Interplay between Spin and Phonon Fluctuations in the double-exchange model for the manganites.

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We present exact solutions, mainly analytical, for the two-site double-exchange-Holstein model, that allow us to draw a complete picture of the role of both phonon and spin quantum fluctuations in determining the short-range correlations in the manganites. We provide analytical solutions of the model for arbitrary electron-phonon coupling and phonon frequency, for \( S = 1/2 \) and for the classical spin limit \( S = \infty \), and compare these results with numerical diagonalization of the realistic \( S = 3/2 \) case. The comparison reveals that the realistic case \( S = 3/2 \) is not well described by the classical spin limit, which is often used in literature. On the other hand, the phonon fluctuations, parametrized by the phonon frequency \( \omega_0 \), stabilize ferromagnetic phases with respect to the adiabatic limit. We also provide a complete analysis on the polaron crossover in this model.

1. INTRODUCTION

It is known from the 1950s that the double exchange mechanism \( \text{(1)} \) is at the basis of the magnetic properties of the manganese perovskites \( R_{1-x}A_x\text{MnO}_3 \) (where \( R \) is a rare earth element (e.g., La), and \( A \) is a divalent element like Sr or Ca). In this compounds, the \( d \)-levels of each Mn ion host \( 4 - x \) electrons. Three of them occupy the three low-lying \( t_{2g} \) levels, with aligned spins due to the Hund’s rule. These electrons are basically localized, and only the \( 1 - x \) electrons in the \( e_g \) level contribute to the transport properties. The discovery of the so-called “colossal” magnetoresistance \( \text{(2)} \) has originated an enormous revival of studies on these compounds, both on the theoretical and the experimental side. The systematic experimental investigations of the last few years have underlined some weakness in the previous understanding, unveiling a surprisingly rich phase diagram where a lot of competing phases are stabilized by varying doping, temperature and chemical nature of the dopants. One of the most relevant new theoretical trends is the suggestion that the transport properties cannot be fully understood on the basis of the double exchange alone, and that the interplay of this mechanism and a significant electron-phonon (e-ph) interaction leading to a Jahn-Teller effect is the key for the explanation of these properties \( \text{(3)} \). The prominent role of e-ph effects has also been firmly established experimentally by various groups and techniques \( \text{(4)} \).

The complex entanglement between charge, orbital and lattice degrees of freedom represents a hard theoretical challenge, that is far from being solved. Many approximate solutions and numerical results have been proposed, but the complexity of the phase diagram has naturally forced various authors to many uncontrolled simplifications. In this work we make a step back, and focus on a system for which \textit{analytical} exact results can be obtained. Namely, we solve the double exchange model for a single electron on two sites in the presence of a local Holstein e-ph coupling \( \text{(5)} \). Since we are interested in the relevant physics determining the interplay between lattice and spin quantum fluctuations, we consider the simple Holstein model, instead of a more involved Jahn-Teller coupling. This choice does not imply a loss of generality since we are not discussing the role of orbital degrees of freedom.

The two site system has been extensively studied as the minimal system able to capture the key features of polaron formation from the point of view of ground state \( \text{(6)} \) as well as spectral properties \( \text{(7,8)} \). Recently also the interplay between e-ph and e-e correlations have been studied semi-analytically within the same model \( \text{(9)} \). However, as far as magnetic properties are concerned, it is quite obvious that a two-site system does not allow for long-range order and phase transitions. Nevertheless, it shows \textit{short-range} (nearest neighbors) correlations, that give substantial indications on the actual long-range properties of the system, at least in strong coupling. In the context of the models for the manganites it is in fact believed that the finite-size effects play a little role \( \text{(10)} \). In the following we will define “first order transitions”, the level crossing between phases with different symmetry, and “second order transitions” the continuous transitions. We will discuss the relevance of our results to large systems in the following.

Due to the simplicity of the model, we can give com-
plete exact phase diagrams without approximations. One of our main results is a complete characterization of the role of the quantum fluctuations of the core $t_{2g}$ spins. In a microscopic model of the manganites, the core spins have $S = 3/2$, and this value is usually thought to be large enough to get rid of quantum fluctuations and treat them as classical variables. We will explicitly test this assumption by comparing the limiting cases $S = \infty$ (classical spins) and the extreme quantum case $S = 1/2$, where the effect of quantum fluctuations is maximum, with the realistic $S = 3/2$ case.

Analogously, we will discuss the role of lattice quantum fluctuations, releasing the adiabatic approximation on the phonon degrees of freedom. The role of quantum phonon fluctuations is not trivial, as already known for e-ph models alone \[16\]–\[18\]. For $S = \infty$ and $S = 1/2$ we give analytical exact solutions of the model, exploiting an exact analytical solution of the two-site Holstein model. For $S = 3/2$ we use standard exact diagonalization to solve the model. Also in this case no approximation is introduced and all the regimes are accessible.

A similar study has been reported in Ref. \[19\], where the two-site double exchange model for classical spins is solved by perturbation theory around a variational reference state obtained by a Lang Firsov canonical transformation. Our work overcomes some limitations of Ref. \[19\], namely the classical spin limit. Contrary to Ref. \[19\], we are also able to explore the adiabatic regime $\omega_0 \ll t$, well beyond the region in which the Lang Firsov result is a good reference state.

The paper is organized as follows: In Section II we introduce the model and the methods for our analytical solutions; In Section III we present the phase diagram of the model and discuss the role of quantum fluctuations; due to the complexity of the phase diagram the discussion is divided in three subsections: in the first the effect of the magnetic degrees of freedom on the polaron crossover is considered; in the second we discuss the effect of the e-ph interaction on the magnetic phase diagram of the model, and in the third subsection the full phase diagram is presented.

In Section IV we discuss the relevance of our results for larger size systems and for the experimental scenario. Finally we give concluding remarks in Section V.

