Asteroseismology of the double-radial mode $\delta$ Scuti star BP Pegasi

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ABSTRACT

Using the ASAS data, we determine the pulsational frequencies of the high-amplitude $\delta$ Sct star BP Pegasi. The analysis revealed only the two known, independent frequencies that we use to perform the seismic analysis of the star. On the basis of the photometric and spectroscopic datasets, we independently find that both frequencies can only be associated with radial modes which, according to the frequency ratio, are fundamental and first overtone modes. The models fitting the two frequencies depend strongly on the model atmosphere parameters. For low values of the mixing length parameter $\alpha_{\text{MLT}} \approx 0.5$, only the OPAL seismic models in the post-main sequence phase of evolution are caught within the observed error box. Seismic models computed with the OPAL and OPLIB data are much less luminous and cooler. They can only reach the error box if we increase the convection efficiency to at least $\alpha_{\text{MLT}} = 2.0$. Then, including the non-adiabatic parameter $f$ into our modelling, we constrain the value of $\alpha_{\text{MLT}}$. Computing an extensive grid of seismic OPAL models and employing Monte Carlo-based Bayesian analysis, we obtain constraints on various parameters. In particular, the efficiency of envelope convection can be parametrized by $\alpha_{\text{MLT}} \in (0.5, 1.0)$, depending on the atmospheric microturbulent velocity that amounts to $\xi_t = 4$ or $8 \text{ km s}^{-1}$.

Key words: stars: evolution – stars: oscillation – stars: convection – atomic data: opacities – stars: individual: BP Pegasi

1 INTRODUCTION

High-amplitude $\delta$ Scuti (HADS) stars are pulsating variables with the V-band magnitude ranging from 0.3 to 0.5 mag and constitute in some aspects a subclass of the $\delta$ Sct pulsators. They are usually in an advanced phase of main-sequence evolution or already in a post-main sequence phase (e.g., Breger 2000). The HADS stars pulsate in just one or two frequencies which are assigned to radial modes. Although, radial pulsations for the HADS stars are most probable because of high amplitudes and the period ratio in case of double-mode pulsators, there is some chance that non-radial modes can be present as well. Therefore, the mode identification should be confirmed by independent observables from photometric and/or spectroscopic time-series observations. Unfortunately, a few such efforts can be found in the literature. For example, Ulusoy et al. (2013) used the $UBVRI$ time-series photometry for the high-amplitude $\delta$ Scuti star V2367 Cyg but they did not get a unique identification of the mode degree $l$ for any of the three detected frequencies. Likewise, Casas et al. (2006) did not obtain unambiguous determination of $l$ from multi-colour diagnostic diagrams for two frequencies of RV Ari. Recently, Daszyńska-Daszkiewicz et al. (2020) applied the method of simultaneous determination of $l$ and the nonadiabatic parameters $f$ for the two frequencies of the prototype SX Phoenicis. They used the amplitudes and phases in the Strömgren photometry and successfully identified the two modes as radial ones.

The detection of two radial modes in any star imposes very strong constraints on its mean parameters and global chemical composition. Despite this, there are not many papers that present detailed seismic modelling of the HADS stars. Petersen & Christensen-Dalsgaard (1996) analyzed pulsational models for a few HADS stars and concluded that these are post-main sequence objects which explains their high amplitudes. On the other hand Breger (2000) suggested that high amplitudes are related to the slow rotation of these stars, which is typically $V \sin i < 40 \text{ km s}^{-1}$. Casas et al. (2006) presented a more advanced modelling for RV Ari including the effects of non-adiabaticity and rotation. Some tentative seismic studies were published for V2367 Cyg by Balona et al. (2012). More recently, Niu et al. (2017) constructed seismic models of the HADS star AE UMa, using the two radial-mode frequencies as well as the period changes of the dominant mode. They obtained that AE UMa is in the post-main sequence stage of evolution.

Xue et al. (2018) made an attempt to construct seismic models for the double-mode HADS star VX Hyd using MESA evolutionary models and the adiabatic pulsation code ADIPLS (Christensen-Dalsgaard 2008). They concluded that only post-main sequence models are suitable for this star. Thus, the seismic modelling of $\delta$ Scuti stars pulsating in two radial modes should be...
definitely intensified and the number of studied stars should be increased.

In our previous paper we presented the complex seismic modelling of the prototype SX Phoenicis (Daszyńska-Daszkiewicz et al. 2020). This analysis consisted of a simultaneous fitting of the two radial-mode frequencies and the corresponding values of the bolometric flux amplitude (the parameter $f$). The effects of various model parameters were investigated. In particular, we showed that seismic models of SX Phe are strongly affected by the choice of opacity data. Our extensive seismic modelling indicated a preference for OPAL tables and the post-main sequence phase of evolution. Besides the important constraints on the efficiency of convection in the envelope of SX Phe, described by the mixing length parameter $\alpha_{MLT} < 0.7$, we also obtained constraints on the microturbulent velocity in the atmosphere $\xi_t \in (4, 8) \ \text{km s}^{-1}$.

Here, we present results for the double-mode HADS star BP Pegasi applying a similar approach as in the case of SX Phe. In Sect. 2 we give the information on BP Peg. Sect. 3 contains the analysis of the ASAS and Strömgren $uvby$ data as well as the frequency determination. Mode identification based on the multicolour photometry for the two frequencies is presented in Sect. 4. In Sect. 5, we give the results of seismic modelling of BP Peg using the three sources of the opacity data and preliminary constraints on the efficiency of envelope convection. Finally, in Sect. 6, we present an extensive seismic modelling with Monte Carlo-based Bayesian analysis. The last section summarizes all our results.

2 BP PEGASI

BP Pegasi is a variable star with the average visual brightness of $V = 12.05$ mag and the Gaia DR2 parallax of $\pi = 0.8848\pm0.361$ mas. We discarded the Gaia EDR3 parallax $\pi = 0.7959\pm0.300$ mas because a quality indicator, the so-called RUWE (the renormalised unit weight error) amounts to 2.1 that is much larger than 1.0. The RUWE quality indicator, the so-called RUWE (the renormalised unit weight error) amounts to 2.1 that is much larger than 1.0. The RUWE significantly greater than 1.0 (usually the limit 1.4 is adopted) could indicate that the source is non-single or otherwise problematic for the astrometric solution (Lindgren et al. 2021).

The spectral type of BP Peg in the Catalogue of SIMBAD is A0, whereas in a more detailed study by Rodriguez et al. (1994) there is given a range A5–F0. The effective temperature of BP Peg was determined by several authors and its values are as follow: $T_{\text{eff}} \approx 7500$ K (Andreasen 1983), $T_{\text{eff}} \in (7130, 8050)$ K (Kim et al. 1989), $T_{\text{eff}} \in (7190, 8100)$ K (Rodriguez et al. 1992) and $T_{\text{eff}} \in (6860, 8000)$ K (Peña et al. 1999). For our studies, we adopted the whole range (6860, 8100) K that converts into log $T_{\text{eff}} = 3.8724\pm0.361$. The atmospheric metallicity is $[\text{m/H}] = -0.02$ according to McNamara (1997). $[\text{m/H}] = -0.08\pm0.05$ according to Kim et al. (1989) whereas Rodriguez et al. (1992) estimated the value of $[\text{m/H}] = +0.2$. These estimates are based on the $\delta$m1 index.

Using the Gaia DR2 parallax, the extinction $A_V = 0.216 \pm 0.012$ from the Bayestar 2019 reddening map (Green et al. 2019) and the bolometric correction from Flower (1996), we arrived at the luminosity range $(1.137, 1.361)$ that is log $L/L_\odot = 1.249(112)$. Using the bolometric correction from Kurucz models and taking into account different values of the atmospheric metallicity $[\text{m/H}]$ and microturbulent velocity $\xi_t$, we derived the luminosity range $(1.127, 1.367)$ that is log $L/L_\odot = 1.247(120)$. This value includes determinations for $[\text{m/H}] = -0.1, 0.0, 0.2, 0.3$ and $\xi_t = 2, 4 \ \text{km s}^{-1}$ and the error in the bolometric correction of about 0.015. For the further analysis, we adopted log $L/L_\odot = 1.247(120)$ because it takes into account all possible uncertainties and includes the range derived with the Flower bolometric correction.

