Electromagnetic and strong isospin breaking in light meson masses

Ángel S. Miramontes\textsuperscript{a}, Reinhard Alkofer\textsuperscript{b}, Christian S. Fischer\textsuperscript{c}, Hélios Sanchis-Alepuz\textsuperscript{d}

\textsuperscript{a}Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Morelia, Michoacán 58040, Mexico
\textsuperscript{b}Institute of Physics, University of Graz, NAWI Graz, Universitätsplatz 5, 8010 Graz, Austria
\textsuperscript{c}Institut für Theoretische Physik, Justus-Liebig-Universität Giessen, 35392 Giessen, Germany; Helmholtz Forschungsakademie Hessen für FAIR (HFHF) GSI Helmholtzzentrum für Schwerionenforschung, Campus Gießen, 35392 Gießen, Germany
\textsuperscript{d}Silicon Austria Labs GmbH, Inffeldgasse 33, 8010 Graz, Austria

Abstract

We study electromagnetic as well as strong isospin breaking effects in the isospin mass splittings of light pseudoscalar and vector mesons. To this end we employ a coupled system of quark Dyson-Schwinger and meson Bethe-Salpeter equations whose interaction kernels contain gluon, pion and photon exchange interactions. In bound states, QCD-induced isospin breaking is manifest on different levels. On the one hand, a different explicit up- and down-quark mass directly affects the propagators of the constituent quarks. On the other hand, it leads to different interaction kernels within the isospin multiplets. In addition, electromagnetic isospin breaking is induced via a photon exchange diagram. Using the kaon iso-doublet and the charged pion masses as input to determine the up, down and strange quark masses we find for the pion, kaon and rho meson mass splittings different patterns each. In particular, our results provide evidence that the effects from two sources of mass splittings, the different quark masses and the different quark charges, do not add up linearly.

1. Introduction

The approach to QCD via functional methods, most prominently with Dyson-Schwinger equations, Bethe-Salpeter equations and/or the functional renormalisation group has reached a degree of sophistication that allows to systematically study sub-leading effects in the hadron spectrum. Arguably the most important of these effects is the breaking of isospin symmetry. It plays an important role in baryon physics, notably in the mass splitting between the proton and the neutron which is related to fundamental questions like the existence and stability of ordinary matter in our universe. It also plays an important role in meson physics allowing for a range of interesting and important isospin violating decays such as the one of the exotic X(3872). Furthermore, its inclusion is mandatory for precision determinations of hadronic contributions to the anomalous magnetic moment of the muon \cite{1,2}, and of the CKM matrix elements $V_{ud}$ and $V_{us}$ (see, e.g., \cite{3} for a recent update on the respective analysis of $\pi$ and $K$ decays in view of the apparent $3\sigma$ violation of the first-row CKM unitarity condition). Thus, a better understanding of isospin breaking effects, besides being interesting in its own right, might play a significant role in the detection of new physics via high-precision experiments.

The mass difference of the charged to the neutral pions have been analysed already before the advent of QCD \cite{4} with the conclusion that the observed pion mass difference does not arise from the mass difference of the two light quark flavours but from the electromagnetic interaction. Based on this, when considering three flavours, one arrives at the conjecture for the electromagnetically induced kaon mass difference

\begin{equation}
(m_{K^\pm}^2 - m_{K^0}^2)_{\text{em}} = m_{\pi^+}^2 - m_{\pi^0}^2
\end{equation}

which is also known as Dashen’s theorem \cite{5}. However, this relation is already significantly violated
of the up- and down-quark. Diagrammatically, this effect is isospin breaking due to the charge difference of the up- and down-quark. The second, in many cases competing, effect is isospin breaking due to the mass difference of the up- and down-quark. The dressed quark propagators, \( S_f(p) \), will be of the form

\[
S_f(p) = \frac{1}{A_f(p^2)} \frac{-ip + M_f(p^2)}{p^2 + M_f(p^2)} \tag{5}
\]

which, for each flavour, is determined by two dressing functions, \( A_f(p^2) \) and \( M_f(p^2) \). The latter are the dynamically generated mass functions, and we

In this section we discuss our heuristic approach to isospin breaking using the formalism of Dyson-Schwinger (DSE) and Bethe-Salpeter (BSE) equations. We briefly summarise the key elements of the DSE/BSE approach (for more details see e.g. \[20, 18, 21, 22\]) and introduce the two mechanisms for isospin breaking that we explore in this paper.

Formally, the quark-DSE for a quark with flavour \( f \) is given by

\[
S^{-1}_f(p) = Z_2 Z_{1f} S_{0, f}^{-1}(p) - Z_{1f} g^2 C_F \times \int \frac{d^4q}{(2\pi)^4} i\gamma^\nu S_f(q) \Gamma^\nu_{86, f}(q, p) D^{\mu\nu}(k),
\]

where \( Z_2 \) and \( Z_{1f} = Z_q Z_2 Z_{3f}^2 / 2 \) are (flavour-dependent) renormalisation constants for the quark propagator and quark-gluon vertex, and \( C_F = 4/3 \) is the colour Casimir for \( N_c = 3 \). The inverse tree-level propagator is given by \( S_{0, f}^{-1}(p) = ip + Z_m m_f \), where \( m_f \) is the renormalized quark mass from the QCD action. The dressed quark-gluon vertex \( \Gamma_{86, f} \) has a very rich structure including gluonic but also effective meson exchange contributions detailed below.

