Abstract

The domination number of a graph $G$, denoted $\gamma(G)$, is the minimum size of a dominating set of $G$, and the independent domination number of $G$, denoted $i(G)$, is the minimum size of a dominating set of $G$ that is also independent. Let $k \geq 4$ be an integer. Generalizing a result on cubic graphs by Lam, Shiu, and Sun, we prove that $i(G) \leq \frac{k-1}{2k-1} |V(G)|$ for a connected $k$-regular graph $G$ that is not $K_k$, which is tight for $k = 4$. This answers a question by Goddard et al. in the affirmative.

We also show that $\frac{i(G)}{\gamma(G)} \leq \frac{k^2 - 3k + 2}{2k^2 - 6k + 2}$ for a connected $k$-regular graph $G$ that is not $K_k$, strengthening upon a result of Knor, Škrekovski, and Tepeh. In addition, we prove that a graph $G$ with maximum degree at most 4 satisfies $i(G) \leq \frac{5}{9} |V(G)|$, which is also tight.

**KEYWORDS**

domination, independent domination, regular graphs

1 | INTRODUCTION

Let $G$ be a finite simple graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, of $G$. A dominating set of $G$ is a subset $S$ of $V(G)$ such that each vertex not in $S$ has a neighbor in $S$. The domination number of $G$, denoted $\gamma(G)$, is the minimum size of a dominating set of $G$. Domination is an extensively studied classic topic in graph theory.

A dominating set that is also an independent set is an independent dominating set. The independent domination number of $G$, denoted $i(G)$, is the minimum size of an independent dominating set of $G$. Note that every graph has an independent dominating set, as a maximal independent set is equivalent to an independent dominating set. This concept appears in the literature as early as 1962 by Berge [4] and Ore [14]. For a survey regarding independent domination, see [10].

We focus on finding the maximum (constant) ratio of the independent domination number and the number of vertices for regular graphs. Surprisingly, not much is known for $k$-regular graphs.
when \( k \geq 4 \). We are also interested in the class of graphs with bounded maximum degree. We first lay out related literature for the independent domination number of regular graphs.

For a connected \( k \)-regular graph \( G \) where \( k \geq 1 \), Rosenfeld [16] showed that \( i(G) \leq \frac{|V(G)|}{2} \), which is tight only for the balanced complete bipartite graph \( K_{k,k} \). We are interested in lowering the upper bound on the independent domination number when the balanced complete bipartite graph is excluded. Note that there is no connected 1-regular graph when \( K_{1,1} \) is excluded. When \( k = 2 \), so \( G \) is a cycle, one can easily calculate that \( i(G) \leq \frac{3}{7}|V(G)| \) holds except for \( K_{2,2} \). Extending this pattern, Lam, Shiu, and Sun [13] showed the below result for cubic graphs, which are 3-regular graphs:

**Theorem 1.1** (Lam et al. [13]). If \( G \) is a cubic graph on at least eight vertices, then \( i(G) \leq \frac{2}{3}|V(G)| \), and the bound is tight by \( C_5 \Box K_2 \). See the left graph in Figure 1.

Since the only cubic graph on at most six vertices that do not satisfy the above theorem is \( K_{3,3} \), one can reinterpret the statement of Theorem 1.1 as the following: if \( G \) is a cubic graph that is not \( K_{3,3} \), then \( i(G) \leq \frac{2}{3}|V(G)| \), and equality holds for \( C_5 \Box K_2 \). Furthermore, Goddard and Henning [10] conjectured that if \( G \) is a connected cubic graph that is neither \( K_{3,3} \) nor \( C_5 \Box K_2 \), then \( i(G) \leq \frac{2}{3}|V(G)| \). This conjecture is still open, and see [7] for a partial result. For other conjectures and related results regarding the independent domination number of subclasses of cubic graphs, see [1, 8, 10, 11].

Unlike cubic graphs, little was known for \( k \)-regular graphs where \( k \geq 4 \). Let \( H \) be the 4-regular expansion of a 7-cycle, see the right graph in Figure 1. Goddard et al. [11] observed that \( H \) satisfies \( i(H) = \frac{3}{7}|V(H)| \), and asked the following question:

**Question 1.2** (Goddard et al. [11]). If \( G \) is a connected 4-regular graph that is not \( K_{4,4} \), then does \( i(G) \leq \frac{3}{7}|V(G)| \) hold? Our first result answers the above question in affirmative. We actually prove a theorem that applies to all \( k \)-regular graphs where \( k \geq 3 \), so our result also encompasses Theorem 1.1.

**Theorem 1.3.** For \( k \geq 3 \), if \( G \) is a connected \( k \)-regular graph that is not \( K_{k,k} \), then \( i(G) \leq \frac{k-1}{2k-1}|V(G)| \).

**Figure 1** The graph \( C_5 \Box K_2 \) and the 4-regular expansion \( H \) of a 7-cycle
To our knowledge, this is the best upper bound on the independent domination number for general $k$. Note that the graphs in Figure 1 demonstrate that the bound in Theorem 1.3 is tight for $k \in \{3, 4\}$. Whether the bound is tight or not for $k \geq 5$ is unknown as we were unable to construct such examples.

We turn our attention to the ratio of the independent domination number and the domination number for connected regular graphs. Note that the ratio can be arbitrarily large, as it is for the complete bipartite graph, so we seek to obtain a bound that depends on the regularity. Note that the independent domination number and the domination number are identical for $k$-regular graphs when $k \leq k/2$.

