Anti-magic Labeling of Card House Graphs

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Abstract. An anti-magic labeling of a finite simple undirected graph $G$ is a bijection from the set of edges $E(G)$ to the set of integers $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all edges incident to such vertex. A graph is called anti-magic if it admits an anti-magic labeling. In this paper, we established an anti-magic labeling of Card House Graphs.

Keywords: Anti-magic, Anti-magic labeling, Card house graph.

1. Introduction

In this paper we consider only simple undirected finite graphs with $V(G)$ to be the set of vertices of the graph $G$ and $E(G)$ to be the set of edges of the graph $G$. Let us give definition of an anti-magic graph.

Definition 1.1. A bijective function $f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ is said to be labeling of graph $G$. The vertex sum function associated with a labeling $f$ of graph $G$ is a function $S : V(G) \rightarrow \mathbb{N}$ defined as

$$S(u) = \sum_{v \in N(u)} f(uv).$$

That is, $S(u)$ is the sum of labels of the edges incident with $u$. A labeling $f$ of a graph $G$ is said to be anti-magic if the vertex sum function associated with $f$ is an injective function. A graph having an anti-magic labeling is said to be anti-magic.

Hartsfield and Ringel [2] conjectured that all connected graphs except $K_2$ are anti-magic. An excellent survey on graph labeling can be found in [1]. Let us define an $n$-card house.

Definition 1.2. An $n$-card house graph is a graph, denoted by $CH(n)$, with the set of vertices and set of edges as follows:

$$V(CH(n)) = \{v_1, \ldots, v_{p(n)}\} \text{ and } E(CF(n)) = \bigcup_{i=1}^{n} F_i,$$

where $F_i = \bigcup_{k=1}^{i} R(i, k)$ and for $1 \leq i \leq n, 1 \leq k \leq i$

$$R(i, k) = \{v_{p(i-2)+k}v_{p(i-1)+k}, v_{p(i-2)+k}v_{p(i-1)+(k+1)}, v_{p(i-1)+k}v_{p(i-1)+(k+1)}\}.$$ 

$$p(0) = 1, p(i) = \frac{(i+1)(i+2)}{2}.$$
Note that the set $F_i$ is referred to as the $i^{th}$ floor. The cycle $R(i, k)$ is referred as the $k^{th}$ room on the $i^{th}$ floor. We call the first edge of this cycle as the left wall, the second edge as the right wall, and the third edge as the flooring of the room number $R(i, k)$, the vertex $V_{p(i-2)+k}$ as the roof vertex of the room $R(i, k)$. We call the collection of the left wall of all rooms $R(i, 1)$ for $1 \leq i \leq n$ as the left border, the collection of the right wall of all rooms $R(i, i)$ for $1 \leq i \leq n$ as the right border, and the collection of the flooring of all rooms $R(n, k)$ for $1 \leq k \leq n$ as the base border.

From the above definition it can be seen that the vertices of the card house are numbered from left to right and then from top to bottom in the natural order. The total number of vertices in $CH(n)$ is $p(n) = \frac{(n+1)(n+2)}{2}$. Clearly, for any $i$ such that $1 \leq i \leq n$, the $i$-card house graph, $CH(i)$, is nothing but the graph consists all the floors from the first floor, $F_1$, to the $i^{th}$ floor, $F_i$, of the $CH(n)$ graph. Hence at any floor $F_i$ the right-most roof vertex of the room $R(i, i)$ is $v_{p(i-1)}$. The $i^{th}$ floor, $F_i$, has $i$ number of rooms. Thus, the total number of rooms in $CH(n)$ is $1 + 2 + \cdots + n = n(n+1)/2 = p(n-1)$. By the construction of the card house graph, it can be seen that triplets of edges in formation of any two rooms of the card house are room-wise distinct. The total number of edges in $CH(n)$ is $3p(n-1) = \frac{3n(n+1)}{2}$.

2. Card House Graph

In this section, we show that the card house graphs $CH(n)$ are anti-magic for all $n \geq 1$. Note that the labeling function for $CH(n)$ will be same for all values of $n$. However, to prove that labeling is anti-magic, we prove the result based on value of $n$. Since all the roof vertices of all rooms in $CH(1)$ and $CH(2)$ are lying on the left border and the right border of the card house and there is only single roof vertex in $CH(3)$ not lying on the left or right border of the house, an anti-magic labeling for $CH(1), CH(2)$ and $CH(3)$ are shown separately in the theorem. Then we prove the result for $n \geq 4$.

