Weather forecasts, Weather derivatives, Black-Scholes, Feynmann-Kac and Fokker-Planck

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Abstract

We investigate the relationships between weather forecasting, weather derivatives, the Black-Scholes equation, Feynmann-Kac theory and the Fokker-Planck equation. There is one useful result, but on the whole the relations we present seem to be more interesting than practically useful.

1 Introduction

Weather forecasting is all about predicting what the weather is going to do next, or over the next 10 days. Weather derivatives are a way of insuring oneself against adverse weather conditions. The Black-Scholes equation is a financial model for the pricing of certain kinds of financial contracts. Feynmann-Kac theory is an esoteric result from the study of partial differential equations and stochastic calculus. The Fokker-Planck equation is a result from statistical physics concerning the probability density of the location of a particle. We describe, in an informal way, how these five topics are related.

2 Temperature forecasts as Brownian motion

Today is Monday, and we start by considering a weather forecast that attempts to predict the average temperature for next Saturday i.e. a 5 day forecast. To be specific, we will define average temperature, as is convention, to be the midpoint of the minimum and maximum temperatures during the 24 hour period of Saturday. We will consider a single-valued forecast, rather than a forecast of the whole distribution of possible temperatures. How should such a forecast be interpreted in statistical terms? The most obvious interpretation is that the forecast represents the expectation, or mean, of the distribution of possible temperatures.

The new forecast will most likely have a different value from today’s forecast, and will probably be more accurate, at least on average over many trials, since it is predicting less far into the future. Come Wednesday we will have a new forecast again, and so on, until the final forecast is delivered on Saturday, and the final measurements become available at the end of the day Saturday and we can look back and judge how well our forecasts have performed.

How do the forecasts we are considering change from day to day? It seems natural that they could go either up or down. Furthermore, if we were sure, on Tuesday, that the Wednesday forecast was going to be higher, then we should really incorporate that information into our Tuesday forecast. Having incorporated all the information we have into the Tuesday forecast, it would seem likely that the Wednesday forecast could be either higher or lower, but would be the same on average. This suggests we could model the changes in the forecast as a random change with mean zero. In statistics such a stochastic process that doesn’t go up or down on average, but jumps randomly, is known as a martingale. We offer three explanations for why we think that it is reasonable to model weather forecasts as martingales:

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• The intuitive explanation given above: if they were not martingales then they would have a predictable component, which contains information that could have been included in the current forecast. We assume that the forecasts are produced by a rational and efficient process, and hence that this information is included in the forecast and so it is a martingale. We have previously called this assumption the ‘efficient forecast hypothesis’, as a parody on the ‘efficient market hypothesis’, which is a similar assumption used to justify the modelling of jumps in share prices as being random.

• A more general intuitive explanation that any expectation, of anything, can only change randomly, otherwise it cannot be an expectation.

• A mathematical proof known variously as the ‘tower law’ (Baxter and Rennie, 1996) or the ‘law of iterated expectations’ (Bjork, 1998, page 33) that shows that all expectations are martingales.

Temperature is very often reasonably close to normally distributed, and so are temperature forecasts. If temperature forecasts are normally distributed, then changes in temperature forecasts are also normally distributed. This makes our model for temperature forecasts more specific: they are gaussian martingales.

In fact, there is only one gaussian martingale and that is Brownian motion, sometimes known as a Wiener process or a continuous random walk. So now we have arrived at the conclusion that weather forecasts can be modelled using Brownian motion. We will write this as the stochastic differential equation (SDE):

$$d\mu = \sigma dB$$

(1)

where $$\mu$$ is the weather forecast, $$B$$ is Brownian motion and $$\sigma$$ is a volatility.

If we knew $$\sigma$$ we could integrate this equation forward in time as an ensemble to give possible future values for forecasts between now and Saturday, and also the possible future values for temperature on Saturday.

So the next question is: what is $$\sigma$$, and how does it vary in time?

