Dressed States Approach to Quantum Systems

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Using the non-perturbative method of dressed states previously introduced in [9], we study effects of the environment on a quantum mechanical system, in the case the environment is modeled by an ensemble of non interacting harmonic oscillators. This method allows to separate the whole system into the dressed mechanical system and the dressed environment, in terms of which an exact, non-perturbative approach is possible. When applied to the Brownian motion, we give explicit non-perturbative formulas for the classical path of the particle in the weak and strong coupling regimes. When applied to study atomic behaviours in cavities, the method accounts very precisely for experimentally observed inhibition of atomic decay in small cavities [10,11].

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I. INTRODUCTION

Quantum mechanical systems remain stable in absence of interaction. When interacting with an environment they lose stability as a consequence of the interaction. A material body, for instance an excited atom or molecule, or an excited nucleon, changes of state in reason of its interaction with the environment, the atom-electromagnetic field coupling, in the case of an atom, or the quark-gluon interaction for a nucleon inside a nucleus. The understanding of the nature of the destabilization mechanism is important but in general not an easy task, due to the fact it is in a large extent modeled by the method, in general approximate, used to study the system. A very complete account on the subject, in particular applied to the study of the Brownian motion can be found in Refs. [1,2]. From a general point of view, in modern physics apart from computer calculations in lattice field theory, the only available method to treat the physics of interacting bodies, except for a few special cases, is perturbation theory. The perturbative solution to the problem, is obtained by means of the introduction of bare, non interacting fields, to which are associated bare quanta, the interaction being introduced order by order in powers of the coupling constant in the perturbative expansion for the observables. The perturbative method gives remarkably accurate results in Quantum Electrodynamics and in Weak interactions. In high energy physics, asymptotic freedom allows to apply Quantum Chromodynamics in its perturbative form and very important results have been obtained in this way in the last decades. However, in spite of its wide applicability, there are situations where the use of perturbation theory is not possible, as in the low energy domain of Quantum Chromodynamics where confinement of quarks and gluons takes place, or are of little usefulness, as for instance in Atomic physics, in resonant effects associated to the coupling of atoms with strong radiofrequency fields. These situations have led since a long time ago to attempts to circumvent the limitations of perturbation theory, in particular in situations where strong effective couplings are involved. In some non perturbative approaches in statistical physics and constructive field theory, general theorems can be derived using cluster-like expansions and other related methods [3]. In some cases, these methods allow the rigorous construction of field theoretical models (see for instance [4] and other references therein), but, in spite of the rigor and in some cases the beauty of demonstrations, they are not of great usefulness in calculations of a predictive character.

As a matter of principle, due to the non vanishing of the coupling constant, the idea of a bare particle associated to a bare matter field is actually an artifact of perturbation theory and is physically meaningless. A charged physical particle is always coupled to the gauge field, in other words, it is always “dressed” by a cloud of quanta of the gauge field (photons, in the case of Electrodynamics). In what the Brownian motion is concerned, there are usually two equivalent ways of modeling the environment (the thermal bath) to which the particle is coupled: to represent the thermal bath by a free field, as is done in the classical work of Ref. [1], or to consider the thermal bath as a reservoir composed of a large number of non-interacting harmonic oscillators (see for instance [5,6,7,8]). In both cases, exactly the same type of argument given above in the case of a charged particle applies mutatis mutandis to this system, we may speak of a “dressing” of the Brownian particle by the ensemble of the particles in the thermal bath. The Brownian particle should be always “dressed” by a cloud of quanta of the thermal bath. This should be true

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in general for any system in which a material body is coupled to an environment, no matter the specific nature of environment and interaction involved.

In what follows we use the term "particle" in a general manner, a particle may refer for instance to an atom coupled to a field, or to a Brownian particle coupled to a thermal bath, the two situations where we apply our formalism in this paper.

In recent publications [9–11] a method (dressed coordinates and dressed states) has been introduced that allows a non-perturbative approach to situations of the type described above, provided they can be approximated by a linear coupling. More precisely, the method applies for all systems that can be described by an Hamiltonian of the form,

\[ H = \frac{1}{2} \left( p^2 + \omega_0^2 q^2 + \sum_{k=1}^{N} \left( p_k^2 + \omega_k^2 q_k^2 \right) \right) - q_0 \sum_{k=1}^{N} c_k q_k, \tag{1.1} \]

where the subscript 0 refers to the "material body" and \( k = 1, 2, \ldots, N \) refer to the harmonic environment modes. A Hamiltonian of this type, describing a linear coupling of a particle with an environment, has been used in [2] to study the quantum Brownian motion of the particle with the path-integral formalism. The limit \( N \to \infty \) is understood. In the case of the coupled atom field system, this formalism recovers the experimental observation that excited states of atoms in sufficiently small cavities are stable. It allows to give formulas for the probability of an atom to remain excited for an infinitely long time, provided it is placed in a cavity of appropriate size [1]. For an emission frequency in the visible red, the size of such cavity is in very good agreement with experimental observations [12,13].

We give a non-perturbative treatment to the system introducing some dressed coordinates that allow to divide the coupled system into two parts, the dressed material body and the dressed environment, which makes unnecessary to work directly with the concepts of bare material body, bare environment and interaction between them. In terms of these new coordinates dressed states are defined, which allow a non-perturbative approach. We investigate the behaviour of the system as a function of the strenght of the coupling between the particle and the bath. In particular we give explicitly non-perturbative formulas for the decay probability and for the classical path of the particle in the weak and strong coupling regimes.