**Theorem 2.1.** The card house graph $CH(n)$ is anti-magic for $n \leq 3$.

![Figure 1. Labeling of $CH(n)$ for $n \leq 3$](image)
Proof. We prove this result by considering the following three cases.

Case (i) $n = 1$.

The labeling of the card house graph $CH(1)$ is shown in the Fig. 1(a). Thus, the vertex sum function is given as

\[ S(v_1) = 3, \quad S(v_2) = 4 \quad \text{and} \quad S(v_3) = 5. \]

Case (ii) $n = 2$.

The labeling of the card house graph $CH(2)$ is shown in the Fig. 1(b). Thus, the vertex sum function is given as follows:

\[ S(v_1) = 3, \quad S(v_2) = 13, \quad S(v_3) = 20, \]
\[ S(v_4) = 10, \quad S(v_5) = 27 \quad \text{and} \quad S(v_6) = 17. \]

Case (iii) $n = 3$.

The labeling of the card house graph $CH(3)$ is shown in the Fig. 1(c). Thus, the vertex sum function is given as

\[ S(v_1) = 3, \quad S(v_2) = 13, \quad S(v_3) = 20, \]
\[ S(v_4) = 22, \quad S(v_5) = 51, \quad S(v_6) = 63 \quad \text{and} \quad S(v_{10}) = 35. \]

It is clear that in each case, the vertex sum function $S$ is injective on $V(CH(n))$. Hence the card house graph $CH(n)$ is anti-magic for $n \leq 3$.

We now prove result for $n \geq 4$.

**Theorem 2.2.** The card house graph $CH(n)$ is anti-magic for $n \geq 4$.

Proof. Let $n \geq 4$. Label left wall as 1, right wall as 2 and floor edge as 3 of the room with the roof vertex $v_1$. Continuing in the similar way label left wall as $3x - 2$, right wall as $3x - 1$ and floor edge as $3x$ of the room with the roof vertex $v_x$ for $1 \leq x \leq p(n - 1)$. Thus, the label function, $f$, for all the edges in the room $R(1, 1)$ is given by $f(v_1v_2) = 1$, $f(v_1v_3) = 2$ and $f(v_2v_3) = 3$. Moreover, for $2 \leq i \leq n$, $1 \leq k \leq i$ the label function, $f$, for all the edges in the room $R(i, k)$ is given as follows:

\[ f(v_{i-2+k}v_{i-1+k}) = 3[p(i - 2) + k] - 2 \]
\[ f(v_{i-2+k}v_{i-1+k+1}) = 3[p(i - 2) + k] - 1 \]
\[ f(v_{i-1+k}v_{i-1+k+1}) = 3[p(i - 2) + k]. \]

Using the fact that $E(CH(n)) = \cup_{i=1}^{p(n-1)} \{R(i, k)|1 \leq k \leq i\}$ and $p(n - 2) + n = p(n - 1)$, it is easy to prove that the above function $f$ is a bijection from $E(CH(n))$ to $\{1, 2, \ldots, 3p(n - 1)\}$. The vertices $v_1, v_{p(n-1)+1}$ and $v_{p(n)}$ are three corners for this graph with each degree 2. The vertex
sum function $S$ associated with this labeling for these corner vertices are given as follows:

$$S(v_1) = f(v_1v_2) + f(v_1v_3) = 3$$

$$S(v_{p(n-1)+1}) = f(v_{p(n-2)+1}v_{p(n-1)+1}) + f(v_{p(n-1)+1}v_{p(n-1)+2})$$
$$= \{3[p(n-2)+1] - 2\} + \{3[p(n-2)+1]\}$$
$$= 6p(n-2) + 4$$
$$= 3n^2 - 3n + 4$$

$$S(v_{p(n)}) = f(v_{p(n-1)}v_{p(n)}) + f(v_{p(n-1)+n}v_{p(n)})$$
$$= \{3[p(n-1)] - 1\} + \{3[p(n-1)]\}$$
$$= 6p(n-1) - 1$$
$$= 3n^2 + 3n - 1.$$  

