Notes on parafermionic QFT’s with boundary interaction

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Abstract

The main result of these notes is an analytical expression for the partition function of the circular brane model [1] for arbitrary values of the topological angle. The model has important applications in condensed matter physics. It is related to the dissipative rotator (Ambegaokar-Eckern-Schön) model [2] and describes a “weakly blocked” quantum dot with an infinite number of tunneling channels under a finite gate voltage bias. A numerical check of the analytical solution by means of Monte Carlo simulations has been performed recently in Ref. [3]. To derive the main result we study the so-called boundary parafermionic sine-Gordon model. The latter is of certain interest to condensed matter applications, namely as a toy model for a point junction in the multichannel quantum wire [4].
1 Introduction

Since the seminal work by Fateev and Zamolodchikov [5] CFT models with parafermionic symmetry were extensively studied. In the original formulation the parafermionic current algebra appears as an extended conformal symmetry of self-dual multicritical points of $Z_k$ symmetric statistical systems [6–9]. Later this algebra has been employed in string theory [10]. Much effort has been devoted to the parafermionic models with a boundary [11–15].

In conformal field theory there is a special class of Conformally invariant Boundary Conditions (CBC’s) [11]. In the case of parafermionic models some of CBC’s are easily visualized in terms of the original $Z_k$ symmetric statistical systems [11–13]. The fluctuating variables (“spins”) in such systems can be thought of as a set of $k$ special points on the unit circle,

$$\sigma \in \left\{ \omega^n \right\}_{n=0}^{k-1} \quad \text{with} \quad \omega = e^{\frac{2\pi i}{k}}.$$

The simplest microscopic boundary conditions are those for which all spin boundary values are the same, say $\sigma_B = \omega^n$. The scaling limit of the self-dual multicritical $Z_k$ system depicted in Fig. 1 is described in terms of the minimal parafermionic model [5] on the half line $x < 0$ with the so-called fixed CBC imposed at $x = 0$. With some abuse of notation we refer to these CBC’s, as well as the corresponding RG fixed points, as $B_{n,n}$ ($n = 0, 1, \ldots k - 1$).

![Figure 1: $Z_k$ symmetric statistical system with fixed boundary condition. The spins $\sigma$ are located at the vertices of the half-infinite square lattice. For an explicit form of self-dual Boltzmann weights see Ref. [9].](image)

Yet another simple type of CBC is the “free” one. In this case, the microscopic boundary spins are free to take any value from the set (1.1). In the present work, we denote such CBC and the associated RG fixed point by $B_{\text{free}}$. The universal ratio [12–15],

$$\frac{g_{\text{free}}}{g_{\text{fixed}}} = \sqrt{k},$$

of the corresponding boundary degeneracies [16] is greater than one. Thus a unitary boundary RG flow $R^{(k)}$ from $B_{\text{free}}$ to $B_{n,n}$ seems to exist, or at least does not contradict the
The g-theorem \cite{16,17}. Indeed, $R^{(2)}$ (the Ising model case) is a textbook example of boundary flow \cite{18}. For an arbitrary $k$ the existence of $R^{(k)}$ can be advocated within the general approach \cite{20}. This boundary flow has been also argued in Refs. \cite{14,19}.

In this paper we study the QFT model underlying the boundary flow $R^{(k)}$. In Section 2.1 we introduce its Hamiltonian $H^{(k)}$. It is crucial that $H^{(k)}$ depends on an additional dimensionless angular parameter, $\theta \equiv \theta + 2\pi k$. The Hamiltonian $H^{(k)}$ suffers from a specific ultraviolet divergence, which is very similar to the “small instanton” divergence in the 2D $O(3)$ nonlinear $\sigma$-model \cite{21,22}. In Section 2.2 we consider a certain non-perturbative regularization of $H^{(k)}$ involving an additional Bose field which does not suffer from this problem. The regularized model is referred to as the Boundary Parafermionic (BP) sinh-Gordon model. The sinh-Gordon parameter $b$ plays the role of a regularization parameter and the divergence shows up when $b \to 0$. In Section 2.3 we propose an analytical expression (2.36) for the partition function of the BP sinh-Gordon model in terms of solutions of a certain second order linear differential equation. Our motivation behind Eq. (2.36) follows closely along the lines of Al.B. Zamolodchikov’s unpublished notes \cite{23}. Namely, in a view of Appendices A and B, the proposed formula is a straightforward generalization of Zamolodchikov’s result for the conventional (without parafermions) boundary sinh-Gordon model. Unfortunately, even though Eq. (2.36) can be easily guessed, its derivation from first principles is still lacking. For this reason we perform some consistency checks for (2.36) in Appendix C.

The main quantity we are interested in, is the partition function $Z_{\theta}^{(k)}$ corresponding to the Hamiltonian $H^{(k)}$. It can be extracted from the limiting $b^2 \to +0$ behavior of the BP sinh-Gordon partition function. An analytical expression for $Z_{\theta}^{(k)}$ is given in Section 2.4. Using this result, we describe a qualitative picture of the boundary flow governed by $H^{(k)}$ in Section 2.5. It turns out that for each $\theta$ from the open segment $(2n-1)\pi < \theta < (2n+1)\pi$, the boundary flow is terminated at the fixed point $B_{n,n}$. At the same time the RG trajectories associated with $\theta = \pi(2n+1)$ with $n = 1, \ldots, k-1$ posses nontrivial infrared fixed points whose boundary degeneracies are all the same and equal to $2 \cos(\frac{\pi}{k+2})$ fixed. Hence, the infrared physics of the boundary flow $R^{(k)}$ essentially depends on the parameter $\theta$. For this reason we shall include an additional symbol in the boundary flow notation: $R^{(k)}_{\theta}$.

In the rest of the main body of the paper we focus on the large-$k$ limit. In Section 3 we identify $Z_{\theta} = \lim_{k \to \infty} Z_{\theta}^{(k)}$ with the partition function of the circular brane model \cite{1}, and the parameter $\theta$ is understood as the topological angle. In Section 4 a particular ultraviolet regularization of the circular brane model is considered. The regularized theory is known as the dissipative quantum rotator model \cite{2} and commonly used in describing the Coulomb charging in quantum dots. The applicability of our results to the quantum dot in weakly blockaded regime is briefly discussed in Conclusion.

Numerous technical details of the work are relegated to appendices. In Appendices A and B we study the model which, from a very formal point of view, is the BP sinh-Gordon model considered at purely imaginary values of the sinh-Gordon parameter $b$. In essence, these appendices constitute work done in 1999 and which was never published before. It follows closely the approach developed in the series of papers \cite{24–28}. In particular, some of the key statements involved have been already proven. In those cases we omit the proofs and refer the reader for details to \cite{26–28}. Among results obtained in Appendices A and B is equation (A.80), which is interesting in its own right. It generalizes a conjecture of
P. Fendley and H. Saleur [4] for the DC conductance in the multichannel quantum wire to the case of finite temperature. It should be emphasized that, although the style of our presentation is somewhat sketchy, it seems possible to transform the derivation of (A.80) into a rigorous mathematical proof.

In Appendix C we perform some consistency checks of the proposed exact analytical expression for the partition function of the BP sinh-Gordon model. Finally, for reference purposes, we collect some formulas concerning the low-temperature expansion of the partition function $Z_{\theta}^{(k)}$ in Appendix D.

2 Boundary flow $\mathcal{R}_{\theta}^{(k)}$

2.1 Hamiltonian $H_{\theta}^{(k)}$

Let us consider the minimal parafermionic model on a half lane, $x \leq 0$, constrained by the free CBC. By employing the magic of modular transformation [13] to the parafermionic characters [10, 29], one can observe operators of the scaling dimensions

$$\Delta_n = \frac{n(k - n)}{k} \quad (n = 0, \ldots, k - 1)$$

among boundary fields of the theory. Below they are denoted as

$$\{ \Psi_n(\tau) \}_{n=0}^{k-1}.$$  \hspace{1cm} (2.2)

The variable $\tau$ labels points along the boundary $x = 0$ (see Fig. 1). Notice that $\Psi_0$ coincides with the unit operator $I$. We also reserve special notations for the two relevant boundary fields of the lowest nontrivial scaling dimension $\Delta_1 = \Delta_{k-1} = 1 - \frac{1}{k}$, namely

$$\Psi_+ \equiv \Psi_1, \quad \Psi_- \equiv \Psi_{k-1}.$$ \hspace{1cm} (2.3)

The boundary fields $\Psi_n(\tau)$ can be thought of as Fateev-Zamolodchikov chiral parafermionic currents brought to the boundary point $\tau$.\footnote{A precise meaning of this statement is discussed in Section B.2.} Thus, their boundary Operator Product Expansions (OPE’s) have a form similar to those of the chiral parafermionic algebra:

$$\Psi_j(\tau_1) \Psi_m(\tau_2) = \sum_{j+m} C_{jm} \Psi_{j+m}(\tau_2) + \cdots.$$ \hspace{1cm} (2.4)

Here the sum $j + m$ should be understood modulo $k$ and the structure constants are the same as in [5]. With a properly normalized $\Psi_j$, the structure constants read explicitly as

$$C_{jm} = \frac{(j + m)!k!(k - j)!(k - m)!}{j!m!k!(k - m - j)!}.$$ \hspace{1cm} (2.5)

Let $\mathcal{H}_{\text{free}}^{(k)}$ and $\mathcal{H}_n^{(k)}$ be the spaces of states of minimal parafermionic models on the half line constrained, respectively, by the free ($\mathcal{B}_{\text{free}}$) and fixed ($\mathcal{B}_{n,n}$) CBC’s. Obviously, all the spaces $\mathcal{H}_n^{(k)}$ are formally isomorphic for different integers $n$:

$$\mathcal{H}_0^{(k)} \simeq \mathcal{H}_1^{(k)} \simeq \cdots \simeq \mathcal{H}_{k-1}^{(k)}.$$ \hspace{1cm} (2.6)
By virtue of the simple microscopic nature of the fixed and free CBC’s (see Introduction), it is expected that

$$\mathcal{H}_{\text{free}}^{(k)} \subset \bigoplus_{n=0}^{k-1} \mathcal{H}_n^{(k)}.$$  \hfill (2.7)

Then, the fields $\Psi_{\pm}(\tau)$, being considered as operators acting in $\mathcal{H}_{\text{free}}^{(k)}$, intertwine the linear subspaces $\mathcal{H}_n^{(k)} \cap \mathcal{H}_{\text{free}}^{(k)}$ and $\mathcal{H}_{n\pm1\text{(mod} k)}^{(k)} \cap \mathcal{H}_{\text{free}}^{(k)}$. Heuristically, one can think about $\Psi_{\pm}(\tau)$ as operators creating discontinuities in the fixed boundary condition $\mathcal{B}_{n,n} \rightarrow \mathcal{B}_{n\pm1,n\pm1}$ at the boundary point $\tau$ (see heuristic Fig.2).

Figure 2: Discontinuity diagram in the fixed boundary conditions corresponding to the insertion of the boundary parafermions.

In this work we study the model described by the Hamiltonian

$$H_\theta^{(k)} = H_{\text{free}}^{(k)} - \mu \left[ e^{\frac{i\theta}{k}} \Psi_+ + e^{-\frac{i\theta}{k}} \Psi_- \right],$$  \hfill (2.8)

where $H_{\text{free}}^{(k)}$ is the Hamiltonian of the minimal parafermionic model subject to the free CBC, $\Psi_{\pm} \equiv \Psi_{\pm}(0)$, and, hence,

$$H_\theta^{(k)} : \mathcal{H}_{\text{free}}^{(k)} \rightarrow \mathcal{H}_{\text{free}}^{(k)}.$$  \hfill (2.9)

The parameter $\mu$ in (2.8) carries the dimension $[\text{length}]^{-\frac{1}{2}}$, so that $\mu^k$ sets the physical energy scale (the Kondo temperature) in the theory. It is useful to keep in mind that the OPE’s (2.4), (2.5) specify the normalization of the boundary fields (2.2) modulo global $\mathbb{Z}_k$ transformations,

$$\Psi_j \rightarrow \omega^{aj} \Psi_j \quad (a = 1, \ldots, k-1).$$  \hfill (2.10)

For this reason the angular parameter $\theta \equiv \theta + 2\pi k$ in Eq. (2.8) is not completely unambiguous; it is defined modulo transformations

$$\theta \rightarrow \theta - 2\pi a \quad (a = 1, \ldots, k-1).$$  \hfill (2.11)
The partition function

\[ Z^{(k)}_\theta = \text{Tr}_{\mathcal{H}_{\text{free}}} \left[ \exp \left( -\frac{H^{(k)}_\theta}{T} \right) \right] \]  

(2.12)
is an even $2\pi$-periodic function of $\theta$. Therefore the uncertainty in $\theta$ does not affect the thermodynamics of the model, and we may consider $\theta$ in the domain $0 \leq \theta \leq \pi$ only.

2.2 The BP sinh-Gordon model

The key idea behind our calculation of (2.12) is that $H^{(k)}_\theta$ appears in a certain limit of a more general Hamiltonian, involving additional bosonic degrees of freedom. The generalized model is referred to below as the Boundary Parafermionic (BP) sinh-Gordon model.

Let $\Phi(x)$ and $\Pi(x)$ be Bose field operators obeying canonical commutation relations

\[ [\Pi(x), \Phi(x')] = -2\pi i \delta(x - x') . \]

(2.13)
The Hamiltonian of the BP sinh-Gordon model is described as follows:

\[ H^{(k)}_{\text{bshg}} = H^{(k)}_{\text{free}} + H^{(\Phi)}_{\text{free}} + h \Phi_B - \mu \left( \Psi_+ e^{\frac{b}{\sqrt{k}} \Phi_B} + \Psi_- e^{-\frac{b}{\sqrt{k}} \Phi_B} \right) , \]

(2.14)

where

\[ H^{(\Phi)}_{\text{free}} = \frac{1}{4\pi} \int_{-\infty}^{0} dx \left( \Pi^2 + (\partial_x \Phi)^2 \right) . \]

(2.15)

We do not impose any constraint on the boundary values of the canonical fields (2.13). Hence the Hamiltonian (2.14) acts in the space

\[ \mathcal{H} = \mathcal{H}^{(k)}_{\text{free}} \otimes \mathcal{H}^{(\Phi)}_{\text{free}} , \]

(2.16)

where $\mathcal{H}^{(k)}_{\text{free}}$ is the same as in Eq.(2.7) and $\mathcal{H}^{(\Phi)}_{\text{free}}$ is the space of states of the one component boson $\Phi(x)$ with no constraint at the boundary $x = 0$. The parameter $h$ in (2.14) plays the role of an external field coupled to the boundary values $\Phi_B \equiv \Phi(x)|_{x=0}$. To give an unambiguous definition of the coupling constant $\mu$, one should specify the normalization of the boundary fields

\[ V_\pm = \Psi_\pm e^{\frac{b}{\sqrt{k}} \Phi_B} . \]

(2.17)

In what follows these composite fields are assumed to be canonically normalized with respect their short-distance behavior, i.e.

\[ V_\pm(\tau_1) V_\mp(\tau_2) = |\tau_1 - \tau_2|^{-2d} \times I + \ldots , \]

(2.18)

and

\[ V_\pm(\tau_1) V_\pm(\tau_2) \cdots V_\pm(\tau_k) = \frac{k!}{k^{\frac{k}{2}}} \prod_{i<j} |\tau_i - \tau_j|^{2(1-d)} \times e^{\pm b \sqrt{k} \Phi_B} + \ldots , \]

(2.19)
while
\[ \pm \sqrt{k}\Phi_B(\tau_1) e^{\pm \sqrt{k}\Phi_B(\tau_2)} = |\tau_1 - \tau_2|^2 b^2 \times I + \ldots. \]  
(2.20)

Here
\[ d = 1 - \frac{1}{k} - \frac{b^2}{k} \]  
(2.21)
is the scaling dimension of the boundary fields \( V_\pm \). Also notice that the factor \( k^! / k^{k/2} \) in (2.19) is a product \( \prod_{j=1}^{k-1} C_{1j} \) of the structure constants (2.5).

Formally, when the dimensionless parameter \( b^2 \) in (2.14) is equal to zero, the Hamiltonian \( H_{\text{bsg}}^{(k)} \) becomes the sum of a free Bose Hamiltonian and \( H_\theta^{(k)} \). In fact, because of the particular divergence generated by the interaction term in (2.14) at \( b^2 \to 0 \), the limiting procedure is not a trivial matter. The divergence appears due to the leading term in the OPE (2.19). Keeping this in mind, we rewrite the Hamiltonian (2.14) as a sum of two operators
\[ H_{\text{bsg}}^{(k)} = H_1[\Phi] + H_2[\Psi, \Phi_B], \]  
(2.22)
with
\[ H_1[\Phi] = H_\text{free}^{(\Phi)} + h \Phi_B + \frac{2}{kb^2} \left( A \cosh \left( b\sqrt{k}\Phi_B \right) - B I \right) \]  
(2.23)
and
\[ H_2[\Psi, \Phi_B] = H_\text{free}^{(k)} - \mu \left( \Psi_+ e^{\frac{b}{\sqrt{k}}\Phi_B} + \Psi_- e^{\frac{-b}{\sqrt{k}}\Phi_B} \right) - \frac{2}{kb^2} \left( A \cosh \left( b\sqrt{k}\Phi_B \right) - B I \right). \]  
(2.24)

Our basic assumption is that the coefficients \( A \) and \( B \) for the local boundary counterterms in (2.24) can be adjusted in such a way that the limit of the operator \( H_2 \) does exist when \( b^2 \to +0 \). In other words, we expect that the singular behavior at \( b^2 \to +0 \) is fully controlled by the operator \( H_1 \), which is just a Hamiltonian of the conventional boundary sinh-Gordon model.

The \( b^2 \to +0 \) limit is the classical limit for \( H_1 \). Indeed, after the field redefinition \( \varphi = b\sqrt{k} \Phi \) this becomes particularly striking. Since quantum fluctuations of \( \varphi \) are suppressed as \( b^2 \to 0 \) we may apply the saddle point approximation to account for their contribution. The bulk Hamiltonian for \( \varphi \) is free and massless, and, consequently, the saddle point is achieved for some constant classical field configuration which we denote as \( i\theta \). Now, since all matrix elements of \( H_2 \) are expected to be finite at \( b^2 = 0 \), we can safely replace \( \Phi_B \) in (2.24) by the \( c \)-number \( \frac{i\theta}{b\sqrt{k}} \). Obviously, the limit
\[ [H_\theta^{(k)}]_{\text{reg}} = \lim_{b^2 \to +0} H_2[\Psi, \frac{i\theta}{b\sqrt{k}}] \]  
(2.25)
should be treated as the regularized version of the Hamiltonian \( H_\theta^{(k)} \).

The saddle point configuration \( \Phi = \frac{i\theta}{b\sqrt{k}} \) corresponds to a minimum of the boundary potential \( U[\Phi_B] \), which is a sum of the last two terms in the boundary sinh-Gordon Hamiltonian (2.23). The boundary potential is minimized at
\[ \sin(\theta) = i b\sqrt{k} \frac{h}{2A}. \]  
(2.26)
We should also take into account the effect of Gaussian fluctuations around the classical saddle-point configuration. Let us write \( \Phi(x) = i \frac{\delta}{\sqrt{k}} + \delta \Phi(x) \). Then, expanding the boundary potential near its minima, one gets

\[
U[\Phi_B] = \frac{2}{k_3^2} (A S(\theta) - B) I + A \cos(\theta) (\delta \Phi_B)^2 + O((\delta \Phi_B)^3),
\]

with

\[
S(\theta) = \theta \sin(\theta) + \cos(\theta).
\]

Let \( Z_{\text{Gauss}} \) be the partition function of the Gaussian theory with the quadratic boundary potential, \( Z^{(k)}_\theta(\kappa) \) be the partition function corresponding to the regularized Hamiltonian \( \mathcal{H}_\theta \), and \( Z^{(k)}_{\text{bsgh}}(h) \) be the partition function of the BP sinh-Gordon model:

\[
Z^{(k)}_{\text{bsgh}}(h) = \text{Tr} \left[ \exp \left( -\mathcal{H}^{(k)}_{\text{bsgh}} / T \right) \right],
\]

where the trace is taken over the space of states \( \mathcal{H}_{\text{bsgh}} \). Then, the above consideration implies the following relation between these three partition functions as \( b^2 \to +0 \):

\[
Z^{(k)}_{\text{bsgh}} \left( \frac{2A}{16\pi k} \sin \theta \right) \bigg|_{b^2 \to +0} \to e^{-\frac{2}{k_3^2} \left( \frac{A}{T} S(\theta) - \frac{\theta}{T} \right)} Z_{\text{Gauss}} Z^{(k)}_\theta.
\]

Finally, according to Ref. [32],

\[
Z_{\text{Gauss}} = g_D \frac{\Gamma(1 + 2 A T \cos \theta)}{\sqrt{4 \pi \frac{A}{T} \cos(\theta)}} \left( \frac{C}{T} \right)^{-2 \frac{A}{T} \cos(\theta)}.
\]

Here \( g_D = 2^{-\frac{1}{4}} \) is the boundary degeneracy associated with the Dirichlet CBC of the uncompactified Bose field and \( C \) is some nonuniversal dimensionful constant.

