ON THE STABILIZER OF WEIGHT ENUMERATORS OF LINEAR CODES

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Abstract. This paper investigates the relation between linear codes and the stabilizer in $\text{GL}_2(\mathbb{C})$ of their weight enumerators. We prove a result on the finiteness of stabilizers and give a complete classification of linear codes with infinite stabilizer in the non-binary case. We present an efficient algorithm to compute explicitly the stabilizer of weight enumerators and we apply it to the family of Reed-Muller codes to show that some of their weight enumerators have trivial stabilizer.

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1. Introduction

Counting $\mathbb{F}_q$-rational points on hypersurfaces is, in general, a very difficult task. In [E06], Noam Elkies uses Coding Theory to give the point counts of cubic surfaces in the 3-dimensional projective space over $\mathbb{F}_q$. His idea was later generalized by Nathan Kaplan in his PhD thesis [K13]. The starting point of their method is that a counting of points of varieties belonging to a family $\mathcal{F}$ can be deduced by the determination of the weight enumerator of a linear code associated to $\mathcal{F}$. In fact, such code is related to well-known ones in Coding Theory, namely the Reed-Muller codes (in their affine or projective version).

One of the most remarkable theorems in Coding Theory is Andrew Gleason’s 1970 Theorem [G70], which classifies the weight enumerators of self-dual doubly-even codes. The crucial argument in the proof this theorem is to observe that self-duality and divisibility give interesting invariants for the weight enumerator of self-dual codes: if $p(x, y) \in \mathbb{Z}[x, y]$ is the homogeneous weight enumerator of a self-dual doubly-even binary code, then

$$p(x, y) = p(x, iy) = p \left( \frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}} \right).$$

This fact has a lot of significant implications on the shape of $p(x, y)$ and consequently on the family of self-dual doubly-even codes. Along those lines, we look for invariances of weight enumerators of Reed-Muller codes. A divisibility condition is proved by James Ax already in 1964 [A64] for the affine version of Reed-Muller codes. The counterpart for the projective version can be easily deduced. On the other hand, Reed Muller codes are not, in general, self-dual. An analogue of
the condition coming from self-duality is much harder to be derived, since the nature of weight enumerator is essentially combinatorial and not geometric, as remarked in [Ken94].

We decided to approach the problem with a different strategy: since some weight enumerators of Reed-Muller codes are known, we wanted to calculate their invariants directly, with the hope to find general invariants of a bigger family of codes. This happened to be not so easy in practice, because the calculation of all invariants directly is computationally complex. So we developed a new method based on the action of $\text{PGL}_2(\mathbb{C})$ on the projective line $\mathbb{P}^1(\mathbb{C})$ (which is simply 3-transitive).

This trick gives a huge restriction on potential elements in $\text{GL}_2(\mathbb{C})$.

Using this technique, we were able to prove the following.

**Theorem 1.** The homogeneous weight enumerator of linear code has a finite number of invariants if and only if its non-homogeneous version has at least 3 distinct complex roots.

We designed an algorithm to calculate approximated invariants, and this allowed us to prove the existence of Reed Muller codes whose weight enumerator have trivial stabilizer: for example, the Reed Muller codes which encodes conics in the affine plane over $\mathbb{F}_3$.

**Theorem 2.** The following codes have weight enumerator with trivial stabilizer in $\text{PGL}_2(\mathbb{C})$, i.e. with stabilizer in $\text{GL}_2(\mathbb{C})$ consisting only of scalar matrices:

\[
\mathcal{R}M_4(2,2), \quad \mathcal{R}M_4(3,2), \quad \mathcal{R}M_5(2,2),
\]
\[
\mathcal{P}\mathcal{R}M_5(3,2), \quad \mathcal{P}\mathcal{R}M_5(3,2)^i = \mathcal{P}\mathcal{R}M_5(5,2).
\]

This suggests that no general non-trivial invariant can be found for Reed Muller codes. However, three main questions arise from what we have done.

**Question 1.** What are the codes whose weight enumerators have at most 2 distinct complex roots?

**Question 2.** Can we find, with the algorithm described, new families of codes for which there exist non-trivial invariants?

**Question 3.** If the answer to Question 2 is positive, can we determine unknown weight enumerators of large codes?

The present paper gives an almost complete answer to Question 1. In particular, we prove the following classification theorem.

**Theorem 3.** Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a linear code with weight enumerator $W_\mathcal{C}(x) \in \mathbb{Z}[x]$ having at most two distinct complex roots. Then only the following possibilities may hold:

(a) $W_\mathcal{C}(x) = x^n$ and $\mathcal{C} = \langle \mathbf{1} \rangle$;
(b) $W_\mathcal{C}(x) = (x + (q - 1))^n$ and $\mathcal{C} = \mathbb{F}_q^n$;
(c) $n$ is even, $W_\mathcal{C}(x) = (x^2 + (q - 1))^{n/2}$ and, if $q \neq 2$, $\mathcal{C} \cong \bigoplus_{i=1}^{n/2} \langle (1,1) \rangle_{x_q}$.

Question 2 and 3 are of course more general, and harder to answer. However, while the approach fails for Reed-Muller codes, it does not imply that the answer to these questions is negative.