The dressed quark propagators, \( \Gamma_{86, f} \), will be of the form

\[
S_f(p) = \frac{1}{A_f(p^2)} \frac{-ip + M_f(p^2)}{p^2 + M_f(p^2)} \tag{5}
\]

QED effect appears in the form of photon loop and exchange diagrams which have to be added to the self-energies and interaction kernels.

In this work, we study contributions of strong and electromagnetic isospin breaking effects on the mass splitting of the isospin multiplets of light pseudoscalar and vector mesons. Based on previous work on the pion electromagnetic form factor \[19\], our truncation scheme takes explicitly into account that the non-perturbative interaction between light quarks becomes flavour dependent away from the isospin limit. We summarise technical details of our scheme in section 2 and discuss our results for the mass splittings and the associated up- and down-quark masses in section 3. In section 4 we conclude with an outlook to future applications.

2. Formalism

In this section we discuss our heuristic approach to isospin breaking using the formalism of Dyson-Schwinger (DSE) and Bethe-Salpeter (BSE) equations. We briefly summarise the key elements of the DSE/BSE approach (for more details see e.g. \[20, 18, 21, 22\]) and introduce the two mechanisms for isospin breaking that we explore in this paper.

Formally, the quark-DSE for a quark with flavour \( f \) is given by

\[
S^{-1}_f(p) = Z_2 Z_{1f} S_{0, f}^{-1}(p) - Z_{1f} g^2 C_F \times \int \frac{d^4q}{(2\pi)^4} i\gamma^\nu S_f(q) \Gamma^\nu_{86, f}(q, p) D^{\mu\nu}(k),
\]

where \( Z_2 \) and \( Z_{1f} = Z_q Z_2 Z_{3f}^2 / 2 \) are (flavour-dependent) renormalisation constants for the quark propagator and quark-gluon vertex, and \( C_F = 4/3 \) is the colour Casimir for \( N_c = 3 \). The inverse tree-level propagator is given by \( S_{0, f}^{-1}(p) = ip + Z_m m_f \), where \( m_f \) is the renormalized quark mass from the QCD action. The dressed quark-gluon vertex \( \Gamma_{86, f} \) has a very rich structure including gluonic but also effective meson exchange contributions detailed below.

The dressed quark propagators, \( \Gamma_{86, f} \), will be of the form

\[
S_f(p) = \frac{1}{A_f(p^2)} \frac{-ip + M_f(p^2)}{p^2 + M_f(p^2)} \tag{5}
\]

QED effect appears in the form of photon loop and exchange diagrams which have to be added to the self-energies and interaction kernels.

In this work, we study contributions of strong and electromagnetic isospin breaking effects on the mass splitting of the isospin multiplets of light pseudoscalar and vector mesons. Based on previous work on the pion electromagnetic form factor \[19\], our truncation scheme takes explicitly into account that the non-perturbative interaction between light quarks becomes flavour dependent away from the isospin limit. We summarise technical details of our scheme in section 2 and discuss our results for the mass splittings and the associated up- and down-quark masses in section 3. In section 4 we conclude with an outlook to future applications.

2. Formalism

In this section we discuss our heuristic approach to isospin breaking using the formalism of Dyson-Schwinger (DSE) and Bethe-Salpeter (BSE) equations. We briefly summarise the key elements of the DSE/BSE approach (for more details see e.g. \[20, 18, 21, 22\]) and introduce the two mechanisms for isospin breaking that we explore in this paper.

Formally, the quark-DSE for a quark with flavour \( f \) is given by

\[
S^{-1}_f(p) = Z_2 Z_{1f} S_{0, f}^{-1}(p) - Z_{1f} g^2 C_F \times \int \frac{d^4q}{(2\pi)^4} i\gamma^\nu S_f(q) \Gamma^\nu_{86, f}(q, p) D^{\mu\nu}(k),
\]

where \( Z_2 \) and \( Z_{1f} = Z_q Z_2 Z_{3f}^2 / 2 \) are (flavour-dependent) renormalisation constants for the quark propagator and quark-gluon vertex, and \( C_F = 4/3 \) is the colour Casimir for \( N_c = 3 \). The inverse tree-level propagator is given by \( S_{0, f}^{-1}(p) = ip + Z_m m_f \), where \( m_f \) is the renormalized quark mass from the QCD action. The dressed quark-gluon vertex \( \Gamma_{86, f} \) has a very rich structure including gluonic but also effective meson exchange contributions detailed below.

The dressed quark propagators, \( \Gamma_{86, f} \), will be of the form

\[
S_f(p) = \frac{1}{A_f(p^2)} \frac{-ip + M_f(p^2)}{p^2 + M_f(p^2)} \tag{5}
\]

QED effect appears in the form of photon loop and exchange diagrams which have to be added to the self-energies and interaction kernels.