II. THE EIGENFREQUENCIES SPECTRUM AND THE DIAGONALIZING MATRIX

We consider for a moment as in [8], the problem of a harmonic oscillator \( q_0 \) coupled to \( N \) other oscillators. In the limit \( N \to \infty \) we recover our original situation of the coupling particle-bath after redefinition of divergent quantities, in a manner analogous as renormalization is done in field theories. The Hamiltonian (1.1) can be turned to principal axis by means of a point transformation, \( q_\mu = t'_\mu Q_r, \quad p_\mu = t'_\mu P_r \), performed by an orthonormal matrix \( T = (t'_\mu) \), \( \mu = (0, k), \quad k = 1, 2, \ldots, N, \quad r = 0, \ldots, N \). The subscript 0 and \( k \) refer respectively to the particle and the harmonic modes of the bath and \( r \) refers to the normal modes. The transformed Hamiltonian in principal axis reads,

\[ H = \frac{1}{2} \sum_{r=0}^{N} (P_r^2 + \Omega_r^2 Q_r^2), \tag{2.1} \]

where the \( \Omega_r \)'s are the normal frequencies corresponding to the possible collective oscillation modes of the coupled system. The matrix elements \( t'_\mu \) are given by [8]

\[ t'_{k} = \frac{c_k}{(\omega_k^2 - \Omega_r^2)} t'^{0}_0, \quad t'^{0}_0 = \left[ 1 + \sum_{k=1}^{N} \frac{c_k^2}{(\omega_k^2 - \Omega_r^2)^2} \right]^{-\frac{1}{2}} \tag{2.2} \]

with the condition,

\[ \omega_0^2 - \Omega_r^2 = \sum_{k=1}^{N} \frac{c_k^2}{\omega_k^2 - \Omega_r^2}. \tag{2.3} \]

We take \( c_k = \eta (\omega_k)^n \). In this case environments are classified according to \( n > 1, n = 1, \) or \( n < 1 \), respectively as supraohmic, ohmic or subohmic. For a subohmic environment the sum in Eq. (2.3) is convergent and the frequency \( \omega_0 \) is well defined. For ohmic and supraohmic environments the sum diverges and a renormalization procedure is needed. In this case, after some subtraction steps it can be seen that Eq. (2.3) can be rewritten in the form,
\[ \omega^2 - \Omega_r^2 = \eta^2 \sum_{k=1}^{N} \frac{E[n]}{\omega_k^2 - \Omega_r^2}. \]  

(2.4)

We take the constant \( \eta \) as \( \eta = \sqrt{\frac{2g}{\Delta \omega}} \), \( \Delta \omega \) being the interval between two neighbouring bath frequencies (supposed uniform) and where \( g \) is some constant [with dimension of \( \text{(frequency)}^2 \)]. In the above equation we have defined the renormalized frequency \( \bar{\omega} \) by,

\[ \bar{\omega}^2 = \omega_0^2 - \delta \omega^2 \quad \delta \omega^2 = \frac{\eta^2}{4} \sum_{k=1}^{N} \sum_{\alpha=1}^{E[n]} \Omega^{2(\alpha-1)} \omega_k^{2(\alpha-\alpha)}, \]  

(2.5)

with \( E[n] \) standing for the smallest integer containing \( n \). From an analysis of Eq. (2.3) it can be seen that if \( \omega_0^2 > \delta \omega^2 \) Eq. (2.3) yields only positive solutions for \( \Omega_r^2 \), while if \( \omega_0^2 < \delta \omega^2 \), Eq. (2.3) has a negative solution \( \Omega_r^2 \). This means that there is a damped collective normal mode that does not allows stationary configurations. We will not consider this last situation. Nevertheless it should be remarked that in a different context, it is precisely this runaway solution that is related to the existence of a bound state in the Lee-Friedrichs model. This solution is considered in Ref. [14] in the framework of a model to describe qualitatively the existence of bound states in particle physics. For reasons that will become apparent later, we restrict ourselves to the physical situations in which the environment frequencies \( \omega_k \) can be written in the form

\[ \omega_k = 2k\pi/L, \quad k = 1, 2, \ldots . \]  

(2.6)

Then using the formula,

\[ \sum_{k=1}^{N} \frac{1}{(k^2 - u^2)} = \left[ \frac{1}{2u^2} - \frac{\pi}{u} \cot(\pi u) \right], \]  

(2.7)

Eq. (2.4) can be written in closed form,

\[ \cot\left(\frac{L\Omega}{2c}\right) = \frac{\Omega^3}{\pi g\Omega E[n]} + \frac{c}{L\Omega} \left(1 - \frac{\bar{\omega}^2 L \Omega^2}{\pi gc \Omega^2 E[n]} \right). \]  

(2.8)

For an ohmic environment we have \( c_k = \eta \omega_k \) and \( \delta \omega^2 = N\eta^2 \). Taking in Eq. (2.5) \( \omega_0^2 > N\eta^2 \), the renormalized oscillator frequency \( \bar{\omega} \) is given by,

\[ \bar{\omega}_{\text{ohmic}} = \sqrt{\omega_0^2 - N\eta^2}, \]  

(2.9)

and the eigenfrequencies spectrum for an \textit{ohmic} environment is given by the equation,

\[ \cot\left(\frac{L\Omega}{2c}\right) = \frac{\Omega}{\pi g} + \frac{c}{L\Omega} \left(1 - \frac{\bar{\omega}^2 L}{\pi gc}\right), \]  

(2.10)

The solutions of Eq. (2.10) or Eq. (2.8) with respect to \( \Omega \) give the spectrum of eigenfrequencies \( \Omega_r \) corresponding to the collective normal modes.

The transformation matrix elements turning the material body-bath system to principal axis is obtained, after some rather long but straightforward manipulations analogous as it has been done in [9]. They read,

\[ t_{r0}^0 = \frac{\eta \Omega_r}{\sqrt{(\Omega_r^2 - \bar{\omega}^2)^2 + \frac{\eta^2}{2}(3\Omega_r^2 - \bar{\omega}^2) + \pi^2 g^2 \Omega_r^2}}, \quad t_{r0}^k = \frac{\eta \omega_k}{\omega_k^2 - \Omega_r^2} t_{r0}^0. \]  

(2.11)

