Coadjoint structure of Borel subgroups and their nilradicals

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ABSTRACT: Let $G$ be a complex simply-connected semisimple Lie group and let $\mathfrak{g} = \text{Lie}G$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$. One readily has that $\text{Cent} \ U(\mathfrak{n})$ is isomorphic to the ring $S(\mathfrak{n})^\mathfrak{n}$ of symmetric invariants. Using the cascade $\mathcal{B}$ of strongly orthogonal roots, some time ago we proved that $S(\mathfrak{n})^\mathfrak{n}$ is a polynomial ring $\mathbb{C}[\xi_1, \ldots, \xi_m]$ where $m$ is the cardinality of $\mathcal{B}$. Using this result we establish that the maximal coadjoint of $N = \text{exp} \mathfrak{n}$ has codimension $m$.

Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ so that the corresponding subgroup $B$ is a Borel subgroup of $G$. Let $\ell = \text{rank} \mathfrak{g}$. Then in this paper we prove

Theorem. The maximal coadjoint orbit of $B$ has codimension $\ell - m$ so that the following statements are equivalent:

1. $-1$ is in the Weyl group of $G$ (i.e., $\ell = m$)
2. $B$ has a nonempty open coadjoint orbit.

Remark. A nilpotent or a semisimple group cannot have a nonempty open coadjoint orbit. Celebrated examples where a solvable Lie group has a nonempty open coadjoint orbit are due to Piatetski–Shapiro in his counterexample construction of a bounded complex homogeneous domain which is not of Cartan type.

In a subsequent paper we will apply the results above to a split real form $G_{\mathbb{R}}$. If $\ell = 1$ this gives rise (E. Stein) to the Hilbert transform.

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1. Introduction

Part A: Recollection of some of the relevant results of [K]

1.1. Let $G$ be a complex semisimple Lie group and let $\mathfrak{g} = \text{Lie}G$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$ and let $\ell$ be the rank of $\mathfrak{g}$. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{b}_- = \mathfrak{n}_- + \mathfrak{h}$. Let $B, H, N$ be the subgroups of $G$ corresponding, respectively, to $\mathfrak{b}, \mathfrak{h} \mathfrak{n}$, so that $B$ is a Borel subgroup of $G$ and one has the familiar semidirect product $B = H \rtimes N$. 

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Let $K$ be the Killing form $(x, y)$ on $g$. We may identify the dual $b^*$ of $b$ with $b_-$, where $v \in b^*$ and $z \in b$; then $(v, z) = (v, z)$. With the obvious similar definitions the dual of $n^*$ is identified with $n_-$. Let $\Phi_b : g \to b_-$ be the projection defined by the decomposition $g = b_- \oplus n$, and let $\Phi_n : g \to n_-$ be the projection defined by the decomposition $g = n_- \oplus b$. The coadjoint representation $\text{Coad}_b$ of $B$ may be given by

$$\text{Coad}_b(g)v = \Phi_b \text{Ad}_g v,$$

where $g \in B$ and $v \in b_-$. For simplicity we will write $g \cdot v$ for $\text{Coad}_b(g)v$. The coadjoint representation $\text{Coad}_n$ of $N$ is given similarly where $n$ replaces $b$, $n_-$ replaces $b_-$, $N$ replaces $B$ and $\cdot$ by $\cdot$.

In order to prove the open $B$ coadjoint theorem mentioned in the abstract, we will need the results of [K]. The results of [K] depend upon definitions and properties of the cascade of orthogonal roots. These will be used freely now, but for the convenience of the reader some of the definitions will be here recalled in Part A of the Introduction.

