Trade-off between Squashed Entanglement and Concurrence in Bipartite Quantum States

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Abstract
In this article, we investigate the unitary dynamics of squashed entanglement and concurrence measures in Werner state and maximally entangled mixed states (MEMS) under two different Hamiltonians. The aim of the present study is twofold. The first part of the study deals with the dynamics under Heisenberg Hamiltonian and the second part deals under bi-linear bi-quadratic Hamiltonian which is the extension of the first Hamiltonian. In both parts, we investigate the dynamical trade-off and equilibrium points for squashed entanglement and concurrence. During the study, we also found the results of entanglement sudden death (ESD) with Heisenberg Hamiltonian in Werner state under concurrence measure. In the second part, we investigate the special result for the bi-linear bi-quadratic Hamiltonian which does not disturb squashed entanglement and concurrence in both the states and exhibits the robust character for both of the states.

Keywords Squashed entanglement · Concurrence · Werner state · Maximally entangled mixed states (MEMS) · Entanglement sudden death · Equilibrium points · Bi-linear-bi-quadratic Hamiltonian · Heisenberg Hamiltonian

1 Introduction
Quantum information and computation is an emerging area that has a potential impact on future quantum technologies [1]. This area has done tremendous progress in recent years and currently many commercial companies have the interest to explore the cloud services for Noisy Intermediate Scale Quantum Computers (NISQ) [2] by 2025. These quantum computers are also rephrased as Near Term Quantum Computers, which will be equipped with

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50 to 100 qubits. Even though quantum information and computation are very emerging but on the other hand, this area also has theoretical hindrances which are still unexplored [3]. A physical system incorporating superposition, prolonged entanglement, and minimum decoherence plays an important role in quantum information. The scientific community always have a keen interest to discover quantum materials [4] which can perform the quantum computation at room temperature. To date, quantum computer functions at very low temperature and dilute refrigerator is the primary requirement for these computers.

Exploring the entanglement dynamics in quantum information needs to consider varieties of quantum states in the background. Since the quantum states are the building blocks for quantum technologies which have vast applications in quantum teleportation [5–7], quantum sensing [8], quantum cryptography [9, 10] etc. Entanglement is a very fragile phenomenon and very sensitive towards external agents; also it is very difficult to maintain for a long time in varieties of physical systems. Recently investigating the influence of quantum information scrambling on entanglement [11] is also a new area for research and experimental manifestation for prolonged entanglement in varieties of quantum states is on the way. On the theoretical side, it is also difficult to characterize the multipartite entanglement by following the quantum resource theory [12]. The dynamical study of quantum correlations and investigating their sustainability in varieties of quantum states is an important regime for quantum information. The various theoretical entanglement measures like Wootters concurrence [13] and quantum discord [14–16] are tested in varieties of spin chains along with experimental manifestations. In this direction, it is also important to proceed with theoretical studies with other quantum correlation measures.

In the current article, we focus on the dynamical aspects of quantum correlation measures such as squashed entanglement [17] and concurrence [13] in bipartite Werner state [18] and maximally entangled mixed states (MEMS) [19]. Here it is mentioned that both states have potential applications in quantum information theory. In literature, their special properties are already investigated in many domains of quantum information like teleportation, spin squeezing and quantum magnetometry etc. The states are quite significant in quantum information theory to study their different dynamical properties under different situations [20, 21]. The present study is carried out under two different Hamiltonians called, Heisenberg Hamiltonian [22–24] and bi-linear bi-quadratic Hamiltonian [25, 26]. The bi-linear bi-quadratic Hamiltonian is the non-linear version of Heisenberg Hamiltonian and both the Hamiltonians support $SU(2)$ symmetry. The entanglement dynamics under Heisenberg Hamiltonian have been studied in varieties of spin chains configured in thermal and nonthermal conditions [29, 30]. But in literature, the study on bi-linear bi-quadratic Hamilton is very limited.

