INTERSECTION OF $(1,1)$-CURRENTS AND THE DOMAIN OF DEFINITION OF THE MONGE-AMPÈRE OPERATOR

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ABSTRACT. We study the Monge-Ampère operator within the framework of Dinh-Sibony’s intersection theory defined via density currents. We show that if $u$ is a plurisubharmonic function belonging to the Blocki-Cegrell class, then the Dinh-Sibony $n$-fold self-product of $dd^c u$ exists and coincides with the classically defined Monge-Ampère measure $(dd^c u)^n$.

1. INTRODUCTION

Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $u$ be a plurisubharmonic (p.s.h. for short) function on $\Omega$. A question of central importance in pluripotential theory and its applications is whether one can define the Monge-Ampère measure $(dd^c u)^n$ in a meaningful way. Recall that $d^c := \frac{i}{2} (\partial - \bar{\partial})$ and $dd^c = \frac{i}{4} \partial \bar{\partial}$.

For bounded p.s.h. functions, the definition of $(dd^c u)^n$ and the study of its fundamental properties are due to Bedford-Taylor [BT76]. The problem of finding the largest class of p.s.h. functions where the Monge-Ampère operator is suitably defined and continuous under decreasing sequences was studied for a long time and a complete characterization of this class was finally achieved by Cegrell [Ceg04] and Blocki [Blo06]. We denote this class by $D(\Omega)$ and call it the Blocki-Cegrell class.

The question of defining $(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u$ is an instance of the fundamental problem of intersection of currents. Indeed, if we set $T := dd^c u$, then $T$ is a positive closed $(1,1)$-current on $\Omega$ and $(dd^c u)^n$ is the self-intersection $T^n = T \wedge \cdots \wedge T$. The intersection theory of currents has been quite well-developed thanks to the work of many authors. The case of bi-degree $(1,1)$-currents is more accessible due to the existence of p.s.h. functions as local potentials. For this reason, this case was soon developed, see [CLN69, BT76, FS95, Dem]. Later on, other notions of intersection were introduced, such as the non-pluripolar product [BT87, BEGZ10] (see also [Vu20-2] for recent developments) and the Andersson-Wulcan product of $(1,1)$-currents with analytic singularities [AW14]. All these generalized notions differ from classical ones by the fact that, in one way or another, one removes the singular set of the currents before intersecting them. As a drawback, there is a mass loss in this procedure.

A general intersection theory for currents of higher bi-degree was developed only later. Most notably, Dinh-Sibony proposed two different notions of intersection, one using what they call superpotentials [DS09] and, more recently, another one based on the notion of density currents, that we consider here. We refer to the original paper [DS18] and also [Vu19] for generalizations and simplified arguments. Both approaches have already found many applications in dynamical systems and foliation theory.

The main goal of the present paper is to study the Monge-Ampère operator from the point of view of theory of density currents. We now briefly recall this notion. More details are given in Section 2.

Let $X$ be a complex manifold and let $T_1, \ldots, T_m$ be positive closed currents on $X$. Consider the Cartesian product $X^m$ and the positive closed current $T = T_1 \otimes \cdots \otimes T_m$ on $X^m$. Let $\Delta = \{(x, \ldots, x) : x \in X\} \subset X^m$ be the diagonal and $N \Delta$ be its normal bundle inside $X^m$. Using a certain type of
local coordinates $\tau$ in $X^m$ around $\Delta$ with values in $N\Delta$, which are called admissible maps, we can consider the current $\pi_*T$ defined around the zero section of $N\Delta$.

For $\lambda \in \mathbb{C}^*$, let $A_\lambda : N\Delta \to N\Delta$ be the fiberwise multiplication by $\lambda$. A density current $R$ associated with $(T_1,\ldots,T_m)$ is a positive closed current on $N\Delta$ such that there exists a sequence of complex numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ converging to $\infty$ for which $R = \lim_{k \to \infty} (A_{\lambda_k})_* \pi_* T$, for every admissible map $\tau$. We then say that the Dinh-Sibony product $T_1 \wedge \cdots \wedge T_m$ of $T_1,\ldots,T_m$ exists if there is only one density current $R$ associated with $(T_1,\ldots,T_m)$ and $R = \pi_* S$ for some positive closed current on $\Delta$, where $\pi : N\Delta \to \Delta$ is the canonical projection. In that case we define

$$T_1 \wedge \cdots \wedge T_m := S.$$

Our main result is the following, see Theorem 4.5 below.

**Main Theorem.** Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $u_1, u_2, \ldots, u_m$, $1 \leq m \leq n$ be plurisubharmonic functions in the Blocki-Cegrell class. Then, the Dinh-Sibony product of $\mathfrak{dd}^c u_1, \ldots, \mathfrak{dd}^c u_m$ is well-defined and

$$\mathfrak{dd}^c u_1 \wedge \cdots \wedge \mathfrak{dd}^c u_m = \mathfrak{dd}^c u_1 \wedge \cdots \wedge \mathfrak{dd}^c u_m.$$\n
In particular, for every $u$ in the Blocki-Cegrell class, the operator $u \mapsto (\mathfrak{dd}^c u)^{\wedge n} := \mathfrak{dd}^c u \wedge \cdots \wedge \mathfrak{dd}^c u$ is well-defined and coincides with the usual Monge-Ampère operator.

The last hand side of (1.1) is obtained in the standard way, i.e., by considering sequences of smooth p.s.h. functions decreasing to $u_j$, $j = 1,\ldots,m$, similarly to the case where $m = n$ and all the $u_j$ are equal. The fact that this mixed product is well-defined and independent of the chosen sequence is the content of Proposition 4.2 below. Although not explicitly stated in the literature, this fact might be well-known among experts. The case $m = n$ can be found in [Ceg04] and the case $m < n$ covered by Proposition 4.2 follows from simple modifications of arguments from [Ceg04] and [Blo06].

Our main theorem is a strengthening of Theorem 1.1 in [KV19] proven by the last two authors and it follows from a more general result, see Theorem 3.1. This yields an optimal result that covers the most general case where the classical Monge-Ampère operator is well-defined and continuous with respect to decreasing sequences.

The proof of Theorem 3.1 in the present work uses different techniques than the ones in [KV19], providing new and more clear arguments. In particular, the integrability assumption on the $u_j$’s required in [KV19] cannot be dropped with the techniques used there, so they cannot be applied in our situation. Here, we bypass this difficulty and show that these assumptions are actually unnecessary, yielding the optimal result. In order to achieve that, we obtain almost everywhere vanishing properties for Lelong numbers with respect to singular currents in the considered class (cf. Lemma 3.2). Also, by systematically working with a well-chosen set of test forms (cf. Definition 3.3), we can easily get the compactness of the family of dilated currents (Lemma 3.8 below), which was overlooked in [KV19]. We refer to the end of this introduction and the comments after Theorem 3.1 for an overview of the arguments and the new ingredients of the proof.

It is worth mentioning that the many notions of intersection quoted here are related to one another. As mentioned before, the present paper together with [KV19] show that the intersection via density currents cover all known classical products of $(1,1)$-currents. For higher bi-degree currents it is also known that the density product generalizes the product of currents with continuous super-potentials, see [DNV18]. Concerning generalized notions of products of $(1,1)$-currents, such as the non-pluripolar product and the Andersson-Wulcan product, some comparison results were obtained in [KV19]. The general phenomenon is that, in some sense, these products are
dominated by the corresponding density currents. Moreover, the ideas of the present paper were further developed in [Vu20] to prove that the Dinh-Sibony product of $(1,1)$-currents of full mass intersection in Kähler classes on a compact Kähler manifolds exists and is equal to their (relative) non-pluripolar product.

To end this introduction, let us outline the structure and the key points of the proof of our main theorem. Our main result will follow from a more general theorem showing that if a mixed product is well-defined in the sense that it is obtained via decreasing sequences of smooth p.s.h. functions, then the corresponding Dinh-Sibony product exists and coincides with the classical one. This is the content of Theorem 3.1 below. Its proof is obtained by induction on the number of currents involved and uses the following key facts: the vanishing of Lelong numbers with respect to the currents obtained in previous steps (Lemma 3.2), the interpretation of Lelong numbers as the mass of dilated currents (Lemma 3.4) and the observation that the dilation procedure in the definition of density currents yields canonical regularizations via decreasing sequences of p.s.h. functions (cf. the proof of Lemma 3.7). A more detailed outline is given below, after the statement of Theorem 3.1. Finally, in Theorem 4.5, we prove our main theorem by verifying that functions in the Błocki-Cegrell class satisfy the assumptions of Theorem 3.1.

