Interacting boundaries of 3D topological crystalline insulators: effective field theories through bosonisation

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Here, we analyse two Dirac fermion species in two spatial dimensions in the presence of general quartic contact interactions. By employing functional bosonisation techniques, we demonstrate that depending on the couplings of the fermion interactions the system can be effectively described by a rich variety of topologically massive gauge theories. Among these effective theories, we obtain an extended Chern-Simons theory with higher order derivatives as well as two coupled Chern-Simons theories. Our formalism allows for a general description of interacting fermions emerging, for example, at the gapped boundary of three-dimensional topological crystalline insulators.

I. INTRODUCTION

Time-reversal-invariant topological insulators are among the most well studied topological phases of matter. In three dimensions, they are characterised by suitable topological numbers in the bulk that guarantee the existence of topologically protected massless Dirac fermions on the boundary [1]. Although the topological invariant is a $\mathbb{Z}_2$ number, on the slab geometry, it has been shown that robust surface states are given by an odd number of Dirac fermions per boundary [2]. The situation changes in the case of three-dimensional topological crystalline insulators (TCIs), namely topological insulators characterised by further crystalline symmetries, such as mirror and rotation symmetries [3–7]. In particular, for three-dimensional TCIs protected by a single mirror symmetry, one can define the so called mirror Chern number $n_M$ on a given two-dimensional plane, which is invariant under the mirror symmetry. These phases host $n = |n_M|$ Dirac cones on each boundary [4]. Recently, mirror-invariant boundary interactions in these systems have been intensively studied by employing several approaches, such as non-linear sigma models [8, 9], the coupled-wire method for $n_M = 2$ [10], Higgs phases for $n_M = 4$ [11] and symmetry arguments for $n_M = 8$ [12].

Bosonisation represents another important quantum-field-theory approach to study interacting Dirac fermions. It was originally formulated in 1+1 dimensions to map the massive Thirring model to the Sine-Gordon theory [13, 14] and then extended in higher-dimensional relativistic systems under the name of functional bosonisation [15]. Although this method has numerous implications that are relevant to condensed matter physics, it has been mainly employed in interacting systems involving a single emergent gauge field.

The goal of this work is to analyse gapped and mirror-broken boundary states in presence of quartic contact interactions between several pieces of fermions. We introduce an external magnetic field orthogonal to the surface to induce a Dirac mass that breaks both time-reversal and mirror symmetries and consider generic intra- and inter-species interactions. For simplicity, we fix $n_M = 2$ and employ functional bosonisation. This approach will allow us to map the self-interacting fermion model to free bosonic models. We are interested in obtaining the low energy topological properties of these effective bosonic models for various configuration of inter and intra-species interactions of the original fermionic model.

Our analysis shows that all the resulting effective models contain topological Chern-Simons terms that usually emerge in a variety of $T$-broken systems such as the quantum Hall states [16, 17], surface states of three-dimensional topological insulators [1] and graphene coupled to external magnetic fields [18]. However, differently from these previous works, we show the existence of a new exotic phase, characterised by a higher-derivative Chern-Simons term [19], when one of the intra-species interaction is switched off. This phase supports a massive $U(1)$ boson and a ghost mode, which is completed decoupled from the bosonic mode, and thus it is a “good ghost” [20, 21]. Moreover, we show the existence of another phase in which the bosonic theory comprises two massive $U(1)\times U(1)$ bosons, with a mutual Chern-Simons term, which generalises the well-known Chern-Simons-Maxwell theory to multi-field gauge fields. The topological sector of this phase resembles the effective action studied in Ref. [22] in the context of thin-film topological insulators. Importantly, our approach is quite general and can be directly extended to $n_M > 2$. This will allow to identify novel topological crystalline phases in presence of very general contact interactions.
II. TWO-FERMION INTERACTING SYSTEM

