Algebraic Structures
of Topological Yang-Mills Theory

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Abstract

We discuss the algebraic structure of the various BRST symmetries associated with topological Yang-Mills theory as a generalization of the BRS analysis developed for the non-Abelian anomaly in the local Yang-Mills theory. We show that our BRST algebra leads to an extended Russian formula and descent equations, which contains the descent equation of Yang-Mills theory as sub-relations. We propose the non-Abelian anomaly counterpart in Topological Yang-Mills theory using the extended descent equation. We also discuss the geometrical structure of our BRST symmetry and some explicit solutions of the extended descent equation are calculated.

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1. Introduction

The topological Yang-Mills theory [1] (TYM in short) is a field theoretical interpretation of Donaldson’s polynomial invariants of smooth four manifolds [2] and the first example of the topological field theories (TFT’s) in the cohomological nature.

Originally, Witten has constructed TYM using a relativistic generalization of Floer’s theory of three manifolds [3] following Atiyah’s conjecture [4]. He also proposed that the theory might be obtained as a BRST quantized version of underlying general covariant theory with higher symmetry. It has been discovered that the theory can be obtained by a BRST quantization of the underlying action which is the Chern class or simply zero [5][6][7]. The classical action has local gauge symmetry as well as topological symmetry, and Witten’s action can be recovered if one quantizes the underlying action such that the configuration space is the instanton moduli space, while preserving the local gauge symmetry.

The remaining gauge symmetry should also be properly fixed, and there are two different approaches. One is to introduce another BRST operator \( \delta_{\text{BRST}} \) which is equivalent to the conventional operator \( \delta_W \) in addition to \( \delta_W \) [8]. And the other is to fix the entire symmetry using a single BRST operator \( \delta_T \) [8][9], which can be roughly decomposed as \( \delta_T \sim \delta_W + \delta_{\text{BRST}} \). Although both approaches have some problems, it has been demonstrated that they are equivalent [10]. The algebraic and geometrical structure of \( \delta_T \) algebra have been analyzed in ref.[8][11]. Kanno [8] in particular has suggested that the correct geometrical framework of \( \delta_T \) algebra is the universal bundle [12].

In the second approach, one identifies the Faddev-Popov ghost with the ghost \( c \) of \( \delta_T \) algebra, and regard \( \delta_T \) as an unification of \( \delta_W \) and \( \delta_{\text{BRST}} \). In this paper, we suggest that this interpretation is somewhat misleading and an alternative way is possible, which naturally leads to an unified formalism of \( \delta_W \) and \( \delta_{\text{BRST}} \) as well as \( \delta_T \) algebras. The main step of our approach is to identify the correct BRS sector among the generalized BRST symmetry relevant in TYM following the traditional method dealing with the local Yang-Mills theory [13][14][15]. Then, we suggest an extended Russian formula and the corresponding extended descent equation.

The Witten’s observables for the Donaldson polynomial invariant can be interpreted as the Abelian anomaly counterparts in TYM [16][17]. On the other hand, it is well known

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1 We shall use the same terminology as ref.[8] for \( \delta_{\text{BRST}}, \delta_W \) and \( \delta_T \).
that the Russian formula with the descent equation of the Yang-Mills theory is an systematic algebraic method to find the 2\(n\)-dimensional non-Abelian anomaly from the Abelian anomaly of two high dimension. Thus, our formalism can be served as an algebraic method for studying the non-Abelian anomaly counterpart in TYM. Historically, the first application of the universal bundle formalism in physics was the geometrical interpretation of the non-Abelian anomaly\[12\][18]. It will be very interesting to investigate the universal bundle formalism whether the structure of non-Abelian anomaly survives in TYM.

In sect. 2, we give a brief review of the algebraic structures of \(\delta_{\text{BRS}}\) algebra[19] and the anomalies of Yang-Mills theory, which are the starting points of this article. In sect. 3, we discuss some difficulties of \(\delta_r\) algebra and propose a unified \(\delta\) algebra which can be decomposed as \(\delta = \delta_r + \delta_{\text{BRS}}\) following the method introduced in sect. 2. We show that \(\delta_{\text{BRS}}\) algebra is identical to the additional BRS algebra in Horne’s approach[9][10]. Then, we give a extension of the Russian formula which leads to an extended descent equation. We shall see that the descent equations of Yang-Mills theory are sub-relations of the extended descent equations, which shows that TYM has surprisingly rich structures. In sect. 4, we suggest that our descent equation can be used to study the non-Abelian anomaly counterpart in TYM. We show that there is a rich class of \(\delta_{\text{BRS}}\) invariant quantities, which contains the consistent anomaly counterparts in TYM.

The conclusion with some open questions is given in the final section. In appendix A we discuss the geometrical origins of the various BRST symmetries and the extended Russian formula based on the universal bundle formalism. This appendix is largely complementary to sect. 3. Some solutions of the extended descent equation are presented in Appendix B.

2. The BRS Algebra and Anomalies

In this section, we will briefly review the algebraic structures of the BRS algebra and the anomalies\[2\] for later use. Let \(M\) be a compact oriented 2\(n\)-dimensional Riemann manifold and \(P\) a principal G-bundle over \(M\). We denote \(\mathcal{U}\) to the space of all connections on \(P\) and \(\mathcal{G}\) to the gauge group, which is the bundle automorphism leaving a base point in \(P\). Let \(A\) denote a Lie algebra valued (Ad(\(P\))-valued) vector potential (connection one-form) over \(M\). The curvature two-form \(F\) is given by

\[
F = dA + A^2,
\]

(2.1)

2 The materials in this section are all standard and I have generally followed Zumino[13][14] with the same conventions.
which satisfies the Bianchi identity

\[ dF + [A, F] = 0. \] (2.2)

Under the gauge transformation

\[ A \rightarrow g^{-1}Ag + g^{-1}dg, \]
\[ F \rightarrow g^{-1}Fg, \] (2.3)

where \( g(x, \lambda) \in \mathcal{G} \) is an element of the gauge group, which is a function of the space-time variables \( x^\mu \) and of additional parameters \( \lambda^i \) which specify the particular element of the gauge group. Thus for a given vector potential \( A(x) \) transformed one \( \mathcal{A} \) defined by

\[ \mathcal{A} \equiv g^{-1}Ag + g^{-1}dg, \] (2.4)

depends on the parameter \( \lambda^i \). The gauge transformed curvature \( \mathcal{F} \) is given by

\[ \mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = g^{-1}Fg. \] (2.5)

Now one can distinguish the exterior derivative in the direction of \( x \) from that in the direction of the group such that

\[ d = dx^\mu \frac{\partial}{\partial x^\mu}, \]
\[ \delta_{\text{BRS}} = d\lambda^i \frac{\partial}{\partial \lambda^i}, \] (2.6)

which satisfy

\[ d^2 = \delta_{\text{BRS}}^2 = d\delta_{\text{BRS}} + \delta_{\text{BRS}}d = 0. \] (2.7)

Though there was no given vector potential in the group direction, it can be generated as the pure gauge, which is a Lie algebra valued one-form (Maurer-Cartan form) in \( \mathcal{G} \)

\[ v = g^{-1}\delta_{\text{BRS}}g. \] (2.8)

Then, it follows that

\[ \delta_{\text{BRS}}\mathcal{A} = -dv - \{\mathcal{A}, v\} \equiv -d_\mathcal{A}v, \]
\[ \delta_{\text{BRS}}v = -v^2, \] (2.9)

which is nothing but the BRS transformations with \( v \) being the Faddev-Popov ghost.
Then we can find that the gauge transformed field strength $F$ satisfies the **Russian formula**;

$$F ≡ dA + A^2 = (d + \delta_{\text{BRS}})(A + v) + (A + v)^2, \quad (2.10)$$

and the Bianchi identity

$$dF + [A, F] = (d + \delta_{\text{BRS}})F + [A + v, F] = 0. \quad (2.11)$$

Now we can define a symmetric invariant polynomial$^3$ of degree $n$

$$P(F^n) = P(F^n), \quad (2.12)$$

satisfying

$$dP(F^n) = (d + \delta_{\text{BRS}})P(F^n) = 0, \quad (2.13)$$

which follows from $^{(2.11)}$. By the Poincaré lemma

$$(d + \delta_{\text{BRS}})\omega_2 = d\omega_1 + \delta_{\text{BRS}}\omega_0.$$

Expanding $\omega_2(A + v, F)$ to the power of $v$

$$\omega_2(A + v, F) = \omega_2^0(A, F) + \omega_2^1 + \cdots + \omega_2^{2n-1}, \quad (2.15)$$

where the superscript indicates the power of $v$. Using eq. $^{(2.14)}$, we can get the descent equations

$$
\begin{align*}
    d\omega_2^1 + \delta_{\text{BRS}}\omega_2^0 &= 0, \\
    d\omega_2^2 + \delta_{\text{BRS}}\omega_2^1 &= 0, \\
    &\vdots \\
    d\omega_2^{2n-2} + \delta_{\text{BRS}}\omega_2^{2n-3} &= 0, \\
    d\omega_2^{2n-1} + \delta_{\text{BRS}}\omega_2^{2n-2} &= 0, \\
    \delta_{\text{BRS}}\omega_2^{2n-1} &= 0.
\end{align*} \quad (2.16)$$

The second relation of the above set of equation is the Wess-Zumino consistency condition$^{[20]}$

$$d\omega_2^2 + \delta_{\text{BRS}}\omega_2^1 = 0, \quad (2.17)$$

that is,

$$\int_M \omega_2^1, \quad (2.18)$$

$^3$ See appendix B for convention.
gives the non-Abelian anomaly in the $2n - 2$ dimension ($\dim(M) = 2n - 2$).

