New Temperature Dependent Configurational Probability Diffusion Equation for Diluted FENE Polymer Fluids: Existence of Solution Results

Ionel Sorin Ciuperca¹,² · Liviu Iulian Palade¹,³

Dedicated to the memory of late Professor Geneviève Raugel, Université Paris-Sud, in fond remembrance.

Received: 22 November 2019 / Revised: 7 January 2021 / Accepted: 16 January 2021 / Published online: 15 February 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC part of Springer Nature 2021

Abstract
The theory for the non-isothermal rheology of polymer fluids proposed in Curtiss and Bird (Adv Polym Sci 125:1–101, 1996) used several approximations including the so-called linear gradient approximations for the temperature field and Brownian forces. While it had the significant advantage of dealing with linear equations, the approximations involved may have led to several non-physical predictions. This work is a continuation of Curtiss and Bird (1996) in that it obtains the corresponding non-linear configurational probability density equation in dimensionless form without the linear gradient approximations for the temperature field and Brownian forces. It does so for incompressible diluted polymer solutions with polymer molecules being modeled as FENE (Finitely Extensible Nonlinear Elastic) chains. Next we prove the existence of temperature dependent, positive variational solutions for the probability density equation of the FENE model.

Keywords FENE polymer chain models · Non-isothermal polymer kinetic theory · Dimensionless configurational probability diffusion equation · Solution existence results

Mathematics Subject Classification Primary 35Q35 · Secondary 35A15

Liviu Iulian Palade
liviu-iulian.palade@insa-lyon.fr

1 CNRS, Institut Camille Jordan UMR 5208, Université de Lyon, Lyon, France
2 Bât. Bracconier, Université Lyon 1, 43 Boulevard du 11 Novembre 1918, 69622 Villeurbanne, France
3 INSA-Lyon, Pôle de Mathématiques, Bât. Leonard de Vinci No. 401, 21 Avenue Jean Capelle, 69621 Villeurbanne, France
1 Introduction

All polymeric liquid flows of practical importance and in everyday life—e.g. injection molding, hydrodynamics of biological fluids, thin film flows, lubrication (to name only a very few)—are subjected to significant heat transfer processes. Therefore all these flows are non-isothermal and require to be studied as such (see e.g. the recent work [16,24] and references cited therein). However, such an undertaking is rendered more complicated by the necessity to focus, in addition to the momentum balance and fluid constitutive equations, on the temperature equation as well.

In the kinetic theory of polymer dynamics one is interested in producing constitutive equations by taking into account the dominant interactions between fluid constituents (polymer–polymer, polymer–solvent molecules) which ultimately govern the macroscopic physical properties. In doing so, a configurational (conformation) probability diffusion equation—hereafter referred to as the CPD equation—is obtained; its solution, denoted $\Psi$, enters both the expression of the momentum balance (via the stress tensor expression) and the temperature balance equation. If the temperature is not held constant, all three aforementioned governing equations are interrelated, forming a system of equations.

Curtiss and Bird undertook in [14] to extend existing isothermal polymer kinetic theory results to non-isothermal flows. In order to present simpler forms for the balance law equations, the velocity, temperature and concentration fields are approximated via first order truncated Taylor expansions (higher order derivatives being deemed as negligible), hence the so-called linear gradients approximations; this assumption is listed as the 5th simplifying hypothesis, see page 85 of Section 17 in [14]. Consequently, the temperature is expressed as a first order approximation in equation (12.3) on page 49 in [14], and its impact on the Brownian force appears in equation (12.16) on page 53 of the same. The theory’s appealing elegance and acclamation notwithstanding, some of the approximations involved (like the linear gradients approximation) may have led to some nonphysical predictions, such as infinite viscosity at a finite extensional rate or the absence of influence of temperature gradients orthogonal to the direction of flow, as noted in [14]. As a matter of fact, the Authors themselves invite on page 85 of [14] for further work as the “...major assumptions can and should be challenged”.

This paper is a first step and exploratory in nature continuation of the work in [14] in that it focuses on the theory without using linear gradients approximations. Specifically, it is devoted to:

- Obtaining the dimensionless form of the CPD equation without the linear gradient approximation for the temperature field and Brownian forces; for the later a classical, Boltzmann description, is assumed instead
- Proving the existence of positive variational solutions to the Finitely Extensible Nonlinear Elastic (FENE) model CPD equation using Schauder’s fixed-point theory

As to the second goal, the importance of studying the full system of equations consisting of momentum balance CPD (and of the closely related stress tensor) and the heat diffusion equations notwithstanding (see for instance [1,2,5,18,22]), it is to be noted here that it is common matter in the rheology literature (as per [4,21,25]) to first focus on the CPD alone, see e.g. [5–13,17,19,27,28].

The paper is organized as follows:

- Section 2 is devoted to obtaining the CPD equation in dimensionless form for the FENE model
Section 3 deals with proving the existence of variational solutions to the CPD equation of the FENE model.

Section 3.1 introduces the mathematical problem under scrutiny.
Section 3.2 gives the proof of the existence of positive solutions to the regularized problem.
Section 3.3 gives estimates uniform in $\epsilon$.
Section 3.4 summarizes the final results and gives the main existence result.

### 2 Modeling the Non-isothermal Dynamics: The CPD Equation for FENE Polymer Chains

The notations are akin to those of [3,4] (see also [15]). The fluids are assumed incompressible.

A polymer molecule in a diluted solution is here modeled as a FENE chain, that is as an elastic dumbbell with the polymer mass concentrated at its two extremities (see Chapter 13 in [4] for a detailed description). Let $x$ be the Eulerian (macroscopic) variable, $Q$ the end-to-end microscopic vector, and $F^{(c)}$ the non-linear elastic recoil spring force. Denote by $\Psi = \Psi(x, Q, t)$ the configurational probability, where $\nabla_y$ is the gradient in the $y$-direction.

Let $\kappa = \nabla_x v(x, t)$ be the macroscopic velocity gradient. Moreover, $\kappa \cdot x = k_{ij} x_j$ by Einstein’s convention for repeated indices. The solvent is here assumed to be a classical, with shear rate independent but temperature dependent $\eta_s$ viscosity Newtonian fluid.

Let $\nu = 1, 2$ label each individual bead. In the absence of external forces, the (vector) force balance equation for each bead reads:

$$F^{(\phi)}_\nu + F^{(h)}_\nu + F^{(b)}_\nu = 0, \quad \nu = 1, 2 \quad (2.1)$$

In the above, $F^{(\phi)}_\nu$ is the intramolecular force that accounts for the polymer molecule entropic elasticity, here formally modeled by the FENE spring and assumed to be a potential force (i.e. $F^{(\phi)}_\nu = -\nabla_{r_\nu} \Phi$, where $\Phi$ is a given potential function and $r_\nu$ is the position vector of bead $\nu$).

Next, $F^{(h)}_\nu$ is the hydrodynamic drag force and $F^{(b)}_\nu$ is the Brownian force caused by thermal fluctuations (that pushes beads to jostle about randomly). Their expressions are:

$$F^{(h)}_\nu = -\zeta \left[ [\dot{r}_\nu] - v_\nu \right], \quad \nu = 1, 2 \quad (2.2)$$

where $\zeta$ is the hydrodynamic drag coefficient, $\left[ ] \right.$ stands for a velocity-space average (as in equation 13.1-4 of [4]), and

$$F^{(b)}_\nu = -k_B \nabla_{r_\nu} (T (r_\nu, t) \ln \Psi (x, Q, t)) , \quad \nu = 1, 2 \quad (2.3)$$

with $k_B$ the Boltzmann’s constant, and $T$ denoting the temperature.

Writing $v_\nu = v_0 + \kappa \cdot r_\nu$ (basically a 1st order expansion of $v_\nu$ about $v_0$), with the help of (2.2)–(2.3) the force balance Eq. (2.1) reads:

$$-\zeta \left[ [\dot{r}_\nu] - v_0 - \kappa \cdot r_\nu \right] - k_B \nabla_{r_\nu} (T (r_\nu, t) \ln \Psi (x, Q, t)) + F^{(\phi)}_\nu = 0, \quad \nu = 1, 2 \quad (2.4)$$

Summing over $\nu$ in (2.4), and because $F^{(\phi)}_1 + F^{(\phi)}_2 = 0$, it leads to:

$$-\zeta \left[ [\dot{r}_1 + \dot{r}_2] - 2v_0 - \kappa \cdot (r_1 + r_2) \right] - k_B \left[ \nabla_{r_1} (T (r_1, t) \ln \Psi (x, Q, t)) + \nabla_{r_2} (T (r_2, t) \ln \Psi (x, Q, t)) \right] = 0 \quad (2.5)$$
We have that $[\dot{r}_1 + \dot{r}_2] = 2[\dot{r}_c]$, and $\kappa \cdot (r_1 + r_2) = 2\kappa \cdot (r_c)$. In order to relate the microscopic molecular scale to the macroscopic fluid flow scale, we make the following homogenization assumption: that $r_c$ and the outer $x$ Eulerian variable are the same. Now, since $r_1 (r_c, Q) = r_c - \frac{1}{2} Q$ and $r_2 (r_c, Q) = r_c + \frac{1}{2} Q$, then $\nabla_{r_1} = \frac{1}{2} \nabla_x - \nabla_Q$ and $\nabla_{r_2} = \frac{1}{2} \nabla_x + \nabla_Q$.

Moreover, we Taylor expand $T(r_{1,2}, t)$ to get, respectively, $T(r_1, t) = T \left( r_c - \frac{1}{2} Q, t \right) \simeq T(x, t) - \frac{1}{2} Q \cdot \nabla_x T(x, t)$ and $T(r_2, t) = T \left( r_c + \frac{1}{2} Q, t \right) \simeq T(x, t) + \frac{1}{2} Q \cdot \nabla_x T(x, t)$.

