0, \quad (L_V)(s) \rightarrow L_0 \quad (s \rightarrow \infty), \quad (2.200) \]
and the convergence is so fast that Theorem \(3\) is applicable. This shows that potential \(V\) is an irrelevant term which vanishes asymptotically. Therefore, all \(L_V\)’s are in the same universality class as \(L_0\), i.e., all scalar field models with arbitrary smooth potential \(V\) exhibits the same critical behaviour with the same critical exponent as the minimally coupled massless scalar field.

## III. Equations of Motion.

We now begin our concrete analysis of gravitational collapse of perfect fluid, as a case study of renormalization group ideas.

### A. Equations of Motion

The line element of any spherically symmetric spacetime is written as
\[ ds^2 = -\alpha^2(t, r) dt^2 + a^2(t, r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.1) \]

We assume the matter content is a perfect fluid having energy-momentum tensor \(T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b)\), where \(\rho\) is the density, \(p\) is the pressure, and \(u^a\) is a unit timelike vector whose components are given by:
\[ u_t = \frac{-\alpha}{\sqrt{1 - V^2}}, \quad u_r = \frac{a V}{\sqrt{1 - V^2}}. \quad (3.2) \]

Here \(V\) is the 3-velocity of fluid particles. Introducing locally \(U \equiv V/\sqrt{1 - V^2}\) and \(W \equiv 1/\sqrt{1 - V^2}\), the equations of motion (EOM) are given in this coordinate by:
\[ \frac{a_r}{a} = 1 - \frac{a^2}{2r} + 4\pi r a^2 (\rho W^2 + pU^2) \quad (3.3a) \]
\[ \frac{\alpha r}{a} = \frac{a^2 - 1}{2r} + 4\pi r a^2 (\rho U^2 + pW^2) \quad (3.3b) \]
\[ \frac{a_t}{a} = -4\pi r a (\rho + p) U W \quad (3.3c) \]
\[ (a p W)_x + p(a W)_r + \frac{(r^2 a p U)_r + p(r^2 a U)_r}{r^2} = 0 \quad (3.3d) \]
\[ (a p U)_x + p(a U)_r + (a p W)_r + p(a W)_r = 0 \quad (3.3e) \]

In the above, first three are obtained by taking linear combinations of \((0, 0), (1, 1), (0, 1)\) components of Einstein equation. The last two are obtained by taking linear combinations of the “Bianchi identity” \(T^{\mu \nu} \nu = 0\). We have to supplement the above by the equation of state for the fluid. We consider the case
\[ p = (\gamma - 1) \rho \quad (3.4) \]
where \(\gamma \in (1, 2)\) is a constant (adiabatic index).

In terms of variables \(s = -\ln(-t), x = \ln(-r/t)\), and introducing
\[ N \equiv \frac{\alpha}{a e^x}, \quad A \equiv a^2, \quad \omega \equiv 4\pi r^2 a^2 \rho, \quad (3.5) \]
we can write the equations of the system in an autonomous form, which makes the scale invariance of the system transparent:

\[
\frac{A_x}{A} = 1 - A + \frac{2\omega (1 + (\gamma - 1)V^2)}{1 - V^2} \tag{3.6a}
\]

\[
\frac{N_x}{N} = -2 + A - (2 - \gamma)\omega \tag{3.6b}
\]

\[
\frac{A_s + A_x}{A} = -\frac{2\gamma N\omega}{1 - V^2} \tag{3.6c}
\]

\[
\frac{\omega_x}{\omega} + \frac{\gamma VV_x}{1 - V^2} + (1 + NV)\frac{\omega_x}{\omega} + \frac{\gamma(N + V)V_x}{1 - V^2} = \frac{3(2 - \gamma)}{2}NV - \frac{2 + \gamma}{2}ANV + (2 - \gamma)NV\omega \tag{3.6d}
\]

\[
(\gamma - 1)V\frac{\omega_x}{\omega} + \frac{\gamma V_x}{1 - V^2} + (\gamma - 1)(N + V)\frac{\omega_x}{\omega} + \frac{\gamma(1 + NV)V_x}{1 - V^2}
\]

\[
= - (2 - \gamma)(\gamma - 1)N\omega + \frac{7\gamma - 6}{2}N + \frac{2 - 3\gamma}{2}AN \tag{3.6e}
\]

Only four out of the above five equations are independent. More precisely, the eqn. (3.6d) is automatically satisfied by solutions of the set (3.6a), (3.6b), (3.6c) and (3.6d), as long as they satisfy a boundary condition

\[
A(s, -\infty) = 1, V(s, -\infty) = \omega(s, -\infty) = 0. \tag{4.1a}
\]

In view of this, we pick up the above set of four equations as our basic equations of motion, and use (3.6d) as an auxiliary equation at appropriate stages in the following.

**B. Gauge degrees of freedom**

The only coordinate transformation which preserves the form of (3.4) is

\[
t \mapsto t' \equiv F^{-1}(t), \tag{3.7}
\]

which corresponds to

\[
(s, x) \mapsto (f^{-1}(s), x - s + f^{-1}(s)), \tag{3.8}
\]

where \( F^{-1}(t) = -e^{-f^{-1}(s)} \) with \( \dot{f} \equiv df/ds \neq 0 \). Under this transformation, the variables \( h \) transform to \( \tilde{h} \), where

\[
\tilde{h}(s, x) = \begin{cases} h(s, x) & (h = A, \omega, V) \\ \tilde{f}(s)h(s, x) & (h = N) \end{cases} \tag{3.9}
\]

Eqs. (3.6) are of course invariant under this transformation. Fixing the value of \( N \) at a point, for example, \( x = 0 \), of each constant \( s \) line determines the coordinate system completely. We shall retain the degree of freedom for the time being.

One note: The gauge degree of freedom present in our system enables us to take \( \beta = 1 \) in our formulation of Sec. II A, without introducing new variables.

**IV. THE CRITICAL (SELF-SIMILAR) SOLUTION.**

To carry out the first step of our scenario, we first have to find out fixed points (i.e. self-similar solutions) of RGT. In this section, we find out (almost) all of the self-similar solutions of the EOM. Because this paper is intended to present the “renormalization group” approach, we here mildly try to exhaust all possible self-similar solutions.

**A. Equations for self-similar space-time**

We first require that the space-time is self-similar, i.e. that \( N \) and \( A \) depend only on \( x \): \( N = N_{ss}(x), A = A_{ss}(x) \). Conversely, it can be shown (see Appendix ??) that any spherically symmetric self-similar space-times can be expressed in that form if a freedom of coordinate transformation (3.7) is used. Then it follows from (3.8) that \( \omega_{ss} \) and \( V_{ss} \) are also functions of \( x \) only:

\[
\omega = \omega_{ss}(x), V = V_{ss}(x). \tag{4.1b}
\]

In this sense, the space-time we are interested in is a fixed point of the RGT.

The equations for a self-similar solution, \( \dot{R}(U^*) = 0 \), are then given by omitting terms containing derivatives of \( s \) in (3.6):

\[
\frac{A_x}{A} = 1 - A + \frac{2\omega (1 + (\gamma - 1)V^2)}{1 - V^2} \tag{4.1a}
\]

\[
\frac{N_x}{N} = -2 + A - (2 - \gamma)\omega \tag{4.1b}
\]

\[
\frac{A_s + A_x}{A} = -\frac{2\gamma N\omega}{1 - V^2} \tag{4.1c}
\]

\[
(1 + NV)\frac{\omega_x}{\omega} + \frac{\gamma(N + V)V_x}{1 - V^2} = \frac{3(2 - \gamma)}{2}NV - \frac{2 + \gamma}{2}ANV + (2 - \gamma)NV\omega \tag{4.1d}
\]

\[
(\gamma - 1)(N + V)\frac{\omega_x}{\omega} + \frac{\gamma(1 + NV)V_x}{1 - V^2}
\]

\[
= - (2 - \gamma)(\gamma - 1)N\omega + \frac{7\gamma - 6}{2}N + \frac{2 - 3\gamma}{2}AN \tag{4.1e}
\]

from (4.1a) and (4.1b), one obtains an algebraic identity

\[
1 - A + \frac{2\omega (1 + (\gamma - 1)V^2)}{1 - V^2} = -\frac{2\gamma N\omega}{1 - V^2}. \tag{4.2}
\]

One could eliminate one variable (e.g. \( A \)) from (4.1) using the above identity. However, to avoid making equations too complicated, we keep using four variables...
\((A, N, \omega, V)\) in the following, using the above (4.2) as a check at appropriate stages of our numerical calculation.

Under the self-similar ansatz, the coordinate freedom (4.1) reduces to \(t \rightarrow t = kt\) with constant \(k\), which corresponds to the translation of \(x\) [the other transformations alter the constant \(x\) lines in \((t, r)\) space]. This freedom of coordinate transformation allows one to adjust the value of \(N\) at a given point arbitrarily. We make use of this freedom, and fix the coordinate system by requiring that the sonic point (see below) be at \(x = 0\).

**B. Conditions on self-similar solutions**

The behaviour of self-similar solutions has been extensively discussed by Bogoyavlenskii in [31,32], followed by other works [33,34]. We here briefly outline our analysis, in our coordinate system, for completeness.

We are interested in self-similar solutions which satisfy the following two properties:

(i) **The self-similar solution is analytic (or at least smooth, in the sense of having an asymptotic expansion to all orders) for all \(x \in \mathbb{R}\).**

(ii) **The space-time and the matter are regular, \(A = 1\) and \(V = 0\), at the center \((x = -\infty)\).**

The reason why we require these may be summarized as follows. In this paper, we are interested in self-similar solutions which could represent the space-time structure of a black hole. Such a critical space time is formed from the sonic point (explained in section IV C). So, there are three types of possible singularities, which will be repeatedly used in subsequent sections.

**Lemma 5** Any solution of (4.1) is monotonically non-decreasing, as long as \(A < 1\).

A direct consequence of the Lemma is that any solution, which is regular at the center (i.e. \(A \rightarrow 1\) as \(x \rightarrow -\infty\)), must satisfy \(A \geq 1\) for all \(x \in \mathbb{R}\).