Following the literature, the present work pursues the study under two different kinds of Hamiltonians and we have found an interesting property that bi-linear bi-quadratic Hamiltonian does not affect the quantum correlations in both the states (Werner state and MEMS). In 2011, squashed entanglement [17, 31] attracted much attention of the quantum community, but the simulations of squashed entanglement in larger Hilbert spaces are also missing to date. So, the present study is new in this direction to the best of our knowledge.

The outlines of the paper are sketched in 8 sections. In Section 2, we present the brief introduction of Werner state, MEMS and characterization of quantum correlations with squashed entanglement and concurrence. Section 3 highlights the Hamiltonians applied in the work and approach of unitary time evolution. Section 4 is devoted to the initial state preparation of quantum states used in the work. In Section 5, we obtain the mathematical functions of squashed entanglement and concurrence with time evolution under Heisenberg Hamiltonian. The dynamics of quantum correlations under Heisenberg Hamiltonian
are explored in Section 6. Section 7 is devoted to the study under the bi-linear bi-quadratic Hamiltonian. The last section explores the conclusion of the work.

2 Werner State, MEMS and Quantum Correlation Measures

In this section, we present the description of the Werner state and MEMS with their corresponding density matrices. Further, we give the overview of two quantum correlation measures named squashed entanglement and concurrence, which we have used in the current work.

Werner state [18] is a widely studied quantum state in quantum information theory and it has its great importance in this field. This is a bipartite quantum state in \( d \times d \) dimensional Hilbert space which is invariant under all unitary operators and satisfies the equation given below,

\[
\rho = (U \otimes U) \rho (U^\dagger \otimes U^\dagger).
\]

Where \( \rho \) is the density matrix of the system. Dealing with two qubits Werner state in \( 2 \times 2 \) dimensional Hilbert space; the Werner state adopt the form as given below,

\[
\rho_{WS} = \gamma |\psi^-\rangle\langle \psi^-| + (1 - \gamma) I_4.
\]

Where \( |\psi^-\rangle \) represents a singlet state and \( I \) is the \( 4 \times 4 \) dimensional identity matrix.

MEMS is another bipartite quantum state, which has a close connection with the bipartite Werner state. The two qubits MEMS was investigated by Munro et al. [19] which is more entangled than two qubits Werner state in terms of concurrence measure and it is experimentally verified also [32]. The density matrix of two qubits bipartite MEMS can be written as,

\[
\rho^{MEMS} = \begin{bmatrix}
g(\gamma) & 0 & 0 & \frac{\gamma}{2} \\
0 & 1 - 2g(\gamma) & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{\gamma}{2} & 0 & 0 & g(\gamma)
\end{bmatrix}.
\]

This state incorporates a function \( g(\gamma) \) with the following conditions,

\[
g(\gamma) = \delta = \begin{cases}
\frac{1}{3}, & 0 \leq \gamma < \frac{2}{3} \\
\frac{\gamma}{2}, & \frac{2}{3} \leq \gamma \leq 1
\end{cases}.
\]

For simplicity throughout this article we consider \( g(\gamma) = \delta \). In continuation of the above discussion on quantum states, we would like to bring attention to quantum correlation measures that we use in our work. Squashed entanglement is a well-known quantum correlation measure in quantum information theory. This multipartite entanglement measure satisfies convexity, additivity and super additivity over a tensor product in general [31]. Further, this measure can also be reduced to the entanglement entropy for pure states. For a quantum state, \( \rho^{AB} \) defined over the bipartite Hilbert space and squashed entanglement is defined as below,

\[
SE(\rho^{AB}) = \inf \left\{ \frac{1}{2} I(A; B|E) : \rho^{ABE} \hookrightarrow \rho^{AB} \right\}.
\]

Here \( \hookrightarrow \) represents ‘extension of’ and mathematically reads as \( \rho^{ABE} = \rho^{AB} \otimes \rho^E \). Further, the infimum will be taken over the set equipped with \( \rho^{ABE} \). With the definition of squashed entanglement, the following condition is satisfied,

\[
\rho^{AB} = Tr_E(\rho^{ABE}).
\]
The expression $I(A; B|E)$ in the definition of squashed entanglement reads,

$$I(A; B|E) = S(AE) + S(BE) - S(ABE) - S(E).$$

(6)

Where $S(.)$ represents the Von Neumann entropy in the above expression. Here we mention that the squashed entanglement is invariant under the tripartite exchange permutation symmetry [17].