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2. Preliminaries on density currents

In this section, we recall the definition and basic properties of tangent and density currents. For details, the reader is refered to the original paper [DS18] and to [KV19], [Vu19], [DNV18] for more material.

Let $X$ be a complex manifold of dimension $n$ and $V$ be a smooth complex submanifold of $X$ of dimension $\ell$. Let $T$ be a positive closed $(p,p)$-current on $X$ with $0 \leq p \leq n$.

Let $NV$ be the normal bundle of $V$ in $X$ and denote by $\pi : NV \to V$ the canonical projection. We identify $V$ with the zero section of $NV$. Let $U$ be an open subset of $X$ with $U \cap V \neq \emptyset$. An admissible map on $U$ is a smooth diffeomorphism $\tau$ from $U$ to an open neighbourhood of $V \cap U$ in $NV$ such that $\tau$ is the identity map on $V \cap U$ and the restriction of its differential $d\tau$ to $NV|_{V \cap U}$ is the identity. Using a Hermitian metric on $X$, we can always find an admissible map defined on a small tubular neighbourhood of $V$, see [DS18] Lemma 4.2. This map is not holomorphic in general. However, if one only works on a small open set of $X$, it is easy to obtain holomorphic admissible maps.

For $\lambda \in \mathbb{C}^*$, let $A_\lambda : NV \to NV$ be the multiplication by $\lambda$ along the fibers of $NV$. Consider the family of currents $(A_\lambda)_* \tau_* T$ on $NV|_{V \cap U}$ parametrized by $\lambda \in \mathbb{C}^*$. Following [DS18] [KV19] [Vu19], we have:

Definition 2.1. A tangent current of $T$ along $V$ is a positive closed current $R$ on $NV$ such that there exist a sequence $(\lambda_k)_{k \geq 1}$ in $\mathbb{C}^*$ converging to $\infty$ and a collection of holomorphic admissible maps $\tau_j : U_j \to NV$, $j \in J$ whose domains cover $V$ such that

$$R = \lim_{k \to \infty} (A_{\lambda_k})_* (\tau_j)_* T$$

on $\pi^{-1}(U_j \cap V)$ for every $j \in J$. 


Before continuing, let us make some comments on Definition 2.1. In [DS18], the authors considered the situation where \( X \) is Kähler and \( \text{supp} T \cap V \) is compact. There, a tangent current to \( T \) along \( V \) is defined as a limit current of the family \((A_\lambda), \tau_\ast T\) as \( |\lambda| \to \infty \), where \( \tau \) is a global admissible map, defined on an open tubular neighborhood of \( V \); see [DS18, Definition 4.5]. Then, they proceed to show, cf. [DS18, Proposition 4.4], that tangent currents are independent of the choice of global admissible maps and can be localized in the following sense: if \( \lambda_k \to \infty \) and \( \lambda_k \neq 0 \) for all \( k \), then there is a sequence \((\pi_k)\) of the choice of global admissible maps and can be localized in the following sense: if \( \lambda_k \to \infty \), then \( \pi_k \) converges to some current \( R \) on \( \pi^{-1}(U \cap V) \). Therefore, our definition of tangent currents is equivalent to that of [DS18] when \( \text{supp} T \cap V \) is compact and \( X \) is Kähler. Note also that, in this situation, it is shown in [DS18] that tangent currents always exist.

However, in the cases we consider here \( \text{supp} T \cap V \) is not necessarily compact. Therefore, it is unclear whether we can use the original definition of [DS18]. This is because using only global admissible maps it is hard to ensure that the family \((A_\lambda), \tau_\ast T\) is compact and that the limit currents are independent of \( \tau \). That is why we adopt a more flexible definition using local holomorphic admissible maps.

As in the compact setting, tangent currents depend in general on the sequence \((\lambda_k)_{k \geq 1}\). The existence of tangent currents in the local setting is a more delicate matter and we have to prove it in our particular situation. However, if such currents exist, they are still independent of the choice of admissible maps.

Lemma 2.2. [KV19, Proposition 2.5] Let \( \tau : U \to NV \) be a holomorphic admissible map. Assume that there is a sequence \((\lambda_k)_{k \geq 1}\) tending to \( \infty \) such that \((A_\lambda)\_\ast, \tau_\ast T\) converges to some current \( R \) on \( \pi^{-1}(U \cap V) \). Then, for any other admissible map \( \tau' : U' \to NV \), we have

\[
R = \lim_{k \to \infty} (A_\lambda)_\ast \tau'_\ast T
\]

on \( \pi^{-1}(U \cap U' \cap V) \).

A density current is a particular type of tangent current where \( V \) is the diagonal inside a product space. More precisely, let \( m \geq 1 \) and let \( T_j \) be positive closed \((p_j, p_j)\)-currents for \( 1 \leq j \leq m \) on \( X \). We usually assume that \( p = p_1 + \cdots + p_m \leq n \). Let \( T = T_1 \otimes \cdots \otimes T_m \) be their tensor product. Then \( T \) is a positive closed \((p, p)\)-current on \( X^m \). Let \( \Delta = \{(x, \ldots, x) : x \in X\} \subset X^m \) be the diagonal. A density current associated with \( T_1, \ldots, T_m \) is a tangent current of \( T \) along \( \Delta \). By definition, a tangent current is a positive closed \((p, p)\)-current on the normal bundle \( N\Delta \) of \( \Delta \) inside \( X^m \).

Let \( \pi : N \Delta \to \Delta \) be the canonical projection. The following definition is given in [DS18].

Definition 2.3. We say that the Dinh-Sibony product \( T_1 \wedge \cdots \wedge T_m \) of \( T_1, \ldots, T_m \) exists if there is a unique density current \( R \) associated with \( T_1, \ldots, T_m \) and \( R = \pi^\ast S \) for some current \( S \) on \( \Delta = X \). In this case we define

\[
T_1 \wedge \cdots \wedge T_m := S.
\]
\( \{1, \ldots, m-1\} \), there is a current \( R_J \) on \( \Omega \) so that, for any open set \( U \subset \Omega \), for \( j \in J \) and any sequence of smooth p.s.h. functions \( (u^\ell_j)_{\ell \in \mathbb{N}} \) decreasing to \( u_j \) on \( U \) as \( \ell \to \infty \), one has

\[
(3.1) \quad \text{dd}^c u^\ell_{j_1} \wedge \cdots \wedge \text{dd}^c u^\ell_{j_k} \wedge T \to R_J \quad \text{on } U \quad \text{as } \ell \to \infty.
\]

We then define \( \text{dd}^c u^\ell_{j_1} \wedge \cdots \wedge \text{dd}^c u^\ell_{j_k} \wedge T \) as the current \( R_J \). If \( J = \emptyset \), we set \( R_J := T \).

Then, the Dinh-Sibony product of \( \text{dd}^c u_1, \ldots, \text{dd}^c u_{m-1}, T \) is well-defined and one has

\[
(3.2) \quad \text{dd}^c u_1 \wedge \cdots \wedge \text{dd}^c u_{m-1} \wedge T = \text{dd}^c u_1 \wedge \cdots \wedge \text{dd}^c u_{m-1} \wedge T.
\]

Before stating the preparatory results, let us briefly outline the structure of the proof. From the definition of density product we have to show that \( R_{1,\lambda} \to \pi^* R_1 \) as \( \lambda \to \infty \), where \( R_{1,\lambda} \) is the dilation of the tensor product of \( T \) and \( \text{dd}^c u_j, j = \ldots, m-1 \) along the diagonal. We will argue by induction on \( m \). After fixing a suitable coordinate system on \( (\mathbb{C}^n)^m = \mathbb{C}^n \times \cdots \times \mathbb{C}^n \) it is enough to work with two types of test forms \( \Phi \). For each type, the estimates are of a different nature.

- \textit{forms of type I}: \( \Phi = \phi_1(y^1) \wedge \phi_2(y^2) \wedge \cdots \wedge \phi_m(y^m) \), where \( \phi_j \) are positive \( (p_j, p_j) \)-forms on \( (\mathbb{C}^n, y^j) \) and at least one among \( \phi_1, \ldots, \phi_m \) is not of top degree. In this case we have that \( \langle R_{1,\lambda}, \Phi \rangle \to 0 \) as \( \lambda \to \infty \), see Lemma \[3.5\]. Here we use Lemma \[3.2\] saying that the Lelong numbers of \( u_j \) are negligible with respect to the currents obtained in previous steps, and a characterization of Lelong numbers in terms of dilated currents (Lemma \[3.4\]).

