Existence, uniqueness and comparison theorem on unbounded solutions of scalar super-linear BSDEs

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Abstract
This paper is devoted to the existence, uniqueness and comparison theorem on unbounded solutions of a scalar backward stochastic differential equation (BSDE) whose generator grows (with respect to both unknown variables \(y\) and \(z\)) in a super-linear way like \(|y| \ln|y|^{(\lambda+1/2)\lambda^1} + |z| \ln|z|^{\lambda}\) for some \(\lambda \geq 0\). For the following four different ranges of the growth power parameter \(\lambda\): \(\lambda = 0\), \(\lambda \in (0, 1/2)\), \(\lambda = 1/2\) and \(\lambda > 1/2\), we give reasonably weakest possible different integrability conditions of the terminal value for the existence of an unbounded solution to the BSDE. In the first two cases, they are stronger than the \(L \ln L\)-integrability and weaker than any \(L^p\)-integrability with \(p > 1\); in the third case, the integrability condition is just some \(L^p\)-integrability for \(p > 1\); and in the last case, the integrability condition is stronger than any \(L^p\)-integrability with \(p > 1\) and weaker than any \(\exp(L^\varepsilon)\)-integrability with \(\varepsilon \in (0, 1)\).

We also establish the comparison theorem, which yields naturally the uniqueness, when either generator of both BSDEs is convex (concave) in both unknown variables \((y, z)\), or satisfies a one-sided Osgood condition in the first unknown variable \(y\) and a uniform continuity condition in the second unknown variable \(z\).

Keywords: Backward stochastic differential equation, Existence and uniqueness, Super-linear growth, Comparison theorem, One-sided Osgood condition, Uniform continuity condition.

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1. Introduction

We fix a positive real number \(T > 0\) and a positive integer \(d\), and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \((B_t)_{t \in [0,T]}\) an \(\mathbb{R}^d\)-valued standard Brownian motion defined on this space, and \((\mathcal{F}_t)_{t \in [0,T]}\) the natural filtration generated by \(B\) and augmented by all \(\mathbb{P}\)-null sets of \(\mathcal{F}\). Any progressively measurability with respect to processes will refer to this filtration in this paper.

Let \(\mathbb{R}_+\) be the set of all nonnegative real numbers, \(x \cdot y\) the scalar inner product of two vectors \(x, y \in \mathbb{R}^d\), \(1_A\) the indicator function of set \(A\), and \(\text{sgn}(x) := 1_{x>0} - 1_{x \leq 0}\). We recall that a real-valued

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and progressively measurable process \((X_t)_{t \in [0,T]}\) belongs to class (D) if the family of random variables \(\{X_t : \tau\) is any \((\mathcal{F}_t)\) -stopping time taking values in \([0,T]\)\} is uniformly integrable.

For any real \(p \geq 1\), let \(L^p\) denote the set of all real-valued and \(\mathcal{F}_T\)-measurable random variables \(\xi\) satisfying \(\mathbb{E}[|\xi|^p] < +\infty\), \(L^p\) the set of all real-valued and progressively measurable processes \((X_t)_{t \in [0,T]}\) such that
\[
\|X\|_{L^p} := \left\{ \mathbb{E} \left[ \left( \int_0^T |X_t|dt \right)^p \right] \right\}^{1/p} < +\infty,
\]
\(S^p\) the set of all real-valued, progressively measurable and continuous processes \((Y_t)_{t \in [0,T]}\) satisfying
\[
\|Y\|_{S^p} := \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^p \right] \right)^{1/p} < +\infty,
\]
and \(M^p\) the set of all \(\mathbb{R}^d\)-valued and progressively measurable processes \((Z_t)_{t \in [0,T]}\) such that
\[
\|Z\|_{M^p} := \left\{ \mathbb{E} \left[ \left( \int_0^T |Z_t|^2dt \right)^{p/2} \right] \right\}^{1/p} < +\infty.
\]

We consider the following backward stochastic differential equation (BSDE for short):
\[
Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0,T], \tag{1.1}
\]
where the terminal condition \(\xi\) is a real-valued \(\mathcal{F}_T\)-measurable random variable, and the generator
\[
g(\omega,t,y,z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}
\]
is jointly continuous in \((y,z)\) and progressively measurable for each \((y,z)\). A solution of (1.1) is a pair of processes \((Y_t,Z_t)_{t \in [0,T]}\) with values in \(\mathbb{R} \times \mathbb{R}^d\), which is progressively measurable such that \(\mathbb{P}\)-a.s., the functions \(t \mapsto Y_t\) is continuous, \(t \mapsto Z_t\) is square-integrable, \(t \mapsto g(t,Y_t,Z_t)\) is integrable, and Equation (1.1) is satisfied. We denote by BSDE \((\xi,g)\) the BSDE with the terminal value \(\xi\) and the generator \(g\).