III. THE DRESSED PARTICLE IN AN OHMIC ENVIRONMENT

To fix our framework and to give precise applications of our formalism, we study in this paper an \textit{ohmic} environment. The normalized eigenstates of our system (eigenstates of the Hamiltonian in principal axis) can be written in terms of normal coordinates,
\[ \langle Q | n_0, n_1, \ldots \rangle \equiv \phi_{n_0n_1n_2\ldots}(Q, t) = \prod_s \left[ \frac{2^{n_s}}{\sqrt{n_s!}} H_{n_s} \left( \sqrt{\frac{\Omega_s}{\hbar}} Q_s \right) \right] \Gamma_0(Q) e^{-\frac{1}{2} \sum_s n_s \Omega_s t}, \]

where \( H_{n_s} \) stands for the \( n_s \)-th Hermite polynomial and \( \Gamma_0 \) is the normalized vacuum eigenfunction. Next we intend to divide the system into the \textit{dressed} particle and the \textit{dressed} environment by means of some conveniently chosen \textit{dressed} coordinates, \( q_0' \) and \( q_j' \) associated respectively to the \textit{dressed} particle and to the \textit{dressed} oscillators composing the environment. These coordinates will allow a natural division of the system into the \textit{dressed} (physically observed) particle and into the \textit{dressed} environment. The dressed particle will contain automatically all the effects of the environment. These coordinates will allow a natural division of the system into the \textit{dressed} states, whose wavefunctions are,

\[ \psi_{\kappa_0\kappa_1\ldots}(q') = \prod_{\mu} \left[ (2^{-\kappa_\mu} \kappa_\mu !)^{-\frac{1}{2}} H_{\kappa_\mu} \left( \sqrt{\frac{\bar{\omega}_\mu}{\hbar}} q_\mu' \right) \right] \Gamma_0(q') , \]

where \( q_\mu' = q_0', q_i', \bar{\omega}_\mu = (\bar{\omega}, \omega_i) \) and \( \Gamma_0 \) is the invariant ground state eigenfunction introduced in Eq. (3.1). Note that the above wavefunctions will evolve in time in a more complicated form than the unitary evolution of the eigenstates (3.1), since these wavefunctions are not eigenstates of the diagonal Hamiltonian (1.1). It is precisely the non unitary evolution of these wavefunctions that will allow (see below) a non-perturbative study of the radiation and dissipation processes of the particle.

In order to satisfy the physical condition of vacuum stability (invariance under a transformation from normal to dressed coordinates) we remember that the the ground state eigenfunction of the system has the form,

\[ \Gamma_0(Q) \propto e^{-\frac{1}{2} \sum_{i=0}^N \Omega_i Q_i^2} , \]

and we require that the ground state in terms of the \textit{dressed} coordinates should have the form

\[ \Gamma_0(q') \propto e^{-\frac{1}{2} \sum_{\mu=0}^N \bar{\omega}_\mu (q'_\mu)^2} . \]

From Eqs. (3.3) and (3.4) it can be seen that the vacuum invariance requirement is satisfied if we define \textit{dressed} coordinates by,

\[ \sqrt{\bar{\omega}_\mu q'_\mu} = \sum_{r=0}^N \sigma_{\mu r} \sqrt{\Omega_r} Q_r . \]

As we have already mentioned above our \textit{dressed} states, given by Eq. (3.2), are \textit{collective} but \textit{non stable} states, linear combinations of the (stable) eigensates (3.1) defined in terms of the normal modes. The coefficients of these combinations are given in Eq. (4.3) below and explicit formulas for these coefficients for an interesting physical situation are given in Eq. (4.11). This gives a complete and rigorous definition of our dressed states. Moreover, our dressed states have the very interesting property of distributing the energy initially in a particular dressed state, among itself and all other dressed states with precise and well defined probability amplitudes (4). We \textit{choose} these dressed states as physically meaningful and we test successfully this hypothesis by studying the radiation process by an atom in a cavity. In both cases, of a very large or a very small cavity, our results are in agreement with experimental observations.

Having introduced \textit{dressed} coordinates and \textit{dressed} states, in the next sections we will apply these concepts to study the time evolution of the expectation value of the particle coordinate.

**IV. BROWNIAN MOTION AT ZERO TEMPERATURE**

As a first application of our formalism we consider the study of the Brownian Motion. The Brownian particle is modeled by a harmonic oscillator coupled to an \textit{ohmic} environment. The whole system being described by the Hamiltonian (1.1). This model for the Brownian motion is in fact not new and has been implemented using the path integral formalism in for instance Refs. (3,2). In this approach an effective action for the Brownian particle is obtained, which in general is very complicated and non local in time. From this effective action, an equation for the classical path of the Brownian particle can be derived. However this equation obtained from the effective action, is a
very complicated integro-differential equation that can not be solved analytically. But in general terms it describes
the expected damped behaviour of the particle.

We will approach this problem using the dressed states introduced in the previous section, and we will treat in detail
the case in which the environment is at zero temperature (what corresponds to consider the environment initially in
its ground state). Our method will account for the expected behaviour in a much more simpler way than the usual
path integral approach.

We assume as usual, that initially the Brownian particle and the environment are decoupled and that the coupling
is turned on suddenly at some given time, that we choose at \( t = 0 \). Since we treat here the case in which the
environment is at zero temperature our assumption is that the initial system can be described by a pure dressed
state. The environment state at zero temperature should be described by its dressed ground state. Thus we can write
the initial state of the system particle-environment in the form,

\[
| \lambda, n'_1, n'_2, \ldots; t = 0 \rangle = | \lambda \rangle \times | n'_1, n'_2, \ldots \rangle . \tag{4.1}
\]

In the above equation \( | \lambda \rangle \) is the initial dressed state of the particle and \( | n'_1, n'_2, \ldots \rangle \) is the initial dressed state of the
environment (after we will take \( n'_1 = n'_2 = \ldots = 0 \), corresponding to the environment at zero temperature). To proceed
we recall that the classical path in the case of the quantum harmonic oscillator is given by the mean value of the
operator position in a coherent state. In our formalism, we define \( | \lambda \rangle \) as a dressed coherent state given by,

\[
| \lambda \rangle = e^{-|\lambda|^2/2} \sum_{n'_0=0}^{\infty} \frac{\langle \lambda | n'_0 \rangle}{\sqrt{n'_0!}} | n'_0 \rangle , \tag{4.2}
\]

and accordingly the classical path of the Brownian particle should be given by the time evolution of the dressed
particle position operator in the dressed coherent state \(| \lambda \rangle \). It is useful to examine firstly the time evolution of
the initial coherent dressed state as given by Eq. (4.1). Replacing Eq. (4.2) in Eq. (4.1) we obtain,

\[
| \lambda, n'_1, n'_2, \ldots; t = 0 \rangle = e^{-|\lambda|^2/2} \sum_{n'_0=0}^{\infty} \frac{\lambda^{n'_0}}{\sqrt{n'_0!}} | n'_0 n'_1, \ldots \rangle . \tag{4.3}
\]

