GLOBAL SOLUTIONS TO THE INITIAL BOUNDARY PROBLEM OF 3-D COMPRESSIBLE NAVIER-STOKES-POISSON ON BOUNDED DOMAINS

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ABSTRACT. The initial boundary value problems for compressible Navier-Stokes-Poisson is considered on a bounded domain in \( \mathbb{R}^3 \) in this paper. The global existence of smooth solutions near a given steady state for compressible Navier-Stokes-Poisson with physical boundary conditions is established with the exponential stability. An important feature is that the steady state (except velocity) and the background profile are allowed to be of large variation.

Keywords: Global regularity near boundaries, Navier-Stokes-Poisson systems, Exponential stability, The initial boundary value problem.

AMS Subject Classifications. 35Q35, 35B40

1. INTRODUCTION

It is well-known that the compressible Navier-Stokes-Poisson (NSP) system consists of the Navier-Stokes equations coupled with the self-consistent Poisson equations, which is used in the simulation of the motion of charged particles (electrons or holes, see [29] for more explanations). In three dimensional space, the NSP system of one carrier type takes the following form

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho (u_t + (u \cdot \nabla)u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + \nabla p &= \rho \nabla \Phi, \\
\Delta \Phi &= \rho - \bar{\rho},
\end{align*}
\]

where \( \rho > 0 \), \( u = (u^1, u^2, u^3) \) and \( p \) denote the density, the velocity field of charged particles, the pressure, respectively. The self-consistent electric potential \( \Phi = \Phi(x, t) \) is coupled with the density through the Poisson equation. The pressure \( p \) is expressed by

\[
p(\rho) = \rho^\gamma,
\]

where \( \gamma \geq 1 \) is a constant. As usual, the constant viscosity coefficients \( \mu \) and \( \lambda \) should satisfy the following physical conditions

\[
\mu > 0, \quad \lambda + \frac{2}{3} \mu \geq 0.
\]

And \( \bar{\rho} = \bar{\rho}(x) > 0 \) is the background profile, the sum of the background ion density and the net density of impurities, which is assumed to be given and immobile.

The object of this paper is to investigate the global existence and long-time behavior of the solutions to the initial boundary value (IBV) problem of (1.1) in \( (t, x) \in [0, +\infty) \times \Omega \), where \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain \( \Omega \subset \mathbb{R}^3 \), with the following initial condition

\[
(\rho, u)(x, t = 0) = (\rho_0, u_0)(x),
\]

and boundary condition

\[
u |_{\partial \Omega} = 0, \quad \nabla \Phi \cdot \nu |_{\partial \Omega} = 0,
\]

where \( \nu \) is the unit outer normal to \( \partial \Omega \).

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The large-time behavior of solutions to compressible Navier-Stokes equations has been investigated extensively in [2, 3, 14, 17, 25, 26, 28]. While for the Cauchy problem (initial value problem without boundaries) of the above Navier-Stokes-Poisson system, recently the decay rate of solutions was studied, see [9, 13, 18, 21, 28] for instance and the references therein, which has been proved that the electric field plays an important role on the large time behavior of solution and the solution will approach to constant state at an algebraic decay rate. For the compressible Navier-Stokes-Poisson of self-gravitating fluids, free boundary problems are a very active research subject, for which the gravitational force plays a different role, compared with the electric forces. One may refer to [22–24] for this topic.

In the presence of physical boundaries, the regularity of solutions near boundaries is a very subtle and important issue for fluids and plasma equations. For this, the classical global existence of smooth solutions to the initial boundary value problem for 3-D compressible Navier-Stokes equations was due to Matsumura & Nishida [28] for initial date being small perturbations of constant states without convergence rate. Recently, [20] has shown that the radially symmetric solutions exist globally to compressible Navier-Stokes-Poisson equations with the large initial data on a domain exterior to a ball in any dimensional space, moreover, the global existence of smooth solution near a given constant steady state for 3-D compressible NSP equations with damping term on an exterior domain has been established with the exponential stability.

It should be noted that Guo & Strauss [6] established the asymptotic stability of the stationary solution of Euler-Poisson equations for the general doping profile in a bounded domain. For the initial boundary problem of Navier-Stokes-Poisson equations considered in this paper, the viscous term creates difficulties in the analysis in the presence of physical boundaries, compared with the inviscid case considered in [3]. Inspired by [6, 20, 28], the steady states about space variable, instead of the constant steady state, begin to be considered. Compared with [20], we prove in this paper that the solution exists globally and stabilizes exponentially to the steady state without the damping term for bounded domains.

Now we state the main result of this paper:

**Theorem 1.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^3$ and $\bar{\rho}(x) > 0$ be a smooth function on $\Omega$. Let $\bar{\rho}(x) > 0$, $\bar{u} \equiv 0$ and $\bar{\Phi}(x)$ be a smooth steady state solution of (1.1) such that $\frac{\partial \bar{\Phi}}{\partial \nu} = 0$ on $\partial \Omega$. Then there exists a constant $\delta > 0$ such that if the initial data $(\rho_0, u_0)$ satisfies

$$
\int_{\Omega} (\rho_0 - \bar{\rho}) dx = 0 \quad (1.5)
$$

and

$$
\|(\rho_0 - \bar{\rho}, u_0)\|_2^2 \leq \delta^2,
$$

then there exists a smooth global solution $(\rho, u, \Phi)(t, x)$ to the initial boundary value problem (1.1)-(1.4). Moreover, there are positive constants $C$ and $\sigma$ such that

$$
\left\{ \|(\rho - \bar{\rho}, u, \nabla(\Phi - \bar{\Phi}))\|_2^2 + \|\rho_t\|_2^2 + \|u_t\|_2^2 \right\}(t) \leq Ce^{-\sigma t}\left\{ \|(\rho - \bar{\rho}, u, \nabla(\Phi - \bar{\Phi}))\|_2^2 + \|\rho_t\|_2^2 + \|u_t\|_2^2 \right\}(0).
$$

**Remark 1.2.** An important feature of this paper is that the profile $\rho(x)$ and steady state $\bar{\rho}(x), \bar{\Phi}(x)$ are allowed to of large variation.

**Remark 1.3.** The condition (1.5) persists in time, and is the necessary condition of solvability of the Poisson equation with Neumann boundary.

The rest of this paper is organized as follows. In Section 2, some useful elliptic estimates have been recalled firstly. Secondly, a steady state $(\bar{\rho}, \bar{u}, \bar{\Phi})(x)$ of (1.1) is established appropriately, which help us to reconstruct the IBV problem for the perturbation variables $(q, u, \phi)(t, x)$. Section 3 is devoted to show that the global existence and exponential convergence to the steady state of smooth solutions. Different from the Navier-Stokes equations, the electric field $\nabla \Phi$, located at the
momentum equation, should be taken into account. The key to this is to consider the quantity
\( \nabla (\gamma \tilde{\rho}^{-2} q - \phi) \) to use the Stokes equation in Lemma 3.10, and apply Lemma 2.1 to Poisson equation
(2.2)_3 and (2.2)_5 to obtain the corresponding elliptic estimates. On the other hand, we cannot
generally designate a coordinate system over all of \( \Omega \) such that the directions are consistent with
the normal and tangential directions on the boundary \( \partial \Omega \). In order to overcome this difficult point,
we divide the estimates of the solution into two parts: over the region away from the boundary and
the near the boundary \( \partial \Omega \), see Lemmas 3.7-3.9. In particular near the boundary, the estimates are
quite involved. Using the local geodesic polar coordinates, we obtain the estimates for tangential
derivatives (Lemma 3.8), and then that for normal derivatives (Lemma 3.9).

Notations
Throughout this paper, \( C \) will be used as a generic constant independent of time \( t \).

(i) \( \frac{df}{dt} = f_t + u \cdot \nabla f \) denotes the material derivative.

(ii) \( \partial_x f = D_i f \), \( \frac{\partial^2 f}{\partial x_i \partial x_j} = D_{ij} f \), \( \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} = D_{ijk} f \), \( i, j, k = 1, 2, 3 \). Moreover,

\[ D^k f := \{ D^\alpha f \mid |\alpha| = k, k \in \mathbb{N} \} \]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is a multi-index, \( D^\alpha := \frac{\partial^{\alpha} f}{\partial x_{\alpha_1} \partial x_{\alpha_2} \partial x_{\alpha_3}} \), \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \) and \( \alpha_i \geq 0 \).

(iii) \( H^m \) is used to denote the Sobolev space with the following norm

\[ \|f\|_m \equiv \|f\|_{H^m(\Omega)} = \left( \sum_{l=0}^{m} \|D^{l} f\|^2 \right)^{1/2}, \quad \text{and} \quad \|f\| \equiv \|f\|_{L^2(\Omega)}. \]

(vi) The Einstein’s summation convention taken for \( i, j, k = 1, 2, 3 \) will be used sometimes in this paper.

2. Preliminaries and the reformulation of the problem

In this section, we first recall some estimates of elliptic equations, which will be used in the
subsequent. Then we reformulate the problem in terms of perturbations.

The classical regularity theory for the Neumann problem of elliptic equation is as follows (see [1]).

Lemma 2.1. (Neumann problem) Given an \( f \in H^k(\Omega) (k \in \mathbb{N}) \) and a \( g \in H^{k+1-1/2}(\partial \Omega) \) such that

\[ \int_{\Omega} f \, dx = \int_{\partial \Omega} g \, ds, \]

then there exists a \( v \in H^{k+2}(\Omega) \) satisfying

\[ \begin{aligned}
\Delta v &= f, & \text{in} \quad \Omega, \\
\frac{\partial v}{\partial n} &= g, & \text{on} \quad \partial \Omega,
\end{aligned} \]

and

\[ \|\nabla v\|_{k+1} \leq C \left( \|f\|_k + \|g\|_{k+1-1/2} \right). \]

A steady state \( (\tilde{\rho}(x), \tilde{u}(x), \tilde{\Phi}(x)) \) of (1.1) can be obtained as:

Proposition 2.2. Let \( \tilde{\rho}(x) > 0 \) in \( \Omega \). Then there exists a smooth steady state solution \( (\tilde{\rho}(x), 0, \tilde{\Phi}(x)) \)
to the problem (1.1) such that \( \tilde{\rho}(x) > 0 \) in \( \Omega \).
Proof. A steady state with \( u \equiv 0 \) must satisfy the following equations:
\[
\nabla p(\rho) - \rho \nabla \Phi = 0, \quad \Delta \Phi = \bar{\rho} \rho
\]
with the boundary condition \( \nabla \Phi \cdot \nu = 0 \) on \( \partial \Omega \). Hence, the proof is the same as that for the Proposition 3 in [3] and is omitted. \( \square \)

Let \( (\bar{\rho}(x), 0, \bar{\Phi}(x)) \) be a given steady state solution by Proposition 2.3, that is
\[
\nabla p(\bar{\rho}) - \bar{\rho} \nabla \bar{\Phi} = 0, \quad \Delta \bar{\Phi} = \bar{\rho} \rho
\] (2.1)

with
\[
\nabla \Phi \cdot \nu|_{\partial \Omega} = 0,
\]
and
\[
\int_{\Omega} (\bar{\rho}(x) - \bar{\rho}(x)) dx = \int_{\partial \Omega} \nabla \bar{\Phi} \cdot \nu ds = 0, \quad \bar{\rho}(x) > 0, \quad x \in \Omega.
\]

Denote the perturbation \( (q, u, \phi)(t, x) \) as
\[
\begin{align*}
q(t, x) &= \rho(t, x) - \bar{\rho}(x), \quad u(t, x) = u(t, x), \quad \phi(t, x) = \Phi(t, x) - \bar{\Phi}(x).
\end{align*}
\]
Then the initial boundary value problem for \( (q, u, \phi) \) is
\[
\begin{align*}
\begin{cases}
q_t + \rho \text{div} u + u \cdot \nabla \bar{\rho} &= f^0, \\
\rho u_t - \mu \Delta u - (\mu + \lambda) \text{div} u + \nabla (\gamma \bar{\rho}^{\gamma-1} q) - \bar{\rho} \nabla \phi - q \nabla \bar{\Phi} &= f, \\
\Delta \phi &= q,
\end{cases}
\end{align*}
\] (2.2)

where the nonlinear terms on the right-hand side are described as:
\[
f^0 = -\text{div}(qu),
\]
\[
f = -\rho (u \cdot \nabla) u + q \nabla \phi + \nabla h(q),
\]
\[
h(q) = (q + \bar{\rho})^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} q = O(q^2).
\]