In §2 we present the necessary background of Coding Theory and we define invariants for weight enumerators. The classification of linear codes with weight enumerator with at most two distinct complex roots is given in §3. In §4 we prove Theorem 1 about the finiteness of the stabilizer. Finally, §5 and §6 are devoted to the presentation of the algorithm and its applications.

## 2. Background

A **linear code** $\mathcal{C}$ is a subspace of $\mathbb{F}_q^n$, where $n$ is a positive integer called the length of the code. A **generator matrix** of a linear code $\mathcal{C}$ is a matrix whose rows generates $\mathcal{C}$. Elements of $\mathcal{C}$ are called **codewords**. The support of a codeword $c \in \mathcal{C}$, denoted $\text{supp}(c)$, is defined as follows:

$$\text{supp}(c) := \{i \in \{1, \ldots, n\} \mid c_i \neq 0\}.$$
The weight \( \text{wt}(c) \) of a codeword \( c \) is the cardinality its support. The (one-variable) weight enumerator of \( C \leq \mathbb{F}_q^n \) is the polynomial

\[
W_C(x) := \sum_{c \in C} x^{n-\text{wt}(c)} = \sum_{i=0}^n a_i x^i, \quad a_i := \# \{ \text{codewords of weight } n-i \}.
\]

We will call homogeneous weight enumerator the homogeneous version of \( W_C(x) \), i.e.

\[
W_C(x, y) = \sum_{c \in C} x^{n-\text{wt}(c)} y^{\text{wt}(c)} = \sum_{i=0}^n a_i x^i y^{n-i}.
\]

If \( C \leq \mathbb{F}_q^n \) and \( D \leq \mathbb{F}_q^n \) are linear codes with generator matrices \( C \) and \( D \) respectively, their direct sum is the vector space \( C \oplus D \) naturally embedded in \( \mathbb{F}_q^{n+m} \), i.e. the code with generator matrix

\[
\begin{bmatrix}
C & 0 \\
0 & D
\end{bmatrix}.
\]

Observe that

\[
W_{(C \oplus D)} = W_C \cdot W_D
\]

for the one-variable and homogeneous weight enumerators.

A monomial transformation \( f : \mathbb{F}_q^n \to \mathbb{F}_q^n \) is a linear transformation of the form

\[
f : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n \\
v \longmapsto D P v
\]

where \( D \) is an \( n \times n \) diagonal matrix with non-zero diagonal entries, and \( P \) is an \( n \times n \) permutation matrix. Two codes are said to be equivalent if one is the image of the other under a monomial transformation. Observe that in the case \( q = 2 \), a monomial transformation is just a permutation. It is easy to observe that two equivalent codes have the same weight enumerator.

Convention: Since we are interested in weight enumerators, we will usually identify codes up to equivalence. In particular, a generator matrix of a code \( C \) will mean a generator matrix of some code equivalent to \( C \).

\( \text{GL}_2(\mathbb{C}) \) acts naturally on \( \mathbb{C}[x, y] \) as follows:

\[
A \cdot p(x, y) = p(ax + by, cx + dy) \quad \text{for} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{C}), \quad \text{and} \quad p \in \mathbb{C}[x, y].
\]

Every element in the stabilizer \( \text{Stab}_{\text{GL}_2(\mathbb{C})}(p) \) of \( p \) is called invariant of \( p \).

There are two main properties which give rise to invariants of weight enumerators.

1. Let \( \Delta > 1 \) be an integer. A linear code \( C \) is divisible by \( \Delta \) if the weight of every codeword of \( C \) is divisible by \( \Delta \). A code is divisible, if it is divisible by some \( \Delta > 1 \). In terms of weight enumerator, a linear code \( C \) is divisible by \( \Delta \) if and only if

\[
D_\Delta := \begin{bmatrix} 1 & 0 \\ 0 & \zeta_\Delta \end{bmatrix} \in \text{Stab}_{\text{GL}_2(\mathbb{C})}(W_C(x, y)),
\]

where \( \zeta_\Delta \) is a primitive \( \Delta \)-root of unity.

2. The dual of a linear code \( C \leq \mathbb{F}_q^n \) (denoted \( C^\perp \)) is the orthogonal space with respect to the standard inner product of \( \mathbb{F}_q^n \), \( \langle x, y \rangle = \sum_{i=1}^n x_i y_i \) for \( x, y \in \mathbb{F}_q^n \), i.e.

\[
C^\perp := \{ v \in \mathbb{F}_q^n \mid \langle v, c \rangle = 0 \text{ for all } c \in C \}.
\]

The relation between the weight enumerator of some code \( C \) and its dual \( C^\perp \) is given by MacWilliams’ Theorem.

Theorem (MacWilliams, [MS77]). Let \( C \leq \mathbb{F}_q^n \) be a linear code, and \( C^\perp \) its dual. Then

\[
W_{C^\perp}(x, y) = \frac{1}{\#C} W_C(x + (q-1)y, x - y).
\]
A linear code is self-dual if $C^\perp = C$. In that case we have $\#C = \#C^\perp = q^{n/2}$, and so MacWilliams' Theorem implies that

$$S_q := q^{-1/2} \begin{bmatrix} 1 & q - 1 \\ 1 & -1 \end{bmatrix} \in \text{Stab}_{\text{GL}_2(\mathbb{C})}(W_C(x, y)) \quad (\star).$$

A linear code which satisfy $(\star)$ is called formally self-dual.