Now, since the eigenstates \(| n_0, n_1, \ldots \rangle \) form a complete basis [stable states having eigenfunctions given by Eq. (3.3)],
we can write Eq. (4.3) as

\[
| \lambda, n'_1, n'_2, \ldots; t = 0 \rangle = e^{-|\lambda|^2/2} \sum_{n'_0=0}^{\infty} \sum_{n_r=0}^{\infty} \frac{\lambda^{n'_0}}{\sqrt{n'_0! n_r!}} T_{n'_0,n'_1,\ldots}^{n_0,n_1,\ldots} | n_0, n_1, \ldots \rangle , \tag{4.4}
\]

where \( \{ n_r \} = \{ n_0, n_1, n_2, \ldots \} \) and

\[
T_{n'_0,n'_1,\ldots}^{n_0,n_1,\ldots} = \langle n_0, n_1, \ldots | n'_0, n'_1, \ldots \rangle = \int dQ \phi_{n_0,n_1,\ldots}(Q) \psi_{n'_0,n'_1,\ldots}(Q) . \tag{4.5}
\]

Since \(| n_0, n_1, \ldots \rangle \) are eigenvectors of the Hamiltonian \(|\lambda\rangle\), the time evolution of Eq. (4.4) is given by

\[
| \lambda, n'_1, n'_2, \ldots; t \rangle = e^{-|\lambda|^2/2} \sum_{n'_0=0}^{\infty} \sum_{n_r=0}^{\infty} \frac{\lambda^{n'_0}}{\sqrt{n'_0! n_r!}} T_{n'_0,n'_1,\ldots}^{n_0,n_1,\ldots} e^{-\iota t \sum_r (n_r + 1/2) \Omega_r} | n_0, n_1, \ldots \rangle . \tag{4.6}
\]

Now we can compute \( q_\lambda(t) \), the time dependent mean value for the dressed oscillator position operator, i.e., the
mean value of the dressed particle position operator taken in the dressed coherent state \(| \lambda \rangle \).

\[
q_\lambda(t) = \langle \lambda, n'_1, n'_2, \ldots; t | q_0 | \lambda, n'_1, n'_2, \ldots; t \rangle = e^{-|\lambda|^2} \sum_{n'_0, m_0} \sum_{n_r, m_r} \frac{(\lambda)^{n'_0}}{\sqrt{n'_0! m_0!}} T_{n'_0,n'_1,\ldots}^{n_0,m_0,\ldots} T_{n'_0,n'_1,\ldots}^{n_0,m_0,\ldots} \Omega_r (n_r + m_r) e^{-\iota t \sum_r (n_r + m_r) \Omega_r} (m_0, m_1, \ldots | q_0 | n_0, n_1, \ldots) . \tag{4.7}
\]

Using Eq. (3.3) for \( \mu = 0 \) and

\[
\langle m_\alpha | Q_\alpha | n_\alpha \rangle = \sqrt{\frac{h}{2\Omega_\alpha}} \left( \sqrt{n_\alpha} \delta_{m_\alpha,n_\alpha - 1} + \sqrt{n_\alpha + 1} \delta_{m_\alpha,n_\alpha + 1} \right) . \tag{4.8}
\]
As we have mentioned above, the situation in which the environment is at zero temperature corresponds to cases in which the function $f$ (in Eq. (4.7)) it is easy to obtain, 
\[ q'_{\lambda}(t) = e^{-|\lambda|^2} \sqrt{\frac{\hbar}{2\omega}} \sum_n \sum_{n_0, n_0', r} \sum t^n s \sqrt{n_0 + 1} T^{n_0, n_1, ..., n_2, ..., n_r, n_1, ..., n_r}_{0, n_0, n_0', r} \times \left[ (\lambda^*)^m_0 \lambda_{0}^n \sqrt{n_0!} \right. \left. \sqrt{m_0!} e^{-\Omega t} + (\lambda^*)^m_0 \lambda_{0}^n \sqrt{n_0!} \sqrt{m_0!} e^{\Omega t} \right]. \] \tag{4.9}

As we have mentioned above, the situation in which the environment is at zero temperature corresponds to $n'_1 = n'_2 = ... = 0$. In this case from Eqs. (4.5), (3.2), (3.5), (3.1) and with the help of the theorem [15],
\[ \frac{1}{n_0^1!} \left[ \sum_r (t^r_{\mu})^2 \right]^2 H_{n_0} \left( \frac{t^r_{\mu} \sqrt{\omega \hbar Q_r}}{\sqrt{\sum_r (t^r_{\mu})^2}} \right) = \sum_{m_0 + m_1 + \ldots = n_0} (t^0_{\mu})^m_0 (t^1_{\mu})^m_1 \ldots H_{m_0} \left( \sqrt{\frac{\hbar Q_0}{\Omega}} \right) H_{m_1} \left( \sqrt{\frac{\hbar Q_1}{\Omega}} \right) \ldots, \] \tag{4.10}

we get,
\[ T^{n_0, n_1, n_2, ..., n_r}_{0, n_0, n_0', r} = \sqrt{\frac{n_1^{n'_1} n_1^{n'_1} \ldots}{n_0! n_1! \ldots}} (t^0_{\mu})^n_0 (t^1_{\mu})^n_1 \ldots \delta_{m_0', n_0 + 1, n_1 + n_2, ...}. \] \tag{4.11}

Replacing Eq. (4.11) in Eq. (4.9) we obtain after some straightforward calculations,
\[ q'_{\lambda}(t) = \sqrt{\frac{\hbar}{2\omega}} \left[ \lambda f^{00}(t) + \lambda^* f^{00*}(t) \right], \] \tag{4.12}

where
\[ f^{00}(t) = \sum_s (t^s_0)^2 e^{-\Omega_0 t}. \] \tag{4.13}

From Refs. [3,13] we recognize the function $f^{00}(t)$ as the probability amplitude that at time $t$ the dressed particle still be excited, if it was initially (at $t = 0$) in the first excited level. We see that underlying to our dressed states formalism there is an unified way to study two physically different situations, the radiation process and the Brownian motion. In next section we shall investigate the radiation process of the dressed particle.