In this paper, let us always assume that \(\beta, \gamma \geq 0\) are two nonnegative real number and \((\alpha_t)_{t \in [0,T]}\) is an \(\mathbb{R}_+\)-valued progressively measurable process. We are interested in the unbounded solution of BSDE (1.1) with a super-linearly growing generator \(g\), which satisfies, roughly speaking, that \(d\mathbb{P} \times dt\)-a.e.,
\[
\forall \ (y,z) \in \mathbb{R} \times \mathbb{R}^d, \quad |g(\omega,t,y,z)| \leq \alpha_t(\omega) + \beta|y|(|y|\mathbf{1}_{|y|>1} + \gamma|z| \ln |z|)^\lambda \tag{1.2}
\]
for some \((\delta, \lambda) \in (0,1] \times [0, +\infty)\).

The generator \(g\) is said to be of a linear growth when \(\delta = \lambda = 0\). In this case, if the terminal condition \((\xi,\alpha)\) belongs to \(L^p \times L^p\) for some \(p > 1\), then BSDE \((\xi,g)\) admits a solution in the space \(S^p \times M^p\), and the solution is unique in this space if \(g\) further satisfies the uniformly Lipschitz condition in \((y,z)\). The reader is referred to [30, 16, 28, 10, 22, 18] for details.

Recently, the authors in [26, 14, 19] found step by step a weakest integrability condition for \((\xi,\alpha)\) to guarantee the existence of a solution to a linearly growing BSDE \((\xi,g)\), where the terminal condition \((\xi,\alpha)\) only needs to satisfy an \(L \exp \left( \mu \sqrt{2 \ln(1 + L)} \right)\)-integrability condition for a positive parameter
\( \mu = \gamma \sqrt{T} \). They showed that this integrability condition is weaker than the usual \( L^p \)-integrability \( (p > 1) \) and stronger than the usual \( L \ln L \)-integrability, and that the preceding integrability for a positive parameter \( \mu < \gamma \sqrt{T} \) is not sufficient to ensure the existence of a solution. They also established the uniqueness of the unbounded solution under the conditions that the generator \( g \) satisfies the monotone condition in the state variable \( y \) and the uniformly Lipschitz condition in the state variable \( z \), and then extended some results mentioned in the above paragraph.

Furthermore, for quadratic BSDEs, namely, the generator \( g \) has a quadratic growth in the second unknown variable \( z \), much progress has been made in the last two decades. Roughly speaking, to have the existence of a solution to BSDE \((\xi, g)\), the boundedness or at least some exponential integrability of the terminal condition \((\xi, \alpha)\) are required. The reader is referred to \([27, 12, 13, 15, 11, 9, 25, 18, 21, 20]\) among others. In addition, we mention that a class of quadratic BSDEs with \( L^p \)-integrable \( (p > 1) \) terminal conditions are investigated in several recent works, see for example \([2, 31, 1, 8]\).

Finally, we would like to especially mention that Bahlali et al. \([3, 7]\) proved the existence of a solution to BSDE \((\xi, g)\) in the space \( S^p \times M^2 \) for some sufficiently large \( p > 2 \), when the terminal condition \((\xi, \alpha)\) belongs to \( L^p \times L^p \) and the generator \( g \) satisfies (1.2) with \( \delta = 1 \) and \( \lambda = 1/2 \). They also established the uniqueness of the solution when \( g \) further satisfies a locally monotonicity condition in \((y, z)\). Some related works on BSDEs of super-linearly growing generators are available in \([4, 6, 5]\).

