Modeling approaches for precise relativistic orbits I: Analytical, Lie-series, and pN approximation

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Abstract

Accurate orbit determination and modeling plays a key role in contemporary and future space missions. To fully exploit the technological capabilities, and to correctly interpret all measurements, relativistic orbital effects need to be taken into account.

Within the theory of General Relativity, the equations of motion for freely falling test objects, such as satellites orbiting the Earth, are given by the geodesic equation. We analyze and compare different methods to solve this equation in a spherically symmetric background, i.e. for the Schwarzschild spacetime, as a test bed. We investigate satellite orbits around the Earth and use direct numerical orbit integration as well as the semi-analytical Lie-series approach. The results are compared to the exact analytical solution in terms of elliptic functions, which serves as the reference. For a set of exemplary orbits, we determine the respective accuracy of the different methods.

Within the post-Newtonian approximation of General Relativity, modified orbital equations are obtained by adding relativistic corrections to the Newtonian equations of motion. We analyze the accuracy of this approximation with respect to the fully relativistic setting. Therefore, we solve the post-Newtonian equation of motion using the eXtended High Performance Satellite dynamics Simulator (XHPS). For the same initial conditions, we compare orbits in the Schwarzschild spacetime to those in its post-Newtonian approximation. Moreover, we compare the magnitude of the relativistic contributions to several typical perturbations of satellite orbits due to, e.g., solar radiation pressure, Earth’s albedo, and atmospheric drag.

Keywords

Relativistic geodesy · post-Newtonian theory · Relativistic effects · Satellite orbits · orbit propagation

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1 Introduction

Contemporary and future high precision geodesy and gravimetry space missions require a precise modeling of satellite orbits. Missions such as GRACE-FO, the successor of the long-lasting gravity field recovery mission GRACE, aim at nanometer accuracy in the change of the spatial distance between two spacecraft [1], [2]. At this level of accuracy, relativistic effects need to be taken into account. Hence, precise orbit modeling and orbit propagation tools, incorporating relativistic equations of motion, are needed to consistently interpret measurements at the best possible level of accuracy. These tools usually use a numerical integration procedure, and our main goal is to quantify the accuracy of this approach in two different ways outlined below.

Within the theory of General Relativity (GR), freely falling test bodies move on timelike geodesics and the equation of motion (EOM) is given by the geodesic
equation, which involves quantities derived from the spacetime metric, see, e.g., [3]. For a certain class of spacetimes, which are exact solutions of Einstein’s field equation with a sufficient amount of symmetries, this equation can be solved analytically, see [4], [5], [6], [7]. Here, we choose one of these spacetimes to build trust into numerical and semi-analytical solution methods, which can be used for more complicated situations where we do not have analytical solutions at hand. These methods will be checked against the analytical reference solution, and we investigate their respective accuracy for satellite orbits around the Earth. This approach is a first step to tackle complex but more realistic situations later on.

The post-Newtonian (pN) EOM used in orbit simulation and propagation tools is an approximation of the exact General Relativistic equation. The eXtended High Performance Satellite dynamics Simulator (XHPS) [8] is an orbit propagation tool that numerically solves the EOM, and it is also capable of simulating the entire space environment as well as detailed satellite properties. In a second step, we therefore include pN corrections into the XHPS and compare its numerical integrator to the direct numerical integration method applied to the EOM in full GR, which we checked before against the analytical solution.

The general purpose of this work is therefore twofold: for the application within relativistic geodesy, we aim at the comparison of different solution methods for the relativistic EOM. Moreover, we investigate the accuracy of the first order pN approximation of GR for satellite orbits. The pN framework yields a modified Newtonian EOM, and allows to compare the magnitude of relativistic effects to various non-gravitational perturbations along satellite orbits.

We use a spherically symmetric gravitational field as a test bed. The general relativistic spacetime is then described by the Schwarzschild metric, and its first order pN approximation involves the Newtonian gravitational potential of a point mass. This approach only includes the dominant relativistic effects on the orbits, which should, however, be sufficient for a first quantification of the accuracy of orbit simulations within the XHPS and similar tools. For the Schwarzschild spacetime, the exact solutions of the EOM are well-known and given in terms of elliptic functions [4]. All necessary notions are introduced in Sec. [2] and the EOM is introduced in Sec. [3].

To construct orbits in the Schwarzschild spacetime, we use direct numerical integration, the semi-analytical Lie-series approach, and the exact analytic solution in terms of the Weierstrass elliptic function. For a predefined set of test orbits, the analytical solution serves as the reference to test the accuracy of the other methods. The test orbits and solution methods are introduced in Sec. [4].

In Sec. [5] the results of the two different methods are compared to the exact analytical solution of the geodesic equation, which exists due to sufficiently many constants of motion. By analyzing the deviation from the analytic solution, we obtain the accuracy of the different methods.

We solve the pN EOM using the XHPS that was refined for use within the German collaborative research center “geo-Q”, and now includes relativistic corrections in the orbit propagation model at the first order pN level. In Sec. [6] we access the accuracy of the pN approximation by analyzing the difference between the XHPS results and the corresponding orbit, with the same initial conditions, in the Schwarzschild spacetime. Finally, in Sec. [7] we use the XHPS and select one test orbit, for which we assume a GRACE-like satellite model, to calculate the relativistic accelerations along one full orbital revolution. The result is compared to non-gravitational accelerations due to solar radiation pressure, Earth’s albedo, thermal radiation pressure, and atmospheric drag.

Note that for the first order pN approximation of the Schwarzschild spacetime, the modified Keplerian equations of motion can also be solved analytically to test the accuracy of the pN approximation. However, we use the XHPS since it allows to calculate the magnitude of various non-gravitational perturbations due to the space environment and to show that relativistic effects must be taken into account for high-precision space missions, at least at a pN level.

