An upper bound for the least energy of a sign-changing solution to a zero mass problem

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Abstract

We give an upper bound for the least energy of a sign-changing solution to the nonlinear scalar field equation

$$-\Delta u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N),$$

where $N \geq 5$ and the nonlinearity $f$ is subcritical at infinity and supercritical near the origin. More precisely, we establish the existence of a nonradial sign-changing solution whose energy is smaller than $12c_0$ if $N = 5, 6$ and smaller than $10c_0$ if $N \geq 7$, where $c_0$ is the ground state energy.

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1 Introduction

The aim of this note is to give an upper bound for the least energy of a sign-changing solution to the problem

$$-\Delta u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N),$$

(1.1)

where $N \geq 5$ and the nonlinearity $f$ is subcritical at infinity and supercritical near the origin. More precisely, we assume

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(f1) \( f \in C^{1,0}_{\text{loc}}(\mathbb{R}) \) with \( \alpha \in \left( \frac{N}{2(N-2)}, 1 \right] \), and there exist \( a_1 > 0 \) and \( 2 < p < 2^* := \frac{2N}{N-2} < q \) such that, for \( \kappa = -1, 0, 1 \),

\[
|f(s)| \leq \begin{cases} 
  a_1 |s|^{p-(\kappa+1)} & \text{if } |s| \geq 1, \\
  a_1 |s|^{q-(\kappa+1)} & \text{if } |s| \leq 1,
\end{cases}
\]  

(1.2)

where \( f^{(-1)} := F, f^{(0)} := f, f^{(1)} := f' \), and \( F(s) := \int_0^s f(t) \, dt \).

(f2) There is a constant \( \theta > 2 \) such that \( 0 \leq \theta F(s) \leq f(s)s - f'(s)s^2 \) for all \( s > 0 \).

(f3) \( f \) is odd.

In their seminal paper [1], Berestycki and Lions showed that problem (1.1) has a ground state solution which is positive, radially symmetric and decreasing in the radial direction. One or multiple positive solutions for a similar equation involving a scalar potential that decays to zero at infinity, both in the whole space and in an exterior domain, have been obtained, for instance, in [3, 7, 8, 12].

The existence of nonradial sign-changing solutions to (1.1) was recently shown by Mederski in [15]. It is readily seen that the energy of any sign-changing solution must be greater than twice the energy of the ground state. But, to our knowledge, there are no upper estimates for the least energy of a sign-changing solution to (1.1).

Our aim is to prove the following result.

**Theorem 1.1.** Assume that \( f \) satisfies (f1) – (f3). Then, there exists a nonradial sign-changing solution \( \tilde{\omega} \) to the problem (1.1) whose energy satisfies

\[
2c_0 < \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \tilde{\omega}|^2 - \int_{\mathbb{R}^N} F(\tilde{\omega}) < \begin{cases} 
  12c_0 & \text{if } N = 5, 6, \\
  10c_0 & \text{if } N \geq 7,
\end{cases}
\]  

(1.3)

where \( c_0 \) is the ground state energy of (1.1). Furthermore, for each \( (z_1, z_2, y) \in \mathbb{R}^N \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \),

(a) \( \tilde{\omega}(z_1, z_2, y) = \tilde{\omega}(e^{2\pi i j/5} z_1, e^{2\pi i j/5} z_2, y) \) for all \( j = 0, \ldots, 4 \),

(b) \( \tilde{\omega}(z_1, z_2, y) = -\tilde{\omega}(z_2, z_1, y) \),

(c) \( \tilde{\omega}(z_1, z_2, y_1) = \tilde{\omega}(z_1, z_2, y_2) \) if \( |y_1| = |y_2| \),

and \( \tilde{\omega} \) has least energy among all nontrivial solutions satisfying (a), (b), (c).

In the positive mass case, for the subcritical pure power nonlinearity \( f(u) = |u|^{p-2}u \) with \( 2 < p < 2^* \), an estimate for the least energy of a sign-changing solution was obtained in [11] Theorem 1.1], and recently improved in [10] Corollary 1.2]. On the other hand, it was shown in [9] Theorem 1.1] that the same estimates as in Theorem 1.1] hold true for the critical pure power nonlinearity \( f(u) = |u|^{2^*-2}u \).

As in [15], to prove the existence of a sign-changing solution to (1.1) we take advantage of suitable symmetries that produce a change of sign by construction. The
symmetries introduced in [15], however, have only infinite and trivial orbits. This
does not allow estimating the energy of the solution. Here, in contrast, we consider
symmetries given by a finite group. This makes it harder to show existence due to the
lack of compactness but, once the existence of a solution is established, one immediately
gets an upper estimate for its energy.

As in [11, 10, 9] we use concentration compactness techniques to establish a condi-
tion for the existence of a symmetric minimizer for the variational problem associated
to (1.1). Then, we consider a suitable ansatz given as a sum of positive and negative
copies of the ground state solution placed along some orbit in a convenient way. But,
unlike in the subcritical case where the decay of the ground state is exponential, here
the decay is polynomial. A careful estimate for the interaction among the terms of the
ansatz, that applies to this situation, is provided by Lemma 3.3.

Another delicate issue is produced by the fact that our nonlinearity is not the pure
power one. This asks for a proper estimate of its lack of additivity. Lemma B.2 gives
such an estimate for a nonlinearity \( f \) satisfying milder regularity assumptions than
those considered in [1].

The condition \( \alpha \in \left( \frac{N^2}{2(N-2)}, 1 \right) \) in assumption \( (f_1) \) implies that \( N \geq 5 \). This condition
is needed to ensure that some terms measuring the deviation from additivity of the
nonlinearity are lower order terms, as pointed out in the proof of Proposition 3.2.

This paper is organized as follows. In Section 2 we study the symmetric variational
problem and give a condition for the existence of a symmetric minimizer. In Section
3 we show that this condition is satisfied for some particular symmetries and we
prove Theorem 1.1. In the appendices we prove some lemmas required to achieve
that purpose.

