$c = r_G$ Theories of $W_G$-gravity:
the set of observables
as a model of simply laced $G$

A.Marshakov, A.Mironov

Theory Department,
P.N.Lebedev Physical Institute,
Leninsky prospect, 53, Moscow, 117 924,

A.Morozov, M.Olshanetsky

Institute of Theoretical and Experimental Physics,
Bol.Cheremushkinskaya st., 25, Moscow, 117 259

---

1E-mail address: theordep@sci.fian.msk.su
2E-mail address: morozov@itep.msk.su
3E-mail address: olshanetsky@itep.msk.su
ABSTRACT

We propose to study a generalization of the Klebanov-Polyakov-Witten (KPW) construction for the algebra of observables in the $c = 1$ string model to theories with $c > 1$. We emphasize the algebraic meaning of the KPW construction for $c = 1$ related to occurrence of a model of $SU(2)$ as original structure on the algebra of observables. The attempts to preserve this structure in generalizations naturally leads to consideration of $W$-gravities. As a first step in the study of these generalized KPW constructions we design explicitly the subsector of the space of observables in appropriate $W_G$-string theory, which forms the model of $G$ for any simply laced $G$. The model structure is confirmed by the fact that corresponding one-loop Kac-Rocha-Caridi $W_G$-characters for $c = r_G$ sum into a chiral (open string) $k = 1$ $G$-WZW partition function.
1 Introduction

Two recent papers [1,2] began a systematic study of the what can be called the *algebras of observables* in $c = 1$ theories of 2$d$-gravity. These algebras should be considered as main invariant characteristics of topological (and/or string) models (in particular they should not depend on the order of perturbation expansion). In most cases string models are constructed starting from particular conformal field theory (CFT) by coupling to and integration over 2-dimensional metric on Riemann surfaces. In practice, one usually proceeds with a kind of BRST formalism instead of explicit integration: (i) extra Liouville and ghost fields are added to degrees of freedom of original CFT in order to create a new conformal model with vanishing Virasoro central charge and (ii) the action of Virasoro algebra is factorized out\footnote{Really these two items are related: when the central charge is equal to zero one can pull out Virasoro action from descendants to the vacuum and cancell corresponding correlators, then only correlators of primaries are non-zero.}, i.e. only the classes of BRST cohomology are identified with the observables in a string model. Moreover, at least, in many cases certain Virasoro-primary fields of original CFT, appropriately “dressed” by Liouville and ghost fields, can be taken for representatives of the cohomology classes. This is a considerable simplification, because the set of primaries can possess additional structures or symmetries which are much more obscure in the entire huge set of all fields of CFT. As an example of such structure, refs.[1,2] associated the set of all Virasoro primaries $SU(2)$-invariant $c = 1$ CFT to the model of $SU(2)$, i.e. to the collection of all the unitary highest-weight representations of $SU(2)$ where each representation appears once and only once. This model structure easily survives Liouville dressing and characterizes (a certain subset of) the set of observables in the corresponding string model (most transparently in the chiral or open-string sector). Moreover, it predetermines, to some extent, the form of the algebra of observables in the latter theory.

The main logical steps in the scheme of [1,2] may be described as follows:

1) Consider first the matter (CFT) sector. Take one free scalar field $X$, compactified on a circle of radius $r = \sqrt{2}$ (i.e. $X \sim X + 2\pi r = X + 2\pi \sqrt{2}$), with a stress tensor $-\frac{1}{2}(\partial X)^2$. At this value of radius the original symmetry $U(1) \times U(1)$ enlarges to $SU(2) \times SU(2)$.
therefore one can consider chiral sector with symmetry $SU(2))$. This correspond to the self-dual point of the $c = 1$ Gaussian model [3,4] where $SU(2) \times SU(2)$ acts naturally on the Virasoro primaries of the theory\textsuperscript{3}. In what follows we would rather consider the holomorphic or open-string sector of the theory (like in [1]) which can be naturally decomposed with respect to single $SU(2)$ action. Then all the primary “chiral” vertex operators in such matter theory form a model of $SU(2)$: $M[SU(2)]$.

Comment: Thus, the loop chiral algebra of this theory, considered as CFT, $J_\pm = e^{\pm i \sqrt{2} x}$ and $J_0 = i \sqrt{2} \partial x$ ($x$ is the holomorphic part of the scalar field $X$). Then the generators $Q_\pm = \oint J_\pm$, $Q_0 = \oint J_0$ form usual $SU(2)$ algebra\textsuperscript{4}. The highest-weight operators (i.e. those annihilated by $Q_+$) are given by $\psi_{J,J} = e^{i \sqrt{2} J x}$ with any (half-)integer $J$. Since $SU(2)$ generators commute with the stress tensor, their action transform Virasoro primaries into primaries, and whole sequence $\psi_{J,m} \equiv Q_- \psi_{J,m+1}$ : beginning from $\psi_{J,J}$ consists of primary fields. For any (half-)integer $J$ this sequence is finite: $\psi_{J,-J} = e^{-i \sqrt{2} J x}$ is the lowest-weight operator (i.e. is annihilated by $Q_-$). The set $\{ \psi_{J,m} \}$ with $|m| \leq J$ and given $J$ forms a spin $J$ representation of $SU(2)$. All these operators $\psi_{J,m}$ are the Virasoro primaries and there are no other primaries\textsuperscript{4}. This last statement is not quite obvious since it is not easy to exclude any other candidates for the role of highest weight fields as well as the possible occurrence of non-highest-weight representation. The simplest way\textsuperscript{5} to confirm that no primaries are overlooked is to reproduce the holomorphic (or open-string) partition function from a sum over all characters of irreducible representations of the Virasoro algebra associated with the announced set of primaries\textsuperscript{6}.

Conclusion: Thus at the starting point one already discovers a model of $SU(2)$: it is formed by the set of primaries of corresponding conformal theory. This set is, of course, not the same as the entire spectrum of the matter conformal theory, since operator product expansion (OPE) in CFT contain also descendants, which have nothing to do with the structure of $M[SU(2)]$. In order to get a $M[SU(2)]$ structure one has to get rid of all

\textsuperscript{2}Thus, one can single out the self-dual radius as a point of enhanced symmetry group whose model we will be trying to reproduce.

\textsuperscript{3}Since $X \sim X + 2\pi \sqrt{2}$, $Q_0$ does not need to vanish.

\textsuperscript{4}For other compactification radii the set of primaries is different.

\textsuperscript{5}Alternative (and long) way is, for example, to thoroughly investigate the operator algebra.

\textsuperscript{6}An alternative description of the same chiral theory is in terms of the level $k = 1$ WZW model.
Virasoro-descendants.

2) In order to achieve this goal let us couple the matter sector to the $2d$ gravity, i.e. consider the corresponding string model. There is a well defined subsector in the space of observables, which consists of surface (for holomorphic part contour-) integrals of ($\text{matter} + \text{Liouville}$) primaries, provided the full dimension is one.

Comment: The drastic reduction of the space of vertex operators is a general phenomenon occurring whenever a string model is build from CFT. Let us remind first the situation with the critical string. In the simplest treatment, one neglects Liouville field at all, but this is not the only thing one does. There are three additional requirements to physical vertex operators in critical string models:

a) They do not contain ghost fields (in certain and natural picture)\(^7\).

b) They are Virasoro primaries (with respect to “full” Virasoro). (All the descendants, present in CFT are “gauged out”. The reason is that coupling to $2d$ gravity implies gauging the Virasoro algebra and thus all the descendants are eliminated as gauge-non-invariant operators. They really decouple from the correlators\(^8\)).

c) They are integrals (in the picture consistent with a)) of operators of conformal dimension one\(^9\).

Naive generalization of these three principles to the case of non-critical string has been suggested in [5,6], with the only modification made at the point b), where one is supposed to take primaries of the full ($\text{matter} + \text{Liouville}$) Virasoro algebra, in the form of ($\text{matter primaries}$)×($\text{Liouville primaries}$) and Liouville primaries being pure exponentials. It is, however, not quite true that such a simple generalization is valid. A problem arises at the very beginning point a). In order to derive honestly this requirement one should prove that in every BRST cohomology class there is a ghost free representative. According to [7,2] this is not true in non-critical case: there are two additional representatives of the cohomology classes, which unavoidably (and non-trivially) contain ghost fields. In the particular case of $c = 1$ model, the first ones were interpreted in [2] as a ground ring.

\(^7\)Except for one delicate point, concerning dilaton operator.

\(^8\)It is just no-ghost theorem.

\(^9\)Let us note that other (equivalent) pictures may be obtained by multiplication of dimension one integrands (constrained by a) and b)) with ghost field instead of integration - see also sect.3.
another one (the “ghost number two”) still has no nice interpretation. It seems, however, that the subsector, described by principles a),b),c) is closed by itself under OPE (modulo descendants, \textit{i.e.} fields vanishing in the correlation functions). Another delicate point is that the requirement c), \textit{i.e.} that the full dimension is one, permits two different choices of Liouville primaries associated with a given matter operator. In what follows we consider a subsector with one specific choice of these two. This subsector also seems to be closed under OPE in the above sense.

\textit{Conclusion:} In any string model (\textit{i.e.} CFT coupled to 2d gravity) a certain subset of observables is in one-to-one correspondence with the set of primaries of the matter theory. Therefore, if we start from the matter model, described in 1), the space of observables contains a subset, which is a \textit{model} of $SU(2)$: it is formed by the set of primaries of $M[SU(2)]$ and is closed under OPE modulo descendants. The corresponding vertex operators are of the form

$$q_{J,m} = \oint \psi_{J,m}(x)e^{(J-1)\sqrt{2}\phi}.$$  \hfill (1)

3) These operators form a Lie algebra $G$ (in contrast to the OPE in original CFT, which is not a Lie algebra), its structure constants being defined by 3-point correlation functions on a sphere.

\textit{Conclusion:} Thus, starting from 1) we obtain an embedding of $M[SU(2)]$ into a certain subset of algebra of observables, which turns to be a \textit{Lie} algebra $G[SU(2)]$. Moreover, according to 1) this embedding $M[SU(2)] \hookrightarrow G[SU(2)]$ is in fact a representation (\textit{i.e.} respects the structure of a \textit{model}):

a) $q_{1,m} = Q_m$, and thus form the \textit{adjoint} representation of $SU(2)$ in $G[SU(2)]$;

b) since $Q_m$ do not act on the Liouville field $\phi$, $\{q_{J,m}\}$ form \textit{model} of $SU(2)$: it is formed by the set of primaries of a spin $J$ representation generated by $\{Q_m\}$, in $G[SU(2)]$.

Thus, Klebanov-Polyakov-Witten (KPW) construction gives rise to \textit{model} $M[SU(2)]$ as representation of the Lie algebra $G[SU(2)]$. This algebra defines an additional structure: while on $M[SU(2)]$ (which is a priori a collection of representations of $SU(2)$ with every representation appearing exactly once) only these commutators are defined where, at least, one of the items is $Q_m = q_{1,m}$, in $G[SU(2)]$ the commutation of \textit{any} two elements
$q_{J',m'}$ and $q_{J'',m''}$ make sense.

