INVOLUTIONS AND THE JACOBIAN CONJECTURE

VERED MOSKOWICZ

Abstract. The famous Jacobian conjecture asks if an endomorphism $f$ of $K[x,y]$ ($K$ is a characteristic zero field) that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ is invertible.

Let $\alpha$ be the exchange involution on $K[x,y]$: $\alpha(x) = y$ and $\alpha(y) = x$. An $\alpha$-endomorphism $f$ of $K[x,y]$ is an endomorphism of $K[x,y]$ that preserves the involution $\alpha$: $f \alpha = \alpha f$. It was shown in [7, Proposition 4.1] that if $f$ is an $\alpha$-endomorphism that satisfies $\text{Jac}(f(x), f(y)) \in K^*$, then $f$ is invertible. Based on this, we bring more results that imply that a given endomorphism $f$ that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ and additional conditions involving involutions, is invertible.

1 Introduction

Let $K$ be a characteristic zero field. The famous Jacobian conjecture asks if an endomorphism $f : K[x,y] \rightarrow K[x,y]$ that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ is invertible.

We suggest some partial answers to this conjecture, based on the following previous result [7, Proposition 4.1]: “If $f$ is an $\alpha$-endomorphism that satisfies $\text{Jac}(f(x), f(y)) \in K^*$, then $f$ is invertible”.

Where $\alpha$ is the exchange involution on $K[x,y]$, $\alpha(x) = y$, $\alpha(y) = x$, and an $\alpha$-endomorphism is an endomorphism of $K[x,y]$ that preserves the involution $\alpha$: $f \alpha = \alpha f$.

Some of the results we bring here are mentioned, without a proof, in [8], while others are new.

2 Two equivalent conjectures to the Jacobian conjecture

In Theorem 2.7 we show that the Jacobian conjecture is equivalent to the $\gamma, \delta$ conjecture 2.5 and to the $g, h$ conjecture 2.6.

By an involution on $K[x,y]$ we mean an automorphism of order 2 (since $K[x,y]$ is commutative, any anti-automorphism is an automorphiam). We wish to consider not only the exchange involution $\alpha$, but also any involution on $K[x,y]$.

Lemma 2.1. Assume $\gamma$ is any involution on $K[x,y]$. Then there exists an automorphism $g$ of $K[x,y]$ such that $\gamma = g^{-1} \alpha g$.

In [3] J. Bell has sketched a proof for the analogue result in the first Weyl algebra. His sketch of proof is applicable to $K[x,y]$.

Proof. Use [5, Proposition 8.9] (or [9] with the fact that triangular automorphisms has infinite order), and a direct calculation that any linear automorphism of order 2 is conjugate to $\alpha$.

Definition 2.2. Let $f$ be an endomorphism of $K[x,y]$ and let $\gamma$ and $\delta$ be any two involutions on $K[x,y]$. We say that $f$ is

2010 Mathematics Subject Classification. Primary 14R15, 16W20.

The author was partially supported by an Israel-US BSF grant #2010/149.
• a $\gamma$-endomorphism of $K[x,y]$, if $f\gamma = \gamma f$.
• a $\gamma, \delta$-endomorphism of $K[x,y]$, if $f\gamma = \delta f$.

(If $\delta = \gamma$, then a $\gamma, \gamma$-endomorphism is just a $\gamma$-endomorphism).

Notice that in the above definition, it is not assumed that the endomorphism $f$ satisfies $\text{Jac}(f(x),f(y)) \in K^*$.

An immediate generalization of [7, Proposition 4.1] is as follows:

**Theorem 2.3.** Assume $\gamma$ and $\delta$ are two involutions on $K[x,y]$. Assume $f$ is a $\gamma, \delta$-endomorphism of $K[x,y]$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$. Then $f$ is invertible.

**Proof.** From Lemma 2.1, there exists an automorphism $g$ of $K[x,y]$ such that $\gamma = g^{-1}\alpha g$, and there exists an automorphism $h$ of $K[x,y]$ such that $\delta = h^{-1}\alpha h$.

Let $f$ be a $\gamma, \delta$-endomorphism, so by definition, $f\gamma = \delta f$. So, $fg^{-1}\alpha g = h^{-1}\alpha hf$. Hence, $(hf^{-1})\alpha = \alpha(hfg^{-1})$. Therefore, $hfg^{-1}$ is an $\alpha$-endomorphism. From the chain rule we have

$$\text{Jac}((hfg^{-1})(x),(hfg^{-1})(y)) \in K^*$$

$(g$ and $h$ are automorphisms of $K[x,y]$, so $\text{Jac}(g^{-1}(x),g^{-1}(y)) \in K^*$ and $\text{Jac}(h(x),h(y)) \in K^*$.

$\text{Jac}(f(x),f(y)) \in K^*$ by assumption).

Apply [7, Proposition 4.1] to $hfg^{-1}$ and get that $hfg^{-1}$ is invertible, hence $f$ is invertible. \qed

**Remark 2.4.** Given two rings with involution $(R_1, \epsilon_1)$ and $(R_2, \epsilon_2)$, we say that $f$ is an involutive endomorphism from $(R_1, \epsilon_1)$ to $(R_2, \epsilon_2)$, if $f\epsilon_1 = \epsilon_2 f$.

In particular, if $R_1 = R_2$ then such $f$ is just an $\epsilon_1, \epsilon_2$-endomorphism.

Hence, Theorem 2.3 says the following: Let $\gamma$ and $\delta$ be two involutions on $K[x,y]$. Let $f$ be an involutive endomorphism from $(K[x,y], \gamma)$ to $(K[x,y], \delta)$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$. Then $f$ is invertible.