Returning to the study of the Brownian particle, we see that to obtain an expression for the classical path we have to perform the sum appearing in Eq. (4.13) and replace the result in Eq. (4.12). If we assume, as it is currently done in studies of the Brownian motion, that the environment distributes itself over the whole free space, its frequencies $\omega_k$ should have a continuous distribution. This continuum can be realized simply taking the limit $L \to \infty$ in Eq. (2.6). In this case the matrix elements $t^r_0$, given by Eq. (2.11) become,
\[ t^r_0 = \lim_{\Delta \Omega \to \infty} \frac{\sqrt{2g^2\Omega \Delta \Omega}}{\sqrt{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}} \] \tag{4.14}

and the function $f^{00}(t)$ in Eq. (4.13) can be written in the form,
\[ f^{00}(t) = \int_0^\infty \frac{2g^2 \Omega e^{-i\Omega t} d\Omega}{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}. \] \tag{4.15}

Before going ahead let us define a "driving parameter" $\kappa$ by,
\[ \kappa = \sqrt{\omega^2 - \frac{\pi^2 g^2}{4}} \] \tag{4.16}

and let us study the above integral $f^{00}(t)$ in the different cases, a) $\kappa^2 > 0$, b) $\kappa^2 = 0$ and c) $\kappa^2 < 0$. The extreme cases in a) and c), $\kappa^2 \gg 0$ or $\kappa^2 \ll 0$ correspond respectively to the situations of a weak coupling between the particle and the environment ($g \ll \omega$) or of a strong coupling ($g \gg \omega$). We get for the above situations, a) $\kappa^2 > 0$.
In the above equations, we have written
\[
\lambda
\]
b) \( \kappa^2 = 0 \)
\[
f^{00}(t) = \left(1 - \frac{i\pi g}{2\kappa}\right) e^{-i\omega t - \pi g t/2} + 2iJ(t), \quad (4.17)
\]
and
\[
f^{00}(t) = \left(1 - \frac{\pi g t}{2}\right) e^{-\pi g t/2} + 2iJ(t), \quad (4.18)
\]
c) \( \kappa^2 < 0 \)
\[
f^{00}(t) = \frac{1}{2} \left[ \left(1 + \frac{\pi g}{2\omega}\right) e^{-\pi g t/2} + 2iJ(t) \right] \quad (4.19)
\]
where,
\[
J(t) = 2ig \int_0^\infty dy \frac{y^2 e^{-yt}}{(y^2 + \omega^2)^2 - \pi^2 g^2 y^2}. \quad (4.20)
\]
Replacing the above equations in Eq. (4.12) we obtain for the classical path at zero temperature the following expressions,
\[
q^\prime_\lambda(t) = \sqrt{\frac{\hbar}{2\omega}} \left\{ 2 \cos(\kappa t + \delta) - \frac{\pi g}{\kappa} \sin(\kappa t + \delta) \right\} e^{-\pi g t/2} + 2 \sin \delta J(t) \quad (\kappa > 0), \quad (4.21)
\]
\[
q^\prime_\lambda(t) = \sqrt{\frac{\hbar}{2\omega}} \left\{ 2 \cos \delta \left(1 - \frac{\pi g}{2\kappa} t\right) e^{-\pi g t/2} + 2(\sin \delta) J(t) \right\} \quad (\kappa = 0), \quad (4.22)
\]
\[
q^\prime_\lambda(t) = \sqrt{\frac{\hbar}{2\omega}} \left(2 \cos \delta \left(\cosh |\kappa| t - \frac{\pi g}{2|\kappa|} \sinh |\kappa| t\right) e^{-\pi g t/2} + 2(\sin \delta) J(t) \right\} \quad (\kappa < 0). \quad (4.23)
\]
In above equations we have written \( \lambda = \sqrt{\bar{n}}e^{-i\delta} \), with \( \bar{n} \) being the mean value for the number operator in the coherent state. Eqs. (4.17) to (4.23) give the expected behaviour for the classical path of the Brownian particle. Apart from a parcel containing the integral \( J(t) \), these equations describe the behaviour of a damped oscillator in the three regimes corresponding to \( \kappa > 0, \kappa = 0 \) and \( \kappa < 0 \), with a damping coefficient equal to \( \pi g \). The above formulas describe the exact behaviours for \( \delta = 0 \), which corresponds to a real value of the coherence parameter \( \lambda \). The integral \( J(t) \) in equations (4.17, 4.19) can be evaluated for large times, \( t >> 1/\bar{\omega} \). We obtain,
\[
J(t) \approx \frac{4g}{\bar{\omega}^3 t^3}; \quad (t \gg \frac{1}{\bar{\omega}}). \quad (4.24)
\]
Using Eq. (4.24) in Eq. (4.12) and remarking that for very large times the power behaviour \( \sim t^{-3} \) dominates over the exponential decay, we obtain identical asymptotic behaviours in the three regimes above,
\[
q^\prime_\lambda(t) \approx \sqrt{\frac{\hbar}{2\omega}} \frac{8g}{\bar{\omega}^3 t^3} \sin \delta \quad (\kappa > 0, \kappa = 0, \kappa < 0; \quad t \gg \frac{1}{\bar{\omega}}). \quad (4.25)
\]
The path behaviour in the different coupling regimes can be obtained from Eqs. (4.21) to (4.23) and Eq. (4.16). In the strong coupling regime, \( \kappa^2 < 0 \) \( (g \gg \bar{\omega}) \), we obtain
\[
q^\prime_\lambda(t) \approx \sqrt{\frac{\hbar}{2\omega}} \cos \delta \left(\frac{2\omega^2}{\pi g}\right) e^{-\pi g t/2} + 2(\sin \delta) J(t), \quad (4.26)
\]
and in the weak coupling regime, \( \kappa^2 > 0 \) \( (g \ll \bar{\omega}) \) we obtain from Eqs. (4.16) and (4.21),
\[
q^\prime_\lambda(t) \approx \sqrt{\frac{\hbar}{2\omega}} \left[ \left(2 \cos \omega t + \delta - \frac{\pi g}{2\omega} \sin(\omega t + \delta)\right) e^{-\pi g t/2} + 2(\sin \delta) J(t) \right]. \quad (4.27)
\]
We see that the behaviours are quite different in the two situations, for not very large values of the time \( t \), for which the exponential decay dominates over the power law decay of \( J(t) \): an oscillatory damped behaviour with time in the weak coupling regime, while in the strong coupling regime the expected dressed coordinate value has an exponential decay. Again, asymptotically \( J(t) \) dominates and both behaviours are identical obeying a power law decay \( \sim t^{-3} \).