Here we have used first order expansions - a common practice in physical modeling - as it leads to a simpler model; the influence of second order terms will be studied in a following paper. Next, using all these facts, the temperature gradients in (2.5) can, up to a first order approximation, be rewritten as:

$$\nabla_{r_1} (T (r_1, t)) = \frac{1}{2} \nabla_x (T \ln \Psi) - \frac{1}{4} \nabla_x \left[ (Q \cdot \nabla_x T) \ln \Psi \right] - T \left( \nabla_Q \ln \Psi \right)$$

$$+ \frac{1}{2} \nabla_Q \left[ (Q \cdot \nabla_x T) \ln \Psi \right]$$

and

$$\nabla_{r_2} (T (r_2, t)) = \frac{1}{2} \nabla_x (T \ln \Psi) + \frac{1}{4} \nabla_x \left[ (Q \cdot \nabla_x T) \ln \Psi \right] + T \left( \nabla_Q \ln \Psi \right)$$

$$+ \frac{1}{2} \nabla_Q \left[ (Q \cdot \nabla_x T) \ln \Psi \right]$$

Therefore, with $\kappa = \kappa(x, t)$, (2.5) implies that

$$[\dot{x}] (x, Q, t) = v_0 + \kappa(x, t) \cdot x$$

$$- \frac{k_B}{2\zeta} \left\{ \nabla_x \left( T(x, t) \ln \Psi(x, Q, t) \right) + \nabla_Q \left[ (Q \cdot \nabla_x T(x, t)) \ln \Psi(x, Q, t) \right] \right\}$$

$$+ \frac{1}{2} Q \cdot \nabla_x T(x, t)$$

(2.6)

Subtracting over $v$ in (2.4), and because $F_1^{(\phi)} - F_2^{(\phi)} = 2 F^{(c)}$, $F^{(c)}$ being the connector force, leads after calculations similar to above to:

$$[\dot{Q}] (x, Q, t) = \kappa(x, t) \cdot Q$$

$$- \frac{k_B}{\zeta} \left\{ \frac{1}{2} \nabla_x \left[ (Q \cdot \nabla_x T(x, t)) \ln \Psi(x, Q, t) \right] + T(x, t) \left( \nabla_Q \ln \Psi(x, Q, t) \right) \right\}$$

$$- \frac{2}{\zeta} F^{(c)} (Q)$$

(2.7)

The configurational probability density PDE in its general form is (see [4]):

$$\frac{D \Psi}{Dt} (x, Q, t) = - \left( \nabla_x \cdot [\dot{x}] \Psi \right) (x, Q, t) - \left( \nabla_Q \cdot [\dot{Q}] \Psi \right) (x, Q, t)$$

(2.8)
In the above, \( \frac{D}{Dt} = \frac{\partial}{\partial t} + (\nabla_x \cdot \mathbf{v}) \Psi \) is the material derivative for incompressible fluids. Making use of Eqs. (2.6)–(2.7) and upon performing the required calculations leads to

\[
\frac{D\Psi}{Dt} = \frac{k_B}{2\zeta} \nabla_x \cdot \left\{ \left[ \nabla_x \left( T \ln \psi \right) + \nabla_Q (\left( Q \cdot \nabla_x T \right) \ln \psi) \right] \Psi \right\} \\
- \nabla_Q \cdot \left\{ \kappa \cdot Q \Psi - \frac{k_B}{\zeta} \left[ \frac{1}{2} \nabla_x ((Q \cdot \nabla_x T) \ln \psi) + T \left( \nabla_Q \ln \psi \right) \right] \Psi - \frac{2}{\zeta} F^{(c)} \Psi \right\}
\]

(2.9)

The above Eq. (2.9) preserves the normalization condition for the probability density \( \psi \). Clearly, the above Eq. (2.9) is nonlinear in \( \psi \). As an aside, its nonlinear pattern contrasts with that of equation (13.17) of [14] (obtained for a multicomponent/mixture of different polymer chains of Rouse-type with varying temperature and concentration gradients) which is linear in \( \psi \): it is reprinted below for sake of clarity:

\[
\frac{D\Psi}{Dt} = - \sum_j \nabla_Q \cdot (\kappa \cdot Q_j \Psi) + \frac{k_B}{\zeta} T \sum_{j,k} A_{jk} \nabla_Q \cdot \left( \nabla_{Q_k} \Psi + \frac{1}{k_B T} \nabla_{Q_k} \phi^{(c)} \right) \\
+ \frac{k_B}{\zeta} T \sum_{j,k,l} \nabla_{Q_j} \cdot \left( \nabla_x \ln T \cdot D_{jkl} Q_l \nabla_{Q_k} \Psi \right)
\]

(2.10)

where, in the above, \( A_{jk} \) and \( D_{jkl} \) are the Rouse model’s matrices and \( \phi^{(c)} \) is the (given) elastic force potential function. Moreover, as \( \Psi \) is considered independent of \( x \), (2.10) does not contain any derivative of \( \Psi \) with respect to \( x \), while our (2.9) does so.

We are now going to rewrite Eq. (2.9) in dimensionless form. Before proceeding further, we notice from Chapter 2 in [26] that for a given system of differential equations there are alternative ways of non-dimensionalizing it, depending on the nature of the problem to be studied. To achieve this goal, dimensionless quantities (identified by starred notation) and relevant dimensionless numbers need first be introduced.

- Let \( L \) scale the length in the flow direction, \( V \) scale velocity, \( T_0 \) be the reference temperature, \( l_0 = \sqrt{k_B T_0 / H} \) scale the microscopic length scale, \( Q_0 \) is the maximum spring stretch. Therefore, \( x^* = \frac{x}{L}, \quad v^* = \frac{v}{V}, \quad t^* = \frac{t}{L/V}, \quad T^* = \frac{T}{T_0}, \quad Q^* = \frac{Q}{Q_0} \). Moreover, \( \nabla x^* = L \nabla_x \) and \( \nabla Q^* = l_0 \nabla_Q \).

- The dimensionless Deborah’s number \( \text{De} \) is here taken as \( \text{De} = \frac{\zeta V}{LH} = \frac{\zeta V l_0^2}{k_B T_0 L} \), with \( H \) the spring constant. However, because of the interplay between two concomitant different scales (an outer or micro- and an inner or macro-one), Deborah’s number \( \text{De} \) may also be introduced as \( \text{De} = \frac{\zeta V}{LH} = \frac{\zeta V L}{k_B T_0} \), however with both being essentially the same quantity. As an aside: if one is tempted to introduce a length scaling factor \( s_f = l_0 / L \), then the 1st and 3rd term of the equation’s r.h.s. will be multiplied by a factor \( s_f^2 \), which is very small, hence they can be neglected, thus profoundly altering the nature of the equation. Actually, that \( s_f = l_0 / L \) is a small quantity indeed may be inferred from the following: while \( L \) is a characteristic macroscopic flow length, of order of (say) meters, \( l_0 \) is the macromolecule stretch (a.k.a. the end-to-end distance of the unravelled molecule), which is in the sub-micron range, say about \( 10^{-7} \) m, for polymers of industrial importance. Therefore, as \( s_f \sim 10^{-7} \div 10^{-6} \) m, can be safely taken as a “small” enough factor and the corresponding terms be dropped.
As it is often common, in order to keep notations simpler, we subsequently drop the \((\cdots)\) notation. Then, (2.9) in dimensionless form becomes:

\[
\frac{D\Psi}{Dt} = \frac{1}{2} \frac{De}{\De} \nabla_x \cdot \left\{ \left[ \nabla_x (T \ln \Psi) + \nabla_Q ((Q \cdot \nabla_x T) \ln \Psi) \right] \Psi \right\} \\
- \nabla_Q \cdot (\kappa \cdot Q \Psi) + \frac{1}{2} \frac{De}{\De} \nabla_Q \cdot \left[ \Psi \nabla_x ((Q \cdot \nabla_x T) \ln \Psi) \right] \\
+ \frac{1}{De} \nabla_Q \cdot (T \nabla \Psi) + \frac{2}{De} \nabla_Q \cdot \frac{Q \Psi}{1 - \|Q\|^2/Q_0^2}
\]

(2.11)

Sidebar: using as scaling factor \(L\) for quantities \(Q, Q_0\) and \(\nabla_Q\) instead of \(l_0\) leads to the same dimensionless form (2.11).

### 3 Existence Results for the CPD Equation

#### 3.1 Introducing the Problem and the Main Result

Let \(d \in \{2, 3\}\). Assume \(x \in \Omega \subset \mathbb{R}^d, \Omega_T := \Omega \times (0, T)\). Let the ball \(B(0, Q_0) \subset \mathbb{R}^d, Q_0 > 0,\) and \(Q \in B(0, Q_0)\). Denote \(\Sigma := \Omega \times B(0, Q_0) \subset \mathbb{R}^{2d}, \Sigma_T := \Sigma \times (0, T) \subset \mathbb{R}^{2d+1}\). Also \(De > 0\).

Let \(v : \Omega_T \mapsto \mathbb{R}^d\) denote a smooth enough velocity field s.t. \(\nabla_x \cdot v = 0\) and \(v |_{\partial \Omega} \cdot v = 0\), where \(v\) is the outward normal on \(\partial \Omega\). \(\kappa : \Omega_T \mapsto M_d(\mathbb{R})\) is the smooth enough velocity gradient, \(\theta : \Omega_T \mapsto (0, +\infty)\) a given (known) smooth enough temperature field.

We search for \(f : \Sigma_T \mapsto [0, +\infty), f = f(x, Q, t)\), solution of (see also (2.11))

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{2} \frac{De}{\De} \nabla_x \cdot \left\{ \left[ \nabla_x (\theta \ln f) + \nabla_Q ((Q \cdot \nabla_x \theta) \ln f) \right] f \right\} \\
- \nabla_Q \cdot (\kappa \cdot Q f) + \frac{1}{2} \frac{De}{\De} \nabla_Q \cdot \left[ f \nabla_x ((Q \cdot \nabla_x \theta) \ln f) \right] \\
+ \frac{1}{De} \nabla_Q \cdot (\theta \nabla Q f) + \frac{2}{De} \nabla Q \cdot \frac{Q f}{1 - \|Q\|^2/Q_0^2}
\]

(3.1)

complying with the boundary condition

\[
f |_{\partial \Sigma_T \times (0, T)} = 0
\]

(3.2)

and with the initial condition

\[
f(t = 0) = f_0, \ f_0 \text{ given}
\]

(3.3)

With the (convenient) change of variable \(Q = q Q_0, q \in B(0, 1)\) and letting \(\Sigma = \Omega \times B(0, 1)\) and \(\Sigma_T = \Sigma \times (0, T)\), then the above introduced problem (we shall stick with the same notations for \(f\) and \(f_0\)) can be restated as following: investigate the existence of a solution \(f : \Sigma_T \mapsto [0, +\infty), f = f(x, q, t)\), to the equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{2} \frac{De}{\De} \nabla_x \cdot \left\{ \left[ \nabla_x (\theta \ln f) + \nabla_q ((q \cdot \nabla_x \theta) \ln f) \right] f \right\} \\
- \nabla_q \cdot (\kappa \cdot q f) + \frac{1}{2} \frac{De}{\De} \nabla_q \cdot \left[ f \nabla_x ((q \cdot \nabla_x \theta) \ln f) \right] \\
+ \frac{1}{Q_0 De} \nabla_q \cdot (\theta \nabla q f) + \frac{2}{De} \nabla q \cdot \frac{q f}{1 - \|q\|^2}
\]