The descent equation (2.16) is a systematic algebraic method to get non-Abelian anomalies in $2n - 2$ dimension from a $2n$ dimensional Abelian anomaly, which is given by the Atiyah-Singer index theorem;

\[ n_+ - n_- = \int \frac{1}{n!} \left( \frac{i}{2\pi} \right)^n \Tr F^n, \]  

where $n_+ (n_-)$ is the number of fermion zero modes of positive (negative) chirality. The non-Abelian anomaly is normalized with an additional factor of $2\pi \sqrt{\text{i}}$ such that

\[ \frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}} \omega_{2n-2}, \]  

gives the non-Abelian anomaly in the $2n - 2$ dimension. In ref.[21], the non-Abelian anomaly has been derived from the families of index theorems of two higher dimensions. They have also discussed the topological origin of non-Abelian anomalies. The general solutions of the descent equation (2.16) can be obtained by various methods[13][14][15][22].

3. BRST Algebras of Topological Yang-Mills theory

3.1. Motivations

Kanno has been suggested[8] that that the natural geometrical framework of $\delta_T$ algebra of [5][6] is the universal bundle constructed by Atiyah-Singer[12]. It is possible to calculate the total curvature of the universal bundle over $M \times U / G$ and the corresponding invariant polynomials, one of which leads to Witten’s observables in TYM theory for Donaldson’s invariants. However, $\delta_T$ algebra is not identical to the universal bundle formalism up to the terms which can be regarded as being similar to the usual BRS transformation. It has been suggested[23][24] that $\delta_T$ algebra does not come from the universal bundle but from a fullback bundle over $M \times U$.

Following[24], consider a principal $G$-bundle $Q$ over $M \times U$. Then one can decompose an arbitrary fixed connection (Lie algebra $\mathfrak{g}$-valued) one-form $\hat{A}$ into

\[ \hat{A} = A + c, \]  

where $A$ denotes an $(1,0)$-form tangent to $M$ and $c$ denotes $(0,1)$-form tangent to $U$. Similarly, we can decompose arbitrary Lie algebra valued $r$-form into $(p, q)$-form

\[ A^r(\mathfrak{g}) = \sum_{r=p+q} A^{p,q}(\mathfrak{g}), \]  

5
where $A^r(g)$ and $A^{p,q}(g)$ denote the space of $g$-valued $r$-forms and $(p,q)$-forms, respectively. We can also decompose the exterior derivative $\hat{d}$ into

$$\hat{d} = d + \delta, \quad d : A^{p,q} \to A^{p+1,q}, \quad \delta : A^{p,q} \to A^{p,q+1}. \quad (3.3)$$

From $\hat{d}^2 = 0$, we get

$$d^2 = \delta^2 = d\delta + \delta d = 0. \quad (3.4)$$

The curvature two-form $\hat{F}$ is decomposed as

$$\hat{F} = (d + \delta)(A + c) + (A + c)^2,$$

$$= \hat{F}^{2,0} + \hat{F}^{1,1} + \hat{F}^{0,2} \quad (3.5)$$

$$= F + \psi + \phi,$$

where

$$\hat{F}^{2,0} = F = dA + A^2,$$

$$\hat{F}^{1,1} = \psi = \delta A + dc + \{A, c\},$$

$$\hat{F}^{0,2} = \phi = \delta c + c^2, \quad (3.6)$$

and satisfies the Bianchi identity

$$(d + \delta)\hat{F} + [A + c, \hat{F}] = 0. \quad (3.7)$$

From $(3.4)(3.5)(3.6)$, we can get the $\delta_r$ algebra of $[5][6]$:

$$\delta A = \psi - dA - \{A, c\},$$

$$\delta c = \phi - c^2,$$

$$\delta \psi = -[c, \psi] - d\phi - [A, \phi],$$

$$\delta \phi = -[c, \phi]. \quad (3.8)$$

Note that if we set $\psi = \phi = 0$ the $\delta_r$ algebra $(3.8)$ looks like the ordinary BRS algebra $(2.9)$, if we identify $c$ with the Faddev-Popov ghost and $\delta$ with the BRS operator $\delta_{b,r}$. The $\delta_r$ algebra also contains Witten’s topological BRST ($\delta_w$) algebra which is given by $[1]$

$$\delta_w A = \psi, \quad \delta_w \psi = -d\phi - [A, \phi], \quad \delta_w \phi = 0. \quad (3.9)$$

The condition $\psi = \phi = 0$ is known as the horizontality and, then, eq. $(3.5)$ reduces to

$$(d + \delta)(A + c) + (A + c)^2 = dA + A^2, \quad (3.10)$$
which is equivalent to the Russian formula (2.10). Thus, the Russian formula is nothing but imposing the horizontality condition in this approach. It follows that there can be no analogue of the Russian formula in TYM - that is, the full BRST symmetry (3.8) of TYM is obtained if we do not impose the horizontality condition, and instead of the Russian formula we have non-vanishing extra components of curvature, which will contribute to the Abelian-anomaly counterpart in TYM. Consequently, there is no analogue of the non-Abelian anomaly counterpart in TYM. The Abelian anomaly of TYM absorbs the non-Abelian anomaly of YM!

However, we will see throughout this paper that the above approach is not correct. Note that we did not impose the extra components of curvature originated from the transition \((d, A) \rightarrow (d + \delta_{\text{BRS}}, A + \nu)\) to vanish by offhand in the approach described in sect. 2, and the BRS algebra (2.9) and the Russian formula (2.10) were just byproducts of the geometrical structures of the Faddev-Popov ghost \(\nu\) and the BRS operator \(\delta_{\text{BRS}}\). It is also quite strange to say that the extra components \(\psi, \phi\) vanish, while \(c\) does not vanish identically - unless \(c\) is the pure gauge, because \(c\) is the connection one-form which defines the curvature \(\phi\) (3.6). Thus, we should consider the gauge transformed connection one-form of \(A + c\) as sect. 2. to incorporate the correct BRS structure. Then, we will see that there is an analogue of the Russian formula in TYM as a natural extension of eq. (2.10). We should not be confused with the notion of the horizontality condition, which can be stated precisely as ‘the extra components of curvature induced by \((\delta_{\text{BRS}}, \nu)\) system vanish identically.’ And, it is evident that the structure of BRS symmetry involving the quantization of Witten’s original action of TYM [1], which has the local gauge symmetry, should be equivalent to the usual BRS structure of YM [2].

Finally, there is no need in practice to consider the gauge transformed connection one-form and the fine points noted previously as long as the set of Witten’s observables is involved, which is given the invariant polynomial \(P(\hat{F}^n)\) [8]. This situation is similar to the 2n-dimensional Abelian anomaly in the Yang-Mills theory, which is also given by an invariant polynomial degree \(n\). However, the non-Abelian anomaly arises from the two-high dimensional Abelian anomaly via the descent equation, where the BRS structure of Yang-Mills theory is crucial. Then, discovering the correct BRS structure will illuminate the counterpart of the non-Abelian anomaly in TYM theory.
3.2. Algebra

In this subsection we reformulate the construction of ref.\[8\] following the methods reviewed in sect. 2. A geometrical construction which is equivalent and somewhat complementary to algebraic one of this subsection is presented in the appendix A.

