I. Introduction

II. Sp(2N) Yang-Mills theories
   A. The lattice

III. Scale setting and Topology
   A. The Wilson flow
   B. Topological susceptibility on the lattice

IV. Numerical results
   A. Setting the scale
   B. The topological charge

V. Summary and discussion

Acknowledgments

A. Scale setting data for \( N_c = 2, 4, 8 \)

References

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Sp(2N) Yang-Mills theories on the lattice: scale setting and topology

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We study Yang-Mills lattice theories with Sp(\(N_c\)) gauge group, with \(N_c = 2N\), for \(N = 1, \ldots, 4\). We show that if we divide the renormalised couplings appearing in the Wilson flow by the quadratic Casimir \(C_2(F)\) of the Sp(\(N_c\)) group, then the resulting quantities display a good agreement among all values of \(N_c\) considered, over a finite interval in flow time. We use this scaled version of the Wilson flow as a scale-setting procedure, compute the topological susceptibility of the Sp(\(N_c\)) theories, and extrapolate the results to the continuum limit for each \(N_c\).

I. INTRODUCTION

Lattice studies of Sp(\(N_c = 2N\)) gauge theories aim at quantitatively appraising, in the strong coupling regime, what are their distinctive features in respect to theories based upon SU(\(N_c\)) gauge groups. For example, a long list of recent investigations of the SU(2) \(\sim\) Sp(2) theories \[11,9\], and of the Sp(2N) theories for \(N > 1\) \[10,17\], are motivated by their paradigm changing potential for applications in contemporary high energy physics.

Prominently, enhanced global symmetry and symmetry breaking patterns arise in Sp(2N) theories in the presence of matter fields. This feature is exploited in new-physics model-building exercises such as in the minimal model in Ref. \[18\], which combines a composite Higgs model (CHM)—in which the Higgs fields originate as pseudo-Nambu-Goldstone bosons (PNGBs) in a new strongly coupled sector \[19,21\]—with partial top compositeness \[22\]. For recent reviews see Refs. \[23,25\], and the summary tables in Refs. \[20,28\]. In the different context of dark matter models emerging from strongly-coupled dynamics \[29,32\], Sp(2N) gauge theories have also been attracting increasing interest \[33,40\].

There are more general, theoretical reasons to study Sp(2N) gauge theories. Pioneering studies of the pure gauge cases \[41\] aimed at qualifying the role of the centre symmetry in the confinement/deconfinement transition at finite temperature. In the light of the conjectured existence of dualities between large-\(N_c\) gauge theories and theories of gravity in higher dimension \[44,45\], the appeal of Sp(\(N_c\)) theories derives from the common features that they share with SU(\(N_c\)) theories. For example, extensive studies of the spectrum of glueballs and strings confirm...
that universal features emerge at large $N_c$, common to
$SU(N_c)$, $Sp(N_c)$, and $SO(N_c)$ theories
[14 15 46 50].

The topological susceptibility, $\chi$ (to be defined in the
body of the paper), plays a central role in our under-
standing of QCD, for several intercorrelated reasons of
historical significance. It enters the Witten-Veneziano
formula [27 55] for the large-$N_c$ behaviour of the mass of the
$\eta'$ meson, and the solution to the $U(1)_A$ problem.
Allowing the gauge coupling to be complex, and expand-
ing in powers of (small) $\theta$, $\chi$ appears as the coefficient at
$O(\theta^2)$ in the free energy. Indirectly, it might hence have
implications for the strong-CP problem, for the physics of
putative new particles such as the axion, and for the
electric dipole moments of hadrons. Many interesting
lattice calculations of $\chi$ exist—see Refs. [40 55 59 71],
and Tables 1 and 2 of the review in Ref. [72].

Accounting for $\chi$ might provide new insight in the
role of instantons and other non-perturbative objects.
A precise knowledge of $\chi$ might have unexpectedly im-
portant consequences also in the aforementioned subfield
encompassing modern phenomenological applications of
strongly-coupled $Sp(N_c)$ theories—models of composite
Higgs, partial top compositeness, dark matter, or even
early universe physics. In general, precise calculations of
$\chi$ might sharpen our understanding both of commonal-
ities and differences between $Sp(N_c)$ and $SU(N_c)$ theo-
ries, starting in the Yang-Mills (pure gauge) theories.

Motivated by such considerations, and as an important
step in the programme of study of $Sp(N_c)$ lattice gauge
theories conducted by our collaboration, in this paper we
compute the topological susceptibility of $Sp(N_c)$ Yang-
Mills theories for $N_c \leq 8$. We made available preliminary
results in contributions to Conference Proceedings [73]
[74], but this analysis is much improved, and based on
larger statistics. In a dedicated publication, we compare
our results for $Sp(N_c)$ with the $SU(N_c)$ literature, and
discuss the large-$N_c$ extrapolation [75].

The paper is organised as follows. In Sect. [II] we define
the lattice theories of interest. In Sect. [III] we discuss how
to use the gradient flow and its lattice implementation,
the Wilson flow [76 77], as a scale setting procedure,
and define the topological charge and susceptibility. Sect. [IV]
is the main body of the paper, in which we present our
numerical results. We conclude with the summary in
Sect. [V] We relegate some useful details to Appendix [A]

II. $Sp(2N)$ YANG-MILLS THEORIES

We define the continuum $Sp(2N)$ gauge theories, in
4-dimensional Euclidean space, in terms of the action

$$ S_{YM} = - \frac{1}{2g_0^2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu}, $$

(1)

where $g_0$ is the gauge coupling, $F_{\mu\nu} = \sum_A F_{\mu\nu}^A T^A$, with

$$ F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f^{ABC} A_\mu^B A_\nu^C, $$

(2)

and the trace is over the color index on the fundamen-
tal representation, while $A = 1, \cdots, N(2N+1)$. The
Hermitian matrices $\tau^A \in \mathbb{C}^{2N \times 2N}$ are the generators of
the algebra associated to the Lie group $Sp(2N)$ in the
fundamental representation. They satisfy the relations

$$ [\tau^A, \tau^B] = if^{ABC}\tau^C, $$

(3)

and are normalised according to $\text{Tr}(\tau^A \tau^B) = \frac{1}{2} \delta^{AB}$.

