Wide-Sense 2-Frameproof Codes

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Abstract

Various kinds of fingerprinting codes and their related combinatorial structures are extensively studied for protecting copyrighted materials. This paper concentrates on one specialised fingerprinting code named wide-sense frameproof codes in order to prevent innocent users from being framed.

Let $Q$ be a finite alphabet of size $q$. Given a $t$-subset $X = \{x^1, \ldots, x^t\} \subseteq Q^n$, a position $i$ is called undetectable for $X$ if the values of the words of $X$ match in their $i$th position: $x^1_i = \cdots = x^t_i$. The wide-sense descendant set of $X$ is defined by $\text{wdesc}(X) = \{y \in Q^n : y_i = x^i_i, i \in U(X)\}$, where $U(X)$ is the set of undetectable positions for $X$. A code $C \subseteq Q^n$ is called a wide-sense $t$-frameproof code if $\text{wdesc}(X) \cap C = X$ for all $X \subseteq C$ with $|X| \leq t$.

The paper establishes a lower bound on the size of a wide-sense 2-frameproof code by constructing 3-uniform hypergraphs and evaluating their independence numbers. The paper also improves the upper bound on the size of a wide-sense 2-frameproof code by applying techniques on non 2-covering Sperner families and intersecting families in extremal set theory.

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1 Introduction

Fingerprinting codes are combinatorial objects that have been studied for more than 20 years due to their applications in digital data copyright protection and their combinatorial interest. Let $Q$ be a finite alphabet of size $q$. In order to protect a copyrighted digital product, a dealer inserts a fingerprint in each copy and then distributes copies to all registered users, where a fingerprint is a string $x = (x_1, \ldots, x_n)$ over $Q$. The goal of inserting the fingerprint is to personalize the copy given out to the user, and to rule out redistribution. Clearly, an individual user will be deterred from releasing an unauthorized copy. However, a coalition of users may collude in order to produce an unauthorized copy. The goal of the coalition is to create a fingerprint of the illegal copy that is unable to identify users from them. We assume that the members of a coalition can only alter those coordinates of the fingerprint in which at least two of their fingerprints differ, and refer to this as the Marking Assumption. In this paper we concentrate on $t$-frameproof

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codes, which have the property that no coalition of at most \( t \) users can frame a user not in the coalition.

Given a \( t \)-subset \( X = \{x^1, \ldots, x^t\} \subseteq Q^n \) we now define the set of descendants of \( X \). We write \( x^i \) for the \( i \)-th component of \( x^j \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq t \). A position \( i \) is called undetectable for \( X \) if the values of the words of \( X \) match in their \( i \)-th position: \( x^1_i = \cdots = x^t_i \). Denote by \( U(X) \) the set of positions undetectable for \( X \). By the marking assumption, the coalition cannot change the values of undetectable positions. If the position is detectable, then there are several options for the coalition to fill it. We will consider wide-sense descendant set defined by

\[
\text{wdesc}(X) = \{ y \in Q^n : y_i = x^1_i, i \in U(X) \},
\]

in contrast to the narrow-sense descendant set

\[
\text{desc}(X) = \{ y \in Q^n : y_i \in \{ x^1_i, \ldots, x^t_i \} \}.
\]

In the literature the wide-sense descendant set could be found under the name of envelope \([3]\), feasible set \([9, 30]\), or Boneh-Shaw descendant \([6]\).

A code \( C \subseteq Q^n \) of size \( |C| = m \) is called an \((n, m, q)\) code. We define \( C \) to be a wide-sense \( t \)-frameproof code, or \((n, m, q)\) \( t \)-wFP code, if

\[
\text{wdesc}(X) \cap C = X
\]

for all \( X \subseteq C \) with \( |X| \leq t \). Replacing \( \text{wdesc}(X) \) with \( \text{desc}(X) \) defines a narrow-sense \( t \)-frameproof code.

Boneh and Shaw \([9]\) were the first to define a \( t \)-frameproof code, where they adopted the wide-sense model of descendant sets. In the narrow-sense model, variants of frameproof codes were extensively studied by many researchers. Named after the security properties it guarantees, the following types of fingerprinting codes are well-known: frameproof codes, secure frameproof codes, identifiable parent property codes, traceability codes, and anti-collusion codes, see \([1, 3]-[8], [10]-[14], [17, 19], [26]-[32], [34, 35]\).

It is clear that \( \text{wdesc}(X) \) always strictly contains \( \text{desc}(X) \) if \( 2 \leq |X| < q \). Just as Blackburn \([6]\) said, “This is one reason why the problem of constructing analogues of the codes for Boneh-Shaw descendants is often more difficult than the original problem.” To the best of our knowledge, frameproof codes is the only type of fingerprinting codes that was ever studied in the wide-sense model. This paper establishes a lower bound on the size of a 2-wFP code on one hand and improves its known upper bounds on the other hand.

The paper is organized as follows. In Section 2 we evaluate the lower bounds on the sizes of 2-wFP codes. 3-Uniform hypergraphs are constructed by imposing constraints of 2-wFP codes and the sizes of their independent sets are applied to establish a lower bound for 2-wFP codes. In Section 3 we improve the known upper bounds on the sizes of 2-wFP codes, which was previously established by Panoui in PhD dissertation \([25]\). Non 2-covering Sperner families generated by all codewords are considered and better upper bounds are established by developing many results on Sperner families and intersecting families. In Section 4 we conclude the paper.
2 Hypergraph approach and a lower bound

Graphs with a large independent set are often applied to evaluate the lower bounds of codes, see Levenshtein [23], Jiang and Vardy [20], Vu and Wu [33], Gao et al. [18]. Similarly, Yang et al. [35] provided a hypergraph theoretical approach to evaluate the lower bounds of certain combinatorial structures. In this section we adopt hypergraph approach to prove a lower bound on the size of a wide-sense 2-frameproof code.

A hypergraph $H$ is a pair $(V, E)$, where $V$ is a finite set of vertices and $E$ is a family of subsets of $V$ called edges. A graph is then a hypergraph where each edge is a two-element subset. An independent set of $H$ is a set $S$ of vertices if no two vertices of $S$ are contained in any edges. The independence number of $H$, denoted by $\alpha(H)$, is the size of its largest independent set. For any vertex $v$, define the degree of $v$ to be the number of edges containing $v$, denoted by $d(v)$. The maximum degree of $H$, denoted by $\Delta(H)$, is the maximum degree of all its vertices. If all the edges have the same size $k$, then $H$ is called $k$-uniform.

Given the length $n$ and given the alphabet set $Q$ of size $q$, a code can be regarded as a subset of $Q^n$ satisfying some constraints. Let $Q^n$ be the vertex set of a hypergraph and let any subset of vertices constitute an edge if their simultaneous appearance do violate one or more constraints of the code. Here the edges are chosen to be minimal in the sense that the deletion of any vertex in the subset makes it no longer a violating subset. Thus the problem of finding a code with large size turns into finding a large independent set in its corresponding hypergraph. We will make use of the following result on the independence number of hypergraphs.

