Composite p-branes in Various Dimensions

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Abstract

We review an algebraic method of finding the composite p-brane solutions for a generic Lagrangian, in arbitrary spacetime dimension, describing an interaction of a graviton, a dilaton and one or two antisymmetric tensors. We set the Fock–De Donder harmonic gauge for the metric and the “no-force” condition for the matter fields. Then equations for the antisymmetric field are reduced to the Laplace equation and the equation of motion for the dilaton and the Einstein equations for the metric are reduced to an algebraic equation. Solutions composed of $n$ constituent p-branes with $n$ independent harmonic functions are given. The form of the solutions demonstrates the harmonic functions superposition rule in diverse dimensions. Relations with known solutions in $D = 10$ and $D = 11$ dimensions are discussed.
1 Introduction

Recent remarkable developments in superstring theory led to the discovery that the five known superstring theories in ten dimensions are related by duality transformations and to the conjecture that M-theory underlying the superstring theories and 11 dimensional supergravity exists \[1\]-\[3\]. Duality requires the presence of extremal black holes in the superstring spectra. A derivation of the Bekenstein-Hawking formula for the entropy of certain extreme black holes was given by using the D-brane approach \[4\]-\[6\].

In all these developments the study of p-brane solutions of the supergravity equations play an important role \[10\]-\[22\]. To clarify the general picture of p-brane solutions it seems useful to have solutions in arbitrary spacetime dimension.

In this talk results obtained in our recent works \[23, 25, 29\] will be presented. We use a systematic algebraic method of finding p-brane solutions in diverse dimensions. We start from an ansatz for a metric on a product manifold and use the Fock–De Donder harmonic gauge. It leads to a simple form for the Ricci tensor. Then we consider an ansatz for the matter fields and use the ”no-force” condition. This leads to a simple form of the stress-energy tensor and moreover the equation for the antisymmetric fields are reduced to the Laplace equation. The Einstein equations for the metric and the equation of motion for the dilaton under these conditions are reduced thence to an algebraic equation for the parameters in the Lagrangian.

2 Ricci tensor in the Fock–De Donder gauge

Let us be given a D-dimensional manifold of the product form \(M^D = M^q \times M^{r_1} \times M^{r_2} \times \cdots \times M^{r_n} \times M^{s+2}\). We will use the following ansatz for a metric on \(M^D\):

\[
ds^2 = e^{2A} \eta_{\mu\nu} dy^\mu dy^\nu + \sum_{i=1}^{n} e^{2F_i} (dz_i^m)^2 + e^{2B} (dx^\gamma)^2
\]

(1)

Here \(\eta_{\mu\nu}\) is a Minkowski metric, \(\mu, \nu = 0, \ldots, q - 1\); \(m_i = 1, \ldots, r_i\); \(\gamma = 1, \ldots, s + 2\),

\[
q + \sum_{i=1}^{n} r_i + s + 2 = D \tag{2}
\]

and the functions \(A, F_i\) and \(B\) depend only on \(x\). The metric (1) is the sum of \(n + 2\) blocks. We shall call it the \(n + 2\)-block p-brane metric.

The Ricci tensor for the above metric reads

\[
R_{\mu\nu} = -\eta_{\mu\nu} e^{2(A-B)}[\Delta A + q(\partial A)^2 + \\
\sum_{i=1}^{N} r_i (\partial A \partial F_i)] + s(\partial A \partial B),
\]

\[
R_{m_i n_i} = -\delta_{m_i n_i} e^{2(F_i-B)}[\Delta F_i + q(\partial A \partial F_i) + \\
\sum_{i=1}^{N} r_i (\partial A \partial F_i)] + s(\partial A \partial B),
\]

where \(\Delta\) is the Laplace operator on the relevant manifold.
\[
\sum_{j=1}^{N} r_j(\partial F_j \partial F_i)] + s(\partial F_i \partial B),
\]

\[
R_{\alpha\beta} = -q \partial_{\alpha} \partial_{\beta} A + \sum_{i=1}^{N} r_i \partial_{\alpha} \partial_{\beta} F_i - s \partial_{\alpha} \partial_{\beta} B -
\]

\[
- q \partial_{\alpha} A \partial_{\beta} A - \sum_{i} r_i \partial_{\alpha} F_i \partial_{\beta} F_i +
\]

\[
s \partial_{\alpha} B \partial_{\beta} B + q (\partial_{\alpha} A \partial_{\beta} B + \partial_{\alpha} B \partial_{\beta} A) +
\]

\[
\sum_{i} r_i (\partial_{\alpha} B \partial_{\beta} F_i + \partial_{\alpha} F_i \partial_{\beta} B) -
\]

\[
\delta_{\alpha\beta} [\Delta B + q (\partial A \partial B) + \sum_{i=1}^{N} r_i (\partial B \partial F_i)] + s (\partial B)^2
\]

We shall use the Fock-De Donder gauge

\[
\partial_M (\sqrt{-g} g^{MN}) = 0
\]

(3)

For the metric (11) this leads to the following important condition

\[
qA + \sum r_i F_i + sB = 0
\]

(4)

