Subleading collinear operators and their matrix elements

Andri Hardmeier\textsuperscript{*1}, Enrico Lunghi\textsuperscript{†1}, Dan Pirjol\textsuperscript{‡2}, and Daniel Wyler\textsuperscript{§1}

\textsuperscript{1} Institut für Theoretische Physik, Universität Zürich, 8057 Zürich, Switzerland
\textsuperscript{2} Department of Physics, The Johns Hopkins University, 3400 N. Charles Street, Baltimore MD 21218, U.S.A.

Abstract

We discuss the most general form of the leading power suppressed collinear operators in the soft-collinear effective theory. Such operators appear in the description of power corrections to exclusive heavy flavor decays into energetic light hadrons. Reparameterization invariance in the SCET provides powerful constraints on the Wilson coefficients of the subleading collinear operators. We present explicit results for the matrix elements of these operators on pseudoscalar and vector mesons, which are expressed in terms of twist-2 and twist-3 light-cone wave functions. We consistently include the effects of three-particle light-cone distribution amplitudes and find that their impact could be of phenomenological relevance.

\textsuperscript{*} Electronic address: andri@physik.unizh.ch
\textsuperscript{†} Electronic address: lunghi@physik.unizh.ch
\textsuperscript{‡} Electronic address: dpirjol@pha.jhu.edu
\textsuperscript{§} Electronic address: wyler@physik.unizh.ch
I. INTRODUCTION

The soft-collinear effective theory (SCET) \[1, 2, 3, 4\] has been proposed as a systematic framework for the study of processes involving energetic light quarks and gluons. Possible applications include the decays of heavy hadrons into light particles in the kinematical regions where the final products are very energetic, and hard scattering processes involving light hadrons, such as deep inelastic scattering and exclusive hadron form factors at large momentum transfer. SCET provides a natural framework for establishing a systematic expansion in $\Lambda/Q$ where $\Lambda$ is the QCD scale and $Q$ is the typical large energy of the particles involved. In particular, it provides a convenient tool to establish factorization theorems and study power corrections.

The observation that lies at the basis of SCET is that all the kinematical singularities appearing in these processes are connected to the exchange of collinear and soft particles ("long distance modes"). Therefore, an explicit description of these processes in terms of these degrees of freedom offers the usual advantages of an effective theory approach. On the one hand, the power counting is greatly simplified since it can be performed at the operator level; on the other hand, standard renormalization group techniques can be easily applied.

For definiteness we consider the decay of a heavy quark of mass $m$ which emits light quarks and gluons of energy $Q \approx m$. The possible external states are the soft spectator (whose momentum is of order $\Lambda$) and the decay products (light mesons and leptons), in addition to the heavy quark which cannot give rise to singularities and can be considered as an external source. By the Coleman-Norton theorem \[5\], the infra-red singularities of Feynman diagrams describing these amplitudes correspond to the propagation of on-shell particles. The idea underlying SCET is to identify all the possible on-shell modes that can appear in the initial and final state (and that are, hence, responsible for all the infra-red singularities) and to write the effective theory of their interactions as an expansion in $\Lambda/Q$.

In the following we use the standard light-cone decomposition of momenta

$$p^\mu = \frac{1}{2} n^\mu \bar{n} \cdot p + \frac{1}{2} \bar{n}^\mu n \cdot p + p^\perp_\perp \equiv (p_+, p_-, p_\perp),$$

where $n$ and $\bar{n}$ are light-cone vectors satisfying $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$. In any given process $n$ and $\bar{n}$ are chosen to be aligned to the final state collinear momenta. We also introduce the dimensionless parameter $\lambda$ that will serve as the expansion parameter of SCET. Its precise definition is discussed below.

In the description of heavy meson decays, there are three relevant kinematical configurations.

a) **Soft quarks** ($q_s, h_v^*$) and **gluons** ($A_\mu^s$) with momenta $p_s \simeq \Lambda = Q(\lambda, \lambda, \lambda)$ (where we defined $\lambda = \Lambda/Q$). The propagation of these modes is described by the non-perturbative regime of QCD ($p_s^2 \simeq \Lambda^2$) and is therefore calculable. Therefore, the exchange of soft particles can only be parameterized and results in the non-factorizable contributions to heavy-to-heavy and heavy-to-light form factors and the $B$ meson wave function.

b) **Collinear quarks** ($\xi_n$) and **gluons** ($A_\mu^c$) with momenta $p_c \simeq Q(1, \lambda^2, \lambda)$. These modes appear in the description of the constituents of a fast-moving light meson. They

\[h_v\] is the usual heavy quark degree of freedom that, after removing the fast oscillating components, is equivalent to a soft field.
are also non-perturbative and their exchanges can only be parameterized in terms of the light-cone wave functions of the final state mesons.

c) **Hard collinear quarks** ($\Xi_n$) and **gluons** ($A_{hc}^\mu$) with momenta $p_{hc} \approx Q(1, \lambda, \sqrt{\lambda})$.

These modes are necessary to describe inclusive jets (e.g. the $X_s$ system near the end-point region of the photon spectrum in the inclusive $B \to X_s\gamma$ decay [1]) and interactions of soft fields with collinear particles (e.g. the $B$-meson soft spectator after being struck by the energetic photon in $B \to \gamma e\nu$ [1, 2, 3, 4, 5]). Hard collinear modes have virtuality of order $p_{hc}^2 \approx \Lambda Q \gg \Lambda^2$, can be integrated out perturbatively and result in so-called jet-functions.

Further intermediate 'hard' collinear modes with momenta $Q(1, \lambda, \lambda)$ occur at tree level (for instance in the coupling of a soft and a collinear field). It is not necessary to introduce explicitly such fields, because they are connected to hard collinear via a reparameterization transformation. In Refs. [10, 11], three different invariances under changes in the light cone vectors $n$ and $\bar{n}$ were introduced:

- **type I**: $n^\mu \to n^\mu + \Delta^\mu, \bar{n}^\mu \to \bar{n}^\mu$
- **type II**: $\bar{n}^\mu \to \bar{n}^\mu + \epsilon^\mu, n^\mu \to n^\mu$
- **type III**: $n^\mu \to n^\mu \alpha, \bar{n}^\mu \to \bar{n}^\mu / \alpha$

where $\Delta^\mu \sim O(\lambda)$, $(\epsilon^\mu, \alpha) \sim O(1)$ and $\lambda$ refers to the expansion parameter relevant to the theory considered. In our case with two types of collinear fields, there are actually two expansion parameters, $\sqrt{\Lambda/Q}$ and $\Lambda/Q$ for the collinear and hard collinear fields, respectively. Thus we must have two invariances, related to the corresponding fields. In particular, the hard collinear sector must satisfy a type-I reparameterization invariance with $\Delta^\mu \sim O(\sqrt{\lambda})$. This invariance connects the $(1, \lambda, \sqrt{\lambda})$ and $(1, \lambda, \lambda)$ modes.

From a technical point of view, it is always possible to completely integrate out the hard collinear modes because in all applications they always appear as internal modes. This is obvious in processes like $B \to \gamma e\nu$. In more complicated situations like heavy-light semileptonic decays $B \to \pi e\nu$, a two step procedure has been proposed [14]. In the first step an effective theory is formulated (SCET-I) containing only soft and hard collinear modes, which is matched into a second step onto the final effective theory (SCET-II) containing the collinear and soft modes. [In the alternative treatment of Ref. [13] only the modes a) and b) are introduced and one matches directly from QCD onto SCET-II.] On the other hand, in inclusive decays (like $B \to X_s\gamma$) one can always write the decay width using the optical theorem and again hard collinear modes can be viewed as internal particles.

For the purpose of this paper, we take the point of view that problems associated with the integration over the hard collinear modes at subleading order have been cleared and proceed to the analysis of power suppressed contributions. Once the process is specified, the integration of the hard collinears gives, order by order in $\alpha_s$, all the relevant SCET operators (that will involve only soft and collinear modes). The matrix elements of these operators between initial and final states fall in two groups: those factorizable in terms of "conventional" form factors and light-cone wave functions, and others that require the introduction of new non-perturbative objects. In particular, the light-cone wave functions enter through matrix elements of SCET operators (involving two quarks and an infinite number of gluons) between the vacuum and a meson state.

In order to trust this perturbative computation, one also has to show that higher loop contributions will not introduce new non-perturbative structures at a given order in $\lambda$. This can be achieved by writing the most general set of operators allowed by gauge and reparameterization invariance and showing that their matrix elements factorise.
In this paper we classify all the possible SCET collinear operators that appear at leading and subleading order in $\lambda$ and compute their matrix elements in terms of the usual light cone distribution amplitudes of pseudoscalar [20, 21] and vector [23, 24, 25] mesons. These operators are necessary for the SCET analysis of any process involving energetic light mesons. In particular, we find that for decays involving transverse polarized vector mesons, only subleading operators contribute and that, in this case, it is not possible to neglect the contribution of three-particle distribution amplitudes.

Note, finally, that there are two equivalent formulations of SCET, that are usually denoted as hybrid [1, 2] and coordinate space [10, 13], respectively. In the former, strongly oscillating collinear modes are removed from the theory by a partial Fourier transformation; in the latter, the slowly varying soft fields are multipole expanded. These manipulations are necessary in order to fully expand the Lagrangian in powers of $\lambda$. A failure in achieving correctly the complete $\lambda$-expansion would result in a theory whose infrared behaviour does not reproduce the one of full QCD [28]. We will work in the hybrid formalism and show how to readily translate our results into coordinate-space.

We start by summarising briefly the main ingredients of the hybrid formulation of SCET. In Sec. II we present the complete list of leading and subleading operators built only of collinear fields. In Sec. III and IV we show how to extract the matrix elements of the various SCET operators and present our results. In Sec. V we show that reparameterization invariance (RPI) gives strong constraints on the Wilson coefficients of the subleading collinear operators. In Sec. VI we present a sample application for the subleading collinear operators constructed in this paper, discussing final state photon emission in weak annihilation. We check our findings for the matrix elements of these operators by verifying an exact Ward identity; this illustrates the numerical impact of the three-particle contributions which are usually neglected in practical computations. We collect our conventions in Appendix A and in Appendix B we give the parameters of a minimal set of light cone wave functions satisfying the equations of motion in QCD used in the numerical evaluations in Sec. VI.

II. SUBLEADING COLLINEAR OPERATORS IN SCET

In this Section, we present a complete set of collinear quark operators $\bar{\xi}_n \cdots \xi_n$ up to subleading order in $\lambda$. Such operators are important in their own right, as interpolating fields for meson states, and as building blocks for more complicated operators in the effective theory. In constructing a complete basis of operators we use constraints from the collinear gauge invariance, Dirac structure of the collinear quark fields $\xi_n$, and from reparameterization invariance of the effective theory. A similar procedure was recently used in Ref. [11] to construct the complete set of heavy-to-light currents in SCET.

The operators to be constructed in this Section contain the collinear quark $\xi_{n,p}$ and gluon $A_{n,q}$ fields, together with the collinear covariant derivative $iD_\mu^c = P_\mu^c + gA_\mu^c$. In order to be able to perform the power counting of the various contributions at the operator level, it is convenient to assign a $\lambda$ scaling to the fields requiring the kinetic terms to be of order $O(1)$. In this way, one obtains $\xi_n \sim \lambda$ and $A_\mu^c \sim (\lambda^2, 1, \lambda)$.

In the hybrid formulation of SCET, one achieves a correct $\lambda$ expansion by extracting the large Fourier modes from each field:

$$\phi_c(x) = \sum_{\hat{p}_c} e^{-i\hat{p}_c.x} \hat{\phi}_{c,\hat{p}_c}(x)$$  \hspace{1cm} (5)
where \( \tilde{p}_c = Q(0, 1, \lambda) \), are labels and the new field \( \phi_{c, \tilde{p}_c} \) is responsible for fluctuations with momenta of order \( Q(\lambda^2, \lambda^2, \lambda^2) \).

It is convenient to introduce a “label” operator \( \mathcal{P}^\mu \) which acts on the collinear fields and picks up their large momentum: \( \mathcal{P}^\mu \xi_{n,p} = (\frac{\tilde{n} \cdot p}{2} n^\mu + \frac{\tilde{p}_c \cdot p}{2} \xi_{n,p} \) and \( \mathcal{P}^\mu \), respectively. When acting on a product of several fields, these operators give the difference between the total label carried by the fields minus the total label of the complex conjugated fields.

We will also use a special notation which associates a momentum label index to an arbitrary product of collinear fields. Our convention is

\[
\chi_{n,\omega} \equiv [W^\dagger \xi_n]_\omega = [\delta(\omega - \tilde{n} \cdot \mathcal{P})W^\dagger \xi_n], \quad [W^\dagger iD_{\perp\,c}W]_\omega = [\delta(\omega - \tilde{n} \cdot \mathcal{P})W^\dagger iD_{\perp\,c}W]
\]

where \( \delta(\omega - \tilde{n} \cdot \mathcal{P}) \) acts only inside the square brackets.

The collinear operators that we write in the following can include a nontrivial flavor structure. When required, this will be denoted by a superscript showing the quark flavours. Note, finally, that for each operator we can have the singlet and octet colour structure. We will write explicitly only the former since the matrix elements of any octet operator between the vacuum and a meson state vanishes.

