On the cohomology of $s\ell(m+1, \mathbb{R})$ acting on differential operators and $s\ell(m+1, \mathbb{R})$-equivariant symbol

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Abstract

Let $\mathcal{D}_{\lambda\mu}^k(\mathbb{R}^m)$ denote the space of differential operators of order $\leq k$ from the space of $\lambda$-densities of $\mathbb{R}^m$ into that of $\mu$-densities and let $\mathcal{S}_\delta^k(\mathbb{R}^m)$ be the space of $k$-contravariant, symmetric tensor fields on $\mathbb{R}^m$ valued in the $\delta$-densities, $\delta = \mu - \lambda$.

Denote by $s\ell_{m+1}$ the projective embedding of $s\ell(m+1, \mathbb{R})$ as a subalgebra of the Lie algebra of vector fields of $\mathbb{R}^m$.

One computes the cohomology of $s\ell_{m+1}$ with coefficients in the space of differential operators from $\mathcal{S}_\delta^p(\mathbb{R}^m)$ into $\mathcal{S}_\delta^q(\mathbb{R}^m)$. It is non vanishing only for some critical values of $\delta$. For $m=1$, these are the values pointed out by H. Gargoubi in a completely different context [6].

This allows to determine the condition under which the short exact sequence of $s\ell_{m+1}$-modules

$$0 \to \mathcal{D}_{\lambda\mu}^{k-1}(\mathbb{R}^m) \to \mathcal{D}_{\lambda\mu}^k(\mathbb{R}^m) \xrightarrow{\sigma} \mathcal{S}_\delta^k(\mathbb{R}^m) \to 0$$

is split ($\sigma$ is the symbol map).

From this one recovers and generalizes useful results about the structure of the $s\ell_{m+1}$-module $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m) = \oplus_k \mathcal{D}_{\lambda\mu}^k(\mathbb{R}^m)$ [3, 5, 6, 8].

The cohomology of the latter is also computed.

Key words: Lie algebra, differential operators, cohomology, infinitesimal homographies.

Classification numbers: 58B99, 17B56, 17B65, 47E05.

Titre courant: cohomology of $s\ell(m+1, \mathbb{R})$ on differential operators.

Rsum

Soient $\mathcal{D}_{\lambda\mu}^k(\mathbb{R}^m)$ l’espace des opérateurs différentiels d’ordre $\leq k$ sur les $\lambda$-densités de $\mathbb{R}^m$ valeurs dans les $\mu$-densités de $\mathbb{R}^m$ et $\mathcal{S}_\delta^k(\mathbb{R}^m)$ l’espace des champs de tenseurs $k$-contravariants symétriques sur $\mathbb{R}^m$ valeurs dans les $\delta$-densités, $\delta = \mu - \lambda$.

Soit également $s\ell_{m+1}$ le plongement projectif de $s\ell(m+1, \mathbb{R})$ dans l’algèbre de Lie des champs de vecteurs de $\mathbb{R}^m$.

On calcule la cohomologie de $s\ell_{m+1}$ coefficients dans les opérateurs différentiels de $\mathcal{S}_\delta^p(\mathbb{R}^m)$ dans $\mathcal{S}_\delta^q(\mathbb{R}^m)$. Elle n’est non nulle que pour certaines valeurs,
dites critiques, de δ. Pour m = 1, ce sont les valeurs mises en évidence par H. Gargoubi dans un tout autre contexte [3].

À l’aide des résultats obtenus, on détermine quelle condition la courte suite exacte de \( sℓ_{m+1}-\text{modules} \)
\[
0 \to D^{k-1}_{\lambda\mu}(\mathbb{R}^m) \to D^k_{\lambda\mu}(\mathbb{R}^m) \to S^k_δ(\mathbb{R}^m) \to 0
\]
est scindée (\( σ \) est le passage au symbole).

Cela permet de retrouver et de généraliser différents résultats utiles concernant la structure de \( sℓ_{m+1}-\text{module} \) de \( D_{\lambda\mu}(\mathbb{R}^m) = \oplus_k D^k_{\lambda\mu}(\mathbb{R}^m) \) [3, 5, 6, 8].

La cohomologie de ce dernier est également calculée.

1 Introduction

Let \( D_{\lambda\mu}(M) \) denote the space of differential operators from the space of scalar \( \lambda \)-densities over a smooth manifold \( M \) into the space of scalar \( \mu \)-densities.

The Lie derivation with respect to vector fields equips \( D_{\lambda\mu}(M) \) with a natural structure of module over the Lie algebra of vector fields \( Vect(M) \) of \( M \). The study of that module has been started in [3, 5, 6, 8, 9]. See also [2] for pseudodifferential operators on the complex line. As a vector space, \( D_{\lambda\mu}(M) \) is isomorphic to the graded space \( S_δ(M) = \bigoplus_{k \geq 1} D^k_{\lambda\mu}(M) \) associated to the filtration of \( D_{\lambda\mu}(M) \) by the order \( k \) of differentiation. The \( k \)-th term of this sum is nothing else but the space \( S^k_δ(M) = \Gamma(\bigwedge^k T M \otimes Δ^δ M) \) of \( k \)-symmetric contravariant tensor fields valued in the scalar densities of weight \( δ = \mu - \lambda \) (\( Δ^δ M \) denotes the vector bundle of \( δ \)-densities of \( M \)). The isomorphism between \( D_{\lambda\mu}(M)/D^k_{\lambda\mu}(M) \) and \( S^k_δ \) is of course induced in a natural way by the symbol map
\[
σ : D^k_{\lambda\mu}(M) \to S^k_δ(M).
\]

This map commutes with the Lie derivation of operators on the left and of tensor fields on the right. But, however, \( D_{\lambda\mu}(M) \) and \( S_δ(M) \) are not isomorphic as \( Vect(M) \)-modules : \( D_{\lambda\mu}(M) \) is better viewed as a non trivial deformation of \( S_δ(M) \).

For \( λ = \mu \) and \( m > 1 \), it has been shown [3], nevertheless, that \( D_{\lambda\mu}(M)(\mathbb{R}^m) \) and \( S_0(\mathbb{R}^m) \) are isomorphic as \( sℓ_{m+1}-\text{modules} \), where \( sℓ_{m+1} \subset Vect(\mathbb{R}^m) \) is the canonical embedding of \( sℓ(m+1, \mathbb{R}) \) as the algebra of infinitesimal linear fractional local transformations of \( \mathbb{R}^m \) (note that it is a maximal subalgebra of the algebra of polynomial vector fields on \( \mathbb{R}^m \)).

A similar result has been obtained in the one dimensional case (cf. [2] for pseudodifferential operators, [3] for differential operators) : except for special critical values of \( λ \) and \( μ \), \( D_{\lambda\mu}(M) \) is again \( sℓ_2 \)-isomorphic to \( S_δ(M) \). For the critical values, \( D_{\lambda\mu}(M) \) is a non trivial deformation of \( S_δ(M) \) and it is described in [3] in terms of a family of cocycles, tailored on the occasion, of \( sℓ_2 \) acting on the space of differential operators from \( Γ(Δ^{-n/2} M) \) into \( Γ(Δ^{(n+2)/2} M) \).

In both cases, \( m = 1 \) and \( m > 1 \), the \( sℓ_{m+1}-\text{module} \) structure of \( D_{\lambda\mu}(\mathbb{R}^m) \) proved to be a powerful tool in studying its \( Vect(\mathbb{R}^m) \)-structure.
In some sense, the possible difference between the $L$-modules $D_{\lambda\mu}(M)$ and $S_\delta(M)$, $L = Vect(M)$ or $s\ell_{m+1}$, follows from the fact that the short exact sequence of modules

$$0 \to D_{\lambda\mu}^{k-1}(M) \xrightarrow{i} D_{\lambda\mu}^k(M) \xrightarrow{\sigma} S_\delta^k(M) \to 0$$

($i : \text{the inclusion}$) maybe is not split. The obstruction against splitting is a member of

$$H^1(L, \text{Hom}(S_\delta^k(M), D_{\lambda\mu}^{k-1}(M))).$$

Composing with the symbol map $\sigma : D_{\lambda\mu}^{k-1}(M) \to S_\delta^{k-1}(M)$, one gets a sort of first order approximation of this obstruction, namely an element of

$$H^1(L, \text{Hom}(S_\delta^k(M), S_\delta^{k-1}(M))).$$

In this paper, we compute the cohomology space

$$H(s\ell_{m+1}, D(S_\delta^p(\mathbb{R}^m), S_\delta^q(\mathbb{R}^m))).$$

This allows us not only to recover the critical values of $[\mathbb{F}]$ but also to find critical values for $m > 1$. To do that, we compute the spaces

$$H(s\ell_{m+1}, D(D_{\lambda\mu}^k(\mathbb{R}^m), S_\delta^q(\mathbb{R}^m)))$$

($D^k(A, B)$ denotes the space of differential operators of order $\leq k$ from $A$ into $B$).