Then the Ricci tensor for the above metric takes the simple form

\[
R_{\mu\nu} = -\eta_{\mu\nu} e^{2(A-B)} \Delta A,
\]

(5)

\[
R_{m,n} = -\delta_{m,n} e^{2(F_i - B)} \Delta F_i
\]

(6)

\[
R_{\alpha\beta} = -q \partial_{\alpha} A \partial_{\beta} A - \sum_{i} r_i \partial_{\alpha} F_i \partial_{\beta} F_i +
\]

\[
s \partial_{\alpha} B \partial_{\beta} B - \delta_{\alpha\beta} \Delta B
\]

(7)

We shall consider the Einstein equations in the form

\[
R_{MN} = \mathcal{G}_{MN}
\]

(8)

\[
\mathcal{G}_{MN} = T_{MN} - \frac{1}{D-2} g_{MNT}
\]

(9)
In this section we shall consider the following action

\[ I = \int d^D x \sqrt{-\tilde{g}} (R - \frac{(\nabla \phi)^2}{2} - \frac{e^{-\alpha \phi}}{2(d+1)!} F_{d+1}^2) \]  

(10)

It describes the interaction of the gravitation field \( g_{MN} \) with the dilaton \( \phi \) and with one antisymmetric field: \( F_{d+1} \) is a closed \( d+1 \)-differential form. The stress energy tensor for the action (10) is

\[ T_{MN} = \frac{1}{2} \delta_{MN} (\partial \phi)^2 + \frac{e^{-\alpha \phi}}{2d!} (F_{MM_1...M_d} F_{N}^{M_1...M_d} - \frac{g_{MN}}{2(d+1)} F^2) \]  

(11)

In this section we shall consider an electric ansatz for the action (10) which leads to a \( n+2 \)-block metric. Let us take the ansatz (1) with \( r_i = r > 1 \). Then (2) takes the form

\[ D = q + nr + s + 2 \]  

(12)

We use the following ansatz for the \( d \)-form \( A \)

\[ A = \omega_0^q \wedge [\omega_1^r h_1 e^{C_1} + ... + \omega_n^r h_n e^{C_n}] \]  

(13)

where \( d = q + r \),

\[ \omega_0 = dy^0 \wedge dy^1 \wedge ... \wedge dy^{q-1} \]

\[ \omega_i^r = dz_1^r \wedge ... \wedge dz_i^r \]

\( C_i \) are functions of \( x \) and \( h_i \) are some constants. More general ansatz has been considered in [31, 30]. For the stress-energy tensor we get

\[ T_{\mu \nu} = -\eta_{\mu \nu} e^{2(A-B)} \frac{1}{4} (\partial \phi)^2 + \sum_{i=1}^{n} \frac{h_i^2}{4} e^{-2qA-2rF_i-2C_i} (\partial C_i)^2 \]

\[ T_{mni} = \delta_{m,n} e^{2(F_i-B)} \frac{1}{4} (\partial \phi)^2 - \frac{h_i^2}{4} e^{-2qA-2rF_i+2C_i} (\partial C_i)^2 + \sum_{j \neq i}^{n} \frac{h_j^2}{4} e^{-2qA-2rF_j+2C_j} (\partial C_j)^2 \]

\[ T_{\alpha \beta} = \frac{1}{2} [\partial_\alpha \partial_\beta \phi - \frac{1}{2} \delta_{\alpha \beta} (\partial \phi)^2] - \sum_{i=1}^{n} \frac{h_i^2}{2} e^{-2qA-2rF_i+2C_i} [\partial_\alpha C_i \partial_\beta C_i - \frac{\delta_{\alpha \beta}}{2} (\partial C_i)^2] \]
We set the following ("no-force") condition
\[- \alpha \phi - 2qA - 2rF_i - 2C_i = 0, \quad i = 1, \ldots n \] (14)

Then the form of $T_{MN}$ simplifies drastically and the Einstein equations (8) take the form
\[
\Delta A = \sum_i th_i^2 (\partial C_i)^2,
\] (15)
\[
\Delta F_i = th_i^2 (\partial C_i)^2 - \sum_{j \neq i} u h_j^2 (\partial C_j)^2,
\] (16)
\[-q \partial_\alpha A \partial_\beta A - \sum_i r_i \partial_\alpha F_i \partial_\beta F_i \]
\[-s \partial_\alpha B \partial_\beta B - \delta_{\alpha\beta} \Delta B = \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi - \sum_{i=1}^n h_i^2 \left[ \frac{1}{2} \partial_\alpha \partial_\beta C_i - u \delta_{\alpha\beta} (\partial C_i)^2 \right],
\] (17)

where
\[
t = \frac{D - 2 - q - r}{2(D - 2)}, \quad u = \frac{q + r}{2(D - 2)}.
\] (18)

The equation of motion for the antisymmetric field,
\[
\partial_M (\sqrt{-g} F^{MM_{1\ldots M_d}}) = 0,
\] (19)
under conditions (4) and (14) for the ansatz (13) reduces to the Laplace equation
\[
\partial_\alpha \partial_\alpha (e^{-C_i}) = 0 \text{ or } \Delta C_i = (\partial C_i)^2.
\] (20)