4) Moreover, due to specific selection rules (dictated by the properties of Liouville sector) $[q_{J',m'}, q_{J'',m''}]$ contain a single representation $q_{J,m}$ with $J = J' + J'' - 1$ (and $m = m' + m''$):

$$[q_{J',m'}, q_{J'',m''}] = C_{J', J''}^{J' + J'' - 1} q_{J' + J'' - 1, m' + m''}.$$  \hspace{1cm} (2)

The coefficients $C[SU(2)]$ are $3j$-symbols (Clebsh-Gordon coefficients) of $SU(2)$ in a certain basis.

5) This Lie algebra with structure constants defined by $3j$-symbols $C_{J', J''}^{J' + J'' - 1}$ may be alternatively interpreted as an algebra of area-preserving diffeomorphisms of a 2-dimensional plane $\mathbb{R}^2 \sim \mathbb{C}$ (i.e. as an algebra of Hamiltonian vector fields on a plane, which can be also identified with a realization of $W_\infty$-algebra).

6) $\mathcal{G}[SU(2)]$ may be also identified with the algebra of derivatives of the ground ring which is formed by the other piece of the algebra of observables: of dimension zero and ghost number zero physical vertex operators [2,7]. The ground ring is, in fact, isomorphic to the ring of Hamiltonians (polynomials on $\mathbb{R}^2 \sim \mathbb{C}$).

This was a brief description of the original KPW construction for the case of $G = SU(2)$. If we omit all the details, the result is as follows:

Consideration of OPE of physical operators (modulo fields vanishing in physical correlators) in a certain $c = 1$ string model (1,2) suggests the representation of a model $M[SU(2)]$ in a Lie algebra $\mathcal{G}[SU(2)]$ (3), with structure constants defined by certain $3j$-symbols of $SU(2)$ (4) and isomorphic to an algebra $W_\infty$ of area-preserving diffeomorphisms of $\mathbb{R}^2 \sim \mathbb{C}$ (5), while the ring of Hamiltonians may be interpreted as the ground ring of dim = 0, #ghost = 0 physical operators (6).

This is a more or less nice description of the algebra of observables in the $SU(2)$-symmetric $c = 1$ string model and can serve as a sample example of what one should try to achieve when studying algebras of observables in other string models. Some fragments of

---

10 See Appendix.
such description are already available in $c < 1$ case, where an essential piece of the algebra of observables is given by the Virasoro- and $W$-constraints, which are the appropriate counterparts (and, in fact, fragments) of $W_\infty$ in the $c = 1$ case.

Another appealing property of the $c = 1$ example is its pure algebraic aspect: the possibility to introduce a new algebraic structure in the model of a simple Lie algebra $G = SU(2)$, which in this particular case is just a Lie-algebra commutator with the structure constants given by appropriate $3j$-symbols of original $G$, and, moreover, possessing an interpretation as an algebra of diffeomorphisms of an orbit of $G$.

It is very natural to look for generalizations of these results, which seem interesting both from physical and mathematical points of view. What we propose to do is to start from any simply laced algebra $G$ and try to work out the analogue of KPW construction as far as possible. This paper concerns with steps 1) and 2) of the above scheme, which are rather straightforward, but unavoidably involve the concepts of $W_G$-gravity and $W_G$-strings, which are still very poorly understood. We believe that their natural occurrence in the KPW construction demonstrates once again that $W_G$-gravities are not artificial “game of the mind” in the context of string theory for $c > 1$ and should attract much more attention to the interesting field of $W$-geometry [8-13].

In section 2 we consider the spectrum of physical fields in the chiral sector of $c = r_G$ $W_G$-string theory. We prove in sect.2.2 that the fact that they form a model of $G$ can be explicitely checked by the one-loop partition function calculations in chiral (or open) theory. In other words, we prove that the chiral (or open string) sector of $k = 1$ $G$-WZW theory forms, in fact, a model of compact Lie group $G$. In sect.2.3 we discuss the spectrum of matter theory by BRST methods and in section 3 consider coupling to $W_G$-gravity. Finally, the Appendix contains some facts about connection of model structure with symplectomorphism algebras on various coadjoint orbits of $G$. 

2 Primary fields for compactifications on root lattices

There are two different approaches to the problem – we present both of them. The first is technically simpler, the second – more sophisticated, but instead it emphasizes the important relation to the structure of Verma modules and null-vectors. We consider these two approaches separately.

2.1 Circle compactification (the case of $SU(2)$)

The model structure. Consider a theory of one free scalar field $X$, compactified on a circle, with Lagrangian $\int \partial X \bar{\partial} X$. Its chiral algebra usually contains $\hat{U}(1) \times \hat{U}(1)$, generated by $J_0 = \partial X$, and Virasoro algebra, generated by $T = -\frac{1}{2}(\partial X)^2$ (times their conjugate). The second one belongs to the $\hat{U}(1)$ universal envelope. Normally the set of primary fields in this theory is given by $e^{ipx}e^{i\bar{p}\bar{x}}$, where

$$ p + \bar{p} = \frac{n}{R}, \quad p - \bar{p} = 2mR, \quad (3) $$

$n,m \in \mathbb{Z}$, where $R$ is so-called radius of compactification [3,4] (to avoid misunderstanding we stress that it is really a half of real compactification radius: $x \sim x + 2\pi R$ and $\bar{x} \sim \bar{x} + 2\pi R$, but $x + \bar{x} = X \sim X + 2\pi r$, where $r = 2R$). However, sometimes the holomorphic chiral algebra is enlarged to become $S\hat{U}(2)_{k=1}$ generated by $J_{\pm} = e^{\pm i\sqrt{2}x}$ and $J_0$. This happens in the self-dual point when $R = 1/\sqrt{2}$ (this theory is, of course, related to the $\hat{SU}(2)_{k=1}$ WZW model, see below). In this case the set of primaries should be $SU(2) \times SU(2)$-invariant, for holomorphic or open-string theory it means that with any $e^{ipx}$ the spectrum should contain all non-vanishing $(Q_-)^k e^{ipx}$ (if $p > 0$, or $(Q_+)^k e^{ipx}$ if $p < 0$; $Q_{\pm} = \oint J_{\pm}$ are generators of $SU(2)$ and commute with the stress tensor). Whenever $p = integer \times \sqrt{2}$ this sequence is finite: $k \leq |p|/\sqrt{2}$ and form a spin $J$ ($J = |p|/2$) representation of $SU(2)$. Clearly every representation appears exactly once and we obtain a model of $SU(2)$.

$x$ and $\bar{x}$ denote holomorphic and anti-holomorphic parts of $X$ respectively.
Holomorphic partition functions as a character of a model. This conclusion is confirmed by the formula for one-loop partition function of the theory. Indeed, it equals \([3,4,14]\) for \(R = 1/\sqrt{2}\), \((q = e^{2\pi i \tau})\)

\[
Z(\tau, \bar{\tau}) = \frac{|\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau)|^2 + |\theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (2\tau)|^2}{|\eta(q)|^2}
\]

where \(\theta\) are ordinary Jacobi theta-functions, and coincides with the partition function of \(SU(2)_{k=1}\) WZW theory. Let us introduce the quantity

\[
Z(\tau) = \frac{\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau)}{\eta(q)} + \frac{\theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (2\tau)}{\eta(q)} .
\]

This can be interpreted as a “holomorphic square root” of partition function or, what is essentially the same, as partition functions of the corresponding open-string models [15-17]. This holomorphic partition function can be represented as a certain sum of the Virasoro characters over all the Virasoro primaries, appearing in the spectrum of the theory. In accordance with our arguments above (2) exactly the model of all representations of \(SU(2)\):

\[
Z(\tau) = \sum_{J \in \mathbb{Z}+/2} (2J + 1) \chi_J(\tau)
\]

Here \(\chi_J(\tau)\) denote the Kac characters of representations Virasoro algebra for \(c = 1\) [18]:

\[
\chi_J(\tau) = \frac{\eta(J^2) - \eta(J^2+1)^2}{\eta(q)} .
\]

and multiplicities \((2J + 1) = \dim R_J\) reflect that we indeed have the model of \(SU(2)\).

Substitution of (7) into (6) gives:

\[
\eta(q)Z(\tau) = \frac{1}{2} \left\{ \sum_{J \in \mathbb{Z}+/2} (2J + 1) \chi_J(\tau) + (J \rightarrow -J - 1) \right\} = \sum_{n=-\infty}^{\infty} q^{(n/2)^2} = \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\tau/2) = \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau) + \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (2\tau),
\]

\(^{12}\)This name may be to some extent motivated by the famous examples of \(E(8)\) and \(SO(32)\) when the weight lattices are equivalent to the root ones (even self dual lattices \(\Gamma_8\) and \(\Gamma_{16}\) respectively) and there is only a single term in the sum (1).
in accordance with (5).

**KRC-formula from a limiting procedure.** As to the formula (7), it may be easily derived\textsuperscript{13} from the well-known expressions [19] for the Rocha-Caridi characters of the Virasoro algebra for \( c = 1 - \frac{6(p - p')^2}{pp'} \equiv 1 - 12\alpha_0^2 \),

\[
\chi_{m,n}(\tau) = \frac{q^{-\frac{1}{4pp'}}}{\eta(q)} [\theta \left( \frac{mp - np'}{2pp'} \right) (0|2pp'\tau) - \theta \left( \frac{mp + np'}{2pp'} \right) (0|2pp'\tau)] = \frac{1}{\eta(q)} \left[ q^{\frac{1}{4pp'}}((mp - np')^2 - 1) - q^{\frac{1}{4pp'}}((mp + np')^2 - 1) + \ldots \right],
\]

in the limit \( p' = p + 1 \rightarrow \infty \) (condition \( p' = p + 1 \) is imposed in order to guarantee the unitarity\textsuperscript{14}). Irreducible representations of Virasoro algebra are unambiguously labeled by their dimensions and the character (5) corresponds to

\[
\Delta_{m,n} = \frac{(m\alpha_+ + n\alpha_-)^2 - (\alpha_+ + \alpha_-)^2}{8}, \tag{10}
\]

Here \( \alpha_\pm = \alpha_0 \pm \sqrt{2 + \alpha_0^2} \), \( \alpha_+ = \sqrt{2p/p'} \), \( \alpha_- = -\sqrt{2p'/p} \), \( \alpha_+\alpha_- = -2 \), \( \alpha_+ + \alpha_- = 2\alpha_0 \). In the limit \( p' = p + 1 \rightarrow \infty \) these parameters turn into \( \alpha_\pm \rightarrow \pm \sqrt{2} \), \( \alpha_0 \rightarrow 0 \) and \( \Delta_{m,n} \rightarrow \tilde{\Delta}_{m,n} \equiv \frac{(n - m)^2}{4} \).