For cubic graphs, Goddard et al. [11] proved that if $G$ is a connected cubic graph, then $i(G)/\gamma(G) \leq 3/2$, and equality holds if and only if $G = K_{3,3}$. Southey and Henning [17] extended the result by showing that $i(G)/\gamma(G) \leq 4/3$ when $G \neq K_{3,3}$, and equality holds if and only if $G = C_3 \Box K_2$.

Suil and West [18] constructed an infinite family of connected cubic graphs $G$ such that $i(G)/\gamma(G) = 5/4$, and asked if $i(G)/\gamma(G) \leq 5/4$ for a sufficiently large connected cubic graph $G$. Among other results, Cho et al. [5] provided the first partial result regarding this question by showing that it is true for all cubic graphs without 4-cycles.

Consider $k$-regular graphs with $k \geq 4$. Babikir and Henning [3] showed that the statement of Goddard et al. in the previous paragraph also holds for $k$-regular graphs when $k \in \{4, 5, 6\}$. Very recently, Knor, Škrekovski, and Tepeh [12] generalized the result to all $k \geq 3$; namely, it is now known that for all $k \geq 3$, if $G$ is a connected $k$-regular graph, then $i(G)/\gamma(G) \leq k/2$, and equality holds if and only if $G = K_{k,k}$.

It is natural to ask if there exists a better bound than $k/2$ when $K_{k,k}$ is excluded, as it is the case for cubic graphs. Using Theorem 1.3, we are able to provide a better upper bound than $k/2$.

**Theorem 1.4.** For $k \geq 4$, if $G$ is a connected $k$-regular graph that is not $K_{k,k}$, then

$$i(G)/\gamma(G) \leq \frac{k^3 - 3k^2 + 2}{2k^2 - 6k + 2}.$$  

Note that $\frac{k^3 - 3k^2 + 2}{2k^2 - 6k + 2} < \frac{k}{2}$ for all $k \geq 4$. In particular, the bound becomes $\frac{9}{2}$ when $k = 4$. To our knowledge, this is the first partial answer to the following question asked in [11]: does $i(G)/\gamma(G) \leq \frac{3}{2}$ hold for a connected 4-regular graph $G$ that is not $K_{4,4}$? If the aforementioned question is true, then it is tight by the 4-regular expansion of a 7-cycle (and also an 8-cycle).

We now switch gears and consider the family of graphs with bounded maximum degrees. Since an isolated vertex must belong to every independent dominating set, we consider the class of isolate-free graphs, which are the graphs without isolated vertices. Akbari et al. [2] proved that if $G$ is an isolate-free graph with a maximum degree at most 3, then $i(G) \leq \frac{|V(G)|}{2}$, and they also characterized all graphs where equality holds. In this vein, we extend their result by proving a sharp upper bound on the independent domination number for isolate-free graphs with maximum degree at most 4.

**Theorem 1.5.** If $G$ is an isolate-free graph with a maximum degree at most 4, then $i(G) \leq \frac{5}{9}|V(G)|$, and equality holds for $H(3, 2)$, where $H(q, p)$ is the graph obtained by attaching $p$ pendant vertices to every vertex of a complete graph on $q$ vertices.
We actually think the family of graphs $H\left(\left\lceil \frac{D}{2} \right\rceil + 1, \left\lfloor \frac{D}{2} \right\rfloor \right)$ has the maximum independent domination number among isolate-free graphs with maximum degree at most $D$, so we put forth the following conjecture:

**Conjecture 1.6.** If $G$ is an isolate-free graph with maximum degree $D \geq 1$, then

$$i(G) \leq \begin{cases} \frac{D^2 + 4}{(D + 2)^2}|V(G)| & \text{if } D \text{ is even}, \\ \frac{D^2 + 3}{(D + 1)(D + 3)}|V(G)| & \text{if } D \text{ is odd.} \end{cases}$$

If Conjecture 1.6 is true, then $H\left(\left\lceil \frac{D}{2} \right\rceil + 1, \left\lfloor \frac{D}{2} \right\rfloor \right)$ demonstrates that the bound is tight. Note that one can easily check that Conjecture 1.6 is true for $D = 2$, and we remark that Conjecture 1.6 is true for $D = 3$ and $D = 4$ by the result of Akbari et al. [2] and by Theorem 1.5, respectively. In addition, we checked that the conjecture holds for $D \in \{5, 6, 7, 8\}$, but we decided not to include the proofs as it mainly consists of tedious case checking. After this manuscript was submitted, Conjecture 1.6 was shown to be true in [6].

For results regarding the ratio of the independent domination number and the domination number for graphs with bounded maximum degree, see [9, 15].

In Section 2, we prove Theorem 1.3. The proof essentially boils down to one (implicit) inequality, which was inspired by an idea in [13]. However, unlike their proof, we use discharging to prove that inequality holds. Using Theorem 1.3 and an idea in [17], we prove Theorem 1.4 in Section 3. In Section 4, we prove a statement slightly stronger than Theorem 1.5, where the proof adopts the approach of [7].

We end the introduction with some notation and terminology used in this paper. Given a graph $G$, let $N_G(v)$ denote the set of neighbors of a vertex $v$, and let $N_G[v] = N_G(v) \cup \{v\}$. For each $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v \in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$. For a vertex $v$ and $X \subseteq V(G)$, an $X$-neighbor of $v$ is a neighbor of $v$ in $X$. A minimum dominating set of $G$ is a dominating set of $G$ with size $\gamma(G)$, and a minimum independent dominating set of $G$ is an independent dominating set of $G$ with size $i(G)$.