In view of Theorem 2.3, given any endomorphism $f$ of $K[x,y]$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$, one wishes to be able to find two involutions $\gamma$ and $\delta$ on $K[x,y]$, such that $f$ is a $\gamma, \delta$-endomorphism. Hence we suggest the following conjectures:

**Conjecture 2.5** (The $\gamma, \delta$ conjecture). Assume $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$. Then there exist involutions $\gamma$ and $\delta$ on $K[x,y]$ such that $f$ is a $\gamma, \delta$-endomorphism.

**Conjecture 2.6** (The $g,h$ conjecture). Assume $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$. Then there exist automorphisms $g$ and $h$ of $K[x,y]$ such that $hfg^{-1}$ is an $\alpha$-endomorphism.

We have:

**Theorem 2.7.** TFAE:

1. The Jacobian conjecture.
2. The $\gamma, \delta$ conjecture.
3. The $g,h$ conjecture.

Where $\gamma$ and $\delta$ are involutions on $K[x,y]$, and $g$ and $h$ are automorphisms of $K[x,y]$.

**Proof.** Let $f$ be an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(f(x),f(y)) \in K^*$.

1. $\Rightarrow$ 2.: $f$ is invertible. Define $\gamma := f^{-1}\alpha f$ and $\delta := \alpha$, and get $f\gamma = f(f^{-1}\alpha f) = \alpha f = \delta f$.

2. $\Rightarrow$ 3.: There exist involutions $\gamma$ and $\delta$ on $K[x,y]$ such that $f$ is a $\gamma, \delta$-endomorphism, namely $f\gamma = \delta f$. From Lemma 2.1, there exists an automorphism
g of $K[x,y]$ such that $\gamma = g^{-1} \circ g$, and there exists an automorphism $h$ of $K[x,y]$ such that $\delta = h^{-1} \circ h$

Then $f\gamma = \delta f$ becomes $(hfg^{-1})\alpha = \alpha(hfg^{-1})$, so $hfg^{-1}$ is an $\alpha$-endomorphism.

(3) $\implies$ (1): There exist automorphisms $g$ and $h$ of $K[x,y]$ such that $hfg^{-1}$ is an $\alpha$-endomorphism. Clearly, $\text{Jac}((hfg^{-1})(x),(hfg^{-1})(y)) \in K^*$.

Apply [7, Proposition 4.1] to $hfg^{-1}$ and get that $hfg^{-1}$ is invertible, hence $f$ is invertible.

The following Lemma is an analogue of [8, Lemma 2.5]; here we add the condition that $\text{Jac}(f(x), f(y)) \in K^*$, because we do not know how to prove that a $\gamma$, $\delta$-endomorphism is invertible without the condition on the Jacobian.

**Lemma 2.8.** Assume $f$ is an endomorphism of $K[x,y]$ that satisfies

$\text{Jac}(f(x), f(y)) \in K^*$. Then: $f$ is a $\gamma$, $\delta$-endomorphism, where $\gamma$ and $\delta$ are involutions on $K[x,y] \iff f$ is invertible.

The proof of this lemma actually appers in the above proof of Theorem 2.7.

**Proof.** $\implies$: There exist involutions $\gamma$ and $\delta$ on $K[x,y]$ such that $f\gamma = \delta f$, $\gamma = g^{-1} \circ g$ and $\delta = h^{-1} \circ h$, for some automorphisms $g$ and $h$ of $K[x,y]$. Then $hfg^{-1}$ is an $\alpha$-endomorphism that satisfies $\text{Jac}((hfg^{-1})(x),(hfg^{-1})(y)) \in K^*$ (here we use the assumption $\text{Jac}(f(x), f(y)) \in K^*$).

By [7, Proposition 4.1] $hfg^{-1}$ is invertible, hence $f$ is invertible.

$\iff$: Take $\gamma := f^{-1} \circ \alpha f$ and $\delta := \alpha$, and get $f\gamma = f(f^{-1} \alpha f) = \alpha f = \delta f$. $\square$

## 3 Extension and restriction conditions

In view of Theorem 2.7, our (hopefully possible) mission is to prove that the $\gamma$, $\delta$ conjecture is true or to prove that the $g$, $h$ conjecture is true.

If each endomorphism $f$ of $K[x,y]$ that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ also satisfies the extension condition, then the $\gamma$, $\delta$ conjecture is true, see Theorem 3.3.

If each endomorphism $f$ of $K[x,y]$ that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ also satisfies the restriction condition, then the $\gamma$, $\delta$ conjecture is true, see Theorem 3.6.

From now on we use the following notations: $K$ will continue to denote a characteristic zero field, except in some results where we demand it to be the field of complex numbers.

Given an endomorphism $f$ of $K[x,y]$ (not necessarily satisfying $\text{Jac}(f(x), f(y)) \in K^*$) we denote $P := f(x)$ and $Q := f(y)$. Denote by $T$ the image of $K[x,y]$ under $f$, namely $T = K[P,Q]$. $T$ is a subalgebra of $K[x,y]$. If $\text{Jac}(P,Q) \in K^*$, then $T$ is isomorphic to $K[x,y]$. Assume $\text{Jac}(P,Q) \in K^*$ and denote by $\sigma_0$ the involution on $T$ which exchanges $P$ and $Q$, namely $\sigma_0(P) = Q$, $\sigma_0(Q) = P$ (extended in the obvious way to all of $T$).

**Remark 3.1.** Let $f$ be an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. We do not know if $\sigma_0$ can be extended to an endomorphism of $K[x,y]$.

**Definition 3.2** (The extension condition). Let $f$ be an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. We say that $f$ satisfies the extension condition if the involution $\sigma_0$ on $T$ can be extended to an involution on $K[x,y]$.

