Analytical Approach of Matter Effect on (3+1) Neutrino Oscillation

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Abstract

We study the analytic expression for four flavor neutrino oscillation in the presence of matter. We calculate the time evolution operator on flavor and mass basis. We find the matter dependent mass square difference and neutrino transition probabilities for (3+1) four flavor neutrino oscillation.

Keywords: Neutrino oscillation, MSW effect

1 Introduction

At present there are four types of experiments for neutrino oscillation; solar, atmospheric, accelerator and reactor neutrino experiments [1, 2, 3, 4, 5, 6, 7, 8]. The presence of sterile neutrino given by LSND and MiniBoone results [7, 8]. LSND experiment gives solar neutrino mass square difference $0 < \Delta_{21} < 100 eV^2$[9] and MiniBooNe experiments gives solar mass square difference in the given range $0.01 < \Delta_{21} < 1.0 eV^2$[10]. Both this experiments gives new possibility of sterile neutrinos. The neutrino oscillation parameter in vacuum can be modified by considering when neutrino interact with matter. The well known MSW effect[11, 12], which can be explain by modified Hamiltonian. In four flavor neutrino oscillation, the three active neutrinos interacts by neutral current interaction. But the sterile neutrino has no any interaction. When neutrino passing through in matter, the matter effect change the neutrino mass square differences and mixing angles. Earlier many author work on three flavor neutrino oscillation in matter[13, 14, 15, 18]. In this work, we calculate the time evolution operator on four flavor neutrino oscillation. We derive the expression for the matter dependent neutrino mass square difference using Cayley–Hamilton theorem. The article outline is as follows. In Section 2, Four Flavor Neutrino Oscillation in Vacuum. In Section 3, neutrino mass square difference and transition probability in matter are driven for four flavor and the conclusion is given in Section 4.

2 Four Flavor Neutrino Oscillation in Vacuum

In this section, we consider with four flavor framework $(3+1)$ by assuming the sterile neutrino of $eV$ range and the mixing of this sterile neutrino with three different neutrinos. By adding one sterile neutrinos [16], there is an increment in mixing angles and CP violating phases in the PMNS matrix $U_{4x4}$ which is given by [17],

\begin{equation}
U = R_{34}(\theta_{34}, \delta_{34})R_{24}(\theta_{34})R_{14}(\theta_{14}, \delta_{14})R_{23}(\theta_{23})R_{13}(\theta_{13}, \delta_{13})R_{12}(\theta_{12}),
\end{equation}

where the matrices $R_{ij}$ are rotations in $ij$ space,

\[
R_{ij}(\theta_{ij}, \delta) = \begin{pmatrix}
c_{ij} & s_{ij}e^{-i\delta} \\
-s_{ij}e^{i\delta} & c_{ij}
\end{pmatrix},
\]

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$.

Note that in four flavor there are three Dirac CP-violating phase $\delta_{ij}$. The explicit form of $U$ is
The effective Hamiltonian for four flavor neutrino mixing is \[ H_{\text{vacuum}} = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix} \] (4) where \( E_k (k = 1, 2, 3, 4) \) are the energies of the neutrino mass eigenstates \( k \) with mass \( m_k \); \[ E_k = \sqrt{m_k^2 + p_k^2} \approx p_k + m_k^2/2p \] (5) we consider the momentum \( p \) is the same for all mass eigenstates. When neutrino interact with matter by weak interaction (charged and neutral current). Sterile neutrino itself not take any participation of weak interaction. The effective Hamiltonian for four flavor neutrino mixing is [12] \[ H_{\text{vacuum}} = \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 & 0 & 0 \\ 0 & m_2^2 & 0 & 0 \\ 0 & 0 & m_3^2 & 0 \\ 0 & 0 & 0 & m_4^2 \end{pmatrix} \] (6)
When neutrinos propagate in matter, there is an additional term coming from the presence of electrons in the matter [5]. This term is diagonal in the flavor state basis and is given by

\[ A_f = \frac{1}{2E} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A' \end{pmatrix} \] (7)

where,

\[ A(eV^2) = 2\sqrt{2}G_FN_eE_\nu, \]

\[ A'(eV^2) = -\sqrt{2}G_FN_nE_\nu, \]

and \( N_e \) and \( N_n \) is the density of electron and neutron [17]. Since, we can switch from flavor state to mass eigenstate via transformation. Thus, the matter induce Hamiltonian [17, 18] is \( H_m = U^T A_f U \), where \( U \) is the PMNS matrix. Now the total Hamiltonian is given by,

\[ H_m = U^T A_f U \]

\[ H = H_{\text{vacuum}} + H_m \]

\[ \frac{1}{2E} \left[ \tilde{U}^T \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 & 0 \\ 0 & 0 & \tilde{m}_3^2 & 0 \\ 0 & 0 & 0 & \tilde{m}_4^2 \end{pmatrix} \right] \tilde{U} = H = \frac{1}{2E} \left[ \begin{pmatrix} m_1^2 & 0 & 0 & 0 \\ 0 & m_2^2 & 0 & 0 \\ 0 & 0 & m_3^2 & 0 \\ 0 & 0 & 0 & m_4^2 \end{pmatrix} + U^T \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A' \end{pmatrix} U \right] \]

