TOWARDS COMPLETE INTEGRABILITY OF
TWO DIMENSIONAL POINCARÉ GAUGE GRAVITY

by

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Abstract

It is shown that gravity on the line can be described by the two dimensional (2D) Hilbert–Einstein Lagrangian supplemented by a kinetic term for the coframe and a translational boundary term. The resulting model is equivalent to a Yang–Mills theory of local translations and frozen Lorentz gauge degrees. We will show that this restricted Poincaré gauge model in 2 dimensions is completely integrable. Exact wave, charged black hole, and ‘dilaton’ solutions are then readily found. In vacuum, the integrability of the general 2D Poincaré gauge theory is formally proved along the same line of reasoning.

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1. Introduction

Recently, 2–dimensional models of gravity have attracted some attention as a conceptual “laboratory” for future studies of gravity in higher dimensions and as a basis of string theory. As is well–known, the Hilbert–Einstein Lagrangian \( s = \text{signature} \)

\[
V_{HE} = (-1)^s \frac{1}{2} R^{(1)\alpha\beta} \wedge \eta_{\alpha\beta}
\]

of GR does not yield any Einstein–type equations in two spacetime dimensions. (For \( n=2 \), no inverse fundamental length \( \ell^{-1} \) occurs in (1.1) as a coupling constant; in \( n \) dimensions this factor would be \( \ell^{2–n} \).) Therefore, in the approach of Teitelboim [1] and Jackiw (TJ model) [2,3], one had to resort to a dynamical model with constraint in which the field equation of constant or even vanishing [4] (scalar) curvature is enforced by means of a Lagrange multiplier. This teleparallelism constraint of the TJ model we will put in this paper in its proper perspective: Effectively, it yields a gauge theory of spacetime translations.

In fact, in \( n=4 \) dimensions, a theory of gravity with the constraint of vanishing Riemann–Cartan curvature \( R^{\alpha\beta} \) is known as teleparallelism theory [5,6]. It is a gauge theory of local translations [7,8] and empirically indistinguishable from Einstein’s theory of general relativity. Moreover, this model remains nontrivial in \( n = 2 \) dimensions and, as it turns out, has many salient features of the TJ model. In the context of the string theory, 2D teleparallel models were actually studied previously [9,10,11].

In this paper we demonstrate the complete integrability of 2D teleparallelism in vacuum. In accordance with old mechanical knowledge on general coordinates [12], the Lagrange multiplier \( \lambda \) of the constraint \( R^{\alpha\beta} = 0 \) converts into one of the two coordinates of our exact black hole solution.

The coupling to gauge, scalar, and spinor matter is also studied. It is a peculiar but common feature of two dimensions that all these fields have vanishing 2D spin currents \( \tau_{\alpha\beta} \). Thus the material energy–momentum current is symmetric and covariantly conserved with respect to the Riemannian connection. This already indicates that in two dimensions a decoupling from the Lorentz connection \( \Gamma^{\alpha\beta} \) occurs. It considerably facilitates the integrability of gravitationally coupled matter.

Constrained dynamical systems tend to become liberated classically or, ultimately, by quantum fluctuations. Nevertheless, we will show for the first time that the general Poincaré gauge (PG) field equations [13] can be formally solved in two dimensions. For a complete proof of integrability, the gauge field momenta have to be invertible with respect to torsion \( T^{\alpha} \) and curvature \( R^{\alpha\beta} \). This puts only very mild restrictions on the form of the gravitational gauge Lagrangian. As an application, we demonstrate that the general \( R + T^2 + R^2 \) Lagrangian is completely integrable and has black hole type solutions [14,15]. In contrast to a previous proof of Katanayev and Volovich [16], see also Ref.17, we do not have to rely on specific gauges, such as the conformal gauge for the coframe.
Our paper is organized as follows: In Sect.2 the geometrical structure of Riemann–Cartan spacetime and some of its peculiarities in two dimensions is exhibited for both signatures of the metric. The transition from the Hilbert–Einstein Lagrangian to teleparallelism is motivated in Sect.3. The resulting field equations are reduced in Sect.4 in order to facilitate the proof of complete integrability in Sect.5. In general, we obtain a black hole solution, whereas a constant torsion leads to the 2D “gravitational waves” of Sect.6. The generalization to charged black holes is straightforward. As shown in Sect. 7, the gravitationally coupled Yang–Mills system is still completely integrable. The coupling to scalar fields is notoriously difficult; nevertheless, an exact dilaton type solution has been obtained in Sect.8 in the static massless case. For the Dirac field of Sect.9, a complete decoupling from the gravitational field equations occurs at least for massless fermions. In Sect.10, the conserved Noether currents are presented such that the identification of the integration constant as the mass of the 2D black hole becomes finally established transparent in Sect.11. For an arbitrary PG gauge Lagrangian \( V \) the general field equations are formally solved completely in Sect.12. The former role of the Lagrange multiplier as a coordinate is now taken over by the momentum conjugate to the curvature. Lagrangians with invertable gauge field momenta turn out to be completely integrable. This new result is exemplified for the \( R + T^2 + R^2 \) Lagrangian in Sect.13.

2. Riemann–Cartan spacetime in \( n \) and in 2 dimensions

The geometrical arena consists of a \( n \)-dimensional differentiable manifold \( M \) together with a metric

\[
g = g_{ij} \, dx^i \otimes dx^j
\]  

(2.1)

and an orthonormal frame and coframe field, respectively,

\[
e_\alpha = e^i_\alpha \, \partial_i, \quad \vartheta^\beta = e^j_\beta \, dx^j.
\]  

(2.2)

They are reciprocal to each other with respect to the interior product \( \cdot \), i.e.,

\[
e_\alpha \vartheta^\beta = e^i_\alpha \, e^j_\beta = \delta^\beta_\alpha.
\]  

(2.3)

In the following, we adhere to the conventions (cf. Ref.[18]) that \( \alpha, \beta, \gamma... = 0,1...n-1 \) are anholonomic or frame indices, \( i, j, k... = 0,1...n-1 \) are holonomic or world indices, \( \partial_i \) are the tangent vectors, and \( \wedge \) denotes the exterior product.

Table I: Gauge field strengths, matter currents, and \( \eta \)-basis

|       | valuedness | p–form | components          | \( n = 4 \) | 3  | 2  |
|-------|------------|--------|---------------------|------------|----|----|
| \( T^\alpha \) | vector     | 2      | \( n^2(n-1)/2 \)   | 24         | 9  | 2  |
| \( R^{\alpha\beta} \) | bivector  | 2      | \( n^2(n-1)/4 \)   | 36         | 9  | 1  |
| \( \Sigma^\alpha \) | vector    | \( n-1 \) | \( n^2 \)           | 16         | 9  | 4  |
| \( \tau^\alpha_{\beta} \) | bivector  | \( n-1 \) | \( n^2(n-1)/2 \)   | 24         | 9  | 2  |
| \( \eta^\alpha \) | vector    | \( n-1 \) | \( n^2 \)           | 16         | 9  | 4  |
Anholonomic indices are lowered by means of the metric with signature $s$ with respect to an orthonormal frame:

$$o_{\alpha\beta} = e_i^\alpha e_j^\beta g_{ij}, \quad (o_{\alpha\beta}) = \text{diag}(\underbrace{-1, \cdots, 1}_s, \underbrace{1, \cdots, 1}_{n-s}).$$  \hfill (2.4)

For $s=1$ we have Minkowskian and for $s=0$ Euclidean signature. In order to be able to relate the pointwise attached tangent spaces to each other in a differentiable manner, we introduce a linear connection $\Gamma = \Gamma_{\alpha\beta}^L$ with values in the Lie algebra of the $n$–dimensional rotation group $\text{SO}(n)$ or “Lorentz” group $\text{SO}(s, n-s)$, respectively. In a holonomic basis, the connection 1–forms can be expanded as

$$\Gamma_{\alpha\beta} = \Gamma_{i}^{\alpha\beta} dx^i = -\Gamma_{\beta\alpha}. \hfill (2.5)$$

Similarly as in the 4–dimensional Poincaré gauge theory [13], the coframe $\vartheta^\alpha$ and the connection $\Gamma_{\alpha\beta}$ are regarded as gauge potentials of local translations and local Lorentz transformations, respectively. The corresponding field strengths are given by the torsion 2–form

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta = \frac{1}{2} T_{ij}^\alpha dx^i \wedge dx^j, \hfill (2.6)$$

and the curvature 2–form

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_\gamma^\beta = \frac{1}{2} R_{ij}^{\alpha\beta} dx^i \wedge dx^j = -R^{\beta\alpha}. \hfill (2.7)$$

In an RC–space in an orthonormal frame, the curvature, like the connection, is antisymmetric in $\alpha$ and $\beta$. For the irreducible decomposition of torsion and curvature in exterior form notation, see Ref. [19].

In order to isolate the Riemannian part of our Riemann–Cartan (RC) spacetime, we decompose the Riemann–Cartan connection into the Levi–Civita connection $\Gamma_{\alpha\beta}^{\{}$ and the contortion 1–form $K_{\alpha\beta} = -K_{\beta\alpha}$:

$$\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}^{\{} - K_{\alpha\beta}. \hfill (2.8)$$

Algebraically, the contortion is equivalent of the torsion according to $T^\alpha = \vartheta^\beta \wedge K_{\beta}^{\alpha}$. Then the curvature decomposes into Riemannian and contortion pieces as follows:

$$R_{\alpha\beta} = R_{\alpha\beta}^{\{} - DK_{\alpha\beta} + K_{\alpha}^{\gamma} \wedge K_{\gamma\beta}. \hfill (2.9)$$

We have developed the general geometrical formalism for arbitrary $n$ dimensions. However, as we saw already in Table I, for $n = 2$, namely for the 2–dimensional RC-space, we have 2 translation and 1 rotation generators. This allows us to introduce
a Lie (or right) duality operation, that is, a duality with respect to the Lie–algebra indices, which maps a vector into a covector and vice versa:

\[ \psi^* = \eta_{\alpha\beta} \psi^\beta \iff \psi^\alpha = (-1)^s \eta^{\alpha\beta} \psi^*_\alpha, \]

(2.10)

The complete antisymmetric tensor is defined by \( \eta_{\alpha\beta} := \sqrt{\det o_{\mu\nu}} \epsilon_{\alpha\beta} \), where \( \epsilon_{\alpha\beta} \) is the Levi-Civita symbol normalized to \( \epsilon_{01} = +1 \); for details of the \( \eta \)-basis, see Appendix A. For \( \psi^\beta = \partial^\beta \) we get \( \partial^*_\alpha = \eta_{\alpha}^* = \partial_{\alpha}^* \). In the case of a bivector–valued \( p \)-form \( \psi_{\alpha\beta} = -\psi_{\beta\alpha} \), the Lie dual is defined by

\[ \psi^* := \frac{1}{2} \eta_{\alpha\beta} \psi^\alpha_{\beta} \iff \psi_{\alpha\beta} = (-1)^s \eta_{\alpha\beta} \psi^*, \]

(2.11)

In two dimensions, we can appreciably compactify formulas, according to the notation given in Table II:

**Table II: 2D geometrical objects**

| n=2  | valuedness | p–form | components |
|------|------------|--------|------------|
| \( \Gamma^* := (1/2) \eta_{\alpha\beta} \Gamma^\alpha_{\beta} \) | scalar | 1 | 2 |
| \( t^\alpha := *T^\alpha \) | vector | 0 | 2 |
| \( t^2 := o_{\alpha\beta} t^\alpha t^\beta \) | scalar | 0 | 1 |
| \( R^* = d\Gamma^* \) | scalar | 2 | 1 |
| \( R := e_{\alpha}[e_{\beta}] R_{\alpha\beta} \) | scalar | 0 | 1 |

For \( n = 2 \) torsion is irreducible and contains only the vector piece (vector–valued 0–form, see Appendix B for further details)

\[ T^\alpha := d\partial^\alpha + (-1)^s \eta^\alpha \wedge \Gamma^* = (-1)^s t^\alpha \eta. \]

(2.12)

Since the curvature 2–form has only one irreducible component, it can be expressed in terms of the curvature scalar:

\[ R^{\alpha\beta} = -\frac{1}{2} R \partial^\alpha \wedge \partial^\beta. \]

(2.13)

Let us confine ourselves to the case \( s = 1 \) up to the end of this section. The local Lorentz transformations are defined by the \( 2 \times 2 \) matrices \( \Lambda_{\beta}^\alpha(x) \in SO(1,1) \), and read for the basic gravitational variables

\[ \partial^\alpha = \Lambda_{\beta}^{-1\alpha} \partial^\beta, \quad \Gamma^\beta_{\alpha} = \Lambda_{\gamma}^\alpha \Gamma_{\beta}^\gamma \Lambda_{\delta}^{-1\beta} - \Lambda_{\beta}^\gamma d\Lambda_{\gamma}^{-1\beta}. \]

(2.14)

With respect to the parametrization

\[ \Lambda_{\alpha}^\beta = \delta_{\alpha}^\beta \cosh \omega + \eta_{\alpha}^\beta \sinh \omega, \]

(2.15)

Eqs.(2.14) can be rewritten as follows:

\[ \partial^\alpha = \partial^\alpha \cosh \omega - \eta^\alpha \sinh \omega, \]

(2.16)

\[ \Gamma^\beta_{\alpha} = \Gamma_{\alpha}^\beta + \eta_{\alpha}^\beta d\omega, \quad \text{or} \quad \Gamma^* = \Gamma^* - d\omega. \]

(2.17)
3. Teleparallel 2D gravity

We regard gravity as a Yang–Mills type gauge theory of translations [7]. In this approach the coframe $\vartheta^\alpha$ and the torsion $T^\alpha$ are the associated gauge potentials and gauge field strengths, respectively. [The intricate details of such a (generalized) affine gauge approach are spelled out in Ref. [8]. There local translations are consider as a “hidden” gauge symmetry such that no need for a “central extension” [4] arises.]

In our new model, the two dimensional Hilbert–Einstein Lagrangian is supplemented by a kinetic term for the coframe, a cosmological term and a boundary term. Since 2–forms are constructed solely from the translational gauge potential $\vartheta^\alpha$, conventional general relativity appears to be rather “minimally” modified. Thus we consider, instead of (1.1), the 2D Lagrangian

$$V_\infty = V_{HE} + (-1)^s \frac{1}{2} T^\alpha \ast T_\alpha + \Lambda \eta - (-1)^s d(\vartheta^\alpha \wedge \ast T_\alpha) - R^{\alpha\beta} \wedge \lambda_{\alpha\beta}.$$ (3.1)

The last term depending on the Lagrange multiplier 0–form $\lambda_{\alpha\beta}$ will enforce the constraint $R^{\alpha\beta} = 0$ of vanishing Riemann–Cartan curvature on the residual Lorentz degrees of freedom. This corresponds to the teleparallelism condition and will replace the Teitelboim–Jackiw constraint of constant or, recently, vanishing Riemannian curvature $R^{\{\alpha\beta\}}$. In order to fully recognize the Yang–Mills type structure of our new Lagrangian, we employ a geometric identity (see Eq. (5.4) of Ref.[6]) which relates GR to its teleparallelism equivalent GR in $n \geq 2$ dimensions. Since the torsion 2–form is already irreducible for $n = 2$, this identity reduces rather drastically to

$$-\frac{1}{2} R^{\alpha\beta} \eta_{\alpha\beta} + \frac{1}{2} R^{\{\alpha\beta\}} \eta_{\alpha\beta} \equiv d(\vartheta^\alpha \ast T_\alpha).$$ (3.2)

Then, our new Lagrangian (3.1) can be rewritten such that the total Lagrangian reads

$$L = V_\infty + L_{\text{mat}} = (-1)^s \frac{1}{2} T^\alpha \ast T_\alpha + \Lambda \eta + (-1)^s \frac{1}{2} R^{\alpha\beta} \eta_{\alpha\beta} - R^{\alpha\beta} \lambda_{\alpha\beta} + L_{\text{mat}}. \quad (3.3)$$

This presentation of the Lagrangian clearly exhibits the leading Yang–Mills term for the translational field strength, whereas $\Lambda \eta = (\Lambda/2) \eta_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta$ formally corresponds to a mass term for the coframe. Observe that the Einstein–Cartan term $(1/2)R^{\alpha\beta} \eta_{\alpha\beta} = R^* = dT^*$ is also a boundary term in two dimensions and will, consequently, not contribute to the field equations.
4. Field equations

The gravitational field equations resulting from varying (3.3) with respect to $\vartheta^\alpha$, $\Gamma^\alpha_{\beta\gamma}$ and $\lambda^\alpha_{\beta\gamma}$ are

\begin{align*}
D^* T_\alpha - \frac{1}{2} (e_\alpha | T^\beta)^* T_\beta + (-1)^s \Lambda \eta_\alpha = (-1)^{s+1} \Sigma_\alpha, & \quad (1st) \\
D \lambda^\alpha_{\beta\gamma} - (-1)^s \vartheta_\alpha [^* T_\beta] = \tau_\alpha_{\beta\gamma}, & \quad (2nd) \\
R^\alpha_{\beta\gamma} = 0. & \quad (4.3)
\end{align*}

Observe that the Einstein–Cartan piece in (3.3) does not give a contribution due to $D \eta^\alpha_{\beta\gamma} = 0$ in two dimensions. By relaxing this teleparallelism constraint, one would obtain a more complicated model (the quadratic theory with Yang–Mills type terms in the Riemann-Cartan curvature and torsion was analyzed in [16]). We will defer the analysis of the general theory to Sect. 12. The right-hand sides are the current 1–forms $\Sigma_\alpha$ and $\tau_\alpha_{\beta\gamma}$ of energy–momentum and spin, respectively, of hypothetical 2–dimensional matter. Eqs. (4.1) and (4.2) represent four and two independent components, respectively.

The integrability condition for the second field equation is identically satisfied, because

\begin{equation*}
DD \lambda^\alpha_{\beta\gamma} = -2R_\alpha [^\gamma \lambda_\gamma_{\beta\gamma}] = 0 \quad (4.4)
\end{equation*}

in a teleparallel (Weitzenböck) spacetime, whereas

\begin{equation*}
D \left( \tau_\alpha_{\beta\gamma} + (-1)^s \vartheta_\alpha [^* T_\beta] \right) = 0 \quad (4.5)
\end{equation*}

follows from the ‘weak’ Noether identity (10.2) for matter and gravitational gauge fields, together with the first field equation. Thus, the second field equation determines (non–uniquely) the Lagrange multiplier $\lambda^\alpha_{\beta\gamma}$.

In order to simplify the field equations, we substitute $^* T^\alpha = t^\alpha$ into the field equations (4.1) and (4.2), respectively, and recall the formula $^* (\Phi \wedge \vartheta_\alpha) = e_\alpha | ^* \Phi$, which is valid for any p–form $\Phi$. Moreover note that the torsion square piece in the Lagrangian is proportional to $t^2$:

\begin{equation*}
T^\alpha ^* T_\alpha = (-1)^s t^2 \eta. \quad (4.6)
\end{equation*}

Then we find

\begin{align*}
Dt_\alpha - (-1)^s \left( \frac{1}{2} t^2 - \Lambda \right) \eta_\alpha = (-1)^{s+1} \Sigma_\alpha, & \quad (4.7) \\
D \lambda^\alpha_{\beta\gamma} - (-1)^s \vartheta_\alpha [^* t_\beta] = \tau_\alpha_{\beta\gamma}. & \quad (4.8)
\end{align*}

Let us represent the Lagrange multiplier as $\lambda^\alpha_{\beta\gamma} = (\lambda/2) \eta^\alpha_{\beta\gamma}$, where $\lambda = (-1)^s 2 \lambda^*$ according to the notation in (2.10). Then, in the last equation, it is more economical to switch over to its Lie dual by multiplying it with $\eta^{\alpha\beta}$:

\begin{equation*}
\frac{1}{2} d\lambda + (-1)^s \frac{1}{2} t_\beta \eta^\beta = \tau^*. \quad (4.9)
\end{equation*}
We do not lose any of its 4 components, if we multiply (4.7) by $\vartheta^\beta$ from the right and employ the formula $\eta_\alpha \wedge \vartheta^\beta = -\delta_\alpha^\beta \eta$:

$$
D(t_\alpha \vartheta^\beta) - (-1)^s \left( t_\alpha t^\beta - \frac{1}{2} \delta_\alpha^\beta (t^2 - 2\Lambda) \right) \eta = (-1)^{s+1} \Sigma_\alpha \wedge \vartheta^\beta .
$$

(4.10)

Thereby, the energy–momentum current of the gravitational field is nicely represented. The trace of (4.10), on substitution of (B.5) of Appendix B, reads

$$
(-1)^s d^* T + 2\Lambda \eta = \Sigma_\alpha \wedge \vartheta^\alpha .
$$

(4.11)

In a similar move, we substitute (B.4) into (4.9):

$$
d\lambda + T = 2\tau^* .
$$

(4.12)

A very useful condition for the torsion–squared function $t^2$ can be derived by transvecting (4.7) with $t^\alpha$:

$$
\frac{1}{2} dt^2 - \left( \frac{1}{2} t^2 - \Lambda \right) T = (-1)^{s+1} t^\alpha \Sigma_\alpha .
$$

(4.13)

We eliminate $T$ by means of (4.12) and find:

$$
dt^2 + (t^2 - 2\Lambda) \ d\lambda = 2 \left[ (-1)^{s+1} t^\alpha \Sigma_\alpha + (t^2 - 2\Lambda) \ \tau^* \right] .
$$

(4.14)

Let us now specialize to the vacuum field equations. They read

$$
dt^2 = - (t^2 - 2\Lambda) \ d\lambda ,
$$

(4.15)

$$
d^* T = (-1)^{s+1} 2\Lambda \ \eta ,
$$

(4.16)

$$
d\lambda = - T ,
$$

(4.17)

$$
R_{\alpha\beta} = 0 .
$$

(4.18)

In two dimensions, the volume 2-form $\eta$ equips the spacetime manifold $M$ with a symplectic structure. In vacuo, the volume 2-form turns out, via the field equation $\eta = (-1)^{s+1} d^* T/(2\Lambda) = (-1)^s d(\vartheta^\alpha t_\alpha)/(2\Lambda)$, to be an exact form, as it was conjectured by Cangemi and Jackiw (Eq.(2.A7a) of Ref.[4]). Since this volume 2–form appears explicitly in the Lagrangian (3.1), the cosmological term in (3.1) turns out to be “weakly” equivalent to the boundary term $(-1)^{s+1} d^* T/2$. Thus, to some extent, Machian ideas are realized: In fact, the total volume $\mu$ of our 2D “world” is, due to Stokes’ theorem, given by the integral of the dual torsion 1–form along the boundary:

$$
\mu(M) = \frac{(-1)^{s+1}}{2\Lambda} \int_{\partial M} ^* T .
$$

(4.19)
On the other hand the cosmological term cannot completely compensate the explicit “topological” term \( d (\vartheta^\alpha \wedge \ast T_\alpha) = -d^\ast T \) in (3.1). Observe also that, according to Ref. [4], the 1–form \( \ast T \) seems to be related to a gauge 1–form \( a \) associated with the central extension of the 2D Poincaré algebra.