**Theorem 2.1.** [15] Let $H = (V, E)$ be a $k$-uniform hypergraph on $v$ vertices satisfying $\Delta(H) \leq p^{k-1}$, $k \ll p$. For any integer $j$, $2 \leq j \leq k-1$, denote by $s_j(H)$ the number of unordered pairs of edges $\{e, e'\}$ intersecting at exactly $j$ vertices. If

$$s_j(H) \leq v \cdot p^{2k-j-1-\gamma}$$

holds for $j = 2, 3, \ldots, k-1$ and some $\gamma > 0$, then

$$\alpha(H) \geq c(k, \gamma) \cdot \frac{v}{p} \cdot (\ln p) \frac{1}{1-\gamma},$$

where $c(k, \gamma)$ is a constant depending only on $k$ and $\gamma$. In particular, choosing $p = (\Delta(H))^{1/(k-1)},$

$$\alpha(H) \geq c(k, \gamma) \cdot \frac{v}{(\Delta(H))^{1/(k-1)}} \cdot (\ln \Delta(H))^{1/(k-1)}.$$

For convenience we denote $[n] := \{1, 2, \ldots, n\}$. For a subset $F$ of $[n]$, denote $\overline{F} = [n] \setminus F$.

**Theorem 2.2.** For sufficiently large $n$, there exists a wide-sense $(n, m, q)$ 2-frameproof code provided that

$$m \leq \frac{cq^n \sqrt{n}}{(q^2 - q + 1)^2}$$

where $c$ is a constant.
Proof. Construct a 3-uniform hypergraph $H = (V, E)$ with vertex set $V = Q^n$. Let $C \subseteq V$. Define any 3-subset of vertices to be an edge of $E$ if their simultaneous appearance in $C$ cause $C$ not forming a 2-wFP code. In particular, $\{a, b, c\} \in E$ if and only if one of the followings holds: (1) $c \in \text{wdesc}(a, b)$, (2) $b \in \text{wdesc}(a, c)$, or (3) $a \in \text{wdesc}(b, c)$.

For any vertex $a$, first choose a vertex $b$ which agrees with $a$ in exactly $n - i$ coordinates where $1 \leq i \leq n$. Denote by $R$ the set of these $n - i$ positions, that is, $a_k = b_k$ for all $k \in R$ and $a_k \neq b_k$ for all $k \in \overline{R}$. There are totally $\binom{n}{i}(q - 1)^i$ choices of $b$ for each $a$. For a given pair $\{a, b\}$, count the number of vertices $c$ such that $\{a, b, c\}$ forms an edge. Considering three forms of an edge we define

$A = \{c \in V : c \in \text{wdesc}(a, b), c \neq a, b\}$,

$B = \{c \in V : a \in \text{wdesc}(b, c), c \neq a, b\}$,

$C = \{c \in V : b \in \text{wdesc}(a, c), c \neq a, b\}$.

It is obvious that

$A = \{c : c_k = a_k, \forall k \in R; c_k \in Q, \forall k \in \overline{R}; c \neq a; c \neq b\}$.

Thus $|A| = q^i - 2$. Clearly

$B = \{c : c_k \in Q, \forall k \in R; c_k \neq b_k, \forall k \in \overline{R}; c \neq a\}$.

So we have $|B| = q^{n-i}(q - 1)^i - 1$. Similarly

$C = \{c : c_k \in Q, \forall k \in R; c_k \neq a_k, \forall k \in \overline{R}; c \neq b\}$,

and $|C| = |B| = q^{n-i}(q - 1)^i - 1$. Moreover,

$A \cap B = \{c : c \in \text{wdesc}(a, b), a \in \text{wdesc}(b, c), c \neq a, b\}$

$= \{c : c_k = a_k, \forall k \in R; c_k \neq b_k, \forall k \in \overline{R}; c \neq a\}$.

Thus $|A \cap B| = (q - 1)^i - 1$. Similarly, $|A \cap C| = (q - 1)^i - 1$.

$B \cap C = \{c : a \in \text{wdesc}(b, c), b \in \text{wdesc}(a, c), c \neq a, b\}$

$= \{c : c_k \in Q, \forall k \in R; c_k \neq a_k, c_k \neq b_k, \forall k \in \overline{R}\}$,

giving $|B \cap C| = q^{n-i}(q - 2)^i$.

$A \cap B \cap C = \{c : c \in \text{wdesc}(a, b), a \in \text{wdesc}(b, c), b \in \text{wdesc}(a, c), c \neq a, b\}$

$= \{c : c_k = a_k, \forall k \in R; c_k \neq a_k, c_k \neq b_k, \forall k \in \overline{R}\}$,

yielding $|A \cap B \cap C| = (q - 2)^i$.

Now for a given pair $\{a, b\}$ we have that the number $N_i$ of choices of vertex $c$ such that $\{a, b, c\} \in E$ is

$$N_i = |A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 2q^{n-i}(q - 1)^i - q^{n-i}(q - 2)^i + q^i + (q - 2)^i - 2(q - 1)^i - 2.$$
Every edge \{a, b, c\} incident with a is counted twice in this way. As a result, applying the binomial theorem yields

\[
\Delta(H) = d(a) = \frac{1}{2} \sum_{i=1}^{n} \binom{n}{i} (q - 1)^i N_i
\]

\[
= \frac{1}{2} \left( \sum_{i=0}^{n} \binom{n}{i} (q - 1)^i N_i - N_0 \right)
\]

\[
= \frac{1}{2} \left( 2((q - 1)^2 + q)^n - ((q - 1)(q - 2) + q)^n + (q(q - 1) + 1)^n + ((q - 1)(q - 2) + 1)^n 
\right.

\[
- 2((q - 1)^2 + 1)^n - 2q^n - (q^n - 2) \right)
\]

\[
= \frac{3}{2} (q^2 - q + 1)^n - \frac{3}{2} (q^2 - 2q + 2)^n + \frac{1}{2} (q^2 - 3q + 3)^n - \frac{3}{2} q^n + 1
\]

\[
\sim (q^2 - q + 1)^n.
\]

We then apply Theorem 2.1. Note that \(s_2(H)\) is the number of pairs of edges \(\{e, e'\}\) intersecting at exactly two vertices. By similar simplifications as above, we have that

\[
s_2(H) = \frac{1}{2} \cdot q^n \sum_{i=1}^{n} \binom{n}{i} (q - 1)^i \left( \frac{N_i}{2} \right)
\]

\[
\sim q^n \sum_{i=0}^{n} \binom{n}{i} (q - 1)^i \left( q^{n-i} (q - 1)^i \right)^2
\]

\[
= q^n \left( q^2 + (q - 1)^3 \right)^n.
\]

