THE PLANAR TREE PACKING THEOREM*

Markus Geyer,† Michael Hoffmann,‡ Michael Kaufmann,† Vincent Kusters,‡ and Csaba D. Tóth§

Abstract. Packing graphs is a combinatorial problem where several given graphs are being mapped into a common host graph such that every edge is used at most once. In the planar tree packing problem we are given two trees \( T_1 \) and \( T_2 \) on \( n \) vertices and have to find a planar graph on \( n \) vertices that is the edge-disjoint union of \( T_1 \) and \( T_2 \). A clear exception that must be made is the star which cannot be packed together with any other tree. But according to a conjecture of García et al. from 1997 this is the only exception, and all other pairs of trees admit a planar packing. Previous results addressed various special cases, such as a tree and a spider tree, a tree and a caterpillar, two trees of diameter four, two isomorphic trees, and trees of maximum degree three. Here we settle the conjecture in the affirmative and prove its general form, thus making it the planar tree packing theorem. The proof is constructive and provides a polynomial time algorithm to obtain a packing for two given nonstar trees.

1 Introduction

The packing problem is to find a graph \( G \) on \( n \) vertices that contains a given collection \( G_1, \ldots, G_k \) of graphs on \( n \) vertices each as edge-disjoint subgraphs. This problem has been studied in a wide variety of scenarios (see, e.g., [1, 4, 8]). Much attention has been devoted to the packing of trees, stimulated by the famous tree packing conjectures by Ringel [23], Gyárfás [16], and Erdős-Sós [7]. Hedetniemi et al. [17] proved that any two nonstar trees can be packed into \( K_n \). Teo and Yap [25] showed, extending an earlier result by Bollobás and Eldridge [2], that any two graphs of maximum degree at most \( n - 2 \) with a total of at most \( 2n - 2 \) edges pack into \( K_n \) unless they are one of thirteen specified pairs of graphs. Maheo et al. [19] characterized triples of trees that can be packed into \( K_n \).

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† Universität Tübingen, {geyer, mk}@informatik.uni-tuebingen.de
‡ ETH Zürich, hoffmann@inf.ethz.ch, supported by the ESF EUROCORES programme EuroGIGA, CRP GraDR and the Swiss National Science Foundation, SNF Project 20GG21-134306
§ California State University Northridge and Tufts University, cdtoth@acm.org, supported by the NSF awards CCF-1422311 and CCF-1423615
In the planar packing problem the graph $G$ is required to be planar. García et al. [11] conjectured in 1997 that there exists a planar packing for any two nonstar trees, that is, for any two trees with diameter greater than two. The assumption that none of the trees is a star is necessary, since a star uses all edges incident to one vertex and so there is no edge left to connect that vertex in the other tree. García et al. proved their conjecture when one of the trees is a path and when the two trees are isomorphic. Oda and Ota [22] addressed the case that one of the trees is a caterpillar or that one of the trees is a spider of diameter at most four. A caterpillar is a tree that becomes a path when all leaves are deleted and a spider is a tree with at most one vertex of degree greater than two. Frati et al. [10] gave an algorithm to construct a planar packing of any spider with any tree. Frati [9] proved the conjecture for the case that both trees have diameter at most four. Finally, Geyer et al. [13] proved the conjecture for binary trees (maximum degree three). In this paper we settle the general conjecture in the affirmative:

**Theorem 1.** Every two nonstar trees of the same size admit a planar packing.

**Related work.** Finding subgraphs with specific properties within a given graph or more generally determining relationships between a graph and its subgraphs is one of the most studied topics in graph theory. The subgraph isomorphism problem [6, 12, 27] asks to find a subgraph $H$ in a graph $G$. The graph thickness problem [20] asks for the minimum number of planar subgraphs which the edges of a graph can be partitioned into. The arboricity problem [5] asks to determine the minimum number of forests which a graph can be partitioned into. Another related classical combinatorial problem is the $k$ edge-disjoint spanning trees problem which dates back at least to Tutte [26] and Nash-Williams [21], who gave necessary and sufficient conditions for the existence of $k$ edge-disjoint spanning trees in a graph. The interior edges of every maximal planar graph can be partitioned into three edge-disjoint trees, known as a Schnyder wood [24]. Gonçalves [15] proved that every planar graph can be partitioned in two edge-disjoint outerplanar graphs.

The study of relationships between a graph and its subgraphs can also be done the other way round. Instead of decomposing a graph, one can ask for a graph $G$ that encompasses a given set of graphs $G_1, \ldots, G_k$ and satisfies some additional properties. This topic occurs with different flavors in the computational geometry and graph drawing literature. It is motivated by applications in visualization, such as the display of networks evolving over time and the simultaneous visualization of relationships involving the same entities. In the simultaneous embedding problem [3] the graph $G = \bigcup G_i$ is given and the goal is to draw it so that the drawing of each $G_i$ is plane. The simultaneous embedding without mapping problem [3] is to find a graph $G$ on $n$ vertices such that: (i) $G$ contains all $G_i$’s as subgraphs, and (ii) $G$ can be drawn so that the induced drawing of each $G_i$ is plane.

**2 Notation and Overview**

A rooted tree is a directed tree $T$ with exactly one vertex of outdegree zero: its root, denoted by $\uparrow(T)$. Every vertex $v \neq \uparrow(T)$ has exactly one outgoing edge $(v, p_T(v))$. The target $p_T(v)$ is the parent of $v$ in $T$, and conversely $v$ is a child of $p_T(v)$. In figures we denote the root of
a tree by an outgoing upward or downward arrow. For a vertex $v$ of a rooted tree $T$, denote by $t_T(v)$ the subree rooted at $v$, that is, the subtree of $T$ induced by the vertices from which $v$ can be reached on a directed path. The subscript is sometimes omitted if $T$ is clear from the context. A subtree of (or below) $v$ is a tree $t_T(c)$, for a child $c$ of $v$ in $T$. For a tree $T$, denote by $|T|$ the size (number of vertices) of $T$. We denote by $\deg_T(v)$ the degree (indegree plus outdegree) of $v$ in $T$. For a graph $G$ we denote by $E(G)$ the edge set of $G$.

A star is a tree on $n$ vertices that contains at least one vertex of degree $n - 1$. Such a vertex is a center of the star. A star $S$ on $n \neq 2$ vertices has a unique center denoted by $\odot(S)$. For a star on two vertices, both vertices act as a center. When considered as a rooted tree, there are two different rooted stars on $n \geq 3$ vertices. A star rooted at a center is called central, whereas a star rooted at a leaf that is not a center is called dangling. In particular, every star on one or two vertices is a central star. A nonstar is a graph that is not a star. A substar of a graph is a subgraph that is a star. A one-page book embedding of a graph $G$ is an embedding of $G$ into a closed halfplane such that all vertices are placed on the bounding line. This line is called the spine of the book embedding.

We embed vertices along the positive $x$-axis and refer to them by their $x$-coordinate, that is, $P = \{1, \ldots, n\}$. To avoid notational clutter we identify points from $P$ with vertices embedded at them. An interval $[i,j]$ in $P$ is a sequence of the form $i,i+1,\ldots,j$, for $1 \leq i \leq j \leq n$, or $i,i-1,\ldots,j$, for $1 \leq j \leq i \leq n$. Observe that we consider an interval $[i,j]$ as oriented and so we can have $i > j$. Denote the length of an interval $[i,j]$ by $|[i,j]| = |i-j| + 1$. A suffix of an interval $[i,j]$ is an interval $[k,j]$, for some $k \in [i,j]$.

Although our final packing does not form a book embedding in general, it is still convenient to describe it using a terminology similar to book embeddings. In this vein, we refer to the $x$-axis as the spine of the embedding. As a default, edges are drawn as proper arcs, which we may think of geometrically as semicircles with both endpoints on the spine. If we say that an edge is drawn above or below the spine, then this means that this edge is drawn as a proper arc above or below the spine, respectively. Alternatively, an edge may be drawn as a biarc, that is, a curve that crosses the spine exactly once. Geometrically we may think of a biarc as a composition of two proper arcs, one above and one below the spine (refer to the figure on the title page for illustration).

**Overview.** We construct a plane drawing of two $n$-vertex trees $T_1$ and $T_2$ on the point set $P = [1,n]$. We call $T_1$ the blue tree; its edges are referred to as blue edges and shown as solid blue arcs in figures. The tree $T_2$ is called the red tree; its edges are referred to as red edges and shown as dotted red arcs in figures. As a default, blue edges are drawn above the spine and red edges are drawn below the spine.

The algorithm first computes a preliminary one-page book embedding of $T_1$ onto $P$ (the blue embedding) using a simple recursive procedure described in Section 3. The resulting blue embedding is not set in stone. We may adapt it later and possibly change it drastically. The role of this step is merely to provide some initial structure to work with in the following step.

In the second step we recursively construct a red embedding for the red tree to pair up with the blue embedding. In principle we follow a similar strategy as in the first step,
but we take the constraints imposed by the blue embedding into account. Specifically, we must ensure to not create a red edge that is already used by the blue tree. During this process we may reconsider and change the blue embedding locally. For instance, we may flip the embedding of some subtree of $T_1$ on an interval $[i, j]$, that is, reflect the embedded tree at the vertical line $x = \frac{i+j}{2}$ through the midpoint of $[i, j]$. In some cases we perform more drastic changes to the blue embedding. In particular, the blue embedding may not be a one-page book embedding in the final packing.

Although neither of the two trees $T_1$ and $T_2$ we start with is a star, it is possible—in fact, unavoidable—that stars appear as subtrees during the recursion. We have to deal with stars explicitly whenever they arise, because the general recursive step works for nonstars only. We introduce the necessary concepts and techniques in Section 4 and give the actual proof in Section 5.

3 A canonical one-page tree embedding

In this section, we describe a simple recursive algorithm to construct a one-page book embedding $\pi : V \rightarrow [i, j]$ for a rooted tree $T = (V, E)$ on $n = |[i, j]|$ vertices. Recall that our intervals are oriented and so we may have $i < j$ or $i > j$. Each recursive step works as follows.

We place $r := \uparrow(T)$ at position $i$. If $n = 1$, then there is nothing else to do. Otherwise, we recursively embed the subtrees of $r$ on pairwise disjoint subintervals of $[i, j] \setminus \{i\}$. The embedding is guided by two rules illustrated in Figure 1.

- The larger-subtree-first rule (LSFR) dictates that for any two subtrees of $r$, the larger of the subtrees must be embedded on an interval closer to $r$. Ties are broken arbitrarily.

- The one-side rule (1SR) dictates that for every vertex, all neighbors are mapped to the same side. That is, if $N_T(v)$ denotes the set of neighbors of $v$ in $T$ (including its parent), then either $\pi(u) < \pi(v)$ for all $u \in N_T(v)$ or $\pi(u) > \pi(v)$ for all $u \in N_T(v)$.

These rules imply that every subtree $T' \subseteq T$ is embedded onto an interval $[i', j'] \subseteq [i, j]$ so that $\{i', j'\}$ is an edge of $T'$ and either $i'$ or $j'$ is the root of $T'$. Together with $\pi(r) = i$, these rules define the embedding (up to tiebreaking). An example is depicted in Figure 1c. We refer to an embedding produced by this algorithm as a canonical embedding. The blue embedding we start with is always canonical. In some cases we also use a canonical embedding for a red subtree.
4 A red tree and a blue forest

As common with inductive proofs, we prove a stronger statement than necessary. This stronger statement does not hold unconditionally but we need to impose some restrictions on the input. The goal of this section is to derive this more general statement—formulated as Theorem 3—from which Theorem 1 follows easily.

Our algorithm receives as input a nonstar subtree $R$ of the red tree and an interval $I = [i, j]$ of size $|R|$ along with a blue graph $B$ embedded on $I$. In the rest of the paper, we will assume without loss of generality (using symmetry) that $i < j$. The notions left, right, before and after are used accordingly, with respect to the order along the spine.

In the initial call $B$ is a tree, but in a general recursive call $B$ is a blue forest that may consist of several components. For $k \in [i, j]$ let $B(k)$ denote the component of $B$ that contains $k$. For $[x, y] \subseteq [i, j]$ let $B[x, y]$ denote the subgraph of $B$ induced by the vertices in $[x, y]$. Observe that $B = B[i, j]$. For $k \in [x, y]$ let $B[x, y](k)$ denote the component of $B[x, y]$ that contains $k$.

In general the algorithm sees only a small part of the overall picture because it has access to the vertices in $I$ only. However, blue vertices in $I$ may have edges to vertices outside of $I$ and also vertices of $R$ may have neighbors outside of $I$. We have to ensure that such outside edges are used by one tree only and can be routed without crossings. In order to control the effect of outside edges, we allow only one vertex in each component—that is, the root of $R$ and the root of each component of $B$—to have neighbors outside of $I$. Whenever we change the blue embedding we need to maintain the relative order of these roots so as to avoid crossings among outside edges.

**Conflicts.** Typically $r := \uparrow(R)$ has at least one neighbor outside of $I$: its parent $p_{T_2}(r)$. But $r$ may also have children in $T_2 \setminus R$. We assume that all neighbors—parent and children—of $r$ in $T_2 \setminus R$ are already embedded outside of $I$ when the algorithm is called for $R$. There are two principal obstructions for mapping $r$ to a point $v \in I$:

- A vertex $v \in I$ is in edge-conflict with $r$ if $\{v, v'\} \in E(T_1)$ for some neighbor $v'$ of $r$ in $T_2 \setminus R$. Mapping $r$ to $v$ would make $\{v, v'\}$ an edge of both $T_1$ and $T_2$ (Figure 2a–2b). In figures we mark vertices in edge-conflict with $r$ by a lightning symbol $\mathbf{\text{\|}}$.

- A vertex $v \in I$ is in degree-conflict with $r$ on $I$ if $\deg_R(r) + \deg_B(v) \geq |I|$. If we map $r$ to $v$, then no child of $r$ in $R$ can be mapped to a neighbor of $v$ in $B$. With only $|I| - 1$ vertices available there is not enough room for both groups (Figure 2c).

We cannot hope to avoid conflicts entirely and we do not need to. It turns out that is sufficient to avoid a very specific type of conflicts involving stars.

- An interval $[i, j]$ is in edge-conflict (degree-conflict) with $R$ if $B^* := B(i)$ is a central star and $\uparrow(B^*)$ is in edge-conflict (degree-conflict) with $\uparrow(R)$ (Figure 3).

- An interval $I$ is in conflict with $R$ if $I$ is in edge-conflict or degree-conflict with $R$ (or both).
Theorem 3. Let $R$ be a nonstar tree with $r = \uparrow(R)$ and let $B$ be a nonstar forest with $|R| = |B| = n$, together with an ordering $b_1, \ldots, b_k$ of the $k \in \{1, \ldots, n\}$ roots of $B$ and a set $C \subseteq \{b_1, \ldots, b_k\}$. Suppose that (i) $\uparrow_B(b_1)$ is not a central star or (ii) $b_1 \notin C$ and $\deg_R(r) + \deg_B(b_1) < n$. Then there is a plane packing $\pi$ of $B$ and $R$ onto any set $P$ of $n$ points in the plane such that...
• \( \pi(r) \notin \pi(C) \) and

• \( b_1, \ldots, b_k, r \) appear in this order on the outer face of \( \pi \), that is, we can add a new vertex \( v \) in the outer face of \( \pi \) and route an edge to each of \( b_1, \ldots, b_k, r \) such that the resulting multigraph is plane and the circular order of neighbors around \( v \) is \( b_1, \ldots, b_k, r \). (If \( r = b_i \), for some \( i \in \{1, \ldots, k\} \), then two distinct edges must be routed from \( v \) to \( r \) so that the result is a non-simple plane multigraph.)

Such a packing \( \pi \) we call an ordered plane packing of \( B \) and \( R \) onto \( P \).

![Figure 4: An ordered packing of a forest \( B \) and a tree \( R \) as defined in Theorem 3.](image)

Theorem 3 is a strengthening of Theorem 1 and so we obtain Theorem 1 as an easy corollary.

**Proof of Theorem 1 from Theorem 3.** Select roots for \( T_1 \) and \( T_2 \) arbitrarily. Then use Theorem 3 with \( B = T_1, R = T_2, k = 1, b_1 = \uparrow(B), \) and \( C = \emptyset \). By assumption \( T_1 \) is not a star and so (i) holds. Therefore we can apply Theorem 3 and obtain the desired plane packing of \( T_1 \) and \( T_2 \).

To prove Theorem 3 we can work with an interval \( I = [1, n] \): Let \( p_1, \ldots, p_n \) be the sequence of points from \( P \) in lexicographic order. The polyline \( p_1, \ldots, p_n \) serves as the spine, and we identify \( i = p_i, \) for \( i \in I \).

**Justification for the conditions in Theorem 3.** Let us show why the conditions (i) and (ii) in Theorem 3 are necessary. If both (i) and (ii) are violated, then \( t_B(b_1) \) is a central star and either \( b_1 \in C \) or \( \deg_R(r) + \deg_B(b_1) \geq n \) (or both). The following two propositions present counterexamples to the corresponding relaxed statements. These examples are presented here to justify and help to understand the statement of Theorem 3, they are not relevant for the proof of Theorem 3.

**Proposition 4.** If in Theorem 3 the requirement \( b_1 \notin C \) is dropped from condition (ii) (and so \( B_1 \) may be a central star with \( b_1 \in C \)), then \( B \) and \( R \) may not admit an ordered plane packing.

**Proof.** Let \( k \geq 2 \) be an integer, and consider the blue forest \( B_1, \ldots, B_{k+1} \), where \( B_1 \) is a central star on \( k + 1 \) vertices and \( |B_i| = 1 \) for \( i \in \{2, \ldots, k + 1\} \). Let all roots of the trees from \( B \) be in \( C \), and let \( R \) be a union of two central stars \( R_1^* \) and \( R_2^* \) on \( k \) vertices each, with a single vertex \( r = \uparrow(R) \) such that the roots of \( R_1^* \) and \( R_2^* \) are the children of \( r \) (Figure 5a). Then \( |B| = |R| = 2k + 1 \). Suppose that \( B \) and \( R \) admit an ordered plane packing on \([1, 2k + 1]\).
Due to the conflicts, \( r \) must be placed at a leaf \( \ell \) of \( B_1 \). Since \( \ell \) and \( b_1 \) are adjacent in \( B \), neither \( \odot(R_i^*) \) nor \( \odot(R_2^i) \) can be put at \( b_1 \). One of the stars below \( r \), say \( R_1^* \), has to use \( b_1 \). As all other vertices in \( B_1 \) are adjacent to \( b_1 \) in \( B \), we have \( \odot(R_1^i) = b_1 \), for some \( i \in \{2, \ldots, k+1\} \).

Consider the triangle \( \delta \) formed by the two red edges \( \{b_i, \ell\} \) and \( \{b_i, b_1\} \), and the blue edge \( \{b_1, \ell\} \). Together with the outside blue arcs \( g_1 \) and \( g_\ell \) going upward from \( b_1 \) and \( b_\ell \), respectively, and the outward red arc \( g_r \) going downward from \( \ell \), the triangle \( \delta \) partitions the plane into four regions: \( F_\ell \) to the left of the curve \( (g_\ell, \ell, b_1, g_1) \), \( F_\ell \) above the curve \( (g_1, b_1, b_\ell, g_\ell) \), \( F_r \) to the right of the curve \( (g_\ell, \ell, b_\ell, g_1) \), and the region \( F_\delta \) bounded by \( \delta \) (Figure 5b).

Note that no edge can connect vertices in two different regions because doing so would either create an edge crossing (with an edge of \( \delta \)) or one of the vertices \( b_1, b_\ell \), or \( r = \ell \) would be cut off from the outer face. Therefore, the star \( R_2^* \) must be embedded properly inside one of the four regions. But \( F_\ell \) cannot be reached by an edge from \( \ell \), and we claim that each of the other three regions contains at most \( k - 1 \) vertices in its interior. In other words, such an embedding is impossible and hence no ordered packing exists.

It remains to prove the claim. The vertices \( b_2, \ldots, b_{i-1} \) lie inside \( F_\ell \), whereas \( b_{i+1}, \ldots, b_{i+1} \) lie inside \( F_r \). The only wildcards are the remaining \( k - 1 \) leaves of \( B_1 \). As \( F_r \) cannot be reached by an edge from \( b_1 \), these leaves are distributed among the regions \( F_\ell, F_\delta \), and \( F_\ell \). Even if all of these leaves are in \( F_\ell \) or \( F_\delta \), this yields no more than \( k - 1 \) vertices in that region, as claimed. The maximum number of vertices in \( F_r \) is obtained for \( i = 2 \), which yields only \( k - 1 \) vertices. This proves the claim and completes the proof of the proposition.

**Proposition 5.** If in Theorem 3 the requirement \( \deg_{R}(r) + \deg_{B}(b_1) < n \) is dropped from condition (ii) (and so \( B_1 \) may be a central star with \( \deg_{R}(r) + \deg_{B}(b_1) \geq n \)), then \( B \) and \( R \) may not admit an ordered plane packing.

**Proof.** Let \( k \geq 2 \) be an integer, and let \( B \) consist of the trees \( B_1, B_2, \) and \( B_3 \), where \( B_1 \) is a central star on \( 2k + 1 \) vertices, \( |B_2| = |B_3| = 1 \), and \( C = \{b_2, b_3\} \). Let \( R \) be a union of \( k + 1 \) paths on two vertices each, attached to a common root \( r \) (Figure 5c). Then \( |B| = |R| = 2k + 3 \).

Suppose that \( B \) and \( R \) admit an ordered plane packing on \([1, 2k + 3]\). As \( r \) has
$k + 1 \geq 3$ children, it cannot be mapped to $b_1$, which has only two non-neighbors in $B$. Due to the conflicts with $b_2$ and $b_3$, it follows that $r$ has to be mapped to a leaf $\ell$ of $B_1$. One of the $k + 1$ paths in $R$ has to use $b_1$, along with a non-neighbor of $b_1$ in $B$, which leaves $b_2$ and $b_3$ as the only choices. It follows that for some $i \in \{2, 3\}$, we have $\{b_i, b_1\} \in E(R)$ and $\{b_i, \ell\} \in E(R)$. Define the triangle $\delta$ and the regions $F_\ell, F_t, F_r,$ and $F_\delta$ is in the proof of Proposition 4 (Figure 5b).

It remains to embed the remaining $k$ paths of $R$ and to place the remaining $2k - 1$ leaves of $B_1$. As $F_r$ cannot be reached by an edge from $b_1$, it follows that $i = 3$. (If $i = 2$, then $b_3$ would be the only vertex in the interior of $F_r$ and we could not use $b_3$ for any of the remaining paths of $R$.) But then $b_2$ lies in the interior of $F_t$, which cannot be reached by an edge from $b_1$. Therefore, $b_2$ cannot be used for any of the remaining paths of $R$ and there is no ordered packing.

\section*{Runtime analysis.}

The algorithm is parameterized with a subtree $R$ of $T_2$ and an interval $I \subseteq [1, n]$, which $R$ is to be packed onto together with an already embedded subforest of $T_1$. If we represent the abstract graph $T_1$ as an adjacency matrix and the embeddings as arrays, then after an $O(n^2)$ time initialization we can test in constant time for the presence of an edge between $i, j \in I$. To represent $T_2$ we use an adjacency list where the children are sorted by the size of their subtrees, which can be precomputed in $O(n \log n)$ time. Then at each step, the algorithm spends $O(|I|)$ time and makes at most two recursive calls with disjoint sub-intervals of $I$, which yields $O(n^2)$ time overall.

\section{Embedding the red tree: fundamentals}

In this section we discuss some fundamental tools for our recursive embedding algorithm to prove Theorem 3. First we formulate four invariants that hold for every recursive call of the algorithm. Next we present three tools that are specific types of embeddings to handle a “large” substar of $B$ or $R$. All of these embeddings rearrange the given embedding of $B$ to make room for the center of the star. Finally, we conclude with an outline of the algorithm.

\subsection*{5.1 Invariants}

In the algorithm we are given a red tree $R$ with $r = \uparrow (R)$, a blue forest $B$ with roots $b_1, \ldots, b_k$, an interval $I = [i, j] \subseteq [1, n]$ with $|I| = |R| = |B|$, and a set $C$ of vertices of $B$ that $r$ must not be embedded on. Without loss of generality we may suppose $C \supseteq \{b_2, \ldots, b_k\}$ and $b_1 \in C$ if and only if $t(b_1)$ is not a central star. Recall that we assume $i < j$ throughout.

As a first step, we embed $B$ onto $I$ by embedding $t(b_1), \ldots, t(b_k)$ in this order from left to right, each time using the canonical embedding algorithm from Section 3, starting from the right endpoint of the corresponding subinterval. This step reinstates the notion of a blue embedding that we abandoned in the statement of Theorem 3. Considering $B$ as an ordered forest rather than an embedded forest emphasizes that we do not assume that $B$ stems from a global canonical embedding of $T_1$. 
Observation 6. We may assume that $R$, $B$ and $I = [i, j]$ satisfy the following invariants:

(I1) $I$ is not in conflict with $R$. (peace invariant)

(I2) Every component of $B$ satisfies LSFR and 1SR. All edges of $B$ are drawn in the upper halfplane (above the $x$-axis). All roots of $B$ are visible from above (that is, a vertical ray going up from $b_x$ does not intersect any edge of $B$). (blue-local invariant)

(I3) $i$ is not in edge-conflict with $r$. (placement invariant)

Proof. (I1) follows from the assumption (i) or (ii) in Theorem 3. (I2) is achieved by using a canonical embedding. Regarding (I3) note that $i$ is not in edge-conflict with $r$ unless $i = \uparrow(B\{i\})$. As we start the canonical embeddings from the right endpoints of the corresponding subintervals, the only way to make $i = \uparrow(B\{i\})$ is to have $B\{i\} = \{i\}$. But then $B\{i\}$ is a central star and $i$ is not in conflict with $r$ by (I1).

Theorem 3 ensures that all roots of $B$ along with $r$ appear on the outer face in the specified order. We cannot assume that we can draw an edge to any other vertex of $B$ or $R$ without crossing edges of the embedding given by Theorem 3. Therefore it is important that whenever the algorithm is called recursively,

(I4) only the roots $b_1, \ldots, b_k$ and $r$ are incident to outside edges. (locality invariant)

Assuming 1SR for $B$ helps when splitting intervals for recursive treatment.

Observation 7. If $B$ satisfies (I2) and (I4) on an interval $I$, then both invariants also hold for $B[x,y]$ on $[x,y]$, for every subinterval $[x,y] \subseteq I$.

Proof. Denote $B' = B[x,y]$. Concerning (I2) it is clear that all edges of $B'$ are drawn in the upper halfplane and that 1SR holds.

If a local root $r \in [x,y]$ is not visible from above in $B'$, then it is covered by an edge $\{a,b\} \in E(B')$, with $r \in [a,b] \subseteq [x,y]$. (We say that an edge $e$ covers a vertex $v$ if the vertical upward ray emanating from $v$ intersects $e$.) As $B'$ is a subgraph of $B$ and all roots of $B$ are visible from above, $r$ is not a root of $B$. Hence there is a parent $p_B(r) \in I \setminus [x,y]$. The edge $\{r,p_B(r)\} \in E(B)$ is drawn in the upper halfplane by assumption and, therefore, crosses the edge $\{a,b\} \in E(B)$, in contradiction to $B$ being plane. So $r$ is visible from above.

To see that LSFR holds, suppose to the contrary that there is a vertex $v \in [x,y]$ that violates LSFR in $B'$. Then there are children $c_1$ and $c_2$ of $v$ in $B'$ so that $c_1 \in [v,c_2] \subseteq [x,y]$ and $|t_{B'}(c_2)| > |t_{B'}(c_1)|$. Neither $c_1$ nor $c_2$ have a neighbor in $B \setminus [v,c_2]$: Due to 1SR, for such a neighbor $q$ we would have $v \in [q,c_2]$ and the edge between $c_1$ or $c_2$ and $q$ would cover $v$, which is a local root in $B[q,c_2]$. But as we have shown above, local roots of subforests induced by intervals are visible from above. Hence, neither $c_1$ nor $c_2$ have a neighbor in $B \setminus [v,c_2]$, and so $t_{B'}(c_i) = t_B(c_i)$, for $i \in \{1,2\}$. As $c_1$ is closer to $v$ than $c_2$ in $B$ (just as in $B'$), LSFR is violated at $v$ in $B$, contrary to our assumption. So there is no vertex in $B'$ that violates LSFR.
Finally, consider (I4) and suppose to the contrary that there is a vertex \( v \in [x, y] \) that is not a local root in \( B' \) but has a neighbor \( q \in B \setminus [x, y] \). (Note that \( v \) cannot have a neighbor outside of \( B \) because it is not a local root of \( B \) and (I4) holds for \( B \).) As \( v \) is not a local root in \( B' \), we have \( p := p_B(v) \in [x, y] \) and so \([p, v] \subseteq [x, y]\). By 1SR it follows that \( p \in [q, v] \) so that the edge \{\( q, v \}\} \in E(B)\) covers the local root \( p \) of \( B[q, v] \), contrary to what we have shown above. Hence there is no such vertex \( v \) and (I4) holds for \( B' \).

Note that if \( B \) satisfies (I4) but does not satisfy 1SR on an interval \( I \), it is easy to find examples where \( B[x, y] \) does not satisfy (I4), for some interval \([x, y] \subset I\). During our algorithm, we frequently flip selected subtrees of \( B \). It is straightforward to observe yet very useful that flipping locally maximal subtrees does not affect the invariants (I1), (I2), and (I4).

**Observation 8.** If \( B \) satisfies (I1), (I2) and (I4) on an interval \( I \), then all three invariants also hold for the tree on \( I \) that is obtained from \( B \) by flipping \( B[x] \), for some \( x \in I \).

Note that the statement of Observation 8 does not hold when flipping an arbitrary subforest \( B[x, y] \), for \([x, y] \subset I\). For such general flips, it is easy to find examples where the resulting blue forest violates 1SR and LSFR.

In the remainder of the proof we will ensure and assume that invariants (I1)–(I4) hold for every call of the algorithm. For the initial instance of packing \( T_1 \) and \( T_2 \), we know that (I1)–(I3) hold by Observation 6 and (I4) holds trivially because there are no vertices outside of \( I \) and, therefore, no outside edges.

### 5.2 Blue-star embedding

The blue-star embedding is useful to handle the center \( \sigma \) of a substar \( B^* = t_B(\sigma) \) of \( B \). Either \( B^* \) is a component of \( B \) or \( \tau := p_B(\sigma) \in [i, j] \). The blue-star embedding explicitly embeds a subtree \( A \) of \( R \) onto a part of \( B \) that includes \( \sigma \). It may use some of the leaves of \( B^* \). After taking care of \( \sigma \), any unused leaf of \( B^* \) appears as a locally isolated vertex in the remaining interval of vertices.