The Killing form $K$ induces a nonsingular bilinear form $(\mu, \nu)$ on $h^*$. Let $\Delta \subset h^*$ be the set of roots corresponding to $(h, g)$. For each $\varphi \in \Delta$ let $e_\varphi \in g$ be a corresponding root vector. The root vectors can and will be chosen so that $(e_\varphi, e_{-\varphi}) = 1$ for all roots $\varphi$. If $s \subset g$ is any subspace stable under $\text{ad} h$ let

$$\Delta(s) = \{ \varphi \in \Delta \mid e_\varphi \in s \}.$$

The set $\Delta_+$ of positive roots is then chosen so that $\Delta_+ = \Delta(n)$, and one puts $\Delta_- = -\Delta_+$.

Let $\mathcal{B} \subset \Delta_+$ be the cascade of orthogonal roots; See §1 in [K]. Then $\text{card} \mathcal{B} = m$ where $m$ is the maximal number of strongly orthogonal roots. See Theorem 1.8 in [K]. As in §2.1 in [K], let $\mathfrak{r} = \sum_{\beta \in \mathcal{B}} \mathbb{C} e_\beta$ so that $\mathfrak{r}$ is an $m$-dimensional commutative Lie subalgebra of $n$. Let $R$ be the $m$-dimensional commutative unipotent subgroup of $N$ corresponding to $\mathfrak{r}$. As in §2.1 of [K], let $\mathfrak{r}_- \subset n_-$ be the span of $e_-\beta$ for $\beta \in \mathcal{B}$. For any $z \in \mathfrak{r}_-$, $\beta \in \mathcal{B}$, let $a_\beta(z) \in \mathbb{C}$ be defined so that

$$z = \sum_{\beta \in \mathcal{B}} a_\beta(z) e_-\beta$$

and let

$$\mathfrak{r}^\times_- = \{ \tau \in \mathfrak{r}_- \mid a_\beta(\tau) \neq 0, \ \forall \beta \in \mathcal{B} \}.$$

As an algebraic subvariety of $n_-$, clearly

$$\mathfrak{r}^\times_- \cong (\mathbb{C}^\times)^m.$$
For any $v \in n_-$ let $O_v$ be the $N$-coadjoint orbit containing $v$. Let $N_v \subset N$ be the coadjoint isotropy subgroup at $v$ and let $n_v = \text{Lie } N_z$. Since the action is algebraic, $N_v$ is connected and as $N$-spaces

$$O_v \cong N/N_v.$$  

The following result appears as Theorems 2.3 and 2.5 in [K].

**Theorem 1.1.** Let $\tau \in r^-$. Then (independent of $\tau$) $N_{\tau} = R$ so that (1.1) becomes

$$O_{\tau} \cong N/R.$$  

(1.2)

Furthermore if $\tau, \tau' \in r^-$ are distinct, then $O_{\tau} \cap O_{\tau'} = \emptyset$ so that one has a disjoint union

$$N \cdot r^- = \bigcup_{\tau \in r^-} O_{\tau}.$$  

(1.3)

Since $B$ normalizes $N$ there is a natural action on $B$ on $n_-$. We refer to this as the $n_-$ action of $B$. Explicitly since the latter extends $\text{Coad}_n$ we denote the $n_-$ action of $B$ of $b \in B$ on $v \in n_-$ by $b \cdot v$, and note that

$$b \cdot v = \Phi_n \circ \text{Ad} b \cdot v.$$  

(1.4)

The Zariski open subvariety $r^- \subset r_-$ is stable under the $n_-$ action of $H \subset B$ and in fact $H$ operates transitively on $r^-_-$ so that $r^-_-$ is isomorphic to a homogeneous space for $H$.

The action of $H$ on $r^-_-\subset r_-$ extends to an action of $H$ on the corresponding set $\{ O_{\tau}, \tau \in r^-_-\}$ of $N$-coadjoint orbits. Since $H$ normalizes $N$ the following statement is obvious. See §2.2 in [K].