- \textit{forms of type II}: \( \Phi = \phi_1(y^1) \wedge \phi_2(y^2) \wedge \cdots \wedge \phi_m(y^m) \), where \( \phi_j \) is a radial \( (n, n) \)-form for every \( j = 1, \ldots, m-1 \). In this case \( \langle R_{1,\lambda}, \Phi \rangle \to \langle R_1, \pi_* \Phi \rangle \) as \( \lambda \to \infty \), see Lemma \[3.7\]. Here we use that the sequence of dilations yields canonical regularizations via decreasing sequences of p.s.h. functions.

Working with forms of type I allows us to prove that the limit currents have minimal horizontal dimension and, therefore, are the pullback of some current \( R^h_{1,\infty} \) in the diagonal, cf. Lemma \[3.11\]. On the other hand, working with forms of type II let us recognize the current \( R^h_{1,\infty} \) as being \( R_1 \).

We now present the auxiliary lemmas we’ll need. The proof of Theorem \[3.1\] is given in the end of this section.

**Lemma 3.2.** Let the notations and the hypothesis be as in Theorem \[3.1\]. Then for every \( J \subset \{1, \ldots, m-1\} \) and every \( 1 \leq k \leq m-1 \) such that \( k \notin J \) we have that \( \nu(u_k, \cdot) = 0 \) almost everywhere with respect to the trace measure of \( R_J \).

**Proof.** We work locally. Let \( J \subset \{1, \ldots, m-1\} \). Let \((u^\ell_j)_{\ell \in \mathbb{N}}\) be a sequence of smooth p.s.h. functions decreasing to \( u_j \) as \( \ell \to \infty \) for \( j \in J \). Let \( k \in \{1, \ldots, m-1\} \setminus J \). Let \((u^\ell_k)_{\ell \in \mathbb{N}}\) be a sequence of locally bounded p.s.h. functions decreasing to \( u_k \). We claim that

\[
(3.3) \quad \text{dd}^c u^\ell_k \wedge R_J \to R_{J \cup \{k\}} \quad \text{as } \ell \to \infty.
\]

Consider first the case where \( u^\ell_k \) is smooth. Let \( \Phi \) be a test form with compact support and \( \epsilon > 0 \) a constant. Using \( (3.1) \) and the fact that \( u^\ell_k \) is smooth we can find, for each \( \ell \geq 1 \) an index \( s^\ell_k \) satisfying

\[
(3.4) \quad |\langle \text{dd}^c u^\ell_k \wedge R_J - \text{dd}^c u^\ell_k \wedge \bigwedge_{j \in J} \text{dd}^c u^{s^\ell_j} \wedge T, \Phi \rangle| = |(R_J - \bigwedge_{j \in J} \text{dd}^c u^{s^\ell_j} \wedge T, \text{dd}^c u^\ell_k \wedge \Phi)| \leq \epsilon.
\]

We can choose \((s^\ell_k)_{\ell} \) so that it increases to \( \infty \) as \( \ell \to \infty \). Hence, \( u^{s^\ell_j} \) decreases to \( u_j \) for \( j \in J \) and by hypothesis, one obtains

\[
\text{dd}^c u^\ell_k \wedge \bigwedge_{j \in J} \text{dd}^c u^{s^\ell_j} \wedge T \to R_{J \cup \{k\}}
\]
as \( s \to \infty \). It follows that
\[
|\langle dd^c u_k^\ell \wedge \bigwedge_{j \in J} dd^c u_j^{\delta_\ell} \wedge T - R_{J \cup \{k\}} \rangle , \Phi \rangle | \leq \epsilon
\]
for \( \ell \) is big enough. Combining this with \((3.7)\) gives
\[
|\langle dd^c u_k^\ell \wedge R_J - R_{J \cup \{k\}} \rangle , \Phi \rangle | \leq 2\epsilon
\]
for \( \ell \) big enough. Therefore, \((3.3)\) follows if \( u_k^\ell \) is smooth.

The case of general \( u_k^\ell \) follows from a regularization argument. Let \( u_k^{\ell,\delta} \) be a standard smooth regularization of \( u_k^\ell \) obtaining from convolution against a smoothing kernel, so that \( u_k^{\ell,\delta} \) decreases to \( u_k^\ell \) as \( \delta \to 0 \). For each \( \ell \), let \( \delta_\ell \) be small enough such that
\[
(3.5) \quad |\langle dd^c u_k^\ell \wedge R_J - dd^c u_k^{\ell,\delta} \wedge R_J \rangle , \Phi \rangle | \leq \epsilon.
\]
We can choose \( \delta_\ell \) to be decreasing in \( \ell \). Hence, \( u_k^{\ell,\delta} \) are smooth p.s.h. functions decreasing to \( u_k \) as \( \ell \to \infty \). By the first part of the proof, we see that \( dd^c u_k^{\ell,\delta} \wedge R_J \) converges to \( R_{J \cup \{k\}} \). This combined with \((3.5)\) gives
\[
|\langle dd^c u_k^\ell \wedge R_J - R_{J \cup \{k\}} \rangle , \Phi \rangle | \leq 2\epsilon
\]
for \( \ell \) big enough. Hence, \((3.3)\) follows.

Recall that our goal is to prove that \( R_J \) has no mass on \( \{ \nu(u_k, \cdot) > 0 \} \). Let \( w(x) = \|x\|^2 \), where \( x \) is the standard coordinate system on \( \mathbb{C}^n \). Let \( N \) be a large constant and set
\[
u(u_k, \cdot) > 0 \}
\[
(3.3) \quad \|u_k^\ell - Nw - \log \ell \| \leq \epsilon.
\]
Then, the \( u_k^\ell \) are locally bounded p.s.h. functions that decrease to \( u \) as \( \ell \to \infty \). Suppose that \( R_J \) has positive mass on \( V := \{ \nu(u_k, \cdot) > 0 \} \), that is \( 1_V R_J \neq 0 \). Notice that \( u_k = -\infty \) on \( V \) implies that \( u_k^\ell := Nw - \log \ell \) on \( V \). It follows that
\[
|dd^c u_k^\ell \wedge R_J - dd^c u_k^\ell \wedge (1_V R_J) - dd^c (u_k^\ell 1_V R_J) = Ndd^c w \wedge (1_V R_J).
\]
Let \( K \) be a fixed compact set that is charged by \( 1_V R_J \). Then, the mass of \( Ndd^c w \wedge (1_V R_J) \) over \( K \) equals \( cN \) for some constant \( c > 0 \) independent of \( \ell \). By the above inequality and \((3.3)\) one gets that the mass of \( R_{J \cup \{k\}} \) on \( K \) is \( \geq cN \). Choosing \( N \) large enough gives a contradiction. This finishes the proof. \( \square \)

Let \((x^1, \ldots, x^m) \) be the canonical coordinate system in \( \Omega^m \) and \( \Delta \) be the diagonal of \( \Omega^m \). Put \( y^j := x^j - x^m \) for \( 1 \leq j \leq m-1 \) and \( y^m := x^m \). Then, \((y^1, \ldots, y^{m-1}, y^m) \) forms a new coordinate system on \( \Omega^m \) and \( \Delta = \{ y^j = 0 : 1 \leq j \leq m-1 \} \) is identified with \( \Omega \). Using these coordinates, we identify naturally the normal bundle of \( \Delta \) with the trivial bundle \( \pi : (\mathbb{C}^n)^{m-1} \times \Omega \to \Omega \). Observe that the change of coordinates \( \varrho : \Omega^m \to (\mathbb{C}^n)^{m-1} \times \Omega \) given by
\[
\varrho(x^1, \ldots, x^m) = (x^1 - x^m, \ldots, x^{m-1} - x^m, x^m) := (y^1, \ldots, y^m):= y
\]
is a holomorphic admissible map. By Lemma \ref{lem:holomorphic_map} it will be enough to work only with \( \varrho \).

For \( 1 \leq j \leq m-1 \), let \( T_j := dd^c u_j, \tilde{T} := \pi^* T \) and
\[
\tilde{u}_j(y^1, y^m) := \varrho_* u_j(y^1, y^m) = u_j(y^1 + y^m).
\]
We can check that \( \tilde{u}_j \) is locally integrable with respect to \( dd^c \tilde{u}_{j+1} \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T} \) for \( j = m-1, \ldots, 1 \) and for every sequence \((u_j^\ell)_{\ell \in \mathbb{N}} \) of smooth p.s.h. functions decreasing to \( u_j \) and \( \tilde{u}_j^\ell := \varrho_* u_j^\ell \), we have
\[
(3.7) \quad dd^c \tilde{u}_1^\ell \wedge \cdots \wedge dd^c \tilde{u}_{m-1}^\ell \wedge \tilde{T} \to dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T}
\]
as \( \ell \to \infty \). For the meaning of the right-hand side, see Definition \ref{def:interior} below. The above assertions follow from a reasoning similar to the one in \cite[Lemma 2.3]{KV19}. Consequently, we get
\[
(3.8) \quad \varrho_*(T_1 \otimes \cdots \otimes T_{m-1} \otimes T) = dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T}.
\]
We will do that by testing the center at the origin in \(C^3\) and for positive split form \(a\) this is clear when the currents. We denote by \(\beta\) the standard volume form on \(C^3\).