### 2.3 Exact expression for the partition function \( Z^{(k)}_{\text{bsgh}}(h) \)

Appendixes A-C are meant to demonstrate that the partition function \( \mathcal{Z}_{\text{bsgh}}^{(k)}(h) \) can be expressed in terms of the Wronskian \( W[\Theta_+, \Theta_-] = \Theta_+ \partial_u \Theta_- - \Theta_- \partial_u \Theta_+ \) of two particular solutions of the Schrödinger equation

\[
\left[ -\partial_u^2 + \kappa^2 \left( e^{-b u} + e^{b u} \right)^k + \xi^2 \right] \Theta(u) = 0,
\]

where

\[
Q = b + b^{-1}.
\]

More precisely, let \( \Theta_+ \) and \( \Theta_- \) be regular solutions of \( \mathcal{H} \) as \( u \to +\infty \) and \( u \to -\infty \), respectively. They are specified unambiguously via their own WKB asymptotics:

\[
\Theta_{\pm}(u) \to (2\kappa)^{-\frac{1}{2}} \exp \left( F(b^{\pm1} | \pm u) \right) \quad \text{as} \quad u \to \pm \infty,
\]
where $F$ is expressed in terms of the conventional hypergeometric function,

$$F(b \mid u) = -\frac{k}{4Q} u - \frac{2bQ}{k} e^{\frac{k}{2}u} \, _2F_1\left(-\frac{k}{2Q}, \frac{-k}{2}; 1 - \frac{k}{2Q} \mid -e^{-u}\right).$$  \hspace{1cm} (2.35)$$

Then, the partition function (2.29) is given by

$$Z_{bshg}^{(k)}(h) = gD \, g_{\text{fixed}} \, W[\Theta_+, \Theta_-].$$ \hspace{1cm} (2.36)

The parameters of the Hamiltonian (2.14) are related to $\xi$ and $\kappa$ in (2.32) as follows:

$$\xi = \frac{\sqrt{k}}{2Q} \frac{h}{T},$$ \hspace{1cm} (2.37)

$$\kappa = \frac{1}{2\pi T} \frac{k}{bQ} \left[\frac{2\pi \mu}{\sqrt{k}T(1 - \frac{k^2}{k})}\right]^{\frac{1}{2Q}}.$$ \hspace{1cm} (2.38)

Notice that the Wronskian $W[\Theta_+, \Theta_-]$ can be viewed as a spectral determinant of the Schrödinger operator (2.32). Indeed the spectrum of (2.32) is simple, discrete and positive:

$${\text{Spect}} = \{\xi_n^2\}_{n=1}^{\infty} \quad (\xi_n^2 > \xi_m^2 > 0 \quad \text{if} \quad n > m),$$ \hspace{1cm} (2.39)

and, as follows from the Bohr-Sommerfeld quantization condition,

$$\xi_n = \frac{\pi k}{Q^2} \frac{n}{\log n} \left[1 + O\left(\frac{\log(\log n)}{\log n}\right)\right] \quad \text{as} \quad n \to \infty.$$ \hspace{1cm} (2.40)

Therefore the spectral determinant can be defined through the equation

$$D(\kappa, \xi) = D_0(\kappa) \prod_{n=1}^{\infty} \left(1 - \frac{\xi_n^2}{\xi_n^2}\right),$$ \hspace{1cm} (2.41)

where $D_0(\kappa)$ is some overall $\xi$-independent factor. At the same time, $W[\Theta_+, \Theta_-]$ is an entire function of the variable $\xi^2$ which shares the same set of zeros as (2.41). Using the standard WKB methods [33], one can show that $\log W[\Theta_+, \Theta_-] = o(\xi^2)$ as $\xi^2 \to \infty$. Hence, as follows from the Liouville theorem, the ratio $D/W$ does not depend on $\xi$, and we can always adjust the overall normalization factor $D_0$ in (2.41) to make it equal to one, i.e.,

$$D(\kappa, \xi) = W[\Theta_+(u), \Theta_-(u)].$$ \hspace{1cm} (2.42)

\hspace{1cm} 2Eq. (2.36) is a generalization of the original proposal from [23] to the case $k > 1$. For $k = 1$ Al.B. Zamolodchikov presented a set of convincing arguments in support of (2.36). In particular, the Wronskian $W[\Theta_+, \Theta_-]$ was expressed in terms of the solution of the massless thermodynamic Bethe ansatz equation associated with the boundary sinh-Gordon model [23].

3Let $\xi^2$ be a zero of $W[\Theta_+, \Theta_-]$. In this case the solution $\Theta_+(u)$ and $\Theta_-(u)$ are linearly dependent. It just means that the Schrödinger equation possesses a solution which is both regular at $x \to +\infty$ and $x \to -\infty$, hence $\xi^2 \in \text{Spect}$. It is also obvious that $W[\Theta_+, \Theta_-]|_{\xi^2=\xi_n^2} = 0$. 

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2.4 Exact expression for the partition function $Z_\theta^{(k)}$

With the exact expression for $Z_{\text{bshg}}(h)$, we can now check the semiclassical behavior (2.30).

Consider the spectral determinant $D(\kappa, \kappa \sin \theta)$ in the limit $b^2 \to +0$. There is no problem with the limit for the differential equation (2.32) and for the solution $\Theta_-$ (2.34). Namely,

$$\Theta_+(u) \to \exp \left[ \left( \frac{2}{kb^2} + c_+ \right) \kappa \sin^2 \left( \frac{\pi(k-1)}{2} \right) \right] \Xi_+(u) \quad \text{as} \quad b^2 \to +0 \, ,$$

(2.43)

where $\Xi_+$ is a solution of

$$\left[ -\partial_u^2 + \kappa^2 \left( 1 + e^u \right)^k - \kappa^2 \sin^2(\theta) \right] \Xi(u) = 0$$

(2.44)

defined by the asymptotic condition

$$\Xi_+(u) \to \frac{e^{-u+\frac{k}{2}}}{\sqrt{2\kappa (1+e^u)^\frac{k}{2}}} \exp \left\{ -\kappa \int_0^u \frac{dz}{z} (1+z)^{\frac{k}{2}} - 1 \right\} \quad \text{as} \quad u \to +\infty \, .$$

(2.45)

The constant $c_+$ in (2.43) reads explicitly

$$c_+ = \psi \left( 1 + \frac{k}{2} \right) + \gamma_E \, ,$$

(2.46)

where $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ and $\gamma_E$ is Euler’s constant. Notice that the argument of the exponential function in (2.43) vanishes for any odd $k$.

The asymptotic condition (2.34) for $\Theta_-$ is singular as $b^2 \to +0$. For this reason the limiting behavior of this solution is a slightly delicate issue. It can be analyzed using the WKB approximation for $\Theta_-$:

$$\Theta_-^{(\text{wkb})}(u) = \frac{1}{\sqrt{2\kappa \mathcal{P}(u)}} \exp \left\{ \kappa \int_{-\infty}^u dv \left( \mathcal{P}(v) - e^{-\frac{2b}{\kappa} v} - \frac{2Q_k}{kb} e^{-\frac{b}{2k^2} u} \right) \right\} \, ,$$

(2.47)

where $\mathcal{P}(x)$ is given by

$$\mathcal{P}(u) = \sqrt{\left( e^{-\frac{bu}{2k^2}} + e^{\frac{bu}{2k^2}} \right)^k - \sin^2(\theta)} \quad \text{with} \quad \sin(\theta) = \xi / \kappa \, .$$

(2.48)

Notice that the subtraction term $e^{-\frac{b}{2k^2} u}$ in the integrand comes from the asymptotic conditions (2.34) and ensures convergence of the integral in (2.47). Consider now the argument $\{ \ldots \}$ of the exponential in (2.47) as $b^2 \to +0$. It is easy to see that $\{ \ldots \} \to F(u) - \frac{2}{kb^2} \kappa S(\theta)$, where $S(\theta)$ is given by (2.28), while $F(u) \to u \kappa \cos(\theta)$ as $u \to -\infty$. Then

$$\Theta_-(u) \to \Xi_-(u) \frac{e^{-c_- \kappa \cos \theta}}{\sqrt{2\kappa \cos(\theta)}} e^{-\frac{2}{kb^2} \kappa S(\theta)} \quad \text{as} \quad b^2 \to +0 \, ,$$

(2.49)

where $\Xi_-$ is a solution of the differential equation (2.44) such that

$$\Xi_-(u) \to e^{(u+c_-)\kappa \cos \theta} \quad \text{as} \quad u \to -\infty \, ,$$

(2.50)

and $c_-$ is an arbitrary constant. It will be convenient to choose

$$c_- = \psi \left( \frac{k}{2} \right) + \gamma_E + 2 \log 2 \, .$$

(2.51)
We emphasize that $\Xi_-$ can be defined by means of the asymptotic condition (2.50) for $0 \leq \theta < \frac{\pi}{2}$ only. For $\frac{\pi}{2} < \theta \leq \pi$, the solution $\Xi_-$ grows at large negative $u$ and the asymptotics (2.45) does not define $\Xi_-$ unambiguously. Fortunately, the function $\frac{\Xi_-}{\Gamma(1 + 2\kappa \cos \theta)}$ is an entire function of the complex variable $\zeta = \cos(\theta)$ for $u < 0$. So the solution $\Xi_-(u)$ can be introduced within $\frac{\pi}{2} \leq \theta \leq \pi$ through the analytic continuation with respect to the variable $\theta$ from the domain $0 \leq \theta < \frac{\pi}{2}$.

In a view of relations (2.38), the singular behavior of $D(\kappa, \kappa \sin \theta)$ as $b^2 \to +0$ matches exactly the structure (2.30) provided

$$A = T\kappa, \quad B = T\kappa \sin^2(\frac{\pi(k-1)}{2}) . \quad (2.52)$$

Furthermore, comparing the pre-exponent in (2.30) with a similar factor for $D(\kappa, \kappa \sin \theta)$, we finally obtain the partition function for the regularized Hamiltonian (2.25)

$$\tilde{Z}_{\theta}^{(k)}(\kappa) = g_{\text{fixed}} \kappa^{2\kappa \cos \theta} \sqrt{2\pi} \frac{W[\Xi_+, \Xi_-]}{\Gamma(1 + 2\kappa \cos \theta)} . \quad (2.53)$$

Two important comments are in order here. First, $\tilde{Z}^{(k)}$ is the function of the dimensionless variable $\kappa$, which is in fact the inverse temperature measured in the units of the RG invariant scale $\bar{E}_*$:

$$\kappa = \frac{E_*}{T} . \quad (2.54)$$

As follows from Eq. (2.38) considered at $b^2 = 0$

$$E_* = \frac{k^{1-\frac{k}{2}}}{2\pi} \left[ \frac{2\pi \mu}{\Gamma(1 - \frac{1}{k})} \right]^k . \quad (2.55)$$

The parameter $\mu$ should be understood now as the coupling of the Hamiltonian (2.25).

Second, the partition function for the regularized Hamiltonian (2.25) is defined modulo an overall factor $c^{\kappa \cos \theta}$, where $c_-$ is an arbitrary constant. By fixing the value of $c_-$ via Eq. (2.51) we have enclosed a particular normalization condition for the regularized ground state energy

$$\bar{E}_{\theta}^{(k)} = - \lim_{T \to 0} \left[ T \log \tilde{Z}_{\theta}^{(k)} \right] . \quad (2.56)$$

Namely, if $c_-$ is chosen as in (2.51), then $\bar{E}_{\theta}^{(k)}$ satisfies the condition

$$\frac{\partial^2}{\partial \theta^2} \bar{E}_{\theta}^{(k)} \bigg|_{\theta=0} = 0 . \quad (2.57)$$

In the case of odd $k$ the regularized ground state energy satisfying (2.57) is unambiguously defined. In particular, the dimensionful constant

$$\bar{E}_{\theta}^{(k)} \bigg|_{\theta=0} = -2 \left( 1 - \frac{1}{k} \right) E_* \quad (2.58)$$

can be viewed as a universal physical energy scale in the model (2.8). Since the constant $B$ (2.52) does not vanish for even values of $k$, the regularized Hamiltonian (2.25) contains an
additional counterterm of unit operator. In this case only the difference \( \bar{E}_\theta^{(k)} - \bar{E}_{\theta=0}^{(k)} \) turns out to be a universal quantity. For even \( k \) we choose Eq. (2.58) as an extra normalization condition for the regularized ground state energy.

Whereas \( \bar{E}_\theta^{(k)} \) satisfying (2.57) and (2.58) is a universal scaling function, the ground state energy \( E_\theta^{(k)} \) for the original Hamiltonian \( H_\theta^{(k)} \) (2.8) does depend on details of the regularization procedure. More precisely, the above consideration suggests (see Eqs. (2.24) and (2.25)) the following general form for \( E_\theta^{(k)} \)

\[
E_\theta^{(k)} = -E_s \left[ L_1^{(k)} \cos(\theta) - L_2^{(k)} \sin^2 \left( \frac{\pi (k-1)}{2} \right) \right] + \bar{E}_\theta^{(k)},
\]

where \( L_{1,2}^{(k)} \) are some dimensionless nonuniversal constants which are expected to absorb all divergences in the theory (2.8). Hence, the partition function (2.12) has the form

\[
Z_\theta^{(k)} = \exp \left[ \kappa L_1^{(k)} \cos(\theta) - \kappa L_2^{(k)} \sin^2 \left( \frac{\pi (k-1)}{2} \right) \right] \tilde{Z}_\theta^{(k)}(\kappa),
\]

where \( \tilde{Z}_\theta^{(k)}(\kappa) \) is given by Eq. (2.53). For the regularization described above \( L_{1,2}^{(k)} = \frac{1}{kb^2} + O(1) \) with \( b \to +0 \). In general, the divergent constants \( L_{1,2}^{(k)} \) are expressed in terms of some UV cutoff equipped with the theory.

One should note that some particular cases of \( Z_\theta^{(k)} \) have been already discussed in the literature. E.g., the case \( k = 2 \) was studied in Ref. [18]. The differential equation (2.44) for \( k = 2 \) can be transformed to Kummer’s equation, so that \( \tilde{Z}_{\theta}^{(2)} \) has the especially simple form,

\[
\tilde{Z}_{\theta}^{(2)}(\kappa) = \frac{g_{\text{fixed}} \sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + 2\kappa \cos^2\left(\frac{\theta}{2}\right)\right)} \left(2\kappa\right)^{2\kappa \cos^2\left(\frac{\theta}{2}\right)} e^{-\kappa}.
\]

Eq. (2.61) is in complete agreement with the result of [18]. It should be also mentioned that for \( \sin(\theta) = 0 \) the partition function is described in terms of the differential equation originally introduced in [31] in the context of the so-called paperclip boundary flow. The difference between \( \theta = 0 \) and \( \theta = \pi \) appears in a choice of \( \Xi_- \)-solution. Namely, \( \Xi_-^{(\theta=0)} \) and \( \Xi_-^{(\theta=\pi)} \) are the two Bloch-wave solutions of (2.44) with \( \sin(\theta) = 0 \) [34]. We refer the reader to those papers for a wealth of data about the differential equation (2.44) with \( \sin(\theta) = 0 \). In particular, the thermodynamic ansatz equations describing the partition function \( Z_{\theta}^{(k)} \) (2.12) for \( \theta = 0 \) and \( \theta = \pi \) can be found in Ref. [34].

### 2.5 Qualitative pattern of the boundary flow \( \mathcal{R}_{\theta}^{(k)} \)

Our prime interest here is in the high- and low-temperature behaviors of \( Z_{\theta}^{(k)} \). With (2.58) it is straight-forward to check that the boundary entropy \( S_{\theta}^{(k)} = \left(1 + T \frac{d}{dT}\right) \log Z_{\theta}^{(k)} \), has the following \( \theta \)-independent limit as \( T \to \infty \):

\[
\lim_{T \to \infty} S_{\theta}^{(k)}(T) = \log(g_{\text{free}}).
\]

\footnote{A very similar phenomenon was discussed in [30] for the minimal parafermionic models perturbed by the parafermionic currents in the bulk.}
At the same time,

\[
\lim_{T \to 0} S^{(k)}_\theta (T) = \begin{cases} 
\log(g_{\text{fixed}}) & \text{as } 0 \leq \theta < \pi \\
\log(g_2) & \text{as } \theta = \pi 
\end{cases} .
\]  

(2.63)

Here the boundary degeneracy \( g_2 \) is given by the formula

\[
g_s = g_{\text{fixed}} \frac{\sin \left( \frac{k \pi}{k+2} \right)}{\sin \left( \frac{\pi}{k+2} \right)} ,
\]  

(2.64)

with \( s = 2 \). Whereas (2.62) is just a simple consequence of the fact that \( R^{(k)}_\theta \) starts from \( B_{\text{free}} \), Eq. (2.63) is much less trivial. It shows that within \( 0 \leq \theta < \pi \) the infrared behavior of the theory is defined by the fixed CBC, but for \( \theta = \pi \) the RG flow is terminated at some nontrivial fixed point.

The CBC associated with the boundary degeneracy \( g_2 \) can be easily identified. Recall that the minimal parafermionic model possesses the Cardy states [13] with the additional conformal symmetry, namely \( W_k \) symmetry introduced in Refs. [35–37].\(^5\) They are in one-to-one correspondence with the \( k(k+1)/2 \) number of \( W_k \) primary fields of the model. Let us denote the CBC’s corresponding to the \( W_k \) invariant Cardy states by \( B_{l,n} \) with \( 0 \leq n \leq l \leq k - 1 \). Their boundary degeneracies are given by Eq. (2.64) with \( s = l - n + 1 \).\(^6\) In particular the boundary degeneracies associated with \( B_{0,1}, \ldots, B_{k-2,k-1} \) and \( B_{k-1,0} \) are all the same and given by the constant \( g_2 = 2 \cos \left( \frac{\pi}{k+2} \right) g_{\text{fixed}} \).

Now, in view of the global \( \mathbb{Z}_k \) invariance (2.10), (2.11), it is easy to imagine the whole qualitative picture of the boundary RG flow \( R^{(k)}_\theta \) for \( 0 \leq \theta < 2\pi k \). In the case \( k = 4 \) it is depicted in Fig. 3. Among the flow lines shown in Fig. 3 there are trajectories adjacent to the big circle associated with zero temperature. In Appendix D we present an exact formula (D.3) for the normalized ground state energy (2.56), (2.57). It turns out that \( E^{(k)}_\theta \) is a singular function at \( \theta = \pi \) (see formula (D.2)):

\[
E^{(k)}_\theta \propto |\theta - \pi|^{-\frac{1}{d_\varepsilon}} ,
\]  

(2.65)

where

\[
d_\varepsilon = \frac{2}{k + 2} .
\]  

(2.66)

It is worth noting that \( d_\varepsilon \) is the scaling dimension of the so-called first energy boundary operator, \( \varepsilon \) [5]. Hence, an effective Hamiltonian, describing the system in a vicinity of the nontrivial infrared fixed points \( B_{n+1,n} \), contains this relevant boundary field along with irrelevant conformal and \( W_k \) descendents of \( \varepsilon \) and \( I \). Notice that the coupling constant for \( \varepsilon \) vanishes at \( \theta = \pi, 3\pi \ldots \), so the leading low-temperature corrections come from the first \( W_3 \) descendent of \( \varepsilon \) of the scaling dimension \( 1 + d_\varepsilon \). The low-temperature expansion of (2.54) for the radial trajectories \( \theta = \pi, 3\pi \ldots \) was studied in Ref. [34]. More details about the low-temperature expansions of \( Z^{(k)}_{\theta} \) for \( \theta \neq \pi \) are presented in Appendix D.

\(^5\)See Section B.2 for the brief review of \( W_k \) symmetry and \( W_k \) invariant CBC’s in the minimal parafermionic models.

\(^6\)In string language [15], \( B_{l,n} \) are the so-called “A-branes”. The fixed CBC’s \( B_{n,n} \) are D0-branes coinciding with the points \( \omega^n \) on the unit circle. The CBC’s \( B_{l,n} \) with \( l > n \) are interpreted as \( D1 \)-branes, the chords stretched between the points \( \omega^n \) and \( \omega^l \). The “lightest B-brane” from [15] is identified with the free CBC \( B_{\text{free}} \).
Figure 3: The boundary flow $\mathcal{K}_\theta^{(k=4)}$ in the 2D plane of coupling constants. Polar coordinates $(\rho, \phi)$ in this plane are some properly chosen “running” couplings, $\rho = \rho(T, \theta)$, $\phi = \phi(T, \theta)$, such that $\lim_{T \to \infty} \rho(T, \theta) = 0$ and $\rho_{\text{max}} = \lim_{T \to 0} \rho(T, \theta)$. The parameter $\theta$ is constant along each trajectory at $T \neq 0$.