## 2 | INDEPENDENT DOMINATION OF REGULAR GRAPHS

In this section, we prove Theorem 1.3. For $I \subseteq V(G)$, let $G_I$ be the spanning bipartite subgraph of $G$ obtained from $G$ by deleting the edges joining two vertices in $I$ and the edges joining two vertices in $V(G) \setminus I$. For brevity, denote $N_{G_I}(v)$ and $\deg_{G_I}(v)$ by $N_I(v)$ and $\deg_I(v)$, respectively.

For $k = 3$, the theorem holds by Lam, Shiu, and Sun [13]. Fix $k \geq 4$. Let $G \neq K_{k,k}$ be a connected $k$-regular graph. Choose a minimum independent dominating set $I$ of $G$ that

1. minimizes the number of subgraphs in $G_I$ isomorphic to $K_{k-1,k}$, and
2. maximizes the number of pendent vertices $v \in V(G) \setminus I$ of $G_I$ such that the $I$-neighbor $w$ of $v$ has a neighbor $x \in V(G) \setminus I$ satisfying $\deg_I(x) = k$, subject to (1).
Since $I$ is an independent set of $G$, all vertices in $I$ have degree $k$ in $G_I$. Let $J = V(G) \setminus I$ and $J_i = \{v \in J | N_G(v) = i\}$ for each $i \in [k]$. Note that $J_1, \ldots, J_k$ form a partition of $J$, since $I$ is a dominating set of a $k$-regular graph $G$. For two integers $s, t$ with $1 \leq s \leq t \leq k$, let $J_{[s,t]}$ denote $\bigcup_{i \in [s, \ldots, t]} J_i$.

For each $v \in J_k$, let $X(v)$ be the set of vertices in $J \setminus \{v\}$ whose $I$-neighbors are in $N_G(v)$, namely,

$$X(v) = \{w \in J \setminus \{v\} | N_I(w) \subseteq N_G(v)\}.$$

For each $v \in J_{[1,k-1]}$, let $Y(v)$ be the set of vertices $w \in J_k$ such that $v$ belongs to $X(w)$, namely,

$$Y(v) = \{w \in J_k | v \in X(w)\}.$$

For $u \in J_{[1,k-1]}$ and $v \in J_k$, note that $u \in X(v)$ if and only if $v \in Y(u)$. Using this terminology, (2) can be rephrased as the following:

(2) maximizes the number of vertices $v \in J_1$ such that $Y(v) \neq \emptyset$, subject to (1).

**Claim 2.1.** The following hold:

(i) If $v \in J_{[1,k-1]}$, then $|Y(v)| \leq k - 1$.

(ii) If $v \in J_i$ for $i \in [k]$, then there are at least $i$ vertices $w \in J$ such that $N_I(w) \subseteq N_G(v)$.

In particular, if $v \in J_k$, then $|X(v)| \geq k - 1$.

**Proof.** Since $G$ is $k$-regular, (i) follows from the definition of $Y(v)$. To show (ii), let $v \in J_i$ for some $i \in [k]$. The set $I' = (I \setminus N_G(v)) \cup \{w \in J | N_I(w) \subseteq N_G(v)\}$ contains an independent dominating set of $G$. By the minimality of $I$, it follows that $|\{w \in J | N_I(w) \subseteq N_G(v)\}| \geq i$, so (ii) holds. \qed

For $v \in J_k$ and $u \in J_1$, $u$ is $v$-special if $|Y(u)| \leq k - 3$ and $\bigcup_{w \in J_{k-1} \cap N_G(u)} Y(w) = \{v\}$.

**Claim 2.2.** Consider a vertex $v \in J_k$ where $|X(v) \cap J_i| \leq k - 3$.

(i) If $|X(v) \cap J_{k-1}| = k - 1$ and the vertices in $X(v) \cap J_{k-1}$ have the same neighborhood in $G_I$, then the vertices in $X(v) \cap J_{k-1}$ have a common $J_1$-neighbor $u$ in $G$ that is $v$-special.

(ii) If $X(v) \cap J_{[2,k-2]} = \emptyset$ and either $X(v) \cap J_1 = \emptyset$ or $X(v) \cap J_1 = \{w\}$ where $|Y(w)| = k - 1$, then $X(v) \setminus J_1$ consists of $k - 1$ distinct vertices in $J_{k-1}$ with a common $J_1$-neighbor $u$ in $G$ that is $v$-special.

**Proof.** Let $w', v'_1, \ldots, v'_{k-1}$ be the $I$-neighbors of $v$.

(i) Assume $X(v) \cap J_{k-1} = \{v_1, \ldots, v_{k-1}\}$ and $N_I(v_1) = \cdots = N_I(v_{k-1}) = \{v'_1, \ldots, v'_{k-1}\}$.

Now, $I' = (I \setminus \{v'_1, \ldots, v'_{k-1}\}) \cup \{v_1, \ldots, v_{k-1}\}$ contains an independent dominating set of $G$.