If $G \leq \text{GL}_2(\mathbb{C})$ is a group of matrices, the ring

$$\mathbb{C}[x, y]^G := \{ p(x, y) \in \mathbb{C}[x, y] \mid A \cdot p(x, y) = p(x, y) \text{ for all } A \in G \}$$

is called the invariant ring of $G$.

A beautiful result by Gleason [G70] states that the weight enumerator of a self-dual binary linear code which is doubly-even, i.e. divisible by 4, lies in the ring

$$\mathbb{C}[x^8 + 14x^4y^4 + y^8, x^4y^4(x^4 - y^4)^4],$$

which is the invariant ring of $G := (D_4, S_2)$. This result has a lot of consequences on the family $\mathcal{F}$ of self-dual doubly-even binary linear codes. Among the others, the length of a code in $\mathcal{F}$ is always divisible by 8 and one can derive upper bounds on the minimum non-zero weight of such codes, depending only of the length.

As we said in the introduction, inspired by this result, we look for invariances of weight enumerators of Reed-Muller codes. Let us briefly introduce this family of codes.

The Reed-Muller code $\mathcal{RM}_q(r, m)$ on $m$ variables, of degree $r$ and defined over $\mathbb{F}_q$, is the code

$$\mathcal{RM}_q(r, m) := \{(p(v))_{v \in \mathbb{F}_q^m} \mid p \in \mathbb{F}_q[x_1, \ldots, x_m]^r\},$$

where $\mathbb{F}_q[x_1, \ldots, x_m]^r$ is the ring of polynomials in $m$ variables, with coefficient in $\mathbb{F}_q$ and of degree at most $r$. Clearly, $\mathcal{RM}_q(r, m)$ encodes all the hypersurfaces in $\mathbb{A}^m(\mathbb{F}_q)$ of degree at most $r$. So, determining the weight enumerator of such a code is equivalent to counting $\mathbb{F}_q$-rational points of hypersurfaces in the affine space.

Similarly, we can define the projective Reed-Muller code $\mathcal{PRM}_q(r, m)$ on $m$ variables, of degree $r$ and defined over $\mathbb{F}_q$ in the following way:

$$\mathcal{PRM}_q(r, m) := \{(p(v))_{v \in R} \mid p \in \mathbb{F}_q[x_0, \ldots, x_m]^h\} \cup \{0\},$$

where $\mathbb{F}_q[x_0, \ldots, x_m]^h$ is the ring of degree $r$ homogeneous polynomials in $m + 1$ variables, with coefficient in $\mathbb{F}_q$, and $R$ is a set of representatives of all the points of $\mathbb{P}^m(\mathbb{F}_q)$. Observe that changing the set of representatives $R$ gives rise to an equivalent code.

Both Reed Muller codes and projective Reed Muller codes are divisible codes, as a consequence of a theorem by Ax [A64]. In general, they are not self-dual codes.

3. Weight enumerators with at most two distinct complex roots

One of the main problem in Coding Theory is to determine a code with a prescribed weight enumerator. The problem in general is extremely difficult, since, as remarked in §1, codes are geometric objects while weight enumerators are combinatorial objects. In this section we will consider very particular weight enumerators, which are important for our purposes, that is polynomials with at most two distinct complex roots. Note that all roots of weight enumerators are algebraic integers, since every weight enumerator is a monic polynomial with integer coefficients. We will give an almost complete classification of codes with such a weight enumerator.

We start with a quite technical lemma which is fundamental for our classification.

**Lemma 3.1.** Let $C$ be a code over $\mathbb{F}_q$ with $q \neq 2$. Assume all codewords of $C$ have even weight. Let $c \in C$ be a codeword of weight two, and $x \in C$ an arbitrary codeword. Let $\text{supp}(c) = \{i, j\}$. Then there exists $\lambda \in \mathbb{F}_q$ such that $(x_i, x_j) = (\lambda c_i, \lambda c_j)$. 
Proof. Suppose that \((x_i, x_j) \neq (\lambda c_i, \lambda c_j)\) for every \(\lambda \in \mathbb{F}_q\). In particular, \((x_i, x_j) \neq (0,0)\). Without lost of generality, we assume that \(x_i \neq 0\). If \(x_j = 0\), then there exists \(\mu \in \mathbb{F}_q^*\) such that 
\[
x_i + \mu c_i \neq 0 \quad \text{and} \quad \mu c_j = x_j + \mu c_j \neq 0.
\]
Indeed, it suffices to take any \(\mu \in \mathbb{F}_q^* \setminus \{-x_i c_i^{-1}\}\), that is non-empty because \(q > 2\). Then 
\[
\text{supp}(x + \mu c) = \text{supp}(x) \cup \{j\}
\]
so that \(x + \mu c\) has odd weight, which gives a contradiction. If \(x_j \neq 0\), then 
\[
\text{supp}(x - (x_i c_i^{-1}) c) = \text{supp}(x) \setminus \{i\},
\]
so that \(x - (x_i c_i^{-1}) c\) has odd weight, which gives again a contradiction. \(\square\)

An immediate consequence is the following.

