A MODEL STRUCTURE ON \textbf{Cat}\textsubscript{Top}

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Abstract. In this article, we construct a cofibrantly generated Quillen model structure on the category of small topological categories \textbf{Cat}\textsubscript{Top}. It is Quillen equivalent to the Joyal model structure of \((\infty, 1)\)-categories and the Bergner model structure on \textbf{Cat}\textsubscript{sSet}.

INTRODUCTION

In the section \cite{1} we construct a Quillen model structure on the category of small topological categories \textbf{Cat}\textsubscript{Top} \cite{1}. The main advantage is the fact that all objects in \textbf{Cat}\textsubscript{Top} are fibrant. We show that this model structure is Quillen equivalent to the model structure on the category of small simplicial categories \textbf{Cat}\textsubscript{sSet} defined in \cite{3}.

Why we are interested on topological categories? In \cite{8}, it is shown that any model category \textbf{M} is naturally enriched over \textbf{sSet} or \textbf{Top}. The enrichment gives us a higher homotopical information about \textbf{M}.

In the topological setting, the cohomology theories are defined directly from the mapping space in the model category of topological spectra. Our future goal is to define algebraic \(K\)-theory \cite{2} for a larger class of categories.

1. CATEGORY OF SMALL TOPOLOGICAL CATEGORIES.

In this article, the category of weakly Hausdorff compactly generated topological spaces will be denoted by \textbf{Top} which is simplicial monoidal model category. Before to start the main theorem of this section we will introduce some notations and definitions.

A topological category is a category enriched over \textbf{Top}. The Category of all (small) topological categories is denoted by \textbf{Cat}\textsubscript{Top}. The morphisms in \textbf{Cat}\textsubscript{Top} are the enriched functors. It is complete and cocomplete category.

\textbf{Theorem 1.1.} \cite{1} The category \textbf{Cat}\textsubscript{Top} admit a cofibrantly generated model structure defined as follows. The weak equivalences \(F : C \rightarrow D\) satisfy the following conditions.

WT1 : The morphism \(\text{Map}_C(a, b) \rightarrow \text{Map}_D(Fa, Fb)\) is a weak equivalence in the category \textbf{Top}.

WT2 : The induced morphism \(\pi_0 F : \pi_0 C \rightarrow \pi_0 D\) is a categorical equivalence in \textbf{Cat}.

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The fibrations are the morphisms $F : C \to D$ which satisfy:

FT1: The morphism $\text{Map}_C(a, b) \to \text{Map}_D(Fa, Fb)$ is a fibration in $\text{Top}$.

FT2: For each objects $a$ and $b$ in $C$, and a weak equivalence of homotopy $e : F(a) \to b$ in $D$, there exists an object $a_1$ in $C$ and a weak homotopy equivalence $d : a \to a_1$ in $C$ such that $Fd = e$.

More over, the set $I$ of generating cofibrations is given by:

CT1: $|U\partial\Delta^n| \to |U\Delta^n|$, for $n \geq 0$.

CT2: $\emptyset \to \{x\}$, where $\emptyset$ is the empty topological category and $\{x\}$ is the category with one object and one morphism.

The set $J$ of generating acyclic cofibrations is given by:

ACT1: $|U\Lambda^n_i| \to |U\Delta^n|$, for $0 \leq n$ and $0 \leq i \leq n$.

ACT2: $\{x\} \to |H|$ where $\{H\}$ as defined in [3].

Remark 1.2. All objects in $\text{Cat}_{\text{Top}}$ are fibrant.

2. Proof of the main theorem

We start by a useful lemma which gives us conditions to transfer a model structure by adjunction.

Lemma 2.1. [13, proposition 3.4.1] Let an adjunction

$$\begin{eqnarray*}
M & \xrightarrow{G} & C \\
\xleftarrow{F} & & \xleftarrow{F} \end{eqnarray*}$$

where $M$ is cofibrantly generated model category, with $I$ generating cofibrations and $J$ generating trivial cofibrations. We pose

(1) $W$ The class of morphisms in $C$ such the image by $F$ is a weak equivalence in $M$.

(2) $F$ The class of morphisms in $C$ such the image by $F$ is a fibration in $M$.

We suppose that the following conditions are verified:

(1) The domain of $G(i)$ are small with respect to $G(I)$ for all $i \in I$ and the domains of $G(j)$ are small with respect to $G(J)$ for all $j \in J$.

(2) The functor $F$ commutes with directed colimits i.e.,

$$F\text{colim}(\lambda \to C) = \text{colim}F(\lambda \to C).$$

(3) Every transfinite composition of weak equivalences in $M$ is a weak equivalence.

(4) The pushout of $G(j)$ by any morphism $f$ in $C$ is in $W$.

Then $C$ form a model category with weak equivalences (resp. fibrations) $W$ (resp. $F$). More over it is cofibrantly generated with generating cofibrations $G(I)$ and generating trivial cofibrations $G(J)$.

We prove the main theorem using [24]

Lemma 2.2. The pushout of $|UA^n_i| \to |U\Delta^n|$ by a morphism $F : |UA^n_i| \to D$ is a weak equivalence.

Proof. See [5, 6] \qed

Lemma 2.3. The pushout of $\{x\} \to |H|$ by $\{x\} \to C$ is a weak equivalence for all $C \in \text{Cat}_{\text{Top}}$. 

Proof. Let $\mathcal{O}$ the set of objects of $\mathbf{C}$ without the object \{x\} touched by the morphism \{x\} $\to$ $\mathbf{C}$. We note by $x$, $y$ objects of $|\mathcal{H}|$. The goal is to prove that $h$ defined in the following pushout is a weak equivalence

$$
\begin{array}{c}
\{x\} \to \\
\downarrow \\
|\mathcal{H}| \to \\
\end{array}
\xrightarrow{h}
\begin{array}{c}
\mathbf{C} \\
\downarrow \\
\mathbf{D} \\
\end{array}
$$

Observe that there is an other double pushout

$$
\begin{array}{c}
\{x\} \sqcup \mathcal{O} \\
\downarrow \\
\mathbf{C} \\
\end{array}
\xrightarrow{i}
\begin{array}{c}
\{x, y\} \sqcup \mathcal{O} \\
\downarrow \\
\mathbf{C} \sqcup \{y\} \\
\end{array}
\xrightarrow{h'}
\begin{array}{c}
|\mathcal{H}| \sqcup \mathcal{O} \\
\downarrow \\
\mathbf{D}. \\
\end{array}
$$

Which is a consequence of:

$$
|\mathcal{H}| \sqcup \mathcal{O} \bigsqcup_{\mathcal{O} \sqcup \{x, y\}} \mathbf{C} \sqcup \{y\} = |\mathcal{H}| \bigsqcup_{\{x, y\}} \mathbf{C} \sqcup \{y\} = |\mathcal{H}| \bigsqcup_{\{y\}} \mathbf{C} = \mathbf{D}.
$$

The morphism $h'$ is a natural extension of $h$, i.e., $h' \circ i = h$.

On the other hand, the counity $c : |\text{sing}\mathbf{C}| \to \mathbf{C}$ is a weak equivalence. Consider the following pushout in $\mathbf{Cat}_{\text{sSet}}$:

$$
\begin{array}{c}
\{x\} \sqcup \mathcal{O} \\
\downarrow \\
\text{sing}\mathbf{C} \\
\end{array}
\xrightarrow{i}
\begin{array}{c}
\{x, y\} \sqcup \mathcal{O} \\
\downarrow \\
\text{sing}(\mathbf{C} \sqcup \{y\}) \\
\end{array}
\xrightarrow{f'}
\begin{array}{c}
\mathcal{H} \sqcup \mathcal{O} \\
\downarrow \\
\mathbf{D}'. \\
\end{array}
$$

Since $\mathbf{Cat}_{\text{sSet}}$ is a model category, we have that $f = f' \circ i$ is a weak equivalence. Consequently $|f|$ is a weak equivalence in $\mathbf{Cat}_{\text{Top}}$.

As before $f'$ is an extension of $f$.

