THE LIMITING SPECTRA OF GIRKO’S BLOCK-MATRIX

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Abstract. To analyze the limiting spectral distribution of some random block-matrices, Girko [7] uses a system of canonical equations from [6]. In this paper, we use the method of moments to give an integral form for the almost sure limiting spectral distribution of such matrices.

1. Introduction and main result

A random block-matrix is a matrix whose entries are random matrices. In [6], Girko studied the spectra of large dimensional random block-matrices by introducing a system of equations, called the system of canonical equations, to analyze the spectra. This system of canonical equations was used later by Girko [7] to study a model for which the system is solvable. The model studied there has many restrictive conditions.

In the current paper, we are going to study the same model under different conditions for the blocks. The main tool of the proof is the method of moments. We will follow the proof of the main theorem by propositions, as applications to the theorem, in which the blocks are made of some known ensembles like the Gaussian unitary ensemble and the Wishart random matrix. Free probability theory is used to prove these propositions.

The spectral measure of an \( n \times n \) Hermitian matrix \( A \) is

\[
\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},
\]

where \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) are the eigenvalues of \( A \). In this paper we consider random matrices, i.e., matrices where the entries are random variables on some probability space. In this case, \( \mu_A \) is, of course, a random measure.

We will denote the weak convergence of probability measures by \( \lim_{n \to \infty} \mu_n \xrightarrow{D} \mu \) or

\[
\mu_n \xrightarrow{D} \mu \quad \text{as} \quad n \to \infty.
\]

If the moments of measures converge,

\[
\lim_{n \to \infty} \int_{\mathbb{R}} x^k \mu_n(dx) = \int_{\mathbb{R}} x^k \mu(dx)
\]

for all \( k \geq 1 \), we will write

\[
\mu_n \xrightarrow{m} \mu \quad \text{as} \quad n \to \infty.
\]

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If \( \{ \mu_n \} \) is a sequence of random measures which converges in one of the above senses almost surely, we will append the abbreviation "a.s." to the above notation. We note that for \( k \geq 1 \), the \( k^{th} \) moment of \( \mu_A \) is

\[
(1) \quad \int x^k \mu_A(dx) = \text{tr}_n(A^k),
\]

where \( \text{tr}_n(A) := \frac{1}{n} \sum_{i=1}^{n} A_{ii} \).

The Kronecker product \( \otimes \) of two matrices \( A = (a_{ij})_{i,j=1}^{k} \) and \( B = (b_{ij})_{i,j=1}^{n} \) is defined to be the \( nk \times nk \) matrix given by

\[
A \otimes B = \begin{pmatrix}
  a_{11}B & a_{12}B & \ldots & a_{1k}B \\
  a_{21}B & a_{22}B & \ldots & a_{2k}B \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k1}B & a_{k2}B & \ldots & a_{kk}B
\end{pmatrix}.
\]

Among the properties of the Kronecker product, we will need the identity

\[
\text{tr}_{kn}(A \otimes B) = \text{tr}_k(A)\text{tr}_n(B).
\]

Also, if \( A \) and \( C \) are two \( k \times k \) matrices and \( B \) and \( D \) are two \( n \times n \) matrices, then \( (A \otimes B)(C \otimes D) = AC \otimes BD \). Finally, \( I_k \) is the \( k \times k \) identity matrix.

Now we are ready to state the main theorem.

**Theorem 1.** For \( n, k \geq 1 \), consider the double array of random block-matrices \( \{B_{n,k}\} \) whose terms are given by \( B_{n,k} = I_k \otimes A_n + W_k \otimes B_n \), where for \( n \geq 1 \), the matrices \( A_n, B_n, \) and \( W_n \) are Hermitian random matrices of order \( n \), and satisfy the following hypotheses:

(i) There exists a compactly supported probability measure \( \mu_\omega \) such that

\[
(2) \quad \mu_{W_n} \to \mu_\omega \text{ as } n \to \infty \quad \text{a.s.}
\]

(ii) For real \( t \), there exist probability measures \( \psi(t,:) \) such that

\[
(\mu_{A_n+B_n} \to \psi(t,:) \text{ as } n \to \infty \quad \text{a.s.})
\]

and \( \psi(t,:) \) has a support that is uniformly bounded for \( t \) in any compact subset of \( \mathbb{R} \).

