CONDENSING OPERATORS AND PERIODIC SOLUTIONS OF INFINITE DELAY IMPULSIVE EVOLUTION EQUATIONS

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Abstract. By showing the existence of the fixed point of the condensing operators in the phase space $C_{\mu}$ for the Cauchy problem for impulsive evolution equations with infinite delay in a Banach space $X$:

$$x'(t) + A(t)x(t) = \mathfrak{J}(t, x(t), x_1), \quad t > 0, \ t \neq t_i,$$
$$x(s) = \varphi(s), \ s \leq 0,$$
$$\Delta x(t_i) = I_i(x(t_i)), \ i = 1, 2, \ldots , 0 < t_1 < t_2 < \cdots < \infty,$$

where $\mathfrak{A}(t)$ is $\omega$-periodic, the operator $A(t)$ is unbounded for each $t > 0$, $x_i(s) = x(t + s), \ s \leq 0$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $\mathfrak{J}$, $\varphi$ and $I_i$ ($i = 1, \ldots , n$) are given functions, we derive periodic solutions from bounded solutions. The new periodic solution existence results obtained here extend earlier results in this area for evolution equations without impulsive conditions or without infinite delay.

1. Introduction. It is known that compact operators or contractive operators or Fredholm operators play a key role in the study of qualitative properties of differential equations, such as the existence of solutions, the periodic solutions and the almost periodic solutions (cf., e.g., [1-16] and references therein). For differential equations without delay or with finite delay, some major results are already achieved with the help of fixed point method or coincidence degree method involving compactness or contractility.

For the more advanced and complex differential equations, some challenges are met. For example, consider the Cauchy problem for following infinite delay evolution

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equation
\[ x'(t) + A(t)x(t) = \mathcal{F}(t, x(t), x_1), \quad t > 0, \tag{1} \]
\[ x(s) = \varphi(s), \quad s \leq 0, \tag{2} \]
in a general Banach space \( (X, \|\cdot\|) \), where \( A(t) \) is an unbounded operator for each \( t > 0 \) and \( \mathcal{F} \) is a continuous function in its variables, \( x_t(s) = x(t + s), \quad s \leq 0, \)
\( x_t \in C((-\infty, 0], X) \) (the space of continuous functions on \((-\infty, 0]\) with values in \(X\)), and \( \phi : (-\infty, 0] \to X \) is a given function.

Recently, there have been some significant developments in the study of evolution equations with impulsive conditions, which are the combinations of the traditional initial value problems and the short-term perturbations whose duration can be negligible in comparison with the duration of the process. In particular, in \([8, 9, 10]\), we present existence results for the periodic solutions of the Cauchy problem for impulsive equations without delay \((r = 0)\) or with finite delay \((r > 0)\):

\[ x'(t) + A(t)x(t) = \mathcal{F}(t, x(t), x_1), \quad t > 0, \quad t \neq t_i, \]
\[ x(s) = \varphi(s), \quad s \in [-r, 0], \]
\[ \Delta x(t_i) = \mathcal{I}_i(x(t_i)), \quad i = 1, 2, \cdots, \quad 0 < t_1 < t_2 < \cdots < \infty, \]
where \( \Delta x(t_i) = x(t_i^+) - x(t_i^-) \) is impulsive, \( \phi : [-r, 0] \to X \) and \( \mathcal{I}_i : X \to X \) \((i = 1, \cdots, n)\) are given functions. Moreover, in \([8, 9, 10]\), a series of techniques are developed in order to handle impulsive conditions for equations without delay or with finite delay.

A natural question to ask is what will happen for the periodicity if impulsive conditions are imposed to the problem \((1)-(2)\), that is, what can we obtain for the periodicity of the solutions to the Cauchy problem for following impulsive evolution equations with infinite delay,

\[ x'(t) + A(t)x(t) = \mathcal{F}(t, x(t), x_1), \quad t > 0, \quad t \neq t_i, \tag{3} \]
\[ x(s) = \varphi(s), \quad s \leq 0, \tag{4} \]
\[ \Delta x(t_i) = \mathcal{I}_i(x(t_i)), \quad i = 1, 2, \cdots, \quad 0 < t_1 < t_2 < \cdots < \infty. \tag{5} \]
where \( A(\cdot) \) and \( \mathcal{F}(\cdot, u, w) \) are \( \omega \)-periodic.

As noted in \([11]\), the infinite delay problem \((1)-(2)\) is itself difficult because most fixed point theorems were not applicable as they would require some compactness which was not obtainable for equations with infinite delay. Moreover, as can be seen from \([8, 9, 10]\) that impulsive conditions are also difficult to deal with. So now, the combination of the two major difficulties creates a system that is significantly more difficult.