Take \(p = (\Delta(H))^\frac{1}{2}\) and let \(\gamma\) be a small positive real in Theorem 2.1. For sufficiently large \(n\), we want to have that

\[
s_2(H) \leq q^n (\Delta(H))^{\frac{3-\gamma}{2}} < q^n (\Delta(H))^\frac{3}{4},
\]

that is,

\[
(q^2 + (q - 1)^3)^{2n} < (q^2 - q + 1)^{3n},
\]

which holds clearly for any positive integer \(q\). Thus applying Theorem 2.1 yields that the independence number

\[
\alpha(H) \geq c' \cdot \frac{v}{(\Delta(H))^\frac{1}{2}} \cdot (\ln \Delta(H))^{\frac{1}{2}}
\]

\[
= \frac{c \cdot \sqrt{n} \cdot q^n}{(q^2 - q + 1)^\frac{3}{2}}.
\]

Now the hypergraph \(H\) has a independent set with \(\alpha(H)\) vertices. By the construction of \(H\), we have an \((n, m, q)\) 2-wFP code of size \(m = \left\lfloor \frac{c \cdot \sqrt{n} q^n}{(q^2 - q + 1)^\frac{3}{2}} \right\rfloor\).

It is evident that a wide-sense \(t\)-frameproof code is a narrow-sense \(t\)-frameproof code by their definitions. Hence, we have a direct corollary to Theorem 2.2 which fills a large gap for alphabet size \(q\) in [35, Theorem 11] (where only \(q = 3\) applies).
Corollary 2.3. For sufficiently large \( n \), there exists a narrow-sense \((n, m, q)\) 2-frameproof code provided that
\[
m \leq \frac{cq^n \sqrt{n}}{(q^2 - q + 1)^2}
\]
where \( c \) is a constant.

3 Sperner families and upper bounds

Stinson and Wei \[30\] were the first to establish the relationship between Sperner families and \(t\)-wFP codes and then proved that \( m \leq \left(\frac{n}{2}\right) + 1 \) for \((n, m, 2)\) 2-wFP codes by applying Sperner’s Theorem (\[30\], Theorem 5.2). Panoui \[25\] developed this idea and presented the equivalence between a 2-wFP code and the non 2-covering Sperner families generated by all codewords. The upper bounds on the sizes of 2-wFP codes were then improved as follows.

Lemma 3.1. \[25\], Theorem 6.3.8] Let \( C \) be an \((n, m, q)\) 2-wFP code.

1. If \( n \) is even, then \( m \leq \left(\frac{n}{2} - 1\right) + 1 \).
2. If \( n \) is odd, then \( m \leq \left(\frac{n}{2}\right) - \frac{n-1}{2} \).

The aim of this section is to improve the above upper bounds. We first introduce related definitions in extremal set theory and recall or develop some useful results.

Let \( \mathcal{F} \) be a family of finite sets. If any two distinct sets of \( \mathcal{F} \) are incomparable, that is, \( A \not\subseteq B \) for any \( A, B \in \mathcal{F} \), then \( \mathcal{F} \) is called an antichain or a Sperner family. To the other extreme, a chain is a set family \( \mathcal{F} \) in which every pair of sets is comparable.

Theorem 3.2. (Sperner’s Theorem) \[2\] Let \( \mathcal{F} \) be a Sperner family over an \( n \)-set. Then
\[
|\mathcal{F}| \leq \left(\frac{n}{\left\lfloor\frac{n}{2}\right\rfloor}\right).
\]

The size of a Sperner family which contains a singleton is easily obtained from Sperner’s Theorem.

Proposition 3.3. \[25\], Proposition 6.3.4] Let \( \mathcal{F} \) be a Sperner family over an \( n \)-set. If there exists a set \( F \in \mathcal{F} \) such that \( |F| = 1 \), then
\[
|\mathcal{F}| \leq \left(\frac{n-1}{\left\lfloor\frac{n-1}{2}\right\rfloor}\right) + 1.
\]

Let \( C = \{A_1, A_2, \ldots, A_k\} \) be a chain of subsets of an \( n \)-set, i.e., \( A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k \). This chain is symmetric if \( |A_1| + |A_k| = n \) and \( |A_{i+1}| = |A_i| + 1 \) for all \( i = 1, 2, \ldots, k-1 \).

Theorem 3.4. \[21\], Theorem 8.3] The family of all subsets of an \( n \)-set can be partitioned into \( \left(\frac{n}{\left\lfloor\frac{n}{2}\right\rfloor}\right) \) mutually disjoint symmetric chains.

A family \( \mathcal{F} \) of sets is called \( k \)-intersecting \((k \geq 1)\), if \( |A \cap B| \geq k \) for all \( A, B \in \mathcal{F} \). An intersecting family is a 1-intersecting family. Call the families \( \mathcal{A} \) and \( \mathcal{B} \) cross-\( k \)-intersecting if
Lemma 3.7. Let $A \cap B \geq k$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting if they are cross-1-intersecting. Let $\mathcal{F}$ be a family of subsets of a ground set $E$. Then $\mathcal{F}$ is called non 2-covering if for every pair of sets $A, B \in \mathcal{F}$ we have $A \cup B \neq E$.

**Theorem 3.5.** [24] If $\mathcal{F}$ is a $k$-intersecting Sperner family over an $n$-set, then

$$|\mathcal{F}| \leq \left(\frac{n}{n-k+1}\right).$$

Let $\mathcal{C} = \{c^1, c^2, \ldots, c^m\}$ be an $(n, m, q)$ code. For any $1 \leq i, j \leq m$ and $i \neq j$, define $I(i, j)$ to be the coincidence set of $c^i$ and $c^j$, i.e.,

$$I(i, j) = \{k : c^i_k = c^j_k, 1 \leq k \leq n\}.$$

For any $1 \leq i \leq m$, define

$$\mathcal{X}_i = \{I(i, j) : i \neq j, 1 \leq j \leq m\}$$

to be the coincidence family generated by the codeword $c^i \in \mathcal{C}$. Clearly $|\mathcal{C}| = |\mathcal{X}_i| + 1$ for $1 \leq i \leq m$.

**Theorem 3.6.** [23, Lemma 6.3.2, Corollary 6.3.3] Let $\mathcal{C} = \{c^1, c^2, \ldots, c^m\}$ be an $(n, m, q)$ code. Then, $\mathcal{C}$ is a 2-wFP code if and only if $\mathcal{X}_i$ is a non 2-covering Sperner family for any $1 \leq i \leq m$.

We have a simple but useful result on the coincidence sets.

