The Complexity of Nonconvex-Strongly-Concave Minimax Optimization

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March 31, 2021

Abstract

This paper studies the complexity for finding approximate stationary points of nonconvex-strongly-concave (NC-SC) smooth minimax problems, in both general and averaged smooth finite-sum settings. We establish nontrivial lower complexity bounds of $\Omega(\sqrt{\kappa \Delta L \epsilon^{-2}})$ and $\Omega(n + \sqrt{n \kappa \Delta L \epsilon^{-2}})$ for the two settings, respectively, where $\kappa$ is the condition number, $L$ is the smoothness constant, and $\Delta$ is the initial gap. Our result reveals substantial gaps between these limits and best-known upper bounds in the literature. To close these gaps, we introduce a generic acceleration scheme that deploys existing gradient-based methods to solve a sequence of crafted strongly-convex-strongly-concave subproblems. In the general setting, the complexity of our proposed algorithm nearly matches the lower bound; in particular, it removes an additional polylogarithmic dependence on accuracy present in previous works. In the averaged smooth finite-sum setting, our proposed algorithm improves over previous algorithms by providing a nearly-tight dependence on the condition number.

1 Introduction

In this paper, we consider general minimax problems of the form $(n, d_1, d_2 \in \mathbb{N}^+)$:

$$\min_{x \in \mathbb{R}^{d_1}} \max_{y \in \mathbb{R}^{d_2}} f(x, y), \quad (1)$$

as well as their finite-sum counterpart:

$$\min_{x \in \mathbb{R}^{d_1}} \max_{y \in \mathbb{R}^{d_2}} f(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x, y), \quad (2)$$

where $f, f_i$ are continuously differentiable and $f$ is $L$-Lipschitz smooth jointly in $x$ and $y$. We focus on the setting when the function $f$ is $\mu$-strongly concave in $y$ and perhaps nonconvex in $x$, i.e., $f$ is nonconvex-strongly-concave (NC-SC). Such problems arise ubiquitously in machine learning, e.g., GANs with regularization [Sanjabi et al., 2018, Lei et al., 2020], Wasserstein robust models [Sinha et al., 2018], robust learning over multiple domains [Qian et al., 2019], and off-policy reinforcement learning [Dai et al., 2017, 2018, Huang and Jiang, 2020]. Since the problem is nonconvex in general, a natural goal is to find an approximate stationary point $\bar{x}$, such that $\|\nabla \Phi(\bar{x})\| \leq \epsilon$, for a given accuracy $\epsilon$, where $\Phi(x) \triangleq \max_y f(x, y)$ is the primal function. This goal is meaningful for the aforementioned applications, e.g., in adversarial models the primal function quantifies the worst-case loss for the learner, with respect to adversary’s actions.

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| Setting           | Our Lower Bound          | Our Upper Bound          | Previous Upper Bound          |
|-------------------|--------------------------|--------------------------|--------------------------------|
| NC-SC, general    | $\Omega(\sqrt{n}\Delta L e^{-2})$ Theorem 3.1 | $\tilde{O}(\sqrt{n}\Delta L e^{-2})$ Section 4.2 | $O(\kappa^2 \Delta L e^{-2})$ Lin et al., 2020a |
|                   |                          |                          | $\tilde{O}(\sqrt{n}\Delta L e^{-2} \log^2 \frac{1}{\Delta})$ Lin et al., 2020b |
| NC-SC, FS, AS¹    | $\Omega(n + \sqrt{n}\kappa \Delta L e^{-2})$ Theorem 3.2 | $\tilde{O}\left(\frac{n}{\kappa^2 \Delta} \cdot \sqrt{\kappa^2 \Delta L e^{-2}}\right)$ Section 4.2 | $\tilde{O}(n + \sqrt{n}\kappa^2 \Delta L e^{-2})$ $n \geq \kappa^2$ |
|                   |                          |                          | $O\left((nk + \kappa^2) \Delta L e^{-2}\right)$ $n \leq \kappa^2$ Luo et al., 2020, Xu et al., 2020a |

¹ FS: finite-sum, AS: averaged smooth; see Section 2 for definitions.

Table 1: Upper and lower complexity bounds for finding an approximate stationary point. Here $\tilde{O}()$ hides poly-logarithmic factor in $L, \mu$ and $\kappa$. $L$: Lipschitz smoothness parameter, $\mu$: strong convexity parameter, $\kappa$: condition number $\frac{1}{\mu}$; $\Delta$: initial gap of the primal function.

There exists a number of algorithms for solving NC-SC problems in the general setting, including GDmax [Nouiehed et al., 2019], GDA [Lin et al., 2020a], alternating GDA [Yang et al., 2020a, Bot and Böhm, 2020, Xu et al., 2020b], Minimax-PPA [Lin et al., 2020b]. Specifically, GDA and its alternating variant both achieve the complexity of $O(\kappa^2 \Delta L e^{-2})$ [Lin et al., 2020a, Yang et al., 2020a], where $\kappa \leq \frac{L}{\mu}$ is the condition number and $\Delta \leq \Phi(x_0) - \inf_x \Phi(x)$ is the initial function gap. Recently, Lin et al., 2020b provided the best-known complexity of $O(\sqrt{n}\kappa \Delta L e^{-2} \cdot \log^2 \frac{1}{\Delta})$ achieved by Minimax-PPA, which improves the dependence on the condition number but suffers from an extra poly-logarithmic factor in $\frac{1}{\Delta}$.

In the finite-sum setting, several algorithms have been proposed recently, e.g., SGDmax [Jin et al., 2020], PGSMD [Rafique et al., 2018], Stochastic GDA [Lin et al., 2020a], SREDA and its variants [Luo et al., 2020, Xu et al., 2020a]. In particular, [Lin et al., 2020a] proved that Stochastic GDA attains the complexity of $O(\kappa^3 e^{-4})$. Luo et al., 2020 recently showed the state-of-the-art result achieved by SREDA: when $n \geq \kappa^2$, the complexity is $\tilde{O}(n \log \frac{1}{\kappa} + \sqrt{n}\kappa^2 \Delta L e^{-2})$, which is sharper than the batch Minimax-PPA algorithm; when $n \leq \kappa^2$, the complexity is $O\left((nk + \kappa^2) \Delta L e^{-2}\right)$, which is sharper than Stochastic GDA.

Despite this active line of research, whether these state-of-the-art complexity bounds can be further improved remains elusive. As a special case by restricting the domain of $y$ to a singleton, lower bounds for nonconvex smooth minimization, e.g., Carmon et al., 2019a,b, Fang et al., 2018, Zhou and Gu, 2019, Arjevani et al., 2019, hardly capture the dependence on the condition number $\kappa$, which plays a crucial role in the complexity for general NC-SC smooth minimax problems. In many of the aforementioned machine learning applications, the condition number is often proportional to the inverse of the regularization parameter, and could be quite large in practice. For example, in statistical learning, where $n$ represents the sample size, the optimal regularization parameter (i.e. with optimal empirical/generalization trade-off) leads to $\kappa = \Omega(\sqrt{n})$ Shalev-Shwartz and Ben-David, 2014.

This motivates the following fundamental questions: What is the complexity limit for NC-SC problems in the general and finite-sum settings? Can we design new algorithms to meet the performance limits and attain optimal dependence on the condition number?

### 1.1 Contributions

Our contributions, summarized in Table 1, are as follows:

- We establish nontrivial lower complexity bounds for finding an approximate stationary point of nonconvex-strongly-concave (NC-SC) minimax problems. In the general setting, we prove an $\Omega(\sqrt{n}\Delta L e^{-2})$ lower complexity bound which applies to arbitrary deterministic linear-span algorithms interacting with the classical first-order oracle. In the finite-sum setting, we prove an $\Omega\left(n + \sqrt{n}\kappa \Delta L e^{-2}\right)$ lower complexity bound (when $\kappa = \Omega(n)^3)$ for the class of averaged smooth functions and arbitrary linear-span algorithms interacting with a (randomized) incremental first-order oracle (precise definitions in Sections 2 and 3).

¹ A concurrent work by Han et al. [2021] appeared on arXiv two weeks ago, and provided a similar lower bound result for finite-sum NC-SC problems under probabilistic arguments based on geometric random variables.
Our lower bounds build upon two main ideas: first, we start from an NC-SC function whose primal function mimics the lower bound construction in smooth nonconvex minimization [Carmon et al., 2019a]. Crucially, the smoothness parameter of this primal function is boosted by an $\Omega(\kappa)$ factor, which strengthens the lower bound. Second, the NC-SC function has an alternating zero-chain structure, as utilized in lower bounds for convex-concave settings [Ouyang and Xu, 2019]. The combination of these features leads to a hard instance for our problem.

- To bridge the gap between the lower bounds and existing upper bounds in both settings, we introduce a generic Catalyst acceleration framework for NC-SC minimax problems, inspired by [Lin et al., 2018a, Yang et al., 2020b], which applies existing gradient-based methods to solving a sequence of crafted strongly-convex-strongly-concave (SC-SC) minimax subproblems. When combined with the extragradient method, the resulting algorithm achieves an $\tilde{O}(\sqrt{\kappa}DL\epsilon^{-2})$ complexity in terms of gradient evaluations, which tightly matches the lower bound in the general setting (up to logarithmic terms in constants) and shaves off the extra poly-logarithmic term in $\frac{1}{\epsilon}$ required by the state-of-the-art [Lin et al., 2020b]. When combined with stochastic variance-reduced method, the resulting algorithm achieves an overall $\tilde{O}((n + n^{3/4}\sqrt{\kappa})DL\epsilon^{-2})$ complexity for averaged smooth finite-sum problems, which has nearly-tight dependence on the condition number and improves on the best-known upper bound when $n \leq \kappa^4$.

1.2 Related Work

**Lower bounds for minimax problems.** Information-based complexity (IBC) theory [Traub et al., 1988], which derives the minimal number of oracle calls to attain an approximate solution with a desired accuracy, is often used in lower bound analysis of optimization algorithms. Unlike the case of minimization [Nemirovski and Yudin, 1983, Nesterov, 2018, Agarwal et al., 2009, Woodworth and Srebro, 2016, Foster et al., 2019, Carmon et al., 2019a,b, Fang et al., 2018, Zhou and Gu, 2019, Arjevani et al., 2019], lower bounds for minimax optimization are far less understood; only a few recent works provided lower bounds for finding an approximate saddle point of (strongly)-convex-(strongly)-concave minimax problems [Ouyang and Xu, 2019, Zhang et al., 2019, Ibrahim et al., 2020, Xie et al., 2020, Yoon and Ryu, 2021]. Instead, this paper considers lower bounds for NC-SC minimax problems of finding an approximate stationary point, which requires different techniques for constructing zero-chain properties. Note that there exists another line of research on the purely stochastic setting, e.g., [Rafique et al., 2018, Luo et al., 2020, Xu et al., 2020a]; constructing lower bounds in that setting is out of the scope of this paper.

**Complexity of making gradient small.** In nonconvex optimization, most lower and upper complexity bound results are presented in terms of the gradient norm (see a recent survey [Danilova et al., 2020] and references therein for more details). For convex optimization, the optimality gap based on the objective value is commonly used as the convergence criterion. The convergence in terms of gradient norm, albeit easier to check, are far less studied in the literature until recently; see e.g., [Nesterov, 2012, Allen-Zhu, 2018, Foster et al., 2019, Carmon et al., 2019b, Diakonikolas and Guzmán, 2021] for convex minimization and [Diakonikolas, 2020, Diakonikolas and Wang, 2021, Yoon and Ryu, 2021] for convex-concave smooth minimax problems.

**Nonconvex minimax optimization.** In NC-SC setting, as we mentioned, there has been several substantial works. Among them, [Lin et al., 2020b] achieved the best dependency on condition number by combining proximal point algorithm with accelerated gradient descent. [Luo et al., 2020] introduced a variance reduction algorithm, SREDA, and [Xu et al., 2020a] enhanced the analysis to allow bigger stepsizes. [Yuan et al., 2021, Guo et al., 2020] provided algorithms for NC-SC minimax formulation of AUC maximization problems with an additional assumption that the primal function satisfies Polyak-Lojasiewicz condition. In addition, nonconvex-concave minimax optimization, i.e., the function $f$ is only concave in $y$, is extensively explored by [Zhang et al., 2020, Ostrovskii et al., 2020, Thekumparampil et al., 2019, Zhao, 2020, Nouiehed et al., 2019, Yang et al., 2020b]. Recently, [Daskalakis et al., 2020] showed that for general smooth nonconvex-nonconcave objectives the computation of approximate first-order locally optimal solutions is intractable. Therefore, another
line of research is devoted to searching for solutions under additional structural properties [Yang et al., 2020c, Zhou et al., 2017, Yang et al., 2020a, Song et al., 2020, Mertikopoulos et al., 2019, Malitsky, 2019, Diakonikolas et al., 2020, Lin et al., 2018b].

Catalyst acceleration. The catalyst framework was initially studied in [Lin et al., 2015] for convex minimization and extended to nonconvex minimization in [Paquette et al., 2018] to obtain accelerated algorithms. A similar idea to accelerate SVRG appeared in [Frostig et al., 2015]. These work are rooted on the proximal point algorithm (PPA) [Rockafellar, 1976, Güler, 1991] and inexact accelerated PPA [Güler, 1992]. Recently, [Yang et al., 2018b] generalized the idea and obtained state-of-the-art results for solving strongly-convex-concave and nonconvex-concave minimax problems. In contrast, this paper introduces a new catalyst acceleration scheme in the nonconvex-strongly-concave setting, which relies on completely different parameter settings and stopping criterion.

2 Preliminaries

Notations Throughout the paper, we use \( \text{dom} F \) as the domain of a function \( F, \nabla F = (\nabla_x F, \nabla_y F) \) as the full gradient, \( \| \cdot \| \) as the \( \ell_2 \)-norm. We use 0 to represent zero vectors or scalars, \( e_i \) to represent unit vector with the \( i \)-th element being 1. For nonnegative functions \( f(x) \) and \( g(x) \), we say \( f = O(g) \) if \( f(x) \leq cg(x) \) for some \( c > 0 \), and further write \( f = O(g) \) to omit poly-logarithmic terms on constants \( L, \mu \) and \( \kappa \), while \( f = \Omega(g) \) if \( f(x) \geq cg(x) \) (see more in Appendix A).

We introduce definitions and assumptions used throughout.

Definition 2.1 (Primal and Dual Functions) For a function \( f(x, y) \), we define \( \Phi(x) \equiv \max_y f(x, y) \) as the primal function, and \( \Psi(y) \equiv \min_x f(x, y) \) as the dual function. We also define the primal-dual gap at a point \( (\bar{x}, \bar{y}) \) as \( \text{gap}_{\bar{x}, \bar{y}} = \max_{y \in \mathbb{R}^2} f(\bar{x}, y) - \min_{x \in \mathbb{R}^n} f(x, \bar{y}) \).

Definition 2.2 (Lipschitz Smoothness) We say a function \( f(x, y) \) is \( L \)-Lipschitz smooth (L-S) jointly in \( x \) and \( y \) if it is differentiable and for any \( (x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \), \( \| \nabla_x f(x_1, y_1) - \nabla_x f(x_2, y_2) \| \leq L (\| x_1 - x_2 \| + \| y_1 - y_2 \|) \) and \( \| \nabla_y f(x_1, y_1) - \nabla_y f(x_2, y_2) \| \leq L (\| x_1 - x_2 \| + \| y_1 - y_2 \|) \), for some \( L > 0 \).

Definition 2.3 (Average / Individual Smoothness) We say \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \) or \( \{ f_i \}_{i=1}^n \) is \( L \)-averaged smooth (L-AS) if each \( f_i \) is differentiable, and for any \( (x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \), we have
\[
\frac{1}{n} \sum_{i=1}^n \| \nabla f_i(x_1, y_1) - \nabla f_i(x_2, y_2) \|^2 \leq L^2 (\| x_1 - x_2 \|^2 + \| y_1 - y_2 \|^2).
\]

We say \( f \) or \( \{ f_i \}_{i=1}^n \) is \( L \)-individually smooth (L-IS) if each \( f_i \) is \( L \)-Lipschitz smooth.

Average smoothness is a weaker condition than the common Lipschitz smoothness assumption of each component in finite-sum / stochastic minimization [Fang et al., 2018, Zhou and Gu, 2019]. Similarly in minimax problems, the following proposition summarizes the relationship among these different notions of smoothness.

Proposition 2.1 Let \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \). Then we have: (a) If the function \( f \) is L-IS or L-AS, then it is L-S. (b) If \( f \) is L-IS, then it is (2L)-AS. (c) If \( f \) is L-AS, then \( f(x, y) + \frac{\tau_y}{2} \| x - \bar{x} \|^2 - \frac{\tau_x}{2} \| y - \bar{y} \|^2 \) is \( \sqrt{2} (L + \max \{ \tau_x, \tau_y \}) \)-AS for any \( \bar{x} \) and \( \bar{y} \).

Definition 2.4 (Strong Convexity) A differentiable function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) is convex if \( g(x_2) \geq g(x_1) + \langle \nabla g(x_1), x_2 - x_1 \rangle \) for any \( x_1, x_2 \in \mathbb{R}^d \). Given \( \mu \geq 0 \), we say \( f \) is \( \mu \)-strongly convex if \( g(x) - \frac{\mu}{2} \| x \|^2 \) is convex, and it is \( \mu \)-strongly concave if \( -g \) is \( \mu \)-strongly convex.

Next we introduce main assumptions throughout this paper.
We further assume that $f(x, y)$ in (1) is a nonconvex-strongly-concave (NC-SC) function such that $f$ is $L$-S, and $f(x, \cdot)$ is $\mu$-strongly concave for any fixed $x \in \mathbb{R}^{d_1}$; for the finite-sum case, we further assume that $\{f_i\}_{i=1}^n$ is $L$-AS. We assume that the initial primal suboptimality is bounded: $\Phi(x_0) = \inf_x \Phi(x) \leq \Delta$.

Under Assumption 2.1, the primal function $\Phi(\cdot)$ is differentiable and $2\kappa L$-Lipschitz smooth [Lin et al., 2020b, Lemma 23] where $\kappa \triangleq \frac{L}{\mu}$. Throughout this paper, we use the stationarity of the primal function as the convergence criterion.

Definition 2.5 (Convergence Criterion) For a differentiable function $\Phi$, a point $\bar{x} \in \text{dom} \Phi$ is called an $\epsilon$-stationary point of $\Phi$ if $\|\nabla \Phi(\bar{x})\| \leq \epsilon$.

Another commonly used criterion is the stationarity of $f$, i.e., $\|\nabla_x f(\bar{x}, \bar{y})\| \leq \epsilon, \|\nabla_y f(\bar{x}, \bar{y})\| \leq \epsilon$. This is a weaker convergence criterion. We refer readers to [Lin et al., 2020a, Section 4.3] for the comparison of these two criteria.

3 Lower Bounds for NC-SC Minimax Problems

In this section, we establish lower complexity bounds (LB) for finding approximate stationary points of NC-SC minimax problems, in both general and finite-sum settings. We first present the basic components of the oracle complexity framework [Nemirovski and Yudin, 1983] and then proceed to the details for each case. For simplicity, in this section only, we denote $x_d$ as the $d$-th coordinate of $x$ and $x^t$ as the variable $x$ in the $t$-th iteration.

3.1 Framework and Setup

We study the lower bound of finding primal stationary point under the well-known oracle complexity framework [Nemirovski and Yudin, 1983], here we first present the basics of the framework.

Function class We consider the nonconvex-strongly-concave (NC-SC) function class, as defined in Assumption 2.1, with parameters $L, \mu, \Delta > 0$, denoted by $\mathcal{F}_{\text{NCSC}}$.

Oracle class We consider different oracles for the general and finite-sum settings. Define $z \triangleq (x, y)$.

- For the general setting, we consider the first-order oracle (FO), denoted as $\mathcal{O}_{\text{FO}}(f, \cdot)$, that for each query on point $z$, it returns the gradient $\mathcal{O}_{\text{FO}}(f, z) \triangleq (\nabla_x f(x, y), \nabla_y f(x, y))$.

- For the finite-sum setting, incremental first-order oracle (IFO) is often used in lower bound analysis [Agarwal and Bottou, 2015]. This oracle for a function $f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y)$, is such that for each query on point $z$ and given $i \in [n]$, it returns the gradient of the $i$-th component, i.e., $\mathcal{O}_{\text{IFO}}(f, z, i) \triangleq (\nabla_x f_i(x, y), \nabla_y f_i(x, y))$. Here, we consider averaged smooth IFO and individually smooth IFO, denoted as $\mathcal{O}_{\text{IFO}}^{L, \text{AS}}(f)$ and $\mathcal{O}_{\text{IFO}}^{L, \text{IS}}(f)$, where $\{f_i\}_{i=1}^n$ is $L$-AS or $L$-IS, respectively.