Let $A$ and $c$ be the components of some fixed total connection over $M \times U$. Consider the gauge transformed connection

$$A + c \rightarrow g^{-1}Ag + g^{-1}dg + g^{-1}cg + g^{-1}\delta g,$$

where we have naturally extends the action of $g$ to $c$. Let

$$\begin{align*}
\mathcal{A} &= g^{-1}Ag + g^{-1}dg, \\
\mathcal{C} &= g^{-1}cg + g^{-1}\delta g,
\end{align*}$$

such that we replace $A + c$ to $\mathcal{A} + \mathcal{C}$. Let $\hat{\mathcal{F}}$ denote the transformed total curvature

$$\hat{\mathcal{F}} = (d + \delta)(\mathcal{A} + \mathcal{C}) + (\mathcal{A} + \mathcal{C})^2$$

$$= g^{-1}\hat{\mathcal{F}}g,$$

which can be written in the components

$$\begin{align*}
\hat{\mathcal{F}}^{2,0} &= \mathcal{F} = d\mathcal{A} + \mathcal{A}^2, \\
\hat{\mathcal{F}}^{1,1} &= \Psi = \delta\mathcal{A} + d\mathcal{C} + \{\mathcal{A}, \mathcal{C}\}, \\
\hat{\mathcal{F}}^{0,2} &= \Phi = \delta\mathcal{C} + \mathcal{C}^2.
\end{align*}$$

The transformed total curvature also satisfies the Bianchi identity

$$(d + \delta)\hat{\mathcal{F}} + [\mathcal{A} + \mathcal{C}, \hat{\mathcal{F}}] = 0.$$  

From (3.4)(3.14)(3.15), we get

$$\begin{align*}
\delta\mathcal{A} &= \Psi - d\mathcal{C} - \{\mathcal{A}, \mathcal{C}\}, \\
\delta\mathcal{C} &= \Phi - \mathcal{C}^2, \\
\delta\Psi &= -[\mathcal{C}, \Psi] - d\Phi - [\mathcal{A}, \Phi], \\
\delta\Phi &= -[\mathcal{C}, \Phi],
\end{align*}$$

\(8\)
One can see that the above algebra is formally identical to $\delta_i$-algebra in ref. [8]. However, there are some important differences. Note that we did not start from the orbit space $\mathcal{U}/\mathcal{G}$ but the space of all connection $\mathcal{U}$. Thus, we can decompose $\delta$ into

$$
\delta = \delta_i + \delta_{\text{BRS}},
$$

$$
\delta^2_i = \delta^2_{\text{BRS}} = \delta_i \delta_{\text{BRS}} + \delta_{\text{BRS}} \delta_i = 0,
$$

and $C$ into

$$
C = g^{-1}cg + g^{-1}\delta_i g + g^{-1}\delta_{\text{BRS}} g = C + v,
$$

where $v = g^{-1}\delta_{\text{BRS}} g$ is the Faddev-Popov ghost and $\delta_{\text{BRS}}$ denote the exterior derivatives along the gauge group as in sect. 2.

Then, from eq. (3.16) or direct computations, we find the $\delta_{\text{BRS}}$-algebra

$$
\delta_{\text{BRS}} A = -dv - \{A, v\},
$$

$$
\delta_{\text{BRS}} v = -v^2,
$$

$$
\delta_{\text{BRS}} \Psi = -[v, \Psi],
$$

$$
\delta_{\text{BRS}} \Phi = -[v, \Phi].
$$

and $\delta_i$-algebra

$$
\delta_i A = \Psi - dC - \{A, C\},
$$

$$
\delta_i C = \Phi - C^2,
$$

$$
\delta_i \Psi = -[C, \Psi] - d\Phi - [A, \Phi],
$$

$$
\delta_i \Phi = -[C, \Phi],
$$

$$
\delta_i v = -\delta_{\text{BRS}} C - \{v, C\}.
$$

Note that the set of equations of eq. (3.19) contains the usual BRS algebra (2.9), and they are identical to the extra BRS algebra introduced by Horne [9] to fix the remaining local gauge symmetry in Witten’s original action of TYM theory [10]. The last term of eq. (3.20) implies that $\delta_i$ and $\delta_{\text{BRS}}$ are not completely decoupled. On the other hand, one can see in appendix A that the correct variation operator of the universal bundle is not $\delta_i$ operator but $\delta_W$ operator

$$
\delta_W A = \Psi,
$$

$$
\delta_W \Psi = -d\Phi - [A, \Phi],
$$

$$
\delta_W \Phi = 0,
$$

$$
\delta_W v = 0.
$$
One can see that the ghost numbers of \((A, \Psi, \Phi, v)\) for \(\delta_W\) (\(\delta_T\)) algebra are \((0, 1, 2, 0)\), while \((0, 0, 0, 1)\) in terms of \(\delta_{BRS}\)-algebra - \(C\) has \(\delta_T\) ghost number 1 and \(\delta_{BRS}\) ghost number 0. Thus we have a natural bigrading structure of the ghost numbers, which should be preserved independently.

It should be stressed that, if we set \(C = 0\), we can recover the usual \(\delta_{BRS}\) algebra of eq. (2.9) plus some extra relations

\[
\begin{align*}
\delta_T A &= \Psi, \\
\delta_T \Psi &= 0, \\
\delta_{BRS} \Psi &= -[v, \Psi], \\
\delta_T v &= 0,
\end{align*}
\]

(3.22)

Note that \(\delta_T\) algebra in (3.20) can be read as

\[
\begin{align*}
\delta_T A + \{C, A\} + dC &= \Psi = \delta_W A, \\
\delta_T \Psi + [C, \Psi] &= -d\Phi - [A, \Phi] = \delta_W \Psi, \\
\delta_T \Phi + [C, \Phi] &= 0 = \delta_W \Phi.
\end{align*}
\]

(3.23)

The last two relations imply that the \(\delta_W\) operator is the covariant derivative as already noted by [24], while \(\delta_T\) is the exterior derivative. The first relation of eq. (3.23) seems to be problematic at first sight due to the extra term \(dC\). The reader can see, however, in appendix A that the horizontal part of \(\delta_T A\) is \(\Psi\), which means that \(\delta_W\) operator is the covariant derivative. Note that \(\Phi\) vanishes identically for \(C = 0\), and in this case \(\delta_W\) operator is equivalent to \(\delta_T\).

Some readers may be confused with our approach, because it was sometimes believed that \(\delta_T\) algebra reduces to \(\delta_W\) algebra (3.3) for \(C = 0\) and reduces to \(\delta_{BRS}\) algebra (2.9) for \(\Psi = \Phi = 0\). The difference is that we did not identify \(C\) as the Faddev-Popov ghost. We have shown that for \(C = 0\) the \(\delta\) operator decouple completely into \(\delta_{BRS}\) plus \(\delta_T\) operators, and the later reduces to \(\delta_W\) algebra with \(\Phi = 0\). It is obvious that a covariant derivative is equivalent to ordinary one when a gauge orbit of a connection form vanishes. Note that there is an additional term \(\delta_{BRS} \Psi = -[v, \Psi]\) in our \(\delta_{BRS}\) algebra (3.22) for \(C = 0\) unlike the original \(\delta_{BRS}\) algebra of (2.9). It is, however, unnecessary and indeed not correct to set \(\delta_T A = \Psi = 0\) to recover the usual BRS algebra, because there is no reason to set \(\Psi\) (tangent to a local cross section of \(U/G \to U\)) to zero. We just do not use the extra
component $\Psi$ and its BRS variation in the BRS quantization of the Yang-Mills theory. More detailed analysis can be found in appendix A.

Finally, we remind the readers that Horne’s approach is correct method in the quantization of TYM\cite{9}. In the next subsection, we will see that the real utility of $\delta_r$ algebra is its ability of local trivialization.

### 3.3. Russian formula and descent equation

Note that

\begin{equation}
\hat{F} = (d + \delta_T + \delta_{\text{BRS}})(A + C + v) + (A + C + v)^2
= (d + \delta_r)(A + C) + (A + C)^2,
\end{equation}

where we have used eqs. (3.19)-(3.20). One can see that the above equation is a generalization of the Russian formula (2.10). In the limit of $C = 0$ it is not entirely identical to eq. (2.10);

\begin{equation}
(d + \delta_T + \delta_{\text{BRS}})(A + v) + (A + v)^2 = dA + A^2 + \delta_r A = F + \Psi,
\end{equation}

however, after throwing away $\delta_r$ we get desired result.\footnote{In preparing the revised version of this manuscript I have found, rather surprisingly, that a similar formula for eq. (3.25) is already appeared in ref. [25]. The BRST symmetry of TYM had been almost discovered!}

From the Bianchi identity (3.15)

\begin{equation}
(d + \delta_T + \delta_{\text{BRS}})\hat{F} + [A + C + v, \hat{F}] = 0,
\end{equation}

we can define an symmetric invariant polynomial of degree $n$, $P(\hat{F}^n)$;

\begin{equation}
(d + \delta_T + \delta_{\text{BRS}})P(\hat{F}^n) = 0.
\end{equation}