They are computed by induction on $k$, using the short exact sequence associated to the inclusion of the operators of order $k-1$ into these of order $k$. The critical values occur when $p - q \in \mathbb{N}_0$. They are then those for which the connecting homomorphism of the corresponding exact sequence in cohomology vanishes. Equivalently, they are these for which

$$H(s\ell_{m+1}, D(S_\delta^p(\mathbb{R}^m), S_\delta^q(\mathbb{R}^m))) \neq 0.$$  

We also compute the space $\text{Hom}_{s\ell_{m+1}}(S_\delta^p(\mathbb{R}^m), S_\delta^q(\mathbb{R}^m))$. This is quite hard because one is left to show that a mapping from $S_\delta^p(\mathbb{R}^m)$ into $S_\delta^q(\mathbb{R}^m)$ is local knowing only that it is commuting with the $L_X$, $X \in s\ell_{m+1}$. As a corollary, we get generalization of a result of $[\mathbb{F}]$ : if $(m + 1)\delta - m \notin \mathbb{N}_0$, then a linear map from $S_\delta(\mathbb{R}^m)$ is $s\ell_{m+1}$-equivariant if and only if its restriction to each $S_\delta^k(\mathbb{R}^m)$, $k \in \mathbb{N}$, is a constant multiple of the identity.

As a consequence of our computations, we find the necessary and sufficient condition for $[\mathbb{F}]$ to be split as a sequence of $s\ell_{m+1}$-modules (with $M = \mathbb{R}^m$). In particular, we show that if $(m + 1)\delta - m \notin \mathbb{N}_0$, then the $s\ell_{m+1}$-modules $D_{\lambda\mu}(\mathbb{R}^m)$ and $S_\delta(\mathbb{R}^m)$ are canonically isomorphic. This also generalizes a useful result of $[\mathbb{F}]$.

In fact, the above mentioned cohomology classes constructed in $[\mathbb{F}]$ are valued in $D_{-\frac{\pi}{2}, 1+\frac{\pi}{2}}(\mathbb{R})$. We recover them as a consequence of our computations due to the fact that $D(S_\delta^p(\mathbb{R}), S_\delta^q(\mathbb{R}))$ is isomorphic to $D_{\delta-p, \delta-q}(\mathbb{R}^m)$.

Besides, very little more is needed then to compute the cohomology of $s\ell_{m+1}$ with coefficients in $D_{\lambda\mu}(\mathbb{R}^m)$, $m \geq 1$. This is done in the last section. The difference between the cases $m > 1$ and $m = 1$ is then really significant.
2 The projective embedding of \( s\ell(m+1, \mathbb{R}) \)

For each \( T \in s\ell(m+1, \mathbb{R}) \) we denote by \( T^* \) the fundamental vector fields on \( P_m \mathbb{R} \) associated to \( T \) and to the canonical action of \( SL(m+1, \mathbb{R}) \) by linear projective transformations. We also denote by \( T^* \) its restriction to \( \mathbb{R}^m \), viewed as the open set of points of \( P_m \mathbb{R} \) having a non vanishing \((m+1)\)-th projective coordinate.

The mapping \( T \to T^* \) is an injective homomorphism of Lie algebras from \( s\ell(m+1, \mathbb{R}) \) into the Lie algebra \( \text{ Vect}(\mathbb{R}^m) \) of vector-fields on \( \mathbb{R}^m \).

Following [8], we call it the projective embedding of \( s\ell(m+1, \mathbb{R}) \). We now describe it in a way well suited to our purpose.

We realize \( s\ell(m+1, \mathbb{R}) \) as \( \mathbb{R}^m \oplus g\ell(m, \mathbb{R}) \oplus \mathbb{R}^m \ast \), where \( \mathbb{R}^m \) and \( \mathbb{R}^m \ast \) are abelian subalgebras, where the adjoint action of \( A \in g\ell(m, \mathbb{R}) \) is the natural representation of \( g\ell(m, \mathbb{R}) \) and where the bracket of \( h \in \mathbb{R}^m \) and \( \alpha \in \mathbb{R}^m \ast \) is given by

\[
[h, \alpha] = \alpha(h) 1 + h \otimes \alpha
\]

(1 is the unit matrix). It is easily seen that the fundamental vector fields associated to \( h = (h^i) \in \mathbb{R}^m \), \( A = (A^i_j) \in g\ell(m, \mathbb{R}) \) and \( \alpha = (\alpha_i) \in \mathbb{R}^m \ast \) are

\[
h^* = -h^i \partial_i, \quad A^* = -A^i_j x^j \partial_i \quad \text{and} \quad \alpha^* = \alpha(x)x^i \partial_i
\]

(sums over repeated indices are understood and \( \partial_i \) denotes the partial derivative with respect to \( x^i \)).

Note that \( s\ell(m + 1, \mathbb{R}) \) is a graded Lie algebra, \( \mathbb{R}^m \), \( g\ell(m, \mathbb{R}) \) and \( \mathbb{R}^m \ast \) being the homogeneous components of degree \(-1, 0 \) and \( 1 \) respectively.

For the sake of brevity, we denote \( s\ell_{m+1} \) the above realization of \( s\ell(m+1, \mathbb{R}) \).

3 Cohomology of \( s\ell_{m+1} \) associated to a representation of \( g\ell(m, \mathbb{R}) \)

Let \((V, \rho)\) be a finite dimensional representation of \( g\ell(m, \mathbb{R}) \).

The space \( C_\infty(\mathbb{R}^m, V) \) of smooth \( V \)-valued functions on \( \mathbb{R}^m \) is a representation of \( \text{ Vect}(\mathbb{R}^m) \). The action \( L^\rho_X \) of a vector field \( X \) on a function \( f \) is given by

\[
L^\rho_X f = X.f - \rho(DX)f
\]

where \( X.f \) is the usual derivative \( X^i \partial_i f \) and where \( DX = (\partial_j X^i) \) is the differential of \( X \).

In fact, \( C_\infty(\mathbb{R}^m, V) \) is the space of sections of the (trivial) bundle associated to the linear frame bundle of \( \mathbb{R}^m \) and to \( V \) and \( L^\rho \) is the corresponding natural Lie derivation.

Our purpose is to compute the Chevalley-Eilenberg cohomology of the restriction of \( L^\rho \) to \( s\ell_{m+1} \).

Recall that \( L^\rho \) induces a representation of \( \mathcal{L}^\rho \) on the space of cochains. It is defined by

\[
\mathcal{L}^\rho_X = i_X \circ \partial^\rho + \partial^\rho \circ i_X, \quad \forall X \in s\ell_{m+1},
\]
where $\partial^\rho$ is the coboundary operator. One has also

$$(L^\rho_X c)(X_0, \ldots, X_{s-1}) = L^\rho_X(c(X_0, \ldots, X_{s-1})) - \sum_{0 \leq i < s} c(X_0, \ldots, [X, X_i], \ldots, X_{s-1})$$

for each $s$-cochain $c$.

**Proposition 3.1** For each $u \in \{0, \ldots, m\}$, there exists a unique linear $\chi^u : \Lambda(g\ell(m, \mathbb{R}), \Lambda^u(\mathbb{R}^{m*}, V)) \to \Lambda(s\ell_{m+1}, C(\mathbb{R}^m, V))$ such that

1. $L^\rho_h \circ \chi^u = 0$, $i_{h^*} \circ \chi^u = 0$, $\forall h \in \mathbb{R}^m$,
2. $(\chi^u(\gamma))(A^*_0, \ldots, A^*_{t-1}, \alpha^*_i, \ldots, \alpha^*_u) = (\gamma(A_0, \ldots, A_{t-1}))(\alpha_1, \ldots, \alpha_u)$ for all $A_i \in g\ell(m, \mathbb{R})$ and all $\alpha_j \in \mathbb{R}^{m*}$,
3. $(\chi^u(\gamma))(X_0, \ldots, X_{t+u-1})$ is an homogeneous polynomial of degree $-u + \sum_i$ degree $(X_i)$, for all $X_i \in s\ell_{m+1}$.

**Proof.** It is easy to verify that the mapping that sends $\gamma$ onto the cochain

$$(X_0, \ldots, X_{t+u-1}) \mapsto \frac{(-1)^t}{t! u!(m+1)^u} \sum_{\nu} \text{sign}(\nu)(\gamma(DX_{\nu_0}, \ldots, DX_{\nu_t}))(d\text{tr}(DX_{\nu_t}), \ldots, d\text{tr}(DX_{\nu_{t+u-1}}))$$

has the required properties ($\nu$ describes the permutations of $t + u$ elements and $\text{sign}(\nu)$ is the signature of the permutation $\nu$). Hence the existence.

Let $c \in \Lambda^{t+u}(s\ell_{m+1}, C(\mathbb{R}^m, V))$ be such that $L^\rho_h c = 0$, $i_{h^*} c = 0$ for each $h \in \mathbb{R}^m$. Suppose also that $c(X_0, \ldots, X_{t+u-1})$ is an homogeneous polynomial of degree $-u + \sum_i$ degree $(X_i)$ for all $X_i$. We claim that if, in addition,

$$c(A^*_0, \ldots, A^*_{t-1}, \alpha^*_1, \ldots, \alpha^*_u) = 0$$

for all $A_i \in g\ell(m, \mathbb{R})$ and all $\alpha_j \in \mathbb{R}^{m*}$, then $c = 0$, thus proving the uniqueness of $\chi^u$. We show that $c(X_0, \ldots, X_{t+u-1}) = 0$ by induction on $k = -u + \sum_i$ degree $(X_i)$. It is true for $k \leq 0$, by assumption. If it is true for $k < \ell$ then for $k = \ell$, it follows from $L^\rho c = 0$ that

$$h^*(c(X_0, \ldots, X_{t+u-1})) = 0, \quad \forall h \in \mathbb{R}^m.$$ 

The homogeneous polynomial $c(X_0, \ldots, X_{t+u-1})$ of degree $\ell > 0$ is thus constant. Hence it vanishes. 

