On Well-Founded and Recursive Coalgebras

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Abstract This paper studies fundamental questions concerning category-theoretic models of induction and recursion. We are concerned with the relationship between well-founded and recursive coalgebras for an endofunctor. For monomorphism preserving endofunctors on complete and well-powered categories every coalgebra has a well-founded part, and we provide a new, shorter proof that this is the coreflection in the category of all well-founded coalgebras. We present a new more general proof of Taylor’s General Recursion Theorem that every well-founded coalgebra is recursive, and we study under which hypothesis the converse holds. In addition, we present a new equivalent characterization of well-foundedness: a coalgebra is well-founded iff it admits a coalgebra-to-algebra morphism to the initial algebra.

1 Introduction

What is induction? What is recursion? In areas of theoretical computer science, the most common answers are related to initial algebras. Indeed, the dominant trend in abstract data types is initial algebra semantics (see e.g. [23]), and this approach has spread to other semantically-inclined areas of the subject. The approach in broad slogans is that, for an endofunctor $F$ describing the type of algebraic operations of interest, the initial algebra $\mu F$ has the property that for every $F$-algebra $A$, there is a unique homomorphism $\mu F \to A$, and this is recursion. Perhaps the primary example is recursion on $\mathbb{N}$, the natural numbers. Recall that $\mathbb{N}$ is the initial algebra for the set functor $FX = X + 1$. If $A$ is any set, and $a \in A$ and $\alpha : A \to A$ are given, then initiality tells us that there is a unique $f : \mathbb{N} \to A$ such that for all $n \in \mathbb{N},$

$$f(0) = a \quad f(n + 1) = \alpha(f(n)).$$

(1.1)

Then the first additional problem coming with this approach is that of how to “recognize” initial algebras: Given an algebra, how do we really know if it is

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initial? The answer – again in slogans – is that initial algebras are the ones with “no junk and no confusion.”

Although initiality captures some important aspects of recursion, it cannot be a fully satisfactory approach. One big missing piece concerns recursive definitions based on well-founded relations. For example, the whole study of termination of rewriting systems depends on well-orders, the primary example of recursion on a well-founded order. Let \((X, R)\) be a well-founded relation, i.e. one with no infinite sequences \(\cdots x_2 R x_1 R x_0\). Let \(A\) be any set, and let \(\alpha: \mathcal{P}A \to A\). (Here and below, \(\mathcal{P}\) is the power set functor, taking a set to the set of its subsets.) Then there is a unique \(f: X \to A\) such that for all \(x \in X\)

\[
f(x) = \alpha(\{f(y) : y R x\}).
\]

(1.2)

The main goal of this paper is the study of concepts that allow to extend the algebraic spirit behind initiality in (1.1) to the setting of recursion arising from well-foundedness as we find it in (1.2). The corresponding concepts are those of well-founded and recursive coalgebras for an endofunctor, which first appear in work by Osius [26] and Taylor [27, 28], respectively. In his work on categorical set theory, Osius [26] first studied the notions of well-founded and recursive coalgebras (for the power-set functor on sets and, more generally, the power-object functor on an elementary topos). He defined recursive coalgebras as those coalgebras \(\alpha: A \to \mathcal{P}A\) which have a unique coalgebra-to-algebra homomorphism into every algebra (see Definition 3.2).

Taylor [27, 28] took Osius’ ideas much further. He introduced well-founded coalgebras for a general endofunctor, capturing the notion of a well-founded relation categorically, and considered recursive coalgebras under the name ‘coalgebras obeying the recursion scheme’. He then proved the General Recursion Theorem that all well-founded coalgebras are recursive for every endofunctor on sets (and on more general categories) preserving inverse images. Recursive coalgebras were also investigated by Eppendahl [14], who called them algebra-initial coalgebras. Capretta, Uustalu, and Vene [12] further studied recursive coalgebras, and they showed how to construct new ones from given ones by using comonads. They also explained nicely how recursive coalgebras allow for the semantic treatment of (functional) divide-and-conquer programs. More recently, Jeannin et al. [18] proved the general recursion theorem for polynomial functors on the category of many-sorted sets; they also provide many interesting examples of recursive coalgebras arising in programming.

Our contributions in this paper are as follows. We start by recalling some preliminaries in Section 2 and the definition of (parametrically) recursive coalgebras in Section 3 and of well-founded coalgebras in Section 4 (using a formulation based on Jacobs’ next time operator [17], which we extend from Kripke polynomial set functors to arbitrary functors). We show that every coalgebra for a monomorphism-preserving functor on a complete and well-powered category has a well-founded part, and provide a new proof that this is the coreflection in the category of well-founded coalgebras (Proposition 5.5), shortening our previous proof [6]. Next we provide a new proof of Taylor’s General Recursion The-
Theorem (Theorem 7.2), generalizing this to endofunctors preserving monomorphisms on a complete and well-powered category having smooth monomorphisms (see Definition 2.14). For the category of sets, this implies that “well-founded ⇒ recursive” holds for all endofunctors, strengthening Taylor’s result. We then discuss the converse: is every recursive coalgebra well-founded? Here the assumption that \( F \) preserves inverse images cannot be lifted, and one needs additional assumptions. In fact, we present two proofs: one assumes the functor has a prefixed point and universally smooth monomorphisms (see Theorem 8.1). Under these assumptions we also give a new equivalent characterization of recursiveness and well-foundedness: a coalgebra is recursive if it has a coalgebra-to-algebra morphism into the initial algebra (which exists under our assumptions), see Corollary 8.2. This characterization was previously established for finitary functors on sets [4]. The other proof of the above implication is due to Taylor [27] and presented for the convenience of the reader. Taylor’s proof uses the concept of a subobject classifier (Theorem 8.6). It implies that ‘recursive’ and ‘well-founded’ are equivalent concepts for all set functors preserving inverse images. We also prove that a similar result holds for the category of vector spaces over a fixed field (Corollary 8.13).

Finally, we show in Section 6 that well-founded coalgebras are closed under coproducts, quotients and, assuming mild assumptions, under subcoalgebras.

2 Preliminaries

We start by recalling some background material. Except for the definitions of algebra and coalgebra in Section 2.1, the subsections below may be read as needed. We assume that readers are familiar with notions of basic category theory; see e.g. [3] for everything which we do not detail.

2.1 Algebras and Coalgebras. We are concerned throughout this paper with algebras and coalgebras for an endofunctor. This means that we have an underlying category, usually written \( \mathcal{A} \); frequently it is the category of sets or of vector spaces over a fixed field, and that a functor \( F : \mathcal{A} \to \mathcal{A} \) is given. An \( F \)-algebra is a pair \((A, \alpha)\), where \( \alpha : FA \to A \). An \( F \)-coalgebra is a pair \((A, \alpha)\), where \( \alpha : A \to FA \). We usually drop the functor \( F \). Given two algebras \((A, \alpha)\) and \((B, \beta)\), an algebra homomorphism from the first to the second is \( h : A \to B \) in \( \mathcal{A} \) such that the diagram below commutes:

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
\downarrow{Fh} & & \downarrow{h} \\
FB & \xrightarrow{\beta} & B
\end{array}
\]

That is \( h \cdot \alpha = \beta \cdot Fh \). An algebra is initial if it has a unique morphism to every algebra. Recall that by Lambek’s Lemma [20], whenever an initial algebra \( i : F(\mu F) \to \mu F \) exists, then \( i \) is an isomorphism. Thus, \( \mu F \) can always be regarded as a coalgebra \((\mu F, \iota^{-1})\).
Similarly, given coalgebras \((A, \alpha)\) and \((B, \beta)\), a homomorphism of \(F\)-coalgebras from the first to the second is \(h: A \to B\) in \(
abla\) such that \(F h \cdot \alpha = \beta \cdot h\). Moreover, a terminal coalgebra is one with the property that every coalgebra has a unique morphism into it. The category of \(F\)-coalgebras is denoted by \(\text{Coalg}_F\).

**Example.**

1. The power set functor \(\mathcal{P}: \text{Set} \to \text{Set}\) takes a set \(X\) to the set \(\mathcal{P}X\) of all subsets of it; for a morphism \(f: X \to Y\), \(\mathcal{P}f: \mathcal{P}X \to \mathcal{P}Y\) takes a subset \(S \subseteq X\) to its direct image \(f[S]\). Coalgebras \(\alpha: X \to \mathcal{P}X\) may be identified with directed graphs on the set \(X\) of vertices, and the coalgebra structure \(\alpha\) describes the edges: \(b \in \alpha(a)\) means that there is an edge \(a \to b\) in the graph.

2. Let \(\Sigma\) be a signature, i.e. a set of operation symbols, each with a finite arity. The polynomial functor \(H\Sigma\) associated to \(\Sigma\) assigns to a set \(X\) the set \(H\Sigma X = \bigcoprod_{n \in \mathbb{N}} \Sigma_n \times X^n\), where \(\Sigma_n\) is the set of operation symbols of arity \(n\). This may be identified with the set of all terms \(\sigma(x_1, \ldots, x_n)\), for \(\sigma \in \Sigma_n\), and \(x_1, \ldots, x_n \in X\). Algebras for \(H\Sigma\) are the usual \(\Sigma\)-algebras.

3. Deterministic automata over an input alphabet \(\Sigma\) are coalgebras for the functor \(FX = \{0, 1\} \times X^{\Sigma}\). Indeed, given a set \(S\) of states, the next-state map \(S \times \Sigma \to S\) may be curried to \(\delta: S \to S^{\Sigma}\). The set of final states yields the acceptance predicate \(a: S \to \{0, 1\}\). So the automaton may be regarded as \(\langle a, \delta \rangle: S \to \{0, 1\} \times S^{\Sigma}\).

4. Labelled transitions systems are coalgebras for \(FX = \mathcal{P}(\Sigma \times X)\).

5. To describe linear weighted automata, i.e. weighted automata over the input alphabet \(\Sigma\) with weights in a field \(K\), as coalgebras, one works with the category \(\text{Vec}_K\) of vector spaces over \(K\). A linear weighted automaton with the input alphabet \(\Sigma\) is then a coalgebra for \(FX = K \times X^{\Sigma}\).

**Remark.**

1. Recall that an epimorphism \(e: A \to B\) is called strong if it satisfies the following diagonal fill-in property: given a monomorphism \(m: C \hookrightarrow D\) and morphisms \(f: A \to C\) and \(g: B \to D\) such that \(m \cdot f = g \cdot e\) (i.e. the outside of the square below commutes) then there exists a unique \(d: B \to C\) such that the diagram below commutes:

   \[
   \begin{array}{ccc}
   A & \xrightarrow{e} & B \\
   f \downarrow & & \downarrow g \\
   C & \xrightarrow{m} & D
   \end{array}
   \]  

(2.1)

2. A complete and well-powered category \(\nabla\) has factorizations of morphisms \(f\) as \(f = m \cdot e\), where \(e\) is a strong epimorphism and \(m\) is a monomorphism. It follows from Adámek et al. [3, Theorem 14.17 and dual of Exercise 14C(d)] that every complete and well-powered category has such factorizations. We call the subobject \(m\) the image of \(f\).

3. We indicate monomorphisms by \(\hookrightarrow\) and strong epimorphisms by \(\twoheadrightarrow\).
2.2 Preservation Properties. Recall that an intersection of two subobjects \( s_i : S_i \hookrightarrow A \) (\( i = 1, 2 \)) of a given object \( A \) is given by their pullback. Analogously, (general) intersections are given by wide pullbacks. Furthermore, the inverse image of a subobject \( s : S \hookrightarrow B \) under a morphism \( f : A \rightarrow B \) is the subobject \( t : T \rightarrow A \) obtained by a pullback of \( s \) along \( f \).

Example 2.3. The condition that a functor preserves intersections is an extremely mild one for set functors:

1. Every polynomial functor preserves intersections and inverse images.
2. The power-set functor \( \mathcal{P} \) preserves intersections and inverse images.
3. The collection of set functors which preserve intersections is closed under products, coproducts, and compositions. A subfunctor \( m : G \hookrightarrow F \) of an intersection preserving functor \( F \) preserves intersections whenever \( m \) is a cartesian natural transformation, i.e. all naturality squares are pullbacks (being a pullback is indicated by the “corner” symbol):

\[
\begin{array}{ccc}
GX & \xrightarrow{m_X} & FX \\
\downarrow Gf & & \downarrow Ff \\
GY & \xrightarrow{m_Y} & FY
\end{array}
\]

Similarly, for inverse images.

4. The functor \( C_{01} : \text{Set} \rightarrow \text{Set} \) is defined by \( C_{01}\emptyset = \emptyset \) and \( C_{01}1 = 1 \) for \( X \neq \emptyset \). \( C_{01} \) clearly preserves monomorphisms but it does not preserve finite intersections. Indeed, the empty intersection of \( \{0\}, \{1\} \hookrightarrow \{0, 1\} \) is mapped to \( \emptyset \); however the intersection of those subsets under \( C_{01} \) is 1, not \( \emptyset \).

5. Consider next the set functor \( R \) defined by \( RX = \{(x, y) \in X \times X : x \neq y\} + \{d\} \) for sets \( X \). For a function \( f : X \rightarrow Y \) put

\[
Rf(d) = d \quad \text{and} \quad Rf(x, y) = \begin{cases} d & \text{if } f(x) \neq f(y) \\ (f(x), f(y)) & \text{else.} \end{cases}
\]

This functor preserves finite intersections, since it preserves the above intersection of \( \{0\}, \{1\} \hookrightarrow \{0, 1\} \), and so it is (naturally isomorphic to) its Trnková hull. However, \( R \) does not preserve inverse images; consider e.g. the pullback diagram (under \( R \)):

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{\text{const}_1} & \{0\} \\
\downarrow & & \downarrow \\
\{0, 1\} & \xrightarrow{\text{const}_1} & \{0, 1\}
\end{array}
\]

(For \( (0, 1) \in R\{0, 1\} \) and \( d \in R\emptyset \) are merged in the right-hand \( R\{0, 1\} \), yet there is no suitable element in \( R\emptyset \).)

6. “Almost” all finitary set functors preserve intersections. In fact, the Trnková hull of a finitary set functor preserves intersections (see Proposition 2.6).
Some of our results require $F$ to preserve finite (or all) intersection or inverse images. For set functors these are rather mild requirements, as we now explain.

**Proposition 2.4** [31]. For every set functor $F$ there exists an essentially unique set functor $\bar{F}$ which coincides with $F$ on nonempty sets and functions and preserves finite intersections (whence monomorphisms).

For the proof see Trnková [31, Propositions III.5 and II.4]; for a more direct proof see Adámek and Trnková [9, Theorem III.4.5]. We call the functor $\bar{F}$ the *Trnková hull* of $F$.