The configuration space of this theory can be partition-
ted into sectors, characterised by the value of the topo-
logical charge $Q$, defined as follows:

$$ Q \equiv \int d^4x \, q(x), $$

(4)

where

$$ q(x) \equiv \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu}(x) F_{\rho\sigma}(x). $$

(5)

The topological susceptibility, $\chi$, is defined as

$$ \chi \equiv \int d^4x \, \langle q(x)q(0) \rangle. $$

(6)

The possible values of $Q$ belong to the third homotopy
group of the gauge group. Since $Sp(2N)$ is compact,
connected and simple, one finds that

$$ \pi_3(Sp(2N)) = \mathbb{Z}, $$

(7)
as in the case of $SU(N_c)$ gauge theories.

A. The lattice

A lattice regularisation of the theory defined in Eq. [1]
allows to characterise quantitatively its non-perturbative
features. We adopt a 4-dimensional Euclidean hyper-
cubic lattice $\Lambda$, with lattice spacing $a$. The sites of
the lattice are denoted by their Cartesian coordinates
$x = \{x_\mu\}$ and the links by $(x, \mu)$, where $\mu = 0, \cdots, 3$
and $x_\mu = n_\mu a$. For a lattice of length $L_\mu$ in the $\mu$
directions, with $n_\mu = 0, \cdots, L_\mu/a - 1$, the total number of sites is thus $V = a^4 = L_0 L_1 L_2 L_3 a^4$. The lattices used
in our calculations are isotropic in the four directions,
$L_\mu = L$, and we impose periodic boundary conditions
in all directions. The elementary degrees of freedom of the
theory are called link variables, and defined as

$$ U_\mu(x) \equiv \exp \left( i \int_x^{x+\hat{\mu}} d\lambda^\mu \tau^A A_\mu^A(\lambda) \right), $$

(8)

where $\hat{\mu}$ is the unit vector in direction $\mu$. The link vari-
bles are $2N \times 2N$ matrices that, under the action of a
gauge transformation $g(x) \in Sp(2N)$, transform as

$$ U_\mu(x) \rightarrow g(x) U_\mu(x) g^\dagger(x + \hat{\mu}). $$

(9)

The trace of a path-ordered product of link variables de-
defined along a closed lattice path is hence gauge invariant.
The simplest such closed path on the lattice defines the elementary plaquette $\mathcal{P}_{\mu\nu}$:

$$\mathcal{P}_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x), \quad (10)$$

and is used to define the Wilson action $S_W$ of the lattice gauge theory (LGT):

$$S_W \equiv \beta \sum_x \sum_{\mu < \nu} \left( 1 - \frac{1}{2N} \Re \text{Tr} \mathcal{P}_{\mu\nu} \right), \quad (11)$$

where the inverse coupling $\beta$ is defined as

$$\beta \equiv \frac{4N}{g_0^2}. \quad (12)$$

Another operator that is useful in lattice calculations is the clover-leaf plaquette operator, defined as $[78, 79]$

$$\mathcal{C}_{\mu\nu}(x) \equiv \frac{1}{8} \left\{ U_\mu(x)U_\nu(x + \hat{\mu})U_\nu^\dagger(x + \hat{\nu})U_\mu^\dagger(x) + U_\mu(x)U_\nu^\dagger(x + \hat{\nu})U_\nu(x - \hat{\mu})U_\mu^\dagger(x - \hat{\mu}) + U_\mu^\dagger(x - \hat{\mu})U_\nu^\dagger(x - \hat{\nu} - \hat{\mu})U_\nu(x - \hat{\nu})U_\mu(x - \hat{\nu} + \hat{\mu})U_\mu^\dagger(x) - \text{h.c.} \right\}. \quad (13)$$

This operator is used in the literature as a way to improve the Yang-Mills lattice action, particularly in the presence of fermions. In the context of this paper, it serves two purposes: we use it to test the regularisation dependence of our scale-setting procedure, but also in the definition of the lattice counterparts of $Q$ and $\chi$.

Vacuum expectation values of operators $\mathcal{O}(U_\mu)$ built of link variables are formally defined as ensemble averages:

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}U_\mu e^{-S_W} \mathcal{O}(U_\mu)}{Z(\beta)}, \quad (14)$$

where $\mathcal{D}U_\mu \equiv \prod_x dU_\mu(x)$, $dU_\mu(x)$ being the Haar measure on $Sp(2N)$, while

$$Z(\beta) \equiv \int \mathcal{D}U_\mu e^{-S_W} \quad (15)$$

is the partition function of the system.

For a given value of $\beta$ and $L/a$, ensembles are generated by a Markovian process that updates the values of the link variables in a configuration. The update algorithm must respect detailed balance and have equilibrium distribution $e^{-S_W}$. An update of all the links of the lattice is called a lattice sweep. It is customary to repeat the update process, subsequent configurations $i$ and $i+1$ in the ensemble being separated by a fixed number $N_{\text{sw}}$ of sweeps. The ensemble average takes the simpler form

$$\langle \mathcal{O} \rangle = \lim_{M \to \infty} \sum_{i=1}^M \mathcal{O}_i, \quad (16)$$

with $\mathcal{O}_i$ the observable $\mathcal{O}$ evaluated on configuration $i$.

The algorithm we adopt combines local heat bath (HB) and over-relaxation (OR) updates, implemented in an openly-available [80] adaptation of the HiRep code [81] to $Sp(2N)$ groups [15].