In this work, we study contributions of strong and electromagnetic isospin breaking effects on the mass splitting of the isospin multiplets of light pseudoscalar and vector mesons. Based on previous work on the pion electromagnetic form factor \[19\], our truncation scheme takes explicitly into account that the non-perturbative interaction between light quarks becomes flavour dependent away from the isospin limit. We summarise technical details of our scheme in section 2 and discuss our results for the mass splittings and the associated up- and down-quark masses in section 3. In section 4 we conclude with an outlook to future applications.
Figure 1: Coupled system of DSEs for up and down quarks (upper two equations) including explicit pion backreaction diagrams and a loop including a photon. Gluons are represented by curly lines, mesons by dashed lines and photons by wiggly lines. In each diagram the left vertex is bare (but renormalised) and the vertex on the right is dressed. Kaon and eta backreaction diagrams are neglected (see main text). Correspondingly, the DSE for the strange quark (lower equation) contains gluon and photon diagrams only.

will use below $M_f(p^2 = 0)$ to demonstrate that in the solutions of the DSE the isospin breaking effects due to the current mass differences are enhanced.

In the BSE framework, bound states of a quark and an antiquark can be described by Bethe-Salpeter amplitudes $\Gamma$ which are obtained as solutions of a homogeneous BSE,

$$\Gamma_{fg}(p,P) = \sum_{f'g'} \int_q K_{fg}^{g'f}(p,q) \left( S_{f'}(k_1) \times \Gamma_{g'f'}(q,P) S_{g'}(k_2) \right), \tag{6}$$

with $P$ the total meson momentum, $p$ is the relative momentum between quark and antiquark, $q$ is an internal relative momentum which is integrated over. For pseudoscalar and vector mesons, the Dirac part of the Bethe-Salpeter amplitude $\Gamma_{fg}$ for a meson with constituents of flavour $f,g$ can be expanded in a tensorial basis with four and eight elements, respectively. The interaction kernel $K$ encodes all possible interactions between quarks and anti-quarks, and the quark propagators $S_f(p)$ are given by the solution of the quark DSE in Eq. (4).

In this work we consider three types of contributions to the quark self-energy and the BSE interaction kernel: (i) a flavour-blind dressed quark-antiquark gluon exchange that provides the necessary interaction strength to form mesonic bound states, (ii) a meson-exchange mechanism that provides flavour mixing and, (iii) a dressed photon exchange describing electromagnetic interactions between the quark and the antiquark. The latter is responsible for the electromagnetically induced isospin breaking. These contributions are shown explicitly in Figs. 1 and 2. In the following, we explain each contribution in turn.

2.1. Explicit Isospin Breaking and Flavour Mixing

Let us consider first the non-electromagnetic contributions to the quark self-energy and the BSE interaction kernel. As indicated above and shown in Figs. 1 and 2 the strongly-interacting part consists of a gluon loop/exchange term and meson loop/exchange terms.

For the gluon term we use the rainbow-ladder truncation of the quark-gluon vertex, whereby the effect of (the leading term of) the dressed quark-gluon vertex $\Gamma_{\nu,QCD}(q,p) = i\gamma^\nu \Gamma(k^2)$ and the Landau-gauge dressed gluon propagator $D_{\mu\nu}(k) = T_{\mu\nu}Z(k^2)/k^2$ with transverse projector $T$ is combined into an effective coupling that only depends on the gluon momentum $k$.

$$\alpha(k^2) = \frac{Z_{1f}}{Z_{2f}} \frac{g^2}{4\pi} Z(k^2) \Gamma(k^2). \tag{7}$$

For the effective coupling we use the Maris-Tandy model

$$\alpha_{QCD}(k^2) = \pi \eta^2 \left( \frac{k^2}{\Lambda^2} \right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}} + \frac{2\pi\gamma_m (1 - e^{-k^2/\Lambda_{QCD}^2})}{\ln(e^2 - 1 + (1 + k^2/\Lambda_{QCD}^2)^2)}, \tag{8}$$

where the Gaussian term provides binding strength and enables dynamical chiral symmetry breaking; the second term reproduces the one-loop QCD behavior of the quark propagator in the ultraviolet when used in the quark DSE. In this model
the scale $\Lambda_t = 1$ GeV is introduced for technical reasons. For the anomalous dimension we use $\gamma_m = 12/(11N_C - 2N_f) = 12/25$ with $N_f = 4$ flavours and $N_c = 3$ colours. For the QCD scale we use $\Lambda_{QCD} = 0.234$ GeV. The parameters $\eta$ and $\Lambda$ are discussed below.

Note that such a truncation of the gluon-interaction guarantees that the axial-vector WTI (axWTI) is respected and, hence, that the pion is massless in the chiral limit.

