Asymptotically cylindrical manifolds with holonomy Spin(7). I

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Abstract

We construct examples of asymptotically cylindrical Riemannian 8-manifolds with holonomy group Spin(7). To our knowledge, these are the first such examples. The construction uses an extension to asymptotically cylindrical setting of Joyce’s existence result for torsion-free Spin(7)-structures. One source of examples arises from ‘Fano-type’ Kähler 4-orbifolds with smooth anticanonical Calabi–Yau 3-fold divisors and with compatible antiholomorphic involution. We give examples using weighted projective spaces and calculate basic topological invariants of the resulting Spin(7)-manifolds.

The holonomy group Spin(7) occurs in 8 dimensions as one of the two special cases in Berger’s classification of the holonomy of the Levi–Civita connections; the other being the holonomy group $G_2$ in 7 dimensions. Both special holonomy groups Spin(7) and $G_2$ are naturally related to the algebra of octonions, or Cayley numbers; the group Spin(7) arises as the stabilizer of the triple cross product on $\mathbb{R}^8$ [HL, SW]. One the other hand, the holonomy representation of Spin(7) is isomorphic to the spin representation.

The metrics with holonomy Spin(7) and $G_2$ are Ricci-flat and are the only holonomy groups in Berger’s list corresponding to Ricci-flat metrics that are not Kähler. More explicitly, the other Ricci-flat holonomy groups are $SU(n)$ and $Sp(n) \subset SU(2n)$; these induce a parallel, integrable complex structure on a base manifold. The proper inclusion of Riemannian holonomy groups $SU(4) \subset Spin(7)$ accounting for holonomy reduction with no compatible complex structure is unique to real dimension 8.

The first examples of Riemannian manifolds with holonomy Spin(7) were obtained by Bryant [Br]. Complete examples were subsequently constructed by Bryant and Salamon [BS] and by Gibbons, Page and Pope [GPP]. The first compact manifolds with holonomy Spin(7) are due to Joyce [Jo1, Jo2].

In this paper, we construct examples of complete asymptotically cylindrical 8-manifolds with holonomy equal to Spin(7). To the author’s knowledge, these are the first such examples. The previously known complete non-compact holonomy Spin(7)-metrics are of asymptotically conical type with a different (non-linear polynomial) volume growth.

Examples of asymptotically cylindrical Calabi–Yau manifolds with holonomy $SU(n)$ were constructed in [Ko1, KL] and [CHNP]. A construction of asymptotically cylindrical manifolds with holonomy $G_2$ was given in [KN].
The metrics with holonomy $\text{Spin}(7)$ can be determined via a differential self-dual 4-form $\Phi$ point-wise modelled on the ‘Cayley form’ on $\mathbb{R}^8$. Any such 4-form $\Phi$ induces a $\text{Spin}(7)$-structure on an 8-manifold, hence a metric as $\text{Spin}(7)$ is a subgroup of $\text{SO}(8)$. The holonomy of this metric is contained in $\text{Spin}(7)$ precisely when the $\text{Spin}(7)$-structure is torsion-free, a condition equivalent to $\Phi$ being a closed 4-form. In practical examples, the task of constructing a suitable 4-form leads to a non-linear PDE. A criterion for an asymptotically cylindrical torsion-free $\text{Spin}(7)$-structure to have the full holonomy $\text{Spin}(7)$ was proved by Nordström [No2]; for simply-connected examples constructed here it amounts to the vanishing of the first Betti number of the cross-section of cylindrical end.

The asymptotically cylindrical $\text{Spin}(7)$-manifolds in this paper are obtained by modifying Joyce’s construction of compact $\text{Spin}(7)$-manifolds in [Jo2]. Joyce uses in $\text{op.cit}.$ quotients of Calabi–Yau complex 4-dimensional orbifolds with suitable isolated singularities by anti-holomorphic involution and resolves the singularities to obtain smooth 8-manifolds with $\text{Spin}(7)$-structures having small torsion. If, instead, a Kähler 4-orbifold has a smooth Calabi–Yau 3-fold divisor in the anticanonical class, then the complement of this divisor, under suitable conditions, admits an asymptotically cylindrical metric with holonomy $\text{SU}(4)$ cf. [Ko1] [HHN]. It is also possible to choose a ‘compatible’ anti-holomorphic involution to obtain asymptotically cylindrical $\text{Spin}(7)$-orbifolds. Furthermore, a large part of the resolution of singularities and the existence argument for torsion-free $\text{Spin}(7)$-structures in [Jo3] Chap. 13 and 15] is carried out in way which can be extended to asymptotically cylindrical setting with little further work. The situation here has some analogy with [KN] §3, where a generalization similar type is provided to obtain an existence result for asymptotically cylindrical $G_2$-structures. In particular, the main additional argument required in both the $\text{Spin}(7)$ and $G_2$ cases is to show an exponential rate of convergence to cylindrical asymptotic structure. The task of applying the existence result in examples then amounts to finding suitable orbifolds (see Definition 5.8) and this can be treated as a question in complex algebraic geometry.

The paper is organized as follows. We begin with introducing the necessary background results on torsion-free $\text{Spin}(7)$-structures and holonomy in §1 and on asymptotically cylindrical manifolds including aspects of Hodge theory in §2. Then, in §3 we explain that the holonomy group of a simply-connected asymptotically cylindrical manifold is topologically determined. This includes a criterion for full holonomy $\text{Spin}(7)$ and a description of the moduli space of asymptotically cylindrical $\text{Spin}(7)$-manifolds, the results due to Nordström [No2]. We also give an interpretation of some results of [HHN] in the context of $\text{Spin}(7)$-structures. An asymptotically cylindrical extension of Joyce’s existence theorem for $\text{Spin}(7)$ structures is proved in §4. The class of Kähler 4-orbifolds required for making asymptotically cylindrical $\text{Spin}(7)$ manifolds is introduced in §5 which also contains justification of the construction. Finally, in §6 we show how to implement the construction. We restrict attention to simple examples and calculate their basic topological invariants and the dimension of respective moduli space.

The asymptotically cylindrical $\text{Spin}(7)$-manifolds in this paper have Betti numbers $b^1 = b^2 = b^3 = 0$ and with their cross-sections having $b^1 = b^2 = 0$. This is in fact the maximal set of vanishing Betti numbers possible for asymptotically cylindrical $\text{Spin}(7)$-manifolds, whether or not the holonomy group is all of $\text{Spin}(7)$, as the cohomology class of $\text{Spin}(7)$-form is never zero and induces non-trivial harmonic 3-form defining a $G_2$-structure on the (compact) cross-section ‘at infinity’. On the other hand, the necessary condition for $\text{Spin}(7)$ holonomy for 8-manifolds of this type is the vanishing of $b^1$ for both the 8-manifold and its cross-section.
Further, the holonomy of the cross-section induced by the $G_2$-structure at infinity may be exactly $G_2$ or a proper subgroup, depending on whether or not the fundamental group is finite. It is infinite in the present examples and the holonomy the $G_2$-manifold at infinity is $\mathbb{Z}_2 \times SU(3)$.

The author has a method which produces examples of asymptotically cylindrical holonomy $Spin(7)$ manifolds where the $G_2$-manifold at infinity is irreducible with full holonomy $G_2$. These developments and some applications are treated in the companion paper [Ko3].

1 Spin(7)-structures on 8-manifolds

We give a short summary of some background results on the Riemannian geometry in dimension 8 associated with the structure group $Spin(7)$. For more details, see [HL, Jo3, SW, S].

The key role in defining the $Spin(7)$ holonomy is played by a particular 4-form, sometimes called the Cayley 4-form, on the Euclidean $SO(8)$ subgroup of $G_2$. The form $\Phi = \Phi_0$ with the fibre $\mathbb{R}^8$ called the Cayley 4-form, on the Euclidean $SO(8)$ subgroup of $G_2$. The form $\Phi_0$ is self-dual with respect to the Euclidean metric.

For an oriented 8-manifold $M$, define a subbundle of admissible 4-forms $\mathcal{A}M \subset \Lambda^4 T^*M$ with the fibre $\mathcal{A}_p M$ at each $p \in M$ consisting of 4-forms that can be identified with $\Phi_0$ via an orientation-preserving isomorphism $T_p M \to \mathbb{R}^8$. The fibres of $\mathcal{A}M$ are diffeomorphic to the orbit $GL_+(8, \mathbb{R})/Spin(7)$ of $\Phi_0$, a 43-dimensional submanifold of the 70-dimensional vector space $\Lambda^4(\mathbb{R}^8)^*$. Every admissible 4-form $\Phi \in \Gamma(\mathcal{A}M)$ defines a $Spin(7)$-structure on $M$, hence a metric $g = g(\Phi)$ and orientation and a Hodge star $\ast \Phi$, so that $\ast \Phi = \Phi$. We shall sometimes slightly inaccurately say that that $\Phi$ is a $Spin(7)$-structure.

Here are two foundational results on the $Spin(7)$-structures, the first is taken from [S, Lemma 12.4] and the second from [Bo].

**Theorem 1.1.** Let $M$ be an 8-manifold and $\Phi \in \Gamma(\mathcal{A}M)$ a $Spin(7)$-structure on $M$. Then

(i) the holonomy of $g(\Phi)$ is contained in $Spin(7)$ if and only if $d\Phi = 0$, and

(ii) if the holonomy of a metric $g$ on $M$ is contained in $Spin(7)$, then $g$ is Ricci-flat.

The $Spin(7)$-structure induced by a closed admissible 4-form $\Phi$ is said to be torsion-free; the condition $d\Phi = 0$ in this case is equivalent to vanishing of the intrinsic torsion. We shall call an 8-manifold $(M, \Phi)$ endowed with a torsion-free $Spin(7)$-structure a $Spin(7)$-manifold.

Each $Spin(7)$-structure on an 8-manifold $M$ defines point-wise orthogonal decompositions of the bundles of differential $r$-forms corresponding to irreducible components of the induced representation of $Spin(7)$ on $\Lambda^r(\mathbb{R}^8)^*$ ([Br] or [Fe]). In particular,

\[ \Lambda^2 T^*M = \Lambda^2_1 T^*M \oplus \Lambda^2_2 T^*M, \]  
\[ \Lambda^3 T^*M = \Lambda^3_1 T^*M \oplus \Lambda^3_2 T^*M, \]  
\[ \Lambda^4 T^*M = \Lambda^4_1 T^*M \oplus \Lambda^4_2 T^*M \oplus \Lambda^4_3 T^*M, \]  
\[ \Lambda^4_1 T^*M = \Lambda^4_1 T^*M \oplus \Lambda^4_2 T^*M \oplus \Lambda^4_3 T^*M \]  
\[ \Lambda^4_2 T^*M = \Lambda^4_1 T^*M \oplus \Lambda^4_2 T^*M \oplus \Lambda^4_3 T^*M \]
where the lower indices indicate ranks of subbundles and the fibres of $\Lambda^4_{\pm}$ are the $\pm 1$-eigenspaces of $*_{\Phi}$. We may sometimes abbreviate $\Lambda^4_j T^* M$ as $\Lambda^4_j$ if no confusion is likely.

The subbundles of 2-forms in (2a) may be determined by

$$\Lambda^2_{\pm} = \{ \alpha : * (\Phi \wedge \alpha) = 3 \alpha \}, \quad \Lambda^2_{21} = \{ \alpha : * (\Phi \wedge \alpha) = -\alpha \}. \quad (3)$$

The fibres of $\Lambda^4_1$ are spanned by $\Phi$ and $\Lambda^2_3$ is the bundle of 3-forms $v.\Phi$ for $v \in TM$. The inner product $g(\Phi)$ is determined by

$$\langle u, v \rangle \Phi \wedge \Phi = 2 (u.\Phi) \wedge *_{\Phi}(v.\Phi).$$

For each $\Phi_p \in A_p M$, $p \in M$, the tangent space to the orbit $(AM)_p \subset \Lambda^4 T^*_p M$ at $\Phi_p$ is $(\Lambda^4_1 \oplus \Lambda^4_2 \oplus \Lambda^4_{35},)_p$ and the normal space is the orthogonal complement $(\Lambda^4_{27},)_p$, where the subspaces $(\Lambda^4_{k},)_p$ are determined by the Spin(7) structure $\Phi_p$. Using this, one can construct a tubular open $\varepsilon$-neighbourhood $\mathcal{J} M$ of $AM$ in $\Lambda^4 T^* M$ consisting of 4-forms $\chi_p = \Phi_p + \psi_p$ such that $\Phi_p \in A_p M$, $\psi_p \in \Lambda^4_{35}$ and $|\psi_p| < \varepsilon$ (with $(\Lambda^4_{27})_p$ and the point-wise norm again induced by $\Phi_p$). Furthermore, for each $\chi_p \in \mathcal{J} M$, $\Phi_p$ and $\psi_p$ are uniquely determined by the latter conditions, if $\varepsilon > 0$ was chosen sufficiently small. The $\mathcal{J} M$ is a fibre bundle of 4-forms (it is not a vector bundle) and there is a well-defined smooth projection

$$\Theta : \chi_p = \Phi_p + \psi_p \in \mathcal{J} M \to \Phi_p \in AM.$$

The definition of $\Theta$ does not use any ‘background’ Spin(7)-structure on $M$.

We note for later use that the map $\Theta$ has a local expansion at each Spin(7)-structure $\Phi$,

$$\Theta(\Phi + \psi) = \Phi + \pi_1(\psi) + \pi_7(\psi) + \pi_{35}(\psi) - F(\psi), \quad \psi \in \Omega^4(M), \quad \|\psi\|_{C^0} < \varepsilon,$$

where $\pi_i$ denotes the $\Lambda^4_i$ component of a 4-form in the decomposition (2) determined by $\Phi$ and the norm is computed using the metric $g(\Phi)$. The remainder $F$ satisfies quadratic estimates

$$|F(\psi') - F(\psi'')| < C_1 |\psi' - \psi''| (|\psi'| + |\psi''|), \quad (4)$$

$$|\nabla(F(\psi') - F(\psi''))| < C_2 (|\psi' - \psi''| (|\psi'| + |\psi''|)) d\Phi + |\nabla(\psi' - \psi'')| (|\psi'| + |\psi''|)$$

$$+ |\psi' - \psi''| (|\nabla(\psi') + |\nabla(\psi'')|), \quad (5)$$

with $C_1 > 0$, $C_2 > 0$ independent of $\psi', \psi''$ [Jo3] Prop. 10.5.9].

