THE CAUCHY PROBLEM FOR HEAT EQUATION WITH FRACTIONAL LAPLACIAN AND EXPONENTIAL NONLINEARITY

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Abstract. We consider the Cauchy problem for heat equation with fractional Laplacian and exponential nonlinearity. We establish local well-posedness result in Orlicz spaces. We derive the existence of global solutions for small initial data. We obtain decay estimates for large time in Lebesgue spaces.

1. Introduction. This paper concerns the Cauchy problem for the following heat equation

\[
\begin{align*}
\begin{cases}
  u_t + (-\Delta)^{\beta/2} u = f(u), & t > 0, x \in \mathbb{R}^n, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

where $u$ is a real-valued unknown function, $0 < \beta \leq 2$, $n \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ having an exponential growth at infinity ($f(u) \sim e^{\|u\|^p}$, $p > 1$, for large $u$) with $f(0) = 0$. Hereafter, $\| \cdot \|_q$ ($1 \leq q \leq \infty$) stands for the usual $L^q(\mathbb{R}^n)$-norm.

When $f(u) = |u|^{p-1}u$, the Lebesgue spaces are adapted to study our problem (cf. [4, 15, 16, 17]). By analogy, we consider the Orlicz spaces [3] in order to study heat equations with exponential nonlinearities. The Orlicz space

$$\text{exp } L^p(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n); \int_{\mathbb{R}^n} \left( \exp \left( \frac{|u(x)|^p}{\lambda^p} \right) - 1 \right) dx < \infty, \text{ for some } \lambda > 0 \right\},$$

don the Luxemburg norm

$$\| u \|_{\text{exp } L^p(\mathbb{R}^n)} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^n} \left( \exp \left( \frac{|u(x)|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\}$$

is a Banach space. For the local well-posedness we use the space

$$\text{exp } L^p_0(\mathbb{R}^n) = \left\{ u \in \text{exp } L^p(\mathbb{R}^n); \text{there exists } \{ u_n \}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^n) \right\}$$

such that $\lim_{n \to \infty} \| u_n - u \|_{\text{exp } L^p(\mathbb{R}^n)} = 0$.

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It is also known (see Ioku, Ruf, and Terraneo [9], Majdoub et al. [10, 11]) that
\[
\exp L^p_0(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n); \int_{\mathbb{R}^n} (\exp (|u(x)|^p) - 1) \, dx < \infty, \text{ for every } a > 0 \right\}.
\]

When \( \beta = 2 \) (i.e. the standard heat equation) and \( p = 2 \), Ioku [8] proved the existence of global solutions in \( \exp L^2(\mathbb{R}^n) \) of (1.1) under the condition (1.4) below with \( m = 1 + \frac{4}{n} \). Later, Ioku et al. [9] studied the local nonexistence of solutions of (1.1) for certain data in \( \exp L^2(\mathbb{R}^2) \), and the well-posedness of (1.1) in the subspace \( \exp L^2_{\text{loc}}(\mathbb{R}^2) \) under the condition (1.3) below. In [6], Furioli et al. considered the asymptotic behavior and decay estimates of the global solutions of (1.1) in \( \exp L^2(\mathbb{R}^n) \) when \( f(u) = |u|^{4/n} u e^{u^2} \). Next, Majdoub et al. [10] proved the local well-posedness in \( \exp L^2_{\text{loc}}(\mathbb{R}^n) \) (if \( f \) satisfies (1.3) below with \( m \geq 1 + \frac{4}{n} \)) and the global existence under small initial data in \( \exp L^2(\mathbb{R}^n) \) (if \( f \) satisfies (1.4) below) for the biharmonic heat equation (i.e. \( u_t + \Delta^2 u = f(u) \)). Finally, when \( \beta = 2, p > 1 \) and \( m \geq 1 + \frac{2p}{n} \), Majdoub and Tayachi [11] proved not only the local well-posedness in \( \exp L^p_{\text{loc}}(\mathbb{R}^n) \) but also the global existence of solutions, when \( \frac{n(p-1)}{2} \geq p \), under small initial data in \( \exp L^p(\mathbb{R}^n) \) of (1.1) and analyzed their decay estimates, while the case of \( \frac{n(p-1)}{2} < p \) is recently completed in [12]. In this paper, we generalize the papers of [11, 12] to the fractional Laplacian case.

In order to state our main results, we note that the linear semigroup \( e^{-t(-\Delta)^{\beta/2}} \) is continuous at \( t = 0 \) in \( \exp L^p_0(\mathbb{R}^n) \) (see Proposition 2) which is not the case in \( \exp L^p(\mathbb{R}^n) \) (cf. [9] in the case of \( \beta = 2 \)), therefore, we have to define two kinds of mild solutions, the standard one where the space \( \exp L^p_{\text{loc}}(\mathbb{R}^n) \) is used, and the weak-mild solution where we use the space \( \exp L^p(\mathbb{R}^n) \).

**Definition 1.1** (Mild solution). Given \( u_0 \in \exp L^p_0(\mathbb{R}^n) \) and \( T > 0 \). We say that \( u \) is a mild solution for the Cauchy problem (1.1) if \( u \in C([0, T]; \exp L^p_{\text{loc}}(\mathbb{R}^n)) \) satisfies
\[
\begin{align*}
u(t) = e^{-t(-\Delta)^{\beta/2}}u_0 + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds, \tag{1.2}
\end{align*}
\]
where \( e^{-t(-\Delta)^{\beta/2}} \) is defined in (2.7) below.

**Definition 1.2** (Weak-mild solution). Given \( u_0 \in \exp L^p(\mathbb{R}^n) \) and \( T > 0 \). We say that \( u \) is a weak-mild solution for the Cauchy problem (1.1) if \( u \in L^\infty((0, T); \exp L^p(\mathbb{R}^n)) \) satisfying the associated integral equation (1.2) in \( \exp L^p(\mathbb{R}^n) \) for almost all \( t \in (0, T) \) and \( u(t) \to u_0 \) in the weak* topology as \( t \to 0 \).

We recall that \( u(t) \to u_0 \) in weak* sense if and only if
\[
\lim_{t \to 0} \int_{\mathbb{R}^n} [u(t, \varphi) \varphi(x) - u_0(x) \varphi(x)] \, dx = 0, \quad \text{for every } \varphi \in L^1([\ln L]^{1/p}(\mathbb{R}^n)),
\]
where
\[
L^1([\ln L]^{1/p}(\mathbb{R}^n)) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n); \int_{\mathbb{R}^n} |f(x)| \ln^{1/p}(2 + |f(x)|) \, dx < \infty \right\}
\]
is a predual of \( \exp L^p(\mathbb{R}^n) \) (see [2, 13]).

First, we interest in the local well-posedness. We assume that \( f \) satisfies
\[
f(0) = 0, \quad |f(u) - f(v)| \leq C|u - v|(e^{\lambda|u|^p} + e^{\lambda|v|^p}), \quad \text{for all } u, v \in \mathbb{R}, \tag{1.3}
\]
for some constants \( C > 0, p > 1, \) and \( \lambda > 0 \). Typical example satisfying (1.3) is:
\[
f(u) = \pm u|u|^p.
\]
**Theorem 1.3** (Local well-posedness). Let \( n \geq 1, \ p > 1 \), and \( 0 < \beta < 2 \). Suppose that \( f \) satisfies (1.3). Given \( u_0 \in \exp L^p_0(\mathbb{R}^n) \), there exist a time \( T = T(u_0) > 0 \) and a unique mild solution \( u \in C([0, T]; \exp L^p_0(\mathbb{R}^n)) \) to (1.1).

Next, our second interest is the global existence and the decay estimate. In this case, the behaviour of \( f(u) \) near \( u = 0 \) plays a crucial role, therefore the following behaviour near zero will be allowed \( |f(u)| \sim |u|^m \), where \( n(m-1) \geq p \). More precisely, we suppose that

\[
f(0) = 0, |f(u) - f(v)| \leq C|u - v|(|u|^{m-1} e^\lambda |u|^p + |v|^{m-1} e^\lambda |v|^p), \quad \text{for all} \ u, v \in \mathbb{R},
\]

where \( n(m-1) \geq p > 1 \), \( C > 0 \), and \( \lambda > 0 \) are constants. Typical example satisfying (1.4) is: \( f(u) = \pm |u|^{m-1} u e^{|u|^p} \) where \( m > 1 + \frac{2p}{n} \); we note that the global existence in the case \( m = 1 + \frac{2p}{n} \) is presented in [6, Section 8] without any proof.

**Theorem 1.4** (Global existence). Let \( n \geq 1, \ 1 < p \leq \frac{n(m-1)}{\beta} \), and \( 0 < \beta < 2 \). Suppose that \( f \) satisfies (1.4) for \( m \geq p \). Then there exists a positive constant \( \varepsilon > 0 \) such that every initial data \( u_0 \in \exp L^p(\mathbb{R}^n) \) with \( \|u_0\|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon \), there exists a global weak-mild solution \( u \in L^\infty((0, \infty); \exp L^p(\mathbb{R}^n)) \) to (1.1) satisfying

\[
\lim_{t \to 0} \left\| u(t) - e^{-t(\Delta)^{\beta/2}} u_0 \right\|_{\exp L^p(\mathbb{R}^n)} = 0.
\]

Moreover, there exists a constant \( C > 0 \) such that

\[
\|u(t)\|_{L^\sigma(\mathbb{R}^n)} \leq C t^{-\sigma}, \quad \text{for all} \ t > 0,
\]

where \( \sigma = \frac{1}{m-1} - \frac{n}{2q} > 0 \), and \( (\cdot)_{+} \) stands for the positive part.

**Remark 1.** In Theorem 1.4, we have to distinguish 3 cases: \( \beta < \frac{n(p-1)}{p} \), \( \beta > \frac{n(p-1)}{p} \), and \( \beta = \frac{n(p-1)}{p} \). We note that in the case of \( \beta > \frac{n(p-1)}{p} \) we have to take \( m > p \).

Indeed, if \( m = p \), it follows that \( \beta > \frac{n(m-1)}{m} \), but \( n(m-1)/\beta \geq p \), which implies that \( \beta \leq \frac{n(m-1)}{m} \), therefore \( \frac{n(p-1)}{p} < \frac{n(p-1)}{m} \); contradiction.

This paper is organized as follows: in Section 2, we present several preliminaries. Section 3 contains the proof of the local well-posedness theorem (Theorem 1.3). Finally, we prove the global existence theorem (Theorem 1.4) in Section 4.

2. Preliminaries.

2.1. Orlicz spaces: basic properties. In this section we present the definition of the so-called Orlicz spaces on \( \mathbb{R}^n \) and some related properties. More details and complete presentations can be found in [1, 13, 14].
**Definition 2.1** (Orlicz space). Let \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a convex increasing function such that
\[
\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty.
\]
The Orlicz space \( L^\phi(\mathbb{R}^n) \) is defined by
\[
L^\phi(\mathbb{R}^n) = \left\{ u \in L^1_{loc}(\mathbb{R}^n); \int_{\mathbb{R}^n} \phi \left( \frac{|u(x)|}{\lambda} \right) \, dx < \infty, \text{ for some } \lambda > 0 \right\},
\]
endowed with the Luxemburg norm
\[
\|u\|_{L^\phi(\mathbb{R}^n)} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^n} \phi \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]
On the other hand, we denote by
\[
L^\phi_0(\mathbb{R}^n) = \left\{ u \in L^1_{loc}(\mathbb{R}^n); \int_{\mathbb{R}^n} \phi \left( \frac{|u(x)|}{\lambda} \right) \, dx < \infty, \text{ for every } \lambda > 0 \right\}.
\]
It can be shown (as in Ioku et al. [9]) that
\[
L^\phi_0(\mathbb{R}^n) = \frac{C_0^\infty(\mathbb{R}^n)}{\| \cdot \|_{L^\phi}} = \text{the closure of } C_0^\infty(\mathbb{R}^n) \text{ in } L^\phi(\mathbb{R}^n).
\]
It is known that \((L^\phi(\mathbb{R}^n), \| \cdot \|_{L^\phi(\mathbb{R}^n)})\) and \( (L^\phi_0(\mathbb{R}^n), \| \cdot \|_{L^\phi(\mathbb{R}^n)}) \) are Banach spaces.