The algebraic sum at the r.h.s. of (5) arises because one needs to eliminate the sub-modules from the Verma module, which are generated by null-vectors. Such null-vectors arise at the levels \( n \cdot m \), \( (p - n) \cdot (p' - m) \), etc. in the original module and are naturally decomposed into two different sets. The levels from one set depend strongly on \( p' \) and/or \( p \) and go to infinity when \( p' = p + 1 \rightarrow \infty \). This leads to a drastic simplification in the case of \( c = 1 \): all intersections of sub-modules disappear, the remaining sub-modules form a single “tower” of embedded modules, and instead of an infinite algebraic sum in (5) a single subtraction survives: \( \chi_{m,n}(\tau) \rightarrow \tilde{\chi}_{m,n}(\tau) \equiv \frac{1}{\eta(q)} \left[ q^{\frac{(n-m)^2}{4}} - q^{\frac{(n+m)^2}{4}} \right] \). This is, however, not the whole story: \( \tilde{\chi}(\tau) \) does not need to be a character of an irreducible

\textsuperscript{13}This derivation of a character for a trivial case from a non-trivial one is of course not the most clever thing to do, but these Rocha-Caridi characters arise in the study of minimal models and thus their \( c < r_G \) analogues are much better known than the analogues of \( c = 1 \) characters. What is even more important, for \( W \)-algebras the counterparts of these “minimal” Rocha-Caridi characters are easily available [23], and it will be simpler for us below to take their limits as \( c \) approaches \( r_G \), than perform an independent derivation of \( W_G \)-characters for \( c = r_G \).

\textsuperscript{14}Really we only need the difference \( p' - p \) to be finite when \( p \rightarrow \infty \).
representation at \( c = 1 \). This is explained by appearance of new null-vectors in the limit \( c = 1 \), which were absent for all values of \( c < 1 \) (\( \alpha_0 \neq 0 \)). This is indeed the case, unless either \( m \) or \( n \) is equal to 1. In fact, this is already clear from the fact, that all dimensions of primaries \( \Delta_{m,n} \) (10) with given \( n - m \) coincide in the limit \( p' = p + 1 \rightarrow \infty \), and thus all such representations of the Virasoro algebra should become identical. It means that representations with \( n \) (or \( m \)) \( \neq 1 \) should acquire new null-vectors when \( c \) becomes 1.

To make this statement a bit more formal, let us introduce a primary field

\[
\Phi_{m,n} = \exp\left\{ \frac{i}{2}[(m + 1)\alpha_+ + (n + 1)\alpha_-]x \right\}
\]

(11)

associated with the character \( \chi_{m,n} \). Denote by \( L_{-N} = L_{-N} + \ldots \) the algebraic combination of Virasoro generators \( L_{-i} \) \( (i > 0) \), which can create the null-vector in the \( \Phi_{m,n} \) module at level \( N \). Then for any minimal model there exists a primary field at level \( N = mn = 2Jn + n^2 \), but only for \( \alpha_0 = 0 \) a new primary field appears also at \( N = m = 2J + 1 \) (here \( 2J = m - n \)); in terms of zero norms it means that \( \|L_{-N}\Phi_{m,n}\|^2 = 0 \) for \( N = mn = 2Jn + n^2 \), while \( \|L_{-N}\Phi_{m,n}\|^2 \sim \alpha_0^2 \) for \( N = 2J + 1 \). Thus, a new primary field of this kind appears exactly in the limit \( \alpha_0 \rightarrow 0 \), i.e. when \( c \rightarrow 1 \). For example, in the simplest case of \( N = 2 \) the possible candidate to be the new primary field at the second level should have the form

\[
[a(\partial x)^2 + b\partial^2 x)e^{i\alpha x}.
\]

(12)

One has to require the OPE of (12) with the stress-tensor \( -\frac{1}{2}(\partial x)^2 + i\alpha_0\partial^2 x \) being of the usual form for the primary field, in particular this means that third- and fourth-order poles in the OPE vanish, what corresponds to

\[
\begin{cases}
-a + 2i(3\alpha_0 - \alpha)b = 0 \\
i(2\alpha_0 - \alpha)a + b = 0
\end{cases}
\]

(13)

The system (13) has non-zero solutions only with zero determinant, i.e.

\[
6\alpha_0^2 - 5\alpha_0\alpha + \alpha^2 = \frac{1}{2}
\]

(14)

with obvious solutions
\[ \alpha = \frac{5}{2} \alpha_0 \pm \frac{1}{2} \sqrt{\alpha_0^2 + 2} . \]  

(15)

Comparing (15) (for example for the positive sign) with the formula (11):

\[ \alpha = \alpha_{m,n} = \frac{1}{2} [(m + 1)\alpha_+ + (n + 1)\alpha_-] = \frac{m + n + 2}{2} \alpha_0 + \frac{m - n}{2} \sqrt{\alpha_0^2 + 2} , \]  

(16)

we get two distinguished cases: for \( \alpha_0 \neq 0 \), \( m = 2, n = 1 \) give the only solution, while for \( \alpha_0 = 0 \) we get the only restriction on \( m \) and \( n \): \( m - n = 1 \), which means that in the modules for \( (m,n) = (n+1,n) \) for \( n > 1 \) new primary fields appear when \( \alpha_0 \to 0 \) at the second level.

In terms of the null-vector condition this means that

\[ \mathcal{L}_{-2}\Phi_{m,n} = [T - (1 + \alpha_0 \alpha_+) \partial^2] e^{i \alpha_{m,n} x} = \]

\[ = \left[ -\frac{1}{2} (\partial x)^2 + i(\alpha_0 + \alpha_{m,n}) \partial^2 x - \right. \]

\[ - (1 + \alpha_0^2 + \alpha_0 \sqrt{\alpha_0^2 + 2})(-\alpha_{m,n}^2 (\partial x)^2 + i\alpha_{m,n} \partial^2 x) \left. \right] e^{i \alpha_{m,n} x} = \]

\[ = \left[ \frac{1}{2} \{(m - n)^2 - 1\}(\partial x)^2 \right] e^{i \alpha_{m,n} x} + \]

\[ + \alpha_0 \times \left\{ \left[ \alpha_0 \left( \frac{(m + n + 2)^2}{2} \right)^2 + \left( \frac{m - n}{2} \right)^2 \right] + \right. \]

\[ + \frac{1}{2} \sqrt{\alpha_0^2 + 2}(m + n + 2)(m - n) (1 + \alpha_0^2 + \alpha_0 \sqrt{\alpha_0^2 + 2}) + \]

\[ + \frac{1}{2} (m - n)^2(\alpha_0 + \sqrt{\alpha_0^2 + 2}) \left\} (\partial x)^2 + \]

\[ + (1 - \alpha_{m,n}(\alpha_0 + \sqrt{\alpha_0^2 + 2})) i\partial^2 x \right\} e^{i \alpha_{m,n} x} . \]

(17)

For \( \alpha_0 \to 0 \) (and only in this case) this has additional zeros for \( m - n = 1 \):

\[ \mathcal{L}_{-2}\Phi_{n+1,n} = \alpha_0 \left\{ \frac{1}{2} \sqrt{2}(2n + 3) (\partial x)^2 \right\} e^{i \alpha_{n+1,n} x} + O(\alpha_0^2) \to 0 . \]  

(18)

Therefore

\[ \| \mathcal{L}_{-2}\Phi_{n+1,n} \|^2 \sim \alpha_0^2 \]  

(19)

for \( N = 2J + 1 = 2 \), where \( 2J = m - n = 1 \), and for any \( n \). The only kind of representation which does not acquire any new null-vectors at the point \( c = 1 \) has \( n = 1 \) (in our
conventions when \( m > n \), and the corresponding \( \tilde{\chi}_{m,1}(\tau) = \frac{1}{\eta(q)}[q^{\tilde{\Delta}_{m,1}} - q^{\tilde{\Delta}_{m,-1}}] \) becomes exactly the Kac-Rocha-Caridi characters (9) with \( 2J = m - 1 \).

Our next purpose is to generalize this more or less widely known construction from the case of \( SU(2) \) to other simply laced algebras (only simply laced algebras are just so simply related to the torus compactifications: in the case of simply laced algebras the Kac-Moody algebra with \( k = 1 \) possesses an obvious free-field representation in terms of \( r_G \) scalar fields).

2.2 Torus compactifications

*The origin of the model structure. Relevance of \( W_G \)-algebra.* Proceed now to the case of arbitrary simply laced algebra \( G \). Namely, consider the set of \( r_G = \text{rank}(G) \) free fields \( X = \{X_1, \ldots, X_{r_G}\} \) with Lagrangian \( \partial X \bar{\partial} X \). Usually the chiral algebra in the \( r_G \)-dimensional free field theory is \( \hat{U}(1)^{r_G} \times \hat{U}(1)^{r_G} \) and the Virasoro algebra is generated by \( T = -\frac{1}{2} \partial X \bar{\partial} X \). However, now it is reasonable to consider not only a Virasoro sub-algebra of the chiral-algebra, but larger (higher-spin) \( W_G \)-algebra, essentially generated by operators like \( \sum (\mu \partial X)^{\sigma} \). Compactification is defined by \( r \)-dimensional lattice \( \Gamma = \{\gamma\} \): \( X \sim X + 2\pi \gamma \). Naive Virasoro primaries are \( e^{px} e^{\bar{p}x} \), where

\[
\begin{align*}
p &= \gamma^* + \frac{1}{2} \gamma, \\
\bar{p} &= \gamma^* - \frac{1}{2} \gamma; \\
\gamma &\in \Gamma, \quad \gamma^* \in \Gamma^* ,
\end{align*}
\]

where \( \Gamma^* \) is dual lattice to \( \Gamma \), *i.e.* \( \gamma \gamma^* = \text{integer} \). However, for particular compactification on lattice, the chiral algebra may become \( G_k=1 \) (of course, any subalgebra \( H \subset G \) is also allowed), generated by \( J_\alpha = e^{i\alpha x}, \ H_\nu = \nu \partial x \), where \( \alpha \) are all the roots of \( G \) and \( \nu \) form some basis in the Cartan (hyper)plane. This happens whenever \( \Gamma \) (\( \gamma \in \Gamma \)) is the *root lattice* of \( G \); then the charges \( Q_\alpha = \oint J_\alpha \) and \( Q_\nu = \oint H_\nu \), which generate the action of the global symmetry algebra \( G \) commute with the stress tensor, and the Virasoros primaries should form representations of \( G \). This implies, that just as it was in the \( SU(2) \) case along with the “naive” primaries (or tachyons) \( e^{px} e^{\bar{p}x} \) there should be also “non-naive”, arising from the naive ones by the action of \( G \). However, when \( r_G > 1 \)
this is not the whole story: the set of non-tachyon Virasoro primaries is much bigger. (In critical string theory this is a well-known phenomenon – there are much more primary fields than only tachyons: gravitons and other higher spin excitations.) Therefore in order to diminish the set of primary fields and make it related to the *model* of $G$ (where every representation appears exactly once), it is reasonable to consider a set of $W_G$-primaries. Operators of $W_G$-algebra are essentially given by $\sum_{a=0}^{\nu_a=0} (\nu_a \partial x)^n$ with $n = 1, \ldots, r_G$ ($\nu_a$ are certain vectors in the Cartan (hyper-) plane, related to the fundamental weights, see [20]). The $W_G$-algebra (or its universal envelope) can be defined as a piece of the chiral algebra (universal envelope of $\hat{G}_1$ in our case), which commutes with all the charges $Q_\alpha, Q_\nu$. Therefore all $W_G$-primaries still need to form multiplets of $G$, and the set of $W_G$-primaries form the *model* $M[G]$. In order to demonstrate that there are no more $W_G$-primaries (which is not true for example for the only Virasoro-primaries) we refer to the formulas for the one-loop partition function (just as we did in the $SU(2)$ case):

