From Hurwitz numbers to Kontsevich-Witten tau-function: a connection by Virasoro operators

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In this letter we present our conjecture on the connection between the Kontsevich-Witten and the Hurwitz tau-functions. The conjectural formula connects these two tau-functions by means of the $GL(\infty)$ group element. The important feature of this group element is its simplicity: it is built of only generators of the Virasoro algebra. If proved, this conjecture would allow to derive the Virasoro constraints for the Hurwitz tau-function, which remain unknown in spite of existence of several matrix model representations, as well as to give an integrable operator description of the Kontsevich-Witten tau-function.

Introduction

While it is commonly assumed that the generating functions in enumerative geometry, which constitute a particular subclass of the string theory partition functions, are the tau-functions of integrable hierarchies, possessing nice matrix model representations, a number of cases for which this statement is established is relatively small (see e.g. [1,2] and references therein). For example, integrable hierarchies behind the full generating functions of Gromov-Witten invariants are known only for the simplest compact manifolds (a point and a two dimensional sphere) and for the certain classes of their deformations/modifications. This is why it is important to establish the common properties of the corresponding tau-functions and to investigate relations between them. In this letter we conjecture a new, integrable, reformulation of the relation between two important tau-functions, which belong to the domain of enumerative geometry: Kontsevich-Witten tau-function and a generating function of single Hurwitz numbers, which we will call the Hurwitz tau-function for the simplicity.

Since the early nineties, when Kontsevich’s matrix integral [3] was constructed for the partition function of the two-dimensional gravity partition function (generating function for the Gromov-Witten invariants of a point) investigated by Witten [4], Kontsevich-Witten tau-function became an inevitable part of mathematical physics [5]. It can arguably be considered as an elementary building block for more complicated partition functions of string theory, which unifies various topological/combinatorial invariants

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possessing universal integrable properties. As a consequence of its special role the Kontsevich-Witten partition function is very well studied and a lot of elements of universal description, which are still lacking for more complicated models, are known in this case. Among such interrelated elements are Virasoro constraints, integrable properties, moment variables description, random partitions representation, spectral-curve-based description, free fermion representation, a vast net of connections with other models and, of course, Kontsevich matrix integral representation (see, e.g., [4][7][8] and references therein).

Another classical subject of enumerative geometry is the theory of the Hurwitz numbers. The generating function of the Hurwitz numbers can be represented in terms of the cut-and-join operator [9][10]. This operator belongs to the (central extension of the) $gl(\infty)$ algebra, acting transitively on the space of KP solutions, what guarantees the KP integrability of the generating function. This generating function can be restored by topological recursion [11][12], and several matrix models representations of this generating function are known [13–15].

In this letter we conjecture a novel relation between the Kontsevich-Witten tau-function and the Hurwitz tau-function. These two functions are very well known to be deeply related [16–19] and our conjecture is based on these known relations. In particular, the Hurwitz numbers are related to the Hodge integrals via profound Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula [20]. On the other side, a generating function of the Hodge integrals is a deformation of the correspondent Gromov-Witten generating function by a Givental operator [21–24]. The first part of this connection is known to be given, up to the linear change of variables, by a $GL(\infty)$ operator, which respects integrability. Integrable properties of the second part of this relation are more subtle. Namely, the generating function of the Hodge integrals and the Kontsevich-Witten tau-function are connected with each other by a Givental operator, which does not belong to the $GL(\infty)$ group, supplemented by a linear change of variables, which is not equivalent to the local change of the spectral parameter. Our conjecture is that, while neither of two elements (Givental operator and change of variables) respects KP integrability, their combination can be substituted by a simple $GL(\infty)$ operator built of only negative components of Virasoro algebra, which does respect the KP integrability. Thus a $GL(\infty)$ operator connecting two tau-functions is particulary simple: it consists of all components of the Virasoro subalgebra of $gl(\infty)$.

$$Z_K(t_k; \beta) = e^{\hat{Q}_0 \cdot \hat{Q}_+ \cdot e^{-\sum_{k>0} \frac{k-1}{k} \hat{L}_k^2} Z_H(t_k; \beta)}$$

(1)

where

$$\hat{Q}_0 = a_0 L_0$$

$$\hat{Q}_\pm = \sum_{k>0} a_{\pm k} \hat{L}_\pm$$

(2)

are linear combinations of the Virasoro operators [14] and $a_k$ are the rational numbers. Let us stress that l.h.s. of (2) does not depend on $\beta$ and even times $t_{2k}$.