**Remark 2.5.** In fact, Trnková gave a construction of $\bar{F}$: she defined $\bar{F}\emptyset$ as the set of all natural transformations $C_{01} \to F$, where $C_{01}$ is the set functor with $C_{01}\emptyset = \emptyset$ and $C_{01}X = 1$ for all nonempty sets $X$. For the empty map $e : \emptyset \to X$ with $X \neq \emptyset$, $\bar{F}e$ maps a natural transformation $\tau : C_{01} \to F$ to the element given by $\tau_X : 1 \to FX$.

Preservation of all intersections can be achieved for *finitary* set functors. Intuitively, a functor on sets is finitary if its behavior is completely determined by its action on *finite* sets and functions. For a general functor, this intuition is captured by requiring that the functor preserve filtered colimits [8]. For a set functor $F$ this is equivalent to being *finitely bounded*, which is the following condition: for each element $x \in FX$ there exists a finite subset $M \subseteq X$ such that $x \in F[i]FM$, where $i : M \hookrightarrow X$ is the inclusion map [7, Rem. 3.14].

**Proposition 2.6** [5, p. 66]. The Trnková hull of a finitary set functor preserves all intersections.

*Proof.* Let $F$ be a finitary set functor. Since $\bar{F}$ is finitary and preserves finite intersections, for every element $x \in \bar{F}X$, there exists a least finite set $m : Y \hookrightarrow X$ with $x$ contained in $\bar{F}m$. Preservation of all intersections now follows easily: given subsets $v_i : V_i \hookrightarrow X$, $i \in I$, with $x$ contained in the image of $\bar{F}v_i$ for each $i$, then $x$ also lies in the image of the finite set $\cap_i v_i$, hence $m \subseteq \cap_i v_i$ by minimality. This proves $m \subseteq \cap_{i \in I} v_i$, thus, $x$ lies in the image of $\bar{F}(\cap_{i \in I} v_i)$, as required.

### 2.3 Factorizations

Every complete and well-powered category $\mathcal{A}$ has the following factorizations of morphisms: every morphism $f$ may be written as $f = m \cdot e$, where $e$ is a strong epimorphism and $m$ is a monomorphism [10, Prop. 4.4.3]. We call the subobject $m$ the *image* of $f$. It follows from a result in Kurz’ thesis [19, Prop. 1.3.6] that factorizations of morphisms lift to coalgebras.

**Proposition 2.7** (*Coalg $F$ inherits factorizations from $\mathcal{A}$*). Suppose that $F$ preserves monomorphisms. Then the category $\text{Coalg } F$ has factorizations of homomorphisms $f$ as $f = m \cdot e$, where $e$ is carried by a strong epimorphism and $m$ by a monomorphism in $\mathcal{A}$. The diagonal fill-in property holds in $\text{Coalg } F$.

**Remark 2.8.** By a subcoalgebra of a coalgebra $(A, \alpha)$ we mean a subobject in $\text{Coalg } F$ represented by a homomorphism $m : (B, \beta) \hookrightarrow (A, \alpha)$, where $m$ is monic in $\mathcal{A}$. Similarly, by a strong quotient of a coalgebra $(A, \alpha)$ is represented by a homomorphism $e : (A, \alpha) \twoheadrightarrow (C, \gamma)$ with $e$ strongly epic in $\mathcal{A}$.
2.4 Subobject Lattices.

Notation 2.9. For every object $A$ we denote by $\text{Sub}(A)$ the poset of subobjects of $A$. The top of this poset is represented by $\text{id}_A$, and the bottom $\bot_A$ is the intersection of all subobjects of $A$.

Now suppose that $\mathcal{A}$ is a complete and well-powered category.

Remark 2.10. Note that $\text{Sub}(A)$ is a complete lattice: it is small since $\mathcal{A}$ is well-powered, and a meet of subobjects $m_i : A_i \to A$, $i \in I$, is their intersection, obtained by forming their wide pullback. It follows that $\text{Sub}(A)$ has all joins as well.

We shall need that forming inverse images, i.e. pulling back along a morphism, is a right adjoint.

Notation 2.11. For every morphism $f : B \to A$ we have two operators:

1. The inverse image operator

   $$\leftarrow f : \text{Sub}(A) \to \text{Sub}(B),$$

   assigning to every subobject $s : S \ni A$ its inverse image under $f$ obtained by the following pullback

   $$\begin{array}{ccc}
   P & \xrightarrow{f} & S \\
   \downarrow & & \downarrow s \\
   B & \xrightarrow{f} & A
   \end{array}$$

2. The (direct) image operator

   $$\rightarrow f : \text{Sub}(B) \to \text{Sub}(A),$$

   assigning to every subobject $t : T \ni B$ the image of $f \cdot t$:

   $$\begin{array}{ccc}
   T & \rightarrow & S \\
   \downarrow \rightarrow f & & \downarrow \rightarrow f(t) \\
   B & \rightarrow f & A
   \end{array}$$

Remark 2.12. (1) A monotone map $r : X \to Y$ between posets, regarded as a functor from $X$ to $Y$ considered as categories, is a right adjoint iff there exists a monotone map $\ell : Y \to X$ such that

   $$\ell(y) \leq x \quad \text{iff} \quad y \leq r(x) \quad \text{for every } x \in X \text{ and } y \in Y.$$ 

(2) Moreover, a monotone map $r : \text{Sub}(B) \to \text{Sub}(A)$ is a right adjoint iff it preserves intersections. Indeed, the necessity follows since right adjoints preserve limits. For the sufficiency, suppose that $r$ preserves intersections, and define $\ell : \text{Sub}(A) \to \text{Sub}(B)$ by

   $$\ell(m) = \bigwedge_{m \leq r(m)} m \quad \text{for every } m \in \text{Sub}(A).$$
Then $\ell$ is clearly monotone, and for every $m'$ in $\text{Sub}(B)$ we have

$$\ell(m) \leq m' \iff m \leq r(m').$$

Thus, $\ell$ is the desired left adjoint of $r$.

**Proposition 2.13.** If $\mathcal{A}$ is complete and well-powered, then for every morphism $f : B \to A$ we have an adjoint situation:

$$\text{Sub}(A) \overset{\ell}{\leftarrow} \text{Sub}(B).$$

In other words: $\overrightarrow{f}(t) \leq s$ iff $t \leq \overleftarrow{f}(s)$ for all subobjects $s : S \to A$ and $t : T \to B$.

**Proof.** In order to see this we consider the following diagram:

![Diagram](image)

By the universal property of the lower middle pullback square and the diagonal fill-in property, we have the dashed morphism on the left iff we have the one on the right. Thus, $t \leq \overrightarrow{f}(s)$ iff $\overleftarrow{f}(t) \leq s$, as desired. $\square$

### 2.5 Chains.

By a *transfinite chain* in a category $\mathcal{A}$ we understand a functor from the ordered class $\text{Ord}$ of all ordinals into $\mathcal{A}$. Moreover, for an ordinal $\lambda$, a $\lambda$-chain in $\mathcal{A}$ is a functor from $\lambda$ to $\mathcal{A}$. A category has colimits of chains if for every ordinal $\lambda$ it has a colimit of every $\lambda$-chain. This includes the initial object $0$ (the case $\lambda = 0$).

**Definition 2.14.** (1) A category $\mathcal{A}$ has *smooth monomorphisms* if for every $\lambda$-chain $C$ of monomorphisms a colimit exists, its colimit cocone is formed by monomorphisms, and for every cone of $C$ formed by monomorphisms, the factorizing morphism from $\text{colim} C$ is monic. In particular, every morphism from $0$ is monic.

(2) $\mathcal{A}$ has *universally smooth monomorphisms* if $\mathcal{A}$ also has pullbacks, and for every morphism $f : X \to \text{colim} C$, the functor $\mathcal{A}/\text{colim} C \to \mathcal{A}/X$ forming pullbacks along $f$ preserves the colimit of $C$. This implies that the initial object $0$ is *strict*, i.e. every morphism $f : X \to 0$ is an isomorphism. Indeed, consider the empty chain ($\lambda = 0$).

**Example 2.15.** (1) $\text{Set}$ has universally smooth monomorphisms. More generally, every Grothendieck topos does.
(2) Vecₖ has smooth monomorphisms, but not universally so because the initial object is not strict.

(3) Categories in which colimits of chains and pullbacks are formed “set-like” have universally smooth monomorphisms. These include the categories of posets, graphs, topological spaces, presheaf categories, and many varieties, such as monoids, graphs, and unary algebras.

(4) Every locally finitely presentable category $\mathcal{A}$ with a strict initial object has smooth monomorphisms. This follows from [8, Prop. 1.62]. Moreover, since pullbacks commute with colimits of chains, it is easy to prove that colimits of chains are universal. Indeed, suppose that $c_i: C_i \to C$ is the colimit cocone of some chain of objects $C_i$, $i < \lambda$, and let $f: B \to C$ be a morphism. Form the pullback of every $c_i$ along $f$:

$$
\begin{array}{ccc}
B & \xrightarrow{f_i} & C_i \\
\downarrow^{b_i} & & \downarrow^{c_i} \\
B & \xrightarrow{f} & C
\end{array}
$$

Then $b_i: B_i \to B$ is a colimit cocone. Indeed, in the category of commutative squares in $\mathcal{A}$, the chain of the above pullbacks squares has as a colimit the following pullback square

$$
\begin{array}{ccc}
\colim B_i & \xrightarrow{\colim f_i} & \colim C_i = C \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & C
\end{array}
$$

Unfortunately, the example of rings demonstrates that the assumption of strictness of 0 cannot be lifted. In fact, the collections of monomorphisms is not smooth in the category of rings since there exist non-injective homomorphisms whose domain is the initial ring $\mathbb{Z}$.

(5) The category $\mathbf{CPO}$ of complete partial orders (i.e. partially ordered sets with joins of $\omega$-chains) does not have smooth monomorphisms. Indeed, consider the $\omega$-chain of linearly ordered sets $A_n = \{0, \ldots, n\} + \{\top\}$ (where $\top$ is a top element) with inclusion maps $A_n \to A_{n+1}$. Its colimit is the linearly ordered set $\mathbb{N} + \{\top, \top'\}$ of natural numbers with two added top elements $\top' < \top$. For the sub-cpo $\mathbb{N} + \{\top\}$, the inclusions of $A_n$ are monic and form a cocone. But the unique factorizing morphism from the colimit is not monic.

**Remark 2.16.** If $\mathcal{A}$ is a complete and well-powered category, then $\text{Sub}(A)$ is a complete lattice. Now suppose that $\mathcal{A}$ has smooth monomorphisms.

(1) In this setting, the unique morphism $\bot_A: 0 \to A$ is a monomorphism and therefore the bottom element of the poset $\text{Sub}(A)$.

(2) Furthermore, a join of a chain in $\text{Sub}(A)$ is obtained by forming a colimit. More precisely, given an ordinal $k$ and an $k$-chain $m_i: A_i \to A$ of subobjects ($i < k$), we have the diagram of objects $(A_i)_{i<k}$, where for all $i \leq j < k$ the
connecting morphisms \( a_{ij} : A_i \rightarrow A_j \) are the unique factorizations witnessing \( m_i \leq m_j \):

\[
A_i \xrightarrow{a_{ij}} A_j \\
\downarrow m_i \quad \downarrow m_j \quad \downarrow a_{ij} \\
A
\]

The colimit \( B \) of this diagram is formed by monomorphisms \( b_i : A_i \rightarrow B \), \( i < k \), and the unique monomorphism \( m : B \rightarrow A \) with \( m \cdot b_i = m_i \) for all \( i < k \) is the join of all \( m_i \), in symbols: \( m = \bigvee_{i<k} m_i \).

(3) If \( \mathcal{A} \) has universally smooth monomorphisms, then for every morphism \( f : A \rightarrow B \), the operator \( \overleftarrow{id} : \text{Sub}(B) \rightarrow \text{Sub}(A) \) preserves unions of chains.

Indeed, suppose that \( c : C \rightarrow A \) is the union of a chain of subobjects \( a_i : A_i \rightarrow A \) in \( \text{Sub}(A) \). Then \( C \) is the colimit of the (chain of connecting morphisms between the) \( A_i \) with colimit injections \( c_i : A_i \rightarrow C \), say. For the morphism \( p = \overleftarrow{id}(c) : P \rightarrow B \) we paste two pullback squares for every \( i \) as shown below:

\[
\begin{array}{ccc}
B_i & \xrightarrow{f_i} & A_i \\
p_i \downarrow & & \downarrow c_i \\
B \downarrow f & & A \\
p \downarrow & & \downarrow \end{array}
\]

The outside is then the pullback square stating that \( b_i = \overleftarrow{id}(a_i) \). By universality, \( P = \text{colim} B_i \) with colimit injections \( p_i \). Thus, by the constructivity of monomorphisms \( p \) is the union of the subobjects \( b_i \) in \( \text{Sub}(B) \); in symbols: \( \bigvee_i \overleftarrow{id}(a_i) = \overleftarrow{id} \left( \bigvee_i a_i \right) \) as desired.

**Remark 2.17.** (1) Suppose that \( \mathcal{A} \) has colimits of chains. Recall [2] that every endofunctor \( F : \mathcal{A} \rightarrow \mathcal{A} \) gives rise to an essentially unique chain \( W : \text{Ord} \rightarrow \mathcal{A} \), the *initial-algebra chain*, of objects \( W_i = F^{0}0, i \in \text{Ord} \) and connecting morphisms \( w_{ij} : F^{0}0 \rightarrow F^{j}0, i \leq j \in \text{Ord} \). They are defined by transfinite recursion:

\[
\begin{align*}
W_0 &= 0, \\
W_{j+1} &= FW_j \quad \text{for all ordinals } j, \\
W_j &= \text{colim}_{i<j} W_i \quad \text{for all limit ordinals } j,
\end{align*}
\]

and

\[
\begin{align*}
w_{0,1} : 0 &\rightarrow W_0 \quad \text{is unique}, \\
w_{j+1,k+1} &= FW_{j,k} : FW_j \rightarrow FW_k, \\
w_{i,j} (i < j) &\text{ is the colimit cocone for limit ordinals } j.
\end{align*}
\]

(2) Now suppose that \( \mathcal{A} \) has smooth monomorphisms and that \( F : \mathcal{A} \rightarrow \mathcal{A} \) has a *pre-fixed point*, i.e. an object \( A \) with a monomorphism \( \alpha : FA \rightarrow A \). Then an initial algebra exists. This follows from results by Trnková et al. [29] as we now
briefly recall. Let \( \alpha : FA \to A \) be a pre-fixed point. Then there is a unique cocone \( \alpha_i : W_i \to B \) satisfying \( \alpha_{i+1} = \alpha \cdot F\alpha_i \). Moreover, each \( \alpha_i \) is monomorphic. Since \( A \) has only a set of subobjects, there is some \( \lambda \) such that for every \( i > \lambda \), all of the morphisms \( \alpha_i \) represent the same subobject of \( A \). Consequently, \( w_{\lambda, \lambda+1} \) is an isomorphism. Then \( \mu F = F^{\lambda 0} \) with the structure \( \iota = w_{\lambda, \lambda+1}^{-1} : F(\mu F) \to \mu F \) is an initial algebra.

