Spontaneous Magnetization of Solid Quark-cluster Stars

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Pulsar-like compact stars have strong magnetic fields, with the strength from $\sim 10^9$ to $\sim 10^{12}$ Gauss. How can such strong magnetic fields be generated and maintained is still an unsolved problem, and in principle it is related to the interior structure of compact stars. In this paper we propose that, under the Coulomb repulsion, the electrons in solid quark-cluster stars could spontaneously magnetized, and hence electrons could contribute non-zero net magnetic momentum to the whole star. We find that, for most cases in solid quark-cluster stars, the amount of net magnetic momentum, which is proportional to spin polarization $\xi = (n_+ - n_-)/n_e$ and depends on the number density of electrons $n_e = n_+ + n_-$, could be significant with non-zero $\xi$, and the corresponding magnetic moment per unit mass could be higher than $10^{-4}$ Gauss cm$^3$ g$^{-1}$. Therefore the net magnetic moments of electron system in solid quark-cluster stars could be large enough to induce the observed magnetic fields for pulsars.

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I. INTRODUCTION

The states of matter of pulsar-like compact stars is a long-standing problem, although the discovery of pulsars dates back to nearly half a century ago. Some efforts have been made to understand the nature of pulsars, among which the model of quark-cluster stars has been proposed. With a stiff equation of state, the model of quark-cluster stars suggests the exist of high mass ($> 2M_\odot$) pulsars to be favored by the discoveries of massive pulsars \cite{2,3}. For traditional models of neutron stars, however, there are two problems to be solved: the so called “hyperon puzzle” and the quark-deconfinement \cite{4}. It is worth noting out that no such kind of embarrassment exists in quark-cluster star model since quark-clusters would be hadron-like. Composed of clustered quarks and solidified at low enough temperatures, the solid quark-cluster stars could have stiff equation of states naturally, and consequently they could have high masses. In addition, glitch phenomenon, including the energy releases during glitches, could be understood in the solid quark-cluster star model \cite{5}. In spite of those successes, what about the strong magnetic field of a solid quark-cluster star?

In fact, the origin of strong magnetic fields of pulsar-like compact stars is also a long-standing problem. The simplest and most popular hypothesis is that, the conservation of magnetic flux resulting from the frozen of magnetic filed to the star’s surface magnify the strength of magnetic field by some orders of magnitude. However, the fossil fields could not be adequate because only a very small fraction of the progenitors have magnetic fields high enough to produce significant fossil fields for pulsars. Therefore, the fossil hypothesis is not a convincing solution generally accepted.

The dissipation of inherited magnetic fields should also be taken into account. The time-scale of diffuse of magnetic field for a typical pulsar could be estimated as

$$\tau = \frac{L^2}{\eta_m} \sim 10^5 \text{ yr} \cdot \left(\frac{R}{10 \text{ km}}\right)^2 \left(\frac{\sigma}{10^8 \text{ cm}^2 \text{ s}^{-1}}\right),$$

(1)

where $L$ is the length-scale of the star, $\eta_m$ is the magnetic viscosity coefficient, $R$ is the radius of the star, and $\sigma$ is the conductivity. Typically for compact stars, the time-scale $\tau$ might not be sufficiently large, which means that the strong magnetic fields inherited from progenitors would not maintain for a long enough time.

If the magnetic field is intrinsically originated via symmetry broken spontaneously, then the dissipation process would not be a problem. The magnetic moment of electrons is much larger than that of nuclei, so electrons could significantly contribute magnetic moment to the whole star. If we take electrons as the possible way giving rise to the macroscopic magnetism, then it seems to be similar to the case of ferromagnetism of normal material.

The ferromagnetism of normal material is studied extensively in condensed matter physics. The origin of ferromagnetism is the correlation between electrons under Coulomb interaction. For quark-cluster stars, the situation is in fact more simple because electrons are not confined to “nuclei” (i.e. the quark-clusters), and all of the electrons are itinerant, and a quark-cluster star could serve as an ideal system where the itinerant ferromagnetism might occur under the Stoner model \cite{7}. The physical picture of ferromagnetism in repulsive Fermi gases can be understood as the result of the competition between the repulsive interaction and the Pauli exclusion principle. Electrons tend to have unbalanced spins to save interaction energy, and on the other hand they tend to have balanced spins to save kinetic energy.