The analysis of spectra by Kim et al. (1989) led them to conclude that the star shows very sharp spectral lines from which they derived an upper limit of the projected rotational velocity, $V \sin i$, of 18 km s$^{-1}$. Thus, BP Pegasi is a slow rotator if the inclination angle is not far from $i = 90^\circ$.

In Fig. 1, we showed the Hertzsprung-Russell diagram with the error box of BP Peg in a comparison with the evolutionary tracks computed for masses $M = 1.8, 1.9, 2.0, 2.1 \ M_\odot$, using the OPAL opacity tables (Iglesias & Rogers 1996), the OPAL2005 equation of state (Rogers et al. 1996; Rogers & Nayfonov 2002) and the solar mixture from Asplund et al. (2009), hereafter AGSS09. We adopted the metallicity $Z = 0.020$, the initial hydrogen abundance $X_0 = 0.70$ and the zero value of the parameter describing overshooting from the convective core $\alpha_{\text{ov}} = 0.0$. Moreover, for the model with a mass $M = 1.9 \ M_\odot$, we depicted also the tracks with $Z = 0.028, X_0 = 0.73$ and $\alpha_{\text{ov}} = 0.2$. The evolution were computed with the Warsaw-New Jersey code (e.g., Pamyatnykh et al. 1998; Pamyatnykh 1999), which takes into account the mean effect of the centrifugal force, assuming a solid-body rotation and constant global angular momentum during evolution. The convective transport in the envelope is treated in the framework of the standard mixing-length theory. The tracks in Fig. 1 were computed with the mixing length parameter $\alpha_{\text{MLT}} = 0.5$ and the initial velocity of rotation $V_{\text{init},0} = 25 \ \text{km s}^{-1}$.

The variability of BP Pegasi was firstly reported by Masani & Broglia (1954). Then, Broglia (1959) showed that it pulsates in two modes with a dominant period of 0.10954337 d. The range of a brightness changes was about 0.45 mag. Moreover, Broglia (1959) found a modulation period on the order of 0.37 d. Based on the visual observations Figer (1983) derived the values of these periods at $P_1 = 0.109543375 \ \text{d}$ and $P_2 = 0.084510 \ \text{d}$. The $uvby\beta$ photometry and the first time series spectroscopy was gath-
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3 DATA ANALYSIS

As we have mentioned in the Introduction Rodriguez et al. (1992) derived nine frequencies from the Broglia (1959) data. We listed them in Table 1. The data of Broglia (1959) consisted of 861 observational points and spanned 127 days which corresponds to the Rayleigh resolution of $1/T = 0.008 \text{d}^{-1}$. BP Peg was also observed in the framework of the All-Sky Automated Survey (ASAS, Pojmanski 2002), specifically in the phase ASAS-3.

Here we analyzed both the Strömgren photometry (Rodriguez et al. 1992) as well as the ASAS-3 V-band photometry. We start from the ASAS data that span 2350 days (see Fig. 2, top panel) which gives the Rayleigh resolution of 0.0004 d$^{-1}$. The ASAS-3 photometry was constructed with five different apertures and each observational point has been assigned a quality flag (the best points have the A flag, the worst - the D flag). We started from removing points with the quality flag C and D. Then, we recalculated amplitude periodograms by means of a discrete Fourier transform (Deeming 1975; Kurtz 1985) for photometric data from two apertures which had the smallest mean errors. Periodograms were calculated with the resolution of 0.000004 d$^{-1}$ up to 40 d$^{-1}$. As a noise level we adopted the mean amplitude in the periodogram calculated for the data before subtraction of a considered frequency. In the periodogram for the data with the second smallest mean error the signal to noise ratio ($S/N$) for the highest amplitude peak was slightly higher. Therefore, we chose this data set for further analysis. However, we note that frequencies from data from both apertures agree within the errors. Moreover, in the residuals we found seven obvious outliers that were also removed from the original data. Finally, we were left with 357 data points. For this cleaned data set we recalculated the periodogram (see Fig. 3, top panel) and performed the standard prewhitening procedure. Although daily aliases are high, the highest peaks in subsequent periodograms are in agreement with frequencies found by Rodriguez et al. (1992) in Broglia (1959) data (compare Table 1 and 2). As a significant frequency peaks we considered those with $S/N > 4$ (Breger 1993, Kuschnig et al. 1997). Periodograms for data after subtraction two and four frequency peaks are shown in Fig. 3 (second and third panel from the top, respectively). The frequency peak seen in the third panel from the top in Fig. 3 is slightly below our significance threshold. However, it agrees with the third harmonic of $v_1$. Therefore we accepted it as a real one. Finally, we detected 5 frequencies: two independent, first and second harmonics of $v_1$ and combination $v_1 + v_2$ (see Table 2). The ASAS data phased with $v_1$ are shown as blue dots in the third panel from the top in Fig 2.

In the next step, we reanalysed Strömgren photometry of Rodriguez et al. (1992). These data consist of 299 observations in each filter that span 365 days that gives the Rayleigh resolution of 0.003 d$^{-1}$. Observations in the $v$ filter are shown in the second panel from the top in Fig. 2. In each Strömgren filter we performed the same periodogram analysis as in the case of ASAS data. However, this time in all four filters we found the highest peak at a frequency of about 10.115 d$^{-1}$, i.e., at the position of a daily alias of $v_1$. As an example we depict the periodogram in the Strömgren $v$ filter (the second panel from the bottom in Fig. 3). In the case of the $u$ data only this frequency (or its aliases) satisfy our significance criterion. In the $v$ and $y$ filters we found the second significant frequency at

| ID | Frequency $[\text{d}^{-1}]$ | $A_V$ [mmag] | $\phi_V$ [rad] |
|----|--------------------------|-------------|-------------|
| $v_1$ | 9.12891 | 190 | 1.929 |
| $2v_1$ | 18.2582 | 60 | 0.07 |
| $3v_1$ | 27.3873 | 19 | 4.48 |
| $v_2$ | 11.8329 | 16 | 1.48 |
| $v_1 + v_2$ | 20.9620 | 14 | 5.98 |
| $4v_1$ | 36.1614 | 8 | 2.9 |
| $v_2 - v_1$ | 2.7038 | 7 | 3.9 |
| $2v_1 + v_2$ | 30.0911 | 6 | 4.2 |
| $3v_1 + v_2$ | 39.2202 | 5 | 2.2 |

| ID | Frequency $[\text{d}^{-1}]$ | $S/N$ | $A_V$ [mmag] | $\phi_V$ [rad] |
|----|--------------------------|-----|-------------|-------------|
| $v_1$ | 9.128797 | 13.23 | 208 | 5.3 |
| $2v_1$ | 18.25759 | 8.01 | 60 | 0.9 |
| $v_2$ | 11.83315 | 5.84 | 34 | 4.4 |
| $v_1 + v_2$ | 20.96188 | 4.66 | 26 | 2 |
| $3v_1$ | 27.38633 | 3.99 | 23 | 4 |

Table 1. Frequencies, amplitudes and phases found by Rodriguez et al. (1992) in the Broglia (1959) observations of BP Peg.

Table 2. The value of the observed frequencies, amplitudes and phases found by us in the ASAS data.
Figure 2. Photometric observations of BP Peg. The two top panels show the ASAS-3-V and Strömgren $v$ data from Rodriguez. The bottom panel contains the ASAS and Strömgren $uvby$ observations phased with $v_1 = 9.128797$ d$^{-1}$.

Observations in the Strömgren $uvby$ filters phased with $v_1$ are shown in the bottom panel of Fig. 2. However, we note that there is a large phase shift between the $uvby$ light curves and the ASAS light curve. This is most probably due to miscounting of time in one of the data sets. We would also like to point out that our values of frequencies are in agreement (within the Rayleigh resolution of 0.003 d$^{-1}$) with those found by Rodriguez et al. (1992) in Broglia (1959) data. On the other hand, the values of $2v_1$ in the $u$ and $b$ filters found by Rodriguez et al. (1992) in its own Strömgren photometry differ more than the Rayleigh resolution (0.003 d$^{-1}$) from our ASAS frequencies. The same is true for $v_2$ in the $uv$ filters, $v_1 + v_2$ in the $by$ filters as well as for $3v_1$ in the $aby$ filters.