2 The symmetric variational setting

We assume throughout that \( f \) satisfies \((f_1)\), \((f_2)\) and \((f_3)\).

Let \( G \) be a closed subgroup of the group \( O(N) \) of linear isometries of \( \mathbb{R}^N \) and
denote by

\[
G_x := \{gx : g \in G\} \quad \text{and} \quad G_x := \{g \in G : gx = x\}
\]

the \( G \)-orbit and the \( G \)-isotropy group of a point \( x \in \mathbb{R}^N \). The \( G \)-orbit \( Gx \) is \( G \)-
homeomorphic to the homogeneous space \( G/G_x \). So both have the same cardinality,
i.e., \( |Gx| = |G/G_x| \).

Let \( \phi : G \to \mathbb{Z}_2 := \{-1, 1\} \) be a continuous homomorphism of groups satisfying
\((A_\phi)\) If \( \phi \) is surjective, then there exists \( \zeta \in \mathbb{R}^N \) such that \( (\ker \phi)\zeta \neq G\zeta \),
where \( \ker \phi := \{g \in G : \phi(g) = 1\} \). A function \( u : \mathbb{R}^N \to \mathbb{R} \) such that

\[
u(gx) = \phi(g)u(x) \quad \text{for all} \quad g \in G, \quad x \in \mathbb{R}^N,
\]

will be called \( \phi \)-equivariant. If \( \phi \equiv 1 \) is the trivial homomorphism then \( u \) is \( G \)-invariant,
i.e., it is constant on every \( G \)-orbit, while if \( \phi \) is surjective and \( u \neq 0 \) then \( u \) is nonradial
and changes sign. If \( K \) is a closed subgroup of \( G \) we write \( \phi|K : K \to \mathbb{Z}_2 \) for the restriction of \( \phi \) to \( K \). Note that \( \phi|K \) satisfies \((A_\phi|K)\) if \( \phi \) satisfies \((A_\phi)\).

Let \( D^{1,2}(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N) \} \), with its standard scalar product and norm

\[
(u,v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \quad \|u\| := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2},
\]

and set

\[\phi(\mathbb{R}^N):= \{ u \in D^{1,2}(\mathbb{R}^N) : u \text{ is } \phi\text{-equivariant} \} .\]

Assumption \((A_\phi)\) guarantees that \(D^{1,2}(\mathbb{R}^N)\) has infinite dimension, see [5].

By the principle of symmetric criticality [16], the \( \phi\)-equivariant solutions to the problem (1.1) are the critical points of the functional \( J : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[ J(u) := \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(u), \]

where \( F(u) := \int_0^u f(s) \, ds \). This functional is well defined and of class \( C^2 \), with derivative

\[ J'(u)v = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} f(u)v, \quad u,v \in D^{1,2}(\mathbb{R}^N); \]

see [2] Proposition 3.8 or [3] Lemma 2.6. The nontrivial \( \phi \)-equivariant solutions belong to the set

\[ N^\phi := \{ u \in D^{1,2}(\mathbb{R}^N) : u \neq 0, J'(u)u = 0 \}, \]

which is a closed \( C^1 \)-submanifold of \( D^{1,2}(\mathbb{R}^N) \) and a natural constraint for \( J \), and

\[ c^\phi := \inf_{u \in N^\phi} J(u) > 0; \]

see [7] Lemma 3.2.

Next, we give a description of the minimizing sequences for \( J \) on \( N^\phi \). We need the following lemmas. Set \( B_R(y) := \{ x \in \mathbb{R}^N : |x - y| < R \} \).

**Lemma 2.1.** If \((u_k)\) is bounded in \( D^{1,2}(\mathbb{R}^N) \) and there exists \( R > 0 \) such that

\[ \lim_{k \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_k|^2 \right) = 0, \]

then \( \lim_{k \to \infty} \int_{\mathbb{R}^N} f(u_k)u_k = 0. \)

**Proof.** See [7] Lemma 3.5. \( \square \)

**Lemma 2.2.** If \( u_k \rightharpoonup u \) weakly in \( D^{1,2}(\mathbb{R}^N) \) then, passing to a subsequence,

\[ (a) \int_{\mathbb{R}^N} |f(u_k) - f(u)||\varphi| = o(1) \text{ for every } \varphi \in C_c^\infty(\mathbb{R}^N), \]
For the sequence $(k)$, closed subgroup $K$ of $G$ such that, up to a subsequence, the following statements hold true:

Proof. See [7, Lemma 3.8].

Lemma 2.3. For any given sequence $(y_k)$ in $\mathbb{R}^N$, there exist a sequence $(\xi_k)$ in $\mathbb{R}^N$ and a closed subgroup $K$ of $G$ such that, up to a subsequence, the following statements hold true:

(a) $\text{dist}(Gy_k, \xi_k) \leq C_0$ for all $k \in \mathbb{N}$ and some positive constant $C_0$.

(b) $G\xi_k = K$ for all $k \in \mathbb{N}$.

(c) If $|G/K| < \infty$ then $\lim_{k \to \infty} |g\xi_k - \xi_k| = \infty$ for any pair $g, \xi \in G$ with $\xi g^{-1} \notin K$.

(d) If $|G/K| = \infty$ then, for each $n \in \mathbb{N}$, there exists $g_n \in G$ such that $\lim_{k \to \infty} |g_m\xi_k - g_n\xi_k| = \infty$ for any $m, n \in \mathbb{N}$ with $m \neq n$.

Proof. These statements follow from [6, Lemma 3.2].

The proof of the next statement follows [11, Theorem 2.1]. We give the details for the sake of completeness.

Proposition 2.4. If

$$c^\phi < |G/\xi| c^{\phi|\xi|} \quad \text{for every } \xi \in \mathbb{R}^N \text{ with } G\xi \neq G,$$

then $c^\phi$ is attained by $J$ on $N^\phi$.