In vacuo, \( T \) is also an exact form. If it were chosen as one basis 1-form, it would be a natural 1-form, that is, the Lagrange multiplier \( \lambda \) could be interpreted as a coordinate. Such a transmutation of \( \lambda \) from a “constraining force” to a generalized coordinate is known from mechanics [12] and quantum cosmology [20]. However, in our model, the vacuum field equations (4.16) and (4.17) impose the Klein–Gordon equation

\[
\square \lambda := (-1)^s [\ast d^\ast d + d^\ast d^\ast] \lambda = (-1)^s 2 \Lambda \tag{4.20}
\]
on the “would–be” coordinate \( \lambda \). Fortunately it turns out, see the next Section, that this is merely a condition on a so far unspecified metric function. Thus, Eq. (4.20) resembles the harmonic gauge condition in 4D general relativity.

Formally, Eq.(4.20) has the solution

\[
\lambda = (-1)^s \square^{-1} \Lambda, \tag{4.21}
\]
such that the constraining part of the Lagrangian (3.1) takes the form

\[
(-1)^s R^\ast \lambda = 2 R^\ast \square^{-1} \Lambda. \tag{4.22}
\]

By imposing the additional constraint \( R = \Lambda \), one can obtain the “weak” relation:

\[
(-1)^s R^\ast \lambda \asymp 2 R^\ast \square^{-1} R = -(R \square^{-1} R) \eta. \tag{4.23}
\]

In Riemannian spacetime, this term is easily recognized as Polyakov’s ‘string inspired’ [21] Lagrangian.

5. Black hole solution and complete integrability

The general quadratic Poincaré gauge theories in two dimensions (in absence of matter) are known to be completely integrable [16,17]. Usually this fact is established with the help of a convenient choice of coordinates, such as the light-cone or conformal ones. We will demonstrate that the model under consideration is also completely integrable. Again the choice of coordinates will be an essential step, but we will use an approach discussed by Solodukhin [22].

Before integrating the gravitational equations it is worth to notice that the flat (Minkowski) spacetime arises when both the Riemann-Cartan curvature and torsion are zero. The former is described by (4.18), but it is clear that torsion cannot be zero in case of a non–trivial cosmological term in (3.1), (4.15), (4.16). Hence the flat Minkowski space-time is not a vacuum solution of the theory. It is evident also,
that in general $t^2$ is non-zero — again the cosmological constant prevents its identical vanishing.

The vacuum equation (4.15), i.e.

$$ \frac{dt^2}{d\lambda} = -t^2 + 2\Lambda, $$

(5.1)
can easily be solved for $t^2 \neq 2\Lambda$ to give the square of torsion as a function of the Lagrange multiplier

$$ t^2 = 2\Lambda + (-1)^s 2M_0 e^{-\lambda}, $$

(5.2)
where $M_0$ denotes an integration constant which, for Minkowskian signature $s = 1$, will later be identified with the active gravitational mass of the configuration. Observe that for $M_0 = 0$ we recover the special solution $t^2 = 2\Lambda$ which will be analyzed in the next section.

The field equation (4.17) suggests to interpret the Lagrange multiplier $\lambda$ as a coordinate, such that $T = -d\lambda$ is one leg (‘Bein’) of an orthogonal coframe. Let us first construct the frame dual to the coframe. Define the vector field

$$ \xi^* = -\frac{t^2}{t^2} e_\alpha, $$

(5.3)
which is dual to $^*T$, i.e.,

$$ [\xi^*, ^*T] = 1, $$

(5.4)
cf. (B.9) and (B.10). In view of (B.12) the equation (4.17) yields the constancy of the $\lambda$ variable along the vector field $\xi^*$,

$$ \ell_{\xi^*} \lambda = \xi^* d\lambda = \xi^*(\lambda) = 0, $$

(5.5)
where $\ell_{\xi} = \xi^* d + d^* \xi$ is the Lie derivative. This fact is crucial, since (5.5) allows to introduce a second coordinate, say $\rho$, defined by the integral lines of the vector field $\xi^*$. In view of (5.5) the $(\lambda, \rho)$ system is orthogonal, and hence the form $^*T$ should be proportional to $d\rho$, while $T$ in view of (4.17) is already proportional to $d\lambda$.

The leg orthogonal to $T$ is $^*T$. Thus we introduce the orthogonal coordinate system $(\lambda, \rho)$. Then

$$ ^*T = B(\lambda, \rho) d\rho, $$

(5.6)
Because of the orthogonality, there enters no term proportional to $d\lambda$. We substitute the Ansatz (5.6) into (4.16), use the explicit expression (B.20) of the volume 2–form, and find

$$ \frac{\partial B(\lambda, \rho)}{\partial \lambda} = 2\Lambda \frac{B(\lambda, \rho)}{t^2}. $$

(5.7)
The same relation could be obtained from the Klein–Gordon equation (4.20) for $\lambda$. Upon integration we obtain:

$$ B(\lambda, \rho) = B_0(\rho) t^2 e^\lambda. $$

(5.8)
In terms of the frame (B.15) or (B.19) of Appendix B, the metric reads explicitly

\[ g = \delta_{\alpha\beta} \theta^\alpha \otimes \theta^\beta = (-1)^s \frac{d\lambda^2}{t^2} + \frac{B^2 d\rho^2}{t^2}, \tag{5.9} \]

or, after substituting \( t^2 \) and absorbing \( B_0(\rho) \) according to the coordinate transformation \( d\tilde{\rho} := B_0 d\rho \),

\[ g = (-1)^s 2e^{2\lambda}(M_0 e^{-\lambda} + (-1)^s \Lambda) d\tilde{\rho}^2 + \frac{d\lambda^2}{2(M_0 e^{-\lambda} + (-1)^s \Lambda)}. \tag{5.10} \]

It is remarkable to notice that this metric has the form of the black hole in the two-dimensional dilaton (string motivated) gravitational theories, widely discussed in the literature (cf. [14,15,23,24,25]). We will see in Sect.11 that the integration constant \( M_0 \) is in fact related to the mass of this black hole.

Along with the metric (5.10) one can construct the explicit form of the torsion in coordinates \( x^i = (x^0 = \rho, x^1 = \lambda) \),

\[ T^i = e^i_\alpha T^\alpha = \frac{1}{2} T_{jk}^i dx^j \wedge dx^k. \tag{5.11} \]

From (B.19) one readily obtains the frame

\[ e_\alpha = -\frac{t_\alpha}{B} \partial_\rho + (-1)^s \eta_{\alpha\beta} t^\beta \partial_\lambda, \tag{5.12} \]

and thus the components of the torsion tensor read

\[ T_{01}^0 = (-1)^s, \quad T_{01}^1 = 0. \tag{5.13} \]

Using the definition of the torsion 2-form (2.6), one can express the two-dimensional Lorentz connection (2.5), according to Table II, in the following convenient dual form

\[ \Gamma^* = (*d\vartheta^\alpha)\vartheta_\alpha + *T. \tag{5.14} \]

In order to complete the analysis of the integrability of the model under consideration one should also study the equation (4.3),(4.18) and verify that this is fulfilled on the above described solutions. In general this is a non trivial problem, especisally in presence of matter, see Sect. 7.

Notice that the above solutions completely describe the behavior of the torsion: equation (5.13) gives the torsion’s components with respect to the local coordinates \((\rho, \lambda)\), while the torsion square was obtained explicitly in (5.2). Its local Lorentz (i.e. with respect to the local orthonormal frame) components seem to remain undetermined, but this is clearly related to the gauge freedom of the model, which means that a vector at any point can be arbitrarily rotated with the help of the local Lorentz
transformations (2.14). Let us demonstrate this explicitly. Since $t^2$ is the known function of $\lambda$, one can assume the general ansatz for the local Lorentz torsion components:

\[ t^0 = t \sinh u, \quad t^1 = t \cosh u, \quad (5.15) \]

where $t = \sqrt{t^2}$, and $u = u(\rho, \lambda)$ is some function of both, spatial and time, coordinates which is real for $s = 1$ and purely imaginary for $s = 0$. Substituting this into (B.19) and differentiating, one finds

\[ d\partial^\alpha = (t^\alpha t^B \partial_\lambda (\frac{B}{t}) + \frac{1}{t} \partial_\rho u) - \eta^\alpha_\beta t^\beta \partial_\lambda u) \eta, \quad (5.16) \]

and hence the dual Lorentz connection (5.14) is calculated to be

\[ \Gamma^* = [B - t \partial_\lambda (\frac{B}{t})]d\rho - \partial_\rho ud\rho - \partial_\lambda ud\lambda = \]

\[ = [B - \partial_\lambda B + \frac{B}{2} \frac{\partial_\lambda t^2}{t^2}]d\rho - du. \quad (5.17) \]

Evidently the last term represents the local Lorentz transformation (2.21) and can be discarded by choosing in (5.15) the gauge $u = 0$. While calculating the two-dimensional Riemann–Cartan curvature, one notices that the last term in (5.17) does not contribute. Therefore (4.18), i.e., the vanishing of the curvature 2-form

\[ R^* = d\Gamma^* = 0, \quad (5.18) \]

reduces to the condition

\[ \partial_\lambda [B - \partial_\lambda B + \frac{B}{2} \frac{\partial_\lambda t^2}{t^2}] = 0 \quad (5.19) \]

Using the vacuum solution (5.8), one finds

\[ B - \partial_\lambda B + \frac{B}{2} \frac{\partial_\lambda t^2}{t^2} = - \frac{B}{2} \frac{\partial_\lambda t^2}{t^2} = (-1)^s M_0 B_0. \quad (5.20) \]

Since Eq. (5.19) holds for the solution (5.10), the proof of the integrability of the vacuum equations (4.15)–(4.18) is completed.