In this paper, with further detailed analysis, we are able to combine the techniques developed in \([8, 9, 10]\), the treatments in \([11]\), and of course some new ideas in this paper, to attack the problem \((3)-(5)\) and derive fixed points and then periodic solutions.

The procedure is to study the problem \((3)-(5)\) in a phase space \(C_\mu\), and carefully handle the impulsive conditions and overcome the related difficulties so as to prove that the Poincare operator given by \( \mathcal{P}(\varphi) = x_\varphi(\varphi) \) \((i.e., \varphi \) units along the unique solution \( x(\varphi) \) determined by the initial function \( \varphi \)) is a condensing operator with respect to the Kuratowski’s measure of non-compactness in \(C_\mu\). Then as an application, we derive periodic solutions from bounded solutions by using Sadovskii’s Fixed Point Theorem. The new results obtained here extend some earlier results.
in this area for evolution equations without impulsive conditions or without infinite delay.

2. Preliminary results. We start with some basic definitions, settings and results here.

Definition 2.1. Let $\alpha$ be the Kuratowski’s measure of non-compactness and let $\mathcal{P}$ be a continuous and bounded operator on $X$. If

$$\alpha(\mathcal{P}(B)) < \alpha(B)$$

for any bounded $B$ of $E$ with $\alpha(B) > 0$, then $\mathcal{P}$ is called a condensing operator.

Our purpose of this paper is to derive a condensing operator associated with the problem (3)-(5), so that we need to use the following Sadovskii’s Fixed Point Theorem to study the periodic solutions.

Lemma 2.2. (Sadovskii’s Fixed Point Theorem) Let $\mathcal{P}$ be a condensing operator on $X$ and $E \subset X$ be a closed convex bounded set. If $\mathcal{P}(E) \subseteq E$, then $\mathcal{P}$ has a fixed point in $E$.

Assumption 2.3. (1). $\mathcal{F}(\cdot, u, w)$ and $\mathcal{A}(\cdot)$ are $\varpi$ periodic, that is,

$$\mathcal{F}(t + \varpi, u, w) = \mathcal{F}(t, u, w), \quad \mathcal{A}(t + \varpi) = \mathcal{A}(t), \quad t \geq 0.$$

$\mathcal{F}(t, u, w)$ is bounded for all $(t, u, w)$.

(2). The domain $D(\mathcal{A}(t)) = D$ for all $t \in [0, \varpi]$, and $D = X$.

(3). For any $t \geq 0$ and $\text{Re}\lambda \leq 0$,

$$|(\lambda I - \mathcal{A}(t))^{-1}| \leq \text{Constant } (|\lambda| + 1)^{-1}$$

(where we use $|\cdot|$ to denote the operator norm) and $(\lambda I - \mathcal{A}(t))^{-1}$ is a compact operator.

(4). For any $s, t, r \in [0, \varpi]$,

$$|(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}(r)^{-1}| \leq \text{Constant } |t - s|^{\theta},$$

for some $0 < \theta \leq 1$.

If Assumption 2.3 holds, then it is known (cf., e.g., [12]) that there exists a unique evolution family $\mathcal{U}(t, s)$, $0 \leq s \leq t \leq \varpi$ such that $\mathcal{U}(t, s)$ is strongly continuous, and

$$| \frac{\partial}{\partial t} \mathcal{U}(t, s) | \leq \frac{C}{t - s}, \quad 0 \leq s < t \leq \varpi,$$

(6)

where $C > 0$ is a constant.

Moreover, by Assumption 2.3 we have

$$\mathcal{U}^* = \max_{0 \leq s \leq \varpi} | \mathcal{U}(t, s) |,$$

(7)

$$\mathcal{F}^* = \max \{ \| \mathcal{F}(\xi, u, w) \| : \text{ all } (\xi, u, w) \}$$

(8)

are finite numbers. Let

$$0 < \Upsilon < 1, \quad 0 < \tau < 1$$

and $\tau > 0$ be constants. Then there exists an integer $m_0 > 1$ such that

$$\mathcal{U}^* + \Upsilon < \Upsilon^{-m_0} \tau \quad \text{and} \quad 2\varpi \mathcal{U}^* \mathcal{F}^* < \Upsilon^{-m_0} m_0 \tau.$$

(9)

It is easy to see that there exists a function $\mu$ on $(-\infty, 0]$ such that

$$\mu(0) = 1, \quad \mu(-\infty) = \infty,$$
\( \mu \) is decreasing on \((-\infty, 0]\), and for \( t \geq \frac{\omega}{m_0} \):

\[
\sup_{s \leq 0} \frac{\mu(s)}{\mu(s-t)} \leq \Upsilon. \tag{10}
\]

Let \( PC((-\infty, 0], X) \) be the space of piecewise continuous functions from \((-\infty, 0]\) to \( X \). For the function \( \mu \) given above, we define

\[
C_\mu = \left\{ \varphi : \varphi \in PC((-\infty, 0], X) \text{ and } \sup_{s \leq 0} \frac{\|\varphi(s)\|}{\mu(s)} < \infty \right\}, \tag{11}
\]

and

\[
|\varphi|_\mu = \sup_{s \leq 0} \frac{\|\varphi(s)\|}{\mu(s)}, \quad \varphi \in C_\mu. \tag{12}
\]

Then \( C_\mu \) is a Banach space.