Table 1 shows an overview of the different solution methods that we use to solve the EOM.

| method       | Schwarzschild pN approximation |
|--------------|-------------------------------|
|              | geodesic                      | modified Kepler                |
| analytic     | reference n.a.                | n.a.                           |
| Lie-series   | test n.a.                     | reference                      |
| numerical    | test reference                 |                               |
| XHPS         | n.a. test                     |                               |

Table 1 To construct orbits in the Schwarzschild spacetime, we use the analytical solution as a reference and check the accuracy of a) the semi-analytical Lie-series method and b) the direct numerical integration. To investigate the accuracy of the pN approximation, we solve the modified Keplerian EOM using the XHPS and check the result against a direct numerical integration of the geodesic equation in the Schwarzschild spacetime for identical initial conditions and suitable coordinates. In the table, n.a. means that we do not consider the respective points in this work.
2 Geometry and notation

2.1 General relativistic spacetime

Within the theory of GR, the curved spacetime geometry is described by a metric \( g \). In a given coordinate system, the metric components are denoted as \( g_{\mu\nu} \), where we use greek indices as spacetime indices taking values 0, 1, 2, 3. The metric itself is to be found as a solution of Einstein’s field equation \[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \] Here, \( R_{\mu\nu} \) is the Ricci tensor and \( R \) is the Ricci scalar. Both are constructed from the spacetime metric, whereas the source term is given by the energy-momentum tensor \( T_{\mu\nu} \). The tensorial equation above is a second order non-linear partial differential equation for the metric components. Moreover, Newton’s gravitational constant \( G \) and the speed of light \( c \) are the only dimensions that enter as dimensional factors of proportionality.

Outside a given mass (energy) distribution, one has to solve the vacuum field equation

\[ R_{\mu\nu} = 0. \]

The Schwarzschild spacetime is the most famous vacuum solution of Einstein’s field equation. It describes the spacetime geometry outside a spherically symmetric mass distribution and is a member of the Weyl class of spacetimes. This spacetime possesses a monopolie moment only, which gives the total mass of the gravitating source. According to Birkhoff’s theorem, the Schwarzschild spacetime is the unique solution with these properties, and spherical symmetry implies that the spacetime must be static.

Using spherical coordinates \((x^0, r, \vartheta, \varphi)\) and the metric signature convention \(-, +, +, +\), the Schwarzschild metric reads

\[ ds^2 = -A(r)(dx^0)^2 + A(r)^{-1}dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2, \]

where the metric function \( A(r) \) is given by

\[ A(r) = 1 - 2m/r =: 1 - rs/r. \]

The coordinates \( x^0 \) and \( r \) have the dimension of a length, whereas \( \vartheta \) and \( \varphi \) are the usual angles on the two-sphere \( S_2 \). The parameter \( m \) is the mass of the gravitating source (in natural units). It is related to the mass \( M \) in SI-units by

\[ m = \frac{GM}{c^2}. \]

The quantity \( r_s = 2m \) denotes the Schwarzschild radius (gravitational radius), i.e. the radius to which one would have to compress all the mass of the object to form a black hole. For the Earth, the Schwarzschild radius is below one centimeter, \( m_{\oplus} \approx 0.88 \) cm.

The spacetime possesses a monopole moment \( M_0 = M \), and all higher order moments vanish identically. We may explicitly rewrite the Schwarzschild metric \[ r \]
in SI-units, including \( G \) and \( c \), and we introduce the coordinate time \( t \) by \( x^0 := c t \) to obtain

\[ ds^2 = -(1 - \frac{2GM}{c^2r}) c^2 dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2. \]

The time coordinate \( t \) has the dimension of a time measured in seconds. It will become important in the following sections as a parameter along timelike geodesics of the spacetime, and allows to reproduce some well-known pN results.

Note that the Schwarzschild radial coordinate \( r \) is an area coordinate; spheres with a radius \( r = r_0 \) have a surface area \( 4\pi r_0^2 \), as can be read off from the metric \[ r \]. The difference \( \Delta r := r_2 - r_1 \) is not the proper spatial distance between two events on the radial line on \( t = \text{const.} \) hypersurface.

2.2 Post-Newtonian approximation

Whenever the gravitational field, inside and in the neighborhood of a central object, is weak and all velocities are small compared to the speed of light, the pN framework is applicable. It is a method to solve Einstein’s field equation to a given order of accuracy. For an overview of the pN framework, we recommend the books \[ 9, 10, 11 \], and the references therein. Modern conventions of the International Astronomical Union (IAU) and the International Earth Rotation and Reference Systems Service (IERS) use a first order pN spacetime, see Refs. \[ 12 \] and \[ 13 \].

For the first order stationary pN approximation of a general relativistic spacetime outside the Earth, we have to use the metric

\[ g_{00} = -\left(1 - \frac{2U}{c^2} + \frac{2U^2}{c^4}\right) + O(c^6), \]

\[ g_{0i} = -\frac{4U^i}{c^3} + O(c^5), \]

\[ g_{ij} = \delta_{ij} \left(1 + \frac{2U}{c^2}\right) + O(c^4), \]

\[ \text{[7a] \quad [7b] \quad [7c]} \]
where the potentials $U, U^i$ satisfy the equations
\begin{align}
\Delta U(X) &= -4\pi G \rho(X) , \\
\Delta U^i(X) &= -4\pi G \rho^i(X) = -4\pi G \rho v^i(X) .
\end{align}
(8a, 8b)

The energy (mass) density $\rho$ and the energy density flux $\rho^i$ are related to the energy-momentum tensor of the Earth by $\rho = (T^{00} + T^{ii})/c^2$ and $\rho^i = T^{0i}/c$, evaluated in the Geocentric Celestial Reference System (GCRS) with Cartesian coordinates $(T, X, Y, Z)$, and $v^i$ is the gravitating matter’s velocity. For the scalar and vector potentials one obtains \[ U(X) = G \int d^3 X' \frac{\rho(X')}{|X-X'|} , \quad \text{and} \quad U^i(X) = G \int d^3 X' \frac{\rho v^i(X')}{|X-X'|} . \]
(9a, 9b)

To construct the pN approximation of the Schwarzschild spacetime, we have to use the Newtonian gravitational potential of a spherically symmetric mass distribution$^2$

\[ U = GM/R , \]
(10)

where $R = \sqrt{X^2 + Y^2 + Z^2}$ is the distance to the center of mass in the GCRS. The vector potential $U^i$ vanishes identically because there are no mass currents present. Hence, the pN metric \[ g_{\mu\nu} = \left(1 - \frac{2m}{r} + \frac{2m^2}{(r^2+\epsilon^2)} \right) c^2 dt^2 + \left(1 + \frac{2m}{c^2 r} \right) \left(dX^2 + dY^2 + dZ^2 \right) . \]
(11)

Now, we introduce spherical coordinates $(R, \Theta, \Phi)$, by the usual relations to $(X, Y, Z)$, and rewrite the metric \[ g_{\mu\nu} = \left(1 - \frac{2m}{R} + \frac{2m^2}{R^2} \right) c^2 dt^2 + \left(1 + \frac{2m}{R} \right) \left(dR^2 + R^2 d\Theta^2 + R^2 \sin^2 \Theta d\Phi^2 \right) , \]
(12)

where we used the relation \[ \frac{1}{2} \] between $M$ and $m$.