**Lemma 3.7.** Let $\mathcal{C}$ be an $(n, m, q)$ code. For any three codewords $c^i, c^j, c^k \in \mathcal{C}$, we have

$$I(i, j) \cap I(i, k) \subseteq I(j, k) \subseteq (I(i, j) \cap I(i, k)) \cup I(i, j) \cup I(i, k).$$

**Proof.** Firstly let $p \in I(i, j) \cap I(i, k)$. Then we have $c^i_p = c^j_p$ and $c^i_p = c^k_p$. Hence $p \in I(j, k)$ and

$$I(i, j) \cap I(i, k) \subseteq I(j, k).$$

Secondly let $p \in I(j, k)$ and $p \notin I(i, j) \cap I(i, k)$. Clearly we have $c^i_p \neq c^j_p = c^k_p$. Hence $p \notin I(i, j) \cup I(i, k)$ and $p \in I(i, j) \cup I(i, k)$. It follows that

$$I(j, k) \subseteq (I(i, j) \cap I(i, k)) \cup I(i, j) \cup I(i, k).$$

This completes the proof. \qed

Let $\mathcal{F}$ be a family of subsets of an $n$-set $E$. Let

$$l = \min\{|F| : F \in \mathcal{F}\},$$

$$u = \max\{|F| : F \in \mathcal{F}\}$$

be the minimum size and maximum size of subsets of $\mathcal{F}$. For $r \geq u$ and $s \leq l$, the families

$$\nabla_r(\mathcal{F}) = \{B \subseteq E : |B| = r, \exists F \in \mathcal{F}, F \subseteq B\},$$

$$\Delta_s(\mathcal{F}) = \{B \subseteq E : |B| = s, \exists F \in \mathcal{F}, B \subseteq F\}$$

are called the $r$-shade and $s$-shadow of $\mathcal{F}$, respectively. When $\mathcal{F}$ is a family of $k$-subsets, the $(k + 1)$-shade and the $(k - 1)$-shadow are simply written as $\nabla(\mathcal{F})$ or $\Delta(\mathcal{F})$. 

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Lemma 3.8. [22] If \( A \) is an intersecting family of \( k \)-subsets of an \( n \)-set, then \( \Delta A \geq |A| \).

Lemma 3.9. [16] Corollary 2.3.2] Let \( F \) be a family of \( k \)-subsets of an \( n \)-set where \( k < n \) and \( n \geq 3 \).

1. If \( k \geq \left\lceil \frac{n}{2} \right\rceil \), then \( |\Delta F| - |F| \geq k - 1 \geq \left\lceil \frac{n}{2} \right\rceil \).

2. If \( k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \), then \( |\nabla F| - |F| \geq n - k - 1 \geq \left\lfloor \frac{n}{2} \right\rfloor \).

Theorem 3.10. Let \( F \) be a Sperner family over an \( n \)-set and let \( l \leq \frac{n}{2} \leq u \), where \( l \) and \( u \) are the minimum size and the maximum size of subsets in \( F \), respectively. Then

\[
|F| \leq \begin{cases} \left( \frac{n}{2} \right) - (u-l) \frac{n}{2} - \left\lfloor \frac{(u-l+1)^2}{4} \right\rfloor, & \text{if } n \text{ is even}, \\ \left( \frac{n}{2} \right) - (u-l-1) \frac{n}{2} - \left\lfloor \frac{(u-l)^2}{4} \right\rfloor, & \text{if } n \text{ is odd}. \end{cases}
\]

Proof. This is an adaptation of [16 Corollary 2.3.3]. The cases \( n = 1, 2 \) are trivial, hence we let \( n \geq 3 \). Replace \( F \) by \( F_1 = (F \setminus G) \cup \nabla (G) \) where \( G = F \cap \binom{[n]}{l+1} \). Because \( F \) is Sperner, we have that \( (F \setminus G) \cap \nabla (G) = \emptyset \) and that \( F_1 \) is a Sperner family for which by Lemma 3.9, we have

\[
|F_1| = |F| - |G| + |\nabla (G)| \geq |F| + n - l - 1.
\]

If \( l + 1 \leq \frac{n}{2} - 1 \) then replace \( F_1 \) by \( F_2 = (F_1 \setminus G_1) \cup \nabla (G_1) \) where \( G_1 = F_1 \cap \binom{[n]}{l+1} \). After this we obtain a Sperner family \( F_2 \) for which by Lemma 3.9

\[
|F_2| \geq |F_1| + n - l - 2 \geq |F| + (n - l - 1) + (n - l - 2).
\]

Repeat doing like this until we raise the minimum size of the subsets to \( \left\lceil \frac{n}{2} \right\rceil \) and we obtain a Sperner family \( F_{\left\lceil \frac{n}{2} \right\rceil - l} \) satisfying

\[
|F_{\left\lceil \frac{n}{2} \right\rceil - l}| \geq |F| + (n - l - 1) + (n - l - 2) + \cdots + \left\lceil \frac{n}{2} \right\rceil. \tag{1}
\]

Now we begin to decrease the maximum size of subsets of \( F_{\left\lceil \frac{n}{2} \right\rceil - l} \) step by step. Replace \( F_{\left\lceil \frac{n}{2} \right\rceil - l} \) by \( F_{\left\lceil \frac{n}{2} \right\rceil - l+1} = (F_{\left\lceil \frac{n}{2} \right\rceil - l} \setminus \mathcal{H}) \cup \Delta (\mathcal{H}) \) where \( \mathcal{H} = F_{\left\lceil \frac{n}{2} \right\rceil - l} \cap \binom{[n]}{u} \). Then we obtain a Sperner family \( F_{\left\lceil \frac{n}{2} \right\rceil - l+1} \) for which by Lemma 3.9, we have

\[
|F_{\left\lceil \frac{n}{2} \right\rceil - l+1}| \geq |F_{\left\lceil \frac{n}{2} \right\rceil - l}| + u - 1.
\]

If \( u - 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1 \) then replace \( F_{\left\lceil \frac{n}{2} \right\rceil - l+1} \) by \( F_{\left\lceil \frac{n}{2} \right\rceil - l+2} = (F_{\left\lceil \frac{n}{2} \right\rceil - l+1} \setminus \mathcal{H}_1) \cup \Delta (\mathcal{H}_1) \) where \( \mathcal{H}_1 = F_{\left\lceil \frac{n}{2} \right\rceil - l+1} \cap \binom{[n]}{u-1} \). Similarly we have

\[
|F_{\left\lceil \frac{n}{2} \right\rceil - l+2}| \geq |F_{\left\lceil \frac{n}{2} \right\rceil - l+1}| + u - 2 \geq |F_{\left\lceil \frac{n}{2} \right\rceil - l}| + (u - 1) + (u - 2).
\]

Repeat this process until we obtain a Sperner family \( F_{\left\lceil \frac{n}{2} \right\rceil - l+u-\left\lceil \frac{n}{2} \right\rceil} \) with maximum size of the subsets being \( \left\lceil \frac{n}{2} \right\rceil \) (and all sizes of the subsets being \( \left\lfloor \frac{n}{2} \right\rfloor \) or \( \left\lceil \frac{n}{2} \right\rceil \)) and we have

\[
\left( \frac{n}{\left\lceil \frac{n}{2} \right\rceil} \right) \geq |F_{\left\lceil \frac{n}{2} \right\rceil - l+u-\left\lceil \frac{n}{2} \right\rceil}| \geq |F_{\left\lceil \frac{n}{2} \right\rceil - l}| + (u - 1) + (u - 2) + \cdots + \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}
\]

