Optimal Local and Remote Controllers with Unreliable Communication

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Abstract—We consider a decentralized optimal control problem for a linear plant controlled by two controllers, a local controller and a remote controller. The local controller directly observes the state of the plant and can inform the remote controller of the plant state through a packet-drop channel. We assume that the remote controller is able to send acknowledgments to the local controller to signal the successful receipt of transmitted packets. The objective of the two controllers is to cooperatively minimize a quadratic performance cost. We provide a dynamic program for this decentralized control problem using the common information approach. Although our problem is not a partially nested LQG problem, we obtain explicit optimal strategies for the two controllers. In the optimal strategies, both controllers compute a common estimate of the plant state based on the common information. The remote controller’s action is linear in the common estimated state, and the local controller’s action is linear in both the actual state and the common estimated state.

I. INTRODUCTION

Networked control systems (NCS) are distributed systems that consist of several components (e.g., physical systems, controllers, smart sensors, etc.) and the communication network that connects them together. With the recent interest in cyber-physical systems and the Internet of Things (IoT), NCS have received considerable attention in the recent years (see [1] and references therein). In contrast to traditional control systems, the interconnected components in NCS are linked through unreliable channels with random packet drops and delays. In the presence of unreliable communication in NCS, the implicit assumption of perfect data exchange in classical estimation and control system fails [2]. Therefore, efficient operation of NCS requires decentralized decision-making while taking into account the unreliable communication among decision-makers.

In this paper, we consider an optimal control problem for a NCS consisting of a linear plant and two controllers, namely the local controller and the remote controller, connected through an unreliable communication link as shown in Fig. [1]. The local controller directly observes the state of the plant and can inform the remote controller of the plant state through a channel with random packet drops. We consider a TCP structure so that the remote controller is able to send acknowledgments to the local controller to signal the successful receipt of transmitted packets. The objective of the two controllers is to cooperatively minimize the overall quadratic performance cost of the NCS. The problem is motivated from applications that demand remote control of systems over wireless networks where links are prone to failure. The local controller can be a small local processor proximal to the system that measures the status of the system and can perform limited control. The remote controller can be a more powerful controller that receives information from the local processor through a wireless channel.

Similar setups of NCS has been investigated in the literature with only the remote controller present. Various communication protocols including the TCP (where acknowledgments are available) and the UDP (where acknowledgments are not available) and variations have been investigated [3–7]. For NCS with two decision-makers, [8], [9] have studied the problem when the local controller is a smart sensor and the remote controller is an estimator. When the linear plant is controlled only by the remote controller and the local controller is a smart sensor or encoder, [10–14] have shown that the separation of control and estimation holds for the remote controller under various communication channel models.

The problem considered in this paper is different from previous works on NCS because our problem is a two-controller decentralized problem where both controllers can control the dynamics of the plant. Finding optimal strategies for two-controller decentralized problems is generally difficult (see [15–17]). In general, linear control strategies are not optimal, and even the problem of finding the best linear control strategies is not convex [18]. Existing optimal solutions of two-controller decentralized problems require either specific information structures, such as static [19], partially nested [20–25], stochastically nested [26], or other specific properties, such as quadratic invariance [27] or substitutability [28]. None of the above properties hold in our problem due to either the unreliable communication or the nature of dynamics and cost function. In spite of this, we solve the two-controller decentralized problem and provide explicit optimal strategies for the local controller and the remote controller. In the optimal strategies, both controllers compute a common estimate of the plant state based on the common information. The remote controller’s action is linear in the common estimated state, and the local controller’s action is linear in both the actual state and the common estimated state.

A. Organization

The rest of the paper is organized as follows. We introduce the system model and formulate the two-controller optimal control problem in Section [II]. In Section [III], we provide a dynamic program for the decentralized control problem using the common information approach. We solve the dynamic program in Section [IV]. Section [V] concludes the paper.
Fig. 1. Two-controller system model. The binary random variable $\Gamma_t$ indicates whether packets are transmitted successfully.

Notation

Random variables/vectors are denoted by upper case letters, their realization by the corresponding lower case letter. For a sequence of column vectors $X, Y, Z, \ldots$, the notation $\text{vec}(X, Y, Z, \ldots)$ denotes vector $[X^T, Y^T, Z^T, \ldots]^T$. The transpose and trace of matrix $A$ are denoted by $A^T$ and $\text{tr}(A)$, respectively. In general, subscripts are used as time index while superscripts are used to index controllers. For time indices $t_1 \leq t_2$, $X_{t_1:t_2}$ (resp. $g_{t_1:t_2}(\cdot)$) is the short hand notation for the variables $(X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ (resp. functions $(g_{t_1}(\cdot), \ldots, g_{t_2}(\cdot))$). The indicator function set of event $E$ is denoted by $\mathbb{1}_E(\cdot)$, that is, $\mathbb{1}_E(x) = 1$ if $x \in E$, and 0 otherwise. $\mathbb{P}(\cdot)$, $\mathbb{E}[\cdot]$, and $\text{cov}(\cdot)$ denote the probability of an event, the expectation of a random variable/vector, and the covariance matrix of a random vector, respectively. For random variables/vectors $X$ and $Y$, $\mathbb{P}(\cdot | Y = y)$ denotes the probability of an event given that $Y = y$, and $\mathbb{E}[X | y] := \mathbb{E}[X | Y = y]$. For a strategy $g$, we use $\mathbb{P}_g(\cdot)$ (resp. $\mathbb{E}_g[\cdot]$) to indicate that the probability (resp. expectation) depends on the choice of $g$. Let $\Delta(\mathbb{R}^n)$ denote the set of all probability measures on $\mathbb{R}^n$. For any $\theta \in \Delta(\mathbb{R}^n)$, $\theta(E) = \int_{\mathbb{R}^n} \mathbb{1}_E(x) \theta(dx)$ denotes the probability of event $E$ under $\theta$. The mean and the covariance of a distribution $\theta \in \Delta(\mathbb{R}^n)$ are denoted by $\mu(\theta)$ and $\text{cov}(\theta)$, respectively, and are defined as $\mu(\theta) = \int_{\mathbb{R}^n} x \theta(dx)$ and $\text{cov}(\theta) = \int_{\mathbb{R}^n} (x - \mu(\theta))(x - \mu(\theta))^\top \theta(dx)$.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Consider the discrete-time system with two controllers as shown in Fig. 1. The linear plant dynamics are given by