Concurrence [13] is another quantum correlation measure that is widely used to determine the entanglement of bipartite quantum systems. Concurrence of a bipartite quantum system of a density matrix $\rho$ can be defined as,

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}.$$  

(7)

Where $\lambda_i$’s are the decreasing order square root of eigenvalues of $\rho \tilde{\rho}$ and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y).$$  

(8)

Where $\tilde{\rho}$ is the result of the spin-flip operation applied on density matrix $\rho$. Here $\sigma_y$ is the Pauli Y matrix and $\rho^*$ is the complex conjugate of the density matrix $\rho$.

3 Hamiltonian and Unitary Time Evolution

In this section, we present Heisenberg [22–24] and bi-linear bi-quadratic Hamiltonian [25, 26] used for the study. Further, we also present the unitary time evolution by using the time-dependent Schrodinger equation.

In this paper, we consider Hamiltonians with the XXX model in one dimension without a magnetic field and following no boundary conditions. Under the above assumption, the Heisenberg Hamiltonian reads,

$$H_1 = -j \sum_{i=1}^{N} \sigma_z^i \sigma_z^{i+1}.$$  

(9)

Next, we consider the non-linear extension of the above Hamiltonian named as bi-linear bi-quadratic Hamiltonian. This Hamiltonian is given below,

$$H_2 = -j \sum_{i=1}^{N} \left[ (\sigma_z^i \sigma_z^{i+1}) + (\sigma_z^i \sigma_z^{i+1})^2 \right].$$  

(10)

Where $j$ is the coupling constant, $\sigma_z^i$ denotes the Pauli Z matrix and $(\sigma_z^i \sigma_z^{i+1})^2$ is the non-linear term. These Hamiltonians are used to investigate the unitary dynamics in the bipartite system.

As per the postulate of quantum mechanics, the unitary time evolution of the physical system is governed by the time-dependent Schrodinger equation given below,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = E |\psi(t)\rangle.$$  

(11)

Where $E$ is the real energies of the physical system. The solution of this equation is obtained as,

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle.$$  

(12)

The density matrix version of the above equation can be framed as,

$$\rho(t) = U(t) \rho(0) U(t)^\dagger.$$  

(13)
Where \( U(t) = e^{-iHt/H} \) is the unitary matrix which includes the Hamiltonian \( H \) in exponential. Here we assume \( \hbar = 1 \) to maintain the simplicity for the present study. This (13) is used in current work to develop the dynamics of quantum correlations in Werner state and MEMS.

4 Initial State Preparation of Werner State and MEMS

In this section, we prepare the initial states according to the requirement of quantum correlations expressed in (4), (7).

As per the definition of squashed entanglement, we add an additional qubit \((E)\) to the bipartite system. This additional qubit produces the extended density matrix \( \rho^{ABE} \). Here we mention that a larger domain of extension provides better accuracy in squashed entanglement measure. But the larger domain of the system makes the calculations of squashed entanglement very difficult, infect it may be considered as NP-Complete/NP-Hard problem\(^1\) [27, 28]. In our work, we consider the simplest case by considering the auxiliary qubit as an external domain for squashed entanglement. To begin with, we express the state vector of the additional qubit as below,

\[
|E\rangle = \alpha|0\rangle + \beta|1\rangle
\]  
(14)

with

\[
|\alpha|^2 + |\beta|^2 = 1.
\]  
(15)

The density matrix of the additional qubit \(|E\rangle\) is expressed as,

\[
\rho^E = \begin{bmatrix}
|\alpha|^2 & \alpha\beta \\
\alpha\beta & |\beta|^2
\end{bmatrix}.
\]  
(16)