Equation of motion for the dilaton reads
\[
\Delta \phi - \frac{\alpha}{2} \sum_i h_i^2 (\partial C_i)^2 = 0
\] (21)
or by using (20)
\[
\Delta (\phi - \frac{\alpha}{2} \sum_i h_i^2 C_i) = 0
\] (22)

We take the following solution of (22)
\[
\phi = \frac{\alpha}{2} \sum_i h_i^2 C_i
\] (23)

Analogously we take the following solutions of (15) and (16)
\[
A = t \sum_i h_i^2 C_i,
\] (24)
\[ F_i = \theta h_i^2 C_i - u \sum_{j \neq i} h_j^2 C_j \]  

(25)

To cancel out the terms proportional to \( \delta_{\alpha\beta} \) in (17) we set

\[ B = -u \sum_i h_i^2 C_i \]  

(26)

Note that if we want to solve equations for arbitrary harmonic functions \( H_i = \exp(-C_i) \) we have also to assume the conditions which follow from (14), (4) and nondiagonal part of (17). Equations (14) lead to the relations

\[
\left[ \frac{\alpha^2}{4} + (q + r)t \right] h_i^2 C_i + \\
\left[ \frac{\alpha^2}{4} + qt - ru \right] \sum_{j \neq i} h_j^2 C_j = C_i, \quad i = 1, \ldots, n.
\]  

(27)

If \( n \neq 1 \) then under the assumption of independence of \( C_i \) the relations (27) yield

\[ \frac{\alpha^2}{4} = ru - qt, \]  

(28)

\[ \frac{\alpha^2}{4} + (q + r)t h_i^2 = 1. \]  

(29)

If \( n = 1 \) we get only (28). Since \( t \) and \( u \) are given by (18) the condition (28) leads to the following equation

\[ \left( \frac{\alpha^2}{2} + q \right)(D - 2) = d^2 \]  

(30)

Equation (30) plays the central role in our approach. For given parameters \( D, d \) and \( \alpha \) in the Lagrangian (10) we have to find a positive integer \( q \) which solves equation (30). In this sense we can interpret equation (30) as “quantization” of the parameter \( \alpha \) in the Lagrangian (see also [23, 24]).

Note that under this condition the formula (18) takes the form

\[ u = \frac{2q + \alpha^2}{4(q + r)}, \quad t = \frac{2r - \alpha^2}{4(q + r)} \]  

(31)

The LHS of (29) for \( t \) and \( u \) given by formulae (31) can be represented as \( r h_i^2 / 2 \), and, therefore, equation (29) gives

\[ h_i^2 = h^2 \equiv \frac{2}{r}. \]  

(32)

By straightforward calculations one can check that for \( \alpha, q, r \) and \( D \) satisfying relation (30) and \( h \) given by (32), equations (14) as well as equations giving a compensation of terms \( \partial_\alpha C_j \partial_\beta C_i \) in the both sides of equation (17) are fulfilled.
This calculation shows that the metric
\[ ds^2 = (H_1 H_2 \ldots H_n)^{\frac{4}{n}} \] (33)

\[ [(H_1 H_2 \ldots H_n)^{-\frac{2}{n}} \eta_{\mu\nu} dy^\mu dy^\nu + \sum_{i=1}^{n} H_i^{-\frac{2}{n}} (dz^m)^2 + (dx^\gamma)^2], \]

and matter fields in the form
\[ \exp \phi = (H_1 H_2 \ldots H_n)^{-\alpha/r} \] (34)

\[ A = \sqrt{\frac{2}{r}} \omega_0 \wedge [\omega_1 H_1^{-1} + \ldots + \omega_n H_n^{-1}] \] (35)

is a solution of the theory (10) if \( H_i(x), i = 1, \ldots n, \) are harmonic functions and the parameters in the Lagrangian \( D, d \) and \( \alpha \) are such that equations (12) and (30), i.e.

\[ d = q + r, \quad D = q + nr + s + 2, \]

\[ (\alpha^2 + q)(D - 2) = d^2 \] (36)

admit solutions with positive integers \( q, n, r \) and \( s + 2 \).

Let us note that the formula (33) proves the harmonic superposition rule for the ansatz (13). Indeed, \( u \) and \( r \) can be written as

\[ u = \frac{d}{2(D - 2)}, \quad r = d(1 - \frac{d}{D - 2}) + \frac{\alpha^2}{2} \]

One can easily see that the exponents in the formula are defined by the two-block solution (14)

\[ ds^2 = H_{\mu\nu}[H^{-\frac{2}{n}} \eta_{\mu\nu} dy^\mu dy^\nu + dx^\gamma dx^\gamma] \]

\( \hat{\mu}, \hat{\nu} = 1, \ldots d - 1, \quad \hat{\gamma} = 1, \ldots D - d. \) Therefore, having the simplest two-block solution one can produce \( n \)-block solution. Let us note that this prescription works only for the case when the characteristic equation (30) is satisfied. The harmonic function rule was formulated in (17). for the case \( D = 10, \quad D = 11. \) The formula (33) demonstrates that the harmonic superposition rule holds for arbitrary dimension.