### 2.3 Radial coordinates

For the pN approximation of the Schwarzschild metric, see Eq. \[ 12 \], an isotropic radial coordinate $R$ is used. Hence, the spatial part of the metric is conformally flat, i.e. it appears to be the Minkowski line element modified by a coordinate dependent factor. To compare the pN metric to the Schwarzschild metric \[ 12 \], we have to either transform the metric \[ 12 \] to area coordinates, or to transform the metric \[ 6 \] to isotropic coordinates. To do the latter, we must have
\[ (1 - \frac{2m}{r})^{-1} \frac{dr}{r} = \frac{d\lambda}{\lambda} , \]
(14)

which is solved by
\[ r = \lambda \left(1 + \frac{m}{2\lambda} \right)^2 . \]
(15)

The Schwarzschild metric in isotropic coordinates now reads
\[ g = - \left(1 - \frac{m/(2\lambda)}{1 + m/(2\lambda)} \right)^2 c^2 dt^2 + \left(1 + \frac{m}{2\lambda} \right)^4 (d\lambda^2 + \lambda^2 d\Theta^2 + \lambda^2 \sin^2 \Theta d\Phi^2) . \]
(16)

Here, $\epsilon := m/\lambda$ is a small quantity, and for the region outside the Earth’s surface it is less than $10^{-9}$. The pN metric can be obtained now by expanding the spatial part in Eq. \[ 16 \] to first order and $g_{tt}$ so second order in $\epsilon$. In this way, we get indeed the pN metric \[ 12 \] after the identification $\lambda \equiv R$ at the given level of accuracy.

### 3 Equations of motion

In GR, massive test bodies move on timelike geodesics. The EOM for a test mass is given by the geodesic equation
\[ \ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = 0 . \]
(17)

The worldline of the object is described by $x^\mu(\tau)$, and the overdot denotes derivatives w.r.t. the proper time $\tau$. The proper time is defined by the normalization of the four-velocity $u = \dot{x}$ according to
\[ g(u, u) = g_{\mu\nu} u^\mu u^\nu = -c^2 . \]
(18)
The Christoffel symbols $\Gamma^n_{\nu\sigma}$ can be calculated from the metric by

$$\Gamma^n_{\nu\sigma} = \frac{1}{2} g^{nk} \left( \partial_k g_{n\lambda} + \partial_{\nu} g_{k\lambda} - \partial_{\lambda} g_{\nu k} \right).$$  \hspace{1cm} (19)$$

The EOM can also be derived using, e.g., the Lagrange or the Hamilton formalism. The Lagrangian for the motion of point-like test bodies is

$$2\mathcal{L} := g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu.$$  \hspace{1cm} (20)$$

We now introduce a general framework that covers all cases which we are going to discuss in the following. The EOM for the Schwarzschild spacetime and its pN approximation are then contained as special cases. Therefore, we introduce the general metric

$$g = g_{tt}(\xi) d\tau^2 + g_{11}(\xi) d\vartheta^2 + g_{22}(\xi) d\varphi^2 + g_{33}(\xi, \vartheta) \, d\varphi^2,$$  \hspace{1cm} (21)$$

where we use $\xi$ as a radial coordinate and angles $\vartheta, \varphi \in S_2$. The results derived in the following are valid for the Schwarzschild spacetime, in this case $\xi$ is either the area coordinate $r$, or the isotropic coordinate $\lambda$, and for the first order pN approximation of the Schwarzschild spacetime, $\xi$ is the geocentric radial coordinate $R$. Note that, according to our sign convention, $g_{00} < 0$, and all $g_{ij} > 0$. The Lagrangian for the metric (21) is

$$2\mathcal{L} = g_{tt}(\xi) \dot{\tau}^2 + g_{11}(\xi) \dot{\vartheta}^2 + g_{22}(\xi) \dot{\varphi}^2 + g_{33}(\xi, \vartheta) \dot{\varphi}^2.$$  \hspace{1cm} (22)$$

The symmetry of the situation at hand allows to restrict the motion to the equatorial plane. Hence, $\vartheta = \pi/2$, $\dot{\vartheta} = 0$, $\ddot{\vartheta} = 0$, \hspace{1cm} (23)$$

and all quantities in the following are assumed to be evaluated at $\vartheta = \pi/2$. The induced 3-dimensional metric in the equatorial plane is

$$g^{(3)} = g_{tt}(\xi) d\tau^2 + g_{11}(\xi) d\xi^2 + g_{33}(\xi) d\varphi^2.$$  \hspace{1cm} (24)$$

Since $\partial_t$ and $\partial_\varphi$ are Killing vector fields of the spacetime, there are two constants of motion, which are related to the energy $E$ and angular momentum $L$:

$$E := -g_{tt}(\xi) \dot{\tau},$$  \hspace{1cm} (25a)$$

$$L := g_{33}(\xi) \dot{\varphi}.$$  \hspace{1cm} (25b)$$

Using the canonical conjugated momenta $p_\mu = \partial \mathcal{L} / \partial \dot{x}^\mu$ and the inverse metric components $g^{\mu\nu}$, we can also construct the Hamiltonian

$$2\mathcal{H} = g^{\mu\nu} p_\mu p_\nu.$$  \hspace{1cm} (26)$$

For the metric (24), we obtain ($p_0 = 0$)

$$p_t = -E,$$  \hspace{1cm} (27a)$$

$$p_\varphi = L,$$  \hspace{1cm} (27b)$$

$$p_\xi = g_{11}(\xi) \dot{\xi},$$  \hspace{1cm} (27c)$$

Hence, two of the momenta are given by constants of motion, and the Hamiltonian (in the equatorial plane) becomes

$$2\mathcal{H} = g^{11}(\xi) p_\varphi^2 + g^{11}(\xi) p_\xi^2 + g^{33}(\xi) L^2.$$  \hspace{1cm} (28)$$

3.1 Proper time parametrization

The EOM, parameterized by the proper time of the respective test body, is given by, e.g., the Euler-Lagrange equations for the Lagrangian (22). For the azimuthal motion we obtain, see Eq. (25),

$$\dot{\varphi} = \frac{d\varphi}{d\tau} = \frac{L}{g_{33}(\xi)}.$$  \hspace{1cm} (29)$$

The normalization of the four-velocity according to Eq. (18) gives a first order differential equation for the radial motion, that is

$$\ddot{\xi} = -\frac{1}{g_{11}(\xi)} \left( \dot{\xi}^2 + \frac{E^2}{g_{tt}(\xi)} + \frac{L^2}{g_{33}(\xi)} \right),$$  \hspace{1cm} (30)$$

where we used the constants of motion from Eq. (25). Thereupon, we can construct a second order differential equation by taking one more derivative with respect to proper time $\tau$,

$$\ddot{\xi} = -\frac{1}{2} \frac{d}{d\xi} \left( \frac{1}{g_{11}(\xi)} \left( \dot{\xi}^2 + \frac{E^2}{g_{tt}(\xi)} + \frac{L^2}{g_{33}(\xi)} \right) \right).$$  \hspace{1cm} (31)$$