Since $I$ and $I'$ have the same size but $I$ was chosen over $I'$, according to condition (1) of the choice of $I$, the number of subgraphs of $G_{I'}$ isomorphic to $K_{k-1,k}$ in $G_{I'}$ is at least that in $G_I$. Since $v, v_1, \ldots, v_{k-1}, v'_1, \ldots, v'_{k-1}$ form a graph isomorphic to $K_{k-1,k}$ in $G_I$, it follows that $v_1, \ldots, v_{k-1}$ are part of a subgraph of $G_{I'}$ isomorphic to $K_{k-1,k}$. Thus $v_1, \ldots, v_{k-1}$ have a common $J$-neighbor $u$. Since $Y(v_i) = \{v\}$ for all $i \in [k - 1]$, we have $\bigcup_{w \in J_{k-1} \cap N_G(u)} Y(w) = \{v\}$. Since $I$ must dominate $u$, we know $u \in J_1$, and let $u'$ be the $I$-neighbor of $u$. 


Now, $I'' = (I \setminus \{ w', v'_1, ..., v'_{k-1}, u' \}) \cup \{ v \} \cup (X(v) \cap J_I) \cup (N_G(w') \setminus Y(u))$ contains an independent dominating set of $G$, since $u$ and $v$ dominate $w', v'_1, ..., v'_{k-1}, u'$, and every vertex that is not dominated by $\{ u', v \} \cup (I \setminus \{ w', v'_1, ..., v'_{k-1}, u' \})$ belongs to $(X(v) \cap J_I) \cup (N_G(w') \setminus Y(u))$. Note that $|I''| = |I| - (k + 1) + 1 + |X(v) \cap J_I| + 1 - |Y(u)|$. If $|Y(u)| \geq k - 2$, then $|I''| < |I|$, which is a contradiction to the choice of $I$. Hence, $|Y(u)| \leq k - 3$, and therefore, $u$ is $v$-special.

(ii) We first consider the case when $X(v) \cap J_I = \emptyset$, so $X(v) \subseteq J_{k-1,k}$. Since $G \neq K_{k,k}$, $X(v)$ contains a vertex $w \in J_{k-1,k}$. By Claim 2.1(ii), there are at least $k - 1$ vertices $u \in X(v) \cap J_{k-1,k}$ such that $N_I(w) = N_I(u)$. Since each vertex in $N_G(v)$ has degree $k$, it follows that $|X(v) \cap J_{k-1,k}| = k - 1$, and the vertices in $X(v) \cap J_{k-1,k}$ have the same neighborhood in $G_I$. By (i), the vertices in $X(v)$ have a common $J_1$-neighbor $u$ in $G$ that is $v$-special.

Now we consider the case when $X(v) \cap J_I = \{ w \}$ where $|Y(w)| = k - 1$. Assume $w'$ is the $I$-neighbor of $w$. Suppose that $X(v) \setminus \{ w \} \subseteq J_{k-1,k}$. Since $|Y(w)| = k - 1$, each $z \in N_G(w') \setminus \{ w \}$ is in $J_{k-1,k}$, so $z \notin X(v)$. Thus, a vertex in $X(v) \setminus \{ w \}$ cannot be adjacent to $w'$, so the $I$-neighbors of each vertex in $X(v) \setminus \{ w \}$ are $v'_1, ..., v'_{k-1}$. By applying Claim 2.1(ii) to a vertex in $X(v) \cap J_{k-1,k}$, we obtain $|X(v) \setminus \{ w \}| = |X(v) \cap J_{k-1,k}| = k - 1$. By (i), the vertices in $X(v) \setminus \{ w \}$ have a common $J_1$-neighbor $u$ in $G$ that is $v$-special. We will complete the proof by showing that $X(v) \setminus \{ w \} \subseteq J_{k-1,k}$ always holds.

Suppose to the contrary that there is a vertex $v_1 \in X(v) \cap J_k$. Since $X(v) \cap J_I = \{ w \}$ and $X(v) \cap J_{2,k-2} = \emptyset$, every vertex in $X(v) \setminus \{ w, v_1 \}$ is in $J_{k-1,k}$. Note that $X(v) \setminus \{ w, v_1 \} \neq \emptyset$ by Claim 2.1(ii) since $k \geq 4$. If $(X(v) \setminus \{ w, v_1 \}) \cap J_{k-1,k} \neq \emptyset$, then $|(X(v) \setminus \{ w, v_1 \}) \cap J_{k-1,k}| \geq k - 1$ by Claim 2.1(ii). However, by counting the number of edges between $X(v) \setminus \{ w, v_1 \}$ and $N_G(v)$, we obtain $|(X(v) \setminus \{ w, v_1 \}) \cap J_{k-1,k}| \leq k - 2$, which is a contradiction. See the left figure of Figure 2.

Thus, we may assume that every vertex in $X(v) \setminus \{ w \}$ is in $J_k$. By Claim 2.1(ii), $|X(v) \setminus \{ w \}| \geq k - 1$, which further implies that there are exactly $k - 2$ vertices $v_1, ..., v_{k-2}$ in $X(v) \cap J_k$ since $G$ is $k$-regular. In particular, $X(v) = \{ w, v_1, ..., v_{k-2} \}$. See the right figure of Figure 2. Note that $I^* = (I \setminus \{ w' \}) \cup \{ w \}$ is an independent dominating set of $G$ with the same size as $I$. Let $J^* = V(G) \setminus I^*$ and $J_i^* = \{ v \in J_i^* | \text{len}(v) = i \}$ for each $i \in [k]$. Define $X^*(v)$ and $Y^*(v)$ analogously.