Notice that in the above definition we demand that $\sigma_0$ can be extended not just to an endomorphism of $K[x,y]$, but to an involution on $K[x,y]$ (an automorphism of $K[x,y]$ of order 2).

**Theorem 3.3** (The extension theorem). Assume $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. Then: $f$ satisfies the extension condition $\iff f$ is a $\gamma$, $\delta$-endomorphism of $K[x,y]$, where $\gamma$ and $\delta$ are involutions on $K[x,y]$. 

Proof. \(\Rightarrow\): If \(f\) satisfies the extension condition, so the involution \(\sigma_{0}\) on \(T\) can be extended to an involution on \(K[x, y]\), call it \(\sigma\). Therefore, we get that \(f\) is an \(\alpha, \sigma\)-endomorphism of \(K[x, y]\), because:

\[
(f\alpha)(x) = f(y) = Q = \sigma_{0}(P) = \sigma(P) = \sigma(f(x)) = (\sigma f)(x)
\]

and

\[
(f\alpha)(y) = f(x) = P = \sigma_{0}(Q) = \sigma(Q) = \sigma(f(y)) = (\sigma f)(y).
\]

\(\Leftarrow\): \(f\) is a \(\gamma, \delta\)-endomorphism of \(K[x, y]\), where \(\gamma\) and \(\delta\) are involutions on \(K[x, y]\).

Lemma 2.8 implies that \(f\) is invertible, so \(T = K[x, y]\).

By definition, \(\sigma_{0}\) is the involution on \(T\) given by \(\sigma_{0}(P) = Q, \sigma_{0}(Q) = P\), hence \(\sigma_{0}\) is an involution on \(K[x, y]\), so \(f\) satisfies the extension condition (the extension of \(\sigma_{0}\) to \(K[x, y]\) is \(\sigma_{0}\) itself).

(Remark: \(T = K[x, y]\) so \(x = \sum a_{ij}P^{\delta}Q^{j}\) and \(y = \sum b_{ij}P^{i}Q^{j}\). Therefore,

\[
\sigma_{0}(x) = \sigma_{0}(\sum a_{ij}P^{\delta}Q^{j}) = \sum a_{ij}(\sigma_{0}(P))^{j}(\sigma_{0}(Q))^{i} = \sum a_{ij}Q^{i}P^{j}.
\]

And similarly, \(\sigma_{0}(y) = \sum b_{ij}Q^{i}P^{j}\).

\(\square\)

From Lemma 2.8 and the above proof it is clear that if \(f\) is an endomorphism of \(K[x, y]\) that satisfies \(\text{Jac}(P, Q) \in K^*\), then: \(f\) satisfies the extension condition \(\iff\) \(f\) is invertible. We also suggest to consider the following restriction condition.

**Definition 3.4** (The restriction condition). Let \(f\) be an endomorphism of \(K[x, y]\) that satisfies \(\text{Jac}(P, Q) \in K^*\). We say that \(f\) satisfies the restriction condition if \(\alpha(P) \in T\) and \(\alpha(Q) \in T\). Equivalently, we say that \(f\) satisfies the restriction condition if the exchange involution \(\alpha\) on \(K[x, y]\) when restricted to \(T\) is an involution on \(T\).

**Remark 3.5.** Let \(f\) be an endomorphism of \(K[x, y]\) that satisfies \(\text{Jac}(P, Q) \in K^*\). We do not know if necessarily \(\alpha(P) \in T\) and \(\alpha(Q) \in T\).

**Theorem 3.6** (The restriction theorem). Assume \(f\) is an endomorphism of \(K[x, y]\) that satisfies \(\text{Jac}(P, Q) \in K^*\). Then: \(f\) satisfies the restriction condition \(\iff\) \(f\) is invertible.

Proof. \(\Rightarrow\): \(f\) satisfies the restriction condition, so \(\alpha\) restricted to \(T\) is an involution on \(T\).

Denote the restriction of \(\alpha\) to \(T\) by \(\alpha_{0}\).

Since \(T\) is isomorphic to \(K[x, y]\) it follows from Lemma 2.1 that every involution on \(T\) is conjugate (by an automorphism of \(T\)) to one chosen involution on \(T\) (equivalently, any two involutions are conjugate):

in particular, there exists an automorphism \(g_{0}\) of \(T\) such that \(\sigma_{0} = g_{0}^{-1}\alpha_{0}g_{0}\).

Therefore,

\[
(f\alpha)(x) = f(y) = Q = \sigma_{0}(P) = \sigma_{0}(f(x)) = (g_{0}^{-1}\alpha_{0}g_{0})(f(x)) = (g_{0}^{-1}\alpha_{0}g_{0}f)(x)
\]

and

\[
(f\alpha)(y) = f(x) = P = \sigma_{0}(Q) = \sigma_{0}(f(y)) = (g_{0}^{-1}\alpha_{0}g_{0})(f(y)) = (g_{0}^{-1}\alpha_{0}g_{0}f)(y).
\]

Therefore, \(f\alpha = (g_{0})^{-1}\alpha_{0}g_{0}f\).

Then, \(g_{0}f\alpha = \alpha_{0}g_{0}f\), so \(g_{0}f\alpha = \alpha_{0}f\), namely \(g_{0}f\) is an \(\alpha\)-endomorphism of \(K[x, y]\).

Since the Jacobian of \(g_{0}(P), g_{0}(Q)\) with respect to \(P, Q\), denote it by \(a\), is a non-zero scalar \((g_{0}\) is an automorphism of \(T\)) and the Jacobian of \(f(x), f(y)\) with respect to \(x, y\), denote it by \(b\), is a non-zero scalar (by assumption \(\text{Jac}(f(x), f(y)) \in K^*\)), we get that \(\text{Jac}((g_{0}f)(x), (g_{0}f)(y)) = ab \in K^*\). \((\text{Jac}((g_{0}f)(x), (g_{0}f)(y))\) is the Jacobian of \((g_{0}f)(x), (g_{0}f)(y)\) with respect to \(x, y\).
By [7, Proposition 4.1] $g_0f$ is an automorphism of $K[x,y]$, so $K[g_0(P),g_0(Q)] = K[x,y]$.

