A NEW FOUR PARAMETER $q$–SERIES IDENTITY AND ITS PARTITION IMPLICATIONS

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Abstract. We prove a new four parameter $q$-hypergeometric series identity from which the three parameter identity for the Göllnitz theorem due to Alladi, Andrews, and Gordon follows as a special case by setting one of the parameters equal to 0. The new identity is equivalent to a four parameter partition theorem which extends the deep theorem of Göllnitz and thereby settles a problem raised by Andrews thirty years ago. Some consequences including a quadruple product extension of Jacobi’s triple product identity, and prospects of future research are briefly discussed.

1. Introduction

One of the most fundamental results in the theory of partitions is the 1926 theorem of Schur [24] which we state in the following form:

**Theorem 1.** (Schur)

Let $P_1(n)$ denote the number of partitions of $n$ into distinct parts $\equiv -2^1, -2^0$ (mod 3).

Let $G_1(n)$ denote the number of partitions of $n$ such that the difference between the parts $\geq 3$, with equality only if a part is $\equiv -2^1, -2^0$ (mod 3). Then

$$G_1(n) = P_1(n).$$

In 1967, Göllnitz [22] established a deep partition theorem which we prefer to state as follows:

**Theorem 2.** (Göllnitz)

Let $P_2(n)$ denote the number of partitions of $n$ into distinct parts $\equiv -2^2, -2^1, -2^0$ (mod 6).

Let $G_2(n)$ denote the number of partitions of $n$ into parts $\not\equiv 1$ or 3, such that the difference between the parts $\geq 6$, with equality only if a part is $\equiv -2^2, -2^1, -2^0$ (mod 6). Then

$$G_2(n) = P_2(n).$$
Although Theorem 1 is not a special case of Theorem 2, the result of Göllnitz can be viewed as the next higher level extension of Schur’s theorem especially in the forms in which we have stated these results. While attempting to find new partition identities via a computer search in 1971, Andrews [16] raised the question whether there exists a partition theorem that goes beyond Theorem 2 in the sense that Theorem 2 may be viewed as going beyond Theorem 1. We settle this problem in the affirmative by proving the following result along with its refinement and generalization (Theorem 3) stated in §3:

**Theorem 3.** Let \( P_3(n) \) denote the number of partitions of \( n \) into distinct parts \( \equiv -2^1, -2^2, -2^3 \), \( -2^0 \) (mod 15).

Let \( G_3(n) \) denote the number of partitions of \( n \) into parts \( \not\equiv 2^3, 2^2, 2^1, 2^0 \) (mod 15), such that the difference between the parts \( \not\equiv 0 \) (mod 15) is \( \geq 15 \), with equality only if a part is \( \equiv -2^1, -2^2, -2^3, -2^0 \) (mod 15), parts which are \( \not\equiv -2^3, -2^2, -2^1, -2^0 \) (mod 15) are \( > 15 \), the difference between the multiples of 15 is \( \geq 60 \), and the smallest multiple of 15 is

\[
\begin{align*}
\geq 30 + 30\tau, & \quad \text{if } 7 \text{ is a part}, \\
\geq 45 + 30\tau, & \quad \text{otherwise},
\end{align*}
\]

where \( \tau \) is number of non–multiples of 15 in the partition. Then

\[
G_3(n) = P_3(n).
\]

Theorem 3 can be seen in §3 as a special case of the following new remarkable four parameter key identity

\[
\sum_{i,j,k,l} A^i B^j C^k D^l \sum_{i,j,k,l - \text{constraints}} q^{T_r + T_{ab} + T_{ac} + T_{ad} + T_{bc} - bd - cd + 4T_{Q-1} + 3Q + 2Q\tau} \frac{(q)_a(q)_b(q)_c(q)_d(q)_{ab}(q)_{ac}(q)_{ad}(q)_{bc}(q)_{bd}(q)_{cd}(q)_q}{(q)_a(q)_b(q)_c(q)_d(q)_{ab}(q)_{ac}(q)_{ad}(q)_{bc}(q)_{bd}(q)_{cd}(q)_q}
\]

\[
\cdot \{(1 - q^a) + q^{a+bc+bd+Q}(1 - q^b) + q^{a+bc+bd+Q+b+c+d}\}
\]

\[
= (-Aq)_\infty (-Bq)_\infty (-Cq)_\infty (-Dq)_\infty
\]

under the transformations

\[
\begin{align*}
&\left\{ \begin{array}{l}
\text{(dilation)} \ q \mapsto q^{15}, \\
\text{(translations)} \ A \mapsto q^{-8}, \ B \mapsto q^{-4}, \ C \mapsto q^{-2}, \ D \mapsto q^{-1}.
\end{array} \right.
\end{align*}
\]

**Warning:** It is to be noted that in (1.2) and everywhere, \( ab, ac, ad, bc, bd, \) and \( cd \) are parameters and that \( ab \) is not \( a \) multiplied by \( b \), with similar interpretation for \( ac, ad, bc, bd \) and \( cd \).

In (1.2) we have made use of the standard notations

\[
(a)_n = (a; q)_n = \begin{cases} 
\prod_{j=0}^{n-1} (1 - aq^j), & \text{if } n > 0, \\
1, & \text{if } n = 0, \\
\prod_{j=1}^{-n} (1 - aq^{-j})^{-1}, & \text{if } n < 0,
\end{cases}
\]

and

\[
(a)_\infty = \lim_{n \to \infty} (a)_n, \quad \text{when } |q| < 1.
\]

(1.5)
In addition, in (1.2) \( T_n = \frac{n(n+1)}{2} \) is the \( n \)-th triangular number, and the \( i, j, k, l \)-constraints on the summation variables \( a, b, c, d, ab, \ldots, cd, Q \) are

\[
\begin{align*}
i & = a + ab + ac + ad + Q, \\
j & = b + ab + bc + bd + Q, \\
k & = c + ac + bc + cd + Q, \\
l & = d + ad + bd + cd + Q.
\end{align*}
\] (1.6)

The quantities \( a, b, c, d, ab, \ldots, cd, \) and \( Q \) may be interpreted as the number of parts occurring in certain colors as will become clear in §3. Finally in (1.2), \( \tau = a + b + c + d + ab + ac + ad + bc + bd + cd \).

Extracting the coefficients of \( A^1 B^1 C^k D^l \) in (1.2), we can rewrite (1.2) in the equivalent form as

\[
\sum_{i, j, k, l \text{ constraints}} \frac{q^{T_r + T_{ab} + T_{ac} + \cdots + T_{cd} - bc - bd - cd + 4T_{Q-1} + 3Q + 2Q\tau}}{(q)a(q)b(q)c(q)d(q)a(q)b(q)c(q)d(q)b(q)d(q)c(q)d(q)q} \\
\cdot \left\{ (1 - q^i)^a + q^{a+bc+bd+Q} (1 - q^j)^b + q^{a+bc+bd+Q+b+cd} \right\} \\
= \frac{q^{T_r + T_{ab} + T_{ac} + T_{cd}}}{(q)^{i}(q)^{j}(q)^{k}(q)^{l}} \left( 1 - q^{\tau} \right),
\] (1.7)

where we note that the sum in (1.7) is actually a finite 7-fold sum.

In the next section we will first describe a two parameter refinement of Schur’s theorem due to Alladi and Gordon [13] (Theorem 4 of §2) who used the notion of partitions into colored integers. The analytic form of Theorem 4 is the identity (2.11) of §2. Next in §2 we will describe the generalization and three parameter refinement of Göllnitz’s theorem due to Alladi, Andrews, and Gordon [6] (Theorem 5 of §2), also obtained by using the notion of partitions into colored integers and by extending the method in [13]. The analytic form of Theorem 5 is the identity (2.16). An advantage of this approach is that it shows clearly that the three parameter colored generalization of Theorem 2 is an extension of the two parameter colored generalization of Theorem 1. That is, Theorem 4 is a special case of Theorem 5 and (2.11) a special case of (2.16) when any one of the parameters \( i, j, \) or \( k \) is set equal to zero.

Pursuing the notion of reformulation in terms of partitions involving colored integers, we will construct in §3 a four parameter colored generalization and refinement of Theorem 3 which is Theorem 6. In §4 we will follow the development in [9] and show that Theorem 6 is the combinatorial interpretation of the new identity (1.7). The proof of this remarkable identity is given in §5 and §6. If any one of the parameters \( i, j, k, \) or \( l \) is set equal to 0 in (1.7), we get (2.11); similarly, Theorem 6 yields Theorem 5 as a special case. Thus the question of Andrews is settled.

It is to be noted that in going from Theorem 1 to Theorem 2 or from Theorem 4 to Theorem 5, the nature of the gap conditions does not change. Still the proof of Theorem 6 is much deeper compared to that of Theorem 4. We wish to emphasize that in going from Theorem 2 to Theorem 3 or from Theorem 5 to Theorem 6 the extension is non–routine because of two very new conditions involving the multiplies of 15 in Theorem 3 and the parts in quaternary color in Theorem 6. More precisely, while the multiplies of 15 (respectively parts in quaternary color) satisfy special gap conditions among themselves, there is no interaction between multiplies of
15 (respectively parts in quaternary color) and the other parts as far as gap conditions are concerned. Also, there is a subtle lower bound condition (1.4) in Theorem 3 (respectively (3.3) in Theorem 6) on the multiplies of 15 (respectively parts in quaternary color). These new conditions might have been the reason that the problem posed by Andrews remained unsolved for so long.