A gluon exchange as interaction mechanism is flavour diagonal. In order to introduce flavour mixing we complement the interaction kernel with a meson exchange mechanism (see Fig. 2). The BSE kernel, defined in [25, 26], and representing the exchange of a pion with Bethe-Salpeter amplitude $\Gamma_\pi$ and propagator $D_\pi(p) = 1/(p^2 + m_\pi^2)$, reads

$$
K^{(t)}_{\pi s} (q, p; P) =
\frac{C}{4} \left[ \Gamma_\pi \right]_{rs} \left( \frac{p + q - P}{2} ; p - q \right) Z_2 \gamma_5 |_{rs} D_\pi(p - q)
+ \frac{C}{4} \left[ \Gamma_\pi \right]_{ru} \left( \frac{p + q - P}{2} ; q - p \right) Z_2 \gamma_5 |_{rs} D_\pi(p - q)
+ \frac{C}{4} \left[ Z_2 \gamma_5 \right]_{rs} \left( \frac{p + q + P}{2} ; p - q \right) D_\pi(p - q)
+ \frac{C}{4} \left[ Z_2 \gamma_5 \right]_{rs} \left( \frac{p + q + P}{2} ; q - p \right) D_\pi(p - q).
$$

(9)

The quark DSE, correspondingly, is extended with the addition of further loop diagrams (third and fourth terms in the quark DSEs in Fig. 1). As discussed in detail in [25, 26], the presence of these loops can be motivated and derived from the underlying Dyson-Schwinger equation of the quark-gluon vertex, where meson exchange diagrams ap-
peared as approximation of a momentum dependent
four-Fermi interaction. In a complete treatment
of these effects, we would need to include meson
diagrams involving pions, kaons and in the DSE for
the strange quark even the $\eta'/\eta$ mesons. However,
already pion backraction effects are subleading on
the 10-20 percent level compared to gluon effects
[25, 26]. Since kaon and $\eta'/\eta$ effects are further
suppressed by factors of $m^2_{\pi}/m^2_{K,\eta}$ it seems safe to ne-
glect these in a first approach. A similar argument
applies to potential contributions from mesons with
other quantum numbers.

In contrast to the gluon exchange, the meson
exchange mechanism does not immediately pre-
serve the axWTI and can, potentially, violate chi-
rnal symmetry. It was shown in [25, 26] how an
axWTI-preserving meson-exchange kernel can be con-
structed. We use this kernel, in particular, we
set $C = -3/2$, in this work. Note that consequently
the meson exchange kernels do not introduce any
additional parameters.

The resulting diagrams in the BSEs are shown in
Fig. 2. It is interesting to note that due to the
flavour structure, the meson exchange diagrams
only appear in the flavour diagonal BSEs for the
$u\bar{u}$ and $d\bar{d}$ components. Corresponding diagrams
in the two BSEs for the charged pions are diquark
exchange diagrams. Again, the effects of these di-
agrams are suppressed by factors of $m^2_{\pi}/m^2_{\eta'}$ and we neglect these due to the heavy mass of the lowest
lying scalar diquark of the order of 800-900 MeV
[18, 27].

Furthermore, we need to point out a technical
difficulty we face in the present calculation. In the
meson-exchange kernels shown in Fig. 2 [see also
Eq. 9] the quark-meson interaction vertex is given
by the meson BS amplitude. Moreover, in order to
access the time-like region, where bound states fulfill
$P^2 = -M^2$, with $P$ the total bound-state momentum
and $M$ the bound-state mass, we need to work
with complex momenta in the BSE, since we work
in Euclidean spacetime. This implies, in particular,
that the BS amplitudes used as vertices need to be evaluated for complex relative quark momenta
(i.e. the momenta $p$ in Eq. (9)). Solving BSEs for
complex relative momenta is currently only possible
in the RL approximation. Therefore, the BS am-
plitudes used as vertices are obtained from a RL-
truncated BSE calculation. Unfortunately, we have
no way of estimating at the moment how big the
quantitative effect of such an approximation on our
results is. We do not expect, however, that any
qualitative statements change.

Finally, let us point out that neither the gluon ex-
change nor the meson exchange interaction me-
chanisms induce any isospin breaking on their own.
However, if explicit isospin breaking is introduced
by a choice of different up- and down-quark masses,
the meson exchange terms will cease to be degener-
ate and introduce non-trivial mixing of the different
BSE amplitudes. Thus isospin breaking occurs on
different levels: it results in different dressing func-
tions for the up- and down-quark propagators but
it also results in a flavour dependent dressing func-
tions for the up- and down-quark propagators but
it also results in a flavour dependent QCD interaction
manifest in the different meson exchange terms in
our truncation.

2.2. Electromagnetically-induced Isospin Breaking

To describe electromagnetic effects we consider
da dressed photon exchange term in the BSE ker-
nel and the corresponding term in the quark DSE
(see Figs. 1 and 2). The quark-photon interaction
is described by the fully-dressed quark-photon ver-
tex. Even though the non-perturbative structure of
the quark-photon vertex is by now reasonably well
known [28, 29, 30, 31, 32, 33], for the qualitative
study of this work we will choose a simpler model.