By using the (15), we can rewrite the above density matrix as below,

\[
\rho^E = \begin{bmatrix}
|\alpha|^2 & \alpha\sqrt{1 - |\alpha|^2} \\
\alpha\sqrt{1 - |\alpha|^2} & 1 - |\alpha|^2
\end{bmatrix}
\]  
(17)

Here we recall that the additional qubit expressed in (14) works as an external domain for \( \rho^{AB} \) which is the density metrics of the bipartite quantum state. Initially the state \( \rho^{AB} \) is prepared in Werner State and MEMS respectively, given in equations (2), (3). After Heisenberg interaction between qubits \( B \) and \( E \), the quantum correlations may change in \( \rho^{AB} \) with the advancement of time. The density matrices of the initial tripartite system prepared in Werner state and MEMS can be expressed respectively as below,

\[
\rho^W(0) = \rho^{WS} \otimes \rho^E
\]  
(18)

and

\[
\rho^M(0) = \rho^{MEMS} \otimes \rho^E.
\]  
(19)

The above equations will be used for the study in the next sections to explore the dynamics of quantum correlations expressed in (4), (7) for Werner state and MEMS. The dynamical study is carried out under two different Hamiltonians given in (9), (10) in two subsequent sections. In the very next section, we explore the mathematical expressions of the dynamics of quantum correlations for quantum states and discuss our results under Heisenberg Hamiltonian. In further section, we investigate the same study under the bi-linear bi-quadratic Hamiltonian.

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\(^1\)In computer science the problems whose domain is very large becomes very difficult to solve; such problems are categorized as NP-Complete/NP-Hard.
5 Quantum Correlations Under Heisenberg Hamilton: The Mathematical Expressions

In this section, we investigate the quantum dynamics of quantum correlations in the tri-partite system presented in (18), (19). Further, we use the time evolution given in (13) under Heisenberg Hamiltonian. Here we present the mathematical expressions obtained for Squashed entanglement and concurrence in Werner states and MEMS in two successive subsections.

5.1 Squashed Entanglement and Concurrence for Werner State

We obtained the time evolution of squashed entanglement for Werner state under Heisenberg Hamiltonian as given below,

\[
SE_W(H_1) = 0.5 + 2a^{-1}b \left[ (a - G) \log(c) + (a + G) \log(c') - (a - H) \log(d) - (a + H) \log(d') \right] + 3b \left[ (f) \log(0.25f) + (b' + \gamma) \log(v) \right].
\] (20)

Where,

\[
G = \sqrt{\alpha^2 gg' + a^2(1 - 2a^2 g)}, \quad H = \sqrt{a^2 f'gg' + a^2(1 - 2a^2 f'g)},
\]

\[
c = (0.5 - 0.5a^{-1}G), \quad c' = (0.5 + 0.5a^{-1}G),
\]

\[
d = (0.25 - 0.25a^{-1}H), \quad d' = (0.25 + 0.25a^{-1}H)
\]

and

\[
a = e^{2ijt}, \quad b = 0.180337, \quad b' = 0.333333, \quad f = (1 - \gamma), \quad f' = (1 - \gamma^2),
\]

\[
f'' = (1 + \gamma^*), \quad f''' = (1 + \gamma), \quad \gamma = (1 - a^2),
\]

\[
g' = (1 + a^4), \quad h = (1 - \delta), \quad h' = (1 - 2\delta), \quad v = (0.25 + 0.75\gamma),
\]

\[
m = (\gamma - 2\delta), \quad m' = (\gamma + 2\delta), \quad n = (-0.5\gamma + \delta), \quad n' = (0.5\gamma + \delta).
\] (21)

The symbols expressed in (21) have been used throughout the paper.