Under these conditions we have

\[
(3) \quad \lim_{k \to \infty} \lim_{n \to \infty} \mu_{B_{n,k}} \xrightarrow{d} \lim_{k \to \infty} \lim_{n \to \infty} \mu_{B_{n,k}} \xrightarrow{d} \nu \quad \text{a.s.,}
\]

where the probability measure \( \nu \) is defined as

\[
(4) \quad \nu(dx) = \int_{\mathbb{R}} \psi(t;dx) \mu_\omega(dt).
\]

**Remark 1.** Since the support of \( \psi(t;dx) \) is uniformly bounded for \( t \) in \( \text{supp}(\mu_\omega) \) (the support of \( \mu_\omega \)), then the probability measure \( \nu(dx) \), introduced in (4), is compactly supported.

As mentioned above, matrices of the form \( I_k \otimes A_n + W_k \otimes B_n \) were analyzed by Girko [7]. In [7] Theorem 3), Girko assumes that \( \{A_n\} \) and \( \{B_n\} \) are two sequences of real symmetric non-random matrices, \( B_n \) is a positive definite matrix for each \( n \geq 1 \), and the entries of the symmetric matrix \( W_k \) are independent \( \pm 1 \) with probability \( \frac{1}{2} \). He shows under these assumptions that the spectral probability distribution \( \overline{F}_{B_{n,k}}(x) := \mu_{B_{n,k}}((-\infty,x]) \) of the sequence of random block-matrices...
\{B_{n,k}\} converges, for almost all \(x\)'s and with probability one, as both \(k\) and \(n\) go to infinity to a non-random distribution function that follows from a complicated equation given in [7].

In Theorem[1] our assumptions allow us to identify the limit. In the course of our proof, we are also able to derive Girko's SS-Law (a sum of semi-circular law), see Proposition[4].

2. Proof of Theorem[7]

We need the following lemma.

**Lemma 1.** [5, p.94, Lemma 3.1] Fix \(k \in \mathbb{N}\), let \(T = \{t_0, t_1, \ldots, t_k\}\) be a set of distinct points in \(\mathbb{R}\) and \(P_n(t) = a_{0,n} + a_{1,n}t + \cdots + a_{k,n}t^k\) be a polynomial with \(a_{i,n} \in \mathbb{C}\) for every \(i\) and \(n\). If \(P_n(t)\) converges for every \(t \in T\) as \(n \to \infty\), then the limit is a polynomial of degree \(\leq k\), say it is \(P(t) = a_0 + a_1t + \cdots + a_k t^k\). Moreover, the convergence is uniform on every compact subset of \(\mathbb{R}\). Furthermore, \(\lim_{n \to \infty} a_{i,n} = a_i\) for every \(i\).

**Proof of Theorem[7]** The proof is based on the method of moments. Using the aforementioned properties of the Kronecker product, the \(m\)th moment of the spectral measure of \(B_{n,k}\) is given by

\[
\text{tr}_{nk}(B_{n,k}^m) = \text{tr}_{kn}((I_k \otimes A_n + W_k \otimes B_n)^m)
\]

(5)

\[
= \sum_{j=0}^{m} \text{tr}_k(W_k^j) \text{tr}_n(\phi(A_n, B_n; m-j, j)),
\]

where \(\phi(A_n, B_n; m-j, j)\) is the sum of all the noncommutative monomials in which \(B_n\) appears \(j\) times and \(A_n\) appears \(m-j\) times. Let the \(j\)th moment of \(\mu_\omega\) be \(\omega_j, j \geq 1\). By (1) and (2), \(\lim_{k \to \infty} \text{tr}_k(W_k^j) = \omega_j\) a.s. Therefore,

\[
\lim_{k \to \infty} \text{tr}_{nk}(B_{n,k}^m) = \sum_{j=0}^{m} \omega_j \text{tr}_n(\phi(A_n, B_n; m-j, j)).
\]