The non-transverse part of the quark-photon ver-
tex is given by the Ball-Chiu construction [28] in
terms of the quark dressing functions $A_f(p^2)$ and
$B_f(p^2)$, with $S_f^{-1}(p) = -i\gamma A_f(p^2) + B_f(p^2)$ the
inverse quark propagator. In this work we simplify
the calculation further, employing the model used
in [34], where only the $\gamma^\mu$ component of the Ball-
Chiu vertex is used. In this way, the QED contrib-
ution takes the same form as the gluon rainbow-
ladder term and can be incorporated quite easily by
adding

$$\alpha_{\text{QED}}(k^2) = \alpha_0 Q_f Q_f Z_{\text{QED}}(k^2),$$

(10)
to $\alpha_{\text{QCD}}$ in Eq. 8. Here, $\alpha_0 \approx 1/137$ corresponds to
the electromagnetic coupling, $Q_f, g$ are the charges
of the two quarks attached to the photon line with
flavour $f, g$ and $Z_{\text{QED}}$ the dressing of the leading
tensorial structure of the quark-photon vertex. It
is parametrised by

$$Z_{\text{QED}}(k^2) = \frac{A_f(k^2) + A_g(k^2)}{2},$$

(11)
where $i, j$ represent the two quarks at both ends of
the photon. This flavour averaging has no effect on
the self-energy in the quark-DSE, but is appropriate
for the kernels in flavour non-diagonal meson states.
Solving the coupled system of DSE and BSE using different masses and charges for up and down quarks we are now in a position to study the effects of the different sources of isospin breaking.

3. Results

First we need to fix the QCD-parameters, the quark masses \(m_u, m_d\) and \(m_s\), and the parameters of our model interaction, the Maris-Tandy parameters \(\Lambda\) and \(\eta\). The model parameter \(\Lambda\) sets the scale and is adjusted such that the exponential constant of the charged pion is reproduced. Furthermore, it is well known from pure RL calculations \([13]\) that within a certain range, experimental quantities are fairly insensitive to the value of the other model parameter, \(\eta\). In our calculations we choose \(\eta = 1.42\) and \(\Lambda = 0.74\).

This leaves the up-, down- and strange-quark masses to be determined. We adjust these such that a complete calculation with rainbow-ladder gluon exchange, meson exchange and photon exchange (cf. Figs. 1 and 2) reproduces the experimental masses for the charged pion and the neutral and charged kaon. The corresponding results are displayed in the rightmost column of Tab. 1. We find \(m_u = 6.3\) MeV, \(m_d = 8.1\) MeV and \(m_s = 82.8\) MeV for the quark masses in a MOM-scheme evaluated at a renormalisation scale \(\mu = 19\) GeV. All other results for meson masses are then model predictions. Note that the masses of the vector meson in general come out too small by about 50 MeV; this is a well-known artefact of the Maris-Tandy model and is remedied only in a full calculation taking into account corrections from the gluon self-interaction \([35]\) and from the decay into two pions \([36]\).

Here we like to point out the well-known result that the dynamically generated mass function \(M_f(p^2)\), cf. Eq. 5, displays a significant enhancement of the mass splitting for values of \(p^2\) being in the sub-GeV region. Taking into account only the flavour-blind gluon-mediated interaction one obtains

\[
M_u(0) - M_d(0) \approx 5(m_u - m_d)
\]  

(12)

Including the pion exchange for the two light flavours lowers \(M_{u,d}(0)\) both by 37 MeV, and consequently the relation \([12]\) stays valid. The photon exchange increases \(M_\pi(0)\) by slightly less than one MeV, resp., 0.3 %, which is of the expected size due to \(Q^2_{\alpha_0} = 0.003\). Due to the lower absolute value of the quark charge \(M_q(0)\) is increased by approximately a quarter MeV (and \(M_s(0)\) by half a MeV). Clearly, these small additional mass increases leave us more or less with the mass difference \([12]\).

The enhancement \([12]\) from \(m_u - m_d = 1.8\) MeV to \(M_u(0) - M_d(0) \approx 9\) MeV is important for an understanding of the induced mass difference of the neutral to the charged kaons in terms of the quark masses. The strong-interaction induced mass difference for the \(K\) mesons is expected to be due to binding effects smaller than the mass difference of the two different light valence quarks. As a matter of fact, from our results for the \(K\) mesons, see Tab. 1, we infer that it is smaller than \(M_u(0) - M_d(0)\) but due to \([12]\) it is significantly larger than the current mass difference \(m_u - m_d\).

In order to analyse our results for the meson masses and in particular the separate effects of isospin breaking due to different up-/down-quark masses and the electromagnetic interaction in the context of the available literature we adopt the notation of the FLAG review \([9]\). As described above, the quark masses are then determined by the system of equations

\[
M_{u+}(m_u, m_d, m_s, \alpha_0) = M_{\pi^+}^{\exp},
\]

(13)

\[
M_{K^+}(m_u, m_d, m_s, \alpha_0) = M_{K^+}^{\exp},
\]

(14)

\[
M_{K^0}(m_u, m_d, m_s, \alpha_0) = M_{K^0}^{\exp},
\]

(15)

where the right hand sides are the experimental values and \(\alpha_0 = 1/137\) is precise enough for our study. All calculated masses are functions of \(m_u, m_d, m_s, \alpha_0\), and since isospin breaking effects are small, in a hadronic quantity \(X\) they may be described to first order by expansions in \(\delta m\) and \(\alpha_0\) with

\[
X(m_u, m_d, m_s, \alpha_0) = \bar{X}(m_{ud}, m_s)
+ \delta m A_X(m_{ud}, m_s)
+ \alpha_0 B_X(m_{ud}, m_s)
+ \delta m \alpha_0 C_X(m_{ud}, m_s)
\]