$$X_{t+1} = AX_t + B^LU_t + B^RU_t + W_t, t = 0, \ldots, T$$

where $X_t \in \mathbb{R}^{n_x}$ is the state of the plant at time $t$, $U_t \in \mathbb{R}^{n_u}$ is the control action of the local controller $C^L$, $U_t \in \mathbb{R}^{n_u}$ is the control action of the remote controller $C^R$, and $A, B^L, B^R$ are matrices with appropriate dimensions. $X_0$ is a random vector with distribution $\pi_{X_0}$, $W_t \in \mathbb{R}^{n_w}$ is a zero mean noise vector at time $t$ with distribution $\pi_{W_t}$. $X_0, W_0, W_1, \ldots, W_T$ are independent random vectors with finite second moments.

At each time $t$ the local controller $C^L$ perfectly observes the state $X_t$ and sends the observed state to the remote controller $C^R$ through an unreliable channel with packet drop probability $p$. Let $\Gamma_t$ be Bernoulli random variable describing the nature of this channel, that is, $\Gamma_t = 0$ when the link is broken and otherwise, $\Gamma_t = 1$. We assume that $\Gamma_t$ is independent of all other variables before time $t$. Furthermore, let $Z_t$ be the channel output, then,

$$\Gamma_t = \begin{cases} 1 & \text{with probability } (1 - p), \\ 0 & \text{with probability } p. \end{cases}$$

$$Z_t = \begin{cases} X_t & \text{when } \Gamma_t = 1, \\ 0 & \text{when } \Gamma_t = 0. \end{cases}$$

We assume that the channel output $Z_t$ is perfectly observed by $C^R$. The remote controller sends an acknowledgment when it receives the state. Thus, effectively, $Z_t$ is perfectly observed by $C^L$ as well. The two controllers select their control actions after observing $Z_t$. We assume that the links from the controllers to the plant are perfect.

Let $H_t^L$ and $H_t^R$ denote the information available to $C^L$ and $C^R$ to make decisions at time $t$, respectively. Then,

$$H_t^L = \{X_{0:t}, Z_{0:t}, U_{0:t-1}^L, U_{0:t-1}^R\}, \quad H_t^R = \{Z_{0:t}, U_{0:t-1}^R\}.$$  \hspace{1cm} (4)

Let $H_t^L$ and $H_t^R$ be the spaces of all possible information of $C^L$ and $C^R$ at time $t$, respectively. Then, $C^L$ and $C^R$’s actions are selected according to

$$U_t^L = g_t^L(H_t^L), \quad U_t^R = g_t^R(H_t^R),$$

where the control strategies $g_t^L : H_t^L \mapsto \mathbb{R}^{n_L}$ and $g_t^R : H_t^R \mapsto \mathbb{R}^{n_R}$ are measurable mappings.

The instantaneous cost $c_t(X_t, U_t^L, U_t^R)$ of the system is a general quadratic function given by

$$c_t(X_t, U_t^L, U_t^R) = S_t R_t S_t^\top,$$

where $S_t = \text{vec}(X_t, U_t^L, U_t^R), R_t = \begin{bmatrix} R_{XX} & R_{XL} & R_{XR} \\ R_{XL} & R_{LL} & R_{LR} \\ R_{XR} & R_{LR} & R_{RR} \end{bmatrix},$ and $R_t$ is a symmetric positive definite (PD) matrix.

The performance of strategies $g_t^L := g_{0:T}^L$ and $g_t^R := g_{0:T}^R$ is the total expected cost given by

$$J(g_l^L, g_r^R) = \mathbb{E}^{g_l^L, g_r^R} \left[ \sum_{t=0}^T c_t(X_t, U_t^L, U_t^R) \right].$$  \hspace{1cm} (6)

Let $G^L$ and $G^R$ denote all possible control strategies of $C^L$ and $C^R$ respectively. The optimal control problem for $C^L$ and $C^R$ is formally defined below.

**Problem 1.** For the system described by (1)-(6), determine control strategies $g^L$ and $g^R$ that minimize the performance cost of (6).

Problem 1 is a two-controller decentralized optimal control problem. Note that Problem 1 is not a partially nested LQG problem. In particular, the local controller $C^L$’s action $U_{t-1}^L$ at $t - 1$ affects $X_t$, and consequently, it affects $Z_t$. Since $Z_t$ is a part of the remote controller $C^R$’s information $H_t^R$ at $t$ but $H_{t-1}^L \not\subset H_t^R$, the information structure in Problem 1 is

$1^\top U_{t-1}^R$ is not directly observed by $C^L$ at time $t$, but $C^L$ can obtain $U_{t-1}^R$ because $U_{t-1}^R = g_t^R(H_{t-1}^R)$ and $H_{t-1}^L \subset H_t^L$. 


not partially nested. Therefore, linear control strategies are not necessarily optimal for Problem [1].

Our approach to Problem [1] is based on the common information approach [29] for decentralized decision-making. We identify the common belief of the system state for $C^L$ and $C^R$. The common belief can serve as an information state that leads to a dynamic program for optimal strategies of the two-controller problem.

**Remark 1.** The results of [29] are developed only for finite spaces. Therefore, we cannot directly apply the results of [29] to Problem [7].