Hence, $x = \sum a_{ij}(g_0(P))^i(g_0(Q))^j = g_0(\sum a_{ij}P^iQ^j)$ and 
$y = \sum b_{ij}(g_0(P))^i(g_0(Q))^j = g_0(\sum b_{ij}P^iQ^j)$, where $a_{ij}, b_{ij} \in K$.

This shows that $x, y \in T$, so $T = K[x,y]$, and we are done.

$\iff$: $f$ is invertible, so $T = K[x,y]$. Hence $f$ satisfies the restriction condition, since trivially $\alpha(P) \in K[x,y] = T$ and $\alpha(Q) \in K[x,y] = T$. \hfill $\Box$

One can generalize both the extension condition and the restriction condition to the following conditions and have results similar to Theorem 3.3 and Theorem 3.6.

More precisely: Let $\epsilon_0$ be an involution on $T$.

**Definition 3.7** (The $\epsilon_0$ extension condition). Let $f$ be an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. We say that $f$ satisfies the $\epsilon_0$ extension condition if the involution $\epsilon_0$ on $T$ can be extended to an involution on $K[x,y]$.

According to this definition, our previous extension condition is just the $\sigma_0$ extension condition.

**Theorem 3.8** (The $\epsilon_0$ extension theorem). Assume $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. Then: $f$ satisfies the $\epsilon_0$ extension condition \iff $f$ is a $\gamma, \delta$-endomorphism of $K[x,y]$, where $\gamma$ and $\delta$ are involutions on $K[x,y]$ \iff $f$ is invertible.

Let $\epsilon$ be an involution on $K[x,y]$.

**Definition 3.9** (The $\epsilon$ restriction condition). Let $f$ be an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. We say that $f$ satisfies the $\epsilon$ restriction condition if $\epsilon(P) \in T$ and $\epsilon(Q) \in T$. Equivalently, we say that $f$ satisfies the $\epsilon$ restriction condition if the involution $\epsilon$ on $K[x,y]$, when restricted to $T$ is an involution on $T$.

According to this definition, our previous restriction condition is just the $\alpha$ restriction condition.

**Theorem 3.10** (The $\epsilon$ restriction theorem). Assume $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$. Then: $f$ satisfies the $\epsilon$ restriction condition \iff $f$ is invertible.

Summarizing, if $f$ is an endomorphism of $K[x,y]$ that satisfies $\text{Jac}(P,Q) \in K^*$, then TFAE:

- $f$ satisfies the $\epsilon_0$ extension condition.
- $f$ satisfies the $\epsilon$ restriction condition.
- $f$ is invertible.

4 More results

Recall the following result of Cheng-Mckay-Wang [4, Theorem 1]: “Let $K$ be the field of complex numbers. Assume $A, B \in K[x,y]$ satisfy $\text{Jac}(A,B) \in K^*$. If $R \in K[x,y]$ satisfies $\text{Jac}(A,R) = 0$, then $R \in K[\sigma_0]$”.

Its analogue result in the first Weyl algebra over any characteristic zero field, not necessarily the field of complex numbers, can be found in [6, Theorem 2.11]; instead of the Jacobian take the commutator.

We shall use [4, Theorem 1] in the proofs of Theorem 4.2, Theorem 4.4, Theorem 4.7 and Theorem 4.9.

Therefore, in Theorem 4.2, Theorem 4.4, Theorem 4.7 and Theorem 4.9 we will demand that $K$ will be the field of complex numbers.

Actually, in the proofs of those theorems we use the $\epsilon$ restriction theorem 3.10.
Hence, Jac($P, \alpha$) and Jac($Q, \alpha$) so [4, Theorem 1] implies that $\alpha \text{Jac}(P, Q) \neq 0$. Assume w.l.o.g that Jac($P, \alpha$) is a generalized $\alpha$-endomorphism of $K[x, y]$.

**Definition 4.1 (A generalized $\alpha$-endomorphism).** Let $f$ be an endomorphism of $K[x, y]$. We say that $f$ is a generalized $\alpha$-endomorphism if the following two conditions are satisfied:

1. $\text{Jac}(P, Q) \in K^*$.
2. $\text{Jac}(P, \alpha(P)) \in K^*$ or $\text{Jac}(Q, \alpha(Q)) \in K^*$.

Notice that a generalized $\alpha$-endomorphism $f$ is indeed a generalization of an $\alpha$-endomorphism $f$ of $K[x, y]$ that satisfies $\text{Jac}(P, Q) \in K^*$. Let $f$ be an $\alpha$-endomorphism of $K[x, y]$ that satisfies $\text{Jac}(P, Q) \in K^*$. Of course, $\alpha(P) = \alpha(f(x)) = (\alpha f)(x) = (f \alpha)(x) = f(y) = Q$. Hence, $\text{Jac}(P, \alpha(P)) = \text{Jac}(Q, P) \in K^*$, as desired (Actually also $\text{Jac}(Q, \alpha(Q)) = \text{Jac}(Q, P) \in K^*$).

**Theorem 4.2.** Assume $K$ is the field of complex numbers. If $f$ is a generalized $\alpha$-endomorphism, then $f$ is invertible.

**Proof.** $f$ is a generalized $\alpha$-endomorphism, hence by definition: $\text{Jac}(P, Q) \in K^*$ and $\text{Jac}(P, \alpha(P)) \in K^*$ or $\text{Jac}(Q, \alpha(Q)) \in K^*$.

