Refinement of Bethe ansatz string and its alternative

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The well known string solution to Bethe ansatz equations is shown to be inconsistent in its widely accepted form. A valid refinement demanding higher order corrections in subsequent roots is identified. A new alternative string solution is proposed for finite long chains, consistent with the non-Bethe strings observed earlier.

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Seventy years ago a phenomenal paper by Hans Bethe [1], opened up a new direction in physics, namely the theory of exactly solvable quantum systems. The pioneering concepts of coordinate Bethe ansatz (CBA) and string solution to Bethe ansatz equations (BAE) at the thermodynamic limit, introduced in this paper for the isotropic spin-$1/2$ chain, proved to be an universal tool for almost all such models discovered in subsequent years. Next development, though came rather late, solved a number of important problems using the same method. Some of such milestone works, where a wide range of one dimensional models were exactly solved at the finite chain as well as in the thermodynamic limit may be listed as follows. Bose gas interacting through $\delta$-function potential [2]; $XXZ$-spin chain [3] along with the application of thermodynamic Bethe ansatz (TBA) [4]; repulsive $\delta$-function fermion gas [5,6]; 1D Hubbard model [7]; fully anisotropic $XYZ$ spin chain [8] etc. In most of these models the Bethe string plays a crucial role, especially in the TBA treatment. A systematic application of TBA to a variety of models has been considered in details in [9]. Another remarkable achievement is the algebraic formulation of Bethe ansatz (ABA) [10,11], which is more suitable for quantum field models like nonlinear Schrödinger equation (NLS) [12], sine-Gordon (SG) [13] and Liouville models [14]. Though ABA differs considerably from its coordinate formulation (CBA), the end equations, e.g. BAE turn out to be equivalent. Therefore the Bethe string solutions may appear also in such integrable field models forming quantum solitons or breathers [10,13]. The widely accepted and frequently quoted form of the Bethe string is given by

$$\lambda_l = \lambda_0 + i(l - \frac{r+1}{2}) + iO(e^{-\alpha N}), \quad l = 1, \ldots , r,$$

(1)

with $\alpha$ positive, which is supposed to satisfy the Bethe ansatz equations

$$\left(\frac{\lambda_l + \frac{1}{2}}{\lambda_l - \frac{1}{2}}\right)^N = \prod_{n \neq l} \frac{\lambda_l - \lambda_n + i}{\lambda_l - \lambda_n - i}, \quad l = 1, \ldots , r,$$

(2)

for large $N$. It is evident from the above solution that exponentially small corrections: $O(e^{-\alpha N})$ are needed for large but finite values of $N$. Moreover, the form (1) suggests that these corrections are of the same order for all the roots, i.e. they should be at least as $\delta_0 e^{-\alpha N}$. Consequently, in the first approximation, i.e. for $N = \infty$, all corrections must vanish making the string solution exact and given in the obvious form

$$\lambda_l = \lambda_0 + i(l - \frac{r+1}{2}), \quad l = 1, \ldots , r$$

(3)

We however find surprisingly that both the above well known and well accepted forms (1) and (3) are, strictly speaking, not consistent with the BAE (2) for higher $r$ ($r \geq 2$) values. We observe further that a fine-tuned form with carefully defined approximations, where higher and higher order corrections appear in the subsequent roots, can only pass for a valid solution. Such inconsistency in the commonly accepted form of the string solution and the necessity for its refinement perhaps have never been addressed before. On the other hand, in early eighties a series of papers reported results [15,16,19], where for long but finite chains some non-Bethe string solutions were found to appear. We intend to propose here a new string solution also for finite long chains, which is consistent with the BAE without finely adjusted orders of correction. Roots within such strings may group together in sets of two, three and four in the thermodynamic limit, providing an explanation for the similar non-Bethe string structures observed earlier.

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An important feature of this alternative string is that, the spacings between the imaginary parts of its roots, in contrast to the Bethe string, are no longer same for all solutions but depends on the spin excitation number \( r \). As a consequence the string length does not grow linearly with \( r \) as in the standard case, but saturates as \( \frac{1}{r} \) giving only short strings.