At leading order in \( \lambda \) there are only three independent collinear operators, which can be chosen as

\[
\mathcal{J}_V(\bar{\omega}) = \bar{\chi}_{n,\omega_1} \frac{\sqrt{2}}{2} \chi_{n,\omega_2},
\]

\[
\mathcal{J}_A(\bar{\omega}) = \bar{\chi}_{n,\omega_1} \frac{\sqrt{2}}{2} \gamma_5 \chi_{n,\omega_2},
\]

\[
\mathcal{J}_T^\dagger(\bar{\omega}) = \bar{\chi}_{n,\omega_1} \frac{\sqrt{2}}{2} \gamma_\perp \chi_{n,\omega_2},
\]

where \( \gamma_\perp \equiv \gamma^\alpha - n^\alpha \tilde{\gamma}/2 - \tilde{n}^\alpha \tilde{\gamma}/2 \) and \( \bar{\omega} = (\omega_1, \omega_2) \). Their transformation properties under charge conjugation are

\[
\mathcal{J}_{\gamma T}^{(ud)}(\omega_1, \omega_2) \to -\mathcal{J}_{\gamma T}^{(du)}(-\omega_2, -\omega_1),
\]

\[
\mathcal{J}_A^{(ud)}(\omega_1, \omega_2) \to \mathcal{J}_A^{(du)}(-\omega_2, -\omega_1).
\]

At subleading order in \( \lambda \), the number of allowed structures is much larger. It is convenient to choose a basis of collinear operators with simple transformation properties under charge conjugation. We choose the following four chiral-even collinear operators

\[
\mathcal{V}_1^\alpha(\bar{\omega}) = \left[ \bar{\xi}_n \frac{\sqrt{2}}{2} (i \mathcal{D}_{\perp\,c})^\dagger W_n \right]_{\omega_1} \frac{1}{\tilde{n} \cdot \mathcal{P}} \gamma^\alpha \chi_{n,\omega_2} + \bar{\chi}_{n,\omega_1} \gamma^\alpha \frac{1}{\tilde{n} \cdot \mathcal{P}} \left[ W_n^\dagger i \mathcal{D}_{\perp\,c} \frac{\tilde{\gamma}}{2} \xi_n \right]_{\omega_2},
\]

\[
\mathcal{V}_2^\alpha(\bar{\omega}) = \left[ \bar{\xi}_n \frac{\sqrt{2}}{2} (i \mathcal{D}_{\perp\,c})^\dagger W_n \right]_{\omega_1} \frac{1}{\tilde{n} \cdot \mathcal{P}} \gamma^\alpha \chi_{n,\omega_2} + \bar{\chi}_{n,\omega_1} \frac{1}{\tilde{n} \cdot \mathcal{P}} \left[ W_n^\dagger i \mathcal{D}_{\perp\,c} \frac{\tilde{\gamma}}{2} \xi_n \right]_{\omega_2},
\]

\[
\mathcal{A}_1^\alpha(\bar{\omega}) = \left[ \bar{\xi}_n \frac{\sqrt{2}}{2} (i \mathcal{D}_{\perp\,c})^\dagger W_n \right]_{\omega_1} \frac{1}{\tilde{n} \cdot \mathcal{P}} \gamma^\alpha \gamma_5 \chi_{n,\omega_2} + \bar{\chi}_{n,\omega_1} \gamma^\alpha \gamma_5 \frac{1}{\tilde{n} \cdot \mathcal{P}} \left[ W_n^\dagger i \mathcal{D}_{\perp\,c} \frac{\tilde{\gamma}}{2} \xi_n \right]_{\omega_2},
\]

\[
\mathcal{A}_2^\alpha(\bar{\omega}) = \left[ \bar{\xi}_n \frac{\sqrt{2}}{2} (i \mathcal{D}_{\perp\,c})^\dagger W_n \right]_{\omega_1} \frac{1}{\tilde{n} \cdot \mathcal{P}} \gamma^\alpha \gamma_5 \chi_{n,\omega_2} - \bar{\chi}_{n,\omega_1} \gamma_5 \frac{1}{\tilde{n} \cdot \mathcal{P}} \left[ W_n^\dagger i \mathcal{D}_{\perp\,c} \frac{\tilde{\gamma}}{2} \xi_n \right]_{\omega_2}.
\]
and the three chiral-odd operators

\[
S(\bar{\omega}) = \left[ \bar{\xi}_{n} \frac{i}{\sqrt{2}} (i \mathcal{D}_{\perp c}) \mathcal{W}_{n} \right]_{\omega_{1}} \frac{1}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \chi_{n,\omega_{2}} + \bar{\chi}_{n,\omega_{1}} \frac{1}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \left[ W_{n}^{\dagger} i \mathcal{D}_{\perp c} \frac{i}{\sqrt{2}} \xi_{n} \right]_{\omega_{2}}, \tag{16}
\]

\[
P(\bar{\omega}) = \left[ \bar{\xi}_{n} \frac{i}{\sqrt{2}} (i \mathcal{D}_{\perp c}) \mathcal{W}_{n} \right]_{\omega_{1}} \frac{\gamma_{5}}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \chi_{n,\omega_{2}} + \bar{\chi}_{n,\omega_{1}} \frac{\gamma_{5}}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \left[ W_{n}^{\dagger} i \mathcal{D}_{\perp c} \frac{i}{\sqrt{2}} \xi_{n} \right]_{\omega_{2}}, \tag{17}
\]

\[
T^{\alpha\beta}(\bar{\omega}) = \left[ \bar{\xi}_{n} \frac{i}{\sqrt{2}} (i \mathcal{D}_{\perp c}) \mathcal{W}_{n} \right]_{\omega_{1}} \frac{\gamma_{\perp}^{\alpha}}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \chi_{n,\omega_{2}} - \bar{\chi}_{n,\omega_{1}} \frac{\gamma_{\perp}^{\alpha}}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \left[ W_{n}^{\dagger} i \mathcal{D}_{\perp c} \frac{i}{\sqrt{2}} \xi_{n} \right]_{\omega_{2}}, \tag{18}
\]

go together with the corresponding colour octet operators which will be denoted by the same letter and a colour index \( a \). We include the factors \( 1/\sqrt{n \cdot \mathcal{P}} \) and \( 1/\sqrt{n \cdot \mathcal{P}^{\dagger}} \) in the definition of the operators, Eqs. (12)-(18), to make them invariant under the transformation \( n \rightarrow n\alpha, \ n \rightarrow \bar{n}/\alpha \) (type-III reparameterization invariance).

These operators are not the most general collinear gauge invariants at \( O(\lambda) \). In analogy to the heavy-to-light current considered in Ref. [11], it is possible to write also three-particle operators, which contain three collinear gauge invariant factors. Their Dirac structure is again restricted by the effective theory constraint \( \not{\xi}_{n} = 0 \), which leaves two possible chiral-even operators

\[
\mathcal{V}_{3}^{a}(\bar{\omega}) = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} W^{\dagger} D_{\perp}^{a} W \right]_{\omega_{1}} \chi_{n,\omega_{2}} = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} \right]^{2} W^{\dagger} i \gamma_{5} \bar{n}_{\beta} G^{\beta\alpha} W \chi_{n,\omega_{2}} \tag{19}
\]

\[
\mathcal{A}_{3}^{a}(\bar{\omega}) = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \gamma_{5} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} W^{\dagger} D_{\perp}^{a} W \right]_{\omega_{1}} \chi_{n,\omega_{2}} = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \gamma_{5} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} \right]^{2} W^{\dagger} i \gamma_{5} \bar{n}_{\beta} G^{\beta\alpha} W \chi_{n,\omega_{2}} \tag{20}
\]

and a single chiral-odd operator

\[
\mathcal{T}_{3}^{a}(\bar{\omega}) = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \gamma_{\perp}^{\alpha} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} W^{\dagger} D_{\perp}^{\alpha} W \right]_{\omega_{1}} \chi_{n,\omega_{2}} = \bar{\chi}_{n,\omega_{1}} \frac{i}{\sqrt{2}} \gamma_{\perp}^{\alpha} \left[ \frac{1}{\sqrt{n \cdot \mathcal{P}}} \right]^{2} W^{\dagger} i \gamma_{5} \bar{n}_{\beta} G^{\beta\alpha} W \chi_{n,\omega_{2}} \tag{21}
\]

The factor \( 1/\sqrt{n \cdot \mathcal{P}} \) assures again invariance under type-III reparameterization invariance.

The transformation properties of the subleading operators under charge conjugation are

\[
\mathcal{V}_{1/2}^{(ud)\alpha}(\omega_{1}, \omega_{2}) \rightarrow -\mathcal{V}_{1/2}^{(du)\alpha}(\omega_{2}, \omega_{1}), \tag{22}
\]

\[
\mathcal{A}_{1/2}^{(ud)\alpha}(\omega_{1}, \omega_{2}) \rightarrow \mathcal{A}_{1/2}^{(du)\alpha}(\omega_{2}, \omega_{1}), \tag{23}
\]

\[
\mathcal{S}^{(ud)\alpha}(\omega_{1}, \omega_{2}) \rightarrow -\mathcal{S}^{(du)\alpha}(\omega_{2}, \omega_{1}), \tag{24}
\]

\[
\mathcal{P}^{(ud)\alpha}(\omega_{1}, \omega_{2}) \rightarrow -\mathcal{P}^{(du)\alpha}(\omega_{2}, \omega_{1}), \tag{25}
\]

\[
\mathcal{T}^{(ud)\alpha\beta}(\omega_{1}, \omega_{2}) \rightarrow -\mathcal{T}^{(du)\alpha\beta}(\omega_{2}, \omega_{1}). \tag{26}
\]

The corresponding operators of opposite charge conjugation properties can be constructed by changing the relative sign of the two terms in Eqs. (12)-(18). They will be denoted with a tilde, e.g.

\[
\mathcal{\tilde{V}}_{1}^{a}(\bar{\omega}) = \left[ \bar{\xi}_{n} \frac{i}{\sqrt{2}} (i \mathcal{D}_{\perp c}) \mathcal{W}_{n} \right]_{\omega_{1}} \frac{1}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \chi_{n,\omega_{2}} - \bar{\chi}_{n,\omega_{1}} \frac{1}{\sqrt{\mathcal{N} \cdot \mathcal{P}}} \left[ W_{n}^{\dagger} i \mathcal{D}_{\perp c} \frac{i}{\sqrt{2}} \xi_{n} \right]_{\omega_{2}}, \tag{27}
\]
and similarly for the remaining 6 operators. They are not independent and can be related to the above ones by using the Dirac identities in the Appendix A. For example, using Eq. (A6) (written in terms of the transverse antisymmetric tensor $\varepsilon_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \tilde{n}^\alpha n^\beta$)

$$\frac{\not{g}}{2} \gamma_\perp \gamma_5 = g_{\mu\nu} \not{g} \frac{\not{g}}{2} \gamma_5 ,$$

one obtains the relations

$$V_1^\mu = V_2^\mu + i \varepsilon_{\perp\nu} \tilde{A}_{2\nu} , \quad A_1^\mu = A_2^\mu + i \varepsilon_{\perp\nu} \tilde{V}_{2\nu} ,$$

$$\tilde{V}_1^\mu = \tilde{V}_2^\mu + i \varepsilon_{\perp\nu} A_{2\nu} , \quad \tilde{A}_1^\mu = \tilde{A}_2^\mu + i \varepsilon_{\perp\nu} V_{2\nu} .$$

They can be solved for $\tilde{V}_i , \tilde{A}_i , \overline{\tilde{V}}_i , \overline{\tilde{A}}_i$ with the results

$$\tilde{V}_1^\alpha = i \varepsilon_{\perp\beta} A_{1\beta} , \quad \tilde{A}_1^\alpha = i \varepsilon_{\perp\beta} V_{1\beta} ,$$

$$\tilde{V}_2^\alpha = i \varepsilon_{\perp\beta} (A_{1\beta} - A_{2\beta}) , \quad \tilde{A}_2^\alpha = i \varepsilon_{\perp\beta} (V_{1\beta} - V_{2\beta}) .$$

For the chiral-odd operators there are two identities following from their definition

$$g_{\alpha\beta} ^{\perp} T^{\alpha\beta}(\bar{\omega}) = \bar{S}(\bar{\omega}) , \quad g_{\alpha\beta} ^{\perp} T^{\alpha\beta}(\bar{\omega}) = S(\bar{\omega}) ,$$

and two other identities following from Eq. (A8)

$$\mathcal{P} = i \varepsilon_{\mu\nu} ^{\perp} T^{\mu\nu} , \quad \bar{\mathcal{P}} = i \varepsilon_{\mu\nu} T^{\mu\nu} .$$

Let us finally discuss the way these operators appear in explicit calculations. A given QCD operator $O_{\text{QCD}}$ is matched onto SCET operators containing the subleading collinear bilinears introduced above

$$O_{\text{QCD}} = \cdots + \int d\omega_1 d\omega_2 C_1(\omega_1, \omega_2) \{ \cdots \} V_i(\omega_1, \omega_2)$$

$$+ \int d\omega_1 d\omega_2 d\omega_3 C_2(\omega_1, \omega_2, \omega_3) \{ \cdots \} V_i(\omega_1, \omega_2, \omega_3)$$

where the ellipses $\{ \cdots \}$ denote possible soft fields which were omitted in writing the SCET operators. The Wilson coefficients $C_{1,2}(\omega_i)$ depend on the momentum labels of the collinear bilinears.

After factorization, the matrix elements of the collinear operators $V_i(\omega_i)$ between a light meson and vacuum lead to non-perturbative functions $\langle M(p_M) | V_i(\omega_1, \omega_2) | 0 \rangle \simeq \varphi_i(u)$. It is convenient to implement momentum conservation $\omega_1 - \omega_2 = p_M$ by introducing the momentum fraction $u$ by $(\omega_1, \omega_2) = (u, 1-u)\vec{n} \cdot p_M$, with $u = (0, 1)$. The charge-conjugation transformation properties of the collinear operators $V_i(\omega_1, \omega_2)$ (see Eqs. \[10\], \[11\], \[22\]-\[26\]), taken together with the $C$ quantum number of the state $| M(p_M) \rangle$, fixes the symmetry property of matrix elements under the substitution $u \rightarrow \bar{u}$. For example, taking $C = -1$ as appropriate for the $J/\psi$ meson, one has

$$\langle \rho(p, \eta) | V_i^{\text{even(odd)}}(\omega_1, \omega_2) | 0 \rangle \sim \varphi_\rho^\text{odd(even)}(u) ,$$

such that only the odd (even) part of the corresponding Wilson coefficient $C(\omega_1, \omega_2)$ will give a non-vanishing contribution to the given matrix element of Eq. (34). The matrix elements of Eq. (34) will be given with the integration measure $\Pi_i d\omega_i$ replaced with $du$ for the 2-parton operators, and with $d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \Sigma_{i=1}^3 \alpha_i)$ for the 3-parton operators.
III. STRATEGY FOR THE COMPUTATION OF THE MATRIX ELEMENTS

In this section we explain the procedure that we use to extract the matrix elements between the vacuum and a state with one vector or pseudo–scalar meson of the leading and subleading collinear operators introduced in Sec. III. We will see that they can be expressed in terms of the vector and pseudo–scalar mesons’ light-cone wave functions.