The holonomy of a Spin(7)-manifold $M$ may be contained in a subgroup $SU(4) \subset \text{Spin}(7)$. In the notation of [1] the holonomy group $SU(4)$ is the subgroup of $GL(8, \mathbb{R})$ preserving the standard symplectic form $\omega_0 = dx_1 \wedge dx_2 + \ldots + dx_7 \wedge dx_8$ and the real part of the 4-form $\theta_0 = (dx_1 + idx_2) \wedge \ldots \wedge (dx_7 + idx_8)$. An 8-manifold $M$ with holonomy in $SU(4)$ carries a parallel integrable complex structure with trivial canonical bundle and a Ricci-flat Kähler metric, i.e. $M$ is a Calabi-Yau complex 4-fold. The parallel forms corresponding to $\omega_0$ and $\theta_0$ are the Kähler form and a non-vanishing holomorphic $(4,0)$-form, sometimes called a holomorphic volume form, normalized so that $3 \theta \wedge \bar{\theta} = 2 \omega^4$. These induce a Spin(7)-structure expressed as

$$\Phi(\omega, \theta) = \frac{1}{2} \omega \wedge \omega + \text{Re} \theta, \quad (6)$$

cf. [Jo3] Prop. 13.1.4. A straightforward computation

$$*(\omega \wedge \Phi) = \frac{1}{2} * \omega^3 = 3 \omega, \quad (7)$$
shows that the Kähler form is a section of the subbundle $\Lambda^2_7$ determined by $\Phi(\omega, \theta)$.

It will be useful to note one more instance when the holonomy of a Spin(7)-manifold $M$ reduces to a proper subgroup of Spin(7). The Cayley 4-form on $\mathbb{R}^8$ may be written as $\Phi_0 = dx_1 \wedge \varphi_0 + *_7 \varphi_0$, where $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ and $*_7$ the Hodge star on the Euclidean $\mathbb{R}^7$ with coordinates $x_2, \ldots, x_8$. The stabilizer of $\varphi_0$ in the $GL(7, \mathbb{R})$ action is a subgroup of Spin(7) fixing a non-zero vector. This subgroup is isomorphic to the exceptional Lie group $G_2$, which also is the stabilizer of the 4-form $*_7 \varphi_0$.

For a 7-manifold $Y$, a 3-form $\varphi \in \Omega^3(Y)$ will be called stable, if for each $y \in Y$ there is a linear isomorphism $j_y : \Lambda^3 T^*_y Y \to \mathbb{R}^7$ smoothly depending on $y$ such that $j_y^* \varphi = \varphi_y$. Every stable form $\varphi$ induces a $G_2$-structure on $Y$ and we shall sometimes say that $\varphi$ is a $G_2$-structure. As $G_2 \subset SO(7)$, every $G_2$-structure determines a metric $g(\varphi)$, an orientation and a Hodge star $*_\varphi$ on $Y$. The holonomy of $g(\varphi)$ is contained in $G_2$ if and only if

$$d\varphi = 0 \text{ and } d*_\varphi \varphi = 0.$$

[S, Lemma 11.5]. In this case, a $G_2$-structure $\varphi$ on $Y$ is said to be torsion-free. A 7-manifold $(Y, \varphi)$ endowed with a torsion-free $G_2$-structure will be called a $G_2$-manifold.

We shall call a Spin(7)-structure $\Phi_\infty$ on $M = \mathbb{R} \times Y$ cylindrical if it induces a product metric $g(\Phi_\infty) = dt^2 + g_Y$, where $t$ is the coordinate on $\mathbb{R}$. Equivalently, $\Phi_\infty$ is invariant under translations in $t$. Every cylindrical Spin(7)-structure can be written as

$$\Phi_\infty = dt \wedge \varphi + *_\varphi \varphi, \quad (8)$$

for some $G_2$-structure $\varphi$ on $Y$ independent of $t$, where $g(\varphi) = g_Y$ and $*_\varphi$ is the Hodge star of $g(\varphi)$. A Spin(7)-cylinder $M$ has a parallel vector field $\frac{\partial}{\partial t}$ and the holonomy of $M$ is identified with the holonomy of the 7-manifold $Y$. It is not difficult to check that a cylindrical Spin(7)-structure [S] on $\mathbb{R} \times Y$ is torsion-free if and only if the $G_2$-structure $\varphi$ on $Y$ is so. Cf. [J63, Prop. 13.1.3].

## 2 Asymptotically cylindrical metrics

Let $M$ be a manifold with cylindrical ends, i.e. $M$ is decomposed as a union of a compact manifold $M_0$ with boundary and a half-cylinder $M_\infty = \mathbb{R}_+ \times Y$, with $M_0$ and $M_\infty$ identified along the common boundary $\{0\} \times Y \cong \partial M_0$. The closed manifold $Y$ is called the cross-section of $M$. If $Y$ is connected, we say that $M$ has a single cylindrical end. We shall use $t$ to denote a smooth function on $M$ which is equal to the standard $\mathbb{R}_+$-coordinate on the cylindrical end $M_\infty$ and is non-positive on $M_0$.

A Spin(7)-structure $\Phi \in AM$ will be called asymptotically cylindrical with rate $\lambda$ if there is a cylindrical Spin(7)-structure $\Phi_\infty$ on $M_\infty$ such that

$$\sup_{\{t\} \times Y} |\nabla^k_\infty (\Phi - \Phi_\infty)| < C_k e^{-\lambda t}, \quad t > 0, \quad k = 0, 1, 2, \ldots. \quad (9)$$

Here the point-wise norms and the covariant derivative are those of $g_\infty = g(\Phi_\infty) = dt^2 + g_Y$. It follows that the metric $g(\Phi)$ approaches a Riemannian product metric $g_\infty$ at an exponential rate,

$$\sup_{\{t\} \times Y} |\nabla^k_\infty (g - g_\infty)| < C_k e^{-\lambda t}, \quad t > 0, \quad k = 0, 1, 2, \ldots,$$
so \((M, g(\Phi))\) is an asymptotically cylindrical Riemannian manifold, with the same rate \(\lambda\). Recall that \(\Phi_\infty\) is determined, via \((\mathcal{N})\), by a torsion-free \(G_2\)-structure on \(Y\) and \(g_Y = g(\varphi)\).

We shall call an 8-manifold with cylindrical ends endowed with a torsion-free \(\text{Spin}(7)\)-structure satisfying \((\mathcal{M})\) an \textit{asymptotically cylindrical \(\text{Spin}(7)\)-manifold} \((M, \Phi)\) and the \(G_2\)-manifold \((Y, \varphi)\) the \textit{cross-section of \(M\) at infinity}.

In this paper, unless stated otherwise, the metric on the cross-section of any asymptotically cylindrical manifold will always be understood as the metric at infinity defined above.

There is a natural class of asymptotically translation-invariant differential operators defined on manifolds with cylindrical ends, with coefficients approaching, at an exponential rate, the coefficients of \(t\)-independent (translation-invariant) operators on the respective cylinder. Any differential operator canonically associated to an asymptotically cylindrical manifold \((M, g)\) will automatically be asymptotically translation-invariant, in particular the Hodge Laplacian \(\Delta\) has this property. The respective translation-invariant operator is the Laplacian \(\Delta\) on the cross-section.

A general theory of the asymptotically translation-invariant elliptic operators is developed in [Lo, LM, MP, Mc]. It includes a generalization of the Hodge theory isomorphism, between harmonic forms and the de Rham cohomology classes on closed compact manifolds, to the in \([\text{Lo, LM, MP, Mc}]\). It includes a generalization of the Hodge theory isomorphism, between harmonic forms and the de Rham cohomology classes on closed compact manifolds, to the asymptotically cylindrical manifolds.

A manifold \(M\) with cylindrical ends is non-compact and there are two standard versions of the de Rham complex, depending on whether one uses spaces of all differential forms or subspaces of the forms with compact support. We shall write \(H^r(M)\) and \(H^r_c(M)\) for the respective cohomology groups; both are finite-dimensional vector spaces and are related via the long exact sequence

\[ \ldots \to H^{r-1}(Y) \to H^r_c(M) \to H^r(M) \to H^r(Y) \to \ldots. \quad (10) \]

where \(Y\), as before, denotes the cross-section of \(M\). The Poincaré duality gives isomorphisms \(H^r_c(M) \cong H^{\text{dim} M-r}(M)\).

It will be useful to define \(H^r_0(M) \subset H^r(M)\) to be the image of \(H^r_c(M)\) under the ‘inclusion homomorphism’ in \((10)\), that is, \(H^r_0(M)\) is the subspace of classes represented by compactly supported closed \(r\)-forms. We denote \(b_0^r(M) = \dim H^r_0(M)\).

Let \(H^r(M)\) denote the space of all \textit{bounded harmonic} \(r\)-forms on \(M\).

**Theorem 2.1.** Suppose that \(M\) is an asymptotically cylindrical oriented manifold with rate \(\lambda > 0\). Let \(\varepsilon > 0\) be such that \(\varepsilon < \lambda\) and \(\varepsilon^2\) is less than any positive eigenvalue of the Hodge Laplacian on differential forms on \(Y\) of \(M\). Then

(a) every \(\alpha \in H^r(M)\) is smooth and can be asymptotically expressed on the end of \(M\) as

\[ \alpha|_{M_\infty} = dt \wedge \alpha_1 + \alpha_r + O(e^{-\varepsilon t}), \quad (11) \]

for some harmonic forms \(\alpha_\nu \in H^{r-1}(Y), \alpha_r \in H^r(Y)\) (pulled back via the natural projection) and some \(\varepsilon > 0\), with \(O(e^{-\varepsilon t})\) understood in the strong sense of decaying to zero with all derivatives as \(t \to \infty\) uniformly on \(Y\).
(b) every $\alpha \in \mathcal{H}^r(M)$ is closed and co-closed. Define

$$\mathcal{H}^r_{\alpha}(M) = \{ \alpha \in \mathcal{H}^r(M) : \alpha_\nu = 0 \}$$

with $\alpha_\nu = 0$ defined by (11). Then the map

$$\alpha \in \mathcal{H}^r_{\alpha}(M) \rightarrow [\alpha] \in H^r(M)$$

assigning to a bounded harmonic form its de Rham cohomology class is a linear isomorphism.

The image under (12) of the subspace $\mathcal{H}^r_{\alpha}(M) \subset \mathcal{H}^r(M)$ of $L^2$ harmonic forms is $H^r_{\alpha}(M)$.

Part (a) is a direct consequence of the general result in [MP, §6] on the kernel elements of asymptotically translation-invariant elliptic operators, cf. also [Mc, Prop. 6.14 and 6.18], cf. also [APS, Prop. 4.9] and [Lo, Theorem 7.9] for the last claim. The results in [Mc] are given for ‘exact $b$-metrics’ which are equivalent to asymptotically cylindrical metrics having asymptotic expansions at $t \to \infty$, but the arguments we require carry over with only cosmetic changes to the present setting.

**Corollary 2.2.** Let $M$ be an asymptotically cylindrical oriented manifold with cross-section $Y$. Then the dimension of $\mathcal{H}^r(M)$ is $b^r(M) + b^r_0(M) - b^r_0(M)$.

**Proof.** Define $\mathcal{H}^r_{\alpha}(M) = \{ \alpha \in \mathcal{H}^r(M) : \alpha_\tau = 0 \}$ with $\alpha_\tau$ defined in (12). We deduce from (12) that $\ast \mathcal{H}^r_{\alpha}(M) = \mathcal{H}^\dim M - r(M)_{\nu}$, then $\dim \mathcal{H}^r_{\alpha}(M) = b^\dim M - r(M) = b^r(M)$ from Theorem 2.1(b) and the Poincaré duality.

For $\alpha \in \mathcal{H}^r(M)$, the image of $[\alpha]$ under homomorphism $i^* : H^r(M) \rightarrow H^r(Y)$ in (10) is $[\alpha_\tau]$, from (11) and the Hodge theory on compact $Y$. Then from Theorem 2.1(b) and the exactness of (10) we find that there exists $\alpha' \in \mathcal{H}^r_{\alpha}$ with $i^* [\alpha'] = [\alpha_\tau] = i^* [\alpha]$, so $\alpha - \alpha' \in \mathcal{H}^r_{\alpha}(M)$. Thus $\mathcal{H}^r(M) = \mathcal{H}^r_{\alpha}(M) + \mathcal{H}^r_{\alpha}(M)$. The intersection $\mathcal{H}^r_{\alpha}(M) \cap \mathcal{H}^r_{\alpha}(M) = \mathcal{H}^r_{\alpha}(M)$ has dimension $b^r_0(M)$ and the result follows by linear algebra.

There is a version of the Hodge decomposition theorem for asymptotically cylindrical manifolds.

**Proposition 2.3** (cf. [No1, p. 328]). Let $M$, $\lambda$ and $\epsilon$ be as in Theorem 2.1. Then for each $k = 0, 1, 2, \ldots$, every differential form $\xi \in C^k(\Omega^r(M))$ satisfying the exponential decay condition

$$\sup_{\{ t \} \times Y} |\nabla^j \xi| < C_j e^{-\epsilon t} \text{ for } t > 0, \ j = 0, \ldots, k, \tag{13}$$

can be written as

$$\xi = d\alpha + d^* \beta + \gamma$$

with $\Delta \gamma = 0$, for some unique $d\alpha$, $d^* \beta$, $\gamma$ satisfying (13) and pair-wise $L^2$-orthogonal.

In light of Theorem 2.1(a), $\gamma$ is equivalently an $L^2$ harmonic form.

When $(M, \Phi)$ is an asymptotically cylindrical Spin(7)-manifold the Hodge Laplacian maps the sections $\Omega^r_k(M)$ of each Spin(7)-invariant subbundle $\Lambda^r_k$ of $r$-forms [2] into itself [Jo3, §3.5]. Then there is a well-defined decomposition $\mathcal{H}^r(M) = \oplus_k \mathcal{H}^r_k(M)$ into spaces of bounded harmonic forms $\mathcal{H}^r_k(M) \subset \Omega^r_k(M)$.

