ANGULAR MOMENTUM TRANSFER IN A PROTOLUNAR DISK
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Received 2000 December 23; accepted 2001 May 29

ABSTRACT

We numerically calculated angular momentum transfer processes in a dense particulate disk within the Roche limit by global N-body simulations, up to $N = 10^5$, for parameters corresponding to a protolunar disk generated by a giant impact on a proto-Earth. In the simulations, both self-gravity and inelastic physical collisions are included. We first formalized expressions for angular momentum transfer rate including self-gravity and calculated the transfer rate with the results of our N-body simulations. Spiral structure is formed within the Roche limit by self-gravity and energy dissipation of inelastic collisions, and angular momentum is effectively transferred outward. Angular momentum transfer is dominated by both gravitational torque due to the spiral structure and particles' collective motion associated with the structure. Since formation and evolution of the spiral structure is regulated by the disk surface density, the angular momentum transfer rate depends on surface density, but not on particle size or number, so that the timescale of evolution of a particulate disk is independent of the number of particles $(N)$ that is used to represent the disk, if $N$ is large enough to represent the spiral structure. With $N = 10^5$, the detailed spiral structure is resolved, while it is only poorly resolved with $N = 10^3$; however, we found that calculated angular momentum transfer does not change as long as $N \gtrsim 10^3$.

Subject headings: accretion, accretion disks — Moon — planets: rings — planets and satellites: formation

1. INTRODUCTION

Angular momentum transfer is essential in the evolution and structure formation of accretion disks such as galactic disks, protoplanetary disks, and planetary rings. As angular momentum is transferred outward, inner material falls to the central body and outer material migrates outward (Lynden-Bell & Pringle 1974). In a particulate disk within the Roche limit such as a planetary ring or a protolunar disk (see below), angular momentum is transferred through mutual collisions and self-gravitational interactions. In the present paper, we focus on angular momentum transfer in such a disk, in particular, a protolunar disk.

The "giant impact hypothesis" for the origin of the Moon (Hartmann & Davis 1975; Cameron & Ward 1976) assumes that a Mars-sized protoplanet collided with the early Earth, some fraction of the impactor's mantle materials were flung away to form a circumterrestrial debris disk (a protolunar disk), and the Moon accreted from the disk. Within the Roche limit, which is about three Earth radii, accretion of large bodies is inhibited by the tidal effect from the Earth, and only materials beyond the Roche limit can form the Moon (e.g., Canup & Esposito 1995). Therefore, outward mass transfer from the Roche limit regulates the timescale of lunar formation (Ida, Canup, & Stewart 1997, hereafter ICS97) if most materials in the protolunar disk are initially confined within the Roche limit. ICS97 followed the evolution of a protolunar disk by direct N-body simulations with $N = 1000$–$3000$. More detailed calculations were performed by Kokubo, Ida, & Makino (2000, hereafter KIM00) with $N = 10,000$. ICS97 and KIM00 found that lunar formation timescale is very short (~a month to a year) after condensation of disk materials. As the protolunar disk evolves, density spiral arms are quickly developed within the Roche limit. KIM00 suggested that the spiral arms regulate the angular momentum transfer. Also, in the local N-body simulations of Saturn's ring, Salo (1995, hereafter S95) and Daisaka & Ida (1999, hereafter DI99) show similar wakelike structure.

When such structure develops, disk particles move collectively. The motion depends on the surface density of the disk, independent of individual particle sizes, as shown later. If collective motion regulates angular momentum transfer, the results in N-body simulations would be independent of how many particles are used for simulations, as long as particle number is enough to resolve the spiral structure. Both physical collisions and self-gravity play important roles in the formation of the structure (S95; DI99).

Angular momentum transfer in a planetary ring has mostly been studied in non-self-gravitating systems assuming spatial uniformity. Goldreich & Tremaine (1978, hereafter GT78) analytically solved the Boltzmann equation and derived an angular momentum transfer rate on the analogy of molecular viscosity. Araki & Tremaine (1986, hereafter AT86) included the effect of a finite size of particles. During a physical collision, angular momentum is transferred from one particle to another by compressive waves through the interior of particles. They showed that this type of angular momentum transfer is dominant when spatial density of particles is high enough (see § 3.1). Wisdom & Tremaine (1988, hereafter WT88) formulated the angular momentum transfer in a particle disk in local coordinates, taking into account physical collisions, and numerically simulated a particle disk to calculate the transfer rate. Their results are in good agreement with those by GT78 and AT86. Petit & Greenberg (1996) also showed similar results, numerically calculating the evolution of the velocity ellipsoid.

When self-gravity is included, the particles are no longer distributed uniformly (Richardson 1994; S95; DI99). Bories, Goldreich, & Tremaine (1983) included the perturbation by a satellite and investigated the dynamics near a
works by ICS97 (1997) and Borders et al. (1985) derived formula for angular momentum transfer by self-gravity with streamline approximation for a ring. Angular momentum transfer due to gravitational torque has been studied also in spiral galaxies or star formation. With trailing spirals, the angular momentum is transferred outward according to the amplitude and wavenumber of the spirals (Lynden-Bell & Kalnajs 1972). The torque exerted by the spirals transfers angular momentum effectively (Larson 1984). However, the theories for galaxies do not include physical collisions.

Ward & Cameron (1978) argued the evolution of a protolunar disk within the Roche limit. They assumed that clump formation by gravitational instability and destruction by tidal force would be repeated on an orbital timescale. By estimating the energy damping rate due to inelastic collisions, they predicted a timescale of disk evolution consistent with the N-body simulations by ICS97 and KIM00. Lin & Pringle (1987) used dimensional analysis to derive a similar timescale for an accreting gas disk with turbulent viscosity induced by self-gravitational instability. The timescale is much shorter than that predicted by GT78 and AT86 without self-gravity. The self-gravity would play an essential role in the evolution of a dense particle disk system.

We formalized the angular momentum transfer in a dense particle disk (in global coordinates) including both self-gravity and collisions, starting from the Boltzmann equation and generalizing WT88’s formula. We performed global N-body simulations including both effects, with parameters of a protolunar disk, and evaluated the angular momentum transfer rate with the above formalism. We used particles up to \( N = 10^5 \) to represent the protolunar disk.

The protolunar disk and Saturn’s ring differ in disk mass relative to the host planet mass: the ratio is 0.01–0.05 for the former and \( \sim 10^{-7} \) for the latter. The radial scale of spiral structure is given by Toomre’s critical wavelength (S95; DI99), which is \( \sim 2\pi (M_{\text{disk}}/M_e) r_{\text{disk}} \) (eq. [39]), where \( M_{\text{disk}} \) and \( r_{\text{disk}} \) are mass and radius of the disk and \( M_e \) is the planet mass. Since the protolunar disk has much larger \( M_{\text{disk}}/M_e \) than the Saturnian ring, the scale of spiral structure is not too small compared with the disk size, so that global calculation with \( N = 10^4–10^5 \) resolves detailed spiral structure. The angular momentum transfer in local simulations with parameters of Saturn’s ring is shown in a companion paper (Daisaka, Tanaka, & Ida 2001).

We found that when \( N \gtrsim 10^3 \), the gravitational torque and collective motion regulate the angular momentum transfer. They are both determined by the surface mass density of the disk. When \( N \) is smaller, the spiral arms are less clear, and particle motion is less collective. However, angular momentum is transferred at nearly the same rate by gravitational interactions and random motion, as long as optical depth is not too small, which corresponds to \( N \gtrsim 10^3 \), for typical parameters of a protolunar disk. Therefore the result of previous works by ICS97 (\( N = 1000–3000 \)) and KIM00 (\( N = 10,000 \)) that the Moon formation timescale after condensation of disk materials is about a month to a year is not changed in our present calculation with \( N = 10^5 \), although the spirals are much more clearly resolved up to fine structure in our calculation. Actually, we performed calculations with different \( N = 10^5, 3 \times 10^3, 10^4, 3 \times 10^4, 10^5 \) and confirmed that the angular momentum transfer rate is almost independent of \( N \).

Note that we must be careful to apply the above results to the evolution of a very massive disk. Since a very massive disk such as a protolunar disk evolves very rapidly, collisional heating would dominate radiative cooling, resulting in revaporation of particles (Thompson & Stevenson 1988). We will discuss this problem later (also see the discussions by KIM00 and Kokubo, Canup, & Ida 2000). For less massive disks such as planetary rings, we need not worry about this problem.

In §§ 2 and 3 we show formulation of angular momentum transfer and how to calculate it from data of N-body simulations. In § 4 we explain methods and models of our numerical simulations. In § 5 we show the results of our simulations and discuss the angular momentum transfer in the protolunar disk. We compare the results to those by the local N-body simulations by WT88 and Daisaka et al. (2001). In § 6 we give conclusions and discuss the problem of vaporization due to collisional heating.

2. ANGULAR MOMENTUM TRANSFER

2.1. Boltzmann Equation

In this section we formulate angular momentum transfer in a particulate disk, including both physical collisions and self-gravity, starting from the Boltzmann equation. The angular momentum is carried by (1) particles’ random motion (like molecular viscosity) or transferred by (2) physical collisions and (3) gravitational forces. During a physical collision, angular momentum is transferred by compressible waves inside the particles. When a disk is dense enough, this contribution is large (AT86). We present the formulation in particle image so that the transfer rates are easily evaluated by N-body simulations. We extend the formulation by WT88 to include the effect of gravitational torque. Borders et al. (1985) included gravitational torque in their formulation. Although they adopted a streamline approximation in which a disk is divided into fluid ringlets, their formulation may be essentially the same as the formulation presented here.