The equations of (2.2) are equivalent to the following equations:
\[
L^0 \equiv \frac{dq}{dt} + \text{div}(\bar{\rho} u) = -q \text{div} u \equiv g^0,
\] (2.3)
\[
L \equiv u_t - \mu \frac{1}{\bar{\rho}} \Delta u - (\mu + \lambda) \frac{1}{\bar{\rho}^2} \nabla (\text{div} u) + \nabla (\gamma \bar{\rho}^{\gamma-2} q) - \nabla \phi = g,
\] (2.4)

where the nonlinear terms are given by
\[
g \equiv -(u \cdot \nabla) u + \mu \left( \frac{1}{q + \bar{\rho}} - \frac{1}{\bar{\rho}} \right) \Delta u + (\mu + \lambda) \left( \frac{1}{q + \bar{\rho}} - \frac{1}{\bar{\rho}} \right) \nabla \text{div} u + k(q),
\]
and
\[
k(q) = \begin{cases}
\nabla \left( \ln \rho - \ln \bar{\rho} - \bar{\rho}^{-1} \right), & \text{if } \gamma = 1, \\
\nabla \left( \frac{\gamma \rho^{\gamma-1}}{\gamma-1} - \frac{\bar{\rho}^{\gamma-1}}{\gamma-1} - \gamma \bar{\rho}^{\gamma-2} q \right), & \text{if } \gamma > 1,
\end{cases}
\]
\[
= O(1)q^2 + O(1)q \nabla q
\]
\[
= \gamma (\gamma - 2) \bar{\rho}^{\gamma-2} q \nabla q + O(q^2) \nabla q + O(q^2) \nabla \bar{\rho}.
\]
Next, we note some elliptic estimates of the elliptic system of equations for our domain. The standard estimates of the Stokes equations, we have:

\[
\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u) = \rho u_t + \nabla (\gamma \tilde{\rho}^{-1} q) - \tilde{\rho} \nabla \phi - q \nabla \tilde{\Phi} - f,
\]

\[
u|_{\partial \Omega} = 0.
\]

Applying the standard elliptic estimates and the smoothness of \(\tilde{\rho}, \tilde{\Phi}\) on \(\Omega\), we have

**Lemma 2.3.** For \(k = 0, 1, 2\), it holds

\[
\|D^{k+2} u\|^2 \leq C \left\{ \|u_t\|_k^2 + \|q u_t\|_k^2 + \|\nabla q\|_k^2 + \|\nabla \phi\|_k^2 + \|f\|_k^2 \right\}.
\]

**Proof.** Let \(U = \tilde{\rho}^{-1} u\), then (2.3) and (2.4) can be rewritten as:

\[
\text{div} U = \tilde{\rho}^{-2} \left( -\frac{dq}{dt} - 2 u \cdot \nabla \tilde{\rho} + g \right),
\]

\[
-\mu \Delta U + \nabla (\gamma \tilde{\rho}^{-2} q - \phi) = -u_t + (\mu + \lambda) \tilde{\rho}^{-1} \nabla \left( \tilde{\rho}^{-1} \left( g - \frac{dq}{dt} - \nabla \tilde{\rho} \cdot u \right) \right) + g
\]

\[
+ \mu \left( 2 \nabla \tilde{\rho}^{-1} \cdot \nabla u + \Delta \tilde{\rho}^{-1} u \right),
\]

\[
U|_{\partial \Omega} = 0.
\]

By the standard estimates of the Stokes equations, we have

\[
\|D^{k+2} U\|^2 + \|D^{k+1} (\gamma \tilde{\rho}^{-2} q - \phi)\|^2 \leq C \left\{ \left\| \frac{dq}{dt} \right\|_{k+1}^2 + \|u\|_{k+1}^2 + \|g\|_{k+1}^2 + \|u_t\|_{k}^2 + \|g\|_{k}^2 \right\},
\]

which implies (2.3) due to the smoothness of \(\tilde{\rho}\).

**3. Proof of the main result**

Theorem 1.1 will be proved in this section. The local-in-time well-posedness in the smooth norm is quite standard, following the arguments in [27], therefore to prove Theorem 1.1, it suffices to prove the following a priori estimates. For clarity, we introduce

\[
\mathcal{E}(t) = \|(q, u, \nabla \phi)\|_3^2 + \|q_t\|_2^2 + \|u_t\|_1^2,
\]

and

\[
\mathcal{D}(t) = \|(q, \nabla \phi)\|_3^2 + \|u\|_4^2 + \|q_t\|_2^2 + \|u_t\|_2^2.
\]

**Proposition 3.1.** (a priori estimates) Let \((q, u, \phi)\) be a solution to the initial boundary value problem (2.2) in time interval \(t \in [0, T]\). Then there exists positive constants \(C, \delta\), and \(\sigma\) which are independent of \(t\), such that if

\[
\sup_{0 \leq t \leq T} \mathcal{E}(t) \leq \delta^2,
\]

then there holds, for any \(t \in [0, T]\),

\[
\mathcal{E}(t) \leq C \mathcal{E}(0) e^{-\sigma t}.
\]

We will prove Proposition 3.1 in the following Lemmas. To begin with, we have the following basic energy estimate which is quite standard.
Lemma 3.2. Suppose that the conditions in Proposition 3.1 hold, there is a positive constant $C$ independent of $t$, such that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho|u|^2 + \gamma \tilde{\rho}^{-2} q^2 + |\nabla \phi|^2 \right) dx + C \left\{ \| \nabla u \|^2 + \left\| \frac{d}{dt} q \right\|^2 \right\} \leq C \delta \mathcal{D}(t). \quad (3.1)$$

Proof. Rewrite the momentum equation (2.1) as

$$\rho(u_t + u \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \text{div}u + \nabla (\gamma \tilde{\rho}^{-1} q) - \tilde{\rho} \nabla \phi - q \nabla \tilde{\Phi} = q \nabla \phi - \nabla h(q). \quad (3.2)$$

Multiplying (3.2) and (2.1) by $u$ and $\gamma \tilde{\rho}^{-2} q$ respectively, summing up them and integrating by parts with $u|_{\partial \Omega} = 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho|u|^2 + \gamma \tilde{\rho}^{-2} q^2 \right) dx + \mu \int_{\Omega} |\nabla u|^2 dx + (\mu + \lambda) \int_{\Omega} |\text{div}u|^2 dx + \int_{\Omega} \gamma \tilde{\rho}^{-2} q \cdot \nabla \tilde{\rho} dx - \int_{\Omega} q \nabla \tilde{\Phi} \cdot u dx - \int_{\Omega} \tilde{\rho} \nabla \phi \cdot u dx$$

$$= \int_{\Omega} \left\{ (q \nabla \phi - \nabla h(q)) \cdot u + \gamma \tilde{\rho}^{-2} q f^0 \right\} dx. \quad (3.3)$$

Noting that the first two terms on the second row of (3.3) can cancel each other because of equation (2.1). Moreover, integrating by parts, using equations (2.1) and (2.2), we have

$$- \int_{\Omega} \tilde{\rho} \nabla \phi \cdot u dx = \int_{\Omega} \phi \text{div}(\tilde{\rho} u) dx = - \int_{\Omega} \phi q_t dx + \int_{\Omega} f^0 \phi dx$$

$$= - \int_{\Omega} \phi \Delta \phi_t dx + \int_{\Omega} f^0 \phi dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi|^2 dx + \int_{\Omega} f^0 \phi dx.$$

Putting the above equation into (3.3) yields the basic energy estimate

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho|u|^2 + \gamma \tilde{\rho}^{-2} q^2 + |\nabla \phi|^2 \right) dx + \int_{\Omega} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) dx = \int_{\Omega} A_0(x,t) dx, \quad (3.4)$$

where

$$A_0(x,t) = (q \nabla \phi - \nabla h(q)) \cdot u + (\gamma \tilde{\rho}^{-2} q - \phi) f^0.$$

And it is clear that

$$\int_{\Omega} A_0(x,t) dx \leq C \delta \mathcal{D}(t). \quad (3.5)$$

Moreover,

$$\frac{d}{dt} q = - \rho \text{div}u - u \cdot \nabla \tilde{\rho},$$

and the Poincaré’s inequality for $u$ give

$$\left\| \frac{d}{dt} q \right\|^2 \leq C (|\text{div}u|^2 + |\nabla u|^2),$$

which together with (3.4) and (3.5) implies the desired result (3.1). \qed

The next lemma is $L^2$-estimates for $t$-derivatives of $(q, u, \nabla \phi)$. 

Lemma 3.3. Under the assumptions in Proposition 3.1, there exists a constant $C > 0$ independent
of $t$ such that
$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho |u_t|^2 + \gamma \tilde{\rho}^{-2} q_t^2 + |\nabla \phi_t|^2 \right) dx + C \left\{ \| \nabla u_t \|^2 + \left\| \left( \frac{dq}{dt} \right)_t \right\|^2 \right\} \leq C \delta \mathcal{D}(t).
$$

Proof. Notice that differentiation of the system (2.2) with respect to $t$ will keep the boundary
conditions (2.3) and (2.4). Estimating the integral for
$$
\int_{\Omega} \left\{ \partial_t (2.2)_1 \gamma \tilde{\rho}^{-2} q_t + \partial_t (2.2)_2 \cdot u_t \right\} dx = 0,
$$
and noting that
$$
\int_{\Omega} \partial_t(\rho u_t) \cdot u_t dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \frac{1}{2} \int_{\Omega} \rho_t |u_t|^2 dx,
$$
then, using the similar way as in Lemma 3.2 shows
$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho |u_t|^2 + \gamma \tilde{\rho}^{-2} q_t^2 + |\nabla \phi_t|^2 \right) dx + C \int_{\Omega} (|\nabla u_t|^2 + |\text{div} u_t|^2) dx
$$
$$
= -\frac{1}{2} \int_{\Omega} \rho_t |u_t|^2 dx + \int_{\Omega} A_1(x,t) dx
$$
where
$$
\int_{\Omega} A_1(x,t) dx = \int_{\Omega} \left\{ f_t \cdot u_t + \gamma \tilde{\rho}^{-2} f_t^0 q_t - f_t^0 \phi_t \right\} dx \leq C \delta \mathcal{D}(t),
$$
which gives the desired result (3.3) by using the following estimate
$$
\left\| \left( \frac{dq}{dt} \right)_t \right\|^2 \leq C \left( \| \text{div} u_t \|^2 + \| \nabla u_t \|^2 + \delta \mathcal{D}(t) \right).
$$

Lemma 3.4. Suppose that the conditions in Proposition 3.1 hold, there is a positive constant $C$
independent of $t$, such that
$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dx - \frac{d}{dt} \int_{\Omega} \left( \rho \nabla \phi \cdot u + qu \cdot \nabla \Phi + \gamma \tilde{\rho}^{-1} q \text{div} u \right) dx
$$
$$
+ C \left( \| q_t \|^2 + \| u_t \|^2 \right) \leq C \| \nabla u \|^2 + C \delta \mathcal{D}(t).
$$

Proof. Testing (2.2)_1 and (2.2)_2 with $\gamma \tilde{\rho}^{-2} q_t$ and $u_t$ respectively, summing up them, we obtain
$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dx + \int_{\Omega} \gamma \tilde{\rho}^{-2} q_t^2 dx + \int_{\Omega} \rho |u_t|^2 dx
$$
$$
+ \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u q_t dx + \int_{\Omega} \nabla (\gamma \tilde{\rho}^{-1} q) \cdot u_t dx
$$
$$
+ \int_{\Omega} \gamma \tilde{\rho}^{-2} u \cdot \nabla \rho q_t dx - \int_{\Omega} q \nabla \Phi \cdot u_t dx - \int_{\Omega} \rho \nabla \phi \cdot u_t dx
$$
$$
= \int_{\Omega} \left\{ \gamma \tilde{\rho}^{-2} f^0 q_t + f \cdot u_t \right\} dx.
$$
Integrating by parts gives
\[ \int_{\Omega} \nabla(\gamma \rho^{-1} q) \cdot u_t \, dx = - \int_{\Omega} \gamma \rho^{-1} q \frac{d}{dt} \rho u_t \, dx \]

\[ = - \frac{d}{dt} \int_{\Omega} \gamma \rho^{-1} q \, dx + \int_{\Omega} \gamma \rho^{-1} q_t \, dx. \tag{3.9} \]

By equation (2.1), it indicates
\[ \int_{\Omega} \gamma \rho^{-2} u \cdot \nabla \rho q_t \, dx - \int_{\Omega} q \nabla \Phi \cdot u_t \, dx = \int_{\Omega} q_t u \cdot \nabla \Phi \, dx - \int_{\Omega} q \nabla \Phi \cdot u_t \, dx \]

\[ = - \frac{d}{dt} \int_{\Omega} q \nabla \Phi \cdot u \, dx + 2 \int_{\Omega} q_t u \cdot \nabla \Phi \, dx. \]

In view of (2.2) and (2.3), one has
\[ - \int_{\Omega} \rho \nabla \phi \cdot u_t \, dx = - \frac{d}{dt} \int_{\Omega} \rho \nabla \phi \cdot u \, dx + \int_{\Omega} \rho \nabla \phi_t \cdot u \, dx \]

\[ = - \frac{d}{dt} \int_{\Omega} \rho \nabla \phi \cdot u \, dx + \int_{\Omega} \nabla \phi_t q_t \, dx - \int_{\Omega} f_0 \phi_t \, dx \]

\[ = - \frac{d}{dt} \int_{\Omega} \rho \nabla \phi \cdot u \, dx - \int_{\Omega} |\nabla \phi_t|^2 \, dx - \int_{\Omega} f_0 \phi_t \, dx. \]