Further for concurrence, we need to obtain the spectrum of eigenvalues with unitary time evolution under Heisenberg Hamiltonian. This spectrum for Werner state is obtained below,

\[
\lambda_W(H_1) = \left\{ \frac{1}{4} X f, \frac{1}{4} X f, \frac{1}{16} a^{-1}(p - p'), \frac{1}{16} a^{-1}(p + p') \right\}.
\] (22)

Where,

\[
p = -4\gamma(Z + Y\alpha^2 - Z\alpha^2) - 4a^2\gamma(Y - Y\alpha^2 + Z\alpha^2) + af''f''',
\]

\[
p' = \sqrt{p^2 - 16 \left[ 4a^2\gamma^2 gg' - a^2(1 + 2\gamma - (3 - 8a^2g)\gamma^2) \right]} \left[ YZ - \frac{1}{16}(f'')^2 \right]
\]

and

\[
X = \frac{1}{2}(a^2 f + gf)^*, \quad Y = -\frac{1}{2}(a^2\gamma + a^{-1}g\gamma)^*, \quad Z = -\frac{1}{2}(a^{-1}\alpha^2 + ag\gamma)^*.
\]

Here, rest of the symbols are considered from (21).
5.2 Squashed Entanglement and Concurrence for MEMS

In this section, we obtain the squashed entanglement for MEMS under unitary time evolution with Heisenberg Hamiltonian. The mathematical expression is obtained as below,

\[
SE_M(H_1) = 2a^{-1}b\left[(a - Q)\log(k) + (a + Q)\log(k') \right] + 8b(0.5 - \delta)\log(h') \\
- (4bh)\log(h) - (4b\delta)\log(\delta) + a^{-1}[(4bP - 2abh)\log(l) - (4bP \\
+ 2abh)\log(l') - (4a\delta)\log(\delta)] - 2b\left[(m)\log(n) - (m')\log(n') \right].
\] (23)

Where,

\[
P = \sqrt{a^2 \left[0.25 - (0.5 + 2a^2g)\delta + (0.25 + 4a^2g)\delta^2 \right] + a^2\delta gg'h'},
\]

\[
Q = \sqrt{4a^2h\delta gg' + a^2(1 - 8a^2gh\delta)},
\]

\[
k = (0.5 - 0.5a^{-1}Q), \quad k' = (0.5 + 0.5a^{-1}Q),
\]

\[
l = (0.5 - a^{-1}P - 0.5\delta), \quad l' = (0.5 + a^{-1}P - 0.5\delta)
\]

and other symbols taken from (21).

Further to obtain the concurrence we need to calculate the eigenvalue spectrum which is given below,

\[
\lambda_M(H_1) = \left\{0, 0, 0.25a^{-1}(q - 2q'), 0.25a^{-1}(q + 2q')\right\}.
\] (24)

Where,

\[
q = S + Ra^2 - Sa^2 + a^2(R - Ra^2 + Sa^2) + 4a\delta^*,
\]

\[
q' = \sqrt{-w(0.5\alpha a^2\gamma + 0.5\alpha^{-1}g\gamma)^*(0.5\alpha^{-1}\alpha^2\gamma + 0.5a\gamma)^* + w(\delta^*)^2 + 0.25q'^2},
\]

and

\[
w = \alpha^2\gamma^2 gg' + a^2[(1 - 2a^2g)\gamma^2 - 4\delta^2],
\]

\[
R = 0.5\gamma(\alpha a^2\gamma + a^{-1}g\gamma)^*, \quad S = 0.5\gamma(a^{-1}\alpha^2\gamma + a\gamma)^*.
\]

Here, remaining symbols expressed in (21).

6 Dynamics of Quantum Correlations Under Heisenberg Hamiltonian

In this section, we present the dynamical results of squashed entanglement and concurrence for Werner state and MEMS under Heisenberg Hamiltonian. The dynamics of Squashed entanglement for Werner State and MEMS is governed by the (20), (23) respectively and for Concurrences it is governed by the set of eigenvalues presented in (22), (24). We note that the squashed entanglement and concurrence are the functions of the parameters \(\alpha, \gamma, j\) and \(t\). Further, we have shown the comparative study of squashed entanglement and concurrence in graphical results with varying values of the parameters \(\alpha, \gamma, j, t\). We divide this study into different cases to understand the dynamical behavior of quantum correlations concerning different parameters. During the discussion, we consider the coupling constant.
$j$ and time $t$ as a single variable $jt$. The study is explored in five different cases, these cases are given in successive subsections.

