Casimir energy for self-interacting scalar field in a spherical shell

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Abstract

In this paper we calculate the Casimir energy for spherical shell with massless self-interacting scalar field which satisfying Dirichlet boundary conditions on the shell. Using zeta function regularization and heat kernel coefficients we obtain the divergent contributions inside and outside of Casimir energy. The effect of self-interacting term is similar with existing of mass for filed. In this case some divergent part arises. Using the renormalization procedure of bag model we can cancel these divergent parts.
1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory\cite{1,2}. Since its first prediction by Casimir in 1948\cite{3} this effect has been investigated for different fields having different boundary geometries\cite{4-7}. The Casimir effect can be viewed as the polarization of vacuum by boundary conditions or geometry. Therefore, vacuum polarization induced by a gravitational field is also considered as Casimir effect.

Casimir effect for spherical shells in the presence of the electromagnetic fields has been calculated several years ago\cite{9, 10}. A recent simplifying account of it for the cases of electromagnetic and scalar fields with both Dirichlet and Neumann boundary conditions on sphere is given in\cite{11}. The dependence of Casimir energy on the dimension of space for electromagnetic and scalar fields with Dirichlet boundary conditions in the presence of a spherical shell is discussed in\cite{12, 13}. The Casimir energy for odd and even space dimensions and different fields, including the spinor field, and all the possible boundary conditions have been considered in\cite{14}. There it is explicitly shown that although the Casimir energy for interior and exterior of a spherical shell are both divergent, irrespective of the number of space dimensions, the total Casimir energy of the shell remains finite for the case of odd space dimensions. More recently a new method have been developed for the scalar Casimir stress on the D-dimensional sphere, in \cite{15}. In this reference the regularized vacuum expectation values for the scalar field energy-momentum tensor inside and outside a spherical shell and in the region between two concentric spheres have been calculated. Of some interest are cases where the field is confined to the inside of a spherical shell. This is sometimes called the bag boundary condition. The application of Casimir effect to the bag model is considered for the case of massive scalar field \cite{16} and the Dirac field \cite{17}. We use the renormalization procedure in the above cases for our problem.

Casimir effect for interacting fields has not been studied extensively. This is due to the lack of knowledge of true vacuum of an interacting quantum theory \cite{23}. In the other hand the study of a massless scalar field with quartic self-interaction is very important in different subject of physics, for example in the Winberg-Salam model of weak interaction, fermions masses generation, in solid state physics \cite{19, 20}, inflationary models \cite{21}, solitons \cite{22, 18} and Casimir effect \cite{23, 24}.

In this paper we calculate the Casimir energy for spherical shell with massless self-interacting scalar filed which satisfying Dirichlet boundary conditions on the shell. We use the heat-kernel technique to drive the zeta function for massless self-interacting scalar field operator. Heat kernel coefficients and zeta function of the Laplace operator on a D-dimensional ball with different boundary conditions, both of them useful tools to calculate Casimir energies, have been calculated in \cite{25, 26}. The problem of calculating the determinant of a Laplacian-like operator $A$ on a manifold $M$ is very important in mathematics and physics\cite{4, 3, 26}. In the cases which $A$ has a discrete spectrum the determinant of $A$ is generally divergent. The zeta function regularization is an appropriate way for these calculations. The zeta function method is a particular useful tool for the determination of effective action, where one-loop effective action is given by $\frac{1}{2} \ln \det A$. Using the relation between zeta function and heat-kernel for operator $A$, one can find the zeta function.

The paper is organized as follows: in the second section we briefly review the Casimir energy inside and outside of spherical shell in terms of zeta function. Then in section 3
we obtain the heat kernel coefficients for massless self-interacting scalar field inside and outside of spherical shell, after that we obtain the divergent part of Casimir energy and introduce the classical part of total energy of system, then using the bag model renormalization procedure [16, 17] the renormalized Casimir energy can be obtained. Section is devoted to conclusion.

2 Casimir energy inside and outside of spherical shell

We consider a massless self-interacting scalar field satisfying Dirichlet boundary conditions on a spherical shell in Minkowski space-time. The Casimir energy $E$ is the sum of Casimir energies $E_{\text{in}}$ and $E_{\text{out}}$ for inside and outside of the shell. The Casimir energies in-side and out-side of the shell are divergent individually. For free massless scalar field when we calculate the total Casimir energy, we add interior and exterior energies to each other. Now in odd space dimensions, divergent parts will cancel each other out [14].