Definition 3.3. Let \(\Phi\) be a differential form on \((\mathbb{C}^n, y^1) \times (\mathbb{C}^n, y^2) \times \cdots \times (\mathbb{C}^n, y^m)\). We say that \(\Phi\) is a positive split form if it can be written as

\[
\Phi = \phi_1(y^1) \wedge \phi_2(y^2) \wedge \cdots \wedge \phi_m(y^m),
\]

where \(\phi_j\) are positive \((p_j, p_j)\)-forms on \((\mathbb{C}^n, y^j)\).

An \((n, n)\)-form \(\phi_j\) on \((\mathbb{C}^n, y^j)\) is radial if it’s rotation invariant, namely, if it is of the form

\[
\phi_j(y^j) = \chi(\|y^j\|^2) \cdot i^n dy^j \wedge d\bar{y}^j
\]

for some smooth function \(\chi\).

In the sequel we will need the following expression of the Lelong number in terms of tangent currents. We denote by \(\beta\) the standard Kähler form of \(\mathbb{C}^n\) and by \(B_\rho(0)\) the open ball of radius \(\rho\) center at the origin in \(\mathbb{C}^n\).

Lemma 3.4. Let \(S\) be a positive closed \((p, p)\)-current defined near the origin in \(\mathbb{C}^n\). Let \(A_\lambda(z) = \lambda z\) and set \(S_\lambda := (A_\lambda)_* S\). Let \(\lambda_k\) be an increasing sequence tending to \(\infty\) such that \(S_\lambda\) converges to \(S_\infty\). Let \(\sigma_{S_\infty} = S_\infty \wedge \frac{1}{(n-p)} S^{n-p}\) be the trace measure of \(S_\infty\) and \(\nu(S; 0)\) be the Lelong number of \(S\) at the origin. Then, there is a constant \(c_\rho > 0\) depending only on \(p\) such that \(\nu(S; 0) = \lim_{\lambda_k \to \infty} c_\rho \sigma_{S_\lambda}(B_1(0)) = c_\rho \sigma_{S_\infty}(B_1(0))\) and the limit is decreasing.

Proof. To simplify the notation we may assume that the limit of \((A_\lambda)_* S\) as \(\lambda\) tends to infinity exists and is equal to \(S_\infty\). Set \(c_\rho := \frac{(n-p)p}{\pi^{n-p}}.\) Then, by the definition of Lelong number \([\text{Dem}]\) we have

\[
\nu(S; 0) = \lim_{\rho \to 0} \frac{1}{\pi^{n-p} r^{2n-2p}} \int_{B_\rho(0)} S \wedge \beta^{n-p} = \lim_{|\lambda| \to \infty} \frac{1}{\pi^{n-p} |\lambda|^{2n-2p}} \int_{B_1(0)} S \wedge \beta^{n-p} = \lim_{|\lambda| \to \infty} \frac{1}{\pi^{n-p}} \int_{B_1(0)} (A_\lambda)_* S \wedge \beta^{n-p} = \lim_{|\lambda| \to \infty} \frac{1}{\pi^{n-p}} \int_{B_1(0)} (A_\lambda)_* S \wedge \beta^{n-p} \geq \frac{1}{\pi^{n-p}} \int_{B_1(0)} S_\infty \wedge \beta^{n-p} = c_\rho \sigma_{S_\infty}(B_1(0)).
\]
In the second equality we have used that \( A_j^* \beta = |\lambda|^2 \beta \) and in the last inequality we have used the fact that if a sequence of measures \( m_\lambda \) converges to \( m \) and \( U \) is open, then \( \liminf_\lambda m_\lambda(U) \geq m(U) \). Hence

\[
\nu(S; 0) \geq c_p \sigma_S\mathcal{m}(B_1(0)).
\]

Repeating the above argument on closed balls and using the fact that if a sequence of measures \( m_\lambda \) converges to \( m \) then \( \limsup_\lambda m_\lambda(K) \leq m(K) \) for all closed sets \( K \), we get that \( \nu(S; 0) \leq \limsup_\lambda \sigma_S\mathcal{m}(B_1(0)) \). Now, the current \( S_\infty \) is invariant by \( (A_t)_* \) for every \( t \in \mathbb{C}^* \) (see [DS18]), hence its mass is homogeneous, namely \( \sigma_S\mathcal{m}(B_p(0)) = \rho^{2n-2p}\sigma_S\mathcal{m}(B_1(0)) \) for every \( p > 0 \) and similarly for the closed ball. For \( 0 < \rho < 1 \) this gives

\[
\nu(S; 0) \leq c_p \sigma_S\mathcal{m}(B_1(0)) = c_p \rho^{2p-2n}\sigma_S\mathcal{m}(B_p(0)) \leq c_p \rho^{2p-2n}\sigma_S\mathcal{m}(B_1(0)).
\]

Letting \( \rho \to 1 \) gives \( \nu(S; 0) \leq c_p \sigma_S\mathcal{m}(B_1(0)) \). Together with (3.11), this gives the desired result.

It is a standard fact that all the above limits are decreasing as \( r \to 0 \), or equivalently as \( \lambda \to \infty \), see [Dem, III.5].

Theorem 3.1 will be proved by induction on \( m \). The induction step will make use of next lemma. Let \( u_1, \ldots, u_{m-1} \) and \( T \) be as in Theorem 3.1. For \( J \subset \{1, \ldots, m-1\} \), let \( R_{J, \lambda} \) be the current defined in (3.9) and \( R_J = \wedge_{j \in J} dd^c u_j \wedge T \), defined as in (i) of Theorem 3.1.

**Lemma 3.5.** With the above notation and the hypothesis of Theorem 3.1 assume that \( R_{J, \lambda} \to \pi^* R_J \) as \( \lambda \to \infty \) for every \( J \subset \{1, \ldots, m-1\} \) such that \( |J| \leq m - 2 \).

Let \( \Phi \) be a positive split test form with compact support on \((\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)\). Assume that \( \Phi \) is not of bi-degree \((n, n)\) on \( y^k \) for some \( 1 \leq k \leq m - 1 \). Then \( \langle R_{1, \lambda}, \Phi \rangle \to 0 \) as \( \lambda \to \infty \).

**Proof.** By assumption \( \Phi = \phi_1(y^1) \wedge \cdots \wedge \phi_{m-1}(y^{m-1}) \wedge \phi_m(y^m) \), where each \( \phi_j \) is positive and compactly supported. Observe that we only need to consider forms \( \Phi \) such that \( R_{1, \lambda} \wedge \Phi \) has full degree, otherwise the last product vanishes and the result is trivial.

Notice that the current \( R_{1, \lambda} \) has only terms of degree 0, 1 or 2 on each \( y^j \), \( j = 1, \ldots, m - 1 \). Therefore, it suffices to consider the case where \( \phi_j \) has bi-degree \((n - 1, n - 1)\) or \((n, n)\) for every \( j = 1, \ldots, m - 1 \). Set

\[
J = \{ j \in \{1, \ldots, m-1\} : \phi_j \text{ has bi-degree } (n, n) \}
\]

and

\[
K = \{ k \in \{1, \ldots, m-1\} : \phi_k \text{ has bi-degree } (n-1, n-1) \}.
\]

It follows from the assumption on \( \Phi \) that \( K \) is non-empty and \( |J| \leq m - 2 \). Hence, by hypothesis

\[
\lim_{|J| \to \infty} R_{J, \lambda} = \pi^* R_J.
\]

Set \( \phi_J = \wedge_{j \in J} \phi_j \) and \( \phi_K = \wedge_{k \in K} \phi_k \). Since \( J \cup K = \{1, \ldots, m-1\} \), we have that

\[
\Phi = \phi_J \wedge \phi_K \wedge \phi_m.
\]

It follows from (3.10) that

\[
R_{1, \lambda} \wedge \Phi = \bigwedge_{k \in K} \left( dd^c u_k(\lambda^{-1} y^k + y^m) \right) \wedge \phi_K \wedge R_{J, \lambda} \wedge \phi_J \wedge \phi_m.
\]