(3.4)
With the help of the following calculations

\[
\nabla_x \cdot \left[ f \nabla_x (\theta \ln f) \right] = \nabla_x \cdot (\theta \nabla_x f) + \nabla_x \cdot [(\nabla_x \theta) \cdot (f \ln f)]\\%
\n\nabla_x \cdot \left[ f \nabla_q ((q \cdot \nabla_x \theta) \ln f) \right] = \nabla_x \cdot [(\nabla_x \theta) \cdot (f \ln f)] + \nabla_x \cdot [(q \cdot \nabla_x \theta) \cdot \nabla_q f]\\%
\n\n\nabla_q \cdot \left[ f \nabla_x ((q \cdot \nabla_x \theta) \ln f) \right] = \nabla_q \cdot \left[ (\nabla^2_x \theta \cdot q) \cdot (f \ln f) \right] + \nabla_q \cdot [(q \cdot \nabla_x \theta) \cdot \nabla_x f] \tag{3.5}
\]

the problem under scrutiny is restated below:

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{2} \text{De} \nabla_x \cdot (\theta \nabla_x f) + \frac{1}{\text{De}} \nabla_x \cdot [(\nabla_x \theta) \cdot (f \ln f)]\\%
\]

\[
+ \frac{1}{\text{De}} \nabla_x \cdot [q \cdot (\nabla_x \theta) \cdot \nabla_q f] + \frac{1}{\text{De}} \nabla_q \cdot [(q \cdot \nabla_x \theta) \cdot \nabla_x f]\\%
\]

\[
+ \frac{1}{\text{De}} \nabla_q \cdot \left[ (\nabla^2_x \theta \cdot q) \cdot (f \ln f) \right] + \frac{1}{Q_0^2} \nabla_q \cdot (\theta \nabla_q f)\\%
\]

\[
- \nabla_q \cdot (\kappa \cdot qf) + \frac{2}{\text{De}} \nabla_q \cdot \left( \frac{qf}{1 - \|q\|^2} \right) \tag{3.6}
\]

\[
f |_{\partial \Sigma \times (0, T)} = 0 \tag{3.7}
\]

\[
f(t = 0) = f_0(x, q) \tag{3.8}
\]

where \( f_0 : \Sigma \mapsto (0, +\infty) \) is being given.

Assume

- \( \theta \in L^\infty (0, T; W^{2, \infty}(\Omega)) \), \( \theta(x, t) \geq \theta_{\min} > 0 \) a.e. \( (x, t) \in \Sigma_T \), where \( \theta_{\min} \) is chosen s.t.

\[
\theta_{\min}^2 - 2c_H \theta_{\min} - \|\nabla_x \theta\|^2_{L^\infty(\Sigma_T)} > 0 \tag{3.9}
\]

with \( c_H \) being a Hardy’s inequality constant s.t.

\[
\left\| \frac{\psi}{1 - \|q\|^2} \right\|_{L^2(B(0, 1))} \leq c_H \left\| \nabla_q \psi \right\|_{L^2(B(0, 1))}, \quad \forall \psi \in H^1_0(B(0, 1)) \tag{3.10}
\]

Remark 3.1 The assumption (3.9) is valid insofar either \( Q_0 \) is small enough, or \( \theta_{\min} \) is sufficiently large.

- \( \kappa \in L^\infty (\Omega_T; M_4(\mathbb{R})) \)
- \( \nu \in L^\infty (0, T, H^0_{\text{div}}(\Omega)) \), where \( H^0_{\text{div}} := \{ u \in L^2(\Omega) : \text{div} (u) = 0, \ u \cdot v = 0 \ \text{on} \ \partial \Omega \times (0, T) \} \), where \( v \) is the outward normal
- \( f_0 \in L^2 (\Sigma) \).

We introduce the continuous function \( E : [0, +\infty) \mapsto \mathbb{R} \), such as

\[
E(y) = \begin{cases} 
\ln y & \text{for } y > 0 \\
0 & \text{for } y = 0
\end{cases} \tag{3.11}
\]
A variational formulation of (3.6)–(3.8) is the following: find \( f \in L^2(0, T; H^1_0(\Sigma)) \), \( f \geq 0 \), such that for any \( \varphi \in C^1(0, T) \) with \( \varphi(T) = 0 \), and for any \( \psi \in H^1_0(\Sigma) \) we have

\[
- \int_0^T \int_\Sigma f' \varphi' \psi(x, q) dx dq dt - \int_\Sigma f_0(x, q) \varphi(0) \psi(x, q) dx dq \\
+ \int_0^T \int_\Sigma (v \cdot \nabla f) \varphi \psi dx dq dt + \frac{1}{De} \int_0^T \int_\Sigma \theta (\nabla_x f \cdot \nabla_x \psi) \varphi dx dq dt \\
+ \frac{1}{Q_0^2 De} \int_0^T \int_\Sigma \left( q \cdot \nabla_x \theta \right) (\nabla_q f \cdot \nabla_q \psi) \varphi dx dq dt + \frac{1}{De} \int_0^T \int_\Sigma \left( q \cdot \nabla_x \theta \right) (\nabla_q f \cdot \nabla_q \psi) \varphi dx dq dt \\
+ \frac{1}{Q_0^2 De} \int_0^T \int_\Sigma \nabla_x^2 \theta (q E(f) \cdot \nabla_q \psi) \varphi dx dq dt - \int_0^T \int_\Sigma (\kappa \cdot q f \cdot \nabla_q \psi) \varphi dx dq dt \\
+ \frac{2}{De} \int_0^T \int_\Sigma \left( \frac{q f}{1 + \|q\|^2} \cdot \nabla_x q \psi \right) \varphi dx dq dt = 0
\] (3.12)

Let the functional space \( X_T \) be defined in the following way:

\[
X_T := \left\{ f \in L^2(0, T; H^1_0(\Sigma)) : \frac{df}{dt} \in L^2(0, T; H^{-1}(\Sigma)) \right\}
\]

It is a Banach space endowed with the norm

\[
\| f \|_{X_T} = \| f \|_{L^2(0, T; H^1_0(\Sigma))} + \left\| \frac{df}{dt} \right\|_{L^2(0, T; H^{-1}(\Sigma))}
\]

The hardcore result of this paper is stated below:

**Theorem 3.1** (Main Existence Result) *There exists at least one solution \( f \in X_T, f \geq 0 \) to the problem stated in (3.12).*

We now introduce a regularization to the aforementioned problem (3.6)–(3.8). First, for any small enough \( \epsilon > 0 \), consider the function \( g_{\epsilon} : \mathbb{R} \mapsto \mathbb{R} \),

\[
g_{\epsilon}(z) = \begin{cases} 
\ln \left( \frac{1}{\epsilon} \right) & \text{for } z \geq \frac{1}{\epsilon} \\
\ln z & \text{for } \frac{1}{\epsilon} \leq z \leq \frac{1}{\epsilon} \\
\ln \epsilon & \text{for } \frac{1}{\ln \epsilon} \leq z \leq \epsilon \\
\frac{1}{z} & \text{for } z \leq \frac{1}{\ln \epsilon}
\end{cases}
\] (3.13)
Denote $E_\epsilon : \mathbb{R} \mapsto \mathbb{R}$, $E_\epsilon(z) = zg_\epsilon(z)$. The announced regularized problem reads: find $f_\epsilon : \Sigma_1 \mapsto \mathbb{R}$ that solves

$$
\frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = \frac{1}{\text{De}} \nabla_x \cdot [(q \cdot \nabla_x \theta) \nabla_q f_\epsilon] + \frac{1}{\text{De}} \nabla_x \cdot [(q \cdot \nabla_x \theta) \nabla_q f_\epsilon] + \frac{1}{\text{De}} \nabla_x \cdot [(\nabla_x \theta) E_\epsilon(f_\epsilon)]
$$

$$
+ \frac{1}{\text{De}} \nabla_q \cdot [(\nabla^2_x \theta) q \epsilon f_\epsilon(\tilde{f})] - \nabla_q \cdot (\kappa \cdot q f_\epsilon) + \frac{2}{\text{De}} \nabla q \cdot \left( \frac{f_\epsilon q}{\epsilon + 1 - \|q\|^2} \right) \quad (3.14)
$$

with

$$
f_\epsilon = 0, \text{ on } \partial \Sigma \times (0, T) \quad (3.15)
$$

$$
f_\epsilon(t = 0) = f_0 \text{ on } \Sigma. \quad (3.16)
$$

Of notice: the variational formulation in (3.14) is the same as the one in (3.12) whereupon replacing $f$ by $f_\epsilon$, $E(f)$ by $E_\epsilon(f_\epsilon)$ and $1 - \|q\|^2$ by $1 - \|q\|^2$.

### 3.2 Proof of the Existence of a Solution to the Regularized Problem for Any Small $\epsilon > 0$

We shall make use of a fixed point theorem method.

Let the operator $S_\epsilon : L^2(\Sigma_T) \mapsto L^2(\Sigma_T)$, $S_\epsilon(\tilde{f}) = f_\epsilon$, where $f_\epsilon$ solves the following linear problem:

$$
\frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = \frac{1}{\text{De}} \nabla_x \cdot [(q \cdot \nabla_x \theta) \nabla_q f_\epsilon] + \frac{1}{\text{De}} \nabla_x \cdot [(q \cdot \nabla_x \theta) \nabla_q f_\epsilon] + \frac{1}{\text{De}} \nabla_x \cdot [(\nabla_x \theta) f_\epsilon E_\epsilon(\tilde{f})]
$$

$$
+ \frac{1}{\text{De}} \nabla_q \cdot [(\nabla^2_x \theta) q \epsilon f_\epsilon(\tilde{f})] - \nabla_q \cdot (\kappa \cdot q f_\epsilon) + \frac{2}{\text{De}} \nabla q \cdot \left( \frac{f_\epsilon q}{\epsilon + 1 - \|q\|^2} \right) \quad (3.17)
$$

with

$$
f_\epsilon = 0, \text{ on } \partial \Sigma \times (0, T) \quad (3.18)
$$

$$
f_\epsilon(t = 0) = f_0 \text{ on } \Sigma. \quad (3.19)
$$

In order to avoid cumbersome notations, we denote by $f_\epsilon$ the solution of (3.17) as well as that of (3.14).