The discretised topological charge density can be defined in several different ways [67,82], that differ by terms proportional to a power of $a$. For the body of this paper, we use the clover-leaf discretisation,

$$q_L(x) \equiv \frac{1}{32\pi^2} e^{\nu_\rho\sigma} \text{Tr} \mathcal{C}_{\mu\nu}(x) \mathcal{C}_{\rho\sigma}(x). \quad (17)$$

Both clover-leaf and elementary plaquette definitions of $q_L(x)$—the latter obtained by replacing $\mathcal{C}_{\mu\nu}(x)$ with $\mathcal{P}_{\mu\nu}$ in Eq. (17)—converge to $q(x)$ in Eq. (5), as $a \to 0$. But the clover-leaf definition treats all lattice directions symmetrically. The (lattice) topological charge is thus

$$Q_L = \sum_x q_L(x), \quad (18)$$

and its susceptibility is

$$\chi_L = \sum_x \langle q_L(x)q_L(0) \rangle. \quad (19)$$

Estimates of physical quantities obtained for given values of $\beta$ and $L/a$ are affected by several types of systematic errors. Finite size (or volume) effects arise when probing the system over physical distances that are not much smaller than $L$. This systematic error becomes insignificant if an increase in $L/a$ has an effect that is smaller than statistical fluctuations. Studies of the topology in $SU(N_c)$ gauge theories show that finite size effects are negligible provided $\sqrt{\sigma L} \gtrsim 3$, where $\sigma$ is the string tension—see, e.g., Figs. 3 and 4 of Ref. [64]. We use earlier analysis of the $Sp(N_c)$ spectrum [15] to identify regions of parameter space satisfying this condition.

The evaluation of $\chi$ via lattice methods is particularly challenging, affected by specific systematic effects. First, the configuration space of the lattice theory is simply connected. Topological sectors, and discrete topological charges, are recovered only in the vicinity of the continuum limit [83], while $Q_L$ is not integer, which affects $\chi_L$.

Second, it is challenging to control the continuum extrapolation. $\chi_L$ is particularly sensitive to discretisation
effects, quantum UV fluctuations yielding both additive and multiplicative renormalisation \[67, 72, 82\]. We extract \( \chi_L \) from Wilson-flowed configurations, as we describe in Sec. \[81\] hence adopting a scale-setting procedure that is also used to smoothen out such divergences.

Third, the evaluation of \( \chi_L \) in \( SU(N_c) \) theories is hindered by the divergence of the integrated autocorrelation time \( \tau_Q N_{sw}^\star \) as \( a \to 0 \) \[94\]. This phenomenon, known as topological freezing, descends from the intrinsic difficulty of evolving with a local update algorithm a global property such as the topological charge. We expect the same challenge to arise in \( Sp(2N) \) theories. Several ideas have been put forward to overcome topological freezing \[69, 70, 84–86\], but we defer their use to future high precision studies. Here, we limit ourselves to monitoring the values of \( \tau_Q \), and discarding compromised ensembles.

### III. SCALE SETTING AND TOPOLOGY

The definition of the continuum limit requires the implementation of a scale-setting procedure. A scale is introduced by selecting a dimensional quantity that can (in principle) be measured both in the physical limit and on the lattice. All physical quantities \( \langle \mathcal{O} \rangle \) are expressed in terms of such scale, and measurements are repeated by varying the lattice parameters (in the present case, \( \beta \)). The extrapolation towards \( a \to 0 \) yields then a finite value of \( \langle \mathcal{O} \rangle \), in the chosen units.

The string tension, \( \sigma \), of Yang-Mills theories is defined as the coefficient of the linear term of the potential between an infinitely massive, static pair of fermion and anti-fermion transforming in the fundamental representation, in the regime of asymptotically large separation. On the lattice, it can be extracted from the asymptotic behaviour of appropriately defined 2-points correlators in Euclidean time. Thanks to its direct physical interpretation, \( \sigma \) is often used for scale-setting in studies of the properties of the confining phase of pure gauge theories—see, e.g., Refs. \[46, 52, 69, 68, 75\]. However, it suffers from the effect of both systematic and statistical errors, that limit the precision of its measurement. Most importantly, the definition of \( \sigma \) is problematic in the presence of string breaking effects, which would emerge in the presence of matter fields. In this paper we adopt an alternative strategy, which could be adapted to more general gauge theories with fermionic matter field content. We will return to using \( \sigma \) to set the scale for the topological susceptibility in Ref. \[75\], as it will help in the comparison with measurements of \( \chi \) in \( SU(N_c) \) Yang-Mills theories.

The gradient flow \( B_\mu(x, t) \) \[76, 77\] is defined unambiguously in the continuum as on the lattice, with or without matter fields, and it can be determined precisely from simple averages of lattice observables. It is introduced as the solution to the differential equation

\[
\frac{dB_\mu(x, t)}{dt} = D_\mu G_{\nu\mu}(x, t),
\]

with boundary conditions \( B_\mu(x, 0) = A_\mu(x) \). The independent variable \( t \) is known as flow time, while \( D_\mu \equiv \partial_\mu + [B_\mu, \cdot] \), with

\[
G_{\mu\nu}(t) = [D_\mu, D_\nu].
\]

The defining properties of the gradient flow make it suitable as a smoothening procedure for UV fluctuations. Since \( \frac{d}{dt} S_{YM} \leq 0 \), a representative configuration \( A_\mu(x) \) at \( t = 0 \) is driven, along the flow, towards a classical configuration. In the perturbative regime, the flow equation can be shown, at leading order in \( g_0 \), to generate a Gaussian smoothening operation with mean-square radius \( \sqrt{\tau} \).

The renormalised coupling \( \alpha \) at scale \( \mu = 1/\sqrt{\tau} \) is

\[
\alpha(\mu) \equiv k_\alpha t^2 \langle E(t) \rangle \equiv k_\alpha E(t),
\]

where \( k_\alpha \) is a (perturbatively) calculable constant, and

\[
E(t) \equiv \frac{1}{2} \text{Tr} G_{\mu\nu}(t) G_{\mu\nu}(t).
\]

The evolution of \( \alpha(1/\sqrt{\tau}) \) defines implicitly the scale \( 1/\sqrt{\tau_0} \) of the system, by the requirement

\[
E(t)|_{t=t_0} = E_0,
\]

where \( E_0 \) is a reference value, chosen for convenience.

Alternatively, one defines the observable \[87\]

\[
W(t) \equiv t \frac{d}{dt} \{ t^2 \langle E(t) \rangle \},
\]

and the scale \( w_0 \) is defined implicitly from

\[
W(t)|_{t=w_0^2} = W_0,
\]

where again \( W_0 \) is a reference constant value. While \( E(t) \) is expected to be sensitive to the fluctuations of the gauge configurations on scales down to \( 1/\sqrt{\tau_0} \), \( W(t) \) only depends on fluctuations around \( 1/\sqrt{\tau_0} \).

