A MEAN-FIELD FORMULATION FOR MULTI-PERIOD ASSET-LIABILITY MEAN-VARIANCE PORTFOLIO SELECTION WITH PROBABILITY CONSTRAINTS

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Abstract. This paper is concerned with studying an optimal multi-period asset-liability mean-variance portfolio selection with probability constraints using mean-field formulation without embedding technique. We strictly derive its analytical optimal strategy and efficient frontier. Numerical examples shed light on efficiency and accuracy of our method when dealing with this class of multi-period non-separable mean-variance portfolio selection problems.

1. Introduction. Mean-variance portfolio selection refers to the design of optimal portfolios balancing gain with risk, which are in expressions of expectation and variance of the terminal return, respectively. After Markowitz’s seminal work for a single-period, research on mean-variance portfolio selection problems have been well developed. For instance, by an embedding technique, Li-Ng [11] extended Markowitz’s model to a multi-period setting and derived analytical optimal portfolio and efficient frontier; Zhang-Li [23] generalized their work to the case with uncertain time horizon when returns are serially correlated. Zhou-Li [24] studied continuous-time mean-variance problem using the similar technique and obtained closed forms of optimal portfolio and efficient frontier; Li-Xie [13] gave a generalization of their problem in several aspects. Fu-Lavassani-Li [7] and Li-Zhou-Lim [12] investigated dynamic mean-variance portfolio selection with borrowing constraint and no-shorting one, respectively. Cui-Gao-Li-Li [4] tackled multi-period
mean-variance portfolio problem under no-shorting constraint using a parameterized method. Cui-Li-Li [5] proposed a mean-field formulation to analyze the multi-period mean-variance portfolio selection again, and Yi-Wu-Li-Cui [21] extended this formulation to consider the portfolio problem with an uncertain exit time.

Asset-liability management is a financial tool for investors that set out to maximize their wealth. The aim of asset-liability management is to reduce risk as well as increase returns, and it has been used successfully for banks, pension funds, insurance companies and wise individuals. A judicious investment considers assets and liabilities simultaneously. A financial institution taking liabilities into account can operate more soundly and lucratively. Krouse [8] noticed that lots of mean-variance models concentrated only upon assets with little or no effort being directed to the liabilities. The mean-variance framework of asset-liability management was first investigated by Sharp-Tint [15] in a single-period setting. Leippold-Trojani-Vanini [9] derived the closed form optimal policies and mean-variance frontiers under exogenous and endogenous liabilities using a geometric approach. Chiu-Li [3] employed the stochastic optimal control theory to analytically solve the asset-liability management in a continuous time setting. Xie-Wang [19] considered the situation in an incomplete market by using the general stochastic linear-quadratic control technique. Yi-Li-Li [20] investigated the case with uncertain investment horizon. Chen-Yang [2] studied the case with regime switching. Zeng-Li [22] investigated the models under benchmark and mean-variance criteria in a jump diffusion market. Wu-Li [17] considered the regime switching and cash flow together in the model.

Due to the volatility of the financial market, it is impossible to eliminate the possibility of bankruptcy in a multi-period investment setting. Bankruptcy occurs when the surplus (total wealth minus liability) falls below a preset level. Once an investor goes bankrupt, he/she will suffer a great loss such as retrieve part of his/her wealth (even take nothing back), high liability and low credit. It is crucial for a successful investment to take bankruptcy into account. Zhu-Li-Wang [25] generalized the multi-period mean-variance model by considering a good risk control over bankruptcy. Bielecki-Jin-Pliska-Zhou [1] studied the continuous-time mean-variance problem with bankruptcy prohibition. Wei-Ye [16] studied the multi-period optimization portfolio with bankruptcy control when the random returns of risky assets depend on the state of the stochastic market. Wu-Zeng [18] investigated the case in a regime switching market. Li-Li [10] considered the multi-period portfolio optimization for liability management with bankruptcy control.

In this paper, we consider the optimal multi-period asset-liability mean-variance portfolio selection with probability constraints. Probability constraints of all periods require that the amount of assets is possibly larger than the liabilities at any period. In general, all the investors would take this fundamental policy. In fact, the bankruptcy control can be regarded as a kind of probability constraints, where we can utilize a preset probability as the bankruptcy prohibition constraint. Since the probabilistic constraint is not easy to conquer in dynamic portfolio selection, we turn it to its upper bound by Tchebycheff inequality. Our model is developed from the model in [10]. In Li-Li [10]’s work, they used the embedding technique to study the problem, and could not directly derive analytical optimal portfolio because of the bankrupt risk control. They first formulated the primal problem to its dual problem by Lagrangian dual approach, then solved the nonseparable dual problem by constructed a class of auxiliary linear quadratic optimal stochastic control
problems. Finally they searched for the Lagrangian multiplier using primal-dual method. The construction of auxiliary problems and the inexplicit optimal objective function may possibly involve some unnecessary and complicated expressions or computational errors, resulting in complicated or even inaccurate formulas. We employ mean-filed formulation to successfully reformulate the nonseparable problem to a mean-field linear-quadratic stochastic control problem which can be solvable by the classical dynamic programming approach, and strictly derive analytical optimal strategy of this problem and its efficient frontier. Compared with Li-Li [10], our results have much simpler formulas.

The rest of the paper is organized as follows. In section 2, we present the mean-field formulation for the multi-period asset-liability mean-variance portfolio selection with probability constraints. We strictly derive the optimal strategy and the corresponding efficient frontier in section 3. Numerical examples are presented in section 4 to show the efficiency of the mean-field formulation to solve the nonseparability of multi-period mean-variance portfolio selection problem with probability constraints.

2. Formulation. Assume that the capital market consists of one risk-free asset, $n$ risky assets and one liability. An investor joining the market at the beginning of period $0$ with an initial wealth $x_0$ and initial liability $l_0$, plans to invest his/her wealth within a time horizon $T$. He/she can reallocate his/her portfolio at the beginning of each following $T - 1$ consecutive periods. At time period $t$, the given deterministic return of the risk-free asset, the random returns of the $n$ risky assets, and the random return of the liability are denoted by $s_t$ $(> 1)$, vector $e_t = [e^1_t, \cdots, e^n_t]'$ and $q_t$, respectively. The random vector $e_t = [e^1_t, \cdots, e^n_t]'$ and the random variable $q_t$ are defined over the probability space $(\Omega, \mathcal{F}, P)$ and are supposed to be statistically independent at different time period. We further assume that the only information known about $e_t$ and $q_t$ are their first two unconditional moments, $\mathbb{E}[e_t] = (\mathbb{E}[e^1_t], \cdots, \mathbb{E}[e^n_t])'$, $\mathbb{E}[q_t]$ and $(n + 1) \times (n + 1)$ positive definite covariance

$$
\text{Cov} \left( \begin{bmatrix} e_t \\ q_t \end{bmatrix} \right) = \mathbb{E} \left[ \begin{bmatrix} e_t \\ q_t \end{bmatrix} \begin{bmatrix} e_t' & q_t' \end{bmatrix} \right] - \mathbb{E} \left[ \begin{bmatrix} e_t \\ q_t \end{bmatrix} \right] \mathbb{E} \left[ \begin{bmatrix} e_t' \\ q_t' \end{bmatrix} \right].
$$