3 Circular brane partition function

3.1 $Z^{(k)}_\theta$ in the limit $k \to \infty$

The regularized partition function (2.53) possesses a finite limit as $k \to \infty$ provided the physical energy scale $E_\star$ is kept fixed. Indeed, let us shift the variable $u$ in the differential equation (2.44):

$$u = y - \log(k).$$

Then, the large-$k$ limit brings (2.44) to the form,

$$\left[ - \partial_y^2 + \kappa^2 \exp(e^y) - \kappa^2 \sin^2(\theta) \right] \Xi(y) = 0.$$  

The solutions $\Xi_+$ and $\Xi_-$ remain finite for any fixed $y$ when $k \to \infty$. Thus

$$\bar{Z}_\theta^{(k)}(\kappa) = \lim_{k \to \infty} Z^{(k)}_\theta(\kappa)$$

can be calculated using the same Eq.(2.53) with $W[\Xi_+, \Xi_-]$ being the Wronskian of two solutions of the differential equation (3.2) which are fixed by the asymptotic conditions

$$\Xi_-(y) \to \left(2e^{\gamma_E}\right)^{\kappa \cos \theta} e^{\kappa \cos \theta} \quad \text{as} \quad y \to -\infty,$$
and
\[ \Xi_+(y) \to (2\kappa)^{-\frac{1}{2}} \exp \left[ -\frac{1}{4} e^y - \kappa \operatorname{Ei}\left(\frac{1}{2} e^y\right) \right] \quad \text{as} \quad y \to +\infty , \] (3.5)
with \( \operatorname{Ei}(z) = \int_{-z}^{\infty} \frac{e^{-x}}{x} \, dx \). Again, the asymptotic condition (3.4) is applicable for \( 0 \leq \theta < \frac{\pi}{2} \) only. Within the domain \( \frac{\pi}{2} \leq \theta < \pi \) the solution \( \Xi_- \) should be defined through the analytic continuation with respect to the variable \( \theta \).

Let us assume now that \( k \) is odd and consider the partition function (2.60) at \( k \to \infty \). One has
\[ \lim_{k \to \infty} Z^{(k)}_{\theta} = e^{\kappa \cos(\theta)} L \operatorname{\bar{Z}}(\kappa) , \] (3.6)
where \( L \) is the limiting value of the nonuniversal constant \( L^{(k)}_1 \). As follows from (2.60), the large-\( k \) limit taken among the even \( k \) may cause the result to differ from (3.6) by an extra multiplicative factor \( e^{\kappa \text{const}} \). Obviously, this subtlety does not affect the physical content of the limiting theory and we shall ignore it below.

### 3.2 Circular brane model

Here we identify the QFT describing the large-\( k \) limit of the model (2.8).

For finite \( k \) the minimal parafermionic model defined on the semi-infinite cylinder \( \tau \equiv \tau + 1/T, \ x \leq 0 \), contains two holomorphic \( \psi_+ (\tau, x) = \psi_+ (\tau - ix) \) and two antiholomorphic \( \bar{\psi}_+ (\tau, x) = \bar{\psi}_+ (\tau + ix) \) currents of conformal dimensions \( \Delta_1 = \bar{\Delta}_1 = 1 - \frac{1}{k} \). When \( k \to \infty \), the unlocal parafermionic currents turn into chiral spin-1 currents. The latter can be bosonized in terms of a pair of fields satisfying free massless equations of motion in the bulk \( x < 0 \):
\[ \partial \bar{\partial} X = \partial \bar{\partial} Y , \] (3.7)
with \( \partial = \frac{1}{2} (\partial_\tau + i \partial_x) \) and \( \bar{\partial} = \frac{1}{2} (\partial_\tau - i \partial_x) \). Namely,
\[ \lim_{k \to \infty} \psi_\pm = \frac{\partial X \pm i \partial Y}{\sqrt{2}} , \quad \lim_{k \to \infty} \bar{\psi}_\pm = \frac{\bar{\partial} X \pm i \bar{\partial} Y}{\sqrt{2}} . \] (3.8)
Hence, the large-\( k \) limit of (2.8) is described by the theory with the bulk Euclidean action
\[ A_{\text{bulk}} = \frac{1}{\pi} \int_0^{1/T} d\tau \int_{-\infty}^0 dx \left( \partial X \bar{\partial} X + \partial Y \bar{\partial} Y \right) . \] (3.9)

In view of the heuristic picture of the boundary interaction in (2.8) (see Fig 2) one may expect the boundary values \( X|_{x=0} \equiv X_B \) and \( Y|_{x=0} \equiv Y_B \) satisfy some \( O(2) \)-invariant boundary condition. This boundary condition has been already identified for \( \theta = 0, \pi \) in Refs. [1, 34]. In both cases the limit (3.6) coincides with the partition function of the circular brane model. The bulk action of this two-dimensional model is indeed given by (3.9), while the boundary values of the Bose fields are subjected to a nonlinear constraint
\[ X_B^2 + Y_B^2 = \frac{1}{g_0} . \] (3.10)
Let us recall some facts about the circular brane model \[1\]. Due to the nonlinear boundary condition (3.10), this theory needs renormalization. It has to be equipped with the ultraviolet (UV) cut-off, and consistent removal of the UV divergences requires that the bare coupling \(g_0\) be given a dependence on the cut-off energy scale \(\Lambda\), according to the Renormalization Group (RG) flow equation

\[
\Lambda \frac{dg_0}{d\Lambda} = -2 g_0^2 - 4 g_0^3 + \cdots. \tag{3.11}
\]

The leading two terms of the \(\beta\)-function written down in (3.11) were computed in [38] and [39], and indeed agree with the more general calculations in [40] and [41]. Eq. (3.11) implies that

\[
E_* \simeq \Lambda g_0^{-1} e^{-\frac{1}{2g_0}} \left(1 + \sum_{m=1}^{\infty} c_m g_0^m\right), \tag{3.12}
\]

where the symbol \(\simeq\) stands for asymptotic equality. The coefficients \(c_m\) in the asymptotic series (3.12) are not universal, i.e., they depend on the details of the regularization.

It is important that the general definition of the circular brane model involves an additional parameter, the topological angle \(\theta\). The configuration space for the two component Bose field \(X^\mu = (X, Y)\) consists of sectors characterized by an integer \(w\), the number of times the boundary value \((X_B, Y_B)\) winds around the circle (3.10) when one goes around the boundary at \(x = 0\). The contributions from the topological sectors can be weighted with the factors \(e^{i w \theta}\). Thus, in general

\[
Z_\theta = \sum_{w=-\infty}^{\infty} e^{i w \theta} Z^{(w)}, \tag{3.13}
\]

where \(Z^{(w)}\) is the path integral

\[
Z^{(w)} = \int_{(w)} D X D Y \, e^{-A_{\text{bulk}}[X,Y]}, \tag{3.14}
\]

evaluated over the fields from the sector \(w\) only. As it was pointed out in [42],\(^7\) the functional integral (3.14) has a particular nonperturbative divergence due to the small-size instantons [45,46], which cannot be absorbed into the renormalization of the coupling constant \(g_0\). Due to this effect, the partition function (3.13) is expected to have the form (3.6), where

\[
L = 2 \log \left(\frac{\bar{\Lambda}}{E_*}\right), \tag{3.15}
\]

with \(\bar{\Lambda}\) some ultraviolet cut-off regularizing the small instanton divergence, and \(\kappa\) the inverse temperature measured in units of the physical scale (3.12).\(^8\) This suggests to extend the

\(^7\)The circular brane model shows many similarities with the \(O(3)\) nonlinear \(\sigma\)-model (or \(n\)-field) [43,44], where a very similar effect of the small instanton divergence was known for a while [21, 22].

\(^8\)Notice that the instanton cut-off \(\bar{\Lambda}\) is different from the perturbative cut-off \(\Lambda\) in the RG flow equation (3.11), i.e. their ratio \(\bar{\Lambda}/\Lambda\) is some non-universal function of the bare coupling \(g_0\). In Section 4 we find this ratio for some particular UV regularization of the circular brane model.
results of Refs. [1, 34] and identify (3.6) with the partition function of the circular brane model $Z_\theta$ where $\theta$ plays the role of the topological angle:

$$Z_\theta = \exp \left( 2\kappa \cos(\theta) \log \left( \frac{\bar{\Lambda}}{\bar{E}} \right) \right) \tilde{Z}_\theta(\kappa).$$

### 3.3 High-temperature expansion

Here we discuss the high-temperature behavior of the partition function $Z_\theta$.

First and foremost we recall a simple qualitative picture described in Refs. [1, 31]. As follows from Eq. (3.11), the circular brane model is a asymptotically free theory – at short distances the effective size of the circle (3.10) becomes large. For small but nonzero values of the “bare” coupling constant $g_0$ we can locally ignore the curvature effects and write the Bose field $X^\mu(\tau, x) = X_0^\mu + \delta X^\mu(\tau, x)$. It makes sense to split the fluctuational part, $\delta X^\mu$, into the components normal and tangent to the circle (3.10) at the point $X_0^\mu$. The components must satisfy the Dirichlet and the Neumann BC, respectively. Therefore the “microscopic” boundary degeneracy for $g_\theta \ll 1$ does not depend on $\theta$ and coincides with the product of the boundary degeneracies associated with the Dirichlet ($g_D$) and the Neumann CBC ($g_N = g_D \frac{C}{2\pi}$, where $C = 2\pi/\sqrt{g_0}$ is the full length of the circle (3.10)). The above consideration supplemented by the common wisdom of the renormalization group suggests that as $E_\star \ll T \ll \Lambda$, the partition function must develop the following leading high-temperature behavior:

$$Z_\theta = g_{\text{fixed}} \sqrt{g(T)} \left[ 1 + O(g(T)) \right].$$

In writing the above equation, we replace the bare coupling $g_0$ in the “microscopic” boundary degeneracy $g_D^2/\sqrt{g_0}$ by the “running” coupling constant $g(T)$. As follows from the two-loop $\beta$-function (3.11), $g = g(T)$ can be defined by the equation:

$$\kappa \equiv \frac{E_\star}{T} = g^{-1} e^{-\frac{1}{2g}}. \quad (3.18)$$

Also, the factor $g_{\text{fixed}}$ in (3.17) should be understood as

$$g_{\text{fixed}} = \frac{g_D^2}{g}. \quad (3.19)$$

The exact partition function (3.16) indeed possesses the leading short distance behavior (3.17). The systematic high-temperature expansion for $\theta = 0$ was obtained in Ref. [1]. That result can be easily generalized to $\theta \neq 0$, yielding an expansion of the form

$$Z_\theta \simeq g_{\text{fixed}} \sqrt{g} \left( \frac{\bar{\Lambda}}{T} \right)^{2\kappa \cos\theta} g^{-\kappa \cos\theta} \sum_{n=0}^\infty \kappa^n z_n(g, \theta), \quad (3.20)$$

where the coefficients $z_n(g, \theta)$ are power series in the running coupling constant (3.18). To describe some coefficients in (3.20) explicitly, it is convenient to represent $Z_\theta$ as

$$Z_\theta = \frac{g_{\text{fixed}}}{\sqrt{g}} e^{\frac{1}{T} \kappa \cos\theta} \tilde{Z}_\theta. \quad (3.21)$$
Here
\[ \chi = -T \frac{\partial^2}{\partial \theta^2} \log(Z_\theta) \bigg|_{\theta=0}, \tag{3.22} \]

and therefore the factor \( \tilde{Z}_\theta \) is subject to the constraint
\[ \frac{\partial^2 \tilde{Z}_\theta}{\partial \theta^2} \bigg|_{\theta=0} = 0. \tag{3.23} \]

It is possible to show that the topological susceptibility \( \chi \) expands explicitly as
\[ \chi \simeq E_* \left[ 2 \log \left( \frac{A}{T} \right) - \log g + 3 \gamma_E + \log 2 - 2 \gamma_E g + O(g^2) - \frac{2\pi^2}{3} \kappa \left( 1 + O(g) \right) + O(\kappa^2) \right]. \tag{3.24} \]

The expansion coefficients of \( \tilde{Z}_\theta \) in (3.21) are somewhat simpler than \( z_n \) in (3.20). For example,
\[ \tilde{Z}_0 = 1 - (1 + \gamma_E) g + O(g^2, \kappa^2), \tag{3.25} \]
and
\[ \frac{\tilde{Z}_\theta}{\tilde{Z}_0} = 1 - \frac{4\pi^2}{3} \left( 1 + O(g^2) \right) \sin^4 \left( \frac{\theta}{2} \right) \kappa^2 + O(\kappa^2). \tag{3.26} \]

Notice that the coefficients \( \frac{2\pi^2}{3} \) and \( \frac{4\pi^2}{3} \) in Eqs. (3.24), (3.26) are in agreement with the result of two-instanton calculation from Ref. [47].

### 3.4 Low-temperature expansion

For \( k = \infty \), the boundary degeneracy \( g_{\text{fixed}} \) in (2.63) should be understood as in Eq. (3.19). It implies that the circular brane boundary flow is terminated at the fixed point corresponding to the Dirichlet CBC \( X_B^\mu = 0 \) for any \( \theta \) within the domain \( 0 \leq \theta < \pi \). At the same time, for \( \theta = \pi \) the flow possesses a nontrivial IR fixed point whose boundary degeneracy is given by
\[ g_2 = 2 g_D^2. \tag{3.27} \]

(see Eq. (2.64) with \( s = 2 \) and \( k = \infty \)). The last equation suggests that the boundary values of \( X^\mu \) are constrained to two points. We refer the reader to Ref. [34] for a comprehensive description of this nontrivial RG fixed point.

Here we describe the low-temperature expansion for \( \theta \neq \pi \). In this case the free energy, \( F_\theta = -T \log Z_\theta \), admits an asymptotic expansion in terms of \( T^2 \):
\[ F_\theta \simeq -2E_* \cos(\theta) \log \left( \frac{A}{E_*} \right) + E_\theta - T \log(g_D^2) - \sum_{l=1}^{\infty} F_l(\theta) \frac{T^{2l}}{E_*^{2l-1}}. \tag{3.28} \]

The regularized ground state energy \( \tilde{E}_\theta \) is a large-\( k \) limit of the function (3.28). It explicitly reads
\[ \frac{\tilde{E}_\theta}{E_*} = C_0 - (\gamma_E + \log 2) \cos \theta + \int_0^1 \! dx \left[ \sqrt{\pi} \cos^2(\theta) \frac{\Gamma(x + \frac{3}{2})}{\Gamma(x)} 3F_2 \left( x + \frac{1}{2}, \frac{1}{2}, 1, 2, \frac{3}{2} | \cos^2(\theta) \right) - \frac{2}{3} \cos^3(\theta) x 3F_2 \left( x + 1, 1, 1, 2, \frac{5}{2} | \cos^2(\theta) \right) \right]. \tag{3.29} \]
Here \( _3F_2 \) is the generalized hypergeometric function and

\[
C_0 = -\sqrt{\pi} \int_0^1 dx \frac{\Gamma(x - \frac{1}{2})}{\Gamma(x)} = -1.44142 \ldots . \tag{3.30}
\]

The function \( (3.29) \) is plotted in Fig. 4. Notice that it develops a non-analytical behavior at \( \theta = \pi \):

\[
\frac{\bar{E}_\theta}{E_*} = 2 + \frac{\pi (\pi - \theta)}{\log(\pi - \theta)} + O\left(\frac{\pi - \theta}{\log^2(\pi - \theta)}\right). \tag{3.31}
\]

![Figure 4: The regularized ground state energy (3.29) in the circular brane model.](image)

The first two coefficients \( F_i \) in the expansion \( (3.28) \) explicitly read as follows:

\[
F_1(\theta) = \frac{\theta}{12 \sin \theta}, \tag{3.32}
\]

and

\[
F_2(\theta) = \frac{1}{720 \sin^3 \theta} \int_0^\theta dt \left[ 6 \log^2 \left( \frac{\sin \theta}{\sin t} \right) - 7 \log \left( \frac{\sin \theta}{\sin t} \right) - 5 \right]. \tag{3.33}
\]

4 The dissipative quantum rotator model

The circular brane model has useful interpretation in terms of Brownian dynamics of a quantum rotator. It was noticed a while ago in Ref. [48] that the free massless bulk dynamics \( (3.9) \) is equivalent to the Caldeira-Leggett model of a quantum thermostat [49]. Upon fixing the boundary values \( X^B(\tau) \) and integrating out the bulk part of the field \( X^\mu(\tau, x) \), Eqs. \( (3.13), (3.14) \) reduce to

\[
Z_\theta = g_{\text{fixed}} \int \mathcal{D}\eta \, e^{-A_{\text{diss}}[\eta] - A_\theta[\eta]}, \tag{4.1}
\]
\[ A_{\text{diss}}[\eta] = \frac{T^2}{2g_0} \int_0^\frac{1}{T} d\tau \int_0^\frac{1}{T} d\tau' \frac{\sin^2 \left( \frac{\eta(\tau) - \eta(\tau')}{2} \right)}{\sin^2(\pi T(\tau - \tau'))}, \]  

(4.2)

and

\[ A_\theta[\eta] = -\frac{i\theta}{2\pi} \int_0^\frac{1}{T} d\tau \partial_\tau \eta(\tau). \]  

(4.3)

The variable \( \eta(\tau) \) is the angular field defined through

\[(X_B, Y_B) = \frac{1}{\sqrt{g_0}} (\cos \eta, \sin \eta).\]  

(4.4)

The path integral (4.1) requires some UV regularization. Among a large variety of regularizations there is one of special interest for applications \([2, 50]\):

\[ A_{\text{DQR}}[\eta] = A_{\text{diss}}[\eta] + A_\theta[\eta] + A_C[\eta], \]  

(4.5)

where

\[ A_C[\eta] = \frac{1}{4E_C} \int_0^\frac{1}{T} d\tau \eta_\tau^2. \]  

(4.6)

The functional (4.5) amounts to the Caldeira-Leggett action for the dissipative quantum rotator. In the weak coupling regime \((g_0 \gg 1)\) the additional term (4.6) just provides an explicit UV cutoff of the dissipative action (4.2) with the cutoff energy \( \Lambda \sim E_C/g_0 \). Now, once the regularization scheme is chosen, all coefficient \( c_l \) in the asymptotic expansion (3.12) are determined unambiguously within the standard perturbation theory, in particular \([1]\)

\[ c_1 = -\frac{3}{4} \pi^2. \]  

(4.7)

The additional term (4.6) not only makes the circular brane model perturbatively well-defined, but also regularizes the small instanton divergence. For example, the one-instanton contribution to the topological susceptibility \([32, 22]\) in the Gaussian approximation reads as follows \([42, 51]\):

\[ \chi^{(\text{pert})} = \frac{E_C}{2\pi^2g_0^2} e^{-2\pi \frac{1}{E_0}} \left[ 2 \int_{a a^* < 1} d a \wedge d a^* \frac{e^{-A_C^{(1)}}}{1 - a a^*} + O(g_0) \right]. \]  

(4.8)

Here \( A_C^{(1)} \) is the functional (4.6) evaluated for the one instanton solution. The exponential factor \( e^{-A_C^{(1)}} \) makes the integral over the instanton moduli \(|a| < 1\) finite. Indeed, using the explicit form of the one instanton solution \( \eta^{(1)} \) \([45, 46]\),

\[ \exp \left( i\eta^{(1)}(\tau) \right) = \frac{z - a}{1 - a^* z}, \quad \text{where} \quad z = e^{2\pi T \tau}, \]  

(4.9)
one finds
\[ A_C^{(1)} = \frac{\pi^2 T}{E_C} \frac{1 + aa^*}{1 - aa^*}. \] (4.10)

Now we can integrate over the instanton moduli explicitly:
\[ \chi^{(\text{pert})} = E_C \frac{e^{-\frac{\bar{\Lambda}}{T}}}{2\pi^2 g_0} \left[ 2 \log \left( \frac{E_C e^{-\gamma_E}}{2\pi^2 T} \right) + O(g_0, T/E_C) \right]. \] (4.11)

It is instructive to compare (4.11) with the high-temperature expansion (3.22). In essence, Eq. (3.22) requires that the bare coupling constant expansion should be of the form
\[ \chi^{(\text{pert})} = E_\ast \left[ 2 \log \left( \frac{\bar{\Lambda}}{T} \right) - \log g \ast + 3\gamma_E + \log 2 + O(g_0, T/\Lambda) \right]. \] (4.12)

Combining Eqs. (4.11) and (4.12), (3.12) yields the following relations between the perturbative cut-off ($\Lambda$), the small instanton cut-off ($\bar{\Lambda}$), and the energy scale $E_C$ in (4.6):
\[ \Lambda = \frac{E_C}{2\pi^2 g_0}, \] (4.13)
and
\[ \bar{\Lambda} = E_C \frac{e^{-\frac{\gamma_E}{2}}}{\sqrt{8\pi^2}} \sqrt{g_0}. \] (4.14)

