ALGEBRAIC PROPERTIES OF THE GROUP
OF GERMS OF DIFEOMORPHISMS

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ABSTRACT. We establish some algebraic properties of the group Diff(C^n,0) of germs of analytic diffeomorphisms of C^n, and its formal completion \( \hat{\text{Diff}}(C^n,0) \). For instance we describe the commutator of Diff(C^n,0), but also prove that any finitely generated subgroup of Diff(C^n,0) is residually finite; we thus obtain some constraints of groups that embed into Diff(C^n,0). We show that Diff(C^n,0) is an Hopfian group, and that \( \hat{\text{Diff}}(C^n,0) \) and Diff(C^n,0) are not co-Hopfian. We end by the description of the automorphism groups of \( \hat{\text{Diff}}(C,0) \), and Diff(C,0).

INTRODUCTION

Let \( M^n \) be a complex manifold of dimension \( n \), and let \( \mathcal{F} \) be a codimension \( p \) holomorphic foliation \( M^n \) with singular locus \( \text{Sing}(\mathcal{F}) \). Suppose that \( N \subset M \) is a submanifold of dimension \( n - p \) invariant by \( \mathcal{F} \). Then there exists a natural morphism

\[
\text{Hol} : \pi_1(N \setminus \text{Sing}(\mathcal{F}),\ast) \to \text{Diff}(C^p,0)
\]

the so-called holonomy representation. Suppose that \( \ker \text{Hol} \) is non trivial, and let \( \gamma \) an element of \( \pi_1(N \setminus \text{Sing}(\mathcal{F}),\ast) \setminus \{\text{id}\} \) such that \( \text{Hol}(\gamma) = \text{id} \). Then, \( \gamma \) can be lifted in the leaves near \( N \) to some loops which are homotopically non trivial in these leaves. In that situation the holonomy representation gives topological and dynamical informations on the foliation. Another interesting fact is the following. Suppose that \( p = 1 \), and that the image of \( \text{Hol} \) is an abelian linearisable group; then \( \mathcal{F} \) is defined in a neighborhood of \( N \setminus \text{Sing}(\mathcal{F}) \) by a closed meromorphic 1-form (\cite{CM82, Per22}).

As a consequence, the study of such representations is an important problem, and requires the knowledge of algebraic properties of the groups Diff(C^p,0). In \cite{Die55} Dieudonné describes the geometric and algebraic properties of the classical linear groups GL(n,k). This text is part of a similar perspective by highlighting some properties that we considered important.

Notations. If \( n \) is an integer, then \( G^n_0 \) denotes Diff(C^n,0), and \( \hat{G}^n_0 \) denotes its formal completion. Furthermore, \( G^n_k \) is the set of elements of \( G^n_0 \) tangent to the identity at order \( k \), and \( \hat{G}^n_k \) is its formal completion.

Structure of the paper. In \S2\ we establish some consequences of Poincaré linearisation theorem:

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Theorem A. For any $k \geq 1$ we have

$$[G^n_0, G^n_k] = G^n_k,$$

and

$$[\hat{G}^n_0, \hat{G}^n_1] = \hat{G}^n_1.$$

As a consequence, we obtain:

$$[G^n_0, G^n_0] = \{ f \in G^n_0 \mid \text{det} Df(0) = 1 \}.$$

Finding the finitely generated groups that embed into $G^n_0$ is a problem related to the foliations theory; in §4 we deal with this question and we get:

Theorem B. Any finitely generated subgroup of $G^n_0$ (resp. $\hat{G}^n_0$) is residually finite.

Hence, if $H$ is a finitely generated and non residually finite group, then $H$ does not embed into $G^n_0$.

But any finitely generated residually finite group is a Hopfian group ([Mal40, Mal65]); as a result, any finitely generated subgroup of $G^n_0$ (resp. $\hat{G}^n_0$) is a Hopfian group. In §5 we refine this result, and also look at the co-Hopfian property:

Theorem C. The group $\hat{G}^1_0$ is a Hopfian group.

The groups $G^1_0$ and $\hat{G}^1_0$ are not co-Hopfian groups.

Inspired by [D06] we study in §7 the automorphism groups of $G^1_0$ and $\hat{G}^1_0$:

Theorem D. The group $\text{Aut}(\hat{G}^1_0)$ is generated by the inner automorphisms and the automorphisms of the field $\mathbb{C}$. In other words

$$\text{Out}(\hat{G}^1_0) \simeq \text{Aut}(\mathbb{C}, +, \cdot)$$

where $\text{Out}(\hat{G}^1_0)$ denotes the non-inner automorphisms of $\hat{G}^1_0$.

The group $\text{Out}(G^1_0)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

1. Notations and Definitions

Let $n$ be a positive integer; consider

$$G^n_0 = \text{Diff}(\mathbb{C}^n, 0) = \{ f : \mathbb{C}^n_0 \rightarrow \mathbb{C}^n_0 \text{ holomorphic} \mid Df(0) \in \text{GL}(\mathbb{C}^n) \}.$$

Denote by $\hat{G}^n_0$ the formal completion of $G^n_0$; in other words, $\hat{G}^n_0$ is the set of formal diffeomorphisms $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$ where

- $\hat{f}_i$ is a formal series,
- $\hat{f}(0) = 0$,
- and $D\hat{f}(0)$ belongs to $\text{GL}(\mathbb{C}^n)$.

More generally, let $k$ be a positive integer; consider $G^n_k$ the set of elements of $G^n_0$ tangent to the identity at order $k$ (i.e. $f$ belongs to $G^n_k$ if and only if $f = \text{id} + \text{terms of order} \geq k + 1$), and $\hat{G}^n_k$ be the formal completion of $G^n_k$. 
Each $G^n_k$'s (resp. $\hat{G}^n_k$'s) is a normal subgroup of $G^n_0$ (resp. $\hat{G}^n_0$). The quotients $G^n_0/G^n_k$ and $\hat{G}^n_0/\hat{G}^n_k$ are isomorphic, and can be identified with the group of polynomial maps

$$\text{Pol}^n_k = \{ f: \mathbb{C}^n \to \mathbb{C}^n \text{ polynomial} \ | \ f(0) = 0, \ \text{deg} \ f \leq k, \ Df(0) \in \text{GL}(\mathbb{C}^n) \}$$

whose the law group is the law of composition truncated to order $k+1$.

Remark that $G^n_0/G^n_k$ acts faithfully on the space $\mathbb{C}[x_1, x_2, \ldots, x_n]_k$ of polynomials of degree less or equal than $k$ (still by truncated composition). Therefore, the $G^n_0/G^n_k$'s can be identified with subgroups of $\text{GL}(\mathbb{C}[x_1, x_2, \ldots, x_n]_k)$.

For elements $g$ and $h$ of a group $H$, the commutator of $g$ and $h$ is $[g, h] = ghg^{-1}h^{-1}$. The derived subgroup $[H, H]$ (also called the commutator subgroup) of $H$ is the subgroup generated by all the commutators. This construction can be iterated:

$$\begin{cases} H^{(0)} := H \\ H^{(n)} := [H^{(n-1)}, H^{(n-1)}] & n \in \mathbb{N} \end{cases}$$

The groups $H^{(2)}$, $H^{(3)}$, ... are called the second derived subgroup, third derived subgroup, and so forth, and the descending normal series

$$\cdots \triangleleft H^{(2)} \triangleleft H^{(1)} \triangleleft H^{(0)} = H$$

is called the derived series. A solvable group is a group whose derived series terminates in the trivial subgroup.

The groups $G^1_0$ and $\hat{G}^1_0$ contain free subgroups ([BCLN96]). It’s a bit surprising since $\hat{G}^1_0$ appears as the limit of solvable groups $G^1_0/G^1_k$. To construct free subgroups the authors use the following deep result ([Coh95]): consider the two following homeomorphisms of $\mathbb{R}$

$$f_1(x) = x + 1, \quad f_2(x) = x^3,$$

then the group generated by $f_1$ and $f_2$ is a free group.