Since \( R_{1, \lambda} \wedge \Phi \) is a current of top degree in \((\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)\), it must have bi-degree \((n, n)\) on each \( y^j \) (otherwise \( R_{1, \lambda} \wedge \Phi = 0 \) and the lemma is trivial). Hence, for \( j \in J \) only the derivatives of \( u_j \) with respect to \( y^m \) will contribute, while for \( k \in K \), only the derivatives of \( u_k \) with respect to \( y^k \) will contribute. This gives

\[
R_{1, \lambda} \wedge \Phi = \bigwedge_{k \in K} \left( dd^c y^k u_k(\lambda^{-1} y^k + y^m) \right) \wedge \phi_K \wedge R_{J, \lambda} \wedge \phi_J \wedge \phi_m.
\]
Here, the symbol $dd^c_y u$ means that we only consider the (weak) derivatives with respect to the $y^k$ variables. The fact that the above wedge product is well-defined is obvious when the $u_j$ are smooth. This is less obvious for non-smooth functions, but it can be justified as in [KV19 Lemma 2.3]. Now, for fixed $y^m$ and $k \in K$ we have that

$$\left| \int_{y^k} dd^c_y u_k(\lambda^{-1} y^k + y^m) \wedge \phi_k(y^k) \right| \leq c_k \int_{B_k} dd^c_y u_k(\lambda^{-1} y^k + y^m) \wedge \beta^{n-1}(y^k),$$

where $c_k > 0$ is a constant independent of $y^m$, $\beta$ is the standard Kähler form on $(\mathbb{C}^n, y^k)$ and $B_k$ is a ball in $(\mathbb{C}^n, y^k)$ containing the support of $\phi_k$. By Lemma 3.4, the integral on the right-hand side of the above inequality decreases to a constant independent of $y^m$ times the Lelong number of the $(1, 1)$-current $dd^c_y u_k(y^k + y^m)$ at $y^k = 0$, which is equal to $\nu(u_k, y^m)$. Here we use that the Lelong number of a positive closed $(1, 1)$-current coincides with the Lelong number of any of its local potential (cf. [Dem III.8.9]). Hence, for every $y^m$ one has

$$\limsup_{|\lambda| \to \infty} \left| \int_{y^k} dd^c_y u_k(\lambda^{-1} y^k + y^m) \wedge \phi_k(y^k) \right| \leq \nu(u_k, y^m).$$

Combining this with (3.13), the hypothesis that $R_{I, \lambda} \to \pi^* R_J$ as $\lambda \to \infty$ and Lemma 3.6 below, one obtains

$$\limsup_{|\lambda| \to \infty} |(R_{1, \lambda} \wedge \Phi)| \leq \int y^m \left( \prod_{k \in K} \nu(u_k, y^m) \right) R_J \wedge \phi_m.$$  

The last integral in the above inequality vanishes because, by Lemma 3.2, $\nu(u_k, \cdot) = 0$ almost everywhere with respect to $R_J$. Therefore

$$\limsup_{|\lambda| \to \infty} |(R_{1, \lambda} \wedge \Phi)| = 0,$$

concluding the proof of the Lemma. \qed

We have used the following well known result.

**Lemma 3.6.** Let $X$ be a locally compact Hausdorff space. Let $m_{\lambda}$ be a sequence of Radon measures on $X$ whose supports are contained in a fixed compact subset of $X$. Assume that $m_{\lambda} \to m$ as $\lambda \to \infty$. Then for any sequence $(f_{\lambda})_{\lambda}$ of continuous functions decreasing pointwise to a function $f$, we have that

$$\limsup_{\lambda \to \infty} \int_X f_{\lambda} \, dm_{\lambda} \leq \int_X f \, dm.$$

**Lemma 3.7.** Under the assumptions of Theorem 3.1 let $\Phi = \phi_1(y^1) \wedge \cdots \wedge \phi_{m-1}(y^{m-1}) \wedge \phi_m(y^m)$ be a positive split test form with compact support on $(\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)$. Assume that $\phi_j$ is a radial $(n, n)$-form for every $j = 1, \ldots, m - 1$. Then

$$(R_{1, \lambda}, \Phi) \to (R_1, \pi, \Phi) \quad as \ \lambda \to \infty.$$  

Proof. After multiplying $\Phi$ by a positive constant, we can assume that $\int_{(\mathbb{C}^n, y^j)} \phi_j = 1$ for every $j = 1, \ldots, m - 1$. Notice that $\pi_\ast \Phi = \phi_m$ and $\phi_m$ has bidegree $(n - m - p + 1, n - m - p + 1)$.

For $j = 1, \ldots, m - 1$, define

$$u_j^\lambda(y^m) := \int_{(\mathbb{C}^n, y^j)} u_j(\lambda^{-1} y^1 + y^m) \phi_j(y^j).$$

Observe that $u_j^\lambda$ is a convolution against a (radially symmetric) smoothing kernel on a disc of radius $|\lambda|^{-1}$ centered at $y^m$. Hence $u_j^\lambda$ is a smooth p.s.h. function on $C^n, y^m$) decreasing pointwise to $u_j(y^m)$ as $\lambda \to \infty$ (see [Dem I.4.18]). By (3.1) we get that

$$dd^c u_1^\lambda \wedge \cdots \wedge dd^c u_{m-1}^\lambda \wedge T \xrightarrow{\lambda \to \infty} R_1.$$
Recall from (3.10) that
\[ R_{1,\lambda} = dd^c u_1 (\lambda^{-1} y^1 + y^m) \wedge \ldots \wedge dd^c u_{m-1} (\lambda^{-1} y^{m-1} + y^m) \wedge T(y^m). \]

Using the fact that the bidegree of each \( \phi_j, j = 1, \ldots, m - 1 \) is maximal, one has
\[ dd^c u_j (\lambda^{-1} y^j + y^m) \wedge \phi_j = dd^c_{y^m} u_j (\lambda^{-1} y^j + y^m) \wedge \phi_j \quad j = 1, \ldots, m - 1. \]

Hence,
\[ R_{1,\lambda} \wedge \Phi = \]
\[ dd^c_{y^m} u_1 (\lambda^{-1} y^1 + y^m) \wedge \phi_1 (y^1) \wedge \ldots \wedge dd^c_{y^m} u_{m-1} (\lambda^{-1} y^{m-1} + y^m) \wedge \phi_{m-1} (y^{m-1}) \wedge T(y^m) \wedge \phi_m (y^m). \]

Taking the integral of both sides of the above equality and using Fubini’s Theorem, one obtains
\[
\langle R_{1,\lambda}, \Phi \rangle = \int_{(\mathbb{C}^n, y^m)} \left( \left( \int_{(\mathbb{C}^n, y^1)} dd^c y^m u_1 (\lambda^{-1} y^1 + y^m) \wedge \phi_1 (y^1) \right) \wedge \ldots \wedge \left( \int_{(\mathbb{C}^n, y^{m-1})} dd^c y^m u_{m-1} (\lambda^{-1} y^{m-1} + y^m) \wedge \phi_{m-1} (y^{m-1}) \right) \right) \wedge T(y^m) \wedge \phi_m (y^m) 
\]
\[ = \int_{(\mathbb{C}^n, y^m)} dd^c u_1 (y^m) \wedge \ldots \wedge dd^c u_{m-1} (y^m) \wedge T(y^m) \wedge \phi_m (y^m) 
\]
\[ = \langle dd^c u_1 \wedge \ldots \wedge dd^c u_{m-1} \wedge T, \phi_m \rangle. \]

By (3.15), the last quantity tends to \( \langle R_1, \phi_m \rangle = \langle R_1, \pi_\ast \Phi \rangle \) as \( \lambda \to \infty \). This finishes the proof. \( \square \)

The following result is an important consequence of the previous lemmas.

**Lemma 3.8.** Under the assumptions of Lemma 3.5 the mass of \( R_{1,\lambda} \) on compact sets is uniformly bounded.

**Proof.** Let \( \omega := \sum_{k=1}^n i dy^k \wedge \overline{dy^k} \) be the standard Kähler form on \( (\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m) \) and set \( \Theta := \omega^{m-m+1-p} \). In order to prove the desired assertion, using the fact that \( R_{1,\lambda} \) is positive, it is enough to check that the mass of the trace measure \( R_{1,\lambda} \wedge \Theta \) is uniformly bounded on compact subsets of \( \Omega^m \).