The paper is devoted to the existence, uniqueness and comparison theorem for unbounded solutions of BSDE\((\xi, g)\) with generator \( g \) having a super-linear growth (1.2). We distinguish four different situations: (i) \( \lambda = 0 \) and \( \delta = 1/2 \), (ii) \( \lambda \in (0, 1/2) \) and \( \delta = \lambda + 1/2 \), (iii) \( \lambda = 1/2 \) and \( \delta = 1 \), (iv) \( \lambda > 1/2 \) and \( \delta = 1 \), and put forward four reasonably weakest possible integrability conditions of the terminal value \((\xi, \alpha)\) to ensure the existence of an unbounded solution to the BSDE. In the first two cases, they are stronger than the \( L \ln L \)-integrability and weaker than any \( L^p \)-integrability with \( p > 1 \); in the third case, the integrability condition is just some \( L^p \)-integrability for \( p > 1 \); and in the last case, the integrability condition is stronger than any \( L^p \)-integrability with \( p > 1 \) and weaker than any \( \exp(L^\varepsilon) \)-integrability with \( \varepsilon \in (0, 1) \). We also establish the comparison theorem, which yields naturally the uniqueness, when either generators of both BSDEs is convex (concave) in both unknown variables \((y, z)\), or satisfies a one-sided Osgood condition in the first unknown variable \( y \) and a uniform continuity condition in the second unknown variable \( z \). We would like to remark that all these results obtained in this paper are not covered by any existing results.

The paper is organized as follows. In next section, we introduce the whole strategy of the paper and establish three useful propositions, which will play an important role later, and in subsequent four sections we establish, respectively for four kinds of different cases mentioned above, the existence, uniqueness and comparison theorem for unbounded solutions of BSDEs in four kinds of different spaces. In the last section, we conclude the whole paper.
2. The whole strategy and three useful propositions

2.1. Existence

For the existence of an unbounded solution to BSDE \((\xi, g)\) with generator \(g\) satisfying (1.2), our whole idea is to establish some uniform a priori estimate and then apply the localization procedure put forward initially in [12]. To this end, we first establish a general stability result for the solutions of BSDEs.

We assume that \(\xi\) is a terminal condition and the generator \(g\) is continuous in \((y, z)\) and satisfies the following general growth assumption:

(EX1) The generator \(g\) has a general growth in \(y\) and a quadratic growth in \(z\), i.e., there exists a real-valued progressively measurable process \((f_t)_{t \in [0,T]}\) with \(\mathbb{P}\)-a.s., \(\int_0^T f_t dt < +\infty\), a nonnegative real function \(H(x)\) defined on \(\mathbb{R}_+\) with \(H(0) = 0\), and a positive real \(c > 0\) such that \(d\mathbb{P} \times dt\)-a.e., for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\),

\[
|g(\omega, t, y, z)| \leq f_t(\omega) + H(|y|) + c|z|^2.
\]

For each pair of positive integers \(n, p \geq 1\), set

\[
\xi^{n,p} := \xi^+ \land n - \xi^- \land p \quad \text{and} \quad g^{n,p}(\omega, t, y, z) := g^+(\omega, t, y, z) \land n - g^-(\omega, t, y, z) \land p, \tag{2.1}
\]

where and hereafter, \(a^+\) stands for the maximum of \(a\) and 0, \(a^- := (-a)^+\) and \(a \land b\) represents the minimum of \(a\) and \(b\). It follows from [27] that for each \(n, p \geq 1\), the following BSDE\((\xi^{n,p}, g^{n,p})\) admits a minimal (maximal) bounded solution \((Y_t^{n,p}, Z_t^{n,p})_{t \in [0,T]}\) such that \(Y_t^{n,p}\) is bounded and \(Z_t^{n,p} \in \mathcal{M}^2\):

\[
Y_t^{n,p} = \xi^{n,p} + \int_t^T g^{n,p}(s, Y_s^{n,p}, Z_s^{n,p}) ds - \int_t^T Z_s^{n,p} \cdot dB_s, \quad t \in [0, T]. \tag{2.2}
\]

By the comparison theorem, \(Y_t^{n,p}\) is nondecreasing in \(n\) and non-increasing in \(p\). Furthermore, we have the following monotone stability theorem, which slightly generalizes those of [27] and [13].