One can also find the another Bianchi identity

\begin{equation}
(d + \delta_r)\hat{F} + [A + C, \hat{F}] = 0,
\end{equation}

which leads

\begin{equation}
(d + \delta_r)P(\hat{F}^n) = 0.
\end{equation}
Note that \( P(\hat{F}^n) \) is identical to \( P(\hat{\mathcal{F}}^n) \). A simple consequence of the above identity (3.29) is so called the topological descent equation \( \delta \). That is, if we expand \( P(\hat{F}^n) \) in powers of the \( \delta_r \) ghost number such that
\[
P(\hat{F}^n) = \hat{\mathcal{W}}_{2n}^{0,0} + \hat{\mathcal{W}}_{2n-1}^{0,1} + \cdots + \hat{\mathcal{W}}_0^{0,2n},
\]
(3.30)
where the superscripts indicate \( \delta_{BRS} \) and \( \delta_r \) ghost numbers, respectively, and the subscript indicate the space-time form degree, eq. (3.29) leads the topological descent equation
\[
d\hat{\mathcal{W}}_{2n}^{0,0} = 0,
\]
\[
d\hat{\mathcal{W}}_{2n-1}^{0,1} + \delta_r \hat{\mathcal{W}}_{2n}^{0,0} = 0,
\]
\[
d\hat{\mathcal{W}}_{2n-2}^{0,2} + \delta_r \hat{\mathcal{W}}_{2n-1}^{0,1} = 0,
\]
\[
\vdots
\]
\[
d\hat{\mathcal{W}}_0^{0,2n} + \delta_r \hat{\mathcal{W}}_1^{0,2n-1} = 0,
\]
\[
\delta_r \hat{\mathcal{W}}_0^{0,2n} = 0.
\]
(3.31)

It is well-known that the integration of \( \hat{\mathcal{W}}_{2n-\ell}^{0,\ell} \) in (3.30) over \( 2n - \ell \) cycle \( \gamma_{2n-\ell} \) of \( M \) is the Witten’s observable \( \tilde{\mathcal{W}}_{2n-\ell}^{0,\ell} \) for Donaldson’s polynomial invariant \[1][8]:
\[
\int_{\gamma_{2n-\ell}} \hat{\mathcal{W}}_{2n-\ell}^{0,\ell} \equiv \tilde{\mathcal{W}}_{2n-\ell}^{0,\ell}.
\]
(3.32)

From the topological descent equation (3.31) we can see that the Witten’s observable \( \tilde{\mathcal{W}}_{2n-\ell}^{0,\ell} \), \( (k = 0, \ldots, 2n) \) is \( \delta_r \) closed
\[
\delta_r \int_{\gamma_{2n-\ell}} \hat{\mathcal{W}}_{2n-\ell}^{0,\ell} = \delta_r \tilde{\mathcal{W}}_{2n-\ell}^{0,\ell} = 0,
\]
(3.33)
as well as \( \delta_{BRS} \) closed, which follows from eq. (3.27) (3.29).

Note that our convention is the somewhat different from the usual one in denoting a Witten’s observable. In Donaldson-Witten theory the dimension of \( M \) is four, \( 0 \leq \dim(Y) \leq 4 \) and there is no restriction such that we should only consider second-rank invariant polynomial, as everybody knows. Thus, if we restrict \( M \) is four dimensional, \( P(\hat{F}^n) \) can be regarded as an element of \( H^{2n}(M \times U/G) \). Then a Witten’s observable \( \tilde{\mathcal{W}}_{2n-\ell}^{0,\ell} \) should reads as \( \tilde{\mathcal{W}}_{\ell'}^{0,2n-\ell'} \), where \( 0 \leq \ell' \leq 4 \), which is an element of \( H^{2n-\ell'}(U/G) \). Similarly, the cycle \( \gamma_{2n-\ell} \) should reads as \( \gamma_{\ell'} \). I hope the readers may not be confused with our conventions.
However, this is not the end of the story. Eq. (3.27)-(3.29) imply that we have an identity by the Poincaré lemma

\[ P(\hat{F}^n) = (d + \delta_r + \delta_{BRS}) W_{2n-1}(A + C + v, \hat{F}) = (d + \delta_r) W_{2n-1}^0 (A + C, \hat{F}), \]  

(3.34)

where \( W_{2n-1}(A + C + v, \hat{F}) \) denotes the extended Chern-Simons form. It should be noted that the above formula valid only after local trivialization and does not imply the Witten’s observables are trivial [10]. Note also that we can not replace \( \delta_r \) with \( \delta_w \) in eq. (3.34), while we can do for eq. (3.31)-(3.33). Expanding \( W_{2n-1}(A + C + v, \hat{F}) \) with powers of \( v \)

\[ W_{2n-1}(A + C + v, \hat{F}) = W_{2n-1}^0 (A + C, \hat{F}) + W_{2n-2}^1 + \cdots + W_0^{2n-1}, \]  

(3.35)

where the superscript indicates the power of \( v \) (the \( \delta_{BRS} \) ghost number) and the subscript indicates the space-time form degree plus the \( \delta_r \) ghost number. Thus we can get an descent equations which have the same form and origin with eq. (2.16)

\[
\begin{align*}
(d + \delta_r) W_{2n-2}^1 (A + C, v, \hat{F}) + \delta_{BRS} W_{2n-1}^0 (A + C, \hat{F}) &= 0, \\
(d + \delta_r) W_{2n-3}^2 (A + C, v, \hat{F}) + \delta_{BRS} W_{2n-2}^1 (A + C, v, \hat{F}) &= 0, \\
& \vdots \\
(d + \delta_r) W_0^{2n-1} (v) + \delta_{BRS} W_1^{2n-2} (A + C, v) &= 0, \\
\delta_{BRS} W_0^{2n-1} (v) &= 0.
\end{align*}
\]  

(3.36)

We shall call the above relations extended descent equations.

We can expand the each relations of (3.36) in powers of \( \delta_r \) ghost number. For an example, consider the first relation of (3.36)

\[(d + \delta_r) W_{2n-2}^1 (A + C, v, \hat{F}) + \delta_{BRS} W_{2n-1}^0 (A + C, \hat{F}) = 0,
\]

which leads

\[
\begin{align*}
d W_{2n-2}^{1,0} + \delta_{BRS} W_{2n-1}^{0,0} &= 0, \\
d W_{2n-3}^{1,1} + \delta_r W_{2n-2}^{1,0} + \delta_{BRS} W_{2n-2}^{0,1} &= 0, \\
& \vdots \\
d W_0^{1,2n-2} + \delta_r W_1^{1,2n-3} + \delta_{BRS} W_1^{0,2n-2} &= 0, \\
\delta_r W_0^{1,2n-2} + \delta_{BRS} W_0^{0,2n-1} &= 0.
\end{align*}
\]  

(3.37)

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after using the expansions in terms of $\delta_r$ ghost number

$$W_{2n-1-k}^k = \sum_{k'=0}^{2n-1-k} W_{2n-1-k-k',k'}, \quad (3.38)$$

such that the superscripts $k$ and $k'$ of $W_{2n-1-k-k',k'}$ denote $\delta_{\text{BRS}}$ and $\delta_r$ ghost numbers, respectively, and the subscript $2n-1-k-k'$ indicate the space-time form degree. Similarly, the second relation of (3.36)

$$(d + \delta_r)W_{2n-3}^2(A + C, v, \hat{F}) + \delta_{\text{BRS}} W_{2n-2}^1(A + C, \hat{F}) = 0, \quad (3.39)$$

leads

$$dW_{2n-3}^2,0 + \delta_{\text{BRS}} W_{2n-2}^1,0 = 0,$$

$$dW_{2n-4}^2,1 + \delta_r W_{2n-3}^2,0 + \delta_{\text{BRS}} W_{2n-3}^1,1 = 0,$$

$$dW_{2n-5}^2,2 + \delta_r W_{2n-4}^2,1 + \delta_{\text{BRS}} W_{2n-4}^1,2 = 0,$$

$$
\vdots
$$

$$dW_0^2,2n-3 + \delta_r W_1^2,2n-4 + \delta_{\text{BRS}} W_1^1,2n-3 = 0,$$

$$\delta_r W_0^2,2n-3 + \delta_{\text{BRS}} W_0^1,2n-2 = 0. \quad (3.40)$$

Note that the first relations of (3.37) and (3.40) are identical to the first and second relations of eq. (2.16) respectively. Similarly, if we expand the remaining relations of (3.36), the first relation of each of the sub-descent equations reduces to the relations in the descent equation of Yang-Mills (2.16) subsequently. Note also, for an example, that the last relations of (3.37), (3.40) and the other sub-descent equations of (3.36) lead

$$\delta_r W_0^{1,2n-2} + \delta_{\text{BRS}} W_0^{0,2n-1} = 0,$$

$$\delta_r W_0^{2,2n-3} + \delta_{\text{BRS}} W_0^{1,2n-2} = 0,$$

$$
\vdots
$$

$$\delta_r W_0^{2n-1,0} + \delta_{\text{BRS}} W_0^{2n-2,1} = 0,$$

$$\delta_{\text{BRS}} W_0^{2n-1,0} = 0. \quad (3.41)$$

Clearly, the above descent equation is originated from a part of the extended Russian formula (3.24)

$$\Phi = \delta_r C + C^2 = (\delta_r + \delta_{\text{BRS}})(C + v) + (C + v)^2, \quad (3.42)$$

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and from a part \( P(\Phi^n) = \tilde{W}_0^{0,2n} \) of the invariant polynomial \( P(\tilde{F}^n) \) after the similar procedure discussed in sect. 2.