Notice that the form \( \Theta \) is a linear combination of positive split forms. Therefore, in order to obtain the above bound, it will be enough to prove that \( \langle R_{1,\lambda}, \Phi \rangle \) is uniformly bounded for any fixed positive split test form \( \Phi = \phi_1 (y^1) \wedge \ldots \wedge \phi_m (y^m) \) with compact support.

If \( \phi_j \) is not of top degree for some \( j = 1, \ldots, m - 1 \), then, by Lemma 3.5 we have that \( |\langle R_{1,\lambda}, \Phi \rangle| \to 0 \) as \( \lambda \to \infty \). In particular, \( |\langle R_{1,\lambda}, \Phi \rangle| \) is uniformly bounded. Hence, we can assume that \( \phi_j \) has bidegree \((n, n)\) for every \( j = 1, \ldots, m - 1 \). In this case, since \( \phi_j \) is always bounded by some radial positive test form, we can assume furthermore that \( \phi_j \) is radial for every \( j \). From this observation and Lemma 3.7, we have \( \langle R_{1,\lambda}, \Phi \rangle \to \langle R_1, \pi_\ast \Phi \rangle \) as \( \lambda \to \infty \). In particular, \( |\langle R_{1,\lambda}, \Phi \rangle| \) is uniformly bounded. This finishes the proof of the Lemma. \( \square \)

We now recall from [DS18, Section 3] the notion of horizontal dimension of currents on vector bundles. Actually, the authors consider projective fibrations, that is, the projective compactification \( \overline{P}(E) \) of a given holomorphic vector bundle \( E \). Here, we phrase the definitions and results for vector bundles instead. The proofs can be easily adapted from the ones in [DS18].

Let \( V \) be a Kähler manifold of dimension \( \ell \) with Kähler form \( \omega_V \) and let \( \pi : E \to V \) be a holomorphic vector bundle over \( V \).

**Definition 3.9.** Let \( S \) be a non-zero positive closed current on \( E \). The **horizontal dimension** (\( h \)-dimension for short) of \( S \) is the largest integer \( j \) such that \( S \wedge \pi^\ast \omega_V^j \neq 0 \).

We will need the following characterization of currents of minimal \( h \)-dimension.
Lemma 3.10. Let $S$ be a positive closed $(p, p)$-current on $E$ with $p \leq \ell$. Assume that the $h$-dimension of $S$ is smaller or equal to $\ell - p$. Then the $h$-dimension of $S$ is equal to $\ell - p$ and there is a positive closed $(p, p)$-current $S^h$ on $V$ such that $S = \pi^* (S^h)$.

Proof. See [DS18, Lemma 3.4].

Now let $V = \Omega \subset (C^n, y^m)$, $\omega_V = \sum_{k=1}^n dy_k^m \wedge d\bar{y}_k^m = \beta(y^m)$ be the standard Kähler form on $\Omega$ and $E$ be the trivial bundle $\pi : (C^n)^{m-1} \times \Omega \to \Omega$, $\pi(y', y^m) = y^m$.

Recall from Lemma 3.8 that $(R_{1, \lambda})_\lambda$ is a relatively compact family of positive closed $(m - 1 + p, m - 1 + p)$-currents on $E$.

Lemma 3.11. In the assumptions of Lemma 3.5, let $R_{1, \infty}$ be a limit point of the family $R_{1, \lambda}$ as $\lambda \to \infty$. Then the $h$-dimension of $R_{1, \infty}$ is minimal, equal to $n - m + 1 - p$. In particular there is a positive closed $(m - 1 + p, m - 1 + p)$-current $R_{1, \infty}^h$ on $\Omega$ such that $R_{1, \infty} = \pi^* R_{1, \infty}^h$.

Proof. Let $\lambda_j$ be a sequence tending to $\infty$ such that $R_{1, \lambda_k} \to R_{1, \infty}$. By Lemma 3.10, we only need to show that $R_{1, \infty} = \pi^* R_{1, \infty}^h$. By Lemma 3.5, one only needs to show that $R_{1, \infty} \wedge \pi^* \beta_{n-m-p+2}(y^m) = 0$. To do this, it is enough to verify that

$$(R_{1, \infty} \wedge \pi^* \beta_{n-m-p+2}(y^m), \Phi) = 0$$

for every positive split test form $\Phi$.

Let $\Phi = \phi_1(y^1) \wedge \ldots \wedge \phi_m(y^m)\wedge \phi_m(y^m)$ be such a form. As in the beginning of the proof of Lemma 3.5, we may assume that each $\phi_j$, $j = 1, \ldots, m - 1$ has bidegree $(n, n)$ or $(n - 1, n - 1)$. Since the total bidegree of $\Phi$ is $(p', p')$, where

$$p' = nm - (n - m + 1 - p) = nm - n - 1,$$

at least one of $\phi_j$, $j = 1, \ldots, m - 1$ has bidegree $(n - 1, n - 1)$. In this case, by Lemma 3.5, one has $(R_{1, \lambda} \wedge \pi^* \beta_{n-m-p+2}(y^m), \Phi) \to 0$ as $\lambda \to \infty$. This finishes the proof.

We are now in position to prove Theorem 3.1.

End of proof of Theorem 3.1. Recall our notation

$$(R_{j, \lambda}) = (A_{\lambda})_\lambda (dd^c \bar{v}_j \wedge \ldots \wedge dd^c \bar{v}_{m-1} \wedge \bar{T})$$

$(1 \leq j \leq m - 1)$,

$$(R_{J, \lambda}) := (A_{\lambda})_\lambda (\bigwedge_{j \in J} dd^c \bar{u}_j \wedge \bar{T})$$

$(J \subset \{1, \ldots, m-1\})$,

$$(R_j) := R_{\{j, \ldots, m-1\}} = dd^c u_j \wedge \ldots \wedge dd^c u_{m-1} \wedge T$$

$(1 \leq j \leq m - 1)$.

Recall also that proving 3.2 is equivalent to proving that

$$R_{1, \lambda} \xrightarrow[|\lambda| \to \infty]{} \pi^* R_1.$$

We'll proceed by induction on $m$. When $m = 1$ the result is obvious. Now let $m \geq 2$ and assume that $R_{J, \lambda} \to \pi^* R_J$ as $\lambda \to \infty$ for every $J \subset \{1, \ldots, m - 1\}$ such that $|J| \leq m - 2$. When $m = 2$ this assumption is vacuous. Then, the hypothesis of Lemma 3.5 is satisfied. By Lemma 3.8, the family $(R_{1, \lambda})_\lambda$ is relatively compact. Let $R_{1, \infty} = \lim_{\lambda \to \infty} R_{1, \lambda}$ be one of its limit points. By Lemma 3.11 there is a positive closed $(m - 1 + p, m - 1 + p)$-current $R_{1, \infty}^h$ on $\Omega$ such that $R_{1, \infty} = \pi^* R_{1, \infty}^h$. We need to show that $R_{1, \infty} = R_1$.

Let $\phi_m$ be a test form on $(C^n, y^m)$. Take $\phi_1, \ldots, \phi_m$ positive radial $(n, n)$-forms with compact support such that $\int (C^n, y^m) \phi_j = 1$ for every $j = 1, \ldots, m - 1$. Then $\Phi := \phi_1 \wedge \ldots \wedge \phi_{m-1} \wedge \phi_m$ is such that $\pi_* \Phi = \phi_m$. Using Lemma 3.7 we get

$$\langle R_{1, \infty}^h, \phi_m \rangle = \langle R_{1, \infty}^h, \pi_* \Phi \rangle = \langle \pi^* R_{1, \infty}^h, \Phi \rangle = \langle R_{1, \infty}, \Phi \rangle = \lim_{\lambda \to \infty} \langle R_{1, \lambda}, \Phi \rangle = \langle R_1, \pi_* \Phi \rangle = \langle R_1, \phi_m \rangle.$$

Since $\phi_m$ is arbitrary, we get $R_{1, \infty}^h = R_1$. This concludes the proof of the Theorem.
Remark 3.12. In the statement of Theorem 3.11 if we consider the case where \( m-1+p > n \), then the arguments in the above proof still work and we obtain that the associated density current vanishes.

Let \( R \) be a positive closed current and \( v \) be a p.s.h. function. If \( v \) is locally integrable with respect to (the trace measure) of \( R \), we define, following Bedford-Taylor,

\[
\ddc v \wedge R := \ddc(vR).
\]

For a collection \( v_1, \ldots, v_s \) of p.s.h. functions, we can apply the above definition recursively, as long as the integrability conditions are satisfied.