5 Conclusion: Charge fluctuations in quantum dot

The dissipative rotator model (4.5) was introduced in Ref. [2] as an effective field theory describing tunneling of quasiparticles between superconductors. Later it has been involved in a surge of activity in the study of a low-capacitance metallic island (quantum dot), connected to an outside lead by a tunnel junction [39, 42, 46, 47, 51–53]. A remarkable feature of such a device (see Fig. 5, borrowed from Ref. [53]) is that the essentially many-body phenomena may be described via a single variable $\eta$ (conjugate to the number of excess charges on the island). As was shown in Refs. [2, 46], the fermionic degrees of freedom can be formally integrated out yielding the Matsubara effective action $A_{\text{eff}}$ for $\eta$. The Coulomb part of this action coincides with $A_\theta + A_C$ (4.3), (4.6) provided the energy scale $E_C$ is understood now as the single electron charging energy, i.e.,
\[ E_C = \frac{e^2}{2C}, \] (5.1)
where $C$ is the island capacitance. The parameter
\[ n_g = \frac{\theta}{2\pi} \] (5.2)
has also the simple physical meaning of a dimensionless applied gate voltage $V_g$: $n_g = C_g V_g/e$, where $C_g$ is the gate capacitance (see Fig. 5).
In addition to the Coulomb terms, the effective action contains a highly nontrivial dissipative piece which essentially depends on details of the island-lead interaction. In fact, the transmission coefficients \( \{ T_a \}_{a=1}^N \) in the island-lead interface are coupling constants of \( \mathcal{A}_{\text{eff}} \) (\( N \) is a number of spinless electron channels). The most general effective action appears to be too complicated for practical purposes. It simplifies drastically in the special large-\( N \) limit when each individual transmission coefficient \( T_a \) vanishes, but the dimensionless (measured in units \( e^2/h \) tunneling conductance

\[
\alpha = \frac{1}{2\pi^2} \sum_{a=1}^{N} T_a , \quad (5.3)
\]

remains finite. In this limit \( \mathcal{A}_{\text{eff}} \) takes the form of the dissipative rotator action \( (5.5) \) with

\[
g_0 = \frac{1}{2\pi^2 \alpha} , \quad (5.4)
\]

so the basic thermodynamic quantities can be extracted from the corresponding partition function, \( Z_{\text{DQR}} \). The renormalized charging energy \( \tilde{E}_C \)

\[
\tilde{E}_C = -2\pi^2 T \frac{\partial^2}{\partial \theta^2} \log ( Z_{\text{DQR}} ) \quad (5.5)
\]

is of special interest. As it has been argued above, the quantum dissipative rotator model possesses the universal scaling behavior provided \( g_0 \ll 1 \) and \( E_C \gg T \). This corresponds to the regime of almost perfect transmission for the low-capacitance metallic island. In the scaling regime \( Z_{\text{DQR}} \) turns into the circular brane partition function \( Z_\theta \). Thus the exact results obtained in this work give a complete description of the renormalized charging energy in the scaling limit. In particular at \( T = 0 \),

\[
\left. \frac{\tilde{E}_C}{2\pi^2 E_*} \right|_{T=0} \rightarrow 2 \cos(\theta) \log \left( \frac{E_C e^{-\frac{3}{4\pi \alpha}}} {4\pi^3 E_* \sqrt{\alpha}} \right) + \frac{\partial^2}{\partial \theta^2} \left( \frac{\tilde{E}_\theta}{E_*} \right) \quad (\alpha, E_C \rightarrow \infty ) . \quad (5.6)
\]

Recall that the function \( \tilde{E}_\theta/E^* \) is given by Eq. \( (3.29) \) and plotted in Fig. 4, while

\[
E_* = 2\pi^2 E_C \alpha^2 \left( 1 - \frac{3}{8\alpha} + O(\alpha^{-2}) \right) e^{-\pi \alpha^2} . \quad (5.7)
\]
It is worth noting that the quoted analytical results for $E_C$ with $\theta = 0$ have been numerically checked in Ref. [3] via Monte Carlo simulations of $Z_{DQR}$.

The quantitative description of the quantum dot with finite number of channels remains a challenging problem. For finite $N$, one would still expect some sort of universal scaling behavior characterized by the energy scale [46]

$$E_s \propto E_C \prod_{a=1}^{N} \sqrt{1 - T_a} .$$  \hspace{1cm} (5.8)

In the case of $N = 2$ this was explicitly demonstrated in Ref. [53]. Matveev argued that the universal behavior of the two-channel quantum dot is described by the Hamiltonian $H_{\theta}^{(k)}$ [2.8] with $k = 2$. Thus the thermodynamic properties of the system can be extracted from the Chatterjee-Zamolodchikov partition function (2.61). With regard to $N = 2$ and $N = \infty$ “integrable” cases, it would be interesting to explore the possibility of using $H_{\theta}^{(k)}$ [2.8] as a QFT Hamiltonian underlying the universal scaling behavior of the $N$-channel quantum dot provided $N = k$.

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A Appendix: The BP sine-Gordon model

In this appendix we shall study the BP sine-Gordon model. It provides an essential step toward the main result (2.53).

A.1 Definition of the model

Let $H_{\text{free}}$ be a Hamiltonian of the boundary CFT model which contains two noninteracting sectors, namely the minimal parafermionic CFT and the Gaussian theory of the free uncompactified Bose field:

$$H_{\text{free}} = H_{\text{free}}^{(k)} + H_{\text{free}}^{(\Phi)} .$$  \hspace{1cm} (A.1)

Suppose the parameter $b$ in Eq.(2.21) is purely imaginary,

$$b = i \beta ,$$  \hspace{1cm} (A.2)
so that the boundary fields (2.17) take the form:

\[ V_\pm = \Psi_\pm e^{\pm i\frac{\beta}{\sqrt{k}} \Phi} . \]  

(A.3)

Their scaling dimension is given by

\[ d = 1 - \frac{1}{k} + \frac{\beta^2}{k} . \]  

(A.4)

For \( 0 < \beta^2 < 1 \), \( V_\pm \) are relevant boundary fields normalized as in (2.18), and in what follows we will focus on the model described by the Hamiltonian

\[ H_{\text{bsg}}^{(k)} = H_{\text{free}} - \mu \left( e^{iVt} V_+ + e^{-iVt} V_- \right) , \]  

(A.5)

where \( \mu \) and \( V \) are parameters. The parameter \( \mu \) carries the dimension of \([\text{length}]^{d-1}\).

The boundary interaction contains an explicit time dependence through the factors \( e^{\pm iVt} \). Below we always assume that \( V > 0 \). Notice that, although by means of the formal change of the field variable, \( \Phi \to \Phi - \frac{\sqrt{k}}{\beta} t \), the Hamiltonian (A.5) can be brought to the form (2.14), effects of the external fields \( V \) in (A.5) and \( h \) in (2.14) are very different. Whereas the system (2.14) possesses the thermal equilibrium state even for non-zero values of \( h \), the system (A.5), at a nonzero \( V \) and a temperature \( T \), develops a nonequilibrium steady state which can be thought of as the result of an infinite time evolution of the equilibrium state of the corresponding “free” system, with the interaction term adiabatically switched on. We denote by \( \langle O \rangle_{\text{bsg}}^{(k)} \) the expectation value of an observable \( O \) over this nonequilibrium steady state.

The model (A.5) was already discussed in the literature. In particular in the absence of the parafermionic sector \( (k = 1) \) it coincides with the massless boundary sine-Gordon model \([55, 56]\). For this reason we shall call (A.5) the BP sine-Gordon model. Besides being an interesting model of Quantum Field Theory on its own, the theory finds important applications in condensed matter physics. In Ref. \([4]\) the model (A.5) was suggested to describe a point-like impurity in the “multichannel quantum wire”\(^9\). In this context the integer \( k \) is the number of spinless electron channels, \( \alpha_0 = k\beta^2 \) is the conductance (in units \( e^2/h \)) of the wire without impurity, while \( V \) is proportional to the voltage drop across the impurity. The quantities of special interest are correlation functions of the Heisenberg boundary operators

\[ V_\pm(t) = e^{\pm iVt} U^{-1}(t) V_\pm U(t) , \]  

(A.6)

where \( U(t) \) is the time evolution operator corresponding to the Hamiltonian (A.5). For instance, the energy dissipation rate \( \dot{E} \) for the steady state, produced by the time dependent boundary perturbation (A.5), can be expressed in terms of the expectation values \( \langle V_\pm \rangle_{\text{bsg}}^{(k)} \). Indeed, as follows from (A.5):

\[ \dot{E} = -J_D V \quad \text{with} \quad J_D = i\mu \left( \langle V_+ \rangle_{\text{bsg}}^{(k)} - \langle V_- \rangle_{\text{bsg}}^{(k)} \right) . \]  

(A.7)

Notice that the quantity

\[ \alpha(V, T) = \alpha_0 \left( 1 + 2\pi \frac{\beta^2}{k} J_D/V \right) \]  

(A.8)

is interpreted as a DC conductance of the multichannel quantum wire \([4]\).

\(^9\)The cases \( k = 1 \) (without the parafermionic sector) and \( k = 2 \) were previously studied in Refs. \([57–59]\) and \([60, 61]\), respectively.
A.2 Perturbative expansions for $\langle V^\pm \rangle_{bsg}^{(k)}$

If $V > 0$ the system (A.5) evolves towards a steady state which is characterized by the expectation values $\langle V^\pm \rangle_{bsg}^{(k)}$. This is not a thermodynamic equilibrium state and no simple explicit expression for its density matrix is known. Nevertheless, one can describe the definition of this state in terms of the real-time perturbation theory [54]. It is useful to represent the perturbative expansions for $\langle V^\pm \rangle_{bsg}^{(k)}$ in the form

$$\langle V^\pm \rangle_{bsg}^{(k)} = T \frac{\partial}{\partial \mu} \log A^\pm,$$

(A.9)

where $\log A^\pm$ are formal power series expansions,

$$\log A^\pm = \sum_{n=1}^{\infty} a^{(\pm)}_n s^n \quad \text{with} \quad s = \mu^2 (2\pi T)^{2d-2}.$$

(A.10)

For $k = 1$ the compact formula for the perturbative coefficients $a^{(\epsilon)}_n$ were obtained in Refs. [27, 62]. In essence the derivation from [27] can be applied without change to the BP sine-Gordon model. As a result, the following formula emerges:

$$a^{(\epsilon)}_n = \epsilon (-1)^{n+1} 2^{2n} \frac{\pi}{n} \sum_{\epsilon_1, \ldots, \epsilon_{2n-1}} \prod_{j=1}^{2n-1} \sin(\pi d \eta_j) J_{\epsilon, \epsilon_1, \ldots, \epsilon_{2n-1}},$$

(A.11)

where the sum is taken over all arrangements of the “charges” $\epsilon_1, \ldots, \epsilon_{2n-1} = \pm 1$ with zero total charge $\epsilon + \sum_{s=1}^{2n-1} \epsilon_s = 0$, and

$$\eta_j = \epsilon + \sum_{s=1}^{j-1} \epsilon_s.$$

(A.12)

The coefficients $J_{\epsilon, \epsilon_1, \ldots, \epsilon_{2n-1}}$ in (A.11) are expressed through the $(2n - 1)$-fold integrals over the real-time correlation functions:

$$J_{\epsilon, \epsilon_1, \ldots, \epsilon_{2n-1}} = \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{2n-2}} dt_{2n-1} e^{iV \sum_{j=1}^{2n-1} \epsilon_j t_j} \times (2\pi T)^{2n(1-d)} e^{-i\pi d \sum_{j=1}^{2n-1} \epsilon_j \eta_j} \left\langle \left\langle V^{(int)}_{\epsilon_1}(0) V^{(int)}_{\epsilon_1}(t_1) \cdots V^{(int)}_{\epsilon_{2n-1}}(t_{2n-1}) \right\rangle \right\rangle_0.$$

(A.13)

Here $V^{(int)}_{\pm}(t) = e^{itH_{\text{free}}} V_\pm e^{-itH_{\text{free}}}$ are the boundary operators in the interaction representation and

$$\left\langle \left\langle \cdots \right\rangle \right\rangle_0 \equiv \text{Tr}_{\mathcal{H}} \left[ \cdots e^{-\frac{H_{\text{free}}}{\beta}} \right] / \text{Tr}_{\mathcal{H}} \left[ e^{-\frac{H_{\text{free}}}{\beta}} \right]$$

(A.14)

denotes an expectation value over the equilibrium thermal state of the free system.

As follows from the real-time OPE

$$V^{(int)}_{\pm}(t)V^{(int)}_{\mp}(t') \to (i(t - t'))^{-2d}, \quad 0 < t - t' \to 0,$$

(A.15)
the integrals \((A.13)\) can be taken literally only if the scaling dimension \(d < \frac{1}{2}\). For \(\frac{1}{2} < d < 1\) we assume the “analytic regularization” common in conformal perturbation theory, i.e. the integrals should be understood as analytically continued from the convergence domain \(\Re d < \frac{1}{2}\). For the sake of illustration we consider explicitly the first coefficient \(a_1^{(\pm)}\). In this case the correlators appearing in the integrand of \((A.13)\) are given by

\[
e^{i\pi d} \left\langle V^{(\text{int})}_{\pm}(0) V^{(\text{int})}_{\mp}(t) \right\rangle_0 = \left[ \frac{\pi T}{\sinh(-\pi T t_1)} \right]^{2d} (t_1 < 0) .
\]

(A.16)

Calculating the elementary integral over \(t_1\), one obtains

\[
a_1^{(\pm)} = 4\pi \sin(\pi d) \frac{\Gamma(1 - 2d) \Gamma(d \pm 2p)}{\Gamma(1 - d \pm 2p)} \quad \text{with} \quad p = -i \frac{V}{4\pi T} .
\]

(A.17)

A.3 The system with \(q\)-oscillator

Remarkably, the formal power series \((A.10)\) can be interpreted as high-temperature expansions of the equilibrium free energy for certain systems which differ from \((A.5)\) in that they involve additional boundary degrees of freedom. Originally this was demonstrated in Ref. [27] for the boundary sine-Gordon model \((k = 1)\), but the same reasoning can be applied to the generalized model \((A.5)\).

Let us define the Hamiltonian

\[
H_+ = H_{\text{free}} - V h - \mu \left( a_- V_+ + a_+ V_- \right) .
\]

(A.18)

Here we use the same notations as in \((A.5)\). Formula \((A.18)\) includes a new ingredient, namely, the operators \(h\), \(a_+, a_-\) which commute with all the parafermionic and bosonic degrees of freedom. They form among themselves the so-called “\(q\)-oscillator algebra”

\[
[h, a_{\pm}] = \pm a_{\pm} , \quad q a_+ a_- - q^{-1} a_- a_+ = q - q^{-1} ,
\]

(A.19)

with

\[
q = e^{i\pi d} .
\]

(A.20)

Recall that the parameter \(1 - \frac{1}{k} < d < 1\) is the scaling dimension of the boundary fields \(V_{\pm}\). Let \(\rho_+\) be some representation of \((A.19)\) such that the spectrum of \(\rho_+(h)\) is real and bounded from above. The Hamiltonian \((A.18)\) acts in the space

\[
H_+ = H \otimes \rho_+ ,
\]

(A.21)

with \(H\) given by \((2.16)\). For \(V > 0\) this Hamiltonian is bounded from below. Then, the system \((A.18)\) possesses a thermal equilibrium state described by the standard density matrix,

\[
Z_+^{-1}(\mu, V) e^{-\frac{H_+}{T}} .
\]

Strictly speaking the partition function

\[
Z_+(\mu, V) = \text{Tr}_{\mathcal{H}_+} \left[ e^{-\frac{H_+}{T}} \right]
\]

(A.22)
is ill-defined even for the non-interacting system. Indeed, since the space of states \( \mathcal{H}_+ \) has a structure of tensor product (see Eqs. (2.16), (A.21)), the partition function \( Z_+(\mu, V)|_{\mu=0} \) factorizes into a product of three terms. The first one coincides with the boundary degeneracy \( g_{\text{free}} \) (1.2). The component \( \rho_+ \) in (A.21) gives rise to the factor \( \text{Tr}_{\rho_+} \left[ e^{Vh} \right] \) which is well-defined if the spectrum of \( \rho_+(h) \) is bounded from above. The problem comes from a divergent contribution of the zero-mode of the uncompactified Bose field subjected by the free CBC.

To cancel this formal divergence we shall consider below the ratio \( Z_+(\mu, V)/Z_+(0, V) \).

Carrying out the finite temperature Matsubara procedure, one obtains the weak coupling expansion of the form

\[
Z_+(\mu, V) = Z_+(0, V) + \sum_{n=1}^{\infty} A_n^{(+)} s^n \quad \text{with} \quad s = \mu^2 (2\pi T)^{2d-2},
\]

where the expansion coefficients are given by

\[
A_n^{(+)} = \sum_{\epsilon_1, \ldots, \epsilon_{2n} = 0} T_{\epsilon_1 \ldots \epsilon_{2n}} G_{\epsilon_1 \ldots \epsilon_{2n}}.
\]

Whereas \( T_{\epsilon_1 \ldots \epsilon_{2n}} \) are expressed in terms of traces over the representation \( \rho_+ \),

\[
T_{\epsilon_1 \ldots \epsilon_{2n}} = \frac{\text{Tr}_{\rho_+} \left[ e^{Vh} a_{-\epsilon_1} \cdots a_{-\epsilon_{2n}} \right]}{\text{Tr}_{\rho_+} \left[ e^{Vh} \right]},
\]

the coefficients \( G_{\epsilon_1 \ldots \epsilon_{2n}} \) are multiple integrals:

\[
G_{\epsilon_1 \ldots \epsilon_{2n}} = \int_0^{T} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} d\tau_{2n} e^{V \sum_{j=1}^{2n} \epsilon_j \tau_j} \times (2\pi T)^{2n(1-d)} \langle \langle V_\epsilon^{(\text{Mats})}(\tau_1) \cdots V_\epsilon^{(\text{Mats})}(\tau_{2n}) \rangle \rangle_0.
\]

Here \( V_\epsilon^{(\text{Mats})}(\tau) = e^{\tau H_{\text{free}}} V_\pm e^{-\tau H_{\text{free}}} \) are unperturbed Matsubara operators associated with the boundary operators (A.3), and \( \langle \langle \cdots \rangle \rangle_0 \) is defined in (A.14). As well as in formula (A.13), we should assume the analytic regularization of \( G_{\epsilon_1 \ldots \epsilon_{2n}} \) in the case \( \frac{1}{2} < d < 1 \). One might also note that all the coefficients \( T_{\epsilon_1 \ldots \epsilon_{2j}} \) in (A.24) are determined unambiguously via the commutation relations (A.19) and the cyclic property of trace. Thus, they do not depend on a particular choice of the representation \( \rho_+ \) provided \( \text{Tr}_{\rho_+} \left[ e^{Vh} \right] \) exist.

Now, following along the lines of Ref. [27], it is possible to prove that the formal power series \( A_+ \) defined in equation (A.10) coincides with (A.23), i.e.,

\[
A_+ = \frac{Z_+(\mu, V)}{Z_+(0, V)}.
\]

Of course, \( A_- \) (A.10) is also related to a certain equilibrium-state partition function. Namely, let \( \rho_- \) be any representation of (A.19) such that the spectrum of \( \rho_- (h) \) is bounded from below. Consider the Hamiltonian

\[
H_- = H_{\text{free}} + Vh - \mu \left( a_+ V_+ + a_- V_- \right)
\]
acting in $\mathcal{H}_- = \mathcal{H} \otimes \rho_-$ and the associated partition function

$$Z_-(\mu, V) = \text{Tr}_{\mathcal{H}_-} \left[ e^{-\mu \frac{H}{T}} \right] .$$  \hspace{1cm} (A.29)

Similarly to (A.27), one can prove the relation

$$A_- = \frac{Z_-(\mu, V)}{Z_-(0, V)} .$$  \hspace{1cm} (A.30)

### A.4 Characteristic properties of $A_{\pm}$

Equation (A.10) shows that $\log A_{\pm}$ are formal power series of the dimensionless variable

$$s = \mu^2 \left( \frac{2\pi T}{2d-2} \right) ,$$  \hspace{1cm} (A.31)

while the expansion coefficients $a_{n}^{(\pm)}$ depend on the dimensionless ratio $V/T$. In what follows, we treat $a_{n}^{(\pm)}$ as functions of the complex variable

$$p = -i \frac{V}{4\pi T} .$$  \hspace{1cm} (A.32)

Here are the main properties of $A_{\pm}(s, p)$.

- $\log A_{\pm}(s, p)$ are formal power series of the form (A.10), or, equivalently,

$$A_{\pm}(s, p) = 1 + \sum_{n=1}^{\infty} A_{n}^{(\pm)}(p) s^n .$$  \hspace{1cm} (A.33)

The coefficients $a_{n}^{(\pm)}(p)$ (A.10) and $A_{n}^{(\pm)}(p)$ (A.33) are meromorphic functions in the whole complex plane of $p$ and

$$A_{n}^{(-)}(p) = A_{n}^{(+)}(-p) , \quad a_{n}^{(-)}(p) = a_{n}^{(+)}(-p) .$$  \hspace{1cm} (A.34)

- Both $a_{n}^{(+)}(p)$ and $A_{n}^{(\pm)}(p)$ are analytic in the right half-plane,

$$\Re p > p_n \quad \text{with} \quad 2p_n = (1 - d) n - 1 .$$  \hspace{1cm} (A.35)

For $n = 1, 2, \ldots, k$, they have a simple pole located at $p = p_n$ of residue

$$\text{Res}_{p=p_n} \left[ A_n(p) \right] = \text{Res}_{p=p_n} \left[ a_n(p) \right] = \frac{2\pi}{\sqrt{k} \Gamma(d)} \left[ 2^n k! \Gamma(1 - n(1 - d)) \right] \frac{1}{2n!(k-n)! \Gamma(n(1-d))} .$$  \hspace{1cm} (A.36)

There is no singularity at $p = p_n$ for $n > k$, therefore the domain of analyticity (A.35) can be extended to the half plane $\Re p > p_n - \delta_n$ with some finite $\delta_n > 0$.  