Combining with inequality \((1)\) yields

\[
\left( \frac{n}{\left\lceil \frac{n}{2} \right\rceil} \right) \geq |F| + (n - l - 1) + (n - l - 2) + \cdots + \left\lfloor \frac{n}{2} \right\rfloor + (u - 1) + (u - 2) + \cdots + \left\lceil \frac{n}{2} \right\rceil. \tag{3}
\]

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Whenever $n$ is even, we further bound (3) by
\[
\left(\frac{n}{2}\right) \geq |\mathcal{F}| + (u-l)\frac{n}{2} + (1 + 2 + \cdots + \left\lfloor \frac{u-l}{2} \right\rfloor) + (1 + 2 + \cdots + \left\lfloor \frac{u-l}{2} \right\rfloor)
\]
\[
= |\mathcal{F}| + (u-l)\frac{n}{2} + \left\lfloor \frac{(u-l+1)^2}{4} \right\rfloor.
\]
Whenever $n$ is odd, similarly we have
\[
\left(\frac{n-1}{2}\right) \geq |\mathcal{F}| + (u-l-1)\frac{n+1}{2} + (1 + 2 + \cdots + \left\lfloor \frac{u-l-1}{2} \right\rfloor) + (1 + 2 + \cdots + \left\lfloor \frac{u-l-1}{2} \right\rfloor)
\]
\[
= |\mathcal{F}| + (u-l-1)\frac{n+1}{2} + \left\lfloor \frac{(u-l-1)^2}{4} \right\rfloor.
\]

Then the conclusion follows immediately. \qed

**Corollary 3.11.** Let $\mathcal{F}$ be a Sperner family over an $n$-set. Let $l$ and $u$ be the minimum size and the maximum size of subsets in $\mathcal{F}$, respectively. For $l \leq i \leq u$, define
\[
\mathcal{F}_i^+ = \{B : B \in \mathcal{F}, |B| \geq i\},
\]
\[
\mathcal{F}_i^- = \{B : B \in \mathcal{F}, |B| \leq i\}.
\]

(i) If $l \leq \left\lfloor \frac{u}{2} \right\rfloor$, then $|\nabla_i(\mathcal{F}_i^-)| \geq |\mathcal{F}_i^-| + (i-l)(n-i)$ for any $l \leq i \leq \left\lfloor \frac{u}{2} \right\rfloor$.

(ii) If $u \geq \left\lceil \frac{u}{2} \right\rceil$, then $|\Delta_i(\mathcal{F}_i^+)| \geq |\mathcal{F}_i^+| + i(u-i)$ for any $\left\lceil \frac{u}{2} \right\rceil \leq i \leq u$.

**Proof.** (i) Analogous to the proof of inequality (1) of Theorem 3.10, we increase the minimum size of the subsets of $\mathcal{F}_i^+$ to $i$ step by step and then we have
\[
|\nabla_i(\mathcal{F}_i^-)| \geq |\mathcal{F}_i^-| + (n-l-1) + (n-l-2) + \cdots + (n-i) \geq |\mathcal{F}_i^-| + (i-l)(n-i).
\]

(ii) Analogous to the proof of inequality (2) of Theorem 3.10, we decrease the maximum size of the subsets of $\mathcal{F}_i^+$ to $i$ step by step and then we have
\[
|\Delta_i(\mathcal{F}_i^+)| \geq |\mathcal{F}_i^+| + (u-1) + (u-2) + \cdots + i \geq |\mathcal{F}_i^+| + i(u-i).
\]

\qed

### 3.1 Length even

**Lemma 3.12.** Let $n \geq 6$ be even and $\mathcal{F}$ a non 2-covering Sperner family over an $n$-set. Denote $l$ and $u$ to be the minimum size and the maximum size of subsets in $\mathcal{F}$, respectively.

(i) If $u \geq \frac{n}{2} + 1$, then $|\mathcal{F}| \leq \left(\frac{n}{2}\right) - \frac{n}{2}$.

(ii) If $l \leq \frac{n}{2} - 2$, then $|\mathcal{F}| \leq \left(\frac{n}{2}\right) - \frac{n}{2} - 1$.

(iii) If $u = l = \frac{n}{2}$, then $|\mathcal{F}| \leq \frac{1}{2}(\frac{n}{2})$. 

Proof. Let $\mathcal{F} = A \cup B$ be a non 2-covering Sperner family on $[n]$, where
\[
A = \{A : A \in \mathcal{F}, |A| \geq \frac{n}{2}\},
\]
\[
B = \{B : B \in \mathcal{F}, |B| \leq \frac{n}{2} - 1\}.
\]

(i) Let $u \geq \frac{n}{2} + 1$. By Corollary 3.11, we have
\[
|\Delta_{\frac{n}{2}}(A)| \geq |A| + (u - \frac{n}{2}) \frac{n}{2} \geq |A| + \frac{n}{2}.
\]
Denote $\mathcal{P} = \Delta_{\frac{n}{2}}(A)$. Because $\mathcal{F}$ is non 2-covering, we have that $\mathcal{P}$ is intersecting. Then by Lemma 3.8 we have
\[
|\Delta(\mathcal{P})| \geq |\mathcal{P}| \geq |A| + \frac{n}{2}.
\]
By Theorem 3.4 all subsets of $[n]$ can be partitioned into $\binom{n}{\frac{n}{2}}$ mutually disjoint symmetric chains. If $B \neq \emptyset$, then replace each $B \in B$ with $B'$ in the same symmetric chain, where $B \subseteq B'$ and $|B'| = \frac{n}{2} - 1$. Then we produce from $B$ a new Sperner family $B'$ of $\binom{n}{\frac{n}{2} - 1}$-subsets. Note also that $\Delta(\mathcal{P}) \cap B' = \emptyset$ because $\mathcal{F} = A \cup B$ is Sperner. (Let $B' = \emptyset$ if $B = \emptyset$.) As a result,
\[
|B| = |B'| \leq \left(\frac{n}{2} - 1\right) - |\Delta(\mathcal{P})|.
\]
It follows that
\[
|\mathcal{F}| = |A| + |B| \leq |A| + \left(\frac{n}{2} - 1\right) - |\Delta(\mathcal{P})| \leq \left(\frac{n}{2} - 1\right) - \frac{n}{2}.
\]