We assume that the proto-Earth is spherical, neglecting the deformation. Nonaxisymmetric deformation may remain for a while after a giant impact, which produces extra angular momentum transfer. We will discuss the effect in § 6. We count orbital angular momentum and neglect the spin angular momentum of particles, since the latter is usually much smaller than the former (KIM00). We thus adopt the free-slip condition that the tangential restitution coefficient is unity.

A number density distribution function \( f(x, v, m) \) satisfies the Boltzmann equation, where \( x, v, \) and \( m \) denote position, velocity, and mass of particles, respectively:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v_x} = \left( \frac{\partial \Phi}{\partial t} \right)_c + \left( \frac{\partial f}{\partial t} \right)_g, \tag{1}$$

where the derivatives with suffix and mean (eq. [c]) is due to mutual collisions and gravitational interactions. \( \Phi \) is the external potential from the central body (planet). We multiply equation (1) by \( m \) and integrate it over \( m \). Since \( \frac{\partial \Phi}{\partial x} \) does not depend on \( m \), an equation for mass density \( g \) is

$$\frac{\partial g}{\partial t} + v_x \frac{\partial g}{\partial x} + \frac{\partial \Phi}{\partial x} \frac{\partial g}{\partial v_x} = \left( \frac{\partial \Phi}{\partial t} \right)_c + \left( \frac{\partial g}{\partial t} \right)_g, \tag{2}$$
where $g(x, v) \equiv \int mf dm$. In this paper, we will simulate an equal-mass ($m_q$) system, then $g = m_q f$.

We integrate equation (2) in cylindrical coordinates over $z$, $\theta$, and $v$. After some partial integrations, we obtain the equation of continuity:

$$\frac{\partial}{\partial t} (2\pi \Sigma u_t) + \frac{1}{r} \frac{\partial}{\partial r} (2\pi r \Sigma u_r) = \left( \frac{\partial}{\partial t} 2\pi \Sigma \right)_c + \left( \frac{\partial}{\partial r} 2\pi \Sigma \right)_g , \quad (3)$$

where $\Sigma(r, t)$ and $u(r, t)$ are surface mass density and averaged velocity defined by

$$\Sigma(r, t) \equiv \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} dz \int dv u_r v_\theta , \quad (4)$$

$$\Sigma u(r, t) \equiv \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} dz \int dv u_r v_\theta g . \quad (5)$$

Since collision and gravitational force do not change the location of particles discontinuously, the right-hand side of equation (3) is 0.

Similarly, equation (2) multiplied by $v_\theta$ with integrations over $z$, $\theta$, and $v$ leads to the $\theta$ component of equation of motion,

$$\frac{\partial}{\partial t} (2\pi \Sigma u_\theta) + \frac{1}{r} \frac{\partial}{\partial r} (2\pi r \Sigma u_\theta) = \left( \frac{\partial}{\partial t} 2\pi \Sigma u_\theta \right)_c + \left( \frac{\partial}{\partial r} 2\pi \Sigma u_\theta \right)_g . \quad (6)$$

Multiplying equation (6) by $r^2$, we obtain the equation of angular momentum,

$$\frac{\partial}{\partial t} (2\pi r^2 \Sigma u_\theta) = - \frac{\partial F_{AM}}{\partial r} , \quad (7)$$

where

$$F_{AM} = F_{\text{trans}} + F_{\text{col}} + F_{\text{grav}} + M_{\text{disk}} h , \quad (8)$$

$$F_{\text{trans}} \equiv \int_0^{2\pi} \int_0^\infty \int_{-\infty}^{\infty} dz \int dv u_r v_\theta g - M_{\text{disk}} h , \quad (9)$$

$$F_{\text{col}} = - \int_{\min}^r dr \left( \frac{\partial}{\partial t} 2\pi r^2 \Sigma u_\theta \right)_c , \quad (10)$$

$$F_{\text{grav}} = - \int_{\min}^r dr \left( \frac{\partial}{\partial t} 2\pi r^2 \Sigma u_\theta \right)_g . \quad (11)$$

We introduced specific angular momentum $h \equiv ru_\theta$ and advective term $M_{\text{disk}} = 2\pi r \Sigma u_\theta$, which expresses angular momentum carried by mean radial flow. Since $F_{AM}$ appears in equation (7) as $\partial F_{AM}/\partial r$, the minimum range of integration $r_{\min}$ is a free parameter. We choose $r_{\min}$ as the radius of the inner boundary of the disk, so that $F_{\text{col}}$ and $F_{\text{grav}}$ are 0 inside the inner boundary and outside the outer boundary of the disk. Since $M_{\text{disk}} = 2\pi r \Sigma u_\theta = \int_0^{2\pi} r d\theta \int_{-\infty}^{\infty} dz \int dv u_r v_\theta g$,

$$F_{\text{trans}} = \int_0^{2\pi} r d\theta \int_{-\infty}^{\infty} dz \int dv (v_r - u_r)(v_\theta - u_\theta)g . \quad (12)$$

This term is angular momentum transfer due to random motion of particles, analogous to that due to molecular viscosity (GT78). WT88 and AT86 called it “local” and “translational” transfer, respectively. Since we will use the term “local” to mean something else, we will call this flux “translational” angular momentum flux and use the suffix “trans.” Note that in local simulation, we can set $u_\theta$ to be 0 without loss of generality (WT88), but $u_\theta$ is not generally 0 in global simulations. GT78 neglected self-gravity and the size of particles, so that they assumed both terms in the right-hand side of equation (6) to be 0. In a similar sense, AT86 neglected the second term in the right-hand side of equation (6). Here both terms are nonzero and $F_{\text{col}}$ and $F_{\text{grav}}$ have nonzero values. Combining equation (3) and equation (7),

$$M_{\text{disk}} \frac{\partial h}{\partial r} = - 2\pi r^2 \Sigma \frac{\partial h}{\partial r} - \frac{\partial}{\partial r} (F_{\text{trans}} + F_{\text{col}} + F_{\text{grav}}) . \quad (13)$$

As long as $M_{\text{disk}} \ll M_p$, $u_\theta$ is Keplerian, $u_\theta = r\Omega = \sqrt{GM_p/r}$, and $\partial h/\partial t = 0$. Then

$$M_{\text{disk}} = - \frac{1}{(\partial h/\partial r)} (F_{\text{trans}} + F_{\text{col}} + F_{\text{grav}}) , \quad (14)$$

where $\partial h/\partial r = \Omega r/2$.

Angular momentum transfer is often expressed in terms of viscosity $\nu$ (e.g., Pringle 1981). In a viscous accretion disk,

$$\dot{M}_{\text{disk}} = - 3\pi (\Sigma v + 2r \frac{\partial \Sigma v}{\partial r}) , \quad (15)$$

$$F_{\text{AM}} = M_{\text{disk}} h + 3\pi r^2 \Sigma v . \quad (16)$$

Comparing equations (8) and (14) with equations (15) and (16), effective viscosity $\nu$ is given by

$$3\pi \Sigma v^2 = F_{\text{trans}} + F_{\text{col}} + F_{\text{grav}} . \quad (17)$$

The effective viscosities corresponding to $F_{\text{trans}}$, $F_{\text{col}}$, and $F_{\text{grav}}$ will hereafter be denoted by $\nu_{\text{trans}}$, $\nu_{\text{col}}$, and $\nu_{\text{grav}}$, respectively.

### 2.2. Bulk and Local Random Velocities

We further split $F_{\text{trans}}$ into two parts. In an optically thick planetary ring system, collective particle motion exists associated with wake-like structure (S95; DI99). Following S95 and DI99,

$$v = v_{\text{bulk}} + v_{\text{local}} , \quad (18)$$

where “bulk” velocity $v_{\text{bulk}}$ is the locally averaged velocity (which expresses collective motion) and “local” velocity $v_{\text{local}}$ is a deviation from $v_{\text{bulk}}$. S95 and DI99 showed that $v_{\text{bulk}}$ is regulated by the disk surface density but not by particle sizes. We will discuss how to separate $v_{\text{bulk}}$ and $v_{\text{local}}$ in § 2.3. Note that $v_{\text{bulk}}$ is averaged over a small range only in the vicinity of the particle, while $\nu$ is averaged over $\theta$ from 0 to $2\pi$.

Using equation (18), the translational angular momentum flux is divided as

$$F_{\text{trans}} = \int_0^{2\pi} r d\theta \int_{-\infty}^{\infty} dz \int dv (v_r - u_r)(v_\theta - u_\theta)g$$

$$= \int_0^{2\pi} r d\theta \int_{-\infty}^{\infty} dz \int dv (v_{\text{bulk}} - u_\theta)v_{\text{local}} - u_\theta g$$

$$+ \int_0^{2\pi} r d\theta \int_{-\infty}^{\infty} dz \int dv v_{\text{local}} r_{\text{local}} g$$

$$= F_{\text{bulk}} + F_{\text{local}} . \quad (19)$$
Since $N$-body simulations show that $\mathbf{v}_{\text{bulk}}$ and $\mathbf{v}_{\text{local}}$ are not correlated, we neglected cross terms of $\mathbf{v}_{\text{bulk}}$ and $\mathbf{v}_{\text{local}}$ as well as those of $\mathbf{u}$ and $\mathbf{v}_{\text{local}}$.