Note that $v, v_1, ..., v_{k-2}, w', v'_1, ..., v'_{k-1}$, $w'$ form a graph isomorphic to $K_{k-1,k}$ in $G_I$, but not in $G_{I^*}$. By condition (1) of the choice of $I$, it follows that $w$ is a vertex of some subgraph $H$ isomorphic to $K_{k-1,k}$ in $G_{I^*}$; let the partite sets of $H$ be $\{ w_1, ..., w_{k-1} \}$ and $\{ w'_1, ..., w'_{k-1}, w \}$. So, the number of subgraphs isomorphic to $K_{k-1,k}$ in $G_{I^*}$ is equal to that of $G_I$. We will reach a contradiction by showing that the number of vertices $v^* \in J_i^*$ such that $Y^*(v^*) \neq \emptyset$ in $G_{I^*}$ is greater than that in $G_I$.

For each $i \in [k - 1]$, let $w_i'$ be the neighbor of $w'_i$ not in $\{ w_1, ..., w_{k-1} \}$. Note that by the minimality of $I$, $I'' = (I \setminus \{ w'_1, ..., w'_{k-1}, w, v'_1, ..., v'_{k-1} \}) \cup \{ w_1, ..., w_{k-1}, v, v_1, ..., v_{k-2} \}$ is not an independent dominating set of $G$. Since $I''$ is an independent set, it is not a dominating

![Figure 2](Illustrations for $(X(v) \setminus \{ w, v_1 \}) \cap J_{k-1} \neq \emptyset (left)$ and for $(X(v) \setminus \{ w \}) \subseteq J_k (right)$)
set of $G$. By our assumption, there is a vertex $w''$ such that $N_I(w'') \subseteq \{w'_1, \ldots, w'_{k-1}\}$, say $w'' = w''_1$. If $w''_1 \notin J_i$, then the set $(I^* \setminus N_G(w''_1)) \cup \{w''_1\}$ is a smaller independent dominating set of $G$ than $I$, which is a contradiction. Thus $w''_1 \in J_i$ (and therefore $w''_1 \in J_i^*$) and $N_I(w''_1) = \{w'_1\}$ (and also $N_I(w''_1) = \{w'_1\}$). Then clearly, $Y(w''_1) = \emptyset$, but $Y^*(w''_1) = \{w_1, \ldots, w_{k-1}\} \neq \emptyset$. Moreover, note that $w' \in J_i^*$ and $Y^*(w') = \{w_1, \ldots, w_{k-1}\} \neq \emptyset$. This is a contradiction to condition (2) of the choice of $I$, which completes the proof.

Now, suppose to the contrary that $|I| = i(G) > \frac{k-1}{2k-1}|V(G)|$, which implies $(k - 1)|V(G)| - (2k - 1)|I| < 0$. For each vertex $v$, define the initial charge $\mu(v)$ of each vertex $v$ to be

$$\mu(v) = \begin{cases} 0 & \text{if } v \in I, \\ k - 1 - i & \text{if } v \in J_i. \end{cases}$$

The sum of the initial charge is negative, since $\sum_i i|J_i| = k|I|$ by counting the edges in $E(G_I)$, so

$$\sum_{i=1}^{k} (k - 1 - i)|J_i| = \sum_{i=1}^{k} (k - 1)|J_i| - \sum_{i=1}^{k} i|J_i|$$

$$= (k - 1)|I| - k|I|$$

$$= (k - 1)|V(G)| - |I| - k|I|$$

$$= (k - 1)|V(G)| - (2k - 1)|I| < 0.$$

We distribute the initial charge according to the discharging rules, which are designed so that the total charge is preserved, to obtain the final charge $\mu^*(v)$ at each vertex $v$. The following are the discharging rules. See Figure 3.

[R1] For $i \in \{2, \ldots, k - 2\}$, every vertex $w \in J_i$ with $Y(w) \neq \emptyset$ sends $\frac{k - 1 - i}{|Y(w)|}$ to each vertex in $Y(w)$.

[R2] Let $w \in J_1$.

[R2-1] $u \in Y(w)$ ($|Y(w)| = k - 1$)

[R2-2] $u \in Y(w)$ ($|Y(w)| \leq k - 2$)

[F I G U R E 3] Illustrations for the discharging rules
[R2-1] If $|Y(w)| = k - 1$, then $w$ sends $\frac{k - 2}{k - 1}$ to each vertex in $Y(w)$.

[R2-2] If $|Y(w)| \leq k - 2$, then $w$ sends 1 to each vertex in $Y(w)$.

[R2-3] If $w$ is $x$-special for some $x \in J_k$, then $w$ sends 1 to the vertex $x$.

We obtain a contradiction by showing that the sum of the final charge is nonnegative, by proving that the final charge of each vertex is nonnegative.

**Claim 2.3.** For every vertex $v$, the final charge $\mu^*(v)$ is nonnegative.

**Proof.** Each vertex in $I \cup J_{k-1}$ is not involved in the discharging rules, so its final charge is 0. If $v \in J_i$ for $i \in \{2, ..., k - 2\}$, then $\mu^*(v) \geq (k - 1 - i) - (k - 1 - i) = 0$ by [R1]. Suppose that $v \in J_1$. If $k - 2 \leq |Y(v)| \leq k - 1$, then $\mu^*(v) \geq (k - 1 - 1) - (k - 2) = 0$ by [R2-1] and [R2-2]. If $|Y(v)| \leq k - 3$, then there is at most one vertex $x$ (in $J_k$) for which $v$ is $x$-special, therefore $\mu^*(v) \geq (k - 1 - 1) - |Y(v)| - 1 \geq 0$ by [R2-2] and [R2-3]. Now it remains to check the final charge of a vertex $v$ in $J_k$. Since the initial charge of $v$ is $-1$, in order for the final charge of $v$ to be nonnegative, $v$ must receive charge at least 1 by the discharging rules.