Let $x_t$ and $l_t$ be the wealth and liability of the investor at the beginning of period $t$ respectively, then $x_t - l_t$ is the surplus. At period $t$, if $\pi^i_t$, $i = 1, 2, \cdots, n$ is the amount invested in the $i$-th risky asset, then, $x_t - \sum_{i=1}^n \pi^i_t$ is the amount invested in the risk-free asset. We assume in this paper that the liability is exogenous, which means it is uncontrollable and cannot be affected by the investor’s strategies. Denote the information set at the beginning of period $t$, $t = 1, 2, \cdots, T - 1$, as $\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1}, q_0, q_1, \cdots, q_{T-1})$ and the trivial $\sigma$-algebra over $\Omega$ as $\mathcal{F}_0$, where $\mathbf{P}_t = (P^1_t, \cdots, P^n_t)' = (e^1_t - s_t, \cdots, e^n_t - s_t)'$ is the excess return vector of risky assets. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is just the unconditional expectation $\mathbb{E}[\cdot]$. We confine all admissible investment strategies to be $\mathcal{F}_t$-adapted Markov controls, i.e., $\pi_t = (\pi^1_t, \pi^2_t, \cdots, \pi^n_t)' \in \mathcal{F}_t$. Then, $\mathbf{P}_t$ and $\pi_t$ are independent, $\{x_t, l_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t, l_t)$.

The mean-variance model for multi-period asset and liability portfolio selection with probability constraints is to seek the best strategy, $\pi^*_t = [(\pi^1_t)^*, (\pi^2_t)^*, \cdots, (\pi^n_t)^*)]', t = 0, 1, \cdots, T - 1$, which is the optimizer of the following stochastic optimal
control problem,

\[
\begin{aligned}
\min & \ Var(x_T - l_T) - wE[x_T - l_T], \\
\text{s.t.} & \ x_{t+1} = \sum_{i=1}^{n} e_i^t \pi_i^t + \left(x_t - \sum_{i=1}^{n} \pi_i^t\right) s_t \\
& \ = s_t x_t + P_t' \pi_t, \\
& \ l_{t+1} = q_l l_t, \\
& \ Pr(x_t \leq l_t) \leq a_t, \quad t = 1, 2, \ldots, T - 1,
\end{aligned}
\]  

(1)

where \(w > 0\) is the trade-off parameter between the mean and the variance, \(a_t\) is the probability of the wealth less than the liability at period \(t\). Since the probabilistic constraint \(Pr(x_t \leq l_t)\) is not easy to conquer in dynamic portfolio selection, we turn it to its upper bound \(Var(x_t - l_t)/(E[x_t - l_t])^2\) by Tchebycheff inequality. Then the mean-variance model (1) can be equivalently re-written to the following problem,

\[
\begin{aligned}
\min & \ Var(x_T - l_T) - wE[x_T - l_T], \\
\text{s.t.} & \ x_{t+1} = s_t x_t + P_t' \pi_t, \\
& \ l_{t+1} = q_l l_t, \\
& \ Var(x_t - l_t) \leq a_t(E[x_t - l_t])^2, \quad t = 1, 2, \ldots, T - 1.
\end{aligned}
\]  

(2)

The optimal solution to problem (2) is feasible in problem (1), thus serving as an approximated solution to problem (1). To solve problem (2), we consider the following Lagrangian minimum problem,

\[
\begin{aligned}
\min & \ Var(x_T - l_T) - wE[x_T - l_T] + \sum_{t=1}^{T-1} \lambda_t \left(Var(x_t - l_t) - a_t(E[x_t - l_t])^2\right), \\
\text{s.t.} & \ x_{t+1} = s_t x_t + P_t' \pi_t, \\
& \ l_{t+1} = q_l l_t, \quad t = 1, \ldots, T - 1,
\end{aligned}
\]  

(3)

where \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{T-1}) \in \mathbb{R}_+^{T-1}\) is the vector of Lagrangian multipliers.

Due to the variance operation does not satisfy the smoothing property, problem (3) is nonseparable in the sense of dynamic programming, i.e., it can not be decomposed by a stage-wise backward recursion and then is difficult to solve directly. We tackle it by mean-field formulation. For \(t = 0, 1, \cdots, T - 1\), taking the expectation operator of the dynamic system specified in (3) we can drive

\[
\begin{aligned}
\mathbb{E}[x_{t+1}] = s_t \mathbb{E}[x_t] + \mathbb{E}[P_t'] \mathbb{E}[\pi_t], \\
\mathbb{E}[l_{t+1}] = \mathbb{E}[q_l] \mathbb{E}[l_t], \\
\mathbb{E}[x_0] = x_0, \\
\mathbb{E}[l_0] = l_0,
\end{aligned}
\]  

(4)

since \(P_t\) and \(\pi_t\), \(q_l\) and \(l_t\) are independent. Combining the dynamic system of (3) and (4) yields the following for \(t = 0, 1, \cdots, T - 1\),
\[
\begin{align*}
    x_{t+1} - E[x_{t+1}] &= s_t(x_t - E[x_t]) + P_t'\pi_t - E[P_t'E]\pi_t, \\
    &= s_t(x_t - E[x_t]) + P_t'(\pi_t - E[\pi_t]) + (P_t' - E[P_t'])E[\pi_t], \\
    l_{t+1} - E[l_{t+1}] &= q_t l_t - E[q_t E[l_t]] \\
    &= q_t(l_t - E[l_t]) + (q_t - E[q_t])E[l_t].
\end{align*}
\]

Then the state space \((x_t, l_t)\) and the control space \((\pi_t)\) are enlarged into \((E[x_t], x_t - E[x_t], l_t - E[l_t])\) and \((E[\pi_t], \pi_t - E[\pi_t])\), respectively. Although we can select the control vector \(E[\pi_t]\) and \(\pi_t - E[\pi_t]\) independently at time \(t\), they should be chosen such that

\[
E(\pi_t - E[\pi_t]) = 0, \quad t = 0, 1, \ldots, T - 1,
\]

and then

\[
E(x_t - E[x_t]) = 0, \quad t = 0, 1, \ldots, T - 1,
\]

is satisfied. We also confine admissible investment strategies \((E[\pi_t], \pi_t - E[\pi_t])\) to be \(\mathcal{F}_t\)-measurable Markov controls.

Problem (3) can be now reformulated as the following mean-field type of linear quadratic optimal stochastic control problem

\[
\begin{align*}
    \min \quad & E[(x_T - l_T - E[x_T - l_T])^2] - wE[x_T - l_T] \\
    & + \sum_{t=1}^{T-1} \left\{ \lambda_t E\left[(x_t - l_t - E[x_t - l_t])^2\right] - \lambda_t a_t E[(x_t - l_t)]^2 \right\}, \\
\text{s.t.} \quad & \{E[x_t], E[l_t], E[\pi_t]\} \text{ satisfies dynamic equation (4),} \\
    & \{x_t - E[x_t], l_t - E[l_t], \pi_t - E[\pi_t]\} \text{ satisfies dynamic equation (5),} \\
    & E(\pi_t - E[\pi_t]) = 0, \quad t = 0, 1, \ldots, T - 1.
\end{align*}
\]

It is indeed a separable linear quadratic optimal stochastic control problem which can be solved by classic dynamic programming approach.