The blue-star embedding consists of several steps: It rearranges some vertices of \( B \), moves some edges of \( B \) below the \( x \)-axis, and introduces edges that straddle both halfplanes above and below the \( x \)-axis (Figure 6).

![Figure 6: Blue-star embedding A onto a part of B.](image)

Suppose that \( A \) is a subtree of \( R \) with \( a := \uparrow(A) \) (possibly \( a = r \)) and \( \sigma \in [i, j] \) is the center of a central star \( B^* = t_B(\sigma) \). Either \( \sigma \) is the root of \( B(\sigma) \) or \( \tau = p_B(\sigma) \in [i, j] \). Denote by \( B^+ \) the subgraph of \( B \) induced by \( \sigma \) and all its neighbors (parent and children) on
I. Note that either $B^+ = B^*$ or $B^+ = B^* \cup \{\tau\}$. Put $d = \deg_A(a)$ and let $\varphi = (v_1, \ldots, v_d)$ be an arbitrary sequence of vertices from $B \setminus B^+$.

Next let us give an intuitive description of the blue-star embedding. We put $a = \sigma$, and then map the children of $a$ to the vertices of $\varphi \subseteq B \setminus B^+$, which are not adjacent to $\sigma$ in $B$. The remaining parts of $A$ (subtrees below the children of $a$) can then be embedded onto the leaves of $B^*$, whose only neighbor in $B$ is $a = \sigma$. Some leaves of $B^*$ may remain, along with some vertices of $B \setminus B^*$. On these remaining vertices we have to embed $R \setminus A$. Therefore, we want the vertices that are not used for the embedding of $A$ to form a subinterval of $I$.

More formally, we require the following four conditions to hold:

1. Condition (BS1) is easy to handle if $a \neq r$ because then we only need to pay attention to where we put $p_R(a)$.

   Let $c_1, \ldots, c_d$ denote the children of $a$ in $A$ such that $|t_R(c_1)| \geq \ldots \geq |t_R(c_d)|$. Partition the leaves of $B^*$ into $d + 1$ groups $G_1, \ldots, G_{d+1}$ such that $|G_k| = |t_R(c_k)| - 1$, for $k \in \{1, \ldots, d\}$, and $|G_{d+1}| = |B^*| - 1 - \sum_{k=1}^d |G_k|$. We intend to embed the vertices of $t_R(c_k) \setminus \{c_k\}$ on the leaves in $G_k$. Note that some (possibly all but $G_{d+1}$, in case $A$ is a central star) of the sets $G_k$ may be empty. Also note that $\sum_{k=1}^d |G_k| = \sum_{k=1}^d (|t_R(c_k)| - 1) = |A| - (d + 1)$, where the $+1$ accounts for $a$. Therefore, $|G_{d+1}| = (|B^*| - 1) - (|A| - d - 1) = |B^*| + d - |A|$ is nonnegative by (BS2) and so our assignment is well-defined.

   If $B \setminus (B^* \cup \varphi)$ does not form an interval, then by (BS4) $A$ is not a central star and so $|G_1| \geq 1$. In this case, we move one leaf from $G_1$ to $G_{d+1}$ and add $\tau$ to $G_1$ instead.

   The blue-star embedding of $A$ from $\sigma$ with $\varphi$ proceeds in four steps, as detailed below. The first two steps rearrange the embedding of $B$ to make room for the embedding of $A$ in the third step. The fourth step ensures that the remaining unused vertices appear in a form that allows to further process them.

**Step 1 (Flip)** We draw all edges of $B^*$ below the spine. All edges of $B \setminus B^*$ remain above the spine (Figure 7a).
Step 2 (Mix) Leaving $\sigma$ where it is, we distribute the leaves of $B^*$ between the vertices in $\varphi$ as follows: for $k \in \{1, \ldots, d\}$, move the vertices of $G_k$ so that they appear as a contiguous subsequence immediately next to (either to the left or to the right of) $v_k$ (Figure 7b). If $B \setminus (B^* \cup \varphi)$ does not form an interval, then we have $\tau$ in $G_1$. As $\tau$ is not a leaf of $B^*$, we cannot move it around so easily. Fortunately, no relocation is necessary because by (BS4) $\tau$ appears right next to $v_1$ in $I$. We flip the blue edge $\{\sigma, \tau\}$ so that it is drawn below the spine and place any remaining vertices of $G_1$ between $\tau$ and $v_1$.

![Figure 7: The example from Figure 6 in detail. We blue-star embed $A$ from $\sigma = j - 1$ with $\varphi = (i + 2, i + 3, j)$.](image)

Step 3 (Complete) Embed $A$ by first mapping $a$ to $\sigma$, which is possible by (BS1). Next map $c_i$ to $v_i$, for $i \in \{1, \ldots, d\}$, drawing the edge to $\sigma$ below the spine. Then embed each subtree $t_{R(c_i)}$ explicitly (using a canonical embedding and drawing all edges above the spine) on the interval of $|t_{R(c_i)}|$ locally isolated vertices immediately to the right of $c_i$ (Figure 7c). Note that $G_1 \cup \{v_1\}$ is locally isolated even if $B \setminus (B^* \cup \varphi)$ does not form an interval because by (BS4) we have $\{v_1, \tau\} \notin E(B)$.

It remains to describe the embedding for $G_{d+1}$. Before we do this, let us consider the properties that we want the embedding to fulfill. Note that the blue-star embedding—as far as described—does not use any of the invariants (I1)–(I2) other than that we start from a one-page book embedding. However, if (I1)–(I2) hold for $B$, then we would like to maintain these invariants also for the part $B' := B \setminus (\{\sigma\} \cup \varphi \cup \bigcup_{k=1}^{d} G_k)$ of $B$ that is not yet used by $R$ after the blue-star embedding. A necessary prerequisite is that $B'$ forms an interval, that is, the vertices of $B'$ appear as a contiguous subsequence of $[i, j]$. Given that we are still free to place the vertices in $G_{d+1}$, it is enough that the vertices in $B' \setminus G_{d+1}$ form a subinterval of $[i, j]$ that is reachable from $\sigma$ (without crossing edges).

Step 4 (Cleanup) Suppose without loss of generality that $\sigma$ is to the right of $B' \setminus G_{d+1}$. (If $\sigma$ is to the left of $B' \setminus G_{d+1}$, replace all occurrences of “right” by “left” in the following
paragraph.)

Move the vertices of $G_{d+1}$ so that they appear as a contiguous subsequence immediately to the right of the rightmost vertex $z$ of $B' \setminus G_{d+1}$. In order to establish that all edges are drawn above the spine, we cannot draw the edges between $\sigma$ and $G_{d+1}$ in the same way as we did for $G_1, \ldots, G_d$ above. Instead we route all edges between $\sigma$ and $G_{d+1}$ as biarcs that leave $\sigma$ below the spine, then cross the spine just to the right of the rightmost vertex of $G_{d+1}$, and finally enter their destination from above (Figure 7d). As a result, for the purpose of embedding some part of $R$ onto $[i, j - |A|]$, the vertices of $G_{d+1}$ become isolated roots; each is connected with a single edge to the outside that is (locally) routed in the upper halfplane.

This completes the description of the blue-star embedding. Below is a formal statement summarizing the pre- and postconditions.

**Proposition 9.** Let $A = t_R(a)$ be a subtree of $R$, let $\sigma \in [i, j]$ be the center of a star $B^* = t_B(\sigma)$, and let $\varphi$ be a sequence of $\deg_A(a)$ pairwise distinct vertices from $B \setminus B^+$, where $B^+$ denotes the subgraph of $B$ induced by $\sigma$ and all its neighbors. If $A$ and $\sigma$ fulfill (BS1)–(BS4), then the blue-star embedding of $A$ from $\sigma$ with $\varphi$ provides an ordered plane packing of $A$ and some induced subforest $B[U]$ of $B$, with $\varphi \cup \{\sigma\} \subseteq U \subseteq \varphi \cup B^+$, onto $U$, such that $[i, j] \setminus U$ is a subinterval $[i', j'] \subset [i, j]$.

Furthermore, $\{x, \sigma\} \not\in E(B)$ after the blue-star embedding, where $x = i'$ if $\sigma > j'$, and $x = j'$ if $\sigma < i'$. The leaves of $B^* \setminus U$ (if any) appear in $[i', j']$ as a sequence of isolated vertices on the side of $[i', j']$ opposite to $x$.

Finally, if the embedding of $B$ on $[i, j]$ initially satisfies (I2) and (I4), then after the blue-star embedding the modified embedding of $B$ on $[i', j']$ satisfies (I2) and (I4).

**Proof.** The packing for $A$ and $B[U]$ is immediate by construction. Only leaves of $B^*$ are moved around. The order of the roots of the blue subtrees is maintained and these roots remain visible from above. The root $a$ of $A$ is visible from below and (BS1) ensures that possible edge-conflicts for $a$ are respected. Therefore, we obtain an ordered plane packing, as claimed.

Let us next argue that after the blue-star embedding we are left with an interval $[i', j'] \subset [i, j]$. We distinguish two cases.

If $B \setminus (B^* \cup \varphi)$ is an interval, then the embedding uses exactly the vertices of $(B^* \cup \varphi) \setminus G_{d+1}$, and the vertices of $G_{d+1}$ are placed so that they extend the interval $B \setminus (B^* \cup \varphi)$.

Otherwise, $B \setminus (B^* \cup \varphi)$ does not form an interval. Then the embedding uses exactly the vertices of $(B^+ \cup \varphi) \setminus G_{d+1}$ (where one vertex originally in $G_1$ is moved to $G_{d+1}$). By (BS3) we know that $B \setminus (B^+ \cup \varphi)$ forms an interval and the vertices of $G_{d+1}$ are placed so that they extend this interval.

Next we argue that $\{x, \sigma\} \not\in E(B)$. By (BS2) we have $|B \setminus B^+| = |R| - |B^+| \geq d + 1$. As $\varphi$ consists of $d$ vertices, at least one vertex in $B \setminus B^+$ is not in $\varphi$. Due to the way we run the cleanup step, it follows that the vertex of $[i', j']$ furthest from $\sigma$ is in $B \setminus B^+$ (whereas the closest vertex may be in $G_{d+1}$, which is adjacent to $\sigma$). By construction no vertex of $B \setminus B^+$ is adjacent to $\sigma$ in $B$. The description of $[i', j']$ holds by construction.
It remains to argue that if $B$ satisfy (I2) and (I4), then so does $B'$. The blue-star embedding does not change the order of the vertices in $B \setminus B'$; in fact, these vertices form a subinterval of $[i', j']$, for which (I2) and (I4) follow by Observation 7. The remaining vertices of $B^*$ (if any) appear as vertices in $G_{d+1}$; they become isolated roots in $B'$, for which (I2) and (I4) hold trivially. Given the way the edges incident to vertices of $G_{d+1}$ have been drawn, they do not affect the visibility of the roots in $B' \setminus G_{d+1}$. □

5.3 Red-star embedding

There is a natural counterpart to the blue-star embedding that we call red-star embedding. It embeds a red central star onto a blue tree.

Consider an interval $I = [i, j]$ on which we wish to embed a subtree $A^*$ of $R$ that is a central star with $a := \uparrow(A^*)$. Consider some $\sigma \in \{i, j\}$ such that $\sigma$ is the root of $B(\sigma)$. Let $k := \deg_B(\sigma)$ and let $v_1, \ldots, v_k$ denote the children of $\sigma$ in $B$, ordered by increasing distance to $\sigma$ (Figure 9a). We require that

(RS1) $a$ is not in edge-conflict with $\sigma$ and

$$\deg_{A^*}(a) + \deg_B(\sigma) \leq |I|.$$ 

Note that (RS1) and (RS2) are analogous to (BS1) and (BS2), but only one inequality is needed in (RS2). In the blue-star embedding, we need (BS3) and (BS4) to handle central stars whose parent is also present in the interval under consideration. In the red-star embedding, we have no requirements on $B$ other than (RS1) and (RS2).

**Step 1 (Embed)** First embed $a$ at $\sigma$. This works by (RS1). Let $d := \deg_{A^*}(a)$ and let $c_1, \ldots, c_d$ denote the children of $a$ in $A$. By (RS2) the interval $I$ contains enough vertices not adjacent to $\sigma$ in order to embed $c_1, \ldots, c_d$. Let $N$ be the set of the $d$ closest non-neighbors of $\sigma$ in $I$. Embed $c_1, \ldots, c_d$ onto $N$. We next describe how to draw the red edges from $c_1, \ldots, c_d$ to $a$. Consider a vertex $c_x$ and let $v$ be the vertex of the blue forest we embedded $c_x$ onto. Refer to Figure 9b. If $v \in t_B(v_1)$, then draw $\{c_x, a\}$ as a semi-circle in the lower halfplane. If $v \in t_B(v_t)$ with $1 < t < k$ then draw $\{c_x, a\}$ as a biarc that is in the upper halfplane near $a$, in the lower halfplane near $c_x$, and crosses the spine between $v_{t-1}$ and $t_B(v_t)$. Finally, if $v \not\in t_B(\sigma)$, then draw $\{c_x, a\}$ as a biarc that is in the upper halfplane near $a$, in the lower halfplane near $c_x$, and crosses the spine right after $t_B(\sigma)$. Afterwards, the vertices of $B \setminus N$, that is, the blue vertices that are not mapped to any $c_x$, are still visible from below.

**Step 2 (Cleanup)** In general, the vertices of $B \setminus N$ do not form an interval. Assume without loss of generality that $\sigma$ is the rightmost vertex of $I$. Let $N^+ = N \cup \{\sigma\}$. We rearrange the vertices of $I$: from left to right, we first place all vertices of $B \setminus N^+$ (maintaining their relative order) and then all vertices of $N^+$ (maintaining their relative order). Refer to Figure 9c. In particular, $\sigma$ is still at the rightmost position after this rearrangement. The edges of $B \setminus N^+$ are drawn as before, as are the edges of $N^+$. We must redraw the edges that have one end
vertex in $N^+$ and one in $B \setminus N^+$. The edges $\{v_x, \sigma\}$ are drawn as triarcs: the edge is in the upper halfplane near $v_x$ and $\sigma$. Its first spine intersection is to the right of the rightmost vertex of $B \setminus N^+$. Its second spine intersection is such that it maintains the cyclic order of edges leaving $\sigma$ (as before the rearrangement). The other edges are drawn similarly.

The pre- and postconditions of the red-star embedding are summarized by the following proposition.

**Proposition 10.** Let $I = [i, j]$ be an interval for which $B$ satisfies (I2) and (I4). Let $A^* = t_R(a)$ be a subtree of $R$ that is a central star. Let $\sigma \in \{i, j\}$ with $\sigma = \uparrow(B(\sigma))$.

If $B$, $A^*$ and $\sigma$ fulfill (RS1) and (RS2), then the red-star embedding of $A^*$ from $\sigma$ on $I$ provides an ordered plane packing of $A^*$ and an induced subforest $B[U]$, for some $U \subset I$.

After the red-star embedding, the set $I \setminus U$ forms an interval that satisfies (I2) after Step 2.

**Proof.** As argued above, Step 1 produces a plane packing of $A^*$ and $B[U]$, where $U := N \cup \{\sigma\}$, by (RS1) and (RS2). Any remaining vertices of $B(\sigma)$ remain visible from below. Furthermore, if a subtree of $B(\sigma)$ is embedded onto a (directed) interval $[x, y]$ with the root at $x$, then Step 1 embeds children of $a$ on a (possibly empty) suffix of $[x+1, y]$. Since Step 2 does not change the relative position of the remaining vertices of $B(\sigma)$ nor the relative position of the other vertices in $I$, the set $I \setminus U$ satisfies (I2) after Step 2.

5.4 Leaf-isolation shuffle

While we are at discussing how to deal with red stars, let us introduce another basic operation that will turn out useful in this context. Suppose we want to embed a substar $A^* \subset R$ onto a subinterval $[a, b] \subset [i, j]$. Then we need to pair $\odot(A^*)$ with an isolated vertex in $B[a, b]$. If there is no such vertex, we need to “create” one. Typically, $[a, b]$ is a proper subinterval of $[i, j]$, just as $A^*$ is a proper subtree of $R$. If we pick some subtree of $B$ that contains $a$, say, $B[a-1, c]$, for some $c \in [a, b]$, then it would be useful to change the embedding of $B[a-1, c]$ so that $a$ becomes a leaf and its parent is at $a-1$. Then $a$ would be locally isolated in $B[a, b]$ and we could easily embed $A$ onto $[a, b]$ with $\odot(A) = a$.

Figure 10c shows the result of performing such a leaf-isolation shuffle on the tree depicted in Figure 10a that is embedded on the interval $[1, 9]$. The goal is to put a leaf at 2 and its parent at 1 so that 2 is locally isolated in $[2, 9]$. The proposition below guarantees that such a leaf-isolation shuffle is always possible. Note that we do not care about the invariants here because we cannot use a recursive embedding for a star, anyway. There is
one part of the invariant that we need to maintain, though, which is the visibility of the blue root from above.

\[(a)\]
\[(b)\]
\[(c)\]

Figure 10: A leaf is shuffled into position 2, with its parent at 1. In the first step of the algorithm (described in the proof of Proposition 11), the closest neighbor of vertex 1 is 4; the subtree on [1, 4] is flipped, resulting in the tree depicted in (b). In the second and final step of the algorithm, the closest neighbor of 1 is 3; the subtree on [1, 3] is flipped, resulting in the tree depicted in (c), which has the desired properties.

**Proposition 11.** Every rooted tree \(T\) on \(|T| \geq 2\) vertices admits a one-page book embedding onto \([1, |T|]\) such that \(q := \uparrow(T)\) is visible from above, 2 is a leaf \(\ell\) of \(T\), and 1 is the parent of \(\ell\). Moreover, if \(T\) is a central star, then \(q = 1\); otherwise, \(q = |T|\) and the edge \(\{1, |T|\}\) is not used by the embedding.

**Proof.** We use induction on \(n = |T|\). Clearly the statement holds for \(n = 2\). For \(n \geq 3\) we start by constructing a one-page book embedding for \(T\) with a modified version of a canonical embedding where we keep 1SR but invert the order of subtrees, that is, we use a “smaller subtree first rule” (SSFR). By starting from \(q\) and placing it at \(|T|\) we ensure that it is visible from above. As \(T\) is a tree, this embedding uses the edge \(\{1, |T|\}\). If 1 is a leaf of \(T\), then \(q\) is its parent and by SSFR \(T\) is a central star. Therefore, flipping \(T\) yields the desired embedding. Otherwise, let \(i \in \{2, \ldots, |T| - 1\}\) denote the smallest (index) neighbor of 1 and obtain the desired embedding inductively for \(B[1, i]\) (whose root is 1). The root of this subtree \(B[1, i]\) ends up at either 1 or \(i\), both of which are visible from above. Therefore, we can complete the embedding by routing all edges from 1 or \(i\) to the existing forest on \([i + 1, |T|]\). Figure 10 illustrates the execution of the leaf-isolation shuffle on an example.

The root \(q\) is at 1 and if and only if \(T\) is a central star; otherwise, it remains at \(|T|\). Similarly, edge \(\{1, |T|\}\) is used by the embedding if and only if \(T\) is a central star. \(\square\)

### 5.5 Algorithm outline

Recall that we are given a red nonstar tree \(R = t(r)\), a blue nonstar forest \(B\) with roots \(b_1, \ldots, b_k\), and an interval \(I = [i, j] \subseteq [1, n]\) with \(|I| = |R| = |B|\). Note that \(B\) being a nonstar does not preclude \(B\) from having components that are stars; it only states that \(B\) as a whole is not a star.

Let \(s\) denote a child of \(r\) that minimizes \(|t_R(c)|\) among all children \(c\) of \(r\) in \(R\). Denote \(S = t_R(s)\) and \(R^- = R \setminus S\). If \(|R^-| \geq 2\), then \(R^-\) cannot be a central star: if it were, then \(|S| = 1\) and \(R\) would be a star. The following statements are easy consequences of the choice of \(s\).

**Lemma 12.** If \(\deg_R(r) \geq 2\), then \(|R^-| \geq |S| + \deg_{R^-}(r)|.\)
Proof. Set \( d := \deg_R(r) = \deg_{R^-}(r) + 1 \) and suppose to the contrary that \(|R^-| - |S| \leq \deg_{R^-}(r) - 1 = d - 2\). Adding \( 2|S| \) to both sides of the inequality yields \(|R| \leq d + 2|S| - 2\). By the minimality of \( S \) we have \(|S| \leq (|R| - 1)/d\). Solving for \(|R|\) and combining with the previous inequality yields
\[
d|S| + 1 \leq |R| \leq d + 2|S| - 2 \implies (d - 2)|S| \leq d - 3,
\]
which is impossible, given that \(|S| \geq 1\).

Observation 13. \(|R^-| \neq 2\).

Proof. If \(|R^-| = 2\), then by the minimality of \( S \) we have \(|S| = 1\). It follows that \(|R| = 3\) and so \( R \) is a star, contrary to our assumption.

We would like to recursively embed \( S \) onto \([j, j - |S| + 1]\) and \( R^- \) onto \([i, j - |S|]\) (Figure 11a). But in general the invariants may not hold for the recursive subproblems. For instance, some of the subgraphs could be stars, or if \( \{i, j\} \in E(B) \), then placing \( r \) at \( i \) may put \([j, j - |S| + 1]\) in edge-conflict with \( S \). Therefore, we explore a number of alternative strategies, depending on which—if any—of the four forests \( R^-\), \( S \), \( B[i, j - |S|] \) and \( B[j - |S| + 1, j] \) in our decomposition is a star.

![Figure 11: Our recursive strategy in an ideal world.](image)

To complete the proof of Theorem 3 we distinguish seven cases. In each of these seven cases, we follow the notation and assume the preconditions as discussed above. First, in Section 6 we discuss the general case, where none of the four forests is a star. Then, in Section 7 and Section 8 we handle the special cases \( \deg_R(r) = 1 \) and \(|S| = 1\), respectively. The final four sections each correspond to one of the four forests being a star. Capturing the general intuition we refer to \( R^- \) as “large” and to \( S \) as “small”, although they may have almost the same size, and if \( \deg_R(r) = 1 \) and \(|R| \geq 3\), then \( S \) is actually larger than \( R^- \).

In most subcases, some subtrees of \( R \) (for instance, \( S \) or \( R^- \)) are embedded recursively, that is, using the same algorithm that we describe, onto some subinterval of \( I \). For every such recursive call we must ensure that the invariants hold for the corresponding subproblem. Sometimes we also embed a subtree of \( R \) explicitly. Typically, this happens in two scenarios:

(i) When embedding a substar \( R^* \) of \( R \), we cannot use the algorithm (because it requires the red tree to be a nonstar). Instead, we need to identify a locally isolated blue vertex where we can put \( \odot(R^*) \). Once we found a safe location for \( \odot(R^*) \), there is nothing left to do because all the other local vertices are necessarily leaves of \( R^* \).
(ii) When embedding a subtree of $R$ onto an interval where the induced blue subforest forms an independent set, we can embed the red tree in any way we like, for instance, using a canonical embedding as described in Section 3.

In contrast to a recursive embedding, for an explicit embedding we do not need to worry about the invariants.

6 Embedding the red tree: the general case

In the general case, we suppose that none of the subtrees in our current decomposition is a star.

Lemma 14. If none of $S$, $R^-$, $B[i, j - |S|]$, and $B[j - |S| + 1, j]$ is a star, then there is an ordered plane packing of $B$ and $R$ onto $I$.

Proof. As $S$ is a minimum size subtree of $r$ in $R$, and neither $S$ nor $R^-$ is a star, we know that $r$ has at least one more subtree other than $S$ and every subtree of $r$ in $R$ has size at least four. (All trees on three or less vertices are stars.) It follows that

$$\deg_{R^-}(r) \leq (|R^-| - 1)/4.$$  

(1)

The general plan is to use one of the following two options. In both cases we first embed $R^-$ recursively onto $[i, j - |S|]$. Then we conclude as follows.

Option 1: Embed $S$ recursively onto $[j, j - |S| + 1]$ (Figure 11a).

Option 2: Embed $S$ recursively onto $[j - |S| + 1, j]$ (Figure 11b).

In some cases neither of these two options works and so we have to use a different embedding.

As we embed $S$ after $R^-$, the (final) mapping for $s$ is not known when embedding $R^-$. However, we need to know something about the position of $s$ in order to avoid a possible edge-conflict caused by the edge $\{r, s\}$. Therefore, before embedding $R^-$ we consider the interval we intend to embed $S$ onto and specifically the local subtree $B_s$ that contains the relevant interval endpoint (which by the placement invariant must be a safe place to put $s$ in the recursive embedding of $S$). In Option 1, we have $B_s = B[j, j - |S| + 1]$ and in Option 2, we have $B_s = B[j - |S| + 1, j]$.

In case that $B_s$ is a central star, we provisionally embed $s$ at $\uparrow(B_s)$. That is, for the recursive embedding of $R^-$ we pretend that some neighbor of $r$ is embedded at $\uparrow(B_s)$. In this way we ensure that, regardless of how the algorithm embeds $R^-$, there is no edge-conflict for the recursive embedding of $S$ on its assigned interval. On the flipside, we may have to deal with a corresponding edge-conflict in the recursive embedding of $R^-$ (if $s$ is placed provisionally and $\{i, \uparrow(B_s)\} \in E(B)$). If $B_s$ is not a central star, we do not place $s$ provisionally. Regardless of a provisional placement, the final position for $s$ is then determined by the recursive embedding of $S$, knowing the definite position of its parent $r$. 

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For the recursive embeddings to work, we need to show that the invariants (I1), (I2) and (I4) hold. (I3) then follows as in Observation 6. Since (I2) and (I4) hold for [i, j], by Observation 7 they also hold for all subintervals of [i, j]. Of course, this argument only works as long as we do not change the embedding of B. Or rather, whenever we change the embedding of B, we need to be careful to maintain these invariants for all subintervals on which we want to recursively embed some subtree of R. As we do not change the blue embedding in Option 1 and 2, it remains to ensure (I1). So suppose that for both options, (I1) does not hold for at least one of the two recursive embeddings. There are two possible obstructions for (I1): edge-conflicts and degree-conflicts. We discuss both types of conflicts, starting with edge-conflicts.

**Case 1** \([i, j - |S|]\) is not in degree-conflict with \(R^-\) and \([j, j - |S| + 1]\) is not in degree-conflict with \(S\). Then Option 1 works, unless \([i, j - |S|]\) is in edge-conflict with \(R^-\). Note that \([j, j - |S| + 1]\) is not in edge-conflict with \(S\) after embedding \(R^-\) onto \([i, j - |S|]\) due to the provisional placement of \(s\).

We claim that an edge-conflict between \(R^-\) and \([i, j - |S|]\) implies \([i, j]\) \(\in\) \(E(B)\). To prove this claim, suppose that \([i, j - |S|]\) is in edge-conflict with \(R^-\). Then \(B[i, j - |S|]/\langle i \rangle\) is a central star whose root \(c\) is in edge-conflict with \(r\). If \(c = i\), then by (I3) there was no such conflict initially (for \(R\) and \([i, j]\)). So, as claimed, the conflict can only come from a blue edge to \(s\) (provisionally placed) at \(j\). Otherwise, \(c > i\) and by 1SR there is no edge in \(B\) from \(c\) to any point in \([c + 1, j]\). It follows that \(B[i, j - |S|]/\langle i \rangle = B/\langle i \rangle\). The conflict between \(c\) and \(r\) does not come from the edge to \(s\) but from an edge to a vertex outside of \([i, j]\). This contradicts (I1) for \(R\) and \([i, j]\), which proves the claim.

The presence of the edge \([i, j]\) implies that \(B\) is a tree and by (I4) only (the root) \(i\) or \(j\) may have edges out of \([i, j]\). Consider Option 2, which embeds \(S\) onto \([j - |S| + 1, j]\). Due to the provisional placement of \(s\) and the hypothesis of Case 1, there are only two possible obstructions: an edge-conflict for \(R^-\) or a degree-conflict for \(S\). As before, an edge-conflict for \(R^-\) can only come from a blue edge to the provisionally placed \(s\), where \(\Uparrow(B_s)\) is the center of a central star. Therefore, regardless of the presence of an edge- or a degree-conflict, we face a central star \(B^* = B[j - |S| + 1, b]\) with center \(b \in [j - |S| + 1, j - 1]\). (We have \(b \neq j\) because \(B[j - |S| + 1, j]\) is not a star.) Due to 1SR and \(\{i, j\} \in E(B)\), we know that \(b = \Uparrow(B[j - |S| + 1, j](j - |S| + 1))\). We distinguish three cases.

**Case 1.1** \(\{i, b\} \in E(B)\). Then we consider a third option: provisionally place \(s\) at \(j\), embed \(R^-\) recursively onto \([j - |S|, i]\) and then \(S\) onto \([j, j - |S| + 1]\) (Figure 12a). The edge \([i, b]\) of \(B\) and the fact that \(B[i, j - |S|]\) is not a star prevent any edge-conflict between \([j - |S|, i]\) and \(R^-\) (and, as before, for \(S\)). Given that we assume in Case 1 that \([j, j - |S| + 1]\) is not in degree-conflict with \(S\), we are left with \([j - |S|, i]\) being in degree-conflict with \(R^-\) as a last possible obstruction. Then the tree \(B[i, j - |S|]/\langle j - |S|\rangle\) is a central star \(A^*\) with root \(a\) such that

\[
\text{deg}_{A^*}(a) + \text{deg}_{R^-}(r) \geq |R^-|.
\]  

(2)

Combining Lemma 12 with (2) we get \(|A^*| = \text{deg}_{A^*}(a) + 1 \geq |S| + 1 \geq 5\). Note that \(A^*\) can be huge, but we know that it does not include \(i\) (because \(B[i, j - |S|]\) is not a star).
also know that \( a \neq j - |S| \): If \( a = j - |S| \), then by 1SR we have \( p_B(a) \in [i, j - |S| - 1] \), in contradiction to \( a = \lceil B[i, j - |S|][j - |S|] \rfloor \). Therefore \( a = j - |S| - |A^*| + 1 \) and by 1SR we have \( p_B(a) \geq j - |S| + 1 \). Due to \( \{i, b\} \in E(B) \) and since \( B[j - |S| + 1, b] \) is a tree rooted at \( b \), we have \( p_B(a) = b \). As \( A^* \) is a subtree of \( b \) in \( B \) on at least five vertices, by LSFR \( b \) cannot have a child that is a leaf at \( b - 1 \). Therefore, the star \( B[j - |S| + 1, j][j - |S| + 1] \) consists of a single vertex only, that is, \( b = j - |S| + 1 \) (Figure 12b). We consider two subcases. In both the packing is eventually completed by recursively embedding \( S \) onto \( [j, j - |S| + 1] \).