Alladi [1], [3] and more recently Alladi and Berkovich [10], [11] obtained several important consequences of (2.16) including new proofs and interpretations of Jacobi’s celebrated triple product identity as well as many new weighted partition identities. In a similar spirit, (1.2) leads to an extension of Jacobi’s triple product identity which is stated as (7.1) in § 7. The new identity (1.2) raises the exciting possibility of four parameter refinements of partition theorems of Capparelli [20] and of Andrews–Bessenrodt–Olsson [17]. We discuss this briefly in § 7 and plan to pursue this in detail later.

The new results (the identity (1.7), Theorem 3 and Theorem 6) were first announced without proofs in [8]. The broad historical and mathematical significance of these results were discussed in the survey article [4]. Our main object here is to provide the detailed proof of (1.7) and to show that (1.7) is the analytic form of Theorem 6.

2. Colored reformulation and refinement of the Schur and Göllnitz theorems

We first describe the colored reinterpretation and refinement of Schur’s theorem due to Alladi and Gordon [13].

For this purpose we assume that all positive integers occur in two primary colors $A$ and $B$ and that integers $\geq 2$ occur also in the secondary color $AB$. Let $A_n, B_n$ and $AB_n$ denote the integer $n$ occurring in colors $A, B$, and $AB$, respectively.

In order to discuss partitions involving the colored integers, we need an ordering among them, and the one we choose is

$$A_1 < B_1 < AB_2 < A_2 < B_2 < AB_3 < A_3 < B_3 < AB_4 < \cdots.$$  \hfill (2.1)

When the substitutions

$$\begin{cases} A_n \mapsto 3n - 2, & B_n \mapsto 3n - 1, \text{ for } n \geq 1, \\ AB_n \mapsto 3n - 3, & \text{for } n \geq 2 \end{cases}$$ \hfill (2.2)

are made, the ordering (2.1) becomes

$$1 < 2 < 3 < 4 < \ldots,$$

the natural ordering among the positive integers. Theorem 4 stated below involving partitions into colored integers yields Theorem 1 under the substitutions (2.2). Note that for a given integer $n$, the ordering is

$$AB_n < A_n < B_n.$$ \hfill (2.3)

In general, there are six orderings generated by the six permutations of the symbols $A_n, B_n$, and $AB_n$ as shown by Alladi–Gordon [13], but for the reasons given above, we have chosen (2.1).
In order to state Theorem 4, we define Type–1 partitions to be those of the form
\[ \pi : m_1 + m_2 + \cdots + m_r \] such that \( m_i - m_{i+1} \geq 1 \), where equality holds if either
\[
\begin{align*}
&\text{m}_i \text{ and } m_{i+1} \text{ are of the same primary color,} \\
&\text{or } m_i \text{ is of a higher order color as given by } (2.3).
\end{align*}
\] (2.4)

It is easy to check that under the substitutions (2.2), the gap conditions in (2.4) translate to the difference conditions defining \( G_1(n) \) in Theorem 1.

Next, for any partition \( \pi \) into colored integers, let \( \nu_A(\pi), \nu_B(\pi), \) and \( \nu_{AB}(\pi) \) denote the number of parts of \( \pi \) in colors \( A \), \( B \), and \( AB \), respectively. We are now in a position to state the following result.

**Theorem 4.** *(Alladi–Gordon [13]*)

For given integers \( i, j \geq 0 \), let \( P_4(n; i, j) \) denote the number of partitions of \( n \) into \( i \) distinct parts all in color \( A \), and \( j \) distinct parts all in color \( B \).

Let \( G_4(n; a, b, ab) \) denote the number of Type–1 partitions \( \pi \) of \( n \) with \( a = \nu_A(\pi), b = \nu_B(\pi), \) and \( ab = \nu_{AB}(\pi) \). Then
\[
\sum_{i=a+ab, j=b+ab} G_4(n; a, b, ab) = P_4(n; i, j).
\]

It is clear that
\[
\sum_n P_4(n; i, j) q^n = \frac{q^{T_i + T_j}}{(q)_i(q)_j}. \tag{2.5}
\]

It turns out that (see [13])
\[
\sum_n G_4(n; a, b, ab) q^n = \frac{q^{T_{a+b+ab} + T_{ab}}}{(q)_a(q)_b(q)_{ab}}. \tag{2.6}
\]

Thus Theorem 4 is equivalent to the identity
\[
\sum_{i=a+ab, j=b+ab} q^{T_{a+b+ab} + T_{ab}} (q)_a(q)_b(q)_{ab} = \frac{q^{T_i + T_j}}{(q)_i(q)_j}. \tag{2.7}
\]

A combinatorial proof of (2.7) was given in [13]. We remark that the identity (2.7) is a special case of \( q \)-Chu-Vandermonde summation formula ([21], (II.6), p. 236).

Multiplying (2.7) by \( A^i B^j \) and summing the result over parameters \( i \) and \( j \) we can rewrite (2.7) in the equivalent form as
\[
\sum_{a,b,ab} A^{a+ab} B^{b+ab} q^{T_{a+b+ab} + T_{ab}} \frac{(q)_a(q)_b(q)_{ab}}{(q)_a(q)_b(q)_{ab}} = (-Aq)_{\infty}(-Bq)_{\infty}. \tag{2.8}
\]

Note that the transformations
\[
\begin{align*}
&\text{(dilation) } q \mapsto q^3, \\
&\text{(translations) } A \mapsto q^{-2}, \ B \mapsto q^{-1},
\end{align*}
\] (2.9)

are equivalent to the substitutions (2.2) on the colored integers. Since under these substitutions, the gap conditions in (2.4) translate to the difference conditions defining \( G_1(n) \) in Theorem 1, we see that under the transformations (2.9), the identity (2.8) becomes the analytic version of Theorem 1.
Pursuing the idea of partitions into colored integers, Alladi, Andrews, and Gordon [6] reinterpreted and refined Göllnitz’s theorem as we describe next.

Let us assume that the positive integers occur in three primary colors \( \mathbb{A}, \mathbb{B}, \) and \( \mathbb{C}, \) and that integers \( \geq 2 \) occur also in the three secondary colors \( \mathbb{AB}, \mathbb{AC}, \) and \( \mathbb{BC}. \) Each of the symbols \( \mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{AB}_n, \mathbb{AC}_n, \) and \( \mathbb{BC}_n, \) will represent the integer \( n \) in the corresponding color.

As in the case of the colored reinterpretation of Schur’s theorem, an ordering of these symbols is required here and the one we choose is

\[ A_1 < B_1 < C_1 < AB_2 < AC_2 < A_2 < BC_2 < B_2 < C_2 < AB_3 < AC_3 < \ldots \quad (2.10) \]

One reason for this choice is that under the substitutions

\[ A_n \mapsto 6n - 4, \quad B_n \mapsto 6n - 2, \quad C_n \mapsto 6n - 1, \quad \text{for } n \geq 1, \]

\[ AB_n \mapsto 6n - 6, \quad AC_n \mapsto 6n - 5, \quad BC_n \mapsto 6n - 3, \quad \text{for } n \geq 2, \quad (2.11) \]

the ordering (2.10) becomes

\[ 2 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12 < \ldots, \]

the natural ordering among the positive integers not equal to 1 and 3. Another reason is that Theorem 5 stated below implies Theorem 2 under the substitutions (2.11) as we shall soon see. We wish to comment that here too one can consider other orderings of the colored integers and we refer the reader to [6] for a discussion of the companion results to Theorem 4 that these other orderings yield.

Note that for any given integer \( n, \) the ordering is

\[ AB_n < AC_n < A_n < BC_n < B_n < C_n. \quad (2.12) \]

In this case Type–1 partitions are defined as those of the form \( \pi : m_1 + m_2 + \cdots + m_\nu, \) where the \( m_i \) are colored integers from the list (2.10) such that \( m_i - m_{i+1} \geq 1 \) with equality only if

\[ \begin{align*}
& \text{if } m_i \text{ and } m_{i+1} \text{ are of the same primary color, or} \\
& \text{if } m_i \text{ is of a higher order color given by (2.12).}
\end{align*} \quad (2.13) \]

We can now state the following:

**Theorem 5.** (Alladi–Andrews–Gordon [6])

For given integers \( i, j, k \geq 0, \) let \( P_5(n; i, j, k) \) denote the number of partitions of \( n \) into \( i \) distinct parts in color \( A, \) \( j \) distinct parts in color \( B, \) and \( k \) distinct parts in color \( C. \)

Let \( G_5(n; a, b, c, ab, ac, bc) \) denote the number of Type–1 partitions \( \pi \) of \( n \) such that \( \nu_A(\pi) = a, \nu_B(\pi) = b, \nu_C(\pi) = c, \nu_{AB}(\pi) = ab, \nu_{AC}(\pi) = ac \) and \( \nu_{BC}(\pi) = bc. \) Then

\[
\sum_{\begin{subarray}{c}
i = a + ab + ac \\
j = b + ab + bc \\
k = c + ac + bc
\end{subarray}} G_5(n; a, b, c, ab, ac, bc) = P_5(n; i, j, k).
\]

Here too the notation involving \( \nu \) and the parameters \( ab, ac, \) and \( bc \) is as explained earlier in this section. It is clear that

\[
\sum_n P_5(n; i, j, k) q^n = \frac{q^{T_i + T_j + T_k}}{(q)_i(q)_j(q)_k}. \quad (2.14)
\]
It is shown in [6] that
\[
\sum_n G_5(n, a, b, c, ab, ac, bc) q^n = \frac{q^{T_3 + T_{ab} + T_{ac} + T_{bc} - 1}(1 - q^a + q^{a+bc})}{(q)_{a}(q)_{b}(q)_{c}(q)_{ab}(q)_{ac}(q)_{bc}}, \tag{2.15}
\]
where \( s = a + b + c + ab + ac + bc \) is the total number of parts in each Type–1 partition \( \pi \). Thus Theorem 5 is equivalent to the identity
\[
\sum_{i=a+b+ac, j=b+ab+bc, k=c+ac+bc} q^{T_i + T_{ab} + T_{ac} + T_{bc} - 1}(1 - q^a + q^{a+bc}) = \frac{q^{T_i + T_j + T_k}}{(q)_{i}(q)_{j}(q)_{k}}. \tag{2.16}
\]

The first proof of (2.16) in [6] utilized Whipple’s \( q \)-analogue of Watson’s transformation \( \phi_2 \rightarrow \phi_3 \) (21), (III.18), p. 242), the \( q \)-sum of Bailey (21), (III.33), p. 239) as well as a transformation formula \( 3\phi_2 \rightarrow 3\phi_2 \) (21), (III.9), p. 241). Subsequently in [5], Alladi and Andrews simplified the proof of (2.16). This latter proof required only a special case of Jackson’s \( 241 \). Subsequently in [5], Alladi and Andrews simplified the proof of (2.16). This section how to construct a four parameter colored partition theorem, which reduces to Theorem 5 when one of the parameters is set equal to 0 and will explain why this Theorem is equivalent to the new identity (1.7).