3. The optimal strategy. Before deriving the main results, we present some useful lemmas.

**Lemma 3.1** (Sherman-Morrison formula). Suppose that \(A\) is an invertible square matrix and \(\mu\) and \(\nu\) are two given vectors. If

\[
1 + \nu'A^{-1}\mu \neq 0,
\]

then the following holds,

\[
(A + \mu\nu')^{-1} = A^{-1} - \frac{A^{-1}\mu\nu'A^{-1}}{1 + \nu'A^{-1}\mu}.
\]

Denote

\[
\begin{align*}
    B_t & \triangleq E[P_t'E][\pi_t][P_t'P_t']E[P_t], \\
    \tilde{B}_t & \triangleq E[q_tP_t'E][\pi_t][P_t'P_t']E[P_t], \\
    \bar{B}_t & \triangleq E[q_tP_t'E][P_t'P_t']E[q_t].
\end{align*}
\]
where $\mathbb{E}^{-1}[P_tP_t']$ is the inverse of matrix $\mathbb{E}[P_tP_t'] (> 0)$. Similar to [5], we have $0 < B_t < 1$ and $\mathbb{E}[q_t^2] - B_t > 0$ for all $t = 0, 1, \cdots, T - 1$.

**Lemma 3.2.** Suppose that $c_1(1 - B_t) + c_2B_t \neq 0$ holds. Then

(i) \[
\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_t] = \frac{1}{c_1(1 - B_t) + c_2B_t} \mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_t];
\]

(ii) \[
\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_tq_t] = \frac{(1 - \frac{c_2}{c_1})\hat{B}_t}{c_1(1 - B_t) + c_2B_t} \mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_tq_t];
\]

(iii) \[
\mathbb{E}[P_t']\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_tq_t] = \frac{\hat{B}_t}{c_1(1 - B_t) + c_2B_t} + \frac{1}{c_1}\hat{B}_t.
\]

**Proof.** (i) Applying Sherman-Morrison formula yields

\[
\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_t] = \left( c_1^{-1}\mathbb{E}^{-1}[P_tP_t'] + c_1^{-1}(c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t']c_1^{-1}\mathbb{E}^{-1}[P_tP_t'] \right)\mathbb{E}[P_t] = \frac{1}{c_1(1 - B_t) + c_2B_t} \mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_t].
\]

(ii) Applying Sherman-Morrison formula yields

\[
\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_tq_t] = \left( c_1^{-1}\mathbb{E}^{-1}[P_tP_t'] + c_1^{-1}(c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t']c_1^{-1}\mathbb{E}^{-1}[P_tP_t'] \right)\mathbb{E}[P_tq_t] = \frac{(1 - \frac{c_2}{c_1})\hat{B}_t}{c_1(1 - B_t) + c_2B_t} \mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_tq_t] + \frac{1}{c_1}\mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_tq_t].
\]

(iii) Applying the above (ii) yields

\[
\mathbb{E}[P_t']\left( c_1\mathbb{E}[P_tP_t'] - (c_1 - c_2)\mathbb{E}[P_t]\mathbb{E}[P_t'] \right)^{-1} \mathbb{E}[P_tq_t] = \frac{(1 - \frac{c_2}{c_1})\hat{B}_t}{c_1(1 - B_t) + c_2B_t} \mathbb{E}^{-1}[P_tP_t']\mathbb{E}[P_tq_t] + \frac{1}{c_1}\hat{B}_t = \frac{\hat{B}_t}{c_1(1 - B_t) + c_2B_t}\hat{B}_t.
\]
(iv) Applying the above (ii) yields
\[
E[q_tP_t'] \left( c_1 E[P_tP_t'] - (c_1 - c_2) E[P_t]E[P_t'] \right)^{-1} E[P_tq_t]
\]
\[
= \frac{(1 - \frac{c_2}{c_1}) \tilde{B}_t}{c_1(1 - B_t) + c_2 B_t} \tilde{B}_t + \frac{1}{c_1} \hat{B}_t
\]
\[
= \frac{1 - \frac{c_2}{c_1}}{c_1(1 - B_t) + c_2 B_t} \tilde{B}_t^2 + \frac{1}{c_1} \hat{B}_t.
\]

Assume that the returns of assets and liability are correlated at every period. For simplicity, we define the following backward recursions for eight deterministic sequences of parameters, \{\xi_t\}, \{\eta_t\}, \{\epsilon_t\}, \{\beta_t\}, \{\zeta_t\}, \{\theta_t\}, \{\delta_t\} and \{\psi_t\}, as

\[
\xi_t = \xi_{t+1}s_t^2(1 - B_t) + \lambda_t,
\]
\[
\eta_t = \eta_{t+1}s_t(E[q_t] - \hat{B}_t) + \lambda_t,
\]
\[
\epsilon_t = \epsilon_{t+1}E[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}} \tilde{B}_t + \lambda_t,
\]
\[
\beta_t = \beta_{t+1}s_t^2 - \frac{\beta_{t+1} s_t^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} - \lambda_t a_t,
\]
\[
\zeta_t = \zeta_{t+1}s_t - \frac{\beta_{t+1}s_t(\zeta_{t+1} B_t + 2\eta_{t+1}(\tilde{B}_t - E[q_t] B_t)E[l_t] + 2\lambda_t a_t E[l_t])}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} + 2\lambda_t a_t E[l_t],
\]
\[
\theta_t = \theta_{t+1}E[q_t] - \frac{\eta_{t+1}(\tilde{B}_t - E[q_t] B_t)}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t},
\]
\[
\delta_t = \delta_{t+1}(E[q_t])^2 + \epsilon_{t+1}(E[q_t^2] - (E[q_t])^2)
\]
\[
- \frac{\eta_{t+1}^2}{\xi_{t+1}} \left( \frac{1 - \frac{\beta_{t+1}}{\xi_{t+1}} \tilde{B}_t^2 - 2E[q_t] B_t + (E[q_t])^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} + \frac{1}{\xi_{t+1}} \hat{B}_t \right) - \lambda_t a_t,
\]
\[
\psi_t = \psi_{t+1} - \frac{1}{4} \frac{\zeta_{t+1}^2}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t},
\]
for \(t = T - 1, T - 2, \ldots, 0\), with terminal conditions
\[
\lambda_T = 0, \ \zeta_T = 1, \ \eta_T = -1, \ \epsilon_T = 1, \ \beta_T = 0, \ \zeta_T = -w, \ \theta_T = w, \ \delta_T = 0, \ \psi_T = 0,
\]
where \(\lambda_0 = 0\).

**Remark 1.** When the returns of assets and liability are uncorrelated, which is to say, \(\tilde{B}_t = E[q_t] B_t, \ \tilde{B}_t = (E[q_t])^2 B_t\), parameters \{\eta_t\}, \{\epsilon_t\}, \{\zeta_t\}, \{\theta_t\} and \{\delta_t\} reduce to

\[
\eta_t = \eta_{t+1}s_t E[q_t](1 - B_t) + \lambda_t,
\]
\[
\epsilon_t = \epsilon_{t+1}E[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}} (E[q_t])^2 B_t + \lambda_t,
\]
\[
\zeta_t = \zeta_{t+1}s_t - \frac{\zeta_{t+1} \beta_{t+1} s_t B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} + 2\lambda_t a_t E[l_t],
\]
\[
\theta_t = \theta_{t+1}E[q_t],
\]
\[
\delta_t = \delta_{t+1}(E[q_t])^2 + \epsilon_{t+1}(E[q_t^2] - (E[q_t])^2) - \lambda_t a_t.
\]
Theorem 3.3. Assume that the returns of assets and liability are correlated at every period. Then, the optimal strategy of problem (3) is given by