Let $\chi$ denote the mapping from $\Lambda(g\ell(m, \mathbb{R}), \Lambda(\mathbb{R}^{m*}, V))$ into $\Lambda(s\ell_{m+1}, C(\mathbb{R}^m, V))$ that reduces to $\chi^u$ when restricted to $\Lambda(g\ell(m, \mathbb{R}), \Lambda^u(\mathbb{R}^m, V))$. Observe also that $\Lambda(\mathbb{R}^{m*}, V))$ is a representation of $g\ell(m, \mathbb{R})$, its action being deduced in a natural way from the given $\rho$ and the natural action on $\mathbb{R}^{m*}$. We denote also by $\partial^\rho$ the Chevalley-Eilenberg coboundary operator associated to that representation.

**Lemma 3.2** The mapping $\chi$ is an homomorphism of differential spaces.
Proof. Let $\gamma$ be given. Then $L^p_{\delta^*}\partial^p\chi^u(\gamma) = \partial^pL^p_{\delta^*}\chi^u(\gamma) = 0$ and $i_{h^*}\partial^p\chi^u(\gamma) = L^p_{\delta^*}\chi^u(\gamma) - \partial^p i_{h^*}\chi^u(\gamma) = 0$. In addition, it is clear that

$$(\partial^p\chi^u(\gamma))(A_i^0, \ldots, A_i^*, \alpha_i^*, \ldots, \alpha_u^*) = ((\partial^\rho\gamma)(A_0, \ldots, A_t))(\alpha_1, \ldots, \alpha_u)$$

for all $A_i \in g\ell(m, \mathbb{R})$ and all $\alpha_j \in \mathbb{R}^{m*}$. Finally, a careful examination of the various terms of $(\partial^\rho\chi^u(\gamma))(X_0, \ldots, X_{t+u})$ shows that it is an homogeneous polynomial of degree $-u+\sum_i$ degree $(X_i)$. In view of the uniqueness property of $\chi^u$, it follows that $\partial^\rho\chi^u(\gamma) = \chi^u(\partial^\rho\gamma)$.

We are now in position to compute the cohomology of $s\ell_{m+1}$ associated to the representation $L^p$.

**Theorem 3.3** The mapping

$$\chi^p_2 : H(g\ell(m, \mathbb{R}), \Lambda(\mathbb{R}^{m*}, V)) \rightarrow H(s\ell_{m+1}, C_\infty(\mathbb{R}^m, V))$$

is a bijection.

**Proof.** We use the Hochschild-Serre spectral sequence $E_1^{p,q}$ associated to the subalgebra $\mathbb{R}^m$ of $s\ell_{m+1}$. Recall [4] that $E_1^{p,q}$ is the $q$-th cohomology space of the Chevalley-Eilenberg complex of the representation of $\mathbb{R}^m$ induced by $h \rightarrow L_{h^*}^p$, on the space $\Lambda^p(s\ell_{m+1}/\mathbb{R}^m, C_\infty(\mathbb{R}^m, V))$ or, equivalently, on the space of the elements of $\Lambda^p(s\ell_{m+1}, C_\infty(\mathbb{R}^m, V))$ that are cancelled out by each $i_{h^*}, h \in \mathbb{R}^m$.

Once evaluated on $X = (X_1, \ldots, X_p) \in s\ell_{m+1}^p$, a $q$-cochain $c$ defines a differential $q$-form on $\mathbb{R}^m$, namely

$$\omega^c_X : (h_1, \ldots, h_q) \rightarrow (c(h_1^*, \ldots, h_q^*))((X_1, \ldots, X_p).$$

As easily seen, if $\omega^c_X = 0$ for degree $(X) = \sum_i$ degree $(X_i) < u$, then for degree $(X) = u$, one has

$$d\omega^c_X = \omega^{d\rho^p}_X,$$

where $d$ is the de Rham differential. An easy induction on degree $(X)$ allows thus to show that

(a) If $q > 0$, then $E_1^{p,q} = 0$.

Now, a 0-cocycle is an element $c \in \Lambda^p(s\ell_{m+1}, C_\infty(\mathbb{R}^m, V))$ such that $L_{h^*}^p c = 0$ and $i_{h^*} c = 0$ for each $h \in \mathbb{R}^m$. In particular, if $c(X) = 0$ when degree $(X) < u$, then $c(X)$ is constant for degree $(X) = u$. Thus, if $\gamma \in \Lambda^{p-u}(g\ell(m, \mathbb{R}), \Lambda^u(\mathbb{R}^{m*}, V))$ is defined by

$$(\gamma(A_1, \ldots, A_{p-u}))(\alpha_1, \ldots, \alpha_u) = c(A_1^*, \ldots, A_{p-u}^*, \alpha_1^*, \ldots, \alpha_u^*)$$

then $c - \chi^u(\gamma)$ vanishes on arguments $X$ of degree $\leq u$. It follows by induction on $u$ that $c$ belongs to the image of $\chi$. It is clear that $\chi$ is injective. Therefore,

(b) $\chi : \Lambda(g\ell(m, \mathbb{R}), \Lambda(\mathbb{R}^{m*}, V)) \rightarrow \oplus_{p\geq 0} E_1^{0,p}$ is an isomorphism of differential spaces.

We deduce from (a) that the higher differentials $d_i$, $i > 1$, of the spectral sequence are vanishing. The result then follows from (b).
4 Deforming the representations of $\mathfrak{gl}(m, \mathbb{R})$

Let again $(V, \rho)$ be a finite dimensional representation of $\mathfrak{gl}(m, \mathbb{R})$. For each $\lambda \in \mathbb{R}$, define $\rho_\lambda$ by

$$\rho_\lambda : A \to \rho(A) - \lambda \text{tr}(A)id$$

where $\text{tr}$ denotes the trace and $id$ the identity from $V$ into $V$.

We want to compute the cohomology of the representation $(V, \rho_\lambda)$ of $\mathfrak{gl}(m, \mathbb{R})$ in terms of that of $(V, \rho)$.

Let $\Phi : (A, \partial) \to (B, \partial)$ be an homomorphism of differential spaces. The mapping

$$\partial \Phi : (a, b) \mapsto (\partial a, \Phi(a) - \partial b)$$

is a differential on $A \oplus B$.

**Lemma 4.1** The cohomology space $H(A \oplus B, \partial \Phi)$ is isomorphic to $\ker \Phi^* \oplus H(B, \partial)/\text{im} \Phi^*$.

*Proof.* Indeed, the connecting homomorphism of the short exact sequence of differential spaces

$$0 \to B \xrightarrow{j} A \oplus B \xrightarrow{i} A \to 0$$

where $iy = (0, -y)$ and $j(x, y) = x$, is $\Phi^*$. $\blacksquare$

**Proposition 4.2** The Chevalley-Eilenberg cohomology of $(V, \rho_\lambda)$ is isomorphic to

$$\ker \rho_\lambda(1)_* \oplus H(\mathfrak{s}\mathfrak{l}(m, \mathbb{R}), V)/\text{im} \rho_\lambda(1)_*$$

where $\rho_\lambda(1)_*$ is the map induced in cohomology by

$$c \in \Lambda(\mathfrak{s}\mathfrak{l}(m, \mathbb{R}), V) \to \rho_\lambda(1) \circ c \in \Lambda(\mathfrak{s}\mathfrak{l}(m, \mathbb{R}), V).$$

*Proof.* Indeed, the map

$$c \to (c|_{\mathfrak{s}\mathfrak{l}(m, \mathbb{R})}, (i_1 c)|_{\mathfrak{s}\mathfrak{l}(m, \mathbb{R})})$$

is an isomorphism of differential spaces between $\Lambda(\mathfrak{g}\mathfrak{l}(m, \mathbb{R}), V)$ equipped with the differential $\partial^\rho$ and $(\Lambda d\mathfrak{l}(m, \mathbb{R}))^2$ equipped with the differential

$$(c', c'') \to (\partial^\rho c', \rho_\lambda(1) \circ c' - \partial^\rho c'').$$

Hence the result, in view of Lemma 4.1. $\blacksquare$

We shall now apply the above proposition when $V \subset \otimes^a_b \mathbb{R}^m$ and when $\rho$ is the natural representation of $\mathfrak{g}\mathfrak{l}(m, \mathbb{R})$ on the tensors. For the sake of simplicity, $V_\lambda$ will denote the representation $(V, \rho_\lambda)$.

**Proposition 4.3** Let $V$ be a subspace of $\otimes^a_b \mathbb{R}^m$ stable under the natural representation of $\mathfrak{g}\mathfrak{l}(m, \mathbb{R})$ on the tensors. If $\lambda \neq (a - b)/m$, then $H(\mathfrak{g}\mathfrak{l}(m, \mathbb{R}), V_\lambda) = 0$. If $\lambda = (a - b)/m$, then $H(\mathfrak{g}\mathfrak{l}(m, \mathbb{R}), V_\lambda)$ is isomorphic to

$$(\Lambda \mathfrak{g}\mathfrak{l}(m, \mathbb{R})^*)_g^{-\text{inv}} \otimes V_s^{-\text{inv}}$$

where $g^{-\text{inv}}$ and $s^{-\text{inv}}$ denote the invariance with respect to $\mathfrak{g}\mathfrak{l}(m, \mathbb{R})$ and $\mathfrak{s}\mathfrak{l}(m, \mathbb{R})$ respectively.
Hence the result when \( m > 1 \), by \((\Lambda s\ell)(\Lambda s\ell)^*_{s-inv} \otimes V_{s-inv}\). On the latter, \( \rho_s(1) \) is the multiplication by \( a-b-\lambda m \). Moreover, \( \langle 2 \rangle \) pulls \((\Lambda s\ell)(\Lambda s\ell)^*_{s-inv} \) back onto \((\Lambda g\ell(m, \mathbb{R})^*_{g-inv}\). Hence the result when \( m > 1 \). For \( m = 1 \), the result follows from an immediate direct computation, \( \langle 3 \rangle \) being just \( V \) in this case.