### 3 Weather forecast volatility

We start by considering the volatility between Monday and Tuesday. Our first model for this 5-day forecast to 4-day forecast volatility is that it is constant at all times of year. However, a cursory investigation of forecasts shows that they show a larger variance in winter, corresponding to the larger variance of temperatures at that time. As a result of this, the changes in forecasts from day to day show a larger variance, and so the volatility is larger. This is an entirely predictable effect. Our second model for the volatility is therefore that the volatility is deterministic, with a seasonal cycle corresponding to the seasonal cycle of variance of observed temperatures. We have used this model previously in Jewson (2003f).

We will now question our second model of volatility. Is the volatility really completely predictable, as we have assumed above, or does it change from day to day? (to be clear, at this point we mean: does the 5 day forecast to 4 day forecast volatility change as the starting and target day of the forecast move in time...we address the question of whether the 5 day forecast to 4 day forecast volatility differs from the 4 day forecast to 3 day forecast volatility later).

Are not some weather situations more predictable than others, and doesn’t that lead to periods of lower volatility in the forecast? It has been shown that weather forecasts can predict, with some skill, that the variance of the conditional distribution of temperature varies with time (for example, see Palmer and Tibaldi, 1988). Another way of saying this is that the variance of forecast errors varies in time. We have extended this to show that it is possible to predict the volatility with skill in advance. So from this we see that the volatility shows days to day variations in size which overlay the seasonal changes, and also that part of these variations are predictable.

So where does this leave us? We have a volatility that varies seasonally, in a way that is predictable infinitely far in advance. It also varies on shorter timescales in a way that could be considered completely unpredictable from a long distance in advance, but becomes partly predictable over short time scales using weather forecast models. In fact, however, our own research seems to show that the predictable short time-scale variations in forecast uncertainty are rather small, and do not add much, if anything, to the skill of temperature forecasts (see Jewson et al, 2003 and Jewson, 2003c). As a result, short term changes in the volatility of the forecasts are also small. Because they are small, and for the sake of being able to make some progress with simple mathematics, we are going to ignore them. This means we can write the volatility simply as a fixed function of time.
We now consider variations in the size of the volatility between Monday and Saturday i.e. the variation with lead time (now we are comparing the 5 day forecast to 4 day forecast volatility with the 4 day forecast to 3 day forecast volatility). Are the sizes of the changes in forecasts between Monday and Tuesday any different from the sizes of changes between Friday and Saturday? In other words, does the volatility change as we approach the target day? It is interesting to pose this question to a few meteorologists: we have done so, and found no consistency at all between the answers that we received. Our (albeit brief) data analysis, however, seems to show, rather interestingly, that these volatilities are roughly constant with lead time (Jewson, 2002a). Maybe there is some underlying reason for this, we don’t know. Either way, it certainly simplifies the modelling. Combined with the model of the changes in the forecasts as being Brownian motion, this gives us a linear variation in forecast mean square error, which has been noted many times (and is also more or less unexplained).

To summarise this section: we now have a model that says that a temperature forecast for a single target day follows a Brownian motion with deterministic volatility as we approach the target day. The volatility varies with forecast day (because it depends on the time of year) but does not vary with lead time. Our arguments in this section have been somewhat ad-hoc: as an alternative one could consider trying to develop models for the volatility based on a detailed analysis of the statistics of the variability of daily temperatures and daily forecasts, and their dependencies. We have not tried this. It may be possible, but we suspect it will be very hard indeed, given how hard it is to model just temperatures alone (see our attempts to solve this much simpler problem in [Caballero et al., 2002] and Jewson and Caballero (2003)).