(16)

\[
= \bar{X} + X^{SU(2)} + X^\gamma + X^{SU(2);\gamma}.
\]

Here, \(m_{ud} = 1/2(m_u + m_d)\) is the average light-quark mass. Higher order effects in this expansion are traditionally expected to be small \([9]\). In order to gauge this expectation, we also included the lowest order mixed quantity proportional to \(\delta m \alpha_0\) to facilitate our analysis. The size of this additional correction can be estimated from comparing results for \(X\) and \(\bar{X}\) with results obtained from individually setting \(\delta m\) or \(\alpha_0\) to zero.
| $M$ [MeV] | $X$ | $m_u = m_d$ | $m_u \neq m_d$ | $m_u = m_d$ | $X$ | $m_u \neq m_d$ |
|---|---|---|---|---|---|---|
| $\pi^0$ | 134.5 | 132.5 | 136.0 | 133.4 |
| $\pi^\pm$ | 134.5 | 134.2 | 139.6 | 139.7 |
| $\pi^\pm - \pi^0$ | 0 | 1.7 | 3.6 | 6.3 |
| $K^0$ | 494.7 | 497.5 | 495.2 | 497.7 |
| $K^\pm$ | 494.7 | 492.1 | 497.2 | 493.7 |
| $K^0 - K^\pm$ | 0 | 5.4 | -2.0 | 4.0 |
| $\rho^0$ | 720.3 | 721.5 | 721.1 | 721.4 |
| $\rho^\pm$ | 720.3 | 719.9 | 722.0 | 720.9 |
| $\rho^\pm - \rho^0$ | 0 | -1.6 | 0.9 | -0.5 |

Table 1: Pion, kaon and rho masses $M$ for the full system depicted in Figs. 3 and 2 with quark masses $m_u = 6.3$ MeV, $m_d = 8.1$ MeV, $m_s = 82.8$ MeV (third and fifth column), average quark masses $m_{ud} = m_u = m_d = 7.2$ MeV (second and fourth column) and electromagnetic isospin breaking switched on and off. A dagger indicates that the quark masses where fitted to obtain those meson masses.

Our results for the isospin doublet kaons and the isospin triplet pions and $\rho$-mesons as well as their mass differences are shown in Tab. 1. We find that the kaon mass splitting is dominated by strong isospin breaking, $(m_{K^0} - m_{K^\pm})^{SU(2)} = 5.4$ MeV with a much smaller electromagnetic breaking with opposite sign $(m_{K^0} - m_{K^\pm})^\gamma = -2.0$ MeV and thus following the expectation derived from Eq. (2).

In contrast, the pion mass splitting is dominated by the electromagnetic effect $(m_{\pi^\pm} - m_{\pi^0})^\gamma = 3.6$ MeV, and an addition due to the strong isospin breaking of only half the size, $(m_{\pi^\pm} - m_{\pi^0})^{SU(2)} = 1.7$ MeV.

In both cases we also observe that the two sources of breaking do not add linearly, but there are non-negligible effects due to mixed higher order terms in the expansion Eqs. (16). We find $(m_{\pi^\pm} - m_{\pi^0})^{SU(2)} = 1.0$ MeV for the pions and $(m_{K^0} - m_{K^\pm})^{SU(2)} = 0.6$ MeV for the kaons. The mass of the neutral pion is only slightly affected by isospin breaking, most of the difference stems from the shift in the mass of the charged pions. The biggest single effect is the electromagnetically induced mass increase of the charged pions, and all isospin breaking effects are adding up. (NB: It should be noted, however, that our calculation overestimates the pion mass difference, we obtain 6.3 MeV instead of the observed 4.6 MeV [8].) For the kaons we note the partial cancellation of electromagnetic and strong isospin breaking effects. For the charged kaons this cancellation is almost complete, whereas for the neutral kaon there is a larger mass shift due to the strong interaction as the one due to the electromagnetic interaction.

A somewhat different pattern can be observed for the rho meson masses. First of all, the observed individual shifts are much smaller than the ones for the pions or the kaons, and in addition they largely cancel each other due to opposite signs. But again, we observe that the two sources of isospin breaking do not add linearly.

4. Conclusions and outlook

In this work, we determined the effects of isospin breaking from the strong interaction, via different masses for the up- and down-quark, and the electromagnetic interaction, via different charges of the up- and down-quark, on the pseudoscalar meson isospin triplet, strange pseudoscalar meson doublet and the vector meson triplet. We find that the pion splitting is mainly induced by the different quark charges, but we also observe some effects due to different masses and even noticeable mixed, i.e. non-linear, effects of both. The corresponding splitting in the rho triplet is very small which is due to relatively small individual mass shifts as well as competing mass and charge splitting effects of opposite sign. Again, the non-linear effects are relevant to obtain the resulting mass splitting. A special role is played by the kaon doublet splitting which results from quite a number of mass shifts of similar magnitude but opposite sign.

