Limits of Quotients of Polynomial Functions in Three Variables

Juan D. Vélez, Juan P. Hernández, Carlos A. Cadavid.

Abstract

A method for computing limits of quotients of real analytic functions in two variables was developed in [4]. In this article we generalize the results obtained in that paper to the case of quotients $q = f(x, y, z)/g(x, y, z)$ of polynomial functions in three variables with rational coefficients. The main idea consists in examining the behavior of the function $q$ along certain real variety $X(q)$ (the discriminant variety associated to $q$). The original problem is then solved by reducing to the case of functions of two variables. The inductive step is provided by the key fact that any algebraic curve is birationally equivalent to a plane curve. Our main result is summarized in Theorem 19.

In Section 4 we describe an effective method for computing such limits. We provide a high level description of an algorithm that generalizes the one developed in [4], now available in Maple as the limit/multi command.

1 Introduction

Algorithms for computing limits of functions in one variable are studied in [12]. Similar algorithms have been developed in [10] and [11]. Computational methods dealing with classical objects, like power series rings and algebraic curves, have been developed by several authors during the last two decades, [1] and [19]. A symbolic computation algorithm for computing local parametrization of analytic branches and real analytic branches of a curve in $n$-dimensional space is presented in [2].

In [4] Vélez, Cadavid and Molina developed a method for analyzing the existence of limits $\lim_{(x,y) \to (a,b)} q(x, y)$, where $q(x, y)$ is a quotient of two real analytic functions $f$ and $g$, under the hypothesis that $(a, b)$ is an isolated zero of $g$. In the case where $f$ and $g$ are polynomial functions with rational coefficients, the techniques developed in that article provide an algorithm for the computation of such limits, now available in Maple as the limit/multi command [17].

An alternative method for computing limits of quotients of functions in several variables has been recently developed in [21]. Their approach is completely different from ours, relying on Wu’s algorithm as the main tool.

In this article we generalize the methods presented in [4] to the case of quotients of polynomials in three variables, under the same assumption that $g$ is a function with an isolated zero at the point $(a, b)$. The main idea consists in reducing the problem of determining the existence of limits of the form

$$\lim_{(x,y,z) \to (a,b,c)} f(x, y, z)/g(x, y, z)$$

(1)

to the problem of determining the limit along some real variety $X(q)$ associated to $q$ (the discriminant variety of $q$). In order to achieve this one needs to study the topology of the irreducible components of the singular locus of $X(q)$. The original problem is then solved by reducing to
the case of functions of two variables. The inductive step is provided by the key fact that any algebraic curve is birationally equivalent to a plane curve. Our main result is summarized in Theorem 1. In Section 4 we provide a high level description of a potential algorithm capable of determining the existence of (1), and if the limit exists, it would be able to determine its value. Any of the Groebner Basis packages available may serve as a computational engine to implement such an algorithm. In Section 5 we present two examples that illustrate some the computation that would be needed in a typical problem of determining and computing a limit of this sort.

2 Preliminaries

2.1 Dimension of algebraic sets and its singular locus

In this article we consider complex affine varieties defined by polynomials with real coefficients. If \( I \) is an ideal in the polynomial ring \( S = \mathbb{R}[x_1, \ldots, x_n] \), by \( X = V(I) \) we will denote the complex affine variety defined by \( I \), i.e., the common zeros of \( I \) in \( \mathbb{C}^n \). The dimension of \( X \) is the Krull dimension of the ring \( \mathbb{C} \otimes_{\mathbb{R}} S/I \). Since \( S/I \subset \mathbb{C} \otimes_{\mathbb{R}} S/I \) is a faithfully flat extension of rings, the dimension of \( X \) coincides with the dimension of \( R[\overline{x_1}, \ldots, \overline{x_n}] \), the real affine ring of \( X \).

It is well known that if \( X \) is irreducible, defined by some prime ideal \( P \subset S \), then the dimension of the domain \( R = S/P \) coincides with the transcendence degree of the field extension \( R \subset L \) (denoted by trdeg\(_K L \)), where \( L \) denotes the fraction field of \( R \).

We recall the definition of the singular locus of an equidimensional affine variety.

**Definition 1.** Let \( Y \subset \mathbb{C}^n \) be an affine variety, and let \( R = \mathbb{C}[x_1, \ldots, x_n]/I(Y) \) be its ring of coordinates. Suppose that \( R \) is equidimensional of dimension \( r \) (i.e., \( \text{ht}(P) = r \), for all the minimal primes \( P \) containing \( I(Y) \)). Let’s choose arbitrary generators \( f_1, \ldots, f_k \) for \( I(Y) \). The singular locus of \( Y \), denoted by \( \text{Sing}(Y) \), is the closed subvariety of \( Y \) defined by the ideal \( J = I(Y) + \text{the ideal of all} \ (n-r) \times (n-r) \text{minors of the Jacobian matrix} \).

**Remark 2.**

1. The above criterion to determine \( \text{Sing}(Y) \) does not depend on the generators one chooses for \( I(Y) \).
2. The singular locus \( \text{Sing}(Y) \) is a proper closed subvariety of \( Y \), defined by those points \( p \in Y \) for which the rank of the Jacobian matrix \( [\partial f_i/\partial x_j](p) \) is less that \( n-r \).
3. \( \text{dim}(\text{Sing}(Y)) < \text{dim}(Y) \)

(See [7], Section 16.5 and [11], Chapter I, Section 5).

We will mainly focus in the following simple case: Suppose that \( X \subset \mathbb{C}^3 \) is an affine variety of dimension 2 defined by a prime ideal \( P \subset \mathbb{R}[x, y, z] \). In this case \( P \subset \mathbb{R}[x, y, z] \) must be a prime ideal of height 1, and so it has to be principal, i.e., \( P = (h) \), where \( h \in \mathbb{R}[x, y, z] \) is some real irreducible polynomial. Therefore, \( X = V(h) \). In this case \( \text{Sing}(X) \) is the complex affine variety defined by the ideal \( IS = (h, \partial h/\partial x, \partial h/\partial y, \partial h/\partial z) \subset \mathbb{R}[x, y, z] \).

2.2 The discriminant variety

The existence of \( \lim_{(x,y,z) \to (a,b,c)} f(x, y, z)/g(x, y, z) \) does not depend on the particular choice of local coordinates. Hence, after an appropriate translation we may always assume that \( p = (a, b, c) \).
functions. However, these minors can be written as

\[ \begin{vmatrix} x & y & z \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \end{vmatrix} \]

Strictly speaking, the $2 \times 2$ minors of $A$, $x_i \frac{\partial q}{\partial x_j} - x_j \frac{\partial q}{\partial x_i}$, are not necessarily polynomial functions. However, these minors can be written as

\[ x_i \frac{\partial q}{\partial x_j} - x_j \frac{\partial q}{\partial x_i} = \frac{x_i (g \partial f/\partial x_j - f \partial g/\partial x_j) - x_j (g \partial f/\partial x_i - f \partial g/\partial x_i)}{g^2}, \]

and therefore, if we let

\[ f_{x_i,x_j} = x_i (g \partial f/\partial x_j - f \partial g/\partial x_j) - x_j (g \partial f/\partial x_i - f \partial g/\partial x_i), \]

then the variety $X(q)$ can be defined as the zeros of the ideal $J = (f_{x,y}, f_{x,z}, f_{y,z})$.