Using the fact that the functor $| - |$ commutes with colimits, the diagram of the following double pushout permit to conclude:

$$
\begin{array}{c}
|\text{sing}\mathbf{C}| \\
\downarrow \\
\mathbf{C} \\
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\{x, y\} \sqcup \mathcal{O} \\
\downarrow \\
|\text{sing}(\mathbf{C} \sqcup \{y\})| \\
\end{array}
\xrightarrow{c}
\begin{array}{c}
\mathbf{C} \sqcup \{y\} \\
\downarrow \\
\mathbf{C} \sqcup \{y\} \\
\end{array}
\xrightarrow{h'}
\begin{array}{c}
|\mathcal{H}| \sqcup \mathcal{O} \\
\downarrow \\
|h'| \\
\end{array}
\xrightarrow{m}
\begin{array}{c}
|\mathbf{D}'| \\
\downarrow \\
\mathbf{D}. \\
\end{array}
$$
In Fact,

\[ m : D = (|\mathcal{H}| \sqcup O) \star \text{sing}(C \sqcup \{y\}) \rightarrow (|\mathcal{H}| \sqcup O) \star (C \sqcup \{y\}) = D' \]

is a weak equivalence by 5.8. We have seen that \(|f|\) is a weak equivalence, so by the property "2 out of 3" we conclude that \(h\) is a weak equivalence. \(\square\)

**Lemma 2.4.** The functor \(\text{sing}\) commutes with directed colimits.

**Proof.** Let \(\lambda\) be an ordinal and let

\[ C = \text{colim}_\lambda C_\lambda, \]

a directed colimit in \(\text{Cat}_{\text{Top}}\). If \(a'\) and \(b'\) are two objects in \(C\), then by definition, there exists an index \(t\) such that they are represented by \(a, b \in C_t\), and \(\text{Map}_{C_t}(a', b')\) is a colimit of the following diagram:

\[ \text{Map}_{C_t}(a, b) \rightarrow \cdots \text{Map}_{C_s}(a_s, b_s) \rightarrow \text{Map}_{C_{s+1}}(a_{s+1}, b_{s+1}) \rightarrow \cdots, \]

where \(C_{a,b}^t\) is a full subcategory of \(C_t\) with only two objects \(a, b\). Since the functor \(\text{Ob} : \text{Cat} \rightarrow \text{Set}\) and the functor \(\text{sing} : \text{Top} \rightarrow \text{sSet}\) commute with directed colimits, we have that \(\text{sing} : \text{Cat}_{\text{Top}} \rightarrow \text{Cat}_{\text{sSet}}\) commutes with directed colimits. \(\square\)

**Lemma 2.5.** The objects \(|U\Lambda^n|\), \(|U\Delta^n|\) and \(|\mathcal{H}|\) are small in \(\text{Cat}_{\text{Top}}\).

**Proof.** It is a consequence of the fact that \(U\Lambda^n\), \(U\Delta^n\), \(\mathcal{H}\) are small in \(\text{Cat}_{\text{sSet}}\) and \(\text{sing} : \text{Cat}_{\text{Top}} \rightarrow \text{Cat}_{\text{sSet}}\) commutes with directed colimits. \(\square\)

**Lemma 2.6.** The transfinite composition of weak equivalences in \(\text{Cat}_{\text{sSet}}\) is a weak equivalence.

**Proof.** It is a consequence that the transfinite composition of weak equivalences in \(\text{sSet}\) and \(\text{Cat}\) is a weak equivalence. Note that \(\pi_0 : \text{Cat}_{\text{sSet}} \rightarrow \text{Cat}\) commutes with colimits because it admits a right adjoint: the Forgetful functor which correspond to each topological enriched category \(\mathbf{C}\) an trivially enriched category i.e., we forget the topology of \(\mathbf{C}\). \(\square\)

**Corollary 2.7.** The category \(\text{Cat}_{\text{Top}}\) is a cofibrantly generated model category Quillen equivalent to \(\text{Cat}_{\text{sSet}}\).

3. Graphs and Categories

In this paragraph, we define an adjunction between \(\text{Cat}_{\text{Top}}\) and the categories of enriched graphs on \(\text{Top}\). This adjunction is constructed in the particular case where the set of objects is fixed. We will denote \(\mathcal{O} = \text{Cat}_{\text{Top}}\) the category of small enriched categories over \(\text{Top}\) with fixed set of objects \(\mathcal{O}\), the morphisms are those functors which are identities on objects. By the same way, we define the category of small graphs enriched over \(\text{Top}\) by \(\mathcal{O} = \text{Graph}_{\text{Top}}\) with a fixed set of vertices \(\mathcal{O}\) There exists an adjunction between \(\mathcal{O} = \text{Cat}_{\text{Top}}\) and \(\mathcal{O} = \text{Graph}_{\text{Top}}\) given by the forgetful functor and the free functor. Before starting, we define the free functor between graphs and categories. First we study the case where \(\mathcal{O}\) is a set with one element.

**Lemma 3.1.** There exists a right adjoint to the forgetful functor \(U : \text{Mon} \rightarrow \text{Top}\) where \(\text{Mon}\) is the category of topological monoids.
Proof. Let $X$ in $\textbf{Top}$, we define

$$L(X) = * \sqcup X \sqcup (X \times X) \sqcup (X \times X \times X) \sqcup \ldots;$$

it is a functor from $\textbf{Top}$ to topological monoids.

It is easy to see that $L: \textbf{Top} \to \textbf{Mon}$ is a well defined functor. In fact, it is the desired functor. Let $M$ be a topological monoids, a morphism of monoid $L(X) \to M$ is given by a morphism of non pointed topological spaces $X \to U(M).$ This morphism extends in an unique way in a morphism of monoids if we consider the following morphisms in $\textbf{Top}$:

$$X \times X \cdots \times X \to M \times M \cdots \times M \to M.$$

We conclude that:

$$\text{hom}_{\textbf{Top}}(X, U(M)) = \text{hom}_{\textbf{Mon}}(L(X), M).$$

□

For a generalization of the precedent adjunction to an adjunction between $\textbf{O} - \text{Cat}_{\textbf{Top}}$ and $\textbf{O} - \text{Graph}_{\textbf{Top}}$, we do ass follow: We pose $\textbf{O}$ the trivial category with set of object $\textbf{O}$. For each graph $\Gamma$ in $\textbf{O} - \text{Graph}_{\textbf{Top}}$ we define the set of the following categories indexed by a pair of element $a, b \in \textbf{O}$

$$\Gamma_{a,b}(c,d) = \begin{cases} \Gamma(c,d) & \text{if } c = a \neq b = d \\ L(\Gamma(c,d)) & \text{if } a = c = b = d \\ \emptyset & \text{if } c \neq d \text{ and } a \neq c \land b \neq d \\ * = id & \text{else} \end{cases}$$

Let $\Gamma$ a graph in $\textbf{O} - \text{Graph}_{\textbf{Top}}$, we define the free category induced by the graph as a free product in the category $\textbf{O} - \text{Cat}_{\textbf{Top}}$ of all categories of the form $\Gamma_{a,b}$, more precisely

$$L(\Gamma) = \ast_{(a,b)\in \textbf{O} \times \textbf{O}} \Gamma_{a,b}.$$

By the free product, we mean the following colimit in $\textbf{Cat}_{\textbf{Top}}$:

$$\text{colim}_{(a,b)\in \textbf{O} \times \textbf{O}} \Gamma_{a,b}.$$

4. Realization

Let $\textbf{M}$ be a simplicial model category (i.e., tensored and cotensored in a suitable way). The category $[\Delta^{op}, \textbf{M}]$ is a model category with Reedy model structure (cf [7]) where the weak equivalences are defined degrewise.

Definition 4.1. The realization functor

$$| - |: [\Delta^{op}, \textbf{M}] \to \textbf{M}$$

is defined as follow:

$$\bigcup_{\phi: [n] \to [m]} M_m \otimes \Delta^n \xrightarrow{d_0} \bigcup_{[n]} M_n \otimes \Delta^n \xrightarrow{d_1} |M_*|$$

\(\alpha d_0 = \phi^* \otimes id\) and \(d_1 = id \otimes \phi\).

Lemma 4.2. Since $\textbf{M}$ is a simplicial category, the functor $| - |$ admit a right adjoint:

$$(-)^\Delta: \textbf{M} \to [\Delta^{op}, \textbf{M}]: M \mapsto M^\Delta.$$
Lemma 4.3. [7, VII, proposition 3.6] Let $M$ a simplicial model category and $[\Delta^{op}, M]$ a Reedy model category, then the realization functor

$$| - | : [\Delta^{op}, M] \to M$$

is a left Quillen functor.

Now, we specify to $M = \text{Top}$. In this particular case, $[\Delta^{op}, \text{Top}]$ is a monoidal category (the monoidal structure is defined degree wise form the monoidal structure of $\text{Top}$). So, the realization functor $| - | : [\Delta^{op}, \text{Top}] \to \text{Top}$ commutes with the monoidal product (cf [6], chapitre X, proposition 1.3).

Corollary 4.4. The realization functor $| - | : [\Delta^{op}, \text{Top}] \to \text{Top}$ preserve the homotopy equivalences.

In the practice, the lemma 4.3 is difficult to use. It is quite-difficult to show that an object in $[\Delta^{op}, M]$ is Reedy cofibrant. In l’appendice A of [12], Segal gives us an alternative solution in the particular case of $[\Delta^{op}, \text{Top}]$.