Putting all the above identities into (3.8), one obtains
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |u|^2 + (\mu + \lambda)|\nabla u|^2 \right) \, dx - \frac{d}{dt} \int_{\Omega} \rho \nabla \phi \cdot u \, dx - \frac{d}{dt} \int_{\Omega} q \nabla \Phi \cdot u \, dx \]

\[ - \frac{d}{dt} \int_{\Omega} \gamma \rho^{-1} q \, dx + \int_{\Omega} \gamma \rho^{-2} q_t^2 \, dx + \int_{\Omega} \rho |u|^2 \, dx \]

\[ = \int_{\Omega} \left( |\nabla \phi_t|^2 - 2 q_t \nabla \Phi \cdot u - 2 \gamma \rho^{-1} q_t \div u \right) \, dx + \int_{\Omega} \left( \gamma \rho^{-2} f_0 q_t + f \cdot u_t + f^0 \phi_t \right) \, dx. \tag{3.10} \]

Now, the terms on the right-hand side of (3.10) will be estimated. By using the equation \( \Delta \phi_t = - \div (\rho u) \) and Poincaré’s inequality for \( u \) with \( u|_{\partial \Omega} = 0 \), we infer the following important estimate:
\[ \| \nabla \phi_t \|^2 \leq C \|\rho u\|^2 \leq C \|u\|^2 \leq C \| \nabla u \|^2. \tag{3.11} \]

Utilizing Cauchy’s inequality and Poincaré’s inequality yields
\[ \int_{\Omega} q_t \nabla \Phi \cdot u \, dx \leq \varepsilon \int_{\Omega} q_t^2 \, dx + C \int_{\Omega} |u|^2 \, dx \]

\[ \leq \varepsilon \|q_t\|^2 + C \| \nabla u \|^2 \]

and
\[ \int_{\Omega} \gamma \rho^{-1} q_t \, dx \leq \varepsilon \|q_t\|^2 + C \| \nabla u \|^2 \]

Finally, the following nonlinear term is controlled by
\[ \int_{\Omega} \left\{ \gamma \rho^{-2} f_0 q_t + f \cdot u_t + f^0 \phi_t \right\} \, dx \leq C \delta D^2(t). \]

Therefore, the proof of Lemma 3.4 is completed. \( \Box \)
Lemma 3.5. Under the assumptions in Proposition 3.4, there exists a constant $C > 0$ independent of $t$ such that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u_t|^2 + (\mu + \lambda) |\text{div} u_t|^2 \right) dx - \frac{d}{dt} \int_{\Omega} \gamma \tilde{\rho}^{-1} q_t \text{div} u_t dx + C \left( \|q_t\|^2 + \|u_t\|^2 \right) \leq C \left( \|\nabla u_t\|^2 + \|\nabla u\|^2 + \|u_t\|^2 + \|q_t\|^2 \right) + C \delta \mathcal{D}(t). \tag{3.12}
\]

Proof. Taking $\partial_t$ to \((2.3)_1, (2.3)_2,\) multiplying the resulting identities by $\gamma \tilde{\rho}^{-2} q_t$ and $u_t$ respectively, and summing up them, the following equation is arrived:
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u_t|^2 + (\mu + \lambda) |\text{div} u_t|^2 \right) dx + \int_{\Omega} \gamma \tilde{\rho}^{-2} q_t^2 dx + \int_{\Omega} \rho_t |u_t|^2 dx + \int_{\Omega} \rho_t u_t \cdot u_t dx \\
+ \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u_t q_t dx + \int_{\Omega} \nabla (\gamma \tilde{\rho}^{-1} q_t) \cdot u_t dx \\
+ \int_{\Omega} \gamma \tilde{\rho}^{-2} u_t \cdot \nabla \rho_t dx - \int_{\Omega} q_t \nabla \Phi \cdot u_t dx - \int_{\Omega} \tilde{\rho} \nabla \phi_t \cdot u_t dx \\
= \int_{\Omega} (\gamma \tilde{\rho}^{-2} f_t^0 q_t + f_t \cdot u_t) dx. \tag{3.13}
\]
The second row on the left-hand side of \((3.13)\) becomes
\[
\int_{\Omega} \left[ \gamma \tilde{\rho}^{-1} \text{div} u_t q_t + \nabla (\gamma \tilde{\rho}^{-1} q_t) \cdot u_t \right] dx = \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u_t q_t dx - \int_{\Omega} \gamma \tilde{\rho}^{-1} q_t \text{div} u_t dx \\
= -\frac{d}{dt} \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u_t q_t dx + 2 \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u_t q_t dx \tag{3.14}
\]
\[
\geq -\frac{d}{dt} \int_{\Omega} \gamma \tilde{\rho}^{-1} \text{div} u_t q_t dx - \varepsilon \|q_t\|^2 - C \|\nabla u_t\|^2.
\]
Different from that in Lemma 3.4, the third row on the left-hand side of \((3.13)\) is bounded by
\[
\int_{\Omega} \gamma \tilde{\rho}^{-2} u_t \cdot \nabla \rho_t dx - \int_{\Omega} q_t \nabla \Phi \cdot u_t dx - \int_{\Omega} \tilde{\rho} \nabla \phi_t \cdot u_t dx \\
\leq \varepsilon \left( \|q_t\|^2 + \|u_t\|^2 \right) + C \left( \|u_t\|^2 + \|\nabla \phi_t\|^2 + \|q_t\|^2 \right) \tag{3.15}
\]
\[
\leq \varepsilon \left( \|q_t\|^2 + \|u_t\|^2 \right) + C \left( \|u_t\|^2 + \|\nabla u\|^2 + \|q_t\|^2 \right),
\]
where we have used Cauchy’s inequality and the fact \((3.11)\). Consequently, combining \((3.13)-(3.13)\) yields the desired result \((3.13)\). \hfill \Box

We will use the momentum equation in view of \((2.4)\) to get the $H^1$-norm of $q$ in the following lemma.

Lemma 3.6. Under the conditions in Proposition 3.4, it holds that
\[
\|\nabla \phi\|^2 + \|q\|^2 + \|\nabla q\|^2 \leq C \left( \|u_t\|^2 + \|D^2 u\|^2 \right) + C \delta \mathcal{D}(t). \tag{3.16}
\]

Proof. The momentum equation \((2.4)\) could be rewritten as
\[
- \nabla \phi + \nabla (\gamma \tilde{\rho}^{-2} q) = -u_t + \frac{\mu}{q + \tilde{\rho}} \Delta u + \frac{\mu + \lambda}{q + \tilde{\rho}} \nabla \text{div} u - (u \cdot \nabla) u + k(q). \tag{3.17}
\]
Taking the inner product of (3.17) with $-\nabla \phi$, and using the boundary condition $\nabla \phi \cdot \nu |_{\partial \Omega} = 0$, the left-hand side becomes
\[
\int_{\Omega} |\nabla \phi|^2 dx + \gamma \int_{\Omega} \tilde{\rho}^{\gamma-2} q^2 dx
\]
due to
\[
- \int_{\Omega} \nabla (\gamma \tilde{\rho}^{\gamma-2} q) \cdot \nabla \phi dx = \int_{\Omega} \gamma \tilde{\rho}^{\gamma-2} q \Delta \phi dx = \int_{\Omega} \gamma \tilde{\rho}^{\gamma-2} q^2 dx.
\]
Therefore, we obtain
\[
\frac{1}{2} \int_{\Omega} |\nabla \phi|^2 dx + \gamma \int_{\Omega} \tilde{\rho}^{\gamma-2} q^2 dx \leq C \left( \|u_t\|^2 + \|D^2 u\|^2 \right) + C \left( \|(u \cdot \nabla) u\|^2 + \|k(q)\|^2 \right)
\]
\[
\leq C \left( \|u_t\|^2 + \|D^2 u\|^2 \right) + C \delta \mathcal{D}(t)
\]
by using (3.17) and Cauchy’s inequality. Furthermore,
\[
\|\nabla q\|^2 \leq C \left( \|u_t\|^2 + \|D^2 u\|^2 \right) + C \delta \mathcal{D}(t),
\]
by means of
\[
\nabla (\tilde{\rho}^{\gamma-2} q) = (\gamma - 2) \tilde{\rho}^{\gamma-3} \nabla \tilde{\rho} q + \tilde{\rho}^{\gamma-2} \nabla q,
\]
and
\[
\|\nabla (\tilde{\rho}^{\gamma-2} q)\|^2 \leq C \left( \|\nabla \phi\|^2 + \|u_t\|^2 + \|D^2 u\|^2 \right) + C \delta \mathcal{D}(t).
\]
Hence, we have finished the proof of this lemma. \qed

From now on, we shall separate the estimates into that away from the boundary and that near the boundary. Let $\chi_0(x)$ be any fixed $C^\infty(\Omega)$ cut-off function such that $\text{supp} \chi_0 \equiv K \subset \subset \Omega$, and $\chi_0 \equiv 1$ in $K_1 \subset \subset K$. With the help of $\chi_0(x)$, we have the estimates in the interior domain.

**Lemma 3.7.** Assume that the conditions in Proposition 3.1 hold, then for any positive $\epsilon$, it holds that
\[
\frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{\alpha-4} |Dq|^2 dx - \frac{d}{dt} \left\{ \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q \cdot \nabla \phi dx - \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla (\gamma \hat{\rho}^{\alpha-2}) \cdot \nabla q dx \right\}
\]
\[
+ C \left\{ \|\chi_0 D(\gamma \hat{\rho}^{\alpha-2} q - \phi)\|^2 + \|\chi_0 D^2 u\|^2 + \|\chi_0 D\frac{du}{dt}\|^2 \right\}
\]
\[
\leq \epsilon \|Dq\|^2 + C \left\{ \|q_t\|^2 + \|Du\|^2 + \|u_t\|^2 \right\} + C \delta \mathcal{D}(t),
\]
(3.18)
and the estimates of derivatives of high order:
\[
\frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{\alpha-4} |D^2 q|^2 dx - \frac{d}{dt} \left\{ \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} D_{ij} q D_{ij} \phi dx \right\}
\]
\[
- 2 \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} D_{i} (\gamma \hat{\rho}^{\alpha-2}) \cdot D_{ij} q D_{ij} dx - \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} D_{ij} (\gamma \hat{\rho}^{\alpha-2}) D_{ij} q dx \right\}
\]
\[
+ C \left\{ \|\chi_0 D^2 (\gamma \hat{\rho}^{\alpha-2} q - \phi)\|^2 + \|\chi_0 D^3 u\|^2 + \|\chi_0 D^2 \frac{du}{dt}\|^2 \right\}
\]
\[
\leq \epsilon \|D^2 q\|^2 + C \left\{ \|q_t\|^2 + \|D^2 u\|^2 + \|Du_t\|^2 \right\} + C \delta \mathcal{D}(t),
\]
(3.19)
Proof. Testing $\nabla \{2.2\}_1, \{2.4\}$ with $(2\mu + \lambda)\chi_0^2 \hat{\rho}^{-2} \nabla (\gamma \hat{\rho}^{-2} q - \phi)$ and $\chi_0^2 \nabla (\gamma \hat{\rho}^{-2} q - \phi)$ respectively, then integrating over $\Omega$, one obtains

\[
\begin{align*}
(2\mu + \lambda) \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q_t \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx &+ \int_{\Omega} \chi_0^2 |\nabla (\gamma \hat{\rho}^{-2} q - \phi)|^2 dx \\
= \int_{\Omega} \mu \chi_0^2 \hat{\rho}^{-1} (\Delta u - \nabla (\text{div} u)) \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx &+ \int_{\Omega} \chi_0^2 u_t \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx \\
&- \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} (\nabla \Phi \text{div} u + \nabla (u \cdot \nabla \hat{\rho})) \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx \\
&+ \int_{\Omega} \chi_0^2 \{ (2\mu + \lambda) \hat{\rho}^{-2} \nabla f^0 + g \} \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx.
\end{align*}
\]

The first term on the left-hand side of (3.21) has the following lower bound:

\[
\begin{align*}
\int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q_t \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) dx &
\geq \gamma \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-4} |\nabla q|^2 dx + \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla (\gamma \hat{\rho}^{-2} q) \cdot \nabla q dx &- \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla (\gamma \hat{\rho}^{-2} q) \cdot \nabla q_t dx \\
&- \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q \cdot \nabla \phi_t dx &+ \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q \cdot \nabla \phi dx \\
&\geq \gamma \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-4} |\nabla q|^2 dx + \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla (\gamma \hat{\rho}^{-2} q) \cdot \nabla q dx \\
&- \frac{d}{dt} \int_{\Omega} \chi_0^2 \hat{\rho}^{-2} \nabla q \cdot \nabla \phi dx - \varepsilon \|\nabla q\|^2 - C\left(\|q_t\| + \|\nabla u\|^2\right).
\end{align*}
\]
where we have used the estimate $\|\nabla \phi_l\|^2 \leq C\|\nabla u\|^2$ in the last step. The first term on the right-hand side (3.21) of can be estimated as

$$
\int_{\Omega} \mu \frac{\partial}{\partial t} (\Delta u - \nabla (\text{div}u)) \cdot \nabla (\gamma \rho_{\gamma}^{-2}q - \phi) dx
$$

$$
= -\mu \int_{\Omega} \partial_x \left( \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi) \right) u_x dx + \mu \int_{\Omega} \partial_x \left( \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi) \right) u_t dx
$$

$$
= -\mu \int_{\Omega} \partial_x \left( \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi) \right) u_x dx + \mu \int_{\Omega} \partial_x \left( \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi) \right) u_t dx
$$

$$
\leq \frac{1}{4} \int_{\Omega} \chi_0^2 |\nabla (\gamma \rho_{\gamma}^{-2}q - \phi)|^2 dx + C \int_{\Omega} |\nabla u|^2 dx.
$$

Putting all the above inequalities into (3.21), it implies

$$
\gamma \frac{d}{dt} \int_{\Omega} \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q)^2 dx + \gamma (\gamma - 2) \frac{d}{dt} \int_{\Omega} \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q) \nabla \cdot \nabla \rho dx
$$

$$
- \frac{d}{dt} \int_{\Omega} \chi_0 \frac{\partial}{\partial t} q \nabla \phi dx + \frac{1}{4} \int_{\Omega} \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi)^2 dx
$$

$$
\leq \epsilon \|q_t\|^2 + C (\|q_t\|^2 + \|\nabla u\|^2 + \|u_t\|^2) dx + C\delta \mathcal{D}(t)
$$

after using Cauchy’s inequality.