Proof. Let $u_k \in N^\phi$ be such that $J(u_k) \to c^\phi$. Then,

$$\int_{\mathbb{R}^N} f(u_k)u_k = \|u_k\|^2 \geq 2J(u_k) \geq c^\phi > 0$$

for large enough $k$. Thus, after passing to a subsequence, Lemma 2.1 yields $\delta > 0$ and $y_k \in \mathbb{R}^N$ such that

$$\int_{B_1(y_k)} |u_k|^2 = \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |u_k|^2 \geq \delta \quad \text{for all } k \in \mathbb{N}. \quad (2.1)$$

For the sequence $(y_k)$ we choose a sequence $(\xi_k)$ in $\mathbb{R}^N$, a closed subgroup $K$ of $G$ and $C_0 > 0$ as in Lemma 2.3 and we set $w_k(x) := u_k(x + \xi_k)$. Then, $w_k \in N^{\phi|K}$ and $J(w_k) \to c^\phi$. From assumption $(f_2)$ we get that $(w_k)$ is bounded in $D^{1,2}(\mathbb{R}^N)$; see [7, Lemma 3.6]. So, after passing to a subsequence, $w_k \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$, $w_k \to w$.
a.e. in $\mathbb{R}^N$ and $w_k \to w$ strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$. As $B_1(g_k y_k) \subset B_{C_k + 1}(\zeta_k)$ for some $g_k \in G$ and $|u_k|$ is $G$-invariant, (2.1) yields

$$\int_{B_{C_k + 1}(0)} |w_k|^2 = \int_{B_{C_k + 1}(\zeta_k)} |u_k|^2 \geq \int_{B_1(g_k y_k)} |u_k|^2 = \int_{B_1(y_k)} |u_k|^2 \geq \delta \quad \text{for all } k \in \mathbb{N}.$$  

Therefore $w \neq 0$ and, using Ekeland’s variational principle and Lemma 2.2 we see that $w$ solves (1.1). Note that, as each $w_k$ is $\phi|K$-equivariant, $w$ is also $\phi|K$-equivariant.

Let $g_1, \ldots, g_n \in G$ be such that $|g_i \xi_k - g_j \xi_k| \to \infty$ if $j \neq i$. Then, for each $j \in \{1, \ldots, n\}$

$$\phi(g_j) \left( w_k \circ g_j^{-1} \right) - \sum_{i=j+1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k + g_j \xi_k) \to \phi(g_j) \left( w \circ g_j^{-1} \right)$$

weakly in $D^{1,2}(\mathbb{R}^N)$, where the sum is defined to be zero if $j = n$. It follows that

$$\left\| \phi(g_j) \left( w_k \circ g_j^{-1} \right) - \sum_{i=j+1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k + g_j \xi_k) \right\|^2$$

$$= \left\| \phi(g_j) \left( w_k \circ g_j^{-1} \right) - \sum_{i=j}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k + g_j \xi_k) \right\|^2 + \left\| \phi(g_j) \left( w \circ g_j^{-1} \right) \right\|^2 + o(1).$$

Since $u_k$ is $\phi$-equivariant, performing the change of variable $y = z - g_i \xi_k$ and recalling that $w_k(x) = u_k(x + \xi_k)$, we obtain

$$\left\| u_k - \sum_{i=j+1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right\|^2$$

$$= \left\| u_k - \sum_{i=j}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right\|^2 + \|w\|^2 + o(1),$$

and iterating this identity for $j = 1, \ldots, n$, we deduce that

$$\|u_k\|^2 - \left\| u_k - \sum_{i=1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right\|^2 + n\|w\|^2 + o(1).$$

Similarly, using Lemma 2.2 we get

$$\int_{\mathbb{R}^N} F(u_k) = \int_{\mathbb{R}^N} F \left( u_k - \sum_{i=1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right) + n \int_{\mathbb{R}^N} F(w) + o(1),$$

$$\int_{\mathbb{R}^N} f(u_k)u_k = \int_{\mathbb{R}^N} f \left( u_k - \sum_{i=1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right) \left[ u_k - \sum_{i=1}^{n} \phi(g_i) \left( w \circ g_i^{-1} \right) (\cdot - g_i \xi_k) \right]$$

$$+ n \int_{\mathbb{R}^N} f(w)w + o(1).$$
As \( u_k, w \in \mathcal{N}^\phi \) we derive
\[
\left\| u_k - \sum_{i=1}^n \phi(g_i)(w \circ g_i^{-1})(\cdot - g_i \xi_k) \right\|^2
= \int_{\mathbb{R}^N} f\left(u_k - \sum_{i=1}^n \phi(g_i)(w \circ g_i^{-1})(\cdot - g_i \xi_k)\right) \left[u_k - \sum_{i=1}^n \phi(g_i)(w \circ g_i^{-1})(\cdot - g_i \xi_k)\right],
\]
and using (f2) we obtain
\[
\epsilon^\phi \geq \lim_{k \to \infty} J(u_k) \geq nJ(w) \geq n\epsilon^\phi/|G|.
\]
Then, \(|G/K| < \infty\) by Lemma 2.3. If \( K \neq G \), taking \( n = |G/K| \) we get that \( \epsilon^\phi \geq |G/K| \epsilon^\phi/|G| \), contradicting our assumption. Therefore, \( K = G \), so \( w \in \mathcal{N}^\phi \) and \( J(w) = \epsilon^\phi \), as claimed. \( \square \)

3 An upper bound for the energy of symmetric minimizers

In [4, Theorem 4] Berestycki and Lions established the existence of a ground state solution \( \omega \in C^2(\mathbb{R}^N) \) to (1.1), which is positive, radially symmetric and decreasing in the radial direction. It satisfies the decay estimates
\[
0 < b_1(1 + |x|)^{(N-2)} \leq \omega(x) \leq b_2(1 + |x|)^{(N-2)},
\]
\[
|\nabla \omega(x)| \leq b_3(1 + |x|)^{(N-1)},
\]
for all \( x \in \mathbb{R}^N \); see [17, Theorem 1.1 and Corollary 1.2].

