Invariance of the Hamilton-Jacobi tunneling method for black holes and FRW model

Yi-Xin Chen
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, China
E-mail: yxchen@zimp.zju.edu.cn

Kai-Nan Shao
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, China
E-mail: shaokn@gmail.com

Abstract: In this paper we revisit the topic of Hawking radiation as tunneling. We show that the imaginary part of the action of the tunneling particle should be reconstructed in a covariant way, as a line integral along the classical forbidden trajectory of tunneling particles. As the quantum tunneling phenomenon, the probability of tunneling is related to the imaginary part of the action for the classical forbidden trajectory. We do the calculations for massless and massive particles, in Schwarzschild coordinate and Painlevé coordinate. The construction of particle action is invariant under coordinate transformations, so this method of calculation black hole tunneling does not have the so called “factor 2 problem”. As an application, we find that the temperature of Hawking temperature of apparent horizon in a FRW universe is $T = \frac{\kappa}{2\pi}$. Based on this result, we briefly discuss the unified first law of apparent horizon in FRW universe.

Keywords: Black Holes, Classical Theories of Gravity, Cosmology of Theories beyond the SM.
1. Introduction

Since Hawking discovered the remarkable fact of black hole radiation\cite{Hawking:1974sw} in 1975, much work have done to calculate and understand this quantum effect\cite{Hawking:1975vc}. In \cite{Kraus:1993jz, Wilczek:1993ve}, Kraus and Wilczek introduced a semiclassical treatment of Hawking radiation, by interpreting the exponent of the classical action as the modes of the system. Subsequently, in \cite{Hawking:1982ga} Hawking radiation was interpreted as a tunneling phenomenon. This interpretation has been extensively studied\cite{Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf, Parikh:1999mf}, and applied on various models of black holes\cite{Banerjee:2004rz, Majhi:2004rr, Majhi:2004rr, Majhi:2004rr, Majhi:2004rr}. There are two methods to calculate the imaginary part of the action: one is Parikh-Wilczek’s radial null geodesic method\cite{Parikh:1999mf}, and the other is the Hamilton-Jacobi method\cite{Banerjee:2004rz, Majhi:2004rr, Majhi:2004rr, Majhi:2004rr}. Based on the Hamilton-Jacobi method, Banerjee and Majhi\cite{Banerjee:2004rz} developed the tunneling method beyond semiclassical approximation to include quantum corrections. This method has been applied to calculate quantum corrections to black hole entropy\cite{Banerjee:2004rz, Majhi:2004rr, Majhi:2004rr, Majhi:2004rr, Majhi:2004rr}. At first the tunneling amplitude calculated by the Hamilton-Jacobi method differed by a factor of 2 with the standard Hawking temperature, which is the so called “factor of 2 problem”\cite{Banerjee:2004rz, Majhi:2004rr}. Later on, it is pointed out in \cite{Banerjee:2004rz, Majhi:2004rr, Majhi:2004rr} that by adding the temporal contribution to the imaginary part of the action, one can resolve this problem. On the other hand, \cite{Banerjee:2004rz} proposed a method to obtain the standard result, by introducing an integration constant into the action, setting the incoming probability to unity, and taking the ratio of the outgoing and incoming probabilities. In this paper, we revisit the Hamilton-Jacobi method of calculating black hole tunneling. Inspired by the calculation in \cite{Banerjee:2004rz}, we construct the imaginary part of the action of the tunneling particle in an invariant way, as an integration along the classical forbidden trajectory of the particle. By interpreting Hawking radiation as a quantum tunneling phenomenon, the tunneling probability is related to the imaginary part of the action of the classical forbidden
trajectory\cite{34}. Once again the tunneling probability is related to the Boltzmann factor for the emission at the Hawking temperature. We perform the calculation of black hole tunneling in Schwarzschild coordinate for massless and massive particle, and in Painlevé coordinate. Since the action is constructed invariantly, this calculation does not have the “factor of 2 ” problem, and the temporal contribution has been intrinsically included. The result shows that Hawking radiation can indeed be viewed as a tunneling phenomenon.