Note that $f_2$ is invariant by the involution $\frac{1}{x}$; hence the group generated by $\tilde{f}_1(x) = \frac{x}{1+x}$ and $f_2(x) = x^3$ is free. As a consequence, the group generated by the germs of analytic diffeomorphisms at the origin of $\mathbb{R}$

$$\tilde{f}_1(x) = \frac{x}{1+x}, \quad \text{and} \quad \tilde{f}_2(x) = f_2^{-1}\tilde{f}_1f_2(x) = \frac{x}{(1+x^2)^{1/3}}$$

is a free subgroup; it induces a free subgroup of $G^1_0$ (and $\hat{G}^1_0$).

Denote by $p_k: G^n_0 \to G^n_0/G^n_k$ the projection.

Let $\mathfrak{M}_n$ be the maximal ideal of $O(\mathbb{C}^n, 0)$ given by

$$\mathfrak{M}_n = \{ f \in O(\mathbb{C}^n, 0) \ | \ f(0) = 0 \}.$$
Denote by $\chi^0_n$ the set of germs of holomorphic vector fields at the origin of $\mathbb{C}^n$, and by $\hat{\chi}^0_n$ its formal completion. Let $\chi^n_k$ be the subspace of $\chi^0_n$ made up of vector fields that vanish at 0 at order $k-1$, that is $\chi^n_k = \mathcal{M}^n_k \chi^0_n$. Finally, let us denote by $\hat{\chi}^n_k$ the formal completion of $\chi^n_k$.

We recall some classical facts; proofs can be found for instance in [CCD13]. Let $X$ be an element of $\hat{\chi}^n_2$; denote by $\exp_t X \in \hat{G}^1_n$ the one-parameter subgroup associated to $X$. By definition $\exp_t X$ is solution of the O.D.E.

$$\frac{\partial}{\partial t} \varphi_t(x) = X(\varphi_t(x))$$

with initial condition $\varphi_0(x) = x$. It is easy too see that $\exp_t X$ is "polynomial in the parameter $t"$, that is:

$$\exp_t X = \text{Id} + \sum_{i=1}^{\infty} t^i A_i$$

where the $A_i$'s belong to $(\mathcal{M}^{n+1}_n)^n$. In particular, $\exp X$ is well defined in $\hat{G}^n_1$.

**Proposition 1.1.** The following properties hold.

- The map $\exp : \hat{\chi}^n_2 \to \hat{G}^1_n$ is bijective.
- Let $X, Y$ be two elements of $\hat{\chi}^n_2$. Then, $f = \exp X$ and $g = \exp Y$ commute if and only if $X$ and $Y$ commute.

**Remark 1.2.** The first property is no longer true in the holomorphic case: if $h$ belongs to $G^1_1$ and $h = \exp X$, then $X$ is most of the time divergent ([É75]).

In dimension 1, there are normal forms for elements of $\chi^1_2$ and $\hat{\chi}^1_2$ as follows:

**Proposition 1.3.** Let $X$ be an element of $\chi^1_2$ (resp. $\hat{\chi}^1_2$). Then, $X$ is holomorphically (resp. formally) conjugated to a vector field of the type $x^{k+1} \frac{\partial}{\partial x}$, $k \in \mathbb{N}$, $k \geq 1$, $\lambda \in \mathbb{C}$.

The proof is a direct application of the implicit function theorem ([CCD13]). For instance the one-parameter subgroup of $x^2 \frac{\partial}{\partial x}$ is $\exp t x^2 \frac{\partial}{\partial x} = \frac{x}{1-tx}$.

## 2. Poincaré Linearisation Theorems and Their Consequences

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a $n$-tuple of non-zero complex numbers. We say that $\lambda$ is **without resonance** if the equality $\lambda_1^{p_1} \lambda_2^{p_2} \ldots \lambda_n^{p_n} = \lambda_j$, $p_j \in \mathbb{N}$, implies $p_j = 1$, and $p_\ell = 0$ for $\ell \neq j$. If $A \in \text{GL}(n, \mathbb{C})$, we denote by $\text{Spec}(A)$ the non-ordered set of its eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

**Theorem 2.1** (Formal Poincaré Theorem, [Arn88]). Let $f$ be an element of $\hat{G}^n_0$. Set $A = Df(0)$.

Assume that $\text{Spec}(A)$ is without resonance.

Then, $f$ is formally conjugate to $A$, i.e. there exists $\varphi \in \hat{G}^n_1$ such that $f = \varphi A \varphi^{-1}$. 


**Theorem 2.2** (Holomorphic Poincaré Theorem, [Arn88]). Let \( \phi \) be an element of \( G_0^n \). Set \( A = D\phi(0) \).

Assume that \( \text{Spec}(A) \) is without resonance, and that \( \text{Spec}(A) \) is contained either in the open unit disk \( \mathbb{D}(0,1) \), or in its complement \( \overline{\mathbb{C}} \setminus \mathbb{D}(0,1) \).

Then, \( \phi \) is holomorphically conjugate to \( A \), i.e. there exists \( \varphi \in G_1^n \) such that \( \phi = \varphi A \varphi^{-1} \).

We say that \( \phi \) is **formally linearisable** (resp. **holomorphically linearisable**) in the formal (resp. holomorphic) case. In both cases the diffeomorphism \( \varphi \) is called the **linearising map**.

**Remark 2.3.** There are generalisations of the previous result; the first are due to Siegel: when \( \text{Spec}(A) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is contained neither in \( \mathbb{D}(0,1) \), nor in \( \mathbb{C} \setminus \mathbb{D}(0,1) \); it requires diophantine conditions controlling \( |\lambda_1^n \lambda_2^p \ldots \lambda_n^{p^n} - \lambda_j| \) that produce the convergence of the linearising maps ([Sie42, Sie52]).

**Remark 2.4.** In the Poincaré Theorems assume that \( \phi \) can be written as \( Ah \) where \( h = \text{id} + \ldots \in G_k^n \); then the linearising map \( \varphi \) can be chosen in \( G_k^n \) (resp. \( \hat{G}_k^n \) in the formal case).

Curiously while the proof of Poincaré Theorem comes from analysis (in its holomorphic version) we deduce it from algebraic properties.

**Theorem 2.5.** We have \( [G_0^n, G_1^n] = G_1^n \), and similarly \( [\hat{G}_0^n, \hat{G}_1^n] = \hat{G}_1^n \).

**Remark 2.6.** Theorem 2.5 has been proved in [CCD13] for \( n = 1 \).

**Proof.** Let us first remark that if \( f, g \) are elements of \( G_0^n \) and \( G_1^n \) respectively, then the commutator \([f, g] = fgf^{-1}g^{-1}\) is an element of \( G_1^n \).

Let \( A \) be an element of \( \text{GL}(n, \mathbb{C}) \) that satisfies the assumptions of Poincaré Theorem, for instance \( A = \lambda \text{id} \) with \( 0 < |\lambda| < 1 \). If \( h \) belongs to \( G_1^n \), then \( hA \) is linearisable, that is there exists \( \varphi \in G_1^n \) such that \( \varphi A \varphi^{-1} = hA \).

As a result, \([\varphi, A] = h\), and we get the result.

A similar proof works in the formal case. \( \square \)

**Corollary 2.7.** The commutator of \( G_0^n \) is given by:

\[
[G_0^n, G_0^n] = \{ f \in G_0^n | \det Df(0) = 1 \}.
\]

**Proof.** The inclusion

\[
\{ f \in G_0^n | \det Df(0) = 1 \} \subset [G_0^n, G_0^n].
\]

holds. So, we just need to prove that any element \( f \) such that \( \det Df(0) = 1 \) is a product of commutators of \( G_0^n \). Write \( f = Ah \) with \( A = Df(0) \in \text{SL}(n, \mathbb{C}) \) and \( h \in G_1^n \). On the one hand, \( A \in \text{SL}(n, \mathbb{C}) \) is a product of commutators of \( \text{GL}(n, \mathbb{C}) \subset G_0^n \); and, one the other hand, \( h \) is a product of commutators of \( G_0^n \) (Theorem 2.5). Consequently, \( f \) is a product of commutators of \( G_0^n \). \( \square \)

Theorem 2.5 can be generalised as follows:
Theorem 2.8. Any element of $G_k^n$ is the commutator of an element of $G_0^n$ and an element of $G_k^n$, i.e. 
$[G_0^n, G_k^n] = G_k^n$.