**Case 1: For parameters $\alpha = 0$ and $jt = 0$** In this case, we present the study of quantum correlations in Werner state and MEMS for initial condition i.e. at $\alpha = 0$, $jt = 0$. The values of the parameters ($\alpha = 0$, $jt = 0$) maps the tripartite system to the pure bipartite initial state given as in (2), (3). The initial behavior of quantum correlations for $\gamma$ is shown in Fig. 1. In this figure, $SE_W$ represents squashed entanglement of Werner state, $SE_M$ corresponds to squashed entanglement of MEMS, $C_W$ denotes the concurrence of Werner state and $C_M$ represents the concurrence of MEMS. These notations will be used in all the figures throughout this article.

To proceed with the investigation, first we focus on the comparison of squashed entanglement in both the states; next we compare the concurrence in both the states and finally squashed entanglement vs. concurrence comparison is done.

The squashed entanglement in both the states is plotted in Fig. 1 by solid red and blue color. Looking at the graphs we have found that it rises exponentially in both of the states. Werner state achieves the maximum squashed entanglement amplitude as $QC \simeq 1$; while for MEMS, it is $QC = 0.4428$. It is investigated that, MEMS does not exhibit the squashed entanglement after the value of the parameter $\gamma = \frac{2}{3}$. The maximum squashed entanglement amplitude in MEMS is always less than the Werner state. At $\gamma = 0$, there is no squashed entanglement in Werner state; while for MEMS it has a certain value $QC = 0.126365$.

Now we compare the concurrence in both the states which are plotted in the figure by dashed purple and green color. Here it is noted that the concurrence grows linearly. The figure shows that the Werner state does not have concurrence between the limit $0 \leq \gamma < \frac{1}{3}$; while on the other hand, MEMS has the concurrence for all the range of the parameter $\gamma$. Both the states achieve the maximum amplitude as $QC = 1$.

![Plot of quantum corrections (QC) vs. $\gamma$](image.png)
Next, we discuss squashed entanglement and concurrence in both states. For Werner state, we have compared both the quantum correlations and found that the squashed entanglement is rising exponentially for the range $0 \leq \gamma \leq 1$; while the concurrence is increasing linearly between the limit $\frac{1}{3} \leq \gamma \leq 1$. Whereas concurrence does not exist for the limit $0 \leq \gamma < \frac{1}{3}$. It is noted that at the initial value of $\gamma$, the squashed entanglement dominates the concurrence. But after the value of $\gamma = 0.4630$, the trade-off is reversed i.e. the concurrence dominates the squashed entanglement and achieves the higher amplitude for $0.4630 < \gamma \leq 1$. At the value of $\gamma = 0.4630$, both the quantum correlations have the same amplitude and we call this point as Squashed entanglement-Concurrence Equilibrium (SCE) point of Werner state.

For MEMS, we compare both the squashed entanglement and concurrence with an interesting trade-off. It is found that the concurrence exists for the whole range of $\gamma$; while the squashed entanglement exists in the range $0 \leq \gamma \leq \frac{2}{3}$. Further for the range of $\frac{2}{3} < \gamma \leq 1$ squashed entanglement does not exist. It is also noticed that the SCE point of MEMS exists at $\gamma = 0.139$. We also observe that for MEMS with $0 \leq \gamma < 0.139$; the squashed entanglement dominates the concurrence and beyond $\gamma = 0.139$, the concurrence dominates the squashed entanglement.

**Case 2: For a fixed value of $\alpha$ and different values of $jt$** In this case, we fix the value of $\alpha$ and plot the graphical results with varying values of $\gamma$ and $jt$. Few Important results are plotted in Fig. 2 with $\alpha = 0.600001$.