The usual starting point are matrix elements of nonlocal operators of the form

\[ \langle M | \bar{q}(x) \Gamma W[x, y] q(y) | 0 \rangle \quad \text{or} \quad \langle M | \bar{q}(x) \Gamma W[x, z] G_{\mu\nu}(z) W[z, y] q(y) | 0 \rangle . \quad (36) \]

The \( W[\cdot, \cdot] \) terms are Wilson lines required by gauge invariance and are explicitly given by

\[ W[x, y] = P \exp \left[ i g (x - y)^\mu \int_0^1 dt A^\mu(tx + ty) \right] , \quad (37) \]

where \( t = 1 - t \). In the following, the presence of the appropriate Wilson lines between fields evaluated at different space-time points is understood.

The general procedure that we adopt is rather simple. We first project the QCD operators onto SCET keeping leading and subleading contributions; then we use the standard definitions of the light-cone wave functions given in Refs. [20, 21, 23, 25, 27] to identify the matrix elements of the various SCET operators.

The presence of Wilson lines and of fields situated at different space-time positions, causes the appearance of arbitrarily suppressed operators in the projection of the QCD operators onto SCET. In order to simplify the extraction of the leading and subleading operators we expand the QCD operators around the transverse direction:

\[ \bar{q}(x) \Gamma q(y) = \bar{q}(x) \Gamma q(y) \bigg|_{x_\perp = 0} + x_\perp^\mu \frac{\partial}{\partial x_\perp^\mu} \bar{q}(x) \Gamma q(y) \bigg|_{x_\perp = 0} \]
\[ + y_\perp^\mu \frac{\partial}{\partial y_\perp^\mu} \bar{q}(x) \Gamma q(y) \bigg|_{y_\perp = 0} + \text{higher powers of } x_\perp, y_\perp , \quad (38) \]

where

\[ \frac{\partial}{\partial x_\perp^\mu} \bar{q}(x) \Gamma q(y) = \bar{q}(x) \frac{\partial}{\partial x_\perp^\mu} \Gamma q(y) - i \int_0^1 dt t z^\alpha \bar{q}(x) g G_{\alpha\mu}(tx + ty) \Gamma q(y) , \quad (39) \]
\[ \frac{\partial}{\partial y_\perp^\mu} \bar{q}(x) \Gamma q(y) = \bar{q}(x) \Gamma D_{\perp\mu} q(y) - i \int_0^1 dt \bar{q}(x) g G_{\alpha\mu}(tx + ty) \Gamma q(y) , \quad (40) \]

and we define \( z = x - y \).

All the terms of this expansion count as \( O(1) \) (in fact, we only require \( x - y \) to be on the light-cone without any constraint on the direction of this vector) but the projections of the various terms start with SCET operators of increasing \( \lambda \) suppression:

\[ \bar{q}(x) W[x, y] \Gamma q(y) \bigg|_{x_\perp = 0, y_\perp = 0} \rightarrow \mathcal{O}_0 + \mathcal{O}_1 + \cdots , \]
\[ \frac{\partial}{\partial x_\perp^\mu} \bar{q}(x) W[x, y] \Gamma q(y) \bigg|_{x_\perp = 0, y_\perp = 0} \rightarrow \mathcal{O}_1 + \mathcal{O}_2 + \cdots , \quad (41) \]
\[ \frac{\partial}{\partial x_\perp^\mu} \frac{\partial}{\partial x_\perp^\nu} \bar{q}(x) W[x, y] \Gamma q(y) \bigg|_{x_\perp = 0, y_\perp = 0} \rightarrow \mathcal{O}_2 + \mathcal{O}_3 + \cdots , \]
where $O_n$ are generic SCET operators suppressed by a factor $\lambda^n$ compared to $O_0$. Since we are interested in leading and subleading operators only, we just need to consider the few terms explicitly written in Eq. (38).

Let us now show in detail how to perform the SCET projection of a given QCD operator. The starting point is to express the QCD quark field in terms of collinear and soft fields

$$q(x) \rightarrow \sum_{\bar{p}_c} e^{-i\bar{p}_c \cdot x} \left[ 1 + \frac{1}{i\hat{n} \cdot D_c} i\bar{p}_c \cdot \frac{\not{\chi}}{2} \right] \xi_{n,\bar{p}_c}(x) + \sum_{\bar{p}_c} e^{-i\bar{p}_c \cdot x} q_{\bar{p}_c}(x).$$

(42)

Since all the terms in the expansion (38) are evaluated at $x_\perp = 0$ we obtain

$$q(x) \rightarrow \sum_{\bar{n},\bar{p}_c} e^{-i\bar{n} \cdot \bar{p}_c \cdot x} \left( 1 + \frac{1}{i\hat{n} \cdot D_c} i\bar{p}_c \cdot \frac{\not{\chi}}{2} \right) \xi_{n,\bar{p}_c}(x),$$

(43)

where we do not include the soft quark because, as argued below, its contribution receives an additional power suppression and it is negligible for the analysis of leading and subleading operators.

Let us now consider the leading order projection of the QCD operator $\bar{q}(x) W[x, y] \gamma^\mu q(y)$. The matrix element of the $\xi n$ term is given by

$$\langle M(p_M)| \int D^2 \bar{\omega} \bar{\chi}_{n,\omega_1} n^\mu \frac{\not{\chi}}{2} \chi_{n,\omega_2}|0\rangle = \int_0^1 du e^{i(u_p x + \bar{u} p y)} \langle M|\bar{\chi}_{n, u p_M} n^\mu \frac{\not{\chi}}{2} \chi_{n, \bar{u} p_M}|0\rangle \langle \chi_{n, \omega_1} n^\mu \frac{\not{\chi}}{2} \chi_{n, \omega_2}\rangle,$$

(44)

$$\propto n^\mu \int_0^1 du e^{i(u_p x + \bar{u} p y)} \varphi(u),$$

where

$$\int D^2 \bar{\omega} \equiv \int d\omega_1 d\omega_2 e^{i(\omega_1 x - \omega_2 y)},$$

(45)

and $\bar{u} = 1 - u$. The $W_n$ factors (contained in the $\chi_n$ fields) come from the projection of the full QCD Wilson line, Eq. (37), in the limit $x_\perp = 0 = y_\perp$. In Eq. (11), we replaced the discrete sum of Eq. (43) by the integral over $\omega_i$. Once momentum conservation is imposed, the sum of the two labels must be equal to the total momentum of the meson: $\omega_1 - \omega_2 = \bar{n} \cdot p_M$ and the fractions of momentum carried by the two collinear quarks are $\omega_1 = u p_M$ and $\omega_2 = -\bar{u} p_M$, respectively (this assumes that the parton carrying momentum $\omega_1(\omega_2)$ ends up as a quark (antiquark) in $M$). The equality in Eq. (44) is obtained by direct comparison with the definition of the QCD light-cone wave functions for the $\bar{q}\gamma^\mu q$ current. The result is

$$\langle M|\bar{\chi}_{n,\omega_1} n^\mu \frac{\not{\chi}}{2} \chi_{n,\omega_2}|0\rangle \propto \varphi(u).$$

(46)

Note that although in principle mixed soft-collinear terms $\bar{q} n$ can be generated too, their matrix elements on a collinear meson state require insertions of the soft-collinear subleading Lagrangian and are suppressed relative to those of the diagonal operators $\xi n \xi n$ we consider here.

Before concluding this section, let us explain how to translate the above arguments into the coordinate-space formulation of SCET. In the first place note that our leading and subleading operators do not involve soft covariant derivatives. This implies that each operator
can be trivially translated into coordinate-space formalism by replacing the label operators with ordinary derivatives: $P_{\mu}^\perp + g A_{\perp \perp}^{\mu} \rightarrow i \partial_{\mu}^\perp + g A_{\perp \perp}^{\mu}$ and $\bar{n} \cdot \mathcal{P} \rightarrow i \bar{n} \cdot \partial$. The latter substitution might rise concerns about the invariance of the coordinate-space version of Eqs. (12)–(21) under soft gauge transformations \[11, 12\]. From the discussion in Ref. \[12\] it follows that the expansion in $\lambda$ is well defined only if the gauge transformations of the fields are homogeneous (\i.e. the theory must be gauge invariant order by order in $\lambda$). This can always be achieved via appropriate redefinitions of the collinear fields. In Ref. \[12\] it has been shown that homogeneous soft gauge transformations cannot depend on $\bar{n} \cdot x$, hence, $i \bar{n} \cdot \partial$ derivatives can be inserted without spoiling soft gauge invariance. From these considerations it becomes clear that matrix elements of the coordinate-space version of our operators will have the form

$$\langle M(p_M) | \bar{\xi}_n(x) \Gamma \xi_n(y) | 0 \rangle \propto \int_0^1 du \, e^{i (u p_M \cdot x + \bar{u} p_M \cdot y)} \varphi(u). \quad (47)$$

IV. COMPUTATION OF THE MATRIX ELEMENTS

In this section we apply the technique introduced above to extract the matrix elements of the leading and subleading SCET operators listed in Sec. II. First we use Eq. (48) to project the three- and two-particle QCD operators onto SCET. Then we compare the resulting expansion with the definition of the light-cone wave functions of pseudo-scalar and vector mesons and extract the SCET matrix elements.

A. SCET decomposition of the QCD operators

Let us now consider the SCET decomposition of the five independent QCD currents and of the relevant three-particle operators, respectively. The matrix elements of the vector, axial and tensor currents involve twist-2 and twist-3 light-cone wave functions and it is therefore necessary to consider the full expansion in Eq. (48). On the other hand, the matrix elements of the scalar, pseudo-scalar and three-particle operators involve only twist-3 and higher wave functions and we need to keep only the leading term in Eq. (48).

The SCET decomposition of the various currents are (with $z = x - y$)

$$\bar{q}(x) q(y) = \int \mathcal{D}^2 \omega S(\omega), \quad (48)$$

$$\bar{q}(x) \gamma^5 q(y) = \int \mathcal{D}^2 \omega \mathcal{P}(\omega), \quad (49)$$

$$\bar{q}(x) \gamma^\mu q(y) = \int \mathcal{D}^2 \omega \left\{ J_V(\omega) n^\mu + V_1^\mu(\omega) 
+ \frac{i}{2} n^\mu \left[ (\omega_1 x_{\perp \perp} - \omega_2 y_{\perp \perp}) V_2^\alpha(\omega) 
+ (\omega_1 x_{\perp \perp} + \omega_2 y_{\perp \perp}) \tilde{V}_2^\alpha(\omega) \right] \right\} 
- \frac{1}{2} n^\mu n \cdot z \int_0^1 dt \left( t x_{\perp \perp} + \bar{t} y_{\perp \perp} \right) \int \mathcal{D}^3 \omega \omega_3^2 V_3^\alpha(\omega), \quad (50)$$
\begin{align}
\bar{q}(x)\gamma^\mu \gamma^5 q(y) &= \int \mathcal{D}^2 \bar{\omega} \left\{ \mathcal{J}_A(\bar{\omega}) n^\mu + \mathcal{A}_1^\mu(\bar{\omega}) \\
&\quad+ \frac{i}{2} n^\mu \left[ (\omega_1 x_{\perp \alpha} - \omega_2 y_{\perp \alpha}) \mathcal{A}_2^\alpha(\bar{\omega}) + (\omega_1 x_{\perp \alpha} + \omega_2 y_{\perp \alpha}) \mathcal{A}_2^\alpha(\bar{\omega}) \right] \right\} \\
&\quad- \frac{1}{2} n^\mu n \cdot z \int_0^1 dt \left( tx_{\perp \alpha} + \bar{t} y_{\perp \alpha} \right) \int \mathcal{D}^3 \bar{\omega} \omega_3^2 \mathcal{A}_3^\alpha(\bar{\omega}), \tag{51}
\end{align}

There are only three 3-particle operators whose matrix elements involve twist-3 light-cone wave functions [23, 25]:

\begin{align}
\bar{q}(x) g_{\mu \nu} (tx + \bar{t} y) \gamma_\alpha q(y) &= -i n_\alpha \left[ n_\mu g_{\perp \nu \gamma} - n_\nu g_{\perp \mu \gamma} \right] \int \mathcal{D}^3 \bar{\omega} \frac{\omega_3^2}{2} \mathcal{V}_3^\gamma(\bar{\omega}), \tag{54}
\end{align}

\begin{align}
\bar{q}(x) g_{\mu \nu} (tx + \bar{t} y) \gamma_\alpha \gamma_5 q(y) &= -i n_\alpha \left[ n_\mu g_{\perp \nu \gamma} - n_\nu g_{\perp \mu \gamma} \right] \int \mathcal{D}^3 \bar{\omega} \frac{\omega_3^2}{2} \mathcal{A}_3^\gamma(\bar{\omega}), \tag{55}
\end{align}

\begin{align}
\bar{q}(x) g_{\mu \nu} (tx + \bar{t} y) \sigma_{\alpha \beta} q(y) &= \left[ n_\mu g_{\perp \nu \gamma} - n_\nu g_{\perp \mu \gamma} \right] \left[ n_\alpha g_{\perp \beta \sigma} - n_\beta g_{\perp \alpha \sigma} \right] \int \mathcal{D}^3 \bar{\omega} \frac{\omega_3^2}{2} \mathcal{T}_3^{\gamma \sigma}(\bar{\omega}). \tag{56}
\end{align}