**Proposition 1.2.** For any $\tau \in r^-_-$ and $a \in H$ one has

$$O_{\text{Ad} a(\tau)} = a \cdot O_{\tau}.$$  

Let

$$X = \bigcup_{\tau \in r^-_-} O_{\tau}$$  

(1.5)

so that, by (1.3), the union (1.5) is disjoint. Furthermore since $B = NH$ we note that $X$ is an orbit of $n_-$ action of $B$ and hence is Zariski open and dense in its closure. The following is one of the main theorems in [K]. See Theorem 2.8. in [K].
Theorem 1.3. $X$ is Zariski dense in $n_-$ so that
\[ \overline{X} = n_- \tag{1.6} \]

Let $W$ be the Weyl group of $\mathfrak{g}$ operating in $\mathfrak{h}$ and $\mathfrak{h}^*$. Reluctantly submitting to common usage, let $w_o$ be the long element of $W$. For any $\beta \in \mathcal{B}$ let $s_\beta \in W$ be the reflection defined by $\beta$. Since the elements in $\mathcal{B}$ are orthogonal to one another, the reflections $s_\beta$ evidently commute with one another. The long element $w_o$ of the Weyl group $W$ is given in terms of the product of these commuting reflections. In fact one has (see Proposition 1.10 in [K]).

Proposition 1.4. One has
\[ w_o = \prod_{\beta \in \mathcal{B}} s_\beta, \tag{1.7} \]
noting that the order of the product is immaterial because of commutativity.

Remark. Note that it is immediate from Proposition 1.4 that $-1 \in W$ if and only if $m = \ell$.

Part B. Statement of new results to be proved in §2.

1.2 With respect to the coadjoint structure of $N$ one has

Theorem 1.5. A maximal coadjoint orbit of $N$ has codimension $m$ in $n^*$ where $m$ is the maximal number of strongly orthogonal roots. In particular it has codimension $\ell$ (the rank of $\mathfrak{g}$) in $n^*$ if and only if $-1 \in W$.

We now deal with the coadjoint representation $\text{Coad}_\mathfrak{b}$ of the Borel subgroup $B$. As noted in §1.1, we have identified $\mathfrak{b}^*$ with $\mathfrak{b}_-$. For any $w \in \mathfrak{b}_-$ let $B_w$ be the $B$-coadjoint isotropy group of $B$ at $w$ and let $\mathfrak{b}_w = \text{Lie} B_w$. We refer to $\mathfrak{b}_w$ as the $B$-coadjoint isotropy algebra at $w$. Let $O_w \subset \mathfrak{b}_-$ be the $B$-coadjoint orbit of $w$ so that
\[ O_w \cong B/B_w. \]

Of course $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{h}^o \subset \mathfrak{h}$ be the orthogonal subspace to $\mathbb{C} \mathcal{B} \subset \mathfrak{h}^*$ in $\mathfrak{h}$. Then
\[ \dim \mathfrak{h}^o = \ell - m. \tag{1.8} \]
Now let \( \Phi_h : g \to h \) be the projection defined by the triangular decomposition \( g = n_- + h + n \). Obviously any \( w \in b_- \) can be uniquely written as
\[
w = v + x, \tag{1.9}
\]
where \( v = \Phi_n w \) and \( x = \Phi_h w \). It is immediate from Theorem 1.3 that \( b'_- \) is open in \( b_- \) where \( b'_- \) is defined by
\[
b'_- = \{ w \in b_- \mid \Phi_n w \in X \}. \tag{1.10}
\]
We will prove

**Theorem 1.6.** Let \( w \in v_\times^- \). Then the \( B \)-coadjoint isotropy algebra of \( b \) at \( w \) is \( h^\circ \). Furthermore \( O_w \) is a maximal \( B \)-coadjoint orbit so that \( \ell - m \) is the codimension of any maximal \( B \)-coadjoint orbit. Furthermore \( O_y \) is a maximal \( B \)-coadjoint orbit of \( B \) for any \( y \in b'_- \).