3.1 Examples

3.1.1 \( \alpha = 0 \)

Note that equations (36) are very restrictive since it has to be solved for integers \( q, d, r \) and \( D. \) Let us present some examples.

For dimensions \( D = 4, 5, 6, 7, 8 \) and 9 there are solutions only with \( r = 0 \) and we have 2-block solutions with \( s = 0. \) In these cases, either the spacetime is asymptotically \( M^q \times Y, \) where \( Y \) is a two-dimensional conical space, or the metric exhibits logarithmic behaviour as \( |x| \to \infty. \)
We get more interesting structures in \( D = 6 \) case. There are four types of solutions of (36) with \( \alpha = 0 \):

\[ i) q = 1, \quad r = 1, \quad n = 2, \quad s = 1, \]
\[ ii) q = 1, \quad r = 1, \quad n = 3, \quad s = 0, \]
\[ iii) q = 1, \quad r = 1, \quad n = 4, \quad s = -1, \]
\[ iv) q = 1, \quad r = 1, \quad n = 5, \quad s = -2. \]

Here we have to assume that different branches with \( r = 1 \) correspond to different gauge field \( A^{(I)} = h dy^0 \wedge dz_i H_i^{-1} \delta_{II}, \quad I = 1, \ldots n \) (otherwise we cannot guarantee the diagonal form of the stress-energy tensor). The solution iv) is identified with the Minkowski vacuum of the theory. The solution iii) separates the 6-dimensional space-time into two asymptotic regions like a domain wall [14]. The metric for the solution with \( s = 0 \) has a logarithmic behaviour, \( H = \sum_a \ln(Q_a/|x - x_a|^2) \).

For \( s = 1 \) we have

\[
ds^2 = (H_1 H_2)^2 (-dy_0^2 + 2H_1^{-2}dy_1^2 + 2H_2^{-2}dy_2^2 + 2(dx_i)^2) + \left(\sum_a Q_a^{(1)} \cdot \frac{1}{|x - x_a^{(1)}|}\right)^2 + \left(\sum_b Q_b^{(2)} \cdot \frac{1}{|x - x_b^{(2)}|}\right)^2, \tag{37}
\]

\[
A_4 = 4\pi \sum_a Q_a^{(1)} Q_a^{(2)} \tag{40}
\]

This confirms an observation [13, 27] that extremal black holes have non-vanishing event horizon in the presence of two or more charges (electric or magnetic). There are more solutions for several scalar fields [21, 27, 28].

For \( D = 10 \) we have two solutions with \( s = 0 \).

\[ i) \quad q = 8, \quad r = 0, \quad s = 0, \quad n = 1; \]
\[ ii) \quad q = 2, \quad r = 2, \quad s = 0, \quad n = 3. \]

There is also solution with \( q = 2, \quad r = 2, \quad s = 2, \quad n = 2, \)

\[
ds^2 = (H_1 H_2)^2(-dy_0^2 + dy_1^2 + K du^2) + H_1^{-1}(dx_1^2 + dz_1^2) + H_2^{-1}(dz_2^2 + dz_3^2) + (dx_i)^2, \tag{41}
\]

\[
H_1 = 1 + \sum_a \frac{Q_a^{(1)}}{|x - x_a|^2}; \quad H_2 = 1 + \sum_b \frac{Q_b^{(2)}}{|x - x_b|^2}.
\]
\[ K = \sum_a \frac{Q_a}{|x - x_a|^2}; \quad u = y_0 + y \]

The area of this horizon is

\[ A_8 = \omega_3 \sum_a (Q_1^a Q_2^a Q_3^a)^{1/2} \]  

(42)

where \( \omega_3 = 2\pi^2 \) is the area of the unit 3-dimension sphere. In this case the different components of the same gauge field act as fields corresponding to different charges.

For \( D = 11 \) we have the following solution with \( s > 0 \).

i) \( q = 1, \ r = 2, \ s = 2, \ n = 3 \),

\[ ds^2 = (H_1 H_2 H_3)^{1/3} \left[ - (H_1 H_2 H_3)^{-1} dy_0^2 + H_1^{-1} (dz_1^2 + dz_2^2 + dz_3^2) \right] \]

\[ H_c = 1 + \sum_a \frac{Q_a^{(c)}}{|x - x_a|^2}; \quad c = 1, 2, 3. \]  

(43)

For \( H_3 = 1 \) this solution reproduces a solution found in [17]. We get non-zero area of the horizons \( x = x_a \)

\[ A_9 = \omega_3 \sum_a (Q_1^a Q_2^a Q_3^a)^{1/2} \]  

(44)

ii) \( q = 4, \ r = 2, \ s = 1, \ n = 2 \),

\[ ds^2 = (H_1 H_2)^{2/3} \left[ - (H_1 H_2)^{-1} dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + \]

\[ H_1^{-1} (dz_1^2 + dz_2^2) + H_2^{-1} (dz_3^2 + dz_4^2) + \sum_{i=1}^3 dx_i^2 \]  

(45)

where \( H_1 \) and \( H_2 \) are given by (39). This solution has been recently found in [17]. The area of the horizon \( x_a = x_b \) is equal to zero.