**Corollary 4.4** Under the assumptions of Proposition \( \langle 4, 3 \rangle \), \( H(g\ell(m, \mathbb{R}), V_\lambda) \neq 0 \) only if \( \lambda = (a-b)/m \) is an integer.

**Proof.** It is obvious if \( m = 1 \). If \( m > 1 \), it follows from the description of \((\otimes^a \mathbb{R}^m)_{s-inv}, \langle 1 \rangle \) : \( T \in \otimes^a \mathbb{R}^m \) is \( s\ell(m, \mathbb{R}) \)-invariant if and only if \( T(\xi_1, \ldots, \xi_m, X_1, \ldots, X_b) \), as a function of \( \xi_i \in \mathbb{R}^{m*} \) and \( X_j \in \mathbb{R}^m \), is a linear combination of products of contractions \( \langle X_j, \xi_i \rangle \) and of determinants of the form \( \det(\xi_{i_1}, \ldots, \xi_{i_m}) \) and \( \det(X_{j_1}, \ldots, X_{j_m}) \).

### 5 Two examples

In this section, we illustrate the previous results by computing two cohomologies of \( s\ell_{m+1} \) that will be needed later.

Let us first consider \( S^k_m \mathbb{R}^m \), the space \( \forall^k \mathbb{R}^m \) of \( k \)-contravariant symmetric tensors equipped with the representation \( \rho_\lambda \) (where \( \rho \) is the natural representation of \( g\ell(m, \mathbb{R}) \) on tensors).

**Proposition 5.1** (i) Assume that \( m > 1 \). If \( (\lambda, k) \in \{(0,0), (1,0), (1,1)\} \), then \( H(s\ell_{m+1}, C_\infty(\mathbb{R}^n, S^k_m \mathbb{R}^m)) \) is isomorphic to \((\Lambda g\ell(m, \mathbb{R})^*_{g-inv}\) otherwise it is vanishing.

(ii) If \( (\lambda, k) \in \{(k,0), (k,1), (k+1,1), (k+1,2)\} \) then \( H(s\ell_2, C_\infty(\mathbb{R}, S^k_m \mathbb{R})) \) is isomorphic to \( \mathbb{R} \) otherwise it vanishes.

**Proof.** It follows from Theorem \( \langle 3, 3 \rangle \) and Proposition \( \langle 4, 3 \rangle \) that \( H^u(s\ell_{m+1}, C_\infty(\mathbb{R}^n, S^k_m \mathbb{R}^m)) \) is non vanishing only if \( \lambda = (j+k)/m \) for some \( j \in \{0, \ldots, m\} \) and that it is then isomorphic to

\[
(\Lambda^{u-j} g\ell(m, \mathbb{R})^*_{g-inv} \otimes A^j(\mathbb{R}^{m*}, \forall^k \mathbb{R}^m)_{s-inv}. \tag{4}
\]

Using the description of \((\otimes^a \mathbb{R}^m)_{s-inv}, \langle 1 \rangle \) recalled in the proof of Corollary \( \langle 4, 4 \rangle \), the latter is non vanishing if and only if \( (j,k) \) equals \( (0,0), (m,0) \) or \( (m-1,1) \) in which case, \( \langle 4 \rangle \) is respectively spanned by the mappings

\[
(A_0, \ldots, A_{l-1}) \rightarrow \gamma(A_0, \ldots, A_{l-1}),
\]

\[
(A_0, \ldots, A_{l-1}, \alpha_1, \ldots, \alpha_m) \rightarrow \gamma(A_0, \ldots, A_{l-1}) \det(\alpha_1, \ldots, \alpha_m)
\]

or

\[
(A_0, \ldots, A_{l-1}, \alpha_1, \ldots, \alpha_{m-1}) \rightarrow [\xi \rightarrow \gamma(A_0, \ldots, A_{l-1}) \det(\alpha_1, \ldots, \alpha_{m-1}, \xi)]
\]

\( t = u-j \), \( A_i \in g\ell(m, \mathbb{R}), \alpha_i, \xi \in \mathbb{R}^{m*}, \gamma \in (\Lambda^t g\ell(m, \mathbb{R})^*_{g-inv}) \). Hence (i).

For (ii), it suffices to note that \( A^j(\mathbb{R}, \forall^t \mathbb{R})_{s-inv} = \mathbb{R} \).
Corollary 5.2 The cohomology of $\mathfrak{sl}_{m+1}$ acting on the space of scalar $\lambda$-densities of $\mathbb{R}^m$ vanishes if $\lambda \neq 0$ and $\lambda \neq 1$ otherwise, it is isomorphic to $(\text{Ag}(m, \mathbb{R})^*)_g$-

Proof. It is the cohomology of the representation $C_\infty(\mathbb{R}^m, S^0_\delta(\mathbb{R}^m))$ of $\mathfrak{sl}_{m+1}$. ■

Let us now consider the case of the space $S^k_p \mathbb{R}^m$ of $k$-contravariant symmetric tensors valued in the space of linear maps from $\mathbb{R}^p \otimes \mathbb{R}^m$ into $\mathbb{R}^\gamma m$:

$$S^k_p \mathbb{R}^m = \mathbb{R}^k m \otimes \text{Hom}(\mathbb{R}^p \otimes \mathbb{R}^m, \mathbb{R}^\gamma m)$$

$$\simeq \mathbb{R}^k m \otimes \mathbb{R}^\gamma m \otimes \mathbb{R}^p \otimes \mathbb{R}^m.$$  

We equip it with the natural representation of $\mathfrak{gl}(m, \mathbb{R})$.

Proposition 5.3 If $p \in \{k + q, k + q + 1\}$ then $H(\mathfrak{sl}_{m+1}, C_\infty(\mathbb{R}^m, S^k_p \mathbb{R}^m))$ is isomorphic to $(\text{Ag}(m, \mathbb{R})^*)_g$-

Proof. The proof is similar to that of the previous proposition. One is now left to study

$$(\Lambda^{u-j} \mathfrak{gl}(m, \mathbb{R})^*)_g \otimes \Lambda^{j} (\mathbb{R}^m, S^k_p \mathbb{R}^m)_{s-g}$$

knowing that $p = j + k + q$ (since $\lambda = 0$). This space is non vanishing if and only if $j = 0$ or $j = 1$, $\mathfrak{B}$ being then respectively spanned by the mappings

$$(A_0, \ldots, A_{t-1}) \rightarrow [P \rightarrow \gamma(A_0, \ldots, A_{t-1})(\eta D_\xi)^k P]$$

or

$$(A_0, \ldots, A_{t-1}, \alpha) \rightarrow [P \rightarrow \gamma(A_0, \ldots, A_{t-1})(\alpha D_\xi)^k P]$$

$t = u - j, A_i \in \mathfrak{gl}(m, \mathbb{R}), \alpha \in \mathbb{R}^m, \gamma \in (\Lambda^{l} \mathfrak{gl}(m, \mathbb{R})^*)_g, P \in \mathbb{R}^{k \mathbb{R}^m}.$

Here, we view an element of $S^k_p \mathbb{R}^m$ as being a homogeneous polynomial of degree $k$ in $\eta \in \mathbb{R}^m$ valued in the space of linear mappings from the space of homogeneous polynomials of degree $p$ in $\xi \in \mathbb{R}^p$ into the space of these of degree $q$. Moreover, $\alpha D_\xi$ and $\eta D_\xi$ denote the derivations with respect to $\xi$ in the direction $\alpha \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^m$ respectively. These conventions will be helpful in the sequel. ■

6  Cohomology of $\mathfrak{sl}_{m+1}$ valued in $\mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$

Recall from the introduction that the structure of the $\mathfrak{sl}_{m+1}$-module $\mathcal{D}_{\lambda}(\mathbb{R}^m)$ is related to the cohomology of $\mathfrak{sl}_{m+1}$ with coefficients in $\mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$, $\delta = \mu - \lambda$.

We first compute the cohomology of $\mathfrak{sl}_{m+1}$ with coefficients in $\mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$, $k \in \mathbb{N}$. We proceed by induction on $k$, using the short exact sequence

$$0 \rightarrow \mathcal{D}^{k-1}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \rightarrow \mathcal{D}^{k}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \rightarrow C_\infty(\mathbb{R}^m, S^k_p \mathbb{R}^m) \rightarrow 0$$
that induces an exact triangle
\[ H(s\ell_{m+1}, D^{k-1}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \]
\[ \theta \uparrow \]
\[ \sigma \]
\[ H(s\ell_{m+1}, C_\infty(\mathbb{R}^m, S^{k,q}_p(\mathbb{R}^m))) \]

where the connecting homomorphism \( \theta \) is of degree 1 ([7]). Note that the cohomology of the quotient is that computed in the previous section. From Proposition 5.3, we immediately deduce

**Theorem 6.1**  (a) If \( p < q \), then \( H(s\ell_{m+1}, D^k(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) = 0 \)

(b) If \( n \geq 0 \), then

(i) \( k < n - 1 \Rightarrow H(s\ell_{m+1}, D^k(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) = 0 \)

(ii) \( H(s\ell_{m+1}, D^{n-1}(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \) is isomorphic to \( (\Lambda g(\ell(m, \mathbb{R}^*))_{g-inv} \)

(iii) The inclusion of \( D^n(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \) into \( D^k(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \) induces an isomorphism

\[ H(s\ell_{m+1}, D^n(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \rightarrow H(s\ell_{m+1}, D^k(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \]

for each \( k > n \).