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• The following leading power-low asymptotic holds in the domain of analyticity,

$$a_n^{(+)}(p) \rightarrow C_n \ p^{1-2n(1-d)} \ \text{as} \ \ p \rightarrow \infty \quad (\Re p > p_n), \quad (A.37)$$

where $C_n$ is some constant.

• The formal power series $A_\pm(s, p)$ obey the so-called “quantum Wronskian” relation,

$$e^{2\pi i p} A_+(qs, p) A_-(q^{-1}s, p) - e^{-2\pi i p} A_+(q^{-1}s, p) A_-(qs, p) = 2i \sin(2\pi p), \quad (A.38)$$

with $q = e^{i\pi d}$.

The proof of functional relation (A.38) is based on results of the work [26]. It is outlined in Appendix B. All other properties readily follow from formulas (A.11) and (A.23). For example the integrals (A.26) are entire functions of the variable $p$ (A.32). Thus, singularities of the coefficients $A_n^{(+)}(p)$ are due to the traces over the representation $\rho_+$. A brief inspection of (A.26) shows that $A_n^{(+)}(p)$ possesses only simple poles. They may be located only at points $2p = -(d(n+l)+m)$ where $l = 0, 1, \ldots n-1$ and $m = 0, \pm 1, \pm 2, \ldots$. Hence, $A_n^{(+)}$ are meromorphic functions in the whole complex plane of $p$. The same, of course, holds true for the coefficients $a_n^{(+)}(p)$. Furthermore, since (A.34) holds trivially for pure imaginary $p$, the claim is also true for any complex $p$.

The analyticity of $a_n^{(+)}(p)$ in the right half-plane $\Re p > p_n$ for some $p_n$ and the asymptotic formula (A.37) follow immediately from Eqs. (A.11), (A.13). To determine $p_n$ and calculate the residue one needs to examine these equations in more details. It is possible to show that the boundary of the domain of analyticity is defined by the integral (A.34) the “cumulative charges” $\eta_j$ (A.12) have maximum admissible values$^{10}$, i.e.

$$\epsilon_1 = \ldots = \epsilon_{n-1} = +1, \quad \epsilon_n = \ldots = \epsilon_{2n-1} = -1. \quad (A.39)$$

Moreover, as $p \rightarrow p_n$, the dominant contribution to this multiple integral comes from the region of integration where the operators $V_{\epsilon_j}^{(int)}$ with the same “charges” $\epsilon_j$ combine into two well-separated clusters: $|t_1|T \sim \cdots \sim |t_{n-1}|T \sim 1$ and $|t_n|T \sim \cdots \sim |t_{2n-1}|T \gg 1$. Therefore, to determine the singular behavior as $p \rightarrow p_n$ we invoke the cluster property and replace the correlation function in the integrand (A.13) by

$$e^{i\pi nd} \langle \langle V_+^{(int)}(t_0) \cdots V_+^{(int)}(t_{n-1}) V_-^{(int)}(t_n) \cdots V_-^{(int)}(t_{2n-1}) \rangle \rangle_0 \rightarrow F(t_0, \ldots, t_{n-1}) e^{2\pi T d_n(t_n-t_0)} F^*(t_n, \ldots, t_{2n-1}), \quad (A.40)$$

where $t_0 = 0$ and$^{11}$

$$d_n = n (n(d-1) + 1) \quad (A.41)$$

is the scaling dimension of the boundary operator $\Psi_n e^{\frac{i\pi d}{\sqrt{h}} \Phi_B}$ (see Fig.6). The latter is a

$^{10}$Note that because of the factor $\prod_{j=1}^{2n-1} \sin(\pi d \eta_j)$ the sum (A.11) enjoys a remarkable property, namely, it contains only the terms where none of the “cumulative charges” $\eta_j$ ($j = 1, \ldots, 2n-1$) vanish.

$^{11}$Here it is assumed that $n \leq k$.  

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Figure 6: The amplitude defining the boundary $p_n$ of the domain of analyticity $\Re p > p_n$ for $a_n^{(\pm)}(p)$ ($n \leq k$). The incoming and outgoing legs correspond to the insertions of boundary fields $V_{\text{int}}^-(t)$ and $V_{\text{int}}^+(t)$, respectively. For $n \leq k$ the intermediate state carries the scaling dimension (A.41) and $p_n = -\frac{d_n}{2n}$.

The product of the bosonic exponential field and the boundary field $\Psi_n$ (2.2). The correlation function appearing in (A.40),

$$F(t_0, \ldots, t_{n-1}) = (2\pi T)^{-d_n} \times \langle \langle V_{\text{int}}^-(t_0) \cdots V_{\text{int}}^-(t_{n-1}) \Psi_B^{(n)} e^{\frac{in \Phi_B}{\sqrt{k}} (-\infty)} \rangle \rangle_0,$$

has a very simple form in the case $n \leq k$. Namely, for $t_1 > \ldots > t_n$

$$F(t_1, \ldots, t_n) = (2\pi T)^{nd} \prod_{m=1}^{n-1} C_{1m} \prod_{m<j} \left( 2i \sinh (\pi T (t_m - t_j)) \right)^{2d-2}.$$

(A.43)

Here $C_{1m} = \sqrt{\frac{(k-m)(m+1)}{k}}$ are the structure constants (2.3) with $j = 1$. Combining (A.40) with (A.11) one finds that the coefficients $a_n(p)$ ($n = 1, \ldots k$) possess a simple pole singularity at $p = p_n = -\frac{d_n}{2n}$ and

$$\text{Res}_{p=p_n}[a_n^{(\pm)}(p)] = (-1)^{n+1} \frac{2^{2n-1} \pi k! n!}{n^2 k^n (k-n)!} \frac{\sin(\pi nd) S^2}{\prod_{j=1}^{n-1} \sin^2(\pi jd)}.$$

(A.44)

where $S$ is the Selberg integral [63, 64],

$$S = \int_{-\infty}^{0} du_1 \int_{-\infty}^{u_1} du_2 \cdots \int_{-\infty}^{u_{2n-2}} du_{2n-1} \prod_{i<j} \left[ 2 \sinh \left( \frac{u_i - u_j}{2} \right) \right]^{2d-2} = (-1)^{1+\frac{n(n+1)}{2}} \left[ \frac{\pi}{\Gamma(d)} \right]^n \frac{\Gamma(1-n(1-d))}{\pi(n-1)! \prod_{j=1}^{n-1} \sin(\pi jd)}.$$

(A.45)

Plugging (A.45) into (A.44) one arrives at equation (A.36).

Finally it is easy to see that for $n > k$ there is no pole at $p = p_n$ and $a_n^{(\pm)}(p)$ is analytic in some half plane $\Re p > p_n - \delta_n$ where $\delta_n$ is a finite positive constant.
A.5 Uniqueness of the solution of the Riemann-Hilbert problem

The above-described properties of the formal power series expansions $A_\pm(s, p)$ constitute a certain Riemann-Hilbert problem defining these series uniquely. For $k = 1$ this was shown in the Appendix of the work [27]. Here we generalize the result of [27] to the case $k > 1$.

Let us substitute the formal power series \((A.10)\) into the quantum Wronskian relation \((A.38)\). This yields an infinite set of relations for the meromorphic functions $a_n^{(+)}(p)$ of the following form

$$
\sin(\pi nd + 2\pi p) a_n^{(+)}(p) - \sin(\pi nd - 2\pi p) a_n^{(+)}(-p) = R_n(p),
$$

(A.46)

where $R_n(p)$ are expressed through $a_i^{(+)}(p)$ with $l = 1, \ldots, n - 1$ only. For example,

$$
R_1(p) = 0,
$$

(A.47)

$$
R_2(p) = \left( q a_1^{(+)}(p) + q^{-1} a_1^{(+)}(-p) \right)^2 e^{4\pi ip} \sin(2\pi p)/2.
$$

Introduce the function $r_n(p)$ such that

$$
r_n(p) = \Gamma(n - 1 - nd + 2p) / \Gamma(2 - n + nd + 2p) a_n^{(+)}(p),
$$

(A.48)

then \((A.46)\) takes the form

$$
r_n(p) - r_n(-p) = \frac{(-1)^{n-1}}{\pi} \times \Gamma(n - 1 - nd + 2p) / \Gamma(2 - n + nd + 2p) R_n(p).
$$

(A.49)

In the case $n = 1$, the RHS in \((A.49)\) vanishes and this equation supplemented by the analyticity condition \((A.35)\) implies that $r_1(p)$ is an even entire function of the variable $p$. Moreover, in view of the asymptotic condition \((A.37)\), one should conclude that $r_1 = \text{const.}$ The constant is defined by the residue condition \((A.36)\). This yields formula \((A.17)\).

We shall prove the uniqueness of the solution of \((A.46)\) for $n > 1$ by using induction. Let us assume all the functions $a_m^{(+)}(p)$ for $m = 1, \ldots, n - 1$ are already determined, so the LHS’s of equations \((A.46)\) and \((A.49)\) are given. Suppose also for the moment that the complex parameter $d$ belongs to the strip $\frac{n-1}{n} - \delta < \Re d < \frac{n-1}{n}$, where $\delta > 0$ is some small number. If $\Re \left( \frac{n-1}{n} - d \right)$ is sufficiently small, then $r_n(p)$ is analytic in the half plane $\Re p \geq 0$ except for a simple pole at $p = p_n \equiv \frac{1}{2} \left( n(1 - d) - 1 \right)$. The residue of $r_n(p)$ at this pole is fixed by the condition \((A.39)\). Also, it is easy to see that $r_n(p) \to 0$ as $p \to \infty$ within $\Re p > 0$. Hence, relation \((A.49)\) supplemented by the above quoted requirements for $r_n(p)$ constitutes a simple factorization problem. The uniqueness of this Riemann-Hilbert problem follows immediately from the Liouville theorem. To complete the induction step we note that the function $r_n(p)$ within $\Re d < 1$ can be obtained via the analytic continuation in the variable $d$ from the strip $\frac{n-1}{n} - \delta < \Re d < \frac{n-1}{n}$.

Thus we see that \((A.46)\) provides a recursion for the evaluation of $a_n^{(+)}(p)$ which allows one to express $a_n^{(+)}(p)$ in terms of an $(n - 1)$-fold integral. Explicit formulas for $a_2^{(+)}$ and $a_3^{(+)}$ in the case $k = 1$ can be found in the Appendix of [27]. Those formulas can be easily generalized to $k > 1$. It is worth noting that such integral representations prove to be convenient for numerical calculations because the integrals converge very fast at infinity.
A.6 Exact expressions for $A_\pm$

It was observed some years ago in the work [65], that $A_\pm(s,p)$ for $k = 1$ and some particular values of $p$ can be related exactly to the eigenvalue problem of a certain ordinary differential operator. This remarkable observation was generalized and proven for any $p$ in Ref. [28]. Here we extend the result of [28] to all integers $k > 1$.

Let us consider the Schrödinger equation

$$\left\{-\frac{\partial^2}{\partial u^2} + \kappa^2 \left[ \exp \left( \frac{\beta^2 u}{1 - \beta^2} \right) + \exp \left( \frac{u}{1 - \beta^2} \right) \right]^k - \xi^2 \right\} \Theta(u) = 0. \quad (A.50)$$

For $0 < \beta^2 < 1$ the potential term decays at $u \to -\infty$ and therefore for real $\xi > 0$ there is a Jost solution which is asymptotically plane wave as $u \to -\infty$:

$$\Theta_-(u, \xi) \to e^{-i\xi u}. \quad (A.51)$$

As a matter of fact, this particular solution is a meromorphic function in the whole complex plane of the parameter $\xi$, so it solves (A.50) for all values $\xi$ except points where $\Theta_-(u, \xi)$ has simple poles (see, e.g., [66]). Notice that $\Theta_-(u, -\xi)$ is another, linearly independent (for $\xi \neq 0$) solution of (A.50).

Since the potential in (A.50) grows rapidly as $u \to +\infty$, the Schrödinger equation admits also a solution which decays at large positive $u$. We denote this solution by $\Theta_+(u, \xi)$ and fix its normalization by the condition

$$\Theta_+(u, \xi) \to (2\kappa)^{-\frac{1}{2}} e^{F(i|\beta|u)} \quad \text{as} \quad u \to +\infty, \quad (A.52)$$

where $F(b \mid u)$ is defined in Eq. (2.35). $\Theta_+(u, \xi)$ is an entire function of $\xi^2$ and solves differential equation (A.50) for all values of this complex parameter.

The main objects of our interest are properly normalized Wronskians of the above-introduced solutions, namely,

$$D_\pm(\kappa, \xi) = \sqrt{\frac{2\pi (1 - d)}{\Gamma(1 \mp 2\xi(1 - d))}} \left( \kappa(1 - d) \right)^{\mp 2\xi(1 - d)} W[\Theta_+(u, \xi), \Theta_-(u, \pm \xi)]. \quad (A.53)$$

Here $d$ is given by (A.4) and the overall factor provides the normalization condition

$$\lim_{\kappa \to 0} D_\pm(\kappa, \xi) = 1. \quad (A.54)$$

In what follows we will show that for $0 < \beta^2 < 1$ and all values of $\kappa$ and $\xi$

$$A_\pm(s, p) = D_\pm(\kappa, \xi), \quad (A.55)$$

provided the variables are identified according to the relations:

$$\xi = \frac{ip}{1 - d}, \quad \kappa = \frac{1}{1 - d} \left[ \frac{2\pi \sqrt{s}}{\sqrt{k\Gamma(d)}} \right]^{1/d}. \quad (A.56)$$

More specifically, we will check that $D_\pm$ considered as functions of the variables $s$ and $p$, obey the same set of conditions (A.33)–(A.37) as $A_\pm(s, p)$. Thus (A.55) will follow from the uniqueness of the solution of the Riemann-Hilbert problem.
In order to examine properties of $D_{\pm}$ it is useful to make a change of the variable in (A.50),
\begin{equation}
  z = \frac{u}{2-2d} + \log \left(2\kappa (1-d)\right),
\end{equation}
and rewrite the differential equation using the notations $s$, $p$ (A.56) and $d$ (A.4):
\begin{equation}
  \left(-\partial_z^2 + e^{2z} + 4p^2 + \delta U(z)\right) \tilde{\Theta}(z) = 0,
\end{equation}
with
\begin{equation}
  \delta U(z) = \sum_{m=1}^{k} \frac{k!}{m!(k-m)!} \left(\frac{2^{2-d}\pi}{\sqrt{k}\Gamma(d)}\right)^{2m} s^m e^{2z(1-(1-d)m)}.
\end{equation}
The solutions $\Theta_{\pm}$ (A.51), (A.52) should be considered now as functions of the complex parameters $s$, $p$ provided $1 - \frac{1}{k} < d < 1$ and $z$ is real. In fact, it is convenient for the present discussion to change slightly their normalizations:
\begin{align}
  \tilde{\Theta}_{-}(z,p) &= \left(2\kappa (1-d)\right)^{\mp 2\xi(1-d)} \Theta_{-}(u,\xi), \\
  \tilde{\Theta}_{+}(z,p) &= \frac{1}{\sqrt{1-d}} \Theta_{+}(u,\xi).
\end{align}
Then formula (A.53) takes the form:
\begin{equation}
  D_{\pm}(s,p) = \sqrt{\frac{\pi}{2}} \frac{2^{\mp 2p}}{\Gamma(1 \pm 2p)} W[\tilde{\Theta}_{+}(z,p) ; \tilde{\Theta}_{-}(z,\pm p)].
\end{equation}
Both $\tilde{\Theta}_{+}$ and $\tilde{\Theta}_{-}$ are entire functions of $s$. In addition, $\tilde{\Theta}_{-}$ is a meromorphic function of $p$, while $\tilde{\Theta}_{+}$ is an even entire function of this complex parameter. Hence $D_{\pm}(s,p)$ can be expanded in a power series of $s$ similar to (A.33), and the expansion coefficients, $D_{n}\left(s\right) : D_{n}(^{+})(p) = D_{n}(^{-})(-p)$, are meromorphic functions of $p$.

To proceed further we will need the following conventional formula for $D_{+}(s,p)$, valid over the domain $\Re p \geq 0$,
\begin{equation}
  \frac{1}{D_{+}(s,p)} = 1 - \int_{-\infty}^{\infty} dz \ I_{2p}(e^{z}) \ \delta U(z) \ \chi(z).
\end{equation}
Here $I_{p}(x)$ is the modified Bessel function and $\chi(z)$ solves the Lipman-Schwinger equation,
\begin{equation}
  \chi(z) = K_{2p}(e^{z}) - \int_{-\infty}^{\infty} dz' \ G(z,z') \ \delta U(z') \ \chi(z'),
\end{equation}
where $G(z,z')$ is the Green function of (A.58) with $\delta U = 0$ subject of the asymptotic condition $\lim_{z \to \infty} G(z,z') = 0$. Notice that $\chi(z)$ differs from the solution $\tilde{\Theta}_{+}(z,p)$ by some overall $z$-independent factor only. The coefficients $D_{n}(^{+})(p)$ can be calculated
within standard perturbation theory developed with respect to $\delta U$. In fact, to determine $D_n^{(+)}(p)$ for given $n$, one needs to perform $n-1$-perturbative iterations in the Lipman-Schwinger equation. In particular,

$$
D_+ = 1 + \int_{-\infty}^{\infty} dz \ I_{2p}(e^z) K_{2p}(e^z) \delta U(z) + O((\delta U)^2). 
$$

(A.64)

Making use of (A.59) and the table integral

$$
\int_{0}^{\infty} dx \ x^{2\alpha-1} I_{2p}(x) K_{2p}(x) = \frac{\Gamma(\frac{1}{2} - \alpha) \Gamma(\alpha) \Gamma(\alpha + 2p)}{4\sqrt{\pi} \Gamma(1 - \alpha + 2p)},
$$

(A.65)

one finds for $n \leq k$:

$$
D_n^{(+)}(p) = \left( \frac{2^{2-d\pi}}{\sqrt{k \Gamma(d)}} \right)^{2n} \frac{k!}{n!(k-n)!} \frac{\Gamma(1 - n(1-d)) \Gamma(1 - n(1-d) + 2p)}{4\sqrt{\pi} \Gamma(n(1-d) + 2p)} + \ldots.
$$

(A.66)

Here the dots mean the terms corresponding to the higher-order perturbative in $\delta V$ contributions. Further, one may argue that the first-order contribution (A.66) is analytic in the half-plane $\Im p > p_n = \frac{1}{2}(n(1-d) - 1)$, whereas the omitted terms have a wider domain of analyticity. Hence, $D_n^{(+)}(p)$ with $n \leq k$ is an analytic function for $\Im p > p_n$ and has a simple pole located at $p = p_n$ with the same residue as in (A.36). Similarly, it is possible to show that $D_n^{(+)}(p)$ with $n > k$ is analytic in the half-plane $\Im p > p_n - \delta_n$ with some $\delta_n > 0$.