**Proof of Lemma 5.** Follows from (4.1a); The last term on the RHS is nonnegative, so we have

\[
\bar{A}_x \geq 1 - A. \quad (4.3)
\]

**C. Conditions at the sonic point**

1. **The sonic point**

Eqs. (4.1d) and (4.1e) can be written in the form:

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\omega, x \\
V, x
\end{pmatrix}
=
\begin{pmatrix}
e \\
f
\end{pmatrix}
\quad (4.4)
\]

where \(a, b, c, d, e, f\) are functions which do not depend on derivatives of \(\omega\) and \(V\). Solving the above in favor of \(\omega, x\) and \(V, x\), the resulting equation violates the Lipschitz condition at the sonic point, where the “determinant” of the coefficient matrix of \(\omega, x\) and \(V, x\) vanishes:

\[
\det \equiv ad - bc \\
\propto \frac{\gamma\{(1 + NV)^2 - (\gamma - 1)(N + V)^2\}}{1 - V^2} = 0. \quad (4.5)
\]

Physically, the sonic point is characterized by the fact that the velocity of fluid particles seen from the observer on the constant \(x\) line is equal to the speed of sound \(\sqrt{\gamma - 1}\).

2. **Expansion at the sonic point**

Because generic solutions at the sonic point can be singular, the analytic condition (i) required at the sonic point places a severe constraint on possible form of solutions and in the end leaves only one free parameter, as we now explain.

The sonic point is characterized by the vanishing of the determinant, (4.5). In order to have finite derivatives, \(\omega, x\) and \(V, x\), at the sonic point, the rows of (4.4) must be proportional to each other:

\[
a f - e c = 0. \quad (4.6)
\]

These two conditions, (4.3) and (4.6), together with the identity (4.2), between \(A, N, \omega, V\), enable us to express values of \(A, N, \omega\) at the sonic point in terms of that of \(V_0 \equiv V_{ss}\) (we write \(A_0, N_0, V_0\), etc. for values of these functions at the sonic point).

To find out the explicit forms of these expressions, we first list up the following properties to be satisfied at the sonic point:
1. $N_0 \geq 0$, because of its definition in terms of lapse function.

2. $A_0 \geq 1$, from Lemma 3 of Section [IV B]

3. The derivatives $\omega, x$ and $V, x$ should exist at the sonic point. In other words, first order expansion coefficients, in particular those of $\omega$ and $V$ must exist as real numbers.

Note that (4.5), being quadratic in $N_0$, in general allows two solutions for $N_0$. However, one of them leads to $A_0 < 1$, and is excluded from the second requirement above. The first and third requirements together restrict allowed values of $V_0$ to $-1 \leq V_0 \leq \sqrt{\gamma - 1}$. As a result, the zeroth order expansion coefficients are given by:

$$ N_0 = \frac{1 - \sqrt{\gamma - 1}V_0}{\sqrt{\gamma - 1} - V_0}, \quad (4.7) $$

$$ A_0 = \frac{\gamma^2 + 4\gamma - 4 + 8(\gamma - 1)^{3/2}V_0 - (3\gamma - 2)(2 - \gamma)V_0^2}{\gamma^2(1 - V_0^2)} \quad (4.8) $$

$$ \omega_0 = \frac{2\sqrt{\gamma - 1}(\sqrt{\gamma - 1} - V_0)(1 + \sqrt{\gamma - 1}V_0)}{\gamma^2(1 - V_0^2)}. \quad (4.9) $$

Once these values at the sonic point are given, we can then determine the higher coefficients of power series expansion of $A, N, \omega,$ and $V$ with respect to $x$ using the above conditions and (4.3). To avoid making presentation too complicated, we do not give explicit expressions, but only remark that the equations for the first order expansion coefficients, given in terms of $A_0, N_0, \omega_0, V_0, \gamma$, have two solutions in general; some of which are not allowed for some values of $V_0$ due to the third requirement above.

The self-similar solution which is regular at the sonic point is thus characterized by a single parameter $V_0 \equiv V_{ss}$.

D. Asymptotic behaviour of solutions as $x \to \pm \infty$

In searching for desired self-similar solutions, it becomes desirable to understand the correct asymptotic behaviour of solutions as $x \to \pm \infty$.

1. Behaviour at the center, $x = -\infty$

Behaviour of solutions of ODE’s (4.1), as $x \to -\infty$, can be most easily understood by introducing a new variable

$$ M \equiv NV \quad (4.10) $$

and rewriting (4.1) in terms of $A, M, \omega, V$. It is easily seen that

$$ A = 1, \quad M = -\frac{2}{3\gamma}, \quad \omega = V = 0 \quad (4.11) $$

is a stationary (fixed) point of (4.1), i.e. all the $x$-derivatives vanish. Moreover, linear analysis around this stationary point shows that there is one unstable direction for this stationary point, namely in the direction of $M + \frac{2}{3\gamma}$. Thus, a generic solution which satisfies the analyticity condition at the sonic point will diverge as $x \to -\infty$, and only a small number of solutions with carefully chosen values of $V_0$ can satisfy the regularity condition at $x = -\infty$; thus leaving only discrete acceptable solutions.

By similar reasoning, it is easily seen that any self-similar solution which satisfies our boundary condition at the center (i.e. has a smooth derivative), allows a (possibly only asymptotic) power series expansion in $\xi \equiv e^x$ at $\xi = 0$, and it is easily seen that the asymptotic behaviour is

$$ A_{ss, \infty}(x) \sim 1 + A_{-\infty}e^{2x}, \quad N_{ss}(x) \sim N_{-\infty}e^{-x}, \quad \omega_{ss} \sim \omega_{-\infty}e^{2x}, \quad V_{ss} \sim V_{-\infty}e^{x}, \quad (4.12) $$

with coefficients $A_{-\infty}, N_{-\infty}, \omega_{-\infty}, V_{-\infty}$ satisfying the relation:

$$ A_{-\infty} = \frac{2\omega_{-\infty}}{3}, \quad N_{-\infty}V_{-\infty} = -\frac{2}{3\gamma}. \quad (4.13) $$

2. Behaviour at $x = \infty$

It is easily seen that

$$ N = 0, \quad A = 1 + \frac{2\omega(1 + (\gamma - 1)V^2)}{1 - V^2} \quad (4.14) $$

is a stationary point (parameterized by two free parameters $\omega, V$) of (4.1). Linear analysis around the stationary point shows that this stationary point becomes an attracting node for

$$ \omega \leq \frac{1 - V^2}{\gamma(1 + V^2)}. \quad (4.15) $$

As will be explained in Section [IV F], most of the self-similar solutions which are analytic for $x < \infty$ found numerically satisfy this condition, and thus are seen to be attracted to this stationary point.

E. Numerical Method

Based on considerations presented in the above, we proceed as follows in order to find out desired self-similar solutions.
We first fix the value $V_0$ at the sonic point $x = 0$, and use the power series expansion there (Section IV C 2) to derive the derivatives. Then starting from the sonic point $x = 0$, we solve ODE’s (4.1) towards $x = -\infty$, using Runge–Kutta fourth order integrator, with reliable error estimates. If at some $x < 0$ (1) either $A < 1$ or (2) $det = 0$, we stop solving and conclude that this $V_0$ does not give rise to a desired self-similar solution. The first case ($A < 1$) is excluded based on the Lemma 3 of section IV D. The second case implies the existence of another sonic point between $x = 0$ and $x = -\infty$. Such a solution with more than one (but with finite number of) sonic points can always be found by searching for its part between its left-most sonic point and $x = -\infty$, and then solving (4.1) from the left-most sonic point to the right (towards $x = \infty$). However, it is quite unlikely that such solutions with more than one sonic points exist, because the existence of only one fixed point already reduces the freedom of parameters to discrete ones, and other possible sonic point will further reduce it.

Our numerical results show that each allowed $V_0$ ends in either of these two possibilities. Moreover it is seen, both numerically and mathematically rigorously, that a desired self-similar solution exists at those values of $V_0$, where $V_0 + 0$ leads to the case (1) and $V_0 - 0$ leads to the case (2).

F. (Numerical) results

We have searched for self-similar solutions which satisfy the above conditions (i) and (ii) numerically, for all the allowed values of $V_0$. As has been extensively studied in [29], there exists a sequence of self-similar solutions, each of which is characterized by the number of zeros of $V$. The number of zeros is an integer starting from 1, and the one with exactly one zero is the solution cited in [30], which we call the Evans–Coleman solution. Due to a numerical difficulty, we could find only self-similar solutions with odd number of zeros of $V$.

In Fig. 6, we show the profile of the Evans–Coleman solution, which will play the central role in later sections. This is the same as Fig. 1 of [30] except for $\alpha$ which is due to the coordinate condition for $t$.

V. PERTURBATION

We studied perturbations around self-similar solutions by two different methods. The first one is to directly find out eigenmodes by solving the system of ODE’s (5.13) below, which characterizes the eigenmodes. The second one is to apply the so–called Lyapunov analysis. The former has the advantage that it reduces the problem to that of solving ODE’s, and enables us fairly accurate estimate of eigenvalues, but has the disadvantage that it is almost impossible to search for every possible eigenvalues. The latter has an advantage that it is quite suited to finding out eigenvalues in descending order of their real part, but has disadvantages that (1) eigenvalue estimates are not so accurate, and (2) we have to solve a PDE. However, we here emphasize that the PDE we have to solve for the Lyapunov analysis is a very regular one, and can be handled by standard techniques for solving PDE’s.

In this section, we explain the first method, that is, direct search for perturbations by shooting method. To simplify the notation, we express perturbations in terms of perturbations of $A \equiv \ln(A), \bar{N} \equiv \ln(N), \bar{\omega} = \ln(\omega)$. For example, the $\bar{A}$-version of (5.3) in fact means

$$\bar{A}(s, x) = \bar{A}_{ss}(x) + \epsilon \bar{A}_{ss}(x) \quad (5.1)$$

where

$$\bar{A}(s, x) \equiv \ln(A(s, x)), \quad \bar{A}_{ss}(x) \equiv \ln(A_{ss}(x)). \quad (5.2)$$

A. Perturbation equations and the gauge degrees of freedom

Perturbation equations are obtained by taking the first order variation in (5.3) from the self-similar solution $H_{ss}$ considered:

$$h(s, x) = H_{ss}(x) + \epsilon h\var_{ss}(s, x), \quad (5.3)$$

where $h$ represents each of ($\bar{A}, \bar{N}, \bar{\omega}, \bar{V}$). In the following, to simplify the notation, we write $V$ for $V_{ss}$ and write $V\var$ for perturbations.