#### 3.2.1 Proof of the Existence and Uniqueness of a Variational Solution to Eqs. (3.17)–(3.19)

First we take on to obtaining the variational formulation that corresponds to (3.17)–(3.19). To achieve this, we first introduce the application $a_\epsilon : (0, T) \times L^2(\Sigma) \times H^1_0(\Sigma) \times H^1_0(\Sigma) \mapsto \mathbb{R}$,
\[ a_e (t, r, \psi, \xi) = \int_\Sigma v \cdot \nabla_x \psi \xi + \frac{1}{\text{De}} \int_\Sigma \theta \nabla_x \psi \cdot \nabla_x \xi + \frac{1}{\text{De}} \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_q \psi \cdot \nabla_x \xi) \]
\[ + \frac{1}{\text{De}} \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_x \psi \cdot \nabla_q \xi) + \frac{1}{Q_0^2 \text{De}} \int_\Sigma \theta \nabla_q \psi \cdot \nabla_q \xi + \frac{1}{\text{De}} \int_\Sigma (\nabla_x \theta) \psi g_e (r) \cdot \nabla_x \xi \]
\[ + \frac{1}{\text{De}} \int_\Sigma (\nabla_x \theta) q \psi g_e (r) \cdot \nabla_x \xi - \int_\Sigma \kappa \cdot q \psi \cdot \nabla_q \xi + \frac{2}{\text{De}} \int_\Sigma \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla_q \xi \]

(3.20)

for any \( t \in (0, T), r \in L^2 (\Sigma), \psi, \xi \in H^1_0 (\Sigma). \)

We now use the fact that \( \theta \in W^{2, \infty} (\Omega), \kappa \in L^\infty (\Omega), g_e (r) \in L^\infty (\Sigma), \) and the function \( q \mapsto \frac{1}{\epsilon + 1 - \|q\|^2} \in L^\infty (B (0, 1)). \) By making use of the Cauchy-Schwarz inequality on all terms making up \( a_e \) it is easily deduced that

\[ |a_e (t, \tilde f, \psi, \xi)| \leq c \|\psi\|_{H^1_0 (\Sigma)} \|\xi\|_{H^1_0 (\Sigma)} \; \forall (t, \psi, \xi) \in (0, T) \times (H^1_0 (\Sigma))^2 \]

(3.21)

where \( c \) is a constant independent of \( t \) (but dependent on \( \epsilon \)).

Let \( b : (0, T) \times H^1_0 (\Sigma) \times H^1_0 (\Sigma) \to \mathbb{R} \) be such that:

\[ b (t, \psi, \xi) = \frac{1}{\text{De}} \int_\Sigma \theta \nabla_x \psi \cdot \nabla_x \xi + \frac{1}{\text{De}} \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_q \psi \cdot \nabla_x \xi) \]
\[ + \frac{1}{\text{De}} \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_x \psi \cdot \nabla_q \xi) + \frac{1}{Q_0^2 \text{De}} \int_\Sigma \theta \nabla_q \psi \cdot \nabla_q \xi, \; \forall (t, \psi, \xi) \in (0, T) \times (H^1_0 (\Sigma))^2 \]

We have:

**Lemma 3.1** There exist \( c_1 > 0, c_2 \in \mathbb{R} \) (not independent of \( \epsilon \)) such that

\[ a_e (t, \tilde f, \psi, \psi) \geq c_1 \|\psi\|_{H^1_0 (\Sigma)}^2 - c_2 \|\psi\|_{L^2 (\Sigma)}^2, \forall \psi \in H^1_0 (\Sigma), \forall t \in (0, T). \]

**Proof** Notice now that

\[ b (t, \psi, \psi) = \frac{1}{\text{De}} \int_\Sigma \theta \|\nabla_x \psi\|^2 + \frac{2}{\text{De}} \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_x \psi \cdot \nabla_q \psi) + \frac{1}{Q_0^2 \text{De}} \int_\Sigma \theta \|\nabla_q \psi\|^2 \]

Next, by Cauchy-Schwarz one gets

\[ \left| \int_\Sigma (q \cdot \nabla_x \theta) (\nabla_x \psi \cdot \nabla_q \psi) \right| \leq \|\theta\|_{W^{1, \infty} (\Omega)} \|\nabla_x \psi\|_{L^2 (\Sigma)} \|\nabla_q \psi\|_{L^2 (\Sigma)} \]

from which one obtains

\[ b (t, \psi, \psi) \geq \frac{\theta_{\text{min}}}{\text{De}} \|\nabla_x \psi\|_{L^2 (\Sigma)}^2 + \frac{\theta_{\text{min}}}{Q_0^2 \text{De}} \|\nabla_q \psi\|_{L^2 (\Sigma)}^2 \]

\[ - \frac{2}{\text{De}} \|\nabla_x \theta\|_{L^\infty (\Omega)} \|\nabla_x \psi\|_{L^2 (\Sigma)} \|\nabla_q \psi\|_{L^2 (\Sigma)} \]

Let the symmetric matrix

\[ A = \begin{pmatrix} \frac{\theta_{\text{min}}}{\text{De}} & -\frac{1}{\text{De}} \|\nabla_x \theta\|_{L^\infty (\Omega)} \\ -\frac{1}{\text{De}} \|\nabla_x \theta\|_{L^\infty (\Omega)} & \frac{\theta_{\text{min}}}{Q_0^2 \text{De}} \end{pmatrix} \]
with the help of which the above inequality can be re-written as
\[ b(t, \psi, \psi) \geq \langle Az, z \rangle \]
with
\[ \mathbb{R}^2 \ni z = \left( \frac{\| \nabla_x \psi \|_{L^2(\Sigma)}}{\| \nabla_q \psi \|_{L^2(\Sigma)}} \right) \]

Clearly \( A \) is a symmetric and positive definite matrix due to (3.9).

Invoking the above assumption leads to the existence of a \( \tilde{\lambda}_m > 0 \) depending on the problem data and such that
\[ b(t, \psi, \psi) \geq \tilde{\lambda}_m \| \psi \|_{H^1_0(\Sigma)}^2, \quad \forall \psi \in H^1_0(\Sigma) \]

Using now Poincaré’s inequality one deduces the existence of a \( \lambda_m > 0 \) such that
\[ b(t, \psi, \psi) \geq \lambda_m \| \psi \|_{H^1_0(\Sigma)}^2, \quad \forall \psi \in H^1_0(\Sigma) \] (3.22)

We now proceed to upper bound the terms in \( a_\epsilon(t, \psi, \psi) \) that do not appear in the definition of \( b(t, \psi, \psi) \). Upon using the Cauchy-Schwarz’s inequality it is easily seen there exists \( c_\epsilon \) such that
\[
\left| \int_\Sigma (v \cdot \nabla_x \psi) \psi + \frac{1}{\text{De}} \int_\Sigma (\nabla_x \theta) \psi g_e(\tilde{f}) \cdot \nabla_x \psi + \frac{1}{\text{De}} \int_\Sigma (\nabla^2_x \theta) \psi g_e(\tilde{f}) q \cdot \nabla_x \psi 
- \int_\Sigma \kappa \cdot q \psi \cdot \nabla_q \psi + \frac{2}{\text{De}} \int_\Sigma \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla_q \psi \right| 
\leq c_\epsilon \| \psi \|_{L^2(\Sigma)} \| \psi \|_{H^1(\Sigma)}^2, \quad \forall (t, \psi) \in (0, T) \times H^1_0(\Sigma)
\]

Next on, for any \( \eta > 0 \), by the Young’s inequality we obtain
\[ c_\epsilon \| \psi \|_{L^2(\Sigma)} \| \psi \|_{H^1(\Sigma)} \leq \eta \| \psi \|_{H^1(\Sigma)}^2 + \frac{1}{4\eta} c_\epsilon^2 \| \psi \|_{L^2(\Sigma)}^2 \]

We then deduce that
\[
\int_\Sigma (v \cdot \nabla_x \psi) \psi + \frac{1}{\text{De}} \int_\Sigma (\nabla_x \theta) \psi g_e(\tilde{f}) \cdot \nabla_x \psi + \frac{1}{\text{De}} \int_\Sigma (\nabla^2_x \theta) \psi g_e(\tilde{f}) q \cdot \nabla_x \psi 
- \int_\Sigma \kappa \cdot q \psi \cdot \nabla_q \psi + \frac{2}{\text{De}} \int_\Sigma \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla_q \psi 
\geq -\eta \| \psi \|_{H^1(\Sigma)}^2 - \frac{1}{4\eta} c_\epsilon^2 \| \psi \|_{L^2(\Sigma)}^2
\]

Taking now \( \eta = \frac{\lambda_m}{2} \) and using (3.22) we obtain the result taking \( c_1 = \frac{\lambda_m}{2} \) and \( c_2 = \frac{c_\epsilon^2}{2\lambda_m} \).

A variational formulation of (3.17)–(3.19) is the following: find \( f_\epsilon \in L^2(0, T; H^1_0(\Sigma)) \cap L^\infty(0, T; L^2(\Sigma)) \) solution to
\[
\frac{d}{dt} (f_\epsilon, \psi)_{L^2(\Sigma)} + a_\epsilon(t, \tilde{f}, f_\epsilon, \psi) = 0, \quad \forall \psi \in H^1_0(\Sigma) \] (3.23)

with
\[ f_\epsilon(t = 0) = f_0 \] (3.24)
Remark that (3.23) is to be understood in the sense of distributions. For any \( \psi \in H_0^1(\Sigma) \), and for any \( \varphi \in \mathcal{C}^1_0(0, T) \), \( \varphi(T) = 0 \), using (3.24) one has:

\[
-(f_0, \psi)_{L^2(\Sigma)} \varphi(0) - \int_0^T (f_\varepsilon, \psi)_{L^2(\Sigma)} \varphi(t)\,dt + \int_0^T a_\varepsilon(t, \tilde{f}_\varepsilon, \psi) \varphi(t)\,dt = 0 \tag{3.25}
\]

Theorem 4.1 on page 257 together with Remark 4.3 on page 258 of [23] grants the existence of a unique solution to (3.25) (due to (3.21) and Lemma 3.1).

Remark that we can introduce the function \( A_\varepsilon : (0, T) \times L^2(\Sigma) \times H_0^1(\Sigma) \mapsto H^{-1}(\Sigma) \), \( A_\varepsilon = A_\varepsilon(t, \tilde{f}, \psi) \), in the following way: \( H_0^1(\Sigma) \ni \xi \mapsto A_\varepsilon \in L^2(0, T; H^{-1}(\Sigma)) \) due to (3.21). Then the mapping \( t \mapsto A_\varepsilon(t, \tilde{f}(t), f_\varepsilon(t)) \) is an element of \( L^2(0, T; H^{-1}(\Sigma)) \) because \( f_\varepsilon \in L^2(0, T; H_0^1(\Sigma)) \). Then Eq. (3.23) can be re-written as

\[
\frac{d}{dt} f_\varepsilon + A_\varepsilon(t, \tilde{f}, f_\varepsilon) = 0 \tag{3.26}
\]

hence \( \frac{d}{dt} f_\varepsilon \in L^2(0, T; H^{-1}(\Sigma)) \).

Therefore the solution \( f_\varepsilon \) is such that \( f_\varepsilon \in X_T \).

Moreover, (3.24) is meaningful because of the continuous embedding \( X_T \subset \mathcal{C} ((0, T); L^2(\Sigma)) \). Therefore the mapping \( S_\varepsilon \) is well defined. Remember that we also have the compact embedding \( X_T \subset L^2(\Sigma_T) \).