**Corollary 3.1.** Let \(C\) be a code over \(\mathbb{F}_q\) with \(q \neq 2\). Assume all codewords of \(C\) have even weight. Let \(c_1, \ldots, c_r \in C\) of weight 2 such that \(c_i\) is not in \((c_j)_{\mathbb{F}_q}\) for any \(i \neq j\). Then 
\[
\text{supp}(c_i) \cap \text{supp}(c_j) = \emptyset,
\]
for every \(i \neq j\).

So, we can get the first classification result.

**Lemma 3.2.** Let \(C\) be a linear code of even length \(n\) over \(\mathbb{F}_q\) with \(q \neq 2\). Suppose that 
\[
W_C(x) = (x^2 + a)^{n/2}, \quad a \in \mathbb{R} \setminus \{0\}.
\]
Then \(a = q - 1\) and 
\[
C \cong \bigoplus_{i=1}^{n/2} \langle (1,1) \rangle_{\mathbb{F}_q}.
\]

**Proof.** If \(n = 2\), then clearly \(C = \langle (1,1) \rangle_{\mathbb{F}_q}\). Write \(W_C(x) = \sum_{i=0}^n a_i x^i\), so that \(a_i\) is the number of codewords of weight \(n - i\). Expanding the above expression, we see that \(C\) has no codewords of odd weight. Moreover, the number of codewords of length 2 is \(a_{n-2} = an/2 \neq 0\). Let \(r := a_{n-2}/(q-1)\) and let \(c_1, \ldots, c_r\) be a set of codewords of weight 2 such that \(c_i\) is not in \((c_j)_{\mathbb{F}_q}\) for any \(i \neq j\). They have disjoint supports by Corollary 3.1.

Let \(S := \bigcup_i \text{supp} c_i\) and let \(C_S := \langle c_1, \ldots, c_r \rangle_{\mathbb{F}_q}\). Every codeword \(x \in C\) can be written as a sum 
\[
x = y + z,
\]
with \(\text{supp}(y) \subset S\) and \(\text{supp}(z) \cap S = \emptyset\). By Lemma 3.1, \(y \in C_S \subset C\), so that \(z \in C\). Consequently, \(C\) is the direct sum 
\[
C = C_S \oplus C_{S^c},
\]
where \(C_{S^c} = \{c \in C \mid \text{supp}(c) \cap S = \emptyset\}\). This implies that \(W_C(x) = W_{C_S}(x) \cdot W_{C_{S^c}}(x)\).

Now observe that \(C_S\) is monomially equivalent to the code \(\bigoplus_{i=1}^{n/2} \langle (1,1) \rangle_{\mathbb{F}_q}\), and hence its weight enumerator is \(W_{C_S}(x) = (x^2 + (q - 1))^r\). Therefore, we must have \(a = (q - 1)\). By induction, 
\[
C_{S^c} \cong \bigoplus_{i=1}^{n/2-r} \langle (1,1) \rangle_{\mathbb{F}_q}\]
so that \(C \cong \bigoplus_{i=1}^{n/2} \langle (1,1) \rangle_{\mathbb{F}_q}\), as desired. \(\square\)

Let us now prove the classification theorem for codes whose weight enumerator has at most two distinct roots in \(\overline{\mathbb{Z}}\) (the set of all algebraic integers).

**Theorem 3.1.** Let \(C \in \mathbb{F}_q^n\) be a linear code with weight enumerator \(W_C(x) \in \mathbb{Z}[x]\) having at most two distinct roots in \(\overline{\mathbb{Z}}\). Then only the following possibilities may hold:

(a) \(W_C(x) = x^n\) and \(C = \{0\}\);
(b) \(W_C(x) = (x + (q - 1))^n\) and \(C = \mathbb{F}_q^n\);
(c) \(n\) is even, \(W_C(x) = (x^2 + (q - 1))^{n/2}\) and, if \(q \neq 2\), \(C \cong \bigoplus_{i=1}^{n/2} \langle (1,1) \rangle\).
Proof. Let \(-a, -b\) be the roots of \(W_C(x)\) in \(\mathbb{Z}\), so that \(W_C(x) = (x + a)^r(x + b)^{n - r}\) for \(r \in \mathbb{N}\). The number of codewords in \(C\) of weight one is then \(c := ra + (n - r)b\).

