Nonabelian Bosonization as Duality†

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Abstract

Applying the techniques of nonabelian duality to a system of Majorana fermions in 1+1 dimensions, invariant under a nonabelian group $O_L(N) \times O_R(N)$, we obtain the level-one Wess-Zumino-Witten model as the dual theory. This makes nonabelian bosonization a particular case of a nonabelian duality transformation, generalizing our previous result for the abelian case.

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1. Introduction

Since its first formulation by Witten [1] close to ten years ago nonabelian bosonization has proven to be a very powerful tool for analyzing two-dimensional fermionic field theories. Its central result states the equivalence between a theory of $N$ majorana fermions, and a nonlinear sigma model whose fields take values in the group $O(N)$.

There is, however, a conceptual drawback to this equivalence as it is usually presented. The drawback is that the usual derivation is not constructive. The form of the required bosonic sigma model can be motivated from the properties of the current algebra of the fermionic theory, but once such arguments have been used to intuit its form, the main line of reasoning is devoted to establishing the equivalence of the known fermionic and bosonic theories. A more direct, constructive, procedure would start from either theory and derive the other without any foreknowledge of its form. Such a method would have the obvious advantage of lending itself to potential generalization to other systems, for which the equivalent theory is not already known.

A first step in providing such a foundation for nonabelian bosonization was recently taken in Ref. [2], where it was shown how abelian bosonization could be viewed as a special case of a wider class of techniques — collectively known as duality transformations — for proving relations among quantum field theories. This procedure has become a well-defined prescription for constructing an equivalent quantum field theory from any given one which has an abelian global symmetry (for a recent review see [3]). The extension of this result to nonabelian bosonization, using the recent extension [4] of the dualization prescription to theories with nonabelian symmetries, is the purpose of the present note. In so doing we intend to furnish a systematic and constructive formulation of the nonabelian bosonization technique.

2. The Fermionic Theory

Our starting point, as was the case for abelian bosonization, is the fermionic theory. We work in 1+1 spacetime dimensions, and take a theory of $N$ free and massless two-
component majorana fermions, $\psi$. At the classical level this theory enjoys an $O_L(N) \times O_R(N)$ global flavour symmetry under which the left- and right- handed fermions rotate amongst themselves: $\psi \to (L \gamma_L + R \gamma_R) \psi$. Only the diagonal (vectorlike) $O_v(N)$ subgroup, $L = R$, is anomaly free, however, and so survives quantization.

For our present purposes, we imagine studying the correlations of the Noether currents for the classical chiral $O_L(N) \times O_R(N)$ transformations. We may write the generating functional for these correlations in the following way:

$$Z[a] = \int [d\psi] \exp \left\{ -\frac{i}{2} \int d^2x \overline{\psi} \gamma^\mu (\partial_\mu - ia_\mu) \psi \right\} = \int [d\psi] \exp \left\{ -\frac{i}{2} \int d^2x \overline{\psi} \left[ \gamma^+ \gamma_R D_+ + \gamma^- \gamma_L D_- \right] \psi \right\}. \quad (1)$$

Here $a_\mu = a_\mu^a t_a$ are matrix-valued external fields, with $t_a$ being the generators of the $O_v(N)$ symmetry. We take these generators to be normalized according to $\text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$. $D_\pm \psi = (\partial_\pm - ia_\pm) \psi$ similarly represent the background-covariant derivatives acting on $\psi$. Notice that the couplings of the external fields promote the global classical chiral flavour invariance to a local symmetry, provided that they have the transformation rules: $a_+ \to R a_+ R^\dagger - i \partial_+ R R^\dagger$ and $a_- \to L a_- L^\dagger - i \partial_- L L^\dagger$.