Lemma 4.5. There exist a functor $|| - || : [\Delta^{op}, \text{Top}] \to \text{Top}$, called good realization with the following properties:

1. Let $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $[\Delta^{op}, \text{Top}]$ such that if $f_n : X_n \to Y_n$ is a weak equivalence for all $n \in \mathbb{N}$, then $||f_\bullet|| : ||X_\bullet|| \to ||Y_\bullet||$ is a weak equivalence in $\text{Top}$;

2. There exists a natural transformation $N : || - || \to | - |$, with the property that for all good simplicial topological space $X_\bullet$, the natural morphism:

$$N_{X_\bullet} : ||X_\bullet|| \to |X_\bullet|$$

is a weak equivalence in $\text{Top}$;

3. The natural morphism $||X_\bullet \times Y_\bullet|| \to ||X_\bullet|| \times ||Y_\bullet||$ is a weak equivalence in $\text{Top}$.

For the details we refer to [12].

Lemma 4.6. There exists an endofunctor $\tau : [\Delta^{op}, \text{Top}] \to [\Delta^{op}, \text{Top}]$ and a natural transformation $Q : \tau \to \text{id}$ with the following properties:

1. $\tau X_\bullet$ is a good simplicial topological space for all $X_\bullet \in [\Delta^{op}, \text{Top}]$;

2. The natural morphism $Q_n : \tau_n(X_\bullet) \to X_n$ is a weak equivalence for all $n \in \mathbb{N}$;

3. The natural morphism $||X_\bullet|| \to |\tau(X_\bullet)||$ is a weak equivalence;

4. Finally, we have $\tau_0(X_\bullet) = X_0$.

Corollary 4.7. Let $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $[\Delta^{op}, \text{Top}]$, such that $f_n$ is a weak equivalence for all $n$, then

$$|\tau(f_\bullet)| : |\tau(X_\bullet)| \to |\tau(Y_\bullet)|$$

is a weak equivalence of topological spaces.

Proof. It is a direct consequence from 4.5 and 4.6.

We can see the functor $\tau$ as kind of cofibrant replacement. It is useful to know how to describe the functor $\tau$. 
Definition 4.8. [12]. Appendice A] Let $A_\bullet$ a simplicial topological space and $\sigma$ a subset of $\{1, \ldots, n\}$. We pose:

1. $A_{n,i} = s_i A_n$.
2. $A_{n,\sigma} = \cap_{i \in \sigma} A_{n,i}$.
3. $\tau_n(A_\bullet)$ is a union of all subsets $[0,1]^\sigma \times A_{n,\sigma}$ of $[0,1]^n \times A_n$.

The morphism $\tau(A_\bullet) \to A_\bullet$ collapses $[0,1]^\sigma$ and inject $A_{n,\sigma}$ in $A_n$.

Lemma 4.9. The functor $\tau$ sends homotopy equivalences to homotopy equivalences.

Proof. Let $h : X_\bullet \times [0, 1] \to Y_\bullet$ be a homotopy between $t$ and $s$. By definition of $\tau$, we have

$$\tau_n(X_\bullet \times [0,1]) = \bigcup_{\sigma \in \{1, \ldots, n\}} [0,1]^\sigma \times (X_\bullet \times [0,1])_{n,\sigma}$$

$$= \bigcup_{\sigma \in \{1, \ldots, n\}} ([0,1]^\sigma \times X_{n,\sigma} \times [0,1])$$

$$= (\bigcup_{\sigma \in \{1, \ldots, n\}} [0,1]^\sigma \times X_{n,\sigma}) \times [0,1]$$

$$= \tau_n(X_\bullet) \times [0,1].$$

Consequently $\tau(h) : \tau(X_\bullet) \times [0,1] \to \tau(Y_\bullet)$ is a homotopy between $\tau(t)$ and $\tau(s)$. □

Definition 4.10. a strong section $f : X \to Y$ is a continues application $i : Y \to X$ such that $f \circ i = id_Y$ and such that there exists a homotopy between $i \circ f$ and $id_X$ which fix $Y$.

Corollary 4.11. The functor $\tau$ preserve strong sections.

Proof. It is a consequence of the lemma 4.9 and that $\tau$ is a functor so it preserves the identities. □

Corollary 4.12. If $X$ is a constant simplicial topological space, then $Q_X : \tau(X) \to X$ admit a strong section.

Proof. The section $i : X \to \tau(X)$ is induced by the identity on $X$. To show that it is a strong section, it is suffisant to see that $\tau_n(X) = [0,1]^n \times X$ by definition. □

5. Pushouts in $\textbf{Cat}_V$

We define and compute some (simple) pushouts in the category of small enriched categories $\mathbf{V} - \textbf{Cat}$. In our example $\mathbf{V}$ is the category $\mathbf{sSet}$ or $\mathbf{Top}$. For more details see ([11], A.3.2).

Definition 5.1. Let $U : \mathbf{V} \to \textbf{Cat}_V$ be a functor defined as follow:

For each object $S \in \mathbf{V}$, $U(S)$ is the enriched category with two objects $x$ and $y$ such that $\text{Map}_{U(S)}(x, y) = S$.

Let $f : S \to T$ be a morphism in $\mathbf{V}$ and $C$ an enriched category on $\mathbf{V}$. We want to describe explicitly the following pushout diagram:
It is enough clear that the objects of $\mathbf{C}$ and $\mathbf{D}$ are the same. The difficult part is to define $\text{Map}_\mathbf{D}$.

Let $w, z \in \mathbf{C}$ and define the following sequence of objects in $\mathbf{V}$:

- $M^0_C = \text{Map}_\mathbf{C}(w, z)$.
- $M^1_C = \text{Map}_\mathbf{C}(y, z) \times T \times \text{Map}_\mathbf{C}(w, x)$.
- $M^2_C = \text{Map}_\mathbf{C}(y, z) \times T \times \text{Map}_\mathbf{C}(y, x) \times T \times \text{Map}_\mathbf{C}(w, x)$.

... 

More generally, an object of $M^k_C$ is given by a finite sequence of the form

$$(\sigma_0, \tau_1, \sigma_1, \tau_2, \ldots, \tau_k, \sigma_k)$$

where $\sigma_0 \in \text{Map}_\mathbf{C}(y, z)$, $\sigma_k \in \text{Map}_\mathbf{C}(w, x)$, $\sigma_i \in \text{Map}(y, x)$ for $0 < i < k$, and $\tau_i \in T$ for $0 < i \leq k$.

We define $\text{Map}_\mathbf{D}(w, z)$ as a quotient $\bigsqcup_k M^k_C$ relative to the following relations:

$$(\sigma_0, \tau_1, \ldots, \sigma_k) \sim (\sigma_0, \tau_1, \ldots, \tau_{j-1}, \sigma_{j-1} \circ h(\tau_j) \circ \sigma_j, \tau_{j+1}, \ldots, \sigma_k),$$

when $\tau_j$ is an element of $S \subset T$.

The category $\mathbf{D}$ is equipped with the following associative composition:

$$(\sigma_0, \tau_1, \ldots, \tau_k, \sigma_k) \circ (\sigma'_0, \tau'_1, \ldots, \tau'_i) = (\sigma_0, \tau_1, \ldots, \tau_k, \sigma_k \circ \sigma'_0, \tau'_1, \ldots, \tau'_i).$$

Observe that there is a natural filtration on $\text{Map}_\mathbf{D}(w, z)$:

$$\text{Map}_\mathbf{D}(w, z) = \text{Map}_\mathbf{D}(w, z)^0 \subset \text{Map}_\mathbf{D}(w, z)^1 \subset \ldots$$

where $\text{Map}_\mathbf{D}(w, z)^k$ is defined as image of $\bigsqcup_{0 \leq i \leq k} M^i_C$ in $\text{Map}_\mathbf{D}(w, z)$ and

$$\bigsqcup_k \text{Map}_\mathbf{D}(w, z)^k = \text{Map}_\mathbf{D}(w, z).$$

The most important fact is that $\text{Map}_\mathbf{D}(w, z)^k \subset \text{Map}_\mathbf{D}(w, z)^{k+1}$ is constructed as pushout of the inclusion: $N^{k+1}_C \subset M^{k+1}_C$, where $N^{k+1}_C$ is a subobject of $M^{k+1}_C$ of $(2m + 1)$-tuples $(\sigma_0, \tau_1, \ldots, \sigma_m)$ such that $\tau_i \in S$ for at least one $i$.

5.1. Monads. The main goal of this section is to generalize the section 2 de l’article [5] to the categories enriched over $\text{Top}$.