The Levi–Civita connection of $g(\Phi)$ on $\Lambda^r T^* M$ also preserves the decomposition (2) determined by a parallel Spin(7)-structure $\Phi$. Let $\mathcal{H}^r_k(M)$ denote the space of parallel forms in $\Omega^r_k(M)$. The following result will be useful in the next section.
Proposition 2.4. Let $M$ be an asymptotically cylindrical Spin(7)-manifold with cross-section $Y$ and with the Spin(7)-structure $\Phi$ asymptotic to $\Phi_\infty = dt \wedge \varphi + \ast_\varphi(\varphi)$ for a torsion-free $G_2$-structure $\varphi$ on $Y$. Suppose that $b^1(M) = 0$. Then $\hat{H}^2_\alpha(M) = \mathbb{H}^2_\alpha(M)$ and the map
\[ \alpha \in \hat{H}^2_\alpha(M) \mapsto \alpha_\nu \in \mathcal{H}^1(Y) \] (14)
is linear isomorphism, where $\alpha_\nu$ is defined by (11).

Proof. Every $\alpha \in \Omega^2_\alpha(M)$ on a Spin(7)-manifold $M$ satisfies the Weitzenböck formula
\[ \Delta \alpha = \nabla^* \nabla \alpha. \] (15)
\[(\text{[Jo3 Prop. 10.6.5]})\], so every $\alpha \in \hat{H}^2_\alpha(M)$ is a bounded harmonic form, $\alpha \in \mathcal{H}^2_\alpha(M)$.

Conversely, let $\alpha \in \mathcal{H}^2_\alpha(M)$. Then by Theorem 2.1 $\alpha$ is asymptotic to a translation-invariant harmonic form $\alpha_\infty = dt \wedge \alpha_\nu + \alpha_\varphi$ on the cylindrical end $M_\infty$. Let $\Lambda^2 T^* M_\infty = (\Lambda^2_2)^\infty \oplus (\Lambda^2_2)^\infty$ denote the Spin(7)-invariant ‘type decomposition’ of 2-forms induced by $\Phi_\infty$. As $\alpha \in \Omega^2_\alpha(M)$ with respect to $\Phi$ and $\Phi$ is asymptotic to $\Phi_\infty$, the $(\Lambda^2_2)^\infty$ component of $\alpha$ is decaying to zero as $t \to \infty$. On the other hand, both $\alpha_\infty$ and $\Phi_\infty$ are translation-invariant and so is the $(\Lambda^2_2)^\infty$ component of $\alpha_\infty$. Therefore, $\alpha_\infty$ is a section of $\Omega^2_\infty$.

In terms of the $G_2$-structure $\varphi$ and the forms on $Y$, the bundles $(\Lambda^2_2)^\infty$ can be written as
\[ (\Lambda^2_2)^\infty = \{ dt \wedge \ast_\varphi(\ast_\varphi(\varphi) \wedge v_\varphi) + 3v_\varphi \wedge \varphi | v \in TY \}, \] (16)
\[ (\Lambda^2_2)^\infty = \{ dt \wedge \ast_\varphi(\ast_\varphi(\varphi) \wedge \alpha) - \alpha | \alpha \in \Lambda^2 T^* Y \} \] (17)
(cf. [SW Theorem 11.4]), where we also noted that a $G_2$-structure $\varphi$ induces a $G_2$-invariant rank 7 subbundle of 2-forms $\Lambda^2 T^* Y = \{ v_\varphi : v \in TY \}$. The map
\[ v_\varphi \mapsto \ast_\varphi(\ast_\varphi(\varphi) \wedge v_\varphi) \] (18)
defines a $G_2$-equivariant isometry of bundles $\Lambda^2_2 T^* Y \to T^* Y$ and gives an isomorphism between subspaces of parallel forms in $\Lambda^2 T^* Y$ and $T^* Y$. As the $G_2$-manifold $Y$ is Ricci-flat the 1-forms on $Y$ satisfy the Weitzenböck formula (15), without curvature terms. Then the integration by parts on a compact $Y$ shows that $\mathcal{H}^1(Y)$ is precisely the space of parallel 1-forms.

We obtain that the leading asymptotic term $\alpha_\infty$ of $\alpha$ is linearly determined by $\alpha_\nu$ via (16) and $\alpha_\infty$ is parallel as $\alpha_\nu \in \mathcal{H}^1(Y)$. Then $\nabla \alpha$ decays to zero along the end of $M$, so the integration by parts argument is valid for $\alpha$ and shows that $\nabla \alpha = 0$. It follows that $\mathcal{H}^2_\alpha(M) = \hat{H}^2_\alpha(M)$ and (14) is a well-defined injective linear map.

For the surjectivity, it suffices to show that the dimension of the space of parallel forms in $\Omega^2_\alpha(M)$ is $b^1(Y)$. We have dim $\mathcal{H}^2(M) = b^1(Y) + b^2(M)$ as the hypothesis $b^1(M) = 0$ yields $b^2_\alpha(M) = b^1(Y) + b^2_\alpha(M)$ from the exact sequence (10). On the other hand, dim $\mathcal{H}^2(M) = \text{dim} \hat{H}^2_\alpha(M) + \text{dim} \mathcal{H}^2_2(M)$ from previous arguments. We see from (17) and Theorem 2.1 that the asymptotic term $\alpha_\infty$ for each $\alpha \in \mathcal{H}^2_2(M)$ is determined by $\alpha_\tau \in \mathcal{H}^2(Y)$ and if $\alpha_\tau = 0$ then $\alpha$ is an $L^2$ form. It follows find that $\mathcal{H}^2_2(M)$ can be identified with the direct sum of the images of homomorphisms $H^2(M) \to H^2(Y)$ and $H^2_\alpha(M) \to H^2_\alpha(M)$ in (10) and therefore dim $\mathcal{H}^2_2(M) \leq b^2(M)$. As dim $\hat{H}^2_\alpha(M) \leq b^1(Y)$ by the above work, we find that the equalities are attained, dim $\mathcal{H}^2_2(M) = b^2(M)$ and dim $\hat{H}^2_\alpha(M) = b^1(Y)$. \[ \square \]

The next result would be needed in §6.1 below.
Proposition 2.5. Let M be an asymptotically cylindrical Spin(7)-manifold with cross-section Y and with the Spin(7)-structure Φ asymptotic to Φ∞ = dt ∧ ϕ + *ϕ(ϕ) for a torsion-free G2-structure ϕ on Y. Suppose that b1(M) = 0.

Then \( H^3_3(M) = \{0\} \) and for each \( α \in H^3(M) \) the asymptotic component \( α_e \in H^3(Y) \) defined by (11) is orthogonal to ϕ at each point in Y with the metric \( g(ϕ) \).

Proof. The vector bundles \( T^*M \) and \( Λ^3_3 T^*M \) are associated to isomorphic representations of Spin(7) and it follows from Weitzenböck formula that \( H^3_3(M) \cong H^1(M) \). On the other hand, as \( b^1(M) = 0 \) it follows from Corollary 2.2 and the exact sequence (10) that \( H^1(M) = \{0\} \). Thus \( H^2_3(M) = \{0\} \) and \( H^3(M) = H^3_{45}(M) \).

For the second claim, an argument similar to that in Proposition 2.4 shows that for each \( α \in H^3_{45}(M) \) the asymptotic term \( α_∞ \) is in \( H^3_{45}(R \times Y) \) relative to the cylindrical Spin(7)-structure \( Φ_∞ \). That \( α_e \) is pointwise orthogonal to ϕ on Y now follows directly from the determination of Spin(7)-invariant subbundle \( Λ^3_3 T^*(R \times Y) \) in terms of the G2-invariant subbundles of \( Λ^2_3 T^*Y \) given in [SW] Theorem 8.4 and 11.4.

\( \square \)

3 Holonomy and deformations of asymptotically cylindrical Spin(7)-manifolds

The holonomy group of Spin(7)-manifolds can be a proper subgroup of Spin(7), even when the manifold is compact and simply-connected; one source of examples is provided by Calabi–Yau manifolds. On the other hand, the following general criterion was proved by Bryant.

Theorem 3.1 ([BE, cf. pages 565–566]). Let M be a simply-connected Spin(7)-manifold. Then the holonomy of M is Spin(7) if and only if M admits no non-zero parallel 1-forms or 2-forms.

For the asymptotically cylindrical Spin(7)-manifolds, the holonomy is determined topologically from Betti numbers of the cross-section as the next result shows.

Theorem 3.2. If M is a simply-connected asymptotically cylindrical Spin(7)-manifold M with non-empty cross-section Y then the holonomy group of M is one of Spin(7), \( SU(4) \), \( G_2 \) and \( SU(2) \times SU(2) \).

(a) The holonomy of M is Spin(7) if and only if \( b^0(Y) = 1 \) (i.e. M has a single end) and \( b^1(Y) = 0 \).

(b) The holonomy of M is \( G_2 \) if and only if M is a cylinder \( Y \times R \) with a product metric, where \( Y \) is a compact simply-connected 7-manifold with holonomy \( G_2 \).

(c) The holonomy of M is \( SU(4) \) if and only if \( b^1(Y) = 1 \).

(d) The holonomy of M is \( SU(2) \times SU(2) \) if and only if \( b^1(Y) = 3 \). In this case, \( M = S_1 \times S_2 \) is the Riemannian product of a Ricci-flat Kähler K3 surface \( S_1 \) and a simply-connected asymptotically cylindrical Ricci-flat Kähler surface \( S_2 \) with cross-section homeomorphic to the 3-torus.

Remark 3.3. We construct examples realizing the possibility (a) in this paper. The case (b) is implemented by any simply-connected \( G_2 \)-manifold, e.g. [Jo3] or [Ko1]. Regarding (c) and (d), a method producing examples of asymptotically cylindrical Calabi–Yau metrics is given in [Ko1]. See also [HHN] and [BM]. It is shown in [He] that the flat 3-torus cross-section at infinity of \( S_2 \) in (d) need not in general be isometric to a Riemannian product of \( S^1 \) and a 2-torus [He].
Much of Theorem 3.2 is known in some form. The criterion (a) is a direct consequence of the result of Nordström [No2, Theorem 4.1.19] proved using spinors and (c) is obtained, again by spinor methods, in [III, Theorem B] which also excludes the holonomy $Sp(n)$.

We give a self-contained argument for a simply-connected $M$ using the octonions algebra and parallel differential forms without appealing to spinors or holomorphic geometry.

**Proof of Theorem 3.2.** As every Spin(7)-manifold is Ricci-flat, the Cheeger–Gromoll splitting theorem [CG] implies that a connected asymptotically cylindrical Spin(7)-manifold $M$ has either one end or two ends (see also [Gi]).

Furthermore, if $M$ has two ends, then it is necessarily a cylinder $M = \mathbb{R} \times Y$ with a product metric. Then $Y$ is a compact simply-connected $G_2$-manifold and by [163, Prop. 10.2.2] the holonomy group of $Y$, hence also of $M$, is $G_2$. Conversely, if the holonomy of $M$ is $G_2$, then an application of the de Rham theorem [dR] shows that $M$ is the Riemannian product of a complete $G_2$-manifold $Y$ and a 1-manifold. As $M$ is simply-connected with (only) cylindrical ends, we deduce that $M = \mathbb{R} \times Y$ and $Y$ is compact simply-connected with holonomy $G_2$. This completes the proof of (b).

Suppose now that $M$ has a single end. Applying the de Rham theorem again we find that the holonomy group of $M$ cannot act trivially on any subspace of $\mathbb{R}^8$ of positive dimension. The only connected subgroups of Spin(7) with the latter property and allowed by Berger’s classification are Spin(7), $SU(4)$, $Sp(2)$ and $SU(2) \times SU(2)$.

Let $b^1(Y) = 0$. If the holonomy of $M$ is not Spin(7), then it is a subgroup of $SU(4)$ and $M$ has a parallel 2-form, the Kähler form $\omega$. But as we saw in (7) then $\omega$ is in $\Omega^2_{\mathbb{R}}(M)$ and we obtain $b^1(Y) > 0$ by Proposition 2.4 a contradiction. Conversely, if the holonomy of $M$ is Spin(7), then by Theorem 3.1 there are no non-zero parallel 2-forms, hence $b^1(Y) = 0$ by Proposition 2.4 This completes the proof of (a).

If the holonomy of $M$ is contained in $SU(4)$, then $M$ has a parallel complex structure $I$ and there is an $SU(4)$-invariant decomposition of real 2-forms on $M$ refining the Spin(7)-invariant decomposition (2),

$$
\Lambda^2_5 = \Lambda^2_7 \oplus \Lambda^2_6, \quad \Lambda^2_{21} = \Lambda^2_{15} \oplus \Lambda^2_6, \quad (19)
$$

according to the types (1,1) and (2,0) $\oplus$ (0,2) with respect to $I$. More explicitly, the fibres of the two $\Lambda^2_5$ subbundles in (19) are, respectively, the $\pm 2$-eigenspaces of $\ast(\cdot \wedge \text{Re}\, \theta_0)$, where $\theta_0$ is the $(4,0)$ form in (6), cf. [SW, Theorem 11.4]. These two rank 6 subbundles are isomorphic, being associated to isomorphic representations of $SU(4)$. The sections of $\Lambda^2_5$ satisfy the Weitzenböck formula (15) with vanishing curvature term and the argument of Proposition 2.4 applies to show that the bounded harmonic sections of each $\Lambda^2_6$ are parallel.

If the holonomy of $M$ is exactly $SU(4)$ then the only parallel 2-forms on $M$ are constant sections of $\Lambda^2_5$ (constant multiples of the Kähler form) and $b^1(Y) = 1$.

If the holonomy of $M$ is a proper subgroup of $SU(4)$, then it is contained in $Sp(2)$ and $M$ admits three parallel complex structures and three respective Kähler forms which are sections of $\Lambda^2_5$. These Kähler forms are point-wise orthogonal to each other, thus two of them are sections of $\Lambda^2_5 \subset \Lambda^2_5$ and, respectively, there are further two parallel sections of $\Lambda^2_5 \subset \Lambda^2_{21}$. We thus obtain a 5-dimensional space of parallel 2-forms on $M$. As the holonomy representation of $Sp(2)$ stabilizes only three linearly independent 2-forms on $\mathbb{R}^8$, we deduce that the holonomy of $M$ reduces to a proper subgroup of $Sp(2)$ and therefore must be $SU(2) \times SU(2)$.