Multiplying (2.16) by \( AB^3C^k \) and summing the result over parameters \( i, j \) and \( k \), we can rewrite (2.16) in the equivalent form as
\[
\sum_{a, b, c, ab, ac, bc} A^{a+b+ac} B^{b+ab+bc} C^{c+ac+bc} q^{T_i + T_{ab} + T_{ac} + T_{bc} - 1}(1 - q^a + q^{a+bc}) \frac{(q)_{a}(q)_{b}(q)_{c}(q)_{ab}(q)_{ac}(q)_{bc}}{(q)_{i}(q)_{j}(q)_{k}}, \tag{2.17}
\]
with \( s = a + b + c + ab + ac + bc \), as before. It is easy to verify that under the substitutions (2.11), the gap conditions (2.18) become the difference conditions defining \( G_2(n) \) in Theorem 2. Therefore, when transformations
\[
\begin{align*}
\{ & \text{(dilation)} \ q \mapsto q^6, \\
\{ & \text{(translations)} \ A \mapsto q^{-4}, \ B \mapsto q^{-2}, \ C \mapsto q^{-1},
\end{align*}
\tag{2.18}
\]
are applied to the identity (2.17), it becomes the analytic version of Theorem 2.

When any one of the parameters \( i, j, \) or \( k \) is set equal to 0, (2.16) reduces to (2.1), and Theorem 5 to Theorem 4. Pursuing this approach, we will describe in the next section how to construct a four parameter colored partition theorem, which reduces to Theorem 5 when one of the parameters is set equal to 0 and will explain why this Theorem is equivalent to the new identity (1.7).

It is to be noted that in Theorem 5 the ternary color \( \text{ABC} \) is not utilized and so only a proper subset of the complete alphabet of 7 colors \( A, B, C, AB, AC, BC, ABC \) is used. Although Andrews did not utilize the viewpoint of partitions into colored integers, starting with Schur’s theorem he was able to construct \( 14, 15 \) infinite hierarchies of partition theorems. His construction was based on choosing \( r \) distinct residue classes \( \text{(mod } 2^r - 1 \text{)} \) and forming all possible sums of these residues to get the full set of residue classes \( \text{(mod } 2^r - 1 \text{)}. \) From the point of view presented here, the \( r \) residue classes Andrews started with could be treated as \( r \) primary colors, and the residue classes obtained by summation as secondary, ternary, \( \ldots , \) colors depending on how many classes are summed. In other words, Andrews’ construction is based on the complete alphabet of colors. In contrast, our goal here
is to construct a partition theorem starting with four primary colors $A, B, C, D$ and utilize only a proper subset of the alphabet of 15 colors – 4 primary, 6 secondary, 4 ternary, and 1 quaternary. It is due to the emphasis on selecting a proper subset of colors that one does not know how long the hierarchy beyond Göllnitz’s Theorem extends. Indeed one did not know until recently whether such a partition theorem existed in case of the four primary colors, and if it did, whether it would reduce to Theorem 5 when one of the colors is eliminated. The main realization in [8] was that in addition to the primary and secondary colors only the quaternary color $ABCD$ has to be retained, but all ternary colors $ABC, ABD, ACD,$ and $BCD$ are to be discarded. Without further ado we now describe the resolution of the Andrews problem along these lines.

3. A new four parameter partition theorem

We assume that all positive integers occur in four primary colors $A, B, C, D$, that integers $\geq 2$ also occur in the six secondary colors $AB, AC, AD, BC, BD,$ and $CD$, and that integers $\geq 4$ occur also in the quaternary color $ABCD$. As before the symbols $A_n, \ldots, CD_n, ABCD_n$ will represent the integer $n$ occurring in the corresponding colors. We need now an ordering of the colored integers.

In the cases where we had 2 or 3 primary colors, Schur’s theorem and Göllnitz’s theorem helped us to select the orderings. More precisely orderings (2.1) and (2.10) were chosen because under the substitutions (2.2) and (2.11) they yielded the natural ordering of the positive integers, and also the general Theorem 4 and Theorem 5 reduced to Theorem 1 and Theorem 2, respectively. In contrast, here in four dimensions, we do not (yet) have a partition theorem to help us to select an ordering. So what we do is to take the ordering (2.12) and extend it as follows:

$$
\begin{align*}
\text{if } m < n & \text{ as ordinary (uncolored) positive integers,} \\
\text{then } m \text{ in any color } & < n \text{ in any color, and} \\
\text{if the same integer } n & \text{ occurs in two different colors, the order is given by} \\
ABCD_n < AB_n < AC_n < AD_n < A_n < BC_n < BD_n < B_n < CD_n < C_n < D_n.
\end{align*}
$$

(3.1)

With this ordering we define Type–1 partitions as those of the form $\pi : m_1 + m_2 + \cdots + m_\nu$, where the $m_i$ are colored integers ordered as in (3.1), and such that parts in the non–quaternary colors differ by $\geq 1$ with equality only if

$$
\begin{align*}
\text{parts are of the same primary color, or} \\
\text{if the larger part is in a higher order color as given by (3.1),}
\end{align*}
$$

(3.2)

and the gap between the parts in quaternary color is $\geq 4$, with the added condition that the least quaternary part is

$$
\begin{align*}
\geq 3 + 2\tau, & \text{ if } A_1 \text{ is a part,} \\
\geq 4 + 2\tau, & \text{ otherwise.}
\end{align*}
$$

(3.3)

Here $\tau$ is the number of non–quaternary parts in the partition $\pi$.

The first important realization in our construction of Type–1 partitions is that all ternary parts are to be discarded. The second major realization is that the quaternary parts do not interact with parts in the other colors in terms of gap conditions but only in terms of the lower bound (3.3). For instance, for $n = 12$ one
The nice thing about these substitutions is that they convert the ordering (3.1) to
ABCD may have the Type–1 partition \( A_6 + B_6 \), but not \( A_6 + A_7 \). It is this special role played by the parts in quaternary color that causes a substantial increase in depth and difficulty in going beyond Göllnitz’s theorem. We are now in a position to state the new partition theorem.

**Theorem 6.** For given integers \( i, j, k, l \geq 0 \), let \( P_6(n; i, j, k, l) \) denote the number of partitions of \( n \) into \( i \) distinct parts in color \( A \), \( j \) distinct parts in color \( B \), \( k \) distinct parts in color \( C \), and \( l \) distinct parts in color \( D \).

Let \( G_6(n; a, b, c, d, ab, \cdots, cd, Q) \) denote the number of Type–1 partitions \( \pi \) of \( n \) such that \( \nu_A(\pi) = a, \ldots, \nu_{CD}(\pi) = cd \), and \( \nu_{ABCD}(\pi) = Q \), where \( \nu_k(\pi) \), \( \ldots, \nu_{CD}(\pi), \nu_{ABCD}(\pi) \) denote the number of parts of \( \pi \) in colors \( A, \ldots, CD, A_{BCD} \), respectively. Then

\[
G_6(n; i, j, k, l) = P_6(n; i, j, k, l),
\]

where

\[
G_6(n; i, j, k, l) = \sum_{i, j, k, l-\text{constraints}} G_6(n; a, b, c, d, ab, \cdots, cd, Q)
\]

and the \( i, j, k, l - \text{constraints} \) on the parameters \( a, b, \ldots, cd, Q \) are as in (1.6).

For example, for \( n = 6, i = j = k = l = 1 \), there are ten pairs of partitions satisfying the conditions in Theorem 6, as summarized in Table 1.

| \( G_6(6; 1, 1, 1, 1) = 10 \) | \( P_6(6; 1, 1, 1, 1) = 10 \) |
|-----------------|-----------------|
| \( A_{BCD}6 \)  | \( A_1 + B_1 + C_1 + D_3 \) |
| \( AB_2 + CD_4 \) | \( A_1 + B_1 + D_1 + C_3 \) |
| \( AC_2 + BD_4 \) | \( A_1 + C_1 + D_1 + B_3 \) |
| \( AD_2 + BC_4 \) | \( B_1 + C_1 + D_1 + A_3 \) |
| \( BC_2 + AD_4 \) | \( A_1 + B_1 + C_2 + D_2 \) |
| \( BD_2 + AC_4 \) | \( A_1 + C_1 + B_2 + D_2 \) |
| \( CD_2 + AB_4 \) | \( A_1 + D_1 + B_2 + C_2 \) |
| \( A_1 + B_2 + CD_3 \) | \( B_1 + C_1 + A_2 + D_2 \) |
| \( A_1 + BC_2 + D_3 \) | \( B_1 + D_1 + A_2 + C_2 \) |
| \( A_1 + BD_2 + C_3 \) | \( C_1 + D_1 + A_2 + B_2 \) |

Table 1.