**Definition 2.4.** A *mild solution* \( x(\cdot) \) of the problem \((3)-(5)\) with \( x_0 = \varphi \in PC((-\infty, 0], X) \) (i.e., \( x(s) = \varphi(s), s \leq 0 \)) is a piecewise continuous function with points of discontinuity \( t_i \) where \( u \) is left continuous and has the right limits, and satisfies

\[
x(t) = U(t, 0)\varphi(0) + \int_0^t U(t, \xi)\mathcal{G}(\xi, x(\xi), x_\xi) \, d\xi + \sum_{0 < t_i < t} U(t, t_i)\mathcal{I}_i(x(t_i)), \quad t \geq 0. \tag{13}
\]

It is easily seen that mild solutions satisfy the impulsive condition \((5)\). With some additional conditions on the Lipschitzian operators \( \mathcal{I}_i \), the proof in \([11]\) can be modified to show that mild solutions do exist on \([0, \alpha)\) for some \( \alpha > 0 \). In this paper, we focus our attention to the existence of fixed points of condensing operators for infinite delay impulsive evolution equations and then the existence of periodic solutions, we suppose that for each \( \varphi \in C_\mu \), the problem \((3)-(5)\) has a unique mild solution \( x(\cdot, \varphi) \) existing on \([0, \infty)\). Since everything in the whole paper will be clear, we will use “solutions” to mean “mild solutions”.

**Assumption 2.5.** (1) \( \mathcal{G}(t, u, w) : \mathbb{R}^+ \times X \times C_\mu \rightarrow X \) is continuous and Lipschitzian in \( u \) and in \( w \). \( \mathcal{I}_i, i = 1, 2, \ldots, \) are Lipschitzian and compact.

(2) \( 0 < t_1 < t_2 < \cdots < t_p < \omega < t_{p+1} \) and \( t_{p+k} = t_k + \omega \), \( \sum_{p+k} k = 3k \), \( k \geq 1 \).

The following lemma from \([8]\) is also needed here.

**Lemma 2.6.** \([8]\) Assume that \( M_0 \geq 0, p \geq 0, \beta_i \geq 0, K > 0 \) are constants, and

\[
w(t) \leq M_0 + p \int_0^t w(s) \, ds + \sum_{0 < t_i < t} \beta_i w(t_i), \quad 0 \leq t \leq K, \tag{14}
\]

where \( w(t) \geq 0 \) is integrable on \([0, K]\). Then,

\[
w(t) \leq M_0 \prod_{0 < t_i < t} (1 + \beta_i) e^{pt}, \quad 0 \leq t \leq K.
\]

Throughout this paper, we write

\[ a \land b = \max\{a, b\}, \]

for any real numbers \( a \) and \( b \).
3. Condensing operator for (3)-(5).

**Theorem 3.1.** *(Boundedness)* Let Assumptions 2.3 and 2.5 hold, and \( P_0 \subset C_\mu \) be bounded. Then the solutions of the problem (3)-(5) with \( \varphi \in P_0 \) (if any) are bounded on \([0, \infty)\).

**Proof.** First, we prove that if \( x \) and \( y \) are two solutions of the problem (3)-(5) with initial value \( \varphi \) and \( \psi \) respectively on \( (-\infty, \infty) \), \( \varphi > 0 \), then

\[
|x_t - y_t|_{\mu} \leq c_1 |\varphi - \psi|_{\mu} \prod_{0 < t_i < t} (1 + \beta_i)e^{c_2 t}, \quad t \in [0, \infty],
\]

where \( c_1, c_2 \) and \( \beta_i \) are some constants.