Explicitly, we have:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_s & B_s \\ 0 & 0 & C_s & D_s \end{pmatrix} \partial_s + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A_x & B_x \\ 0 & 0 & C_x & D_x \end{pmatrix} \partial_x \begin{pmatrix} \bar{A}\var \\ \bar{N}\var \\ \bar{\omega}\var \\ V\var \end{pmatrix}$$
\[
\begin{pmatrix}
G_1 & 0 & G_3 & G_4 \\
H_1 & 0 & H_3 & 0 \\
E_1 & E_2 & E_3 & E_4 \\
F_1 & F_2 & F_3 & F_4
\end{pmatrix}
\begin{pmatrix}
\bar{A}_{\text{var}} \\
\bar{N}_{\text{var}} \\
\bar{\omega}_{\text{var}} \\
V_{\text{var}}
\end{pmatrix}
\]

(5.4)

with coefficients given in terms of self-similar solutions as

\[
A_s \equiv 1, \quad B_s \equiv \frac{\gamma V}{1 - V^2},
\]

\[
C_s \equiv (\gamma - 1)V, \quad D_s \equiv \frac{-\gamma}{1 - V^2},
\]

(5.5)

\[
A_x \equiv 1 + NV, \quad B_x \equiv \frac{\gamma (N + V)}{1 - V^2},
\]

\[
C_x \equiv (\gamma - 1)(N + V), \quad D_x \equiv \frac{\gamma (1 + NV)}{1 - V^2},
\]

(5.6)

\[
E_1 \equiv -\frac{(\gamma + 2)}{2}ANV,
\]

\[
E_2 \equiv \frac{6 - 3\gamma}{2}NV - 2 + \gamma ANV + (2 - \gamma)\omega NV - NV\bar{\omega}_x - \frac{\gamma NV_x}{1 - V^2},
\]

(5.7a)

\[
E_4 \equiv \frac{6 - 3\gamma}{2}N - 2 + \gamma AN + (2 - \gamma)\omega N - N\bar{\omega}_x - \frac{\gamma (1 + 2NV + V^2)V_x}{(1 - V^2)^2},
\]

(5.7b)

\[
F_1 \equiv \frac{2 - 3\gamma}{2}AN,
\]

\[
F_2 \equiv \frac{(2 - \gamma)(\gamma - 1)\omega N + \frac{7\gamma - 6}{2}N + \frac{2 - 3\gamma}{2}AN - (\gamma - 1)N\bar{\omega}_x - \frac{\gamma NV_x}{1 - V^2}}{-N\bar{\omega}_x - \frac{\gamma NV_x}{1 - V^2}},
\]

(5.7c)

\[
F_3 \equiv \frac{(2 - \gamma)(\gamma - 1)\omega N}{N\bar{\omega}_x - \frac{\gamma NV_x}{1 - V^2}},
\]

\[
F_4 \equiv \frac{-(\gamma - 1)\bar{\omega}_x - \frac{\gamma (N + 2V + NV^2)V_x}{(1 - V^2)^2}}{N\bar{\omega}_x - \frac{\gamma NV_x}{1 - V^2}},
\]

(5.7d)

\[
G_1 \equiv -A, \quad G_3 \equiv \frac{2(1 + (\gamma - 1)V^2)\omega}{1 - V^2},
\]

\[
G_4 \equiv \frac{4\gamma \omega V}{(1 - V^2)^2},
\]

(5.8)

\[
H_1 \equiv A, \quad H_3 \equiv (\gamma - 2)\omega.
\]

(5.9)

We also have

\[
\partial_s A_{\text{var}} + \partial_x A_{\text{var}} = \frac{-2\gamma NV\omega}{1 - V^2} (\bar{N}_{\text{var}} + \bar{\omega}_{\text{var}})
\]

\[
-\frac{2\gamma N\omega (1 + V^2)}{(1 - V^2)^2} V_{\text{var}}
\]

(5.11)

We note that there is a gauge freedom due to the coordinate transformation of order \(\epsilon\), namely \((3.8)\) with \(f(s) = s + \epsilon f_1(s)\). The transformation \((3.9)\) now becomes [taking \(O(\epsilon)\) terms; \(\gamma' = \frac{d\gamma}{ds}, \gamma = \frac{d\gamma}{ds}\)]

\[
\bar{h}_{\text{var}}(s, x) = \begin{cases}
\bar{h}_{\text{var}}(s, x) + f_1(s)h_{ss}(x) & (\bar{A}, \bar{\omega}, V), \\
\bar{h}_{\text{var}}(s, x) + f_1(s)h_{ss}(x) + f_1(s) & (h = N)
\end{cases}
\]

(5.12)

This in particular means that one can always require \(\bar{N}_{\text{var}}(s, x_0) \equiv 0\) for a fixed \(x_0\).

### B. EOM for eigenmodes

We consider eigenmodes of the form \(h_{\text{var}}(s, x) = h_p(x)e^{\kappa s}\), with \(\kappa \in \mathbb{C}\) being a constant. Substituting this form into \((5.4)\) yields a set of linear, homogeneous first order ODE’s for \((\bar{N}_p, \bar{A}_p, \bar{\omega}_p, V_p)\). Explicitly, we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A_x & B_x \\
0 & 0 & C_x & D_x
\end{pmatrix}
\begin{pmatrix}
\bar{A}_p \\
\bar{N}_p \\
\bar{\omega}_p \\
V_p
\end{pmatrix}
= \kappa \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & A_s & B_s \\
0 & 0 & C_s & D_s
\end{pmatrix}
\begin{pmatrix}
\bar{A}_p \\
\bar{N}_p \\
\bar{\omega}_p \\
V_p
\end{pmatrix}
\]

(5.13)

and

\[
\kappa \bar{A}_p + \partial_x \bar{A}_p = \frac{-2\gamma NV\omega}{1 - V^2} (\bar{N}_p + \bar{\omega}_p)
\]

(5.14)

Eq. \((5.14)\) and the first row of \((5.13)\) provide an algebraic identity:

\[
(\kappa - A)\bar{A}_p + \left(\frac{2\gamma NV\omega}{1 - V^2}\right) \bar{N}_p + \left(\frac{2\gamma N\omega(1 + V^2)}{(1 - V^2)^2}\right) V_p = 0,
\]

(5.15)

just as in the case of self-similar solutions [cf. \((1.2)\)]. As in our treatment of self-similar solutions, we here treat \(\bar{A}_p, \bar{N}_p, \bar{\omega}_p, V_p\) as four unknown functions, and use \((5.15)\) as a check at appropriate steps of computation.

We require \(\bar{N}_p(s, 0) = 0\) to fix the coordinate freedom \((5.12)\).
C. Conditions on perturbations

As in the case of self-similar solutions, we require (i) that the perturbations are analytic for all \( x \in \mathbb{R} \), and (ii) that the perturbed space-times are regular at the center \( (\bar{A}_p = 0 \text{ at } x = -\infty) \).

D. The sonic point

It can be seen that apart from the overall multiplicative factor, the perturbation solutions which satisfy the analyticity condition (i) at the sonic point are specified by one free parameter \( \kappa \), as we now explain.

(1) First we note that the sonic point is a regular singular point for the perturbations. That is, the system of ODE’s for perturbations \( (5.13) \) is singular where the determinant of the matrix on the LHS of \( (5.13) \) vanishes: \( A_x D_x - B_x C_x = 0 \); this is identical with the sonic point condition for self-similar solutions, \( (4.3) \).

(2) Next, in order to have a smooth solution at the sonic point, the third and fourth row of \( (5.13) \) must be proportional to each other, like in the case of self-similar solution, \( (4.6) \). This yields an algebraic relation between \( \bar{A}_p(x), \bar{N}_p(x), \bar{\omega}_p(x) \) and \( V_p(x) \) at the sonic point.

(3) Third, we have an algebraic identity between \( \bar{A}_p(x), \bar{N}_p(x), \bar{\omega}_p(x) \) and \( V_p(x) \), given by \( (5.13) \).

(4) The above two algebraic relations, together with our choice of gauge \( \bar{N}_p(0) = 0 \) enables us to express, e.g. \( \bar{\omega}_p(0) \) and \( V_p(0) \) in terms of \( \bar{A}_p(0) \); and then higher order expansion coefficients of perturbations are given in terms of \( \bar{A}_p(0) \). Because the system is linear and homogeneous, overall normalization is irrelevant. We thus see that the solution which satisfies (i) is characterized by a single parameter \( \kappa \).

This, together with the regularity condition (ii) at the center, in general allows only discrete values for \( \kappa \) as now be explained.

E. Asymptotic behaviour as \( x \to \pm \infty \)

Searching numerically for eigenmodes of perturbations is in some sense easier than searching for self-similar solutions, because the ODE’s are now linear and homogeneous in unknown functions. In particular, asymptotic behaviour of solutions as \( x \to \pm \infty \) is easily obtained as follows.

1. Asymptotic behaviour as \( x \to -\infty \)

Substituting the asymptotic behaviour, \( (5.12) \), of self-similar solutions as \( x \to -\infty \), the system of ODE’s for perturbations \( (5.13) \) take on a simple form, given by

\[
\begin{pmatrix}
\partial_x (\bar{A}_p) \\
\bar{N}_p \\
\bar{\omega}_p \\
V_p
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\frac{2-3\gamma}{\Delta(\gamma-1)} & 0 & 0 & 0 \\
0 & 0 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
\bar{A}_p \\
\bar{N}_p \\
\bar{\omega}_p \\
V_p
\end{pmatrix}.
\]

This homogeneous linear ODE with constant coefficients has four independent solutions, given by

\[
\begin{pmatrix}
0 \\
0 \\
e^{-2x} \\
e^{-x}
\end{pmatrix}.
\]

The last one should be excluded in view of the identity \( (5.15) \), and thus asymptotic behaviour of arbitrary solutions of ODE’s is described by suitable linear combinations of the first three independent solutions. Now, one of them (the third one) blows up as \( x \to -\infty \). Therefore we see that any solution, which satisfies our boundary condition, should thus be obtained by choosing \( \kappa \) so as to eliminate the unwanted expanding mode.

2. Asymptotic behaviour as \( x \to \infty \)

Similarly, the asymptotic behaviour as \( x \to \infty \) is obtained. It is now seen that there are three independent solutions [after taking into account the identity \( (5.15) \)], whose \( x \)-dependence are

\[
1, \quad e^{-\kappa x}, \quad e^{-\kappa x}
\]

as \( x \to \infty \). It is thus seen that every solution stays bounded as long as \( \text{Re} \kappa > 0 \), whereas for \( \text{Re} \kappa < 0 \) it blows up as \( x \to \infty \).

