Potential–density pairs for axisymmetric galaxies: 
the influence of scalar fields

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Abstract

We present a formulation for potential-density pairs to describe axisymmetric galaxies in the Newtonian limit of scalar-tensor theories of gravity. The scalar field is described by a modified Helmholtz equation with a source that is coupled to the standard Poisson equation of Newtonian gravity. The net gravitational force is given by two contributions: the standard Newtonian potential plus a term stemming from massive scalar fields. General solutions have been found for axisymmetric systems and the multipole expansion of the Yukawa potential is given. In particular, we have computed potential–density pairs of galactic disks for an exponential profile and their rotation curves.

Keywords: stellar dynamics – galaxy: kinematics and dynamics – galaxy: halo– galaxy: disk – galaxy: structure

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1 Introduction

In recent years there has been much interest in the study of dark matter and energy in a cosmological and astrophysical context. This has been motivated by recent independent observational data in the Cosmic Microwave Background Radiation (CMBR) at various angular scales \cite{1}, type Ia supernovae \cite{2}, and the 2dF Galaxy Redshift survey \cite{3}, among others. These suggest that $\Omega = \Omega_\Lambda + \Omega_m \approx 1$, with $\Omega_\Lambda \approx 0.7$ and $\Omega_m \approx 0.3$, implying the existence of dark energy and dark matter, respectively. In this way galaxies are expected to possess dark components and, in accordance with the rotation curves of stars and gas around the centers of spirals, they might be in the form of halos \cite{4} and
must contribute to at least 3 to 10 times the mass of the visible matter \[5\]. The origin of these dark components is, however, unknown.

The theoretical framework to explain the existence of dark components finds its origin in theories of elementary particles physics, with the addition of the action of gravity. There exist many theories (grand unification schemes, string theories, braneworlds, etc) that involve such physics, but scalar–tensor theories (STT) of gravity are typically found to represent classical effective descriptions of such original theories \[6\]. In this way, the scalar fields of these theories are the natural candidates to be the quintessence field \[7\], as a remnant of some cosmological function. It has been even suggested that the quintessence field is the scalar field that also acts on local planetary scales \[8\] or on galactic scales \[9\]. Moreover, massive scalar fields might account to the dark matter components of galaxies in the form of halos.

Motivated by the above arguments, we have recently studied some STT effects in galactic systems\[10\]. We have computed spherical potential–density pairs \[11\] coming from such theories in their Newtonian approximation. In the present report we compute general potential–density pairs and take some limiting cases. Then, we consider the general axisymmetric case and take, as an example, an exponential profile, that typically describes stars on the disk. Also, rotation curves of such a model can be directly obtained.

This paper is organized as follows: in section 2 we present the equations of the Newtonian approximation of a general scalar–tensor theory of gravity and solutions are given in terms of integrals of Green functions. In section 3, we discuss solutions for point-like mass distributions and show how a multipole expansion can be done for the Newtonian limit contribution of the scalar field. In section 4, computations are done for general axisymmetric potentials and rotation velocities of stars in galaxies. Then, an exponential disk is considered. In section 5, we present our conclusions.

2 Scalar Fields and the Newtonian Approximation

In references \[10\] \[12\] are derived the following differential equations, that are valid for a general scalar–tensor theory in its Newtonian limit,

\[
\begin{align*}
\nabla^2 \psi & = 4\pi \rho , \\
\nabla^2 \tilde{\phi} - m^2 \tilde{\phi} & = -8\pi \alpha \rho ,
\end{align*}
\]

in which we consider deviations of the scalar field \(\tilde{\phi}\) from some average value, given by the inverse of the Newtonian constant, \(1/G\); \(\alpha\) is a some free constant of the theory and \(m\) the mass of some boson particle, see details in \[10\].

The standard Newtonian potential, \(\psi\), is obtained when the perturbation is set to zero, \(\tilde{\phi} = 0\) and \(\alpha = 0\). Otherwise, the new Newtonian potential is given by

\[
\Phi_N = \psi - \frac{1}{2} \tilde{\phi} ,
\]
The next step is to find solutions for this new Newtonian potential given a density profile, that is, to find the so-called potential–density pairs. General solutions to Eqs. (1) and (2) can be found in terms of the corresponding Green functions

\[ \psi = -\int dr_s \frac{\rho(r_s)}{|r-r_s|} + \text{B.C.}, \quad (4) \]

\[ \bar{\phi} = 2\alpha \int dr_s \frac{\rho(r_s)e^{-m|r-r_s|}}{|r-r_s|} + \text{B.C.}, \quad (5) \]

and the new Newtonian potential is

\[ \Phi_N = \psi - \frac{1}{2} \bar{\phi} = -\int dr_s \frac{\rho(r_s)}{|r-r_s|} - \alpha \int dr_s \frac{\rho(r_s)e^{-m|r-r_s|}}{|r-r_s|} + \text{B.C.} \quad (6) \]

The first term of Eq. (6), given by \( \psi \), is the contribution of the usual Newtonian gravitation (without scalar fields), while information about the scalar field is contained in the second term, that is, arising from the influence function determined by the Helmholtz Green function, where the coupling \( \alpha \) enters as part of a source factor.