First, assume that \(c \neq 0\) and let \(m = c/(q - 1)\). Taking arbitrary linear combinations of the codewords of weight one gives a copy of \(\mathbb{F}_q^m\) in \(C\). Therefore, \(C = C_1 \oplus C_2\), with \(C_1 = \mathbb{F}_q^m\) and \(C_2 := \{c \in C \mid \text{supp}(c) \cap \text{supp}(d) = \emptyset \ \forall d \in C_1\}\). Hence

\[
W_C(x) = (x + (q - 1))^m \cdot W_{C_2}(x).
\]

Consequently, \(-(q - 1)\) is a root of \(f\) and so either \(a\) or \(b\) is equal to \((q - 1)\). Assume \(a = (q - 1)\) without loss of generality. We get

\[
m = \frac{ra + (n - r)b}{q - 1} = r + \frac{(n - r)b}{q - 1} \geq r.
\]

Hence, either \(b = 0\) and \(r = n\), or \((x + (q - 1))\) divides \((x + b)^{n - r}\), which implies \(b = q - 1\). Both cases give that \(W_C(x) = (x + (q - 1))^n\). But this implies \(#C = W_C(1) = q^n = \#\mathbb{F}_q^n\), whence \(C = \mathbb{F}_q^n\), as desired.

Now assume \(C\) has no codewords of weight one, i.e. \(c = ra + (n - r)b = 0\). If \(a\) is real then so is \(b\), and both must be non-negative: indeed, since \(W_C(x)\) is non-zero and has positive coefficients, \(W_C(y) > 0\) for any positive real \(y\), so \(W_C\) has only non-positive roots. Since \(ra = -(n - r)b\), we must have \(a = b = 0\) whence \(W_C\) has one root and \(C = \{0\}\).

If \(a\) is non-real, then \(a b\) are complex conjugate algebraic integers, and we must have \(r = s\), which is possible only if \(n\) is even. Consequently,

\[
W_C(x) = (x^2 + \text{Tr}(a)x + N(a))^{n/2},
\]

where \(\text{Tr}(a) = a + \bar{a}\) and \(N(a) = a\bar{a}\). The fact that \(C\) has no codeword of weight one implies that \(\text{Tr}(a) = 0\), and hence,

\[
W_C(X) = (x^2 + N(a))^{n/2}.
\]

If \(q \neq 2\), Lemma 3.2 gives the desired conclusion about \(C\). If \(q = 2\), then we must show that \(N(a) = 1\). Since \(a\) is an algebraic integer, \(N(a) \in \mathbb{Z}\). Since \(q = 2\), the number of codewords of weight \(n\) is

\[
N(a)^{n/2} = 1
\]

so that \(N(a) = \pm 1\). If \(N(a) = -1\) we have negative coefficients in \(W_C(X)\), which is not possible. \(\square\)

Note that Theorem 3.1 almost classify, up to monomial equivalence, all linear codes with weight enumerator with at most two distinct roots in \(\mathbb{Z}\). The case \(q = 2\) is left unsolved and it seems quite difficult to be settled. If \(q = 2\), the sum of two codewords of weight 2 cannot have weight 3, so that the argument in the proof of Lemma 3.2 does not work.

**Question 3.1.** Is it possible to classify all the binary codes of length \(n\) with weight enumerator \((x^2 + 1)^{n/2}\)?

Let \(C\) and \(C'\) two codes with weight enumerator \((x^2 + 1)^{n/2}\) and \((x^2 + 1)^{n'/2}\) respectively. Then \(C \oplus C'\) has weight enumerator \((x^2 + 1)^{(n+n')/2}\). Hence, if we denote

\[
\mathcal{M} := \{\text{binary codes of length } n \text{ and weight enumerator } (x^2 + 1)^{n/2} \mid n \in 2\mathbb{N}\},
\]

we have that \((\mathcal{M}, \oplus)\) is a semigroup: in order to answer positively to Question 3.1 it suffices to find all irreducible elements in \(\mathcal{M}\), which means to find a minimal set of generators of \((\mathcal{M}, \oplus)\). We consider, as usual, elements in \(\mathcal{M}\) as classes of codes up to equivalence.

Clearly, the generator with minimum length is the \([2, 1, 2]\) code \(X_1 := \{(1, 1)\} \). Furthermore, every element in \(\mathcal{M}\) is formally self-dual and all formally self-dual codes up to length 16 are classified in [BH01]. From an analysis of the tables in the paper, we have that, up to length 16, there are exactly 4 other irreducible elements of \(\mathcal{M}\), namely the formally self-dual (but not self-dual) \([6, 3, 2]\) code \(X_2\) with generator matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
and three \([14,7,2]\) codes, which we call \(X_3, X_4\) and \(X_5\), with generator matrices \([I|X_3],[I|X_4]\) and \([I|X_5]\) respectively, where

\[
X_3 := \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}, \quad X_4 := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad X_5 := \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ \end{bmatrix}
\]

and \(I\) is the \(7 \times 7\) identity matrix. It is not clear how to construct other generators and it seems already too complex for software like MAGMA [BCP97]. It is not evident if there are infinitely many such generators or not.

We conclude this section showing a relation between our result and the Gleason-Pierce Theorem (cf. [Slo79] for a proof). Recall that a code is divisible if there exists an integer \(\Delta > 1\) such that the weight of every codeword of \(C\) is divisible by \(\Delta\).