The following proposition states that in order to determine the existence of the limit, it suffices to analyze the behavior of the function $q(x, y, z)$ along the discriminant variety $X(q)$.

**Proposition 3.** The limit $\lim_{(x,y,z) \to (0,0,0)} q(x, y, z)$ exists, and equals $L \in \mathbb{R}$, if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $(x, y, z) \in X(q)$ with $0 < \{(x, y, z)\} < \delta$ the inequality $|q(x, y, z) - L| < \epsilon$ holds.

**Proof.** The method of Lagrange multipliers applied to the function $q(x, y, z)$ with the constraint $x^2 + y^2 + z^2 = r^2$, $r > 0$ guarantees that if $C_r(0) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ then the extreme values of $q(x, y, z)$ on $C_r(0)$ are taken at those points $p = (a, b, c) \in C_r(0)$ for which $(\partial q/\partial x(p), \partial q/\partial y(p), \partial q/\partial z(p)) = \lambda (a, b, c)$, i.e., at those points in $X(q)$. Suppose that given $\epsilon > 0$ there is $\delta > 0$ such that for every $(x, y, z) \in X(q) \cap D^*_\delta$ the inequality $|q(x, y, z) - L| < \epsilon$ holds, where $D^*_\delta = \{(x, y, z) \in \mathbb{R}^3 : 0 < \sqrt{x^2 + y^2 + z^2} < \delta\}$. Let $(x, y, z) \in D^*_\delta$ and $r = \sqrt{x^2 + y^2 + z^2}$. If $t(r)$, $s(r) \in C_r(0)$ are respectively the maximum and minimum values of $q(x, y, z)$, subject to $C_r(0)$, then

\[ q(s(r)) - L \leq q(x, y, z) - L \leq q(t(r)) - L. \]

As $t(r)$ and $s(r) \in X(q) \cap C_r(0) \subset X(q) \cap D^*_\delta$, one sees that $-\epsilon < q(s(r)) - L$, and henceforth $q(t(r)) - L < \epsilon$. Thus, $|q(x, y, z) - L| < \epsilon$.

The reciprocal is obvious. \(\square\)

### 2.3 Birational equivalence of curves

We intend to reduce the problem of determining the existence of the limit to a problem in fewer variables. In order to achieve this we will use the fact that any algebraic curve is birationally equivalent to a plane curve. This result follows from the following standard result:

**Proposition 4** (existence of primitive elements). Let $K$ be a field of characteristic zero and let $L$ be a finite algebraic extension of $K$. Then there is $z \in L$ such that $L = K(z)$ ([5], Page 75).
This immediately implies the following corollary:

**Corollary 5.** Let $X$ be an irreducible algebraic curve over a field $k$ of characteristic zero, and let $K$ be the quotient field of the ring of coordinates of $X$. Then for any $x \in K - k$ which is not algebraic over $k$, $K$ is algebraic over $k(x)$, and there is an element $y \in K$ such that $K = k(x, y)$.

The next theorem is a well known fact. Notwithstanding, we give a proof since we will need the explicit construction of the isomorphism denoted by $\mu$ in the following theorem.

**Theorem 6.** Let $X$ be an irreducible space curve $X$ in $\mathbb{C}^3$ defined by polynomials with real coefficients, and such that the origin of $\mathbb{C}^3$ is a point of $X$. Then there exists an irreducible affine plane curve $Y \subset \mathbb{C}^2$ and a field isomorphism $\varphi : K(Y) \to K(X)$ so that $X$ is birationally equivalent to $Y$. After removing a finite set of points $Z \subset Y$, if $Y_0 = Y \setminus Z$, then there is a morphism $\mu : X \to Y_0$ such that $\mu$ restricted to $X_0 = \mu^{-1}(Y_0)$ is an isomorphism onto $Y_0$. Both $\mu$ and its inverse can be explicitly constructed.

**Proof.** Suppose that $X = V(P)$, where $P \subset \mathbb{R}[X, Y, Z]$ is a prime ideal. Since $X$ is an irreducible algebraic curve $\dim(X) = \dim(\mathbb{R}[X, Y, Z]/P) = 1$. Denote by $\mathbb{R}(x, y, z)$ the fraction field of $\mathbb{R}[X, Y, Z]/P$. Recall that $\dim(X) = \text{trdeg}_K\mathbb{R}(x, y, z)$.

For dimensional reasons some of the variables $x, y$ or $z$ has to be transcendental over $\mathbb{R}$. Suppose without loss of generality that $x$ is transcendental over $\mathbb{R}$. Corollary 5 implies that $\mathbb{R}(x) \subset \mathbb{R}(x, y, z)$ is an algebraic extension. By Proposition 1 one can always find $u = y + \lambda z$, for some $\lambda \in \mathbb{R}(x)$, such that $\mathbb{R}(x, y, z) = \mathbb{R}(x, u)$. Moreover, since this is true for almost all $\lambda$, this element can be taken to be any real constant, except for finitely many choices. Define $\varphi : \mathbb{R}[S, T] \to \mathbb{R}[x, u] \subset \mathbb{R}(x, y, z)$ as the $\mathbb{R}$-algebra homomorphism that sends $S \rightarrow x$ and $T \rightarrow u$. Clearly $\varphi$ is surjective, and therefore, if $J = \ker(\varphi)$, there is an isomorphism of $\mathbb{R}$-algebras $\varphi \circ J \to \mathbb{R}[x, u]$. Consequently, $J \subset \mathbb{R}[S, T]$ is a prime ideal. Denote $V(J)$ by $Y$. The last isomorphism induces a field isomorphism $\varphi : \mathbb{R}(Y) \cong \mathbb{R}(x, u) \to \mathbb{R}(x, y, z)$ defined as $\varphi(x) = x, \varphi(u) = y + \lambda z$. Therefore, $\dim(Y) = \dim(X) = 1$. Hence, $Y = V(J)$ is an irreducible algebraic plane curve which is birationally equivalent to $X$.

The morphism $\varphi : \mathbb{R}(Y) \to \mathbb{R}(X)$ induces a morphism of varieties $\mu : X \to Y$ given by $\mu(a, b, c) = (a, b + \lambda c).

Notice that since $(0, 0, 0) \in X$, then, obviously, $(0, 0) \in Y$. Since $Y$ is an irreducible plane curve, $J$ must be a height one prime ideal. Thus, $J = (h)$, for some $h(X, U) \in \mathbb{R}[X, U]$.

We can assume that the polynomial $h(a, U)$ obtained by replacing the variable $X$ by $a \in \mathbb{C}$ is not identically zero: If $h(a, U) = 0$ we would have $h(X, U) = (X - a)^m t(X, U)$, with $t(a, U) \neq 0$. But $(0, 0) \in Y$ implies $a = 0$, and henceforth $h(X, U) = X^m t(X, U)$. Thus, $h(X, U) = X$ or $h(X, U) = t(X, U)$, since $Y$ is irreducible. Finally, we note that $h(X, U) = X$ contradicts the fact that $x$ is transcendental over $\mathbb{R}$.