Such a system of free fermions is sufficiently simple to permit the explicit evaluation of the functional integrals [5] over the fermions. Since we require this result later, we pause to record it here. Defining the group-valued Wilson-line variables, $\ell$ and $r$, according to $a_+ = i r^\dagger \partial_+ r$ and $a_- = i \ell^\dagger \partial_- \ell$, as well as the field-independent constant $Z_0 = Z[a = 0]$, we have:

$$\frac{Z[a]}{Z_0} = \left[ \frac{\det D_+}{\det D_-} \right]^{1/2} = \exp \left\{ -i \Gamma \left[ \ell r^\dagger \right] \right\}. \quad (2)$$

The quantity $\Gamma$ — a.k.a. the Wess-Zumino-Witten (WZW) action [6], [1] — which appears in this equation represents the following expression:

$$\Gamma[g] = \frac{1}{16\pi} \int_M d^2x \, \text{tr} \left( g^\dagger \partial_\mu g \, g^\dagger \partial^\mu g \right) + \frac{2}{3} \int_B d^3x \, \varepsilon^{\mu \nu \lambda} \, \text{tr} \left( g^\dagger \partial_\mu g \, g^\dagger \partial_\nu g \, g^\dagger \partial_\lambda g \right), \quad (3)$$

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1 Our conventions are: $x^0 = t$, $x^1 = x$, $x^{\pm} = \frac{1}{\sqrt{2}}(x \pm t)$, $\eta^{11} = -\eta^{00} = -\varepsilon^{01} = 1$, $\gamma_0 = i \sigma_1$, $\gamma_1 = \sigma_2$, $\gamma_3 = \gamma^0 \gamma^1 = \sigma_3$, and $\gamma_5 = \frac{1}{2}(1 + \gamma_3)$. 
where \( M \) denotes the (1+1)-dimensional spacetime, and \( B \) is a three-dimensional region having \( M \) as its boundary.

The symmetry of eq. (2) under vectorlike background gauge transformations is clear, given the transformation rules which \( \ell \) and \( r \) inherit from \( a_{\pm} : \ell \rightarrow \ell L^\dagger \) and \( r \rightarrow r R^\dagger \). The properties of the WZW action are also such as to ensure that eq. (2) properly reproduces the fermion anomaly for chiral \( O_L(N) \times O_R(N) \) rotations.

Among the remarkable properties that are satisfied by \( \Gamma \), there is an identity, due to Polyakov and Wiegmann [5], that is particularly useful in what follows. This is:

\[
\Gamma [gh^\dagger] = \Gamma [g] + \Gamma [h^\dagger] - \frac{1}{4\pi} \int_M d^2x \ \text{tr} \left[ g^\dagger \partial_+ g \ h^\dagger \partial_- h \right].
\] (4)

### 3. Dualization

In order to dualize this theory, we follow Refs. [7] and [4] (see also [2]) and promote the background \( O_V(N) \) invariance into a \textit{bona fide} gauge symmetry, by introducing a functional integration over its gauge potential. We therefore rewrite our path integral for \( Z[a] \) in the following way:

\[
Z[a] = \int [d\psi] [dA_\mu] [dA] \ \exp \left\{ i \int d^2x \ \left[ -\frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi + \varepsilon^{\mu\nu} \ \text{tr} \ (\Lambda V_{\mu\nu}) \right] \right\} \delta[G] \Delta
\]

\[
= \int [d\psi] [dA_+] [dA_-] [dA] \ \exp \left\{ i \int d^2x \ \left[ -\frac{1}{2} \bar{\psi} \left( \gamma^+ \gamma_R D_+ + \gamma^- \gamma_L D_- \right) \psi + 2 \ \text{tr} \ (\Lambda V_{+-}) \right] \right\} \delta[G] \Delta,
\] (5)

in which the covariant derivatives are defined with respect to both the background field, \( a_\mu \), and the new quantum field, \( A_\mu \), \textit{i.e.:} \( D_\pm = \partial_\pm - i(a + A)_\pm \).

Besides the quantum gauge potential, \( A_\mu \), there are four other new quantities in eq. (5) which need to be defined: \( \Lambda, V_{\mu\nu}, \delta(G) \) and \( \Delta \).

1. \( \Lambda \) is a Lagrange multiplier field, which takes its values in the Lie algebra of \( O_V(N) \).