The graphical plots show that as the value of $jt$ increases the maximum value of all quantum correlations decreases and the rate of amplitude decrement of concurrence is higher in comparison to the amplitude decrement of squashed entanglement in both states. It is noted

![Graphs showing quantum corrections vs. $\gamma$]
that the sustainability of squashed entanglement in both the state w.r.t the axis of $\gamma$ does not change; it remains the same as described in case 1. On the other hand, the sustainability of the concurrence along the axis of $\gamma$ in Werner state changes as the value of the parameter $jt$ increases but for MEMS it does not change. Further, we have noticed that for $\gamma = 0$, the squashed entanglement of MEMS has a certain value, but as time moves forward this value decreases, and the difference between the squashed entanglement of both the states also decreases.

**Case 3: For the fixed value of $jt$ and different values of $\alpha$** In this case, we describe the quantum correlations for a constant value of $jt$ and varying values of $\alpha$ and $\gamma$. We have shown few of the plots for $jt = 0.600001$ and different values of $\alpha$ in Fig. 3.

Here we notice that as the value of $jt$ is fixed, the amplitude of the concurrence of quantum states becomes freeze at $QC = 0.363468$. On the contrary, the increment of the value of $\alpha$ decreases the amplitude of squashed entanglement. It is also noted that the amplitude of squashed entanglement and their gap in both of the states become freeze at $\alpha \geq 0.600001$.

**Case 4: For the fixed value of $\alpha$ and different values of $\gamma$** In this section, we explore the study by fixing the value of the parameter $\alpha$ at $0.600001$ with the varying values of $\gamma$ and $jt$ and present the graphical results in Fig. 4. In the initial study, we discuss the comparison of squashed entanglement in both states. Then the comparison of concurrence in both of the states is explored. Finally, the squashed entanglement and concurrence are compared for each of the states.

The squashed entanglement is higher in MEMS than Werner state and amplified in both the states as the value of the parameter $\gamma$ increases. As the value of parameter $jt$ advances, the amplitude of squashed entanglement decreases in both states.

![Fig. 3](image-url)  
*Fig. 3* Plot of quantum corrections (QC) vs. $\gamma$ with different values of $\alpha*
Comparing the concurrence in both states, we have found that the concurrence is also higher in MEMS than Werner state. On the other hand, we have found the important result of Entanglement sudden death (ESD) [33–40] for the concurrence in Werner state with the increasing value of $\gamma$. But during the ESD, the concurrence exists in MEMS and this state shows a more robust character compared to the Werner state. We have found as the value of $\gamma$ increases the ESD zone squeezes.

Finally, we compare the squashed entanglement and concurrence in both states. We have found at $jt = 0$, the amplitude of squashed entanglement is always maximum. As the parameter $jt$ advances, both the quantum correlations have a decreasing tendency. It is also noted that at $jt = 0.787579$, the concurrence of MEMS becomes zero but at this point, squashed entanglement exists. For Werner state with $\gamma = 0.200001$, the concurrence does not exist. But the advancing value of $\gamma$ produces the concurrence in the Werner state with the ESD effect. It is important to note that in the absence of concurrence the squashed entanglement always exists in Werner state.

**Case 5: For the fixed value of $\gamma$ and different values of $\alpha$** In this case, we present the study of the dynamics of quantum correlations by fixing the value of the parameter as $\gamma = 0.600001$ with the varying values of $\alpha$ and $jt$. The graphical results are shown in Fig. 5.

Here we have found that the maximum amplitude of all the quantum correlations freeze at $jt = 0$ for all values of $\alpha$. We also noticed that the ESD zone of Werner state starts at $jt = 0.623712$ and zone width freeze with the advancement of time.
7 Quantum Correlations Under Bi-Linear Bi-Quadratic Hamiltonian: The Mathematical Expressions

In this section, we explore the dynamics of quantum correlations in both the states under the bi-linear bi-quadratic Hamiltonian presented in (10). This Hamiltonian incorporates a non-linear term as \((\sigma_i^Z \sigma_{i+1}^Z)^2\) along with the Heisenberg Hamilton. This term is responsible to cancel out the parameter \(jt\) while calculating the time evolution of the system governed by (13). Hence bi-linear bi-quadratic Hamiltonian preserves the quantum correlations in the system. Here in the following subsections, we present the mathematical functions calculated for quantum correlations in both states.