Certainly the state of compact stars is different from normal matter, although we have not enough knowledge about the former. In solid quark-cluster stars, the quarks are localized in clusters by strong interaction, so they
would not make contribution to the total magnetic moment although they are Fermions. Although both of the origin of magnetic fields of compact stars and the nature of compact stars are uncertain to us, we propose here a possible way to solve the former, and hope it could give us some hints to the latter. If the electrons in solid quark-cluster star could spontaneously magnetization and give rise to enough strong magnetic fields, then we can give some constraints about the electric charge per baryon $Y_e$, which could reflect some properties of strong interactions.

Defining the amount of polarization

$$\xi = \frac{n_+ - n_-}{n_+ + n_-} = \frac{n_+ - n_-}{n_e},$$

where $n_+$ and $n_-$ denote the number density of spin-up and spin-down electrons, respectively. We find that, in solid quark-cluster stars, $\xi$ could be non-zero which means the non-zero net macroscopic magnetic moment. In some cases, the corresponding magnetic moment per unit mass $\mu_0$ could be higher than $10^{-4}$ Gauss cm$^3$ g$^{-1}$, which is large enough to induce the observed magnetic fields of pulsars.

This paper is arranged as follows: In §II we will show the basic properties of quark-clusters and electrons in solid quark-cluster stars. Based on the solid quark-cluster star model, in §III we will demonstrate the spontaneous magnetization of electrons in solid quark-cluster stars, using a toy model and then a more quantitative model, and show that the induced magnetic fields could be high enough for pulsars. Conclusions and discussions are made in §IV.

II. QUARK-CLUSTERS AND ELECTRONS IN SOLID QUARK-CLUSTER STARS

The average baryon number density $n_0$ of a pulsar-like compact star is about $3n_0$, where $n_0$ is the baryon number density of saturated nuclear matter. We have proposed that pulsar-like compact stars could be quark-cluster stars [1, 3], because the strongly interacting quarks could be grouped into quark-clusters [10, 11]. At low enough temperatures, quark-clusters could crystallize into solid state, just like the phase transition of normal matter from liquid state to solid state. The number of quarks in each quark-cluster, $N_q$, is seem as a free parameter which is related to the interaction. Based on the symmetric consideration, the most possible value of $N_q$ is 18, which means that quark-clusters are singlets of spins, flavors and colors. Certainly the choice of $N_q$ could have many possibilities, e.g. the $H$-cluster stars, composed of $H$-dibaryons (an $H$-dibaryon is the bound state of two $A$-particles), was proposed as a kind of quark-cluster stars. [12], in which case $N_q = 6$. The constrains of $N_q$ by the maximum mass of pulsars was also studied [13], and we find that the $2M_\odot$ pulsar PSR 1614-2230 [2] infers $N_q \lesssim 10^3$.

A quark star could be considered as a gigantic nucleus, with electrons inside, but changing from two-flavor ($u$, $d$) to three-flavor ($u$, $d$ and $s$) symmetries [10]. The $H$-cluster star model was proposed based on three-flavor symmetries [12]. If the star is composed of equal numbers of $u$, $d$ and $s$ quarks, no electrons will exist. In this case, the three-flavor symmetry may result in a ground state of matter, as Bodmor-Witten conjecture said [14, 15]. The mass difference between $u$, $d$ and $s$ quarks would break the symmetry, but on the other hand, the interaction between quarks would lower the effect of mass difference and try to restore the symmetry. The amount of symmetry breaking was estimated based on perturbative calculations [10], which found the electric charge per baryon $n_e/n_b = Y_e$ is smaller than $10^{-4}$. Although it is difficult for us to calculate how strong the interaction between quarks is, the non-perturbative nature and the energy scale of the system make it reasonable to assume that the degree of the light flavor symmetry breaking is small. In the following calculations, we assume $Y_e$ is ranged from $10^{-6}$ to $10^{-4}$.

With small $Y_e$, most of quark-clusters are electric neutral, but the quarks inside them have electric charge. Electrons among lattices could feel the electric charge of individual quarks, so the interaction between electrons could be screened by quarks, unless there is no lattice between them, i.e. their distance is smaller than $d$. Therefore in §III we take the scattering length $a$ of electrons to be the lattice length $d = n_L^{-1/3} \sim 2$ fm $(n_b/3n_0)^{-1/3}(N_q/18)^{1/3}$, where $n_L$ is the number density of lattices (i.e. quark-clusters).