Next, we deal with $\nabla (\frac{\partial}{\partial t} \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q - \phi) + \nabla (\chi_0 \frac{\partial}{\partial t} \gamma \rho_{\gamma}^{-2}q - \phi))$, and integrate the yielding result over $\Omega$ to have

$$
\int_{\Omega} \chi_0 \nabla q_t \cdot \nabla (\gamma \rho_{\gamma}^{-2}q - \phi) dx + \int_{\Omega} \chi_0 \nabla u_t \cdot \nabla (\rho u^i) dx
$$

$$
+ \int_{\Omega} \left\{ \chi_0 \nabla \text{div}(\rho u) \cdot \nabla (\gamma \rho_{\gamma}^{-2}q - \phi) + \chi_0 \nabla \partial_x (\gamma \rho_{\gamma}^{-2}q - \phi) \cdot \nabla (\rho u^i) \right\} dx
$$

$$
- \int_{\Omega} \left\{ \mu \chi_0 \nabla (\rho_{\gamma}^{-1} \Delta u^i) \cdot \nabla (\rho u^i) + (\mu + \lambda) \chi_0 \nabla (\rho_{\gamma}^{-1} \partial_x (\text{div}u)) \cdot \nabla (\rho u^i) \right\} dx
$$

$$
= \int_{\Omega} \chi_0 \nabla f^0 \cdot \nabla (\gamma \rho_{\gamma}^{-2}q - \phi) dx + \int_{\Omega} \chi_0 \nabla g^i \cdot \nabla (\rho u^i) dx.
$$

By the same argument as (3.22), it shows us that

$$
\int_{\Omega} \chi_0 \nabla q_t \cdot \nabla (\gamma \rho_{\gamma}^{-2}q - \phi) dx
$$

$$
\geq \gamma \frac{d}{dt} \int_{\Omega} \chi_0 \frac{\partial}{\partial t} (\gamma \rho_{\gamma}^{-2}q)^2 dx + \frac{d}{dt} \int_{\Omega} \chi_0 \nabla (\gamma \rho_{\gamma}^{-2}q) \cdot \nabla q dx
$$

$$
- \frac{d}{dt} \int_{\Omega} \chi_0 \nabla q \cdot \nabla \phi dx - \epsilon \|Dq\|^2 - C (\|q_t\|^2 + \|Du\|^2).\]
It is easy to see that
\[
\int_{\Omega} \chi_0^2 \nabla u_i \cdot \nabla (\hat{\rho}u^j) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_0^2 |\nabla u|^2 \, dx + \int_{\Omega} \chi_0^2 \nabla u_i \cdot \nabla \hat{\rho}u^j \, dx
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_0^2 |\nabla u|^2 \, dx - \int_{\Omega} \hat{\rho} \nabla u_i \cdot \left( \chi_0^2 \partial_{x_j} \hat{\rho}u^j \right) \, dx
\]
\[
\geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_0^2 |\nabla u|^2 \, dx - C(\|Du\|^2 + \|u_t\|^2).
\]
by Cauchy’s inequality and Poincaré’s inequality. While for the following terms of (3.24), one has
\[
\int_{\Omega} \left\{ \chi_0^2 \nabla (\hat{\rho}u) \cdot \nabla (\gamma \hat{\rho}^{-2} q - \phi) + \chi_0^2 \nabla \partial_{x_i} (\gamma \hat{\rho}^{-2} q - \phi) \cdot \nabla (\hat{\rho}u^j) \right\} \, dx
\]
\[
= - \int_{\Omega} \partial_{x_i} \chi_0^2 \nabla (\gamma \hat{\rho}^{-2} q - \phi) \cdot \nabla (\hat{\rho}u^j) \, dx
\]
\[
\geq - \frac{1}{8} \int_{\Omega} \chi_0^2 |\nabla (\gamma \hat{\rho}^{-2} q - \phi)|^2 \, dx - C\|Du\|^2
\]
due to Cauchy’s inequality and Poincaré’s inequality. In the meantime, we have
\[
- \int_{\Omega} \chi_0^2 \partial_{x_i} (\hat{\rho}^{-1} \Delta u) \cdot \partial_{x_i} (\hat{\rho}u^j) \, dx \geq \frac{1}{2} \int_{\Omega} \chi_0^2 |D^2 u|^2 \, dx - C\|Du\|^2,
\]
by utilizing integration by parts, Cauchy’s inequality and the elliptic estimate for bounded domain to have $C^{-1}\|D^2 u\|^2 \leq \|\Delta u\|^2 \leq C\|D^2 u\|^2$. It is clear that
\[
- \int_{\Omega} \chi_0^2 \partial_{x_i} (\hat{\rho}^{-1} \partial_{x_i} \text{div} u) \cdot \partial_{x_j} (\hat{\rho}u^j) \, dx \geq \frac{1}{2} \int_{\Omega} \chi_0^2 |D\text{div} u|^2 \, dx - C\|Du\|^2
\]
\[
\geq C \int_{\Omega} \chi_0^2 \left| \frac{D\phi}{Dt} \right|^2 \, dx - C\|Du\|^2.
\]
Putting all the above inequalities into (3.24), it implies that
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{\gamma}{2} \chi_0^2 \hat{\rho}^{-2} |\nabla q|^2 + \frac{1}{2} \chi_0^2 |\nabla u|^2 + \gamma (\gamma - 2) \chi_0^2 \hat{\rho}^{-5} q \nabla q \cdot \nabla \hat{\rho} - \chi_0^2 |\nabla q| \cdot \nabla \phi \right) \, dx
\]
\[
+ \int_{\Omega} \left\{ \chi_0^2 |D^2 u|^2 + \chi_0^2 |D\text{div} u|^2 \right\} \, dx
\]
\[
\leq \frac{1}{8} \int_{\Omega} \chi_0^2 |\nabla (\gamma \hat{\rho}^{-2} q - \phi)|^2 \, dx + C\left( \|q_t\|^2 + \|\nabla u\|^2 + \|u_t\|^2 \right) + C\delta D(t),
\]
which together with (3.28) yields (3.18).

The estimate (3.13) for second-order derivatives could be obtained similarly, i.e., estimating the following two integrals
\[
\int_{\Omega} \left\{ D_{ij} \left( \frac{2}{2} \right) (2\mu + \lambda) \chi_0^2 \hat{\rho}^{-2} D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) + D_{ij} \left( \frac{2}{2} \right) i \chi_0^2 D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) \right\} \, dx = 0, \quad (3.25)
\]
and
\[
\int_{\Omega} \left\{ D_{jk} \left( \frac{2}{2} \right) \chi_0^2 D_{jk} (\gamma \hat{\rho}^{-2} q - \phi) + D_{jk} \left( \frac{2}{2} \right) i \chi_0^2 D_{jk} (\hat{\rho}u^i) \right\} \, dx = 0. \quad (3.26)
\]
Here, we only show how to handle with the following terms for the first integral \((3.25)\). While other terms for \((3.25)\) and \((3.26)\) could be controlled similarly as \((3.18)\).

\[
(2\mu + \lambda) \int \chi_0^2 (\hat{\rho}^2 D_{ij} q_i D_{ij} (\gamma \hat{\rho}^{-2} q - \phi)) dx + \int \Omega \chi_0^2 |D_{ij} (\gamma \hat{\rho}^{-2} q - \phi)|^2 dx
\]

\[
= \int \mu \chi_0^2 D_j \left( \hat{\rho}^{-1} \left( \Delta u^i - D_i (\text{div} u) \right) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx - \int \Omega \chi_0^2 D_j u^i D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
- (2\mu + \lambda) \int \Omega \chi_0^2 D_j \left( \hat{\rho}^{-2} \left( D_i \rho \text{div} u + D_i (u \cdot \nabla \rho) \right) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
+ \int \Omega \chi_0^2 \left( (2\mu + \lambda) \hat{\rho}^{-2} D_{ij} f^0 + D_{ij} g^i \right) \cdot D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
- (4\mu + 2\lambda) \int \Omega \chi_0^2 \hat{\rho}^{-2} D_{ij} \rho D_{ij} \text{div} u D_{ij} (\gamma \hat{\rho}^{-2} q - \phi).
\]

The first term on the left-hand side of \((3.27)\) becomes

\[
\int \Omega \chi_0^2 \hat{\rho}^{-2} D_{ij} q_i D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
\geq \frac{\gamma}{2} \frac{d}{dt} \int \Omega \chi_0^2 \hat{\rho}^{-4} |D^2 q|^2 dx - \frac{d}{dt} \int \Omega \chi_0^2 \hat{\rho}^{-2} D_{ij} q \cdot D_{ij} \phi dx
\]

\[
+ \gamma (\gamma - 2) \frac{d}{dt} \int \Omega \chi_0^2 \hat{\rho}^{-5} q D_{ij} q \cdot D_{ij} \hat{\rho} dx + 2\gamma (\gamma - 2) \frac{d}{dt} \int \Omega \chi_0^2 \hat{\rho}^{-6} q D_{ij} q \cdot D_{ij} \hat{\rho} dx
\]

\[
\geq \gamma (\gamma - 2)(\gamma - 3) \frac{d}{dt} \int \Omega \chi_0^2 \hat{\rho}^{-7} q D_{ij} q \cdot D_{ij} \hat{\rho} dx - \varepsilon \|D^2 q\|^2 - C \|q_i\| + \|\nabla q_i\|^2,
\]

where we have used the elliptic estimate \(\|D^2 \phi_i\|^2 \leq C \|q_i\|^2\). The first term on the right-hand side of \((3.27)\) could be bounded by

\[
\mu \int \Omega \chi_0^2 D_j \left( \hat{\rho}^{-1} \left( \Delta u^i - D_i (\text{div} u) \right) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
= \mu \int \Omega \chi_0^2 \hat{\rho}^{-1} D_j \left( \left( \Delta u^i - D_i (\text{div} u) \right) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
+ \mu \int \Omega \chi_0^2 D_{ij} \hat{\rho}^{-1} \left( \Delta u^i - D_i (\text{div} u) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
= - \mu \int \Omega D_k (\chi_0^2 \hat{\rho}^{-1}) (D_{jk} u^i - D_{ij} u^k) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
+ \mu \int \Omega \chi_0^2 D_{ij} \hat{\rho}^{-1} \left( \Delta u^i - D_i (\text{div} u) \right) D_{ij} (\gamma \hat{\rho}^{-2} q - \phi) dx
\]

\[
\leq \frac{1}{4} \int \Omega \chi_0^2 |D^2 \hat{\rho}^{-2} q - \phi|^2 dx + C \int \Omega |D^2 u|^2 dx.
\]

At last, the third-order derivatives \((3.24)\) can be proved similarly by estimating the following integrals:

\[
\int \Omega \left\{ D_{ijk} \left[ \chi_0^2 (2\mu + \lambda) \chi_0^2 \hat{\rho}^{-2} D_{ijk} (\gamma \hat{\rho}^{-2} q - \phi) + D_{jk} (2\mu) \chi_0^2 D_{ijk} (\gamma \hat{\rho}^{-2} q - \phi) \right] \right\} dx = 0.
\]
and
\[ \int_\Omega \{ D_{jkl}(\bar{\chi}_{0}^2D_{jkl}(\gamma\hat{\rho}^{-2}q - \phi) + D_{jkl}(\bar{\chi}_{0}^2D_{jkl}(\hat{\rho}u^i)) \} \, dx = 0. \]

Therefore, the proof is complete. \( \square \)

Our next goal is to establish the estimates near the boundary \( \partial\Omega \). For this purpose, we choose a finite number of bounded open sets \( \{O_j\}_{i=1}^N \) in \( \mathbb{R}^3 \) such that
\[ \partial\Omega \subset \bigcup_{j=1}^N O_j, \]

Following the idea of (28), local coordinates \( (\xi, \zeta, r) \) will be set up in each set \( O_j \) as follows:

(i) The boundary \( O_j \cap \Omega \) is the image of smooth functions \( z = z^i(\xi, \zeta) \) satisfying
\[ |z| = 1, \quad \xi z_\zeta = 0, \quad |z_\zeta| \geq \tau > 0, \]
where \( \tau \) is some positive constant independent of \( j = 1, 2, \ldots, N \).