2.3. Angular Momentum Transfer in Particle Image

In this subsection, we derive expressions of $F_{\text{local}}$, $F_{\text{col}}$, and $F_{\text{grav}}$ in a particle disk, generalizing the procedure by WT88. Particle $i$ is represented as $\delta(x - x_i)\delta(v - v_i)\delta(m_i - m)$ in phase space $(x, v, m)$, where $x_i$, $v_i$, and $m_i$ are the position of the mass, the velocity, and the mass of particle $i$, and $\delta$ is delta function. Mass density distribution $g(x, v)$ is

$$g = \int dm = \sum_i m_i \delta(x - x_i)\delta(v - v_i).$$

With equation (20), we rewrite $F_{\text{local}}$, $F_{\text{col}}$, and $F_{\text{grav}}$ in particle image as below.

2.3.1. Translational Angular Momentum Flux

Substituting equation (20) into equation (12), we obtain

$$F_{\text{trans}}(r) = \frac{1}{r_0} \int_{r_0 - 1/2r_0}^{r_0 + 1/2r_0} r \, d\theta \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\psi g(r_{\text{local}} v_{\text{local}} - u_{\text{local}})$$

$$= \sum_i m_i [v_i - u_i] r [v_i - u_i] \delta(r - r_i).$$

It is convenient to average $F_{\text{trans}}$ over a finite range $[r - \frac{1}{2}r_0, r + \frac{1}{2}r_0]$, where $r_p \leq r_0 \leq r$ ($r_p$: particle physical radius). In our numerical simulation, we adopt $r_0 = 0.02$ $R_{\text{Roche}}$, where $R_{\text{Roche}}$ is the Roche limit radius defined by equation (44). To see the sensitivity of the result to the choice of $r_0$, we also performed a calculation with $r_0 = 0.05$ $R_{\text{Roche}}$ and $r_0 = 0.1$ $R_{\text{Roche}}$. There is no significant difference between the cases with $r_0 = 0.02$ $R_{\text{Roche}}$ and $r_0 = 0.05$ $R_{\text{Roche}}$. However, in the case with $r_0 = 0.10$ $R_{\text{Roche}}$, $F_{\text{trans}}$ at $r = 0.4$ $R_{\text{Roche}}$ (where $r_0/r$ is only $\frac{1}{2}$) differs from $F_{\text{trans}}$ with $r_0 = 0.02$ $R_{\text{Roche}}$ by a factor 2. Thus, $r_0$ should be $\leq 0.05$ $R_{\text{Roche}}$. If $r_0$ is too small, fluctuation is large, so that we have adopted $r_0 = 0.02$ $R_{\text{Roche}}$. Note that the other fluxes are less sensitive to the choice of $r_0$.

The averaged translational angular momentum transfer is

$$F_{\text{trans}}(r) = \frac{1}{r_0} \int_{r_0 - 1/2r_0}^{r_0 + 1/2r_0} r \, d\theta \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\psi g(r_{\text{local}} v_{\text{local}} - u_{\text{local}})$$

$$= \sum_i m_i [v_i - u_i] r [v_i - u_i] \delta(r - r_i).$$

where $u_i$ is averaged velocity at $r_i$.

We must define the averaged velocity $u_i$. In a planetary ring, deviation of $u_i$ from Keplerian velocity is usually much smaller than random velocity, so that we may replace it by Kepler velocity, $(u_i, u_{\psi}) = (0, r_i \Omega(r_i))$. Then, equation (22) corresponds to the local angular momentum flux in WT88. However, since in our simulation of a protolunar disk, $|u_i|$ can not be neglected, and we do not replace $u_i$ by Kepler velocity. Substituting equation (20) into equations (4), and (5), we obtain $\Sigma(r)$ and $m(r)$ in particle image. We average them over $r_0$. The averaged surface density $\Sigma = \Sigma(r)$ and averaged velocity $u = \langle \mathbf{u} \rangle$ are

$$\Sigma = \frac{1}{2\pi r_0} \int_{r_0 - 1/2r_0}^{r_0 + 1/2r_0} \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} dz \int d\psi = \frac{1}{2\pi r_0} \sum_j m_j,$$

$$u = \frac{1}{2\pi r_0} \int_{r_0 - 1/2r_0}^{r_0 + 1/2r_0} \int_{-\infty}^{\infty} \rho^2 \int_{-\infty}^{\infty} d\psi.$$
2.3.3. Gravitational Angular Momentum Transfer

Substituting equation (20) into equation (11), we obtain \( F_{\text{grav}} \) in particle image as

\[
F_{\text{grav}}(r) = -\int_{r_{\min}}^{r} dr' \int_{0}^{2\pi} d\theta' \int_{-\infty}^{\infty} dz' d\mathbf{v}_{o}' \left( \frac{\partial}{\partial t} \right)_{r' \theta' z'} g(r' \theta' z')
\]

\[
= -\sum_{i} m_{i} r_{i} \left( \frac{\partial}{\partial t} \mathbf{v}_{0,i} \right)_{g} = -\sum_{i} N_{i}^{\text{grav}},
\]

where \( N_{i}^{\text{grav}} \) is the torque exerted on particle \( i \) by mutual gravity. Apparently, \(-F_{\text{grav}}\) is total torque exerted on particles inside the radius \( r \). Since the same amount of torque is exerted on particles outside \( r \) as recoils, \( F_{\text{grav}} \) is the angular momentum flux through a cylinder of radius \( r \). We also average \( F_{\text{grav}} \) over the range \([r - \frac{1}{2} r_{0}, r + \frac{1}{2} r_{0}]\).

\[
\overline{F_{\text{grav}}}(r) = -\sum_{i} N_{i}^{\text{grav}} - \sum_{i} \frac{r_{i} - (r - 1/2 r_{0})}{r_{0}} N_{i}^{\text{grav}},
\]

where the first summation is taken over all particles with \( r_{i} < r - \frac{1}{2} r_{0} \) and the second summation is taken over all particles in the range \([r - \frac{1}{2} r_{0}, r + \frac{1}{2} r_{0}]\).

3. ANALYTICAL ESTIMATION OF ANGULAR MOMENTUM FLUX

3.1. Estimation of Angular Momentum Flux without Wake-like Structure

In the case of a non-self-gravitating particle disk, \( v_{\text{trans}} \approx v_{\text{local}} (v_{\text{bulk}} \approx 0 \) in this case), and it is analytically derived by GT78:

\[
v_{\text{trans}} \approx \frac{\sigma^{2}}{\Omega} \left( \tau + \frac{1}{\tau} \right)^{-1},
\]

where \( \sigma \) is one-dimensional random velocity and \( \tau \) is optical depth defined by

\[
\tau = \frac{\pi r_{p}^{2}}{4/3 \pi \rho_{p} r_{p}^{3}},
\]

where \( \rho_{p} \) and \( n_{p} \) are the particle’s internal surface density and surface number density. A simple explanation of equation (33) is as follows. When collision frequency \( \omega_{c} \gg \Omega \), radial mean free path is \( \sigma/\omega_{c} \). Then \( v_{\text{trans}} \approx \sigma \times \sigma/\omega_{c} = \sigma^{2}/\omega_{c} \). When collision frequency \( \omega_{c} \ll \Omega \), radial mean free path is truncated at radial excursion of epicycle motion, \( \sigma/\Omega \). Since only a fraction of particles actually transfers angular momentum during one epicycle, \( \sigma/\Omega \) must be multiplied. Then, \( v_{\text{local}} \approx \sigma(\sigma/\Omega)(\sigma/\Omega) \approx \sigma^{2} \omega_{c}/\Omega^{2} \). Replacing \( \omega_{c} \) by \( \tau \Omega \) (GT78),

\[
v_{\text{trans}} \approx \begin{cases} 
\frac{\sigma^{2}}{\Omega} & \text{for } \tau \gg 1, \\
\frac{\sigma^{2}}{\tau} & \text{for } \tau \ll 1.
\end{cases}
\]

Interpolation of equation (35) gives equation (33). Greenberg (1988) gives a more detailed description.

AT86 introduced the effect of particle size in their analytical estimation and found that at high optical depth \( v_{\text{col}} > v_{\text{trans}} \). Their result is consistent with numerical simulations by WT88. Their result can be fitted as (Schmit & Tscharnu-ter 1995)

\[
v = v_{\text{col}} + v_{\text{trans}} = C \frac{\sigma^{2}}{\Omega} \tau^{x},
\]

where \( C = 0.78 \) and \( x \approx 1.26 \).

These formulae are derived with assumption that the spatial distribution of particles is uniform and the distribution in velocity space is a stationary Gaussian distribution. When wake-like structure develops in a self-gravitational disk, particles are no longer distributed uniformly and spatially nearby particles move collectively with the velocity ellipsoid–like rotating bar structure (DJ99). With the excitation of bulk motion, \( v_{\text{trans}} \) should be much enhanced.