This type of connection between two KP tau-functions, which was investigated in [25], is one of the simplest possible among all Bäcklund transformations, and it naturally generalizes a notion of equivalent hierarchies [24]. Namely, instead of an operator with positive components of the Virasoro algebra, which describes a connection between equivalent hierarchies, here there appears an operator consisting of all components of the Virasoro algebra.

The organization of this paper is as follows. In Section 1 we remind the reader some details about the Kontsevich-Witten tau-function and a construction of the cut-and-join-type representation for it. This representation is not quite satisfactory, because the operator does not belong to $GL(\infty)$ group. In Section 2 we remind the reader some basic facts about $gl(\infty)$ symmetry algebra of the KP hierarchy. In Section 3 we define the Hurwitz tau-function in terms of the cut-and-join operator and give some related matrix
model representations of this function. Section 4 is devoted to the formulation of our conjectural formula. In Section 5 we give our conclusions and open questions.

1 Kontsevich-Witten tau-function

In this section we basically follow a matrix model theory point of view and do not give an explanation of the enumerative geometry origin of the Kontsevich-Witten tau-function, which can be found elsewhere. Kontsevich matrix integral over Hermitian matrix $\Phi$

$$Z_K(\tau_k) = \frac{\int [d\Phi] \exp \left(-\frac{1}{g} \text{Tr} \left( \frac{\Phi^3}{3!} + \Lambda \Phi^2 \right) \right)}{\int [d\Phi] \exp \left(-\frac{1}{g} \text{Tr} \Lambda \Phi^2 \right)}$$

(3)

gives the Kontsevich-Witten tau-function dependent of the times given by the Miwa variables

$$\tau_k = \frac{g}{(2k + 1)} \text{Tr} \frac{1}{\Lambda^{2k+1}}$$

(4)

Infinite size of the matrix $\Phi$ guarantees that all $\tau_k$ can be considered as independent variables. This tau-function has a natural genus expansion:

$$Z_K(\tau_k) = \exp \left( \sum_{h=0}^{\infty} g^{2h-2} \mathcal{F}_K^{(h)}(\tau_k) \right)$$

(5)

It satisfies to an infinite set of the differential equations known as the Virasoro constraints

$$\hat{L}_n Z_K = \frac{\partial}{\partial \tau_{n+1}} Z_K, \quad n \geq -1$$

(6)

where the operators

$$\hat{L}_m = \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tau_k \frac{\partial}{\partial \tau_{k+m}} + \frac{g^2}{8} \sum_{k=0}^{m-1} \frac{\partial^2}{\partial \tau_k \partial \tau_{m-k-1}} + \frac{g^2}{2} \delta_{m,-1} + \frac{1}{16} \delta_{m,0}, \quad m \geq -1$$

(7)

constitute a subalgebra of the Virasoro algebra:

$$[\hat{L}_n, \hat{L}_m] = (n - m) \hat{L}_{n+m}$$

(8)

These Virasoro constraints allow us to derive a cut-and-join-type representation for the Kontsevich-Witten tau-function [27]:

$$Z_K = e^{\hat{W}_K} \cdot 1$$

(9)

where

$$\hat{W}_K = \frac{2}{3} \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tau_k \hat{L}_{k-1}$$

$$= \frac{2}{3} \sum_{k,m \geq 0}^{\infty} \left( k + \frac{1}{2} \right) \left( m + \frac{1}{2} \right) \tau_k \tau_m \frac{\partial}{\partial \tau_{k+m-1}}$$

$$+ \frac{g^2}{12} \sum_{k,m \geq 0} \left( k + m + \frac{5}{2} \right) \tau_{k+m+2} \frac{\partial^2}{\partial \tau_k \partial \tau_m} + \frac{1}{g^2} \frac{\tau_3^3}{3!} + \frac{\tau_1}{4}$$

(10)
The most important disadvantage of this representation is that the operator $\hat{W}_K$ does not belong to the $\mathfrak{gl}(\infty)$ algebra so that the KdV integrability is hidden. The main result of this note is a conjecture, which represents the Kontsevich-Witten tau-function in terms of the finite number of simple $GL(\infty)$ operators acting on the trivial tau-function.