The starting point of our construction is a \((2 + 1)\)-dimensional system of two interacting fermion species \(\psi\) and \(\chi\) living on the boundary of 3D topological crystalline insulator with bulk mirror Chern number \(n_M = 2\). The corresponding effective action is given by

\[
S[\chi, \psi] = \int d^3x \left[ \bar{\chi} (i\gamma^\mu \partial_\mu + m) \chi + \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi + \frac{V_\chi}{2} \bar{\chi} \gamma^\mu \gamma^\nu \gamma^\rho \chi + V_\psi \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \psi + \frac{V_\psi}{2} \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \psi \right],
\]

where \(m = B_2 \sigma_3\) is the time-reversal broken mass induced by an external magnetic field \(B_2\) orthogonal to the surface of the 3D TCI defined on the \(xy\)-plane. Here, we use the convention for the Minkowski metric \(\eta_{\mu\nu} = \text{diag}(-, +, +)\). The gamma matrices are defined in terms of the Pauli matrices as

\[
\gamma^0 = \sigma_3, \quad \gamma^1 = i \sigma_1 \gamma^2 = i \sigma_2 \quad \text{and} \quad \bar{\psi} = \psi^\dagger \gamma^0,
\]

the Dirac conjugate is \(\bar{\psi} = \psi^\dagger \gamma^0\), and the Clifford algebra has the form \(\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\gamma_2\). For convenience in the presentation we choose the intra-species coupling constants to be given by \(V_\chi = e_\chi^2\) and \(V_\psi = e_\psi^2 + \xi \alpha^2\), where \(e_\chi\), \(e_\psi\) and \(\alpha\) are real constants, and \(\xi = \pm 1\), while \(V_\chi \psi = e_\chi e_\psi\) is the inter-species coupling constants.

To analytically determine the behaviour of this interaction system we employ functional bosonisation. This is a powerful approach that will allows us to identify the equivalent bosonic theory describing our model in the low-energy regime. By defining \(k^\mu = \bar{\chi} \gamma^\mu \chi\), \(j^\mu = \bar{\psi} \gamma^\mu \psi\), the corresponding generating functional has the form

\[
Z = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ i \int d^3x \left[ \bar{\chi} (i\gamma^\mu \partial_\mu + m) \chi + \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi + \frac{1}{2} (e_\chi k^\mu + e_\psi j^\mu) (e_\chi k_\mu + e_\psi j_\mu) \right. \right.
\]

\[
+ \left. \frac{\xi \alpha^2}{2} j^\mu j_\mu \right]\right\}.
\]

In order to integrate out the fermion field \(\chi\), we follow [15, 23] and express the third term in the action as

\[
\exp \left\{ \frac{i}{2} \int d^3x (e_\chi k^\mu + e_\psi j^\mu) (e_\chi k_\mu + e_\psi j_\mu) \right\} = \int \mathcal{D}a \ exp \left\{ i \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu + a^\mu (e_\chi k^\mu + e_\psi j^\mu) \right] \right\},
\]

where \(a_\mu\) is an Hubbard-Stratonovich vector field. By replacing this back into the generating functional \(Z\), we obtain

\[
Z = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi\mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}a \ \exp \left\{ i \int d^3x \left[ \bar{\chi} (\gamma^\mu (i\partial_\mu + e_\chi a_\mu) + m) \chi \right. \right.
\]

\[
+ \left. \bar{\psi} (\gamma^\mu (i\partial_\mu + e_\psi a_\mu) + m) \psi + \frac{\xi \alpha^2}{2} j^\mu j_\mu - \frac{1}{2} a^\mu a_\mu \right\}.
\]

We now integrate out \(\chi\) to obtain an effective bosonic action \(\Gamma[a]\). In the large mass limit, it can be approximated as [24–26]

\[
\Gamma[a] = -i \log \left| \det (\gamma^\mu (i\partial_\mu + e_\chi a_\mu) + m) \right| \approx \frac{s_m e_\chi^2}{8\pi} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho,
\]

where \(s_m = \frac{m}{|m|} = \text{sign}(m)\) and \(\epsilon^{\mu\nu\rho}\) is the \((2+1)\)-dimensional Levi-Civita symbol with \(\epsilon^{012} = 1\). Therefore we can write

\[
Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}a \ \exp \left\{ i S_{\text{eff}}[\psi, a] \right\},
\]

where

\[
S_{\text{eff}}[\psi, a] = \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu + \frac{s_m e_\chi^2}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \bar{\psi} (\gamma^\mu (i\partial_\mu + e_\psi a_\mu) + m) \psi + \frac{\xi \alpha^2}{2} j^\mu j_\mu \right].
\]
III. PAULI TERM AND HIGHER-DERIVATIVE CHERN-SIMONS ACTION