$$Z(\tau, \bar{\tau}) = |\eta(q)^{-r_G}|^2 \sum_{\nu \in \Gamma^* / \Gamma} \sum_{\epsilon} \left| \Theta \left[ \nu + \epsilon \atop 0 \right] (\tau) \right|^2$$

with $\epsilon$ running over set of vectors $\{1/2e_i\}$ and $0$ ($\{e_i\}$ being the basis of lattice $\Gamma$) [14]. The item with $\epsilon = 0$,

$$Z(\tau, \bar{\tau}) = |\eta(q)^{-r_G}|^2 \sum_{\nu \in \Gamma^* / \Gamma} \left| \Theta \left[ \nu \atop 0 \right] (\tau) \right|^2,$$

is modular invariant by itself and is, in fact, a one-loop partition function of the $k = 1$ WZW model for a simply laced $G$ [21]. The corresponding 1-loop partition function in “chiral” or “open” sector is

$$Z(\tau) \equiv \eta(q)^{-r_G} \sum_{\nu \in \Gamma^* / \Gamma} \Theta \left[ \nu \atop 0 \right] (\tau) = \sum_{\Lambda \in \Gamma^+} D_{\Lambda} \chi_{\Lambda}(\tau).$$

Here we label the highest weight representations $R_G[\Lambda]$ of $G$ by the highest weight vectors $\Lambda$, lying in a “positive” Weyl chamber $\Gamma^+$. Dimension of $R_G[\Lambda]$ is given by a product over all *positive* roots $\alpha$ [22]:

15
\[ D_\Lambda = \prod_{\alpha \in \Delta_+} \frac{\langle \Lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \]  

(\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha, \langle , \rangle \text{ is the scalar product in the Cartan (hyper-)plane). According to our above reasoning the same } \Lambda \text{ can be used to label the } W_G \text{-primaries, and thus the irreducible representations } R_{W_G}[\Lambda] \text{ of the } W_G \text{-algebra with } c = r_G. (Note that } R_{W_G}[\Lambda] \text{ is certainly not the same as } R_G[\Lambda]: \text{ these are representations of different algebras!). For the sake of brevity we use the symbol } \chi_\Lambda(\tau) \text{ instead of } \chi[R_{W_G}[\Lambda]](\tau) \text{ to denote the analogues of the Virasoro Kac-Rocha-Caridi characters of the irreducible representation } R_{W_G}[\Lambda] \text{ with conformal dimensions } \Delta_\Lambda = \frac{1}{2} \Lambda^2:

\[ \chi_\Lambda(\tau) = \eta(q)^{-r_G} \sum_{\sigma \in \mathcal{W}} \det(\sigma) q^{\frac{1}{2}(\Lambda + \rho - \sigma(\rho))^2}. \]  

(24)

Here } \mathcal{W} \text{ is the finite Weyl group of the algebra } G \text{ and } \det(\sigma) \text{ stands for the determinant of transformation } \sigma \in \mathcal{W}, \text{ (for } G = SU(2) \text{ the Weyl group is } \mathcal{W} = \mathbb{Z}_2, \text{ thus there are only two items in the sum (24) and we return to eq.(7)). These characters depend only on dimension } \Delta \text{ and therefore are the same for all the primaries } D_\Lambda, \text{ belonging to representation } R[\Lambda]. \text{ Thus they give equal contributions to the sum (22), that is why the factors } D_\Lambda \text{ appear in front of corresponding } \chi' \text{'s. It is eq.(22) that confirms our statement that } W_G \text{-primaries form model of } G. \text{ Thus it remains to explain the origin of eqs.(21-22) and (24).}

Proof of eq.(22). The proof is a bit more subtle but very close to that of the } SU(2) \text{ case.}

Let us calculate the sums at the r.h.s. of eq.(22). First of all, note, that these sum can be taken over entire weight lattice. Indeed, as a simple corollary of eqs.(23) and (24) we have for any } \sigma \in \mathcal{W} \text{ and weight } \nu:

\[ D_\nu \chi_\nu(\tau) = D_{\nu \circ} \chi_{\nu \circ}(\tau), \]  

(25)

where } \nu \circ = \sigma(\nu) + \sigma(\rho) - \rho. \text{ This implies that for any lattice } \mathcal{T}

\[ \sum_{\nu \in \mathcal{T}_+} D_\nu \chi_\nu(\tau) = \frac{1}{\text{ord}\mathcal{W}} \left\{ \sum_{\nu \in \mathcal{T}} D_\nu \chi_\nu(\tau) \right\}, \]  

(26)
where $\text{ord} \mathcal{W}$ is the order (the number of elements) of Weyl group, $\mathcal{T}_+$ is intersection of $\mathcal{T}$ with the Weyl chamber, and $\hat{\mathcal{T}}$ is the union of specifically shifted images of $\mathcal{T}_+$ under all the transformations from the Weyl group:

$$\hat{\mathcal{T}} = \bigcup_{\sigma \in \mathcal{W}} [\sigma(\mathcal{T}_+) + \sigma(\rho) - \rho]. \quad (27)$$

In general, $\hat{\mathcal{T}}$ does not need to coincide with original lattice $\mathcal{T}$. This is true for root lattice $\hat{\Gamma} = \Gamma$, but for the case we consider now: $\hat{\Gamma}^* \neq \Gamma^*$. (In the simplest example of $SU(2)$ $\Gamma^* = \{n/\sqrt{2}, n \in \mathbb{Z}\}$, $\rho = 1/\sqrt{2}$, $\Gamma_+^* = \{n/\sqrt{2}, n \in \mathbb{Z}, n \geq 0\}$ and $\hat{\Gamma}_+^* = \{n/\sqrt{2}, n \in \mathbb{Z}, n \neq -1\}$. Thus the difference between $\Gamma_+^*$ and $\hat{\Gamma}_+^*$ consists of a single point $\nu = -\rho$, and, according to (23) this is exactly the point where $D_{-\rho} = 0$, i.e. which does not contribute to the sum at the r.h.s. of (26), so that it can be taken over the whole lattice $\Gamma_+^*$. The last statement is also true for all other simply laced algebras $G$: in general the difference between $\Gamma^* + \hat{\Gamma}^*$ and $\Gamma^* + \hat{\Gamma}^*$ is no longer a point, but consists of hyperplanes of codimension 1, such that for any $\nu \in \Gamma^* - \hat{\Gamma}^*$ the sum $\nu + \rho$ is orthogonal at least to one of the positive roots and thus, according to (23) the corresponding $D_{\nu} = 0$, and the sum at the r.h.s. of (26) can be taken over entire $\Gamma^*$ instead of $\hat{\Gamma}^*$. We conclude that (up to a factor $\text{ord} \mathcal{W}$) the sum at the r.h.s. of eq.(26) is over entire lattices $\Gamma^*$.

Now the calculation can be done as follows. First, we have

$$Z(\tau) = \sum_{\nu \in \Gamma_+^*} D_{\nu} \chi_{\nu}(\tau) = \frac{1}{\text{ord} \mathcal{W}} \sum_{\nu \in \Gamma^*} D_{\nu} \chi_{\nu}(\tau) = \frac{1}{\text{ord} \mathcal{W}} \sum_{\nu \in \Gamma^*} D_{\nu} \chi_{\nu}(\tau), \quad (28)$$

when, we have also used that $D_{\nu} = 0$ for $\nu \in \Gamma^* - \hat{\Gamma}^*$. Substituting (28), and changing the summation variable $\Lambda = \nu + \rho - s(\rho)$ one gets

$$\eta(q)^{\tau c} Z(\tau) = \frac{1}{\text{ord} \mathcal{W}} \sum_{\nu \in \Gamma^*} \sum_{s \in \mathcal{W}} \prod_{\alpha \in \Delta_+} \frac{\langle \nu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \det(s) q^{1/2[\nu + \rho - s(\rho)]^2} = \frac{1}{\text{ord} \mathcal{W}} \sum_{s \in \mathcal{W}} \det(s) \sum_{\Lambda \in \Gamma^*} \prod_{\alpha \in \Delta_+} \frac{\langle \Lambda + s(\rho), \alpha \rangle}{\langle \rho, \alpha \rangle} q^{1/2 \Lambda^2} = \frac{1}{\text{ord} \mathcal{W}} \sum_{s \in \mathcal{W}} \det(s) \sum_{\Lambda \in \Gamma^*} q^{1/2 \Lambda^2} \quad (29)$$

17
(we do not need to take care about boundary terms since we sum now over entire lattice $\Gamma^*$), and it has been used that $s^2 = 1$ for $s \in W$, as well as:

$$\sum_{\sigma \in W} \text{det}(\sigma) \frac{\langle \Lambda + \sigma(\rho), \alpha \rangle}{\langle \rho, \alpha \rangle} = \text{ord} W$$

for any $\Lambda$. Finally, the r.h.s. of eq.(29) is equal to

$$\sum_{\nu \in \Gamma^*} q^{\nu^2/2} = \sum_{\nu \in \Gamma^*/\Gamma} \left\{ \sum_{\lambda \in \Gamma} q^{(\lambda + \nu)^2/2} \right\} = \sum_{\nu \in \Gamma^*/\Gamma} \Theta \left[ \frac{\nu}{0} \right] (\tau)$$

where the lattice $\Theta$-function is defined as a sum over the root lattice $\Gamma$. This proves eq.(22) provided (24) is correct.

Characters (24) as the limit of Fateev-Lukyanov characters. In order to derive the formula (24) for the characters of irreducible $W_G$-representations we will just apply the same trick as we demonstrated in the case of $SU(2)$. Namely, let us make use of the known $W_G$-characters for the “minimal” series, arising for certain values of $c < r_G$

$$c = r_G - 12\alpha_0^2 \rho^2 = r_G - 6(\rho - \rho')^2$$

and then take the limit $c \to r_G$, or $\alpha_0 \to 0$. According to [23]

$$\chi_{\Lambda_1, \Lambda_2}(\tau) = \eta(q)^{-r_G} \sum_{s_1, s_2 \in W} \text{det}(s_1) \text{det}(s_2) \text{ord} W \Theta \left[ \frac{ps_1 \Lambda_1 - p's_2 \Lambda_2}{0} \right] (pp') =$$

$$= \eta(q)^{-r_G} \sum_{s \in W} \text{det}(s) \sum_{\alpha \in \Gamma} \exp \left( \frac{i\pi \tau}{pp'} (ps \Lambda_1 - p' \Lambda_2 - 2pp' \alpha)^2 \right)$$

After one takes the limit $p \to \infty$, $p' \to \infty$ so that $p' - p$ is fixed and finite only the term with $\alpha = 0$ from the whole sum over root lattice survives in (33). After a conventional redefinition $\Lambda_i \to \Lambda_i + \rho$ and setting $\Lambda_1 = 0$ (the analog of $n = 1$ condition in the $SU(2)$ case) and $\Lambda_2 \equiv \Lambda$ one recognizes the formula (24).