Algorithm class In this work, we consider the class of linear-span algorithms interacting with oracle $\mathcal{O}$, denoted as $\mathcal{A}(\mathcal{O})$. These algorithms satisfy the following property: if we let $(z^t)_t$ be the sequence of queries by the algorithm, where $z^t = (x^t, y^t)$; then for all $t$, we have

$$z^{t+1} \in \text{Span}\{z^0, \ldots, z^t; \mathcal{O}(f, z^0), \ldots, \mathcal{O}(f, z^t)\}. \quad (4)$$

For the finite-sum case, the above protocol fits with many existing deterministic and randomized linear-span algorithms. We distinguish the general and finite-sum setting by specifying the used oracle, which is $\mathcal{O}_{\text{FO}}$ or $\mathcal{O}_{\text{IFO}}$, respectively. Most existing first-order algorithms, including simultaneous and alternating update algorithms,
can be formulated as linear-span algorithms. It is worth pointing out that typically the linear span assumption is used without loss of generality, since there is a standard reduction from deterministic linear-span algorithms to arbitrary oracle based deterministic algorithms [Nemirovsky, 1991, 1992, Ouyang and Xu, 2019]. We defer this extension for future work.

**Complexity measures** The efficiency of algorithms is quantified by the oracle complexity [Nemirovski and Yudin, 1983] of finding an $\epsilon$-stationary point of the primal function: for an algorithm $A \in A(\mathcal{O})$ interacting with a FO oracle $\mathcal{O}$, an instance $f \in \mathcal{F}$, we define

$$T_\epsilon(f, A) \triangleq \inf \{ T \in \mathbb{N} \| \nabla \Phi(x^T) \| \leq \epsilon \}$$

as the minimum number of oracle calls $A$ makes to reach stationarity convergence. For the general case, we define the worst-case complexity

$$\text{Compl}_\epsilon(\mathcal{F}, A, \mathcal{O}) \triangleq \sup_{f \in \mathcal{F}} \inf_{A \in A(\mathcal{O})} T_\epsilon(f, A).$$

For the finite-sum case, we consider the randomized complexity [Braun et al., 2017]:

$$\text{Compl}_\epsilon(\mathcal{F}, A, \mathcal{O}) \triangleq \sup_{f \in \mathcal{F}} \inf_{A \in A(\mathcal{O})} \mathbb{E} T_\epsilon(f, A).$$

Following the motivation of analysis discussed in Section 1.1, we will use the zero-chain argument for the analysis. First we define the notion of (first-order) zero-chain [Carmon et al., 2019b] and activation as follows.

**Definition 3.1 (Zero Chain, Activation)** A function $f : \mathbb{R}^d \to \mathbb{R}$ is a first-order zero-chain if for any $x \in \mathbb{R}^d$,

$$\text{supp}(x) \subseteq \{ 1, \cdots, i - 1 \} \Rightarrow \text{supp}(\nabla f(x)) \subseteq \{ 1, \cdots, i \},$$

where $\text{supp}(x) \triangleq \{ i \in [d] \mid x_i \neq 0 \}$ and $[d] = \{ 1, \cdots, d \}$. For an algorithm initialized at $0 \in \mathbb{R}^d$, with iterates $\{x^t\}_t$, we say coordinate $i$ is activated at $x^s$, if $x^t_i \neq 0$ and $x^s_i = 0$, for any $s < t$.

### 3.2 General NC-SC Problems

First we consider the general NC-SC (Gen-NC-SC) minimax optimization problems. Following the above framework, we choose function class $\mathcal{F}_{NCSC}$, oracle $\mathcal{O}_{FO}$, linear-span algorithms $A$, and we analyze the complexity defined in (6).

**Hard instance construction** Inspired by the hard instances constructed in [Ouyang and Xu, 2019, Carmon et al., 2019b], we introduce the following function $F_d : \mathbb{R}^{d+1} \times \mathbb{R}^{d+2} \to \mathbb{R}$ ($d \in \mathbb{N}^+$) and

$$F_d(x, y; \lambda, \alpha) \triangleq \lambda_1 \langle B_d x, y \rangle - \lambda_2 \| y \|^2 - \frac{\lambda_1^2 \sqrt{\alpha}}{2 \lambda_2} (e_1, x) + \frac{\lambda_1^2 \alpha}{2 \lambda_2} \sum_{i=1}^{d} \Gamma(x_i) - \frac{\lambda_1^2 \alpha}{4 \lambda_2} x_{d+1}^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4 \lambda_2},$$

where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ is the parameter vector, $e_1 \in \mathbb{R}^{d+1}$ is the unit vector with the only non-zero element in the first dimension, $\Gamma : \mathbb{R} \to \mathbb{R}$ and $B_d \in \mathbb{R}^{(d+2) \times (d+1)}$ are

$$B_d = \begin{bmatrix} 1 & 1 & 1 \\ \vdots & \ddots & \vdots \\ 1 & -1 & -1 \\ \sqrt[\alpha]{2} & -1 & \sqrt{\alpha} \end{bmatrix}, \quad \Gamma(x) = 120 \int_1^x \frac{t^2(t-1)}{1+t^2} dt.$$
Matrix $B_d$ essentially triggers the activation of variables at each iteration, and function $\Gamma$ introduces non-convexity in $x$ to the instance. By the first-order optimality condition of $F_d(x, \cdot; \lambda, \alpha)$, we can compute its primal function, $\Phi_d$:  
\[
\Phi_d(x; \lambda, \alpha) \triangleq \max_{y \in \mathbb{R}^{d+1}} F_d(x, y; \lambda, \alpha) = \frac{\lambda^2}{2\lambda^2} \left( \frac{1}{2} x^\top A_d x - \sqrt{\alpha} x_1 + \frac{\sqrt{\alpha}}{2} + \alpha \sum_{i=1}^{d} \Gamma(x_i) + \frac{1-\alpha}{2} x_{d+1}^2 \right),
\]
where $A_d \in \mathbb{R}^{(d+1) \times (d+1)}$ is
\[
A_d = (B_d^\top B_d - e_{d+1}e_{d+1}^\top) = \begin{bmatrix} 1 + \sqrt{\alpha} & -1 & \cdots & \cdots & -1 \\ -1 & 2 & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 2 & -1 \\ -1 & -1 & \cdots & \cdots & 1 \end{bmatrix}.
\]

The resulting primal function resembles the worst-case functions used in lower bound analysis of minimization problems [Nesterov, 2018, Carmon et al., 2019b].

Zero-Chain Construction  First we summarize key properties of the instance and its zero-chain mechanism. We further denote $\mathbf{\hat{e}} \in \mathbb{R}^{d+2}$ as the unit vector for the variable $y$ and define $(k \geq 1)$
\[
\mathcal{X}_k \triangleq \text{Span}\{e_1, e_2, \cdots, e_k\}, \quad \mathcal{X}_0 \triangleq \{0\}, \\
\mathcal{Y}_k \triangleq \text{Span}\{\mathbf{\hat{e}}_{d+2}, \mathbf{\hat{e}}_{d+1}, \cdots, \mathbf{\hat{e}}_{d-k+2}\}, \quad \mathcal{Y}_0 \triangleq \{0\},
\]
then we have the following properties for $F_d$.

Lemma 3.1 (Properties of $F_d$) For any $d \in \mathbb{N}^+$ and $\alpha \in [0, 1]$, $F_d(x, y; \lambda, \alpha)$ in (9) satisfies:

(i) The function $F_d(x, \cdot; \lambda, \alpha)$ is $L_F$-Lipschitz smooth where $L_F = \max \left\{ \frac{200\lambda^2_1}{\lambda_2^2}, 2\lambda_1, 2\lambda_2 \right\}$.

(ii) For each fixed $x \in \mathbb{R}^{d+1}$, $F_d(x, \cdot; \lambda, \alpha)$ is $\mu_F$-strongly concave where $\mu_F = 2\lambda_2$.

(iii) The following properties hold:

a) $x = y = 0 \implies \nabla_x F_d \in \mathcal{X}_1$, $\nabla_y F_d = 0$.  

b) $x \in \mathcal{X}_k, \ y \in \mathcal{Y}_k \implies \nabla_x F_d \in \mathcal{X}_{k+1}$, $\nabla_y F_d \in \mathcal{Y}_k$.

c) $x \in \mathcal{X}_{k+1}, \ y \in \mathcal{Y}_k \implies \nabla_x F_d \in \mathcal{X}_{k+1}$, $\nabla_y F_d \in \mathcal{Y}_{k+1}$.

(iv) For $L \geq \mu > 0$, if $\lambda = \lambda^* = (\lambda^*_1, \lambda^*_2) = (\frac{1}{2}, \frac{3}{2})$ and $\alpha \leq \frac{\mu}{100L}$, then $F_d$ is $L$-Lipschitz smooth. Moreover for any fixed $x \in \mathbb{R}^{d+1}$, $F_d(x, \cdot; \lambda, \alpha)$ is $\mu$-strongly concave.

The proof of Lemma 3.1 is deferred to Appendix C.1.1. The first two properties show that function $F_d$ is Lipschitz smooth and NC-SC; the third property above suggests that, starting from $(x, y) = (0, 0)$, the activation process follows an "alternating zero-chain" form [Ouyang and Xu, 2019]. That is, for a linear-span algorithm, when $x \in \mathcal{X}_k$, $y \in \mathcal{Y}_k$, the next iterate will at most activate the $(k+1)$-th coordinate of $x$ while keeping $y$ fixed; similarly when $x \in \mathcal{X}_{k+1}$, $y \in \mathcal{Y}_k$, the next iterate will at most activate the $(d-k+1)$-th element of $y$. We need the following properties of $\Phi_d$ for the lower bound argument.

Lemma 3.2 (Properties of $\Phi_d$) For any $\alpha \in [0, 1]$ and $x \in \mathbb{R}^{d+1}$, if $x_d = x_{d+1} = 0$, we have:
(i) \( \| \nabla \Phi_d(x; \lambda, \alpha) \| \geq \frac{\lambda^2}{3n}\alpha^{3/4}. \)

(ii) \( \Phi_d(0; \lambda, \alpha) - \inf_{x \in \mathbb{R}^n+1} \Phi_d(x; \lambda, \alpha) \leq \frac{\lambda^2 \sqrt{d}}{3n} \left( \frac{\sqrt{d}}{2} + 10\alpha d \right). \)

We defer the proof of Lemma 3.2 to Appendix C.1.2. This lemma indicates that, starting from \((x, y) = (0, 0)\) with appropriate parameter settings, the primal function \(\Phi_d\) will not approximate stationarity until the last two coordinates are activated. Now we are ready to present our final lower bound result for the general NC-SC case.

**Theorem 3.1 (LB for Gen-NC-SC)** For the linear-span first-order algorithm class \(\mathcal{A}\), parameters \(L, \mu, \Delta > 0\), and accuracy \(\epsilon\) satisfying \(\epsilon^2 \leq \min \left( \frac{\Delta L}{6400}, \frac{\Delta L \sqrt{\kappa}}{38400} \right)\), we have

\[
\text{Compl}_{\mathcal{L}_{\text{NCSC}}, \mathcal{A}, \mathcal{O}_{\text{IFO}}} = \Omega(\sqrt{\kappa} \Delta L \epsilon^{-2}).
\]

The hard instance in the proof is established based on \(F_d\) in (9). We choose the scaled function \(f(x, y) = \eta^2 f_d(\frac{x}{\eta}, \frac{y}{\eta}; \lambda^*, \alpha)\) as the final hard instance, which preserves the smoothness and strong convexity (by Lemma B.3), while appropriate setting of \(\eta\) will help to fulfill the requirements on the initial gap and large gradient norm (before thorough activation) of the primal function. The detailed statement and proof of Theorem 3.1 are presented in Appendix C.1.3.

**Remark 3.1 (Tightness of Theorem 3.1)** The best-known upper bounds for general NC-SC problems are \(O(\Delta L n^2 \epsilon^{-2})\) [Lin et al., 2020a, Boţ and Böhm, 2020] and \(O(\Delta \sqrt{\kappa} L \epsilon^{-2} \log^2 \frac{1}{\epsilon})\) [Lin et al., 2020b]. Therefore, our result exhibits significant gaps in terms of the dependence on \(\epsilon\) and \(\kappa\). In order to mitigate these gaps, we propose faster algorithms in Section 4. On the other hand, compared to the \(\Omega(\Delta L \epsilon^{-2})\) lower bound for nonconvex smooth minimization [Carmon et al., 2019a], our result reveals an explicit dependence on \(\kappa\).

### 3.3 Finite-Sum NC-SC Problems

The second case we consider is finite-sum NC-SC (FS-NC-SC) minimax problems, for the function class \(\mathcal{F}_{\text{NCSC}}^{L, \mu, \Delta}\), the linear-span algorithm class \(\mathcal{A}\) and the averaged smooth IFO class \(\mathcal{O}_{\text{IFO}}^{L, \text{AS}}\). The complexity is defined in (7).

**Hard instance construction** To derive the finite-sum hard instance, we modify \(F_d\) in (9) with orthogonal matrices defined as follows.

**Definition 3.2 (Orthogonal Matrices)** For positive integers \(a, b, n \in \mathbb{N}^+\), we define a matrix sequence \(\{U^{(i)}\}_{i=1}^n \in \text{Orth}(a, b, n)\) if for each \(i, j \in \{1, \cdots, n\}\) and \(i \neq j\), \(U^{(i)}, U^{(j)} \in \mathbb{R}^{a \times b}\) and \(U^{(i)}(U^{(i)})^\top = I \in \mathbb{R}^{a \times a}\) and \(U^{(i)}(U^{(j)})^\top = 0 \in \mathbb{R}^{a \times b}\).

Here the intuition for the finite-sum hard instance is combining \(n\) independent copies of the hard instance in the general case (9), then appropriate orthogonal matrices will convert the \(n\) independent variables with dimension \(d\) into one variable with dimension \(n \times d\), which results in the desired hard instance. To preserve the zero chain property, for \(\{U^{(i)}\}_{i=1}^n \in \text{Orth}(d+1, n(d+1), n), \{V^{(i)}\}_{i=1}^n \in \text{Orth}(d+2, n(d+2), n), \forall n, d \in \mathbb{N}^+\) and \(x \in \mathbb{R}^{n(d+1)}, y \in \mathbb{R}^{n(d+2)}\), we set \(U^{(i)}\) and \(V^{(i)}\) by concatenating \(n\) matrices:

\[
U^{(i)} = \begin{bmatrix} 0_{d+1} & \cdots & 0_{d+1} & I_{d+1} & 0_{d+1} & \cdots & 0_{d+1} \end{bmatrix},
\]

\[
V^{(i)} = \begin{bmatrix} 0_{d+2} & \cdots & 0_{d+2} & I_{d+2} & 0_{d+2} & \cdots & 0_{d+2} \end{bmatrix},
\]

where \(0_d, I_d \in \mathbb{R}^{d \times d}\) are the zero and identity matrices respectively, while the \(i\)-th matrix above is the identity matrix. Hence, \(U^{(i)}x\) will be the \((id + d + 1)\)-th to the \((id)\)-th elements of \(x\), similar property also holds for \(V^{(i)}y\).

The hard instance construction here follows the idea of that in the deterministic hard instance (9), the basic motivation is that its primal function will be a finite-sum form of the primal function \(\Phi_d\) defined in the
We defer the proof of Lemma 3.3 (Properties of \(\bar{f}\)) in Appendix 3.2. Define the index set 
\[ I \]
\[ \{i\} \subseteq \{1, \ldots, n\}, \quad i \neq j \quad \forall i, j \in I \]

\[ L \]
\[ \{1, \ldots, n\}, \quad i \neq j \quad \forall i, j \in I \]

We choose the following functions 
\[ H_d : \mathbb{R}^{d+1} \times \mathbb{R}^{d+2} \to \mathbb{R}, \quad \Gamma_d^n : \mathbb{R}^{n(d+1)} \to \mathbb{R} \]

\[ H_d(x, y; \lambda, \alpha) \triangleq \lambda_1(B_d x, y) - \lambda_2\|y\|^2 - \frac{\lambda^2 \sqrt{\alpha}}{2\lambda_2} (\epsilon_1, x) - \frac{\lambda^2 \alpha_2}{4\lambda_2} x_{d+1}^2 + \frac{\lambda^2 \sqrt{\alpha}}{4\lambda_2}, \]

\[ \Gamma_d^n(x) \triangleq \sum_{i=1}^n \sum_{j=i+(d+1)-d}^n \Gamma(x_j), \]

\[ \bar{f}_i(x, y) \triangleq H_d \left( \mathbf{U}^{(i)} x, \mathbf{V}^{(i)} y; \lambda, \alpha \right) + \frac{\lambda^2 \alpha}{2n\lambda_2} \Gamma_d^n(x), \]

\[ \bar{f}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^n \bar{f}_i(x, y) = \frac{1}{n} \sum_{i=1}^n \left[ H_d \left( \mathbf{U}^{(i)} x, \mathbf{V}^{(i)} y; \lambda, \alpha \right) + \frac{\lambda^2 \alpha}{2n\lambda_2} \Gamma_d^n(x) \right], \]

note that by denoting 
\[ \Gamma_d(x) \triangleq \sum_{i=1}^d \Gamma(x_i), \]

we can easily find that

\[ \Gamma_d^n(x) = \sum_{i=1}^n \sum_{j=i+(d+1)-d} \Gamma(x_j) = \sum_{i=1}^n \Gamma_d \left( \mathbf{U}^{(i)} x \right) = \sum_{i=1}^n \sum_{j=1}^d \Gamma \left( \left( \mathbf{U}^{(i)} x \right)_j \right). \]

Define \( u^{(i)} \triangleq \mathbf{U}^{(i)} x \), we summarize the properties of the above functions in the following lemma.

**Lemma 3.3 (Properties of \(\bar{f}\))** For the above functions \(\{\bar{f}_i\}_i\) and \(\bar{f}\) in (17), they satisfy that:

(i) \(\{\bar{f}_i\}_i\) is \(L_F\)-AS where 
\[ L_F = \sqrt{\frac{1}{n} \max \left\{ 16\lambda_1^2 + 8\lambda_2^2, \frac{C^2\lambda_1^4\alpha^2}{n\lambda_2^2} + \frac{\lambda_1\alpha^2}{\lambda_2^2} + 8\lambda_1^2 \right\}.} \]

(ii) \(\bar{f}\) is \(\mu_F\)-strongly concave on \(y\) where 
\[ \mu_F = \frac{2\lambda_2}{n} \]

(iii) For \(n \in \mathbb{N}^+\), \(L \geq 2n\mu > 0\), if we set \(\lambda = \lambda_* = (\lambda_1^*, \lambda_2^*) = \left( \sqrt{\frac{C}{4n}}, \frac{\mu}{L} \right)\), \(\alpha = \frac{nu}{\min\{d, n\}} \in [0, 1]\), then \(\{\bar{f}_i\}_i\) is \(L\)-AS and \(\bar{f}\) is \(\mu\)-strongly concave on \(y\).

(iv) Define \(\tilde{\Phi}(x) \triangleq \max_y \bar{f}(x, y)\), then we have

\[ \tilde{\Phi}(x) = \frac{1}{n} \sum_{i=1}^n \tilde{\Phi}_i(x), \quad \text{where} \quad \Phi_i(x) \triangleq \Phi_d \left( \mathbf{U}^{(i)} x \right), \]

while \(\Phi_d\) is defined in (11).

We defer the proof of Lemma 3.3 to Appendix C.2.1. From Lemma 3.2, we have

\[ \tilde{\Phi}(0) - \inf_{x \in \mathbb{R}^{n(d+1)}} \tilde{\Phi}(x) = \sup_{x \in \mathbb{R}^{n(d+1)}} \frac{1}{n} \sum_{i=1}^n \left( \tilde{\Phi}(0) - \tilde{\Phi}_i(x) \right) \leq \frac{1}{n} \sum_{i=1}^n \sup_{x \in \mathbb{R}^{d+1}} \left( \tilde{\Phi}(0) - \tilde{\Phi}_i(x) \right) \]

\[ = \frac{1}{n} \sum_{i=1}^n \left( \sup_{x \in \mathbb{R}^{d+1}} \left( \tilde{\Phi}_d(0) - \tilde{\Phi}_d \left( \mathbf{U}^{(i)} x \right) \right) \right) \leq \frac{\lambda_1^2}{2\lambda_2} \left( \frac{\sqrt{\alpha}}{2} + 10\alpha d \right). \]
orthogonality and Lemma 3.2 we have

\[ \| \nabla \Phi(x) \|^2 = \left( \frac{1}{n} \sum_{i=1}^{n} \nabla \Phi_i(x) \right) = \left( \frac{1}{n} \sum_{i=1}^{n} \nabla \Phi_d(U^{(i)}_x) \right) = \left( \frac{1}{n^2} \sum_{i=1}^{n} (U^{(i)})^\top \nabla \Phi_d(U^{(i)}_x) \right) \]

\[ \geq \frac{1}{n^2} \frac{\lambda_2^2}{8 \lambda_2} \left( \alpha \frac{\lambda_1}{2} \right)^2 = \frac{\lambda_1^2}{128n \lambda_2^2} \alpha \frac{\lambda_1}{2}. \]

Now we arrive at our final theorem for the averaged smooth FS-NC-SC case as follows.