In this section we show that for a particular value of the couplings the fermionic system (1) can be described in the low energy limit by a single fermion field non-minimally coupled to an effective $U(1)$ gauge field. In particular, this coupling configuration gives rise to a higher-derivative Chern-Simons theory [19]. We start our analysis of (7) by considering the interpolating action

$$S_I[\psi, a, A] = \int d^3x \left\{ -\frac{1}{2} \epsilon^{\mu\nu\rho} a_\mu + e^{\mu\nu\rho} \partial_\mu A_\rho - \frac{2\pi s_m}{\epsilon^2} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + e_\psi a_\mu j_\mu + \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi + \frac{\xi \alpha^2}{2} j^\mu j_\mu \right\}, \quad (8)$$

which is given in terms of the Dirac fermion field $\psi$, the vector field $a_\mu$ and a new gauge field $A_\mu$. The path integral of $S_I[\psi, a, A]$ is equivalent to the functional integral associated to the effective action (7). By integrating out the field $A_\mu$ in (8), we obtain

$$Z_I = \int D\bar{\psi} D\psi D\bar{a} DA \exp \{ iS_I[\psi, a, A] \} = \int D\bar{\psi} D\psi Da \exp \{ iS_{\text{eff}}[\psi, a] \}, \quad (9)$$

where $S_{\text{eff}}$ is given by (7). On the other hand, by integrating out the vector field $a_\mu$ in the interpolating action (8) we find

$$Z_I = \int D\bar{\psi} D\psi DA \exp \{ iS_{\text{eff}}^\text{dual}[\psi, A] \}, \quad (10)$$

where the action $S_{\text{eff}}^\text{dual}[\psi, A]$ is given by

$$S_{\text{eff}}^\text{dual}[\psi, A] = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{2\pi s_m}{\epsilon^2} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi + e_\psi \bar{\psi} \gamma^\mu \psi + \frac{1}{2} \left( \epsilon^2 + \xi \alpha^2 \right) j^\mu j_\mu \right\}. \quad (11)$$

In this dual effective action $S_{\text{eff}}^\text{dual}[\psi, A]$ the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength associated to the gauge potential $A_\mu$ and we have also used the standard definition $\epsilon_{\mu\nu\rho} = \frac{1}{4} [\gamma_\mu, \gamma_\nu] = \frac{1}{2} \epsilon_{\mu\nu\rho} \gamma^\rho$.

The action $S_{\text{eff}}^\text{dual}[\psi, A]$ is dual to $S_{\text{eff}}[\psi, a]$, so it faithfully describes the original system (1). The advantage of $S_{\text{eff}}^\text{dual}[\psi, A]$ is that it is given in terms of the gauge field $A_\mu$ rather than the vector field $a_\mu$ and thus it is easier to identify its topological character. From (11) we observe that the effective action (7) can be dualised to a Chern-Simons-Maxwell model coupled to the fermion field $\psi$ by means of the Pauli term. Note that, after using the interpolating action, the coupling of the self interaction for $\psi$ has been shifted back to the original value it had in (1). By directly comparing (1) and (11), we see that we can interpret the Pauli coupling as the low energy description of the mixed interaction term $V_{\psi}\psi^\gamma \gamma^\mu \psi \gamma^\rho \chi$. The duality between (11) and (7) has been previously established on-shell by eliminating the field $a_\mu$ or $A_\mu$ from the interpolating action (8) by means of their corresponding field equations [29, 30].