II. METHODS OF SOLUTION

We consider the Holstein-Double Exchange model on a two-site cluster for a single electron:

$$
H = -t \sum_{\sigma} (c_{1,\sigma}^\dagger c_{2,\sigma} + c_{2,\sigma}^\dagger c_{1,\sigma}) + 
-J_H \sum_{i=1,2} \sigma_i \cdot S_i + J_1 S_1 \cdot S_2 + 
-g(n_1 - n_2)(a + a^\dagger) + \omega_0 a^\dagger a,
$$

where $S_i$ ($i = 1, 2$) is a local spin associated to the localized $t_{2g}$ electrons on each site, $c_{i,\sigma} (c_{i,\sigma}^\dagger)$ destroys (creates) an electron of spin $\sigma$ on site $i$, $n_i = \sum_\sigma c_{i,\sigma}^\dagger c_{i,\sigma}$ is the number operator on each site, $\sigma_i = c_{i,\uparrow}^\dagger \sigma_\alpha c_{i,\downarrow}$ is the spin operator on each site ($\sigma$ are the Pauli matrices). $a (a^\dagger)$ is the destruction (creation) operator for a lattice distortion that couples to the difference of density between the two sites. The lattice displacement $X$ is given by $X = \sqrt{\hbar/2m_0 \omega_0 (a + a^\dagger)}$. We could have started from a standard Holstein model with a phonon mode per each site, coupled to the local density. It is in fact easy to show that, in this case, the symmetric combination of the two phonon modes $A = 1/\sqrt{2}(a_1 + a_2)$ couples to the total density, giving rise to a trivial term, and the only term left is the one we introduced, where the phonon mode may be written in terms of the local ones as $a = 1/\sqrt{2}(a_1 - a_2)$.

We explicitly consider, besides the hopping between the two sites and the Hund’s rule term ($J_H$) that couples ferromagnetically the conduction electrons to the localized ones, an antiferromagnetic superexchange term ($J_1$) between the core electrons. $J_H$ is always taken to be the largest energy scale, consistently with the physics of the manganites. This latter term, even though $J_1$ is significantly smaller than $J_H$, has crucial importance on the magnetic properties of the manganites \[3\]. We also consider a Holstein coupling ($g$) between the electron and a dispersionless mode of frequency $\omega_0$.

In this paper, we present the exact solution for the model \[3\]. In particular for the classical spin case $S = \infty$ and the extreme quantum case $S = 1/2$ we provide analytical solutions for arbitrary values of both the electron-phonon coupling and of the phonon frequency $\omega_0$, exploiting an exact solution of the two-site Holstein model reported in Appendix A. In the $S = 3/2$ case, a numerically exact solution by means of exact diagonalization is instead presented. In the following subsections, we describe the analytical solutions for $S = 1/2$ and $S = \infty$.

A. Exact Solution for $S = \infty$

Following Ref. \[3\], in the classical spins case we can write

$$
H(\theta) = J_1 \cos \theta - H_H(\theta) + E_{Hund}
$$

(2)

where $E_{Hund}$ is the contribution of the Hund’s term to the total energy and $H_H$ is the Hamiltonian of a 2-site Holstein model in which the hopping $t$ is replaced by $T = t \cos(\theta/2)$. The canting angle $\theta$, that measures the relative orientation of the core spins fully characterizes the magnetic arrangement. If $\theta = 0$ the spins are aligned and a ferromagnetic (FE) state is found, whereas for $\theta = \pi$, an antiferromagnetic state (AF) is found. Intermediate values of $\theta$ describe canted (CA) states. There-
Therefore, the solution of the classical spins two-sites Holstein double-exchange model can be obtained by minimizing on $\theta$, once the eigenvalues of $H_H$ are known, as shown in Appendix A. For $J_H \gg J_1$ the extremal condition then gives

$$\sin(\theta/2)(-2J_1 \cos(\theta/2) - \frac{t}{2} \frac{\partial E_H}{\partial t}) = 0. \quad (3)$$

which shows that the ferromagnetic state $\theta = 0$ is always an extrema of $E(\theta)$ \[^2\]. Then the transition from $FE$ to $CA$ state is of “second order”. The critical coupling $J_1$ for this transition is given by the vanishing of the term in parentheses in Eq. (3)

$$J_1^c = -\frac{E_{kin}^{kin}}{4} \quad (4)$$

where $E_{kin}^{kin} = t \partial E_H / \partial t$ is the kinetic energy of the Holstein model, i.e., the kinetic energy of the system with $J_1 = 0$ (Notice that $E_{kin}^{kin}$ is a negative quantity). The effect of el-ph interaction on the $FE \rightarrow CA$ transition is therefore the substitution $t \rightarrow -E_{kin}$.

Now let us consider the “first order” $FE \rightarrow AF$ transition. We have to compare the $FE$ and $AF$ energies obtained by Eq. (2) respectively with $\theta = 0$ and $\theta = \pi$. The critical coupling is given by

$$J_1^c = \frac{E_H(0) - E_H(t)}{2} \quad (5)$$

where $E_H(0) = -g^2/\omega_0$ is the energy of the atomic Holstein model.

### B. Exact Solution for $S = 1/2$

In this case, neglecting for the moment the phonon degrees of freedom, the electronic Hilbert space (including the core spins) is in principle made by 16 states, that reduce to 8 if the symmetry for inversion of all the spins is considered. As shown in Appendix B, this problem can be simplified, and the largest subspace to deal with is a $3 \times 3$ sector, but the remaining problem is still not trivial if we switch on the coupling with the phonons. Fortunately, in the limit $J_H \gg t$, a further simplification occurs (also shown in Appendix B), leading to the possibility to express the eigenvalues of the model in terms of the two-site Holstein model. The details of the solution are reported in Appendix B. In such a way, we can characterize the condition for the only possible transition, i.e., the transition from $FE$ to $AF$ ground state. The transition is discontinuous and can be obtained from the comparison of the energies of the different phases, that we compute in Appendix B. The critical coupling for the $FE \rightarrow AF$ transition is then given by

$$J_1^c = \frac{4(E_H(t/2) - E_H(t))}{3} \quad (6)$$

where $E_H$ is the energy of the 2-sites Holstein model. Contrary to the classical spin case quantum spin fluctuations allow for non zero effective hopping $\tilde{t} = t/2$ in the AF phase.

### III. RESULTS

In this section, we present the phase diagrams of the model \[^4\], with a particular emphasis on the interplay between the role of phonon and spin quantum fluctuations (measured respectively by $\omega_0/t$ and the value of the “local” spin of the $t_{2g}$ electrons $S$). Due to the relatively large number of parameters that determine the phase diagram, we organize the discussion of the results in subsections: in the first subsection we discuss the polaron crossover in the different regimes, providing a unifying picture of the effect of both phonon and spin fluctuations, and of electron-spin correlation, that generalizes in a consistent way the conditions for the small polaron crossover in the simplest Holstein model \[^4\] \[^5\]. In the second subsection, we discuss the magnetic transitions occurring in our model. Both the nature of the magnetic phases and the relation between the magnetic state and the occurrence of polaronic behavior are strongly dependent on the value of the spin $S$. Finally, in the last subsection, we present complete phase diagrams in the $\lambda$-$J_1$ plane, where the role of lattice and magnetic degrees of freedom is highlighted.