A strong four parameter refinement of Theorem 8 follows from Theorem 6 under the substitutions

\[
\begin{aligned}
A_n &\rightarrow 15n - 8, B_n \rightarrow 15n - 4, C_n \rightarrow 15n - 2, D_n \rightarrow 15n - 1, \text{ for } n \geq 1, \\
\text{and consequently } AB_n &\rightarrow 15n - 12, AC_n \rightarrow 15n - 10, AD_n \rightarrow 15n - 9, \\
BC_n &\rightarrow 15n - 6, BD_n \rightarrow 15n - 5, CD &\rightarrow 15n - 3, \text{ for } n \geq 2, \\
\text{and } A_{BCD}n &\rightarrow 15n - 15, \text{ for } n \geq 4.
\end{aligned}
\]

(3.4)

The nice thing about these substitutions is that they convert the ordering \( 7 < 11 < 13 < 14 < 18 < 20 < 21 < 22 < 24 < 25 < 26 < 27 < 28 < 29 < \ldots \),
the natural ordering among integers \( \not\equiv 2^3, 2^2, 2^1, 2^0 \pmod{15} \) and \( \not\equiv 3, 5, 6, 9, 10, 12, 15, 30 \). The substitutions \( \text{(3.4)} \) imply that the primary colors correspond to the residue classes

\[-2^3, -2^2, -2^1, -2^0 \pmod{15}.
\]

Since

\[2^3 + 2^2 + 2^1 + 2^0 = 15,
\]

the ternary colors correspond to the residue classes

\[2^3, 2^2, 2^1, 2^0 \pmod{15},
\]

which are precisely the four residue classes not considered in Theorem 3. Also since the residue classes relatively prime to 15 are \( \pm 2^3, \pm 2^2, \pm 2^1, \pm 2^0 \pmod{15} \), it follows that the integers in secondary colors correspond to the non–multiples of 15 that are not relatively prime to 15. Finally the integers in quaternary color correspond to the multiples of 15, which are \( \geq 45 \). These features make Theorem 3 especially appealing.

In the next section we show that Theorem 6 is combinatorially equivalent to \( \text{(1.7)} \).

### 4. Combinatorial interpretation of the identity \( \text{(1.7)} \)

In this section we will slightly generalize the analysis in §4 of \( \text{[9]} \) to show that the identity \( \text{(1.7)} \) is an analytic version of Theorem 6. First we observe that if the colors \( A, B, C, D \) are ordered in some fashion such as in \( \text{(3.1)} \), then it is clear that

\[
\sum_n P_6(n; i, j, k, l)q^n = \frac{q^{T_i + T_j + T_k + T_l}}{(q)_i(q)_j(q)_k(q)_l), \tag{4.1}
\]

because for any integer \( i \geq 0, \)

\[
\frac{q^{T_i}}{(q)_i}
\]

is the generating function of partitions into \( i \) distinct positive parts. We will now show that the generating function of Type–1 partitions \( \pi \) with \( \nu_A(\pi) = a, \ldots, \nu_{CD}(\pi) = cd \), and \( \nu_{ABCD}(\pi) = Q \) is the summand on the left of the identity \( \text{(1.7)} \). To this end we first observe that for \( i \geq 0, \)

\[
\frac{q^{T_i - i}}{(q)_i}
\]

is the generating function of partitions into \( i \) distinct nonnegative parts and

\[
\frac{1}{(q)_i} \cdot \frac{q^i}{(q)_i} \cdot \frac{1}{(q)_{i-1}} = \frac{1 - q^i}{(q)_i}
\]

are the generating functions of partitions into \( i \) nonnegative parts with the least part \( \geq 0 \), the least part \( > 0 \), the least part \( = 0 \), respectively. Since the discussion of the least part will play a prominent role, we introduce the notation \( \lambda(\pi) \) for the least part of the partition \( \pi \).

The discussion of Type–1 partitions \( \pi \) splits into three cases.

**Case 1:** \( \lambda(\pi) = A_1 \)

Decompose \( \pi \) as \( \tilde{\pi} \cup \pi' \), where \( \tilde{\pi} \) has all the parts of \( \pi \) in primary and secondary colors, and \( \pi' \) has the remaining parts all in quaternary color. Note that \( \nu(\tilde{\pi}) = \tau = a + b + c + d + ab + \cdots + cd \), and \( \nu(\pi') = Q \).
Subtract 1 from the smallest part of \( \tilde{\pi} \), 2 from the second smallest part of \( \tilde{\pi} \),..., and \( \tau \) from the largest part of \( \tilde{\pi} \). We call this process the Euler subtraction. It accounts for the term \( q^{T \tau} \) on the left side of (4.7). Note that Euler subtraction preserves the ordering among the colored parts and therefore can be easily reversed.

After this subtraction, \( \tilde{\pi} \) can be decomposed into ten monochromatic partitions \( \tilde{\pi}_A, \tilde{\pi}_B, \ldots, \tilde{\pi}_{CD} \), with the color indicated by the subscript, and satisfying the following conditions:

\[
\begin{align*}
\lambda(\tilde{\pi}_A) &= 0, & \nu(\tilde{\pi}_A) &= a, \\
\lambda(\tilde{\pi}_B) &\geq 0, & \nu(\tilde{\pi}_B) &= b, \\
\lambda(\tilde{\pi}_C) &\geq 0, & \nu(\tilde{\pi}_C) &= c, \\
\lambda(\tilde{\pi}_D) &\geq 0, & \nu(\tilde{\pi}_D) &= d, \\
\lambda(\tilde{\pi}_{AB}) &\text{ has } ab \text{ distinct (positive) parts,} \\
\lambda(\tilde{\pi}_{AC}) &\text{ has } ac \text{ distinct (positive) parts,} \\
\lambda(\tilde{\pi}_{AD}) &\text{ has } ad \text{ distinct (positive) parts,} \\
\lambda(\tilde{\pi}_{BC}) &\text{ has } bc \text{ distinct non–negative parts,} \\
\lambda(\tilde{\pi}_{BD}) &\text{ has } bd \text{ distinct non–negative parts,} \\
\lambda(\tilde{\pi}_{CD}) &\text{ has } cd \text{ distinct non–negative parts.}
\end{align*}
\] (4.2)

Note that in a Type–1 partition the gap between the parts in the same secondary color is \( \geq 2 \), therefore after the Euler subtraction each monochromatic partition with the parts in secondary color will have distinct parts. However, since the secondary colors \( BC, BD, \) and \( CD \) are of higher order than the primary color \( A \) (see (3.1)), the partitions \( \tilde{\pi}_{BC}, \tilde{\pi}_{BD} \) and \( \tilde{\pi}_{CD} \), could have least part equal to 0. For example, if from the Type–1 partition \( A_1BC_2 \) we subtract 1 from \( A_1 \) and 2 from \( BC_2 \), we get \( A_0B_0C_0 \). Such a possibility does not arise with the colors \( AB, AC, \) and \( AD \) which are of lower order than \( A \). Hence the conditions (4.3) are slightly different from the conditions (4.4).

Our most crucial observation concerning these monochromatic partitions is that they are independent of each other. Thus we may calculate the generating function of each of these monochromatic partitions. The conditions (4.2) imply that the generating functions of \( \tilde{\pi}_A, \tilde{\pi}_B, \tilde{\pi}_C, \) and \( \tilde{\pi}_D \) are

\[
\frac{1}{(q)_a} \frac{1}{(q)_b} \frac{1}{(q)_c} \frac{1}{(q)_d},
\] (4.5)

respectively. Similarly, (4.3) implies that the generating functions of \( \tilde{\pi}_{AB}, \tilde{\pi}_{AC}, \) and \( \tilde{\pi}_{AD} \) are

\[
\frac{q^{T_{ab}}}{(q)_{ab}} \frac{q^{T_{ac}}}{(q)_{ac}} \frac{q^{T_{ad}}}{(q)_{ad}},
\] (4.6)

respectively. Finally, (4.4) implies that the generating functions of \( \tilde{\pi}_{BC}, \tilde{\pi}_{BD}, \) and \( \tilde{\pi}_{CD} \) are

\[
\frac{q^{T_{bc} – bc}}{(q)_{bc}} \frac{q^{T_{bd} – bd}}{(q)_{bd}} \frac{q^{T_{cd} – cd}}{(q)_{cd}},
\] (4.7)

respectively. The parts in quaternary color have no interaction with the other parts in primary and secondary colors and so act independently except for the lower bound, which for \( \lambda(\pi) = A_1 \) is

\[\lambda(\pi') \geq 3 + 2\tau.\]
Since the gap between the parts in quaternary color is \( \geq 4 \), the generating function of \( \pi' \) is given by

\[
q^{4T_{\pi'_{-1}}+3Q+2Q^r}(q)_Q.
\]

(4.8)

Taking the product of the terms in (4.5)–(4.8) along with \( q^{T_r} \) (to account for the Euler subtraction) we see that if \( a, b, c, d, ab, \ldots, cd, \) and \( Q \) are specified, then the generating function of Type–1 partitions with \( \lambda(\pi) = B_1 \) is

\[
q^{T_r+T_{ab}+\cdots+T_{cd}-bc-bd-cd}(1-q^b)q^{4T_{\pi'_{-1}}+3Q+2Q^r}q^{a+bc+bd+Q}(q)_{a(q)b(q)c(q)d(q)ac(q)ad(q)bc(q)bd(q)cd(q)}.
\]

(4.9)

Case 2: \( \lambda(\pi) = B_1 \).