2.3 Virasoro and $W$-primaries from BRST formalism

There exists a somewhat different way to observe the complicated structure of primary fields in $c = 1$ and $c = r_G$ string theories (see for example [2], another example was in fact given in sect.2.1. above when we discussed the occurrence of new null-vectors in the limit
of $c = 1$). Since it illustrates some new features of these theories we mention this kind of ideas here without going into many details. Let us start from uncompactified theory where any (and not essentially rational) values of momenta are admissible. For generic momenta (and dimension $p^2/2$) there is a single Virasoro or $W$-primary: $\Phi_p = e^{ipX}$. Occurrence of non-trivial primaries for certain values of $p$ can be observed as follows. Apply some algebraic combination of Virasoro and/or $W$-operators to $\Phi_p$ and look at the norm of the resulting operator, $\| P(p) \|_2$ as a function of $p$. If this function is vanishing at the point $p_o$, one can introduce a new field of non-vanishing norm, $\lim_{p \to p_o} \{ |p - p_o|^{-1} P(p) \Phi_p \}$, which is a new non-naive primary field. We present an example of this phenomenon (to be added to those already given in [2]).\footnote{Note that this mechanism of generating new primaries as null-vectors is not the only possible. In fact we consider cokernel of the chiral algebra. If we discuss Virasoro (rather than appropriate $W$-) algebra for $r_G > 1$ the cokernel is very large and there is plenty of Virasoro primaries which are neither tachyonic operators, nor can be described as null-vectors, as we have in critical string theory.}

First, we demonstrate how it works for $c = 1 - 12\alpha_0^2$ and illustrate the occurrence of non-naive Virasoro primaries at appropriate values of momentum $p$. Again we shall discuss only the simplest non-trivial example of the null-vector at level $N = 2$. Consider the Virasoro null-vector in the module of $e^{ipX}$. The non-positive Virasoro generators act as

\begin{align}
L_0 e^{ipX} &\sim (p^2/2 - \alpha_0 p) e^{ipX}; \\
L_{-1} e^{ipX} &\sim ip\partial X e^{ipX}; \\
L_{-2} e^{ipX} &\sim \left[ i(p + \alpha_0) \partial^2 X - \frac{1}{2} (\partial X)^2 \right] e^{ipX}; \\
L_{-1}^2 e^{ipX} &\sim \left[ ip\partial^2 X - p^2 (\partial X)^2 \right] e^{ipX}.
\end{align}

(34)

Then

\begin{align}
L_{-2}^{(p)} e^{ipX} = \{ L_{-2} + aL_{-1}^2 \} e^{ipX} = \left\{ i[p - \alpha_0 + ap] \partial^2 X + \left[ -\frac{1}{2} - ap^2 \right] (\partial X)^2 \right\} e^{ipX}.
\end{align}

(35)

To get in (35) zero, one has to put

15Note that this mechanism of generating new primaries as null-vectors is not the only possible. In fact we consider cokernel of the chiral algebra. If we discuss Virasoro (rather than appropriate $W$-) algebra for $r_G > 1$ the cokernel is very large and there is plenty of Virasoro primaries which are neither tachyonic operators, nor can be described as null-vectors, as we have in critical string theory.
\[ ap = -(p - \alpha_0), \]
\[ p^2 = -\frac{1}{2a}. \]  
The formulae (36) certainly coincide with usual second level null-vector conditions (see, for example [24])

\[ c = 1 - 12\alpha_0^2 = 2\Delta(5 - 8\Delta)/(2\Delta + 1) = 1 - (4\Delta - 1)^2/(2\Delta + 1); \]
\[ a = -3/(2\Delta + 1); \alpha_0 = (4\Delta - 1)/2\sqrt{3(2\Delta + 1)} \]  
for \( \Delta = p^2/2 - \alpha_0 p. \)

Now we shall turn to the example which is the simplest possible, involving higher \( W \)-operators. Let us take \( G = SU(3), \) \( c = 2 (\alpha_0 = 0), \) level \( N = 2. \) Then

\[ T = -\frac{1}{2} \sum_{\nu}(\nu \partial X)^2 = -\frac{1}{2}(\partial X)^2; \quad W = \sum_{\nu}(\nu \partial X)^3. \]  
The Virasoro and \( W \)-modes act as

\[ L_{-1} e^{pX} = ip\partial X e^{pX}; \]
\[ L_{-2} e^{pX} = \left[i p\partial^2 X - \frac{1}{2}(\partial X)^2\right] e^{pX}; \]
\[ L_{-1}^2 e^{pX} = \left[i p\partial^2 X - (p\partial X)^2\right] e^{pX}. \]  
Now the only parameter \( a \) is not enough ensure vanishing of the sum \( \{L_{-2} + aL_{-1}^2\} e^{pX} \) for any \( p. \) Thus in order to get null-vectors we need to consider “mixed” Virasoro–\( W \) action (\( i.e. \) there is no null-vector built from only the Virasoro modes). For the \( W \)-operators one has:

\[ W_{-1} e^{pX} = 3 \sum_{\nu}(\nu \partial X)(ip\nu)^2 e^{pX}; \]
\[ W_{-2} e^{pX} \sim 3 \sum_{\nu}[(\nu \partial^2 X)(ip\nu)^2 + (\nu \partial X)^2(ip\nu)] e^{pX}; \]
\[ W_{-1}^2 e^{pX} \sim 9 \sum_{\nu,\nu'} (\nu \partial X)^2 (\nu \nu')(ip\nu)^2 e^{pX}; \]
\[ L_{-1} W_{-1} e^{pX} \sim 3 \sum_{\nu}[(\nu \partial^2 X)(ip\nu)^2 + (\nu \partial X)(ip\partial X)(ip\nu)^2] e^{pX}. \]  
The formulae for the particular case of \( SU(3) \) read:
\[ W \sim (\partial X_1)^3 - 3(\partial X_1)(\partial X_2)^2; \]
\[ \frac{1}{3}W_{-1}e^{pX} \sim [(p_2^2 - p_1^2)\partial X_1 + 2p_1p_2\partial X_2]e^{pX}; \]
\[ \frac{1}{3}L_{-1}W_{-1}e^{pX} \sim [(p_2^2 - p_1^2)\partial^2 X_1 + 2p_1p_2\partial^2 X_2 + \]
\[ + \{(p_2^2 - p_1^2)\partial X_1 + 2p_1p_2\partial X_2\}(ip\partial X)]e^{pX}; \]
\[ \frac{1}{9}W_{-1}^2e^{pX} \sim \{(p_2^2 - p_1^2)\partial X_1 + 2p_1p_2\partial X_2\}^2 + \]
\[ + [(p_2^2 - p_1^2)[(\partial X_1)^2 - (\partial X_2)^2] - 4p_1p_2(\partial X_1\partial X_2)] + \]
\[ + [2(p_2^2 - p_1^2)[ip_1\partial^2 X_1 - ip_2\partial^2 X_2] - 4p_1p_2[ip_2\partial^2 X_1 + ip_1\partial^2 X_2]]e^{pX}; \]
\[ \frac{1}{3}W_{-1}e^{pX} \sim \{(p_2^2 - p_1^2)\partial^2 X_1 + 2p_1p_2\partial^2 X_2 + ip_1[(\partial X_1)^2 - (\partial X_2)^2] - \]
\[ - 2ip_2(\partial X_1\partial X_2)\}e^{pX}. \]

Combining (41), we see that
\[ [L_{-2} - L_{-1}^2]e^{pX} = \left[(p\partial X)^2 - \frac{1}{2}(\partial X)^2\right]e^{pX}. \tag{42} \]

and
\[ [(\alpha/3)L_{-1}W_{-1} + (\beta/9)W_{-1}^2 + (\gamma/3)W_{-2}]e^{pX} \sim \]
\[ \sim \left\{ \partial^2 X_1[\alpha(p_2^2 - p_1^2) + \gamma(p_2^2 - p_1^2) + 6\beta(p_2^2 - p_1^2)p_1] + \right. \]
\[ + \partial^2 X_2[2\alpha p_1p_2 + 2\gamma p_1p_2 - 6\beta(p_2^2 - p_1^2)p_2] + \]
\[ + (\partial X_1)^2[\alpha p_1(p_2^2 - p_1^2) + \beta p_1^2p_2] + (\partial X_1)^2[\gamma p_1^2(p_2^2 - p_1^2)] + \]
\[ + \partial X_1\partial X_2[\alpha p_2(p_2^2 - p_1^2) + 2i\alpha p_1^2p_2 + 2i\gamma p_1] + \right\} e^{pX}. \tag{43} \]

The solution to the null-vector conditions (43) exists, when
\[ \beta = 0, \gamma = -2\alpha^2 = \sqrt{6}/4. \tag{44} \]

(in our conventions \(X_1\) is chosen along \(\mu = (0, \sqrt{2/3})\)) and looks like
\[ p_1 = \pm 1/\sqrt{6}; p_2 = \pm 1/\sqrt{2}; \]
\[ p_1 = \pm \sqrt{2/3}; p_2 = 0 \tag{45} \]
and corresponds to six fundamental weights of 3 and $\bar{3}$.

## 3 Coupling to $W_G$-gravity

We demonstrated in the previous section that the $W_G$-primaries in the matter theory, compactified on the root lattice of the Lie algebra $G$, form the *model* of $G$. However, the spectrum of this matter theory which is just an ordinary conformal theory is by no means exhausted by the primaries: there are many $W_G$-descendants, and these do not respect in any obvious way the structure of the *model*, which is of the main interest for us. Therefore, if we want to distinguish this *model* structure we need to get rid of descendants. The simplest way to do this is just to gauge the $W_G$-algebra, *i.e.* consider the $W_G$-string model: by coupling to $W_G$-gravity we will cancel the contribution of all the $W_G$-descendants to the physical correlation functions. In this short section we demonstrate that this transition, which essentially involves dressing of matter primary fields by appropriate $W$-Toda fields, indeed preserves the $G$-*model* structure, which therefore becomes the feature of the set of observables in this string theory (again we will consider a kind of “chiral” theory). The reason for this is very simple: all the $W$-Toda fields just commute with the action of $G$-charges $Q_\alpha, Q_\nu$, and thus dressing does not affect any structures, related to the algebra $G$ (but may give a correct selection rule).

