MORPHOLOGY OF THE UNIVERSAL FUNCTION FOR THE CRITICAL CIRCLE MAP  
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Abstract  
We describe the morphology of the universal function for the critical (cubic) circle map at the golden mean, paying particular attention to the birth of inflection points and their reproduction. In this way one can fully understand its intricacies.

Introduction  
At the critical point, the circle map has a cubic order of inflection. Near the golden mean winding number the self-reproducing (renormalization group) properties of the Fibonacci mode-locked steps [1] leads to the notion of a universal circle map function \(g(x)\), which has received a great deal of study in the literature [2, 3, 4]. It obeys the compatible [5] pair of functional equations,

\[
\begin{align*}
g(g(x)) &= g(\alpha^2 x)/\alpha, \quad (1) \\
g(g(x)/\alpha^2) &= g(\alpha x)/\alpha^2. \quad (2)
\end{align*}
\]

The universal circle map constant \(\alpha = -1.288575..\) [2] can be determined to extremely high accuracy [6] by examining the equations over a restricted range of \(x\). In this paper however we focus on the behaviour of \(g(x)\) over large \(x\), in order to make sense of its morphology. Some years ago we tried a similar analysis for the period-doubling universal function [7]; our aim here is to comprehend how the ever-increasing number of inflection points are spawned as we go out to ever larger \(|x|\). We shall see that they can be discovered directly by a close look at the functional equations and their derivatives, starting from previously known series of inflection points.

In the next section we quickly recapitulate the origin and character of the functional equations and derive the principal features, including the first family of inflection points. In the last section we locate the higher families of inflection points of \(g\) and explain how successive ones are derived theoretically. In this way one can better appreciate the morphology of the universal circle map function and comprehend its complicated structure.

Origin and Main features  
The approach to the irrational golden mean winding number in nonlinear maps of an angular variable \(\theta\) is normally made [1] via a succession of Fibonacci number ratios: \(F_{n-1}/F_n\) as \(n \to \infty\). In particular for the critical circle map,

\[
\theta \to \theta' = f(\theta, \Omega) \equiv \Omega + \theta - \sin(2\pi \theta)/2\pi, \quad (3)
\]
with a cubic inflection point at the origin, it is conventional to work out the superstable values of the driving frequencies \( \Omega_n \) which lead to the rational
\[
\frac{F_n - 1}{F_n},
\]
(Above we are using the notation, \([f]^N(x)\) to denote the \(N\)-fold composition, \(f(f(\cdots f(x)\cdots))\).) Correspondingly,
\[
\Delta \theta_n \equiv [f]^{F_n-2}(0, \Omega_n) - F_{n-1}
\]
is the nearest fixed point to the origin. The universal Feigenbaum constants \( \delta, \alpha \) of the map are then obtained \[2\] as
\[
\delta = \lim_{n \to \infty} \Delta \Omega_n / \Delta \Omega_{n-1} = -2.833612\ldots; \quad \Delta \Omega_n \equiv \Omega_n - \Omega_{n-1},
\]
\[
\alpha = \lim_{n \to \infty} \Delta \theta_n / \Delta \theta_{n-1} = -1.288575\ldots
\]
Parenthetically, we note the numerical value \( \Omega_\infty = 0.6066610635\ldots \). By considering the set of functions
\[
\phi_r(\theta) = \alpha^n ([f]^n(\Omega_{n+r}, \theta/\alpha^n) - F_{n-1})
\]
in the limit of large \( n \) and taking the limit as \( r \to \infty \), one may establish that the limiting (Shenker) function \( \phi \), evaluated at \( \Omega_\infty \), obeys two compatible functional equations,
\[
\phi (\phi(\theta)/\alpha) = \phi(\alpha \theta)/\alpha^2 \quad \text{and} \quad \phi (\alpha \phi(\theta)) = \phi(\alpha^2 \theta)/\alpha.
\]
Numerical approximations to (8) indicate that the function \( \phi(x) \) (called \( f(x) \) by Shenker) is a monotonically increasing function of its argument; therefore the alternative function \( g(x) = \alpha \phi(x) \) decreases monotonically with \( x \) and one readily establishes that it satisfies the universal equations (1,2). At the same time one may easily show that the scaled function \( G(x) = g(\rho x)/\rho \) obeys the self-same equation pair (9,10); thus one is at liberty to set the normalization scale at will. We shall fix the unique zero of \( g(x) \) at \( x = 1 \), that is \( g(1) = 0 \), and thereby find the value \( g(0) \equiv \lambda = 1.2452\ldots \) to be the intercept at the origin. Because of the cubic nature of inflection, \( g'(0) = g''(0) = 0 \).