(1) Suppose $X(v) \cap J_{[1,k-2]} = \emptyset$.

By Claim 2.2(ii), $X(v)$ consists of $k - 1$ distinct vertices in $J_{k-1}$ with a common $J_1$-neighbor $u$ in $G$ that is $v$-special. Thus $u$ sends 1 to $v$ by [R2-3].

(2) Suppose $X(v) \cap J_{[1,k-2]} \neq \emptyset$.

(2-1): $X(v)$ contains a vertex $v_1 \in J_i$.

If $|Y(v_1)| \leq k - 2$, then $v_1$ sends 1 to $v$ by [R2-2], so suppose $|Y(v_1)| = k - 1$. If there is a vertex $v_2 \in (X(v) \setminus \{v_1\}) \cap J_{[1,k-2]}$, then $v_2$ sends at least $\frac{1}{k - 1}$ to $v$ and $v_1$ sends $\frac{k - 2}{k - 1}$ to $v$ by [R1] and [R2], so $v$ receives at least 1.

If $(X(v) \setminus \{v_1\}) \cap J_{[1,k-2]} = \emptyset$, then by Claim 2.2(ii), $X(v) \setminus J_1$ consists of $k - 1$ distinct vertices in $J_{k-1}$ with a common $J_1$-neighbor $u$ in $G$ that is $v$-special. Thus $u$ sends 1 to $v$ by [R2-3].

(2-2): $X(v) \cap J_1 = \emptyset$.

Let $i$ be the maximum integer such that $X(v) \cap J_{[1,i]} = \emptyset$. Since $X(v) \cap J_{[1,k-2]} \neq \emptyset$, there is a vertex $v_1 \in X(v) \cap J_{i+1}$ where $1 \leq i \leq k - 3$. Moreover, by Claim 2.1(ii), there are $i + 1$ vertices $v_1, ..., v_{i+1} \in X(v) \cap J_{i+1}$ such that $N_j(v_1) = \cdots = N_j(v_{i+1})$ since $X(v) \cap J_{[1,i]} = \emptyset$. Since $G$ is $k$-regular, $|Y(v_j)| \leq k - 1 - i$ for all $j \in [i + 1]$, so by [R1], $v$ receives from $v_1, ..., v_{i+1}$ at least

$$\left(\frac{k - 1 - (i + 1)}{|Y(v_j)|}\right) \cdot (i + 1) \geq \left(\frac{k - 1 - (i + 1)}{k - 1 - i}\right) \cdot (i + 1) = \frac{i(k - 3 - i) + (k - 2)}{k - 1 - i} \geq 1,$$

where the last inequality holds since $1 \leq i \leq k - 3$.

In every case, $v$ receives charge at least 1, so $\mu^*(v) \geq -1 + 1 = 0$. \qed
3 | RATIO OF INDEPENDENT DOMINATION AND DOMINATION FOR REGULAR GRAPHS

In this section, we prove Theorem 1.4. Fix $k \geq 4$, and let $G$ be a connected $k$-regular graph on $n$ vertices that is not $K_{k,k}$. For simplicity, let $n_0(G)$ denote the number of isolated vertices of $G$, and let $n_1(H) = |V(H)| - n_0(H)$ for every graph $H$. We prove that the following statement holds.

Claim 3.1. For a dominating set $D$ of $G$, $i(G) \leq |D| + (k - 3) \cdot n_1(G[D])$.

Proof. We use induction on $n_1(G[D])$. If $n_1(G[D]) = 0$, then $D$ is an independent dominating set of $G$, so the statement holds since $i(G) \leq |D|$.

Now, assume $n_1(G[D]) > 0$. Take a vertex $v \in D$ with a maximum degree in $G[D]$, and let $\deg_{G[D]}(v) = d$. Note that $d \geq 1$, and let $P = \{u \in V(G) \setminus N_G[u] \cap D = \{v}\}$. Take a maximal independent set $P'$ of $G[P]$, and let $D' = (D \setminus \{v\}) \cup P'$. Note that $D'$ is a dominating set of $G$ by the maximality of $P'$. Since $G$ is $k$-regular, $|P'| \leq k - d$, so $|D'| = |D| + |P'| - 1 \leq |D| + k - d - 1$. Note that all vertices in $P'$ are isolated vertices in $G[D']$, and therefore $n_1(G[D']) \leq n_1(G[D]) - 1 - q$, where $q$ is the number of pendant neighbors of $v$ in $G[D]$. If $d = 1$, then the neighbor of $v$ in $G[D]$ is also a pendant vertex in $G[D]$, so $q = 1$. Hence, since $k \geq 4$, it holds that $-d + 2 - q(k - 3) \leq -d + 2 - q \leq 0$.

By the induction hypothesis,

$$i(G) \leq |D'| + (k - 3) \cdot n_1(G[D']) \leq (|D| + k - d - 1) + (k - 3)(n_1(G[D]) - 1 - q) = |D| + (k - 3) \cdot n_1(G[D]) - d + 2 - q(k - 3) \leq |D| + (k - 3) \cdot n_1(G[D]).$$

Let $D$ be a minimum dominating set of $G$, so $|D| = \gamma(G)$. Let $c_k = \frac{k^2 - 4k + 2}{k^2 - 2k}$. We have two cases.