Let us investigate some of the properties of our black hole solutions and, in particular, compare these with the dilaton gravity black holes. In a first step, one can try to find a new coordinate $\tilde{\lambda}$ such that also $\theta^1$ can be represented as a natural leg:

\[ \theta^0 = d\tilde{\lambda} = d\lambda/\sqrt{2(M_0 e^{-\lambda} + (-1)^s \Lambda)}. \quad (5.21) \]
Clearly, one has to distinguish the different cases: $M_0 e^{-\lambda} + (-1)^s \Lambda > 0$, $= 0$ or $< 0$. Here we restrict ourselves to the first case. Then the coordinate transformation reads:

$$\lambda = - \log \left( \frac{(-1)^s \Lambda}{M_0} \sinh^{-2}(-\sqrt{(-1)^s \frac{\Lambda}{2} \tilde{\lambda}}) \right).$$  \hspace{1cm} (5.22)

Substitution into the metric (5.10) yields

$$g = \frac{M_0^2}{2\Lambda} \sinh^2(-\sqrt{(-1)^s 2\Lambda} \tilde{\lambda}) \, d\tilde{\rho}^2 + d\tilde{\lambda}^2.$$ \hspace{1cm} (5.23)

Then, the further coordinate transformation

$$\tilde{\lambda} = \frac{2}{\sqrt{(-1)^s 2\Lambda}} \arctanh \left( \frac{1}{2} \sqrt{2\Lambda((-1)^s x^2 + y^2)} \right).$$ \hspace{1cm} (5.24)

$$\tilde{\rho} = \left\{ \begin{array}{ll}
\frac{1}{M_0} \arctan \left( \frac{y}{x} \right) & \text{for } s = 0 \\
\frac{1}{M_0} \arctanh \left( \frac{y}{x} \right) & \text{for } s = 1
\end{array} \right.$$

converts the metric (5.23) into the explicit \textit{conformally flat} form

$$g = \frac{(-1)^s \, dx^2 + dy^2}{1 - \frac{1}{2}((-1)^s x^2 + y^2)}.$$ \hspace{1cm} (5.26)

6. \textbf{Gravitational waves}

In order to exhibit the propagating degrees of freedom of our model, we consider the vacuum field equations. For \textit{nonvanishing} “cosmological” constant $\Lambda$, we obtain from the 1st field equation

$$D^* \vartheta^\alpha = D \eta^\alpha = -d \log(t^2 - 2\Lambda) \wedge \eta^\alpha,$$ \hspace{1cm} (6.1)

due to the constraint (4.3), and

$$\Box \vartheta^\alpha := (-1)^s \left[ D^* D + D^* D^* \right] \vartheta^\alpha = (-1)^s \left( t^2 - 2\Lambda \right) \vartheta^\alpha.$$ \hspace{1cm} (6.2)

These gauge-covariant nonlinear \textit{Proca type equations} for the coframe are \textit{exact} consequences of our “topological” gauge model (3.1) with teleparallelism.

For the special solution $t^2 = 2\Lambda$, which has been left out in Sect. 5, these equations simplify to a wave equation for the coframe:

$$\Box \vartheta^\alpha = 0.$$ \hspace{1cm} (6.3)
We can almost adopt the solution (5.10) of the previous section for \( M_0 = 0 \), but use the coordinate freedom to put \( B_0 = e^{\pm \rho} \). Then we find the metric

\[
g = (-1)^s \frac{1}{2\Lambda} d\lambda^2 + 2\Lambda e^{2(\lambda \pm \rho)} d\rho^2
\]

of a “left– or right moving” wave solution. It is the analogue of the plane fronted gravitational wave solution

\[
g = -d\lambda^2 + L^2(\lambda - z)(e^{2\beta(\lambda - z)}dx^2 + e^{-2\beta(\lambda - z)}dy^2) + dz^2
\]

in 4D gravity, cf. [26,p.975].

It can easily be shown that the Cauchy problem for (6.3) is well–posed: In 2 dimensions, the coframe \( \theta^\alpha = e_j^\alpha dx^j \) has \( 2 \times 2 = 4 \) components. Two degrees of freedom get fixed by considering coframes in the conformal gauge \( \theta^\alpha = \Omega dx^\alpha \). Moreover, the one local Lorentz degree of freedom \( \Lambda^0^1 \) in the transformation formula (2.14) has also to be subtracted. Then for Minkowskian signature \( s = 1 \), Eq. (6.3) constitutes a hyperbolic wave equation for the conformal factor \( \Omega \) as the only remaining dynamical degree of freedom.

Thus our model contains only a massless “spin–2” mode, i.e. a “topological graviton” in two dimensions. Quantization will be straightforward. Moreover, by relaxing the teleparallelism constraint, the extended model with a Yang–Mills type curvature squared term appears to be renormalizable [27].

7. Charged black holes

Let us add to our gravitational Lagrangian (3.3) the standard Maxwell Lagrangian

\[
L_M = (-1)^s \frac{1}{2} F \wedge * F,
\]

where \( F = dA \) is the field strength for the abelian gauge potential \( A \). In two dimensions there is no magnetic field: the only component of the Maxwell tensor describes the electric field along the unique spatial direction. This is expressed by introducing the scalar \( f \) dual to the Maxwell field strength, i.e.,

\[
f := * F, \quad F = (-1)^s f \eta.
\]

The energy–momentum current in (4.1) reads

\[
\Sigma_\alpha := e_\alpha \vert L_M - (-1)^s (e_\alpha \vert F) \wedge * F = -(-1)^s \frac{1}{2} (e_\alpha \vert F) \wedge * F = -\frac{1}{2} f^2 \eta_\alpha ,
\]

whereas the spin current vanishes, i.e., \( \tau_{\alpha\beta} = 0 \), since the Lorentz connection does not couple to the Maxwell field. Observe that the energy–momentum trace, in contrast to four dimensions, does not vanish:

\[
\theta^\alpha \wedge \Sigma_\alpha = -f^2 \eta.
\]
The Maxwell equations are obtained from the variation of (7.1) with respect to $A$, and read as usually

$$d^* F = df = 0,$$  (7.5)

In two dimensions these are easily integrated to give $f = \text{const} = Q$. Moreover, this constant is indeed the conserved total electric charge.

As a result, the field equations (4.11)–(4.13) are completely integrable along the same lines, and the relevant charged black hole solutions are obtained by the following simple shift of the cosmological constant:

$$\Lambda \rightarrow \Lambda = \Lambda - \frac{1}{2} Q^2.$$  (7.6)

It is straightforward to see that this result is also valid for a Yang-Mills field with an arbitrary gauge group: After replacing $F$ by the non-abelian Lie algebra–valued form $F^A$, and the Lagrangian (7.1) by

$$L_{YM} = (-1)^s \frac{1}{2} F^A \wedge * F_A,$$

one obtains, with the aid of (7.6), the same “charged” black holes for which

$$Q^2 = f^A f_A, \quad f_A = * F_A.$$

[Note that $f_A$ is not constant in view of the nonlinear nature of the Yang-Mills equations $D^* F_A = df_A + c_{ABC} A^B f^C = 0$, but its square is conserved].

8. Coupling to a scalar field

Let us now consider a gravitationally coupled scalar field $\phi$ for which $L_{\text{mat}}$ in (3.3) is given by

$$L_\phi = (-1)^s \frac{1}{2} \partial^2 + U(\phi).$$  (8.1)

The potential $U = U(\phi)$ may include the mass term $\frac{1}{2} m^2 \phi^2$ as well as a nonlinear selfinteraction of scalar matter. Introducing the notations

$$\partial_\alpha \phi := e_\alpha [d\phi, \quad (\partial^2) = g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \quad \mathcal{L} := * L_\phi = \frac{1}{2} (\partial^2) + (-1)^s U,$$  (8.2)

where $\mathcal{L}$ is just the Lagrangian function, one finds for the sources of the gravitational field

$$\Sigma_\alpha = (-1)^s \left[ (\partial_\alpha \phi)(\partial^\beta \phi) - \delta_\alpha^\beta \mathcal{L} \right] \eta_\beta, \quad \tau_{\alpha\beta} = 0.$$  (8.3)
The gravitational field equations (4.1)–(4.3) have to be supplemented by the equation of motion of the scalar matter:

\[ *d* d\phi - \frac{dU}{d\phi} = 0. \]  

\[ (8.4) \]

As a first step towards a solution of the highly nonlinear system (4.1)–(4.3), (8.4), we will confine ourselves to the static case, such that

\[ \partial_{\rho} \phi \sim t^{\alpha} \partial_{\alpha} \phi = 0. \]  

\[ (8.5) \]

Then, the basic equations (4.15), (4.16) and (4.17) are modified as follows:

\[ dt^{2} = (-t^{2} + 2(\Lambda + (-1)^{s}\mathcal{L})) d\lambda, \]  

\[ (8.6) \]

\[ d*T = (-1)^{s+1}2(\Lambda + U)\eta, \]  

\[ (8.7) \]

\[ d\lambda = -T. \]  

\[ (8.8) \]

An important case is obtained, when the scalar field potential has non-trivial local extrema (normally, minima), say \( \phi_0 \), at which

\[ \frac{dU}{d\phi} = 0. \]  

\[ (8.9) \]

Evidently the constant configuration \( \phi = \text{const} = \phi_0 \) is a solution of (8.4) and the remaining gravitational field equations (8.5)–(8.7) are reduced to those of the “vacuum case”, except that the cosmological constant is shifted to

\[ \Lambda \rightarrow \overline{\Lambda} = \Lambda + U(\phi_0). \]  

\[ (8.10) \]

The solutions are thus again the black holes (5.10).

For the general case, the scheme of reasoning is the same as in the Sect.5: the coordinates \((\rho, \lambda)\) are introduced, such that (5.6) and (5.9) hold. However, (4.17) is replaced by (8.6), and this yields for the function \( B(\rho, \lambda) \) the condition

\[ \frac{t^{2} \partial B}{B} \partial_{\lambda} = 2(\Lambda + U). \]  

\[ (8.11) \]

In these coordinates the Klein-Gordon equation for the matter field reads explicitly

\[ \partial^{2}_{\lambda} \phi + \frac{1}{B} (\partial_{\lambda} B) \partial_{\lambda} \phi - \frac{1}{t^{2}} \frac{dU}{d\phi} = 0. \]  

\[ (8.12) \]

For \( \Lambda = U = 0 \), similarly as in an exact Einstein–dilaton field solution in four dimensions [28], the term \( \mathcal{L} = (1/2)C^2 \) effectively replaces the cosmological constant. In two dimensions this leads to a a conformally invariant model, for which the exact ‘dilaton’ solution

\[ \phi = C\lambda, \quad B_0 = B_0(\rho), \]  

\[ g = \frac{d\lambda^2}{C^2 - Ae^{-\lambda}} + (-1)^{s} \frac{B_0^2 d\rho^2}{C^2 - Ae^{-\lambda}} \]  

\[ (8.13) \]

can be obtained. In general, the integration is more difficult mainly due to the necessity to fulfil the zero curvature constraint (4.18). This imposes on \( B \) and \( t^2 \) the additional non–trivial constraint (5.18). Consequently, the system (8.5), (8.10), (8.11), (5.18) may admit a dilaton black hole type solutions only for particular forms of the potential \( U(\phi) \). However, it is likely that generic two-dimensional black holes have no ”scalar hair”. 
9. Coupling to Dirac matter

In this section and in the Appendix A we consider only Minkowski spacetime \((s = 1)\). The theory of spinors in two-dimensions can be formally constructed along the same lines as in \(n\) dimensions, see Ref.[29] and Appendices A and C. However, there are certain peculiarities due to the abelian nature of the two-dimensional Lorentz group. The most unusual feature is the absence of coupling of the 2D Dirac field to the local Lorentz connection.