### III. Common Belief and Dynamic Program

From [4], $H^L_t$ is the common information among the two controllers. Consider fixed strategies $g_{0:t-1}^L, g_{0:t-1}^R$ until time $t - 1$. Given any realization $h^R_t \in H^R_t$ of the common information, we define the common belief $\theta_t \in \Delta(\mathcal{R}^{n_x})$ as the conditional probability distribution of $X_t$ given $h^R_t$. That is, for any measurable set $E \subset \mathcal{R}^{n_x}$

$$\theta_t(x_t \in E) = \mathbb{P}_{\theta_{0:t-1}^L, \theta_{0:t-1}^R}(X_t \in E|h^R_t). \quad (7)$$

Using ideas from the common information approach [29], the common belief $\theta_t$ could serve as an information state for decentralized decision-making. We proceed to show that $\theta_t$ is indeed an information state that can be used to write a dynamic program for Problem [1].

The following Lemma provides a structural result for $C^L$.

**Lemma 1.** Let $\hat{H}_t^L = \text{vec}(X_t, h^R_t)$, and $\hat{H}_t^L$ be the space of all possible $\hat{H}_t^L$. Let $\hat{G}_t^L = \{g^L : g^L_t \text{ is measurable from } \hat{H}_t^L \text{ to } \mathcal{R}^{n_L}\}$. Then,

$$\inf_{g^L \in \hat{G}_t^L, \theta \in \mathcal{R}^R} J(g^L_t, h^R_t) = \inf_{g^L \in \hat{G}_t^L, \theta \in \mathcal{R}^R} J(g^L_t, h^R_t). \quad (8)$$

From Lemma [1] we only need to consider strategies $g^L \in \hat{G}_t^L$ for the local controller $C^L$. That is, $C^L$ only needs to use $\hat{H}_t^L = \text{vec}(X_t, h^R_t)$ to make the decision at $t$.

For any strategy $g^L \in \hat{G}_t^L$ we provide a representation of $g^L$ using the space $\hat{Q}^\theta$ defined below.

**Definition 1.** For any $\theta \in \Delta(\mathcal{R}^{n_x})$, define a set of mappings $\hat{Q}^\theta = \left\{q : \mathcal{R}^{n_x} \mapsto \mathcal{R}^{n_L} \text{ measurable}, \int_{\mathcal{R}^{n_x}} q(x) \theta(dx) = 0 \right\}. \quad (9)$

**Lemma 2.** For any strategies $g^L \in \hat{G}_t^L$ and $g^R \in \mathcal{R}^R$, let $\theta_t$ be the conditional probability distribution defined in (7), then at any time $t$ there exists $\hat{g}_t^L : \hat{H}_t^L \mapsto \mathcal{R}^{n_L}$ and $\hat{\theta}_t : \hat{H}_t^L \mapsto \hat{Q}^\theta$ such that $\hat{g}_t^L$ is measurable and

$$g^L_t(x_t, h^R_t) = \hat{g}_t^L(h^R_t) + q_t(x_t), \quad q_t = \hat{\theta}_t(h^R_t). \quad (10)$$

**Proof of Lemma 2** Define

$$\hat{g}_t^L(h^R_t) = \mathbb{E}^\theta_{0:t-1} \left[ g^L_t(X_t, h^R_t)|X_t = h^R_t \right], \quad (11)$$

$$q_t(\cdot) = \hat{g}_t^L(h^R_t)(\cdot) = g^L_t(\cdot, h^R_t) - \hat{g}_t^L(h^R_t). \quad (12)$$

Since $g^L_t(x_t, h^R_t)$ is measurable, $\hat{g}_t^L(h^R_t)$ is also measurable. For each $h^R_t \in H^R_t$, $q_t(\cdot) = \hat{g}_t^L(h^R_t)(\cdot)$ is a measurable function because $g^L_t(x_t, h^R_t)$ is measurable. Furthermore,

$$\int_{\mathcal{R}^{n_x}} q_t(x) \theta_t(dx) = \int_{\mathcal{R}^{n_x}} g^L_t(x, h^R_t) \theta_t(dx) - \mathbb{E}^\theta_{0:t-1} \left[ g^L_t(X_t, h^R_t)|h^R_t \right] = 0. \quad$$

The last equality follows from (7). Therefore, $q_t \in \hat{Q}^\theta$. □

Note that $q_t$ belongs to $\hat{Q}^\theta$ and is itself a function of $h^R_t$.

From Lemma [2] for any strategies $g^L \in \hat{G}_t^L$ and $g^R \in \mathcal{R}^R$ we have a corresponding representation of the strategy $g_t^L$ of $C^L$ in terms of $\hat{g}_t^L$ and $\hat{\theta}_t^L$.

Using the above representation of $C^L$’s strategy, we can show that the common belief $\theta_t$ is an information state with a sequential update function.

For any $x \in \mathcal{R}^{n_x}$, let $\delta_x \in \Delta(\mathcal{R}^{n_x})$ denote the Dirac delta distribution at point $x$, that is, for any measurable set $E \subset \mathcal{R}^{n_x}$, $\delta_x(E) = 1$ if $x \in E$, and otherwise $\delta_x(E) = 0$. Define $\varphi : \mathcal{R}^{n_x} \mapsto \Delta(\mathcal{R}^{n_x})$ such that $\varphi(x) = \delta_x$ for any $x \in \mathcal{R}^n$.