3.1.2 \( \alpha \neq 0 \)

Let us present some examples of solution of equation (39). For \( D = 10 \) we have \( q = 1, \ r = 3, \ alpha = \pm \sqrt{2}, \ s = 1, \ n = 2 \) and the corresponding metric has the form

\[ ds^2 = (H_1 H_2)^{1/3} \left[ - (H_1 H_2)^{-1} dy_0^2 + H_1^{2/3} (dz_1^2 + dz_2^2 + dz_3^2) + \]

\[ H_2^{2/3} (dz_4^2 + dz_5^2 + dz_6^2) + \sum_{i=1}^3 dx_i^2 \]  

(46)

where \( H_1 \) and \( H_2 \) are given by (39). This solution corresponds to IIA supergravity. The area of the horizon \( x = x_a = x_b \) is equal to zero.
4 Three block solution

Let us consider the following action with two gauge fields

\[ I = \int d^Dx \sqrt{-g} (R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2(q+1)!} e^{-\alpha \phi} F_{q+1}^2 - \frac{1}{2(d+1)!} e^{\beta \phi} G_{d+1}^2). \]  

(47)

Here \( F_{q+1} \) is a closed \( q+1 \)-differential form and \( G_{d+1} \) is a closed \( d+1 \)-differential form. There is the following solution \[ ds^2 = H_1^{-2a_1} H_2^{-2a_2} \eta_{\mu \nu} dy^\mu dy^\nu + H_1^{-2b_1} H_2^{-2b_2} dz_n^2 + H_1^{-2b_1} H_2^{-2b_2} dx_\gamma^2, \]  

(48)

where \( \eta_{\mu \nu} \) is a flat Minkowski metric, \( \mu, \nu = 0, ..., q-1; m, n = 1, 2, ..., d-q \), and \( \gamma = 1, ..., D-d \). For definitness we assume that \( D > d \geq q \).

The parameters \( a_i \) and \( b_i \) in the solution (48) are rational functions of the parameters in the action (47):

\[ a_1 = \frac{2\tilde{q}}{\alpha^2(D-2) + 2q\tilde{q}}, \quad b_1 = -\frac{q}{\tilde{q}} a_1, \]  

(49)

\[ a_2 = \frac{\alpha^2(D-2)}{\alpha^2d(D-2) + 2dq^2}, \quad b_2 = -\frac{d}{\tilde{d}} a_2, \]  

(50)

where

\[ \tilde{d} = D - d - 2, \quad \tilde{q} = D - q - 2. \]  

(51)

The solution (48) is valid only if the following relation between parameters in the action is satisfied

\[ \alpha \beta = \frac{2q\tilde{d}}{D-2}. \]  

(52)

There are two arbitrary harmonic functions \( H_1 \) and \( H_2 \) of variables \( x^\gamma \) in (48),

\[ \Delta H_1 = 0, \quad \Delta H_2 = 0. \]  

(53)

Non-vanishing components of the differential form are given by

\[ A_{\mu_1...\mu_q} = h^{\epsilon_{\mu_1...\mu_q}} H_1^{-1}, \quad F = dA, \]  

(54)

\[ B_{I_1...I_d} = k^{\epsilon_{I_1...I_d}} H_2^{-1}, \quad G = dB. \]  

(55)

Here \( I = 0, ...d-1, \epsilon_{123...q} = 1, \epsilon_{123...d} = 1 \) and \( h \) and \( k \) are given by the formulae

\[ h^2 = \frac{4(D-2)}{\alpha^2(D-2) + 2q\tilde{q}}, \]  

(56)

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The dilaton field is
\[ \phi = \frac{1}{2} \beta k^2 \ln H_2 - \frac{1}{2} \alpha h^2 \ln H_1. \] (58)

We have obtained [25] the solution (48) by reducing the Einstein equations to the system of algebraic equations. To this end we have assumed the "no-force" condition, that is an analog of relations (14). This condition gives a linear dependence between functions in the Ansatz (54), (55). To satisfy a nonlinear relation that follows from the Einstein equations we have to assume the relation (52).

4.1 Dual Action

Let us consider the following "dual" action
\[ \tilde{I} = \int d^Dx \sqrt{-g} (R - \frac{1}{2} (\nabla \phi)^2 - e^{-\alpha \phi} F_{q+1}^2 - \frac{e^{\beta \phi}}{2(q+1)!} G_{s+1}^2), \] (59)

where \( G_{s+1} \) is a closed \( s + 1 \)-differential form. If \( s \) is related to \( d \) by
\[ s = D - d - 2, \quad \text{i.e.} \quad s = \tilde{d} \] (60)

and
\[ \tilde{\beta} = -\beta \] (61)

then the solution for the metric (48) with the differential form \( F \) (54) and the dilaton (58) is valid also for the action (59). An expression for the antisymmetric field \( G \) will be different, namely
\[ G^{\alpha_1 \ldots \alpha_{\tilde{d}+1}} = k H_1^{\sigma_1} H_2^{\sigma_2} e^{\alpha_1 \ldots \alpha_{\tilde{d}+1} \beta} \partial_\beta H_2^{-1}. \] (62)

Here \( \epsilon_{123 \ldots \tilde{d}+2} = 1 \) and
\[ \sigma_1 = \frac{\alpha \beta h^2}{2} (1 - \frac{1}{s}), \quad \sigma_2 = \frac{\beta k^2}{2} (\frac{1}{s} - 1) \] (63)

Equations of motion for the case of one form corresponds to equation of motion for ansatz (7), (54) and (62) for the dual action (59) when \( \alpha = \beta \) and \( q = s \). This three-block p-branes solution for the Lagrangian with one differential form for various dimensions of the space-time was found in [23]. It contains previously known D=10 case [17, 8].