**Theorem 3.2 (LB for AS FS-NC-SC)** For the linear-span algorithm class \( A \), parameters \( L, \mu, \Delta > 0 \) and component size \( n \in \mathbb{N}^+ \), if \( L \geq 2n\mu > 0 \), the accuracy \( \epsilon \) satisfies that \( \epsilon^2 \leq \min \left( \frac{\sqrt{\pi} L^2 \Delta}{1280m^2 \mu}, \frac{\alpha L^2 \Delta}{1280m^2 \mu} \right) \) where \( \alpha = \frac{\mu}{3\lambda_2 L} \in [0, 1] \), then we have

\[ \text{Compl}_\epsilon \left( \mathcal{F}_{\text{NCSC}}^{L, \mu, \Delta}, A, O(\text{IFO}) \right) = O(n + \sqrt{n \kappa \Delta L \epsilon^{-2}}). \]  

(22)

The theorem above indicates that for any \( \mathbf{A} \in A \), we can construct a function \( f(x,y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x,y) \), such that \( f \in \mathcal{F}_{\text{NCSC}}^{L, \mu, \Delta} \) and \( \{ f_i \} \) is L-AS, and \( \mathbf{A} \) requires at least \( \Omega(n + \sqrt{n \kappa \Delta L \epsilon^{-2}}) \) IFO calls to attain an approximate stationary point of its primal function (in terms of expectation). The hard instance construction is based on \( f \) and \( \tilde{f} \), above (17), combined with a scaling trick similar to the one in the general case. Also we remark that lower bound holds for small enough \( \epsilon \), while the requirement on \( \epsilon \) is comparable to those in existing literature, e.g. [Zhou and Gu, 2019, Han et al., 2021]. The detailed statement and proof of the theorem are deferred to Appendix C.2.2.

**Remark 3.2 (Tightness of Theorem 3.2)** The state-of-the-art upper bound for NC-SC finite-sum problems is \( \tilde{O}(n + \sqrt{n \kappa^2 \Delta L \epsilon^{-2}}) \) when \( n \geq \kappa^2 \) and \( O((nk + \kappa)^2 \Delta L \epsilon^{-2}) \) when \( n \leq \kappa^2 \) [Luo et al., 2020, Xu et al., 2020a]. Note that there is still a large gap between upper and lower bounds on the dependence in terms of \( \kappa \) and \( n \), which motivates the design of faster algorithms for FS-SC case, we address this in Section 4. Note that a weaker result on the lower bound of nonconvex finite-sum averaged smooth minimization is \( \Omega(n + \sqrt{n \kappa \Delta L \epsilon^{-2}}) \) [Fang et al., 2018, Zhou and Gu, 2019, Li et al., 2020]; here, our result presents explicitly the dependence on \( \kappa \).

### 4 Faster Algorithms for NC-SC Minimax Problems

In this section, we introduce a generic Catalyst acceleration scheme that turns existing optimizers for (finite-sum) SC-SC minimax problems into efficient, near-optimal algorithms for (finite-sum) NC-SC minimax optimization. Rooted in the inexact accelerated proximal point algorithm, the idea of Catalyst acceleration was introduced in Lin et al. [2015] for convex minimization and later extended to nonconvex minimization in Paquette et al. [2018] and nonconvex-concave minimax optimization in Yang et al. [2020b]. In stark contrast, we focus on NC-SC minimax problems.

The backbone of our Catalyst framework is to repeatedly solve regularized subproblems of the form:

\[ \min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^s} \left( f(x,y) + L \| x - \tilde{x}_t \|^2 - \frac{\tau}{2} \| y - \tilde{y}_t \|^2 \right), \]

where \( \tilde{x}_t \) and \( \tilde{y}_t \) are carefully chosen prox-centers, and the parameter \( \tau \geq 0 \) is selected such that the condition numbers for \( x \)-component and \( y \)-component of these subproblems are well-balanced. Since \( f \) is \( L \)-Lipschitz smooth and \( \mu \)-strongly concave in \( y \), the above auxiliary problem is \( L \)-strongly convex in \( x \) and \( (\mu + \tau) \)-strongly concave in \( y \). Therefore, it can be easily solved by a wide family of off-the-shelf first-order algorithms with linear convergence rate.
Our Catalyst framework, presented in Algorithm 1, consists of three crucial components: an inexact proximal point step for primal update, an inexact accelerated proximal point step for dual update, and a linear-convergent algorithm for solving the subproblems.

**Inexact proximal point step in the primal.** The $x$-update in the outer loop, $\{x^0_t\}_{t=1}^T$, can be viewed as applying an inexact proximal point method to the primal function $\Phi(x)$, requiring to solve the following sequence of auxiliary problems:

$$
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^d} \left[ f^*_t(x, y) \triangleq f_t(x, y) + L\|x - x^0_t\|^2 \right].
$$

Inexact proximal point methods have been explored in minimax optimization in several work, e.g. [Lin et al., 2020b, Rafique et al., 2018]. Our scheme is distinct from these work in two aspects: (i) we introduce a new subroutine to approximately solve the auxiliary problems \((\ast)\) with near-optimal complexity, and (ii) the inexactness is measured by an adaptive stopping criterion using the gradient norms:

$$
\|\nabla f_t(x_0^{t+1}, y_0^{t+1})\|^2 \leq \alpha_t\|\nabla f_t(x_0^t, y_0^t)\|^2,
$$

where $\{\alpha_t\}_t$ is carefully chosen. Using the adaptive stopping criterion significantly reduces the complexity of solving the auxiliary problems. We will show that the number of steps required is only logarithmic in $L, \mu$ without any dependence on target accuracy $\epsilon$. Although the auxiliary problem is $(L, \mu)$-SC-SC and can be solved with linear convergence by algorithms such as extragradient, OGDA, etc., these algorithms are not optimal in terms of the dependency on the condition number when $L > \mu$ [Zhang et al., 2019].

**Inexact accelerated proximal point step in the dual.** To solve the auxiliary problem with optimal complexity, we introduce an inexact accelerated proximal point scheme. The key idea is to add an extra regularization in $y$ to the objective such that the strong-convexity and strong-concavity are well-balanced. Therefore, we propose to iteratively solve the subproblems:

$$
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^d} \left[ f^*_t(x, y) \triangleq \hat{f}_t(x, y) - \frac{\tau}{2}\|y - z_k\|^2 \right],
$$

where $\{z_k\}_k$ is updated analogously to Nesterov’s accelerated method [Nesterov, 2005] and $\tau \geq 0$ is the regularization parameter. For example, by setting $\tau = L - \mu$, the subproblems become $(L, L)$-SC-SC and can be approximately solved by extragradient method with optimal complexity, to be discussed in more details in next section. Finally, when solving these subproblems, we use the following stopping criterion $\|\nabla \hat{f}_{t,k}(x, y)\|^2 \leq \epsilon_k^* \infty$ with time-varying accuracy $\epsilon_k^*$ that decays exponentially with $k$.

**Linearly-convergent algorithms for SC-SC subproblems.** Let $\mathcal{M}$ be any algorithm that solves the subproblem \((\ast\ast)\) (denoting $(x^*, y^*)$ as the optimal solution) at a linear convergence rate such that after $N$ iterations:

$$
\|x_N - x^*\|^2 + \|y_N - y^*\|^2 \leq \left(1 - \frac{1}{\Lambda_{L, \mu}^*(\tau)}\right)^N \left[\|x_0 - x^*\|^2 + \|y_0 - y^*\|^2\right],
$$

if $\mathcal{M}$ is a deterministic algorithm; or taking expectation to the left-hand side above if $\mathcal{M}$ is randomized. The choices for $\mathcal{M}$ include, but are not limited to, extragradient (EG) [Tseng, 1995], optimistic gradient descent ascent (OGDA) [Gidel et al., 2018], SVRG [Balamurugan and Bach, 2016], SPDI-VR [Tan et al., 2018], SVRE [Chavdarova et al., 2019], Point-SAGA [Luo et al., 2019], and variance reduced prox-method [Carmon et al., 2019c]. For example, in the case of EG, $\Lambda_{L, \mu}^*(\tau) = \frac{L + \max\{2L, \tau\}}{4 \min\{L, \mu + \tau\}}$ [Tseng, 1995].
**Algorithm 1** Catalyst for NC-SC Minimax Problems

**Input:** objective $f$, initial point $(x_0, y_0)$, smoothness constant $L$, strong-concavity const. $\mu$, and param. $\tau > 0$.

1. Let $(x^0_0, y^0_0) = (x_0, y_0)$ and $q = \frac{\mu}{\mu + \tau}$.
2. **for all** $t = 0, 1, ..., T$ **do**
   3. Let $z_1 = y^0_0$ and $k = 1$.
   4. Let $\hat{f}_t(x, y) \triangleq f(x, y) + L\|x - x^t_0\|^2$.
   5. **repeat**
      6. Find inexact solution $(x^t_k, y^t_k)$ to the problem below by algorithm $M$ with initial point $(x^t_{k-1}, y^t_{k-1})$:
         $$\min_{x \in \mathbb{R}^d_1} \max_{y \in \mathbb{R}^d_2} \left[ \hat{f}_{t, k}(x, y) - f(x, y) + L\|x - x^t_0\|^2 - \frac{\tau}{2}\|y - z^t_k\|^2 \right]$$
         (***)  
         such that $\|\nabla \hat{f}_{t, k}(x^t_k, y^t_k)\|^2 \leq \epsilon^t_k$.
   7. Let $z_{k+1} = y^t_k + \frac{\tau}{\sqrt{\alpha_t \tau + \eta}}(y^t_k - y^t_{k-1})$, $k = k + 1$.
   8. **until** $\|\nabla \hat{f}_{t, k}(x^t_k, y^t_k)\|^2 \leq \alpha_t \|\nabla \hat{f}_{t, k}(x^t_0, y^t_0)\|^2$  
   9. Set $(x^{t+1}_0, y^{t+1}) = (x^t_k, y^t_k)$.
10. **end for**

**Output:** $\hat{x}_T$, which is uniformly sampled from $x^1_0, ..., x^T_0$.

### 4.1 Convergence Analysis

In this section, we analyze the complexity of each of the three components we discussed. Let $T$ denote the outer-loop complexity, $K$ the inner-loop complexity, and $N$ the number of iterations for $M$ (expected number if $M$ is randomized) to solve subproblem (***) of Algorithm 1. The total complexity of Algorithm 1 is computed by multiplying $K, T$ and $N$. Later, we will provide a guideline for choosing parameter $\tau$ to achieve the best complexity, given an algorithm $M$.

**Theorem 4.1 (Outer loop)** Suppose function $f$ is NC-SC with strong convexity parameter $\mu$ and $L$-Lipschitz smooth. If we choose $\alpha_t = \frac{\mu}{\sqrt{\mu + \tau}}$ for $t > 0$ and $\alpha_0 = \frac{\sqrt{\mu}}{\sqrt{\mu} \max\{1, L\}}$, the output $\hat{x}_T$ from Algorithm 1 satisfies

$$\mathbb{E}\|\nabla \Phi(\hat{x}_T)\|^2 \leq \frac{268L}{5T}\Delta + \frac{28L}{5T}D^0_y,$$

where $\Delta = \Phi(x_0) - \inf_x \Phi(x)$, $D^0_y = \|y_0 - y^*(x_0)\|^2$ and $y^*(x_0) = \arg\max_{y \in \mathbb{R}^d_2} f(x_0, y)$.

This theorem implies that the algorithm finds an $\epsilon$ stationary point of $\Phi$ after inexactly solving (*) for $T = O\left(L(\Delta + D^0_y)\epsilon^{-2}\right)$ times. The dependency on $D^0_y$ can be eliminated if we select the initialization $y_0$ close enough to $y^*(x_0)$, which only requires an additional logarithmic cost by maximizing a strongly concave function.

**Theorem 4.2 (Inner loop)** Under the same assumptions in Theorem 4.1, if we choose $\epsilon^t_k = \frac{\sqrt{\mu}}{\sqrt{\mu} + \tau} \text{gap}_k \hat{f}_k(x^t_0, y^t_0)$ with $\rho < \sqrt{q} = \sqrt{\frac{\mu}{\mu + \tau}}$, we have

$$\|\nabla \hat{f}_{t, k}(x^t_k, y^t_k)\|^2 \leq \frac{5508L^2}{\mu^2(\sqrt{q} - \rho)^2} + \frac{18\sqrt{2}L^2}{\mu}(1 - \rho)^k \|\nabla \hat{f}_{t, k}(x^t_0, y^t_0)\|^2.$$

Particularly, setting $\rho = 0.9\sqrt{q}$, Theorem 4.2 implies after inexactly solving (***) for $K = \tilde{O}\left(\sqrt{(\tau + \mu) / \mu} \log \frac{1}{\epsilon^t_k}\right)$ times, the stopping criterion (23) is satisfied. This complexity decreases with $\tau$. However, we should not choose $\tau$ too small, because the smaller $\tau$ is, the harder it is for $M$ to solve (**). The following theorem captures the complexity for algorithm $M$ to solve the subproblem.
Theorem 4.3 (Complexity of solving subproblems (⋆⋆)) Under the same assumptions in Theorem 4.1 and the choice of $\epsilon_k^j$ in Theorem 4.2, the number of iterations (expected number of iterations if $\mathcal{M}$ is stochastic) for $\mathcal{M}$ to solve (⋆⋆) such that $\|\nabla f_{L,k}(x,y)\|^2 \leq \epsilon_k^j$ is

$$N = O\left(\Lambda_{\mu,L}^\mathcal{M}(\tau) \log \left(\frac{\max\{1, L, \tau\}}{\min\{1, \mu\}}\right)\right).$$

The above result implies that the subproblems can be solved within constant iterations that only depends on $L, \mu, \tau$ and $\Lambda_{\mu,L}^\mathcal{M}$. This largely benefits from the use of warm-starting and stopping criterion with time-varying accuracy. In contrast, other inexact proximal point algorithms in minimax optimization, such as [Yang et al., 2020b, Lin et al., 2020b], fix the target accuracy, thus their complexity of solving the subproblems usually has an extra logarithmic factor in $1/\epsilon$.

The overall complexity of the algorithm follows immediately after combining the above three theorems:

Corollary 4.1 Under the same assumptions in Theorem 4.1 and setting in Theorem 4.2, the total number (expected number if $\mathcal{M}$ is randomized) of gradient evaluations for Algorithm 1 to find an $\epsilon$-stationary point of $\Phi$, is

$$\tilde{O}\left(\frac{\Lambda_{\mu,L}^\mathcal{M}(\tau)L(\Delta + D_y^0)}{\epsilon^2} \sqrt{\frac{\mu + \tau}{\mu}}\right).$$

In order to minimize the total complexity, we should choose the regularization parameter $\tau$ that minimizes $\Lambda_{\mu,L}^\mathcal{M}(\tau)\sqrt{\mu + \tau}$.

### 4.2 Specific Algorithms and Complexities

In this subsection, we discuss specific choices for $\mathcal{M}$ and the corresponding optimal choices of $\tau$, as well as the resulting total complexities for solving NC-SC problems.

**Catalyst-EG/OGDA algorithm.** When solving NC-SC minimax problems in the general setting, we set $\mathcal{M}$ to be either extra-gradient method (EG) or optimistic gradient descent ascent (OGDA). Hence, we have $\Lambda_{\mu,L}^\mathcal{M}(\tau) = \frac{L + \max\{2L, \tau\}}{4\min\{L, \mu + \tau\}}$ [Tseng, 1995, Gidel et al., 2018, Azizian et al., 2020]. Minimizing $\Lambda_{\mu,L}^\mathcal{M}(\tau)\sqrt{\mu + \tau}$ yields that the optimal choice for $\tau$ is $L - \mu$. This leads to a total complexity of

$$\tilde{O}\left(\sqrt{rL(\Delta + D_y^0)}\epsilon^{-2}\right).$$

Remark 4.1 The above complexity matches the lower bound in Theorem 3.1, up to a logarithmic factor in $L$ and $\kappa$. It improves over Minimax-PPA [Lin et al., 2020b] by $\log^2(1/\epsilon)$, GDA [Lin et al., 2020a] by $\kappa^2$ and therefore achieves the best of two worlds in terms of dependency on $\kappa$ and $\epsilon$. In addition, our Catalyst-EG/OGDA algorithm does not require the bounded domain assumption on $y$, unlike [Lin et al., 2020b].

**Catalyst-SVRG/SAGA algorithm.** When solving NC-SC minimax problems in the averaged smooth finite-sum setting, we set $\mathcal{M}$ to be either SVRG or SAGA. Hence, we have $\Lambda_{\mu,L}^\mathcal{M}(\tau) \propto n + \left(\frac{L + \sqrt{\max\{2L, \tau\}}}{\min\{L, \mu + \tau\}}\right)^2$ [Balamurugan and Bach, 2016]. Minimizing $\Lambda_{\mu,L}^\mathcal{M}(\tau)\sqrt{\mu + \tau}$, the best choice for $\tau$ is (proportional to) $\max\left\{\frac{\mu}{\sqrt{\kappa}} - \mu, 0\right\}$, which leads to the total complexity of

$$\tilde{O}\left(\left(n + n^2\sqrt{\kappa}\right)\Delta + D_y^0\epsilon^{-2}\right).$$

Remark 4.2 According to the lower bound established in Theorem 3.2, the dependency on $\kappa$ in the above upper bound is nearly tight, up to logarithmic factors. Recall that SREDA [Luo et al., 2020] and SREDA-boost [Xu et al., 2020a] achieve the complexity of $\tilde{O}\left(\kappa n^2\epsilon^{-2} + n + (n + \kappa)\log(\kappa)\right)$ for $n \geq \kappa^2$ and $O\left((\kappa^2 + \kappa n)\epsilon^{-2}\right)$ for $n \leq \kappa^2$. Hence, our Catalyst-SVRG/SAGA algorithm attains better complexity in the regime $n \leq \kappa^2$.\footnote{Although Balamurugan and Bach [2016] assumes individual smoothness, their analysis can be extended to average smoothness.}
Particularly, in the critical regime $\kappa = \Omega(\sqrt{n})$ arising in statistical learning [Shalev-Shwartz and Ben-David, 2014], our algorithm performs strictly better.

5 Conclusion

In this work, we take an initial step towards understanding the fundamental limits of minimax optimization in the nonconvex-strongly-concave setting for both general and finite-sum cases, and bridge the gaps between lower and upper bounds. It remains interesting to investigate whether the dependence on $n$ can be further tightened in the complexity for finite-sum NC-SC minimax optimization.
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A Notations

For convenience, we summarize some of the notations used in the paper.