On the other hand, since $x$ is transcendental over $\mathbb{R}$, by Corollary 5 the extension $\mathbb{R}(x) \subset \mathbb{R}(x)_{(u)}$ is algebraic. Therefore, since $y \in \mathbb{R}(x, u)$ one can write $y$ as:

$$y = \frac{a_0(x)}{b_0(x)} + \frac{a_1(x)}{b_1(x)} u + \cdots + \frac{a_r(x)}{b_r(x)} u^r,$$

where $r$ is smaller than the degree of the field extension $[\mathbb{R}(x)(u) : \mathbb{R}(x)]$. Taking $b(x) = b_0(x) \cdots b_r(x)$ we can rewrite the last equation as

$$y = \frac{c_0(x) + c_1(x) u + \cdots + c_r(x) u^r}{b(x)}, \quad (4)$$

for certain $c_i(x)$. Therefore, we have $y = f_1(x, u)/g_1(x)$ and $z = f_2(x, u)/g_2(x)$. Consider $Z = \{(a, b) \in Y : g_1(a) = 0 \text{ or } g_2(a) = 0\}$, which is a Zariski closed subset of $Y$. 


Let us see that $Z$ is a finite set. Indeed, the polynomials $g_1$ and $g_2$ have finitely many roots. Therefore, if $a_1, \ldots, a_k \in \mathbb{C}$ are these roots, for each $a_i$, $(a_i, b) \in Y$ if and only if $h(a_i, b) = 0$, where $Y = V(J)$ with $J = (h)$. Notice that the polynomial $t(U) = h(a_i, U) \in \mathbb{C}[U]$ has finitely many roots. Hence, there are only finitely many elements $(a_i, b)$ with $g_1(a_i) = 0$ or $g_2(a_i) = 0$, and such that $h(a_i, b) = 0$. Thus, we conclude that $Z$ is finite.

Consider the open subset $Y_0 = Y \setminus Z$ of $Y$. Let $X_0 = \mu^{-1}(Y_0)$. Define $\tau : Y_0 \to X_0$ as $\tau(a, c) = (d, \frac{f_1(d, c)}{g_1(d)}, \frac{f_2(d, c)}{g_2(d)})$. This last morphism induces an $\mathbb{R}$-algebra homomorphism $\psi : \mathbb{R}(X) \to O_Y(Y_0)$ given by $\psi(x) = s$, $\psi(y) = f_1(s, t)/g_1(s)$, and $\psi(z) = f_2(s, t)/g_2(s)$. Clearly, $\varphi \circ \psi(x) = x$, $\varphi \circ \psi(y) = \frac{f_1(x, u)}{g_1(x)} = y$, $\varphi \circ \psi(z) = \frac{f_2(x, u)}{g_2(x)} = z$. (5)

Therefore, $\varphi \circ \psi = Id_{\mathbb{R}(X)}$, and consequently $\varphi \circ \psi|_X : O_X|_X \to O_Y|_{Y_0}$ is the identity. On the other hand, $\psi \circ \varphi(s) = \psi(x) = s$ and $\psi \circ \varphi(t) = \psi(u)$. By (5) we have $\varphi \circ \psi(u) = u$ and $\varphi(t) = u$, which implies that $t = \psi(u)$, since $\varphi$ is injective. Hence, $\psi \circ \varphi(t) = t$ and therefore $\psi \circ \varphi|_{Y_0} : O_Y|_{Y_0} \to O_X|_X$ is the identity. Hence, $\psi : O_X|_X \to O_Y|_{Y_0}$ is the inverse of the morphism $\varphi : O_Y|_{Y_0} \to O_X|_X$. Thus, the homomorphism $\tau : Y_0 \to X_0$ induced by $\psi$ is the inverse of $\mu : X_0 \to Y_0$.

Finally, it is clear that the morphism $\mu : X_0 \to Y_0$ sends the real part of $X_0$ into the real part of $Y_0$, and since $\mu^{-1} = \tau : Y_0 \to X_0$ is determined by the polynomials $f_1, f_2, g_1$ and $g_2$, which are all real polynomials, then $\mu^{-1} = \tau$ also sends the real part of $Y_0$ into the real part of $X_0$.

**Remark 7.** $X_0$ is obtained from $X$ by removing finitely many points.

**Proof.** In fact, a point $(a, b, c) \in X$ does not belong to $X_0$ if and only if $\mu(a, b, c) = (a, b + \lambda c) \notin Y_0$, i.e., if $a \in \mathbb{R}$ and $b + \lambda c$ such that $(a, b + \lambda c) \notin Y_0$. Fix any values for $a$ and for $\eta = b + \lambda c$. If $f_1(x, y, z), \ldots, f_k(x, y, z)$ are generators for $P$ then, clearly, $f_1(a, \eta - \lambda c, e) = 0$. But each polynomial $g_i(z) = f_i(a, \eta - \lambda z, z)$ can only have finitely many roots. This proves the claim.

This Remark tells us that the problem of determining (and computing) the limit of a function along the varieties $X$ and $Y$ is equivalent to the same problem when one approaches the origin along $X_0$ and $Y_0$.

### 2.4 Groebner bases

In this section we collect some basic properties and results on Groebner bases and Elimination Theory that will be needed later for the development of an algorithm that computes $\mathfrak{I}$. The main reference for this section is [2], Chapter 15.

By $S = K[x_1, \ldots, x_n]$ we denote the polynomial ring in $n$-variables with coefficients in a field $K$. We denote the set of monomials of $S$ by $M$. By a term in $S$ is meant a polynomial of the form $cm$, where $c \neq 0 \in K$ and $m \in M$.

**Definition 8.** A monomial order in $S$ is a total order on $S$ satisfying $nm_1 > nm_2 > m_2$, for every monomial $n \neq 1$, and for any pair of monomials $m_1$ and $m_2$ satisfying $m_1 > m_2$.

Every monomial order is Artinian which means that every subset of $S$ has a least element.

For a fixed monomial order $> \in S$, the initial term of $p \in S$ is the term of $p$ whose monomial is the greatest with respect to $>$. It is usually denoted by $\text{in}(p)$. Given an ideal $I \subset S$, its ideal of initial terms, $\text{in}(I)$, is defined as the ideal generated by the set $\{\text{in}(p) : p \in I\}$.
Definition 9. Let $I \subset S$ be any ideal, and fix a monomial order in $S$. We say that a set of elements $\{f_1, \ldots, f_k\}$ of $I$ is a Groebner basis for $I$ iff $\text{in}(I) = (\text{in}(f_1), \ldots, \text{in}(f_k))$.

We list some basic facts about Groebner bases.

Remark 10.  
1. The set of monomials not in the ideal in$(I)$ forms a basis for the $K$-vector space $S/I$.