F. The gauge mode

In searching for possible eigenmodes, one has to pay attention to a “gauge mode” which emerges from a coordinate transformation applied to the self-similar solution, as we now explain.

Suppose one has a self-similar solution \( h(s, x) = H_{ss}(x) \). Seen from another coordinate [related by the coordinate transformation \( (5.8) \)], its perturbation is given by setting \( h_p \equiv 0 \) in \( \tilde{h} \) of \( (5.12) \):

\[
\tilde{h}_{\text{var}}(s, x) = \left\{ \begin{array}{c}
f_1(s)h'_{ss}(x) \\
f_1(s)h_{ss}(x) + \tilde{f}_1(s) \end{array} \right\} (h = \bar{A}, \bar{\omega}, V) \quad (5.19)
\]

For general \( f_1 \), this does not behave like an eigenmode. However, with the choice of \( f_1(s) \equiv e^{\kappa s} \) (\( \kappa \) arbitrary), \( \tilde{h} \) does behave as an eigenmode with eigenvalue \( \bar{\kappa} \):

\[
\tilde{h}_{\text{gauge}}(s, x; \bar{\kappa}) = e^{\kappa s} \left( \begin{array}{c}
h'_{ss}(x) \\
h_{ss}(x) + \bar{\kappa}
\end{array} \right) (h = \bar{A}, \bar{\omega}, V) \quad (5.20)
\]
The mode emerges via the coordinate transformation of the self-similar solution \( h_{ss} \), and thus can be considered as a result of pure gauge transformation, like a “zero mode” in translation invariant systems. Because \( \bar{\kappa} \in \mathbb{C} \) is arbitrary (with the same eigenvalue \( \kappa \)), this pure gauge mode forms a one-parameter family.

It can also be seen that other (non-gauge) modes are transformed under the gauge transformation as follows:

Eq. (5.19) for \( h_{\text{var}}(s, x) = h_p(x)e^{\kappa s} \) becomes

\[
\tilde{h}_{\text{var}}(s, x) = \begin{cases} 
    h_p(x)e^{\kappa s} + f_1(s)h_{ss}'(x) & (h = \tilde{A}, \tilde{\omega}, V) \\
    h_p(x)e^{\kappa s} + f_1(s)h_{ss}'(x) + \tilde{f}_1(s) & (h = N)
\end{cases}
\]

(5.21)

As long as \( h_p \neq 0 \), by taking \( f_1(s) = e^{\kappa s} \), this means transformed \( \tilde{h} \) is also an eigenmode, with the same \( \kappa \). Thus we have a one-parameter family of eigenmodes (with the same eigenvalue \( \kappa \)), which are mutually related by the gauge mode (5.20).

### G. Numerical Results

Based on the above observations, we searched for desired eigenmodes as follows. We first fix a value of \( \kappa \), and then starting from \( x = 0 \) (the sonic point), integrate (5.13) to \( x = -\infty \). When there appear nonzero components of the expanding modes (for sufficiently large \( |x| \)), we judge that this \( \kappa \) does not give a desired eigenmode and stopped.

In this way, we searched for eigenmodes in the range of \( \Re \kappa \geq -1.5, |\Im \kappa| \leq 14 \). We found several eigenmodes which are explained below.

1. **The relevant mode for the Evans–Coleman solution**

The profile of the eigenmode with the largest \( \Re \kappa \) is shown in Fig. 7. It has the eigenvalue \( \kappa \simeq 2.81055255 \), which corresponds to the exponent value \( \beta_{BH} \simeq 0.35580192 \), according to our scenario of section [11] and which we believe to be exact to the last digit. This is in good agreement with the value of [8].

2. **The gauge mode**

In our gauge, where \( N_{p}(s, 0) = 0 \), we observe the gauge mode with \( \bar{\kappa} = -\frac{dN_{ss}}{dx}(0) \simeq 0.35699 \), as explained in section [11].

3. **Other modes for the Evans–Coleman solution**

To confirm our scenario, we performed a thorough search of other eigenmodes in the region \( 0 \leq \Re \kappa \leq 15, |\Im \kappa| \leq 14 \), and found none, except for the above mentioned relevant mode and the gauge mode. We also performed a less complete search in the region \( -1.5 \leq \Re \kappa < 0, |\Im \kappa| < 2 \). There is an eigenmode with \( \Re \kappa \lesssim -1.4 \), which is consistent with the results of the Lyapunov analysis in Sec. [VI].

It is not a priori obvious whether the eigenmodes form a complete set of basis functions. Due to the complicated structure of the equations of motion, we have not found a beautiful argument which can restrict possible eigenvalues (like that of [8] and references therein).

Our search of eigenmodes has a serious drawback that it is theoretically impossible to cover the whole values of \( \kappa \in \mathbb{C} \) (unless, of course, one employs more sophisticated mathematical techniques). To fill this gap, and to further confirm our scenario, we performed a Lyapunov analysis, as explained in Sec. [VI].

---

\footnote{We take this opportunity to clarify the confusion expressed in “Note Added” of [11], which states that the gauge mode reported in [11] (same as reported here) “seems to be erroneous.” The fact is that our report in [11] (and of course here) is correct. The confusion seems to be due to different gauges used in the analysis. We here and in [11] use the gauge \( N_{p}(s, 0) = 0 \) at the sonic point, while Maison [11] and we in Sec. [VI] use the gauge \( N_{p}(s, -\infty) = 0 \) at the origin. The former gauge gives the gauge mode at \( \kappa \simeq 0.35699 \), while the latter gives at \( \kappa = 1 \); the difference in the values of \( \kappa \) is well understood in view of (5.20).}
H. Modes for other self-similar solutions

As has been stated in Sec. [V.F], there exists a series of self-similar solutions (specified by the number of zeros of $V$), in which the Evans–Coleman solution can be considered as the first one (exactly one zero for $V$). We have searched relevant eigenmodes for the first several self-similar solutions and found more than one relevant modes (see Table I). This implies that the other self-similar solutions are irrelevant for the generic critical behaviour. However, we here admit that our analysis is less complete for these higher self-similar solutions, because due to a difficulty in coding, (1) we have not done the analysis for (possible) self-similar solutions with even number of zeros of $V$, (2) there can be more relevant modes than is reported in the Table.

| self-similar solution | $\kappa$         |
|----------------------|------------------|
| 1 (Evans–Coleman)    | 2.8105525488     |
| 3                    | 8.456, 3.464, 1.665, 0.497 |
| 5                    | 15.80, 7.13, 3.22, 1.51, 0.500 |
| 7                    | 15.97, 6.92, 3.20, 1.50 |

TABLE I. Relevant modes found for other self-similar solutions, which are labeled by the number of zeros of $V$.

VI. LYAPUNOV ANALYSIS

To further confirm the uniqueness of the relevant mode we performed a Lyapunov analysis around the critical solution. A Lyapunov analysis and the shooting method for an ordinary differential equation adopted in [10] and in the previous section are complementary to each other for the following reasons: (1) The former extracts eigenmodes in the descending order of its real part, whereas the latter affords information only of a finite region of complex $\kappa$ plane. (2) The Lyapunov analysis is useful even if the (discrete) eigenmodes do not form a complete set. (3) It is easier in the latter than in the former to numerically obtain accurate values of the eigenvalues $\kappa$, hence the critical exponent $\beta_{BH}$. We here present our method of analysis and its results.

A. Numerical Methods

The Lyapunov analysis is a method of extracting eigenvalues in descending order. It involves time integration of the linearized EOM (5.4) about the self-similar solution. It takes advantage of the fact that the eigenmodes with large $Re\kappa$ dominates at late times and that the volume, defined by an (arbitrarily chosen) inner product on $\Gamma$, of the parallelepiped spanned by the eigenmodes corresponding to $\kappa_1, \kappa_2, ..., \kappa_n$ approaches $e^{\kappa_1+\kappa_2+...+\kappa_n}$. During the integration, vectors in $\Gamma$ are orthonormalized by the Gram–Schmidt procedure. This is essential for avoiding numerical overflows and underflows. For details see Appendix D. We can also have information on the imaginary part of eigenvalues by the period of oscillation in time $\gamma$.

We have performed the above-described calculation by numerically solving the PDE’s for the perturbations, (5.4). We emphasize that solving the PDE’s for perturbations, (5.4), is much easier than solving the full EOM close to the critical point, because perturbations are well behaved in contrast with near-critical solutions whose derivatives are diverging in $(t, r)$ coordinate.

We used the first order Lax scheme, and the second order Lax–Wendroff scheme for time evolution. We show the results from the Lax–Wendroff scheme. To simplify coding, we employed a new gauge, $N_{\text{var}}(s, -\infty) = 0$. In this gauge we observe the gauge mode at $\kappa = 1$. We have made appropriate coordinate transformations to stabilize the integration schemes.

We imposed the free boundary condition at the sonic point $\xi = 1$. This can be done because no information can come in from outside the sonic point (as is easily seen by studying the characteristic curves), for spherically symmetric metric perturbations contains no gravitational waves and are determined by the matter degrees of freedom.

B. Numerical results

Fig. 8 shows the dependence of $Re\kappa$ on $\gamma$. The largest eigenvalue is real and agrees well with the first eigenvalue (which is real) found by solving two-point boundary problem of an ODE in Sec. D. The second largest eigenvalue is that of the gauge mode, which should be exactly equal to one. For $\gamma \leq 1.2$ the third and the fourth largest eigenvalues are real, and the fifth is imaginary. For $\gamma = 1.3, 1.4$ the third is real and the fourth and the fifth is imaginary, which are complex conjugate. those for $\gamma \geq 1.4$ are complex. We can consider that there is a cross-over of Lyapunov exponents corresponding to the real eigenvalue and the complex pair between $\gamma = 1.2$ and $\gamma = 1.3$. For $\gamma \geq 1.5$ the third and the fourth are a complex conjugate pair, so that there is another cross-over of Lyapunov exponents corresponding to the real eigenvalue and the complex pair between $\gamma = 1.4$ and $\gamma = 1.5$. For
large \( \gamma \) it seems that many eigenmodes with close values of \( \text{Re} \kappa \) are present. Our preliminary computation of ten modes for \( \gamma = 1.889 \) shows that there at least nine modes in the range \(-0.6 \leq \text{Re} \kappa \leq -0.4\), though it seems that not all of them are degenerate in \( \text{Re} \kappa \).