In \cite{35}, by assuming a temperature $T = \frac{1}{2\pi R_H}$ and the entropy $S = \frac{A}{4}$, where $R_H$ and $A$ are the radius and area of the apparent horizon, one can introduce the first law of thermodynamics associated with the apparent horizon, and show that the Friedmann equation which describes the dynamics of the FRW universe can be derived from it. There is also another proposal of the unified first law\cite{36} of the FRW model, which assigns the temperature $T = \frac{\kappa}{2\pi}$. The tunneling method provides a way to calculate the Hawking temperature associated with the apparent horizon in the FRW universe. The original use of tunneling method on FRW universe gave the result $T = \frac{\kappa}{2\pi}$ in \cite{37, 38}. However, in their calculation, the action of tunneling particle was not constructed in the invariant way. Later on, Hayward\cite{33} used the invariant Hamilton-Jacobi tunneling method on dynamical black holes, FRW model with $k = 0$, gave the result $T = \frac{\kappa}{2\pi}$, and discussed the unified first law. In dynamical black hole models, the Hawking temperature of black hole radiation should be proportional to surface gravity on the horizon\cite{39, 40}. The metric of FRW universe model can be viewed as an example case of dynamical black holes. A direct thought is that their temperatures and unified first laws should coincide. In this paper, based on the invariant Hamilton-Jacobi method of black hole tunneling, we shall calculate the Hawking temperature for the FRW universe with general $k$. We get the result $T = \frac{\kappa}{2\pi}$, and show that by demanding the Friedmann equation to satisfy, the first law of thermodynamics of the apparent horizon in FRW universe coincides with the unified first law of dynamical black holes discussed in \cite{39}. Another calculation of the temperature for FRW model, see\cite{41}.

The paper is organized as follows. In Section 2, we revisit Hamilton-Jacobi method of black hole tunneling. We show that the imaginary part of the action of tunneling particles can be construction in an invariant way, as a line integral along the classical forbidden trajectory. To illustrate the method, we perform the calculation in Schwarzschild coordinate for massless and massive particle, and in Painlevé coordinate. In Section 3, we calculate the temperature associated with the apparent horizon in FRW universe, obtain the result $T = \frac{\kappa}{2\pi}$, and then we discuss the first law based on this temperature. Section 4 is for discussions.

2. The invariant Hamilton-Jacobi tunneling method for black holes

In the tunneling interpretation of Hawking radiation, the probability of radiation is related to the imaginary part of the tunneling particle via

$$\Gamma \sim e^{-2imI},$$

\hspace{3cm} (2.1)
which is in turn related to the Boltzmann factor for the emission at the Hawking temperature
\[ \Gamma \sim e^{-\frac{\omega}{\hbar}} \quad . \]

The semiclassical wave function of the particle is written as
\[ \phi = e^{-\frac{i}{\hbar}I} \quad . \]

The wave function satisfies the corresponding wave equation: Klein-Gordon equation for scalar particles, Dirac equation for spin-1/2 particles, etc.

The action is reconstructed in an invariant way\[33, 42\], as a line integral along the classical forbidden trajectory \( \gamma \) of the tunneling particle
\[ I = \int_\gamma \partial_t I \, dx^i \quad . \]

Now we take the scalar particle as the example to calculate the black hole tunneling probability.

2.1 Schwarzschild coordinate

The static spherical symmetric metric is written as
\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2 \quad . \]

This metric can describe a black hole whose horizon is \( r_0 \), with \( f(r_0) = 0 \).

The wave equation \[23\] for scalar particle satisfies Klein-Gordon equation
\[ -\frac{\hbar^2}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) \phi = 0 \quad . \]

The leading order in \( \hbar \) gives the Hamilton-Jacobi equation
\[ g^{\mu\nu} \partial_\mu I \partial_\nu I = 0 \quad . \]

For radial trajectory only the \((t, r)\) sector of the metric \[25\] is relevant,
\[ -f(r)^2 (\partial_r I)^2 + (\partial_t I)^2 = 0 \quad , \]

i.e.
\[ \partial_r I = \frac{1}{f(r)} \partial_t I \quad . \]

For outgoing modes we should take the + sign. On the other hand, the timelike Killing vector is \( K = \frac{\partial}{\partial t} \), the energy of particle is
\[ \omega = K^t \partial_t I = \partial_t I \quad . \]

The trajectory of massless particle in metric \[25\] is the null geodesics, described by
\[ ds^2 = 0 \quad , \]
\[ -f(r)dt^2 + \frac{1}{f(r)}dr^2 = 0 \quad . \]
i.e.
\[ dt = \frac{1}{f(r)} dr \] (2.12)

Here we have chosen the + sign, i.e., the outgoing mode, which is the classical forbidden one (Classically particles are not allowed to fall out of the black hole horizon). The curve of trajectory can be parametrized by the variable \( r \) as \( \gamma(\tau) = (t(\tau), r) \), where \( t(r) \) is determined by (2.12). Now the action is reconstructed as a line integral along \( \gamma \)

\[
I = \int_\gamma dx^i \partial_i I = \int_\gamma (\partial_t I dt + \partial_r I dr) = \int_\gamma \left( (\partial_t I, \partial_r I) \cdot \frac{d}{dr} \gamma(r) \right) dr = \int_\gamma \left( \frac{\omega}{f(r)} dr + \frac{1}{f(r)} \omega dr \right) = \int_\gamma \left( \frac{2\omega}{f(r)} \right) dr .
\] (2.13)