The EOM (29) and (30) or (31) can now be solved using different methods. The initial conditions ($\xi_0, \varphi_0$) and ($\dot{\xi}_0, \dot{\varphi}_0$) at some $\tau = \tau_0$ must be specified. They are related to the constants of motion, as we will show in the next sections. Note that the advantage of Eq. (31) over the first order equation (30) is that it automatically takes care of all turning points along the orbit, which are, e.g., at the perigee and apogee of any bound orbit. Circular orbits can be found by equating Eqs. (30) and (31) to zero at the same time.
3.2 Coordinate time parametrization

To compare the solutions to the EOM in the Schwarzschild spacetime to its pN approximation (and to Newtonian Kepler orbits) later on, we need an EOM that is parametrized by the coordinate time $t$. 

We can reparametrize all orbits using $d\xi/dt = \xi/\dot{t}$ and $d\varphi/dt = \varphi/\dot{t}$. We define derivatives w.r.t. the coordinate time by the symbol $\dot{\xi} := dx/dt$. Hence, we obtain

$$\dot{\varphi} := \frac{d\varphi}{dt} = -\frac{L}{E} g_{tt}(\xi),$$

$$\ddot{\xi}^2 := \left(\frac{d\xi}{dt}\right)^2 = \frac{g_{tt}(\xi)}{g_{11}(\xi)} \left(\frac{c^2 g_{tt}(\xi)}{E^2} + \frac{L^2 g_{tt}(\xi)}{E^2 g_{33}(\xi)} + 1 \right).$$

Again, taking one more derivative with respect to coordinate time $t$, we get the second order differential equation

$$\dddot{\xi} = \frac{1}{2} \frac{d}{d\xi} \left[ \frac{g_{tt}(\xi)}{g_{11}(\xi)} \left(\frac{c^2 \varrho tt(\xi)}{E^2} + \frac{L^2 \varrho tt(\xi)}{E^2 g_{33}(\xi)} + 1 \right) \right].$$

(33)

3.3 Initial conditions

We can relate the constants of motion $L$ and $E$ to orbital parameters. For the perigee $\xi_p$ and apogee $\xi_o$ of a bound orbit we introduce the well known formulae

$$\xi_p = (1 - e)a, \quad \xi_o = (1 + e)a,$$

by which we define a semi-major axis $a$ and an eccentricity $e$ also in the relativistic setting. These radii mark the turning points of a bound orbit, $\xi|_{\xi_p} = 0 = \xi|_{\xi_o}$. Therefore, we obtain

$$L^2 = c^2 \frac{g_{tt}(\xi_p) - g_{tt}(\xi_o)}{g_{11}(\xi_p) - g_{11}(\xi_o)},$$

$$E^2 = -\frac{g_{tt}(\xi_p)}{g_{33}(\xi_p)} \left(c^2 + \frac{L^2}{g_{33}(\xi_p)}\right).$$

Given an initial position $(\xi_0, \varphi_0)$, we can also derive the initial velocities w.r.t. proper time

$$\dot{\xi}_0 = \sqrt{\frac{1}{g_{11}(\xi_0)} \left(c^2 + \frac{E^2}{g_{tt}(\xi_0)} + \frac{L^2}{g_{33}(\xi_0)}\right)},$$

$$\dot{\varphi}_0 = \frac{L}{g_{33}(\xi_0)},$$

and w.r.t. coordinate time we get

$$\dot{\xi}_0 = \sqrt{\frac{g_{tt}(\xi_0)}{g_{11}(\xi_0)} \left(\frac{c^2 g_{tt}(\xi_0)}{E^2} + \frac{L^2 g_{tt}(\xi_0)}{E^2 g_{33}(\xi_0)} + 1 \right)},$$

$$\dot{\varphi}_0 = \frac{L}{E} g_{tt}(\xi_0).$$

Note that $g_{tt}$ is negative such that there is no problem with the square root and for bound orbits, the entire expression under the square root must be positive.

3.4 Orbits in the Schwarzschild spacetime

In the following, we apply the general results of the previous section to the Schwarzschild spacetime. Since the relativistic EOM is usually considered in a proper time parametrization, we do only explicitly give this result. However, the EOM parametrized by the coordinate time can be easily deduced from Eqs. (32) and (33).

Written in the form of Eq. (24), where we identify $\xi = r$, the induced Schwarzschild metric in the equatorial plane reads

$$g_{00} = -\left(1 - \frac{2m}{r}\right) c^2 = -A(r) c^2,$$

$$g_{11} = \left(1 - \frac{2m}{r}\right)^{-1} = A(r)^{-1},$$

$$g_{33} = r^2.$$

The Lagrangian for a particle moving in the Schwarzschild spacetime now becomes

$$2\mathcal{L} = -A(r) c^2 \dot{r}^2 + A(r)^{-1} \dot{r}^2 + r^2 \dot{\varphi}^2.$$

(39)

The two constants of motion, related to energy $E$ and angular momentum $L$, are

$$E = c^2 A(r) \dot{r},$$

$$L = r^2 \dot{\varphi}.$$

(40)

The canonical conjugated momenta become

$$p_r = -E,$$

$$p_\varphi = L,$$

$$p_r = A(r)^{-1} \dot{r},$$

and the Hamiltonian reads

$$2\mathcal{H} = -\frac{1}{c^2} A(r)^{-1} p_r^2 + A(r) p_\varphi^2 + \frac{1}{r^2} p_\varphi^2.$$
The motion of the test body is described by
\[ \dot{\varphi} = \frac{L}{r^2}, \]
\[ \ddot{\varphi} = - \frac{2 \ddot{r} \dot{\varphi}}{r}, \] (43b)
\[ r^2 = - A(r)^{-1} \left( \frac{c^2}{\varepsilon^2 A(r)} + \frac{L^2}{r^2} \right), \] (43c)
\[ \ddot{r} = - \frac{1}{2} \frac{d}{dr} \left[ A(r)^{-1} \left( \frac{c^2}{\varepsilon^2 A(r)} + \frac{L^2}{r^2} \right) \right]. \] (43d)

The constants of motion are related to the initial conditions by
\[ \dot{r}_0 = \frac{L}{r_0^2}, \] (44a)
\[ \ddot{r}_0 = - A(r_0) \left( \frac{c^2}{\varepsilon^2 A(r_0)} + \frac{L^2}{r_0^2} \right), \] (44b)
and the relations to the orbital elements \((r_p, r_a)\) are
\[ \frac{L^2}{c^2} = A(r_p) - A(r_a), \] (45a)
\[ \frac{E^2}{c^2} = \left( \frac{L^2}{r_a^2} + c^2 \right) A(r_a). \] (45b)

### 3.5 Orbits in the post-Newtonian approximation

In this section, we apply the general results to the first order pN approximation of the Schwarzschild spacetime. In the pN framework, orbits are usually parameterized by the coordinate time. Doing so, the pN EOM appears to look like a relativistically modified Newtonian EOM.