(ii) Any \( x \) in \( O_j \) is represented by
\[ x^i = x^i(\xi, \zeta, r) = rn^i(\xi, \zeta) + z^i(\xi, \zeta), \quad (3.29) \]
where \( n^i(\xi, \zeta) \) is the external unit normal vector at the point of the boundary coordinate \( (\xi, \zeta) \).

Here and in what follows we omit the suffix \( l \) for simplicity. Bases on \( z^i \), we introduce the unit vectors \( e_1^i \) and \( e_2^i \) as \( e_1^i = z_\zeta^i, e_2^i = z_\xi^i/|z_\zeta^i| \). Thanks to Frenet’s formula, there exists smooth functions \( (m_1, m_2, m_3, m_1', m_2', m_3') \) of \( (\xi, \zeta) \) such that
\[ \frac{\partial}{\partial \xi} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix} = \begin{pmatrix} 0 & -m_3 & -m_1 \\ m_3 & 0 & -m_2 \\ m_1 & m_2 & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix}, \]
\[ \frac{\partial}{\partial \zeta} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix} = \begin{pmatrix} 0 & -m_3' & -m_1' \\ m_3' & 0 & -m_2' \\ m_1' & m_2' & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix}. \]

Hence the Jacobian \( J \) of the transformation \( (3.29) \) is given by
\[ J = |x_\zeta \times x_\xi| = |z_\zeta| + (m_1|z_\zeta| + m_2')r + (m_1m_2 - m_2m_1')r^2. \quad (3.30) \]

From (3.30), the transformation \( (3.29) \) is regular choosing \( r \) small if needed, which implies the functions \( (\xi, \zeta, r)_{x_i}(x) \) make senses and we have
\[ \xi_{x_i} = \frac{1}{J} (x_\zeta \times x_r)_i = \frac{1}{J} (Ae_1^i + Be_2^i), \]
\[ \zeta_{x_i} = \frac{1}{J} (x_r \times x_\zeta)_i = \frac{1}{J} (Ce_1^i + De_2^i), \]
\[ r_{x_i} = \frac{1}{J} (x_\zeta \times x_\xi)_i = n_i, \quad (3.31) \]
where \( A = |z_\zeta| + m_2' r, B = -m_1' r, C = -m_2 r, D = 1 + m_1 r \) and \( J = AD - BC > 0 \). So (3.31) gives us
\[ \partial_{x_i} = \frac{1}{J} (Ae_1^i + Be_2^i) \partial_\xi + \frac{1}{J} (Ce_1^i + De_2^i) \partial_\zeta + n_i \partial_r. \]

Denote the tangential derivatives by \( \bar{\partial} = (\partial_\xi, \partial_\zeta) \), then the following estimate for the commutator hold:
\[ \| [\partial_{x_i}, \bar{\partial}] v \|^2 \leq C \| \nabla v \|^2, \quad \text{for any function } v. \quad (3.32) \]

Let \( \chi_l \) \((1 \leq l \leq N)\) be any fixed cut-off function in \( C^\infty_0(O_l) \), we would like to derive the estimates for tangential derivatives of order up to three.
Lemma 3.8. Assume that the conditions in Proposition 3.1 hold, then for any positive $\epsilon$, it holds that

\[
\frac{1}{2} \frac{d}{dt} \left\{ \gamma \int_{\Omega} \chi_i^2 \bar{\rho}^{-2} |\bar{q}|^2 dx + \int_{\Omega} \chi_i^2 \bar{\rho} |\bar{u}|^2 dx + \int_{\Omega} \chi_i^2 |\bar{\rho} \nabla \phi|^2 dx \right\} \\
+ \frac{d}{dt} \int_{\Omega} \chi_i^2 \partial_t (\gamma(\rho^{-2})q \bar{\rho}) q dx + C \left\{ \|\chi_i \bar{\partial} \nabla u\|^2 + \left\| \chi_i \frac{\partial q}{\partial t} \right\|^2 \right\} \\
\leq \epsilon \left( \|q\|^2 + \|\nabla q\|^2 + \|\nabla \phi\|^2 \right) + C \left( \|Du\|^2 + \|u_t\|^2 + \|q_t\|^2 \right) + C\delta D(t),
\tag{3.33}
\]

\[
\frac{1}{2} \frac{d}{dt} \left\{ \gamma \int_{\Omega} \chi_i^2 \bar{\rho}^{-2} |\bar{q}|^2 dx + \int_{\Omega} \chi_i^2 \bar{\rho} |\bar{u}|^2 dx + \int_{\Omega} \chi_i^2 |\bar{\rho} \nabla \phi|^2 dx \right\} \\
+ \frac{d}{dt} \left\{ 2 \int_{\Omega} \chi_i^2 \bar{\rho} (\gamma(\rho^{-2})q \bar{\rho})^2 q dx + \int_{\Omega} \chi_i^2 \bar{\rho} (\gamma(\rho^{-2})q \bar{\rho})^3 q dx \right\} \\
+ C \left\{ \|\chi_i \bar{\partial}^2 \nabla u\|^2 + \left\| \chi_i \bar{\partial}^2 \frac{dq}{dt} \right\|^2 \right\} \\
\leq \epsilon \|D^2 q\|^2 + C \left( \|q\|^2 + \|Du\|^2 + \|D^2 u_t\|^2 + \|q_t\|^2 \right) + C\delta D(t),
\tag{3.34}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \left\{ \gamma \int_{\Omega} \chi_i^2 \bar{\rho}^{-2} |\bar{q}|^2 dx + \int_{\Omega} \chi_i^2 \bar{\rho} |\bar{u}|^2 dx + \int_{\Omega} \chi_i^2 |\bar{\rho} \nabla \phi|^2 dx \right\} \\
+ \frac{d}{dt} \left\{ 3 \int_{\Omega} \chi_i^2 \bar{\partial} (\gamma(\rho^{-2})q \bar{\rho}) q \bar{\rho}^3 q dx + 3 \int_{\Omega} \chi_i^2 \bar{\partial}^2 (\gamma(\rho^{-2})q \bar{\rho})^2 q dx + \int_{\Omega} \chi_i^2 \bar{\partial}^3 (\gamma(\rho^{-2})q \bar{\rho})^3 q dx \right\} \\
+ C \left\{ \|\chi_i \bar{\partial}^2 \nabla u\|^2 + \left\| \chi_i \bar{\partial}^2 \frac{dq}{dt} \right\|^2 \right\} \\
\leq \epsilon \|D^3 q\|^2 + C \left( \|q\|^2 + \|\nabla u\|^2 + \|D^2 u_t\|^2 + \|q_t\|^2 \right) + C\delta D(t).
\tag{3.35}
\]

Proof. Estimating the integral for

\[
\int_{\Omega} \left\{ \bar{\partial}(L^0 - g^0) \chi_i^2 \bar{\partial} (\gamma(\rho^{-2})q) + \bar{\partial}(L^i - g^i) \chi_i^2 \bar{\partial}(\bar{\rho} u^i) \right\} dx = 0.
\tag{3.36}
\]

The terms involved $u_t$ and $q_t$ become

\[
\int_{\Omega} \chi_i^2 \bar{\partial} \frac{dq}{dt} \bar{\partial}(\gamma(\rho^{-2})q) dx \geq \int_{\Omega} \chi_i^2 \bar{\partial} q_t \bar{\partial}(\gamma(\rho^{-2})q) dx - C\delta D(t) \\
= \frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \rho^{-2} |\bar{q}|^2 dx + \frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \bar{\partial}(\rho^{-2}) \bar{\rho} q dx - \int_{\Omega} \chi_i^2 \bar{\partial}(\rho^{-2}) q \bar{\rho} q dx - C\delta D(t) \\
\geq \frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \rho^{-2} |\bar{q}|^2 dx + \frac{\gamma}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \bar{\partial}(\rho^{-2}) \bar{\rho} q dx - \epsilon \|\nabla q\|^2 - C\|q_t\|^2 - C\delta D(t),
\]
and

$$\int_{\Omega} \chi_i^2 \partial u_i \partial (\bar{\rho} u^i) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \partial \bar{\rho} |u|^2 dx + \int_{\Omega} \chi_i^2 \partial \bar{\rho} \partial u_i u^i dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \partial \bar{\rho} |u|^2 dx - \int_{\Omega} u^i_l \partial (\chi_l^2 \partial \bar{\rho} u^i) dx$$

$$\geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \partial \bar{\rho} |u|^2 dx - C \left\{ \| \nabla u \|^2 + \| u_t \|^2 \right\}.$$

Next, the following terms will be estimated by applying the inequality (3.32) for commutators, integration by part, Cauchy’s inequality and Poincaré’s inequality. Firstly, we have

$$- \int_{\Omega} \chi_i^2 \partial x_i \partial \bar{\rho} (\bar{\rho} u^i) dx = - \int_{\Omega} \chi_i^2 \partial x_i \partial \bar{\rho} (\bar{\rho} u^i) dx - \int_{\Omega} \chi_i^2 [\partial, \partial x_i] \partial \bar{\rho} (\bar{\rho} u^i) dx$$

$$\geq \int_{\Omega} \chi_i^2 \partial u_i \partial \Phi_i (\bar{\rho} u^i) dx - \epsilon \| \nabla \Phi_i \|^2 - C \| \nabla u \|^2$$

$$\geq \int_{\Omega} \chi_i^2 \partial u_i \partial \Phi_t (\bar{\rho} u^i) dx - \epsilon \| \nabla \Phi_t \|^2 - C \| \nabla u \|^2 - C \delta D(t)$$

$$\geq - \int_{\Omega} \chi_i^2 \partial \Phi (\bar{\rho} u^i) dx - \epsilon \| \nabla \Phi \|^2 - C \| \nabla u \|^2 - C \delta D(t)$$

$$\geq - \int_{\Omega} \chi_i^2 \partial \Phi (\bar{\rho} x_j) \partial \Phi_t (\bar{\rho} u^i) dx - \int_{\Omega} \chi_i^2 \partial \Phi (\bar{\rho} x_j) \partial x_j \Phi_t dx - \epsilon \| \nabla \Phi \|^2 - C \| \nabla u \|^2 - C \delta D(t)$$

$$\geq \int_{\Omega} \chi_i^2 \partial x_j \partial \Phi_t (\bar{\rho} u^i) dx - \epsilon \| \nabla \Phi_t \|^2 - C \| \nabla u \|^2 - C \| q_t \|^2 - C \delta D(t)$$

$$\geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi_i^2 \partial \Phi_t (\bar{\rho} u^i) dx - \epsilon \| \nabla \Phi_t \|^2 - C \| \nabla u \|^2 - C \| q_t \|^2 - C \delta D(t),$$

where we have used the fact $\| [\partial, \partial x_j] \partial x_j \Phi_t \|^2 \leq C \| \nabla^2 \Phi_t \|^2 \leq C \| q_t \|^2$. Similarly, one can get

$$\int_{\Omega} \chi_i^2 \partial \Phi_t (\bar{\rho} u^i) dx + \int_{\Omega} \chi_i^2 \partial \Phi_t (\bar{\rho} u^i) \cdot \partial (\bar{\rho} u^i) dx \geq -\epsilon (\| q_t \|^2 + \| \nabla q \|^2) - C \| \nabla u \|^2.$$
While the principal terms could be dealt with as follows:

\[
- \int_{\Omega} \chi_t^2 \bar{\partial} \Delta (\bar{\rho}^{-1} u^i) \bar{\partial} (\bar{\rho} u^i) dx
= - \int_{\Omega} \chi_t^2 \partial_{x_j} (\bar{\rho}^{-1} u^i) \partial_j (\bar{\rho} u^i) dx - \int_{\Omega} \chi_t^2 (\bar{\partial}, \partial_{x_j}) \partial_{x_j} (\bar{\rho}^{-1} u^i) \partial_j (\bar{\rho} u^i) dx
\geq \int_{\Omega} \chi_t^2 \partial_{x_j} (\bar{\rho}^{-1} u^i) \partial_{x_j} (\bar{\rho} u^i) dx + \int_{\Omega} \chi_t^2 \partial_{x_j} (\bar{\rho}^{-1} u^i) [\bar{\partial}, \partial_{x_j}] (\bar{\rho} u^i) dx - \varepsilon \|\chi_t \bar{\partial} \nabla u\|^2 - C \|\nabla u\|^2
\geq \int_{\Omega} \chi_t^2 \bar{\partial} \partial_{x_j} (\bar{\rho}^{-1} u^i) [\bar{\partial}, \partial_{x_j}] (\bar{\rho} u^i) dx
+ \int_{\Omega} \chi_t^2 \partial_{x_j} (\bar{\rho}^{-1} u^i) [\bar{\partial}, \partial_{x_j}] (\bar{\rho} u^i) dx - \varepsilon \|\chi_t \bar{\partial} \nabla u\|^2 - C \|\nabla u\|^2
\geq \frac{1}{2} \int_{\Omega} \chi_t^2 \|\bar{\partial} \nabla u\|^2 dx - C \|\nabla u\|^2,
\]

and

\[
- \int_{\Omega} \chi_t^2 \bar{\partial} (\bar{\rho}^{-1} \partial_{x_j} \text{div} u) \bar{\partial} (\bar{\rho} u^i) dx \geq \frac{1}{2} \int_{\Omega} \chi_t^2 \|\bar{\partial} \text{div} u\|^2 dx - C \|\nabla u\|^2
\geq C \int_{\Omega} \chi_t^2 \|\partial_q\|^2 dt dx - C \|\nabla u\|^2.
\]

Finally, the rest of nonlinear terms for (3.36) could be dominated by \(C \delta D(t)\). Therefore, combining the above inequalities with (3.36), it induces the desired result (3.33).