#### A. The polaron crossover

Models with electron-phonon interaction quite generally exhibit a polaronic ground state when the coupling strength is large enough. The transformation of the free electron into a small polaron is not a phase transition, but a continuous crossover. For the Holstein model and a single particle, it has been shown that the condition for the crossover significantly depends upon the ratio between the phonon frequency $\omega_0$, and the typical electronic energy scale $t$. In the adiabatic regime $\omega_0 \ll t$, the crossover occurs for $\lambda = g^2/\omega_0 t \approx 1$, whereas in the antiadiabatic regime $\omega_0 \gg t$, the crossover is controlled by the (purely phononic) variable $\alpha = g/\omega_0 \ll 1$. As a matter of fact, the crossover coupling $\lambda_{pol}$ is pushed to larger values of $\lambda$ as the frequency is increased. Moreover, the crossover becomes smoother and smoother as $\omega_0/t$ gets larger.

The conditions for a polaron crossover in the adiabatic and antiadiabatic regimes can be understood on basic physical grounds. In the adiabatic regime, the key condition is that a bound state can be formed. The condition $\lambda > 1$ expresses this property, since it just implies that the polaron binding energy $E_{pol} = g^2/\omega_0$ exceeds the kinetic energy of a free electron $\sim t$. On the other hand, in the antiadiabatic limit, the electronic energy scale $t$ is...
not the largest scale, and the polaron crossover is ruled by the condition \(\alpha^2 > 1\), that corresponds to the excitation of a significant number of phonons (or, equivalently, to a sizeable lattice distortion). The crossover conditions we have just described are based on simple, model independent, physical insights, and are therefore expected to basically hold, with some marginal changes, also for more complicated models like the double-exchange model we are considering.

It must be noted that, since the formation of a polaron is not a phase transition with an associated broken symmetry, there is some ambiguity in determining a physically sensible clear-cut criterion for the polaron crossover. In most previous studies, including that of Ref. [19], the crossover line has been drawn as the locus of the points in which some relevant expectation values, like the electron-lattice correlation function \(1/N \sum_i <n_iX_i>/t\) or the average number of phonons in the ground state change their behavior. This kind of characterization has no problem in the adiabatic regime, where the crossover is rather sharp, but it is more questionable in the antiadiabatic regime. In this work we use a much more definite criterion, that is based on a qualitative difference between polaronic and non-polaronic states. Namely, we study the (quantum) probability distribution function for the displacement operator \(P(X) = \langle 0|X\rangle\langle X|0\rangle\), where \(|0\rangle\) is the ground state wave function and \(|X\rangle\) denotes the state with displacement \(X\). In the adiabatic limit \(\omega_0 = 0\), the phonon degrees of freedom are described by classical variables, and no quantum fluctuations are present. The solution of the model involves a minimization of the electronic groundstate as a function of \(X\). As a result, the probability distribution is a single (or a few) \(\delta\)-function, centred at the values that minimize the energy. More explicitly, if the system is not polaronic, a single value of \(X\) minimizes the energy, while in the polaronic regime, two different minima are obtained. The polaron crossover is then associated with the coupling value in which a single \(\delta\)-function leaves place to two symmetric peaks. As soon as the quantum fluctuations of the lattice are restored by introducing a finite phonon frequency \(\omega_0\), the \(\delta\)-functions broaden, but the qualitative features do not change. The polaronic regime is characterized by a bimodal distribution, while the non-polaronic state present a unimodal distribution. In the particular case of a single particle on two sites, the polaronic regime presents two symmetric peaks at \(X = \pm X_0\), and the non-polaronic state in non-distorted, so that \(P(X)\) is peaked at \(X = 0\).

Fig. [3] shows the evolution of \(P(X)\) varying \(\lambda\) in the two-site Holstein model for \(\omega_0/t = 0.1\) (representative of the adiabatic regime) and \(\omega_0/t = 4\) (representative of the antiadiabatic regime). In both cases a smooth crossover occurs between a quasi-free electron state (unimodal distribution) and a polaronic state (bimodal distribution). The figure also clearly shows that the crossover in indeed much sharper in the adiabatic case than in the antiadiabatic one. Furthermore, \(P(X = 0)\) in the polaronic region rapidly vanishes soon after the crossover in the adiabatic case, while it stays finite in the antiadiabatic, despite it is a local minimum in both cases.

\[
P(\lambda) = \frac{1}{\sqrt{2\pi \omega}} e^{-\frac{\lambda^2}{2\omega^2}}
\]

It is worth to emphasize that that the monomodal to bimodal crossover of \(P(X)\) has also been reported as signature to a crossover toward a "polaronic" state also in studies of the Holstein model in the thermodynamic limit using Dynamical Mean Field Theory [22], [23].

### B. Magnetic Correlations

In this section we discuss the behavior of the magnetic correlations in the two-site double exchange model. It should be clear that a such a small system can not undergo phase transitions, and that we are only able to describe short range correlations. We parametrize the magnetic correlations between the two sites by means of the scalar product \(<S_1 \cdot S_2>\) between the core spins.

Since our interest in the model is motivated by the manganites, we will always assume that \(J_H\) is the largest energy scale. In the \(S = \infty\) case we let \(J_H\) go to infinity, and in the finite spin case, we take \(J_H = 10t\).

In the absence of electron-phonon coupling, and assuming that \(J_H\) is the largest energy scale, the direct antiferromagnetic exchange between the core electrons determines the magnetic properties of the system. For zero and small \(J_1\), the spins are ferromagnetically aligned due to the Hund’s coupling. Increasing \(J_1\), antiferromagnetic correlations tend to appear.

The nature of the spin correlations depends crucially on the value of the spin \(S\), since the latter rules the possible values of \(<S_1 \cdot S_2>\). More explicitly, in the classical spin case, \(<S_1 \cdot S_2> = S^2 \cos(\theta)\), where the canting angle
\( \theta \) between the spins is a continuous variable. The ferrromagnet continuously evolves into a canted state as \( J_1 \) is enhanced. The canting angle asymptotically tends to \( \pi \), corresponding to the antiferromagnetic state, as \( J_1 \) is enhanced. Panel (a) in Fig. 2 displays the dependence of \( \langle S_1 \cdot S_2 \rangle \) on \( J_1 S^2 \) for the classical spin case for \( J_H = \infty \).