Assume w.l.o.g that $\text{Jac}(P, \alpha(P)) \in K^*$.

Denote $\text{Jac}(P, Q) = a$ and $\text{Jac}(P, \alpha(P)) = b$, where $a, b \in K^*$.

$$0 = 1 - 1 = \text{Jac}(P, \alpha(P))/b - \text{Jac}(P, Q)/a = \text{Jac}(P, \alpha(P))/b - Q/a,$$

so [4, Theorem 1] implies that $\alpha(P)/b - Q/a = H(P)$, where $H(t) \in K[t]$. Then $\alpha(P) = (bQ)/a + bH(P) \in T$.

From the chain rule, since $\text{Jac}(P, Q) = a$ and $\text{Jac}(\alpha(x), \alpha(y)) = -1$, we get $\text{Jac}(\alpha(P), \alpha(Q)) = -a$.

$$0 = \text{Jac}(\alpha(P), \alpha(Q))/(-a) - \text{Jac}(\alpha(P), P)/(-b) = \text{Jac}(\alpha(P), \alpha(Q))/(-a) - P/(-b),$$

so [4, Theorem 1] implies that $\alpha(Q)/(-a) - P/(-b) = G(\alpha(P))$, where $G(t) \in K[t]$. Then,

$$\alpha(Q) = (aP)/b - aG(\alpha(P)) = (aP)/b - aG((bQ)/a + bH(P)) \in T.$$ 

The restriction theorem 3.6 implies that $f$ is invertible. \qed

An obvious generalization is as follows:

**Definition 4.3 (A generalized $\epsilon$-endomorphism).** Let $\epsilon$ be an involution on $K[x, y]$. Let $f$ be an endomorphism of $K[x, y]$.

We say that $f$ is a generalized $\epsilon$-endomorphism if the following two conditions are satisfied:

1. $\text{Jac}(P, Q) \in K^*$.
2. $\text{Jac}(P, \epsilon(P)) \in K^*$ or $\text{Jac}(Q, \epsilon(Q)) \in K^*$.
A generalized $\epsilon$-endomorphism is not a generalization of an $\epsilon$-endomorphism that satisfies $\text{Jac}(P, Q) \in K^*$; for example, $f(x) = x + y^2$ and $f(y) = y$ is a $\beta$-endomorphism (automorphism) that satisfies $\text{Jac}(f(x), f(y)) = 1$, where $\beta$ is the involution given by $\beta(x) = x$ and $\beta(y) = -y$. But,

$$\text{Jac}(f(x), \beta(f(x))) = \text{Jac}(x + y^2, x + y^2) = 0$$

and

$$\text{Jac}(f(y), \beta(f(y))) = \text{Jac}(y, -y) = 0.$$ 

Notice that $f$ is a generalized $\alpha$-endomorphism, since

$$\text{Jac}(f(y), \alpha(f(y))) = \text{Jac}(y, x) = -1 \in K^*.$$ 

**Theorem 4.4.** Assume $K$ is the field of complex numbers. If there exists an involution $\epsilon$ on $K[x, y]$ such that $f$ is a generalized $\epsilon$-endomorphism, then $f$ is invertible.

**Proof.** $f$ is a generalized $\epsilon$-endomorphism, hence by definition: $\text{Jac}(P, Q) \in K^*$ and $\text{Jac}(P, \epsilon(P)) \in K^*$ or $\text{Jac}(Q, \epsilon(Q)) \in K^*$.

Assume w.l.o.g that $\text{Jac}(P, \epsilon(P)) \in K^*$.

Denote $\text{Jac}(P, Q) = a$ and $\text{Jac}(P, \epsilon(P)) = b$, where $a, b \in K^*$.

$$0 = \text{Jac}(P, \epsilon(P)/b - \text{Jac}(P, Q/a) = \text{Jac}(P, \epsilon(P)/b - Q/a),$$

so [4, Theorem 1] implies that $\epsilon(P)/b - Q/a = H(P)$, where $H(t) \in K[t]$. Then $\epsilon(P) = (bQ)/a + bH(P) \in T$.

From the chain rule, since $\text{Jac}(P, Q) = a \in K^*$ and $\text{Jac}(\epsilon(x), \epsilon(y)) \in K^*$, we get $\text{Jac}(\epsilon(P), \epsilon(Q)) \in K^*$.

$$0 = \text{Jac}(\epsilon(P), \epsilon(Q)/(-a)) - \text{Jac}(\epsilon(P), P/(-b)) = \text{Jac}(\epsilon(P), \epsilon(Q)/(-a) - P/(-b),$$

so [4, Theorem 1] implies that $\epsilon(Q)/(-a) - P/(-b) = G(\epsilon(P))$, where $G(t) \in K[t]$. Then,

$$\epsilon(Q) = (aP)/b - aG(\epsilon(P)) = (aP)/b - aG((bQ)/a + bH(P)) \in T.$$

The $\epsilon$ restriction theorem 3.10 implies that $f$ is invertible. $\square$

The converse of Theorem 4.2 is not true; namely, an automorphism of $K[x, y]$ need not be a generalized $\alpha$-endomorphism. For example, $P := h(x) = x + y$ and $Q := h(y) = x - y$ is an automorphism of $K[x, y]$ which is not a generalized $\alpha$-endomorphism, since

$$\text{Jac}(P, \alpha(P)) = 0, \text{Jac}(Q, \alpha(Q)) = 0.$$ 

$h$ is not a generalized $\alpha$-endomorphism, but it is a generalized $\beta$-endomorphism, where $\beta$ is the involution on $K[x, y]$ given by $\beta(x) = x, \beta(y) = -y$. Indeed, $\text{Jac}(P, \beta(P)) = \text{Jac}(x + y, x - y) \in K^*$

(and $\text{Jac}(Q, \beta(Q)) = \text{Jac}(x - y, x + y) \in K^*$).