The figure establishes the uniqueness of the relevant mode in collapse of perfect fluids with \( p = (\gamma - 1)\rho \), \( 1 < \gamma \lesssim 1.889 \). from this we can conclude that the critical behaviour must be observed for all of these models with \( 1 < \gamma \lesssim 1.889 \).

### VII. SUMMARY OF RESULTS

We have performed the analysis presented in Secs. IV–VI for perfect fluid, with \( 1 < \gamma \leq 2 \). As has been noted in [31, 32], a self-similar solution of Evans–Coleman type (i.e. with one zero of \( V \)) ceases to exist for \( \gamma \gtrsim 1.889 \). For Evans–Coleman type self-similar solutions, the Lyapunov analysis of Sec. IV establishes that there is a unique relevant mode, and thus we can observe the critical behaviour (Fig. 5). Then, the precise values of the relevant eigenvalue \( \kappa \) are obtained by the shooting method of Sec. V, and the critical exponent is given by \( \beta_{\text{BH}} = 1/\kappa \).

The result is shown in Table II. This in particular shows that the relevant eigenmodes found by Maison [11] are actually unique, and thus are responsible for the critical behaviour. The value of the critical exponent \( \beta_{\text{BH}} \) depends strongly on \( \gamma \). Moreover, the limit \( \gamma \to 1 \) seems to be discontinuous. (For the dust, \( \gamma = 1 \), we expect a trivial critical behaviour with \( \beta_{\text{BH}} = 1 \).) This may be because the domain of attraction of the Evans–Coleman type self-similar solution vanishes as \( \gamma \to 1^+ \).

| \( \kappa \) | \( \beta_{\text{BH}} \) |
|---|---|
| 1.00001 | 9.4629170 | 0.1056756 |
| 1.0001 | 9.45592488 | 0.1057538 |
| 1.001 | 9.3866322 | 0.1065348 |
| 1.01 | 8.74868715 | 0.11430286 |
| 1.03 | 7.61774326 | 0.13127247 |
| 1.04 | 7.16334221 | 0.13959964 |
| 1.05 | 6.76491004 | 0.14782163 |
| 1.06 | 6.41269915 | 0.15594058 |
| 1.08 | 5.81789124 | 0.17188358 |
| 1.1 | 5.33438518 | 0.18746398 |
| 1.12 | 4.93282886 | 0.2027343 |
| 1.15 | 4.44250509 | 0.22510605 |
| 1.18 | 4.0481584 | 0.247076 |
| 1.2 | 3.82545008 | 0.26140715 |
| 1.22 | 3.62729455 | 0.27568756 |
| 1.25 | 3.36750228 | 0.29695600 |
| 1.28 | 3.1437431 | 0.31812947 |
| 1.3 | 3.00990875 | 0.33223599 |
| 1.32 | 2.88714829 | 0.34636253 |
| 1.34 | 2.68838221 | 0.3580192 |
| 1.36 | 2.57025726 | 0.3745891 |
| 1.38 | 2.47850858 | 0.3906611 |
| 1.4 | 2.39245265 | 0.4034684 |
| 1.42 | 2.31507228 | 0.4179811 |
| 1.44 | 2.23517329 | 0.4328151 |
| 1.46 | 2.16301995 | 0.4479260 |
| 1.48 | 2.09467339 | 0.4621659 |
| 1.5 | 2.0290722 | 0.4774140 |
| 1.52 | 1.9619716 | 0.49265763 |
| 1.55 | 1.93841621 | 0.5188508 |
| 1.58 | 1.85338883 | 0.5395219 |
| 1.6 | 1.79980976 | 0.5555893 |
| 1.62 | 1.74873002 | 0.5718436 |
| 1.64 | 1.69974510 | 0.5883235 |
| 1.66 | 1.65278973 | 0.6050376 |
| 1.68 | 1.60773076 | 0.6219947 |
| 1.7 | 1.56446268 | 0.6392038 |
| 1.72 | 1.52282404 | 0.6566747 |
| 1.74 | 1.48276003 | 0.674418 |
| 1.76 | 1.44415717 | 0.6924454 |
| 1.78 | 1.40924242 | 0.7107703 |
| 1.8 | 1.37097467 | 0.7294081 |
| 1.82 | 1.2383842 | 0.8075039 |
| 1.84 | 1.22595959 | 0.8156709 |
| 1.86 | 1.21244458 | 0.8406966 |

### VIII. CONCLUSIONS AND DISCUSSION

The aim of this paper was twofold: First, in Sec. II we presented a mathematically rigorous framework, under which most of the critical behaviour observed in gravitational collapse are expected to be analyzed. We have in particular shown rigorous sufficient conditions under which (1) critical behaviour with a continuous self-similar solution is observed and the critical exponent is given exactly in terms of the unique relevant Lyapunov exponent \( \kappa \) by \( \beta_{\text{BH}} = \beta/\kappa \) (Sec. II C), (2) critical behaviour with a discrete self-similar solution is observed and the critical exponent is given by the same formula (Sec. II D),
and (3) different models exhibit the same critical behaviour, i.e. universality in the sense of statistical mechanics (Sec. II E). The sufficient conditions for (1) and (2) are essentially the uniqueness of the relevant mode, together with some regularity on the dynamics. The condition for (3) is essentially that the difference of EOM is relatively small in the critical region.

The key idea behind the analysis is the renormalization group. It naturally casts a system described by partial differential equations into a dynamical system on an infinite-dimensional phase space, where continuously and discretely self-similar solutions of the original equations are described by fixed points and limit cycles. This opened the way to the abstract analysis on dynamical systems described in Sec. II. With the help of techniques developed in statistical mechanics to study critical behaviour, we have been able to obtain detailed estimates on the time evolution of the deviation from the critical solutions (or the stable manifold). We thus obtained general and precise results, including rigorous relations between the critical exponent and Lyapunov exponent.

Another advantage of the renormalization group idea is that it enables us to treat systems with different EOM by naturally introducing a space of dynamical systems, as described in Sec. II E. Each system is in general driven along a renormalization group flow in this space, and this enabled us to clarify terms of EOM which are “relevant” for the phenomena we are interested in.

These point of view provide us a natural understanding on why critical behaviour are often observed in gravitational collapse.

We then proceeded to confirm the picture for a concrete model of perfect fluid in Secs. II–VII. We employed essentially two methods to show that the main assumption, uniqueness of the relevant mode, is satisfied: (1) usual shooting method, and (2) Lyapunov analysis. Their results agree well, and strongly suggest that the relevant mode is unique. We obtained values of the critical exponent $\beta_{BH}$ in Sec. VII.

It should be emphasized that the general framework presented in this paper is useful for both mathematical and numerical analysis on concrete models of critical behaviour in gravitational collapse, because it tells that in which case one can rely on the intuitive picture of the phase space shown in the schematic diagrams Figs. I, II, and the argument presented in III, which states that the linear perturbation is enough to understand all aspects of the critical behaviour, apart from pathological behaviour mainly due to the infinite dimensionality of the phase space. For mathematical proof of the existence of the critical behaviour and estimation of the critical exponent, the essential step is to find the norm $|| \cdot ||$, which defines the functional space or the phase space $\Gamma$, that satisfies the Assumptions in the Sec. II C. This should be highly nontrivial and depends on each model considered. For numerical analysis the nonlinear simulation is needed only to confirm Assumptions G1–3 in Sec. II C, especially Assumption G1. This is qualitative one rather than quantitative (because only the finiteness of the time before blowup is needed), and requires the time evolution of very small region of initial data, essentially a set of initial data which is the fixed point plus the small relevant mode. The calculation of this e.g. in $(s, x)$ coordinate should be much easier than adaptive mesh refinement calculation in $(t, r)$ coordinates for wide region of the phase space and should allow a closer control of the numerical errors. This together with close analysis of the spectrum of the linear operator $T_\sigma$ around the fixed point may lead to a computer–assisted proof of Theorem I.

There remain several open questions to be answered. First, it should be emphasized that our analysis in Secs. II–VII confirms only the local behaviour of the flow, around the specific self-similar solutions. Although self-similar solutions with more than one zeros of $V$ seem to be irrelevant for generic critical behaviour (because they seem to have more than one relevant modes), it would be interesting to know the global behaviour of the flow, in particular, how these self-similar solutions are related.

Second, now that various kinds of critical behaviour for different models have been observed, it would be interesting to ask what happens in a mixed system (e.g. perfect fluid + scalar fields). We performed a preliminary analysis in this direction. That is, we considered a “mixed” system of radiation fluid with a real scalar field, whose energy-momentum tensor is given by $T_{ab} = \rho u_a u_b + p (u_a u_b + g_{ab}) + \nabla_a \phi \nabla_b \phi - (1/2) g_{ab} \nabla^c \phi \nabla_c \phi$, and studied the linear stability of the Evans–Coleman self-similar solution in this mixed system. (It is easily seen that eigenmodes decouple.) Lyapunov analysis in Sec. VII applied to this mixed system, shows that the Evans–Coleman self-similar solution has the unique relevant mode (given by the relevant mode of the radiation fluid), and all scalar eigenmodes are irrelevant. This means that the Evans–Coleman self-similar solution is stable under the scalar perturbation, and we will observe the same radiation fluid critical behaviour for the mixed system (at least when the scalar field is sufficiently weak). It would be interesting to investigate the behaviour of the system without the restriction that the scalar field is weak; in particular, where does the cross-over between two types (fluid, scalar) of critical behaviour occur?

As is suggested from our general framework in Sec. II, a “critical behaviour” such as observed in gravitational collapse is not in fact limited to self-gravitating systems. Similar, but slightly simpler, phenomena can be observed in much simpler systems, such as nonlinear heat equation, where we can rigorously carry out the analysis of Sec. II. However if a solution of EOM blows up in such simple systems, it usually means that the equation is not physically applicable in the blow-up region, and the blow-up is an artifact of bad approximation. General relativity provides rare examples where the blow-up of solutions does
have a physical meaning — formation of a black hole.