**Proposition 2.1.** Assume that \(\xi\) is a terminal condition and the generator \(g\) is continuous in the state variables \((y, z)\) and satisfies (EX1). For each pair of positive integers \(n, p \geq 1\), let \(\xi^{n,p}\) and \(g^{n,p}\) be defined in (2.1), and \((Y_t^{n,p}, Z_t^{n,p})\) be the minimal (maximal) bounded solution of (2.2). If there exists an \(\mathbb{R}_+\)-valued, progressively measurable and continuous process \((X_t)_{t \in [0,T]}\) such that

\[
d\mathbb{P} \times dt\text{-a.e., } \forall n, p \geq 1, \quad |Y_t^{n,p}| \leq X_t,
\]

then there exists an \(\mathbb{R}^d\)-valued progressively measurable process \((Z_t)_{t \in [0,T]}\) such that \((Y := \inf_p \sup_n Y_t^{n,p}, Z)\) is a solution to BSDE \((\xi, g)\).

**Proof.** We use the localization procedure to construct the desired solution. For each integer \(m \geq 1\), define the following stopping time:

\[
\sigma_m := \inf \left\{ t \in [0, T] : X_t + \int_0^t f_s ds \geq m \right\} \land T
\]

with the convention that \(\inf \emptyset = +\infty\). Then \((Y_t^{m,p}(t), Z_t^{m,p}(t)) := (Y_t^{n,p}, Z_t^{n,p}1_{t \leq \sigma_m})\) solves the BSDE:
\[ Y^{n,p}_m(t) = Y^{n,p}_\sigma + \int_t^T 1_{s \leq \sigma_m} g^{n,p}(s, Y^{n,p}_m(s), Z^{n,p}_m(s))ds - \int_t^T Z^{n,p}_m(s) \cdot dB_s, \quad t \in [0, T]. \]

Note that for fixed \( m \geq 1 \), \( Y^{n,p}_m(\cdot) \) is nondecreasing in \( n \), non-increasing in \( p \) and bounded by \( m \), and that \( dP \times dt \)-a.e., the sequence \( (g^{n,p})_{n,p} \) converges locally uniformly in \((y, z)\) to the generator \( g \) as \( n, p \to \infty \). Furthermore, in view of the facts that \(|g^{n,p}| \leq |g|\) and \( g \) satisfies (EX1), it follows that \( dP \times ds \rightarrow a.e. \)

\[ \forall (y, z) \in [-m, m] \times \mathbb{R}^d, \quad \sup_{n, p \geq 1} (1_{s \leq \sigma_m} |g^{n,p}(s, y, z)|) \leq 1_{s \leq \sigma_m} f_s + H(m) + c|z|^2, \]

with \( P \)-a.s., \( \int_0^T 1_{s \leq \sigma_m} f_s ds \leq m \). Thus, we can apply the stability property for bounded solutions of BSDEs (see e.g. Proposition 3.1 in [29]). Setting \( Y_m(t) := \inf_p \sup_n Y^{n,p}_{t \wedge \sigma_m} \), then \( Y_m(\cdot) \) is continuous and there exists a process \( Z_m(\cdot) \) such that

\[ \lim_{n, p \to \infty} Z^{n,p}_t 1_{t \leq \sigma_m} = Z_m(t) \quad \text{in} \quad \mathcal{M}^2 \]

and

\[ Y_m(t) = \inf_p \sup_n Y^{n,p}_m + \int_t^T 1_{s \leq \sigma_m} g(s, Y_m(s), Z_m(s))ds - \int_t^T Z_m(s) \cdot dB_s, \quad t \in [0, T]. \]

Finally, in view of the assumptions of Proposition 2.1, the stability of stopping time \( \sigma_m \) and the fact that \( dP \times dt \)-a.e., for each \( m \geq 1 \),

\[ Y_{m+1}(t \wedge \sigma_m) = Y_m(t \wedge \sigma_m) = \inf_p \sup_{n \geq 1} Y^{n,p}_{t \wedge \sigma_m} \]

and

\[ Z_{m+1} 1_{t \leq \sigma_m} = Z_m 1_{t \leq \sigma_m} = \lim_{n, p \to \infty} Z^{n,p}_t 1_{t \leq \sigma_m}, \]

the conclusion of Proposition 2.1 follows immediately by picking

\[ Z_t := \sum_{m=1}^{+\infty} Z_m(t) 1_{t \in [\sigma_{m-1}, \sigma_m)}, \quad t \in [0, T] \]

with \( \sigma_0 := 0 \). The proof is complete.