Our results agree qualitatively with those of other continuum approaches, mainly the $\chi$PT results, see, e.g., [6, 7, 37], and with those of lattice QCD see,
In particular, we find that the traditional view that the pion mass difference does not arise from the quark mass difference but practically only from the different quark charges is an unjustified oversimplification. We also note that our results do not support Dashen’s theorem \([5]\) for the electromagnetically induced kaon mass splitting, instead we have

\[
(m_{K^+}^2 - m_{K^0}^2)_{\text{em}} \approx 2.0 (m_{\pi^+}^2 - m_{\pi^0}^2)_{\text{em}}, \tag{17}\]

in agreement with the \(\chi PT\) result, Eq. \([2]\).

Acknowledgements

We thank Jan Bonnet and Richard Williams for collaboration and discussions in early stages of this work. This work has been supported by Silicon Austria Labs (SAL), owned by the Republic of Austria, the Styrian Business Promotion Agency (SFG), the federal state of Carinthia, the Upper Austria, the Styrian Business Promotion Agency Austria Labs (SAL), owned by the Republic of Austria, and the Austrian Association for the Electric and Electronics Industry (FEII). A.S. Miramontes acknowledges CONACYT for financial support.

References

[1] T. Aoyama, et al., The anomalous magnetic moment of the muon in the Standard Model, Phys. Rept. 887 (2020) 1–166. \[arXiv:2006.04822\] doi:10.1016/j.physrep.2020.07.006

[2] S. Borsanyi, et al., Leading hadronic contribution to the muon magnetic moment from lattice QCD, Nature 593 (7857) (2021) 51–55. \[arXiv:2002.12347\] doi:10.1038/s41586-021-03418-1

[3] C.-Y. Seng, D. Galván, W. J.Marciano, U.-G. Meißner, Update on \(|V_{ud}|\) and \(|V_{us}/V_{ud}|\) from semileptonic kaon and pion decay. \[arXiv:2107.14708\]

[4] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, J. E. Young, Electromagnetic mass difference of pions, Phys. Rev. Lett. 18 (1967) 759–761. doi:10.1103/PhysRevLett.18.759

[5] R. F. Dashen, Chiral SU(3) x SU(3) as a symmetry of the strong interactions, Phys. Rev. 183 (1969) 1245–1260. doi:10.1103/PhysRev.183.1245

[6] J. Bijnens, J. Prades, Electromagnetic corrections for pions and kaons: Masses and polarizabilities, Nucl. Phys. B 490 (1997) 239–271. \[arXiv:hep-ph/9610360\] doi:10.1016/S0550-3213(97)00107-7

[7] G. Amoros, J. Bijnens, F. Talavera, QCD isospin breaking in meson masses, decay constants and quark mass ratios, Nucl. Phys. B 692 (2001) 87–108. \[arXiv:hep-ph/0101127\] doi:10.1016/S0550-3213(01)00121-3

[8] P. A. Zyla, et al., Review of Particle Physics, PTEP 2020 (8) (2020) 083C01. doi:10.1093/ptep/ptaa104

[9] Y. Aoki, et al., FLAG Review 2021. \[arXiv:2111.09849\]

[10] R. Horsley, et al., QED effects in the pseudoscalar meson sector, JHEP 04 (2016) 003. doi:10.1007/JHEP04(2016)003

[11] S. Basak, et al., Lattice computation of the electromagnetically induced kaon and pion masses, Phys. Rev. D 99 (3) (2019) 034503. \[arXiv:1807.05556\] doi:10.1103/PhysRevD.99.034503

[12] Z. Fodor, C. Hoelbling, S. Krieg, L. Lellouch, T. Lippert, A. Portelli, A. Sastre, K. K. Szabo, L. Varnhorst, Up and down quark masses and corrections to Dashen’s theorem from lattice QCD and quenched QED, Phys. Rev. Lett. 117 (8) (2016) 082001. \[arXiv:1604.07112\] doi:10.1103/PhysRevLett.117.082001

[13] D. Giusti, V. Lubicz, C. Tarantino, G. Martinelli, F. Sanfilippo, S. Simula, L. Tantalo, Leading isospin-breaking corrections to pion, kaon and charmed-meson masses with Twisted-Mass fermions, Phys. Rev. D 95 (11) (2017) 114504. \[arXiv:1704.06651\] doi:10.1103/PhysRevD.95.114504

[14] A. Z. N. Yong, et al., Near-Physical Point Lattice Calculation of Isospin-Breaking Corrections to \(K_{2/3}\). \[arXiv:2111.09849\]

[15] P. Jain, H. J. Munczek, q anti-q bound states in the Bethe-Salpeter formalism, Phys. Rev. D 48 (1993) 5403–5411. \[arXiv:hep-ph/9307221\] doi:10.1103/PhysRevD.48.5403

[16] M. Harada, M. Kurachi, K. Yamawaki, p+ - p0 mass difference from the Bethe-Salpeter equation, Phys. Rev. D 70 (2004) 03009. \[arXiv:hep-ph/0403120\] doi:10.1103/PhysRevD.70.033009