**Case 1.1.1** \( \{x, b\} \in E(B) \), for some \( x \in [i + 1, a - 1] \) (Figure 13a). Select \( x \) to be maximal with this property. Then we exchange the order of the two subtrees \( t_B(x) \) and \( A^* \) of \( b \) (Figure 13b). This may violate LSFR for \( B \) at \( b \), but (I2) holds for both \( B[i, j - |S|] \) and \( B[j - |S| + 1, j] \). Clearly there is still no edge-conflict for \( [j - |S|, i] \) with \( R^- \) after this change. We claim that there is no degree-conflict anymore, either.

To prove the claim, note that by LSFR at \( b \) we have \( |t(x)| \leq |A^*| \). As the size of both subtrees combined is at most \( |R^-| \), we have \( |t(x)| \leq |R^-|/2 \). Then, using (1), \( |t(x)| - 1 + \deg_{R^-}(r) < |R^-|/2 + \deg_{R^-}(r) < 3|R^-|/4 < |R^-| \). Therefore after the exchange \( [j - |S|, i] \) is not in degree-conflict with \( R^- \), which proves the claim and concludes this case.

**Case 1.1.2** \( i \) and \( a \) are the only neighbors of \( b \) in \( B \). We claim that in this case \( A^* \) extends all the way up to \( i + 1 \), that is, \( A^* = B[i + 1, j - |S|] \). To prove this claim, suppose to the contrary that \( a \geq i + 2 \). Then there is another subtree of \( i \) to the left of \( a \) and, in particular, \( \{i, a - 1\} \in E(B) \). By LSFR this closer subtree is at least as large as \( A^* \). Using (1) and (2) we get \( |[i + 1, a - 1]| + |A^*| \geq 2|A^*| > 2(|R^-| - \deg_{R^-}(r)) > 3|R^-|/2 > |R^-| \), in contradiction to \( |[i + 1, a - 1]| + |A^*| < |R^-| \). Therefore \( a = i + 1 \), as claimed (Figure 13c).

The vertex \( a \) has high degree in \( B \) but it is not adjacent to \( i \). Therefore, we can embed \( R^- \) as follows: put \( r \) at \( j - |S| \) and embed an arbitrary subtree \( Y \) of \( r \) onto \([i, i + |Y| - 1]\) recursively or, if it is a star, explicitly, using the locally isolated vertex at \( i \) for the center (and \( i + |Y| - 1 \) for the root in case of a dangling star). As \( i \) is isolated in \( B[i, i + |Y| - 1] \), there is no conflict between \([i, i + |Y| - 1]\) and \( Y \). As \(|Y| \geq |S| \geq 4 \), the remaining graph
$B[i + |Y|, j - |S| - 1]$, if any, consists of isolated vertices only, on which we can explicitly embed any remaining subtrees of $r$ using the algorithm from Section 3.

**Case 1.2** $\{i, b\} \notin E(B)$ and $b = p_B(j - |S|)$. Then due to 1SR we know that $j - |S|$ is a locally isolated vertex in $B[i, j - |S|]$, whose only neighbor in $B$ is at $b \notin B[j - |S| + 1, j](j)$. Therefore, we can provisionally place $s$ at $j$ so that $[j - |S|, i]$ is not in conflict with $R^-$. By the assumption of Case 1 $[j, j - |S| + 1]$ is not in degree-conflict with $S$. Therefore, we obtain the claimed packing by recursively embedding first $R^-$ onto $[j - |S|, i]$ and then $S$ onto $[j, j - |S| + 1]$.

**Case 1.3** $\{i, b\} \notin E(B)$ and $b \neq p_B(j - |S|)$. As $\{i, b\} \notin E(B)$ and $s$ is provisionally placed at $b$, the interval $[i, j - |S|]$ is not in edge-conflict with $R^-$. Thus, Option 2 (Figure 11b) succeeds unless $[j - |S| + 1, j]$ is in degree-conflict with $S$. Hence suppose

$$\deg_S(s) + \deg_{B^*}(b) \geq |S|. \tag{3}$$

By Lemma 2 we have $|B^*| \geq 3$. As $b \neq p_B(j - |S|)$, by LSFR $b$ has exactly one neighbor in $B$ outside of $B^*$: its parent $p_B(b) \in [i + 1, j - |S|]$ (Figure 14). Let $B^+ = B^* \cup \{p_B(b)\}$. We blue-star embed $S$ starting from $b$ with $\varphi = (v_1, \ldots, v_d) = (j, \ldots)$ so that $\varphi$ takes the vertices of $I \setminus B^+$ from right to left. Let us argue that the conditions for the blue-star embedding hold.

\begin{enumerate}[(a)]
\item \includegraphics[width=0.2\textwidth]{case1_2a.png}
\item \includegraphics[width=0.2\textwidth]{case1_2b.png}
\item \includegraphics[width=0.2\textwidth]{case1_2c.png}
\item \includegraphics[width=0.2\textwidth]{case1_2d.png}
\item \includegraphics[width=0.2\textwidth]{case1_2e.png}
\item \includegraphics[width=0.2\textwidth]{case1_2f.png}
\end{enumerate}

Figure 14: Explicit embedding of $S$ in Case 1.3. The edge $\{p_B(b), j\}$ need not be in $B$.

(BS1) holds due to $\{i, b\} \notin E(B)$ and $i = \uparrow(B[i, j - |S|](i))$. For the first inequality of (BS2) we have to show $|S| \leq |B^*| + \deg_S(s)$, which is immediate from (3). For the second inequality of (BS2) we have to show $|B^+| + \deg_S(s) \leq |I| - 1$. This follows from $|B^*| + 1 + \deg_S(s) \leq (|S| - 1) + 1 + (|S| - 1) \leq |S| + (|R^-| - 1) - 1 = |I| - 2$. Regarding (BS3) note that in $\varphi$ we take the vertices of $I \setminus B^+$ from right to left. If $\varphi$ reaches beyond $p_B(b)$, then $B \setminus (B^* \cup \varphi)$ forms an interval (Figure 14b); otherwise, $B \setminus (B^* \cup \varphi)$ forms an interval (Figure 14c). Conversely, if $B \setminus (B^* \cup \varphi)$ does not form an interval, then $\varphi$ reaches beyond $p_B(b)$. In particular, in that case $\varphi$ includes $p_B(b) - 1$ and we may simply move $p_B(b) - 1$ to the front of $\varphi$, establishing the second condition in (BS4). Regarding the remaining two conditions it suffices to note that $S$ is not a star by assumption and that $p_B(b) - 1$ is not a neighbor of $b$ in $B$ because $p_B(b)$ is the only neighbor of $b$ outside of $B^*$.
Therefore, we can blue-star embed $S$ as claimed. By construction and Proposition 9 that leaves us with an interval $[i', j']$, where $i = i'$. This “new” interval is obtained from the interval $[i, j - |S|]$ before the blue-star embedding by replacing some suffix of vertices by a corresponding number of locally isolated vertices. In particular, $B[i', j'](i')$ is a subtree of $B[i, j - |S|](i)$ and $i' = \uparrow(B[i', j'](i'))$.

We complete the packing by recursively embedding $R^-$ onto $[i', j']$. This interval is not in edge-conflict with $R^-$ by (I3), $\{i, b\} \notin E(B)$ and $i' = \uparrow(B[i', j'](i'))$. We claim that it is not in degree-conflict with $R^-$, either. Suppose towards a contradiction that $[i', j']$ is in degree-conflict with $R^-$. Then $B[i', j'](i')$ is a central star and so by LSFR also $B[i, j - |S|](i)$ is a central star on at least this many vertices before the blue-star embedding. This contradicts the assumption of Case 1 that $[i, j - |S|]$ is not in degree-conflict with $R^-$. Therefore, $[i', j']$ is not in degree-conflict with $R^-$ and we can complete the packing as described. This completes the proof for Case 1.

**Case 2** $[i, j - |S|]$ is in degree-conflict with $R^-$ (and $[j, j - |S| + 1]$ may or may not be in degree-conflict with $S$). Then $B[i, j - |S|](i)$ is a central star $B[i, x]
\begin{equation}
\text{with } \deg_{R^-}(r) + (x - i) \geq |R^-| (4)
\end{equation}
and $|B[i, x]| = x - i + 3 \geq 3$ by Lemma 2. We distinguish two cases.

**Case 2.1** $B(i) = B[i, x]$. Then $B(i) \neq B(j)$. If necessary, flip $B(i)$ to put its center at $i$. If $B(j)$ is a central star on $\geq 3$ vertices, then—if necessary—flip $B(j)$ to put its center at $j$. We use a blue-star embedding for $R^-$ starting from $\sigma = i$ with $\varphi = (x + 1, \ldots)$. As $\varphi$ consists of $d := \deg_{R^-}(r)$ vertices, we have $[i, j] \setminus (B[i, x] \cup \varphi) = [x + d + 1, j]$. Note that (4) implies $x + d + 1 \geq j - |S| + 2$. If $B[x + d + 1, j]$, a central star on $\geq 3$ vertices, then use $\varphi = (j, x + 1, \ldots)$ instead (and note that $\uparrow(B[x + d + 1, j](j)) = j$).

In the notation of the blue-star embedding we have $B^* = B^+ = B[i, x]$. We need to show that the conditions for this embedding hold. (BS1) holds by (I1) (for embedding $R$ onto $[i, j]$). For (BS2) we have to show $|R^-| \leq |B^*| + \deg_{R^-}(r) \leq |R| - 1$. The first inequality holds by (4) and $|B^*| = x - i + 1$. The second inequality holds due to (I1) (for embedding $R$ onto $[i, j]$), which implies $\deg_{R}(r) + (x - i) \leq |R| - 1$. As $|B[i, x]| = x - i + 1$ and $\deg_{R}(r) = \deg_{R^-}(r) + 1$, (BS2) follows. (BS3) is obvious by the choice of $\varphi$ and (BS4) is trivial for $B^* = B^+$ due to (BS3). That leaves us with an interval $[i', j']$, where $j' \in \{j, j - 1\}$. We claim that $[j', i']$ is not in conflict with $S$.

In the notation of the blue-star embedding we have $B^* = B^+ = B[i, x]$. We need to show that the conditions for this embedding hold. (BS1) holds by (I1) (for embedding $R$ onto $[i, j]$). For (BS2) we have to show $|R^-| \leq |B^*| + \deg_{R^-}(r) \leq |R| - 1$. The first inequality holds by (4) and $|B^*| = x - i + 1$. The second inequality holds due to (I1) (for embedding $R$ onto $[i, j]$), which implies $\deg_{R}(r) + (x - i) \leq |R| - 1$. As $|B[i, x]| = x - i + 1$ and $\deg_{R}(r) = \deg_{R^-}(r) + 1$, (BS2) follows. (BS3) is obvious by the choice of $\varphi$ and (BS4) is trivial for $B^* = B^+$ due to (BS3). That leaves us with an interval $[i', j']$, where $j' \in \{j, j - 1\}$. We claim that $[j', i']$ is not in conflict with $S$.

![Figure 15: Handling a degree-conflict for $R^-$ in Case 2.1.](image)

To prove the claim we consider two cases. If $j' = j - 1$, then initially $B[x + d + 1, j](j)$ was a central star on $\geq 3$ vertices rooted at $j$. By the choice of $\varphi$ a leaf of this star is at $j'$

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whose only neighbor in $B$ is at $j \neq i = r$. Therefore $[j', i']$ is not in edge-conflict with $S$. As $B[j', i'][j']$ is an isolated vertex, by Lemma 2 there is no degree-conflict between $[j', i']$ and $S$, either, which proves the claim.

Otherwise, $j' = j$ and $B[j', i'][j'] = B[x + d + 1, j][j]$ is not a central star on at least 3 vertices. Therefore by Lemma 2 there is no degree-conflict between $[j', i']$ and $S$. In order to show that there is no edge-conflict, either, it is enough to show that $B[j', i'][j']$ is not adjacent to $i = r$ in $B$. If $B[j', i'][j'] \neq j'$ this follows from 1SR. Otherwise $B[j', i'][j'] = j' = j$, and $\{i, j\} \notin E(B)$ because $B(i)$ is a star and by LSFR the presence of the edge $\{i, j\}$ would make $B$ star, which we know it is not. Therefore the claim holds and we can complete the packing by recursively embedding $S$ onto $[j', i']$.

**Case 2.2** $B[i] \neq B[i, x]$. By 1SR this means that $i = p_B(x)$ and $i$ has at least one more neighbor in $[i, j]\B[i, x]$. Since by assumption $B[i, j-|S|]$ is not a star, we have $x \leq j - |S| - 1$. Since $B[i, x]$ is a central star and $x \leq j - |S| - 1$, by LSFR for $i$ the only neighbor of $i$ in $B$ outside of $B[i, x]$ is its parent $p_B(i) \in [j-|S| + 1, j]$. We claim that such a configuration is impossible. To prove the claim, note that $p_B(i)$ has at least two children in $B[i, j-|S|]$ because $x \leq j - |S| - 1$ and $p_B(i) \geq j - |S| + 1$. By LSFR, the corresponding subtrees have size at least $|B[i, x]| = x - i + 1$, and so $|R| \geq |B[i, p_B(i)]| \geq 2|B[i, x]| + 1 \geq 2(|R| - \deg_{R^-}(r) + 1) + 1$, where the last inequality uses (4). Rewriting and using (1) yields

$$|R^-| \leq \frac{|R| - 1}{2} + \deg_{R^-}(r) - 1 < \frac{|R| - 1}{2} + \frac{|R^-|}{4}.$$ 

It follows that $|R^-| < \frac{3}{2}(|R| - 1)$ and hence that $|S| > \frac{1}{3}(|R| - 1)$. Since $S$ is a smallest subtree of $r$ in $R$, this means that $r$ is binary in $R$ and thus unary in $R^-$. This, finally, contradicts the degree-conflict for $[i, j-|S|]$ with $R^-$ because $x < j - |S|$ and hence $\deg_{R^-}(r) + \deg_{B[i, x]}(i) = 1 + (x - i) < 1 + (j - |S|) - i = |R^-|$.

**Case 3** $[j, j-|S| + 1]$ is in degree-conflict with $S$ and $[i, j-|S|]$ is not in degree-conflict with $R^-$. Then $B[j - |S| + 1, j][j]$ is a central star $Z$ with $|Z| \geq 3$ by Lemma 2 and

$$\deg_S(s) + (|Z| - 1) \geq |S|.$$ 

(5)

**Case 3.1** $\{i, j\} \notin E(B)$. Then we claim that we may assume $\uparrow(Z) = j$ and $Z = B(j)$.

Let us prove this claim and denote $z = \uparrow(Z)$. If $z = j - |Z| + 1$, then by 1SR it does not have any neighbor in $B \setminus Z$. Flipping $Z = B(j)$ establishes the claim. Otherwise, $z = j$. Suppose that $z$ has a neighbor $y \in B \setminus Z$. As $z$ is the root of $Z = B[j - |S| + 1, j][j]$, it does not have a neighbor in $[j - |S| + 1, j - |Z|]$ and therefore $y \in [i + 1, j - |S|]$. By LSFR and because $B[j - |S| + 1, j]$ is not a star, $y = p_B(z)$. In particular, since $|Z| \geq 3$, LSFR for $y$ implies $\{y, y + 1\} \notin E(B)$. It follows that after flipping $B(j)$ the resulting subtree $B[j - |S| + 1, j][j]$ is not a central star anymore and so there is no conflict for embedding $S$ onto $[j, j - |S| + 1]$ anymore. Therefore we can proceed as above in Case 1 (the conflict situation for $R^-$ did not change because $B(i)$ remains unchanged). Hence we may suppose that there is no such neighbor $y$ of $z$, which establishes the claim.

We blue-star embed $S$ starting from $\sigma = j$ with $\varphi = (j - |Z|, j - |Z| - 1, \ldots)$. Possible edge-conflicts caused by the blue edges incident to $j$ we will address when embedding $R^-$. 

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In the terminology of the blue-star embedding we have $B^* = B^+ = Z$. Let us argue that the conditions for the embedding hold. (BS1) is trivial because no neighbor of $s$ is embedded yet. For (BS2) we have to show $|S| \leq |Z| + \deg_S(s) \leq |R| - 1$. The first inequality holds by (5) and the second by $|S| \leq (|R| - 1)/\deg_R(r) \leq (|R| - 1)/2$, which implies $|Z| + \deg_S(s) \leq 2(|S| - 1) \leq |R| - 3$. (BS3) is obvious by the choice of $\varphi$ and (BS4) is trivial, given $B^* = B^+$. That leaves us with an interval $[i', j']$, where $i' = i$. Denote the resulting blue forest on $[i, j']$ by $B'$. Note that $B'$ consists of a prefix $[i, k]$, for some $k \in [i, j - |S| - 1]$, so that $B'[i, k] = B[i, k]$, and the remaining vertices in $B'[k + 1, j']$ are isolated vertices that correspond to leaves of $Z$.

The plan is to recursively embed $R^-$ onto $[i, j']$. This works fine, unless $[i, j']$ and $R^-$ are in conflict. So suppose that they are in conflict. Then there is a central star $Y = B'(i)$. Considering how $\varphi$ consumes the vertices in $I$ from right to left, we have $Y = B'[i, y] = B[i, y]$, for some $y \in [i, k]$.

We claim that $\Uparrow(Y) = i$. To prove the claim, suppose to the contrary that $\Uparrow(Y) = y = p_B(i)$. Then by 1SR $Y$ is a component of $B$ and $Y = B(i)$. Thus, a degree-conflict contradicts the assumption of Case 3 that $[i, j - |S|]$ is not in degree-conflict with $R^-$, and an edge-conflict contradicts (11) for embedding $R$ onto $[i, j]$: Indeed, by 1SR $y$ is not adjacent to any vertex outside of $Y$ in $B$—in particular not to $j$, where $s$ was placed. Thus, any edge-conflict between $R^-$ and $[i, j']$ would also be an edge-conflict between $R$ and $[i, j]$. This proves the claim and, furthermore, that $p_B(i)$ appears in $\varphi$ (if not, then $p_B(i)$ would also appear in $B'$ and $i \neq \Uparrow(Y)$). In $B$ we also have $p_B(i) \leq j - |S|$ (otherwise, $Y = B[i, j - |S|][i]$, contrary to our assumption that $[i, j - |S|]$ is not in conflict with $R^-$). Therefore, $p_B(i) \in [y + 1, j - |S|]$.

By (B3) for embedding $R$ onto $[i, j]$ and since $\{i, j\} \notin E(B)$, we know that $[i, j']$ and $R^-$ are not in edge-conflict. So they are in degree-conflict. In particular, $\deg_Y(i) + \deg_{R^-}(r) \geq |R^-|$.

Undo the blue-star embedding, that is, go back to the original embedding of $B$ and the red tree $R$ to be embedded. We claim $\{i, y + 1\} \in E(B)$. To prove the claim, suppose to the contrary that $\{i, y + 1\} \notin E(B)$. Then $\{y + 1, p_B(i)\} \in E(B)$ because in $B$ the vertex $y + 1$ lies below the edge $\{i, p_B(i)\}$. By LSFR the subtree of $p_B(i)$ rooted at $y + 1$ is at least as large as $Y$. Therefore,$$
|t_B(p_B(i))| \geq 2|Y| + 1 = 2 \deg_Y(i) + 3 \geq 2(|R^-| - \deg_{R^-}(r)) \geq \frac{3}{2}|R^-|,$$
where the last inequality uses (1). This is in contradiction to $p_B(i) \leq j - |S|$, which implies $|t_B(p_B(i))| \leq |R^-|$. Therefore, the claim holds and $\{i, y + 1\} \in E(B)$.

Flip $B[i, y + 1]$ and denote the resulting blue forest by $B^f$. Perform the blue-star embedding again, with the same parameters, except with the blue forest $B^f$ instead of the original $B$. Note that ISR may be violated at $y + 1$ in $B^f$ (was $i$ in $B$), unless $y + 1 = p_B(i)$ in $B$. However, this is of no consequence for the blue-star embedding because it does not require ISR. Denote by $B''$ the resulting blue forest and by $[i'', j'']$ the resulting interval after the blue-star embedding. As $Y = B'(i) = B[i, y]$, we know that $y + 1$ appears in $\varphi$ and so $Y$ is not part of $B''$ and the vertices in $B''[i, y]$ (the leaves of $Y$ plus the vertex at $y + 1$ in $B$) are locally isolated. In particular, $i$ is isolated in $B''$ and its only neighbor
in $B^f$ is at $y+1 \neq j$. Therefore, $[i'', j'']$ and $R^-$ are not in conflict, unless $y + 1 = p_B(i)$ initially and $p_B(i) (= i''$ in $B''$) is in edge-conflict with $r$.

In other words, it remains to consider the case where $B(i) = B[i, y + 1] = t_B(y + 1)$ is a dangling star whose root is in edge-conflict with $r$ (Figure 16a). Then $[i'', j'']$ is an independent set in $B''$ that consists of leaves of the two stars $Y$ and $Z$ plus the isolated vertex at $i$. Yet we cannot simply embed $R^-$ using the algorithm from Section 3 because $i''$ is and $j''$ may be in edge-conflict with $r$. Given that $|Y| \geq 3$ and $\varphi$ gets to $y + 1 = \circ(Y)$ only, at least two leaves of $Y$ remain in $B''$ and so, in particular, $i'' + 1$ is not in conflict with $r$. We explicitly embed $R^-$ as follows (Figure 16b): place $r$ at $i'' + 1$ and a child $c$ of $r$ in $R^-$ at $i$. Then collect $|t_R(c)|$ leaves from $Z$ and/or $Y$ and put them right in between $i''$ and $i'' + 1$. First—from left to right—the leaves of $Z$ whose blue edges leave them upwards to bend down and cross the spine immediately to the right of the vertices of the red subtree rooted at $\circ(Y)$ (the leftmost subtree of $S$) and then reach $j''$ from below. Next come the leaves of $Y$ whose blue edges to $\circ(Y)$ are drawn as arcs in the upper halfplane. In order to make room for those leaves, the blue edge $\{i'', \circ(Y)\}$ is re-routed to leave $i''$ downwards to bend up and cross the spine immediately to the left of $i'' + 1$ in order to reach $\circ(Y)$ from above. Using the algorithm from Section 3 we can now embed $t_R(c)$ onto these leaves and any remaining subtrees of $r$ can be embedded explicitly on the vertices $i'' + 2, \ldots$ (ignoring the change of numbering caused by the just discussed repositioning of leaves).

![Figure 16: (a)–(b): Relocating some leaves of the stars $Y$ and $Z$ in Case 3.1. One subtree of $t_R(c)$ of $R^-$ is embedded at $i''$ and the leaves to the right of $i''$; all other subtrees of $R^-$ are embedded to the right of $r$. Both from $\circ(Y)$ and from $\circ(Z)$ we can route as many blue edges as desired to either of these “pockets”. (c): Evading a degree-conflict for $S$ in Case 3.2.1.](image)

**Case 3.2** $\{i, j\} \in E(B)$. Then $\uparrow(Z) = j$ because $j - |Z| + 1$ is enclosed by $\{i, j\}$ and therefore cannot be the root of $Z$. Moreover, $\uparrow(B) = i$ by LSFR and since $B$ is not a star. By LSFR $j$ does not have any child in $B \setminus Z$ and as $B[j - |S| + 1, j]$ is not a star, $Z \subseteq B[j - |S| + 2, j]$. In particular, $j$ is not adjacent to any vertex in $B[i + 1, j - |S| + 1]$. We provisionally place $s$ at any vertex in $[i + 1, |R^-|]$, say at $j - |R^-|$. Then $[j, j - |R^-| + 1]$ is not in edge-conflict with $R^-$. We claim that it is not in degree-conflict, either. As $Z$ is a star on $|Z| \leq |S| - 1$ vertices, by Lemma 12 we have $\deg_{R^-}(r) + (|Z| - 1) \leq \deg_{R^-}(r) + |S| - 2 \leq |R^-| - 2$ and the claim follows. We recursively embed $R^-$ onto $[j, j - |R^-| + 1]$, treating all local roots of (the components of) $B[j - |R^-| + 1, j]$ other than $j$ as in edge-conflict with $r$. It remains to recursively embed $S$ onto $[j - |R^-|, i] = [i + |S| - 1, i]$.

Suppose towards a contradiction that $[j - |R^-|, i]$ is in conflict with $S$. Then there is a central star $X = B[x, j - |R^-|] = B[i, j - |R^-|]$. Due to $\{i, j\} \in E(B)$ and
Let us argue that the conditions for the blue-star embedding hold. \( \sigma(BS2) \) we have to show
\[
\left\{ \begin{array}{l}
\sigma(BS2) \quad \text{we have to show}
|B_i| = |B_j| + 1, i \notin B_j \text{ for all } i, j \in B
\end{array} \right.
\]

As there are not enough vertices in \( B \) when embedding \( R^- \) and \( R^- \) are not in conflict. Therefore \( |B| \geq 3 \) by Lemma 2 and
\[
\deg_S(s) + \deg_X(x) \geq |S|.
\]

Depending on the placement of \( p_B(x) \) we consider two final subcases.

**Case 3.2.1** \( p_B(x) \in [j - |R^-| + 1, j - |Z|] \) (Figure 16c). Then the edge \( \{x, p_B(x)\} \) encloses \( j - |R^-| + 1 \) so that, in particular, \( \{i, j - |R^-| + 1\} \notin E(B) \). We provisionally place \( s \) at \( i \) and claim that \( \langle j - |R^-| + 1, j \rangle \) and \( R^- \) are not in conflict.

To prove the claim, consider \( W^* := B[j - |R^-| + 1, j](j - |R^-| + 1) \) and suppose it is a central star. (If it is not, then we are done.) If \( \uparrow(W^*) > j - |R^-| + 1 \), then by ISR and \( \{x, p_B(x)\} \in E(B) \) we have \( p_B(\uparrow(W^*)) = x \), in contradiction to LSFR for \( x \). Therefore \( \uparrow(W^*) = j - |R^-| + 1 \). In order for \( j - |R^-| + 1 \) to be the local (on \( j - |R^-| + 1, j \)) root for \( W^* \) in the presence of \( \{x, p_B(x)\} \in E(B) \), it follows that \( p_B(j - |R^-| + 1) = x \) and so by ISR \( |W^*| = 1 \). Therefore by Lemma 2 there is no degree-conflict between \( j - |R^-| + 1, j \) and \( R^- \).

As \( \{x, p_B(x)\} \in E(B) \) prevents any connection in \( B \) from \( j - |R^-| + 1 \) to \( i \) and to vertices outside of \( [i, j] \), there is no edge-conflict between \( j - |R^-| + 1, j \) and \( R^- \), either. This proves the claim. Recursively embed \( R^- \) onto \( \langle j - |R^-| + 1, j \rangle \). Recall that \( \uparrow(B) = i \). There is no conflict for embedding \( S \) onto \( \{i, j - |R^-|\} \) since \( \{i, r\} \notin E(B) \) (due to the provisional placement of \( s \) when embedding \( R^- \)) and \( B[i, j - |R^-|](i) \) is not a central star of size at least 2 by LSFR at \( i \). Finish the packing by recursively embedding \( S \) onto \( \{i, j - |R^-|\} \).

**Case 3.2.2** \( p_B(x) = j - |R^-| + 1 \). Then by ISR \( p_B(x) \) is the only neighbor of \( x \) outside of \( X \) in \( B \). We provisionally place \( r \) at \( j \) and employ a blue-star embedding for \( S \), starting from \( \sigma = x \) with \( \varphi = (i, \ldots) \), that is, \( \varphi \) takes vertices from left to right, skipping over \( [x, p_B(x)] \).

Let us argue that the conditions for the blue-star embedding hold.

In the terminology of the blue-star embedding we have \( B^* = X \) and \( B^+ = X \cup \{p_B(x)\} \). (BS1) holds because \( \{x, j\} \notin E(B) \). For the first inequality of (BS2) we have to show \( |S| \leq |X| + \deg_S(s) \), which is immediate from (6). For the second inequality of (BS2) we have to show \( |X| + 1 + \deg_S(s) \leq |I| - 1 \). This follows from \( |X| + 1 + \deg_S(s) \leq |S| + (|S| - 1) \leq |R^-| + |S| - 1 = |I| - 1 \). Regarding (BS3) note that in \( \varphi \) we take the vertices of \( B \setminus B^+ \) from left to right. As there are not enough vertices in \( [i, x - 1] \) to embed the neighbors of \( s \) (which causes the degree-conflict), \( \varphi \) reaches beyond \( p_B(x) \) and so \( B \setminus (B^+ \cup \varphi) \) forms an interval. In particular, \( \varphi \) includes \( p_B(x) + 1 \) and \( B \setminus (B^+ \cup \varphi) \) is not a neighbor of \( x \) in \( B \) because \( p_B(x) \) is the only neighbor of \( x \) outside of \( X \).

Therefore, we can blue-star embed \( S \) as claimed, which leaves us with an interval \( [i', j'] \), where \( j = j' \). As \( \{x, j\} \notin E(B) \) and \( j \neq \uparrow(B) \) (\( i \) is), there is no edge-conflict between \( [j', i] \) and \( R^- \). As there is no degree-conflict between \( [j, j - |R^-| + 1] \) and \( R^- \) and the
number of neighbors of $j$ in $B[i',j']$ can only decrease compared to $B[j-|R^-|+1,j]$ (if they appear in $\varphi$), there is no degree-conflict between $[j',i']$ and $R^-$, either. Therefore, we can complete the packing by embedding $R^-$ onto $[j',i']$ recursively.

7 Embedding the red tree: a unary root

In this section we handle all cases where $r := \uparrow(R)$ is unary.

**Proposition 15.** If $\text{deg}_R(r) = 1$ and $S$ is a star, then there is an ordered plane packing of $B$ and $R$ onto $I$.

**Proof.** Since $\text{deg}_R(r) = 1$ and $R$ is not a star by assumption, $S$ must be a dangling star. Thus, we know exactly what $R$ looks like: it is rooted at $r$, which has a single child $s$, which has a single child $q$, which finally has zero or more leaf children.

**Case 1** $\{i, j\} \not\in E(B)$. We consider three cases.

**Case 1.1** $i$ and $j$ are both isolated in $B$. Embed $r$ to $i$, $s$ to $i+1$, $q$ to $j$, and the children of $q$ onto $[j-1,i+2]$. See Figure 17a. Note that $i$ is not in edge-conflict with $r$ due to the placement invariant. Every red edge is incident to $i$ or $j$ and hence does not occur in $B$ by assumption.