Its functional integration enforces the constraint that \( V_{\mu\nu} \) vanish. This constraint,
together with the gauge condition (more about which below) is chosen to have the unique solution \( A_+ = A_- = 0 \), in the absence of nontrivial spacetime topology.\(^2\) As a result, the integration over \( \Lambda \) and \( A_\mu \) simply ensures that \( A_\mu \) may be set to zero throughout the path-integral integrand, and this establishes the equivalence of eq. (5) with the original fermionic theory, eq. (1).

2. The tensor \( V_{\mu\nu} \) is defined to be the difference between the field strengths, \( F_{\mu\nu} \) and \( f_{\mu\nu} \), that are constructed from the two gauge potentials, \( (a + A)\mu \) and \( a_\mu \). Explicitly:
\[
V_{+\nu} \equiv F_{+\nu} - f_{+\nu} = D_+ A_\nu - D_- A_\nu - i [A_+, A_\nu].
\]
In using this as our constraint we generalize slightly the procedure of Ref. [4], which uses the field strength built directly from \( A_\mu \) alone. Our purpose in so doing is to keep manifest the invariance with respect to vectorlike gauge transformations of the background fields. Notice that in order to have this invariance, the Lagrange-multiplier field must acquire the transformation law \( \Lambda \rightarrow \mathcal{G} \Lambda \mathcal{G}^\dagger \), where \( \mathcal{G} \equiv \mathcal{L} = \mathcal{R} \) is the common group element for the background \( O_V(N) \) gauge transformations.

3. Finally, the factor \( \delta(\mathcal{G}) \) is a functional delta function which imposes an appropriate gauge condition, \( \mathcal{G}(x) = 0 \), throughout spacetime. \( \Delta \) represents the corresponding Fadeev-Popov-DeWitt determinant. In what follows we will choose the background-covariant gauge, \( \mathcal{G} = A_+ = 0 \), for which we may take \( \Delta = 1 \) (up to irrelevant field-independent factors).

We now proceed to evaluate the functional integrals over \( \psi \) and \( A_\pm \), leaving the Lagrange-multiplier field, \( \Lambda \), as the bosonized variable. We do so in the following four steps.

- **1. The Fermion Integral:**

The fermion integral may be directly performed using eq. (2). In order to use this expression, we require a definition of the Wilson-line variables for the quantum field, \( A_\mu \). We take:
\[
(a + A)_+ = i(Rr)^\dagger \partial_+(Rr) \quad \text{and} \quad (a + A)_- = i(Lr^\dagger (Lr^\dagger). \]
Together with the previous definitions, \( a_+ = ir^\dagger \partial_+ r \) and \( a_- = il^\dagger \partial_- l \), we therefore have \( A_+ = ir^\dagger (R^\dagger \partial_+ R)r \)

\(^2\) Some topological issues arising in duality are addressed in Refs. [7], [8] and [2].
and \( A_- = i \ell \dagger (L^\dagger \partial_+ L) \ell \). Clearly the new variables, \( R \) and \( L \), do not transform under background gauge transformations.

With these definitions, performing the fermion integrations gives (ignoring, as always, a field-independent overall factor):

\[
\left[ \frac{\det D_+ \det D_-}{\det \partial_+ \det \partial_-} \right]^{1/2} = \exp \left\{ -i \Gamma \left[ L \ell \ell \dagger R \dagger \right] \right\}.
\]  

(6)

\* 2. Changes of Variables I:

We next change variables from \( A_\pm \) to \( L \) and \( R \). The Jacobian of the transformation from \([dA_+][dA_-]\) to the group-invariant measure, \([dR][dL]\), is \([5]\):

\[
J = \left[ \frac{\det \hat{D}_+ \det \hat{D}_-}{\det \partial_+ \det \partial_-} \right] = \exp \left\{ -i \kappa \Gamma \left[ L \ell r \dagger R \dagger \right] \right\},
\]  

(7)

in which the ‘hat’ over the gauge-covariant derivative is meant to indicate that this derivative is to be taken in the adjoint representation: \( \hat{D}_\pm X \equiv \partial_\pm X - i[\sigma (a + A)_\pm, X] \). The constant \( \kappa \) here accounts for the difference in normalization between the generators in the fundamental and the adjoint representations, as well as for the absence of the overall square root of the determinants in eq. (7). If the adjoint generators, \( T_a \), satisfy \( \text{tr}(T_a T_b) = \lambda \delta_{a b} \), then \( \kappa = 4 \lambda \).