(6)

On another hand, for all \(t \in \mathbb{R}\)

\[
\text{tr}_n((A_n + tB_n)^m) = \sum_{j=0}^{m} t^j \text{tr}_n(\phi(A_n, B_n; m-j, j)).
\]

(7)

Therefore,

\[
\int_\mathbb{R} \text{tr}_n((A_n + tB_n)^m)\mu_\omega(dt) = \sum_{j=0}^{m} \omega_j \text{tr}_n(\phi(A_n, B_n; m-j, j)).
\]

So by (8),

\[
\lim_{k \to \infty} \text{tr}_{nk}(B_{n,k}^m) = \int_\mathbb{R} \text{tr}_n((A_n + tB_n)^m)\mu_\omega(dt).
\]

But,

\[
\lim_{n \to \infty} \text{tr}_n((A_n + tB_n)^m) = \int_\mathbb{R} x^m \psi(t; dx)
\]

and by Lemma[1] this limit is uniform in \(t\) as \(t\) varies over the compact set \(\text{supp}(\mu_\omega)\).

Therefore,

\[
\lim_{n \to \infty} \lim_{k \to \infty} \text{tr}_{nk}(B_{n,k}^m) = \int_\mathbb{R} \int_\mathbb{R} x^m \psi(t; dx) \mu_\omega(dt).
\]
By Fubini’s Theorem

$$\lim_{n \to \infty} \lim_{k \to \infty} \tr_{n,k}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}} x^m \left( \int_{\mathbb{R}} \psi(t; dx) \mu_\omega(dt) \right).$$

The other iterated limit follows from the observation that (2) and (7) imply

$$\tr_{n,k}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}} \tr_n((A_n + tB_n)^m)\mu_{\psi_k}(dt)$$

for every $n$ and $k$. Since $\mu_{\psi_k}$ is a discrete measure,

$$\lim_{n \to \infty} \tr_{n,k}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}} \int_{\mathbb{R}} x^m \psi(t; dx)\mu_{\psi_k}(dt).$$

By Lemma 1, $\int_{\mathbb{R}} x^m \psi(t; dx)$ is a polynomial in $t$ and since $\mu_{\psi_k}$ converges in moments, it follows that

$$\lim_{k \to \infty} \lim_{n \to \infty} \tr_{n,k}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}} x^m \left( \int_{\mathbb{R}} \psi(t; dx) \mu_\omega(dt) \right).$$

Now, since $\nu(dx) := \int_{\mathbb{R}} \psi(t; dx) \mu_\omega(dt)$ has a bounded support, the result follows.

$$\square$$

**Remark 2.** The integral in (4) always exists because $\psi(t; \cdot)$ is measurable in $t$. This can be seen as follows. The characteristic function of $\psi(t; \cdot)$ is analytic, as $\psi(t; \cdot)$ has a compact support for each $t$. So the characteristic function is measurable in $t$ as a pointwise limit of the series in the moments $\int_{\mathbb{R}} x^k \psi(t; dx)$, $k \geq 1$; the latter are polynomials in $t$ by Lemma 1. Therefore the inversion formula of the characteristic function implies the measurability of $\psi(t; (-\infty, a])$ for any $a$.

### 3. Applications

In this section, we apply Theorem 1 to some well-studied ensembles of random matrices. To do so, we introduce these ensembles and review the pertinent topics from free probability theory.

#### 3.1 Random Matrix Theory.

We call an $n \times n$ Hermitian matrix $A = (A_{ij})_{i,j=1}^n$ a Wigner matrix if it is a random matrix whose upper-diagonal entries are independent and identically distributed complex random variables such that $E(A_{ij}) = 0$ and $E(|A_{ij}|^2) = \frac{1}{n}$ for all $i < j$. Moreover, the diagonal entries are independent and identically distributed real random variables such that $E(A_{ii}) = 0$ and $E(A_{ii}^2) = \frac{1}{n}$. We will denote all such Wigner matrices of order $n$ by $\text{{Wigner}}(n)$.