For concreteness we consider the Heisenberg ferromagnet with nearest neighbor interactions: \( H_s = -\frac{i}{r} \sum_{n=1}^{N} \sigma_n \cdot \sigma_{n+1} \). We look briefly into the Bethe ansatz method for solving its eigenvalue problem to see how BAE arise, before concentrating on the string solution. The Bethe ansatz for the eigenfunction in \( r \)-down spin state: \( \Psi_r = \sum_{(m_i)} a(m_1, m_2, \ldots, m_r) m_1, m_2, \ldots, m_r \) is chosen as

\[
a(m_1, m_2, \ldots, m_r) = \sum_P \exp \left( \sum_{i=1}^{r} k_{P_i} m_i + \frac{1}{2} \sum_{i<n} \phi_{P_i P_n} \right),
\]

where \( P \) is any permutation of the numbers 1, 2, \ldots, \( r \) and \( P_l \) is the number that replaces \( l \) under this permutation. It is easily seen that if none of these spin excitations are adjacent, they become noninteracting and the Schrödinger equation \( H \Psi_r = E_r \Psi_r \) leads to

\[
E_r a(m_1, m_2, \ldots, m_r) = \sum_{j=1}^{r} (a(\ldots, m_j + 1, \ldots) + a(\ldots, m_j - 1, \ldots) - 2a(\ldots, m_j, \ldots)).
\]

The ansatz (3) representing a superposition of plane waves is clearly an exact solution of (3) for arbitrary phases. However as soon as any two of the spins become adjacent, say \( m_{l+1} = m_l + 1 \), interaction sets in making the equation different from (4). These two sets of equations can only become consistent if the condition on the amplitude \( \lambda \) is easily seen by taking its modulus. RHS = \( \lambda \phi \) Therefore it is enough to check Eqn (2) only for \( \lambda = 1 \) and \( \lambda = 2 \) for demonstration consider the case \( r = 4 \) in some detail, where the Bethe string obtained from (4) is of the form

\[
\cos \frac{l \phi}{2} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i \phi_n}, \quad l = 1, \ldots, r,
\]

known as BAE. Introducing rapidity variables as

\[
\lambda_l = \frac{1}{2} \cot \frac{k_l}{2}, \quad \text{giving} \quad e^{\imath \phi_n} = \frac{\lambda_l - \lambda_n + i}{\lambda_l - \lambda_n - i}
\]

one can easily rewrite BAE (3) in the form (2). It is worth noting that in the ABA method the BAE appear automatically in the form (2), where \( \lambda_i \)'s correspond to fixed values of the spectral parameter. The solutions for \( \lambda_l \) to BAE (3) must be equivalent to the solutions \( k_l \) for (4), though their ranges are different. The total number of solutions of BAE, as shown in (4), must be given by real as well as complex roots. To find the complex solutions for \( k_l = u_l + iv_l \), we notice that for positive (negative) values of \( v_l \) the LHS of (3) exponentially vanishes (diverges) for large \( N \). Therefore at least one of the \( \phi_n \)'s must have positive (negative) imaginary part dominatingly large as \( O(N) \).

The well known \( r \)-Bethe string solution, which is supposed to fulfill this criterion is given in the form (4) or (3) for variables \( \lambda_l \), as mentioned above.

Let us however check the validity of this solution for all values of \( r \) by direct insertion into BAE (3). In the simplest case of \( r = 2 \) we get from (3) in the first approximation: \( \lambda_1 = \lambda_0 - i \frac{1}{2}, \lambda_2 = \lambda_1, \) forming a complex conjugate pair. Therefore it is enough to check Eqn (2) only for \( l = 1 \), where LHS = \( \left( \frac{\lambda_0}{\sqrt{N}} \right)^N \) gives a vanishing term at \( N = \infty \). This is easily seen by taking its modulus. RHS = \( \frac{\lambda_0 - \lambda_1 + i}{\lambda_0 - \lambda_1 - i} \) on the other hand vanishes due to \( \lambda_1 - \lambda_2 = -i \), proving thus the 2-string as a valid solution. For \( r = 3 \) with \( \lambda_2 = \lambda_0 \) real and \( \lambda_1 = \lambda_3 = \lambda_0 - i \), we have to check equations for \( l = 1, 2 \). Here again for the neighboring pairs \( \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = -i \) and consequently, the equation for \( l = 1 \) holds on similar grounds as above at large \( N \). For \( l = 2 \) on the other hand, zeros appearing in the RHS from the terms with \( n = 1 \) and \( n = 3 \) get canceled giving a nonvanishing term. This however does not lead to any contradiction, since \( \lambda_2 \) being real the LHS is a pure phase and does not vanish for any \( N \).