(ii) Let $l \leq \frac{n}{2} - 2$. We form a partition $B = B_1 \cup B_2$ where
\[
B_1 = \{B : B \in \mathcal{F}, |B| = \frac{n}{2} - 1\}, \quad B_2 = \{B : B \in \mathcal{F}, |B| \leq \frac{n}{2} - 2\}.
\]
By Corollary 3.11, we have
\[
|\nabla_{\frac{n}{2} - 1}(B_2)| \geq |B_2| + \frac{n}{2} + 1.
\]
In the decomposition of the power set of $[n]$ into symmetric chains, if $A \neq \emptyset$, then replace each $A \in A$ by $A'$ of the same symmetric chain where $|A'| = \frac{n}{2}$ to obtain a new family $A'$ of $\frac{n}{2}$-sets. Since $\mathcal{F}$ is non 2-covering, $A'$ is intersecting and thus $|\Delta(A')| \geq |A'| = |A|$ by Lemma 3.8. Furthermore, it is easy to see that $\Delta(A'), B_1$, and $\nabla_{\frac{n}{2} - 1}(B_2)$ are pairwise disjoint because $\mathcal{F}$ is Sperner. (Let $A' = \emptyset$ if $A = \emptyset$.) It follows that
\[
|\mathcal{F}| = |A| + |B_1| + |B_2| \leq |\Delta(A')| + |B_1| + |\nabla_{\frac{n}{2} - 1}(B_2)| - \frac{n}{2} - 1 \leq \left(\frac{n}{2} - 1\right) - \frac{n}{2} - 1.
\]

(iii) If $u = l = \frac{n}{2}$, then $|\mathcal{F}| \leq \frac{1}{2}\binom{n}{\frac{n}{2}}$ because $\mathcal{F}$ is non 2-covering. \qed

For a family $\mathcal{F}$ of subsets of $[n]$, we define its complement by $\overline{\mathcal{F}} = \{F : F \in \mathcal{F}\}$.

**Theorem 3.13.** Let $n$ be even and $n \geq 8$. Suppose that $\mathcal{C}$ is an $(n, m, q)$ 2-wFP code. Then
\[
m \leq \left(\frac{n}{2} - 1\right) - \frac{n}{2} + 1.
\]
Proof. For $1 \leq i \leq m$, let $\mathcal{X}_i$ be the coincidence family generated by the codeword $c^i \in \mathcal{C}$. Then each $\mathcal{X}_i$ is a non 2-covering Sperner family by Theorem 3.6. Take a fixed $i \in [m]$ and let $l$ and $u$ be the minimum size and the maximum size of subsets in $\mathcal{X}_i$, respectively.

By Lemma 3.12 if $u \geq \frac{n}{2} + 1$, then $|X_i| \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2}$; if $l \leq \frac{n}{2} - 2$, then $|X_i| \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2} - 1$; if $u = l = \frac{n}{2}$, then $|X_i| \leq \left(\frac{n}{4}\right) = \left(\frac{n-1}{4}\right)$. It is easy to show that $m = |X_i| + 1 \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2} + 1$ for these cases. So, to prove the conclusion, we only need to consider two cases $u = l = \frac{n}{2} - 1$ and $(u, l) \neq \left(\frac{n}{2}, \frac{n}{2}\right)$. Thus we only need to let $l = \frac{n}{2} - 1$. Let

$$A = \{i : A \in \mathcal{X}_i, |A| = \frac{n}{2}\}, \quad B = \{i : A \in \mathcal{X}_i, |A| = \frac{n}{2} - 1\}.$$ 

Case 1: Let $u = l = \frac{n}{2} - 1$. Then $\mathcal{X}_i = B$. We evaluate the upper bound of $m$ by considering whether $B$ is intersecting.

If $B$ is intersecting, then consider its complement $\overline{B} = \{B : B \in \mathcal{B}\}$. For any $\overline{A}, \overline{B} \in \overline{B}$,

$$|\overline{A} \cap \overline{B}| = n - |A \cup B| = n - (|A| + |B| - |A \cap B|) = n - (n - 2 - |A \cap B|) \geq 3.$$ 

Consequently $\overline{B}$ is a 3-intersecting Sperner family. By Theorem 3.5 we have

$$m = |X_i| + 1 = |\overline{B}| + 1 \leq \left(\frac{n}{4} - 2\right) + 1.$$ 

If $B$ is not intersecting, then there exist $B_1, B_2 \in B$ such that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 = I(i, j)$ and $B_2 = I(i, k)$ where $i \neq j, k$ and $1 \leq j, k \leq m$. By Lemma 3.7,

$$I(j, k) \subseteq (B_1 \cap B_2) \cup \overline{B_1 \cup B_2} = B_1 \cup B_2$$

and hence $|I(j, k)| \leq 2$. If $|I(j, k)| = 0$ then $X_j$ is not Sperner, contradicting to Theorem 3.6. So we have $|I(j, k)| = 1, 2$, meaning that $X_j$ contains a set of size 1 or 2. Obviously if $X_j$ contains a singleton, then by Proposition 3.3 we have $|X_j| \leq \left(\frac{n-1}{2}\right) + 1$ and hence $m = |X_j| + 1 \leq \left(\frac{n-1}{2}\right) + 2$. If $|I(j, k)| = 2$, then the minimum size $l_j$ of elements of $X_j$ satisfies $l_j \leq 2 \leq \frac{n}{2} - 2$ whenever $n \geq 8$. Hence $|X_j| \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2} - 1$ by Lemma 3.12. Thus $m \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2}$. Comparing the upper bounds of $m$ for $n \geq 8$ shows $m \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2}$ in Case 1.

Case 2: Let $u = \frac{n}{2}$ and $l = \frac{n}{2} - 1$. Then $\mathcal{X}_i = A \cup B$.

If $A$ is not intersecting, then we have $m \leq \left(\frac{n}{4} - 1\right) - \frac{n}{2}$ by similar discussions in Case 1.

If $A$ and $B$ are not cross-intersecting, then there exist $A \in A$ and $B \in B$ such that $A \cap B = \emptyset$. Suppose that $A = I(i, j)$ and $B = I(i, k)$ where $i \neq j, k$ and $1 \leq j, k \leq m$. By Lemma 3.7 $|I(j, k)| \leq |A \cup B| = 1$. Since $X_j$ is Sperner, we have $|I(j, k)| = 1$. Then by Proposition 3.3 we have $|X_j| \leq \left(\frac{n-1}{4}\right) + 1$ and hence $m \leq \left(\frac{n-1}{4}\right) + 2$.

What remains to bound $m$ is the subcase that $B$ is intersecting and that $A$ and $B$ are cross-intersecting. By Lemma 3.9, $|\nabla(B)| \geq |B| + \frac{n}{2}$. Note that $X_i$ is Sperner and non 2-covering. As a result, $A \cap \nabla(B) = \emptyset$ and that $\mathcal{F} := \mathcal{A} \cup \nabla(B)$ is a non 2-covering family of $\frac{n}{2}$-subsets. Hence

$$\frac{1}{2}\left(\frac{n}{4}\right) \geq |\mathcal{F}| = |\mathcal{A} \cup \nabla(B)| = |\mathcal{A}| + |\nabla(B)| \geq |\mathcal{A}| + |\mathcal{B}| + \frac{n}{2},$$

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yielding that
\[ m = |\mathcal{X}_i| + 1 = |\mathcal{A}| + |\mathcal{B}| + 1 \leq \frac{1}{2} \left( \frac{n}{4} \right) - \frac{n}{2} + 1 = \left( \frac{n}{2} - 1 \right) - \frac{n}{2} + 1. \]

It follows that \( m \leq \left( \frac{n}{2} - 1 \right) + 2 \) in Case 2.