Now if the holonomy of $M$ is $SU(2) \times SU(2)$, then applying the de Rham theorem again we obtain that $M = S_1 \times S_2$ is a Riemannian product of two 4-manifolds with holonomy $SU(2)$, i.e. $S_i$ are Ricci-flat Kähler surfaces with a trivial canonical bundle. Both $S_i$ are
simply-connected and one of these, $S_1$ say, is compact, hence a K3 surface, and the metric on $S_2$ is asymptotically cylindrical. The 3-dimensional cross-section of $S_2$ at infinity then is a flat 3-torus, thus $b^1(Y) = 3$ in this case. This completes the proof of Theorem 3.2.

Joyce investigated the deformation theory of torsion-free Spin(7)-structures on a compact 8-manifold $M$ and proved that these have a smooth moduli space of dimension $b^4_4(M) + b^2_2(M) + 1$, which becomes $b^4_4(M) + 1$ if $M$ has full holonomy Spin(7). (Here $b^r_i(M)$ denote the dimensions of the spaces of harmonic forms of respective times.) Cf. [Jo3, Theorem 10.7.1].

Nordström obtained an analogue of Joyce’s result for 8-manifolds with cylindrical ends, where the moduli space is defined using diffeomorphisms isotopic to the identity and asymptotically to those preserving the product cylindrical structure of the ends. Recall from §2 the definition of subspaces $H^r_0$ in the de Rham cohomology.

**Theorem 3.4** ([No2 Theorem 4.3.2 and Prop. 4.3.3]). Let $(M, \Phi)$ be a connected asymptotically cylindrical Spin(7)-manifold. Then the moduli space of torsion-free asymptotically cylindrical Spin(7)-structures on $M$ is a smooth manifold of dimension $b^4(M) - b^4_4(M) + b^2_2(M) + 1$, where $b^4_4(M)$ is the dimension of a maximal positive subspace for the cup-product on $H^4_0(M)$.

For a simply-connected asymptotically cylindrical $M$ with holonomy equal to Spin(7), the dimension of the moduli space therefore is $b^4(M) - b^4_4(M) + 1$. It can be checked using the argument of Corollary 2.2 and the Poincaré duality that $b^4(M) - b^4_4(M)$ is the dimension of the space $H^4(M) \subset \Omega^4_+(M)$ of bounded harmonic anti-self-dual 4-forms on $M$.

## 4 Existence of asymptotically cylindrical torsion-free Spin(7)-structures

In order to construct asymptotically cylindrical metrics with holonomy Spin(7) on 8-manifolds with cylindrical ends, we shall obtain Spin(7)-structures (more precisely, 1-parameter families) with arbitrary small torsion, satisfying Theorem 4.1 which is the main result of this section. We modify Joyce’s existence result [Jo3 Theorem 13.6.1] (and Proposition 15.2.13 op.cit.) for torsion-free Spin(7)-structures on compact manifolds, extending it to manifolds with cylindrical ends. The strategy of the proof of Theorem 4.1 has some analogy with [KN, Theorem 3.1] for asymptotically cylindrical torsion-free $G_2$-structures on 7-manifolds.

**Theorem 4.1.** Let $\lambda$, $\mu$, $\nu$ be positive constants. Then there exist positive constants $\kappa$, $K$ and $K'$ such that whenever $0 < s \leq \kappa$ the following is true.

Let $M$ be an 8-manifold with cylindrical end $M_\infty = \mathbb{R}_+ \times Y$, and suppose that an admissible 4-form $\tilde{\Phi}$ defines on $M$ a Spin(7)-structure which is asymptotically cylindrical with rate $\delta_M$. Suppose that $\tilde{\Phi}$ is closed on $M_\infty$ and there is a smooth compactly supported 4-form $\psi$ on $M$ satisfying $d\tilde{\Phi} + d\psi = 0$. If

\begin{align}
&\|\psi\|_{L^2} \leq \lambda s^{13/3} \quad \text{and} \quad \|d\psi\|_{L^{10}} \leq \lambda s^{7/5},\tag{20}
\end{align}

(b) the injectivity radius $\delta(\Phi)$ of the metric $g(\Phi)$ satisfies $\delta(\Phi) \geq \mu s$, and

(c) the Riemann curvature $R(\Phi)$ of the metric $g(\Phi)$ satisfies $\|R(\Phi)\|_{C^0} \leq \nu s^{-2}$.
then there is a smooth anti-self-dual 4-form η on M, exponentially decaying to zero with all derivatives as \( t \to \infty \), such that \( \| \eta \|_{C^0} \leq K s^{1/3} \) and

\[
d\eta = d\psi + dF(\eta) \tag{21}
\]

and \( \Phi = \Theta(\tilde{\Phi} + \eta) \) is a torsion-free Spin(7)-structure, with \( \| \Phi - \tilde{\Phi} \|_{C^0} \leq K' s^{1/3} \). The exponential decay rate of \( \eta \) can be taken to be \( \delta = \frac{1}{2} \min\{\delta_M, \delta_Y\} \), where \( \delta_Y^2 \) is the smallest positive eigenvalue of the Hodge Laplacian on \( \Omega^*(Y) \) for the metric of the cross-section of \( M \) at infinity.

The difference between Theorem 4.1 and [Jo3, Theorem 13.6.1] is that in the present case \( M \) is non-compact with a cylindrical end and we added appropriate hypotheses on the asymptotical behaviour of \( \tilde{\Phi} \) and \( \Phi \) on the end and are claiming an asymptotically cylindrical property of the torsion-free \( \Phi \). Formally, Theorem 4.1 includes the statement of [Jo3, Theorem 13.6.1], corresponding to the case when \( Y = \emptyset \) (so \( M \) is compact).

In the rest of this section we prove Theorem 4.1. Firstly, we note that it suffices to show the existence of exponentially decaying anti-self-dual smooth 4-form \( \eta \), with ‘small’ uniform norm, satisfying (21). The argument is local and readily carries over from [Jo3, Theorem 13.6.1]. We then adapt the contraction mapping method used by Joyce [Jo3, §13.7] on compact 8-manifolds to find an anti-self-dual form \( \eta \) solving (21) on asymptotically cylindrical \( M \) and use elliptic regularity to show that the solution \( \eta \) is smooth. In fact, the method will also show that \( \eta \) decays along the cylindrical end \( \mathbb{R}_+ \times Y \subset M \) at \( t \to \infty \), uniformly in the \( Y \) variables. Our last claim that the solution decays at an exponential rate has no analogue in the compact case. To prove this claim we interpret (21) on the end of \( M \) as an infinite-dimensional flow in \( t \) and use the results about hyperbolic stationary points.

Suppose that \( \eta \in \Omega^4_+(M) \) satisfies (21) and \( \| \eta \|_{C^0} \leq K s^{1/3} \) with \( s \) sufficiently small. Then, by the discussion in [Jo3, Proposition 13.7.1] on compact 8-manifolds to find an anti-self-dual form \( \eta \) solving (21) on asymptotically cylindrical \( M \) and use elliptic regularity to show that the solution \( \eta \) is smooth. In fact, the method will also show that \( \eta \) decays along the cylindrical end \( \mathbb{R}_+ \times Y \subset M \) at \( t \to \infty \), uniformly in the \( Y \) variables. Our last claim that the solution decays at an exponential rate has no analogue in the compact case. To prove this claim we interpret (21) on the end of \( M \) as an infinite-dimensional flow in \( t \) and use the results about hyperbolic stationary points.

Thus we wish to find an appropriate solution of (21). The following is a version of the contraction mapping result [Jo3, Prop. 13.7.1] adapted to the asymptotically cylindrical base manifold.

**Proposition 4.2.** Assume that an 8-manifold \( M \) with cylindrical end and a Spin(7) structure \( \Phi \) on \( M \) satisfy all the hypotheses of Theorem 4.1.

Then there exist positive constants \( \kappa \), \( C \) and \( K \) depending only on \( \lambda \) in (20) such that if \( s \leq \kappa \) then there exists a sequence \( \eta_j \in \Omega^4_+(M) \), \( \eta_0 = 0 \), of smooth forms exponentially decaying with all derivatives with rate \( \delta \) (defined in Theorem 4.1) and satisfying, for each \( j = 1, 2, \ldots \), the equation

\[
d\eta_j = d\psi + dF(\eta_{j-1}) \tag{22}
\]

and the inequalities

(a) \( \| \eta_j \|_{L^2} \leq 4\lambda s^{13/3} \),
(b) \( \| \nabla \eta_j \|_{L^{10}} \leq C s^{2/15} \),
(c) \( \| \eta_j \|_{C^0} \leq K s^{1/3} \)
(d) \( \| \eta_j - \eta_{j-1} \|_{L^2} \leq 4\lambda 2^{-j} s^{13/3} \),
(e) \( \| \nabla (\eta_j - \eta_{j-1}) \|_{L^{10}} \leq C 2^{-j} s^{2/15} \),
\( \| \eta_j - \eta_{j-1} \|_{C^0} \leq K2^{-j}s^{1/3}. \)

**Proof.** The proof proceeds by induction on \( j \). Assume that there exist \( \eta_0, \ldots, \eta_{j-1} \) satisfying all the assertions of Proposition 4.2. Then \( \eta_{j-1} \in \Omega^4(M) \) is exponentially decaying and satisfies the uniform estimate (c) with small \( s \), so \( F(\eta_{j-1}) \) is well-defined and satisfies the quadratic estimate (4). Therefore, \( \psi + F(\eta_{j-1}) \) decays with the same exponential rate as \( \eta_{j-1} \). Applying Proposition 2.3 we can write \( \psi + F(\eta_{j-1}) = d\alpha + d^*\beta + \gamma \) and

\[
d\psi + dF(\eta_{j-1}) = dd^*\beta = d(d^*\beta - *d^*\beta + \tilde{\gamma}),
\]

for some exponentially decaying \( d^*\beta \) and an arbitrary \( L^2 \) harmonic form \( \tilde{\gamma} \). By Theorem 2.1(a)), \( \tilde{\gamma} \) is closed and may be chosen to represent any given cohomology class in \( H^4_0(M) \). We choose \( \tilde{\gamma} \) so that \( \tilde{\gamma} - \gamma \in \mathcal{H}^4_0(M) \), where we denoted by \( \mathcal{H}^4_0 = L^2\Omega^4_\pm(M) \cap \mathcal{H}^4(M) \) the spaces of self- and anti-self-dual \( L^2 \) harmonic 4-forms. Note that the forms in \( \mathcal{H}^4_\pm \) represent exactly the classes in a maximal positive, respectively negative, subspace \( H^4_\pm(M) \) for the cup-product on \( H^4_\pm(M) \). Now define \( \eta_j := d^*\beta - *d^*\beta + \gamma \) and this is a smooth anti-self-dual 4-form exponentially decaying with all derivatives and satisfying (22).

The method used in \([103, \S 13.7]\) to prove the estimates (a)–(f) for a compact base manifold \( M \) applies mutatis-mutandis to the present situation. The role of harmonic 4-forms and their cohomology classes in the compact case is now taken by the exponentially decaying (equivalently, \( L^2 \)) harmonic 4-forms on \( M \) and the subspace \( H^4_M(M) \subset H^4(M) \). The respective integrals in the proof of (a) and (d) are finite on \( M \) as each \( \eta_j \) decays exponentially with all derivatives along the cylindrical end. The proof of (b),(c),(e),(f) in \([103, \S 13.7]\) uses estimates for Sobolev norms on complete Riemannian \( 8 \)-manifolds (it is at this point that the hypotheses (b) and (c) of Theorem 4.1 are required) and the quadratic estimates (4), (5) for and carries over without changes. We therefore omit the details.

When \( s \) is sufficiently small the sequence \( \{ \eta_j \} \) given by Proposition 4.2 is Cauchy and converges in the \( L^2 \) norm and also in the \( L^2 \) and \( C^0 \) norms on \( M \), to \( \eta \in \Omega^4(M) \). The limit \( \eta \) is a weak \( L^1_{10} \) solution to the (21) and satisfies

\[
\| \psi \|_{L^2} \leq Ks^{13/3}, \quad \| \psi \|_{C^0} \leq Ks^{13/3} \quad \| \nabla \psi \|_{L^{10}} \leq Ks^{7/5},
\]

for some \( K > 0 \) independent of \( 0 < s < \kappa \).

**Proposition 4.3.** If \( s > 0 \) is sufficiently small, then

(i) \( \eta \) is in \( L^1_{10}(M) \), for each \( k > 0 \), therefore smooth on \( (M) \), and

(ii) \( \eta \) on the cylindrical end of \( M \) converges to zero with all derivatives, uniformly on \( Y \), as \( t \to \infty \).

**Proof.** For part (i), we note that the exterior derivative on \( \Omega^4(M) \) is an overdetermined elliptic operator, i.e. has injective symbol, and the standard interior elliptic \( L^p \) estimates still hold for it. (More explicitly, it is a restriction of the elliptic operator arising from elliptic complex \( 0 \to \Omega^4(M) \to \Omega^5(M) \to \ldots \)) We may write \( dF(\eta) \) satisfies a quadratic estimate we can write the term \( dF(\eta) \) in (21) as \( G(\eta, \nabla \eta) \), where \( G \) is a smooth function, linear in the second argument and \( G(0, \cdot) \equiv 0 \). Then equation satisfied by \( \eta \) takes the form

\[
d\eta - G(\eta, \nabla \eta) = d\psi.
\]
We regard the left-hand side as a linear operator in $P_\zeta(\eta) := d\eta - G(\zeta, \nabla \eta)$ with $\zeta = \eta$. Assuming the $\kappa$ is sufficiently small, we have the $C^0$ norm of $\zeta$ sufficiently small. Then $P_\zeta$ is overdetermined elliptic as this is an open condition.

If $\eta \in L^1_k(M)$ for some $k \geq 0$, then the coefficients of $P_\zeta$ are also in $L^1_k(M)$, hence in $C^{0,1/5}(M)$ by Sobolev embedding. Applying [Mo] Theorems 6.2.5, 6.2.6] we deduce that $\eta$ is locally in $L^1_k(M)$. Furthermore, the estimate of the $L^1_k$ norm on local neighbourhoods can be taken uniformly on $M$ as the metric on $M$ is asymptotically cylindrical. It follows that $\eta \in L^1_k(M)$ and then in $C^\infty(M)$ by induction on $k$.

