The Transplanckian Question and the Casimir Effect

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Abstract

It is known that, through inflation, Planck scale phenomena should have left an imprint in the cosmic microwave background. The magnitude of this imprint is expected to be suppressed by a factor $\sigma^n$ where $\sigma \approx 10^{-5}$ is the ratio of the Planck length to the Hubble length during inflation. While there is no consensus about the value of $n$, it is generally thought that $n$ will determine whether the imprint is observable. Here, we suggest that the magnitude of the imprint may not be suppressed by any power of $\sigma$ and that, instead, $\sigma$ may merely quantify the amount of fine tuning required to achieve an imprint of order one. To this end, we show that the UV/IR scale separation, $\sigma$, in the analogous case of the Casimir effect plays exactly this role.

1 Introduction

The so-called transplanckian question is concerned with low energy phenomena whose calculation appears to require the validity of standard quantum field theory (QFT) at energies beyond the Planck scale. The issue first arose in the context of black holes: the derivation of Hawking radiation is based on the assumption that standard QFT is valid even at scales beyond the Planck scale. For example, the typical low-energy Hawking photons that an observer might detect far from the horizon are implied to have possessed proper frequencies that were much larger than the Planck frequency close to the event horizon, even at distances from the horizon that are farther than a Planck length. This led to the question if Planck scale effects could influence or even invalidate the prediction of Hawking radiation. Numerous studies have investigated the issue and the current consensus is that Hawking radiation is largely robust against modifying QFT in the ultraviolet (UV). This is plausible since general thermodynamic considerations already constrain key properties of Hawking radiation. See, e.g., \cite{1,2}.

More recently, the transplanckian question arose in the context of inflationary cosmology: according to most inflationary models, space-time inflated to the extent that
fluctuations which are presently of cosmological size started out with wavelengths that were shorter than the Planck length. The derivation of the inflationary perturbation spectrum therefore assumes the validity of standard QFT beyond the Planck scale. Unlike in the case of black holes, no known thermodynamic reasons constrain the properties of the inflationary perturbation spectrum so as to make it robust against the influence of physics at the Planck scale. It is, therefore, very actively being investigated if future precision measurements of the cosmic microwave background (CMB) intensity and polarization spectra could in this way offer an experimental window to Planck scale phenomena. See e.g. [3].

It is generally expected that the imprint of Planck scale physics on the CMB is suppressed by a factor $\sigma^n$ where $\sigma$ is defined as the ratio of the UV and IR scale. In inflation, this ratio is $\sigma \approx 10^{-5}$ since modes evolve nontrivially only from the Planck scale to the Hubble scale, $L_{\text{Hubble}} \approx 10^5 L_{\text{Planck}}$, after which their dynamics freezes until much later when they reenter the horizon to seed structure formation. We note that if the UV scale is the string scale, $\sigma$ could be as large as $\sigma \approx 10^{-3}$. Regarding the value of the power, $n$, in $\sigma^n$, no consensus has been reached. It is generally expected however, that the value of $n$ decides whether the imprint of Planck scale physics in the CMB could ever become measurable.

Concrete studies in this field often model the influence of Planck scale physics on QFT through dispersion relations that become nonlinear at high energies. This approach is motivated by the fact that the natural ultraviolet cutoff in condensed matter systems characteristically affects the dispersion relations there. See, e.g., [2]. It has been shown that while some ultraviolet-modified dispersion relations would affect the inflationary predictions for the CMB to the extent that effects might become measurable, other modified dispersion relations would have a negligible effect on the CMB. It is so far not fully understood which properties of Planck scale modifications to the dispersion relation decide whether or not an observable effect is induced. In order to clarify if and how an imprint of Planck scale effects in the CMB are suppressed by $\sigma$ it would be most interesting, therefore, to find and study the operator which maps arbitrary ultraviolet-modified dispersion relations directly into the correspondingly modified CMB perturbation spectra.

Here, we will investigate the simpler transplanckian question for the Casimir force. As is well-known, the Casimir force arises due to quantum fluctuations of the electromagnetic field and occurs between neutral conducting objects. Similar to Hawking radiation and inflationary fluctuations, the Casimir force can be seen as a vacuum effect which involves modes of arbitrarily short wave lengths. In fact, naively it appears that modes contribute the more the shorter their wave length is. This suggests that, in principle, the predicted Casimir force could be influenced by Planck scale physics. The Casimir effect is simple enough so that we will be able to completely answer its transplanckian question when modelling Planck scale physics through ultraviolet-modified dispersion relations. Namely, we will find the explicit operator which maps generic ultraviolet-modified dispersion relations into the corresponding Casimir force functions. The properties of this operator reveal that and how ultraviolet-modified dispersion relations can strongly affect the Casimir force even in the ‘infrared’ i.e. at practically measurable distances. Interestingly, the extreme ratio $\sigma \approx 10^{-28}$ between the effective UV and IR scales in the Casimir effect does not suppress the possible
strength of Planck scale effects in the Casimir force at macroscopic distances. We find that, instead, the extreme value of $\sigma$ implies that UV-modified dispersion relations that lead to a large IR effect merely need to be extremely fine-tuned, which suppresses the a priori likelihood that such a dispersion relation should arise from an underlying theory of quantum gravity. This is of interest because if the situation in inflation is analogous, the imprint of Planck space physics in the CMB may not be suppressed in strength by any power $\sigma^n$ of $\sigma$. Instead, the $\sigma$ of inflation, $\sigma \approx 10^{-5}$ or $\sigma \approx 10^{-3}$, may determine the amount of fine-tuning required to achieve an imprint of order one. Thus, $\sigma$ would be related to the a priori likelihood for an observable imprint to arise from an underlying theory of quantum gravity. In inflation, this likelihood would not be extremely small since the UV and IR scales in inflation are not extremely separated.

## 2 The Casimir force and ultraviolet-modified dispersion relations

The Casimir effect arises when reflecting surfaces pose boundary conditions on the modes of the electromagnetic field. For example, two perfectly reflecting parallel plates impose boundary conditions such that the set of electromagnetic modes in between them is discretized. The spacing of the modes, and therefore the vacuum energy that each mode contributes, depends on the distance between the plates. This distance-dependence of the vacuum energy leads to the Casimir force between the plates. In general, the force is a function of both the distance and the shape of the reflecting surfaces, and the force can be both attractive or repulsive.

The Casimir effect was first predicted, by Casimir, in 1948, see [4]. In the meanwhile, the Casimir force has been calculated for several types of geometries and in various dimensions. Also, effects of imperfect conductors, rough surfaces and finite temperatures have been considered, see [5]. In addition, detailed calculations have been carried out to account for higher order corrections due to virtual electrons and their interaction with the boundaries [6]. For recent reviews see [7] and for precision measurements of the effect see e.g. [8].

For our purposes, the essential features of the Casimir effect are captured already when working with a massless real scalar field between two perfectly conducting parallel plates. For simplicity, we will consider the simple case of just one space dimension, in which case the reflecting plates are mere points. We place these points at $x = 0$ and $x = L$, i.e., we impose the boundary conditions $\hat{\phi}(0, t) = 0 = \hat{\phi}(L, t)$ for all $t$. In order to fulfill these boundary conditions we expand the quantum field between the plates using the Fourier sine series:

$$\hat{\phi}(x, t) = \sum_{n=1}^{\infty} \hat{\phi}_n(t) \sin(k_n x), \quad k_n = \frac{n\pi}{L}$$  \hspace{1cm} (1)