2. There always exists a Groebner basis for an ideal $I \subset S$. As $S$ is a Noetherian ring, the ideal $I$ is finitely generated, let’s say, $I = (f_1, \ldots, f_k)$. Consider the ideal $J = (\text{in}(f_1), \ldots, \text{in}(f_k))$. If $J = \text{in}(I)$ then $\{f_1, \ldots, f_k\}$ is a Groebner basis for $I$.

3. If $\{f_1, \ldots, f_k\}$ is a Groebner basis for $I$ then $I = (f_1, \ldots, f_k)$.

4. There is a criterion that allows to compute algorithmically a Groebner basis for an ideal $I \subset S$. This criterion is known as Buchberger’s algorithm ([7], Page 332).

5. Let $I, J$ be ideals of $S$ such that $I \subset J$. If $\text{in}(I) = \text{in}(J)$ then $I = J$.

An example of a monomial order is the lexicographic order, defined in the following way: Fix any total order for the variables, for instance $x_1 > x_2 > \cdots > x_n$, and define $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} > x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ if for the first $j$ with $a_j \neq b_j$ one has $a_j > b_j$. (The lexicographic order will be the monomial order that we will use in this article.)

Now we discuss a basic result that will be needed in Sections 4 and 5.

Let $I$ be an ideal of the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_s]$. Given a Groebner basis for $I$ we want to compute a Groebner basis for $I \cap K[x_1, \ldots, x_n]$. For this, we have to introduce the notion of an elimination order:

Definition 11. A monomial order in $K[x_1, \ldots, x_n, y_1, \ldots, y_s]$ is called an elimination order if the following condition holds: $f \in K[x_1, \ldots, x_n, y_1, \ldots, y_s]$ with $\text{in}(f) \in K[x_1, \ldots, x_n]$ implies $f \in K[x_1, \ldots, x_n]$.

Lemma 12. Let $I \subset K[x_1, \ldots, x_n, y_1, \ldots, y_s]$ be an ideal, and let $\mathcal{B} = \{f_1, \ldots, f_k\}$ be a Groebner basis for $I$ with respect to an elimination order. Assume that $f_1, \ldots, f_t$ with $t \leq k$ are all elements of $\mathcal{B}$ such that $f_1, \ldots, f_t \in K[x_1, \ldots, x_n]$. Then $\{f_1, \ldots, f_t\}$ is a Groebner basis for $I \cap K[x_1, \ldots, x_n]$.

Proof. (See [7], Page 380).

Remark 13. Suppose that $\varphi : K[x_1, \ldots, x_n] \rightarrow K[y_1, \ldots, y_s]/J$ is a ring homomorphism defined as $\varphi(x_1) = f_1$. Consider $F_1 \in K[y_1, \ldots, y_s]$ such that $F_1 = f_1$ in $K[y_1, \ldots, y_s]/J$, and define the ideal $I = JT + (F_1 - x_1, \ldots, F_s - x_n) \subset T$, where $T = K[x_1, \ldots, x_n, y_1, \ldots, y_s]$. Then $\ker \varphi = I \cap K[x_1, \ldots, x_n]$. Therefore the above lemma implies that $\ker \varphi$ can be computed algorithmically.

Proof. (See [7], Page 358).

3 Reduction to the case of functions of two variables

Let $q(x, y, z) = f(x, y, z)/g(x, y, z)$ be the quotient of two polynomials. We recall (Section 2.2) that the discriminant variety associated to $q$, $X(q) \subset \mathbb{C}^3$, is the affine variety defined by the $2 \times 2$ minors of the matrix we denoted by $A$. As a variety, $X(q)$ may be decomposed into its irreducible components in $\mathbb{C}^3$, let’s say $X(q) = X_1 \cup X_2 \cup \cdots \cup X_k$.

We are only interested in those components that contain the origin. These will be called the relevant components. Suppose these are $X_1, X_2, \ldots, X_k$, $k \leq n$. We consider three possible cases:
1. \( \dim X_i = 0 \): In this case, if \( X_i = V(P_i) \), then \( \mathbb{R}[x, y, z]/P_i \) is a field and \( X_i \) is just the origin \( \{O\} \). Hence, \( X_i \) does not contribute to any trajectory in \( \mathbb{R}^3 \) that approaches \( O \), and can be discarded.

2. \( \dim X_i = 1 \): In this case \( X_i \) is an irreducible algebraic curve.

3. \( \dim X_i = 2 \): In this case \( X_i \) is a hypersurface, i.e., \( X_i = V(P_i) \), where \( P_i \) is a principal ideal.

We only have to study Cases 2 and 3.

We deal first with the case of an irreducible space curve in \( \mathbb{C}^3 \). Let us see that the problem of determining the limit of \( q(x, y, z) \) along \( X \), as well as its computation can be reduced to the case of a real plane curve, a question already addressed in [4].

By Theorem \[6\] there is a plane curve \( Y \) which is birationally equivalent to \( X \), and therefore a local isomorphism \( \mu: X_0 \rightarrow Y_0 \), where \( X_0 \) and \( Y_0 \) are as in Theorem \[6\]. There we observed that the existence of the limit of \( q(x, y, z) \) as \( (x, y, z) \rightarrow (0, 0, 0) \) along \( X_0 \) is equivalent to the existence of the limit of \( q \circ \mu^{-1} \) as \( (u, v) \rightarrow (0, 0) \) along \( Y_0 \). Thus,

\[
\lim_{(x, y, z) \rightarrow O} q(x, y, z) = \lim_{(u, v) \rightarrow O} q \circ \mu^{-1}(u, v). \tag{6}
\]

Summarizing:

**Proposition 14.** Let \( X \subset \mathbb{C}^3 \) be an irreducible component of \( X(q) \) of dimension 1 containing \( O \). Let \( \mu: X_0 \rightarrow Y_0 \) be the local isomorphism defined in Theorem \[6\]. Then, the limit of \( q(x, y, z) \) as \( (x, y, z) \rightarrow O \) along \( X \) exists if and only if exists along the irreducible plane curve \( Y \) as \( (u, v) \rightarrow (0, 0) \). The corresponding limits are related by \( (6) \).

In the sequel we will denote by \( X_i \) and \( Y_i \) the two open subsets \( X_0 \subset X \) and \( Y_0 \subset Y \) defined in Theorem \[6\]. For the purpose of analyzing the limit along the space curve \( X \) it is only necessary to consider those cases where the real trace of the birationally isomorphic curve \( Y \) turns out to be a plane curve containing the origin. By \( \mu_X \), we denote the corresponding isomorphism between \( X_i \) and \( Y_i \) already constructed.

Now we analyze Case 3. This is a lot more subtle, and requires a careful analysis of the topology of the corresponding two dimensional component. A key ingredient is a celebrated theorem of Whitney [20] about the number of connected components of an affine algebraic variety.

In the following discussion we will show how one can reduce the analysis of the 2-dimensional irreducible components to Case 2.

Suppose that we have a rational function \( q(x, y, z) = f(x, y, z)/g(x, y, z) \) defined on an irreducible hypersurface \( X = V(h) \), where \( h \) is a real polynomial function of three variables and \( q \) has an isolated zero at 0. Let \( S = \text{Sing}(X) \) be the singular locus of \( X \). By Remark \[2\] \( S \) must be a variety of dimension strictly less than two. Hence, if \( S \) contains the origin, the limit of \( q \) as \( (x, y, z) \rightarrow O \) along \( S \) can be computed as in Case 2.