Part follows directly from the argument of [KN, Cor. 3.7] which exploits the finiteness of $L^1_k$ norm of $\eta$ and a uniform bound for the local embedding $L^1_k \subset C^k$ on geodesic balls of fixed radius in $M$. □

The exponential rate of decay requires additional work. We consider the ‘Spin(7) equation’ on the cylindrical end $\mathbb{R}_+ \times Y \subset M$ as an evolution equation on $Y$ with the zero solution corresponding to a stationary point. The decaying solution then may be considered as a path of forms parameterised by $t$ on the cross-section 7-manifold $Y$. The key property is that the stationary point zero is hyperbolic. The exponential decay property is well-known for hyperbolic stationary points of flows in finite-dimensional spaces. For an infinite-dimensional setting, a suitable argument is given in [MMR, Chap. 5, Theorem 5.2.2]. In fact, the result is slightly more general and asserts that every solution path converging to a stationary point is exponentially asymptotic to a path in an invariant ‘centre manifold’ — when the stationary point is hyperbolic, a centre manifold is just this stationary point.

For each $p$, the space of $p$-forms on $\mathbb{R}_+ \times Y$ is naturally isomorphic to the one-parameter families $\mathbb{R} \to (\Omega^{p-1} \oplus \Omega^p)(Y)$. For anti-self-dual 4-forms on $\mathbb{R}_+ \times Y$ we may write
\[
\eta|_{M_\infty} = dt \wedge (\eta_3(t) + \varphi(t, x, \eta_3) - *_Y \eta_3(t)), \tag{23}
\]
where $\varphi(t, x, \eta_3)$ is linear in $\eta_3$, smooth in $x \in Y$ and $O(e^{-\delta_M t})$ with all derivatives, for $t > 0$.

As $\psi$ is compactly supported, we may take $\psi = 0$ on $M_\infty$, so the equation satisfied by $\eta$ becomes
\[
d\eta = dF(\eta). \tag{24}
\]
We know that $\eta_3(t) \to 0$ uniformly on $Y$ as $t \to \infty$.

Write $F(\eta) = dt \wedge F_3(\eta) - F_4(\eta)$ and then [Kn] implies that $F_3, F_4$ on $\{t\} \times Y$ satisfy quadratic estimates in $\eta_3(t)$ for each $t$,
\[
|F_i(\eta_3) - F_i(\eta_3')| < C_2|\eta_3' - \eta_3||(\eta_3')^2| + |\eta_3''|, \quad i = 3, 4,
\]
with $C_2$ independent of $t$.

Differentiating (23) and $F(\eta)$, plugging in (24)
\[
-dt \wedge (dY \eta_3 + \frac{\partial}{\partial t}(*_Y \eta_3)) - dY *_Y \eta_3 = -dt \wedge (dY F_3 + \frac{\partial}{\partial t} F_4) - dY F_4
\]
and equating the components, we obtain
\[
dY(*_Y \eta_3 - F_4) = 0, \quad \frac{\partial}{\partial t}(*_Y \eta_3 - F_4) = dY(F_3 - \eta_3 - \varphi), \tag{25}
\]
so $*_Y \eta_3(t) - F_4(\eta_3(t))$ is exact on $\{t\} \times Y$ for each $t > 0$. 

The function assigning to \( \eta_3 \in \Omega^3(Y) \) the co-closed component of \( *_Y \eta_3 - F_4(\eta_3) \), in the Hodge decomposition on compact \( Y \), is smooth and has surjective derivative at \( \eta_3 = 0 \). By the implicit function theorem in Banach spaces the set

\[
\mathcal{Y}_\varepsilon = \{ \eta_3 \in L^2 \Omega^3(Y) : \| \eta_3 \| < \varepsilon, *_Y \eta_3 - F_4(\eta_3) \text{ is exact} \}
\]

for some \( \varepsilon > 0 \) is a graph of smooth function \( d_Y^* \left( L^2_1 \Omega^4(Y) \right) \to \text{Ker} \left( d \cap L^2 \Omega^3(Y) \right) \). In particular \( \mathcal{Y}_\varepsilon \) is a Banach submanifold of \( L^2 \Omega^3(Y) \) with the tangent space at \( \eta_3 = 0 \)

\[
T_0 \mathcal{Y}_\varepsilon = d_Y^* \left( L^2_1 \Omega^4(Y) \right).
\]

By the Hodge theory, the space of co-exact forms \( T_0 \mathcal{Y}_\varepsilon \) for a compact \( Y \) is a closed subspace of \( L^2 \) 3-forms, thus a Banach space. Further, the \( L^2 \)-orthogonal projection \( \text{pr}_Z : L^2 \Omega^3(Y) \to T_0 \mathcal{Y}_\varepsilon \) is a bounded linear map. Denote \( \tilde{\eta}_3 := \text{pr}_Z(\eta_3) \). The restriction of \( \text{pr}_Z \) to \( \mathcal{Y}_\varepsilon \) is invertible with \( \eta_3 = \tilde{\eta}_3 - \tilde{F}_3(\tilde{\eta}_3) \) and \( \tilde{F}_3 \) satisfying a quadratic estimate.

Substituting the latter formula in the second equation of \( (26) \) we obtain that \( \tilde{\eta}_3 \) satisfies a flow equation in on a neighbourhood of 0 in the space of co-exact \( L^2 \) 3-forms on \( Y \)

\[
\frac{\partial \tilde{\eta}_3}{\partial t} = -*_Y d_Y \tilde{\eta}_3 + Q(\tilde{\eta}_3) + \varphi(t, \tilde{\eta}_3),
\]

with \( Q(\tilde{\eta}_3) \) of quadratic order in \( \tilde{\eta}_3 \) and \( \varphi \) of linear order in \( \tilde{\eta}_3 \) and \( \varphi(t, \tilde{\eta}_3) = O(e^{-\delta_M t}) \) for \( t > 0 \).

The operator \( *_Y d_Y \) defines a linear isomorphism of the space \( d_Y^* \Omega^4(Y) \) onto itself and is formally self-adjoint. Its square is the (restriction of) Hodge Laplacian. The space of co-exact 3-forms splits into \( L^2 \)-orthogonal direct sum \( d_Y^* \Omega^4(Y) = T_+ \oplus T_- \) of positive and negative eigenspaces of \( -*_Y d_Y \) and \( \eta_3 = 0 \) is thus a hyperbolic stationary point of the infinite-dimensional flow \( (26) \).

**Proposition 4.4.** Let \( \tilde{\eta}_3(t), t \geq t_0, \) be a path of co-exact 3-forms on \( Y \) satisfying the equation \( (26) \) and such that \( \tilde{\eta}_3(t) \to 0 \) uniformly on \( Y \) as \( t \to \infty \).

Then \( \| \tilde{\eta}_3 \|_{L^p_k((t) \times Y)} < C_{p,k} e^{-(6/2)t} \) for \( t \geq t_0 \) (here \( p > 1, k = 1,2,\ldots \)) with \( \delta \) defined in Theorem \( 3.4 \).

**Proof.** This is an instance of a general property for partially hyperbolic fixed point of flows. Our argument is similar to that in [MMR §5.4], but in the present case the linear part has no pure imaginary eigenvalues and some details are simplified. It will suffice to consider \( p = 2 \) as any given \( L^p \) norm of a smooth \( \tilde{\eta}_3 \) on \( Y \) is controlled by its \( L^k \) norm for a sufficiently large \( k \).

The hypothesis implies that there is, for each \( E_0 > 0 \) and \( k = 1,2,\ldots, a T_{0,k} > 0, \) so that \( \sup_{t \in [T_{0,k}, \infty)} \| \tilde{\eta}_3 \|_{L^2_k((t) \times Y)} < E_0 \). Given a \( \tilde{\eta}_3 \in T_0 \mathcal{Y}_\varepsilon \) we can write \( \tilde{\eta}_3 = \tilde{\eta}_3^+ + \tilde{\eta}_3^- \), for some unique \( \tilde{\eta}_3^+ \in T^+ \) and \( \tilde{\eta}_3^- \in T^- \).

We claim that for a sufficiently small \( E_0 \), the paths \( \tilde{\eta}_3^+(t) \) satisfy the differential inequalities

\[
\frac{\partial}{\partial t} \| \tilde{\eta}_3^+ \|_{L^k} \geq \delta(\| \tilde{\eta}_3^+ \|_{L^k} - \frac{1}{4} \| \tilde{\eta}_3 \|_{L^k} - C'e^{-\delta_M t})
\]

\[
\frac{\partial}{\partial t} \| \tilde{\eta}_3^- \|_{L^k} \leq -\delta(\| \tilde{\eta}_3^- \|_{L^k} - \frac{1}{4} \| \tilde{\eta}_3 \|_{L^k} - C'e^{-\delta_M t})
\]

for \( t \in [T_{0,k}, \infty) \). The arguments for the two inequalities are very similar and we show \( (27b) \).

We have
\[-\frac{1}{2} \frac{\partial}{\partial t} \| \tilde{\eta}_3^\pm \|_{L_k^4}^2 = - \langle \tilde{\eta}_3^\pm, \frac{\partial}{\partial t} \tilde{\eta}_3^\pm \rangle_k = - \langle \tilde{\eta}_3^\pm, \frac{\partial}{\partial t} \tilde{\eta}_3 \rangle_k = \langle \tilde{\eta}_3^\pm, *_{Y} d_{Y} \tilde{\eta}_3 - Q(\tilde{\eta}_3) - \tilde{\psi}(t, \tilde{\eta}_3) \rangle_k \geq \delta \| \tilde{\eta}_3^\pm \|_{L_k^4}^2 - \frac{1}{4} \| \tilde{\eta}_3^\pm \|_{L_k^4} \| \tilde{\eta}_3^- \|_{L_k^2}^2 - C' e^{-\delta m t} \| \tilde{\eta}_3^- \|_{L^2}\]

where $\langle \cdot, \cdot \rangle_k$ denotes the $L^2_k$ inner product on $Y$. This implies (27b) for all $t$ where $\tilde{\eta}_3^+ (t) \neq 0$ and, by continuity, on the closure of the set of such $t$. The complement is an open set where $\tilde{\eta}_3^+ (t) \equiv 0$ and the inequality is obvious.

We next check that $\| \tilde{\eta}_3^+ \|_{L_k^4} < \| \tilde{\eta}_3^- \|_{L_k^4}$ for each sufficiently large $t$. Subtracting (27b) from (27a) yields

$$
\frac{\partial}{\partial t}(\| \tilde{\eta}_3^+ \|_{L_k^4} - \| \tilde{\eta}_3^- \|_{L_k^4}) \geq \frac{\delta}{2} \| \tilde{\eta}_3 \|_{L_k^4},
$$

so $\| \tilde{\eta}_3^+ \|_{L_k^4} - \| \tilde{\eta}_3^- \|_{L_k^4}$ is non-decreasing in $t$. If our claim is false, then $\| \tilde{\eta}_3^+ \|_{L_k^4} - \| \tilde{\eta}_3^- \|_{L_k^4} \geq 0$ on some semi-infinite interval $t > T_0$. But then

$$
\frac{\partial}{\partial t} \| \tilde{\eta}_3^+ \|_{L_k^4} \geq \delta(\| \tilde{\eta}_3^+ \|_{L_k^4} - \frac{1}{4} \| \tilde{\eta}_3^- \|_{L_k^4}) \geq \delta(\| \tilde{\eta}_3^+ \|_{L_k^4} - \frac{1}{2} \| \tilde{\eta}_3^- \|_{L_k^4})
$$

and $\| \tilde{\eta}_3^+ \|_{L_k^4}$ is exponentially increasing, a contradiction. Finally,

$$
\frac{\partial}{\partial t} \| \tilde{\eta}_3^- \|_{L_k^4} \leq -\delta(\| \tilde{\eta}_3^- \|_{L_k^4} + \frac{1}{4} \| \tilde{\eta}_3^+ \|_{L_k^4}) \leq -\delta(\| \tilde{\eta}_3^- \|_{L_k^4} + \frac{1}{2} \| \tilde{\eta}_3^- \|_{L_k^4})
$$

implies the required $L^2_k$ estimate of $\tilde{\eta}_3|_{\{t\} \times Y}$.

\[\square\]

5 A class of asymptotically cylindrical Calabi–Yau orbifolds

In order to construct examples of asymptotically cylindrical holonomy-Spin(7) manifolds by application of Theorem 4.1, we use complex projective 4-dimensional orbifolds equipped with asymptotically cylindrical Calabi–Yau metrics.

Recall that the definition of an $n$-dimensional orbifold, $V$ say, is analogous to that of a smooth manifold, but a neighbourhood $U_p$ of a point $p$ is homeomorphic to $\mathbb{R}^n/\Gamma_p$, where $a_p: \Gamma_p \times \mathbb{R}^n \to \mathbb{R}^n$ is an effective action of a finite group $\Gamma_p$ (the ‘local isotropy group’) on $\mathbb{R}^n$. The covering map $\mathbb{R}^n \to U_p$ is called a local uniformizing chart centred at $p$ and the coordinates on the domain $\mathbb{R}^n$ are local uniformizing coordinates. (Cf. [15a].)

If $\Gamma_p = \{1\}$ then $p$ is a smooth point of $V$, otherwise $p$ is a singular point. The set of all smooth points of $V$ is a manifold and will be denoted $V^s$. We shall be interested in orbifolds whose singular points are isolated, with each $\gamma \neq 1$ in $\Gamma_p$ only fixing 0 for each point $p$.

The concepts of smooth differential forms and, more generally, sections of vector bundles associated to $TV$ admit natural generalizations to orbifolds: in a neighbourhood of singular point these are defined by the respective $G$-invariant objects in uniformizing coordinates.

A Riemannian metric $g$ on $V^s$ is called a smooth orbifold metric, and $(V, g)$ is called a Riemannian orbifold, if the pull-back of $g$ to a local uniformizing chart extends smoothly to $0 \in \mathbb{R}^n$ and $a_p$ acts by isometries. In particular $a_p$ gives a representation of $\Gamma_p$ in $O(n)$, the orthogonal group of $T_0 \mathbb{R}^n$. The holonomy group of a Riemannian orbifold $V$ is defined as the holonomy group of its smooth part, the Riemannian manifold $V^s$. 
In a similar manner one defines \( n \)-dimensional complex orbifolds and Kähler orbifold metrics, with each \( \Gamma_p \) now assumed to be a subgroup of \( U(n) \). On the other hand, the canonical bundle of a complex orbifold will be a well-defined orbifold complex line bundle if \( \Gamma_p \subset SL(n, \mathbb{C}) \) for each \( p \) and then it makes sense to speak of the anticanonical Weil divisors.

Our construction of asymptotically cylindrical Spin(7)-manifolds will start from asymptotically cylindrical Calabi–Yau complex 4-folds obtained using the following result.