We are using units such that $\hbar = c = 1$. Recall that in a Fourier sine series all $n$ and therefore all wave numbers $k_n$ are positive. The reason is that the sine functions form a complete eigenbasis of the square of the momentum operator, $\hat{p}^2 = -d^2/dx^2$, all of whose eigenvalues are of course positive. (Recall that the momentum operator of a
particle in a box is not self-adjoint and not diagonalizable, see e.g. [10]). The usual ansatz
\[ \hat{\phi}_n = \sqrt{\frac{1}{\omega(k_n) L}} \left( e^{i\omega(k_n)t} a_n^\dagger + e^{-i\omega(k_n)t} a_n \right) \] (2)
with \([a_n, a_m^\dagger] = \delta_{n,m}\) diagonalizes the Hamiltonian:
\[ \hat{H} = \sum_{n=1}^{\infty} \omega(k_n) \left( a_n^\dagger a_n + \frac{1}{2} \right) \] (3)
Thus, with the usual linear dispersion relation
\[ \omega(k) = k, \] (4)
the vacuum energy between plates of distance \(L\) is divergent:
\[ E_{\text{in}}(L) = \frac{1}{2} \sum_{n=0}^{\infty} \omega(k_n) \] (5)
\[ = \frac{\pi}{2L} \sum_{n=0}^{\infty} n = \infty \] (6)
We notice that modes appear to contribute the more the shorter their wavelength, i.e. the larger \(k\) and \(n\) are. One proceeds by regularizing the divergence and by then calculating the change in the regularized total energy (of a large region that contains the plates) when varying \(L\). As is well-known, the resulting expression for the Casimir force remains finite after the regularization is removed, and reads:
\[ F(L) = -\frac{\pi}{24L^2} \] (7)
It has been shown that this result does not depend on the choice of regularization method. Our aim now is to re-calculate the Casimir force within standard quantum field theory while modelling the onset of Planck scale phenomena at high energies through general nonlinear modifications to the dispersion relation. The goal is to calculate the operator which maps arbitrary modified dispersion relations \(\omega(k)\) into the resulting Casimir force functions \(F(L)\). To this end, let us begin by writing generalized dispersion relations in the form:
\[ \omega(k) = k_c f \left( \frac{k}{k_c} \right) \] (8)
Here, \(k_c > 0\) is a constant with the units of momentum, say the Planck momentum so that its inverse is the Planck length: \(L_c = k_c^{-1}\). The function \(f\) encodes unknown Planck scale physics and for now we will make only these minimal assumptions:
- \(f(0) = 0\), and \(f(x) \approx x\) if \(x \ll 1\) (regular dispersion at low energies)
- \(f(x) \geq 0\) when \(x \geq 0\) (stability: each mode carries positive energy)
We will use the term dispersion relation for both \(\omega(k)\) and \(f(x)\).
3 Exponential regularization

For generically modified dispersion relations the vacuum energy (5) must be assumed to be divergent and therefore in need of regularization. Let us therefore regularize (5) by introducing an exponential regularization function, parametrized by \( \alpha > 0 \), i.e. we define the regularized vacuum energy between the plates as:

\[
E_{\text{reg}}^{\text{in}}(L) = \frac{1}{2} \sum_{n=0}^{\infty} k_c f \left( \frac{n\pi}{k_c L} \right) \exp \left[ -\alpha k_c f \left( \frac{n\pi}{k_c L} \right) \right]
\] (9)

In order to calculate the regularized vacuum energy density outside the plates we notice that the right and left outside regions are half axes and that the energy density in a half axis can be calculated from (9) by letting \( L \) go to infinity:

\[
\mathcal{E}_{\text{reg}} = \lim_{L \to \infty} \frac{E_{\text{reg}}^{\text{in}}(L)}{L}
\] (10)

The expression for the vacuum energy density outside the plates, (10), is conveniently rewritten as a Riemann sum by defining \( \Delta x = \frac{1}{L} \):

\[
\mathcal{E}_{\text{reg}} = \lim_{\Delta x \to 0} \left\{ \frac{1}{2} \sum_{n=0}^{\infty} \Delta x k_c f \left( \frac{n\Delta x \pi}{k_c} \right) \exp \left[ -\alpha k_c f \left( \frac{n\Delta x \pi}{k_c} \right) \right] \right\}
\]

\[
= \frac{k_c^2}{2\pi} \int_{0}^{\infty} dx f(x) \exp \left[ -\alpha k_c f(x) \right].
\] (11)

Notice that we are here implicitly restricting attention to dispersion relations for which exponential regularization is sufficient to render the energy densities outside and between the plates finite. This excludes, for example, the dispersion relation \( f(x) = \ln(1 + x) \) which would require a regularization function such as \( \exp(-f(x)^2) \). We will later be able to lift this restriction on the dispersion relations, namely by allowing the use of arbitrary regularization functions. Indeed, as we will prove in Sec. 8 our results only depend on the dispersion relation and are independent of the choice of regularization function, as long as the regularization function does regularize the occurring series and integrals, obeys certain mild smoothness conditions and recovers the original divergent series of (5) in the limit \( \alpha \to 0 \).

In order to calculate the Casimir force, let us now consider a very large but finite region, say of length \( M \), which contains the two plates. The total energy in this region is finite and consists of the energy between the plates, (9), plus the energy density outside the plates, (11), multiplied by the size of the region outside, namely \( M - L \). Note that by choosing \( M \) large enough ensures that the energy density outside the plates does not depend on \( L \). Thus, the total energy in this region is given by \( E_{\text{reg}}^{\text{in}}(L) + (M - L)\mathcal{E}_{\text{reg}} \). The regularized Casimir force is the derivative of this energy with respect to a change in the distance of the plates:

\[
\mathcal{F}_{\alpha}(L) = -\frac{\partial}{\partial L} E_{\text{reg}}^{\text{in}} + \mathcal{E}_{\text{reg}}.
\] (12)
The total length $M$ of the region under consideration has dropped out, as it should be. Hence, before removing the regularization (i.e. before letting $\alpha \to 0^+$), the Casimir force in the presence of a nonlinear dispersion relation is given by:

$$F_\alpha(L) = \frac{1}{2} k_c \left\{ \sum_{n=0}^{\infty} \frac{1}{L} \left[ \left( \frac{n\pi}{k_c L} \right) f' \left( \frac{n\pi}{k_c L} \right) \right] \exp \left[ -\alpha k_c f \left( \frac{n\pi}{k_c L} \right) \right] \times \right.$$ \n
$$\left. \times \left( 1 - \alpha k_c f \left( \frac{n\pi}{k_c L} \right) \right) \right\} + \frac{k_c}{\pi} \int_{0}^{\infty} dx \ f(x) \exp \left[ -\alpha k_c f(x) \right]$$ \n
(13)

Here, $f'$ stands for differentiating $f$ with respect to the variable $x = \frac{n\pi}{k_c L}$.

4 Application of the Euler-Maclaurin formula

It will be convenient to collect the terms that constitute the argument of the series in a new definition:

$$\varphi_\alpha(t) := \frac{t\pi}{k_c L} f' \left( \frac{t\pi}{k_c L} \right) \exp \left[ -\alpha k_c f \left( \frac{t\pi}{k_c L} \right) \right] \left( 1 - \alpha k_c f \left( \frac{t\pi}{k_c L} \right) \right)$$ \n
(14)

Thus, (13) becomes:

$$F_\alpha(L) = \frac{k_c}{2L} \sum_{n=0}^{\infty} \varphi_\alpha(n) + \frac{k_c^2}{2\pi} \int_{0}^{\infty} dx \ f(x) \ e^{-\alpha k_c f(x)}$$ \n
(15)

We notice that if the first term in (15) were an integral instead of a series then the two terms in (15) would exactly cancel another:

$$\frac{k_c}{2L} \int_{0}^{\infty} \varphi_\alpha(t) \ dt = \frac{k_c}{2L} \frac{\pi}{k_c L} \int_{0}^{\infty} \varphi_\alpha(t) \ \frac{\pi}{k_c L} \ dt$$ \n
(16)

$$= \frac{k_c^2}{2\pi} \int_{0}^{\infty} dx \ x f(x) e^{-\alpha k_c f(x)} \ (1 - \alpha k_c f(x))$$ \n
(17)

$$= \frac{k_c^2}{2\pi} \left[ x f(x) e^{-\alpha k_c f(x)} \right]_{0}^{\infty} - \frac{k_c^2}{2\pi} \int_{0}^{\infty} dx \ f(x) e^{-\alpha k_c f(x)}$$ \n
(18)

$$= 0 - \frac{k_c^2}{2\pi} \int_{0}^{\infty} dx \ f(x) e^{-\alpha k_c f(x)}.$$ \n
(19)

In (15), the boundary terms are zero because at $x = 0$ the dispersion relation yields $f(0) = 0$ and because for $x \to \infty$ the finiteness of (11) implies that its integrand decays faster than $1/x$.

In order to compute the Casimir force, let us now use the Euler-Maclaurin sum formula, see e.g. [11], to express the series of $\varphi_\alpha$ as an integral of $\varphi_\alpha$ plus corrections. As we just saw, the integral will then cancel in (15) and the correction terms will constitute the Casimir force. To this end, recall that if the $(k + 1)$st derivative of a
function $\xi$ is continuous, i.e., if $\xi \in \mathcal{C}^{k+1}$, then:

$$\sum_{a < n \leq b} \xi(n) = \int_a^b \xi(t) \, dt + \sum_{r=0}^k \frac{(-1)^{r+1} B_{r+1}}{(r+1)!} \left( \xi^{(r)}(b) - \xi^{(r)}(a) \right) + \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(t) \xi^{(k+1)}(t) \, dt$$

(20)

Here, the superscript at $\xi^{(r)}$ denotes the $r$’th derivative of the function $\xi$, the $B_s$ are the Bernoulli numbers and $B_s(t)$ is the $s$’th Bernoulli periodic function, i.e. the periodic extension of the $s$’th Bernoulli polynomial from the interval $[0, 1]$.