The proofs for (3.34) and (3.35) are achieved by estimating the following integrals

\[
\int_{\Omega} \left\{ \bar{\partial}^2 (L^0 - g^0) \chi_t \bar{\partial}^2 (\gamma \bar{\rho}^{-2} q) + \bar{\partial}^2 (L - g) \cdot \chi_t \bar{\partial}^2 (\bar{\rho} u) \right\} dx = 0,
\]

\[
\int_{\Omega} \left\{ \bar{\partial}^3 (L^0 - g^0) \chi_t \bar{\partial}^3 (\gamma \bar{\rho}^{-2} q) + \bar{\partial}^3 (L - g) \cdot \chi_t \bar{\partial}^3 (\bar{\rho} u) \right\} dx = 0,
\]

respectively, and utilizing the similar argument as (3.33). Thus, this lemma has been completed. \(\square\)

Next, we estimate the normal derivatives and the mixed (tangential-normal) derivatives of the solutions. For this purpose, in each \(O_j\), rewriting the equations (2.3), (2.4) by local coordinates
where we have used div $u = \rho^{-1}(g^0 - u \cdot \nabla \tilde{\rho} - \frac{\partial q}{\partial t})$.

One can rewrite $\partial_r (\bar{L}^0 - g^0) = 0$ and $n^i(\bar{L}^i - g^i) = 0$ as:

\[
\begin{align*}
\left( \frac{dq}{dt} \right)_r + \left\{ \frac{1}{J}(Ae_1 + Be_2)(\frac{\partial u}{\partial r}) \right\} &+ \frac{1}{J}(Ce_1 + De_2)(\frac{\partial u}{\partial \zeta}) + n^i(\frac{\partial u}{\partial \zeta})_r + \frac{1}{J}(Ce_1 + De_2)(\frac{\partial u}{\partial \zeta}) + n^i(\frac{\partial u}{\partial \zeta})_r \\
= (\mu + \lambda) \frac{1}{\rho} \left( \frac{1}{\rho} g^0 \right)_{x_i} + g^i, \quad i = 1, 2, 3,
\end{align*}
\]

where we have used $\rho^{-1}(g^0 - u \cdot \nabla \tilde{\rho} - \frac{\partial q}{\partial t})$,

and

\[
\begin{align*}
n^i u_i^r - \mu \frac{1}{\rho} \left\{ \frac{1}{J^2}(A^2 + B^2)n^i u_i^x + \frac{2}{J^2}(AC + BD)n^i u_i^x + \frac{1}{J^2}(C^2 + D^2)n^i u_i^x + n^i u_i^r \right\} \\
+ \text{first order and zero order terms of } u
\end{align*}
\]

\[
\begin{align*}
+ (\mu + \lambda) \frac{1}{\rho} \left( \frac{1}{\rho} g^0 \right)_{x_i} + g^i
\end{align*}
\]

(3.37)

(3.38)

where we have used the following equality

\[
\left( \frac{1}{\rho} g^0 \right)_{x_i} = \frac{1}{J}(Ae_1 + Be_2)(\frac{1}{\rho} g^0)_{\xi} + \frac{1}{J}(Ce_1 + De_2)(\frac{1}{\rho} g^0)_{\zeta} + n^i(\frac{1}{\rho} g^0)_{x_i}.
\]

Eliminating the term $n^i u_i^r$ from (3.37) and (3.38), one has

\[
\begin{align*}
(2\mu + \lambda) \frac{1}{\rho^2} \left( \frac{dq}{dt} \right)_r + (\gamma \rho^{-2}q - \phi) = -n^i u_i^r + \mu \frac{1}{\rho} \left\{ \frac{1}{J^2}(A^2 + B^2)n^i u_i^x + \frac{2}{J^2}(AC + BD)n^i u_i^x + \frac{1}{J^2}(C^2 + D^2)n^i u_i^x \\
- \frac{1}{J}(Ae_1 + Be_2)u_i^x - \frac{1}{J}(Ce_1 + De_2)u_i^x \right\} \\
+ \text{first order and zero terms of } u + \mu \frac{1}{\rho^2} g^0 + (\mu + \lambda) \frac{1}{\rho} \left( \frac{1}{\rho} g^0 \right)_{x_i} + n^i g^i.
\end{align*}
\]

(3.39)

Lemma 3.9. Under the assumptions in Proposition 3.4, then for any positive $\epsilon$, it holds that
(i) the estimate of normal derivative:

\[
\frac{\gamma d}{2 dt} \int_{\Omega} \chi_i^2 \rho^{-4} |q_r|^2 dx - \frac{d}{dt} \left\{ \int_{\Omega} \chi_i^2 \rho^{-2} q_r \phi_r dx - \int_{\Omega} \chi_i^2 \rho^{-2} (\gamma \rho^{-2}) q_r q dx \right\} \\
+ C \left\{ \left\| \chi_i (\gamma \rho^{-2} q - \phi) \right\|^2 + \left\| \chi_i \left( \frac{dq}{dt} \right) \right\|^2 \right\} \\
\leq \epsilon \| Dq \|^2 + C \left( \| q_t \|^2 + \| Du \|^2 + \| u_t \|^2 + \| \chi_i \partial Dq \|^2 \right) + C \delta \mathcal{D}(t).
\]

(ii) For \( k + m = 1 \), it holds that

\[
\frac{d}{dt} \left( \frac{\gamma}{2} \int_{\Omega} \chi_i^2 \rho^{-4} |\tilde{\partial}_r q_r|^2 dx \right) - \frac{d}{dt} \left\{ \int_{\Omega} \chi_i^2 \rho^{-2} \tilde{\partial}_r q_r \phi_r dx \right\} \\
\leq C \left( \left\| \chi_i \tilde{\partial}_r q_r \right\|^2 + \left\| q_t \right\|^2 + \left\| q_t \right\|^2 + \| Du_t \|^2 + \| Du \|^2 + \| \chi_i \tilde{\partial}_r u \|^2 \right) \\
+ \epsilon \| D^2 q \|^2 + C \delta \mathcal{D}(t).
\]

(iii) For \( k + m = 2 \), it holds that

\[
\frac{\gamma d}{2 dt} \int_{\Omega} \chi_i^2 \rho^{-4} |\tilde{\partial}_r^{m-1} q_r|^2 dx - \frac{d}{dt} \left\{ \int_{\Omega} \chi_i^2 \rho^{-2} \tilde{\partial}_r^{m-1} q_r \tilde{\partial}_r q_r \phi_r dx - \int_{\Omega} G_{k,m} dx \right\} \\
+ C \left\{ \left\| \chi_i \tilde{\partial}_r^{m-1} q_r \right\|^2 + \left\| \chi_i \tilde{\partial}_r^{m-1} \left( \frac{dq}{dt} \right) \right\|^2 \right\} \\
\leq C \left( \left\| \left( \frac{dq}{dt} \right) \right\|^2 + \left\| q_t \right\|^2 + \left\| q_t \right\|^2 + \| D^2 u_t \|^2 + \| Du \|^2 + \| \chi_i \tilde{\partial}_r^{m-1} \tilde{\partial}_r Du \|^2 \right) \\
+ \epsilon \| D^3 q \|^2 + C \delta \mathcal{D}(t),
\]

where

\[
G_{2,0} = \chi_i^2 \rho^{-2} \tilde{\partial}_r q_r \left\{ (\gamma \rho^{-2}) \tilde{\partial}^2 q + 2 \tilde{\partial}(\gamma \rho^{-2}) \tilde{\partial}_q + \tilde{\partial}^2 (\gamma \rho^{-2}) q + \tilde{\partial}(\gamma \rho^{-2}) \tilde{\partial}_q + \tilde{\partial}^2 (\gamma \rho^{-2}) q dx \right\},
\]
\[
G_{1,1} = \chi_i^2 \rho^{-2} \tilde{\partial}_r q_r \left\{ \tilde{\partial}(\gamma \rho^{-2}) q + 2(\gamma \rho^{-2}) \tilde{\partial}_q + \tilde{\partial}(\gamma \rho^{-2}) q + \tilde{\partial}(\gamma \rho^{-2}) q + \tilde{\partial}(\gamma \rho^{-2}) q dx \right\},
\]
\[
G_{0,2} = \chi_i^2 \rho^{-2} \tilde{\partial}_r q_r \left\{ 3(\gamma \rho^{-2}) q + 3(\gamma \rho^{-2}) q + (\gamma \rho^{-2}) q dx \right\}.
\]
For simply, we denote the estimate of normal-normal derivative (i.e. (3.44) with $k = 0$, $m = 1$) by

$$
\frac{d}{dt}(H_3 + F_3) + C \left\{ \| \chi_t(\gamma \tilde{\rho}^{-2} q - \phi)_r \|^2 + \left\| \chi_t \left( \frac{dq}{dt} \right)_r \right\|^2 \right\}
$$

\begin{align*}
\leq C \left( \left\| \left( \frac{dq}{dt} \right)_r \right\|^2 + \| q_t \|^2 + \| q_t \|^2 + \| Du_t \|^2 + \| Du \|^2 + \| \chi_t \partial_r \partial_r Du \|^2 \right) \\
+ \epsilon \| D^2 q \|^2 + C \delta D(t).
\end{align*}

Denote the estimate of tangential-normal-normal derivative (i.e. (3.43) with $k = 1$, $m = 1$) by

$$
\frac{d}{dt}(H_5 + F_5) + C \left\{ \| \chi_t \tilde{\rho}(\gamma \tilde{\rho}^{-2} q - \phi)_r \|^2 + \left\| \chi_t \partial_r \left( \frac{dq}{dt} \right)_r \right\|^2 \right\}
$$

\begin{align*}
\leq C \left( \left\| \left( \frac{dq}{dt} \right)_r \right\|^2 + \| q_t \|^2 + \| q_t \|^2 + \| D^2 u_t \|^2 + \| Du \|^2 + \| \chi_t \partial_r \partial_r Du \|^2 \right) \\
+ \epsilon \| D^2 q \|^2 + C \delta D(t),
\end{align*}

and the estimate of normal-normal-normal-normal derivative (i.e. (3.42) with $k = 0$, $m = 2$) by

$$
\frac{d}{dt}(H_6 + F_6) + C \left\{ \| \chi_t \partial^3_r(\gamma \tilde{\rho}^{-2} q - \phi) \|^2 + \left\| \chi_t \partial^3_r \left( \frac{dq}{dt} \right)_r \right\|^2 \right\}
$$

\begin{align*}
\leq C \left( \left\| \left( \frac{dq}{dt} \right)_r \right\|^2 + \| q_t \|^2 + \| q_t \|^2 + \| D^2 u_t \|^2 + \| Du \|^2 + \| \chi_t \partial_r \partial_r \partial_r Du \|^2 \right) \\
+ \epsilon \| D^3 q \|^2 + C \delta D(t).
\end{align*}

**Proof.** Taking the inner product of (3.33) with $\chi_t^2(\gamma \tilde{\rho}^{-2} q - \phi)_r$, then the left-hand side is

$$
(2\mu + \lambda) \int \chi_t^2 \frac{1}{\tilde{\rho}^2} \left( \frac{dq}{dt} \right)_r (\gamma \tilde{\rho}^{-2} q - \phi)_r dx + \| \chi_t(\gamma \tilde{\rho}^{-2} q - \phi)_r \|^2 = LHS_1 + LHS_2 + \| \chi_t(\gamma \tilde{\rho}^{-2} q - \phi)_r \|^2
$$

(3.46)

A simple calculation gives

$$
LHS_1 = \int \chi_t^2 \frac{1}{\tilde{\rho}^2} \left( \frac{dq}{dt} \right)_r (\gamma \tilde{\rho}^{-2} q)_r dx
$$

$$
= \gamma \int \chi_t^2 \tilde{\rho}^{-4} q_{t+r} dx + \int \chi_t^2 \tilde{\rho}^{-2} (\gamma \tilde{\rho}^{-2}) q_{t+r} q dx + \int \chi_t^2 \tilde{\rho}^{-2} (u \cdot \nabla q)_r (\gamma \tilde{\rho}^{-2} q), dx
$$

$$
\geq \frac{\gamma}{2} \frac{d}{dt} \int \chi_t^2 \tilde{\rho}^{-4} |q_r|^2 dx + \frac{d}{dt} \int \chi_t^2 \tilde{\rho}^{-2} (\gamma \tilde{\rho}^{-2}) q_r q dx - \epsilon \| \nabla q \|^2 - C \| q_t \|^2 - C \delta D(t),
$$
and

\[\text{LHS}_2 \equiv - \int_\Omega \chi_l^2 \frac{1}{\bar{\rho}^2} \left( \frac{dq}{dt} \right)_r \phi_r \, dx\]
\[= - \frac{d}{dt} \int_\Omega \chi_l^2 \bar{\rho} \phi_r \, dx + \int_\Omega \chi_l^2 \bar{\rho}^{-2} q_r \phi_r \, dx - \int_\Omega \chi_l^2 \bar{\rho}^{-2} (u \cdot \nabla q)_r \phi_r \, dx\]
\[\geq - \frac{d}{dt} \int_\Omega \chi_l^2 \bar{\rho} \phi_r \, dx - \epsilon \| \nabla q \|^2 - C \| \nabla u \|^2 - C \delta D(t),\]

where we have used (3.11) in the last inequality.