3 Recursive Coalgebras

**Assumption 3.1.** We work with a standard set theory (e.g. Zermelo-Fraenkel), assuming the Axiom of Choice. In particular, we use transfinite induction on several occasions. (We are not concerned with constructive foundations in this paper.)

Throughout this paper we assume that \( \mathcal{A} \) is a complete and well-powered category \( \mathcal{A} \) and that \( F : \mathcal{A} \to \mathcal{A} \) preserves monomorphisms.

For \( \mathcal{A} = \text{Set} \) the condition that \( F \) preserves monomorphisms may be dropped. In fact, preservation of nonempty monomorphism is sufficient in general (for a suitable notion of nonempty monomorphism) [25, Lemma 2.5], and this holds for every set functor.

The following definition of recursive coalgebras was first given by Osius [26]. Taylor [28] speaks of coalgebras obeying the recursion scheme. Capretta et al. [12] extended the concept to parametrically recursive coalgebra by dualizing completely iterative algebras [24].

**Definition 3.2.** A coalgebra \( \gamma : C \to FC \) is called recursive if for every algebra \( \alpha : FA \to A \) there exists a unique coalgebra-to-algebra morphism \( h : C \to A \), i.e. a unique morphism such that the square below commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{h} & A \\
\downarrow{\gamma} & & \downarrow{\alpha} \\
FC & \xrightarrow{Fh} & FA
\end{array}
\]

**Examples 3.3.** (1) The first examples of recursive coalgebras are well-founded relations. Recall that a binary relation \( R \) on a set \( X \) is well-founded if there is no infinite descending sequence

\[
\cdots R x_3 R x_2 R x_1 R x_0.
\]

Now a binary relation \( R \subseteq X \times X \) is essentially a graph on \( X \), equivalently the coalgebra structure \( \alpha : X \to \mathcal{P}X \) with \( \alpha(x) = \{y \mid y R x\} \) (cf. Example 2.1(1)). Osius [26] showed that for every well-founded relation the associated \( \mathcal{P} \)-coalgebra is recursive. Shortly: a graph regarded as a coalgebra for \( \mathcal{P} \) is recursive iff it has no infinite path.

(2) If \( \mu F \) exists, then it is a recursive coalgebra.

(3) The initial coalgebra \( 0 \to F0 \) is recursive.
(4) If $(C, \gamma)$ is recursive so is $(FC, F\gamma)$, see [12, Prop. 6].

(5) Every colimit of recursive coalgebras in $\text{Coalg} F$ is recursive. This is easy to prove, using that colimits of coalgebras are formed on the level of the underlying category.

(6) It follows from items (3)–(5) that in the initial-algebra chain from Remark 2.17 all coalgebras $w_{i,i+1}: F^i 0 \to F(F^i 0)$, $i \in \text{Ord}$, are recursive.

By an argument similar to the proof of the (dual of) Lambek’s Lemma, we see that a terminal recursive $F$-coalgebra is a fixed point of $F$, and we have

**Corollary 3.4** [12, Prop. 7]. The initial algebra is precisely the same as the terminal recursive coalgebra.

Capretta et al. [12] study the notion of a parametrically recursive coalgebra dualizing the notion of a completely iterative algebra [24].

**Definition 3.5.** A coalgebra $(A, \alpha)$ is parametrically recursive if for every morphism $e: FX \times A \to X$ there is a unique morphism $e^!: A \to X$ so that the square below commutes:

$$
\begin{array}{c}
A \xrightarrow{e^!} X \\
\downarrow{\langle \alpha, A \rangle} \\
FA \times A \xrightarrow{Fe^! \times A} FX \times A
\end{array}
$$

The dual statement of [24, Thm. 2.8] states that the initial algebra is, equivalently, the terminal parametrically recursive coalgebra. Of course, every parametrically recursive coalgebra is recursive. (To see this, form for a given $e: FX \to X$ the morphism $e' = e \cdot \pi$, where $\pi: FX \times A \to FX$ is the projection.) In Corollaries 8.2 and 8.7 we will see that the converse often holds. However, in general the converse fails:

**Example 3.6** [1]. Let $R: \text{Set} \to \text{Set}$ be the functor defined in Example 2.3, part (5). Also, let $C = \{0, 1\}$, and define $\gamma: C \to RC$ by $\gamma(0) = \gamma(1) = (0, 1)$. Then $(C, \gamma)$ is a recursive coalgebra. Indeed, for every algebra $\alpha: RA \to A$ the constant map $h: C \to A$ with $h(0) = h(1) = \alpha(d)$ is the unique coalgebra-to-algebra morphism.

However, $(C, \gamma)$ is not parametrically recursive. To see this, consider any morphism $e: RX \times \{0, 1\} \to X$ such that $RX$ contains more than one pair $(x_0, x_1)$, $x_0 \neq x_1$ with $e((x_0, x_1), i) = x_i$ for $i = 0, 1$. Then each such pair yields $h: C \to X$ with $h(i) = x_i$ making (3.1) commute. Thus, $(C, \gamma)$ is not parametrically recursive.

The situation in Example 3.6 is relatively rare and artificial because for functors preserving inverse images, recursive and parametrically recursive coalgebras coincide (see Corollary 8.2 and Corollary 8.7).

We conclude this section with a few examples explaining how recursive coalgebras capture familiar recursive function definitions as well as functional divide-and-conquer programs.
Examples 3.7. (1) The functor $FX = X + 1$ has unary algebras with a constant as algebras, and coalgebras for $F$ may be identified with partial unary algebras. The initial algebra for $F$ is the set of natural numbers $\mathbb{N}$ with the structure given by the successor function and the constant 0. The inverse of the initial $F$-algebra is the coalgebra given by the partial unary operation $n \mapsto n - 1$ (defined iff $n > 0$). This coalgebra is parametrically recursive. Hence every function
\[ e = [u, v]: FX \times \mathbb{N} \cong \mathbb{N} + X \times \mathbb{N} \to X \]
defines a unique sequence $e^1: \mathbb{N} \to X$, $e^1(n) = x_n$ such that (3.1) commutes. This means that $x_0 = v(0)$ and $x_{n+1} = u(x_n, n + 1)$. For example, the factorial function is then given by the choice $X = \mathbb{N}; u(n, m) = n \cdot m$ and $v(0) = 1$.

(2) For the set functor $F$ given by $FX = X \times X + 1$, coalgebras $\gamma: C \times C + 1$ are deterministic systems with a state set $C$, a binary input and with halting states (expressed by $\gamma^{-1}(1)$).

The coalgebra $\mathbb{N}$ of natural numbers with halting states 0 and 1 and input structure $\gamma: n \mapsto (n - 1, n - 2)$ for $n \geq 2$ is parametrically recursive (see Example 7.4).

For example, to define the Fibonacci sequence starting with $a_0, a_1 \in \mathbb{N}$, consider the morphism $e: F\mathbb{N} \times \mathbb{N} \cong \mathbb{N}^3 + \mathbb{N} \to \mathbb{N}$ with
\[ e(i, j, k) = i + j \quad \text{and} \quad e(n) = \begin{cases} a_0 & \text{if } n = 0, \\ a_1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \]

We know that there is a unique sequence $e^1: \mathbb{N} \to \mathbb{N}$ such that the diagram (3.1) commutes, which means $e^1(0) = a_0$, $e^1(1) = a_1$ and $e^1(n + 2) = e^1(n + 1) + e^1(n)$.

(3) Capretta et al. [13] showed how to obtain Quicksort using parametric recursivity. Let $A$ be any linearly ordered set (of data elements). Then Quicksort is usually defined as the recursive function $q: A^* \to A^*$ given by
\[ q(\varepsilon) = \varepsilon \quad \text{and} \quad q(aw) = q(w_{\leq a}) \ast (aq(w_{> a})), \]
where $A^*$ is the set of all lists on $A$, $\varepsilon$ is the empty list, $\ast$ is the concatenation of lists and $w_{\leq a}$ denotes the list of those elements of $w$ which are less than or equal to $a$; analogously for $w_{> a}$.

Now consider the functor $FX = 1 + A \times X \times X$ on $\text{Set}$, where $1 = \{\bullet\}$, and form the coalgebra $s: A^* \to 1 + A \times A^* \times A^*$ given by
\[ s(\varepsilon) = \bullet \quad \text{and} \quad s(aw) = (a, w_{\leq a}, w_{> a}) \quad \text{for } a \in A \text{ and } w \in A^*. \quad (3.2) \]

We shall see that this coalgebra is recursive in Example 7.4. Thus, for the $F$-algebra $m: 1 + A \times A^* \times A^* \to A^*$ given by
\[ m(\bullet) = \varepsilon \quad \text{and} \quad m(a, w, v) = w \ast (av) \]
there exists a unique function $q$ on $A^*$ such that $q = m \cdot Fq \cdot s$. Notice that the last equation reflects the idea that Quicksort is a divide-and-conquer algorithm.
The coalgebra structure $s$ divides a list into two parts $w_{\leq a}$ and $w_{>a}$. Then $Fq$ sorts these two smaller lists, and finally in the combine- (or conquer-) step, the algebra structure $m$ merges the two sorted parts to obtain the desired whole sorted list.

Similarly, functions defined by parametric recursivity (cf. Diagram (3.1)), can be understood as divide-and-conquer algorithms, where the combine-step is allowed to access the original parameter additionally. For instance, in the current example the divide-step $\langle s, id_{A^*} \rangle$ produces the pair consisting of $(a, w_{\leq a}, w_{>a})$ and the original parameter $aw$, and the combine-step, which is given by an algebra $FX \times A^* \to X$ will, by the commutativity of (3.1), get $aw$ as its right-hand input.

Jeannin et al. [18, Sec. 4] provide a number of recursive functions arising in programming that are determined by recursivity of a coalgebra, e.g. the gcd of integers, the Ackermann function, and the Towers of Hanoi.

4 The Next Time Operator and Well-Founded Coalgebras

As we have mentioned in the Introduction, the main issue of this paper is the relationship between two concepts pertaining to coalgebras: recursiveness and well-foundedness. The concept of well-foundedness is well-known for directed graphs: it means that the graph has no infinite directed paths. Similarly for relations: for example, the elementhood relation $\in$ of set theory is well-founded; this is precisely the Foundation Axiom.

Taylor [28, Def. 6.2.3] gave a more general category theoretic formulation of well-foundedness. We observe here that his definition can be presented in a compact way, by using an operator that generalizes the way one thinks of the semantics of the ‘next time’ operator of temporal logics for non-deterministic (or even probabilistic) automata and transitions systems. It is also strongly related to the algebraic semantics of modal logic, where one passes from a graph $G$ to a function on $\mathcal{P}G$. Jacobs [17] defined and studied the ‘next time’ operator on coalgebras for Kripke polynomial set functors, which can be generalized to arbitrary functors as follows.

Recall that $\text{Sub}(A)$ denotes the complete lattice of subobjects of $A$.

**Definition 4.1** [5, Def. 8.9]. Every coalgebra $\alpha: A \to FA$ induces an endofunction on $\text{Sub}(A)$, called the **next time operator**

$$\bigcirc: \text{Sub}(A) \to \text{Sub}(A), \quad \bigcirc(s) = \alpha^{-1}(Fs) \quad \text{for } s \in \text{Sub}(A).$$

In more detail: we define $\bigcirc s$ and $\alpha(s)$ by the following pullback:

$$\begin{array}{c}
\bigcirc S \rightarrow & FS \\
\bigcirc \downarrow & \downarrow F \alpha \\
A \rightarrow & FA
\end{array}$$

(4.1)
In words, \( \bigcirc \) assigns to each subobject \( s : S \rightarrow A \) the inverse image of \( F s \) under \( \alpha \). Since \( F s \) is a monomorphism, \( \bigcirc s \) is a monomorphism and \( \alpha(s) \) is (for every representation \( \bigcirc s \) of that subobject of \( A \)) uniquely determined.

**Example 4.2.** (1) Let \( A \) be a graph, considered as a coalgebra for \( \mathcal{P} : \text{Set} \rightarrow \text{Set} \). If \( S \subseteq A \) is a set of vertices, then \( \bigcirc S \) is the set of vertices all of whose successors belong to \( S \).

(2) For the set functor \( F X = \mathcal{P}(\Sigma \times X) \) expressing labelled transition systems the operator \( \bigcirc \) for a coalgebra \( \alpha : A \rightarrow \mathcal{P}(\Sigma \times A) \) is the semantic counterpart of the next time operator of classical linear temporal logic, see e.g. Manna and Pnueli [22]. In fact, for a subset \( S \hookrightarrow A \) we have that \( \bigcirc S \) consists of those states whose next states lie in \( S \), in symbols:

\[
\bigcirc S = \{ x \in A \mid (s, y) \in \alpha(x) \text{ implies } y \in S, \text{ for all } s \in \Sigma \}.
\]

The next time operator allows a compact definition of well-foundedness as characterized by Taylor [28, Exercise VI.17] (see also [6, Corollary 2.19]):

**Definition 4.3.** A coalgebra is well-founded if \( \text{id}_A \) is the only fixed point of its next time operator.

**Remark 4.4.** (1) Let us call a subcoalgebra \( m : (B, \beta) \rightarrow (A, \alpha) \) cartesian provided that the square below is a pullback.

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow{\mu} & & \downarrow{m} \\
A & \xrightarrow{\alpha} & FA
\end{array}
\] (4.2)

Then \((A, \alpha)\) is well-founded iff it has no proper cartesian subcoalgebra. That is, if \( m : (B, \beta) \rightarrow (A, \alpha) \) is a cartesian subcoalgebra, then \( m \) is an isomorphism. Indeed, the fixed points of next time are precisely the cartesian subcoalgebras (see Lemma 4.19 for a more refined statement).

(2) A coalgebra is well-founded iff \( \bigcirc \) has a unique pre-fixed point \( \bigcirc m \leq m \). Indeed, since \( \text{Sub}(A) \) is a complete lattice, the least fixed point of a monotone map is its least pre-fixed point. Taylor’s definition [28, Def. 6.3.2] uses that property: he calls a coalgebra well-founded iff \( \bigcirc \) has no proper subobject as a pre-fixed point.