As a consequence we now have

**Theorem 1.7.** A Borel subgroup of a complex semisimple Lie group has a nonempty (and hence unique) open coadjoint orbit if and only if \( m = \ell \), that is if and only if \( -1 \) is in the Weyl group \( W \). Furthermore, explicitly, the unique nonempty open coadjoint orbit of \( B \) is \( b'_- \).

## 2. Proofs of stated results and additional results

**2.1.** The proof of Theorem 1.5 depends on the density of \( X \) and standard arguments about dimensions of orbits.

**Proof of Theorem 1.5.** Let \( v \in X \). Since \( \dim R = m \) it follows from (1.2) and (1.5) that the codimension of \( O_v \) in \( n_- \) is \( m \). But if \( w \in n_- \), then by (1.6) there exists a sequence \( v_k \in X \) which converges to \( w \). But then by the compactness of subspaces of dimension \( m \) there exists a subsequence \( v_{k'} \) such that \( n_{v_{k'}} \) converges to an \( m \)-dimensional subspace of \( n_w \). Hence the codimension of \( O_w \) in \( n_- \) is greater than or equal to \( m \). QED

Recall (1.4).

**Theorem 2.1.** Let \( w \in v_\times^- \). Then the isotropy subalgebra of \( b \) at \( w \) with respect to the \( n_- \) action of \( B \) is \( h^\circ + r \).
**Proof.** Let $c_w$ be the isotropy subalgebra of $b$ at $w$ with respect to the $n_-$ action of $B$. Since $B = NH$ and since the $n_-$ action of $B$ extends Coad$_n$ clearly $(\mathfrak{h}^o + r) \subset c_w$. But

$$B \cdot w = X$$  \hspace{1cm} (2.1)

by Proposition 1.2 and Theorem 1.1. But $\dim (\mathfrak{h}^o + r) = \ell$ since clearly

$$\dim \mathfrak{h}^o = \ell - m$$  \hspace{1cm} (2.2)

and $\dim r = m$. But $\dim X = \dim n$ by (1.6). Thus $\dim c_w = \ell$. This implies $\mathfrak{h}^o + r = c_w$. QED

Comparing Coad$_b$ with the $n_-$ action of $B$, we note that if $w \in n_-$ and $b \in B$ (recall the notation of §1.1.), one has

$$b \cdot w = b \cdot w + \Phi_b b \cdot w$$  \hspace{1cm} (2.3)

This is immediate since $\Phi_b = \Phi_n + \Phi_b$. This latter sum also yields the following infinitesimal analogue of (2.3). Let $x \in b$ and $w \in n_-; \text{ then }$

$$x \cdot w = x \cdot w + \Phi_b x \cdot w$$  \hspace{1cm} (2.4)

where $x \cdot w = \Phi_b [x, w]$ and $x \cdot w = \Phi_n [x, w]$. But now (2.1) and (2.3) imply

**Proposition 2.2.** Let $w \in r^\times$. Then

$$\Phi_n B \cdot w = X.$$  \hspace{1cm} (2.5)

We also note that the fixed point set $b^B_-$ of the coadjoint action of $B$ is immediately given by

$$b^B_- = \mathfrak{h}.$$  \hspace{1cm} (2.6)

One now has

**Theorem 2.3.** Let $w \in r^\times$. Then the isotropy algebra, $b_w$, of $b$ at $w$, with respect to the coadjoint action of $B$ is $\mathfrak{h}^o$ so that, by (2.2), $\ell - m$ is the codimension of the coadjoint orbit $B \cdot w$.

