A simple scheme for expanding a polarization-entangled $W$ state by adding one photon

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Abstract

We propose a simple scheme for expanding a polarization-entangled $W$ state. By mixing a single photon and one of the photons in an $n$-photon $W$ state at a polarization-dependent beam splitter (PDBS), we can obtain an $(n+1)$-photon $W$ state after post-selection. Our scheme also opens the door to generating $n$-photon $W$ states using single photons and linear optics.

Entanglement not only plays a central role in fundamental quantum physics [1, 2], but also has wide applications in quantum information processing, such as quantum teleportation [3], dense coding [4], quantum cryptography [5] and quantum computation [6]. While bipartite entanglement has been well understood, multipartite entanglement offers a very complicated structure. For example, it was shown that genuine three-particle entanglement can be classified into two classes by the equivalence under stochastic local operations and classical communication (SLOCC) [7]. One is the Greenberger–Horne–Zeilinger (GHZ) state [8]

$$|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

(1)

The other is the $W$ state

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

(2)

The GHZ state is usually taken as a ‘maximally entangled’ state in some senses, for instance, it violates Bell inequalities maximally. However, it is also maximally fragile, i.e., if one or more particles are lost or discarded, all the entanglement is destroyed. While, the $W$ state is very robust against the loss of one of the particles, namely, two-particle entanglement can be observed after one particle is lost or measured.

The entanglement persistency property can easily be obtained from the representation of the $n$-particle $W$ state

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|00\cdots 01\rangle + |00\cdots 10\rangle + \cdots + |01\cdots 00\rangle + |10\cdots 00\rangle)$$

$$= \frac{1}{\sqrt{n}}|n-1, 1\rangle,$$

(3)

where $|n-1, 1\rangle$ denotes the (unnormalized) totally symmetric state including $n-1$ particles in state $|0\rangle$ and one particle in state $|1\rangle$, for instance, $|3, 1\rangle = |001\rangle + |010\rangle + |010\rangle + |100\rangle$. We can see that any particle is entangled with the other particles and that all the particles are equivalent. In fact, it was shown that the $W$ state has the maximum degree of entanglement between any pair of particles [9]. These interesting features lead the $W$-class states to applications in a variety of quantum information processing tasks, such as quantum teleportation [10–12], dense coding [13] and quantum secret communication [14].

Linear optical systems have supplied a broad field for experimental implementation of multipartite entangled states. There have been many proposals [15–23] and experimental implementations [24–27] for producing $W$ states. Quite recently, Tashima et al have introduced an interesting optical gate for expanding polarization-entangled $W$ states [28]. In their scheme, after the operation of the gate on one of the photons in an $n$-photon $W$ state, an $(n+2)$-photon $W$ state can be obtained after post-selection.

In this paper, using a similar expanding principle with that in [28], we propose a simple scheme for expanding a polarization-entangled $W$ state by adding a single photon to the existing state, rather than adding two photons in [28]. Our scheme needs only a polarization-dependent beam splitter (PDBS), where one of the photons in an $n$-photon $W$ state interferes with a single photon and after post-selection an $(n+1)$-photon $W$ state can be obtained.

Before introducing our scheme we would like to note that the qubits here are all encoded in polarization states of
horizontally (H) and vertically (V) polarized photons, respectively.

One of the photons in the W state is input in mode a, and a single photon in state \(|H\rangle\) is added in mode c. A half-wave plate (HWP) oriented at 0° can introduce a phase shift of \(\pi\) between H and V polarized photons. This scheme succeeds in the case of two-fold coincidence detection in the output modes c and d.

single photons so that \(|0\rangle \equiv |H\rangle\) and \(|1\rangle \equiv |V\rangle\), where \(|H\rangle\) ((|V\rangle)) denotes the horizontal (vertical) polarization state. Our scheme for adding a single photon to an n-photon W state is depicted in figure 1. The key of our scheme is an element of PDBS, with reflectivities of

\[
\eta_H = \frac{5 - \sqrt{5}}{10} \quad \text{and} \quad \eta_V = \frac{5 + \sqrt{5}}{10},
\]

for horizontally (H) and vertically (V) polarized photons, respectively. Such a class of elements has been used in several experiments [29–32]. One of the photons in state \(|W_n\rangle\) and a single photon in state \(|H\rangle\) meet at the PDBS, which are input in modes a and b, respectively. If they are indistinguishable except for the degrees of path and polarization (four-order interference will happen for the same polarization photons), the state transformations at the PDBS can be expressed as

\[
|H\rangle_a \rightarrow \sqrt{\eta_H} |H\rangle_c + \sqrt{1 - \eta_H} |H\rangle_d,
\]

\[
|V\rangle_a \rightarrow \sqrt{\eta_V} |V\rangle_c + \sqrt{1 - \eta_V} |V\rangle_d,
\]

\[
|H\rangle_b \rightarrow \sqrt{1 - \eta_H} |H\rangle_c - \sqrt{\eta_H} |H\rangle_d.
\]