\[ \left[ \tilde{U}^T \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 & 0 \\ 0 & 0 & \tilde{m}_3^2 & 0 \\ 0 & 0 & 0 & \tilde{m}_4^2 \end{pmatrix} \right] \tilde{U} = H = \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 & 0 \\ 0 & 0 & \Delta_{31} & 0 \\ 0 & 0 & 0 & \Delta_{41} \end{pmatrix} + U^T \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A' \end{pmatrix} U \right] \] (11)

where,

\[ \tilde{m}_{ij} = m_i^2 - m_j^2, \] and \( \Delta_{ij} = m_i^2 - m_j^2 \)

The explicit form of Hamiltonian are,

\[ H_m = \begin{pmatrix} AU_{e1}^2 + A'U_{s1}^2 & AU_{e1}U_{e2} + AU_{s1}U_{s2} & AU_{e1}U_{e3} + A'U_{s1}U_{s3} & AU_{e1}U_{e4} + A'U_{s1}U_{s4} \\ AU_{e1}U_{e2} + A'U_{s1}U_{s2} & AU_{e2}^2 + A'U_{s2}^2 & AU_{e2}U_{e3} + AU_{s2}U_{s3} & AU_{e2}U_{e4} + A'U_{s2}U_{s4} \\ AU_{e1}U_{e3} + A'U_{s1}U_{s3} & AU_{e2}U_{e3} + AU_{s2}U_{s3} & AU_{e3}^2 + AU_{s3}^2 & AU_{e3}U_{e4} + A'U_{s3}U_{s4} \\ AU_{e1}U_{e4} + A'U_{s1}U_{s4} & AU_{e2}U_{e4} + AU_{s2}U_{s4} & AU_{e3}U_{e4} + A'U_{s3}U_{s4} & AU_{e4}^2 + A'U_{s4}^2 \end{pmatrix} \]

\[ H = \begin{pmatrix} AU_{e1}^2 + A'U_{s1}^2 & AU_{e1}U_{e2} + A'U_{s1}U_{s2} & AU_{e1}U_{e3} + A'U_{s1}U_{s3} & AU_{e1}U_{e4} + A'U_{s1}U_{s4} \\ AU_{e1}U_{e2} + A'U_{s1}U_{s2} & AU_{e2}^2 + A'U_{s2}^2 & AU_{e2}U_{e3} + AU_{s2}U_{s3} & AU_{e2}U_{e4} + A'U_{s2}U_{s4} \\ AU_{e1}U_{e3} + A'U_{s1}U_{s3} & AU_{e2}U_{e3} + AU_{s2}U_{s3} & AU_{e3}^2 + AU_{s3}^2 & AU_{e3}U_{e4} + A'U_{s3}U_{s4} \\ AU_{e1}U_{e4} + A'U_{s1}U_{s4} & AU_{e2}U_{e4} + A'U_{s2}U_{s4} & AU_{e3}U_{e4} + A'U_{s3}U_{s4} & AU_{e4}^2 + A'U_{s4}^2 \end{pmatrix} \]

(12)

The propagation of neutrinos or change of flavor in oscillation at some distance \( L \) is governed by the time evolution of a wavefunction \( \Phi_m(t) \), which is basically a plane wave solution of Schrödinger equation, given by,

\[ \Phi_m(x,t) = e^{-iHt} \varphi_m(x) \]

where, \( \varphi_m(t), H_m \) are the wavefunction and Hamiltonian in mass eigenstate basis. At \( t = L \), we have,

\[ \Phi_m(x,L) = e^{-iHL} \varphi_m(x) \]

(15)

We can simply switch to flavor eigen basis by applying the unitary operator. Then eq. (15) becomes,

\[ \Phi_m(x,t) = U e^{-iHL} \varphi_m(x) \]
\[ \Phi_f(x, t) = U e^{-iH L} U^T U \varphi_m(x) \]
\[ \Phi_f(x, t) = e^{-iH j L} \varphi_f(x) \]  
(16)

Now we are finding the exponential terms in mass eigenstate and flavor eigenstate basis, \( e^{-iH L} \) and \( e^{-iH j L} \), respectively. We can write any 4 × 4 matrix in term of traceless matrix, i.e.
\[ A = A_0 + \frac{1}{4} (\text{Trace } A) I \]  
(17)

where, \( A_0 \) is a matrix with zero trace. Then,
\[ e^{-iH L} = \Omega e^{-iLT} = e^{-iH L - \frac{1}{4} (\text{Trace } H) L} \]

where,
\[ \Omega = e^{-iL \frac{1}{4} (\text{Trace } H)} \quad \text{and} \quad T = H - \frac{1}{4} (\text{Trace } H) I \]
\[ e^{-iLT} = \sum_{n=1}^{4} \frac{(-iLT)^n}{n!} = -iLT - \frac{L^2 T^2}{2!} + \frac{i L^3 T^3}{3!} + \frac{L^4 T^4}{4!} + \ldots \]  
(18)