Any vertex on the left border except corners is of degree 4. The vertex sum of $v_{p(i-2)+1}$ for $2 \leq i \leq n$ is given as follows:

$$S(v_{p(i-2)+1}) = f(v_{p(i-2)+1}v_{p(i-1)+1}) + f(v_{p(i-2)+1}v_{p(i-1)+2})$$
$$+ f(v_{p(i-2)+1}v_{p(i-2)+2}) + f(v_{p(i-3)+1}v_{p(i-2)+1})$$
$$= \{3[p(i-2)+1] - 2\} + \{3[p(i-2)+1]\}$$
$$+ \{3[p(i-3)+1]\} + \{3[p(i-3)+1]\}$$
$$= 6i^2 - 12i + 13.$$  

Any vertex on the right border except corners is of degree 4. The vertex sum of $v_{p(i-1)}$ for $2 \leq i \leq n$ is given as follows:

$$S(v_{p(i-1)}) = f(v_{p(i-1)}v_{p(i)}) + f(v_{p(i-1)}v_{p(i-1)+1}) + f(v_{p(i-2)+i-1}v_{p(i-1)}) + f(v_{p(i-2)}v_{p(i-1)})$$
$$= \{3[p(i-1)] - 1\} + \{3[p(i-1)] - 2\} + \{3[p(i-2)]\} + \{3[p(i-2)] - 1\}$$
$$= 6i^2 - 4.$$  

Any vertex on the base border except corners is also of degree 4. The vertex sum of $v_{p(n-1)+k}$ for $2 \leq k \leq n$ is given as follows:

$$S(v_{p(n-1)+k}) = f(v_{p(n-1)+k-1}v_{p(n-1)+k}) + f(v_{p(n-2)+k-1}v_{p(n-1)+k})$$
$$+ f(v_{p(n-2)+k}v_{p(n-1)+k}) + f(v_{p(n-1)+k}v_{p(n-1)+k+1})$$
$$= \{3[p(n-2) + k - 1]\} + \{3[p(n-2) + k - 1]\}$$
$$+ \{3[p(n-2) + k - 2]\} + \{3[p(n-2) + k]\}$$
$$= 6n^2 - 6n + 12k - 9.$$  

Now all the remaining vertices (non-border) are of degree 6. The labeling of all rooms with a common non-border vertex $v_{p(i-2)+k}$ is shown in the Fig. 2. The vertex sum of $v_{p(i-2)+k}$ for
3 \leq i \leq n \text{ and } 2 \leq k \leq i - 1 \text{ is given as follows:}

\begin{align*}
S(v_{p(i-2)+k}) &= f(v_{p(i-2)+k-1}v_{p(i-2)+k}) + f(v_{p(i-3)+k-1}v_{p(i-2)+k}) \\
&+ f(v_{p(i-3)+k}v_{p(i-2)+k}) + f(v_{p(i-2)+k}v_{p(i-2)+k+1}) \\
&+ f(v_{p(i-2)+k}v_{p(i-1)+k+1}) + f(v_{p(i-2)+k}v_{p(i-1)+k}) \\
&= \{3[p(i-3) + k - 1]\} + \{3[p(i-3) + k - 1] - 1\} \\
&+ \{3[p(i-3) + k] - 2\} + \{3[p(i-3) + k]\} \\
&+ \{3[p(i-2) + k] - 1\} + \{3[p(i-2) + k] - 2\} \\
&= 12p(i-3) + 6p(i-2) + 18k - 12 \\
&= 9i^2 - 21i + 18k. \quad (7)
\end{align*}

We consider the following cases to show that if $x, y \in V(CH(n))$ such that $x \neq y$, then $S(x) \neq S(y)$.

**Case (i)** $x = v_1$ and $y = v_k$ for $2 \leq k \leq p(n)$.

By Equation 1, $S(v_1) = 3$. Since $f(v_k) \geq 2$ and $\text{deg}(v_k) \geq 2$, we get $S(v_1) < S(v_k)$ for all $k$ with $2 \leq k \leq p(n)$. Thus, $S(x) \neq S(y)$.