**Lemma 3.** For any strategies $g^L \in \hat{G}_t^L$ and $g^R \in \mathcal{R}^R$, let $(\hat{g}_t^L, \hat{\theta}_t^L)$ be the representation of $g_t^L$ given by Lemma [2]. Then the common beliefs $\{\theta_t, t = 0, 1, \ldots, T\}$, defined by (7), can be sequentially updated according to

$$\theta_0 = \begin{cases} \pi_{X_0} & \text{if } z_0 = 0, \\ \varphi(x_0) & \text{if } z_0 = x_0. \end{cases} \quad (13)$$

$$\theta_{t+1}(x_t) = \psi_{t}(x_t, u^L_t, u^R_t, q_t, z_{t+1}), \quad (14)$$

where $u^L_t, u^R_t$ and $q_t$ are functions of the common information $h^R_t$ given by

$$u^L_t = g^L_t(h^R_t), \quad u^R_t = \hat{g}_t^L(h^R_t), \quad q_t = \hat{\theta}_t(h^R_t). \quad (15)$$

Furthermore, $\psi_t(\theta_t, u^L_t, u^R_t, q_t, z_{t+1}) = \varphi(x_{t+1})$ and $\psi_t(\theta_t, u^L_t, u^R_t, q_t, \theta(0))$ is a distribution on $\mathcal{R}^{n_x}$ such that for any measurable set $E \subset \mathcal{R}^{n_x}$,

$$\psi_t(\theta_t, u^L_t, u^R_t, q_t, \theta(0))(E) = \int_{\mathcal{R}^{n_x}} \int_{\mathcal{R}^{n_x}} 1_E(Ax_t + B^L(u^L_t + q_t(x_t)) + B^R u^R_t + w_t) \theta_t(dx_t)\pi_{W_t}(dw_t). \quad (16)$$

Using the common belief and its update function, we define a class of strategies which select actions depending on the common belief $\theta_t$ instead of the entire common information $h^R_t$.

**Definition 2.** We define the set of common belief based strategies $\mathcal{G}^C \subset \mathcal{G}^L \times \mathcal{G}^R$. For any $(g^L, g^R) \in \mathcal{G}^C$ we have the following. At any time $t$, for each $h^R_t$, let $\theta_t$ be the common belief constructed by (15) in Lemma [2]. Then, there exists $g^L, g^R : \Delta(\mathcal{R}^{n_x}) \mapsto \mathcal{R}^{n_L}$ and $\hat{g}^L, \hat{g}^R : \Delta(\mathcal{R}^{n_x}) \mapsto \hat{Q}^\theta$ such that

$$g^L(h^R_t) = g^R_C(\theta_t), \quad (17)$$

$$g^R(h^R_t) = g^R_C(\theta_t) + \hat{g}^R(\theta_t)(x_t). \quad (18)$$
Our main result of this section is the dynamic program
provided in the theorem below.

**Theorem 1.** Suppose there are value functions \( V_t : \Delta(\mathbb{R}^{n_x}) \to \mathbb{R} \) for \( t = 0, 1, \ldots, T+1 \) such that \( V_{T+1} = 0 \), and for each time \( t \) and for each \( \theta_t \in \Delta(\mathbb{R}^{n_x}) \)
\[
V_t(\theta_t) = \min_{q_t \in Q^x_t} \left\{ \min_{u_t^L \in \mathbb{R}^{n_{u_t}}, u_t^+ \in \mathbb{R}^{n_{u_t}}} \int_{\mathbb{R}^{n_x}} c_t(x_t, u_t^L + q_t(x_t), u_t^+) \psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t(\theta_t)) (dx_t) + pV_{t+1}(\varphi(x_{t+1})) \psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t(\theta_t)) (dx_{t+1}) + pV_{t+1}(\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t(\theta_t))) \right\},
\]
(19)

If there are strategies \((g_t^{L*}, g_t^{R*}) \in G^C\) with
\[
\begin{align*}
\tilde{g}_t^{R*}(h_t^R) &= g_t^{R,C*}(\theta_t), \\
g_t^{L*}(h_t^R) &= \tilde{g}_t^{L,C*}(\theta_t) + \tilde{g}_t^{L,C*}(\theta_t)(x_t)
\end{align*}
\]
(20, 21)
such that for each \( h_t^R \)
\[
\begin{align*}
u_t^{R*} &= g_t^{R,C*}(\theta_t), \\
\tilde{u}_t^{L*} &= \tilde{g}_t^{L,C*}(\theta_t), \quad q_t^{*} = \tilde{g}_t^{L,C*}(\theta_t),
\end{align*}
\]
(22)
achieve the minimum in the definition of \( V_t(\theta_t) \), where \( \theta_t \) is the common belief constructed by [13, 15] in Lemma 2. Then \( g_t^{L*}, g_t^{R*} \) are optimal.

Theorem 1 provides a dynamic program to solve the two-controller problem. However, there are two challenges in solving the dynamic program. First, it is a dynamic program on the belief space \( \Delta(\mathbb{R}^{n_x}) \) which is infinite dimensional. Second, each step of the dynamic program involves a functional optimization over the functional space \( Q^\theta \). Nevertheless, in the next section, we show that it is possible to find an exact solution to the dynamic program of Theorem 1 and provide optimal strategies for the controllers.

**IV. OPTIMAL CONTROL STRATEGIES.**

In this section, we identify the structure of the value function in the dynamic program (19). Using the structure, we explicitly solve the dynamic program and obtain the optimal strategies for Problem 1. For a vector \( x \) and a matrix \( G \), we use
\[
QF(G, x) = x^T G x = \text{tr}(G xx^T)
\]
(23)
to denote the quadratic form.

The main result of this section, stated in the theorem below, presents the structure of the value function and an explicit optimal solution of the dynamic program (19).