**Examples 4.5.** (1) A coalgebra for \( \mathcal{P} \) regarded as a graph (see Example 2.1) is well-founded iff it has no infinite directed path, see [28, Example 6.3.3].

(2) If \( \mu F \) exists, then as a coalgebra it is well-founded. Indeed, in every pullback (4.2), since \( \epsilon^{-1} \) (as \( \alpha \)) is invertible, so is \( \beta \). The unique algebra homomorphism from \( \mu F \) to the algebra \( \beta^{-1} : F B \rightarrow B \) is clearly inverse to \( m \).

(3) If a set functor \( F \) fulfils \( F \emptyset = \emptyset \), then the only well-founded coalgebra is the empty one. Indeed, this follows from the fact that the empty coalgebra is a fixed point of \( \bigcirc \). For example, a deterministic automaton over the input alphabet \( \Sigma \), as a coalgebra for \( F X = \{0, 1\} \times X^\Sigma \), is well-founded if it is empty.
(4) A non-deterministic automaton may be considered as a coalgebra for the set functor \( FX = \{0, 1\} \times (\mathcal{P}X)^\Sigma \). It is well-founded iff its state transition graph is well-founded (i.e. has no infinite path). This follows from Corollary 4.13 below.

(5) Well-founded linear weighted automata. A linear weighted automaton, i.e. a coalgebra \((A, \alpha)\) for \( FX = K \times X^\Sigma \) on \( \mathcal{V}ec_K \), is well-founded iff every path in its state transition graph eventually leads to 0. This means that every path starting in a state \( s \in A \) leads to the state 0 after finitely many steps (where it stays). In fact, denote by \( A^* \): \( A \) the subset of all states with that property. Clearly, \( A^* \) is a subspace of \( A \). Furthermore, \( \bigcup \) preserves joins of \( \omega \)-chains in \( \text{Sub}(A) \) (see Remark 6.9(2)). Hence, it follows from Kleene’s fixed point theorem that the least fixed point of \( \bigcup \) is \( \bigcup_{n \in \mathbb{N}} \bigcap^n(\bot_A) \). We also know that \( \bot_A \) is the 0-subspace, and for every subspace \( s: S \hookrightarrow A \), \( \bigcap s \) is the space of all nodes whose successors are in \( S \). Therefore \( \bigcap^n(\bot_A) \) consists of precisely those states from which every path reaches 0 in at most \( n \) steps. Thus \( A^* = \bigcup_{n \in \mathbb{N}} \bigcap^n(\bot_A) \). It follows that \((A, \alpha)\) is well-founded iff \( A = A^* \).

We next show that to every coalgebra for a set functor \( F \) one may associate a graph, in a canonical way. Moreover, if \( F \) preserves intersections, then a coalgebra is well-founded if and only if so is its canonical graph.

**Notation 4.6.** Given a set functor \( F \), we define for every set \( X \) the map \( \tau_X: FX \to \mathcal{P}X \) assigning to every element \( x \in FX \) the intersection of all subsets \( m: M \hookrightarrow X \) such that \( x \) lies in the image of \( Fm \):

\[
\tau_X(x) = \bigcap \{ m \mid m: M \hookrightarrow X \text{ satisfies } x \in Fm[Fm]\}. \tag{4.3}
\]

**Definition 4.7.** Let \( F \) be a set functor. For every coalgebra \( \alpha: A \to FA \) its **canonical graph** is the following coalgebra for \( \mathcal{P} \):

\[
A \xrightarrow{\alpha} FA \xrightarrow{\tau_A} \mathcal{P}A.
\]

**Examples 4.8.** (1) Given a graph as a coalgebra \( \alpha: A \to \mathcal{P}A \), the condition \( \alpha(x) \in \mathcal{P}m[\mathcal{P}M] \) states precisely that all successors of \( x \) lie in the set \( M \). The least such set is \( \alpha(x) \). Therefore, the canonical graph of \((A, \alpha)\) is itself (see [28, Example 6.3.3]).

(2) For the type functor of \( FX = \{0, 1\} \times X^\Sigma \) of deterministic automata, we have

\[
\tau_X(i, t) = \{ t(s) : s \in \Sigma \} \quad \text{for } i = 0, 1 \text{ and } t: \Sigma \to X.
\]

Thus, the canonical graph of a deterministic automaton \( A \) is precisely its state transition graph (forgetting the labels of transitions and the finality of states), i.e. we have an edge \((a, a')\) iff \( a' = \delta(a, s) \) for some \( s \in \Sigma \), where \( \delta \) is the nextstate function of \( A \).

Similarly, for the type functor \( FX = \{0, 1\} \times (\mathcal{P}X)^\Sigma \) of non-deterministic automata we have

\[
\tau_X(i, g) = \bigcup_{s \in \Sigma} t(s) \quad \text{for } i = 0, 1 \text{ and } t: \Sigma \to \mathcal{P}X.
\]
(3) For the functor $FX = \mathcal{P}(\Sigma \times X)$ whose coalgebras are labeled transition systems we have

$$\tau_X = (\mathcal{P}(\Sigma \times X) \xrightarrow{\mathcal{P}\pi_X} \mathcal{P}X),$$

where $\pi_X : \Sigma \times X \to X$ is the projection. Again, the canonical graph of a labeled transition system is its state transition graph. Thus $(a, a')$ is an edge iff some action leads from state $a$ to $a'$.

Recall that a functor preserves intersections if it preserves (wide) pullbacks of families of monomorphisms. Gumm [16, Theorem 7.3] observed that for a set functor preserving intersections, the maps $\tau_X : FX \to \mathcal{P}X$ in (4.3) form a “subnatural” transformation from $F$ to the power-set functor $\mathcal{P}$. Subnatural-ity means that (although these maps do not form a natural transformation in general) for every monomorphism $i : X \to Y$ we have a commutative square:

$$
\begin{array}{c}
FX \\
\downarrow F_i \\
FY
\end{array} 
\xrightarrow{\tau_X} 
\begin{array}{c}
\mathcal{P}X \\
\downarrow \mathcal{P}i \\
\mathcal{P}Y
\end{array}
$$

(4.4)

For many set functors this is even a pullback square:

**Theorem 4.9** [16, Thm. 7.4] and [27, Prop. 7.5]. A set functor $F$ preserves intersections iff the squares in (4.4) above are pullbacks.

**Theorem 4.10** [16, Thm. 8.1] and [27, Prop. 7.5]. Let $F$ be a set functor which preserves inverse images and intersections. Then $\tau : F \to \mathcal{P}$ is a natural transformation.

**Example 4.11.** To see that $\tau$ is not a natural transformation in general, we use the set functor $R$ from Example 2.3, part (5). Let $X = \{0, 1\}$, $Y = \{0\}$, and $f : X \to Y$ the unique function. Then $(0, 1) \in FX$, and $\tau_X(0, 1) = X$. Further, $\mathcal{P}f(X) = Y$. But $Rf(0, 1) = d$, and $\tau_Y(d) = \emptyset$.

**Lemma 4.12.** For every set functor $F$ preserving intersections, the next time operator of a coalgebra $\langle A, \alpha \rangle$ coincides with that of its canonical graph.

**Proof.** In the diagram below the outside is a pullback if and only if so is the left-hand square:

$$
\begin{array}{c}
\circ A' \\
\downarrow \alpha(m) \\
\alpha(m)
\end{array} 
\xrightarrow{\tau_{A'}} 
\begin{array}{c}
FA' \\
\downarrow Fm \\
FA
\end{array} 
\xrightarrow{\tau_A} 
\begin{array}{c}
\mathcal{P}A' \\
\downarrow \mathcal{P}m \\
\mathcal{P}A
\end{array}
$$

Taylor [28, Rem. 6.3.4] proved the following result for functors preserving intersections and inverse images; the latter assumption is not needed.
Corollary 4.13 [28]. A coalgebra for a set functor preserving intersections is well-founded iff its canonical graph is well-founded.

Examples 4.14. (1) A coalgebra for the identity functor $FX = X$ on Set is a set $A$ equipped with a function $\alpha: A \to A$. The canonical graph of $(A, \alpha)$ is the graph of the function $\alpha$, i.e., the graph with edges $(a, \alpha(a))$ for all $a \in A$. Hence, $(A, \alpha)$ is well-founded iff it is empty (see Examples 4.5(3)).

(2) For $FX = X + 1$ coalgebras are sets $A$ equipped with a partial function $\alpha: A \to A$, and the canonical graph is the graph of $\alpha$. This functor has many nonempty well-founded coalgebras. For example, the initial $F$-algebra, considered as the coalgebra on $\mathbb{N}$ with the structure given by the partial function $n \mapsto n - 1$, for $n > 0$ (cf. Examples 3.7(1)), is well-founded since its canonical graph is so.

(3) For a (deterministic or non-deterministic) automaton, the canonical graph has an edge from $s$ to $t$ iff there is a transition from $s$ to $t$ for some input letter. Thus, we obtain the characterization of well-foundedness as stated in Examples 4.5(3) and (4).

(4) Consider the functor $FX = X \times X + 1$ and a coalgebra $\alpha: A \to A \times A + 1$. The edges in its canonical graph are all of the pairs $(a, a_1)$ and $(a, a_2)$ such that $a \in A$ and $\alpha(a) = (a_1, a_2)$. For example, the coalgebra $(\mathbb{N}, \gamma)$ from Examples 3.7(2) has the canonical graph with edge set $\{(n, n - 1), (n, n - 2) : n \geq 2\}$, which is clearly well-founded, and therefore so is the coalgebra.

Similarly, for the functor $FX = 1 + A \times X \times X$, the coalgebra $(A^n, s)$ in Examples 3.7(3) is easily seen to be well-founded via its canonical graph. Indeed, this graph has for every list $w$ one outgoing edge to the list $w_{\leq a}$ and one to $w_{> a}$ for every $a \in A$. Hence, this is a well-founded graph.

(5) More generally, for a polynomial functor $H_\Sigma: \text{Set} \to \text{Set}$ associated to a finitary signature $\Sigma$, a coalgebra $\alpha: A \to \prod_{n \in \mathbb{N}} \Sigma_n \times A^n$ has the canonical graph where every vertex $a \in A$ has an outgoing edge $(a, a')$ for every $a' \in A$ occurring in the tuple $\alpha(a) \in \Sigma_n \times A^n$ for some $n < \omega$.

Thus, the coalgebra $(A, \alpha)$ is well-founded iff for every $a \in A$ its tree-unfolding, i.e., its image under the unique homomorphism $h: A \to \nu F$, is a finite $\Sigma$-tree. In particular, if the signature $\Sigma$ does not contain any constant symbols, then the only well-founded $H_\Sigma$-coalgebra is $A = \emptyset$.

For further use we now compare well-founded and recursive coalgebras for a given set functor $F$ with those of its Trnková hull $\hat{F}$ (see Proposition 2.4). Since empty coalgebras are (trivially) well-founded and recursive, we can restrict ourselves to the nonempty ones. Observe that $\text{Coalg} F$ and $\text{Coalg} \hat{F}$ have the same nonempty objects, and these categories are isomorphic.

Lemma 4.15. Let $(A, \alpha)$ be a nonempty coalgebra for a set functor $F$. If it is well-founded or (parametrically) recursive, then it also has those properties as a coalgebra for the Trnková hull $\hat{F}$.

Proof. (1) Let $(A, \alpha)$ be well-founded for $F$. Nonempty subcoalgebras of $(A, \alpha)$ for $\hat{F}$ and for $\hat{F}$ coincide. Thus, we only need to show that the left-hand square
below, where $r_A: \emptyset \to X$ denotes the empty map, is not a pullback:

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{r_{\emptyset}} & \tilde{F}\emptyset \\
\downarrow r_A & & \downarrow \tilde{F}r_A \\
A & \xrightarrow{\alpha} & FA = F\tilde{A} \\
\end{array}
\quad
\begin{array}{ccc}
\emptyset & \xrightarrow{r_{\emptyset}} & F\emptyset \\
\downarrow r_A & & \downarrow Fr_A \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

Since $(A, \alpha)$ is well-founded, we know that the right-hand square is not a pullback. Thus, there exist $a \in A$ and $x \in F\emptyset$ with $\alpha(a) = Fr_A(x)$. For the functor $C_{01}$ of Remark 2.5, define a natural transformation $\tau: C_{01} \to \tilde{F}$ by $\tau_X = F\tau_X(x) \in FX$. Then $\alpha(a) = \tau_A$, and we know that $\tau$ lies in $\tilde{F}\emptyset$ so that $\tau_A = \tilde{F}r_A(\tau)$. Consequently, we have $\alpha(a) = Fr_A(\tau)$, which proves that the left-hand square above is not a pullback.

(2) Let $(A, \alpha)$ be a nonempty recursive coalgebra for $F$. Given an algebra $e: \tilde{F}X \to X$ we know that $X \neq \emptyset$, for otherwise the existence of a unique coalgebra-to-algebra morphism $A \to X$ would force $A$ to be empty. But then the unique coalgebra-to-algebra morphism from $(A, \alpha)$ to $(X, e)$ w.r.t. $F$ is also one for $\tilde{F}$.

We now collect a few properties of the next time operator we will need in the following.

**Lemma 4.16.** The next time operator is monotone: if $m \leq n$, then $\Box m \leq \Box n$.

*Proof.* Suppose that $m: A' \hookrightarrow A$ and $n: A'' \hookrightarrow A$ are subobjects such that $m \leq n$, i.e. $n \cdot x = m$ for some $x: A' \hookrightarrow A''$. Then we obtain the dashed arrow in the diagram below using that its lower square is a pullback:

\[
\begin{array}{ccc}
\Box m & \xrightarrow{\alpha(m)} & FA' \\
\downarrow \alpha & & \downarrow F\tilde{m} \\
\Box n & \xrightarrow{\alpha(n)} & FA'' \\
\downarrow \alpha & & \downarrow Fn \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

This shows that $\Box m \leq \Box n$. \hfill \Box

The following lemma will be useful when we establish the universal property of the well-founded part of a coalgebra in the next section.