**Theorem 5.1.** Let \( \overline{W} \) be a compact Kähler 4-dimensional orbifold with Kähler form \( \omega_{\overline{W}} \) and with the singular locus of \( \overline{W} \) consisting of finitely many points with local isotropy groups contained in \( SU(4) \). Suppose that \( D \in | - K_{\overline{W}}^e \) is a smooth 3-fold in the anticanonical class containing no singular points of \( \overline{W} \) and such that the normal bundle \( N_{D/\overline{W}} \) is holomorphically trivial. Suppose further that \( D \) and the smooth locus \( \overline{W}^* \subset \overline{W} \) and \( \overline{W}^* \setminus D \) are simply-connected.

Then \( W = \overline{W} \setminus D \) admits a complete Ricci-flat Kähler metric \( g \) with holonomy \( SU(4) \) and a non-vanishing holomorphic volume \((4,0)\)-form. These are exponentially asymptotic to the product cylindrical Ricci-flat Kähler structure on \( D \times S^1 \times \mathbb{R}_{>0} \), via a diffeomorphism \( U \setminus D \simeq D \times S^1 \times \mathbb{R}_{>0} \), where \( U \) is a neighbourhood of \( D \) in \( \overline{W} \) and \( z = \exp(-t-i\vartheta) \), \( t > 0 \), \( \vartheta \in S^1 \) extends over \( D \) to give a holomorphic coordinate on \( U \subset \overline{W} \) vanishing to order one precisely on \( D \). The Kähler form \( \omega_g \) of \( g \) may be written near \( D \) as

\[
\omega_{g}|_{U \setminus D} = dt \wedge d\vartheta + \omega_D + d\psi.
\]

and a holomorphic volume form (up to a constant factor) as

\[
\Omega_g|_{U \setminus D} = (dt + i d\vartheta) \wedge \theta_D + d\Psi.
\]

Here \( \omega_D \) is the Ricci-flat Kähler metric on \( D \) in the cohomology class \( [\omega_{\overline{W}}]|_D \) and \( \theta_D \) is a non-vanishing holomorphic \((3,0)\)-form on \( D \) satisfying \( 3i \theta_D \wedge \theta_D = 4 \omega_D^3 \). The forms \( \psi, \Psi \) decay at the rate \( O(e^{-\lambda t}) \), \( t \to \infty \), with all derivatives, for some \( \lambda > 0 \) depending on \( \omega_D \).

The metric \( g \) is uniquely determined by the above properties.

Theorem 5.1 in the case of non-singular \( \overline{W} \) is an adapted from [Ko1 \S\S 2–3], [Ko2 pp. 142–143] and [HHN § 4]. The changes required in the presence of isolated orbifold singularities away from \( D \) are straightforward as the foundational results used in the proof carry over to orbifolds [Ba, Sat]. Note also that anticanonical divisor \( D \) admits Ricci-flat Kähler metrics as \( c_1(D) = 0 \) by the adjunction formula.

The simply-connected condition on \( D \) is used for showing that the holonomy of \( W \) is \( SU(4) \) (cf. [Ko1 Theorem 2.7]). The hypothesis can be weakened to \( b^1(D) = 0 \) as the cross-section \( D \times S^1 \) then has \( b^1 = 1 \); this requires an orbifold version of Theorem 3.2.

Recall that an antiholomorphic involution of a complex orbifold is defined as a diffeomorphism \( \rho \) of the underlying real orbifold such that \( \rho^2 = \text{id} \) and \( \rho^* (J) = -J \), where \( J \) is an almost complex structure.

**Proposition 5.2.** Assume the hypotheses and notation of Theorem 5.1. Let \( \rho \) be an antiholomorphic involution of \( \overline{W} \), such that \( \rho \) fixes precisely the singular points in \( \overline{W} \) and maps \( D \) onto itself.

Then there exists an asymptotically cylindrical Calabi–Yau structure \((\omega, \theta)\) on the orbifold \( W \) satisfying the assertions of Theorem 5.1 and such that

\[
\rho^* (\omega) = -\omega \quad \text{and} \quad \rho^* (\theta) = \bar{\theta}.
\]
Furthermore, the 4-form
\[ \Phi = \frac{1}{2} \omega \wedge \omega + \text{Re} \theta \]
induces an asymptotically cylindrical torsion-free Spin(7)-structure on \( W \) which is invariant under \( \rho \).

Proof. Let \( g' \) be the Kähler metric of \( \omega_W \). We may assume without loss that \( g \) is invariant under \( \rho \), by passing if necessary to \( \frac{1}{2}(g + \rho^*(g)) \). Then the antiholomorphic involution of \( D \) defined by the restriction of \( \rho \) preserves the Kähler metric on \( D \) induced by \( g \). The argument in \[Jo3\] Prop. 15.2.2] applies with only cosmetic changes to \( D \) show that \( \rho^*(\omega_D) = -\omega_D \) and \( \rho^*(\theta) = \bar{\theta} \).

Let \( g \) be the asymptotically cylindrical Calabi–Yau metric on \( W \) given by Theorem 5.1 with Kähler form asymptotic to \( \omega_D \). It is not difficult to check using \[Ko2\] Prop. 1.1] that the Kähler metric \( \rho^*(g) \) is also asymptotic to \( \omega_D \) in the sense of Theorem 5.1. Then \( \rho^*(g) = g \) by the uniqueness of \( g \).

Remark 5.3. Note also that \( \rho \) induces an antiholomorphic involution on the canonical bundles of \( \overline{W} \) and \( D \), hence by adjunction on the conormal bundle of \( D \). It can be checked that a local holomorphic coordinate \( z \) vanishing to order 1 on \( D \) may be chosen so that \( z \circ \rho = \bar{z} \). Thus the action of \( \rho \) on the cylindrical end of \( \overline{W} \) is asymptotic to the antiholomorphic involution \( \rho_\infty \) on \( (\mathbb{R} \times S^1) \times D \) acting as \( t + i\vartheta \mapsto t - i\vartheta \) on the first factor and as \( \rho|_D \) on the second factor. The asymptotic torsion-free Spin(7)-structure is
\[ \Phi_\infty = dt \wedge d\vartheta \wedge \omega_D + dt \wedge \text{Re} \theta_D - d\vartheta \wedge \text{Im} \theta_D + \frac{1}{2} \omega_D \wedge \omega_D, \]
induced by the Calabi–Yau structure \( (\omega_D, \theta_D) \) on the 3-fold \( D \) (cf. \[Jo3\] Prop. 13.1.2]), and is invariant under \( \rho_\infty \).

The next proposition will be useful for providing examples of orbifolds satisfying the hypotheses of Theorem 5.1.

Proposition 5.4 (cf. \[Ko1\] Prop. 6.42]). Let \( V \) be a compact Kähler 4-dimensional orbifold and suppose that a 3-fold \( D \subset V^* \) is a smooth anticanonical divisor \( D \in |-K_V| \). Let \( \Sigma \) be a connected smooth surface in \( D \) representing the self-intersection \( D \cdot D \) in the Chow ring of \( V \) and denote by \( \sigma : \tilde{V} \to V \) the blow-up of \( V \) along \( \Sigma \).

Then the closure \( \tilde{D} \) of \( \sigma^{-1}(D \setminus \Sigma) \) is a smooth anticanonical divisor on \( \tilde{V} \) and \( \tilde{D} \cdot \tilde{D} = 0 \), so the normal bundle of \( \tilde{D} \) is holomorphically trivial. Furthermore, \( \sigma \) restricts to give an isomorphism \( \tilde{D} \to D \) of complex 3-folds and this isomorphism identifies the Kähler metric on \( \tilde{D} \subset \tilde{V} \) with the restriction to \( \tilde{D} \) of some Kähler metric on \( V \).

Remark 5.5. If the orbifold \( V \) satisfies the hypotheses of Proposition 5.4 and both \( V^* \) and \( D \) are simply-connected, then \( \tilde{V} \setminus \tilde{D} \) is also simply-connected because the exceptional divisor on \( \tilde{V} \), which is the projective normal bundle of \( \Sigma \), has a fibre \( \mathbb{C}P^1 \) meeting the proper transform \( \tilde{D} \) in exactly one point.

Remark 5.6. Proposition 5.4 is easily generalized to the case when \( \Sigma \) is a finite sequence of connected smooth surfaces \( \Sigma_1, \ldots, \Sigma_m \). These need not necessarily be distinct, i.e. the argument allows an effective divisor on \( D \) with smooth components of multiplicity > 1. One then recursively blows up each \( \Sigma_i \) and lifts the remaining \( \Sigma_{i+1}, \ldots, \Sigma_m \) to the respective proper transform of \( D \). Cf. \[KL\] p. 199].
Assuming further that the singular locus of $V$ is finite with each local isotropy group in $SU(4)$, we can apply Theorem 5.1 to $\overline{W} = \overline{V}$ and obtain on $W = \overline{V} \setminus \overline{D}$ an asymptotically cylindrical Ricci-flat Kähler metric $g$ with holonomy $SU(4)$. Then $W$ is also an instance of asymptotically cylindrical Spin(7)-manifold and $g$ is induced by the Spin(7)-structure defined via (31).

Now suppose that $V$ also has an antiholomorphic involution $\rho$ fixing precisely the singular points of $V$ and mapping each of $D$, $\Sigma$ onto itself. Then $\rho$ lifts to an antiholomorphic involution $\tilde{\rho}$ of $\overline{V}$ so that

$$
\begin{array}{ccc}
\tilde{V} & \overset{\tilde{\rho}}{\longrightarrow} & \tilde{V} \\
\sigma & \searrow & \sigma \\
V & \overset{\rho}{\longrightarrow} & V
\end{array}
$$

(31)

is a commutative diagram. Further, $\tilde{\rho}$ defines by restriction an antiholomorphic involution of $\overline{D}$. Therefore, by Proposition 5.2, the asymptotically cylindrical Spin(7)-structure on $W = \overline{V} \setminus \overline{D}$ can be taken to be $\tilde{\rho}$-invariant and descends to the quotient

$$
M_0 = (\overline{V} \setminus \overline{D})/\tilde{\rho},
$$

(32)

and $M_0$ is an asymptotically cylindrical Spin(7)-orbifold. As $\overline{V}^* \setminus D$ is simply-connected and $\tilde{\rho}$ acts freely the fundamental group of the smooth locus $M_0^* \subset M_0$ is $\mathbb{Z}_2$.

The $G_2$-structure $\varphi$ ‘at infinity’ on the cross-section $Y_W = S^1 \times D$ of $W$ is induced by the Calabi–Yau structure on the divisor,

$$
\varphi = d\theta \wedge \omega_D + \text{Re } \theta_D.
$$

The antiholomorphic involution $\rho$ acts freely on the 3-fold $D$ and as reflection on $S^1$, thus freely of $Y_W$ preserving the ‘product’ $G_2$-structure, $\rho^* \varphi = \varphi$. As $b^1(D) = 0$ we obtain that the cross-section $Y \cong (D \times S^1)/\rho$ of $M_0$ has $b^1(Y) = 0$ and infinite $\pi_1(Y)$. The holonomy of the metric $g(\varphi)$ induced by the $G_2$-structure on $Y$ is a semi-direct product of $\mathbb{Z}_2$ and $SU(3)$.

We next impose a condition on the singularities of $V$. As $D$ contains no singularities of $V$ the fixed points of $\tilde{\rho}$ are preimages of the fixed points of $\rho$ and the respective singularities of $V$ and $\tilde{V}$ are isomorphic. The type of singularities and the method of resolution of these are adapted from [Jo3, Chap. 15], the only difference being that we shall preform the resolution on asymptotically cylindrical, rather than compact Calabi–Yau 4-folds.

We require that the orbifold singularities of $V$ have local isotropy group $\mathbb{Z}_4$, so that the action of a generator of $\mathbb{Z}_4$ is given in uniformizing coordinates by

$$
(z_1, z_2, z_3, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4).
$$

(33)

With this arrangement, $\mathbb{Z}_4$ acts as a subgroup of $SL(4, \mathbb{C})$, so the canonical bundle of $V$ (and of $\tilde{V}$) is a well-defined ‘orbifold line bundle’ and $c_1(V)$ is a well-defined class in $H^2(V, \mathbb{Z})$.

The resulting singularities of $M_0$ have non-abelian local isotropy group $G$ on two generators coming from $\mathbb{Z}_4$ and $\rho$. In fact, the singularities of $M_0$ are of the same type; the action of the generators of each is isomorphic to $G \subset \text{Spin}(7)$ generated by the unitary maps $\overline{W}$ and

$$
(z_1, z_2, z_3, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3),
$$

(34)

with the real and imaginary parts of $\partial/\partial z_1, \ldots, \partial/\partial z_4$ corresponding to an orthonormal frame at $p_i$ in uniformizing real coordinates (cf. [Jo3, Prop. 15.2.3]).
The orbifold singularities modelled on \( \mathbb{R}^8/G \) can be resolved using a method given in [Jo3], by forming at each singular point \( p_i \) (\( i = 1, \ldots, k \)) a generalized connected sum of \( M_0 \) and an asymptotically locally Euclidean (ALE) Spin(7)-manifold asymptotic to \( \mathbb{R}^8/G \). There are two suitable choices of ALE Spin(7)-manifolds \((X_1, \Phi_1), (X_2, \Phi_2)\) with \( \pi_1(X_i) = \mathbb{Z}_2 \) for \( i = 1, 2 \) ([Jo3] §15.1]). The respective Spin(7)-compatible generalized connected sums of \( M_0 \# X_i \) correspond to different identifications on the cross-section of the neck \( S^7/G \) and the resulting spaces are topologically distinct.

**Proposition 5.7** (cf. [Jo3, Prop. 15.2.9 and 15.2.13]). The asymptotically cylindrical Spin(7)-orbifold defined in \( 32 \) admits a simply-connected resolution of singularities \( M_0 \# (X^{(1)} \sqcup \ldots \sqcup X^{(k)}) \) with \( X^{(j)} \) are ALE Spin(7)-manifolds asymptotic to \( \mathbb{R}^8/G \).