[17] M. Harada, M. Kurachi, K. Yamawaki, The \(p^+ - p^0\) mass difference and the \(S\) parameter in large \(N_c\) QCD, Prog. Theor. Phys. 115 (2006) 765–795. doi:10.1103/PhysRevD.95.114504

[18] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, C. S. Fischer, Baryons as relativistic three-quark bound states, Phys. Rev. D 99 (3) (2019) 034503. doi:10.1103/PhysRevD.99.034503

[19] A. Z. N. Yong, M. Harada, H. Sanchis Alepuz, R. Alkofer, Elucidating the effect of intermediate resonances in the quark interaction kernel on the timelike electromagnetic pion form factor, Phys. Rev. D 101 (11) (2020) 116006. doi:10.1103/PhysRevD.103.116006

[20] I. C. Cloet, C. D. Roberts, Explanation and Prediction of Observables using Continuum Strong QCD, Prog. Part. Nucl. Phys. 92 (2018) 1–21. doi:10.1016/j.ppnp.2014.02.001

[21] M. Q. Huber, Nonperturbative properties of Yang–Mills theories, Phys. Rept. 879 (2020) 1–92. doi:10.1016/j.physrep.2018.05.022

[22] H. Sanchis-Alepuz, R. Williams, Recent developments in bound-state calculations using the Dyson–Schwinger and Bethe–Salpeter equations, Comput. Phys. Commun. 232 (2018) 1–21. doi:10.1016/j.cpc.2018.06.020

[23] F. Maris, C. D. Roberts, Pi- and K meson Bethe-Salpeter amplitudes, Phys. Rev. C 60 (1999) 3369–3383. \[arXiv:nucl-th/9905056\] doi:10.1103/PhysRevC.60.3369

[24] F. Maris, P. C. Tandy, Bethe-Salpeter study of vector meson masses and decay constants, Phys. Rev. C 60 (1999) 055214. \[arXiv:nucl-th/9905056\] doi:10.1103/PhysRevC.60.055214

\[\chi PT\]
[25] C. S. Fischer, D. Nickel, J. Wambach, Hadronic unquenching effects in the quark propagator, Phys. Rev. D 76 (2007) 094009. arXiv:0705.4407 doi:10.1103/PhysRevD.76.094009

[26] C. S. Fischer, D. Nickel, R. Williams, On Gribov’s supercriticality picture of quark confinement, Eur. Phys. J. C 60 (2009) 47–61. arXiv:0807.3486 doi:10.1140/epjc/s10052-008-0821-1

[27] M. Y. Barabanov, et al., Diquark correlations in hadron physics: Origin, impact and evidence, Prog. Part. Nucl. Phys. 116 (2021) 103835. arXiv:2008.07630 doi:10.1016/j.ppnp.2020.103835

[28] J. S. Ball, T.-W. Chiu, Analytic Properties of the Vertex Function in Gauge Theories. 1., Phys. Rev. D 22 (1980) 2542. doi:10.1103/PhysRevD.22.2542

[29] M. R. Frank, Nonperturbative aspects of the quark - photon vertex, Phys. Rev. C 51 (1995) 987–998. arXiv:nucl-th/9403009 doi:10.1103/PhysRevC.51.987

[30] P. Maris, P. C. Tandy, The Quark photon vertex and the pion charge radius, Phys. Rev. C 61 (2000) 045202. arXiv:nucl-th/9910033 doi:10.1103/PhysRevC.61.045202

[31] L. Chang, Y.-X. Liu, C. D. Roberts, Dressed-quark anomalous magnetic moments, Phys. Rev. Lett. 106 (2011) 072001. arXiv:1009.3458 doi:10.1103/PhysRevLett.106.072001

[32] G. Eichmann, Probing nucleons with photons at the quark level, Acta Phys. Polon. Supp. 7 (3) (2014) 597. arXiv:1404.4149 doi:10.5506/APhysPolBSupp.7.597

[33] C. Tang, F. Gao, Y.-X. Liu, Practical scheme from QCD to phenomena via Dyson-Schwinger equations, Phys. Rev. D 100 (5) (2019) 056001. arXiv:1902.01679 doi:10.1103/PhysRevD.100.056001

[34] J. Bonnet, Effects of isospin symmetry breaking in light mesons and application to the anomalous magnetic moment of the muon, Ph.D. thesis, Giessen U. (2018).

[35] C. S. Fischer, R. Williams, Probing the gluon self-interaction in light mesons, Phys. Rev. Lett. 103 (2009) 122001. arXiv:0905.2291 doi:10.1103/PhysRevLett.103.122001

[36] R. Williams, Vector mesons as dynamical resonances in the Bethe–Salpeter framework, Phys. Lett. B 708 (2019) 134943. arXiv:1804.11161 doi:10.1016/j.physletb.2019.134943

[37] J. Bijnens, P. Gosdzinsky, P. Talavera, Vector meson masses in chiral perturbation theory, Nucl. Phys. B 501 (1997) 495–517. arXiv:hep-ph/9704212 doi:10.1016/S0550-3213(97)00391-X