An important example of a Wigner matrix is the Gaussian Wigner matrix for which $\{\Re A_{ij} : 1 \leq i \leq j \leq n\} \cup \{\Im A_{ij} : 1 \leq i < j \leq n\}$ is a family of independent random Gaussian variables such that $A_{ii} \sim N(0, \frac{1}{n})$ for every $i$ and $\Re A_{ij}$, $\Im A_{ij} \sim N(0, \frac{1}{2n})$ for every $i < j$. We will denote all such Gaussian matrices of order $n$ by $G(n)$.

We call the random matrix $B = X^*X$ a Wishart matrix if $X$ is a $p_n \times n$ matrix whose entries are complex independent Gaussian random variables such that $\Re X_{ij}$, $\Im X_{ij} \sim N(0, \frac{1}{2n})$ for every $1 \leq i, j \leq n$. Here $X^*$ is the conjugate transpose of $X$. We will denote all such Wishart matrices of order $n$ and shape parameter $p_n$ by $\text{{Wishart}}(n, p_n)$. See [2, 4] for more details and references.
For these type of random matrices, the limiting spectral distributions are known. If $A_n$ is Wigner$(n)$, then by Wigner’s Theorem (cf. [2]),

$$\mu_{A_n} \xrightarrow{D} \gamma_{0,1} \text{ as } n \to \infty \quad \text{a.s.}$$

where $\gamma_{0,\sigma^2}$ is the semicircular law centered at $\alpha$ and of variance $\sigma^2$ which is given as

$$\gamma_{0,\sigma^2}(dx) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x - \alpha)^2} \ 1_{[\alpha - 2\sigma, \alpha + 2\sigma]}(x) dx.$$

On the other hand, if $B_n$ is Wishart$(n,p_n)$ for all $n$ and $\lim_{n \to \infty} \frac{p_n}{n} = \alpha > 0$, then (cf. [2])

$$\mu_{B_n} \xrightarrow{D} \rho_{\alpha} \text{ as } n \to \infty \quad \text{a.s.}$$

where $\rho_{\alpha}$ is the Marchenko-Pastur law with mean $\alpha > 0$ which is given as

$$\rho_{\alpha}(dx) = (1 - \alpha)^+ \delta_0(dx) + \frac{\sqrt{(x - (\sqrt{\alpha} - 1)^2)((\sqrt{\alpha} + 1)^2 - x)}}{2\pi x} \ 1_{[(\sqrt{\alpha} - 1)^2, (\sqrt{\alpha} + 1)^2]}(x) dx.$$

The following two propositions are consequences of Theorem 1.

**Proposition 1.** Let $\{W_n\}$, $\{A_n\}$ and $\{B_n\}$ be three independent sequences of random matrices such that $W_n$ is Wigner$(n)$ and $A_n, B_n$ are $G(n)$ for all $n$. Then the almost sure limiting spectral distribution $\nu$ of $B_{n,k} = I_k \otimes A_n + W_k \otimes B_n$, see [3], is absolutely continuous with the probability density function

$$g(x) = \begin{cases} g_1(x) & \text{ whenever } 2 \leq |x| \leq 2\sqrt{5} \\ g_2(x) & \text{ whenever } |x| \leq 2 \end{cases}$$

where

$$g_1(x) = \frac{1}{2\pi^2} \int_0^2 \frac{\sqrt{4(1 + t^2) - x^2}}{\sqrt{4 - t^2}} \ dt,$$

and

$$g_2(x) = \frac{1}{2\pi^2} \int_0^2 \frac{\sqrt{4(1 + t^2) - x^2}}{t^2} \ dt.$$

![Figure 1](image-url) The probability density function corresponding to the limiting spectral distribution of $B_{n,k}$ when $A_n$ and $B_n$ are Gaussian matrices.