**Lemma 4.17.** For every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$ we have $\Box_\beta \cdot \tilde{f} \leq \tilde{f} \cdot \Box_\alpha$,

where $\Box_\alpha$ and $\Box_\beta$ denote the next time operators of the coalgebras $(A, \alpha)$ and $(B, \beta)$, respectively, and $\leq$ is the pointwise order.
Proof. Let \( s : S \to A \) be a subobject. We see that \( \bar{f} (\bigcirc s) \) is obtained by pasting two pullback squares as shown below:

\[
\begin{array}{c}
T \xrightarrow{t} \bigcirc \alpha S \xrightarrow{\alpha(s)} FS \\
\bigcirc \alpha S \xrightarrow{\bigcirc \alpha} \bigcirc \alpha S \xrightarrow{F \alpha} FA
\end{array}
\]

(4.5)

In order to show that \( \bigcirc \beta (\bar{f} (s)) \leq \bar{f} (\bigcirc s) \), we consider the following diagram:

\[
\begin{array}{c}
\bigcirc \beta U \xrightarrow{\beta(s)} FU \xrightarrow{Fu} FS \\
\bigcirc \beta (s) \xrightarrow{\bigcirc \beta} FB \xrightarrow{F \beta} FA
\end{array}
\]

(4.6)

The upper left-hand part is the pullback square defining \( \bigcirc \beta (\bar{f} (s)) \), and the upper right-hand one is that defining \( \bar{f} (s) \), with \( F \) applied. On the bottom, we use that \( f \) is a coalgebra homomorphism. Thus, the outside of the diagram commutes. Since the outside of the diagram in (4.5) is a pullback, we have some \( g : \bigcirc \beta U \to T \) such that \( \bigcirc \beta (\bar{f} (s)) = \bar{f} (\bigcirc s) \cdot g \), which proves the desired inequality. \( \square \)

**Corollary 4.18.** For every coalgebra homomorphism \( f : (B, \beta) \to (A, \alpha) \) we have \( \bigcirc \beta \cdot \bar{f} = \bar{f} \cdot \bigcirc \alpha \) provided that either

(1) \( f \) is a monomorphism in \( \mathcal{A} \) and \( F \) preserves finite intersections, or
(2) \( F \) preserves inverse images.

**Proof.** Indeed, under either of the above conditions, the upper right-hand part in Diagram (4.6) is a pullback. Thus, pasting this part with the pullback in the upper left of (4.6) and using that the lower part commutes, we see that \( \bigcirc \beta (\bar{f} (s)) \) is obtained by pulling back \( Fs \) along \( f \cdot \alpha \). This implies the desired equality since this is how \( \bar{f} (\bigcirc s) \) is obtained (see (4.5)). \( \square \)

**Lemma 4.19.** Let \( \alpha : A \to FA \) be a coalgebra and \( m : B \to A \) be a monomorphism.

(1) There is a coalgebra structure \( \beta : B \to FB \) for which \( m \) gives a subcoalgebra of \( (A, \alpha) \) iff \( m \leq \bigcirc m \).
(2) There is a coalgebra structure \( \beta : B \to FB \) for which \( m \) gives a cartesian subcoalgebra of \( (A, \alpha) \) iff \( m = \bigcirc m \).

**Proof.** We prove the left-to-right directions of both assertions first, and then the right-to-left ones.
Suppose first that there exists \( \beta : B \to FB \) such that \( m : (B, \beta) \to (A, \alpha) \) is a coalgebra morphism. Then the fact that \( \bigcirc B \) is given by a pullback yields a morphism \( x : B \to \bigcirc B \) such that, inter alia, \( \bigcirc m \cdot x = m \). It follows that \( m \leq \bigcirc m \). If \((B, \beta)\) is a cartesian subcoalgebra, then we have a pullback square

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
m \downarrow & & \downarrow Fm \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]

So clearly \( m = \bigcirc m \) in \( \text{Sub}(A) \).

Conversely, suppose that \( m \leq \bigcirc m \) via \( x : B \to \bigcirc B \). Then \( \alpha(\bigcirc m) \cdot x : B \to FB \) is a coalgebra, and \( m : B \to A \) is a homomorphism:

\[
\begin{array}{ccc}
B & \xrightarrow{x} & \bigcirc B & \xrightarrow{\alpha(\bigcirc m)} & FB \\
m \downarrow & & \downarrow \bigcirc m & & \downarrow Fm \\
A & \xrightarrow{\alpha} & FA & & FA
\end{array}
\]

If in addition \( m = \bigcirc m \), i.e. \( x \) is an isomorphism, we see that \( m \) is a cartesian subcoalgebra. \( \square \)

We close this section with a characterization result: \( F \) preserves intersections if and only if the following “generalized next time” operators are right adjoints. Given a morphism \( f : A \to FB \), we have the operator \( \bigcirc_f : \text{Sub}(B) \to \text{Sub}(A) \) that maps \( m : B' \to B \) to the pullback of \( Fm \) along \( f \):

\[
\begin{array}{ccc}
\bigcirc_f A' & \xrightarrow{f(m)} & FB' \\
\downarrow \bigcirc m & & \downarrow \bigcirc Fm \\
A & \xrightarrow{f} & FB
\end{array}
\]

**Proposition 4.20 [32].** The functor \( F \) preserves intersections if and only if every generalized next time operator \( \bigcirc_f \) is a right adjoint.

*Proof.* For the “if”-direction, choose \( f = id_{FY} \). Then \( \bigcirc id_{FY} : m \mapsto Fm \) is a right adjoint and so preserves all meets, i.e. \( F \) preserves intersections.

The converse follows from the easily established fact that intersections are stable under inverse image, i.e. for every morphism \( f : X \to Y \) and every family \( m_i : S_i \to Y \) of subobjects, the intersection \( m : P \to X \) of the inverse images of the \( m_i \) under \( f \) yields a pullback

\[
\begin{array}{ccc}
P & \xrightarrow{m} & \bigcap_i S_i \\
\downarrow m & & \downarrow \bigcap_i m_i \\
X & \xrightarrow{f} & Y
\end{array}
\]

Hence, if \( F \) preserves intersections, then so does every operator \( \bigcirc_f \). Equivalently, \( \bigcirc_f \) is a right adjoint. \( \square \)
5 The Well-Founded Part of a Coalgebra

We introduced well-founded coalgebras in Section 4. We now discuss the well-founded part of a coalgebra, i.e. its largest well-founded subcoalgebra. We prove that this is the least fixed point of the next time operator. Then we prove that the well-founded part is the coreflection of a coalgebra in the category of well-founded coalgebras.

Definition 5.1 [5]. The well-founded part of a coalgebra is its largest well-founded subcoalgebra.

The well-founded part of a coalgebra always exists and is the coreflection in the category of well-founded coalgebras [6, Prop. 2.27]. We provide a new, shorter proof of this fact. The well-founded part is obtained by the following:

Construction 5.2 [6, Not. 2.22]. Let $\alpha : A \to FA$ be a coalgebra. We know that $\text{Sub}(A)$ is a complete lattice and that the next time operator $\Box$ is monotone (see Lemma 4.19). Hence, by the Knaster-Tarski fixed point theorem, $\Box$ has a least fixed point, which we denote by $a^* : A^* \to A$.

Moreover, by Lemma 4.19(2), we know that there is a coalgebra structure $\alpha^* : A^* \to FA^*$ so that $a^* : (A^*, \alpha^*) \to (A, \alpha)$ is the smallest cartesian subcoalgebra of $(A, \alpha)$.

Proposition 5.3. For every coalgebra $(A, \alpha)$, the coalgebra $(A^*, \alpha^*)$ is well-founded.

Proof. Let $m : (B, \beta) \to (A^*, \alpha^*)$ be a cartesian subcoalgebra. By Lemma 4.19, $a^* \cdot m : B \to A$ is a fixed point of $\Box$. Since $a^*$ is the least fixed point, we have $a^* \leq a^* \cdot m$, i.e. $a^* = a^* \cdot m \cdot x$ for some $x : A^* \to B$. Since $a^*$ is monic, we thus have $m \cdot x = id_{A^*}$. So $m$ is a monomorphism and a split epimorphism, whence an isomorphism. $\Box$

Example 5.4. Consider the coalgebra $G$ for $\mathcal{P}$ depicted as the following graph:

\[ a \longrightarrow b \quad c \longrightarrow d \]

We list all subcoalgebras below (the structures are the obvious ones given by the picture of $G$). Those are $\emptyset$, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, and \{a, b, c, d\}. Of these, the cartesian subcoalgebras of $G$ are \{a, b\}, and \{a, b, c, d\}. The well-founded part of $G$ is the least cartesian subcoalgebra, namely \{a, b\}.

We know from Proposition 5.3 that for every coalgebra $(A, \alpha)$ its subcoalgebra represented by $a^* : A^* \to A$ is well-founded. We now prove that, categorically, this subcoalgebra is characterized uniquely up to isomorphism by the following universal property: every homomorphism from a well-founded coalgebra into $(A, \alpha)$ factorizes uniquely through $a^*$. In particular, this implies that $a^* : A^* \to A$ is the largest well-founded subcoalgebra of $A$, viz. the well-founded part of $A$. 

Proposition 5.5. The full subcategory of $\text{Coalg}_F$ given by well-founded coalgebras is coreflective. In fact, the well-founded coreflection of a coalgebra is its well-founded part $a^*: (A^*, a^*) \to (A, \alpha)$.

Proof. We are to prove that for every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$, where $(B, \beta)$ is well-founded, there exists a coalgebra homomorphism $f^*: (B, \beta) \to (A^*, a^*)$ such that $a^* \cdot f^* = f$. It is unique since $a^*: A^* \to A$ is a monomorphism. It then follows that $a^*: (A^*, a^*) \to (A, \alpha)$ is the largest well-founded subcoalgebra.

For the existence of $f^*$, we first observe that $\overleftarrow{f} (a^*)$ is a pre-fixed point of $\circ \beta$: indeed, using Lemma 4.17 we have

$$\circ \beta (\overleftarrow{f} (a^*)) \leq \overleftarrow{f} (\circ \alpha (a^*)) = \overleftarrow{f} (a^*).$$

By Remark 4.4(2), we therefore have $id_B = b^* \leq \overleftarrow{f} (a^*)$ in $\text{Sub}(B)$. Using the adjunction in Proposition 2.13, we have $\overleftarrow{f} (id_B) \leq a^*$ in $\text{Sub}(A)$. Now let

$$f = (B \xrightarrow{e} C \xrightarrow{m} A)$$

be the factorization of $f$ as in Remark 2.2(2). This implies that $\overleftarrow{f} (id_B) = m$. Thus we obtain

$$m = \overleftarrow{f} (id_B) \leq a^*,$$

i.e. there exists a morphism $h: C \to A^*$ such that $a^* \cdot h = m$. Thus, $f^* = h \cdot e: B \to A^*$ is a morphism satisfying

$$a^* \cdot f^* = a^* \cdot h \cdot e = m \cdot e = f.$$

It follows that $f^*$ is a coalgebra homomorphism from $(B, \beta)$ to $(A^*, a^*)$ since $f$ and $a^*$ are and $F$ preserves monomorphisms. \qed

6 Closure Properties of Well-Founded Coalgebras

In this section we will see that strong quotients and subcoalgebras (see Remark 2.8) of well-founded coalgebras are well-founded again. For subcoalgebras we need to assume more about $\mathscr{F}$ and $F$. We present two variants in Proposition 6.4 and Theorem 6.12.

We mention the following corollary to Proposition 5.5. For endofunctors on sets preserving inverse images this was stated by Taylor [28, Exercise VI.16]:

Corollary 6.1. The subcategory of $\text{Coalg}_F$ formed by all well-founded coalgebras is closed under strong quotients and coproducts in $\text{Coalg}_F$.

This follows from a general result on coreflective subcategories [3, Thm. 16.8]; the category $\text{Coalg}_F$ has the factorization system of Proposition 2.7, and its full subcategory of well-founded coalgebras is coreflective with monomorphic coreflections (see Proposition 5.5). Consequently, it is closed under strong quotients and colimits.
Remark 6.2. We prove next that, for an endofunctor preserving finite intersections, well-founded coalgebras are closed under subcoalgebras provided that \( \text{Sub}(A) \) forms a frame. This assumption is not needed provided that monomorphisms are universally smooth (see Theorem 6.12). Recall that \( \text{Sub}(A) \) is a frame if for every subobject \( m : B \to A \) and every family \( m_i \ (i \in I) \) of subobjects of \( A \) we have
\[
m \land \bigvee_{i \in I} m_i = \bigvee_{i \in I} (m \land m_i).
\]
Equivalently, \( \overline{m} : \text{Sub}(A) \to \text{Sub}(B) \) has a right adjoint \( m_* : \text{Sub}(B) \to \text{Sub}(A) \) (use the dual of Remark 2.12).

Examples 6.3. (1) \( \text{Set} \) has the property that all \( \text{Sub}(A) \) are frames. In fact, given subsets \( S \) and \( S_i \ (i \in I) \) of \( A \) the equality \( S \cap (\bigcup_{i \in I} S_i) = \bigcup_{i \in I} (S \cap S_i) \) clearly holds.

(2) This property is shared by categories such as posets and monotone maps, graphs and homomorphisms, unary algebras and homomorphisms, topological spaces and continuous maps, and presheaf categories \( \text{Set}^e \), with \( e \) small. This follows from the fact that joins and meets of subobjects of an object \( A \) are formed on the level of subsets of the underlying set of \( A \).

(3) For every Grothendieck topos, the posets \( \text{Sub}(A) \) are frames. In fact, it is sufficient for a topos to have all coproducts or intersections to satisfy this requirement.

(4) The category \( \text{Vec}_K \) does not have the above property. For example, for \( K = \mathbb{R} \) and two distinct lines \( m_1 : \mathbb{R} \to \mathbb{R}^2 \), the desired equation fails. Indeed, for every line \( m : \mathbb{R} \to \mathbb{R}^2 \) different from \( m_1, m_2 \) we have that
\[
m \land (m_1 \lor m_2) = m \neq 0 = (m \land m_1) \lor (m \land m_2).
\]

(5) The category \( \text{CPO} \) does not have the above property: for the cpo \( A = \mathbb{N}^\top \) of natural numbers with a top element \( \top \) (linearly ordered) the lattice \( \text{Sub}(A) \) is not a frame. Consider the subobjects given by inclusion maps \( m_i : \{0, \ldots, i\} \hookrightarrow \mathbb{N}^\top \) for \( i \in \mathbb{N} \), with domains linearly ordered. It is easy to see that \( \bigvee_{i \in \mathbb{N}} m_i = \text{id}_A \).

For the inclusion map \( m : \{\top\} \hookrightarrow \mathbb{N}^\top \) we have \( m \land m_i = 0 \ (i \in I) \), the empty subobject. Thus, \( \bigvee_{i \in \mathbb{N}} (m \land m_i) = 0 \neq m = m \lor \bigvee_{i \in I} m_i \).

Proposition 6.4. Suppose that \( F \) preserves finite intersections, and let \((A, \alpha)\) be a well-founded coalgebra such that \( \text{Sub}(A) \) a frame. Then every subcoalgebra of \((A, \alpha)\) is well-founded.