In the next section we apply our formalism to the study of the radiation process.
V. THE RADIATION PROCESS

In this section we study the radiation process of the dressed particle when it is prepared in such a way that initially it is in its first excited state. We shall consider two situations, the particle in free space and the particle confined in a cavity of diameter $L$.

A. The particle in free space

In this case the spectrum of the frequencies $\omega_k$ has a continuous distribution as we have seen in the last section, and the function $f^{00}(t)$ is given by Eqs. (4.12)-(4.15). Combining these equations with Eq. (4.24) we obtain for the probability that the dressed particle still remain in its first excited state at a time $t \gg 1/\bar{\omega}$, the following expressions,

$$|f^{00}(t)|^2 = \left(1 + \frac{\pi^2 g^2}{4\kappa^2}\right) e^{-\pi g t} - \frac{8g}{\omega^4 t^4} \left(\sin \kappa t + \frac{\pi g}{2\kappa} \cos \kappa t\right) + \frac{16g^2}{\omega^8 t^8} \quad (\kappa > 0),$$

$$|f^{00}(t)|^2 = \left(1 - \frac{\pi g}{2} t\right)^2 e^{-\pi g t} + \frac{16g^2}{\omega^8 t^8} \quad (\kappa = 0),$$

and

$$|f^{00}(t)|^2 = \left(\cosh |\kappa| t - \frac{\pi g}{2|\kappa|} \sinh |\kappa| t\right) \left(\cosh |\kappa| t + \frac{\pi g}{2|\kappa|} \sinh |\kappa| t\right) e^{-\pi g t} + \frac{16g^2}{\omega^8 t^8} \quad (\kappa < 0).$$

In the weak coupling regime $\kappa \gg 0$, we obtain from Eq. (5.3) that the probability that the particle be still excited at time $t \gg 1/\bar{\omega}$ if it was in the first excited level at $t = 0$, obeys the well known exponential decay law,

$$|f^{00}(t)|^2 \approx e^{-\pi g t}. \quad (5.4)$$

In the strong coupling regime, $\kappa \ll 0$, we obtain from Eq. (5.3),

$$|f^{00}(t)|^2 \approx \left(\frac{\bar{\omega}}{\pi^2 g^2}\right) e^{-\frac{2\bar{\omega}^2 t}{\pi^2 g^4}}. \quad (5.5)$$

We can see that the decay of the particle is considerably enhanced for a strong coupling as compared to the weak coupling. We emphasize that the different behaviours described in Eqs. (5.4) and (5.5) are due to the fact that in the two situations the system obey different decay laws and that this fact can not be inferred from perturbation theory. It is a consequence of the dressed states approach.

B. Behaviour of the confined system

Let us now consider the ohmic system in which the particle is placed in the center of a cavity of diameter $L$, in the case of a very small $L$, i.e., that satisfies the condition of being much smaller than the coherence length, $L \ll 2c/g$. We note that from a physical point of view, $L$ stands for either the diameter of a spherical cavity or the spacing between infinite parallel mirrors. To fix our framework we consider a spherical cavity. To obtain the eigenfrequencies spectrum, we remark that from a graphical analysis of Eq. (2.10) it can be seen that in the case of a small values of $L$, its solutions are very near the frequency values corresponding to the asymptotes of the curve cot(2$k$), which correspond to the environment modes $\omega_i = i2\pi c/L$, except from the first eigenfrequency $\Omega_0$. The only exception is the smallest solution $\Omega_0$. As we take larger and larger solutions, they are nearer and nearer to the values corresponding to the asymptotes. For instance, for a value of $L$ of the order of $2 \times 10^{-2} m$ and $\bar{\omega} \sim 10^{10}/s$, only the lowest eigenfrequency $\Omega_0$ is significantly different from the field frequency corresponding to the first asymptot, all the other eigenfrequencies $\Omega_k$, $k = 1, 2, ...$ being very close to the field modes $k2\pi c/L$. For higher values of $\bar{\omega}$ (and lower values of $L$) the differences between the eigenfrequencies and the field modes frequencies are still smaller. Thus to solve Eq. (2.10) for the larger eigenfrequencies we expand the function cot(2$k$) around the values corresponding to the asymptotes. We write,

$$\Omega_k = \frac{2\pi c}{L}(k + c_k), \quad k = 1, 2, .. \quad (5.6)$$
with $0 < \epsilon_k < 1$, satisfying the equation,

$$\cot(\pi \epsilon_k) = \frac{2c}{gL}(k + \epsilon_k) + \frac{1}{(k + \epsilon_k)}(1 - \bar{\omega}^2L/2\pi gc^2).$$

(5.7)

But since for a small value of $L$ every $\epsilon_k$ is much smaller than 1, Eq. (5.7) can be linearized in $\epsilon_k$, giving,

$$\epsilon_k = \frac{4\pi gcLk}{2(4\pi^2c^2k^2 - \bar{\omega}^2L^2)}.$$  

(5.8)

Eqs. (5.6) and (5.8) give approximate solutions to the eigenfrequencies $\Omega_k, \ k = 1, 2,...$

To solve Eq. (2.11) with respect to the lowest eigenfrequency $\Omega_0$, let us assume that it satisfies the condition $\Omega_0/2c << 1$ (we will see below that this condition is compatible with the condition of a small $L$ as defined above). Inserting the condition $\Omega_0/2c << 1$ in Eq. (2.10) and keeping up to quadratic terms in $\Omega$ we obtain the solution for the lowest eigenfrequency $\Omega_0$,

$$\Omega_0 = \frac{\bar{\omega}}{\sqrt{1 + \frac{2qL}{2c}}}.$$  

(5.9)