The 8-manifold \( M \) admits a 1-parameter family of asymptotically cylindrical Spin(7)-structures \( \Phi^s \), \( 0 < s < \varepsilon \), which are torsion-free on the cylindrical end. There are positive constants \( \lambda, \mu, \nu \), such that

(a) the injectivity radius \( \delta(g) \) satisfies \( \delta(g) \geq \mu s \),

(b) the Riemann curvature \( R(g) \) satisfies \( \|R(g)\|_{C^0} \leq \nu s^{-2} \),

(c) \( \|\psi\|_{L^2} \leq \lambda s^{24/5} \) and \( \|d\psi\|_{L^{10}} \leq \lambda s^{36/25} \),

for all \( 0 < s < \varepsilon \), with the norms defined using the metric \( g(\Phi^s) \).

Parts (a), (b), (c) of Proposition 5.7 imply the hypotheses (a), (b) and (20) of the existence Theorem 4.1 and taking a sufficiently small \( s > 0 \) we obtain an asymptotically cylindrical torsion-free structure \( \Phi \) on \( M \). Since the cross-section of \( M \) has \( b^1 = 0 \) and \( M \) has single end, the induced metric \( g(\Phi) \) has holonomy Spin(7) by Theorem 3.2(a).

Putting together the conditions imposed on \( V \) we make the following.

**Definition 5.8.** We say that \((V, D, \Sigma, \rho)\) is an admissible orbifold configuration if the following properties hold.

(i) \( V \) is a compact Kähler 4-dimensional orbifold. The singular locus of \( V \) is non-empty and consists of with finitely many isolated singular points \( \{p_1, \ldots, p_k\} \). Each singularity is isomorphic to \( 33 \) with the local isotropy group \( \mathbb{Z}_4 \).

(ii) The smooth locus \( V^* \subset V \) is simply-connected and there exists a simply-connected smooth 3-fold \( D \subset V^* \) in the anticanonical class \( D \in |-K_V| \) and \( V^* \setminus D \) is simply-connected. The self-intersection \( D \cdot D \), in the sense of the Chow ring, is represented by an effective divisor \( \Sigma = \sum_i n_i \Sigma_i \), where each \( \Sigma_i \) is a smooth complex surface in \( D \).

(iii) There is an antiholomorphic involution \( \rho \) of \( V \) fixing the singular locus and no other points in \( V \) and such that \( \rho(D) = D \) and \( \rho(\Sigma_i) = \Sigma_i \), for each \( i \).

We call a 4-orbifold \( V \) admissible if \( V \) is part of some admissible orbifold configuration.

Summarizing the work in this section, we obtain.

**Theorem 5.9.** Suppose that \((V, D, \Sigma, \rho)\) is an admissible orbifold configuration and let \( \tilde{V} \) be obtained from \( V \) by successive blow-up of the components of \( \Sigma \) according to multiplicities. Let \( \tilde{D} \) be the proper transform of \( D \) and \( \tilde{\rho} \) the lift of \( \rho \) to \( \tilde{V} \).

Then the real 8-dimensional orbifold \( M_0 = (\tilde{V} \setminus \tilde{D})/\tilde{\rho} \) admits a simply-connected resolution of singularities \( M \), so that there is an asymptotically cylindrical metric on \( M \) with holonomy equal to Spin(7). The cross-section of \( M \) ‘at infinity’ is isometric to \((D \times S^1)/\rho \) and has holonomy \( \mathbb{Z}_2 \ltimes SU(3) \), where \( D \) is taken with the Ricci-flat Kähler metric determined by the restriction of the Kähler class of \( V \).

We shall refer to \( M \) in this theorem as the Spin(7)-manifold constructed from \((V, D, \Sigma, \rho)\).
6 Examples of the construction

By the results of the previous section, in order to construct asymptotically cylindrical $\mathsf{S}$-manifolds with holonomy $\mathsf{Spin}(7)$ it suffices to find admissible orbifolds. In this section, we give simple examples using the geometry of weighted complex projective spaces.

For a set of positive integers $a = \{a_0, \ldots, a_n\}$ with highest common factor 1, the complex weighted projective space $\mathbb{C}P^n_a$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation

$$(z_0, \ldots, z_n) \sim (u^{a_0} z_0, \ldots, u^{a_n} z_n), \text{ for each } u \in \mathbb{C}^*$$

(when all $a_i = 1$ this gives the usual complex projective space). We shall denote by $[z_0, \ldots, z_n]$ the equivalence classes. Each $\mathbb{C}P^n_a$ is a projective complex orbifold and admits Kähler metrics. The smooth locus $(\mathbb{C}P^n_a)^s$ is simply-connected [DD] and the rational cohomology ring of $\mathbb{C}P^n_a$ is $\mathbb{Q}[x]/x^{n+1}$ generated, like in the case of the usual complex projective space, by an element $x$ of degree 2 [Di Prop. B13].

A number of standard properties of varieties in $\mathbb{C}P^n$ extends to weighted projective spaces. For the proof and further details of results used below, we refer to [Dq], [Di, Appendix B] and [Ia].

If a polynomial $f(z_0, \ldots, z_n)$ satisfies a weighted homogeneous condition of degree $d$

$$f(u^{a_0} z_0, \ldots, u^{a_n} z_n) = u^d f(z_0, \ldots, z_n) \text{ for each } u, z_0, \ldots, z_n \in \mathbb{C},$$

then the zero locus of $f$ is a well-defined hypersurface $S$ in $\mathbb{C}P^n_a$ and $d$ is the degree of $S$. Hypersurfaces and complete intersections of (complex) dimension $\geq 2$ in a weighted projective space are simply-connected.

A hypersurface $S$ is called a linear cone if the defining weighted homogeneous polynomial may be written as $f = z_i + g$ for some $i$. In this case, $\{f = 0\}$ is isomorphic to an $(n-1)$-dimensional weighted projective space with weights $a_0, \ldots, \hat{a}_i, \ldots, a_n$. Here and later in this section the notation means this element is omitted.

If there are no points $(z_0, \ldots, z_n) \neq 0$ such that $f(z_0, \ldots, z_n) = 0$ and $\partial f / \partial z_j(z_0, \ldots, z_n) = 0$, then $S$ has no singularities outside the singular locus of $\mathbb{C}P^n_a$ and any singularity of $S$ is of orbifold type with a cyclic group [Dq, Theorem 3.1.6].

Each $\mathbb{C}P^n_a$ may be considered as an algebraic variety and the definition of sheaves $\mathcal{O}(m)$ extends to weighted projective spaces. Furthermore, the anticanonical class of a hypersurface or complete intersection, $Z$ say, is determined by application of the adjunction formula if one assumes that the preimage of $Z$ is a smooth cone in $\mathbb{C}^{n+1} \setminus \{0\}$ (the quasismooth condition) and $Z$ is well-formed in $\mathbb{C}P^n_a$. For hypersurfaces of degree $d$, the latter condition means

- $\gcd(\ldots, \hat{a}_i, \ldots) = 1$ for all $i$;
- $\gcd(\ldots, \hat{a}_i, \ldots)$ divides $d$, for all $i \neq j$.

For a complete intersection of two hypersurfaces, of degrees $d_1$ and $d_2$, the condition becomes

- $\gcd(\ldots, \hat{a}_i, \ldots) = 1$ for all $i$;
- $\gcd(\ldots, \hat{a}_i, \ldots)$ divides both $d_1$ and $d_2$, for all $i \neq j$;
- $\gcd(\ldots, \hat{a}_i, \ldots)$ divides either $d_1$ or $d_2$, for all distinct $i, j, k$.

A general definition requires more notation and can be found in [Ia, §6].

It can be shown that the sum of linear cones $\{z_i = 0\}$, for all $i = 0, \ldots, n$, defines an anticanonical (Weil) divisor on $\mathbb{C}P^n_a$. The anticanonical sheaf on a well-formed complete
intersection $S$ of $k$ hypersurfaces in $\mathbb{CP}_a^n$ then is given by

$$-K_S = \mathcal{O}(\sum_{j=0}^n a_j - \sum_{i=0}^k d).$$  \hfill (35)

Thus $\sum_{i=0}^k d < \sum_{j=0}^n a_j$ is a necessary condition for $S$ to have effective anticanonical divisors.

There is version of Lefschetz hyperplane theorem.

**Theorem 6.1** ([Di], Theorem B22). If $Z$ is a hypersurface or complete intersection in $\mathbb{CP}_a^n$, then the homomorphism $H^r(\mathbb{CP}_a^n; \mathbb{Q}) \to H^r(Z, \mathbb{Q})$ induced by the inclusion is an isomorphism for $r < \dim_{\mathbb{C}} Z$ and is injective for $r = \dim_{\mathbb{C}} Z$.

### 6.1 Considerations of Cohomology

We next determine the de Rham cohomology of Spin(7)-manifolds obtainable via the procedure of [5] when $V$ is an orbifold in a weighted projective space. Recall from [2] and Theorem 3.4 the definitions of $b_0^i$ and $b_{1/2}^i$.

**Proposition 6.2.** Assume $(V, D, \Sigma, \rho)$ is an admissible orbifold configuration and a hypersurface or complete intersection $V \subset \mathbb{CP}_a^n$ is quasismooth and well-formed. Then the Betti numbers of the Spin(7)-manifold $M$ with cross-section $Y$ constructed from $(V, D, \Sigma, \rho)$ are

- $b_1^i(Y) = b_2^i(Y) = 0$,
- $b_3^i(Y) = 2 + h^{2,1}(D)$,
- $b_3^i(M) = b_4^i(M) = b_5^i(M) = 0$,
- $b_4^i(M) = b_0^i(M) + b_3^i(Y)$,
- $b_0^i(M) = \frac{1}{2}(\chi(\Sigma) + \chi(V) + 3k) - 4$,

where $k$ is the number of singular points in $V$.

**Remark 6.3.** Every asymptotically cylindrical Spin(7)-manifold $M$ has $b_1^i(M) > 0$ as the closed admissible 4-form $\Phi$ is non-decaying bounded harmonic and defines a non-trivial cohomology class on a cross-section of cylindrical end. The cross-section $Y$ is a compact $G_2$-manifold and $b_0^1(Y) > 0$ as there is a non-trivial harmonic 3-form $\varphi$ inducing the $G_2$-structure. Thus no further Betti numbers of $M$ and $Y$ in Proposition 6.2 can vanish.

**Proof of Proposition 6.2.** The vanishing of first Betti numbers follows at once as $M$ is simply-connected and admits holonomy Spin(7) metrics. We have $H^1(\mathbb{CP}_a^n) \cong H^3(\mathbb{CP}_a^n)$ for each $j$ and then $b_1^i(V) = b_1^i(D) = b_1^i(\Sigma) = b_1^i(\mathbb{CP}_a^n) = 0$ and $b_2^i(V) = b_2^i(D) = b_2^i(\mathbb{CP}_a^n) = 1$ and $b_3^i(V) = b_3^i(\mathbb{CP}_a^n) = 0$ by application of the Lefschetz hyperplane theorem. The cohomology $H^* (V/\rho)$ is the $\rho$-invariant part $H^* (V)^\rho \subset H^* (V)$ and the cohomology in degree 2 is generated by the Kähler form on which the antiholomorphic involution acts as $-1$. Thus $b_2^2(V)^\rho = b_2^2(D)^\rho = 0$.

The cohomology of the blow-up may be written as $H^* (\tilde{V}) \cong H^* (V) \oplus (H^* (E)/\sigma^* H^* (\Sigma))$, where $E$ is the exceptional divisor [6, pp. 605–608]. The cohomology of $E$ is generated as a cup-product algebra over $H^* (\Sigma)$ by the Chern class $\zeta = c_1(\langle [E]\rangle)_E$ with a relation $\zeta^2 - c_1(N_{\Sigma/M}) \zeta + c_2(N_{\Sigma/M}) = 0$, where $N_{\Sigma/M}$ is the normal bundle. The pull-back $\sigma^*$ is injective on $H^* (\Sigma)$, so $H^3(E)/\sigma^* (\Sigma)$ is generated by $\zeta \wedge \sigma^* H^{j-2}(\Sigma)$ and the blow-up adds $b_1^{j-2}(\Sigma)$ to the $j$-th Betti number. Now $\tilde{\rho}^* \zeta = -\zeta$ as $c_1(\langle [E]\rangle)$ is Poincaré dual to the cycle $\langle [E]\rangle$ defined...
by complex 3-dimensional submanifold on which $\tilde{\rho}_* \chi$ changes the orientation. We deduce that $b^j(\tilde{V})^\rho = b^j(V)^{\tilde{\rho}} + b^{i-2}(\Sigma)^{\tilde{\rho}}$ and

$$b^1(\tilde{V})^\rho = b^2(\tilde{V})^\rho = b^3(\tilde{V})^\rho = 0, \quad b^2(\tilde{V})^{-\tilde{\rho}} = h^{1,1}(\tilde{V})^{-\tilde{\rho}} = 2,$$

$$b^4(\tilde{V})^\rho = b^4(V)^{\rho} + b^2(\Sigma)^{-\rho}.$$  

The Betti numbers in middle dimension are now determined by the Euler characteristic, $b^2(\Sigma)^{-\rho} = b^2(\Sigma) - b^2(\Sigma)^\rho = \frac{1}{2} \chi(\Sigma)$ as $\rho$ acts freely on $\Sigma$ and $b^1(\Sigma) = 0$. Similarly, $b^4(V)^{\rho} = \frac{1}{2} (\chi(V) + k) - 2$.

For the Calabi–Yau 3-fold $D$, we have $b^3(D)^{\rho} = b^3(D)^{-\rho} = \frac{1}{2} b^3(D) = 1 + h^{2,1}(D)$ as $\rho$ interchanges $H^{i,j}(D)$ and $H^{2,j}(D)$. Then for $Y \cong (D \times S^1)/\rho$, recalling that $b^1(Y) = 0$ and that $\rho$ acts on $S^1$ as reflection, we obtain

$$b^2(Y) = b^2(D)^{\rho} = 0, \quad b^3(Y) = 2 + h^{2,1}(D)$$

as $\rho^*\omega_D = -\omega_D$ from Proposition 5.2 and [30].