Now, to apply the preceding stability theorem, we need the uniform a priori bound \( X \) for the processes \( Y^{n,p}_m \), which is crucial. Our strategy is to first search for an appropriate function \( \phi(s, x) \) and then apply Itô-Tanaka’s formula to \( \phi(s, Y^{n,p}_s) \) on the time interval \( s \in [t, \tau_m] \) with \( t \in (0, T] \) and \( \tau_m \) being an \((\mathcal{F}_t)\)-stopping time valued in \([t, T]\). More specifically, we need to find a constant \( \delta \geq 0 \) and a nonnegative smooth function \( \phi(\cdot, \cdot) : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \phi_x(s, x) > 0, \phi_{xx}(s, x) > \delta \) and

\[ -\phi_x(s, x) (\beta x (\ln x)^\gamma 1_{x>1} + \gamma |\ln x|) + \frac{1}{2} (\phi_{xx}(s, x) - \delta) |z|^2 + \phi_z(s, x) \geq 0 \quad (2.3) \]

for all \((s, x, z) \in (0, T] \times \mathbb{R}_+ \times \mathbb{R}^d \). Here and hereafter, \( \phi_z \) is the first-order partial derivative of \( \phi \) with respect to the first variable, and \( \phi_x \) and \( \phi_{xx} \) are the first- and second-order partial derivatives of \( \phi \) with respect to the second variable.

To find the function \( \phi \) satisfying inequality (2.3), our next idea is to find, for the term in the left hand side of (2.3), a perfect lower bound \( \Phi(s, x) \) without containing the variable \( z \), and transfer (2.3) to
the inequality \( \Phi(s, x) \geq 0 \). For the case \( \lambda = 0 \), we use the inequality \( 2ab \leq a^2 + b^2 \) to get that for each \((s, x, z) \in (0, T] \times \mathbb{R}_+ \times \mathbb{R}^d\),

\[
- \gamma \phi_x(s, x) |z| + \frac{1}{2} (\phi_{xx}(s, x) - \delta) |z|^2 = \frac{1}{2} \left( \phi_{xx}(s, x) - \delta \right) \left( -2 \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta} |z| + |z|^2 \right) \geq -\frac{\gamma^2}{2} \frac{(\phi_x(s, x))^2}{\phi_{xx}(s, x) - \delta}.
\]

Hence, (2.3) holds if the function \( \phi(\cdot, \cdot) \) satisfies the following condition:

\[
\forall (s, x) \in (0, T] \times \mathbb{R}_+, \quad -\beta \phi_x(s, x) x [\ln x]^4 1_{x > 1} - \frac{\gamma^2}{2} \frac{(\phi_x(s, x))^2}{\phi_{xx}(s, x) - \delta} + \phi_x(s, x) \geq 0. \tag{2.4}
\]

However, for the other case of \( \lambda \in (0, +\infty) \), the situation seems to be more complicated.

We first establish the following inequality.

**Proposition 2.2.** Given \( \lambda > 0 \). For each \( k > 1 \), there is a positive constant \( C_{k, \lambda} > 0 \) depending only on \((k, \lambda)\) such that

\[
\forall x, y > 0, \quad 2xy \ln y \leq x^2 \left( k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda} + C_{k, \lambda} \right) + y^2. \tag{2.5}
\]

Moreover, for each \( k \leq 1 / 4^{(\lambda - 1)^+} \), there is no constant \( C_{k, \lambda} \) to satisfy the last inequality.