We will compute the value of \( t_q \) and \( w_0 \) in \( Sp(N_c) \) theories for different values of \( N_c \), with the implicit intention of exploring the \( N_c \to \infty \) limit at constant ’t Hooft coupling \( \lambda \equiv 4\pi N_c \alpha \). From the perturbative relation between \( E(t) \) and the gradient flow coupling \[76\], we obtain the leading-order expression

\[
E(t) = \frac{3\lambda}{64\pi^2} C_2(F),
\]

with \( C_2(F) = \frac{N_c+1}{2} \), the quadratic Casimir operator of the fundamental representation of \( Sp(N_c) \). In order to compare different \( Sp(N_c) \) theories, we scale \( E_0 \) and \( W_0 \) according to the following relations:

\[
E_0(N_c) = c_e C_2(F), \quad W_0(N_c) = c_w C_2(F),
\]

where \( c_e \) and \( c_w \) are constants. From Eq. \[27\], leading-order perturbation theory gives

\[
c_e = \frac{3\lambda}{64\pi^2},
\]
showing that \( c_\alpha \) determines the fixed-\( \lambda \) trajectory along which to take the \( N_c \to \infty \) limit.\(^1\) Whether the scaling law in Eq. (28) holds outside of the domain of validity of perturbation theory is a question we return to in Sec. IV.

A. The Wilson flow

The lattice incarnation of the gradient flow is based on the Wilson action \( S_W \) in Eq. (11), and is known as the Wilson flow. \( V_\mu(x, t) \) is defined by solving

\[
\frac{\partial V_\mu(x, t)}{\partial t} = -g^2 \{\partial_\mu, V^\text{flow}_\mu \} V_\mu(x, t),
\tag{30}
\]

where \( V_\mu(x, 0) = U_\mu(x) \). The properties of the Wilson flow are naturally inherited from the continuum formulation. Moreover, a numerical integration can be set up to obtain \( V_\mu(x, t) \) from \( U_\mu(x) \) explicitly, using for example a Runge-Kutta integration scheme, as detailed in Ref. [76]. Observables can then be constructed from \( V_\mu(x, t) \).

To use the Wilson flow as a scale setting procedure requires the computation of \( E(t) \) or \( W(t) \), for which purpose two alternative lattice discretisations of \( G_{\mu\nu}(t) \) can be used. One is the elementary plaquette operator defined in Eq. (10), computed from \( V_\mu(x, t) \). The other is the four-plaquette clover-leaf in Eq. (13). By comparing numerically the values of \( E(t) \) and \( W(t) \), as well as the two different discretisations, we can assess the magnitude of discretisation errors in approaching the continuum limit.

B. Topological susceptibility on the lattice

As anticipated in Section IV, we compute the topological susceptibility on the lattice from Wilson-flowed configurations. At time \( t \), the topological charge density can be obtained from

\[
q_L(t, x) \equiv \frac{1}{32\pi^2} e^{\mu\rho\sigma\tau} \text{Tr} C_{\mu\rho}(x, t) C_{\rho\sigma}(x, t),
\tag{31}
\]

where \( C_{\mu\rho}(x, t) \) is the clover operator computed from \( V_\mu(x, t) \). The topological charge is \( Q_L(t) = \sum_x q_L(x, t) \).

On the lattice, the values of the topological charge are quasi-integers, affecting the measurement of \( \chi_L \). Following Ref. [66], we reduce this systematical error by redefining \( \tilde{Q}_L \) as follows:

\[
\tilde{Q}_L(t) \equiv \text{round} \left( \hat{\alpha} \sum_x q_L(x, t) \right),
\tag{32}
\]

where \( \hat{\alpha} \) is a numerical factor determined by minimising the \( t \)-dependent quantity

\[
\Delta(\hat{\alpha}) = \left\langle |\hat{\alpha} Q_L - \text{round}(\hat{\alpha} Q_L)|^2 \right\rangle.
\tag{33}
\]

\(^{1}\) For unitary groups \( SU(N_c) \), \( C_2(F) = (N_c^2 - 1)/(2N_c) \). The choice \( c_\alpha = 9/40 \) would yield \( \varepsilon_0 = 0.3 \) for \( SU(3) \) [76].

### Table I: Ensembles used for scale setting.

| \( N_c \) | \( L/a \) | \( \beta \) | \( N_{sw} \) | \( N_{tot} \) | \( \tau_\xi \) |
|---|---|---|---|---|---|
| 2 | 20 | 2.55 | 50 | 3999 | 0.512(30) |
| 2 | 24 | 2.60 | 100 | 3999 | 0.512(30) |
| 2 | 32 | 2.65 | 100 | 4003 | 0.561(33) |
| 2 | 32 | 2.70 | 100 | 4003 | 0.729(43) |
| 4 | 20 | 7.7 | 50 | 4000 | 0.644(38) |
| 4 | 20 | 7.72 | 50 | 4000 | 0.672(40) |
| 4 | 20 | 7.76 | 50 | 4000 | 0.779(52) |
| 4 | 20 | 7.8 | 40 | 4002 | 1.079(80) |
| 4 | 20 | 7.8 | 80 | 4021 | 0.691(41) |
| 4 | 20 | 7.85 | 70 | 4002 | 1.104(82) |
| 4 | 24 | 8.2 | 3500 | 3898 | 0.550(33) |
| 6 | 18 | 15.75 | 60 | 4000 | 0.848(57) |
| 6 | 16 | 15.9 | 100 | 4006 | 0.959(64) |
| 6 | 16 | 16.1 | 400 | 4011 | 1.170(87) |
| 6 | 20 | 16.3 | 800 | 4001 | 1.47(12) |
| 8 | 16 | 26.5 | 600 | 3924 | 0.617(37) |
| 8 | 16 | 26.7 | 400 | 4061 | 1.27(10) |
| 8 | 16 | 27.0 | 1200 | 3887 | 1.50(13) |
| 8 | 16 | 27.2 | 3000 | 4107 | 1.245(99) |

We will provide an illustrative example of the choice of numerical factor \( \alpha \sim O(1) \) when presenting our results. The topological susceptibility at flow time \( t \) is then

\[
\chi_L(t)a^4 = \frac{1}{L^4} \left\langle \tilde{Q}_L(t)^2 \right\rangle.
\tag{34}
\]

IV. NUMERICAL RESULTS

We generated and stored ensembles of thermalised configurations for the set of bare parameters listed in the three left-most columns of Table I. From a comparison with the results obtained for \( \sigma \) in Ref. [13], we know that \( \sqrt{\sigma L} \geq 3 \) for all such ensembles, and neglect finite-volume effects. In the following, we present and discuss our numerical results for the scale-setting procedure, and for the topological susceptibility. All data presented, as well as underlying raw data, are available at Ref. [88], and the analysis code used to prepare the main figures and tables are shared at Ref. [80].

