A CONVEX SET WITH A RICH DIFFERENCE

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Abstract. We construct a convex set A with cardinality 2n and with the property that an element of the difference set A − A can be represented in n different ways. We also show that this construction is optimal by proving that for any convex set A, the maximum possible number of representations an element of A − A can have is ⌊|A|/2⌋.

1. Introduction

A finite set A ⊂ R is said to be convex if the consecutive differences are strictly increasing. That is, if we write A = {a1 < a2 < ⋯ < an}, A is convex if

\[ a_i - a_{i-1} < a_{i+1} - a_i \]

holds for all 2 ≤ i ≤ n − 1. One can also use the equivalent formulation that a set A is convex if we can write A = f({1, 2, ⋯, n}) for some strictly convex function f. The convexity of f disrupts the additive structure of the pre-image \{1, 2, ⋯, n\}, and this leads us to expect that a convex set cannot have much additive structure.

This principle can be quantified in different ways, and one such way is to prove that the difference set

\[ A - A := \{a - b : a, b \in A\} \]

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is large. The current state of the art for this problem is a result of Schoen and Shkredov [6], proving that the bound
\[ |A - A| \gg |A|^{8/5 - o(1)} \]
holds for any convex set \( A \).

Another approach is to consider the additive energy
\[ E(A) := \left| \{(a, b, c, d) \in A^4 : a - b = c - d\} \right|, \]
which can also be expressed as
\[ E(A) = \sum_x r^2_{A-A} \]
where \( r_{A-A}(x) := \left| \{(a, b) \in A \times A : a - b = x\} \right| \). The bound
\[ E(A) \ll |A|^{5/2} \tag{1} \]
was proven using incidence theory by Konyagin [3] and using elementary methods by Garaev [2]. See also [4] for an alternative presentation of a proof of (1). A further improvement was later given by Shkredov [7], using additional higher energy tools from additive combinatorics.

One might even expect that a qualitatively stronger statement than (1) holds; namely that \( r_{A-A}(x) \) is guaranteed to be small for all \( x \neq 0 \). Indeed, if one knew, for instance, that \( r_{A-A}(x) \leq |A|^{1-c} \) holds for all \( x \neq 0 \), this immediately implies the non-trivial bound \( E(A) \ll |A|^{3-c} \), which in turn implies the non-trivial bound \( |A - A| \gg |A|^{1+c} \).

However, a construction of Schoen [5] shows that such a uniform upper bound for the representation function \( r_{A-A}(x) \) is not possible. Schoen constructed a convex set with \( n \) elements and some \( x \neq 0 \) with \( r_{A-A}(x) \geq n/4 \).

The main purpose of this note is to give a construction of a convex set with a rich difference which improves the construction of Schoen. We prove the following result.

**Theorem 1.** For every \( m \in \mathbb{N} \), there exists a convex set \( A \subseteq \mathbb{R} \) of size \( 2m \) and a non-zero element \( d \in A - A \) such that \( r_{A-A}(d) \geq m \).

We also show that this construction is optimal, proving that, for any convex set with cardinality \( n \) and any \( d \neq 0 \),
\[ r_{A-A}(d) \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

\[ ^1 \] Throughout this note, the notation \( X \gg Y \) and \( Y \ll X \), are equivalent and mean that \( X \geq cY \) for some absolute constant \( c > 0 \).

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2. The construction

Proof of Theorem 1. We give a concrete construction of the set

\[ A = \{a_1 < a_2 < \cdots < a_{2m}\}, \]

which is made up of two halves. The set \( A \) begins with 0, and then has gaps \( 1 + (i-1)\delta \), for some very small \( \delta > 0 \) which will be specified later. The first half of \( A \) is filled like this. That is, for \( 1 \leq k \leq m+1 \), we define

\[ a_k := (k-1) + \delta \frac{(k-2)(k-1)}{2}, \]

and so the first \( m+1 \) elements of \( A \) are the elements of the set

\[ A_1 := \{0, 1, 2 + \delta, 3 + 3\delta, \ldots, m + \delta \frac{m(m-1)}{2}\}. \]

Fix

\[ d := m + \delta \frac{m(m-1)}{2} = a_{m+1}. \]

The rest of \( A \) is defined iteratively. For \( 1 \leq i \leq m-1 \), we set

\[ a_{m+1+i} := a_{1+2i} + d. \]

This immediately gives rise to the system of equations

\[ d = a_{m+1} - a_1 = a_{m+2} - a_3 = \cdots = a_{2m} - a_{2m-1}. \]

We therefore have \( r_{A-A}(d) \geq m \).

It remains to check that this set is convex. Note that the first part of \( A \), namely \( A_1 = \{a_1, \ldots, a_{m+1}\} \), is convex, since the consecutive difference increase by \( \delta \) at each step.

We will prove by induction on \( i \) that the set

\[ \{a_1, a_2, \ldots, a_{m+2+i}\} \]

is convex for \( 0 \leq i \leq m-2 \).