2 $\mathfrak{gl}(\infty)$ algebra

$GL(\infty)$ group is generated by normal ordered powers $\hat{J}(z)^k$ of the bosonic current on a sphere

$$\hat{J}(z) = \sum_{k=1}^{\infty} \left( k t_k z^{k-1} + g^2 \frac{z^{k+1}}{2^k \partial t_k} \right)$$

For instance, for the spin $k = 1$

$$\hat{J}(z) := \hat{J}(z)$$

so that the corresponding group elements are given by constant time shift $\exp(\alpha_k \partial t_k)$ and by multiplication by $\exp(\alpha_k t_k)$, which obviously preserve the KP equations. The key role in our construction is played by the Virasoro algebra, which appears at $k = 2$:

$$\hat{J}(z)^2 := 2 g^2 \sum_{k=-\infty}^{\infty} \frac{\hat{L}_k}{z^{k+2}}$$

where

$$\hat{L}_k = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + g^2 \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b} + \frac{1}{2 g^2} \sum_{a+b=-m} ab t_a t_b$$

These operators constitute a central extension of the Virasoro algebra with central charge $c = 1$ [28]:

$$\left[ \hat{L}_n, \hat{L}_m \right] = (n-m) \hat{L}_{n+m} + \frac{1}{12} (n^3 - n) \delta_{n,-m}$$

Let us stress that the Virasoro operators (11) for the Witten-Kontsevich tau function can be given in terms of the current

$$\hat{J}(z) = \sum_{k=1}^{\infty} (2k + 1) t_k z^{2k} + \frac{g^2}{2z^{2k+2}} \frac{\partial}{\partial t_k}$$

which almost coincides with the odd part of (11), namely:

$$\hat{J}(z)^2 := 2 g^2 \sum_{k=-\infty}^{\infty} \frac{\hat{L}_k}{z^{2k+2}} - \frac{g^2}{8z^2}$$

This is the different pre-factor in front of the second term in this current what is partially responsible for the difference between $t_k$ and $t_{2k+1}$ in (23).

Among higher spin operators the most important for us is an operator with $k = 3$

$$\hat{J}(z)^3 := 3 g^2 \sum_{k=-\infty}^{\infty} \frac{W_k}{z^{k+3}}$$

a component of which

$$W_0 = \sum_{i,j \geq 1} i j t_i t_j \frac{\partial}{\partial t_{i+j}} + g^2 (i + j) t_{i+j} \frac{\partial^2}{\partial t_i \partial t_j}$$

is the cut-and-join operator of the Hurwitz tau-function.
3 Hurwitz tau-function

A geometrical definition of the Hurwitz numbers and their generating function can be found elsewhere. Here, to fix our notations, we give a representation of the generating function of the simple Hurwitz numbers in terms of the cut-an-join operator \(19\), namely \([9, 10]\):

\[
Z_H(t_k; \beta) = \exp \left( \frac{\beta}{2} \hat{W}_0 \right) \cdot \exp \left( \frac{t_1}{g^2} \right)
\]

(20)

For this generating function several matrix model representations are known, among which let us mention the simplest ones. Namely, the Hurwitz tau-function can be represented as a Hermitian matrix integral \([14]\):

\[
Z_H(t_k; \beta) = \int_{N \times N} [d\mu(\Phi)] \exp \left( -\frac{1}{2\beta} \text{Tr} \Phi^2 + \text{Tr} \left( e^{\Phi - N\beta/2} \psi \right) \right)
\]

(21)

where the measure can be represented in terms of the standard flat measure as follows:

\[
[d\mu(\Phi)] = \sqrt{\det \sinh \left( \frac{Y \otimes 1 - 1 \otimes Y}{2} \right)} [d\Phi]
\]