Every adjunction define a monad and a comonad. We are interested on the particular adjunction

$$\mathcal{O} \dashv \text{Graph}_{\text{Top}} \xrightarrow{L} \mathcal{O} \dashv \text{Cat}_{\text{Top}}$$

We have a monad $T = UL$ and a comonad $F = LU$. The multiplication on $T$ is denoted by $\mu : TT \to T$ and the unity $\eta : id \to T$, the comultiplication by $\psi : F \to FF$ and finally the counity by $\phi : F \to id$. The $T-$algebras are exactly those graphs which have a structure of a category (composition).
Notation 5.2. The category of small categories enriched over $\mathbf{Top}$ and with fixed set of objects $\mathcal{O}$ is noted by $\mathcal{O} - \mathbf{Cat}_{\text{Top}}$. We note by $\mathcal{O} - \mathbf{sCat}_{\text{Top}}$ the category of presheaves $[\Delta^{op}, \mathcal{O} - \mathbf{Cat}_{\text{Top}}]$ and $\mathcal{O} - \mathbf{sGraph}_{\text{Top}}$ the category of presheaves $[\Delta^{op}, \mathcal{O} - \mathbf{Graph}_{\text{Top}}]$. If we note $[\Delta^{op}, \mathbf{Top}]$ by $s\mathbf{Top}$ then we have $\mathcal{O} - \mathbf{sCat}_{\text{Top}} = \mathcal{O} - \mathbf{Cat}_{s\mathbf{Top}}$, and $\mathcal{O} - \mathbf{sGraph}_{\text{Top}} = \mathcal{O} - \mathbf{Graph}_{s\mathbf{Top}}$.

5.1.1. Simplicial resolution. Let $C$ be an object of $\mathcal{O} - \mathbf{Cat}_{\text{Top}}$. We define the iterated composition of $F$ by:

$$F^k = F \circ F \cdots \circ F.$$  

The comonad $F$ gives us a simplicial resolution $C$ (cf [5]) defined as follow:

$$F_k C = F^{k+1} C,$$

With faces and degeneracies:

$$F_k C \xrightarrow{d_i = F^i \phi^{k-1}} F_{k-1} C$$

$$F_k C \xrightarrow{s_i = F^i \psi^{k-1}} F_{k+1} C$$

The category of compactly generated spaces $\mathbf{Top}$ is a simplicial model category (tensted and cotensored over $s\mathbf{Set}$) So we have:

1. In $\mathcal{O} - \mathbf{sCat}_{\text{Top}}$ we have the morphism $f : F_{\bullet} C \to C$, where $C$ is sow as a constant object in $\mathcal{O} - \mathbf{sCat}_{\text{Top}}$ and $t f_k = \phi^{k+1}$.
2. The morphism $f$ admit a section $i : C \to F_{\bullet} C$ in the category $\mathbf{Graph}_{s\mathbf{Top}}$. The section $i$ is induced by the unity of the monad $T$ i.e., $\eta_{UC} : UC \to ULUC$.
3. The adjunction $[\Delta^{op}, \mathbf{Top}] \xrightarrow{(-)^{\Delta}} \mathbf{Top}$, induce the following adjunction $\mathcal{O} - \mathbf{Cat}_{s\mathbf{Top}} \xrightarrow{(-)^{\Delta}} \mathcal{O} - \mathbf{Cat}_{\text{Top}}$, since the realization functor is monoidal.
4. The realization of the morphism $f$ in $\mathcal{O} - \mathbf{sCat}_{\text{Top}}$ induce a weak equivalence i.e., $[f] : \mathbf{Map}_{F_{\bullet} C}(a, b) \to \mathbf{Map}_{C}(a, b)$ is a weak equivalence in $\mathbf{Top}$ for all $a, b \in \mathcal{O}$.

Remark 5.3. The realization functor $| - |$ does not ”see” the category structure, but only the graph structure.
More generally, for all \( C, D \) in \( \mathcal{O} - \mathbf{Cat}_{\text{Top}} \) the following morphism:

\[
F_a(C) \ast D \longrightarrow C \ast D
\]

admit a strong section \( C \ast D \to F_a(C) \ast D \) in the category \( \mathcal{O} - \mathbf{sGraph}_{\text{Top}} \). In fact, The category \( \mathcal{O} - \mathbf{Graph}_{\text{Top}} \) is monoidal (nonsymmetric) with monoidal product \( \times_{\mathcal{O}} \) which is a generalization of \(([10], II, 7)\). A topologically enriched category is a monoid with respect to this monoidal product. The free product \( C \ast D \) is constructed in \( \mathcal{O} - \mathbf{Graph}_{\text{Top}} \) as

\[
\mathcal{O}^c \sqcup_{\mathcal{O}} C \sqcup_{\mathcal{O}} D \sqcup_{\mathcal{O}} \left( C' \times_{\mathcal{O}} C' \right) \sqcup_{\mathcal{O}} \left( D' \times_{\mathcal{O}} D' \right) \sqcup_{\mathcal{O}} \left( C' \times_{\mathcal{O}} D' \right) \sqcup_{\mathcal{O}} \left( D' \times_{\mathcal{O}} C' \right) \ldots
\]

where \( C' \) (resp. \( D' \)) is a correspondent graph of \( C \) (resp. \( D \)) without identities and \( \mathcal{O}^c \) is the trivial category obtained from the set \( \mathcal{O} \). So \( C \ast D \to F_a(C) \ast D \) is induced by the section \( i : C \to F_a(C) \) and \( id : D \to D \). consequently the morphism

\[
\text{Map}_{C \ast D}(a, b) \to \text{Map}_{F_a(C) \ast D}(a, b) = \text{Map}_{F_a(C) \ast D}(a, b)
\]

is a weak equivalence in \( \mathbf{Top} \) for all objects \( a, b \in \mathcal{O} \).

**Lemma 5.4.** Let \( C \to D \) a weak equivalence in \( \mathcal{O} - \mathbf{Cat}_{\text{Top}} \) and let \( \Gamma \) a graph in \( \mathcal{O} - \mathbf{Graph}_{\text{Top}} \), the the induced morphism :

\[
L(\Gamma) \ast C \to L(\Gamma) \ast D
\]

is a weak equivalence in the category \( \mathcal{O} - \mathbf{Cat}_{\text{Top}} \).

**Proof.** It is enough to prove that \( C' = L(\Gamma)_{a,b} \ast C \to L(\Gamma)_{a,b} \ast D = D' \) is an equivalence for all \( (a, b) \in \mathcal{O} \times \mathcal{O} \). If \( a \neq b \), it is a direct consequence of the lemma [5.6] where we replace \( S \) by \( \emptyset \) and \( T \) by \( X \). So \( \text{Map}_{C'}(w, z) = \bigsqcup_k M^k_C \) and respectively \( \text{Map}_{D'}(w, z) = \bigsqcup_k M^k_D \). But \( M^k_C \) is equivalent to \( M^k_D \) since \( C \) is equivalent to \( D \). We conclude that \( \text{Map}_{C'}(w, z) \) is equivalent to \( \text{Map}_{D'}(w, z) \).

If \( a = b \), we note the edges from \( a \) to \( a \) of the graph \( \Gamma \) by \( X \). Then we use the precedent case if we remark that \( C' = L(\Gamma)_{a,a} \ast C \) is simply the following pushout:

\[
\begin{array}{ccc}
U(\emptyset) & \xrightarrow{f} & C \\
\downarrow{g} & & \downarrow{h} \\
U(X) & \xrightarrow{\alpha} & C'
\end{array}
\]

The morphism \( f \) send the two objects of \( U(\emptyset) \) to \( a \in C \), so, by the lemma [5.6] we have that \( L(\Gamma)_{a,a} \ast C \to L(\Gamma)_{a,a} \ast D \) is a weak equivalence. consequently \( L(\Gamma) \ast C \to L(\Gamma) \ast D \) is a weak equivalence by a possibly transfinite composition of weak equivalences. \( \square \)

**Lemma 5.5.** Let \( i : X \to Y \) an inclusion and a weak equivalence of topological spaces and \( i(X) \) closed in \( Y \) such that there exists a homotopy \( H : Y \times [0, 1] \to Y \) which verify the following conditions:

1. \( H(-, 0) = id_Y \)
2. \( H(i(x), t) = i(x) \) for all \( x \in X \).
3. \( H(-, 1) = s \) with \( s \circ i = id_X \).
then the morphism $g$ of the pushout:

$$
\begin{array}{ccc}
X & \overset{\psi}{\longrightarrow} & Z \\
\downarrow{s} & & \downarrow{g} \\
Y & \xrightarrow{\alpha} & D
\end{array}
$$

is a weak equivalence.

**Proof.** We remind that $D = Y \cup_X Z$. To simplify notation be denote the image of $y \in Y$ in $D$ by $y$, respectively $z$ for the image of $z \in Z$ in $D$.

Since $i$ admit a retraction, $g$ admit also a retraction noted by $s'$ and induced by $s$. It means that we have an inclusion of $Z$ in $D$ via $g$ because of $s' \circ g = \text{id}_Z$. In fact, $s' : D \rightarrow Z$ is defined as follow:

1. $s'(z) = z$ for $z \in Z$.
2. $s'(y) = s(y)$ for $y \in Y$.