Our analysis in Secs. II is not mathematically rigorous. In view of the general framework given in Sec. I, and in view of the above remark that the critical behaviour and blow-up of solutions are in fact physically meaningful, it is extremely desirable to develop a mathematically rigorous analysis of critical behaviour for a physically interesting model such as radiation fluid or scalar fields. We are planning to come back to this problem in the near future.

Our intuition on gravitational collapse still seems to be heavily based on few exact solutions, especially the limiting case of pressureless matter. The critical behaviour may provide a different limiting case that the final mass is small compared to the initial mass for more realistic and wider range of matter contents. It will be of great help to settle the problems in gravitational collapse such as cosmic censorship conjecture.

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APPENDIX A: SKETCH OF THE PROOF OF LEMMA 2

The proof consists of several steps, and is somewhat complicated. Because the Lemma itself is of auxiliary importance, and because the proof uses techniques which have been explained in detail in the proofs of other lemmas and propositions already proved in Secs. II, III, here we sketch only the essence of each step.

Step 1. $\bar{W}_r$ is a continuous curve, connecting $U_0$ and $U_{n_1+1}$, with $\|U_0 - U^\ast\|_u = r$, $\|U_{n_1+1} - U^\ast\|_u > e_2$.

We write $U_0 \equiv U^\ast + rF^{rel}$, and $U_n \equiv R_{n\sigma}(U^\ast + rF^{rel})$. Because $U_0$ satisfies the assumptions of Lemma 1,

- $\|U_n - U^\ast\|_u$ keeps growing until it finally exceeds $\delta_1$. This also shows $n_1(U_0; e_2) < \infty$.
- $\|U_n - U^\ast\|_u < \|U_n - U^\ast\|_u$ for $n \leq n_1(U_0; e_2) + 1$.

Now $D_r$ is a straight line connecting $U_0$ and $U_1$; and $\bar{W}_r$ is a union of images of $D_r$ under $R_{n\sigma}$ for $n \leq n_1(U_0; e_2) + 1$:

$$\bar{W}_r = \bigcup_{n=0}^{n_1} D_{r,n}, \quad D_{r,n} \equiv R_{n\sigma}(D_r). \quad (A1)$$

Because $R_{n\sigma}$ is continuous and because $n_1$ is finite, $R_{n\sigma}$ is also continuous (for $n \leq n_1$). Thus $D_{r,n}$ is a continuous set. Because endpoints of $D_{r,n}$ are overlapping, this proves the continuity of $\bar{W}_r$. Bounds on the norm follow immediately from the definition of $n_1$.

Step 2. $\bar{W}_r$ is “one-dimensional”, and any $U, U' \in \bar{W}_r$ satisfy $\|U - U'\|_u \leq C_4\|U - U^\ast\|_u$.

We first prove the claim for $U, U' \in D_{r,n}$ by induction in $n$, and then extend the result for general $U, U' \in \bar{W}_r$. First, the claim is satisfied for $n = 0$, because $D_{r,0} = D_r$ is a straight line connecting $U_0$ and $U_1$. Now suppose the claim holds for $U, U' \in D_{r,n}$, and try to prove it for $R_{n\sigma}(U), R_{n\sigma}(U') \in D_{r,n+1}$. [Any $U \in D_{r,n+1}$ should have at least one inverse image in $D_{r,n}$, so this is sufficient.] We compute as we did in deriving (2.83)–(2.86) using (2.33), (write $F = U' - U$)

$$R_{n\sigma}(U') = R_{n\sigma}(U + F) = R_{n\sigma}(U) + T_{n\sigma}(F) + O(6K_1\|U - U^\ast\|_u F) + O(K_1\|F\|_u^2). \quad (A2)$$

Now, by the inductive assumption, $\|F\|_u < \|F\|_u = \|F\|_u$. So we have

$$\|R_{n\sigma}(U') - R_{n\sigma}(U)\|_u \leq e^{-\kappa\|F\|_u} + K_1\|U - U^\ast\|_u \|F\|_u \|F\|_u \quad (A3)$$

$$\|R_{n\sigma}(U') - R_{n\sigma}(U)^\ast\|_u \geq [e^{\kappa\|F\|_u} - 6K_1\|U - U^\ast\|_u - K_1\|F\|_u] \|F\|_u \quad (A4)$$

Taking $\|U - U^\ast\| < \epsilon_2$ and $\|F\| < \epsilon_2$ into account, this leads to

$$\frac{\|R_{n\sigma}(U') - R_{n\sigma}(U)\|_u}{\|R_{n\sigma}(U') - R_{n\sigma}(U)^\ast\|_u} \leq \frac{e^{-\kappa\|F\|_u} \|U - U^\ast\|_u}{e^{\kappa\|F\|_u} - 7K_1\epsilon_2 \|U - U^\ast\|_u + 7K_1\epsilon_2} \quad (A5)$$

It is now easily seen that this recursion preserves

$$\frac{\|U - U^\ast\|_u}{\|U - U^\ast\|_u} \leq C_4, \quad (A6)$$

and $C_4 < 1$ follows from our choice of $\delta_2, \epsilon_2$, i.e. (2.63) and (2.66).

All this proves the claim for $U, U' \in D_{r,n}$ for some $n$. Because $\|U_n - U^\ast\|_u$ is strictly increasing, we can combine this bound for each $D_{r,n}$, to conclude $\|U - U^\ast\|_u \leq C_4\|U - U^\ast\|_u$ for $U, U' \in \bar{W}_r$ which are not necessarily in some $D_{r,n}$.

Step 3. Contraction in $E^\alpha$-direction. We now consider the distance between a flow $R_{n\sigma}(U)$ and $\bar{W}_r$, and show that their distance in $E^\alpha$-direction is decreasing. To be more precise, we introduce a decomposition (see Fig. 3 (c)):

$$U^{(n\sigma)} = \tilde{U}_n + G_n^\alpha, \quad (A7)$$

where this time $\tilde{U}_n \in \bar{W}_r(U^\ast)$ and $F^n \in E^\alpha(U^\ast)$. Because we have already shown that $\bar{W}_r(U^\ast)$ is one-dimensional and extends from $r$ to $\epsilon_2$ in $E^\alpha$-direction, we can always decompose this way, as long as $r \leq \|U^{(n\sigma)} - U^\ast\|_u \leq \epsilon_2$.

To derive a recursion for $\|G_n^\alpha\|$, we start as usual
where 
\[ \Delta = \tilde{O}(6K_1\|\hat{U}_n - U^*\|\cdot\|G_n\|) + \tilde{O}(K_1\|G_n\|^2). \]

We have to define \( \hat{U}_{n+1} \in \tilde{W}_r \) so that \( \Delta \) is decomposed as 
\[ \Delta = \hat{U}_{n+1} - \mathcal{R}_\sigma(\hat{U}_n) + F^s \quad (F^s \in E^s(U^*)) \] 
so that the entire expression can be written as \( \tilde{U}_{n+1} + G_{n+1}^s \) with 
\[ G_{n+1}^s \equiv \mathcal{T}_{\sigma,U^*}(G_n^s) + F_{n+1}^s. \]

Now (A10) implies 
\[ \|\Delta\|_u = \|\hat{U}_{n+1} - \mathcal{R}_\sigma(\hat{U}_n)\|_u \] 
and 
\[ \|F^s\| = \|F^s\|_u \leq \|\hat{U}_{n+1} - \mathcal{R}_\sigma(\hat{U}_n)\|_u + \|\Delta\|_u. \]

However, we have shown in Step 2 that \( \tilde{W}_r \) is in some sense “parallel” to \( E^s(U^*) \), i.e. 
\[ \|\hat{U}_{n+1} - \mathcal{R}_\sigma(\hat{U}_n)\|_u \leq C_4\|\hat{U}_{n+1} - \mathcal{R}_\sigma(\hat{U}_n)\|_u \] 
Combining (A12), (A14) implies 
\[ \|F^s\| \leq (1 + C_4)\|\Delta\| \] 
and we finally have 
\[ \|G_{n+1}^s\| \leq e^{-\kappa'\sigma}\|G_n\| + (1 + C_4) \times \left[ \tilde{O}(6K_1\|\hat{U}_n - U^*\|\cdot\|G_n\|) + \tilde{O}(K_1\|G_n\|^2) \right]. \]

By taking \( \|\hat{U}_n - U^*\| \) and \( \|G_n\| \) sufficiently small as in (2.65) and (2.66), we can bound the right hand side by 
\[ e^{-\kappa''\sigma}\|G_n\| \] 
with some \( \kappa'' > 0 \).

This in particular shows that intersections of \( \tilde{W}_r \) and the plane \( U^* + E^s(U^*) + aF^{\text{rel}} \) for various \( r \)'s are Cauchy sequences, and thus the limit \( r \searrow 0 \) exists: existence of \( W_{r,\text{loc}}(U^*) \).

Finally, repeating the above argument replacing \( \tilde{W}_r \) by \( W_{r,\text{loc}}(U^*) \), we get the contraction around \( W_{r,\text{loc}}(U^*) \) as well.

\[ \square \]

**APPENDIX B: SOLVING RECURSIVE INEQUALITIES**

We here provide a lemma, which effectively solves recursive inequalities encountered in Sec. 3 such as (2.87), (2.88), (2.174), (2.175).

**Lemma 6** Let \( \Lambda, \bar{\Lambda}, C, \delta \) and \( \epsilon \) be positive constants satisfying \( 1 < \bar{\Lambda}^{-1} \leq \Lambda, C \geq 1, \) and 
\[ \delta + \epsilon < \frac{\Lambda - 1}{2C}. \]

Let \( b_n, f_n \) be nonnegative sequences satisfying 
\[ 0 < b_0 < \epsilon, \quad f_0 = 0, \]
\[ b_{n+1} \leq \Lambda b_n + C[g_n(f_n \vee b_n) + (f_n \vee b_n)^2] \]
\[ f_{n+1} \leq \Lambda f_n + C[g_n(f_n \vee b_n) + (f_n \vee b_n)^2], \]
where \( g_n (n \geq 0) \) satisfies 
\[ 0 \leq g_n \leq \delta. \]

Define \( n_2 \equiv \max\{n \in \mathbb{Z} | b_k \leq \epsilon \text{ for } k \leq n\} \).