The characteristic equation of matrix \( H \) is given by,
\[ \det(-iHL - \lambda I) = 0 \]  
(19)
\[ \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 I = 0 \]  
(20)

Using Cayley-Hamilton’s theorem \( \lambda \) is change by matrix \( H \), i.e
\[ H^n + a_{n-1} H^{n-1} + \ldots + a_1 H + a_0 I = 0 \]
\[ H^n = -a_{n-1} H^{n-1} - \ldots - a_1 H - a_0 I \]
\[ H^n = c_{n-1} H^{n-1} + \ldots + c_1 H + c_0 I \]  
(21)

Using eq. (18) and eq. (21), we get
\[ e^{-iLT} = c_0 I + c_1 (-iLT) + c_2 (-iLT)^2 + c_3 (-iLT)^3 \]
\[ e^{-iLT} = c_0 I - ic_1 (LT) - c_2 L^2 T^2 + ic_3 L^3 T^3 \]  
(22)
\[ e^{-iHL} = \Omega (c_0 I - ic_1 (LT) - c_2 L^2 T^2 + ic_3 L^3 T^3) \]  
(23)

The Trace of Matrix \( H \) is,
\[ \text{Trace } H = \Delta_{21} + \Delta_{31} + \Delta_{41} + A + A' \]

and the matrix \( T \) is given by,
\[ T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} \]  
(24)

With,
\[ T_{11} = AU_{e1}^2 + A' U_{e1}^2 - \frac{1}{4} (\Delta_{21} + \Delta_{31} + \Delta_{41}) - \frac{1}{4} (A + A') \]  
(25)
\[ T_{22} = AU_{e2}^2 + A' U_{e2}^2 - \frac{1}{4} (-\Delta_{21} + \Delta_{32} + \Delta_{42}) - \frac{1}{4} (A + A') \]  
(26)
\[ T_{33} = AU_{e3}^2 + A' U_{e3}^2 - \frac{1}{4} (-\Delta_{32} - \Delta_{33} + \Delta_{43}) - \frac{1}{4} (A + A') \]  
(27)
\[ T_{44} = AU_{e4}^2 + A' U_{e4}^2 - \frac{1}{4} (-\Delta_{42} - \Delta_{43} - \Delta_{44}) - \frac{1}{4} (A + A') \]  
(28)
And
\[
T_{12} = T_{21} = AU_{e1}U_{e2} + A'U_{s1}U_{s2}
\]
\[
T_{13} = T_{31} = AU_{e1}U_{e3} + A'U_{s1}U_{s3}
\]
\[
T_{14} = T_{41} = AU_{e1}U_{e4} + A'U_{s1}U_{s4}
\]
\[
T_{23} = T_{32} = AU_{e2}U_{e3} + A'U_{s2}U_{s3}
\]
\[
T_{24} = T_{42} = AU_{e2}U_{e4} + A'U_{s2}U_{s4}
\]
\[
T_{34} = T_{43} = AU_{e3}U_{e4} + A'U_{s3}U_{s4}
\]

We can see from eq. (25-29) that the matrix \( T \) is Traceless symmetrix matrix i.e
\[
\sum_{i=1}^{4} T_{ii} = 0 \quad ; \quad T_{ij} = T_{ji}
\]

Characteristic equation of matrix \( T \) is
\[
a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0
\]
where the coefficients \( a_4, a_3, a_2, a_1 \) and \( a_0 \) are given in Appendix A.
\[
P = \frac{3a_2^2 - 8a_0a_4}{12a_4^2} , \quad X = -a_3^2 + 4a_3a_2a_4 - 8a_1a_4^2,
\]
\[
L = a_2^2 - 3a_3a_1 + 12a_0a_4 , \quad M = 2a_3^2 - 9a_3a_1a_2 + 27a_0^2a_2 + 27a_1^2a_4 - 72a_0a_4
\]
\[
Q = \sqrt{P + \frac{1}{3a_4} \left[ \left( \frac{M}{2} + \sqrt{\left( \frac{M}{2} \right)^2 - L^3} \right)^{1/3} + L \left( \frac{M}{2} + \sqrt{\left( \frac{M}{2} \right)^2 - L^3} \right)^{-1/3} \right]}
\]

The roots of eq.(31), gives matter dependent mass square \( m^2_{m1}, m^2_{m2}, m^2_{m3}, m^2_{m4} \)
\[
\lambda_1 = m^2_{m1} = -\frac{a_3}{4} - \frac{Q}{2} - \frac{1}{2} \sqrt{3P - Q^2 - \frac{X}{4Q}},
\]
\[
\lambda_2 = m^2_{m2} = -\frac{a_3}{4} - \frac{Q}{2} + \frac{1}{2} \sqrt{3P - Q^2 - \frac{X}{4Q}},
\]
\[
\lambda_3 = m^2_{m3} = -\frac{a_3}{4} + \frac{Q}{2} - \frac{1}{2} \sqrt{3P - Q^2 + \frac{X}{4Q}},
\]
\[
\lambda_4 = m^2_{m4} = -\frac{a_3}{4} + \frac{Q}{2} + \frac{1}{2} \sqrt{3P - Q^2 + \frac{X}{4Q}}
\]