This new section $s'$ is well defined by $s'(\psi(x)) = \psi(x)$ and $s'(i(x)) = i(x)$ but in $D$ we have $i(x) = \psi(x)$ for all $x \in X$. We resume the situation in the following diagram

$$
\begin{array}{ccc}
X & \overset{\psi}{\longrightarrow} & Z \\
\downarrow{s} & & \downarrow{g} \\
Y & \xrightarrow{\alpha} & Y \cup_X Z
\end{array}
$$

$$
\begin{array}{c}
\xrightarrow{\alpha}
\end{array}

\begin{array}{c}
\xrightarrow{id}
\end{array}

\begin{array}{c}
\xleftarrow{\psi \circ s}
\end{array}

\begin{array}{c}
\xrightleftharpoons{\psi \circ s}
\end{array}

\begin{array}{c}
Z
\end{array}
$$

We construct the homotopy $H' : D \times [0,1] \rightarrow D$ as follow:

1. $H'(-,0) = \text{id}_D$.
2. $H'(z,t) = z$ if $z$ is in $Z$.
3. $H'(y,t) = H(y,t)$ for all $y$ in $Y$.

This homotopy is well defined. In fact, it is enough to prove that the gluing operation is well define. We have $\psi(x) = i(x)$ in $D$, then $H'(i(x),t) = H(i(x),t) = i(x)$ by definition, on the other hand $H'(\psi(x),t) = \psi(x)$. Since $i(X)$ is closed in $Y$, then $i(X)$ is closed in $D$. We conclude that $H'$ is well defined. More over $H'(y,0) = H(y,0) = y$ and so $H'(-,0)$ is the identity.

By simple computation of $H'(-,1) : D \rightarrow D$ we have that $H'(z,1) = z$ for all $z \in Z$ and $H'(y,1) = H(y,1) = s(y)$ for all $y \in Y$. So, $H'(-,1) = s'$. That means the morphism $s' : D \rightarrow Z \subset D$ is a weak equivalence since it is homotopic to the identity. Consequently $g$ est aussi is a homotopy equivalence because $s' \circ g = \text{id}$.

\[\square\]

**Lemma 5.6.** With the precedent notation of graphs if we pose $f : S = |\Delta^n| \rightarrow T = |\Delta^n|$, then, $\text{Map}_C(w,z) \subset \text{Map}_D(w,z)$ is a weak equivalence $\forall w, z \in C$.

**Proof.** We remind here that $\mathbf{V} = \text{Top}$. Since all objects in $\text{Top}$ are fibrant, so $f$ admit a section $s$. On the other hand, the inclusion $N^{k+1}_C \subset M^{k+1}_C$ is a weak
equivalence and admit also a section. We will do the demonstration for the case
$k = 2$. We use the following notations:

\[(5.1) \quad A_0 = \text{Map}_C(y, z) \times S \times \text{Map}_C(y, x) \times S \times \text{Map}_C(w, x)\]
\[(5.2) \quad A_1 = \text{Map}_C(y, z) \times S \times \text{Map}_C(y, x) \times T \times \text{Map}_C(w, x)\]
\[(5.3) \quad A_2 = \text{Map}_C(y, z) \times T \times \text{Map}_C(y, x) \times S \times \text{Map}_C(w, x)\]

The evident inclusions are weak equivalences which admit sections induced by $s$
$A_0 \to A_i$, $i = 1, 2$.

We define the complement of $N^2_C$, which consist on tuples $(a, s_1, b, s_2, c)$ in
$\text{Map}_C(y, z) \times T \times \text{Map}_C(y, x) \times T \times \text{Map}_C(w, x)$ such that $s_1, s_2 \notin S$. We
will do our argument in low dimension $n = 1$, the rest is similar. The space
$T \times S \cup S \times T$ is a gluing of two intervals $[0, 1]$ at the point 0 and $T \times T$
is simply $[0, 1] \times [0, 1]$. If we pose $f : X = T \times S \cup S \times T \to T \times T = Y$, we are
exactly in the situation of the lemma 5.5 i.e., there exist a homotopy between $X$ and $Y$
which is identity map on $X$. If we rewrite $N^2_C$ by

$N^2_C = A_1 \bigcup_{A_0} A_2 = X \times \text{Map}_C(y, z) \times \text{Map}_C(y, x) \times \text{Map}_C(w, x),$

and $M^2_C$ by

$M^2_C = Y \times \text{Map}_C(y, z) \times \text{Map}_C(y, x) \times \text{Map}_C(w, x),$

The the induced morphism $N^2_C \to M^2_C$ verify the condition of the lemma 5.5.
Consequently, the pushout of $N^2_C \subset M^2_C$ by $N^2_C \to \text{Map}_D(w, z)^1$ is also a weak
equivalence. Which means that the inclusion $\text{Map}_D(w, z)^1 \subset \text{Map}_D(w, z)^2$ is a
weak equivalence. By the same argument we prove the statement for all $k$ and use
the fact that a transfinite composition of weak equivalences is a weak equivalence. So

$\text{Map}_C(y, z) \cdot \cdot \cdot \subset \text{Map}_D(w, z)^k \subset \text{Map}_D(w, z)^{k+1} \cdot \cdot \cdot \subset \text{Map}_D(w, z)$

is a weak equivalence. \hfill \qed

Corollary 5.7. Let $M$ in $\mathcal{O} - \text{Cat}_{\text{Top}}$, then $F_i M \ast C \to F_i M \ast D$
is a weak equivalence in $\mathcal{O} - \text{Cat}_{\text{Top}}$ for all $0 \leq i$.

Proof. It is enough to see that $F = LU$ and applied the lemma 5.4 by putting
$\Gamma = UM.$ \hfill \qed

Lemma 5.8. Let $C, D$ and $M$ in $\mathcal{O} - \text{Cat}_{\text{Top}}$, and $C \to D$ a weak equivalence.
Then

$M \ast C \to M \ast D$

is a weak equivalence.

Proof. We have seen by 5.7 that

$h_i : F_i(M) \ast C \to F_i(M) \ast D$
A model structure on $\text{Cat}_{\text{Top}}$

is a weak equivalence for all $0 \leq i$. Consider the following commutative diagram in $\mathcal{O} - \mathbf{Graph}_{s\text{Top}}$:

\[ \begin{array}{ccc}
\tau(F_\bullet(M \star C)) & \xrightarrow{\tau(h_\bullet)} & \tau(F_\bullet(M \star D)) \\
\downarrow{\tau(t)} & & \downarrow{\tau(s)} \\
F_\bullet(M \star C) & \xrightarrow{h_\bullet} & F_\bullet(M \star D) \\
\downarrow{\tau(h)} & & \downarrow{g} \\
M \star C & \xrightarrow{h} & M \star D
\end{array} \]

The morphism $t$ and $s$ are homotopy equivalences. By 4.4, the morphisms $|t|$ and $|s|$ are also homotopy equivalences (of underlying graphs). The morphisms $\tau(t)$ and $\tau(s)$ are homotopy equivalences by 4.11. And by 4.4, the morphisms $|\tau(t)|$ and $|\tau(s)|$ are homotopy equivalences.

The morphism $|\tau(h_\bullet)|$ is a weak equivalence 4.7. By the property "2 out of 3" $|\tau(h)|$ is a weak equivalence.

The morphisms $f$ and $g$ are homotopy equivalences by 4.12. So $|f|$ and $|g|$ are also homotopy equivalences by 4.4.

We conclude by the property "2 out of 3" that $|h|$ is a weak equivalence and so $h$ is a weak equivalence.

\[ \square \]

6. $\infty$-categories (quasi-categories)

In the mathematical literature, there are many models for $\infty$-categories, for example the enriched categories on Kan complexes $[3]$, The categories enriched over $\text{Top}$ as we saw before, and the the quasi-categories defined by Joyal. More precisely Joyal constructed a new model structure on $s\text{Set}$, voir $[9]$, where the fibrant object are by definition quasi-categories ($\infty$-categories). We introduce the notion of quasi-groupode which generalize the notion of groupoids in the classical setting of categories. We remind also the definition of coherent nerve for the enriched categories on $s\text{Set}$ and $\text{Top}$.

**Definition 6.1.** Une quasi-category is a simplicial set $X$ which has a lifting property for all $0 < i < n$:

\[
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\vee} & X \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\exists} & \bullet
\end{array}
\]

It is important to remark that the condition $0 < i < n$ codify the law composition up to homotopy. Sometimes, we will call such simplicial complexes by weak Kan complexes. For example, if $C$ is a classical category, then the nerve $N_\bullet C$ is a quasi-category with an additional property: The lifting, is in fact, unique (cf $[11]$,
proposition 1.1.2.2). More over a simplicial set is isomorphic to the nerve of a category \( C \) if and only if the lifting exists and it is unique.