**Lemma 2.2** ([11], Lemma 2.3). For every \( 1 \leq q \leq p \), we have \( L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^p_0(\mathbb{R}^n) \hookrightarrow \exp L^p(\mathbb{R}^n) \), more precisely
\[
\|u\|_{\exp L^p(\mathbb{R}^n)} \leq \frac{1}{(\ln 2)^{1/p}} (\|u\|_q + \|u\|_\infty).
\]  

Similarly, we have

**Lemma 2.3.** Let \( \phi(s) = e^{s^p} - 1 - s^p, \ p > 1 \). For every \( q \leq 2p \), we have \( L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow L^0_0(\mathbb{R}^n) \hookrightarrow L^\phi(\mathbb{R}^n) \), more precisely
\[
\|u\|_{L^\phi(\mathbb{R}^n)} \leq C(p)(\|u\|_q + \|u\|_\infty).
\]  

**Proof.** Let \( g(s) = e^{s^p} - s^p; \ g \) is a strictly increasing. Let \( \alpha \geq C(p)(\|u\|_q + \|u\|_\infty) \) where \( C(p) := 1/g^{-1}(2) \), then
\[
\int_{\mathbb{R}^n} \left( \exp \left( \frac{|u(x)|^p}{\alpha^p} \right) - 1 - \left( \frac{|u(x)|^p}{\alpha^p} \right) \right) \, dx
\]
\[
= \sum_{k=2}^{\infty} \frac{1}{k! \alpha^k} \|u\|_p^k
\]
It means that the solution of (2.5) with any initial data
\[ u(x,0) = u_0(x) \]
where we have used the interpolation inequality \( \|u\|_r \leq \|u\|_\infty^{1-q/r} \|u\|_\infty^{1-q} \leq \|u\|_q + \|u\|_\infty \) for all \( q \leq r \leq \infty \) and all \( u \in L^q \cap L^\infty \). Therefore
\[ [C(p)](\|u\|_q + \|u\|_\infty); \|u\|_\infty \leq 1 \}
which implies that
\[ \|u\|_{L^q(\mathbb{R}^n)} = \inf \left\{ \alpha > 0; \int_{\mathbb{R}^n} \phi \left( \frac{|u(x)|}{\alpha} \right) \; dx \leq 1 \right\} \]
\[ \leq \inf \left( \alpha > 0 \right) \alpha \in [C(p)](\|u\|_q + \|u\|_\infty); \|u\|_\infty \}
\[ = C(p)(\|u\|_q + \|u\|_\infty). \]

**Lemma 2.4** ([11], Lemma 2.4). For every \( 1 \leq p \leq q < \infty \), we have \( L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \), more precisely
\[ \|u\|_q \leq \left( \Gamma \left( \frac{q}{p} + 1 \right) \right)^{1/q} \|u\|_{C(\mathbb{R}^n)}, \] (2.4)
where \( \Gamma \) is the gamma function.

Next, we present some definitions and results concerning the fractional Laplacian that will be used hereafter. The fundamental solution \( S_\beta \) of the usual linear fractional diffusion equation
\[ u_t + (-\Delta)^{\beta/2} u = 0, \quad \beta \in (0,2], \quad x \in \mathbb{R}^n, \quad t > 0, \] (2.5)
can be represented via the Fourier transform by
\[ S_\beta(t)(x) := S_\beta(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i x \cdot \xi - t|\xi|^\beta} \; d\xi. \] (2.6)
It means that the solution of (2.5) with any initial data \( u(0) = u_0 \) can be written as
\[ u(x,t) = S_\beta(x,t) * u_0(x) =: e^{-t(-\Delta)^{\beta/2}} u_0, \] (2.7)
where \( e^{-t(-\Delta)^{\beta/2}} \) is a strongly continuous semigroup on \( L^p(\mathbb{R}^n) \), \( p > 1 \), generated by the fractional power \( (-\Delta)^{\beta/2} \). Moreover, \( S_\beta \) satisfies
\[ S_\beta(1) \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad S_\beta(x,t) \geq 0, \quad \int_{\mathbb{R}^n} S_\beta(x,t) \; dx = 1, \] (2.8)
for all \( x \in \mathbb{R}^n \) and \( t > 0 \). Hence, using Young’s inequality for the convolution and the following self-similar form \( S_\beta(x,t) = t^{-\beta/2} S_\beta(xt^{-1/\beta}, 1) \), we get the \( L^r - L^q \) estimate
\[ \left\| e^{-t(-\Delta)^{\beta/2}} v \right\|_r \leq C t^{-\frac{\beta}{2}(\frac{q}{r} - \frac{1}{4})} \|v\|_r, \] (2.9)
for all \( v \in L^r(\mathbb{R}^n) \) and all \( 1 < r < q \leq \infty, t > 0, \) where \( C > 0 \) is a positive constant depending only on \( n. \) In particular, using Young's inequality for the convolution and (2.8), we have
\[
\left\| e^{-t(-\Delta)^{\beta/2}} v \right\|_q = \| S_\beta(x, t) * v \|_q \leq \| S_\beta(t) \|_1 \| v \|_q = \| v \|_q, \tag{2.10}
\]
for all \( v \in L^q(\mathbb{R}^n) \) and all \( 1 \leq q \leq \infty, t > 0. \)

The following proposition is a generalization of Proposition 3.2 in [11] and it is presented (without proof) by Furioli et al. [6, Lemma 3.1].

**Proposition 1.** Let \( 1 \leq q \leq p, 1 \leq r \leq \infty, \) and \( 0 < \beta \leq 2. \) Then the following estimates hold.

(i) \( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq \| \varphi \|_{\exp L^p(\mathbb{R}^n)}, \) for all \( t > 0, \varphi \in \exp L^p(\mathbb{R}^n). \)

(ii) \( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq C t^{-\beta/2} \left( \ln(t^{-\beta/2} + 1) \right)^{-1/p} \| \varphi \|_q, \) for all \( t > 0, \varphi \in L^q(\mathbb{R}^n). \)

(iii) \( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq \frac{1}{(\ln 2)^{\beta/2}} \left[ C t^{-\beta/2} \| \varphi \|_r + \| \varphi \|_q \right], \) for all \( t > 0, \varphi \in L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n). \)

**Proof.** We start by proving (i). For any \( \lambda > 0, \) using (2.10) and Taylor expansion, we have
\[
\int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx
\]
\[
= \sum_{k=1}^{\infty} \left\| \frac{e^{-t(-\Delta)^{\beta/2}} \varphi}{|\lambda^p|} \right\|_{pk}^{pk} \leq \sum_{k=1}^{\infty} \left\| \frac{\varphi}{|\lambda^p|} \right\|_{pk}^{pk} = \int_{\mathbb{R}^n} \left( \exp \left( \frac{|\varphi|^p}{\lambda^p} \right) - 1 \right) dx.
\]
Then
\[
\left\{ \lambda > 0; \int_{\mathbb{R}^n} \left( \exp \left( \frac{|\varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\}
\]
\[
\subseteq \left\{ \lambda > 0; \int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\},
\]
and therefore
\[
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\}
\]
\[
\leq \inf \left\{ \lambda > 0; \int_{\mathbb{R}^n} \left( \exp \left( \frac{|\varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\} = \| \varphi \|_{\exp L^p(\mathbb{R}^n)}.
\]
This proves (i). Similarly, to prove (ii), we use again (2.9) and Taylor expansion. For any \( \lambda > 0, \) we have
\[
\int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx
\]
we conclude that

$$
\sum_{k=1}^{\infty} \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{k! \lambda^p k} 
\leq \sum_{k=1}^{\infty} C \frac{k^{\beta/2} \left( \frac{1}{n} - \frac{1}{p} \right)^p \varphi^p}{k! \lambda^p} = t^\frac{\beta}{2} \left( \exp \left( \frac{C t^{\frac{n}{\lambda}} \|\varphi\|^q}{\lambda} \right) - 1 \right).
$$

As

$$
t^\frac{\beta}{2} \left( \exp \left( \frac{C t^{\frac{n}{\lambda}} \|\varphi\|^q}{\lambda} \right) - 1 \right) \leq 1 \iff \lambda \geq C t^{-\frac{n}{\lambda}} \left( \ln(\lambda) + 1 \right)^{-1/p} \|\varphi\|^q,
$$

we conclude that

$$
\left\{ \lambda > 0, \lambda \in \left[ C t^{-\frac{n}{\lambda}} \left( \ln(\lambda) + 1 \right)^{-1/p} \|\varphi\|^q \right] \cap \infty \right\}
\subseteq \left\{ \lambda > 0, \int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\};
$$

whereupon

$$
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{exp \ L^p(\mathbb{R}^n)} = \inf \left\{ \lambda > 0, \int_{\mathbb{R}^n} \left( \exp \left( \frac{|e^{-t(-\Delta)^{\beta/2}} \varphi|^p}{\lambda^p} \right) - 1 \right) dx \leq 1 \right\}
\leq \inf \left\{ \lambda > 0, \lambda \in \left[ C t^{-\frac{n}{\lambda}} \left( \ln(\lambda) + 1 \right)^{-1/p} \|\varphi\|^q \right] \cap \infty \right\}
= C t^{-\frac{n}{\lambda}} \left( \ln(\lambda) + 1 \right)^{-1/p} \|\varphi\|^q.
$$

This proves (ii). Finally, to prove (iii), we use the embedding \( L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow exp L^p(\mathbb{R}^n) \) (2.2); we get

$$
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{exp \ L^p(\mathbb{R}^n)} \leq \frac{1}{(\ln 2)^{1/p}} \left( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_q + \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_\infty \right).
$$

Using the \( L^r - L^\infty \) and \( L^q - L^q \) estimates (2.9), we conclude that

$$
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{exp \ L^p(\mathbb{R}^n)} \leq \frac{1}{(\ln 2)^{1/p}} \left( \left\| \varphi \right\|_q + C t^{-\frac{n}{\lambda}} \left\| \varphi \right\|_r \right).
$$

We will also need the following smoothing results.

**Proposition 2.** If \( \varphi \in exp L^0_0(\mathbb{R}^n) \), then \( e^{-t(-\Delta)^{\beta/2}} \varphi \in C(0, \infty); exp L^p_0(\mathbb{R}^n) \).

**Proof.** The proof of this proposition follows the same one of [10, Proposition 3.7] by making the appropriate modifications. To be self-contained, we will present it in details. Let \( \varphi \in exp L^0_0(\mathbb{R}^n) \). By (i) of Proposition 1 and the definition of \( exp L^0_0(\mathbb{R}^n) \), we have \( e^{-t(-\Delta)^{\beta/2}} \varphi \in exp L^0_0(\mathbb{R}^n) \) for every \( t > 0 \). Thus, by the linearity of the semigroup \( e^{-t(-\Delta)^{\beta/2}} \), it remains to prove the continuity at \( t = 0 \),

$$
\lim_{t \to 0} \left\| e^{-t(-\Delta)^{\beta/2}} \varphi - \varphi \right\|_{exp \ L^p(\mathbb{R}^n)} = 0.
$$

Since \( \varphi \in exp L^0_0(\mathbb{R}^n) \), there exists a sequence \( (\varphi_n)_n \subseteq C^\infty(\mathbb{R}^n) \) such that

$$
\lim_{n \to \infty} \left\| \varphi_n - \varphi \right\|_{exp \ L^p} = 0.
$$
By (2.2), and estimation (i) of Proposition 1, we obtain
\[
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi - \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq \left\| e^{-t(-\Delta)^{\beta/2}} (\varphi - \varphi_n) \right\|_{\exp L^p} + \left\| e^{-t(-\Delta)^{\beta/2}} \varphi_n - \varphi_n \right\|_{\exp L^p} + \left\| \varphi_n - \varphi \right\|_{\exp L^p} 
\]
\[
\leq \frac{1}{(\ln 2)^{1/p}} \left( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi_n - \varphi_n \right\|_p + \left\| e^{-t(-\Delta)^{\beta/2}} \varphi_n - \varphi_n \right\|_\infty \right) + 2 \left\| \varphi_n - \varphi \right\|_{\exp L^p}. 
\]
Since \( \varphi_n \in C_0^\infty (\mathbb{R}^n) \), using the fact that \( e^{-t(-\Delta)^{\beta/2}} \) is a strongly continuous semigroup on \( L^r(\mathbb{R}^n) \) \( (1 < r \leq \infty) \), we have
\[
\lim_{t \to 0} \left( \left\| e^{-t(-\Delta)^{\beta/2}} \varphi_n - \varphi_n \right\|_p + \left\| e^{-t(-\Delta)^{\beta/2}} \varphi_n - \varphi_n \right\|_\infty \right) = 0. 
\]
Hence
\[
\limsup_{t \to 0} \left\| e^{-t(-\Delta)^{\beta/2}} \varphi - \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq 2 \left\| \varphi_n - \varphi \right\|_{\exp L^p}, 
\]
for every \( n \in \mathbb{N} \). This finishes the proof of the proposition. \( \square \)

It is known that \( e^{-t(-\Delta)^{\beta/2}} \) is a \( C_0 \)-semigroup on \( L^p(\mathbb{R}^n) \). By Proposition 2, it is a \( C_0 \)-semigroup on \( \exp L^p_0(\mathbb{R}^n) \).