(i) \( n_2 \) is finite and the following hold for \( n \leq n_2 \):
\[ 0 \leq f_n \leq b_n, \]
\[ b_n \leq \epsilon \left( \frac{\Lambda + 1}{2} \right)^{-(n_2-n)}. \]

(ii) Suppose in addition that \( g_n \) satisfies 
\[ \sum_{n=0}^{\infty} g_n < \infty. \]
Then the following hold for \( n \leq n_2 \):
\[ \frac{2\epsilon}{3\Lambda - 1} \leq b_{n_2} \leq \epsilon, \]
\[ \frac{\epsilon}{C'} \leq b_0 \Lambda^{n_2} \leq C' \epsilon, \]
where \( C' \) is some positive constant.

The proof is an elementary exercise in calculus, which is done in several steps in a bootstrapping manner. We only consider \( n \leq n_2 \), which in particular means \( 0 \leq b_n \leq \epsilon \).

(i) \( \text{Step 1. Rough estimates and } f_n \leq b_n. \) We first prove by induction \( f_n \leq b_n \) (i.e. (B9)), and 
\[ \frac{\Lambda + 1}{2} b_n \leq b_{n+1} \leq \frac{3\Lambda - 1}{2} b_n. \]

These are satisfied for \( n = 0 \), because \( f_0 = 0 \), and because of our choice of \( \epsilon \). Now suppose they hold for \( n \).
Our inductive assumption in particular includes \( f_n \leq b_n \), which reduces the recursion (B3) to 
\[ b_{n+1} \leq \Lambda b_n + C b_n (g_n + b_n) \]
\[ \leq b_n [\Lambda + C(\delta + \epsilon)] \]
where we used uniform bounds on \( g_n \) and \( b_n \). Eq. (B11) immediately follows from our choice of \( \delta \) and \( \epsilon \).

On the other hand, \( f_n \leq b_n \) reduces the recursion (B4) to
\[ f_{n+1} \leq \tilde{A}f_{n} + Cb_{n}(g_{n} + b_{n}) \leq b_{n} \left[ A + C(\delta + \epsilon) \right] \leq \left[ A + C(\delta + \epsilon) \right] \frac{2}{\Lambda + 1} b_{n+1} \]  

(B14)

where in the last step we used (B11) just proved. The factor before \( b_{n+1} \) is easily seen to be less than one, and (B10) is proved.

**Step 2. Rough estimate on \( b_{n} \).** First note that (B11) for \( n = n_{2} \) and the definition of \( n_{2} \) implies

\[ \epsilon \leq b_{n_{2}+1} \leq \frac{3\Lambda - 1}{2} b_{n_{2}}, \]  

(B15)

e.i. (B9). Also solving (B11) backwards from \( n_{2} \) to \( n \), we have

\[ \frac{b_{n}}{b_{n_{2}}} \leq \left( \frac{\Lambda + 1}{2} \right)^{n-n_{2}}, \]  

(B16)

which, in view of \( b_{n_{2}} \leq \epsilon \), is nothing but (B7).

**Step 3. Refined estimate on \( b_{n} \).** For later use, we note the following refined estimate, which is valid without the additional assumption (B8). First note that \( g_{n} \leq \delta \), \( b_{n} \leq \epsilon \), and \( f_{n} \leq b_{n} \) guarantee \( \lambda - C(g_{k} + b_{k}) \geq (\Lambda + 1)/2 > 1 \). So the recursion (B12) implies \( n \leq n_{2} \)

\[ \frac{b_{n}}{b_{0}} \leq \prod_{k=0}^{n-1} \left[ \Lambda + C(g_{k} + b_{k}) \right] \leq \Lambda^{n} \prod_{k=0}^{n-1} \left[ 1 \pm \frac{C(g_{k} + b_{k})}{\Lambda} \right] \leq \Lambda^{n} \exp \left[ \pm \frac{(2\ln 2)C}{\Lambda} \sum_{k=0}^{n-1} (g_{k} + b_{k}) \right]. \]  

(B17)

In the second step above we used \(|\ln(1 + x)| \leq (2 \ln 2)|x|\), which is valid for \(|x| \leq 1/2\). [Note that \( C(g_{k} + b_{k})/\Lambda \leq (\Lambda - 1)/(2\Lambda) < 1/2\].]

(ii) Given the above estimate (B17), the rest is easy. Making use of the assumption (B8) on \( g_{n} \), and the estimate (B3) on \( b_{k} \) proven so far, it can be easily seen that the sum in the exponent of (B17) is bounded by a constant depending only on \( C, \Lambda, \Lambda \) and \( \sum_{k=0}^{\infty} g_{k} \), uniformly in \( n \) and \( n_{2} \) as long as \( n \leq n_{2} \). Taking (B14) into account, (B17) with \( n = n_{2} \) gives (B11). \( \square \)

**APPENDIX C: SPHERICALLY SYMMETRIC, SELF-SIMILAR SPACE-TIME**

Defriese and Carter (see [34]) showed that the line element of any self-similar, spherically symmetric spacetime is written as

\[ ds^{2} = e^{-2\eta}( -G_{1}(y)dy^{2} + G_{2}(y)dy^{2} + e^{2\eta}(d\theta^{2} + \sin \theta d\phi^{2}) ). \]  

\[ (C1) \]

Let \( t = -e^{-\eta}Y(y) \) and \( r = e^{\eta-\eta}, \) where

\[ Y^{2}(y) = \int_{y_{\text{constant}}}^{y} dy G_{1}(y) \frac{e^{2y}}{2G_{2}}. \]  

(C2)

One has

\[ ds^{2} = -\alpha^{2}dt^{2} + \alpha^{2}dr^{2} + r^{2}(d\theta^{2} + \sin \theta d\phi^{2}), \]  

(C3)

where \( \alpha \) and \( a \) are functions of \( y \). Since \( r/( -t) \) is a function of \( y \) only, \( \alpha \) and \( a \) are functions of \( r/( -t) \).

**APPENDIX D: THE LYAPUNOV ANALYSIS**

The Lyapunov method has been extensively used in nonlinear analysis. To make our presentation self-contained and transparent, we present a concise mathematical description of the method, restricted to the situation we are interested in.

Let us consider the linearization of a flow on a real Hilbert space (or a real Hilbert manifold) around an orbit. It is determined by an equation

\[ \frac{df}{ds}(s) = A(s)f(s), \]  

(D1)

where each \( f(s) \) is an element of real Hilbert space \( V \), a complete vector space with inner product, and \( A(s) \) is a real function determined by the (background) orbit which we are considering. Let \( (\cdot, \cdot) \) denote the inner product and \( \| \cdot \| = \sqrt{(\cdot, \cdot)} \) denote the norm defined by the inner product. We have \( A(s) = A \) in (D1) when the orbit is a fixed point, and \( A(s + \Delta) = A(s) \) when it is a periodic orbit with periodicity \( \Delta \). We concentrate on these cases below.

Let us define time evolution operator \( T_{s} : V \to V \) by

\[ f(0) = F, \quad f(s) = T_{s}F; \]  

(D2)

where \( f \) is a solution of (D1). We wish to find eigenvalues \( \kappa \) and eigenvectors \( E^{\kappa} \) of \( T_{s} \) satisfying

\[ T_{s}E^{\kappa} = e^{\kappa s}E^{\kappa}, \]  

(D3)

in particular those with large Re\( \kappa \). Here \( \kappa \) is a complex number and \( E^{\kappa} \) is a complex vector with Re\( E^{\kappa} \), Im\( E^{\kappa} \in V \). In (D3) and below, \( s \) (and \( \sigma \)) can be any number in the case of a fixed point, whereas it is restricted to be integer multiple of \( \Delta \) in the case of a periodic orbit.

We assume that the eigenvalue problem (D3) has a complete set of discrete eigenvectors, denoted by \( E^{\kappa} = \{ E_{i}^{\kappa} \}_{i=1, 2,...} \), with the corresponding eigenvalues \( \{ \kappa_{i} \}_{i=1,2,...} \), Re\( \kappa_{1} \geq \text{Re} \kappa_{2} \geq \text{Re} \kappa_{3} \geq ... \). Each \( \kappa_{i} \) is called the \( i \)th Lyapunov exponent. (Here we implicitly assume that we can always find \( \kappa_{i} \) with the largest real part after we have defined \( \kappa_{1}, ..., \kappa_{i-1} \).) We observe that if \( \kappa_{i} \) is not real there is an integer \( i^{*} \neq i \) with \( \kappa_{i^{*}} = \kappa_{i}^{*} \) and \( E_{i}^{\kappa_{i}} = E_{i}^{\kappa_{i}^{*}} \), where asterisks (except for those on \( i \) or... \)
j) denote complex conjugate in this appendix. We define \( \mathcal{E} = \{E_i\}_{i=1,2,...} \) by \( E_i = E_i^* \) if \( \kappa_i \) is real, and \( E_i = \text{Re} E_i^* \), \( E_i = \text{Im} E_i^* \) if \( \kappa_i \) is not real, \( i \leq \kappa \), and \( E_i^* = E_i^* \). Any real vector \( F \) can be expanded by \( \mathcal{E} \) or \( \mathcal{E}^c \) as

\[
F = \sum_{i=1}^{\infty} f_i E_i = \text{Re} \sum_{i=1}^{\infty} f_i^i E_i^c, \tag{D4}
\]

where \( f_i = f_i^* \) is \( (1/2)(f_i + \sqrt{-1} f_i^*) \). Without loss of generality, we assume all eigenvectors are normalized so that \( \| \text{Re} E_i^* \|^2 + \| \text{Im} E_i^* \|^2 = 1 \), although \( E_i \)'s are not necessarily orthogonal with respect to the inner product \( \langle , \rangle \).

Let \( \mathcal{F} = \{F_i\}_{1 \leq i \leq n} \) denote an n-frame of (linearly independent, real) vectors and let \( \text{span}\mathcal{F} \) be the subspace of \( V \) spanned by \( \mathcal{F} \). Below we often omit \( 1 \leq i \leq n \) in \( \mathcal{F} = \{F_i\}_{1 \leq i \leq n} \) if no confusion occurs.

We define several operations on frames. First given an operator \( X : V \rightarrow V \) we define \( \mathcal{X} \), an operator on frames induced from \( X \), by \( \mathcal{X} \mathcal{F} = \{X(F_i)\}_i \). In the following consider the cases where \( X \) is a linear operator (such as \( T_{\sigma} \)), or the normalization operator \( N \): \( NF \equiv F/\|F\| \).