### 3.2.2 Estimates for the Solution to the Problem (3.17)–(3.19)

Since \( f_\varepsilon \in L^2(0, T; H_0^1(\Sigma)) \) and because \( L^2(0, T; H^{-1}(\Sigma)) \) is the dual space of \( L^2(0, T; H_0^1(\Sigma)) \), we apply (3.26) to \( f_\varepsilon \). One has

\[
\left\langle \frac{df_\varepsilon}{dt}, f_\varepsilon \right\rangle_{\mathcal{C}^0(0, T)} = \frac{1}{2} \frac{d}{dt} \left( \| f_\varepsilon \|^2_{L^2(\Sigma)} \right)
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \left( \| f_\varepsilon \|^2_{L^2(\Sigma)} \right) + a_\varepsilon(t, f_\varepsilon, f_\varepsilon) = 0
\]

Using the result stated in Lemma 3.1 gives

\[
\frac{d}{dt} \left( \| f_\varepsilon \|^2_{L^2(\Sigma)} \right) + 2c_1 \| f_\varepsilon \|^2_{H^1(\Sigma)} + 2c_2  \| f_\varepsilon \|^2_{L^2(\Sigma)} \leq 2c_2 \| f_\varepsilon \|^2_{L^2(\Sigma)} \tag{3.27}
\]

Further use of Gronwall’s inequality on (3.27) entails

\[
\| f_\varepsilon \|^2_{L^2(\Sigma)} \leq \| f_0 \|^2_{L^2(\Sigma)} e^{2c_2 T}, \forall t \in (0, T)
\]

Next, integrating (3.27) w.r.t. \( t \in (0, T) \) gives

\[
\| f_\varepsilon \|_{L^2(0, T; H^1(\Sigma))} \leq \sqrt{\frac{T}{2c_1}} \| f_0 \|_{L^2(\Sigma)} e^{c_2 T}
\]

and with the help of (3.26) one gets

\[
\left\| \frac{df_\varepsilon}{dt} \right\|_{L^2(0, T; H^{-1}(\Sigma))} \leq c_3.
\]
We then deduce the existence of a constant $c = c(\epsilon)$ s.t.:

$$\|f_\epsilon\|_{X_T} \leq c$$  \hfill (3.28)

### 3.2.3 Proof of the Fixed Point Result

Schauder’s fixed-point Theorem is used to proving the existence of at least one variational solution to the problem (3.14)–(3.16). From (3.28) we have that $S_\epsilon (L^2(\Sigma_T))$ is relatively compact in $L^2(\Sigma_T)$. All constitutive requirements of Schauder’s fixed-point Theorem are met save for the continuity of $S_\epsilon$, fact we shall ascertain in the following.

**Lemma 3.2** $S_\epsilon$ is a continuous mapping from $L^2(\Sigma)$ to $L^2(\Sigma)$.

**Proof** Let $\tilde{f} \in L^2(\Sigma_T)$ be a fixed element, and consider a converging sequence

$$L^2(\Sigma_T) \ni f_k \xrightarrow{k \to \infty} \tilde{f}$$

Denote $f_{\epsilon,k} = S_\epsilon (f_k)$ and $f_\epsilon = S_\epsilon (\tilde{f})$. We need to prove that

$$f_{\epsilon,k} \xrightarrow{k \to \infty} f_\epsilon$$

We have, as in (3.28), that

$$\|f_{\epsilon,k}\|_{X_T} \leq c$$

where $c$ may depend on $\epsilon$ but not on $k$. From the property of compactness we infer there exist $\hat{f}_\epsilon \in X_T$, and a subsequence (also denoted by) $f_{\epsilon,k}$ s.t.

$$f_{\epsilon,k} \xrightarrow{k \to \infty} \hat{f}_\epsilon$$

We now prove that passing to the limit in (3.29), for $k \to \infty$, leads to obtaining (3.25) with $f_\epsilon$ being replaced by $\hat{f}_\epsilon$. All the limit related calculations are obvious due to the established weak convergence $f_{\epsilon,k} \xrightarrow{k \to \infty} \hat{f}_\epsilon$, excepting the following convergences:

$$\int_0^T \int_{\Sigma} \nabla_x \theta_{f_{\epsilon,k},g_e} \left(f_k\right) \cdot \nabla_x \xi \varphi(t) \xrightarrow{k \to \infty} \int_0^T \int_{\Sigma} \nabla_x \theta_{f_\epsilon,g_e} \left(\tilde{f}\right) \cdot \nabla_x \xi \varphi(t)$$  \hfill (3.30)

$$\int_0^T \int_{\Sigma} \nabla_x^2 \theta_{f_{\epsilon,k},g_e} \left(f_k\right) \cdot \nabla_x \xi \varphi(t) \xrightarrow{k \to \infty} \int_0^T \int_{\Sigma} \nabla_x^2 \theta_{f_\epsilon,g_e} \left(\tilde{f}\right) \cdot \nabla_x \xi \varphi(t)$$  \hfill (3.31)
The above convergences hold true in wake of the strong convergence $g_\epsilon (\tilde{f}_k) \xrightarrow{k \to \infty} g_\epsilon (\tilde{f})$, which is manifest in view of the fact that the function $z \mapsto g_\epsilon (z)$ is an element of $W^{1,\infty} (\mathbb{R})$.

We eventually obtain the desired limit problem being satisfied by $\hat{f}_\epsilon$. Moreover, the uniqueness of $\hat{f}_\epsilon$ tells that all sequences $\{f_{\epsilon,k}\}_{k \in \mathbb{N}}$ converge towards $\hat{f}_\epsilon$, fact that ends the proof.

Therefore, by Schauder’s fixed-point Theorem we have a variational solution to the regularized problem (3.14)–(3.16).

### 3.3 Estimates Uniform in $\epsilon$

We draw some inspiration from [20] and from [7] for the obtention of $L^1$–type estimates. Here we obtain $\epsilon$-free estimates for the solution $f_\epsilon$ that solves (3.14)–(3.16), in order to calculate the limit for $\epsilon \to 0$. Moreover, the solution $f_\epsilon$ is in fact a variational solution for it solves (see also (3.25) and (3.12))

\[
- (f_0, \psi)_{L^2(\Sigma)} \varphi(0) - \int_0^T (f_\epsilon, \psi)_{L^2(\Sigma)} \varphi(t) dt + \int_0^T a_\epsilon (t, f_\epsilon, f_\epsilon, \psi) \varphi(t) dt = 0
\]

for any $\psi \in H^1_0 (\Sigma)$, and for any $\varphi \in C^1 (0, T)$ s.t. $\varphi(T) = 0$. This means that $f_\epsilon \in X_T$ solves

\[
\frac{df_\epsilon}{dt} + A_\epsilon (t, f_\epsilon, f_\epsilon) = 0
\]

#### 3.3.1 $L^1 (\Sigma)$ Estimates

For any $\eta > 0$ we introduce the functions:

- an approximation of the function $\text{sgn} (y)$
  \[
  \beta_\eta : \mathbb{R} \mapsto \mathbb{R}, \quad \beta_\eta (y) := \frac{y}{\sqrt{y^2 + \eta}}
  \]

- an approximation of function $|y|$
  \[
  \gamma_\eta : \mathbb{R} \mapsto \mathbb{R}, \quad \gamma_\eta (y) = \sqrt{y^2 + \eta}
  \]

Remark that $\beta_\eta, \gamma_\eta \in C^\infty (\mathbb{R})$ and that

\[
\gamma'\eta = \beta_\eta, \quad \beta'\eta (y) = \frac{\eta}{(y^2 + \eta)^{3/2}} > 0, \quad \forall y \in \mathbb{R}
\]

Making use of (3.33) upon $\beta_\eta (f_\epsilon)$ gives

\[
\left( \frac{df_\epsilon}{dt}, \beta_\eta (f_\epsilon) \right) + a_\epsilon (t, f_\epsilon, f_\epsilon, \beta_\eta (f_\epsilon)) = 0
\]

Also remark that as $\beta_\eta \in W^{1,\infty} (\mathbb{R})$, $\beta_\eta (0) = 0$, it implies $\beta_\eta (f_\epsilon) \in L^2 (0, T; H^1_0 (\Sigma))$. 

\textcopyright Springer
Since \( \gamma'_{\eta} (f_{\epsilon}) = \beta_{\eta} (f_{\epsilon}) \), then
\[
\left\langle \frac{d f_{\epsilon}}{d t}, \beta_{\eta} (f_{\epsilon}) \right\rangle = \frac{d}{d t} \int_{\Sigma} \gamma_{\eta} (f_{\epsilon}) \, dx \, dq
\]
We now have
\[
b_{\epsilon} (t, f_{\epsilon}, \beta_{\eta} (f_{\epsilon})) = \frac{1}{De} \int_{\Sigma} \beta'_{\eta} (f_{\epsilon}) \| \nabla x f_{\epsilon} \|^2 + \frac{2}{De} \int_{\Sigma} \beta'_{\eta} (f_{\epsilon}) (q \cdot \nabla x \theta) (\nabla q f_{\epsilon} \cdot \nabla x f_{\epsilon})
\]
\[
+ \frac{1}{Q_{\eta}^2 De} \int_{\Sigma} \beta'_{\eta} (f_{\epsilon}) \| \nabla q f_{\epsilon} \|^2
\]
\[
\geq \int_{\Sigma} \beta'_{\eta} (f_{\epsilon}) \left[ \frac{1}{De} \theta_{\min} \| \nabla x f_{\epsilon} \|^2 - \frac{2}{De} \| \nabla x \theta \|_{L^\infty(\Omega)} \| \nabla x f_{\epsilon} \| \| \nabla q f_{\epsilon} \|
\right]
\]
\[
+ \frac{1}{Q_{\eta}^2 De} \theta_{\min} \| \nabla q f_{\epsilon} \|^2
\]
Due to the assumption (3.9) we have
\[
b_{\epsilon} (t, f_{\epsilon}, \beta_{\eta} (f_{\epsilon})) \geq 0
\]
Remark that, owing to the assumption made on \( v \), one has
\[
\int_{\Sigma} v \cdot \nabla x f_{\epsilon} \beta_{\eta} (f_{\epsilon}) = \int_{\Sigma} v \cdot \nabla x \gamma_{\eta} (f_{\epsilon}) = - \int_{\Sigma} \nabla x \cdot v \gamma_{\eta} (f_{\epsilon})
\]+\int_{\partial \Sigma} (v \cdot \nu) \gamma_{\eta} (f_{\epsilon}) = 0
\]
Then, from (3.34) we get
\[
\frac{d}{d t} \int_{\Sigma} \gamma_{\eta} (f_{\epsilon}) \, dx \, dq + \frac{1}{De} \int_{\Sigma} f_{\epsilon} g_{\eta} (f_{\epsilon}) \cdot \beta'_{\eta} (f_{\epsilon}) \nabla x f_{\epsilon} \cdot (2 \nabla x \theta + q \nabla^2 x \theta)
\]
\[
- \int_{\Sigma} \kappa \cdot q f_{\epsilon} \beta'_{\eta} (f_{\epsilon}) \nabla x f_{\epsilon} + \frac{1}{De} \int_{\Sigma} \frac{f_{\epsilon} q}{\epsilon + 1 - \|q\|^2} \cdot \beta'_{\eta} (f_{\epsilon}) \nabla q f_{\epsilon} \leq 0
\]
We now integrate w.r.t. \( t \) from 0 to a arbitrarily fixed \( t \in (0, T) \) and take the limit \( \eta \to 0 \).
We shall make repeated use of Lemma 3.2 of [7] with \( h = f_{\epsilon} \). We deduce that
\[
\lim_{\eta \to 0} \left\{ \int_{\Sigma} f_{\epsilon} g_{\eta} (f_{\epsilon}) \cdot \beta'_{\eta} (f_{\epsilon}) \nabla x f_{\epsilon} \cdot (2 \nabla x \theta + q \nabla^2 x \theta) \frac{1}{De}
\]
\[
- \kappa \cdot q \cdot \nabla x f_{\epsilon} + \frac{2}{De} \frac{q}{\epsilon + 1 - \|q\|^2} \cdot \nabla q f_{\epsilon} \right\} = 0
\]
We also have that
\[
\lim_{\eta \to 0} \int_{\Sigma} \gamma_{\eta} (f_{\epsilon}) \, dx \, dq = \int_{\Sigma} |f_{\epsilon}| \, dx \, dq, \text{ a.e.} t \in (0, T)
\]
By the use of Lebesgue’s Dominated Convergence Theorem we have that
\[
\| f_{\epsilon} \|_{L^\infty(0, T; L^1(\Sigma))} \leq \| f_0 \|_{L^1(\Sigma)}
\]
3.3.2 Estimates Uniform in $\epsilon$ in Functional Space $X_T$