- SC / C / NC / WC: strongly convex, convex, nonconvex, weakly-convex.
- FS: finite-sum.
- L-S: L-Lipschitz smooth. L-IS / AS: L-Lipschitz individual / averaged smoothness.
- SOTA: state-of-the-art, LB / UB: lower / upper bound
- FO / IFO: first-order oracle, incremental first-order oracle, denoted by \( \mathcal{O}_{FO} \) and \( \mathcal{O}_{IFO} \).
- \( A \): linear-span first-order algorithm class.
- \( \Phi(x), \Psi(y) \): primal and dual functions of \( f(x, y) \).
- \( \nabla_f \), \( \nabla_y f \): gradients of a function \( F \) with respect to \( x \) and \( y \). Also we set \( \nabla f = (\nabla_x f, \nabla_y f) \).
- \( \nabla_x^2 f, \nabla_y^2 f, \nabla_x^2 f, \nabla_y^2 f \): the Hessian of \( F(x, y) \) with respect to different components.
- \( \{U^{(i)}\}_{i=1}^{n} \in \text{Orth}(a, b, n) \): a matrix sequence where if for each \( i, j \in [1, n] \) and \( i \neq j \), \( U^{(i)}, U^{(j)} \in \mathbb{R}^{a \times b} \) and \( U^{(i)}(U^{(i)})^\top = I \in \mathbb{R}^{a \times a} \) and \( U^{(i)}(U^{(j)})^\top = 0 \in \mathbb{R}^{a \times a} \).
- \( e_i \): unit vector with the \( i \)-th element as 1.
- \( 0 \): zero scalars or vectors.
- \( X_k = \text{Span}\{e_1, e_2, \cdots, e_k\}, Y_k = \text{Span}\{e_{d+1}, e_d, \cdots, e_{d-k+2}\}, X_0 = Y_0 = \{0\} \).
- \( a \lor b = \max\{a, b\}, a \land b = \min\{a, b\} \).
- \( \|\cdot\|_2 \): \( \ell_2 \)-norm.
- \( \mathbb{N}^+ \): all positive integers.
- \( \mathbb{N} \): all nonnegative integers.
- \( \text{dom} f \): the domain of a function \( f \).
- \( d_1, d_2 \in \mathbb{N}^+ \): dimension numbers of \( x \) and \( y \).
- \( x_d \): the \( d \)-th coordinate of \( x \), \( x^t \): the variable \( x \) in the \( t \)-th iteration (in Section 3 and Appendix C only)

B Useful Lemmas and Proofs of Section 2

Lemma B.1 (Lemma B.2 [Lin et al., 2020b]) Assume \( f(\cdot, y) \) is \( \mu_x \)-strongly convex for \( \forall y \in \mathbb{R}^{d_2} \) and \( f(x, \cdot) \) is \( \mu_y \)-strongly concave for \( \forall x \in \mathbb{R}^{d_1} \) (we will later refer to this as \( (\mu_x, \mu_y)\)-SC-SC)) and \( f \) is L-Lipschitz smooth. Then we have

\[
\begin{align*}
    a) \quad y^*(x) &= \arg\max_{y \in \mathbb{R}^{d_2}} f(x, y) \quad \text{is} \quad \frac{L^2}{\mu_y} \text{-Lipschitz}; \\
    b) \quad \Phi(x) &= \max_{y \in \mathbb{R}^{d_2}} f(x, y) \quad \text{is} \quad \frac{2L^2}{\mu_y} \text{-Lipschitz smooth and} \quad \mu_x \text{-strongly convex with} \quad \nabla \Phi(x) = \nabla_x f(x, y^*(x)); \\
    c) \quad x^*(y) &= \arg\min_{x \in \mathbb{R}^{d_1}} f(x, y) \quad \text{is} \quad \frac{L^2}{\mu_x} \text{-Lipschitz}; \\
    d) \quad \Psi(y) &= \min_{x \in \mathbb{R}^{d_1}} f(x, y) \quad \text{is} \quad \frac{2L^2}{\mu_y} \text{-Lipschitz smooth and} \quad \mu_y \text{-strongly concave with} \quad \nabla \Psi(y) = \nabla_y f(x^*(y), y).
\end{align*}
\]

Lemma B.2 Under the same assumptions as Lemma B.1, we have

\[
\begin{align*}
    a) \quad \text{gap}_f(x, y) \leq \frac{L^2}{\mu_y} \|x - x^*\|^2 + \frac{L^2}{\mu_x} \|y - y^*\|^2, \quad \text{where} \quad (x^*, y^*) \quad \text{is} \quad \text{the optimal solution} \quad \text{to} \quad \min_{x \in \mathbb{R}^{d_1}} \max_{y \in \mathbb{R}^{d_2}} f(x, y).
\end{align*}
\]
b) \( \text{gap}_f(x,y) \leq \frac{1}{2\mu_y} \|\nabla_x f(x,y)\|^2 + \frac{1}{2\mu_y} \|\nabla_y f(x,y)\|^2 \).

c) \( \frac{\mu_x}{2}\|x-x^*\|^2 + \frac{\mu_y}{2}\|y-y^*\|^2 \leq \text{gap}_f(x,y) \).

d) \( \|\nabla_x f(x,y)\|^2 + \|\nabla_y f(x,y)\|^2 \leq 4L^2(\|x-x^*\|^2 + \|y-y^*\|^2) \).

Proof

a) Because \( \Phi(x) \) is \( \frac{2L^2}{\mu_y} \)-smooth by Lemma B.1 and \( \nabla \Phi(x^*) = 0 \), we have \( \Phi(x) - \Phi(x^*) \leq \frac{L^2}{\mu_y} \|x-x^*\|^2 \). Similarly, because \( \Psi(y) \) is \( \frac{2L^2}{\mu_x} \)-smooth and \( \Psi(y^*) = 0 \), we have \( \Psi(y^*) - \Psi(y) \leq \frac{L^2}{\mu_y} \|y-y^*\|^2 \). We reach the conclusion by noting that \( \text{gap}_f(x,y) = \Phi(x) - \Psi(y) \) and \( \Phi(x^*) = \Psi(y^*) \).

b) Because \( f(\cdot,y) \) is \( \mu_x \)-strongly-convex and \( \nabla_x f(x^*(y),y) = 0 \), we have \( f(x,y) - \min_x f(x,y) \leq \langle \nabla_x f(x^*(y),y), x-x^*(y) \rangle + \frac{1}{2\mu_x} \|\nabla_x f(x^*(y),y)\|^2 \). Similarly, we have \( \max_y f(x,y) - f(x,y) \leq \frac{1}{2\mu_y} \|\nabla_y f(x,y)\|^2 \). Then we note that \( \text{gap}_f(x,y) = \max_y f(x,y) - f(x,y) + f(x,y) - \min_x f(x,y) \).

c) Because \( \Phi(x) \) is \( \mu_x \)-strongly-convex and \( \nabla \Phi(x^*) = 0 \), we have \( \Phi(x) \geq \Phi(x^*) + \frac{\mu_x}{2}\|x-x^*\|^2 \). Similarly, because \( \Psi(y) \) is \( \mu_y \)-strongly-concave and \( \nabla \Psi(y^*) = 0 \), we have \( \Psi(y^*) - \Psi(y) \geq \frac{\mu_y}{2}\|y-y^*\|^2 \).

d) By definition of Lipschitz smoothness, \( \|\nabla_x f(x,y)\|^2 = \|\nabla_x f(x,y) - \nabla_x f(x^*,y^*)\|^2 \leq L^2(\|x-x^*\|^2 + \|y-y^*\|^2) \) and \( \|\nabla_y f(x,y)\|^2 = \|\nabla_y f(x,y) - \nabla_y f(x^*,y^*)\|^2 \leq L^2(\|x-x^*\|^2 + \|y-y^*\|^2) \).

Proof of Proposition 2.1

Proof (a) and (b) directly follow from the definition of averaged smoothness and individual smoothness.

c) Denote

\[
\bar{f}(x,y) = f(x,y) + \frac{\tau_x}{2}\|x-\bar{x}\|^2 - \frac{\tau_y}{2}\|y-\bar{y}\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ f_i(x,y) + \frac{\tau_x}{2}\|x-\bar{x}\|^2 - \frac{\tau_y}{2}\|y-\bar{y}\|^2 \right] \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x,y),
\]

where \( f_i(x,y) = f_i(x,y) + \frac{\tau_x}{2}\|x-\bar{x}\|^2 - \frac{\tau_y}{2}\|y-\bar{y}\|^2 \). Note that for any \( (x_1,y_1) \) and \( (x_2,y_2) \),

\[
\begin{align*}
\|\nabla_x f_i(x_1,y_1) - \nabla_x f_i(x_2,y_2)\|^2 &\leq 2\|\nabla_x f_i(x_1,y_1) - \nabla_x f_i(x_2,y_2)\|^2 + 2\tau_x^2\|x_1-x_2\|^2, \\
\|\nabla_y f_i(x_1,y_1) - \nabla_y f_i(x_2,y_2)\|^2 &\leq 2\|\nabla_y f_i(x_1,y_1) - \nabla_y f_i(x_2,y_2)\|^2 + 2\tau_y^2\|y_1-y_2\|^2.
\end{align*}
\]

Therefore,

\[
\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_1,y_1) - \nabla f_i(x_2,y_2)\|^2 \leq 2\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_1,y_1) - \nabla f_i(x_2,y_2)\|^2 + 2[\tau_x^2\|x_1-x_2\|^2 + \tau_y^2\|y_1-y_2\|^2] \leq (2L^2 + 2\max\{\tau_x^2,\tau_y^2\}) (\|x_1-x_2\|^2 + \|y_1-y_2\|^2).
\]

An important trick to transform the basic hard instance into the final hard instance is scaling, which will preserve the smoothness of the original function while extend the domain of the function to a high dimension, i.e., enlarging \( d \), which helps to increase the lower bound. The properties of scaling is summarized in the following lemma.

Lemma B.3 (Scaling and Smoothness) For a function \( \bar{g}(x,y) \) defined on \( \mathbb{R}^d_1 \times \mathbb{R}^d_2 \), if \( \bar{g} \) is \( L \)-smooth, then for the following scaled function:

\[
g(x,y) = \eta^2 \bar{g} \left( \frac{x}{\eta}, \frac{y}{\eta} \right),
\]

(29)
then \( g \) is also \( L \)-smooth. Furthermore if the function \( \bar{g} \) has a finite-sum form: \( \bar{g}(x, y) = \frac{1}{n} \sum_{i=1}^{n} \bar{g}_i(x, y) \), if \( \{\bar{g}_i\}_{i=1}^{n} \) is \( L \)-averaged smooth, then for the following functions:

\[
g_i(x, y) = \eta^2 \bar{g}_i \left( \frac{x}{\eta}, \frac{y}{\eta} \right), \quad \text{and} \quad g(x, y) = \frac{1}{n} \sum_{i=1}^{n} g_i(x, y) = \frac{1}{n} \sum_{i=1}^{n} \eta^2 \bar{g}_i \left( \frac{x}{\eta}, \frac{y}{\eta} \right),
\]

\( \{g_i\}_{i=1}^{n} \) is also \( L \)-averaged smooth. If we further assume \( \{\bar{g}_i\}_{i=1}^{n} \) is \( L \)-individually smooth, then \( \{g_i\}_{i=1}^{n} \) is also \( L \)-individually smooth.

**Proof** For the first statement, note that \( \nabla g(x, y) = \eta \nabla \bar{g} \left( \frac{x}{\eta}, \frac{y}{\eta} \right) \), so for any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d\),

\[
\| \nabla_x g(x_1, y_1) - \nabla_x g(x_2, y_2) \| = \left\| \eta \nabla_x \bar{g} \left( \frac{x_1}{\eta}, \frac{y_1}{\eta} \right) - \eta \nabla_x \bar{g} \left( \frac{x_2}{\eta}, \frac{y_2}{\eta} \right) \right\| 
\]

\[
\leq \eta L \left( \left\| \frac{x_1}{\eta} - \frac{x_2}{\eta} \right\| + \left\| \frac{y_1}{\eta} - \frac{y_2}{\eta} \right\| \right) = L \left( \| x_1 - x_2 \| + \| y_1 - y_2 \| \right),
\]

similar conclusion also holds for \( \nabla_y g \), which verifies the first conclusion.

For the averaged smooth finite-sum statement, note that \( \nabla g_i(x, y) = \eta \nabla \bar{g}_i \left( \frac{x}{\eta}, \frac{y}{\eta} \right) \), so for any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d\),

\[
E \left[ \| \nabla g_i(x_1, y_1) - \nabla g_i(x_2, y_2) \|^2 \right] 
\]

\[
= E \left[ \| \eta \nabla \bar{g}_i \left( \frac{x_1}{\eta}, \frac{y_1}{\eta} \right) - \eta \nabla \bar{g}_i \left( \frac{x_2}{\eta}, \frac{y_2}{\eta} \right) \|^2 \right] 
\]

\[
= \eta^2 E \left[ \| \nabla \bar{g}_i \left( \frac{x_1}{\eta}, \frac{y_1}{\eta} \right) - \nabla \bar{g}_i \left( \frac{x_2}{\eta}, \frac{y_2}{\eta} \right) \|^2 \right] 
\]

\[
\leq \eta^2 L^2 \left( \left\| \frac{x_1}{\eta} - \frac{x_2}{\eta} \right\|^2 + \left\| \frac{y_1}{\eta} - \frac{y_2}{\eta} \right\|^2 \right) = L^2 \left( \| x_1 - x_2 \|^2 + \| y_1 - y_2 \|^2 \right),
\]

so \( \{g_i\}_{i=1}^{n} \) is \( L \)-averaged smooth.

For the individually smooth case statement, note that each \( g_i \) is a scaled version of \( \bar{g}_i \), which is \( L \)-smooth, by the conclusion for the first statement, it implies that \( g_i \) is also \( L \)-smooth, which concludes the proof.

**C Proof of NC-SC Lower Bound**

Similar to Section 3 in the main text, here in this section only, we denote \( x_d \) as the \( d \)-th coordinate of \( x \) and \( x^t \) as the variable \( x \) in the \( t \)-th iteration.

**C.1 Deterministic NC-SC Lower Bound**

We start from the proof several important lemmas, then proceed to the analysis of Theorem 3.1.

**C.1.1 Proof of Lemma 3.1**

**Proof** Recall the definition of \( F_d \) in (9), define \( \Gamma_d(x) \triangleq \sum_{i=1}^{d} \Gamma(x_i) \), note that \( x_d^2 = x^\top e_i e_i^\top x \), and

\[
\nabla_x F_d(x, y; \lambda, \alpha) = \lambda_1 B_d^\top y - \frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} e_i + \frac{\lambda_2^2 \alpha}{2\lambda_2} \nabla \Gamma_d(x) - \frac{\lambda_2^2 \alpha}{2\lambda_2} e_{d+1} e_{d+1}^\top x 
\]

\[
\nabla_y F_d(x, y; \lambda, \alpha) = \lambda_1 B_d x - 2\lambda_2 y,
\]

22
where $\nabla \Gamma_d(x) = (\nabla \Gamma(x_1), \nabla \Gamma(x_2), \ldots, \nabla \Gamma(x_d))^\top$. Then for the matrix norm of $B_d$, note that $\alpha \in [0, 1]$ and

\[
\|B_d\| = \sqrt{x_{d+1}^2 + (x_d - x_{d-1})^2 + \cdots + (x_1 - x_0)^2 + (\sqrt{\alpha}x_1)^2}
\]

\[
\leq \sqrt{x_{d+1}^2 + 2(x_d^2 + x_{d+1}^2 + x_{d-1}^2 + \cdots + x_2^2 + x_1^2 + x_0^2) + x_1^2}
\]

\[
\leq \sqrt{4(x_{d+1}^2 + x_d^2 + x_{d-1}^2 + \cdots + x_2^2 + x_1^2)} = 2\|x\|,
\]

similarly we have $\|B_d^\top y\| \leq 2\|y\|$. Denote $C_\gamma \triangleq 360^3$ so because $0 \leq \alpha \leq 1$ and $\|B_d\| \leq 2$, we have ($\|\cdot\|$ here denotes the spectral norm of a matrix)

\[
\|\nabla^2_{xx} F_d\| \leq \frac{\lambda_1^2}{2\lambda_2} (C_\gamma \alpha + \alpha) \leq \frac{400\lambda_1^2 \alpha}{2\lambda_2} = \frac{200\lambda_1^2 \alpha}{\lambda_2}, \quad \|\nabla^2_{xy} F_d\| \leq 2\lambda_1, \quad \|\nabla^2_{yx} F_d\| \leq 2\lambda_1, \quad \|\nabla^2_{yy} F_d\| = 2\lambda_2,
\]

which proves the first two statements (i) and (ii).

For (iii), due to the structure of $B_d$ and concerning the activation status defined in $X_k$ and $Y_k$, it is easy to verify that if $x \in X_{k_1}, y \in Y_{k_2}$ for $k_1, k_2 \in \mathbb{N}$ and $k_1, k_2 \leq d$, we have

$$B_d x \in Y_{k_1}, \quad B_d^\top y \in X_{k_2+1}.$$  

Since the remaining components in the gradient do not affect the activation with the initial point $(0, 0) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+2}$, this proves (iii).

For (iv), by substituting the parameter settings, we have $\frac{200\lambda_1^2 \alpha}{\lambda_2} = L$, $2\lambda_1 = L$ and $2\lambda_2 = \mu$, so the function $F_d$ is $\mu$-strongly concave in $y$ and $L$-Lipschitz smooth, which concludes the proof.

\[\square\]

C.1.2 Proof of Lemma 3.2

Proof Recall the primal function $\Phi_d$ of $F_d$ (9):

$$
\Phi_d(x; \lambda, \alpha) = \frac{\lambda_1^2}{2\lambda_2} \left( \frac{1}{2} x^\top A_d x - \sqrt{\alpha}x_1 + \frac{\sqrt{\alpha}}{2} + \alpha \sum_{i=1}^d \Gamma(x_i) \right) + \frac{(1-\alpha)\lambda_1^2}{4\lambda_2} x_{d+1}^2. \quad (36)
$$

For the first statement, because $x_d = x_{d+1} = 0$, we have

$$
\nabla \Phi_d(x; \lambda, \alpha) = \nabla \Phi_{d1}(x; \lambda, \alpha) + \nabla \Phi_{d2}(x; \lambda, \alpha) = \nabla \Phi_{d1}(x; \lambda, \alpha), \quad (37)
$$

which corresponds to the hard instance in [Carmon et al., 2019b, Equation 9] with an extra coefficient $\frac{\lambda_1^2}{2\lambda_2}$, then we apply [Carmon et al., 2019b, Lemma 3] therein to attain the desired large gradient norm result, i.e.

$$
\|\nabla \Phi_d(x; \lambda, \alpha)\| \geq \frac{\lambda_1^2}{2\lambda_2} \times \frac{\alpha^2}{4} = \frac{\lambda_1^2}{8\lambda_2} \alpha^4. \quad (38)
$$

---

\[\text{3} \text{The choice of } C_\gamma \text{ follows the setting in [Zhou and Gu, 2019, Proposition 3.11], which is an upper bound of the Lipschitz smoothness parameter of } F_d(x) \text{ in [Carmon et al., 2019b, Lemma 2].}\]
For the second statement, we have

\[
\Phi_d(0; \lambda, \alpha) - \inf_{x \in \mathbb{R}^{d+1}} \Phi_d(x; \lambda, \alpha) \\
= \Phi_{d1}(0; \lambda, \alpha) - \inf_{x \in \mathbb{R}^{d+1}} [\Phi_{d1}(x; \lambda, \alpha) + \Phi_{d2}(x; \lambda, \alpha)] \\
\leq \Phi_{d1}(0; \lambda, \alpha) - \inf_{x \in \mathbb{R}^{d+1}} \Phi_{d1}(x; \lambda, \alpha) \\
\leq \frac{\lambda_1^2}{2\lambda_2} \left( \frac{\sqrt{\alpha}}{2} + 10\alpha d \right),
\]

where the first inequality uses that \( \Phi_{d2}(x; \lambda, \alpha) \geq 0 \) because \( \alpha \in [0, 1] \), and the last inequality applies [Carmon et al., 2019b, Lemma 4], which proves the second statement.

\[\square\]

C.1.3 Proof of Theorem 3.1

The complexity for deterministic nonconvex-strongly-concave problems is defined as

\[
\text{Compl}_{\mathcal{F}_{\text{NCSC}}, \mathcal{A}, \mathcal{O}_F} \triangleq \sup_{f \in \mathcal{F}_{\text{NCSC}}} \inf_{\alpha \in \mathcal{A}(\mathcal{O}_F)} T_\epsilon(f, A)
\]

(39)

As a helper lemma, we first discuss the primal function of the scaled hard instance.

**Lemma C.1 (Primal of the Scaled Hard Instance)** With the function \( F_d \) defined in (9), \( \Phi_d \) defined in (11) and any \( \eta \in \mathbb{R} \), for the following function:

\[
f(x, y) = \eta^2 F_d \left( \frac{x}{\eta}, \frac{y}{\eta}; \lambda, \alpha \right),
\]

(41)

then for its primal function \( \Phi(x) \triangleq \max_{y \in \mathbb{R}^{d+2}} f(x, y) \), we have

\[
\Phi(x) = \eta^2 \Phi_d \left( \frac{x}{\eta}; \lambda, \alpha \right).
\]

(42)

**Proof** Check the scaled function,

\[
f(x, y) \\
= \eta^2 \left( \lambda_1 \left< B_d, \frac{x}{\eta}, \frac{y}{\eta} \right> - \lambda_2 \left\| \frac{y}{\eta} \right\|^2 - \frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} \left< e_1, \frac{x}{\eta} \right> + \frac{\lambda_1^2 \alpha}{2\lambda_2} \sum_{i=1}^{d} \Gamma \left( \frac{x_i}{\eta} \right) - \frac{\lambda_1^2 \alpha}{4\lambda_2} \left( \frac{x_{d+1}}{\eta} \right)^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4\lambda_2} \right)
\]

(43)

\[
= \lambda_1 \left< B_d, x, y \right> - \lambda_2 \left\| y \right\|^2 + \eta^2 \left( - \frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} \left< e_1, x \right> + \frac{\lambda_1^2 \alpha}{2\lambda_2} \sum_{i=1}^{d} \Gamma \left( \frac{x_i}{\eta} \right) - \frac{\lambda_1^2 \alpha}{4\lambda_2} \left( \frac{x_{d+1}}{\eta} \right)^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4\lambda_2} \right),
\]

check the gradient over \( y \) and set it to be 0 to solve for \( y^*(x) \), we have

\[
\nabla_y f(x, y^*(x)) = \lambda_1 B_d x - 2\lambda_2 y^*(x) = 0 \quad \Rightarrow \quad y^*(x) = \frac{\lambda_1}{2\lambda_2} B_d x,
\]

(44)
so the primal function is

\[ \Phi(x) = f(x, y^*(x)) \]

\[ = \lambda_1(B_d x, y^*(x)) - \lambda_2 \|y^*(x)\|^2 + \eta^2 \left( -\frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} \left\langle e_1, \frac{x}{\eta} \right\rangle + \frac{\lambda_1^2 \alpha}{2\lambda_2} \sum_{i=1}^{d} \Gamma \left( \frac{x_i}{\eta} \right) - \frac{\lambda_2^2 \alpha}{4\lambda_2} \left( \frac{x_{d+1}}{\eta} \right)^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4\lambda_2} \right) \]

\[ = \frac{\lambda_1^2}{4\lambda_2} \|B_d x\|^2 + \eta^2 \left( -\frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} \left\langle e_1, \frac{x}{\eta} \right\rangle + \frac{\lambda_1^2 \alpha}{2\lambda_2} \sum_{i=1}^{d} \Gamma \left( \frac{x_i}{\eta} \right) - \frac{\lambda_2^2 \alpha}{4\lambda_2} \left( \frac{x_{d+1}}{\eta} \right)^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4\lambda_2} \right) \]

\[ = \eta^2 \left( \frac{\lambda_1^2}{4\lambda_2} \|B_d x\|^2 - \frac{\lambda_1^2 \sqrt{\alpha}}{2\lambda_2} \left\langle e_1, \frac{x}{\eta} \right\rangle + \frac{\lambda_1^2 \alpha}{2\lambda_2} \sum_{i=1}^{d} \Gamma \left( \frac{x_i}{\eta} \right) - \frac{\lambda_2^2 \alpha}{4\lambda_2} \left( \frac{x_{d+1}}{\eta} \right)^2 + \frac{\lambda_1^2 \sqrt{\alpha}}{4\lambda_2} \right) \]

\[ = \eta^2 \Phi_d \left( \frac{x}{\eta}; \lambda, \alpha \right), \]

which concludes the proof. ■

Now we come to the formal statement and proof of the main theorem.