Plots of \( g(x) \) in the ranges -8 to -4, -4 to +4, +4 to +8 are respectively given in Figures 1, 2 and 3. These plots are approximate and they were obtained from equation (8) by going to high order \( (n = 14) \) at the accumulation point \( \Omega_\infty \) and scaling appropriately in \( x \), as indicated above. (The range -8 to 8 was chosen so that we have sufficiently many inflection points to make the investigation of the morphology worthwhile, but not too many to confuse the subsequent discussion.) Two significant points may be noted from these plots:
• they exhibit a series of inflection points which are self-similar and which proliferate on ever smaller (and larger) scales as we go out in $x$,

• the function behaves asymptotically as $g(x) \simeq \alpha x$.

The last fact is readily verified from the initial functional equations (1) and (2); our goal is to understand the first fact, namely the origin and families of inflection points. Before doing so let us derive a number of useful facts about $g$ at particular locations, some of which are required later. By evaluating (1) at $x = 1/\alpha^2$ and (2) at $x = 1/\alpha$, one readily discovers that

$$g(1/\alpha^2) = 1, \quad g(1/\alpha) = \alpha^2;$$

the former also follows from (2) evaluated at $x = 1/\alpha^2$. Taking (1) and (2) at $x = 1$, leads to

$$g(\alpha^2) = \alpha \lambda, \quad g(\alpha) = \alpha^2 \lambda,$$

the former following from (1a) evaluated at $x = 1/\alpha$. Finally, we note that working out (1) and (2) at $x = 0$, gives

$$g(\lambda) = \lambda/\alpha, \quad g(\lambda/\alpha^2) = \lambda/\alpha^2.$$

The latter condition demonstrates that $\lambda/\alpha^2$ is the (single) fixed point of the universal circle function! Further relations between $g(\lambda \alpha^M)$ can be found by substituting suitable $x = \alpha^N$ in eqs. (1), but we will not need them below.

As the next step, differentiate equations (1,2) to obtain the related pair,

$$g'(x)g'(g(x)) = \alpha g'(\alpha^2 x) \quad (11)$$

$$g'(x)g'(g(x)/\alpha^2) = \alpha g'(\alpha x). \quad (12)$$

Remember that where $g$ has zero first derivative, its second derivative also vanishes (because the map is cubic), so equations (11,12) provide information about the inflection points. A number of interesting values of the derivatives can be obtained immediately. Setting $x = 1/\alpha^2$ in (11) and $x = 1/\alpha$ in (12) yields

$$g'(1/\alpha^2) = \alpha, \quad g'(1/\alpha) = \alpha$$

and one can check the correctness of these values from the graphs. Also, taking the limit as $x \to 0$ in (11,12), one deduces that

$$g'(\lambda) = \alpha^5, \quad g'(\lambda/\alpha^2) = \alpha^3,$$

because the inflection point is cubic. By similar means one may establish relations between derivatives of $g$ at various points $x = \alpha^M \lambda$ which provide useful checks on numerical work.

But one can go further through the following observation: if $\xi$ corresponds to an inflection point, (11) and (12) ensure that $\alpha \xi$ and $\alpha^2 \xi$ are also inflection points; by induction one generates an entire sequence of such points, viz.
\(a^n \xi, \ n = 0, 1, 2, \ldots\) However equation (11) at \(x = 1\) and (12) at \(x = 0\), inform us that

\[
g'(\alpha^2) = 0, \quad g'(\alpha) = 0,
\]

so we conclude that \(\xi_n \equiv \alpha^n, \ n = 1, 2, 3, \ldots\) represent a family of inflection points, seeded by \(\xi_1 = \alpha\); for our purposes we shall regard them as the primary or parent series. These points can be picked out in Table 1, where we have listed all the inflection points that occur between \(x = -8\) to +8 and they can also be spotted in the figures as marked dots. This table contains many more inflection points and the question is: how are they seeded? The answer will be provided in the next section.