A. Setting the scale

Each configuration in the ensembles in Table I sets the initial conditions for the numerical integration of the Wilson flow, which obeys Eq. (30). Following Ref. [76], we use a third-order Runge-Kutta integrator (implemented by HiRep [80]) to evaluate \( V_\mu(x, t) \) in the range...
The quantities $E(t)$ (top panel) and $W(t)$ (bottom panel), defined in Eqs. (22) and (23), and in Eq. (25), respectively, in the $N_c = 6$ ensembles with $\beta = 15.9$ and $\beta = 16.3$, as functions of the flow time $t$. Computations adopt the alternative choices of discretisation provided by the elementary plaquette (pl.) and the clover-leaf plaquette (cl.). The horizontal dashed line represents the choice $E_0 = 0.5$, $W_0 = 0.5$.

$0 < t < t_{\text{max}}$, with $t_{\text{max}}$ such that $\sqrt{8t_{\text{max}}} \lesssim L/2$, to avoid large finite size effects.

The quantity $E(t)$ is obtained from the definitions in Eqs. (22) and (23), by computing $G_{\mu\nu}$ from $B_{\mu}$. We use two alternative discretised expressions for $E(t)$, provided by the plaquette (pl.), $P_{\mu\nu}$, or the clover-leaf (cl.), $C_{\mu\nu}$.

We then compute $W(t)$ according to Eq. (25). The resampled results for $E(t)$ and $W(t)$, as functions of $t$, are displayed in the two panels of Fig. 1 in the case $N_c = 6$, with $\beta = 15.9$ and $\beta = 16.3$ and for the two different discretizations (pl. and cl.). For each value of $t$, the vertical thickness of the curves represents the error of $E(t)$ and $W(t)$, computed by bootstrapping. The picture is qualitatively the same for other values of the bare parameters.
and is similar to the $SU(N_c)$ case.

From $E(t)$ and $W(t)$ we can extract the scales $t_0$ and $w_0$, according to the definitions in Eqs. (24) and (26), once we make a choice for the reference values $E_0$ and $W_0$. For illustrative purposes, the choices $E_0 = 0.5$ and $W_0 = 0.5$ are represented as horizontal dashed lines in the top and bottom panel of Fig. 1 though we do not use this choice in the analysis. The comparison between $E(t)$ and $W(t)$ provides a first assessment of the magnitude of discretisation effects in the calculation. For each ensemble, the difference between the curves corresponding to the plaquette and clover discretisations tends to a constant at large $t$. This difference is the smallest in the ensemble with the largest value of $\beta$—the closest to the continuum limit—and is smaller for $W(t)$ than $E(t)$, as anticipated in Section III.

A more refined assessment of discretisation effects can be obtained by studying the value of the scales $\sqrt{t_0}/a$ and $w_0^2/a$, obtained for a range of choices of $E_0$ and $W_0$, for each available ensemble, which we report in Tables II and III and display in Fig. 2 for $Sp(6)$. The difference be-

### Table II: The gradient flow scale $\sqrt{t_0}/a$ for different choices of $\beta$ and $E_0$, for $N_c = 6$. We report the results for both the plaquette and clover discretisations.

| $\beta$ | $E_0$ | $\sqrt{t_0}/a$ (pl.) | $\sqrt{t_0}/a$ (cl.) |
|---|---|---|---|
| 15.75 | 0.3 | 1.14525(18) | 1.13768(17) |
| 15.9 | 0.3 | 1.20901(29) | 1.19500(29) |
| 16.1 | 0.3 | 1.22707(40) | 1.21698(38) |
| 16.3 | 0.3 | 1.34793(34) | 1.34454(32) |
| 15.75 | 0.4 | 1.20835(21) | 1.20689(20) |
| 15.9 | 0.4 | 1.26876(33) | 1.26784(33) |
| 16.1 | 0.4 | 1.34754(46) | 1.34701(44) |
| 16.3 | 0.4 | 1.42652(39) | 1.42624(37) |
| 15.75 | 0.5 | 1.26030(23) | 1.26195(22) |
| 15.9 | 0.5 | 1.32419(37) | 1.32572(37) |
| 16.1 | 0.5 | 1.40712(51) | 1.40852(48) |
| 16.3 | 0.5 | 1.48995(43) | 1.49123(41) |
| 15.75 | 0.6 | 1.35458(25) | 1.35079(24) |
| 15.9 | 0.6 | 1.37126(40) | 1.37421(40) |
| 16.1 | 0.6 | 1.45758(55) | 1.46009(53) |
| 16.3 | 0.6 | 1.54353(47) | 1.54572(45) |

### Table III: The gradient flow scale $w_0^2/a$ for different choices of $\beta$ and $W_0$, for $N_c = 6$. We report the results for both the plaquette and clover discretisations.