3.2. Estimation of Angular Momentum Flux with Wake-like Structure

Gravitational angular momentum transfer in a disk with a spiral pattern is analytically given by Lynden-Bell & Kalnajs (1972), under WKB approximation. The outward angular momentum flux by self-gravitational torque is (Appendix)

\[
F_{\text{grav}} = \frac{\pi}{2} G r_{c} \sin i \cos^{2} i,
\]

where \( i, H, \) and \( r_{c} \) are pitch angle, density amplitude, and radial wave length of spirals (Larson 1984). In the linear theory, the wave length is (e.g., Binney & Tremaine 1987)

\[
\lambda_{\text{ct}} = \frac{2 \pi^{2} G \Sigma}{\Omega^{2}}.
\]

Note that

\[
\lambda_{\text{ct}} = 2 \pi \frac{\pi \Sigma r^{2}}{M_{e}} \sim 2 \pi \frac{M_{\text{disk}}}{M_{e}} r,
\]

which is consistent with the results of our N-body simulations (Fig. 3). In our simulations, \( H \sim \Sigma \) when \( \tau \sim O(1) \), and \( i \) is from \( 15^\circ \) to \( 45^\circ \) (§ 5.2). Here we simply put \( \sin i \cos^{2} i \sim O(1) \). Though derivation of equation (37) uses the assumption that the spiral is tightly winding \((\lambda_{c} \ll r)\), the assumption is often a good approximation even for relatively loosely winding spirals with \( \lambda_{c} \sim r/2 \) (e.g., Binney & Tremaine 1987). Thus, \( F_{\text{grav}} \) and the corresponding viscosity are

\[
F_{\text{grav}}^{\text{est}} \approx \frac{\pi^{3} G^{2}}{\Omega^{2}} r^{3} \Sigma^{3},
\]

\[
v_{\text{grav}}^{\text{est}} \approx \frac{1}{3} \frac{\pi^{2} G^{2}}{\Omega^{3}} \Sigma^{2}.
\]

A similar formula can be derived by the results of Borderies et al. (1985) if the results of N-body simulations are used. In their model, gravitational interactions between streamlines are calculated. The streamlines are perturbed by external potential and have the form \( r(r, \phi) = a[1 - e(a) \cos m(\phi + \Delta(a))] \), where \( a \) and \( m \) are the unperturbed semimajor axis and the azimuthal mode of perturbation potential. With negative \( da/d\alpha \), these streamlines form a trailing density spiral. Borderies et al. (1985) showed that in the tightly winding limit, angular momentum transfer due to gravitational torque is

\[
F_{\text{grav}}^{\text{est}} \approx \pi^{2} G^{2} a^{3} m e^{2}.
\]
N-body simulations show that when spiral structure develops, $v_{\text{bulk}} \gg v_{\text{local}}$ and $v_{\text{local}} \sim 2 \pi G \Sigma / \Omega$ (S95; D199; eq. [57] below), which means $v \sim 2 \pi G \Sigma / \Omega a$. With this empirical relation and $m \sim (2 \pi a / a_{\odot}) \tan i$, equation (42) reads as

$$F_{\text{grav}} = 4 \pi^3 \frac{G^2}{\Omega^2} a^2 \Sigma^3 R_{\odot},$$

(43)

This is similar to equation (40).

4. Numerical Methods and Model

We numerically simulated the evolution of protolunar disks by N-body simulations, including physical collisions and self-gravity. We performed several sets of simulations with different initial surface density distribution. For each set, we performed five runs with different particle numbers. We started the calculation from an impact-debris disk whose mass is mostly within the Roche limit. For simplicity, we considered no heat generation and vaporization here.

We followed collisions and gravitational interactions step by step and calculated $F_{\text{trans}}$, $F_{\text{col}}$, and $F_{\text{grav}}$ defined as equations (22), (30), and (32) at different times. We also calculated $F_{\text{bulk}}$ and $F_{\text{local}}$, given by equations (25) and (26). The sampling time was about $10^{-1}$ Kepler time, and we averaged them over 2 Kepler times.

We set material density of the proto-Earth $\rho_{\oplus}$ as $5.5$ g cm$^{-3}$ and that of disk particles $\rho_d$, as $3.3$ g cm$^{-3}$, which are the present Earth's and lunar bulk densities, respectively. With these densities, the Roche limit radius is

$$R_{\text{Roche}} = 2.456 \left( \frac{\rho_d}{\rho_{\oplus}} \right)^{-1/3} R_{\odot} = 2.9 R_{\odot},$$

(44)

where $R_{\odot}$ is the radius of the Earth. We adopt the proto-Earth mass as the present Earth mass $M_{\oplus} = 5.97 \times 10^{27}$ g. In this case, Kepler time $T_K$ at $r = R_{\text{Roche}}$ is about 7 hr.

We use $T_K$, $M_{\oplus}$, and $R_{\text{Roche}}$, as units of time, mass, and distance. The physical radius of a particle with mass $m$ is

$$r_p = \left( \frac{m}{M_{\oplus}} \right)^{1/3} \left( \frac{\rho_d}{\rho_{\oplus}} \right)^{-1/3} R_{\odot} = \frac{1}{2.456} \left( \frac{m}{M_{\oplus}} \right)^{1/3} R_{\text{Roche}}.$$  

(45)

4.1. Integration Method

We follow the orbits of all particles, numerically integrating the equation of motion,

$$\frac{du_i}{dt} = -G M_\odot \frac{x_i}{|x_i|^3} - \sum_{j \neq i}^N G m_i \frac{x_i - x_j}{|x_i - x_j|^3}.$$  

(46)

We use a fourth-ordered Hermite scheme with an individual time-step scheme. Full description of the scheme is given in Makino & Aarseth (1992). Let $d$ be the distance between the mass centers of two particles. When we detect $d$ smaller than the sum of particle radii $(r_p + r_d)$, we change velocities according to the restitution coefficient. For detailed adjustment of collision and rebounding to avoid numerical difficulty, we follow Richardson (1994).

Since the particle number is limited in N-body simulations, fragmentation cannot be included in practice. As shown below, typical random velocity of particles is $\sim \pi G \Sigma / \Omega$, which ranges from a few hundred meters s$^{-1}$ to a few kilometers s$^{-1}$. A collision with such energy may result in a catastrophic disruption (e.g., Benz & Asphaug 1999) unless particles are small enough that material strength is important. However, the angular momentum transfer depends on surface density, but not on each particle size, as shown below. The disruption would not affect the angular momentum transfer process. If a disruptive collision is supposed to occur, we represent energy dissipation during the disruption as an effective restitution coefficient for inelastic collision. Since we do not know the value of the effective restitution coefficient, we perform runs with different values.

We assume that tangential restitution coefficient $e_\parallel = 1$, neglecting the exchange between orbital angular momentum and spin angular momentum (KIM00). For simplicity, we use a normal restitution coefficient $e_\perp$ that is independent of collision velocity. The mean reduction of relative velocity due to inelasticity is (Canup & Esposito 1995)

$$1 - e = 1 - \left( \frac{v_{\parallel}^2 + v_{\perp}^2}{v_{\parallel}^2 + v_{\perp}^2} \right)^{1/2}.$$  

(47)

Assuming that mean tangential and normal collision velocities, $v_\parallel$ and $v_\perp$, are similar, the effective restitution coefficient $e$ is about 0.7, for example, if $e_\perp = 0.1$. In this case we assumed that about half (~0.7%) the fraction of kinetic energy of collision is given to fragments. In nominal cases, we adopt $e_\perp = 0.1$, which may be a rather dissipative one. We also study the effect of the restitution coefficient with an additional set of simulations with different $e_\perp$ from 0.1 to 0.8. We will show that angular momentum transfer rates are almost independent of $e_\perp$, except for highly elastic $e_\perp$ ($e_\perp \geq 0.6$); in the highly elastic cases, velocity distribution is so high that spiral arms are not developed.

If the relative velocity after a collision is sufficiently small, the collision results in gravitationally bounded particles. At $r \gg R_{\text{Roche}}$, the particles are gravitationally bounded if Jacobi energy is negative and particles are within a Hill sphere (Ohtsuki 1993; Canup & Esposito 1995; Kokubo et al. 2000). We adopt the rubble pile model, in which no mergers are allowed. In this case, gravitationally bounded particles form particle aggregates. When the aggregates are scattered to inside of the Roche limit, particle size overflows the Hill sphere and tidal force disrupts the aggregates (Ohtsuki 1993; Canup & Esposito 1995; Kokubo et al. 2000).

4.2. Parameters and Initial Conditions

We simulate disks with equal-mass particles. The initial surface density $\Sigma$ is distributed as $\Sigma \propto a^2$ in the range $a_{\min} < a < a_{\max}$ where $a$ is semimajor axis. The initial conditions of the disks are shown in Table 1. We set $a_{\max}$ near the Roche limit, so that most particles are initially within the Roche limit. Sets 1–4 start from centrally condensed disks with $a = -3$, with different total disk masses. Runs with constant surface density ($a = 0$) are investigated in set 5 for comparison. We simulated the disks by runs with $N = 10^3$, $3 \times 10^3$, $10^4$, $3 \times 10^4$, and $10^5$ for each set. Totally, we performed 25 runs with $e_\perp = 0.1$. We call a run with $j \times 1000$ particles in set $k$ “run j-kK”. We also performed additional simulations (set 6), with fixed initial particle number $N = 3 \times 10^4$, changing the normal restitution coefficient as $e_\perp = 0.2, 0.4, 0.6, and 0.8$. For each $e_\perp$, we perform four runs with different initial surface density distribution similar to sets 1–4 (set 6 includes 16 runs).

