H. CARTAN’S THEORY FOR RIEMANN SURFACES

XIANJING DONG

ABSTRACT. We generalize the H. Cartan’s theory of holomorphic curves for a general open Riemann surface. Besides, a vanishing theorem for jet differentials and a Bloch’s theorem for Riemann surfaces are obtained.

1. Introduction

1.1. Main results.

The value distribution theory \[21, 23\] for holomorphic mappings has arisen strong research interest after R. Nevanlinna \[18\] who founded two celebrated fundamental theorems for meromorphic functions in one complex variable in 1925. Many well-known results were obtained by extending source manifolds and target manifolds, see H. Cartan \[8\], L. V. Ahlfors \[11\], Carlson-Griffiths \[6\], Griffiths-King \[14\], E. I. Nochka \[20\], J. Noguchi \[21\], M. Ru \[23, 24, 25\], etc.. Particularly, H. Cartan’s theory for holomorphic curves is an important branch of Nevanlinna theory, which was devised by H. Cartan in 1933. Let us review some key developments of this theory related closely to our study of the paper. In 1970, H. Wu \[29\] gave a generalization of H. Cartan’s theory for the parabolic Riemann surfaces (admitting a parabolic exhaustion function), see also B. V. Shabat \[27\]. In 1983, E. I. Nochka \[20\] established the Second Main Theorem of holomorphic curves into \(\mathbb{P}^n(\mathbb{C})\) intersecting hyperplanes in \(N\)-subgeneral position. Recently in 2019, Ru-Sibony \[25\] studied this theory for the discs in \(\mathbb{C}\).

Due to the lack of Green-Jensen formula, difficulties arise in extending H. Cartan’s theory to an arbitrary Riemann surface by employing the classical approaches. So, we are motivated by such problem: What is the H. Cartan’s theory for Riemann surfaces? We aim to consider this problem following the stochastic technique initiated by T. K. Carne \[7\] and developed by A. Atsuji \[2, 3\]. In this paper, we give an extension of H. Cartan’s theory for a general Riemann surface. Note that the classical Second Main Theorem for the discs

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was presented in terms of incomplete metric induced from Euclidean metric on \( \mathbb{C} \), see [25]. This incomplete metric produces a boundary term (main error term) in the Second Main Theorem, and hence it cannot describe the value distribution for holomorphic curves defined on a general hyperbolic Riemann surface without boundary, and even it fails for the multi-connected domains of \( \mathbb{C} \) though their universal covering is the unit disc. In the paper, we adopt a complete metric and receive a Second Main Theorem of holomorphic curves defined on the discs. Instead of the classical boundary condition, we receive a defect relation by means of a geometric condition. As applications, one also obtains a vanishing theorem for jet differentials as well as a Bloch’s theorem for Riemann surfaces. In what follows, we introduce the main results.

Let \( S \) be an open (connected) Riemann surface. Due to the uniformization theorem, one could equip \( S \) with a complete Hermitian metric 
\[
\dd s^2 = 2g(z)dz \bar{dz},
\]
such that the Gauss curvature \( K_S \leq 0 \) associated to \( g \), here \( K_S \) is given by
\[
K_S = -\frac{1}{4} \Delta_S \log g = -\frac{1}{g} \frac{\partial^2 \log g}{\partial z \partial \bar{z}}.
\]
Clearly, \((S, g)\) is a complete Kähler manifold with associated Kähler form 
\[
\alpha = g \frac{\sqrt{-1}}{2\pi} dz \wedge d \bar{z}.
\]
Fix \( o \in S \) as a reference point. Denote by \( D(r) \) the geodesic disc centered at \( o \) with radius \( r \), and by \( \partial D(r) \) the boundary of \( D(r) \). By Sard’s theorem, \( \partial D(r) \) is a submanifold of \( S \) for almost all \( r > 0 \). Set
\[
(1) \quad \kappa(t) = \min \{ K_S(x) : x \in \overline{D(t)} \},
\]
which is a non-positive, decreasing and continuous function on \([0, \infty)\).

**Theorem 1.1.** Let \( f: S \to \mathbb{P}^n(\mathbb{C}) \) be a linearly non-degenerate holomorphic curve. Let \( H_1, \cdots, H_q \) be hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in \( N \)-subgeneral position. Then
\[
(q - 2N + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + O \left( \log T_f(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]
The term \( \kappa(r)r^2 \) comes from geometric nature of \( S \). Let \( S = \mathbb{C} \), we obtain \( \kappa(r) \equiv 0 \) and \( T_f(r) \geq O(\log r) \) as \( r \to \infty \). Thus, it covers Nochka’s result:

**Corollary 1.2 (Nochka, [20]).** Let \( H_1, \cdots, H_q \) be hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in \( N \)-subgeneral position. Let \( f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be a linearly non-degenerate
holomorphic curve. Then

$$(q - 2N + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + O(\log T_f(r))\|.$$ 

If $S$ is the Poincaré disc $\mathbb{D}$, then $\kappa(r) \equiv -1$. We can receive a more precise estimate $O(r)$ instead of $O(r^2)$.

**Corollary 1.3.** Let $f : \mathbb{D} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve. Let $H_1, \cdots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position. Then

$$(q - 2N + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + O(\log T_f(r) + r)\|.$$ 

The value distribution of holomorphic curves defined on a disc was studied by Ru-Sibony [25]. But, differently here, we use a complete metric while Ru-Sibony used the incomplete one induced from $\mathbb{C}$. We here point out that our result has the same accuracy with Ru-Sibony’s one. It’s sufficient to compare the main error terms. Let $r(z)$ denote the poincaré distance of $z$ from 0 for $z \in \mathbb{D}$, and let $r, r_E$ be the Poincaré and Euclidean radius respectively. Note that

$$r(z) = \log \frac{1 + |z|}{1 - |z|}. \quad (2)$$

In the classical case, the main error term is $O(\log(1 - r_E)^{-1})$. By (2)

$$\log \frac{1}{1 - r_E} \leq r = \log \frac{1 + r_E}{1 - r_E} < \log \frac{1}{1 - r_E} + \log 2.$$ 

Therefore, the two main error terms are equivalent.

**Corollary 1.4** (Defect relation). Assume the same conditions as in Theorem 1.1. If $f$ satisfies the growth condition

$$\lim_{r \to \infty} \frac{\kappa(r)r^2}{T_f(r)} = 0,$$

then

$$\sum_{j=1}^{q} \delta_f^{[n]}(H_j) \leq 2N - n + 1.$$ 

In particular, if $S$ is the Poincaré disc, then the conclusion holds only if

$$\lim_{r \to \infty} \frac{r}{T_f(r)} = 0.$$ 

We introduce another two results. Let $M$ be a compact complex manifold with an ample divisor $A$ on $M$. 
**Theorem 1.5.** Let $\omega$ be a logarithmic $k$-jet differential on $M$ which vanishes on $A$. Let $f : S \to M$ be a holomorphic curve such that $f(S)$ is disjoint from the log-poles of $\omega$. If $f$ satisfies the growth condition
\[
\liminf_{r \to \infty} \frac{\kappa(r)r^2}{T_{f,A}(r)} = 0,
\]
then $f^*\omega \equiv 0$ on $S$. In particular, if $S$ is the Poincaré disc, then the conclusion holds only if
\[
\limsup_{r \to \infty} \frac{r}{T_{f,A}(r)} = 0.
\]

Theorem 1.5 gives an extension of the classical vanishing theorem referred to Green-Griffiths [12], Siu-Yeung [28] and Demailly [10], etc. Recently, Ru-Sibony [25] proved a version for discs with incomplete metric.

Let $T$ be an $n$-dimensional complex torus with the universal covering $\mathbb{C}^n$. The standard coordinate $z = (z_1, \ldots, z_m)$ of $\mathbb{C}^n$ induces a global coordinate on $T$, still denoted by $z$. Denote $\beta := \frac{dd^c\|z\|^2}{2\pi}$ which is a positive $(1, 1)$-form on $T$. As a consequence of Theorem 1.5, a generalized Bloch’s theorem can be obtained as follows

**Theorem 1.6.** Let $f : S \to T$ be a holomorphic curve such that
\[
\liminf_{r \to \infty} \frac{\kappa(r)r^2}{T_{f,\beta}(r)} = 0.
\]

Then either $f(S)$ is the translate of a subtorus of $T$, or there exists a variety $W$ of general type and a mapping $R : f(S) \to W$ such that $R \circ f$ does not satisfy (3). In particular, if $S$ is the Poincaré disc, then the conclusion holds only if
\[
\limsup_{r \to \infty} \frac{r}{T_{f,\beta}(r)} = 0.
\]

1.2. **Brownian motions.**

Let $(M, g)$ be a Riemannian manifold with Laplace-Beltrami operator $\Delta_M$ associated to $g$. For $x \in M$, we denote by $B_x(r)$ the geodesic ball centered at $x$ with radius $r$, and denote by $S_x(r)$ the geodesic sphere centered at $x$ with radius $r$. Apply Sard’s theorem, $S_x(r)$ is a submanifold of $M$ for almost all $r > 0$. A Brownian motion $X_t$ in $M$ is a heat diffusion process generated by $\Delta_M/2$ with transition density function $p(t, x, y)$ being the minimal positive fundamental solution of the heat equation
\[
\frac{\partial}{\partial t}u(t, x) - \frac{1}{2}\Delta_Mu(t, x) = 0.
\]

We denote by $\mathbb{P}_x$ the law of $X_t$ started at $x \in M$ and by $\mathbb{E}_x$ the corresponding expectation with respect to $\mathbb{P}_x$. 
A. Co-area formula

Let $D$ be a bounded domain with smooth boundary $\partial D$ in $M$. Fix $x \in D$, we use $d\pi^D_x$ to denote the harmonic measure on $\partial D$ with respect to $x$. This measure is a probability measure. Set

$$\tau_D := \inf\{t > 0 : X_t \not\in D\}$$

which is a stopping time. Denote by $g_D(x, y)$ the Green function of $\Delta_M/2$ for $D$ with Dirichlet boundary condition and a pole at $x$, namely

$$-\frac{1}{2} \Delta_M g_D(x, y) = \delta_x(y), \ y \in D; \ g_D(x, y) = 0, \ y \in \partial D,$$

where $\delta_x$ is the Dirac function. For $\phi \in C_\delta(D)$ (space of bounded continuous functions on $D$), the co-area formula [5] asserts that

$$E_x \left[ \int_0^{\tau_D} \phi(X_t) dt \right] = \int_D g_D(x, y) \phi(y) dV(y).$$

From Proposition 2.8 in [5], we also have the relation of harmonic measures and hitting times that

$$(4) \quad E_x[\psi(X_{\tau_D})] = \int_{\partial D} \psi(y) d\pi^D_x(y)$$

for any $\psi \in C_\delta(D)$. Thanks to the expectation $E_x$, the co-area formula and (4) still work in the case when $\phi$ or $\psi$ has a pluripolar set of singularities.