**Case 1.2** $i$ is not isolated in $B$. If $B(i)$ is a central star, flip it if necessary to put its root (which is not in edge-conflict with $r$ by the peace invariant) at $i$. Otherwise, use the leaf-isolation shuffle to put a leaf at $i+1$ and its parent at $i$; this will place the root of $B(i)$ at some position $x > i+1$ by Proposition 11 since $B(i)$ is not a central star. In both cases, embed $r$ onto $i$, $s$ onto $j$, $q$ onto $i+1$, and the children of $q$ onto $[i+2,j-1]$. See Figure 17b. The edge $\{r,s\}$ is not used by $B$ since $\{i,j\} \not\in E(B)$ by assumption (and the leaf-isolation shuffle cannot change that by Proposition 11 even if $B = B(i)$). The red edges incident to $q$ are not used by $B$ since the only neighbor of $i+1$ in $B$ is $i$.
Case 1.3 \(i\) is isolated and \(j\) is not isolated in \(B\). Flip \(B(j)\) if its root is currently at \(j\). Note that \([j,i]\) is not in degree-conflict with \(R\); this would imply that \(B\) is a star since \(\deg_{B}(r) = 1\). If \([j,i]\) is not in edge-conflict with \(R\), either, then invariants (11)-(14) hold for \([j,i]\), and since \(j\) is not isolated, we can apply Case 1.2 by embedding \(R\) on \([j,i]\) instead of \([i,j]\). Otherwise, \(B(j)\) is a central star on at least two vertices.

If \(|B(j)| = 2\), then embed \(r\) onto \(j\), \(s\) onto \(i\), \(q\) onto \(j - 1\), and the children of \(q\) onto \([j - 2, i + 1]\). See Figure 17c. If \(|B(j)| \geq 3\), then embed \(r\) onto \(j\), \(s\) onto \(j - 1\), \(q\) onto \(i\), and the children of \(q\) onto \([i + 1, j - 2]\). See Figure 17d. This works because the root of \(B(j)\) is not at \(j\) (so \(j\) is not in edge-conflict with \(r\)), the size of the star \(B(j)\) is at least three (so \([j - 1, j]\) \(\notin E(B)\)), and \(i\) is isolated in \(B\) (so the red edges incident to \(q\) are not used by \(B\)).

Case 2 \(\{i,j\} \in E(B)\). In this case, \(B\) is a tree. Denote \(b := \uparrow(B)\). We claim that (1) some vertex of \(B\) has distance at least three to \(b\) or (2) \(\deg_{B}(b) \geq 2\). To prove the claim, suppose that all vertices in \(B\) have distance at most two to \(b\) and that \(b\) is unary. Then the child of \(b\) has distance one to all other vertices of \(B\); hence \(B\) is a star centered at the child of \(b\), a contradiction. We perform a case analysis on whether (1) or (2) holds.

Case 2.1 Some vertex \(v\) of \(B\) has distance at least three to \(b\). Let \(b'\) be the child of \(b\) that contains \(v\) in its subtree \(B'\). Let \(w\) be the size of \(B'\). We re-embed \(B\) as follows. \(B'\) is not a central star by choice of \(v\). Hence, by Proposition 11, we can use the leaf-isolation shuffle to embed \(B'\) on \([i,i+w-1]\), placing a leaf at \(i+1\), its parent at \(i\), and the root \(b'\) at some position in \([i+2,i+w-1]\). Complete this embedding of \(B'\) to any one-page book embedding of \(B\). Note that this embedding does not use the edge \(\{i,j\}\). Embed \(r\) at \(i\), \(s\) at \(j\), \(q\) at \(i+1\), and the children of \(q\) at \([i+2,j-1]\). See Figure 17e. This works because \(b\) is not at \(i\) (so \(i\) is not in edge-conflict with \(r\)), \(B\) does not use the edge \(\{i,j\}\) (so \(\{r,s\} \notin E(B)\)), and \(i+1\) is isolated in \(B[i+1,j]\) (so the red edges incident to \(q\) are not used by \(B\)).

Case 2.2 \(\deg_{B}(b) \geq 2\). Since \(B\) is not a star, some vertex \(v\) has distance at least two to \(b\) in \(B\). Let \(b'\) be the child of \(b\) that contains \(v\) in its subtree \(B'\). Let \(w\) be the size of \(B'\). We re-embed \(B\) as follows. Use the leaf-isolation shuffle to embed \(B'\) together with \(b\) on \([i,i+w]\), placing a leaf at \(i+1\), its parent at \(i\), and \(b\) at \(i+w\). Complete this embedding to any one-page book embedding of \(B\). Note that this embedding does not use the edge \(\{i,j\}\). Finish by using the same embedding for \(R\) as in Case 2.1. See Figure 17f. 

We continue with the case in which \(S\) is not a star. The following two propositions address separately the scenarios in which edge \(\{i,j\}\) does or does not belong to \(B\).

Proposition 16. If \(\deg_{R}(r) = 1\), \(S\) is not a star, and \(\{i,j\} \in E(B)\), then there is an ordered plane packing of \(B\) and \(R\) onto \(I\).

Proof. Note that \(\{i,j\} \in E(B)\) implies that \(B\) is a tree. Flip \(B\) if necessary to put its root at \(j\). The general plan is to embed \(r\) onto \(i\) and \(S\) recursively onto \([i+1,j]\). This works unless (1) \([i+1,j]\) is a star, (2) \(\{i,i+1\} \in E(B)\), or (3) there is a conflict for embedding \(S\) onto \([i+1,j]\). Below, we find an ordered plane packing under a weaker condition than (1) to allow for reuse in cases (2) and (3). In case (2), by LSFR, \(B[i,j-1] \langle i \rangle\) is a central
star on at least two vertices. In case (3), $B[i + 1, j](i + 1)$ is a central star. We deal with these cases below.

**Case 1** $B[i + 1, j]$ is a star or $B[i, j - 1]$ is a star. If $B[i, j - 1]$ is a star, then we flip $B$ to reduce to the case that $B[i + 1, j]$ is a star. Thus, in the following, assume that $B[i + 1, j]$ is a star and that $\uparrow(B)$ may be either at $i$ or at $j$. We know exactly what $B$ looks like: since $B$ is not a star, the star $B[i + 1, j]$ must be centered at $i + 1$ and rooted at $j$. Flip the blue embedding at $[i + 1, j]$: this puts the star-center at $j$. Note that $\{i, j\} \not\in E(B)$. Embed $r$ onto $j$. The interval $[i, j - 1]$ is in edge-conflict with $S$ if $\uparrow(B)$ is now at $i + 1$. Hence, we embed $S$ explicitly. Embed $s$ onto $i$. Since $S$ is not a star, it must have a subtree of size $k \geq 2$. Embed this subtree explicitly at $[i + k, i + 1]$. Embed the other subtrees of $s$ explicitly on the remainder. See Figure 18a.

![Case 1](image1.png)  ![Case 2](image2.png)  ![Case 3](image3.png)

**Figure 18:** The case analysis in the proof of Proposition 16.

**Case 2** $B[i, j - 1](i)$ is a central star on at least two vertices. Let $x$ be such that $B[i, j - 1](i) = B[i, x]$. We may assume that $x \leq j - 2$, since $x = j - 1$ would imply that $B$ is a star. By LSFR at $i$ and by the choice of $x$, $B[x + 1, j]$ is a tree. Flip $B[x + 1, j]$. Since $x \leq j - 2$, the root of $B$ is no longer at $j$. Embed $r$ onto $i$ and $S$ recursively onto $[j, i + 1]$. See Figure 18b. Since $\{i, j\} \not\in E(B)$ after flipping and $i + 1$ is isolated in $B[i + 1, j]$, this works unless $\{j, i + 1\}$ is in conflict with $S$. Then $B[i + 1, j](j)$ is a central star that is rooted at $\uparrow(B)$. But this contradicts LSFR at $j$ before flipping. Hence, $[j, i + 1]$ is not conflict with $S$.

**Case 3** $B[i + 1, j](i + 1)$ is a central star. Let $x$ be such that $B[i + 1, x] = B[i + 1, j](i + 1)$. Since $\{i, j\} \in E(B)$, $B[i + 1, x]$ is rooted and centered at $x$ and the parent of $x$ is at $i$. Hence $B[i, x]$ is a dangling star. By Case 1 we may assume that $x \leq j - 2$. Flip $B[i, x]$. Embed $r$ at $i$ and $S$ recursively at $[j, i + 1]$. See Figure 18c. Since $\{i, j\} \not\in E(B)$ after flipping and $i + 1$ is isolated in $B[i + 1, j]$, this works unless $[j, i + 1]$ is in conflict with $S$. Then $B[i + 1, j](j)$ is a central star that is rooted at $\uparrow(B)$. But this contradicts LSFR at $j$ before flipping. Hence, there is no conflict with $S$.

**Proposition 17.** If $\deg_R(r) = 1$, $S$ is not a star, and $\{i, j\} \not\in E(B)$, then there is an ordered plane packing of $B$ and $R$ onto $I$.

**Proof.** The general plan is to embed $r$ onto $i$ and $S$ recursively onto $[j, i + 1]$. Since $\{i, j\} \not\in E(B)$ and $S$ is not a star, this works unless (1) $B[i + 1, j]$ is a star or (2) $[j, i + 1]$ is in conflict with $S$. In case (2), the star $B^* := B[j, i + 1](j)$ is either in edge-conflict or in degree-conflict with $S$. If it is in edge-conflict, then there must be an edge from $\uparrow(B^*)$ to $r$. By 1SR, $\uparrow(B^*)$ must be at $j$. But that means that $\{i, j\} \in E(B)$, a contradiction. Thus, in
case (2), there is a degree-conflict for embedding $S$ onto $[j, i + 1]$. We deal with these cases below.

**Case 1** $B[i + 1, j]$ is a star. Since $\{i, j\} \not\in E(B)$ and by 1SR, vertex $i$ is isolated in $B$. Flip $B(j) = B[i + 1, j]$ if necessary to put its center at $j$. If $\uparrow(B[i + 1, j])$ is at $i + 1$, then embed $r$ onto $j$ and $S$ recursively onto the independent set $[i, j - 1]$. Since $i$ is isolated in the blue embedding, $[i, j - 1]$ is not in conflict with $S$. If $\uparrow(B[i + 1, j])$ is at $j$, then flip the blue embedding at $[j - 1, j]$. This places the root at $j - 1$ and a leaf of the star $B[i + 1, j]$ at $j$. After flipping, the interval $[i, j - 1]$ still satisfies invariants (I1)–(I4). Embed $r$ onto $j$ (which is not in edge-conflict with $r$) and $S$ recursively onto $[i, j - 1]$. See Figure 19a. Since $i$ is isolated in the blue embedding, $[i, j - 1]$ is not in conflict with $S$.

![Figure 19: The case analysis in the proof of Proposition 17.](image)

**Case 2** There is a degree-conflict for embedding $S$ onto $[j, i + 1]$. Let $y$ be such that $B[j, i + 1]\langle j \rangle = B[y, j]$. Due to the degree-conflict, $B[y, j]$ is a central star on at least three vertices. Since $B[j, i + 1]\langle j \rangle = B[y, j]$, the root of $B[y, j]$ is not adjacent to any vertex in $[i + 1, y - 1]$. By 1SR, if it were adjacent to $i$, then $\uparrow(B[y, j])$ must be at $j$: this however, violates the assumption that $\{i, j\} \not\in E(B)$. Hence, $B[y, j] = B\langle j \rangle$. Since $B(j) = B[y, j]$ is a tree and $y \leq j - 2$, $B[i, j - 1]$ is not a star. We distinguish two cases.

**Case 2.1** If we embed $r$ onto $j$, then $S$ has no conflict with $[i, j - 1]$. Flip $B[y, j]$ if necessary to put its root at $y$. This preserves invariants (I1)–(I4) on $[i, j]$. Since $B[y, j]$ is not a star and its root is not at $j$, $r$ has no conflict with $j$. Embed $r$ onto $j$ and $S$ recursively onto $[i, j - 1]$. See Figure 19b. This works by the assumption that there is no conflict for embedding $S$ onto $[i, j - 1]$ before flipping $B[y, j]$ and since $B(i) \neq B\langle j \rangle$ due to $\{i, j\} \not\in E(B)$.

**Case 2.2** If we embed $r$ onto $j$, then $S$ has a conflict with $[i, j - 1]$. By the 1SR and the fact that $\{i, j\} \not\in E(B)$, there $S$ has a degree-conflict with $[i, j - 1]$ if $r$ is embedded onto $j$. Let $x$ be such that $B[i, j - 1]\langle i \rangle = B[i, x]$. By the same argumentation that proved $B[y, j] = B\langle j \rangle$ we have $B[i, x] = B(i)$. Thus, we can divide $B$ into three disjoint parts: $B[i, x]$ (a central star), $B[x + 1, y - 1]$ (about which we know nothing), and $B[y, j]$ (a central star). For notational convenience, let $k_1 = \lceil i, x \rceil$, $k_2 = \lceil x + 1, y - 1 \rceil$, and $k_3 = \lceil y, j \rceil$ be the corresponding interval sizes. Let $d = \deg_S(s)$ and let $v_1, \ldots, v_d$ be the children of $s$, ordered by increasing size of their subtrees ($t_S(v_d)$ is the largest). Since $S$ is not a star, $|t_S(v_d)| \geq 2$. Let $\lambda$ be the number of leaf children of $s$. Then $|t_S(v_\ell)| = 1$ if and only if $\ell \leq \lambda$.

Flip $B[i, x]$ if necessary to put the root (and center) at $i$ and flip $B[y, j]$ if necessary to put the root (and center) at $y$. We first explain how to embed $R$ and then prove that it always
works. Refer to Figure 19c for the case \( k_2 > 0 \) and Figure 19d for the case \( k_2 = 0 \). Embed \( r \) onto \( i \) and \( s \) onto \( j \). This works so far: by the peace invariant the root of \( B[i, x] = B(i) \) is not in conflict \( R \) and \( \{i, j\} \notin E(B) \). Next, embed \( t(v_d) \) recursively onto \([y - 1, y + |t(v_d)| - 2]\). Since \( \{y - 1, j\} \notin E(B) \) and \( y - 1 \) is isolated in \([y - 1, y + |t(v_d)| - 2]\), it follows that \( t(v_d) \) is not in conflict with \([y - 1, y + |t(v_d)| - 2]\); consequently this embedding works provided \( t(v_d) \) fits inside \([y - 1, j - 1]\), that is, provided \( |t(v_d)| \leq |[y - 1, j - 1]| = |[y, j]| = k_3 \). Next, embed a leaf child of \( s \) on each vertex in \([x + 1, y - 2]\) (this interval may be empty). This embeds the children \( v_1, \ldots, v_{k_2 - 1} \) and works provided that \( \lambda \geq k_2 - 1 \). This leaves two disjoint intervals to embed the remaining subtrees \( t(v_{\max\{1, k_2\}}, \ldots, t(v_{d - 1})) \) of \( s \): \( I_1 : = [i + 1, \min\{x, y - 2\}] \) and \( I_2 : = [y + |t(v_d)| - 1, j - 1] \). Thus, it remains to prove that \( i \leq k_3 \), \( ii \lambda \geq k_2 - 1 \), and that \( iii \) we can distribute the remaining subtrees over \( I_1 \) and \( I_2 \).

We begin by showing that \( d \) must be large. Since there is a degree-conflict for embedding \( S \) onto \([j, i + 1]\) we have \( k_1 + k_2 - 1 < d \), and since there is a degree-conflict for embedding \( S \) onto \([i, j - 1]\) we have \( k_2 + k_3 - 1 < d \):

\[
\begin{align*}
k_1 + k_2 & \leq d; \\
k_2 + k_3 & \leq d.
\end{align*}
\]

(7)

(8)

Recall that \( k_1 + k_2 + k_3 = |R| = |S| + 1 \). Adding (7) and (8) yields \( 2d \geq k_1 + k_2 + k_3 + k_2 = |S| + 1 + k_2 \) and so

\[
d \geq \frac{|S| + 1 + k_2}{2},
\]

(9)

**Proof of (i)** We must show that \( |t(v_d)| \leq k_3 \). Using (7) we get \( \sum_{\ell = 1}^{d - 1} |t(v_\ell)| \geq d - 1 \geq k_1 + k_2 - 1 = |S| - k_3. \) Since the total size of the subtrees rooted at the children of \( S \) is \( |S| - 1 \) we have \( |t(v_d)| = |S| - 1 - \sum_{\ell = 1}^{d - 1} |t(v_\ell)| \leq |S| - 1 - |S| + k_3 = k_3 - 1 < k_3 \), which completes the proof of (i).

**Proof of (ii)** We must show that \( \lambda \geq k_2 - 1 \). Since \( |t(v_\ell)| \geq 2 \) for all \( \ell \), where \( \lambda + 1 \leq \ell \leq d \), we have \( 2(d - \lambda) + \lambda \leq |S| - 1 \) and so

\[
\lambda \geq 2d - |S| + 1 \geq (|S| + 1 + k_2) - |S| + 1 = k_2 + 2.
\]

**Proof of (iii)** It remains to prove that we can distribute \( t(v_{\max\{1, k_2\}}, \ldots, t(v_{d - 1})) \) over the disjoint intervals \( I_1 : = [i + 1, \min\{x, y - 2\}] \) and \( I_2 : = [y + |t(v_d)| - 1, j - 1] \). We use the following observation on partitioning natural numbers.

**Observation 18.** Let \( n \) and \( t \) be positive integers with \( t \geq \lceil n/2 \rceil + 1 \) and let \( a_1 \leq \cdots \leq a_t \) be positive integers with \( \sum_{i=1}^t a_i = n \). Then for all \( 0 \leq k \leq n \) there exists a set \( J_k \subseteq [1, t] \) such that \( \sum_{i \in J_k} a_i = k \).
Proof. We prove the statement by induction on n. The statement is true for n = 1: in this case we must have t = 1 and a₁ = 1, and so J₀ = ∅ and J₁ = {1} work. Suppose that the statement holds for all positive integers smaller than n. It suffices to prove the statement for k ≥ ⌈n/2⌉ since we can choose Jₖ = [1, t] \ Jₙ₋ₖ for k < ⌈n/2⌉. If aₜ = 1 then a₁ = ⋯ = aₜ = 1 and we choose Jₖ = [1, k]. Otherwise, by the assumption on t we have aₜ = n − ∑ᵢ=₁ᵗ−¹ aᵢ ≤ n − t + 1 ≤ ⌈n/2⌉ and hence k − aₜ ≥ 0. By the assumption on t and since aₜ ≥ 2 we have t − 1 ≥ ⌈n/2⌉ ≥ ⌊(n − aₜ)/2⌋ + 1. Hence, by the induction hypothesis, there exists a set Jₖ−ₐₜ ∈ [1, t − 1] with ∑ᵢ∈Jₖ−ₐₜ aᵢ = k − aₜ. Choose Jₖ = Jₖ−ₐₜ ∪ {t} to complete the proof.

On one hand, the total size of the remaining subtrees is n := |S| − 1 − ∑ₖ₂−¹ t(vᵢ) − |t(v_d)| ≤ |S| − 1 − max{0, k₂ − 1} − 2 = |S| − 2 − max{1, k₂} since |t(v_d)| ≥ 2. On the other hand, the number of remaining subtrees is

\[ t := d − (k₂ − 1) − 1 = d − k₂ \overset{(9)}{=} \frac{|S| + 1 + k₂}{2} − k₂ = \frac{|S| − 1 − k₂}{2} + 1 \geq \frac{n}{2} + 1, \]

where the last step uses that |S| − 1 − k₂ ≥ |S| − 2 − k₂ for k₂ ≥ 1 and |S| − 2 − k₂ = |S| − 2 for k₂ = 0. Hence, n and t satisfy the precondition of Observation 18. We apply the observation with k = |I₁|. This gives us a set Jₖ such that ∑ ℓ ∈ Jₖ |t(vₖ)| = |I₁| and ∑ ℓ ∈ [1, d − 1] \ Jₖ |t(vₖ)| = |I₂|.

Since B[I₁] and B[I₂] have no internal edges and no edges to the position of s at j, we can embed the subtrees t(vₖ) with ℓ ∈ Jₖ explicitly from left to right on I₁ and the remaining subtrees explicitly from left to right on I₂. This completes the proof. □

Proposition 15, Proposition 16, and Proposition 17 together prove the following.

Lemma 19. If degᵣ(r) = 1, then there is an ordered plane packing of B and R onto I.

8 Embedding the red tree: a singleton subtree

Here we completely handle the case |S| = 1.

Lemma 20. If |S| = 1, then R and B admit an ordered plane packing onto [i, j].

Proof. We distinguish two cases.

Case 1 R⁻ is not a star. We first describe an embedding that works whenever B[i, j − 1] is a star. Flip B[i, j − 1] if necessary to put its center at j − 1. In addition to the star at [i, j − 1], the blue embedding may use the edge {i, j}. Note that it cannot use {j − 1, j}, as this would imply that B is a star. Thus, j is isolated in B[i + 1, j]. Embed r onto i + 1 and s onto i. Let U be a largest subtree of r in R⁻. Note that |U| ≥ 2, otherwise R⁻ would be a star. We embed U onto [j, j − |U| + 1] as follows: If U is not a star, then embed U recursively on [j, j − |U| + 1]; this works since U has no conflict with [j, j − |U| + 1] since j is locally isolated in B[j, j − |U| + 1] and j is not adjacent to i + 1 (which is where we embedded r). If U is a star, we embed it explicitly: If |U| = 2, then we embed its root onto
and its leaf onto \( j - 1 \); otherwise \(|U| \geq 3\), and we embed its center at \( j \) and its root onto either \( j \) (if \( U \) is a central star) or \( j - |U| + 1 \) (if \( U \) is a dangling star). Finally, embed the remaining subtrees of \( r \) in \( R^- \) explicitly on \([i + 2, j - |U|]\). This works since \( B[i + 1, j - |U|] \) consists of isolated vertices (that is, the leaves of the \( B[i, j - 1] \)).

Assume now that \( B[i, j - 1] \) is not a star. If \( \{i, j\} \not\in E(B) \), then we embed \( s \) at \( j \), and recursively embed \( R^- \) onto \([i, j - 1]\). \( R^- \) has no edge-conflict with \([i, j - 1]\) by the peace invariant, and by the placement of \( s \). It also has no degree-conflict with \([i, j - 1]\), otherwise \( R \) would already have had a degree-conflict with \([i, j]\).

So assume that \( \{i, j\} \in E(B) \). Note that \( B \) is a tree in this case. Flip \( B \) if necessary to put its root at \( j \). If \( B[i, j - 1] \) is a star now, then use the embedding described in the first paragraph to find an ordered plane packing. Otherwise the general plan is to embed \( s \) at \( j \) and \( R^- \) recursively onto \([j - 1, i] \). Since \( B \) is not a star and \( B \) is rooted at \( j \), the edge \([j - 1, j]\) is not used by \( B \). Hence, this works unless \( R^- \) is in conflict with \([j - 1, i]\). A conflict means in particular that \( B[i, j - 1](j - 1) \) is a central star \( B^* = B[x, j - 1] \) for some \( i + 1 \leq x \leq j - 2 \). See Figure 20a. Due to the presence of the edge \( \{i, j\} \) and since \( x \geq i + 1 \), \( \UParrow(B^*) = \odot(B^*) \) must be at \( x \).

Case 20: Case 1 in the proof of Lemma 20.

Case 1.1 \( x \geq i + 2 \). Flip \( B[x, j] \). Note that afterwards \( \{i, j\} \not\in E(B) \) and \( B[i, j - 1] \) satisfies 1SR and LSFR. Embed \( s \) onto \( j \) and \( R^- \) recursively onto \([i, j - 1]\). See Figure 20b. Since \( |B^*| \geq 2 \), the interval \([i, j - 1]\) contains at least one leaf of \( B^* \) and so \( B[i, j - 1] \) is not a star. Hence, this works unless there is a conflict for embedding \( R^- \) onto \([i, j - 1]\). In that case, note that \( B[i, x] \langle i \rangle \) is now formed by \( \UParrow(B) \) and its subtrees other than \( B^* \). Since \( B[i, x] \langle i \rangle \) is a central star, it follows that the subtrees of the root \( b \) of \( B \) other than \( B^* \) are all leaves. Flip \( B[x, j] \) again to restore the original embedding. Embed \( s \) onto \( i \) and \( R^- \) recursively onto \([i + 1, j]\). See Figure 20c. Since \( x \geq i + 2 \), \( B[i + 1, j] \) is a tree that is not a star and \( \{i, i + 1\} \not\in E(B) \). Hence, the peace invariant holds for \( R^- \), and \( R^- \) has no conflict with \([i + 1, j]\).

Case 1.2 \( x = i + 1 \). Flip \( B[x, j] \). Embed \( r \) onto \( i \) and \( s \) onto \( j \). Embed the remaining subtrees of \( r \) in \( R \) explicitly onto the independent set \( B[i + 1, j - 1] \), putting the largest one (which has size at least two) next to \( i \). See Figure 20d.

Case 2 \( R^- \) is a star. Then \( \deg_{R^-}(r) = 2 \) and the child \( q \) of \( r \) in \( R^- \) is the root and center of a star \( Q = t(q) \).

Case 2.1 \( \{i, j\} \in E(B) \). Denote \( b := \UParrow(B) \). Flip \( B \) if necessary to put its root at \( j \). If \( \deg_{B}(b) = 1 \), then \( j \) is isolated in \( B[i + 1, j] \) and \( \{i, i + 1\} \not\in E(B) \) since \( B \) is not a star and
by LSFR. Embed \( r \) onto \( i + 1, s \) onto \( i, q \) onto \( j \), and the children of \( q \) onto \([j - 1, i + 2]\). See Figure 21a.

![Figure 21: Case 2 in the proof of Lemma 20.](image)

If \( \deg_B(b) \geq 2 \), then let \( x \) be such that \( B[i, j - 1]i = B[i, x] \), which is a smallest subtree of \( b \) by LSFR. Since \( B \) is not a star, \( B[x + 1, j] \) is not a central star. Flip \( B[x + 1, j] \). This puts the root \( b \) at \( x + 1 \). Use a leaf-isolation shuffle on \( B[x + 1, j] \) to embed a leaf at \( j - 1 \), its parent at \( j \), and the root at \( x + 1 \). This works by Proposition 11. Embed \( r \) onto \( i, s \) onto \( j, q \) onto \( j - 1 \) and the children of \( q \) onto \([j - 1, i + 1]\). See Figure 21b.

**Case 2.2** \( \{i, j\} \notin E(B) \). Then \( B(i) \neq B(j) \). If \( |B(j)| \geq 2 \), then perform a leaf-isolation shuffle to put a leaf at \( j - 1 \) and its parent at \( j \). Since \( B(i) \neq B(j) \), this does not touch the blue vertex at \( i \). Embed \( r \) onto \( i, s \) onto \( j, q \) onto \( j - 1 \), and the children of \( q \) onto \([j - 2, i + 1]\). See Figure 21c.

If \( |B(j)| = 1 \) and \( B(i) \) is not a central star, then flip \( B(i) \) if necessary to put its root at \( i \). Since it is not a central star, \( \{i, i + 1\} \notin E(B) \). Embed \( r \) onto \( i + 1, s \) onto \( i, q \) onto \( j \), and the children of \( q \) onto \([j - 1, i + 2]\). See Figure 21d.

Finally, if \( |B(j)| = 1 \) and \( B(i) \) is a central star, then let \( x \) be such that \( B[i, x] = B(i) \). We have \( x \leq j - 2 \) by the peace invariant. Flip \( B[i, x] \) if necessary to put its root at \( i \). By the peace invariant, \( i \) is not in edge-conflict with \( r \). Embed \( r \) onto \( i, s \) onto \( x + 1, q \) onto \( j \), and the children of \( q \) onto \([j - 1, x + 2]\) and \([x, i + 1]\). See Figure 21e.

9 Embedding the red tree: a large blue star

In this and the following section we handle the case that \( B[i, j - |S|] \) is a star. The graphs \( S, R^-, \) and \( B[j - |S| + 1, j] \) may or not be stars. The case that we actually handle is more general, as specified in the following

**Lemma 21.** If \( B[i, x] \) is a star, for \( x \in [j - |S|, j - 1] \), then \( R \) and \( B \) admit an ordered plane packing onto \([i, j]\).

**Proof.** By Lemma 19 and Lemma 20, we may assume \( \deg_R(r) \geq 2 \) and \( |S| \geq 2 \). We have \( |R^-| \geq 3 \) by Observation 13 and by \( \deg_R(r) \geq 2 \). Select \( x \) maximally so that \( B^* = B[i, x] \) is a star, and let \( d = \deg_{R^-}(r) \geq 1 \). Note that \( |B[i, x]| \geq |R^-| \geq 3 \). We distinguish two cases.

**Case 1** \( B^* \) is a central star. Then by LSFR we have \( B(i) = B^* \). If necessary, flip \( B^* \) to put its root and center at \( i \). We will use a blue-star embedding to embed \( R^- \) from \( \sigma = i \) with \( \varphi = (x + 1, \ldots, x + d) \). Let us first check the conditions for the blue-star
embedding. (BS1) holds by (I1) for embedding $R$ onto $[i, j]$. For (BS2) we must show $|R^-| \leq |B^*| + d$ and $|B^*| + d \leq |I| - 1$. We wish to argue that at least one leaf of $B^*$ remains after the blue-star embedding, and thus we need to show $|R^-| < |B^*| + d$. This inequality holds since $|R^-| = j - |S| \leq x = |B^*|$ and $d \geq 1$. For the second inequality, by (I1), $\deg_B(i) + \deg_R(r) \leq |I| - 1$ and so $|B^*| + d = (\deg_B(i) + 1) + (\deg_R(r) - 1) \leq |I| - 1$. (BS3) and (BS4) hold since $B \setminus (B^* \cup \varphi)$ forms an interval. Hence, by Proposition 9, the blue-star embedding succeeds and leaves an interval $[i', j'] = [i', j]$ such that $j$ is not in edge-conflict for embedding $s$ and a non-empty prefix of $[i', j]$ consists of isolated vertices that are in edge-conflict for embedding $s$ (these are leaves of $B^*$). Recursively embed $S$ onto $[j, i']$. This works unless $S$ is a star or $B[i', j] \cup \varphi$ is in conflict (which must be a degree-conflict) with $S$.