In view of this it seems natural to ask the following question: Given an automorphism $g$ of $K[x, y]$, is there exists an involution $\epsilon$ on $K[x, y]$ such that $g$ is a generalized $\epsilon$-endomorphism?

Thus far we only managed to show that for every generator of the group of automorphisms of $K[x, y]$ the answer to this question is positive. Indeed,

- Let $g$ be linear with $g(x) = ax + by$, $a, b \in K$. If $a = 0$, then $g(x) = by$, so $g$ is a generalized $\alpha$-endomorphism:

$$\text{Jac}(g(x), \alpha(g(x))) = \text{Jac}(by, bx) \in K^*.$$
If \( b = 0 \), then \( g(x) = ax \), so \( g \) is a generalized \( \alpha \)-endomorphism:

\[
\text{Jac}(g(x), \alpha(g(x))) = \text{Jac}(ax, ay) \in K^*.
\]

If both \( a \) and \( b \) are non-zero, then \( g(x) = ax + by \), so \( g \) is a generalized \( \beta \)-endomorphism (\( \beta(x) = x \) and \( \beta(y) = -y \)):

\[
\text{Jac}(g(x), \beta(g(x))) = \text{Jac}(ax + by, ax - by) = -2ab \in K^*.
\]

- Let \( g \) be triangular. Then it is clear that \( g \) is a generalized \( \alpha \)-endomorphism.

**Remark 4.5.** A given endomorphism can be a generalized \( \epsilon \)-endomorphism, for more that one involution \( \epsilon \). For example, let \( g \) be the automorphism given by \( g(x) = ax + by \) and \( g(y) = cy + dy \), where \( ad - bc \neq 0 \) and \( a \neq \pm b \). It is easy to see that

\[
\text{Jac}(g(x), \alpha(g(x))) = a^2 - b^2 \in K^*.
\]

So \( g \) is a generalized \( \alpha \)-endomorphism and we have already seen that \( g \) is a generalized \( \beta \)-endomorphism (\( \beta(x) = x \) and \( \beta(y) = -y \)).

### 4.2 \( P \) is symmetric or skew-symmetric

Denote the set of symmetric elements (with respect to \( \alpha \)) by \( S_\alpha \) and denote the set of skew-symmetric elements (with respect to \( \alpha \)) by \( K_\alpha \). The set of symmetric elements is \( K \)-linearly spanned by \( \{x^n y^m + x^m y^n | n \geq m\} \), while the set of skew-symmetric elements is \( K \)-linearly spanned by \( \{x^n y^m - x^m y^n | n > m\} \).

**Lemma 4.6.** (1) If \( a \in S_\alpha \) and \( b \in S_\alpha \), then \( \text{Jac}(a, b) \in K_\alpha \).

(2) If \( a \in K_\alpha \) and \( b \in K_\alpha \), then \( \text{Jac}(a, b) \in K_\alpha \).

(3) If \( a \in S_\alpha \) and \( b \in K_\alpha \), then \( \text{Jac}(a, b) \in S_\alpha \).

The analogue result in the first Weyl algebra is also true (and is easier to prove), where instead of the Jacobian take the commutator.

**Proof.** We shall only prove (1); the proofs of (2) and (3) are similar. Write

\[
a = \sum a_{ij} (x^i y^j + x^j y^i), \quad b = \sum b_{kl} (x^k y^l + x^l y^k).
\]

The Jacobian is \( K \)-linear, so

\[
\text{Jac}(a, b) = \sum \sum a_{ij} b_{kl} \text{Jac}(x^i y^j + x^j y^i, x^k y^l + x^l y^k).
\]

Since the sum of skew-symmetric elements is skew-symmetric, it suffices to show that each \( \text{Jac}(x^i y^j + x^j y^i, x^k y^l + x^l y^k) \) is skew-symmetric. Indeed, a direct computation yields:

\[
\text{Jac}(x^i y^j + x^j y^i, x^k y^l + x^l y^k) = (i - k)(j + i - l - 1) + (ik - jl)(x^{i+k-l}y^{i+j-l} + x^{k+l-1}y^{i+j-1}).
\]

\( \square \)

We again assume that \( K \) is the field of complex numbers, because in our proof we wish to use [4, Theorem 1].

**Theorem 4.7.** Assume \( K \) is the field of complex numbers. Assume \( f \) is an endomorphism of \( K[x, y] \) that satisfies \( \text{Jac}(P, Q) \in K^* \). Assume that one of the following conditions is satisfied:

- \( P \) is symmetric.
- \( P \) is skew-symmetric.
- \( Q \) is symmetric.
- \( Q \) is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to \( \alpha \). Then \( f \) is invertible.
**Remark 4.8.** By Lemma 4.6 the Jacobian of two symmetric or two skew-symmetric elements is skew-symmetric, hence it is impossible to have both $P$ and $Q$ symmetric or both $P$ and $Q$ skew-symmetric.

Actually, if $P$ is symmetric and $Q$ is skew-symmetric (or vice-versa), then it is immediate that such $f$ (= an endomorphism of $K[x, y]$ that satisfies $\text{Jac}(P, Q) \in K^*$) is invertible:

Write $s := P$ and $k := Q$ ($s$ is symmetric and $k$ is skew-symmetric). Define $g(x) := s + k$ and $g(y) := s - k$. It is easy to see that $g$ is an $\alpha$-endomorphism of $K[x, y]$ that satisfies

$\text{Jac}(g(x), g(y)) \in K^*$: $g$ is an endomorphism of $K[x, y]$ that satisfies

$\text{Jac}(g(x), g(y)) \in K^*$: Let $h(x) = x + y$ and $h(y) = x - y$. We have

$$(fh)(x) = f(h(x)) = f(x + y) = f(x) + f(y) = P + Q = s + k = g(x),$$

and

$$(fh)(y) = f(h(y)) = f(x - y) = f(x) - f(y) = P - Q = s - k = g(y),$$

so $g = fh$.