Apply (3.3) to $f_\epsilon$ to get

$$\frac{1}{2} \frac{d}{dt} \int_{\Sigma} f_\epsilon^2 + a_\epsilon (t, f_\epsilon, f_\epsilon, f_\epsilon) = 0$$  \hfill (3.36)

Due to the assumption on $v$ one has

$$\int_{\Sigma} (v \cdot \nabla x f_\epsilon) f_\epsilon = \frac{1}{2} \int_{\Sigma} v \cdot \nabla (f_\epsilon)^2 = -\frac{1}{2} \int_{\Sigma} (\nabla_x \cdot v) f_\epsilon^2 + \frac{1}{2} \int_{\partial \Sigma} (v \cdot \nabla_x f_\epsilon) f_\epsilon^2 = 0$$  \hfill (3.37)

Next, integrating (3.10) on $\Omega$ gives

$$\left\| \frac{\psi}{1 - \|q\|^2} \right\|_{L^2(\Sigma)} \leq c_H \left\| \nabla q \psi \right\|_{L^2(\Sigma)}, \forall \psi \in H_0^1(B(0, 1))$$  \hfill (3.38)

Then we have

$$\left| \int_{\Sigma} \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla q \psi \right| \leq \int_{\Sigma} \frac{1}{1 - \|q\|^2} |\psi| \left\| \nabla q \psi \right\| \leq \left\| \frac{\psi}{1 - \|q\|^2} \right\|_{L^2(\Sigma)} \left\| \nabla q \psi \right\|_{L^2(\Sigma)}$$  \hfill (3.39)

Then

$$b(t, \psi, \psi) + \frac{2}{De} \int_{\Sigma} \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla q \psi \geq \frac{\theta_{\min}}{De} \int_{\Sigma} \| \nabla x \|_{L^\infty(\Omega)} \int_{\Sigma} \| \nabla_x \psi \| \| \nabla q \psi \| + \left( \frac{\theta_{\min}}{Q_0^2 De} - \frac{2 c_H}{De} \right) \int_{\Sigma} \| \nabla q \psi \|^2$$

where

$$B = \begin{pmatrix} \frac{\theta_{\min}}{De} & -\| \nabla x \|_{L^\infty(\Omega)} \frac{\theta_{\min}}{Q_0^2 De} & -\frac{\theta_{\min}}{De} \frac{2 c_H}{De} \end{pmatrix}$$

Clearly $B$ is a symmetric positive definite matrix due to assumption (3.9).

Then one deduces there exists a $\Lambda_M > 0$ s.t.

$$b(t, \psi, \psi) + \frac{2}{De} \int_{\Sigma} \frac{\psi q}{\epsilon + 1 - \|q\|^2} \cdot \nabla q \psi \geq \Lambda_M \| \psi \|^2_{H^1(\Sigma)}, \forall \psi \in H_0^1(\Sigma)$$  \hfill (3.40)

From the above, together with (3.36), (3.37), one obtains

$$\frac{1}{2} \frac{d}{dt} \int_{\Sigma} f_\epsilon^2 + \Lambda_M \| f_\epsilon \|^2_{H^1(\Sigma)} \leq \int_{\Sigma} \kappa \cdot q f_\epsilon \cdot \nabla q f_\epsilon + \frac{1}{De} \int_{\Sigma} f_\epsilon g_\epsilon (f_\epsilon) \cdot \nabla x f_\epsilon (2 \nabla_x \theta + q \nabla_x^2 \theta)$$  \hfill (3.41)
Observe now that for any \( \delta > 0 \), there exists a \( c(\delta) \geq 0 \) (independent of \( \epsilon \)), s.t.
\[
z | \ln z| \leq c(\delta) + z^{1+\delta}, \forall z > 0
\]

Since
\[
|g_\epsilon (z)| \leq | \ln z|, \forall z > 0
\]
then
\[
z |g_\epsilon (z)| \leq c(\delta) + z^{1+\delta}, \forall z > 0
\]

On the other hand now,
\[
z |g_\epsilon (z)| \leq 1, \forall z < 0
\]

From the above it follows that: for any \( \delta > 0 \), there exists a \( c(\delta) \geq 0 \) (independent of \( \epsilon \)), s.t.
\[
z g_\epsilon (z) | \leq c(\delta) + |z|^{1+\delta}, \forall z \in \mathbb{R}
\]

Now, from (3.41) and capitalizing on (3.42) gives
\[
\frac{1}{2} \frac{d}{dt} \int_\Sigma f_\epsilon^2 + \Lambda_M \| f_\epsilon \|^2_{H^1(\Sigma)} \leq \frac{\Lambda_M}{4} \| q f_\epsilon \|^2_{L^2(\Sigma)} + \frac{1}{\Lambda_M} \| \kappa \|^2_{L^\infty(\Omega)} \| f_\epsilon \|^2_{L^2(\Sigma)}
\]

Using the fact that
\[
\int_\Sigma \| \nabla x f_\epsilon \| \leq |\text{mes}(\Sigma)|^{1/2} \| \nabla x f_\epsilon \|_{L^2(\Sigma)}
\]
and that
\[
\frac{3}{\text{De}} c(\delta) |\text{mes}(\Sigma)|^{1/2} \| \theta \|_{W^{2,\infty}(\Omega)} \| \nabla x f_\epsilon \|_{L^2(\Sigma)} \leq \frac{\Lambda_M}{4} \| \nabla x f_\epsilon \|^2_{L^2(\Sigma)}
\]

Next, using (3.43) we further obtain
\[
\frac{d}{dt} \int_\Sigma f_\epsilon^2 + \Lambda_M \| f_\epsilon \|^2_{H^1(\Sigma)} \leq c_4 \| f_\epsilon \|^2 + c_5 + \frac{3}{\text{De}} \| \theta \|_{W^{2,\infty}(\Omega)} \int_\Sigma |f_\epsilon|^{1+\delta} |f_\epsilon| \| \nabla x f_\epsilon \| \|
\]

By selecting now a \( \delta \) s.t. \( 0 < \delta < 1/2 \), one gets
\[
\int_\Sigma |f_\epsilon|^{1+\delta} |f_\epsilon| \| \nabla x f_\epsilon \| \leq \| f_\epsilon \|^{1+\delta}_{L^{1/\delta}(\Sigma)} \| f_\epsilon \| \| \nabla x f_\epsilon \|_{L^2(\Sigma)}
\]

Since, by (3.35)
\[
\| f_\epsilon \|^2_{L^{1/\delta}(\Sigma)} = \| f_\epsilon \|^{\delta}_{L^1(\Sigma)} \leq \| f_0 \|^\delta_{L^1(\Sigma)}
\]
then
\[
\int_\Sigma |f_\epsilon|^{1+\delta} |f_\epsilon| \| \nabla x f_\epsilon \| \leq \| f_0 \|^{\delta}_{L^1(\Sigma)} \| f_\epsilon \|^{1+\delta}_{L^{1/\delta}(\Sigma)} \| \nabla x f_\epsilon \|_{L^2(\Sigma)}
\]
Next, by Sobolev’s inclusions, taking a $\delta > 0$ small enough, there exists a $\delta_1 \in (0, 1)$ s.t.
\[
\| f_\varepsilon \|_{L^{\frac{2}{1+\delta}}(\Omega)} \leq c(\delta_1) \| f_\varepsilon \|_{H^{\delta_1}(\Omega)}, \quad c(\delta_1) > 0
\]

By interpolation we have
\[
\| f_\varepsilon \|_{H^{\delta_1}(\Omega)} \leq c(\delta_2) \| f_\varepsilon \|_{L^2(\Sigma)}^{1-\delta_1} \| f_\varepsilon \|_{H^1(\Sigma)}^{\delta_1}
\]

Now, from (3.44) and (3.45)
\[
\frac{d}{dt} \int_\Sigma f_\varepsilon^2 + \Lambda M \| f_\varepsilon \|_{L^2(\Sigma)}^2 \leq c_4 \| f_\varepsilon \|_{L^2(\Sigma)}^2 + c_5 + 3 \| f_0 \|_{H^{\delta_1}(\Omega)} \| f_\varepsilon \|_{H^{\delta_1}(\Sigma)} \| f_\varepsilon \|_{H^{1+\delta_1}(\Sigma)}
\]