Proof. Let \( m : (B, \beta) \to (A, \alpha) \) be a subcoalgebra. We will show that the only pre-fixed point of \( \circ_\beta \) is \( \text{id}_B \) (cf. Remark 4.4(2)). Suppose \( s : S \to B \) fulfills \( \circ_\beta(s) \leq s \). Since \( F \) preserves finite intersections, we have
\[
\overline{m} \cdot \circ_\alpha = \circ_\beta \cdot \overline{m}
\]
by Corollary 4.18(1). The counit of the adjunction \( \overline{m} \dashv m_* \) yields \( \overline{m}(m_*(s)) \leq s \), so that we obtain
\[
\overline{m}(\circ_\alpha(m_*(s))) = \circ_\beta(\overline{m}(m_*(s))) \leq \circ_\beta(s) \leq s.
\]
Using again the adjunction \( \overline{m} \dashv \cdot \), we have equivalently that \( \otimes_{\alpha}(m_{\ast}(s)) \leq m_{\ast}(s) \), i.e. \( m_{\ast}(s) \) is a pre-fixed point of \( \otimes_{\alpha} \). Since \( (A, \alpha) \) is well-founded, Corollary 4.18(1) implies that \( m_{\ast}(s) = \text{id}_A \). Since \( \overline{m} \) is also a right adjoint and therefore preserves the top element of \( \text{Sub}(B) \), we thus obtain

\[ \text{id}_B = \overline{m}(\text{id}_A) = \overline{m}(m_{\ast}(s)) \leq s, \]

which completes the proof. \( \square \)

**Remark 6.5.** Given a set functor \( F \) preserving inverse images, a much better result was proved by Taylor [28, Corollary 6.3.6]: for every coalgebra homomorphism \( f : (B, \beta) \to (A, \alpha) \) with \( (A, \alpha) \) well-founded so is \( (B, \beta) \). In fact, our proof above is essentially Taylor’s who (implicitly) uses Corollary 4.18(2) instead.

**Corollary 6.6.** If a set functor preserves finite intersections, then subcoalgebras of well-founded coalgebras are well-founded.

Trnková proved [30] that every set functor preserves all nonempty finite intersections. However, this does not suffice for Corollary 6.6:

**Example 6.7.** A well-founded coalgebra for a set functor can have non-well-founded subcoalgebras. Let \( F\emptyset = 1 \) and \( FX = 1 + 1 \) for all nonempty sets \( X \), and let \( Ff = \text{inl} : 1 \to 1 + 1 \) be the left-hand injection for all maps \( f : \emptyset \to X \) with \( X \) nonempty. The coalgebra \( \text{inr} : 1 \to F1 \) is not well-founded because its empty subcoalgebra is cartesian. However, this is a subcoalgebra of \( \text{id} : 1 + 1 \to 1 + 1 \) (via the embedding \( \text{inr} \)), and the latter is well-founded.

The fact that subcoalgebras of a well-founded coalgebra are well-founded does not necessarily need the assumption that \( \text{Sub}(A) \) is a frame. Using the construction of the least fixed point \( a^\ast \) of \( \otimes \) provided by the (proof of the) Knaster-Tarski fixed point theorem, it is essentially sufficient that \( \overline{m} \) in the proof of Proposition 6.4 preserves joins of unions of chains in \( \text{Sub}(A) \). We now discuss this in more detail.

Recall (universally) smooth monomorphisms from Definition 2.14.

**Construction 6.8 [6, Not. 2.22].** Let \( (A, \alpha) \) be a coalgebra. We obtain \( a^\ast \), the least fixed point of \( \otimes \), as the join of the following transfinite chain of subobjects \( a_i : A_i \to A, i \in \text{Ord} \). First, put \( a_0 = \perp_A \), the least subobject of \( A \). Given \( a_i : A_i \to A \), put \( a_{i+1} = \bigcirc a_i : A_{i+1} = \bigcirc A_i \to A \). For every limit ordinal \( j \), put \( a_j = \bigvee_{i < j} a_i \). It follows from the proof of the Knaster-Tarski fixed point theorem that there exists an ordinal \( i \) such that \( a_i = a^\ast : A^\ast \to A \).

**Remark 6.9.** (1) Note that, whenever monomorphisms are smooth, we have \( A_0 = 0 \) and the above join \( a_j \) is obtained as the colimit of the chain of the subobject \( a_i : A_i \to A, i < j \) (see Remark 2.16).

(2) If \( F \) is a finitary functor on a locally finitely presentable category, then the least ordinal \( i \) with \( a^\ast = a_i \) is at most \( \omega \). Indeed, \( \bigcirc \) preserves joins of \( \omega \)-chains in \( \text{Sub}(A) \) because \( F \) does, since these joins are obtained as chain colimits (see [8, Prop. 1.62]) and so does \( \bigvee \) since colimits of chains are universal (cf. Example 2.15(4)). By Kleene’s fixed point theorem \( a^\ast = \bigvee_{i \in \mathbb{N}} a_i \).
(3) The same holds for a finitary functor on a category with universally smooth monomorphisms. However, in general one needs transfinite iteration to reach a fixed point (see Remark 8.4).

**Example 6.10.** Let \((A, \alpha)\) be a graph regarded as a coalgebra for \(\mathcal{P}\) (see Example 2.1). Then \(A_0 = \emptyset, A_1\) is formed by all leaves, i.e. those nodes with no neighbours, \(A_2\) by all leaves and all nodes such that every neighbour is a leaf, etc. We see that a node \(x\) lies in \(A_{i+1}\) iff every path starting in \(x\) has length at most \(i\). Hence \(A^* = A_\omega\) is the set of all nodes from which no infinite paths start.

**Notation 6.11.** For every pair \(i \leq j\) or ordinals, we denote by \(a_{ij}: A_i \rightarrowtail A_j\) the unique morphism witnessing \(a_i \leq a_j\), i.e. \(a_i = a_j \cdot a_{ij}\). Note that these arise by transfinite recursion as well: \(a_{0i}\) is obtained by initiality, at limit steps use the colimit morphisms, and at successor steps one uses the pullback property. That is, in the following diagram (in which all vertical morphisms are monomorphisms)

![Diagram](image)

the outside commutes by the definitions of \(A_{i+1}, a_{i+1},\) and \(\alpha(a_i)\); also the triangle on the right commutes by induction hypothesis on \(i\). Since the bottom square is a pullback, we obtain \(a_{i+1,j+1}\) as desired.

**Theorem 6.12.** Let \(\mathcal{A}\) be a complete and well-powered category with universally smooth monomorphisms. Then for endofunctors preserving finite intersections, every subcoalgebra of a well-founded coalgebra is well-founded itself.

**Proof.** Let \(\alpha: A \rightarrowtail FA\) be well-founded. Recall the subobjects \(a_i: A_i \rightarrowtail A\) from Construction 6.8. Let

![Diagram](image)

be a subcoalgebra and denote by \(b_i: B_i \rightarrowtail B\) the subobjects of \(B\) provided by Construction 6.8). There is an ordinal \(\lambda\) such that \(a_\lambda\) is invertible, and we shall prove that \(b_\lambda\) is also invertible; thus, \((B, \beta)\) is well-founded. It is sufficient to prove by transfinite induction that the following squares are pullbacks, for
suitable monomorphisms $m_i$:

\[
\begin{array}{c}
B_i \xrightarrow{b_i} B \\
m_i \\
A_i \xrightarrow{a_i} A
\end{array}
\]

In other words we prove that for every $i$ we have

\[b_i = \lambda m(a_i)\].

For $i = 0$, the statement $b_0 = \lambda m(a_0)$ means that the square below is a pullback:

\[
\begin{array}{c}
0 \xrightarrow{b_0} B \\
m
0 \xrightarrow{a_0} A
\end{array}
\]

which is trivial since 0 is a strict initial object (see Remark 2.16(1)).

For the isolated step we use the induction hypothesis and Corollary 4.18(1) to obtain:

\[b_{i+1} = \bigcirc \beta \lambda \alpha(a_i) = \lambda m(\bigcirc \alpha(a_i)) = \lambda m(a_{i+1}).\]

For a limit ordinal $j$, we use Remark 2.16(3) to obtain

\[b_j = \bigvee_{i<j} b_i = \bigvee_{i<j} \lambda m(a_i) = \lambda m(\bigvee_{i<j} a_i) = \lambda m(a_j).\]

\[\square\]

7 The General Recursion Theorem

The main consequence of well-foundedness is parametric recursivity. This is Taylor’s General Recursion Theorem [28, Theorem 6.3.13]. Taylor assumed that $F$ preserves inverse images. We present a new proof for which it is sufficient that $F$ preserves monomorphisms, assuming those are smooth. In the next section, we discuss the converse implication in Theorem 8.1 and Theorem 8.6.

Remark 7.1. Recall from Remark 2.17 the initial-algebra chain for $F$. If $\mathcal{A}$ has smooth monomorphisms and $F$ preserves monomorphisms, then all $w_{i,j}$ in the initial-algebra chain are monic. This follows from an easy transfinite induction.

Theorem 7.2 (General Recursion Theorem). Let $\mathcal{A}$ be a complete and wellpowered category with smooth monomorphisms. For $F : \mathcal{A} \to \mathcal{A}$ preserving monomorphisms, every well-founded coalgebra is parametrically recursive.
Proof. (1) Given an arbitrary coalgebra \((A, \alpha)\) we use the chain of subobjects \(a_i: A_i \to A\) from Construction 6.8.\(^4\) We also have the initial-algebra chain \(W_i = F^00\) with connecting morphisms \(w_{ji}\) (see Remark 2.17). We obtain a natural transformation 
\[h_i: A_i \to W_i \quad i \in \text{Ord},\]
by transfinite recursion as follows: \(h_0 = id_0\), and given \(h_i: A_i \to W_i\), let
\[h_{i+1} = (A_{i+1} \xrightarrow{\alpha(a_i)} FA_i \xrightarrow{Fh_i} FW_i = W_{i+1}).\]
Finally, for a limit ordinal \(i\), \(h_i\) is uniquely determined by the universal property of the colimit \(A_i\).

We must verify that for \(j \leq i\) the naturality square below commutes:

\[
\begin{array}{ccc}
A_j & \xrightarrow{h_j} & W_j \\
\downarrow{a_{ji}} & & \downarrow{w_{ji}} \\
A_i & \xrightarrow{h_i} & W_i
\end{array}
\]  \hspace{1cm} (7.1)

The proof is by transfinite induction on \(i\). The base case for \(i = 0\) is trivial, and the step when \(i\) is a limit ordinal follows from the fact that we use colimits to define both \(A_i\) and \(W_i\). We are left with the successor step \(i + 1\). Here we again use transfinite induction on \(j\). The verification amounts to assuming (7.1) for \(i\) and \(j\) and showing the same equation for \(j + 1\) and \(i + 1\). For this, consider the diagram below:

\[
\begin{array}{cccccc}
A_{j+1} & \xrightarrow{a_{j+1,i+1}} & A_{i+1} \\
& \xrightarrow{\alpha(a_j)} & \downarrow{\alpha(a_i)} & \\
FA_{j+1} & \xrightarrow{Fa_{ji}} & FA_{i+1} & \xrightarrow{\alpha(a_i)} & \\
& \xrightarrow{Fh_j} & \downarrow{Fh_i} & \downarrow{h_{i+1}} & \\
W_{j+1} = F^{j+1}0 & \xrightarrow{w_{j+1,i+1}} & F^{i+1}0 = W_{i+1}
\end{array}
\]  \hspace{1cm} (7.2)

The region at the top is also the top square of (6.1), the triangles commute by the definition of \((h_i)\), and the region at the bottom commutes by the induction hypothesis and the fact that \(FW_{ji} = w_{j+1,i+1}\). Thus the outside commutes, as desired.

(2) Now suppose that \((A, \alpha)\) is a well-founded coalgebra. We prove that \((A, \alpha)\) is recursive, i.e. for every algebra \(e: FX \to X\) we present a coalgebra-to-algebra morphism \(e^\dagger\) and prove that it is unique.

\(^4\) One might object to this use of transfinite recursion, since Theorem 7.2 itself could be used as a justification for transfinite recursion. Let us emphasize that we are not presenting Theorem 7.2 as a foundational contribution. We are building on the classical theory of transfinite recursion, extending that result by categorifying it.
For every ordinal $i$, the coalgebra $w_{i,i+1}: W_i \rightarrow FW_i$ is recursive (see Examples 3.3(6)). Hence we have a morphism $f_i: W_i \rightarrow X$ such that the square on the bottom below commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha(a_i)} & FA \\
\downarrow h_i & & \downarrow Fh_i \\
W_i & \xrightarrow{w_{i,i+1}} & FW_i \\
\downarrow f_i & & \downarrow Ff_i \\
X & \xleftarrow{e} & FX \\
\end{array}
$$

(7.3)

Since $(A, \alpha)$ is well-founded, there exists an ordinal $i$ such that $A = A_i = A_{i+1}$ (see Construction 6.8). Then we have $\alpha(a_i) = \alpha$, so that the upper triangle commutes by definition of $h_{i+1}$. Moreover, the lower triangle is an instance of (7.1) using the fact that $a_{i,i+1} = id$. Thus the outside of the diagram commutes, and so $f_i \cdot h_i$ is the desired coalgebra-to-algebra morphism.

(3) For the uniqueness, suppose that $e^\dagger$ is any coalgebra-to-algebra morphism from $\alpha$ to $e$, i.e. in the diagram below the lower square commutes:

$$
\begin{array}{ccc}
A_{i+1} & \xrightarrow{\alpha(a_i)} & FA_i \\
\downarrow a_{i+1} & & \downarrow Fa_i \\
A & \xrightarrow{\alpha} & FA \\
\downarrow e^\dagger & & \downarrow Fe^\dagger \\
X & \xleftarrow{e} & FX \\
\end{array}
$$

(7.4)

Moreover, the upper one is the square defining $\alpha(a_i)$ (see Definition 4.1).

We verify by induction on $j$ that $e^\dagger \cdot a_j = f_j \cdot h_j \cdot a_j$. Then for the above ordinal $i$ with $a_i = id_A$, we have $e^\dagger = f_i \cdot h_i$ as desired. For the base case $j = 0$, the equation trivially holds, and for limit ordinals $j$ we use the universal property of the colimit $A_j$. For the successor step we use that (7.4) and (7.3) commute (with $j$ substituted for $i$). By pasting (7.3) and the upper square of (7.4) we obtain

$$
e \cdot F(f_j \cdot h_j \cdot a_j) \cdot \alpha(a_j) = f_j \cdot h_j \cdot a_{j+1}
$$

(7.5)

This yields the desired equality:

$$
e^\dagger \cdot a_{j+1} = e \cdot F(e^\dagger \cdot a_j) \cdot \alpha(a_j) = e \cdot F(f_j \cdot h_j \cdot a_j) \cdot \alpha(a_j) = f_j \cdot h_j \cdot a_{j+1}
$$

(7.6)

(4) Finally, we prove that the coalgebra $(A, \alpha)$ is a parametrically recursive.