In the form \([24]\), where we identify \( \xi = R \), the pN metric in the equatorial plane is given by
\[ g_{00} = -c^2 \left( 1 - \frac{2m}{R} + \frac{2m}{R^2} \right), \] (46a)
\[ g_{11} = \left( 1 + \frac{2m}{R} \right), \] (46b)
\[ g_{33} = r^2 \left( 1 + \frac{2m}{R} \right). \] (46c)

We use the small parameter \( \epsilon = m/R \) for order counting. Being consistent to \( \mathcal{O}(\epsilon) \), the EOM can be written as the Keplerian orbital equation with pN correction terms. We get
\[ \ddot{R} = - \frac{\epsilon^2}{R^2} + \frac{d}{R^2} \frac{1}{\varepsilon^2} \left( 4GM + 3R \ddot{R} - R \dot{\varphi}^2 \right) \] (47a)
\[ \ddot{\varphi} = - \frac{2 \ddot{r} \dot{\varphi}}{R} \left( 1 - \frac{1}{2GM} \right). \] (47b)

The well-known Keplerian EOM is recovered in the limit \( \epsilon \to 0 \), i.e., \( c \to \infty \).

Above, we derived the pN EOM for the first order approximation of the Schwarzschild spacetime in spherical coordinates. Introducing the Cartesian position vector \( \mathbf{x} = (X, Y, Z) \) in the GCRS, the two equations can be combined to
\[ \ddot{\mathbf{x}} = \frac{-GM}{R^3} \mathbf{x} \]
\[ + \frac{1}{c^2 R^3} \left( \left( 4GM \right) \mathbf{x} - \dot{\mathbf{x}} \cdot \mathbf{x} \right) \mathbf{x} + 4 \left( \mathbf{x} \cdot \dot{\mathbf{x}} \right) \dot{\mathbf{x}}, \] (48)
where \( R = ||\mathbf{x}||_2 \). This equation is a special case of
\[ \ddot{\mathbf{x}} = \nabla U + c^{-2} ( - 4U \nabla U - 4(\nabla U \cdot \dot{\mathbf{x}}) \dot{\mathbf{x}} + (\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) \nabla U - 4 \dot{\mathbf{x}} \times (\nabla \times U) ), \] (49)
which is valid for a general potential \( U \) and also includes gravitomagnetic (Lense-Thirring) effects caused by the vector potential \( \mathbf{U} \), see, e.g., \([9], [15]\), and references therein.

### 4 Test orbits and solution methods

In the following, we consider different methods to solve the EOM in the Schwarzschild spacetime and its first order pN approximation. We use a set of satellite test orbits with different eccentricities and altitudes to test the solution methods and to quantify their respective accuracy by comparison to the exact analytical reference solution.

#### 4.1 Set of test orbits

To compare the different solution methods for the EOM, we consider the orbits shown in Tab.\([2]\). All orbits are assumed to lie in the equatorial plane, \( \theta = \pi/2 \), such that the orbit modeling reduces to a 2-dimensional problem. All values for orbital elements such as the semi-major axis or the perigee are taken to be defined using the Schwarzschild area coordinate \( r \).

\[^3\] The first order pN approximation is usually defined by order counting in the EOM. Terms of order \( \mathcal{O}(\epsilon^n) \) are proportional to \( c^{-2n} \) and said to be at the \( n \)-th pN order.
| orbit | type | eccentricity $e$ | semi-major axis $a$ [m] |
|-------|------|----------------|------------------------|
| #1    | e, MEO | 0             | 2.79776 · 10\(^7\)     |
| #2    | e, MEO | 0.162         | 2.79776 · 10\(^7\)     |
| #3    | e, MEO | 0.300         | 2.79776 · 10\(^7\)     |
| #4    | e, MEO | 0.450         | 2.79776 · 10\(^7\)     |
| #5    | e, MEO | 0.600         | 2.79776 · 10\(^7\)     |
| #6    | e, MEO | 0.750         | 2.79776 · 10\(^7\)     |
| #7    | e, LEO | 0.2           | 8.5 · 10\(^6\)         |
| #8    | e, LEO | 0.001         | 6.8 · 10\(^6\)         |

Table 2 Orbital parameters for the eight different test orbits. These orbits are used to compare the different solution methods for the EOM. The shape of the orbit is either circular (c) or elliptical (e), and we use Low Earth Orbits (LEO) as well as Medium Earth Orbits (MEO). The orbital parameters (perigee, apogee) are defined in Eq. (43), and the semi-major axis as well as the eccentricity are defined in standard Schwarzschild area coordinates, see Eq. (34) with $\xi \equiv r$.

Without loss of generality, we can assume that all considered orbits start at their respective perigee, such that $r_0 = r_p$. Hence, initially $r_0 = 0$ for all cases. Furthermore, we assume initially $\phi_0 = 0$ to fix the argument of the perigee.

Orbit #1 is a circular orbit with a radius $r_0 = 27977600\,\text{m}$, which is the altitude of the Galileo satellites. Orbit #2 is characterized by a small eccentricity and corresponds roughly to the orbit of one of the Galileo satellites 5 and 6, which were not successfully launched into a circular orbit. The remaining orbits #3–#6 have larger eccentricities but we keep the same semi-major axis. Orbit #7 is a low Earth orbit (LEO) with a moderate eccentricity, whereas orbit #8 is an almost circular LEO at very low altitude. Choosing these orbits, we aim to cover a broad range of possible scenarios to test the different solution methods.

For the equatorial radius of the Earth we use $r_\oplus = 6378137\,\text{m}$. From the eccentricity and semi-major axis, see Table 2 the initial angular velocity and the constants of motion can be calculated using Eqs. (44) and (45). These initial conditions are kept the same for all solution methods that we describe in the following.