Second, we define the orthogonalization operator \( \mathcal{O}\mathcal{F} \rightarrow \mathcal{F}' = \{F'_i\}_{1 \leq i \leq n} \), where \( \mathcal{F}' \) is defined by

\[
F'_1 \equiv F_1; \\
F'_i \equiv F_i - \sum_{j=1}^{i-1} \langle F'_j, F_i \rangle F'_j, \quad 2 \leq i \leq n. \tag{D5}
\]

We adopt the convention that \( \mathcal{X} \mathcal{F}_i \equiv \mathcal{X} \mathcal{F} \) denote the ith vector of n-frame \( \mathcal{X} \mathcal{F} \), when \( \mathcal{X} \) is one of the operations on a frame defined above. Distinct from \( \mathcal{X} \) induced from \( X \) : \( V \rightarrow V \), \( \mathcal{X} \mathcal{F}_i \) in general depends on \( F_j \) with \( j \neq i \). Note that \( \text{span}\mathcal{O}\mathcal{F} = \text{span}\mathcal{F} \).

**Proposition.** For almost every \( \mathcal{F} \), i.e., except for “measure zero” cases whose meaning is given by (D10), we have

\[
\text{Re} \kappa_i = \lim_{m \to \infty} \frac{1}{m \sigma} \ln \frac{\| \mathcal{O} \mathcal{T}_{ms} \mathcal{F}_i \|}{F_i} \tag{D6}
\]

\[
= \lim_{m \to \infty} \frac{1}{m \sigma} \sum_{i=1}^{m} \lambda_i(l, \sigma), \tag{D7}
\]

for \( 1 \leq i \leq n \), where

\[
\lambda_i(l, \sigma) \equiv \ln \| \mathcal{O} \mathcal{T}_{l} (\mathcal{N} \mathcal{T}_{\sigma} \mathcal{F}_i) \|^{-1} \| F_i \|. \tag{D8}
\]

**Remark.** The first expression (D6) of the Proposition is mathematically straightforward, but is not suitable for numerical calculations, due to serious overflow and underflow problems in large time integration. To overcome this difficulty, we employ the second expression of the Proposition, which is performed in the following manner in practice.

(0) Prepare \( \mathcal{F} \) and let \( \Lambda_i = 0 \).

(1) Evolve \( \mathcal{F} \) in time by \( T_{\sigma} \), and define \( \mathcal{F}' = T_{\sigma} \mathcal{F} \).

(2) Find \( \mathcal{F}'' = \mathcal{O} T_{\sigma} \mathcal{F} \) and \( \mathcal{F}''' = \mathcal{N}\mathcal{O} T_{\sigma} \mathcal{F} \) by the Gram–Schmidt procedure:

\[
F''_1 = F'_1, \quad F'''_1 = F''_1/\|F''_1\|; \\
F''_i = F'_i - \sum_{j=1}^{i-1} \langle F''_j, F'_i \rangle F''_j, \quad F'''_i = F''_i/\|F''_i\|, \tag{D9}
\]

\( i \geq 2 \).

(3) Add \( \lambda_i = \ln \|F''_i\| \) to \( \Lambda_i \).

(4) Define new \( \mathcal{F} = \mathcal{F}''' \) and go back to (1).

\( \Lambda_i/m\sigma \), where \( m \) is the number of iteration, gives the Lyapunov exponent \( \text{Re} \kappa_i \) according to the Proposition, because \( \mathcal{N}\mathcal{O} \) is equivalent to the Gram–Schmidt procedure.

Before proceeding to the proof of the Proposition we list several properties of operations on frames.

**Lemma.** Let \( \mathcal{X}, \mathcal{Y} \) be operations on frames.

(i) If \( \mathcal{X} \) is induced from a linear operator \( X : V \rightarrow V \), then as operators on frames, \( \mathcal{X}\mathcal{O} = \mathcal{O}\mathcal{X} \). In particular,

\[
\mathcal{O}^2 = \mathcal{O}, \quad \mathcal{O} T_{\sigma} \mathcal{O} = \mathcal{T}_{\omega} \mathcal{T}_{\sigma}. \tag{D10}
\]

(ii) If \( \mathcal{X}, \mathcal{Y} \) are induced from operators \( X, Y \) on \( V \), we have \( \mathcal{X}\mathcal{Y} \mathcal{F} = \{X(Y(F_i))\}_1 \leq i \leq n \). In particular,

\[
\mathcal{N}^2 = \mathcal{N}, \quad \mathcal{N} T_{\sigma} \mathcal{N} = \mathcal{T}_{\omega} \mathcal{T}_{\sigma}. \tag{D11}
\]

\( \mathcal{T}_{\omega} \mathcal{F} = \{\|F_i\|^{-1} T_{\omega}(F_i)\}_1 \leq i \leq n \). \tag{D12}

(iii) For \( c_i \neq 0 \) (1 \( \leq i \leq n \)) we have

\[
\mathcal{O}(\{c_i F_i\}_1 \leq i \leq n) = \{c_i \mathcal{O} F_i\}_1 \leq i \leq n. \tag{D13}
\]

and

\[
\mathcal{N}\mathcal{O} = \mathcal{N}\mathcal{O}. \tag{D14}
\]

**Proof.** (i) By the definition of \( \mathcal{O} \), any frame \( \mathcal{F} \) is written as \( \mathcal{F} = \{\mathcal{O} F_i + \sum_{j=1}^{i-1} a_j^1 \mathcal{O} F_j\}_1 \leq i \leq n \) with suitable scalars \( a_j^1 \)'s. Since \( X \) is linear we have \( \mathcal{X}\mathcal{F} = \{\mathcal{X}\mathcal{O} F_i + \sum_{j=1}^{i-1} a_j^1 \mathcal{X}\mathcal{O} F_j\}_1 \leq i \leq n \). However, for scalars \( a_j^1 \)'s,

\[
\mathcal{O}(\{F_i + \sum_{j=1}^{i-1} a_j^1 F_j\}_1 \leq i \leq n) = \mathcal{O}(\{F_i\}_1 \leq i \leq n). \tag{D15}
\]
which can be shown by induction in $i$. So we have $O\mathcal{X}\mathcal{F} = O\mathcal{X}O\mathcal{F}$.

(ii) is just a definition.

(iii) can be shown by induction in $n$, noting that $O\mathcal{F}_i$ depends only on $F_i (j \leq i)$.

Proof of the Proposition. We prove the proposition in two steps. First, we prove (D16), and then prove the equivalence of expressions in (D17) and (D18).

We assume $\mathcal{F}$ to be generic in the sense that

$$\text{span}\{F_j\}_{j \leq i} \cap (\text{span}\{E_j\}_{j \geq i} \backslash \text{span}\{E_j\}_{j \geq i+1} \neq \{0\}),$$

$$1 \leq i \leq n. \quad (D16)$$

The set of all nongeneric $\mathcal{F}$’s is of measure zero in the space of $n$-frames, because each nongeneric $\mathcal{F}$ contains at least one such $i$ that $F_i$ lies in a subspace of $V$ of codimension one, span{E_j}_{j \neq i}. from (D10) we can define $\tilde{\mathcal{F}} = \{\tilde{F}_i\}_i$ such that (for $1 \leq i \leq n$)

$$\tilde{F}_i \in \text{span}\{E_j\}_{j \geq i} \backslash \text{span}\{E_j\}_{j \geq i+1},$$

$$F_i - \tilde{F}_i \in \text{span}\{F_j\}_{j \leq i-1}. \quad (D17)$$

Note that we have

$$O\tilde{\mathcal{F}} = O\mathcal{F}. \quad (D18)$$

Let us expand $\tilde{\mathcal{F}}$ in terms of $\mathcal{E}$ as

$$\tilde{F}_i = \sum_{j=1}^{\infty} f^j_i E_j = \Re \left[ \sum_{j=1}^{\infty} f^j_i E_j^c \right], \quad (D19)$$

where $f^j_i$’s are defined similarly as $f^j_i$’s in (D4). Condition (D16) implies that $f^j_i \neq 0$ for all $i \leq n$, and that no $f^j_i$’s appear for $j < i$. Time evolution of $\tilde{F}_i$ is given by

$$T_s \tilde{F}_i = \Re \left[ \sum_{j=1}^{\infty} e^{\kappa j} f^j_i E_j^c \right] = \Re \left[ \tilde{E}_i(s) + O(e^{-\delta s}) \right], \quad (D20)$$

where

$$\tilde{E}_i(s) = \Re \left[ \sum_{j=1, \text{Re} \sigma_j = \text{Re} \kappa_j}^{\infty} e^{i(\text{Im} \kappa_j)s} f^j_i E_j^c \right], \quad (D21)$$

and $\delta$ is some positive constant. Since $f^j_i \neq 0$, $||\tilde{E}_i(s)||$ is bounded and nonzero uniformly in $s$. Operating $O$ on both sides of (D21), we thus have by (D13),

$$O T_s \tilde{F}_i = \Re \left[ O \tilde{E}_i(s) + O(e^{-\delta s}) \right] \quad (D22)$$

for sufficiently large $s$. The left hand side can be replaced by $O T_s \mathcal{F}_i$ leading to

$$O T_s \mathcal{F}_i = \Re \left[ O \tilde{E}_i(s) + O(e^{-\delta s}) \right], \quad (D23)$$

because we have from (D10) and (D18) that $O T_s \mathcal{F} = O T_s O \mathcal{F} = O T_s \mathcal{F} = O T_s \mathcal{F}$ for any $s$. Thus we have

$$C_1 \leq \frac{||O T_s F_i||}{e^{\Re \kappa} s} \leq C_2 \quad (D24)$$

for sufficiently large $s$, where $C_1$, $C_2$ are positive constants. Eqs. (D20) and (D24) prove (D9).

Now we turn to the proof of equivalence of (D9) and (D7). Using (D11) and (D14) we have

$$N O T N = N O T N = N O N = N O T$$

and thus

$$(N O T)^e = N(O T)^e.$$

So by the definition of $\lambda_i(l, \sigma)$, we have

$$\lambda_i(l, \sigma) = \ln ||O T\sigma N(O T\sigma)^{l-1} F_i||$$

$$= \ln \frac{||O T\sigma F_i||}{||O T\sigma(l-1)^{-1} F_i||}, \quad (D25)$$

where in the last step we used (D12) and (D13). We thus have

$$\sum_{i=1}^{m} \lambda_i(l, \sigma) = \ln \frac{||O T\sigma F_i||}{||F_i||}. \quad (D26)$$

and the proposition is proved. \qed

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