3 Point–like masses and multipole expansion

Now we present solutions for point–like masses and the multipole expansion of the potentials which are useful for numerical simulations of galaxies. Substituting the following distribution of \( N \) point-like masses in Eq. (6)

\[ \rho(r) = \sum_{s=1}^{N} m_s \delta(r - r_s) \quad (7) \]

one obtains

\[ \Phi_N = -\sum_{s=1}^{N} \frac{m_s}{|r-r_s|} - \alpha \sum_{s=1}^{N} \frac{m_s e^{-m|r-r_s|}/\lambda}{|r-r_s|} \quad (8) \]

with \( m_s \) a source mass at spatial position \( r_s \), and the total gravitational force on a particle of mass \( m_i \) is

\[ \sum F = -\nabla \Phi_N = m_i a, \quad (9) \]

where \( \lambda = \hbar/mc \) is the Compton wavelength of the effective mass (\( m \)) of some elementary particle (boson) determined by specific particle physics models. In what follows we will use \( \lambda \) instead of \( m^{-1} \). This length can have a range of values depending on particular particle physics models. The first right hand term in Eq. (8) is the pure Newtonian part and the second one is the dark matter contribution which is of the Yukawa type. There are two limits: On the
one hand, for if \( r \gg \lambda \) (or \( \lambda \to 0 \)) one recovers the Newtonian theory of gravity. On the other hand, for if \( r \ll \lambda \) (or \( \lambda \to \infty \)) one again obtains the Newtonian theory, but now with a rescaled Newtonian constant, \( G \to G_N (1 + \alpha) \). There are stringent constraints on the possible \( \lambda - \alpha \) values determined by measurements on local scales [14].

In the past the above solutions have been used to solve the missing mass problem in spirals [15, 16] as an alternative to considering a distribution of dark matter. This was done assuming that most of the galactic mass is located in the galactic center, and then considering the center to be a point source. In our present investigation we do not avoid dark matter, since our model predicts that bosonic dark matter produces, through a scalar field associated to it, a modification of Newtonian gravity theory. This dark matter is presumably clumped in the form of dark halos. Therefore we will consider in what follows that a dark halo is axisymmetrically distributed along an observable spiral and beyond, having some density profile. Next, we compute the potentials, and some astrophysical quantities, for general halo density distributions.

The standard gravitational potential due to a distribution of mass \( \rho(r) \), in a point exterior to the distribution and without considering the boundary condition, can be expanded as [17]

\[
\psi(r) = -\sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{4\pi}{2l+1} q_{ln} \frac{Y_{ln}(\theta, \phi)}{r^{l+1}},
\]

with the expansion

\[
\frac{1}{|r - r_s|} = 4\pi \sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{ln}^*(\theta',\phi')Y_{ln}(\theta,\phi),
\]

where \( r_< \) is the smaller of \( |r| \) and \( |r_s| \), and \( r_> \) is the larger of \( |r| \) and \( |r_s| \); \( Y_{ln}(\theta,\phi) \) are the spherical harmonics.

Here we are interested in the potential outside the distribution of mass, consequently, the coefficients of the expansion of \( \psi \), and known as multipoles are given by

\[
q_{ln} = \int d\mathbf{r}' Y_{ln}(\theta',\phi') r^l \rho(\mathbf{r}').
\]

In the case of the scalar field, with the expansion

\[
\frac{\exp(-m|\mathbf{r} - \mathbf{r}_s|)}{|\mathbf{r} - \mathbf{r}_s|} = 4\pi m \sum_{l=0}^{\infty} \sum_{n=-l}^{l} i_l(m r_<) k_l(m r_> Y_{ln}^*(\theta',\phi') Y_{ln}(\theta,\phi),
\]

the contribution of the scalar field to the Newtonian gravitational potential can be written as

\[
\frac{1}{2\alpha} \bar{\phi}(r) = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{4\pi}{2l+1} \bar{q}_{ln} k_l(m r) Y_{ln}(\theta,\phi),
\]
where \( i_l(x) \) and \( k_l(x) \) are the modified spherical Bessel functions and we have defined the multipoles for the scalar field as

\[
\bar{q}_l = m \int \mathrm{d}r' Y_l(\theta', \phi') i_l(mr') \rho(r'). \tag{15}
\]