**Remark 6.2** In (ii), the isomorphism is obtained by composing the isomorphism of Proposition 5.3 and the map induced in cohomology by the symbol map \( \sigma \) from \( D^{n-1}(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \).

To complete Theorem 6.1, we need to compute the connecting homomorphism \( \theta \).

**Lemma 6.3** If \( n > 0 \), identifying \( H(s\ell_{m+1}, C_\infty(\mathbb{R}^m, S^{n,q}_{q+n}(\mathbb{R}^m))) \) and \( H(s\ell_{m+1}, D^{n-1}(S^{q+n}_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) \) with \( (\Lambda g(\ell(m, \mathbb{R}^*))_{g-inv} \) using Proposition 5.3 and Theorem 6.1, one has

\[ \theta(\gamma) = (-1)^{l+1}n[(m+1)\delta - (2q+n+m)]\gamma, \quad \forall \gamma \in (\Lambda g(\ell(m, \mathbb{R}^*))_{g-inv}. \]

**Proof.** To perform the computation, it is convenient to identify the spaces \( D^k(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \) and \( \oplus_{0 \leq i \leq k} C_\infty(\mathbb{R}^m, S^{i,q}_p(\mathbb{R}^m)) \) by representing the operator

\[ D : T \rightarrow \sum_{|\alpha| \leq k} A_\alpha(D^\alpha_T) \]

where \( A_\alpha \in C_\infty(\mathbb{R}^m, Hom(\vee^p \mathbb{R}^m, \vee^q \mathbb{R}^m)) \), by

\[ T_D : (\eta, P) \in \mathbb{R}^{m*} \times \vee^p \mathbb{R}^m \rightarrow \sum_{|\alpha| \leq k} \eta^\alpha A_\alpha(P) \in C_\infty(\mathbb{R}^m, \vee^q \mathbb{R}^m). \]
(cf. the proof of Proposition 5.3). It follows then easily that, the components of $\zeta \in \mathbb{R}^{m*}$ representing symbolically the partial derivatives of $X \in \text{Vect}(\mathbb{R}^m)$,

$$
\mathcal{T}_{LX}(\eta, P) = (X, \mathcal{T}_D)(\eta, P) + \langle X, \eta \rangle \mathcal{T}_D(\eta, P) + \delta \langle X, \zeta \rangle \mathcal{T}_D(\eta, P)
$$

- $X(\zeta D\xi)\mathcal{T}_D(\eta, P) - \mathcal{T}_D(\eta + \zeta, \langle X, \eta \rangle P + \delta \langle X, \zeta \rangle P - X(\zeta D\xi)P) \quad (7)$

(see \[3\] where similar symbolical computations are used).

Now $\gamma \in (\Lambda gl(m, \mathbb{R})^*)_g$ defines an element in $H(s\ell_{m+1}, C_\infty(\mathbb{R}^m, S^{n,q}_{q+n}\mathbb{R}^m))$, namely the class of the cocycle $c = 0$ (see \[9\] where similar symbolical computations are used).

To compute the constant term of the isomorphism (ii) of Theorem 6.1 (see Remark 6.2). To compute $\gamma'$, it suffices to evaluate $(\partial c)(A^*_0, \ldots, A^*_{t-1}, \alpha^*)$, $A_i \in gl(m, \mathbb{R})$, $\alpha \in \mathbb{R}^m$, at $x = 0$. That is the same as to compute the constant term of

$$
(1)^tL_{\alpha^*}(c(A^*_0, \ldots, A^*_{t-1}))
$$

where $L_{\alpha^*}$ is the derivation of differential operators in the direction of $\alpha^*$. Observe that

$$
c(A^*_0, \ldots, A^*_{t-1}) = \gamma(A_0, \ldots, A_{t-1})(\eta D\xi)^n
$$

has constant coefficients so that we need only the terms of $\langle 8 \rangle$ involving second order derivatives of the coefficients of $\alpha^*$. It follows from $\langle 7 \rangle$ that if $X = \alpha^*$, the terms of second order in $X$ in $\mathcal{T}_{LX}(\eta, P)$ are given by

$$
-(\eta D\eta)(\alpha D\eta)\mathcal{T}_D(\eta, P) - (m + 1)\delta(\alpha D\eta)\mathcal{T}_D(\eta, P) + (\alpha D\eta)\mathcal{T}_D(\eta, (\xi D\xi)P) + \sum_i D\eta_i \mathcal{T}_D(\eta, \xi_i(\alpha D\xi)P).
$$

With $\mathcal{T}_D(\eta, P) = (\eta D\xi)^nP$ and $P \in \mathbb{R}^{q+n}\mathbb{R}^m$, this gives

$$
-n[(m + 1)\delta - (2q + n + m)](\alpha D\xi)(\eta D\xi)^{n-1}.
$$

Hence the lemma.

Theorem 6.4 If $n > 0$, the space $H(s\ell_{m+1}, \mathcal{D}^n(S^{q+n}_q(\mathbb{R}^m), S^q_0(\mathbb{R}^m)))$ is isomorphic to $(\Lambda gl(m, \mathbb{R})^*)_g$ if $\delta = (2q + n + m)/(m + 1)$ and vanishes otherwise. If $n = 0$, it is isomorphic to $(\Lambda gl(m, \mathbb{R})^*)_g$.

Proof. This follows immediately from $\langle 7 \rangle$ and previous results.

We are now able to compute the cohomology of $s\ell_{m+1}$ with coefficients in $\mathcal{D}(S^p_0(\mathbb{R}^m), S^q_0(\mathbb{R}^m))$. 
Theorem 6.5 If \( p < q \), then \( H(s\ell_{m+1}, \mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))) = 0 \). Otherwise, the inclusion

\[
\mathcal{D}^n(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \overset{i}{\to} \mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))
\]

induces an isomorphism in cohomology, where \( n = p - q \geq 0 \).

Proof. For the sake of simplicity, we denote here by \( \mathcal{D} \) and \( \mathcal{D}^k \) the spaces \( \mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \) and \( \mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \) respectively.

Observe that a cochain on \( s\ell_{m+1} \) valued in the space of differential operators necessarily takes its values in the space of operators of a certain order \( k \). This is because \( s\ell_{m+1} \) is finite dimensional.

This implies first that \( H(s\ell_{m+1}, \mathcal{D}) = 0 \) if \( p < q \) because, in this case, the spaces \( H(s\ell_{m+1}, \mathcal{D}^k) \) are all vanishing.

Second, this also implies that if \( p = q + n \), \( n \geq 0 \),

\[
i_\sharp : H(s\ell_{m+1}, \mathcal{D}^n) \to H(s\ell_{m+1}, \mathcal{D})
\]

is an isomorphism. Indeed, if \( i_\sharp[S] = 0 \), then \( S \) is the coboundary of some \( \mathcal{D}^k \)-valued cochain \( T \). If \( k \leq n \), then \( [S] = 0 \) because \( \mathcal{D}^k \subset \mathcal{D}^n \). If \( k > n \), then again \( [S] = 0 \), due to Theorem 6.1 (b) (iii). This proves that \( i_\sharp \) is injective. A similar discussion shows that it is onto. \( \blacksquare \)

Remark 6.6 When \( p > q \), the cohomology of \( s\ell_{m+1} \) with coefficients in \( \mathcal{D}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m)) \) is non vanishing if and only if \( \delta = \frac{m+p+q}{m+1} \). This is why we say that \( \frac{m+p+q}{m+1} \) is a critical value for \( \delta \). When \( m = 1 \), these critical values are the special values pointed out in [6] in classifying the \( s\ell_2 \)-module \( \mathcal{D}^k_{\lambda\mu}(\mathbb{R}) \). In [6], they were called “resonant” instead of critical.

For \( n \geq 0 \), we define

\[
\tau_n : s\ell_{m+1} \to \mathcal{D}(S^{*+n}_\delta(\mathbb{R}^m), S^{*}_{\delta}(\mathbb{R}^m))
\]

by

\[
\tau_n(X)P = - \sum_{j,i_1,\ldots,i_n} \partial_j X^{i_1} \partial_{i_1} \ldots \partial_{i_n} D_{\xi_{i_1}} \ldots D_{\xi_{i_n}} P
\]

(recall that \( \partial_i \) denotes the partial derivation with respect to the \( i \)-th coordinate and that \( D_{\xi_{j}} \) denotes the derivation with respect to the \( j \)-th component of \( \xi \in \mathbb{R}^{m*} \)).

Using the notations of the proof of Lemma 6.3, one can also write

\[
\tau_n(X) = - \langle X, \zeta \rangle (\eta D_\xi)^n
\]

where \( \eta \) and \( \zeta \) represent the derivatives acting on \( P \) and \( X \) respectively.