In order to state the following proposition we first define the following functions:

$$h_1(x; t) = 2 + 27t^2 - 3tx - 3t^2x^2 + 2t^3x^3,$$
for all \( n \) \( W \) \( \text{Proposition 2.} \)\n
The Stieltjes inversion formula is given by \( R \) \( \text{E} \) for every continuity set \( G \) \( 6 \) TAMER ORABY \( 3.2. \) \( \text{possesses the following properties:} \)

\[
\begin{align*}
4 + 27 t^2 - 6 t x - x^2 - 6 t^2 x^2 + 2 t x^3 + 4 t^3 x^3 - t^2 x^4 &= 0 \\
(10) & \text{(see Proposition 3 for details) and the probability density function} \\
(11) & f(x; t) = \frac{1}{2 \sqrt{3 \pi t}} \left( H(x; t) - \frac{h_2(x; t)}{h_1(x; t)} \right) 1_{[s_1(t), s_2(t)]}(x).
\end{align*}
\]

Proposition 2. Let \( \{W_n\}, \{A_n\} \) and \( \{B_n\} \) be three independent sequences of random matrices such that \( W_n \) is \( \text{Wigner}(n) \), \( A_n \) is \( \text{G}(n) \) and \( B_n \) is \( \text{Wishart}(n, p_n) \) for all \( n \) and \( \lim_{n \to \infty} \frac{p_n}{n} = 1 \). Then the almost sure limiting spectral distribution \( \nu \) of \( \mathbb{B}_{n,k} = I_k \otimes A_n + W_k \otimes B_n \), see \( 3 \), is absolutely continuous with the probability density function

\[
g(x) = \int_{\mathbb{R}} f(x; t) \gamma_{0,1}(dt)
\]

where \( f(x; t) \) is given in Equation (11).

We simply need to verify the hypothesis of Theorem 1. We use the machinery of free probability (specifically free additive convolution) to do this.

3.2. Free Additive Convolution. Let \( \mu \) be a probability measure with a compact support in \( \mathbb{R} \). We define its corresponding Cauchy (Stieltjes) transform to be \( G_{\mu}(z) := \int_{\mathbb{R}} \frac{1}{z-x} \mu(dx) \), for \( z \in \mathbb{C} \) such that \( \Im(z) > 0 \). The Cauchy transform \( G_{\mu}(z) \) possesses the following properties:

\( \begin{align*} 
\text{(i) } & \Im(G_{\mu}(z)) < 0 \text{ whenever } \Im(z) > 0. \\
\text{(ii) } & \lim_{|z| \to \infty} zG_{\mu}(z) = 1. \\
\text{(iii) } & G_{\mu}(z) \text{ is analytic in a neighborhood of } \infty.
\end{align*} \)

The Stieltjes inversion formula is given by

\[
(12) \quad \mu(E) = -\frac{1}{\pi} \lim_{y \to 0} \int_{E} \Im(G_{\mu}(x + iy))dx
\]

for every continuity set \( E \subset \mathcal{B}(\mathbb{R}) \) (the \( \sigma \)-field of Borel subsets of \( \mathbb{R} \)). The R-transform of \( \mu \) is defined as \( R_{\mu}(z) = K_{\mu}(z) - \frac{1}{z} \) where \( K_{\mu}(z) \) is the inverse function of the Cauchy transform \( G_{\mu}(z) \), i.e., \( G_{\mu}(K_{\mu}(z)) = z \). The two functions \( K_{\mu}(z) \) and \( R_{\mu}(z) \) are well defined in \( 0 < |z| < r \) and \( 0 \leq |z| < r \), respectively, for some \( r > 0 \).