We deduce from it the following statement that can be useful:

Proposition 2.9. Let $H$ be a group, and let $\tau: G_0^n \to H$ be a group homomorphism.
Assume that there exists $f$ in $\ker \tau$ such that $Df(0) = A$ satisfies the assumptions of Poincaré Theorem. Then $G_1^n$ is contained in $\ker \tau$.

Proof. Let $h$ be an element of $G_1^n$. Since $A$ satisfies the assumptions of Poincaré Theorem, then $hf$ and $f$ are conjugate: there exists $\varphi$ in $G_1^n$ such that $hf = \varphi f \varphi^{-1}$. As a consequence, $h = [\varphi, f]$ and $h$ belongs to $\ker \tau$. \hfill \Box

Remark 2.10. Since $A$ and $f \in \ker \tau$ are conjugate, $A$ belongs to $\ker \tau$, and so does the normal subgroup of $GL(n, \mathbb{C})$ generated by $A$. The quotient $G_0^n/\ker \tau \simeq \text{im} \tau$ can thus be identified with the quotient $GL(C^n)/GL(C^n) \cap \ker \tau$.

Poincaré theorem gives the description of germs of diffeomorphisms with generic linear part. When the linear part is not generic, for instance in dimension 1, the formal classification is relatively easy whereas the holomorphic one is rather difficult. There are many contributions ([É75, PM95, Yoc95, Mal82, AI88, Arn88]).

3. Finite groups

Finite subgroups of $G_0^n$ are described by the following classical result:

Theorem 3.1. Let $H$ be a finite subgroup of $G_0^n$. Denote by 
$H_0 = \{Df(0) | f \in H\} \subset GL(n, \mathbb{C})$
the linear part of $H$. Then, $H$ is holomorphically conjugate to $H_0$, i.e. there exists $\varphi \in G_0^n$ such that $H = \varphi^{-1}H_0\varphi$.

Proof. The proof relies on a classical average argument (like in the Cartan-Bochner’s theorem about the local linearisation of the action of a compact Lie group near a fixed point). Consider the map $\varphi = \sum_{h \in H} (Dh(0))^{-1} \circ h$; it is holomorphic, and $D\varphi(0) = (\#H)\text{id}$, so $\varphi$ belongs to $G_0^n$. Finally, let us remark that for any $f \in H$ one has 
$\varphi \circ f = \sum_{h \in H} (Dh(0))^{-1} \circ h \circ f = Df(0) \sum_{h \circ f \in H} (D(h \circ f)(0))^{-1} \circ (h \circ f) = Df(0) \circ \varphi$. \hfill \Box
4. RESIDUALLY FINITE GROUPS

There are a number of equivalent definitions of residually finite groups; we will use the following one:

**Definition 4.1.** A group $H$ is residually finite if for every element $h$ in $H \setminus \{\text{id}_H\}$ there exists a group morphism $\varphi : H \to F$ from $H$ to a finite group $F$ such that $\varphi(h) \neq \text{id}_F$.

The groups $\mathbb{Z}$ and $\text{SL}(n, \mathbb{Z})$ are residually finite (by reduction modulo $p$). Subgroups of residually finite groups are residually finite. Conversely, a non-finite simple group is not residually finite, and the Baumslag-Solitar group $BS(2,3) = \langle a, b \mid a^{-1}b^2a = b^3 \rangle$ is not residually finite (\cite{BS62}).

A group is linear if it is isomorphic to a subgroup of $\text{GL}(n, \mathbb{k})$ where $\mathbb{k}$ is a field.

Malcev established the following fundamental result:

**Theorem 4.2** (\cite{Mal40, Mal65}). A finitely generated linear group is residually finite.

In particular, there is no faithful linear representation of a non-residually finite, finitely generated group.

**Remark 4.3.** The assumption "finitely generated" turns out to be essential. For instance, $\mathbb{Q}$ is not residually finite. If $\xi$ is a positive transcendental number, then $\xi^\mathbb{Q} = \left\{ \xi^q \mid q \in \mathbb{Q} \right\}$ is isomorphic to $\mathbb{Q}$. As a subgroup of a residually finite group is residually finite, we thus get that $\text{GL}(1, \mathbb{C}) \simeq \mathbb{C}^*$, $\text{GL}(n, \mathbb{C})$, $G_0^n$ and $\hat{G}_0^n$ are not residually finite.

The following statement, that is mentioned in \cite{CLPT19} without detail in the 1-dimensional and formal case, is a direct consequence of Theorem 4.2:

**Theorem 4.4.** Any finitely generated subgroup of $G_0^n$ (resp. $\hat{G}_0^n$) is residually finite.

**Proof.** Let $H$ be a finitely generated subgroup of $G_0^n$, and $h$ be an element of $G_0^n \setminus \{\text{id}\}$. There exists an integer $k$ such that $p_k(h)$ is non-trivial in the quotient group $G_0^n / G_k^n$. Recall that $G_0^n / G_k^n$ is isomorphic to a subgroup of a linear group; Theorem 4.2 applied to the group $p_k(H)$ asserts the existence of a morphism $\varphi_k : p_k(H) \to F_k$ from $p_k(H)$ to a finite group $F_k$ such that $\varphi_k(p_k(h)) \neq \text{id}_{F_k}$. Then, the morphism $\varphi_k \circ p_k|_H : H \to F_k$ suits. \hfill $\square$

Finding the finitely generated subgroups that embed into $G_0^n$ is an important problem, in particular related to the theory of foliations (representations of holonomy, and cycles in leaves). For instance, in \cite{CCGS20} one can find:

**Theorem 4.5** (\cite{CCGS20}). The fundamental group of a compact surface $\Sigma_g$ of genus $g$ embeds into $G_1^n$, and so into $G_0^n$.

In particular, there are surfaces $S$ with a foliation $\mathcal{F}$ by curves having an invariant curve $\Sigma_g$ withfaithfull holonomy representation.
Conversely we get the following statement:

**Corollary 4.6.** The Baumslag-Solitar group $BS(2,3)$ does not embed into $G_0^n$.

More generally, if $H$ is a finitely generated and non residually finite group, then $H$ does not embed into $G_0^n$.

Theorem 4.4 has direct applications in the theory of holomorphic foliations. Toledo constructs smooth complex projective varieties with fundamental groups which are not residually finite, answering to some Serre’s question ([To93]). Assume that $\mathcal{F}$ is, for instance, a codimension one holomorphic foliation on the complex manifold $M$ having an invariant variety $N \subset M$ satisfying Toledo’s property, that is $\pi_1(N,*)$ not residually finite. Then the holonomy representation ([CD13])

$\text{Hol} : \pi_1(N,*) \to \text{Diff}(\mathbb{C},0)$

is not faithfull. As we have seen previously, there exist families of cycles in the leaves of $\mathcal{F}$ near the invariant manifold $N$.

In [DS05] Drutu and Sapir construct residually finite groups that are not linear. One of their examples is the group $\hat{G} = \langle a, b | b^2ab^{-2} = a^2 \rangle$; we prove that $\hat{G}$ can not be embeded into $G_0^1$ (or $\hat{G}_0^1$):

**Proposition 4.7.** There is no faithfull representation of $\hat{G} = \langle a, b | b^2ab^{-2} = a^2 \rangle$ into $\hat{G}_0^1$ (resp. $G_0^1$).