Using matter dependent mass square \( m^2_{m1}, m^2_{m2}, m^2_{m3}, m^2_{m4} \), we can write matter dependent mass square difference for four flavor neutrino oscillation
\[
\Delta^m_{21} = m^2_{m2} - m^2_{m1} = \sqrt{3P - Q^2 - \frac{X}{4Q}},
\]
\[
\Delta^m_{31} = m^2_{m3} - m^2_{m1} = Q + \frac{1}{2} \left( \sqrt{3P - Q^2 - \frac{X}{4Q}} - \sqrt{3P - Q^2 + \frac{X}{4Q}} \right),
\]
\[
\Delta^m_{41} = m^2_{m4} - m^2_{m1} = Q + \frac{1}{2} \left( \sqrt{3P - Q^2 - \frac{X}{4Q}} + \sqrt{3P - Q^2 + \frac{X}{4Q}} \right)
\]

Now, from eq.(22) we have
\[
e^{-iL\lambda_1} = c_0I - ic_1L\lambda_1 - c_2L^2\lambda_1^2 + ic_3L^3\lambda_1^3
\]
\[
e^{-iL\lambda_2} = c_0I - ic_1L\lambda_2 - c_2L^2\lambda_2^2 + ic_3L^3\lambda_2^3
\]
\[ e^{-iL\lambda_3} = c_0 I - ic_1 L\lambda_3 - c_2 L^2 \lambda_3^2 + ic_3 L^3 \lambda_3^3 \]
\[ e^{-iL\lambda_4} = c_0 I - ic_1 L\lambda_4 - c_2 L^2 \lambda_4^2 + ic_3 L^3 \lambda_4^3 \]

\[
\begin{pmatrix}
  e^{-iL\lambda_1} \\
  e^{-iL\lambda_2} \\
  e^{-iL\lambda_3} \\
  e^{-iL\lambda_4}
\end{pmatrix} =
\begin{bmatrix}
  1 & -iL\lambda_1 & -L^2 \lambda_1^2 & iL^3 \lambda_1^3 \\
  1 & -iL\lambda_2 & -L^2 \lambda_2^2 & iL^3 \lambda_2^3 \\
  1 & -iL\lambda_3 & -L^2 \lambda_3^2 & iL^3 \lambda_3^3 \\
  1 & -iL\lambda_4 & -L^2 \lambda_4^2 & iL^3 \lambda_4^3
\end{bmatrix}^{-1}
\begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3
\end{pmatrix}
\]

(35)

\[
e = \mathcal{L}e
\]

(37)

Where,

\[
\mathcal{L}_{11} = \frac{1}{4}(L^6(\lambda_4(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2) + \lambda_3(\lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_4^2) + \lambda_2(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2))
\]
\[
\mathcal{L}_{12} = \frac{1}{4}(L^6(\lambda_4(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2) + \lambda_3(\lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_4^2) + \lambda_2(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2))
\]
\[
\mathcal{L}_{13} = \frac{1}{4}(L^6(\lambda_4(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2) + \lambda_3(\lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_4^2) + \lambda_2(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2))
\]
\[
\mathcal{L}_{14} = \frac{1}{4}(L^6(\lambda_4(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2) + \lambda_3(\lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_4^2) + \lambda_2(\lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2))
\]

(38)

\[
D = (L^6(\lambda_4(\lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2))
\]
\[
+ \lambda_3(\lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2)
\]
\[
+ \lambda_2(\lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2)
\]
\[
+ \lambda_1(\lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2 - \lambda_3^2 \lambda_4^2)
\]

(39)

Then coefficients are,

\[ c_0 = \mathcal{L}_{11}e^{-iL\lambda_1} + \mathcal{L}_{12}e^{-iL\lambda_2} + \mathcal{L}_{13}e^{-iL\lambda_3} + \mathcal{L}_{14}e^{-iL\lambda_4} \]

(40)

\[ c_1 = \mathcal{L}_{21}e^{-iL\lambda_1} + \mathcal{L}_{22}e^{-iL\lambda_2} + \mathcal{L}_{23}e^{-iL\lambda_3} + \mathcal{L}_{24}e^{-iL\lambda_4} \]

(41)

\[ c_2 = \mathcal{L}_{31}e^{-iL\lambda_1} + \mathcal{L}_{32}e^{-iL\lambda_2} + \mathcal{L}_{33}e^{-iL\lambda_3} + \mathcal{L}_{34}e^{-iL\lambda_4} \]

(42)

\[ c_3 = \mathcal{L}_{41}e^{-iL\lambda_1} + \mathcal{L}_{42}e^{-iL\lambda_2} + \mathcal{L}_{43}e^{-iL\lambda_3} + \mathcal{L}_{44}e^{-iL\lambda_4} \]

(43)