Let us then show why the electrons in quark-cluster stars are completely itinerant. The Fermi momentum of electrons with number density $n_e$ is

$$p_F = (3\pi^2 n_e)^{1/3} \hbar c \simeq \frac{10 \text{ MeV}}{c} \cdot \left(\frac{Y_e}{10^{-5}}\right)^{1/3} \left(\frac{n_b}{3n_0}\right)^{1/3},$$

so electrons are relativistic, with Fermi energy of about 10 MeV. The Coulomb attraction by quark-clusters on electrons is

$$E_e = \alpha \cdot n_e^2 \simeq 10^{-2} \text{ MeV} \cdot \left(\frac{Y_e}{10^{-5}}\right)^{2/3} \left(\frac{n_b}{3n_0}\right)^{1/3},$$

($\alpha = 1/137$ is the fine structure constant) which means the Coulomb attraction cannot bound electrons because of the kinetic energy is much larger than the binding energy, and all the the electrons are itinerant. This is quite different from the electrons in normal solid.

In summary, the differences from the electrons in solid quark-cluster stars and normal solid are at least the following two aspects: (1) the electrons are relativistic in solid quark-cluster stars and non-relativistic in normal solid; and (2) in solid quark-cluster stars all of the electrons are nearly freely moving around rather than bond by the lattices.
III. SPONTANEOUS MAGNETIZATION OF ELECTRONS

In solid quark-cluster stars, electrons repulse each other because of Coulomb interaction. Electrons tend to have unbalanced spins to save interaction energy, and they also tend to have balanced spins to save kinetic energy. If the amount of polarization $\xi \neq 0$, there are net macroscopic magnetic moments. The star could be composed of many magnetic domains, which has net macroscopic magnetic moments due to the unbalanced spins. Applying an external magnetic field (e.g. a fossil field), the directions of magnetic domains will tend to align, and the maximum magnetic field will achieve when all of the magnetic domains align perfectly.

In the following, we will first show a toy model which qualitatively give the amount of polarization which could induce the magnetic fields of pulsars, and then we will show a more quantitative way to demonstrate this.

A. A toy model

A pulsar with a dipole magnetic field $B \sim 10^{12}$ Gauss and radius $R \sim 10$ km has the magnetic dipole moment $\sim 5 \times 10^{39}$ Gauss cm$^3$. If the baryon density $n_b = 3n_0$ and the electric charge per baryon $Y_e = 10^{-5}$, the total number of electrons is about $10^{52}$. We then try to estimate if the the polarization of electrons could account for such magnetic moment. Because the electrons with unbalanced spins will save interaction energy, the electrons on the Fermi surface, within the momentum depth of $E_c/c$, will tend to have the same spin. From Eq. 3 and 4, the total number of such electrons is

$$\frac{4\pi p_F^2 E_c}{3} \sim 10^{39},$$

which is just the number of unbalanced electrons, and we can see that the amount of polarization $\xi \sim 10^{49}/10^{52} \sim 10^{-3}$. The star will have a maximum magnetic dipole moment $\mu_d$ if all of the magnetic moments of $10^{49}$ electrons align perfectly

$$\mu_d \simeq \frac{4\pi}{3} \mu_c n_e \xi R^3 \sim 2 \times 10^{29} \text{ Gauss cm}^3,$$

where $\mu_c = 9 \times 10^{-21}$ Gauss cm$^3$ is the Bohr magneton of electrons. Therefore the qualitative estimation convinces us that the spontaneous magnetization of electrons could account for the origin of strong magnetic fields of pulsars.

B. Spin-alignment of electrons

Coulomb repulsion is responsible to the spontaneous magnetization of electrons, as indicated above. Now we will demonstrate this in a more quantitative way. In fact, it is generally believed in condensed matter physics that, a dilute Fermi gas with repulsive interactions can undergo a ferromagnetic phase transition to a spin-polarized state, which will happen when the number density of electrons reaches a critical value. To show this, the Coulomb interaction between two electrons is simplified as

$$v_{ij} = C \delta(\vec{r}_i - \vec{r}_j),$$

where $C$ is related to the scattering length $a$ and the mass of electrons $m$ via

$$C = \frac{4\pi a}{m}.$$  

Although the above relation between $C$ an $a$ is come from low energy scattering of electrons, we assume that it can be extrapolate to high energy case.