**Lemma 6.2.** A category \( C \) is a groupoid iff \( N \bullet C \) is a Kan complex.

**Proof.** If \( C \) is a groupoid, then \( N \bullet C \) admit a lifting with respect to \( \Lambda^n_0 \rightarrow \Delta^n \) and \( \Lambda^n_0 \rightarrow \Delta^n \) simply because all arrows in \( C \) are invertible. So \( N \bullet C \) is a Kan complex. If \( N \bullet C \) is a Kan complex, we have a lifting with respect to \( \Lambda^2_0 \rightarrow \Delta^2 \). That means, every diagram in \( C \)

\[
\begin{array}{ccc}
  f & \quad & x \\
  \downarrow g & & \downarrow \text{id} \\
  y & \quad & x
\end{array}
\]

can be completed by a unique arrow \( f : y \rightarrow x \), so \( g \) is right invertible. We show that \( g \) is left invertible using the lifting property with respect to \( \Lambda^2_0 \rightarrow \Delta^2 \). So \( C \) is a groupoid. \(\square\)

The precedent lemma suggest us a definition for an \( \infty \)-groupode.

**Definition 6.3.** An \( \infty \)-category (quasi-category) \( X \) is an \( \infty \)-groupoid (quasi-groupoid) if it is a Kan complex.

**Example 6.4.** Let \( Y \) be a topological space, the simplicial set \( \text{sing} Y \) is a Kan complex. So we can see every topological space as an \( \infty \)-groupoid.

**Theorem 6.5.** [9] (section 6.3) The category \( \text{sSet} \) admit a model structure where

- the cofibrations are the monomorphisms,
- the fibrant objects are the quasi-categories,
- the fibrations are the pseudo-fibrations and the weak equivalences are the categorical equivalences.

This is a cartesian closed model structure. This new structure is noted by \( (\text{sSet}, Q) \).

We don’t know if the this new model structure is cofibrantly generated! We will explain later what we mean by categorical equivalences, but we don’t describe explicitly the pseudo-fibration. For each quasi-category \( X \) (fibrant object in \( (\text{sSet}, Q) \)), we can associate its homotopy category (in a classical sens) noted \( \text{Ho} X \). This theory was developed by Joyal, see for example [9].

### 7. Some Quillen adjunctions

In this paragraph, we describe different Quillen adjunction between \( \text{sSet} \rightarrow \text{Cat} \), \( (\text{sSet}, Q) \) and \( (\text{sSet}, K) \).

#### 7.1. \( \text{sSet} \rightarrow \text{Cat} \) vs \( (\text{sSet}, Q) \).

The first adjunction is described in details in [11]. We start by some analogies between classical categories and simplicial sets.

\[ \text{sSet} \xrightarrow{\tau} \text{Cat}, \]

The right adjoint is the nerve and the left adjoint associate to each simplicial set its fundamental category. Note that this adjunction is not a Quillen adjunction for the two known model structure on \( \text{Cat} \) (Thomason structure and Joyal structure). We remind the nerve functor is fully faithful and \( \tau N \bullet = \text{id} \). The basic idea is to "extend" this adjunction to an adjunction between \( (\text{sSet}, Q) \) and the category \( \text{Cat}_{\text{sSet}} \). If we use the standard nerve for the enriched categories on simplicial
sets, by remembering only the 0-simplices, the we lose all the higher homotopical information. Because of that, we use an other strategy. First we define a left adjoint as follow

\[ \Xi : (\text{sSet}, Q) \to \text{sSet} - \text{Cat} \]

On \( \Delta^n \), then we apply the left Kan extension.

**Definition 7.1.** [11] (1.1.5.1) The enriched category \( \Xi(\Delta^n) \) has as objects the 0-simplices of \( \Delta^n \), and

\[ \Xi(\Delta^n)(i,j) = \begin{cases} N \cdot P_{i,j} & \text{if } i \leq j \\ \emptyset & \text{if } i > j \end{cases} \]

Where \( P_{i,j} \) is the set partially ordered by inclusion:

\[ \{ I \subseteq J : (i, j) \in I \land (\forall k \in I)[i \leq k \leq j] \} \].

**Definition 7.2.** The right adjoint to the functor \( \Xi \) is called the coherent nerve and noted by \( \tilde{N}_* \). It is defined by the following formula:

\[ \tilde{N}_n C = \text{hom}_{\text{sSet}}(\tilde{\Delta}^n, C) := \text{hom}_{\text{sSet} - \text{Cat}}(\Xi(\Delta^n), C) \].

Now, we can define the categorical equivalences used in the model structure \( (\text{sSet}, Q) \). We call a morphism of simplicial sets \( f : X \to Y \) an equivalence if \( \Xi(f) : \Xi(X) \to \Xi(Y) \) is an equivalence of enriched categories, i.e., if \( \text{Map}_{\text{set}}(a, b) \to \text{Map}_{\text{set}}(\Xi(f)a, \Xi(f)b) \) is a weak equivalence of simplicial sets for all \( a, b \) and \( \pi_0 \Xi(f) : \pi_0 \Xi(X) \to \pi_0 \Xi(Y) \) is an equivalence of classical categories.

**Theorem 7.3.** The following adjunction is a Quillen equivalence between the Joyal model structure \( (\text{sSet}, Q) \) [9], and the model category on \( \text{Cat}_{\text{sSet}} \) defined in [3]

\[ \text{sSet} \xrightarrow{\Xi} \text{sSet} - \text{Cat} \]

For the proof we refer to [11] theorem 2.2.5.1.

**Corollary 7.4.** Let \( C \) an enriched category on Kan complexes, then the counity

\[ \Xi\tilde{N}_* C \to C \]

is a weak equivalence of enriched categories.

### 7.2. \( (\text{sSet}, Q) \) vs \( (\text{sSet}, K) \)

In this paragraph, we describe the Quillen adjunction Between Joyal model structure on simplicial sets and the classical model structure on \( \text{sSet} \) which we note by \( (\text{sSet}, K) \), \( K \) for Kan complexes.

**Definition 7.5.** The functor \( k : \Delta \to \text{sSet} \) is defined by \( k[n] = \tilde{\Delta}^n \) for all \( n \geq \), where \( \tilde{\Delta}^n \) is the nerve of the free groupoid generated by the category \( [n] \). If \( X \) is a simplicial set, we define the functor \( k^! : \text{sSet} \to \text{sSet} \) by:

\[ k^!(X)_n = \text{hom}_{\text{sSet}}(\tilde{\Delta}^n, X) \].

The functor \( k^! \) has a left adjoint \( k_! \) which is the left Kan extension of \( k \). From the inclusion \( \Delta^n \subset \tilde{\Delta}^n \) we obtain, for all \( n \), a set morphism \( k^!(X)_n \to X_n \) which is \( n \)-level of a simplicial morphism \( \beta_X : k^!(X) \to X \). More precisely, \( \beta : k^! \to id \) is a natural transformation. Dually, we define a natural transformation \( \alpha : id \to k_! \).
Theorem 7.6. The adjoint functors

\[(\text{sSet}, \text{Kan}) \overset{k_!}{\longrightarrow} (\text{sSet}, \text{Q}) \]

is a Quillen adjunction. Moreover, \(\alpha_X : X \to k_!(X)\) is an equivalence for each \(X\).

Proof. For the proof, see ([9], 6.22). \(\square\)

7.3. \(\infty\)-groupoids. In this paragraph, we define a notion of groupoid for categories enriched on simplicial sets or topological spaces, Which we compare with the notion of \(\infty\)-groupoid defined for quasi-categories.

Definition 7.7. An enriched category \(C\) on \(\text{sSet}\) (or \(\text{Top}\)) is an \(\infty\)-groupode if \(\pi_0 C\) is a groupoid in the classical sense of categories. If \(C\) is enriched on \(\text{sSet}\) (\(\text{Top}\)), the \(\infty\)-groupoid \(C'\) associated to \(C\) is a fibred product in \(\text{Cat}_{\text{sSet}}\) (or \(\text{Cat}_{\text{Top}}\)):

\[
C' = \text{iso} \pi_0 C \times_{\pi_0 C} C \xrightarrow{k_!} \text{iso} \pi_0 C \xrightarrow{\alpha} \pi_0 C.
\]

We remark that the functor \(\pi_0 : \text{Cat}_{\text{sSet}} \to \text{Cat}\) is a left adjoint, so it does not commute with limits in general. But the evident projection \(pr : \pi_0 C' \to \text{iso} \pi_0 C\) is an isomorphism. In fact, if \(w_1\) and \(w_2\) are weak equivalences in \(\text{Map}_C(a, b)\) and \(h\) is a homotopy between them (i.e. \(\text{un 1-simplex in } \text{Map}_C(a, b)\) such that the borders are \(w_1, w_2\) Then \(h\) is also a homotopy in \(\text{Map}_C'(a, b)\) This prove that the projection \(pr\) is fully faithful. the essential surjectivity of \(pr\) est evident.

We note by \(G\) the functor which associate to \(C\) its \(\infty\)-groupoid \(C'\). The full subcategory of \(\text{Cat}_{\text{sSet}}\) of \(\infty\)-groupoids is noted by \(\text{Grp}_{\text{sSet}}\).