**Case 1.1** $S$ is a star. If $S$ is a dangling star then embed $s$ onto $j$, the child $s'$ of $s$ onto $i'$ (which is locally isolated), and the children of $s'$ onto $[i' + 1, j - 1]$. Otherwise, $S$ is a central star. If there is a locally isolated vertex in $B[i', j]$ that is not in edge-conflict, then use this vertex to embed $s$ and embed the children of $s$ on the remainder. Otherwise, undo the blue-star embedding. Consider the blue vertex at $j$, which does not get consumed by the blue-star embedding. Since it was not isolated after the blue-star embedding, it is not isolated now. By the choice of $x$ and by LSFR, we have $B[j] = B[x + 1, j]$ and $w$. Perform a leaf-isolation shuffle on $B[j]$ to place a leaf $\ell$ at $j - 1$ and its parent at $j$. Perform the original blue-star embedding, but now with $\varphi = (j, x + 1, \ldots, x + d - 1)$ if $d \geq 2$ and $\varphi = (j)$ if $d = 1$. The conditions of the blue-star embedding still hold. The resulting interval $[i', j']$ contains the now isolated vertex $\ell$ and we embed $S$ by placing $s$ onto $\ell$ and embedding the children of $s$ on the remainder.

**Case 1.2** $B[i', j] \cup \varphi$ is a central star that raises a degree-conflict. Note that $[i', j]$ is composed of some locally isolated vertices plus some suffix of the interval $[i, j]$ before the blue-star embedding. Undo the blue-star embedding. Now $B[z, j]$ is a central star for some minimal $z$. We claim that we may assume that $B[z, j]$ is rooted at $j$. Indeed, if $B[z, j]$ is rooted at $z$, then by ISR we have $B[z, j] = B[j]$ and we can flip $B[j]$ to establish the claim. Perform the original blue-star embedding for $R^-$, but now with $\varphi = (j, x + 1, \ldots, x + d - 1)$ if $d \geq 2$ and $\varphi = (j)$ if $d = 1$. In the remaining interval $[i', j']$, the vertex $j'$ is a leaf of what was the central star $B[z, j]$ before the blue-star embedding. Hence, $j'$ is locally isolated and not in edge-conflict with $s$. Recursively embed $S$ onto $[j', i']$ to complete the embedding.

**Case 2** $B^*$ is a dangling star. In this case (I1) does not tell us anything about the size of $B^*$ (because it applies to central stars only). If $\hat{(B^*)}$ is at $x$, then its center is at $i$ and by ISR $i$ is the only neighbor of $x$ in $B$. Hence by flipping $B^*$ we may assume that $\hat{(B^*)}$ is at $i$. Note that $i$ may have more neighbors, in addition to $\hat{B^*}$ at $x$. Also note that $i$ may be in conflict with $r$, in case we flipped $B^*$ (the original vertex at $i$ cannot be in conflict by $(I3)$). We distinguish several cases.

**Case 2.1** $x = j - 1$. In this case we know almost completely what $B$ looks like: $B[i, j - 1]$ is a star rooted at $i$ and centered at $j - 1$ and the edge $\{i, j\}$ may or may not be in $E(B)$. We embed $R$ explicitly as follows. Since $\deg_R(r) \geq 2$, there is a subtree $W = t(w)$ of $r$
different from \( S \). Embed \( r \) onto \( i + |W| \) and embed \( W \) explicitly onto the independent set at \([i, i + |W|] - 1\). Since \(|S| \geq 2\), we know that \(|R'| < |B[i, j - 1]|\), and hence \( r \) is not embedded at \( j - 1 \). If \( S \) is not a star, embed it recursively onto \([j, j - |S| + 1]\). This works because \( j \) is locally isolated in \( B[j - |S| + 1, j] \) and \( j \) is not adjacent to \( i + |W| \) (which is where we embedded \( r \)). If \( S \) is a central star, embed \( s \) onto \( j \) and its children onto \([j - 1, j - |S| + 1]\).

If \( S \) is a dangling star, embed \( s \) onto \( j - |S| + 1 \), the child \( s' \) of \( s \) onto \( j \), and the children of \( s' \) onto \([j - 1, j - |S| + 2]\). Embed the remaining subtrees (if any) of \( r \) on the remaining interval \([i + |W| + 1, j - |S|]\), which forms a locally independent set, none of whose vertices is adjacent to \( r \).

**Case 2.2** \( x \leq j - 2 \) and \( B[j - |S| + 1, j]\) is a central star \( B^{**} \) with \(|B^{**}| \geq |S| - \deg_S(s) + 1\) (in particular, this holds if \( S \) has a degree-conflict with \([j, j - |S| + 1]\)). We distinguish two subcases.

**Case 2.2.1** \( \{i, j\} \in E(B) \). Then the root and center \( b^{**} \) of \( B^{**} \) must be at \( j \) and cannot be the root of \( B \) because then LSFR would imply that \( B \) is a star. Therefore \( i = \uparrow(B) \) (Figure 22a). Since \( \{i, j\} \in E(B) \), \( i \) is the original root of \( B^* \) (that is, \( B^* \) was not flipped), and \( i \) is not in conflict with \( r \) due to \((I3)\). Note that \( B^{**} \) has at least one leaf since \(|B^{**}| \geq |S| - \deg_S(s) + 1 \geq 2\), and at most \(|S| - 1\) leaves since \(|B^{**}| \leq |I| - |B^*| \leq |S| \). We modify the embedding of \( B \) as follows: Move \( b^{**} \) to \( j - |S| \) and all leaves of \( B^{**} \) immediately to the right of \( b^{**} \), at position \( j - |S| + 1 \) and onward, shifting all vertices between there and \( j \) to the right accordingly. Draw the edge \( \{i, b^{**}\} \) below the spine to avoid crossings, and all other edges incident to \( b^{**} \) above the spine (Figure 22b).

![Figure 22: \( \{i, j\} \in E(B) \) (Case 2.2.1)](image)

We place \( r \) at \( i \) and explicitly embed \( R^- \) onto \([i, j - |S|]\), which in \( B \) consists of a single edge \( \{i, j - |S|\} \) with isolated vertices (at least one because \(|R^-| \geq 3\) in between. Recall that \( R^- \) is not a central star and so a largest subtree of \( R^- \) has \( y \geq 2 \) vertices. Embed a largest subtree of \( R^- \) onto \([j - |S| - y + 1, j - |S|]\), and embed all other subtrees on \([i + 1, j - |S| - y]\) in an arbitrary order. It remains to embed \( S \) onto \([j - |S| + 1, j]\). As \( j - |S| + 1 \) is a leaf of \( B^{**} \), which is isolated on \([j - |S| + 1, j]\), there is no conflict for this embedding and \( B[j - |S| + 1, j] \) is not a star. Therefore, if \( S \) is not a star, then we can complete the packing recursively by embedding \( S \) onto \([j - |S| + 1, j]\).

It remains to consider the case that \( S \) is a star. If \( S \) is a central star, then we can put this center at the locally isolated vertex \( j - |S| + 1 \). Otherwise, \( S \) is a dangling star with \(|S| \geq 3\). As \(|B^{**}| \geq |S| - \deg_S(s) + 1 = |S| \geq 3\), we have at least two locally isolated vertices (leaves of \( B^{**} \)) at \( j - |S| + 1 \) and \( j - |S| + 2 \). We put \( \uparrow(S) \) at \( j - |S| + 1 \) and the center at \( j - |S| + 2 \) to complete the packing.
Case 2.2.2 \{i, j\} \notin E(B). Recall that \(B^{**} = B[j, j - |S| + 1] \langle j \rangle\) is a central star. We claim that \(B^{**} = B \langle j \rangle\). Indeed, if the root \(b^{**}\) of \(B^{**}\) is on the left, then by ISR \(B^{**} = B \langle j \rangle\). Otherwise, \(b^{**}\) is at \(j\). By the definition of \(B^{**}\), \(b^{**}\) has no neighbors in \(B[j, j - |S| + 1] \setminus B^{**}\). Since \(B^* = B[i, x]\) is a star and \(x \geq j - |S|\), the only remaining possible neighbor of \(b^{**}\) would be \(i\), but this is excluded by the assumption. We conclude that \(B^{**} = B \langle j \rangle\). If necessary, flip \(B^{**}\) to put its root (and center) at \(j\).

Case 2.2.2.1 \(x = j - |S|\) (Figure 23a). Then we change the embedding of \(B\) by moving one leaf \(\ell\) of \(B^{**}\) to \(i\), and drawing the blue edge \(\{j, \ell\} = \{i, j\}\) below the spine. As \(\ell\) is a leaf of \(B^{**}\), it is not in conflict with \(r\), and so we can map \(r\) to \(\ell = i\) and embed \(R^*\) explicitly onto the locally independent set \(B[i, j - |S|]\). If \(S\) is not a star, then we recursively embed \(S\) onto \([j - |S| + 1, j]\) (Figure 23b). Note that \(j - |S| + 1\) is \(\odot(B^*)\), which is isolated in \(B[j - |S| + 1, j]\) and not adjacent to the leaf of \(B^{**}\) at \(i\). Therefore, \(B[j - |S| + 1, j]\) is not a star and \(S\) has neither degree- nor edge-conflict with \([j - |S| + 1, j]\).

Figure 23: \(x = j - |S|\) (Case 2.2.2.1).

It remains to consider the case that \(S\) is a star. If \(S\) is a central star, then the center can be embedded on the isolated vertex at \(j - |S| + 1\). Otherwise, \(S\) is a dangling star with \(|S| \geq 3\). Then at least one more leaf of \(B^{**}\) remains at \(j - 1\), where we can embed \(\uparrow(S)\). The \(\odot(S)\) is again mapped to the isolated vertex \(j - |S| + 1\) and the edge \(\{j - |S| + 1, j\}\) is drawn as a biarc, crossing the spine between \(j - 2\) and \(j - 1\).

Case 2.2.2.2 \(x \geq j - |S| + 1\) (Figure 24a). Then we change the embedding of \(B\) by simultaneously moving \(\uparrow(B^*)\) to \(x\) and all other vertices of \(B^*\) to the left by one (Figure 24b). Embed \(r\) at \(i\); and we need to embed the subtrees of \(r\) onto \([i + 1, j]\). Since Note that \(B[i + 1, j - |S| + 1]\) consists of isolated vertices (as \(x \geq j - |S| + 1\), and \(B[i + 1, j] \langle j \rangle = B^{**}\)). We will use a blue-star embedding with respect to \(B[i + 1, j]\) to embed \(S\) from \(\sigma = j\), where \(\varphi\) consists of the rightmost \(\deg_{S}(s)\) non-neighbors of \(j\) in \(B\) from right to left. Since \(B[i + 1, j] \langle j \rangle = B^{**}\), we have \(B^+ = B^{**}\) for the blue-star embedding with respect to \(B[i + 1, j]\). Let us check the conditions for the blue-star embedding. \((BS1)\) holds because \(|R^-| \geq 3\) and so \(i\) is a leaf of \(B^*\) that is adjacent to \(x - 1 \neq j\) only. \((BS3)\) and \((BS4)\) hold because \(B[i + 1, j] \setminus (B^{**} \cup \varphi)\) forms an interval. For \((BS2)\) we need to show \(|S| \leq |B^{**}| + \deg_{S}(s)\) and \(|B^{**}| + \deg_{S}(s) \leq |I| - 2\). The first inequality follows from the assumption of Case 2.2. For the second inequality, we have \(|B^{**}| \leq |B[x + 1, j]| \leq |S| - 1\) and \(\deg_{S}(s) \leq |S| - 1\), and so \(|B^{**}| + \deg_{S}(s) \leq 2(|S| - 2) < |I| - 2\), since \(|S| < |I|/2\) given that \(\deg_{R}(r) \geq 2\). Hence, the conditions for the blue-star embedding are satisfied.

Since \(\deg_{S}(s) \geq |S| - |B^{**}| + 1\), we have \(\varphi \supset B[j - |S|, j] \setminus B^{**}\), and hence the blue-star embedding embeds a child of \(s\) onto \(\uparrow(B^*)\), which was embedded at \(x\), and onto \(\odot(B^*)\), which was embedded at \(x - 1\). Therefore, the remaining vertices not used for the
embedding of \( S \) form an independent set in \( B \) and we can explicitly embed \( R^- \) on them.

**Case 2.3** \( x \leq j - 2 \) and \( B^{**} = B[j - |S| + 1, j] \( \langle j \rangle \) is not a central star of size \( |B^{**}| \geq |S| - \deg_S(s) + 1 \) (that is, \( B^{**} \) is not a central star or \( |B^{**}| < |S| - \deg_S(s) + 1 \)). Note that \( x \leq j - 2 \) implies \( |S| \geq 2 \), hence \( |R| \geq 5 \). We first prove a claim and then distinguish two subcases.

**Claim:** We may assume that \( S \) is a star or \( x = j - |S| \). To prove the claim, suppose that \( x \geq j - |S| + 1 \). Then \( j - |S| \) is a leaf of \( B^* \) and we can explicitly embed \( R^- \) onto the independent set \([j - |S|, i]\) such that \( r \) embeds onto \( j - |S| \). As \( x \neq j \) is the only neighbor of \( j - |S| \) in \( B \), we have \( \{j - |S|, j\} \notin E(B) \) and so there is no edge-conflict for embedding \( S \) onto \([j, j - |S| + 1]\). By assumption there is no degree-conflict for this embedding, either, and \( B[j - |S| + 1, j] \) is not a star because \( (B^*) \) is part of it but \( B^* \neq B \). The only remaining obstruction for the recursive embedding of \( S \) onto \([j, j - |S| + 1]\) is \( S \) being a star. This proves the claim.

**Case 2.3.1** \( \{i, x + 1\} \notin E(B) \). Then we rearrange the embedding of \( B \) as follows: move \( c := \odot(B^*) \) to \( j - |S| + 1 \) and the vertex \( b' \) at \( x + 1 \) (the leftmost vertex not in \( B^* \)) to \( j - |S| \). In order to avoid crossings with the edge(s) incident to \( b' \), draw all edges between \( c \) and its neighbors in \([i, j - |S| - 1]\) below the spine, whereas edges to neighbors in \([j - |S| + 2, j]\) remain above the spine (Figure 25). After this transformation \( B[i, j - |S|] \) is an independent set, on which we can embed \( R^- \) explicitly. However, we have to be cautious because of the blue edges drawn below the spine and the (possibly) conflicting root \( i \). Without loss of generality suppose that \( (B^*) \) at \( i \) is in conflict with \( r \).

Recall that \( R^- \) is not a central star and so there is at least one non-leaf child \( u \) of \( r \) in \( R^- \). Denote \( U = t(u) \) and map both \( u \) and the conflicting root of \( B^* \) to \( j - |S| - |U| + 1 \) (by exchanging the order of leaves of the dangling star \( B^* \) in \( B[i, j - |S| - 1] \)). As \( |U| \geq 2 \), we have \( j - |S| - |U| + 1 \leq j - |S| - 1 \) and so the local order for the roots of subtrees from \( B \) is maintained. On the other hand, we have \( j - |S| = i + |R^-| - 1 \) and \( |U| \leq |R^-| - 1 \),
which imply $j - |S| - |U| + 1 \geq (i + |R^-| - 1) - (|R^-| - 1) + 1 = i + 1$. Therefore (the leaf now at) $i$ is not in conflict with $r$.

As $\{i, x + 1\} \notin E(B)$ initially, after the transformation we have $\{j - |S| - |U| + 1, j - |S|\} \notin E(B)$ and so $\{j - |S| - |U| + 1, j - |S|\}$ is an independent set in $B$. Therefore, we can embed $U$ onto $[j - |S| - |U| + 1, j - |S|]$ explicitly, drawing all edges above the spine, and complete the embedding of $R^-$ by embedding $R^- \setminus U$ onto $[i, j - |S| - |U|]$ explicitly, again drawing all edges above the spine. After these changes to the embedding of $B$, the only neighbor of $i$ in $B$ is $j - |S| + 1$. Combined with $|S| \geq 2$ it follows that $S$ has no edge-conflict with $[j, j - |S| + 1]$. We also know that $B[j, j - |S| + 1]$ is not a star because it contains part of $B^*$ (at least the center at $j - |S| + 1$) and at least one more vertex not connected to that part of $B^*$: the vertex at $j$. (As $|S| \geq 2$, there were at least two such vertices initially, but one, the vertex $b'$, has been moved and used for embedding $R^-$.) Two possible obstructions for the recursive embedding of $S$ onto $[j, j - |S| + 1]$ remain: a degree-conflict or $S$ is a star. We conclude by considering both cases.

**Case 2.3.1.1** $S$ is a star. Undo the rearrangement of $B$ described in the first paragraph of Case 2.3.1. We will redo the rearrangement after some other modifications.

Suppose first that $|B[x + 1, j](x + 1)| \geq 2$ or $|B[x + 1, j](x + 2)| = 1$. In the former case (Figure 26), use a leaf-isolation shuffle on $B[x + 1, j]$ if necessary to place a leaf at $x + 2$ and its parent at $x + 1$. We can apply the shuffle because $|B[x + 1, j](x + 1)| \geq 2$ implies $|B[x + 1, j]| \geq 2$. Perform now the rearrangement described in the first paragraph of Case 2.3.1), and then proceed as follows. If $S$ is a central star, then $s$ can be placed at $x + 2$, which is adjacent to $j - |S|$ only and therefore locally isolated in $B[j - |S| + 1, j]$ (Figure 26b). Otherwise, $S$ is a dangling star, and so $|S| \geq 3$. Then either there is a (non-root) leaf of $B^*$ in $[j - |S| + 1, j]$ or $c := \circ(B^*)$ is isolated in $[j - |S| + 1, j]$. In either case, we put $\circ(S)$ at $x + 2$. In the former case, we put $\uparrow(S)$ at $j - |S|$ (the leftmost leaf of $B^*$ in $[j - |S| + 1, j]$), and draw the edge $\{c, x + 2\}$ (between the leftmost leaf of $B^*$ in $[j - |S| + 1, j]$ and $\circ(S)$) above the spine (Figure 26c). In the latter case, we have $x = j - |S| \leq j - 3$ hence $x + 2 < j$, and we can put $\uparrow(S)$ at $j$. Either way, we can complete the star $S$ and the embedding of $R^-$ works just as before.

**Figure 26:** $|B[x + 1, j](x + 1)| \geq 2$ and $S$ is a central star (b) or a dangling star (c) (Case 2.3.1.1).

Otherwise, $|B[x + 1, j](x + 1)| = 1$ and $|B[x + 1, j](x + 2)| \geq 2$ (Figure 27a). Since $\{i, x+1\} \notin E(B)$ by the assumption of Case 2.3.1, we know that $B(x + 1) = B[x + 1, j](x + 1)$ and hence also $B(x + 2) = B[x + 1, j](x + 2)$. It follows that $x \leq j - 3$ and hence $|S| \geq 3$. Perform a leaf-isolation shuffle on $B(x + 2)$ to place a leaf at $x + 3$ and its parent at $x + 2$.  

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**Figure 26:**
Rearrange the embedding of $B$ as follows: move $c := \ominus(B^*)$ to $j - |S| + 1$, the vertex $b'$ at $x + 1$ to $j - |S| - 1$, and the vertex $b''$ at $x + 2$ to $j - |S|$. Draw the blue edges as explained in the first paragraph of Case 2.3.1. We embed $S$ analogously to the previous paragraph, using $x + 3$ as the location for the star-center (Figure 27b–27c). To embed $R^-$, let us consider the embedding $B[i, j - |S|]$. It is again an independent set. As opposed to Case 2.3.1, however, we have local roots at $j - |S| - 1$ and at $j - |S|$. Fortunately, since $|S| \geq 3$ and $S$ is a smallest subtree, also $|U| \geq 3$, and hence $j - |S| - |U| + 1 \leq j - |S|- 2$, as required. Hence, we can embed $R^-$ explicitly, analogously to the second paragraph of Case 2.3.1.

Figure 27: $|B[x + 1, j](x + 1)| = 1$ and $|B[x + 1, j](x + 2)| \geq 2$ (Case 2.3.1.1).

**Case 2.3.1.2** There is a degree-conflict for embedding $S$ onto $[j, j - |S| + 1]$. Then this conflict must have been created by the rearrangement of the embedding of $B$. Before this rearrangement there was no degree-conflict by assumption (Case 2.2 handles this scenario). In other words, $b'$ is the root of a star $B[x + 1, j]$ in the initial embedding whose center is at $j$. After moving $b'$ out of the interval $[j - |S| + 1, j]$, $j$ became the local root, which raised the degree-conflict. By our claim and the preceding Case 2.3.1.1 we may suppose that $x = j - |S|$ (Figure 28a).

Undo the rearrangement of $B$ described in the first paragraph of Case 2.3.1. We use a different, explicit embedding as follows: flip the star $B[x + 1, j]$ so that its root is at $j$ and the center is at $x + 1$ and draw all edges below the spine. Next move the center at $x + 1$ to $i$ instead, shifting all vertices in between to the right by one. Then put $r$ at $i$ (not being the root of $B[x + 1, j]$ it is not in conflict), and explicitly embed $R^-$ onto the (now) independent set $[i, x]$. Finally, explicitly embed $S$ onto the (now) independent set $[x + 1, j]$ (Figure 28b). Note that $B$ might be a tree (in which case the two roots in the figure are actually connected), but the embedding works also in this case.

Figure 28: A new degree-conflict for $S$ on $[j, j - |S| + 1]$ (Case 2.3.1.2).

**Case 2.3.2** $\{i, x + 1\} \in E(B)$ and $x \geq j - |S| + 1$. Then by our claim we may suppose that $S$ is a star. We embed $R^-$ explicitly onto $[j - |S|, i]$, noting that $j - |S|$ is a non-root leaf of $B^*$ and, therefore, not in conflict with $r$. If $S$ is a central star, then we put the center at $x + 1$,
which is connected to $i$ only and therefore locally isolated on $[j - |S| + 1, j]$. Otherwise, $|S| \geq 3$ and $S$ is a dangling star. Given that $x \leq j - 2$, we have $x + 1 \neq j$ and therefore can put $\uparrow(S)$ on $j$ and the center on $x + 1$.

**Case 2.3.3** $\{i, x + 1\} \in E(B)$ and $x = j - |S|$. We distinguish two final subcases.

**Case 2.3.3.1** $\uparrow(B[i, x + 1])$ is at $i$. Then we change the embedding of $B$ by moving the vertex at $x + 1$ to $i$ and shifting the vertices in between to the right by one. Then $B[i, j - |S|]$ is an independent set except for the single edge $\{i, i + 1\}$. If $R^-$ is not a star, we can embed it recursively onto $[i, j - |S|]$. If $R^-$ is a dangling star, we can embed it explicitly by placing its root $r$ at $i$ and its center at $j - |S|$. Then if $S$ is a central star, we embed $\odot(S)$ at $j - |S| + 1$, which is an isolated vertex in $[j - |S| + 1, j]$. If $S$ is a dangling star, then we embed $\uparrow(S)$ at $j$ and $\odot(S)$ at $j - |S| + 1$. Otherwise, $S$ is not a star and we recursively embed $S$ onto $[j - |S| + 1, j]$. Recall that $j - |S| + 1$ is a locally isolated vertex and $\{i, j - |S| + 1\} \notin E(B)$ (because $i$ is a leaf whose only neighbor is at $i + 1 \neq j - |S| + 1$). Therefore, $S$ is not in conflict with $[j - |S| + 1, j]$, and $B[j - |S| + 1, j]$ is not a star.

**Case 2.3.3.2** $\uparrow(B[i, x + 1])$ is at $x + 1$ (and possibly in conflict with $r$; Figure 29a).

If $S$ is a central star, then we change the embedding of $B$ as follows: First flip $B^*$ so that its center is at $i$ and then exchange the vertices at $i$ ($\odot(B^*)$) and $i + 1$ (a leaf of $B^*$). Put $r$ at $i$, which is a leaf of $B^*$ and therefore not in conflict. Then put $s$ at $x + 1$, whose only neighbor is (now) at $x$ (originally at $i$), drawing the edge $\{r, s\} = \{i, x + 1\}$ above the spine. Next put a leaf of $S$ on $i + 1$ ($\odot(B^*)$), again drawing the edge $\{i + 1, x + 1\}$ above the spine. Put any remaining leaves of $S$ on the vertices $[x + 2, j - 1]$, drawing the edges to $s$ below the spine. This leaves us with a set of isolated vertices $[i + 2, x] \cup \{j\}$, all accessible from below the spine, on which we can complete an explicit embedding of $R^-$ (Figure 29b).

![Figure 29: $\{i, x + 1\} \in E(B), x = j - |S|$, and $\uparrow(B[i, x + 1]) = x + 1$ (Case 2.3.3.2).](image)

Otherwise, $S$ is not a central star (it may be a dangling star). Then we modify the embedding of $B$ by drawing all edges of $B^*$ below the spine and exchanging $x$ and $x + 1$. Explicitly embed $R^-$ onto $[i, j - |S|]$. This is possible because $B[i, j - |S|]$ is an independent set except for the edge $\{i, j - |S|\}$ and $R^-$ is not a central star. Note that $j - |S| + 1 = x + 1$ is a locally isolated vertex in $B[j - |S| + 1, j]$, and so $B[j - |S| + 1, j]$ is not a star. We embed $S$ onto $[j, j - |S| + 1]$ (Figure 29c). There is no degree-conflict by assumption (Case 2.2 handles this scenario) and—as opposed to Case 2.3.1.2—we do not change $B[j]$ here. By assumption $\{i, j\} \notin E(B)$ and so there is no edge-conflict for the recursive embedding of $S$, either. If $S$ is not a star, we can embed $s$ recursively on $[j, j - |S| + 1]$. If $S$ is a dangling star, then we embed $S$ explicitly on $[j, j - |S| + 1]$, placing its root at $j$ and it center at

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\[ j - |S| + 1. \]

10 Embedding the red tree: a large red star

In this section we handle the case where \( R^- \) is a star with \(|R^-| \geq 2\). Recall that the case \(|R^-| = 1\) was handled in Lemma 19. If \( R^- \) is a star, then it must be a dangling star: otherwise, by the choice of \( S \) as a smallest subtree, \( R \) would be a star. Let \( q \) be the child of \( r \) in \( R^- \) and let \( Q = t(q) \). Then \( Q \) is a central star. As the case \(|S| = 1\) was handled in Lemma 20, we can assume \(|S| \geq 2\) and so \(|Q| \geq |S| \geq 2\). Our default approach in this case is to explicitly embed \( Q \) and recursively embed \( S^+ := R \setminus Q \). Note that \( \deg_{S^+}(r) = 1 \). Consequently, when we try to recursively embed \( S^+ \) onto some interval \([x, y]\), there is no degree-conflict: If \( B' := B[x, y] \) is not a star, then for every \( b \in [x, y] \) we have \( \deg_{B'}(b) < |B'| - 1 \) and so \( \deg_{B'}(b) + \deg_{S^+}(r) < |B'| \). Hence, for a recursive embedding of \( S^+ \) it suffices to check that we are not embedding against a star and that there is no edge-conflict.

Proposition 22. If \( R^- \) is a star, \( \{i, j\} \notin E(B) \), and \( B[i, x] \) is a star, for some \( x \in [i + |S|, j - 1] \), then \( R \) and \( B \) admit an ordered plane packing onto \([i, j]\).

Proof. Let \( x \) be maximal so that \( B[i, x] =: B^* \) is a star. By Lemma 21 we may suppose \( x < j - |S| \leq j - 2 \). Note that \( B^* \) has \( x - i \) leaves and that in general \( B(i) \neq B^* \). Also note that \( \odot(B^*) \) is not in conflict with \( r \). If \( \odot(B^*) = i \), then this is guaranteed by (I3); otherwise \( \odot(B^*) = x \) and if \( x = \uparrow(B^*) \), then by ISR also \( x = \uparrow(B(i)) \), which is not in conflict with \( r \) by (I1). We distinguish two cases depending on \( j \). In both cases we will change the embedding of \( B^* \) and then construct an explicit embedding for \( R \).

Case 1 \( j \) is isolated in \( B \).

Case 1.1 \( x > i + |S| \), that is, \( |B^*| > |S^+| \). Then we embed \( B^* \) so that all the vertices of \([i, i + |S|]\) are leaves, that is, we put \( \odot(B^*) = i + |S| + 1 \). The edges from \( \odot(B^*) \) to those leaves are drawn below the spine. If \( B^* = B(i) \) and \( B^* \) is not central, then we put \( \uparrow(B^*) \) at \( i + |S| \). It follows that after this change there is no conflict between \( r \) and \( i \) and \( B[i, i + |S|] \) consists of isolated vertices. Therefore we can embed \( S^+ \) onto \([i, i + |S|]\) explicitly with \( r = i \), drawing its edges above the spine, and \( Q \) onto \([i + |S| + 1, j]\) with \( q = j \), drawing the edges below the spine (Figure 30a).

Case 1.2 \( x = i + |S| \) (that is, \( |B^*| = |S^+| \)) and \( i = \odot(B^*) \). Then by the choice of \( x \) we have \( \{i, x + 1\} \notin E(B) \). We move a nonroot leaf \( \ell \) of \( B^* \) to \( x + 1 \), shifting all vertices in \([\ell + 1, x + 1]\) to the left by one. The edge \( \{i, \ell\} \) is drawn below the spine. In the resulting blue forest \( B' \) we have \( \{i, x\} \notin E(B') \) and \( B'[i + 1, i + |S|] \) consists of isolated vertices. Therefore we can put \( r = i \), explicitly embed \( S \) onto \([x, i + 1]\) and \( Q \) onto \([i + |S| + 1, j]\) with \( q = j \) (Figure 30b).

Case 1.3 \( x = i + |S| \) (that is, \( |B^*| = |S^+| \)) and \( x = \odot(B^*) \). Recall that \( \odot(B^*) \) is not in conflict with \( r \). Therefore, if \( \uparrow(B^*) = x \), then we can flip \( B(i) \) and proceed as in Case 1.2.
It remains to consider the case that $\uparrow(B^*) = i$. Then draw all edges of $B^*$ below the spine and exchange $x$ and $x + 1$ in $B$. In the resulting blue forest $B'$ we know that $B'[i + 1, x]$ consists of isolated vertices. Therefore, we can put $r = i$, explicitly embed $S$ onto $[i + 1, x]$ with $\uparrow(S) = i + 1 < x$, drawing the edges above the spine, and $Q$ onto $[i + |S| + 1, j]$ with $q = j$, drawing the edges below the spine (Figure 30d). Note that possibly $\{i, x + 1\} \in E(B)$, that is, $\{i, x\} \in E(B')$, which is not a problem.