Note that we use the terms semi-major axis and eccentricity here as defined by the area coordinate $r$ in the Schwarzschild spacetime, i.e. as defined by Eq. (45) and Eq. (34), where $\xi \equiv r$. Hence, they do not fully coincide with (post-)Keplerian orbital elements. To have the same initial conditions for the pN orbits, the area coordinate needs to be transformed to an isotropic radial coordinate, and the initial angular velocity needs to be given w.r.t. the coordinate time, see Sec. 6.

4.2 Solution methods

4.2.1 Analytical solution

The analytical solution of the EOM can be given in terms of elliptic functions. Using $\text{dr}/\text{d}\varphi = \dot{r}/\dot{\varphi}$ yields a differential equation for $r(\varphi)$ that can be solved in terms of the Weierstrass elliptic function $\wp$. The solution is given by

$$r(\varphi) = \frac{m}{2\wp(\varphi - \varphi_\text{in}) + 1/6},$$

(50)

where $\varphi_\text{in}$ is related to the initial conditions according to

$$\varphi_\text{in} = \varphi_0 + \int_{y_0}^{\infty} \frac{\text{d}z}{\sqrt{4z^3 - g_2z - g_3}}, \quad y_0 = \frac{1}{2} \left( \frac{m}{r_0} - \frac{1}{6} \right).$$

(51)

The Weierstrass invariants $g_2$ and $g_3$ are determined by the constants of motion as follows:

$$g_2 = \frac{1}{12} - \frac{c^2m^2}{L^2},$$

(52a)

$$g_2 = \frac{1}{12} - \frac{1}{12} \frac{c^2m^2}{L^2} - \frac{m^2}{4L^2} \left( \frac{L^2}{c^2} - c^2 \right).$$

(52b)

For details on the analytic solution and possible applications, we refer the reader to the seminal paper by Hagihara [4] and the work in Refs. [16] and [17].

The analytical solution serves as the reference to check the accuracy of all other solution methods. If a solution in terms of $r(\varphi)$ is obtained, Eq. (50) can be used to calculate the actual value of the radius $r$ for a given value of the azimuthal angle $\varphi$. Thereupon, we can calculate the radial deviation from the analytical solution along the orbit.

4.2.2 Numerical solution

For the numerical integration of the EOM, we solve Eqs. (43) using a Runge-Kutta (RK) integrator on a fixed grid of proper time values $\tau$, with a working precision of 32 digits. The same grid is used for the semi-analytical Lie-series method in the following and is specified in the corresponding section. Here, we use the numerical RK integrator that is implemented in the Mathematica computer algebra system.

4.2.3 Lie series approach

The semi-analytical Lie series approach is based on the Hamilton formulation of the EOM. The Lie-series formalism was first applied to Newtonian orbital equations by Lelgemann [18] in 1983. The approach was developed towards a second order analytical orbital theory.

Since we consider a spherically symmetric gravitational field, the Earth is modeled as a sphere.
by Cui [19]. In Ref. [20], it is shown how the approach can be improved by using parallel computing techniques for the series coefficient calculation. Since the Lie-series approach has proven to be very useful in Newtonian dynamics, we employ the method also for the relativistic case. Therefore, we use the Hamiltonian \[42\] that generates the EOM. For the Lie series, we recursively define coefficients \( f_{\mu,(k)} \) and \( h_{\mu,(k)} \) by the Poisson brackets

\[
\begin{align*}
  f_{\mu,(k+1)} &= \partial_\tau f_{\mu,(k)} + \{ f_{\mu,(k)}, \mathcal{H} \}, \quad (53a) \\
  h_{\mu,(k+1)} &= \partial_\tau h_{\mu,(k)} + \{ h_{\mu,(k)}, \mathcal{H} \}, \quad (53b)
\end{align*}
\]

where \( \mu \) is a spacetime index, labeling the coordinates and the components of the canonical momenta, and \( k \) gives the order of the Lie-series approximation. For the multi-dimensional Poisson bracket on the phase space, see, e.g., [20], we have

\[
\begin{align*}
  \{ f_{\mu,(k)}, \mathcal{H} \} &= \frac{\partial f_{\mu,(k)}}{\partial x^\nu} \frac{\partial \mathcal{H}}{\partial p_\nu} - \frac{\partial f_{\mu,(k)}}{\partial p_\nu} \frac{\partial \mathcal{H}}{\partial x^\nu}, \quad (54a) \\
  \{ h_{\mu,(k)}, \mathcal{H} \} &= \frac{\partial h_{\mu,(k)}}{\partial x^\nu} \frac{\partial \mathcal{H}}{\partial p_\nu} - \frac{\partial h_{\mu,(k)}}{\partial p_\nu} \frac{\partial \mathcal{H}}{\partial x^\nu}. \quad (54b)
\end{align*}
\]

Note that \( f_{\mu,(k)} \) and \( h_{\mu,(k)} \) are functions of the phase space coordinates. The initial conditions for the recursive definitions are given by

\[
\begin{align*}
  f_{\mu,(0)} &= x^\mu, \quad (55a) \\
  h_{\mu,(0)} &= p_\mu. \quad (55b)
\end{align*}
\]

Hence, all the coefficients \( f_{\mu,(k)} \) and \( h_{\mu,(k)} \) can be calculated by nested Poisson brackets. Thereupon, the solution of the EOM is obtained by the Lie-series

\[
\begin{align*}
  x^\mu(\tau_0 + \Delta \tau) &= \sum_{k=0}^{\infty} \frac{(\Delta \tau)^k}{k!} f_{\mu,(k)}(\tau_0), \quad (56a) \\
  p_\mu(\tau_0 + \Delta \tau) &= \sum_{k=0}^{\infty} \frac{(\Delta \tau)^k}{k!} h_{\mu,(k)}(\tau_0). \quad (56b)
\end{align*}
\]

Introducing a finite upper limit of summation \( k_{\text{max}} \) gives an approximation to the full solution. A step size \( \Delta \tau \) needs to be chosen, and the summation is then performed numerically. Hence, the entire approach is said to be semi-analytic. The advantage over the direct numerical integration of the EOM is given by the expression of all series coefficients as analytical functions of the phase space coordinates. Therefore, the method may prove useful in the spectral domain, where individual terms can be studied and their impact on the orbit, as well as relativistic effects, can be quantified.

### 4.2.4 The XHPS integrator

The eXtended High Performance Satellite dynamics Simulator (XHPS) is designed to simulate multi-satellite missions incorporating gravity and also non-gravitational perturbations based on Newtonian mechanics. Its modular design allows to easily set up simulations at all levels of detail. Different numeric integration schemes are implemented, using, e.g., Runge-Kutta and multistep integrators. Due to the C++ code basis, the GNU MPFR library [21] with variable-precision data types can be utilized, so that results can in general reach every desired numerical accuracy.