4 IDENTIFICATION OF THE MODE DEGREE $\ell$

The period ratio of the two modes of BP Pegasi amounts to 0.77146. As mentioned in Sect. 2, this strongly suggests that these are two radial modes; more specifically fundamental and first overtone. Here, we want to independently confirm this identification using photometric observables.

To make the mode identification from the photometric amplitudes and phases, we rely on the method based on a simultaneous
Figure 3. Periodograms from photometric observations of BP Peg. Panels from top to bottom show: periodograms for the ASAS-3 data (on the original data, after a subtraction of 2 terms and after a subtraction of 4 terms), periodograms for the ν data (on the original data, after a subtraction of 2 terms). Black triangles mark significant frequencies. Red triangles mark known frequencies but insignificant in the given periodogram. Red horizontal lines indicate the 4σ/N level.

Table 3. The photometric amplitudes and phases determined by Rodriguez et al. (1992) in the uvby filters. Frequencies listed in the second column are repeated from Table 1.

| ID   | Frequency | \(A_u\) | \(\phi_u\) | \(A_v\) | \(\phi_v\) | \(A_b\) | \(\phi_b\) | \(A_y\) | \(\phi_y\) |
|------|-----------|---------|-----------|---------|-----------|---------|-----------|---------|-----------|
| ν₁   | 9.1291    | 231     | 3.764     | 287     | 3.732     | 241     | 3.714     | 200     | 3.694     |
|      | ±5        | ±0.023  | ±0.012    | ±3      | ±0.014    | ±3      | ±0.017    | ±3      | ±0.017    |
| 2ν₁  | 18.2582   | 74      | 3.779     | 93      | 3.638     | 85      | 3.645     | 63      | 3.708     |
|      | ±5        | ±0.73   | ±0.037    | ±3      | ±0.040    | ±3      | ±0.053    | ±3      | ±0.053    |
| ν₂   | 11.8329   | 56      | 4.499     | 54      | 4.387     | 47      | 4.392     | 38      | 4.229     |
|      | ±5        | ±0.094  | ±0.062    | ±3      | ±0.069    | ±3      | ±0.083    | ±3      | ±0.083    |
| ν₁ + ν₂ | 20.9620 | 34      | 4.890     | 35      | 4.662     | 32      | 4.711     | 27      | 4.770     |
|      | ±5        | ±0.152  | ±0.095    | ±3      | ±0.100    | ±3      | ±0.117    | ±3      | ±0.117    |
| 3ν₁  | 27.3873   | 27      | 3.918     | 32      | 3.644     | 29      | 3.642     | 25      | 3.548     |
|      | ±5        | ±0.189  | ±0.104    | ±3      | ±0.108    | ±3      | ±0.125    | ±3      | ±0.125    |
determination of the mode degree $\ell$, the intrinsic mode amplitude $\varepsilon$ multiplied by $Y^\ell_m(i, 0)$ and the non-adiabatic parameter $f$ for a given frequency (Daszyńska-Daszkiewicz et al. 2003, 2005). The value of $\varepsilon$ defines the relative radial displacement at the surface caused by a pulsational mode with the angular frequency $\omega$. The factor $Y^\ell_m(i, 0)$ is the spherical harmonic that depends on the inclination angle $i$. In the case of radial modes we get the $\varepsilon$ value itself, because $Y^\ell_0(i, 0) = 1$. The non-adiabatic parameter $f$ is the ratio of the relative flux variation to the relative radial displacement at the photosphere level. Both, $\varepsilon$ and $f$ have to be regarded as complex numbers because pulsations are non-adiabatic. The method requires models of stellar atmospheres and in this paper we use complex numbers because pulsations are non-adiabatic. The method requires models of stellar atmospheres and in this paper we use

\begin{equation}
\chi^2 = \frac{1}{2N - N_p} \sum_{i=1}^{N} \left| \mathcal{A}^{ob}_i - \mathcal{A}^{cal}_i \right|^2, \end{equation}

where $\mathcal{A}^{ob}$ and $\mathcal{A}^{cal}$ denote the complex observational and the calculated amplitudes from the empirical values of $\varepsilon$ and $f$, respectively. $N$ is the number of passbands and $N_p$ is the number of parameters to be determined. The method yields two complex parameters, $\varepsilon$ and $f$, thus $N_p = 4$. The observational errors $\sigma_i$ are expressed as

\begin{equation}
|\sigma_i|^2 = \sigma^2(A_i) + \sigma^2(\varepsilon_i), \end{equation}

where $A_i = |\mathcal{A}_i|$ and $\varepsilon_i = \text{arg}(\mathcal{A}_i)$, are the values of the amplitude and phase, respectively. The value of $f$ and associated empirical values of $\varepsilon$ and $f$ are taken as the most probable for the $\ell = \ell_1$ at which the $\chi^2(\ell_1)$ reaches a minimum.

In Fig. 4, we show the values of the discriminant $\chi^2$ as a function of $\ell$ for the dominant frequency $\nu_1 = 9.128797$ d$^{-1}$. We considered several values of $T_{rad}$ and $\log g$ to include the entire observed error box. Besides, we checked the effect of the atmospheric parameters, [m/H] and $\xi$. As one can see for all values of $T_{rad}$ and $\log g$ and all considered pairs of $([\text{m/H}], \xi)$ the clear minimum of $\chi^2$ is at $\ell = 0$. Thus, there is no doubt that the dominant mode is radial.

The similar plots were drawn for the second frequency in Fig. 5. Also in this case there is a strong indication for $\ell = 0$. Thus, we obtained an independent and unique identification of the two modes of BP Peg and confirmed that they are radial as suggested by the period ratio.

Besides, the empirical values of $f$ can be compared with the theoretical counterparts from linear computations of stellar pulsations. This will be done in the next Section. In contrast, the empirical values of $\varepsilon$ cannot be compared with theoretical predictions because the radius amplitude is not determinable in the framework of the linear theory of stellar pulsations which we use here. However, from the empirical values of $\varepsilon$ we can estimate the amplitude of radial velocity variations, $A(V_{rad})$, due to pulsations and, in the case of radial modes, we get an absolute value of the radius changes for a given pulsational mode.

In Table 4, we give the derived values of $|\varepsilon|$ and $A(V_{rad})$ of the two modes for two different values of the effective temperature, surface gravity, atmospheric metallicity and microturbulent velocity. As one can see all these parameters have a significant effect on $|\varepsilon|$ and $A(V_{rad})$. The predicted amplitude of the relative radius changes for the dominant mode is between 1.1% and 2.1%, what results in the predicted amplitude of the radial velocity variations in the range $9.8 - 18.5$ km s$^{-1}$. For the second mode we obtained the radius change of $(0.5\% - 1.3\%)$ and the radial velocity amplitude of $5.3 - 11.9$ km s$^{-1}$. Kim et al. (1989) obtained the total range of radial velocity variations of $36 \pm 3$ km s$^{-1}$, which gives an amplitude from $16.5$ km s$^{-1}$ to $19.5$ km s$^{-1}$. This observed amplitude results from pulsations in the two radial modes and is consistent with our estimates for $\nu_1$ and $\nu_2$.