Using eq. (40-43), eq.(23) becomes

\[
e^{-i\mathcal{H}L} = \Omega(e^{-iL\lambda_1}(\mathcal{L}_{11}I - i\mathcal{L}_{21}LT - \mathcal{L}_{31}L^2T^2 + i\mathcal{L}_{41}L^3T^3)
\]
\[
+ e^{-iL\lambda_2}(\mathcal{L}_{12}I - i\mathcal{L}_{22}LT - \mathcal{L}_{32}L^2T^2 + i\mathcal{L}_{42}L^3T^3)
\]
\[
+ e^{-iL\lambda_3}(\mathcal{L}_{13}I - i\mathcal{L}_{23}LT - \mathcal{L}_{33}L^2T^2 + i\mathcal{L}_{43}L^3T^3)
\]
\[
+ e^{-iL\lambda_4}(\mathcal{L}_{14}I - i\mathcal{L}_{24}LT - \mathcal{L}_{34}L^2T^2 + i\mathcal{L}_{44}L^3T^3))
\]

(44)

\[
e^{-i\mathcal{H}L} = \Omega(e^{-iL\lambda_1}(\mathcal{L}_{11}I + \mathcal{L}_{21}T - \mathcal{L}_{31}T^2 - \mathcal{L}_{41}T^3)
\]
\[
+ e^{-iL\lambda_2}(\mathcal{L}_{12}I + \mathcal{L}_{22}T - \mathcal{L}_{32}T^2 - \mathcal{L}_{42}T^3)
\]
\[
+ e^{-iL\lambda_3}(\mathcal{L}_{13}I + \mathcal{L}_{23}T - \mathcal{L}_{33}T^2 - \mathcal{L}_{43}T^3)
\]
\[
+ e^{-iL\lambda_4}(\mathcal{L}_{14}I + \mathcal{L}_{24}T - \mathcal{L}_{34}T^2 - \mathcal{L}_{44}T^3))
\]

(45)
The above equation is the evolution operator in mass eigenstate basis. Thus the evolution operator for the neutrinos in the flavor basis is given by,

$$e^{-i H_j L} = U e^{-i H L} U^T$$

(46)

$$e^{-i H_j L} = \Omega (e^{-i L \lambda_1} (L_{11} I + L_{21} \hat{T} - L_{31} \hat{T}^2 - L_{41} \hat{T}^3))$$

$$+ e^{-i L \lambda_2} (L_{12} I + L_{22} \hat{T} - L_{32} \hat{T}^2 - L_{42} \hat{T}^3)$$

$$+ e^{-i L \lambda_3} (L_{13} I + L_{23} \hat{T} - L_{33} \hat{T}^2 - L_{43} \hat{T}^3)$$

$$+ e^{-i L \lambda_4} (L_{14} I + L_{24} \hat{T} - L_{34} \hat{T}^2 - L_{44} \hat{T}^3)$$

(47)

$$e^{-i H_j L} = \Omega \sum_{j=1}^{4} (e^{-i L \lambda_j} (L_{1j} I + L_{2j} \hat{T} - L_{3j} \hat{T}^2 - L_{4j} \hat{T}^3))$$

(48)

where

$$\hat{T} = U T U^T$$

(49)

The matrix element for matrix $T^2, T^3$ and $\hat{T}, \hat{T}^2, \hat{T}^3$ are given in Appendix B. Using equation (48), we are in stage to define the probability amplitude for neutrino oscillation from one flavor to another flavor, i.e.

$$A M_{\alpha \beta} = \langle \beta | e^{-i H_j L} | \alpha \rangle$$

$$A M_{\alpha \beta} = \Omega \sum_{j=1}^{4} (e^{-i L \lambda_j} (L_{1j} I + L_{2j} \hat{T} - L_{3j} \hat{T}^2 - L_{4j} \hat{T}^3))$$

(50)

Thus the transition probability is given by,

$$P(\nu_\alpha \rightarrow \nu_\beta) = A M_{\alpha \beta}^* A M_{\alpha \beta} = |A M_{\alpha \beta}|^2$$

$$P(\nu_\alpha \rightarrow \nu_\beta) = \Omega^* \sum_{j=1}^{4} (e^{-i L \lambda_j} (L_{1j} I + L_{2j} \hat{T} - L_{3j} \hat{T}^2 - L_{4j} \hat{T}^3)^\dagger) \times \Omega$$

(51)

where, $L$ are given in eq. (38). For $A = A' = 0$, we have vacuum transition probability,

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha \beta} - 4 \sum_{i<j} \text{Re}(U_{\alpha i} U_{\beta j}^* U_{\alpha i}^* U_{\beta j}^* \sin^2 \Delta_{ij}) + 2 \sum_{i<j} \text{Im}(U_{\alpha i} U_{\beta j}^* U_{\alpha i}^* U_{\beta j}^* \sin 2 \Delta_{ij}), \quad \alpha, \beta = e, \mu, \tau, s$$

(51)

Analogous to eq. (51) we can write a transition probability for $(3+1)$ neutrino oscillation in matter. i.e.