4 Forecasts for the monthly mean

We will now consider a slightly different weather forecast. It is Monday 30th November. Tomorrow is the first day of December. We are interested in a forecast for the average temperature in December (defined, for the sake of being a little precise, as the arithmetic mean of the daily averages). Today’s estimate of that average temperature is based on a single valued forecast for the expected temperature that extends out for the next 10 days, followed by estimates of the expected temperature for the remainder of the month based on historical data. Tomorrow (December 1st) we will update this forecast: we will have one more day of forecasts relevant for the month of December, and will need to use one fewer day of historical data. The day after tomorrow we will get an observed value for December 1st (it won’t be the final, quality controlled value, as so is still subject to change, but we will ignore that). We will then use 10 days of forecast from the 2nd to the 11th, and historical data from the 12th to the 31st. The day after that we will get one more day of historical data, our forecast will move forward one more step, and so on. Then at some point the forecast starts to drop off the end, and a few days after that we get the final observed data point for December 31st.

We can now think about how such a forecast changes in time. In fact, the changes in the monthly forecast are made up of sums of changes in daily forecasts, which we have already considered. Since a sum of Brownian motions is a Brownian motion, we conclude that we can model the monthly forecast as a Brownian motion too. The shape of the volatility for this monthly forecast Brownian motion, is, however, a little more interesting than before. When the forecast first starts to impinge upon December the volatility is going to be low. The forecast does not initially have much effect on the monthly mean, and so the day to day changes in the monthly forecast are small. As more and more forecast takes effect the volatility grows. During the bulk of the contract all of the forecast is relevant, and the volatility has a relatively constant value. Finally, at the end of the contract, the volatility starts to reduce as less and less of the forecast is used. What model could we use to represent this ramping up, constant level, and then ramping down, of the volatility? The actual shape of the ramps depends on the sizes of the volatilities for daily forecasts, and of the correlations between changes of the forecasts at different leads. To keep things simple we will make two assumptions: firstly, that the sizes of the volatilities for single day forecasts are constant...the same assumption we made in section 3. This is partly justified by analysis of data. Secondly, we assume that these changes are uncorrelated. This is not really true...the changes in forecasts at different leads do show some, albeit weak, correlations (see Jewson (2002a)). But it is not too far wrong, at least, and allows us to make some progress. In particular, it means that the ramping up and down is given by a straight line in volatility squared. We will call this the ‘trapezium’ model. We have described this model in detail in Jewson (2002b) and Jewson (2003d), and used it in Jewson (2003c) and Jewson (2003d).
5 Statistical properties of weather forecasts

We have concluded that both daily and monthly forecasts can be modelled using Brownian motion. In the daily case the volatility is constant with lead, but varies seasonally. In the monthly case the volatility squared follows a trapezium shape. We can now derive a number of further results from these models. We start with a consideration of transition probabilities.

We first ask the question: if our forecast says $\mu(t)$ today (at time $t$) then what is the distribution of the different things it might say at some point in the future (at time $T$)? This can be solved by integrating equation 1 forward in time. This gives:

$$\mu \sim N(\mu(t), \int_t^T \sigma^2 dt)$$

i.e. the distribution of future values of our forecast is a normal distribution with mean $\mu(t)$ and variance given by the integral of the daily volatilities squared. Alternatively, the probabilities themselves as a function of the value of the forecast $x$ and time $t$ are given by solutions of the Fokker-Planck equation (also known as the Kolmogorov forward equation):

$$\frac{\partial p}{\partial t} = \frac{\partial^2}{\partial x^2} (\sigma^2 p)$$

along with a boundary condition based on what the forecast currently says. This equation is a diffusion type of equation. The probabilities diffuse outwards as time moves forwards.

Another related question is: if our forecast is going to say $x$ in the future, what are the possible things it could be saying today? These probabilities are given by the backward Fokker-Planck equation (also known as the Kolmogorov backward equation):

$$\frac{\partial p}{\partial t} = -\sigma^2 \frac{\partial^2 p}{\partial x^2}$$

along with a boundary condition that specifies the final condition. This equation is diffusion in reverse i.e. starting at the final forecasts, the probabilities diffuse outwards as time moves backwards.

For more details on these equations, see, for example, Gardiner (1985) or Bjork (1998).