Orbital eccentricities and inclinations are given by Rayleigh distribution with given mean values $\langle e^2 \rangle^{1/2}$ and $\langle i^2 \rangle^{1/2}$. The specific choice of initial $\langle e^2 \rangle^{1/2}$ and $\langle i^2 \rangle^{1/2}$ changes according to the restitution coefficient. For detailed adjustment of collision and rebounding to avoid numerical difficulty, we follow Richardson (1994).
5. RESULTS

5.1. Overall Disk Evolution

We simulated the evolution of a protolunar disk and investigated angular momentum transfer, which causes mass redistribution of the disk. First we show the overall evolution of the disk. In Figures 1a–1f we show snapshots of disk evolution for set 2 with $1 \times 10^5$ particles (run 2–100K). We also show snapshots in $r$-$z$ plane in Figures 2a–2f. At $t = 0$ (Fig. 2a), particles are distributed azimuthally uniformly. In set 2 we give an initially high random velocity so that the disk scale height is large (Fig. 2a), to avoid self-gravitational instability initially. Inelastic collisions quickly damp the random motion. Since the orbital rotation time is shorter and optical depth is higher in inner region, the damping is faster there. At $t = 2T_K$ (Fig. 2b), the spirals begin to develop in the inner region. However, as we will discuss later, clear spiral structure does not develop in the innermost region. At $t = 6T_K$ (Fig. 2c) and $t = 10T_K$ (Fig. 2d), spirals extend to the outer region, and disk material is transferred beyond the Roche limit. At $t = 20T_K$ (Fig. 2e), solar seeds (large aggregates) are formed beyond the Roche limit. At $t = 30T_K$ (Fig. 2f), two large solar seeds grow further. The largest and the second largest seeds consist of about 5000 and 2400 particles, respectively, in this stage. These solar seeds grow to a single moon. This is typical evolution of the protolunar disk (ICS97; KIM00).

In Figures 3a–3e, we show snapshots of runs 1–100K, 2–100K, 3–100K, 4–100K, with different $M_{\text{disk}}$ from about 1.5–5 $M_J$. The radial wave length of spirals is larger in heavier disks, as predicted by equation (38). Equation (38) agrees well with numerical results, as shown in § 5.2.

We show the time evolution of the surface density distributions of runs 2–100K and 5–30K in Figures 4a and 4b. In run 5–30K surface density is initially flat. After several Kepler times, a quasi-equilibrium state is achieved in both cases, and disk mass is rather steadily transferred. In the outer region, mass is transferred outward to form solar seeds, which correspond to a small peak of surface density beyond the Roche limit. In the inner part, particles fall onto the Earth in compensation, and surface density decreases.

As we will show in the following sections, effective viscosity is proportional to $r_n^{1/2} \Sigma^{3/2}$ in the region $r \leq 0.6 R_{\text{Roche}}$ (§ 5.3.1) and to $\Sigma^3$ in the region $r \geq 0.6 R_{\text{Roche}}$ (§ 5.3.2). Collisonal angular momentum transfer is dominant at $r \leq 0.6 R_{\text{Roche}}$, while other two processes are dominant at $r \geq 0.6 R_{\text{Roche}}$. In Figures 4c and 4d, we show the surface density distribution at $t = 6T_K$ and $t = 18T_K$ of set 2 with various $N$. In the inner region, the peak height of surface density is lower for smaller $N$, because $\nu \propto r_n^{1/2} \Sigma^{3/2}$ increases with decrease in $N$. On the other hand, the surface density distribution in the outer region is almost independent of $N$, because effective viscosity is independent of physical size of $r$.

In this region, angular momentum transfer is regulated by collective effects, so that $\nu$ is dependent only on $\Sigma$, and does not depend on $r$.

The accretion processes are regulated by mass and angular momentum transfer near $R_{\text{Roche}}$, as explained in ICS97 and KIM00. Figures 4e and 4d indicate that they are almost independent of $r$ or particle number $N$ of the N-body simulation.

We show magnified snapshots of set 2 in Figures 5a and 5b and set 3 in Figures 5c and 5d. Figures 5a and 5c are $N = 1 \times 10^5$ cases and Figures 5b and 5d are $N = 1 \times 10^4$ cases. Spiral structure with high density contrast develops only in the outer region. To clarify quantitative features of spiral structure, we adopt an autocorrelation analysis.

5.2. Autocorrelation Analysis for Spiral Structure

We calculated autocorrelation directly by superposing the relative locations of particles, following S95 and D199 with a little modification. The autocorrelation of spatial distribution function $n$ is

$$\text{Corr} (\Delta r, \Delta \theta, r) = \frac{1}{f_{n} \int_{-1/2}^{1/2} dr_{r} \int_{0}^{2\pi} d\theta} \times n(r' + \Delta r, \theta + \Delta \theta)n(r', \theta),$$

(48)

where $\Delta r, \Delta \theta$, and $f_{n}$ are the relative locations and normalizing factor. As shown below Corr is a function of $r$. Substituting $n(r, \theta) = \sum m_{i} \delta(r - r_{i}) \delta(\theta - \theta_{i})/r_{i}$ into equation (48), Corr ($\Delta r, \Delta \theta, r$)

$$= \frac{1}{f_{n} \sum_{i} m_{i} \int_{-1/2}^{1/2} dr_{r} \int_{-1/2}^{1/2} dr_{r} \delta(r' + \Delta r - r_{i})}{r' + \Delta r} \times \delta(r' - r_{i}) \int_{-1/2}^{1/2} d\theta \delta(\theta + \Delta \theta - \theta_{i}) \delta(\theta - \theta_{i})$$

$$= \frac{1}{f_{n} \sum_{i} m_{i} \int_{-1/2}^{1/2} dr_{r} \delta[\Delta r - (r_{i} - r_{j})] \delta[\Delta \theta - (\theta_{i} - \theta_{j})]}$$

(49)

where $j$ is summed over particles in the region $[r - \frac{1}{2} r_{i}, r + \frac{1}{2} r_{i}]$ for each reference radius $r$, while summation over $i$ is taken for all particles. Equation (49) is a superposition of the particles’ relative distribution. We used a $60 \times 200$ mesh in $(\Delta r, \Delta \theta)$ space in the region $\Delta r = [-0.3, 0.3]$, and $\Delta \theta = [-1, 1]$ (radian) and averaged Corr in each mesh. Since the radial density gradient is large, we choose the following
normalizing factor,
\[ f_n = 2\pi \int_{r-1/2r_0}^{r+1/2r_0} dr' r' \Sigma(r' + \Delta r) \Sigma(r') . \] (50)

With the above normalizing factor, Corr is 1 everywhere if surface density is a function only of \( r \).

Since unstable wave length depends on \( \Sigma \), Corr depends on surface density. In a global simulation, \( \Sigma \) changes with...
time, so that we choose snapshot data from different runs with similar surface density and averaged them. We choose only snapshots with initial $N \geq 3 \times 10^4$ and $\varepsilon_0 \leq 0.2$. We show the contour maps of Corr in Figure 6. We averaged 10–30 Corr snapshots data to draw each figure. The straight dotted lines represent pitch angles of 15°, 30°, and 45°. We show Corr at $r = 0.7 R_{\text{Roche}}$ with different surface density of $\Sigma = 0.005–0.006$, $\Sigma = 0.008–0.010$, and $\Sigma = 0.012–0.016$ in Figures 6a–6c, respectively. The pitch angle is always $\sim 20^\circ$, and it does not depend on surface density within the range.
Fig. 3.—Snapshots of (a) run 1–100 K, (b) 2–100 K, and (c) 3–100 K at $t = 6T_K$ and (d) 4–100 K at $t = 4T_K$

of our simulation. However, we found that pitch angle increases with $r$. A similar tendency is also shown in local simulations (S95 and DI99). Figures 6d–6f show Corr at $r = 0.5\ R_{\text{Roche}}, 0.6\ R_{\text{Roche}}$, and $0.9\ R_{\text{Roche}}$, with $\Sigma = 0.012$–0.016, 0.012–0.016, and 0.004–0.005. In the innermost region (Fig. 6d), there is no clear structure, as shown in Figure 5, and Corr is almost featureless. In the outer region (Figs. 6e and 6f), a clear barlike structure appears at the center. The pitch angle is about $15^\circ$ at $r = 0.6\ R_{\text{Roche}}$ and about $45^\circ$ at $r = 0.9\ R_{\text{Roche}}$. In the analytical estimate in the following sections, we adopt $30^\circ$ as an averaged pitch angle.

In Figures 6a or 6b we also recognize the bars next to the central bars. The radial separation increases with $\Sigma$. The corresponding $\lambda_{\text{cr}}$ (eq. [38]) for Figures 6a–6c are 0.046 $R_{\text{Roche}}$, 0.061 $R_{\text{Roche}}$ and 0.095 $R_{\text{Roche}}$. Our numerical result and linear theory agree well.

5.3. Angular Momentum Transfer

Figures 7a–7d show angular momentum fluxes $F_{\text{col}}, F_{\text{grav}}$, and $F_{\text{trans}}$ in runs 2–100K, 2–10K, 3–100K, and 3–10K, after spiral patterns fully develop but before large lunar seeds grow, which may perturb the entire disk. The horizontal axis is the distance from the center of the Earth, and vertical axis is angular momentum flux. In the inner region ($r \lesssim 0.6\ R_{\text{Roche}}$), collisional angular momentum flux $F_{\text{col}}$ dominates. In outer region ($0.6\ R_{\text{Roche}} \lesssim r$), $F_{\text{grav}} > F_{\text{trans}} > F_{\text{col}}$. The difference should reflect the fact that spiral structure develops in the outer region while it does not in the inner region.