For \( n > 0 \), we also introduce the map

\[
\gamma_n : s\ell_{m+1} \to \mathcal{D}(S^{*+n}_\delta(\mathbb{R}^m), S^{*}_{\delta}(\mathbb{R}^m))
\]

given by

\[
\gamma_n(X)P = \frac{1}{m+1} \sum_{j,i_1,\ldots,i_n} \partial_{i_1} \partial_j X^{i_1} \partial_{i_2} \ldots \partial_{i_n} D_{\xi_{i_1}} \ldots D_{\xi_{i_n}} P.
\]

It is symbolically given by

\[
\gamma_n(X) = \frac{1}{m+1} \langle X, \zeta \rangle (\zeta D_\xi)(\eta D_\xi)^{n-1}.
\]
Corollary 6.7 Let $p = q + n$. The space $H^1(s\ell_{m+1}, \mathcal{D}(S^p_\delta(\mathbb{R}^m)), S^q_\delta(\mathbb{R}^m))$ is spanned by $[\tau_0]$ if $n = 0$ and by $[\tau_n]$ and $[\gamma_n]$ if $n > 0$ and $\delta = \frac{m+p+q}{m+1}$. It vanishes otherwise.

Proof. This easily follows from the previous results. \hfill \blacksquare

7 The space $\text{Hom}_{s\ell_{m+1}}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$

In this section, we compute the 0-th cohomology space of $s\ell_{m+1}$ with coefficients in $\text{Hom}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$. It is the space $\text{Hom}_{s\ell_{m+1}}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$ of $s\ell_{m+1}$-equivariant linear mappings from $S^p_\delta(\mathbb{R}^m)$ into $S^q_\delta(\mathbb{R}^m)$.

For each $n \in \mathbb{N}$, we define $T_n : S^{p+n}_\delta(\mathbb{R}^m) \rightarrow S^p_\delta(\mathbb{R}^m)$ by

$$T_nP = \sum_{i_1, \ldots, i_n} \partial_{i_1} \cdots \partial_{i_n} D_{\xi_{i_1}} \cdots D_{\xi_{i_n}} P.$$

With the notations of the proof of Lemma 6.3, $T_n$ is represented by the polynomial $(\eta D\xi)^n$. Moreover, it follows from that proof that

$$L_X \circ T_n - T_n \circ L_X = -n[(m+1)\delta - (m+2p+n)]\gamma_n(X).$$

In particular, $T_n : S^{p+n}_\delta(\mathbb{R}^m) \rightarrow S^p_\delta(\mathbb{R}^m)$ is $s\ell_{m+1}$-equivariant if and only if $n[(m+1)\delta - (m+2p+n)] = 0$.

Observe that

$$\tau_n(X) = -\langle X, \zeta \rangle T_n$$

and

$$\gamma_n(X) = \frac{1}{m+1} \langle X, \zeta \rangle (\zeta D\xi) \circ T_{n-1}.$$ 

Moreover,

$$\gamma_{n+i}(X) = \gamma_n(X) \circ T_i$$

since $T_i \circ T_j = T_{i+j}$.

Lemma 7.1 If $p \geq q$ and if $A \in \text{Hom}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$ commute with the Lie derivations in the direction of $\partial_1, \ldots, \partial_m$ and of $E = x^i \partial_i$ then $A$ is a differential operator.

Proof. It has been shown in [8] that if $\delta = 0$, then under the assumptions of the lemma, $A$ is a local operator. In fact, the proof of [8] works also if $\delta \neq 0$. Thus, we may assume that $A$ is local. From the theorem of Peetre [10], it is then locally a differential operator. Since it commutes with $L_{\partial_1}, \ldots, L_{\partial_m}$, its order is bounded. \hfill \blacksquare

Theorem 7.2 If $p \geq q$ and if $A \in \text{Hom}_{s\ell_{m+1}}(S^p_\delta(\mathbb{R}^m), S^q_\delta(\mathbb{R}^m))$ is non vanishing then either $p = q$ and $A$ is a constant multiple of the identity or $p > q$, $\delta = \frac{m+p+q}{m+1}$ and $A$ is a constant multiple of $T_{p-q}$.

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Proof. Indeed, since $A$ is a differential operator, it is a $0$-cocycle of $s\ell_{m+1}$ valued in $\mathcal{D}(S^p_0(\mathbb{R}^m), S^q_0(\mathbb{R}^m))$. The theorem then follows immediately from Theorem 7.3.

The case $p < q$ is more difficult. We state directly the theorem. Indeed, we do not know whether a lemma similar to Lemma 7.1 holds true in this case.

**Theorem 7.3** If $p < q$ and $A \in \text{Hom}_{s\ell_{m+1}}(S^p_0(\mathbb{R}^m), S^q_0(\mathbb{R}^m))$, then $A = 0$.

Proof. It suffices to show that $A$ is local (and thus a differential operator, proceeding like in the proof of Lemma 7.1). Indeed, it follows from Theorem 7.3 that a $0$-cocycle of $s\ell_{m+1}$ with values in $\mathcal{D}(S^p_0(\mathbb{R}^m), S^q_0(\mathbb{R}^m))$ is vanishing, because $p < q$.

The proof has three parts. In the first, $\delta$ has not the critical value $\frac{m+p+q}{m+1}$ and we show directly that $A = 0$. In the second, $\delta = \frac{m+p+q}{m+1}$ and $m > 1$ while in the third, $\delta = \frac{m+p+q}{m+1}$ but $m = 1$. We set again $q = p + n$.

(i) **The non critical case.** We make use of the Casimir operator $C^p_\delta$ of the $s\ell_{m+1}$-module $S^p_0(\mathbb{R}^m)$. Recall from [7] that it is the $s\ell_{m+1}$-equivariant linear map from $S^p_0(\mathbb{R}^m)$ into itself defined by

$$C^p_\delta = \sum_i L_{X_i} \circ L_{Y_i}$$

where $\{X_i : i \leq m(m+2)\}$ is any basis of $s\ell(m+1, \mathbb{R})$ and $\{Y_i : i \leq m(m+2)\}$ is its dual with respect to the Killing form $K$ of $s\ell(m+1, \mathbb{R})$ (i.e. $K(X_i, Y_j) = \delta_{ij}$).

By Theorem 7.2, $C^p_\delta$ is a multiple $c^p_\delta$ of the identity. It is easy to compute. One gets

$$2(m+1)c^p_\delta = m(m+1)\delta^2 - (m+2p)(m+1)\delta + 2p(m+p).$$

Since $A$ is $s\ell_{m+1}$-equivariant, one has $A \circ C^p_\delta = C^{p+n}_\delta \circ A$. But

$$c^{p+n}_\delta - c^p_\delta = -n \left( \delta - \frac{m+2p+n}{m+1} \right).$$

Therefore, if $\delta \neq \frac{m+2p+n}{m+1}$, then $A = 0$.

(ii) **The critical case** $\delta = \frac{m+2p+n}{m+1}$, $m > 1$. This case is more delicate to handle. We need some preparations. For each $\alpha \in \mathbb{R}^{m*}$, $\hat{\alpha}$ denotes the function $x \to \alpha(x)$ on $\mathbb{R}^m$ (it is $\alpha$ but viewed as an element of $C_\infty(\mathbb{R}^m)$). Moreover, $I \in S^0_0(\mathbb{R}^m)$ is defined by

$$I(\xi)x = \xi(x), \quad \forall x \in \mathbb{R}^m, \quad \forall \xi \in \mathbb{R}^{m*}.$$ One has

$$L_{\alpha^*}Q = L_E(\hat{\alpha}Q) + (\delta - 1)\hat{\alpha}Q - I(\alpha D\xi)Q, \quad \forall Q \in S^0_0(\mathbb{R}^m).$$

Using this, the $s\ell_{m+1}$-equivariance of $A$ and the assumption $\delta = \frac{m+2p+n}{m+1}$, one gets, after some computations,

$$(m + p - 1)A(\hat{\alpha}Q)_0 = A(I(\alpha D\xi)Q)_0, \quad \forall Q \in S^p_0(\mathbb{R}^m),$$

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the index 0 denoting the evaluation at \(x = 0\). This implies easily that
\[
A(\hat{\alpha} I^k Q)_0 = \frac{1}{m + p - (k + 1)} A(I^{k+1}(\alpha D_\xi)Q)_0.
\]
Hence, by induction on \(k\),
\[
A(\hat{\alpha}_1 \ldots \hat{\alpha}_k Q)_0 = \frac{1}{(m + p - 1) \ldots (m + p - k)} A(I^k(\alpha_1 D_\xi) \ldots (\alpha_k D_\xi)Q)_0
\]
for each \(Q \in S_\delta^p(\mathbb{R}^m)\) and for all \(\alpha_1, \ldots, \alpha_k \in \mathbb{R}^{m*}\).

We now prove that \(A\) is local. Suppose that \(P \in S_\delta^p(\mathbb{R}^m)\) vanishes in a neighborhood of \(y \in \mathbb{R}^m\). We want to show that \(A(P)_y = 0\).

Assume first that \(y = 0\). There exists then \(Q_{i_0 \ldots i_p} \in S_\delta^p(\mathbb{R}^m)\) such that \(P = \hat{\epsilon}_{i_0} \ldots \hat{\epsilon}_{i_p} Q_{i_0 \ldots i_p}\) \((\epsilon^1, \ldots, \epsilon^m\) denotes canonical basis of \(\mathbb{R}^{m*}: \epsilon^i(x) = x^i, \forall x \in \mathbb{R}^m\)). Then
\[
A(P)_0 = \frac{1}{(m + p - 1) \ldots (m - 1)} A(I^{p+1} D_{\xi_{i_0}} \ldots D_{\xi_{i_p}} Q_{i_0 \ldots i_p})_0 = 0
\]
because the \(Q_{i_0 \ldots i_p}\)'s are homogeneous of order \(p\) in \(\xi \in \mathbb{R}^{m*}\).

One reduces the case \(y \neq 0\) to the case \(y = 0\) just by replacing above \(x\) by \(x - y\) in \(\hat{\alpha}, I, E, \alpha^*, \) etc.