Let \(L_{\text{mat}}\) in (3.3) be now the Dirac Lagrangian
\[
L_{\psi} = i \left( \bar{\psi} \gamma^\alpha \gamma^\beta \psi \right) - \frac{1}{2} \bar{\psi} \psi \eta, \tag{9.1}
\]
where \(\gamma := \gamma^\alpha \theta^\alpha\) is the matrix–valued 1-form of the Dirac algebra in 2D satisfying \(\gamma \cdot \gamma = -2\eta \gamma_5\). (For the details on spinors and realization of the Dirac algebra in two dimensions see the Appendices A and C.)

The covariant exterior derivative is defined by
\[
D\psi = d\psi + \Gamma \psi, \quad D\bar{\psi} = d\bar{\psi} - \Gamma \bar{\psi}, \tag{9.2}
\]
where
\[
\Gamma := \frac{i}{4} \Gamma_{\alpha\beta} \sigma_{\alpha\beta} = \frac{1}{4} \eta_{\alpha\beta} \Gamma^\alpha_5 = \frac{1}{2} \gamma_5 \Gamma^1 \tag{9.3}
\]
is the \(SO(1,1)\)–valued connection.

In two dimensions, the connection is not only abelian but also involves the \(\gamma_5\)–matrix. This implies that the spin current
\[
\tau_{\alpha\beta} := \frac{\partial L_{\psi}}{\partial \Gamma_{\alpha\beta}} = \frac{1}{8} \bar{\psi} (\gamma \gamma_5 + \gamma_5 \gamma) \psi = \frac{1}{8} \eta_{\alpha\beta} \bar{\psi} \psi = 0 \tag{9.4}
\]
vanishes identically on account of (C.4) and (C.5). Consequently, the Dirac Lagrangian (9.1) reduces to
\[
L_{\psi} = i \left( \bar{\psi} \gamma^\alpha \gamma^\beta \psi \right) - \frac{1}{2} \bar{\psi} \psi \eta,
\]
\[
= \frac{i}{2} \left( \bar{\psi} \gamma^\alpha \gamma^\beta \psi \right) + \frac{1}{2} \bar{\psi} \psi \eta. \tag{9.5}
\]

Variation of the Dirac action (9.1) with respect to \(\bar{\psi}\) yields the Dirac equation
\[
\gamma \cdot \gamma^\beta \psi + \frac{1}{2} \eta_{\alpha\beta} T^\beta \psi - m \psi \eta = 0. \tag{9.6}
\]
Using (9.3) and (5.14) one can verify that (9.6) does not contain torsion: the apparent term actually is cancelled by those “hidden” in the exterior covariant derivative. After defining \(D_{\alpha} = e_{\alpha} \square D\), Eq. (9.6) is equivalent to
\[
\gamma^\alpha D_{\alpha} \psi - m \psi = 0. \tag{9.7}
\]
This again proves the absence of any coupling of 2D Dirac spinors to the local Lorentz connection.

Variation of (9.5) with respect to the coframe yields the energy-momentum current
\[
\Sigma_\alpha \equiv -\frac{i}{2} (\bar{\psi}\gamma^\beta \partial_\alpha \psi - \partial_\alpha \bar{\psi}\gamma^\beta \psi) \eta_\beta,
\]
(9.8)
where we took into account that the Dirac Lagrangian vanishes “weakly”, i.e. \( L_\psi \cong 0 \), on account of the equation of motion (9.6).

Similarly as in the case of the scalar matter, we are not attempting to find the general solution, but restrict ourselves to the static case, when
\[
\partial_\rho \psi \sim t^\alpha \partial_\alpha \psi = 0.
\]
(9.9)
Then
\[
\Sigma_\alpha t^\alpha \cong 0, \quad \Sigma_\alpha \wedge \vartheta^\alpha \cong im\bar{\psi}\psi \eta,
\]
(9.10)
where again \( L_\psi \cong 0 \) is used.

Hence the gravitational field equations (4.1)–(4.3) are reduced to the following system
\[
dt^2 = (2\Lambda - t^2) T,
\]
(9.11)
\[
d^* T = (2\Lambda - im\bar{\psi}\psi) \eta,
\]
(9.12)
\[
d\lambda = -T,
\]
(9.13)
\[
R_{\alpha\beta} = 0.
\]
(9.14)

In the massless case \( m = 0 \), a static Dirac field completely decouples from the gravitational field equations. Hence, Eqs. (9.11)–(9.14) reduce to the vacuum case (4.15)–(4.18) and thus give rise to the same black hole and wave solutions. The massive spinor case will be discussed elsewhere.

10. Noether identities and conserved currents

The sources for the gravitational gauge fields are the material energy–momentum current \( \Sigma_\alpha := \delta L_{\text{mat}}/\delta \vartheta^\alpha \) and the spin current \( \tau_{\alpha\beta} := \delta L_{\text{mat}}/\delta \Gamma^{\alpha\beta} \), which are both \((n-1)\)-forms in \( n \) dimensions. In fact, in 2 dimensions \( \Sigma_\alpha \) represents stress – and this is a well-known concept of a force distributed over a (1-dimensional spacelike) line element. In 4 dimensions, however, \( \Sigma_\alpha \) describes energy–momentum distributed in a (3-dimensional) volume element. Accordingly, \( \Sigma_\alpha \) corresponds to the intuitive notions of a line–stress and energy–momentum density in 2 and 4 dimensions, respectively. This convinces us of the correctness of the interpretation of the \((n-1)\)-form \( \Sigma_\alpha \). An analogous consideration applies to \( \tau_{\alpha\beta} \) as spin moment stress and spin angular momentum density, respectively.
From local “Poincaré” invariance $R^n \cong SO(s, n - s)$ one finds [5,13], for $n \geq 2$, the 1st and the 2nd Noether identity

$$D\Sigma_{\alpha} \cong (e_{\alpha} [T^\gamma] \wedge \Sigma_{\gamma} + (e_{\alpha} [R]^\delta) \wedge \tau_{\gamma\delta}) \quad (10.1)$$

and

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha \wedge \Sigma_{\beta}]} \cong 0 \quad . \quad (10.2)$$

These equations having $n$ and $n(n - 1)/2$ independent components, respectively, hold only “weakly”, denoted by $\cong$, i.e., provided the matter field equation $\delta L/\delta \psi = 0$ is fulfilled.

In two dimensions, the Noether identities can be rewritten as

$$D\Sigma_{\alpha} \cong (-1)^s \eta_{\alpha} \wedge (t^\gamma \Sigma_{\gamma} - R\tau^*) \quad (10.3)$$

and

$$d\tau^* - \frac{1}{2} \eta_{\beta} \wedge \Sigma^\beta \cong 0 \quad . \quad (10.4)$$

Note that the Lie dual $\tau^*$ of the spin current is also given by

$$\tau^* = \frac{(-1)^s}{2} \frac{\delta L_{\text{mat}}}{\delta \Gamma^*} \quad . \quad (10.5)$$

For spinless matter, the energy–momentum becomes symmetric and covariantly conserved with respect to the Riemannian connection [30]:

$$\vartheta_{[\alpha \wedge \Sigma_{\beta}]} \cong 0 \quad , \quad D\Sigma_{\alpha}^{(1)} \cong 0 \quad . \quad (10.6)$$

If a spacetime admits symmetries, we can construct from the Noether currents a set of invariant conserved quantities, one for each symmetry. We consider Killing symmetries, where the vector field $\zeta = \zeta^\alpha e_\alpha$ is a generator of a one–parameter group of diffeomorphisms. Then it obeys the generalized Killing equation

$$\mathcal{L}_\zeta g = (\mathcal{L}_\zeta g_{\alpha\beta} + 2g_{\gamma(\alpha} e_{\beta)} \mathcal{L}_\zeta \vartheta^\gamma) \vartheta^\alpha \otimes \vartheta^\beta = 0 \quad , \quad \mathcal{L}_\zeta \Gamma_{\alpha\beta} = 0 \quad , \quad (10.7)$$

where $\mathcal{L}_\zeta$ is the usual Lie derivative and $\mathcal{L}_\zeta := \zeta [D + D\zeta]$ the gauge–covariant version for exterior forms.

As it was shown in Ref. [31], the current 1-form

$$\varepsilon_{\text{RC}} := \zeta^\alpha \Sigma_{\alpha} + (e_{\beta} [D\zeta^\gamma]) \tau_{\gamma\beta} \quad , \quad (10.8)$$

involving the exterior covariant derivative $\widehat{D}$ with respect to the transposed connection

$$\widehat{\Gamma}_{\alpha\beta} := \Gamma_{\alpha\beta} + e_{\alpha} [T^\beta] \quad , \quad (10.9)$$

is ‘weakly’ such a closed form:

$$d\varepsilon_{\text{RC}} \cong 0 \quad . \quad (10.10)$$

Thus $\varepsilon_{\text{RC}}$ is a globally conserved energy–momentum current in the presence of space-time symmetries.
11. Mass of the black hole solution

In order to apply these results to our exact black solution, observe that the metric (5.10) is independent of the coordinate $\tilde{\rho}$. The corresponding “timelike” Killing vector field $\partial_{\tilde{\rho}}$ can be immediately expanded in terms of the frame:

$$\xi^* | T = 1 \quad \Rightarrow \quad \zeta = \partial_{\tilde{\rho}} = B \xi^* = -e^{-\lambda} t^\alpha e_\alpha.$$  \hfill (11.1)

The material spin current of our exact solution vanishes [32], i.e., $\tau_{\alpha \beta} = 0$ such that (10.8) reduces to $\varepsilon_{\text{RC}} := \zeta^\alpha \Sigma_\alpha$. This is, in fact, a general feature of all the matter sources considered in this paper: the Yang–Mills bosons, the dilaton and even a Dirac field. On the other hand, for a nonzero mass $M_0$, the material energy–momentum current cannot vanish everywhere, but needs to have delta type concentration at the origin, i.e.,

$$\Sigma_\alpha \sim \delta(0) \eta_\alpha.$$ \hfill (11.2)

For the derivation of the related weakly conserved current (10.8), it is convenient to substitute for $\Sigma_\alpha$ the field equation (4.7). Due to (4.9), i.e., $t_\alpha \eta^\alpha = (-1)^s T = (-1)^{s+1} d\lambda$, we easily obtain

$$\varepsilon_{\text{RC}} = \zeta^\alpha \Sigma_\alpha = (-1)^s e^{\lambda} \left[ dt^2 + (t^2 - 2\Lambda) \lambda \right].$$ \hfill (11.3)

Since this current can be derived via

$$\varepsilon_{\text{RC}} = dM$$ \hfill (11.4)

from the superpotential

$$M = (-1)^s e^{\lambda} \left( \frac{1}{2} t^2 - \Lambda \right),$$ \hfill (11.5)

the current $\varepsilon_{\text{RC}}$ is conserved, indeed. If we go to the “mass shell”, we discover (5.2) with

$$M = M_0.$$ \hfill (11.6)
12. Integrability of the general PG equations in two dimensions

For the quadratic Poincaré gauge model (PG) in two dimensions its complete integrability in vacuum has been established [16,17,22]. However, these proofs rely on certain choices of a gauge. In this section, we will extend this result to the case of the general 2D Poincaré gauge theory without imposing any gauge condition.