**Theorem C.1 (Lower Bound for General NC-SC, Restate Theorem 3.1)** For any linear-span first-order algorithm \( \mathcal{A} \in \mathcal{A} \) and parameters \( L, \mu, \Delta > 0 \), with a desired accuracy \( \epsilon > 0 \), for the following function \( f : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R} \):

\[ f(x, y) \triangleq \eta^2 F_d \left( \frac{x}{\eta}, \frac{y}{\eta}; \lambda^*, \alpha \right), \]

where \( F_d \) is defined in (9), with a primal function \( \Phi(x) \triangleq \max_{y \in \mathbb{R}^{d+1}} f(x, y) \), for a small enough \( \epsilon > 0 \) satisfying

\[ \epsilon^2 \leq \min \left( \frac{\Delta L}{64000}, \frac{\Delta L \sqrt{\kappa}}{38400} \right), \]

if we set

\[ \lambda^* = \left( \frac{L}{2}, \frac{\mu}{2} \right), \quad \eta = \frac{16\mu}{L^2} \beta^{-3/4} \epsilon, \quad \alpha = \frac{\mu}{100L} \in [0, 1], \quad d = \left\lfloor \frac{\Delta L \sqrt{\kappa}}{12800} \epsilon^{-2} \right\rfloor \geq 3, \]

we have

- The proposed function \( f \in \mathcal{F}^{L, \mu, \Delta}_{\text{NC-SC}} \).

- To obtain a point \( \hat{x} \in \mathbb{R}^{d+1} \) such that \( \|\nabla \Phi(\hat{x})\| \leq \epsilon \), the number of FO queries required by the algorithm \( \mathcal{A} \in \mathcal{A} \) is at least \( 2d - 1 = \Omega \left( \sqrt{\kappa} \Delta L \epsilon^{-2} \right) \), namely,

\[ \text{Compl}_{\left( \mathcal{F}^{L, \mu, \Delta}_{\text{NC-SC}}, \mathcal{O}_{\text{FO}} \right)} = \Omega \left( \sqrt{\kappa} \Delta L \epsilon^{-2} \right). \]

**Proof** First, we verify the smoothness and strong concavity of the function \( f \). According to Lemma 3.1, \( \alpha < \frac{\mu}{100L} \) implies that \( F_d(x, y; \lambda^*, \alpha) \) is \( L \)-smooth and \( \mu \)-strongly concave in \( y \). Given that \( f \) is a scaled version of \( F_d \), by Lemma B.3, it is easy to verify that \( f \) is also \( L \)-smooth and \( \mu \)-strongly concave in \( y \).

Then by Lemma C.1, we have

\[ \Phi(x) = \eta^2 \Phi_d \left( \frac{x}{\eta}; \lambda^*, \alpha \right), \]

where \( \Phi_d \) is defined in (11). Next we check the initial primal function gap, by Lemma 3.2 and parameter substitution,

\[ \Phi(0) - \inf_x \Phi(x) = \eta^2 \left( \Phi_d(0) - \inf_x \Phi_d(x) \right) \leq \frac{\eta^2 L^2}{4\mu} \left( \frac{\sqrt{\alpha}}{2} + 10\alpha d \right) \leq \frac{64\mu}{L^2} \left( \frac{1}{2\alpha} + \frac{10d}{\sqrt{\alpha}} \right) \epsilon^2, \]

(50)
by substituting $\alpha$ and $d$ into the RHS above, we have
\begin{equation}
\frac{64\mu}{L^2} \left( \frac{1}{2\alpha} + \frac{10d}{\sqrt{\alpha}} \right) \epsilon^2 \leq \frac{64\mu}{L^2} \left( \frac{50L}{\mu} + 100 \sqrt{L \mu \Delta L \sqrt{\epsilon} \epsilon^{-2}} \right) \epsilon^2 \\
\leq \frac{64}{L} \left( 50 + \frac{\Delta L}{128} \epsilon^{-2} \right) \epsilon^2 \leq \frac{64}{L} \left( \frac{\Delta L}{64} \epsilon^{-2} \right) \epsilon^2 = \Delta.
\end{equation}

The second inequality holds because $\epsilon$ above is set to be small enough than $\frac{\Delta L}{6400}$. We conclude that $f \in \mathcal{F}_{NCSC}^{L,\mu,\Delta}$.

We now discuss the lower bound argument. Based on Lemma 3.2 and the setting of $\eta$, we have when $x_d = x_{d+1} = 0$,
\begin{equation}
\|\nabla \Phi(x)\| = \eta \left\| \nabla \Phi_d \left( \frac{x}{\eta}, \lambda^*, \alpha \right) \right\| \geq \frac{\eta L^2}{16\mu} \alpha^{3/4} = \epsilon.
\end{equation}

So starting from $(x, y) = (0, 0) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+2}$, we cannot get the primal stationarity convergence at least until $x_d \neq 0$. By the “alternating zero-chain” mechanism\footnote{Also known as the “Domino argument” in Ibrahim et al. [2020].} in Lemma 3.1, each update with the linear-span algorithm interacting with the FO oracle call will activate exactly one coordinate alternatively between $x$ and $y$. Therefore the algorithm $A$ requires at least $2d - 1$ queries to FO to activate the $d$-th element of $x$, i.e., $x_d$, which implies the lower bound is (note that $\epsilon$ is small enough such that $d \geq 3$)
\begin{equation}
2d - 1 = \Omega\left( \sqrt{\epsilon} \Delta L \epsilon^{-2} \right),
\end{equation}

which concludes the proof. Notice that this argument works even for randomized algorithms, as long as they satisfy the linear-span assumption.

\section{C.2 Averaged Smooth Finite-Sum NC-SC Lower Bound}

Similar to the deterministic NC-SC case, here we still start from several important lemmas and proceed to the proof of Theorem 3.2.

\subsection{C.2.1 Hard Instance Construction}

Recall the (unscaled) hard instance in averaged smooth finite-sum case in (17): $H_d : \mathbb{R}^{d+2} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}$, $\Gamma_d^n : \mathbb{R}^{n(d+1)} \rightarrow \mathbb{R}$ and
\begin{equation}
H_d(x, y; \lambda, \alpha) \triangleq \lambda_1 (B_d x, y) - \lambda_2 \|y\|^2 - \frac{\lambda_2^2 \sqrt{\alpha}}{2\lambda_2} \langle e_1, x \rangle - \frac{\lambda_2^2 \alpha}{4\lambda_2} x_{d+1}^2 + \frac{\lambda_2^2 \sqrt{\alpha}}{4\lambda_2},
\end{equation}
\begin{equation}
\Gamma_d^n(x) \triangleq \sum_{i=1}^{n} \sum_{j=i(d+1) \rightarrow d} \Gamma(x_j),
\end{equation}
then $\bar{f}_i, \bar{f} : \mathbb{R}^{n(d+1)} \times \mathbb{R}^{n(d+2)} \rightarrow \mathbb{R}$, $\{U^{(i)}\}_{i=1}^{n} \in \text{Orth}(d+1, n(d+1), n)$, $\{V^{(i)}\}_{i=1}^{n} \in \text{Orth}(d+2, n(d+2), n)$ and
\begin{equation}
\bar{f}_i(x, y) \triangleq H_d \left( U^{(i)}x, V^{(i)}y; \lambda, \alpha \right) + \frac{\lambda_2^2 \alpha}{2n \lambda_2} \Gamma_d^n(x),
\end{equation}
\begin{equation}
\bar{f}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} \bar{f}_i(x, y) = \frac{1}{n} \sum_{i=1}^{n} \left[ H_d \left( U^{(i)}x, V^{(i)}y; \lambda, \alpha \right) + \frac{\lambda_2^2 \alpha}{2n \lambda_2} \Gamma_d^n(x) \right].
\end{equation}
i.e., by denoting $u^{(i)} \triangleq U^{(i)}x$ and note that $\|y\|^2 = \sum_{i=1}^{n} \|V^{(i)}y\|^2$,

$$
\bar{f}(x, y) = \frac{1}{n} \sum_{i=1}^{n} \left[ \lambda_i \langle B_d U^{(i)}x, V^{(i)}y \rangle - \lambda_2 \|V^{(i)}y\|^2 - \frac{\lambda_i^2 \sqrt{\alpha}}{2\lambda_2} \langle e_1, U^{(i)}x \rangle + \frac{\lambda_i^2 \alpha}{2n\lambda_2} \Gamma_d^n(x) - \frac{\lambda_i^2 \alpha}{4\lambda_2} (u^{(i)}_{d+1})^2 + \frac{\lambda_i^2 \sqrt{\alpha}}{4\lambda_2} \right] 
$$

(56)

$$
= - \frac{\lambda_2}{n} \|y\|^2 + \frac{1}{n} \sum_{i=1}^{n} \left[ \lambda_i \langle B_d U^{(i)}x, V^{(i)}y \rangle - \frac{\lambda_i^2 \sqrt{\alpha}}{2\lambda_2} \langle e_1, U^{(i)}x \rangle + \frac{\lambda_i^2 \alpha}{2n\lambda_2} \Gamma_d^n(x) - \frac{\lambda_i^2 \alpha}{4\lambda_2} (u^{(i)}_{d+1})^2 + \frac{\lambda_i^2 \sqrt{\alpha}}{4\lambda_2} \right],
$$

so $\bar{f}$ is $\frac{2\lambda_2}{n}$-strongly concave in $y$. Recall the gradient of $f_i$:

$$
\nabla_x f_i(x, y) = \lambda_i (U^{(i)})^T B_d^T V^{(i)} y - \frac{\lambda_i^2 \sqrt{\alpha}}{2\lambda_2} (U^{(i)})^T e_1 + \frac{\lambda_i^2 \alpha}{2n\lambda_2} \nabla \Gamma_d^n(x) - \frac{\lambda_i^2 \alpha}{2\lambda_2} (U^{(i)})^T e_{d+1} e_{d+1}^T U^{(i)} x,
$$

$$
\nabla_y f_i(x, y) = \lambda_i (V^{(i)})^T B_d U^{(i)} x - 2\lambda_2 (V^{(i)})^T V^{(i)} y,
$$

then we discuss the smoothness of $\{\bar{f}_i\}$.

**Lemma C.2 (Properties of $\bar{f}$)** For $n \in \mathbb{N}^+$, $L \geq 2n\mu > 0$, if we set

$$
\lambda = \lambda^* = (\lambda_1^*, \lambda_2^*) = \left( \sqrt{\frac{n}{40}} L, \frac{n\mu}{2} \right) \quad \text{and} \quad \alpha = \frac{n\mu}{50L},
$$

(58)

then the function $\{\bar{f}_i\}$ is $L$-averaged smooth, and $\bar{f}(x, \cdot)$ is $\mu$-strongly concave for any fixed $x \in \mathbb{R}^{d+1}$.

**Proof** For the strong concavity, note that $\bar{f}$ is $\frac{2\lambda_2}{n}$-strongly concave, so by substitution we have $\bar{f}$ is $\mu$-strongly concave in $y$. Then for the average smoothness, by definition, we have for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1},

$$
\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_1, y_1) - \nabla f_i(x_2, y_2)\|^2
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left[ \|\nabla_x f_i(x_1, y_1) - \nabla_x f_i(x_2, y_2)\|^2 + \|\nabla_y f_i(x_1, y_1) - \nabla_y f_i(x_2, y_2)\|^2 \right],
$$

(59)

then note that $\Gamma_d^n$ and $\Gamma_d$ enjoys the same Lipschitz smoothness parameter as that of $\Gamma$, so we have

$$
\|\nabla_x f_i(x_1, y_1) - \nabla_x f_i(x_2, y_2)\|^2
$$

$$
\leq 4 \left\| \lambda_i (U^{(i)})^T B_d^T V^{(i)} (y_1 - y_2) \right\|^2 + 4 \left\| \frac{\lambda_i^2 \alpha}{2n\lambda_2} (\nabla \Gamma_d^n(x_1) - \nabla \Gamma_d^n(x_2)) \right\|^2
$$

$$
+ 4 \left\| \frac{\lambda_i^2 \alpha}{2\lambda_2} (U^{(i)})^T e_{d+1} e_{d+1}^T U^{(i)} (x_1 - x_2) \right\|^2
$$

(60)

$$
= 4\lambda_i^2 \left\| B_d^T V^{(i)} (y_1 - y_2) \right\|^2 + \frac{\lambda_i^2 \alpha^2}{n^2\lambda_2^2} \|\nabla \Gamma_d^n(x_1) - \nabla \Gamma_d^n(x_2)\|^2 + \frac{\lambda_i^2 \alpha^2}{\lambda_2^2} \|e_{d+1} e_{d+1}^T U^{(i)} (x_1 - x_2)\|^2
$$

$$
\leq 16\lambda_i^2 \left\| V^{(i)} (y_1 - y_2) \right\|^2 + \frac{C_2 \lambda_i^4 \alpha^2}{n^2\lambda_2^2} \|x_1 - x_2\|^2 + \frac{\lambda_i^4 \alpha^2}{\lambda_2^2} \|U^{(i)} (x_1 - x_2)\|^2,
$$

27
and

\[
\|\nabla_y f_i(x_1, y_1) - \nabla_y f_i(x_2, y_2)\|^2 = \left\| \lambda_1 \left( V^{(i)} \right)^\top B_d U^{(i)}(x_1 - x_2) - 2\lambda_2 \left( V^{(i)} \right)^\top V^{(i)}(y_1 - y_2) \right\|^2 \\
\leq 2 \left\| \lambda_1 \left( V^{(i)} \right)^\top B_d U^{(i)}(x_1 - x_2) \right\|^2 + 2 \left\| 2\lambda_2 \left( V^{(i)} \right)^\top V^{(i)}(y_1 - y_2) \right\|^2 \\
\leq 8\lambda_1^2 \left\| U^{(i)}(x_1 - x_2) \right\|^2 + 8\lambda_2^2 \left\| V^{(i)}(y_1 - y_2) \right\|^2 ,
\]

so we have

\[
\frac{1}{n} \sum_{i=1}^n \left\| \nabla_y f_i(x_1, y_1) - \nabla_y f_i(x_2, y_2) \right\|^2 \\
\leq \frac{1}{n} \sum_{i=1}^n \left[ (16\lambda_1^2 + 8\lambda_2^2) \left\| V^{(i)}(y_1 - y_2) \right\|^2 + \frac{(\lambda_1^4\alpha^2}{\lambda_2^2} + 8\lambda_1^2) \left\| U^{(i)}(x_1 - x_2) \right\|^2 + \frac{C_2^2\lambda_1^4\alpha^2}{n^2\lambda_2^2} \|x_1 - x_2\|^2 \right] \\
= \frac{1}{n} \left( 16\lambda_1^2 + 8\lambda_2^2 \right) \|y_1 - y_2\|^2 + \frac{1}{n} \left( \frac{\lambda_1^4\alpha^2}{\lambda_2^2} + 8\lambda_1^2 \right) \|x_1 - x_2\|^2 + \frac{C_2^2\lambda_1^4\alpha^2}{n^2\lambda_2^2} \|x_1 - x_2\|^2 \\
\leq \frac{1}{n} \max \left\{ 16\lambda_1^2 + 8\lambda_2^2, \frac{C_2^2\lambda_1^4\alpha^2}{n^2\lambda_2^2} + \frac{\lambda_1^4\alpha^2}{\lambda_2^2} + 8\lambda_1^2 \right\} \left( \|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 \right) ,
\]

then note that \( \alpha \in [0, 1] \) because we set \( L \geq 2n\mu \geq \frac{1}{36}n\mu \), so substitute parameters into the above, we have

\[
\frac{1}{n} \max \left\{ 16\lambda_1^2 + 8\lambda_2^2, \frac{C_2^2\lambda_1^4\alpha^2}{n^2\lambda_2^2} + \frac{\lambda_1^4\alpha^2}{\lambda_2^2} + 8\lambda_1^2 \right\} \\
= \frac{1}{n} \max \left\{ 16\lambda_1^2 + 2n^2\mu^2, \frac{4C_2^2\lambda_1^4\alpha^2}{n^3\mu^2} + \frac{\lambda_1^4\alpha^2}{n^2\mu^2} + 8\lambda_1^2 \right\} \\
\leq \frac{1}{n} \max \left\{ 16\lambda_1^2 + 2n^2\mu^2, 1000000\alpha^2 \frac{\lambda_1^4}{n^3\mu^2} + 8\lambda_1^2 \right\} \\
= \frac{1}{n} \max \left\{ \frac{16nL^2}{40} + 2n^2\mu^2, 1000000\cdot \frac{n^2\mu^2}{2500L^2} \cdot \frac{n^2L^4}{1600n^3\mu^2} + \frac{8nL^2}{40} \right\} \\
\leq \max \left\{ \frac{2L^2}{5} + \frac{L^2}{2}, \frac{L^2}{4} + \frac{L^2}{5} \right\} \\
\leq \max \left\{ \frac{9L^2}{10}, \frac{9L^2}{20} \right\} \leq L^2 ,
\]

where the first inequality is attained by the computation with the value of \( C_\gamma = 360 \), the second inequality comes from the assumption \( L \geq 2n\mu \geq 2\sqrt{n}\mu \); the last equality is attained by parameter substitution, which verifies the conclusion.

Next we discuss the primal function of the finite-sum hard instance.

**Lemma C.3 (Primal of Averaged Smooth Finite-Sum Hard Instance)** For the function \( \bar{f} = \frac{1}{n} \sum_{i=1}^n \bar{f}_i \) defined in (17), define \( \bar{\Phi}(x) \triangleq \max_y \bar{f}(x, y) \), then we have

\[
\bar{\Phi}(x) = \frac{1}{n} \sum_{i=1}^n \bar{\phi}_i(x), \quad \text{where} \quad \bar{\phi}_i(x) \triangleq \bar{\phi}_d \left( U^{(i)}x \right) ,
\]
while $\Phi_d$ is defined in (11).

**Proof** By the expression of $\tilde{f}$ in (56), take the gradient over $y$ and set it as 0, denote the maximizer as $y^*(x)$, we have

$$-\frac{2\lambda_2}{n} y^*(x) + \frac{1}{n} \sum_{i=1}^{n} \lambda_1 \left(V^{(i)}\right)^\top B_d U^{(i)} x = 0 \implies y^*(x) = \frac{\lambda_1}{2\lambda_2} \sum_{i=1}^{n} \left(V^{(i)}\right)^\top B_d U^{(i)} x,$$

so we have

$$\Phi(x) = \tilde{f}(x, y^*(x))$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\lambda_1^2}{4\lambda_2} \|B_d U^{(i)} x\|^2 - \frac{\lambda_1^2\sqrt{\alpha}}{2\lambda_2} \langle e_1, U^{(i)} x \rangle + \frac{\lambda_1^2\alpha}{2n\lambda_2} \Gamma_d(x) - \frac{\lambda_1^2\alpha}{4\lambda_2} \left(u_{d+1}^{(i)}\right)^2 + \frac{\lambda_1^2\sqrt{\alpha}}{4\lambda_2} \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\lambda_1^2}{4\lambda_2} \|B_d U^{(i)} x\|^2 - \frac{\lambda_1^2\sqrt{\alpha}}{2\lambda_2} \langle e_1, U^{(i)} x \rangle + \frac{\lambda_1^2\alpha}{2n\lambda_2} \sum_{j=1}^{n} \Gamma_d \left(U^{(i)} x\right) - \frac{\lambda_1^2\alpha}{4\lambda_2} \left(u_{d+1}^{(i)}\right)^2 + \frac{\lambda_1^2\sqrt{\alpha}}{4\lambda_2} \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\lambda_1^2}{2\lambda_2} \left(\frac{1}{2} \left(U^{(i)} x\right)^\top A_d U^{(i)} x - \sqrt{\alpha} \langle e_1, U^{(i)} x \rangle + \alpha \Gamma_d \left(U^{(i)} x\right) + \frac{1-\alpha}{2} \left(u_{d+1}^{(i)}\right)^2 + \frac{\sqrt{\alpha}}{2}\right) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Phi_d \left(U^{(i)} x\right)$$

where the third equality follows from (18), and $A_d$ and $\Phi_d$ are defined in (12) and (11), which concludes the proof.