To sum up it is immediate that for even \( n \geq 8 \) we have
\[ m \leq \left( \frac{n}{2} - 1 \right) - \frac{n}{2} + 1. \]

This completes the proof.

\[ \square \]

### 3.2 Length odd

**Lemma 3.14.** Let \( n \) be odd and \( n \geq 7 \). Suppose that \( \mathcal{F} \) is a non 2-covering Sperner family on \([n]\). If \(|F| \geq \frac{n+1}{2}\) for all \( F \in \mathcal{F} \), then
\[ |\mathcal{F}| \leq \left( \frac{n}{n+3} \right). \]

**Proof.** Since \( \mathcal{F} \) is a non 2-covering Sperner family, for any \( A, B \in \mathcal{F} \) we have
\[ n - 1 \geq |A \cup B| = |A| + |B| - |A \cap B| \geq n + 1 - |A \cap B|, \]

implying \(|A \cap B| \geq 2\). As a result \( \mathcal{F} \) is a 2-intersecting Sperner family. Then applying Theorem 3.5 yields the conclusion.

**Lemma 3.15.** Let \( n \) be odd and \( n \geq 7 \). Suppose that \( \mathcal{C} \) is an \((n, m, q)\) 2-wFP code and \( \mathcal{X}_i \) is the coincidence family generated by the codeword \( c^i \in \mathcal{C} \). If there is \( i \in [m] \) such that \(|F| \leq \frac{n-1}{2}\) for all \( F \in \mathcal{X}_i \), then
\[ m \leq \left( \frac{n}{n+3} \right) + 1. \]

**Proof.** By Theorem 3.6 \( \mathcal{X}_i \) is a non 2-covering Sperner family. Let \( \mathcal{X}_i = \mathcal{A}_0 \cup \mathcal{A}_1 \), where
\[ \mathcal{A}_0 = \{ A : A \in \mathcal{X}_i, |A| \leq \frac{n-3}{2} \}, \]
\[ \mathcal{A}_1 = \{ A : A \in \mathcal{X}_i, |A| = \frac{n-1}{2} \}. \]

If \( \mathcal{A}_1 \) is not intersecting, then there exist \( I(i, j), I(i, k) \in \mathcal{A}_1 \) \((1 \leq j, k \leq m, i \neq j, k)\) such that \( I(i, j) \cap I(i, k) = \emptyset \). Hence by Lemma 3.7 we have \( I(j, k) \subseteq \overline{I(i, j)} \cup \overline{I(i, k)} \). Because \( \mathcal{X}_i \) is Sperner, we know that \(|I(j, k)| = 1\). Apply Proposition 3.3 to have
\[ m = |\mathcal{X}_j| + 1 \leq \left( \frac{n-1}{n-2} \right) + 2. \]
If \( A_1 \) is intersecting, then let \( F = A_0 \cup A_1 \). Similarly to the proof of Lemma 3.14, we can check that \( F \) forms a 2-intersecting Sperner family. Hence by Theorem 3.5 we have

\[
m = |X_i| + 1 = |F| + 1 \leq \left( \frac{n}{n+3} \right) + 1.
\]

Noting that \( \left( \frac{n-1}{2} \right) \leq \left( \frac{n}{n+3} \right) \) if \( n \geq 7 \) yields the conclusion. \( \square \)

**Theorem 3.16.** Let \( C \) be an \((n, m, q)\) 2-wFP code with \( n \) odd and \( n \geq 7 \). Then

\[
m \leq \begin{cases} 
\left( \frac{n}{2} \right) - \frac{n^2-9}{8} - \lfloor \frac{(n+3)^2}{64} \rfloor, & \text{if } n \equiv 1 \pmod{4}, \\
\left( \frac{n}{2} \right) - \frac{(n+1)^2-8}{8} - \lfloor \frac{(n+5)^2}{64} \rfloor, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** Let \( C \) be an \((n, m, q)\) 2-wFP code and \( X_i \) be the non 2-covering Sperner family generated by the codeword \( c^i \). Denote

\[
l_i = \min\{|A| : A \in X_i\}, \\
u_i = \max\{|A| : A \in X_i\}, \\
d_i = u_i - l_i.
\]

If there is \( 1 \leq i \leq m \) such that \( u_i \leq \frac{n-1}{2} \) or \( l_i \geq \frac{n+1}{2} \), then by Lemmas 3.14 and 3.15 we have

\[
m = |X_i| + 1 \leq \left( \frac{n}{n+3} \right) + 1. \quad (4)
\]

Next we let \( l_i \leq \frac{n-1}{2} \) and \( u_i \geq \frac{n+1}{2} \) for all \( 1 \leq i \leq m \). Denote \( d = \min\{d_i : 1 \leq i \leq m\} \) and assume w.l.o.g. \( d_1 = d \). In the case that \( d \geq \frac{n+1}{2} \), by Theorem 3.10 we have

\[
m = |X_1| + 1 \leq \left( \frac{n}{n-2} \right) - \frac{n^2 - 1}{4} - \lfloor \frac{(n+1)^2}{16} \rfloor + 1. \quad (5)
\]

In the following proof we let \( d \leq \frac{n-1}{2} \) and then bound \( m \). Let \( X_1 = A_1 \cup A_2 \cup B \), where

\[
A_1 = \{A : A \in X_1, |A| \leq \frac{n-3}{2}\}, \\
A_2 = \{A : A \in X_1, |A| = \frac{n+1}{2}\}, \\
B = \{B : B \in X_1, |B| \geq \frac{n+1}{2}\}.
\]

We consider three cases as follows.

**Case 1:** Let \( A_2 \neq \emptyset \) be not intersecting. Then there exist \( A, B \in A_2 \) such that \( A \cap B = \emptyset \) and \( |A \cup B| = n - 1 \). Suppose that \( A = I(1, j) \) and \( B = I(1, k) \). Then by Lemma 3.7 we have \( I(j, k) \subseteq (A \cap B) \cup A \cup B = A \cup B \), meaning that \( X_j \) contains a singleton or \( \emptyset \). Since \( X_j \) is Sperner, it contains a singleton and hence by Proposition 3.3 we have

\[
m = |X_1| + 1 \leq \left( \frac{n-1}{n-2} \right) + 2. \quad (6)
\]

**Case 2:** Let \( A_1 \neq \emptyset \) and let \( A_1 \) and \( B \) be not cross-intersecting. Then there exist \( A \in A_1, B \in B \) such that \( A \cap B = \emptyset \). Clearly we have \( |A| \geq \frac{n+1}{2} - d \) \((2 \leq d \leq \frac{n-1}{2})\) and \( |B| \geq \frac{n+1}{2} \). Suppose
that \( A = I(1,j) \) and \( B = I(1,k) \). Then by Lemma 3.7 we have \( I(j,k) \subseteq A \cup B \). It follows that \( |I(j,k)| \leq n - \left( \frac{n+1}{2} - d + \frac{n+1}{2} \right) = d - 1 \), meaning that \( X_j \) contains an \( r \)-subset where \( 1 \leq r \leq d - 1 \). Hence we have \( d_j \geq \frac{n+1}{2} - d + 1 \). Now we apply Theorem 3.10 to bound \( m \). Let \( d_0 \) be an integer with \( 1 \leq d_0 \leq \frac{n-3}{2} \).