The XHPS uses the International Celestial Reference System (ICRS), and the International Terrestrial Reference System (ITRS) as Cartesian coordinate systems derived from the before mentioned BCRS and GCRS definitions [13]. In these frames, relativistic effects can be considered as pN corrections in the EOM. The pN correction term for a spherically symmetric gravitational field, see Eqs. [48], in Cartesian GCRS coordinates is

\[
a_{\text{pN}} = \frac{GM}{c^2 R^3} \left[ \left( \frac{AGM}{R} - \frac{\Phi}{x} \right) x + 4 \left( \frac{\Phi}{x} \right)^2 \right]. \quad (57)
\]

See Refs. [13] and references therein for an overview of the IERS conventions that are used for the XHPS. Here, the vector \( x = (X, Y, Z) \) is the three component Cartesian position vector in the GCRS and \( \Phi \) its coordinate time derivative. \( a_{\text{pN}} \) can be added to all other gravitational (Earth, Sun and other bodies) and non-gravitational (environmental) accelerations in the EOM that are implemented in the XHPS.

For all orbit calculations, the data type was set to 64 digits, and the 8th order Dormand-Prince RK integrator was used with a constant step size, resulting in an absolute numerical precision of at least 25 digits, or about \( 10^{-17} \) m.

As mentioned already at the beginning of this section, the orbital parameters in Tab. [2] are defined using the Schwarzschild area coordinate \( r \) and have to be transformed to be used in the (post-)Newtonian framework. For the pN orbits, the XHPS needs as initial conditions the angular velocity w.r.t. the coordinate time, \( \Phi \), and the initial radius \( R_0 = R_p \) (perigee) in isotropic coordinates. These are related to \( \Phi \) and \( r_0 \) by

\[
\frac{\Phi}{r_0} = \frac{\dot{\Phi}}{r_0^2} \frac{c^2 A(r_0)}{r_0^2}, \quad r_0 = R_0 \left( 1 + \frac{m}{2 R_0} \right)^2. \quad (58a)
\]
5 Comparison of different methods

5.1 Numerical integration

Figure [1] shows the difference between the numerical orbit integration of Eq. (43) and the analytical solution for orbits #2 and #6, i.e. for the orbits with the smallest and largest eccentricity, respectively. With a working precision of 32 digits, the numerical orbits where obtained using a simple Runge-Kutta integrator on a fixed grid with 2001 equidistant grid points along the orbit. We used the computer algebra system Mathematica and the implemented functions for integrating second order differential equations to obtain the solutions for \( r(\tau) \) and \( \varphi(\tau) \). For all considered cases, see Tab. [2], the accuracy of this numerical approach is in the sub-nanometer regime along one full orbital arc.

The difference to the analytical solution is maximal for orbit #6, which has the largest eccentricity of \( \epsilon = 0.75 \). But even in this case, the deviation is below \( 10^{-3} \) nm. Hence, direct numerical integration of the EOM, with the given settings, yields an accuracy in the sub-nanometer regime after only a few seconds of integration time for one full orbit, and is well suited to integrate the fully relativistic EOM for high precision results. Here, the purpose was to test and verify the applicability of the direct numerical integration method. Increasing the working precision and refining the numerical grid yields even more accurate orbits. However, to date the experimental capabilities do not allow for a more precise orbit determination, and the state of the art accuracy is achieved by distance measurements using laser interferometry in a GRACE-FO like mission scheme [2].

5.2 Lie-series approach

All test orbits are constructed using the semi-analytical Lie-series approach with a maximal order \( k_{\text{max}} = 9 \) and \( k_{\text{max}} = 12 \), which appear to yield (sub-)nanometer accuracy for all cases.

The step size \( \Delta \tau \) is chosen to be the same as for the direct numerical integration of the EOM. We have chosen an equidistant grid of 2001 points with a maximal integration time of \( \tau_{\text{end}} = 4.67 \cdot 10^7 \text{s} \) for the orbits #1 to #6, while \( \tau_{\text{end}} = 7.81 \cdot 10^7 \text{s} \) and \( \tau_{\text{end}} = 5.57 \cdot 10^7 \text{s} \) for orbits #7 and #8, respectively. The time of integration is chosen such that in each case at least one full revolution is obtained and it is roughly the orbital period. Hence, the first grid point is at \( \tau = 0 \) and the last grid point is at \( \tau = \tau_{\text{end}} \).

To summarize the results for the semi-analytical Lie-series approach, we can say that for all considered test orbits the deviation from the analytical solution is in the (sub-)nanometer regime, with an upper bound for the Lie-series order at \( k_{\text{max}} = 9 \). For \( k_{\text{max}} = 12 \), the deviation from the analytical solution is at least three orders of magnitude smaller, i.e. the method becomes better with increasing order. Note, however, that there is no proof of convergence for the Lie-series [50].

In Fig. [2] we show the results for \( k_{\text{max}} = 9 \) and orbit #2 as well as orbit #6, to depict two exemplary results. The maximal deviation from the analytical solution is found for orbit #6, and its magnitude is about \( \pm 4 \) nm. For all other cases, the deviation is found to be orders of magnitude smaller. Hence, the semi-analytical Lie-series approach appears to work very well for solving the relativistic EOM using the Hamiltonian of the system and nested Poisson brackets. One drawback is the much larger computation time (minutes) compared to the direct numerical integration, which delivers results at the same level of accuracy. However, all Lie series coefficients are known analytically and their influence on the orbit and relativistic effects will be studied in separate paper.
6 Accuracy of the post-Newtonian approximation

To access the accuracy of the first order pN approximation of the relativistic EOM, we have implemented the pN correction term \( a_{\text{pN}}\), see Eq. (57), into the XHPS. Since we analyze the spherically symmetric situation, we specify the Newtonian gravitational potential to be the pure monopole potential given by Eq. (10). In general, the XHPS can model detailed satellite properties and the coupling to various gravitational and environmental perturbations. Here, we simplify matters and test the accuracy of the first order pN approximation w.r.t. the fully relativistic solution. Therefore, the satellite is modeled as a point mass, which allows for neglecting effects due to a possible spinning or tumbling.

The reason to use the XHPS to investigate the pN orbits is twofold: this approach allows to i) test the accuracy of the first order pN approximation by comparing the orbit to the solution in the Schwarzschild spacetime, and ii) to compare the acceleration caused by the pN correction terms to various disturbing forces, such as solar radiation pressure, at a later stage; see the next section.