In the quantum case the total spin is given by

\[
\langle S_1 \cdot S_2 \rangle = \frac{1}{2} (S(S + 1) - S_1(S_1 + 1) - S_2(S_2 + 1))
\]

and assumes only a few values. For \( S_1 = S_2 = 1/2, S = 0 \) and 1 are the two only possible values. For \( S_1 = S_2 = 3/2, \) we can have four values \( (S = 0, 1, 2, 3) \). It must be noted anyway, that the total spin operator \( S^2 \) does not commute with the Hamiltonian \( H \), so that the energy eigenstates have no reason to be eigenstates of \( S^2 \).

An inspection to the results in the absence of electron-phonon coupling shows indeed that, for \( S = 1/2 \), the magnetic state abruptly varies, for \( J_1 = J_1(FE - AF) \), from a fully polarized ferromagnet (FE) to an antiferromagnetic (AF) state, which is not fully polarized. The transition is a level crossing between two states with different symmetry. The exact value of \( \langle S_1 \cdot S_2 \rangle \) in this state depends on both \( J_H \) and \( J_1 \), even if the dependence on \( J_1 \) (for \( J_1 > J_1(FE - AF) \)) is really weak, as it appears in panel (c) of Fig. 2.

The \( S = 3/2 \) case is strictly analogous to \( S = 1/2 \), and shows the fully polarized FE state, followed by three different combinations of the AF states. Also in this case, the precise values of \( \langle S_1 \cdot S_2 \rangle \) in the three states depend on \( J_H \) and \( J_1 \), and the dependence on \( J_1 \) is really weak in each region. Moreover, the state with the largest negative correlation is really close to the full antiferromagnet. We label the two intermediate spin phases as canted 1 (CA1) and canted 2 (CA2), and the “most antiferromagnetic” as antiferromagnetic (AF) tout court. The dependence of \( \langle S_1 \cdot S_2 \rangle \) on \( J_1 \) is shown in Fig. 2 (b). We notice that the scale of \( J_1 \) associated with the change in the magnetic structure is consistent with the experimental estimates [24].

**C. The phase diagram**

In this section we discuss the interplay between the magnetic properties and the e-ph coupling and finally determine the phase diagram of our model. We tune the relevance of lattice and magnetic degrees of freedom, by varying the strength of the electron-phonon coupling \( \lambda \), and of the antiferromagnetic coupling between the core spins \( J_1 \). Then we draw various phase diagrams in the \( \lambda - J_1 \) plane. Each of the diagrams is characterized by the values of the spin \( S \) and of the phonon frequency \( \omega_0 \), that parametrize the relevance of quantum spin and lattice fluctuations respectively. We consider the \( S = \infty, S = 3/2 \) and \( S = 1/2 \) cases, and \( \omega_0/t = 0.1 \) (adiabatic regime) and 4 (antiadiabatic regime). Finally, we always assume that the Hund’s rule coupling \( J_H \) is the largest energy scale. In the classical case, we take \( J_H = \infty \), and in the quantum cases, we use \( J_H = 10t \).

We denote “first order transitions” (level crossings between states with different symmetries) with full lines and “second order transitions” (crossovers between phases with the same symmetry) with dashed lines.

1. The effect of \( J_1 \) on the polaron crossover

In section [III A], we have briefly described the conditions ruling the polaron crossover in the Holstein model. In the adiabatic regime \( \omega_0/t \ll 1 \), the condition for a polaron ground state is that the polaron binding energy \( E_{pol} = g^2/\omega_0 \) is larger than the free electron kinetic energy, measured by \( t \). It is quite natural to generalize this condition to the double-exchange model, at least when

![FIG. 2. \( \langle S_1 \cdot S_1 \rangle \) for \( S = \infty, 3/2, 1/2 \) (from top to bottom) as a function of \( J_1 \).](image)
the polaron crossover occurs between two phases that share a common magnetic state. In this case, we can replace the bare hopping $t$ by the “magnetically renormalized” kinetic energy at $\lambda = 0$ \((7)\). Thus the crossover condition is determined by the condition

$$
\lambda_{mg} = \frac{g^2}{\omega(t)} \simeq 1.
$$

(8)

In the $S = \infty$ case the magnetic hopping is given by

$$
\tilde{t} = t \cos(\frac{\theta}{2}).
$$

(9)

where $\theta$ is the canting angle between the $t_{2g}$ spins. In the quantum cases $S = 1/2$ and $S = 3/2$, one can view the canting angle as a quantized quantity, that can assume only a few discrete values. We anticipate that these are not the quantized values of the semiclassical approximation.

Regardless the value of $S$ and $\omega_0/t$, for small values of $J_1$ the ground state is always ferromagnetic due to the Hund’s rule. A crossover occurs between a ferromagnetic itinerant electron and a ferromagnetic polaron. Within this region, the magnetic hopping is fixed to the free value $\tilde{t} \equiv t$, and does not depend on $J_1$. As a result, the model is completely equivalent to a two-site Holstein model, and the crossover is associated with a vertical line in the $\lambda - J_1$ diagram, as shown in all the phase diagrams (Figs. 3, 4, 5, 6). The crossover value of $\lambda$ depends only on the ratio $\omega_0/t$, and moves from the $\lambda \simeq 1$ in the extreme adiabatic limit, to $\lambda \simeq 1.2$ for $\omega_0/t = 0.1$, to a substantially larger value ($\lambda \simeq 3.46$) for $\omega_0/t = 4$, where the condition for the polaron crossover is close to $\alpha^2 \simeq 1$ (that implies $\lambda \simeq 4$).

Increasing $J_1$, phases with antiferromagnetic correlation between the core spins appear. The nature of this phases depends on the value of $S$, as shown in section III B. We start from the quantum cases, that present sharp level crossings at $\lambda = 0$, where $\tilde{t}$ sharply jumps following the magnetic correlations shown in Fig. 3. If we neglect the really weak dependence on $J_1$ of $\langle S_1 \cdot S_2 \rangle$ within a given magnetic phase, the polaron crossover is controlled by the condition \(\theta \), where $\tilde{t}$ is the value corresponding to the actual magnetic phase.