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha \beta} - 4 \sum_{i<j} \text{Re}(A M_{\alpha \beta}^* A M_{\alpha \beta} \sin^2 \Sigma_{ij}) + 2 \sum_{i<j} \text{Im}(A M_{\alpha \beta}^* A M_{\alpha \beta} \sin 2 \Sigma_{ij}), \quad \alpha, \beta = e, \mu, \tau, s$$

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha \beta} - 4 \sum_{i<j} \text{Re}((\sum_{j=1}^{4} (L_{1j} I + L_{2j} \hat{T} - L_{3j} \hat{T}^2 - L_{4j} \hat{T}^3)^\dagger)^2 \sin^2 \Sigma_{ij}) + 2 \sum_{i<j} \text{Im}((\sum_{j=1}^{4} (L_{1j} I + L_{2j} \hat{T} - L_{3j} \hat{T}^2 - L_{4j} \hat{T}^3)^\dagger)^2 \sin 2 \Sigma_{ij})$$

where $\Sigma_{ij} = \Delta_{ij} L / 2E$, baseline length of particular experiment is L.

4 Conclusions

The main results of our analysis is to determine all neutrino mass square difference for four flavor neutrino, when neutrinos passing through matter with constant density in Eq.(32-34). We have also calculated the transition probabilities for neutrino oscillation in matter. We used Cayley-Hamilton theorem be used to derive neutrino oscillation probability. In Appendix A and B, we given coefficients and all T matrix components, which is useful for determine neutrino oscillation probability in matter.
APPENDIX A:

Coefficients $a_4$, $a_3$, $a_2$, $a_1$ and $a_0$

$$a_4 = 1$$

$$a_3 = -\text{Trace } T$$

$$a_3 = -(T_{11} + T_{22} + T_{33} + T_{44}) = 0$$

$$a_2 = - [(T_{12})^2 + (T_{13})^2 + (T_{14})^2 + (T_{23})^2 + (T_{24})^2 + (T_{34})^2] + T_{11}(T_{22} + T_{33} + T_{44}) + T_{22}(T_{33} + T_{44}) + T_{33}T_{44}$$

$$= \frac{1}{8} [-3 \sum_{i=1}^{4} \Delta_{i1}^2 + 2(\Delta_{21}\Delta_{31} + \Delta_{21}\Delta_{41} + \Delta_{31}\Delta_{41}) + 2A(\Delta_{21}(1 - 4U_{c2}^2) + \Delta_{31}(1 - 4U_{c3}^2) + \Delta_{41}(1 - U_{c4}^2)) + 2A\sum_{i=1}^{4}U_{s1}(1 - U_{c1}^2) - 6AA\sum_{i=1}^{4}U_{c1}U_{s1}^2 - 16AA\sum_{i=1<j}U_{c1}U_{c2}U_{s1}U_{s2} - 3A^2 - 3A^2]$$

$$a_1 = T_{11}[(T_{23})^2 + (T_{24})^2 + (T_{32})^2 - T_{22}T_{33} - T_{22}T_{44} - T_{33}T_{44}] + T_{22}[(T_{13})^2 + (T_{14})^2 + (T_{34})^2 - T_{33}T_{44}] + T_{33}[(T_{12})^2 + (T_{14})^2 + (T_{24})^2] + T_{44}[(T_{12})^2 + (T_{13})^2 + (T_{23})^2] - 2(T_{12}T_{13}T_{23} + T_{12}T_{14}T_{24} + T_{13}T_{14}T_{34} + T_{23}T_{24}T_{34})$$

$$a_0 = \text{det}(T)$$

APPENDIX B:

Matrix Element for matrix $T^2, T^3$ and $\hat{T}, \hat{T}^2, \hat{T}^3$

$$(T^2)_{11} = A^2U_{c1}^2 + A^2U_{s1}^2 + 2AA'U_{c1}U_{s1}(U_{c2}U_{s2} + U_{c3}U_{s3} + U_{c4}U_{s4}) - \frac{1}{4}(\Delta_{21} + \Delta_{31} + \Delta_{41} + A + A') \left( \frac{1}{4}(\Delta_{21} + \Delta_{31} + \Delta_{41} + A + A') + 2(AU_{c1}^2 + A'U_{s1}^2) \right)$$

$$(T^2)_{22} = A^2U_{c2}^2 + A^2U_{s2}^2 + 2AA'U_{c2}U_{s2}(U_{c1}U_{s1} + U_{c3}U_{s3} + U_{c4}U_{s4}) - \frac{1}{4}(\Delta_{21} + \Delta_{32} + \Delta_{42}) \left( \frac{1}{4}(\Delta_{21} + \Delta_{32} + \Delta_{42} + A + A') + 2(AU_{c2}^2 + A'U_{s2}^2) \right)$$

$$(T^2)_{33} = A^2U_{c3}^2 + A^2U_{s3}^2 + 2AA'U_{c3}U_{s3}(U_{c1}U_{s1} + U_{c2}U_{s2} + U_{c4}U_{s4}) - \frac{1}{4}(\Delta_{32} + \Delta_{33} + \Delta_{34} + A + A') \left( \frac{1}{4}(\Delta_{32} + \Delta_{33} + \Delta_{34} + A + A') + 2(AU_{c3}^2 + A'U_{s3}^2) \right)$$