Consider the coalgebra $(\alpha, id_A): A \rightarrow FA \times A$ for $F(-) \times A$. This functor preserves monomorphisms since $F$ does and monomorphisms are closed under
products. The next time operator $\bigcirc$ on $\sub(A)$ is the same for both coalgebras since the square (4.1) is a pullback if and only if the square below is one:

$$
\begin{array}{c}
\circ S \\
\downarrow \alpha \\
\circ A \\
\downarrow \langle \alpha, A \rangle \\
A \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
F S \times A \\
\rightarrow \\
F A \times A \\
\end{array}
$$

Since $id_A$ is the unique fixed point of $\bigcirc$ w.r.t. $F$, it is also the unique fixed point of $\bigcirc$ w.r.t. $F(-) \times A$. Thus, $(A, \langle \alpha, id_A \rangle)$ is a well-founded coalgebra for $F(-) \times A$. By point (2), it is thus recursive for $F(-) \times A$. This states equivalently that $(A, \alpha)$ is a parametrically recursive coalgebra for $F$. □

**Corollary 7.3.** For every endofunctor on $\text{Set}$ or $\text{Vec}_K$ (vector spaces and linear maps), every well-founded coalgebra is parametrically recursive.

**Proof.** For $\text{Set}$, we apply Theorem 7.2 to the Trnková hull $\bar{F}$ (see Proposition 2.4), noting that $F$ and $\bar{F}$ have the same (non-empty) coalgebras. By Lemma 4.15 the desired result follows. For $\text{Vec}_K$, observe that monomorphisms split and are therefore preserved by every endofunctor $F$. □

**Example 7.4.** For the set functor $FX = X \times X + 1$ the coalgebra $(\mathbb{N}, \gamma)$ from Examples 4.14(4) is well-founded. Hence it is parametrically recursive.

Similarly, we saw that for $FX = 1 + A \times X \times X$ the coalgebra $(A, s)$ from Examples 3.7(3) is well-founded, and therefore it is (parametrically) recursive.

**Example 7.5.** Well-founded coalgebras need not be recursive when $F$ does not preserve monomorphisms. We take $\mathcal{A}$ to be the category of sets with a predicate, i.e. pairs $(X, A)$, where $A \subseteq X$. Morphisms $f: (X, A) \to (Y, B)$ satisfy $f[A] \subseteq B$. Denote by $\mathbbm{1}$ the terminal object $(1, 1)$. We define an endofunctor $F$ by $F(X, \emptyset) = (X + 1, \emptyset)$, and for $A \neq \emptyset$, $F(X, A) = 1$. For a morphism $f: (X, A) \to (Y, B)$, put $F = f + id$ if $A = \emptyset$; if $A \neq \emptyset$, then also $B \neq \emptyset$ and $FF$ is $id: \mathbbm{1} \to \mathbbm{1}$.

The terminal coalgebra is $id: \mathbbm{1} \to \mathbbm{1}$, and it is easy to see that it is well-founded. But it is not recursive: there are no coalgebra-to-algebra morphisms into an algebra of the form $F(X, \emptyset) \to (X, \emptyset)$.

We close with a general fact on well-founded parts of fixed points (i.e. (co)algebras whose structure is invertible). The following result generalizes [18, Cor. 3.4], and it also appeared before for functors preserving finite intersections [5, Theorem 8.16 and Remark 8.18]. Here we lift the latter assumption:

**Theorem 7.6.** Let $\mathcal{A}$ be a complete and well-powered category with smooth monomorphisms. For $F$ preserving monomorphisms, the well-founded part of every fixed point is an initial algebra. In particular, the only well-founded fixed point is the initial algebra.

**Proof.** Let $\alpha: A \to FA$ be a fixed point of $F$. By Remark 2.17(2) we know that the initial algebra $(\mu F, \iota)$ exists. Now let $a^*: (A^*, \alpha^*) \to (A, \alpha)$ be the well-founded part of $A$ given in Proposition 5.3. This is a cartesian subcoalgebra,
i.e. we have a pullback square

\[
\begin{array}{c}
A^* \\ \alpha^* \\
\downarrow \alpha \\
A \\
\end{array} \rightarrow 
\begin{array}{c}
FA^* \\ Fa^* \\
\downarrow F \alpha \\
FA \\
\end{array}
\]

Since \(\alpha\) is an isomorphism, so is \(\alpha^*\).

By initiality, we have an algebra homomorphism \(h: (\mu F, \iota) \rightarrow (A^*, (\alpha^*)^{-1})\), i.e. a coalgebra homomorphism

\[
\begin{array}{c}
\mu F \\ \iota^{-1} \\
\downarrow h \\
A^* \\
\end{array} \rightarrow 
\begin{array}{c}
F(\mu F) \\ Fh \\
\downarrow Fh \\
FA^* \\
\end{array}
\]

Since both horizontal morphisms are invertible, this square is a pullback. By Theorem 7.2, \((A^*, \alpha^*)\) is recursive. Thus, we have a coalgebra homomorphism \(k: (A^*, \alpha^*) \rightarrow (\mu F, \iota^{-1})\) by Corollary 3.4. By the universal property of \(\mu F\), we obtain \(k \cdot h = id_{\mu F}\), whence \(h\) is a split monomorphism. Thus the above square exhibits \((\mu F, \iota^{-1})\) as a cartesian subcoalgebra of \((A^*, \alpha^*)\). By Remark 4.4(1), we conclude that \(h\) is an isomorphism. \(\square\)

**Example 7.7.** We illustrate that for a set functor \(F\) preserving monomorphisms, the well-founded part of the terminal coalgebra is the initial algebra. Consider \(FX = A \times X + 1\). The terminal coalgebra is the set \(A^\infty \cup A^*\) of finite and infinite sequences from the set \(A\). The initial algebra is \(A^*\). It is easy to check that \(A^*\) is the well-founded part of \(A^\infty \cup A^*\).

**8 The Converse of the General Recursion Theorem**

We prove a converse to Theorem 7.2; “recursive \(\implies\) well-founded”. Related results appear in Taylor [27, 28], Adámek et al. [4] and Jeannin et al. [18].

For this, one needs to assume more than preservation of finite intersections. In fact, we will assume that \(F\) preserves inverse images. But even this is not enough. We additionally assume that either

1. The underlying category \(\mathcal{A}\) has universally smooth monomorphisms and the endofunctor \(F\) has a pre-fixed point (see Remark 2.17(2)).
2. The underlying category \(\mathcal{A}\) has a subobject classifier.

The first of these possible assumptions leads to Theorem 8.1, the second is a theorem of Taylor [27]. Finally, at the end of this section we prove the above converse implication for every functor on vector spaces preserving inverse images (see Theorem 8.12). This last result is not covered by the previous two results since \(\text{Vec}_K\) neither has universally constructive monomorphims nor a subobject classifier.
Theorem 8.1. Let \( \mathcal{A} \) be a complete and wellpowered category with universally smooth monomorphisms, and suppose that \( F : \mathcal{A} \to \mathcal{A} \) preserves inverse images and has a pre-fixed point. Then every recursive \( F \)-coalgebra is well-founded.

Proof. First observe that an initial algebra exists by Remark 2.17(2). Now suppose that \( (A, \alpha) \) is a recursive coalgebra. Then there exists a unique coalgebra homomorphism \( h : (A, \alpha) \to (\mu F, \iota^{-1}) \). Let us abbreviate \( w_i \lambda \) by \( c_i : A_i \to A \) from Construction 6.8. We are going to prove by transfinite induction that for every \( i \in \text{Ord} \), \( a_i \) is the inverse image of \( c_i \) under \( h \), i.e. we have a pullback square

\[
\begin{array}{c}
A_i \xrightarrow{h_i} W_i \\
\downarrow a_i \quad \downarrow c_i \\
A \xrightarrow{h} \mu F
\end{array}
\]

for some morphism \( h_i : A_i \to W_i \); (8.1)

in symbols: \( a_i = \overleftarrow{h}(c_i) \) for all ordinals \( i \). Then it follows that \( a_\lambda \) is an isomorphism, since so is \( c_\lambda \), whence \( (A, \alpha) \) is well-founded. In the base case \( i = 0 \) the above square clearly is a pullback since \( A_0 = W_0 = 0 \) is a strict initial object (see Remark 6.9(1)).

For the isolated step we compute the pullback of \( c_{i+1} : W_{i+1} \to \mu F \) along \( h \) using the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
A_{i+1} \xrightarrow{\alpha(a_i)} FA_i \xrightarrow{Fh_i} FW_i \\
\downarrow a_{i+1} \quad \downarrow Fh \quad \downarrow c_{i+1} \\
A \xrightarrow{\alpha} FA \xrightarrow{Fh} F(\mu F) \xrightarrow{\iota} \mu F
\end{array}
\end{array}
\]

By the induction hypothesis and since \( F \) preserves inverse images, the middle square above is a pullback. Since the structure map \( \iota \) of the initial algebra is an isomorphism, it follows that the middle square pasted with the right-hand triangle is also a pullback. Finally, the left-hand square is a pullback by the definition of \( a_{i+1} \). Thus, the outside of the above diagram is a pullback, as required.

For a limit ordinal \( j \), we know that \( a_j = \bigvee_{i<j} a_i \) and similarly, \( c_j = \bigvee_{i<j} c_i \) since \( W_j = \text{colim}_{i<j} W_j \) and monomorphisms are smooth (see Remark 2.16(2)). Using Remark 2.16(3) and the induction hypothesis we thus obtain

\[
\overleftarrow{h}(c_j) = \overleftarrow{h} \left( \bigvee_{i<j} c_i \right) = \bigvee_{i<j} \overleftarrow{h}(c_i) = \bigvee_{i<j} a_i = a_j.
\]

\[\square\]

Corollary 8.2. Let \( \mathcal{A} \) and \( F \) satisfy the assumptions of Theorem 8.1. Then the following properties of a coalgebra are equivalent:

(1) well-foundedness,

(2) parametric recursiveness,
(3) recursiveness,
(4) existence of a homomorphism into \((\mu F, \iota^{-1})\),
(5) existence of a homomorphism into a well-founded one.

Proof. We already know (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). Since \(F\) has an initial algebra (as proved in Theorem 8.1), the implication (3) \(\Rightarrow\) (4) follows from Corollary 3.4. In Theorem 8.1 we also proved (4) \(\Rightarrow\) (1). The implication (4) \(\Rightarrow\) (5) follows from Examples 4.5(2). Finally, it follows from [6, Remark 2.40] that \((\mu F, \iota^{-1})\) is a terminal well-founded coalgebra. Thus, (5) \(\Rightarrow\) (4), which completes the proof. \(\square\)

Example 8.3. (1) The category of many-sorted sets satisfies the assumptions of Theorem 8.1, and polynomial endofunctors on that category preserve inverse images. Thus, we obtain Jeannin et al.'s result [18, Thm. 3.3] that (1)–(4) in Corollary 8.2 are equivalent as a special instance.

(2) Recall from Example 2.15(2) that vector spaces fail to have universally smooth monomorphisms. The implication (4) \(\Rightarrow\) (3) in Corollary 8.2 does not hold for vector spaces. In fact, for the identity functor on \(\text{Vec}_K\) we have \(\mu \text{Id} = (0, \text{id})\). Hence, every coalgebra has a homomorphism into \(\mu \text{Id}\). However, not every coalgebra is recursive, e.g. the coalgebra \((K, \text{id})\) admits many coalgebra-to-algebra morphisms to the algebra \((K, \text{id})\). Similarly, the implication (4) \(\Rightarrow\) (1) does not hold. In fact, a coalgebra \(\alpha: A \rightarrow A\) is well-founded iff for every \(x \in A\) there exists a natural number \(n\) with \(\alpha^n(x) = 0\) (cf. Examples 4.5(5)). Clearly, not every coalgebra satisfies this property. In contrast, see Corollary 8.13.

Remark 8.4. Coming back to Remark 6.9(3), we see from the proof of Theorem 8.1 that in general one needs transfinite iteration to obtain the least fixed point \(\ominus\). Indeed, for \((A, \alpha) = (\mu F, \iota^{-1})\) we have \(h = \text{id}\) in (8.1) and therefore \(a_i = c_i\). Now for \(FX = X^R + 1\) on \(\text{Set}\) we have that \(\mu F\) is carried by the set of all (ordered) well-founded countably-branching trees. Furthermore, it is easy to show that \(\mu F = W_\omega\), where \(\omega\) is the first uncountable ordinal, and each \(W_i, i < \omega\) is a proper subset.

In Theorem 8.1, we assumed that the endofunctor has a pre-fixed point. For set functors, this assumption may be lifted. Indeed, whenever a category has a subobject classifier, then every recursive coalgebra is well-founded, as shown by Taylor [27, Rem. 3.8]. We present this in all details for convenience of the reader.

Remark 8.5. (1) Let us recall the definition of a subobject classifier originating in [21] and prominent in topos theory. This is an object \(\Omega\) with a subobject \(t: 1 \rightarrow \Omega\) such that for every subobject \(b: B \rightarrow A\) there is a unique \(\bar{b}: C \rightarrow \Omega\) such that the square below is a pullback:

\[
\begin{array}{ccc}
B & \xrightarrow{1} & 1 \\
\downarrow{b} & & \downarrow{t} \\
A & \xrightarrow{\bar{b}} & \Omega
\end{array}
\]

(8.2)
By definition, every elementary topos has a subobject classifier, in particular every category \( \text{Set} \) with \( \mathcal{C} \) small.

(2) \( \text{Set} \) has a subobject classifier given by \( \Omega = \{t, f\} \) with the evident \( t : 1 \to \Omega \). Indeed, subsets \( b : B \to A \) are in one-to-one correspondence with characteristic maps \( \hat{b} : B \to \Omega \).

(3) Our standing assumption that \( \mathcal{A} \) is a complete and well-powered category is not needed for the next result: finite limits are sufficient.

**Theorem 8.6 (Taylor [27]).** Let \( F \) be an endofunctor preserving inverse images on a finitely complete category with a subobject classifier. Then every recursive \( F \)-coalgebra is well-founded.

**Proof.** Let \( (A, \alpha) \) be a recursive coalgebra. Clearly, \( id_A \) is a fixed point of \( \Box \), and we prove below that it is the unique one. Thus, \( (A, \alpha) \) is well-founded.