Here the discussion is identical to Case 1 except that after the Euler subtraction the smallest permissible value of parts in colors \( A, B, C, \) and \( D \), will be 1 because these colors are of lower order compared to \( B \) (but they are not of lower order than \( A \)). Also by the second case of (3.3) the smallest permissible value of a part in quaternary color is now \( 4 + 2r \) which is one higher than in Case 1. Finally, the term

\[
\frac{1}{(q)_{a-1}} = \frac{1 - q^b}{(q)b^{-1}}
\]

(4.10)

will now replace \( \frac{1}{(q)_{a-1}} \), because \( \lambda(\tilde{\pi}_A) = 0 \) and \( \lambda(\tilde{\pi}_B) \geq 0 \). Thus if \( a, b, c, d, ab, \ldots, cd, \) and \( Q \) are specified, then from (4.9), (4.10) and the preceding observations we see that the generating function of Type–1 partitions with \( \lambda(\pi) = B_1 \) is

\[
q^{T_r+T_{ab}+\cdots+T_{cd}-bc-bd-cd}(1-q^b)q^{4T_{\pi'_{-1}}+3Q+2Q^r}q^{a+bc+bd+Q}(q)_{a(q)b(q)c(q)d(q)ac(q)ad(q)bc(q)bd(q)cd(q)}.
\]

(4.11)

Case 3: \( \lambda(\pi) > B_1 \) (all cases other than Cases 1 and 2)

Note that although integers \( n \geq 2 \) can occur in colors other than the primary color, the integer 1 can occur only in the primary colors \( A < B < C < D \). Thus for a Type–1 partition, the smallest part can be either \( A_1 \) (which is Case 1), or \( B_1 \) (which is Case 2), or \( > B_1 \) (which is Case 3). In this final case, the discussion is the same as in Case 1 except that after the Euler subtraction the smallest permissible value of parts in colors \( A, B, C, D \), and \( Q \) will be 1. Thus if \( a, b, c, d, ab, \ldots, cd, \) and \( Q \) are specified, then the generating function of Type–1 partitions with \( \lambda(\pi) > B_1 \) is

\[
q^{T_r+T_{ab}+\cdots+T_{cd}-bc-bd-cd}q^{4T_{\pi'_{-1}}+3Q+2Q^r}q^{a+bc+bd+Q}(q)_{a(q)b(q)c(q)d(q)ac(q)ad(q)bc(q)bd(q)cd(q)}.
\]

(4.12)

Finally, adding the expressions in (4.9), (4.11), and (4.12) and summing the result over parameters \( a, b, c, d, \ldots, cd, \) and \( Q \) satisfying the \( i, j, k, l- \) constraints (1.0) yields the sum on the left in (1.7). This is precisely the generating function of the sum over Type–1 partitions in Theorem 6. Comparing this with (4.1) we see that Theorem 6 is the combinatorial version of the identity (1.7).

Thus to prove Theorem 6 we need to establish (1.2), which is equivalent to (1.7). This we will accomplish in the next two sections.
5. The key identity \((1.2)\) as a constant term identity

In this section we will follow the development in \(\S 4\) of [3] and transform \((1.2)\) into the equivalent constant term identity \((5.19)\) below. To state this identity and for later use, we need the following notations and definitions:

\[
[z^m]P(z) = \text{coefficient of } z^m \text{ in the Laurent expansion of } P(z), \quad (5.1)
\]

\[
(A_1, A_2, \ldots, A_r; q)_n = (A_1, A_2, \ldots, A_r)_n = (A_1)_n (A_2)_n \cdots (A_r)_n, \quad (5.2)
\]

\[
r_+^\psi_r \left( \frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_r}; q, z \right) = \sum_{n=\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_r)_n} z^n, \quad (5.3)
\]

\[
r_+^\phi_r \left( \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1})_n}{(q, b_1, \ldots, b_r)_n} z^n. \quad (5.4)
\]

A \(r_+^\phi_r\) series is called balanced if \(z = q\) and \(a_1 \cdots a_{r+1} q = b_1 \cdots b_r\). We will also need very-well-poised series defined as

\[
r_+^\phi_r (a_1; a_4, \ldots, a_{r+1}; q, z) = r_+^\phi_r \left( \frac{a_1, q \sqrt[4]{a_1}, -q \sqrt[4]{a_1}, a_4, \ldots, a_{r+1}}{q, \sqrt[4]{a_1}, -\sqrt[4]{a_1}, a_4, \ldots, a_{r+1}; q, z} \right). \quad (5.5)
\]

Next, we will make use of the relation among triangular numbers

\[
T_r + 4T_{Q-1} + (3 + 2\tau)Q = T_{r+2Q} \quad (5.6)
\]

and get rid of the condition that

\[
\tau = a + b + c + d + ab + ac + ad + bc + bd + cd \quad (5.7)
\]

in \((1.2)\) by rewriting it in the form

\[
[z^0] \sum_{a, b, c, d, Q \geq 0} \frac{q^{T_r + T_{ab} + \cdots + T_{cd} - bc - bd - cd}}{(q)_a (q)_b \cdots (q)_c (q)_d) q^Q \left(1 - q^a + q^{a+bc+bd+Q} (1 - q^b) + q^{a+bc+bd+Q+b+cd} \right) \cdot \left(1 - q^a + q^{a+bc+bd+Q} (1 - q^b) + q^{a+bc+bd+Q+b+cd} \right) \right) \quad (5.8)
\]

At this point we observe that all twelve sums in \((5.8)\) over the variables \(t, a, b, \ldots, Q\) can be evaluated to be infinite products. To deal with the sum over \(t\), we use Jacobi’s triple product identity \(\text{(21), (II.28), p. 239)}\)

\[
\sum_{t=-\infty}^{\infty} q^{T_r z^{-t}} = (q, -z, -q/z)_\infty. \quad (5.9)
\]

To deal with the sums over \(a, b, c, d, Q\) and \(ab, ac, ad, bc, bd, cd\), we use two identities of Euler

\[
\sum_{n \geq 0} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}, \quad (5.10)
\]

and

\[
\sum_{n \geq 0} \frac{q^{T_n} z^n}{(q)_n} = (q z)_\infty, \quad (5.11)
\]
respectively. This way we derive

\[
[z^0] \left\{ A_z \frac{(q, -z, -\frac{z}{2}, -ABqz, -ACqz, -ADqz, -BCz, -BDz, -CDz)_\infty}{(Az, Bz, Cz, Dz, ABCDz^2, -Aq, -Bq, -Cq, -Dq)_\infty} + (1 + BCDz^2) \frac{(q, -z, -\frac{z}{2}, -ABqz, -ACqz, -ADqz, -BCqz, -BDqz, -CDqz)_\infty}{(Az, Bz, Cz, Dz, ABCDqz^2, -Aq, -Bq, -Cq, -Dq)_\infty} \right\} = 1. \tag{5.12}
\]

To proceed further we recall the famous Bailey \(6\psi_6\) summation \([21]\), (II.33), p. 239)

\[
6\psi_6 = 6\psi_6 \left( q\sqrt{-z}, -q\sqrt{-z}, -A^{-1}, -B^{-1}, -C^{-1}, -D^{-1} : q, ABCDz^2 \right) = (q, -qz, -\frac{z}{2}, -ABqz, -ACqz, -ADqz, -BCqz, -BDqz, -CDqz)_\infty, \tag{5.13}
\]

provided \(|qz^2ABCD| < 1\). Thanks to \([5.18]\) we can rewrite \([5.12]\) as

\[
[z^0] \left\{ R(A, B, C, D, z) \frac{(1 + z)}{(1 - Az)(1 - Bz)(1 - Cz)(1 - Dz)} 6\psi_6 \right\} = 1, \tag{5.14}
\]

where

\[
R(A, B, C, D, z) = (1 - Az)(1 + BCDz^2)
+ \frac{A_z}{1 - ABCDz^2}(1 + BCz)(1 + BDz)(1 + CDz). \tag{5.15}
\]

Next, we use the easily verifiable relation

\[
(a)_n = (-1)^n a^{-n} q^{T_n} \frac{(\frac{a}{z})_n}{(z)_n}, n > 0 \tag{5.16}
\]

to decompose the function \(6\psi_6\) in \([5.14]\) as

\[
6\psi_6 = \sum_{n \geq 1} \frac{1 + z q^{2n}}{1 + z} \frac{(-A^{-1}, -B^{-1}, -C^{-1}, -D^{-1})_n (ABCDz^2)_n}{(Az, Bz, Cz, Dz)_n} = \frac{(1 - Az)(1 - Bz)(1 - Cz)(1 - Dz)}{1 + z} \sum_{n \geq 1} (z + q^{2n}) \frac{\left( \frac{q}{A_z}, \frac{q}{Bz}, \frac{q}{Cz}, \frac{q}{Dz} \right)_n}{(-Aq, -Bq, -Cq, -Dq)_n} \cdot (ABCD)^{n-1} q^n z^{2n-4}. \tag{5.17}
\]

Note that the first sum on the right of \([5.17]\) contains only positive powers of \(z\) and therefore, can be discarded in \([5.14]\). Since

\[
[z^0] \left\{ R(A, B, C, D, z) \frac{(1 + z)}{(1 - Az)(1 - Bz)(1 - Cz)(1 - Dz)} \right\} = 1, \tag{5.18}
\]

we derive from \([5.14]\) the desired constant term identity

\[
[z^0] \left\{ R(A, B, C, D, z) \sum_{n \geq 1} (z + q^{2n})(ABCD)^{n-1} q^n z^{2n-4} \cdot \frac{\left( \frac{q}{A_z}, \frac{q}{Bz}, \frac{q}{Cz}, \frac{q}{Dz} \right)_n}{(-Aq, -Bq, -Cq, -Dq)_n} \right\} = 0. \tag{5.19}
\]
where $R(A, B, C, D, z)$ is defined in (5.15). Notice that the sum on the left of (5.19) can be rewritten as

$$\frac{q(z + q^2)}{z^2(1 + Aq)(1 + Bq)(1 + Cq)(1 + Dq)}W_7(-\frac{q^2}{z}; q, \frac{q}{Az}, \frac{q}{Bz}, \frac{q}{Cz}, \frac{q}{Dz}; q, q^2; F),$$

where $F = ABCD$. If we use the transformation formula (III.24) in [21] together with (5.19) and (5.20) we obtain

$$[z^0] \left\{ R(A, B, C, D, z) \left( -\frac{q^2}{z}, q^2 z ABC, q^2 z ABD, q^2 z ACD, q^2 z BCD \right)_\infty \right\} = 0.$$  

It is interesting to observe that $W_7(-q^2 z F; -qA, -qB, -qC, -qD, qz^2 F; q, q)$ in (5.21) is very-well-poised and balanced at the same time.