B. Dynkin formula

Let $u \in C^2(M)$ (space of bounded $C^2$-class functions on $M$), we have the famous Itô formula (see [2, 16, 17, 19])

$$u(X_t) - u(x) = B \left( \int_0^t \|\nabla_M u\|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \ P_x - a.s.$$

where $B_t$ is the standard Brownian motion in $\mathbb{R}$ and $\nabla_M$ is gradient operator on $M$. Take expectation of both sides of the above formula, it follows Dynkin formula (see [2, 19])

$$E_x[u(X_T)] - u(x) = \frac{1}{2} E_x \left[ \int_0^T \Delta_M u(X_t) dt \right]$$

for a stopping time $T$ such that each term makes sense. Notice that Dynkin formula still holds for $u \in C^2(M)$ if $T = \tau_D$. In further, it also works when $u$ is of a pluripolar set of singularities, particularly, for a plurisubharmonic function $u$. 
2. First Main Theorem

Let \((S,g)\) be an open Riemann surface with Kähler form \(\alpha\) associated to the Hermitian metric \(g\). Fix \(o \in S\) as a reference point, we denote by \(g_r(o,x)\) the Green function of \(\Delta_S/2\) for \(D(r)\) with Dirichlet boundary condition and a pole at \(o\), and by \(d\pi^r_o(x)\) the harmonic measure on \(\partial D(r)\) with respect to \(o\). Let \(X_t\) be the Brownian motion with generator \(\Delta_S/2\) starting from \(o \in S\). Moreover, we set the stopping time

\[
\tau_r = \inf \{ t > 0 : X_t \not\in D(r) \}.
\]

2.1. Nevanlinna’s functions.

Let \(f : S \to M\) be a holomorphic curve into a compact complex manifold \(M\). We introduce the generalized Nevanlinna’s functions over Riemann surface \(S\). Let \(L \to M\) be an ample holomorphic line bundle equipped with Hermitian metric \(h\). We define the characteristic function of \(f\) with respect to \(L\) by

\[
T_{f,L}(r) = \pi \int_{D(r)} g_r(o,x) f^* c_1(L,h) = -\frac{1}{4} \int_{D(r)} g_r(o,x) \Delta_S \log h \circ f dV(x),
\]

where \(dV(x)\) is the Riemannian volume measure of \(S\). It can be easily known that \(T_{f,L}(r)\) is independent of the choices of metrics on \(L\), up to a bounded term. Since a holomorphic line bundle can be represented as the difference of two ample holomorphic line bundles, the definition of \(T_{f,L}(r)\) can extend to an arbitrary holomorphic line bundle. Apply co-area formula, we have

\[
T_{f,L}(r) = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log h \circ f(X_t) dt \right].
\]

For simplicity, we use notation \(T_{f,D}(r)\) to stand for \(T_{f,L_D}(r)\) for a divisor \(D\) on \(M\). Similarly, a divisor can also be written as the difference of two ample divisors. The Weil function of \(D\) is well defined by

\[
\lambda_D(x) = -\log \| s_D(x) \|
\]

up to a bounded term, here \(s_D\) is the canonical section associated to \(D\). We define the proximity function of \(f\) with respect to \(D\) by

\[
m_f(r,D) = \int_{\partial D(r)} \lambda_D \circ f d\pi^r_o(x).
\]

A relation between harmonic measures and hitting times shows that

\[
m_f(r,D) = \mathbb{E}_o[\lambda_D \circ f(X_{\tau_r})].
\]
Locally, we write $s_D = \tilde{s}_De$, where $e$ is a local holomorphic frame of $(L_D, h)$. The counting function of $f$ with respect to $D$ is defined by

$$N_f(r, D) = \pi \sum_{x \in f^*D \cap D(r)} g_r(o, x)$$

$$= \pi \int_{D(r)} g_r(o, x) dd^c \left[ \log |\tilde{s}_D \circ f(x)|^2 \right]$$

$$= \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

in the sense of distributions or currents.

Our definition of Nevanlinna’s functions in above is natural. When $S = \mathbb{C}$, the Green function is $(\log \frac{r}{|z|})/\pi$ and the harmonic measure is $d\theta/2\pi$. So, by integration by part, we observe that they agree with the classical ones.

2.2. First Main Theorem.

With the previous preparatory work, we are ready to prove the First Main Theorem of a holomorphic curve $f : S \rightarrow M$ such that $f(o) \not\in \text{Supp}D$, where $D$ is a divisor on $M$. Apply Dynkin formula to $\lambda_D \circ f(x)$, it yields that

$$E_o[\lambda_D \circ f(X_t)] - \lambda_D \circ f(o) = \frac{1}{2} E_o \left[ \int_0^{r_v} \Delta_S \lambda_D \circ f(X_t) dt \right].$$

The first term on the left hand side of the above equality is equal to $m_f(r, D)$, and the term on the right hand side equals

$$\frac{1}{2} E_o \left[ \int_0^{r_v} \Delta_S \lambda_D \circ f(X_t) dt \right] = \frac{1}{2} \int_{D(r)} g_r(o, x) \Delta_S \log \frac{1}{\|s_D \circ f(x)\|^2} dV(x)$$

due to co-area formula. Since $\|s_D\|^2 = h|\tilde{s}_D|^2$, where $h$ is a Hermitian metric on $L_D$, then we get

$$\frac{1}{2} E_o \left[ \int_0^{r_v} \Delta_S \lambda_D \circ f(X_t) dt \right] = -\frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log h \circ f(x) dV(x)$$

$$- \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

$$= T_{f,D}(r) - N_f(r, D).$$

Therefore, we obtain the First Main Theorem:

**Theorem 2.1.** Let $f : S \rightarrow M$ be a holomorphic curve with $f(o) \not\in \text{Supp}D$. Then

$$T_{f,D}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

**Corollary 2.2** (Nevanlinna’s inequality). We have

$$N_f(r, D) \leq T_{f,D}(r) + O(1).$$
3. TWO KEY LEMMAS

Let \((S, g)\) be a simply-connected and complete open Riemann surface with Gauss curvature \(K_S \leq 0\) associated to \(g\). Since uniformization theorem, there is a nowhere-vanishing holomorphic vector field \(X\) over \(S\). In fact, it is known from \([15]\), a nowhere-vanishing holomorphic vector field always exists for an arbitrary open Riemann surface.

3.1. Calculus Lemma.

Let \(\kappa\) be defined by (1). As is noted before, \(\kappa\) is a non-positive, decreasing continuous function on \([0, \infty)\). Associate the ordinary differential equation
\[
G''(t) + \kappa(t)G(t) = 0; \quad G(0) = 0, \quad G'(0) = 1.
\]
We compare (5) with \(y''(t) + \kappa(0)y(t) = 0\) under the same initial conditions, \(G\) can be easily estimated as
\[
G(t) = t \quad \text{for } \kappa \equiv 0; \quad G(t) \geq t \quad \text{for } \kappa \not\equiv 0.
\]
This implies that
\[
G(r) \geq r \quad \text{for } r \geq 0; \quad \int_1^r \frac{dt}{G(t)} \leq \log r \quad \text{for } r \geq 1.
\]
On the other hand, we rewrite (5) as the form
\[
\log' G(t) \cdot \log G'(t) = -\kappa(t).
\]
Since \(G(t) \geq t\) is increasing, then the decrease and non-positivity of \(\kappa\) imply that for each fixed \(t\), \(G\) must satisfy one of the following two inequalities
\[
\log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t > 0; \quad \log G'(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t \geq 0.
\]
Since \(G(t) \to 0\) as \(t \to 0\), by integration, \(G\) is bounded from above by
\[
G(r) \leq r \exp \left( r \sqrt{-\kappa(r)} \right) \quad \text{for } r \geq 0.
\]

The main result of this subsection is the following Calculus Lemma:

**Theorem 3.1 (Calculus Lemma).** Let \(k \geq 0\) be a locally integrable function on \(S\) such that it is locally bounded at \(o \in S\). Then for any \(\delta > 0\), there exists a constant \(C > 0\) independent of \(k, \delta\), and a subset \(E_\delta \subseteq (1, \infty)\) of finite Lebesgue measure such that

\[
\mathbb{E}_o[k(X_{\tau_r})] \leq \frac{F(\hat{k}, \kappa, \delta)e^{r\sqrt{-\kappa(r)}} \log r}{2\pi C \mathbb{E}_o \left[ \int_0^{\tau_r} k(X_t)dt \right]} \quad \text{holds for } r > 1 \text{ outside } E_\delta, \quad \text{where } \kappa \text{ is defined by (1) and } F \text{ is defined by}
\]
\[
F(\hat{k}, \kappa, \delta) = \left\{ \log^+ \hat{k}(r) \cdot \log^+ \left( re^{r\sqrt{-\kappa(r)}} \hat{k}(r) \left\{ \log^+ \hat{k}(r) \right\}^{1+\delta} \right) \right\}^{1+\delta}
\]
with
\[ \hat{k}(r) = \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^{r_e} k(X_t) dt \right]. \]

Moreover, we have the estimate
\[ \log F(\hat{k}, \kappa, \delta) \leq O \left( \log \log E_o \left[ \int r_0 \kappa(X_t) dt \right] + \log \sqrt{-\kappa(r)} + \log \log \right). \]

To prove theorem 3.1 we need to prepare some lemmas.

Lemma 3.2 (3). Let \( \eta > 0 \) be a constant. Then there is a constant \( C > 0 \) such that for \( r > \eta \) and \( x \in B_o(r) \setminus \overline{B_o(\eta)} \)
\[ g_r(o, x) \int_{\eta}^{r} dt G(t) \geq C \int_{r(x)}^{r} dt G(t) \]
holds, where \( G \) is defined by (5).