Putting the above two inequalities with (3.46), and using Cauchy’s inequality, one obtains that

\[\frac{d}{dt} \left\{ \frac{\gamma}{2} \int_\Omega \chi_l^2 \bar{\rho}^{-4} |q_r|^2 \, dx + \gamma (\gamma - 2) \int_\Omega \chi_l^2 \bar{\rho}^{-5} \bar{\rho} q_r q_r \, dx - \int_\Omega \chi_l^2 \bar{\rho}^{-2} q_r \phi_r \, dx \right\}\]
\[+ \frac{1}{2} \| \chi_l (\gamma \bar{\rho}^{-2} q - \phi)_r \|^2 \]
\[\leq \epsilon \| \nabla q \|^2 + C \left( \| q_r \|^2 + \| \nabla u \|^2 + \| u_t \|^2 + \| \chi_l \bar{k} D \|^2 \right) + C \delta D(t).\]

Meanwhile, taking the inner product of (3.39) with \(\chi_l^2 \left( \frac{dq}{dt} \right)_r\), a similar argument as (3.47) gives the desired estimate (3.40). Furthermore, the estimates (3.41) and (3.42) can be obtained in a similar way as before, we thus omit the proof. \(\Box\)

At last, taking \(\chi_l \bar{k} \partial_k (k = 1, 2)\) to equations (2.3), (2.4) in a similar manner to Lemma 2.4, one obtains

**Lemma 3.10.** For \(k = 1, 2\) and \(k + m = 1, 2\), it holds

\[\| \chi_l D^{m+2} \bar{k} u \|^2 + \| \chi_l D^{m+1} \bar{k} (\gamma \bar{\rho}^{-2} q - \phi) \|^2\]
\[\leq C \left( \| \chi_l \bar{k} \|_{m+1}^2 \| q \|^2 + \| u_t \|^2_{k+m} + \| g \|^2_{k+m} + \| Dq \|^2_{k+m-1} + \| Du \|^2_{k+m} \right).

Now we are ready to prove Proposition 3.1 by the following steps.
Proof of Proposition 3.1. Step 1. Adding the results of Lemma 3.2, Lemma 3.4, (3.18), (3.33), and (3.40) with some small suitable constants, then one obtains that

\[
\frac{1}{2} \frac{d}{dt} \left\{ \int_\Omega \left( \rho |u|^2 + \gamma \bar{\rho}^{-2} q^2 + |\nabla \phi|^2 \right) dx + \int_\Omega \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dx + \gamma \int_\Omega \chi_0 \bar{\rho}^{-4} |\nabla q|^2 dx \right. \\
+ \frac{\gamma}{2} \int_\Omega \chi_i \bar{\rho}^{-2} |\partial q|^2 dx + \int_\Omega \chi_i \bar{\rho} |\partial u|^2 dx + \int_\Omega \chi_i^2 |\partial \nabla \phi|^2 dx + \frac{\gamma}{2} \int_\Omega \chi_i^2 \bar{\rho}^{-4} |q_t|^2 dx \right\} \\
- \frac{d}{dt} \left\{ \int_\Omega \bar{\rho} \nabla \phi \cdot u dx + \int_\Omega q u \cdot \nabla \bar{\Phi} dx + \int_\Omega \gamma \bar{\rho}^{-2} q \text{div} u dx - \int_\Omega \chi_i^2 \bar{\rho}^{-2} q \nabla \cdot \nabla (\gamma \bar{\rho}^{-2}) dx \\
+ \int_\Omega \chi_i^2 \bar{\rho}^{-2} q_t \phi_t dx \right\} + C \left\{ \|Du\|^2 + \|q_t\|^2 + \|u_t\|^2 + \| \frac{dq}{dt} \|_1^2 \right\} \\
\leq \epsilon (\|q\|^2 + \|\nabla q\|^2) + C \delta D(t). 
\] (3.48)

Further, utilizing Lemma 2.4 with \( k = 0 \), Lemma 3.6 and Poincaré’s inequality, we have

\[
\|q\|^2 + \|\nabla q\|^2 + \|\nabla \phi\|^2 \leq C \left\{ \|D^2 u\|^2 + \|D(\gamma \bar{\rho}^{-2} q - \phi)\|^2 + \delta D(t) \right\} \\
\leq C \left\{ \left\| \frac{dq}{dt} \right\|_1^2 + \|u\|^2 + \|u_t\|^2 + \delta D(t) \right\} \\
\leq C \left\{ \left\| \frac{dq}{dt} \right\|_1^2 + \|\nabla u\|^2 + \|u_t\|^2 + \delta D(t) \right\}. 
\]

Denoting the time derivative of (3.48) by \( \frac{d}{dt} H_1(t) + \frac{d}{dt} F_1(t) \), then for \( \epsilon \) small enough, (3.48) gives

\[
\frac{d}{dt} \left( H_1(t) + F_1(t) \right) + C \left\{ \|q\|^2_1 + \|q_t\|^2_1 + \|\nabla \phi\|^2 + \|Du\|^2_1 + \|u_t\|^2 + \left\| \frac{dq}{dt} \right\|_1^2 \right\} \leq C \delta D(t). 
\] (3.49)

Step 2. In view of Lemma 3.3, Lemma 3.7, Lemma 3.8 and Lemma 3.9 with \( k = 1, m = 0 \), one has
Denoting the time derivative of (3.50) by \( \frac{d}{dt} H_2(t) \) and \( \frac{d}{dt} F_2(t) \), then (3.51) and (3.49) infer that

\[
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + H_2(t) + F_2(t) \right\} + C \left\{ \| q \|_1^2 + \| q_t \|_1^2 + \| \nabla \phi \|_1^2 + \| D u \|_1^2 + \| u_t \|_1^2 \right\} \\
\quad + \left\| \frac{dq}{dt} \right\|_1^2 + \left( \frac{dq}{dt} \right)_r^2 + \chi_0 D^2(\gamma \rho \gamma^2 q - \phi)^2 + \| \chi_0 D^3 u \|_2^2 + \left\| \chi_0 D \frac{dq}{dt} \right\|_1^2 \\
\quad + \| \chi_i \bar{\partial}^2 D u \|_1^2 + \| \chi_i \bar{\partial} \frac{dq}{dt} \|_1^2 + \| \chi_i \bar{\partial}(\gamma \rho \gamma^2 q - \phi)_r \|_1^2 + \left\| \chi_i \bar{\partial} \left( \frac{dq}{dt} \right)_r \right\|_1^2 \right\} \leq \epsilon \| D^2 q \|_1^2 + C \delta^2 \mathcal{D}(t).
\] (3.51)

**Step 3.** Lemma 3.3 with \( k = 0, m = 1 \), i.e. (3.43) tells that

\[
\frac{d}{dt} (H_3(t) + F_3(t)) + C \left\{ \| \chi_i (\gamma \rho \gamma^2 q - \phi)_r \|_1^2 + \chi_i \left( \frac{dq}{dt} \right)_{rr} \right\} \leq \epsilon \| D^2 q \|_1^2 + C \left\{ \left( \frac{dq}{dt} \right)_{rr}^2 + \| q \|_1^2 + \| q_t \|_1^2 + \| D u_t \|_1^2 + \| D u \|_1^2 + \| \chi_i \partial_r D u \|_1^2 \right\} + C \delta^2 \mathcal{D}(t) \] (3.52)

\[
\leq \epsilon \| D^2 q \|_1^2 + C \left\{ \left( \frac{dq}{dt} \right)_r^2 + \| q \|_1^2 + \| q_t \|_1^2 + \| D u_t \|_1^2 + \| D u \|_1^2 \right\} + C \left\| \chi_i \partial \frac{dq}{dt} \right\|_1^2 + C \delta^2 \mathcal{D}(t).
\]
where we have used Lemma 3.10 with $k = 1, m = 0$ in the last inequality. Then (3.52) and (3.51) imply that, for $\eta_3$ small, it holds

$$
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + \eta_2 H_2(t) + \eta_2 F_2(t) + \eta_3 H_3(t) + \eta_3 F_3(t) \right\}
\leq \epsilon \| D^2 q \|^2 + C \delta D(t),
$$

that is

$$
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + \eta_2 H_2(t) + \eta_2 F_2(t) + \eta_3 H_3(t) + \eta_3 F_3(t) \right\}
\leq \epsilon \| D^2 q \|^2 + C \delta D(t).
$$

On the other hand, using Lemma 2.4 with $k = 1$, one has

$$
\| D^3 u \|^2 + \| D^2 (\gamma \tilde{p}^{-2} q - \phi) \|^2 \leq C \left\{ \left\| \frac{dq}{dt} \right\|_2^2 + \| u \|_2^2 + \| g^0 \|_2^2 + \| u_t \|_1^2 + \| g \|_1^2 \right\},
$$

which together with the elliptic estimate $\| D^2 \phi \|^2 \leq C \| q \|^2$ gives

$$
\| D^2 q \|^2 \leq C \left\{ \| D^2 (\gamma \tilde{p}^{-2} q) \|^2 + \| q \|^2 \right\} \leq \left\{ \| D^2 (\gamma \tilde{p}^{-2} q - \phi) \|^2 + \| q \|^2 \right\}
\leq C \left\{ \left\| \frac{dq}{dt} \right\|_2^2 + \| u \|_2^2 + \| u_t \|_1^2 + \| g \|_1^2 \right\} + C \delta D(t),
$$

Therefore, for $\epsilon$ small enough, (3.53) becomes

$$
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + \eta_2 H_2(t) + \eta_2 F_2(t) + \eta_3 H_3(t) + \eta_3 F_3(t) \right\}
\leq C \delta D(t).
$$
Step 4. Adding the results on Lemma 3.7, Lemma 3.8 and Lemma 3.9 with \( k = 2, m = 0 \), then for \( \eta \) small, it holds that

\[
\frac{1}{2} \frac{d}{dt} \left\{ \int_{\Omega} \left( \mu |Du_t|^2 + (\mu + \lambda) |\text{div}u_t|^2 \right) dx + \gamma \int_{\Omega} \chi_0^2 \rho^2|q|^2 dx \right. \\
+ \gamma \int_{\Omega} \chi_1^2 \rho^2|q|^2 dx + \int_{\Omega} \chi_2^2 \rho^2|q|^2 dx + \int_{\Omega} \chi_3^2 |\partial^3 \psi|^2 dx + \eta \gamma \int_{\Omega} \chi_4^2 \rho^2|q|^2 dx \left\} \\
+ \frac{d}{dt} \left\{ - \int_{\Omega} \gamma \rho^{-1} \partial q dx - \int_{\Omega} \frac{2}{\partial \rho^2} D_{ijkl} \partial_{ijkl} q dx \\
+ 3 \int_{\Omega} \chi_0^2 \rho^2 D_i (\gamma \rho^{-2}) \cdot D_{ijkl} q dx + 3 \int_{\Omega} \chi_0^2 \rho^2 D_i (\gamma \rho^{-2}) D_{ijkl} q dx + 3 \int_{\Omega} \chi_1^2 \partial (\gamma \rho^{-2}) \partial^2 q dx + 3 \int_{\Omega} \chi_1^2 \partial (\gamma \rho^{-2}) \partial^2 q dx \\
+ 3 \int_{\Omega} \chi_0^2 \partial (\gamma \rho^{-2}) \partial^2 q dx + \int_{\Omega} \chi_0 ^2 \partial (\gamma \rho^{-2}) \partial^2 q dx \\
- \int_{\Omega} \chi_1 \partial^2 (\gamma \rho^{-2}) \partial^2 q dx + 3 \int_{\Omega} \chi_1 \partial^2 (\gamma \rho^{-2}) \partial^2 q dx \\
+ \partial^2 (\gamma \rho^{-2}) \partial^2 q + \partial (\gamma \rho^{-2}) \partial^2 q + \partial^2 (\gamma \rho^{-2}) \partial^2 q dx \right\} (3.55)
\]

\[
\leq \epsilon \| D^3 q \|^2 + C \left\{ \| Du_t \|_{L^2}^2 + \| u_t \|_{L^2}^2 + \| q_t \|_{L^2}^2 + \| q_{tt} \|_{L^2}^2 + \left( \frac{dq}{dt} \right)_{L^2}^2 \right\} + C \delta D(t).
\]