We can now choose $\xi = \varphi_\alpha$, set $a = 0$ and take the limit $b \to \infty$. Since the vacuum energy density, (11), is finite it follows that (19) is finite and therefore also (16). This in turn implies that $\lim_{x \to \infty} \varphi_\alpha(x) = 0$ and $\lim_{x \to \infty} \varphi^{(n)}_\alpha(x) = 0$ for all $n \geq 1$. Hence, the series involving the Bernoulli numbers simplifies and we obtain for arbitrary $k \in \mathbb{N}$ this Euler-Maclaurin formula for $\varphi_\alpha$:

$$\sum_{n=0}^\infty \varphi_\alpha(n) = \int_0^\infty \varphi_\alpha(t) \, dt - \sum_{r=1}^k \frac{B_{2r}}{(2r)!} \varphi^{(2r-1)}_\alpha(0) + \Omega_k[\varphi_\alpha]$$

(21)

Here, $\Omega_k[\varphi_\alpha]$ represents the remainder integral:

$$\Omega_k[\varphi_\alpha] = \frac{(-1)^k}{(k+1)!} \int_0^\infty B_{k+1}(t) \varphi^{(k+1)}_\alpha(t) \, dt$$

(22)

Using $\varphi_\alpha(0) = 0$ and the fact that, except for $B_1$, all Bernoulli numbers $B_s$ with odd indices $s$ are zero, we obtain:

$$\sum_{n=0}^\infty \varphi_\alpha(n) = \int_0^\infty \varphi_\alpha(t) \, dt - \sum_{r=1}^k \frac{B_{2r}}{(2r)!} \varphi^{(2r-1)}_\alpha(0) + \Omega_k[\varphi_\alpha]$$

(23)

Equation (20) expresses the series as an integral plus corrections, as desired. Applied to the expression (16) for the regularized Casimir force, $F_\alpha(L)$, the integrals then cancel and we obtain for the regularized Casimir force:

$$F_\alpha(L) = -\frac{k_c}{2L} \sum_{r=1}^k \frac{B_{2r}}{2r!} \varphi^{(2r-1)}_\alpha(0) + \frac{k_c}{2L} \Omega_k[\varphi_\alpha]$$

(24)

The actual Casimir force, $F(L)$, is obtained by removing the regularization:

$$F(L) = \lim_{\alpha \to 0^+} \left\{ -\frac{k_c}{2L} \sum_{r=1}^k \frac{B_{2r}}{2r!} \varphi^{(2r-1)}_\alpha(0) + \frac{k_c}{2L} \Omega_k[\varphi_\alpha] \right\}$$

(25)
5 The Casimir force for polynomial dispersion relations

In order to further evaluate this expression for the Casimir force let us restrict attention to dispersion relations that are sufficiently well behaved so that \( \phi_\alpha(t) \) is \( C^\infty \) with respect to both \( \alpha \) and \( t \). The simplest case is that of dispersion relations which are polynomial:

\[
f(x) = \sum_{n=0}^{\infty} \nu_n x^n
\]

We are assuming that \( \phi_\alpha(t) \in C^\infty \) which here allows us to take the limit \( \alpha \to 0 \) in \( \phi_\alpha(t) \) before differentiating it. From (14) we then have

\[
\phi_0(t) = \lim_{\alpha \to 0} \phi_\alpha(t) = x(t)f'(x(t))
\]

where \( x(t) = \frac{\pi}{k_c L} \), and where ' stands for \( d/dx \). Thus, iterated differentiation yields

\[
\frac{d^n \phi_0(t)}{dt^n} = n \left( \frac{\pi}{k_c L} \right)^n \frac{d^n f(x)}{dx^n} + x \left( \frac{\pi}{k_c L} \right)^{n+1} \frac{d^{n+1} f(x)}{dx^{n+1}}
\]

and therefore the terms in the series in (25) read:

\[
\phi_0^{(n)}(t)|_{t=0} = \frac{n}{k_c L}
\]

We now show that the remainder term \( \Omega_k[\phi_\alpha] \) does not contribute. Assuming for the moment that the dispersion relation is polynomial, \( \phi_\alpha(t) \) is a polynomial times the exponential regularization function \( e^{-\alpha k_c f} \) which tends to 1 as \( \alpha \to 0 \). Therefore, after sufficiently many differentiations, i.e., when choosing \( k \) large enough, \( \phi_\alpha^{(k+1)}(t) \to 0 \) as \( \alpha \to 0 \) for all fixed \( t \). In order to evaluate \( \Omega_k[\phi_\alpha] \), let us now split into two integrals: \( \int_0^\infty = \int_0^b + \int_b^\infty \). For all finite \( b > 0 \) the first integral commutes with the limit \( \alpha \to 0 \) to yield for large enough \( k \):

\[
\lim_{\alpha \to 0} \int_0^b B_{k+1}(t) \phi_\alpha^{(k+1)}(t) \, dt = \int_0^b \lim_{\alpha \to 0} B_{k+1}(t) \phi_\alpha^{(k+1)}(t) \, dt = 0
\]

Further, we notice that, since \( f \) is polynomial and the exponential regularization function is positive, \( \phi_\alpha(t) \) does not change sign for all \( t > b \) if \( b \) is chosen sufficiently large. Since the periodic Bernoulli functions are bounded from above by their Bernoulli numbers we therefore obtain:

\[
\left| \int_b^\infty B_{k+1}(t) \phi_\alpha^{(k+1)}(t) \, dt \right| \leq |B_{k+1}| \left| \int_b^\infty \phi_\alpha^{(k+1)}(t) \, dt \right|
\]

\[
\leq |B_{k+1}| \left| \phi_\alpha^{(k)}(t) \right| \left|_b^\infty \right|
\]

\[
= |B_{k+1}| \left| \phi_\alpha^{(k)}(b) \right|
\]

\[
\to 0 \quad \text{as} \quad \alpha \to 0
\]
Thus, when choosing $k$ large enough, the remainder term disappears so that, using (28), we obtain for the Casimir force for arbitrary polynomial dispersion relations:

$$F(L) = -\frac{k_c}{2L} \sum_{r=1}^{k} \frac{(2r-1)B_{2r}}{2r!} f^{(2r-1)}(0) \left( \frac{\pi}{k_c L} \right)^{2r-1}$$

(31)

Further, since $f^{(s)}(0) = s! \nu_s$, we obtain:

$$F(L) = -\frac{k_c}{2L} \sum_{r=1}^{k} \frac{(2r-1)B_{2r}}{2r!} \nu_{2r-1} \left( \frac{\pi}{k_c L} \right)^{2r-1}$$

(32)

We notice that, interestingly, the even powers in a nonlinear dispersion relation, i.e. the coefficients $\nu_{2r}$, do not contribute to the Casimir force.

As a consistency check, let us now choose the usual linear dispersion relation $f(x) = x$. Since $B_2 = \frac{1}{6}$, we obtain

$$F(L) = -\left( \frac{k_c}{2L} \right) \left( \frac{\pi}{k_c L} \right) \frac{1}{2 \cdot 6} = -\frac{\pi}{24L^2},$$

(33)

which is the well-known usual result for the Casimir force, as it should be.

6 Generic dispersion relations

Considering our results for the Casimir force with polynomial dispersion relations, (31,32) we notice that the addition of mode energies translates into the addition of the corresponding Casimir forces: if two dispersion relations are added, $f_t(x) = f_1(x) + f_2(x)$, then the two corresponding Casimir forces are added:

$$F_t = F_1 + F_2$$

(34)

This shows that the operator, $\mathcal{K}$, that we have been looking for, namely the operator which maps arbitrary dispersion relations into their corresponding Casimir forces, $\mathcal{K} : f \mapsto F$, is a linear operator:

$$\mathcal{K}[f_1 + f_2] = \mathcal{K}[f_1] + \mathcal{K}[f_2]$$

(35)

Because of its linearity, we can straightforwardly extend the action of $\mathcal{K}$ to arbitrary dispersion relation, $f$, which are given by power series in $x$:

$$f(x) = \sum_{s=0}^{\infty} \nu_s x^s$$

(36)

The radius of convergence of the power series must be infinite since the dispersion relation needs to be evaluated for all $x$, i.e., $f$ is an entire function. The linearity of $\mathcal{K}$ yields the corresponding Casimir force function $F$ as a power series in $1/L$:

$$\mathcal{K}[f](L) = F(L) = -\frac{k_c}{2L} \sum_{r=1}^{\infty} \frac{(2r-1)B_{2r}}{2r!} \nu_{2r-1} \left( \frac{\pi}{k_c L} \right)^{2r-1}$$

(37)
We need to determine under which conditions the resulting power series for the Casimir force function is convergent. Interestingly, as we will show in Sec. 7, the convergence, i.e. the well-definedness of the Casimir force, generally depends on the plate separation $L$. When the power series possesses a finite radius of convergence, i.e. when there is a largest allowed value for $1/L$, this means that there is a smallest allowed value for the length $L$. This is beautifully consistent with the expectation that dispersion relations that arise from an underlying quantum gravity theory can imply a finite minimum length scale.