The subject of $W$-strings (or closely related $W$-gravity) has been already discussed in the literature [8-13]. Attempts were made in order to work out an analogue of the DDK approach (or [25]). It has been also confirmed by the BRST calculation that observable operators, obtained in this way, are indeed annihilated by BRST charge [26]. Of course the DDK-like construction, which assumes an oversimplified dependence on the ghost and Liouville fields, does not need to give all the observables. But even in the sector which

---

16 It means that Liouville and ghost contributions just cancel Dedekind $\eta$-factor in (22).
17 Just as it happens in the $SU(2)$ case, where along with “dimension zero, ghost number one” (or “dimension one, ghost number zero”) operators implied by the DDK formalism and considered in [1], there are also less trivial “dimension zero, ghost numbers zero and two” observable operators [7,1,2]. Another example is provided by minimal $c < 1$ series, where (at least in some versions of the formalism) Liouville field does not usually appear in the pure exponential form (as Virasoro primary in the Liouville sector) as it is implied in what we call the DDK approach.
it does describe, the DDK approach is less conclusive than in the case of ordinary strings. The main difference arising in consideration of $W_G$-gravity as compared to the ordinary “Virasoro gravity”, is that observable operators are no longer represented as integrals of (at least any simple) ghost free operators of dimension one. Instead the adequate ghost free operators have dimension $\Delta^{(G)} = 2\rho^2 = \frac{1}{6} C_V [G] \dim G$. Even triple correlators of such observables involve non-trivial correlators of ghosts (the moduli space of $W_G$-gravity is already non-trivial for a sphere with 3 punctures), which need a reasonable understanding of $W$-geometry in order to be defined. This is the main obstacle on the way to work out the structure of algebra of observables (reminiscent of operator product expansion) and thus to understand the points 3)–6) of the KPW construction, as described in the Introduction. While resolution of these problems is beyond the scope of this paper, in this section we present a bit more details about DDK formalism for $W$-strings to illustrate what we just said about it and also to demonstrate that it preserves the model structure of our $c = r_G$ matter theory from the previous section.

The analogue of conventional Liouville action is provided by conformal $G$-Toda action:

$$\int_{d^2z} \left\{ |\partial \phi|^2 + \beta_0 R \rho \phi + \sum_{i=1}^{\eta_i e^{\alpha_i \phi}} \right\},$$

(46)

the sum is over $r_G$ simple roots of $G$. In the framework of the KPW construction we consider a point where all $\eta_i = 0$. Besides the $r_G$-component $W$-Toda field $\phi$ there are also $r_G$ species of ghosts: Grassmann $b,c$-systems with spins $j \in S_G$ with the first-order action

$$\int_{d^2z} \sum_{j \in S_G} \{b_j \partial c_{1-j} + c.c.\}.$$

(47)

Occurrence of ghosts is obvious from the geometrical meaning of $W$-fields (describing certain flag structures in the jet bundle over Riemann surface [10]) and, more technically, from interpretation of (12) as a result of the Drinfeld-Sokolov reduction of WZW model [27-30]. The set $S_G$ is nothing but the set of $G$-invariants or the Casimir orders, which is different for the three series $A$, $D$ and $E$ of the simply laced algebras: for $SU(r + 1) - j = 2, \ldots, r_G + 1$ (and $1 - j = -1, \ldots, -r_G$); $SO(2r) - j = 2, 4, \ldots, 2r - 2$ and $r$; $E_6 - j = 2, 5, 6, 8, 9, 12$; $E_7 - j = 2, 6, 8, 10, 12, 14, 18$; and for $E_8 - j = 2, 8, 12, 14, 18, 20, 24, 30$. 23
Thus the central charge of ghosts is equal to

$$c_{\text{ghosts}} = \sum_{j \in S_G} [-2(6j^2 - 6j + 1)] = -48\rho^2 - 2r_G.$$  

(48)

The central charge of the $W$-Toda fields equals $c_\phi = r_G + 48\beta_0^2\rho^2$. Since the total central charge $c_{\text{matter}} + c_\phi + c_{\text{ghosts}} = 0$, we have: $48(\beta_0^2 - 1)\rho^2 = c_{\text{matter}} - r_G$; and, therefore, for $c_{\text{matter}} = r_G - \beta_0 = \pm 1$.

The naive DDK approach implies the following algorithm for building up the observables in the $W_G$-string model, which arises by gauging $W_G$-symmetry of the matter sector.

A) Pick up any $W_G$-primary from the matter sector. In our particular model from sect.2 it is labeled by two indices: $\Psi_{\nu,\xi}(x) = \prod_{i=1}^{\nu} (Q - \alpha_i)^{\nu} \Psi_{\nu,0}$. $\Psi_{\nu,0} = e^{i\nu x}$, where $\nu$ (a is a vector in the weight lattice) labels representations $R_{\nu}[G]$ of $G$ ($\nu$ is just its highest weight), while $\xi \equiv \xi_R$ labels the element of this representation. The conformal dimension $\Delta_{\nu,\xi} = \nu^2/2$ is independent of $\xi$.

B) Dress this matter field by the $W$-Toda exponent: $\Xi_{\nu,\xi}(x, \phi) = \Psi_{\nu,\xi}(x)e^{B\beta\nu\phi}$, so that $\Xi_{\nu,\xi}$ has the fixed dimension $\Delta^{(G)}$. This requirement restricts the value of $\beta_{\nu}$:

$$\Delta_{\nu,\xi} - \frac{1}{2}\beta_{\nu}^2 - 2\beta_0\beta_{\nu}\rho = \Delta^{(G)},$$

(49)

or, in our particular model with $\Delta_{\nu,\xi} = \frac{1}{2}\nu^2$ and $\beta_0 = 1$,

$$\frac{1}{2}\nu^2 = \frac{1}{2}(\beta_{\nu} + 2\rho)^2 + (\Delta^{(G)} - 2\rho^2).$$

(50)

Of course, there are many solutions of this 1-component (scalar) equation for the $r_G$-component (vector) $\beta_{\nu}$ once $\nu$ is fixed. However, it is clear that a distinguished set of solutions arises, if

$$\Delta^{(G)} = 2\rho^2$$

(51)

and

$$\beta_{\nu} = \nu - 2\rho.$$  

(52)
C) Add extra ghost factor in order to compensate for non-vanishing $\Delta^{(G)}$ and create an operator of dimension zero.

In conventional Liouville theory (2d gravity) $\Delta^{(SU(2))} = 1$ and it is enough to multiply $\Xi(x, \phi)$ by an ordinary reparameterization ghost $c_{-1} \equiv c$ to create an observable operator

$$O_{\nu, \xi}(x, \phi, c) = \Xi_{\nu, \xi}(x, \phi)c_{-1} = \psi_{\nu, \xi}(x)e^{\beta_{\nu}\phi}c_{-1}. \quad (53)$$

Correlators of observables are evaluated with additional insertions of the form

$$\prod_{\alpha=1}^{N^{(2)}} \int_{\mathbb{D}_z} b_2 \mu^{(2)}_{\alpha}, \quad (54)$$

where $\mu^{(2)}_{\alpha}$ are the Beltrami differentials, associated with the moduli of complex structure of the surface, and $N^{(2)}$ is dimension of the module space. An alternative and conceptually simpler description of essentially the same operator as (53) in Liouville theory may be given without any reference to ghosts and BRST formalism: observable may be defined as

$$\hat{O}_{\nu, \xi}(x, \phi) = \int dz \Xi_{\nu, \xi}(x, \phi) = \int dz \psi_{\nu, \xi}(x)e^{\beta_{\nu}\phi}, \quad (55)$$

i.e. as an integral of ghost free operator of dimension one. When the correlators are evaluated in this representation the integration contours are deformed and integrals automatically pick up all contributions from the points where other operator are inserted and as well as from the handles on the surface. Equivalence between correlators, evaluated in the two representations (53) and (55) may be achieved by explicit calculation of the ghost contribution when using representation (53) – then the integrals over moduli corresponding to the marked points on Riemann surface turn into the contour- (surface-) integral of representation (55).

However, again for $G \neq SU(2)$ the situation is much more complicated. No natural representation of observables of the form (63) is available (at least at the moment), so one needs to rely upon a less transparent BRST formulation, analogous to (53). Generalization

18Such distinguished choice of W-Toda dressing, corresponding to a particular solution of the equation, as well as the ghost dressing below, has been already proposed, for example, in [12].
is obvious: instead of dressing \( \Xi(x, \phi) \) by multiplication with a single ghost field \( c_{-1} \), now one needs to multiply with a whole combination of ghosts:

\[
O_{\nu, \xi}(x, \phi, c) = \Xi_{\nu, \xi}(x, \phi) \prod_{j \in S_j} \left\{ c_{1-j} \partial c_{1-j} \partial^2 c_{1-j} \ldots \partial^{j-2} c_{1-j} \right\} = \psi_{\nu, \xi}(x)e^{(\nu-2\rho)\phi}e^{i\sum (j-1)\varphi_j}.
\]

In the last equation we substituted our explicit formula (52) for \( \beta_{\nu} \) and “bosonized” ghost fields, i.e. \( b_j = e^{-i\varphi_j} \), \( c_{1-j} = e^{i\varphi_j} \). The bosonized ghost action (47) is

\[
\int d^2 z \sum_{j \in S_G} \{ b_j \bar{c}_{1-j} + c.c. \} = \int d^2 z \sum_{j \in S_G} \left[ \partial \varphi_j \bar{\partial} \varphi_j + t(j - \frac{1}{2})\varphi_j \mathcal{R} \right].
\]

The peculiar combination \( \{ c_{1-j} \partial c_{1-j} \partial^2 c_{1-j} \ldots \partial^{j-2} c_{1-j} \} =: (c_{1-j})^{j-1} := e^{i(j-1)\varphi_j} \) has dimension \( \Delta_j = -j(j-1)/2 \), and the entire product in (64) is of dimension

\[
\sum_{j \in S_G} \Delta_j = \frac{1}{24} \sum_{j \in S_G} [-2(6j^2 - 6j + 1) + 2] = \frac{1}{24} (c_{\text{ghosts}} + 2r_G) = -2\rho^2.
\]

Therefore the observable operator indeed has vanishing dimension, but instead it is of incredibly large ghost charge. This ghost charge is compensated, when correlators of observables are evaluated, by a product

\[
\prod_{j \in S_G} \left\{ \prod_{\alpha=1}^{\nu(j)} \int d^2 z \ b_j \mu_{\alpha}^{(j)} \right\},
\]

which now involves Beltrami differentials associated with all the “moduli of \( W \)-structure”. Unfortunately, the meaning of this last notion is much worse understood than its \( W_2 \)-counterpart, which is known to be associated with moduli of the complex structure of the surface (so, that the Beltrami differentials \( \mu^{(2)} \), associated with punctures, just describe the shift of the puncture position). As explained in [10] a holomorphic (chiral) fragment of \( W \)-structure is associated with a certain flag in the jet bundle over the surface. However, there are still unresolved serious difficulties in understanding the entire (non-chiral)
The non-trivial “W-moduli” appear already on the 3-punctured sphere, thus
turning even evaluation of 3-point functions and thus the algebra of observables into a
problem. Resolution of this problem is crucial for proceeding with the KPW construction
in our $c = r_G$ model. We hope to return to these intriguing problem elsewhere.