\* 3. Light-Cone Gauge:

We next choose to work within the background-covariant light-cone gauge, \( A_+ \equiv 0 \). In terms of the Wilson-line variables we may implement this gauge with the choice: \( R \equiv 1 \), for which the Lagrange-multiplier term in eq. (5) simplifies considerably:

\[
S_{LM} \equiv 2 \int d^2 x \text{ tr} (\Lambda V_{+-})
= 2 \int d^2 x \text{ tr} \left( \Lambda \hat{D}_+ A_- \right)
= -2i \int d^2 x \text{ tr} \left[ (\hat{D}_+ \Lambda) \ell \dagger (L^\dagger \partial_- L) \ell \right],
\]  

(8)
where the last equality requires an integration by parts. We have again introduced a ‘hat’ on the background-covariant derivative to emphasize that it here acts in the adjoint representation.

4. Changes of Variables II:

The final step that is required in order to proceed is a judicious change of variables for the Lagrange-multiplier field, Λ. In order to take advantage of the Polyakov-Wiegmann identity, eq. (4), we wish to transform to a group-valued variable, X, for which we may rewrite the quantity $\hat{D}_+ \Lambda$ in terms of the combination $X^\dagger \partial_+ X$. There are two considerations which can be used to pin down the required change of variables: (i) Any relation between $\hat{D}_+ \Lambda$ and $X^\dagger \partial_+ X$ should be consistent with the vectorlike background gauge invariance; and (ii) since $\hat{D}_+ \Lambda$ is independent of the variable $\ell$, so must be $X$.

Given that background transformations take $\hat{D}_+ \Lambda$ into $\mathcal{G} \hat{D}_+ \Lambda \mathcal{G}^\dagger$, we see that a background-covariant choice for the desired change of variables is:

$$\hat{D}_+ \Lambda = i \xi \ r^\dagger (X^\dagger \partial_+ X) \ r$$

$$= i \xi \ [(Xr)^\dagger \partial_+ (Xr) - r^\dagger \partial_+ r].$$

(9)

Here $X$ is completely neutral under background gauge transformations, with the usual transformation rule for $r$ – i.e. $r \rightarrow r \mathcal{G}^\dagger$ – providing the proper transformation property for the right-hand side. The parameter, $\xi$, is at present an arbitrary number which is to be chosen to simplify later results.

With this choice, the Lagrange-multiplier term of eq. (8) becomes:

$$S_{LM} = 2 \xi \int d^2 x \ tr \{(X^\dagger \partial_+ X) r^\dagger (L^\dagger \partial_- L) \ell r^\dagger \}$$

$$= 2 \xi \int d^2 x \ tr \{(X^\dagger \partial_+ X) [(L \ell r^\dagger)\dagger \partial_- (L \ell r^\dagger) - (\ell r^\dagger)\dagger \partial_- (\ell r^\dagger)] \}$$

$$= -8 \pi \xi \left\{ \Gamma \left[ L \ell r^\dagger X^\dagger \right] - \Gamma \left[ L \ell r^\dagger \right] - \Gamma \left[ \ell r^\dagger X^\dagger \right] + \Gamma \left[ \ell r^\dagger \right] \right\}.$$  

(10)

This final form follows after using the Polyakov-Wiegmann identity, eq. (4).