The above two lemmas proves the statements in Lemma 3.3. Before we present the main theorem, we first discuss the behavior of the scaled hard instance, which will be used in the final lower bound analysis.

**Lemma C.4 (Primal of the Scaled Finite-Sum Hard Instance)** With the function $\tilde{f}(x, y)$ and $\tilde{f}_i(x, y)$ defined in (17), $\Phi(x) \triangleq \max_y \tilde{f}(x, y)$, then for any $\eta \in \mathbb{R}$ and the following function:

$$f(x, y) = \frac{1}{n} \sum_{i=1}^{n} \tilde{f}_i(x, y) = \frac{1}{n} \sum_{i=1}^{n} \eta^2 \tilde{f}_i \left(\frac{x}{\eta}, \frac{y}{\eta}\right),$$

then for its primal function $\Phi(x) \triangleq \max_{\eta \in \mathbb{R}^{d+1}} f(x, y)$, we have

$$\Phi(x) = \frac{1}{n} \sum_{i=1}^{n} \Phi_i(x), \quad \text{where} \quad \Phi_i(x) = \eta^2 \tilde{\Phi}_i \left(\frac{x}{\eta}\right) = \eta^2 \Phi_d \left(\frac{1}{\eta} U^{(i)} x\right).$$
Proof Based on (56), we can write out the formulation of $f$:

$f(x, y) = \eta^2 f\left(\frac{x}{\eta}, \frac{y}{\eta}\right)$

\[
= \eta^2 \left(-\frac{\lambda_2}{n} \left\| y \right\|^2 + \frac{1}{n} \sum_{i=1}^{n} \left[ \lambda_1 \left\langle B_d U^{(i)} x, V^{(i)} y \right\rangle - \frac{\lambda_2^2 \sqrt{\alpha}}{2 \lambda_2} \left\langle e_1, U^{(i)} x \right\rangle + \frac{\lambda_2^2 \alpha}{2 n \lambda_2} \Gamma_d \left(\frac{x}{\eta}\right) \right]\right)
\]

\[
= -\frac{\lambda_2}{n} \left\| y \right\|^2 + \frac{1}{n} \sum_{i=1}^{n} \lambda_1 \left\langle B_d U^{(i)} x, V^{(i)} y \right\rangle + \eta^2 \left(\frac{1}{n} \sum_{i=1}^{n} \left[ -\frac{\lambda_2^2 \sqrt{\alpha}}{2 \lambda_2} \left\langle e_1, U^{(i)} x \right\rangle + \frac{\lambda_2^2 \alpha}{2 n \lambda_2} \Gamma_d \left(\frac{x}{\eta}\right) \right] \left(\frac{u^{(i)}_{d+1}}{\eta}\right)^2 + \frac{\lambda_2^2 \sqrt{\alpha}}{4 \lambda_2} \right) \right).
\]

check the gradient over $y$ and set it to be 0 to solve for $y^*(x)$, we have

\[
\nabla_y f(x, y^*(x)) = -\frac{2\lambda_2}{n} y^*(x) + \frac{\lambda_1}{n} \sum_{i=1}^{n} \left( V^{(i)} \right)^\top B_d U^{(i)} x = 0 \implies y^*(x) = \frac{\lambda_1}{2 \lambda_2} \sum_{i=1}^{n} \left( V^{(i)} \right)^\top B_d U^{(i)} x,
\]

which implies that

\[
V^{(i)} y^*(x) = \frac{\lambda_1}{2 \lambda_2} \sum_{j=1}^{n} V^{(i)} \left( V^{(j)} \right)^\top B_d U^{(j)} x = \frac{\lambda_1}{2 \lambda_2} B_d U^{(i)} x
\]

\[
\left\| y^*(x) \right\|^2 = \frac{\lambda_1^2}{4 \lambda_2^2} \sum_{i=1}^{n} \left\| B_d U^{(i)} x \right\|^2,
\]

so the primal function is

\[
\Phi(x) = f(x, y^*(x)) = \eta^2 \left(\frac{x}{\eta}, \frac{y^*(x)}{\eta}\right)
\]

\[
= \frac{\lambda_1^2}{4 \lambda_2^2} \sum_{i=1}^{n} \left\| B_d U^{(i)} x \right\|^2 + \frac{\eta^2}{n} \sum_{i=1}^{n} \left[ -\frac{\lambda_2^2 \sqrt{\alpha}}{2 \lambda_2} \left\langle e_1, U^{(i)} x \right\rangle + \frac{\lambda_2^2 \alpha}{2 n \lambda_2} \Gamma_d \left(\frac{x}{\eta}\right) \left(\frac{u^{(i)}_{d+1}}{\eta}\right)^2 + \frac{\lambda_2^2 \sqrt{\alpha}}{4 \lambda_2} \right) \right]
\]

\[
= \frac{\eta^2}{n} \sum_{i=1}^{n} \left[ \frac{\lambda_1^2}{4 \lambda_2^2} \left\| B_d U^{(i)} x \right\|^2 - \frac{\lambda_2^2 \sqrt{\alpha}}{2 \lambda_2} \left\langle e_1, U^{(i)} x \right\rangle + \frac{\lambda_2^2 \alpha}{2 n \lambda_2} \Gamma_d \left(\frac{x}{\eta}\right) \left(\frac{u^{(i)}_{d+1}}{\eta}\right)^2 + \frac{\lambda_2^2 \sqrt{\alpha}}{4 \lambda_2} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \eta^2 \Phi_d \left(\frac{1}{\eta} U^{(i)} x \right) \right),
\]

where the last equality directly applies the conclusion in Lemma C.3, which concludes the proof.

\[\square\]

C.2.2 Proof of Theorem 3.2

Recall that the complexity for averaged smooth finite-sum nonconvex-strongly-concave problems is defined as

\[
\text{Compl}_c \left( F_{NCSC}^{L, \mu, \Delta}, A, O_{IFO}^{L, \mu, \Delta} \right) \triangleq \sup_{f \in F_{NCSC}^{L, \mu, \Delta}} \inf_{A \in A(O_{IFO}^{L, \mu, \Delta})} \mathbb{E} T_c(f, A)
\]

\[
= \sup_{f \in F_{NCSC}^{L, \mu, \Delta}} \inf_{A \in A(O_{IFO}^{L, \mu, \Delta})} \mathbb{E} \inf \left\{ T \in \mathbb{N} \mid \left\| \nabla \Phi(x^T) \right\| \leq \epsilon \right\}.
\]
Based on the discussion of the properties of the hard instance, we come to the final statement and proof of the theorem.

**Theorem C.2 (Lower Bound for Finite-Sum AS NC-SC, Restate Theorem 3.2)** For any linear-span first-order algorithm $A \in \mathcal{A}$, and parameters $L, \mu, \Delta > 0$ with a desired accuracy $\epsilon > 0$, for the following function $f : \mathbb{R}^{(d+1)} \times \mathbb{R}^{(d+1)} \to \mathbb{R}$:

$$f_i(x, y) = \eta^2 \bar{f}_i \left( \frac{x}{\eta}, \frac{y}{\eta} \right), \quad f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y) \tag{74}$$

where $\bar{f}_i$ is defined as (17) and $\{U^{(i)}\}_{i=1}^{n} \in \text{Orth}(d+1, (d+1)n, n)$ is defined in (15), with its primal function $\Phi(x) \triangleq \max_{y \in \mathbb{R}^{d+1}} f(x, y)$, for small enough $\epsilon > 0$ satisfying

$$\epsilon^2 \leq \min \left( \frac{\sqrt{nL^2 \Delta}}{76800n\mu}, \frac{\alpha L^2 \Delta}{1280n\mu}, \frac{L^2 \Delta}{\mu} \right), \tag{75}$$

if we set $L \geq 2n\mu > 0$ and

$$\lambda^* = \left( \sqrt{\frac{n}{40}}, \frac{n\mu}{2} \right), \quad \eta = 160\sqrt{2n\mu} L^{-\frac{4}{3}} \epsilon, \quad \alpha = \frac{n\mu}{50L}, \quad d = \left\lfloor \frac{\sqrt{nL^2 \Delta}}{25600n\mu} \epsilon^{-2} \right\rfloor \geq 3, \tag{76}$$

we have

- The function $f \in \mathcal{F}_{\text{L, NCSC}}, \{f_i\}_{i=1}^{n}$ is $L$-averaged smooth.
- In the worst case, the algorithm $A$ requires at least $\Omega(n + \sqrt{n\kappa \Delta L \epsilon^{-2}})$ IFO calls to attain a point $\hat{x} \in \mathbb{R}^{d+1}$ such that $\mathbb{E}\|\nabla \Phi(\hat{x})\| \leq \epsilon$, i.e.,

$$\text{Comp}_\epsilon \left( \mathcal{F}_{\text{L, NCSC}}, A, O_{\text{IFO}} \right) = \Omega(n + \sqrt{n\kappa \Delta L \epsilon^{-2}}). \tag{77}$$

**Proof** We divide our proof into two cases.

**Case 1** The first case builds an $\Omega(n)$ lower bound from a special case of NC-SC function. Consider the following function: $x, y \in \mathbb{R}^d$ and

$$h_i(x, y) \triangleq \theta \langle v_i, x \rangle + L \langle x, y \rangle - \frac{\mu}{2} \|y\|^2, \quad h(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} h_i(x, y), \tag{78}$$

where $\theta \leq \sqrt{\frac{2L^2n^2 \Delta}{\mu d}}, \quad 0 < \mu \leq L$, the dimension number $d$ is set as a multiple of $n$, and $v_i \in \mathbb{R}^d$ is defined as

$$v_i \triangleq \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T, \tag{79}$$

such that elements with indices from $\frac{1}{n} d + 1$ to $\frac{1}{n} d$ are 1 and the others are all 0, namely, there are $\frac{d}{n}$ non-zero elements.

It is easy to see that the function $h_i$ is $\mu$-strongly convex and $L$-smooth in both $x$ and $y$. For the initial value gap, denote $\varphi \triangleq \max_y h$. We have

$$\varphi(x) = \frac{1}{n} \sum_{i=1}^{n} \left( \theta \langle v_i, x \rangle + L \frac{\|x\|^2}{2\mu} \right) = \frac{L^2}{2\mu} \|x\|^2 + \frac{\theta}{n} \sum_{i=1}^{n} \langle v_i, x \rangle, \tag{80}$$

which is a strongly convex function, and its optimal point $x^*$ is

$$x^* = -\frac{\mu \theta}{L^2 n} \sum_{i=1}^{d} v_i, \quad \varphi^* = -\frac{\mu \theta^2}{2L^2 n^2} \left\| \sum_{i=1}^{n} v_i \right\|^2. \tag{81}$$
Based on the setting of \( \theta \),
\[
\varphi(0) - \varphi^* = \frac{\mu \theta^2}{2L^2n^2} \left\| \sum_{i=1}^{n} v_i \right\|^2 = \frac{\mu \theta^2 d}{2L^2n^2} \leq \Delta. \tag{82}
\]
Hence, we have \( h \in \mathcal{F}_{NCSC}^{L, \mu, \Delta} \). Then based on the expression of \( \nabla_x h_i \) and \( \nabla_y h_i \), we have that, starting from \((x, y) = (0, 0)\) and denoting \( \{i_1, i_2, \cdots, i_t\} \) as the index of IFO sequence for \( t \) queries, then the output \((\hat{x}_t, \hat{y}_t)\) will be
\[
\hat{x}_t, \hat{y}_t \in \text{Span}\{v_{i_1}, v_{i_2}, \cdots, v_{i_t}\}. \tag{83}
\]
then note that each \( v_i \) contains only \( \frac{d}{n} \) non-zero elements, by the expression of the gradient of the primal function \( \nabla \varphi \), we have that if \( t \leq n/2 \), then there must be at least \( \frac{d}{n} \times \frac{d}{n} = \frac{d^2}{n^2} \) zero elements in \( \hat{x}_t \), which implies that for \( \epsilon^2 \leq \frac{L^2 \Delta}{\mu} \),
\[
\|\nabla \varphi(\hat{x}_t)\| \geq \frac{L^2}{\mu} \hat{x}_t + \frac{\theta}{n} \sum_{i=1}^{n} v_i \geq \frac{\theta}{n} \sqrt{\frac{d}{n}} \geq \epsilon,
\tag{84}
\]
where we follow the setting of \( \theta \) above. So we proved that it requires \( \Omega(n) \) IFO calls to find an \( \epsilon \)-stationary point.

**Case 2** The second case provides an \( \Omega(\sqrt{n\kappa \Delta L e^{-2}}) \) lower bound concerning the second term in the result. Throughout the case, we assume \( L \geq 2n\mu > 0 \) as that in Lemma C.2.

Here we still use the hard instance constructed in (17), note that \( \nabla f_i(x, y) = \eta \nabla \hat{f}_i \left( \frac{x}{\eta}, \frac{y}{\eta} \right) \) is a scaled version of \( \hat{f}_i \), which is \( L \)-averaged smooth by Lemma C.2, so by Lemma B.3 we have \( \{f_i\}_i \) is also \( L \)-average smooth. The for the strong concavity, note that \( \hat{f} \) is \( \mu \)-strongly concave on \( y \), so as the scaled version, \( f \) is also \( \mu \)-strongly concave on \( y \).

Then for the primal function of \( f \), let \( \Phi(x) \triangleq \max_y f(x, y) \), by Lemma C.3 and Lemma C.4, we have
\[
\Phi(x) = \eta^2 \hat{\Phi}\left(\frac{x}{\eta}\right) = \frac{1}{n} \sum_{i=1}^{n} \eta^2 \hat{\Phi}_i \left( \frac{x}{\eta} \right),
\tag{85}
\]
where \( \hat{\Phi} \) and \( \hat{\Phi}_i \) follow the definition in Lemma C.3.

We first justify the lower bound argument by lower bounding the norm of the gradient. Recall the definition of \( \mathcal{I} \) (see (21)), which is the index set such that \( u_d^{(i)} = u_d^{(i+1)} = 0, \forall i \in \mathcal{I} \) while \( u^{(i)} = U^{(i)}x \). By substituting the parameters in the statement above into (21) and Lemma 3.2, we have that when the size of the set \( \mathcal{I} \), i.e., \( |\mathcal{I}| > n/2 \) (note that scaling does not affect the activation status),
\[
\|\nabla \Phi(x)\|^2 = \left\| \eta \nabla \hat{\Phi}\left( \frac{x}{\eta} \right) \right\|^2 = \eta^2 \left\| \nabla \hat{\Phi}\left( \frac{x}{\eta} \right) \right\|^2
\geq \frac{51200n\mu^2}{L^4} \alpha^{-2} \epsilon^2 \cdot \frac{\lambda_1^4}{128n\lambda_2^2} \alpha^2 = \frac{L^4}{51200n\mu^2} \alpha^2 \epsilon^2 = \epsilon^2.
\tag{86}
\]
Next, we upper bound the starting optimality gap. By substitution of parameter settings and the initial gap
of $\bar{\Phi}$ in (20), also recall the setting of $\epsilon$, we have

$$\Phi(0) - \Phi^* = \eta^2 \left( \bar{\Phi}(0) - \inf_{x \in \mathbb{R}^{d+1}} \bar{\Phi}(x) \right) = \frac{51200n \mu^2}{L^4} \alpha - \frac{2}{2 \alpha} \epsilon^2 + \frac{640n \mu^2}{\alpha L^2} + \frac{12800n \mu \epsilon^2}{L^2 \sqrt{\alpha}}$$

$$\leq \frac{640n \mu}{\alpha L^2} \frac{\alpha L^2 \Delta^2}{1280n \mu} + \frac{12800n \mu^2}{L^2 \sqrt{\alpha}} \frac{\sqrt{\alpha} L^2 \Delta^2}{25600n \mu} \epsilon^{-2}$$

$$\leq \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta,$$

so we conclude that $f \in F_{NC\text{-}SC}^{L, \mu, \Delta}$, i.e. the function class requirement is satisfied.

To show the lower bound, by previous analysis and the choice of (15), the activation process for each component will also mimic the "alternating zero-chain" mechanism (see Lemma 3.1) independently. So we have, by the lower bound argument (21), it requires to activate at least half of the components through until their $d$-th elements (or at least half of $\{u^{(i)}\}$, are not activated through until the $d$-th element, note that each $u^{(i)}$ corresponds to an unique part of $x$ with length $(d+1)$) for the primal stationarity convergence of the objective function, which takes (note that $2| x| - 1 \geq x$ when $x \geq 3$)

$$T = \frac{n}{2} (2d - 1) \geq \frac{n}{2} \frac{\sqrt{\alpha} L^2 \Delta}{25600n \mu} \epsilon^{-2} = \Omega \left( \sqrt{\alpha} L \kappa \epsilon^{-2} \right) = \Omega \left( \sqrt{n \kappa} \Delta \epsilon^{-2} \right)$$

IFO oracle queries. So we found that for any fixed index sequence $\{i_t\}_{t=1}^T$, the output $z^{t+1}$ from a randomized algorithm\textsuperscript{5} must not be an approximate stationary point, which verifies the $\Omega \left( n \vee \sqrt{n \kappa} \Delta \epsilon^{-2} \right)$ or $\Omega \left( n + \sqrt{n \kappa} \Delta \epsilon^{-2} \right)$ lower bound by combining the two cases discussed above together. We conclude the proof by applying Yao’s minimax theorem [Yao, 1977], the lower bound will also hold for a randomized index sequence incurred by IFOs.

\textbf{D} \hspace{1em} \textbf{Proof of NC-SC Catalyst}

\textbf{D.1 Outer-loop Complexity}

In this section, we first introduce a few useful definitions. The Moreau envelop of a function $F$ with a positive parameter $\lambda > 0$ is:

$$F_{\lambda}(x) = \min_{z \in \mathbb{R}^d} F(z) + \frac{1}{2\lambda} \|z - x\|^2.$$  

We also define the proximal point of $x$:

$$\text{prox}_{\lambda F}(x) = \arg\min_{z \in \mathbb{R}^d} \left\{ F(z) + \frac{1}{2\lambda} \|z - x\|^2 \right\}.$$  

When $F$ is differentiable and $\ell$-weakly convex, for $\lambda \in (0, 1/\ell)$ we have

$$\nabla F(\text{prox}_{\lambda F}(x)) = \nabla F_{\lambda}(x) = \lambda^{-1}(x - \text{prox}_{\lambda F}(x)).$$

Thus a small gradient $\|\nabla F_{\lambda}(x)\|$ implies that $x$ is near a point $\text{prox}_{\lambda F}(x)$ that is nearly stationary for $F$. Therefore $\|\nabla F_{\lambda}(x)\|$ is also commonly used as a measure of stationarity. We refer readers to [Drusvyatskiy and Paquette, 2019] for more discussion on Moreau envelop.

In this subsection, we use $(x^t, y^t)$ as a shorthand for $(x^t_t, y^t_0)$. We will denote $(\hat{x}^t, \hat{y}^t)$ as the optimal solution to the auxiliary problem $(\ast)$ at $t$-th iteration: $\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^d} \left[ \hat{f}_t(x, y) \triangleq f(x, y) + L \|x - x^t\|^2 \right]$. It is easy

\textsuperscript{5}Note that randomization does not affect the lower bound, as long as the algorithm satisfies the linear-span assumption.

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to observe that $\hat{x}^t = \text{prox}_{\Phi_t/2L}(x^t)$. Define $\hat{\Phi}_t(x) = \max_y f(x, y) + L\|x - x^t\|^2$. In the following theorem, we show the convergence of the Moreau envelop $\|\nabla \Phi_{t/2L}(x)\|^2$ when we replace the inexactness measure (23) by another inexactness measure $\text{gap}_f(x^{t+1}, y^{t+1}) \leq \beta_t(\|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2)$. Later we will show this inexactness measure can be implied by (23) with our choice of $\beta_t$ and $\alpha_t$.