6 The Feynmann-Kac formula

We now take a diversion from discussing weather forecasts, and present the Feynmann-Kac formula. In subsequent sections we will then see that this formula is related to the fair price of weather options. Consider the following partial differential equation (PDE) for $F(x, t)$:

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2} = 0$$

with the boundary condition:

$$F(x, T) = \Phi(x)$$

This PDE is very ”special”: because of the particular nature of the terms in the equation we can solve it by taking expectations of a function of Brownian motion. The Brownian Motion we must use is fixed by the coefficients in the PDE:

$$dX = \mu dt + \sigma dB$$

with the initial condition:

$$X = x$$

and the expectation we must calculate is

$$F(x, t) = E[\Phi(X_T)]$$

What we see from this is that one interpretation of what the solution of the PDE actually does is to somehow take expectations of the final condition, at different points in time, and according to some stochastic process. The proof of this relation is easy, and is given in, for example, Bjork (1998).
Weather derivative fair prices are martingales

We will now link our discussion of weather forecasts to the pricing of weather derivatives. Weather derivatives are financial contracts that allow entities to insure themselves against adverse weather. They are based on a weather index, such as monthly mean temperature in December, and have a payoff function which converts the index into a financial amount that is to be paid from the seller of the contract to the buyer of the contract. The payoff function can in principle have any form, but a certain number of piecewise linear forms known as swaps, puts and calls are most common (see, for example, Jewson (2003a)).

The fair price of a weather derivative is defined, by convention, as the expectation of the distribution of possible payoffs from the contract. The fair price can form the basis of negotiations of what the actual price should be. Typically the seller of the contract (who sells such contracts as a business) would want to charge more than the fair price, otherwise they could not expect to make money on average. The buyer may be willing to pay above the fair price because they are grateful for the chance to insure themselves against potentially dire weather situations.

We can calculate the fair price using whatever relevant historical data and forecasts are available to estimate the final distribution of outcomes of the contract. This is the usual approach (details are given in Jewson et al. (2002) and Brix et al. (2002)). Alternatively, we can calculate the fair price by integrating equation (1) forward to the end of the contract to give us the final distribution of settlement indices.

As we move through a contract, our estimate of the fair price will change in time as the available historical data and forecasts change. The fair price is an expectation, by definition, and we have argued in section 7 that expectations are martingales, so the fair price should change as a martingale.

Deriving the fair price PDE

It is useful to think of the conditional estimate of the fair price at some point during a weather contract as being dependent on the conditional estimate of the expected index, rather than, say, on the temperature so far during the contract. The former is much more convenient in many situations because the expected index is a Brownian motion while the temperature is very highly autocorrelated and hard to model. Thus we have:

\[ V = V(\mu, \sigma_\tau) \]  

(10)

where \( \mu \) and \( \sigma_\tau \) are the conditional mean and standard deviation of the settlement index.

Differentiating this with respect to time gives:

\[
\begin{align*}
dV &= \frac{\partial V}{\partial \mu} d\mu + \frac{\partial V}{\partial \sigma} d\sigma + \frac{1}{2} \frac{\partial^2 V}{\partial \mu^2} d\mu^2 + ... \\
&= \delta d\mu + \zeta d\sigma + \frac{1}{2} \gamma d\mu^2 + ... \\
&= \delta dB + \theta dt + \frac{1}{2} \gamma \sigma^2 dB^2 + ... \\
&= \delta dB + \theta dt + \frac{1}{2} \gamma \sigma^2 dt + ... \\
&= \delta dB + dt(\theta + \frac{1}{2} \sigma^2 \gamma) + ... \\
&\approx \delta dB + dt(\theta + \frac{1}{2} \sigma^2 \gamma)
\end{align*}
\]

This last expression is the Ito derivative, and we have used the standard definitions \( \delta = \frac{\partial V}{\partial \mu}, \zeta = \frac{\partial V}{\partial \sigma}, \theta = \frac{\partial V}{\partial t} \) and \( \gamma = \frac{\partial^2 V}{\partial \mu^2} \) (see Jewson (2003b)). We see that changes in \( V \) are driven by stochastic jumps (the \( dB \) term) and by a deterministic drift (the \( dt \) term).