Since the Bethe string in the form (3) is valid, as shown above, for \( r = 2 \) and \( r = 3 \), it is probably natural to assume that it will be valid for arbitrary \( r \), by analogy. This assumption is generally made in most of the works in proving Bethe string solution for higher \( r \). We show however that this assumption is not true in general and for demonstration consider the case \( r = 4 \) in some detail, where the Bethe string obtained from (3) is of the form

\[
\sigma \cdot \sigma_{n+1} \]

\[
\phi \]

\[
\Psi \]

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E \]

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H \]

\[
H_s = -\frac{i}{r} \sum_{n=1}^{N} \sigma_n \cdot \sigma_{n+1} \]

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H_s = -\frac{i}{r} \sum_{n=1}^{N} \sigma_n \cdot \sigma_{n+1} \]

\[
\phi \]

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\Psi \]

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E \]

\[
H \]
\[ \lambda_1 = \lambda_0 - i \frac{3}{2} + iO(e^{-\alpha N}), \quad \lambda_2 = \lambda_0 - i \frac{1}{2} + iO(e^{-\alpha N}), \quad \text{with } \lambda_4 = \lambda_1^*, \quad \lambda_3 = \lambda_2^*, \]  

(8)

and one has to check only for the independent solutions with \( l = 1, 2 \). Defining for convenience a function \( V_s(x) = \frac{x^{s+i\alpha}}{s+i\alpha} \) with the obvious property

\[ V_1 \left( \mp i + O(e^{-\alpha N}) \right) = O(e^{\mp \alpha N}), \]  

(9)

we see for \( l = 1 \) that the LHS of (3): \( V^N_1(\lambda_1) = \left( \frac{\lambda_1 - i}{\lambda_0 - 2i} \right)^N \) vanishes exponentially for large \( N \), while the RHS is a product of three factors \( V_1(\lambda_1 - \lambda_2)V_1(\lambda_2 - \lambda_3)V_1(\lambda_1 - \lambda_4) \). Since only neighboring rapidities can give \( \lambda_1 - \lambda_2 = -i + O(e^{-\alpha N}) \), we conclude immediately that the RHS also vanishes exponentially due to the single factor \( V_1(\lambda_1 - \lambda_2) = O(e^{-\alpha N}) \) and the remaining giving finite contributions. This clears the string solution for \( l = 1 \). However, as we see below, the contradiction starts from \( l = 2 \). The LHS in this case: \( V^N_2(\lambda_2) = \left( \frac{\lambda_2}{\lambda_0 - 1} \right)^N \) is again exponentially small and the RHS contains three factors \( V_1(\lambda_2 - \lambda_1)V_1(\lambda_2 - \lambda_3)V_1(\lambda_2 - \lambda_4) \). However the important point is that, there are now two neighboring roots of \( \lambda_2 \) contributing as \( \lambda_2 - \lambda_3 = -i + O(e^{-\alpha N}) \) and \( \lambda_2 - \lambda_1 = i + O(e^{-\alpha N}) \) and consequently using property (1) we get \( V_1(\lambda_2 - \lambda_3) = O(e^{-\alpha N}) \), while \( V_1(\lambda_2 - \lambda_1) = O(e^{\alpha N}) \) and \( V_1(\lambda_2 - \lambda_4) \) as a finite term. Therefore the important conclusion we arrive at is that, all singularities in the terms of RHS get mutually canceled leaving only a finite contribution even at \( N \to \infty \), contrasting the LHS. It is apparent now from the above reasoning that this inconsistency would always appear for higher \( r \), in the Bethe string solution in its accepted form (3) for all its roots \( \lambda_l \), with \( l = 2, \ldots, r - 1 \), having two nearest neighbors, giving

\[ \text{LHS} = V^N_1(\lambda_l) \sim O(e^{-\alpha_l N}), \]  