![Figure 30: Case analysis in the proof of Proposition 22.](image)

**Case 2** $j$ is not isolated in $B$. We apply a leaf-isolation shuffle on $B(j)$ ($\neq B(i)$ because $\{i, j\} \notin E(B)$) to put a leaf $\ell$ at $j - 1$ and its parent $p$ at $j$. Recall that $\odot(B^*)$ is not in conflict with $r$. Therefore we can put (or leave) $\odot(B^*) = i$ and move $|S| - 1 \leq x - i - 1$ of the nonroot leaves of $B^*$ to the interval between $\ell$ and $p$ (interval $[j - 1, j]$), drawing the edges to $i = \odot(B^*)$ below the spine. There are this many leaves in $B^*$ even if $B^*$ is dangling because $|B^*| = x - i + 1 \geq |S| + 1$. Denote the resulting blue forest by $B'$. Then $B'[j - |S| + 1, j]$ consists of isolated vertices (leaves from $B^*$ and $p$) and $\{i, j\} \notin E(B')$. Therefore we can put $r = i$, explicitly embed $S$ onto $[j, j - |S| + 1]$, drawing the edge $\{i, j\}$ below the spine, and all other edges above the spine; and explicitly embed $Q$ onto $[i + 1, j - |S|]$ with $q = \ell = j - |S|$ (Figure 30d).

**Proposition 23.** If $R^-$ is a star, $\{i, j\} \notin E(B)$ and $B(j)$ is a central star with $2 \leq |B(j)| \leq |S| + 1$, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$. □

**Proof.** By Proposition 22 we may suppose that $B[i, i + |S|]$ is not a star. As $\{i, j\} \notin E(B)$, we have $B(i) \neq B(j)$. We rearrange $B(j)$ by moving all its $k := |B(j)| - 1$ leaves to the interval between $i + |S| - k$ and $i + |S| - k + 1$. Note that $1 \leq k \leq |S|$ and so $i \leq i + |S| - k \leq i + |S| - 1$. The edges from $j$ to those leaves are drawn as biarcs that leave $j$ below the spine and cross the spine immediately after (what was) $i + |S| - k$ (before this change). Denote the resulting blue forest by $B'$ (Figure 31a). Then $j$ is isolated in $B'[i + |S| + 1, j]$ and we can explicitly embed $Q$ onto $[i + |S| + 1, j]$ with $q = j$. Finally, we complete the packing as follows: If $S^+$ is a star, then it is dangling and we put $r = i$ and $\odot(S^+) = i + |S|$, which is isolated in $B'[i, i + |S|]$; otherwise, $S^+$ is not a star and we recursively embed it onto $[i, i + |S|]$. □

**Proposition 24.** If $R^-$ is a star, $S^+$ is not a star, and there is some $x \in [i + |S| + 1, j]$ with $B(x) = \{x\}$ (that is, $x$ is isolated in $B$), then $R$ and $B$ admit an ordered plane packing onto $[i, j]$. □

**Proof.** Clearly $B(x) = \{x\}$ implies $\{i, j\} \notin E(B)$. Hence by Proposition 22 we may suppose that $B[i, i + |S|]$ is not a star. We explicitly embed $Q$ onto $[i + |S| + 1, j]$ with $q = x$ and
In the terminology of the blue-star embedding we have 
\[ B \neq \phi \] is not a star and so either 
\[ B \cup \{x\} \in \mathcal{E}(B) \] does not conflict with \( i \) or \( j \).

Lemma 2 and the upper bound uses Proposition 22. Due to the degree-conflict we have 
\[ |\mathcal{E}(B)| = \deg \mathcal{S}(s) + (x - i) \geq |S| - 1. \]

Therefore due to \( \tau(B(j)) \neq j \) we know that \( B[j - |S|, j] \) consists of isolated vertices, which proves the claim.

If \( S \) is a central star, then \( S^+ \) is a (dangling) star. As we assume that \( S^+ \) is not a star, it follows that if \( S \) is a star, then \( S \) is dangling. Denote \( s := \tau(S) \). We distinguish four cases.

Case 1 \( S \) is not a star and \( S \) is in degree-conflict with \([i, i + |S| - 1]\). Then \( B[i, i + |S| - 1], [\tau(B(i)) \neq j \) we know that \( B[j - |S|, j] \) consists of isolated vertices, which proves the claim.

If \( S \) is a central star, then \( S^+ \) is a (dangling) star. As we assume that \( S^+ \) is not a star, it follows that if \( S \) is a star, then \( S \) is dangling. Denote \( s := \tau(S) \). We distinguish four cases.

Case 1 \( S \) is not a star and \( S \) is in degree-conflict with \([i, i + |S| - 1]\). Then \( B[i, i + |S| - 1], [\tau(B(i)) \neq j \) we know that \( B[j - |S|, j] \) consists of isolated vertices, which proves the claim.

If \( S \) is a central star, then \( S^+ \) is a (dangling) star. As we assume that \( S^+ \) is not a star, it follows that if \( S \) is a star, then \( S \) is dangling. Denote \( s := \tau(S) \). We distinguish four cases.

Case 1 \( S \) is not a star and \( S \) is in degree-conflict with \([i, i + |S| - 1]\). Then \( B[i, i + |S| - 1] \) is a central star with \( i + 2 \leq x \leq i + |S| - 1 \), where the lower bound is by Lemma 2 and the upper bound uses Proposition 22. Due to the degree-conflict we have \( \deg \mathcal{S}(s) + (x - i) \geq |S| \) and due to the upper bound on \( x \) we have \( |B^*| = x - i + 1 \leq |S| \).

Possibly flip \( B(i) \) so that its root is at \( i \). By Proposition 22 we can assume that \( B[i, i + |S|] \) is not a star and so either \( B[i, x] = B(i) \) or \( i \) has one neighbor in \( B[i + |S|, j - 1] \): its parent \( p_B(i) \).

We put \( r = j \) and use a blue-star embedding for \( S \) starting from \( \sigma = i \). Selecting vertices \( \varphi = (x + 1, \ldots) \) from left to right but skipping over \( p_B(i) \) (if it exists).

In the terminology of the blue-star embedding we have \( A = S, B^* = B[i, x] \), and either \( B^+ = B^* \) or \( B^+ = B^* \cup \{\tau\} \), where \( \tau = p_B(i) \). Let us check the conditions for the blue-star embedding.

(BS1) holds because \( \{i, j\} \notin \mathcal{E}(B) \). Regarding the first inequality of (BS2) note that
\[ |S| \leq \deg \mathcal{S}(s) + (x - i) = \deg \mathcal{S}(s) + |B^*| - 1, \]

where the inequality is due to the degree-conflict. There is a slack of at least one compared to the requirement in (BS2), which means that at least one leaf of \( B^* \) is not used by the
blue-star embedding. This leaf appears as an isolated blue vertex \(i + |S| - 1\) in the resulting interval \([i + |S| - 1, j]\) of remaining vertices. Therefore, we can conclude the packing by putting \(q = i + |S| - 1\) and in this way explicitly embed \(R^-\) onto \([i + |S| - 1, j]\).

Regarding the second inequality of (BS2) note that

\[
\deg_S(s) + |B^+| \leq (|S| - 1) + (|B^*| + 1) \leq 2|S| \leq |R| - 1.
\]

(BS3) holds due to the way we select vertices in \(\varphi\) from left to right. Regarding (BS4) note that \(S\) is not a star by assumption and if \(B \setminus (B^* \cup \varphi)\) is not an interval, then \(\varphi\) skips over \(\tau\) and therefore we can set \(v_1 = \tau + 1\) without loss of generality, where \(\{\tau, \tau + 1\} \notin E(B)\) due to \(\{i, \tau\} \in E(B)\) and 1SR for \(\tau\). Therefore, by Proposition 9 we can apply the blue-star embedding and then explicitly embed \(R^-\) onto the resulting interval as described (Figure 32a).

\[
\begin{align*}
\text{(a) Case 1} & \quad \text{(b) Case 2.1} & \quad \text{(c) Case 2.2} & \quad \text{(d) Case 2.4} \\
n & \quad n & \quad n & \quad n
\end{align*}
\]

Figure 32: Packings in the proof of Proposition 25 (1/3).

**Case 2** \(i + |S|\) has no edge to the right in \(B\) (that is, either \(i + |S|\) is isolated in \(B\) or it has an edge to the left in \(B\), and so by 1SR no edge to the right). Denote \(H := B[i, i + |S|] \cap (i + |S|)\). We distinguish seven cases.

**Case 2.1** \(S\) is not a star. Then we put \(r = j\), explicitly embed \(Q\) onto \([i + |S|, j - 1]\) with \(q = i + |S|\), and recursively embed \(S\) onto \([i, i + |S| - 1]\) (Figure 32b). As \(\{i, j\} \notin E(B)\), there is no edge-conflict and since we are not in Case 1 there is no degree-conflict for the recursive embedding.

**Case 2.2** \(S\) is a star and \(H\) is not a central star. Note that by definition no vertex of \(H\) other than \(\uparrow(H)\) has an edge to vertices outside of \(H\). We apply a leaf-isolation shuffle on \(H\) to put a leaf at \(i + |S| - 1\) and its parent at \(i + |S|\). Denote the resulting blue forest by \(B'\) and the resulting embedding of \(H\) by \(H'\). By Proposition 11 we have \(\uparrow(H') \neq i + |S|\) and so \(\{i + |S|, j\} \notin E(B')\). Therefore we can put \(r = j\) and explicitly embed \(Q\) onto \([i + |S|, j - 1]\) with \(q = i + |S|\) and \(S\) onto \([i, i + |S| - 1]\) with \(s = i\) and \(\odot(S) = i + |S| - 1\) (Figure 32c).

**Case 2.3** \(S\) is a star and \(H\) is a central star on \(\geq 2\) vertices with \(\{\uparrow(H), j\} \notin E(B)\). Then we can use the same packing as in Case 2.2, however here \(\uparrow(H') = i + |S|\) and \(\{i + |S|, j\} \notin E(B')\) by assumption.

**Case 2.4** \(S\) is a star, \(H\) is a central star on \(\geq 2\) vertices with \(\{\uparrow(H), j\} \in E(B)\), and \(i + |S| + 1\) has no edge to the right in \(B\). We explicitly embed \(Q\) onto \([i + |S| + 1, j]\) with \(q = i + |S| + 1\) and then recursively embed \(S^+\) onto \([i, i + |S|]\) (Figure 32d). The embedding of \(Q\) is fine because \(i + |S| + 1\) is isolated in \(B[i, i + |S| + 1, j]\) by 1SR. Let us argue why the recursive
embedding for $S^+$ works as described. For once, by Proposition 22 we know that $B[i, i + |S|]$ is not a star, in particular, $i \notin H$. As $\deg_S r = 1$, there is no degree-conflict for embedding $S^+$. Finally, there is no edge-conflict for embedding $S^+$ on $[i, i + |S|]$ due to (11) and because $i + |S| + 1$ lies below the edge $\{\uparrow(H), j\} \in E(B)$ and, therefore, $\{i, i + |S| + 1\} \notin E(B)$.

**Case 2.5** $S$ is a star, $H$ is a central star on $\geq 2$ vertices with $\{\uparrow(H), j\} \in E(B)$, and $i + |S| + 1$ has an edge to the right in $B$. Denote $x := \uparrow(H)$ (Figure 33a). By Proposition 22 we can assume $i < x$ and by 1SR for $x$ we have $\{i, x\} \notin E(B)$. By LSFR for $x$ and given that by 1SR for $i + |S| + 1$ we have $\{x, i + |S| + 1\} \notin E(B)$, it follows that $j$ is the only neighbor (in $B$) of $x$ outside of $[x + 1, i + |S|]$ and so $\deg_B(x) = [x + 1, i + |S|] + 1 \leq |S|$. Therefore we can put $r = i$ and explicitly embed $Q$ using all vertices of $[i + 1, x]$ with $q = x$ and putting the remaining leaves onto a prefix of $[i + |S| + 1, j - 1]$. Let $y$ be the rightmost vertex of $[i + |S| + 1, j - 1]$ that is used by the embedding of $Q$. Next we move all leaves of $H$ to the region between $y$ and $y + 1$ on the spine, drawing the edges to $x$ as biarcs that leave $x$ below the spine and cross the spine between $y$ and $y + 1$ (Figure 33b). Then we complete the packing by explicitly embedding $S$ onto the remaining vertices with $s = j$ and $\odot(S)$ put onto a leaf of $H$. (There is at least one such leaf because $|H| \geq 2$.)

![Figure 33: Packings in the proof of Proposition 25 (2/3).](image)

**Case 2.6** $S$ is a star and both $i$ and $i + |S|$ are isolated in $B$. We put $r = j$, explicitly embed $Q$ onto $[i + |S|, j - 1]$ with $q = i + |S|$, and explicitly embed $S$ onto $[i, i + |S| - 1]$ with $s = i + |S| - 1$ and $\odot(S) = i$ (Figure 33c).

**Case 2.7** $S$ is a star, $i + |S|$ is isolated in $B$, and $i$ is not isolated in $B$. We apply a leaf-isolation shuffle to $B(\downarrow i)$ to put a leaf at $i + 1$ and its parent at $i$. Then we move the leaf from $i + 1$ to the position immediately to the right of $i + |S|$, drawing the edge to $i$ below the spine. Denote the resulting blue forest by $B'$. Then $i + |S| - 1$ is isolated in $B'$ (this vertex corresponds to $i + |S|$ in $B$) and $i + |S|$ is isolated in $B'(i + |S|, j)$ because its parent and only neighbor resides at $i$. We put $r = j$, explicitly embed $Q$ onto $[i + |S|, j - 1]$ with $q = i + |S|$, and explicitly embed $S$ onto $[i, i + |S| - 1]$ with $s = i$ and $\odot(S) = i + |S| - 1$ (Figure 33d).

**Case 3** $i + |S|$ has an edge to the right in $B$ (and so by 1SR no edge to the left) and $B^- := B[i + |S|, j][i + |S|]$ is a central star where $y := \uparrow(B^-)$ has a neighbor in $B[i, i + |S| - 1]$. By 1SR it follows that $y \neq i + |S|$, that is, $|B^-| \geq 2$. If $y = \uparrow(B(i + |S|))$, then we flip $B(i + |S|)$ (which is a central star $\neq B(i)$ by Proposition 22 and $\neq B(j)$ because $\uparrow(B(j)) \neq j$) and proceed as in Case 2. Otherwise, $p := p_B(y) \in [i, i + |S| - 1]$. Note that if $y$ has children in $[p + 1, i + |S| - 1]$, then by LSFR and due to the leaf $i + |S|$ of $B^-$ these children are leaves
of $B$, that is, $t_B(y)$ is a star. By Proposition 22 we may assume that $B[i, y]$ is not a star. We consider two subcases.

**Case 3.1** $|B^-| = 3$ (that is, $y \geq i + |S| + 2$). We first flip $B[p, y]$ and then exchange the order of the subtrees of $p$ to the left of $p$ if necessary to ensure that $t_B(y)$ is the subtree of $p$ that is embedded closest to $p$ on the left. Then in the resulting blue forest $B'$ the vertex of $B$ that was at $y$ before is at $y' = y - |B_B(y)| \leq y - 3$ and $p$ is at $p':= y$ in $B'$. Note that $B'$ may violate 1SR and LSFR at $p'$, but only there. In particular, $B'[i, i + |S|]$ satisfies (I2). Also note that due to $|t_B(y)| \geq |B^-| \geq 3$ we have at least two leaves of $t_B(y)$, at $i + |S|$ and $i + |S| + 1$, both of which are isolated in $B'[i + |S|, j]$ (Figure 34a). We explicitly embed $Q$ onto $[i + |S| + 1, j]$ with $q = i + |S| + 1$, and then recursively embed $S^+$ onto $[i, i + |S|]$. As $B[i, y]$ is not a star, we have $i < y'$ in $B'$ and so $(i, i + |S| + 1) \notin E(B')$. Therefore, there is no edge-conflict for the recursive embedding of $S^+$. Given that $\deg_{S^+}(r) = 1$, there is no degree-conflict for embedding $S^+$, either.

![Figure 34: Packings in the proof of Proposition 25 (3/3).](image)

**Case 3.2** $|B^-| = 2$ (that is, $y = i + |S| + 1$ and $i < p$). Then by 1SR $y = i + |S| + 1$ is isolated in $B[i + |S| + 1, j]$. We explicitly embed $Q$ onto $[i + |S| + 1, j]$ with $q = i + |S| + 1$ and then recursively embed $S^+$ onto $[i, i + |S|]$ (Figure 34b). As $i < p$, we have $(i, i + |S| + 1) \notin E(B)$ and so there is no edge-conflict for the recursive embedding of $S^+$. Given that $\deg_{S^+}(r) = 1$, there is no degree-conflict for embedding $S^+$, either.

**Case 3.3** $|B^-| = 2$ (that is, $y = i + |S| + 1$ and $i = p$. As $B[i, y]$ is not a star, there is an $x \in [i + 1, i + |S| - 1]$ so that $(x, y) \notin E(B)$. Select $x$ to be maximal. Then $B[x + 1, i + |S| + 1]$ is a star rooted at $y = i + |S| + 1$. We put $r = j$ and embed $Q$ explicitly onto $x \cup [i + |S| + 1, j]$ with $q = i + |S| + 1$. Then take all vertices in $[x + 1, i + |S|]$, which are leaves of $B[x + 1, i + |S| + 1]$ (there is at least one such leaf) and move them immediately to the left of $x$, draw the edges to $i + |S| + 1$ as biars that leave $i + |S| + 1$ below the spine and cross the spine immediately to the left of $x$. Denote the resulting blue forest by $B'$. If $S$ is not a star, then we recursively embed $S$ onto $[i + |S| - 1, i]$ (Figure 34c). As $i + |S| - 1$ is locally isolated in $B'[i, i + |S| - 1]$ and $(i, i + |S| - 1, j) \notin E(B')$, there is no conflict. Otherwise, $S$ is a star and we explicitly embed it with $s = i$ and $\circ(S) = i + |S| - 1$.

**Case 4** $i + |S|$ has an edge to the right in $B$ (and so by 1SR no edge to the left) and either (i) $B^- := B[i + |S|, j](i + |S|)$ is not a central star or (ii) $\uparrow(B^-)$ has no neighbor in $B[i, i + |S| - 1]$. We use a leaf-isolation shuffle on $B^-$ to put a leaf of $B^-$ at $i + |S| + 1$ and its parent at $i + |S|$. Denote the blue forest after the shuffle by $B'$. Then $i + |S| + 1$ is isolated in $B'[i + |S| + 1, j]$ and $i + |S|$ is isolated in $B'[i, i + |S|]$; In Case (ii) this is obvious
and in Case (i) it follows because in $B'$ we have $i + |S| \neq \uparrow(B^-)$ by Proposition 11. We explicitly embed $Q$ onto $[i + |S| + 1, j]$ with $q = i + |S| + 1$, and then recursively embed $S^+$ onto $[i, i + |S|]$ (Figure 34d). By assumption, neither $B[i, i + |S|]$ nor $S^+$ is a star and there is no edge-conflict because $i + |S|$ is the only neighbor (in $B$) of $i + |S| + 1$ in $[i, i + |S|]$. □

**Proposition 26.** If $R^-$ and $S^+$ are both stars and $\{i, j\} \notin E(B)$, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$.

**Proof.** As $S^+$ is a star, $S$ is a central star. Denote $s := \uparrow(S)$. Select $x \in [i, j]$ to be maximal so that $B(x) \neq \{x\}$, or $x := i - 1$, if there is no such vertex. We distinguish six cases.

**Case 1** $x \leq j - |S| - 1$. We put $r = i$, and explicitly embed $S$ onto $[j, j - |S| + 1]$ with $s = j$ and $Q$ onto $[j - |S|, i + 1]$ with $q = j - |S|$ (Figure 35a).

**Case 2** $x \geq j - |S| + 1$ and $x \in B(i)$. Then $x < j$ because $\{i, j\} \notin E(B)$ by assumption. Possibly flip $B(i)$ so that $x = \uparrow(B(i))$. Note that a flip does not affect the invariants. Without loss of generality we may suppose that all vertices in $[x, j]$ are in edge-conflict with $r$. By Proposition 22 we know that $B(i)$ is not a star (noting that $j - i \geq 2|S|$ and so $i + |S| \leq j - |S|$). We distinguish four subcases.

**Case 2.1** $x = j - |S|$. As $B(i)$ is not a star, there is a subtree of $x$ (in $B$) that is not a single vertex. Possibly change the order of subtrees below $x$ in $B(i)$ so that $i$ is not a leaf. Then flip $B[i, x - 1](i)$ and call the resulting blue forest $B'$. Note that $\{i, x\} \notin E(B')$. We put $r = i$ and explicitly embed $S$ onto $[x, j - 1]$ with $s = x$, drawing the edge $\{i, x\}$ above the spine and all other edges below the spine. Then we explicitly embed $Q$ onto $[i + 1, x - 1] \cup \{j\}$ with $q = j$ (Figure 35b).

![Figure 35: Packings in the proof of Proposition 26 (1/4).](image)

**Case 2.2** $x \geq j - |S| + 1$ and $\deg_B(x) = 1$. Flip $B(i)$ so that its root is at $i$. We put $r = j - |S|$, which is in $B(i) = B[i, x]$ and therefore not in edge-conflict. Then we explicitly embed $S$ onto $[j - |S| + 1, j]$ with $s = j$ and $Q$ onto $[i, j - |S| - 1]$ with $q = i$ (Figure 35c).

**Case 2.3** $x \geq j - |S| + 1$, $\deg_B(x) \geq 2$, and $i$ is a leaf in $B(i)$. We arrange the subtrees of $x$ in $B(i)$ so that the second subtree from the left is a leaf $\ell$. Denote the resulting blue forest by $B'$, where $\ell \in [i + 1, x - 1]$. We put $r = i$ (not $\uparrow(B'(i)) = x$ and therefore not in edge-conflict with $r$) and explicitly embed $S$ with $s = j$ and putting the leaves of $S$ so that (i) all vertices in $[x, j - 1]$ are used (which is possible because $j - x \leq |S| - 1$ by assumption) and (ii) possible additional leaves are put in $[i + 1, x - 1] \setminus \{\ell\}$ so that the remaining unused
vertices form an interval $I'$ with $\ell \in I'$. Then we explicitly embed $Q$ onto $I'$ with $q = \ell$, drawing the edge $\{i, q\}$ above the spine and all other edges below the spine (Figure 35d).

**Case 2.4** $x \geq j - |S| + 1$, $\deg_B(x) \geq 2$ and $i$ is not a leaf in $B(i)$. Set $y$ so that $B[i, x-1]i) = B[i, y]$. Flip $B'[i, y]$ and denote the resulting blue forest by $B'$. Denote $\hat{B} := B'[i, y]$ and let $B^{-} := B'(i) \setminus \hat{B} = B'[y+1, x] = B[y+1, x]$, where $y + 1 < x$ (because $\deg_B(x) \geq 2$). We put $r = i$ and explicitly embed $S$ so that $s = x$ and the leaves of $S$ use all of $J := [x+1, j-1]$ along with some non-neighbors of $x$ in $B^-$. The edges from $s$ to $[x+1, j-1]$ are drawn below the spine. An edge from $s$ to a vertex $t \in [y+1, x-1]$ is drawn as follows: If there is no neighbor (in $B$) of $x$ in $[t+1, x-1]$, then the edge is drawn as an arc below the spine; otherwise, the edge is drawn as a biarc that leaves $x$ above the spine and crosses the spine between $z-1$ and $z$, where $z$ is the leftmost neighbor of $x$ in $B[t+1, x-1]$. In this way the neighbors of $x$ remain accessible from the halfplane below the spine, to be used by the embedding of $Q$ that follows. We complete the packing by explicitly embedding $Q$ onto the remaining vertices, with $q = j$ (Figure 36a).

It remains to argue why there are enough vertices in $B^-$ that are not neighbors of $x$ to embed $S$ as described. Eligible to host a leaf of $S$ are all vertices in $J$ and all vertices in $B^-$ other than $x$ and its neighbors. Denote $d := \deg_B(x) = \deg_B(x) + 1$. Therefore we want to show that

$$|B^-| - d + |J| \geq |S| - 1.$$  \hfill (10)

Let us prove (10). As $i$ is not a leaf of $B(i)$, by LSFR for $x$ we know that every subtree of $x$ in $B(i)$ has at least two vertices. Therefore, if $|S| \leq d - 1$, then $|B^-| \geq 2(d - 1) + 1 = 2d - 1 \geq |S| + d$, which proves (10).

Otherwise, $|S| \geq d$. By LSFR for $x$ we have $|B(i)| = |\hat{B}| \leq (|B(i)| - 1)/d$ and so

$$|B^-| \geq \frac{d - 1}{d} |B(i)| + \frac{1}{d} = \frac{d - 1}{d}(|B| - |J| - 1) + \frac{1}{d} \geq \frac{d - 1}{d}(|B| - 1) - |J| + \frac{1}{d}.$$  

Together with $|B| \geq 2|S| + 1$ we obtain

$$|B^-| + |J| \geq \frac{2(d - 1)}{d} |S| + \frac{1}{d} = |S| + \frac{d - 2}{d} |S| + \frac{1}{d} \geq |S| + (d - 2) + \frac{1}{d} > |S| + d - 2,$$

proving (10), given that $|S|$ and $d$ are integral.

![Figure 36: Packings in the proof of Proposition 26 (2/4).](image)

**Case 3** $x \notin B(i)$ and $x = j - |S|$. We distinguish two subcases.
Case 3.1 \( B(i) = \{i\} \). Possibly flip \( B(x) \) so that \( x \neq \uparrow(B(x)) \). We put \( r = x \), explicitly embed \( S \) onto \([j, j - |S| + 1]\) with \( s = j \), and explicitly embed \( Q \) onto \([i, x - 1]\) with \( q = i \) (Figure 36b).

Case 3.2 \( B(i) \neq \{i\} \). Apply a leaf-isolation shuffle on \( B(i) \) to put a leaf at \( i + 1 \) and its parent at \( i \). Then move the leaf at \( i + 1 \) all the way to \( j \), shifting all of \([i + 2, j]\) by one position to the left, and drawing the edge \( \{i, j\} \) below the spine. Denote the resulting blue forest by \( B' \). By Proposition 11 we have \( \uparrow(B'(i)) = i \) if and only if \( B(i) \) is a central star. By (11) we conclude that \( i \in B' \) is not in edge-conflict with \( r \). Therefore we may put \( r = i \), explicitly embed \( S \) onto \([j, j - |S| + 1]\) with \( s = j - 1 \), and explicitly embed \( Q \) onto \([i + 1, j - |S|]\) with \( q = j - |S| \) (Figure 36c).

Case 4 \( x \notin B(i) \) and \( x = j - |S| + 1 \). Apply a leaf-isolation shuffle on \( B(x) \) to put a leaf at \( x - 1 \) and its parent at \( x \). We put \( r = i \), explicitly embed \( S \) onto \([j, j - |S| + 1]\) with \( s = j \), and explicitly embed \( Q \) onto \([i + 1, j - |S|]\) with \( q = j - |S| \) (Figure 36d).

Case 5 \( x \notin B(i), x \geq j - |S| + 2, \) and \( \deg_y(x) \geq |Q| \). Possibly flip \( B(x) \) so that \( x = \uparrow(B(x)) \). As \( x \notin B(i) \), a possible flip does not change \( B(i) \). Define \( y \) so that \( B(x) = B[y, x] \). Then \( y \) is a leaf of \( B(x) \) because otherwise by LSFR for \( x \) all subtrees have \( \geq 2 \) vertices and then \( |B(x)| \geq 2|Q| + 1 \), in contradiction to \( |B| = |S| + |Q| + 1 \leq 2|Q| + 1 \). We distinguish two subcases.

Case 5.1 \( x \in [j - |S| + 2, j - 1] \). We put \( r = i \), explicitly embed \( S \) onto \([j, j - |S| + 1]\) with \( s = j \), and then explicitly embed \( Q \) onto \([i + 1, j - |S|]\) with \( q = y \) (Figure 37a).

![Figure 37: Packings in the proof of Proposition 26 (3/4).](image)

Case 5.2 \( x = j \). If \( B(y - 1) = \{y - 1\} \), then let \( z := y - 1 \). Otherwise, possibly flip \( B(y - 1) \) so that \( z := \uparrow(B(y - 1)) \neq y - 1 \). Possibly exchange the order of subtrees below \( x \) in \( B[y, x] \) so that the second subtree of \( x \) from the right is a leaf \( \hat{y} \). (Note that \( |Q| \geq |S| \geq 2 \) and so there are at least two subtrees below \( x \).) We put \( r = x - 1 \) and explicitly embed \( Q \) with \( q = z \) and so that the leaves of \( Q \) use all vertices in \([i, z - 1] \cup \{x\}\), and vertices needed in addition are taken from \([y, x - 1]\). An edge from \( z \) to \([i, z - 1]\) is drawn below the spine, the edge \( \{z, x\} \) is drawn above the spine, and an edge from \( z \) to a vertex \( t \in [y, x - 1] \) is drawn as a biarc that leaves \( z \) above the spine and crosses the spine between \( y - 1 \) and \( y \). (In case \( z = y - 1 \) the edges from \( z \) to \([y, x - 1]\) can be drawn as arcs below the spine.) As there are \( \geq |Q| - 2 \) neighbors in \( B \) of \( x \) to the left of \( \hat{y} \) and both \( i \) and \( x \) are also used for the embedding of \( Q \), no leaf of \( Q \) is put at \( \hat{y} \). Therefore we can complete the packing by explicitly embedding \( S \) with \( s = \hat{y} \) on the remaining vertices (Figure 37b).
Case 6.1 $z < y$. Then all vertices in $[y + 1, x - 1]$ are used by leaves of $S$ and so $y$ is isolated in $B$ among the remaining vertices. Therefore, we can complete the packing by explicitly embedding $Q$ onto the remaining vertices with $q = y$ (Figure 37c).

Case 6.2 $z > y$ and $z - 1$ has an edge to the right in $B$. Then by 1SR $z - 1$ is isolated in $B[i, z - 1]$. Therefore, we can complete the packing by explicitly embedding $Q$ onto $[i + 1, z - 1]$ with $q = z - 1$ (Figure 38a).

Case 6.3 $z > y$, $z - 1$ has an edge to the left in $B$, and $\ell - 1$ is isolated in $B$. We change the embedding by putting $r = z - 1$, and complete the packing by explicitly embedding $Q$ onto $[i, z - 2]$ with $q = \ell - 1$ (Figure 38b).