• The large $p$ behavior of the expansion coefficients $d_n^{(+)}(p)$ for the series $\log D_n^{(+)}(s,p)$ can be explored by means of standard semiclassical methods. A simple WKB analysis of the Schrödinger equation (A.58) shows that $d_n^{(+)}(p) \to C_n \ p^{1-2n(1-d)}$ as $p \to \infty$ and $\Im p > p_n$, with the constants $C_n$ given by

$$
C_n = \frac{(-1)^{n+1} k}{\Gamma(n - (1-d) k n) \Gamma(1 - (1-d) n) \Gamma(-\frac{1}{2} + (1-d) n)} \times
$$

$$
\left[ \frac{2\pi}{\sqrt{k \Gamma(d)}} \right]^{2n} \frac{\Gamma(n - (1-d) k n) \Gamma(1 - (1-d) n) \Gamma(-\frac{1}{2} + (1-d) n)}{2\sqrt{\pi} n! \Gamma(1 - (1-d) k n)}.
$$

(A.67)

• Finally, let us prove that $D_\pm(s,p)$ (A.61) obey the quantum Wronskian relation. We start with an observation that the following transformations of the variable $z$ and the parameters $s$ and $p$,

$$
\hat{\Lambda} : z \to z, \quad s \to s, \quad p \to -p,
$$

$$
\hat{\Omega} : z \to z + i\pi, \quad s \to q^{-2} s, \quad p \to p,
$$

(A.68)

leave the differential equation (A.58) unchanged while acting nontrivially on its solutions. It has been mentioned already, that the transformation $\hat{\Lambda}$ applied to the solution $\tilde{\Theta}_-(z,p)$ (A.60) yields another solution, and the pair

$$
\tilde{\Theta}_- = \tilde{\Theta}_-(z,p), \quad \hat{\Lambda}\tilde{\Theta}_- = \tilde{\Theta}_-(z,-p),
$$

(A.69)
forms a basis in the space of solutions of \((A.58)\). Indeed,
\[
W[\tilde{\Theta}_-, \hat{\Lambda} \tilde{\Theta}_-] = -4p , \tag{A.70}
\]
therefore the solutions \((A.69)\) are linearly independent provided \(p \neq 0\). The solution \(\tilde{\Theta}_+ \) \((A.60)\) can be always expanded in the basis \((A.69)\),
\[
\tilde{\Theta}_+ = c_1 \tilde{\Theta}_- + c_2 \hat{\Lambda} \tilde{\Theta}_- . \tag{A.71}
\]
Making use of \((A.70)\) we have
\[
c_2 = \frac{1}{4p} W(s, p) , \tag{A.72}
\]
with \(W(s, p) = W[\tilde{\Theta}_+, \tilde{\Theta}_-]\). The solution \(\tilde{\Theta}_+\) is an even function of \(p\), therefore
\[
c_1 = -\frac{1}{4p} W(s, -p) . \tag{A.73}
\]
Let us apply now the transformation \(\hat{\Omega} \) \((A.68)\) to both sides of \((A.71)\). Since
\[
\hat{\Omega} \tilde{\Theta}_-(z, \pm p) = e^{\pm 2\pi i p} \tilde{\Theta}_-(z, \pm p), \nonumber
\]
so
\[
\hat{\Omega} \tilde{\Theta}_+ = \frac{1}{4p} \left( e^{-2\pi i p} W(sq^{-2}, -p) \hat{\Lambda} \tilde{\Theta}_- - e^{2\pi i p} W(sq^{-2}, p) \tilde{\Theta}_- \right) . \tag{A.74}
\]
Now, it is not difficult to check that
\[
W_z[\tilde{\Theta}_+, \hat{\Omega} \tilde{\Theta}_+] = -2i . \tag{A.75}
\]
Combining formulae \((A.71)\), \((A.74)\) and \((A.75)\) one obtains the relation
\[
e^{2\pi i p} W(s, p) W(q^{-2} s, -p) - e^{-2\pi i p} W(q^{-2} s, p) W(s, -p) = 8i p . \tag{A.76}
\]
Hence \(D_{\pm}(s, p) \) \((A.61)\) satisfy the quantum Wronskian relation \((A.38)\) with \(A_{\pm}(s, p)\) replaced by \(D_{\pm}(s, p)\).

Notice that once the relation \((A.55)\) is established we may assert that series \((A.23)\), \((A.33)\) have an infinite radius of convergence.

**A.7 Formula for the DC conductance**

Since the parafermionic boundary sine-Gordon model \((A.5)\) has been of a certain interest in the quantum impurity problem \([4]\), we present here an exact formula for the DC conductance \((A.8)\) which follows immediately from the above consideration.

Using equation \((A.9)\), the conductance can be written in the form
\[
\frac{\alpha(V, T)}{\alpha_0} = 1 + \beta^2 \frac{s}{2pk} \partial_s \log \left( \frac{A_+}{A_-} \right) . \tag{A.77}
\]
This formula has a nice interpretation in terms of the Schrödinger problem (A.50). Indeed, as \( v \to -\infty \), the solution \( \Theta_+(v, \xi) \) (A.52) develops the asymptotic behavior:

\[
\Theta_+(u, \xi) \to \text{const} \left( e^{i\xi u} + S(\xi, \kappa) e^{-i\xi u} \right),
\]

where

\[
S(\xi, \kappa) = -\frac{W[\Theta_+(u, \xi), \Theta_-(u, -\xi)]}{W[\Theta_+(u, \xi), \Theta_-(u, \xi)]} \quad (A.79)
\]

is the reflection scattering amplitude for the Schrödinger equation (A.50). Therefore, in view of (A.53) and (A.55), the DC conductance is expressed in terms of \( S(\xi, \kappa) \) as follows,

\[
\frac{\alpha(V, T)}{\alpha_0} = 1 + \frac{k}{4i \xi (\beta - 1 - \beta)^2} \kappa \partial_\kappa \log S(\xi, \kappa), \quad V/T = \frac{4\pi \xi}{k} (1 - \beta^2). \quad (A.80)
\]

Recall that \( \beta^2 = \alpha_0/k \) is the dimensionless conductance per channel of the wire without impurity, while \( \kappa \) in (A.80) is the dimensionless inverse temperature measured in unit of the Kondo scale: \( \kappa = E^*/T, \ E^* \propto \mu k^{-\alpha_0} \).

Formula (A.80) extends the result of [28] to the case \( k > 1 \). We note also that as \( T \to 0 \), the dimensionless variable \( p = -iV/(4\pi T) \to \infty \). In this limit, the coefficients \( a_n^{(+)}(p) \) develop the simple asymptotic behavior (A.37) where the \( C_n \) are given by (A.67). This immediately yields the large-\( V \) expansion of the conductance at \( T = 0 \) proposed in Ref. [4]:

\[
\frac{\alpha(V, 0)}{\alpha_0} = 1 + \sqrt{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - kn\gamma) \Gamma(1 - n\gamma)}{n! \Gamma(1 - kn\gamma) \Gamma(\frac{3}{2} - n\gamma)} \left[ \frac{4\pi E^*_\gamma}{V} \right]^{2n\gamma}, \quad (A.81)
\]

where \( \gamma = \frac{k - \alpha_0}{k^2} \).

### B Appendix: Quantum Wronskian relation

The purpose of this appendix is twofold. First, it is intended to sketch how the quantum Wronskian relation (A.38) can be proven for the perturbative series \( A_+ \) (A.27), (A.30). Second, it provides a short review of some well-known issues of CFT which are useful for understanding the models under consideration.

Our previous discussion has always dealt with the so-called open string picture (channel). In this Hamiltonian picture the variable \( x \leq 0 \) plays the role of the space coordinate, while the coordinate \( \tau \) is treated as the compact Euclidean time. We now want to switch over to the “closed string” channel. In this channel the world-sheet coordinate \( x \) is viewed as the Euclidean time, so that the CFT lives in the finite volume \( 1/T \). The closed string channel Hilbert space is embedded in a tensor product of the chiral spaces of states of right and left movers. Below the following notational convention is applied: If the open string Hilbert space was denoted by \( \mathcal{H} \), then the right and left movers’ spaces of states in the closed string channel will be denoted (with some abuse of notations) as \( \tilde{\mathcal{H}} \) and \( \overline{\mathcal{H}} \) respectively. Of course, the spaces \( \tilde{\mathcal{H}} \) and \( \overline{\mathcal{H}} \) are isomorphic to each other. We will focus mainly on the “right” component \( \tilde{\mathcal{H}} \).
In the closed string channel an effect of the boundary at \( x = 0 \) can be described in terms of the boundary state which incorporates all information about boundary conditions [13, 55, 67, 68]. In the case of the “free” CFT (A.1) the corresponding conformal boundary state \(| B \rangle_{\text{free}}\) takes the form,

\[
| B \rangle_{\text{free}} = | B \rangle^{(\Phi)}_{\text{free}} \otimes | B \rangle^{(k)}_{\text{free}}, \tag{B.1}
\]

where

\[
| B \rangle^{(\Phi)}_{\text{free}} \subset \mathcal{H}^{(\Phi)} \otimes \overline{\mathcal{H}^{(\Phi)}} \tag{B.2}
\]

is the boundary state of the Gaussian CFT subject to the von Neumann boundary condition, while

\[
| B \rangle^{(k)}_{\text{free}} \subset \mathcal{H}^{(k)} \otimes \overline{\mathcal{H}^{(k)}} \tag{B.3}
\]

is the boundary state of the minimal parafermionic model with free boundary condition.

As an important step toward the derivation of (A.38), we will show that the power series \( A_\pm(s, p) \) are vacuum eigenvalues of certain operators acting in the chiral Hilbert space \( \mathcal{H}^{(\Phi)} \otimes \overline{\mathcal{H}^{(k)}} \). With this aim in mind we need to describe explicitly \( \mathcal{H}^{(\Phi)} \) and \( \overline{\mathcal{H}^{(k)}} \), as well as the boundary states (B.2) and (B.3).

### B.1 Chiral space \( \mathcal{H}^{(\Phi)} \) and boundary state \( | B \rangle^{(\Phi)}_{\text{free}} \)

A general solution to the bulk equation of motion \( \Delta \Phi = 0 \) has the form:

\[
\Phi(\tau, x) = \phi(\tau - ix) + \bar{\phi}(\tau + ix), \tag{B.4}
\]

where \( \phi(v) \) and \( \phi(\bar{v}) \) are right and left chiral fields respectively. In the case of the uncompactified Bose field \( \Phi \): \( \Phi(\tau + 1/T, x) = \Phi(\tau, x) \), the chiral fields admit the Fourier mode expansions of the form:

\[
\phi(v) = \frac{1}{2} \Phi_0 + \pi v T \Pi_0 + i \sum_{m \neq 0} \frac{\phi_m}{m} e^{-2\pi i m v T} \quad (v = \tau - ix), \tag{B.5}
\]

and

\[
\bar{\phi}(\bar{v}) = \frac{1}{2} \Phi_0 - \pi \bar{v} T \Pi_0 + i \sum_{m \neq 0} \frac{\bar{\phi}_m}{m} e^{2\pi i m \bar{v} T} \quad (\bar{v} = \tau + ix). \tag{B.6}
\]

The canonical quantization procedure leads to the following nontrivial commutation relations:

\[
[\Phi_0, \Pi_0] = i; \quad [\phi_n, \phi_m] = [\bar{\phi}_n, \bar{\phi}_m] = \frac{i}{2} \delta_{n+m,0}. \tag{B.7}
\]

Let \( \mathcal{F}_P \) (\( \bar{\mathcal{F}}_P \)) be the Fock space, i.e. the space generated by a free action of the operators \( \phi_n \) (\( \bar{\phi}_n \)) with \( n < 0 \) on the “vacuum” vector \( | P \rangle \) which satisfies

\[
\phi_n | P \rangle = \bar{\phi}_n | P \rangle = 0, \quad \text{for} \quad n > 0; \quad \Pi_0 | P \rangle = P | P \rangle, \quad \Pi_0 | P \rangle = P | P \rangle. \tag{B.8}
\]
Then the closed string space of states coincides with the direct integral $\int_p \mathcal{F}_P \otimes \bar{\mathcal{F}}_P$, and the corresponding chiral Hilbert space is given by

$$\hat{\mathcal{H}}^{(\Phi)} = \int_p \mathcal{F}_P .$$  \hspace{1cm} (B.9)

The boundary state (B.2) associated with the von Neumann boundary condition obeys the equation,

$$\left. \left( \partial_x \Phi(\tau, x) \right) \right|_{x=0} | B \rangle^{(\Phi)}_{\text{free}} = 0 .$$  \hspace{1cm} (B.10)

It reads explicitly as follows [67, 68]

$$| B \rangle^{(\Phi)}_{\text{free}} = g_D \exp \left( \sum_{n=1}^{\infty} \frac{2}{n} (-1)^n \phi_n \bar{\phi}_n \right) | \text{vac} \rangle^{(\Phi)} ,$$  \hspace{1cm} (B.11)

where $| \text{vac} \rangle^{(\Phi)} = | 0 \rangle \otimes | 0 \rangle$ is the ground state of the Gaussian theory in the closed string channel. Now, in view of (B.11) it is easy to see that for $1/T > \tau_1 > \cdots > \tau_n > 0$ the following relation, involving the correlators of the boundary exponential fields, holds:

$$\langle B | e^{ia_1 \Phi_B(\tau_1)} \cdots e^{ia_n \Phi_B(\tau_n)} | \text{vac} \rangle^{(\Phi)} = e^{-2\pi a P \sum_{j=1}^{n} \tau_j} \langle P | e^{2i a_1 \phi(\tau_1)} \cdots e^{2i a_n \phi(\tau_n)} | P \rangle .$$  \hspace{1cm} (B.12)

Here

$$e^{2ia\phi(\tau)} = (2i\pi T)^{\frac{1}{2}} e^{2a \sum_{m=1}^{\infty} \phi_m} \exp \left( 2a \sum_{m>0} \frac{\phi_m}{m} e^{2\pi i m \tau T} \right) \exp \left( ia \Phi_0 + 2i\pi a \tau T \Pi_0 \right) \times$$

$$\exp \left( -2a \sum_{m>0} \frac{\phi_m}{m} e^{-2\pi i m \tau T} \right)$$

is an intertwining operator for the Fock spaces:

$$e^{2ia\phi} : \mathcal{F}_P \rightarrow \mathcal{F}_{P+a} .$$  \hspace{1cm} (B.13)

### B.2 Chiral space $\hat{\mathcal{H}}^{(k)}$ and boundary state $| B \rangle^{(k)}_{\text{free}}$

#### B.2.1 Parafermionic and $W_k$ current algebras

Here we review shortly some well-known facts about the chiral Hilbert space of the minimal parafermionic models. The details can be found in Refs. [5, 36, 37].

Again, let the variable $v$ be a complex coordinate, $v = \tau - ix$, on the 2D cylinder. The chiral parafermionic algebra contains a set of currents $\{ \psi_n(v) \}_{n=1}^{k-1}$ of conformal dimensions

$$\psi_n \rightarrow \omega^{na} \psi_n$$

with $a = 0, \ldots k-1$, and its $\mathbb{Z}_k$ invariant component can be generated by the currents with the lowest conformal dimension, i.e., $\psi_+ \equiv \psi_1$ and $\psi_- \equiv \psi_{k-1}$, through the OPE:

$$\psi_+(u) \psi_-(v) = (u-v)^{\frac{k+2}{2k}} \left\{ I + \frac{k+2}{2k} (u-v)^{2} \left( W_2(u) + W_2(v) \right) + \right.$$

$$\left. \frac{1}{2k^2} (u-v)^{3} \left( W_3(u) + W_3(v) \right) + \ldots \right\} .$$  \hspace{1cm} (B.15)
The spin-2 current $W_2$ \[ (B.15) \], in turn, generates the Virasoro algebra with the central charge

$$c_k = \frac{2(k-1)}{k+2}. \quad (B.16)$$

The higher-order terms involve higher-spin currents, $W_3, W_4 \ldots$ forming the $W_k$ algebra introduced in [36, 37] with the special value \[ (B.16) \] of the central charge. Notice that the higher currents can also be produced recursively from singular parts of OPE's of the lower currents. In this sense the chiral $W_k$ algebra is generated by two basic currents $W_2$ and $W_3$.

The parafermionic algebra has a set of irreducible representations (irreps),

$$\{ \mathcal{M}_J \mid J = 0, \frac{1}{2}, \ldots \frac{k}{2} \}, \quad (B.17)$$

obtained through the action of the $\psi_+$ and $\psi_-$ modes on the highest vectors (see Ref. [5] for details). The irrep $\mathcal{M}_J$ decomposes into a direct sum of subspaces specified by the $\mathbb{Z}_k$ charge $\ell$ (number of $\psi_+$ modes minus number of $\psi_-$ modes modulo $k$):

$$\mathcal{M}_J = \bigoplus_{\ell=0}^{k-1} \mathcal{M}_J^{(\ell)}. \quad (B.18)$$

Each component $\mathcal{M}_J^{(\ell)}$ in \[ (B.18) \] is an irrep of the $W_k$ algebra.

Recall that an irrep of the $W_k$ algebra is obtained by factorizing a highest weight modules over submodules of “null-vectors” [36,37]. In the case under consideration, the highest weight can be thought of as a pair of eigenvalues $(\Delta, w)$ of mutually commuting operators

$$\frac{c_k}{24} + (2\pi T)^{-1} \int_0^{1/T} \frac{dv}{2\pi} W_2(v), \quad (2\pi T)^{-2} \int_0^{1/T} \frac{dv}{2\pi} W_3(v), \quad (B.19)$$

corresponding to the highest weight vector. In fact, it is most convenient to label the highest weight vectors by means of another pair of numbers $(J, M)$ such that

$$\Delta_{(J,M)} = \frac{J(J+1)}{k+2} - \frac{M^2}{k}, \quad w_{(J,M)} = \frac{M}{3\sqrt{k}} \left( \frac{2(3k+4)}{k} M^2 - 6 J(J+1) + k \right). \quad (B.20)$$

If $W_{(J,M)}$ denotes the irrep of highest weight $(\Delta_{(J,M)}, w_{(J,M)})$ \[ (B.20) \], then the $W_k$ structure of the parafermionic irreps \[ (B.18) \] is described as follows:

$$\mathcal{M}_0^{(0)} \simeq W_{(0,0)}, \quad \mathcal{M}_0^{(j)} \simeq W_{(\frac{k}{k-j})} \quad (j = 1, \ldots k-1), \quad (B.21)$$

$$\mathcal{M}_J^{(\ell)} \simeq W_{(J,J-\ell)} \quad \text{for} \quad \ell < 2J \quad (J > 0),$$

and

$$\mathcal{M}_J^{(\ell)} \simeq \mathcal{M}_{\frac{\ell}{k-j}}^{(\ell)} \quad \text{for} \quad \ell - \ell' = 2J \pmod k. \quad (B.22)$$
Let $\Sigma$ be a set containing $\frac{k(k+1)}{2}$ number of pairs $(J, M)$ of the form:

$$\Sigma = \{ (J, M) \mid J = 0, \frac{1}{2}, \ldots, \frac{k}{2}; \ M = -J, -J+1, \ldots, J; \ (J, -J) \equiv (\frac{k}{2} - J, \frac{k}{2} - J) \}.$$  \hfill (B.23)

As implied above, the chiral parafermionic algebra can be thought of as an algebra of intertwiners in the set

$$\{ \mathcal{W}_{(J,M)} \}_{(J,M) \in \Sigma} \hfill (B.24)$$

of “integrable” irreps of $W_k$. In particular,

$$\psi_{\pm}(v) : \mathcal{W}_{(J,M)} \rightarrow \mathcal{W}_{(J,M \pm 1)}.$$

(B.25)

Notice that the pairs $(J, -J)$ and $(\frac{k}{2} - J, \frac{k}{2} - J)$ are subject to the equivalence relation in (B.23) since the corresponding highest weights $(\Delta, w)$ (B.20) are the same, i.e., $\mathcal{W}_{(J,-J)} \simeq \mathcal{W}_{(\frac{k}{2} - J, \frac{k}{2} - J)}$. Finally, the closed string Hilbert space in the case of minimal parafermionic models is given by the direct sum $\bigoplus_{(J,M) \in \Sigma} \mathcal{W}_{(J,M)} \otimes \overline{\mathcal{W}}_{(J,M)}$, while the corresponding chiral Hilbert space can be described as follows:

$$\mathcal{H}^{(k)} = \bigoplus_{(J,M) \in \Sigma} \mathcal{W}_{(J,M)}.$$  \hfill (B.26)

### B.2.2 $W_k$ invariant boundary states

The $W_k$ algebra possess an automorphism of the form

$$\mathbb{C} : W_s \rightarrow (-1)^s W_s, \quad s = 2, 3, \ldots.$$  \hfill (B.27)

Thus [13, 68], there are two types of $W_k$ invariant CBC’s. In the string theory they are referred to as A and B-branes [15]. The corresponding boundary states obey the equations

$$\text{A-brane} : \quad [W_s(\tau) - (-1)^s \overline{W}_s(\tau)]_{x=0} |B\rangle = 0,$$

(B.28)

and

$$\text{B-brane} : \quad [W_s(\tau) - \overline{W}_s(\tau)]_{x=0} |B\rangle = 0.$$  \hfill (B.29)

Let $|\mathcal{I}_{(J,M)}\rangle_A$ be the A-type Ishibashi state [13, 15, 68] corresponding to the integrable irreps $\mathcal{W}_{(J,M)}$ from the set (B.24). In other words, it is a solution of (B.28) in the space $\mathcal{W}_{(J,M)} \otimes \overline{\mathcal{W}}_{(J,M)}$, that is unique, provided

$$\{ \langle (J, M) | \otimes \overline{\langle (J, M) |} \} : |\mathcal{I}_{(J,M)}\rangle_A = 1,$$

(B.30)

where $| (J, M) \rangle$ is the highest weight vector of $\mathcal{W}_{(J,M)}$. Now, following the formalism of Ref. [13], we can construct a set of states satisfying both the Cardy consistency condition and (B.28):

$$|B_{(j,m)}\rangle = \sum_{(J,M) \in \Sigma} \sqrt{\frac{S^{(J,M)}_{(j,m)}}{S^{(0,0)}_{(J,M)}}} |\mathcal{I}_{(J,M)}\rangle_A, \quad (j, m) \in \Sigma.$$  \hfill (B.31)
Here \( S^{(J,M)}_{(j,m)} \) is a modular matrix of the characters of the integrable irreps.\(^{12}\) The common wisdom then is that the \( \frac{k(k+1)}{2} \) amount of Cardy states \( |\{ B_{\frac{k}{2}}(\frac{j}{k} l, s) \}_{s=0}^{k-1} \rangle \) are the boundary states corresponding to some local, \( \hat{W}_k \) invariant boundary conditions. In Section 2.5 these CBC’s have been referred to as \( \mathcal{B}_{l,n} \) (0 \( \leq n \leq l \leq k - 1 \)), where the integers \( l, n \) are related to the pair \( (j, m) \) in \( (B.31) \) as follows: \( l = k - j - m \) and \( n = j - m \). In particular, the \( k \) amount of the boundary states \( \{ | B_{\frac{k}{2}}(\frac{j}{k} l, s) \rangle \}_{s=0}^{k-1} \) correspond to the \( k \) possible types of the fixed CBC’s, while \( \{ | B_{\frac{k}{2}}(\frac{j}{k} l, s) \rangle \}_{s=0}^{k-1} \) are the boundary states associated with the non-trivial fixed points of the boundary flow \( \mathcal{R}_g(k) \).