**Proof.** Let $x \in b_-$. Then, by Theorem 2.1 and (2.4) one has $x \in b_w$ if and only if (1) $x \in \mathfrak{h}^o + r$ and (2) $\Phi_b [x, w] = 0$. But clearly $\mathfrak{h}^o \subset b_w$. Thus to prove $b_w = \mathfrak{h}^o$ it suffices to show that if

$$0 \neq x \in r \text{ then } \Phi_b [x, w] \neq 0.$$  \hspace{1cm} (2.7)
But since all the coefficients $a_\beta(w)$ in (1.1) are not zero and the set $\{\beta^\vee \in \mathfrak{h} \mid \beta \in B\}$ are linearly independent, one has (2.7). QED

2.2. As an immediate consequence of (2.6) one has

**Proposition 2.4.** Let $w \in \mathfrak{b}_-$ and $z \in \mathfrak{h}$; then for any $v \in O_w$ one has $v + z \in O_{w+z}$ and the map

$$O_w \to O_{w+z}, \ v \mapsto v + z$$

is a $B$-isomorphism of $B$-coadjoint orbits.

As an immediate consequence of Proposition 2.4, one has

**Proposition 2.5.** Let $w \in \mathfrak{b}_-$ and $z \in \mathfrak{h}$. Then

$$\mathfrak{b}_w = \mathfrak{b}_{w+z}$$

so that obviously in particular

$$\dim \mathfrak{b}_w = \dim \mathfrak{b}_{w+z}.$$  \hspace{1cm} (2.10)

From the open density of $X$ in $\mathfrak{n}_-$, it is obvious, recalling (1.10), that $\mathfrak{b}'_-$ is Zariski open and dense in $\mathfrak{b}_-$.

**Theorem 2.6** For any $y \in \mathfrak{b}'_-$ the dimension of $\mathfrak{b}_y$ in $\mathfrak{b}$ is $\ell - m$ where we recall that $m$ is the cardinality of a maximal set of strongly orthogonal roots. Furthermore $O_y$ is a maximal $B$-coadjoint orbit and has codimension $\ell - m$ in $\mathfrak{b}_-$. That is, the codimension of any $B$-coadjoint orbit $O$ is greater than or equal to $\ell - m$.

**Proof.** Let $v = \Phi_n y$ so that $v \in X$. But then by (2.1) and (2.3) there exists $w \in \mathfrak{r}_-^*$ and $b \in B$ such that $\Phi_n b \cdot w = v$. But by conjugacy and Theorem 2.3, $\dim \mathfrak{b}_b \cdot w = \ell - m$. But if $z = y - b \cdot w$, then $z \in \mathfrak{h}$. But then $\dim \mathfrak{b}_y = \ell - m$ by Proposition 2.5. But then using the limit argument in the proof of Theorem 1.5, one has $\dim \mathfrak{b}_x \geq \ell - m$ for any $x \in \mathfrak{b}_-$.

Applying Theorem 2.6 to the case where $\ell = m$ we come to the statement and proof of our main theorem.

**Theorem 2.7.** Let $G$ be a complex semisimple Lie group and let $B$ be a Borel subgroup of $G$. Then $B$ has a (necessarily unique) nonzero open coadjoint orbit, $O$,
if and only if \(-1\) is in the Weyl group of \(G\), or equivalently, there exists \(\ell\) orthogonal roots where \(\ell\) is the rank of \(G\). Furthermore in the notation above

\[ O = b' \]

(2.11)

where \(b'_\) is given by (1.10). In particular \(v_\subset O\). (See §1.1.)

**Proof.** By Theorem 2.6 \(B\) has an open coadjoint orbit \(O\) in \(b_\) if and only if \(m = \ell\). The orbit is necessarily unique since the action of \(B\) is linear algebraic and \(b_\) is affine irreducible. But now \(b'_\subset O\) by Theorem 2.6. On the other hand, \(O \subset b'_\) by Proposition 2.2. This proves (2.11).

**Remark 2.8.** If \(G\) is simple, then \(B\) has a nonempty open coadjoint orbit if and only if \(\mathfrak{g}\) is of type \(B_\ell, C_\ell, D_\ell\) with \(\ell\)-even, \(G_2, F_4, E_7, E_8\). Except for the case \(D_\ell\), with \(\ell\)-even, the other cases follow from the fact that there exists no outer automorphism. This fact readily implies \(-1\) is in the Weyl group.

**References**

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