Let \( b : B \to FB \) be any fixed point of \( \Box \). Consider the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{\alpha(b)} & FB \\
\downarrow^{b} & & \downarrow^{Fb} \\
A & \xrightarrow{\alpha} & FA
\end{array}
\end{array}
\begin{array}{ccc}
& & 1 \\
F1 & \xrightarrow{F1} & FA \\
\downarrow^{t} & & \downarrow^{t} \\
\Omega & \xleftarrow{\hat{b}} & \Omega
\end{array}
\]

The square on the left is a pullback because \( b = \Box b \). The central square is \( F \) applied to the pullback square (8.2) for \( b : B \to A \). The square on the right is the pullback square (8.2) for \( F1 : F1 \to F\Omega \). The upper morphism is \( ! : B \to 1 \), and so the lower one is \( \hat{b} \). Thus the outside rectangle is again a pullback. In particular,

\[ \hat{b} = F1 \cdot F\hat{b} \cdot \alpha. \]

So we have a coalgebra-to-algebra morphism

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow^{\hat{b}} & & \downarrow^{F\hat{b}} \\
\Omega & \xleftarrow{\hat{b}} & F\Omega
\end{array}
\]

Since \( (A, \alpha) \) is recursive, this means that \( \hat{b} \) is uniquely determined by \( \alpha \), independent of which fixed point \( b \) of \( \Box \) was used in our argument. Thus \( \hat{b} = id_A \), as desired.

**Corollary 8.7.** For every set functor preserving inverse images, the following properties of a coalgebra are equivalent:

- well-foundedness \( \iff \) parametric recursiveness \( \iff \) recursiveness.
Example 8.8. The hypothesis in Theorem 8.1 and Theorem 8.6 that the functor preserves inverse images cannot be lifted. In order to see this, we consider the functor \( R: \text{Set} \to \text{Set} \) of Example 2.3(5). It preserves monomorphisms but not inverse images. The recursive coalgebra \((C, \gamma)\) in Example 3.6 is not well-founded: \(\emptyset\) is a cartesian subcoalgebra.

We have seen that for set functors well-founded coalgebras are recursive, and the converse holds for functors preserving inverse images. Moreover, the latter requirement cannot be lifted as we just saw in Example 8.8. Recall that an initial algebra \((\mu F, \iota)\) is also considered as a coalgebra \((\mu F, \iota^{-1})\). Taylor [27, Cor. 9.9] showed that, for functors preserving inverse images, the terminal well-founded coalgebra is the initial algebra. Surprisingly, this result is true for all set functors.

Theorem 8.9 [6, Thm. 2.46]. For every set functor, a terminal well-founded coalgebra is precisely an initial algebra.

The proof is nontrivial, and we are not going to present it. It is based on properties of well-founded coalgebras in locally presentable categories. The fact that no assumptions on \( F \) are needed seems very special to \( \text{Set} \). On the one hand, Theorem 8.9 can be proved for every locally finitely presentable base category \( \mathcal{A} \) having a strict initial object and every endofunctor on \( \mathcal{A} \) preserving finite intersections [6, Theorem 2.36]. On the other hand, without this last assumptions, Theorem 8.9 does not even generalize from \( \text{Set} \) to the category of graphs as the following example shows.

Example 8.10. Let \( \text{Gra} \) be the category of graphs, i.e. the category of presheaves over the category \( \{ \bullet \rightrightarrows \bullet \} \) given by two parallel morphisms. Here is a simple endofunctor \( F \) on \( \text{Gra} \) whose initial algebra is infinite and whose terminal well-founded coalgebra is a singleton graph: On objects \( A \) put \( FA = 1 \) (the terminal graph) if \( A \) has edges. For a graph \( A \) without edges, let \( FA \) be the graph \( A + 1 \) without edges. The definition of \( F \) on morphisms \( h: A \to B \) is as expected: \( Fh \) maps the additional vertex of \( A \) to that of \( B \) in the case where \( B \) has no edges. Then \( \mu F \) is the graph of natural numbers without edges. However, the terminal well-founded coalgebra is \( F1 \cong 1 \).

As the last result of this section we now turn to showing the implication “recursive \( \Rightarrow \) well-founded” for functors on the category \( \text{Vec}_K \) preserving inverse images. This follows neither from either Theorem 8.1 (since monomorphism are not universally smooth in \( \text{Vec}_K \)) nor from Theorem 8.6 (since \( \text{Vec}_K \) does not have a subobject classifier).

Recall first that the kernel of a linear map \( f: X \to Y \) is the subspace \( \ker f = \{ x \in X : f(x) = 0 \} \). A functor \( F: \text{Vec}_K \to \text{Vec}_K \) preserves kernels if for every linear map \( f: X \to Y \) its kernel \( s: \ker f \to X \) is mapped to the kernel of \( Ff \), shortly \( Fs = \ker Ff \).

Remark 8.11. (1) Observe that for every linear map \( f: X \to Y \) its kernel \( s: \ker f \to X \) is the inverse image of the least subobject \( z: 0 \to Y \), shortly \( \ker f = f^{-1}[0] \).
If \( F: \text{Vec}_K \to \text{Vec}_K \) preserves inverse images and \( F0 = 0 \), then it preserves kernels. Indeed, \( FS \) is then the inverse image of \( Fz \) under \( Ff \), and \( Fz: 0 = F0 \to FY \) is the zero map. Thus \( FS \) is the kernel of \( Ff \) as desired.

(2) Conversely, if \( F \) preserves kernels, then \( F0 = 0 \) (the terminal object) and \( F \) preserves inverse images. In fact, \( F \) preserves finite limits: by [15, Thm. 3.12], a functor preserving kernels is additive, and for an additive functor preservation of kernels is equivalent to preservation of finite limits (see [11, Prop. 1.11.2]).

(3) Every subspace \( s: S \to X \) induces a quotient space \( X/S \), and we denote the corresponding canonical quotient map by \( \text{coker } s: S \to X/S \).

(4) Every linear map \( f: X \to Y \) induces an isomorphism \( X \cong \ker f + [f(X)] \), where \( [f(X)] \) denotes the image of \( f \) in \( Y \).

(5) For a linear map \( f: X \to Y \) and a subspace \( s: S \to Y \) let \( t: T = f^{-1}[S] \to Y \). Then there exists a unique monomorphism \( u: X/T \to Y/S \) such that the following diagram commutes:

\[
\begin{array}{ccc}
T & \to & S \\
\downarrow f & & \downarrow s \\
X & \to & Y \\
\text{coker } t & \downarrow & \text{coker } s \\
X/T & \to & Y/S \\
\end{array}
\]

Indeed, \( u \) exists by the universal property of \( \text{coker } t \). Moreover, we see that \( u \) is injective: if \( x+T \) satisfies \( u(x+T) = 0 \), i.e. \( f(x) + S = 0 \), then we have \( f(x) \in S \), thus \( x \in T \).

**Theorem 8.12.** Let \( F \) be an endofunctor on \( \text{Vec}_K \) preserving inverse images. Then every recursive \( F \)-coalgebra is well-founded.

**Proof.** Let \( \alpha: A \to FA \) be a recursive coalgebra and let \( a^*: (A^*, \alpha^*) \to (A, \alpha) \) be its well-founded part.

(1) Assume first that \( F0 = 0 \). Then \( F \) preserves zero maps and kernels by Remark 8.11(1). Then \( Fa^* \) is the kernel of \( F(\text{coker } a^*) \) as shown in the following diagram:

\[
\begin{array}{ccc}
A^* & \xrightarrow{a^*} & FA^* \\
\downarrow \alpha^* & & \downarrow \alpha \\
A & \xrightarrow{\alpha} & FA \\
\text{coker } a^* & \downarrow & F(\text{coker } a^*) \\
A/A^* & \xrightarrow{u} & F(A/A^*) \\
\end{array}
\]

Since \( \text{coker } a^* \) is epimorphic so is \( F(\text{coker } a^*) \) since epimorphisms split in \( \text{Vec}_K \). Thus, we have \( F(\text{coker } a^*) = \text{coker}(Fa^*) \), and by Remark 8.11(5) we obtain the unique monomorphism \( u: A/A^* \to F(A/A^*) \) such that the diagram above commutes. Choose a splitting \( e: F(A/A^*) \to A/A^* \), i.e. \( e \cdot u = \text{id} \). It follows
that \( q = \text{coker } a^* \) is a coalgebra-to-algebra morphism from \((A, \alpha)\) to \((A/A^*, e)\).

Indeed, we obtain
\[
e \cdot Fq \cdot \alpha = e \cdot u \cdot q = q.
\]
Since \( F \) preserves zero morphisms, the zero morphism \( z: A \to A/A^* \) is also a coalgebra-to-algebra morphism. Consequently, \( q = z \), which is equivalent to \( a^* \) being an isomorphism \( A \cong A^* \) as desired.

(2) Let \( F \) be arbitrary, and put \( R = F0 \). Then there is an endofunctor \( G \) on \( \text{Vec}_K \) with \( G0 = 0 \) and preserving inverse images such that \( FX = R \times GX \).

Indeed, for every vector space \( X \), let \( t_X: X \to 0 \) denote the zero map, and let \( k_X: GX \to FX \) be the kernel of \( Ft_X \). For every linear map \( f: X \to Y \) the equality \( t_X = t_Y \cdot f \) implies that \( Ff \) yields a linear map \( Gf \) making the following square commutative:

\[
\begin{array}{ccc}
GX & \xrightarrow{k_X} & FX \\
Gf & \downarrow & \downarrow Ff \\
GY & \xrightarrow{k_Y} & FY
\end{array}
\]

It is easy to verify that \( G \) is an endofunctor and \( k: G \to F \) a natural transformation. Observe that \( t_X \) is a split epimorphism (whose splitting is the unique \( s_X: 0 \to X \)), whence \( Ft_X \) is a split epimorphism with splitting \( Fs_X: R \to FX \).

Using Remark \( 8.11(5) \), this implies that \( FX \cong R + GX \) with coproduct injections \( Fs_X \) and \( k_X \). Since \( + \) is also product, we obtain \( FX \cong R \times GX \) as desired.

(3) We prove that \( G \) preserves kernels. By Remark \( 8.11(2) \), \( G \) then preserves finite limits, whence inverse images. Suppose that \( s = \text{ker } f \) so that we have the pullback on the left below

\[
\begin{array}{ccc}
S & \xrightarrow{s} & 0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\quad
\begin{array}{ccc}
GS & \xrightarrow{Gs} & 0 \\
\downarrow & & \downarrow \\
GX & \xrightarrow{Gf} & GY
\end{array}
\]

It is our task to prove that the square on the right above is a pullback. Since \( F \) preserves inverse images, applying it to left-hand square yields the following pullback square:

\[
\begin{array}{ccc}
R \times GS & \xrightarrow{\pi} & R \\
\downarrow & & \downarrow \\
R \times GX & \xrightarrow{R \times Gf} & R \times GY
\end{array}
\]

Note that since \( G0 = 0 \) the upper morphism is the left-hand product projection and the right-hand one the left-hand coproduct injection.

Now suppose we have \( g: Z \to GX \) with \( Gf \cdot g = z \), where \( z: Z \to 0 \to GY \) is the zero morphism. Then for the zero morphism \( z': Z \to R \) we clearly have
\[
(R \times Gf) \cdot \langle z', g \rangle = \langle z', z \rangle = i \cdot z',
\]

since the latter two are both the zero morphism \( Z \to R \times GY \). Therefore, there is a unique morphism \( h: Z \to R \times GS \) with \( (R \times GS) \cdot h = \langle z', g \rangle \) and \( \pi \cdot h = z' \).
This implies that \( h = (z', h') \) for a unique morphism \( h' \colon Z \to GS \) such that \( GS \cdot h' = g \), which proves the claim.

(4) Observe that \( \mathcal{G} \mathcal{O} = 0 \), thus we can apply part (1). Our recursive coalgebra \( \alpha = (\alpha_1, \alpha_2) \colon A \to R \times GA \) yields a coalgebra \( \alpha_2 \colon A \to GA \), and we prove that it is recursive, too. Indeed, given any algebra \( \beta \colon GB \to B \), we have an algebra

\[
R \times GB \cong R + GB \xrightarrow{[z, \beta]} GB
\]

for \( F \). Now observe that a morphism \( h \colon A \to B \) is a coalgebra-to-algebra morphism for \( F \)

\[
\begin{array}{ccc}
A & \xrightarrow{(\alpha_1, \alpha_2)} & R \times GA \\
h \downarrow & & \downarrow \nu_{\times Gh} \\
B & \xleftarrow{[0, \beta]} & R \times GB
\end{array}
\]

iff it is a coalgebra-to-algebra morphism from \((A, \alpha_2)\) to \((B, \beta)\) for \( G \). Since the former exists uniquely, so does the latter. This proves that \((A, \alpha_2)\) is recursive.

By part (1) the coalgebra \((A, \alpha_2)\) is well-founded for \( G \). Its next time operator \( \circ \) is the same as that of the \( F \)-coalgebra \((A, \alpha)\) because in the diagram below the outside is a pullback iff the left-hand square is:

\[
\begin{array}{ccc}
\circ S & \xrightarrow{\alpha_2} & R \times GS & \xrightarrow{\pi_S} & GS \\
\circ A & \downarrow R \times GS & \nu_{GS} & \downarrow GS \\
A & \xrightarrow{\alpha} & R \times GA & \xrightarrow{\pi} & GA
\end{array}
\]

Since \( \text{id}_A \) is the unique fixed point of \( \circ \) w.r.t. \( G \), it is also the unique fixed point w.r.t. \( F \). Thus \((A, \alpha)\) is well-founded for \( F \) as desired. \( \square \)

**Corollary 8.13.** For every functor on \( \text{Vec}_K \) preserving inverse images, the following properties of a coalgebra are equivalent:

\[
\text{well-foundedness} \iff \text{parametric recursiveness} \iff \text{recursiveness}.
\]

### 9 Conclusions

Well-founded coalgebras introduced by Taylor [28] have a compact definition based on an extension of Jacobs’ ‘next time’ operator. Our main contribution is a new proof of Taylor’s General Recursion Theorem that every well-founded coalgebra is recursive, generalizing this result to all endofunctors preserving monomorphisms on a complete and well-powered category with smooth monomorphisms. For functors preserving inverse images, we also have seen two variants of the converse implication “recursive \( \Rightarrow \) well-founded”, under additional hypothesis: one due to Taylor for categories with a subobject classifier, and the second one provided that the category has universally smooth monomorphisms and the functor has a pre-fixed point. Various counterexamples demonstrate that all our hypotheses are necessary.
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