**Theorem 2.** For any \( \theta_t \) and any time \( t \), the value function of the dynamic program (19) in Theorem 1 is given by
\[
V_t(\theta_t) = QF(P_t, \mu(\theta_t)) + \text{tr} \left( \tilde{P}_t \text{cov}(\theta_t) \right) + e_t,
\]
(24)
\[
e_t = \sum_{s=t}^T \text{tr} \left( ((1-p)P_{s+1} + p\tilde{P}_{s+1}) \text{cov}(\pi_{W_s}) \right),
\]
(25)
and the optimal solution is given by
\[
\begin{align*}
\tilde{u}_t^{L*} &= \tilde{g}_t^{L,C*}(\theta_t) \\
u_t^{R*} &= g_t^{R,C*}(\theta_t)
\end{align*}
\]
(26)
\[
\begin{align*}
q_t^{*} &= g_t^{L,C*}(\theta_t)(x_t) \\
\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t(\theta_t)) = \psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t^{*}(\theta_t))
\end{align*}
\]
(27)
The matrices \( P_t, G_t, H_t, \tilde{P}_t, \tilde{G}_t, \tilde{H}_t \) defined recursively below are symmetric positive semi-definite (PSD); \( G_t \) and \( \tilde{G}_t \) are symmetric positive definite (PD).
\[
P_{T+1} = \tilde{P}_{T+1} = \mathbf{0}, \text{ the all zeros matrix},
\]
(28)
\[
P_t = G_t^{XX}
\]
(29)
\[
- \left[ G_t^{XL} G_t^{XR} \right]^{-1} \left[ G_t^{LX} G_t^{RX} \right],
\]
(30)
\[
H_t = [A, B^L, B^R] \tilde{P}_{t+1} [A, B^L, B^R]^T.
\]
(31)
\[
\tilde{P}_t = \tilde{G}_t^{XX} - \tilde{G}_t^{XL} (\tilde{G}_t^{LL})^{-1} \tilde{G}_t^{LX},
\]
(32)
\[
\tilde{G}_t = \left[ \begin{array}{llll}
G_t^{XX} & G_t^{XL} & G_t^{XR} \\
G_t^{LX} & G_t^{LL} & G_t^{LR} \\
G_t^{RX} & G_t^{RL} & G_t^{RR}
\end{array} \right],
\]
(33)
\[
= R_t + (1-p) H_t + p \tilde{H}_t,
\]
\[
\tilde{H}_t = [A, B^L, B^R] \tilde{P}_{t+1} [A, B^L, B^R].
\]
(34)
The proof of Theorem 2 relies on the following lemma for quadratic optimization problems.

**Lemma 4.** Let \( G = \left[ \begin{array}{ll}
G^{XX} & G^{XU} \\
G^{UX} & G^{UU}
\end{array} \right] \) be a PD matrix and \( P := G^{XX} - G^{XU} (G^{UU})^{-1} G^{UX} \) be the Schur complement of \( G^{UU} \) of \( G \).

(a) For any constant vector \( x \in \mathbb{R}^n \),
\[
\min_{u \in \mathbb{R}^m} QF(G, \text{vec}(x, u)) = QF(P, x)
\]
(35)
with optimal solution
\[
u^* = - (G^{UU})^{-1} G^{UX} x.
\]
(36)
(b) For any \( \theta \in \Delta(\mathbb{R}^n) \), let \( X^\theta \) be a random variable with distribution \( \theta \), then
\[
\min_{q \in Q^\theta} \text{tr} \left( G \text{cov} \left( \text{vec}(X^\theta, q(X^\theta)) \right) \right) = \text{tr} \left( P \text{cov} (X^\theta) \right)
\]
(37)
with optimal solution
\[
q^*(X^\theta) = - (G^{UU})^{-1} G^{UX} (X^\theta - \mu(\theta)).
\]
(38)

Using Lemma 4, we present a sketch of the proof of Theorem 2. The complete proof is in the Appendix.
Sketch of the proof of Theorem 2. The proof is done by induction. Suppose the result is true at \( t + 1 \), then at time \( t \):

- Show that \( G_t, \hat{G}_t \) are PD.
- Apply the induction hypothesis for \( q_t \) and the sequential update of common belief in Lemma 3 to obtain

\[
V_t(\theta_t) = \min_{q_t \in Q^h_t} \left\{ \min_{\hat{u}_t^L, \hat{u}_t^R} \left\{ QF \left( G_t, \mathbb{E} \left[ S_t^h \right] \right) + \text{tr} \left( \hat{G}_t \text{cov} \left( S_t^h \right) \right) \right\} \right\}.
\]

In the above equation, \( S_t^h := \text{vec}(X_t^h, \bar{u}_t^L + q_t(X_t^h), \bar{u}_t^R) \) where \( X_t^h \) is a random vector with distribution \( \theta_t \).

- Since \( q_t \in Q^h_t \), \( \mathbb{E}[q_t(X_t^h)] = 0 \) and consequently, \( \mathbb{E} \left[ S_t^h \right] \) depends only on \( u_t^R, \bar{u}_t^L \). Furthermore, \( \text{cov} \left( S_t^h \right) \) depends only on \( q_t \). Hence, (39) is equivalent to solving the following optimization problems

\[
\begin{align*}
\min_{u_t^R, \bar{u}_t^L} QF \left( G_t, \text{vec}(\mathbb{E}[X_t^h], \bar{u}_t^L, u_t^R) \right), \\
\min_{q_t \in Q^h_t} \text{tr} \left( \hat{G}_t \text{cov} \left( \text{vec}(X_t^h, q_t(X_t^h), 0) \right) \right).
\end{align*}
\]

- Apply Lemma 4 to problems (40) and (41), then we get (24) and the optimal solution at \( t \).

From Theorem 1 and Theorem 2 we can explicitly compute the optimal strategies for Problem 1. The optimal strategies of controllers \( C^L \) and \( C^R \) are shown in the following theorem.