$$(T^2)_{44} = A^2U_{c4}^2 + A^2U_{s4}^2 + 2AA'U_{c4}U_{s4}(U_{c1}U_{s1} + U_{c2}U_{s2} + U_{c3}U_{s3}) - \frac{1}{4}(\Delta_{42} + \Delta_{43} + \Delta_{44} + A + A') \left( \frac{1}{4}(\Delta_{42} + \Delta_{43} + \Delta_{44} + A + A') + 2(AU_{c4}^2 + A'U_{s4}^2) \right)$$

$$(T^2)_{12} = (T^2)_{21} = A^2U_{c1}U_{c2}(U_{s2}^2 + U_{s1}^2) + A^2U_{s1}U_{s2}(U_{c3}^2 + U_{c4}^2) + AA'(U_{c3}U_{s3} + U_{c4}U_{s4})(U_{c1}U_{s1} + U_{c2}U_{s2}) + (AU_{c1}U_{c2} + A'U_{s1}U_{s2})(T_{11} + T_{22})$$

$$(T^2)_{13} = (T^2)_{31} = A^2U_{c1}U_{c3}(U_{c2}^2 + U_{c4}^2) + A^2U_{s1}U_{s3}(U_{c2}^2 + U_{c4}^2) + AA'(U_{c2}U_{s2} + U_{c4}U_{s4})(U_{c1}U_{s1} + U_{c3}U_{s3}) + (AU_{c1}U_{c3} + A'U_{s1}U_{s3})(T_{11} + T_{33})$$

$$(T^2)_{14} = (T^2)_{41} = A^2U_{c1}U_{c4}(U_{c2}^2 + U_{c3}^2) + A^2U_{s1}U_{s4}(U_{c2}^2 + U_{c3}^2) + AA'(U_{c2}U_{s2} + U_{c3}U_{s3})(U_{c1}U_{s1} + U_{c4}U_{s4}) + (AU_{c1}U_{c4} + A'U_{s1}U_{s4})(T_{11} + T_{44})$$
$$(T^2)_{23} = (T^2)_{32} = A^2 U_{e_3} U_{e_3} (U_{e_1}^2 + U_{e_4}^2) + A^2 U_{e_2} U_{e_3} (U_{e_1}^2 + U_{e_4}^2) + A U_{e_2} U_{e_3} (U_{e_1}^2 + U_{e_4}^2) + A A' (U_{e_1} U_{e_3} + U_{e_4} U_{e_3}) (U_{e_2} U_{e_3} + U_{e_3} U_{e_2})$$

$$(T^2)_{24} = (T^2)_{42} = A^2 U_{e_2} U_{e_4} (U_{e_1}^2 + U_{e_3}^2) + A^2 U_{e_2} U_{e_3} (U_{e_1}^2 + U_{e_4}^2) + A A' (U_{e_1} U_{e_3} + U_{e_4} U_{e_3}) (U_{e_2} U_{e_4} + U_{e_4} U_{e_2})$$

$$(T^2)_{34} = (T^2)_{43} = A^2 U_{e_3} U_{e_4} (U_{e_1}^2 + U_{e_3}^2) + A^2 U_{e_3} U_{e_4} (U_{e_1}^2 + U_{e_4}^2) + A A' (U_{e_1} U_{e_4} + U_{e_3} U_{e_4}) (U_{e_3} U_{e_4} + U_{e_4} U_{e_3})$$

$$(T^3)_{11} = A U_{e_1}^2 + A' U_{e_1}^2 - \frac{1}{2} (\Delta_{21} + \Delta_{31} + \Delta_{41} + A + A')((A^2 U_{e_1}^2 - \frac{1}{16}) + A^2 U_{e_1}^2 - \frac{1}{16})$$

$$-2A A' ((\frac{1}{16}) (\Delta_{21} + \Delta_{31} + \Delta_{41} + 1) - U_{e_1} U_{e_1} (U_{e_2} U_{e_2} + U_{e_3} U_{e_3} + U_{e_4} U_{e_4}))$$

$$-\frac{1}{16} (\Delta_{21} + \Delta_{31} + \Delta_{41})^2 + (T^2)_{12} + (T^2)_{13} + (T^2)_{14}$$

$$(T^3)_{22} = A U_{e_2}^2 + A' U_{e_2}^2 - \frac{1}{2} (\Delta_{21} + \Delta_{32} + \Delta_{42} + A + A')((A^2 U_{e_2}^2 - \frac{1}{16}) + A^2 U_{e_2}^2 - \frac{1}{16})$$

$$-2A A' ((\frac{1}{16}) (\Delta_{21} + \Delta_{32} + \Delta_{42} + 1) - U_{e_2} U_{e_2} (U_{e_1} U_{e_1} + U_{e_3} U_{e_3} + U_{e_4} U_{e_4}))$$

$$-\frac{1}{16} (\Delta_{21} + \Delta_{32} + \Delta_{42})^2 + (T^2)_{21} + (T^2)_{23} + (T^2)_{24}$$

$$(T^3)_{33} = A U_{e_3}^2 + A' U_{e_3}^2 - \frac{1}{2} (\Delta_{31} + \Delta_{32} + \Delta_{41} + A + A')((A^2 U_{e_3}^2 - \frac{1}{16}) + A^2 U_{e_3}^2 - \frac{1}{16})$$