A. Higher derivative Chern-Simons theory: the $\epsilon^2 + \xi \alpha^2 = 0$ case

We now consider the action (7) for the case where $\alpha^2 = \epsilon^2 \neq 0$ and $\xi = -1$, which corresponds to $V_{\psi} = 0$ in (1). In that case, the action (7) does not describe free fermions so that they cannot be integrated out. Moreover for $\xi = -1$ we cannot employ a similar relation to (3) in order to linearise the interactions. In this case the self interaction in the dual action (11) vanishes and the effective theory takes the form of a fermion non-minimally coupled to the field strength $F_{\mu\nu}$ by means of the Pauli term. Defining the Hodge dual of the curvature $F_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho}$ the action takes the form

$$S_{\text{eff}}^\text{dual}[\psi, A] = \int d^3x \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{2\pi s_m}{\epsilon^2} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \bar{\psi} \left[ \gamma^\mu (i\partial_\mu + e_\psi F_\mu) + m \right] \psi \right\}. \quad (12)$$

In other words, the Pauli term couples the magnetic moment of the fermions with the magnetic field [31–33]. Note that in $2+1$ dimensions the magnetic moment is a scalar leading to the coupling term $e_\psi k^\mu F_\mu$ seen in (12).
To analyse the properties of (12) we integrate out the fermions \( \psi \). As shown in [34–36] the Pauli term can be obtained starting from the standard minimal coupling in the Dirac action \( \bar{\psi} \gamma^\mu A_\mu \psi \) and shifting the gauge field \( A_\mu \) into the generalised connection \( A_\mu \rightarrow A_\mu + e_\psi F_\mu \). We can then integrate out \( \psi \) in (12) by using the result (5) for the generalised connection and then set \( A_\mu = 0 \), i.e.

\[
\Gamma[A_\mu + e_\psi F_\mu]_{A_\mu=0} = -i \log \left[ \det (\gamma^\mu (i \partial_\mu + A_\mu + e_\psi F_\mu) + m) \right]_{A_\mu=0} \approx \frac{s_m e_\psi^2}{8\pi} \int d^3x \epsilon^{\mu\nu\rho} F_\mu \partial_\nu F_\rho. \quad (13)
\]

By using this result, which is also compatible with Ref. [37], the corresponding effective action takes the form

\[
S_{\text{eff}}^{\text{dual}}[A] = \int d^3x \left[ \frac{1}{2} F_\mu F_\mu - \frac{2\pi s_m}{e_\chi^2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{s_m e_\psi^2}{8\pi} \epsilon^{\mu\nu\rho} F_\mu \partial_\nu F_\rho \right]. \quad (14)
\]

It is important to remark that, even though the higher-derivative term in the above action looks like a Chern-Simons form, it is not topological as it depends on the space-time metric. Indeed, as shown in [19], up to boundary terms one can write

\[
\epsilon^{\mu\nu\rho} F_\mu \partial_\nu F_\rho = \epsilon^{\mu\nu\rho} \Box A_\mu \partial_\nu A_\rho. \quad (15)
\]

Thus, this term leads to a non-vanishing contribution to the energy-momentum tensor, which is a signature of its non-topological nature.

As it has been shown in [19], the action (14) includes a ghost mode. Now we will show that this model admits a description in which the ghost is decoupled from the physical degree of freedom. In order to do so, we follow [38] and decompose the vector potential in terms of new variables \( X \) and \( Y \) as follows

\[
A_0 = \frac{1}{\sqrt{-\nabla^2}} X, \quad A_i = \frac{1}{\sqrt{-\nabla^2}} \varepsilon_{ij} \partial_j Y. \quad (16)
\]

The effective action (14) then becomes

\[
S_{\text{eff}}^{\text{dual}}[X,Y] = \int d^3x \left[ \frac{1}{2} Y \Box Y + \frac{s_m e_\psi^2}{8\pi} X \Box Y - \frac{2\pi s_m}{e_\chi^2} X Y + \frac{1}{2} X^2 \right]. \quad (17)
\]

We can now integrate out the field \( X \) in the corresponding partition function \( Z = \int DXDY e^{S_{\text{eff}}^{\text{dual}}[X,Y]} \), which yields an effective action for \( Y \) given by

\[
S_{\text{eff}}^{\text{dual}}[Y] = -\frac{1}{2} \left( \frac{e_\psi^2}{8\pi} \right)^2 \int d^3x Y \left( \Box - m_\varphi^2 \right) Y, \quad m_\varphi^2 = \frac{1}{2} \left( \frac{8\pi}{e_\psi^2} \right)^2 \left( 1 + \frac{e_\psi^2}{2e_\chi^2} \right) \left( 1 + \frac{e_\chi^2}{e_\psi^2} \right). \quad (18)
\]