**Proof.** Assume by contradiction that there exists a faithfull representation $\varphi$ of $\hat{G}$ into $\hat{G}_0^1$. Set $A = \varphi(a)$, and $B = \varphi(b)$. From $B^2AB^{-2} = A^2$ we get that $A$ and $A^2$ are conjugate (by $B^2$). In particular, $A$ is tangent to the identity, that is $A$ belongs to $\hat{G}_0^1$; so there exists a formal vector field $X$ of order at least 2 such that $A = \exp X$ (Proposition 1.1). Since $\exp X$ and $\exp 2X$ are conjugate, the vector fields $X$ and $2X$ are conjugate (by $B^2$). One can assume, up to conjugacy, that

$$X = \frac{x^{\nu+1}}{1 - \lambda x} \frac{\partial}{\partial x}$$

with $\nu \geq 1$ and $\lambda \in \mathbb{C}$. Let $\mu x$ be the linear part of $B^2$; note that $\mu x$ has to conjugate the first non-zero jet $x^{\nu+1} \frac{\partial}{\partial x}$ of $X$ to the first non-zero jet $2x^{\nu+1} \frac{\partial}{\partial x}$ of $2X$. Hence $\mu^x = 2$, and the linear part of $B$ is a 2\nu-th root of 2; we thus can linearise $B$, i.e. assume that $B = \alpha x$ where $\alpha = 2^{\frac{1}{2\nu}}$. Let $Y = h(x) \frac{\partial}{\partial x}$ be a vector field of order $\nu + 1$ such that $B^2Y = 2Y$, that is such that $\alpha^{-2}h(\alpha^2x) = 2h$; in other words we have the equality

$$h(\alpha^2x) = 2\alpha^2h.$$  \ (4.1)

Write $h$ as $h = \sum_{\ell \geq \nu + 1} h_{\ell} x^\ell$; then \ (4.1) yields to $\alpha^{2\ell}h_{\ell} = 2\alpha^{2h_{\ell}}$, i.e. $\alpha^{2(\ell-1)}h_{\ell} = 2h_{\ell}$. For any $h_{\ell} \neq 0$ we get $\alpha^{2(\ell-1)} = 2$; in other words $2 \frac{\nu^{\ell} - \nu^{\ell-1}}{\nu} = 1$, and so $2 \frac{\nu^{\ell-1} - 1}{\nu} = 1$; as a consequence, $\ell = \nu + 1$. As a result, in the linearising coordinate for $B$, we have: $B = \alpha x$ and $A = \exp cx^{\nu+1} \frac{\partial}{\partial x}$ for some $c$. In particular the group generated by $A$ and $B$ is linear whereas $\hat{G}$ is not ([DS05]): contradiction. \ \ \ \Box

In [CL98] the authors prove the following curious result. Let $\gamma$ be an irreducible curve in $\mathbb{P}_C^2$ of degree $p^s$ with $p$ prime number. If $\varphi: \pi_1(\mathbb{P}_C^2 \setminus \gamma,*) \to G_0^1$ or $\varphi: \pi_1(\mathbb{P}_C^2 \setminus \gamma,*) \to \hat{G}_0^1$ is a morphism, then
the image of \( \varphi \) is a finite group (conjugate to a group of linear rotations, see Theorem [3.1]); moreover, there are some \( \gamma \) such that \( \pi_1(\mathbb{P}^2_\mathbb{C} \setminus \gamma, \ast) \) contains a free group of rank 2. That result is used by the authors to construct holomorphic first integral for codimension one holomorphic foliations in \((\mathbb{C}^n, 0)\), \(n \geq 3\) in special situations generalising "Malgrange-Mattei-Moussu Frobenius theorems with singularities" ([MM80, Mal76]).

**Problem 4.8.** Let \( \gamma \) be a curve in \( \mathbb{P}^2_\mathbb{C} \); is the group \( \pi_1(\mathbb{P}^2_\mathbb{C} \setminus \gamma, \ast) \) a linear group ? is the group \( \pi_1(\mathbb{P}^2_\mathbb{C} \setminus \gamma, \ast) \) a residually finite group ?

## 5. Hopfian and Co-Hopfian Groups

**Definition 5.1.** A group \( G \) is **Hopfian** if every surjective morphism group from \( G \) to \( G \) is an isomorphism. Equivalently, a group is Hopfian if and only if it is not isomorphic to any of its proper quotients.

**Definition 5.2.** A group \( G \) is **co-Hopfian** if every injective morphism group from \( G \) to \( G \) is an isomorphism. Equivalently, a group is co-Hopfian if and only if it is not isomorphic to any of its proper subgroups.

Every finite group is a Hopfian group. Every simple group is a Hopfian group. The group \( \mathbb{Z} \) of integers and the group \( \mathbb{Q} \) of rationals are Hopfian groups. However, \( \mathbb{C}^* \) is not a Hopfian group (the morphisms \( \mathbb{C}^* \to \mathbb{C}^*, x \mapsto x^p \) are not injective), and \( \mathbb{R}^* \) is not a Hopfian group (the morphisms \( \mathbb{R}^* \to \mathbb{R}^*, x \mapsto x^p, p \) even, are not injective). In [D07] the author shows that the group \( \text{Bir}(\mathbb{P}^2_\mathbb{C}) \) of birational self-maps of the complex projective plane \( \mathbb{P}^2_\mathbb{C} \) is Hopfian.

Let us mention an other statement due to Malcev:

**Theorem 5.3 ([Mal40, Mal65]).** Any finitely generated residually finite group is a Hopfian group.

**Corollary 5.4.** Any finitely generated subgroup of \( \hat{G}_n^0 \) (resp. \( \hat{\hat{G}}_n^0 \)) is a Hopfian group.

Let us now establish the following statement: in which the assumption "finitely generated" has been removed ?

**Theorem 5.5.** The group \( \hat{\hat{G}}_1^0 \) is a Hopfian group.

To prove it we will use the following result of finite determination, statement specific to the 1-dimensional and formal case:

**Lemma 5.6.** Let \( h \) be an element of \( \hat{\hat{G}}_1^1 \). There exists an integer \( \ell \) such that if \( g \) belongs to \( \hat{\hat{G}}_1^1 \), then \( h \) and \( hg \) are conjugate in the group \( \hat{\hat{G}}_1^1 \).

In other words, if two elements of \( \hat{\hat{G}}_1^1 \) coincide up to a sufficiently large order, then they are conjugate.

Lemma 5.6 is a direct consequence of Proposition 1.1 and Proposition 1.3.
Proof of Theorem 5.5. Let $\varphi: \hat{G}^1_0 \rightarrow \hat{G}^1_0$ be a surjective morphism. Assume that $\varphi$ is not injective. Let $f \in \hat{G}^1_0 \setminus \{\text{id}\}$ such that $\varphi(f) = \text{id}$. Replacing $f$ by a non-trivial commutator $[a,f]$ (that also belongs to $\ker \varphi$) if needed we can assume that $f$ belongs to $\hat{G}^1_0 \setminus \{\text{id}\}$. Consider $h$ in $\hat{G}^\ell_1$ for $\ell$ sufficiently large. According to Lemma 5.6 the elements $f$ and $hf$ are conjugate, i.e. there exists $g \in \hat{G}^1_0$ such that $gfg^{-1} = hf$. As a consequence, $h = [g,f]$ belongs to $\ker \varphi$. Hence, $\ker \varphi$ contains $\hat{G}^1_0$ for $\ell$ sufficiently large. Since $\varphi$ is surjective, $\hat{G}^1_0$ and $\hat{G}^1_0 / \ker \varphi$ are isomorphic. As $\hat{G}^\ell_1$ is contained in $\ker \varphi$ the morphism $\hat{G}^1_0 / \hat{G}^\ell_1 \rightarrow \hat{G}^1_0 / \ker \varphi$ is surjective. The group $\hat{G}^1_0 / \hat{G}^\ell_1$ is solvable, so does $\hat{G}^1_0 / \ker \varphi \simeq \hat{G}^1_0$: contradiction with the fact that $\hat{G}^1_0$ contains free subgroups (BCLN96). The surjective morphism $\varphi$ is thus injective, and so an isomorphism.

Problems 5.7.  

1) Is the group $G^1_0$ a Hopfian group? One way to answer to this question is to show that if $\tau: G^1_0 \rightarrow G^1_0$ is surjective, then $\tau$ can be extended to a morphism $\bar{\tau}: \hat{G}^1_0 \rightarrow \hat{G}^1_0$ still surjective.

2) Are the groups $G^n_0$ and $\hat{G}^n_0$ Hopfian groups?

Unfortunately, the method used for the proof of Theorem 5.5 turns out to be ineffective for Problems 5.7.

Let us now deal with the notion of co-Hopfian group.

Using transcendence basis it is easy to construct an injective and non-surjective morphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$ of the field $\mathbb{C}$; then $\tau$ induces an injective and non-surjective homomorphism from $\hat{G}^n_0$ into itself defined by

$$\sum A_1 x^l \mapsto \sum \tau(A_1) x^l$$

where $A_1$ belongs to $\mathbb{C}^n$. In particular, $\hat{G}^n_0$ is not co-Hopfian.