In PG theory the total action of interacting matter and gravitational gauge fields reads

\[ W = \int \left[ L(\vartheta^\alpha, \Psi, D\Psi) + V(\vartheta^\alpha, T^\alpha, R^{\alpha\beta}) \right]. \]  

(12.1)

It is a functional of a minimally coupled matter field \( \Psi \), which, in general, may be a \( p \)-form, and of the geometrical variables \( \vartheta^\alpha \) and \( \Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha} \). Their independent variations yield the following field equations:

\[ \frac{\delta L}{\delta \Psi} = \frac{\partial L}{\partial \Psi} - (-1)^p D \frac{\partial L}{\partial D \Psi} = 0, \quad \text{(MATTER)} \]  

(12.2)

\[ DH^\alpha - E^\alpha = \Sigma_\alpha, \quad \text{(FIRST)} \]  

(12.3)

\[ DH^\alpha\beta - E^\alpha\beta = \tau^\alpha\beta. \quad \text{(SECOND)} \]  

(12.4)

Observe that, in two dimensions, the gauge field momenta are 0–forms:

\[ H^\alpha := -\frac{\partial V}{\partial d \vartheta^\alpha} = -\frac{\partial V}{\partial T^\alpha}, \quad \text{and} \quad H^\alpha\beta := -\frac{\partial V}{\partial d \Gamma^{\alpha\beta}} = -\frac{\partial V}{\partial R^{\alpha\beta}}. \]  

(12.5)

The sources of these Yang–Mills type field equations are the 1–forms of material energy–momentum and spin, respectively,

\[ \Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha}, \quad \tau^\alpha\beta := \frac{\delta L}{\delta \Gamma^{\alpha\beta}}. \]  

(12.6)

Due to the universality of the gravitational interaction, the 1–forms of energy–momentum

\[ E^\alpha := \frac{\partial V}{\partial \vartheta^\alpha} = e^\alpha[V + (e^\alpha T^\beta) \wedge H^\beta + (e^\alpha R^{\beta\gamma}) \wedge H^\beta\gamma], \]  

(12.7)

and

\[ E^\alpha\beta := -\mathcal{D}_{[\alpha} H_{\beta]}, \]  

(12.8)

provide a self–coupling of the gravitational gauge field.

The trace of the energy–momentum current (12.7), formed with the aid of the coframe \( \vartheta^\alpha \), in general gives us back the gauge Lagrangian \( V \) amended by Yang–Mills type terms according to:

\[ \vartheta^\alpha \wedge E^\alpha = 2V + 2T^\alpha \wedge H^\alpha + 2R^{\beta\gamma} \wedge H^\beta\gamma. \]

(12.9)
In order to reduce the field equations, we introduce the 1–forms

\[ H := H_\alpha \vartheta^\alpha, \quad *H = H_\alpha \eta^\alpha. \quad (12.10) \]

In the 1st field equation, we do not lose any of its 4 components, if we multiply (12.3) by \( \vartheta^\beta \) from the right:

\[ -D(\vartheta^\beta H_\alpha) + T^\beta H_\alpha - \vartheta^\beta \wedge E_\alpha = \vartheta^\beta \wedge \Sigma_\alpha. \quad (12.11) \]

In this presentation the energy current of the gravitational field comes out nicely. The trace of (12.11), on substitution of (12.9) and (12.10), reads

\[ -dH - T^\alpha H_\alpha - 2V - 2R^\beta\gamma \wedge H_\beta\gamma = \vartheta^\alpha \wedge \Sigma_\alpha. \quad (12.12) \]

In the 2nd field equation it is more economical to switch over to its Lie-dual by multiplying it with \( (1/2) \eta^{\alpha\beta} \) and to introduce the notation

\[ H^* = \frac{1}{2} \eta^{\alpha\beta} H_{\alpha\beta} =: \frac{1}{2} \kappa. \quad (12.13) \]

Then we get

\[ \frac{1}{2} d\kappa - \frac{1}{2} H_\alpha \eta^\alpha = \tau^*, \quad (12.14) \]

or, in view of (12.10),

\[ d\kappa - *H = 2\tau^*. \quad (12.15) \]

Observe that the gravitational energy–momentum current can be rewritten as

\[ E_\alpha = (-1)^s \left[ \mathcal{V} + t^\beta H_\beta - \frac{1}{2} R_K \right] \eta_\alpha =: (-1)^s \tilde{V} \eta_\alpha, \quad (12.16) \]

where \( \mathcal{V} := *V \) is the Lagrangian function (0–form). Then a very useful condition for the squared translational momentum

\[ H^2 := o^{\alpha\beta} H_\alpha H_\beta \quad (12.17) \]

can be derived by transvecting the first field equation (12.3) with \( H^\alpha \):

\[ \frac{1}{2} dH^2 - (-1)^s \tilde{V} *H = H^\alpha \Sigma_\alpha. \quad (12.18) \]

We eliminate \( *H \) by means of (12.15) and find:

\[ dH^2 - (-1)^s 2\tilde{V} d\kappa = 2 [H^\alpha \Sigma_\alpha - (-1)^s 2\tilde{V} \tau^*]. \quad (12.19) \]
Let us now specialize to the vacuum field equations. They read

\begin{align}
  dH^2 &= (-1)^s 2\tilde{\nabla} d\kappa, \quad (12.20) \\
  dH &= (-1)^s \left[ t^\alpha H_\alpha - 2\tilde{\nabla} \right] \eta, \quad (12.21) \\
  d\kappa &= *H. \quad (12.22)
\end{align}

Moreover, in our general PG model, we can derive from the vacuum field equations (12.21) and (12.20) the Klein–Gordon equation

\begin{equation}
  \Box \kappa := (-1)^s \left[ *d^s d + d^* d^s \right] \kappa = (-1)^s \left[ 2\tilde{\nabla} - t^\alpha H_\alpha \right] \quad (12.23)
\end{equation}

for the “would–be” coordinate \( \kappa \).

In order to obtain the general solution, one can proceed along the same lines as in the Sect.5: We introduce a coordinate system \((\rho, \kappa)\) which is related to the translational 1–forms (12.10) via

\begin{equation}
  H = B d\rho, \quad *H = d\kappa, \quad (12.24)
\end{equation}

with some function \( B(\rho, \kappa) \). Similarly as in the teleparallel case, the volume 2–form is, for \( H^2 \neq 0 \), given by

\begin{equation}
  \eta = - \frac{B}{H^2} d\kappa \wedge d\rho, \quad (12.25)
\end{equation}

cf. (B.20) with the torsion 1–form being replaced by \( *T \to (-1)^s*H \).

Insertion of this ansatz into (12.21) together with (12.20) yields

\begin{equation}
  \frac{\partial}{\partial \kappa} \ln B = (-1)^s \left[ 2\tilde{\nabla} - t^\alpha H_\alpha \right] \frac{1}{H^2} = \frac{\partial}{\partial \kappa} \ln H^2 - (-)^s \frac{t^\alpha H_\alpha}{H^2}. \quad (12.26)
\end{equation}

A formal integration of (12.26) straightforwardly leads to the solution

\begin{equation}
  B = B_0(\rho) H^2 \exp \left( (-1)^s + 1 \int d\kappa \frac{t^\alpha H_\alpha}{H^2} \right), \quad (12.27)
\end{equation}

where again \( B_0(\rho) \) is an arbitrary function only of \( \rho \).

Let us conclude with several remarks on the integration of the vacuum field equations (12.3), (12.4) and (12.20)–(12.22). First of all, we will treat the case \( H^2 = 0 \).

In two dimensions curvature has only one non-trivial component, namely the curvature scalar \( R \). Thus, in view of (2.17), the general gravitational action (12.1) has the form

\begin{equation}
  V(\theta^\alpha, T^\alpha, R^\alpha) = V(\theta^\alpha, T^\alpha, R). \quad (12.28)
\end{equation}
The gravitational gauge field momentum (12.13), on account of (12.5), can then be rewritten as
\[ \kappa = 2 \frac{\partial V}{\partial R} . \]  
(12.29)

For \( H^2 = 0 \), Eq.(12.20) yields \( \tilde{V} = 0 \), and, by (12.16), \( E_\alpha = 0 \). Hence in vacuum the 1st field equation (12.3) degenerates to \( DH_\alpha = 0 \). The integrability condition of this equation is the vanishing of the curvature: \( R_{\alpha\beta} = 0 \). If this is fulfilled, we actually return to the teleparallel case, which was analyzed in detail in previous sections. However, in general, \( R_{\alpha\beta} \neq 0 \). Thus \( H_\alpha = 0 \). Then, in combination with the equation \( \tilde{V} = 0 \), we find \( V = R\kappa/2 = R(\partial V/\partial R) \). Summarizing, one is left with two algebraic equations for curvature and torsion
\[ H_\alpha = 0, \quad V - R\frac{\partial V}{\partial R} = 0, \] 
(12.30)
the roots of which yield constant values for \( R \) and \( T \).