The third line is the definition of line integral. To get the fourth line, one can start from the second line, replace \( dt \) with the trajectory equation (2.12) of the particle, replace \( \partial_t I, \partial_r I \) with the energy of the tunneling particle \( \omega \) by Eqs. (2.9) and (2.10), and finally remove the \( \gamma \) below the the symbol of integral. This integral is performed along the outgoing trajectory, i.e., from inside the horizon \( r_{in} \) to outside the horizon \( r_{out} \). It has a pole at the horizon \( r_0 \), where \( f(r) = f(r_0)(r - r_0) + O(r - r_0) \). The other part of the integration is regular, so the imaginary part of the action is

\[
\text{Im} I = \frac{2\pi \omega}{f'(r_0)} .
\] (2.14)

The emission rate of the radiation is associated with the imaginary part of the action, and it is also related with the temperature

\[
\Gamma \sim e^{-\frac{2\pi \omega}{\hbar}} e^{-\frac{\pi \omega}{\kappa H}} .
\] (2.15)

So, we obtain the temperature

\[
T_H = \frac{f'(r_0) \hbar}{4\pi} .
\] (2.16)

On the other hand, the surface gravity on the horizon \( r_0 \) of metric (2.3) is

\[
\kappa_H^2 = \frac{g^{\mu\nu} \partial_\mu (K^2) \partial_\nu (K^2)}{4K^2} \bigg|_{r=r_0} = \frac{g^{rr} \partial_r (-f) \partial_r (-f)}{-4f} \bigg|_{r=r_0} = -\frac{(f'(r_0))^2}{4} ,
\] (2.17)

\[
\kappa_H = \sqrt{-\kappa_H^2} = \frac{f'(r_0)}{2} .
\] (2.18)

Now the temperature can be written as

\[
T_H = \frac{\hbar}{2\pi} \kappa_H .
\] (2.19)
This is the standard Hawking temperature.

In order to illustrate the tunneling method mentioned above, especially the invariant construction of the action of tunneling particle. We next perform the calculation for a massive scalar particle. Its wave function (2.3) satisfies the Klein-Gorden equation

\[ g^{\mu \nu} \partial_{\mu} I \partial_{\nu} I + m^2 = 0 \]  

(2.20)
i.e.

\[ -\frac{1}{f(r)} (\partial_t I)^2 + f(r) (\partial_r I)^2 + m^2 = 0 \]  

(2.21)
One should take the outgoing mode (+ sign)

\[ \partial_r I = \frac{1}{f(r)} \sqrt{-m^2 f(r) + (\partial_t I)^2} \]  

(2.22)

The energy of the particle is also Eq.(2.10). The trajectory of the massive particle is the geodesics in the metric (2.5). To simplify, the equation of geodesic motion can be derived by the following Lagrangian

\[ L = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \]

where \( \dot{x} \) means \( \frac{dx}{d\tau} \). The Euler-Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{d}{d\tau} \left( \frac{\partial L}{\partial x^\mu} \right) = 0 \]  

for \( t \) is

\[ \dot{r} = \sqrt{E^2 - f(r)} \]  

(2.23)
where we have taken the + sign to get the outgoing trajectory. Then we obtain the curve

\[ dt = \frac{E}{f(r) \sqrt{E^2 - f(r)}} dr \]  

(2.24)

Now the action is reconstructed as

\[ I = \int (\partial_t I dt + \partial_r I dr) \]

\[ = \int \left( \omega \frac{E}{f(r) \sqrt{E^2 - f(r)}} dr + \frac{1}{f(r)} \sqrt{-m^2 f(r) + \omega^2} dr \right) \]

\[ = \int \left( \frac{E \omega}{\sqrt{E^2 - f(r)}} + \sqrt{\omega^2 - m^2 f(r)} \right) \frac{dr}{f(r)} \]  

(2.25)
\[ r = r_0 \text{ with } f(r_0) = 0 \] is a pole of the above integration, the imaginary part of the action is obtained by \( \pi \) times the residue of the integrand

\[ \text{Im} I = \frac{2\pi \omega}{f'(r_0)} \]  

(2.26)
Once again the Hawking temperature is the standard result

\[ T_H = \frac{f'(r_0) \hbar}{4\pi} = \frac{\hbar}{2\pi \kappa_H} \]  

(2.27)
2.2 Painlevé coordinate

The metric (2.3) has a coordinate singularity at the horizon $r = r_0$, which can be removed by transforming it to Painlevé coordinate. The coordinate transformation $dt \rightarrow dt - \sqrt{1 - f(r)} dr$ gives

$$ds^2 = -f(r)dt^2 + 2\sqrt{1 - f(r)}dt dr + dr^2 + r^2d\Omega^2.$$  (2.28)

The massless particle moves along the radially null geodesic described by $ds^2 = 0$. The outgoing trajectory is

$$dr = \frac{1 + \sqrt{1 - f(r)}}{f(r)} dr.$$  (2.29)

The Hamilton-Jacobi equation (2.7) in this background is

$$- (\partial I)^2 + 2\sqrt{1 - f(r)} \partial_t I \partial_r I + f(r) (\partial_r I)^2 = 0.$$  (2.30)

One should choose the outgoing mode

$$\partial_r I = \frac{1 - \sqrt{1 - f(r)}}{f(r)} \partial_t I.$$  (2.31)

The energy of the particle is still $\omega = \partial_t I$, as in (2.10).