Lemma 3.3 (23). Let \( T \) be a strictly positive nondecreasing function of \( \mathcal{C}^1 \) class on \( (0, \infty) \). Let \( \gamma > 0 \) be a number such that \( T(\gamma) \geq e \), and \( \phi \) be a strictly positive nondecreasing function such that
\[ c_\phi = \int_e^{\infty} \frac{1}{t \phi(t)} dt < \infty. \]

Then, the inequality
\[ T'(r) \leq T(r) \phi(T(r)) \]
holds for all \( r \geq \gamma \) outside a subset of Lebesgue measure not exceeding \( c_\phi \).
In particular, take \( \phi(t) = \log^{1+\delta} t \) for a number \( \delta > 0 \), we have
\[ T'(r) \leq T(r) \log^{1+\delta} T(r) \]
holds for all \( r > 0 \) outside a subset \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

We are ready to prove Theorem 3.1:

Proof. We follow the arguments of Atsuji (3). The simple-connectedness and non-positivity of Gauss curvature of \( S \) imply the following relation (see [9])
\[ d\pi_o^r(x) \leq \frac{1}{2\pi r} d\sigma_r(x), \]
where $d\sigma_r(x)$ is the induced volume measure on $\partial D(r)$. By Lemma 3.2 and (6), we have

$$
\mathbb{E}_o \left[ \int_0^{\tau_r} k(X_t) dt \right] = \int_{D(r)} g_r(o,x) k(x) dV(x)
= \int_0^r dt \int_{\partial D(t)} g_r(o,x) k(x) d\sigma_t(x)
\geq C \int_0^r \frac{r^G(s)}{G(s)} ds \int_{\partial D(t)} k(x) d\sigma_t(x)
\geq \frac{C}{\log r} \int_0^r dt \int_\delta \frac{ds}{G(s)} \int_{\partial D(t)} k(x) d\sigma_t(x),
$$

$$
\mathbb{E}_o[k(X_{\tau_r})] = \int_{\partial D(r)} k(x) d\pi_r(x) \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x) d\sigma_r(x).
$$

Hence,

$$
\mathbb{E}_o \left[ \int_0^{\tau_r} k(X_t) dt \right] \geq \frac{C}{\log r} \int_0^r \frac{r^G(s)}{G(s)} ds \int_{\partial D(t)} k(x) d\sigma_t(x),
$$

(8)

$$
\mathbb{E}_o[k(X_{\tau_r})] \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x) d\sigma_r(x).
$$

Set

$$
\Lambda(r) = \int_0^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x) d\sigma_t(x).
$$

We conclude that

$$
\Lambda(r) \leq \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^{\tau_r} k(X_t) dt \right] = \hat{k}(r).
$$

Since

$$
\Lambda'(r) = \frac{1}{G(r)} \int_0^r dt \int_{\partial D(t)} k(x) d\sigma_t(x),
$$

then it yields from (8) that

$$
\mathbb{E}_o[k(X_{\tau_r})] \leq \frac{1}{2\pi r} \frac{d}{dr} \left( \Lambda'(r) G(r) \right).
$$

By Lemma 3.3 twice and (7), then for any $\delta > 0$

$$
\frac{d}{dr} \left( \Lambda'(r) G(r) \right)
\leq G(r) \left\{ \log^+ \Lambda(r) \cdot \log^+ \left( G(r) \Lambda(r) \{ \log^+ \Lambda(r) \}^{1+\delta} \right) \right\}^{1+\delta} \Lambda(r)
\leq re^r \sqrt{-\kappa(r)} \left\{ \log^+ \hat{k}(r) \cdot \log^+ \left( re^r \sqrt{-\kappa(r)} \hat{k}(r) \{ \log^+ \hat{k}(r) \}^{1+\delta} \right) \right\}^{1+\delta} \hat{k}(r)
= \frac{F(\hat{k},\kappa,\delta) re^r \sqrt{-\kappa(r)} \log r}{C} \mathbb{E}_o \left[ \int_0^{\tau_r} k(X_t) dt \right]
$$
holds outside a set \( E \subseteq (1, \infty) \) of finite Lebesgue measure. Thus,
\[
E_o[k(X_t)] \leq \frac{F(\hat{k}, \kappa, \delta) e^{\sqrt{-\kappa(r)} \log r}}{2\pi C} \left[ \int_0^{r_t} k(X_t) dt \right].
\]
Hence, we get the desired inequality. Indeed, for \( r > 1 \) we compute that
\[
\log F(\hat{k}, \kappa, \delta) \leq O \left( \log^+ \log^+ \hat{k}(r) + \log^+ r \sqrt{-\kappa(r)} + \log^+ \log r \right)
\]
with
\[
\log^+ \hat{k}(r) \leq \log E_o \left[ \int_0^{r_t} k(X_t) dt \right] + \log^+ \log r + O(1).
\]
Therefore, we have arrived at the required estimate. \( \Box \)

3.2. Logarithmic Derivative Lemma.

Let \( \psi \) be a meromorphic function on \((S, g)\). The norm of the gradient of \( \psi \) is defined by
\[
\|\nabla_S \psi\|^2 = \frac{1}{g} \left| \frac{\partial \psi}{\partial z} \right|^2
\]
in a local holomorphic coordinate \( z \). Locally, we can write \( \psi = \psi_1/\psi_0 \), where \( \psi_0, \psi_1 \) are local holomorphic functions without common zeros. Regard \( \psi \) as a holomorphic mapping into \( \mathbb{P}^1(\mathbb{C}) \) by \( x \mapsto [\psi_0(x) : \psi_1(x)] \). We define
\[
T_\psi(r) = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log (|\psi_0(x)|^2 + |\psi_1(x)|^2) dV(x)
\]
and
\[
T(r, \psi) := m(r, \psi) + N(r, \psi),
\]
where
\[
m(r, \psi) = \int_{\partial D(r)} \log^+ |\psi(x)| d\pi^r_o(x),
\]
\[
N(r, \psi) = \pi \sum_{x \in \psi^{-1}(\infty) \cap D(r)} g_r(o, x).
\]
Let \( i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C}) \) be an inclusion defined by \( z \mapsto [1 : z] \). Via the pull-back by \( i \), we have a \((1,1)\)-form \( i^*\omega_{FS} = dd^c \log(1 + |\zeta|^2) \) on \( \mathbb{C} \), where \( \zeta := w_1/w_0 \) and \([w_0 : w_1]\) is the homogeneous coordinate system of \( \mathbb{P}^1(\mathbb{C}) \). The characteristic function of \( \psi \) with respect to \( i^*\omega_{FS} \) is defined by
\[
\hat{T}_\psi(r) = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log (1 + |\psi(x)|^2) dV(x).
\]
Clearly, \( \hat{T}_\psi(r) \leq T_\psi(r) \). We adopt the spherical distance \( \| \cdot, \cdot \| \) on \( \mathbb{P}^1(\mathbb{C}) \), the proximity function of \( \psi \) with respect to \( a \in \mathbb{P}^1(\mathbb{C}) \) is defined by
\[
m_\psi(r, a) = \int_{\partial D(r)} \log \frac{1}{\|\psi(x), a\|} d\pi^r_o(x).
\]
Again, set
\[ \tilde{N}_\psi(r, a) = \pi \sum_{x \in \psi^{-1}(a) \cap D(r)} g_r(o, x). \]

By the similar arguments as in the proof of Theorem 2.1, we have
\[ \tilde{T}_\psi(r) = \tilde{m}_\psi(r, a) + \tilde{N}_\psi(r, a) + O(1). \]

Note that \( m(r, \psi) = \tilde{m}_\psi(r, \infty) + O(1) \), which yields that
\[ T(r, \psi) = \tilde{T}_\psi(r) + O(1), \quad T\left(r, \frac{1}{\psi - a}\right) = T(r, \psi) + O(1). \]

Hence, we arrive at
\[ T(r, \psi) + O(1) = \tilde{T}_\psi(r) \leq T_\psi(r) + O(1). \]

In order to prove the main results stated in Introduction, the Logarithmic Derivative Lemma (LDL) below is important.

**Theorem 3.4 (LDL).** Let \( \psi \) be a nonconstant meromorphic function on \( S \). Let \( X \) be a nowhere-vanishing holomorphic vector field over \( S \). Then
\[ m\left(r, \frac{X^k(\psi)}{\psi}\right) \leq \frac{3k}{2} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) - \kappa(r)^2 + \log^+ \log r\right) \]
with \( X^j = X \circ X^{j-1} \) and \( X^0 = id \), where \( \kappa \) is defined by (1). In particular, if \( S \) is the Poincaré disc, then
\[ m\left(r, \frac{X^k(\psi)}{\psi}\right) \leq \frac{3k}{2} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) + r\right) \]

**Remark.** LDL is still valid for a general open Riemann surface by lifting \( S \) to the universal covering, see arguments in Section 4.2.

On \( \mathbb{P}^1(\mathbb{C}) \), we take a singular metric
\[ \Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \sqrt{-1} \frac{d\zeta \wedge d\overline{\zeta}}{4\pi^2}. \]

A direct computation gives that
\[ \int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2\pi \psi^* \Phi = \frac{\|\nabla_S \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} \alpha. \]

Set
\[ T_\psi(r, \Phi) = \frac{1}{2\pi} \int_{D(r)} g_r(o, x) \frac{\|\nabla_S \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(x)dV(x). \]
Using Fubini’s theorem,
\[
T_\psi(r, \Phi) = \int_{D(r)} g_r(o, x) \frac{\psi^* \Phi}{\alpha} dV(x)
\]
\[
= \pi \int_{\zeta \in \mathbb{P}_1(C)} \Phi \sum_{x \in \psi^{-1}(\zeta) \cap D(r)} g_r(o, x)
\]
\[
= \int_{\zeta \in \mathbb{P}_1(C)} N(r, 1/(\psi - \zeta)) \Phi \leq T(r, \psi) + O(1).
\]

Then we get
\[
(10) \quad T_\psi(r, \Phi) \leq T(r, \psi) + O(1).
\]

**Proposition 3.5.** Assume that \( \psi(x) \neq 0 \). Then
\[
\frac{1}{2} \mathbb{E}_o \left[ \log^+ \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_r) \right] \leq \frac{1}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r \sqrt{-\kappa(r)} + \log^+ \log r \right),
\]
where \( \kappa \) is defined by (1).