We first check the base case \( i = 0 \). We need to verify that the difference \( a_{m+2} - a_{m+1} \) is sufficiently large, which will give a condition on \( \delta \). We must have

\[ a_{m+2} - a_{m+1} > a_{m+1} - a_m, \]

which upon plugging in the definitions yields

\[ 2 + \delta > 1 + \frac{\delta m(m-1)}{2} - \frac{\delta(m-2)(m-1)}{2}. \]

After simplification, this gives the condition \( \delta < \frac{1}{m-2} \).
Now let $1 \leq i \leq m - 2$. We must verify that
\[ \{a_1, a_2, \ldots, a_{m+2+i}\} \]
is convex, given the induction hypothesis that $\{a_1, a_2, \ldots, a_{m+1+i}\}$ is convex.
All that remains is to check that
\[ a_{m+2+i} - a_{m+1+i} > a_{m+1+i} - a_{m+i}. \]
We use equations (2) to rewrite each side, as
\begin{align*}
a_{m+2+i} - a_{m+1+i} &= a_{1+2(i+1)} - a_{1+2i}, \\
a_{m+1+i} - a_{m+i} &= a_{1+2i} - a_{1+2(i-1)}.
\end{align*}
Note that, since the differences on the right hand side above are then consecutive differences of length two within a convex set, we have
\begin{align*}
a_{m+2+i} - a_{m+1+i} &= a_{1+2(i+1)} - a_{1+2i} \\
&> a_{1+2i} - a_{1+2(i-1)} = a_{m+1+i} - a_{m+i}
\end{align*}
as needed. Here we used the inductive hypothesis that $\{a_1, a_2, \ldots, a_{m+1+i}\}$ is convex as well as the fact that $1 + 2(i + 1) \leq m + 1 + i$. The latter inequality follows from the condition that $i \leq m - 2$. \(\square\)

Note that by taking $\delta$ to be a sufficiently small rational number, and dilating the set $A$ through by common denominators, we can find $A \subseteq \mathbb{Z}$ satisfying Theorem 1.

3. A matching upper bound for the representation function

The next result shows that the construction of Theorem 1 is optimal.

**Theorem 2.** For a convex set $A \subset \mathbb{R}$ and any $d \in \mathbb{R} \setminus \{0\}$,
\[ r_{A-A}(d) \leq \left\lfloor \frac{|A|}{2} \right\rfloor. \]

**Proof.** Write the elements of $A$ in increasing order so that $A = \{a_1 < a_2 < \cdots < a_n\}$. Suppose that $d$ can be represented in $t$ different ways as an element of $A - A$. We can write
\begin{equation}
\begin{align*}
d &= a_{j_1+k_1} - a_{j_1} = a_{j_2+k_2} - a_{j_2} \cdots = a_{j_t+k_t} - a_{j_t},
\end{align*}
\end{equation}
such that the $k$ indices satisfy
\begin{equation}
\begin{align*}
k_1 > k_2 > \cdots > k_t.
\end{align*}
\end{equation}

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Indeed, because $A$ is convex, we cannot have two of the $k$ indices repeating in the list (3). This follows from the fact that, for fixed $k$, the sequence

\[(a_{j+k} - a_j)_{j \in \mathbb{N}}\]

is strictly increasing. Note also that, for fixed $j$, the sequence

\[(a_{j+k} - a_j)_{k \in \mathbb{N}}\]

is strictly increasing. This follows immediately from the fact that the $a_i$ are increasing.

**Claim 1.** For all $1 \leq i \leq t-1$, $j_{i+1} \geq j_i + 2$.

**Proof.** Suppose for a contradiction that $j_{i+1} \leq j_i + 1$. We also have $k_{i+1} \leq k_i - 1$, and so $j_{i+1} + k_{i+1} \leq j_i + k_i$. Therefore $a_{j_{i+1}+k_{i+1}} \leq a_{j_i+k_i}$. But then it follows from (3) that $0 \leq a_{j_i+k_i} - a_{j_{i+1}+k_{i+1}} = a_j - a_{j_i}$, and so

\[j_i \geq j_{i+1}.\]

However, since the sequences (5) and (6) are strictly increasing, it follows that

\[a_{j_{i+1}+k_{i+1}} - a_{j_{i+1}} \leq a_{j_i+k_i} - a_j < a_{j_i+k_i} - a_{j_i}.\]

This contradicts (3). \(\square\)

Applying the claim iteratively yields

\[j_t \geq j_{t-1} + 2 \geq j_{t-2} + 4 \geq \cdots \geq j_1 + 2(t-1) \geq 1 + 2(t-1) = 2t - 1.\]

We also know that $j_t + k_t \leq n$ and $k_t \geq 1$. Therefore, $j_t \leq n - 1$. Combining this with (8) gives $t \leq n/2$.

Finally, since $t$ is an integer, this is equivalent to the bound $t \leq \lfloor n/2 \rfloor$. \(\square\)

**4. Concluding remarks**

Interestingly, the construction cannot be modified to give a rich sum in a convex set. For $x \in \mathbb{R}$, we use the notation

\[r_{A+A}(x) := \left| \{(a, b) \in A \times A : a + b = x\} \right|.\]

In sharp contrast with Theorem 1, the bound

\[r_{A+A}(C) \ll |A|^{2/3}.\]

holds for any convex set $A$ and $C \in \mathbb{R}$. The inequality (9) was also observed by Schoen [5], and can be proved using the Szemerédi-Trotter Theorem.
Another interesting direction is to determine how many $k$-rich representations can occur. A well-known application of the Szemerédi–Trotter Theorem (see for instance [4]) gives the bound

$$|\{d : r_{A-A}(d) \geq t\}| \ll \frac{n^3}{t^3}$$

for any convex set $A$ with cardinality $n$. On the other hand, one can glue together $n/t$ copies of the construction in Theorem 1 with $t$ elements in order to obtain a convex set $A$ with $n$ elements and

$$|\{d : r_{A-A}(d) \geq t\}| \gg \frac{n}{t}.$$  

There is a considerable gap between the upper and lower bounds of (10) and (11) respectively, although the bounds converge as $t$ gets close to $n$.

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