(22)

and times \(t_k\) are given by the Miwa variables

\[
t_k = \frac{1}{k} \text{Tr} \psi_k
\]

(23)

The same tau-function can be given by an integral over normal matrices \([15]\):

\[
Z_H(t_k; \beta) = \int_{N \times N} \frac{|dZ|}{\det (ZZ^\dagger)^{N+\frac{1}{2}}} \exp \left( -\frac{1}{2\beta} \text{Tr} \log^2 ZZ^\dagger + \text{Tr} Z Z^\dagger + \sum_{k=1}^{\infty} t_k \text{Tr} Z^k \right)
\]

(24)

In both formulas it is assumed that the size of the matrices tends to infinity and unimportant \(t_k\)-independent factors are omitted.

4 Relation between two tau-functions

The main result of our “experimental” investigation is a conjectural formula connecting two tau-functions (Kontsevich-Witten and Hurwitz) defined in the previous sections. Namely, on the basis of the explicit calculations we claim that\(^2\)

\[
Z_K(\tau_k = 2^k t_{2k+1}) = \hat{U}_{KH} \cdot Z_H(t_k; \beta)
\]

(25)

with

\[
\hat{U}_{KH} = e^{\hat{Q}_0} \cdot e^{\hat{Q}_+} \cdot e^{\hat{Q}_-} \cdot e^{-\sum_{k>0} \frac{k-1 \cdot k-1 \cdot k}{6 \cdot g^2}}
\]

(26)

and

\[
\hat{Q}_0 = a_0 \log \beta \hat{L}_0
\]

\[
\hat{Q}_\pm = \sum_{k>0} a_{\pm k} \beta^{\pm k} \hat{L}_{\pm k}
\]

\(^2\)We have compared about 150 first terms of both sides of this relation.
The first part of the relation between two tau-functions is known to be given by a $GL(\infty)$ operator \[18\], namely

$$Z_{Hodge}(t) = \exp\left(\sum_{k<0} a_k \beta^{-k} L_k \right) \exp\left(-\sum_{k=1}^{\infty} \frac{k^{k-1} \beta^{(k-1)} t_k}{k! g^2}\right) Z_H(t)$$

$$= 1 + \frac{t_2 \beta^3}{6} + \frac{t_1^2 h + 8 t_1^3 + 6 t_3 g^2}{48 g^2} \beta^4 + \frac{1}{120} t_2 \left(60 t_1 + g^2\right) \beta^5$$

$$+ \frac{1}{1440} \frac{500 h t_2^2 + 414 t_3 h^2 + 80 t_1^3 h + t_1^2 h^2 + 240 t_1^4 + 2160 t_1 t_3 h \beta^6}{h}$$

$$+ \frac{1}{10080} \frac{2 t_2 h^3 + 21224 t_4 h^2 + 15155 t_1^2 t_2 h + 7000 t_1^3 t_2 + 5250 t_2 t_3 h + 1400 t_1^2 h^2 + 16800 t_4 t_1 h \beta^7}{h}$$

$$+ O(\beta^8)$$

(27)

The coefficients $a_k$ for the negative $k$ can be restored from the equation \[18\]

$$\exp\left(\sum_{k>0} a_{-k} \beta^{k+1} \frac{\partial}{\partial z}\right) \cdot z = \frac{z}{1+z} e^{-\frac{z}{1+z}}$$

(28)

The few first coefficients $a_k$ are given in the following table:

| $k$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ | $-8$ | $-9$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $a_k$ | $-2$ | $\frac{1}{2}$ | $\frac{1}{2} \cdot 3$ | $0$ | $\frac{3}{2^3 \cdot 5}$ | $\frac{1}{2^3 \cdot 3^2 \cdot 5}$ | $\frac{211}{2^3 \cdot 3^3 \cdot 7}$ | $\frac{187}{2^3 \cdot 3^3 \cdot 7}$ | $\frac{21751}{2^7 \cdot 3^2 \cdot 5 \cdot 7}$ |