Each term of the fully expanded descent equations can be exactly calculated by the following procedures. Note that the structure of the extended descent equation (3.36) is formally identical to that of the Yang-Mills case (2.16) if we set \( \hat{d} = d + \delta_r \) and \( \hat{A} = A + C \). Thus we can follow the fully developed and well-known methods proposed in [13][14][15][22] to get \( W_{2n−1−k}^k \). Then, by expanding \( W_{2n−1−k}^k \) in terms of the \( \delta_r \) ghost number, we can get \( W_{2n−1−k−k',k'} \). Following the method developed in ref. [14], we describe the detailed procedures and some explicit results of calculation in the appendix B.

4. Non-Abelian Anomaly Counterpart in TYM

In the previous section we have extended the Russian formula and the descent equation, and we have seen that the extended descent equation contains the Yang-Mills descent equation (2.16) as a sub-relation. In particular, eq. (3.39) after the expansion in terms of \( \delta_r \)-ghost number leads the Wess-Zumino consistency condition as a sub-relation, i.e. the first relation of eq. (3.40)

\[
d W_{2n−3}^{2,0} + \delta_{BRS} W_{2n−2}^{1,0} = 0,
\]

which is identical to (2.17).

Then, it is natural to ask whether \( W_{2n−2}^1 \) in general is the non-Abelian anomaly counterpart in topological Yang-Mills theory and whether eq. (3.39) is the corresponding consistency condition. Clearly \( W_{2n−2}^1 \) is also linear in \( v \) as \( \omega_{2n−2}^1 (W_{2n−2}^{1,0}) \), and if we integrate (3.39) by a \( 2n−2 \) dimensional cycle \( \tilde{\gamma}_{2n−2} \) of \( M \times \mathcal{U}/G \) we get

\[
\delta_{BRS} \int_{\tilde{\gamma}_{2n−2}} W_{2n−2}^1 = 0, \tag{4.2}
\]

which can be interpreted as a consistency condition.

To be more precise, consider integration the \( i \)th relation of eq. (3.40) over a \( 2n−1−i \) cycle \( \gamma_{2n−1−i} \) of \( M \)

\[
\delta_{BRS} W_{2n−2}^{1,0} = 0,
\]

\[
\delta_r W_{2n−2−\ell}^{2,\ell−1} + \delta_{BRS} W_{2n−2−\ell}^{1,\ell} = 0, \quad \text{for} \quad \ell = 1, \ldots, 2n−2 \tag{4.3}
\]
where we generally denote
\[ W_{2n-1-k-k'}^{k,k'} = \int_{\gamma_{2n-1-k-k'}} W_{2n-1-k-k'}^{k,k'}, \tag{4.4} \]

where \( \gamma_{2n-1-k-k'} \) is a \( 2n - 1 - k - k' \) dimensional cycle of \( M \). Note that \( W_{2n-2-\ell}^{1,\ell} \) is a zero form over \( M \) and a \( \ell \)-form over \( U/G \), respectively, and \( \delta_r \) is the exterior derivative over \( U/G \). Thus, if we integrate (4.3) over a \( \ell \)-dimensional cycle \( \gamma'_{\ell} \) of \( U/G \), we get
\[ \delta_{\text{BRS}} \int_{\gamma'_{\ell}} W_{2n-2-\ell}^{1,\ell} = 0. \tag{4.5} \]

That is, the resulting \( \int_{\gamma'_{\ell}} W_{2n-2-\ell}^{1,\ell} \) is also linear in \( v \) and satisfies the consistency condition.

Integration of eq. (3.37) over appropriate cycle of \( M \) leads
\[ \delta_r W_{2n-2-j}^{1,j} = -\delta_{\text{BRS}} W_{2n-2-j}^{0,j+1}, \quad \text{for } j = 0, \ldots, 2n - 2 \tag{4.6} \]

Thus we can see that \( W_{2n-2-j}^{1,j} \) is \( \delta_{\text{BRS}} \) (\( \delta_r \)) closed up to \( \delta_r \) (\( \delta_{\text{BRS}} \)) exact term. If we integrate the both sides of (4.6) by a \( j + 1 \) dimensional cycle \( \gamma'_{j+1} \) of \( U/G \), we can see that
\[ \delta_{\text{BRS}} \int_{\gamma'_{j+1}} W_{2n-2-j}^{0,j+1} = 0. \tag{4.7} \]

We can repeat the same procedures for the remaining sub-descent equations of eq. (3.38) after using the expansion (3.38) and conclude that \( \int_{\gamma'_{k'}} W_{2n-1-k-k'}^{k,k'} \) is \( \delta_{\text{BRS}} \) invariant
\[ \delta_{\text{BRS}} \int_{\gamma'_{k'}} W_{2n-1-k-k'}^{k,k'} = 0. \tag{4.8} \]

We can see that the \( \delta_r \) with \( \delta_{\text{BRS}} \) cohomology class of \( W_{2n-1-k-k'}^{k,k'} \) depends only on the homology class of \( \gamma \). That is, if \( \gamma_{2n-1-k-k'} \) is a boundary, say \( \gamma_{2n-1-k-k'} = \partial \beta_{2n-k-k'} \), then
\[ W_{2n-1-k-k'}^{k,k'} = \int_{\gamma_{2n-1-k-k'}} \mathcal{W}_{2n-1-k-k'}^{k,k'} \\
= \int_{\beta_{2n-k-k'}} d\mathcal{W}_{2n-1-k-k'}^{k,k'} \\
= -\delta_r \int_{\beta_{2n-k-k'}} \mathcal{W}_{2n-k-k'}^{k,k'-1} - \delta_{\text{BRS}} \int_{\beta_{2n-k-k'}} \mathcal{W}_{2n-k-k'}^{k-1,k'}. \tag{4.9} \]
It is well known that the $\delta_r$ cohomology class (or $\delta_w$ cohomology class) of Witten’s observables $\tilde{W}_{2n-\ell}0,\ell$ depends only on the homology class of cycle in $M$

$$\tilde{W}_{2n-\ell}0,\ell = \int_{\gamma_{2n-\ell}} \tilde{W}_{2n-\ell}0,\ell = \int_{\beta_{2n+1-\ell}} d\tilde{W}_{2n-\ell}0,\ell$$

$$= -\delta_r \int_{\beta_{2n+1-\ell}} \tilde{W}_{2n+1-\ell}0,\ell-1$$

(4.10)

where $\gamma_{2n-\ell} = \partial\beta_{2n+1-\ell}$. If we further integrate the both sides of eq. (4.9) over a $k'$ dimensional cycle $\gamma'_{k'}$ of $U/G$

$$\int_{\gamma'_{k'}} \int_{\gamma_{2n-1-k-k'k,k'}} W_{2n-1-k-k'k',k} = -\delta_{BRS} \int_{\gamma'_{k'}} \int_{\beta_{2n-k-k'}} W_{2n-k-k'k-1,k'},$$

(4.11)

which show that the $\delta_{BRS}$ cohomology class of $\int_{\gamma'_{k'}} W_{2n-1-k-k'k,k'}$ depends only on homology class of $\gamma$. Note also that, if $\gamma'$ is a boundary of $\beta'$, say $\gamma'_{k'} = \partial \beta'_{k'+1}$, we have

$$\int_{\gamma'_{k'}} W_{2n-1-k-k'k,k} = -\delta_{BRS} \int_{\beta'_{k'+1}} \delta_r W_{2n-k-k'k-1,k'}$$

$$= \delta_{BRS} \int_{\beta'_{k'+1}} \delta_{BRS} W_{2n-k-k'k-2,k'+1}$$

$$= 0,$$

(4.12)

which show that $\int_{\gamma'_{k'}} \int_{\gamma_{2n-1-k-k'k,k'}} \mathcal{W}_{2n-1-k-k'k,k'}$ vanishes if $\gamma'$ is trivial in homology.