**Theorem 3.** The optimal strategies of Problem 1 are given by

\[
\begin{bmatrix}
\hat{U}_t^{L*} \\
\hat{U}_t^{R*}
\end{bmatrix} = -\begin{bmatrix}
G_t^{LL} & G_t^{LR} \\
G_t^{RL} & G_t^{RR}
\end{bmatrix}^{-1} \begin{bmatrix}
G_t^{LX} \\
G_t^{RX}
\end{bmatrix} \hat{X}_t,
\]

\[
U_t^{L*} = U_t^{L*} - \left( \hat{G}_t^{LL} \right)^{-1} \hat{G}_t^{LX} (X_t - \hat{X}_t),
\]

where \( \hat{X}_t \) is the estimate (conditional expectation) of \( X_t \) based on the common information \( H_t^L \). \( \hat{X}_t \) can be computed recursively according to

\[
\begin{align*}
\hat{X}_0 &= \begin{cases}
\mu(\pi_0), & \text{if } Z_0 = \emptyset, \\
X_0, & \text{if } Z_0 = X_0.
\end{cases} \\
\hat{X}_{t+1} &= \begin{cases}
A\hat{X}_t + B^L \hat{U}_t^{L*} + B^R U_t^{R*}, & \text{if } Z_{t+1} = \emptyset, \\
X_{t+1}, & \text{if } Z_{t+1} = X_{t+1}.
\end{cases}
\end{align*}
\]

Theorem 3 shows that the optimal control strategy of \( C^R \) is linear in the estimated state \( \hat{X}_t \), and the optimal control strategy of \( C^L \) is linear in both the actual state \( X_t \) and the estimated state \( \hat{X}_t \). Note that even though the local controller \( C^L \) perfectly observes the system state, \( C^L \) still needs to compute the estimated state \( \hat{X}_t \) to make optimal decisions.

V. Conclusion

We considered a decentralized optimal control problem for a linear plant controlled by two controllers, a local controller and a remote controller. The local controller directly observes the state of the plant and can inform the remote controller of the plant state through a packet-drop channel with acknowledgments. We provided a dynamic program for this decentralized control problem using the common information approach. Although our problem is not partially nested, we obtained explicit optimal strategies for the two controllers. In the optimal strategies, both controllers compute a common estimate of the plant state based on the common information. The remote controller’s action is linear in the common estimated state, and the local controller’s action is linear in both the actual state and the common estimated state.

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APPENDIX

Proof of Lemma [7] Consider an arbitrary but fixed strategy \( g^R \) of \( C^R \). Then the control problem of \( C^L \) becomes a MDP with state \( \bar{H}_t = \text{vec}(X_t, H_t^R) \). From the theory of MDP we know that \( C^L \) can use only \( \bar{H}_t \) to make the decision at \( t \) without loss of optimality.

Proof of Lemma [3] At time \( t = 0 \), \( h^R_0 = z_0 \). According to (7), for any \( E \in \mathbb{R}^{n \times n} \),

\[
\theta_0(X_0) = \mathbb{P}(X_0 \in E|z_0) = \mathbb{P}(X_0 \in E|Z_0 = z_0) = \begin{cases} \mathbb{P}(X_0 \in E|G_0 = 0) = \mathbb{P}(X_0 \in E) = \pi_{X_0}(E) & \text{if } z_0 = \emptyset, \\ \mathbb{P}(X_0 \in E|X_0 = x_0) = \varphi(x_0)(E) & \text{if } z_0 = x_0 \end{cases}
\]

which gives (13). At time \( t + 1 \), for any \( E \in \mathbb{R}^{n \times n} \), if \( z_{t+1} = x_{t+1} \), then

\[
\mathbb{P}^{g^L_{t+1}, g^R_{t+1}}(X_{t+1} \in E|h^R_{t+1}) = \mathbb{P}(X_{t+1} \in E|x_{t+1})
\]

\[
= \mathbb{P}(X_{t+1} \in E|X_{t+1} = x_{t+1}) = \varphi(x_{t+1})(E). \tag{46}
\]
Therefore,

\[
\mathbb{E}^{g'} \left[ \sum_{s=1}^{T} c_s(X_s, U_s^L, U_s^R) | h_t^R \right] \\
= \mathbb{E}^{g'} \left[ V_{t+1}(\mathbb{P}^{g'}(dx_{t+1}|H_{t+1}^R)) | h_t^R \right] \\
+ \mathbb{E}^{g'} \left[ c_t(X_t, U_t^L, U_t^R) | h_t^R \right] \\
= \mathbb{E}^{g'} \left[ V_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, Z_{t+1})) | h_t^R \right] \\
+ \int_{\mathbb{R}^n} c_t(x_t, \tilde{u}_t^L + q_t^* (x_t), u_t^R) \theta_t(dx_t). \\
\tag{51} 
\]

Note that \( X_{t+1} \) is independent of \( \Gamma_{t+1} \). Since \( \mathbb{P}(\Gamma_{t+1} = 0) = 1 - \mathbb{P}(\Gamma_{t+1} = 1) = p, \) the first term in (51) becomes

\[
\mathbb{E}^{g'} \left[ V_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, Z_{t+1})) | h_t^R \right] \\
= p \mathbb{E}^{g'} \left[ V_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, Z_{t+1})) | h_t^R, \Gamma_{t+1} = 0 \right] \\
+ (1 - p) \mathbb{E}^{g'} \left[ V_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, Z_{t+1})) | h_t^R, \Gamma_{t+1} = 1 \right] \\
= pV_{t+1}(\alpha_t) + (1 - p) E^{g'} \left[ V_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, Z_{t+1})) | h_t^R, \Gamma_{t+1} = 0 \right] \\
= pV_{t+1}(\alpha_t) + (1 - p) \int_{\mathbb{R}^n} V_{t+1}(\psi(x_{t+1})) \alpha_t(dx_{t+1}). \\
\tag{52} 
\]

where \( \alpha_t := \psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, \emptyset) \). The third equality in (52) is true because \( X_{t+1} \) is independent of \( \Gamma_{t+1} \). The last equality in (52) follows from Lemma 3.

Combining (51) and (52) we get (48) from the definition of the value function (19). Moreover, since \( g^L_t = \{g_{0:t-1}^L, g_{R:t-1}^R, g_{T:t}^L, g_{T:t}^R\} \) are all measurable functions, \( V_t(\mathbb{P}^{g^L_{0:t-1}, g^R_{R:t-1}, g^L_{T:t}, g^R_{T:t}}(dx_t|h_t^R)) \) equals to the conditional expectation \( \mathbb{E}^{g'} \left[ \sum_{s=1}^{T} c_s(X_s, U_s^L, U_s^R) | h_t^R \right] \) which is measurable with respect to \( h_t^R \).