Now, we restrict our analysis to the nonsingular locus of \( X \), that we denote by \( \mathcal{N} = X \setminus S \). Without loss of generality we may assume that \( \mathcal{N} \) contains the origin, otherwise all of their components would be irrelevant.

Assume \( O \in \mathcal{N} \), and define a family of real ellipsoids \( E_r = \{(x, y, z) \in \mathbb{R}^3 : Ax^2 + By^2 + Cz^2 - r^2 = 0\} \), \( A, B, C > 0 \), \( r \neq 0 \). By \( p_r(x, y, z) \) we will denote the quadratic polynomial \( Ax^2 + By^2 + Cz^2 - r^2 \).
Definition 15. Let $X = V(h) \subset \mathbb{C}^3$ and $E_r = \{(x,y,z) \in \mathbb{R}^3 : Ax^2 + By^2 + Cz^2 - r^2 = 0\}, r \neq 0$ as above. The critical set $C_r(q)$ will be the set of all real points in $E_r \cap X$ where $q(x,y,z)$ attains its maxima and minima. The union $\cup_{r>0} C_r(q)$ of all critical sets will be denote by $\text{Crit}_X(q)$.

Since each $E_r \cap X$ is a compact set, and by hypothesis $O$ is an isolated zero of $q$, the set $\text{Crit}_X(q)$ is a well defined subset of $X$.

We need the following analogue of Proposition 3.

Proposition 16. The limit $\lim_{(x,y,z) \to O} q(x,y,z)$ along $X$ exists and equals $L$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $0 < r < \delta$ the inequality $|q(x,y,z) - L| < \epsilon$ holds for all $(x,y,z) \in C_r$.

Proof. The proof follows identical lines as in Proposition 3. One just have to notice that each point in the critical set must lie in some $E_r$, since $p = (a,b,c)$ is obviously contained in $E_r$, with $r = \sqrt{Ax^2 + By^2 + Cz^2}$.

Our objective is to determine $\text{Crit}_X(q)$. We can decompose this set as the union of $\text{Crit}_\mathcal{N}(q) = \text{Crit}_X(q) \cap N$ and $\text{Crit}_X(q) \cap S$. Since $\text{Crit}_X(q) \cap S \subset S$, and the limit along $S$ can be determined as in Case 2, we just have to focus on $\text{Crit}_X(q)$.

First, we want to determine the nonsingular part of $\text{Crit}_\mathcal{N}(q)$ by using the method of Lagrange Multipliers, as in [H]. For this we define $\mathcal{X} = V(\mathcal{J}) \subset X$ to be the zero set of the ideal $\mathcal{J}$ generated by $h$ and the determinant:

$$d(x,y,z) = \begin{vmatrix} \frac{\partial p_1}{\partial x} & \frac{\partial p_1}{\partial y} & \frac{\partial p_1}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \end{vmatrix}.$$  

As the points of $X$ already satisfy $\nabla q(x,y,z) = \lambda(x,y,z)$ (where $\nabla q$ denotes the gradient of $q$), and since $\nabla p_1(x,y,z) = (2Ax, 2By, 2Cz)$, the affine variety $\mathcal{X}$ must be defined by the ideal generated by $h$ and by the determinant:

$$D(x,y,z) = \begin{vmatrix} Ax & By & Cz \\ x & y & z \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix}.$$  

That is, $\mathcal{X} = V(D,h)$. This variety is precisely the set of regular points of $X$ that are critical points of $q$.

Proposition 17. (Notation as above) Let us assume $O \in \mathcal{N}$. Then it is possible to choose (in a generic way) suitable positive constants $A,B$ and $C$ such that the height of the ideal $\mathcal{J} = (D,h)$ in the polynomial ring $\mathbb{C}[x,y,z]$ is greater than one, and consequently $\dim \mathcal{X} < 2$.

Proof. It suffices to show that for a suitable choice of positive constants $A,B,C$ there is at least one point $p \neq O$ in $\mathcal{N}$ such that $D(p) \neq 0$.

First, let us see that there is at least one point $p \in \mathcal{N}$ different from the origin such that the gradient of $h$ does not point in the direction of $p$, i.e., such that $\nabla h(p) \neq \lambda p$, for all $\lambda \in \mathbb{R}$. Indeed, suppose on the contrary that for every $p \in \mathcal{N}$ there existed $\lambda(p) \neq 0$ such that $\nabla h(p) = \lambda(p)p$. Since each $p$ is a regular point of $X$, one must have $\nabla h(p) \neq 0$. Hence, after making an appropriated change of coordinates that fixes $O$ (a rotation, and then a homothety) we may assume without loss of generality that $\nabla h(0,0,1) = 0$, and that $p = (0,0,1)$. By the implicit function theorem there would exist $U_0 \subset \mathbb{R}^2$, a neighborhood of $(0,0)$, and a smooth function $u(x,y)$ in $U_0$ such that $u(0,0) = 1$, and $h(x,y,u(x,y)) = 0$, for all $(x,y) \in U_0$. Since $\nabla h(p) = \begin{vmatrix} A \end{vmatrix}$, we have $\begin{vmatrix} A \end{vmatrix} 

\begin{vmatrix} u_x & u_y \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} \neq 0$ for all $(x,y) \in U_0$. Since $\nabla h(p)$ is not zero, $\mathcal{X}$ is a nontrivial variety.
\( \lambda(p) p \), one must have \( \partial h/\partial x(0,0,1) = \partial h/\partial y(0,0,1) = 0 \), and consequently \( \partial u/\partial x(0,0) = 0 = \partial u/\partial y(0,0) \).

Let \( W_p \) be the graph \( W_p = \{ (x,y,u(x,y)) : (x,y) \in U_0 \} \). For any \( t \in W_p \), the normal vector at \( t \) is given by

\[
n(t) = \left( \frac{-u_x, -u_y, 1}{\sqrt{u_x^2 + u_y^2 + 1}} \right).
\]

Henceforth, if \( \mu(t) = \lambda(t)/\| \nabla h(t) \| \) one has that \( \nabla h(t) = \mu(t)\| \nabla h(t) \| t \), and consequently \( n(t) \) can be written as

\[
n(t) = \frac{(x, y, u(x,y))}{\sqrt{x^2 + y^2 + u^2(x,y)}}.
\]

From this, we deduce:

\[
\frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} = \frac{u(x,y)}{\sqrt{x^2 + y^2 + u^2(x,y)}} = \frac{x}{\sqrt{x^2 + y^2 + u^2(x,y)}}
\]

and

\[
\frac{-u_x}{\sqrt{u_x^2 + u_y^2 + 1}} = \frac{y}{\sqrt{x^2 + y^2 + u^2(x,y)}}.
\]

This implies \( u_x = -x/u(x,y), \) and \( u_y = -y/u(x,y) \). Hence, \( u(x,y) = \sqrt{1-x^2-y^2} \), since \( u(0,0) = 1 \). We conclude that \( W_p \) would be a neighborhood of \( p \) in \( N \) which is part of a sphere centered at the origin. But on the other hand, a theorem of Whitney asserts that \( N \) can only have finitely many connected components (see [20]). Then this would imply that \( N \) could not contain the origin, a contradiction with our assumption.