Let us now turn to the general case with \( H^2 \neq 0 \). Since \( R \) is scalar, it is clear from (12.28) that, modulo boundary terms, torsion can only appear in \( V \) in the form of the scalar \( t^2 \), i.e.
\[ V(\vartheta^\alpha, T^\alpha, R) = V(\vartheta^\alpha, t^2, R) = (-1)^s V(t^2, R)\eta. \] 
(12.31)
Hence the relevant translational momentum reads
\[ H_\alpha = -2 \frac{\partial V}{\partial t^2} t_\alpha := P(t^2, R)t_\alpha. \] 
(12.32)
Together with \( \kappa = \kappa(t^2, R) \), this function plays a decisive role in the formal integration of the system (12.20)–(12.22). Indeed, since \( t^\alpha H_\alpha = Pt^2 \) and \( H^2 = P^2t^2 \), one recognizes (12.20) as a well-posed equation which involves one dependent (say \( t^2 \)) and one independent (say \( R \)) variable. Provided \( V \), and hence \( P \), is smooth, the solution of this first order ordinary differential equation always exists, thus completing our formal demonstration of the integrability of the general two-dimensional Poincaré gauge theory. Remarkably, the complete vacuum solution (if \( H^2 \neq 0 \)) is again of the black hole type with the metric
\[ g = (-1)^s \frac{d\kappa^2}{H^2} + H^2 \exp \left(-1\right)^{s+1} \int \frac{t^\alpha H_\alpha}{H^2} \right) d\rho^2, \] 
(12.33)
even if it is more complicated than (5.10). Torsion and curvature for our solution are obtained by inverting the definitions in (12.5) of the gauge field momenta, or equivalently, by inverting the relations \( \kappa = \kappa(t^2, R), P = P(t^2, R) \to t^2 = t^2(\kappa, P), R = R(\kappa, P) \). For the solution to be unique, one must assume the relevant Hessian \( \frac{\partial^2 V}{\partial t^2 \partial \kappa}, \frac{\partial^2 V}{\partial t^2 \partial H^2} \) to be non-degenerate. It is straightforward to derive from (5.14) the curvature scalar of our general solution:
\[ R = -(-1)^s \frac{H^2}{B} \frac{\partial}{\partial \kappa} \left( \frac{B}{H^2} \frac{\partial}{\partial \kappa} H^2 \right). \] 
(12.34)
13. Complete integrability of quadratic PG Lagrangians in two dimensions

Let us apply these results of the general Lagrangian to a specific example, namely to a Lagrangian with terms quadratic in torsion and curvature. Since the torsion and curvature possess only one irreducible piece, respectively, the most general quadratic (parity conserving) Lagrangian reads:

\[ L = (-1)^s \left( \frac{a}{2} T_\alpha^* T^\alpha + \frac{1}{2} R^{\alpha \beta} \eta_{\alpha \beta} + \frac{b}{2} R_{\alpha \beta}^* R^{\alpha \beta} \right) + \Lambda \eta + L_{\text{mat}} \]  

(13.1)

Following the prescriptions (12.5), (12.7), and (12.8), respectively, we calculate from (13.1) the gauge field momenta

\[ H_\alpha = -(-1)^s a t_\alpha \]  

(13.2)

and

\[ H_{\alpha \beta} = -\frac{1}{2} (-1)^s (1 - b R) \eta_{\alpha \beta} \]  

(13.3)

as well as the gravitational energy–momentum current

\[ E_\alpha = -\left( \frac{a}{2} t^2 + (-1)^s b \frac{1}{4} R^2 - \Lambda \right) \eta_\alpha \]  

(13.4)

and the gravitational spin current

\[ E_{\alpha \beta} = (-1)^s a \vartheta_{[\alpha} t_{\beta]} \]  

(13.5)

Then the vacuum field equations (12.20)–(12.22) in condensed form read:

\[ a^2 d t^2 = - \left( a t^2 + (-1)^s b \frac{1}{2} R^2 - 2 \Lambda \right) d \kappa, \]  

(13.6)

\[ a \, d^* T = \left( \frac{b}{2} R^2 - (-1)^s 2 \Lambda \right) \eta, \]  

(13.7)

and

\[ d \kappa = b \, d R = -a T. \]  

(13.8)

In the special case \( t^2 = 0 \), we are led a space of constant Riemannian curvature as in the TJ model:

\[ T^\alpha = 0, \quad R^2 = (-1)^s \frac{4}{b} \Lambda. \]  

(13.9)

Otherwise, we find from (13.6) and (13.8) by integration

\[ t^2 = (-1)^s \left( 2 M_0 e^{-(b R/a)} - \frac{b}{2a} R^2 + R + (-1)^s 2 \Lambda \frac{2 \Lambda}{a} - \frac{a}{b} \right). \]  

(13.10)
According to (13.8), the torsion 1–form $T$ is again an exact form:

$$ T = d \left( -\frac{b}{a} R \right). $$

(13.11)

Thus we can repeat the reasoning of Sect.5 and regard $dR$ as one natural leg. Then $R$ is the associate “coordinate” such that $^*T$ is orthogonal to $T$, implying again the ansatz:

$$ ^*T =: B(\rho, R) \, d\rho. $$

(13.12)

Following the steps done in (12.25)–(12.26) with $^*H = -a \, T$ and $H = (-1)^*a \, ^*T$, the unknown function $B$ in (13.12) turns out to be

$$ B(\rho, R) = B_0(\rho) e^{bR/a}. $$

(13.13)

For the black hole solution, we can put $B_0 = 1$ without loss of generality and, eventually, obtain the following new 1-form basis (cf. (B.15) of Appendix B):

$$ \theta^0 := \frac{T}{\sqrt{t^2}} = -\frac{b}{a\sqrt{t^2}} \, dR, $$

$$ \theta^1 := \frac{^*T}{\sqrt{t^2}} = \sqrt{t^2} e^{bR/a} \, d\rho, $$

(13.14)

with the square of the torsion components given by (13.12). Accordingly, the torsion components fulfil the relation

$$ T^0 = 0, \quad T^1 = -\sqrt{t^2} \, \theta^0 \wedge \theta^1. $$

(13.15)

Since the metric is again given by (12.28), the proof of integrability of the general quadratic 2D PG model is formally completed.

This was first done in Ref.[22], but the following essential point was not explicitly demonstrated: Compared to teleparallelism model, where the Lagrange multiplier is an independent field which can “transmute” freely to a coordinate, there seems to be a catch in the case of the general theory. The scalar curvature $R$, regarded as a “coordinate”, may still keep a remembrance of its origin as a derivative of the Riemann–Cartan connection. For our exact solution, the connection 1–form $\Gamma^* \, \Gamma$ contains the leg $dR$ in its expansion. Thus we have to face a highly implicit interrelation between the curvature $R^* = d\Gamma^*$ and the formal coordinate $R$. Fortunately, one can show the selfconsistency of our scheme: inserting (13.9) and (13.12) into (12.34), one can verify that the scalar curvature $R(\rho, R)$ is indeed equal to $R$ regarded as the coordinate. [We checked this also also with the aid of the EXCALC package of REDUCE [33]). This finally concludes the proof of complete integrability of the $R + T^2 + R^2$ model.
Appendix A: (Anti–)selfdual basis for exterior forms in 2 dimensions

The symbol $\wedge$ denotes the exterior product of forms, the symbol $\mathbf{]}$ the interior product of a vector with a form and $\mathbf{*}$ the Hodge star (or left dual) operator, which maps a $p$–form into a $(2 - p)$–form. It has the property that

$$\mathbf{*}\Phi(p) = (-1)^{p(2-p)+s}\Phi(p), \quad (A.1)$$

where $p$ is the degree of the form $\Phi$.

The volume 2–form is defined by

$$\eta := \frac{1}{2} \eta_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta, \quad (A.2)$$

where $\eta_{\alpha\beta} := \sqrt{|\det o_{\mu\nu}|} \epsilon_{\alpha\beta}$, and $\epsilon_{\alpha\beta}$ is the Levi–Civita symbol normalized to $\epsilon_{01} = +1$. Together with $\eta$, the following forms span a basis for the algebra of arbitrary $p$–forms in $n$ dimensions,

$$\eta_\alpha := e_\alpha \mathbf{]} \eta = \mathbf{*} \vartheta_\alpha, \quad \eta_{\alpha\beta} := e_\beta \mathbf{]} \eta_\alpha = \mathbf{*}(\vartheta_\alpha \wedge \vartheta_\beta). \quad (A.3)$$

We will call the forms

$$\{\eta, \eta_\alpha, \eta_{\alpha\beta}\} \quad (A.4)$$

the $\eta$–basis of the 2–dimensional space. In 2 dimensions $\eta_{\alpha\beta}$ is a 0–form which we took for the definition of the Lie dual in (2.10). For the inversion of the Lie dual, we have to use the 2–dimensional relation

$$\eta_{\alpha\gamma} \eta^{\beta\gamma} = (-1)^s \delta_\beta^\alpha. \quad (A.5)$$

From Table I we recognize that 2–forms are of central importance in 2–dimensional gravity. This is also true for $\eta_\alpha$, which is Lie dual to the 1–form $\vartheta_\beta$. Indeed,

$$\eta_\alpha = \eta_{\alpha\beta} \vartheta^\beta = \mathbf{*} \vartheta_\alpha. \quad (A.6)$$

The exterior product of the coframe with the $\eta$–basis satisfies the following relations:

$$\vartheta^\gamma \wedge \eta_\alpha = \delta_\alpha^\gamma \eta, \quad \vartheta^\gamma \eta_{\alpha\beta} = -\delta_\alpha^\gamma \eta_\beta + \delta_\beta^\gamma \eta_\alpha, \quad (A.7)$$

which imply, in particular, that

$$\eta = \frac{1}{2} \vartheta^\beta \wedge \eta_\beta. \quad (A.8)$$
Differentiating the $\eta$’s yields:

$$D\eta_\alpha = T^\gamma \wedge \eta_{\alpha\gamma},$$

$$D\eta_{\alpha\beta} = 0.$$  \hspace{1cm} (A.9)

For $s = 1$, the Lorentz transformation (2.15) suggests to introduce the 1–forms which are irreducible with respect to the connected component of the Lorentz group:

$$^{(\pm)}_\sigma = \vartheta^\alpha \pm \eta^\alpha.$$  \hspace{1cm} (A.10)

These forms are self– and anti–self dual

$$^*^{(\pm)}_\sigma = \pm ^{\mp}\sigma_\alpha,$$  \hspace{1cm} (A.11)

and satisfy the relations

$$^{(\pm)}_\sigma \wedge ^{(\pm)}_\sigma = 0, \quad ^{(\pm)}_\sigma \wedge ^{(\mp)}_\sigma = 2\eta(\mp o^\alpha\beta - \eta^\alpha\beta).$$  \hspace{1cm} (A.12)

Under the $SO_o(1,1)$ transformations (2.20), these objects simply transform as

$$^{(\pm)}_\sigma' = e^{\pm \omega}^{(\pm)}_\sigma.$$  \hspace{1cm} (A.13)

This became manifest in the theory of spinors in two dimensions (see Sect.9). Eqs. (A.11) show that each $\sigma$–form actually has only one independent component which can be denoted as

$$^{(\pm)}_\sigma = \hat{\sigma}^0.$$  \hspace{1cm} (A.14)

For 2D spinors, these are the ”generalized Pauli matrices”.