The assertion that the operator (56) is indeed a proper observable operator in $W_G$-
string $c = r_G$ model is somehow confirmed by the fact, that it belongs to the BRST
cohomology. It was explicitly checked in [26] for the case of $G = SU(3)$. The general
form of the BRST operator

$$Q_{BRST} = \sum T_a c^a - \frac{1}{2} \sum f_{bc}^a b^b c^c ,$$  

(60)

survives even in the case of non-Lie $W$-algebra with quadratic commutation relations, if
one substitutes to (60) the dependent on generators $T_a$ structure constants $f_{bc}^a = f_{ab}^c (T_a)$.

For the case of $SU(3)$ the structure constants are known from [31] and the BRST operator

$$Q_{BRST} = -\Delta c_0^{(-1)} + \sum L_m c_m^{(-1)} - \frac{1}{2} \sum (m - n) : b_{m+n}^{(2)} c_{-m}^{(-1)} c_{-n} : -$$

$$- \sum (2m - n) : b_{m+n}^{(3)} c_{m}^{(-2)} c_{-n} : - \sum W_m c_{-m}^{(-2)} - \frac{1}{2} \beta \sum (m - n) L_{-m_p} b_{m+n+p}^{(2)} c_{-m_n} c_{-n} -$$

$$- \frac{1}{2} \gamma \sum (m - n) \left( \frac{1}{15} (m + n + 2)(m + n + 3) - \frac{1}{6} (m + 2)(n + 2) b_{m+n}^{(2)} c_{-m_n} c_{-n}^{(-2)} \right)$$  

(61)

(with $\beta = \frac{16}{22 + 5c}$, $\gamma = \frac{5c}{1044}$) satisfies nilpotency condition for $c = 100, \Delta = 2 \rho^2 = 4$

[26]. The form of the similar BRST operator for any simply laced $G$ can be got from (60)
by substitution of the structure constants (depending on the generators) of the quantum
algebra $W_G$. Note that these arguments confirm also our suggested choice of $W$-Toda

\footnote{There are at least two distinct(?) problems. One is related to the Serre relations: the Borel subalgebra
is not free for $G \neq SU(2)$, this leads to additional non-linear constraints and, in particular to problems
with quantization (see [10,29]). Another problem is that while the stress tensors in conformal field theory
is essentially holomorphic: $T_{\mu\nu} = \{T_{zz}, T_{z\bar{z}}, T_{\bar{z}z}\}$ and $T_{z\bar{z}} = 0$, there are no a priori reasons why, say,
$W_{zz\bar{z}}$-component of the spin-3 $W_3$-operator should be neglected, though no direct information about it is
contained in the structure of chiral $W$-algebra, which involves only $W_{zz\bar{z}}$. Moreover, the existing geometric
interpretation of $W_G$-gravity [10] involves only chiral (holomorphic or antiholomorphic) structures on the
surface.
[111x702]dressing of eq.(52).

4 Conclusion

To conclude, we discussed in some details the $G$-induced structure on (a subsector of) the space of observables in the $c = r_G W$-string model, arising after gauging the $W_G$-symmetry of the 2-dimensional conformal matter theory, which describes compactification on Cartan torus of a simply laced Lie algebra $G$. The chiral component of this space may be considered as a model of $G$.

This implies that the result of the KPW construction in this case should be some algebraic structure, intimately related to the model, which still remains to be discovered. This algebraic structure should be related to the algebra of observables of the theory, i.e. algebra of OPE of physical operators modulo descendants, or algebra of correlation functions. A systematic approach to this problem requires a better understanding of the $W_G$-geometry, what seems to be also an intriguing subject. In particular it would be nice to have a simple generalization of DDK prescription, allowing one to represent the physical operators involved into correlation functions in a simple form. However, a lot of questions on this way need to be understood much more clear. Finally, we will try to present here at least some of them:

a) The relation between algebra of commutators of chiral operators (contour integral) and correlation functions (surface integrals) should be more clear even in the $SU(2)$ case (a priori they have different symmetry properties even under the permutation of operators).

b) The important remaining problem is complete classification in spirit of [7] the physical spectrum of $W_G$-gravity. In fact, be this classification known a lot of questions we addressed to in the paper will be surely immediately tractable.

c) Even considering of three-point correlation functions may not be enough for higher groups $G$ and higher $W_G$-gravities, though even the commutation relation for $W_G$-algebra itself are quadratic, one could expect the necessity to consider four-point correlation functions in order to determine a non-Lie algebraic structure.

d) After Witten’s proposal [2] it is natural to ask whether any algebra of observables can be decomposed into the commutative (ground) ring and antisymmetric chiral...
algebra, acting on this commutative ring by derivatives, and what is the action of this antisymmetric chiral algebra (see also the item f)).

e) The next question rather concerns the \( c < 1 \) string models. It would be nice to find various reductions of the \( W_\infty \)-algebra in corresponding theories which can be obtained as certain reductions of Hilbert space of the \( c = 1 \) theories. From comparison with matrix models one could expect the appearance of \( W_q \)-algebra's for OPE's in different \((p,p')\)-minimal series (see also [32]).

f) Finally, it is interesting to study algebraic properties of the Clebsh-Gordon coefficients for higher (at least, simply laced) groups. Namely, do the \( 3j \)-symbols

\[
C_{\xi_1,\xi_2,\xi_3}^{\nu_1,\nu_2,\nu_3}
\]

with additional conservation law like \( \nu_3 = \nu_1 + \nu_2 - 2\rho \) or \( \nu_3 = \nu_1 + \nu_2 - \theta \), where \( \theta \) – the highest root of \( G \), satisfy some clear algebraic relations like Jacobi identities? 

The answer to this question would state whether any chiral algebra can be interpreted as universal envelope of some smaller and simpler algebraic structure like \( SU(2) \) case, when Clebsh-Gordon coefficients are the structure constants (or adjoint representation) of the enveloping algebra \( T(\mu) \) of \( SU(2) \) [1] (see also Appendix).

One could try also to guess, what can be this algebraic structure, just generalizing the result of the KPW construction in the \( SU(2) \) case. We add Appendix, describing immediate implication of such attempts: concerning the Lie algebra, defined by symplectomorphisms of the coadjoint orbits. This topic is not quite trivial and is of certain interest by its own, though its relation to the KPW construction for \( G \neq SU(2) \) still remains somewhat obscure.

We are indebted to A.Alekseev, Vl.Dotsenko, A.Gerasimov, D.Juriev, A.Losev, S.Pakuliak and A.Sagnotti for illuminating discussions.

---

21Actually the second choice allows corresponding operators to acquire unit dimension instead of \( 2\rho^2 \) for the first case, as it has been considered above, see also Appendix

22They are antisymmetric for group \( SU(n) \) only at \( n = 2, 5, 6, 9, 10, \ldots \). Thus, in these cases they have a chance to be a structure constants of Lie algebra satisfying the Jacobi identities. At all remaining values of \( n \) Clebsh-Gordon coefficients are symmetric and generate a commutative ring.
Appendix. Symplectomorphisms as a model

This Appendix is devoted to another ingredient of the KPW construction: the interpretation of (the Lie-algebra fragment of) the algebra of observables as the symplectomorphism algebra of a certain manifold. Symplectomorphisms (area-preserving diffeomorphisms or Hamiltonian vector fields) usually form a much wider space than more familiar isometries (since they preserve a non-degenerate antisymmetric symplectic 2-form $\omega_{ij} dx^i \wedge dx^j$ rather than non-degenerate symmetric metric tensor $g_{ij} dx^i dx^j$). The symplectomorphisms algebra depends on the choice of symplectic form $\omega_{ij}$ but is usually infinite (as compared to finite isometry groups). The symplectomorphisms are of special interest to the string theory since they are natural generalizations of the Virasoro algebra\textsuperscript{23}. In particular this is the reason why symplectomorphisms of $\mathbb{C}^2$ arise in membrane theory etc.

The generators of the symplectomorphisms algebra can be represented either as hamiltonians $h(x)$ – functions on a manifold or as hamiltonian vector fields

\[ e_h = \omega^{ij} \partial_i h \partial_j . \quad (A.1) \]

The Lie-algebra commutator is just a commutator of these vector fields

\[ [e_{h_1}, e_{h_2}] = e_{h_3}^{24}, \quad (A.2) \]

or a Poisson bracket of the Hamiltonians

\[ h_3 = \{h_1, h_2\} = \omega^{ij} \partial_i h_1 \partial_j h_2 . \quad (A.3) \]

Below we shall prefer the Poisson bracket representation.

We shall also restrict ourselves to the case of complex spaces $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ and “flat” symplectic form (or complex structure ) $\sum dz_i \wedge d\bar{z}_i$.

The study of this symplectomorphisms in the context of this paper is motivated by the crucial discovery of [1,2] that the Lie-algebra piece of the algebra of observables for $G = \mathbb{C}^2$.

\textsuperscript{23}Indeed, on complex manifolds the complex structure is introduced whenever a certain symplectic form is fixed (which satisfy additional relation $\omega^2 = - I$) and, thus, the transformations which preserve the complex structure are particular examples of symplectomorphisms.

\textsuperscript{24}Generically there can be global obstacles for this relation.
SU(2) coincides with $W_\infty$ represented as symplectomorphism of $\mathbb{C}$. If it is translated to the algebraic language this result implies that the algebra $\mathcal{J}(M)$ of symplectomorphisms of $M = \mathbb{C}$ can be splitted (graduated) with respect to representations of $G = SU(2)$ in such a way that any irreducible highest weight representation appears exactly once, i.e. $\mathcal{J}(\mathbb{C})$ is a representation of a model of SU(2). This is a very nice statement of its own interest and we are going to discuss and generalize it in this Appendix, namely $\mathcal{J}(\mathbb{C}^3)$ is closely related to a representation of a model of SU(3).

What distinguishes the KPW construction for the SU(2) case is that the global SU(2) algebra is identified with the adjoint representation from the SU(2)-model $\mathcal{J}(\mathbb{C})$. In other words the operators $q_{1,m}$ would correspond to both $O_{2\rho,\xi}$ and $O_{\theta,\xi}$ only in the SU(2) case ($\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$, $\theta$ – is the highest positive root) when according to the first representation it has proper conformal dimension while the second reflects its “adjoint” properties. In general $2\rho \neq \theta$, therefore neither Toda fields nor ghosts must decouple for the operator $O_{\theta,\xi}$. However, this identification would still be valid if we use the construction which substitutes $2\rho$ by $\theta$, but it does not seem to be quite adequate to the KPW construction. Usually, for $G \neq SU(2)$ the piece algebra of observables (consisting of $O_{\nu,\xi}$) is a G-model with respect to external action of global $G$ (generated by $Q_\alpha = \int \mathcal{J}_\alpha$ and $Q_\nu = \int \mathcal{H}_\nu$), i.e. it is not identifies with the OPE involving the operators $O_{\theta,\xi}$ which belong to the selected set of observables (we do not insist that this piece of observables form a Lie-algebra!). However, despite all we have said, below in the example for SU(3) we shall show that $J(\mathbb{C}^3)$ is indeed a kind of SU(3)-model with respect to internal action of the adjoint representation from $J(\mathbb{C}^3)$. The relevance of these results to the KPW construction is at least obscure, but we decided to present them here because of their own interest.