**Families of inflection points**

Examination of the figures shows that not all inflection points belong to the primary sequence \(a^N\). The first non-standard inflection point occurs at \(x \simeq 2.25\) and to see where this comes from we examine first the seed values \(x_N \equiv g^{-1}(\alpha^N), \ N = 1, 2, \ldots\). These may be read off from the graph of \(g\) and are tabulated in Table 2. At those locations,

\[
g'(g(x_N)) = g'(\alpha^N) = 0,
\]

and from (11) we are assured that the set \(\alpha^M x_N, \ M = 2, 3, \ldots\) will generate a family of inflection points, some of which may not be new. [For example we note again that the location \(x_2 = 1/\alpha\) simply generates the primary or parent family.] The first daughter family, which is new, is given by the secondary series, \(\alpha^2 x_1, \alpha^3 x_1, \ldots\); another daughter family, which is also new, corresponds to the sequence, \(\alpha x_3, \alpha^2 x_3, \ldots\), etc. and it starts off with \(M = 1\), by virtue of equation (12). However the family \(a^n x_4\) is not new because one may readily establish that \(x_4 = \alpha x_1\) by inserting \(x = x_1\) in equation (12). New secondary families \(\alpha^n x_5, \alpha^n x_6, \ldots\), etc. begin with \(n = 1\), again via equation (12).

While the parent and daughter sequences lead to two generations of inflection points, some of which are listed in Table 1, this does not exhaust the class of such points. Proceeding in a similar fashion to above, define the granddaughter sequence,

\[
x_{MN} \equiv g^{-1}(\alpha^M x_N) = g^{-1}(\alpha^M g^{-1}(\alpha^N)) \tag{13}
\]

Many of these points are not new; by manipulating (8a) and (8b) appropriately, one may prove that some daughters are identical with or related to their aunts. For instance,

\[
\alpha x_{21} = x_3, \quad \alpha x_{31} = x_6, \quad \alpha x_{25} = x_7, \quad x_{15} = x_6, \quad x_{43} = \alpha x_6, \ldots
\]

As well, some of these granddaughters are interrelated; for instance

\[
\alpha x_{13} = x_{41}, \quad \alpha x_{16} = x_{51}, \ldots
\]
Only the independent $\alpha^n x_{MN}$ occurring between -8 and +8 are stated in Table 1.

Practically all the remaining missing points of inflection in that range arise from the next generation, $x_{LMN} \equiv g^{-1}(\alpha^L x_{MN})$. Again we have only listed those that spawn new series. The final ‘missing’ point within -8 to 8, at $x \simeq -7.45$, is noteworthy because it belongs to the ‘fourth’ generation. By following through this procedure one may track all the inflection points over any desired range, with each family being generated from the previous one, though many are identified with previous members; for example the point $\alpha^2 x_{241}$ is the same as its aunt $\alpha x_{61}$. Within the restricted range -8 to 8, there are no other inflection points beyond the ones noted in the table. An interesting observation is that the number of inflection points between $\alpha^n$ and $\alpha^{n+2}$ equals the Fibonacci number $F_n$; while we can readily show that this holds for subcritical maps where $\alpha$ is the inverse of the golden mean ratio, an exact proof eludes us for critical cubic maps.

All told we can see how the functional equations reproduce their structure in a self-similar way, becoming ever more intricate and extended at one and the same time: for instance, the region between -7.7 and -7.4 looks very much like the region between -2 and 2. This feature happens everywhere and is not unlike the morphology of the period-doubling universal function. We expect that the cubic nature of inflection is not vital for the validity of this conclusion and that it is common to all scaling functional equations of the type $[g]^N(x) \propto g(\rho x)$. The exquisite self-similarity on every scale as we go out in $x$ means that it is effectively impossible to make an analytical approximation to $g$ over a wide range of $x$; but of course over a small, limited range it is always possible to do so and indeed this is a sensible way to compute the scaling constant $\alpha$, by focussing on the region $x = 0$ to 1 say.