**Proof.** Note first that (2.5) can be rewritten as

\[
2 \frac{y}{x} |\ln y| + \ln x \leq k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda} + C_{k, \lambda} + \left( \frac{y}{x} \right)^2.
\]

Set \( a := y/x > 0 \). Then, to prove (2.5), it is equivalent to prove that

\[
\forall a, x > 0, \quad 2a |\ln a + \ln x| \leq k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda} + a^2 + C_{k, \lambda}. \tag{2.6}
\]

Now, we let \( k > 1 \) and prove (2.6). By virtue of Young’s inequality, observe that for each \( a, x > 0 \),

\[
|\ln a + \ln x| \leq 2^{(\lambda - 1)^+} |\ln a| + 2^{(\lambda - 1)^+} |\ln x| \tag{2.7}
\]

and

\[
2a \left( 2^{(\lambda - 1)^+} |\ln x| \right) \leq \frac{1}{k} a^2 + k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda}. \tag{2.8}
\]

And, note that the function

\[
f(a; k, \lambda) := 2a \left( 2^{(\lambda - 1)^+} |\ln a| \right) = \left( 1 - \frac{1}{k} \right) a^2
\]

is continuous on \( a \in (0, +\infty) \), and respectively tends to 0 and \(-\infty\) as \( a \) tends to \( 0^+ \) and \(+\infty\), and then there must exist a constant \( C_{k, \lambda} > 0 \) depending only on \((k, \lambda)\) such that

\[
\forall a > 0, \quad f(a; k, \lambda) \leq C_{k, \lambda}. \tag{2.9}
\]

Combining (2.7), (2.8) and (2.9) yields that

\[
2a |\ln a + \ln x| - k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda} - a^2
\]

\[
\leq 2a \left( 2^{(\lambda - 1)^+} |\ln a| \right) - \left( 1 - \frac{1}{k} \right) a^2 + 2a \left( 2^{(\lambda - 1)^+} |\ln x| \right) - k 4^{(\lambda - 1)^+} |\ln x|^{2\lambda} - \frac{1}{k} a^2
\]

\[
\leq f(a; k, \lambda) \leq C_{k, \lambda}, \quad \forall a, x > 0.
\]
This is just (2.6). Thus, we have proved (2.5) for each $k > 1$.

Next, we show the second part of this proposition. In fact, it suffices to prove that (2.6) does not hold for $k = 1/4(\lambda-1)^+$, i.e., there does not exist a positive constant $C_\lambda > 0$ such that

$$\forall a, x > 0, \quad 2a|\ln a + \ln x|^{\lambda} \leq |\ln x|^{2\lambda} + a^2 + C_\lambda.$$  \hfill (2.10)

Let $\bar{a}(x) := |\ln x|^{\lambda}$. Then $2\bar{a}(x)|\ln x|^{\lambda} = |\ln x|^{2\lambda} + \bar{a}(x)$. Thus, if (2.10) holds, then we have

$$g(x; \lambda) := 2\bar{a}(x)|\ln \bar{a}(x) + \ln |\ln x|^{\lambda} - 2\bar{a}(x)|\ln x|^{\lambda} \leq C_\lambda, \quad x > 0,$$

which is impossible since $g(x; \lambda)$ tends to positive infinity as $x \to +\infty$. The proof is then complete. \hfill \Box

With Proposition 2.2 in the hand, coming back to (2.3), observe that for each $k > 1$, $\lambda > 0$, $\gamma > 0$ and $\delta \geq 0$, there exists a constant $C_{k,\lambda} > 0$ such that for each $(s, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$-\gamma \phi_x(s, x)|z||\ln ||z||^{\lambda} + \frac{1}{2} \left(\phi_{xx}(s, x) - \delta\right) |z|^2$$

$$\geq \frac{1}{2} \left(\phi_{xx}(s, x) - \delta\right) \left(-2\frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta} |z||\ln ||z||^{\lambda} + |z|^2\right)$$

$$\geq -\frac{\gamma^2}{2} \left(\phi_x(s, x)\right)^2 \left(k4(\lambda-1)^+ \left|\ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta}\right|^{2\lambda} + C_{k,\lambda}\right).$$

Then, (2.3) holds if for some $k > 1$ and $C_{k,\lambda} > 0$ associated with $k$ by Proposition 2.2, the function $\phi(s, x)$ satisfies that for each $(s, x) \in (0, T] \times \mathbb{R}^+$, with $\gamma > 0$,

$$-\beta \phi_x(s, x) x|\ln x|^{\delta} 1_{x > 1} - \frac{\gamma^2}{2} \left(\phi_x(s, x)\right)^2 \left(k4(\lambda-1)^+ \left|\ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta}\right|^{2\lambda} + C_{k,\lambda}\right) + \phi_x(s, x) \geq 0. \hfill (2.11)$$

In the following four sections, for four different cases, we shall find the constant $\delta \geq 0$ and the function $\phi$ satisfying (2.4) and (2.11), and then the uniform a priori bound $X$ for $Y^{n, p}$ in Proposition 2.1.