But we have already seen in section 7 that \( V \) is a martingale, and hence that there can be no drift term. Thus we must have that the coefficient of \( dt \) in this equation is zero, giving:

\[ \theta + \frac{1}{2} \sigma^2 \gamma = 0 \]

(12)

or, re-expanding in terms of the full notation:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial \mu^2} = 0 \]

(13)
We conclude that the fair price of a weather option satisfies a PDE, which is a backwards diffusion-type of equation. The diffusion coefficient comes from the volatility of weather forecasts.

8.1 Relation to Feynmann-Kac

There is a close relation to what we have just derived, and the Feynmann-Kac theory discussed in section 6. Equation 13 is a particular example of equation 5. Applying the Feynmann-Kac theorem, we see that we can solve this equation by integrating a stochastic process and taking an expectation. The stochastic process is given by equation 11 We have come full circle.

9 The stochastic process for the fair price and the VaR

Given the PDE for the fair price (equation 13) equation 11 now simplifies to:

$$dV = \delta \sigma dB$$

(14)

In other words, changes in the fair price of a weather derivative over short time horizons are normally distributed around the current fair price, and have a volatility given simply in terms of the $\delta$ of the contract, and the $\sigma$ of the underlying expected index.

Holders of financial contracts often like to know how much they could lose, and how quickly. Minimising, or limiting, the amount one could lose is known as risk management. One of the most common measures of risk used in risk management is the market value at risk, or market VaR. Market VaR is defined to be the 5% level of the distribution of possible market prices of a contract at a specified future time. So far we haven’t considered market prices, just fair value, so we will define actuarial VaR as the 5% level of the distribution of possible fair values of a contract at a specified future time.

Equation 14 actually gives us the actuarial VaR over short time horizons $^1$. The distribution of changes in the fair value is given by:

$$dV \sim N(0, \delta \sigma)$$

(15)

and the VaR is a quantile from this distribution.

This equation doesn’t apply over finite time horizons because both $\delta$ and $\sigma$ change with time. We could integrate equation 14 forward in time to get around this and derive the distribution of $dV$ over finite time horizons. However, very small time steps are needed because $\delta$ can change very rapidly in certain situations. It is generally easier to calculate actuarial VaR over longer horizons by integrating the underlying process (equation 11) to give a distribution of future values of the expected index (which can be done analytically) and then calculating the distribution of values of $V$. These issues are discussed in more details in [Jewson, 2003c].

10 Arbitrage pricing, Black-Scholes and Black

Equity options are contracts that have a payoff dependent on the future level of some share price. By trading the shares themselves very frequently one can more or less replicate the payoff structure of an equity option. The cost of doing this replication tell us what the value of the option should be. This is the Black-Scholes theory of option pricing [Black and Scholes, 1973]. If the replication is done using forward contracts on the equity rather than the equity itself then the Black-Scholes theory must be modified as described by [Black, 1976] (although if we set interest rates to zero, as we will, the two theories are the same).

We can apply the Black-Scholes argument to weather options, as described in [Jewson and Zervos, 2003]. $^2$ We imagine hedging the weather option using weather forwards. The final equation is slightly different from the Black equation, because the underlying process we use to model the expectation of the weather index is slightly different from the processes that are used to model share prices. In the case where interest rates are zero we find that the Black equation for weather is the same as the PDE for the fair price, already given in equation 13.

To summarise: in an actuarial pricing world equation 13 gives us the fair price for weather options. In a Black-Scholes world (where frequent trading is possible) equation 13 gives us both the fair price and the market price.