**Proof.** By Jensen’s inequality
\[
\mathbb{E}_o \left[ \log^+ \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_r) \right] \leq \mathbb{E}_o \left[ \log \left( 1 + \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_r) \right) \right] \leq \log^+ \mathbb{E}_o \left[ \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_r) \right] + O(1).
\]

Combine Lemma 3.1 and co-area formula with (10)
\[
\log^+ \mathbb{E}_o \left[ \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_r) \right] \leq \log^+ \mathbb{E}_o \left[ \int_0^{r_r} \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) dt \right] + \log F(\hat{k}, \kappa, \delta) e^{r \sqrt{-\kappa(r)} \log r} \frac{2\pi C}{2}\log^+ \log r + \log T_\psi(r, \Phi) + \log^+ \log \log T(r, \psi) + O(1)
\]
\[
\leq \log^+ \log T(r, \psi) + O\left( \log^+ \log^+ \hat{k}(r) + r \sqrt{-\kappa(r)} + \log^+ \log r \right),
\]
where
\[
\hat{k}(r) = \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^{r_r} \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) dt \right].
\]

Indeed, note that
\[
\hat{k}(r) = \frac{2\pi \log r}{C} T_\psi(r, \Phi) \leq \frac{2\pi \log r}{C} T(r, \psi).
\]

Hence, we have the desired inequality. \(\square\)
We first give LDL for the first-order derivative:

**Theorem 3.6 (LDL).** Let \( \psi \) be a nonconstant meromorphic function on \( S \). Let \( X \) be a nowhere-vanishing holomorphic vector field over \( S \). Then

\[
m\left( r, \frac{X(\psi)}{\psi} \right) \leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) - \kappa(r) r^2 + \log^+ \log r \right),
\]

where \( \kappa \) is defined by (11). In particular, if \( S \) is the Poincaré disc, then

\[
m\left( r, \frac{X(\psi)}{\psi} \right) \leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r \right).
\]

**Proof.** Write \( X = a \frac{\partial}{\partial z} \), then \( \|X\|^2 = g |a|^2 \). We have

\[
m\left( r, \frac{X(\psi)}{\psi} \right) = \int_{\partial D(r)} \log^+ \frac{|X(\psi)|}{|\psi|} (x) d\pi^r_o(x)
\]

\[
\leq \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|X(\psi)|^2}{\|X\|^2 |\psi|^2 (1 + \log^2 |\psi|)} (x) d\pi^r_o(x)
\]

\[
+ \frac{1}{2} \int_{\partial D(r)} \log (1 + \log^2 |\psi(x)|) d\pi^r_o(x) + \frac{1}{2} \int_{\partial D(r)} \log^+ \|X_x\|^2 d\pi^r_o(x)
\]

\[= A + B + C.\]

We handle \( A, B, C \) respectively. For \( A \), it yields from Proposition 3.5 that

\[
A = \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|a|^2 \left| \frac{\partial \psi}{\partial z} \right|^2}{g |a|^2 |\psi|^2 (1 + \log^2 |\psi|)} (x) d\pi^r_o(x)
\]

\[= \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{\|\nabla S \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} (x) d\pi^r_o(x)
\]

\[\leq \frac{1}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r \sqrt{-\kappa(r)} + \log^+ \log r \right).
\]

For \( B \), the Jensen’s inequality implies that

\[
B \leq \int_{\partial D(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi^r_o(x)
\]

\[\leq \log \int_{\partial D(r)} \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi^r_o(x)
\]

\[\leq \log T(r, \psi) + O(1).
\]

Finally, we estimate \( C \). By the condition, \( \|X\| > 0 \). Since \( S \) is non-positively curved and \( a \) is holomorphic, then \( \log \|X\| \) is subharmonic, i.e., \( \Delta_S \log \|X\| \geq 0 \). Clearly, we have

\[
\Delta_S \log^+ \|X\| \leq \Delta_S \log \|X\|
\]

(11)
for $x \in S$ satisfying $\|\mathfrak{X}_x\| \neq 1$. Notice that
\[(12) \quad \log^+ \|\mathfrak{X}_x\| = 0\]
for $x \in S$ satisfying $\|\mathfrak{X}_x\| \leq 1$. Note that Dynkin formula cannot be directly applied to $\log^+ \|\mathfrak{X}\|$, but by virtue of (11) and (12), it is not hard to verify
\[(13) \quad C = \frac{1}{2} \mathbb{E}_o \left[ \log^+ \|\mathfrak{X}(X_{\tau_r})\|^2 \right]\]
\[\leq \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta S \log \|\mathfrak{X}(X_t)\|^2 dt \right] + O(1)\]
\[= \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta S \log g(X_t) dt \right] + \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta S \log |a(X_t)|^2 dt \right] + O(1)\]
\[= - \mathbb{E}_o \left[ \int_0^{\tau_r} K_S(X_t) dt \right] + O(1)\]
\[\leq -\kappa(r) \mathbb{E}_o [\tau_r] + O(1),\]
where we use the fact $K_S = -\left( \Delta S \log g \right)/4$. Thus, we prove the first assertion by $\mathbb{E}_o [\tau_r] \leq 4r^2$, due to Proposition 3.7 below. When $S = \mathbb{D}$ with Poincaré metric, $\kappa(r) \equiv -1$ and $g = 2/(1 - |z|^2)^2$. Take $\mathfrak{X} = \partial/\partial z$, then $C$ in (13) is estimated as follows
\[C = \frac{1}{2} \mathbb{E}_o \left[ \log^+ \|\mathfrak{X}(X_{\tau_r})\|^2 \right]\]
\[= \frac{1}{2} \int_{\partial D(r)} \log \frac{2}{\sqrt{2}} \frac{d\theta}{1 - (e^r - 1)^2/(e^r + 1)^2} \frac{2\pi}{2}\]
\[= \log \frac{\sqrt{2}}{1 - (e^r - 1)^2/(e^r + 1)^2}\]
\[\leq r + O(1).\]
This implies that the second assertion holds. $\square$

**Proposition 3.7.** We have
\[\mathbb{E}_o [\tau_r] \leq 4r^2.\]

**Proof.** The argument follows essentially from Atsuji [3], but here we provide a simpler proof though a rougher estimate. Let $X_t$ be the Brownian motion in $S$ started at $o \neq o_1$, where $o_1 \in D(r)$. Let $r_1(x)$ be the distance function of $x$ from $o_1$. Apply Itô formula to $r_1(x)$
\[(14) \quad r_1(X_t) - r_1(X_0) = B_t - L_t + \frac{1}{2} \int_0^t \Delta s r_1(X_s) ds,\]
here $B_t$ is the standard Brownian motion in $\mathbb{R}$, and $L_t$ is a local time on cut locus of $o$, an increasing process which increases only at cut loci of $o$. Since
S is simply connected and non-positively curved, then
\[ \Delta_S r_1(x) \geq \frac{1}{r_1(x)}, \quad L_t \equiv 0. \]
By (14), we arrive at
\[ r_1(X_t) \geq B_t + \frac{1}{2} \int_0^t ds \frac{1}{r_1(X_s)}. \]
Let \( t = \tau_r \), and take expectation on both sides of the above inequality, then it yields that
\[ \max_{x \in \partial D} (r_1(x)) \geq E{\tau_r}. \]
Let \( o' \to o \), we are led to the conclusion. \( \square \)

Finally, let us prove Theorem 3.4:

**Proof.** Note that
\[ m(r, \mathcal{X}^k) \leq \sum_{j=1}^k m(r, \mathcal{X}^j). \]
We conclude the proof by using Proposition 3.8 below. \( \square \)

**Proposition 3.8.** We have
\[ m(r, \frac{\mathcal{X}^{k+1}}{\mathcal{X}^k}) \leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right), \]
where \( \kappa \) is defined by (11). In particular, if \( S \) is the Poincaré disc, then
\[ m(r, \frac{\mathcal{X}^{k+1}}{\mathcal{X}^k}) \leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r \right). \]

**Proof.** For the first assertion, we claim that
\[ T(r, \mathcal{X}^k) \leq 2^k T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right). \]
By virtue of Theorem 3.6 when \( k = 1 \)
\[ T(r, \mathcal{X}(\psi)) = m(r, \mathcal{X}(\psi)) + N(r, \mathcal{X}(\psi)) \]
\[ \leq m(r, \psi) + 2N(r, \psi) + m(r, \frac{\mathcal{X}(\psi)}{\psi}) \]
\[ \leq 2T(r, \psi) + m(r, \frac{\mathcal{X}(\psi)}{\psi}) \]
\[ \leq 2T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right) \]
holds for \( r > 1 \) outside a set of finite Lebesgue measure. Assuming now that the claim holds for \( k \leq n - 1 \). By induction, we only need to prove the claim
in the case when \( k = n \). By this claim for \( k = 1 \) proved above and Theorem \ref{thm:3.6} repeatedly, we conclude that

\[
T(r, \mathcal{X}^n(\psi)) \leq 2T(r, \mathcal{X}^{n-1}(\psi)) + O\left( \log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r \right)
\]

\[
\leq 2^n T(r, \psi) + O\left( \log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r \right)
\]

\[
\leq 2^n T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right)
\]

\[
\leq 2^n T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right)
\]

\[
\leq 2^n T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right).
\]

So, the claim (15) is proved. Employing Theorem \ref{thm:3.6} and (15) to get

\[
m\left( r, \frac{\mathcal{X}^{k+1}(\psi)}{\mathcal{X}^k(\psi)} \right)
\]

\[
\leq \frac{3}{2} \log T(r, \mathcal{X}^k(\psi)) + O\left( \log^+ \log T(r, \mathcal{X}^k(\psi)) - \kappa(r)r^2 + \log^+ \log r \right)
\]

\[
\leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right).
\]

This proves the first assertion, and the second assertion is proved similarly by replacing \( \kappa(r)r^2 \) by \(-r\) due to the second conclusion of Theorem \ref{thm:3.6} \( \square \)