Now let’s consider (49). Let \( u_t^R, \tilde{u}_t^L, q_t \) be the variables defined by (13) from \( h_t^R \) and \( q^L_1, g^R_0 \). Following an argument similar to that of (50)-(52), we get

\[
\mathbb{E}^{g^L \cdot g^R} \left[ \sum_{s=1}^{T} c_s(X_s, U_s^L, U_s^R) | h_t^R \right] \\
\geq \int_{\mathbb{R}^n} c_t(x_t, \tilde{u}_t^L + q_t(x_t), u_t^R) \theta_t(dx_t) \\
+ (1 - p) \int_{\mathbb{R}^n} V_{t+1}(\psi_t(x_{t+1})) \psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t, \emptyset)(dx_{t+1}) \\
+ pV_{t+1}(\psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t, \emptyset)) \geq V_t(\theta_t). \\
\tag{53} 
\]

The last inequality in (53) follows from the definition of the value function (19). This completes the proof of the induction step, and the proof of the theorem.

\begin{proof}[Proof of Lemma 4]

The proof of the first part of Lemma 4 is trivial.

Now let’s consider the second part of Lemma 4, the functional optimization problem (37). From the property of trace and covariance we have

\[
\text{tr} \left( \mathbb{C}^{\text{vec}} \left( \text{vec} (X^\theta, q(X^\theta)) \right) \right) \\
= \mathbb{E} \left[ QF(G, \text{vec} (X^\theta, q(X^\theta))) - \mathbb{E} \left[ \text{vec} (X^\theta, q(X^\theta)) \right] \right] \\
= \mathbb{E} \left[ QF(G, \text{vec} (X^\theta - \mu(\theta), q(X^\theta))) \right] \tag{54} 
\]

where the last equation in (54) holds because \( \mathbb{E} \left[ q(X^\theta) \right] = 0 \). Since \( \theta \) is the distribution of \( X^\theta \), we have

\[
\mathbb{E} \left[ QF(G, \text{vec} (X^\theta - \mu(\theta), q(X^\theta))) \right] \\
= \int_{\mathbb{R}^n} QF(G, \text{vec} (y - \mu(\theta), q(y))) \theta(dy) \tag{55} 
\]

Note that the function inside the integral of (55) has the quadratic form of the optimization problem (35) with \( x = y - \mu(\theta) \) and \( u = q(y) \). From the results of the first part of Lemma 4, for any \( y \in \mathbb{R}^n \) we have

\[
QF(G, \text{vec} (y - \mu(\theta), q(y))) \\
\geq QF(G, \text{vec} (y - \mu(\theta), q^*(y))) = QF(P, y - \mu(\theta)) \\
\]

where \( q^* \) is the function given by (38). It is clear that \( q^* \) is measurable. Furthermore,

\[
\mathbb{E} \left[ q^*(X^\theta) \right] = \int_{\mathbb{R}^n} - (G^{LU})^{-1} G^{UX} (x - \mu(\theta))(\theta(dx)) = 0. 
\]

Consequently, \( q^* \in \mathcal{Q}^L \). Then \( q^* \) is the optimal solution to problem (37), and the optimal value is given by

\[
\int_{\mathbb{R}^n} QF(G, \text{vec} (y - \mu(\theta), q^*(y))) \theta(dy) \\
= \int_{\mathbb{R}^n} QF(P, y - \mu(\theta))(\theta(dy) = \mathbb{E} \left[ QF \left( P, X^\theta - \mu(\theta) \right) \right] \\
= \text{tr} \left( P \cdot \text{cov} (\theta) \right). \\
\tag{56} 
\]

\end{proof}

\begin{proof}[Proof of Theorem 2]

The proof is done by induction.

At \( t + 1, \) (24) is true since \( P_{t+1} = \hat{P}_{t+1} = 0 \). Suppose \( (24) \) is true at \( t \) and the matrices are all PSD and \( G_{t+1}, \tilde{G}_{t+1} \) are PD.

At time \( t, \) since \( P_{t+1} \) and \( \tilde{P}_t \) are PSD, \( H_t \) and \( \hat{H}_t \) are PSD. Since \( R_t \) is PD, \( G_t = R_t + H_t \) and \( \tilde{G}_t = R_t + (1 - p) H_t + p \hat{H}_t \) are also PD. Then \( P_t \) is PSD because \( P_t \) is the Schur complement of \( G_t^{LL} \) of the matrix \( G_t \).

Similarly, \( \tilde{P}_t \) is PSD because \( \tilde{P}_t \) is the Schur complement of \( G_t^{LL} \) of the matrix \( G_t^{LL} \).

Let’s now compute the value function at \( t \) given by (19) in Theorem 1. For notational simplicity, let \( \alpha_t = \psi_t(\theta_t, u_t^R, \tilde{u}_t^L, q_t^*, q_t^*, \emptyset) \).

We first consider the second term of the value function in (19). From the induction hypothesis we have

\[
(1 - p) \int_{\mathbb{R}^n} V_{t+1}(\psi(x_{t+1})) \alpha_t(dx_{t+1}) \\
= (1 - p) \int_{\mathbb{R}^n} QF(P_{t+1}, x_{t+1}) \alpha_t(dx_{t+1}) + (1 - p) e_{t+1} \\
= (1 - p) QF(P_{t+1}, \theta(\alpha)) \\
+ (1 - p) \text{tr} \left( P_{t+1} \text{cov}(\alpha_t) \right) + (1 - p) e_{t+1}. \tag{57} 
\]

\end{proof}
The last equality in (57) follows from the property of covariance. Similarly, the last term of (19) becomes
\[
pV_{t+1}(\alpha_t) = pQF(P_{t+1}, \mu(\alpha_t)) + p tr(\hat{P}_{t+1} \text{cov}(\alpha_t)) + e_{t+1}.
\]
(58)