Definition 3.13. We say that the intersection of \( \ddc v_1, \ldots, \ddc v_s \), \( R \) is classically well-defined if for every non-empty subset \( J = \{ j_1, \ldots, j_k \} \) of \( \{1, \ldots, s\} \), we have that \( v_{j_k} \) is locally integrable with respect to the trace measure of \( R \) and inductively, \( v_{j_r} \) is locally integrable with respect to the trace measure of \( \ddc v_{j_{r+1}} \wedge \cdots \wedge \ddc v_{j_k} \wedge R \) for \( r = k-1, \ldots, 1 \), and the product \( \ddc v_{j_1} \wedge \cdots \wedge \ddc v_{j_k} \wedge R \) is continuous under decreasing sequences of p.s.h. functions.

The last definition is slightly more restrictive than the one given in [KV19]. We have the following comparison result between the Dinh-Sibony product and the above notion of wedge products. This result is a direct consequence of Theorem 3.1.

Corollary 3.14. Let \( m \geq 2 \) and \( p \geq 0 \) be such that \( m-1+p \leq n \). Let \( u_1, \ldots, u_{m-1} \) be p.s.h. functions on \( \Omega \) and let \( T \) be a positive closed \((p,p)\)-current on \( \Omega \). Assume that \( \ddc u_1 \wedge \cdots \wedge \ddc u_{m-1} \wedge T \) is classically well-defined. Then the Dinh-Sibony product of \( \ddc u_1, \ldots, \ddc u_{m-1}, T \) is well-defined and

\[
\ddc u_1 \wedge \cdots \wedge \ddc u_{m-1} \wedge T = \ddc u_1 \wedge \cdots \wedge \ddc u_{m-1} \wedge T.
\]

We note that [KV19 Theorem 1.1] asserts a similar conclusion, but there's a slip in the proof of the result as stated there.

4. Products in the Blocki-Cegrell class and the domain of definition of the Monge-Ampère operator

In this section we apply Theorem 3.1 to studying the domain of definition of the Monge-Ampère operator via Dinh-Sibony’s intersection product. Let \( \Omega \) be a domain in \( \mathbb{C}^n \). Denote by \( \text{PSH}(\Omega) \) the set of p.s.h. functions on \( \Omega \).

Definition 4.1. The Blocki-Cegrell class on \( \Omega \) is the subset \( \mathcal{D}(\Omega) \) of \( \text{PSH}(\Omega) \) consisting of functions \( u \) with the following property: there exists a measure \( \mu \in \Omega \) such that for every open set \( U \subset \Omega \) and every sequence \( (u_\ell)_\ell \) of smooth p.s.h. functions on \( U \) decreasing to \( u \) pointwise as \( \ell \to \infty \), we have that \( (\ddc u_\ell)^n \) converges to \( \mu \) as \( \ell \to \infty \).

For \( u \in \mathcal{D}(\Omega) \), we define \( (\ddc u)^n := \mu \), where \( \mu \) is the above measure.

The class \( \mathcal{D}(\Omega) \) is the largest subset of \( \text{PSH}(\Omega) \) where we can define a Monge-Ampère operator that coincides with the usual one for smooth p.s.h. functions and which is continuous under decreasing sequences, see [Ceg04, Blo06].

We first need the following result ensuring the existence of the mixed products in the Blocki-Cegrell class.

Proposition 4.2. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( u_1, u_2, \ldots, u_m, 1 \leq m \leq n \) be p.s.h. functions in \( \mathcal{D}(\Omega) \). Then, there exists a positive closed \((m,m)\)-current \( S_m \) such that for every open set \( U \subset \Omega \) and every sequence \( (u_\ell^j)_\ell \) of smooth p.s.h. functions on \( U \) decreasing to \( u_j \) pointwise as \( \ell \to \infty \), we have that

\[
\ddc u_1^\ell \wedge \cdots \wedge \ddc u_m^\ell \to S_m \quad \text{on} \ U \quad \text{as} \ \ell \to \infty.
\]
For $u_1, u_2, \ldots, u_m \in D(\Omega)$, we define their wedge product by
\begin{equation}
\dd^\ast u_1 \wedge \cdots \wedge \dd^\ast u_m := S_m,
\end{equation}
where $S_m$ is the current appearing in the above proposition. In particular, for $u \in D(\Omega)$, one sees that $(\dd^\ast u)^n$ is the Monge-Ampère measure given in Definition 4.1.

As mentioned in the Introduction, Proposition 4.2 is known when $m = n$ and the case $m < n$ might also be known to experts. We give a proof here for completeness, following closely the proof of [Blo06 Theorem 1.1]. A simplifying step in [Blo06] is the fact that it suffices to work with test functions that are p.s.h. on a ball and vanish in its boundary. In the case $m < n$, this step is replaced by the following lemma.

**Lemma 4.3.** Let $B_1 \Subset B_2 \Subset \Omega$ be balls. Let $A$ be the vector space generated by forms of the type $h \dd^\ast v_1 \wedge \cdots \wedge \dd^\ast v_{n-m}$, where $h, v_1, \ldots, v_{n-m}$ are p.s.h. functions on $B_2$ which are continuous up to $\partial B_2$ and vanish on $\partial B_1$. Then, every smooth $(n-m, n-m)$-form $\psi$ compactly supported in $B_1$ is in $A$.

**Proof.** It is a standard fact that every smooth $(n-m, n-m)$-form $\psi$ compactly supported in $B_1$ can be written as a linear combination of forms of type $\gamma_i = h_i \gamma_1 \wedge \cdots \wedge \gamma_{n-m} \wedge \gamma_{n-m}$, where $\gamma_i$ is a smooth function with compact support in $B_1$ and $\gamma_1, \ldots, \gamma_{n-m}$ are $(1, 0)$-forms with constant coefficients, see [Dem, III.1.4]. Hence, it is enough to prove the desired assertion for $\eta$ as above.

Write $\gamma_\ell = \sum_{j=1}^n a_{j\ell} dz_j$, for $1 \leq \ell \leq n-m$, where $a_{j\ell} \in \mathbb{C}$. Observe $i\gamma_\ell \wedge \overline{\gamma_\ell} = \dd^\ast v_\ell$, where $v_\ell(z) := \sum_{j=1}^n a_{j\ell} z_j^2$, where $(z_1, \ldots, z_n)$ are the standard coordinates on $\mathbb{C}^n$. Let $\nu_\ell$ be the envelope constructed from $v_\ell$ as in Lemma 4.4 for $1 \leq \ell \leq n-m$. We have that $\nu_\ell = v_\ell$ on $B_1$, $\nu_\ell \in \mathrm{PSH}(B_2) \cap C^0(\overline{B_2})$, and $\nu_\ell = 0$ on $\partial B_2$. This combined with the fact that $h$ is compactly supported in $B_1$ gives $\eta = h \dd^\ast \nu_1 \wedge \cdots \wedge \dd^\ast \nu_{n-m}$. On the other hand, since $B_2$ is a ball, we can express $h = h_1 - h_2$ where $h_1, h_2$ are smooth p.s.h. functions such that $h_1 = h_2 = 0$ on $\partial B_2$. We deduce that $\eta \in A$. This finishes the proof. \hfill $\square$

For the proof of Proposition 4.2, we need the following result about Monge-Ampère measures of envelopes. The first part is classical (see [BT76, Wal69]), while the second part is contained in the proof of Theorem 1.1 in [Blo06].

**Lemma 4.4.** Let $B_1 \Subset B_2 \Subset \Omega$ be balls compactly contained in $\Omega$. For a negative continuous function $v \in \mathrm{PSH}(\Omega)$, set
\[ \overline{v} := \sup \{ w \in \mathrm{PSH}(B_2) : w < v \text{ on } B_1 \text{ and } w < 0 \text{ on } B_2 \}. \]

Then $\overline{v}$ is a p.s.h. function on $B_2$ which is continuous on $\overline{B_2}$ and satisfies
\begin{enumerate}
\item $\overline{v} = 0$ on $\partial B_2$,
\item $\overline{v} = v$ on $\overline{B_1}$,
\item $(\dd^\ast \overline{v})^n = 0$ on $B_2 \setminus B_1$.
\end{enumerate}

Moreover, if $u \in D(\Omega)$, then for any sequence $(u_\ell)_{\ell \geq 1}$ of smooth p.s.h. functions on $\Omega$ decreasing to $u$, we have
\[ \sup_{\ell \geq 1} \int_{B_2} (\dd^\ast \overline{u_\ell})^n < +\infty. \]

**Proof of Proposition 4.2.** Using Lemma 4.3, the proof is parallel to that of [Blo06 Theorem 1.1]. We include the main differences in the argument for completeness. Since the problem is local, in order to get the desired assertion, it suffices to prove that there exists a current $S_m$ on $\Omega$ such that for every ball $B_1$ and every sequence $(u_\ell')_{\ell \geq 1}$ of smooth p.s.h. functions on $\Omega$ decreasing to $u_j$ for $1 \leq j \leq m$, we have $\dd^\ast u_1' \wedge \cdots \wedge \dd^\ast u_m' \to S_m$ on $B_1$ as $\ell \to \infty$.