In spite of all our attempts to prove (5.19) or (5.21) in a purely $q$-hypergeometric fashion, we succeeded only when one of the parameters, say $D$, is set to be zero. In the latter case, we obtain from (5.21) with $D = 0$ the following

$$[z^0] H(A, B, C, z) = 0,$$

where

$$H(A, B, C, z) = (1 + ABCz)(\frac{q^2}{z}, q^2 z ABC)_{\infty} \phi_2 \left( -qA, -qB, -qC, -qD, qz^2 F; q, q \right).$$

To prove (5.22) we follow [6] and notice that

$$H(A, B, C, z) = -H(A, B, C, -\frac{1}{ABCz}).$$

Our strategy to prove (5.19) in full generality is briefly described below.

First, we multiply both sides of (5.19) by $(-Aq, -Bq, -Cq, -Dq)_\infty$ to obtain

$$[z^0] \left\{ R(A, B, C, D, z) \sum_{n \geq 1} (z + q^{2n}) F^{n-1} q^n z^{2n-4} \right\} = 0.$$  

Next we rewrite (5.25) as

$$\lim_{m \to \infty} S(m) = 0,$$

where

$$S(m) = [z^0] \left\{ R(A, B, C, D, z) \sum_{n=1}^m (z + q^{2n}) F^{n-1} q^n \right\} \cdot z^{2n-4} \left( \frac{q}{Az}, \frac{q}{Bz}, \frac{q}{Cz}, \frac{q}{Dz} \right)_{n-1} \left( -Aq^{n+1}, -Bq^{n+1}, -Cq^{n+1}, -Dq^{n+1} \right)_{m-n}.$$  

In the next section we will establish the identity (6.1) for $S(m)$, which would enable us to prove (5.26) and, as a result, (1.2).
6. Proof of the constant term identity

In this section we will prove that

\[
S(m) = \sum_{2m+1 \leq i+j+k+l} q^{T_i+T_j+T_k+T_l} A^i B^j C^k D^l \left[ \begin{array}{c} m \\ i \\ j \\ k \\ l \end{array} \right] - \sum_{L \geq 1} q^{2L(m+1)} F^L \sum_{i+j+k+l=2m-2L} q^{T_i+T_j+T_k+T_l} A^i B^j C^k D^l \left[ \begin{array}{c} m \\ i \\ j \\ k \\ l \end{array} \right],
\]

where \( F = ABCD \), \( S(m) \) is defined in (5.27), and \( q \)-binomial coefficients are defined by

\[
\left[ \begin{array}{c} m \\ i \end{array} \right] = \begin{cases} \frac{(q)_m}{(q)_i(q)_{m-i}}, & \text{if } 0 \leq i \leq m, \\ 0, & \text{otherwise}. \end{cases}
\]

This is accomplished by showing that both sides in (6.1) satisfy the same recurrence relations and the same initial conditions.

Before we move on, we remark that setting one of the parameters \( A, B, C, D \) to zero will turn the second sum on the right of (6.1) into zero. In other words, the reduction from the four parameter case to the three parameter (Göllnitz) case is rather dramatic.

Next, we note that while it is not immediately obvious from (5.15), the rational function \( R(A, B, C, D, z) \) is actually symmetric in \( A, B, C, D \). To render this symmetry explicit we expand the term \( (1 - ABCDz^2)^{-1} \) in (5.15) to obtain after rearrangement

\[
R(A, B, C, D, z) = 1 - Fz^3 + (A + B + C + D) \sum_{L \geq 1} F^L z^{2L+1} + (ABC + ABD + ACD + BCD) \sum_{L \geq 0} F^L z^{2L+2}. \tag{6.3}
\]

Next from the definition of \( S(m) \) in (5.27) it follows that

\[
S(m) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m)S(m - 1) = \left[ z^0 \right] \left\{ R(A, B, C, D, z)(z + q^{2m})F^{m-1}q^{m-2m-4} \left( \frac{q}{A z}, \frac{q}{B z}, \frac{q}{C z}, \frac{q}{D z} \right)^{m-1} \right\},
\]

because in this subtraction only the term corresponding to \( n = m \) in the sum (5.27) survives. At this stage we expand the product

\[
\left( \frac{q}{A z}, \frac{q}{B z}, \frac{q}{C z}, \frac{q}{D z} \right)^{m-1}
\]

using the \( q \)-binomial theorem

\[
(qz)_N = \sum_{\nu=0}^{N} (-1)^{\nu} z^{\nu} q^{\nu} N^{\nu} = \sum_{\nu=0}^{N} (-1)^{N-\nu} z^{N-\nu} q^{T_N - \nu} \left[ \begin{array}{c} N \\ \nu \end{array} \right],
\]

to get

\[
F^{m-1}q^{m-2m-4} \left( \frac{q}{A z}, \frac{q}{B z}, \frac{q}{C z}, \frac{q}{D z} \right)^{m-1} = \sum_{0 \leq i,j,k,l \leq m-1} q^{T_{m-1-i}+T_{m-1-j}+T_{m-1-k}+T_{m-1-l}+m}
\]
Substituting the above expression in the right hand side of (6.4), we obtain

\[ S(m) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m) S(m - 1) = \]

\[ (z^0) \sum_{0 \leq i,j,k,l \leq m-1} R(A, B, C, D, z) q^{T_{m-1-i} + T_{m-1-j} + T_{m-1-k} + T_{m-1-l} + m} \]

\[ \cdot A^i B^j C^k D^l (-z)^{i+j+k+l-2m} \begin{bmatrix} m-1 \\ i \end{bmatrix} \begin{bmatrix} m-1 \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} m-1 \\ l \end{bmatrix}. \] (6.6)

If now the symmetric expansion for \( R(A, B, C, D, z) \) in (6.3) is substituted in (6.4), then after rearrangement this yields the following recurrence for \( S(m) \):

\[ S(m) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m) S(m - 1) = \]

\[ q^{2m} L_2(m) - L_1(m) - (A + B + C + D) \sum_{L \geq 1} F^L \{ q^{2m} L_{1-2L}(m) - L_{-2L}(m) \} \]

\[ + (ABC + ABD + ACD + BCD) \sum_{L \geq 0} F^L \{ q^{2m} L_{-2L}(m) - L_{-2L-1}(m) \} \]

\[ + F \{ q^{2m} L_{-1}(m) - L_{-2}(m) \}, \] (6.8)

where

\[ L_n(m) = \sum_{i+j+k+l=2(m-1)+n} q^{T_{m-1-i} + T_{m-1-j} + T_{m-1-k} + T_{m-1-l} + m} A^i B^j C^k D^l \]

\[ \cdot \begin{bmatrix} m-1 \\ i \end{bmatrix} \begin{bmatrix} m-1 \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} m-1 \\ l \end{bmatrix}. \] (6.9)

The triangular numbers satisfy the relations

\[ T_n = T_{n-1} \] and \( T_{m+n} = T_m + T_n + mn \) (6.10)

for all integers \( m \) and \( n \). Using (6.10) we can rewrite (6.9) as

\[ L_n(m) = q^{-m(n-1)} \sum_{i+j+k+l=2(m-1)+n} q^{T_i + T_j + T_k + T_l} A^i B^j C^k D^l \]

\[ \cdot \begin{bmatrix} m-1 \\ i \end{bmatrix} \begin{bmatrix} m-1 \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} m-1 \\ l \end{bmatrix}. \] (6.11)

Next we will derive a recurrence relation for the right hand side of (6.4). To this end we define

\[ \sigma_0(m) = \sum_{2m+1 \leq i+j+k+l} q^{T_i + T_j + T_k + T_l} A^i B^j C^k D^l \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix}. \] (6.12)

The \( q \)-binomial coefficients satisfy two recurrence relations, namely,

\[ \begin{bmatrix} m \\ i \end{bmatrix} = \begin{bmatrix} m-1 \\ i \end{bmatrix} + q^{m-i} \begin{bmatrix} m-1 \\ i-1 \end{bmatrix} \] (6.13)

and

\[ \begin{bmatrix} m \\ i \end{bmatrix} = q^i \begin{bmatrix} m-1 \\ i \end{bmatrix} + \begin{bmatrix} m-1 \\ i-1 \end{bmatrix}. \] (6.14)
If we use the first recurrence (6.13) to expand all $q$-binomials in (6.12), we get

$$
\sigma_0(m) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m)\sigma_0(m - 1)
= -L_1(m) - q^nL_2(m) - q^n(A + B + C + D)L_1(m)
+ q^{2m}(ABC + ABD + ACD + BCD)L_0(m) + q^{3m}FL_0(m) + q^{2m}FL_{-1}(m),
$$
(6.15)

where $L_0(m)$ is as in (6.11).