**B. Pseudo-scalar meson matrix elements**

In the following we collect the matrix elements between the vacuum and a pseudo-scalar meson state of all the relevant QCD operators [20, 21], including the contributions from all wave functions of twist 2 and 3:

\begin{align}
\langle P(p)|\bar{q}(x) q(y)|0 \rangle &= 0, \tag{57}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \gamma^5 q(y)|0 \rangle &= -i f_{P\mu P} \int_0^1 du \, e^{i (u p x + \bar{u} p y)} \varphi_p(u), \tag{58}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \gamma^\mu q(y)|0 \rangle &= 0 \tag{59}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \gamma^\mu \gamma^5 q(y)|0 \rangle &= -i f_{P\mu P} \int_0^1 du \, e^{i (u p x + \bar{u} p y)} \varphi_p(u), \tag{60}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \sigma^{\mu \nu} q(y)|0 \rangle &= i f_{P\mu P} \int_0^1 du \, e^{i (u p x + \bar{u} p y)} \varphi_p(u), \tag{61}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \sigma^{\mu \nu} \gamma_5 q(y)|0 \rangle &= i f_{P\mu P} \int_0^1 du \, e^{i (u p x + \bar{u} p y)} \varphi_p(u) \tag{62}
\end{align}

\begin{align}
\langle P(p)|\bar{q}(x) \sigma^{\mu \nu} \gamma_5 g G^{\alpha \beta}(tx + \bar{t} y) q(y)|0 \rangle &= i [p_\mu (p_\alpha g_{\nu \beta} - p_\beta g_{\alpha \nu} - (\mu \leftrightarrow \nu)] f_{3\pi} \int \mathcal{D}^3 \bar{\alpha} \varphi_{3\pi}(\bar{\alpha}) \tag{62}
\end{align}
where $f_P$ is the decay constant, $\mu_P = m_P^2/(m_1 + m_2)$ ($m_i$ denote the masses of the light valence quarks), $f_{3\pi}$ is the three-particle decay constant (with dimension GeV$^2$). The 3-body integration measure is defined as

$$\int \mathcal{D}^3 \vec{\alpha} \equiv \int_0^1 d\alpha_1\,d\alpha_2\,d\alpha_3 \, e^{i(\alpha_1 x + \alpha_2 y + \alpha_3 z)} \delta(1 - \sum \alpha_i). \quad (63)$$

The wave function $\varphi(u)$ is of twist-2 while $\varphi_p(u)$, $\varphi_\sigma(u)$ and $\varphi_{3\pi}(\vec{\alpha})$ are twist-3. In the matrix element of the tensor current one can use partial integration to remove an explicit $n \cdot z$ factor, which gives

$$\langle P(p)|\bar{q}(x)\sigma^{\mu\nu} q(y)|0\rangle = \frac{f_P \mu_P}{6} \int_0^1 du \, e^{i(u x + u y)} \left[ \frac{\bar{u} \cdot p}{2} \varepsilon^{\mu\nu\rho\tau} n_\rho z_\tau \varphi_\sigma(u) - i \varepsilon^{\mu\nu} \varphi'(u) \right], \quad (64)$$

Also, the matrix element of the 3-parton operator can be written as

$$\langle P(p)|\bar{q}(x)\sigma^{\alpha\beta} G^{\mu\nu}(tx + ty) q(y)|0\rangle = \frac{f_{3\pi}}{4} (\bar{u} \cdot p)^2 n_\mu \left( n^\nu \varepsilon^{\alpha\beta\mu\nu} - n^\nu \varepsilon^{\alpha\beta\rho\nu} \right) \int \mathcal{D}^3 \vec{\alpha} \varphi_{3\pi}(\vec{\alpha}). \quad (65)$$

From the comparison with Eqs. (48)-(56) we can easily extract the matrix elements of all the SCET operators introduced in Sec. III

$$\langle P(p)|J_A(\vec{\omega})|0\rangle = -\frac{i}{2} f_P \bar{n} \cdot p \varphi(u), \quad (66)$$

$$\langle P(p)|P(\vec{\omega})|0\rangle = -if_P \mu_P \varphi_p(u), \quad (67)$$

$$\langle P(p)|\bar{P}(\vec{\omega})|0\rangle = -\frac{i}{6} f_P \mu_P \varphi'(u), \quad (68)$$

$$\langle P(p)|T^{\alpha\beta}(\vec{\omega})|0\rangle = -\frac{1}{12} f_P \mu_P \varepsilon_\perp^{\alpha\beta} \left[ \frac{\bar{u} - u}{u \bar{n}} \varphi_\sigma(u) - 6 R \left( \frac{G_{P_2}(u)}{u} + \frac{G_{P_3}(u)}{\bar{u}} \right) \right], \quad (69)$$

$$\langle P(p)|\bar{T}^{\alpha\beta}(\vec{\omega})|0\rangle = -\frac{1}{12} f_P \mu_P \varepsilon_\perp^{\alpha\beta} \left[ \frac{1}{u \bar{n}} \varphi_\sigma(u) - 6 R \left( \frac{G_{P_3}(u)}{u} - \frac{G_{P_3}(u)}{\bar{u}} \right) \right], \quad (70)$$

$$\langle P(p)|T_3^{\alpha\beta}(\vec{\omega})|0\rangle = -\frac{i}{2} f_{3\pi} \varepsilon_\perp^{\alpha\beta} \varphi_{3\pi}(\vec{\alpha}) \frac{\alpha_3}{\alpha_3^2}, \quad (71)$$

where $R = f_{3\pi}/(f_P \mu_P)$, $(\omega_1, \omega_2) = (u, -\bar{u}) \bar{n} \cdot p$, $(\omega_1, \omega_2, \omega_3) = (\alpha_1, -\alpha_2, -\alpha_3) \bar{n} \cdot p$ and

$$G_{P_2}^{(t)}(u) = \frac{d}{du} \int_0^u d\alpha_1 \int_0^{\bar{u}} d\alpha_2 \frac{u - \alpha_1}{\alpha_2^2} \varphi_{3\pi}(\vec{\alpha}), \quad (72)$$

$$G_{P_3}^{(t)}(u) = \frac{d}{du} \int_0^u d\alpha_1 \int_0^{\bar{u}} d\alpha_2 \frac{\bar{u} - \alpha_2}{\alpha_3^2} \varphi_{3\pi}(\vec{\alpha}). \quad (73)$$

All the other matrix elements vanish in the limit in which we neglect twist-4 wave functions. Some comments on the manipulations we used to derive the above results are necessary. The operator $\bar{P}$ does not appear in Eqs. (48)-(56) but is present in the projection of $\bar{q}(x)\sigma^{\mu\nu}\gamma_5 q(y)$, hence it can be expressed in terms of the above introduced wave functions. The functions $G_{P_2}^{(t)}(u)$ and $G_{P_3}^{(t)}(u)$ have been obtained by inserting Eq. (71) into Eq. (52),
using the identities

\[
\int_0^1 dt \int D^3 \vec{\alpha} \mathcal{F}(\vec{\alpha}) = \int_0^1 du e^{i(u \vec{p} \cdot \vec{\alpha} + \bar{u} \vec{p} \cdot \bar{\alpha})} \int_0^u d\alpha_1 \int_0^{\bar{u}} d\alpha_2 \left[ \frac{u - \alpha_1}{\alpha_3} \right] \mathcal{F}(\bar{\alpha}), \quad \text{(74)}
\]

\[
\int_0^1 dt \int D^3 \vec{\alpha} \mathcal{F}(\vec{\alpha}) = \int_0^1 du e^{i(u \vec{p} \cdot \vec{\alpha} + \bar{u} \vec{p} \cdot \bar{\alpha})} \int_0^u d\alpha_1 \int_0^{\bar{u}} d\alpha_2 \left[ \frac{\bar{u} - \alpha_2}{\alpha_3} \right] \mathcal{F}(\bar{\alpha}), \quad \text{(75)}
\]

which are valid for any function \( \mathcal{F}(\vec{\alpha}) \) and, finally, integrating by parts. The subscripts \( x \) and \( y \) indicate that \( G_{P_x}^{(t)} \) and \( G_{P_y}^{(t)} \) stem from terms proportional to \( x_{\perp} \) and \( y_{\perp} \), respectively. The symmetry property of \( \varphi_{3\pi}(\vec{\alpha}) \) under the exchange \( \alpha_1 \leftrightarrow \alpha_2 \) implies relations between these two functions. For the pion, G-parity implies [21]:

\[
\varphi_{3\pi}(\alpha_1, \alpha_2, \alpha_3) = \varphi_{3\pi}(\alpha_2, \alpha_1, \alpha_3) \implies G_{P_y}^{(t)}(1 - u) = -G_{P_x}^{(t)}(u). \quad \text{(76)}
\]

Finally, the equations of motion in QCD imply relations among the twist-3 wave functions. In the approach discussed here they follow from the relations among SCET operators presented in Sec. II. The first such relation is obtained by noting that the matrix element of \( T^{\mu\nu} \) can be directly extracted from the first line in Eq. [52]. Comparing this determination with Eq. [63] we find

\[
\varphi_\sigma'(u) - \frac{1}{\bar{u} - u} \varphi_\sigma(u) + 6R \left[ \frac{G_{P_x}^{(t)}(u)}{u} + \frac{G_{P_y}^{(t)}(u)}{\bar{u}} \right] = 0. \quad \text{(77)}
\]

A second equation of motion follows from using the results [67], [70] in the relation Eq. [33] and reads

\[
\phi_p(u) = \frac{1}{6u\bar{u}} \phi_\sigma(u) - R \left( \frac{1}{u} G_{P_x}^{(t)}(u) - \frac{1}{\bar{u}} G_{P_y}^{(t)}(u) \right). \quad \text{(78)}
\]

Neglecting the terms proportional to the 3-parton wave function \( \varphi_{3\pi} \), these relations give the so-called Wandzura-Wilczek relations for the 2-parton twist-3 wave functions [28]. They can be solved exactly and give the solutions \( \phi_p(u)|_{WW} = 1, \phi_\sigma(u)|_{WW} = 6u\bar{u} \).

C. Vector meson matrix elements

Let us collect the matrix elements between the vacuum and a vector meson state of all the relevant QCD operators [23, 25]. We will work in a reference frame where the momentum of the vector meson has a large component along the \( n \) direction, and a vanishing transverse momentum, thus \( p_{\mu} = \frac{2E_V}{m_V^2} n_{\mu} \). We neglect a term proportional to the mass squared of the meson, because it is irrelevant for the analysis of twist-2 and twist-3 distribution amplitudes.

The large light-cone momentum component is given by \( \bar{n} \cdot p = 2E_V + O(m_V^2/E_V) \). The polarization vector \( \eta \) is decomposed as a sum of longitudinal \( \eta_\parallel \) and transverse \( \eta_\perp \) components

\[
\eta_\parallel^\mu = \frac{\bar{n} \cdot \eta}{2} n^\mu, \quad \eta_\perp^\mu = \eta^\mu - \eta_\parallel^\mu. \quad \text{(79)}
\]

Note that these definitions are slightly different from the conventions used in Refs. [23, 25]; hence, the expressions for the various matrix elements will look slightly different. Our choice of the transverse plane is forced by the structure of the subleading SCET operators.
The QCD matrix elements read (we often integrate by parts to remove $n \cdot z$ factors):

\[
\langle V(p, \eta) | \bar{q}(x) q(y) | 0 \rangle = - \frac{i}{2} f_V^T m_V^2 z \cdot \eta^* \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} h_{\parallel}^s(u) \]

\[
= \frac{1}{2} f_V^T m_V^2 \frac{n \cdot \eta^*}{n \cdot p} \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} h_{\parallel}^s(u) ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) \gamma^5 q(y) | 0 \rangle = 0 ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) \gamma^\mu q(y) | 0 \rangle = f_V m_V \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} \left( \eta^\mu * g_{\perp}^a(u) + p^\mu z \cdot \eta^* (\phi_{\parallel}(u) - g_{\perp}^a(u)) \right)
\]

\[
= f_V m_V \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} \left( \eta_{\perp}^a * g_{\perp}^a(u) + \frac{i}{2} n^\mu \bar{n} \cdot p \eta_{\perp}^a \cdot z F(u) \right) ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) \gamma_{\mu} \gamma^5 q(y) | 0 \rangle = - \frac{1}{4} f_V m_V \epsilon_{\mu \nu \rho \sigma} \eta_{\nu} \bar{p} p_{\rho} z_{\sigma} \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} g_{\perp}^a(u)
\]

\[
= \frac{1}{4} f_V m_V \int_0^1 du \ e^{i(u p - x + \bar{u} p - y)} \left[ i \epsilon_{\perp}^{\mu \nu} \eta_{\perp, \nu} g_{\perp}^a(u) \right]
\]

\[
+ \frac{- n^\mu \bar{n} \cdot p}{2 (n^\mu \bar{n} \cdot p)^2} \bar{D} \cdot \bar{t} (n^\mu \bar{n} \cdot p) (\eta^*_{\perp} n^\nu - n^\nu \eta^*_{\perp}) h_{\perp}^{(t)}(u) - i m^2 \bar{n} \cdot \eta^* (n^\mu z_{\perp}^u - n^\nu z_{\perp}^u) \int_0^u dv (h_{\perp}^{(t)}(v) - \phi_{\perp}(v)) \right] ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) g^{\mu \nu} \gamma^\alpha q(y) | 0 \rangle = \frac{i}{4} f_V m_V (\bar{n} \cdot p)^2 n^\alpha (n^\nu \eta_{\perp}^* - n^\nu \eta_{\perp}^*) \int \bar{D}^3 \bar{\alpha} \bar{V}(\bar{\alpha}) ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) g^{\mu \nu} \gamma^\alpha \gamma^5 q(y) | 0 \rangle = \frac{1}{4} f_V m_V (\bar{n} \cdot p)^2 \epsilon_{\mu \nu \rho \sigma} n^\alpha n^\rho \eta_{\perp, \sigma} \int \bar{D}^3 \bar{\alpha} \bar{A}(\bar{\alpha}) ,
\]

\[
\langle V(p, \eta) | \bar{q}(x) g^{\alpha \beta} q(y) | 0 \rangle = \frac{1}{8} f_V^T m^2 \bar{n} \cdot \eta^* \bar{n} \cdot p \left( n^\alpha n^\mu g_{\perp}^{\beta \nu} - n^\beta n^\mu g_{\perp}^{\alpha \nu} - n^\alpha n^\nu g_{\perp}^{\beta \mu} + n^\beta n^\nu g_{\perp}^{\alpha \mu} \right) \int \bar{D}^3 \bar{\alpha} \bar{T}(\bar{\alpha}) ,
\]

where $F(u) = \int_0^u dv \ [\phi_{\parallel}(v) - g_{\perp}^{(v)}(v)]$ and $f_V$ and $f_V^T$ are the vector meson decay constants, defined as

\[
\langle V(p, \eta) | \bar{q} \gamma_{\mu} q | 0 \rangle = f_V m_V \eta_{\mu}^* , \quad \langle V(p, \eta) | \bar{q} \sigma_{\mu \nu} q | 0 \rangle = - i f_V^T (\eta_{\mu}^* p_{\nu} - \eta_{\nu}^* p_{\mu})
\]
In the above formulae, $\phi_{\parallel}(u)$ and $\phi_{\bot}(u)$ are of twist-2, while all the other wave functions are of twist-3. Since $\eta_{\perp}/\eta_{\parallel} \sim m_{V}/E_{V}$, we keep terms proportional to $m_{V}^{2}$ only when they involve $\eta_{\parallel}$.