**Appendix B: The many faces of 2-dimensional torsion**

We recognize that the translational field momentum (12.5), i.e. $H^\alpha = -\partial V/\partial T^\alpha = (-1)^{s+1}T^\alpha$, is, in 2 dimensions, a 0-form. In 2 dimensions, the torsion has 2 independent components. And so has its Hodge dual, i.e.,

$$t^\alpha := ^*T^\alpha \quad \text{with} \quad T^\alpha = (-1)^s t^\alpha,$$  \hspace{1cm} (B.1)

according to (A.1). Thus, instead of $T^\alpha$, we can express the field equations in terms of the equivalent $t^\alpha = H^\alpha$. This is more convenient, since a 0-form can be handled more easily than a 2-form. The 2-form $T^\alpha$ can also be developed with respect to the volume 2-form $\eta$:

$$T^\alpha = (-1)^s t^\alpha = (-1)^s t^\alpha 1 = (-1)^s t^\alpha * 1 = (-1)^s t^\alpha \eta.$$  \hspace{1cm} (B.2)
The torsion 2-form $T^\alpha$ is not only fully contained in the 0-form $t^\alpha$, but also in a 1-form $T$. This comes about as follows: In n-dimensions, the torsion can be decomposed into 3 irreducible pieces, a tensor, a vector, and an axial vector piece, see [19]. In 2 dimensions torsion is irreducible and only the (co-)vector piece survives:

$$(2)T^\alpha := \vartheta^\alpha \wedge (e_\beta]T^\beta) = \vartheta^\alpha \wedge T \quad \text{with} \quad T := e_\beta]T^\beta. \quad (B.3)$$

The 1-form $T$ can be expressed in terms of $t_\alpha$ as follows:

$$T = e_\beta]T^\beta = (-1)^s t^\beta e_\beta]\eta = (-1)^s t_\beta \eta^\beta. \quad (B.4)$$

Due to $e_\alpha] \ast \Phi = \ast(\Phi \wedge \vartheta_\alpha)$, its dual reads

$$\ast T = \ast(e_\beta]T^\beta) = (-1)^s e_\beta]\ast t^\beta = (-1)^s \ast(t^\beta \vartheta_\beta) = -t_\beta \psi^\beta. \quad (B.5)$$

Now it is easy to show that the vector piece of the torsion coincides with the total torsion:

$$(2)T^\alpha = \vartheta^\alpha \wedge T = (-1)^s \vartheta^\alpha \wedge t_\beta \eta^\beta = (-1)^s t_\beta \vartheta^\alpha \wedge \eta^\beta = (-1)^s t^\alpha \eta = T^\alpha. \quad (B.6)$$

Accordingly, we recognize that the torsion can alternatively presented by the 0-form $t^\alpha$, the two 1-forms $T$ or $\ast T$, or by the standard 2-form $T^\alpha$:

$$t^\alpha := \ast T^\alpha, \quad T := e_\alpha]T^\alpha, \quad \ast T = -\vartheta^\beta \ast T_\beta, \quad T^\alpha, \quad (B.7)$$

with

$$T^\alpha = (-1)^s t^\alpha = (-1)^s t^\alpha \eta = \vartheta^\alpha \wedge T = (-1)^s \eta^\alpha \wedge \ast T. \quad (B.8)$$

The set $\{t^\alpha, T, \ast T, T^\alpha\}$ of equivalent torsion forms turned out to be very useful.

For the presentation of exact 2D solutions it is rather convenient to introduce, instead of $\vartheta^\alpha$ and $e_\beta$, quite generally the new coframe $\{T, \ast T\}$ together with its dual vectors $\{\xi, \xi^\ast\}$. By duality we have:

$$\xi]T = 1 \quad \Rightarrow \quad (-1)^s \xi^\alpha e_\alpha](t_\beta \eta^\beta) = (-1)^s \xi^\alpha t_\beta \eta^\beta = 1, \quad (B.9)$$

$$\xi^\ast]\ast T = 1 \quad \Rightarrow \quad -\xi^\ast e_\alpha](t_\beta \vartheta^\beta) = -\xi^\ast e_\alpha = 1. \quad (B.10)$$

Apart from singular points, the condition $t^2 \neq 0$ holds as a result of the field equation. Then we find

$$\xi = -\frac{\eta^\alpha t^\beta}{t^2} e_\alpha, \quad \xi^\ast = -\frac{t^\alpha}{t^2} e_\alpha. \quad (B.11)$$

Furthermore we can check

$$\xi]\ast T = 0 \quad \text{and} \quad \xi^\ast]T = 0. \quad (B.12)$$
As a final proof that $T$ and $^*T$ formally span a coframe, we display the orthogonality of $\xi$ and $\xi^*$ with the help of the metric $g$. Since the $e_\alpha$ are orthonormal, we have

$$g(\xi, \xi^*) = -\frac{\eta^{\alpha\beta} t{_{\beta} t^\gamma}}{t^4} g(e_\alpha, e_\gamma) = -\frac{\eta^{\alpha\beta} t{_{\alpha} t^\beta}}{t^4} = 0. \quad (B.13)$$

The vectors $\{\xi, \xi^*\}$ are not of unit length, rather

$$\xi^2 := g(\xi, \xi) = (-1)^s \xi^*^2 := (-1)^s g(\xi^*, \xi^*) = \frac{(-1)^s}{t^2}. \quad (B.14)$$

Consequently, the new coframe

$$\theta^\alpha = \{\theta^0, \theta^1\} := \left\{ \frac{T}{\sqrt{t^2}}, \frac{^*T}{\sqrt{t^2}} \right\} = \left\{ (-1)^s \frac{t^\beta}{\sqrt{t^2}} e^\beta, -\frac{t^\beta}{\sqrt{t^2}} \vartheta^\beta \right\}, \quad (B.15)$$

is orthonormal, but not the orthogonal system $\{T, ^*T\}$. The dual frame reads

$$E_\alpha = \{E_0, E_1\} := \left\{ \frac{\eta^{\beta\gamma} t^\alpha}{\sqrt{t^2}} e^\beta, -\frac{t^\beta}{\sqrt{t^2}} e^\beta \right\}, \quad (B.16)$$

that is,

$$E_\alpha | \theta^\beta = \delta^\beta_\alpha \quad \text{and} \quad g(E_\alpha, E_\beta) = o_{\alpha\beta} \quad (B.17)$$

with

$$E_\alpha = E^i_\alpha \frac{\partial}{\partial X^i} \quad \text{and} \quad \theta^\beta = E^i_\beta dX^i, \quad (B.18)$$

where $X^i$ are some (holonomic) coordinates.

A 2-dimensional Lorentz transformation depends only on one parameter. From (B.16) we can read off its inverse

$$\vartheta^\alpha = -\frac{\eta^{\alpha\beta} t^\beta}{\sqrt{t^2}} \left( \frac{T}{\sqrt{t^2}} \right) - \frac{t^\alpha}{\sqrt{t^2}} \left( \frac{^*T}{\sqrt{t^2}} \right). \quad (B.19)$$

The volume 2-form can also be expressed in terms of the coframe (B.15):

$$\eta := \theta^0 \wedge \theta^1 = \frac{(-1)^s}{t^2} T \wedge ^*T. \quad (B.20)$$
14. Appendix C: Spinors in two dimensions

Dirac spinors in two dimensions have two (complex) components,

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]

and, as usually, the spinor space at any point of the space-time manifold is related to the tangent space at this point via the spin-tensor objects: the Dirac and the Pauli matrices.

The Dirac matrices \( \gamma^\alpha \) satisfy the standard relations

\[ \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2 \sigma^{\alpha\beta}, \]

and in 2D these \( 2 \times 2 \) matrices can be chosen to be real.

Further elements of the 2D Clifford algebra are the \( \gamma_5 \) matrix and the \( SO(1,1) \)–generator \( \sigma_{\alpha\beta} \) which are defined by

\[ \gamma_5 := \frac{1}{2} \eta_{\alpha\beta} \gamma^\alpha \gamma^\beta, \quad \sigma_{\alpha\beta} := \frac{i}{2} [\gamma_\alpha, \gamma_\beta]. \]

From (C.2) and (C.3) one can derive the useful relations:

\[ \gamma_\alpha \gamma_\beta \gamma^\alpha = 0, \quad \gamma_5 \gamma^\alpha + \gamma^\alpha \gamma_5 = 0, \quad (\gamma_5)^2 = 1, \]

and

\[ \gamma^\alpha \gamma_5 = \eta^{\alpha\beta} \gamma_\beta, \quad [\gamma_\alpha, \gamma_\beta] = -2 \eta_{\alpha\beta} \gamma_5. \]

If we introduce the matrix-valued 1–form

\[ \gamma = \gamma_\alpha \theta^\alpha, \]

Eqs. (C.2)–(C.5) now can be rewritten in Clifford algebra–valued exterior forms as

\[ \gamma \otimes \gamma = g, \quad \gamma \wedge \gamma = -2 \gamma_5 \eta, \quad \star \gamma = \gamma_5 \gamma. \]

The action of the gauge (local Lorentz) group on spinors is given by

\[ \psi \rightarrow \psi' = S \psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} S^{-1}, \]

where the Dirac conjugation is defined as

\[ \bar{\psi} := \psi^\dagger \gamma^0. \]
Eqs. (C.2) and (C.7) relate the metric structure on a spacetime to the spinor space. Then the Lorentz transformation are converted via the covering homomorphism $\overline{SO}(1,1) \approx SO(1,1)$ to the similarity transformations
\[
\gamma' = S\gamma S^{-1}, \tag{C.10}
\]
of the $\gamma$-matrices, where $\gamma' = \gamma_\alpha \theta'^\alpha$. Substituting (2.14) and using the explicit form of the local Lorentz rotations (2.15), one finds
\[
S = \exp\left(\frac{\omega}{2} \gamma_5\right) = \cosh\left(\frac{\omega}{2}\right) + \gamma_5 \sinh\left(\frac{\omega}{2}\right). \tag{C.11}
\]
This completes the definition of a spinor algebra on a 2D manifold. The next step is to develop the spinor analysis, and the central point is the notion of the so-called spinor covariant derivative $D$. The formal definition of the spinor covariant derivative is given by (9.2), where the connection 1-form, due to the covering homomorphism, has the usual transformation law
\[
\Gamma \to \Gamma' = S\Gamma S^{-1} + SdS^{-1}. \tag{C.12}
\]
The explicit form of the connection (9.3) is obtained from the natural assumption that spinor bilinear terms behave covariantly, that is $\bar{\psi}\psi$, $\bar{\psi}\gamma\psi$ and $\bar{\psi}\gamma \wedge \gamma \psi$ are the 0-, 1- and 2–forms, respectively, on the space-time manifold. This is equivalent to the condition
\[
D\gamma^\alpha = d\gamma^\alpha + \Gamma^\alpha_\beta \gamma^\beta + [\Gamma, \gamma^\alpha] = 0, \tag{C.13}
\]
for which the explicit solution is just (9.3).

The concrete realization of the Clifford algebra (C.2) and (C.7) on a 2D manifold is easily achieved in terms of the $1 \times 1$ Pauli matrices given by (A.14). According to (A.12) these satisfy
\[
\begin{align*}
(+)\sigma \wedge (+)\sigma &= (\sigma)\wedge (\sigma) = 0, & (+)\sigma \wedge (\sigma) = -(\sigma)\wedge (+)\sigma = 2\eta. \tag{C.14}
\end{align*}
\]
Then one can easily prove that
\[
\gamma := \begin{pmatrix}
0 & -(+)\sigma \\
(-)\sigma & 0
\end{pmatrix} \tag{C.15}
\]
are indeed the Dirac algebra in two dimensions. Using (A.14) one can read off from (C.15) the explicit realization in terms of the matrices
\[
\begin{align*}
\gamma^0 &= \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, & \gamma^1 &= \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, & \gamma_5 &= \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}. \tag{C.16}
\end{align*}
\]
In view of (C.10) it is clear that each of the two components of the Dirac spinor (C.1), $\psi_1$ and $\psi_2$ represent the irreducible spinor fields with the simple transformation laws
\[
\begin{align*}
\psi_1' &= e^{\omega/2}\psi_1, & \psi_2' &= e^{-\omega/2}\psi_2, \tag{C.17}
\end{align*}
\]
compare to (A.13).
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