It should be noted that the condition for simplifying Coulomb interaction to Delta function is that $k_F \cdot a \ll 1$, where $k_F$ is the wave number of particles at Fermi level, whereas in condensed matter physics, where the electrons are treated as degenerate non-relativistic Fermions, the condition for ferromagnetism is $k_F \cdot a \gg \pi/2$ [17]. That is to say, the model constructed based on Eq. 7 can only serve as an instructive way to show how the repulsion between Fermions to enhance the spin alignment.

The electrons in a solid quark-cluster star, however, could be seen as a really dilute Fermi gas, because $n_e$ is much smaller than $n_b$. We make the approximation that, due to the screening effect of quarks inside quark-clusters, the scattering length $a$ is equal to the lattice length, i.e. the distance between two neighboring quark-clusters, as discussed in §II. Then the condition making the model valid, $k_F \cdot a \ll 1$, is satisfied.

The corresponding interaction energy for electron system composed of $N$ electrons is

$$E_{int} = \left(\sum_{i<j} v_{ij}\phi \right) = \frac{4\pi a}{m} \frac{N_+ N_-}{V},$$

where $\phi$ is the wave function of electron system, $N_+$ and $N_-$ are the total number of spin-up and spin-down electrons respectively, $N = N_+ + N_-$, and $V$ is the volume of the system. The total energy density of relativistic electrons, taking into account the kinetic energy and interaction energy, is then

$$\epsilon = \frac{3}{8} (3\pi^2)^{\frac{1}{3}} n_e^{\frac{1}{3}} \left[ (1 + \xi)^{\frac{1}{3}} + (1 - \xi)^{\frac{1}{3}} \right] + \frac{\pi a}{m} n_e^2 (1 - \xi^2),$$

where $a = n_L^{-1/3}$, i.e. the lattice length. The condition for ferromagnetism is that a value of $\xi$ in the range $[0, 1]$ minimize $\epsilon$. Inserting the definition of $N_q$ and $Y_e$, this condition is

$$N_q \cdot Y_e^2 > \frac{m^3}{24 \pi^3 n_b} = 4.8 \times 10^{-10} \left( \frac{n_b}{3 n_0} \right)^{-1},$$

which means that larger $N_q$ or larger $Y_e$ would favor ferromagnetism. With non-zero $\xi$ that minimize the energy
density in Eq. [10] the electron system will have non-zero macroscopic magnetic moment under an external field.

To derive the total magnetic moment of the electron system, we can define the magnetic moment per unit mass \( \mu_0 \), so the total magnetic moment \( \mu \) of the a star with mass \( M \) is \( \mu = \mu_0 M \), or

\[
\mu = 3 \times 10^{29} \text{ Gauss cm}^3 \text{ g}^{-1} \left( \frac{\mu_0}{10^{-4} \text{ Gauss cm}^3 \text{ g}^{-1}} \right) \left( \frac{M}{1.5 M_\odot} \right) \tag{12}
\]

Then we define the magnetic moment per unit mass \( \mu_0 \), when all the magnetic moments of electrons point to the same direction, so \( \mu_0 \) is proportional to \( \xi \) that minimize the energy density \( \epsilon \) in Eq. [10]. Like normal ferromagnetic material, the ferromagnetic quark-cluster stars should be composed of magnetic domains. The direction of each domain would not be completely the same with each other, so the real magnetic moment per unit mass of a star \( \tilde{\mu}_0 \) would be smaller than \( \mu_0 \). The relation between \( \tilde{\mu}_0 \) and \( \mu_0 \) depends on many uncertain factors such as the configurations of magnetic fields and the shapes of domains, so it is difficult for us to derive an accurate expression. Here we assume that under an external field, the degree of alignment is sufficient, and \( \tilde{\mu}_0 \sim 0.01 - 0.1 \mu_0 \). If \( \mu_0 \sim 10^{-2} \text{ Gauss cm}^3 \text{ g}^{-1} \), then \( \tilde{\mu}_0 \gtrsim 10^{-4} \text{ Gauss cm}^3 \text{ g}^{-1} \).