5 SEISMIC MODELLING OF BP PEG

5.1 Fitting the two radial modes

Having unambiguous identification of the mode degree $\ell$, we can now construct seismic models that reproduce the two frequencies as the radial modes. We started from finding the models that reproduce the dominant frequency $\nu_1 = 9.128797$ d$^{-1}$. To this aim we use the linear nonadiabatic code for stellar pulsations of Dziembowski (Dziembowski 1977; Pamyatnykh 1999). The code adopts the frozen convection approximation, i.e., the convective flux does not change during the pulsations. The effects of rotation on pulsational frequencies are taken into account up to the second order in the framework of perturbation theory.

| $\nu_1$ = 9.128797 [d$^{-1}$] |
|------------------|---|---|---|---|
| $|\varepsilon|$ | $A(V_{rad})$ |
| $T_{rad}$ | $\log g$ | $\xi \text{ [km s}^{-1}\text{]}$ | $A(V_{rad})$ [km s$^{-1}$] |
| $3.8724$ | $3.86$ | $0.2$ | $2$ | $0.014(4)$ | $12.3(3.4)$ |
| $3.8636$ | $3.86$ | $0.2$ | $2$ | $0.012(2)$ | $10.4(1.7)$ |
| $3.8724$ | $4.01$ | $0.2$ | $2$ | $0.021(4)$ | $15.2(3.2)$ |
| $3.8724$ | $3.86$ | $0.3$ | $2$ | $0.011(3)$ | $9.8(2.7)$ |
| $3.8636$ | $3.86$ | $0.3$ | $2$ | $0.014(2)$ | $12.2(1.3)$ |
| $3.8724$ | $4.01$ | $0.3$ | $2$ | $0.016(4)$ | $12.0(2.7)$ |

| $\nu_2$ = 11.83315 [d$^{-1}$] |
|------------------|---|---|---|---|
| $|\varepsilon|$ | $A(V_{rad})$ |
| $T_{rad}$ | $\log g$ | $\xi \text{ [km s}^{-1}\text{]}$ | $A(V_{rad})$ [km s$^{-1}$] |
| $3.8724$ | $3.86$ | $0.2$ | $2$ | $0.008(2)$ | $8.8(2.6)$ |
| $3.8636$ | $3.86$ | $0.2$ | $2$ | $0.006(3)$ | $7.2(3.0)$ |
| $3.8724$ | $4.01$ | $0.2$ | $2$ | $0.013(3)$ | $11.9(3.2)$ |
| $3.8724$ | $3.86$ | $0.3$ | $2$ | $0.007(2)$ | $8.3(2.7)$ |
| $3.8636$ | $3.86$ | $0.3$ | $2$ | $0.006(3)$ | $6.8(3.2)$ |
| $3.8724$ | $4.01$ | $0.3$ | $2$ | $0.012(4)$ | $11.0(3.3)$ |
| $3.8724$ | $3.86$ | $0.3$ | $4$ | $0.006(2)$ | $6.4(2.8)$ |
| $3.8636$ | $3.86$ | $0.3$ | $4$ | $0.005(3)$ | $5.3(3.2)$ |
| $3.8724$ | $4.01$ | $0.3$ | $4$ | $0.009(4)$ | $8.3(3.3)$ |
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Figure 4. The discriminant $\chi^2$ as a function of $\ell$ for the dominant frequency $v_1$ of BP Peg for five combinations of the effective temperatures $\log T_{\text{eff}}$ and surface gravity $\log g$, and three combinations of the atmospheric metallicity $[m/H]$ and microturbulent velocity $\xi_t$.

Figure 5. The same as in Fig. 4 but for the frequency $v_2$.

We performed computations for the three opacity tables: OPAL (Iglesias & Rogers 1996), OP (Seaton 2005) and OPLIB (Colgan et al. 2015, 2016). Regardless of the period ratio of the two frequencies, all models within the error box clearly indicate that the dominant radial mode can be only fundamental. Thus, the second frequency can only be the radial first overtone. However, the models reproducing the frequency $v_1$ can be in the three evolutionary stages: main sequence (MS), overall contraction (OC) and hydrogen-shell burning (HSB). In Fig. 6, we have drawn the lines of constant period corresponding to the dominant frequency of BP Peg for the models computed with the OPAL (the left panel), OP (the middle panel) and OPLIB (the right panel) data. The example is shown for the following parameters: the initial hydrogen abundance $X_0 = 0.70$, metallicity $Z = 0.020$, initial rotational velocity $V_{\text{rot},0} = 25$ km s$^{-1}$.
the mixing length parameter $\alpha_{\text{MLT}} = 0.5$ and no overshooting from the convective core.

Next, we included the second frequency to the seismic modelling. We started with the OPAL data. The fitting of the two radial modes is best illustrated on the Petersen diagrams. The examples are given in Fig.7. In the top panel we show the effect of mass whereas in the bottom panel the effects of metallicity, convective overshooting, initial hydrogen abundance and rotation are presented. As one can see the lower the mass the lower the frequency ratio $v_1/v_2$. From the bottom panel of Fig.7, we can draw the following conclusions: 1) the higher the metallicity the lower the value of $v_1/v_2$, 2) adding overshooting from the convective core decreases the frequency ratio, 3) the higher the hydrogen abundance, the lower the value of $v_1/v_2$, and 4) the higher the rotational velocity, the greater the value of $v_1/v_2$.

The effect of rotation on the Petersen diagram has been studied more extensively by Suarez et al. (2006).

The first important result is that with $\alpha_{\text{MLT}} = 0.5$ only post-main sequence seismic models have the luminosities that reach the minimum observed value of BP Peg. Secondly, the fitting of the two radial modes gives a strict relation between the mass and metallicity. Thirdly, adding convective overshooting moves seismic models away from the error box towards lower effective temperatures and luminosities. Seismic models with $\alpha_{\text{ov}} \neq 0$ are mostly in the main-sequence phase and have too low luminosities. In the top panel of Fig.8, we put the OPAL seismic models in the HSB phase on the mass-metallicity diagram. We included only HSB models without overshooting because they have the highest luminosities reaching the error box.

With the OPAL opacities, we got $Z \in (0.0155, 0.028)$ for the initial hydrogen abundance $X_0 = 0.70$ and $Z \in (0.013, 0.024)$ for $X_0 = 0.73$. The lower value of $Z$ is limited by the observed value of the luminosity whereas the upper limit of $Z$ results directly from the fitting to the two radial modes. For higher metallicity than $Z_{\text{max}}$ for a given $X_0$ there is no model reproducing the observed frequencies. Then, we repeated seismic modelling with the OP and OPLIB opacity data. It turned out that for $\alpha_{\text{MLT}} = 0.5$ in the wide range of $X_0$, $Z$ there are no models fitting the two frequencies within the error box of BP Peg. The enormous opacity effect when fitting the two radial modes was also demonstrated by Lenz et al. (2007), who performed seismic modelling for the $\delta$ Scut star 44 Tau. The positions of the seismic models of BP Peg on the HR diagram computed with the OPAL, OP and OPLIB data are shown in the bottom panel of Fig.8. As one can see all OP and OPLIB seismic models are far too cool and too less luminous. We will examine this rather unexpected result in more detail in the next subsection.

5.2 The effect of $\alpha_{\text{MLT}}$

Till now we made seismic modelling assuming the one value of the mixing length parameter in the envelope, i.e., $\alpha_{\text{MLT}} = 0.5$. The natural question arrises: what is the effect of this parameter on models fitting the two radial modes? The effect of $\alpha_{\text{MLT}}$ is illustrated in Fig.9. The figure shows the Petersen diagram, with the run of $v_1/v_2$ vs $v_1$ for the OPAL model with $M = 1.73 \, M_\odot$ adopting $\alpha_{\text{MLT}} = 0.5$ and 1.8, and the two values of metallicity $Z = 0.020$ and 0.015.

In the case of model with $\alpha_{\text{MLT}} = 1.8$ and $Z = 0.02$, we have a deep decrease of $v_1/v_2$ when the Terminal Age Main Sequence (TAMS) is approaching. Then, $v_1/v_2$ increases in the overall contraction phase reaching the same maximum value as in the case of $\alpha_{\text{MLT}} = 0.5$. Next, it goes down steeply in hydrogen-shell burning phase. As one can see, this effect depends strongly on the metallicity and for $Z = 0.015$ there is no sharp drop in the frequency ratio. The sudden drop of $v_1/v_2$ for $\alpha_{\text{MLT}} = 1.8$ and $Z = 0.020$ is not caused by our assumption about the frozen convection during pulsations as one could guess. This is because the adiabatic and nonadiabatic frequencies of the radial fundamental and first overtone modes are the same up to the forth decimal place. The reason of such behaviour of $v_1/v_2$ is the change of the internal structure of cooler models for larger values of $\alpha_{\text{MLT}}$, in particular around the local opacity bumps. In Fig.10, we show the 3D plot of the mean opacity as a function of depth, expressed by the temperature and density, for $X_0 = 0.70, Z = 0.020$. The dotted and solid lines correspond to the $M = 1.73 \, M_\odot$ models with $\alpha_{\text{MLT}} = 0.5$ and 1.8, respectively. The blue, red and black lines mark the $M = 1.73 \, M_\odot$ models near the end of MS (no. 125), on TAMS (no. 158) and right after TAMS (no. 170), respectively.