\[
\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}'_t] \mathbb{E}[\mathbf{P}_t] \left( s_t x_t - \frac{(\xi_{t+1} - \beta_{t+1})(1 - B_t)}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} s_t \mathbb{E}[x_t] + \frac{1}{2} \xi_{t+1} + \eta_{t+1} \left( \frac{(1 - \beta_{t+1}) B_t - \mathbb{E}[q_t] \mathbb{E}[l_t]}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \right) \right) - \frac{\eta_{t+1}}{\xi_{t+1}} \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}'_t] \mathbb{E}[\mathbf{P}_t q_t] l_t,
\]

where

\[
\mathbb{E}[x_t] = x_0 \prod_{j=0}^{t-1} \frac{\xi_{j+1}(1 - B_j) s_j}{\xi_{j+1}(1 - B_j) + \beta_{j+1} B_j} - \sum_{k=0}^{t-1} \left( \prod_{j=k+1}^{t-1} \frac{\xi_{j+1}(1 - B_j) s_j}{\xi_{j+1}(1 - B_j) + \beta_{j+1} B_j} \right)
\times \frac{1}{2} \xi_{k+1} B_k + \eta_{k+1} \left( \frac{B_k - \mathbb{E}[q_k] B_k}{\xi_{k+1}(1 - B_k) + \beta_{k+1} B_k} \right) \right) (\prod_{j=0}^{k-1} \mathbb{E}[q_j] l_0)
\]

for \( t = 0, 1, \cdots, T - 1 \).

Proof. We prove the main results by dynamic programming approach. For the information set \( \mathcal{F}_t \), the cost-to-go functional at period \( t \) is computed by

\[
J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t])
= \min_{\{\pi_t \in \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}} \mathbb{E} \left[ J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]) | \mathcal{F}_t \right]
+ \lambda_t (x_t - l_t - \mathbb{E}[x_t - l_t])^2 - \lambda_t a_t (\mathbb{E}[x_t - l_t])^2.
\]

The cost-to-go functional at terminal time \( T \) is

\[
J_T(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T])
= (x_T - l_T - \mathbb{E}[x_T - l_T])^2 - w \mathbb{E}[x_T - l_T]
= \xi_T (x_T - \mathbb{E}[x_T])^2 + 2 \eta_T (l_T - \mathbb{E}[l_T]) (x_T - \mathbb{E}[x_T]) + \epsilon_T (l_T - \mathbb{E}[l_T])^2 + \beta_T \mathbb{E}[x_T] + \theta_T \mathbb{E}[l_T] + \delta_T (\mathbb{E}[l_T])^2 + \psi_T.
\]

Assume that the cost-to-go functional at time \( t + 1 \) is the following expression

\[
J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])
= \xi_{t+1} (x_{t+1} - \mathbb{E}[x_{t+1}])^2 + 2 \eta_{t+1} (l_{t+1} - \mathbb{E}[l_{t+1}]) (x_{t+1} - \mathbb{E}[x_{t+1}])
+ \epsilon_{t+1} (l_{t+1} - \mathbb{E}[l_{t+1}])^2 + \beta_{t+1} (\mathbb{E}[x_{t+1}])^2 + \delta_{t+1} (\mathbb{E}[l_{t+1}])^2 + \psi_{t+1}.
\]

We prove that the above statement still holds at time \( t \). For given information set \( \mathcal{F}_t \), i.e., knowing \( x_t - \mathbb{E}[x_t], \mathbb{E}[x_t], l_t - \mathbb{E}[l_t] \) and \( \mathbb{E}[l_t] \), we have
\[ \mathbb{E} [J_{t+1} \left( \mathbb{E} [x_{t+1}, x_{t+1} - \mathbb{E} [x_{t+1}], \mathbb{E} [l_{t+1}, l_{t+1} - \mathbb{E} [l_{t+1}]] \right) | \mathcal{F}_t] \]

\[ = \mathbb{E} \left[ \xi_{t+1} \left[ s_t (x_t - \mathbb{E} [x_t]) + \mathbb{P}'_t (\pi_t - \mathbb{E} [\pi_t]) + (\mathbb{P}'_t - \mathbb{E} [\mathbb{P}'_t]) \mathbb{E} [\pi_t] \right]^2 \right] \]

\[ + 2 \eta_{t+1} \left[ q_t (l_t - \mathbb{E} [l_t]) + (q_t - \mathbb{E} [q_t]) \mathbb{E} [l_t] \right] \times \left[ s_t (x_t - \mathbb{E} [x_t]) + \mathbb{P}'_t (\pi_t - \mathbb{E} [\pi_t]) + (\mathbb{P}'_t - \mathbb{E} [\mathbb{P}'_t]) \mathbb{E} [\pi_t] \right] \]

\[ + \epsilon_{t+1} \left[ q_t (l_t - \mathbb{E} [l_t]) + (q_t - \mathbb{E} [q_t]) \mathbb{E} [l_t] \right]^2 + \beta_{t+1} (s_t \mathbb{E} [x_t] + \mathbb{E} [\mathbb{P}'_t] \mathbb{E} [\pi_t]) \]

\[ + \zeta_{t+1} (s_t \mathbb{E} [x_t] + \mathbb{E} [\mathbb{P}'_t] \mathbb{E} [\pi_t]) + \theta_{t+1} \mathbb{E} [q_t] \mathbb{E} [l_t] + \delta_{t+1} (\mathbb{E} [q_t] \mathbb{E} [l_t])^2 + \psi_{t+1} \mathcal{F}_t \]

\[ = \xi_{t+1} \left[ s_t^2 (x_t - \mathbb{E} [x_t]) \right]^2 + (\pi_t - \mathbb{E} [\pi_t])' \mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] (\pi_t - \mathbb{E} [\pi_t]) \]

\[ + 2 s_t (x_t - \mathbb{E} [x_t]) \mathbb{E} [\mathbb{P}'_t] (\pi_t - \mathbb{E} [\pi_t]) + (\mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] - \mathbb{E} [\mathbb{P}_t] \mathbb{E} [\mathbb{P}'_t]) \mathbb{E} [\pi_t] \]

\[ + 2 (\pi_t - \mathbb{E} [\pi_t])' \mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] (\pi_t - \mathbb{E} [\pi_t]) \]

\[ + 2 \eta_{t+1} \left[ s_t \mathbb{E} [q_t] (l_t - \mathbb{E} [l_t]) (x_t - \mathbb{E} [x_t]) + \mathbb{E} [q_t \mathbb{P}'_t] (l_t - \mathbb{E} [l_t]) (\pi_t - \mathbb{E} [\pi_t]) \]

\[ + (\mathbb{E} [q_t \mathbb{P}'_t] - \mathbb{E} [q_t] \mathbb{E} [\mathbb{P}'_t]) \left( \mathbb{E} [l_t] (\pi_t - \mathbb{E} [\pi_t]) + (l_t - \mathbb{E} [l_t]) \mathbb{E} [\pi_t] + \mathbb{E} [l_t] \mathbb{E} [\pi_t] \right) \]

\[ + \epsilon_{t+1} \left[ q_t^2 (l_t - \mathbb{E} [l_t])^2 + 2 (\mathbb{E} [q_t^2] - (\mathbb{E} [q_t])^2) (l_t - \mathbb{E} [l_t]) \mathbb{E} [l_t] \right] \]

\[ + (\mathbb{E} [\pi_t])' \mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] \mathbb{E} [\pi_t] + \zeta_{t+1} (s_t \mathbb{E} [x_t] + \mathbb{E} [\mathbb{P}'_t] \mathbb{E} [\pi_t]) + \theta_{t+1} \mathbb{E} [q_t] \mathbb{E} [l_t] \]

\[ + \delta_{t+1} (\mathbb{E} [q_t] \mathbb{E} [l_t])^2 + \psi_{t+1} \]