\(^3\) We thank Luis Alvarez-Gaumé for a key conversation on this point. A general discussion of non-abelian duality using group-valued dual variables may be found in Ref. [9].
All that remains is to find the Jacobian for the change of variables from $\Lambda$ to $X$. For the purposes of doing so it is useful to think of this transformation as happening in two steps, first from $\Lambda$ to $\hat{D}_+\Lambda$, and then from $\hat{D}_+\Lambda$ to $X$. The measures therefore are related by:

$$
[d\Lambda] = [d(\hat{D}_+\Lambda)] \left[ \frac{\det \partial_+}{\det \hat{D}_+(r)} \right]
= [dX] \left[ \frac{\det \hat{D}_+(Xr)}{\det \hat{D}_+(r)} \right],
$$

where the notation $\hat{D}_\pm(g)$ is meant to indicate that the corresponding covariant derivative is constructed using the gauge field $ig^\dagger \partial_+ g$. The Jacobian for transforming from $\hat{D}_+\Lambda$ to $X$ may be recognized as a special case of the Jacobian of eq. (7), for transforming from a gauge potential to the corresponding Wilson-line variable.

A problem presents itself as soon as one tries to make sense out of the determinants which appear in eq. (11). This is because these determinants are not yet unambiguously defined [10], suffering as they do from an anomaly for the vector symmetry group, $O_V(N)$. As a result, although the original measure, $[d\Lambda]$, was supposed to be invariant under vectorlike background gauge transformations — as is $[dX]$ trivially, since $X$ is invariant — the determinants which appear on the right-hand-side of eq. (11) are not.

We may remove these ambiguities by using the following two properties to more completely define the determinants of interest. We firstly require that both determinants be invariant under $O_V(N)$ background gauge transformations. Also, since lorentz-invariance would require an integration over $\hat{D}_-\Lambda$ as well as over $\hat{D}_+\Lambda$, we can think of eq. (11) as having been evaluated in a gauge for which $\hat{D}_-\Lambda = \partial_-\Lambda$. We therefore also require that our $O_V(N)$-invariant result for the determinants agree with the naive application of eq. (2) to eq. (11) in this gauge.

These two conditions introduce an $\ell$-dependence into the result, and uniquely specify the determinants to be:

$$
[d\Lambda] = [dX] \exp \left\{ i\kappa \Gamma [\ell r^\dagger] - i\kappa \Gamma [\ell r^\dagger X^\dagger] \right\}.
$$

8
We may now put the above four results together to simplify our starting expression, eq. (5). We have:

\[
Z[a] = \int [d\psi] [dA_+] [dA_-] [d\Lambda] \exp\left\{i \int d^2 x \left[ -\frac{1}{2} \overline{\psi} (\gamma^+ \gamma_R D_+ + \gamma^- \gamma_L D_-) \psi \\
+ 2 \text{ tr} (\Lambda V_{+,-}) \right]\right\} \delta[A_+]
\]

\[
= \int [dX] [dL] \exp\left\{-i(1 + \kappa) \Gamma [L \ell r\dagger] + i\kappa \left( \Gamma [\ell r\dagger] - \Gamma [\ell r\dagger X\dagger] \right) \\
- 8\pi i \xi \left( \Gamma [L \ell r\dagger X\dagger] - \Gamma [L \ell r\dagger] - \Gamma [\ell r\dagger X\dagger] + \Gamma [\ell r\dagger] \right) \right\}.
\] (13)

Each of the terms in this last expression correspond to one of the items from the above discussion\(^4\). To wit: (1) the factor proportional to \((1 + \kappa)\) originates from the fermion determinant, and the Jacobian for the change of variables from \(A_\pm\) to \(R\) and \(L\) (eqs. (6) and (7)); (2) the terms proportional to \(\kappa\) arise due to the Jacobian of eq. (12) for the change from \(\Lambda\) to \(X\); and (3) the terms proportional to \(\xi\) correspond to the Lagrange-multiplier lagrangian of eq. (10).