Case 1: Suppose that $n_0(G[D]) \geq c_k \cdot n_1(G[D])$. Then

$$\gamma(G) = |D| = n_0(G[D]) + n_1(G[D]) \geq (c_k + 1) \cdot n_1(G[D]),$$

so $\frac{n_1(G[D])}{\gamma(G)} \leq \frac{1}{c_k + 1}$.

By Claim 3.1, $i(G) \leq |D| + (k - 3) \cdot n_1(G[D])$, and therefore,

$$\frac{i(G)}{\gamma(G)} \leq \frac{\gamma(G) + (k - 3) \cdot n_1(G[D])}{\gamma(G)} = 1 + \frac{(k - 3) \cdot n_1(G[D])}{\gamma(G)} \leq 1 + \frac{k - 3}{c_k + 1} = \frac{k^3 - 3k^2 + 2}{2k^2 - 6k + 2}.$$

Case 2: Suppose that $n_0(G[D]) \leq c_k \cdot n_1(G[D])$. Note that $n \leq (k + 1) \cdot n_0(G[D]) + k \cdot n_1(G[D])$ since $D$ is a dominating set, and each vertex whose degree in $G[D]$ is 0 (resp., 1) is adjacent to at most $k$ (resp., $k - 1$) vertices not in $D$. Thus

$$\gamma(G) = n_0(G[D]) + n_1(G[D]) \geq \frac{n - k \cdot n_1(G[D])}{k + 1} + n_1(G[D]) = \frac{n + n_1(G[D])}{k + 1}.$$
Since \( n_0(G[D]) \leq c_k \cdot n_1(G[D]) \), we obtain
\[
\gamma(G) \geq \frac{n}{k + 1} + \frac{n}{(k + 1) \cdot (k + 1) c_k + k} = \frac{n(2k^2 - 6k + 2)}{(2k - 1)(k^2 - 2k - 2)}.
\]

By Theorem 1.3, \( i(G) \leq \frac{n(k - 1)}{2k - 1} \), and therefore,
\[
\frac{i(G)}{\gamma(G)} \leq \frac{k - 1}{2k - 1} \cdot \frac{(2k - 1)(k^2 - 2k - 2)}{2k^2 - 6k + 2} = \frac{k^3 - 3k^2 + 2}{2k^2 - 6k + 2}.
\]

## 4 INDEPENDENT DOMINATION FOR GRAPHS WITH MAXIMUM DEGREE 4

In this section, we prove Theorem 1.5. We actually prove the following slightly stronger statement, whose direct consequence is Theorem 1.5.

**Theorem 4.1.** If \( G \) is a graph with a maximum degree at most 4, then
\[
9i(G) \leq 5|V(G)| + 4n_0(G).
\]

**Proof.** Suppose to the contrary that a graph \( G \) is a minimum counterexample to Theorem 4.1 with respect to the number of vertices. In particular, \( 9i(G) > 5|V(G)| + 4n_0(G) \) and \( 9i(H) \leq 5|V(H)| + 4n_0(H) \) for every proper subgraph \( H \) of \( G \).

Note that \( |V(G)| \geq 2 \) since the theorem holds for the graph with a single vertex. If \( G \) is the disjoint union of two graphs \( G_1 \) and \( G_2 \), then by the minimality of \( G \), we obtain
\[
9i(G) = 9i(G_1) + 9i(G_2) \leq (5|V(G_1)| + 4n_0(G_1)) + (5|V(G_2)| + 4n_0(G_2)) = 5|V(G)| + 4n_0(G),
\]
which is a contradiction. Thus \( G \) is connected, so \( n_0(G) = 0 \), therefore, \( 5|V(G)| \leq 9i(G) \).

For simplicity, denote the set of isolated vertices of \( G - N_G[v] \) by \( I_G(v) \). By counting the number of edges between \( N_G[v] \) and \( G - N_G[v] \), we know \( |I_G(v)| \leq \frac{3\deg_G(v)}{\delta(G)} \), since \( G \) is a connected graph with a maximum degree at most 4. Adding \( v \) to an independent dominating set of \( G - N_G[v] \) is an independent dominating set of \( G \). Thus, by the minimality of \( G \),
\[
5|V(G)| < 9i(G) \leq 9 + 9i(G - N_G[v]) \leq 9 + 5|V(G - N_G[v])| + 4n_0(G - N_G[v]) = 9 + 5|V(G)| - 5(\deg_G(v) + 1) + 4|I_G(v)|,
\]
so \( 4|I_G(v)| \geq 5\deg_G(v) - 3 \) since each term is an integer. Hence, it holds that
\[
\forall v \in V(G), 4 \cdot \frac{3\deg_G(v)}{\delta(G)} \geq 4|I_G(v)| \geq 5\deg_G(v) - 3. \tag{1}
\]
If \( \delta(G) \geq 3 \), then (1) implies that \( \deg_G(v) = 3 \) for every vertex \( v \) so \( G \) is cubic. Now, (1) again implies that \( |I_G(v)| = 3 \) for every vertex \( v \), which is impossible in a cubic graph. If \( \delta(G) = 2 \), then by considering a vertex \( v \) of degree 2, (1) implies that \( |I_G(v)| \geq 2 \). This further implies that \( v \) has a neighbor \( x \) such that \( \deg_G(x) \geq 3 \) and \( |I_G(x)| \leq 1 \), which is a contradiction to (1). Hence, \( \delta(G) = 1 \).

Claim 4.2. If a vertex \( v \) has a pendent neighbor, then \( v \) is a vertex of degree 4 with exactly two pendent neighbors.