**Lemma 2.5** ([5], Lemma 4.1.5). Let \( X \) be a Banach space and \( g \in L^1(0, T; X) \), then \( \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} g(s) \, ds \in C([0, T]; X) \). Moreover
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} g(s) \, ds \right\|_{L^\infty(0, T; X)} \leq \|g\|_{L^1(0, T; X)}. 
\]

The following lemmas are essential for the proof of the global existence (Theorem 1.4).

**Lemma 2.6** ([11], Lemma 2.6). Let \( \lambda > 0, 1 \leq q < \infty \) and \( K > 0 \) be such that \( \lambda q K^p \leq 1 \). Assume that \( u \in \exp L^p(\mathbb{R}^n) \) satisfies
\[
\|u\|_{\exp L^p(\mathbb{R}^n)} \leq K, 
\]
then \( \exp \left( \frac{|u|^p}{\lambda p} \right) - 1 \in L^q(\mathbb{R}^n) \) and
\[
\left\| e^{\lambda |u|^p} - 1 \right\|_{L^r(\mathbb{R}^n)} \leq (\lambda q K^p)^{1/q}. 
\]

**Lemma 2.7**. Let \( p > 1, 0 < \beta \leq 2 \) be such that \( \beta < \frac{n(p-1)}{p} \). Then, for every \( r > \frac{n}{\beta} \), there exists \( C = C(n, p, \beta, r) \) such that
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} g(s) \, ds \right\|_{L^\infty(0, \infty; \exp L^p(\mathbb{R}^n))} \leq C \|g\|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))}, 
\]
for every \( g \in L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \).

**Proof.** By Proposition 1 (ii) with \( q = 1 \), we have
\[
\left\| e^{-(t-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq C \left( t^{-\frac{\beta}{p}} \left( \ln(t^{-\frac{\beta}{p}} + 1) \right)^{-1/p} \right) \|\varphi\|_1, \quad (2.11) 
\]
for all \( t > 0, \varphi \in L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \) \( (\|\varphi\|_{L^1 \cap L^r} = \|\varphi\|_{L^1} + \|\varphi\|_{L^r}) \), while by Proposition 1 (iii) with \( q = 1 \), we obtain
\[
\left\| e^{-(t-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq C \left( t^{-\frac{\beta}{p}} + 1 \right) \|\varphi\|_1, \quad (2.12) 
\]
Combining (2.11) and (2.12), we get
\[
\left\| e^{t(-\Delta)^{\beta/2}} \varphi \right\|_{\exp L^p(\mathbb{R}^n)} \leq \kappa(t) \left[ \| \varphi \|_r + \| g \|_1 \right],
\]
where
\[
\kappa(t) = \min \left\{ C \left( t^{-\frac{n}{\beta}} + 1 \right), C t^{-\frac{n}{\beta}} \left( \ln(t^{-\frac{n}{\beta}} + 1) \right)^{-1/p} \right\}.
\]
Due to the assumptions $\beta < \frac{n(p-1)}{p}$ and $r > \frac{n}{\beta}$, we see that $\kappa \in L^1(0, \infty)$. Thus, for $g \in L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$, we have
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} g(s) ds \right\|_{\exp L^p(\mathbb{R}^n)} \leq \int_0^t \left\| e^{-(t-s)(-\Delta)^{\beta/2}} g(s) \right\|_{\exp L^p(\mathbb{R}^n)} ds
\leq \int_0^t \kappa(t-s) \left( \| g(s) \|_{L^1(\mathbb{R}^n)} + \| g(s) \|_{L^r(\mathbb{R}^n)} \right) ds
\leq \| g \|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))} \int_0^\infty \kappa(s) ds,
\]
for every $t > 0$. This proves Lemma 2.7. \qed

We remark that $\frac{n(p-1)}{p}$ may not included in $(0, 2)$. So if $\frac{n(p-1)}{p} > 2$, we have $\frac{n(p-1)}{p} > \beta$, and this case is recovered by Lemma 2.7. If $\frac{n(p-1)}{p} \leq 2$, we have three cases to study: the case of $\beta < \frac{n(p-1)}{p}$ is done by Lemma 2.7, and the case $\beta > \frac{n(p-1)}{p}$ can be done separately without using any kind of an a priori estimate, so it remains to study the case of $\beta = \frac{n(p-1)}{p}$ where we have a similar result as in Lemma 2.7. For this, we need to introduce an appropriate Orlicz space. Let $L^\phi(\mathbb{R}^n)$ this space, with $\phi(u) = e^{|u|^p} - 1 - |u|^p$, associated with its Luxemburg norm. From the definition of $\| \cdot \|_{L^\phi}$, (2.4), and the standard inequality $e^{\beta s} - 1 \leq \theta(e^s - 1)$, $0 \leq \theta \leq 1$, $s \geq 0$, we can easily get
\[
C_1 \| u \|_{\exp L^p(\mathbb{R}^n)} \leq \| u \|_{L^p(\mathbb{R}^n)} + \| u \|_{L^{\phi}(\mathbb{R}^n)} \leq C_2 \| u \|_{\exp L^p(\mathbb{R}^n)},
\]
for some $C_1, C_2 > 0$.

Lemma 2.8. Let $p > 1$, $0 < \beta \leq 2$ be such that $\beta = \frac{n(p-1)}{p}$. Then, there exists $C = C(n, p)$ such that
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} g(s) ds \right\|_{L^\infty(0, \infty; L^\phi(\mathbb{R}^n))} \leq C \| g \|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-2n}}(\mathbb{R}^n))},
\]
for every $g \in L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-2n}}(\mathbb{R}^n))$.

Proof. On the one hand, by (2.9), we have
\[
\int_{\mathbb{R}^n} \phi \left( \frac{|e^{t(-\Delta)^{\beta/2}} \varphi|}{\lambda} \right) dx \leq \sum_{k=2}^{\infty} \frac{C p! t^{-\frac{p}{2}(1-\frac{n}{p})} k^{p}}{k! \lambda^{pk}} \| \varphi \|_1^{pk} = \sum_{k=2}^{\infty} \frac{C p! t^{-\frac{p}{2}(1-\frac{n}{p})} k^{p}}{k! \lambda^{pk}} \| \varphi \|_1^{pk}
\leq \sum_{k=2}^{\infty} \frac{C p! t^{-\frac{p}{2}(1-\frac{n}{p})} k^{p}}{k! \lambda^{pk}} \| \varphi \|_1^{pk}
\leq t^{\frac{p}{2}} \left( \exp \left( \frac{C t^{-\frac{n}{2}} \| \varphi \|_1}{\lambda} \right)^{2p} - 1 \right),
\]
for all $t > 0$, $\varphi \in L^1(\mathbb{R}^n)$, where we have used the fact that $e^{\lambda x^p} - 1 - |x|^p \leq e^{\xi|x|^{2p}} - 1$, for all $x \in \mathbb{R}$. As
\[
\frac{n}{p} \left( \exp \left( \frac{Ct^{-\frac{n}{p}} \|\varphi\|_1}{\lambda} \right) - 1 \right) \leq 1 \iff \lambda \geq C t^{-\frac{n}{p}} \left( \ln(t^{-\frac{n}{p}} + 1) \right)^{-1/2p} \|\varphi\|_1 ;
\]
hence,
\[
\left\{ \lambda > 0, \lambda \in [C t^{-\frac{n}{p}} \left( \ln(t^{-\frac{n}{p}} + 1) \right)^{-1/2p} \|\varphi\|_1 ; \infty[, \right. \right. \right.
\]
\[
\leq \left\{ \lambda > 0, \int_{\mathbb{R}^n} \phi \left( \frac{e^{-t(-\Delta)^{\beta/2}} \varphi}{\lambda} \right) \, dx \leq 1 \right\} ;
\]
whereupon
\[
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{p}} \left( \ln(t^{-\frac{n}{p}} + 1) \right)^{-1/2p} \|\varphi\|_1 ,
\] (2.14)
for all $t > 0$, $\varphi \in L^1(\mathbb{R}^n)$. On the other hand, from (2.9) and the embedding $L^{2p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow L_0^q(\mathbb{R}^n)$ (see Lemma 2.3), we have
\[
\left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{L^q(\mathbb{R}^n)} \leq \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{L^\infty(\mathbb{R}^n)} + \left\| e^{-t(-\Delta)^{\beta/2}} \varphi \right\|_{L^{2p}(\mathbb{R}^n)}
\]
\[
\leq Ct^{-\frac{n}{p}} \left( \frac{e^{-t}}{\lambda} - 0 \right) \|\varphi\|_{L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)} + \|\varphi\|_{L^{2p}(\mathbb{R}^n)}
\]
\[
= C t^{-\frac{n}{p}} \|\varphi\|_{L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)} + \|\varphi\|_{L^{2p}(\mathbb{R}^n)}
\]
\[
\leq C \left( t^{-\frac{n}{p}} + 1 \right) \left( \|\varphi\|_{L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)} + \|\varphi\|_{L^{2p}(\mathbb{R}^n)} \right) ,
\] (2.15)
for all $t > 0$, $\varphi \in L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)$, where we have used the fact that $\beta = \frac{n(p-1)}{p}$. Now, let $g \in L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n))$, we conclude from (2.14) and (2.15) that
\[
\left\| e^{-t(-\Delta)^{\beta/2}} g(t) \right\|_{L^q(\mathbb{R}^n)} \leq \kappa(t) \left\| g(t) \right\|_{L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)} ,
\]
for all $t > 0$, where
\[
\kappa(t) := \min \left\{ C \left( t^{-\frac{n}{p}} + 1 \right); C t^{-\frac{n}{p}} \left( \ln(t^{-\frac{n}{p}} + 1) \right)^{-1/2p} \right\} .
\]
We can easily check that $\kappa \in L^1(0, \infty)$. Therefore
\[
\left\| \int_0^t e^{-t(-s)(-\Delta)^{\beta/2}} g(s) \, ds \right\|_{L^q(\mathbb{R}^n)}
\]
\[
\leq \int_0^t \left\| e^{-t(-s)(-\Delta)^{\beta/2}} g(s) \right\|_{L^q(\mathbb{R}^n)} \, ds
\]
\[
\leq \int_0^t \kappa(t-s) \left\| g(s) \right\|_{L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n)} \, ds
\]
\[
\leq \left\| g \right\|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^{2p}(\mathbb{R}^n) \cap L^{\frac{2p}{p-\gamma}}(\mathbb{R}^n))} \int_0^\infty \kappa(s) \, ds ,
\]
for every $t > 0$. This proves Lemma 2.8.
Finally, the following proposition is needed for the local well-posedness result in the space $\exp L^p(\mathbb{R}^n)$.

**Proposition 3** ([11], Proposition 2.9). Let $1 \leq p < \infty$ and $u \in C([0, T]; \exp L^p(\mathbb{R}^n))$ for some $T > 0$. Then, for every $\alpha > 0$, it holds

$$e^{|u|^p} - 1) \in C([0, T]; L^r(\mathbb{R}^n)), \quad 1 \leq r < \infty.$$  

**Corollary 1** ([11], Corollary 2.13). Let $1 \leq p < \infty$ and $u \in C([0, T]; \exp L^p(\mathbb{R}^n))$ for some $T > 0$. Assume that $f$ satisfies (1.3). Then, for every $p \leq r < \infty$, it holds $f(u) \in C([0, T]; L^r(\mathbb{R}^n))$.