This higher derivative scalar field action can be expressed in terms of two Klein-Gordon fields \( \varphi_\pm \) defined by

\[
\varphi_\pm = \frac{e_\psi^2}{8\pi} \left( \Box - m_\varphi^2 \right) Y, \quad m_\varphi^2 = \frac{1}{2} \left( \frac{8\pi}{e_\psi^2} \right)^2 \left( 1 + \frac{e_\psi^2}{2e_\chi^2} \right) \left( 1 + \frac{e_\chi^2}{e_\psi^2} \right). \quad (19)
\]

The action then takes the form [20, 21]

\[
S_{\text{eff}}^{\text{dual}}[\varphi_+, \varphi_-] = \int d^3x \left[ \frac{1}{2} \varphi_+ \left( \Box - m_\varphi^2 \right) \varphi_+ - \frac{1}{2} \varphi_- \left( \Box - m_\varphi^2 \right) \varphi_- \right]. \quad (20)
\]

Hence, the field redefinition (19) allows us to express (14) as the action for two decoupled massive Klein-Gordon fields, \( \varphi_+ \) and \( \varphi_- \). The field \( \varphi_+ \) is a physical Klein-Gordon field, while \( \varphi_- \) is a ghost. Since the ghost fields is totally decoupled from the physical degree of freedom, the physical spectrum is not affected by it. In this sense we have a “good” ghost [20] emerging in our theory. From (19) we see that \( m_\varphi^2 > 0 \) for any values of \( e_\psi \) and \( e_\chi \). On the other hand, \( m_\varphi^2 \) can be positive or negative depending on the values of the couplings \( e_\psi \) and \( e_\chi \), implying that the ghost \( \varphi_2 \) can be also a tachyon. Thus, this theory shares similar features with the Chern-Simons-Maxwell theory [39] that describes a single propagating massive bosonic mode. In our case, the effect of the higher-derivative term is to renormalise the topological mass of the boson.
IV. SINGLE AND MUTUAL CHERN-SIMONS THEORIES

In this section we show that, besides the Chern-Simons and Maxwell terms, suitable choices of the parameters in the starting action (1) lead to an effective description of the system that includes a mutual Chern-Simons term [40].

A. Single Chern-Simons theory: the $\alpha = 0$ case

The case $\alpha = 0$ corresponds to interaction couplings in (1) that satisfy $V_\chi V_\psi = V_\chi^2$. In this case we can define the four-spinor $\Psi = (\psi, \chi)^T$ and the corresponding current $J^\mu = \bar{\Psi} \Gamma^\mu \Psi$, where $\Gamma^\mu = \frac{1}{2} \otimes \gamma^\mu$. The generating functional (2) then boils down to

$$Z = \int D\Psi D\bar{\Psi} \exp \left\{ i \int d^3x \left[ \bar{\Psi} (i\partial^\mu + m) \Psi + \frac{1}{2} e^2 \bar{\Psi} J^\mu J_\mu \right] \right\}, \quad (21)$$

where $e^2 = e_\chi^2 + e_\psi^2$. Since this is a standard Thirring model for $\Psi$ we can linearise the interactions by introducing a vector field $a_\mu$ [15], so that by means of Gaussian integration we implement the replacement $e^2 J^\mu J_\mu \rightarrow -\frac{1}{2} a^\mu a_\mu + e_\psi a^\mu J_\mu$ in (21). Using (5) to integrate out $\Psi$, the low energy behaviour of this system is captured by the following effective action

$$S_{\text{eff}}[\psi, a] = \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu + \frac{s_m e_\chi^2}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \bar{\psi} \left( \gamma^\mu (i\partial^\mu + e_\psi a_\mu) + m \right) \psi + \frac{\alpha^2}{2} j^\mu j_\mu \right]. \quad (22)$$

This result can be also obtained from (7) by setting $\alpha = 0$ in and subsequently integrating out $\psi$. Following [27, 28] we can dualise this action to a Chern-Simons-Maxwell theory

$$S_{\text{eff}}^{\text{dual}}[A] = \int d^3x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - M_A \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right]. \quad (23)$$

Hence, for the specific case where $\alpha = 0$ the system becomes formally equivalent to a single species self-interacting fermion that gives rise to a Chern-Simons theory with coupling $M_A = 2\pi s_m/(e_\chi^2 + e_\psi^2)$. This theory describes massive bosons that only mediate short-range interactions [39].