Another remark is that it would be much more interesting to have a generic theory of symplectomorphisms of the coadjoint orbits of $G$ and to understand what is the exact information about $G$, encoded there. Our discussion of $\mathbb{C}^N$ spaces below is just a first step in this direction.

Let us consider the space of Hamiltonians $h(z_i, \bar{z}_i)$ on $\mathbb{C}^N$, i.e. just the space of all polynomials of $z_i$ and $\bar{z}_i$. Introduce the gradation in this space by degrees of homogeneity in holomorphic and anti-holomorphic coordinates. Consider now the degree two
polynomials

\[ h_{ij}^{adj} = z_i \bar{z}_j . \]  \hspace{1cm} (A.4)

Their Poisson brackets

\[ \{ h_{ij}^{adj}, h_{kl}^{adj} \} = \delta_{il} h_{kj}^{adj} - \delta_{jk} h_{il}^{adj} \]  \hspace{1cm} (A.5)

obviously describe the construction of \( GL(N) \) algebra and thus contain the \textit{adjoint} representation of \( SU(N) \) and the \( SU(N) \) scalar \( h_0 = \sum |z_i|^2 \). Associated vector fields

\[ e_{ij}^{adj} = z_i \partial_j - \bar{z}_j \bar{\partial}_i \]  \hspace{1cm} (A.6)

obviously preserve our graduation, and the finite sets of polynomials \( h[m, n] \) form (probably reducible) representation of \( SU(N) \).

In order to select the irreducible representation we need to find out all highest vectors, \textit{i.e.} the Hamiltonians annihilated by the action of \( e_{ij} \) with \( i < j \). Obviously the \textit{holomorphic} functions of \( z_1 \) and \( \bar{z}_N \) are among them

\[ \{ h_{ij}^{adj}, H(z_1, \bar{z}_N) \} = 0 \quad \text{for all} \quad i < j . \]  \hspace{1cm} (A.7)

One can easily check that the highest vector

\[ H_{pq} = z_1^{p} \bar{z}_N^{q} \]  \hspace{1cm} (A.8)

generates an irreducible representation with the highest weight

\[ \nu_{pq} = p \mu_1 + q \mu_{N-1} , \]  \hspace{1cm} (A.9)

where \( \{ \mu_i \}, i = 1, \ldots, N - 1 = r = \text{rank}[SU(N)] \) are the fundamental weights of \( SU(N) \). In particular \( H_{11} = z_1 \bar{z}_N \) is the highest vector of the adjoint representation with \( \mu_1 + \mu_{N-1} = \theta \). Thus, we proved that \( J(C^N) \) contains a piece of \textit{model} of \( SU(N) \), consisting of all the representation of the form (A9), with every such representation arising \textit{exactly once} (unless \( \mu_{N-1} \neq \mu_1 \)) This would be all irreducible representation for \( N = 3 \), \textit{i.e.} for \( G = SU(3) \) (since all the representations are represented as \( \nu = \sum_{i=1}^r a_i \mu_i \) and for \( SU(3) \) \( a_2, \ldots, a_{N-2} \) do not need to vanish).
Several remarks are in order now. First, $H(z_1, \bar{z}_N)$ do not exhaust the entire set of Hamiltonians, annihilated by all $e_{ij}$ with $i < j$, the simplest counterexample being the scalar $h_0 = \sum |z_i|^2$. Therefore the entire $J(C^N)$ is somewhat bigger that the model of representations of the form (A9): it can contain the extra scalars and thus extra representations arising by multiplication with these scalars. So, a kind of factorization over these scalars is required to make any $SU(N)$ representation appearing only once\footnote{Without such factorization $J(C^N)$ seems to contain many copies of the model.}

Second, the above construction can be reduced by identification

$$z_1 = \bar{z}_N \quad (A.10)$$

which does not break $SU(N)$ commutation relations (A5), but eliminates the extra scalars. Note that this reduction does break our original gradation and preserves only the degree of homogeneity in $z_i$ and $\bar{z}_i$ together rather than in $z_i$ and $\bar{z}_i$ separately. However, this reduction eliminates also half of highest vectors: $H(z_1, \bar{z}_N) \rightarrow H(z_1)$ and we obtain a “reduced” model of representations of the form

$$\nu = p\mu_1 \quad (A.11)$$

In the particular case of $SU(2)$ this is exactly what is necessary to eliminate (since $\mu_1 = \mu_{N-1}$ for $N = 2$). Therefore the model of $SU(2)$ is just represented by

$$\hat{J}(C) = J(C)/\{z_1 = \bar{z}_2\}, \quad (A.12)$$

i.e. by Hamiltonians of the form

$$h^{SU(2)}_{J,m} = z^{J+m}\bar{z}^{J-m} \quad (A.13)$$

with (half-)integer $J = \nu/\sqrt{2} \geq 0$ and $m = \xi/\sqrt{2}, |m| \leq J$. The adjoint representation is given by

$$h^{adj}_{1,m} = \{z^2, z\bar{z}, \bar{z}^2\} \quad (A.14)$$

the highest vectors are
and thus \( \hat{J}(C) \) contains all the representations of (half-)integer spin \( J \), i.e. is the model of \( SU(2) \).

Third, as we already discussed, while in the case of \( G = SU(2) \) the above algebra \( \hat{J}(C) \) is indeed the algebra of observables arising in the KPW construction, it is not precisely \( J(C^3) \) in the case \( G = SU(3) \) (despite that \( J(C^3) \) is essentially the model of \( SU(3) \), note also that in contrast to the \( SU(2) \) case \( \hat{J}(C^2) = J(C^3)/\{z_1 = \bar{z}_3\} \) is less than a model of \( SU(3) \)). In particular, the commutator of some vectors from the fundamental representation from \( J(C^3) \) if defined by the Poisson relations in \( J(C^3) \) is vanishing, but this is far from being obvious in the algebra of observables. For example, for \( \nu = \mu_1, \nu' = \mu_2 \) in general one has \( 3 \times 3 = 6 + 3 \), and according to the rule \( \nu'' = \nu + \nu' = \nu - \theta \), only 6 is eliminated by conservation law in the \( W_G \)-Toda sector. Note, however, that this particular discrepancy can be connected with a specific choice of symplectic structure \( \omega_{ij} = \delta_{ij} \) which does not include any other invariant tensors, which should certainly appear in the chiral KPW construction.

References

[1] I.Klebanov, A.Polyakov Interaction of Discrete states in Two-Dimensional String Theory, Preprint PUPT-1281, September 1991

[2] E.Witten Ground Ring of Two Dimensional String Theory, Preprint IASSNS-HEP-91/51

[3] R.Dijkgraaf, E.Verlinde, H.Verlinde Comm.Math.Phys. 115 (1988) 649

[4] P.Ginsparg Nucl.Phys. B295 [FS21] (1988) 153

[5] F.David Mod.Phys.Lett. A3 (1988) 1651

[6] J.Distler, H.Kawai Nucl.Phys. B312 (1989) 509
[7] B.Lian, G.Zuckerman *Phys.Lett.* B254 (1991) 417
B.Lian, G.Zuckerman *Phys.Lett.* B266 (1991) 21
P.Bouwknegt, J.McCarthy, K.Pilch *Fock Space resolutions of the Virasoro Highest Weight Modules with c ≤ 1*, Preprint CERN-TH-6196/91

[8] E.Bergshoeff, C.Pope, L.Romans, E.Sezgin, X.Shen, K.Stelle *Phys.Lett.* B243 (1990) 350

[9] G.Sotkov, M.Stanishkov *Nucl.Phys.* B356 (1991) 439
G.Sotkov, M.Stanishkov, C.J.Zhu *Nucl.Phys.* B356 (1991) 245

[10] A.Gerasimov et al. *Nucl.Phys.* B360 (1991) 537

[11] A.Bilal, V.Fock, I.Kogan *Nucl.Phys.* B359 (1991) 635

[12] P.Mansfield, B.Spence *Nucl.Phys.* B362 (1991) 294

[13] J.-L.Gervais, Y.Matsuo Preprint LPTENS-91/35, NBI-HE-91-50

[14] A.Morozov, M.Olshanetsky *Nucl.Phys.* B299 (1988) 389
A.Gerasimov et al. *Int.J.Mod.Phys.* A5 (1990) 2495

[15] A.Morozov, A.Rosly *Phys.Lett.* B214 (1988) 522

[16] J.Cardy *Nucl.Phys.* B324 (1989) 581

[17] A.Sagnotti *Recent Developments in Open-String Theories*, Preprint ROM2F-91/11 and references therein

[18] V.Kac *Lect.Notes in Physics* 94 (1979) 441

[19] A.Rocha-Caridi in: *Vertex Operators in Mathematics and Physics*, MSRI Publ. 3 (Springer, Heidelberg, 1984) 451

[20] A.Morozov *Nucl.Phys.* B357 (1991) 619

[21] D.Gepner, E.Witten *Nucl.Phys.* B278 (1986) 493

[22] D.Zhelobenko *Compact Lie Groups and Their Representations*, Moscow, Nauka, 1970
[23] S.Lukyanov, V.Fateev *Additional Symmetries in Two-Dimensional Conformal Field Theory and Exactly Solvable Models*, Preprints ITP-88-74R,75R,76R, Kiev 1988

[24] Vl.Dotsenko *Adv.Stud.in Pure Math.* 16 (1988) 123

[25] V.Knizhnik, A.Polyakov, A.Zamolodchikov *Mod.Phys.Lett.* A3 (1988) 819

[26] J.Thierry-Mieg *Phys.Lett.* B197 (1987) 368

[27] P.Mansfield *Phys.Lett.* B248 (1990) 387

[28] M.Bershadsky, H.Ooguri *Comm.Math.Phys.* 126 (1989) 49

[29] A.Marshakov, A.Morozov *Nucl.Phys.* B339 (1990) 79

[30] P.Forgacs, A.Wipf, J.Balog, L.Feher, L.O’Raifeartaigh *Phys.Lett.* 227B (1989) 214

J.Balog, L.Feher, L.O’Raifeartaigh, P.Forgacs, A.Wipf *Ann.Phys.* 203 (1990) 76

[31] V.Fateev, A.Zamolodchikov *Nucl.Phys.* B280 (1987) 644

[32] D.Kutasov, E.Martinec, N.Seiberg *Ground rings and their modules in 2d gravity with c ≤ 1 matter*, Preprint PUPT-1293, RU-91-49, November 1991