From now on we consider the following symmetries.

**Example 3.1.** We write \( \mathbb{R}^N \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \) and a point in \( \mathbb{R}^N \) as \( (z_1, z_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}. \) For \( m \in \mathbb{N} \), let
\[
\mathbb{Z}_m := \left\{ e^{2\pi i j/m} : j = 0, \ldots, m - 1 \right\},
\]
\( G_m \) be the group generated by \( \mathbb{Z}_m \cup \{ \tau \} \), acting on \( \mathbb{R}^N \) as
\[
e^{2\pi i j/m}(z_1, z_2, y) := (e^{2\pi i j/m} z_1, e^{2\pi i j/m} z_2, y), \quad \tau(z_1, z_2, y) := (z_2, z_1, y),
\]
and \( \phi : G_m \rightarrow \mathbb{Z}_2 \) be the homomorphism satisfying \( \phi(e^{2\pi i j/m}) = 1 \) and \( \phi(\tau) = -1 \). Set \( \zeta := (1, 0, 0) \), and for each \( R > 1 \) define
\[
\hat{\sigma}_R(x) := \sum_{g \in G_m} \phi(g) \omega(x - Rg \zeta), \quad x \in \mathbb{R}^N.
\]
Note that \( \hat{\sigma}_R(gx) = \phi(g)\hat{\sigma}_R(x) \) for every \( g \in G_m, \ x \in \mathbb{R}^N \). As in [8, Lemmas 4.6 and 4.7] one shows that there exists \( R_0 > 0 \) and for each \( R \geq R_0 \) a unique \( t_R > 0 \) such that
\[
\sigma_R := t_R \hat{\sigma}_R \in \mathcal{N}^\phi.
\]
Furthermore, \( t_R \to 1 \) as \( R \to \infty \).
Our next goal is to prove the following result.

**Proposition 3.2.** If

\[ m \geq \sqrt{2\pi} \left( \frac{\pi}{\sqrt{2}} \right)^{\frac{1}{d+1}} \]  

(3.2)

then, for \( R \) large enough,

\[ c^\theta \leq J(\sigma_R) < 2mc_0, \]

where \( c_0 \) is the ground state energy of (1.1).

We start with some lemmas. The first one is a refinement of [7 Lemma 4.1].

**Lemma 3.3.** Let \( y_1, \ldots, y_n \) be \( n \) different points in \( \mathbb{R}^N \) and \( \vartheta_1, \ldots, \vartheta_n \) be positive numbers with \( \vartheta := \vartheta_1 + \cdots + \vartheta_n > N \). Then there exists \( C = C(\vartheta, N) > 0 \) such that

\[ \int_{\mathbb{R}^N} \prod_{i=1}^n (1 + |x - R y_i|)^{-\vartheta_i} \, dx \leq C d^{-\mu} R^{-\mu} \]

for all \( R \geq 1 \), where \( d := \min \{|y_i - y_j| : i, j = 1, \ldots, n, i \neq j\} \) and \( \mu := \min \{\vartheta - \vartheta_i, \vartheta - N : i = 1, \ldots, n\} \).

**Proof.** Set \( \rho := \frac{1}{2} d > 0 \) and

\[ H_i := \{z \in \mathbb{R}^N : |z - R y_j| \geq |z - R y_i| \text{ for every } j \neq i\}. \]

Note that \( B_{\rho R}(R y_i) \subset H_i \) and \( H_1 \cup \cdots \cup H_n = \mathbb{R}^N \).

Henceforth \( C \) will denote different positive constants depending on \( \vartheta, \vartheta_i \) and \( N \). If \( |x - R y_j| \leq \rho R \), then \( |x - R y_i| \geq \rho R \) for every \( i \neq j \). Thus, for each \( j = 1, \ldots, n \) we have

\[ \int_{B_{\rho R}(R y_i)} \prod_{i=1}^n (1 + |x - R y_i|)^{-\vartheta_i} \, dx \leq \int_{B_{\rho R}(R y_i)} \frac{dx}{(1 + |x - R y_j|)^{\vartheta_j}(\rho R)^{\vartheta - \vartheta_j}} \]

\[ \leq (\rho R)^{-(\vartheta - \vartheta_j)} \int_{B_{\rho R}(0)} \frac{dx}{(1 + |x|)^{\vartheta_j}} \leq C \left[ (\rho R)^{-(\vartheta - \vartheta_j)} + (\rho R)^{N - \vartheta} \right] \leq C (\rho R)^{-\mu}. \]

Setting \( x = R z \) we obtain

\[ \int_{H_i \setminus B_{\rho R}(R y_i)} \prod_{i=1}^n (1 + |x - R y_i|)^{-\vartheta_i} \, dx \leq \int_{H_i \setminus B_{\rho R}(R y_i)} \frac{dx}{(1 + |x - R y_j|)^{\vartheta_j}} \]

\[ \leq \int_{H_i \setminus B_{\rho R}(y_j)} \frac{R^N \, dz}{(R |z - y_j|)^{\vartheta_j}} \leq CR^{N-\vartheta} \int_0^{+\infty} \frac{r^{N-1}}{r^\vartheta} \, dr \leq C (\rho R)^{-\mu}. \]

This completes the proof. \( \square \)

**Lemma 3.4.** There are positive constants \( C_0 \) and \( \tilde{C}_0 \) such that

\[ \lim_{|y| \to \infty} |y|^{N-2} \int_{\mathbb{R}^N} f(\omega(x)) \omega(x - y) \, dx = C_0 \]

\[ \lim_{|y| \to \infty} |y|^{N-2} \int_{\mathbb{R}^N} |\omega(x)|^{2^*-1} \omega(x - y) \, dx = \tilde{C}_0. \]
Proof. This is proved in Appendix A. \hfill \Box