Let us consider now equation \( (B.29) \). Again, to construct nontrivial boundary states of the B-type one must draw on the B-type Ishibashi states \( |\mathcal{I}_{(J,M)}\rangle_B \), i.e., the unique normalized solutions of \( (B.29) \) in the space \( \mathcal{W}_{(J,M)} \otimes \overline{\mathcal{W}_{(J,M)}} \). Obviously \( |\mathcal{I}_{(J,M)}\rangle_B \) exists only if \( J \) and \( M \) satisfy the condition
\[
 w_{(J,M)} = 0 \, ,
\]
where the polynomial \( w_{(J,M)} \) is given by \( (B.20) \). The following solutions of the Cardy consistency condition \( [13] \) can be easily verified \([12–15]):\(^{13}\)
\[
 | B_j \rangle = \sqrt{k} \sum_{J=0}^{\left[\frac{j}{k}\right]} \frac{S^{(J,0)}_{(j,0)}}{\sqrt{S^{(J,0)}_{(0,0)}}} |\mathcal{I}_{(J,0)}\rangle_B \quad (j = 0, 1, 2, \ldots \left[\frac{j}{k}\right] ) \, .
\]

(B.33)

There are strong reasons to believe \([12–14]\) that the solution \( | B_j \rangle \) with \( j = 0 \) corresponds to the free CBC:
\[
 | B \rangle^{(k)}_{\text{free}} = | B_0 \rangle \, .
\]

(B.34)

By virtue of the above described structure of this boundary state, one can show that the following relations are satisfied when \( \sum_{s=1}^{n} \epsilon_s = 0 \) (mod \( k \)):
\[
^{(k)}_{\text{free}} \langle B | \psi_{\epsilon_1}(v_1) \ldots \psi_{\epsilon_n}(v_n) | \text{vac} \rangle^{(k)} = g_{\text{free}} \langle (0,0) | \psi_{\epsilon_1}(v_1) \ldots \psi_{\epsilon_n}(v_n) | (0,0) \rangle \, ,
\]

(B.35)

and
\[
^{(k)}_{\text{free}} \langle B | \bar{\psi}_{\bar{\epsilon}_1}(\bar{v}_1) \ldots \bar{\psi}_{\bar{\epsilon}_n}(\bar{v}_n) | \text{vac} \rangle^{(k)} = g_{\text{free}} \langle (0,0) | \bar{\psi}_{\bar{\epsilon}_1}(\bar{v}_1) \ldots \bar{\psi}_{\bar{\epsilon}_n}(\bar{v}_n) | (0,0) \rangle \, .
\]

(B.36)

Here we use the notation \( | \text{vac} \rangle^{(k)} \equiv | (0,0) \rangle \otimes | (0,0) \rangle \) for the closed string ground state of the minimal parafermionic model.

It is essential to point out that correlators \( (B.35) \) and \( (B.36) \) are not single-valued functions of their variables. To resolve the phase ambiguity in \( (B.35) \) we require that, (a)

\(^{12}\)In the case under consideration the modular matrix is given by \([10, 29] \),

\[
 S^{(J,M)}_{(j,m)} = \frac{2 \omega^{2 M m}}{\sqrt{k(k+2)}} \sin \left( \frac{\pi (2j+1)(2J+1)}{k+2} \right) \, .
\]

\(^{13}\)Equation \( (B.33) \) requires special consideration for even \( k \) and \( j = \frac{k}{4} \) (see Ref. \([15]\) for details)
when the arguments are real, \( v_s = \tau_s \), then \((B.35)\) is a real function within the domain 
\( 1/T > \tau_1 > \ldots > \tau_n > 0 \), and (b) the short distance expansions of \((B.35)\) must be consistent with the OPE's \((2.4)\) provided \( \Psi_n(\tau) \equiv \psi_n(\tau - ix)|_{x=0} \). Making use of the relation between parafermions and \( SU_k(2) \) WZW currents \([5]\), it is possible to show that both conditions can be indeed satisfied. Similarly, one can resolve the phase ambiguity of \((B.36)\). As a result, the boundary values of the correlation functions \((B.35)\) and \((B.36)\) at real \( v_s = \bar{v}_s = \tau_s \) coincide within the domain \( 1/T > \tau_1 > \ldots > \tau_n > 0 \).

### B.3 Baxter’s operators

In the boundary state formalism, thermal expectation values \((A.14)\) are expressed in terms of correlators of the form:

\[
\langle\langle \cdots \rangle\rangle_0 = \left< \frac{\text{free}}{B} | \cdots | \text{vac} \right>_{\text{free}}, \quad |\text{vac}\rangle = |\text{vac}\rangle^{(\Phi)} \otimes |\text{vac}\rangle^{(k)},
\]

with \(|B\rangle_{\text{free}}\) defined in \((B.1)\). Our previous analysis implies, therefore, that the integrand in \((A.26)\) can be written as an expectation value over the highest vectors \(|P, (J,M)\rangle = |P\rangle \otimes |(J,M)\rangle\) with \( J = M = 0 \):

\[
e^V \sum_{j=1}^{2n} \epsilon_j \tau_j \left( \left\langle \left\langle V^{(\text{Mats})}_{\epsilon_1}(\tau_1) \cdots V^{(\text{Mats})}_{\epsilon_{2n}}(\tau_{2n}) \right\rangle \right\rangle_0 = \langle (0,0), P | U_{\epsilon_1}(\tau_1) \cdots U_{\epsilon_n}(\tau_n) | P, (0,0) \rangle .
\]

Here the parameter \( V \) is related to the zero-mode momentum \( P \):

\[
V = 2\pi i \frac{T \beta P}{\sqrt{k}}, \quad (B.39)
\]

and

\[
U_{\pm}(\tau) = \psi_{\pm} e^{\pm \frac{2\pi i}{\sqrt{k}} \Phi(\tau)} . \quad (B.40)
\]

From the mathematical point of view, \( U_{\pm}(\psi) \) are intertwining operators (chiral vertex operators) of the conformal dimension \( d \) \((A.4)\). They act in the chiral Hilbert space

\[
\tilde{\mathcal{H}}_{\text{free}} = \int_P \otimes_{(J,M)} \mathcal{V}_{P,(J,M)} \quad \text{with} \quad \mathcal{V}_{P,(J,M)} \equiv \mathcal{F}_P \otimes \mathcal{W}_{(J,M)} \quad (B.41)
\]

in accordance with the formula

\[
U_{\pm}(\tau) : \mathcal{V}_{P,(J,M)} \rightarrow \mathcal{V}_{P_{\pm \frac{2}{\sqrt{k}}} (J,M_{\pm 1})} . \quad (B.42)
\]

Equation \((B.38)\), in turn, implies that the power series \( A_{\pm}(s,p) \) \((A.24), (A.30)\) can be thought of as expectation values over the state \(| P, (0,0) \rangle\) with \( P = 2\sqrt{k} P/\beta \) of the operators

\[
A_{\pm}(s) = \left( \text{Tr}_{\rho_{\pm}} \left[ e^{\pm 2\pi i \sqrt{k} \Pi_{\pm} h} \right] \right)^{-1} \times \quad (B.43)
\]

\[
\text{Tr}_{\rho_{\pm}} \left[ e^{\pm 2\pi i \sqrt{k} \Pi_{\pm} h} \tau \exp \left\{ \mu \int_0^{1/T} d\tau K_{\pm}(\tau) \right\} \right] .
\]

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Here $K_{\pm}(\tau) = a_{\pm} U_+(\tau) + a_{\pm} U_-(\tau)$ and $s = \mu^2 (2\pi T)^{2d-2}$.

Some explanations of formula (B.43) are in order at this point. First, it contains the ordered exponential (the symbol $T$ denotes the path ordering) which is defined in terms of the power series in $\mu$ as follows,

$$
T \exp \left\{ \mu \int_0^{1/T} d\tau \ K_{\pm}(\tau) \right\} = 1 + \sum_{n=1}^{\infty} \mu^n \times \int_{1/T > \tau_1 > \tau_2 > \ldots > \tau_n > 0} \prod_{k=1}^{n} d\tau_k \ K_{\pm}(\tau_1) K_{\pm}(\tau_2) \ldots K_{\pm}(\tau_n) .
$$

In fact, only even powers of $\mu$ survive in the definition (B.43), so $A_{\pm}$ are formal series in $s \propto \mu^2$ whose expansion coefficients are operators acting in the chiral Hilbert space (B.41). Second, the factor $\omega^{2Mh}$ appearing in (B.43) was undefined yet. As follows from the commutation relations (A.19), only integer powers of the operator $\omega^{2M}$ are involved in (B.43). The action of $\omega^{2M}$ in the chiral Hilbert space (B.41) is defined by the formula

$$
\omega^{2M} \ Y_{P,(J,M)} = \omega^{2M} \ Y_{P,(J,M)} . 
$$

Notice that, because of the isomorphism $Y_{(J,\pm J)} \simeq Y_{(\frac{1}{2}-J,\frac{1}{2}+J)}$, the operator $M$ itself is ill-defined.

In spite of the fact that the factor $\omega^{2Mh}$ does not contribute to the matrix element $\langle P, (0, 0) | A_{\pm} | P, (0, 0) \rangle$, it plays an important role in the definition of $A_{\pm}$. Due to this phase factor the operators $A_{\pm}$ commutes with the operator

$$
\mathbb{I}_1 = \int_0^{1/T} d\tau \frac{1}{2\pi} (\partial \phi)^2 + W_2 . 
$$

A proof of the commutation relation

$$
[I_1, A_{\pm}(s)] = 0 ,
$$

is based on the quasiperiodic properties of the intertwining operators $\psi_{\pm}(u)$:

$$
\psi_{\pm}(u + 1/T) = \psi_{\pm}(u) \omega^{1+2M} . 
$$

It follows along the line of Appendix C of Ref. [26]. Notice that the operators $I_1 + \mathbb{I}_1$ and $I_1 - \mathbb{I}_1$ coincide, respectively, with Hamiltonian and the momentum operator in the closed string channel. Hence, commutation relation (B.47) manifests the invariance of $A_{\pm}(s)$ with respect to translations along the $\tau$-direction.

It is worth to note that the operators $A_{\pm}(s)$ (B.43) act invariantly in $Y_{P,(J,M)}$. Furthermore, since each space $Y_{P,(J,M)}$ naturally splits into the direct sum of finite dimensional “level subspaces”

$$
Y_{P,(J,M)} = \bigoplus_{\ell=0}^{\infty} Y_{P,(J,M)}^{(\ell)} ; \quad L \ Y_{P,(J,M)}^{(\ell)} = \ell \ Y_{P,(J,M)}^{(\ell)} ,
$$

and the grading operator $L$ essentially coincides with $I_1$, then $A_{\pm}(s)$ act invariantly in all the level subspaces $Y_{P,(J,M)}^{(\ell)}$. In particular, the highest vectors $| P, (J, M) \rangle$ are eigenvectors for $A_{\pm}(s)$ and

$$
A_{\pm}(s) | P, (0, 0) \rangle = A_{\pm}(s,p) | P, (0, 0) \rangle \quad \text{with} \quad p = \frac{\beta P}{2\sqrt{k}} .
$$
Following the route of the work [26] it is possible to prove that $A_\pm(s)$ commute among themselves for any values of $s$,

$$[A_\pm(s), A_\pm(s')] = [A_+(s), A_-(s')] = 0,$$  \hspace{1cm} (B.51)

and satisfy the operator-valued identity

$$e^{\frac{2\pi i}{\sqrt{k}} \Pi_0} A_+(qs) A_-(q^{-1}s) - \omega^{2M} A_+(q^{-1}s) A_-(qs) = e^{\frac{2\pi i}{\sqrt{k}} \Pi_0} - \omega^{2M},$$  \hspace{1cm} (B.52)

with $q = e^{i\pi d}$. Again, formulae (B.51) and (B.52) should be understood as an infinite set of formal operator-valued relations for the expansion coefficients of $A_\pm$ without any reference to convergence issues. The quantum Wronskian relation (A.38) follows immediately from (B.50) and (B.52).

A key idea of the proof of (B.51) and (B.52) is based on the observation that the formal integrals

$$x_0 = \frac{1}{q - q^{-1}} \int_0^{1/T} du \ U_-(u), \quad x_1 = \frac{1}{q - q^{-1}} \int_0^{1/T} du \ U_+(u)$$  \hspace{1cm} (B.53)

satisfy the Serre relations for the quantum affine algebra $U_q(\widehat{sl}(2))$ and can be identified with the generators of its Borel subalgebra. With this observation the construction elaborated in [26] can be applied without any changes. Finally, we note that the operators $A_\pm$, acting in the CFT chiral Hilbert space, play a role similar to that of the Q-operators of Baxter’s lattice theory [69].

\section{Appendix : Partition function of the BP sinh-Gordon model}

If the parameter $\beta^2 = -b^2$ is negative the Schrödinger equation (A.50) takes the form (2.32). It is quite natural to expect that this differential equation within the parametric domain $b^2 > 0$ is somehow related to the BP sinh-Gordon model (2.14). The precise relation is proposed in Eq. (2.36). Unfortunately, a rigorous derivation of this formula does not currently exist. Here we briefly argue in favour of (2.36) based on the high- and low-temperature expansions of the partition function $Z_{\text{bshg}}^{(k)}(h)$.

\subsection{C.1 High-temperature expansion}

\subsubsection{C.1.1 The BP Liouville model}

First of all, let us discuss the leading high-temperature behavior of (2.29) based on properties of the Hamiltonian (2.14). Qualitatively, one may expect that if $\Re h$ is not too small, the limit $T \to \infty$ of (2.29) is controlled by either the boundary operator $V_+ \equiv \Psi_+ e^{\frac{\sqrt{k} \Phi_+}{\hbar}}$ or $V_- \equiv \Psi_- e^{-\frac{\sqrt{k} \Phi_-}{\hbar}}$ in (2.14) depending on the sign $\Re h$, with some crossover at small $\Re h$.

More precisely, let us assume that $\Re h < 0$. Then we can treat the term $\mu V_-$ in the Hamiltonian (2.14) as a perturbation, and in leading approximation one has

$$Z_{\text{bshg}}^{(k)}(h) \big|_{T \to \infty} \to Z_{\text{bl}}^{(k)}(h) \quad (\Re h < 0),$$  \hspace{1cm} (C.1)
where $Z_{bl}^{(k)}(h)$ is the partition function corresponding to the Hamiltonian of the “BP Liouville” model:

$$H_{bl}^{(k)} = H_{\text{free}} + h \Phi_B - \mu \Psi_+ e^{\frac{h}{\sqrt{k}} \Phi_B}.$$  \hspace{1cm} (C.2)

An important feature of the QFT (C.2) is the conformal invariance. To be precise, the theory can be made scale and conformally invariant by an appropriate redefinition of the RG transformation, namely by supplementing it by a formal field redefinition

$$\Phi \to \Phi + Q \sqrt{k} \delta t_{\text{RG}},$$

where the RG “time” $t_{\text{RG}} \sim \log(1/T)$ and $Q = b + b^{-1}$. In the nomenclature of string theory this corresponds to introducing a linear dilaton which modifies slightly the stress-energy tensor of the model:

$$T^{(2)} = -(\partial \Phi)^2 + \frac{Q}{\sqrt{k}} \partial^2 \Phi + W_2,$$

$$\bar{T}^{(2)} = -\bar{(\partial \Phi)}^2 + \frac{Q}{\sqrt{k}} \bar{\partial}^2 \Phi + \bar{W}_2.$$  \hspace{1cm} (C.3)

The first two terms in each equation (C.3) constitute the corresponding chiral components of the modified stress-energy tensor of the Gaussian theory, while $W_2$ and $\bar{W}_2$ are chiral components of the stress-energy tensor of the minimal parafermionic model (see the OPE (B.15)). The scaling dimension of the boundary operator $\Psi_+ e^{\frac{h}{\sqrt{k}} \Phi_B}$ with respect to the modified stress-energy tensor (C.3) equals one and, hence, the temperature and $\mu$-dependences of the partition function $Z_{bl}^{(k)}(h)$ readily follows from the dimensional analysis:

$$Z_{bl}^{(k)}(h) = (2\pi T)^{\frac{iQ}{\sqrt{k}}} \mu^{-\frac{iP - iT}{k}} G^{(k)}(P | b),$$  \hspace{1cm} (C.4)

with

$$P = i \frac{h}{T}.$$  \hspace{1cm} (C.5)

Here $G^{(k)}$ is some function of the dimensionless parameters $P$, $b$ and $k$.

In a view of the standard heuristic arguments of Ref. [72], it is expected that $G^{(k)}(P | b)$ is an analytical function of the complex variable $P$ in the lower half plane $\Im m P < 0$. Furthermore, following [72] in the “perturbative” calculation of the partition function $Z_{bl}^{(k)}$, one can first integrate out the constant mode of the Bose field $\Phi$. This integration produces simple poles in the variable $P$ at the points $P = i n b \sqrt{k}$ ($n = 0, 1, \ldots$), and the corresponding residues

$$G_n^{(k)} = \text{Res}_{P = i n b \sqrt{k}} \left[ G^{(k)}(P | b) \right] \quad (n = 0, 1, 2 \ldots)$$  \hspace{1cm} (C.6)

are expressed through the integrals of the $k \times n$-points Matsubara correlation functions

$$G_n^{(k)} = \frac{g_D g_{\text{free}}}{2\pi i} (2\pi T)^{n-(kn-1)nb^2} \times$$

$$\int_0^\tau d\tau_1 \ldots \int_0^{\tau_{kn-1}} d\tau_{kn} \langle \langle V_+^{(\text{Mats})}(\tau_1) \ldots V_+^{(\text{Mats})}(\tau_{kn}) \rangle \rangle.'$$  \hspace{1cm} (C.7)

Here $g_{\text{free}}$ is the boundary degeneracy of the free CBC in the minimal parafermionic model (see Eq. (1.2)), and the prime means that the constant mode contribution is excluded from
the thermal averaging \( \langle \cdots \rangle_0 \) \( (A.14) \). The free-field correlators in \( (C.7) \) have an especially simple form for \( n = 1 \). In this case the integral \( (C.7) \) can be brought to the form of the Selberg integral \( [63, 64] \). This yields the explicit formula

\[
G_1^{(k)} = \frac{\mathbb{g}_D \mathbb{g}_{\text{free}}}{2\pi i} \frac{k!}{k^2} \int_0^{2\pi} dv_1 \cdots \int_0^{\nu_{k-1}} dv_k \prod_{i<j} [2 \sin \left( \frac{v_i-v_j}{2} \right)]^{-\frac{1+4b^2}{k}} = \frac{\mathbb{g}_D \mathbb{g}_{\text{free}}}{2\pi i} \Gamma(-b^2) \frac{2\pi}{\sqrt{k\Gamma(1-i\frac{b^2}{k})}}. \tag{C.8}
\]

Another highly nontrivial property of the BP Liouville partition function can be guessed by virtue of the form of stress-energy tensor \( (C.3) \). Namely, it is invariant with respect to the transformation \( b \to b^{-1} \). In the case of the conventional Liouville model a similar phenomenon manifests a remarkable non-perturbative duality of the theory. If one admits that the same symmetry occurs in the BP Liouville model, then, the function \( G^{(k)}(P | b) \) should also possess an additional series of “dual” poles at \( P = i b^{-1} \sqrt{k} n \).

It should be mentioned that the model \( (C.2) \) has been already studied in the works \([70]\) and \([71]\) for the cases \( k = 1 \) and \( k = 2 \), respectively. In both cases the function \( G^{(k)}(P | b) \) looks as follows

\[
G^{(k)}(P | b) = \frac{\mathbb{g}_D \mathbb{g}_{\text{free}}}{2\pi i \Gamma} \Gamma \left( 1 + i \frac{bP}{\sqrt{k}} \right) \Gamma \left( 1 + \frac{iP}{b\sqrt{k}} \right) \frac{2\pi \frac{b^2}{k}}{\sqrt{k\Gamma(1-i\frac{b^2}{k})}}. \tag{C.9}
\]

Notice that \( (C.9) \) satisfies the all above-mentioned conditions even thought \( k > 2 \).