Lemma 7.8. The functor \(G : \text{sSet} \to \text{Grp}_{\text{sSet}}\) is the right adjoint of the inclusion, i.e.,

\[
\text{hom}_{\text{Grp}_{\text{sSet}}}(C, GD) = \text{hom}_{\text{Cat}_{\text{sSet}}}(C, D)
\]

\(\forall C \in \text{Grp}_{\text{sSet}}\) and \(D \in \text{Cat}_{\text{sSet}}\).

Remark 7.9. We can do the same thing for \(\text{Cat}_{\text{Top}}\).

Proof. Let \(C\) be an \(\infty\)-groupoid and let \(D \in \text{sSet} \to \text{Cat}\). A morphism \(f : C \to D\) define in a unique way an adjoint morphism \(g : C \to GD\) given by the universal map

\[
\begin{array}{ccc}
C & \xrightarrow{\exists! g} & GD \\
\downarrow q & & \downarrow \phi \\
\pi_0 C & \xrightarrow{\pi_0 f} & \pi_0 D \\
\pi_0 f \end{array}
\]

The morphism \(\phi = \pi_0 f \circ q\) exists and make the diagram commuting, since \(C\) is an \(\infty\)-groupoid. \(\square\)
Let \([n]'\) denote the groupoid freely generated by the category \([n]\). An example of \(\infty\)-groupoid is the category \(\Xi k\Delta^n\). In fact, \(\Xi k\Delta^n = \Xi N\bullet [n]' \to [n]'\) is a weak categorical equivalence and \([n]'\) is fibrant. Since \([n]'\) is a groupoid groupoid, then \(\pi_0\Xi k\Delta^n\) is also a groupoid.

**Lemma 7.10.** Let \(C\) a fibrant category enriched on \(sSet\), then \(k!\tilde{\Xi}_\bullet C = k!\tilde{\Xi}_\bullet C'\), where \(C'\) is an \(\infty\)-groupoid associated to \(C\).

**Proof.** Using the precedent adjunctions, we have for all \(n \geq 0\)

\[
(k!\tilde{\Xi}_\bullet C)_n = \text{hom}_{sSet}(\Delta^n, k!\tilde{\Xi}_\bullet C)
\]

(7.1)

\[
= \text{hom}_{sSet}(k\Delta^n, \tilde{\Xi}_\bullet C)
\]

(7.2)

\[
= \text{hom}_{sSet-\text{Cat}}(\Xi k\Delta^n, C)
\]

(7.3)

But \(\Xi k\Delta^n\) is an \(\infty\)-groupoid, so

\[
\text{hom}_{sSet-\text{Cat}}(\Xi k\Delta^n, C) = \text{hom}_{sSet-\text{Grp}}(\Xi k\Delta^n, C')
\]

(7.4)

\[
= \text{hom}_{sSet-\text{Cat}}(\Xi k\Delta^n, C')
\]

(7.5)

\[
= \text{hom}_{sSet}(\Delta^n, k!\tilde{\Xi}_\bullet C')
\]

(7.6)

\[
= (k!\tilde{\Xi}_\bullet C')_n
\]

(7.7)

we conclude that \(k!\tilde{\Xi}_\bullet C' = k!\tilde{\Xi}_\bullet C\).

**Definition 7.11.** In Bergner’s model structure on \(\text{Cat}_{sSet}\) a morphism \(F : C \to D\) is a fibration if

1. \(\text{Map}_C(a, b) \to \text{Map}_D(Fa, Fb)\) is a fibration of simplicial sets for all \(a, b \in C\).
2. \(F\) has a lifting property of weak equivalences, i.e. it is Grothendieck fibration for weak equivalences.

**Corollary 7.12.** Let \(C'\) the \(\infty\)-groupoid associated to the enriched category \(C\) over Kan complexes (or \(\text{Top}\)), then

\[
\tilde{\Xi}_\bullet C' \to \text{iso } \pi_0 C
\]

pseudo-fibration (cf. [9]) in \((sSet, Q)\).

**Proof.** Remark that if \(C\) is fibrant, then \(C \to \pi_0 C\) is a fibration. The Bergner’s model structure is right proper so \(C' \to \text{iso } \pi_0 C\) is also a fibration. Moreover, the groupoid iso \(\pi_0 C\) is fibrant, and so \(C'\) is. Consequently \(\tilde{\Xi}_\bullet C' \to \text{iso } \pi_0 C\) is a pseudo-fibration in the category \((sSet, Q)\), so a pseudo fibration between quasi-categories.

But the category \(\pi_0 C\) is a ”constant” simplicial category, so \(\text{iso } \pi_0 C = \text{iso } \pi_0 C\). We conclude that \(\tilde{\Xi}_\bullet C' \to \text{iso } \pi_0 C\) is a pseudo-fibration between quasi-category and a Kan complex, see [9].

Let \(X\) a quasi-category, Joyal defined the homotopy category \(\text{Ho}(X)\) which is a category in the classical sens. The 0-simplexes of \(X\) form the set of objects of \(\text{Ho}(X)\) and the 1-simplexes (modulo the homotopy equivalence) form the morphisms of \(\text{Ho}(X)\). An 1-simplex in \(X\) is called an weak equivalence if it is represented in \(\text{Ho}(X)\) by an isomorphism.
Definition 7.13. Let $p : X \to Y$ a morphism between quasi-categories, and let $w$ a 1-simplex in $X$, then $p$ is called conservative if:

\[ p(w) \text{ a weak equivalence in } Y \Rightarrow w \text{ a weak equivalence in } X. \]

Lemma 7.14. ([9], 4.30) Let $p : X \to Y$ a morphism between quasi-categories, such that $p$ is a pseudo-fibration and conservative. If $Y$ is a Kan complex, then $X$ is.

Lemma 7.15. Let $C \in \text{Cat}_{\text{sSet}}$ fibrant, then $\tilde{N}C'$ is a Kan complex, where $C'$ is the $\infty$-groupoid associated to $C$.

Proof. We have seen by the corollary 7.12 that if $C$ is fibrant, then $\tilde{N}_*C' \to N_!\pi_0C$ is a pseudo-fibration between quasi-categories, and $N_!\pi_0C$ is a Kan complex. We must verify that the morphism is conservative, which is an evident fact because all 0-simplices of $\text{Map}_{\tilde{Q}}(a, b)$ are weak equivalences by definition. By the lemma 7.14, we conclude that $\tilde{N}_*C'$ is a Kan complex.

In [9] (Theorem 4.19), Joyal construct an adjunction between Kan complexes and quasi-categories. If we note by $\text{Kan}$ the full subcategory of $\text{sSet}$ of Kan complexes, and by $\text{QCat}$ the full subcategory of $\text{sSet}$ of quasi-categories, then the inclusion $\text{Kan} \subset \text{QCat}$ admit a right adjoint noted by $J$. The functor can be interpreted as follows: for each quasi-category $X$, $J(X)$ is the quasi-groupoid associated to $X$, and if $X$ is a Kan complex, then $J(X) = X$.

Lemma 7.16. Let $X$ a quasi-category (a fibrant object) in $(\text{sSet}, Q)$. The natural transformation $\beta_X : k^!(X) \to X$ is factored by $\beta_X : k^!(X) \to J(X) \subset X$. Moreover, $\beta_X : k^!(X) \to J(X)$ is a trivial Kan fibration.

Proof. See [9], proposition 6.26.

Corollary 7.17. Let a fibrante category $C \in \text{Cat}_{\text{sSet}}$, and $GC$ the associated $\infty$-groupoid. Then $k!\tilde{N}_*(C) \to \tilde{N}_*(GC)$ is a trivial Kan fibration.

Proof. Since $C$ is fibrant, we have seen that $k!\tilde{N}_*(C) = k!\tilde{N}_*(GC)$, and by the precedent lemma $k!\tilde{N}_*(GC) \to J(\tilde{N}_*(GC))$ is a trivial Kan fibration. But $\tilde{N}_*(GC)$ is a Kan complex, since $GC$ is a fibrant $\infty$-groupoid, so $J(\tilde{N}_*(GC)) = \tilde{N}_*(GC)$.

Now, we can see the analogy between $\tilde{N}_!\pi_0$ in the case of classical categories and the functor $k!\tilde{N}_*$ in the case of enriched categories over $\text{sSet}$. In fact, if $C$ is a classical category, then the functor iso sends $C$ to its associated groupoid $GC$ and so $N_!\pi_0C = N_!\pi_0GC$. If $C$ is a category enriched over Kan complexes, (i.e., $C$ is fibrant in Bergner’s model structure), then the simplicial set $k!N_*C$ is equivalent to $\tilde{N}_*GC$ by the corollary 7.17.