Since any admissible strategy of \((\mathbb{E} [\pi_t], \pi_t - \mathbb{E} [\pi_t])\) satisfies \(\mathbb{E} [\pi_t - \mathbb{E} [\pi_t]] = 0\) and \(\mathbb{E} [l_t - \mathbb{E} [l_t]] = 0\), we have

\[ \mathbb{E} \left[ (\pi_t - \mathbb{E} [\pi_t])' (\mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] - \mathbb{E} [\mathbb{P}_t] \mathbb{E} [\mathbb{P}'_t]) \mathbb{E} [\pi_t] \right] = 0, \]

\[ \mathbb{E} \left[ \mathbb{E} [q_t \mathbb{P}'_t] - \mathbb{E} [q_t] \mathbb{E} [\mathbb{P}'_t] \right] \mathbb{E} [l_t] \left( \pi_t - \mathbb{E} [\pi_t] \right) = 0, \]

\[ \mathbb{E} \left[ \mathbb{E} [q_t \mathbb{P}'_t] - \mathbb{E} [q_t] \mathbb{E} [\mathbb{P}'_t] \right] \left( l_t - \mathbb{E} [l_t] \right) \mathbb{E} [\pi_t] = 0, \]

\[ \mathbb{E} \left[ (\mathbb{E} [q_t^2] - (\mathbb{E} [q_t])^2) (l_t - \mathbb{E} [l_t]) \mathbb{E} [l_t] \right] = 0. \]

We first identify optimal \((\mathbb{E} [\pi_t^*], \pi_t^* - \mathbb{E} [\pi_t^*])\) by minimizing the following equivalent cost functional,

\[ \mathbb{E} [J_{t+1} \left( \mathbb{E} [x_{t+1}, x_{t+1} - \mathbb{E} [x_{t+1}], l_{t+1} - \mathbb{E} [l_{t+1}]] \right) | \mathcal{F}_t] \]

\[ = \xi_{t+1} \left[ s_t^2 (x_t - \mathbb{E} [x_t]) \right]^2 + (\pi_t - \mathbb{E} [\pi_t])' \mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] (\pi_t - \mathbb{E} [\pi_t]) \]

\[ + 2 s_t (x_t - \mathbb{E} [x_t]) \mathbb{E} [\mathbb{P}'_t] (\pi_t - \mathbb{E} [\pi_t]) + (\mathbb{E} [\mathbb{P}_t \mathbb{P}'_t] - \mathbb{E} [\mathbb{P}_t] \mathbb{E} [\mathbb{P}'_t]) \mathbb{E} [\pi_t] \]

\[ + 2 \eta_{t+1} \left[ s_t \mathbb{E} [q_t] (l_t - \mathbb{E} [l_t]) (x_t - \mathbb{E} [x_t]) + \mathbb{E} [q_t \mathbb{P}'_t] (l_t - \mathbb{E} [l_t]) (\pi_t - \mathbb{E} [\pi_t]) \right] \]
\[
\begin{align*}
&+ \epsilon_{t+1} \left[ E[q_t^2] (t_t - E[l_t])^2 + (E[q_t^2] - (E[q_t])^2) (E[l_t])^2 \right] \\
&+ \beta_{t+1} \left[ s_t^2 (E[x_t])^2 + 2 s_t E[x_t] E[P_t^i] E[\pi_t] \right]
+ E[\pi_t^i] E[P_t^i] E[\pi_t] \right]
+ \zeta_{t+1} (s_t E[x_t] + E[P_t^i] E[\pi_t]) + \theta_{t+1} E[q_t] E[l_t] + \delta_{t+1} (E[q_t] E[l_t])^2 + \psi_{t+1} \\
&= \xi_{t+1} \left[ s_t^2 (x_t - E[x_t])^2 + (E[x_t] - E[\pi_t])^2 E[P_t^i] (\pi_t - E[\pi_t]) \right]
+ 2 s_t (x_t - E[x_t]) E[P_t^i] (\pi_t - E[\pi_t]) \right) + E[\pi_t^i] (E[P_t^i] E[\pi_t] \right]
\frac{\eta_{t+1}}{\xi_{t+1}} E^{-1} [P_t^i] E[P_t^i] E[\pi_t] E[l_t] \right] \\
&+ \epsilon_{t+1} \left[ E[q_t^2] (E[l_t])^2 + (E[q_t^2] - (E[q_t])^2) (E[l_t])^2 \right] \\
&+ \beta_{t+1} \left[ s_t^2 (E[x_t])^2 + 2 s_t E[x_t] E[P_t^i] E[\pi_t] \right]
+ \zeta_{t+1} (s_t E[x_t] + E[P_t^i] E[\pi_t]) \\
&+ \theta_{t+1} E[q_t] E[l_t] + \delta_{t+1} (E[q_t] E[l_t])^2 + \psi_{t+1}.
\end{align*}
\]

It is easy to see that \( \pi_t^* - E[\pi_t^*] \) can be expressed by the linear form of states and their expected states, and \( E[\pi_t^*] \) can be constructed by the linear form of the expected states, i.e.,

\[
\pi_t^* - E[\pi_t^*] = - E^{-1} [P_t^i] E[P_t^i] s_t (x_t - E[x_t]) - \frac{\eta_{t+1}}{\xi_{t+1}} E^{-1} [P_t^i] E[P_t^i] E[\pi_t] E[l_t],
\]

(9)

\[
\begin{align*}
E[\pi_t] &= - (\xi_{t+1} E[P_t^i] - (\xi_{t+1} - \beta_{t+1}) E[P_t^i] E[P_t^i])^{-1} \left( \beta_{t+1} s_t E[x_t] E[P_t] \right) \\
&\quad + \frac{1}{2} \zeta_{t+1} E[P_t] + \eta_{t+1} (E[P_t E[q_t] E[\pi_t]) E[l_t]) \\
&= - E^{-1} [P_t^i] E[P_t^i] \frac{\beta_{t+1} s_t E[x_t] + \frac{1}{2} \zeta_{t+1} + \eta_{t+1} (1 - \frac{\delta_{t+1}}{\xi_{t+1}}) E[q_t]) E[l_t]}{\xi_{t+1} (1 - E[l_t]) + \beta_{t+1} E[l_t]} \\
&\quad - \frac{\eta_{t+1} E^{-1} [P_t^i] E[P_t^i] E[\pi_t] E[l_t]}{\xi_{t+1}^2}.
\end{align*}
\]

(10)