3. **Proof of Theorem 1.3.** In this section, we prove Theorem 1.3 i.e. the local existence and the uniqueness of a mild solution to (1.1) in $C([0, T]; \exp L^p(\mathbb{R}^n))$ for some $T > 0$. Throughout this section, we assume that the nonlinearity $f$ satisfies (1.3). In order to find the required solution, we will apply the Banach fixed-point theorem to the integral formulation (1.2), using a decomposition argument developed in [7] and used in [9, 10, 11]. The idea is to split the initial data $v_0 \in \exp L^p(\mathbb{R}^n)$, using the density of $C_0^\infty(\mathbb{R}^n)$, into a small part in $\exp L^p(\mathbb{R}^n)$ and a smooth one. Let $w_0 \in \exp L^p(\mathbb{R}^n)$. Then, for every $\varepsilon > 0$ there exists $v_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|w_0\|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon,$$

where $w_0 := v_0 - v_0$. Now, we split our problem (1.1) into the following two problems. The first one is the fractional semilinear heat equation with smooth initial data:

$$\begin{cases}
  w_t + (-\Delta)^{\beta/2}v = f(v), & t > 0, x \in \mathbb{R}^n, \\
  v(0) = v_0 \in C_0^\infty(\mathbb{R}^n), & x \in \mathbb{R}^n,
\end{cases}$$  

and the second one is a fractional semilinear heat equation with small initial data in $\exp L^p(\mathbb{R}^n)$:

$$\begin{cases}
  w_t + (-\Delta)^{\beta/2}w = f(w + v) - f(v), & t > 0, x \in \mathbb{R}^n, \\
  w(0) = w_0, & \|w_0\|_{\exp L^p} \leq \varepsilon, x \in \mathbb{R}^n.
\end{cases}$$

We notice that if $v$ is a mild solution of (3.1) and $w$ is a mild solution of (3.2), then $u = v + w$ is a solution of our problem (1.2), where the definition of the mild solutions for problems (3.1)-(3.2) are defined similarly as in definition 1.1. We now prove the local existence result concerning (3.1) and (3.2).

**Lemma 3.1.** Let $0 < \beta < 2$, $p > 1$ and $v_0 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exist a time $T = T(v_0) > 0$ and a mild solution $v \in C([0, T]; \exp L^p(\mathbb{R}^n)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^n))$ of (3.1).

**Lemma 3.2.** Let $0 < \beta < 2$, $p > 1$, and $w_0 \in \exp L^p(\mathbb{R}^n)$. Let $T > 0$ and $v \in L^\infty(0, T; L^\infty(\mathbb{R}^n))$ be given by Lemma 3.1. Then, for $\|w_0\|_{\exp L^p} \leq \varepsilon$, with $\varepsilon \ll 1$ small enough, there exist a time $\bar{T} = \bar{T}(w_0, \varepsilon, v) > 0$ and a mild solution $w \in C([0, T]; \exp L^p(\mathbb{R}^n))$ to problem (3.2).

**Proof of Lemma 3.1.** In order to use the Banach fixed-point theorem, we introduce the following complete metric space

$$Y_T := \left\{ v \in C([0, T]; \exp L^p(\mathbb{R}^n)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^n)) : \|v\|_{Y_T} \leq 2\|v_0\|_{L^p \cap L^\infty} \right\},$$
where \( \|v\|_{Y_T} := \|v\|_{L^\infty(0,T;L^p)} + \|v\|_{L^\infty(0,T;L^\infty)} \) and \( \|v_0\|_{L^p \cap L^\infty} := \|v_0\|_{L^p} + \|v_0\|_{L^\infty} \).

For \( v \in Y_T \), we define \( \Phi(v) \) by
\[
\Phi(v) := e^{-t(-\Delta)^{\beta/2}}v_0 + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}f(v(s)) \, ds.
\]

We will prove that if \( T > 0 \) is small enough, then, \( \Phi \) is a contraction from \( Y_T \) into itself.

- \( \Phi : Y_T \to Y_T \). Let \( v \in Y_T \). As \( v_0 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), then, by Lemma 2.2, we conclude that \( v_0 \in \exp L^p_0(\mathbb{R}^n) \). Then, using Proposition 2, \( e^{-t(-\Delta)^{\beta/2}}v_0 \in C([0,T];\exp L^p_0(\mathbb{R}^n)) \). Next, for \( q = p \) or \( q = \infty \), we have
\[
\|f(v)\|_{L^q} \leq Ce^{\lambda\|v\|_{L^p}}\|v\|_{L^q} \leq Ce^{\lambda\|v\|_{L^p}}(2\|v_0\|_{L^p \cap L^\infty}),
\]
which implies, using again Lemma 2.2, that \( f(v) \in \exp L^p_0(\mathbb{R}^n) \) and more precisely \( f(v) \in L^1(0,T;\exp L^p_0(\mathbb{R}^n)) \). It follows, by density and smoothing effect of the fractional semigroup \( e^{-t(-\Delta)^{\beta/2}} \) (Lemma 2.5), that
\[
\int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}f(v(s)) \, ds \in C([0,T];\exp L^p_0(\mathbb{R}^n)).
\]

So \( \Phi(v) \in C([0,T];\exp L^p_0(\mathbb{R}^n)) \). Moreover, using (2.10) and (3.3), we have
\[
\|\Phi(v)\|_{Y_T} \leq 2TCE^{\lambda\|v_0\|_{L^p}}(2\|v_0\|_{L^p \cap L^\infty})^p \leq 2\|v_0\|_{L^p \cap L^\infty},
\]
for \( T > 0 \) small enough, namely \( 4TCE^{\lambda\|v_0\|_{L^p}}(2\|v_0\|_{L^p \cap L^\infty})^p \leq 1 \). This proves that \( \Phi(v) \in Y_T \).

- \( \Phi \) is a contraction. Let \( v_1, v_2 \in Y_T \). For \( q = p \) or \( q = \infty \), we have
\[
\|f(v_1) - f(v_2)\|_{L^q} \leq 2C\|v_1 - v_2\|_{q} e^{\lambda\|v_1\|_{L^p} + \lambda\|v_2\|_{L^p}}
\leq 2C\|v_1 - v_2\|_{Y_T} e^{\lambda\|v_0\|_{L^p \cap L^\infty}}
\leq 2C\|v_1 - v_2\|_{Y_T} e^{\lambda\|v_0\|_{L^p \cap L^\infty}}.
\]

By (2.10), it holds
\[
\|\Phi(v_1) - \Phi(v_2)\|_{Y_T} \leq 2TCE^{\lambda\|v_0\|_{L^p \cap L^\infty}} \leq \frac{1}{2} \|v_1 - v_2\|_{Y_T}.
\]

This finishes the proof of Lemma 3.1. \( \square \)

**Proof of Lemma 3.2.** To prove Lemma 3.2, we need the following result.

**Lemma 3.3** ([11], Lemma 4.4). Let \( v \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \) for some \( T > 0 \). Let \( 1 < p \leq q < \infty \), and \( w_1, w_2 \in \exp L^p(\mathbb{R}^n) \) with \( \|w_1\|_{\exp L^p}, \|w_2\|_{\exp L^p} \leq M \) for sufficiently small \( M > 0 \) (namely \( 2^p\lambda q M^p \leq 1 \), where \( \lambda \) is given in (1.3)). Then, there exists a constant \( C = C(q) > 0 \) such that
\[
\|f(w_1 + v) - f(w_2 + v)\|_q \leq Ce^{2^{p-1}\lambda\|v\|_{L^\infty}}\|w_1 - w_2\|_{\exp L^p}.
\]

For \( \bar{T} > 0 \), we define
\[
W_{\bar{T}} := \left\{ w \in C([0,\bar{T}];\exp L^p_0(\mathbb{R}^n)); \|w\|_{L^\infty(0,\bar{T};\exp L^p_0)} \leq 2\varepsilon \right\},
\]
and we consider the map \( \Phi \) defined, for \( w \in W_{\bar{T}} \), by
\[
\Phi(w) := e^{-t(-\Delta)^{\beta/2}}w_0 + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}(f(w(s) + v(s)) - f(v(s))) \, ds.
\]

We will prove that if \( \varepsilon \) and \( \bar{T} > 0 \) are small enough, then, \( \Phi \) is a contraction from \( W_{\bar{T}} \) into itself.
\( \Phi \) is a contraction. Let \( w_1, w_2 \in W_T \). Using Lemma 2.2, i.e. the embedding \( L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^p_0(\mathbb{R}^n) \), we have
\[
\|\Phi(w_1) - \Phi(w_2)\|_{\exp L^p} \leq \frac{1}{(\ln 2)^{1/p}} (\|\Phi(w_1) - \Phi(w_2)\|_p + \|\Phi(w_1) - \Phi(w_2)\|_{L^\infty}) \quad (3.4)
\]

Let \( r > 0 \) be an auxiliary constant such that \( r > \max\{p, \frac{p}{2}\} \). Then
\[
\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty} \leq C \int_0^t (t-s)^{-\frac{p}{r}} \left\| f(w_1(s) + v(s)) - f(w_2(s) + v(s)) \right\| ds,
\]
thanks to the \( L^r - L^\infty \) estimate (2.9). Applying Lemma 3.3 with \( q = r \) and under the condition \( 2p\lambda r(2\varepsilon)^p \leq 1 \), we obtain
\[
\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty} \leq C e^{2p^{-1}\lambda v_0 \mu} \left( \int_0^t (t-s)^{-\frac{p}{r}} ds \right) \|w_1 - w_2\|_{L^\infty(0,\tilde{T};\exp L^p)} \leq C e^{2p^{-1}\lambda v_0 \mu} \tilde{T}^{-\frac{p}{r}} \|w_1 - w_2\|_{L^\infty(0,\tilde{T};\exp L^p)}.
\]

On the other hand, applying again the \( L^p - L^p \) estimate (2.10), and Lemma 3.3 with \( q = p \) under the condition \( 2p\lambda p(2\varepsilon)^p \leq 1 \), we obtain
\[
\|\Phi(w_1) - \Phi(w_2)\|_p \leq \int_0^t \left\| e^{-(t-s)(-\Delta)^{p/2}} (f(w_1(s) + v(s)) - f(w_2(s) + v(s))) \right\|_p ds
\]
\[
\leq \int_0^t \|f(w_1(s) + v(s)) - f(w_2(s) + v(s))\|_p ds
\]
\[
\leq C e^{2p^{-1}\lambda v_0 \mu} \int_0^t \|w_1 - w_2\|_{\exp L^p} ds
\]
\[
\leq C e^{2p^{-1}\lambda v_0 \mu} \tilde{T} \|w_1 - w_2\|_{L^\infty(0,\tilde{T};\exp L^p)}.
\]

Using (3.5) and (3.6) into (3.4), we infer, by choosing \( \varepsilon \ll 1 \) small enough, that
\[
\|\Phi(w_1) - \Phi(w_2)\|_{\exp L^p} \leq C e^{2p^{-1}\lambda v_0 \mu} \left( \tilde{T} + \tilde{T}^{-1} + \frac{\tilde{T}^{-1}}{r} \right) \|w_1 - w_2\|_{L^\infty(0,\tilde{T};\exp L^p)} \leq \frac{1}{2} \|w_1 - w_2\|_{L^\infty(0,\tilde{T};\exp L^p)},
\]

where \( \tilde{T} \ll 1 \) is chosen small enough such that \( C e^{2p^{-1}\lambda v_0 \mu} \left( \tilde{T} + \tilde{T}^{-1} + \frac{\tilde{T}^{-1}}{r} \right) \leq \frac{1}{2} \).