Consistency between Eq. (5.9) and the condition $\Omega_0/2c << 1$ gives a condition on $L$,

$$L \ll \frac{2c}{g} f; \quad f = \frac{\pi}{2} \left( \frac{g}{\bar{\omega}} \right)^2 \left( 1 + \sqrt{1 + \frac{4}{\pi^2} \left( \frac{\bar{\omega}}{g} \right)^2} \right).$$

(5.10)

Let us consider as in the preceding section, the situations of weak coupling, $\kappa^2 \gg 0 \ (g \ll \bar{\omega})$ and of strong coupling, $\kappa^2 \ll 0 \ (g \gg \bar{\omega})$. We define a parameter $\beta$, by $g = \beta \bar{\omega}$ and weak or strong coupling are defined respectively for $\beta \ll 1$ or $\beta \gg 1$. For weak or strong couplings we obtain from Eq. (5.11),

$$f_{weak} \approx \beta; \quad f_{strong} \approx \frac{\pi}{2} \beta^2.$$  

(5.11)

Let us consider the situation where the dressed material body is initially in its first excited level. Then from Eq. (4.13) we obtain the probability that it will still be excited after an elapsed time $t$,

$$|f^{00}(t)|^2 = (t_0^0)^4 + 2\sum_{k=1}^{\infty} (t_0^0)^2 (t_k^0)^2 \cos(\Omega_k - \Omega_0)t +$$

$$\sum_{k,l=1}^{\infty} (t_k^0)^2 (t_l^0)^2 \cos(\Omega_k - \Omega_l)t.$$  

(5.12)

a) Weak coupling

In the case of weak coupling a physically interesting situation is when interactions of electromagnetic type are involved. In this case, we take $\beta = \alpha$, where $\alpha$ is the fine structure constant, $\alpha = 1/137$. Then the factor $f$ multiplying $2c/g$ in Eq. (5.10) is $\sim 0.07$ and the condition $L \ll 2c/g$ is replaced by a more restrictive one, $L \ll 0.07(2c/g)$. For a typical infrared frequency, for instance $\bar{\omega} \sim 2.0 \times 10^{11}/s$, our calculations are valid for a value of $L, L \ll 10^{-3} m$.

From Eqs. (2.11) and using the above expressions for the eigenfrequencies for small $L$, we obtain the matrix elements,

$$\langle 0^0 | 0^0 \rangle \approx 1 - \frac{\pi qL}{2c}; \quad \langle 0^k | 0^k \rangle \approx \frac{gL}{\pi \epsilon k^2}.$$  

(5.13)

To obtain the above equations we have neglected the corrective term $\epsilon_k$, from the expressions for the eigenfrequencies $\Omega_k$. Nevertheless, corrections in $\epsilon_k$ should be included in the expressions for the matrix elements $t_k^0$, in order to avoid spurious singularities due to our approximation. Using Eqs. (5.13) in Eq. (5.12), we obtain

$$|f^{00}(t)|^2 \approx 1 - \pi \delta + 4(\delta^2 - \pi^2) \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(\Omega_k - \Omega_0)t +$$

$$\pi^2 \delta^2 + \frac{4}{\pi^2} \delta^2 \sum_{k,l=1}^{\infty} \frac{1}{k^2 l^2} \cos(\Omega_k - \Omega_l)t.$$  

(5.14)
where we have introduced the dimensionless parameter $\delta = L g / 2 c \ll 1$, corresponding to a small value of $L$ and we remember that the eigenfrequencies are given by Eqs. (5.6) and (5.8). As time goes on, the probability that the mechanical oscillator be excited attains periodically a minimum value which has a lower bound given by,

$$\text{Min}(|f^{00}(t)|^2) = 1 - \frac{5\pi}{3}\delta + \frac{14\pi^2}{9}\delta^2.$$  

(5.15)

For a frequency $\bar{\omega}$ of the order $\bar{\omega} \sim 4.00 \times 10^{14}/s$ (in the red visible), which corresponds to $\delta \sim 0.005$ and $L \sim 1.0 \times 10^{-6}m$, we see from Eq. (5.13) that the probability that the material body be at any time excited will never fall below a value $\sim 0.97$, or a decay probability that is never higher that a value $\sim 0.03$. It is interesting to compare this result with experimental observations in \textit{[42]}. Where stability is found for atoms emitting in the visible range placed between two parallel mirrors a distance $L = 1.1 \times 10^{-6}m$ apart from one another. For lower frequencies the value of the spacing $L$ ensuring quasi-stability of the same order as above, for the excited particle may be considerably larger. For instance, for $\bar{\omega}$ in a typical microwave value, $\bar{\omega} \sim 2.00 \times 10^{10}/s$ and taking also $\delta \sim 0.005$, the probability that the material body remain in the first excited level at any time would be larger than a value of the order of 97%, for a value of $L$, $L \sim 2.0 \times 10^{-2}m$. The probability that the material body remain excited as time goes on, oscillates with time between a maximum and a minimum values and never departs significantly from the situation of stability in the excited state.