For analyzing the convergence properties of the series (37) the presence of the Bernoulli numbers is somewhat cumbersome. It will be useful, therefore, to use the connection between the Bernoulli numbers and the Riemann zeta function, see [12]:

$$B_n = (-1)^{n+1} n \zeta(1-n) \tag{38}$$

Thus:

$$F(L) = \frac{k_c}{2L} \sum_{r=1}^{\infty} (2r-1) \zeta(1-2r) \nu_{2r-1} \left( \frac{\pi}{k_c L} \right)^{2r-1} \tag{39}$$

We can now use the fact that, see [13]:

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cos \left( \frac{1}{2} \pi s \right) \Gamma(s) \zeta(s) \tag{40}$$

In our case, since $s$ is always an integer, the Euler gamma function reduces to a factorial, and the cosine is ±1. Thus:

$$F(L) = \frac{k_c}{L} \sum_{r=1}^{\infty} \frac{(-1)^r}{(2\pi)^{2r}} (2r-1)(2r-1)! \zeta(2r) \nu_{2r-1} \left( \frac{\pi}{k_c L} \right)^{2r-1} \tag{41}$$

Having replaced the Bernoulli numbers by the Riemann zeta function is advantageous because obviously $\zeta(r) \to 1$ very quickly as $r \to \infty$. For example, for $r = 6$, the difference is already at the one percent level. This means that for the purpose of analyzing the convergence properties of the power series we will be able to use that the Riemann zeta function for the arguments that occur is close to 1 and essentially constant.

7 Example with minimum length

Ultraviolet-modified nonlinear dispersion relations which approach the usual linear dispersion relation for small momenta are given, for example, by:

$$f(x) = \exp(x) - 1 \quad \text{and} \quad f(x) = \sinh(x) \tag{42}$$

The odd coefficients, $\nu_{2r-1} = 1/(2r-1)!$ are the same for both the exponential and the sinh dispersion relation, i.e. the two functions differ only by their even part. But we know from [11] that the even components of the dispersion relations do not affect the Casimir force. The two dispersion relations therefore happen to lead to the same
Figure 1: The Casimir force for the exponential dispersion relation \( \omega(k) = k_c(\exp(k/k_c) - 1) \). Note that the Casimir force is defined only for \( L \) larger than the finite minimum length \( L_{\text{min}} = 1/2 \) (in units of \( 1/k_c \)).

Casimir force. It is plotted with the usual Casimir force in Fig.1. We see that the Casimir force matches the usual Casimir force at large \( L \) but is weaker for small \( L \). As the plot also shows, the Casimir force is well defined only for values of \( L \) above a certain value \( L_c \), corresponding to a finite radius of convergence of the power series in \( 1/L \) for the Casimir force. In order to calculate this minimum length \( L_c \), we notice that all the coefficients \( \nu_{2r-1} \) are non-negative, which implies that (41) is an alternating series. Such series converge if and only if their coefficients converge to zero. Hence, for any such dispersion relation, the Casimir force is well defined for all \( L \) which obey:

\[
\lim_{r \to \infty} \left[ \frac{1}{(2\pi)^{2r}} \frac{(2r-1) \ (2r-1)! \ \zeta(2r) \ \nu_{2r-1} \ \left( \frac{\pi}{k_c L} \right)^{2r-1}}{\nu_{2r-1}} \right] = 0 \tag{43}
\]

In the particular case of the two dispersion relations above, we have \( \nu_{2r-1} = 1/(2r-1)! \) and the condition that the Casimir force be well-defined therefore reads

\[
\lim_{r \to \infty} \left[ \frac{\zeta(2r)}{(2\pi)^{2r}} \ (2r-1) \left( \frac{1}{2k_c L} \right)^{2r-1} \right] = 0 \tag{44}
\]

which means that \( \frac{1}{2k_c L} < 1 \). The minimum length implied by this dispersion relation is therefore:

\[
L_c = \frac{1}{2k_c} \tag{45}
\]
This is an example of what we hinted at before, namely that a dispersion relation can in this way reveal an underlying short-distance cutoff.

For general dispersion relations the coefficients $\nu_{2r-1}$ are not necessarily all positive, i.e., the Casimir force need not be given by an alternating series. In this general case the minimum length can be determined by using the fact that the radius of convergence, $R$, of an arbitrary power series $\sum c_r x^r$ is given by:

$$
\frac{1}{R} = \limsup_{r \to \infty} |c_r|^{1/r}.
$$

(46)

For example, in the case of the dispersion relations given in (42), where $\nu_{2r-1} = 1/(2r-1)!$, the Casimir force (41) can be written as a power series $F(L) = \sum_{r=1}^{\infty} c_r \left( \frac{1}{L^2} \right)^{2r-1}$ in $1/L^2$ with the coefficients:

$$
c_r = \frac{(-1)^r k_c (2r-1) \zeta(2r)}{2\pi} \left( \frac{1}{2k_c} \right)^{2r-1}
$$

(47)

Thus, the minimum length obeys

$$
L^2_c = \limsup_{r \to \infty} \left[ \frac{k_c (2r-1) \zeta(2r)}{2\pi} \left( \frac{1}{2k_c} \right)^{2r-1} \right]^{1/2}
$$

(48)

$$
= \lim_{r \to \infty} \left( \frac{1}{2k_c} \right)^{2r-1}
$$

(49)

$$
= \left( \frac{1}{2k_c} \right)^2
$$

(50)

and therefore:

$$
L > L_c = \frac{1}{2k_c}
$$

(51)

As expected, this agrees with the result (45) which we obtained by using the alternating series test.

8 Regularization-function independence

It is known that the prediction for the Casimir force with the usual linear dispersion relation does not depend on the choice of regularization function, as long as the regularization function obeys certain smoothness conditions and is such that it does in fact regularize the integrals and series which occur in the calculation.

In our calculation of the Casimir force for nonlinear dispersion relations we chose an exponential regularization function. We need to prove that our result (31) does not depend on this choice. To see that this indeed the case, assume that we use an arbitrary regularization function, $\gamma_\alpha(x)$, which is a positive function of $x$ that obeys $\lim_{\alpha \to 0^+} \gamma_\alpha(x) = 1$ for all $x$ so that the original divergent series is recovered when the regulator $\alpha$ goes to zero. The regularized energy between the plates then reads:

$$
\tilde{E}_{\text{reg}}^{\alpha \gamma} = \frac{1}{2} \sum_{n=0}^{\infty} k_c f \left( \frac{n\pi}{k_c L} \right) \gamma_\alpha \left[ f \left( \frac{n\pi}{k_c L} \right) \right]
$$

(52)
The regularization function, $\gamma_\alpha$, needs to be chosen such that (52) as well as the energy density are finite, i.e. such that $\lim_{L \to \infty} \tilde{E}_{\text{reg}}(L)/L < \infty$, which means:

$$\int_0^\infty dx f(x)\gamma_\alpha[f(x)] < \infty$$

Finally, in order to be able to use the Euler-Maclaurin sum formula and in it to interchange $d/dt$ and the limit $\alpha \to 0$, we require the regularization functions $\gamma_\alpha$ to be smooth enough so that $\gamma_\alpha \in C^\infty$ as well as $\varphi_\alpha(t) \in C^\infty$ as a function of $\alpha$ and $t$. The above derivation of the Casimir force can then be repeated point by point using the corresponding new definition of $\varphi_\alpha$. In particular, we apply the Euler-Maclaurin sum formula to the expression:

$$\tilde{F}_\alpha(L) = -\frac{k_c}{2L} \left\{ \sum_{n=0}^{\infty} \frac{n\pi}{k_c L} f'' \left( \frac{n\pi}{k_c L} \right) \{ \gamma_\alpha \left[ f \left( \frac{n\pi}{k_c L} \right) \right] \right\} + \frac{k_c L}{\pi} \int_0^\infty f(x)\gamma_\alpha[f(x)] \right\}$$

An integration by parts as in (16) shows that the integrals cancel. Equation (53) ensures that the boundary term vanishes, as before in (18). Hence, we again arrive at (24). We now take the limit $\alpha \to 0$ term by term in the sum, and since $\varphi_\alpha$ is in $C^\infty$, we can again do this before differentiating. Moreover, by the basic assumptions made on $\gamma_\alpha$, we know that $\gamma_\alpha'(x) \to 0$ as $\alpha \to 0$, so that as before:

$$\lim_{\alpha \to 0} \varphi_\alpha(t) = x(t)f'(x(t))$$

The arguments given in the previous section to show that the remainder integral disappears for polynomial dispersion relations and that the coefficients in the Euler-Maclaurin sum are those given in (32) apply unchanged. This proves that our results for the Casimir force are independent of the choice of regularization function, as it should be.