The action of the particle is reconstructed as

$$I = \int (\partial_t I dt + \partial_r I dr)$$

$$= \int \left( \omega \frac{1 + \sqrt{1 - f(r)}}{f(r)} dr + \omega \frac{1 - \sqrt{1 - f(r)}}{f(r)} dr \right)$$

$$= \int \left( \frac{2\omega}{f(r)} \right) dr.$$  (2.32)

The same result as the massless particle in the Schwarzschild coordinate. This is a natural result of the invariant construction of the particle action.

2.3 Discussion and comments on the factor 2 problem

In the original calculation of black hole tunneling [5], the action of particle ImI is constructed as $\text{Im} \int pdr$ in the Painlevé coordinate, and $\Gamma \sim e^{-2\text{Im} \int pdr}$. It is pointed out in [6] that the quantity $\Gamma \sim e^{-2\text{Im} \int pdr}$ is not canonically invariant, and the answer would be different in different canonical frame. He proposed that one should use the canonically invariant formula $\Gamma \sim e^{-\text{Im} \int pdr}$. However, using this canonically invariant formula in Schwarzschild coordinate, the Hawking temperature obtained is twice as the original one [44]. This is the so called “factor of 2 problem” [8]. Later on, this problem was resolved by adding the temporal contribution to the action [31, 32, 9, 45]. The calculation in Painlevé coordinate is correct using $\Gamma \sim e^{-2\text{Im} \int pdr}$, as presented in [9]. Other solutions of this problem is to introduce an integration constant into the action [7, 46], or consider the thermal balance and take the tunneling rate as the ratio between the emission and absorption probabilities [47].
In our paper, we construct the action as (2.4), inspired by the calculation in [33]. As mentioned above, the action is a scalar quantity and should be canonically invariant. Eq. (2.4) is invariant, and gives the correct temperature. In fact, temporal contribution has intrinsically been included in this construction of action. Additionally, we illustrate that the integration in (2.4) is in fact a line integral, and should be performed along the classical forbidden trajectory (here it is the trajectory from inside to outside of black hole horizon) of the test particle. This coincides with the philosophy of quantum tunneling [34], which tells that the propagating rate of quantum tunneling is related to the imaginary part of the action along classical forbidden trajectory.

3. Hawking radiation of apparent horizon in a FRW universe and unified first law

The tunneling method can also be applied to analyze the radiation of dynamical black holes [40, 33, 42] and FRW universe model [37, 38]. The calculation of Hawking radiation of the apparent horizon in FRW method using scalar particles tunneling [37] and fermions tunneling [38] gives the temperature \( T = \frac{\kappa}{2\pi R_H} \). However, the action of the tunneling particles through the apparent horizon are not constructed in the invariant way. Later on, Hayward [10] calculated the temperature of dynamical black holes, and gave the result \( T = \frac{\kappa}{2\pi} \). The temperature of dynamical black hole and FRW universe with \( k = 0 \) were also calculated by the invariant Hamilton-Jacobi method [33]. Using the Hawking temperature, one can discuss the first law of thermodynamics associated with the apparent horizon of dynamical black holes and FRW model. In fact, the FRW metric can be viewed as a certain kind of dynamic black hole. In this section we shall apply the tunneling method illustrated in the above section to the FRW model with general \( k \), obtain its temperature, and discuss the first law on its apparent horizon.

The homogeneous and isotropic universe model is described by the \((3+1)\) dimensional FRW metric

\[
    ds^2 = -dt^2 + \frac{a(t)^2}{1 - kr^2}dr^2 + a(t)^2r^2d\Omega^2,
\]

where the spatial curvature constant \( k = +1, 0 \) and \(-1\), corresponding to a closed, flat and open universe, respectively. Defining \( R = a(t)r \), \( H(t) = \frac{\dot{a}}{a} \) and \( R_A(t) := \frac{1}{\sqrt{H^2 + kR^2}} \), the metric (3.1) can be rewritten as

\[
    ds^2 = -\frac{1-R_A^2}{1-kR^2}dt^2 - \frac{2HR_A}{1-kR^2}dt dR + \frac{1}{1-kR^2}dR^2 + R^2d\Omega^2.
\]

Denote the \((t, R)\) sector of the above metric by \( ds^2 = \gamma_{ij}dx^idx^j \), \((i, j = \{0, 1\})\). Now we can calculate the quantities mentioned in [33].