The exact results obtained as described in section 4 confirm this expectation, and the polaron crossovers among phases with the same magnetic correlation are in fact delimited by vertical dashed lines in all the diagrams for $S = 1/2$ and $S = 3/2$ (Figs. 3, 4, 5, 6). The value of the crossover coupling obviously changes in the different magnetic phases. The $FE$ state is the one with the largest kinetic energy due to the double exchange mechanism, so that the critical $\lambda$ is the highest in this phase, and it decreases by decreasing the value of the magnetic correlation according to Eq. \(\approx \) (see, e.g., Fig. 3). Notice that the effect of the value of $\langle S_1 \cdot S_2 \rangle$ on the crossover coupling is much more evident in the adiabatic limit (Fig. 3), where the competition between the polaron energy and the kinetic energy rules the crossover, than in the nonadiabatic regime (Fig. 4), where the electronic kinetic energy is not the most relevant quantity. In the extreme adiabatic limit $\omega_0/t \rightarrow \infty$ this dependence must completely disappear, since the kinetic energy plays no role in the crossover.

The formation of polaron does not only occur as smooth crossover between states with the same magnetic correlation. Indeed, in the $S = 1/2$ case (see Fig. 3), if we continuously increase $J_1$, we have that, between the vertical dashed lines corresponding to the polaronic crossovers within the $FE$ and the $AF$ phases, a first-order transition (level crossing) occurs from a ferromagnetic non-polaronic state and an antiferromagnetic polaron. The interplay between the localizing effect of both the e-ph and the antiferromagnetic magnetic interaction strongly favors the $AF$ polaronic state with respect to the competing phases.

Similar level-crossings occur for all the finite-spin cases, with more involved details depending upon the value of $S$ and $\omega_0$.

For example, in the nonadiabatic regime $\omega_0/t = 4$, the $FE - AF$ polaron transition occurs only in a narrow range of parameters (see Fig. 3) compared to the adiabatic case. This is simply due to the fact that, in this regime, the crossover values for $\lambda$ in the $FE$ and the $AF$ are very close. In the extreme adiabatic limit $\omega_0/t \rightarrow \infty$ this region would indeed vanish.

In the richer $S = 3/2$ case, the $CA1$ and $CA2$ states intrude between the $FE$ and the $AF$ at weak e-ph coupling. In the adiabatic regime (see Fig. 3), no polaron crossover occurs within the canted phases, and both these phases undergo a first-order transition to the $AF$ polaron. Only the $FE$ and $AF$ phases display the usual polaron crossover. In the one-dimensional case, besides the aforementioned reduction of the regions in which the polaron formation becomes first-order, canted polaronic states are stabilized by the phonon quantum fluctuations. (The “critical” frequency above which canted polaronic states appear is $\omega_0/t \simeq 1$).

In the $S = \infty$ case, when canted phases with a continuous canting angle are stable, the polaron crossover is not represented by a vertical line, since the kinetic energy is a continuous function of $J_1$ with $\tilde{t} = t \cos(\theta(J_1))$, and $\theta(J_1)$ is the value of the canting angle at $\lambda = 0$. Again, all results are consistent with the condition \(\theta \). In this case, the phonon fluctuations play a somewhat qualitative role. In the extreme adiabatic limit the $CA$ state undergoes a level crossing to the $AF$ polaronic state. The e-ph interaction and the antiferromagnetic coupling $J_1$ cooperate to stabilize the $AF$ polaron without forming a canted polaron. As soon as we introduce a finite, but small $\omega_0/t$, a tiny slice of a canted polaronic phase appears to bridge the canted state and the antiferromagnetic po-
laron. In the antiadiabatic regime, the huge quantum fluctuations strongly favor a canted polaronic state, and the antiferromagnetic polaron appears only for \( \lambda > 10 \) and \( J_1 > 2 \).

The above results show that the effect of the magnetic correlations on the small polaron crossover is influenced by the spin quantum fluctuations. In particular, the \( S = 3/2 \) case, which is relevant to the manganites is not qualitatively similar to the classical spin case, that is usually considered for simplicity. Many features of the \( S = 3/2 \) case are in fact direct consequences of the quantum nature of the spins, and are similar to the simplest quantum case \( S = 1/2 \).

2. The effect of e-ph coupling on magnetic transitions

In this section we analyze how the various magnetic transitions described in Section II are influenced by the e-ph interaction. In the previous section we have found analytical results for the transition from \( FE \) to \( AF \) states for \( J_H \to \infty \). The expressions, given by Eqs. (8) and (9), can be recast in the common form

\[
J^c_1 S^2 = \frac{E_H(AF) - E_H(FE)}{(S_1 \cdot S_2)_{FE} - (S_1 \cdot S_2)_{AF}}, \tag{10}
\]

where \( E_H(AF) \) (\( E_H(FE) \)) is the energy of the Holstein model for the antiferromagnetic (ferromagnetic) spin alignment, and \( (S_1 \cdot S_2) \) in the different magnetic phases is computed at \( g = 0 \). In particular, the evaluation of \( E_H \) for a given phase simply amounts to find the groundstate of the Holstein model where the bare hopping \( t \) is replaced by \( \tilde{t} \). This relation also holds for the case of \( S = 3/2 \), once the proper value for the kinetic energy in the antiferromagnetic phase \( \tilde{t} = t/4 \) is used.

Eq. (11) results from the competition between the magnetic energy balance, controlled by \( J_1 \), and the “polaronic” energy, i.e., the energy resulting from the e-ph coupling.

Since the \( AF \) phase has always a smaller hopping with respect to the \( FE \) phase, the first qualitative effect of the e-ph interaction is to lower the energy of the antiferromagnetic phase with respect to the ferromagnetic one, therefore favoring antiferromagnetism. The region of stability of the \( FE \) phase is always shrunk by increasing \( \lambda \). More generally, the e-ph interaction favors phases with smaller values of \( (S_1 \cdot S_2) \), so that the boundaries of the different magnetic phases are always marked by downward curves in the \( J_1-\lambda \) plane.