\textit{b) Strong coupling}

In this case we see from Eq. (5.9) that $\Omega_0 \approx \bar{\omega}$ for $\beta \gg 2c / \pi L \bar{\omega}$. For $\bar{\omega} \sim 4.00 \times 10^{14}/s$ (in the red visible) and $L \sim 1.0 \times 10^{-6}m$ this means $\beta \gg 1$. We obtain from Eq. (2.41),

$$\langle t_0^0 \rangle^2 \approx \frac{1}{1 + \pi \delta / 2}; \quad \langle t^0_1 \rangle^2 \approx \frac{gL}{\pi c k^2}.$$  

(5.16)

Using Eqs. (5.10) in Eq. (5.12), we obtain

$$|f^{00}(t)|^2 \approx \left( \frac{2}{2 + \pi \delta} \right)^2 + \frac{2}{2 + \pi \delta} \sum_{k=1}^{\infty} \frac{2\delta}{\pi k^2} \cos(\Omega_k - \Omega_0) t +$$

$$+ \frac{4}{\pi^2 \delta^2} \sum_{k,l=1}^{\infty} \frac{1}{k^2 l^2} \cos(\Omega_k - \Omega_l) t,$$  

(5.17)

The function (5.17) As time goes on, the probability that the mechanical system be excited attains periodically a minimum value which has a lower bound given by,

$$\text{Min}(|f^{00}(t)|^2) = \left( \frac{2}{2 + \pi \delta} \right)^2 - \left( \frac{2}{2 + \pi \delta} \right) \frac{\pi \delta}{3} - \frac{\pi^2 \delta^2}{9}.$$  

(5.18)

The condition of positivity of (5.18) imposes for fixed values of $\beta$ and $\bar{\omega}$ an upper bound for the quantity $\delta$, $\delta_{\text{max}}$, which corresponds to an upper bound to the diameter $L$ of the cavity, $L_{\text{max}}$ (remember $\delta = L g / 2 c$). Values of $\delta$ larger than $\delta_{\text{max}}$, or equivalently, values of $L$ larger than $L_{\text{max}}$ are unphysical and should not be considered. These upper bounds are obtained from the solution of the inequality Min$\{|f^{00}(t)|^2\} \geq 0$. We have Min$\{|f^{00}(t)|^2\} > 0$ or Min$\{|f^{00}(t)|^2\} = 0$, for respectively $\delta < \delta_{\text{max}}$ or $\delta = \delta_{\text{max}}$. For a frequency $\bar{\omega}$ of the order $\bar{\omega} \sim 4.00 \times 10^{14}/s$ (in the red visible), with $\beta = 10$ ($g = 10 \bar{\omega}$) the lower bound (5.18) above attains zero for a cavity of size $L \sim 1.1 \times 10^{-7}m$. For a typical microwave frequency $\bar{\omega} \sim 2.00 \times 10^{10}/s$, the same vanishing lower bound is attained for a cavity of size $L \sim 1.2 \times 10^{-3}m$. We see that the behaviour of the system for strong coupling is rather different from the weak coupling regime. For appropriate cavity sizes, which are of order $10^{-1}$ of those ensuring stability in the weak regime, we ensure for strong coupling the complete decay of the system to the ground state in a small elapsed time. In other words, strong coupling enhances the system decay in small cavities, contrarily to the inhibition that happens in the weak coupling regime.

VI. CONCLUDING REMARKS

We have presented in this paper a non-perturbative treatment of a quantum system consisting of particle (in the larger sense of a "material body", an atom or a Brownian particle) coupled to an environment modeled by non-interacting oscillators. We have used dressed states which allow to divide the system into the dressed particle and the
dressed environment by means of some conveniently chosen dressed coordinates, \( q'_0 \) and \( q'_j \) associated respectively to the dressed particle and to the dressed oscillators composing the environment. In terms of these coordinates a division of the system into the dressed (physically observed) particle and the dressed environment arises naturally. The dressed particle will contain automatically all the effects of the environment on it. This formalism allows a non-perturbative approach to the time evolution of a system that may be approximated by a particle coupled to its environment, in rather different situations as confinement of atoms in cavities or the Brownian motion. In other words, underlying our dressed states formalism there is an unified way to study two physically different situations, the radiation process and the Brownian motion. We have approached these situations using the dressed states, and in both we have obtained results in good agreement with experimental observations or with expected behaviours. In the Brownian motion we have treated in detail the case in which the environment is at zero temperature (what corresponds to consider the environment initially in its ground state). Our method accounts for the expected damped behaviour of the particle in a much more simpler way than the usual path integral approach. For atomic systems we recover with our formalism the experimental observation that excited states of atoms in sufficiently small cavities are stable. We are able to give formulas for the probability of an atom to remain excited for an infinitely long time, provided it is placed in a cavity of appropriate size. For an emission frequency in the visible red, the size of such cavity is in good agreement with experimental observations (\[10\], \[11\]). The generalization of the work presented in this paper to the case of a generic (supraohmic or subohmic) environment and finite temperature is in progress and will be presented elsewhere.

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[1] W.G. Unruh, W.H. Zurek, Phys. Rev. **D40**, 1071 (1989).
[2] B.L. Hu, Juan Pablo Paz, Yuhong Zhang, Phys. Rev. **D45**, 2843 (1992).
[3] J. Glimm, A. Jaffe, Quantum Physics, a Functional Integral Point of View, Springer-Verlag - Berlin, 2nd Ed. (1987).
[4] C. de Calan, P.A. Faria da Veiga, J. Magnen, R. Sénéor, Phys. Rev. Lett. **66**, 3233 (1991).
[5] P. Ullersma, Physica **32**, 56, (1966); Physica **32**, 74, (1966); Physica **32**, 90, (1966).
[6] F. Haake, R. Reibold, Phys. Rev. **A32**, 2462 (1982).
[7] A.O. Caldeira, A.J. Legget, Ann. Phys. (N.Y) **149**, 374 (1983).
[8] H. Grabert, P. Schramm, G.-L. Ingold, Phys. Rep. **168**, 115 (1988).
[9] N.P. Andion, A.P.C. Malbouisson and A. Mattos Neto, J.Phys.**A34**, 3735, (2001).
[10] A.P.C. Malbouisson, Phys. Lett. **A296**, 65 (2002).
[11] G. Flores-Hidalgo, A.P.C. Malbouisson, Y.W. Milla, ”Stability of excited atoms in small cavities”. To appear in Physical Review A (2002).
[12] W. Jhe, A. Anderson, E.A. Hinds, D. Meschede, L. Moi, S. Haroche, Phys. Rev. Lett. **58**, 666 (1987).
[13] R.G. Hulet, E.S. Hilfer, D. Kleppner, Phys. Rev. Lett. **55**, 2137 (1985).
[14] A.K. Likhoded, G.P. Pronko, Int. Journ. Theor. Phys. **36**, 2335 (1997).
[15] H. Ederlyi et al.; Higher Transcendental Functions, New York, Mc Graw-Hill (1953), p. 196, formula (40).