The dynamical apparent horizons \( \mathcal{H} \) is a marginally trapped surface with vanishing expansion, determined by

\[
    \chi(x) \Big|_H = \gamma^{ij} \partial_i R \partial_j R \Big|_H = 0
\]
i.e.
\[ R_H = \frac{1}{\sqrt{H^2 + \frac{k}{a(t)^2}}} = R_A(t) . \] (3.4)

The Misner-Sharp mass, on the horizon, is
\[ m_H = \frac{1}{2} R (1 - \chi) \bigg|_H = \frac{R_H}{2} . \] (3.5)

In spherical symmetric case, it is possible to introduce the Kodama vector field
\[ K^i = \frac{1}{\sqrt{-\gamma}} \varepsilon^{ij} \partial_j R , \quad K^\theta = 0 = K^\varphi , \] (3.6)
where \( \varepsilon_{ij} = \frac{1}{\sqrt{1 - k R^2 / a^2}} (dt)_i \wedge (dR)_j \). Here \( K^i = \left( \sqrt{1 - k R^2 / a^2}, 0 \right) \). The Kodama vector gives a preferred flow of time and is a dynamic analogue of a stationary Killing vector. By using the Kodama vector, the Misner-Sharp mass can be defined as a conserved quantity [48]. The Kodama vector is timelike, null and spacelike as \( R < R_H, R = R_H \) and \( R > R_H \). One can also introduce a dynamical surface gravity [39, 40, 33] \( \kappa \) defined by
\[ \kappa = \frac{1}{2} \mathcal{D}_R R \ , \] (3.7)
and satisfies similar Killing identity on the horizon
\[ K^a \nabla_b [K_a] \simeq \pm \kappa K_b \ . \] (3.8)

The surface gravity associated with the apparent horizon is
\[ \kappa_H = \frac{1}{2 \sqrt{-\gamma}} \partial_i \left( \sqrt{-\gamma} \varepsilon^{ij} \partial_j R(x) \right) \bigg|_H \\
= \frac{R \left( k + a^2 \left( 2H^2 + \dot{H} \right) \right)}{2a^2} \bigg|_H \\
= -\frac{R}{2} \left( \frac{k}{a^2} + 2H^2 + \dot{H} \right) \bigg|_H \\
= -\frac{R_H}{2} \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right) \\
= \frac{a^2 \left( 2H^2 + \dot{H} \right) + k}{2 \sqrt{a^2H^2 + k}} . \] (3.9)

In [35, 36, 49], by assuming a temperature to the apparent horizon, one can discuss the first law of thermodynamics in the FRW universe. Later on, [37, 38] proposed that one can derive the temperature of the apparent horizon using the tunneling method. The energy of particle tunneling through the apparent horizon is defined using the Kodama vector
\[ \omega = -K^i \partial_i I = -\sqrt{1 - k R^2 a^2} \partial_t I . \] (3.10)
For a massless particle, the Hamilton-Jacobi equation (2.7) is
\[
-(\partial_t I)^2 - 2RH(\partial_R I)(\partial_t I) + \left(1 - \frac{kR^2}{a^2} - R^2H^2\right)(\partial_R I)^2 = 0 .
\]
(3.11)

For the outgoing mode we take the plus sign
\[
\partial_t I = \left(\sqrt{1 - k\frac{R^2}{a^2} - RH}\right)(\partial_R I) .
\]
(3.12)

The trajectory for the massless particle is the radial null geodesics, determined by
\[
0 = ds^2 = \frac{1 - \frac{R^2}{a^2}}{1 - k\frac{R^2}{a^2}} dt^2 - \frac{2HR}{1 - k\frac{R^2}{a^2}} dt dR + \frac{1}{1 - k\frac{R^2}{a^2}} dR^2 .
\]
(3.13)

For the outgoing particle, the curve is
\[
dR = \left(\sqrt{1 - k\frac{R^2}{a^2} + RH}\right) dt .
\]
(3.14)

The curve can be parametrized by the variable \(R\) as \(\tilde{\gamma}(R) = (t(R), R)\), where the function \(t(R)\) is determined by (3.14).

The action of the particle tunneling through the horizon can be reconstructed as a line integral along the incoming trajectory
\[
I = \int dx^i \partial_i I = \int_{\gamma} (dt \partial_t I + dR \partial_R I) .
\]
(3.15)

Combining Eqs. (3.10)(3.12)(3.14), the action can be calculated as
\[
I = \int \left(\frac{-\omega}{\sqrt{1 - k\frac{R^2}{a^2}}} \frac{1}{\sqrt{1 - k\frac{R^2}{a^2} + RH}} - \frac{-\omega}{\sqrt{1 - k\frac{R^2}{a^2}}} \frac{1}{\sqrt{1 - k\frac{R^2}{a^2} - RH}}\right) dR
\]
\[
= -\int \left(\frac{2\omega}{1 - H^2R^2 - \frac{kR^2}{a^2}}\right) dR
\]
(3.16)