9 The operator $\mathcal{K}$ which maps dispersion relations into Casimir force functions

In preparation for our study of the transplanckian question for the Casimir effect in Sec. 10, let us now calculate explicit representations of the operator $\mathcal{K}$ which maps dispersion relations $f$ into Casimir force functions $\mathcal{F}$:

$$\mathcal{K} : f(x) \mapsto \mathcal{F}(L)$$

We already saw that $\mathcal{K}$ is linear. Indeed, from (41), it can be written as a differential operator:

$$\mathcal{K} = \frac{k_c}{2\pi L} \sum_{r=1}^{\infty} (-1)^r (2r-1) \zeta(2r) \left( \frac{1}{2k_c L} \right)^{(2r-1)} \frac{d^{(2r-1)}}{dx^{(2r-1)}} \Big|_{x=0}$$
As we already mentioned, the convergence of the zeta function, \( \zeta(2r) \to 1 \), is very fast as \( r \to \infty \). Since the study of the transplanckian question involves large orders of magnitudes, we will therefore henceforth replace \( \zeta(2r) \) by 1. By this approximation we incur at most a numerical error of a pre-factor of order one which will not affect our later analysis of the question when the ultraviolet modifications to the dispersion relations can or cannot affect the Casimir force in the infrared.

### 9.1 Representation of \( K \) as an integral operator

For the purpose of studying the transplanckian question, the representation of \( K \) as a differentiation operator in (57) is not as suitable as a representation as an integral operator would be. Indeed, as we now show, an equivalent representation of \( K \) is given by

\[
K[f](L) = \mathcal{F}(L) = \frac{k_c^2}{\pi} \text{Im} \int_0^\infty f(ix) \left( 1 - 2k_c Lx \right) e^{-2k_c Lx} \, dx
\]  

(58)

where \( \text{Im} \) stands for taking the imaginary part. To verify that the action of this operator on all polynomial \( f \) agrees with that given in (57), let us begin by introducing variables \( \Lambda = 2k_c L \) and \( \tilde{x} = 2k_c Lx \), to write:

\[
\mathcal{F}(L) = \frac{k_c^2}{\pi \Lambda} \text{Im} \int_0^\infty f \left( i\frac{\tilde{x}}{\Lambda} \right) \left( 1 - \tilde{x} \right) e^{-\tilde{x}} \, d\tilde{x}
\]  

(59)

We claim that iterated integrations by parts yield:

\[
\mathcal{F}(L) = \frac{k_c^2}{\pi \Lambda} \text{Im} \left\{ \sum_{s=0}^{n} e^{-\tilde{x}} (\tilde{x} + s) \frac{d^s}{d\tilde{x}^s} f \left( i\frac{\tilde{x}}{\Lambda} \right) \right|_{\tilde{x}=0}^\infty
\]

\[
- \int_0^\infty e^{-\tilde{x}} (\tilde{x} + n) \frac{d^{n+1}}{d\tilde{x}^{n+1}} f \left( i\frac{\tilde{x}}{\Lambda} \right) \right|_{\tilde{x}=0}
\]

(60)

Integrating (59) by parts once shows that the equation holds for \( n = 0 \). Assuming now that the formula is valid for \( n - 1 \), integration by parts of the remaining integral yields:

\[
\mathcal{F}(L) = \frac{k_c^2}{\pi \Lambda} \text{Im} \left\{ \sum_{s=0}^{n-1} e^{-\tilde{x}} (\tilde{x} + s) \frac{d^s}{d\tilde{x}^s} f \left( i\frac{\tilde{x}}{\Lambda} \right) \right|_{\tilde{x}=0}^\infty
\]

\[
+ e^{-\tilde{x}} (\tilde{x} + n) \frac{d^n}{d\tilde{x}^n} f \left( i\frac{\tilde{x}}{\Lambda} \right) \right|_{\tilde{x}=0}^\infty
\]

\[
- \int_0^\infty e^{-\tilde{x}} (\tilde{x} + n) \frac{d^{n+1}}{d\tilde{x}^{n+1}} f \left( i\frac{\tilde{x}}{\Lambda} \right) \right|_{\tilde{x}=0}
\]

(61) 

(62) 

(63)

The boundary term in (62) becomes the next term in the sum (61) and by induction this completes the proof of (60). In (60), since \( f \) is polynomial, the integral vanishes if \( n \) is chosen large enough. Also, the boundary terms clearly vanish at the upper limit.
Letting $n \to \infty$, we are left with:

$$
\mathcal{F}(L) = \frac{-k_c^2}{\pi \Lambda} \text{Im} \sum_{s=0}^{\infty} s \int \frac{d^s f\left(\frac{x}{\Lambda}\right)}{\partial x^s}_{x=0}
$$

(64)

$$
= \frac{k_c}{2\pi L} \sum_{r=1}^{\infty} (2r-1) (-1)^r \left(\frac{1}{2k_c L}\right)^{2r-1} \int \frac{d^{2r-1}}{dx^{2r-1}} f(x)|_{x=0} \tag{65}
$$

which agrees with (67), up to the zeta function which we omitted since it is close to one. In the step from (64) to (65) we made use of the fact that the imaginary part selects for only the odd powers in the series.

As a consistency check, let us apply the integral representation, (58), of $\mathcal{K}$ to the usual linear dispersion relation $f(x) = x$. Carrying out the integration yields

$$
\mathcal{F}(L) = -\frac{1}{4\pi L^2} \tag{66}
$$

As expected, this differs from the usual result only by the omitted $\zeta$ function pre-factor of $\zeta(2) = \frac{\pi^2}{6}$.

### 9.2 Relation of $\mathcal{K}$ to the Laplace transform

The representation of $\mathcal{K}$ as an integral operator came at the cost of complexifying the analysis by having to integrate the dispersion relation along the imaginary axis.

Fortunately, it is possible to re-express $\mathcal{K}$ as a real integral operator, namely as a slightly modified Laplace transform. To this end, let us use our finding that even powers in the dispersion relations do not contribute to the Casimir force. This means that, without restricting generality, we can assume that the dispersion relation is odd, i.e. that it can be written in the form

$$
f(x) = x \ g(x^2) \tag{66}
$$

for some function $g$. Thus, $f(ix) = i \ x \ g(-x^2)$, and therefore the integral representation of $\mathcal{K}$ now takes the form:

$$
\mathcal{K}[f](L) = \mathcal{F}(L) = \frac{k_c^2}{\pi} \int_0^{\infty} x \ g(-x^2) \ (1 - 2k_c L x) \ e^{-2k_c L x} \ dx \tag{67}
$$

Using the properties of the Laplace transform with respect to differentiation, we can finally conclude that the operator $\mathcal{K}$ which maps dispersion relations into Casimir force functions can be written as a modified Laplace transform:

$$
\mathcal{K}[f](L) = \mathcal{F}(L) = \frac{k_c^2}{\pi} \left(1 + L \frac{d}{dL}\right) \int_0^{\infty} e^{-2k_c L x} x \ g(-x^2) \ dx
$$

$$
= \frac{k_c^2}{\pi} \left(1 + L \frac{d}{dL}\right) \mathcal{L}_\Lambda[\tilde{f}] \tag{68}
$$

In the last line, $\mathcal{L}_\Lambda[\tilde{f}]$ stands for the Laplace transform of $\tilde{f}(x) = x \ g(-x^2)$ with respect to the variable $\Lambda = 2k_c L$. Let us test (68) by applying it to the linear dispersion relation $f(x) = x$. Carrying out the integration yields

$$
\mathcal{L}_\Lambda[\tilde{f}] = \frac{k_c}{2\pi L} \sum_{r=1}^{\infty} (2r-1) (-1)^r \left(\frac{1}{2k_c L}\right)^{2r-1} \int \frac{d^{2r-1}}{dx^{2r-1}} \tilde{f}(x)|_{x=0} \tag{69}
$$

which agrees with (65), up to the zeta function which we omitted since it is close to one. In the step from (68) to (69) we made use of the fact that the imaginary part selects for only the odd powers in the series.

As a consistency check, let us apply the integral representation, (68), of $\mathcal{K}$ to the usual linear dispersion relation $f(x) = x$. Carrying out the integration yields

$$
\mathcal{F}(L) = -\frac{1}{4\pi L^2} \tag{70}
$$

As expected, this differs from the usual result only by the omitted $\zeta$ function pre-factor of $\zeta(2) = \frac{\pi^2}{6}$.
relation, where \( \tilde{f}(x) = x \). Then,

\[
\mathcal{F}(L) = \frac{k^2}{\pi} \left(1 + L \frac{d}{dL}\right) \int_0^\infty e^{-2k_c L x} x \, dx
\]

which indeed agrees with the expected result as obtained at the end of Sec. 9.1.