Case 6.4 $z > y$, $z - 1$ has an edge to the left in $B$, and $\ell - 1$ is not isolated in $B$. Then by 1SR we have $z - 1 > y$. Also, by 1SR and by the definition of $\ell$ we know that $\ell - 1 \notin B(x)$ and hence that $\ell - 1$ has an edge to the left in $B$. We apply a leaf-isolation shuffle on $B(\ell - 1)$ to put a leaf at $\ell - 2$ and its parent at $\ell - 1$. We change the embedding by putting $r = z$ (noting that $\{z, x\} \notin E(B)$ by choice of $y$) and moving the leaf of $S$ that was at $z$ to $\ell - 1$ instead, drawing the edge $\{\ell - 1, x\}$ above the spine. Then we complete the packing by explicitly embedding $Q$ onto $[i, z - 1] \setminus \{\ell - 1\}$ with $q = \ell - 2$ (Figure 38c).

Proposition 27. If $R^-$ is a star and $\{i, j\} \in E(B)$, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$.

Proof. The presence of the edge $\{i, j\} \in E(B)$ implies that $B$ is a tree. In this case, we can exchange the roles of $B$ and $R$: Discard the initial embedding of $B$ and embed $R$ onto $[j, i]$. 

Figure 38: Packings in the proof of Proposition 26 (4/4).
using a canonical embedding instead (that is, putting \(r = j\); see Figure 39a). Then re-embed \(B\) onto \([i, j]\), respecting the existing embedding of \(R\).

Let us first argue that \((I1)\)–\((I3)\) also hold for this “reversed” embedding, as described. For \((I2)\) this is a consequence of using a canonical embedding. Regarding \((I1)\) it suffices to note that \(R\) is not a star and \((I3)\) is guaranteed by putting \(\uparrow(R)\) at \(j\).

Then \(B\) decomposes into a (smallest) subtree \(C\) rooted at \(i\) (the original \(S\)), with \(|C| \geq 2\), and a dangling star \(B^-\) rooted at \(j\) (the original \(R^-\)), with \(|B^-| \geq (|B| + 1)/2\). Set \(x\) so that \(C = B[i, x]\) and \(B^- = B[x + 1, j]\) (Figure 39b). Similarly, \(R\) (the original \(B\)) decomposes into a smallest subtree \(S\) and \(R^- = R \setminus S\), with \(|R^-| \geq (|B| + 1)/2\). Set \(z\) so that \(|R^-| = |[i, z]|\) (Figure 39c) and note that \(z - x = |[i, z]| - |[i, x]| = |R^-| - |C| \geq 1\). Denote \(r = \uparrow(R) = \uparrow(R^-)\). It remains to embed \(R\) onto \([i, j]\).

![Figure 39: If \(B\) is a tree, then we exchange the roles of \(R\) and \(B\) (Proposition 27).](image)

If \(\deg_R(r) = 1\), then Lemma 19 completes the proof. If \(|S| = 1\), then Lemma 20 completes the proof. Hence we may assume \(\deg_R(r) \geq 2\) and \(|S| \geq 2\). Together with \(x < z\) it follows that \(B[i, z]\) consists of two components and \(B[z + 1, j]\) consists of at least two isolated vertices. In particular, neither \(B[i, z]\) nor \(B[z + 1, j]\) is a star. We distinguish two cases.

**Case 1** \(R^-\) is not a star. We explicitly embed \(S\) on the isolated vertices in \([z + 1, j]\), with \(\uparrow(S) = z + 1\). Then \(\{i, \uparrow(S)\} \notin E(B)\) and so \(i\) is not in edge-conflict with \(r\). Finally, we recursively embed \(R^-\) onto \([i, z]\) (Figure 39c). As neither \(R^-\) nor \(B[i, z]\) is a star and there is no edge-conflict between \([i, z]\) and \(R^-\), it remains to show that \([i, z]\) is not in degree-conflict with \(R^-\).

So suppose for the sake of a contradiction that \([i, z]\) and \(R^-\) are in degree-conflict. Then \(C\) is a central star rooted at \(i\) and \(\deg_C(i) + \deg_{R^-}(r) \geq |R^-|\). It follows that

\[
\begin{align*}
(|R| - |B^-| - 1) + \frac{|R^-| - 1}{|S|} & \geq |R^-| \\
\frac{|R| - 3}{2} + \frac{|R| - |S| - 1}{|S|} & \geq |R| - |S| \\
\frac{|R| - |S| - 1}{|S|} & \geq \frac{|R|}{2} - |S| + \frac{3}{2} \\
|R| - |S| - 1 & \geq \frac{|R| \cdot |S|}{2} - |S|^2 + \frac{3|S|}{2}
\end{align*}
\]
\[
|S|^2 - \frac{|R| \cdot |S|}{2} + |R| - \frac{5|S|}{2} - 1 \geq 0 \\
\left(|S| - \frac{|R|}{2}\right) \left(|S| - \frac{5}{2}\right) - \frac{|R|}{4} - 1 \geq 0.
\]

Since \((|S| - |R|/2)\) is negative, the left hand side can be nonnegative only if \(|S| < 5/2\). However, since \(|S| \geq 2\) we only need to consider the case \(|S| = 2\), for which the inequality is easily shown to be false.

**Case 2** \(R^-\) is a star. We use a leaf-isolation shuffle on \(C\) to put a leaf \(\ell\) at \(i + 1\) and its parent \(p\) at \(i\). Next we flip \(B^-\) to put its center at \(j\). Then we take \(|S| - 1\) leaves of \(B^-\) (as \(|B^-| \geq |S| + 1\), there are this many leaves in \(B^-\)) and move them all the way to the left of \(i\), shifting all other vertices to the right and drawing the edges to \(j\) below the spine. Denote the resulting blue forest by \(B'\) (Figure 39d). Note that \(\ell = i + |S|\) and \(p = i + |S| - 1\) in \(B'\) and that \(B'|i, i + |S| - 1\) consists of isolated vertices (\(p\) plus \(|S| - 1\) leaves of \(B^-\)). We explicitly embed \(R^-\) onto \([j, i + |S|]\) with \(r = j\) and \(\circ(R^-) = i + |S|\). Then we explicitly embed \(S\) onto the isolated vertices in \([i + |S| - 1, i]\) with \(\uparrow(S) = i + |S| - 1\), drawing the edge \([i + |S| - 1, j]\) below the spine, and drawing all other edges above the spine. \(\square\)

Proposition 25 (\(R^-\) is a star, \(S^+\) is not a star, and \(\{i, j\} \notin E(B)\)), Proposition 26 (\(R^-\) is a star, \(S^+\) is a star, and \(\{i, j\} \notin E(B)\)), and Proposition 27 (\(R^-\) is a star and \(\{i, j\} \in E(B)\)) together prove the following.

**Lemma 28.** If \(R^-\) is a star, then \(R\) and \(B\) admit an ordered plane packing onto \([i, j]\).

### 11 Embedding the red tree: a small blue star

In this section, we consider the case that \(B[j - |S| + 1, j]\) is a star, but \(B[i, j - |S|]\) is not a star. The size of the star is \(|S|\). Due to Lemma 20, we may assume \(|S| \geq 2\). Note, however, that \(B[j - |S| + 1, j]\) may be part of a larger star within \(B\). Let \(\xi \leq j - |S| + 1\) be minimal so that \(B^{**} := B[\xi, j]\) is a star. Note that the tree \(B(j)\) may be larger than \(B^{**}\). Clearly, we have \(|B^{**}| \geq |B[j - |S| + 1, j]| = |S| \geq 2\). Due to ISR, the center and the root of \(B^{**}\) are each located at either \(j\) or \(\xi\). We distinguish two cases: either \(|S| < |B^{**}|\) (Proposition 29) or \(|S| = |B^{**}|\) (Proposition 30, Proposition 31, and Proposition 32). These cases are tackled below.

**Proposition 29.** If \(B[j - |S| + 1, j]\) is a star and \(2 \leq |S| < |B^{**}|\), then \(R\) and \(B\) admit an ordered plane packing onto \([i, j]\).

**Proof.** By Lemma 21, we may assume that \(B[i, j - |S|]\) is not a star. By Lemma 28, we may assume that \(R^-\) is not a star. As \(|S| < |B^{**}|\), we have \(\circ(B^{**}) = j\) and \(\uparrow(B^{**}) \in \{\xi, j\}\). We start by rearranging the tree \(B^{**}\) such that its center moves to \(j - |S|\); see Figure 40. If \(B^{**}\) is rooted at its center, then the root automatically moves to \(j = |S|\), as well. Otherwise \(B^{**}\) is rooted at a leaf, which is the leftmost vertex of \(B^{**}\) and the root of the entire tree \(B(j)\) due to ISR, and then we move the root of \(B^{**}\) to \(j\). In both cases, \(B[j - |S| + 1, j]\)
consists of \(|S|\) isolated vertices, and \(B[i, j − |S|]\) continues to fulfil invariants \((I2)\) and \((I4)\). We distinguish two cases.

\[
\begin{array}{cccc}
  \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} \\
  \includegraphics[width=0.2\textwidth]{case1a.png} & \includegraphics[width=0.2\textwidth]{case1b.png} & \includegraphics[width=0.2\textwidth]{case1c.png} & \includegraphics[width=0.2\textwidth]{case1d.png}
\end{array}
\]

Figure 40: In the proof of Proposition 29 we move \(\odot(B^{∗∗})\) from \(j\) to \(j − |S|\): when \(\uparrow(B^{∗∗}) = \odot(B^{∗∗})\) \((\text{a–b})\), and when \(\uparrow(B^{∗∗}) \neq \odot(B^{∗∗})\) \((\text{c–d})\).

**Case 1** \(B[i, j − |S|][i]\) is not a central star of size at least \(|R^−| − \deg_{R^−}(r) + 1\). In this case, we embed \(S\) explicitly onto \([j − |S| + 1, j]\) and then \(R^−\) recursively onto \([i, j − |S|]\). Since \(B[j − |S| + 1, j]\) consists of isolated vertices, the embedding of \(S\) always works. It remains to argue that we can recursively embed \(R^−\) onto \([i, j − |S|]\). Recall that \((I2)\) and \((I4)\) hold by construction. Concerning \((I1)\) note that \(j − |S| + 1\) is not adjacent to \(i\), which together with \((I1)\) for \(R\) and \([i, j]\) implies that there is no edge-conflict. A possible degree-conflict is prevented by the assumption of this case.

**Case 2** \(B^∗ := B[i, j − |S|][i]\) is a central star of size at least \(|R^−| − \deg_{R^−}(r) + 1\). We claim that \(B(i) \neq B(j)\). Suppose that \(B(i) = B(j)\) for the sake of contradiction. Then before rearranging \(B^{∗∗}\), we had \(i, j \in E(B)\). By LSFR and since \(B\) is not a star, \(\uparrow(B)\) was not at \(i\). Again by LSFR, the root could have been at \(j\) only if \(B^{∗∗}\) was a dangling star. But then, since \(|B^{∗∗}| > |S|\), \(B[j − |S| + 1, j]\) was not a star to begin with: a contradiction. The claim follows. Since \(|B(j)| \geq |B^{∗∗}| > |S|\), we have \(B[i, j − |S|][i] = B(i)\) and so \(B^∗ = B(i)\).

Since \(R^−\) has a degree-conflict with \([i, j − |S|]\), we follow a different strategy. Possibly flip \(B(i)\) so that its root is at \(i\). We first blue-star embed \(R^−\) from \(σ = i\) with \(ϕ = (i + [B^∗][i, \ldots, i + [B^∗] + d − 1])\), with \(d = \deg_{R^−}(r)\). Then we embed \(S\) on \([j − |S| + 1, j]\). The conditions for the blue-star embedding are met: \((BS1)\) holds by \((I3)\) for embedding \(R\) onto \([i, j]\); for \((BS2)\) on the one hand \(|R^−| \leq |B^∗| + \deg_{R^−}(r) − 1 \leq |B^∗| + \deg_{R^−}(r)\) and on the other hand, by \((I1)\), we have \(|B^∗| \leq |R| − \deg_{R}(r)\) and so \(|B^∗| + \deg_{R^−}(r) \leq |R| − 1\). As \(B^∗ = B(i)\), the vertices in \(B \setminus (B^∗ \cup ϕ)\) form an interval and both \((BS3)\) and \((BS4)\) hold.

By Proposition 9 we are left with an interval \([j − |S| + 1, j]\) that satisfies \((I2)\). Note that \(ϕ\) includes the center \(j − |S|\) of the star \(B^{∗∗}\), but does not include \(j\). Consequently, \(B[j − |S|, j]\) consists of isolated vertices after the blue-star embedding, and \(j\) is not in edge-conflict with \(s\). Hence, we can embed \(S\) explicitly onto \([j, j − |S| + 1]\).

**Proposition 30.** If \(B[j − |S| + 1, j]\) is a star, \(2 \leq |S| = |B^{∗∗}|\), and \(\{i, j\} \notin E(B)\), then \(R\) and \(B\) admit an ordered plane packing onto \([i, j]\).

**Proof.** By Lemma 21, we may assume that \(B[i, j − |S|]\) is not a star. By Lemma 19 and Lemma 28, we may assume that \(\deg_{R^−}(r) \geq 1\) and \(R^−\) is not a star. Due to LSFR, the center and the root of \(B^{∗∗}\) are each located at either \(j − |S| + 1\) or \(j\), but \(B^{∗∗}\) may be either a central star or a dangling star. We distinguish three cases.

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Case 1 \([i, j - |S|]\) is in degree-conflict with \(R^-\). Then \(B^o := B[i, j - |S|] \langle i \rangle\) is a central-star with
\[
\deg_{R^-}(r) + (|B^o| - 1) \geq |R^-|.
\]
Let \(x\) be so that \(B^o = B[i, x]\). Our plan is to blue-star embed \(R^-\) from \(\odot(B^o)\). Recall that \(\{i, j\} \notin E(B)\) and so \(B(i) \neq B(j)\). Given that \(B^{**} = B[j - |S| + 1, j]\) is part of \(B(j)\) it follows that \(B^o = B(i)\). Flip \(B^o\) if necessary to ensure that \(\odot(B^o) = i\). Flip \(B^{**}\) if necessary to ensure that \(\odot(B^{**}) = j - |S| + 1\). The latter flip may result in a violation of 1SR at \(j - |S| + 1\), but this is of no consequence because the blue-star embedding does not require 1SR and we will not use any recursive embedding in this case. We blue-star embed \(R^-\) starting from \(i\) with \(\varphi = (x + 1, \ldots, y)\), where \(y = x + \deg_{R^-}(r)\). By (11) we have
\[
\deg_{R^-}(r) \geq |R^-| - |B^o| + 1 = |x + 1, j - |S|| + 1 = j - |S| - x + 1
\]
and hence \(y = x + \deg_{R^-}(r) \geq j - |S| + 1\). In particular, \(\odot(B^{**}) \in \varphi\). Let us check the conditions of the blue-star embedding. In the terminology of the blue-star embedding we have \(A = R^-\) and \(B^* = B^+ = B^o\).

(BS1) is ensured by (I1) and the first inequality of (BS2) by (11). The second inequality of (BS2) is \(|B^o| + \deg_{R^-}(r) \leq |R| - 1\); it holds because (I1) ensures that \(\deg_{R}(r) + (|B^o| - 1) \leq |R| - 1\). Note that this also implies \(y = x + \deg_{R^-}(r) = (i + |B^o| - 1) + \deg_{R}(r) - 1 \leq i + |R| - 2 = j - 1\). As \(\varphi\) selects the vertices of \(B \setminus B^o\) from left to right, \(B \setminus (B^* \cup \varphi)\) is an interval and so (BS3) holds and (BS4) is trivial. Thus, by Proposition 9 we can blue-star embed \(R^-\) as described.

As \(y \in [j - |S| + 1, j - 1]\) and \(\odot(B^{**}) \in \varphi\), the blue vertices that are not used by the blue-star embedding form an independent set: they are leaves of either \(B^o\) or \(B^{**}\). At least one leaf of \(B^{**}\) remains because \(y \leq j - 1\). Therefore, we can explicitly embed \(S\) with \(s = j\), which is not adjacent to \(i = r\) in the blue forest (Figure 41a).

![Figure 41: The case analysis in the proof of Proposition 30.](image)

Case 2 \([i, j - |S|]\) is not in degree-conflict with \(R^-\) and \(B^{**}\) is a central star. Flip \(B^{**}\) if necessary to ensure that \(\odot(B^{**}) = j\). Observe that a flip maintains (I2) due to 1SR. We blue-star embed \(S\) from \(j\) with \(\varphi = (j - |S|, \ldots, z)\), where \(z = j - |S| - \deg_{S}(s) + 1 = i + |R^-| - \deg_{S}(s)\). Let us check the conditions for the blue-star embedding. In the terminology of the blue-star embedding we have \(A = S\) and \(B^* = B^{**}\).

(BS1) holds trivially because no neighbor of \(s\) is embedded yet. If the blue parent \(p\) of \(j\) is in \(B\) then since \(B^{**}\) is maximal, \(p\) must have a subtree other than \(B^{**}\). By LSFR and as \(\deg_{S}(s) \leq |S| - 1\), we have \(|p, j| \geq 2|S| + 1 \geq |S| + \deg_{S}(s) + 2\) and therefore \(p \leq j - |S| - \deg_{S}(s) - 1 = z - 2\). Hence, regardless of whether or not \(p\) is in \(B\), we know that \(B \setminus (B^{**} \cup \varphi)\) forms an interval, and both (BS3) and (BS4) follow. The first inequality
of (BS2) is \(|S| \leq |B^{**}| + \deg_S(s)\), which is trivial given that \(|B^{**}| = |S|\). The second inequality of (BS2) is \(|B^{**}| + 1 + \deg_S(s) \leq |R| - 1\), which easily follows from \(|B^{**}| = |S|\) and \(\deg_S(s) \leq |S| - 1\).

Therefore, the blue-star embedding works as described. Let \(I'\) denote the interval of remaining vertices. We complete the packing by recursively embedding \(R^-\) onto \(I'\) (Figure 41b). There is no edge-conflict between \(R^-\) and \(I'\) due to (I1) for \(B\) and \(I\) and because \(\{i, j\} \notin E(B)\) (recall that \(s = j\)). There is no degree-conflict between \(R^-\) and \(I'\) by assumption of Case 2.

**Case 3** \([i, j − |S|]\) is not in degree-conflict with \(R^-\) and \(B^{**}\) is a dangling star. As there is no dangling star on two vertices, we have \(|S| = |B^{**}| \geq 3\). Flip \(B^{**}\) if necessary to ensure that \(\uparrow(B^{**}) = j\). Observe that a flip maintains (I2) and \(\{i, j\} \notin E(B)\) due to 1SR. We distinguish three subcases.

**Case 3.1** \(\{j − |S|, j\} \in E(B)\). Then we exchange the order of the two subtrees \(t_B(j − |S|)\) (which is just a single vertex due to 1SR) and \(B[j − |S| + 1, j − 1]\) of \(j\) in \(B\), and denote the resulting blue forest by \(B'\). In \(B'\), we find \(\odot(B^{**})\) at \(j − |S|\) and it is isolated in \(B'[i, j − |S|]\). Similarly, in \(B'[j − |S| + 1, j − 2]\) the interval \([j − |S| + 1, j − 2]\) consists of isolated vertices—at least one because \(|S| \geq 3\)—the leaves of \(B^{**}\) (Figure 41c). In particular, (I2) and (I4) holds for both \(B'[i, j − |S|]\) and \(B'[j − |S| + 1, j]\).

If \(S\) is a star, then we explicitly embed it onto \([j − |S| + 1, j]\) with \(\odot(S) = j − |S| + 1\). If \(S\) is dangling, then we put \(s = j\). Otherwise, \(S\) is not a star and we recursively embed it onto \([j − |S| + 1, j]\). There is no edge-conflict between \(S\) and \([j − |S| + 1, j]\) because no neighbor of \(s\) is embedded yet, and there is no degree-conflict because \(\deg_{B'}(j − |S| + 1) = 1\). Then we complete the packing by recursively embedding \(R^-\) onto \([i, j − |S|]\). There is no edge-conflict between \(R^-\) and \([i, j − |S|]\) due to (I1) and because \(B(\langle i \rangle) \neq B(\langle j \rangle)\). There is no degree-conflict between \(R^-\) and \([i, j − |S|]\) by assumption of Case 3.

**Case 3.2** \(\{j − |S|, j\} \notin E(B)\) and either \(j − |S| \neq \uparrow(B(\langle i \rangle))\) or \(B[i, j − |S| − 1] \langle i \rangle\) is not a central star. Then \(j − |S|\) is isolated in \(B[j − |S|, j]\) by 1SR. We change the embedding of \(B^{**}\) so that all edges are drawn below the spine. Then we exchange the order of the vertices \(j − |S|\) and \(j − |S| + 1\) (Figure 41d). In the resulting blue forest \(B'\), the vertex \(j − |S| (\odot(B^{**}))\) is isolated in \(B'[i, j − |S|]\) and \(B'[j − |S| + 1, j]\) consists of isolated vertices. Therefore we can explicitly embed \(S\) onto \([j − |S| + 1, j]\) with \(s = j\), drawing all edges of \(S\) above the spine. We claim that \(R^-\) and \([i, j − |S|]\) are not in degree-conflict in \(B'\). To see this, note that \(R^-\) and \([i, j − |S|]\) were not in degree-conflict in \(B\). If \(B(\langle i \rangle) = B'(\langle i \rangle)\), then this continues to hold. The only way to make \(B(\langle i \rangle) \neq B'(\langle i \rangle)\) is if \(j − |S| = \uparrow(B(\langle i \rangle))\). However, if this is the case, then the assumption of Case 3.2 guarantees that \(B[i, j − |S| − 1] \langle i \rangle\) is not a central star. Hence, there is no degree-conflict between \(R^-\) and \([i, j − |S|]\), as claimed. There is no edge-conflict, either, by (I1) and because \(\{i, j\} \notin E(B)\) (recall that \(s = j\)). Therefore, we can complete the packing as described.

**Case 3.3** \(\{j − |S|, j\} \notin E(B)\), \(j − |S| = \uparrow(B(\langle i \rangle))\), and \(B[i, j − |S| − 1] \langle i \rangle\) is a central star. Then \(j − |S|\) has at least one subtree other than \(t_B(i)\) because \(B[i, j − |S|]\) is not a star by assumption. Let \(B[x, j − |S| − 1]\) be the closest subtree of \(j − |S|\) in \(B\). We first flip
$B[x, j - |S|]$ and then perform the exchange as in Case 3.2: change the embedding of $B^{**}$ so that all edges are drawn below the spine; then, exchange the order of the vertices $j - |S|$ and $j - |S| + 1$. Denote the resulting blue forest by $B'$.

Observe that both (I2) and (I4) hold for both $B'[i, j - |S|]$ and $B'[j - |S| + 1, j]$: The only violation is at the vertex $x = \uparrow(B'[i, j - |S|]|(i))$, which has neighbors on both sides, but the neighbor on the right is at $j - |S| + 1 \notin B'[i, j - |S|]$. Also note that $j - |S| + 1$ is isolated in $B'[j - |S|, j]$ because it corresponds to the vertex $x \in B(i) \neq B(j)$ in $B$. Therefore, as in Case 3.2 all the vertices in $B'[j - |S| + 1, j]$ are locally isolated and we can explicitly embed $S$ onto $[j, j - |S| + 1]$. Due to the flip of $B[x, j - |S| - 1]$, the subtree $B'[i, j - |S|]|(i)$ is not a central star. Thus, there is no obstruction to recursively embed $R^-$ onto $[i, j - |S|]$.

**Proposition 31.** If $B[j - |S| + 1, j]$ is a star, $2 \leq |S| = |B^{**}|$, $\{i, j\} \in E(B)$, and $S$ is not a star, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$.

**Proof.** The presence of edge $\{i, j\} \in E(B)$ means that $B$ is a tree, rooted at $i$ or $j$. We distinguish two cases based on $\uparrow(B)$. By Lemma 28, we may assume that $R^-$ is not a star.

**Case 1** $B$ is a tree rooted at $i$. We shall flip $B$, and show that $B[j - |S| + 1, j]$ is no longer a star after the flip. By LSFR, $B^{**}$ is a smallest subtree of $i$. The largest subtree of $i$ has size at least $|S| = |B^{**}|$, and so its root is outside of $[i, i + |S| - 1]$. Therefore, $B[i, i + |S| - 1](i)$ is an isolated vertex. Consequently, after flipping $B$, $B[j - |S| + 1, j](j)$ is an isolated vertex, and $B[j - |S| + 1, j]$ cannot be a star. If $B[i, j - |S|]$ is a star now, use Lemma 21 to find an ordered plane packing. Otherwise, none of $S$, $R^-$, $B[i, j - |S|]$ and $B[j - |S| + 1, j]$ are stars, and we can use Lemma 14 to find an ordered plane packing.

![Figure 42](image.png)

**Figure 42:** Case analysis in the proof of Proposition 31: (a) $B^{**} = S$, $\{i, j\} \in E(B)$, and $\uparrow(B) = j$. (b) Two subtrees of $j$ are central stars each with at least 2 vertices. (c) Vertex $j$ has a unique maximal subtree, and all other subtrees are singletons or not central stars.

**Case 2** $B$ is a tree rooted at $j$ (Figure 42a). Since $B$ is not a star, LSFR implies that $B^{**}$ is a dangling star rooted at $j$. That is, $B[j - |S| + 1, j - 1]$ is a central star, and by LSFR it is a largest subtree of $j$. Because $S$ is a smallest subtree of $r$, we have $|S| \leq (|I| - 1)/2$, and so every subtree of $j$ has size at most $|S| - 1 \leq (|I| - 3)/2$. Consequently, $j$ has at least 3 subtrees in $B$. Note that we may assume that $\text{deg}_R(r) \geq 2$ as otherwise a packing is constructed by Lemma 19. We distinguish subcases based on the subtrees of $j$.

Recall that $j$ has a maximal subtree that is a central star ($B[j - |S| + 1, j - 1]$). If $j$ has another maximal subtree, then either this is a central star (Case 2.2) or not (Case 2.1). Otherwise, $B[j - |S| + 1, j - 1]$ is the unique maximal subtree of $j$ and either there exists
another subtree of \( j \) that is a central star on \( \geq 2 \) vertices (Case 2.2) or every other subtree of \( j \) is a singleton or not a central star (Case 2.3).

**Case 2.1** \( j \) has two or more maximal subtrees, but not all of them are central stars. Re-embed \( B \) using a canonical embedding such that the subtree closest to \( j \) is not a central star (we only change the tie-breaking rule). Then \( B[j - |S| + 1, j] \) is no longer a star, and \( B[i, j - |S|] \) does not become a star. Use Lemma 14 to find an ordered plane packing.

**Case 2.2** Two or more subtrees of \( j \) are central stars each with at least 2 vertices. Let \( C_1 := B[j - |S| + 1, j - 1] \), which is central star subtree of \( j \) with at least 2 vertices. Let \( C_2 \) be another subtree of \( j \) that is a central star and has minimal size (possibly 1). We re-embed \( B \) as follows (Figure 42b). Embed \( \uparrow(B) \) at \( j - |C_1| - |C_2| \). Embed \( C_1 \) onto \( [j, j - |C_1| + 1] \) and \( C_2 \) onto \( [j - |C_1|, j - |C_1| - |C_2| + 1] \) each respecting 1SR. Embed all remaining subtrees onto \( [i, j - |C_1| - |C_2| - 1] \) each respecting 1SR. Note that \( B \) violates 1SR at its root because \( \uparrow(B) \) has subtrees on both sides. However, \( B[i, j - |S|] \) and \( B[j - |S| + 1, j] \) each satisfy both LSFR and 1SR (note that \( j - |S| + 1 = j - |C_1| \)). Furthermore, \( B[j - |S| + 1, j](j - |S| + 1) \) is an isolated vertex and neither \( B[j - |S| + 1, j] \) nor \( B[i, j - |S|] \) is a star. To see the latter, suppose to the contrary that \( B[i, j - |S|] \) is a star. Then \( |C_2| = 1 \) and there is only one more subtree \( C_3 \) of \( \uparrow(B) \) of size

\[
|C_3| = j - |S| - i = |R| - |S| - 1 \geq 2|S| + 1 - |S| - 1 = |S| > |S| - 1 = |C_1|
\]

in contradiction to LSFR for \( j \) in \( B \), where \( C_1 \) is the closest subtree of \( j \).

We recursively embed \( S \) onto \( [j - |S| + 1, j] \). There is no edge-conflict for this embedding because no neighbor of \( \uparrow(S) \) is embedded yet. There is no degree-conflict because \( j - |S| + 1 \) is isolated in \( B[j - |S| + 1, j] \). Next, recursively embed \( R^+ \) onto \( [i, j - |S|] \). This works because (1) \( i \) has no neighbor in \( B[j - |S| + 1, j] \) and (2) \( B[i, j - |S|](i) \) is a singleton or not a central star. Indeed, suppose to the contrary that \( B[i, j - |S|](i) \) is a central star with more than one vertex. By construction, \( B[i, j - |S|](i) = B[i, j - |C_1| - |C_2|] \) contains \( \uparrow(B) \). Hence, apart from \( C_1 \) and \( C_2 \), all subtrees of \( \uparrow(B) \) are singletons. By the choice of \( C_2 \), however, \( C_2 \) is also a singleton. Therefore \( \uparrow(B) \) has only one subtree with at least two vertices, contradicting our assumption.

**Case 2.3** \( j \) has a unique maximal subtree, which is a central star, and every other subtree is either a singleton or not a central star. Recall that \( j \) has at least 3 subtrees. Re-embed \( B \) such that its root is at \( j \), an arbitrary smallest subtree is embedded closest to \( j \), and all other subtrees are embedded according to LSFR (all subtrees are embedded recursively by a canonical embedding). In particular, \( B^{**} \) is now the second subtree of \( j \), counting from the right (Figure 42c). As a result, \( B[j - |S| + 1, j] \) is no longer a star, and \( B[i, j - |S|] \) does not become a star. Note also that \( B[j - |S| + 1, j](j - |S| + 1) \) becomes an isolated vertex (it is a leaf of the dangling star \( B^{**} \)); and \( B[i, j - |S|](i) \) is either an isolated vertex or not a central star.

Embed \( S \) recursively onto \( [j - |S| + 1, j] \). This works because \( B[j - |S| + 1, j](j - |S| + 1) \) is locally isolated. Embed \( R^- \) recursively onto \( [i, j - |S|] \). The recursive embedding of \( R^- \) works because \( B[i, j - |S|](i) \) is either an isolated vertex (which is not adjacent to the blue vertex on which \( s \) was embedded) or not a central star. \( \square \)
It remains to consider the case where $B[j - |S| + 1, j]$ is a star, $2 \leq |S| = |B^{**}|$, \{i, j\} $\in E(B)$, and $S$ is a star. We deal with this case by handling the case where $S$ is a star and \{i, j\} $\in E(B)$ in full generality.