Before discussing angular momentum fluxes in each region in detail, we consider the condition for development of a spiral structure. In the linear theory, axisymmetric density perturbations grow if $Q < 1$, where $Q$ is defined by $\sigma\Omega/\pi\Sigma$ (Toomre 1964) with velocity dispersion $\sigma$. We show $Q$-values as a function of $r$ averaged over from $t = 8T_K$ to $t = 10T_K$ for set 2, and from $t = 4T_K$ to $t = 6T_K$ for set 3 in Figure 8. In the outer region, the velocity of collective motion is pumped up by spirals and $Q$ is greater than 1. Local simulations show that $Q \sim 2$ when spirals develop (S95; DI99). In our simulation, $Q$ increases with $r$, up to about 5 at $r = R_{\text{Roche}}$, which may be caused by global effects. In the inner region where optical depth is high, $Q$ is enhanced by incompressibility of the particles. The physical meaning of $Q$ is $\sim \Omega^2/\pi G \rho$, where spatial density $\rho \sim \Sigma/h$.
and disk scale height $h \sim \sigma / \Omega$ are used. In the high optical depth case, $\rho$ is limited by particle material density $\rho_p$ and $Q$ should be $\gtrsim \Omega^2/\pi G \rho_p$. The minimum value of $Q$ is independent of $\sigma$ and $r_p$ (or $N$), and increases with decrease in $r$. Ward & Cameron (1978) argued that mid-plane pressure stabilizes the disk and predicted that gravitational instability does not occur when even if the random velocity is small. A similar criterion is also derived by comparing the particle’s Hill radius with its physical size (DI99). Our result that spiral structure does not develop in the inner region is consistent with these predictions.

5.3.1. Angular Momentum Transfer in the Inner Region

In Figure 9 we plot $v_{\text{col}}/r_p^2 \Omega$ obtained by our simulations at $r = 0.5 R_{\text{Roche}}$, at various times of all runs with various $\Sigma$ and $r_p$. Collisional viscosity $v_{\text{col}}$ is well fitted as

$$v_{\text{col}} \approx 1.35 \Omega r_p^2 \tau^{1.5}.$$  \hfill (51)

We found $v_{\text{col}}$ to be consistent with the results of AT86. In the inner region, one-dimensional velocity dispersion $\sigma \sim r_p \Omega$ (S95). Thus, equation (36) reads

$$v = v_{\text{col}} + v_{\text{local}} \sim 0.78 \Omega r_p^2 \tau^{5/4}.$$  \hfill (52)

In Figure 9 we also plot $(v - v_{\text{local}})/r_p^2 \Omega$ (solid line), where we use equation (33) for $v_{\text{local}}$. The results of our $N$-body simulation agree with equation (52) except for a numerical factor $\sim 2-3$. $v_{\text{col}}$ also agrees with the result of local simulations when the spiral structure does not develop (Daisaka et al. 2001).

Since $\tau \propto \Sigma r_p^{-1}$ (eq. [34]), $v_{\text{col}} \propto r_p^{1/2} \Sigma^{3/2}$. For the same $\Sigma$, $v_{\text{col}}$ decreases with decrease in $r_p$, that is, increase in $N$. From equations (17) and (51),

$$F_{\text{col}} = 3\pi \Sigma r_p^2 \Omega v_{\text{col}} \approx 8.26 \rho \Sigma^{5/2} r_p^{-3/2} r_p^{1/2}.$$  \hfill (53)

Eliminating $r_p$ using equation (34), $F_{\text{col}}$ at $r = 0.5$ is

$$F_{\text{col}} \approx C_c \tau^{-1/2} \left( \frac{\rho \Sigma^2}{\Omega^2} \right) r_p^{2/3},$$  \hfill (54)

where $C_c \approx 1.2$.

5.3.2. Angular Momentum Transfer in the Outer Region

5.3.2.1. Angular Momentum Transfer by Gravitational Torque

In the outer region, where spiral structure develops, $F_{\text{grav}}$ and $F_{\text{trans}}$ overwhelm $F_{\text{col}}$. We show $F_{\text{grav}}$ as a function of surface density at $r = 0.7 R_{\text{Roche}}$ in Figure 10a. We plot the
results of all runs in sets 1–4, during $t = 6T_K$ to $t = 16T_K$ with sampling intervals $2T_K$. Since $\Sigma$ is different between different runs and between different times, we obtain $F_{\text{grav}}$ with various $\Sigma$ at the same $r$. The analytical estimation of $F_{\text{grav}} = (\pi^3 G^2 / \Omega^2 \Sigma^3) \sin i \cos^2 i$ (eq. [40]) with $i = 30^\circ$ is represented by a dashed line. The angular momentum flux $F_{\text{grav}}$ in the numerical results is in agreement with $F_{\text{grav}}$, in particular in the case of high $r$, where spiral structure clearly appears.

We introduce numerical factor $C_g$, which is defined as

$$F_{\text{grav}} = C_g \frac{\pi^3 G^2}{\Omega^2} r^2 \Sigma^3.$$  \hfill (55)

The estimation $F_{\text{grav}}$ for $i = 30^\circ$ corresponds to $C_g = 0.38$. We plot $C_g$ at $r = 0.7 R_{\text{Roche}}$ in numerical results in Figure 10b. When $\tau \approx 0.1$, $C_g$ is $\sim 1$–$2$, independent of $\tau$ and $r_p$, which indicates $F_{\text{grav}}$ has the same functional form as $F_{\text{grav}}$.  

5.3.2.2. Translational Angular Momentum Transfer

We show the relation between $F_{\text{trans}}$ and $\Sigma$ at $r = 0.7 R_{\text{Roche}}$ in Figure 11a. The dashed line is explained below. Similar to as we did in equation (55), we define a numerical factor $C_t$ as

$$F_{\text{trans}} = C_t \frac{\pi^3 G^2}{\Omega^2} r^2 \Sigma^3.$$  \hfill (56)

We show $C_t$ in Figure 11b. Figure 11 shows that $F_{\text{trans}}$ is always almost equal to $F_{\text{grav}}$ at $0.7 R_{\text{Roche}}$ where spiral structure develops. In the outermost region near the Roche limit, the results are more noisy in global simulation. However, $C_g$ and $C_t$ seem to be larger by a factor of 2–3 near the Roche limit in general. $F_{\text{col}}$ becomes less important with increase of $r$ (see Fig. 7). The local $N$-body simulation (Daisaka et al. 2001) also shows a similar tendency for $r$.

In the outer region, $F_{\text{trans}}$ is far greater than that corresponding to equation (33), with $\sigma \sim r_p \Omega$, which was derived under the assumption of spatial uniformity by GT78. The development of spiral structure is responsible for the enhancement of $F_{\text{trans}}$. To see this more clearly, we separate $F_{\text{trans}}$ into the component of bulk motion $F_{\text{bulk}}$ and that of random motion $F_{\text{local}}$, as explained in § 2.3.2.
In Figure 12 we show $F_{\text{bulk}}$ and $F_{\text{local}}$ of runs in set 3, averaged from $t = 4T_K$ to $6T_K$, as a function of $n_{\text{bulk}}$. The dashed lines are $\lambda_{\text{bulk}}$, which is the scale of the region in which $n_{\text{bulk}}$ particles exist. Figures 12a and 12b show $F_{\text{bulk}}$ and $F_{\text{local}}$ at $r = 0.7 R_{\text{Roche}}$ for runs 3–100 K and 3–10 K, respectively. The predicted radial scale of spirals $\lambda_{\text{cr}}$ (eq. [38]) in the cases of Figures 12a and 12b are about 0.08 $R_{\text{Roche}}$. In Figure 12a $F_{\text{bulk}}$ and $F_{\text{local}}$ are almost constant up to $n_{\text{bulk}} \sim 100$, and $\lambda_{\text{bulk}} \sim \lambda_{\text{cr}}$. This means that the particles move collectively as a group with scale $\sim \lambda_{\text{cr}}$, in which about 100 particles exist. In this scale, $F_{\text{local}} \ll F_{\text{bulk}}$, and translational angular momentum transfer is almost wholly due to this bulk motion. This is the case also for run 2–10 K (Fig. 12b). In this case, particles move collectively with about 10 neighboring particles with the scale $\lambda_{\text{bulk}} \sim \lambda_{\text{cr}}$. These results explain why $F_{\text{trans}}$ is enhanced over equation (33) and has the functional dependence predicted by equation (56).

When the structure develops, $Q = \sigma\Omega/\pi G \Sigma \sim O(1)$. Since $F_{\text{bulk}} \gg F_{\text{local}}$, $\sigma \sim v_{\text{bulk}} \gg v_{\text{local}}$. Then,

$$v_{\text{bulk}} \sim \frac{\pi G \Sigma}{\Omega} Q \sim \frac{\pi G \Sigma}{\Omega}.$$  

(57)

Since particles move as groups, these groups may be treated as superparticles with random velocity $v_{\text{bulk}}$. Lifetime of each structure is $\sim \Omega^{-1}$, so that the collision frequency of the superparticles is $\omega_c \sim \Omega$. Thus, translational viscosity
Fig. 7.—Angular momentum fluxes $F_{\text{col}}$, $F_{\text{grav}}$, and $F_{\text{trans}}$ as functions of $r$, averaged over $t = 4-6T_K$: (a) run 2–100 K, (b) run 2–10 K, (c) run 3–100 K, and (d) run 3–10 K.