Therefore, we may assume there exists a point \( p \neq O \) in \( N \) such that \( \nabla h(p) \neq \lambda p \), for all \( \lambda \neq 0 \). After applying a rotation (if necessary) we may also assume that \( a, b, c \) are all nonzero.

After those preliminaries it becomes clear how to choose positive constants \( A, B \) and \( C \) such that the determinant

\[
\begin{vmatrix}
Aa & Bb & Cc \\
\partial h/\partial x(a,b,c) & \partial h/\partial y(a,b,c) & \partial h/\partial z(a,b,c)
\end{vmatrix}
\]

does not vanish: The vectors \( \nabla h(p) \) and \( p = (a,b,c) \) generate a plane \( H \), since they are not parallel. Therefore, it suffices to choose any point \( (\alpha,\beta,\gamma) \) outside \( H \) and such that \( A = \alpha/a, \) \( B = \beta/b, \) and \( C = c/\gamma \) are positive.

As before, for the limit \( \lim_{(x,y,z) \rightarrow O} q(x,y,z) \) to exist along \( X \) it is necessary that it exists along any real curve that contains \( O \). In particular, the limit along each component of \( X \) must exist, and all these limits must be equal. By Proposition 17, \( \dim(X) < 2 \), and henceforth we can reduce this last question to cases 1 and 2.

Let \( \mathcal{Z} \) be the affine variety defined by the ideal generated by \( h \) and by the minors \( 2 \times 2 \) of the matrix

\[
\begin{bmatrix}
Ax & By & Cz \\
\partial h/\partial x & \partial h/\partial y & \partial h/\partial z
\end{bmatrix}.
\]
The set $\mathfrak{Z} \cap E_r \cap \mathcal{N}$ defines the locus of those real points where $E_r$ and $\mathcal{N}$ do not intersect transversely. Outside this set, $E_r \cap \mathcal{N}$ is a 1-dimensional manifold (see [13], Page 30) that we shall denote by $\Sigma$. Clearly, the vanishing of these two by two minors forces the vanishing of the determinant $D(x, y, z)$. Henceforth, $\mathfrak{Z} \subset \mathfrak{X}$, and consequently $\dim(\mathfrak{Z}) < 2$, by Proposition 17. Again, for the existence of the limit $\lim_{(x,y,z) \to O} q(x, y, z)$ it is required, in particular, its existence along any relevant component of $\mathfrak{Z}$, and consequently the problem reduces again to cases 1 and 2. This takes care of the subset of $\text{Crit}_X(q)$ inside $\mathfrak{Z}$.

As for those points in $\text{Crit}_X(q)$ that lie outside $\mathfrak{Z}$, we notice that they are contained in the 1-dimensional manifold $\Sigma$. Then they must be part of $\mathfrak{X}$, since this variety is precisely those regular points where $q$ attains an extreme value. Thus, the points in $\text{Crit}_X(q)$ that lie outside $\mathfrak{Z}$ must be contained in $\mathfrak{X}$. Once again, we have reduced the problem to cases 1 and 2.

The following proposition summarizes this discussion:

**Proposition 18.** Let $X$ be a relevant irreducible component of dimension 2 of the discriminant variety $X(q)$. Consider $\mathcal{S}$, $\mathfrak{X}$, and $\mathfrak{Z}$ as defined above. Then, the limit of $q(x, y, z)$ as $(x, y, z) \to O$ along $X$ exists, and equals $L$, if and only if, the limit of $q(x, y, z)$ as $(x, y, z) \to O$ exists and equals $L$ along each one of the components of the curves $\mathcal{S}$, $\mathfrak{X}$, and $\mathfrak{Z}$.

We are now ready to state our main result.

**Theorem 19.** Let $q(x, y, z) = f(x, y, z)/g(x, y, z)$, where $f$ and $g$ are rational polynomial functions, and where $g$ has an isolated zero at the origin. Let $X(q)$ be the discriminant variety associated to $q$. Denote by $\{X_1, \ldots, X_k\}$ the relevant irreducible components of dimension one of $X(q)$, and by $\{X_{k+1}, \ldots, X_n\}$ the relevant irreducible components of dimension two of $X(q)$. Then, the limit of $q$ as $(x, y, z) \to O$ exists, and equals $L$, if and only if the limit of $q(x, y, z)$ as $(x, y, z) \to O$ along $X_i$ exists, and equals $L$, for all $i = 1, 2, \ldots, n$. Moreover:

1. For the components $X_i$, $i = 1, 2, \ldots, k$, the limit of $q(x, y, z)$ as $(x, y, z) \to (0, 0, 0)$ along $X_i$ is determined as in Proposition 17.
2. For the components $X_j$, $j = k+1, \ldots, n$, the limit of $q(x, y, z)$ as $(x, y, z) \to (0, 0, 0)$ along $X_j$ is determined as in Proposition 17.

4 A high level description of an algorithm for computing the limit

Let $q(x, y, z) = f(x, y, z)/g(x, y, z)$, where $f$ and $g$ are polynomial functions of three variables with rational coefficients, and $g$ has an isolated zero at the origin. Consider $X(q)$, the discriminant variety associated to $q$. We have to decompose $X(q)$ into irreducible components, and then choose only those irreducible components $\{X_1, \ldots, X_n\}$ that are relevant.

The algorithm has to deal with two different cases:

- **D1:** The component $X_i$ has dimension 1. Then as observed before, $X_i$ is birationally equivalent to an irreducible plane curve $Y_i$. Let us denote by $\mathbb{C}(x, y, z)$ the fraction field of the ring of coordinates $\mathbb{C}[X, Y, Z]/I(X_i)$ of $X_i$. As we already noticed we may always assume that $x, y, z$ are transcendental elements over $\mathbb{C}$: If, for instance, $x$ were algebraic over $\mathbb{C}$, then there would exist a polynomial $P(X) \in \mathbb{C}[X]$ such that $P(x) = 0$. This is equivalent to saying that $P(X) \in I(X_i)$.