By Young’s inequality and for any $\eta > 0$,
\[
\frac{3}{\eta \text{De}} \| f_0 \|_{L^1(\Sigma)} \| f_\varepsilon \|_{L^2(\Sigma)} \| f_\varepsilon \|_{H^1(\Sigma)} \| f_\varepsilon \|_{H^{1+\delta_1}(\Sigma)} \leq \frac{1}{2} \left( \eta \| f_\varepsilon \|_{H^{1+\delta_1}(\Sigma)} \right)^{\frac{2}{1+\delta_1}} + \frac{1-\delta_1}{2} \left[ \frac{3}{\eta \text{De}} \| f_0 \|_{L^1(\Sigma)} \| f_\varepsilon \|_{L^2(\Sigma)} \right]^{\frac{2}{1+\delta_1}}
\]

Taking $\eta > 0$ small enough gives
\[
\frac{d}{dt} \int_\Sigma f_\varepsilon^2 + \frac{\Lambda M}{2} \| f_\varepsilon \|_{H^1(\Sigma)}^2 \leq c_5 + c_6 \| f_\varepsilon \|_{L^2(\Sigma)}^2
\]

By Gronwall’s inequality we deduce (proceeding in a classical manner) that there exists a constant $c > 0$ (which is independent of $\varepsilon$ but depending upon $T$) s.t.
\[
\| f_\varepsilon \|_{L^\infty(0,T;L^2(\Sigma))} + \| f_\varepsilon \|_{L^2(0,T;H^1(\Sigma))} \leq c
\]

From (3.46) and (3.33) we can also prove that
\[
\left\| \frac{df_\varepsilon}{dt} \right\|_{L^\infty(0,T;H^{-1}(\Sigma))} \leq c
\]

Actually, to prove (3.47) above, observe that $|a_\varepsilon(t, f_\varepsilon, f_\varepsilon, \psi)| \leq c \| f_\varepsilon \|_{H^1(\Sigma)} \| \psi \|_{H^1(\Sigma)}$ due to Hardy’s inequality (3.38) and to (3.42) in which we set $\delta = 1$.

Eventually, from (3.46) and (3.47), we see that
\[
\| f_\varepsilon \|_{X_T} \leq c
\]

### 3.3.3 Proof of the Non-negativity of $f_\varepsilon$

We make the (physically sound) assumption that
\[
f_0 \geq 0
\]

Let now $f_\varepsilon$ be expressed as
\[
f_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-, \quad f_\varepsilon^+ = \max\{f_\varepsilon, 0\}, \quad f_\varepsilon^- = -\min\{f_\varepsilon, 0\}
\]

Our goal is now to prove that $f_\varepsilon^- = 0$ (so that $f_\varepsilon = f_\varepsilon^+ \geq 0$). We first apply (3.33) to $f_\varepsilon^-$ and this gives
\[
\left\{ \frac{df_\varepsilon^-}{dt}, f_\varepsilon^- \right\} + a_\varepsilon(t, f_\varepsilon, f_\varepsilon, f_\varepsilon^-) = 0
\]
We now have (by a density-type argument) that
\[
\left\langle \frac{df}{dt}, f^{-\epsilon} \right\rangle = -\frac{1}{2} \frac{d}{dt} \| f^{-\epsilon} \|_{L^2(\Sigma)}^2
\]
Next, since \( \nabla_x f^+ \cdot \nabla_x f^- = 0 \)
\[
b \left( f^+, f^- \right) = b \left( f^+ - f^-, f^- \right) = -b \left( f^-, f^- \right)
\]
and
\[
\int_{\Sigma} (v \cdot \nabla_x f^-) f^- = -\int_{\Sigma} (v \cdot \nabla_x f^-) f^- = 0
\]
Next, for any \( \eta > 0 \),
\[
\frac{1}{\text{De}} \left| \int_{\Sigma} g_e (f^-) \nabla_x f^- \cdot (2 \nabla_x \theta + \nabla_x^2 \theta \cdot q) \right|
\]
\[
= \frac{1}{\text{De}} \left| \int_{\Sigma} g_e (f^-) \nabla_x f^- \cdot (2 \nabla_x \theta + \nabla_x^2 \theta \cdot q) \right|
\]
\[
\leq c(\epsilon) \| f^- \|_{L^2(\Sigma)} \| \nabla_x f^- \|_{L^2(\Sigma)} \leq \eta \| \nabla_x f^- \|_{L^2(\Sigma)}^2 + \frac{c(\epsilon)}{4\eta} \| f^- \|_{L^2(\Sigma)}
\]
Using that \( \frac{1}{\epsilon} + 1 - \| q \|^2 \leq \frac{1}{\epsilon} \), then for any \( \eta > 0 \) one also gets
\[
\left| -\int_{\Sigma} \kappa \cdot q f_{\epsilon} \nabla_q f_{\epsilon} + \frac{2}{\text{De}} \int_{\Sigma} \frac{f_{\epsilon}}{\epsilon + 1 - \| q \|^2} q \cdot \nabla_q f_{\epsilon} \right| \leq \| \nabla_x f^- \|_{L^2(\Sigma)}^2
\]
With the help of inequality (3.22) and by taking \( \eta > 0 \) small enough we obtain
\[
\frac{d}{dt} \| f_{\epsilon}^- \|_{L^2(\Sigma)}^2 \leq c(\epsilon) \| f_{\epsilon}^- \|_{L^2(\Sigma)}^2
\]
Since \( f_{\epsilon}^- (t = 0) = 0 \), use of Gronwall’s inequality leads to \( f_{\epsilon}^- = 0 \), or put it differently,
\[
f_{\epsilon} \geq 0 \quad (3.50)
\]

3.4 Performing the Limit \( \epsilon \to 0 \) and Proving the Main Result

It is now undertook to proving Theorem 3.1.

Proof The estimates of Sects. 3.3.2 and 3.3.3 allow to deduce the existence of a \( f \in X_T \), \( f \geq 0 \), s.t. we have (up to a subsequence of \( \epsilon \), for simplicity also denoted by \( \epsilon \))
\[
f_{\epsilon} \overset{L^2(0,T;H^1_0(\Sigma))}{\to} f, \text{ weakly}
\]
\[
f_{\epsilon} \overset{L^\infty(0,T;L^2(\Sigma))}{\to} f, \text{ weakly - *}
\]
\[
\frac{df_{\epsilon}}{dt} \overset{L^2(0,T;H^{-1}(\Sigma))}{\to} \frac{df}{dt}, \text{ weakly}
\]
\[
f_{\epsilon} \overset{L^2(\Sigma_T)}{\to} f, \text{ strongly, by compactness.} \quad (3.51)
\]
We now pass to the limit for $\epsilon \to 0$ in the variational formulation given in (3.14), (3.15), (3.16), which is:

\[
- \int_0^T \int_{\Sigma} f_\epsilon(x, q) \varphi'(t) - \int_\Sigma f_0(x, q) \varphi(0) \varphi(x, q) + \int_0^T \int_{\Sigma} (v \cdot \nabla_x f_\epsilon) \varphi
\]

\[
+ \int_0^T \int_{\Sigma} b(t, f_\epsilon, \psi) + \frac{1}{De} \int_0^T \int_{\Sigma} f_\epsilon g_\epsilon(f_\epsilon) \nabla_x \psi \cdot (2\nabla_x \theta + \nabla^2_x \theta \cdot q) \varphi
\]

\[
- \int_0^T \int_{\Sigma} (\kappa \cdot q f_\epsilon \cdot \nabla_q \psi) \varphi + \frac{2}{De} \int_0^T \int_{\Sigma} \left( \frac{f_\epsilon}{\epsilon + 1 - \|q\|^2} q \cdot \nabla_q \psi \right) \varphi = 0,
\]

\[
\forall \psi \in H_0^1(\Sigma), \forall \varphi \in C^1(0, T), \varphi(T) = 0 \quad (3.52)
\]

Let $\psi \in \mathcal{D}(\Sigma)$ in (3.52). Due to the convergences stated in (3.51) we get

\[
- \int_0^T \int_{\Sigma} f_\epsilon \varphi'(t) + \int_0^T \int_{\Sigma} (v \cdot \nabla_x f_\epsilon) \varphi + \int_0^T b(t, f_\epsilon, \psi) \varphi
\]

\[
- \int_0^T \int_{\Sigma} (\kappa \cdot q f_\epsilon \cdot \nabla_q \psi) \varphi
\]

\[
\to \epsilon \to 0 - \int_0^T \int_{\Sigma} f \varphi' + \int_0^T \int_{\Sigma} (v \cdot \nabla_x f) \varphi
\]

\[
+ \int_0^T b(t, f, \psi) \varphi - \int_0^T \int_{\Sigma} (\kappa \cdot q f \cdot \nabla_q \psi) \varphi \quad (3.53)
\]

Let us now prove the convergence

\[
\int_0^T \int_{\Sigma} E_\epsilon(f_\epsilon) \nabla_x \psi \cdot (2\nabla_x \theta + \nabla^2_x \theta \cdot q) \varphi \to \epsilon \to 0 \int_0^T \int_{\Sigma} E(f) \nabla_x \psi \cdot (2\nabla_x \theta + \nabla^2_x \theta \cdot q) \varphi \quad (3.54)
\]

To achieve this, it suffices to prove the strong convergence

\[
E_\epsilon(f_\epsilon) \overset{L^2(\Sigma_T)}{\to} E(f)
\]

From the strong convergence

\[
f_\epsilon \overset{\epsilon \to 0}{\to} f
\]

we deduce that (up to a subsequence of $\epsilon$)

\[
f_\epsilon(x, t) \overset{L^2(\Sigma_T)}{\to} f(x, t), \text{ a.e. } (x, t) \in (\Sigma_T)
\]

and

\[
|f_\epsilon(x, t)| \leq h(x, t), \text{ a.e. } (x, t) \in \Sigma_T, h \in L^2(\Sigma_T)
\]

Observe function $E$ is decreasing on $(0, 1/e)$, and increasing on $(1/e, +\infty)$. Then,

\[
|E(f_\epsilon(x, t))| \leq \frac{1}{\epsilon}, \text{ for } 0 \leq f_\epsilon(x, t) \leq 1
\]

and

\[
|E(f_\epsilon(x, t))| \leq E(h(x, t)), \text{ for } f_\epsilon(x, t) > 1
\]
and

$$|E_{\epsilon}(z)| \leq |E(z)|, \quad \forall z \geq 0$$

It follows that for any $\delta > 0$, there exists $c(\delta) > 0$ independent of $\epsilon$, s.t.

$$|E_{\epsilon}(f_{\epsilon}(x, t))| \leq c(\delta) + h^\delta(x, t), \quad \text{a.e. } (x, t) \in \Sigma_T$$

(3.55)

Next, consider $(x, t) \in \Sigma_T$ s.t. $f(x, t) > 0$. Taking $\epsilon$ small enough we have

$$g_{\epsilon}(f_{\epsilon}(x, t)) = \ln(f_{\epsilon}(x, t))$$

Then, by continuity

$$f_{\epsilon}(x, t)g_{\epsilon}(f_{\epsilon}(x, t)) \xrightarrow{\epsilon \to 0} f(x, t) \ln(f(x, t)) = E(f(x, t))$$

Let us now consider $(x, t) \in \Sigma_T$ for which $f(x, t) = 0$. Then,

$$|f_{\epsilon}(x, t)g_{\epsilon}(f_{\epsilon}(x, t))| \leq |f_{\epsilon}(x, t)\ln(f_{\epsilon}(x, t))| = |E(f_{\epsilon}(x, t))| \xrightarrow{\epsilon \to 0} E(f(x, t)) = 0$$

Then, for a.e. $(x, t) \in \Sigma_T$ we have

$$E_{\epsilon}(f_{\epsilon}(x, t)) \xrightarrow{\epsilon \to 0} E(f(x, t))$$

With the help of (3.55) and upon using Lebesgue’s Dominated Convergence Theorem one gets

$$E_{\epsilon}(f_{\epsilon}) \xrightarrow{\epsilon \to 0} L^2(\Sigma_T) E(f), \quad \text{strongly}$$

which ends the announced proof for (3.54).