Now define

$$
\sigma_1(m) = \sum_{L \geq 1} q^{2L(m + 1)} F^L \sum_{i + j + k + l = 2m - 2L} q^{T_i + T_j + T_k + T_l} A^i B^j C^k D^l \left[ \begin{array}{cccc} m & m & m & m \\ i & j & k & l \end{array} \right].
$$
(6.16)

Next, we expand all the $q$–binomial coefficients in (6.10) using the second recurrence (6.14). This yields

$$
\sigma_1(m) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m)\sigma_1(m - 1)
= q^{3m}FL_0(m) + (A + B + C + D) \sum_{L \geq 1} F^L \{ q^{2m}L_{-2L}(m) - L_{-2L}(m) \}
- (ABC + ABD + ACD + BCD) \sum_{L \geq 1} F^L \{ q^{2m}L_{-2L}(m) - L_{-2L-1}(m) \}
- q^{-m}FL_{-2}(m).
$$
(6.17)

The difference

$$
\sigma(m) = \sigma_0(m) - \sigma_1(m)
$$

is precisely the right hand side of (6.14). Combining (6.8), (6.15) and (6.17) we find that

$$
(S(m) - \sigma(m)) - (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m)(S(m - 1) - \sigma(m - 1))
= q^{m}(A + B + C + D)L_1(m) - (ABC + ABD + ACD + BCD)L_{-1}(m)
+ q^{m}(1 + q^m)L_2(m) - Fq^{-m}(1 + q^m)L_{-2}(m).
$$
(6.18)

If we show that the right hand side of (6.18) is 0, then it follows that $S(m)$ and $\sigma(m)$ satisfy the same recurrence. So this is what we will do next.

To demonstrate that the right hand side of (6.18) is identically 0, we need to establish that for each $i, j, k, l$, the coefficient of $A^i B^j C^k D^l$ is 0. Using (6.11) and collecting the coefficient of $A^i B^j C^k D^l$ in (6.18) we see that it is sufficient to prove that for $i + j + k + l = 2m$

$$
q^m \left( q^{T_{i-1} + T_j + T_k + T_l} \left[ \begin{array}{cccc} m-1 & m-1 & m-1 & m-1 \\ i-1 & j-1 & k-1 & l \end{array} \right] \right) + \ldots
- q^{2m} \left( q^{T_{i-1} + T_{j-1} + T_k + T_l} \left[ \begin{array}{cccc} m-1 & m-1 & m-1 & m-1 \\ i-1 & j-1 & k-1 & l \end{array} \right] \right) + \ldots
+(1 + q^m) q^{T_i + T_j + T_k + T_l} \left[ \begin{array}{cccc} m-1 & m-1 & m-1 & m-1 \\ i & j & k & l \end{array} \right] + \ldots
-q^{2m} (1 + q^m) q^{T_{i-1} + T_{j-1} + T_{k-1} + T_{l-1}} \left[ \begin{array}{cccc} m-1 & m-1 & m-1 & m-1 \\ i-1 & j-1 & k-1 & l-1 \end{array} \right] = 0,
$$
(6.19)

where the dots inside the first parenthesis on the left indicate that there are three other similar terms in which $i$ is successively interchanged with $j, k, l$, and the dots
in the second parenthesis on the left indicate that there are three other terms in which \( l \) is successively interchanged with \( k, j \) and \( i \).

To help us prove (6.19) we transform the left side using

\[
\begin{bmatrix} m-1 \\ i-1 \end{bmatrix} = \begin{cases} \frac{1-q^i}{1-q^{m-1}} m-1 \\ i \end{cases}, \quad \text{if } m \neq i, \\
1, \quad \text{if } m = i. \tag{6.20}
\]

Since the right hand side of (6.20) is defined in a piecewise fashion, one needs to consider the following three cases.

**Case A:** \( i + j + k + l = 2m \) and exactly two of \( i, j, k, l \) equal \( m. \)

Owing to the symmetry in \( i, j, k, l \), we may assume \( i = j = m. \) Thus \( k = l = 0. \) Obviously, each of the \( q \)-binomial products in (6.19) is 0. Thus we get no contribution in this case.

**Case B:** \( i + j + k + l = 2m \) and exactly one of \( i, j, k, l \) equals \( m. \)

Again by symmetry in \( i, j, k, l \), we assume \( i = m. \) So \( j + k + l = m. \) In this case the penultimate term in (6.19) is clearly 0. Also in the expression in the first parenthesis, all three products after the first are 0 because of the factor \( \binom{m-1}{i} \) with \( i = m. \) Thus using (6.20) we may rewrite the left hand side of (6.19) in the form

\[
q^{T_m + T_j + T_k + T_l} \rho_1(j, k, l, q) \binom{m-1}{j} \binom{m-1}{k} \binom{m-1}{l}, \tag{6.21}
\]

where \( j + k + l = m, \) and

\[
\rho_1(j, k, l, q) = 1 - \frac{q^{m-j-k}(1-q^j)(1-q^k)}{(1-q^{m-j})(1-q^{m-k})} - \frac{q^{m-j-\ell}(1-q^l)(1-q^j)}{(1-q^{m-j})(1-q^{m-\ell})}. \tag{6.22}
\]

It turns out quite miraculously that upon simplification

\[
\rho_1(j, k, l, q) = (q^{j+k+l} - q^m) \frac{(1 - q^m)^2}{(q^j - q^m)(q^k - q^m)(q^l - q^m)}, \tag{6.23}
\]

and therefore

\[
\rho_1(j, k, l, q) = 0, \quad \text{if } j + k + l = m. \tag{6.24}
\]

Thus (6.24) implies that the expression in (6.21) has value 0, and so there is no contribution from Case B either.

**Case C:** \( i + j + k + l = 2m \) and none of \( i, j, k, l \), equals \( m. \)

In this case, again using (6.20), we may write the expression on the left in (6.19) as

\[
q^{T_i + T_j + T_k + T_l} \rho_2(i, j, k, l, q) \binom{m-1}{i} \binom{m-1}{j} \binom{m-1}{k} \binom{m-1}{l}, \tag{6.25}
\]

where \( i + j + k + l = 2m, \) and

\[
\rho_2(i, j, k, l, q) = \left\{ \frac{q^{m-i-j-k}(1-q^j)(1-q^k)}{1-q^{m-i}} + \ldots \right\} \left\{ \frac{q^{m-j-\ell-k}(1-q^\ell)(1-q^k)}{(1-q^{m-j})(1-q^{m-k})} + \ldots \right\} + (1 + q^m) \left\{ \frac{1 - (1-q^j)(1-q^k)(1-q^l)}{(1-q^{m-j})(1-q^{m-k})(1-q^{m-\ell})} \right\}. \tag{6.26}
\]
In (6.26) the dots inside the first parenthesis indicate that there are three other similar summands with $i$ interchanged with $j, k,$ and $l$, in succession, and the dots in the second parenthesis indicate that there are three other similar summands in which the absence of $l$ is replaced successively by the absence of $k, j,$ and $i$. If the expression in (6.26) is simplified, it turns out quite miraculously (for a second time) that

$$\rho_2(i, j, k, l, q) = (6.27)$$

$$q^{2m} - q^{i+j+k+l} \{ (1 + q^m)(1 - q^i)(1 - q^j)(1 - q^k)(1 - q^l) - (1 - q^m)^3 \} / (q^m - q^i)(q^m - q^j)(q^m - q^k)(q^m - q^l)$$

and therefore

$$\rho_2(i, j, k, l, q) = 0, \text{ if } i + j + k + l = 2m. \quad (6.28)$$

Thus there is no contribution from Case C either. Hence, (6.19) is true. Clearly, (6.18) and (6.19) together imply that

$$S(m) - \sigma(m) = (1 + Aq^m)(1 + Bq^m)(1 + Cq^m)(1 + Dq^m)(S(m-1) - \sigma(m-1)). \quad (6.29)$$

Using the definitions (6.1), (6.12) and (6.16) it is easy to verify that

$$S(0) = \sigma(0) = 0. \quad (6.30)$$

From (6.29) and (6.30) it follows that

$$S(m) = \sigma(m) \text{ for all } m \geq 0. \quad (6.31)$$

Finally, observe that (6.12) and (6.16) imply

$$\begin{cases} [q^n] \sigma_0(m) = 0, & \text{if } n < T_{m+1}, \\ [q^n] \sigma_1(m) = 0, & \text{if } n < T_{m+1} + \left\lfloor \frac{m+2}{2} \right\rfloor, \end{cases} \quad (6.32)$$

and therefore

$$\lim_{m \to \infty} \sigma_0(m) = \lim_{m \to \infty} \sigma_1(m) = 0. \quad (6.33)$$

Since $\sigma(m) = \sigma_0(m) - \sigma_1(m)$, (6.33) yields

$$\lim_{m \to \infty} \sigma(m) = 0. \quad (6.34)$$

This combined with (6.31) shows

$$\lim_{m \to \infty} S(m) = \lim_{m \to \infty} \sigma(m) = 0 \quad (6.35)$$

and this completes the proof of (5.19) and, as a result, of the key identity (1.2).