We have found the coefficients $a_k$ for the positive $k$ by a direct calculation, namely

| $k$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $a_k$ | $-\frac{2}{3}$ | $\frac{2^2}{3^2 \cdot 5}$ | $-\frac{2}{3^3 \cdot 5}$ | $\frac{11}{3^3 \cdot 5 \cdot 7}$ | $\frac{11}{3^3 \cdot 5^2}$ | $-\frac{2^4}{3^3 \cdot 5^2}$ | $-\frac{359}{3^3 \cdot 5^3 \cdot 7}$ | $\frac{13 \cdot 3137}{2 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11}$ |

and $a_0 = -\frac{1}{4}$, as it follows from the Theorem 2.3 of \[18\]. Unfortunately, we do not know any simple recursion relation for the $a_k$ with positive $k$. The only observation we made is that

$$f(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^{-k} \frac{\partial}{\partial z}\right) \cdot \frac{z}{1+z} e^{-\frac{z}{1+z}}$$

(29)

is symmetric

$$f(z) = f(-z)$$

(30)

which could be related to the fact that the Kontsevich-Witten tau-function satisfies to the KdV equations, that is 2-reduction of the KP hierarchy. Let us stress that the operator, connecting $Z_K$ and $Z_H$ can be represented in various ways, so most probably the representation (26) can be simplified.
The formula (25) relating two tau-function can be of use for both of them. For the Kontsevich-Witten tau-function it gives an integrable representation in terms of $GL(\infty)$ operators. This relation also could be used for construction of a Kontsevich-Witten tau-function matrix model representation in terms of times, contrary to the Kontsevich matrix integral integral representation, dependent on the external matrix.

For the Hurwitz tau-function this relation allows to derive a set of Virasoro constraints by a simple conjugation of the Virasoro constraints for the Kontsevich-Witten tau-function. Namely

$$\hat{\mathcal{L}}_n Z_H(t_k; \beta) = 0, \quad n \geq -1$$ (31)

where

$$\hat{\mathcal{L}}_n = \hat{U}_K^{-1} \left( \partial \tau_{n+1} \frac{\partial}{\partial \tau_{n+1}} \right) \hat{U}_K$$ (32)

We guess that this transformation can be given in terms of the global current on the Lambert spectral curve. Let us stress that in this formulation the change of the variables is performed by the Virasoro operators, thus a problem with non-invertible change of variables does not appear at all.

5 Conclusion and open questions

In this letter we present a conjectural relation between two important generating functions of enumerative geometry: the Kontsevich-Witten tau-function and the generating function of single Hurwitz numbers. This conjecture, if proved, should help in unification of three overlapping, but different types of operators, which show themselves in modern string theory/enumerative geometry. These three types of operators are Givental operators, $GL(\infty)$ operators and cut-and-join operators.

Givental operators constitute a special class of exponential operators given by quantization of quadratic hamiltonians [23, 24], and they correspond to particular transformations of the bosonic currents on the spectral curve. They allow to express some of non-trivial partition functions of string theory in terms of elementary building blocks, such as Kontsevich-Witten tau-functions, and they are conjectured to be applicable in more general setup. In particular, the Givental operators appear in the matrix model theory [29-34]. Their relation to integrability remains poorly investigated, see, however, [35].

$GL(\infty)$-operators, by definition, preserve the KP-type integrability of the partition functions, that’s why relations in terms of these operators look particularly attractive. They are generated by the powers of the bosonic fields currents on the Riemann surfaces. Some matrix integrals can be related with elementary functions by means of $GL(\infty)$ operators [14, 15].

The cut-and-join operator, appearing in the description of the single and double Hurwitz numbers, also belongs to this $gl(\infty)$ algebra. But, some other cut-and-join operators, appearing in the description of more general Hurwitz partition functions, are not of this integrable form [36-38].

It would be extremely interesting to understand the meaning of our conjectural relation in terms of matrix integrals. Since both tau-functions can be represented as matrix integrals dependent on external matrix it could be possible to clarify our relation in terms of the operators, acting on the eigenvalues. Relations of this type should generalize the connections between different solutions of the Generalized Kontsevich model, given by equivalent hierarchies (for a review see [39]).

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