Note that we should express the above integrand as a function of \(R\), say, \(H(t(R))\) and \(a(t(R))\), where the function \(t(R)\) is determined by (3.14). This integral has a pole at the horizon \(R = R_H\), where
\[
1 - H^2R^2 - \frac{kR^2}{a^2} = \left(1 - \frac{kR_H^2}{a^2} - R_H^2H^2\right) + \left(\frac{-a^2\left(2H^2 + \dot{H}\right) + k}{\sqrt{a^2H^2 + k}}\right)(R - R_H) + O(R - R_H) .
\]

The first term on the right hand side equals zero, due to the definition of the horizon (3.4), and the coefficient of the second term gives \(-2|\kappa_H|\). The imaginary part of the action is then
\[
\text{Im}I = \frac{\pi\omega}{|\kappa_H|} .
\]
(3.17)
Again, the emission rate of the radiation is associated with the imaginary part of the action, and it is also related with the temperature $$\Gamma \sim e^{-2\text{Im} I/\hbar} \sim e^{-\frac{\pi}{T_H}}.$$ So, we obtain the temperature for Hawking radiation of the apparent horizon in FRW universe

$$T_H = \frac{|\kappa_H|}{2\pi}. \quad (3.18)$$

Using the expression (3.18) of the temperature, we can discuss the first law of thermodynamics associated with the apparent horizon. According to [34, 35], the unified first law is written as

$$dm_H = T_H dS + W dV_H, \quad (3.19)$$

where $$S = \frac{A_H}{4}$$, a quarter of the horizon area, and $$W dV$$ is the work term.

In the FRW universe, the energy-momentum tensor has the form of perfect fluid

$$T_{ab} = (\rho + p) U_a U_b + pg_{ab}, \quad (3.20)$$

where $$U^a$$ denotes the four-velocity of the fluid, $$\rho, p$$ are the energy density and pressure, respectively. In the FRW metric (3.1), the components of $$T_{ab}$$ are

$$T_{00} = \rho, \quad T_{ij} = pg_{ij}. \quad (3.21)$$

Write them out explicitly, in the $$(t, r)$$ sector

$$T_{(2)}^{(2)} = \left( \rho, \frac{p a^2}{1 - kr^2} \right), \quad T_{(2)}^{(2)} = \text{diag} \left( -\rho, p \right). \quad (3.22)$$

The work term is given by

$$W = -\text{trace} T^{(2)} = \frac{1}{2} (\rho - p). \quad (3.23)$$

Substitute the expressions

$$m_H = \frac{R_H}{2}, \quad A_H = 4\pi R_H^2$$

$$V_H = \frac{4}{3} \pi R_H^2, \quad \kappa_H = -\frac{R_H}{2} \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right)$$

and the temperature $$T_H = \frac{|\kappa_H|}{2\pi}$$, comparing the coefficients of $$dR_H$$ on both sides, and demanding that the Friedmann equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \rho \quad (3.24)$$

$$\dot{H} - \frac{k}{a^2} = -4\pi (\rho + p) \quad (3.25)$$

one can check that the unified first law (3.19) indeed holds.

Thus, we have obtained the Hawking temperature of the apparent horizon in the FRW universe, and check that this temperature can indeed be consistent with a kind of unified first law of thermodynamics. This is the application of the tunneling method discussed in the previous section in the FRW universe model.
4. Discussions

In this article, we revisited the topic of interpreting Hawking radiation of black holes as an effect of quantum tunneling. We discuss the Hamilton-Jacobi method of black hole tunneling. The main viewpoint of this article is that the action should be constructed as a line integral along the classically forbidden trajectory of the tunneling particle. This invariant construction is mostly inspired by the discussion in [33]. As an example, we illustrate the calculation for massless particles in Schwarzschild coordinate. Additionally, in order to support this method, we perform the calculation for a massive particle, and for a massless particle in Painlevé coordinate. These calculations all give the result of standard Hawking temperature. Since the propagating rate of quantum tunneling is related to the imaginary part of the action along classical forbidden trajectory, the results here support the interpretation of Hawking radiation as a quantum tunneling phenomenon. On the other hand, since the construction of the action is invariant, there is not the “factor of 2 problem” here.

The thermodynamics of the FRW model has been extensively discussed. As an application, we calculate this temperature using the Hamilton-Jacobi tunneling method discussed in this article. We obtain the result \( T = \frac{|\kappa H|}{2\pi} \), and show that the unified first law based on this expression of temperature can be consistent with the Friedmann equations which describe the dynamical properties of the FRW model. This result is different with the temperature \( T = \frac{1}{2\pi H} \) in [37, 38]. In fact, in their calculation the actions of tunneling particles are not constructed in an invariant way. Maybe the result \( T = \frac{1}{2\pi H} \) can be viewed as a certain kind of approximation on the apparent horizon. Another point is that in [37, 38] they chose the in coming mode for the tunneling particles, while in [33] and our article the choice is the outgoing mode. This point still needs further investigation.