The Casimir energy in- and out-side of a spherical shell for massless free scalar field with Dirichlet boundary conditions is given by

$$E_{\text{in}} = \frac{1}{2R}(0.008873 + \frac{0.001010}{\varepsilon}), \quad E_{\text{out}} = -\frac{1}{2R}(0.003234 + \frac{0.001010}{\varepsilon}). \quad (1)$$

where $R$ is a radius of spherical shell. Now we shall restrict our selves to a quartic self-interaction that is

$$V(\hat{\phi}) = \frac{\lambda}{24} \hat{\phi}^4. \quad (2)$$

We wish to consider the Casimir energy for this new case which given by

$$E_0 = \frac{1}{2} \sum_k \lambda_k^{1/2}, \quad (3)$$

where $\lambda_k$ are determined by the eigenvalue equation

$$(-\Delta + V''(\hat{\phi}))\hat{\phi}_k(x) = \lambda_k \hat{\phi}_k(x), \quad (4)$$

which the scalar field satisfy Dirichlet boundary condition on the shell.

Using the zeta function for operator $A$ which given by

$$A = \Box + v''(\phi). \quad (5)$$

and

$$\zeta_A(s) = \sum_k \lambda_k^{-s}, \quad (6)$$

we regularize $E_0$ by

$$E_{\text{reg}} = \frac{1}{2} \zeta(s - 1/2)\mu^{2s}, \quad (7)$$

where we have introduced an unknown scale parameter $\mu$, whith dimensions of mass to keep the zeta function dimensionless. The zeta function (6) is the sum of zeta functions $\zeta_{\text{in}}(s)$ and $\zeta_{\text{out}}(s)$ for inside and outside of the shell, therefore we can write

$$E_{\text{reg}}^{\text{in}} = \frac{1}{2} \zeta_{\text{in}}(s - 1/2)\mu^{2s}, \quad E_{\text{reg}}^{\text{out}} = \frac{1}{2} \zeta_{\text{out}}(s - 1/2)\mu^{2s}. \quad (8)$$
3 Heat kernel coefficients inside and outside of spherical shell

Now we use the heat-kernel technique to drive the zeta function, the relation between zeta function and heat-kernel is given by

\[
\zeta_A(s) = \frac{1}{\Gamma(s)} \sum_k \int_0^\infty dtt^{s-1} \exp(-\lambda_k t) = \frac{1}{\Gamma(s)} \int_0^\infty dtt^{s-1} K(t), \tag{9}
\]

where heat-kernel \( K(x, x', t) \) satisfies the equation

\[
\left( \frac{\partial}{\partial t} + A \right) K(t, x, x') = 0. \tag{10}
\]

Now we write the short-time expansion of the heat-kernel \( K(t) \)

\[
K(t) = \sum_k \exp(-\lambda_k t) \sim (4\pi t)^{-3/2} \sum_{k=0,1/2,1,...}^\infty \left( \int_M d\nu_k + \int_{S^2} d\sigma_k \right) \exp(-tV''(\phi)) t^k. \tag{11}
\]

The short-time expansion contain both volume and a boundary part, the coefficients wit integer numbers are volume parts and the coefficients with half integer come from the boundary conditions, for free field the heat kernel coefficients are as following

\[
B_k = \int_M d\nu_k + \int_{S^2} d\sigma_k, \tag{12}
\]

where \( M \) is inside and outside of spherical shell and \( S^2 \) is the surface of spherical shell. The simplest first of \( a_k \) and \( c_k \) coefficients for a manifold with boundary are given in \[28\], these coefficients in the presence of potential term are given by

\[
a_0 = 1, \tag{13}
\]

\[
a_1 = Q - \frac{1}{6} R \tag{14}
\]

\[
a_2 = (Q - R/6)^2 - \frac{1}{3} \Box Q - \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \frac{1}{15} \Box R + \frac{1}{6} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu}, \tag{15}
\]

where \( Q \) is a potential term, in our problem \( Q \) is given by

\[
Q = V''(\phi) \tag{16}
\]

as one can see the \( a_k \) coefficients are functions of geometric quantities, \( R_{\mu\nu\alpha\beta}, R_{\mu\nu} \) and \( R \) are respectively, Rieman, Ricci and scalar curvature tensor.

\[
\tilde{R}_{\mu\nu} = [\nabla_\mu, \nabla_\nu], \tag{17}
\]

where \( \nabla_\mu \) is covariant derivative. Several first boundary coefficients in asymptotic expansion for Dirichlet boundary condition are as follow \[28\]

\[
c_{1/2} = -\frac{\sqrt{\pi}}{2}, \tag{18}
\]
\[ c_1 = \frac{1}{3} K - \frac{1}{2} f^{(1)}, \]  
\[ c_{3/2} = \frac{\sqrt{\pi}}{2} \left( \frac{1}{6} \tilde{R} - \frac{1}{4} R_{nn}^0 + \frac{3}{32} K^2 - \frac{1}{16} K_{ij} K^{ij} + Q \right) + \frac{5}{16} K f^{(1)} - \frac{1}{4} f^{(2)}, \]