In the limit of small \( t \), the energy difference in the numerator of (11) reduces to the pure kinetic energy of the Holstein model close to the atomic limit. Defining \( \gamma = \tilde{t}/t \), we can write

\[
\lim_{t \to 0} (E_H(\gamma t) - E_H(t)) = \lim_{t \to 0}(1 - \gamma)t \frac{E_H(t + (1 - \gamma)t) - E_H(t)}{(1 - \gamma)t} = \frac{(1 - \gamma)t}{\partial E_H(t)/\partial t} = -E^\text{kin}_H(1 - \gamma). \tag{11}
\]

In the small \( t \) limit the condition (10) reduces then to

\[
J^c_1 = \kappa(S)(-E^\text{kin}_H), \tag{12}
\]

where \( \kappa(S) \) is a constant that contains the factor \( 1 - \gamma \), and depends on the value of the spin \( S \) and on the transition under consideration, such that, in the absence of e-ph interaction the condition is simply \( J^c_1 = \kappa(S)t \). This latter relation is analogous to Eq. (8), since the effect of the e-ph interaction results in the substitution \( t \to -E^\text{kin}_H \), but, contrary to that, it is valid only for \( t \to 0 \).

This result gives valuable informations about the role of the retardation effects in the e-ph coupling. In general terms, the interaction mediated by the phonons is in fact retarded, and becomes instantaneous only if \( \omega_0/t \to \infty \). In such a limit, as we have shown, the kinetic energy rules the magnetic transitions. As soon as the approximation \( t \to 0 \) is released, the retardation effects imply that Eq. (11) must be used. Notice that \( E_H(AF) - E_H(FE) = -E^\text{kin}_H + \Delta \), where \( \Delta \) is the difference between the e-ph interaction energies in two phases, and turns out to be always positive. The overall effect of the retarded e-ph interaction is therefore to reduce the stability of \( FE \) phases with respect to \( AF \) and \( CA \) phases.

In the \( S = 3/2 \) case all the magnetic transitions are of “first order”, as described in section II, so that similar arguments can be applied, and Eq. (11) allows to compute the transition coupling, once the energy of the Holstein model and the magnetic energy of the appropriate phases are known.

The situation is different only for the “second order” transition between the \( FE \) and \( CA \) phases in the classical spin case. In this case the continuity of the transition involves that the energy difference between the phases is infinitesimal, so that \( E^\text{kin}_H = -\partial E_H(t)/\partial t \) rules the transition not only for small \( t \), but for arbitrary values of \( t \), as shown by Eq. (8). This preliminary analysis suggests a really important difference between the classical spin limit \( S = \infty \) and the quantum \( S = 3/2 \) case.

Now we can give some description and interpretation of the exact phase diagrams in light of the above analysis.

In the adiabatic limit \( \omega_0/t = 0 \), for finite value of \( S \), and for \( \lambda < \lambda_{pol} \), the energy of the e-ph model alone does not depend upon \( \lambda \), so that the relative stability of the various magnetic phases is in turn expected to be \( \lambda \)-independent. The magnetic transitions are therefore associated with horizontal lines in the \( \lambda-J_1 \) plane. If we introduce phonon fluctuations, the kinetic energy depends upon \( \lambda \) also before the crossover, and the boundary between the magnetic states acquires a finite slope, that becomes larger and larger by increasing \( \omega_0/t \). This
behavior can be easily seen comparing Fig. 3 with Fig. 4 (or 5 with 6). The case of the classical spin variables, where the magnetization is a continuous variable, can be understood in similar terms. The crossover between the FE and the CA phase in fact closely follows the curve obtained using Eq. (4).

As discussed previously, the phonon fluctuations always favor FE phases with respect to AF and CA phases, as it can be seen comparing the phase diagrams for $\omega_0/t = 0.1$ with the corresponding with $\omega_0/t = 4$ (at the same value of $\lambda$). In we increase the quantum fluctuations of the phonons, by increasing the phonon frequency $\omega_0$, the retardation effects are decreased, so that the localization of the electrons is made more difficult. An enhanced mobility of the electron results in an enhanced stability of the FE phases due to the double exchange mechanism.

FIG. 3. Phase diagram for $S = 3/2$ and $\omega_0/t = 0.1$.

FIG. 4. Phase diagram for $S = 3/2$ and $\omega_0/t = 4$.

FIG. 5. Phase diagram for $S = 1/2$ and $\omega_0/t = 0.1$.

FIG. 6. Phase diagram for $S = 1/2$ and $\omega_0/t = 4$.

FIG. 7. Phase diagram for $S = \infty$ and $\omega_0/t = 0.1$. 
IV. RELEVANCE FOR LARGER SYSTEMS

In this section we discuss the relationship between the two-site model and larger systems. There are not indeed so many solutions of the double exchange model in the presence of e-ph coupling, and most of them are forced to consider the classical limit for both the phonons and the core spins. In the case of numerical calculation the main problem in dealing with quantum phonons and spins is the enlargement of the Hilbert space (that becomes infinite for the case of phonons). This problems are even harder if the orbital degrees of freedom are taken into account. In Ref. [15], a numerical analysis of the double-exchange model in presence of a Jahn-Teller coupling is performed for the three-dimensional structure of the actual compounds, and for a number of electrons corresponding to the stoichiometric LaMnO$_3$. The three-dimensional structure and the realistic hopping matrix elements allow for different antiferromagnetic arrangements of the spins besides the ferromagnetic state. In particular, realistic values of the parameters give rise to the A-type antiferromagnetism found in the experiments [16,20]. Of course our two-site system is not enough to distinguish among the various kind of antiferromagnetic orderings, but the phase diagram in Fig. 2(b) of Ref. [13] and our phase diagram for classical spins and for small phonon frequencies (Fig. 7) are rather similar. The shape of the curves separating the $FE$ phase to the “non-$FE$” phases look very similar, and the crossover line between a $FE$ with no distortions and a $FE$ polaronic phase is very close to the one between the $FE$ and the $FE(JT)$ phase. The agreement is not only qualitative, but also quantitative as it can be checked by direct comparison.

The capability of the two-site cluster to capture the physics of both the polaron crossover and the magnetic transitions in a more realistic system is consistent with previous claims (see, e.g. [14]) of the irrelevance of finite-size effects for these systems.

The $x = 0.5$ system has been considered in Refs. [25,29]. Also in this case, the phase diagram is quite similar to Fig. 1. A first-order transition separates a $FE$ state to non-$FE$ states, and within the $FE$ phase a vertical line denotes a crossover associated to small polaron formation (or Jahn-Teller effect).