Note that the metric (48) describes also the solution for the action with the form \( F_{q+1} \) replaced by its dual \( \tilde{F}_{\tilde{q}+1} \) with \( \tilde{q} + q + 2 = D \) and \( \alpha \rightarrow \tilde{\alpha} = -\alpha \). One can also change two forms \( F \) and \( G \) to their dual version without changing the metric (48).
Below we consider equations of motion for the action (47). The energy-momentum tensor is for the theory with the action (47) has the form

\[
T_{MN} = \frac{1}{2}(\partial_M \phi \partial_N \phi - g_{MN}(\partial \phi)^2)
\]

\[
+ \frac{e^{-\alpha \phi}}{2q!} (F_{M_1...M_q} F^M_1...M_q - \frac{g_{MN}}{2(q+1)} F^2)
\]

\[
+ \frac{e^{-\alpha \phi}}{2s!} (G_{M_1...M_s} G^M_1...M_s - \frac{g_{MN}}{2(s+2)} G^2)
\]

The equation of motion for the antisymmetric fields are

\[
\partial_M (\sqrt{-g}e^{-\alpha \phi} F^{M_1...M_q}) = 0,
\]

\[
(64)
\]

and one has the Bianchi identity

\[
\epsilon^{M_1...M_{q+2}} \partial_{M_1} F_{M_2...M_{q+2}} = 0.
\]

\[
(66)
\]

The equation of motion for the dilaton is

\[
\partial_M (\sqrt{-g}g^{MN} \partial_N \phi) + \frac{\alpha}{2(q+1)!} \sqrt{-g}e^{-\alpha \phi} F^2 + \frac{\beta}{2(s+1)!} \sqrt{-g}e^{-\beta \phi} G^2 = 0.
\]

\[
(68)
\]

We shall solve equations (6), (64)-(68) by using the following Ansatz for the metric

\[
ds^2 = e^{2A(x)} \eta_{\mu\nu} dy^\mu dy^\nu + e^{2F(x)} (dz^\gamma)^2 + e^{2B(x)} (dx^\gamma)^2,
\]

\[
(69)
\]

where \(\mu, \nu = 0,...,q-1\), \(\eta_{\mu\nu}\) is a flat Minkowski metric, \(n=1,2,...,r\) and \(\gamma =1,...,s+2\). Here \(A\), \(B\) and \(C\) are functions on \(x\). Non-vanishing components of the differential forms are

\[
A_{\mu_1...\mu_q} = h \epsilon_{\mu_1...\mu_q} e^{C(x)}, \quad F = dA
\]

\[
(70)
\]

\[
G^{$\alpha_1$...$\alpha_{s+1}$} = \frac{1}{\sqrt{-g}} k e^{\beta \phi} \epsilon^{\alpha_1...\alpha_{s+1}\gamma} \partial_\gamma e^\chi,
\]

\[
(71)
\]

where \(h\) and \(k\) are constants. \((\mu\nu)\)-components of the energy-momentum tensor for this ansatz have the form

\[
T_{\mu\nu} = \eta_{\mu\nu} e^{2(A-B)} \left[ -\frac{1}{4}(\partial \phi)^2 - \frac{h^2}{4} (\partial C)^2 e^{-\alpha \phi - 2qA + 2C} - \frac{k^2}{4} (\partial \chi)^2 e^{2sB + \beta \phi + 2\chi} \right],
\]

\[
12
\]
$(nm)$-components are:

$$T_{mn} = \delta_{mn} e^{2(F-B)} \left[ -\frac{1}{4} (\partial \phi)^2 + \frac{\hbar^2}{4} (\partial C)^2 e^{-\alpha \phi - 2qA + 2C} - \frac{k^2}{4} (\partial \chi)^2 e^{2s B + \beta \phi + 2\chi} \right],$$

and $(\alpha \beta)$-components:

$$T_{\alpha \beta} = \frac{1}{2} [\partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} \delta_{\alpha \beta} (\partial \phi)^2 - \frac{\hbar^2}{2} e^{-\alpha \phi - 2qA + 2C} [\partial_\alpha C \partial_\beta C - \delta_{\alpha \beta} (\partial C)^2] - \frac{k^2}{2} e^{2s B + \beta \phi + 2\chi} [\partial_\alpha \chi \partial_\beta \chi - \frac{\delta_{\alpha \beta}}{2} (\partial \chi)^2],$$

where we use notations $(\partial A \partial B) = \partial_\alpha A \partial_\beta B$ and $D = q + r + s + 2 = d + s + 2$.