The extraction of the SCET matrix elements proceeds in complete analogy with the pseudoscalar meson case. Comparing Eqs. (81)-(89) with Eqs. (48)-(56), we obtain:

\begin{align}
\langle V(p, \eta)|J_{V}(\vec{\omega})|0 \rangle &= \frac{1}{2} f_{V} m_{V} \bar{n} \cdot \eta^{*} \phi_{\parallel}(u), \\
\langle V(p, \eta)|J_{A}(\vec{\omega})|0 \rangle &= 0, \\
\langle V(p, \eta)|J_{T}^{\mu}(\vec{\omega})|0 \rangle &= \frac{1}{4} f_{V}^{T} \eta_{\perp}^{*} \bar{n} \cdot p \phi_{\bot}(u), \\
\langle V(p, \eta)|V_{1}^{\mu}(\vec{\omega})|0 \rangle &= f_{V} m_{V} \eta_{\perp}^{*} g_{\perp}^{(e)}(u), \\
\langle V(p, \eta)|\tilde{V}_{1}^{\mu}(\vec{\omega})|0 \rangle &= \frac{1}{4} f_{V} m_{V} \eta_{\perp}^{*} g_{\bot}^{(e)}(u), \\
\langle V(p, \eta)|V_{2}^{\mu}(\vec{\omega})|0 \rangle &= -\frac{1}{2} f_{V} m_{V} \eta_{\perp}^{*} \left[ \frac{\bar{u} - u}{u \bar{u}} F(u) - \frac{G_{V_{x}}^{(e)}(u)}{u} - \frac{G_{V_{y}}^{(e)}(u)}{\bar{u}} \right], \\
\langle V(p, \eta)|\tilde{V}_{2}^{\mu}(\vec{\omega})|0 \rangle &= -\frac{1}{2} f_{V} m_{V} \eta_{\perp}^{*} \left[ \frac{1}{u \bar{u}} F(u) - \frac{G_{V_{x}}^{(e)}(u)}{u} + \frac{G_{V_{y}}^{(e)}(u)}{\bar{u}} \right], \\
\langle V(p, \eta)|A_{1}^{\mu}(\vec{\omega})|0 \rangle &= \frac{i}{4} f_{V} m_{V} \varepsilon_{\perp}^{\mu \nu} \eta_{\perp \nu}^{*} g_{\perp}^{(a)}(u), \\
\langle V(p, \eta)|\tilde{A}_{1}^{\mu}(\vec{\omega})|0 \rangle &= i f_{V} m_{V} \varepsilon_{\perp}^{\mu \nu} \eta_{\perp \nu}^{*} g_{\bot}^{(a)}(u), \\
\langle V(p, \eta)|A_{2}^{\mu}(\vec{\omega})|0 \rangle &= \frac{i}{2} f_{V} m_{V} \varepsilon_{\perp}^{\mu \nu} \eta_{\perp \nu}^{*} \left[ \frac{\bar{u} - u}{u \bar{u}} g_{\perp}^{(a)}(u) - \frac{G_{V_{x}}^{(a)}(u)}{u} - \frac{G_{V_{y}}^{(a)}(u)}{\bar{u}} \right], \\
\langle V(p, \eta)|\tilde{A}_{2}^{\mu}(\vec{\omega})|0 \rangle &= \frac{i}{2} f_{V} m_{V} \varepsilon_{\perp}^{\mu \nu} \eta_{\perp \nu}^{*} \left[ \frac{1}{u \bar{u}} g_{\perp}^{(a)}(u) - \frac{G_{V_{x}}^{(a)}(u)}{u} + \frac{G_{V_{y}}^{(a)}(u)}{\bar{u}} \right], \\
\langle V(p, \eta)|S(\vec{\omega})|0 \rangle &= \frac{1}{2} f_{V}^{T} m_{V}^{2} \frac{\bar{n} \cdot \eta^{*}}{\bar{n} \cdot p} h_{[s]}^{(s)}(u), \\
\langle V(p, \eta)|\tilde{S}(\vec{\omega})|0 \rangle &= f_{V}^{T} m_{V}^{2} \frac{\bar{n} \cdot \eta^{*}}{\bar{n} \cdot p} h_{[t]}^{(t)}(u), \\
\langle V(p, \eta)|P(\vec{\omega})|0 \rangle &= 0, \\
\langle V(p, \eta)|\tilde{P}(\vec{\omega})|0 \rangle &= 0, \\
\langle V(p, \eta)|T_{\mu \nu}(\vec{\omega})|0 \rangle &= \frac{1}{2} f_{V}^{T} m_{V}^{2} \frac{\bar{n} \cdot \eta^{*}}{\bar{n} \cdot p} g_{\perp}^{\mu \nu} \\
&\quad \times \left[ \frac{\bar{u} - u}{u \bar{u}} \int_{0}^{u} dv \left( h_{[s]}^{(t)}(v) - \phi_{\perp}(v) \right) + \frac{G_{V_{x}}^{(t)}(u)}{2u} + \frac{G_{V_{y}}^{(t)}(u)}{2\bar{u}} \right], \\
\langle V(p, \eta)|\tilde{T}_{\mu \nu}(\vec{\omega})|0 \rangle &= \frac{1}{2} f_{V}^{T} m_{V}^{2} \frac{\bar{n} \cdot \eta^{*}}{\bar{n} \cdot p} g_{\bot}^{\mu \nu} \\
&\quad \times \left[ \frac{1}{u \bar{u}} \int_{0}^{u} dv \left( h_{[s]}^{(t)}(v) - \phi_{\perp}(v) \right) + \frac{G_{V_{x}}^{(t)}(u)}{2u} - \frac{G_{V_{y}}^{(t)}(u)}{2\bar{u}} \right],
\end{align}
\[ \langle V(p, \eta) | \mathcal{V}_3^\mu(\bar{\omega}) \rangle = \frac{1}{2} f_V m_V \eta_\perp^\mu \mathcal{V}(\bar{\alpha}) \alpha_3^3, \] (108)

\[ \langle V(p, \eta) | \mathcal{A}_3^\mu(\bar{\omega}) \rangle = -\frac{i}{2} f_V m_V \eta_\perp^\mu \mathcal{A}(\bar{\alpha}) \alpha_3^3, \] (109)

\[ \langle V(p, \eta) | \mathcal{T}_3^\mu(\bar{\omega}) \rangle = \frac{1}{4} f_V m_V^2 \frac{\bar{\eta} \cdot \eta^*}{\bar{\eta} \cdot p} g_\perp^\mu T(\bar{\alpha}) \alpha_3^2, \] (110)

where

\[ G^{(v,a,t)}_{V_x}(u) = \frac{d}{du} \int_0^u du_1 \int_0^{u_1} du_2 \frac{u - \alpha_1}{\alpha_3^3}(\mathcal{V}, \mathcal{A}, \mathcal{T})(\bar{\alpha}), \] (111)

\[ G^{(v,a,t)}_{V_y}(u) = \frac{d}{du} \int_0^u du_1 \int_0^{u_1} du_2 \frac{\bar{u} - \alpha_2}{\alpha_3^3}(\mathcal{V}, \mathcal{A}, \mathcal{T})(\bar{\alpha}). \] (112)

For the case of the \( \rho \) meson in the limit of isospin symmetry, G-parity requires that the 3-parton wave functions satisfy \( \mathcal{A}(\alpha_1, \alpha_2, \alpha_3) = \mathcal{A}(\alpha_2, \alpha_1, \alpha_3) \) and \( \{ \mathcal{V}, \mathcal{T}\}(\alpha_1, \alpha_2, \alpha_3) = -\{ \mathcal{V}, \mathcal{T}\}(\alpha_2, \alpha_1, \alpha_3) \). This implies the relations \( G^{(v)}_{V_y}(\bar{u}) = -G^{(a)}_{V_x}(u) \) and \( G^{(v,t)}_{V_y}(\bar{u}) = G^{(v,t)}_{V_x}(u) \).

Note that \( \tilde{V}_1 \) and \( \tilde{A}_1 \) do not appear in Eqs. (118-119) but their matrix elements are easily obtained via Eq. (30). The computation of all the other matrix elements is similar to the pseudoscalar meson case.

The insertion of the matrix elements, Eqs. (91)-(107), into Eq. (31) results in the following relations between the chiral-even light-cone wave functions:

\[ 2 \, g_\perp^{(v)}(u) - \frac{1}{4} g_\perp^{(a)}(u) \left( \frac{\bar{u} - u}{u \bar{u}} \right) F(u) = \frac{G^{(v)}_{V_x}(u) - G^{(a)}_{V_x}(u)}{u} + \frac{G^{(v)}_{V_y}(u) + G^{(a)}_{V_y}(u)}{\bar{u}}, \] (113)

\[ \frac{1}{2} g_\perp^{(a)}(u) - \frac{1}{4} \left( \frac{\bar{u} - u}{u \bar{u}} \right) g_\perp^{(a)}(u) + \frac{F(u)}{u \bar{u}} = \frac{G^{(v)}_{V_x}(u) - G^{(a)}_{V_x}(u)}{u} - \frac{G^{(v)}_{V_y}(u) + G^{(a)}_{V_y}(u)}{\bar{u}}. \] (114)

The identities Eq. (32) imply two other relations for the chiral-odd wave functions

\[ h^{(t)}_{\parallel}(u) - \frac{\bar{u} - u}{u \bar{u}} \int_0^u dv(h^{(t)}_{\parallel}(v) - \phi_{\perp}(v)) = \frac{1}{2u} G^{(t)}_{V_x}(u) + \frac{1}{2u} G^{(t)}_{V_y}(u), \] (115)

\[ \frac{d}{du} h^{(s)}_{\parallel}(u) - \frac{2}{u \bar{u}} \int_0^u dv(h^{(t)}_{\parallel}(v) - \phi_{\perp}(v)) = \frac{1}{u} G^{(t)}_{V_x}(u) - \frac{1}{u} G^{(t)}_{V_y}(u). \] (116)

In the limit in which we neglect higher Fock state contributions (i.e., where we set to zero the right-hand sides of Eqs. (113-114)), we obtain Wandzura-Wilczek like relations between twist-2 and twist-3 wave functions [20]. They can be again solved exactly [23, 25] and give the twist-3 wave functions in terms of the twist-2 ones \( \phi_{\parallel}(u), \phi_{\perp}(u) \). For the chiral-even structures one finds

\[ g_\perp^{(v)}(u) \big|_{\text{WW}} = \frac{1}{2} \left[ \int_0^u dv \frac{\phi_{\parallel}(v)}{v} + \int_u^1 dv \frac{\phi_{\parallel}(v)}{v} \right], \] (117)

\[ g_\perp^{(a)}(u) \big|_{\text{WW}} = 2 \left[ \bar{u} \int_0^u dv \frac{\phi_{\parallel}(v)}{v} + u \int_u^1 dv \frac{\phi_{\parallel}(v)}{v} \right]. \] (118)
and for the chiral-odd wave functions

\[
\begin{align*}
    h^{(t)}_{\parallel}(u)_{\text{WW}} &= (2u - 1) \left[ \int_0^u dv \frac{\phi_\perp(v)}{v} - \int_u^1 dv \frac{\phi_\perp(v)}{v} \right] \\
    h^{(s)}_{\parallel}(u)_{\text{WW}} &= 2 \left[ \bar{u} \int_0^u dv \frac{\phi_\perp(v)}{v} + u \int_u^1 dv \frac{\phi_\perp(v)}{v} \right].
\end{align*}
\]

**V. REPARAMETERIZATION INVARIANCE CONSTRAINTS**

The soft-collinear effective theory has an additional symmetry, related to the Lorentz invariance of the full theory, which was explicitly broken by defining the effective theory in terms of the arbitrary light-cone vectors \( n_\mu \) and \( \bar{n}_\mu \). This symmetry manifests itself as an invariance under small changes in the light-cone vectors \( n_\mu \) and \( \bar{n}_\mu \), and is usually called reparameterization invariance (RPI). The RPI has been shown to impose rather stringent constraints on the form of the effective theory Lagrangian and heavy-light currents [10, 11, 13, 15, 16]. For earlier uses of Lorentz invariance in hard scattering processes, see Refs. [30].