Let \( S_t^\theta := \text{vec}(X_t^\theta, \bar{u}_t^L + q_t(X_t^\theta), u_t^R) \) where \( X_t^\theta \) is a random vector with distribution \( \theta \), such that \( X_t^\theta \) and \( W_t \) are independent. Note that from (16) in Lemma 3
\[
Y_t^\theta := [A, B^L, B^R]S_t^\theta + W_t = AX_t^\theta + B^L(\bar{u}_t^L + q_t(X_t^\theta)) + B^R u_t^R + W_t
\]
is a random vector with distribution \( \alpha_t \). Then, combining (57) and (58), the last two terms of the value function becomes
\[
QF(P_{t+1}, \mu(\alpha_t)) + tr \left( \left( ((1-p)P_{t+1} + p\hat{P}_{t+1}) \text{cov}(\alpha_t) \right) + e_{t+1} \right)
\]
(57)

Finally, substituting (66) and (67) into (63) we obtain the (24) at \( t \). This completes the proof of the induction step and the proof of the theorem.

**Proof of Theorem 3.** Let \( \hat{X}_t \) be the estimate (conditional expectation) of \( X_t \) based on the common information \( H^R_t \). Then, for any realization \( h^R_t \) of \( H^R_t \), \( \hat{x}_t = \mu(\theta_t) \). This together with Theorems 1 and 2 result in (42) and (43).

To show (44) and (45), note that at time \( t = 0 \), for any realization \( h^R_t \) of \( H^R_t \),
\[
\hat{x}_0 = \mu(\theta_0) = \int_{R^n} y\theta_0(dy)
\]
(66)

Therefore, (64) is true. Furthermore, at time \( t + 1 \) and for any realization \( h^R_t \) of \( H^R_t \),
\[
\hat{x}_{t+1} = \mu(\theta_{t+1}) = \int_{R^n} y\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t, z_{t+1})dy.
\]
(67)

If \( z_{t+1} = x_{t+1} \), then \( \hat{x}_{t+1} = \int_{R^n} y\varphi(x_{t+1})dy = x_{t+1} \). If \( z_{t+1} = \emptyset \), then,
\[
\hat{x}_{t+1} = \int_{R^n} y\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t, \emptyset)dy = \int_{R^n} y\int_{R^n} y\int_{R^n} \mathbb{1}(y)(Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t)
\]
\[
\theta_t(dx_t)\pi_{W_t}(dw_t)dy = \int_{R^n} \mathbb{1}(y)(Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t)
\]
\[
\theta_t(dx_t)\pi_{W_t}(dw_t) = Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t.
\]
(68)

Now we need to solve the two optimization problems
\[
\min_{\bar{u}_t^L, u_t^R} QF(\hat{G}_t, \text{vec}(\mu(\theta_t), \bar{u}_t^L, u_t^R)),
\]
(64)

\[
\min_{q_t \in \mathbb{Q}^R} tr(\hat{G}_t \text{cov}(\text{vec}(X_t^\theta, q_t(X_t^\theta), 0))).
\]
(65)

Since \( G_t \) is PD, it follows by Lemma 4 that the optimal solution of (64) is given by (26) and
\[
\min_{\bar{u}_t^L, u_t^R} QF(\hat{G}_t, \text{vec}(\mu(\theta_t), \bar{u}_t^L, u_t^R)) = QF(\hat{P}_t, \mu(\theta_t)).
\]
(66)

Similarly, since \( \hat{G}_t \) is also PD, Lemma 4 implies that the optimal solution of (65) is given by (27) and
\[
\min_{q_t \in \mathbb{Q}^R} tr(\hat{G}_t \text{cov}(\text{vec}(X_t^\theta, q_t(X_t^\theta), 0))) = tr(\hat{P}_t \text{cov}(\theta_t)).
\]
(67)

Finally, substituting (66) and (67) into (63) we obtain the (24) at \( t \). This completes the proof of the induction step and the proof of the theorem.

**Proof of Theorem 3.** Let \( \hat{X}_t \) be the estimate (conditional expectation) of \( X_t \) based on the common information \( H^R_t \). Then, for any realization \( h^R_t \) of \( H^R_t \), \( \hat{x}_t = \mu(\theta_t) \). This together with Theorems 1 and 2 result in (42) and (43).

To show (44) and (45), note that at time \( t = 0 \), for any realization \( h^R_t \) of \( H^R_t \),
\[
\hat{x}_0 = \mu(\theta_0) = \int_{R^n} y\theta_0(dy)
\]
(66)

Therefore, (44) is true. Furthermore, at time \( t + 1 \) and for any realization \( h^R_t \) of \( H^R_t \),
\[
\hat{x}_{t+1} = \mu(\theta_{t+1}) = \int_{R^n} y\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t, z_{t+1})(dy).
\]
(67)

If \( z_{t+1} = x_{t+1} \), then \( \hat{x}_{t+1} = \int_{R^n} y\varphi(x_{t+1})(dy) = x_{t+1} \). If \( z_{t+1} = \emptyset \), then,
\[
\hat{x}_{t+1} = \int_{R^n} y\psi_t(\theta_t, u_t^R, \bar{u}_t^L, q_t, \emptyset)(dy) = \int_{R^n} y\int_{R^n} y\int_{R^n} \mathbb{1}(y)(Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t)
\]
\[
\theta_t(dx_t)\pi_{W_t}(dw_t)dy = \int_{R^n} \mathbb{1}(y)(Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t)
\]
\[
\theta_t(dx_t)\pi_{W_t}(dw_t) = Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t.
\]
(68)

where the third equality is true because
\[
\int_{R^n} y\mathbb{1}(y)(Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t)dy
\]
\[
= Ax_t + B^L(\bar{u}_t^L + q_t(x_t)) + B^R u_t^R + w_t.
\]
Furthermore, the last equality is true because \( q_t \in \mathbb{Q}^R \) and \( W_t \) is a zero mean random vector. Therefore, (45) is true and the proof is complete.