The free additive convolution of probability measures with compact supports in \( \mathbb{R} \) arises in free probability theory (cf. [10]). If \( \mu \) and \( \nu \) are two probability measures with compact supports in \( \mathbb{R} \), then their free additive convolution \( \mu \boxplus \nu \) is a probability measure with a compact support in \( \mathbb{R} \), see [10]. The R-transform of \( \mu \boxplus \nu \) is given by \( R_{\mu \boxplus \nu}(z) = R_{\mu}(z) + R_{\nu}(z) \).

Denote the dilation \( D_t \) of a measure \( \mu \) by \( D_t(\mu) \), where \( D_t(\mu)(E) = \mu(E/t) \) for every \( E \) if \( t \neq 0 \) and \( D_t(\mu) = \delta_0 \) if \( t = 0 \). Since \( R_{D_t(\mu)}(z) = tR_{\mu}(tz) \) for every \( t \in \mathbb{R} \) (cf. [10] p.26), therefore

\[
D_t(\mu \boxplus \nu) = D_t(\mu) \boxplus D_t(\nu).
\]
For the limit laws described above, $R_{\gamma_{0,t}}(z) = \sigma^2 z$ and $R_{\rho_t}(z) = \frac{z^2}{1-4z^2}$ (cf. [10]); furthermore $\gamma_{0,1} \boxplus D_t(\gamma_{0,1}) = \gamma_{0,1+t^2}$ for every real $t$. The next proposition computes $\gamma_{0,1} \boxplus D_t(\rho_1)$ for $t > 0$.

**Proposition 3.** If $\mu = \gamma_{0,1}$ and $\nu = \rho_1$ then for $t > 0$

$$\mu \boxplus D_t(\nu)(dx) = f(x;t)dx$$

where $f(x;t)$ is given by Equation (11). Furthermore, for each compact set $C \subset \mathbb{R}$ there exists $M > 0$ such that the support of $f(x;t)dx$ is contained in $[-M,M]$ for all $t$ in $C$.

**Proof.** Fix $t > 0$. The R-transforms of $\mu$ and $\nu$ are given by

$$R_\mu(z) = z \text{ and } R_{D_t(\nu)}(z) = \frac{t}{1-tz}$$

and accordingly

$$R_{\mu \boxplus D_t(\nu)}(z) = z + \frac{t}{1-tz}.$$ 

Therefore, the Cauchy transform $G_{\mu \boxplus D_t(\nu)}(z)$ is the root of the cubic equation

$$tg^3 - g^2 (1+tz) + g - 1 = 0.$$ 

First, in order to show uniqueness, we will show that Equation (13) has only one root for which $\lim_{|z| \to \infty} zg(z) = 1$. This follows from the observation that if $g_1$, $g_2$ and $g_3$ are the roots of Equation (13) then $g_1g_2g_3 = \frac{1}{t}$ and $g_1 + g_2 + g_3 = \frac{1}{t} + z$. Combining both identities results in

$$g_1^2 g_2 + g_1 g_2^2 + \frac{1}{t} = \frac{1}{t} g_1 g_2 + z g_1 g_2$$

Thus if two of the roots, say $g_1$ and $g_2$, are such that $\lim_{|z| \to \infty} zg_1(z) = 1$ and $\lim_{|z| \to \infty} zg_2(z) = 1$, then Equation (14) would lead to the contradiction that $\frac{1}{t} = 0$. It is known, see [3] Corollary 2, Corollary 4, and Proposition 5], that the free convolution of a compactly supported measure with a semicircle law has a smooth and bounded density. Thus by picking the right root and then using the Stieltjes inversion formula [12] we get

$$-\frac{1}{\pi} \lim_{y \downarrow 0} \Im(G_{\mu \boxplus D_t(\nu)}(x + iy)) = f(x;t)$$

where $f(x;t)$ is given by Equation (11).

Second, since we know in advance that the free additive convolution of two probability measures with compact supports in $\mathbb{R}$ has a compact support in $\mathbb{R}$ (cf. [10]), then we can find the support of $\mu \boxplus D_t(\nu)$ by identifying when $f(x;t) = 0$. This last identity leads to Equation (10).