(10)

while

\[ \text{RHS} \sim V_1(\lambda_l - \lambda_{l+1})V_1(\lambda_l - \lambda_{l-1}) = O(e^{-\alpha_l N})O(e^{\alpha_l N}) \sim O(1). \]  

(11)

Only for the end roots having single neighbors or for roots with real values, the form (3) remains consistent. Since for \( r = 2, 3 \) all roots fall into this category, they give valid string solutions.

The next natural question is how to refine the structure of (3), so that we can get valid Bethe string for all values of spin excitations \( r \). For answering this question we see from (14) that the reason for failure of the string form (3) is that, it allows the same order of correction \( O(e^{-\alpha N}) \) for all its roots \( \lambda_l \). Therefore, the refined form for the Bethe string, consistent for arbitrary \( r \) should be given as

\[ \lambda_l = \lambda_0 + i(l - \frac{r+1}{2}) + iO(e^{-\alpha_l N}), \quad \text{and} \quad \lambda_{r+1-l} = \lambda^*_l, \quad l = 1, \ldots, s, \]  

(12)

with \( r = 2s \) for even and \( r = 2s + 1 \) for odd \( r \) with real \( \lambda_{s+1} = \lambda_0 \). The essential point for the validity of this refined string is that, the exponential orders in (12) must be fine-tuned as a strictly growing sequence

\[ 0 < \alpha_1 < \cdots < \alpha_l < \alpha_{l+1} < \cdots < \alpha_s, \]  

(13)

and more precisely they should be given by the recurrence relation

\[ \alpha_l - \alpha_{l-1} = \nu_l, \quad \text{where} \quad \nu_l = \frac{1}{2} \ln(1 + \kappa_l) > 0, \]  

(14)

with \( \kappa_l = \frac{r+1-2l}{2s+1} > 0 \) for \( l = 1, \ldots, s \). To check the validity of the refined form (12) we insert it again in BEA (3) and notice that the LHS \( \sim O(e^{-\nu_l N}) \) at large \( N \) similar to (10), while the RHS is now given by

\[ \text{RHS} \sim V_1(\lambda_l - \lambda_{l+1})V_1(\lambda_l - \lambda_{l-1}) = \left( O(e^{-\alpha_l N}) - O(e^{-\alpha_{l+1} N}) \right) \left( O(e^{-\alpha_l N}) - O(e^{-\alpha_{l-1} N}) \right)^{-1} \approx O(e^{-\alpha_l N})O(e^{-\alpha_{l-1} N}) \sim O(e^{-\nu_l N}), \]  

(15)

which proves the claim. In deriving the expressions in (12) we have used the relation (14) and approximated \( O(e^{-\alpha_l N}) - O(e^{-\alpha_{l+1} N}) \approx O(e^{-\alpha_l N}) \) and \( O(e^{-\alpha_l N}) - O(e^{-\alpha_{l-1} N}) \approx O(e^{-\alpha_l N}) \) neglecting higher order smaller terms due to strict inequalities (13). It is important to note that since all higher orders of correction are present simultaneously in (12), it is not principally possible to approximate this solution order by order for \( r \geq 4 \) cases. For example, one may naively consider the first approximation by putting all corrections \( O(e^{-\alpha_l N}) = 0 \) in (12) reducing it to the form (3), which is however not a valid solution for \( r \geq 4 \), as we have seen above. In any approximation, one can at best neglect the highest order of correction by putting \( O(e^{-\alpha_l N}) = 0 \), keeping however all other corrections as in (12).
The limit \( N = \infty \) should be inserted in the solution only at the end after performing all calculations, even in the first approximation. For \( r = 2, 3 \) however, since \( s = 1 \), we can neglect all corrections in (12) in the first approximation and get (4) as the consistent solution. In fact for these cases only the form (12) coincides with (4).