where
\[ f^{(1)}(z) = \frac{1}{6} + \frac{z^2}{6} (2 + \frac{z^2}{2} - z(z^2 + 6) h(z)), \]
\[ f^{(2)}(z) = -\frac{1}{6} + \frac{z^2}{6} (-4 + \frac{z^2}{2} - 4z^3 h(z)), \]
\[ h(z) = \int_0^\infty \exp(-(x^2 + 2zx)), \]

where \( \tilde{R} \) is the scalar curvature of the boundary, \( K \) is trace of extrinsic curvature tensor on the \( S^2 \),
\[ K_{ij} = \nabla_i N_j, \]

where \( N_j \) is outward unit normal vector.

The heat kernel coefficients \( B_k \) for free scalar field with Dirichlet boundary condition for interior region are given by
\[ B^0_0 = \frac{4\pi}{3} R^3, \]
\[ B^{1/2}_{1/2} = -2\pi^{3/2} R^2, \]
\[ B^1_1 = \frac{8\pi}{3} R, \]
\[ B^{3/2}_{3/2} = -\frac{\pi^{3/2}}{6}, \]
\[ B^2_2 = \frac{16\pi}{315} R. \]

The coefficients for exterior region are
\[ B^\text{ext}_i = B^\text{in}_i, \quad i = 1/2, 3/2, \ldots \]
\[ B^\text{ext}_i = -B^\text{in}_i, \quad i = 0, 1, 2, \ldots \]

In the presence of self-interaction the heat kernel coefficients for inside and outside changed as following
\[ B_k = (\int_M d\nu_k + \int_{S^2} d\sigma_k) (V''(\hat{\phi}))^{3/2 - k - s} \]
then we have
\[ B^0_0 = \int_M d\nu_0 V'^{3/2 - s} = \int_M d\nu \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{3/2 - s}, \]
\[ B^{1/2}_{1/2} = \int_{S^2} d\sigma_{1/2} V^{1-s} = \frac{\sqrt{\pi}}{2} \int_{S^2} \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{1-s} d\sigma, \]
\[ B^1_1 = (\int_M d\nu_1 + \int_{S^2} d\sigma_1) V'^{1/2 - s} = \int_M \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{3/2 - s} d\nu + \frac{2}{3R} \int_{S^2} \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{1/2 - s} d\sigma \]
\[ B^{3/2}_{3/2} = \int_{S^2} d\sigma_{3/2} V'^{-s} = \frac{-\pi^{3/2}}{6} \int_{S^2} d\sigma \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{-s} + \frac{\sqrt{\pi}}{2} \int_{S^2} \left( \frac{\lambda}{2} \hat{\phi}^2 \right)^{1-s} d\sigma, \]
\[
B_2^{in} = (\int_M d\nu a_2 + \int_{S^2} d\nu c_2) V''^{-1/2-s} = \int_M \frac{\lambda^2}{4} \phi^4 \\
- \frac{\lambda}{6} \Box \phi^2 (\frac{\lambda}{2} \phi^2)^{-1/2-s} dv + \int_{S^2} \frac{4}{3 \pi R^3} (\frac{\lambda}{2} \phi^2)^{-1/2-s} ds,
\]

here \(R\) is radius of spherical shell. Now we can rewrite the zeta function (9) for inside and outside of spherical shell.

\[
\zeta_{in}^A(s) = \frac{1}{(4\pi)^{3/2} \Gamma(s)} \sum_{k=0,1/2,1,...} B_k^{in} \Gamma(s + k - 3/2) \quad (38)
\]

Similarly for outside we have

\[
\zeta_{out}^A(s) = \frac{1}{(4\pi)^{3/2} \Gamma(s)} \sum_{k=0,1/2,1,...} B_k^{out} \Gamma(s + k - 3/2) \quad (39)
\]

The first five coefficients of heat kernel expansion contribute to divergences of zeta functions inside and outside of spherical shell. Using (8) we can write

\[
E_{div}^{in} = \frac{\mu^{2s}}{2(4\pi)^{3/2} \Gamma(s-1/2)} [B_0 \Gamma(s-2) + B_{1/2} \Gamma(s-3/2) - B_1 \Gamma(s-1) + B_{3/2} \Gamma(s-3/2) + B_2 \Gamma(s)] \quad (40)
\]

Similarly

\[
E_{div}^{out} = \frac{\mu^{2s}}{2(4\pi)^{3/2} \Gamma(s-1/2)} [-B_0 \Gamma(s-2) + B_{1/2} \Gamma(s-3/2) - B_1 \Gamma(s-1) + B_{3/2} \Gamma(s-3/2) - B_2 \Gamma(s)] \quad (41)
\]