B. Mutual Chern-Simons theories: the $\xi = 1$, $\alpha \neq 0$ case

The choice of parameters $\xi = 1$ and $\alpha \neq 0$ corresponds to the action (1) with $V_\chi V_\psi > V_\chi^2$. In this case the effective action (7) becomes

$$S_{\text{eff}}[\psi, a] = \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu - \frac{1}{2} b^\mu b_\mu + \frac{s_m (e_\chi^2 + e_\psi^2)}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \bar{\psi} \left( \gamma^\mu (i\partial^\mu + e_\psi a_\mu) + m \right) \psi + \frac{\alpha^2}{2} j^\mu j_\mu \right]. \quad (24)$$

so one can integrate out $\psi$ directly. Following similar steps as above, we use (3) to linearise the self interaction in the path integral associated to (24) by introducing a new vector field $b_\mu$. Subsequently, using (5) with the replacement $a_\mu \rightarrow a_\mu + \frac{\alpha}{e_\psi} b_\mu$ leads to

$$S_{\text{eff}}[a, b] = \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu - \frac{1}{2} b^\mu b_\mu + \frac{s_m (e_\chi^2 + e_\psi^2)}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{s_m \alpha^2}{8\pi} \epsilon^{\mu\nu\rho} b_\mu \partial_\nu b_\rho + \frac{s_m e_\psi \alpha}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho \right]. \quad (25)$$

In this action both $a_\mu$ and $b_\mu$ are vector fields. In order to turn them into gauge fields we employ the interpolating action procedure. Consider the interpolating path integral (see Appendix A)

$$Z_I = \int DADBDA dB \exp \left\{ i \int d^3x \left[ -\frac{1}{2} a^\mu a_\mu - \frac{1}{2} b^\mu b_\mu + \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} b_\mu \partial_\nu B_\rho - \frac{m_A}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho - m_I \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho - \frac{m_B}{2} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right] \right\}, \quad (26)$$
where the masses $m_A$, $m_B$ and $m_I$ are to be fixed in term of the couplings constants in (25). Integrating out the fields $A_\mu$ and $B_\mu$ leads exactly to the functional integral of the action (25), i.e.

$$Z_I = \int \mathcal{D}a \mathcal{D}b \exp\{iS_{\text{eff}}[a,b]\},$$

provided the masses $m_A$, $m_B$ and $m_I$ are given by

$$m_A = \frac{4\pi s_m}{e^2_\chi}, \quad m_B = \frac{4\pi s_m}{\alpha^2} \left( 1 + \frac{e^2_\psi}{e^2_\chi} \right), \quad m_I = \frac{4\pi s_m}{\alpha e^2_\chi} e_\psi.$$

The dual theory is obtained by integrating out the vector fields $a_\mu$ and $b_\mu$ in (26), which yields

$$Z_I = \int \mathcal{D}A \mathcal{D}B \exp \left\{ i \int d^3x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} 
- \frac{m_A}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho 
+ m_I \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho 
- \frac{m_B}{2} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right] \right\},$$

where we have introduced a second field strength, $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Interestingly, for $\alpha^2 = e^2_\chi + e^2_\psi$, there appears an emergent $Z_2$ symmetry that exchanges the gauge fields, i.e.

$$Z_2: \ A_\mu \rightarrow B_\mu, \ B_\mu \rightarrow A_\mu.$$ (30)

This theory describes two massive bosons and generalises the Chern-Simons-Maxwell theory [39], which is defined for a single $U(1)$ gauge field and the double-Maxwell-BF theory [41–43]. The latter, defined for $m_A = m_B = 0$, has been employed to study the Meissner effect in two-dimensional superconductors/superfluids that preserve time-reversal symmetry. In this context, the two massive bosons can be interpreted as massive modes related to an effective London penetration length [42].