In this Section we show that reparameterization invariance can be used to derive strict constraints on the Wilson coefficients of the subleading collinear operators considered in this paper. In particular, these constraints fix the coefficients of the three-parton operators of the form \( \bar{\chi}_{n,\omega_1} W^\dagger iD_\perp W \chi_{n,\omega_2} \), in terms of coefficients of the leading two-body operators of type \( \bar{\chi}_{n,\omega_1} \Gamma \chi_{n,\omega_2} \).

The explicit form of the RPI constraints for the subleading collinear operators is in general process dependent, and depends on the SCET operators which are allowed in the expansion of the physical quantity being described (effective Hamiltonian, current, etc.). Despite this diversity, there is one general feature which is common to all these situations: the SCET expansion must contain (at least) one additional vector \( z_\mu \), in addition to \( n_\mu \) and \( \bar{n}_\mu \). This vector can be for example \( v_\mu \), the heavy quark velocity in problems involving heavy quark decay, or a space-time vector \( z_\mu \), describing the nonlocality of a T-product, as for example in hard scattering processes. The presence of the additional vector is required by type III RPI: the Wilson coefficients of the SCET operators must depend on RPI invariant arguments, and it is impossible to form such combinations from \( \bar{n} \cdot \mathcal{P} \) and \( \bar{n} \cdot \mathcal{P}^\dagger \) alone. The situation is different in the presence of an additional vector \( z \), when RPI-III invariant combinations can be formed as \( n \cdot z \bar{n} \cdot \mathcal{P} \) and \( n \cdot z \bar{n} \cdot \mathcal{P}^\dagger \).

In the following we illustrate the form of the RPI constraints using a few examples. We will derive the RPI constraints on the Wilson coefficients in the matching of operators with scalar and vector quantum numbers, which will be left completely unspecified.

**A. Scalar operator**

Consider first a scalar QCD chiral-even operator \( S(z) \), depending on an arbitrary vector \( z_\mu \). Keeping terms of \( O(1, \lambda) \), the most general form for this operator in SCET contains the
terms (integration over \( \omega_i \) is implied on the RHS)

\[
S(z) = C(\omega_1, \omega_2)(n \cdot z)\mathcal{J}_V(\omega_1, \omega_2) + \sum_{i=1}^{2} D_i(\omega_1, \omega_2)(z_a \nu^a_i) + \sum_{i=1}^{2} \tilde{D}_i(\omega_1, \omega_2)(z_a \tilde{\nu}^a_i) + E(\omega_1, \omega_2, \omega_3)(z_a \nu^a_3). \tag{121}
\]

This expansion contains the most general structures transforming as a scalar under Lorentz transformations, and which are chiral-even and invariant under type-III RPI. As mentioned above, the Wilson coefficients \( C, D_i, \tilde{D}, E \) are functions of the type III invariants \( n \cdot z \omega_1 \) and \( n \cdot z \omega_2 \). In the presence of more than one vector \( z \), the possible number of structures on the right-hand side is correspondingly larger. The Wilson coefficients will also depend on all possible type III RPI invariant combinations \( n \cdot z \bar{n} \cdot \mathcal{P}^{(i)} \). The constraints derived below can be straightforwardly generalised to such cases.

We will show in the following that type-I and -II RPI gives stringent constraints on the form of the Wilson coefficients appearing in Eq. (121). There are two constraints following from type-I RPI

(\text{I-1):} \quad \left( 1 + \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - \sum_{i=1,2} D_i(\omega_1, \omega_2) - \frac{1}{2} E(\omega_1, \omega_2, 0) = 0, \tag{122}
\]

(\text{I-2):} \quad \tilde{D}_1(\omega_1, \omega_2) = 0. \tag{123}

and five other constraints from type-II RPI

(\text{II-1):} \quad C(\omega_1, \omega_2) - D_1(\omega_1, \omega_2) = 0, \tag{124}
(\text{II-2):} \quad \left( \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - D_2(\omega_1, \omega_2) = 0, \tag{125}
(\text{II-3):} \quad \tilde{D}_1(\omega_1, \omega_2) = 0, \tag{126}
(\text{II-4):} \quad \left( \omega_1 \frac{\partial}{\partial \omega_1} - \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - \tilde{D}_2(\omega_1, \omega_2) = 0, \tag{127}
(\text{II-5):} \quad C(\omega_1 - \omega_3, \omega_2) - C(\omega_1, \omega_2 + \omega_3) + \omega_3 \left( \frac{\partial}{\partial \omega_1} C(\omega_1 - \omega_3, \omega_2) + \frac{\partial}{\partial \omega_2} C(\omega_1, \omega_2 + \omega_3) \right) + \frac{1}{2} E(\omega_1, \omega_2, \omega_3) = 0. \tag{128}

In the following we present the derivation of these constraints, focusing on the type-II RPI constraints, which are technically more involved. We start by computing the variation of the leading \( O(\lambda^0) \) term in (121) under a type II reparameterization transformation \( \bar{n}_\mu \rightarrow \bar{n}_\mu + \varepsilon^\perp_\mu \). This receives contributions from the change in the Wilson coefficient and in the operator

\[
\delta_{\text{II}}[C(n \cdot z \omega_i)\mathcal{J}_V(\omega_i)] = \delta_{\text{II}}[C(n \cdot z \omega_i)]\mathcal{J}_V(\omega_i) + C(n \cdot z \omega_i)\delta_{\text{II}}[\mathcal{J}_V(\omega_i)]. \tag{129}
\]

The variation of the leading order operator \( \mathcal{J}_V(\omega_i) \) in Eq. (129) can be computed using the
action of the RP transformation on the collinear fields given in [16]. One finds

$$\delta^{(\lambda)} J_V(\omega_i) = \left[ \frac{\bar{\xi}_n}{2} (i \partial_{\perp}) \right] W_{\omega_1} - \int d\omega_3 \left\{ \bar{\chi}_{n,\omega_1+\omega_3} \left[ \frac{1}{n \cdot \mathcal{P}} W^{\dagger} \varepsilon_{\perp} i D_{c\perp} W \right]_{\omega_3} \frac{\delta}{\delta} \chi_{\omega_1} \right\} + \bar{\chi}_{n,\omega_1} \left[ W^{\dagger} (\varepsilon_{\perp} i D_{c\perp}) W \frac{1}{n \cdot \mathcal{P}} \right] \chi_{\omega_2}$$

where the first line comes from the change in the collinear quark fields, and the second from the variation of $W$.

The variation of the Wilson coefficient in Eq. (129) can be found by recalling that $\omega_1 \to \bar{n} \cdot \mathcal{P}$ and $\omega_2 \to n \cdot \mathcal{P}$

$$\delta^{(\lambda)} [C(n \cdot z \omega_i)] J_V = \frac{\partial C(n \cdot z \omega_i)}{\partial \omega_1} \left[ \bar{\chi}_{n,\omega_1} (\varepsilon_{\perp} \mathcal{P}_{\perp}^{\dagger}) \frac{\delta}{\delta} \chi_{\omega_2} \right] + \frac{\partial C(n \cdot z \omega_i)}{\partial \omega_2} \left[ \bar{\chi}_{n,\omega_1} \frac{\delta}{\delta} \chi_{\omega_2} \right]$$

The two terms can further be written in terms of the operators introduced in Sec. II as

$$\bar{\chi}_{n,\omega_1} (\varepsilon_{\perp} \mathcal{P}_{\perp}^{\dagger}) \frac{\delta}{\delta} \chi_{\omega_2} = \left[ \frac{\bar{\xi}_n}{2} (i D_{c\perp}) W \right]_{\omega_1} \frac{\delta}{\delta} \chi_{\omega_2} - \left[ \frac{\bar{\xi}_n}{2} \varepsilon_{\perp} i D_{c\perp} W \right]_{\omega_1} \frac{\delta}{\delta} \chi_{\omega_2}$$

and

$$\bar{\chi}_{n,\omega_1} \frac{\delta}{\delta} \chi_{\omega_2} = \bar{\chi}_{n,\omega_1} \left[ W^{\dagger} \varepsilon_{\perp} i D_{c\perp} \xi_{n} \right]_{\omega_2} - \bar{\chi}_{n,\omega_1} \left[ W^{\dagger} \varepsilon_{\perp} (i D_{c\perp}) \xi_{n} \right]_{\omega_2}$$

Combining everything, one finds the following total result for the variation of the leading term in Eq. (121) under a type-II RPI

$$\delta^{(\lambda)} [C(n \cdot z \omega_i)] J_V(\omega_i) =$$

$$\left[ \frac{\partial}{\partial \omega_1} C(\omega_i) \right] \left[ \frac{1}{2} \omega_3 \varepsilon_{\perp} \mathcal{V}_2 + \varepsilon_{\perp} \tilde{\mathcal{V}}_2 \right] - \int d\omega_3 \omega_3 \varepsilon_{\perp} \mathcal{V}_3(\omega_1 + \omega_3, \omega_2, \omega_3)$$

$$+ \left[ \frac{\partial}{\partial \omega_2} C(\omega_i) \right] \left[ \frac{1}{2} \omega_3 \varepsilon_{\perp} \mathcal{V}_2 - \varepsilon_{\perp} \tilde{\mathcal{V}}_2 \right] - \int d\omega_3 \omega_3 \varepsilon_{\perp} \mathcal{V}_3(\omega_1, \omega_2 - \omega_3, \omega_3)$$

$$+ C(\omega_i) \left[ \frac{1}{2} \varepsilon_{\perp} \mathcal{V}_1(\omega_i) - \int d\omega_3 \left( \varepsilon_{\perp} \mathcal{V}_3(\omega_1 + \omega_3, \omega_2, \omega_3) - \varepsilon_{\perp} \mathcal{V}_3(\omega_1, \omega_2 - \omega_3, \omega_3) \right) \right]$$

This has to be cancelled by the $O(\lambda)$ terms in the variation of the power suppressed terms in Eq. (121). The transformation of all collinear subleading operators $\mathcal{V}_{1,2}, \tilde{\mathcal{V}}_{1,2}$ and $\mathcal{V}_3$ is the same, and is given by

$$\delta^{(\lambda)} \mathcal{V}_i(\omega_i) = - \frac{1}{2} n^a \varepsilon_{\perp} \mathcal{V}_i(\omega_i)$$
Using this relation in Eq. (121), and requiring the cancellation of all independent structures, gives the type-II RPI constraints Eqs. (124)-(128).

The type-I RPI constraints Eqs. (122)-(123) are derived in a very similar way, so we only sketch the relevant steps. Under a type I RPI transformation \( n_{\mu} \rightarrow n_{\mu} + \Delta_{\mu} \), the variation of the leading order term in Eq. (121) is given by

\[
\delta \left[ C(\omega_i) J_V(\omega_i) \right] = (\Delta_{\perp} \cdot z_{\perp}) \left[ \left( \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_i) \right] J_V(\omega_i). \tag{136}
\]

This has to cancel the \( O(\lambda) \) term in the variation of the subleading terms. The action of the type I RPI transformation on the individual operators is easily obtained as

\[
\delta^{(\lambda)} V^\mu_1(\omega_i) = \delta^{(\lambda)} V^\mu_2(\omega_i) = -\Delta_{\mu} J_V(\omega_i), \tag{137}
\]
\[
\delta^{(\lambda)} \tilde{V}^\mu_1(\omega_i) = -i\varepsilon_{\mu\nu} \Delta_{\nu} J_A(\omega_i), \tag{138}
\]
\[
\delta^{(\lambda)} V^\mu_2(\omega_i) = 0, \tag{139}
\]
\[
\delta^{(\lambda)} \tilde{V}^\mu_2(\omega_i \omega_2 \omega_3) = -\frac{1}{2} \Delta_{\mu} J_V(\omega_1 \omega_2) \delta(\omega_3). \tag{140}
\]

Inserting this into Eq. (121), it is easy to see that the total variation of the \( \tilde{V}^\mu_1 \) terms cannot be cancelled by anything else. This gives \( \tilde{D}_1 = 0 \). Requiring the cancellation of the terms proportional to \( (z_{\perp} \cdot \Delta_{\perp}) J_V \) gives Eq. (134). This completes the proof of the RPI constraints.

We note here the remarkable result that the Wilson coefficients of the \( O(\lambda) \) operators in the expansion of a scalar operator are completely fixed by RPI in terms of the Wilson coefficient \( C(\omega_i) \) of the leading operator. This is different from the case for the heavy-light currents studied in Ref. [11], where several of the subleading operators were not constrained from symmetry arguments alone, and their Wilson coefficients have to be determined by explicit matching computations.