**Lemma 3.5.** There exists \( C_1 > 0 \) such that
\[
\left| \int_{\mathbb{R}^N} (tf(u) - f(tu)) v \right| \leq C_1 |t - 1| \int_{\mathbb{R}^N} |u|^{2^*-1} |v| \quad \text{for all } t \in [0, 2], \ u, v \in D^{1,2}(\mathbb{R}^N).
\]

**Proof.** Fix \( u \in \mathbb{R} \) and let \( h(t) := tf(u) - f(tu) \). It follows from (f) that \( |h'(t)| \leq C_1 |u|^{2^*-1} \) for all \( t \in [0, 2] \). Hence, \( |tf(u) - f(tu)| = |h(t) - h(1)| \leq C_1 |u|^{2^*-1}|t - 1| \), which implies the conclusion. \hfill \Box

**Lemma 3.6.** Given \( m \in \mathbb{N} \) and \( \pi > 0 \), there exists \( C_2 > 0 \) such that
\[
\left| F\left( \sum_{i=1}^{2m} u_i - \sum_{j=1}^{2m} f(u_i) \right) \right| \leq C_2 \left( \sum_{i=1}^{2m} |u_i|^\alpha \sum_{i<j} |u_i u_j| + \sum_{i<j<k} |u_i u_j|^\alpha |u_k| \right),
\]
for any \( u_1, \ldots, u_{2m} \in [-\pi, \pi] \).

**Proof.** This is proved in Appendix B. \hfill \Box

**Proof of Proposition 3.2.** We write the \( G_m \)-orbit of \( \zeta = (1, 0, 0) \) as
\[
G_m \zeta = \{ \zeta_1, \ldots, \zeta_{2m} \} \quad \text{with } \zeta_i := e^{2\pi i (i-1)/m} \zeta_0 \quad \text{and} \quad \zeta_{m+i} := \tau \zeta_i, \quad i = 1, \ldots, m,
\]
and set
\[
\omega_{iR}(x) := \begin{cases} \omega(x - R \zeta_i) & \text{for } i = 1, \ldots, m, \\ -\omega(x - R \zeta_i) & \text{for } i = m + 1, \ldots, 2m. \end{cases}
\]
Then, \( \sigma_R = \sum_{i=1}^{2m} t_R \omega_{iR} \); see Example 3.1. Note that \( f(\omega_{iR}) = f(\omega) \). As \( \omega_{iR} \) solves (1.1), from Lemmas 3.5 and 3.4 we derive
\[
(t_R \omega_{iR}, t_R \omega_{jR}) = t_R^2 \int_{\mathbb{R}^N} f(\omega_{iR}) \omega_{jR} \leq \int_{\mathbb{R}^N} f(t_R \omega_{iR}) t_R \omega_{jR} + C |t_R - 1| \int_{\mathbb{R}^N} |\omega_{iR}|^{2^*-1} |\omega_{jR}| \leq \int_{\mathbb{R}^N} f(t_R \omega_{iR}) t_R \omega_{jR} + C |t_R - 1| |\zeta_i - \zeta_j|^{2-N} R^{2-N},
\]
for $R$ large enough. Using this inequality and Lemmas 3.6 and 3.3 we obtain

\[
J(\sigma_R) = \frac{1}{2} \left\| \sum_{i=1}^{2m} t_R \omega_{i|R} \right\|^2 - \int_{\mathbb{R}^N} F(\sum_{i=1}^{2m} t_R \omega_{i|R})
\]

\[
= \frac{2m}{2} \left\| t_R \omega_{i|R} \right\|^2 - \sum_{i=1}^{2m} \int_{\mathbb{R}^N} F(t_R \omega_{i|R}) + \frac{1}{2} \sum_{i,j=1}^{2m} \langle t_R \omega_{i|R}, t_R \omega_{j|R} \rangle
\]

\[
- \int_{\mathbb{R}^N} F(\sum_{i=1}^{2m} t_R \omega_{i|R}) + \sum_{i=1}^{2m} \int_{\mathbb{R}^N} F(t_R \omega_{i|R})
\]

\[
\leq \frac{2m}{2} \sum_{i,j=1}^{2m} \int_{\mathbb{R}^N} f(\omega_{i|R}) \omega_{j|R} + C|t_R - 1| R^{2-N}
\]

\[
+ C \sum_{i,j=1}^{2m} \int_{\mathbb{R}^N} \left| \omega_{i|R} \omega_{j|R} \right|^{1+\frac{1}{2}} + C \sum_{i,j,k=1}^{2m} \int_{\mathbb{R}^N} \left| \omega_{i|R} \omega_{j|R} \right| \left| \omega_{k|R} \right|
\]

\[
\leq 2m c_0 - \frac{t_2^2}{2} \sum_{i,j=1}^{2m} \int_{\mathbb{R}^N} f(\omega_{i|R}) \omega_{j|R} + C|t_R - 1| R^{2-N} + CR^{-\mu_1} + CR^{-\mu_2},
\]  

(3.3)

where $C$ stands for different positive constants that do not depend on $R$, $\mu_1 := \min\{(1 + \frac{1}{2})(N - 2), (2 + \alpha)(N - 2) - N\}$ and $\mu_2 := \min\{(\alpha + 1)(N - 2), 2\alpha(N - 2), (2\alpha + 1)(N - 2) - N\}$. Since, by assumption $(f_1)$, $\alpha > \frac{N}{2(N-2)}$, we have that $\mu_1, \mu_2 > N - 2$.