We notice that the representation of \( \mathcal{K} \) through (68) involves the analytic extension of the function \( g \) from positive arguments, where it encodes the dispersion relation through \( f(x) = x g(x^2) \), to negative arguments where \( g \) is evaluated by the Laplace transform in (68).

This observation about \( \mathcal{K} \) will be useful for answering the transplanckian question in Sec. 10: clearly, the dispersion relation \( f(x) = x g(x^2) \) may be very close to linear, i.e. \( g(y) \) may be close to one for \( y > 0 \), while at the same time the unique analytic extension \( g(y) \) for \( y < 0 \) may be far from linear. This already shows that ultraviolet-modified dispersion relations can easily lead to arbitrarily pronounced nontrivial Casimir forces even at infrared length scales.

9.3 The inverse of \( \mathcal{K} \)

Let us now calculate the inverse of the operator \( \mathcal{K} \) to obtain the operator which maps odd Casimir force functions (recall that the even ones do not contribute to the Casimir force) into the corresponding dispersion relations. To this end, we need to solve for \( \tilde{\mathcal{F}}(L) \):

\[
\frac{k^2}{\pi} \left(1 + L \frac{d}{dL}\right) \tilde{\mathcal{F}}(L) = \mathcal{F}(L) .
\]

The Green’s function for this differential operator satisfies the following equation:

\[
\frac{k^2}{\pi} \left(1 + L \frac{d}{dL}\right) G_{\mathcal{F}}(L, L') = \delta(L - L')
\]

Since the \( \delta \)-function is formally the derivative of the Heavyside step function \( \theta \), an integration on both sides yields

\[
\int G_{\mathcal{F}}(L, L') \, dL + LG_{\mathcal{F}}(L, L') - \int G_{\mathcal{F}}(L, L') \, dL = \frac{\pi}{k_c^2} \theta(L - L') + \kappa(L') ,
\]

where \( \kappa(L') \) is some arbitrary function. Hence,

\[
G_{\mathcal{F}}(L, L') = \frac{1}{L} \left[ \frac{\pi}{k_c^2} \theta(L - L') + \kappa(L') \right] ,
\]

and

\[
\tilde{\mathcal{F}}(L) = \frac{1}{L} \int_{-\infty}^{\infty} \left[ \frac{\pi}{k_c^2} \theta(L - L') + \kappa(L') \right] \mathcal{F}(L') \, dL' .
\]
For the boundary condition, we set \( \tilde{F}(L) \to 0 \) as \( L \to +\infty \), to ensure the correct behavior of \( F \). Hence,

\[
\kappa(L') + \frac{\pi}{k^2_c} = 0 \iff \kappa(L') \equiv -\frac{\pi}{k^2_c}
\]

Thus, the integral in (76) is effectively truncated and we have:

\[
\tilde{F}(L) = -\frac{\pi}{k^2_c L} \int_L^\infty F(L') dL' .
\]  

Eventually, we also need to invert the Laplace transform through a Fourier-Mellin integral, to obtain:

\[
x g(-x^2) = -\frac{1}{2i k^2_c L} \int_\gamma dL e^{xL} \int_L^\infty F(L') dL' .
\]

Here, the integration path \( \gamma \) is to be chosen parallel to the imaginary axis and to the right of all singularities of the integrand. Analytic continuation of \( g \) to the positive reals finally yields the dispersion relation \( K^{-1}[F](x) = f(x) = xg(x^2) \), modulo, of course, even components to the dispersion relations. We will here not go further into the functional analysis of (78) and the inverse of \( K \).

**10 The transplanckian question**

Having calculated \( K \), we are now prepared to address the transplanckian question, namely the question which types of Planck scale modified dispersion relations would significantly affect the predictions for the Casimir force at realistic plate separations.

To this end, let us begin by investigating the lowest order corrections to the dispersion relation, \( f \), namely by including a quadratic and a quartic correction term:

\[
f(x) = x + \nu_2 x^2 + \nu_3 x^3 .
\]

The coefficients \( \nu_2, \nu_3 \) can be as large as of order one, \( \nu_2, \nu_3 \approx 1 \), without appreciably affecting the dispersion relation \( \omega(k) = k_c f(k/k_c) \) at small momenta \( k \ll k_c \). Using our result (41) for \( K \) we find the corresponding Casimir force function:

\[
F(L) = -\frac{\pi}{24 L^2} + \nu_3 \frac{\pi^5}{20 k^2_c L^4} .
\]

The quadratic correction term \( \nu_2 x^2 \) is an even component of \( f \) and therefore does not affect the Casimir force. The quartic correction term does affect the Casimir force, changing the Casimir force from attractive to repulsive at very short distances, as shown in Fig. 2. However, as we can also see in Fig. 2 the Casimir force function converges very rapidly towards the usual Casimir force function for plate separations that are significantly larger than \( L_c = k_c^{-1} \). To be precise, we recall that the standard dispersion relation \( f_{\text{standard}}(x) = x \) implies the standard Casimir force function \( F_{\text{standard}}(L) = -\frac{\pi}{24 L^2} \). The relative size of the correction to the Casimir force depends on the plate separation \( L \) and reads:

\[
\frac{F_{\text{standard}}(L) - F(L)}{F_{\text{standard}}(L)} = \nu_3 \frac{6\pi^4}{5L^2 k^2_c} .
\]
Figure 2: The Casimir force for a lowest order correction to the dispersion relation: \( f(x) = x + x^3 \). The plate separation, \( L \), is measured in multiples of the UV scale \( \frac{k_c^{-1}}{c^2} \).

Let us calculate the orders of magnitude. The dispersion relation \( \omega(k) = k_c f(k/k_c) \) is expected to start to appreciably differ from linearity the latest at the Planck scale, which in 3 + 1 dimensional space-time means that the critical length, \( L_c \), obeys \( L_c = k_c^{-1} \approx 10^{-35}m \). Actual measurements of the Casimir force have been performed at about \( L_m \approx 10^{-7}m \), see e.g. [8]. Therefore, evaluating the relative correction of the Casimir force, \( \frac{F_{\text{standard}}(L_m) - F(L_m)}{F_{\text{standard}}(L_m)} \), at the measurable scale \( L = L_m \) yields

\[
\frac{F_{\text{standard}}(L_m) - F(L_m)}{F_{\text{standard}}(L_m)} = \nu_3 \frac{6\pi^4}{5} \sigma^2
\]

where \( \sigma \) denotes the dimensionless ratio of the ultraviolet length scale \( L_c \) and the infrared length scale \( L_m \):

\[
\sigma = \frac{L_c}{L_m} \approx 10^{-28}
\]

Thus, the effect of the lowest order corrections to the dispersion relation on the Casimir force is extremely small at measurable plate separations.

Naively, one might expect that higher-order corrections to the dispersion relations contribute even less to the Casimir force. If true, this would indicate that the physical processes that happen at these two length scales respectively are very effectively decoupled from another. In fact, however, the two scales are not quite as decoupled. Roughly speaking, the reason is that higher-order corrections to the dispersion relations contribute more rather than less to the Casimir force, as we will now show.
10.1 UV-IR coupling with polynomial dispersion relations

Recall that we here need not be concerned with the even components of dispersion relations since they do not contribute to the Casimir force. Let us, therefore, consider higher order odd polynomial dispersion relations:

\[ f(x) = x + \sum_{r=2}^{N} \nu_{2r-1} x^{2r-1} \] (83)

The coefficients \( \nu_{2r-1} \) can be chosen as large as of order one, \( \nu_{2r-1} \approx 1 \), and \( f \) will still be modified only in the ultraviolet. We showed above that the contribution of the lowest order correction term, \( \nu_{3}x^{3} \), to the Casimir force at the infrared length scale \( L_{m} \) is proportional to \( \sigma^{2} \), i.e. that it is completely negligible. One might expect that higher order terms \( \nu_{2r-1} x^{2r-1} \) in the dispersion relation would contribute even less to the Casimir force. At first sight this expectation appears to be confirmed: \( K \) maps a dispersion relation term \( \sim x^{2n-1} \) into a Casimir force term \( \sim (k_{c}L)^{-2r} \). At the infrared scale, \( L = L_{m} \), the latter term reads:

\[ \left( \frac{1}{k_{c}L} \right)^{2r} = \left( \frac{L_{m}}{L_{m}} \right)^{2r} = \sigma^{2r} \] (84)