Figure 3 shows the accuracy of the first order pN orbits for three exemplary cases. We compare the radial and tangential deviation from the orbit in the Schwarzschild spacetime, which we constructed numerically with sufficient accuracy. To be consistent, we have to use isotropic radial coordinates and a coordinate time parameterization of the orbits in either case. The maximal deviation is found for the elliptical orbit \#6, where it is in the nanometer regime. Hence, the result is as expected and summarized by: the first order pN approach yields orbits at the nm accuracy level for satellite constellations around the Earth.

The way we assure identical initial conditions, at the level of accuracy that is inherent for the first order pN approximation, for orbits in the Schwarzschild spacetime and its pN analogue is as follows. The orbits are confined to the equatorial plane, start at the perigee, and the argument of the perigee is taken to be zero. Both, the Schwarzschild orbit and the pN orbit are parametrized by the respective coordinate time, and the isotropic radial coordinates \( \lambda \) and \( R \) are used, respectively. We read off the eccentricity and semi-major axis, for which the Schwarzschild area coordinate \( r \) is used, from Tab. 2. Then, the perigee and apogee radii are calculated and transformed to the Schwarzschild isotropic coordinate \( \lambda \). We also calculate the initial azimuthal velocity w.r.t. the coordinate time by

\[
\Phi_0 = \dot{\varphi}_0 \frac{c^2 A(r_0)}{r_0^2}.
\]

Hence, the non-vanishing initial conditions for the orbit in the Schwarzschild spacetime are given by \( \lambda_0 \) and \( \Phi_0 \). For the pN orbits, we identify the angular coordinates with those of the Schwarzschild spacetime, and for the radial coordinates we use \( \lambda = R \) at the given level of accuracy. We then use the same numerical values for \( R_0 = \lambda_0 \) and the initial azimuthal velocity for the pN orbit integration with the XHPS.

The identification of the pN spherical coordinates with the Schwarzschild coordinates introduces an a priori error. Nevertheless, this error is of the order \( O(c^{-3}) \) and below the accuracy of the first order pN approximation. From a mathematical point of view, we simply use the same initial values for a differential equation and its approximation.

7 Magnitude of relativistic corrections

To judge the importance and magnitude of relativistic effects, we compare the acceleration (and therefore the force) caused by the pN corrections in the EOM to various non-gravitational perturbations. We use the XHPS and take into account the solar radiation pressure (SRP), the effects of Earth’s albedo, the atmospheric drag, and the thermal radiation pressure (TRP)
Fig. 3 The difference between the pN orbits (calculated using the XHPS) and the Schwarzschild orbits (calculated as geodesics of the Schwarzschild spacetime in isotropic coordinates and parametrized by coordinate time). The results are shown for orbit #2 (smallest eccentricity, top), orbit #6 (largest eccentricity, middle), and the LEO #7, which is used in Fig. 4, (bottom). In either case, we show the radial and tangential difference between both solutions, calculated at the same grid points.

We find the relativistic accelerations in the radial and tangential direction to reach a maximum of about 20 nm/s² and they are, thus, comparable to the non-gravitational perturbations shown in the same figure. Hence, at least the first order pN effects need to be taken into account to accurately model satellite orbits for high precision space missions in an environment around the Earth. The results shown here do agree with those shown in Tab. 3 of Ref. [24]. In this work, the authors estimated the magnitude of several orbital effects, including pN contributions and the space environment, for the LAGEOS satellite.

8 Conclusion

The purpose of this work was to quantify the accuracy of different methods to approximately solve the general relativistic equations of motions. We have compared different methods in the spherically symmetric Schwarzschild spacetime and its first order post-Newtonian approximation. Moreover, we have compared the magnitude of relativistic corrections in a post-Newtonian spacetime to various non-gravitational perturbations of satellite orbits.

To solve the relativistic equations of motion in the Schwarzschild spacetime, we have used direct numerical integration, the semi-analytical Lie-series approach, and the exact analytical solution in terms of the Weierstrass elliptic function. The latter served as the reference solutions and enabled us to test the accuracy of the other methods. To obtain satellite orbits in the pN approximation of the Schwarzschild spacetime, we included relativistic corrections in the XHPS.

When compared to the exact solution of the geodesic equation in the Schwarzschild spacetime, both, the direct numerical integration using a Runge-Kutta scheme on a fixed numerical grid, and the semi-analytical Lie-series approach yield (sub-)nanometer accuracies for a pre-defined set of test orbits. The drawback of using the Lie-series method is the longer computation time due to nested Poisson-brackets that are calculated analytically. However, the analytical part of the Lie-series approach may yield further insight into the dynamics in the spectral domain. Hence, this method may turn out to be an important link between the numerical integration and analytical solutions in later studies.

As high precision space mission we understand scenarios where orbital effects due to the space environment (SRP, Albedo, etc.) need to be taken into account to meet the accuracy goals of the respective mission.
The magnitude of the accelerations due to the first order pN contribution to the EOM, the solar radiation pressure (SRP), Earth’s albedo, atmospheric drag, and the thermal radiation pressure (TRP). The orbit starts at the perigee and has an eccentricity $e \approx 0.2$ and a semi-major axis $a \approx 8.5 \cdot 10^6$ m. All the accelerations were modeled using the XHPS, a GRACE-like model for the satellite properties, and Nadir pointing.

The first order post-Newtonian approximation of the Schwarzschild spacetime was considered and the equations of motion turned out to be modified Keplerian orbital equations that we solved by implementing the relativistic corrections into the XHPS. We verified the accuracy of the post-Newtonian approximation to the nanometer level, and we have shown that for a GRACE-like satellite in a low Earth orbit, the relativistic acceleration is comparable to various environmental perturbations and needs to be taken into account for high precision space missions.

In a follow-up paper, we plan to analyze longer orbital arcs and the relativistic orbital effects in a post-Newtonian approximation of a more complicated gravitational field of the Earth by considering higher multipole moments in the Newtonian gravitational potential.
as well. We will analyze GRACE, GRACE-FO, and similar satellite constellations to compare the relativistic effects to gravitational (other bodies) non-gravitational (environmental) perturbations. Furthermore, we will investigate extended satellites and the coupling of their moments of inertia to the gravitational field with its higher order multipoles. For such a situation, there is no analytical solution to the exact relativistic problem. However, the results that we presented in this work for the spherically symmetric case give strong confidence in the applicability and accuracy of the post-Newtonian approximation of General Relativity and the use of numerical integration tools for orbit propagation such as the XHPS to solve the equations of motion. We will also analyze how to determine the relativistic geoid and properties of the relativistic gravitational field by GRACE-like mission scenarios.

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