For a wide range of $X_0, Z$, it appeared that increasing the value of $\alpha_{\text{MLT}}$ in the OP and OPLIB models, brings them closer to the error box. For $\alpha_{\text{MLT}} \geq 2.0$ it is possible to get seismic models with the accepted effective temperatures and luminosities.

In Table 5, we present examples of seismic models in the three phases of evolution computed with the OPAL, OP and OPLIB opacity tables, adopting the two values of the mixing parameter $\alpha_{\text{MLT}} = 0.5$ and 2.0. For $\alpha_{\text{MLT}} = 0.5$ only the OPAL seismic model in the HSB phase has the effective temperature and luminosity consistent with the observed values. Moreover, in the case of the OP and OPLIB tables there are no models in the MS and OC phases reproducing the two frequencies of BP Peg at all. For $\alpha_{\text{MLT}} = 2.0$, all phases of evolution for each opacity data are possible. Although, the values of $T_{\text{eff}}$ and log $L/L_\odot$ are often marginally consistent with observational determinations.

5.3 Constraints on convection from the parameter $f$

The potential of the parameter $f$ to obtain the information on the efficiency of convective transport has been demonstrated in many works (Daszyńska-Daszkiewicz et al. 2003, 2020, 2005, 2021). Let us recall that this parameter describes the relative amplitude of the radiative flux variations at the photosphere level for a given pulsational mode (see Sect. 4). The theoretical value of $f$ can be derived only in the framework of nonadiabatic theory of stellar pulsations as it is complex because, besides the amplitude of the flux variations, we need also the phase shift between the flux and radius variations. The parameter $f$ is very sensitive to the subphotospheric condition and, in the case of $\delta$ Scuti pulsators, the envelope convection greatly modifies the value of $f$. This fact has profound consequences for the mode identification because the parameter $f$ enters the expression for the photometric amplitudes.

On the other hand, the empirical value of $f$ for a given mode can be obtained from the observed photometric amplitudes and phases simultaneously with the identification of the mode degree $\ell$. In that way, the determination of $\ell$ is independent of the treatment of convection whereas the empirical values of $f$ can be directly compared with the theoretical predictions for different values of the mixing length parameter $\alpha_{\text{MLT}}$, that is for different efficiency of convective transport in the envelope.

Such a comparison for the radial fundamental mode of BP Pegasi is presented in Fig.11 on the complex plane. The theoretical values of $f$ are for the OPAL seismic model reproducing the two radial-mode frequencies and twelve values of the mixing length parameter $\alpha_{\text{MLT}}$. The model parameters are: $X_0 = 0.70, Z = 0.023, M = 1.79 \, M_\odot, \log T_{\text{eff}} \approx 3.8365, \log L/L_\odot \approx 1.164$. The
The parameters of the seismic models that reproduce \( v_2 = 9.128797 \text{ d}^{-1} \) as the radial fundamental mode and \( v_2 = 11.83315 \text{ d}^{-1} \) as the radial first overtone, for the three opacity data and two values of the mixing length parameter \( \alpha_{\text{MLT}} = 0.5, 2.0 \). The other parameters are: \( X_0 = 0.70, Z = 0.020, \alpha_{\text{teff}} = 0.0 \).

### Table 5

| \( \alpha_{\text{MLT}} \) | phase | \( M/M_\odot \) | \( \log T_{\text{eff}} \) | \( \log L/L_\odot \) | \( R/R_\odot \) | age [Gyr] | \( \nu(p_2) \) [d\(^{-1}\)] | \( \nu(p_1)/\nu(p_2) \) | \( \eta(p_1) \) | \( \eta(p_2) \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | MS | 1.674 | 3.7918 | 0.964 | 2.65 | 1.57 | 11.833146 | 0.77146 | 0.073 | 0.022 |
| 0.5 | OC | 1.696 | 3.8064 | 1.027 | 2.66 | 1.58 | 11.833157 | 0.77146 | 0.092 | 0.054 |
| 0.5 | HSB | 1.726 | 3.8362 | 1.152 | 2.68 | 1.52 | 11.833146 | 0.77146 | 0.101 | 0.091 |
| 2.0 | MS | 1.860 | 3.8342 | 1.149 | 2.69 | 1.10 | 11.833217 | 0.77146 | 0.439 | 0.459 |
| 2.0 | OC | 1.797 | 3.8365 | 1.152 | 2.67 | 1.35 | 11.833283 | 0.77145 | 0.338 | 0.337 |
| 2.0 | HSB | 1.745 | 3.8414 | 1.169 | 2.67 | 1.48 | 11.833351 | 0.77145 | 0.198 | 0.195 |

### Table 5

| \( \alpha_{\text{MLT}} \) | phase | \( M/M_\odot \) | \( \log T_{\text{eff}} \) | \( \log L/L_\odot \) | \( R/R_\odot \) | age [Gyr] | \( \nu(p_2) \) [d\(^{-1}\)] | \( \nu(p_1)/\nu(p_2) \) | \( \eta(p_1) \) | \( \eta(p_2) \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | HSB | 1.501 | 3.7994 | 0.920 | 2.55 | 2.36 | 11.833146 | 0.77146 | 0.076 | 0.023 |
| 2.0 | MS | 1.834 | 3.8302 | 1.123 | 2.66 | 1.19 | 11.831851 | 0.77154 | 0.443 | 0.432 |
| 2.0 | OC | 1.777 | 3.8311 | 1.119 | 2.64 | 1.41 | 11.831579 | 0.77156 | 0.458 | 0.437 |
| 2.0 | HSB | 1.685 | 3.8322 | 1.111 | 2.60 | 1.66 | 11.833919 | 0.77141 | 0.480 | 0.446 |

### Table 5

| \( \alpha_{\text{MLT}} \) | phase | \( M/M_\odot \) | \( \log T_{\text{eff}} \) | \( \log L/L_\odot \) | \( R/R_\odot \) | age [Gyr] | \( \nu(p_2) \) [d\(^{-1}\)] | \( \nu(p_1)/\nu(p_2) \) | \( \eta(p_1) \) | \( \eta(p_2) \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | HSB | 1.577 | 3.8044 | 0.999 | 2.60 | 1.94 | 11.833146 | 0.77146 | 0.094 | 0.066 |
| 2.0 | MS | 1.856 | 3.8324 | 1.138 | 2.68 | 1.10 | 11.833146 | 0.77146 | 0.416 | 0.439 |
| 2.0 | OC | 1.805 | 3.8332 | 1.135 | 2.66 | 1.28 | 11.833146 | 0.77146 | 0.430 | 0.455 |
| 2.0 | HSB | 1.708 | 3.8350 | 1.130 | 2.63 | 1.52 | 11.833140 | 0.77146 | 0.414 | 0.427 |

As one can see from Fig.11, the agreement between the theoretical and empirical values of \( f \) for \( v_1 \) can be achieved for the mixing parameters in the range of about \( \alpha_{\text{MLT}} \in (0.7, 1.0) \) if the atmospheric metallicity and microturbulent velocity are \((\text{[m/H]}, \xi_t) = (0.2, 8 \text{ km s}^{-1})\), \((\text{[m/H]}, \xi_t) = (0.3, 8 \text{ km s}^{-1})\) or \((\text{[m/H]}, \xi_t) = (0.5, 4 \text{ km s}^{-1})\). As we have checked, the lower values of \([\text{m/H}]\) and \(\xi_t\) are definitely excluded. Moreover, the lower the metallicity and microturbulent velocity the larger the errors of \( f \). This means that lower values of \([\text{m/H}]\) and \(\xi_t\) give a much worse fit between the calculated and observed photometric amplitudes and phases.