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$^1$this is the useful result we mention in the abstract

$^2$thanks to Anna-Maria Velioti for pointing out that equation 28 in this paper has a sign error in the last term
11 Notes on the relation between Black-Scholes and Weather
Black-Scholes

In the standard Black-Scholes world one way of expressing the price of an option is as the discounted expected payoff with expectations calculated under the risk neutral measure. A curious feature of the weather derivative version of the Black-Scholes model is that the change of measure is not necessary, since the underlying process (given by equation 1) does not have any drift. The objective measure is the risk neutral measure.

12 Summary

We have argued that temperature forecasts for a fixed target day can be modelled as Brownian motion with deterministic volatility. We have also argued that temperature forecasts for a fixed target month can be modelled as a Brownian motion, but with a more complex volatility structure. Standard results for Brownian motion thus apply such as the forward and backward Fokker-Planck equations, and equations for transition probabilities.

We have also argued that the fair price of a weather derivative must be a martingale. As a result, the fair price satisfies both a PDE and an SDE, and the SDE can be used to estimate actuarial VaR over short time horizons.

Finally, under additional assumptions about the weather market the PDE for the fair price is the weather derivative equivalent of the Black-Scholes and Black equations.

References

M Baxter and A Rennie. Financial Calculus. Cambridge University Press, 1996.

T Bjork. Arbitrage Theory in Continuous Time. Oxford University Press, 1998.

F Black. The pricing of commodity contracts. Journal of Financial Economics, 3:167–179, 1976.

F Black and M Scholes. The pricing of options and corporate liabilities. Journal of Political Economy, 81:637–654, 1973.

A Brix, S Jewson, and C Ziehmann. Weather derivative modelling and valuation: a statistical perspective. In Climate Risk and the Weather Market, chapter 8, pages 127–150. Risk Books, 2002.

R Caballero, S Jewson, and A Brix. Long memory in surface air temperature: Detection, modelling and application to weather derivative valuation. Climate Research, 21:127–140, 2002.

C Gardiner. Handbook of Stochastic Methods. Springer, 1985.

S Jewson. Weather derivative pricing and risk management: volatility and value at risk. http://ssrn.com/abstract=405802, 2002a. Technical report.

S Jewson. Weather option pricing with transaction costs. Energy, Power and Risk Management, 2002b.

S Jewson. Closed-form expressions for the pricing of weather derivatives: Part 1 - the expected payoff. http://ssrn.com/abstract=436262, 2003a. Technical report.

S Jewson. Closed-form expressions for the pricing of weather derivatives: Part 2 - the greeks. http://ssrn.com/abstract=436263, 2003b. Technical report.

S Jewson. Horizon value at risk for weather derivatives part 1: single contracts. Submitted to SSRN, 2003c.

S Jewson. Horizon value at risk for weather derivatives part 2: portfolios. Submitted to SSRN, 2003d.

S Jewson. Moment based methods for ensemble assessment and calibration. arXiv:physics/0309042, 2003e. Technical report.

S Jewson. Simple models for the daily volatility of weather derivative underlyings. Submitted to SSRN, 2003f.
S Jewson, A Brix, and C Ziehmann. Risk management. In *Weather Risk Report*, pages 5–10. Global Reinsurance Review, 2002.

S Jewson, A Brix, and C Ziehmann. A new framework for the assessment and calibration of ensemble temperature forecasts. *Atmospheric Science Letters*, 2003. Submitted.

S Jewson and R Caballero. Seasonality in the dynamics of surface air temperature and the pricing of weather derivatives. *Journal of Applied Meteorology*, 11 2003.

S Jewson and M Zervos. The Black-Scholes equation for weather derivatives. <http://ssrn.com/abstract=436282>, 2003. Technical report.

S Jewson and C Ziehmann. Using ensemble forecasts to predict the size of forecast changes, with application to weather swap value at risk. *Atmospheric Science Letters*, 2003. In print.

T Palmer and S Tibaldi. On the prediction of forecast skill. *Mon. Wea. Rev.*, 116:2453–2480, December 1988.

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