### C.1.2 General structure of the high-temperature expansion

Above we have discussed the leading high-temperature behavior of \( Z_{\text{bshg}}^{(k)} \) in the case \( \Re e h < 0 \). Obviously for \( \Re e h > 0 \), \( Z_{\text{bl}}^{(k)}(h) \) in asymptotic formula \( (C.1) \) should be replaced by \( Z_{\text{bl}}^{(k)}(-h) \). Since \( Z_{\text{bl}}^{(k)}(\pm h) \) vanishes in the limit \( T \to \infty \) if \( h \) is taken in the “wrong” half plane (note the factor \( T^{i\frac{i k Q}{2}} \) in \( (C.1) \)), this in turn suggests that the overall \( T \to \infty \) asymptotics of the partition function is correctly expressed by the sum

\[
Z_{\text{bshg}}^{(k)} |_{T \to \infty} \to Z_{\text{bl}}^{(k)}(h) + Z_{\text{bl}}^{(k)}(-h). \tag{C.10}
\]

What can be said about corrections to the leading asymptotic? Again we assume that \( \Re e h < 0 \) and consider the perturbative effect of the term \( \mu V_- \) in \( (2.14) \) to the partition function \( (C.4) \). In the unperturbed BP Liouville model theory the parameter \( \mu \) is a dimensionless constant. Let us eliminate \( \mu \) from the Hamiltonian \( (C.2) \) by shifting the field \( \Phi \). Then the coupling \( \mu \) in front of the \( V_- \) in Eq. \( (2.14) \) is replaced by \( \mu^2 \). Since the anomalous dimension of the boundary operator \( V_- \) with respect to the BP Liouville stress-energy tensor \( (C.3) \) is given by \( 1 - \frac{2Q}{k} \), then \( \mu^2 \sim E_* \kappa \), where \( E_* \) is the physical energy scale of the BP sinh-Gordon model. Hence we deduce that the perturbative corrections to the leading asymptotics \( (C.1) \) should be in a form of power series expansion of the dimensionless parameter \( \kappa \), with \( \kappa = E_* / T \). The case \( \Re e h > 0 \) can be analyzed similarly and one comes to
the same conclusion about the form of perturbative corrections to (C.10). As was mentioned above, it is quite natural to expect that the leading asymptotics (C.10) is invariant with respect to the duality transformation \( b \rightarrow b^{-1} \). If one assumes that the BP sinh-Gordon partition function possesses this property of self-duality as well, then the high-temperature expansion of \( Z_{\text{bshg}}^{(k)} \) should be of the form

\[
Z_{\text{bshg}}^{(k)} = Z_{\text{bl}}^{(k)}(h) \, M(\kappa, h) + Z_{\text{bl}}^{(k)}(-h) \, M(\kappa, -h) .
\]  
(C.11)

Here \( M(\kappa, h) \) is a formal double power series in integer powers of \( \kappa \frac{2Q}{k} \) and \( \kappa \frac{Q}{k} Q \):

\[
M(\kappa, h) \simeq 1 + \sum_{n,m} M_{n,m}(h) \, \kappa^{2Q} (\kappa b + m b^{-1}) .
\]  
(C.12)

### C.1.3 Small \( \kappa \) expansion of the spectral determinant

In order to justify the proposed formula for \( Z_{\text{bshg}}^{(k)} \) (2.36), we should observe the structure (C.11) in the small-\( \kappa \) expansion of the spectral determinant \( D(\kappa, \xi) \) (2.42). When \( \kappa \) goes to zero and \( u \ll 1 \) the potential in the Schrödinger problem (2.32) can be approximated by \( \kappa^2 e^\frac{bQ}{\sqrt{\kappa}} \). To understand the quality of this approximation one can make a change the variable (A.57) with \( d = 1 - bQ/k \) and bring the differential equation (2.32) to the form (A.58). Therefore for \( u \gg 1 \)

\[
\Theta_+(u) \rightarrow \sqrt{2Qb \frac{\pi}{k}} K_{i\frac{bP}{\sqrt{k}}} \left( \frac{2Qb\kappa}{k} e^{\frac{bQ}{\sqrt{k}}} \right) .
\]  
(C.13)

Here and below, in order to write formulae in the most instructive form, we use the notation \( P = 2\sqrt{\kappa} \) rather than \( \xi \) (see relations (C.5) and (2.38)). The overall normalization in (C.13) is chosen to ensure the asymptotic condition (2.34). Within the domain,

\[
-\frac{Q}{kb} \log \left( \frac{1}{\kappa^2} \right) \ll u \ll \frac{Qb}{k} \log \left( \frac{1}{\kappa^2} \right) ,
\]  
(C.14)

the potential in (A.58) develops a wide plateau and the solution \( \Theta_+ \) (C.13) becomes a combination of the two plane waves,

\[
\Theta_+(u) = \sqrt{\frac{Qb}{2\pi k}} \, \Gamma( -i \frac{bP}{\sqrt{k}} ) \left( \frac{Qb\kappa}{k} \right)^{-i \frac{bP}{\sqrt{k}}} A(\kappa, -P \mid b) \, e^{i \sqrt{\kappa} P \frac{u}{2Q}} + \\
\sqrt{\frac{Qb}{2\pi k}} \, \Gamma( i \frac{bP}{\sqrt{k}} ) \left( \frac{Qb\kappa}{k} \right)^{i \frac{bP}{\sqrt{k}}} A(\kappa, P \mid b) \, e^{-i \sqrt{\kappa} P \frac{u}{2Q}} ,
\]  
(C.15)

with

\[
A(\kappa, P \mid b) = 1 + O \left( \kappa \frac{2Qb}{k} \right) .
\]  
(C.16)

When \( \kappa \) is small but finite, corrections to (C.16) can be obtained using the perturbation theory. In view of equation (A.59), these corrections have the form of a power series in \( \kappa \frac{2Qb}{k} \):

\[
A(\kappa, P \mid b) \simeq 1 + \sum_{n=1}^{\infty} A_n(\kappa \mid b) \, \kappa \frac{2Qb}{k} .
\]  
(C.17)
The formal power series $A(\kappa, P \mid b)$ amounts to $A_+(s, p)$ \[^{[A.55]}\] taken at $\beta^2 = -b^2 < 0$ with the variable $s$ related to $\kappa$ as in \[^{[A.56]}\] and $p = i\frac{bP}{2\sqrt{k}}$. We have no reason to expect the convergence of the power series \[^{[C.17]}\] for $b^2 > 0$, i.e., the series should be understood as a formal asymptotic expansion.

To find the form of the solution $\Theta_-$ in the domain \[^{[C.14]}\], we note that the transformation $u \to -u$ and $b \to b^{-1}$ leaves the Schrödinger equation \[^{[2.32]}\] unchanged while interchanging the solutions $\Theta_+$ and $\Theta_-$ \[^{[2.34]}\].

Hence, $\Theta_-$ can be obtained by means of this transformation from \[^{[C.15]}\] within the domain \[^{[C.14]}\]. Now the Wronskian \[^{[2.42]}\] can be calculated at any point from \[^{[C.14]}\], where both $\Theta_\pm$ are combinations of plane waves. It yields the following form of the small-$\kappa$ expansion of the spectral determinant:

$$D(\kappa, \xi) \simeq R(P) A(\kappa, P \mid b) A(\kappa, P \mid b^{-1}) + R(-P) A(\kappa, -P \mid b) A(\kappa, -P \mid b^{-1}) ,$$

with

$$R(P) = \frac{iP}{2\pi\sqrt{k}} \Gamma\left(\frac{iP}{\sqrt{k}}\right) \Gamma\left(\frac{iP}{b\sqrt{k}}\right) b^{\frac{P}{b\sqrt{k}}(b^{-1}-b)} \left(\frac{\kappa Q}{k}\right)^{1/\sqrt{\kappa}} , \quad (C.18)$$

and $P = \frac{2Q}{\sqrt{k}} \xi$.

The high-temperature expansion \[^{(C.18)}\] is in agreement with \[^{(C.11)}\]. Furthermore, it implies that formula \[^{(C.9)}\] holds true for any $k \geq 1$.

### C.2 Low-temperature expansion

Here we elucidate the low-temperature behavior of $Z_{\text{bsgh}}(h)$ using the boundary state formalism described in Appendix B.

In the case of the BP sinh-Gordon model the boundary state $|B\rangle_{\text{bsgh}}$ is some particular vector in the closed string space of state

$$|B\rangle_{\text{bsgh}} \in \int_{p} ^{P} \oplus_{(J,M)} \mathcal{V}_{P,(J,M)} \otimes \bar{\mathcal{V}}_{P,(J,M)} , \quad (C.19)$$

where we use the same notations as in formula \[^{[B.41]}\]. Unlike the conformal boundary states discussed in Appendix B, $|B\rangle_{\text{bsgh}}$ essentially depends on the temperature via the dimensionless parameter $\kappa = E_\star / T$. In the presence of the external field $h$ \[^{[2.14]}\], the partition function $Z_{\text{bsgh}}(h)$ can be expressed in terms of the vacuum overlap of the boundary state \[^{(C.19)}\] analytically continued to pure imaginary values of the variable $P$ \[^{(C.5)}\]:

$$Z_{\text{bsgh}}(h) = _{\text{bsgh}}\langle B | \cdot \left\{ |P, (0,0)\rangle \otimes \bar{|P, (0,0)\rangle} \right\} \right|_{P=i\frac{h}{T}} . \quad (C.20)$$

The motivation behind \[^{(C.20)}\] can be found in Ref. \[^{[1]}\] where a similar relation was exploited in the context of the circular brane model.

\[^{14}\]For this reason the partition function \[^{[2.30]}\] is invariant with respect to the duality transformation $b \to b^{-1}$.
The simplest idea about the infrared fixed point of the BP sinh-Gordon boundary flow is that it corresponds to the fixed CBC for both bosonic and parafermionic sectors of the theory. This scenario implies that

\[
|B\rangle_{\text{bsgh}} \rightarrow e^{-\frac{E_{\text{bsgh}}}{T}} |B\rangle_{D}^{(\Phi)} \otimes |B_{(0,m)}\rangle \quad \text{as} \quad \kappa \rightarrow \infty .
\]  

(C.21)

Here \(E_{\text{bsgh}} \propto E_{*}\) is the ground state energy, \(|B_{(0,m)}\rangle\) is the boundary state corresponding to the fixed CBC of the minimal parafermionic model \((B.31)\), and

\[
|B\rangle_{D}^{(\Phi)} = g_{D} \int_{P} \exp \left( \sum_{n=1}^{\infty} \frac{2}{n} \phi_{-n} \bar{\phi}_{-n} \right) |P\rangle \otimes |\overline{P}\rangle
\]

(C.22)

is the boundary state associated with the Dirichlet CBC \([67, 68]\). The assumption \((C.21)\) leads, in turn, to the leading low-temperature asymptotics for the BP sinh-Gordon partition function:

\[
\log Z_{\text{bsgh}}(h) = -\frac{E_{\text{bsgh}}}{T} + \log (g_{D} g_{\text{fixed}}) + O(T).
\]

(C.23)

To make more definite predictions about the low-temperature expansion of \(Z_{\text{bsgh}}(h)\), one should take into account the integrability of the model. In a forthcoming paper \([73]\) we intend to present a comprehensive discussion of a relevant set of local Integral of Motions (IM’s) in the context of more general QFT model with boundary interaction. Skipping details of the computational complexity we mention here that the boundary state \((C.19)\) is expected to satisfy an infinite set of conditions

\[
(\mathbb{I}_{2l-1} - \mathbb{I}_{2l-2}) |B\rangle_{\text{bsgh}} = 0 \quad \text{for some operators}
\]

(C.24)

for some operators

\[
\mathbb{I}_{2l-1}(x) = \int_{0}^{1/T} \frac{d\tau}{2\pi} T_{2l}(\tau - ix) , \quad \mathbb{I}_{2l-2}(x) = \int_{0}^{1/T} \frac{d\tau}{2\pi} \bar{T}_{2l}(\tau + ix) ,
\]

(C.25)

where \(T_{2l}\) and \(\bar{T}_{2l}\) are chiral currents of the Lorentz spin \(2l\) and \((-2l)\), respectively. As has been explained in Appendix C, the local chiral currents of the parafermionic minimal model constitute \(W_{k}\) algebra. Therefore the local densities \(T_{2l}\) are, in fact, appropriately regularized polynomial in \(\partial \phi\) and \(W_{k}\) currents as well as their derivatives. Similarly, \(\bar{T}_{2l}\) are constructed using \(\bar{\partial} \phi\) and \(\bar{W}_{k}\) currents. Notice that any operators in the form \((C.25)\) are conserving charges in the sense that

\[
\frac{d}{dx} \mathbb{I}_{2l-1}(x) = \frac{d}{dx} \mathbb{I}_{2l-2}(x) = 0 ,
\]

(C.26)

and the meaning of \((C.24)\) is that the boundary neither emits nor absorbs any amount of the combined charges \(\mathbb{I}_{2l-1} - \mathbb{I}_{2l-2}\) \([55]\). For this reason \(\mathbb{I}_{2l-1}\) and \(\mathbb{I}_{2l-2}\) are referred to as local IM’s.

\^\text{15}Recall, in order to fix the value of integer \(m = 0, 1, \ldots k - 1\) in \((C.21)\), one needs to specify the normalization of the boundary fields \(\Psi_{\pm}\) unambiguously \((\text{see Section} \, 2.1)\).
The local IM’s $\mathbb{I}_1$ and $\mathbb{I}_3$ are known in the closed form. Namely, $\mathbb{I}_1$ is given by (B.46) and an explicit bosonized form of the first nontrivial local density $T_4$ is presented in Ref. [74]. There are strong indications that the commutativity conditions

$$[\mathbb{I}_{2l-1}, \mathbb{I}_1] = [\mathbb{I}_{2l-1}, \mathbb{I}_2] = 0 \quad (C.27)$$

(a) fix all the operators $\mathbb{I}_{2l-1}$ for $l > 2$ uniquely up to the normalization, and (b) such defined operators $\mathbb{I}_{2l-1}$ mutually commute:

$$[\mathbb{I}_{2l-1}, \mathbb{I}_{2m-1}] = 0 . \quad (C.28)$$

In order to specify $\mathbb{I}_{2l-1}$ unambiguously, it is convenient to normalize the corresponding local density in such a way that the monomial $(\partial \phi)^{2l}$ appears in $T_{2l}$ with the coefficient equals one, i.e.

$$\mathbb{I}_{2l-1} = \int_0^{1/T} \frac{d\tau}{2\pi} \left((\partial \phi)^{2l} + \ldots\right) . \quad (C.29)$$

We can now take advantage of the general prediction from the Introduction of Ref. [31] about the form of the low-temperature expansion in the case of an RG boundary flow which possesses an infinite set of mutually commuting local IM’s, and terminated at the “trivial” infrared fixed point (the one which in Cardy’s classification [13] corresponds to the identity primary state). In the case under consideration the prediction implies that the low-temperature expansion of $\log Z_{\text{bsg}}(h)$ looks as follows

$$\log Z_{\text{bsg}}(h) \simeq -f_0 \frac{E_r}{T} + \log \left(g_D g_{\text{fixed}}\right) - \sum_{l=1}^{\infty} f_l \left(\frac{T}{E_r}\right)^{2l-1} \mathbb{I}_{2l-1} \left(\frac{h}{T}\right) , \quad (C.30)$$

where $f_l$ are some coefficients and $I_{2l-1}(P)$ are vacuum eigenvalues of the operators $\mathbb{I}_{2l-1}$:

$$\mathbb{I}_{2l-1} | P, (0, 0) \rangle = (2\pi T)^{2l-1} I_{2l-1}(P) | P, (0, 0) \rangle . \quad (C.31)$$

Let us emphasize that the $I_{2l-1}(P)$ are unique polynomials of order $l$ of the variable $P^2$, and, as follows from the normalization condition (C.29)

$$I_{2l-1}(P) = \left(\frac{P}{2}\right)^{2l} + \ldots , \quad (C.32)$$

where the omitted terms contain lower powers of $P^2$. In particular, in view of the explicit formulae for $\mathbb{I}_1$ (B.46) and $\mathbb{I}_3$ [74] it is possible to show that

$$I_1(P) = \left(\frac{P}{2}\right)^2 - \frac{k}{8 (2 + k)} , \quad (C.33)$$

$$I_3(P) = \left(\frac{P}{2}\right)^4 - \frac{5 k}{4 (2 + 3k)} \left(\frac{P}{2}\right)^2 + \frac{k (9 k + 4 Q^2)}{64 (2 + k) (2 + 3k)} ,$$

with $Q = b + b^{-1}$.

Applying the standard WKB iterational scheme [33] to the spectral determinant (2.42), one can reproduce exactly the structure of the low-temperature expansion (C.30) as well as the eigenvalues (C.33). En passant, the WKB calculation provides an explicit form of the the expansion coefficients:

$$f_l = \frac{\Gamma(l - \frac{1}{2})}{\Gamma(k l - \frac{1}{2})} \frac{\Gamma\left(\frac{2l-1}{2} k\right)}{2 \sqrt{\pi}} \frac{\Gamma\left(\frac{2l-1}{2} k b^{-1}\right)}{Q} \left(\frac{\sqrt{k}}{Q}\right)^{2l} . \quad (C.34)$$
D  Appendix : Low-temperature expansion of $\tilde{Z}_\theta^{(k)}$

At low temperature the regularized partition function (2.53) can be studied using the WKB expansion. The leading WKB approximation yields an explicit expression for the regularized ground state energy (2.56):

$$\frac{\bar{E}_\theta^{(k)}}{E_*} = -2\left(1 - \frac{1}{k}\right) + \sum_{m=2}^\infty \frac{\Gamma(m - \frac{1}{2})}{2\sqrt{\pi}m!} \left[ \psi(m - \frac{1}{2}) - \psi(mk - \frac{k}{2}) + \psi\left(\frac{k}{2}\right) - \psi\left(\frac{1}{2}\right) \right] \sin^{2m}(\theta).$$

(D.1)

It is useful to bear in mind that (D.1) is applicable for $0 \leq \theta < \frac{\pi}{2}$ only. It can be analytically continued within the domain $\frac{\pi}{2} \leq \theta < \pi$ by means of the relation:

$$\frac{E_\theta^{(k)}}{E_*} = -E_{\pi - \theta}^{(k)}/E_* + \frac{\sqrt{\pi}}{k} \sum_{l=1}^{k-1} \frac{\Gamma(-\frac{l}{k} - \frac{1}{2})}{\Gamma(1 - \frac{l}{k})} \left[ \sin(\theta) \right]^{1+2l} \frac{1}{2} F_1\left(\frac{l}{k}; 1; \frac{3}{2} + \frac{l}{k} \sin^2(\theta) \right) +$$

$$\frac{4}{k} \sin^2\left(\frac{\pi(k-1)}{2}\right) \left[ (1 - \cos(\theta)) \log \left[ \sin \left(\frac{\theta}{2}\right) \right] + (1 + \cos(\theta)) \log \left[ \cos \left(\frac{\theta}{2}\right) \right] \right].$$

(D.2)

The following formula for the regularized ground state energy holds within $0 \leq \theta < \pi$

$$\frac{\bar{E}_\theta^{(k)}}{E_*} = C_0^{(k)} - (\gamma_E + 2 \log 2 + \psi\left(\frac{k}{2}\right) - \log k) \cos(\theta) +$$

$$\frac{\sqrt{\pi}}{k} \cos^2(\theta) \sum_{l=1}^{k-1} \frac{\Gamma\left(\frac{l}{k} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{k}\right)} \left. \left[ 3F_2\left(\frac{l}{k} + 1; 1, 1; \frac{3}{2} + \frac{l}{k} \cos^2(\theta) \right) - \frac{2}{3k} \cos^3(\theta) \sum_{l=1}^{k-1} \frac{l}{k} \left. 3F_2\left(\frac{l}{k} + 1, 1, 1; \frac{3}{2} + \frac{l}{k} \cos^2(\theta) \right) \right. \right. \right),$$

where

$$C_0^{(k)} = -\frac{\sqrt{\pi}}{k} \sum_{l=1}^{k-1} \frac{\Gamma\left(\frac{l}{k} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{k}\right)} - \frac{2\log 2}{k} \sin^2\left(\frac{\pi(k-1)}{2}\right).$$

(D.4)

The systematic WKB expansion of (2.53) leads to an asymptotic series of the form

$$\log \tilde{Z}_\theta^{(k)} \simeq -\frac{\bar{E}_\theta^{(k)}}{T} + \log(g_{\text{fixed}}) + \sum_{l=1}^{\infty} F_{1l}^{(k)}(\theta) \left( \frac{T}{E_*} \right)^{2l-1},$$

(D.5)

where the coefficient $F_{1l}^{(k)}$ reads explicitly as

$$F_{1l}^{(k)}(\theta) = \frac{k - 1}{12(k + 2)} 2F_1\left(\frac{1}{2}; \frac{1}{2} + \frac{k}{2}; \frac{3}{2} + \frac{1}{k} \sin^2(\theta) \right) =$$

$$-\frac{\sqrt{\pi} k \Gamma\left(\frac{1}{2} + \frac{1}{k}\right)}{24 \Gamma\left(\frac{1}{k} - 1\right)} \left[ \sin(\theta) \right]^{-1 - \frac{2}{k}} - \frac{k - 1}{12k} \cos(\theta) 2F_1\left(1 + \frac{1}{k}, 1; \frac{3}{2} \cos^2(\theta) \right).$$

(D.6)
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