We must mention that for heat kernel coefficients \(B_k\) in Eqs.(40,41) \(s \to s-1\). At this stage we recall \(E_0\) as given by (3) is only one part of total energy. There is also a classical part. We can try to absorb \(E_{div}\) into the classical energy for inside and outside of spherical shell separately. This technique of absorbing an infinite quantity into a renormalized physical quantity is familiar in quantum field theory and quantum field theory in curved space [8]. Here, we use a procedure similar to that of bag model [16, 17] (to see application of this renormalization procedure in Casimir effect problem refer to [29, 30]). The classical energy of spherical shell may be written as,

\[
E_{class} = PR^3 + \sigma R^2 + FR + K + \frac{h}{R}, \quad (42)
\]

where \(P\) is pressure, \(\sigma\) is surface tension and \(F, K, h\) do not have special names. In odd space dimensions, for free massless scalar field divergent parts inside and outside cancel each other out, then we do not need to renormalize the parameters in the classical energy, but in massive case, when we add the interior and exterior energies to each other, there are only two contributions, which are divergent. In our problem there is similar situation, as one can see, in Eqs.(40,41) when we add divergent parts inside and outside, all terms cancel each other, unless \(B_{1/2}\) and \(B_{3/2}\) terms

\[
E_{div} = E_{div}^{in} + E_{div}^{out} = \frac{\mu^{2s}}{(4\pi)^{3/2} \Gamma(s-1/2)} [B_{1/2} \Gamma(s-3/2) + B_{3/2} \Gamma(s-1/2)] \quad (43)
\]
Therefore the Casimir energy for this general case becomes divergent. The total energy of the shell maybe written as
\[ E^{\text{tot}} = E_0 + E_{\text{class}} \] (44)

In order to obtain a well defined result for the total energy, we have to renormalize some parameters of classical energy according to the below:

\[ K \rightarrow K - \frac{\mu^{2s}}{(4\pi)^{3/2}} \left[ \frac{-\pi^{3/2}}{6} \int_{S^2} ds (\frac{\lambda}{2} \phi^2)^{-s} + \frac{\sqrt{\pi}}{2} \int_{S^2} (\frac{\lambda}{2} \phi^2)^{1-s} ds \right] \] (45)

\[ \sigma \rightarrow \sigma + \frac{\mu^{2s}}{16\pi R^2 \Gamma(s - 1/2)} \Gamma(s - 3/2) \left[ -\frac{\sqrt{\pi}}{2} \int_{S^2} (\frac{\lambda}{2} \phi^2)^{1-s} ds \right]. \] (46)

Hence the effect of the self-interacting scalar quantum field is to change, or renormalize some parameters of classical energy of system. Once, the terms \( E_{\text{div}} \) have been removed from \( E_0 \), the remainder is finite and will be called the renormalized Casimir energy. Using Eq.(8) and Eqs.(38,39) we have

\[ E_{\text{ren}} = E_0 - E_{\text{div}} = \frac{\mu^{2s}}{2} (\zeta^{\text{in}}_A (s - 1/2) + \zeta^{\text{out}}_A (s - 1/2)) \] (47)

\[ - E_{\text{div}} = \frac{\mu^{2s}}{(4\pi)^{3/2} \Gamma(s - 1/2)} \sum_{k=5/2,3,...} \Gamma(s + k - 2) (B^{\text{in}}_k + B^{\text{out}}_k). \]

\[ \frac{1}{x} \]

4 Conclusion

In this paper we have considered the Casimir energy for massless self-interacting scalar field in a spherical shell. The scalar field satisfy Dirichlet boundary condition on the shell. Unlike to the main part of previous studies on the scalar Casimir effect, here we adopt the Casimir energy with interacting quantum field. To obtain the divergent parts of energy inside and outside of spherical shell, we calculate the first five heat kernel coefficients for operator \( A = -\Box + V''(\hat{\phi}) \). Previous result of heat kernel coefficients for Laplace operator \([16, 26]\) have been changed for our problem. The new result are given by Eqs.(32-37). In free massless case when we add divergent parts inside and outside, all divergences cancel each other out, but in interacting scalar field case, exactly similar to the massive case, divergent parts inside do not cancel the divergent parts outside of shell. The renormalization procedure which is necessary to apply in this situation is similar to that of the bag model \([16, 17]\). At this stage we introduce the classical energy and try to absorb divergent part into this classical energy. The renormalization can be achieved now by shifting some parameter of classical energy by an amount which cancel the divergent contribution. After the subtraction of divergent contribution, the remainder is finite and will be called the renormalized Casimir energy.

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