In order to get the explicit expression of the cost-to-go functional at time \( t \), we substitute \( \pi_t^* - E[\pi_t^*] \) and \( E[\pi_t^*] \) back and derive

\[
\begin{align*}
J_t &= \min_{\{x_t, x_t - E[x_t], E[l_t], l_t - E[l_t]\}} \left\{ E[J_{t+1}(x_{t+1}, x_{t+1} - E[x_{t+1}], l_{t+1}, l_{t+1} - E[l_{t+1}]) | F_t] \\
&\quad + \lambda_t (x_t - l_t - E[x_t] - E[l_t])^2 - \lambda_t \alpha_t (E[x_t] - E[l_t])^2 \\
&= \xi_{t+1} s_t^2 (x_t - E[x_t])^2 + 2 \eta_{t+1} s_t E[q_t] (l_t - E[l_t]) (x_t - E[x_t]) + \delta_{t+1} s_t E[x_t] + \theta_{t+1} E[q_t] E[l_t].
\end{align*}
\]
\[ + \epsilon_{t+1} \left[ \mathbb{E}[q_{t+1}^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_{t+1}] - (\mathbb{E}[q_{t+1}])^2)(\mathbb{E}[l_t])^2 \right] + \delta_{t+1}(\mathbb{E}[q_{t+1}l_t])^2 + \psi_{t+1} \]

\[ - \xi_{t+1} \left[ - \mathbb{E}[P_t, S_{t+1}](x_t - \mathbb{E}[x_t]) - \frac{n_{t+1}}{\xi_{t+1}} \mathbb{E}[P_t, q_{t+1}](l_t - \mathbb{E}[l_t]) \right] \]

\[ \times \mathbb{E}^{-1}[P_t, P_t'] \left[ - \mathbb{E}[P_t, S_t](x_t - \mathbb{E}[x_t]) - \frac{n_{t+1}}{\xi_{t+1}} \mathbb{E}[P_t, q_{t+1}](l_t - \mathbb{E}[l_t]) \right] \]

\[ - \left[ \beta_{t+1} s_t \mathbb{E}[x_t] \mathbb{E}[P_t] + \frac{1}{2} \zeta_{t+1} \mathbb{E}[P_t] + \eta_{t+1} (\mathbb{E}[P_t, q_t] - \mathbb{E}[q_t]) \mathbb{E}[P_t] \right] \]

\[ \times (\xi_{t+1} \mathbb{E}[P_t, P_t'] - (\xi_{t+1} - \beta_{t+1}) \mathbb{E}[P_t] \mathbb{E}[P_t'])^{-1} \]

\[ \times \left[ \beta_{t+1} s_t \mathbb{E}[x_t] \mathbb{E}[P_t] + \frac{1}{2} \zeta_{t+1} \mathbb{E}[P_t] + \eta_{t+1} (\mathbb{E}[P_t, q_t] - \mathbb{E}[q_t]) \mathbb{E}[P_t] \right] \]

\[ + \lambda_t (x_t - l_t - \mathbb{E}[x_t - l_t])^2 - \lambda_t a_t (\mathbb{E}[x_t - l_t])^2 \]

\[ = \xi_{t+1} s_t^2 (1 - B_t)(x_t - \mathbb{E}[x_t])^2 + 2n_{t+1}s_t(\mathbb{E}[q_t] - \bar{B}_t)(l_t - \mathbb{E}[l_t])(x_t - \mathbb{E}[x_t]) \]

\[ + \left( \epsilon_{t+1} \mathbb{E}[q_{t+1}^2] - \frac{n_{t+1}^2}{\xi_{t+1}} \bar{B}_t \right) (l_t - \mathbb{E}[l_t])^2 \]

\[ + \left( \beta_{t+1} - \frac{\beta_{t+1}^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \right) s_t^2 \mathbb{E}[x_t]^2 \]

\[ + \left( \zeta_{t+1} + \beta_{t+1} \frac{\zeta_{t+1} B_t + 2n_{t+1}(\bar{B}_t - \mathbb{E}[q_t] B_t) \mathbb{E}[l_t]}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \right) s_t \mathbb{E}[x_t] \]

\[ + \left( \theta_{t+1} \mathbb{E}[q_t] - \zeta_{t+1} \frac{n_{t+1}(\bar{B}_t - \mathbb{E}[q_t] B_t)}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \right) \mathbb{E}[l_t] \]

\[ + \left[ \epsilon_{t+1}(\mathbb{E}[q_{t+1}^2] - (\mathbb{E}[q_{t+1}])^2) + \delta_{t+1}(\mathbb{E}[q_{t+1}])^2 \right] \]

\[ - \eta_{t+1}^2 \left( \frac{1 - \beta_{t+1}}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \bar{B}_t - 2 \mathbb{E}[q_t] \bar{B}_t + (\mathbb{E}[q_t])^2 B_t \right) + \frac{1}{\xi_{t+1}} \bar{B}_t \right) \mathbb{E}[l_t]^2 \]

\[ + \psi_{t+1} - \frac{1}{4} \xi_{t+1}(1 - B_t) + \beta_{t+1} B_t + \lambda_t (x_t - l_t - \mathbb{E}[x_t - l_t])^2 \]

\[ - \lambda_t a_t (\mathbb{E}[x_t - l_t])^2 \].

Also,

\[ J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], l_t - \mathbb{E}[l_t]) \]

\[ = \left( \xi_{t+1} s_t^2 (1 - B_t) + \lambda_t \right) (x_t - \mathbb{E}[x_t])^2 \]

\[ + 2 \left( n_{t+1}s_t(\mathbb{E}[q_t] - \bar{B}_t) + \lambda_t \right) (l_t - \mathbb{E}[l_t])(x_t - \mathbb{E}[x_t]) \]

\[ + \left( \epsilon_{t+1} \mathbb{E}[q_{t+1}^2] - \frac{n_{t+1}^2}{\xi_{t+1}} \bar{B}_t + \lambda_t \right) (l_t - \mathbb{E}[l_t])^2 \]

\[ + \left( \beta_{t+1}s_t^2 - \frac{\beta_{t+1} \xi_{t+1}(1 - B_t) + \beta_{t+1} B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} - \lambda_t a_t \right) (\mathbb{E}[x_t])^2 \]

\[ + \left[ \left( \xi_{t+1} s_t^2 - \beta_{t+1} s_t \frac{\xi_{t+1} B_t + 2n_{t+1}(\bar{B}_t - \mathbb{E}[q_t] B_t) \mathbb{E}[l_t]}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} + 2 \lambda_t a_t \mathbb{E}[l_t] \right) \mathbb{E}[x_t] \right] \]
+ \left( \theta_{t+1} \mathbb{E}[q_t] - \xi_{t+1} \eta_{t+1} \left( \bar{B}_t - \mathbb{E}[q_t] \bar{B}_t \right) / \xi_{t+1} (1 - B_t) + \beta_{t+1} B_t \right) \mathbb{E}[l_t]

+ \left[ \delta_{t+1} (\mathbb{E}[q_t])^2 + \epsilon_{t+1} (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2) \right]

- \eta_{t+1}^2 \left( \frac{(1 - \beta_{t+1} / \xi_{t+1}) \bar{B}_t^2 - 2 \mathbb{E}[q_t] \bar{B}_t + (\mathbb{E}[q_t])^2 B_t}{\xi_{t+1} (1 - B_t) + \beta_{t+1} B_t} + \frac{1}{\xi_{t+1}} \bar{B}_t \right) - \lambda_t a_t \right) (\mathbb{E}[l_t])^2

+ \psi_{t+1} - \frac{1}{4} \xi_{t+1} (1 - B_t) + \beta_{t+1} B_t

= \xi_t \left( x_t - \mathbb{E}[x_t] \right)^2 + 2 \eta_t \left( l_t - \mathbb{E}[l_t] \right) \left( x_t - \mathbb{E}[x_t] \right) + \epsilon_t \left( l_t - \mathbb{E}[l_t] \right)^2

+ \beta_t (\mathbb{E}[x_t])^2 + \zeta_t \mathbb{E}[x_t] + \theta_t \mathbb{E}[l_t] + \delta_t (\mathbb{E}[l_t])^2 + \psi_t.