This indeed means that the size of this term decreases exponentially with increasing \( r \). Upon closer inspection, however, we see that, nevertheless, a higher order term \( x^{2r-1} \) in \( f \) can give an arbitrarily large contribution to the Casimir force, in particular if \( r \) is very large. The reason is that \( K \) involves a factorial amplification of higher order terms which eventually overcomes the exponential suppression that we discussed above. Namely, as (11) shows, the precise action of \( K \) on the correction term \( \nu_{2r-1} x^{2r-1} \) reads:

\[ K: \nu_{2r-1} x^{2r-1} \rightarrow \nu_{2r-1} \frac{(-1)^{r} k_{c}^{2}}{\pi} (2r-1)(2r-1)! \zeta(2r) \left( \frac{1}{2k_{c}L} \right)^{2r} \] (85)

Due to the presence of the factorial term \( (2r-1)! \), the coefficients of the Casimir force function grow much faster than those of the dispersion relation. In particular, for the dispersion relation \( f(x) = x + \nu_{2r-1} x^{2r-1} \) the relative change in the Casimir force at the infrared scale \( L_{m} \) reads:

\[ \frac{F_{\text{standard}}(L_{m}) - F(L_{m})}{F_{\text{standard}}(L_{m})} = \nu_{2r-1} \frac{(-1)^{r-1}(2r-1)\zeta(2r)}{4\pi^{2}} (2r-1)! \left( \frac{\sigma}{2} \right)^{2r-2} \] (86)

It is straightforward to apply Stirling’s formula for the factorial, \( n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^{n} \) for \( n \gg 1 \) in order to calculate how large \( r \) needs to be for the factorial amplification to overcome the exponential suppression. We find that a correction term \( \nu_{2r-1} x^{2r-1} \) with \( \nu_{2r-1} \approx 1 \) in the dispersion relation leads to a relative change of order one in the Casimir force at the infrared scale \( L_{m} \) if \( r \) is of the order \( \sigma^{-1} \), i.e. if \( r \approx 10^{28} \).

To summarize: We found that \( K \) is a well-defined but unbounded and therefore discontinuous operator (as are, e.g., the quantum mechanical position and momentum
operators. Namely, a modified dispersion relation of the form \( f(x) = x + \nu_{2r-1}x^{2r-1} \), say with \( r \approx 10^{28} \) and \( \nu_{2r-1} \approx 1 \), is virtually indistinguishable from the linear dispersion relation \( f(x) = x \) at all scales up to the Planck scale, but does lead to a modification of the Casimir force which is very strong (the relative change is of order 100%) even at laboratory length scales. Thus, even though the first order terms contribute extremely little to the Casimir force, very high order corrections to the dispersion relations can contribute significantly to the Casimir force - in fact, the more so the larger \( r \) is.

Realistic candidates for Planck scale modified dispersion relations are given by a series \( f(x) = x + \sum_{n=2}^{\infty} \nu_n x^n \) and such dispersion relations therefore contain terms \( \nu_{2r-1}x^{2r-1} \) for arbitrarily large \( r \). At the same time, the prefactors \( \nu_n \) must of course obey \( \nu_n \to 0 \) as \( n \to \infty \) because this is a necessary condition for the convergence of the series. We conclude that it is this competition between the decay of the coefficients \( \nu_{2r-1} \) and the increasing Casimir effect of terms \( x^{2r-1} \), for \( r \to \infty \), which decides whether or not a given ultraviolet-modified dispersion relation does or does not lead to an appreciable effect on the Casimir force at infrared distances. In practice, to study this competition directly by using the complicated representation of \( K \) in (85) would be a tedious approach to the transplanckian question because, for example, the coefficients of the Casimir force acquire alternating signs. Instead, as we will show in the next section, we will conveniently be able to study the transplanckian question by making use of our representation of \( K \) in terms of the Laplace transform.

### 10.2 UV-IR coupling with generic dispersion relations

Let us write the dispersion relations again in the form \( f(x) = x g(x^2) \) so that, e.g., \( g \equiv 1 \) yields the standard dispersion relation. This allows us to apply the representation of \( K \) in terms of the Laplace transform, (67). We begin by noticing that, since \( x^2 \) is positive, the evaluation of the dispersion relation \( f \) involves evaluating \( g(y) \) only for positive \( y \). Now considering (67) we see that, curiously, the calculation of the Casimir force involves evaluating \( g(y) \) only for negative values of \( y \).

This is surprising because if \( g \) could be any arbitrary function, this would mean that the dispersion relation, which is determined by the behavior of \( g \) on the positive half-axis, and the Casimir force function, which is determined by the behavior of \( g \) on the negative half axis, were unrelated. But of course our \( g \) are not arbitrary functions but are polynomials or power series with infinite radius of convergence, i.e. they are entire functions. Therefore, the behavior of \( g \) on the positive half axis fully determines its behavior also on the negative half axis. The dispersion relations do determine the corresponding Casimir force.

Of crucial importance for the transplanckian question, however, is the fact that there are entire functions \( g \) which are arbitrarily close to one for \( 0 < y < 1 \) and which nevertheless reach arbitrarily large values on the negative half axis. Such functions do not noticeably affect the dispersion relation for momenta up to the Planck scale but do arbitrarily strongly affect the Casimir force. These are the dispersion relations \( f(x) = x g(x^2) \) with

\[
g(y) = 1 + h(y),
\]

where the function \( h \) obeys \( h(y) \approx 0 \) for \( y \in (0, 1) \) while exhibiting large \( |h(y)| \) in some
range of negative values of $y$. Let us now analyze which behavior of $h$ on the negative half axis determines if the Casimir force is affected in the infrared. To this end, let us use (67) and (87) to express the correction in the Casimir force, $\Delta F = F - F_{\text{standard}}$, in terms of the correction $h$ to the dispersion relation:

$$\Delta F(L) = \frac{k^2}{\pi} \int_0^\infty x h(-x^2) \left(1 - 2k_c L x\right) e^{-2k_c L x} dx \quad (88)$$

The integral kernel

$$G(x, L) = (1 - 2k_c L x) e^{-2k_c L x} \quad (89)$$

is positive for $x < (2k_c L)^{-1}$, negative for $x > (2k_c L)^{-1}$ and rapidly decreases to zero for $x \gg (2k_c L)^{-1}$. (We remark that the the integral of the kernel over all $x \in [0, \infty)$ is 0, which expresses the fact that the Casimir force does not depend on the absolute value of the energy.) Thus, for a fixed plate separation $L$, what matters most for the Casimir force is the behavior of $h(y)$ from $y = 0$ to about $y \approx -(k_c L)^{-2}$. As we increase $L$, the interval $y \in (-k_c L)^{-2}, 0)$ on which the integral kernel $G$ is mostly supported is shrinking, see Fig. 3. Thus, there is a significant effect on the Casimir force at realistically large plate separations, such as $L = L_m$, if the function $h$ is either of order one in this small interval close to the origin or it must be exponentially large (so as to compensate the exponential suppression in $G$) in some interval to the left of $-(k_c L)^{-2}$. Of course, both are possible. There are entire functions $h$ which possess either one of these behaviors on the negative half axis and therefore do affect the

![Figure 3: The integral kernel $(1 - xL)e^{-xL}$ for different values of $L$. Note the shift of its zero towards the origin as $L$ grows.](image-url)
Casimir force in the infrared, while being arbitrarily close to zero for $0 < y < 1$, so as to leave the dispersion relation virtually unchanged in the infrared.

There is even the extreme case of functions, $h$, whose corresponding dispersion relation $f$ is arbitrarily little affected at all scales while the Casimir force function is arbitrarily much affected at any scale we wish, say in the infrared. To see this, consider for example the case where $h$ is a Gaussian which is centred around a low negative value $y_0 < 0$ while being so sharply peaked that its tail into the positive half axis is negligibly small. The function that enters into the calculation of the Casimir force, $\tilde{f}_1 = x g(-x^2)$, then features the low-$x$ spike of the Gaussian, implying by our above consideration that the Casimir force is affected in the infrared. At the same time, the dispersion relation itself, $\tilde{f}_2(x) = x g(x^2)$, is virtually unaffected for all $x$.

11 Conclusions

We investigated the effect of ultraviolet corrections to the dispersion relation on the Casimir force. To this end, we calculated the operator $K$ which maps generic dispersion relations, $\omega(k) = k_c f(k/k_c)$, into the corresponding Casimir force functions $F(L)$. Here, $k_c$ is the Planck momentum, $f$ is a power series in $x = k/k_c$ and $L$ is the plate separation. The structure of $K$ showed that the even components of dispersion relations do not contribute to the Casimir force. This implies, for example, that the dispersion relations defined through $f(x) = \sinh(x)$ and $f(x) = \exp(x) - 1$ yield identical Casimir force functions.