**Theorem D.1** Suppose function $f$ is NC-SC with strong convexity parameter $\mu$ and $L$-Lipschitz smooth. If we replace the stopping criterion (23) by $\text{gap}_f(x^{t+1}, y^{t+1}) \leq \beta_t(\|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2)$ with $\beta_t = \frac{\mu^2}{8L^2}$ for $t > 0$ and $\beta_0 = \frac{\mu^2}{8L^2 \max_{t \geq 1} T}$, then iterates from Algorithm 1 satisfy

$$\sum_{i=0}^{T-1} \|\nabla \Phi_{t/2L}(x^t)\|^2 \leq \frac{87L}{5} \Delta_0 + \frac{7L}{5} D_y^0,$$

where $D_y^0 = \|y^0 - y^*(x^0)\|^2$ and $\Delta_0 = \Phi(x^0) - \inf_x \Phi(x)$.

**Proof** Define $b_{t+1} = \text{gap}_f(x^{t+1}, y^{t+1})$. By Lemma 4.3 in [Drusvyatskiy and Paquette, 2019],

$$\|\nabla \Phi_{t/2L}(x^t)\|^2 = 4L^2\|x^t - \text{prox}_{\Phi_t/2L}(x^t)\|^2 \leq 8L[\hat{\Phi}_t(x^t) - \hat{\Phi}_t(\text{prox}_{\Phi_t/2L}(x^t))]$$

$$\leq 8L[\hat{\Phi}_t(x^t) - \hat{\Phi}_t(x^{t+1} + b_{t+1})]$$

$$= 8L[\Phi(x^t) - \Phi(x^{t+1}) + L\|x^{t+1} - x^t\|^2 + b_{t+1}]$$

$$\leq 8L[\Phi(x^t) - \Phi(x^{t+1} + b_{t+1})],$$

where in the first inequality we use $L$-strongly convexity of $\Phi_t$. Then, for $t \geq 1$

$$\|y^t - \hat{y}^t\|^2 \leq 2\|y^t - \hat{y}^{t-1}\|^2 + 2\|y^*(\hat{x}^{t-1}) - y^*(\hat{x}^t)\|^2$$

$$\leq 2\|y^t - \hat{y}^{t-1}\|^2 + 2\left(\frac{L}{\mu}\right)^2 \|\hat{x}^t - \hat{x}^{t-1}\|^2$$

$$\leq 2\|y^t - \hat{y}^{t-1}\|^2 + 4\left(\frac{L}{\mu}\right)^2 \|\hat{x}^t - x^t\|^2 + 4\left(\frac{L}{\mu}\right)^2 \|\hat{x}^t - \hat{x}^{t-1}\|^2$$

$$\leq \frac{8L}{\mu^2} b_t + 4\left(\frac{L}{\mu}\right)^2 \|\hat{x}^t - x^t\|^2,$$

where we use Lemma B.1 in the second inequality, and $(L, \mu)$-SC-SC of $\hat{f}_{i-1}(x, y)$ and Lemma B.2 in the last inequality. Therefore,

$$\|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2 \leq \frac{8L}{\mu^2} b_t + \left(\frac{4L^2}{\mu^2} + 1\right) \|\hat{x}^t - x^t\|^2.$$  

By our stopping criterion and $\|\nabla \Phi_{t/2L}(x^t)\|^2 = 4L^2\|x^t - \hat{x}^t\|^2$, for $t \geq 1$

$$b_{t+1} \leq \beta_t \left[\|x_t - \hat{x}^t\|^2 + \|y_t - \hat{y}^t\|^2\right] \leq \frac{8L\beta_t}{\mu^2} b_t + \beta_t \left(\frac{1}{\mu^2} + \frac{1}{4L^2}\right) \|\nabla \Phi_{t/2L}(x^t)\|^2.$$

Define $\theta = \frac{2}{\mu}$ and $w = \frac{5\mu^2}{112L^2}$. It is easy to verify that as $\beta_t = \frac{\mu^2}{8L^2}$, then $\frac{8L\beta_t}{\mu^2} \leq \theta$ and $\beta_t \left(\frac{1}{\mu^2} + \frac{1}{4L^2}\right) \leq w$. We conclude the following recursive bound

$$b_{t+1} \leq \theta b_t + w \|\nabla \Phi_{t/2L}(x^t)\|^2.$$  

For $t = 0$,

$$\|y^0 - \hat{y}^0\|^2 \leq 2\|y^0 - y^*(x^0)\|^2 + 2\|\hat{y}^0 - y^*(x^0)\|^2 \leq 2\|y^0 - y^*(x^0)\|^2 + 2\left(\frac{L}{\mu}\right)^2 \|x^0 - \hat{x}^0\|^2.$$  

$$\|x^0 - \hat{x}^0\|^2 \leq 2\|x^0 - \hat{x}^0\|^2 + 2\|y^0 - y^*(x^0)\|^2 \leq 2\|y^0 - y^*(x^0)\|^2 + 2\left(\frac{L}{\mu}\right)^2 \|x^0 - \hat{x}^0\|^2.$$  

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Because $\Phi(x) + L\|x - x^0\|^2$ is $L$-strongly convex, we have

$$ (\Phi(\bar{x}^0) + L\|\bar{x}^0 - x^0\|^2) + \frac{L}{2}\|\bar{x}^0 - x^0\|^2 \leq \Phi(x^0) = \Phi^* + (\Phi(x^0) - \Phi^*) \leq \Phi(\bar{x}^0) + (\Phi(x^0) - \Phi^*). $$

This implies $\|\bar{x}^0 - x^0\|^2 \leq \frac{1}{L}(\Phi(x^0) - \Phi^*)$. Then combining with (94)

$$ \|y^0 - y^0\|^2 + \|x^0 - x^0\|^2 \leq \left(\frac{L^3}{\mu^2} + \frac{L}{2}\right)(\Phi(x^0) - \Phi^*) + 2\|y^0 - y^*(x^0)\|^2. $$

Hence, by the stopping criterion,

$$ b_1 \leq \beta_0 \left(\frac{L^3}{\mu^2} + \frac{L}{2}\right)(\Phi(x^0) - \Phi^*) + 2\beta_0\|y^0 - y^*(x^0)\|^2. $$

Define $\theta_0 = \frac{\mu^2}{4L^2\max\{1,L\}}$. With $\beta_0 = \frac{\mu^2}{2L\max\{1,L\}}$, $\beta_0 \left(\frac{L^3}{\mu^2} + \frac{L}{2}\right) \leq \theta_0$ and $2\beta_0 \leq \theta_0$. So we can write

$$ b_1 \leq \theta_0(\Phi(x^0) - \Phi^*) + \theta_0\|y^0 - y^*(x^0)\|^2. $$

Unravelling (93), we have for $t \geq 1$

$$ b_{t+1} \leq \theta^tb_1 + \sum_{k=1}^{t} \theta^t-b\|\nabla\Phi_{1/2L}(x_k)\|^2 \leq \theta^t\theta_0(\Phi(x^0) - \Phi^*) + \theta^t\theta_0\|y^0 - y^*(x^0)\|^2 + w\sum_{k=1}^{t} \theta^t-b\|\nabla\Phi_{1/2L}(x_k)\|^2. $$ (95)

Summing from $t = 0$ to $T - 1$,

$$ \sum_{t=0}^{T-1} b_{t+1} = \sum_{t=1}^{T-1} b_t + b_1 $$

$$ \leq \theta_0 \sum_{t=0}^{T-1} \theta^t(\Phi(x^0) - \Phi^*) + \theta_0 \sum_{t=0}^{T-1} \theta^t\|y^0 - y^*(x^0)\|^2 + w \sum_{t=1}^{T-1} \theta^t \sum_{k=1}^{t} \theta^t-b\|\nabla\Phi_{1/2L}(x_k)\|^2 $$

$$ \leq \theta_0 \sum_{t=0}^{T-1} \theta^t(\Phi(x^0) - \Phi^*) + \theta_0 \sum_{t=0}^{T-1} \theta^t\|y^0 - y^*(x^0)\|^2 + w \sum_{t=1}^{T-1} \frac{1}{1-\theta} \|\nabla\Phi_{1/2L}(x_k)\|^2, $$ (96)

where we use $\sum_{t=1}^{T-1} \sum_{k=1}^{t} \theta^t-b\|\nabla\Phi_{1/2L}(x_k)\|^2 = \sum_{k=1}^{T-1} \sum_{t=k}^{T} \theta^t-b\|\nabla\Phi_{1/2L}(x_k)\|^2 \leq \sum_{k=1}^{T-1} \frac{1}{1-\theta} \|\nabla\Phi_{1/2L}(x_k)\|^2$. Now, by telescoping (91),

$$ \frac{1}{8L} \sum_{t=0}^{T-1} \|\nabla\Phi_{1/2L}(x^0)\|^2 \leq \Phi(x^0) - \Phi^* + \sum_{t=0}^{T-1} b_{t+1}. $$

Plugging (96) in,

$$ \frac{1}{8L} \sum_{t=0}^{T-1} \|\nabla\Phi_{1/2L}(x^t)\|^2 - w \sum_{t=1}^{T-1} \frac{1}{1-\theta} \|\nabla\Phi_{1/2L}(x^t)\|^2 \leq \left(1 + \frac{\theta_0}{1-\theta}\right)(\Phi(x^0) - \Phi^*) + \frac{\theta_0}{1-\theta}\|y^0 - y^*(x^0)\|^2. $$ (97)

Plugging in $w \leq \frac{5}{12L}$, $\frac{1}{1-\theta} = \frac{7}{5}$ and $\theta_0 \leq \frac{1}{16}$

$$ \frac{1}{16L} \sum_{t=0}^{T-1} \|\nabla\Phi_{1/2L}(x^t)\|^2 \leq \frac{87}{80}(\Phi(x^0) - \Phi^*) + \frac{7}{80}\|y^0 - y^*(x^0)\|^2. $$
Proof of Theorem 4.1

**Proof.** We first show that criterion (23) implies the criterion in Theorem D.1. By Lemma B.2, as \( \hat{f}_t \) is \((L, \mu)\)-SC-SC and 3L-smooth,

\[
2\mu \text{gap}_{\hat{f}_t}(x^{t+1}, y^{t+1}) \leq \|\nabla \hat{f}_t(x^{t+1}, y^{t+1})\|^2 \leq \alpha_t \|\nabla \hat{f}_t(x^t, y^t)\|^2 \leq 36L^2 \alpha_t \|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2,
\]

therefore,

\[
\text{gap}_{\hat{f}_t}(x^{t+1}, y^{t+1}) \leq \frac{18L^2 \alpha_t}{\mu} \|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2,
\]

which implies \( \text{gap}_{\hat{f}_t}(x^{t+1}, y^{t+1}) \leq \beta_t \|x^t - \hat{x}^t\|^2 + \|y^t - \hat{y}^t\|^2 \) by our choice of \( \{\beta_t\}_t \) and \( \{\alpha_t\}_t \).

We still use \( b_{t+1} = \text{gap}_{\hat{f}_t}(x^{t+1}, y^{t+1}) \) as in the proof of Theorem (D.1). First, note that

\[
\|\nabla \Phi(x^{t+1})\|^2 \leq 2\|\nabla \Phi(x^{t+1}) - \nabla \Phi(\hat{x}^t)\|^2 + 2\|\nabla \Phi(\hat{x}^t)\|^2 \\
\leq 2 \left( \frac{2L^2}{\mu} \right) \|x^t - \hat{x}^t\|^2 + 2\|\nabla \Phi_{1/2L}(x^t)\|^2 \\
\leq \frac{16L^3}{\mu^2} b_{t+1} + 2\|\nabla \Phi_{1/2L}(x^t)\|^2.
\]

(98)

where in the second inequality we use Lemma B.1 and Lemma 4.3 in [Drusvyatskiy and Paquette, 2019]. Summing from \( t = 0 \) to \( T - 1 \), we have

\[
\sum_{t=0}^{T-1} \|\nabla \Phi(x^{t+1})\|^2 \leq \frac{16L^3}{\mu^2} \sum_{t=0}^{T-1} b_{t+1} + 2 \sum_{t=0}^{T-1} \|\nabla \Phi_{1/2L}(x^t)\|^2.
\]

(99)

Applying (96), we have

\[
\frac{16L^3}{\mu^2} \sum_{t=0}^{T-1} b_{t+1} \leq \frac{16L^3 \theta_0}{\mu^2} \sum_{t=0}^{T-1} \theta^t [\Phi(x^0) - \Phi^*] + \frac{16L^3 \theta_0}{\mu^2} \sum_{t=0}^{T-1} \theta^t \|y^0 - y^*(x^0)\|^2 + \frac{2L^3 w}{\mu^2} \sum_{t=1}^{T-1} \frac{1}{1 - \theta} \|\nabla \Phi_{1/2L}(x^t)\|^2.
\]

Plugging in \( \theta_0 = \frac{1}{16L^2} \), \( \theta = \frac{2}{T} \) and \( w = \frac{5n^2}{112L^2} \).

\[
\frac{16L^3}{\mu^2} \sum_{t=0}^{T-1} b_{t+1} \leq \frac{7L}{5} [\Phi(x^0) - \Phi^*] + \frac{7L}{5} \|y^0 - y^*(x^0)\|^2 + \sum_{t=1}^{T-1} \|\nabla \Phi_{1/2L}(x^t)\|^2.
\]

Plugging back into (99),

\[
\sum_{t=0}^{T-1} \|\nabla \Phi(x^{t+1})\|^2 \leq \frac{7L}{5} [\Phi(x^0) - \Phi^*] + \frac{7L}{5} \|y^0 - y^*(x^0)\|^2 + 3 \sum_{t=0}^{T-1} \|\nabla \Phi_{1/2L}(x^t)\|^2.
\]

Applying Theorem D.1,

\[
\frac{1}{T} \sum_{t=1}^{T} \|\nabla \Phi(x^{t+1})\|^2 \leq \frac{268L}{5T} [\Phi(x^0) - \Phi^*] + \frac{28L}{5T} \|y^0 - y^*(x^0)\|^2.
\]

D.2 Complexity of solving auxiliary problem (⋆) and proof of Theorem 4.2

In this layer, we apply an inexact proximal point algorithm to solve the \((L, \mu)\)-SC-SC and 3L-smooth auxiliary problem: \( \min_x \max_y \hat{f}_t(x, y) \). Throughout this subsection, we suppress the outer-loop index \( t \) without confusion,
i.e. we use $\tilde{f}$ instead of $\hat{f}$ and $\tilde{f}_k = \tilde{f}(x, y) - \frac{\tau}{2}||y - z_k||^2$ instead of $\hat{f}_{t,k}$. Accordingly, we also omit the superscript in $(x_k^t, y_k^t)$ and $\epsilon_k^t$.

Before we prove Theorem 4.2, we present a lemma from [Lin et al., 2018a]. The inner loop to solve $(\star \star)$ can be considered as applying Catalyst for strongly-convex minimization in [Lin et al., 2018a] to the function $-\Psi(y) = -\min_{x \in \mathbb{R}^d} \tilde{f}(x, y)$. The following lemma captures the convergence of Catalyst framework in minimization, which we present in Algorithm 2.

Algorithm 2 Catalyst for Strongly-Convex Minimization

Input: function $\tilde{h}$, initial point $x_0$, strong-convexity constant $\mu$, parameter $\tau > 0$
1: Initialization: $q = \frac{\mu}{\mu + \tau}$, $z_1 = x_0$, $\alpha_1 = \sqrt{q}$.
2: for all $k = 1, 2, ..., K$ do
3: Find an inexact solution $x_k$ to the following problem with algorithm $\mathcal{M}$

$$\min_{x \in \mathbb{R}^d} \tilde{h}_k(x) \triangleq \left[ h(x) + \frac{\tau}{2}||x - z_k||^2 \right]$$

such that

$$\tilde{h}_k(x_k) - \min_{x \in \mathbb{R}^d} \tilde{h}_k(x) \leq \epsilon_k. \quad (100)$$

4: Choose $\alpha_{k+1} \in [0, 1]$ such that $\alpha_{k+1}^2 = (1 - \alpha_{k+1})\alpha_k^2 + q\alpha_{k+1}$.
5: $z_{k+1} = x_k + \beta_k(x_k - x_{k-1})$ where $\beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$.
6: end for

Output: $x_K$.

Lemma D.1 ([Lin et al., 2018a]) Consider the problem $\min_{x \in \mathbb{R}^d} \tilde{h}(x)$. Assume function $\tilde{h}$ is $\mu$-strongly convex. Define $A_k = \prod_{i=1}^k (1 - \alpha_i)$, $\eta_k = \frac{\alpha_k - 2}{1 - q}$ and a sequence $\{v_t\}_t$ with $v_0 = x_0$ and $v_k = x_{k-1} + \frac{1}{\alpha_k}(x_k - x_{k-1})$ for $k > 1$. Consider the potential function: $S_k = h(x_k) - h(x^*) + \frac{\eta_k + \rho_{k+1}}{2K(1 - \alpha_{k+1})}||x^* - v_k||^2$, where $x^*$ is the optimal solution. After running Algorithm 2 for $K$ iterations, we have

$$\frac{1}{A_k}S_k \leq \left( \sqrt{S_0} + 2 \sum_{t=1}^K \sqrt{\epsilon_k} \right)^2. \quad (101)$$

Before we step into the proof of Theorem 4.2, we introduce several notations. We denote the dual function of $\tilde{f}$ by $\overline{\Psi}(y) = \min_{x} \tilde{f}(x, y)$. We denote the dual function of $\tilde{f}_k(x, y)$ by $\overline{\Psi}_k(y) = \min_{x} \tilde{f}_k(x, y) = \min_{x} \tilde{f}(x, y) - \frac{\tau}{2}||y - z_k||^2 = \overline{\Psi}(y) - \frac{\tau}{2}||y - z_k||^2$. Let $y^*_k = \arg\max_y \overline{\Psi}_k(y)$. We also define $(x^*_k, y^*_k)$ as the optimal solution to $\min_{x} \max_y \tilde{f}(x, y)$

Proof of Theorem 4.2

Proof When the criterion $||\nabla \tilde{f}_k(x^k, y^k)||^2 \leq \epsilon_k$ is satisfied, by Lemma B.2,

$$\text{gap}_{f_k}(x_k, y_k) \leq \frac{1}{2\mu}||\nabla \tilde{f}_k(x^k, y^k)||^2 \leq \frac{1}{2\mu} \epsilon_k = \frac{\sqrt{2}}{4}(1 - \rho)^k \text{gap}_{f}(x_0, y_0) = \hat{\epsilon}_k,$$

where we define $\hat{\epsilon}_k = \sqrt{\frac{2}{3}}(1 - \rho)^k \text{gap}_{f}(x_0, y_0)$.

The auxiliary problem $(\star \star)$ can be considered as $\max_y \overline{\Psi}(y)$. We see $\text{gap}_{f_k}(x_k, y_k) \leq \hat{\epsilon}_k$ implies $\max_y \overline{\Psi}_k(y) - \overline{\Psi}_k(y_k) \leq \hat{\epsilon}_k$. By choosing $\alpha_1 = \sqrt{\frac{7}{3}}$ in Algorithm 2, it is easy to check that $\alpha_k = \sqrt{q}$ and $\beta_k = \frac{\sqrt{2} - q}{\sqrt{3} + q}$, for all.
k. So this inner loop can be considered as applying Algorithm 2 to $-\hat{\Psi}(y)$ and Lemma D.1 can guarantee the convergence of the dual function. Define $S_k = \Psi(y^*) - \hat{\Psi}(y_k) + \frac{\eta_k(1-\alpha_{i+1})}{\sqrt{1-\eta^2}}\|y^* - y_k\|^2$ with $\eta_k = \frac{\sqrt{1-\eta^2}}{1-\eta}$. Lemma D.1 gives rise to
\[
\frac{1}{A_K}S_k \leq \left(\sqrt{S_0} + 2 \sum_{k=1}^{K} \sqrt{\frac{\bar{e}_k}{A_k}}\right)^2.
\]  
(102)

Note that $A_k = \prod_{i=1}^{k}(1-\alpha_i) = (1-\sqrt{q})^k$ and
\[
\frac{\eta_k\alpha_k\tau}{2(1-\alpha_k)} = \frac{\sqrt{q}-q}{1-q} = \frac{\sqrt{q}-q}{\tau/(\mu + \tau)2(1-\sqrt{q})} = \frac{q(\mu + \tau)}{2} = \frac{\mu}{2}.
\]

So $S_0 = \hat{\Psi}(y^*) - \hat{\Psi}(y_0) + \frac{\eta_k}{\sqrt{1-\eta^2}}\|y^* - y_0\|^2 \leq 2(\hat{\Psi}(y^*) - \hat{\Psi}(y_0))$. Then with $\epsilon_k = \frac{\sqrt{q}}{4}(1-\rho)^k \text{gap}_f(x_0, y_0)$, and we have
\[
\text{Right-hand side of (102)} \leq \left(\sqrt{2(\hat{\Psi}(y^*) - \hat{\Psi}(y_0))} + \sum_{i=1}^{T} \sqrt{2 \left(\frac{1-\rho}{1-\sqrt{q}}\right)^i \text{gap}_f(x_0, y_0)}\right)^2
\]
(103)
\[
\leq 2 \left(1 + \sum_{k=1}^{K} \left(\frac{1-\rho}{1-\sqrt{q}}\right)^k \text{gap}_f(x_0, y_0)\right)^2
\]
(104)
\[
\leq 2 \left(\frac{(1-\rho)^{K+1}}{(1-\sqrt{q})^{K+1}} \text{gap}_f(x_0, y_0)\right)^2 \leq 2 \left(\frac{1-\rho}{1-\sqrt{q}}\right)^K \text{gap}_f(x_0, y_0).
\]
(105)

Plugging back into (102),
\[
S_K \leq 2 \left(\frac{1}{\sqrt{1-\rho} - 1}\right)^2 (1-\rho)^K \text{gap}_f(x_0, y_0) \leq \frac{8}{(\sqrt{q}-\rho)^2}(1-\rho)^K \text{gap}_f(x_0, y_0),
\]
(106)

where the second inequality is due to $\sqrt{1-x+\frac{x}{2}}$ is decreasing in $[0, 1]$. Note that
\[
\|x_K - x^*\|^2 \leq 2\|x_K - x(y_K)\|^2 + 2\|x^*(y_K) - x^*(y)\|^2
\]
\[
\leq \frac{4}{L}f(x_K, y_K) - \hat{f}(x^*(y_K), y_K) + 18\|y_K - y^*\|^2
\]
\[
\leq \frac{4}{L}\epsilon_k + 18\|y_K - y^*\|^2.
\]
(107)

where in the second inequality we use Lemma B.1. Then,
\[
\|x_K - x^*\|^2 + \|y_K - y^*\|^2 \leq 19\|y_K - y^*\|^2 + \frac{4}{L}\epsilon_k.
\]
(108)

Because $\|y_K - y^*\|^2 \leq \frac{2}{\rho}\|\hat{\Psi}(y^*) - \hat{\Psi}(y_K)\| \leq \frac{2}{\rho}S_K$, by plugging in (106) and the definition of $\epsilon_k$, we get
\[
\|x_K - x^*\|^2 + \|y_K - y^*\|^2 \leq \left(\frac{306}{\rho(\sqrt{q}-\rho)^2} + \frac{\sqrt{2}}{L}\right)(1-\rho)^K \text{gap}_f(x_0, y_0).
\]

By Lemma B.2, we have
\[
\|x_K - x^*\|^2 + \|y_K - y^*\|^2 \geq \frac{1}{36L^2}\|\nabla \hat{f}(x_K, y_K)\|^2 \quad \text{and} \quad \text{gap}_f(x_0, y_0) \leq \frac{1}{2\mu}\|\nabla \hat{f}(x_0, y_0)\|^2.
\]

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Then we finish the proof.