**Case (ii)** $x$ and $y$ are corner vertices.

Equation 2 shows that $S(v_{p(n-1)+1})$ is even. However, Equation 3 implies that $S(v_{p(n)})$ is odd. Thus, $S(x) \neq S(y)$.

**Case (iii)** $x$ and $y$ are on the left border.

Using Equation 4, we get $S(v_{p(i-2)+1}) < S(v_{p(i-2)+1})$ for $2 \leq i < j \leq n$. The sum vertex at left corner, $S(v_{p(n-1)+1})$ and $S(v_{p(i-2)+1})$ have opposite parity for $2 \leq i \leq n$. Thus, $S(x) \neq S(y)$.

**Case (iv)** $x$ and $y$ are on the right border.

Using Equation 5, we get $S(v_{p(i-1)+1}) < S(v_{p(j-1)+1})$ for $2 \leq i < j \leq n$. The sum vertex at right corner, $S(v_{p(n)})$ and $S(v_{p(i-1)+1})$ have opposite parity for $2 \leq i \leq n$. Thus, $S(x) \neq S(y)$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Labeling of All Rooms with a Common Non-border Vertex $v_{p(i-2)+k}$}
\end{figure}
Case (v) x and y are on the base border. Using Equation 6, we get $S(v_{p(n-1)+k}) < S(v_{p(n-1)+m})$ for $2 \leq k < m \leq n$. Also $S(v_{p(n-1)+k}) \equiv 0 \pmod{3}$, $S(v_{p(n-1)+1}) \equiv 1 \pmod{3}$, and $S(v_{p(n)}) \equiv 2 \pmod{3}$. Thus, $S(x) \neq S(y)$.

Case (vi) x and y are on two distinct borders.
If either x or y is the top corner $v_1$, then by Case (i), $S(x) \neq S(y)$. If $x \neq v_1$ and $y \neq v_1$, then by the Equations 2, 3, 4, 5 and 6, we get that $S(x) \equiv a \pmod{3}$ and $S(y) \equiv b \pmod{3}$ such that $a \neq b$ for $0 \leq a \leq 2$ and $0 \leq b \leq 2$. Thus, $S(x) \neq S(y)$.

Case (vii) x is not on the border of the house and y is on the border of the house.
Using Equation 7, we get $S(x) \equiv 0 \pmod{3}$. If y is on the left border, then $S(y) \equiv 1 \pmod{3}$ and if y is on the right bound, then $S(y) \equiv 2 \pmod{3}$. Thus, $S(x) \neq S(y)$. If y is on the base border of the house, then Equations 7 and 6 implies that $S(x)$ and $S(y)$ have opposite parity. Thus, $S(x) \neq S(y)$.

Case (viii) x and y are not on any border of the house.
If x and y are the roof vertices of the rooms on the same floor $F_i$ for $3 \leq i \leq n$. Then without loss of generality assume that $x = v_{p(i-2)+k}$ and $y = v_{p(i-2)+m}$ for some $2 \leq k < m \leq i - 1$. However, the Equation 7 implies that $S(v_{p(i-2)+k}) < S(v_{p(i-2)+m})$ for $2 \leq k < m \leq i - 1$. Thus, $S(x) \neq S(y)$. If x and y are the roof vertices of the rooms on the different floors. Then result follows from the fact that $S(v_{p(i-2)+i-1}) < S(v_{p(i-1)+2})$.

Therefore, the vertex sum function $S$ is injective function on $V(CH(n))$. Thus, $CH(n)$ is anti-magic for $n \geq 4$.

Combining Theorem 2.1 and Theorem 2.2, we get the following result.

**Theorem 2.3.** The card house graph $CH(n)$ is anti-magic for $n \geq 1$.

3. Conclusions
In this paper, we established that the card house graph $CH(n)$ is anti-magic for $n \geq 1$. However, Joshi [3] proved that the card house graph $CH(n)$ is not E-super vertex magic for $n > 1$. Thus, the card house graph $CH(n)$ is open for investigation of many other types of labelings apart from E-super vertex magic labeling and anti-magic labeling. Uses of $CH(n)$ and its labelings (whenever they exist) can be another interesting aspect to investigate.

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