The left hand side in Equation (10) is positive at $x = 0$ and Equation (10) has two real roots and two complex conjugate roots. We prove the existence and uniqueness of its real roots $s_1(t)$ and $s_2(t)$ as follows. Substitute $x = y + \frac{1}{2t} + t$ in Equation (10). Hence, we get another quartic equation in $y$ which reads as

$$y^2 + y^4 = 0$$
By Descartes–Euler theorem [1], Equation (15) has two real roots and two complex conjugate roots (and correspondingly Equation (10) does) if and only if the cubic equation in \( z \) that is given by

\[
(16) \quad - \left( \frac{8}{t} - 8t^3 \right)^2 + (80 + 48t^4) z - \left( \frac{1}{t^2} + 12t^2 \right) z^2 + z^3 = 0
\]

has one nonnegative real root and two complex conjugate roots. To show this we substitute \( z = u + \left( \frac{1}{3t^2} + 4t^2 \right) \) in Equation (16) which in turn reduces to another cubic equation. By Cardano’s theorem, the discriminant of the new reduced equation \( D = \frac{64(1+27t^4)}{27t^8} \) is nonnegative and so it has one real root and two complex conjugate roots and so does Equation (16).

Now, it is left to show that this real root of Equation (16) is nonnegative. This is true since the product of the three roots of Equation (16) is equal to \( (\frac{8}{t} - 8t^3)^2 \) which is nonnegative and consequently the real root is also nonnegative.

We can see from Equation (10) that \( s_1(t) \) and \( s_2(t) \) are continuous and hence uniformly bounded on any compact subset of \( \mathbb{R} \). This completes the proof. □

In Figure 1, we show the graphs of \( f(x;t) \) for \( t = 1, 2 \).

![Graphs of f(x;t) for t = 1, 2](image)

**Figure 2.** The probability density functions corresponding to \( \mu \boxplus D_t(\nu) \) with \( t=1, 2 \).

**Remark 3.** If \( t = 0 \), then \( f(x;0) = \frac{1}{2\pi} \sqrt{4-x^2} 1_{[-2,2]}(x) \) since \( \mu \boxplus \delta_0 = \mu \). In case \( t < 0 \), since \( \mu \boxplus D_t(\nu) = D_{-1}(\mu \boxplus D_{-1}(\nu)) \), then it follows that \( \mu \boxplus D_t(\nu)(dx) = f(-x; -t)dx \).

We will also need the following.

**Theorem 2.** [8 Proposition 4.3.9] Let \( \{A_n\} \) and \( \{B_n\} \) be two sequences of Hermitian random matrices and \( \{U_n\} \) be a sequence of random matrices with the uniform distribution on the unitary group \( U(n) \). Suppose that \( U_n \) is independent of \( (A_n, B_n) \) for all \( n \geq 1 \). If there exist two compactly supported probability distributions \( \mu \) and \( \nu \) such that

\[
\mu_{A_n} \xrightarrow{m} \mu \text{ as } n \to \infty \quad \text{a.s.} \quad \text{and} \quad \mu_{B_n} \xrightarrow{m} \nu \text{ as } n \to \infty \quad \text{a.s.,}
\]

then

\[
\mu_{A_n + U_n \cdot B_n U_n} \xrightarrow{D} \mu \boxplus \nu \text{ as } n \to \infty \quad \text{a.s.}
\]
In particular, if $A_n$, $B_n$ are independent, and the distribution of $B_n$ is unitarily invariant for all $n$, i.e., if $B_n$ and $U_nB_nU_n^*$ have the same distribution for all unitary matrices $U_n$, then Theorem 2 implies

$$
\mu_{A_n+B_n} \xrightarrow{D} \mu \boxplus \nu \quad \text{as } n \to \infty \quad \text{a.s.}
$$

Since the distributions of $G(n)$ and Wishart$(n,p_n)$ matrices are unitarily invariant, we get the following.

Corollary 1.