7.1 Squashed Entanglement and Concurrence for Werner State

The expressions of squashed entanglement and eigenvalue spectrum for concurrence under bi-linear bi-quadratic Hamiltonian with the (13), is obtained as,

\[
SE_W(H_2) = 1 + (3bf) log(0.25f) + b(1 + 3\gamma) log(v) \tag{25}
\]

and

\[
\lambda_W(H_2) = \left\{ \frac{1}{16} ff''', \frac{1}{16} ff''', \frac{1}{16} ff''', \frac{1}{16} (1 + 3\gamma)(1 + 3\gamma^*) \right\} \tag{26}
\]

The required symbols considered in (21).

We have found, the above equations are independent of the parameters \(\alpha\) and \(jt\). The (20) and (22) maps to the (25) and (26) with the condition \(\alpha = 0, \; jt = 0\). Hence the dynamics of
quantum correlations in Werner state under bi-linear bi-quadratic Hamiltonian remain the same as observed in case 1 with the initial condition.

7.2 Squashed Entanglement and Concurrence for MEMS

Here we obtain the expressions of squashed entanglement and eigenvalue spectrum for concurrence under bi-linear bi-quadratic Hamiltonian with (13). These expressions are given below,

\[ SE_M(H_2) = 0.5 \left[ 16b(0.5 - \delta) \log(h') - (16bh) \log(h) - (16b\delta) \log(\delta) 
- 4b \left\{ (m) \log(n) - (m') \log(n') \right\} \right] \] (27)

and

\[ \lambda_M(H_2) = \{0, 0, (0.25\gamma\gamma^* + \delta\delta^* - 0.5T), 0.5(T + 0.5\gamma\gamma^* + 2\delta\delta^*)\}. \] (28)

Where,

\[ T = (\delta\gamma^* + \gamma\delta^*) \]

and other symbols expressed in (21).

Again it is found that the above equations are independent of the parameter \( \alpha \) and \( jt \), and it happens due to the presence of the non-linear term in the bi-linear bi-quadratic Hamiltonian. The (23) and (24) maps to the above (27) and (28) with the initial condition \( \alpha = 0, jt = 0 \). Hence the dynamics of quantum correlations in MEMS remain the same as explained in case 1 under section six with the initial condition.

8 Conclusion

In this article, we have studied the unitary dynamics of quantum correlations for Werner state and MEMS under Heisenberg and the bi-linear bi-quadratic Hamiltonian. For the initial condition with \( \alpha = 0, jt = 0 \), we have found a trade-off between squashed entanglement and concurrence in both of the states. With the range \( 0 \leq \gamma < \frac{1}{3} \), the concurrence vanishes in Werner state, while squashed entanglement exists. On the other hand, with the range \( \frac{2}{3} < \gamma \leq 1 \) the squashed entanglement vanishes in MEMS but concurrence exists.

Dealing with Heisenberg Hamiltonian, we have explored this study by varying the parameters \( \alpha, \gamma \) and \( jt \) in different cases and discuss the impact of every parameter on quantum correlations. We have noticed that the concurrence increases linearly in both of the states while squashed entanglement increases in an exponential fashion. In both states, when either one of the parameters \( \alpha \) or \( jt \) has been increased by keeping another parameter fix; the amplitude of the quantum correlations is decreased. Further, the parameter \( \gamma \) is responsible to amplify the quantum correlations. We have also investigated the phenomenon of entanglement sudden death (ESD) in Werner state with concurrence measure by varying the value of \( \gamma \). But as the value of the parameter \( \gamma \) is fixed, the width of the ESD zone is also fixed but in the absence of concurrence, the squashed entanglement always exits. More interestingly, it has been observed that the Werner state is more fragile than MEMS in terms of concurrence measure with the parameter \( \gamma \).

The most important result we have shown that the bi-linear bi-quadratic Hamiltonian does not contribute to the time evolution for both the quantum states and hence it does not disturb the quantum correlations. The non-linear term in this Hamiltonian plays an important role to preserve the quantum correlations and both the states are quite robust under this Hamiltonian.
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