**Theorem 3.2** (Gleason-Pierce). Let \(C\) be a formally self-dual code divisible by some \(\Delta > 1\). Then

- \(q = 2\) and \(\Delta \in \{2,4\}\),
- \(q = 3\) and \(\Delta = 3\),
- \(q = 4\) and \(\Delta = 2\),
- or \(q\) arbitrary, \(\Delta = 2\) and \(W_C(x) = (x^2 + (q - 1))^{n/2}\).

Hence, Theorem 3.1 implies the following.

**Corollary 3.2.** For \(q > 4\), if \(C\) is a formally self-dual divisible code of length \(2n\), then \(C\) is equivalent to the direct sum of \(n\) copies of \(((1,1))_{F_q}\).

4. PROOF OF THEOREM 1

Let us start from a general lemma.

**Lemma 4.1.** Let \(p(x,y) \in \mathbb{C}[x,y]\) be a homogeneous polynomial and let \(n\) be the number of distinct complex roots of \(p(x,1)\). If \(n \geq 3\), then Then

\[
\# \text{Stab}_{GL_2(\mathbb{C})}(p(x,y)) \leq n! \deg(p(x,1)).
\]

In particular, \(\text{Stab}_{GL_2(\mathbb{C})}(p(x,y))\) is finite if \(n \geq 3\).

**Proof.** Let \(V\) denote the projective variety defined by the vanishing of \(p(x,y)\). Then

\[
V(\mathbb{C}) = \{(x : 1) \in \mathbb{P}^1(\mathbb{C}) \mid f(x) = 0\},
\]

so that \(\#V(\mathbb{C}) \geq 3\). Set \(G := \text{Stab}_{GL_2(\mathbb{C})}(p(x,y))\). Since every \(A \in G\) fixes \(p(x,y)\), \(G\) acts on \(V(\mathbb{C})\), and since scalar matrices fix \(V(\mathbb{C})\) point-wise, this action induces an action of \(G \subset \text{PGL}_2(\mathbb{C})\) on \(V(\mathbb{C})\), where \(G\) is the image of \(G\) in \(\text{PGL}_2(\mathbb{C})\). It is well-known that \(\text{PGL}_2(\mathbb{C})\) acts simply 3-transitively on \(\mathbb{P}^1(\mathbb{C})\). Since \(\#V(\mathbb{C}) \geq 3\), every permutation of \(V(\mathbb{C})\) is realized by at most one \(A \in \text{PGL}_2(\mathbb{C})\), and hence by at most one \(A \in G\).

Such a \(A\) has exactly \(\deg(p(x,1))\) pre-images in \(G\): if \(A \in G\) is such a pre-image, then all the pre-images in \(\text{PGL}_2(\mathbb{C})\) are given by \(\lambda A\) for \(\lambda \in \mathbb{C} \setminus \{0\}\). But

\[
(\lambda A) \cdot p(x,y) = \lambda^{\deg(p(x,1))}(A \cdot p(x,y)) = \lambda^{\deg(p(x,1))}p(x,y),
\]

so that \(\lambda A \in G\) if and only if \(\lambda^{\deg(p(x,1))} = 1\). There are \(n!\) permutations of \(V(\mathbb{C})\), so at most \(n!\) elements in \(G\). It follows that \(\#G \leq n! \deg(p(x,1))\).

\[\square\]

Let us prove Theorem 1, which we restate here for reader convenience.

**Theorem 4.1.** The homogeneous weight enumerator of linear code has a finite number of invariants if and only if its non-homogeneous version has at least 3 distinct roots in \(\mathbb{Z}\).
Proof. Lemma 4.1 gives immediately that if the weight enumerator has at least 3 distinct roots in \( \mathbb{Z} \), then the stabilizer of its homogeneous version is finite. If the weight enumerator of a linear code \( C \) has at most 2 distinct roots, then Theorem 3.1 implies that there exist \( a, m \in \mathbb{N} \setminus \{0\} \) such that \( W_C(x, y) \) is equal to one of the following:

\[
x^m, \quad (x + ay)^m, \quad (x^2 + ay^2)^m.
\]

It is easy to observe that in all three cases we have an infinite stabilizer. \( \square \)

5. The algorithm

The proof of Lemma 4.1 gives an algorithm to find the stabilizer of every weight enumerator. Let \( C \) be a linear code. Suppose that its homogeneous weight enumerator \( W_C(x, y) \) is known and of degree \( n \).

1. Set \( G := \emptyset \).
2. Calculate \( V(C) := \{z_1, \ldots, z_n\} \) the set of roots of \( W_C(x, 1) \).
3. Call \( V(C)_3 \) the set of all ordered 3-subsets of \( V(C) \); we have \( \#V(C)_3 = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n \).
4. For every triple \( \{w_1, w_2, w_3\} \in V(C)_3 \):
   4a. solve the system \( z_i a + b - w_i z_i c - w_i d = 0, \ i \in \{1, 2, 3\} \), where the unknowns are \( a, b, c, d \).
   It has clearly infinitely many solutions depending of one complex parameter \( \lambda \) (the action of \( \text{PGL}_2(\mathbb{C}) \) is simply 3-transitive, as we said). Call \( a, b, c, d \) one solution.
   4b. If \( \{\frac{a}{z_i}, \frac{b}{z_i}, \frac{c}{z_i}, \frac{d}{z_i}\} | z_i \in V(C) \} = V(C) \), then
   4bi. let \( A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).
   4bii. Calculate \( \lambda := \frac{W_C(b, d)}{W_C(1, 1)} \).
   4biii. Let \( G := G \cup \{\zeta_n^{\lambda/m} A | \zeta_n \in \mathbb{C} \text{ s.t. } \zeta_n^n = 1\} \).