The two-site cluster gives therefore results really close to larger systems when the latter are available. This suggests that our results are generically representative of larger sizes, also for parameters that are presently unaccessible to numerical simulations. We emphasize that this agreement is not accidental and can be easily rationalized. As mentioned in the introduction, the small polaron crossover is well represented by a two-site cluster. This is essentially due to the extreme short-range character of the polaronic state. The capability of the two-site cluster to describe the crossover from a delocalized state and a polaronic state is well documented. It must be noticed, however, that our small cluster can not determine whether the polaronic and the delocalized states are Fermi liquids or not.

As far as the magnetic properties are concerned, the two-site cluster has no long-range correlation, but only short-range correlations. Nevertheless the features related to local interactions such as the ones in Hamiltonian (1) are well represented. In this spirit we expect the results of our calculations to be somehow related to a Dynamical Mean Field Theory, where the local quantum fluctuations are exactly taken into account, while the spatial correlations are frozen [27]. Similarly to the Dynamical Mean Field Theory, our exact solutions faithfully reproduce the physics of the model if the local and short-range effects are dominant, as it is the case for the manganites. More specifically the one electron solution of the two site Holstein model (see Appendix A) is surprisingly similar to the one electron solution of the Holstein model for an infinite lattice within Dynamical Mean Field Theory. Looking at a given site, the role played by the “effective quantum medium” of Dynamical Mean Field Theory is here simply played by the other site.

V. CONCLUSIONS

In this work the two-site double exchange model for an electron coupled with phonons is solved exactly for an extremely wide range of parameters and physical regimes. For $S = 1/2$ and $S = \infty$ we give an analytical exact solution for arbitrary e-ph coupling and phonon frequency. For $S = 3/2$ the solution is obtained through standard numerical techniques. The availability of these solutions allows us to study the effect of both phonon and spin quantum fluctuations, and of their mutual interplay.

This study, though limited to the extreme small size of the two-site cluster, is shown to be a good description of
larger systems, since the relevant physics involves local and short-ranged quantities.

One of our main results is a complete characterization of the effect of the double exchange and of an antiferromagnetic coupling between the $t_{2g}$ spins on the small polaron crossover. In this regard, we give an analytical estimate for the crossover coupling, given by $\lambda_{mg} = g^2/\omega_0 \bar{t} \simeq 1$, where $\bar{t}$ is the kinetic energy renormalized by the magnetic effects.

From a complementary point of view, we considered in detail the effect of the e-ph interaction on the magnetic properties of the system. In this case, we give an analytical condition for the magnetic transitions in the presence of a finite $\lambda$, given by Eq. [10]. This relation can be simplified in the limit $t \to 0$, where there is no retardation effect. The comparison between the general case and the atomic limit allows us to quantitatively describe the role of retardation in stabilizing AF (or CA) phases.

A comparison of the realistic $S = 3/2$ case with the classical spin case $S = \infty$ shows that this latter approximation does not reproduce some qualitative features of the phase diagram. A proper study of the manganites should therefore take into account the quantum nature of the core spins.

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APPENDIX A: THE EXACT SOLUTION OF THE TWO-SITE HOLSTEIN MODEL

In this appendix we sketch the solution through a continued fraction expansion of the two site Holstein model. A continued fraction solution has been already reported in literature [28] for a related model in the field of quantum optics. Here we derive the continued fraction expansion of the two site Holstein model.

The Holstein model for an electron on two sites can be written in a pseudo-spin representation in terms of the Pauli matrices

$$H_H = \omega_0 a^\dagger a - g \sigma_x (a + a^\dagger) - t \sigma_x$$  \hspace{1cm} (A1)

where $1$ is the unity matrix. The Hamiltonian can be diagonalized in the electron subspace using a transformation introduced in Ref. [8]

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & (-)^n a^\dagger a \\ -1 & (-)^n a^\dagger a \end{pmatrix}$$  \hspace{1cm} (A2)

and the property

$$(-)^n a^\dagger a(a + a^\dagger)(-)^n a^\dagger = -(a + a^\dagger)^n$$  \hspace{1cm} (A3)

we obtain for $\tilde{H}_H = U H_H U^{-1}$

$$\tilde{H}_H = \begin{pmatrix} \tilde{H}_H(t) & 0 \\ 0 & \tilde{H}_H(-t) \end{pmatrix}$$  \hspace{1cm} (A4)

where

$$\tilde{H}_H(\pm t) = \omega_0 a^\dagger a - g(a + a^\dagger) \mp t(-)^n a^\dagger.$$  \hspace{1cm} (A5)

In each block we have a purely phononic Hamiltonian $H_H(\pm t)$. The eigenvalues and eigenvectors can be determined by continued fraction solution for the resolvent between $|m\rangle$ and $|n\rangle$-phonon states

$$G_{m,n}^\pm(\omega) = \langle m| \frac{1}{\omega - \tilde{H}_H(\pm t)} |n\rangle$$  \hspace{1cm} (A6)

Using

$$\frac{1}{\omega - H} = \frac{1}{\omega - H_0} + \frac{1}{\omega - H_0} H_1 \frac{1}{\omega - H}$$  \hspace{1cm} (A7)

with $H_0 = \omega_0 a^\dagger a \mp t(-)^n a^\dagger a$ and $H_1 = -g(a + a^\dagger)$ we get the recursion

$$G_{m,n}^\pm(\omega) = \delta_{m,n} G_0^\pm(\omega - n \omega_0) - g \sum_p G_0^\pm(\omega - n \omega_0) X_{n,p} G_{p,n}^\pm(\omega)$$  \hspace{1cm} (A8)

where $X_{n,p} = \langle n| a + a^\dagger |p\rangle$. This tri-diagonal recursion can be solved for the diagonal elements through a continued fraction solution [29]

$$G_{n,n}^\pm(\omega) = \frac{1}{\omega - n \omega_0 \pm t - \Sigma_{em} - \Sigma_{abs}}$$  \hspace{1cm} (A9)

where

$$\Sigma_{abs} = \frac{n g^2}{\omega + \omega_0 \mp t - g^2(\omega - n - 1 \omega_0) \pm g^2(\omega - n - 2 \omega_0) \mp \cdots \mp g^2(\omega + n \omega_0 + (\mp)^n t)}$$  \hspace{1cm} (A10)

and

$$\Sigma_{em} = \frac{(n + 1) g^2}{\omega - \omega_0 \mp t - g^2(\omega - n + 1 \omega_0) \pm g^2(\omega - n + 2 \omega_0) \mp \cdots \mp g^2(\omega - n - n \omega_0 - (\mp)^n t)}$$  \hspace{1cm} (A11)