Theorem 5.8. The groups $G^n_1$ and $\hat{G}^n_1$ are not co-Hopfian groups.

Proof. Let us first assume that $n = 1$. The morphism $\tau_1: f \mapsto \tau_1(f)$ defined by $\tau_1(f)(x) = \left( f(x^2) \right)^{1/2}$ is injective but not surjective; indeed, any $\tau_1(f)$ commutes with the involution $x \mapsto -x$ (we choose the determination $\sqrt{1} = 1$), and for instance $x + x^2$ does not commute with the involution $x \mapsto -x$.

Suppose now that $n > 1$. We will use a similar idea considering the application

$$E: (x_1, x_2, \ldots, x_n) \mapsto (x_1^2, x_1 x_2, x_1 x_3, \ldots, x_1 x_n)$$

whose inverse is

$$E^{-1}: (x_1, x_2, \ldots, x_n) \mapsto \left( \sqrt{x_1}, \sqrt{x_2}, \frac{x_3}{\sqrt{x_1}}, \ldots, \frac{x_n}{\sqrt{x_1}} \right).$$

Let us choose the determination of $E^{-1}$ associated to the principal determination of $\sqrt{}$; the application $\tau_n$ defined by

$$\tau_n(f) = \tau_n(f_1, f_2, \ldots, f_n)(x) = E^{-1}(f \circ E)(x)$$
is an injective morphism that is not surjective; indeed the $\tau_n(f)$ commute with the involution $x \mapsto -x$. \hfill $\Box$

**Problem 5.9.** Is the group $G_0^n$ a co-Hopfian group?

**6. Tits Alternative**

A group $H$ satisfies **Tits alternative** if for every finitely generated subgroup $K$ of $H$:

- either $K$ is virtually solvable (i.e. $K$ contains a solvable subgroup of finite index),
- or $K$ contains a non-abelian free subgroup.

Tits proved in [Tit72] that linear groups satisfy Tits alternative.

**Problem 6.1.** Do the groups $G_0^n$ and $\hat{G}_0^n$ satisfy Tits alternative?

This important question is related to the Galois theory of holomorphic foliations ([Cas06 Cas11]). Note that, as it can be seen in [BCLN96], the group $G_0^n$ contains free subgroups of rank $\geq 2$. Furthermore, the solvable non-abelian subgroups of $G_0^1$ and $\hat{G}_0^1$ are classified in [CM82]. Solvable subgroups of $G_0^n$, $n > 1$, have been studied ([MRT14 Rib19]); in particular, Ribón determines the "length of resolubility" of solvable subgroups of $G_0^n$.

**7. Automorphism Groups**

In [Whi63] Whittaker proves the following statement: let $X$ and $Y$ be compact manifolds, with or without boundary, and $\varphi$ be a group isomorphism between the group $\text{Homeo}(X)$ of all homeomorphisms of $X$ into itself and $\text{Homeo}(Y)$, then there exists an homeomorphism $\psi$ of $X$ onto $Y$ such that $\varphi(f) = \psi f \psi^{-1}$ for all $f \in \text{Homeo}(X)$. When $X = Y$ we get that every automorphism of $\text{Homeo}(X)$ is an inner one. In [Fil82] Filipkiewicz gives a similar result in the context of differentiable manifolds: let $M$ and $N$ be smooth manifolds without boundary, and let $\text{Diff}^p(M)$ denote the group of $C^p$-diffeomorphisms of $M$. The author proves that if $\text{Diff}^p(M)$ and $\text{Diff}^q(N)$ are isomorphic as abstract groups, then $p = q$, and the isomorphism is induced by a $C^p$-diffeomorphism from $M$ to $N$. Let us mention that there are similar results in different contexts: see for instance [Ban86 Ban97 D06 ...]. In particular, in [D06] the author proves that any automorphism of the Cremona group $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ of birational self-maps of the complex projective plane is the composition of an inner automorphism and an automorphism of the field of complex numbers. The sketch of the proof is the following. Let $G_1$, $G_2$, ..., $G_\ell$ be maximal abelian uncountable subgroups of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$, and let $\varphi$ be an automorphism of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$; the author proves that, up to inner conjugacy and the action of an automorphism of the field $\mathbb{C}$, we have $\varphi|_{G_k} = \text{id}$ for $1 \leq k \leq \ell$, and deduce from it that $\varphi|_{\text{Bir}(\mathbb{P}^2_\mathbb{C})} = \text{id}$. A similar strategy will be used in this section to describe the automorphism groups of $\hat{G}_0^1$ and of $G_0^1$. 
7.1. **Automorphism groups of $\hat{G}_0^1$**. In this section we establish the description of the automorphism groups of $\hat{G}_0^1$.

**Theorem 7.1.** The group $\text{Aut}(\hat{G}_0^1)$ is generated by the inner automorphisms and the automorphisms of the field $\mathbb{C}$. In other words

$$\text{Out}(\hat{G}_0^1) \simeq \text{Aut}(\mathbb{C}, +, \cdot)$$

where $\text{Out}(\hat{G}_0^1)$ denotes the non-inner automorphisms of $\hat{G}_0^1$.

The rest of this section is devoted to the proof of the Theorem 7.1. The study of the maximal abelian subgroups of $\hat{G}_0^1$ (resp. $G_0^1$) is essential for the understanding of automorphism groups of $\hat{G}_0^1$ (resp. $G_0^1$). Unfortunately we are able to study them only in the case $\hat{G}_0^1$. Once again the essential argument is the following one: if $f$ belongs to $\hat{G}_0^1 \setminus \hat{G}_0^1$, then $f$ is conjugate to $\exp X_{\kappa, \lambda} = \exp \frac{\kappa^{k+1}}{1 + \lambda x^f}$ for a certain $\lambda$. A computation ([CM88, CCD13]) shows that the centralizer

$$\text{Cent}(\exp X_{\kappa, \lambda}, \hat{G}_0^1) = \{ f \in \hat{G}_0^1 \mid f \exp X_{\kappa, \lambda} = \exp X_{\kappa, \lambda} f \}$$

of $\exp X_{\kappa, \lambda}$ in $\hat{G}_0^1$ coincides with the group

$$A_{\kappa, \lambda} = \{ \exp t X_{\kappa, \lambda} \mid t \in \mathbb{C} \} \times \{ x \mapsto \xi x \mid \xi^k = 1 \}.$$ 

This group, which is abelian, and so maximal abelian, contains exactly $(k-1)$ non-trivial torsion elements.

Let $\kappa$ be in $\mathbb{C}^*$; denote by $\kappa$: $x \mapsto \kappa x$ the homothety of ratio $\kappa$. If $\kappa$ is not a root of unity, then

$$\text{Cent}(\kappa, \hat{G}_0^1) = A_0 = \{ \mu \mid \mu \in \mathbb{C}^* \}$$

which is also a maximal abelian subgroup. A subgroup of $\hat{G}_0^1$ whose all elements are periodic is abelian and conjugate to a subgroup of $A_0$ (see [CCD13, Corollary 7.21]); in particular, a maximal abelian subgroup of $\hat{G}_0^1$ contains a non-periodic element. As a consequence, we get:

**Theorem 7.2.** The maximal abelian subgroups of $\hat{G}_0^1$ are the conjugate of the groups $A_0$ and $A_{\kappa, \lambda}$ where $k \geq 1$ is an integer, and $\lambda$ an element of $\mathbb{C}$.

Let us now consider $\sigma$: $\hat{G}_0^1 \to \hat{G}_0^1$ an automorphism of $\hat{G}_0^1$. For instance the inner automorphisms of $\hat{G}_0^1$

$$f \mapsto \tau(f) = \phi f \phi^{-1}$$

are such examples. Let $\text{Aut}(\mathbb{C}, +, \cdot)$ be the automorphism group of the field $\mathbb{C}$. Any automorphism $\tau$ of the field $\mathbb{C}$ induces an automorphism $\sigma^\tau$ that sends $f = \sum_{\ell \geq 1} a_{\ell} x^\ell$ to $\sigma^\tau(f) = \sum_{\ell \geq 1} \tau(a_{\ell}) x^\ell$.