4. An extension of H. Cartan’s theory

4.1. Cartan-Nochka’s approach.

Let \( S \) be an open Riemann surface with a nowhere-vanishing holomorphic vector field \( \mathcal{X} \). Let

\[ f : S \to \mathbb{P}^n(\mathbb{C}) \]

be a holomorphic curve into complex projective space with the Fubini-Study form \( \omega_{FS} \). Locally, we may write \( f = [f_0 : \cdots : f_n] \), a reduced representation, i.e., \( f_0 = w_0 \circ f \), \( \cdots \) are local holomorphic functions without common zeros, where \( w = [w_0 : \cdots : w_n] \) denotes homogenous coordinate system of \( \mathbb{P}^n(\mathbb{C}) \). Set \( \|f\|^2 = |f_0|^2 + \cdots + |f_n|^2 \). Noting that \( \Delta_S \log \|f\|^2 \) is independent of the choices of representations of \( f \), hence it is globally defined on \( S \). The height function of \( f \) is defined by

\[
T_f(r) = \pi \int_{D(r)} g_r(o, x) f^* \omega_{FS} = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log \|f(x)\|^2 dV(x).
\]
Given a hyperplane $H$ of $\mathbb{P}^n(\mathbb{C})$ with defining function $\hat{H}(w) = h_0w_0 + \cdots + h_nw_n$. Set $\|\hat{H}\|^2 = |h_0|^2 + \cdots + |h_n|^2$. The counting function of $f$ with respect to $H$ is defined by

$$N_f(r, H) = \pi \int_{D(r)} g_r(o, x) d\sigma \left[ \log |\hat{H} \circ f(x)|^2 \right] = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log |\hat{H} \circ f(x)|^2 dV(x).$$

We define the proximity function of $f$ with respect to $H$ by

$$m_f(r, H) = \int_{\partial D(r)} \log \frac{\|\hat{H}\|}{|H \circ f(x)|} d\sigma(x).$$

**Proposition 4.1.** Assume that $f_k \not\equiv 0$ for some $k$. We have

$$\max_{0 \leq j \leq n} T\left( r, \frac{f_j}{f_k} \right) \leq T_f(r) + O(1).$$

**Proof.** Clearly, $f_j/f_k$ is well defined on $S$. From (9), we get

$$T\left( r, \frac{f_j}{f_k} \right) \leq T_{f_j/f_k}(r) + O(1) \leq \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log \left( \sum_{j=0}^n |f_j(x)|^2 \right) dV(x) + O(1) = T_f(r) + O(1).$$

This completes the proof. \( \square \)

**Wronskian determinants.** Let $H_1, \cdots, H_q$ be $q$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with defining functions given by

$$\hat{H}_j(w) = \sum_{k=0}^n h_{jk}w_k, \quad 1 \leq j \leq q.$$ 

Assume that $f$ is linearly non-degenerate. We define Wronskian determinant and logarithmic Wronskian determinant of $f$ with respect to $\mathbf{x}$ respectively by

$$W_{\mathbf{x}}(f_0, \cdots, f_n) = \begin{vmatrix} f_0 & \cdots & f_n \\ \mathbf{x}(f_0) & \cdots & \mathbf{x}(f_n) \\ \vdots & \vdots & \vdots \\ \mathbf{x}^n(f_0) & \cdots & \mathbf{x}^n(f_n) \end{vmatrix}, \quad \Delta_{\mathbf{x}}(f_0, \cdots, f_n) = \begin{vmatrix} 1 & \cdots & 1 \\ \mathbf{x}(f_0) & \cdots & \mathbf{x}(f_n) \\ \vdots & \vdots & \vdots \\ \mathbf{x}^n(f_0) & \cdots & \mathbf{x}^n(f_n) \end{vmatrix}.$$
For a \((n+1)\times (n+1)\)-matrix \(A\) and a nonzero meromorphic function \(\phi\) on \(S\), we can check the following basic properties:

\[
\Delta_X(\phi f_0, \cdots, \phi f_n) = \Delta_X(f_0, \cdots, f_n),
\]

\[
W_X(\phi f_0, \cdots, \phi f_n) = \phi^{n+1} W_X(f_0, \cdots, f_n),
\]

\[
W_X((f_0, \cdots, f_n) A) = \det(A) W_X(f_0, \cdots, f_n),
\]

\[
W_X(f_0, \cdots, f_n) = \left(\prod_{j=0}^{n} f_j\right) \Delta_X(f_0, \cdots, f_n).
\]

Clearly, \(\Delta_X(f_0, \cdots, f_n)\) is globally defined on \(S\).

**Nochka weights.** For a subset \(Q = \{i_1, \cdots, i_k\} \subseteq \{1, \cdots, q\}\), we define

\[
\text{rank}(Q) = \text{rank} \begin{pmatrix} h_{i_10} & \cdots & h_{i_1n} \\ \vdots & \ddots & \vdots \\ h_{i_k0} & \cdots & h_{i_kn} \end{pmatrix}.
\]

**Lemma 4.2** \([20]\). Let \(H_1, \cdots, H_q\) be hyperplanes of \(\mathbb{P}^n(\mathbb{C})\) in \(N\)-subgeneral position with \(q > 2N - n + 1\). Then there exists rational constants \(\gamma_1, \cdots, \gamma_q\) satisfying the following conditions:

(i) \(0 < \gamma_j \leq 1\) for \(1 \leq j \leq q\);

(ii) Set \(\gamma = \max_{1 \leq j \leq q} \gamma_j\), we have

\[
\frac{n + 1}{2N - n + 1} \leq \gamma \leq \frac{n}{N}, \quad \gamma(q - 2N + n - 1) = \sum_{j=1}^{q} \gamma_j - n - 1.
\]

(iii) If \(Q \subseteq \{1, \cdots, q\}\) with \(0 < |Q| \leq N + 1\), then \(\sum_{j \in Q} \gamma_j \leq \text{rank}(Q)\).

Here, \(\gamma_1, \cdots, \gamma_q\) are called the Nochka weights and \(\gamma\) is called the Nochka constant.

**Lemma 4.3** \([20]\). Let \(H_1, \cdots, H_q\) be hyperplanes of \(\mathbb{P}^n(\mathbb{C})\) in \(N\)-subgeneral position with \(q > 2N - n + 1\). Let \(\gamma_1, \cdots, \gamma_q\) be Nochka weights for \(H_1, \cdots, H_q\) and let \(\beta_1, \cdots, \beta_q\) be arbitrary constants not less than 1. Then for each subset \(Q \subseteq \{1, \cdots, q\}\) with \(0 < |Q| \leq N + 1\), there are distinct \(j_1, \cdots, j_{\text{rank}(Q)}\) in \(Q\) such that

\[
\text{rank} \left( \{j_1, \cdots, j_{\text{rank}(Q)}\} \right) = \text{rank}(Q), \quad \prod_{j \in Q} \beta_{j_1}^{\gamma_j} \leq \prod_{i=1}^{\text{rank}(Q)} \beta_j.
\]

We need the following preliminary results:

**Proposition 4.4.** Let \(H_1, \cdots, H_q\) be hyperplanes of \(\mathbb{P}^n(\mathbb{C})\) in \(N\)-subgeneral position with \(q > 2N - n + 1\). Let \(\gamma\) and \(\gamma_1, \cdots, \gamma_q\) be Nochka constant and
weights for \( H_1, \cdots, H_q \). Then there exists a constant \( C > 0 \) determined by \( \hat{H}_1 \circ f, \cdots, \hat{H}_q \circ f \) such that

\[
\|f\|^{\gamma(q-2N+n-1)} \leq C \frac{\prod_{j=1}^{q} |\hat{H}_j \circ f|^{\gamma_j}}{|W_{\hat{X}}(f_0, \cdots, f_n)|} \sum_{Q \subseteq \{1, \cdots, q\}, \ |Q| = n+1} \left| \Delta_X(\hat{H}_j \circ f, j \in Q) \right|.
\]

**Proof.** Note from the definition of \( N \)-subgeneral position that for each point \( w \in \mathbb{P}^n(\mathbb{C}) \), there is a subset \( Q \subseteq \{1, \cdots, q\} \) with \( |Q| = q - N - 1 \) such that \( \prod_{j \in Q} \hat{H}_j(w) \neq 0 \). Hence, there is a constant \( C_1 > 0 \) such that

\[
C_1^{-1} < \sum_{|Q| = q - N - 1} \prod_{j \in Q} \left( \frac{|\hat{H}_j(w)|}{\|\hat{H}_j\| \|w\|} \right)^{\gamma_j} < C_1, \quad w \in \mathbb{P}^n(\mathbb{C}).
\]

Set \( R = \{1, \cdots, q\} \setminus Q \) and rewrite

\[
\prod_{j \in Q} \left( \frac{|\hat{H}_j(w)|}{\|\hat{H}_j\| \|w\|} \right)^{\gamma_j} = \prod_{j \in R} \left( \frac{|\hat{H}_j(w)|}{\|\hat{H}_j\| \|w\|} \right)^{\gamma_j} \cdot \frac{\prod_{j=1}^{q} |\hat{H}_j(w)|^{\gamma_j}}{\prod_{j=1}^{q} |\hat{H}_j\| \cdot \|w\|^{\gamma(q-2N+n-1)+n+1}}.
\]

By the property of Wronskian determinant given before, we see that

\[
c(R') := \frac{|W_{\hat{X}}(f_0, \cdots, f_n)|}{|W_{\hat{X}}(\hat{H}_f \circ f, j \in R')|}
\]

is a positive number depending on \( R' \). Hence, it yields from (16) that

\[
\prod_{j \in Q} \left( \frac{|\hat{H}_j \circ f|}{\|\hat{H}_j\| \|f\|} \right)^{\gamma_j} \leq \frac{c(R')}{\|f\|^{\gamma(q-2N+n-1)} \prod_{j=1}^{q} |\hat{H}_j\|} \cdot \frac{\prod_{j=1}^{q} |\hat{H}_j \circ f|^{\gamma_j}}{\prod_{j \in R'} |\hat{H}_j \circ f|} \cdot \frac{|W_{\hat{X}}(\hat{H}_f \circ f, j \in R')|}{\prod_{j=1}^{q} |\hat{H}_j\| \cdot \|w\|^{\gamma(q-2N+n-1)+n+1}}.
\]

where we use the relation between Wronskian determinant and logarithmic Wronskian determinant stated before. The proposition is proved by setting \( C = C_1 \max_{R'} \{c(R')\} / \prod_{j=1}^{q} |\hat{H}_j| \). \( \square \)
Proposition 4.5. Assume the same notations as in Proposition 4.4. Then
\[ \sum_{j=1}^{q} \gamma_j(\hat{H}_j \circ f) - (W_X(f_0, \cdots, f_n)) \leq \sum_{j=1}^{q} \gamma_j \sum_{a \in S} \min \{ \text{ord}_a \hat{H}_j \circ f, n \} \cdot a \]
holds as a divisor on S with rational coefficients.