\( \Phi : W_T \to W_T \). Let \( w \in W_T \). As \( w_0 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), then by Lemma 2.2, we conclude that \( w_0 \in \exp L^p_0(\mathbb{R}^n) \). Then, using Proposition 2,
\[
e^{-t(-\Delta)^{p/2}} w_0 \in C([0,T];\exp L^p_0(\mathbb{R}^n)).
\]

Next, the estimates (3.5)-(3.6) with \( w_1 = w \) and \( w_2 = 0 \), under the condition \( 2p\lambda r(2\varepsilon)^p \leq 1 \), show that the nonlinear term satisfies
\[
\Phi(w) - e^{-t(-\Delta)^{p/2}} w_0 \in L^\infty(0,T;\exp L^p_0(\mathbb{R}^n)),
\]
thanks to the embedding \( L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^p_0(\mathbb{R}^n) \) (Lemma 2.2). By the standard smoothing effect of the fractional semigroup \( e^{t(-\Delta)^{p/2}} \) (Lemma 2.5), it follows that
\[
\Phi(w) - e^{-t(-\Delta)^{p/2}} w_0 \in C([0,T];\exp L^p_0(\mathbb{R}^n)).
\]
So \( \Phi(w) \in C([0,T];\exp L^p_0(\mathbb{R}^n)). \)

Moreover, using Proposition 1, and (3.7) with \( w_1 = w \) and \( w_2 = 0 \) for \( T \ll 1 \), we have
\[
\|\Phi(w)\|_{W_T} \leq \|w_0\|_{\exp L^p} + \frac{1}{2} \|w\|_{L^\infty(0,\tilde{T};\exp L^p)} \leq \varepsilon + \frac{1}{2} (2\varepsilon) = 2\varepsilon.
\]
We conclude that $u$ order. Let us suppose that $\varepsilon$ initial data $C$ for any $\parallel u\parallel_{L^p(\mathbb{R}^n)}$ are two mild solutions of (1.1) for some $t_0$ such that $\parallel u\parallel_{L^\infty(0, T; L^p(\mathbb{R}^n))}$ and satisfy (1.2) on $(0, T - t_0]$ with $u(t_0) = \tilde{v}(t_0) = u(t_0)$. Let

$$t_0 = \sup\{t \in [0, T] \text{ such that } u(s) = v(s) \text{ for every } s \in [0, t]\}.$$ 

Let us suppose that $0 \leq t_0 < T$. Since $u(t)$ and $v(t)$ are continuous in time, we have $u(t_0) = v(t_0)$. Let us denote $\tilde{u}(t) := u(t + t_0)$ and $\tilde{v}(t) := v(t + t_0)$. Then $\tilde{u}, \tilde{v} \in C([0, T - t_0]; \exp L^p_0(\mathbb{R}^n))$ and satisfy (1.2) on $(0, T - t_0]$ with $\tilde{u}(0) = \tilde{v}(0) = u(t_0)$. We will prove that there exists a short positive time $0 < \tilde{t} \leq T - t_0$ such that

$$\sup_{0 < \tilde{t} < \tilde{t}} \parallel \tilde{u}(t) - \tilde{v}(t)\parallel_{\exp L^p} \leq C(\tilde{t}) \sup_{0 < \tilde{t} < \tilde{t}} \parallel \tilde{u}(t) - \tilde{v}(t)\parallel_{\exp L^p}, \tag{3.8}$$

for some $C(\tilde{t}) < 1$, and so $\tilde{u}(t) = \tilde{v}(t)$ for any $t \in [0, \tilde{t}]$. Therefore $u(t + t_0) = v(t + t_0)$ for any $t \in [0, \tilde{t}]$ which is a contradiction with the definition of $t_0$. In order to establish inequality (3.8), we control both the $L^p$-norm and $L^\infty$-norm of $\tilde{u} - \tilde{v}$. Using $L^p - L^p$ estimate (2.10), we obtain

$$\parallel \tilde{u}(t) - \tilde{v}(t)\parallel_p \leq \int_0^t \parallel e^{-(t-s)(-\Delta)^{\gamma/2}}(f(\tilde{u}(s)) - f(\tilde{v}(s)))\parallel_p \, ds \leq \int_0^t \parallel (f(\tilde{u}(s)) - f(\tilde{v}(s)))\parallel_p \, ds.$$ 

By (1.3) and Hölder’s inequality, we get

$$\parallel \tilde{u}(t) - \tilde{v}(t)\parallel_p \leq C \int_0^t \parallel (\tilde{u}(s) - \tilde{v}(s))(e^{\lambda|\tilde{u}|^p} + e^{\lambda|\tilde{v}|^p})\parallel_p \, ds \leq 2 \int_0^t \parallel \tilde{u}(s) - \tilde{v}(s)\parallel_p \, ds + \int_0^t \parallel (\tilde{u}(s) - \tilde{v}(s))(e^{\lambda|\tilde{u}|^p} - 1 + e^{\lambda|\tilde{v}|^p} - 1)\parallel_p \, ds \leq 2 \int_0^t \parallel \tilde{u}(s) - \tilde{v}(s)\parallel_p \, ds + \int_0^t \parallel (e^{\lambda|\tilde{u}|^p} - 1 + e^{\lambda|\tilde{v}|^p} - 1)\parallel_r \, ds,$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. Thanks to Lemma 2.4 and $q \geq p$, we infer that

$$\parallel \tilde{u}(t) - \tilde{v}(t)\parallel_p \leq C \sup_{0 < s < t} \parallel \tilde{u}(s) - \tilde{v}(s)\parallel_{\exp L^p} \int_0^t \parallel (e^{\lambda|\tilde{u}|^p} - 1 + e^{\lambda|\tilde{v}|^p} - 1)\parallel_r \, ds.$$
Moreover, using Proposition 3, we obtain
\[
\sup_{0 < s < T - t_0} \left\| (e^{\lambda \tilde{u}^p} - 1) + (e^{\lambda \bar{v}^p} - 1) \right\|_r \\
\leq \sup_{0 < s < T - t_0} \left( \left( \int_{\mathbb{R}^n} (e^{\lambda \tilde{u}^p} - 1) \, dx \right)^{1/r} + \left( \int_{\mathbb{R}^n} (e^{\lambda \bar{v}^p} - 1) \, dx \right)^{1/r} \right) \\
\leq C(T, t_0, \bar{u}, \bar{v}) < \infty.
\]
(3.9)

Consequently,
\[
\sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{L^p} \leq C(T, t_0, \bar{u}, \bar{v}) t \sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p}.
\]
(3.10)

In a similar way, using \( L^r - L^\infty \) estimate (2.9), we obtain
\[
\| \tilde{u}(t) - \bar{v}(t) \|_{\infty} \leq \int_0^t \left\| e^{-(t-s)(-\lambda)^{b/2}} (f(\tilde{u}(s)) - f(\bar{v}(s))) \right\|_{\infty} \, ds \\
\leq C \int_0^t (t-s) \left\| \tilde{u}(s) - \bar{v}(s) \right\|_{\exp L^p} \, ds,
\]
for some \( r > \max\{p, \frac{q}{\beta}\} \). By (1.3) and Hölder’s inequality, we get
\[
\| \tilde{u}(t) - \bar{v}(t) \|_{\infty} \leq C \int_0^t (t-s) \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p} \, ds \\
\leq 2 \int_0^t (t-s)^{-\frac{r}{\beta}} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p} \, ds \\
+ \int_0^t (t-s)^{-\frac{r}{\beta}} \| (\tilde{u}(s) - \bar{v}(s)) [(e^{\lambda \tilde{u}^p} - 1) + (e^{\lambda \bar{v}^p} - 1)] \|_{\exp L^p} \, ds \\
\leq 2 \int_0^t (t-s)^{-\frac{r}{\beta}} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p} \, ds \\
+ \int_0^t (t-s)^{-\frac{r}{\beta}} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p} \| (e^{\lambda \tilde{u}^p} - 1) + (e^{\lambda \bar{v}^p} - 1) \|_{\exp L^p},
\]
where \( \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = 1 \). Since \( \tilde{q}, \tilde{r} > r > p \), one can apply an estimate similar to (3.9) via Lemma 2.4 and Proposition 3, and obtain that
\[
\sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{L^\infty} \leq C(T, t_0, \bar{u}, \bar{v}) t^{1-\frac{r}{\tilde{r}}} \sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p}.
\]
(3.11)

Finally, the two inequalities (3.10) and (3.11) with the embedding relation \( L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L_0^p(\mathbb{R}^n) \) (Lemma 2.2) imply
\[
\sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p} \leq C(T, t_0, \bar{u}, \bar{v})(t + t^{1-\frac{r}{\tilde{r}}}) \sup_{0 < s < t} \| \tilde{u}(s) - \bar{v}(s) \|_{\exp L^p},
\]
and for \( t \) small enough, we obtain the desired estimate (3.8).

**Remark 2.** The solution in Theorem 1.3 belongs to \( L^\infty_{loc}(0, T; L^\infty(\mathbb{R}^n)) \). Indeed, let \( u \in C([0, T]; \exp L_0^p(\mathbb{R}^n)) \) be a mild solution of (1.1) i.e. a solution of the integral equation (1.2). Using \( L^p - L^\infty \) estimate (2.9) and Lemma 2.4, we get
\[
\| e^{-(t-(\lambda)^{b/2}} u_0 \|_{\infty} \leq C t^{-\frac{r}{\tilde{r}}} \| u_0 \|_p \leq C t^{-\frac{r}{\tilde{r}}} \| u_0 \|_{\exp L^p},
\]
for $0 < t < T$. Hence $e^{-t(-\Delta)^{\beta/2}}u_0 \in L^\infty(\mathbb{R}^n)$ for $0 < t < T$. Thus it remains to estimate the nonlinear term. Fix $r > \max\{p, \frac{n}{\beta}\}$, using $L^r-L^\infty$ estimate (2.9), we get
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_\infty \leq \int_0^t \left\| e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_\infty ds
\]
\[
\leq \int_0^t (t-s)^{-\frac{\beta}{2r}} \left\| f(u(s)) \right\|_r ds
\]
\[
\leq C t^{1-\frac{\beta}{2r}} \sup_{0 \leq t \leq T} \left\| f(u(t)) \right\|_r < \infty,
\]
where we have used Corollary 1. This shows that $u \in L^\infty_{loc}(0, T; L^\infty(\mathbb{R}^n))$. In particular, if $f \in C^1(\mathbb{R}^n)$, the solution $u \in C([0, T]; \exp L^p_0(\mathbb{R}^n)) \cap L^\infty_{loc}(0, T; L^\infty(\mathbb{R}^n))$ satisfies (1.1) in the classical sense, i.e. $C^1$ in time $t \in (0, T)$ and $C^2$ in space $\mathbb{R}^n$.

**Remark 3.** Using the uniqueness, the constructed solution $u$ of (1.1) can be extended to a maximal interval $[0, T_{\max})$ by well known argument (see cf. Cazenave et Haraux [5]) where

$T_{\max} := \sup \{ T > 0; \text{there exist a mild solution } u \in C([0, T]; \exp L^p_0(\mathbb{R}^n)) \text{ to (1.1)} \}$

\[ \leq +\infty. \]

Moreover, if $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \| u(t) \|_{L^p \cap L^\infty(\mathbb{R}^n)} = \infty$.

4. **Proof of Theorem 1.4.**

4.1. **Proof of global existence in Theorem 1.4: case of $\beta < \frac{n(p-1)}{p}$.** In this subsection, we prove the global existence of solution in Theorem 1.4 in the case of $\beta < \frac{n(p-1)}{p}$. We will use the fixed-point theorem. Let us first define the following space

$E_\varepsilon = \left\{ u \in L^\infty(0, \infty; \exp L^p(\mathbb{R}^n)); \| u \|_{L^\infty(0, \infty; \exp L^p(\mathbb{R}^n))} \leq 2\varepsilon \right\},$

where $\varepsilon > 0$ is a positive constant, small enough, that will be chosen later such that $\| u_0 \|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon$. For $u \in E_\varepsilon$, we define $\Phi(u)$ by

$\Phi(u) := e^{-t(-\Delta)^{\beta/2}}u_0 + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds.$

Our goal is to prove that $\Phi : E_\varepsilon \to E_\varepsilon$ is a contraction map.