Next, we estimate the sign of the second summand in the last row. To this end, set $d_{ij} := |\xi_i - \xi_j|$ for $i \neq j$. Note that $d_{12} = 2 \sin(\frac{\pi}{m})$, and $d_{ij} = \sqrt{2}$ if $1 \leq i \leq m < j \leq 2m$. Therefore,

\[
\sum_{i,j=1}^{m} d_{ij}^{-N} - \sum_{i=1}^{m} \sum_{j=m+1}^{2m} d_{ij}^{-N} \geq \sum_{i=1}^{m} (d_{i(i+1)}^{-N} + d_{i(i-1)}^{-N}) - \sum_{i=1}^{m} \sum_{j=m+1}^{2m} d_{ij}^{-N}
\]

\[
= 2m \left(2 \sin\left(\frac{\pi}{m}\right)\right)^{2-N} - m^2 (\sqrt{2})^{2-N} > m \left[2 \left(\frac{2\pi}{m}\right)\right]^{2-N} - m (\sqrt{2})^{2-N} \geq 0
\]

because, by assumption,

\[
m \geq \frac{\sqrt{2}N}{\sqrt{2}N - 1}.
\]
Let $C_0$ be as in Lemma 3.4 and fix $\epsilon \in (0, C_0)$ such that

$$M_0 := 2(C_0 - \epsilon) \sum_{i,j=1 \atop i \neq j}^{m} d_{ij}^{2-N} - 2(C_0 + \epsilon) \sum_{i=1}^{m} \sum_{j=m+1}^{2m} d_{ij}^{2-N} > 0.$$  

Then, for $R$ large enough we have that

$$\sum_{i,j=1 \atop i \neq j}^{2m} \int_{\mathbb{R}^N} f(\omega_{iR}) \omega_{jR} = 2 \sum_{i,j=1 \atop i \neq j}^{m} \int_{\mathbb{R}^N} f(\omega) \omega(\cdot - R(\xi_j - \xi_i))$$

$$- 2 \sum_{i=1}^{m} \sum_{j=m+1}^{2m} \int_{\mathbb{R}^N} f(\omega) \omega(\cdot - R(\xi_j - \xi_i)) \geq M_0 R^{2-N},$$

and we derive from (3.3) that

$$J(\sigma_R) \leq 2mc_0 - \frac{t_R^2}{2} M_0 R^{2-N} + C |t_R - 1| R^{2-N} + o(R^{2-N}).$$

Since $M_0 > 0$ and $t_R \to 1$ as $R \to \infty$, we conclude that $J(\sigma_R) < 2mc_0$ for $R$ large enough, as claimed. \hfill \Box

**Remark 3.7.** The function $\psi(t) := \sqrt{2\pi} \left( \frac{\pi}{\sqrt{2}} \right)^{\frac{1}{2}}$ is decreasing in $t > 0$. Since $\psi(t) \to \sqrt{2\pi}$ as $t \to \infty$ and $\sqrt{2\pi} > 4$, any number $m$ satisfying (3.2) must be greater than or equal to 5.

Direct computation shows that the least integer greater than or equal to $\sqrt{2\pi} \left( \frac{\pi}{\sqrt{2}} \right)^{\frac{1}{2}}$ is 6 if $N = 5, 6$, and it is 5 if $N \geq 7$.

**Proof of Theorem 1.1.** Let $m$ satisfy (3.2). Take $G := G_m$ and $\phi$ as in Example 3.1. For $\xi = (z_1, z_2, \lambda) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$ we have that $|G^\xi| = 2m$ and $G^\xi = \{1\}$ if $z_1 \neq z_2$, $|G^\xi| = m$ and $G^\xi = \{1, \tau\}$ if $z_1 = z_2 \neq 0$, and $G^\xi = G$ if $z_1 = z_2 = 0$. So, according to Proposition 2.4 $c^\phi$ is attained if $c^\phi < \min\{2mc_0, mc_0^{\phi\{1,\tau\}}\}$.

If $u \in \mathcal{N}^{\phi\{1,\tau\}}$ then $u$ changes sign and $u^-(x) = -u^+(\tau x)$ where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Therefore,

$$\|u\|^2 = 2\|u^\pm\|^2, \quad \int_{\mathbb{R}^N} F(u) = 2 \int_{\mathbb{R}^N} F(u^\pm), \quad \int_{\mathbb{R}^N} f(u)u = 2 \int_{\mathbb{R}^N} f(u^\pm)u^\pm,$$

and, as a consequence,

$$u^\pm \in \mathcal{N} := \{v \in D^{1,2}(\mathbb{R}^N) : v \neq 0 \text{ and } \|v\|^2 = \int_{\mathbb{R}^N} F(v)\}$$

and $f(u) = 2f(u^\pm) \geq 2c_0$. It follows that $\min\{2mc_0, mc_0^{\phi\{1,\tau\}}\} = 2mc_0$. 

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Proposition 3.2 asserts that \(2m c_0 > c^\phi\). Therefore \(c^\phi\) is attained at some \(\overline{\omega} \in \mathcal{N}^\phi\). Furthermore, as \(\mathcal{N}^\phi \subset \mathcal{N}^\phi (\{1, \tau\})\), we have that \(\mathcal{N}^\phi (\{1, \tau\}) \leq \mathcal{N}^\phi\). If \(\overline{f(\omega)} = 2c_0\), then \(\overline{f(\omega)} = c_0\), contradicting the maximum principle. This shows that \(c^\phi > 2c_0\).

As \(\overline{\omega} \in \mathcal{N}^\phi\), it has properties \((a)\) and \((b)\). Property \((c)\) holds true after a suitable translation. Indeed, if \(a_i \in \mathbb{R}\) and \(q_i : \mathbb{R}^N \to \mathbb{R}^N\) is the reflection on the hyperplane \(\{(z_1, z_2, y_1, \ldots, y_{N-4}) : y_1 = a_i\}\), the function

\[
\varphi(z_1, z_2, y) := \begin{cases} 
\overline{\omega}(z_1, z_2, y) & \text{if } y_1 > a_i, \\
(\overline{\omega} \circ q_i)(z_1, z_2, y) & \text{if } y_1 < a_i,
\end{cases}
\]

is also in \(\mathcal{N}^\phi\) and \(\overline{f(\varphi)} = c^\phi\). Applying Lopes' method \([14]\) one shows that there exists \(a = (a_1, \ldots, a_{N-4})\) such that \(\overline{\omega}\) is invariant under the reflection on every hyperplane \(y_i = a_i\). It follows that \(\overline{\omega}(z_1, z_2, y) := \overline{\omega}(z_1, z_2, y + a)\) belongs to \(\mathcal{N}^\phi\) and satisfies \((c)\) and \(\overline{f(\varphi)} = c^\phi\).