Fig. 8.—$Q$-value as a function of $r$ in (a) set 2 and (b) set 3 at $t = 6T_K$. 
Collisional viscosity $\nu_{\text{col}}/\Omega r^2$ is given as a function of $\tau$, in the region where spiral structure does not develop. Circles are the results of $N$-body simulations at $r = 0.5 R_{\text{Roche}}$ for different times and runs. The dashed line is a fitted value given by equation (51). The solid line is an analytical estimate by GT78 and AT86.

Note that and We also plot $F_{\text{gravest}}$ in Figure 11a with a dashed line. Considering a disk with turbulence induced by self-gravitational instability, Ward & Cameron (1978) and Lin & Pringle (1987) derived a similar angular momentum transfer rate.

We have studied the cases with $\epsilon_n = 0.1$. This choice may lead to a rather high cooling rate of random velocity, keeping the $Q$-value low. Equilibrium random velocity is determined by heating due to transfer of shear motion to random motion through collisions and damping due to inelastic collisions (GT78; Ohtsuki 1993). However, for $\epsilon_n > 0.67$, the damping is no longer strong enough to attain an equilibrium state and random velocity keeps growing (GT78). Then, $Q$-value should be so high that the spiral structure does not appear. To examine the effect of higher $\epsilon_n$, we performed additional simulations (set 6) with restitution coefficients 0.2, 0.4, 0.6, and 0.8. We show the snapshots of some cases in set 6 in Figure 13. The initial surface density distribution is the same as that in set 3. Figure 13a shows the case with $\epsilon_n = 0.4$ at $t = 8 T_K$. Figure 13b shows the case with $\epsilon_n = 0.6$. For $\epsilon_n \lesssim 0.4$, spiral structure seems similar to the runs with $\epsilon_n = 0.1$. However, when $\epsilon_n = 0.6$, the spiral is less clear. In the case with $\epsilon_n = 0.8$, no structure appears. We show angular momentum transfer rates in Figure 14. These rates are averaged over $t = 6 T_K$ to $t = 8 T_K$. For $\epsilon_n = 0.6$, gravitational transfer is less dominant. This reflects the less clear spiral structure (Fig. 14b).

We show $F_{\text{grav}}$ as a function of $\Sigma$ for different $\epsilon_n$ in Figure 15a. Only data with optical depth larger than 0.3 are chosen. The dashed lines are the best-fitted lines with the assumption that $F_{\text{grav}} \propto \Sigma^3$. While $F_{\text{grav}}$ decreases with increase of $\epsilon_n$, the relation $F_{\text{grav}} \propto \Sigma^3$ remains. $F_{\text{grav}}$ and $F_{\text{col}}$ are plotted in Figures 15b and 15c. Although they decrease with increase of $\epsilon_n$, the decrease is smaller than that in $F_{\text{grav}}$. ($F_{\text{grav}}$ does not show clear dependence as $\Sigma^3$ in the large $\epsilon_n$ case. We omit the fitted line in the case with $\epsilon_n = 0.8$ because of large dispersion.)

Figure 15d shows the corresponding $C_{\epsilon}$, $C_{r}$, and $C_\tau$ as a function of $\epsilon_n$. Error bars represent standard deviations. In general, as $\epsilon_n$ increases, angular momentum transfer, in particular $F_{\text{grav}}$, decreases. However, the results do not change
Fig. 11.—(a) $F_{\text{trans}}$ as a function of surface density $\Sigma$ at $r = 0.7 R_{\text{Roche}}$. The dashed line is $F_{\text{trans}}$ (eq. [59]). (b) $C_r$ as a function of optical depth $\tau$ at $r = 0.7 R_{\text{Roche}}$.

Fig. 12.—$F_{\text{bulk}}$ and $F_{\text{local}}$ as a function of $n_{\text{bulk}}$ at $r = 0.7 R_{\text{Roche}}$ in (a) run 3–100 K and (b) run 3–10 K. Filled circles are $F_{\text{bulk}}$ and circles are $F_{\text{local}}$. The dashed lines are $\lambda_{\text{bulk}}$ corresponding to $n_{\text{bulk}}$.

Fig. 13.—Snapshots for set 6, with (a) $\epsilon_a = 0.4$ and (b) $\epsilon_a = 0.6$. Initial surface density is similar to set 3.
Fig. 14.—Angular momentum fluxes $F_{\text{col}}$, $F_{\text{grav}}$, and $F_{\text{trans}}$ as functions of $r$, with different $\epsilon_a$. (a) $\epsilon = 0.4$ and (b) $\epsilon = 0.6$.

Fig. 15.—Angular momentum fluxes $F_{\text{col}}$, $F_{\text{grav}}$, and $F_{\text{trans}}$ as a function of $\Sigma$, with different $\epsilon_a$. The dashed lines are the best-fitted lines assuming that fluxes are proportional to $\Sigma^2$. (a) $F_{\text{grav}}$, (b) $F_{\text{trans}}$, and (c) $F_{\text{col}}$; (d) shows the relation between $\epsilon_a$ and corresponding $C_g$ and $C_r$. 
so much \((C_g, C_c, \text{ and } C_t)\) change only by factor 2) except for in the highly elastic case \(\varepsilon_n \gtrsim 0.6\), when the spiral structure is not clear.

5.3.4. Summary of Angular Momentum Transfer

In an optically thick particle disk, angular momentum is transferred in different ways in the inner and outer regions. The boundary between the inner and outer regions is about 0.6–0.7 \(R_{\text{Roche}}\). In the inner region, clear spiral structure does not develop, so that the analytical formulae assuming spatial uniformity work well (e.g., GT78; AT86). In this region, collisional angular momentum transfer dominates, which is well fitted by equation (51).

In the outer region, \(F_{\text{grav}}\) and \(F_{\text{trans}}\) are enhanced by the spiral structure and dominate \(F_{\text{col}}\). They are both proportional to \(\Sigma^3\) and independent of \(r\). In the outer region, \(F_{\text{grav}} \approx F_{\text{trans}} > F_{\text{col}}\). Defining \(C_i (i = g, t, c) = F_{\text{grav}} / F_{\text{trans}}\) and \(F_{\text{col}} = C_i (\pi^2 G^2 / \Omega^2) r^5 \Sigma^3\) \((i = g, t, c)\), our numerical results show \(C_{\text{total}} = C_g + C_t + C_c \sim C_g + C_t \sim 4–8\). In local simulations, \(C_{\text{total}} \sim 2\) in the region corresponding to \(r \sim 0.7 R_{\text{Roche}}\) and \(C_{\text{total}} \sim 6\) at \(r \sim 0.9 R_{\text{Roche}}\) (Daisaka et al. 2001). The numerical factor slightly increases with \(r\) in local simulations (Daisaka et al. 2001). A similar tendency is found in our global simulations. As the restitution coefficient increases, numerical factors decrease and the relative importance of \(F_{\text{grav}}\) diminishes. However, the transfer rate changes only by a factor 2 unless \(\varepsilon_n\) is highly elastic \((\varepsilon_n \gtrsim 0.6)\).

We show the time evolution of the surface density distribution in log scale in Figure 16, for \((a)\) run 3–100 K and \((b)\) run 5–30 K. The initial surface density distribution is proportional to \(\Sigma \propto r^{-3}\) in run 3–100K and flat in run 5–30K. The dashed lines in the figures are proportional to \(\Sigma^{-3/2}\). The surface density distribution after initial relaxation but before formation of lunar seeds in our simulations is consistent with \(\Sigma \propto r^{-3/2}\) in the outer region. The relation \(\Sigma \propto r^{-3/2}\) is the steady accretion solution \((dM_{\text{disk}}/dr = 0)\) to equation (15) with constant \(C_{\text{total}}\) (Lin & Pringle 1987).

6. CONCLUSIONS AND DISCUSSION

We have investigated angular momentum transfer and associated mass transfer in a particle disk where physical collisions as well as self-gravity are important. First we presented a formulation of angular momentum transfer in the disk, starting from the Boltzmann equation. Next, we performed global N-body simulation with \(N = 10^3–10^5\) to directly calculate angular momentum transfer fluxes, based on the above formulation. We simulated disks that correspond to a protolunar disk generated by a giant impact on the proto-Earth. The disk has total mass \(\sim 0.02–0.06 M_\odot\), where \(M_\odot\) is the central body mass and most mass is initially within the Roche limit of the central body. In such a dense disk, spiral structure is formed by self-gravity and energy dissipation due to inelastic collisions, except for the innermost region, where tidal force of the central body is too strong.

In the region 0.6 \(R_{\text{Roche}}\) \(\leq r \leq 1 R_{\text{Roche}}\), angular momentum transfer is regulated by gravitational torque exerted by spiral structure and collective motion of particles in the spirals. ICS97 and KIM00 showed that formation of the Moon is regulated by angular momentum transfer in this region. With increasing \(\tau\), spiral structure becomes clearer. When \(\tau \gtrsim 0.2\), angular transfer rate is

\[
F_{\text{grav}} + F_{\text{trans}} + F_{\text{col}} \sim F_{\text{grav}} + F_{\text{trans}}
\]

\[
= (C_g + C_t) \frac{\pi^2 G^2}{\Omega^2} r^2 \Sigma^3 ,
\]

where \(C_g + C_t\) is about 4–8 in the parameters for a protolunar disk. Then, surface density distribution approaches to distribution in steady accretion that is proportional to \(r^{-3/2}\).