Figure 1 shows the relation between \( \mu_0 \) and \( Y_e \), in three cases \( N_q = 6 \) (solid line), 18 (dashed line) and 100 (dash-dotted line), when \( n_0 = 3n_0 \). From this figure we can see that, if \( Y_e \gtrsim 10^{-5} \), the corresponding \( \mu \) could be large enough to account for the magnetic moments of pulsars, as long as \( N_q \gtrsim 6 \). Smaller values of \( Y_e \) need larger values of \( N_q \) to reduce high magnetic moments. For example, \( Y_e \simeq 6 \times 10^{-6} \) and \( N_q = 18 \) can give satisfying value of magnetic moment of a pulsar with magnetic field \( B = 10^{12} \text{ Gauss} \) and radius \( R = 10 \text{ km} \), when \( \tilde{\mu}_0 \sim 0.01 \mu_0 \), as shown in Eq. [12]. If \( N_q < 10^2 \) and \( Y_e < 2 \times 10^{-6} \), spontaneous magnetization is unlikely to happen, consistent with condition [11].

Higher baryon densities means weaker interaction and consequently larger \( Y_e \), so from this figure and the relation [11] we can see that, at higher baryon densities the condition for spontaneous magnetization would be less strict. Because the range of \( n_0 \) in pulsars is about from \( 2n_0 \) to \( 10n_0 \), the spontaneous magnetization could occur in nearly all of the region in the star, leading to a sufficiently large macroscopic magnetic moment. Therefore, the model of spontaneous magnetization of electrons we show in this paper could induce large enough magnetic moments to account for the observed magnetic fields of pulsars.

### IV. CONCLUSIONS AND DISCUSSIONS

In this paper we demonstrate that the strong magnetic fields of pulsars could be originated from the spontaneous magnetization of electrons, in solid quark-cluster star model. Due to the small values of electric charge per baryon \( Y_e \), the electrons in a solid quark-cluster star could be seen as a dilute Fermi gas, so the method used in condensed matter physics to demonstrate ferromagnetism due to Coulomb repulsion between electrons is valid in the solid quark-cluster star model.

The amount of polarization \( \xi \) depends on the number of quarks inside each quark-cluster \( N_q \) and the electric charge per baryon \( Y_e \). Larger \( N_q \) or \( Y_e \) would result in larger \( \xi \) and consequently induce larger magnetic moment per unit mass \( \mu_0 \). \( \tilde{\mu}_0 < \mu_0 \) where \( \mu_0 \) is the value of \( \tilde{\mu}_0 \) when all of the magnetic domains point to the same direction. If \( Y_e > 10^{-5} \), the corresponding \( \mu \) could be large enough to account for the magnetic moments of pulsars. Smaller values of \( Y_e \) need larger values of \( N_q \) to reduce high magnetic moments. If \( N_q < 10^2 \) and \( Y_e < 2 \times 10^{-6} \), spontaneous magnetization is unlikely to happen. Therefore the spontaneous magnetization of electrons could account for the origin of strong magnetic fields of pulsars.

In the model of quark-cluster stars, the values of \( N_q \) and \( Y_e \) depends on the properties of strong interaction at low energy scale. In our previous work, we constrained \( N_q \) by the maximum mass of pulsars, and in this paper we give constrains on \( Y_e \) based on the proposal that the strong magnetic fields of pulsars are originated from the spontaneous magnetization of electrons. Larger \( Y_e \) implies weaker interaction between quarks, since if interaction is weak, the effect of mass difference between \( u, d \) and \( s \) would become more significant and make the number of s quarks to be less. So this picture could tell us that, although the interaction between quarks are strong enough to group quarks into clusters, it may not enough to make \( Y_e \) to be smaller than about \( 10^{-6} \).

Besides magnetic properties, electrons are also important for the radiation properties of pulsars. In quark
star models, the stars may be enveloped in thin electron layers which uniformly surround the entire star. The hydrocyclotron oscillation of electron layers could explain the observed absorption features of some pulsars [19].

It should also be noted that, the properties of relativistic electrons are in fact not very certain to us. Limited by the model used in normal solid containing non-relativistic electrons, we are in lack of the knowledge about the relativistic effects on electrons moving in solid quark-cluster stars, including the scattering length and the influences of lattices on electrons. On this point of view, the model we show in this paper for relativistic electron-system is to some extent an approximation to the electrons in quark-cluster stars. The polarization of electrons could give rise the the strong magnetic fields of pulsars, but the detailed model to demonstrate this based on more reliable ground remains to be constructed.

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