**Proposition 32.** If $S$ is a star and \{i, j\} $\in E(B)$, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$.

**Proof.** Since \{i, j\} $\in E(B)$, $B$ is a tree, rooted at $i$ or $j$, and we can use symmetry by exchanging the roles of $B$ and $R$ (Figure 43a). Remove the embedding of $B$. Embed $R$ using a canonical embedding, placing its root at $i$. Rename $R$ to $B$ and $B$ to $R$. Define $S$ to be a smallest subtree of $R$. Since $B$ is rooted at $j$ and $B$ is not a star, there is no conflict for embedding $R$ onto $[i, j]$.

Embedding $R$ onto $[i, j]$ is handled by Lemma 14, Lemma 19, Lemma 20, Lemma 21, Lemma 28, or Proposition 29 unless the situation after the color exchange is as follows: \deg_{R}(r) \geq 2, |S| \geq 2, $B[i, j - |S|]$ is not a star, $R_{-}$ is not a star, and (i) $S$ is a star with \deg_{B}(i) \geq 2 or (ii) $B[j - |S| + 1, j]$ is a star and the maximal star that contains $B[j - |S| + 1, j]$ has size exactly $|S|$. If $S$ is not a star then (ii) holds and we can use Proposition 31 to find an ordered plane packing.

Otherwise, we are in Case (i) and $S$ is a star. This means that the smallest subtree of both $r$ and $b$ is a star on at least two vertices and both $R$ and $B$ have at least two subtrees each. Denote by $S_B$ a smallest subtree of $B$. By symmetry (possibly exchanging roles again), we may assume $|S_B| \geq |S|$.

Figure 43: In Proposition 32 both $R$ and $B$ are trees where a smallest subtree of the root is a star. We can use symmetry and exchange the roles of $R$ and $B$.

We proceed as follows (Figure 43b). Re-embed $B$ in the upper halfplane, placing $b$ at $i$, $S_B$ as the closest subtree, and the remaining subtrees according to LSFR.

We first explain how to embed $S$. We will do this in such a way that $s$ is embedded on a vertex of $S_B$ at $i + |S| - 1$. If $S_B$ is a central star, this re-embedding places its root and center at $i + |S_B|$. Since $|S_B| \geq |S|$, now $B[i, i + |S| - 1]$ is an independent set. Embed $S$ explicitly onto $[i + |S| - 1, i]$. If $S_B$ is a dangling star, the re-embedding places its root at $i + |S_B|$ and its center at $i + 1$. If $|S| = 2$, then embed $s$ onto $i + 1$ and its child onto $i$. If $|S| \geq 3$ and $S$ is a central star, flip $B[i + 1, i + |S_B|]$ to put $\uparrow(S_B)$ at $i + 1$ and the center at $i + |S_B|$. Further, embed $s$ onto $i + |S| - 1$ and the children of $s$ onto $[i - |S| - 2, i + 1]$. If $|S| \geq 3$ and $S$ is a dangling star, flip $B[i + 1, i + |S_B| - 1]$ to put $\circ(S_B)$ at $i + |S_B| - 1$. Further, embed $s$ onto $i + |S| - 1$, its child $s'$ onto $i$, and the children of $s'$ onto $[i + 1, i + |S| - 2]$. 
Next, embed \( R^- \) recursively onto \([j, j - |R^-| + 1] \). Since \( s \) was not embedded at \( i \), the only obstacle for this recursive embedding is a possible conflict, in which case \( B^* = B[i + |S|, j]\langle j \rangle \) is a central star. Since \( i + |S| - 1 \) (which is where we embedded \( s \)) is adjacent only to vertices of \( S_B \) and possibly \( b \), and since none of these vertices is part of \( B^* \), the conflict must be a degree-conflict. Then \(|B^*| \geq 3 \). As the root \( b \) of \( B \) is not in \([j, j - |R^-| + 1]\), we can reorder the subtrees of \( B \setminus S_B \) arbitrarily without having to worry about LSFR on \([j, j - |R^-| + 1] \). Therefore, we may suppose that all subtrees of \( b \) are central stars on \( \geq 3 \) vertices and each of them leads to a degree-conflict when taking the role of \( B^* = B\langle j \rangle \) above. Given that there are at least two such substars, we may as well choose a smallest one, \( S_B \) to have its center at \( j \). Any other substar can take the role intended for \( S_B \) in Figure 43b originally, its leaves being paired up with \( S \).

We claim that then there is no degree-conflict for embedding \( R^- \) onto \([j, j - |R^-| + 1] \) recursively. For such a degree-conflict to occur we need \( \text{deg}_{R^-}(r) + |S_B| - 1 \geq |R^-| \). So let us argue that this does not happen.

By the choice of \( S \) as a minimal size subtree of \( r \), we have \( \text{deg}_R(r) \leq (|R| - 1)/|S| \). As \( S_B \) is a smallest of at least two subtrees of \( b \), we have \(|S_B| \leq (|B| - 1)/2 = (|R| - 1)/2 \). The two inequalities together yield

\[
\text{deg}_{R^-}(r) + |S_B| = \text{deg}_R(r) - 1 + |S_B| \\
\leq \frac{|R| - 1}{|S|} - 1 + \frac{|R| - 1}{2} \\
= \frac{|R||S| + 2|R| - |S| - 2 - 1}{2|S|} \\
= \frac{|R||S| + 2|R| - 3|S| - 2}{2|S|}
\]

We want to show \( \text{deg}_{R^-}(r) + |S_B| \leq |R^-| \). So consider the expression

\[
|R^-| - (\text{deg}_{R^-}(r) + |S_B|) = |R| - |S| - (\text{deg}_{R^-}(r) + |S_B|) \\
\geq |R| - |S| - \frac{|R||S| + 2|R| - 3|S| - 2}{2|S|} \\
= \frac{|R||S| - 2|R| - 2|S|^2 + 3|S| + 2}{2|S|} \\
= \frac{(|S| - 2)(|R| - 2|S| - 1)}{2|S|},
\]

which is non-negative because \( 2 \leq |S| \leq (|R| - 1)/2 \). This proves our claim and shows that there is no degree-conflict for embedding \( R^- \) onto \([j, j - |R^-| + 1] \) recursively. Therefore at least one of the two options provides an ordered plane packing as claimed.

Lemma 20, Proposition 29, Proposition 30, Proposition 31, and Proposition 32 together prove the following.
Lemma 33. If \( B[j - |S| + 1, j] \) is a star, then \( R \) and \( B \) admit an ordered plane packing onto \([i, j] \).

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Finally, we handle the case where \( S \) is a star with \(|S| \geq 2\). Recall that the case \(|S| = 1\) is handled in Lemma 20. We may assume that \( B[i, j - |S|] \) and \( B[j - |S| + 1, j] \) are not stars, and that \( R^- \) is not a star, either.

Proposition 34. If \( S \) is a star with \(|S| \geq 2\) and \( \{i, j\} \notin E(B) \), then \( R \) and \( B \) admit an ordered plane packing onto \([i, j] \).

Proof. We may assume \( \deg_R(r) \geq 2 \) by Lemma 19. \( S \) can be a central star or a dangling star. We handle these cases separately. Let \( x = j - |S| \) so that \(|R^-| = |[i, x]| \). Flip \( B(j) \) if necessary to put the root at \( j \). In the case analysis we will sometimes embed a part of \( R \) and, as a result, the interval of remaining blue vertices changes. For instance, think of a red-star embedding of \( S \), which may add some isolated blue vertices to one end of the interval. The following observation states that such a change cannot introduce edge-conflicts.

Observation 35. Suppose that we embedded \( s \) on a vertex of \( B(j) \) and that at most \(|[i, x]| - 1\) rightmost vertices of \( B[i, x] \) have been replaced by locally isolated vertices. Then \([i, x]\) is not in edge-conflict with \( R^- \).

Proof. Suppose to the contrary that \([i, x]\) is in edge-conflict with \( R^- \). Let \( y \leq x \) be such that \( B[i, y] = B[i, x](i) \). Then \( \uparrow(B[i, y]) \) is in edge-conflict with \( r \). It cannot be due to an edge to \( s \) since \( B(i) \neq B(j) \). Hence, \( \uparrow(B[i, y]) \) must have an edge to the outside of \([i, j]\). By ISR and LSFR, \( B(i) \) must then also be a (possibly larger) central star whose root is in edge-conflict with \( r \). This contradicts the peace invariant for embedding \( R \) onto \( I \) and thus concludes the proof.

Case 1 \( S \) is a central star. Since \( \{i, j\} \notin E(B) \) we have \( B(i) \neq B(j) \) and hence flipping \( B(j) \) does not change the blue vertex at \( i \). Use the red-star embedding to embed \( s \) onto \( j \) and the children of \( s \) onto the rightmost \( \deg_S(s) \) non-neighbors of \( j \) in \([i + 1, j - 1]\). If \( B[i, x] \) is a star now, then it was also a star before the red-star embedding (although that may have modified \( B[i, x] \)), and we can find an ordered plane packing by Lemma 21. Otherwise, recursively embed \( R^- \) onto \([i, x]\). By the placement invariant for \( R \) and \([i, j]\) and since \( \{i, j\} \notin E(B) \), the placement invariant for the recursive embedding of \( R^- \) holds. Hence, the embedding of \( R^- \) fails only if (1) there is a conflict for embedding \( R^- \) onto \([i, x]\). For the embedding of \( S \), (R S1) holds and so the embedding works unless (R S2) fails, that is, unless (2) \( \deg_S(s) + \deg_B(j) \geq |I| - 1 \). We deal with (1) and (2) next.

Case 1.1 There is a conflict for embedding \( R^- \) onto \([i, x]\). Let \( y \leq x \) be such that \( B[i, y] = B[i, x](i) \). Then \( B[i, y] \) is a central star rooted at a vertex \( b^* \). By Observation 35, the conflict for embedding \( R^- \) onto \([i, x]\) is a degree-conflict. In other words, \( \deg_{S[i, x]}(b^*) + \deg_{R^-}(r) \geq |R^-| \). Consequently, \( |B[i, y]| \geq |R^-| - \deg_{R^-}(r) \). Additionally, \( \deg_{R^-}(r) \geq 2 \) since \( B[i, x] \) is
not a star and \(|B[i, y]| \geq 3\) by Lemma 2. Revert to the original blue embedding (Figure 44a).

Note that \(B[i, y]\) is still a central star. We distinguish two cases.

**Case 1.1.1** \(B(j)\) is not a central star. Then in particular \(|B(j)| \geq 3\). Since \(\deg_{R}(r) \geq 2\) and \(|B[i, y]| \geq |R^-| - \deg_{R^-}(r)\) we get by Lemma 12 that \(|B[i, y]| \geq |S|\). If \(B[i, y]\) is rooted at \(y\) then \(B(i) = B[i, y]\) by 1SR and we can flip \(B(i)\) to put the root (and center) at \(i\). \(B[i, i + |S| - 1]\) is now a small blue star. Flip \(B(j)\) to put its root on the left and embed \(R\) onto \([j, i]\) by Lemma 33. This works because \(j\) is not in edge-conflict with \(r\) and \(B(j)\) is not a central star.

**Case 1.1.2** \(B(j)\) is a central star. Flip \(B(j)\) if necessary to put its root (and center) at \(j\). If \(|B(j)| \geq |S|\) then use Lemma 33 to find an ordered plane packing. Otherwise \(|S| \geq |B(j)| + 1\). We distinguish two cases.

**Case 1.1.2.1** \(B(i)\) is a central star. Let \(z\) be such that \(B[i, z] = B(i)\) and note that \(z \geq y\). If necessary, flip \(B[i, z]\) to put its root at \(i\). By the peace invariant, \(i\) is not in edge-conflict with \(r\). Since \(i\) is in degree-conflict with \(r\) for embedding \(R^-\) onto \([i, x]\) we have \(\deg_{B}(i) + \deg_{R^-}(r) \geq |R^-|\).

In our first attempt at embedding \(R\), we embedded \(S\) from \(j\) using a red-star embedding and tried to embed \(R^-\) onto \([i, x]\). Since \(|S| \geq 2\), the red-star embedding moved all (possibly zero) children of \(j\) in \(B(j)\) to a suffix of \([i, x]\). Since there was a degree-conflict for the embedding of \(R^-\) onto \([i, x]\), it follows that \(\deg_{R^-}(r) > |B(j)| - 1\). Let \(h\) be such that \(B[h, j] = B(j)\) (Figure 44b).

We know that \(\deg_{B}(i) + \deg_{R^-}(r) \leq |I| - 1\) by the peace invariant. It follows that \(|B(i)| + \deg_{R^-}(r) \leq |I| - 1\). Combining this with the degree-conflict at \(i\), we obtain \(|R^-| \leq |B[i, y]| + \deg_{R^-}(r) \leq |B(i)| + \deg_{R^-}(r) \leq |I| - 1\). Hence, (BS2) is satisfied and we can perform a blue-star embedding to embed \(R^-\) onto \([i, j]\) (which will not embed any vertex onto \(j\)). Before doing so, modify the blue embedding by simultaneously shifting \(B[h, j - 1]\) to \([z + 1, z + j - h]\) (redrawing the edges to \(j\) with biarcs) and \(B[z + 1, h - 1]\) to \([z + j - h + 1, j - 1]\) (Figure 44c). Since \(\deg_{R^-}(r) > |B(j)| - 1\), the blue-star embedding will embed a vertex on every child of \(B(j)\). Complete the embedding by placing \(s\) at \(j\) and the children of \(s\) onto the remainder.

**Case 1.1.2.2** \(B(i)\) is not a central star. Let \(w\) be such that \(B(i) = B[i, w]\). Since \(B[i, y]\) is a central star, by 1SR \(B[i, y]\) must be rooted at \(i\) and \(B[i, w]\) must be rooted at \(w\) (Figure 45a).

We claim that \(\deg_{B}(w) = 1\). Towards a contradiction, suppose that \(\deg_{B}(w) \geq 2\). Recall that \(\deg_{B}(i) + \deg_{R^-}(r) \geq |R^-|\). Since \(S\) is a smallest subtree of \(r\) in \(R\), we have \(\deg_{R^-}(r) \leq |R^-| - 1\), which completes the proof.
$|R^-|/|S| \leq |R^-|/|S|$. Hence, $\deg_B(i) \geq |R^-| - \deg_{R^-}(r) \geq (1 - 1/|S|)|R^-|$. By LSFR, $|B\langle w \rangle| \geq 1 + 2(1 + \deg_B(i)) = 3 + 2 \deg_B(i)$ and hence $|B\langle w \rangle| \geq 3 + (2 - 2/|S|)|R^-|$. Since $|R^-| + |S| = |I| \geq |B\langle w \rangle|$ and $|R^-| \geq |S|$, we obtain $|S| \geq 3 + (2 - 2/|S|)|R^-| - |R^-| = 3 + (1 - 2/|S|)|R^-| \geq 3 + |S| - 2 = |S| + 1$, a contradiction. The claim follows.

Figure 45: The case analysis in the proof of Proposition 34 (Part 2/3).

Since $B[i, y]$ is a central star rooted at $i$, by LSFR $B\langle i \rangle = B[i, w]$ is a star centered at $i$ and rooted at $w$. If $w \geq x$ then we can use Lemma 21 to find an ordered plane packing. Otherwise, $w \leq x - 1$.

Since there was a conflict for the original embedding, the red-star embedding of $S$ from $j$ embeds a child of $s$ onto all vertices originally at $[w, x]$. Flip $B[i, w]$ to put its root at $i$ and center at $w$ (Figure 45b). Execute the red-star embedding of $S$ from $j$ again. This embeds a child of $s$ onto the center of $B\langle i \rangle$ at $w$ and hence the remaining vertices of $B\langle i \rangle$ form an independent set. Consider the now-modified blue embedding at $[i, x]$. The leftmost vertex of $B[i, x]$ is the original root of $B\langle i \rangle$ and may be in edge-conflict with $r$. The suffix of $[i, x]$ of size $\deg_B(j) \leq |S| - 2$ is formed by blue vertices adjacent to $j$ (which is where we embedded $s$) that were placed there by the red-star embedding of $S$ from $j$. All of these blue vertices are in edge-conflict with $r$. However, by the original degree-conflict, we know that $\deg_{R^-}(r) \geq 2$ and hence we can find an explicit embedding of $R^-$ onto $[i, x]$ that avoids placing the root at $i$ or at the suffix of size $|S| - 2$. This uses that all subtrees of $r$ in $R^-$ have size at least $|S|$.

**Case 1.2** $\deg_S(s) + \deg_B(j) \geq |I| - 1$. Then $\deg_B(j) \geq |I| - 1 - \deg_S(s) = |I| - |S| = |R^-| \geq (|I| + 1)/2$ and hence $B\langle j \rangle$ has a leaf. Let $h$ be such that $B[h, j] = B\langle j \rangle$. Then $|B[h, j]| > |S|$ and so $h \leq x$. If $B[h, j]$ is a star, then we flip $B[h, j]$ if necessary to put its center at $j$ and use Lemma 33 to find an ordered plane packing. Otherwise, $B[h, j]$ is not a star. We claim that then $h < x$. Indeed, if $h = x$, then $\deg_B(j) \leq |B\langle x, j \rangle| - 2 = |S| + 1 - 2 = |S| - 1$ and so $\deg_S(s) \geq |I| - 1 - \deg_B(j) \geq |I| - 1 - |S| + 1 = |R^-|$, a contradiction. The claim follows. Flip $B[h, j]$ to put the root on the left. This places a leaf at $j$. Embed $s$ onto $j$ and the children of $s$ onto $[j - 1, x + 1]$. Embed $R^-$ recursively onto $[x, i]$ (Figure 45c). The placement invariant holds since $h < x$ and $h$ is the only vertex incident to $j$ (which is where we embedded $s$). By LSFR and since $B[h, j]$ is not a star, $B[i, x, j]$ is not a central star. Hence the peace invariant holds and we can complete the packing.

**Case 2** $S$ is a dangling star. Let $q = \circ(S)$ and $Q = t_R(q)$. We will embed $R$ similarly to Case 1. Let $h$ be such that $B[h, j] = B\langle j \rangle$. We distinguish two cases.

**Case 2.1** Suppose that $B[h, j]$ is not a central star. Then in particular $|B[h, j]| \geq 3$. Let $h'$
be the rightmost neighbor of \( j \) in \([i, j - 1]\). If \( h' \leq x \), then embed \( s \) onto \( x = j - |S| + 1, q \) onto \( j \), and the children of \( q \) onto \([j - 1, j - |S| + 2]\) (Figure 46a). Otherwise, embed \( s \) onto \( h' + 1, q \) onto \( j \), and embed a child of \( q \) onto every vertex of \([h' + 2, j - 1]\). Use the red-star embedding to embed the remaining vertices onto the rightmost \( \deg_Q(q) - [|h' + 2, j - 1]| \) non-neighbors of \( j \) on \([i + 1, h']\). In either case, embed \( R^- \) recursively onto \([i, x]\). The embedding of \( S \) works unless \((RS2)\) fails, that is, unless \((2)\) \( \deg_Q(q) + \deg_B(j) \geq |I| - 2 \).

**Figure 46:** The case analysis in the proof of Proposition 34 (Part 3/3).

**Case 2.1.1** There is a conflict for embedding \( R^- \) onto \([i, x]\). Let \( y \leq x \) be such that \( B[i, y] = B[i, x](i) \). Then \( B[i, y] \) is a central star. By Observation 35, the conflict for embedding \( R^- \) onto \([i, x]\) is a degree-conflict, and hence \( |B[i, y]| \geq |R^-| - \deg_{R^-}(r) \). Following the reasoning in Case 1.1.1, we see that \( |B[i, y]| \geq |S| \) and hence \( B[i, i + |S| - 1] \) is a small blue star after flipping \( B[i, y] \) if necessary. Flip \( B[h, j] \) to put its root at \( h \) and use Lemma 33 to embed \( R \) onto \([j, i]\). This works because \( j \) is not in edge-conflict with \( r \) and \( B(j) \) is not a central star.

**Case 2.1.2** \( \deg_Q(q) + \deg_B(j) \geq |I| - 2 \). This case is similar to Case 2. Since \( |S| \leq \frac{|I| - 1}{2} \) we have \( \deg_Q(q) = |S| - 2 \leq \frac{|I| - 5}{2} \). Then \( \deg_B(j) \geq |I| - 2 - \deg_Q(q) \geq |I| - 2 - \frac{|I| - 5}{2} = \frac{|I| + 1}{2} \). Since \( B[h, j] \) is not a central star, we get \( h < x \) as in Case 2. Let \( \lambda \) be the number of leaf children of \( j \). Then \( 1 + \lambda + 2((\deg_B(j) - \lambda)) \leq |B(j)| \leq |I| - 1 \). Since \( \deg_B(j) \geq \frac{|I| + 1}{2} \), it follows that \( 1 + \lambda + |I| + 1 \leq |I| - 1 \) and hence \( \lambda \geq 3 \). Flip \( B[h, j] \) to put its root at \( h \). Since \( h \) now has \( \lambda \) leaf children in \( B[h, j] \), in particular \( j - 1 \) and \( j \) are leaves. Embed \( s \) onto \( j \), \( q \) onto \( j - 1 \), and the children of \( q \) onto \([j - 2, x + 1]\). Embed \( R^- \) recursively onto \([x, i]\). Since \( B[h, j] \) is not a star by assumption and by LSFR, \( B[i, x](x) \) is not a central star on at least two vertices. Hence the peace invariant holds.

**Case 2.2** Suppose that \( B[h, j] \) is a central star. If \( h \leq x + 1 \) then \( B[x + 1, j] \) is a star and we can find an ordered plane packing by Lemma 33. Otherwise \( h \geq x + 2 \). Flip \( B(h - 1) \) if necessary to put its root at \( h - 1 \). Embed \( s \) onto \( j \), \( q \) onto \( h - 1 \), and a child of \( q \) on every vertex in \([h, j - 1]\). Use the red-star embedding to embed the remaining children of \( q \) onto the rightmost \( \deg_Q(q) - [|h, j - 1]| \) non-neighbors of \( h - 1 \) in \([i + 1, h - 2]\). Embed \( R^- \) recursively onto \([i, x]\). See Figure 46b for the situation before the cleanup step of the red-star embedding. The embedding of \( R^- \) works unless \((1)\) there is a conflict for embedding \( R^- \) onto \([i, x]\). The embedding of \( S \) works unless \((RS2)\) fails, that is, unless \((2)\) \( \deg_Q(q) + \deg_B(h - 1) \geq |I| - 2 \).

**Case 2.2.1** There is a conflict for embedding \( R^- \) onto \([i, x]\). Let \( y \leq x \) be such that \( B[i, y] = B[i, x](i) \). Then \( B[i, y] \) is a central star. By Observation 35, the conflict is a degree-conflict. Revert to the original blue embedding (before the red-star embedding in
Case 2.2) and note that \( B[i, y] \) is still a central star. We proceed similarly to Case 1.1.2.

**Case 2.2.1.1** \( B(i) \) is a central star. Let \( z \) be such that \( B[i, z] = B(i) \) and note that \( z \geq y \). If necessary, flip \( B[i, z] \) to put its root at \( i \). By the peace invariant, \( i \) is not in edge-conflict with \( r \). Since \( i \) is in degree-conflict with \( r \) for embedding \( R^- \) onto \([i, x]\) we have \( \deg_B(i) + \deg_{R^-}(r) \geq |R^-| \).

We blue-star embed \( R^- \) starting from \( i \) with \( \varphi = (z + 1, \ldots) \). Let us argue that the conditions for the blue-star embedding hold. The peace invariant guarantees (BS1) and \( \deg_B(i) + \deg_{R}(r) \leq |I| - 1 \). It follows that \( |B(i)| + \deg_{R^-}(r) \leq |I| - 1 \), which is the second inequality of (BS2). The first inequality of (BS2) holds by the degree-conflict condition. (BS3) holds by construction, making (BS4) trivial. Hence, the conditions are satisfied and we can perform the blue-star embedding as described.

Since we attain the first inequality in (BS2) strictly, the blue-star embedding does not exhaust all vertices in \( B[i, z] \). Indeed, \( \deg_B(i) \geq |R^-| - \deg_{R^-}(r) \), while the blue-star embedding embeds only \( |R^-| - \deg_{R^-}(r) - 1 \) vertices on the neighbors of \( i \). Perform the blue-star embedding of \( R^- \) onto \([i, j]\). This leaves an interval containing \( j \) (since the blue-star-embedding always leaves at least one vertex) and at least one locally isolated vertex (originating from \( B[i + 1, z] \)). Embed \( s \) onto \( j \), \( q \) onto this locally isolated vertex, and the children of \( q \) onto the remainder to complete the embedding.

**Case 2.2.1.2** \( B(i) \) is not a central star. We proceed similarly to Case 1.1.2.2. Let \( w \) be such that \( B[i, w] = B(i) \). The exact same argument as in Case 1.1.2.2 shows that \( B[i, w] \) is a star rooted at \( w \) and centered at \( i \). If \( w \geq x \) then we can use Lemma 21 to find an ordered plane packing. Otherwise, \( w \leq x - 1 \).

Since there was a conflict for the original embedding of \( R^- \) onto \([i, x]\), the red-star embedding of (the remainder of) \( Q \) from \( h - 1 \) embeds a child of \( q \) onto all blue vertices originally at \([w, x]\). Flip \( B[i, w] \) to put its root at \( i \) and center at \( w \). Embed \( s \) onto \( j \), \( q \) onto \( h - 1 \), and a child of \( q \) onto all vertices in \([h, j - 1]\). Execute the red-star embedding of the remainder of \( Q \) from \( h - 1 \) onto \([h - 2, i + 1]\) again. This embeds a child of \( s \) onto the center of \( B(i) \) and hence the remaining vertices form an independent set. Consider the now-modified blue embedding at \([i, x]\). The leftmost vertex of \( B[i, x] \) is the original root of \( B(i) \) and may be in edge-conflict with \( r \). We embedded a child of \( q \) onto all neighbors of \( j \) (which is where we embedded \( s \)), and hence there are no further edge-conflicts. Hence, we can embed \( R^- \) explicitly onto \([x, i]\).

**Case 2.2.2** \( \deg_Q(q) + \deg_B(h - 1) \geq |I| - 2 \). Let \( z \) be such that \( B[z, h - 1] = B(h - 1) \). It is possible that \( z = i \) and \( B(i) = B(h - 1) \). Analogously to Case 2.1.2 we get \( \deg_B(h - 1) \geq (|I| + 1)/2 \) and that \( h - 1 \) has at least 3 leaf children. It follows that \( z < x \). Recall that \( h \geq x + 2 \). Flip \( B[z, h - 1] \) to put its root at \( z \). If \( z = i \) and \( B[i, x] \) is now a star, use Lemma 21 to find an ordered plane packing. Otherwise, flipping \( B[z, h - 1] \) placed a leaf child of \( z \) at \( h - 1 \). Embed \( s \) onto \( j \), \( q \) onto \( h - 1 \), and the children of \( q \) onto \([j - 1, h]\) and \([h - 2, x + 1]\). This works because \( z < x \) and \( h \geq x + 2 \).

We first try to embed \( R^- \) recursively onto \([x, i]\) (Figure 46c). Since \( x < x \), this works unless \( B[i, x] \) is a central star, which implies that \( B[z, h - 1] \) is a central star by LSFR. In
this scenario we already handled the case $z = i$ and so we may assume $B(i) \neq B(z)$. Embed $R^-$ recursively onto $[i, x]$ (Figure 46d). By the placement invariant, this works unless there is a conflict for embedding $R^-$ onto $[i, x]$.

So suppose there is a conflict for embedding $R^-$ onto $[i, x]$. Since $z < x$ and $B(i) \neq B(z)$, we have $B[i, x] \not\subseteq B(i)$ and hence $B(i)$ is a central star. By the peace invariant, the root of $B(i)$ is not in edge-conflict with $r$. Flip $B(i)$ if necessary to put its root at $i$. Then $i$ is in degree-conflict with $r$ and hence $\deg_B(i) + \deg_{R^-}(r) \geq |R^-|$. Adding this inequality to the inequality in the assumption (replacing $h - 1$ by $z$ due to our flipping), we get $\deg_B(i) + \deg_{R^-}(r) + \deg_Q(q) + \deg_B(z) \geq |I| - 2 + |R^-|$. Since $B(i)$, $B(z)$, and $B(j)$ are all different we have $\deg_B(i) + \deg_B(z) \leq |I| - 3$. Hence $\deg_{R^-}(r) + \deg_Q(q) \geq |I| - 2 + |R^-| - |I| + 3 = |R^-| + 1$. Since $|S| = \deg_Q(q) + 2$ we get $|S| + \deg_{R^-}(r) \geq |R^-| + 3$. Since $S$ is a smallest subtree of $r$, we have $\deg_{R^-}(r) \leq (|R^-| - 1)/|S|$ and hence $|R^-| \geq |S| \deg_{R^-}(r)$. It follows that $|S| + \deg_{R^-}(r) \geq |S| \deg_{R^-}(r) + 3$, which has no solution for $|S| \geq 1$ and $\deg_{R^-}(r) \geq 1$. We conclude that there is no conflict for embedding $R^-$ onto $[i, x]$, as desired.

Propositions 32 and 34 together prove the following.

**Lemma 36.** If $S$ is a star, then $R$ and $B$ admit an ordered plane packing onto $[i, j]$.

Finally, Lemmata 14, 21, 28, 33, and 36 together prove Theorem 3.

### 13 Conclusions

The planar tree packing theorem characterizes which pairs of trees can be packed into a planar graph on the same number of vertices. It can be regarded as another instance of TONCAS (the obvious necessary condition—here that no tree is a star—is also sufficient). A number of intriguing open questions remain.

Maheo et al. [19] characterize triples of trees on $n$ vertices that pack into $K_n$. Just looking at the number of edges, three trees on $n$ vertices use $3n - 3$ edges, which is three more than a planar graph can provide. Therefore, it seems natural to consider the scenario of three trees on $n - 1$ vertices each, which use exactly $3n - 6$ edges.

**Open Problem 1.** Characterize the triples of trees on $n - 1$ vertices that can be packed into a planar graph on $n$ vertices.

Woźniak [28] showed that there is no obstruction from a purely combinatorial point of view, that is, any such triple of trees can be packed into $K_n$. Does planarity make a difference? It is not hard to show that three paths on $n - 1$ vertices admit a planar packing. But we are not aware of any further results for the planar case.

Looking beyond trees, Kheddouci et al. [18] showed that any triple of graphs, each on $n$ vertices and with at most $n - 3$ edges, packs into $K_n$. What happens if we restrict to planar graphs?

**Open Problem 2.** Characterize the triples of planar graphs, each on $n$ vertices and with $n - 3$ edges, that can be packed into a planar graph on $n$ vertices.
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