Remark 3.7 completes the proof.

\[\Box\]

A. The proof of Lemma 3.4

We start by establishing the exact decay of the ground state.

**Lemma A.1.** Let \(\omega \in C^2(\mathbb{R}^N)\) be a positive radial ground state solution to (1.1). Then there exists \(c_0 > 0\) such that

\[
\lim_{t \to \infty} t^{N-2} \omega(t) = c_0.
\]

**Proof.** By (3.1) there exist two positive constants \(c_1, c_2\) such that

\[
0 < c_1 \leq r^{N-2} \omega(r) \leq c_2 \quad \text{for all } r \geq 1.
\]

So it suffices to show that there exists \(\rho > 0\) such that the function \(r \mapsto r^{N-2} \omega(r)\) is monotone in the interval \((\rho, \infty)\) or, equivalently, that the function \(v : [1, \infty) \to (0, \infty)\) defined as \(v(r) := \frac{1}{r^{N-2} \omega(r)}\) is monotone in \((\rho, \infty)\). To this aim, observe that

\[
\Delta(\omega^{-(N-2)}) = (\omega^{-(N-2)})'' + \frac{N-1}{r} (\omega^{-(N-2)})' = 0,
\]

so that

\[
0 = \Delta(v \omega) = \omega v'' + 2v' \omega' + vv'' + \frac{N-1}{r} (v' \omega + v \omega')
\]

\[
= \omega \left[ v'' + \left( 2 \frac{\omega'}{\omega} + \frac{N-1}{r} \right) v' \right] + \left( \omega'' + \frac{N-1}{r} \omega' \right) v.
\]

Then, \(v\) satisfies

\[
v'' + b(r)v' + c(r)v = 0, \quad \text{with } b(r) := \frac{2 \omega'}{\omega} + \frac{N-1}{r} \quad \text{and } c(r) := -\frac{f(\omega)}{\omega} < 0.
\]

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If there exists $\rho > 0$ such that $v'(r) \geq 0$ for every $r \geq \rho$, or $v'(r) \leq 0$ for every $r \geq \rho$, then $v$ is monotone in $(\rho, \infty)$. On the other hand, if such $\rho$ does not exist, then $v'$ changes sign infinitely many times in $(1, \infty)$. In particular, $v$ has a local maximum $r_0$ in $[1, \infty)$. But this implies that

$$0 \geq v''(r_0) = -c(r_0)v(r_0) > 0,$$

a contradiction. The proof is complete.

**Remark A.2.** In fact, $v = \frac{1}{r^{N-2}}$ is nonincreasing. Indeed, defining $u := \frac{\omega}{r^{N-2}}$ and using (3.1), for every $\rho > 0$ one has

$$\begin{cases} 
\Delta (u - \omega) = f(\omega) \quad \text{in} \quad (\rho, +\infty), \\
(u - \omega)(\rho) \leq 0.
\end{cases}$$

Then, the maximum principle implies that $\omega > u$ and Hopf's Lemma yields $(\omega - u)'(\rho) < 0$. Therefore $v = \frac{u}{c(\omega)}$ satisfies

$$v'(\rho) = \frac{1}{c_1} \frac{u'\omega - \omega' u}{\omega^2} \leq \frac{1}{c_1} \frac{\omega' \omega - \omega' \color{black}{\omega}}{\omega^2} \leq 0,$$

as $\omega$ is decreasing.

**Proof of Lemma 3.4.** From Lemma A.1 for each fixed $x \in \mathbb{R}^N$ we get that

$$\lim_{|y| \to \infty} |y|^{N-2}f(\omega(x))\omega(x-y) = \lim_{|y| \to \infty} \left| \frac{y}{x-y} \right|^{N-2} f(\omega(x))|x-y|^{N-2} \omega(x-y) = c_0 f(\omega(x)).$$

Note that, as $\omega$ is decreasing in the radial direction,

$$|y|^{N-2} \omega(x-y) \leq \begin{cases} 
|x-y|^{N-2} \omega(x-y) & \text{if } |y| \leq |x-y|, \\
|y|^{N-2} \omega(y) & \text{if } |y| \geq |x-y|.
\end{cases}$$

So from (3.1) we get that $|y|^{N-2} f(\omega(x)) \omega(x-y) \leq C f(\omega(x))$ for every $y \in \mathbb{R}^N$. Assumption (f1) and (3.1) yield $f(\omega(x)) \leq C|x|^{-(q-1)(N-2)}$ and $q-1 > \frac{N+2}{N-2}$. Therefore $f \circ \omega$ is integrable in $\mathbb{R}^N$ and by the dominated convergence theorem

$$\lim_{|y| \to \infty} \int_{\mathbb{R}^N} |y|^{N-2} f(\omega(x))\omega(x-y) \, dx = c_1 \int_{\mathbb{R}^N} f(\omega(x)) \, dx.$$

The other identity is obtained in a similar way. □
B The proof of Lemma 3.6

Lemma B.1. Given \( n \in \mathbb{N}, \pi > 0 \) and \( f \in C^{1,\beta}_{\text{loc}}(\mathbb{R}) \) with \( \beta \in (0,1] \) such that \( f(0) = 0 \), there exists \( b_1 > 0 \) such that

\[
\left| f\left( \sum_{i=1}^{n} u_i \right) - \left( \sum_{i=1}^{n} f(u_i) \right) \right| \leq b_1 \sum_{i,j} |u_i u_j|^{\beta} \quad \text{for any } u_1, \ldots, u_n \in [-\pi, \pi].
\]