Actually, it is not so hard to see that for \( t \in [0, \infty] \),

\[
|x_t - y_t|_{\mu} \leq \sup_{s \in [0, t]} \|x(s) - y(s)\| \wedge |\varphi - \psi|_{\mu}.
\]

By the related Lipschitz continuity, we know that there is a constant \( M > 0 \) such that

\[
\|F(\xi, x(\xi), y(\xi)) - F(\xi, y(\xi), y(\xi))\| \leq M\|x(\xi) - y(\xi)\|,
\]

\[
\|F(\xi, x(\xi), x(\xi)) - F(\xi, x(\xi), y(\xi))\| \leq M\|x(\xi) - y(\xi)\|,
\]

\[
\|I(\xi(t_i)) - I(\xi(y(t_i)))\| \leq M\|x(t_i) - y(t_i)\|.
\]

Hence, for \( s \in [0, t] \),

\[
\|x(s) - y(s)\| = \|U(s, 0)(x(0) - y(0))
+ \int_{0}^{s} U(s, \xi) \left[ F(\xi, x(\xi), x(\xi)) - F(\xi, x(\xi), y(\xi)) \right] d\xi
+ \sum_{0 < t_i < t} U(t, t_i) \left[ I(\xi(t_i)) - I(\xi(y(t_i))) \right]\|
\]

\[
= \|U(s, 0)(x(0) - y(0))
+ \int_{0}^{s} U(s, \xi) \left[ F(\xi, x(\xi), x(\xi)) - F(\xi, x(\xi), y(\xi)) \right] d\xi
+ \int_{0}^{s} U(s, \xi) \left[ F(\xi, x(\xi), y(\xi)) - F(\xi, y(\xi), y(\xi)) \right] d\xi
+ \sum_{0 < t_i < t} U(t, t_i) \left[ I(\xi(t_i)) - I(\xi(y(t_i))) \right]\|
\]

\[
\leq \mu^*\|x(0) - y(0)\| + \int_{0}^{s} \mu^* M\|x(\xi) - y(\xi)\| d\xi
+ \int_{0}^{s} \mu^* M\|x(\xi) - y(\xi)\| d\xi + \sum_{0 < t_i < t} \mu^* M\|x(t_i) - y(t_i)\|
\]

\[
\leq \mu^*|\varphi - \psi|_{\mu} + 2 \int_{0}^{t} \mu^* M\|x(\xi) - y(\xi)\| d\xi
+ \sum_{0 < t_i < t} \mu^* M\|y(t_i) - y(t_i)\|.
\]

Thus it follows from (16) that

\[
|x_t - y_t|_{\mu} \leq (\mu^* + 1)|\varphi - \psi|_{\mu} + 2 \int_{0}^{t} \mu^* M\|x(\xi) - y(\xi)\| d\xi
\]
Now, Lemma 2.6 implies (15).

Let \( \psi \in \mathcal{P}_0 \) be fixed. Then

\[
|x_\psi|_\mu \leq |x_\psi - y_\psi|_\mu + |y_\psi|_\mu,
\]

and hence (15) implies that \( \{x_\psi(\varphi)|_\mu : \varphi \in \mathcal{P}_0 \} \) is bounded. Therefore the result is true by using the definition of the norm in \( C_\mu \). The proof ends then. \( \square \)

Based on the result above, for \( \mathcal{P}_0 \subset C_\mu \) and \( x(\varphi) \) the unique (mild) solution of the problem (3)-(5) with \( \varphi \in \mathcal{P}_0 \), we define

\[
\mathcal{Z}_i(\mathcal{P}_0) = \{ x_i(\varphi) : \varphi \in \mathcal{P}_0 \}
\]

and

\[
\mathcal{Z}_{[\xi,r]}(\mathcal{P}_0) = \{ x_{[\xi,r]}(\varphi) : \varphi \in \mathcal{P}_0 \},
\]

where \( x_{[\xi,r]} \) means the restriction of \( u \) on \([\xi,r]\).

**Theorem 3.2. (Estimation under measure of non-compactness)** Let Assumptions 2.3 and 2.5 hold, and \( \mathcal{P}_0 \subset C_\mu \) be bounded. Suppose that the problem (3)-(5) with \( \varphi \in \mathcal{P}_0 \) has a unique (mild) solution \( x(\varphi) \) existing on \( \mathbb{R}^+ \). Then \( \mathcal{Z}_{[0,\varpi]}(\mathcal{P}_0) \) and \( \mathcal{Z}_r(\mathcal{P}_0) \) are bounded in \( PC([0,\varpi],X) \) and \( C_\mu \) respectively for each \( r \in [0,\varpi] \), and

\[
\alpha(\mathcal{Z}_r(\mathcal{P}_0)) \leq \{ \mathcal{T} \alpha(\mathcal{Z}_{\varpi}(\mathcal{P}_0)) \} \wedge \alpha(\mathcal{Z}_{[\xi,r]}(\mathcal{P}_0)),
\]

where, as in Section 2,

\[
0 < \mathcal{T} < 1, \quad 0 \leq \tau < r \leq \varpi \text{ with } r - \tau \geq \frac{\varpi}{m_0}
\]