| $\beta$ | $W_0$ | $w_0^2/a$ (pl.) | $w_0^2/a$ (cl.) |
|---|---|---|---|
| 15.75 | 0.3 | 1.14525(18) | 1.13768(17) |
| 15.9 | 0.3 | 1.20901(29) | 1.19500(29) |
| 16.1 | 0.3 | 1.22707(40) | 1.21698(38) |
| 16.3 | 0.3 | 1.34793(34) | 1.34454(32) |
| 15.75 | 0.4 | 1.20835(21) | 1.20689(20) |
| 15.9 | 0.4 | 1.26876(33) | 1.26784(33) |
| 16.1 | 0.4 | 1.34754(46) | 1.34701(44) |
| 16.3 | 0.4 | 1.42652(39) | 1.42624(37) |
| 15.75 | 0.5 | 1.26030(23) | 1.26195(22) |
| 15.9 | 0.5 | 1.32419(37) | 1.32572(37) |
| 16.1 | 0.5 | 1.40712(51) | 1.40852(48) |
| 16.3 | 0.5 | 1.48995(43) | 1.49123(41) |
| 15.75 | 0.6 | 1.35458(25) | 1.35079(24) |
| 15.9 | 0.6 | 1.37126(40) | 1.37421(40) |
| 16.1 | 0.6 | 1.45758(55) | 1.46009(53) |
| 16.3 | 0.6 | 1.54353(47) | 1.54572(45) |
between the plaquette and clover discretizations becomes smaller as $\beta$ is increased. This difference has generally a smaller magnitude for the $w_0$ scale than for $\sqrt{t_0}$ scale. A similar picture emerges for $N_c = 2$, $N_c = 4$, and $N_c = 8$, as reported in Appendix A. In view of these considerations, we adopt the clover discretisation in the following.

In Section III, we observed that the functions $E(t)$ and $W(t)$ obey perturbative relations that suggest the scaling behaviour in Eq. (28). The numerical data we have collected allows to test the validity of these scaling relations outside of the domain of perturbation theory. We choose values of $E_0$ and $W_0$ for each value of $N_c$ according to

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**FIG. 4:** The quantities $E(t)/C_2(F)$ (top panel) and $W(t)/C_2(F)$ (bottom panel) computed with the clover-leaf plaquette discretisation of the flow equation on the available ensembles corresponding to the finest and coarsest lattices, for each $N_c$, with $C_2(F) = (N_c + 1)/4$, displayed as a function of the rescaled flow times $t/t_0$ and $t/w_0^2$. The figure adopts the choice $c_e = c_w = 0.225$ (horizontal dashed line).

**FIG. 5:** In the top panel, the topological charge $Q_L$ as a function of the flow time $t/a^2$, for the 100 first configurations of the ensemble with $N_c = 8$, $L/a = 16$, and $\beta = 26.7$. The values of $Q_L$ are also reported as red bullets at $t = t_0$, where $t_0$ is obtained from the choice $c_e = 0.225$. In the bottom panel, the topological susceptibility $\chi_L(t)a^4$ as a function of the flow time $t$, with its 1-σ error band, for $N_c = 8$, at $\beta = 26.7$. The vertical dashed line represents $t = t_0$, where $t_0$ is obtained from $c_e = c_w = 0.225$. The value of $\chi_L(t = t_0)a^4$ obtained at $t = t_0$ is depicted as a red bullet and reported in Table IV.
Eq. (28), with fixed $c_e$ and $c_w$. The corresponding values of $t_0$ and $w_0$ are then used to scale also $t$.

The result of these operations, with the clover-leaf discretisation, and at each fixed value of $N_c$, for the ensembles with the smallest and the greatest values of $\beta$, are displayed in Fig. 3 for the choice $c_e = c_w = 0.5$. The plots exhibit the same qualitative features for both $E(t)/C_2(F)$ and $W(t)/C_2(F)$ as functions of $t/t_0$ and $t/w_0^2$, respectively. We repeat the same process also for the choice $c_e = c_w = 0.225$ (which would yield $E_0 = 0.3$ for $SU(3)$), and display the results in Fig. 4. We labelled the curves by the conveniently defined discretised coupling

$$\tilde{\lambda} = \frac{d_G}{\beta \langle \text{Tr} F_{\mu\nu} \rangle}, \quad (35)$$

where $d_G$ is the dimension of the group.

By construction, the rescaled flows corresponding to different values of $N_c$ coincide when $t/t_0 = 1$ (or $t/w_0^2 = 1$). What is interesting is that the curves agree (within uncertainties) over a sizeable range of $t$ around these points. This might indicate that the validity of the perturbative scaling in Eq. (28) holds also outside of the naive range of the perturbative regime. Small deviations from perfect scaling are nevertheless visible and might be ascribed to a combination of finite-$a$ and finite-$N_c$ effects.

### B. The topological charge

The quantity $Q_L(t)$ is computed from Wilson-flowed configurations at flow time $t$, using Eq. (31), as implemented by HiRep [80]. Fig. 5 is compiled with the first 100 configurations of the ensemble with $N_c = 8$ and $\beta = 26.7$. Integer values for $Q_L(t)$ are only obtained for $a \to 0$. At non-zero $a$, the quantity $Q_L(t)$ tends, at large $t$, towards quasi-integer values. The integer-valued topological charge $Q_L$ is instead obtained for a finite value of $t$, according to Eq. (32). The values of $Q_L$ are displayed in Fig. 5 as red bullets, and for the value of $t = t_0$ identified with the choice $c_e = 0.225$. Effectively this definition of $Q_L$ optimises and expedites the convergence towards the physical, discrete values of the topological charge. Similar conclusions hold for other values of the bare parameters, though we do not report further details here. In the following, we will compute $Q_L$ for every configuration of every ensemble at both $c_e = 0.225$ and $c_e = 0.5$.

Simulation histories of $Q_L$ for the ensembles with the finest and coarsest lattice are displayed in Figs. 6, 7, 8 and 9 for $N_c = 2, 4, 6, 8$, respectively. The frequency histogram for the values of $Q_L$ is also reported, in each case, and is consistent with a Gaussian, symmetric distribution around $Q_L = 0$.

The magnitude of autocorrelations can be evaluated using the Madras-Sokal windowing algorithm [90] on the simulation history of either $Q_L$ or $Q_L$. For our ensembles, we found no significant difference to arise between the two, and no visible dependence on the choice of $c_e$. We thus compute $\tau_Q$ at $t = t_0$, for $c_e = 0.225$. The behaviour of the logarithm of $\tau_Q N_{sw}$ as a function of $\beta$ is displayed in Fig. 10 and reported in Table I. On the basis of known results obtained for $SU(N_c)$ gauge groups, in $\tau_Q N_{sw}$ is expected to be linearly diverging as $\beta \to \infty$ [59], and indeed this is consistent with what we find.