Proof. The proof essentially follows Fujimoto [11]. We observe that
\[ \text{ord}_a \hat{H}_j \circ f = \min \{ \text{ord}_a \hat{H}_j \circ f, n \} + (\text{ord}_a \hat{H}_j \circ f - n)^+ . \]
Hence, the inequality claimed in the proposition is equivalent to
\[ \sum_{j=1}^{q} \gamma_j \sum_{a \in S} (\text{ord}_a \hat{H}_j \circ f - n)^+ \cdot a \leq (W_X(f_0, \cdots, f_n)) . \]
In what follows, we show (17) holds. Take an arbitrary point \( a \in S \) and put
\[ E = \{ j \in \{ 1, \cdots, n \} : \text{ord}_a \hat{H}_j \circ f \geq n + 1 \} . \]
The assumption of N-subgeneral position implies \(|E| \leq N\). We may assume that \( E \neq \emptyset \). Let \( m_1 > \cdots > m_t \geq n + 1 \) be the orders \( \text{ord}_a \hat{H}_j \circ f \), \( j \in E \) in order from the largest to the smallest. Take a sequence of subsets of \( E \)
\[ \emptyset = E_0 \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_t = E \]
so that \( \text{ord}_a \hat{H}_j \circ f = m_k \) for all \( j \in E_k \setminus E_{k-1} \). For each \( E_k \), we take a subset \( F_k \subseteq E_k \) such that \( |F_k| = \text{rank}(F_k) = \text{rank}(E_k) \) and \( F_{k-1} \subseteq F_k \). Whence, it yields that \( |F_k \setminus F_{k-1}| = \text{rank}(E_k) - \text{rank}(E_{k-1}) \). Set \( m'_k = m_k - n \). Then
\[ \sum_{j=1}^{q} \gamma_j (\text{ord}_a \hat{H}_j \circ f - n)^+ = \sum_{j \in E} \gamma_j (\text{ord}_a \hat{H}_j \circ f - n) = \sum_{k=1}^{t} \sum_{j \in E_k \setminus E_{k-1}} \gamma_j m'_k . \]
For the last term, by (iii) in Lemma 4.2 we get
\[ \sum_{k=1}^{t} \sum_{j \in E_k \setminus E_{k-1}} \gamma_j m'_k = (m'_1 - m'_2) \sum_{j \in E_1} \gamma_j + \cdots + m'_t \sum_{j \in E_t} \gamma_j \leq (m'_1 - m'_2) \text{rank}(E_1) + \cdots + m'_t \text{rank}(E_t) = \text{rank}(E_1) m'_1 + \cdots + (\text{rank}(E_t) - \text{rank}(E_{t-1})) m'_t = |F_1| m'_1 + |F_2 \setminus F_1| m'_2 + \cdots + |F_t \setminus F_{t-1}| m'_t . \]
Hence,
\[ \sum_{j=1}^{q} \gamma_j (\text{ord}_a \hat{H}_j \circ f - n)^+ \leq |F_1| m'_1 + |F_2 \setminus F_1| m'_2 + \cdots + |F_t \setminus F_{t-1}| m'_t . \]
On the other hand, since \( X \) is vanishing nowhere, the order of \( W_X(f_0, \cdots, f_n) \) at \( a \) does not change by a linear reversible transformation of \( f_0, \cdots, f_n \). Put
Proposition 4.1 we have replacing $\kappa$ we prove the first assertion, and the second assertion is proved similarly by loss of generality. A simple computation for Wronskian determinant gives
\[
\text{ord}_a W_X(f_0, \cdots, f_n) \geq |F_1|m'_1 + |F_2|F_1|m'_2 + \cdots + |F_t \setminus F_{t-1}|m'_t.
\]
Therefore, (17) is confirmed. \qed

4.2. **Proof of Theorem 1.1**

Let $\pi : \tilde{S} \to S$ be the (analytic) universal covering. By the pull-back of $\pi$, $\tilde{S}$ can be equipped with the induced metric from the metric of $S$. In such case, $\tilde{S}$ is a simply-connected and complete open Riemann surface of non-positive Gauss curvature. Take a diffusion process $\tilde{X}_t$ in $\tilde{S}$ so that $X_0 = \pi(\tilde{X}_0)$, then $\tilde{X}_t$ becomes a Brownian motion with generator $\Delta_{\tilde{S}}/2$ which is induced from the pull-back metric. Let $\tilde{X}_t$ start from $\tilde{\phi} \in \tilde{S}$ with $\phi = \pi(\tilde{\phi})$, then we have
\[
\mathbb{E}_{\tilde{\phi}}[\phi(X_t)] = \mathbb{E}_{\tilde{\phi}}[\phi \circ \pi(\tilde{X}_t)]
\]
for $\phi \in \mathcal{C}_b(S)$. Set
\[
\tilde{\tau}_r = \inf \{ t > 0 : \tilde{X}_t \not\in \tilde{D}(r) \},
\]
where $\tilde{D}(r)$ is a geodesic disc centered at $\tilde{\phi}$ with radius $r$ in $\tilde{S}$. If necessary, one can extend the filtration in probability space where $(X_t, \mathbb{P}_{\tilde{\phi}})$ are defined so that $\tilde{\tau}_r$ is a stopping time with respect to a filtration where the stochastic calculus of $X_t$ works. By the above arguments, we could assume $\tilde{S}$ is simply connected by lifting $f$ to the covering.

**Lemma 4.6.** Let $Q \subseteq \{1, \cdots, q\}$ with $|Q| = n + 1$. If $S$ is simply connected, then we have
\[
m\left( r, \Delta_X(\hat{H}_k \circ f, k \in Q) \right) \leq O\left( \log T_f(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]
where $\kappa$ is defined by (1). In particular, if $S$ is the Poincaré disc, then
\[
m\left( r, \Delta_X(\hat{H}_k \circ f, k \in Q) \right) \leq O\left( \log T_f(r) + r \right).
\]

**Proof.** We write $Q = \{j_0, \cdots, j_n\}$ and suppose that $\hat{H}_{j_0} \circ f \neq 0$ without loss of generality. The property of logarithmic Wronskian determinant indicates
\[
\Delta_X(\hat{H}_{j_0} \circ f, \cdots, \hat{H}_{j_n} \circ f) = \Delta_X\left( 1, \frac{\hat{H}_{j_1} \circ f}{\hat{H}_{j_0} \circ f}, \cdots, \frac{\hat{H}_{j_n} \circ f}{\hat{H}_{j_0} \circ f} \right).
\]
Since $\hat{H}_{j_0} \circ f, \cdots, \hat{H}_{j_n} \circ f$ are linear forms of $f_0, \cdots, f_n$, by Theorem 3.4 and Proposition 4.1 we have
\[
m\left( r, \Delta_X(\hat{H}_k \circ f, k \in Q) \right) \leq O\left( \log T_f(r) - \kappa(r)r^2 + \log^+ \log r \right).
\]
We prove the first assertion, and the second assertion is proved similarly by replacing $\kappa(r)r^2$ by $-r$. \qed
We now prove Theorem 1.1.

Proof. By Proposition 4.4 and Dynkin formula

\[
\gamma(q - 2N + n - 1) \frac{ddc}{T_f(r)} \log \|f\|^2 - \sum_{j=1}^{q} \gamma_j(H_j \circ f) + (W_X(f_0, \cdots, f_n)) 
\]

\[
\leq 2 \frac{ddc}{2} \left[ \log \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} |\Delta_X(\hat{H}_k \circ f, k \in Q)| \right] 
\]

in the sense of currents. Integrating both sides of the above inequality, then it yields from Proposition 4.5 that

\[
(q - 2N + n - 1)T_f(r) - \sum_{j=1}^{q} N_f^{[n]}(r, H_j) 
\]

\[
\leq \frac{2\pi}{\gamma} \int_{D(r)} g_r(o, x) \frac{ddc}{2} \left[ \log \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} |\Delta_X(\hat{H}_k \circ f, k \in Q)| \right] 
\]

\[
= \frac{1}{2\gamma} \int_{D(r)} g_r(o, x) \Delta_S \log \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} |\Delta_X(\hat{H}_k \circ f, k \in Q)| \ dV(x). 
\]

Applying co-area formula and Dynkin formula to the last term, we get

\[
\frac{1}{2} \int_{D(r)} g_r(o, x) \Delta_S \log \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} |\Delta_X(\hat{H}_k \circ f, k \in Q)| \ dV(x) 
\]

\[
= \int_{\partial D(r)} \log \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} |\Delta_X(\hat{H}_k \circ f, k \in Q)| \ d\pi_o(x) + O(1) 
\]

\[
\leq \sum_{Q \subseteq \{1, \cdots, q\}, |Q|=n+1} m \left( r, \Delta_X(\hat{H}_k \circ f, k \in Q) \right) + O(1) 
\]

\[
\leq O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right). 
\]

The last step follows from Lemma 4.6. This proves the theorem. \(\square\)

Corollary 1.3 can be proved similarly just by replacing \(\kappa(r)r^2\) by \(-r\), due to the second conclusion of Lemma 4.6.

Let \(H\) be a hyperplane of \(\mathbb{P}^n(\mathbb{C})\) such that \(H \nsubseteq f(S)\). The \(k\)-defect \(\delta_f^{[k]}(H)\) of \(f\) at \(k\)-level with respect to \(H\) is defined by

\[
\delta_f^{[k]}(H) = 1 - \limsup_{r \to \infty} \frac{N_f^{[k]}(r, H)}{T_f(r)}, 
\]

where \(N_f^{[k]}(r, H)\) is the \(k\)-truncated counting function. By definition, we can derive Corollary 1.3 immediately.
5. Vanishing theorem

5.1. LDL for logarithmic jet differentials.

Let $S$ be an open Riemann surface with a nowhere-vanishing holomorphic vector field $X$. Let $M$ be a complex manifold of complex dimension $n$. First, we introduce jet bundles.