### B. Vector operator

We consider next the reparameterization invariance constraints on the Wilson coefficients in the matching of a chiral-even operator with the quantum numbers of the vector current \( V_\mu(z) \). For this case, the SCET expansion can contain more terms. Keeping again terms up to subleading order we can write again the most general type-III RP invariant structure as

\[
V_\mu(z) = n_{\mu} C(\omega_1, \omega_2) J_V(\omega_1, \omega_2) + \sum_{i=1}^{2} B_i(\omega_1, \omega_2) V_{i\mu} + \sum_{i=1}^{2} \tilde{B}_i(\omega_1, \omega_2) \tilde{V}_{i\mu} \tag{141}
\]

\[
\quad + \frac{n_{\mu}}{n \cdot z} \sum_{i=1}^{2} D_i(\omega_1, \omega_2)(z \cdot \nabla_i) + \frac{n_{\mu}}{n \cdot z} \sum_{i=1}^{2} \tilde{D}_i(\omega_1, \omega_2)(z \cdot \tilde{\nabla}_i)
\]

\[
\quad + E_1(\omega_1, \omega_2, \omega_3) V_{3\mu} + \frac{n_{\mu}}{n \cdot z} E_2(\omega_1, \omega_2, \omega_3)(z \cdot \nabla_3).
\]

We neglect here terms of the form \( z_{\mu} S(z) \) with \( S(z) \) a scalar quantity which is RP invariant by itself. The most general form for such a term has been discussed in the previous section.
The constraints for this case can be derived in analogy with those for the scalar current. We find four constraints from type I RPI

(I-1): \[ C(\omega_1, \omega_2) - \sum_{i=1,2} B_i(\omega_1, \omega_2) - \frac{1}{2} E_i(\omega_1, \omega_2, 0) = 0 , \] (142)

(I-2): \[ \left( \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - \sum_{i=1,2} D_i(\omega_1, \omega_2) - \frac{1}{2} E_i(\omega_1, \omega_2, 0) = 0 , \] (143)

(I-3): \[ \tilde{B}_1(\omega_1, \omega_2) = 0 , \] (144)

(I-4): \[ \tilde{D}_1(\omega_1, \omega_2) = 0 . \] (145)

and four other constraints from type II RPI

(II-1): \[ C(\omega_1, \omega_2) - B_1(\omega_1, \omega_2) - D_1(\omega_1, \omega_2) = 0 , \] (146)

(II-2): \[ \left( \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - B_2(\omega_1, \omega_2) - D_2(\omega_1, \omega_2) = 0 , \] (147)

(II-3): \[ \left( \omega_1 \frac{\partial}{\partial \omega_1} - \omega_2 \frac{\partial}{\partial \omega_2} \right) C(\omega_1, \omega_2) - \tilde{B}_2(\omega_1, \omega_2) - \tilde{D}_2(\omega_1, \omega_2) = 0 , \] (148)

(II-4): \[ C(\omega_1 - \omega_3, \omega_2) - C(\omega_1, \omega_2 + \omega_3) \]
\[ + \omega_3 \left( \frac{\partial}{\partial \omega_1} C(\omega_1 - \omega_3, \omega_2) + \frac{\partial}{\partial \omega_2} C(\omega_1, \omega_2 + \omega_3) \right) \]
\[ + \frac{1}{2} \sum_{i=1,2} E_i(\omega_1, \omega_2, \omega_3) = 0 . \] (149)

It is instructive to compare the RPI constraints Eqs. (142)-(149) with the explicit results for the matching of the nonlocal vector operator \( \bar{q}(x)\gamma_\mu q(y) \) onto SCET operators given in Eq. (50). For simplicity we take \( x_\mu = -y_\mu = z_\mu \) in Eq. (50), which can then be written in a form similar to Eq. (141) provided that one takes

\[ C(\omega_1, \omega_2) = e^{\frac{i}{\sqrt{2}} n \cdot z(\omega_1 + \omega_2)} , \] (150)

\[ B_1(\omega_1, \omega_2) = C(\omega_1) , \quad B_2(\omega_1) = 0 , \quad \tilde{B}_{1,2}(\omega_1) = 0 , \] (151)

\[ D_1(\omega_1, \omega_2) = 0 , \quad D_2(\omega_1) = \frac{i}{2} n \cdot z(\omega_1 + \omega_2) C(\omega_1, \omega_2) , \] (152)

\[ \tilde{D}_1(\omega_1, \omega_2) = 0 , \quad \tilde{D}_2(\omega_1) = \frac{i}{2} n \cdot z(\omega_1 - \omega_2) C(\omega_1, \omega_2) , \] (153)

\[ E_1(\omega_1, \omega_2, \omega_3) = 0 , \quad E_2(\omega_1) = - \int_0^1 dt(2t - 1) \omega_3^2 (n \cdot z)^2 e^{\frac{i}{\sqrt{2}} n \cdot z(\omega_1 + \omega_2 + \omega_3(2t-1))} . \] (154)

Performing the integration in \( E_2(\omega_1) \) one finds

\[ E_2(\omega_1) = -2 \left( 1 - \frac{i}{2} n \cdot z \omega_3 \right) e^{\frac{i}{\sqrt{2}} n \cdot z(\omega_1 + \omega_2 + \omega_3)} + 2 \left( 1 + \frac{i}{2} n \cdot z \omega_3 \right) e^{\frac{i}{\sqrt{2}} n \cdot z(\omega_1 + \omega_2 - \omega_3)} . \] (155)

It is easy to see that the constraints Eqs. (142)-(149) are indeed satisfied with these results for the Wilson coefficients. For the case of the nonlocal vector current, the matching Eq. (50) can be worked out using the equations of motion in QCD and is exact to all orders. The true power of the RPI constraints Eqs. (142)-(149) becomes apparent in those cases where the Wilson coefficients have to be obtained in perturbation theory, and where such constraints can provide a useful check.
VI. APPLICATION: WEAK ANNIHILATION IN $B \to V \gamma$ DECAYS

In this Section we discuss an application where the subleading operators constructed in Sec. IV appear in a situation of physical interest. Consider real photon emission from an energetic $q\bar{q}$ pair, which subsequently hadronizes into a meson. This process is relevant for the rare radiative decay $B \to V \gamma$, where it contributes to the weak annihilation amplitude.

In the following we will match the amplitudes contributing to this process onto SCET operators, including the three-parton operators constructed in Sec. II. The matrix elements of these operators are then computed with the help of the relations in Sec. IV. For the case of a vector current, the corresponding matrix element is fixed by an exact Ward identity [31, 32]. In addition to serving as illustration for the use of the matrix elements computed here, this will provide, at the same time, a strong check for the consistency of our results.

Let us consider the matrix element of the weak current $\bar{d}\gamma^{\mu}P_L u$ ($P_L = (1-\gamma^5)/2$) between the vacuum and a state with a transverse polarised vector meson and a photon:

$$\langle V(p, \eta)\gamma(q, \epsilon) | (\bar{d}\gamma^{\mu}P_L u)(0) | 0 \rangle = -ie \epsilon^*_\alpha \int d^4x \; e^{iq\cdot x} \langle V(p, \eta) | T[(\bar{d}\gamma^{\mu}P_L u)(0), j^a_{\text{em.}}(x)] | 0 \rangle$$

$$\equiv -ie \epsilon^*_\alpha \langle V(p, \eta) | T^\mu\alpha | 0 \rangle . \tag{156}$$

The conservation of the electromagnetic and the weak currents implies in the usual way the following Ward identities for the matrix element of the time-ordered product in Eq. 156 [31].
\[ q_\alpha \langle V(p, \eta)|T^{\mu \alpha}|0\rangle = \frac{i}{2} (Q_d - Q_u) f_V m_V \eta^{\mu*}, \quad (157) \]
\[ (p_\mu + q_\mu) \langle V(p, \eta)|T^{\mu \alpha}|0\rangle = \frac{i}{2} (Q_d - Q_u) f_V m_V \eta^{\alpha*}. \quad (158) \]

We checked that the electromagnetic Ward identity, Eq. (157), is satisfied at leading and subleading order but its check does not require any cancellation between the matrix elements of the various subleading SCET operators. We will therefore focus onto the weak current Ward identity, Eq. (158).

In the following we project the T-product (156) onto SCET operators, and then use the matrix elements computed in the previous section to reproduce Eq. (158). For generality we keep the Dirac structure of the weak current completely general \( \bar{u} \gamma_\mu \alpha \) and define the operator

\[ T^\alpha(q) = \int d^4x e^{iq \cdot x} T[(\bar{d}T \eta)(0), \bar{\eta} \gamma_\alpha(x)]. \quad (159) \]

For definiteness, we take the photon momentum along the \( \bar{n} \) direction \( q_\mu = E_\gamma \bar{n}_\mu \), and the meson moving with a large momentum component along the \( n_\mu \) direction. Expanding the QCD graphs contributing to the \( T^- \) product and keeping terms up to subleading order gives the following tree level matching

\[ T^\alpha = T^\alpha_{(0)} + T^\alpha_{(1)}; \quad (160) \]
\[ T^\alpha_{(0)} = \frac{i Q_d}{\omega_1} \bar{\chi}^{(d)}_{n_\omega_1} \gamma^\alpha \frac{\not{\omega}}{2} \Gamma \chi^{(u)}_{n_\omega_2} + \frac{i Q_u}{\omega_2} \bar{\chi}^{(u)}_{n_\omega_1} \frac{\not{\omega}}{2} \gamma^\alpha \chi^{(u)}_{n_\omega_2}. \quad (161) \]
\[ T^\alpha_{(1)} = i Q_d \left\{ \frac{2}{n \cdot q \omega_1} \left[ \bar{\xi}^{(d)}_{n_\omega_1} (iD^\perp\dagger) W \right]_{\omega_1} \Gamma \chi^{(u)}_{n_\omega_2} + \frac{1}{\omega_1 \omega_2} \bar{\chi}^{(d)}_{n_\omega_1} \gamma^\alpha \frac{\not{\omega}}{2} \Gamma \chi^{(u)}_{n_\omega_2} \right\} - \frac{1}{n \cdot q \omega_1 (1 - \omega_3)} \bar{\chi}^{(d)}_{n_\omega_1} \gamma^\alpha \frac{1}{n \cdot \not{P}} W^\dagger \not{P} W \Gamma \chi^{(u)}_{n_\omega_2} \right\} \]
\[ + i Q_u \left\{ \frac{2}{n \cdot q \omega_2} \bar{\xi}^{(u)}_{n_\omega_1} \Gamma [W^\dagger i D^\perp\dagger W \chi^{(d)}_{n_\omega}]_{\omega_2} + \frac{1}{\omega_1 \omega_2} \bar{\xi}^{(d)}_{n_\omega_1} \frac{\not{\omega}}{2} (iD^\perp\dagger) W \chi^{(d)}_{n_\omega_2} \Gamma \chi^{(u)}_{n_\omega_2} \right\} + \frac{1}{n \cdot q \omega_2 (\omega_2 + \omega_3)} \bar{\chi}^{(d)}_{n_\omega_1} \Gamma \left[ \frac{1}{n \cdot \not{P}} W^\dagger \not{P} W \right] \chi^{(d)}_{n_\omega_2} \chi^{(u)}_{n_\omega_2}. \quad (162) \]

On the right-hand side, integration over \( \omega_i \) is implied.

Some details about this matching computation are perhaps in order. The two-body subleading operators (the first and third lines of \( T^\alpha_{(1)} \)) can be obtained from expanding the graphs with an external \( du \) quark pair (see Fig. 1(a)). The remaining three-body operators (the second and fourth lines of \( T^\alpha_{(1)} \)) are computed by expanding the QCD graphs in Fig. 1(b)-(d) and subtracting the insertion of the two-body operators computed in the previous step (Fig. 2(b)-(d)).

The case of the weak current considered in Eq. (156) is obtained by taking \( \Gamma \rightarrow \gamma^\mu P_L \) and expressing the operators in Eqs. (160) in terms of the subleading collinear operators introduced in Sec. II. We will assume the \( \alpha \) Lorentz index to be strictly orthogonal, corresponding
to real photon emission. This gives

$$T^{\mu \alpha} = T_{(0)}^{\mu \alpha} + T_{(1)}^{\mu \alpha},$$

$$T_{(0)}^{\mu \alpha} = \left[ i g_{\perp}^{\mu \alpha} \left( \frac{Q_d}{\omega_1} + \frac{Q_u}{\omega_2} \right) + \epsilon_{\perp}^{\mu \alpha} \left( \frac{Q_d}{\omega_1} - \frac{Q_u}{\omega_2} \right) \right] \frac{\mathcal{J}_A - \mathcal{J}_V}{2},$$

$$T_{(1)}^{\mu \alpha} = \frac{i}{4} \eta^{\mu} \left( \frac{Q_d}{\omega_1} \left[ g_{\perp}^{(1)}(u) \frac{2}{u} - g_{\perp}^{(a)}(u) \frac{8}{u} \right] - F(u) - G_{\perp}^{(v)}(u) \frac{V(\tilde{\alpha}) + A(\tilde{\alpha})}{2\alpha_1(1 - \alpha_2)} \right) + \frac{i}{4} \eta^{\mu} \omega_2^2 \left( \frac{Q_u}{\omega_1(\omega_1 - \omega_3)} \left[ g_{\perp}^{(1)}(u) \frac{2}{u} - g_{\perp}^{(a)}(u) \frac{8}{u} \right] - F(u) + G_{\perp}^{(v)}(u) \frac{2V(\tilde{\alpha}) - A(\tilde{\alpha})}{2\alpha_2(1 - \alpha_1)} \right)$$

After the insertion of the SCET matrix elements and the contraction with $p^\mu + q^\mu = \frac{n_p}{2} n^\mu + E_\gamma \bar{n} n^\mu$, the matrix element of the leading order term vanishes, $(p_\mu + q_\mu)\langle T^{\mu \alpha}_0 \rangle = 0$. The subleading contribution, on the other hand, gives

$$\langle p_\mu + q_\mu \rangle \langle T^{\mu \alpha}_1 \rangle = \frac{1}{2} Q_d f V_m V_n \eta^{\alpha \ast}_\perp \left[ g_{\perp}^{(1)}(u) \frac{2}{u} - g_{\perp}^{(a)}(u) \frac{8}{u} \right] - F(u) - G_{\perp}^{(v)}(u) \frac{V(\tilde{\alpha}) + A(\tilde{\alpha})}{2\alpha_1(1 - \alpha_2)} + \frac{1}{2} Q_u f V_m V_n \epsilon^{\alpha \beta}_\perp \eta^{\ast \beta}_\perp \left[ g_{\perp}^{(1)}(u) \frac{2}{u} - g_{\perp}^{(a)}(u) \frac{8}{u} \right] - F(u) + G_{\perp}^{(v)}(u) \frac{2V(\tilde{\alpha}) - A(\tilde{\alpha})}{2\alpha_2(1 - \alpha_1)}$$

where the integrations $\int_0^1 \frac{1}{u} du$ and $\int_0^1 \frac{1}{u} du \int_0^{1-\alpha_1} \frac{1}{\alpha_2(1-\alpha_2)}$ are understood. Using the QCD equations of motion [23, 24] it is easy to verify that the $\epsilon^{\alpha \beta}_\perp \eta^{\ast \beta}_\perp$ pieces vanish while the integral over the $\eta^{\alpha \ast}_\perp$ terms reproduces exactly the Ward identity, Eq. (158). In proving this cancellation, the following identities are useful

$$\int_0^1 \frac{1}{u} du \frac{1}{u} G_{\perp}^{(v)}(u) - \int_0^1 \frac{1}{u} du \int_0^{1-\alpha_1} \frac{1}{\alpha_2(1-\alpha_2)} \frac{V(\alpha_1)}{\alpha_1(1-\alpha_2)} = 0,$$

$$\int_0^1 \frac{1}{u} du \frac{1}{u} G_{\perp}^{(a)}(u) - \int_0^1 \frac{1}{u} du \int_0^{1-\alpha_1} \frac{1}{\alpha_2(1-\alpha_2)} \frac{A(\alpha_1)}{\alpha_1(1-\alpha_2)} = 0.$$

Adopting the parameterization of the vector meson wave functions given in Refs. [23, 25], we can get an idea of the numerical impact of the three-particle contributions. First of all, note that the equations of motion of QCD allow to express the twist-3 two-particle wave functions in terms of the twist-2 two-particle and twist-3 three-particle ones. This implies that, in the limit in which we consistently set to zero the functions $V$ and $A$, the Ward identity will be still satisfied. This is a peculiar property of the check that we are performing and it is not a general feature of exclusive decay amplitudes. From the inspection
of Eq. (166) we see that three-particle light-cone wave functions can enter the amplitude either by modifying the SCET matrix elements of the subleading two-particle operators (e.g. $V_2^\mu$, $A_2^\mu$, ...) or with new genuine contributions (e.g. $V_3^\mu$, $A_3^\mu$, ...). In our case both contributions are of order 10% and cancel each other. We expect, therefore, effects of order 20% in generic exclusive amplitudes that start at subleading order in $\lambda$. This is quite common for decays with vector mesons in the final state because, in this case, the usual $V-A$ weak current has no leading projections onto states with a transverse polarised meson. For pseudo-scalar mesons, on the other hand, we do not expect large contributions because the leading SCET operators have non vanishing projection and are expected to give the dominant contribution.