The topological susceptibility in lattice units $\chi_L(t)a^4$ was computed for each ensemble separately, using Eq. (34). The effect of the rounding procedure, Eq. (32), on the topological susceptibility, is displayed in Figure 5 where $\chi_L(t)a^4$ is plotted as a function of the flow time $t/a^2$. For sufficiently large values of the flow time, $\chi_L(t)a^4$ is compatible with a constant within errors. The value of $\chi_L(t)a^4$, computed at the scale $t_0$, obtained from
Our final results are the extrapolations towards the continuum limit of $\chi_{L} w_{0}^{2}$ for each of the $Sp(N_c)$ Yang-Mills theories. We obtain these by assuming the following functional dependence:

$$\chi_L(a) t_0^2 = \chi_L(a = 0) t_0^2 + c_1 a^2 t_0,$$  \hspace{2cm} (36)

for each value of $N_c$, separately. The results of this extrapolation are reported in Table VI (and displayed in Fig. 11) for $c_e = 0.225$ and $c_e = 0.5$, respectively. For completeness, we also report in the same Table VI and Fig. 11 also the extrapolation of $\chi_L w_0^2$, obtained with a similar fitting function which includes one correction term $O\left(\frac{a^2}{w_0^2}\right)$. The uncertainty on the extrapolated values are obtained from the maximum likelihood analysis by including statistical uncertainties only. Ref. [65], which reports $\chi_L t_0^2$ for $SU(N_c)$ with $N_c = 3, 4, 5, 6$, present the results in units of $t_0$, and for the choice $c_e = 0.225$. Unfortunately, a direct comparison is not possible, as the authors of Ref. [65] did not report on the common $SU(2) \sim Sp(2)$ case. We will return to the comparison of the topological susceptibility in $Sp(N_c)$
and $SU(N_c)$ theories in a companion publication [20].

V. SUMMARY AND DISCUSSION

We studied the four-dimensional Yang-Mills theories with group $Sp(N_c)$, for $N_c = 2, 4, 6, 8$, by means of lattice numerical techniques. We used the Wilson flow as a scale-setting procedure. We showed that lattice artefacts are reduced by adopting the clover-leaf plaquette discretisation. We also found that by simultaneously rescaling $\mathcal{E}(t)$ (or alternatively $W(t)$), the reference value of $\mathcal{E}_0$ (or alternatively $W_0$), and $t_0/a^2$ (or alternatively $w_0/a$), the Wilson flows for different gauge groups agree (within numerical errors) over a non-trivial range of $t$. The proposed rescaling is based upon the group-factor dependence of the leading-order perturbative evaluation of $\mathcal{E}(t)$, indicating that the suppression of sub-leading corrections that differentiate the groups holds also non-perturbatively, over a finite range of flow time $t$.

We then computed the topological susceptibility from
FIG. 11: Topological susceptibility per unit volume $\chi_L t_0^2$ as a function of $a^2/t_0$ (top panels) and $\chi_L w_0^4$ as a function of $a^2/w_0^2$ (bottom), in $Sp(N_c)$ Yang-Mills theories with $N_c = 2, 4, 6, 8$. We adopt reference values $c_c = c_w = 0.225$ (left panels) and $c_c = c_w = 0.5$ (right). Our continuum extrapolations are represented as dashed lines. The results are reported in Table VI.

our lattice ensembles, expressed it in units of the Wilson flow scale $t_0$ ($w_0$), and performed the continuum extrapolation for the four gauge groups. We summarise in Table VI our results for the continuum extrapolation of $\chi_t^2$ and $\chi_w^4$ for two different choices of reference values for $E_\sigma$ and $W_0$. In a companion paper, we present our extrapolation of the measurement of the topological susceptibility towards the large-$N_c$ limit (in units of the string tension), and discuss how to compare it with the analogous calculations performed for $SU(N_c)$ Yang-Mills theories.
TABLE V: Topological susceptibilities $\chi_\ell t_0^2$ and $\chi_\ell w_0^2$, with $c_\ell = c_w = 0.5$, and the string tensions $\sigma_0$ and $\sigma w_0^2$, for all available ensembles. These results are displayed as a function of $a^2/t_0$ and $a^2/w_0$ in Figure 11.

| $N_c$ | $\beta$ | $\sigma_0$ | $\chi_\ell t_0^2 \cdot 10^4$ | $\sigma w_0^2$ | $\chi_\ell w_0^2 \cdot 10^4$ |
|-------|---------|------------|-----------------------------|---------------|-----------------------------|
| 2     | 2.55    | 0.2289(19) | 2.547(58) 0.1956(16) 1.881(47) |               |                             |
| 2     | 2.6     | 0.2235(21) | 2.448(49) 0.1978(18) 1.789(35) |               |                             |
| 2     | 2.65    | 0.2233(39) | 2.527(60) 0.1902(33) 1.840(40) |               |                             |
| 2     | 2.7     | 0.2186(37) | 2.388(63) 0.1861(31) 1.736(45) |               |                             |
| 4     | 7.7     | 0.2265(52) | 2.510(60) 0.2070(47) 2.100(51) |               |                             |
| 4     | 7.72    | 0.2281(62) | 2.669(76) 0.2087(56) 2.246(56) |               |                             |
| 4     | 7.76    | 0.2178(63) | 2.647(59) 0.1992(58) 2.225(46) |               |                             |
| 4     | 7.78    | 0.2271(64) | 2.556(61) 0.2080(59) 2.137(57) |               |                             |
| 4     | 7.8     | 0.2186(51) | 2.430(53) 0.2001(47) 2.040(48) |               |                             |
| 4     | 7.85    | 0.2246(31) | 2.507(67) 0.2060(29) 2.113(53) |               |                             |
| 4     | 8.2     | 0.2323(36) | 2.399(62) 0.2146(33) 2.055(47) |               |                             |
| 6     | 15.75   | 0.2261(29) | 2.468(49) 0.2147(28) 2.221(50) |               |                             |
| 6     | 15.9    | 0.2313(30) | 2.317(53) 0.2208(29) 2.103(47) |               |                             |
| 6     | 16.1    | 0.2230(38) | 2.310(56) 0.2136(37) 2.126(53) |               |                             |
| 6     | 16.3    | 0.2254(78) | 2.137(56) 0.2162(75) 1.975(60) |               |                             |
| 8     | 26.5    | 0.2285(31) | 2.380(49) 0.2220(30) 2.248(45) |               |                             |
| 8     | 26.7    | 0.2301(63) | 2.345(70) 0.2243(61) 2.230(65) |               |                             |
| 8     | 27.0    | 0.2285(33) | 2.351(76) 0.2236(32) 2.244(66) |               |                             |
| 8     | 27.2    | 0.2204(32) | 2.149(64) 0.2161(32) 2.057(51) |               |                             |