Proof. We argue by induction on \( n \). As \( f \in C^{1,\beta}_{\text{loc}}(\mathbb{R}) \) and \( f(0) = 0 \), we have that

\[
|f(u + v) - f(u) - f(v)| = \left| \int_{0}^{u} (f'(s + v) - f'(s)) \, ds \right|
\]

\[
\leq \left| \int_{0}^{u} |f'(s + v) - f'(s)| \, ds \right| \leq C|v|^{\beta}|u| \leq C\pi^{1-\beta}|uv|^\beta \quad \text{for any } u, v \in [-\pi, \pi].
\]

Assume the result is true for \( n - 1 \geq 2 \) and let \( u_1, \ldots, u_n \in [-\pi, \pi] \). Then,

\[
\left| f\left( \sum_{i=1}^{n} u_i \right) - \left( \sum_{i=1}^{n} f(u_i) \right) \right|
\]

\[
\leq \left| f\left( \sum_{i=1}^{n-1} u_i \right) - \left( \sum_{i=1}^{n-1} f(u_i) \right) - f(u_n) + f\left( \sum_{i=1}^{n-1} u_i \right) - \left( \sum_{i=1}^{n-1} f(u_i) \right) \right|
\]

\[
\leq C\left( \sum_{i=1}^{n-1} u_i u_n |^\beta + \sum_{i,j} |u_i u_j|^{\beta} \right) \leq b_1 \sum_{i,j} |u_i u_j|^{\beta},
\]

as claimed. \( \square \)

Lemma B.2. Given \( n \in \mathbb{N}, \pi > 0 \) and \( f \in C^{1,\beta}_{\text{loc}}(\mathbb{R}) \) with \( \beta \in (0,1] \) such that \( f(0) = 0 = f'(0) \), there exists \( b_2 > 0 \) such that

\[
\left| F\left( \sum_{i=1}^{n} u_i \right) - \sum_{i=1}^{n} F(u_i) - \sum_{i,j} f(u_i) u_j \right| \leq b_2 \left( \sum_{i=1}^{n} |u_i u_j|^{1+\beta} + \sum_{i,j,k} |u_i u_j|^{\beta} |u_k| \right),
\]

for any \( u_1, \ldots, u_n \in [-\pi, \pi] \), where \( F(u) := \int_{0}^{u} f \).

Proof. We argue by induction on \( n \). Following \[13\] Lemma 3, we define

\[
G(u, v) := F(u + v) - F(u) - F(v) - f(u)v - f(v)u, \quad u, v \in [-\pi, \pi].
\]

Then \( \partial_u \partial_v G(u, v) = f''(u + v) - f''(u) - f''(v) \) and, as \( f \in C^{1,\beta}_{\text{loc}}(\mathbb{R}) \) and \( f'(0) = 0 \),

\[
|\partial_u \partial_v G(u, v)| \leq |f''(u + v) - f''(u)| + |f''(v)| \leq C|v|^{\beta},
\]

\[
|\partial_u \partial_v G(u, v)| \leq |f'(u + v) - f'(u)| + |f'(v)| \leq C|u|^{\beta}.
\]

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Therefore,
\[ |\partial_u \partial_v G(u, v)| \leq C \min\{|u|, |v|\}^\beta. \]

Taking into account that \( G(0, v) = \partial_u G(s, 0) = 0 \) we get
\[ |G(u, v)| = \left| \int_0^u \partial_u G(s, b) \, ds \right| = \left| \int_0^u \int_0^v \partial_v \partial_u G(s, t) \, dt \, ds \right| \leq \left| \int_0^u \int_0^v |\partial_v \partial_u G(s, t)| \, dt \, ds \right| \]
\[ \leq C \left| \int_0^u \int_0^v \min\{|s|, |t|\}^\beta \, dt \, ds \right| \leq C \min\{|u|, |v|\}^\beta |uv| \leq b_2|uv|^{1+\frac{\beta}{2}}, \]
because
\[ \min\{|u|, |v|\}^\beta |uv| = |u|^{1+\frac{\beta}{2}} |v|^{1+\frac{\beta}{2}} \leq |uv|^{1+\frac{\beta}{2}} \quad \text{for } |u| \leq |v|. \]

This proves the statement for \( n = 2 \). Assume the inequality holds true for \( n - 1 \geq 2 \) and let \( u_1, \ldots, u_n \in [-\bar{v}, \bar{v}] \). Then, using Lemma [B.1], we obtain
\[
\left| F\left( \sum_{i=1}^n u_i \right) - \sum_{i=1}^n F(u_i) - \sum_{i \neq j}^n f(u_i) u_j \right|
\]
\[ \leq \left| F\left( \sum_{i=1}^{n-1} u_i \right) - F\left( \sum_{i=1}^{n-1} u_i \right) - F(u_n) - f\left( \sum_{i=1}^{n-1} u_i \right) u_n - \sum_{i=1}^{n-1} u_i f(u_n) \right|
\]
\[ + \left| F\left( \sum_{i=1}^{n-1} u_i \right) - F\left( \sum_{i=1}^{n-1} u_i \right) - \sum_{i \neq j}^{n-1} f(u_i) u_j \right| + \left| f\left( \sum_{i=1}^{n-1} u_i \right) u_n - \sum_{i=1}^{n-1} f(u_i) u_n \right|
\]
\[ \leq C \left( \sum_{i=1}^{n-1} |u_i u_n|^{1+\frac{\beta}{2}} + \sum_{i,j=1}^{n-1} |u_i u_j|^{1+\frac{\beta}{2}} + \sum_{i,j,k=1}^{n-1} |u_i u_j|^{\beta} |u_k| + \sum_{i,j,k=1}^{n-1} |u_i u_j|^{\beta} |u_n| \right)
\]
\[ \leq C \left( \sum_{i,j=1}^{n} |u_i u_j|^{1+\frac{\beta}{2}} + \sum_{i,j,k=1}^{n} |u_i u_j|^{\beta} |u_k| \right), \]
as claimed. \( \square \)

**Remark B.3.** Note that the growth condition on \( f' \) in \((f_1)\) does not imply that \( f \in C^{1,\alpha} \) even in a small interval \([0, \delta]\).

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