8. MAPPING SPACE

The goal of this section is to describe the mapping space of the model category $\text{Cat}_{\text{Top}}$. Before making progress in this direction, we need some introduction to different model on $\text{sSet}$.

Notation 8.1. We will note the category of simplicial sets with Kan model structure by $(\text{sSet}, K)$. The Joyal model structure of quasi-categories will be noted by $(\text{sSet}, Q)$. 
Theorem 8.2. Let a Quillen adjunction of Quillen model categories:

\[
\begin{array}{c}
C \\ \pi_0 C \\
\end{array} \xrightarrow{\sim} \begin{array}{c}
D \\ \pi_0 D \\
\end{array}
\]

The there is a natural isomorphism

\[\text{map}_C(a, RFb) \to \text{map}_D(LGa, b)\]

in Ho(sSet)

8.1. Mapping space in \text{Cat}_{\text{Top}} and \text{Cat}_{sSet}. In this paragraph, we compute \text{map} for the model categories \text{Cat}_{sSet} and \text{Cat}_{\text{Top}}.

Suppose that \(C\) is a small enriched category on \(\text{Top}\). We define the coherent nerve of \(C\) by \(\widetilde{N}_\bullet\text{sing}C\), and we define the corresponding \(\infty\)-groupoid \(C'\) by

\[GC = \text{iso} \pi_0 C \times_{\pi_0 C} C \xrightarrow{\sim} C\]

By applying the functor sing to this diagram, we obtain also a pullback diagram since sing is a right adjoint. We note that \(\text{sing} \pi_0 C = \pi_0 \text{sing} C = \pi_0 C\) and \(\text{sing iso} \pi_0 C = \text{iso} \pi_0 C = \text{iso} \pi_0 \text{sing} C\)

\[G\text{ sing } C = \text{sing}(\text{iso} \pi_0 C \times_{\pi_0 C} C) \xrightarrow{\sim} \text{sing } C\]

We conclude that

\[\text{sing } GC = G\text{ sing } C.\]

More over \(k^!\widetilde{N}_\bullet\text{ sing } C\) is weak equivalent to \(\widetilde{N}_\bullet\text{ sing } GC\). The homotopy type of the mapping space \(\text{map}_{\text{Cat}_{\text{Top}}}(\ast, C)\) is computed easily using the theorem 8.2 and the adjunction

\[
sSet \xrightarrow{\Xi k^!} \text{Cat}_{sSet}.\]

We conclude that for every (fibrant) small category enriched on \(sSet\), we have the following isomorphism in Ho(sSet)

\[k^!\widetilde{N}_\bullet C \sim \text{map}_{sSet}(\ast, k^!\widetilde{N}_\bullet C) \sim \text{map}_{\text{Cat}_{sSet}}(\ast, C)\]

and by the same way, if \(D\) is a small category enriched on \(\text{Top}\), then

\[\text{map}_{\text{Cat}_{\text{Top}}}(\ast, D) \sim k^!\widetilde{N}_\bullet\text{sing } D.\]

by the corollary 7.17, we conclude that

\[\text{map}_{\text{Cat}_{sSet}}(\ast, C) \sim \widetilde{N}_\bullet GC.\]

et

\[\text{map}_{\text{Cat}_{\text{Top}}}(\ast, D) \sim \widetilde{N}_\bullet G\text{sing } D.\]
In the classical setting of $\text{Cat}$, we know that $\text{map}_\text{Cat}(A, B) \sim N^* \text{iso} \text{HOM}_\text{Cat}(A, B)$. If $A$ is the terminal category $\ast$, then $\text{map}_\text{Cat}(\ast, B) \sim N^* \text{iso} B$. More generally, we have that:

$$\text{map}_\text{Cat}(\ast, B) \sim N^* \text{iso} B.$$ 

where $\text{Map}$ is the right adjoint functor to the cartesian product in $\text{sSet}$. Now, the similarity between $\text{Cat}$ and $\text{Cat}_{\text{sSet}}$ is evident.

### 9. Localization

In this paragraph, we show how to construct localization for a topological category with respect to a morphism or a set of morphisms. In the classical setting of small categories, we know how to define the localization in a functorial way. The idea is quite simple: let $C \in \text{Cat}$ and $f$ a morphism in $C$, we want to define a functor $C \rightarrow L_f C$ and having the following universal property: if $F : C \rightarrow D$ is a functor such that $F(f)$ is an isomorphism in $D$ then there is a unique factorization of $F$ as $C \rightarrow L_f C \rightarrow D$.

**Notation 9.1.** In this section, the category with two objects $x$ and $y$ and with one non-trivial morphism from $x$ to $y$ will be denoted $A$.

The category with the same objects $x$ and $y$ and an isomorphism from $x$ to $y$ (resp. from $y$ to $x$) will be denoted $B$.

**Lemma 9.2.** The category $L_f C$ is isomorphic to following pushout in $\text{Cat}$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{\text{inc}} & & \downarrow{\text{s}} \\
B & \rightarrow & M
\end{array}
$$

where $\text{inc}$ is the evident inclusion and $f$ sends the unique arrow in $A$ to the morphism $f$ in $C$.

**Proof.** Suppose that we have a functor $F : C \rightarrow D$ such that the morphism $f$ is sent to an isomorphism. It induce a functor from $B \rightarrow D$. By the pushout property we have a unique functor from $M$ to $D$ which factors the functor $F$. So $L_f C$ is isomorphic to $M$.

**Corollary 9.3.** For any set $S$ of morphism in $C$ the category $L_S C$ exist and it is unique up to isomorphism.

Now, we are interested for the same construction in the enriched setting $\text{Cat}_{\text{Top}}$.

The main difference with the classical case is the existence, we will construct a functorial model for the localization up to homotopy.

**Notation 9.4.** We denote by $A^h$ the topological category $|\Xi_k(A)|$ and by $B^h$ the category $|\Xi_kB|$.

Choosing a morphism $f$ in a topological category $C$ we want to construct a category $L_f C$ with the following property: given a morphism $F : C \rightarrow D$ in $\text{Cat}_{\text{Top}}$ such that $F(f)$ is a weak equivalence in $D$ then $F$ is factored (unique up to homotopy) as $C \rightarrow L_f C \rightarrow D$. 
Lemma 9.5. The category $L_f C$ could be taken as following pushout in $\text{Cat}_\text{Top}$:

$$
\begin{array}{ccc}
A^h & \xrightarrow{f} & C \\
\downarrow\text{inc} & & \downarrow i \\
B^h & \xrightarrow{j} & M
\end{array}
$$

More over $\pi_0 C \rightarrow L_{\pi_0 f} \pi_0 C$ is a localization in $\text{Cat}$.

Proof. First, we note that the inclusion $\text{inc}$ is a cofibration in $\text{Cat}_\text{Top}$. The functor $A^h \rightarrow C$ is constructed as follow: Let $A \rightarrow C$ which sends the only nontrivial morphism of $A$ to $f \in C$. It induces a map of simplicial sets $N_* A \rightarrow \tilde{N}_* \text{sing} C$ and by adjunction a functor $|\Xi N_* A| \rightarrow C$ which is the functor noted $f : A^h \rightarrow C$ in the diagram. The functor $\text{inc} : A^h \rightarrow B^h$ is induced by the functor $\text{inc} : A \rightarrow B$.

Now suppose that we have a functor $C \rightarrow D$ which sends $f$ to a weak equivalence in $D$. The induced functor $A^h \rightarrow D$ factors by $A^h \rightarrow GD \rightarrow D$ where $GD$ is the associated groupoid of $D$ as seen in previews section.

Consider the diagram:

$$
\begin{array}{ccc}
A^h & \rightarrow & GD \\
\downarrow\text{inc} & & \downarrow i \\
B^h & \rightarrow & \ast
\end{array}
$$

and using the adjunctions we have a corresponding diagram in $\text{sSet}$

$$
\begin{array}{ccc}
N_* A & \rightarrow & \tilde{N}_* \text{sing} GD \\
\downarrow\text{inc'} & & \downarrow i' \\
N_* B & \rightarrow & \ast
\end{array}
$$

But now $\text{sing} GD$ is a Kan complex see [1, 15] and $\text{inc'}$ is a trivial cofibration in $\text{sSet}$, so there exist a lifting (not unique) $N_* B \rightarrow \text{sing} GD$. By adjunction we have a lifting $B^h \rightarrow GD \rightarrow D$. So we can define unique morphism (up to homotopy) $M \rightarrow D$ and any functor $C \rightarrow D$ as before factors (uniquely up to homotopy) by $C \rightarrow M \rightarrow D$. So a functorial model for $L_f C$ is $M$ and the localisation map $C \rightarrow L_f C$ is a cofibration and in fact an inclusion of enriched categories. □

Corollary 9.6. For any set $S$ of morphism in a topological category $C$, the topological category $L_S C$ exist and it is unique up to homotopy. More over the localization map $C \rightarrow L_S C$ is a cofibration.

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