Proof. Let \( v \) be a vertex with the maximum number of pendent neighbors such that \( \deg_G(v) = d_1 + d_2 \) where \( d_1 \) denotes the number of pendent neighbors of \( v \). By the choice of \( v \), each neighbor of \( v \) has at most \( d_1 \) pendent neighbors. Since each vertex in \( I_G(v) \) has neighbors only in \( N_G(v) \), by counting the number of edges between \( N_G[v] \) and \( G - N_G[v] \), we know \( |I_G(v)| \leq d_1d_2 + \frac{3d_1 - d_1d_2}{2} \). By (1),

\[
4d_1d_2 + 2(3d_1 - d_1d_2) \geq 4|I_G(v)| \geq 5d_1 + 5d_2 - 3. \tag{2}
\]

Since \( d_1 + d_2 \leq 4 \), this implies \( d_1 \leq 2 \). Thus, \( v \) has at most two pendent neighbors.

Note that by (1), for a pendent vertex \( w \), \( |I_G(w)| \geq 1 \), which implies that every vertex with a pendent neighbor has at least two pendent neighbors. Hence, \( v \) has exactly two pendent neighbors.

Moreover, \( d_1 = 2 \) in (2) results in \( d_2 \geq 2 \), so we conclude that \( v \) must be a vertex of degree 4.

Consider a vertex \( v \) with a pendent neighbor. By Claim 4.2, \( v \) is a vertex of degree 4 with exactly two pendent neighbors. By (1), \( |I_G(v)| \geq 5 \). Since every vertex has at most two pendent neighbors, \( |I_G(v)| \leq 5 \). This further implies that \( G \) is the graph with 10 vertices obtained from a 4-cycle \( v_1v_2v_3v_4v_1 \) by attaching exactly two pendent neighbors to each of \( v_1, v_2, \) and \( v_3 \). One may check easily that this is not a counterexample to our theorem.

**ACKNOWLEDGMENTS**

Eun-Kyung Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1I1A1A0105858711). Ilkyoo Choi was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B0403049), and also by the Hankuk University of Foreign Studies Research Fund. Boram Park was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (NRF-2018R1C1B6003577).

**ORCID**

Ilkyoo Choi  https://orcid.org/0000-0003-1102-7922

**REFERENCES**

1. G. Abrishami and M. A. Henning, *Independent domination in subcubic graphs of girth at least six*, Discrete Math. **341** (2018), no. 1, 155–164.
2. A. Akbari, S. Akbari, A. Doosthosseini, Z. Hadizadeh, M. A. Henning, and A. Naraghi, *Independent domination in subcubic graphs*, J. Comb. Optim. **43** (2022), no. 1, 28–41.

3. A. Babikir and M. A. Henning, *Domination versus independent domination in graphs of small regularity*, Discrete Math. **343** (2020), no. 7, 111727.

4. C. Berge, *The theory of graphs and its applications*, Methuen & Co. Ltd., London, John Wiley & Sons Inc., New York, 1962, Translated by Alison Doig.

5. E.-K. Cho, I. Choi, H. Kwon, and B. Park, *Tight bound for independent domination of cubic graphs without 4-cycles*, ArXiv. page 2112.11720, 2021.

6. E.-K. Cho, J. Kim, M. Kim, and S. Oum, *Independent domination of graphs with bounded maximum degree*, ArXiv. page 2202.09594, 2022.

7. P. Dorbec, M. A. Henning, M. Montassier, and J. Southey, *Independent domination in cubic graphs*, J. Graph Theory. **80** (2015), no. 4, 329–349.

8. W. Duckworth and N. C. Wormald, *On the independent domination number of random regular graphs*, Combin. Probab. Comput. **15** (2006), no. 4, 513–522.

9. M. Furuya, K. Ozeki, and A. Sasaki, *On the ratio of the domination number and the independent domination number in graphs*, Discrete Appl. Math. **178** (2014), 157–159.

10. W. Goddard and M. A. Henning, *Independent domination in graphs: a survey and recent results*, Discrete Math. **313** (2013), no. 7, 839–854.

11. W. Goddard, M. A. Henning, J. Lyle, and J. Southey, *On the independent domination number of regular graphs*, Ann. Comb. **16** (2012), no. 4, 719–732.

12. M. Knor, R. Škrekovski, and A. Tepeh, *Domination versus independent domination in regular graphs*, J. Graph Theory. **98** (2021), no. 3, 525–530.

13. P. C. B. Lam, W. C. Shiu, and L. Sun, *On independent domination number of regular graphs*, Discrete Math. **202** (1999), no. 1–3, 135–144.

14. O. Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, RI, 1962.

15. N. J. Rad and L. Volkmann, *A note on the independent domination number in graphs*, Discrete Appl. Math. **161** (2013), no. 18, 3087–3089.

16. M. Rosenfeld, *Independent sets in regular graphs*, Israel J. Math. **2** (1964), no. 4, 262–272.

17. J. Southey and M. A. Henning, *Domination versus independent domination in cubic graphs*, Discrete Math. **313** (2013), no. 11, 1212–1220.

18. O. Suil and D. B. West, *Cubic graphs with large ratio of independent domination number to domination number*, Graphs Combin. **32** (2016), no. 2, 773–776.

**How to cite this article:** E.-K. Cho, I. Choi, and B. Park, *On independent domination of regular graphs*, J. Graph Theory. 2023;103:159–170. https://doi.org/10.1002/jgt.22912