What is left over now to prove is the validity of the convergence

$$\int_0^T \int_\Sigma \left( \frac{f_{\epsilon}}{\epsilon + 1 - \|q\|^2} q \cdot \nabla q \psi \right) \phi \xrightarrow{\epsilon \to 0} \int_0^T \int_\Sigma \left( \frac{f}{1 - \|q\|^2} q \cdot \nabla q \psi \right) \phi$$

(3.56)

We have

$$\left\| \frac{1}{\epsilon + 1 - \|q\|^2} \nabla q \psi - \frac{1}{1 - \|q\|^2} \nabla q \psi \right\| = \frac{\epsilon}{(1 - \|q\|^2)(\epsilon + 1 - \|q\|^2)} \| \nabla q \psi \|

\leq \frac{\epsilon}{(1 - \|q\|^2)^2} \| \nabla q \psi \|

$$

Since $\psi \in \mathcal{D}(\Sigma)$, then there exists a $\delta_2 > 0$ s.t. $1 - \|q\|^2 \geq \delta_2$ whenever $\nabla q \psi(x, q) \neq 0$. Then the function

$$\frac{1}{(1 - \|q\|^2)^2} \| \nabla q \psi \| \in L^\infty(\Sigma)$$

Therefore

$$\frac{1}{\epsilon + 1 - \|q\|^2} \nabla q \psi \xrightarrow{\epsilon \to 0} \frac{1}{1 - \|q\|^2} \nabla q \psi, \quad \text{strongly}$$
which implies the statement in (3.56). We then obtain from (3.52), (3.53), (3.54), and (3.56), that for any \( \psi \in \mathcal{D}(\Sigma) \) and for any \( \varphi \in \mathcal{C}^1(0, T) \), s.t. \( \varphi(T) = 0 \),

\[
- \int_0^T \int_{\Sigma} f \psi \varphi' - \int_{\Sigma} f_0 \varphi(0) \psi + \int_0^T \int_{\Sigma} (v \cdot \nabla f) \psi \varphi \\
+ \int_0^T \int_{\Sigma} b(t, f, \psi) \varphi + \frac{1}{De} \int_0^T \int_{\Sigma} E(f) \nabla \psi \cdot (2 \nabla \theta + q \nabla^2 \theta) \varphi \\
- \int_0^T \int_{\Sigma} (\kappa \cdot q f \cdot \nabla \psi) \varphi + \frac{2}{De} \int_0^T \int_{\Sigma} \left( \frac{f}{1 - \|q\|^2} q \cdot \nabla \psi \right) \varphi = 0 \tag{3.57}
\]

Remark that \( E(f) \in L^\infty(0, T; L^2(\Omega)) \) (because \( |E(z)| \leq c(1 + z^2) \), for any \( z \geq 0 \)), and that \( \frac{f}{1 - \|q\|^2} \in L^2(\Sigma_T) \) (because \( f \in L^2(0, T; H^1_0(\Sigma)) \)) and due to Hardy’s inequality). Then, because \( \mathcal{D}(\Sigma) \) is densely included into \( H^1_0(\Sigma) \), it allows to deduce that (3.57) is also valid for any \( \psi \in H^1_0(\Sigma) \), fact that ends the proof.

\[ \square \]

4 Conclusions

In this paper we extended the early work of Curtiss and Bird [14] on kinetic theory describing the temperature influence on the dynamics of polymer fluids. Specifically, we derived a new configurational probability equation without the originally proposed “linear gradient approximation”, fact that may account for some shortcomings pointed out in the original work [14]. The resulting transport equation of this work is consequently non-linear and (hence) more complex in nature compared to the original (linear) one. As a first step towards putting it to work for practical purposes we proved the existence of positive variational solutions.

Subsequent work devoted to obtaining the corresponding temperature diffusion equation by accounting for the various molecular interactions responsible for the heat that is conveyed by the fluid is the focus of a forthcoming paper.

Acknowledgements The Authors were deeply saddened by the untimely passing of Professor Geneviève Raugel whom they got to know and took benefit of her outstanding scientific skills and, also, to appreciate her generous human and personality stances. The Authors are grateful to the anonymous Referee for useful and insightful comments on the originally submitted manuscript. L. I. Palade thanks Professor Cătălin Radu Picu, RPI, Troy (NY), for clarifying talks on polymer dynamics.

References

1. Barrett, J.W., Süli, E.: Existence and equilibration of global weak solutions to kinetic models for dilute polymers II: Hookean-type models. Math. Models Methods Appl. Sci. 22(5), 1150024 (2012)
2. Barrett, J.W., Süli, E.: Existence of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers with variable density and viscosity. J. Differ. Equ. 253, 3610–3677 (2012)
3. Bird, R.B., Armstrong, R.C., Hassager, O.: Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics. Wiley, New-York (1987)
4. Bird, R.B., Armstrong, R.C., Hassager, O.: Dynamics of Polymeric Liquids, Vol. 2: Kinetic Theories. Wiley, New-York (1987)
5. Busuioec, A.V., Ciuperca, I.S., Iftimie, D., Palade, L.I.: The FENE dumbbell polymer model: existence and uniqueness of solutions for the momentum balance equation. J. Dyn. Differ. Equ. 26(2), 217–241 (2014)
6. Ciuperca, I.S., Heibig, A.: Existence and uniqueness of a density probability solution for the stationary Doi–Edwards equation. Annales de l’Institut Henri Poincaré - Analyse Non-Linéaire 19, 2039–2064 (2016)
7. Ciuperca, I.S., Heibig, A., Palade, L.I.: Existence and uniqueness results for the Doi–Edwards polymer melt model: the case of the (full) nonlinear configurational probability density equation. Nonlinearity 25(4), 991–1009 (2012)
8. Ciuperca, I.S., Heibig, A., Palade, L.I.: On the IAA version of the Doi–Edwards model versus the K-BKZ rheological model for polymer fluids: a global existence result for shear flows with small initial data. Eur. J. Appl. Math. 28(1), 42–90 (2017)
9. Ciuperca, I.S., Hingant, E., Palade, L.I., Pujo-Menjouet, L.: Fragmentation and monomers lengthening of rod-like polymers, a relevant model of prion proliferation. Discrete Continuous Dyn. Syst. B 17(3), 775–799 (2012)
10. Ciuperca, I.S., Palade, L.I.: The steady state configurational distribution diffusion equation of the standard FENE dumbbell polymer model: existence and uniqueness of solutions for arbitrary velocity gradients. Math. Models Methods Appl. Sci. 33(5), 1353–1373 (2009)
11. Ciuperca, I.S., Palade, L.I.: On the existence and uniqueness of solutions of the configurational probability diffusion equation for the generalized rigid dumbbell polymer model. Dyn. Partial Differ. Equ. 7, 245–263 (2010)
12. Ciuperca, I.S., Palade, L.I.: Asymptotic behavior of the solution of the distribution diffusion equation for FENE dumbbell polymer model. Math. Model. Nat. Phenomena 6(5), 84–97 (2011)
13. Ciuperca, I.S., Palade, L.I.: A turning point asymptotic expansion for a rigid-dumbbell polymer fluid probability configurational equation for fast shear flows. Asympt. Anal. 105(1–2), 45–76 (2017)
14. Curtiss, C.F., Bird, R.B.: Statistical mechanics of transport phenomena: polymeric liquid mixtures. Adv. Polym. Sci. 125, 1–101 (1996)
15. Cleja-Țigoiu, S., Tigoiu, V.: Rheology and Thermodynamics, Part I - Rheology, Editura Universității din București (1998)
16. Fu, Q., Hu, T., Yang, L.: Instability of a weakly viscoelastic film flowing down a heated inclined plane. Phys. Fluids 30, 084102 (2018)
17. Jbara, L.M., Jeffrey-Giacomin, A.: Macromolecular tumbling and wobbling in large-amplitude oscillatory shear flow. Phys. Fluids 31, 021214 (2019)
18. Jourdain, B., Le Bris, C., Lelievre, T., Otto, F.: Long-time asymptotics of a multiscale model for polymeric fluid flows. Arch. Ration. Mech. Anal. 181(1), 97–148 (2006)
19. Kanso, M.A., Jeffrey-Giacomin, A., Saengow, C., Piette, J.H.: Macromolecular architecture and complex viscosity. Phys. Fluids 31, 087107 (2019)
20. Kružkov, S.N.: First order quasilinear equations in several independent variables. Math. USSR-Sbornik 10(2), 217–243 (1970)
21. Lin, Y.-H.: Polymer Viscoelasticity: Basics, Molecular Theories and Experiments, 2nd edn. World Scientific, Singapore (2010)
22. Lin, F., Zhang, P., Zhang, Z.: On the global existence of smooth solution to the 2D FENE-dumbell model. Commun. Math. Phys. 277, 531–553 (2008)
23. Lions, J.-L., Magenes, E.: Problèmes aux Limites Non-homogènes et Applications, vol. 1. Dunod, Paris (1968)
24. Mackay, A.T., Philips, T.N.: On the derivation of macroscopic models for compressible viscoelastic fluids using the generalized bracket framework. J. Nonnewton. Fluid Mech. 266, 59–71 (2019)
25. Morrison, F.A.: Understanding Rheology. Oxford University Press, Oxford (2001)
26. Murdock, A.J.: Perturbations. Theory and Methods. SIAM, Philadelphia (1999)
27. Palade, L.I.: On slow flows of the full nonlinear Doi–Edwards polymer model. Zeitschrift für Angewandte Mathematik und Physik ZAMP 65, 139–148 (2014)
28. Piette, J.H., Jbara, L.M., Saengow, C., Jeffrey-Giacomin, A.: Exact coefficients for rigid dumbbell suspensions for steady shear flow material function expansions. Phys. Fluids 31, 021212 (2019)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.