- $\Phi : E_\varepsilon \to E_\varepsilon$. Let $u \in E_\varepsilon$, we have

$\| \Phi(u) \|_{\exp L^p(\mathbb{R}^n)} \leq \left\| e^{-t(-\Delta)^{\beta/2}}u_0 \right\|_{\exp L^p(\mathbb{R}^n)} + \left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{\exp L^p(\mathbb{R}^n)}$

\[ \leq \| u_0 \|_{\exp L^p(\mathbb{R}^n)} + C \| f(u) \|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))} \]

\[ \leq \varepsilon + C \| f(u) \|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))}, \]

for every $r > \frac{n}{\beta} > 1$, where we have used Proposition 1 and Lemma 2.7. It remains to estimate $f(u)$ in $L^q(\mathbb{R}^n)$ for $q = 1, r$. From the assumption (1.4), we see

$|f(u)| \leq C |u|^m + C |u|^m \left( e^{|u|^p} - 1 \right) + C |u|^m,$

then, by Hölder’s inequality, we obtain

$\| f(u) \|_{L^q(\mathbb{R}^n)} \leq C \| u \|_{L^2m(\mathbb{R}^n)}^m \left( e^{|u|^p} - 1 \right)_{L^2m(\mathbb{R}^n)} + C \| u \|_{L^2m(\mathbb{R}^n)}^m \left( e^{|u|^p} - 1 \right)_{L^2m(\mathbb{R}^n)} + C \| u \|_{\exp L^p(\mathbb{R}^n)}^m,$
where we have used Lemma 2.4 and $m \geq p$. Next, using Lemma 2.6 and the fact that $u \in E_\varepsilon$, we have
\[
\|f(u)\|_{L^q(\mathbb{R}^n)} \leq C(2\varepsilon)^m \left( 1 + (2\lambda q(2\varepsilon)^p)^{1/q} \right) \leq C(2\varepsilon)^m \left( 1 + (2\lambda q(2\varepsilon)^p)^{1/r} \right).  
\] (4.1)
This implies, by choosing $\varepsilon$ small enough, that
\[
\|\Phi(u)\|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon + C(2\varepsilon)^m \left( 1 + (2\lambda q(2\varepsilon)^p)^{1/r} \right) \leq 2\varepsilon,
\]
i.e. $\Phi(u) \in E_\varepsilon$.

- **$\Phi$ is a contraction.** Let $u, v \in E_\varepsilon$, we have
\[
\|\Phi(u) - \Phi(v)\|_{\exp L^p(\mathbb{R}^n)} = \left\| \frac{1}{\varepsilon} \int_0^\varepsilon e^{-(t-s)(-\Delta)^{\beta/2}} (f(u(s)) - f(v(s))) \, ds \right\|_{\exp L^p(\mathbb{R}^n)}
\leq C \|f(u) - f(v)\|_{L^\infty((0,\varepsilon); L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))},
\]
for every $r > \frac{n}{\beta} > 1$, where we have used Lemma 2.7. To estimate $f(u) - f(v)$ in $L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, let $q = 1, r$. We see, using assumption (1.4), that
\[
|f(u) - f(v)| \leq C |u - v| \left( |u|^{m-1} e^{\lambda|u|^p} + |v|^{m-1} e^{\lambda|v|^p} \right)
\leq C |u - v| \left( |u|^{m-1} (e^{|u|^p} - 1) + |v|^{m-1} (e^{|v|^p} - 1) \right) + C |u - v| (|u|^{m-1} + |v|^{m-1}),
\]
then, by Hölder’s inequality, we obtain
\[
C \|f(u) - f(v)\|_{L^q(\mathbb{R}^n)} \leq I + II,
\]
where
\[
I := C \|u - v\|_{L^{mq}(\mathbb{R}^n)} \left( \|u|^{m-1} (e^{|u|^p} - 1)\|_{L^{mq}(\mathbb{R}^n)} + \|v|^{m-1} (e^{|v|^p} - 1)\|_{L^{mq}(\mathbb{R}^n)} \right),
\]
and
\[
II := C \|u - v\|_{L^{mq}(\mathbb{R}^n)} \left( \|u|^{m-1} + |v|^{m-1}\|_{L^{mq}(\mathbb{R}^n)} \right).
\]
Using again Hölder’s inequality, Lemma 2.4, and $m \geq p$, we get
\[
I \leq C \|u - v\|_{\exp L^p(\mathbb{R}^n)} \times \left( \|u|^{m-1} (e^{|u|^p} - 1)\|_{L^{mq}(\mathbb{R}^n)} + \|v|^{m-1} (e^{|v|^p} - 1)\|_{L^{mq}(\mathbb{R}^n)} \right)\leq C \|u - v\|_{\exp L^p(\mathbb{R}^n)} \times \left( \|u|^{m-1} \|e^{|u|^p} - 1\|_{L^{2mq}(\mathbb{R}^n)} \right) + \|v|^{m-1} \|e^{|v|^p} - 1\|_{L^{2mq}(\mathbb{R}^n)} \right) \leq C \|u - v\|_{\exp L^p(\mathbb{R}^n)} \times \left( \|u|^{m-1} \|e^{|u|^p} - 1\|_{L^{2mq}(\mathbb{R}^n)} \right) + \|v|^{m-1} \|e^{|v|^p} - 1\|_{L^{2mq}(\mathbb{R}^n)} \right)\)
\]
Then, using Lemma 2.6 and the fact that $u, v \in E_\varepsilon$, we have
\[
I \leq C^2 \varepsilon^{m-1} \left( \frac{2\lambda m}{m - 1} (2\varepsilon)^p \right)^{\frac{m-1}{2mq}} \|u - v\|_{E_\varepsilon} \leq \frac{1}{8} \|u - v\|_{E_\varepsilon},
\]
for $\varepsilon > 0$ small enough. Similarly,
\[
II \leq C \|u - v\|_{\exp L^p(\mathbb{R}^n)} \left( \|u|^{m-1} \|L^{mq}(\mathbb{R}^n) + \|v|^{m-1} \|L^{mq}(\mathbb{R}^n) \right)
\[ \|u - v\|_{L^p(\mathbb{R}^n)} \leq C \left( \|u\|_{L^p(\mathbb{R}^n)}^{m-1} + \|v\|_{L^p(\mathbb{R}^n)}^{m-1} \right) \]
\[ \leq C 2^{m} \|u - v\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{8} \|u - v\|_{E_\varepsilon}, \]
for \( \varepsilon > 0 \) small enough. We conclude that
\[ C \|f(u) - f(v)\|_{L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)} \leq 2(I + II). \]
Hence,
\[ \|\Phi(u) - \Phi(v)\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{2} \|u - v\|_{E_\varepsilon}. \]
This completes the proof of the existence of global solution in Theorem 1.4 in the case of \( \beta < \frac{n(p-1)}{p} \).

To obtain the decay estimate (1.6), we follow the same calculation as in the part of contraction mapping in the Subsection 4.2 below where we consider, instead of the space \( E_\varepsilon \), the following complete metric space
\[ \left\{ u \in L^\infty(0, \infty; \exp L^p(\mathbb{R}^n)) ; \sup_{t > 0} t^\sigma u(t) \|_{L^q(\mathbb{R}^n)} + \| u \|_{L^\infty(0, \infty; \exp L^p(\mathbb{R}^n))} \leq M \varepsilon \right\}, \]
endowed by the metric \( d \) defined by \( d(u, v) := \sup_{t > 0} t^\sigma u(t) - v(t) \|_{L^q(\mathbb{R}^n)} \), for certain large constant \( M > 0 \), where \( 0 < \varepsilon << 1 \) is a positive constant, small enough, that will be chosen later such that \( \| u_0 \|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon \). The new parameters \( \sigma \) and \( q \) are chosen as follows:
\[ \sigma = \frac{1}{m-1} - \frac{n}{\beta q} > 0, \quad \text{and} \quad \frac{n(m-1)}{\beta} < q < \frac{n(m-1)}{\beta} \left( \frac{1}{2} - m \right). \]

### 4.2. Proof of global existence in Theorem 1.4: case of \( \beta \geq \frac{n(p-1)}{p} \)

This subsection is devoted to prove the existence of global solution in Theorem 1.4 in the case of \( \beta \geq \frac{n(p-1)}{p} \) by using same ideas as in [11] together with Lemma 2.8.

As the last section, we will use a contraction mapping argument in an appropriate complete space. Let us define
\[ B_\varepsilon = \left\{ u \in L^\infty(0, \infty; \exp L^p(\mathbb{R}^n)) ; \sup_{t > 0} t^\sigma u(t) \|_{L^q(\mathbb{R}^n)} + \| u \|_{L^\infty(0, \infty; \exp L^p(\mathbb{R}^n))} \leq M \varepsilon \right\}, \]
for certain large constant \( M > 0 \), where \( 0 < \varepsilon << 1 \) is a positive constant, small enough, that will be chosen later such that \( \| u_0 \|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon \). Using Proposition 2.2 in [11], we can check that \( B_\varepsilon \) is a complete metric space with the distance
\[ d(u, v) := \sup_{t > 0} t^\sigma \| u(t) - v(t) \|_{L^q(\mathbb{R}^n)}. \]

For \( u \in B_\varepsilon \), we define, as above, \( \Phi(u) \) by
\[ \Phi(u) := e^{-t(-\Delta)^{\beta/2}} u_0 + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds. \]

- \( \Phi : B_\varepsilon \to B_\varepsilon \). Let \( u \in B_\varepsilon \). By Proposition 1, we have
\[ \| e^{-t(-\Delta)^{\beta/2}} u_0 \|_{L^p(\mathbb{R}^n)} \leq \| u_0 \|_{\exp L^p(\mathbb{R}^n)} \leq \varepsilon. \]

Moreover, by choosing \( \sigma = \frac{n}{\beta} \left( \frac{1}{m-1} - \frac{1}{q} \right) = \frac{1}{m-1} - \frac{n}{\beta q} > 0 \), for \( q > \frac{n(m-1)}{\beta} \geq p \), and using Lemma 2.4, we get
\[ t^\sigma \| e^{-t(-\Delta)^{\beta/2}} u_0 \|_{L^q(\mathbb{R}^n)} \leq C t^\sigma t^{\frac{n}{\beta} \left( \frac{1}{m-1} - \frac{1}{q} \right)} \| u_0 \|_{L^{\frac{n(m-1)}{\beta}}(\mathbb{R}^n)} \]
\[ = C \| u_0 \|_{L^{\frac{n(m-1)}{\beta}}(\mathbb{R}^n)} \leq C \| u_0 \|_{\exp L^p(\mathbb{R}^n)} \leq C \varepsilon. \]
To estimate the second term in $\Phi(u)$ on exp $L^p(\mathbb{R}^n)$, we start to study the case of $\beta = \frac{n(p-1)}{p}$ by remembering (see (2.13)) that

$$C_1\|u\|_{\exp L^p(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^q(\mathbb{R}^n)} \leq C_2\|u\|_{\exp L^p(\mathbb{R}^n)},$$

for some $C_1, C_2 > 0$, where $\phi(u) = e^{|u|^p} - 1 - |u|^p$. Therefore, it is enough to prove the two following inequalities:

$$\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^\infty(0,\infty; L^p(\mathbb{R}^n))} = O(\varepsilon), \quad (4.2)$$

and

$$\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^\infty(0,\infty; L^q(\mathbb{R}^n))} = O(\varepsilon). \quad (4.3)$$

We start to prove (4.2). As

$$|f(u)| \leq C|u|^m e^{|u|^{\rho}} = C|u|^m \sum_{k=0}^\infty \frac{\lambda_k^k}{k^l}|u|^k = C|u| \sum_{k=0}^\infty \frac{\lambda_k^k}{k^l}|u|^{k+p-1},$$

we have

$$\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^p(\mathbb{R}^n)} \leq C \int_0^t (t-s)^{-\frac{n}{p}(\frac{1}{r} - \frac{1}{q})} \|f(u(s))\|_{L^r(\mathbb{R}^n)} \, ds$$

$$\leq C \sum_{k=0}^\infty \frac{\lambda_k^k}{k^l} \int_0^t (t-s)^{-\frac{n}{p}(\frac{1}{r} - \frac{1}{q})} \|u(s)\|_{L^p(\mathbb{R}^n)} \|u(s)\|^{k+p-1} \, ds,$$

where we have used (2.9) and Hölder’s inequality, with $1 \leq r \leq p$, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{a}$. Then, using Hölder’s interpolation inequality and Lemma 2.4, we have

$$\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^p(\mathbb{R}^n)} \leq C \sum_{k=0}^\infty \frac{\lambda_k^k}{k^l} \left[ \frac{\Gamma \left( \frac{p}{p} + 1 \right) \left( \frac{k+p-1}{p} \right)^{\frac{k+p-1}{p}}}{\frac{k+p-1}{p}} \int_0^t (t-s)^{-\frac{n}{p}(\frac{1}{r} - \frac{1}{q})} \|u(s)\|_{L^p(\mathbb{R}^n)} \|u(s)\|^{k+p-1} \, ds \right]$$