Substituting $\mathbb{E}[\pi^*_t]$ to dynamics of $\mathbb{E}[x_t]$ in (4) yields

$$\mathbb{E}[x_{t+1}] = \frac{\xi_{t+1} (1 - B_t) \bar{s}_t}{\xi_{t+1} (1 - B_t) + \beta_{t+1} B_t} \mathbb{E}[x_t] - \frac{1}{2} \xi_{t+1} B_t + \frac{\eta_{t+1} (\bar{B}_t - \mathbb{E}[q_t] \bar{B}_t) \mathbb{E}[l_t]}{\xi_{t+1} (1 - B_t) + \beta_{t+1} B_t},$$

which implies

$$\mathbb{E}[x_t] = x_0 \prod_{j=0}^{t-1} \frac{\xi_{j+1} (1 - B_j) s_j}{\xi_{j+1} (1 - B_j) + \beta_{j+1} B_j} - \sum_{k=0}^{t-1} \left( \prod_{j=k+1}^{t-1} \frac{\xi_{j+1} (1 - B_j) s_j}{\xi_{j+1} (1 - B_j) + \beta_{j+1} B_j} \right) \frac{1}{\xi_{k+1} (1 - B_k) + \beta_{k+1} B_k} \mathbb{E}[q_k] \mathbb{E}[l_0].$$

Hence, combining with (9) and (10), we derive the desired result (7).

Finally, we show that this optimal strategy satisfies the linear constraints. At time 0, $\mathbb{E}[\pi^*_0] = 0$ is obvious due to $x_0 = \mathbb{E}[x_0]$ and $l_0 = \mathbb{E}[l_0]$. Then, according to the dynamic system of (5), we have $\mathbb{E}[x_1 - \mathbb{E}[x_1]] = 0$ and $\mathbb{E}[l_1 - \mathbb{E}[l_1]] = 0$, which further implies $\mathbb{E}[\pi^*_t - \mathbb{E}[\pi^*_t]] = 0$. Repeating this argument, we have $\mathbb{E}[\pi^*_t - \mathbb{E}[\pi^*_t]] = 0$ holds for all $t$. 

We can simply reformulate the optimal strategy (7) to the following form:

$$\pi^*_t = \bar{c}_t x_t + \bar{c}_t l_t + c_t,$$  

where

$$\bar{c} = -s_t \mathbb{E}^{-1}[\mathbb{P}_t \mathbb{P}_t^t] \mathbb{E}[\mathbb{P}_t],$$

$$\bar{c}_t = -\frac{\eta_{t+1} \mathbb{E}^{-1}[\mathbb{P}_t \mathbb{P}_t^t] \mathbb{E}[\mathbb{P}_t q_t]}{\xi_{t+1}},$$

$$c_t = \frac{-\mathbb{E}^{-1}[\mathbb{P}_t \mathbb{P}_t^t] \mathbb{E}[\mathbb{P}_t]}{\xi_{t+1} (1 - B_t) + \beta_{t+1} B_t} \times \left( (\xi_{t+1} - \beta_{t+1}) (1 - B_t) s_t \mathbb{E}[x_t] + \frac{1}{2} \xi_{t+1} + \eta_{t+1} (1 - \beta_{t+1} / \xi_{t+1}) \bar{B}_t - \mathbb{E}[q_t] \mathbb{E}[l_t] \right).$$

It is obviously that the derived analytical optimal portfolio policy consists of three terms: the investor’s current wealth, current liability and risk attitude specified by $w$ and $a_t$, which are also a function of the initial wealth $x_0$ and the initial liability $l_0$. In other words, it is of a feedback form, but not Markovian. At each period $t$,
Remark 2. When the returns of assets and liability are not correlated, \[
\pi^*_t = -\mathbb{E}^{-1}[P_t P_t']\mathbb{E}[P_t] \left( s_t x_t - \frac{(\xi_{t+1} - \beta_{t+1})(1 - B_t)s_t E[x_t] + \frac{1}{2}\zeta_{t+1}}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t} \right.
- \frac{\eta_{t+1}}{\xi_{t+1}} E[q_t] (l_t - E[l_t]))
\]
where \[
E[x_t] = x_0 \prod_{j=0}^{t-1} \frac{\xi_{j+1} (1 - B_j)s_j}{\xi_{j+1}(1 - B_j) + \beta_{j+1}B_j}
- \frac{1}{2} \sum_{k=0}^{t-1} \left( \sum_{j=k+1}^{t-1} \frac{\xi_{j+1} (1 - B_j)s_j}{\xi_{j+1}(1 - B_j) + \beta_{j+1}B_j} \right) \frac{\zeta_{k+1}B_k}{\xi_{k+1}(1 - B_k) + \beta_{k+1}B_k}.
\]

Based on the proof of Theorem 3.3, the optimal objective of problem (3) is as follows,
\[
J_0(E[x_0], 0, E[l_0], 0) = \beta_0 x_0^2 + \zeta_0 x_0 + \theta_0 l_0 + \delta_0 l_0^2 + \psi_0.
\] (12)

In fact, \(J_0(\cdot)\) is convex in \(\lambda\). From its explicit form, we can find optimal Lagrangian multiplier vector \(\lambda^*\) simply by steepest descent algorithm or interior point algorithm directly in the code via MATLAB. Compared with Li-Li [10] using the embedding scheme, they could not obtain the optimal objective value function \(J_0(\cdot)\) analytically. Then they proposed the prime-dual iterative algorithm to search for the optimal Lagrangian multiplier vector \(\lambda^*\): First, for a given \(\omega\), a system of linear equations were solved to get the embedding parameter vector \(\lambda(\omega)\), then the optimal policy of Lagrangian problem and a feasible decent direction were computed. Finally, a line search along the feasible decent direction was carried out to determine the optimal step-size. The inexplicit optimal objective function might possibly involve some computational errors, resulting in even inaccurate results. By the mean-field formulation, we can derive directly the analytical optimal policy. It is an efficient way to solve the asset-liability problem under mean-variance framework with probability constraints. When there is no liability, Theorem 3.3 reduces to Proposition 2 in Cui-Li-Li [5]. According to (12), we can also derive the variance term as
\[
\text{Var}(x_T - l_T) = \max_{\lambda \in \mathbb{R}^n_{\text{l}}^{-1}} J_0(x_0, 0, l_0, 0) + wE[x_T - l_T].
\]

Theorem 3.4. Assume that the returns of assets and liability are correlated at every period. Then the efficient frontier of problem (3) is given by
\[
\text{Var}(x_T - l_T) = \max_{\lambda \in \mathbb{R}^n_{\text{l}}^{-1}} J_0(x_0, 0, l_0, 0) + wE[x_T - l_T] \geq \zeta_0 x_0 - \theta_0 l_0.
\] (13)

4. Numerical examples. We consider an example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of U.S market and a bank account. Based on the data provided in Elton-Gruber-Brown-Goetzmann [6], Table 1 presents the expected values, standard deviations and correlation coefficients of the annual return rates of these three indices.
Table 1. Data for the asset allocation example

|                | SP   | EM   | MS   | liability |
|----------------|------|------|------|-----------|
| Expected return| 14%  | 16%  | 17%  | 10%       |
| Standard deviation | 18.5% | 30%  | 24%  | 20%       |