Up to now we have developed an analogy with the local Yang-Mills theory and introduced the non-Abelian anomaly counterpart of TYM. We have shown that there are large classes of $\delta_{BRS}$ invariant quantities which $\delta_{BRS}$ cohomology class only depends on homology class of $\gamma$. These properties should be compared with the Witten’s observables, which are $\delta_r$ invariant as well as $\delta_{BRS}$ invariant and their $\delta_r$ cohomology classes are depends only on the homology class of $\gamma$.

It should be stressed that the notions of cohomology associated with a Witten’s observable $\tilde{W}_{2n-k}0,k$ and with $\int_{\gamma'_{k'}} W_{2n-1-k-k'k,k'}$ (as well as $W_{2n-1-k-k'k,k'}$) are indeed different. The later is the integrals of a locally defined density and its cohomology is so called the local cohomology[25], and we can not interchange $\delta_r$ cohomology with $\delta_w$ cohomology unlike the former.

One can easily show that it is non-trivial in the sense of the local cohomology. The non-triviality of $\mathcal{W}_{2n-1-k,k,0}$ in the sense of local cohomology has been prove in sect. 3 of
Following the same procedures after replacing \((d, A)\) to \((d + \delta_\tau, A + C)\), one can easily prove that one can not write

\[
W_{2n-1-k}^k = (d + \delta_\tau)\hat{\eta}_{2n-2-k}^k + \delta_{\text{BRST}} \hat{\theta}_{2n-1-k}^{-1}, \tag{4.13}
\]

where \(\hat{\eta}, \hat{\theta}\) are some local expressions. Furthermore, a mathematical expression for the triviality of \(W_{2n-1-k_{,k'}}\)

\[
W_{2n-1-k_{,k'}} = d\eta_{2n-2-k_{,k'}} + \delta_\tau \xi_{2n-1-k_{,k'}}^{-1} + \delta_{\text{BRST}} \theta_{2n-1-k_{,k'}}^{-1}, \tag{4.14}
\]

can be viewed as an expansion of eq. (4.13), that is impossible.

5. Conclusion and Further Study

At present, there are two important open problems in Donaldson-Witten theory. One is to extend the theory beyond the stable region, and the other is to find some consistent topological symmetry breaking mechanism. It is well known that there are many serious problems beyond the stable region, i.e. we can not avoid the reducible connections which contribute to the singularities and the non-compactness of the space of \(\delta_w\) fixed points \(\tilde{\mathcal{M}}\) - the instanton moduli space \(\mathcal{M}\) with the solution space of \(D\Phi = 0\).

One of the main motivations of this paper is that TFT’s should be a generalizations of the local theories. Clearly TFT’s have no local degrees of freedom that unless the topological symmetry is broken down to the local symmetry, there seems to be no way to describe the local theories. However, we find that the algebraic structure of TYM theory is rich enough to contains that of local Yang-Mills theory, such that the Yang-Mills theory can be regarded as a local sector of TYM theory. In particular, we have suggested the Russian formula, descent equation and non-Abelian anomaly counterparts in TYM, which are natural extensions of those of local Yang-Mills theory.

It remains an open question to know the precise mathematical and physical meanings of the non-Abelian anomaly counterpart in TYM. There is at least one indication of the possible physical application of the non-Abelian anomaly. Note that the zero modes of Faddev-Popov ghost \(v\) will arise due to the reducible connections or due to the Gribov ambiguity\[27\][28]. Then, we have the net violation of \(\delta_{\text{BRST}}\) ghost number as well as that of \(\delta_w\) ghost number zero-modes. Thus, some appropriate set of observables should be inserted to the correlation function of TYM to absorb the both kinds of zero modes. Because the
Witten’s observables have no $\delta_{\text{BRS}}$ ghost number, we need other set of observables which have non-zero $\delta_{\text{BRS}}$ ghost number. Such an observable should be $\delta_{\text{W}}$ as well as $\delta_{\text{BRS}}$ closed and non-trivial in the sense of global topology. If there is no such an observable, not only the topological interpretation of correlation function becomes impossible but also the correlation function itself can not be well defined. A way out of this problem is to include $\delta_{\text{W}}$ non-invariants, but preserving $\delta_{\text{BRS}}$ invariance, into the correlation function. Then, a natural candidate may be the consistent anomaly counterpart in TYM

$$\int_{\gamma_{\ell}} \int_{\gamma'_{2n-\ell}} W_{\ell,1,2n-\ell},$$

which is $\delta_{\text{BRS}}$ invariant and its $\delta_{\text{BRS}}$ cohomology (local cohomology) class depends only on the homology class of $\gamma$.

To be definite, let $\triangle U$ and $\triangle u$ denotes the net violations of $\delta_{\text{W}}$ and $\delta_{\text{BRS}}$ ghost numbers of zero-modes, respectively. Then the correlation function

$$\left\langle \prod_{i=1}^{r} \tilde{W}_{k_i}^{0,2n-k_i} \frac{\triangle u}{\prod_{j=1}^{v} \int_{\gamma'_{2n-\ell_j}} W_{\ell_j,1,2n-\ell_j}} \right\rangle,$$

may be well defined for

$$\triangle U = \sum_{i=1}^{r} (2n - k_i).$$

In the semi-classical limit the above correlation function reduce to an integration over the space of fixed points of $\delta_{\text{W}}$ and $\delta_{\text{BRS}}$ symmetries, and the integrand is certain cohomology class on the fixed points, which can be obtained by replacing the fields of the inserted quantities by their zero-modes. Then, one can immediately see that the above correlation function is factorized

$$\left\langle \prod_{i=1}^{r} \tilde{W}_{k_i}^{0,2n-k_i} \right\rangle_{\delta_{\text{W}} X=0} \left\langle \prod_{j=1}^{v} \int_{\gamma'_{2n-\ell_j}} W_{\ell_j,1,2n-\ell_j} \right\rangle_{\delta_{\text{BRS}} A=0},$$

where $\langle \cdots \rangle_{\delta_{\text{W}} X=0}$ denotes the integration over $\delta_{\text{W}}$ fixed points, which is identical to the original correlation function of TYM. The additional terms $\langle \cdots \rangle_{\delta_{\text{BRS}} A=0}$ is then the integration the local $\delta_{\text{BRS}}$ cohomology class over the space of $v$ zero modes. I do not know what kind of topological meaning, if any, it may have.

Note, however, that if there are zero modes due to the reducible connections, the validity of semi-classical approximation becomes doubtful, the factorization of (5.3) becomes
doubtful and the topological interpretation of the correlation function becomes unclear. Note also that it is quite unpleasant to observe that the consistent anomaly involve the field $C$ in general, which never appears explicitly in the completely fixed action of TYM. It will be also practically impossible to define non-trivial homology cycle in the orbit space $U/G$ or in the instanton moduli space unless the connection is generic. Thus, the use of the consistency anomaly seems to be restricted to the stable region.

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Appendix A. Geometrical Understanding of BRST Symmetries

In this appendix we discuss the geometrical origin of the various BRST algebras - \(\delta_{\text{BRST}}, \delta_{\text{T}}\) and \(\delta_{\text{W}}\) - algebras based on the universal bundle formalism\(^\text{[12]}\). The presentations of this section are motivated by the appendix of the ref.\(^\text{[18]}\), and we generally follow the conventions of ref.\(^\text{[13]}\).

Consider a principal \(G\)-bundle \(P\) over base space \(M\). Let \(U\) denote the affine space of all connections on \(P\) and \(G\) denote bundle automorphism, which acts as the gauge symmetry group, leaving a base point fixed. Now consider a principal \(G\)-bundle over base space \((P \times U)/G\) where \(G\) acts freely; \((P \times U, G, (P \times U)/G)\). The base space of the above bundle itself can be regarded as a principal \(G\)-bundle over \(M \times U/G\), which is called the universal bundle\(^\text{[12]}\)

\[
((P \times U)/G, G, M \times U/G).
\] (A.1)

It is convenient to start from a pull backed bundle \(Q\) over \(M \times U\) from the universal bundle \((A.1)\). One can locally parametrize \(U\) by

\[
A = g^{-1}Ag + g^{-1}dg,
\] (A.2)

where \(A\) denotes a fixed connection one-form tangent to \(M\), and \(g\) is an element of gauge group \(G\) which depends on the space-time coordinates \(x^\mu\), some group parameter \(\lambda^i\).

An arbitrary variation on \(A\) is

\[
\delta A = g^{-1}\delta Ag - d_A(g^{-1}\delta g),
\] (A.3)

where we will interpret the operator \(\delta\) as the exterior derivative over \(U\).