Acknowledgments

We thank C.Cao, Q.J.Cao, Y.J.Du, J.L.Li and Q.Ma for useful discussions. Chen would like to thank the organizer and the participants of the advanced workshop, “Dark Energy and Fundamental Theory” supported by the Special Fund for Theoretical Physics from the National Natural Science Foundation of China with grant No. 10947203, for stimulating discussions and comments. The research is supported by the NNSF of China Grant No. 90503009, No. 9775116, 973 Program Grant No. 2005CB724508, and in part by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, Grant No. KJCX2.YW.W10.

References

[1] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43 (1975) 199–220.
[2] J. B. Hartle and S. W. Hawking, Path Integral Derivation of Black Hole Radiance, Phys. Rev. D13 (1976) 2188–2203.
[3] P. Kraus and F. Wilczek, Self-Interaction Correction to Black Hole Radiance, Nucl. Phys. B433 (1995) 403–420, [arXiv:gr-qc/9408003].
[4] P. Kraus and F. Wilczek, "Effect of selfinteraction on charged black hole radiance," Nucl. Phys. B437 (1995) 231–242, hep-th/9411219.

[5] M. K. Parikh and F. Wilczek, "Hawking radiation as tunneling," Phys. Rev. Lett. 85 (2000) 5042–5045, hep-th/9907001.

[6] B. D. Chowdhury, "Problems with Tunneling of Thin Shells from Black Holes," Pramana 70 (2008) 593–612, hep-th/0605197.

[7] P. Mitra, "Hawking temperature from tunnelling formalism," Phys. Lett. B648 (2007) 240–242, hep-th/0611265.

[8] T. Pilling, "Black hole thermodynamics and the factor of 2 problem," Physics Letters B 660 (2008), no. 4 402 – 406, arXiv:0709.1624.

[9] T. Pilling, "Quasi-classical Hawking Temperatures and Black Hole Thermodynamics," arXiv:0809.2701.

[10] R. Banerjee and B. R. Majhi, "Hawking black body spectrum from tunneling mechanism," Phys. Lett. B675 (2009) 243, arXiv:0903.0253.

[11] R. Banerjee, B. R. Majhi, and E. C. Vagenas, "Quantum tunneling and black hole spectroscopy," Phys. Lett. B686 (2010) 279–282, arXiv:0907.4271.

[12] B. R. Majhi, "Hawking radiation and black hole spectroscopy in Horava- Lifshitz gravity," Phys. Lett. B686 (2010) 49–54, arXiv:0911.3233.

[13] R. Banerjee, B. R. Majhi, and E. C. Vagenas, "A Note on the Lower Bound of Black Hole Area Change in Tunneling Formalism," arXiv:1005.1495.

[14] R. Banerjee and B. R. Majhi, "Connecting anomaly and tunneling methods for Hawking effect through chirality," Phys. Rev. D79 (2009) 064024, arXiv:0812.0497.

[15] S. Kar, "Tunneling between de Sitter and anti de Sitter black holes in a noncommutative D3-brane formalism," Phys. Rev. D74 (2006) 126002, hep-th/0607029.

[16] J.-Y. Zhang and Z. Zhao, "Hawking radiation via tunneling from Kerr black holes," Mod. Phys. Lett. A20 (2005) 1673–1681.

[17] Q.-Q. Jiang, S.-Q. Wu, and X. Cai, "Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes," Phys. Rev. D73 (2006) 064003, hep-th/0512351.

[18] R. Kerner and R. B. Mann, "Fermions Tunnelling from Black Holes," Class. Quant. Grav. 25 (2008) 095014, arXiv:0710.0612.

[19] T. Jian and C. Bing-Bing, "Fermions tunneling from Kerr and Kerr-Newman black holes," Acta Phys. Polon. B40 (2009) 241–250.

[20] K. Srinivasan and T. Padmanabhan, "Particle production and complex path analysis," Phys. Rev. D60 (1999) 024007, gr-qc/9812023.

[21] S. Shankaranarayanan, K. Srinivasan, and T. Padmanabhan, "Method of complex paths and general covariance of Hawking radiation," Mod. Phys. Lett. A16 (2001) 571–578, gr-qc/0007022.

[22] S. Shankaranarayanan, T. Padmanabhan, and K. Srinivasan, "Hawking radiation in different coordinate settings: Complex paths approach," Class. Quant. Grav. 19 (2002) 2671–2688, gr-qc/0010042.
[23] S. Shankaranarayanan, Temperature and entropy of Schwarzschild-de Sitter space-time, Phys. Rev. D67 (2003) 084026, [gr-qc/0301090].

[24] R. Banerjee and B. R. Majhi, Quantum Tunneling Beyond Semiclassical Approximation, JHEP 06 (2008) 095, [arXiv:0805.2220].

[25] R. Banerjee and S. K. Modak, Quantum Tunneling, Blackbody Spectrum and Non-Logarithmic Entropy Correction for Lovelock Black Holes, JHEP 11 (2009) 073, [arXiv:0908.2346].