Suppose we write $P(X) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n)$
Finally, if the limit of \( I(X_i) \) is a prime ideal, some linear factor \( X - \alpha_j \) must belong to \( I(X_i) \). But as \( X_i \) contains the origin, we must have \( \alpha_j = 0 \). Hence, we could write \( I(X_i) = (X, h_1(Y, Z), \ldots, h_m(Y, Z)) \), where \( h_k(Y, Z) \in \mathbb{C}[Y, Z] \), for \( k = 1, 2, \ldots, m \). If we denote by \( I \) the ideal \( (h_1(Y, Z), \ldots, h_m(Y, Z)) \), and by \( X'_i = V(I) \subset \mathbb{C}^2 \) the affine variety defined by \( I \), then the limit of \( q(X, Y, Z) \) as \( (X, Y, Z) \to O \) along \( X_i \) is the same as the limit of \( q(0, Y, Z) \) as \( (Y, Z) \to (0, 0) \) along \( X'_i \). But the existence of this limit, as well as its value, can be computed using the algorithmic method developed in [4]. By Proposition 2.4 and Corollary 2.4 we know that if \( x \) is transcendental over \( \mathbb{C} \) there exists \( \lambda \in \mathbb{C}(x) \) such that \( \mathbb{C}(x, y, z) = \mathbb{C}(x, u) \), where \( u = y + \lambda z \). Also, by Theorem 5 if we consider \( \varphi : \mathbb{C}[X, U] \to \mathbb{C}[X, Y, Z]/I(X) \) defined by \( \varphi(X) = x \), \( \varphi(U) = y + \lambda z \), then \( \ker \varphi \) defines the irreducible plane curve \( Y \) that is birationally equivalent to \( X \). As we observed in Section 2.4, \( \ker \varphi = (I(W) + U(Y + \lambda Z)) \cap \mathbb{C}[X, U] \), where \( T = \mathbb{C}[X, U, Y, Z] \) is computable. On the other hand, the ring homomorphism \( \mathbb{C}[X, U] \to \ker \varphi \to \mathbb{C}[X, Y, Z]/I(X) \) induces an isomorphism of fields \( \varphi : K(X) \to K(Y) \). As we showed in the proof of Theorem 6 since \( y, z \in \mathbb{C}(x, u) \) then one must have \( y = f_1(x, u)/g_1(x) \), and \( z = f_2(x, u)/g_2(x) \), for some \( f_1, f_2, g_1, \) and \( g_2 \) with real coefficients. In that same proof we noticed that the local isomorphism \( \mu : X_0 \to Y_0 \) is determined by those polynomials.

By Proposition 2.4, computing the limit of \( q(X, Y, Z) \) as \( (X, Y, Z) \to O \) along \( X_i \) is equivalent to computing the limit of \( q \circ \mu^{-1}_{X_i}(X, U) \), as \( (X, U) \to (0, 0) \) along \( X_i \), and this last limit can be dealt with using the algorithm developed in [4].

- **D2:** Suppose that \( \dim X_i = 2 \). Then \( X_i \) is an affine variety defined by a principal ideal \( I(X_i) = (h) \). For random positive values \( A, B, C \) the algorithm computes the height of the ideal \( J = (D, h) \), where

\[
D = \begin{vmatrix}
Ax & By & Cz \\
x & y & z \\
\partial h/\partial x & \partial h/\partial y & \partial h/\partial z
\end{vmatrix}
\]

As we saw in the reduction to plane curves, there always exist positive constants \( A, B, \) and \( C \) such that \( \text{ht}(J) \geq 2 \). Since \( \dim(X) \leq 1 \), then \( \mathcal{X} = V(J) \). Henceforth, one can compute the limit of \( q(x, y, z) \) as \( (x, y, z) \to O \) along \( J \) using the prescription in D1. Since \( \mathcal{S} = \text{Sing}(X) \), the affine variety defined by the ideal \( (h, \partial h/\partial x, \partial h/\partial y, \partial h/\partial z) \) must be a proper subset of \( X \). Then \( \mathcal{S} \) is also an algebraic curve, and once again we can compute the limit of \( q(x, y, z) \) as \( (x, y, z) \to O \) along \( \mathcal{S} \) using D1.

Now, the affine variety \( \mathcal{S} \) defined by the ideal generated by the minors \( 2 \times 2 \) of the matrix \[
\begin{bmatrix}
Ax & By & Cz \\
\partial h/\partial x & \partial h/\partial y & \partial h/\partial z
\end{bmatrix}
\]

and the polynomial \( h \), has also dimension less than 2. Hence, the limit of \( q(x, y, z) \) as \( (x, y, z) \to O \) along \( \mathcal{S} \) is also computed using D1.

- **Finally,** if the limit of \( q(x, y, z) \) as \( (x, y, z) \to O \) along each relevant irreducible component of \( X(q) \) of dimension one exists, and equals \( L \), one says that the limit of \( q(x, y, z) \) as \( (x, y, z) \to O \) is \( L \). Otherwise, one says that this limit does not exists.
5 Examples

5.1 Example 1

Suppose that we want to compute the limit:

$$\lim_{(X,Y,Z)\to(0,0,0)} \frac{YX - ZY + ZX}{X^2 + Y^2 + Z^2}.$$

Let $q(X,Y,Z) = YX - ZY + XZ/X^2 + Y^2 + Z^2$. We illustrate the necessary computations, carried out in the program Maple.

1. Using the command `PrimeDecomposition(X(q))` one gets the irreducible components of $X(q)$:

$$V((Y - X + Z)), \quad V((X^2 + Y^2 + Z^2)), \quad V((X + Y, Z - 2X)), \quad V((X + Y, Z + X)) \quad \text{and} \quad V((X + Y, Z^2 + 2X^2)).$$

By using the command `HilbertDimension(Q)` one can see that the irreducible component $V(X+Y, Z+X)$ has dimension 1.

Let us see that for $\lambda = 1$, $\mathbb{C}(x,u) = \mathbb{C}(x,y,z)$, where $u = y + z$. Here, $\mathbb{C}(x,y,z)$ denotes the fraction field of the ring of coordinates of the variety $V(X + Y, Z + X)$. Consider the ideal $I = (X + Y, Z + X)T + (U - (Y + Z))$, where $T = \mathbb{C}[X,Y,Z,U]$. The command `EliminationIdeal(I,plex(Z,Y,U))` gives us a Groebner basis for $I$ with respect to the lexicographic monomial order, with $Z > Y > U$. In this particular case we obtain the following basis: $\{2X+U,Y+X,Z+X\}$. From this basis we deduce that $y = -x$ and $z = -x$ are elements of $\mathbb{C}(x,u)$. Therefore, $\mathbb{C}(x,u) = \mathbb{C}(x,y,z)$, and consequently the ideal $J = (2X + U)$ defines an irreducible plane curve which is birationally equivalent to $V(X + Y, Z + X)$.

Also, $y = -x$ and $z = -x$ determine the isomorphism $\rho : V(2X + U) \to V(X + Y, Z + X)$. Therefore, the limit of $q(X,Y,Z)$, as $(X,Y,Z) \to (0,0,0)$ along $V(X + Y, Z + X)$, is equivalent to the limit of $q \circ \rho(X,U)$ as $(X,U) \to (0,0)$ along $V(2X + U)$. This latter limit can be computed using the algorithm developed in [4]. However, in this case it is easy to see directly that the value of the limit is $-1/3$, since $q \circ \rho(X,U) = q(X,-X,-X) = -1/3$.