The equations of motion (65) for a part of components of the antisymmetric field are identically satisfied and for the other part they are reduced to a simple equation:

$$\partial_\alpha (e^{-\alpha \phi - 2qA + C} \partial_\alpha C) = 0. \quad (72)$$

For $\alpha$-components of the antisymmetric field we also have the Bianchi identity:

$$\partial_\alpha (e^{\alpha \phi + 2Bq + \chi} \partial_\alpha \chi) = 0 \quad (73).$$

The equation of motion for the dilaton has the form

$$\partial_\alpha (e^{qA + sB + Fr} \partial_\alpha \phi) + \frac{\beta k^2}{2} e^{\beta \phi + 2sB + 2\chi (\partial_\alpha \chi)^2} - \frac{\alpha h^2}{2} e^{-\alpha \phi - qA + qB + r F + 2C} (\partial_\alpha C)^2 = 0. \quad (74)$$

In order to get rid of exponents in the above expressions for the energy-momentum tensor we impose the following relations:

$$2\chi + 2sB + \beta \phi = 0, \quad (75)$$

$$2C - 2qA - \alpha \phi = 0. \quad (76)$$

and we also have

$$qA + rF + sB = 0. \quad (77)$$

Then the tensor $G_{MN}$ will have the form

$$G_{\mu \nu} = \eta_{\mu \nu} e^{2(A-B)} [-uh^2 (\partial C)^2 - vk^2 (\partial \chi)^2],$$

$$G_{\mu \nu} = \delta_{\mu \nu} e^{2(F-B)} [th^2 (\partial C)^2 - vk^2 (\partial \chi)^2]$$
\[ G_{\alpha\beta} = \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi - \frac{h^2}{2} \partial_\alpha C \partial_\beta C - \frac{k^2}{2} \partial_\alpha \chi \partial_\beta \chi + \delta_{\alpha\beta} \left[ th^2(\partial C)^2 + wk^2(\partial \chi)^2 \right] \]

Here

\[ u = \frac{1}{2} \left( 1 - \frac{q}{D - 2} \right), \quad v = \frac{1}{2} - \frac{q + r}{2(D - 2)} \] (78)

\[ t = \frac{q}{2(D - 2)}, \quad w = \frac{q + r}{2(D - 2)} \] (79)

The Einstein equations (8) under the conditions (75), (76) and (77) are crucially simplified and take the form

\[ \Delta A = uh^2(\partial c)^2 + vk^2(\partial \chi)^2 \] (80)

\[ \Delta F = -th^2(\partial c)^2 + vk^2(\partial \chi)^2 \] (81)

\[ -q \partial_\alpha A \partial_\beta A - \sum_i r_i \partial_\alpha F_i \partial_\beta F_i + s \partial_\alpha B \partial_\beta B - \delta_{\alpha\beta} \Delta B = -\frac{h^2}{2} \left[ \partial_\alpha C \partial_\beta C - 2t \delta_{\alpha\beta} (\partial C)^2 \right] - \frac{k^2}{2} \left[ \partial_\alpha \chi \partial_\beta \chi - 2w \delta_{\alpha\beta} (\partial \chi)^2 \right] + \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi, \] (82)

Now equations (72), (73) and (74) will have the following forms, respectively,

\[ \partial_\alpha (e^{-C} \partial_\alpha C) = 0, \quad \partial_\alpha (e^{-\chi} \partial_\alpha \chi) = 0, \] (83)

\[ \Delta \phi + \frac{\beta k^2}{2} (\partial_\alpha \chi)^2 - \frac{\alpha h^2}{2} (\partial_\alpha C)^2 = 0. \] (84)

One rewrites (83) as

\[ \Delta C = (\partial C)^2, \quad \Delta \chi = (\partial \chi)^2. \] (85)

Therefore (84) will have the form

\[ \Delta \phi + \frac{\beta k^2}{2} \Delta \chi - \frac{\alpha h^2}{2} \Delta C = 0. \] (86)

We solve (80), (81) and the \( \delta_{\alpha\beta} \) part of (82) as

\[ \phi = \frac{\alpha h^2}{2} C - \frac{\beta k^2}{2} \chi \] (87)

\[ A = uh^2 C + vk^2 \chi, \] (88)

\[ F = -th^2 C + vk^2 \chi, \] (89)

\[ B = -th^2 C - wk^2 \chi. \] (90)
Substituting these expressions into (75) we get a relation on \( \alpha \) and \( \beta \)

\[
\alpha \beta = \frac{2qs}{q + r + s} \quad (91)
\]

and an expression for \( h \)

\[
h = \pm \sqrt{\frac{4(q + r + s)}{\alpha^2(q + r + s) + 2q(s + r)}}, \quad (92)
\]

Substituting these expressions into (76) we get the same relation on \( \alpha \) and \( \beta \) as before as well as

\[
k = \pm \frac{2\alpha(q + r + s)}{\sqrt{s[2\alpha^2(q + r)(q + r + s) + 4q^2s]}} \quad (93)
\]

By straightforward calculations one can check that the non-diagonal part of the Einstein equation is also satisfied under above conditions.