$\text{Jac}(g(x), g(y)) = \text{Jac}((fh)(x), (fh)(y)) \in K^*$,

since $\text{Jac}(f(x), f(y)) \in K^*$ and $\text{Jac}(h(x), h(y)) \in K^*$. Another argument:

$\text{Jac}(g(x), g(y)) = \text{Jac}(s + k, s - k) = 2 \text{Jac}(k, s) \in K^*$.

$g$ preserves $\alpha$:

$$(go)(x) = g(\alpha(x)) = g(y) = s - k = \alpha(s) + \alpha(k) = \alpha(s + k) = \alpha(g(x)) = (\alpha g)(x);$$

and

$$(go)(y) = g(\alpha(y)) = g(x) = s + k = \alpha(s) - \alpha(k) = \alpha(s - k) = \alpha(g(y)) = (\alpha g)(y).$$

From [7, Proposition 4.1] $g$ is invertible, hence $f$ is invertible ($f = gh^{-1}$, $g$ and $h^{-1}$ are automorphisms).

**Proof.** Assume that $P$ is symmetric, namely $\alpha(P) = P \in T$. Clearly,

$$\text{Jac}(P, \alpha(Q)) = \text{Jac}(\alpha(P), \alpha(Q)) \in K^*. $$

Write: $\text{Jac}(P, Q) = a$ and $\text{Jac}(P, \alpha(Q)) = b$, where $a, b \in K^*$. Then,

$$\text{Jac}(P, Q/a - \alpha(Q)/b) = \text{Jac}(P, Q/a) - \text{Jac}(P, \alpha(Q)/b) = 0,$$

so from [4, Theorem 1] we have $Q/a - \alpha(Q)/b = H(P)$ where $H(t) \in K[t]$. Hence, $\alpha(Q) = bQ/a - bH(P) \in T$. The restriction theorem 3.6 implies that $f$ is invertible.

Showing that each of the other three conditions implies that $f$ is invertible is similar.

Let $\epsilon$ be an involution on $K[x, y]$ and let $w \in K[x, y]$. $w$ is symmetric with respect to $\epsilon$ if $\epsilon(w) = w$, and $w$ is skew-symmetric with respect to $\epsilon$ if $\epsilon(w) = -w$.

One way to generalize Theorem 4.7 is as follows:

**Theorem 4.9.** Assume $K$ is the field of complex numbers. Assume $f$ is an endomorphism of $K[x, y]$ that satisfies $\text{Jac}(P, Q) \in K^*$. Assume that one of the following conditions is satisfied:

- $P$ is symmetric.
- $P$ is skew-symmetric.
- $Q$ is symmetric.
- $Q$ is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to some involution $\epsilon$ on $K[x, y]$. Then $f$ is invertible.
Proof. Assume that $P$ is symmetric with respect to $\epsilon$, namely $\epsilon(P) = P \in T$. Clearly,
\[ \text{Jac}(P, \epsilon(Q)) = \text{Jac}(\epsilon(P), \epsilon(Q)) \in K^*. \]
Write: Jac($P, Q$) = a and Jac($P, \epsilon(Q)$) = b, where $a, b \in K^*$. Then,
\[ \text{Jac}(P, Q/a - \epsilon(Q)/b) = \text{Jac}(P, Q/a) - \text{Jac}(P, \epsilon(Q)/b) = 0, \]
so from [4, Theorem 1] we have $Q/a - \epsilon(Q)/b = H(P)$, where $H(t) \in K[t]$. So $\epsilon(Q) = bQ/a - bH(P) \in T$. The restriction theorem 3.6 implies that $f$ is invertible.

Showing that each of the other three conditions implies that $f$ is invertible is similar.

The converse of Theorem 4.7 is not true: for example, $g(x) = x$ and $g(y) = y + x^2$ is invertible, but non of $g(x), g(y)$ is symmetric or skew-symmetric with respect to $\alpha$.

It seems natural to ask the following question: Given an automorphism $g$ of $K[x, y]$, is there exists an involution $\epsilon$ on $K[x, y]$ such that $g(x)$ or $g(y)$ is symmetric or skew-symmetric with respect to $\epsilon$?

Thus far we only managed to show that for every generator of the group of automorphisms of $K[x, y]$ the answer to this question is positive:

- Let $g$ be linear with $g(x) = ax + by$, $a, b \in K$. If $a = 0$, then $g(x) = by$ is symmetric with respect to the involution $x \mapsto -x$ and $y \mapsto y$. If $b = 0$, then $g(x) = ax$ is symmetric with respect to the involution $x \mapsto x$ and $y \mapsto -y$. If both $a$ and $b$ are non-zero, then $g(x) = ax + by$ is symmetric with respect to the following involution $\epsilon$ given by $\epsilon(x) = (b/a)y$ and $\epsilon(y) = (a/b)x$. Indeed,
\[ \epsilon(ax + by) = a\epsilon(x) + b\epsilon(y) = a(b/a)y + b(a/b)x = by + ax. \]
(Obviously,
\[ (\epsilon)^2(x) = \epsilon(\epsilon(x)) = \epsilon((b/a)y) = (b/a)\epsilon(y) = (b/a)(a/b)x = x \]
and
\[ (\epsilon)^2(y) = \epsilon(\epsilon(y)) = \epsilon((a/b)x) = (a/b)\epsilon(x) = (a/b)(b/a)y = y. \]

- Let $g$ be triangular. Then it is clear that there exists an involution on $K[x, y]$ such that $g(x)$ or $g(y)$ is symmetric with respect to it (for example, if $g(x) = x$ and $g(y) = y + x^3$, then take $x \mapsto x$ and $y \mapsto -y$.