Correlation coefficient

|                | SP  | EM  | MS  | liability |
|----------------|-----|-----|-----|-----------|
| SP             | 1   | 0.64| 0.79| \( \rho_1 \) |
| EM             | 0.64| 1   | 0.75| \( \rho_2 \) |
| MS             | 0.79| 0.75| 1   | \( \rho_3 \) |
| liability      | \( \rho_1 \) | \( \rho_2 \) | \( \rho_3 \) | 1         |

Thus, for any time \( t \), we have

\[
E[P_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \quad \text{Cov}(P_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},
\]

\[
E[P_t q_t] = \begin{pmatrix} 0.0423 \\ 0.0454 \\ 0.0459 \end{pmatrix}, \quad \text{Cov}(q_t, P_t) = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.
\]

We consider 5 time periods and annual risk free rate 5% \((r_t = 0.05)\). Assume that the investor has initial wealth \( x_0 = 3 \), initial liability \( l_0 = 1 \), a trade-off parameter \( w = 1 \). Furthermore, for \( t = 0, 1, 2, 3, 4 \), assume that the probability \( a_t = 0.1 \), the correlation of assets and the liability is \( \rho = (\rho_1, \rho_2, \rho_3) \), where

\[
\rho_i = \frac{\text{Cov}(q_t, P'_i)}{\sqrt{\text{Var}(q_t) \cdot \text{Var}(P'_i)}}
\]

is the correlation coefficient of the \( i \)-th asset and the liability. This means

\[
E[P'_t q_t] = E[q_t]E[P'_t] + \rho_i \sqrt{\text{Var}(q_t)} \sqrt{\text{Var}(P'_t)}.
\]

4.1. Correlation example. In this subsection, assume that the returns of the assets and liability are correlated with \( \rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25) \). Hence,

\[
\text{Cov} \left( \begin{pmatrix} P_t \\ q_t \end{pmatrix} \right) = \begin{pmatrix} \text{Cov}(P_t) & \text{Cov}(q_t, P'_t) \\ \text{Cov}(q_t, P'_t) & \text{Var}(q_t) \end{pmatrix} = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix} \succ 0.
\]

Using the above formula of \( E[P'_t q_t] \), we have \( E[P'_t q_t] = (0.0898, 0.1510, 0.1440)' \). Moreover,

\[
K_1 = E^{-1}[P'_t P'_t]E[P_t] = \begin{pmatrix} 1.0580 \\ -0.1207 \\ 1.1052 \end{pmatrix}, \quad K_2 = E^{-1}[P'_t P'_t]E[P'_t q_t] = \begin{pmatrix} -0.2398 \\ 0.4374 \\ 1.7446 \end{pmatrix}.
\]
By interior point algorithm of “fmincon” with the initial point \( \lambda = (0, 0, 0, 0) \), we can obtain \( \lambda^* = (0, 0, 0, 0.4902) \). According to Theorem 3.3, we can derive the optimal strategy of problem (3) as follows,

\[
\begin{align*}
\pi_0^* &= -1.05(x_0 - 3.6997)K_1 + 0.3520K_2l_0, \\
\pi_1^* &= -1.05(x_1 - 3.8847)K_1 + 0.3360K_2l_1, \\
\pi_2^* &= -1.05(x_2 - 4.0789)K_1 + 0.3207K_2l_2, \\
\pi_3^* &= -1.05(x_3 - 4.2829)K_1 + 0.3060K_2l_3, \\
\pi_4^* &= -1.05(x_4 - 3.5243)K_1 + 1.0000K_2l_4.
\end{align*}
\]

The optimal expected surplus levels are \( E(x_5 - l_5) = 3.2005 \) and \( \text{Var}(x_5 - l_5) = 0.5740 \), respectively.

4.2. Uncorrelation example. In this subsection, assume that the returns of the assets and liability are uncorrelated. Hence,

\[
\text{Cov} \left( \begin{pmatrix} P_t \\ q_t \end{pmatrix} \right) = \begin{pmatrix}
\text{Cov}(P_t) & \text{Cov}(q_t, P_t) \\
\text{Cov}(q_t, P_t') & \text{Var}(q_t)
\end{pmatrix}
= \begin{pmatrix}
0.0342 & 0.0355 & 0.0351 & 0 \\
0.0355 & 0.0900 & 0.0540 & 0 \\
0.0351 & 0.0540 & 0.0576 & 0 \\
0 & 0 & 0 & 0.04
\end{pmatrix} \succ 0,
\]

and parameters \( \{\xi_t\}, \{\eta_t\}, \{\epsilon_t\}, \{\beta_t\}, \{\zeta_t\}, \{\theta_t\}, \{\delta_t\} \) and \( \{\psi_t\} \) are defined in Remark 1.

By interior point algorithm of “fmincon” with the initial point \( \lambda = (0, 0, 0, 0) \), we can obtain \( \lambda^* = (0, 0, 0, 0.1775) \). According to Remark 2, the optimal strategy of problem (3) is specified as follows,

\[
\begin{align*}
\pi_0^* &= -1.05(x_0 - 3.3587 + 0.7658l_0)K_1, \\
\pi_1^* &= -1.05(x_1 - 3.5267 + 0.7310l_1)K_1, \\
\pi_2^* &= -1.05(x_2 - 3.7030 + 0.6977l_2)K_1, \\
\pi_3^* &= -1.05(x_3 - 3.8882 + 0.6660l_3)K_1, \\
\pi_4^* &= -1.05(x_4 - 3.6231 + 0.9524l_4)K_1.
\end{align*}
\]

The mean and variance of the final optimal surplus are \( E(x_5 - l_5) = 3.3042 \) and \( \text{Var}(x_5 - l_5) = 0.8157 \), respectively.

It is showed from the two examples that given the first and second moments of the return rates of assets and liability, the trade-off parameter and the probability, the initial wealth and liability, we can derive directly the optimal policy according Theorem 3.3 and Remark 2. By changing the parameters \( a_t \), we can see the impact of the probability on the optimal strategy and efficient frontier which is similar as Li-Li [10]. Given the same variance, the investors with the probability control of wealth less than liability obtain smaller expected terminal surplus compared to those without.

4.3. The impact of correlation coefficient. In this example, we investigate the impact of correlation coefficient of the assets and liability which has not been mentioned in most papers. The efficient frontiers in Figure 1 is computed for \( \omega \) from 0.1 to 1 with a step size 0.01. As it shows, the higher the correlation is, the
better the efficient frontier. It is straightforward. Consider a single period setting and assume that the investor wants to achieve some preset expected return, which is larger than the riskfree return. Due to the linearity of expectation, the investor may choose a fixed long position on risky asset no matter the value of $\rho$. The variance term is computed as $\text{Var}(x_1 - l_1) = \text{Var}(x_1) + \text{Var}(l_1) - 2\text{Cov}(x_1, l_1)$. When the correlation coefficient is higher, the larger the covariance, the smaller the variance. Therefore, the efficient frontier of higher $\rho$ is better.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Efficient frontiers with different correlation coefficients}
\end{figure}

5. **Conclusion.** In this paper, we employ a mean-field formulation to solve an optimal multi-period asset-liability mean-variance portfolio selection with probability constraints. Compared with using embedding scheme to deal with the nonseparable problem, we do not need to construct a system of auxiliary problems and can strictly derive its analytical optimal strategy and efficient frontier. The numerical examples show how to use the theoretical results, and the impact of the correlation of asset and liability on the efficient frontier.

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