To summarize, the action (47) has the solution of the form (48) expressed in terms of two harmonic functions \( H_1 \) and \( H_2 \) if the parameters in the action are related by (91) and the parameters in the Ansatz \( h \) and \( k \) are given by (92), (93).

### 4.2 Examples

There is the relation (11) between parameters \( \alpha \) and \( \beta \) in the action (47). As a result the action corresponds to the bosonic part of a supergravity theory only in some dimensions.

If \( D = 4 \) and \( q = d = 1 \) then one can take \( \alpha = \beta = 1 \) and the action corresponds to the \( SO(4) \) version of \( N = 4 \) supergravity. The solution (7) takes the form

\[
ds^2 = -H_1^{-1}H_2^{-1}dy_0^2 + H_1H_2dx^\gamma dx^\gamma
\]

This supersymmetric solution has been obtained in [13].

If \( \alpha = \beta \) and \( q = d \) then one has the solution

\[
H_2^{-\frac{2}{3}}dz^mdz^m + dx^\gamma dx^\gamma], \quad (95)
\]

This solution was obtained in [23]. It contains as a particular case for \( d = 10, \ q = 2 \) the known solution [17, 8]

\[
ds^2 = H_1^{-\frac{2}{5}}H_2^{-\frac{1}{5}}(-dy_0^2 + dy_1^2) +
(H_1^{-\frac{1}{3}}H_2^{-\frac{1}{5}}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) +
H_1^{\frac{1}{3}}H_2^{-\frac{3}{5}}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).
\]

This solution has been used in the D-brane derivation of the black hole entropy [7, 8]. Note however that the solution (96) corresponds to the action (47) with the 3-form \( F_3 \) and the 7-form \( G_7 \).
5 Conclusion

In conclusion, we have described multi-block p-brane solutions for high dimensional gravity interacting with matter. We have assumed the "electric" ansatz for the field and used "no-force" conditions for local fields \cite{14} together with the harmonic gauge condition \cite{9} to reduce the system of differential equations to a system of non-linear algebraic equations. The found solutions support a picture in which an extremal p-brane can be viewed as a composite of ‘constituent’ branes, each of the latter possessing a corresponding charge.

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References

[1] A.Font, L.Ibanez, D.Lust and F.Quevedo, Phys.Let.B249(1990)35
[2] A.Sen, Int.J.Mod.Phys.A8(1993)2023
[3] C.Hull and P.Townsend, Nucl.Phys. B438 (1995) 109
[4] E.Witten, Nucl.Phys.B443(1995)85
[5] J.H.Schwarz, hep-th/9607201
[6] C.Vafa, hep-th/9602022
[7] A.Strominger and C.Vafa, hep-th/9601029
[8] C.Callan and J.Maldacena, hep-th/9602043
[9] J.Polchinski, hep-th/9511026
[10] A.Dubholkar, G.W.Gibbons, J.A.Harvey, F. Ruiz Ruiz, Nucl.Phys. 340 (1990) 33.
[11] M.J.Duff and K.S.Stelle, Phys. Lett. B 253 (1991) 113.
[12] R.Guven, Phys. Lett. 276 (1992) 49; Phys. Lett. 212 (1988) 277.
[13] R.Kallosh, A.Linde, T.Ortin, A.Peet and A. van Proeyen, Phys. Rev. D46 (1992) 5278.
[14] H.Lu, C.N.Pope, E.Sergin and K.Stelle, hep-th/9508042
[15] M.J.Duff, R.R.Khuri and J.X.Lu, Phys.Rep. 259 (1995) 213.
[16] M.Cvetic and D.Youm, hep-th/9507090 hep-th/951098
[17] A.A.Tseytlin, hep-th/9601177, hep-th/9604033, hep-th/9609212
[18] I.R.Klebanov and A.A.Tseytlin, hep-th/9604166
[19] G.Papadopoulos and P.K.Townsend, hep-th/9603087
[20] K.Behrndt, E.Bergshoeff and B.Janssen, hep-th/9604168
[21] G.Clément and D.V.Galtsov, hep-th/9607043
[22] E.Bergshoeff, R.Kallosh, T.Ortin, hep-th/9605059
[23] A.Volovich, hep-th/9608095
[24] E.Bergshoeff, M.De Roo and S.Panda, hep-th/9609056
[25] I.Ya.Aref’eva, K.S.Viswanathan and I.V.Volovich, hep-th/9609223
[26] R. Emparan, hep-th/9610170.
[27] C.G.Callan, Jr., S.S.Gubser, I.R.Klebanov and A.A.Tseytlin, hep-th/9610172
[28] K.Behrndt, G.L.Cardoso, B.de Wit, R.Kallosh, D.Lust and T.Mohaupt, hep-th/9610103
[29] I.Ya. Aref’eva and A.Volovich, hep-th/9611020
[30] V.D.Ivashchuk and V.N.Melnikov, hep-th/9612085
[31] I.Ya. Aref’eva and O.A.Rytchkov, hep-th/9612236