Whenever \( d \geq d_0 + 1 \), by Theorem 3.10 we have
\[
|X_1| \leq \left( \frac{n}{n-2} \right) - d_0 \left( \frac{n+1}{2} - \left\lfloor \frac{d_0+1}{4} \right\rfloor \right).
\]

Whenever \( d \leq d_0 \), we have \( d_j \geq \frac{n+1}{2} - d_0 + 1 \) and similarly we have
\[
|X_j| \leq \left( \frac{n}{n-2} \right) - (n+1) \left( \frac{n+1}{2} - \left\lfloor \frac{n+3}{2} - d_0 \right\rfloor \right).
\]

In order to get a better upper bound of \( m = |X_1| + 1 = |X_j| + 1 \), we take \( d_0 = \frac{n-1}{4} \) if \( n \equiv 1 \) (mod 4) and \( d_0 = \frac{n+1}{4} \) if \( n \equiv 3 \) (mod 4). By simple reduction we have
\[
m \leq \begin{cases} 
\left( \frac{n}{2} - \frac{n^2-1}{8} - \left\lfloor \frac{(n+3)^2}{64} \right\rfloor + 1, \text{ if } n \equiv 1 \pmod{4}, \\
\left( \frac{n}{2} - \frac{(n+1)^2}{8} - \left\lfloor \frac{(n+5)^2}{64} \right\rfloor + 1, \text{ if } n \equiv 3 \pmod{4}. 
\end{cases}
\tag{7}
\]

Case 3: Let \( A_2 \) be intersecting if \( A_2 \neq \emptyset \) and let \( A_1 \) and \( B \) be cross-intersecting if \( A_1 \neq \emptyset \). Define
\[
\mathcal{F} = \overline{A_1} \cup \overline{A_2} \cup B,
\]
where we let \( \overline{\emptyset} = \emptyset \) if necessary. Then we claim that \( \mathcal{F} \) is a 2-intersecting Sperner family.

Obviously each family of \( \overline{A_1} \), \( \overline{A_2} \) and \( B \) is Sperner. If \( \mathcal{F} \) is not Sperner, then one of the following three possibilities would happen: (a) there is \( A_1 \in A_1 \) and \( A_2 \in A_2 \) such that \( \overline{A_2} \subseteq \overline{A_1} \), yielding \( A_1 \subseteq A_2 \) and contradicting to that \( X_1 \) is Sperner; (b) there is \( A \in A_1 \) and \( B \in B \) such that \( B \subseteq A \) or \( A \subseteq B \), yielding \( A \cap B = \emptyset \) or \( A \cup B \supseteq A \cup \overline{A} = [n] \) and contradicting to that \( A_1 \) and \( \overline{A} \) are cross-intersecting or that \( X_1 \) is non 2-covering; (c) there is \( A \in A_2 \) and \( B \in B \) such that \( \overline{A} \subseteq B \), yielding \( A \cup B \supseteq A \cup \overline{A} = [n] \) and contradicting to that \( X_1 \) is non 2-covering. It follows that \( \mathcal{F} \) is Sperner.

Next we show that \( \mathcal{F} \) is 2-intersecting. Noting the size of subsets in each family \( \overline{A_1} \), \( \overline{A_2} \) and \( B \), we readily check that (a) \( \overline{A_1} \) is 3-intersecting, (b) \( \overline{A_2} \) is 2-intersecting because \( A_2 \) is intersecting, (c) \( B \) is 2-intersecting as \( B \) is non 2-covering, (d) \( \overline{A_1} \) and \( \overline{A_2} \) are cross-2-intersecting because \( |A| + |B| \leq n - 2 \) for \( A \in A_1, B \in A_2 \), and (e) \( \overline{A_1} \cup \overline{A_2} \) and \( B \) are cross-2-intersecting because for all \( A \in A_1 \cup A_2, B \in B \) we have \( A \cap B \neq A \) (\( X_1 \) Sperner) and
\[
|A \cap B| = |B| - |A \cap B| \geq |B| - (|A| - 1) \geq \frac{n+1}{2} - \frac{n-3}{2} = 2.
\]

Now that \( \mathcal{F} \) is a 2-intersecting Sperner family. So by Theorem 3.5 we have
\[
m = |X_1| + 1 = |\mathcal{F}| + 1 \leq \left( \frac{n+3}{2} \right) + 1.
\tag{8}
\]

Comparing the upper bounds in (7)-(8) when \( n \geq 7 \) yields the conclusion. \( \square \)
4 Conclusion

Blackburn in [7] displayed a general upper bound for narrow-sense $t$-frameproof codes. We list the particular result for $t = 2$.

**Theorem 4.1.** [7, Corollary 2] Let $s = 1, 2$ and $s \equiv n \pmod{2}$. If there exists a narrow-sense $(n, m, q)$ 2-frameproof code, then

$$m \leq sq\left\lceil \frac{n}{2} \right\rceil + O(q^{\frac{n}{2}} - 1).$$

By Stirling’s formula, $n! = (1 + o(1))\sqrt{2\pi n}\left(\frac{e}{n}\right)^n$. So approximating $k$ by Stirling’s formula where $k = O(n)$ we have

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} < \frac{n^k}{k!} < \left(\frac{en}{k}\right)^k.$$  

(9)

This paper provides new upper bounds on the sizes of wide-sense 2-frameproof codes in Theorems 3.13 and 3.16 which are better than those of Lemma 3.1. When $n$ approximates infinity, the upper bound of the size of a $(n, m, q)$ 2-wFP code is $O((2e)^{\frac{n}{2}})$ by (9), which is obviously tighter than the narrow-sense case for all $q \geq 6$ in Theorem 4.1.

The paper also establishes a lower bound on the sizes of wide-sense 2-frameproof codes by constructing 3-uniform hypergraphs and evaluating their independence numbers. This fills a large gap in alphabet size $q$ for narrow-sense 2-frameproof codes in [35, Theorem 11]. We tried the standard probabilistic method to achieve lower bounds on the sizes of wide-sense 2-frameproof codes as what was did for narrow-sense codes in [27, 31], but no larger lower bounds were obtained than Theorem 2.2.

For future research, we suggest that the upper bounds in Theorems 3.13 and 3.16 be improved further or wide-sense 2-frameproof codes with the stated sizes be constructed.

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