Denoting the \( t \)- derivative of (3.55) by \( \frac{d}{dt} H_4(t) + \frac{d}{dt} F_4(t) \), then putting (3.55) together with (3.54), it implies that, for \( \eta_1 \) small,

\[
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + \sum_{i=2}^{4} (\eta_i H_i(t) + \eta_i F_i(t)) \right\} \\
+ C \left\{ \| q \|_{L^2}^2 + \| q_t \|_{L^2}^2 + \| D \phi \|_{L^2}^2 + \| Du \|_{L^2}^2 + \| u_t \|_{L^2}^2 + \| q_{tt} \|_{L^2}^2 + \left( \frac{dq}{dt} \right)_{L^2}^2 \right\} \\
+ \| \chi_0 D^3 (\gamma \rho^{-2} - \phi) \|_{L^2}^2 + \| \chi_0 D^4 u \|_{L^2}^2 + \left( \frac{dq}{dt} \right)_{L^2}^2 + \| \chi_0 D^3 \|_{L^2}^2 + \| \chi_1 \partial^2 \|_{L^2}^2 + \left( \frac{dq}{dt} \right)_{L^2}^2 \right\} (3.56)
\]

\[
\leq \epsilon \| D^3 q \|^2 + C \delta D(t).
\]
**Step 5.** Lemma 3.3 with $k = 1, m = 1$ implies that

$$
\frac{d}{dt} (H_5(t) + F_5(t)) + C \left\{ \left\| \chi_l \partial \partial_t (\gamma \bar{\rho}^{-2} q - \phi)_r \right\|^2 + \left\| \chi_l \partial \partial_t \left( \frac{dq}{dt} \right)_r \right\|^2 \right\}
$$

$$
\leq \epsilon \|D^3 q\|^2 + C \left( \left\| \left( \frac{dq}{dt} \right)_r \right\|_1^2 + \|q\|^2_2 + \|q_t\|^2_2 + \|D^2 u_t\|^2 + \|D u\|^2_2 + \|\chi_l \bar{\rho} \partial \partial_t Du\|^2 \right) + C \delta \mathcal{D}(t)
$$

$$
\leq \epsilon \|D^3 q\|^2 + C \left\{ \left\| \left( \frac{dq}{dt} \right)_r \right\|_1^2 + \|q\|^2_2 + \|q_t\|^2_2 + \|u_t\|^2_2 + \|Du\|^2_2 + \|\chi_l \bar{\rho} \partial \partial_t dq\|^2 \right\} + C \delta \mathcal{D}(t),
$$

(3.57)

where we have used Lemma 3.10 with $k = 2, m = 0$. Adding (3.57) and (3.56) implies that, for $\eta_5$ small, it holds

$$
\frac{d}{dt} \left\{ H_1(t) + F_1(t) + \sum_{i=2}^{5} (\eta_i H_i(t) + \eta_i F_i(t)) \right\}
$$

$$
\begin{align*}
&+ C \left\{ \|q\|^2_2 + \|q_t\|^2_2 + \|D\phi\|^2 + \|Du\|^2_2 + \|u_t\|^2_2 + \|u_{tt}\|^2 + \left\| \frac{dq}{dt} \right\|^2_2 + \left\| \left( \frac{dq}{dt} \right)_r \right\|^2_2 \\
&+ \|\chi_0 D^3 (\gamma \bar{\rho}^{-2} q - \phi)\|^2 + \|\chi_0 D^4 u\|^2 + \left\| \chi_0 \partial D^3 \frac{dq}{dt} \right\|^2 + \|\chi_l \partial \partial^3 \frac{dq}{dt} \|^2 \right\} \right\} \leq \epsilon \|D^3 q\|^2 + C \delta \mathcal{D}(t).
\end{align*}
$$

(3.58)

**Step 6.** It is obvious to see that

$$
\frac{d}{dt} (H_6(t) + F_6(t)) + C \left\{ \left\| \chi_l \partial \partial_t^3 (\gamma \bar{\rho}^{-2} q - \phi) \right\|^2 + \left\| \chi_l \partial \partial_t \left( \frac{dq}{dt} \right) \right\|^2 \right\}
$$

$$
\leq \epsilon \|D^3 q\|^2 + C \left( \left\| \left( \frac{dq}{dt} \right)_r \right\|_1^2 + \|q\|^2_2 + \|q_t\|^2_2 + \|D^2 u_t\|^2 + \|Du\|^2_2 + \|\chi_l \partial \partial_t Du\|^2 \right) + C \delta \mathcal{D}(t)
$$

$$
\leq \epsilon \|D^3 q\|^2 + C \left\{ \left\| \left( \frac{dq}{dt} \right)_r \right\|_1^2 + \|q\|^2_2 + \|q_t\|^2_2 + \|u_t\|^2_2 + \|Du\|^2_2 + \|\chi_l \partial \partial_t \frac{dq}{dt}\|^2 \right\} + C \delta \mathcal{D}(t),
$$

(3.59)
from Lemma 3.9 with \( k = 0, m = 2 \), where we have used Lemma 3.10 with \( k = 1, m = 1 \). Therefore, (3.59) and (3.58) imply that, for \( \eta_6 \) small, it holds

\[
\frac{d}{dt}\left\{ H_1(t) + F_1(t) + \sum_{i=2}^{6} (\eta_i H_i(t) + \eta_i F_i(t)) \right\} + C \left\{ \|q\|_2^2 + |q_t|^2 + \|\nabla\phi\|^2 + \|Du\|_2^2 + |u_t|^2 + |q_t|^2 + \|u_t\|^2 + \left\| \frac{dq}{dt} \right\|^2 + \left\| \left( \frac{dq}{dt} \right)_t \right\|^2 \\
+ \|\chi_0 D^3(\gamma \hat{\rho}^{-2} q - \phi)\|^2 + \|\chi_0 D^4 u\|^2 + \left\| \chi_0 \partial^2_t \frac{dq}{dt} \right\|^2 + \left\| \chi_0 \partial^2_t \nabla u \right\|^2 \\
+ \|\chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi)_r\|^2 + \left\| \chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi)_{rr} \right\|^2 + \left\| \chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi) \right\|^2 \right\}
\leq \epsilon \|D^3 q\|^2 + C \delta D(t),
\]

which yields that

\[
\frac{d}{dt}\left\{ H_1(t) + F_1(t) + \sum_{i=2}^{6} (\eta_i H_i(t) + \eta_i F_i(t)) \right\} + C \left\{ \|q\|_2^2 + |q_t|^2 + \|\nabla\phi\|^2 + \|Du\|_2^2 + |u_t|^2 + |q_t|^2 + \|u_t\|^2 + \left\| \frac{dq}{dt} \right\|^2 + \left\| \left( \frac{dq}{dt} \right)_t \right\|^2 \\
+ \|\chi_0 D^3(\gamma \hat{\rho}^{-2} q - \phi)\|^2 + \|\chi_0 D^4 u\|^2 + \left\| \chi_2 \partial^2_t \frac{dq}{dt} \right\|^2 + \left\| \chi_2 \partial^2_t \nabla u \right\|^2 \\
+ \|\chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi)_r\|^2 + \left\| \chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi)_{rr} \right\|^2 + \left\| \chi_2 \partial^2_t (\gamma \hat{\rho}^{-2} q - \phi) \right\|^2 \right\}
\leq \epsilon \|D^3 q\|^2 + C \delta D(t).
\]

On the other hand, using Lemma 2.4 with \( k = 2 \), one has

\[
\|D^1 u\|^2 + \|D^3(\gamma \hat{\rho}^{-2} q - \phi)\|^2 \leq C \left\{ \left\| \frac{dq}{dt} \right\|^2 + \|u\|^2 + |g|^2 + \|u_t\|^2 + |q|^2 \right\},
\]

which together with the elliptic estimate \( \|D^3 \phi\|^2 \leq C \|q\|^2 \) gives

\[
\|D^3 q\|^2 \leq C \left\{ \|D^3(\gamma \hat{\rho}^{-2} q)\|^2 + \|q\|^2 \right\} \leq \left\{ \|D^3(\gamma \hat{\rho}^{-2} q - \phi)\|^2 + \|q\|^2 \right\}
\leq C \left\{ \left\| \frac{dq}{dt} \right\|^2 + \|u\|^2 + |u_t|^2 + \|q|^2 \right\} + C \delta D(t).
\]

Therefore, for \( \epsilon \) small enough, (3.60) is controlled as

\[
\frac{d}{dt}\left\{ H_1(t) + F_1(t) + \sum_{i=2}^{6} (\eta_i H_i(t) + \eta_i F_i(t)) \right\} + C \left\{ \|q\|^2 + |q_t|^2 + \|\nabla\phi\|^2 + \|Du\|^2 + |u_t|^2 + \|q_t|^2 + \|u_t\|^2 + \left\| \frac{dq}{dt} \right\|^2 \right\} \leq C \delta D(t).
\]
Step 7. Let

$$\tilde{E}(t) \equiv H_1(t) + F_1(t) + \sum_{i=2}^{6} (\eta_i H_i(t) + \eta_i F_i(t)),$$

and

$$\tilde{D}(t) \equiv \|q\|_3^2 + \|q_t\|_1^2 + \|\nabla \phi\|^2 + \|Du\|_3^2 + \|u_t\|_1^2 + \|q_{tt}\|^2 + \|u_{tt}\|^2 + \left\| \frac{dq}{dt} \right\|_3^2,$$

then we obtain

$$\frac{d}{dt} \tilde{E}(t) + C \tilde{D}(t) \leq C \delta D(t). \quad (3.62)$$

Recalling the definitions of $E(t)$, $D(t)$, it is directly to see that

$$E(t) \leq D(t). \quad (3.63)$$

Now we claim that

$$D(t) \leq C \tilde{D}(t) + C \delta D(t), \quad (3.64)$$

which implies for $\delta$ small, it holds

$$D(t) \leq C \tilde{D}(t). \quad (3.65)$$

Indeed, we can show that

$$\|D^2 q_t\|^2 \leq C \left\{ \left\| D^2 \frac{dq}{dt} \right\|^2 + \|D^2(u \cdot \nabla q)\|^2 \right\} \leq C \tilde{D}(t) + C \delta D(t).$$

and

$$\|D^2 u_t\|^2 \leq C \left\{ \|u_{tt}\|^2 + \|\nabla q_t\|^2 + \|\nabla \phi_t\|^2 + \|u_t\|^2 + \|f_t\|^2 + \|\nabla u_t\|^2 \right\}$$

$$\leq C \left\{ \|u_{tt}\|^2 + \|\nabla q_t\|^2 + \|u_t\|^2 + \|u_t\|^2 + \|f_t\|^2 + \|\nabla u_t\|^2 \right\}$$

$$\leq C \tilde{D}(t) + C \delta D(t).$$

Therefore, the claim (3.64) is proved.

On the other hand, Hölder’s inequality implies that

$$C^{-1} E_1(t) \leq \tilde{E}(t) \leq C E_1(t). \quad (3.66)$$

where

$$E_1(t) = \|q\|_3^2 + \|q_t\|^2 + \|u\|_1^2 + \|u_t\|_1^2 + \|D\phi\|^2.$$ 

Notice that Lemma 2.1 yields that

$$\|\nabla \phi\|_3^2 \leq C\|q\|_3^2.$$ 

By using Lemma 2.3, one has

$$\|D^2 u\|^2 + \|D^3 u\|^2 \leq CE_1(t) + C \delta D(t).$$

Then, in view of the equation (2.2)1, we can obtain the estimate of \(\|\nabla q_t\|^2\) and \(\|D^2 q_t\|^2\). Therefore,

$$C^{-1} E(t) \leq E_1(t) \leq E(t).$$

Then, putting (3.63), (3.64) and (3.66) into (3.62), we obtain

$$\frac{d}{dt} \tilde{E}(t) + \sigma \tilde{E}(t) \leq 0,$$

which gives

$$\tilde{E}(t) \leq e^{-\sigma t} \tilde{E}(0).$$

Then

$$E(t) \leq C e^{-\sigma t} E(0).$$

The proof is completed. □
Acknowledgements

Liu’s research is supported by National Natural Science Foundation of China (No.11926418). The authors are grateful to Professor Tao Luo for helpful suggestions and discussions.

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