D.3 Complexity of solving subproblem (**) and proof of Theorem 4.3

As in the previous subsection, we suppress the outer-loop index $t$. Define $\hat{f}(x) = \max_y \hat{f}(x,y)$, $\hat{y}(y) = \min_x \hat{f}(x,y)$ and $\hat{\Phi} = \min_x \hat{f}(x)$, $\hat{\Psi}(y) = \max_y \hat{f}(y)$. We still define $\tilde{\Phi}_k(y) = \min_x \tilde{f}(x,y) = \min_x \hat{f}(x,y) - \frac{\tau}{2} \|y - z_k\|^2 = \hat{\Psi}(y) - \frac{\tau}{2} \|y - z_k\|^2$, and $\tilde{\Phi}_k(x) = \max_y \tilde{f}_k(x,y)$. Let $(x^*, y^*)$ be the optimal solution to $\min_x \max_y \hat{f}(x,y)$ and $(x_k^*, y_k^*)$ be the optimal solution to $\min_x \max_y \hat{f}_k(x,y)$. Also, in this subsection, we denote $x^*(y) = \arg\min_x \hat{f}(x,y)$ and $y^*(x) = \arg\max_y \hat{f}(x,y)$. Recall that we defined a potential function $S_k = \hat{\Psi}(y^*) - \tilde{\Phi}(y_k) + \frac{\tau}{2} \|y^* - v_k\|^2$ in the proof of Theorem 4.2.

The following lemma shows that the initial point we choose to solve (**) for $\mathcal{M}$ at iteration $k$ is not far from the optimal solution of (**) if the stopping criterion is satisfied for every iterations before $k$.

Lemma D.2 (Initial distance of the warm-start) Under the same assumptions as Theorem 4.2, with accuracy $\epsilon_k$ specified in Theorem 4.2, we assume that for $\forall i < k$, $\|\nabla \hat{f}_i(x_i, y_i)\|^2 \leq \epsilon_i$. At iteration $k$, solving the subproblem (**) from initial point $(x_{k-1}, y_{k-1})$, we have

$$\|x_k - x_k^*\|^2 + \|y_k - y_k^*\|^2 \leq C_k \epsilon_k,$$

where $C_k = \left[ \frac{72 \sqrt{2}}{\mu^2} + \frac{74 \sqrt{2}}{2(\mu + \tau \rho)} \right] \frac{1}{1 - \rho}$, $C_t = \frac{288 \sqrt{2} \max\{40t^2, g^2, t^2\}}{(\mu + \tau)^2 (\mu + \tau)^2} \frac{1}{1 - \rho}$ for $t > 1$.

Proof We separate the proof into two cases: $k = 1$ and $k > 1$.

Case $k = 1$: Note that $x_1 = y_0$, and therefore the subproblem at the first iteration is

$$\min_x \max_y \left[ \hat{f}_1(x, y) = \hat{f}(x, y) - \frac{\tau}{2} \|y - y_0\|^2 \right].$$

Since $x_1^* = \arg\min_x \hat{f}_1(x, y_1^*) = \arg\min_x \hat{f}(x, y_1^*)$ and $x^* = \arg\min_x \hat{f}(x, y^*)$, by Lemma B.1 we have $\|x^* - x_1^*\| \leq 3\|y^* - y_1^*\|$. Furthermore,

$$\|x_0 - x_1^*\|^2 + \|y_0 - y_1^*\|^2 \leq 2\|x_0 - x^*\|^2 + 2\|x^* - x_1^*\|^2 + \|y_0 - y_1^*\|^2 \leq 2\|x_0 - x^*\|^2 + 18\|y_0 - y_1^*\|^2 + \|y_0 - y_1^*\|^2 \leq 2\|x_0 - x^*\|^2 + 36\|y_0 - y_1^*\|^2 + 37\|y_0 - y_1^*\|^2 \leq \frac{72}{\mu} \text{gap}_f(x_0, y_0) + 37\|y_0 - y_1^*\|^2, \quad \text{(110)}$$

where in the last inequality we use Lemma B.2. It remains to bound $\|y_0 - y_1^*\|$. Since $\hat{\Psi}(y) - \frac{\tau}{2} \|y - y_0\|^2$ is $(\mu + \tau)$ strongly-concave w.r.t. $y$, we have

$$\left( \hat{\Psi}(y_1^*) - \frac{\tau}{2}\|y_1^* - y_0\|^2 \right) - \frac{\tau + \mu}{2} \|y_1^* - y_0\|^2 \geq \hat{\Psi}(y_0) = \hat{\Psi}^* - [\hat{\Psi}^* - \hat{\Psi}(y_0)] \geq \hat{\Psi}(y_1^*) - [\hat{\Psi}^* - \hat{\Psi}(y_0)], \quad \text{(111)}$$

It further implies

$$\left( \frac{\tau}{2} \|y_1^* - y_0\|^2 \right) \|y_1^* - y_0\|^2 \leq \hat{\Psi}^* - \hat{\Psi}(y_0) \leq \text{gap}_f(x_0, y_0). \quad \text{(112)}$$

Plugging back into (110), we have

$$\|x_0 - x_1^*\|^2 + \|y_0 - y_1^*\|^2 \leq \left[ \frac{72}{\mu} + \frac{74 \sqrt{2}}{2(\mu + \tau \rho)} \right] \frac{1}{1 - \rho} \epsilon_1.$$
Case $k > 1$: From the proof of Theorem 4.2, we see that $\|\nabla f_i(x_i, y_i)\|^2 \leq \epsilon_i$ implies $\text{gap}_i(x_i, y_i) \leq \tilde{\epsilon}_i$ where $\tilde{\epsilon}_i = \frac{\sqrt{2}}{4}(1 - \rho^i)\text{gap}_i(x_0, y_0)$. Note that $\tilde{f}_k$ is $(L, \mu + \tau)$-SC-SC and $(L + \max\{2L, \tau\})$-smooth. Then

$$\|x_{k-1} - x_k\|^2 \leq 2\|x_{k-1} - x^*(y_{k-1})\|^2 + 2\|x^*(y_{k-1}) - x^*(y_k)\|^2$$

$$\leq 2\|x_{k-1} - x_k\|^2 + 2\left(\frac{L + \max\{2L, \tau\}}{L}\right)^2 \|y_k - y_{k-1}\|^2. \quad (113)$$

Furthermore,

$$\|x_{k-1} - x^*_k\|^2 + \|y_{k-1} - y^*_k\|^2 \leq \|x_{k-1} - x^*_k\|^2 + 2\|y_{k-1} - y^*_k\|^2 + 2\|y_{k-1} - y^*_k\|^2$$

$$\leq 2\|x_{k-1} - x^*_k\|^2 + 2\|y_{k-1} - y^*_k\|^2 + 2\left(\frac{L + \max\{2L, \tau\}}{L}\right)^2 \|y_k - y^*_k\|^2$$

$$\leq \frac{4\tilde{\epsilon}_k}{\min\{L, \mu + \tau\}} + \max\left\{20, \frac{9\tau^2}{2L^2} + 2\right\} \|y_k - y^*_k\|^2. \quad (114)$$

Now we want to bound $\|y_{k-1} - y^*_k\|$. By optimality condition, we have for all $y$,

$$(y - y_{k-1})^T \nabla \hat{f}(y_{k}) \leq 0, \quad (y - y_{k-1})^T \nabla \Psi_{k-1}(y_{k-1}) \leq 0. \quad (115)$$

Choose $y$ in the first inequality to be $y^*_{k-1}$, $y$ in the second inequality to be $y^*_k$, and sum them together, we have

$$(y^*_k - y^*_{k-1})^T (\nabla \Psi_{k-1}(y^*_{k-1}) - \nabla \Psi_k(y^*_k)) \leq 0. \quad (116)$$

Using $\nabla \tilde{\Psi}_k(y) = \nabla_y \tilde{f}(x^*(y), y) - \tau(y - z_k)$, we have

$$(y^*_k - y^*_{k-1})^T (\nabla_y \tilde{f}(x^*(y^*_{k-1}), y^*_{k-1}) - \tau(y^*_{k-1} - z_{k-1}) - \nabla_y \tilde{f}(x^*(y^*_k), y^*_k) + \tau(y^*_k - z_k)) \leq 0. \quad (117)$$

By strong concavity of $\tilde{\Psi}(y) = \max_x \tilde{f}(x, y)$, we have

$$(y^*_k - y^*_{k-1})^T (\nabla \tilde{\Psi}(y^*_k) - \nabla \tilde{\Psi}(y^*_{k-1})) \leq -\mu \|y^*_k - y^*_{k-1}\|^2. \quad (118)$$

Adding to (117), we have

$$(y^*_k - y^*_{k-1})^T \left[\tau(y^*_k - z_k) - \tau(y^*_{k-1} - z_{k-1})\right] \leq -\mu \|y^*_k - y^*_{k-1}\|^2 \quad (119)$$

Rearranging,

$$\frac{\tau}{\mu + \tau} (y^*_k - y^*_{k-1})^T (z_{k-1} - z_k) \geq \|y^*_k - y^*_{k-1}\|^2. \quad (120)$$

Further with $(y^*_k - y^*_{k-1})^T (z_{k-1} - z_k) \leq \|y^*_k - y^*_{k-1}\|\|z_{k-1} - z_k\|$, we have

$$\|y^*_k - y^*_{k-1}\| \leq \frac{\tau}{\mu + \tau} \|z_{k-1} - z_k\|. \quad (121)$$

From updates of $\{z_k\}_k$, we have for $t > 2$

$$\|z_{k-1} - z_k\| = \left\|yk_{k-1} + \frac{\sqrt{q} - q}{\sqrt{q} + q} (yk_{k-1} - y_{k-2}) - y_{k-2} - \frac{\sqrt{q} - q}{\sqrt{q} + q} (yk_{k-2} - y_{k-3})\right\|$$

$$\leq \left(1 + \frac{\sqrt{q} - q}{\sqrt{q} + q}\right) \|yk_{k-1} - y_{k-2}\| + \frac{\sqrt{q} - q}{\sqrt{q} + q} \|yk_{k-2} - y_{k-3}\|$$

$$\leq 2\|yk_{k-1} - y_{k-2}\| + \|yk_{k-2} - y_{k-3}\|$$

$$\leq 6 \max\{\|yk_{k-1} - y^*\|, \|yk_{k-2} - y^*\|, \|yk_{k-3} - y^*\|\} \quad (122)$$

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Therefore,
\[
\|z_k - z_{k-1}\|^2 \leq 36 \max\{\|y_{k-1} - y^*\|^2, \|y_{k-2} - y^*\|^2, \|y_{k-3} - y^*\|^2\} \\
\leq \frac{72}{\mu} \max\{\hat{\Psi}(y_{k-1}) - \hat{\Psi}^*, \hat{\Psi}(y_{k-2}) - \hat{\Psi}^*, \hat{\Psi}(y_{k-3}) - \hat{\Psi}^*\} \\
\leq \frac{72}{\mu} \max\{S_{k-1}, S_{k-2}, S_{k-3}\},
\]
where in the second inequality we use strongly concavity of $\hat{\Psi}$ and in the last we use $\hat{\Psi}(y_k) - \hat{\Psi}^* \leq S_k$. Combining with (121) and (114), we have
\[
\|x_{k-1} - x_k^*\|^2 + \|y_{k-1} - y_k^*\|^2 \leq \frac{4\hat{\epsilon}_1}{\min\{L, \mu + \tau\}} + \frac{36\tau^2 \max\{40L^2, 9\tau^2 + 4L^2\}}{(\mu + \tau)^2 L^2 \mu} \max\{S_{k-1}, S_{k-2}, S_{k-3}\}. \quad (123)
\]
Plugging in $S_k \leq \frac{8}{(\sqrt{q} - \rho)^2}(1 - \rho)^k\|\text{gap}_f(x_0, y_0)\|$ from the proof of Theorem 4.2 and from definition of $\epsilon_k$ and $\hat{\epsilon}_k$, we have
\[
\|x_{k-1} - x_k^*\|^2 + \|y_{k-1} - y_k^*\|^2 \leq \left\{ \frac{2}{\mu \min\{L, \mu + \tau\}} + \frac{288\sqrt{\tau}^2 \max\{40L^2, 9\tau^2 + 4L^2\}}{(\mu + \tau)^2 L^2 \mu (\sqrt{q} - \rho)^2} \right\} \epsilon_k. \quad (124)
\]
It is left to discuss the case $t = 2$. Similarly, we have
\[
\|z_2 - z_1\| = \left\| y_1 + \frac{\sqrt{q} - q}{\sqrt{q} + q} (y_1 - y_0) - y_0 \right\| = \left( 1 + \frac{\sqrt{q} - q}{\sqrt{q} + q} \right) \|y_1 - y_0\| \leq 4 \max\{\|y_1 - y^*\|, \|y_0 - y^*\|\}
\]
Then
\[
\|z_2 - z_1\|^2 \leq 16 \max\{\|y_1 - y^*\|^2, \|y_0 - y^*\|^2\} \\
\leq \frac{32}{\mu} \max\{\hat{\Psi}(y_1) - \hat{\Psi}^*, \hat{\Psi}(y_0) - \hat{\Psi}^*\} \leq \frac{32}{\mu} \max\{S_1, \text{gap}_f(x_0, y_0)\},
\]
Combining with (121) and (114), we have
\[
\|x_1 - x_2^*\|^2 + \|y_1 - y_2^*\|^2 \leq \frac{4\hat{\epsilon}_1}{\min\{L, \mu + \tau\}} + \frac{16\tau^2 \max\{40L^2, 9\tau^2 + 4L^2\}}{(\mu + \tau)^2 L^2 \mu} \max\{S_1, \text{gap}_f(x_0, y_0)\}. \quad (125)
\]
Plugging in $S_1 \leq \frac{8}{(\sqrt{q} - \rho)^2}(1 - \rho)^2\|\text{gap}_f(x_0, y_0)\|$ and definition of $\epsilon_2$ and $\hat{\epsilon}_1$, we have
\[
\|x_1 - x_2^*\|^2 + \|y_1 - y_2^*\|^2 \leq \left\{ \frac{2}{\mu \min\{L, \mu + \tau\}} + \frac{128\sqrt{\tau}^2 \max\{40L^2, 9\tau^2 + 4L^2\}}{(\mu + \tau)^2 L^2 \mu (\sqrt{q} - \rho)^2} \right\} \epsilon_2. \quad (126)
\]

**Proof of Theorem 4.3**

**Proof** We separate our arguments for the deterministic and stochastic settings. Inside this proof, $(x(i), y(i))$ denotes the $i$-th iterate of $M$ in solving the subproblem: $\min_z \max_y \tilde{f}_k(x, y)$. We use $(x_k^*, y_k^*)$ to denote the optimal solution as before. We pick $(x_{k-1}, y_{k-1})$ to be $(x_{k-1}, y_{k-1})$. 


Deterministic setting. The subproblem is \((L + \max\{2L, \tau\})\)-Lipschitz smooth and \((L, \mu + \tau)\)-SC-SC. By Lemma B.2, after \(N\) iterations of algorithm \(\mathcal{M}\),

\[
\|\nabla \hat{f}_k(x(N), y(N))\|^2 \leq 4(L + \max\{2L, \tau\})^2(\|x(N) - x_k^*\|^2 + \|y(N) - y_k^*\|^2)
\]

\[
\leq 4(L + \max\{2L, \tau\})^2 \left(1 - \frac{1}{\Lambda_{\mu, L}(\tau)}\right)^N [\|x_{k-1} - x_k^*\|^2 + \|y_{k-1} - y_k^*\|^2].
\]

Choosing

\[
N = \Lambda_{\mu, L}(\tau) \log \frac{4(L + \max\{2L, \tau\})^2(\|x_{k-1} - x_k^*\|^2 + \|y_{k-1} - y_k^*\|^2)}{\epsilon_k}
\]

\[
\leq \Lambda_{\mu, L}(\tau) \log \frac{4(L + \max\{2L, \tau\})^2 C_t \epsilon_k}{\epsilon_k} = \Lambda_{\mu, L}(\tau) \log (4(L + \max\{2L, \tau\})^2 C_t),
\]

where \(C_t\) is specified in Lemma D.2, we have \(\|\nabla \hat{f}_k(x(N), y(N))\|^2 \leq \epsilon_k\).

Stochastic setting. With the same reasoning as in deterministic setting and applying Appendix B.4 of [Lin et al., 2018a], after

\[
N = \Lambda_{\mu, L}(\tau) \log \frac{4(L + \max\{2L, \tau\})^2(\|x_{k-1} - x_k^*\|^2 + \|y_{k-1} - y_k^*\|^2)}{\epsilon_k} + 1
\]

iterations of \(\mathcal{M}\), we have \(\|\nabla \hat{f}_k(x(N), y(N))\|^2 \leq \epsilon_k\).

\[\square\]

D.4 Total complexity

Proof of Corollary 4.1

Proof From Theorem 4.1, the number of outer-loop calls to find an \(\epsilon\)-stationary point of \(\Phi\) is \(T = O(L(\Delta + D_0^0)\epsilon^{-2})\). From Theorem 4.2, by picking \(\rho = 0.9\sqrt{\frac{L}{\mu}} = 0.9\sqrt{\mu/(\mu + \tau)}\), we have

\[
\|\nabla \hat{f}_t(x_k^t, y_k^t)\|^2 \leq \left[\frac{5508L^2}{\mu^2(\sqrt{q} - \rho)^2} + \frac{18\sqrt{5L^2}}{\mu}\right](1 - \rho)^k \|\nabla \hat{f}_t(x_0^t, y_0^t)\|^2.
\]  

(127)

Therefore, to achieve \(\|\nabla \hat{f}_t(x_k^t, y_k^t)\|^2 \leq \alpha_t \|\nabla \hat{f}_t(x_0^t, y_0^t)\|^2\), we need to solve (**) times, where \(\alpha_t\) is defined as in Theorem 4.1. Finally, Theorem 4.3 implies that solving (**) needs \(N = O \left(\Lambda_{\mu, L}(\tau) \log \left(\frac{\max\{1, L, \tau\}}{\min\{1, \mu\}}\right)\right)\) gradient oracles. The total complexity is

\[
T \cdot K \cdot N = O \left(\frac{\Lambda_{\mu, L}(\tau)L(\Delta + D_0^0)}{\epsilon^2} \sqrt{\frac{\mu + \tau}{\mu}} \log^2 \left(\frac{\max\{1, L, \tau\}}{\min\{1, \mu\}}\right)\right).
\]  

(128)

\[\square\]