$$-2A A' ((\frac{1}{16}) (\Delta_{31} + \Delta_{32} + \Delta_{41} + 1) - U_{e_3} U_{e_3} (U_{e_1} U_{e_1} + U_{e_2} U_{e_2} + U_{e_4} U_{e_4}))$$

$$-\frac{1}{16} (\Delta_{31} + \Delta_{32} + \Delta_{41})^2 + (T^2)_{32} + (T^2)_{34} + (T^2)_{34}$$

$$(T^3)_{12} = (T^3)_{21} = A U_{e_2}^2 + A' U_{e_2}^2 - \frac{1}{2} (\Delta_{21} + \Delta_{32} + \Delta_{42}) - \frac{1}{2} (A + A') + (T^2)_{12} + (A U_{e_2} U_{e_2} + A' U_{e_2} U_{e_2} + T^2)_{11}$$

$$+ (A U_{e_2} U_{e_3} + A' U_{e_2} U_{e_3}) (T^2)_{13} + (A U_{e_2} U_{e_4} + A' U_{e_2} U_{e_4}) (T^2)_{14}$$

$$(T^3)_{13} = (T^3)_{31} = A U_{e_3}^2 + A' U_{e_3}^2 - \frac{1}{2} (\Delta_{31} + \Delta_{32} + \Delta_{41}) - \frac{1}{2} (A + A') + (T^2)_{13} + (A U_{e_3} U_{e_3} + A' U_{e_3} U_{e_3} + T^2)_{11}$$

$$+ (A U_{e_3} U_{e_2} + A' U_{e_3} U_{e_2}) (T^2)_{12} + (A U_{e_3} U_{e_4} + A' U_{e_3} U_{e_4}) (T^2)_{14}$$

$$(T^3)_{14} = (T^3)_{41} = A U_{e_4}^2 + A' U_{e_4}^2 - \frac{1}{2} (\Delta_{41} + \Delta_{42} + \Delta_{43}) - \frac{1}{2} (A + A') + (T^2)_{14} + (A U_{e_4} U_{e_4} + A' U_{e_4} U_{e_4} + T^2)_{11}$$

$$+ (A U_{e_4} U_{e_2} + A' U_{e_4} U_{e_2}) (T^2)_{12} + (A U_{e_4} U_{e_3} + A' U_{e_4} U_{e_3}) (T^2)_{13}$$

$$(T^3)_{23} = (T^3)_{32} = A U_{e_3}^2 + A' U_{e_3}^2 - \frac{1}{2} (\Delta_{32} + \Delta_{31} + \Delta_{43}) - \frac{1}{2} (A + A') + (A U_{e_3} U_{e_3} + A' U_{e_3} U_{e_3} + T^2)_{22}$$

$$+ (A U_{e_3} U_{e_1} + A' U_{e_3} U_{e_1}) (T^2)_{23} + (T^2)_{24}$$

$$(T^3)_{24} = (T^3)_{42} = A U_{e_2}^2 + A' U_{e_2}^2 - \frac{1}{2} (\Delta_{42} + \Delta_{43} + \Delta_{41}) - \frac{1}{2} (A + A') + (A U_{e_2} U_{e_2} + A' U_{e_2} U_{e_2} + T^2)_{22}$$

$$+ (A U_{e_2} U_{e_4} + A' U_{e_2} U_{e_4}) (T^2)_{23} + (A U_{e_2} U_{e_3} + A' U_{e_2} U_{e_3}) (T^2)_{24} + (T^2)_{24}$$

$$(T^3)_{34} = (T^3)_{43} = A U_{e_3}^2 + A' U_{e_3}^2 - \frac{1}{2} (\Delta_{42} + \Delta_{43} + \Delta_{41}) - \frac{1}{2} (A + A') + (A U_{e_3} U_{e_3} + A' U_{e_3} U_{e_3} + T^2)_{33}$$

$$+ (A U_{e_3} U_{e_4} + A' U_{e_3} U_{e_4}) (T^2)_{23} + (A U_{e_3} U_{e_1} + A' U_{e_3} U_{e_1}) (T^2)_{34} + (T^2)_{34}$$

$$\hat{T}_{\alpha\beta} = \sum_{i=1}^{4} U_{ai} \left( \sum_{j \neq i}^{4} U_{bj} (A U_{e_3} U_{e_j} + A' U_{e_j} U_{e_3}) + U_{bi} T_{\alpha} \right) ; \ \alpha, \beta = e, \mu, \tau, s$$

$$\hat{T}_{\alpha\beta}^2 = \sum_{\alpha, \beta} (\hat{T}_{\alpha\beta} \hat{T}_{\beta\alpha}) ; \ \alpha, \beta = e, \mu, \tau, s$$

$$\hat{T}_{\alpha\beta}^3 = \sum_{\alpha, \beta} (\hat{T}_{\alpha\beta} \hat{T}_{\beta\alpha}) ; \ \alpha, \beta = e, \mu, \tau, s$$
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