Finally, note that the importance of these contributions strongly depends on the form of the hard scattering kernel. Consider, for instance, the operator $V_2^\mu$. We plot in Fig. 3 its matrix element, Eq. (166), with and without the inclusion of the three-particle contributions. In the former case, the shape of the distribution amplitude is changed and configurations closer to the center-point ($u \sim 0.5$) become more important.

VII. CONCLUSIONS

In this paper we classify the leading and subleading SCET operators (in powers of $\lambda$) necessary for the analysis of exclusive processes that involve fast moving light mesons. Some aspects of SCET at subleading order still have to be fully understood, but, in order to perform our analysis, one only needs the field content of the theory and the corresponding gauge transformations. At subleading order, both two-particle ($\sim \xi \bar{\xi}$) and three-particle ($\sim \xi A_1^\mu \xi$) operators are present.

We show how to express the matrix elements of all the SCET operators between the vacuum and a meson state in terms of the standard two- and three-particle meson light-
cone distribution amplitudes. Exact operator identities between various SCET operators result in corresponding identities between different distribution amplitudes. We checked that all these relations are exactly satisfied by the application of the QCD equations of motion. In the limit in which the contribution of the three-particle operators is neglected, we recover several Wandzura–Wilczek relations between wave functions of geometrical twist-2 and twist-3. This reflects the fact that operators with a given $\lambda$ scaling have fixed dynamical twist (i.e. they have a fixed $\Lambda/Q$ suppression); due to the mismatch between dynamical and geometric twist, their matrix elements contain contributions from wave functions of different geometrical twist.

The subleading collinear structures considered here appear as building blocks for SCET operators contributing to power suppressed processes. We showed that Lorentz invariance (manifested in the SCET as reparameterization invariance) severely constrains the Wilson coefficients of such subleading operators, and connects them to the coefficients of the leading operators.

In order to further validate our results and get a feeling for the size of the three-particle contributions, we checked a Ward identity for the final state emission of a photon from a transverse polarized vector meson. Comparing the exact result with the computation in the Wandzura-Wilczek limit (in which the three-particle distribution amplitudes are set to zero), we find that corrections of order $O(20\%)$ at the amplitude level are possible. In general three-particle light-cone wave functions are expected to be more important in processes involving vector mesons in the final state (e.g. $B \to K^* \pi$, ...). In fact, the amplitude for the production of a transverse polarised vector meson does not receive twist-2 contributions (in SCET language, the matrix elements of the leading order operator are vanishing). On the contrary, decays that involve only pseudo-scalar mesons (e.g. $B \to \pi \pi$, ...) start at twist-2 and, therefore, three-particle corrections are small.

ACKNOWLEDGEMENTS

We thank Martin Beneke and Thorsten Feldmann for interesting discussions. D.P. is grateful to Michael Gronau and Iain Stewart for useful discussions. This work is partially supported by the Swiss National Fonds and by the DOE under Grant No. DOE-FG03-97ER40546 and by the US National Science Foundation Grant PHY-9970781.

APPENDIX A: NOTATION AND CONVENTIONS

We use the standard convention for the metric tensor in Minkowski space $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and the totally antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$ is defined with $\varepsilon_{0123} = -\varepsilon^{0123} = 1$. For the Dirac matrices we use the standard conventions, in particular $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and define $\sigma_{\mu\nu}$ by

$$
\sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu].
$$

From this we get the relation $\sigma_{\mu\nu}\gamma^5 = -\frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} \sigma^{\rho\sigma}$, and its inverse $\sigma_{\alpha\beta} = -\frac{i}{2} \varepsilon_{\alpha\beta\mu\nu} \sigma^{\mu\nu}\gamma^5$. The dual gluon field strength tensor is defined by $\tilde{G}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}$.

The hadrons are taken to be moving with large momentum along the light-cone direction $n$. The opposite direction light cone vector $\bar{n}$ is chosen such that $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. 

26
Momentsa are decomposed along the light cone as $p = (p_-, p^+, p_\perp)$ with
\begin{equation}
  p^\mu = \frac{n \cdot p}{2} n^\mu + \frac{n \cdot \bar{n}}{2} \bar{n}^\mu + p_\perp^\mu. \tag{A2}
\end{equation}

We thus define the perpendicular component of the $\gamma$ matrices by
\begin{equation}
  \gamma_\perp^\mu \equiv \gamma^\mu - \frac{\not{\bar{n}}}{2} \bar{n}^\mu - \frac{\not{n}}{2} n^\mu.
\end{equation}

Similarly we write the perpendicular component of the metric tensor:
\begin{equation}
  g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \frac{1}{2} n^\mu \bar{n}^\nu - \frac{1}{2} n^\nu \bar{n}^\mu. \tag{A3}
\end{equation}

We define the perpendicular components of the total antisymmetric tensor as
\begin{equation}
  \varepsilon_\perp^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{n}_\rho n_\sigma. \tag{A4}
\end{equation}

It satisfies the following relations which are useful in practical calculations
\begin{align}
  \varepsilon_\perp^{\mu\alpha} \varepsilon_\perp^{\nu\beta} g_\perp^{\gamma\delta} &= -g_\perp^{\mu\nu}, \tag{A5} \\
  \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu \gamma_\perp^\nu &= \frac{\not{n}}{2} g_\perp^{\mu\nu} - i \varepsilon_\perp^{\mu\nu} \frac{\not{\bar{n}}}{2} \gamma_\perp^5, \tag{A6} \\
  \varepsilon_\perp^{\mu\nu} p_\perp^{\mu\nu} n_\rho &= \varepsilon^{\mu\rho\sigma} p_\perp^{\mu\nu} n_\sigma, \tag{A7} \\
  \frac{\not{n}}{2} \gamma_\perp^\mu \gamma_\perp^5 &= i \varepsilon_\perp^{\mu\nu} \frac{\not{n}}{2} \gamma_\perp^\nu. \tag{A8}
\end{align}

APPENDIX B: WAVEFUNCTION PARAMETERIZATIONS

We collect in this Appendix simple parameterizations of the light-cone wave functions used in the numerical evaluations in the text, following [24, 25]. For more details about their derivation and the computation of the nonperturbative parameters appearing in these formulas, we refer to [18, 19, 20, 21, 23, 24, 25, 26, 33].

We start by listing the pseudoscalar wave functions $\phi_p(u), \phi_\sigma(u), \phi_{3\pi}(\alpha_1, \alpha_2, \alpha_3)$. Working at next-to-leading order in the conformal expansion, they can be written as (with $\xi = 2u - 1$)
\begin{align}
  \phi_p(u) &= 1 + 15R[3\xi^2 - 1], \tag{B1} \\
  \phi_\sigma(u) &= 6u(1-u) \left\{ 1 + (5 - \frac{1}{2} \omega_{1,0}) R \frac{15}{2} \xi^2 - \frac{3}{2} \right\}, \tag{B2} \\
  \phi_{3\pi}(\alpha_i) &= 360 \alpha_1 \alpha_2 \alpha_3 \left( 1 + \frac{1}{2} \omega_{1,0}(7\alpha_3 - 3) \right). \tag{B3}
\end{align}

For the vector meson case, we give only the wave functions corresponding to the $\rho$ meson. Working again at next-to-leading order in the conformal expansion, the twist-2 two-parton wave functions are
\begin{equation}
  \phi_{\parallel,\perp}(u) = 6u(1-u) \left[ 1 + a_2^{\parallel,\perp} \frac{3}{2} [5(2u-1)^2 - 1] \right]. \tag{B4}
\end{equation}
The remaining twist-3 two-parton wave functions \( g^{(v,a)}(u) \) and \( h^{(s,t)}(u) \) can be expressed with the help of the equation of motion in terms of the twist-3 three-parton wave functions. The leading terms in their conformal expansions are

\[
\{V, T\}(\alpha_1, \alpha_2, \alpha_3) = 540 \zeta_3 \omega_3 V^T (\alpha_1 - \alpha_2)\alpha_1 \alpha_2 \alpha_3^2 ,
\]

\[
A(\alpha_1, \alpha_2, \alpha_3) = 360 \zeta_3 \alpha_3 \alpha_1 \alpha_2 \alpha_3^2 [1 + \omega_3^A \frac{1}{2} (7 \alpha_3 - 3)].
\]

We follow here the notations of [24, 25]. The numerical values of the nonperturbative constants are

\[
d_{2}^{\perp} = 0.18 \pm 0.10, d_{2}^{\parallel} = 0.2 \pm 0.1, \zeta_3 = 0.032, \omega_{3}^{V} = 3.8, \omega_{3}^{T} = 7.0, \omega_{3}^{A} = -2.1
\]

(at the scale \( \mu^2 = 1 \text{ GeV}^2 \)).

Using the equations of motion, the chiral-even two-particle twist-3 wave functions are determined as

\[
g^{(a)}_{\perp}(u) = 6 u \bar{u} \left[ 1 + \left( \frac{1}{4} a_{2}^{\parallel} + \frac{5}{3} \zeta_3 [1 - \frac{3}{16} \omega_3^A] + \frac{15}{16} \zeta_3 \omega_3^V \right) (5 \xi^2 - 1) \right],
\]

\[
g^{(v)}_{\perp}(u) = \frac{3}{4} (1 + \xi^2) + \left( \frac{3}{7} a_{2}^{\parallel} + 5 \zeta_3 \right) (3 \xi^2 - 1)
\]

\[
+ \left[ \frac{9}{112} a_{2}^{\parallel} + \frac{15}{64} \zeta_3 (3 \omega_3^V - \omega_3^A) \right] (3 - 30 \xi^2 + 35 \xi^4).
\]

The corresponding results for the chiral-odd two-particle twist-3 wave functions are

\[
h^{(t)}_{\parallel}(u) = 3 \xi^2 + \frac{3}{2} a_{2}^{\perp} \xi^2 (5 \xi^2 - 3) + \frac{15}{16} \zeta_3 \omega_3^T (3 - 30 \xi^2 + 35 \xi^4)
\]

\[
h^{(s)}_{\parallel}(u) = 6 u \bar{u} \left\{ 1 + \left( \frac{1}{4} a_{2}^{\perp} + \frac{5}{8} \zeta_3 \omega_3^T \right) (5 \xi^2 - 1) \right\}.
\]
[19] A. S. Gorsky, Sov. J. Nucl. Phys. 45, 512 (1987) [Yad. Fiz. 45, 824 (1987)].
[20] V. M. Braun and A. Fylianov, Z. Phys. C44 (1989) 157.
[21] V. M. Braun and A. Fylianov, Z. Phys. C48 (1990) 239.
[22] I. E. Halperin, Phys. Rev. D 57, 1680 (1998).
[23] P. Ball, V. M. Braun, Y. Koike and K. Tanaka, Nucl. Phys. B529 (1998) 323;
[24] P. Ball and V. M. Braun, arXiv:hep-ph/9808229.
[25] P. Ball and V. M. Braun, Nucl. Phys. B543 (1999) 201.
[26] P. Ball, JHEP 9901, 010 (1999).
[27] M. Beneke and T. Feldmann, Nucl. Phys. B 592, 3 (2001).
[28] M. Beneke and V. A. Smirnov, Nucl. Phys. B522 (1998) 321.
[29] S. Wandzura and F. Wilczek, Phys. Lett. B72 (1977) 195.
[30] R. K. Ellis, W. Furmanski and R. Petronzio, Nucl. Phys. B 207, 1 (1982); Nucl. Phys. B 212, 29 (1983); R. L. Jaffe and M. Soldate, Phys. Rev. D 26, 49 (1982); I. V. Anikin and O. V. Teryaev, Phys. Lett. B 509, 95 (2001).
[31] B. Grinstein and D. Pirjol, Phys. Rev. D62 (2000) 093002.
[32] A. Khodjamirian and D. Wyler, arXiv:hep-ph/0111249.
[33] V. M. Braun, G. P. Korchemsky and D. Muller, arXiv:hep-ph/0306057.