At zero temperature the Green function of the two site Holstein model defined as
\[ G_{i,j}(\omega) = -i \langle 0 | T_{c_i}(t) e^{i \omega t} |0 \rangle \] (A12)
can be expressed as
\[ G_{1,1}(\omega) = \frac{1}{2} (G_{0,0}^+ (\omega) + G_{0,0}^- (\omega)) \]
\[ G_{1,2}(\omega) = \frac{1}{2} (G_{0,0}^+ (\omega) - G_{0,0}^- (\omega)) \]
\[ \Sigma_1^\pm = \frac{p g^2}{\omega - p \omega_0 + (\mp) pt - \Sigma_{p+1}^\pm} \]
\[ z_p^\pm = (z_{p+1}^\pm - 1) \frac{p g^2}{E_p - p \omega_0 + (\mp) pt - \Sigma_{p+1}^\pm} \]
\[ b_n^\pm = \sqrt{\frac{1}{1 - z_1^\pm}} \]
\[ \text{APPENDIX B: THE EXACT SOLUTION OF THE TWO-SITE HOLSTEIN DOUBLE-EXCHANGE MODEL FOR } S = 1/2 \]

Let us start form the case \( g = 0 \). We choose the following basis set labelling the states according to the total spin \( S_{tot} = S + s \) where \( S \) is the spin of the Mg\(^{2+} \) ion and \( s \) that of the \( e_g \) electron: We have two states in the \( S = 3/2 \) sector:
- \(|A\rangle = |\uparrow\uparrow\rangle|\uparrow\rangle, |A'\rangle = |\downarrow\uparrow\rangle|\uparrow\rangle\)
and six states in the \( S = 1/2 \) sector:
- \(|B\rangle = |\uparrow\downarrow\rangle|\uparrow\rangle, |B'\rangle = |\downarrow\downarrow\rangle|\downarrow\rangle\)
- \(|C\rangle = |\downarrow\uparrow\rangle|\uparrow\rangle, |C'\rangle = |\uparrow\downarrow\rangle|\downarrow\rangle\)
- \(|D\rangle = |\uparrow\uparrow\rangle|\downarrow\rangle, |D'\rangle = |\downarrow\uparrow\rangle|\uparrow\rangle\).

where \(|\uparrow\rangle, |\downarrow\rangle\) represent an up (down) spin state for the core spins and \(|\uparrow\rangle, |\downarrow\rangle\) are the same for the \( e_g \) electrons. The Hamiltonian is invariant for flipping of all the spins so these are all the states we need. The states \(|A\rangle\) and \(|D\rangle\) have \( FE \) character while the states \(|B\rangle\) and \(|C\rangle\) have \( AF \) character. The \( S = 3/2 \) subspace, spanned by the combinations of \(|A\rangle\) and \(|A'\rangle\), decouples from the other states even in the presence of e-ph phonon interaction.

If we consider the symmetric and antisymmetric combinations
\[ |A^\pm\rangle = \frac{1}{\sqrt{2}}(|A\rangle \pm |A'\rangle) \]
\[ |B^\pm\rangle = \frac{1}{\sqrt{2}}(|B\rangle \pm |B'\rangle) \]
\[ |C^\pm\rangle = \frac{1}{\sqrt{2}}(|C\rangle \pm |C'\rangle) \]
\[ |D^\pm\rangle = \frac{1}{\sqrt{2}}(|D\rangle \pm |D'\rangle) \]
\[ H_{3/2} = \begin{pmatrix} \lambda - \mu t & 0 \\ 0 & \lambda - \mu t \end{pmatrix} \]
\[ H_{1/2,+) = \begin{pmatrix} -\frac{\lambda t}{4} - t & \frac{\lambda t}{4} - t \\ \frac{\lambda t}{4} - t & -\frac{\lambda t}{4} - t \end{pmatrix} \]
the last block \( H_{1/2,-} \) can be obtained from Eq. \( (B3) \) by the substitution \( t \to -t \). The e-ph matrix elements couple the subspaces \((A^+, B^+, C^+, D^+)\) and \((A^-, B^-, C^-, D^-)\). The subspace spanned by \(|A^\pm\rangle\) can be diagonalized independently having the same eigenvalues and eigenvectors of a 2 site Holstein model (see Appendix A). The hamiltonian matrix in the \( S_{1/2} \) can be written
\[ H_{1/2} = \begin{pmatrix} H_{1/2,+} + \omega_1 a d a^\dagger & -g a \langle a d a^\dagger \\ -g a^\dagger \langle a d a \rangle & H_{1/2,-} + \omega_1 a d a^\dagger \end{pmatrix} \]
Here \( 1 \) is the \( 3 \times 3 \) unit matrix. We can diagonalize \( H_{1/2} \) in the phonon space by means of the unitary transformation
\[ U = \frac{1}{2} \begin{pmatrix} (1 + (-a^d a)^2) |1 \rangle & (1 + (-a^d a)^2) |1 \rangle \\ (1 + (-a^d a)^2) |1 \rangle & (1 + (-a^d a)^2) |1 \rangle \end{pmatrix} \]
Using the property given in Eq. \[A3\] we get for \[\tilde{H}_{1/2} = UH_{1/2}U^{-1}\]
\[
\tilde{H}_{1/2} = \begin{pmatrix} (\omega_0 a^\dagger a - g(a^\dagger + a)) \mathbb{1} + H_{\text{at}}^{\text{H}} - t(-)^{a^\dagger a} \Delta & \omega_0 a^\dagger a + g(a^\dagger + a) \mathbb{1} + H_{\text{at}}^{\text{H}} + t(-)^{a^\dagger a} \Delta \\
0 & 0 \end{pmatrix} \tag{B6}
\]
where we have splitted \[H_{1/2, \pm} = H_{\text{at}}^{\text{H}} \pm t \Delta\] in the atomic \((t = 0)\) part and in the hopping dependent term
\[
\Delta = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} \tag{B7}
\]
and \[H_{\text{at}}^{\text{H}}\] can be obtained from Eq. \[B3\] with \(t = 0\).

The comparison between the energies of the \(F E\) and \(A F\) phases leads to the condition \[\tilde{B}\] for the magnetic transition.

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