Fix a point $x_0 \in S$. Now consider a holomorphic curve $f : U_1 \to M$ defined in an open neighborhood $U_1$ of $x_0$ with $f(x_0) = y$, and another holomorphic curve $g : U_2 \to M$ defined in an open neighborhood $U_2$ of $x_0$ with $g(x_0) = y$, we define an equivalent relation $f \sim g$ if there exists an open neighborhood $U_3 \subseteq U_1 \cap U_2$ of $x_0$ such that $f = g$ on $U_3$. Denote by $\text{Hol}_{\text{loc}}((S, x_0), (M, y))$ the set of such equivalent classes. Locally, we write $f = (f_1, \cdots, f_n)$, where $f_j = \zeta_j \circ f$ near $x_0$, here $(\zeta_1, \cdots, \zeta_n)$ is a local holomorphic coordinate system near $y$. Put

$$X^j(f) = (\cdots, X^j(f_i), \cdots).$$

For $f, g \in \text{Hol}_{\text{loc}}((S, x_0), (M, y))$, we speak of $f \sim^k g$ with respect to $X$ if

$$X^j f(x_0) = X^j g(x_0), \quad 1 \leq j \leq k.$$ 

This relation is independent of the choices of local holomorphic coordinates, and defines an equivalent relation. Let $j_k, X f$ be this equivalent class. Set

$$J_k(M, X)_y = \left\{ j_k, X f : f \in \text{Hol}_{\text{loc}}((S, x_0), (M, y)) \right\},$$

$$J_k(M, X) = \bigsqcup_{y \in M} J_k(M, X)_y.$$ 

Apparently, $J_k(M, X)_y \cong \mathbb{C}^{nk}$, i.e., each $v \in J_k(M, X)_y$ is represented by

$$(X^j(\zeta_i \circ \phi)(x_0) : 1 \leq i \leq n, \ 1 \leq j \leq k)$$

for some $\phi \in \text{Hol}_{\text{loc}}((S, x_0), (M, y))$. Write $X = a \frac{\partial}{\partial z}$ locally, then $X$ induces a global holomorphic 1-form $\eta := dz/a$ without zeros on $S$. For every $x \in S$, there exists an open neighborhood $U(x)$ of $x$ such that

$$\int_{\gamma} \eta = 0$$

for any closed Jordan curve $\gamma$ in $U(x)$. In fact, it is known [15], there exists a nowhere-vanishing holomorphic 1-form $\eta$ on $S$ such that the contour integral (18) equals 0 along any closed Jordan curve $\gamma$ in $S$. We can choose $X$ induced from $\eta$. For $x \in S$, define

$$\overline{x - x_0} := \int_{x_0}^{x} \eta,$$
which is independent of the paths of integration, and hence is a holomorphic function of \( x \). If \( f \) is defined in an open neighborhood of another point \( x_1 \in S \), then we consider the power series of \( x - x_0 \) that

\[
f_{x_1}(x) := f(x_1) + \mathcal{X}f(x_1) \cdot x - x_0 + \frac{\mathcal{X}^2 f(x_1)}{2!} \cdot (x - x_0)^2 + \cdots
\]

which is convergent in an open neighborhood of \( x_0 \), and holomorphic on this neighborhood. It is trivial to check that

\[
\mathcal{X}^k f_{x_1}(x_0) = \mathcal{X}^k f(x_1), \quad k = 0, 1, \cdots
\]

Hence, \( f \) gives a \( k \)-jet \( j_k, \mathcal{X}f_{x_1}(x_0) \) in \( J_k(M, \mathcal{X}) f(x_0) \) at \( x_1 \). In further, \( f \) induces naturally

\[
J_k, \mathcal{X} f : x \mapsto j_k, \mathcal{X}f(x_0) \in J_k(M, \mathcal{X}) f(x)
\]

for \( x \) in some open neighborhood of \( x_1 \), which is called the \( k \)-jet lift of \( f \).

For every holomorphic curve \( f : S \to M \), \( f \) defines a \( k \)-jet in \( J_k(M, \mathcal{X}) f(x) \) at every \( x \in S \) in a natural way, i.e.,

\[
J_k, \mathcal{X}f(x) = \left( \mathcal{X}^j (\zeta_i \circ f)(x) : 1 \leq i \leq n, \ 1 \leq j \leq k \right), \ \forall x \in S.
\]

We equip a complex structure to make \( \pi_k : J_k(M, \mathcal{X}) \to M \) be a \( \mathbb{C}^{nk} \)-fiber bundle over \( M \), where \( \pi_k \) is the natural projection, and so \( J_k(M, \mathcal{X}) \) becomes a complex manifold. Note that \( J_1(M, \mathcal{X}) \cong T_M \) (holomorphic tangent bundle over \( M \)), but, in general, \( J_k(M, \mathcal{X}) \) is not a vector bundle for \( k > 1 \).

A (holomorphic) jet differential \( \omega \) of order \( k \) and weighted degree \( m \) on an open set of \( M \) (with a local holomorphic coordinate system \( \zeta = (\zeta_1, \cdots, \zeta_n) \)) is a homogeneous polynomial in \( d^j \zeta_j \) \((1 \leq i \leq k, 1 \leq j \leq n)\) of the form

\[
\omega = \sum_{|l_1| + \cdots + |l_k| = m} a_{l_1 \cdots l_k}(\zeta) d^{l_1} \zeta_{l_1} \cdots d^{l_k} \zeta_{l_k}
\]

with holomorphic function coefficients \( a_{l_1 \cdots l_k}(\zeta) \), which is also simply called a \( k \)-jet differential of degree \( m \). We use \( E_{k,m}^G T_M^* \) to denote the sheaf of germs of \( k \)-jet differentials of degree \( m \).

Let \( D \) be a reduced divisor on \( M \). Then a logarithmic \( k \)-jet differential \( \omega \) of degree \( m \) along \( D \) is a \( k \)-jet differential of degree \( m \) with possible logarithmic poles along \( D \). Namely, along \( D \), \( \omega \) is locally a homogeneous polynomial in

\[
d^s \log \sigma_1, \cdots, d^s \log \sigma_r, d^s \sigma_{r+1}, \cdots, d^s \sigma_n, \ 1 \leq s \leq k
\]

of weighted degree \( m \), where \( \sigma_1, \cdots, \sigma_r \) are irreducible, and \( \sigma_1 \cdots \sigma_r = 0 \) is a local defining equation of \( D \). Let \( E_{k,m}^G T_M^*(\log D) \) denote the sheaf of germs of logarithmic \( k \)-jet differentials of degree \( m \) over \( M \). We note that \( \omega(J_k, \mathcal{X}f) \) is a meromorphic function on \( S \) for \( \omega \in H^0(M, E_{k,m}^G T_M^*(\log D)) \).

Suppose that \( M \) is a compact complex manifold with an ample divisor \( A \) on \( M \). In what follows we extend Theorem 3.4 to logarithmic jet differentials.
Theorem 5.1 (LDL for logarithmic jet differentials). Let \( f : S \rightarrow M \) be a holomorphic curve such that \( f(S) \not\subseteq D \), where \( D \) is a reduced divisor on \( M \). Then
\[
m(r, \omega(J_kxf)) \leq O\left( \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right)\]
for \( \omega \in H^0(M, E^{\mathbb{C}G}_{k,m} T_M^*(\log D)) \), where \( \kappa \) is defined by (1). In particular, if \( S \) is the Poincaré disc, then
\[
m(r, \omega(J_kxf)) \leq O\left( \log T_{f,A}(r) + r \right)\]
for \( \omega \in H^0(M, E^{\mathbb{C}G}_{k,m} T_M^*(\log D)) \).

Proof. We can regard \( M \) as an algebraic submanifold of a complex projective space. By Hironaka’s desingularization [13], there exists a brow-up \( \tau : M \rightarrow M \) with center at singular locus of \( D \) such that \( f^{-1}(D) \) is of simple normal crossing type. Let a holomorphic curve \( \hat{f} : S \rightarrow M \) such that \( \tau \circ \hat{f} = f \). Since \( \omega(J_kxf) = \tau^* \omega(J_kx\hat{f}) \), it suffices to certify this assertion for \( \tau^* \omega(J_kx\hat{f}) \) on \( M \). So, we assume that \( D \) has only simple normal crossings. By Ru-Wong’s arguments (23, Page 231-233), there exists a finite open covering \( \{U_\lambda\} \) of \( M \) and rational functions \( w_{\lambda_1}, \ldots, w_{\lambda_n} \) on \( M \) for each \( \lambda \), such that \( w_{\lambda_1}, \ldots, w_{\lambda_n} \) are holomorphic on \( U_\lambda \) as well as
\[
dw_{\lambda_1} \wedge \cdots \wedge dw_{\lambda_n}(x) \neq 0, \quad \forall x \in U_\lambda, \quad D \cap U_\lambda = \{w_{\lambda_1} \cdots w_{\lambda_{\alpha_\lambda}} = 0\}, \quad \exists \alpha_\lambda \leq n.
\]
Hence, for each \( \lambda \) we get
\[
\omega|_{U_\lambda} = P_\lambda\left(\frac{d^i w_{\lambda_j}}{w_{\lambda_j}}, \frac{d^j w_{\lambda_l}}{w_{\lambda_l}}\right), \quad 1 \leq i, h \leq k, \quad 1 \leq j \leq \alpha_\lambda, \quad \alpha_\lambda + 1 \leq l \leq n,
\]
which is a polynomial in variables described above, with coefficients rational on \( M \) and holomorphic on \( U_\lambda \). It yields that
\[
\omega(J_kxf)|_{f^{-1}(U_\lambda)} = P_\lambda\left(\frac{\mathcal{X}^i(w_{\lambda_j} \circ f)}{w_{\lambda_j} \circ f}, \frac{\mathcal{X}^j(w_{\lambda_l} \circ f)}{w_{\lambda_l} \circ f}\right).
\]
Let \( \{c_\lambda\} \) be a partition of unity subordinate to \( \{U_\lambda\} \), which implies that
\[
\omega(J_kxf) = \sum_\lambda c_\lambda \circ f \cdot P_\lambda\left(\frac{\mathcal{X}^i(w_{\lambda_j} \circ f)}{w_{\lambda_j} \circ f}, \frac{\mathcal{X}^j(w_{\lambda_l} \circ f)}{w_{\lambda_l} \circ f}\right)
\]
on \( M \). Since \( P_\lambda \) is a polynomial, then
\[
\log^+ \left| \omega(J_kxf) \right| \leq O\left( \sum_{i,j,\lambda} \log^+ \left|\frac{\mathcal{X}^i(w_{\lambda_j} \circ f)}{w_{\lambda_j} \circ f}\right| \right) + \sum_{h,l,\lambda} \log^+ \left( c_\lambda \circ f \cdot \left|\frac{\mathcal{X}^h(w_{\lambda_l} \circ f)}{w_{\lambda_l} \circ f}\right| \right) + O(1).
\]
Note that $L_A > 0$, by virtue of Theorem 3.4 and Proposition 4.1

$$m(r, \omega(J_k \chi f)) \leq O \left( \sum_{i,j,\lambda} m(r, \chi'(w_{\lambda j} \circ f)) + \sum_{h,l,\lambda} m(r, c_{\lambda} \circ f \cdot \chi^h(w_{\lambda h} \circ f)) \right) + O(1)$$

$$\leq O \left( \log T_{f,A}(r) - \kappa(r)^2 + \log^+ \log r \right) + O \left( \sum_{h,l,\lambda} m(r, c_{\lambda} \circ f \cdot \chi^h(w_{\lambda h} \circ f)) \right).$$