If we decompose the operator \(\delta\) as

\[
\delta = \delta_{\text{T}} + \delta_{\text{BRST}},
\] (A.4)

such that \(\delta_{\text{BRST}}\) is the variation (exterior derivation) along gauge group \(G\)

\[
\delta_{\text{BRST}} = d\lambda^i \frac{\partial}{\partial \lambda^i},
\] (A.5)

\(^6\) Since \(U\) itself can be viewed as a principal bundle \(U \rightarrow U/G\) over \(U/G\), \(g\) depends also on coordinates of \(U/G\) via a local section. That is, \(\delta_{\text{T}} g\) does not vanish in general.
and $\delta_T$ is the exterior derivative over the orbit space $\mathcal{U}/\mathcal{G}$

$$\delta^2 = \delta^2_T = \delta^2_{\text{BRS}} = \delta_T \delta_{\text{BRS}} + \delta_{\text{BRS}} \delta_T = 0. \quad (A.6)$$

Then (A.3) becomes

$$\delta A = g^{-1}\delta_T A g - d_A(g^{-1}\delta_T g) - d_A(g^{-1}\delta_{\text{BRS}} g). \quad (A.7)$$

such that

$$\delta_T A = g^{-1}\delta_T A g - d_A(g^{-1}\delta_T g),$$
$$\delta_{\text{BRS}} A = -d_A(g^{-1}\delta_{\text{BRS}} g), \quad (A.8)$$

where we have used $\delta_{\text{BRS}} A = 0$.

Introducing the connection one-form on $\mathcal{U}$

$$-G_A d_A^* \delta A \equiv C, \quad (A.9)$$

where

$$G_A = (d_A^* d_A)^{-1}.$$ 

The connection one-form $C$ can be also decomposed as

$$C = -G_A d_A^* \delta A = -G_A d_A^* \delta_T A - G_A d_A^* \delta_{\text{BRS}} A, \quad (A.10)$$

Then one can define the Faddev-Popov ghost $v$ as

$$-G_A d_A^* \delta_{\text{BRS}} A = g^{-1}\delta_{\text{BRS}} g \equiv v, \quad (A.11)$$

which is the connection one-form along $\mathcal{G}$, and the BRS algebra naturally follows

$$\delta_{\text{BRS}} A = -dv - A v - v A \equiv -d_A v,$$
$$\delta_{\text{BRS}} v = -v^2. \quad (A.12)$$

The total connection one-form over $M \times \mathcal{U}$ is

$$A + C = A - G_A d_A^* \delta A, \quad (A.13)$$

and total curvature over $M \times \mathcal{U}$ is

$$\hat{\mathcal{F}} = (d + \delta)(A - G_A d_A^* \delta A) + (A - G_A d_A^* \delta A)^2$$
$$= \mathcal{F} + (1 - d_A G_A d_A^*) \delta A - \delta (G_A d_A^* \delta A) + (G_A d_A^* \delta A)^2, \quad (A.14)$$
which can be written in components

\[ \hat{F}^{2,0} \equiv F = dA + A^2, \]
\[ \hat{F}^{1,1} = (1 - d_A G_A d_A^* \delta A), \tag{A.15} \]
\[ \hat{F}^{0,2} = -\delta (G_A d_A^* \delta A) + (G_A d_A^* \delta A)^2. \]

Using the decomposition (A.8) and (A.8)(A.10)(A.11)(A.12) we can get

\[ \hat{F}^{1,1} = (1 - d_A G_A d_A^*) (\delta_r A + \delta_{\text{bns}} A), \tag{A.16} \]
\[ = (1 - d_A G_A d_A^*) \delta_r A, \]
and

\[ \hat{F}^{0,2} = -\delta_r (G_A d_A^* \delta_r A) + (G_A d_A^* \delta_r A)^2 + \delta_{\text{bns}} v + v^2 \]
\[ + \delta_r v - \delta_{\text{bns}} (G_A d_A^* \delta_r A) - \{G_A d_A^* \delta_r A, v\} \]
\[ = -\delta_r (G_A d_A^* \delta_r A) + (G_A d_A^* \delta_r A)^2. \tag{A.17} \]

Thus, \( \hat{F} \) is also given by

\[ \hat{F} = F + (1 - d_A G_A d_A^*) \delta_r A - \delta_r (G_A d_A^* \delta_r A) + (G_A d_A^* \delta_r A)^2, \tag{A.18} \]

which means \( \hat{F} \) is the total curvature over \( M \times U/G \). The identities (A.14)(A.18) an extended Russian formula (B.24). That is, if one restricts the variation of \( A \) in (A.3) to the gauge group direction [18], the above two equation (A.14)(A.18) lead to the well-known Russian formula (2.10).

Note that \( (1 - d_A G_A d_A^*) \) is the horizontal projection. Then

\[ \hat{F}^{1,1} = (1 - d_A G_A d_A^*) \delta_r A \equiv \delta^H A, \tag{A.19} \]

where \( \delta^H \) denotes the operator for horizontal variation. Furthermore, direct calculation shows that

\[ \delta^H (\delta^H A) = -d_A \hat{F}^{0,2}, \quad \delta^H \hat{F}^{0,2} = 0. \tag{A.20} \]

Being the horizontal variation, \( \delta^H A \) should satisfy

\[ d_A^* (\delta^H A) = 0. \tag{A.21} \]

Applying \( \delta^H \) to the above condition, we can get

\[ \delta^H (d_A^* \delta^H A) = [\delta^H * A, \delta^H A] - d_A^* (\delta^H (\delta^H A)) \]
\[ = [\delta^H * A, \delta^H A] + d_A^* d_A^* \Phi \tag{A.22} \]
\[ = 0, \]

23
which can be read as
\[ \Phi = \hat{\mathcal{F}}^{0,2} = -G_A[\delta^H \mathcal{A}, \delta^H \mathcal{A}] \].
\hfill (A.23)

Thus we have obtained Atiyah-Singer’s results\cite{12}.

Note that if we denote \( \hat{\mathcal{F}}^{1,1} \equiv \Psi, \hat{\mathcal{F}}^{0,2} = \Phi \) such that
\[ \hat{\mathcal{F}} = \mathcal{F} + \Psi + \Phi, \]
\hfill (A.24)

and \( \delta^H \equiv \delta_{\psi} \), we can get Witten’s BRST algebra
\[ \delta_{\psi} \mathcal{A} = \Psi, \quad \delta_{\psi} \Psi = -d_A \Phi, \quad \delta_{\psi} \Phi = 0. \]
\hfill (A.25)

Let
\[ C \equiv -G_A d^*_A \delta_T A, \]
\hfill (A.26)
such that (A.10) becomes
\[ C = C + v. \]
\hfill (A.27)

Then (A.16)(A.17) lead to the \( \delta_T \) algebra\cite{6}
\[ \delta_T \mathcal{A} = \Psi - d_A C, \]
\[ \delta_T C = \Phi - C^2, \]
\[ \delta_T \Psi = -[C, \Psi] - d_A \Phi, \]
\[ \delta_T \Phi = -[C, \Phi], \]
\hfill (A.28)

with
\[ \delta_T v = -\delta_{BRS} C - \{C, v\}. \]
\hfill (A.29)

Using
\[ \delta_T \mathcal{A} = \Psi - d_A C, \]
\hfill (A.30)

one can find that
\[ -G_A d^*_A \delta_T \mathcal{A} = -G_A d^*_A \Psi + C. \]
\hfill (A.31)

Then, the following condition is crucial in the self-consistency of \( \delta_T \) algebra
\[ d^*_A \Psi = 0, \]
\hfill (A.32)

which is identical to (A.21). Note also that
\[ \hat{\mathcal{F}}^{1,1} = (1 - d_A G_A d^*_A) (\delta_T \mathcal{A} + \delta_{BRS} \mathcal{A}) = \Psi \equiv \delta^H \mathcal{A}. \]
\hfill (A.33)
Thus, we can see that the horizontal part\[5\] of $\delta A$ is $\Psi$ and the vertical part of $\delta A$ is $-d_A C - d_A v$. Note that the vertical part of $\delta A$ is not just the BRS variation of $A$. Precisely speaking, the BRS variation of $A$ is the vertical part of $\delta A$ if we restrict $\delta$ to $\delta_{\text{BRS}}$ or the component of the vertical part which has $\delta_{\text{BRS}}$ ghost number one. Clearly, for $C = 0$ the vertical part is identical to the BRS variation. Note also that $\delta_w v = 0$ unlike eq. (A.29).

We can also find $\delta_{\text{BRS}}$ algebra from eq. (A.12)(A.15)(A.16)(A.17)

$$
\begin{align*}
\delta_{\text{BRS}} A &= -d_A v, \\
\delta_{\text{BRS}} v &= -v^2, \\
\delta_{\text{BRS}} F &= -[v, F], \\
\delta_{\text{BRS}} \Psi &= -[v, \Psi], \\
\delta_{\text{BRS}} \Phi &= -[v, \Phi],
\end{align*}
$$

which is identical to the additional BRS algebra of ref.[9]. We shall see that this additional BRS structure to $\delta_T$ and $\delta_W$ is crucial for non-Abelian anomalies in TYM.