Note that the image of a maximal abelian subgroup of $\hat{G}_0^1$ by $\sigma$ is still a maximal abelian subgroup of $\hat{G}_0^1$.

**Theorem 7.3.** Let $\sigma$ be an automorphism of $\hat{G}_0^1$. Then, up to a suitable conjugacy, $\sigma(A_0) = A_0$ and $\sigma(A_{\kappa, 0}) = A_{\kappa, 0}$ for any integer $k \geq 1$.

Furthermore, if $\lambda$ is non-zero, then $\sigma(A_{\kappa, \lambda})$ is conjugate to $A_{\kappa, \mu}$ for some $\mu$ in $\mathbb{C}^*$.
Proof. The group \( A_0 \) has an infinite number of torsion elements whereas the \( A_{k,\lambda} \)'s don't; this gives the first assertion. We can thus assume that \( \sigma(A_0) = A_0 \). Let us note that \( A_0 \) acts by conjugacy on the groups \( A_{k,0} \): if \( \mu \) belongs to \( A_0 \), then
\[
\mu \exp t X_{k,0} \mu^{-1} = \exp \mu^{-k} t X_{k,0}.
\]
However, the conjugate of \( A_{k,\lambda} \) by \( \mu \) is \( A_{k,\mu^{-k} \lambda} \), and \( A_0 \) does not act on the \( A_{k,\lambda} \). Consequently, \( A_0 = \sigma(A_0) \) acts by conjugacy on \( \sigma(A_{k,0}) \). Counting the torsion elements we get that \( \sigma(A_{k,0}) \) is conjugate to \( A_{k,\lambda} \) for some \( \lambda \) in \( \mathbb{C} \). In particular
\[
\sigma \left( \exp x^{k+1} \frac{\partial}{\partial x} \right) = \exp X
\]
where \( X \) is a formal vector field conjugate to \( X_{k,\lambda} \) for some \( \lambda \) in \( \mathbb{C} \).

**Lemma 7.4.** One has: \( X = a_k x^{k+1} \frac{\partial}{\partial x} \) for some non-zero complex number \( a_k \).

**Proof of Lemma 7.4.** Let \( \mu \) be a complex number that is not a root of unity. Then, \( \mu' = \mu^\sigma \) is not a root of unity. We have
\[
\sigma \left( \mu \left( \exp x^{k+1} \frac{\partial}{\partial x} \right) \mu^{-1} \right) = \mu' (\exp X) (\mu')^{-1} = \exp \mu' X
\]
and
\[
\sigma \left( \mu \left( \exp x^{k+1} \frac{\partial}{\partial x} \right) \mu^{-1} \right) = \sigma \left( \exp x^{k+1} \frac{\partial}{\partial x} \right) = \sigma \left( \exp \mu^{-k} x^{k+1} \frac{\partial}{\partial x} \right)
\]
From
\[
\sigma \{ \exp t x^{k+1} \frac{\partial}{\partial x} \mid t \in \mathbb{C} \} = \{ \exp s X \mid s \in \mathbb{C} \}
\]
on one gets
\[
\sigma \left( \exp \mu^{-k} x^{k+1} \frac{\partial}{\partial x} \right) = \exp s X
\]
for some \( s \) (dependent on \( \mu \)) which finally implies \( sX = \mu' X \). We write \( X = \sum_{\ell \geq k} a_{k,\ell} \frac{\partial}{\partial x} \), \( a_k \neq 0 \), then
\[
\mu' X = \sum_{\ell \geq k} a_k \mu_{1-k} x^{k+1} \frac{\partial}{\partial x}.
\]
The relation \( sX = \mu' X \) implies \( sa_{k,\ell} = (\mu')^{-1-l} a_{\ell} \) for \( l \geq k \). Since \( a_k \neq 0 \), if \( a_{\ell} \neq 0 \) for an \( \ell > k \), then \( (\mu')^{1-\ell} = s = (\mu')^{1-k} \), and \( \mu' \) is a root of unity: contradiction. \hfill \Box

As a result, \( \sigma(A_{k,0}) = A_{k,0} \) for any \( k \); the torsion elements can optionally be swapped and \( \exp t x^{k+1} \frac{\partial}{\partial x} \) is sent onto \( \exp t a_k x^{k+1} \frac{\partial}{\partial x} \). Let us remark that \( \mathbb{C} \ni t \mapsto t a_k \in \mathbb{C} \) is an additive morphism group. \hfill \Box

Hence an automorphism \( \sigma \) of \( \mathbb{G}_1^0 \) induces a multiplicative isomorphism of \( \mathbb{C}^* \)
\[
\sigma(\lambda) = \lambda^\sigma,
\]
and an additive isomorphism \( \sigma_k \) of \( \mathbb{C} \) for any \( k \)
\[
\sigma_k(t) = t a_k.
\]
Let us come back to the action of $A_0$ on $A_{k,0}$

\[
\left( \lambda, \exp tx^{k+1} \frac{\partial}{\partial x} \right) \mapsto \exp \lambda tx^{k+1} \frac{\partial}{\partial x}
\]

that corresponds to the action of $\mathbb{C}^*$ on $\mathbb{C}$

\[
(\lambda,t) \mapsto \lambda^t.
\]

The action is transformed by the automorphism $\sigma$ into

\[
\left( \lambda^\sigma, \exp t_e x^{k+1} \frac{\partial}{\partial x} \right) \mapsto \exp (\lambda^\sigma)^t x^{k+1} \frac{\partial}{\partial x}
\]

but also into

\[
\left( \lambda^\sigma, \exp t_e x^{k+1} \frac{\partial}{\partial x} \right) \mapsto \exp (\lambda^t) x^{k+1} \frac{\partial}{\partial x}.
\]

Therefore

\[
(\lambda^\sigma)^t x_e = (\lambda^t) x_e ;
\]

in particular, for $t = 1$ we get

\[
\sigma^1 x_e = s e_x.
\]

Hence $s \mapsto s^\sigma$ is an automorphism of the field $\mathbb{C}$, and the additive morphisms $s \mapsto s e_x$ differ from $s^\sigma$ only by a multiplicative constant $1 e_x$. Up to the action of the automorphism of $\hat{\mathcal{G}}^1_0$ associated to this field automorphism we can assume that $s \mapsto s^\sigma$ is the identity, and that $s e_x = \varepsilon e_s$ where $\varepsilon_x$ denotes a non-zero constant.

**Lemma 7.5.** If $\sigma$ belongs to $\text{Aut}(\hat{\mathcal{G}}^1_0)$, then $\sigma(\hat{\mathcal{G}}^1_0) = \hat{\mathcal{G}}^1_0$.

**Proof.** Let $h$ be an element of $\hat{\mathcal{G}}^1_0$; then there exist $\varphi$ in $\hat{\mathcal{G}}^1_0$, $p \geq k$, and $\lambda$ in $\mathbb{C}$ such that

\[
h = \varphi \exp X_p \lambda \varphi^{-1}.
\]

Recall that $\sigma(\exp X_p \lambda) = \exp a X_p \lambda'$ for some $a \in \mathbb{C}^*$ and $\lambda' \in \mathbb{C}$. Hence

\[
\sigma(h) = \sigma(\varphi) \exp a X_p \lambda \sigma(\varphi^{-1})
\]

belongs to $\hat{\mathcal{G}}^1_k$; similarly $\sigma^{-1}(h)$ belongs to $\hat{\mathcal{G}}^1_k$. \qed

**Remark 7.6.** From Lemma 7.5 we get that $\sigma$ is a continuous automorphism of $\hat{\mathcal{G}}^1_0$ endowed with the Krull topology.