It is fair to mention here that, though the Bethe string in the form (12) refined through (13), (14), as far as we know, has not appeared in any of the earlier works, the form presented by Gaudin agrees qualitatively with (12), though the requirement (13), (14) necessary for the validity of the solution was not specified in his paper. Note that the complex string solution for pseudomomentum \( k_i \) equivalent to (12) should be given through the mapping (10), from which it is clear that in the corresponding phase \( i \sum_{n \neq 0} \theta_n \) appearing in (4), the terms \( \phi_{\ell \pm 1} \) must have large imaginary parts having corrections like \( i\beta N \) and \(-i\beta_{l-1} N \), with the inequalities

\[
0 < \beta_1 < \cdots < \beta_{l-1} < \beta_l < \cdots < \beta_s, \quad \beta_l - \beta_{l-1} = v_1, \tag{16}
\]

reflecting (13), (14). Bethe (4) suggested also that at least one of the \( \phi_{ln} \) has to have a very large imaginary part of \( O(N) \). However the essential requirement (10) for the consistent higher \( r \) strings was not emphasized and (4) was taken as the solution for arbitrary \( r \) in (4) (see Eqn (30)). We have shown however that one can not get any solution in the form (4) for \( r \geq 4 \), since the corrections can not be neglected from the beginning, even in the first approximation.

Thus we have identified a consistent but complicated form (12) for the Bethe string, which demands simultaneous involvement of all higher order corrections (up to \( s \)) for its validity. However such fine-tuned structures growing linearly with \( r \) might be on physical grounds less probable to appear in low lying excitations. Earlier findings of short strings with non-Bethe nature seem to support such arguments. This therefore motivates us to find some new type of string solution with desirable properties like shorter string length; possibility of order by order approximation and finally possible explanation of non-Bethe strings observed earlier. The alternative string we propose has the form

\[
\lambda_l = \lambda_0 + i \left( \frac{1}{s} \left( 1 - \frac{1}{2} \right) + v_1 \right) \tag{17}
\]

Note that the corrections appearing in all its roots have the same order and the spacing between the neighboring roots \( d(1) = \frac{1}{s} \) varies inversely with \( r \). In contrast to the Bethe string with constant spacing \( d(1) = i \), the string (17) with more roots is more dense along the imaginary axis and its length: \( L(r) = \frac{2(e-1)}{r} \), measuring the distance between end roots, is bounded between \( 1 \geq L(r) \geq 2 \). Recall that for Bethe string \( L(r) = r - 1 \).

For checking the validity of (17) we put it directly in BAE (4) and observe that while the LHS is similar to (10) giving exponentially vanishing term \( V_2^N (\lambda_l) \) for large \( N \), the RHS behaves differently from the standard case (11). Since now \( d(s) = i \), the pairing partner having the crucial difference \( \lambda_l - \tilde{\lambda}_l = -i + O(e^{-\alpha N}) \) is given by \( \tilde{\lambda}_l = \lambda_{l+s} \). It is also seen easily that in this case for every complex root \( \lambda_l \) such pairing can occur only once, i.e. there can be no other root except \( \tilde{\lambda}_l \) giving either \( i \) or \(-i \). This is in sharp contrast with the standard Bethe string (4), where singularities can come from two neighboring roots. Therefore unlike (4), we get for (17) RHS \( \sim V_2 (\lambda_l - \lambda_{l+s}) = O(e^{-\alpha N}) \), i.e. an exponentially vanishing term like LHS. It is also worth noting that since the alternative string (17) allows order by order approximation one can consider the first approximation by putting \( N = \infty \), when the correction term \( O(e^{-\alpha N}) \) drops out from all roots of (17), making it an exact solution. Recall again that this is not possible for a consistent Bethe string given in the form (12).