$$\leq C \sum_{k=0}^\infty \frac{\lambda_k^k}{k^l} \left[ \frac{\Gamma \left( \frac{p}{p} + 1 \right) \left( \frac{k+p-1}{p} \right)^{\frac{k+p-1}{p}}}{\frac{k+p-1}{p}} \int_0^t (t-s)^{-\frac{n}{p}(\frac{1}{r} - \frac{1}{q})} \|u(s)\|_{L^p(\mathbb{R}^n)} \|u(s)\|^{k+p-1} \, ds \right],$$

where

$$\frac{1}{a(kp + m - 1)} = \frac{\theta}{q} + \frac{1 - \theta}{p}, \quad 0 \leq \theta \leq 1, \quad \text{and } p \leq \rho < \infty.$$
By using the fact that $u \in B_{\varepsilon}$, we get
\[
\left\| \int_{0}^{t} e^{-\Delta(\Delta^{\beta}/2)} f(u(s)) \, ds \right\|_{L^{p}(\mathbb{R}^{n})} \leq C \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \left( \Gamma \left( \frac{\rho}{p} + 1 \right) \right)^{(kp+m-1)(1-\theta)} (M\varepsilon)^{kp+m} \cdot \int_{0}^{t} (t-s)^{-\frac{p}{\rho}\left(\frac{1}{r}-\frac{1}{p}\right)} s^{-\sigma(kp+m-1)\theta} \, ds
\]
\[
= C \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \left( \Gamma \left( \frac{\rho}{p} + 1 \right) \right)^{(kp+m-1)(1-\theta)} (M\varepsilon)^{kp+m} \cdot \int_{0}^{1} (1-s)^{-\frac{p}{\rho}\left(\frac{1}{r}-\frac{1}{p}\right)} s^{-\sigma(kp+m-1)\theta} \, ds
\]
\[
= C \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \left( \Gamma \left( \frac{\rho}{p} + 1 \right) \right)^{(kp+m-1)(1-\theta)} \cdot (M\varepsilon)^{kp+m} \mathcal{B} \left( 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) ; 1 - \sigma(kp + m - 1)\theta \right), \quad (4.4)
\]
where $\mathcal{B}$ is the beta function, under the following conditions:
\[
\frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) < 1, \quad \sigma(kp + m - 1)\theta < 1, \quad \text{and} \quad 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) - \sigma(kp + m - 1)\theta = 0.
\]
It remains to prove the existence of $\theta = \theta_{k}$, $\rho = \rho_{k}$, $k \geq 0$, and $a$. As $q > \frac{(m-1)p}{p-1}$, one can choose
\[
\frac{1}{\sigma(kp+m-1)\theta} < \theta_{k} = \frac{1}{pk+m-1} \min(m-1, \frac{1}{\sigma}) , \quad \text{and as} \quad \sigma = \frac{1}{m-1} - \frac{n}{\beta q} < \frac{1}{m-1};
\]
it follows that $\theta_{k}$ is chosen by
\[
\frac{1 - \frac{n(p-1)}{p^{2}}}{\sigma(pk + m - 1)} < \theta_{k} < \frac{m - 1}{pk + m - 1}.
\]
We note that the lower bound of $\theta_{k}$ is just to be compatible with the condition that $r > 1$. For the choice of $\rho_{k}$, we explain slightly the steps; we need the condition
\[
1 - \frac{\beta}{\frac{1}{r} - \frac{1}{p}} - \sigma(kp+m-1)\theta_{k} = 0, \quad \text{and as} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{\sigma}, \quad \text{so} \quad 1 - \frac{n}{\beta r} - \sigma(kp+m-1)\theta_{k} = 0.
\]
Then, using the fact that $\frac{1}{a(kp+m-1)} = \frac{\theta_{k}}{q} + \frac{1 - \theta_{k}}{\rho_{k}}$ and $\sigma = \frac{1}{m-1} - \frac{n}{\beta q}$, we conclude that $\rho = \rho_{k}$ is chosen such that
\[
\frac{1 - \theta_{k}}{\rho_{k}} = \frac{\beta}{n(kp + m - 1)} - \frac{\beta\theta_{k}}{n(m-1)}.
\]
We note that $\frac{1 - \theta_{k}}{\rho_{k}} \leq \frac{\beta}{n(m-1)} - \frac{\beta\theta_{k}}{n(m-1)} = \frac{\beta(1 - \theta_{k})}{n(m-1)}$, which implies that $\rho_{k} \geq \frac{n(m-1)}{\beta}$, $p$. Finally, we choose $a > 0$ such that
\[
\frac{1}{a(kp + m - 1)} = \frac{\theta_{k}}{q} + \frac{1 - \theta_{k}}{\rho_{k}}.
\]
Moreover, for the choice of parameters,
\[
\mathcal{B} \left( 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) ; 1 - \sigma(kp + m - 1)\theta \right) = \frac{\Gamma \left( 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) \right) \Gamma \left( \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right) \right)}{\Gamma(1)} \leq C, \quad (4.5)
\]
where we have used the fact that $\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, for every $x, y > 0$. We notice also that $\theta_k \to 0$, $\rho_k \to \infty$ as $k \to \infty$, then
\[
\frac{(kp + m - 1)(1 - \theta_k)}{p \rho_k} (1 + \rho_k) \leq k, \quad \text{for all } k \geq 1.
\]
This implies, together with the property $\Gamma(x + 1) \leq C x^{x+\frac{1}{2}}$, for all $x \geq 1$, that
\[
\left( \Gamma \left( \frac{p_k}{p} + 1 \right) \right)^{(kp+m-1)(1-\theta_k)} \leq C^k k!.
\]
Combining (4.4), (4.5) and (4.6), we obtain
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{L^p(\mathbb{R}^n)} \leq C \sum_{k=0}^{\infty} (C \lambda)^k (M\varepsilon)^{kp+m} \leq C(M\varepsilon)^m,
\]
for $\varepsilon$ small enough. This proves (4.2). Next, we prove (4.3). Using the fact that $\beta = \frac{n(p-1)}{p}$ and Lemma 2.8, we have
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{L^\infty(0, \infty; L^s(\mathbb{R}^n))} \leq C \|f(u(s))\|_{L^\infty(0, \infty; L^s(\mathbb{R}^n))} \leq C \|f(u(s))\|_{L^\infty(0, \infty; L^s(\mathbb{R}^n))} \cap L^{\frac{2p}{2-p}}(\mathbb{R}^n)}.
\]
As
\[
|f(u)| \leq C|u|^m e^{\lambda|u|^p} = C|u|^m \left( e^{\lambda|u|^p} - 1 \right) + C|u|^m,
\]
so, using $m \geq p$ and a similar calculation as in the case of $\beta < \frac{n(p-1)}{p}$ (see (4.1)), we conclude that
\[
\|f(u(t))\|_{L^r(\mathbb{R}^n)} \leq C(M\varepsilon)^m,
\]
for $r = 1, 2p, \frac{2p}{2-p} \geq 1$, and all $t > 0$. This proves (4.3).

To estimate the second term in $\Phi(u)$ on $\exp L^p(\mathbb{R}^n)$ in the case of $\beta > \frac{n(p-1)}{p}$, let $b > 0$ be the positive number satisfying $b = 2 \ln(b + 1)$, then we can check that
\[
\left( \ln \left( (t-s)^{-\frac{n}{\beta}} + 1 \right) \right)^{-1/p} \leq 2^{1/p} (t-s)^{n/\beta}, \quad \text{for } 0 \leq s \leq t - b^{-\beta/n}.
\]
If $t \leq b^{-\beta/n}$, similarly to (2.12), we have
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{\exp L^p} \leq \int_0^t \left\| e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{L^p} ds
\]
\[
\leq \int_0^t \left( C(t-s)^{-\frac{n}{\beta}} + 1 \right) (\|f(u(s))\|_r + \|f(u(s))\|_1) ds,
\]
for any $r \geq 1$. Let $r = \frac{p}{p-1} > 1$, we get
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{\exp L^p} \leq \int_0^t \left( C(t-s)^{-\frac{n(p-1)}{p}} + 1 \right) \left( \|f(u(s))\|_{\frac{p}{p-1}} + \|f(u(s))\|_1 \right) ds
\]
\[
\leq \|f(u)\|_{L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^{\frac{p}{p-1}}(\mathbb{R}^n))} \int_0^t \left( C(t-s)^{-\frac{n(p-1)}{p}} + 1 \right) ds
\]
Similarly to (4.8), using \( \beta > \frac{n(p-1)}{p} \), we conclude that

\[
\| f(u(t) \|_{L^r(\mathbb{R}^n)} \leq C(M\varepsilon)^m,
\]

for \( r = 1, \frac{p}{p-1} \geq 1 \), i.e.

\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{\exp L^p} = O(\varepsilon).
\]

If \( t > b^{-\beta/n} \), we have

\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{\exp L^p} \leq \int_0^t \left\| e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{\exp L^p} \, ds
\]

\[
= \left\| e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{\exp L^p} \, ds + \int_{t-b^{-\beta/n}}^t \left\| e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \right\|_{\exp L^p} \, ds
\]

\[
= I + II.
\]

Similarly to (4.8), using \( \beta > \frac{n(p-1)}{p} \) and \( m > p \), we have

\[
II \leq \left\| f(u) \right\|_{L^\infty(0,\infty;L^1(\mathbb{R}^n))} \int_{t-b^{-\beta/n}}^t (C(t-s) \frac{n(p-1)}{np} + 1) \, ds
\]

\[
= \left\| f(u) \right\|_{L^\infty(0,\infty;L^1(\mathbb{R}^n))} \int_{0}^{b^{-\beta/n}} \left( C s^{\frac{n(p-1)}{np}} + 1 \right) \, ds \leq C(M\varepsilon)^m.
\]

On the other hand, using Proposition 1 (ii) and (4.7), we have

\[
I \leq C \int_0^{t-b^{-\beta/n}} (t-s)^{-\frac{n}{2p}} \left( \ln \left( \left( t-s \right)^{-n/\beta} + 1 \right) \right)^{-1/p} \| f(u(s)) \|_{L^a} \, ds
\]

\[
\leq C \int_0^t (t-s)^{-\frac{n}{2p} \left( \frac{1}{2} - \frac{1}{p} \right)} \| f(u(s)) \|_{L^a} \, ds,
\]

where \( 1 \leq a \leq p \). Apply the same calculation done above to obtain (4.2) (with same conditions), we conclude that \( I = O(\varepsilon) \). This implies that

\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{\exp L^p} = O(\varepsilon),
\]

in the case of \( t > b^{-\beta/n} \). Therefore

\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^\infty(0,\infty;\exp L^p)} = O(\varepsilon), \quad \text{for all } t > 0.
\]
It remains to prove that
\[ t^\sigma \left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} f(u(s)) \, ds \right\|_{L^q(\mathbb{R}^n)} = O(\varepsilon), \]
for every \( t > 0 \), to conclude that \( \Phi(u) \in B_\varepsilon \). This follows similarly as in (4.9) below by using the fact that \( f(0) = 0 \).