Here, we give a physical interpretation of our model by neglecting the Maxwell terms and focusing on the topological sector

$$S_{\text{eff}}^{\text{dual}}[A,B] = -\int d^3x \left[ \frac{m_A}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho 
+ m_I \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho 
+ \frac{m_B}{2} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right],$$

which is dominant at large distances. By a suitable rescaling of the gauge fields, this topological action formally coincides with that one derived in Ref. [22] in thin-film topological insulators. In this context, our $T$-broken action would describe an emergent quantum anomalous Hall state induced by interactions. In fact, the presence of $s_m$ in all the three coefficients $m_A, m_B$ and $m_I$ is the signature of the presence of a common Chern number encoded in those terms that changes sign when the external Zeeman field is flipped. There are however important physical differences with respect to Ref. [22]. In that work, the $A_\mu$ field is identified with an external electromagnetic field and the two fermion species live on different boundaries, such that only in the thin-film limit the effective 2D model for the boundary contain both species.

Finally, note that the effective action in (31) can be further reduced by integrating out the gauge field $B_\mu$, which yields the Chern-Simons action we met in (23) with the same mass $M_A = 2\pi s_m / (e_\chi^2 + e_\psi^2)$. Therefore, at the level of the topological affective action, integrating out $B_\mu$ is equivalent to set $\alpha = 0$ in the original action (1). On the other hand, if we choose to integrate the gauge field $A_\mu$, we obtain the Chern-Simons term of (23) for the field $B_\mu$ with $M_B = 2\pi s_m / \alpha^2$. This result corresponds to setting $e_\psi = e_\chi = 0$ in (1), which eliminates the interaction between $\psi$ and $\chi$ and keeping only the Thirring self-interaction for $\psi$ with coupling $\alpha^2$.

V. CONCLUSIONS

In this article we have studied the effect interactions have on two Dirac fermions in $2 + 1$ dimensions. As we are interested in the topological properties of this system we employed the bosonisation method in order to obtain the corresponding effective gauge theories. As we vary the fermion couplings with intra-species interactions, $V_\chi$ and $V_\psi$, and inter-species interactions $V_{\chi\psi}$ we obtain a variety of topological theories that correspond to different phases of the model. When one of the fermionic species does not self-interact, $V_\psi = 0$, then the system is described by a
Here we have defined the masses $m_1$, $m_2$, and $m_3$. In this case the system is described by an emergent quantum anomalous Hall state induced by interactions and the two interacting massive Dirac fermions can be mapped to the two massive bosons. Moreover, for a particular choice of the coupling constants, there appears an emergent $\mathbb{Z}_2$ symmetry. Our method does not have a simple interpretation in the case where $V_1 V_2 < V_2^2$ so an alternative approach needs to be taken. We leave this case for a future investigation. Finally, note that our approach can be naturally generalised in various ways. One can consider multi-species interactions described by multi-$U(1)$ gauge fields. This paves the way to study the interacting boundaries of 3D topological crystalline insulators for $n_M > 2$ through functional bosonisation. Moreover, one can consider multi-$SU(N)$ non-Abelian generalisation of the gauge fields along the lines of Ref. [44].

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Appendix A: Interpolating action

In this Appendix we consider a generalisation of the interpolating action method [23] to the case of an interacting Chern-Simons-Proca model of the form (25):

$$ S[a, b] = \int d^3 x \left[ -\frac{1}{2} a^\mu a_\mu - \frac{1}{2} b^\mu b_\mu + \frac{1}{2m_a} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{1}{2m_b} \epsilon^{\mu\nu\rho} b_\mu \partial_\nu b_\rho + \frac{1}{m_c} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho \right], \quad (A1) $$

where here the constants $m_a$, $m_b$, and $m_c$ are considered as arbitrary. In order to do so, we introduce two gauge fields $A_\mu$ and $B_\mu$ and consider the following general form for an interpolating action