(iii) The critical case \(\delta = \frac{m+2p+n}{m+1}, m = 1\). We denote by \(t\) the canonical coordinate of \(\mathbb{R}\). The map \(A\) is of the form
\[
A : f \left( \frac{d}{dt} \right)^p |dt|^\delta \rightarrow A_f \left( \frac{d}{dt} \right)^{p+n} |dt|^\delta, \quad f, A_f \in C_\infty(\mathbb{R}).
\]

Expressing the fact that \(A\) commutes with \(L_{\frac{d}{dt}}\) and \(L_{\frac{d}{dt}}\) leads immediately to the following relations:
\[
\frac{d}{dt} A_f = A_{\frac{d}{dt}} A_f
\]
and
\[
t \frac{d}{dt} A_f = A_{\frac{d}{dt}} A_f + n A_f.
\]

From this, it follows first that \(A_f = 0\) if \(f\) is a polynomial. Indeed, the first relation shows that if \(f\) is a homogeneous polynomial of degree \(s\), then \(A_f\) is a polynomial of degree \(\leq s\) but the second relation shows that, in the same time, \(A_f\) is homogeneous of order \(s + n > s\).

On the other hand, \(T_n : S_\delta^{p+n}(\mathbb{R}) \rightarrow S_\delta^p(\mathbb{R})\) is \(sl_2\)-equivariant (because \(\delta\) is critical). By Theorem 7.2, it follows that \(A \circ T_n\) is a constant multiple \(a\) of the identity on \(S_\delta^{p+n}(\mathbb{R})\). It is clear that \(a = 0\) because \(A\) vanishes on polynomials. This reads
\[
A_{\frac{d}{dt}} f = 0
\]
for all \(f \in C_\infty(\mathbb{R})\). Therefore, \(A = 0\).
8 The splitting of the short exact sequence (1) for $M = R^m$

Our aim in this section is to obtain a necessary and sufficient condition for the short exact sequence of $s\ell_{m+1}$-modules (1) to be split (with $M = R^m$).

We denote by $\varphi : S_{\lambda}(R^m) \to D_{\lambda\mu}(R^m)$ the canonical right inverse of the symbol map. If $P \in S_{\delta}(R^m)$, then $\varphi(P) \in D_{\lambda\mu}(R^m)$ is the unique homogeneous differential operator of order $k$ such that $\sigma(\varphi(P)) = P$. As known, the coboundary $E_k$ of the restriction of $\varphi$ to $S_{\delta}(R^m)$ takes its values in $Hom(S_{\delta}(R^m), D_{\lambda\mu}(R^m))$. Moreover, its cohomology class characterizes the isomorphism class of the short exact sequence (1). In particular, (1) is split if and only if $E_k$ is a coboundary of some element of $Hom(S_{\delta}(R^m), D_{\lambda\mu}(R^m))$.

**Lemma 8.1** One has $E_k = -u_k \varphi \circ \gamma_1$, where

$$u_k = (m + 1)\lambda + k - 1.\$$

**Proof.** It is just a matter of simple computation. ☐

**Lemma 8.2** As a cocycle of $s\ell_{m+1}$ valued in $Hom(S_{\delta}(R^m), S_{\delta}^{k-n}(R^m))$, $\gamma_n$ is a coboundary if and only if

$$v_n = -n((m + 1)\delta - (m + 2k - n))$$

is non vanishing.

**Proof.** By Corollary 7.7, it suffices to show that if $\gamma_n = \partial T$, then $T \in Hom(S_{\delta}^k(R^m), S_{\delta}^{k-n}(R^m))$ is a differential operator. Since $\gamma_n(\partial_i) = 0$ and $\gamma_n(E) = 0$, this follows immediately from Lemma 7.1. ☐

**Remark 8.3** It is clear that if $v_n \neq 0$, then $\gamma_n$ is the coboundary of $\frac{1}{v_n} T_n$.

In the sequel, for $A = \sum_i A_i t^i \in R[i]$, we denote by $A(\eta\partial\xi)$ the differential operator

$$\sum_i A_i T_i : S^k_{\delta}(R^m) \to \bigoplus_{\sigma \leq k} S^{k-i}_{\delta}(R^m).$$

**Proposition 8.4** If $\gamma_1, \ldots, \gamma_{n-1}$ are coboundaries, then

$$E_k = -\frac{u_k \ldots u_{k-n+1}}{v_1 \ldots v_{n-1}} \varphi \circ \gamma_n + \partial \varphi \circ A_n(\eta\partial\xi)$$

for some polynomial $A_n$ of degree $n - 1$ such that $A_n(0) = 0$.

**Proof.** We proceed by induction on $n$. By Lemma 8.1, $E_k = -u_k \varphi \circ \gamma_1$. Assume that (1) holds true and that $\gamma_n$ is a coboundary. Then, setting

$$a = -\frac{u_k \ldots u_{k-n+1}}{v_1 \ldots v_{n-1}}$$

we have

$$E_k = a \varphi \circ \gamma_n + \partial a \varphi \circ A_n(\eta\partial\xi).$$

By induction, we know that $H_k = \partial a \varphi \circ A_n(\eta\partial\xi)$ is a differential operator. Since $\gamma_n(\partial_j) = 0$ and $\gamma_n(H_k) = 0$, this follows immediately from Lemma 7.1. ☐
for simplicity, one successively gets

\[ (E_k - \partial \varphi \circ A_n(\eta D_\xi))(X) \]
\[ = \frac{a}{v_n} (\varphi \circ L_X \circ T_n - \varphi \circ T_n \circ L_X) \]
\[ = \frac{a}{v_n} (\varphi \circ L_X \circ T_n - L_X \circ \varphi \circ T_n) + \frac{a}{v_n} (L_X \circ \varphi \circ T_n - \varphi \circ T_n \circ L_X) \]
\[ = -\frac{a}{v_n} E_{k-n}(X) \circ T_n + \frac{a}{v_n} (\partial(\varphi \circ T_n))(X) \]
\[ = \frac{a \lambda_{k-n}}{v_n} \varphi \circ \gamma_1(X) \circ T_n + \frac{a}{v_n} (\partial(\varphi \circ T_n))(X) \]
\[ = \frac{a \lambda_{k-n}}{v_n} \varphi \circ \gamma_{n+1}(X) + \frac{a}{v_n} (\partial(\varphi \circ T_n))(X) \]

because \( \gamma_1 \circ T_n = \gamma_{n+1} \).

\[ \text{Theorem 8.5} \]

The short exact sequence of \( sl_{m+1} \)-modules

\[ 0 \to \mathcal{D}_{\lambda \mu}^{-1}(\mathbb{R}^m) \to \mathcal{D}_{\lambda \mu}^k(\mathbb{R}^m) \to \mathcal{S}_{\lambda}^k \to 0 \quad (k \geq 1) \]

is split if and only if

\[ \delta \notin \left\{ \frac{m + k}{m + 1}, \frac{m + k + 1}{m + 1}, \ldots, \frac{m + 2k - 1}{m + 1} \right\} \quad (10) \]

or

\[ \delta = \frac{m + 2k - n}{m + 1} \quad \text{and} \quad \lambda = \frac{i - k}{m + 1} \quad (11) \]

for some \( n \in \{1, \ldots, k\} \) and some \( i \in \{1, \ldots, n\} \).

\[ \text{Proof.} \]

Let \( n \) denote the least integer such that \( \gamma_n \) is not a coboundary, as a cocycle valued in \( \text{Hom}(\mathcal{S}_{\lambda}^k(\mathbb{R}^m), \mathcal{S}_{\lambda}^{k-n}(\mathbb{R}^m)) \) (we set \( n = k + 1 \) if \( \gamma_1, \ldots, \gamma_k \) are coboundaries).

It follows from Lemma \[8.2\] and Proposition \[8.4\] that \( E_k \) is a coboundary if (11), or (10), holds true.

Conversely, assume that \( E_k \) is a coboundary and that \( n \leq k \). Then, \( \delta = \frac{m + 2k - n}{m + 1} \).

Moreover, it follows from (8) that \( u_k \ldots u_{k-n+1} \gamma_n \) is a coboundary. Since \( \gamma_n \) is not a coboundary, \( u_i = 0 \) for some \( i \in \{k - n + 1, \ldots, k\} \). Hence the result. \[ \square \]

\[ \text{Corollary 8.6} \]

If \((m + 1)\delta - m \notin \mathbb{N}_0\), then there exists a unique \( sl_{m+1} \)-equivariant linear bijection \( \sigma^{\lambda \mu} : \mathcal{D}_{\lambda \mu}(\mathbb{R}^m) \to \mathcal{S}_{\lambda}(\mathbb{R}^m) \) such that, for each \( A \in \mathcal{D}_{\lambda \mu}(\mathbb{R}^m) \), the term of highest order of \( \sigma^{\lambda \mu}(A) \) is the symbol \( \sigma(A) \) of \( A \).

\[ \text{Proof.} \]

The existence of \( \sigma^{\lambda \mu} \) follows immediately from Theorem \[8.3\]. For the uniqueness, assume that \( \sigma' \) has the same properties than \( \sigma^{\lambda \mu} \). Then, for each \( k \), the restriction of \( S = \sigma' \circ (\sigma^{\lambda \mu})^{-1} \) to \( \mathcal{S}_{\lambda}^k(\mathbb{R}^m) \) is of the form

\[ P \in \mathcal{S}_{\lambda}^k(\mathbb{R}^m) \to (P, S_1(P), \ldots, S_k(P)) \in \bigoplus_{0 \leq \ell \leq k} \mathcal{S}_{\lambda}^\ell(\mathbb{R}^m) \]

for some \( sl_{m+1} \)-equivariant \( S_i \in \text{Hom}(\mathcal{S}_{\lambda}^k(\mathbb{R}^m), \mathcal{S}_{\lambda}^{k-i}(\mathbb{R}^m)) \), \( i = 1, \ldots, k \). By Theorem \[8.2\], \( S_i = 0 \) for \( i = 1, \ldots, k \). \[ \square \]
Remark 8.7 The existence and uniqueness of $\sigma^{\lambda\mu}$ was shown in [8] for $\lambda = \mu$ and $m > 1$. It has been obtained in [6, 2] for $m = 1$ and non critical values of $\delta$. For critical $\delta$, $D_{\lambda\mu}(\mathbb{R})$ has been described in [6]. We will not discuss here the case of the critical values of $\delta$ in higher dimension. The next result is also a generalization of a useful result of [6, 8].