**Proof.** It follows from Theorem 3.1 that \( \mathcal{Z}_{[0,\varpi]}(\mathcal{P}_0) \) is bounded in \( PC([0,\varpi],X) \). Moreover, in view of the properties of \( \mu \) and (10), we deduce that for every \( x_r \in \mathcal{Z}_r(\mathcal{P}_0) \) \((r \in [0,\varpi])\),

\[
|x_r|_\mu = \sup_{s \leq r} \left\| x(s) \right\| / \mu(s - r)
\]

\[
= \sup_{s \leq 0} \frac{\left\| x(s) \right\|}{\mu(s - r)} \wedge \sup_{0 \leq s \leq r} \frac{\left\| x(s) \right\|}{\mu(s - r)}
\]

\[
\leq \{ |x_0|_\mu \sup_{s \leq 0} \frac{\mu(s)}{\mu(s - r)} \} \wedge \sup_{0 \leq s \leq r} \left\| x(s) \right\|
\]

\[
\leq \{ \mathcal{T} |x_0|_\mu \} \wedge \sup_{0 \leq s \leq r} \left\| x(s) \right\|.
\]

Hence, \( \mathcal{Z}_r(\mathcal{P}_0) \) is bounded in \( C_\mu \) for each \( r \in [0,\varpi] \).

On the other hand, it follows from the properties of \( \mu \) and (10) that for any \( x_r, y_r \in \mathcal{Z}_r(\mathcal{P}_0) \) \((r \in [0,\varpi])\), and for every \( 0 \leq \tau < r \leq \varpi \) with \( r - \tau \geq \frac{\varpi}{m_0} \), we have

\[
|x_r - y_r|_\mu = \sup_{s \leq r} \left\| x(s) - y(s) \right\| / \mu(s - r)
\]

\[
= \sup_{s \leq \tau} \left\| x(s) - y(s) \right\| / \mu(s - r) \wedge \sup_{\tau \leq s \leq r} \left\| x(s) - y(s) \right\| / \mu(s - r)
\]
Let \( \mathcal{P}_0 \subset C_\mu \) be bounded with \( \alpha(\mathcal{P}_0) > 0 \), then \( \mathcal{Z}_{[0,\varpi]}(\mathcal{P}_0) \subset PC([0,\varpi],X) \) is bounded and \( \mathcal{Z}_\xi(\mathcal{P}_0) \subset C_\mu \) is bounded for each \( r \in [0,\varpi] \), and for any \( 0 \leq \xi < r \leq \varpi \) with \( r - \xi \geq \frac{\varpi}{m_0} \),

\[
\alpha(\mathcal{Z}_r(\mathcal{P}_0)) \leq \alpha(\mathcal{Z}_{[\xi,r]}(\mathcal{P}_0)) \land \{ \vartheta(\mathcal{Z}_{\xi}(\mathcal{P}_0)) \}. \tag{20}
\]

On the other hand, we claim that

\[
\alpha(\mathcal{Z}_{[l,r]}(\mathcal{P}_0)) = 0, \quad \text{for any } 0 < l < r \leq \varpi.
\]

Actually, for any \( x \in \mathcal{Z}_{[l,r]}(\mathcal{P}_0) \subset PC([l,r], X) \), we have

\[
x(s,\varphi) = \mathcal{U}(s,0)\varphi(0) + \int_0^s \mathcal{U}(s,\xi)\mathfrak{A}(\xi,x(\xi))d\xi + \sum_{0 < t_i < s} \mathcal{U}(s,t_i)\mathfrak{A}(x(t_i)), \quad s \in [l,r], \quad \varphi \in \mathcal{P}_0.
\]

By our assumptions, it is known that if \( 0 \leq \kappa \leq q < 1 \), then for \( q - \kappa < \vartheta < 1 \), there is a constant \( C(\kappa, q, \vartheta) \) such that

\[
\| \mathcal{U}(t, h) \|_{\kappa, q} \leq C(\kappa, q, \vartheta)(t-h)^{-\vartheta}, \quad 0 \leq h < t \leq \varpi,
\]

where \( \| \cdot \|_{\kappa, q} \) is the norm of \( L(X_\kappa, X_\varphi) \). Here,

\[
X_\kappa = (D(\mathfrak{A}(0)^\kappa), \| \cdot \|_\kappa), \quad 0 \leq \kappa \leq 1,
\]

with

\[
\| x \|_\kappa = \| \mathfrak{A}(0)^\kappa x \|, \quad 0 \leq \kappa \leq 1,
\]

is a Banach space, and \( L(X_\kappa, X_\varphi) \) is the space of bounded linear operators from \( X_\kappa \) to \( X_\varphi \). Thus, we can deduce that \( \{ \mathcal{U}(\cdot, 0)\varphi(0) : \varphi \in \mathcal{P}_0 \} \) is precompact in \( C([l,r], X) \), and hence precompact in \( PC([l,r], X) \).