7. Consequences and prospects

In order to obtain Göllnitz’s theorem from the identity (2.17), the transformations (2.18) were used. The dilation $q \mapsto q^6$ in (2.18) is the minimal one required to convert the six colors (three primary and three secondary) in Theorem 5 to non-overlapping residue classes. If a dilation $q \mapsto q^M$ with $M < 6$ is used, then the six colors will lead to some overlap in the residue classes, that is to say, certain residue classes mod $M$ will represent integers in more than one of the six colors. This will involve a discussion of partitions with weights attached, these weights being polynomials in $A, B,$ and $C$. A general theory of weighted partition identities and its uses can be found in Alladi [2]. In addition Alladi [1], [3], and more recently Alladi and Berkovich [10], [11] discuss weighted partition reformulations of Göllnitz’s theorem obtained from dilations $q \mapsto q^M$ with $M < 6$, and some of their important
consequences. In some instances, values of the parameters \( A, B, C \) can be chosen so that there is a major collapse of the weights. This has led to derivations of various equivalent forms of Jacobi’s celebrated triple product identity (5.9).

To get Theorem 3 from Theorem 6 we used the dilation \( q \mapsto q^{15} \). This dilation led to 11 distinct residue classes mod 15 being used in Theorem 1. Since Theorem 6 involves 11 colors, any dilation \( q \mapsto q^{M} \) with \( M < 11 \) will lead to an overlap in residue classes and therefore to weighted partition identities. It would be worthwhile to discuss weighted partition identities that emerge from Theorem 6 in a manner similar to what was done in [1], [3], [10], [11]. Another very interesting question is whether there exists a quadruple product extension of (5.9). More precisely, observing that (5.9) contains a lacunary expansion of a triple product with one free parameter \( z \), we ask whether there is an expansion of a quadruple product with two free parameters which reduces to (5.9) when one of the parameters is set equal to 0. Recently, by setting \( D = -1 \) in (1.2), Alladi and Berkovich succeeded in reformulating Theorem 6 as a weighted partition identity. It turned out that the choice

\[ C = B^{-1} \]

(motivated by \( z \cdot z^{-1} = 1 \) in (5.9)) led to a significant collapse in the weights, and to the following quadruple product extension of (5.9):

**Proposition:**

\[
\sum_{k,l,m,n \geq 0} q^{T_l} \left( \frac{B_{l+1} + B_{l-1}}{B + 1} \right) \frac{q^{4T_m + 2T_n + k^2 + 2n(l+m)+m(l-k)}(-1)^n A_{n+m} B^{2k-m}}{(q)(q)_{k}} (q)_{m-k} = (-Aq, -Bq, -B^{-1}q, q)_\infty. \tag{7.1}
\]

If we put \( A = 0 \) in (7.1), it forces the collapse of the entire second sum on the left; also the contribution from the first sum in (7.1) is only when \( k = m = n = 0 \). So with \( A = 0 \), (7.1) reduces to

\[
\sum_{l \geq 1} q^{T_l} \left( \frac{B_{l+1} + B_{l-1}}{B + 1} \right) = (-Bq, -B^{-1}q, q)_\infty. \tag{7.2}
\]

which is the same as (5.9) with \( B = z \). Being a theta function identity, (7.2) is very fundamental. It would be interesting to investigate applications of the more general identity (7.1). The derivation of (7.1) as a special case of the key identity (1.2) involves some nontrivial combinatorial analysis and will be presented elsewhere.

In the course of studying vertex operators in the theory of affine Lie algebras, Capparelli was led to the following theorem: The number of partitions \( C^*(n) \) of \( n \) into parts \( \equiv \pm 2, \pm 3 \) (mod 12) equals the number of partitions \( D(n) \) of \( n \) into parts \( > 1 \), and differing by \( \geq 2 \), where the difference is \( \geq 4 \) unless consecutive parts are multiples of 3 or add up to a multiple of 6.

The generating function of \( C^*(n) \) is

\[
\sum_{n=0}^{\infty} C^*(n) q^n = \frac{1}{(q^2, q^3, q^9, q^{12}; q^{12})_\infty}. \tag{7.3}
\]
Alladi, Andrews, and Gordon [7] showed that by rewriting the product in (7.3) as
\((-q^2, -q^3, -q^4, -q^6; q^6)_\infty,\)

it is possible to get a two parameter refinement of Capparelli’s theorem. More precisely, if
\[ C(n; i, j) \]

is defined by
\[ \sum_{n,i,j} C(n; i, j) A^i B^j q^n = (-Aq^2, -q^3, -Bq^4, -q^6; q^6)_\infty, \]  
then there is a two parameter refinement of Capparelli’s theorem in which
\[ C(n; i, j) \]

is equated to
\[ D(n; i, j), \]

subject to the additional restrictions that there are precisely \( i \) parts \( \equiv 2 \ (\mod 3) \) and \( j \) parts \( \equiv 1 \ (\mod 3) \). In [7] a combinatorial bijective proof of this two parameter refined theorem is given.

Notice that the transformations
\[
\begin{align*}
\text{(dilation) } q &\mapsto q^6, \\
\text{(translations) } A &\mapsto Aq^{-4}, B \mapsto Bq^{-2}, C \mapsto Cq^{-3}, D \mapsto D,
\end{align*}
\]

in (1.2) lead to the product
\[ (-Aq^2, -Cq^3, -Bq^4, -Dq^6; q^6)_\infty, \]

which is a four parameter refinement of the product in (7.4). Since \( q \mapsto q^6 \) is a dilation smaller than \( q \mapsto q^{11} \), in view of certain remarks made above, (1.2) and Theorem 6 will lead to a weighted partition theorem in which partitions with parts differing by \( \geq 6 \) and with weights (polynomials in \( A, B, C, D \)) attached are equated to the partition function whose generating function is the product in (7.6). The gap conditions that would emerge from this construction are different from those defining \( D(n) \) in Capparelli’s theorem. It would be worthwhile to investigate whether the gap conditions for our weighted partitions are connected to those defining \( D(n) \), and whether our companion of Capparelli’s theorem has implications in the theory of affine Lie algebras.

Motivated by studies in the representation theory of symmetric groups, Andrews, Bessenrodt, and Olsson [17] considered partitions whose generating function is the product
\[ (-Aq, -Bq^2, -Cq^3, -Dq^4; q^5)_\infty, \]

and related these to partitions satisfying certain difference conditions. We may view the product in (7.7) as one of the form
\[ (-Aq, -Bq^2, -Cq^3, -Dq^4; q^5)_\infty, \]

where there is the relation
\[ AD = BC \]

among the parameters. Thus there are only three free parameters in (7.7). Clearly the transformations
\[ \begin{align*}
\text{(dilation) } q &\mapsto q^5, \\
\text{(translations) } A &\mapsto Aq^{-4}, B \mapsto Bq^{-3}, C \mapsto Cq^{-2}, D \mapsto Dq^{-1},
\end{align*} \]  

convert the product in (1.2) to that in (7.8). Thus in this case, Theorem 6 will yield a result in which a four parameter weighted count of certain partitions into parts differing by \( \geq 5 \) are connected to partitions whose generating function is the product in (7.8). As in the case of Capparelli’s theorem, the gap conditions
we would get are different from those considered in [17], but perhaps there are
connections worth exploring.

In the last decade, many new generalizations of the Rogers–Ramanujan identities
were discovered and proved by McCoy and collaborators (see [18] for a review and
references) using the so called thermodynamic Bethe ansatz (TBA) techniques. It
would be highly desirable to find a TBA interpretation of the new identity (1.7).
Such an interpretation, besides being of substantial interest in physics, may provide
insight into how to extend Theorem 6 to the one with five or more primary colors.

Recently Alladi and Berkovich [9] obtained a bounded version of (2.16):

$$
\sum_{i=a+b+bc} T_i + T_{ab} + T_{ac} + T_{bc} - 1 \cdot 
\left\{ q_{bc} \left[ \begin{array}{c}
L - s + a \\
a
\end{array} \right] \left[ \begin{array}{c}
L - s + b \\
b
\end{array} \right] \left[ \begin{array}{c}
L - s + c \\
c
\end{array} \right] \left[ \begin{array}{c}
L - s \\
ar
\end{array} \right] \left[ \begin{array}{c}
L - s \\
bc
\end{array} \right] 
+ \left[ \begin{array}{c}
L - s + a - 1 \\
a - 1
\end{array} \right] \left[ \begin{array}{c}
L - s + b \\
b
\end{array} \right] \left[ \begin{array}{c}
L - s + c \\
c
\end{array} \right] \left[ \begin{array}{c}
L - s \\
ar
\end{array} \right] \left[ \begin{array}{c}
L - s \\
bc - 1
\end{array} \right] 
\right\} 
= q^{T_i + T_{j} + T_k} \left[ \begin{array}{c}
L - j \\
j
\end{array} \right] \left[ \begin{array}{c}
L - k \\
k
\end{array} \right] \left[ \begin{array}{c}
L - i \\
i
\end{array} \right],
$$

where $s = a + b + c + ab + ac + bc$. Actually in [9], a doubly bounded version of
(2.16) was established. Subsequently, the full triply bounded refinement of (2.16)
was found and proven by Berkovich and Riese in [19].

Alladi and Berkovich [9] noticed that (7.10) can be interpreted combinatorially
to yield a refinement of Theorem 5 with bounds on the sizes of the parts. By
considering dilations $q \mapsto q^M$ with $M < 6$, it can be shown that the bounded
Göllnitz theorem leads to new bounded versions of fundamental $q$–series identities
including (5.9). A detailed treatment of some of these important applications may
be found in [11]. In view of (7.10), we may ask whether there exists a bounded
version of the new four parameter identity (1.7).

Finally, it is important to investigate what kind of $q$–hypergeometric transforma-
tion formulae hidden behind (1.7) and (5.21). This question as well as those
mentioned above indicate that (1.7) opens up several exciting avenues for further
exploration.

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