\* \, \Phi \text{ is a contraction.} \, \text{Let } u, v \in B_\varepsilon. \, \text{By (2.9), we obtain}
\[ t^\sigma \| \Phi(u) - \Phi(v) \|_{L^q(\mathbb{R}^n)} \leq C t^\sigma \int_0^t (t-s)^{-\frac{\beta}{2} \left( \frac{1}{q} - \frac{1}{r} \right)} \| f(u(s)) - f(v(s)) \|_r \, ds, \]
for every \( 1 \leq r \leq q \). From our assumption (1.4), we have
\[ |f(u) - f(v)| \leq C|u - v|(|u|^{m-1} e^{\lambda|u|^p} + |v|^{m-1} e^{\lambda|v|^p}) \]
\[ = C|u - v| \left( |u|^{m-1} \sum_{k=0}^\infty \frac{\lambda^k}{k!} |u|^{kp} + |v|^{m-1} \sum_{k=0}^\infty \frac{\lambda^k}{k!} |v|^{kp} \right) \]
\[ = C \sum_{k=0}^\infty \frac{\lambda^k}{k!} |u - v| \left( |u|^{kp+m-1} + |v|^{kp+m-1} \right). \]

Using Hölder’s inequality and Hölder’s interpolation inequality, we get
\[ \| f(u) - f(v) \|_r \leq C \sum_{k=0}^\infty \frac{\lambda^k}{k!} \left( \| u - v \|_r (|u|^{kp+m-1} + |v|^{kp+m-1}) \right) \]
\[ \leq C \sum_{k=0}^\infty \frac{\lambda^k}{k!} \| u - v \|_q \left( \| u \|_a(kp+m-1) + \| v \|_a(kp+m-1) \right) \]
\[ \leq C \sum_{k=0}^\infty \frac{\lambda^k}{k!} \| u - v \|_q \left( \| u \|_q^{(kp+m-1)(1-\theta)} + \| v \|_q^{(kp+m-1)(1-\theta)} \right), \]
where
\[ \frac{1}{r} = \frac{1}{q} + \frac{1}{a} \text{ and } \frac{1}{a(kp+m-1)} = \frac{\theta}{q} + \frac{1-\theta}{\rho}, \text{ for all } 0 \leq \theta \leq 1. \]

Using Lemma 2.4, assuming that \( p \leq \rho < \infty \), we infer that
\[ \| f(u) - f(v) \|_r \]
\[ \leq C \sum_{k=0}^\infty \frac{\lambda^k}{k!} \left( \Gamma \left( \frac{p}{\rho} + 1 \right) \right) \frac{(kp+m-1)(1-\theta)}{\rho} \| u - v \|_q \]
\[ \times \left( \| u \|_q^{(kp+m-1)(1-\theta)} + \| v \|_q^{(kp+m-1)(1-\theta)} \right). \]

So
\[ t^\sigma \| \Phi(u) - \Phi(v) \|_{L^q(\mathbb{R}^n)} \]
\[ \leq C \sum_{k=0}^\infty \frac{\lambda^k}{k!} \left( \Gamma \left( \frac{p}{\rho} + 1 \right) \right) \frac{(kp+m-1)(1-\theta)}{\rho} \]
\[ \times t^\sigma \int_0^t (t-s)^{-\frac{\beta}{2} \left( \frac{1}{q} - \frac{1}{r} \right)} s^{-(\sigma - \sigma(\sigma) p - \sigma(\sigma) \theta)} \| u - v \|_q^{\sigma(\sigma) p - \sigma(\sigma) \theta} \| u - v \|_q^{\sigma(\sigma) p - \sigma(\sigma) \theta}. \]
× \left( s^\sigma \|u\|_q^{(kp+m-1)\theta} \|u\|_{L^p}^{(kp+m-1)(1-\theta)} + s^\sigma \|v\|_q^{(kp+m-1)\theta} \|v\|_{L^p}^{(kp+m-1)(1-\theta)} \right) ds
\leq Cd(u,v)(\varepsilon M)^{m-1} \sum_{k=0}^{\infty} (\varepsilon M)^{kp} \frac{\lambda_k}{k!} \left( \Gamma\left( \frac{p}{p+1} \right) \right)^{(kp+m-1)(1-\theta)}
\leq B \left( 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{q} \right); 1 - \sigma(1 + (kp + m - 1)\theta) \right),
where we have used the fact that \( u, v \in B_\varepsilon \), under the following conditions:
\[ 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{q} \right) - \sigma(kp + m - 1)\theta = 0, \quad \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{q} \right) < 1, \quad \text{and} \quad \sigma(1 + (kp + m - 1)\theta) < 1. \]

As above, for all \( k \geq 0 \), we choose first \( \theta = \theta_k \geq 0 \) such that
\[ 1 - \frac{n}{\beta} \left( \frac{(m-1)\rho}{p-1} \right) < \theta < \frac{1}{pk + m - 1} \min(m - 1, 1 - \frac{\sigma}{\theta}), \]
where we have used the fact that \( q > \frac{(m-1)p}{p-1} \geq m \). Next, we choose \( \rho = \rho_k \) such that
\[ 1 - \frac{\theta_k}{\rho_k} = \frac{\beta}{n(pk + m - 1)} - \frac{\beta\theta_k}{n(m - 1)}, \]
and finally, we choose \( a > 0 \) such that
\[ \frac{1}{a(pk + m - 1)} = \frac{\theta_k}{q} + \frac{1 - \theta_k}{\rho_k}. \]
To ensure that \( \sigma < 1 \), we also suppose the following condition
\[ q < \frac{n(m - 1)}{\beta(2 - m)_+}, \]
where \((\cdot)_+\) stands for the positive part. Moreover, for these choice of parameters,
\[ B \left( 1 - \frac{n}{\beta} \left( \frac{1}{r} - \frac{1}{p} \right); 1 - \sigma(1 + (kp + m - 1)\theta) \right) = \frac{1}{\Gamma\left( \frac{m-2}{m-1} + \frac{n}{\beta q} \right)} \frac{\Gamma\left( \frac{np}{p+1} \right)}{\Gamma\left( \frac{m-2}{m-1} + \frac{n}{\beta q} \right)} \leq C, \]
and
\[ \left( \Gamma\left( \frac{pk}{p+1} \right) \right)^{(kp+m-1)(1-\theta_k)} \leq Ck! \]
This implies that
\[ t^\sigma \|\Phi(u) - \Phi(v)\|_{L^p(\mathbb{R}^n)} \leq Cd(u,v)(M\varepsilon)^{m-1} \sum_{k=0}^{\infty} (C\lambda)^k (M\varepsilon)^{kp} \leq \frac{1}{2} d(u,v), \quad (4.9) \]
for \( \varepsilon \) small enough. This completes the proof the existence of global solution in Theorem 1.4 in the case of \( \beta \geq \frac{n(p-1)}{p} \). The estimation (1.6) follows from \( u \in B_\varepsilon \).
4.3. Proof of the property (1.5) in Theorem 1.4. We now prove the continuity of solution at zero. Let $q$ be a positive number such that $q > \max \left\{ \frac{n}{p}, 1 \right\}$. From the embedding $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^p(\mathbb{R}^n)$ (Lemma 2.2), and $L^p - L^p$, $L^q - L^\infty$ estimates (2.9), we have

$$\|u(t) - e^{-t(-\Delta)^{\beta/2}}u_0\|_{\exp L^p} \leq \int_0^t \left\| e^{-t(-\Delta)^{\beta/2}}f(u(s)) \right\|_{\exp L^p} \, ds$$

$$\leq C \int_0^t \left\| e^{-t(-\Delta)^{\beta/2}}f(u(s)) \right\|_{L^p} \, ds + C \int_0^t \left\| e^{-t(-\Delta)^{\beta/2}}f(u(s)) \right\|_{L^\infty} \, ds$$

$$\leq C \int_0^t \left\| f(u(s)) \right\|_{L^p} \, ds + C \int_0^t (t - s)^{-\frac{n}{p^*}} \left\| f(u(s)) \right\|_{L^q} \, ds. \quad (4.10)$$

Let us estimate $\|f(u)\|_{L^r}$, for $r = p, q \geq 1$. We have

$$|f(u)| \leq C|u|^m e^{\lambda|u|^p} = C|u|^m \left( e^{\lambda|u|^p} - 1 \right) + C|u|^m,$$

then, by Hölder’s inequality, we obtain

$$\|f(u)\|_{L^r(\mathbb{R}^n)} \leq C \left\| u \right\|_{L^{2mr}(\mathbb{R}^n)}^{m} \left( e^{\lambda|u|^p} - 1 \right)_{L^{2r}(\mathbb{R}^n)} + C \left\| u \right\|_{L^{mr}(\mathbb{R}^n)}^{m}$$

$$\leq C \left\| u \right\|_{\exp L^p(\mathbb{R}^n)}^{m} \left( e^{\lambda|u|^p} - 1 \right)_{L^{2r}(\mathbb{R}^n)} + C \left\| u \right\|_{\exp L^p(\mathbb{R}^n)}^{m},$$

where we have used Lemma 2.4 and $2mr \geq mr \geq m \geq p$. Next, using Lemma 2.6 and the fact that $u \in E_\varepsilon$ (or $u \in B_\varepsilon$), we have

$$\|f(u)\|_{L^r(\mathbb{R}^n)} \leq C \left\| u \right\|_{\exp L^p(\mathbb{R}^n)}^{m} \left( 1 + 2C\lambda r^p \right)^{1/2r} \leq C \left\| u \right\|_{\exp L^p(\mathbb{R}^n)}^{m}. \quad (4.11)$$

Substituting (4.11) in (4.10), we obtain

$$\|u(t) - e^{-t(-\Delta)^{\beta/2}}u_0\|_{L^p} \leq C t \left\| u \right\|_{L^{\infty}(\mathbb{R}^n)}^{m} \left( e^{\lambda|u|^p} - 1 \right)_{L^{2r}(\mathbb{R}^n)} + C t \left\| u \right\|_{L^{\infty}(\mathbb{R}^n)}^{m}$$

$$\leq C t \left\| u \right\|_{L^{\infty}(\mathbb{R}^n)}^{m} \left( e^{\lambda|u|^p} - 1 \right)_{L^{2r}(\mathbb{R}^n)} + C t \left\| u \right\|_{L^{\infty}(\mathbb{R}^n)}^{m} \left( e^{\lambda|u|^p} - 1 \right)_{L^{2r}(\mathbb{R}^n)}$$

This completes the proof of (1.5). \qed

4.4. Proof of the weak* convergence in Theorem 1.4. We complete the proof of Theorem 1.4 by showing the continuity at $t = 0$ in the weak* sense. Let $X := L^1(\ln L)^{1/p}(\mathbb{R}^n)$ be the pre-dual space of $\exp L^p$. It is known that $X$ is a Banach space and $C^\infty_0(\mathbb{R}^n)$ is dense in $X$ (cf. [1]). Let $\varphi \in X$. By Hölder’s inequality for the Orlicz space, we have

$$\left| \int_{\mathbb{R}^n} \left( e^{-t(-\Delta)^{\beta/2}}u_0(x) - u_0(x) \right) \varphi(x) \, dx \right| \leq 2 \left\| u_0 \right\|_{\exp L^p} \left\| e^{-t(-\Delta)^{\beta/2}}\varphi - \varphi \right\|_X.$$

Since $C^\infty_0(\mathbb{R}^n)$ is dense in $X$, so by applying similar calculations as in the proof of Proposition 2, we conclude that

$$\lim_{t \to 0} \left\| e^{-t(-\Delta)^{\beta/2}}\varphi - \varphi \right\|_X = 0.$$

This completes the weak* convergence. \qed
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REFERENCES

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, 2nd edition, Pure and Applied Mathematics (Amsterdam), Vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
[2] C. Bennett and R. Sharpley, Interpolation of Operators, Pure and applied mathematics, Academic Press, 1988.
[3] Z. W. Birnbaum and W. Orlicz, Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, Studia Math., 3 (1931), 1–67.
[4] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math., 68 (1996), 277–304.
[5] T. Cazenave and A. Haraux, Introduction aux Problèmes d’évolution Semi-linéaires, Ellipses, Paris, 1990.
[6] G. Furioli, T. Kawakami, B. Ruf and E. Terraneo, Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity, J. Differ. Equ., 262 (2017), 145–180.
[7] S. Ibrahim, M. Majdoub and N. Masmoudi, Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type, Commun. Pure Appl. Math., 59 (2006), 1639–1658.
[8] N. Ioku, The Cauchy problem for heat equations with exponential nonlinearity, J. Differ. Equ., 251 (2011), 1172–1194.
[9] N. Ioku, B. Ruf and E. Terraneo, Existence, non-existence, and uniqueness for a heat equation with exponential nonlinearity in $\mathbb{R}^2$, Math. Phys. Anal. Geom., 18 (2015), Art. 29, 19 pp.
[10] M. Majdoub, S. Otsmane and S. Tayachi, Local well-posedness and global existence for the biharmonic heat equation with exponential nonlinearity, Adv. Differ. Equ., 23 (2018), 489–522.
[11] M. Majdoub and S. Tayachi, Well-posedness, global existence and decay estimates for the heat equation with general power-exponential nonlinearities, Proc. Int. Cong. Math. Río de Janeiro, 2 (2018), 2379–2404.
[12] M. Majdoub and S. Tayachi, Global existence and decay estimates for the heat equation with exponential nonlinearity, preprint, arXiv:1912.06490v1.
[13] M. M. Rao and Z. D. Ren, Applications of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 250, Marcel Dekker, Inc., New York, 2002.
[14] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Math. Mech., 17 (1967), 473–483.
[15] F. B. Weissler, Semilinear evolution equations in Banach spaces, J. Funct. Anal., 32 (1979), 277–296.
[16] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in $L^p$, J. Indiana Univ. Math., 29 (1980), 79–102.
[17] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math., 38 (1981), 29–40.

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