From the above arguments it is clear that for each \( \lambda_l \) in (14) its partner \( \tilde{\lambda}_l = \lambda_{l+s} \) and its conjugate \( \lambda^*_l = \lambda_{r+1-l} \) are also included in the solution and since the partner of its partner and conjugate of its conjugate is the root itself, the group of four \( \{ \lambda_l, \lambda^*_l, \lambda^*_l, \lambda^*_l \} \) forms a close unit, being the decisive contributors to the equation at large \( N \) and thus becomes almost independent entries. Therefore all the \( r \) roots can be thought of to be broken up into units of \( four \) at the thermodynamic limit, providing an explanation of the quartet formations found earlier. However when the number \( r \) is not divisible by 4, doublet and triplet groups can also occur. For \( l = \frac{r+2}{2} \) when \( r \) even and for \( l = \frac{r+1}{2} \) when \( r \) odd, one gets \( \lambda_l = \lambda^*_l \), and consequently it can group only as a unit of two, forming a two string. For odd \( r \), \( \lambda_{l+1} = \lambda^*_l = \lambda^*_l \) is real and gives a triplet \( \{ \lambda_l, \lambda^*_l, \lambda^*_l \} \), forming a 3-string. So the general rule is that, in the thermodynamic limit \( r = 4n \) results \( n \) quartets, \( r = 4n + 1 \) gives \( n + 1 \) quartets, a doublet and a triplet, \( r = 4n + 2 \), yields \( n \) quartets and a doublet and the case \( r = 4n + 3 \) groups into \( n \) quartets and a triplet. Thus within the framework of a new kind of short-lengthed \( r \)-string (17), we can provide a possible explanation of the non-Bethe string structures that have been observed for long but finite chains (14)(15). For the Bethe string on the other hand, each root generally has two partners as its nearest neighbors, each of which in turn has its own two partners. Therefore the Bethe string can not close into any shorter substrings.

For the alternative \( r \)-string (17) the total momentum and the excitation energy may be given by \( P^{(as)}_r = \sum_{j=1}^r p_j \), with \( \cot \frac{\theta}{\alpha} = \frac{\alpha}{\beta} \), and
\[ E^{(as)}_r = \frac{\partial P}{\partial \lambda_0} = \sum_{j=1}^{\infty} \frac{2g_j}{\lambda_0^2 + g_j^2} \]  

(18)

where the factor \( g_j = g_1 - \frac{1}{2}(j - 1) \), with \( g_1 = \frac{3}{2} \) for odd and \( g_1 = \frac{3}{2} - \frac{1}{2s} \) for even values of \( r \).

We note here that though the eigenfunction (4) for solution (17) does not vanish exponentially in all directions and therefore does not give fully bound states, it should nevertheless be normalizable, since this solution can be considered only for long but finite chains with periodic boundary conditions (see [3]). We believe that the energy (18) of our string is always lower than the energy of \( r \)-free magnons with rapidity \( \lambda_0 \): 

\[ rE_1 = \frac{1}{\lambda_0^2 + g_j^2} \]

We could not prove this conjecture in general, we checked it for a number of cases. Few of such sample cases are \( r = 4, 5, 7, 9, 10 \) with \( \lambda_0 = 1.0 \), which gives \( E^{(as)}_r = 1.94, 1.92, 2.9, 3.9, 4.8 \) and \( rE_1 = 3.2, 4.0, 5.6, 7.2, 8.0 \), correspondingly. Any new string solutions like (17) or non-Bethe excitations observed earlier, if they exist, should possibly arise from the interactions of different Bethe strings at the thermodynamic limit, since the later form a complete set.

Precise formulation of the valid Bethe string (12) with (13), (14) and presentation of a new string (17) are the main results of this paper. Solution (17) can be generalized to strings with nonuniform spacing, since at the thermodynamic limit it splits functionally into smaller groups with not much relevance on their relative distances. Non-Bethe macroscopic string with nonuniform spacing has been considered recently [20]. The results obtained here for the simplest case of XXX-spin chain can be extended also to other integrable models allowing Bethe string solutions, like XXZ-spin chain, 1D Hubbard model, supersymmetric \( t-J \) model etc. and field models like NLS, SG, Liouville model. Therefore, application of the TBA by using the alternative string proposed here to the models investigated earlier with the use of Bethe strings [3] would be an interesting problem.

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