As one can see, the obtained range of \( \alpha_{\text{MLT}} \) is quite narrow and amounts to about \((0.7, 1.0)\). Is this conclusion valid for other seismic models? To answer this question as fully as possible, we will perform more extensive seismic analysis in the next Section.
Figure 7. The Petersen diagrams spanning from the zero-age main sequence to the post-main sequence phase of evolution. The observed value of $(v_1, v_1/v_2)$ is indicated by the black diamond. There is shown the effect of mass (the top panel) and the effects of hydrogen abundance, metallicity, convective overshooting and rotational velocity (the bottom panel). The models were computed with the OPAL tables and the mixing length parameter $\alpha_{\text{MLT}} = 0.5$.

6 ASTEROSEISMIC MODELLING WITH MONTE CARLO-BASED BAYESIAN ANALYSIS

In the previous section, we showed the results of seismic modelling of BP Peg considering only some sets of parameters for the three sources of the opacity data, OPAL, OP and OPLIB. These results allowed to draw the two important conclusions. Firstly, main-sequence seismic models, reproducing the two radial-mode frequencies, are much less likely because their luminosities are too low compared to the observed values. Secondly, only with the OPAL tables it is possible to construct seismic models with the mixing length parameter $\alpha_{\text{MLT}} < 2$ and the parameters ($T_{\text{eff}}, L/L_\odot$) consistent with the observed determinations.

In this section, we present the more advanced approach using the Bayesian analysis based on the Monte Carlo method. For the reasons mentioned above, we will consider only the evolutionary phases of overall contraction and hydrogen-shell burning, and we will use the OPAL opacities.

The analysis is based on the Gaussian likelihood function defined as (e.g., Jørgensen & Lindegren 2005; da Silva et al. 2006; Daszyńska-Daszkiewicz et al. 2006).

Figure 8. The top panel: the OPAL post-MS seismic models in the HSB phase fitting the two frequencies of BP Peg as the radial fundamental and first overtone modes on the diagram “metallicity $Z$ vs. mass $M$”. There is also shown the effect of the initial hydrogen abundance $X_0$ and initial rotational velocity $V_\text{rot,0}$. The mixing length parameter was $\alpha_{\text{MLT}} = 0.5$. The bottom panel: the position of these models on the HR diagram with the observed error box of BP Peg. There are also shown positions of the OP and OPLIB seismic models in the HSB phase.

Figure 9. The Petersen diagram for the OPAL model with the mass $M = 1.73$ M$\odot$. There is shown the effect of the mixing length parameter on the frequency ratio $v_1/v_2$ and the effect of metallicity for $\alpha_{\text{MLT}} = 1.8$. 
In the first grid of simulation, we fixed the mixing parameter \( \alpha_{\text{MLT}} = 0.5 \). The other five parameters of models, i.e., mass, initial rotational velocity, metallicity, hydrogen abundance and convective overshooting parameter were randomly generated during simulation. For each randomly selected set of parameters we calculated evolutionary and pulsational models. Then, we chose the models that had the frequency of the radial fundamental mode fitting the observed frequency \( v_1 \). Thus, the models, we considered, reproduced exactly the dominant frequency. The OC and HSB models were considered separately, that is we ran independent simulations for each of them.

For \( X_0 \) we assumed a beta function \( B(2, 2) \) as a prior probability, since we wanted to restrict its value to the reasonable range, i.e., from 0.6 to 0.8 with \( X_0 = 0.7 \) as the most probable. For other parameters we used uninformative priors, i.e., a uniform distribution. The values of the effective temperature and luminosity with uncertainties were given in Sect. 2. The value of the second frequency corresponding to the first overtone mode was taken from Table 2. However, rather than using the formal error from the least square fitting procedure, we took the Rayleigh resolution of the ASAS data, as a more realistic measure of the error, that is \( \sigma_v \) = 0.0004 day\(^{-1}\).

The important result of our first simulations was that convective overshooting is rather ineffective because in all runs the parameter \( \alpha_{\text{ov}} \) tended to zero. We already received such a hint in Sect. 5 where we found that seismic models with \( \alpha_{\text{ov}} = 0.2 \) have too low luminosities. Therefore, in the following runs we fixed \( \alpha_{\text{ov}} = 0.0 \) while \( \alpha_{\text{MLT}} \) was henceforth treated as a free parameter. Thus, we searched for the best values of the parameters: \( M, X_0, Z, V_{\text{rot}} \) and \( \alpha_{\text{MLT}} \).

The results of our simulations are shown in Fig. 12. In the upper panels we plotted the corner plots for \( M, X_0, Z, V_{\text{rot}} \) and \( \alpha_{\text{MLT}} \). In the bottom panels we showed the corresponding histograms. The left column indicates the OC seismic models, while the right column - the HSB seismic models. The histograms were normalised to 1.0 by the number of all models, thus the numbers on the Y-axis times 100 are the percentage of models with a given parameter range.

Rodrigues et al. 2017; Jiang & Gizon 2021)

\[
\mathcal{L}(E|H) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left( -\frac{(O_i - M_i)^2}{2\sigma_i^2} \right),
\]

(3)

where \( H \) is the hypothesis that represents adjustable model and theory parameters that in our case are: mass \( M \), initial hydrogen abundance \( X_0 \), metallicity \( Z \), rotation \( V_{\text{rot}} \), convective overshooting parameter \( \alpha_{\text{ov}} \) and the mixing length parameter \( \alpha_{\text{MLT}} \). The evidence \( E \) represents the calculated observables \( M_i \), e.g., the effective temperature \( T_{\text{eff}} \), luminosity \( L/L_\odot \), pulsational frequencies, that can be directly compared with the observed parameters \( O_i \) determined with the errors \( \sigma_i \).

Here we used the following observations: effective temperature \( T_{\text{eff}} \), luminosity \( L/L_\odot \), the frequencies of the two radial modes \( v_1 \) and \( v_2 \), and the parameter \( f \) of the dominant mode. Then, we made a huge number (about 80,000) of simulations to maximize the likelihood function given in Eq. (3) in order to constrain the model parameters.

6.1 Simulations without the parameter \( f \)

In the first grid of simulation, we fixed the mixing parameter \( \alpha_{\text{MLT}} = 0.5 \). The other five parameters of models, i.e., mass, initial rotational velocity, metallicity, hydrogen abundance and overshooting parameter were randomly generated during simulation. For each randomly selected set of parameters we calculated evolutionary and pulsational models. Then, we chose the models that had the frequency of the radial fundamental mode fitting the observed frequency \( v_1 \). Thus, the models, we considered, reproduced exactly the dominant frequency. The OC and HSB models were considered separately, that is we ran independent simulations for each of them.

For \( X_0 \) we assumed a beta function \( B(2, 2) \) as a prior probability, since we wanted to restrict its value to the reasonable range, i.e., from 0.6 to 0.8 with \( X_0 = 0.7 \) as the most probable. For other parameters we used uninformative priors, i.e., a uniform distribution. The values of the effective temperature and luminosity with uncertainties were given in Sect. 2. The value of the second frequency corresponding to the first overtone mode was taken from Table 2. However, rather than using the formal error from the least square fitting procedure, we took the Rayleigh resolution of the ASAS data, as a more realistic measure of the error, that is \( \sigma_v \) = 0.0004 day\(^{-1}\).

The important result of our first simulations was that convective overshooting is rather ineffective because in all runs the parameter \( \alpha_{\text{ov}} \) tended to zero. We already received such a hint in Sect. 5 where we found that seismic models with \( \alpha_{\text{ov}} = 0.2 \) have too low luminosities. Therefore, in the following runs we fixed \( \alpha_{\text{ov}} = 0.0 \) while \( \alpha_{\text{MLT}} \) was henceforth treated as a free parameter. Thus, we searched for the best values of the parameters: \( M, X_0, Z, V_{\text{rot}} \) and \( \alpha_{\text{MLT}} \).