Let $B_2 \Subset \Omega$ be a ball containing $\overline{B_1}$. Let $h, v_1, \ldots, v_{n-m} \in \mathrm{PSH}(B_2) \cap C^0(\overline{B_2})$ be functions vanishing on $\partial B_2$. Put $\eta := h \dd^\ast v_1 \wedge \cdots \wedge \dd^\ast v_{n-m}$. Let $\overline{u}_j$ be the envelope constructed from $u_j$ as
in Lemma 4.4 for $1 \leq j \leq m$. We have that

\begin{equation}
\tilde{u}_j = u_j \quad \text{on} \quad \mathbb{B}_1,
\end{equation}

$\tilde{u}_j$ is continuous up to $\partial \mathbb{B}_2$ and is equal to 0 on $\partial \mathbb{B}_2$ for $1 \leq j \leq m$. Put

\[ S^\ell_m := dd^c u_1^\ell \wedge \cdots \wedge dd^c u_m^\ell, \quad \tilde{S}^\ell_m := dd^c \tilde{u}_1^\ell \wedge \cdots \wedge dd^c \tilde{u}_m^\ell. \]

We will prove that $(\tilde{S}^\ell_m, \eta)$ is convergent. By [Ceg04, Corollary 5.6], we have

\[ \int_{B_2} dd^c \tilde{u}_1^\ell \wedge \cdots \wedge dd^c \tilde{u}_m^\ell \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \leq \left( \int_{B_2} (dd^c \tilde{u}_1^\ell)^n \right)^{1/n} \cdots \left( \int_{B_2} (dd^c v_1)^n \right)^{1/n}. \]

This combined with Lemma 4.4 yields that $(\tilde{S}^\ell_m, \eta)$ is of uniformly bounded as $\ell \to \infty$. With this last property, we can follow the exact same arguments from the proof of [Bia06, Theorem 1.1]. This gives that $\lim_{\ell \to \infty} (S^\ell_m, \eta)$ exists and is independent of the choice of the sequences $(u_j^\ell)_{\ell \geq 1}$.

Using this and Lemma 4.3, for every smooth form $\phi$ compactly supported in $\mathbb{B}_1$, we obtain that $(\tilde{S}^\ell_m, \phi)$ converges to a number independent of the choice of $(u_j^\ell)_{\ell \geq 1}$ as $\ell \to \infty$. On the other hand, by (4.3), we get

\[ (S^\ell_m, \phi) = (\tilde{S}^\ell_m, \phi). \]

Consequently, the limit $\lim_{\ell \to \infty} (S^\ell_m, \phi)$ exists and is independent of the choice of $(u_j^\ell)_{\ell \geq 1}$ as $\ell \to \infty$. Hence, for $S_m$ defined by putting $(S_m, \phi) := \lim_{\ell \to \infty} (\tilde{S}^\ell_m, \phi)$ satisfies the desired property. This concludes the proof of Proposition 4.2. \hfill \Box

**Theorem 4.5.** Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $u_1, u_2, \ldots, u_m$, $1 \leq m \leq n$ be functions in $D(\Omega)$. Then, the Dinh-Sibony product of $dd^c u_1, \ldots, dd^c u_m$ is well-defined and

\begin{equation}
dd^c u_1 \wedge \cdots \wedge dd^c u_m = dd^c u_1 \wedge \cdots \wedge dd^c u_m.
\end{equation}

In particular, for $u \in D(\Omega)$, the Dinh-Sibony Monge-Ampère operator $u \mapsto (dd^c u)^n$ is well-defined and coincides with the usual one.

**Proof.** We'll apply Theorem 3.1 to $u_1, u_2, \ldots, u_{m-1}$ and $T = dd^c u_m$. Let $J \subset \{1, \ldots, m-1\}$ Then by Proposition 4.2, the current $R_J = \bigwedge_{j \in J} dd^c u_j \wedge dd^c u_m$ is well-defined. Now we check the hypothesis of Theorem 3.1 for $R_J$. Let $(u_j^\ell_m)_{\ell \in \mathbb{N}}$ be a sequence of smooth p.s.h. functions decreasing to $u_j$ for $j \in J$. We need to show that $\bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^\ell$ converges to $R_J$ as $\ell \to \infty$.

Let $(u_m^\ell)_{\ell \in \mathbb{N}}$ be a sequence of smooth functions decreasing to $u_m$. Let $\Phi$ be a smooth test form with compact support and $\epsilon > 0$ a constant. For every $\ell$, since $dd^c u_m^\ell \to dd^c u_m$, there exists $s_{\ell} \in \mathbb{N}$ such that

\begin{equation}
|\bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^\ell - \bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^m, \Phi| \leq \epsilon.
\end{equation}

We can choose $s_{\ell}$ so that $s_{\ell}$ is decreasing in $\ell$. Hence, $u_m^\ell$ decreases to $u_m$. By Proposition 4.2, we get $\bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^\ell \to R_J$ as $\ell \to \infty$. This combined with (4.5) gives

\[ |\bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^\ell - R_J, \Phi| \leq 2\epsilon \]

for $\ell$ big enough. Hence, $\bigwedge_{j \in J} dd^c u_j^\ell \wedge dd^c u_m^\ell$ converges to $R_J$ as $\ell \to \infty$. In other words, we have checked the hypothesis of Theorem 3.1 for $R_J$. The desired assertion follows. The proof is finished. \hfill \Box
References

[AW14] Mats Andersson and Elizabeth Wulcan. Green functions, Segre numbers, and King's formula. *Ann. Inst. Fourier (Grenoble)*, 64(6):2639–2657, 2014.

[Blo06] Zbigniew Błocki. The domain of definition of the complex Monge-Ampère operator. *Amer. J. Math.*, 128(2):519–530, 2006.

[BEGZ10] Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Monge-Ampère equations in big cohomology classes. *Acta Math.*, 205(2):199–262, 2010.

[BT76] Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37, 1976.

[BT87] Eric Bedford and B. A. Taylor. Fine topology, Šilov boundary, and $(dd^c)^n$. *J. Funct. Anal.*, 72(2):225–251, 1987.

[Ceg04] Urban Cegrell. The general definition of the complex Monge-Ampère operator. *Ann. Inst. Fourier (Grenoble)*, 54(1):159–179, 2004.

[CLN69] S. S. Chern, Harold I. Levine, and Louis Nirenberg. Intrinsic norms on a complex manifold. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 119–139. Univ. Tokyo Press, Tokyo, 1969.

[Dem] Jean-Pierre Demailly. Complex analytic and differential geometry. [http://www.fourier.ujf-grenoble.fr/~demailly](http://www.fourier.ujf-grenoble.fr/~demailly).

[DNV18] Tien-Cuong Dinh, Viêt-Anh Nguyên, and Duc-Viet Vu. Super-potentials, densities of currents and number of periodic points for holomorphic maps. *Adv. Math.*, 331:874–907, 2018.

[DS09] Tien-Cuong Dinh and Nessim Sibony. Super-potentials of positive closed currents, intersection theory and dynamics. *Acta Math.*, 203(1):1–82, 2009.

[DS18] Tien-Cuong Dinh and Nessim Sibony. Density of positive closed currents, a theory of non-generic intersections. *J. Algebraic Geom.*, 27(3):497–551, 2018.

[FS95] John Erik Fornæss and Nessim Sibony. Oka’s inequality for currents and applications. *Math. Ann.*, 301(3):399–419, 1995.

[KV19] Lucas Kaufmann and Duc-Viet Vu. Density and intersection of $(1,1)$-currents. *J. Funct. Anal.*, 277(2):392–417, 2019.

[Vu19] Duc-Viet Vu. Densities of currents on non-Kähler manifolds, 2019, *Int. Math. Res. Not. IMRN*, https://doi.org/10.1093/imrn/rnz270.

[Vu20] Duc-Viet Vu. Density currents and relative non-pluripolar products, 2020. *Bull. London. Math. Soc.*, https://doi.org/10.1112/blms.12451.

[Wal69] J. B. Walsh. Continuity of envelopes of plurisubharmonic functions. *J. Math. Mech.*, 18:143–148, 1968/1969.