The resolution of each singularity of $\tilde{V}/\tilde{\rho}$ is topologically a generalized connected sum $\bar{M}$ of $\tilde{V}/\tilde{\rho}$ and an ALE Spin(7) manifold $X_\text{p}$ and cross-section of the neck is a spherical space-form $S^7/G$. The manifolds $X_\text{p}$ are taken from [Je3] §15.1 and have $b^1(X_\text{p}) = b^2(X_\text{p}) = b^3(X_\text{p}) = 0$, $b^4(X_\text{p}) = 1$, so the resolved compact smooth 8-manifold $\bar{M}$ has $b^1(\bar{M}) = b^1(V)^{\tilde{\rho}} = 0$, for $j = 1, 2, 3$, and

$$b^4(\bar{M}) = b^4(V)^{\tilde{\rho}} + k = b^4(V)^{\rho} + b^2(\Sigma)^{-\rho} + k.$$  

To determine the cohomology of asymptotically cylindrical $M$ with cross-section $Y$ we use the Mayer–Vietoris theorem applied to $\bar{M} = M \cup U$. Here $U$ is a tubular neighbourhood of $\bar{D}/\bar{\rho}$ and $M \cap U$ retracts to $Y$. Firstly, $b^1(M) = 0$ as $M$ is simply-connected and also $b^2(M) = b^2(\bar{M}) = 0$ as $b^1(Y) = b^2(Y) = 0$. In the part of the Mayer–Vietoris exact sequence

$$0 \to H^3(\bar{M}) \to H^3(M) \oplus H^3(D)^{\rho} \to H^3(D)^{\rho} \oplus \mathbb{R}[\varphi]$$  

the last homomorphism maps onto $H^3(D)^{\rho}$ as the image contains $H^3(D)^{\rho}$ and is orthogonal to the $G_3$ 3-form $\varphi$ on $Y$ by Proposition 2.5. We find that $b^3(M) = b^3(\bar{M}) = 0$. Then $b^2_c(M) = 0$, $b^2_c(M) = 0$ and $b^2_0(M) = b^2_0(M) - b^3(Y) = b^3(M) - b^3(Y)$ from the long exact sequence (10) for de Rham cohomology with compact support and the Poincaré duality. Extracting a further part of the Mayer–Vietoris exact sequence

$$0 \to \mathbb{R}[\varphi] \to H^4(\bar{M}) \to H^4(M) \oplus \mathbb{R}[\omega^2_\bar{D}] \to \mathbb{R}[\omega^2_\bar{D}] \oplus H^3(D)^{-\rho} \to 0,$$  

we deduce $b^4(M) = b^4(\bar{M}) + b^3(V)^{-\rho} - 1 = b^4(\bar{M}) + b^3(Y) - 2$. Then $b^4_0(M) = b^4(\bar{M}) - 2 = b^4(V)^{\rho} + b^2(\Sigma)^{-\rho} + k - 2$ and the required formula for $b^4_0(M)$ and $b^2(\Sigma)^{-\rho}$ found above. Proposition 6.2 is proved.

The only non-trivial multiplicative structure on the de Rham cohomology of $M$ is the intersection form on $H^4_c(M)$. A maximal subspace on which the latter form is non-degenerate is isomorphic to $H^4_0(M)$.

**Proposition 6.4.** In the situation of Proposition 6.2, the signature of the intersection form on $H^4_0(M)$ is

$$b^4_+(M) = h^{2,2}(V)^{\rho} + h^{1,1}(\Sigma)^{-\tilde{\rho}} - 2,$$

$$b^4_-(M) = h^{3,1}(V) + h^{2,0}(\Sigma) + k.$$
Note that \( b^4_1(M) + b^4_2(M) = b^4_3(M) \) in Proposition 6.2 because \( h^{2,2}(V) \rho + h^{3,1}(V) = b^4(V) \rho \) and \( h^{1,1}(\Sigma)^{-\rho} + h^{2,0}(\Sigma) = \frac{1}{2} \chi(\Sigma) \).

**Proof.** The signature of Kähler orbifold \( \tilde{V} \) is \( b^4_1(\tilde{V}) \rho = h^{2,2}(\tilde{V}) \rho - h^{1,1}(\tilde{V})^{-\rho} + 1 \), \( b^4_2(\tilde{V}) \rho = h^{3,1}(\tilde{V}) + h^{1,1}(\tilde{V})^{-\rho} - 1 \), determined from the Hodge–Riemann bilinear relations. The ALE Spin(7)-manifolds used in the resolution of singularities of \( \tilde{V} \) have \( b^4_1(X_i) = b^4(X_i) = 1 \) and it follows from the argument of Proposition 6.2 that \( b^4_1(M) = h^{2,2}(V) + h^{1,1}(\Sigma)^{-\rho} - 1 \) and \( b^4_2(M) = h^{3,1}(V) + h^{2,0}(\Sigma) + k + 1 \).

Comparing the exact sequence (37) and

\[ 0 \rightarrow \mathbb{R}[\varphi] \oplus H^3(D) \rho \rightarrow H^3_0(M) \rightarrow H^4(M) \rightarrow \mathbb{R}[\omega^2_D] \oplus H^3(D)^{-\rho} \rightarrow 0 \]

we find that the image of \( H^4(\tilde{M}) \) in \( H^4(M) \) contains \( H^3_0(M) \) as a codimension 1 hyperplane. Its complement is spanned by a class \( \xi' \) such that \( \xi'|_{\{t_0\} \times Y} = [\omega^2_D] \) and \( \xi' \) may be taken orthogonal to \( H^3_0(M) \) with respect to the cup product. The kernel of \( H^4(\tilde{M}) \rightarrow H^4(M) \) is spanned by \( \xi'' = [d\eta(t)\varphi] \) where \( \eta(t) \) is a cut-off function with derivative supported on the cylindrical end of \( M \).

On the other hand, \( H^3_0(M) \) is the subspace of classes in \( H^4(M) \) that are represented by closed 4-forms with compact support in \( M \subset \tilde{M} \). We deduce that the intersection form on \( H^3_0(M) \) may be identified with the intersection form on the orthogonal complement in \( H^4(\tilde{M}) \) of the 2-plane spanned by \( \xi', \xi'' \). As \( \xi' \cup \xi'' \neq 0 \) and \( \xi' \cup \xi'' = 0 \) this latter 2-plane has signature \((1,1)\) and \( H^3_0(M) \) has the complementary signature in \( H^4(\tilde{M}) \).  

### 6.2 An admissible weighted projective space

Let \( V = \mathbb{C}P^4_{1,1,1,1,4} \). This orbifold has unique singular point \( p_0 = [0,0,0,0,1] \) and the local isotropy group is \( \mathbb{Z}_4 \) with the required action (33). The anticanonical sheaf of \( \mathbb{C}P^4_{1,1,1,1,4} \) is \( \mathcal{O}(8) \) and an anticanonical Weil divisor \( D \) may be given by the vanishing of weighted degree 8 homogeneous polynomial

\[ f(z) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2. \]

As \( df(z) \) does not vanish for \( z \neq 0 \) and \( D \) does not contain \( p_0 \) the hypersurface \( D \) is smooth. In fact, \( \mathbb{C}P^4_{1,1,1,1,4} \) is the unique non-smooth weighted projective space with only \( \mathbb{Z}_4 \)-singularities and having a smooth Calabi–Yau 3-fold in the anticanonical class, cf. [31] Theorem 14.3.

The hypersurface \( D \) is well-formed and the Euler characteristic of \( D \) is determined via the Chern class (CLS p. 393) \( c(D) = (1 + x)^4(1 + 4x)(1 + 8x)^{-1} \) from the adjunction formula, whence \( \chi(D) = \langle -148x^3, [D]\rangle = (-148/4) \cdot 8 = -296 \) and \( h^{2,1}(D) = 149 \) (noting also that \( \mathbb{C}P^4 \rightarrow \mathbb{C}P^4_{1,1,1,1,4} \) is a 4-to-1 branched cover).

Choosing another anticanonical divisor \( D' \) defined by the vanishing of

\[ f_1(z) = z_0^8 - z_1^8 + 2z_2^8 - 2z_3^8 + iz_4^2 \]

we obtain \( \Sigma_{16} = D \cap D' \) a well-formed smooth complete intersection surface. From \( c(\Sigma_{16}) = (1 + x)^4(1 + 4x)(1 + 8x)^{-2} \), the Euler characteristic is \( \chi(\Sigma_{16}) = \langle 86x^2, [\Sigma_{16}] \rangle = 86 \cdot (1/4) \cdot 16 = 344 \).

The anti-holomorphic involution of \( \mathbb{C}P^4_{1,1,1,1,4} \) given by

\[ \rho_1 : [z_0, \ldots, z_4] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4] \]
The moduli space for asymptotically cylindrical Spin(7)-structures on $V$ using the nested sequence of orbifolds spaces of $b$ with holonomy Spin(7) in this example has in $z$ graded according to the weights of $C$. The manifold $M_{D}$ has the anticanonical divisor $D$ with the Jacobian ideal $J$ such that as the cross-section has $b^{3}(Y) = 151$ by Proposition 6.2. By Theorem 3.4 and Proposition 6.4 there is a 153-dimensional moduli space for asymptotically cylindrical Spin(7)-structure on $M_{1}$.

6.3 An admissible hypersurface

Consider a hypersurface of weighted degree 8 in a 5-dimensional weighted projective space

$$V = \{ f_{8}(z) = z_{0}^{8} + z_{1}^{8} + z_{2}^{8} + z_{3}^{8} + z_{4}^{2} + z_{5}^{2} = 0 \} \subset \mathbb{C}P_{1,1,1,1,4}^{5}.$$  \hspace{1cm} (38)

The singular locus of $\mathbb{C}P_{1,1,1,1,4}^{5}$ is a copy of the projective line $\mathbb{C}P^{1}$ given by $\{ z_{0} = z_{1} = z_{2} = z_{3} = 0 \}$. Every singularity of $\mathbb{C}P_{1,1,1,1,4}^{5}$ is locally modelled on $(\mathbb{C}^{4}/\mathbb{Z}^{4}) \times \mathbb{C}$. The singular locus of $V$ consists of two isolated points $p_{\pm} = [0,0,0,0,i,\pm 1]$ which are orbifold singularities of type $(\mathbb{C}^{3})$. The smooth locus $V^{*}$ is simply-connected by the results in [DD].

The hypersurface $V$ is well-formed and $-K_{V} = O(4)|_{V}$ by the formula (35) and an anticanonical divisor $D$ on $V$ may be given by the intersection of $V$ with a linear cone $C = \{ z_{4} + z_{5} = 0 \} \cong \mathbb{C}P_{1,1,1,4}^{4}$. The unique singular point of $C$ is $p_{0} = [0,0,0,1,-1]$ and $D = C \cap V$ contains no singular points of $C$ or $V$. It is not difficult to see that the tangent spaces of $C$ or $V$ are transverse at each point of $D$ and the 3-fold $D$ is a smooth complete intersection, isomorphic to the Calabi–Yau 3-fold in 6.2.

The self-intersection of $D$ in the present case is realized by $\Sigma_{8} = V \cap C \cap C'$, with $C' = \{ z_{4} - z_{5} = 0 \}$ and is isomorphic to the degree 8 Fermat surface $\Sigma_{8} = \{ z_{0}^{8} + z_{1}^{8} + z_{2}^{8} + z_{3}^{8} = 0 \}$ in $\mathbb{C}P^{3}$. Its Euler characteristic is $\chi(\Sigma_{8}) = 304$, determined again via the Chern class.

Each of $V, C, C'$, hence also $D$ and $\Sigma_{8}$, are invariant under an antiholomorphic involution of $\mathbb{C}P_{1,1,1,1,4}^{5}$ defined by

$$\rho_{2} : [z_{0}, \ldots, z_{5}] \mapsto [z_{1}, -z_{0}, z_{3}, -z_{2}, z_{5}, z_{4}]$$

and the fixed point set of $\rho_{2}$ on $V$ is precisely the singular locus $p_{\pm}$. The configuration $(V, D, \Sigma_{8}, \rho_{2})$ is admissible.

We calculate $\chi(V) = 306$ via a recursive method given in [Jo3] Ex. 15.3.4, cf. also §15.4, using the nested sequence of orbifolds $V_{j} = V \cap \{ z_{j+1} = \ldots = z_{5} = 0 \}$ and the branched covers $[z_{0}, \ldots, z_{j}] \in V_{j} \to [z_{0}, \ldots, z_{j-1}] \in \mathbb{C}P_{1,1,1,1,4}^{5} \cap \{ z_{j} = \ldots = z_{5} = 0 \}$, for $j = 0, 1, \ldots, 5$. (Note that as $V$ has orbifold singularities the method using the Chern class computes a different quantity known as ‘orbifold Euler characteristic’ which in general is not $\sum_{j}(-1)^{j}b^{j}(V)$ and in the present case is not an integer.) The Hodge numbers of $V$ can be computed by Steenbrink’s result [14, Theorem 7.2] for hypersurfaces in weighted projective spaces, using the Jacobian ideal $J$ of the defining polynomial $f_{8}$ in the ring of polynomials $\mathbb{C}[z_{0}, \ldots, z_{5}]$ graded according to the weights of $z_{i}$ in $\mathbb{C}P_{1,1,1,1,4}^{5}$. It suffices to find $b^{3}(V) = 35$.

We obtain from Propositions 6.2 and 6.4 that the asymptotically cylindrical 8-manifold $M_{2}$ with holonomy Spin(7) in this example has $b^{4}(M_{2}) = 455$ and $b_{0}^{4}(M_{2}) = 304$ and $b_{1}^{4}(M_{2}) = 37$. The moduli space for asymptotically cylindrical Spin(7)-structures on $M_{2}$ has dimension 189. The manifold $M_{2}$ is topologically distinct from the Spin(7)-manifold $M_{1}$ in 6.2.
The Spin(7)-manifold $M_2$ can also be constructed from $\mathbb{CP}^4_{1,1,1,1,4}$ with $D$ as in §6.2 if we choose a second anticanonical divisor to be $D'' = \{z_4^2 = 0\} \subset \mathbb{CP}^4_{1,1,1,1,4}$ a linear cone of multiplicity 2. Then the self-intersection of $D$ is represented by $D \cdot D'' = 2\Sigma_8$, the Fermat surface with multiplicity 2. This admissible configuration is $(\mathbb{CP}^4_{1,1,1,1,4}, D, 2\Sigma_8, \rho_1)$ a modification of the example in §6.2. In the present case, in order to obtain a 4-orbifold $\tilde{V}$ with an anticanonical divisor $\tilde{D}$ having holomorphically trivial normal bundle, two successive blow-ups of $\Sigma_8$ in $C$ are required as in Remark 5.9 (respectively, the Betti numbers of $M_2$ are recovered by applying Proposition 6.2 twice.) The first blow-up produces a 4-orbifold isomorphic to the hypersurface isomorphic to $V$ in (38).

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