Then \( G \) is equal to \( \text{Stab}_{\text{GL}_2(\mathbb{C})}(W_C(x, y)) \).

This algorithm can be implemented easily in MAGMA, but there is a problem for Step 2.: in \( \mathbb{C} \), we do not have access to the exact roots but only to approximations. Of course, one can consider the splitting field of \( W_C(x, 1) \) instead of \( \mathbb{C} \), but this does not give explicit solutions. So, whenever Step 2. is feasible explicitly, the algorithm gives the exact form for the stabilizer. Otherwise, we get an approximate version.

To show that some Reed-Muller codes have weight enumerator with trivial stabilizer, one needs to control the error made by approximating the roots over \( \mathbb{C} \). The following lemmas are useful for this.

Recall that the cross ratio of four points \( (z_1 : 1), \ldots, (z_4 : 1) \) is defined as

\[
[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.
\]

Make the symmetric group \( S_4 \) acts on the cross ratios by permuting the points, and observe that for any \( \sigma \in V_4 := \{\text{id}, (12)(34), (13)(24), (14)(23)\} \), we have \( [z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}] = [z_1, z_2, z_3, z_4] \).

Let \( Z \) be a set of at least four complex points. A 4-tuple of distinct elements

\[
\zeta = (z_1, z_2, z_3, z_4) \in Z^4
\]

is called critical if for any 4-tuple of distinct elements \( (y_1, y_2, y_3, y_4) \in Z^4 \), we have

\[
[z_{1}, z_{2}, z_{3}, z_{4}] = [y_1, y_2, y_3, y_4]
\]

if and only if \( (y_1, y_2, y_3, y_4) = \zeta^\sigma = (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) \) for some \( \sigma \in V_4 \).

The following lemma gives the crucial argument to show that the stabilizer is trivial.

**Lemma 5.1.** Let \( p(x, y) \in \mathbb{C}[x, y] \) be a polynomial with 5 roots \( z_1, z_2, z_3, z_4, z_5 \) of \( p(x, 1) \) such that both \( (z_1, z_2, z_3, z_4) \) and \( (z_1, z_2, z_3, z_5) \) are critical. Then \( \text{Stab}_{\text{GL}_2(\mathbb{C})}(p(x, y)) \) is trivial.
Proof. Every $\bar{A}$ in $\text{PGL}_2(C)$ fixes $\{z_1, z_2, z_3, z_4\}$, for $j = 4, 5$, since it must preserve the cross ratio of these four points. If $A$ sends $z_4$ to $z_j$ for $j \in \{1, 2, 3\}$, then from the fact that $A$ fixes $\{z_1, z_2, z_3, z_4\}$ it follows that some element of this set is also sent to $z_j$, contradicting the injectivity of $\bar{A}$. Thus $\bar{A}z_4 = z_4$. But the only permutation of $(z_1, z_2, z_3, z_4)$ which fixes the cross ratio and sends $z_4$ to $z_4$ is the identity. Thus $\bar{A}$ fixes four points of $\mathbb{P}^1(C)$. Since the action is sharply 3-transitive, $\bar{A} = \text{id}$, and the conclusion follows.

In order to prove the triviality of the stabilizer, Lemma 5.1 implies that it suffices to find 5 roots with the above property. This is feasible even with approximated roots, provided that we can control the errors.

Lemma 5.2. Let $z_1, z_2, z_3, z_4$ and $\delta_1, \delta_2, \delta_3, \delta_4$ be complex numbers. Let $M = \max_i |z_i|$ and $\delta = \max_i |\delta_i|$. Assume $\delta < 1$. Then

$$|(z_1 + \delta_1)(z_2 + \delta_2)(z_3 + \delta_3)(z_4 + \delta_4) - z_1z_2z_3z_4| \leq 15M^3\delta.$$ 

Proof. Expand the expression. The term $z_1z_2z_3z_4$ cancels out, so we are left with 15 terms, all of which contain at least one of the $\delta_i$, and at most three $z_i$’s. Explicitly, all remaining terms are of the form

$$\prod z_i \cdot \prod \delta_j$$

where the first product ranges over at most 3 indices $i$, and the second ranges over at least 1 index $j$. Since $\delta < 1$, all these terms are less than $M^3\delta$ in module, whence the conclusion.

Corollary 5.1. Let $x_1, \ldots, x_8$ be 8 complex numbers, and let $\tilde{x}_1, \ldots, \tilde{x}_8$ be approximations so that

$$\max_j |x_j - \tilde{x}_j| \leq \epsilon < 1/2$$

for a fixed $\epsilon$. Let $N := \max_j |x_j|$. Then:

(1) $\max_{i,j} |(x_i - x_j) - (\tilde{x}_i - \tilde{x}_j)| \leq 2\epsilon$

(2) $|(x_1 - x_3)(x_2 - x_4)(x_5 - x_8)(x_6 - x_7) - (\tilde{x}_1 - \tilde{x}_3)(\tilde{x}_2 - \tilde{x}_4)(\tilde{x}_5 - \tilde{x}_8)(\tilde{x}_6 - \tilde{x}_7)| \leq 60N^3\epsilon$.