References

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Table 1: Positions of successive inflection points of \( g(x) \) to two decimal points, between \( x = -8 \) & +8. ID is the interpretation of their origin in the recursive notation \( x_{MN...} = g^{-1}(\alpha^Mx_{N...}) \).

| \( x \) | \( g \) | ID | \( x \) | \( g \) | ID | \( x \) | \( g \) | ID |
|---|---|---|---|---|---|---|---|---|
| -7.99 | 10.60 | \( \alpha^4x_1 \) | -4.80 | 6.43 | \( \alpha^5x_1 \) | 3.73 | -4.58 | \( \alpha^6x_1 \) |
| -7.76 | 10.10 | \( \alpha^2x_{61} \) | -4.52 | 5.92 | \( \alpha^3x_{13} \) | 4.12 | -5.12 | \( \alpha^3x_6 \) |
| -7.65 | 9.91 | \( \alpha^2x_{61} \) | -4.39 | 5.71 | \( \alpha x_{33} \) | 4.58 | -5.46 | \( \alpha^6 \) |
| -7.56 | 9.83 | \( \alpha^5x_{13} \) | -4.27 | 5.62 | \( \alpha^3x_3 \) | 5.12 | -6.42 | \( \alpha^2x_5 \) |
| -7.45 | 9.63 | \( \alpha x_{3313} \) | -3.98 | 5.26 | \( \alpha x_5 \) | 5.27 | -6.77 | \( \alpha x_{1413} \) |
| -7.40 | 9.56 | \( \alpha^3x_{313} \) | -3.55 | 4.99 | \( \alpha^5 \) | 5.48 | -6.91 | \( \alpha^4x_3 \) |
| -7.35 | 9.52 | \( \alpha^3x_{33} \) | -2.90 | 3.97 | \( \alpha^3x_1 \) | 6.66 | -7.22 | \( \alpha^2x_{331} \) |
| -7.26 | 9.39 | \( \alpha^2x_{233} \) | -2.58 | 3.42 | \( \alpha x_3 \) | 5.74 | -7.37 | \( \alpha^2x_{1313} \) |
| -7.09 | 9.30 | \( \alpha^5x_3 \) | -2.14 | 3.20 | \( \alpha^3 \) | 5.83 | -7.42 | \( \alpha^4x_{131} \) |
| -6.87 | 8.95 | \( \alpha^2x_{43} \) | -1.29 | 2.08 | \( \alpha \) | 5.99 | -7.64 | \( \alpha x_{61} \) |
| -6.78 | 8.78 | \( \alpha^2x_{143} \) | 0 | 1.25 | 1 | 6.19 | -7.78 | \( \alpha^6x_1 \) |
| -6.62 | 8.69 | \( \alpha^3x_5 \) | 1.66 | -1.62 | \( \alpha^2 \) | 6.52 | -8.29 | \( \alpha^3x_{16} \) |
| -6.34 | 8.32 | \( \alpha x_7 \) | 2.25 | -2.67 | \( \alpha^2x_1 \) | 6.67 | -8.53 | \( \alpha x_{36} \) |
| -5.90 | 8.06 | \( \alpha^7 \) | 2.76 | -3.08 | \( \alpha^4 \) | 6.85 | -8.65 | \( \alpha^2x_9 \) |
| -5.31 | 7.06 | \( \alpha^2x_6 \) | 3.30 | -4.07 | \( \alpha^2x_3 \) | 7.20 | -9.09 | \( \alpha x_8 \) |
| -5.07 | 6.63 | \( \alpha^2x_{16} \) | 3.52 | -4.41 | \( \alpha^2x_{13} \) | 7.60 | -9.40 | \( \alpha^5 \) |

Table 2: The primary family of inflection points between -8 & +8 and their corresponding inverses (to two decimal points).

| \( N \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| \( \alpha^N \) | -1.29 | 1.66 | -2.14 | 2.76 | -3.55 | 4.58 | -5.90 | 7.60 |
| \( x_N = g^{-1}(\alpha^N) \) | 1.35 | -0.78 | 1.99 | -1.74 | 3.09 | -3.20 | 4.92 | -5.58 |

Figure Captions
Figure 1. $g(x)$ from $x = -8$ to $x = -4$. This range contains 20 inflection points.
Figure 2. $g(x)$ from $x = -4$ to $x = +4$. This range contains 13 inflection points (only 8 points between -3 to 3).
Figure 3. $g(x)$ from $x = +4$ to $x = +8$. This range contains 15 inflection points.
Figure 2: R Delbourgo