Remark 4.10. For a given endomorphism $g$ of $K[x, y]$, $g(x)$ can be symmetric (or skew-symmetric) with respect to more that one involution. For example, let $g$ be the automorphism given by $g(x) = ax + by$ and $g(y) = cx + dy$, where $ad - bc \neq 0$ and $a \neq \pm b$.

It is easy to see that
\[ \text{Jac}(g(x), \alpha(g(x))) = a^2 - b^2 \in K^*. \]
So $g$ is a generalized $\alpha$-endomorphism and we have already seen that $g$ is a generalized $\beta$-endomorphism ($\beta(x) = x$ and $\beta(y) = -y$).

Proposition 4.11. Assume $f$ is an endomorphism of $K[x, y]$. The following conditions are equivalent:

1) There exists an involution $\epsilon$ on $K[x, y]$ such that $P$ is symmetric (skew-symmetric) with respect to $\epsilon$.

2) There exists an automorphism $g$ of $K[x, y]$ such that $g(P)$ is symmetric (skew-symmetric) with respect to $\alpha$. 

Proof. (1) ⇒ (2): By Lemma 2.1, \( \epsilon = g^{-1} \alpha g \) for some automorphism \( g \) of \( K[x, y] \).

\( P \) is symmetric with respect to \( \epsilon \): \( \epsilon(P) = P \).

Therefore, \( (g^{-1} \alpha g)(P) = P \), hence \( \alpha(g(P)) = g(P) \), so \( g(P) \) is symmetric with respect to \( \alpha \).

(2) ⇒ (1): \( g(P) \) is symmetric with respect to \( \alpha \): \( \alpha(g(P)) = g(P) \).

Hence, \( (g^{-1} \alpha g)(P) = P \). Then \( P \) is symmetric with respect to the involution \( g^{-1} \alpha g \).

The skew-symmetric version can be proved similarly.

Another way to generalize Theorem 4.7 is as follows; notice that:

- \( K \) is not necessarily the field of complex numbers, but any characteristic zero field.
- There is no assumption on the Jacobian of \( f(x) \) and \( f(y) \); however, we assume that there exist two special elements in the image of \( f \) having a non-zero scalar Jacobian.

**Theorem 4.12.** Assume \( f \) is an endomorphism of \( K[x, y] \). If there exist \( s \) and \( k \) in the image of \( f \), \( T \), such that

- \( s \) is symmetric with respect to \( \alpha \).
- \( k \) is skew-symmetric with respect to \( \alpha \).
- \( \text{Jac}(s, k) \in K^* \),

then \( f \) is invertible.

**Proof.** Define \( g(x) := s + k \) and \( g(y) := s - k \). It is easy to see that \( g \) is an \( \alpha \)-endomorphism of \( K[x, y] \) that satisfies

\[ \text{Jac}(g(x), g(y)) = \text{Jac}(s + k, s - k) = 2 \text{Jac}(k, s) \in K^*. \]

From [7, Proposition 4.1] \( g \) is invertible. Hence,

\[ T \supseteq K[s + k, s - k] = K[g(x), g(y)] = K[x, y], \]

namely \( f \) is surjective.

Recall that a surjective endomorphism of \( K[x, y] \) is an automorphism (see [5, page 343]), so \( f \) is an automorphism.

\[ \square \]

5 Acknowledgements

I wish to thank Prof. C. Valqui for working with me on the starred Dixmier conjecture [7], Prof. J. Bell for his note [3] concerning involutions on \( A_1 \), and Prof. L. Rowen and Prof. U. Vishne for being my advisors.

References

[1] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (New Series) 7 (1982), 287-330.

[2] H. Bass, *Differential structure of etale extensions of polynomial algebras*, Commutative Algebra: Proceedings of a Microprogram (June 15-July 2, 1987, MSRI, Berkeley, CA), Math. Sci. Res. Inst. Publ., 15, Springer-Verlag, New York, (1989), 69-108.

[3] J. Bell, a private note sent by email, 8 Apr 2014.

[4] C. C. -A. Cheng, J. H. McKay and S. S. -S. Wang, *Younger mates and the Jacobian conjecture*, Proc. Amer. Math. Soc. 123 (1995), no. 10, 2939-2947.

[5] P. M. Cohn, *Free rings and their relations*, London Mathematical Society Monograph No. 19, Second edition, 1985.

[6] J. A. Guccione, J. J. Guccione and C. Valqui, *On the centralizers in the Weyl algebra*, Proc. Amer. Math. Soc. 140 (2012), no. 4, 1233-1241.

[7] V. Moskowicz, C. Valqui, *The starred Dixmier conjecture for \( A_1 \)*, arXiv:1401.5141v1 [math.RA] 21 Jan 2014, to appear in Communications in Algebra.
[8] V. Moskowicz, *About Dixmier’s conjecture*, arXiv:1406.4368v3 [math.RA] 9 July 2014.
[9] J.P. Serre, trees, *Springer Monographs in Mathematics*, 1980.
[10] A. Van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, 190. Birkhuser Verlag, Basel, 2000.
[11] S. S. -S. Wang, *Extension of derivations*, J. Algebra 65 (1980), 453-494.
[12] D. Wright, *On the Jacobian conjecture*, Illinois J. Math. 25 (1981), 423-440.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.
E-mail address: vered.moskowicz@gmail.com