For the last term in the above inequality, we claim that

$$m(r, c_{\lambda} \circ f \cdot \chi^h(w_{\lambda h} \circ f)) \leq O \left( \log T_{f,A}(r) - \kappa(r)^2 + \log^+ \log r \right).$$

Equip $L_A$ with a Hermitian metric $h$ such that $c_1(L_A, h) > 0$ and set

$$f^* c_1(L_A, h) = \xi \alpha, \quad \alpha = g \sqrt{-1} d\bar{z} \wedge \bar{dz}$$

in a local holomorphic coordinate $z$. $c_1(L_A, h) > 0$ implies that there exists a constant $B > 0$ such that

$$c_2^2 \circ f \cdot \frac{\sqrt{-1}}{2\pi} dw_M \circ f \wedge d\bar{w}_M \circ f \leq B f^* c_1(L_A, h).$$

Write $\chi = a \frac{\partial}{\partial z}$, then $\|\chi\|^2 = g|a|^2$. Combine (19) with (20), we have

$$c_2^2 \circ f \cdot |\chi(w_M \circ f)|^2 \leq B \xi \|\chi\|^2.$$

By this with (13) and Proposition 5.7

$$m \left( r, c_{\lambda} \circ f \cdot \chi(w_{\lambda h} \circ f) \right) \leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( B \xi(X_{\tau_r}) \|\chi(X_{\tau_r})\|^2 \right) \right]$$

$$\leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \xi(X_{\tau_r}) \right) \right] + \frac{1}{2} \mathbb{E}_o \left[ \log^+ \|\chi(X_{\tau_r})\|^2 \right] + O(1)$$

$$\leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\xi(X_{\tau_r})}{\kappa(r)} \right) \right] - \frac{\kappa(r)^2}{2} + O(1).$$
Indeed, it follows from Theorem 3.1 that
\[
E_o \left[ \log^+ \xi(X_{r_t}) \right] \leq \log^+ E_o [\xi(X_{r_t})] + O(1)
\]
\[
\leq \log^+ E_o \left[ \int_0^{r_t} \xi(X_t)dt \right] + O \left( \log^+ \log E_o \left[ \int_0^{r_t} \xi(X_t)dt \right] + r \sqrt{-\kappa(r)} + \log^+ \log r \right)
\]
\[
\leq \log T_{f,A}(r) + O \left( \log^+ \log T_{f,A}(r) + r \sqrt{-\kappa(r)} + \log^+ \log r \right)
\]
due to
\[
\log^+ E_o \left[ \int_0^{r_t} \xi(X_t)dt \right] = \log^+ \int_{D(r)} g_r(\alpha, x) \frac{\mathcal{L}_1(L_A, h)}{\alpha} dV(x)
\]
\[
\leq \log T_{f,A}(r) + O(1).
\]

Hence, for the case \( h = 1 \)
\[
m(\rho, c_\lambda \circ f \cdot \mathcal{X}(w_M \circ f)) \leq O \left( \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right).
\]

For the general case, we use mathematical induction. Assume that the claim holds for the case \( h = k - 1 \), then for \( h = k \)
\[
m(\rho, c_\lambda \circ f \cdot \mathcal{X}(w_M \circ f)) \leq m \left( \rho, \mathcal{X}(w_M \circ f) \right) + m \left( \rho, c_\lambda \circ f \cdot \mathcal{X}(w_M \circ f) \right)
\]
\[
\leq O \left( \log T_{f,A}(r) + \log^+ \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right)
\]
due to Proposition 3.3. Thus, the claim is confirmed. Combining the above, This proves the first assertion, and the second assertion is proved similarly by replacing \( \kappa(r)r^2 \) by \( -r \).

\[
\text{Corollary 5.2. Let } f : C \to M \text{ be a holomorphic curve such that } f(C) \not\subseteq D, \text{ where } D \text{ is a reduced divisor on } M. \text{ Then}
\]
\[
m(\rho, \omega(J_{k,x} f)) \leq O \left( \log T_{f,A}(r) + \log^+ \log r \right)
\]
for \( \omega \in H^0(M, E_{k,m}^{\log}(\mathcal{O}(D))) \).

5.2. Proof of Theorem 1.5

\[
\text{Proof. By lifting } f \text{ to the universal covering, we may assume that } S \text{ is simply connected. Since } A \text{ is ample, then there is an integer } n_0 > 0 \text{ such that } n_0 A \text{ is very ample. Hence, } L_{n_0} \text{ induces a holomorphic embedding of } M \text{ into a complex projective space. Viewing } M \text{ as an algebraic submanifold of } \mathbb{P}^N(C) \text{ with homogeneous coordinate system } [w_0 : \cdots : w_N], \text{ i.e., there is an inclusion } i : M \hookrightarrow \mathbb{P}^N(C) \text{ for some } N. \text{ Let } E \text{ be the restriction of the hyperplane line}
\]
bundle over $\mathbb{P}^N(\mathbb{C})$ to $M$. By virtue of Theorem 11 for any $0 < \epsilon < 1$, there exists a hyperplane $H$ of $\mathbb{P}^N(\mathbb{C})$ such that

$$(1 - \epsilon)T_{f,E}(r) \leq N_f(r, i^*H) + O\left(\log T_{f,E}(r) - \kappa(r)r^2 + \log^+ \log r\right).$$

By replacing $\omega$ by $(\omega/\|s_A\|)^{n_0}i^*\hat{H}$, where $s_A$ is the canonical line bundle associated to $A$ and $\hat{H}$ is the defining function of $H$, we may assume without loss of generality that $A = i^*H$ and $n_0 = 1$. Namely, we have

$$(1 - \epsilon)T_{f,A}(r) \leq N_f(r, A) + O\left(\log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r\right).$$

By the assumption that $\kappa(r)r^2/T_{f,A}(r) \to 0$ as $r \to \infty$, it yields that

$$(21) \quad (1 - \epsilon)T_{f,A}(r) \leq N_f(r, A) + o(T_{f,A}(r)).$$

If $f^*\omega \not\equiv 0$ on $S$, then the condition $\omega|A \equiv 0$ implies that

$$f^*A \leq dd^c [\log |\omega(J_{k,X}f)|^2]$$

in the sense of currents. By Dynkin formula and Theorem 5.1

$$N_f(r, A) \leq m(r, \omega(J_{k,X}f)) + O(1) \leq o(T_{f,A}(r)),$$

which contradicts with (21). This proves the first assertion, and the second assertion is proved similarly by replacing $\kappa(r)r^2$ by $-r$. □

6. Bloch’s theorem

In this section, we extend Bloch’s theorem [1], following Păun-Sibony [22] and Ru-Sibony [25].

Let $T$ be a complex torus of complex dimension $n$, fix a nowhere-vanishing holomorphic vector field $\mathfrak{X}$ over $S$. Then the $k$-jet bundle over $T$ with respect to $\mathfrak{X}$ is trivial, i.e., $J_k(T, \mathfrak{X}) \cong T \times \mathbb{C}^nk$. Let $P(J_k(T, \mathfrak{X}))$ be the projective bundle of $J_k(T, \mathfrak{X})$, then we have $P(J_k(T, \mathfrak{X})) \cong T \times \mathbb{P}^{nk-1}(\mathbb{C})$. Let $X$ denote the Zariski closure of $f(S)$, and $X_k \subseteq P(J_k(T, \mathfrak{X}))$ denote the Zariski closure of $J_k,T,S,f(S)$. Let $\pi_k : X_k \to \mathbb{P}^{nk-1}(\mathbb{C})$ be the second projection.

**Lemma 6.1** (Ueno, [26]). Let $V$ be a subvariety of $T$. There exists a complex torus $T_1 \subseteq T$, a projective variety $W$ and an Abelian variety $A$ such that

(i) $W \subseteq A$ and $W$ is of general type;

(ii) There exists a reduction mapping $R : V \to W$ such that whose general fiber is isomorphic to $T_1$.

**Lemma 6.2** (Sibony-Păun, [22]). Assume that for each $k \geq 1$, the fibers of $\pi_k$ are positive dimensional. Then the subgroup $A_X$ of $X$ defined by

$$A_X := \{a \in T : a + X = X\}$$

is strictly positive dimensional.
Lemma 6.3 (Ru-Sibony, [25]). Assume that $\pi_k : X_k \to \mathbb{P}^{nk-1}(\mathbb{C})$ has finite generic fibers for some $k > 0$. Then there is a $k$-jet differential $\omega$ with values in the dual of an ample line bundle such that $\omega \not\equiv 0$ on $X_k$.

Now we prove a generalized Bloch’s theorem, i.e., Theorem 1.6.

Proof. Assume that $X$ is not the translate of a subtorus. Let $R : X \to W$ be the reduction mapping in Lemma 6.1. Then $W$ is not a point since otherwise $X$ would be the translation of a subtorus. Now we claim that $R \circ f$ cannot satisfy (3). If not, then there will be two possible cases:

(i) For any $k \geq 1$, the fibers of $\pi_k$ are positive dimensional;
(ii) There exists a $k$ such that $\pi_k : X_k \to \mathbb{P}^{mk-1}(\mathbb{C})$ is generically finite.

If the case (i) happens, then from Lemma 6.2, $X$ is stabilized by a positive-dimensional subtorus of $T$. But we assumed that $X$ is of general type, then it cannot happen since the automorphism group of $X$ must be of finite order.

If the case (ii) happens, then $X_k$ is algebraic. By Lemma 6.3, there exists a nonzero $k$-jet differential $\omega \in H^0(X_k, E^G_{k,m}T^*_{X_k} \otimes \mathcal{O}(-A))$ for some ample line bundle $A$, but Theorem 1.5 gives a contradiction. So, (ii) cannot happen. We have arrived at a contradiction in both cases. Thus, $R \circ f$ cannot satisfy (3). $\square$

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Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing, 100190, P. R. China

Email address: xjdong@amss.ac.cn