Now we mainly prove that \( \{ \int_0^t \mathcal{U}(\cdot, \xi)\mathfrak{A}(\xi,x(\xi))d\xi : \varphi \in \mathcal{P}_0 \} \) is precompact in \( C([l,r], X) \).
Let $s \in [l, r]$ be fixed. For $0 < \epsilon < l$, consider

$$\Psi_\epsilon \varphi(s) = \int_0^{s-\epsilon} \Psi(s, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi$$

$$= \Psi(s, s - \epsilon) \int_0^{s-\epsilon} \Psi(s, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi.$$

Based on (7) and (8), we conclude that the set \{\Psi_\epsilon \varphi(s) : \varphi \in \mathcal{P}_0\} is precompact in $X$ since $\Psi(s, s - \epsilon)$ is compact.

Moreover, we have

$$\| \int_0^s \Psi(s, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi - \Psi_\epsilon \varphi(s) \| \leq \int_0^s \| \Psi(s, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) \| d\xi$$

$$\leq \| \Psi(s, s, 0) \tilde{\mathcal{F}}(s, x(\xi), x_\xi) \|.$$ 

which implies that for any $s \in [l, r]$ fixed, \{\int_0^s \Psi(s, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi : \varphi \in \mathcal{P}_0\} is precompact in $X$.

For the equicontinuity, look at, for $l \leq s_1 < s_2 \leq r$,

$$\int_0^{s_2} \Psi(s_2, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi - \int_0^{s_1} \Psi(s_1, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi$$

$$= \int_0^{s_1} [\Psi(s_2, \xi) - \Psi(s_1, \xi)] \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi$$

$$+ \int_{s_1}^{s_2} \Psi(s_2, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi.$$

Let $\epsilon > 0$ be given. We can find $\eta > 0$ such that $\eta < l$ and

$$\| \int_{s_1 - \eta}^{s_1} [\Psi(s_2, \xi) - \Psi(s_1, \xi)] \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi \| \leq 2\| \tilde{\mathcal{F}} \| \leq \frac{\epsilon}{3}.$$ 

For

$$\int_0^{s_1 - \eta} [\Psi(s_2, \xi) - \Psi(s_1, \xi)] \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi,$$

note that $s_1 - h \geq \eta$, so that from (6), there is $q \in [s_1, s_2]$ such that

$$|\Psi(s_2, \xi) - \Psi(s_1, \xi)| = |\frac{\partial}{\partial t} \Psi(q, \xi)| |s_2 - s_1| \leq C \frac{1}{q - h} |s_2 - s_1| \leq \frac{C}{\eta} |s_2 - s_1|,$$

because $q \geq s_1$ and hence $q - h \geq s_1 - h \geq \eta$. Therefore, when $s_1$ and $s_2$ are close, we have

$$\| \int_0^{s_1 - \eta} [\Psi(s_2, \xi) - \Psi(s_1, \xi)] \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi \|$$

$$\leq \| \tilde{\mathcal{F}} \| \frac{C}{\eta} |s_2 - s_1| \leq \frac{\epsilon}{3}.$$ 

Also, when $s_1$ and $s_2$ are close, we have

$$\| \int_{s_1}^{s_2} \Psi(s_2, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi \| \leq \tilde{\mathcal{F}} \| s_2 - s_1 \| \leq \frac{\epsilon}{3}.$$ 

Therefore, \{\int_0^s \Psi(\cdot, \xi) \tilde{\mathcal{F}}(\xi, x(\xi), x_\xi) d\xi : \varphi \in \mathcal{P}_0\} is equicontinuous, and hence precompact in $C([l, r], X)$, and then precompact in $PC([l, r], X)$. 
So, we get, for $\cdot \in \mathbb{R}$