(3) Let $a = (x_1 - x_3)(x_2 - x_4)\ldots(x_5 - x_8)(x_6 - x_7)$, $b = (x_1 - x_4)(x_2 - x_3)\ldots(x_5 - x_7)(x_6 - x_8)$, and let $\tilde{a}$, $\tilde{b}$ denote the same expressions where the $x$’s have been replaced by their approximations. If $|\tilde{a} - \tilde{b}| > 120N^3\epsilon$, then $|a - b| > 0$.

Proof. Part (1) is just the triangle inequality. Part (2) follows from Lemma 5.2 applied to the various differences $(x_i - x_j)$ in place of the $a$’s and the approximations $(\tilde{x}_i - \tilde{x}_j)$ in place of the $\alpha$’s, and noting that the $M$ in Lemma 5.2 is less than twice $N$, and that the maximal module of the $\alpha$’s is less than $2\epsilon$.

Finally, for Part (3) it suffices to observe that $|\tilde{a} - \tilde{b}| \leq |\tilde{a} - a| + |a - b| + |b - \tilde{b}| \iff |\tilde{a} - \tilde{b}| - |\tilde{a} - a| - |b - \tilde{b}| \leq |a - b|$. Now the previous part applied to different permutations of $x_1, \ldots, x_8$ certainly imply that $|\tilde{a} - a|, |b - \tilde{b}| \leq 60N^3\epsilon$.

6. Invariants of some Reed Muller codes

Using this algorithm, we found some non-trivial invariants of particular Reed-Muller codes. This is not useful for determining unknown weight enumerators, but rather shows an application of our procedure.

Proposition 6.1. Let $\zeta_{2^m}$ be a $2^m$-root of unity, and let $u_m := \bar{\zeta}_m^{m+1}/2$. For any $m \geq 3$,

$$\begin{bmatrix} u_m & u_m - 1 \\ u_m - 1 & u_m \end{bmatrix} \in \text{Stab}_{\text{GL}_2(C)}(W_{RM}(m-2, m))(x, y).$$
Proof. The code $\mathcal{RM}_2(m - 2, m)$ is the dual of $\mathcal{RM}_2(1, m)$, the code coming from the evaluation of linear polynomials over $\mathbb{F}_2$ (Theorem 4, Chapter 13 [MS77]). It is easy to see that the weight enumerator of $\mathcal{RM}_2(1, m)$ is

$$W_{\mathcal{RM}_2(1,m)}(x, y) = X^{2^m} + 2(2^m - 1)x^{2^m-1}y^{2^m-1} + y^{2^m}.$$ 

Now by MacWilliams’ Theorem,

$$W_{\mathcal{RM}_2(m-2,m)} = \frac{1}{2^{m+1}} \left( (X - Y)^{2^m} + 2 \cdot (2^m - 1)(X - Y)^{2^m-1}(X + Y)^{2^m-1} + (X + Y)^{2^m} \right).$$ 

It is immediate to check that the above matrix fixes this polynomial. □

On the other hand, Lemma 5.1 and Corollary 5.1 give a way to show that some codes have weight enumerators with trivial stabilizers.

**Theorem 6.1.** The following codes have weight enumerator with trivial stabilizer in $\text{PGL}_2(\mathbb{C})$, i.e. with stabilizer $\text{GL}_2(\mathbb{C})$ consisting only of scalar matrices:

- $\mathcal{RM}_4(2, 2)$,
- $\mathcal{RM}_4(3, 2)$,
- $\mathcal{PRM}_5(3, 2)$,
- $\mathcal{PRM}_5(3, 2)^\perp = \mathcal{RM}_5(5, 2)$.

**Proof.** For every code $\mathcal{C}$ in the list above, we can proceed as follows: it can be checked that $W_{\mathcal{C}}(x, 1)$ has at least 5 roots in $\mathbb{C}$. The goal is to find two critical 4-tuples which differ only in one element, as in Lemma 5.1. The problem lies in the fact that we can only work with approximations. However, a computer software like MAGMA gives the roots of a polynomial up to arbitrarily precision, which will help us finding critical 4-tuples.

Let $x_1, \ldots, x_8$ be complex points, and let $a, b$ be as in part (3) of Corollary 5.1. Easy algebraic manipulations show that for $[x_1, x_2, x_3, x_4] + [x_5, x_6, x_7, x_8]$ if and only if $|a - b| > 0$. Therefore, by choosing the approximations of the roots of $W_{\mathcal{C}}(x, 1)$ close enough, it is possible to get the $\epsilon$ of Corollary 5.1 arbitrarily small. This gives us a way to find critical 4-tuples, as desired. □

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ON THE STABILIZER OF WEIGHT ENUMERATORS OF LINEAR CODES

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