Let us recall the Baker-Campbell-Hausdorff formula applied to the formal vector fields $X = a(x) \frac{\partial}{\partial x}$ and $Y = b(x) \frac{\partial}{\partial x}$ of $\hat{\mathcal{G}}^1_2$ (see for instance [Hal15]): If $Z \in \hat{\mathcal{G}}^1_2$ is a solution of

\[
\exp Z = \exp X \exp Y,
\]

then

\[
Z = X + Y + \frac{1}{2} [X,Y] + \frac{1}{12} [X,[X,Y]] - \frac{1}{12} [Y,[X,Y]] + \ldots
\]
In particular
\[(\exp X)(\exp Y)(\exp -X)(\exp -Y) = \exp ([X,Y] + \text{h.o.t.})\]  
where h.o.t. denotes terms of order \(\geq 3\) in the algebra generated by \(X\) and \(Y\). We thus get
\[(\exp x^{k+1}\frac{\partial}{\partial x})(\exp x^{\ell+1}\frac{\partial}{\partial x})(\exp -x^{k+1}\frac{\partial}{\partial x})(\exp -x^{\ell+1}\frac{\partial}{\partial x}) = \exp \left([\exp x^{k+1}\frac{\partial}{\partial x},\exp x^{\ell+1}\frac{\partial}{\partial x}] + \chi\right)\]
where \(\chi\) is a vector field of the form
\[\chi = x^{\ell+k+1+\inf(\ell,\ell)}a(x)\frac{\partial}{\partial x};\]
in other words
\[\left(\exp x^{k+1}\frac{\partial}{\partial x}\right)\left(\exp x^{\ell+1}\frac{\partial}{\partial x}\right)\left(\exp -x^{k+1}\frac{\partial}{\partial x}\right)\left(\exp -x^{\ell+1}\frac{\partial}{\partial x}\right) = \exp \left((\ell-k)x^{\ell+k+1}\frac{\partial}{\partial x} + \chi\right)\]
\[= \exp \left((\ell-k)x^{\ell+k+1}\frac{\partial}{\partial x}\right)h\]
where \(h\) denotes an element of \(\hat{G}_{\ell+k+\inf(\ell,\ell)}\) (still by Baker-Campbell-Hausdorff formula). Applying \(\sigma\) we get (\(\epsilon_k\) has been introduced just before Lemma 7.5)
\[\left(\exp\epsilon_kx^{k+1}\frac{\partial}{\partial x}\right)\left(\exp\epsilon_\ell x^{\ell+1}\frac{\partial}{\partial x}\right)\left(\exp -\epsilon_kx^{k+1}\frac{\partial}{\partial x}\right)\left(\exp -\epsilon_\ell x^{\ell+1}\frac{\partial}{\partial x}\right)\]
\[= \exp \left(\epsilon_\ell\epsilon_k\left[x^{k+1}\frac{\partial}{\partial x},x^{\ell+1}\frac{\partial}{\partial x}\right] + \tilde{\chi}\right)\]
\[= \exp \left(\epsilon_\ell\epsilon_kx^{k+\ell+1}\frac{\partial}{\partial x}\right)\sigma(h)\]
where \(\tilde{\chi}\) is given by (7.1) and \(\sigma(h)\) is controlled by Lemma 7.5. By truncating to a suitable order we see that \(\epsilon_k\epsilon_\ell = \epsilon_{k+\ell}\). Up to the action by an homothety we can assume that \(\epsilon_1 = 1\); as a result \(\epsilon_\ell = \epsilon_{\ell+1}\). Hence by induction \(\epsilon_k = 1\) for any \(k\). The automorphism \(\sigma\) thus fixes the homotheties and the \(\exp tx^{k+1}\frac{\partial}{\partial x}\).

The quotient groups \(\hat{G}_0^1/\hat{G}_1\) are generated by the projections of the homotheties and by the projections of the \(\exp tx^{k+1}\frac{\partial}{\partial x}, t \in \mathbb{C}, \ell \leq k - 1\). According to Lemma 7.5 the automorphism \(\sigma\) induces an automorphism of the quotient groups \(\hat{G}_0^1/\hat{G}_1\) that coincides with the identity on any \(\hat{G}_0^1/\hat{G}_1\). Therefore, \(\sigma\) coincides with the identity. This ends the proof of Theorem 7.1.

7.2. Automorphisms groups of \(G_0^1\). The aim of this section is to give the following description of the automorphisms groups of \(G_0^1\):

**Theorem 7.7.** The group \(\text{Out}(G_0^1)\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\).

We are going to adapt the above approach to the holomorphic case, i.e. to the description of \(\text{Out}(G_0^1)\). Theorem 7.2 has no analogue but [75] and [CM88] imply:
Theorem 7.8. Let $A$ be an uncountable maximal abelian subgroup $A$ of $G^1_0$ such that $A \cap G^1_1$ is uncountable. Then, up to conjugacy,
\begin{itemize}
  \item either $A = A_0$,
  \item or $A = A_{k,\lambda}$ where $k \geq 1$ denotes an integer, and $\lambda$ a complex number.
\end{itemize}
Moreover, the group $A_0$ is a maximal abelian subgroup of $G^1_0$.

Remarks 7.9. \begin{itemize}
  \item[(1)] The group $G^1_1$ contains many countable abelian maximal subgroups; in fact the group generated by a "generic" element of $G^1_1$ is maximal (E75).
  \item[(2)] There are diffeomorphisms $f = \lambda z + \mathrm{h.o.t.}$, $\lambda = \exp(2i\pi \gamma)$, $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, which are not holomorphically linearisable. The maximal abelian group that contains $f$ is not necessarily conjugate to $A_0$ or $A_{k,\lambda}$. It can be uncountable (see PM95).
\end{itemize}

Let $\sigma$ be an element of $\text{Aut}(G^1_0)$. We can, as in the formal case, characterize the groups $A_0$ and $A_{k,\lambda}$ by their torsion elements. Recall that if $f$ belongs to $G^1_1$, then $f$ is a commutator, and $\sigma(f)$ also. As a consequence, $\sigma(G^1_1) = G^1_1$, and if $H$ is a maximal abelian subgroup of $G^1_1$, then $\sigma(H)$ also. Hence, as in the formal context, $\sigma(A_{k,0})$ is holomorphically conjugate to one of the $A_{k,\lambda}$. We want to prove that $\sigma(A_0)$ is conjugated to $A_0$.

Lemma 7.10. Let $f$ be an element of $G^1_0$. Assume that $A_{k,\lambda}$ is invariant by conjugation by $f$, i.e. $fA_{k,\lambda}f^{-1} = A_{k,\lambda}$. Then
\begin{itemize}
  \item if $\lambda \neq 0$, then $f = \xi \exp \frac{x^{p+1}}{1 + \lambda x^p} \frac{\partial}{\partial x}$ for some $\xi, \xi^p = 1$;
  \item if $\lambda = 0$, then $f = \mu \exp \frac{\partial}{\partial x}$ for some $\mu$ in $\mathbb{C}^*$, and $\xi$ in $\mathbb{C}$.
\end{itemize}

Proof. Left to the reader. \hfill $\square$

Now, let us fixed $k$; by Theorem 7.8 we can assume, up to conjugacy, that $\sigma(A_{k,0}) = A_{k,\lambda}$ for some $\lambda$. But $\sigma(A_0)$ contains periodic elements of all periods so $\lambda = 0$, and $\sigma(A_0)$ is an abelian subgroup of the "affine" group $\{ \mu \exp \frac{\partial}{\partial x} | \mu \in \mathbb{C}^*, \xi \in \mathbb{C} \}$ (Lemma 7.10). Such a group is either conjugate to a subgroup of $A_0$, or conjugate to a subgroup of
\[ \{ \xi \exp \frac{x^{p+1}}{1 + \lambda x^p} \frac{\partial}{\partial x} | \xi, \xi^p = 1 \} . \]
The fact that $\sigma(A_0)$ contains an infinite number of periodic elements implies that $\sigma(A_0)$ is a subgroup of $A_0$. As $\sigma(A_0)$ is maximal, one gets: $\sigma(A_0) = A_0$.

Recall that $\sigma(A_0) = A_0$ acts on all the $\sigma(A_{k,0})$; an argument similar to that used in the formal case (7.11) implies that $\sigma(A_{k,0}) = A_{k,0}$ for all $k \in \mathbb{N}$.

As before $\sigma$: $A_0 \rightarrow A_0$ corresponds to an automorphism of the field $\mathbb{C}$.