Hence, $\mathcal{Z}_{[t,r]}(\mathcal{P}_0) \subset PC([t,r], X)$ is precompact in $PC([t,r], X)$. Thus

$$\alpha(\mathcal{Z}_{[t,r]}(\mathcal{P}_0)) = 0.$$ 

Therefore, by virtue of (20), we obtain

$$\alpha(\Psi(\mathcal{P}_0)) = \alpha(\mathcal{Z}_{\mathbb{R}}(\mathcal{P}_0)) \leq \alpha(\mathcal{Z}_{[\mathbb{R} - \frac{m_0}{\mathbb{R}}, \mathbb{R}]}(\mathcal{P}_0)) \land \gamma(\mathcal{Z}_{\mathbb{R} - \frac{m_0}{\mathbb{R}}}(\mathcal{P}_0)) = \gamma(\mathcal{Z}_{\mathbb{R} - \frac{m_0}{\mathbb{R}}}(\mathcal{P}_0)) \leq \gamma\left(\alpha(\mathcal{Z}_{[\mathbb{R} - \frac{m_0}{\mathbb{R}}, \mathbb{R}]}(\mathcal{P}_0)) \land \left\{\gamma(\mathcal{Z}_{\mathbb{R} - \frac{m_0}{\mathbb{R}}}(\mathcal{P}_0))\right\}\right) = (\gamma)^2(\mathcal{Z}_{[\mathbb{R} - \frac{m_0}{\mathbb{R}}, \mathbb{R}]}(\mathcal{P}_0)) \land \left\{\gamma(\mathcal{Z}_{\mathbb{R} - \frac{m_0}{\mathbb{R}}}(\mathcal{P}_0))\right\}) \leq (\gamma)^{m_0 - 1}(\alpha(\mathcal{Z}_{[0, \mathbb{R} - (m_0 - 1)\frac{m_0}{\mathbb{R}}]}(\mathcal{P}_0)) \land \left\{\gamma(\mathcal{Z}_{\mathbb{R} - \frac{m_0}{\mathbb{R}}}(\mathcal{P}_0))\right\}). \quad (21)$$

Moreover, for $\cdot \in [0, \mathbb{R} - (m_0 - 1)\frac{m_0}{\mathbb{R}}] = [0, \mathbb{R} - m_0]$,

$$\mathcal{Z}_{[0, \mathbb{R} - (m_0 - 1)\frac{m_0}{\mathbb{R}}]}(\mathcal{P}_0) \subseteq \left\{\mathcal{U}(\cdot, 0)\varphi(0) : \varphi \in \mathcal{P}_0\right\}$$

$$\quad + \left\{\int_{0}^{t} \mathcal{U}(\cdot, \xi) \mathfrak{g}(\xi, x(\xi), x_\xi(\varphi)) d\xi : \varphi \in \mathcal{P}_0\right\}$$

$$\quad + \left\{\sum_{0 < t_i < \cdot} \mathcal{U}(\cdot, t_i) \mathfrak{I}_i(x(t_i)) : \varphi \in \mathcal{P}_0\right\}. \quad (22)$$

Clearly,

$$\|\mathcal{U}(t, 0)\varphi(0) - \mathcal{U}(t, 0)\varphi(0)\| = \|\mathcal{U}(t, 0)(\varphi(0) - \varphi(0))\| \leq \mathcal{U}^*\|\varphi(0) - \varphi(0)\|$$

$$\leq \mathcal{U}^*\|\varphi - \varphi\|_\mu, \quad t \in [0, \mathbb{R} - m_0].$$

So, for $\cdot \in [0, \mathbb{R} - m_0],$

$$\alpha\{\mathcal{U}(\cdot, 0)\varphi(0) : \varphi \in \mathcal{P}_0\} \leq \mathcal{U}^*\alpha(\mathcal{P}_0). \quad (23)$$

Hence,

$$\alpha\{\sum_{0 < t_i < \cdot} \mathcal{U}(\cdot, t_i) \mathfrak{I}_i(x(t_i)) : \varphi \in \mathcal{P}_0\} = 0, \quad \cdot \in [0, \mathbb{R} - m_0]. \quad (24)$$

For $\{\int_{0}^{s} \mathcal{U}(\cdot, \xi) \mathfrak{g}(\xi, x(\xi), x_\xi(\varphi)) d\xi : \varphi \in \mathcal{P}_0\}$ on $[0, \mathbb{R} - m_0]$, the interval involves the zero, hence we have to treat it in the following way. For $s \in [0, \mathbb{R} - m_0]$, we have

$$\|\int_{0}^{s} \mathcal{U}(s, \xi) \mathfrak{g}(\xi, x(\xi), x_\xi(\varphi)) d\xi\| \leq \mathcal{U}^*\mathfrak{g}^* \frac{\mathbb{R}}{m_0}.$$ 

So, we get, for $\cdot \in [0, \mathbb{R} - m_0],$

$$\alpha\{\int_{0}^{s} \mathcal{U}(\cdot, \xi) \mathfrak{g}(\xi, x(\xi), x_\xi(\varphi)) d\xi : \varphi \in \mathcal{P}_0\} \leq 2\mathcal{U}^*\mathfrak{g}^* \frac{\mathbb{R}}{m_0}. \quad (25)$$
By (22)-(25), we obtain
\[ \alpha(\mathcal{Z}_{\left[0, \varpi - (K - 1) \frac{\varpi}{m_0}\right]}(P_0)) \leq \mu^* \alpha(P_0) + 2\mu^* \frac{\varpi}{m_0}. \] (26)