The results of our simulations are shown in Fig. 12. In the upper panels we plotted the corner plots for \( M, X_0, Z, V_{\text{rot}} \) and \( \alpha_{\text{MLT}} \). In the bottom panels we showed the corresponding histograms. The left column indicates the OC seismic models, while the right column - the HSB seismic models. The histograms were normalised to 1.0 by the number of all models, thus the numbers on the Y-axis times 100 are the percentage of models with a given parameter range.
As one can see, for both the OC and HSB models, the mass and metallicity are the best constrained parameters. The initial hydrogen abundance $X_0$ for the OC models seems to have two extremes, near $X \approx 0.68$ and $X \approx 0.73$. The HSB models concentrate around $X \approx 0.71$. The least constrained parameters, in both cases, are $V_{\text{rot}}$ and $\alpha_{\text{MLT}}$. The models seem to be weakly dependent on them. The expected values of the parameters from the distributions represented by the histograms in Fig. 12 are given in the first two lines of Table 6. The uncertainties were calculated as the square roots of the variance.

All models from the simulations are also marked in the HR diagram shown in Fig. 13. All of them concentrate near the bottom right corner of the error box. All HSB models lay within $2\sigma$ of the effective temperature and luminosity whereas most of the OC models are within $3\sigma$ errors.

6.2 Simulations with the parameter $f$

In the next step we added more constraints to our simulations. We used the non-adiabatic $f$-parameter of the radial fundamental mode. This parameter is complex so we have two more constraints. As has been earlier mentioned, the theoretical value of $f$ in $\delta$ Sct stellar models strongly depends on the convection efficiency in the envelope. Thus, from a comparison of the theoretical and empirical values of $f$ it is feasible to obtain information on the mixing length parameter $\alpha_{\text{MLT}}$.

We calculated the second grid of models for which the likelihood function was modified by multiplying Eq. (3) by two additional factors that contain the real ($f_R$) and imaginary part ($f_I$) of the parameter $f$ of the radial fundamental mode. Now the results depend on the model atmosphere. As in Sect. 5, we used the NEMO models. We made simulations for a discrete values of the microturbulent velocity $\xi_t=2$, 4, 8 km s$^{-1}$ and the atmospheric metallicity [m/H] was changed consistently with the current value of $Z$.

The results of the simulations for $\xi_t=4$ km s$^{-1}$ are presented on the corner plots in Fig. 14. As one can see, now the values of $\alpha_{\text{MLT}}$ concentrates in a narrow range both for the OC and HSB models.

Our simulations indicate rather small convective efficiency with $\alpha_{\text{MLT}}=0.58(8)$ for the OC seismic models and $\alpha_{\text{MLT}}=0.63(7)$ for the HSB seismic models. The estimated mass of BP Peg is very similar for the two evolutionary phases and amounts to $M=1.80(4)$ $M_\odot$. The initial hydrogen abundance is $X_0=0.70(1)$ if BP Peg is in the OC phase or $X_0=0.67(2)$ in the HSB phase, whereas the metallicity is $Z=0.026(2)$ in the OC phase or $Z=0.029(3)$ in the HSB phase. The seismic models from these simulations are also marked in the HR diagram shown in Fig. 15. As before all models are located near the bottom right corner of the error box. The expected values of other parameters with the errors are given in Table 6. In general, all parameters are now better constrained than in the case of simulations without the parameter $f$. Moreover, one can also see that most parameters are better determined if the microturbulent velocity is $\xi_t=4$ km s$^{-1}$ or $\xi_t=8$ km s$^{-1}$. It is caused by the fact that the theoretical values of $f$ for $\xi_t=2$ km s$^{-1}$ have very large errors if [m/H]<+0.3, as has been shown in Fig. 11 for one model (Sect. 5.3).

For the other values of the microturbulent velocity we got the following ranges of the mixing length parameter:

- $\alpha_{\text{MLT}}=0.71(36)$ for the OC models and $\alpha_{\text{MLT}}=0.16(10)$ for the HSB models if $\xi_t=2$ km s$^{-1}$
- $\alpha_{\text{MLT}}=0.97(16)$ for the OC models and $\alpha_{\text{MLT}}=0.71(8)$ for the HSB models if $\xi_t=8$ km s$^{-1}$. Thus, in all cases the mixing parameter is below 1.0.

The seismic models obtained with $\xi_t=2$ km s$^{-1}$ have the initial hydrogen abundance of $X_0=0.64(2)$ in the OC phase and $X_0=0.78(1)$ in the HSB phase. These values of $X_0$ are unusually low (the OC phase) or high (the HSB phase). Therefore, we treat this solution as much less probable. In the case of the simulations with $\xi_t=8$ km s$^{-1}$ we got $X_0=0.75(2)$ in the OC phase and $X_0=0.68(1)$ in the HSB phase. These abundances of $X_0$ can be rather still acceptable. The metallicity in this case differs significantly between the OC and HSB models, i.e., $Z=0.021(2)$ (OC) vs. $Z=0.018(1)$ (HSB). Because the metallicity correlates with the mass, likewise, there is the large difference in the mass value, i.e., $M=1.92(3)$ $M_\odot$ for the OC models vs. $M=1.62(4)$ $M_\odot$ for the HSB models.

The age of all obtained models is in the range of about (1.43, 1.54) Gyr. The rotational velocity is in the range of about $V_{\text{rot}} \in (12, 33)$ km s$^{-1}$ which is consistent with a crude estimation by Kim et al. (1989).
Figure 12. Upper panels: the corner plots for the parameters: $M$, $X_0$, $Z$, $V_{rot}$ and $\alpha_{MLT}$. Bottom panels: the normalized histograms for the same parameters. The left column corresponds to the OC models, while the right column to the HSB models.

Table 6. The expected values of the parameters of BP Peg obtained from the Monte-Carlo simulations. The uncertainties were calculated as the square roots of the variance. The first two lines correspond to the simulations without the parameter $f$ of the radial fundamental mode and the remaining six lines to the simulations with this parameter $f$. In the second case, the first column contains the microturbulent velocity in the atmosphere $\xi_f$ whereas the atmospheric metallicity [m/H] were changed consistently with $Z$ given in the fourth column.

| $\xi_f$ [km s$^{-1}$] | evolution phase | $M/M_\odot$ | $Z$ | $X_0$ | $V_{rot,0}$ [km s$^{-1}$] | $\alpha_{MLT}$ | log $T_{eff}$ | log $L/L_\odot$ | $R/R_\odot$ | $V_{tot}$ | log (s/yr) |
|------------------------|-----------------|-------------|-----|------|-----------------|---------------|---------------|---------------|-------------|-----------|--------------|
| —                      | OC              | 1.782 (72)  | 0.0256 (49) | 0.686 (35) | 26.5 (11.9) | 1.13 (68) | 3.8300 (120) | 1.132 (57) | 2.69 (4) | 24.3 (10.6) | 9.157 (35) |
| —                      | HSB             | 1.775 (94)  | 0.0200 (39) | 0.715 (24) | 11.6 (7.8)  | 0.95 (58) | 3.8386 (26) | 1.169 (20) | 2.70 (5) | 11.0 (7.3) | 9.181 (16) |
| 2                      | OC              | 1.718 (40)  | 0.0314 (50) | 0.643 (21) | 36.1 (17.9) | 0.71 (36) | 3.8217 (77) | 1.086 (35) | 2.66 (3) | 33.3 (17.4) | 9.169 (20) |
| 2                      | HSB             | 1.951 (47)  | 0.0172 (14) | 0.783 (9)  | 12.4 (7.1)  | 0.16 (10) | 3.8423 (13) | 1.214 (10) | 2.80 (2) | 11.4 (6.5) | 9.189 (7)  |
| 4                      | OC              | 1.803 (23)  | 0.0259 (24) | 0.698 (12) | 36.6 (13.2) | 0.58 (8)  | 3.8276 (50) | 1.128 (23) | 2.71 (1) | 33.8 (12.9) | 9.187 (13) |
| 4                      | HSB             | 1.805 (45)  | 0.0286 (29) | 0.674 (21) | 12.9 (9.0)  | 0.63 (7)  | 3.8359 (14) | 1.162 (10) | 2.71 (2) | 12.1 (8.4) | 9.157 (12) |
| 8                      | OC              | 1.921 (28)  | 0.0211 (18) | 0.752 (15) | 20.5 (10.8) | 0.97 (16) | 3.8406 (53) | 1.200 (24) | 2.78 (2) | 18.9 (9.8) | 9.183 (11) |
| 8                      | HSB             | 1.618 (38)  | 0.0179 (14) | 0.677 (10) | 16.1 (7.1)  | 0.71 (8)  | 3.8367 (9)  | 1.136 (8)  | 2.62 (2) | 15.4 (6.7) | 9.185 (8)  |
sequence phase of evolution. We demonstrated also that the value of $\alpha_{MLT}$ can have a huge effect on the Petersen diagram and in some cases causes a deep decrease of the frequency ratio $v_1/v_2$ near TAMS. How big this effect is depends on mass and metallicity. The decrease of $v_1/v_2$ is a consequence of changes in the internal structure of a star described by the mean opacity profile $\kappa(T, \rho)$.