(i) For $n \geq 1$, let $A_n$ be $Wigner(n)$ and $B_n$ be $G(n)$ such that $A_n$ and $B_n$ are independent. Then for all $t \in \mathbb{R}$,

$$
\mu_{A_n+B_n} \xrightarrow{D} \gamma_{0,1+t^2} \quad \text{as } n \to \infty \quad \text{a.s.}
$$

(ii) For $n \geq 1$, let $A_n$ be $Wigner(n)$ and $B_n$ be Wishart$(n,p_n)$ such that $A_n$ and $B_n$ are independent. If $\lim_{n \to \infty} \frac{p_n}{n} = 1$, then for all $t \in \mathbb{R}$,

$$
\mu_{A_n+B_n} \xrightarrow{D} \mu_t \quad \text{as } n \to \infty \quad \text{a.s.}
$$

where $\mu_t$ has the probability density function given by Equation (11).

Now, Proposition 4 follows directly from Corollary 1 part (i). Proposition 2 follows easily from Corollary 1 part (ii) since $f(x; t)dx$ has a bounded support that is uniformly bounded in $t \in [-2, 2]$ as shown in Proposition 3.

3.3. Comments on limits with respect to one index. Here we remark about additional information about limits with respect to one of the indexes that can be extracted from the proof of Theorem 1.

3.3.1. Formula (9) identifies the limiting spectral distribution of finite dimensional random block-matrices as the size of the blocks goes to infinity. For instance, consider the sequence of $k \times k$ random block-matrices $S_n$, for a fixed $k \in \mathbb{N}$, in which the diagonal blocks are made of $A_n$'s and all the other blocks are made of $B_n$'s. This random block matrix is studied in [9] among other things, using algebraic manipulations of block matrices. Hence, for each $n$ we can write $S_n$ as $I \otimes A_n + W \otimes B_n$, where $I$ is the $k \times k$ identity matrix and $W$ is the $k \times k$ non-random matrix whose entries are 0's on the diagonal and 1's elsewhere. By induction on $k$, one can easily find that $\mu_W = \frac{k-1}{k} \delta_{-1} + \frac{1}{k} \delta_{k-1}$. If $\{A_n\}$ and $\{B_n\}$ satisfy condition (iii) in Theorem 1 then it follows from formula (9) that

$$
\mu_{S_n} \xrightarrow{D} \frac{k-1}{k} \psi(-1,.) + \frac{1}{k} \psi(k-1,.) \quad \text{as } n \to \infty \quad \text{a.s.}
$$

3.3.2. The following result is a generalization of [7] Theorem 4, where it is assumed that the entries of $W_k$ take values $-1$ and $1$ with probabilities $\frac{1}{2}$, and $A$ and $B$ are non-random matrices.

Proposition 4. (The SS-Law) Let $\{W_k\}$ be a sequence of random matrices such that $W_k$ is $Wigner(k)$ for all $k$. Let also $A$ and $B$ be two $n \times n$ Hermitian commuting random matrices which have eigenvalues $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$, respectively. Then

$$
\mu_{B_{n,k}} \xrightarrow{D} \frac{1}{n} \sum_{j=1}^{n} \gamma_{\alpha_j, \beta_j}^2 \quad \text{as } k \to \infty \quad \text{a.s.}
$$

(17)
Proof. Since $A$ and $B$ are Hermitian and commute, then there is a unitary matrix $U$ such that

$$A = UD\text{diag}(\alpha_1, \ldots, \alpha_n) U^*$$

and

$$B = UD\text{diag}(\beta_1, \ldots, \beta_n) U^*.$$ 

It follows directly from equation (8) that

$$\lim_{k \to \infty} \text{tr}_{nk}(B^m_{n,k}) = \frac{1}{n} \sum_{j=1}^{n} (\alpha_j + t\beta_j)^m \gamma_{0,1}(dt)$$

where the last equality follows by changing of variables.

Since the probability measure on the right-hand side of (17) is a finite mixture of semicircle laws, then it has a compact support in $\mathbb{R}$. Therefore, convergence of moments implies weak convergence. □

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