After the PDBS, we use a half-wave plate (HWP) set to 0° to introduce a phase shift of \(\pi\) between H and V polarized photons with the transformations

\[
|H\rangle_c \rightarrow |H\rangle_c, \quad |V\rangle_c \rightarrow -|V\rangle_c.
\]

Therefore, we can obtain the state transformations as follows:

\[
|H\rangle_a |H\rangle_b \rightarrow \frac{1}{\sqrt{2}}(|H\rangle_a |H\rangle_d + \cdots,
\]

\[
|V\rangle_a |H\rangle_b \rightarrow \frac{1}{\sqrt{2}}(|H\rangle_a |V\rangle_d + |V\rangle_a |H\rangle_d) + \cdots,
\]

where, for simplicity, we neglect the amplitudes that would not lead to the successful events, i.e., exactly one photon in each of the output modes c and d.

Next we explain how a single photon can be added to a state \(|W_n\rangle\) through our scheme. Since all the photons in the W state are equivalent, we can choose any photon to inject in mode a, for instance, mode n, so that we can rewrite the W state given by equation (3) as follows:

\[
|W_n\rangle = \frac{1}{\sqrt{n}}|n - 1, 1\rangle
\]

\[
= \frac{1}{\sqrt{n}}(|n - 2, 1\rangle |H\rangle_a + |H\rangle^\otimes(n-1)|V\rangle_n\rangle
\]

\[
\rightarrow \frac{1}{\sqrt{n}}(|n - 2, 1\rangle |H\rangle_a + |H\rangle^\otimes(n-1)|V\rangle_d\rangle.
\]

Then we can write the state evolution of the photon-added process as

\[
|W_n\rangle |H\rangle_b \rightarrow \frac{1}{\sqrt{5n}}(|n - 2, 1\rangle |H\rangle_a |H\rangle_d + |H\rangle^\otimes(n-1) \otimes (|H\rangle_a |V\rangle_d + |V\rangle_a |H\rangle_d) + |\Phi\rangle
\]

\[
= \frac{n + 1}{\sqrt{5n}} |W_{n+1}\rangle + |\Phi\rangle,
\]

where \(|\Phi\rangle\) is an unnormalized state including the amplitudes that would not lead to the successful events.

From equation (12) we can see that the success probability for adding a single photon to a state \(|W_n\rangle\) is \((n+1)/5n\), which approaches a constant 1/5 when n becomes large. It is not difficult to find that if we generate an N-photon W state using single photons and our scheme (see figure 2 for the schematic), the total success probability is \(N/5^{N-1}(N \geq 2)\). However, this is not optimal. By replacing the leftmost PDBS with a balanced polarization-independent beam splitter, namely \(\eta_H = \eta_V = 1/2\), the probability of success can be improved to \(N/(4 \times 5^{N-2})\). Alternatively, if we do not restrict our sources to single photons and EPR states are available, we can get a higher probability. Explicitly, with our scheme, we can first add a single photon to an EPR state ((H V) + |V H\rangle)/\sqrt{2} to get a state \(|W_3\rangle\) and then add single photons one by one as in figure 2. In this case we can obtain the state \(|W_N\rangle\) with the success probability \(N/(2 \times 5^{N-2})\). In particular, the success probability for preparing state \(|W_3\rangle\) is 3/10, which is highest compared with other linear optical schemes, as the most efficient one at present is 3/16 in [28]. The success probability for preparing state \(|W_4\rangle\) is 2/25, which is lower than 1/8 in [28] but still higher than the other linear optical schemes (the most efficient one before the scheme in [28] is 2/27 in [15]). Therefore, we believe that our scheme is experimental feasible for preparing states \(|W_3\rangle\) and \(|W_4\rangle\). However, as the number of photons increases the success probability decreases exponentially, so experimental preparing more-photon W states would still be difficult. Actually, this is a common problem in many linear optical schemes.
Finally, we would like to give a brief discussion on the comparison of our scheme with the scheme in [28]. The two schemes are based on similar expanding principles (this can be seen from equations (9) and (10) and equations (3) and (4) in their paper), but a single photon is added in our scheme rather than two photons are added in their scheme. This leads to different experimental requirements, i.e., we need single photons while they need two-photon Fock states. In their scheme, the success probability is $(2k + 1)2^{-4k}$ for preparing state $|W_{2k+1}\rangle$ and $(k + 1)2^{-4k}$ for preparing state $|W_{2(k+1)}\rangle$. Therefore, our scheme is more efficient for preparing state $|W_3\rangle$ but less efficient for preparing more-photon $W$ state.

In conclusion, we propose a simple scheme to expand a polarization-entangled $W$ state by adding a single photon. This method should be very helpful in quantum information processing in future when quantum memory and nondemolition measurements are available, because we can easily add a single photon to an existing $W$ state to get a larger one. Furthermore, our method gives a new way to prepare a $W$ state using single photons and linear optical elements.

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