[26] S. K. Modak, Corrected entropy of BTZ black hole in tunneling approach, Phys. Lett. B671 (2009) 167–173, [arXiv:0807.0953].

[27] R. Banerjee and S. K. Modak, Exact Differential and Corrected Area Law for Stationary Black Holes in Tunneling Method, JHEP 05 (2009) 063, [arXiv:0903.3321].

[28] T. Zhu, J.-R. Ren, and M.-F. Li, Corrected Entropy of Friedmann-Robertson-Walker Universe in Tunneling Method, JCAP 0908 (2009) 010, [arXiv:0905.1833].

[29] T. Zhu, J.-R. Ren, and M.-F. Li, Corrected entropy of high dimensional black holes, [arXiv:0906.4194].

[30] M. Akbar and K. Saifullah, Quantum corrections to the entropy of Einstein-Maxwell dilaton-axion black holes, [arXiv:1002.3901].

[31] E. T. Akhmedov, T. Pilling, and D. Singleton, Subtleties in the quasi-classical calculation of Hawking radiation, Int. J. Mod. Phys. D17 (2008) 2453–2458, [arXiv:0805.2653].

[32] V. Akhmedova, T. Pilling, A. de Gill, and D. Singleton, Temporal contribution to gravitational WKB-like calculations, Phys. Lett. B666 (2008) 269–271, [arXiv:0804.2289].

[33] R. D. Criscienzo, S. A. Hayward, M. Nadalini, L. Vanzo, and S. Zerbini, Hamilton-jacobi tunneling method for dynamical horizons in different coordinate gauges, Classical and Quantum Gravity 27 (2010), no. 1 015006, [arXiv:0906.1728].

[34] L. Landau, E. Lifshitz, J. Sykes, J. Bell, and M. Rose, Quantum Mechanics, Non-Relativistic Theory: Vol. 3 of Course of Theoretical Physics, Physics Today 11 (1958) 56.

[35] R.-G. Cai and S. P. Kim, First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe, JHEP 02 (2005) 050, [hep-th/0501055].

[36] R.-G. Cai and L.-M. Cao, Unified first law and thermodynamics of apparent horizon in FRW universe, Phys. Rev. D75 (2007) 064008, [gr-qc/0611071].

[37] R.-G. Cai, L.-M. Cao, and Y.-P. Hu, Hawking Radiation of Apparent Horizon in a FRW Universe, Class. Quant. Grav. 26 (2009) 155018, [arXiv:0809.1554].

[38] R. Li, J.-R. Ren, and D.-F. Shi, Fermions Tunneling from Apparent Horizon of FRW Universe, Phys. Lett. B670 (2009) 446, [arXiv:0812.4217].

[39] S. A. Hayward, Unified first law of black-hole dynamics and relativistic thermodynamics, Class. Quant. Grav. 15 (1998) 3147–3162, [gr-qc/9710089].

[40] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini, and S. Zerbini, Local Hawking temperature for dynamical black holes, Class. Quant. Grav. 26 (2009) 062001, [arXiv:0806.0014].

[41] Y.-P. Hu, Hawking temperature of the cosmological horizon in a FRW universe, [arXiv:1007.4044].
[42] R. di Criscienzo, S. Hayward, M. Nadalini, L. Vanzo, and S. Zerbini, *Invariance of the Tunneling Method for Dynamical Black Holes*, arXiv:1006.1590.

[43] R. M. Wald, *General Relativity*. Chicago, Usa: Univ. Pr. (1984) 491p.

[44] E. T. Akhmedov, V. Akhmedova, and D. Singleton, *Hawking temperature in the tunneling picture*, Phys. Lett. B642 (2006) 124–128, hep-th/0608098.

[45] T. Zhu, J.-R. Ren, and D. Singleton, *Hawking-like radiation as tunneling from the apparent horizon in a FRW Universe*, Int. J. Mod. Phys. D19 (2010) 159–169, arXiv:0902.2542.

[46] S. Stotyn, K. Schleich, and D. Witt, *Observer Dependent Horizon Temperatures: a Coordinate-Free Formulation of Hawking Radiation as Tunneling*, Class. Quant. Grav. 26 (2009) 065010, arXiv:0809.5093.

[47] Y.-P. Hu, J.-Y. Zhang, and Z. Zhao, *A note on the Hawking radiation calculated by the quasi-classical tunneling method*, Mod. Phys. Lett. A25 (2010) 295–308, arXiv:0901.2680.

[48] S. A. Hayward, *Gravitational energy in spherical symmetry*, Phys. Rev. D53 (1996) 1938–1949, gr-qc/9408002.

[49] Y. Gong and A. Wang, *The Friedmann equations and thermodynamics of apparent horizons*, Phys. Rev. Lett. 99 (2007) 211301, arXiv:0704.0793.