Therefore, the limit of $q(X,Y,Z)$ as $(X,Y,Z) \to (0,0,0)$ along $V(X + Y, Z + X)$ is $-1/3$. Let $h(X,Y,Z) = Y - X + Z$. One may choose $A = 1$, $B = 2$, and $C = 1$. Using the command `HilbertDimension(P)`, with $P = (f,h)$, where

$$f = \begin{vmatrix} X & 2Y & Z \\ X & Y & Z \\ \partial f \partial x & \partial f \partial y & \partial f \partial z \end{vmatrix},$$

one obtains that the variety defined by the ideal $P = (f,h)$ has dimension 1. In this case $V(P) = V(-XY - YZ, Y - X + Z)$. Using again the command `PrimeDecomposition(P)` one obtains the irreducible components of the variety $V(P)$: $V(P) = V(Y,-X + Z) \cup V(2X - Y, Y + 2Z)$, where each of these components has dimension 1. Therefore, one just needs to compute a limit along irreducible algebraic curves (again, using the main algorithm of [4]). For the variety $V(Y + X, Z + X)$ we may follow an analogous procedure. It is not difficult to see that the limit of $q(X,Y,Z)$, as $(X,Y,Z) \to (0,0,0)$ along the variety $V(Y,-X + Z)$ is equal to 1/2.

Hence, we conclude that

$$\lim_{(X,Y,Z)\to(0,0,0)} \frac{YX - ZY + ZX}{X^2 + Y^2 + Z^2} = \frac{1}{2}.$$
5.2 Example 2

1. We want to compute the limit:

\[
\lim_{{(x,y,z) \to (0,0,0)}} \frac{X^2YZ}{X^2 + Y^2 + Z^2}
\]

Let \( q(X, Y, Z) = X^2YZ/X^2 + Y^2 + Z^2 \).

Using the command \texttt{PrimeDecomposition}(X(q)) one obtains the irreducible components of \( X(q) \):

2. The irreducible components of dimension 1 are: \( X_1 = V(X, Y), \ X_2 = V(X, Z), \ X_3 = V(Y, Z), \ X_4 = V(X, Z - Y), \ X_5 = V(X, Z + Y), \ X_6 = V(X, Y^2 + Z^2), \ X_7 = V(X, 3Z^2 - Y^2), \ X_8 = V(X, 3Z^2 + Y^2), \ X_9 = V(Z, X^2 + Y^2), \ X_{10} = V(Z + Y, -2Z^2 + X^2), \ X_{11} = V(-Z + Y, -2Z^2 + X^2), \ X_{12} = V(-2Z^2 + X^2, 3Z^2 + Y^2) \)\).

3. The irreducible components of dimension 2 are: \( X_{13} = V(X) \), and \( X_{14} = V(X^2 + Y^2 + Z^2) \).

4. There is an irreducible component of dimension 0: \( X_{15} = V(X, Y, Z) \).

We know that each irreducible component of dimension 1 is birationally equivalent to an irreducible plane curve. Now, if any of the variables \( X, Y \) or \( Z \) appears in the ideal that defines the corresponding irreducible component, then one can easily see that such component in \( \mathbb{C}^3 \) is actually contained in \( \mathbb{C}^2 \). Henceforth, it is already a plane curve, and one can use the main algorithm of \cite{4} to compute these (two variable) limits. Hence, one sees that the limits along the varieties \( X_1 = V(X, Y), \ X_2 = V(X, Z), \ X_3 = V(Y, Z), \ X_4 = V(X, Z - Y), \ X_5 = V(X, Z + Y), \ X_6 = V(X, Y^2 + Z^2), \ X_7 = V(X, 3Z^2 - Y^2), \ X_8 = V(X, 3Z^2 + Y^2), \ X_9 = V(Z, X^2 + Y^2) \) are equal to zero.

Now we discuss the limit along the other irreducible components of dimension 1.

5. For \( X_{10} = V(Z + Y, -2Z^2 + X^2) \): We noticed in the proof of the Primitive Element Theorem that for almost all \( \lambda \in \mathbb{R}, \mathbb{R}(x, y, z) = \mathbb{R}(x, u) \), where \( u = y + \lambda z \). In this case, one could take \( \lambda = 2 \). Let \( I = (Z + Y, -2Z^2 + X^2, U - (Y + 2Z)) \subset \mathbb{R}[X, Y, Z, U] \), with the command \texttt{EliminationIdeal}(I, X, U) one gets the plane curve \( V(2U^2 - X^2) \), which is birationally equivalent to \( X_{10} \). On the other hand, by using the command \texttt{Basis}(I, plex(Z, Y, U)) one computes the basis \( \{2U^2 - X^2, Y + U, -U + Z\} \). From this basis we deduce that \( y = -u \) and \( z = u \) as elements of \( \mathbb{R}(x, u) \). Thus, the limit along the component \( X_{10} \) is the same as the limit of \( q(X, -U, U) \) along the irreducible plane curve \( V(2U^2 - X^2) \). This latter limit can be calculated using the the main algorithm of \cite{4}. In this case we obtain the value zero.

6. For \( X_{11} = V(-Z + Y, -2Z^2 + X^2) \), with \( \lambda = 1 \) and following the same procedure, i.e., defining the ideal \( I = (-Z + Y, -2Z^2 + X^2, U - (Y + Z)) \) and then computing \texttt{EliminationIdeal}(I, X, U) and \texttt{Basis}(I, plex(Z, Y, U)), one obtains the irreducible plane curve \( V(U^2 - 2X^2) \), which is birationally equivalent to \( X_{11} \), as well as the basis \( \{U^2 - 2X^2, -U + 2Y, -U + 2Z\} \). From this basis one deduces that \( y = u/2 \) and \( z = u/2 \), and therefore the limit along the component \( X_{11} \) is the same as the limit of \( q(X, U/2, U/2) \) along the plane curve \( V(U^2 - 2X^2) \) which is again a limit in two variables, and can be computed using the methods of \cite{4}. In this case the limit is also zero.
7. For $X_{12} = V(-2Z^2 + X^2, 3Z^2 + Y^2)$, with $\lambda = 1$, by defining the ideal

$$I = (-2Z^2 + X^2, 3Z^2 + Y^2, U - (Y + Z)),$$

and then computing $\text{EliminationIdeal}(I, X, U)$ and $\text{Basis}(I, plex(Z, Y, U))$, one obtains the irreducible plane curve $V(4X^4 + 2X^2U^2 + U^4)$, which is birationally equivalent to $X_{12}$, as well as the basis

$$\{4X^4 + 2X^2U^2 + U^4, -U^3 - 4UX^2 + 4ZX^2 + U^3\}.$$

From this we deduce $y = \frac{u^3 + 4ux^2}{4x^2}$, and $z = \frac{-u}{4x}$, and therefore the limit along the component $X_{12}$ is the same as the limit of $q(X, \frac{u^3 + 4ux^2}{4x^2}, \frac{-u^3}{4x^2})$ along the plane curve $V(4X^4 + 2X^2U^2 + U^4)$, which is again a limit in two variables. This limit is also zero.

8. Now, the components of dimension 2 are $V(X^2 + Y^2 + Z^2)$ and $V(X)$. The first component is precisely the set of points where the rational function $q$ is not defined, and consequently can be discarded. Since the variable $X$ appears in the ideal defining the second variety the limit clearly must be zero.

We conclude that

$$\lim_{(X,Y,Z) \to (0,0,0)} \frac{X^2YZ}{X^2 + Y^2 + Z^2} = 0,$$

since it is zero along each of the irreducible components of the discriminant variety.

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