$$ S_I[a, b, A, B] = \int d^3 x \left[ -\frac{1}{2} a^\mu a_\mu - \frac{1}{2} b^\mu b_\mu + \lambda_1 \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho + \lambda_2 \epsilon^{\mu\nu\rho} b_\mu \partial_\nu B_\rho + \lambda_3 \epsilon^{\mu\nu\rho} b_\mu \partial_\nu A_\rho + \lambda_4 \epsilon^{\mu\nu\rho} a_\mu \partial_\nu B_\rho - \frac{m_A}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - m_T \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho - \frac{m_B}{2} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right], \quad (A2) $$

together with the corresponding path integral

$$ Z_I = \int D A D B D a D b \exp \left\{ i \int d^3 x S_I[a, b, A, B] \right\}, \quad (A3) $$

Here we have defined the masses $m_A$, $m_B$, and $m_I$, which are the coupling constants of the model, while $\lambda_n$ ($n = 1, 2, 3, 4$) stands for a set of real constant coefficients that can be functions $m_A$, $m_B$, and $m_I$.

By independently integrating out the pairs of fields $a_\mu$, $b_\mu$ and $A_\mu$, $B_\mu$, the generating functional (A3) interpolates between the path integral associated to the action (A1) and an effective action for the fields $A_\mu$ and $B_\mu$, which means that up to normalisation factors,

$$ \int D a D b \exp\{i S[a, b]\} = Z_I = \int D A D B \exp \{i S^{\text{dual}}[A, B] \}. \quad (A4) $$
One can show that the action $S_{\text{dual}}[A, B]$ is given by

$$
S_{\text{dual}}[A, B] = \int d^3 x \left[ -\frac{1}{4} (\lambda_1^2 + \lambda_2^2) \mathcal{F}_{\mu
u} \mathcal{F}^{\mu\nu} - \frac{1}{4} (\lambda_2^2 + \lambda_3^2) \mathcal{G}_{\mu
u} \mathcal{G}^{\mu\nu} - \frac{1}{2} (\lambda_1 \lambda_2 + \lambda_2 \lambda_3) \mathcal{F}_{\mu\nu} \mathcal{G}^{\mu\nu} \right. \\
\left. - \frac{m_A}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + m_\pi \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho - \frac{m_B}{2} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right].
$$

(A5)

where we have defined the field strengths $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{G}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and the constants $m_A, m_B, m_\pi$ are determined by the relations

$$
m_A = \frac{m_A m_B - m_\pi^2}{\lambda_2^2 m_A + \lambda_1^2 m_B + 2 \lambda_1 \lambda_2 m_\pi}, \\
m_B = \frac{m_A m_B - m_\pi^2}{\lambda_2^2 m_A + \lambda_1^2 m_B + 2 \lambda_1 \lambda_2 m_\pi}, \\
m_\pi = \frac{m_A m_B m_\pi}{\lambda_2^2 m_A + \lambda_1^2 m_B + (\lambda_1 + \lambda_2 + \lambda_3) m_\pi}.
$$

(A6)

Models with this type of interaction in the for the gauge fields and their corresponding curvatures have been considered in effective descriptions of incompressible quantum fluids and Josephson junction arrays [45–47]. Different choices for the coefficients $\lambda_i$ translate into different choices of gauge fields $A_\mu$ and $B_\mu$. The fields $A_\mu$ and $B_\mu$ considered in (26) correspond to the cases $\lambda_3 = \lambda_4 = 0$, $\lambda_1 = \lambda_2 = 1$. In this case one can solve (A6) to find

$$
m_A = \frac{m_A m_i^2}{m_\pi^2 - m_A m_B}, \\
m_B = \frac{m_B m_i^2}{m_\pi^2 - m_A m_B}, \\
m_\pi = \frac{m_A m_B m_i}{m_\pi^2 - m_A m_B}.
$$

(A7)

Thus, using the values $m_A = 4\pi s_m/(e_\nu^2 + e_\omega^2)$, $m_B = 4\pi s_m/\alpha^2$ and $m_i = 4\pi s_m/e_\nu \alpha$ given in (25), one finds (28).

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