4 General axisymmetric density distributions and their rotation velocities

Given that we are interested in axisymmetric galaxies in what follows we will only consider the case of this symmetry. Additionally, we use flatness boundary conditions (B.C.) at infinity, such that the boundary terms in Eqs. (4)-(5) are zero. Moreover, regularity conditions must be applied to spatial points where the potentials are singular. For axisymmetric systems these conditions mean that \( d\psi/dr = d\bar{\phi}/dr = 0 \) along the symmetry axis. Accordingly, we assume that

\[
\rho(r, z) = \begin{cases} \rho(r, z) & ; \quad l < L \\ 0 & ; \quad l \geq L \end{cases} \tag{16}
\]

where we define the boundary surface by a single-valued function \( r = Z(z) \) with \( z \) in the interval \((z_0, z_1)\) and \( l = \sqrt{r^2 + z^2} \), and the same relation for \( L \) with \((r, z)\) on the boundary surface. We obtain for \( \psi \)

\[
\psi(r, z) = -\int_{z_0}^{z_1} \mathrm{d}z \int_0^{Z(z)} r_s \mathrm{d}r_s \rho(r_s, z_s) \int_{0}^{\infty} \mathrm{d}k I_0(kr_<) K_0(kr_>) \cos k(z - z_s), \tag{17}
\]

and for \( \bar{\phi} \)

\[
\frac{1}{2} \bar{\phi}(r, z) = -\alpha \int_{z_0}^{z_1} \mathrm{d}z \int_0^{Z(z)} r_s \mathrm{d}r_s \rho(r_s, z_s) \int_{0}^{\infty} \mathrm{d}k I_0(\nu r_<) K_0(\nu r_>) \cos (z - z_s), \tag{18}
\]

where \( I_0(x) \) and \( K_0(x) \) are modified Bessel functions of index \( n = 0 \) and \( \nu^2 = k^2 + \lambda^{-2} \). Then, the new Newtonian potential can be obtained using (6).

An important quantity that characterizes the steady state of astrophysical systems is the circular velocity. The circular velocity for a test particle moving on the equatorial plane of an axisymmetric galactic system is given by

\[
v_c^2 = r \frac{d\Phi_N}{dr}, \tag{19}\]

We may consider as an example that the distribution of mass is

\[
\rho(r, z) = A \exp(-r/r_0) \delta(z), \tag{20}\]

where \( r_0 \) is the effective radial extension of the system. This distribution of mass is typically observed in stars of a disk galaxy. In this case the Newtonian
potential is
\[
\Phi_N(r) = -A \left\{ \int_0^r r_s dr_s \exp(-r_s/r_0) \int_0^\infty dk I_0(kr)K_0(kr_0) \cos(kz) \\
+ \int_r^\infty r_s dr_s \exp(-r_s/r_0) \int_0^\infty dk I_0(kr)K_0(kr_0) \cos(kz) \\
+ \alpha \int_0^r r_s dr_s \exp(-r_s/r_0) \int_0^\infty dk I_0(\nu r_s)K_0(\nu r_0) \cos(kz) \\
+ \alpha \int_r^\infty r_s dr_s \exp(-r_s/r_0) \int_0^\infty dk I_0(\nu r)K_0(\nu r_s) \cos(kz) \right\}.
\]  \hspace{1cm} (21)

Using Eq. (19) the rotation curve is obtained.

5 Discussion and Conclusions

We have found potential–density pairs of axisymmetric galactic systems within the context of linearized scalar–tensor theories of gravitation. The influence of massive scalar fields is given by \( \bar{\phi} \), determined by Eq. (18). Circular velocities of stars can be directly obtained in the axisymmetric system; by stars we mean probe particles that follow the dark halo potential. Specifically, these results were used to find potential–density pairs for an exponential profile. In general the contribution due to massive scalar fields is non–trivial, see for instance Eq. (21), and interestingly, forces on circular orbits of stars depend on the parameters \( \lambda \) and \( \alpha \) in a rather complicated way. This means that even when local experiments force \( \alpha \) to be a very small number \([14]\), the amplitude of forces exerted on stars is not necessarily very small and may contribute significantly to the dynamics of stars. Alternatively, one may interpret the local Newtonian constant as given by \( (1 + \alpha) \langle \phi \rangle^{-1} \), instead of being given by \( \langle \phi \rangle^{-1} \) (1 in our convention). In this case, the local measurement constraints are automatically satisfied, and at scales larger than \( \lambda \) one sees a reduction of \( 1/(1 + \alpha) \) in the Newtonian constant.

In the past, different authors have used point solutions, Eq. (6), to solve the missing mass problem encountered in the rotation velocities of spirals and in galaxy cluster dynamics \([15, 16]\). These models were used as an alternative to avoid dark matter. Indeed, for single galaxies one can adjust the parameters \( (\alpha, \lambda) \) to solve these problems without the need for dark matter. However, these models do not provide a good description of the systematics of galaxy rotation curves because they predict the scale \( \lambda \) to be independent of the galactic luminosity, and this conflicts with observations for different galactic sizes \([18]\) unless one assumes various \( \lambda \)'s, and hence various fundamental masses, \( m \), one for each galaxy size. Such a particle spectrum is not expected from theoretical arguments; it represents a considerable fine tuning of masses. This criticism would also apply to our models. The purpose of the present investigation was, however, not to present a model alternative to dark matter in order to solve the missing mass problem, but to compute the influence of scalar–field dark
matter distributed in the form of an axisymmetric dark halo. This contribution, together with the other components of the galaxy, will give rise to a flat velocity curve, in a similar way as in Newtonian mechanics.

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