In the next step, we made an extensive seismic modelling with the Bayesian analysis based on the Monte Carlo simulations. Firstly, we showed that the convective overshooting is inefficient and it is reasonable to assume $\alpha_{ov} = 0.0$. From a huge number of simulations (about 80 000) we constrained the model parameters, i.e., the mass $M$, initial hydrogen abundance $X_0$, metallicity Z, rotational velocity $V_{rot}$. Moreover, we obtained that the efficiency of envelope convection is characterized by the mixing length parameter of about $\alpha_{MLT} = 0.5 - 1.0$ depending on the adopted microturbulent velocity that we estimated at $\xi_t \approx 4$ or 8 km s$^{-1}$. Thus, the convective transport in the envelope of BP Peg is rather moderately efficient. This conclusions is similar to our previous results for SX Phe (Daszyńska-Daszkiewicz et al. 2020) or for the prototype $\delta$ Scuti (Daszyńska-Daszkiewicz et al. 2021). Definitely, such studies have to be continued to collect such seismic results for more $\delta$ Sct stars. Then, perhaps it will be possible to draw more general conclusions and of greater statistical significance, on convection in the envelopes of stars with masses between 1.5 and 2.5 $M_\odot$. Another result that requires careful study is the effect of opacities on seismic models. Why are the OPAL seismic models data so different from the OP and OPLIB seismic models when the two radial-mode frequencies are fitted? Why are the seismic OP AL models better in some cases causes a deep decrease of the frequency ratio $v_1/v_2$ near TAMS. How big this effect is depends on mass and metallicity. The decrease of $v_1/v_2$ is a consequence of changes in the internal structure of a star described by the mean opacity profile $\kappa(T, \rho)$.

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DATA AVAILABILITY

The ASAS observations are available at the website of http://www.astrouw.edu.pl/asas. Theoretical computations will be shared on reasonable request to the corresponding author.

REFERENCES

Andreasen G. K., 1983, A&A, 121, 250
Asplund M., Grevesse N., Sauval A. J., Scott P., 2009, Annu.Rev.Astron.Astrophys., 47, 481
Balona L. A., Evers E. A., 1999, MNRAS, 302, 349
Balona L. A., Lenz P., Antoci V. a., 2012, MNRAS, 419, 3028
Breger M., 2000, in Delta Scuti and Related Stars. p. 3
Broggia P., 1959, Mem. Soc. Astron. Italiana, 30, 57
Canuto V. M., Goldman I., Mazzitelli I., 1996, ApJ, 473, 550
Casas R., Suarez J. C., Moya A., Garrido R., 2006, A&A, 455, 1019
Christensen-Dalsgaard J., 2008, Ap&SS, 316, 113
Colgan J., Kilcrease D. P., Magee N. H., et al. 2015, High Energy Density Physics, 14, 33
Colgan J., Kilcrease D. P., Magee N. H., et al. 2016, ApJ, 817, 116
Cox A. N., King D. S., Hudson S. W., 1979, ApJ, 228, 870
Daszyńska-Daszkiewicz J., Dziembowski W. A., Pamyatnykh A. A., 2003, A&A, 407, 999
Daszyńska-Daszkiewicz J., Dziembowski W. A., Pamyatnykh A. A., et al. 2005, A&A, 438, 653
Daszyńska-Daszkiewicz J., Pamyatnykh A. A., Walczak P., Szewcuk W., 2020, MNRAS, 499, 3034
Daszyńska-Daszkiewicz J., Pamyatnykh A. A., Walczak P., et al. 2021, MNRAS, 505, 88
Deeming T. J., 1975, Ap&SS, 36, 137
Dziembowski W. A., 1977, Acta Astr., 27, 95
Figer A., 1983, GEOS Circular on RR Lyrae Variables, p. 9p.
Fitch W. S., Szeidl B., 1976, ApJ, 203, 616
Flower P. J., 1996, ApJ, 469, 355
Green G. M., Schlaffly E. F., Zucker C., et al. 2019, ApJ, 887, 93
Heiter U., Kupka F., van’t Veen-Mennaret C., et al. 2002, A&A, 392, 619
Iglesias C. A., Rogers F. J., 1996, ApJ, 464, 943
Jiang C., Gizon L., 2021, Research in Astronomy and Astrophysics, 21, 226
Jørgensen B. R., Lindegren L., 2005, A&A, 436, 127
Kim C., McNamara D. H., Joner M. D., 1989, PASP, 101, 1009
Kurtz D. W., 1985, MNRAS, 213, 773
Langer N., Pamyatnykh A. A., Breger M., 2007, AIP Conf. Proc., 948, 201
Lindegren L., Bastian U., Biermann M., et al. 2021, A&A, 649, A4
Masani A., Broggia P., 1954, MmSAI, 25, 59
McNamara D. H., 1997, PASP, 109, 1221
Niu J.-S., Fu J.-N., Li Y., et al. 2017, MNRAS, 467, 3122
Pamyatnykh A. A., 1999, Acta Astr., 49, 119
Pamyatnykh A. A., Dziembowski W. A., Handler G., et al. 1998, A&A, 333, 141
Peña J. H., Gonzales D., Peniche R., 1999, A&AS, 138, 11
Petersen J. O., Christensen-Dalsgaard J., 1996, A&A, 312, 463
Pojmanski G., 2002, Acta Astron., 52, 397
Rodriguez E., et al., 2017, MNRAS, 467, 1433
Rodriguez E., Rolland A., Lopez de Coca P., Garcia-Lobo E., Sedano J. L., 1992, A&AS, 93, 189
Rodriguez E., Lopez de Coca P., Rolland A., et al., 1994, A&A Suppl. Ser., 106, 21
Rodgers F. J., Nayfonov A., 2002, ApJ, 576, 1064

Figure 13. The HR diagram with the position of the seismic models obtained from the simulations. The OC models are marked with magenta circles and the HSB models with orange dots.

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Figure 14. The same as in Fig. 12 but we included also the empirical parameter $f$ for the dominant mode in our simulations. The NEMO model atmospheres were used with the microturbulent velocity $\xi_t = 4 \text{ km s}^{-1}$.

Rogers F. J., Swenson F. J., Iglesias C. A., 1996, ApJ, 456, 902
Seaton M. J., 2005, MNRAS, 362, L1
Suarez J. C., Garrido R., Goupil M. J., 2006, A&A, 447, 649
Ulusoy C., Guinmez T., Stateva I., et al. 2013, MNRAS, 428, 3551
Xue H.-F., Fu J.-N., Foc-Machado L., et al. 2018, ApJ, 861, 96
da Silva L., et al., 2006, A&A, 458, 609

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Figure 15. The same as in Fig. 13 but for the seismic models obtained from the Bayesian analysis including the parameter $f$ for the dominant mode. The NEMO model atmospheres were used with $\xi_t = 4 \text{ km s}^{-1}$. 

with the non-adiabatic parameter $f$