Proposition 8.8 If $(m + 1)\delta - m \not\in \mathbb{N}_0$, then $T : D_{\lambda\mu}(\mathbb{R}^m) \rightarrow D_{\lambda'\mu'}(\mathbb{R}^m)$ is $\mathfrak{sl}_{m+1}$-equivariant if and only if there exists constant $a_k \in \mathbb{R}$ such that, for each $k \in \mathbb{N}$,

$$\sigma^{\lambda'\mu'} \circ T \circ (\sigma^{\lambda\mu})^{-1} = a_k \text{id}$$

on $S^k_{\delta}(\mathbb{R}^m)$.

Proof. This follows immediately from Theorem 7.2 and Theorem 7.3. □

9 Cohomology of $\mathfrak{sl}_{m+1}$ valued in $D_{\lambda\mu}(\mathbb{R}^m)$

The computation of $H(\mathfrak{sl}_{m+1}, D_{\lambda\mu}(\mathbb{R}^m))$ is quite similar to that of $H(\mathfrak{sl}_{m+1}, D(S^k_{\delta}(\mathbb{R}^m), S^k_{\delta}(\mathbb{R}^m)))$. It uses the short exact sequence (1) and Proposition 5.1 to get first the spaces $H(\mathfrak{sl}_{m+1}, D^k_{\lambda\mu}(\mathbb{R}^m)), k \in \mathbb{N}$. In most of the cases, the cohomology of the module $C_\infty(\mathbb{R}^m, S^k_{\delta}(\mathbb{R}^m))$ vanishes. It is however necessary to compute the connecting homomorphism associated to the sequence (1) for some values of $\delta$ and $k$. This leads also to a sort of critical values for $\lambda$ and $\mu$. In view of Proposition 5.1, they are not the same when $m > 1$ as when $m = 1$.

We only summarize the results, leaving the reader to supply the proofs.

A. The case $m > 1$

Theorem 9.1 $\{m > 1\}$ If $\delta \not\in \{0, 1\}$ or if $\delta = 1$ and $\lambda \neq 0$, then $H(\mathfrak{sl}_{m+1}, D_{\lambda\mu}(\mathbb{R}^m)) = 0$.

Theorem 9.2 $\{m > 1\}$ The space $H(\mathfrak{sl}_{m+1}, D^0_{\lambda\lambda}(\mathbb{R}^m))$ is isomorphic to $(\Lambda^{\mathfrak{gl}(m, \mathbb{R})}_*(\mathbb{R}^m))_{g=\text{inv}}$. The inclusion of $D^0_{\lambda\lambda}(\mathbb{R}^m)$ into $D_{\lambda\lambda}(\mathbb{R}^m)$ induces an isomorphism in cohomology.

Theorem 9.3 $\{m > 1\}$ One has a short exact sequence

$$0 \rightarrow H(\mathfrak{sl}_{m+1}, D^0_{01}(\mathbb{R}^m)) \xrightarrow{i_*} H(\mathfrak{sl}_{m+1}, D^1_{01}(\mathbb{R}^m)) \xrightarrow{\sigma_*} H(\mathfrak{sl}_{m+1}, C_\infty(\mathbb{R}^m, S^1_{\delta}(\mathbb{R}^m))) \rightarrow 0$$

where the kernel and the quotient are isomorphic to $(\Lambda^{\mathfrak{gl}(m, \mathbb{R})}_*(\mathbb{R}^m))_{g=\text{inv}}$. The inclusion of $D^1_{01}(\mathbb{R}^m)$ into $D_{01}(\mathbb{R}^m)$ induces an isomorphism in cohomology.

Remark 9.4 If $\lambda \neq 0$, then $H(\mathfrak{sl}_{m+1}, D^0_{\lambda,\lambda+1}(\mathbb{R}^m))$ is isomorphic to $(\Lambda^{\mathfrak{gl}(m, \mathbb{R})}_*(\mathbb{R}^m))_{g=\text{inv}}$ while $H(\mathfrak{sl}_{m+1}, D^1_{\lambda,\lambda+1}(\mathbb{R}^m)) = 0$. 

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B. The case \( m = 1 \)

In this case, the “critical” values are given by \((\lambda, \mu) = (\frac{1-n}{2}, \frac{1+n}{2})\), \(n \in \mathbb{N}_0\). Moreover, the description of the cohomology is very close to that of the module \(\mathcal{D}(S_0^p(\mathbb{R}), S_0^q(\mathbb{R}))\). This follows from the fact that this module is isomorphic to \(\mathcal{D}_{\delta-p,\delta-q}(\mathbb{R})\) since \(S_0^p(\mathbb{R})\) is isomorphic to the space of \((\delta-p)\)-densities. In particular, the critical values introduced in Section 6 occur when \(p - q\) is a positive integer \(n\). They are then given by \(\delta = (p + q + 1)/2\). In this case,

\[
\delta - p = \frac{1-n}{2} \quad \text{and} \quad \delta - q = \frac{1+n}{2}.
\]

Note that these critical values were also obtained in [6], but in a completely different way.

**Theorem 9.5** If \(\delta \notin \mathbb{N}\) or if \(\delta \in \mathbb{N}_0\) and \(\lambda \neq \frac{1-\delta}{2}\), then \(H(s\ell_2, \mathcal{D}_\lambda(\mathbb{R})) = 0\).

**Theorem 9.6** The inclusion of \(\mathcal{D}_0^0(\mathbb{R})\) into \(\mathcal{D}_\lambda(\mathbb{R})\) induces an isomorphism in cohomology. In particular

\[
H^i(s\ell_2, \mathcal{D}_\lambda(\mathbb{R})) = \begin{cases} 
\mathbb{R} & \text{if } i = 0, 1 \\
0 & \text{if } i = 2, 3
\end{cases}
\]

**Theorem 9.7** If \(\lambda = \frac{1-n}{2}\) and \(\mu = \frac{1+n}{2}\) for some \(n \in \mathbb{N}_0\), then for each \(u\), one has a short exact sequence

\[
0 \to H^n(s\ell_2, \mathcal{D}_{\lambda\mu}^{n-1}(\mathbb{R})) \xrightarrow{i} H^n(s\ell_2, \mathcal{D}_{\lambda\mu}^n(\mathbb{R})) \xrightarrow{\partial^n} H^n(s\ell_2, C_\infty(\mathbb{R}, S_n^0(\mathbb{R}))) \to 0
\]

where the kernel (resp. the quotient) is isomorphic to \(\mathbb{R}\) for \(u = 1, 2\) (resp. 0, 1) and is vanishing otherwise. Moreover, the inclusion of \(\mathcal{D}_\lambda^0(\mathbb{R})\) into \(\mathcal{D}_{\lambda\mu}(\mathbb{R})\) induces an isomorphism in cohomology.

**Remark 9.8** 1) One sees in particular that \(H^3(s\ell_2, \mathcal{D}_{\lambda\mu}(\mathbb{R})) = 0\) for all \(\lambda, \mu \in \mathbb{R}\).

2) For \((\lambda, \mu) = \left(\frac{1-n}{2}, \frac{1+n}{2}\right), n \in \mathbb{N}_0\), \(H^1(s\ell_2, \mathcal{D}_{\lambda\mu}(\mathbb{R}))\) is spanned by

\[
\tau'_n \left( f \frac{d}{dt} \right) : g |dt|^{\lambda} \rightarrow \frac{df}{dt} \frac{d^{n}g}{dt^{n}} |dt|^\mu
\]

and

\[
\gamma'_n \left( f \frac{d}{dt} \right) : g |dt|^{\lambda} \rightarrow \frac{d^2f}{dt^2} \frac{d^{n-1}g}{dt^{n-1}} |dt|^\mu.
\]

This follows from Theorem 9.4. The \(\gamma'_n\)’s are exactly the cocycles used in [3] in the study of the \(s\ell_2\)-structure of \(\mathcal{D}_{\lambda\mu}(\mathbb{R})\).

3) If \(p = q+n\) and \(\delta = \frac{p+q}{2}\), through the isomorphism between \(\mathcal{D}(S_0^p(\mathbb{R}), S_0^q(\mathbb{R}))\) and \(\mathcal{D}_{\frac{p+q}{2},\frac{p+q}{2}}(\mathbb{R})\), \(\tau_n\) and \(\gamma_n\) become respectively \(\tau'_n\) and \(\gamma'_n\).
Acknowledgements

We would like to thank very much M. De Wilde, P. Mathonet and V. Ovsienko for helpful discussions.

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