At this point it is worthwhile to make a helpful choice for the parameter \(\xi\). Since the coefficient which premultiplies the factor \(\Gamma [L \ell r\dagger]\) is proportional to \((1 + \kappa - 8\pi \xi)\), it is irresistible to choose \(8\pi \xi = 1 + \kappa\), in which case this entire term vanishes. This has the great advantage of completely decoupling the \([dL]\) integration from all of the others, since \(L\) then only enters the functional integrand through the overall multiplicative factor

\[
\int [dL] \exp\left\{-i(1 + \kappa) \Gamma [L \ell r\dagger X\dagger] \right\}.
\] (14)

The change of variables \(L \rightarrow \hat{L} = L \ell r\dagger X\dagger\), for which \([d\hat{L}] = [dL]\), then shows this to be an irrelevant field-independent constant.

We are left with

\[
Z[a] = \int [dX] \exp\left\{i(\kappa - 8\pi \xi) \left( \Gamma [\ell r\dagger] - \Gamma [\ell r\dagger X\dagger] \right) \right\}
\]

\[
= \int [dX] \exp\left\{-i \left( \Gamma [\ell r\dagger] - \Gamma [\ell r\dagger X\dagger] \right) \right\}.
\] (15)

\(^4\) Notice that implicit in equation (13) there is a nontrivial path integral representation of the \(\delta\)-function as a result of the change of variables II.
This may be put into a more familiar form simply by redefining fields $X \rightarrow g$, with $X^\dagger = r g r^\dagger$, for which $[dX] = [dg]$. With this choice $g$ inherits the transformation rule $g \rightarrow G g G^\dagger$ under the vectorlike symmetry. The final, bosonized, result becomes:

$$Z[a] = \int [dg] \exp\left\{ i\Gamma_{GWZw}(g, a) \right\},$$

(16)

where

$$\Gamma_{GWZw}(g, a) = \Gamma [\ell g r^\dagger] - \Gamma [\ell r^\dagger] = \Gamma [g] + \frac{1}{4\pi} \int d^2 x \, \text{tr} \left[ ig^\dagger \partial_- ga_+ - i\partial_+ g g^\dagger a_- + g a_+ g^\dagger a_- - a_+ a_- \right].$$

(17)

This second way of writing the bosonized action re-expresses the Wilson-line variables $\ell$ and $r$ in terms of the original fields, $a_\mu$. In this form it may be recognized as the gauged Wess-Zumino-Witten action, which is usually derived by ‘gauging’ the global $O$, $O(N)$ symmetry of $\Gamma [g]$, using e.g. the Noether prescription [11]. This action also properly reproduces the fermion anomaly under the chiral $O_L(N) \times O_R(N)$ group provided that $g$ is defined to transform in the standard way: $g \rightarrow L g R^\dagger$.

The expression for $\Gamma_{GWZw}$ in terms of Wilson-line variables is useful in that it makes clear that the remaining integration over $g$ can be explicitly performed. Changing variables to $\hat{g} = \ell g r^\dagger$ (with $[d\hat{g}] = [dg]$) shows that the integration over $\hat{g}$ simply provides an overall field-independent normalization constant. We arrive in this way back to our expected result, eq. (2), for $Z[a]$. Notice also that by looking at the correlation functions, differentiating $Z[a]$ with respect to the background fields $a_\pm$ evaluated at $a_\pm = 0$ we find the known correspondence among the fermionic and the bosonic currents:

$$i \bar{\psi} \gamma_- \psi \leftrightarrow \frac{i}{2\pi} g^\dagger \partial_- g, \quad \text{and:} \quad i \bar{\psi} \gamma_+ \psi \leftrightarrow -\frac{i}{2\pi} \partial_+ g g^\dagger,$$

(18)

together with the delta-function contact terms in the correlations between the left- and right-handed currents that had been found elsewhere using path-integral methods [12].

Finally, following the same arguments as in ref. [2], it is also possible to extend this analysis to include four fermion couplings as well as mass terms, reproducing the results of ref. [1].
4. Conclusions

Our purpose has been to provide a constructive and systematic derivation of the rules for nonabelian bosonization that were first written down by Witten some ten years ago. We have done so by showing that nonabelian bosonization may be considered to be a special case of a nonabelian duality transformation, for which systematic and constructive techniques have recently been formulated. It is our hope that this connection can lead to a wider application and understanding of both nonabelian bosonization and duality transformations.

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