**Appendix B. Solutions of Descent Equation**

Our conventions and the method to find the explicit solutions of the extended descent equation (3.36) are essentially same to those of ref. [14]. However, for the sake of self-consistency, we will sketch the procedure and present some explicit solutions.

We denote

$$P(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \ldots, \hat{\mathcal{F}}_n),$$

(B.1)

to a symmetric invariant polynomial of degree $n$ in the Lie algebra valued variables $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \ldots, \hat{\mathcal{F}}_n$. We shall write (B.1) as

$$P(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}^{n-2})$$

(B.2)

---

7 Eq. (A.19) implies that $\delta_w$ is the exterior covariant derivative for $\delta_T$, while eq. (A.33) implies that $\delta_w$ is the exterior covariant derivative for $\delta = \delta_T + \delta_{\text{BRS}}$. There is no conflict between the two interpretations due to the extended Russian formula (3.24)(A.14)(A.18). It is enough to examine an identity

$$\delta_T \Phi + [C, \Phi] = (\delta_T + \delta_{\text{BRS}}) \Phi + [C + v, \Phi] = \delta_w \Phi = 0$$

.
for $\hat{F}_3 = \hat{F}_4 = \cdots = \hat{F}_n = \hat{F}$.

Note that if we introduce the one-parameter family of the total connection one-forms

$$\hat{A}_t = t(A + C) + v$$

$$= t \hat{A} + v$$

(B.3)

and associated total field strengths

$$\hat{F}_t = (d + \delta + \delta_{BR S}) \hat{A}_t + \hat{A}_t^2$$

$$\equiv (d + \delta_{BR S}) \hat{A}_t + \hat{A}_t^2$$

$$= t \hat{d}\hat{A} + t^2 \hat{A}^2 + (1 - t) \hat{d}v,$$

(B.4)

we can obtain

$$P(\hat{F}_n^1) - P(\hat{F}_0^n) = n(\hat{d} + \delta_{BR S}) \int_0^1 dt \, P(\hat{A}, \hat{F}_t^{n-1})$$

$$= n(\hat{d} + \delta_{BR S}) \int_0^1 dt \, P(\hat{A}, (t \hat{d}\hat{A} + t^2 \hat{A}^2 + (1 - t) \hat{d}v)^{n-1})$$

(B.5)

after following the same procedure as eq. (2.15) - (2.17) in ref. \[14\]. Now we can expand the right-hand side in powers of $\hat{d}v$

$$n \int_0^1 dt \, P(\hat{A}, (t \hat{d}\hat{A} + t^2 \hat{A}^2 + (1 - t) \hat{d}v)^{n-1}) = W_{2n-1}^0 + W_{2n-2}^1 + \cdots W_{n}^{n-1},$$

(B.6)

where the superscript denotes the $\delta_{BR S}$ ghost number and the subscript denotes the spacetime form degree plus the $\delta_r$-ghost number.

Then

$$W_{2n-1-k}^k = \frac{n(n-1)(n-2) \cdots (n-k)}{k!}$$

$$\times \int_0^1 dt \, (1 - t)^k P((\hat{d}v)^k, \hat{A}, (t \hat{d}\hat{A} + t^2 \hat{A}^2)^{n-2}),$$

(B.7)

for $0 \leq k \leq n - 1$. For $k > n - 1$, we introduce a different family of the total connections

$$\hat{A}_t = t v,$$

(B.8)

and associated total field strengths

$$\hat{F}_t = (\hat{d} + \delta_{BR S}) \hat{A}_t + \hat{A}_t^2 = t \hat{d}v + (t^2 - t)v^2.$$

(B.9)
After following the same procedure as eq. (2.27) - (2.30) of ref. [14], we can obtain

\[ W_{n-1-l}^{n+l} = (-1)^l \frac{n!(n-1)!}{(n-1-l)!(n+l)!} P((\hat{d}v)^{n-1-l}, v, (v^2)^l), \]  

(B.10)

for \(0 \leq l \leq n-1\).

The next step is to expand \( W_{2n-1-k}^k \), given by (B.7) and (B.10), in powers of \( \delta \) ghost number such that

\[ W_{2n-1-k}^k = \sum_{k'=0}^{2n-1-k} W_{2n-1-k-k',k'}. \]  

(B.11)

To be explicit, for \(n = 2\)

\[
\begin{align*}
W_2^1 &= c_2 \text{Tr} \left( \hat{d}v \hat{A} \right), \\
W_1^2 &= c_2 \text{Tr} \left( \hat{d}v v \right), \\
W_0^3 &= -\frac{1}{3} c_2 \text{Tr} \left( v^3 \right),
\end{align*}
\]

(B.12)

we get

\[
\begin{align*}
W_2^{1,0} &= c_2 \text{Tr} \left( d v \hat{A} \right), \\
W_1^{1,1} &= c_2 \text{Tr} \left( d v C + \delta_c v \hat{A} \right), \\
W_0^{1,2} &= c_2 \text{Tr} \left( \delta_c v C \right), \\
W_1^{2,0} &= c_2 \text{Tr} \left( d v v \right), \\
W_0^{2,1} &= c_2 \text{Tr} \left( \delta_c v v \right), \\
W_0^{3,0} &= -\frac{1}{3} c_2 \text{Tr} v^3.
\end{align*}
\]

(B.13)

For \(n = 3\)

\[
\begin{align*}
W_4^1 &= \frac{1}{2} c_3 \text{Tr} \left( \hat{d}v (\hat{A} \hat{d} \hat{A} + \hat{d}\hat{A} \hat{A} + \hat{A}^3) \right), \\
W_3^2 &= c_3 \text{Tr} \left( (\hat{d}v)^2 \hat{A} \right), \\
W_2^3 &= c_3 \text{Tr} \left( (\hat{d}v)^2 v \right), \\
W_1^4 &= -\frac{1}{2} c_3 \text{Tr} \left( \hat{d}v v^3 \right), \\
W_0^5 &= \frac{1}{10} c_3 \text{Tr} \left( v^5 \right),
\end{align*}
\]

(B.14)
we find

\[ W_{4}^{1,0} = \frac{1}{2} c_3 \text{Tr} \left( dv \left( A F + F A - A^3 \right) \right), \]
\[ W_{3}^{1,1} = \frac{1}{2} c_3 \text{Tr} \left( dv \left( A \Psi + \Psi A + C F + F C - A^2 C - C A^2 - A C A \right) \right. \]
\[ \left. + \delta_r v \left( A F + F A - A^3 \right) \right), \]
\[ W_{2}^{1,2} = \frac{1}{2} c_3 \text{Tr} \left( dv \left( C \Psi + \Psi C + A \Phi + A \Phi - A C^2 - C^2 A - C A C \right) \right) \]
\[ \left. + \delta_r v \left( A \Psi + \Psi A + C F + F C - A^2 C - C A^2 - A C A \right) \right), \]
\[ W_{1}^{1,3} = \frac{1}{2} c_3 \text{Tr} \left( dv \left( C \Phi + \Phi C - C^3 \right) \right) \]
\[ \left. + \delta_r v \left( C \Psi + \Psi C + A \Phi + A \Phi - A C^2 - C^2 A - C A C \right) \right), \]
\[ W_{0}^{1,4} = \frac{1}{2} c_3 \text{Tr} \left( \delta_r v \left( C \Phi + \Phi C - C^3 \right) \right), \]
\[ W_{3}^{2,0} = c_3 \text{Tr} \left( (dv)^2 A \right), \]
\[ W_{2}^{2,1} = c_3 \text{Tr} \left( (dv \delta_r v + \delta_r v dv) A + (dv)^2 C \right), \]
\[ W_{1}^{2,2} = c_3 \text{Tr} \left( (dv \delta_r v + \delta_r v dv) C + (\delta_r v)^2 C \right), \]
\[ W_{0}^{2,3} = c_3 \text{Tr} \left( (\delta_r v)^2 C \right), \]
\[ W_{2}^{3,0} = c_3 \text{Tr} \left( (dv)^2 v \right), \]
\[ W_{1}^{3,1} = c_3 \text{Tr} \left( (dv \delta_r v + \delta_r v dv) v \right), \]
\[ W_{0}^{3,2} = c_3 \text{Tr} \left( (\delta_r v)^2 v \right), \]
\[ W_{1}^{4,0} = -\frac{1}{2} c_3 \text{Tr} \left( dv v^3 \right), \]
\[ W_{0}^{4,1} = -\frac{1}{2} c_3 \text{Tr} \left( \delta_r v v^3 \right), \]
\[ W_{0}^{5,0} = \frac{1}{10} c_3 \text{Tr} \left( v^5 \right). \]
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