Therefore, it follows from (9), (21) and (26) that
\[ \alpha(P_0) \leq (\Upsilon)^{m_0-1} \left( \{ \mu^* \alpha(P_0) + 2\mu^* \frac{\varpi}{m_0} \} \wedge \{ \Upsilon \alpha(P_0) \} \right) \]
\[ < \tau \alpha(P_0) + (\Upsilon)^{m_0-1} \frac{\varpi}{m_0}. \]

Since \( \tau > 0 \) is arbitrary, we obtain
\[ \alpha(\mathcal{P}(P_0)) \leq \tau \alpha(P_0) < \alpha(P_0). \]

This completes the proof. \( \square \)

4. Periodic solutions to (3)-(5). In this section, as an application of Sadovskii’s Fixed Point Theorem, we study the periodic solutions of the problem (3)-(5). First, we note from [11, 8, 9, 10] that fixed points of \( \mathcal{P} \) give rise to periodic solutions of the problem (3)-(5). Then from Lemma 2.2 and Theorem 3.3, we have

**Theorem 4.1.** Let Assumptions 2.3 and 2.5 hold. Let the operator \( \mathcal{P} \) be defined by (19) in \( C_\mu \). If there exists a convex, closed and bounded set \( E \subset C_\mu \) such that \( \mathcal{P}(E) \subseteq E \), then \( \mathcal{P} \) has a fixed point in \( E \). Moreover, the problem (3)-(5) has a \( \varpi \)-periodic solution.

In order to get the periodic solutions by virtue of Sadovskii’s Fixed Point Theorem, we need the following concept of the Locally Strictly Boundedness of the solutions of the problem (3)-(5): If there exists a constant \( W > 0 \) such that \( |\varphi|_\mu \leq W \) implies that its solution satisfies
\[ \|x(t, \varphi)\| \leq W, \quad t \in [0, \varpi], \]
then we call the solutions of the problem (3)-(5) are Locally Strictly Bounded.

We now present the existence theorem for the periodic solutions of the problem (3)-(5).

**Theorem 4.2.** Let Assumptions 2.3 and 2.5 hold. If the solutions of the problem (3)-(5) are locally strictly bounded (or assume that solutions are non-increasing in norm \( \|\cdot\| \) on \([0, \varpi]\)), then the problem (3)-(5) has a \( \varpi \)-periodic solution.

**Proof.** Let
\[ E = \{ \varphi \in C_\mu : |\varphi|_\mu \leq W \}. \]
Then \( E \) is convex, closed and bounded in \( C_\mu \).

Moreover, for \( x(\cdot) = x(\cdot, \varphi) \) with \( \varphi \in E \), by the Locally Strictly Boundedness, we have
\[ \|x(t)\| \leq W, \quad t \in [0, \varpi]. \]

On the other hand, from the proof of Theorem 3.2, we can see that if \( x \) is a piecewise continuous function on \( (-\infty, \varpi] \) such that \( |x|_\mu \) is finite for every \( t \in [0, \varpi] \), then for any \( 0 \leq \xi < r \leq \varpi \) with \( r - \xi \geq \frac{\varpi}{m_0} \),
\[ |x_r|_\mu \leq \sup_{s \in [\xi, r]} \|x(s)\| \wedge \{ \Upsilon |x_\xi|_\mu \}. \] (27)

By (22)-(25), we obtain
\[ \alpha(\mathcal{Z}_{\left[0, \varpi - (K - 1) \frac{\varpi}{m_0}\right]}(P_0)) \leq \mu^* \alpha(P_0) + 2\mu^* \frac{\varpi}{m_0}. \] (26)

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\[ E = \{ \varphi \in C_\mu : |\varphi|_\mu \leq W \}. \]
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\[ |x_r|_\mu \leq \sup_{s \in [\xi, r]} \|x(s)\| \wedge \{ \Upsilon |x_\xi|_\mu \}. \] (27)
Therefore, for the operator $\mathcal{P}$ defined by (19) in $C_{\mu}$, we infer that
\[
|\mathcal{P}(\varphi)|_{\mu} = |x_{\varphi}(\varphi)|_{\mu} \leq \sup_{s \in [0, \varpi]} \|x(s)\| \wedge \left\{ \Upsilon|\varphi|_{\mu} \right\} \leq W. \tag{28}
\]
Thus, in view of Theorem 4.1, we see that the problem (3)-(5) has a $\varpi$ periodic solution. This completes the proof.

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