The average optical depth is

\[
\tau = \frac{N \rho r^2}{\rho (R_{\text{Roche}}^2 - R_{\oplus}^2)} = \left( \frac{3 M_{\text{disk}}}{4 \pi \rho} \right)^{2/3} \frac{1}{(R_{\text{Roche}}^2 - R_{\oplus}^2)} N^{1/3}.
\]

Since \(\tau\) saturates for \(\tau \gtrsim 0.2\), particle number \(N\) in global simulation must be

\[
N \gtrsim 3500 \left( \frac{M_{\text{disk}}}{3 M_\odot} \right)^{-2}.
\]

The relation between the initial disk mass and the mass of the Moon was derived by ICS97. When the initial mass is
distributed within the Roche limit, the required initial disk mass is about 3 times the present lunar mass. Thus, evolution of the protolunar disk can be followed with a rather small number of particles; $N \gtrsim 3000$ is enough, and $N \gtrsim 1000$ ($\tau \gtrsim 0.15$) may be okay.

In the innermost region, spiral structure does not develop and angular momentum transfer is dominated by collisions between particles, and the corresponding viscosity has a positive dependency on particle size. $N$-body simulations with a limited number overestimate the diffusion process in this region. However, this does not change the lunar formation, since the lunar seed is formed by the angular momentum transfer in outer region. Once a large proto moon is formed, the remaining disk would interact with the lunar seed, and the disk materials would eventually be scattered to fall to the Earth (ICS97; KIM00). Thus, we conclude that an $N$-body simulation of the evolution of a protolunar disk is not affected by a limited number of particles, as long as $N \gtrsim 1000$–3000.

The timescale of viscous evolution is given as $\sim \Delta r^2/\nu$. Since total viscosity is

$$v_{\text{total}} = C_{\text{total}} \frac{\pi^2 G^2}{3 \Omega^2} \Sigma^2,$$  \hspace{1cm} (63)

and the timescale of disk evolution is

$$\tau_{\text{ev}} \approx \frac{500}{C_{\text{total}}} \left( \frac{\Sigma}{0.01 M_\odot R_{\text{Roche}}^2} \right)^{-2} \left( \frac{\Delta r}{R_{\text{Roche}}} \right)^2 \left( \frac{r}{R_{\text{Roche}}} \right)^{-9/2} T_K.$$  \hspace{1cm} (64)

Since $C_{\text{total}} \approx C_g + C_1$ is about 4–8, $\tau_{\text{ev}} \approx 100 T_K$, which is consistent with lunar formation time obtained by ICS97 and KIM00. The development of spiral structure is essential for the rapid evolution of a protolunar disk.

We comment on the physical processes that we neglected and validity of assumptions in our simulations. One process we neglected is fragmentation. Since typical random velocity is large, catastrophic disruption would occur in particle collisions. Thus, realistic particle size would be much smaller. However, as long as random energy damping is sufficient, spiral structure develops as well. Thus fragmentation would not affect the evolution of the disk, since the angular momentum transfer in the outer region has no dependency on particle size, being regulated by the movement of particles as a group.

We simply assumed that all particles have the same size, so that the filling factor is 0.7 at most. If size distribution is included, the effective material density increases. It may expand the region where a clear spiral develops more into the inner region. Also, the random velocity of smaller particles would be larger if size distribution is included. However, size distribution does not prevent the spiral structure from developing in the simulations in (ICS97 and KIM00), so that this would not affect the physics in principle.

We also assumed that the central body is spherical. Objects interact with tidal bulges raised on the Earth, so that materials exterior to the synchronous orbit migrate outward, and interior materials migrate inward. The timescale of tidal evolution of a body with one lunar mass is of order of $10^3$ yr (Canup & Esposito 1996). This is much longer than the timescale of disk evolution, and the tidal effect is not essential for disk evolution. The formed moon would migrate outward, sweeping up remnants of the disk (Canup, Levison, & Stewart 1999). The Earth itself may be deformed considerably without tidal bulges. In smoothed particle hydrodynamics (SPH) simulations of a giant impact, the core of impactor penetrates through the mantle material and accumulates on the proto-Earth, and forms a rotating quadrupole (e.g., Cameron 1997). Though the quadrupole subsides substantially in a day or two, some fraction of the quadrupole component may remain. This would not affect the angular momentum transfer due to local instability, which was discussed in this paper, since the timescale of angular momentum transfer by local instability is very short. However, this may have a considerable effect when the timescale is longer.

We also assumed that a protolunar disk is an entirely particulate disk. Recent SPH simulations (Cameron 1997) suggest that vapor/liquid and solid phase coexist in a protolunar disk after the giant impact, with average temperature above 4000K. Assuming that an initial disk with 3 lunar mass is within the Roche limit, the heat dissipation required for the disk evolution is about $1.5 \times 10^{37}$ ergs, which is about a half the latent heat of the vaporization of silicate of the disk mass. If an entire disk is evaporated, Toomre’s $Q$-value is $\sim 3(T/1000 K)^{1/2} (r/R_{\text{Roche}})^{-3/2} [\Sigma/(0.01 M_\odot R_{\text{Roche}})]^{-1}$, with mean molecular weight 30, and the disk would be stable against gravitational instability. During a short timescale of dynamical disk evolution, it is difficult for radiation to cool down the disk (Thompson & Stevenson 1988). If gravitational instability is aborted by the vaporization, the disk evolution would be regulated by the cooling time of the disk (Thompson & Stevenson 1988).

The disk evolution and Moon accretion would depend on how the disk with vapor/liquid-solid mixture evolves, as discussed in KIM00. If the disk remains within the Roche limit until the disk is sufficiently cooled down to develop gravitational instability, a single moon would be formed just beyond the Roche limit. The disk would be condensed from outer region. Spirals would develop in the condensed region, which results in rapid diffusion of the materials there. A single moon accretes the materials diffused out, staying at the location just beyond the Roche limit. The eventual mass of the moon would be the same as the results of $N$-body simulations neglecting vaporization, because final moon mass is determined by redistribution of disk angular momentum to the moon and materials that fall to the Earth (ICS97; KIM00), although the timescale may be regulated by cooling.

The heat generation problem occurs in a very heavy disk such as a protolunar disk. The result that angular momentum transfer in a particulate disk within the Roche limit is regulated by gravitational instability is applicable to other disk systems, such as planetary rings or small satellite formation. In planetary rings, the only difference is that the wavelength of structure $\lambda_{\text{er}}$ is much smaller since $M_{\text{disk}}/M_\odot \ll 1$ (see eq. [39]). Daisaka et al. (2001) performed the local $N$-body simulations with the parameters of Saturn’s B-ring, and calculated the angular momentum transfer rate in a similar way. They found that the angular momentum transfer in Saturn’s B-ring is also regulated by a wakelike structure and that equation (60) also holds.

We are grateful for helpful comments by Hidekazu Tanaka on formalizations. Discussion with Hiroshi
Daisaka was valuable. We also thank him for his technical comments on numerical calculations. The numerical calculations were carried out on a special purpose computer, GRAPE-4. We thank Junichro Makino and Eiichro Kokubo for technical advice on GRAPE-4. We also thank the anonymous referee for useful comments. This work was supported by Grant-in-Aid for Scientific Research (c) 12640405.

APPENDIX

ANGULAR MOMENTUM FLUX DUE TO GRAVITATIONAL TORQUE

A formula for angular momentum flux due to gravitational torque with a spiral pattern is given by Lynden-Bell & Kalnajs (1972). From the definition (eq. [11]),

\[ F_{\text{grav}} = \int_{r_{\text{min}}}^{r} dr' \int_{0}^{2\pi} r' d\theta' \int_{-\infty}^{\infty} dz \rho \frac{\partial \Phi'}{\partial \theta}, \]  

(A1)

de where \( \Phi' \) is potential due to disk material. Combining equation (A1) with a Poisson equation, it is expressed as

\[ F_{\text{grav}}(r) = \frac{1}{4\pi G} \int_{0}^{2\pi} r d\theta' \int_{-\infty}^{\infty} dz \frac{\partial \Phi'}{\partial r} \frac{\partial \Phi'}{\partial \theta}. \]  

(A2)

We represent surface density distribution as

\[ \Sigma(r, \theta, t) = \Sigma_0(r, t) + \Sigma_1(r, \theta, t) = \Sigma_0(r, t) + H(r, t)e^{i\theta + f_{\sigma}(t)}, \]  

(A3)

where \( \Sigma_0 \) is azimuthally averaged density distribution and \( f_{\sigma}(t) \) is shape function for the \( n \)th arm satisfying

\[ m \theta' + f_{\sigma}(r, t) = \text{constant (mod } 2\pi). \]  

(A4)

Using a tightly winding approximation (e.g., Binney & Tremaine 1987), the potential due to \( \Sigma_1 \) is given by

\[ \Phi'_i(r, \theta, z, t) = -\frac{2\pi G}{|k|} H(r, t) \Re \left\{ e^{i(m \theta + f_{\sigma}(t)) - |k|z} \right\}, \]  

(A5)

where \( k \) is the wavenumber. Note that the \( r \) component of \( k \) is \( \kappa_r = \partial f_{\sigma}/\partial \theta = k \cos \ i \). Substituting equation (A5) into equation (A2), and using \( m = k_r \tan \ i \), where \( i \) is pitch angle, we obtain

\[ F_{\text{grav}} = \frac{\pi^2 Gr \kappa_r}{k^2} \frac{mH^2}{2} = \frac{\pi}{2} Gr^2 \lambda_r \sin \ i \cos^2 \ i, \]  

(A6)

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