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Acknowledgments
In this Appendix, we report the intermediate results of the scale-setting procedure for the $Sp(2)$, $Sp(4)$, and $Sp(8)$ Yang-Mills theories. The presentation mirrors the one for the $Sp(6)$ theory, in the main body of the paper. Tables VII and VIII present our results for $t_0/a^2$ and $w_0/a$, respectively, for various choices of reference values $E_0$ and $W_0$, for $N_c = 2$. We tabulate the results for both the plaquette and clover discretisations. Tables IX and X list the same information, but for $Sp(4)$, while Tables XI and XII refer to $Sp(8)$. Cases in which Eqs. (24) or (26) do not admit a solution, because of extreme choices of $E_0$ or $W_0$, are left blank. The information in Tables VII-XII is also graphically displayed in Figs. 12-17 and available in machine-readable form in Ref. [88].

![Appendix A: Scale setting data for $N_c = 2, 4, 8$](image)

### Table VII: The gradient flow scale $\sqrt{t_0}/a$ for different choices of $E_0$ and $\beta$, for $N_c = 2$. We report the results for both the plaquette and clover discretisations.

| $\beta$ | $E_0$ | $\sqrt{t_0}/a$ (pl.) | $\sqrt{t_0}/a$ (cl.) |
|---------|-------|-----------------------|-----------------------|
| 2.55    | 0.3   | 2.698(30)             | 2.748(29)             |
| 2.60    | 0.3   | 3.181(33)             | 3.224(32)             |
| 2.65    | 0.3   | 3.734(29)             | 3.771(30)             |
| 2.70    | 0.3   | 4.357(48)             | 4.390(46)             |
| 2.55    | 0.4   | 3.084(36)             | 3.083(34)             |
| 2.60    | 0.4   | 3.577(40)             | 3.620(38)             |
| 2.70    | 0.4   | 4.195(36)             | 4.235(36)             |
| 2.70    | 0.5   | 4.892(57)             | 4.926(55)             |
| 2.55    | 0.5   | 3.318(41)             | 3.369(40)             |
| 2.60    | 0.5   | 3.904(46)             | 3.948(43)             |
| 2.65    | 0.5   | 4.576(41)             | 4.614(41)             |
| 2.70    | 0.5   | 5.333(66)             | 5.368(64)             |
| 2.55    | 0.6   | 3.561(47)             | 3.613(45)             |
| 2.60    | 0.6   | 4.186(51)             | 4.230(48)             |
| 2.65    | 0.6   | 4.904(46)             | 4.943(46)             |
| 2.70    | 0.6   | —                     | —                     |

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TABLE XI: The gradient flow scale $\sqrt{t_0/a}$ for different choices of $\epsilon_0$ and $\beta$, for $N_c = 8$. We report the results for both the plaquette and clover discretisations.

| $\beta$ | $\epsilon_0$ | $\sqrt{t_0/a}$ (pl.) | $\sqrt{t_0/a}$ (cl.) |
|---------|--------------|-----------------------|----------------------|
| 26.5    | 0.3          | 0.6428(11)            | 1.04083(17)          |
| 26.7    | 0.3          | 0.91955(26)           | 1.12079(20)          |
| 27.0    | 0.3          | 1.09768(30)           | 1.24650(27)          |
| 27.2    | 0.3          | 1.20530(34)           | 1.33540(35)          |
| 26.5    | 0.5          | 1.08015(25)           | 1.21190(23)          |
| 26.7    | 0.5          | 1.19344(27)           | 1.30764(27)          |
| 27.0    | 0.5          | 1.36119(40)           | 1.45737(38)          |
| 27.2    | 0.5          | 1.47602(46)           | 1.56329(49)          |
| 26.5    | 0.6          | 1.25801(30)           | 1.35615(28)          |
| 26.7    | 0.6          | 1.37719(33)           | 1.46475(34)          |
| 26.5    | 0.6          | 1.55864(49)           | 1.63417(47)          |
| 27.2    | 0.6          | 1.68496(57)           | 1.75412(61)          |

TABLE XII: The gradient flow scale $w_0/a$ for different choices of $W_0$ as a function of $\beta$, for $N_c = 8$. We report the results for both the plaquette and clover discretisations.

| $\beta$ | $W_0$ | $w_0/a$ (pl.) | $w_0/a$ (cl.) |
|---------|-------|---------------|---------------|
| 26.5    | 0.3   | 1.11790(16)   | 1.10726(15)   |
| 26.7    | 0.3   | 1.16124(17)   | 1.15275(18)   |
| 27.0    | 0.3   | 1.22617(25)   | 1.21983(24)   |
| 27.2    | 0.3   | 1.27059(28)   | 1.26524(30)   |
| 26.5    | 0.5   | 1.16496(18)   | 1.16048(17)   |
| 26.7    | 0.5   | 1.21151(19)   | 1.20807(20)   |
| 27.0    | 0.5   | 1.28068(28)   | 1.27826(27)   |
| 27.2    | 0.5   | 1.32779(31)   | 1.32582(34)   |
| 26.5    | 0.6   | 1.20570(19)   | 1.20466(18)   |
| 26.7    | 0.6   | 1.25467(21)   | 1.25407(22)   |
| 27.0    | 0.6   | 1.32710(31)   | 1.32690(29)   |
| 27.2    | 0.6   | 1.37632(34)   | 1.37627(37)   |
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