We also showed that a certain class of UV-modified dispersion relations, such as $f(x) = \sinh(x)$, lead to Casimir force functions that are well defined only down to a finite smallest distance between the plates. Physically, the existence of a finite lower bound for the plate separation, $L$, is indeed what should be expected if the ultraviolet-modified dispersion relation arises from an underlying theory of quantum gravity which possesses a notion of minimum length.

Technically, the phenomenon of a finite minimum $L$ arises because the Casimir force $F(L)$ is always a polynomial or power series in $1/L$, depending on whether the dispersion relation is polynomial or a power series. Therefore, if $F(L)$ is a power series then it can possess a finite radius of convergence, i.e. an upper bound on $1/L$, which then implies a lower bound on $L$. Of course, a finite radius of convergence can occur only for power series but not for polynomials. Interestingly, this means that the existence of a finite lower bound on $L$ cannot arise from polynomial dispersion relations of any degree. An important conclusion that we can draw from this is that if a candidate quantum gravity theory yields a non-polynomial dispersion relation then working with any finite degree polynomial approximation of this dispersion relation may be missing crucial qualitative features, such as the existence of a finite minimum length.

There is a deeper reason for why it is important to apply a nontrivial dispersion relation in the exact form in which it arises from some proposed quantum gravity theory. The reason is that $K$ is an unbounded and therefore also discontinuous operator, which means that arbitrarily small changes to the dispersion relation can lead to arbitrarily large changes to the Casimir force. On the other hand, the action of $K$ is of

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course well-defined, which means that if a candidate quantum gravity theory implies
a particular UV-modified dispersion relation then $K$ can be used to precisely predict
the corresponding Casimir force function.

We proceeded by determining which ultraviolet modifications to the dispersion
relation would appreciably affect the Casimir force function at a large length scale
$L_m$. To this end, it was convenient to express dispersion relations, $f$, in the form
$f(x) = x g(x^2)$ and $g(y) = 1 + h(y)$ where $h$ is an entire function (so that $h \equiv 0$ for the
usual linear dispersion relation). Recall that $y$ is the momentum squared, in units of
$k_c^2 = L_c^2$, i.e., $y = 1$ is the Planck momentum squared. We are interested in dispersion
relations which are essentially unchanged in the infrared, i.e., which obey $h(y) \approx 0$, up
to unmeasurable deviations, for all $y$ in the interval $(0, 1)$. Our analysis of $K$ through
the Laplace transform then showed that if the corresponding Casimir force is to be
affected at an infrared scale, say $L_m$, then the dispersion relation must come from a
function $h$ which obeys one or both of two conditions: (a) either $h$ obeys $|h(y)| = O(1)$
for $y$ in parts of the interval $(-L_c^2/L_m^2, 0) = (-\sigma^2, 0)$, or (b) $h$ is exponentially large
in a finite interval of more negative $y$ obeying $y < -\sigma^2$.

In the case (a), an ultraviolet-modified dispersion relation induces an infrared mod-
ification of the Casimir force if the correction to the dispersion relation, $h(y)$, is essen-
tially zero in all of $(0, 1)$, while it rises very steeply towards the left to amplitudes of
order one within the extremely short interval $(-\sigma^2, 0)$, where we recall that $\sigma \approx 10^{-28}$.
In the case (b), UV/IR coupling arises if $h$ is again essentially zero in the interval $(0, 1)$,
while now needing to reach exponentially large values for a finite stretch of more neg-
ative $y$ values, again resulting in the need for $h$ to rise extremely steeply towards the
left. It is easy to give examples of such $h$, such as the Gaussian $h$ that we discussed.
In fact, we can easily write down $h$ which would lead to no appreciable modification of
the dispersion at low energies and yet to arbitrarily large changes to the Casimir force
even at macroscopically large plate separations. Because of their large slope, however,
such functions $h$ are severely fine-tuned and must therefore be considered unlikely to
arise from an underlying quantum gravity theory. We can conclude, therefore, that the
28 orders of magnitude which separate the effective UV and IR scale do not suppress
UV/IR coupling in strength but instead in likelihood, namely through the need for
extreme fine tuning.

This is interesting because, in inflation, the separation of the effective UV and IR
scales is only about three to five orders of magnitude: Consider the operator $K$ for
inflation, namely the operator which maps arbitrary ultraviolet-modified dispersion
relations into the function that describes the CMB’s tensor or scalar fluctuation spec-
trum. Let us assume that its properties are analogous to that of the operator $K$ which
we here found for the Casimir effect. This would mean that an ultraviolet-modified
dispersion relation that arises from some underlying quantum gravity theory can lead
to effects on the CMB spectrum which are not automatically limited in the strength
by the separation of scales $\sigma \approx 10^{-5}$, or indeed by any power of $\sigma$. Instead, arbitrarily
large effects on the CMB must be considered possible, while it is merely the a priori
likelihood of large effects that is suppressed by the separation of scales. That this is
indeed the case can of course only be confirmed by calculating an explicit expression
for the operator $K$ for inflation.
12 Outlook

The task of finding the operator $\mathcal{K}$ for inflation will be more difficult than it was to calculate $\mathcal{K}$ for the Casimir effect. This is mainly because it is highly nontrivial to identify the comoving modes' initial condition, i.e. their ingoing vacuum state. This problem needs to be solved because a misidentification of the vacuum could mask the infrared effects that one is looking for. The reason is that the mode equations reduce to the mode equations with the usual linear dispersion at late times, namely at large length scales. Therefore, the mode solutions at late times live in the usual solution space. Thus, any effects of ultraviolet-modified dispersion relations in the IR could be masked by an incorrect choice of the initial condition for the mode equation. A further complication is that of possibly strong backreaction, although there are indications that this problem can be absorbed in a suitable redefinition of the inflaton potential, see [14]. Once these points are clarified, $\mathcal{K}$ for inflation can be calculated.

A limitation of our investigation of the Casimir effect has been that we restricted attention to modelling the effects of Planck scale physics on quantum field theory exclusively through UV-modified dispersion relations. This assumes that fields can possess arbitrarily large $k$ and arbitrarily short wavelengths, an assumption which is likely too strong. Indeed, studies of quantum gravity and string theory strongly indicate the existence of a universal minimum length at the Planck or string scale. In particular, it has been suggested that, in terms of first quantization, this natural UV cutoff could possess an effective description through uncertainty relations of the form
\[ \Delta x \Delta p \leq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...), \]
see, e.g., [15]. As is easily verified, such uncertainty relations encode the minimum length as a lower bound, $\Delta x_{\text{min}} = \hbar \sqrt{\beta}$, on the formal position uncertainty, $\Delta x$. It has been shown that this type of uncertainty relations also implies a minimum wavelength and that, therefore, fields possess the sampling property, see [16]: if a field's (number or operator-valued) amplitudes are known only at discrete points then the field’s amplitudes everywhere are already determined - if the average sample spacing is less than the critical spacing, which is given by the minimum length. As a consequence, any theory with this type of uncertainty relation can be written as continuum theory or, fully equivalently, as a discrete theory on any lattice of sufficiently tight spacing. This UV cutoff can also be viewed as an information theoretic cutoff, and it possesses a covariant generalization, see [17].

Indeed, nontrivial dispersion relations also raise the question of local Lorentz invariance. One possibility is that local Lorentz is broken hard or soft and that, e.g., the CMB rest frame is the preferred frame. It has also been suggested that the Lorentz group might be deformed, or that it may be unchanged but represented nonlinearly. Various experimental bounds on Lorentz symmetry breaking are being discussed, e.g., from observations of gamma ray bursts. For the literature, see e.g. [18].

An application of the minimum length uncertainty principle to the Casimir effect has recently been tried, see [19]. There, the Casimir force was found to be a discontinuous function of the plate separation. This problem is due to the fact that, in [19], the plate boundaries are implicitly treated as possessing sharp positions. This is not fully consistent with the assumption that all particles including those that make up the plates can be localized only up to the finite minimum position uncertainty. As a consequence, as the plate separation increases, the energy eigenvalues discontinuously
enter the spectrum of the first quantized Hamiltonian. It should be very interesting to extend these Casimir force calculations while applying the minimum length uncertainty relations to both the field and the plates.

Finally, we note an additional analogy between the Casimir effect and inflation: in the Casimir effect with UV cutoff, as the distance between the plates is increased, new modes enter the space between the plates, thereby changing the vacuum energy. In cosmology, space itself expands and, in the presence of an UV cutoff, new comoving modes (recall that these are the independent degrees of freedom) are continually being created, similar to the Casimir effect. A priori, these new modes arise with vacuum energy. During the expansion, the modes’ vacuum energy becomes diluted but if the dispersion is nonlinear then the balance of new vacuum energy creation and vacuum energy dilution is nontrivial. A paper which addresses this question is in progress, [20].

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