Lemma 7.11. Any $\sigma \in \text{Aut}(G^1_0)$ extends into an automorphism $\bar{\sigma}$: $\bar{G}^1_0 \rightarrow \bar{G}^1_0$. Moreover, $\sigma(G^1_k) = G^1_k$. 
Proof. The projections of the homotheties and the \( \exp x \frac{d}{dx} \)'s, \( \ell \leq k - 1 \), generate \( G_0^k / G_0^k \cong \hat{G}_0^k / G_0^k \). Since \( \sigma \) preserves \( A_0 \) and the \( A_{\ell,0} \)'s, it induces an automorphism

\[
\sigma_k: \hat{G}_0^k / G_0^k \to \hat{G}_0^k / G_0^k
\]

for any \( k \). By construction these automorphisms are compatible with the filtration induced by the \( G_0^k \)'s.

As a result, the \( \sigma_k \)'s determine an automorphism \( \hat{\sigma}: \hat{G}_0^k \to \hat{G}_0^k \). This automorphism extends \( \sigma \) in the following sense: if we fix a coordinate \( z \), we get an embedding \( G_0^k \hookrightarrow \hat{G}_0^k \) that associates to the convergent element \( f = \sum a_n z^n \) the element \( f = \sum a_n z^n \) seen as a formal series. By construction \( \hat{\sigma}(f) = \sigma(f) \), i.e. \( \hat{\sigma} \) sends a holomorphic diffeomorphism onto a holomorphic diffeomorphism. \( \square \)

According to Theorem 7.1 we can assume that \( \hat{\sigma} \) is associated to an automorphism \( \tau \) of the field \( \mathbb{C} \): if \( f = \sum a_n z^n \), then

\[
\hat{\sigma}(f) = \sum_{n \geq 1} \tau(a_n) z^n.
\]

The automorphism \( \tau \) satisfies the following property: if \( \sum a_n z^n \) converges, then \( \sum \tau(a_n) z^n \) also converges.

**Lemma 7.12.** Either \( \tau \) is the identity \( z \mapsto z \), or \( \tau \) is the complex conjugation \( z \mapsto \bar{z} \).

**Proof.** If \( \tau \) preserves \( \mathbb{R} \), that is if \( \tau(\mathbb{R}) = \mathbb{R} \), then \( \tau \) is either the identity, or the complex conjugation.

If \( \tau(\mathbb{R}) \neq \mathbb{R} \), then the image of the unit disk \( \mathbb{D}(0,1) \) is dense in \( \mathbb{C} \) (see [Kes51]). Suppose that \( \tau \) is neither the identity, nor the complex conjugation. By density there exists \( a_n \in \mathbb{D}(0,1), n \geq 2 \), such that \( |\tau(a_n) - n| < 1 \). If \( f = z + \sum a_n z^n \), then on the one hand \( f \) belongs to \( G_0^1 \), and on the other hand \( z + \sum a_n z^n \) diverges: contradiction. \( \square \)

### 8. Automorphisms Groups of \( \hat{G}_0^0 \)

We are not able to prove an analogue of Theorem 7.1 in higher dimension, nevertheless we obtain the following partial result:

**Proposition 8.1.** Up to conjugacy and up to the action of an element of \( \text{Aut}(\mathbb{C},+,\cdot) \), the restriction of an element of \( \text{Aut}(\hat{G}_0^0) \) to \( \text{GL}(\mathbb{C}^n) \) is the identity map.
Denote by
\[ j^1 : \hat{G}_0^n \to \text{GL}(\mathbb{C}^n), \quad f \mapsto Df(0) \]
the map that associates to \( f \) its linear part at 0. Let \( \sigma \) be an element of \( \text{Aut}(\hat{G}_0^n) \), and set \( \tilde{\varphi} = j^1 \circ \sigma \), i.e.
\[ \tilde{\varphi} : \hat{G}_0^n \xrightarrow{\sigma} \hat{G}_0^n \xrightarrow{j^1} \text{GL}(\mathbb{C}^n) \]
\[ f \mapsto \sigma(f) \mapsto D(\sigma(f))(0) \]
and consider \( \varphi = \tilde{\varphi}_{|\text{GL}(\mathbb{C}^n)} \) the restriction of \( \tilde{\varphi} \) to \( \text{GL}(\mathbb{C}^n) \):
\[ \varphi : \text{GL}(\mathbb{C}^n) \to \text{GL}(\mathbb{C}^n), \quad A \mapsto D(\sigma(A))(0). \]

Denote by \( \mathcal{H} \) the subgroup of homotheties \( \mathbb{C}^* \text{id} \) of \( \text{GL}(\mathbb{C}^n) \).

Proposition 8.1 follows from the following facts:

- The normal subgroup \( \ker \varphi \) of \( \text{GL}(\mathbb{C}^n) \) is a subgroup of \( \mathcal{H} \).
- The map \( \varphi \) is an injective morphism.
- Consider a non-periodic homothety \( \lambda \text{id} \). The image \( \varphi(\lambda \text{id}) \) of \( \lambda \text{id} \) by \( \varphi \) is an homothety.
- If \( \sigma \) belongs to \( \text{Aut}(\hat{G}_0^n) \), then up to conjugacy \( \sigma_{|\text{GL}(\mathbb{C}^n)} \) is an automorphism of \( \text{GL}(\mathbb{C}^n) \).

**Remark 8.2.** Proposition 8.1 can be used as follows. Take an element \( A \) in \( \text{GL}(n, \mathbb{C}) \), typically \( A_0 = (\lambda x_1, x_2, x_3, \ldots, x_n) \), and consider the group \( \text{Cent}(A, \hat{G}_0^n) = \{ f \in \hat{G}_0^n | f \circ A = A \circ f \} \). In the case of the example of \( A_0 \), for some generic \( \lambda \), we have
\[ \text{Cent}(A_0, \hat{G}_0^n) = \{ (a(x_2, x_3, \ldots, x_n)x_1, g(x_2, x_3, \ldots, x_n)) \mid a \in \mathbb{C}[[x_2, x_3, \ldots, x_n]], g \in \hat{G}_0^{n-1} \}. \]

If \( \sigma \in \text{Aut}(\hat{G}_0^n) \) is such that \( \sigma_{|\text{GL}(n, \mathbb{C})} = \text{id}_{\text{GL}(n, \mathbb{C})} \), then \( \sigma(\text{Cent}(A, \hat{G}_0^n)) = \text{Cent}(A, \hat{G}_0^n) \), and we can expect for instance with \( A \) of type \( A_0 \) to use an induction on the dimension to prove that the automorphisms group of \( \hat{G}_0^n \) is generated by inner automorphisms, and the automorphisms of the field \( \mathbb{C} \).

9. The \( \mathcal{C}^\infty \)-case

Let \( \text{Diff}^\infty(\mathbb{R}^n, 0) \) be the group of germs of \( \mathcal{C}^\infty \)-diffeomorphisms of \( (\mathbb{R}^n, 0) \), and let \( \widehat{\text{Diff}}^\infty(\mathbb{R}^n, 0) \) be its completion. Consider the map \( \mathcal{T} : \text{Diff}^\infty(\mathbb{R}^n, 0) \to \widehat{\text{Diff}}^\infty(\mathbb{R}^n, 0) \) that sends the diffeomorphism \( f \) onto the infinite Taylor expansion \( \mathcal{T}(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \). Denote by \( \text{Diff}^\infty_{\mathcal{T}}(\mathbb{R}^n, 0) \) the kernel of \( \mathcal{T} \) consisting of diffeomorphisms infinitely tangent to the identity.

**Proposition 9.1.** Let \( G \) be a finitely generated subgroup of \( \text{Diff}^\infty(\mathbb{R}^n, 0) \). If the restriction of \( \mathcal{T} \) to \( G \) is one-to-one, then \( G \) is residually finite.

**Proof.** The group \( G \) is isomorphic to a finitely generated subgroup of \( \text{Diff}^\infty(\mathbb{R}^n, 0) \); hence \( G \) is residually finite. \( \square \)

**Question 9.2.** Let \( G \) be a finitely generated subgroup of \( \text{Diff}^\infty_{\mathcal{T}}(\mathbb{R}^n, 0) \); is \( G \) residually finite?
Let us mention a result attributed to Thurston and "reproved" by Reeb and Schweitzer ([RS78]) which could be useful to answer the question in dimension 1:

**Theorem 9.3 (Thurston).** Let $G$ be a finitely generated subgroup of $\text{Diff}^\infty_\infty(\mathbb{R}^n,0)$. There exists a non-trivial morphism from $G$ to $\mathbb{R}$.

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