2.2. Uniqueness

For the uniqueness and comparison theorem of the unbounded solutions to BSDE $(\xi, g)$ with generator $g$ satisfying (1.2), more assumptions on the generator $g$ are required as usual than those required for the existence.

First, let us introduce the following three assumptions.

(UN1) The generator $g$ satisfies the one-sided Osgood condition in $y$, i.e., there exists a nonnegative, increasing, continuous and concave function $\rho(\cdot)$ defined on $\mathbb{R}^+$ satisfying $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and

$$\int_0^+ \frac{du}{\rho(u)} := \lim_{\varepsilon \to 0^+} \int_0^\varepsilon \frac{du}{\rho(u)} = +\infty$$

such that $dP \times dt$-a.e., for each $(y_1, y_2, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$\text{sgn}(y_1 - y_2)(g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \leq \rho(|y_1 - y_2|).$$
(UN2) The generator $g$ is uniformly continuous in $z$, i.e., there exists a nonnegative, nondecreasing and continuous function $\kappa(\cdot)$ defined on $\mathbb{R}_+$ with $\kappa(0) = 0$ such that $d\mathbb{P} \times dt$-a.e., for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}_d \times \mathbb{R}_d$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \kappa(|z_1 - z_2|).$$

(UN3) $d\mathbb{P} \times dt$-a.e., the generator $g$ is convex or concave in $(y, z)$.

Remark 2.3. Assumptions (UN1) and (UN2) are respectively strictly weaker than the usual monotonicity condition of the generator $g$ in $y$ and the uniformly Lipschitz continuity condition of $g$ in $z$. They are usually used when the generator $g$ has a general growth in $y$ and a linear growth in $z$. See [23, 17, 18] among others for more details. And, to study the uniqueness, Assumption (UN3) seems to be very natural for a non-linear growth function, see for example [13] and [15], where the convexity (concavity) condition of the generator $g$ in $z$ are required to ensure the uniqueness of the solution.

Remark 2.4. Without loss of generality, we always assume that the functions $\rho(\cdot)$ and $\kappa(\cdot)$ defined respectively in (UN1) and (UN2) are of linear growth, i.e., there exists a constant $A > 0$ such that

$$\forall \ u \in \mathbb{R}_+, \ \rho(u) \leq A(u + 1) \ \text{and} \ \kappa(u) \leq A(u + 1).$$

Now, we establish a general comparison theorem for solutions of BSDEs under Assumptions (UN1) and (UN2), which naturally yields the uniqueness of the solution. In particular, it is clear that this comparison theorem strengthens the uniqueness part of Fan and Hu [19, Theorem 3.1].

Proposition 2.5. Let $\xi$ and $\xi'$ be two terminal conditions, $g$ and $g'$ be two generators which are both continuous in the state variables $(y, z)$, and $(Y_t, Z_t)_{t \in [0,T]}$ and $(Y'_t, Z'_t)_{t \in [0,T]}$ be respectively a solution to BSDE $(\xi, g)$ and BSDE $(\xi', g')$ such that both $\psi(|Y|, \mu)$ and $\psi(|Y'|, \mu)$ belongs to class (D), where

$$\psi(x, \mu) := x \exp \left( \mu \sqrt{2 \ln(1 + x)} \right), \ (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+.$$  \hspace{1cm} (2.12)

and $\mu$ is any nonnegative, strictly increasing and continuous function defined on $[0, T]$ with $\mu_0 = 0$.

Assume that $\mathbb{P}$-a.s., $\xi \leq \xi'$. If $g$ (resp. $g'$) verifies Assumptions (UN1) and (UN2), and $d\mathbb{P} \times dt$-a.e.,

$$1_{Y_{t} > Y'_{t}} \ (g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)) \leq 0 \quad (\text{resp.} \ 1_{Y_{t} > Y'_{t}} \ (g(t, Y_t, Z_t) - g'(t, Y_t, Z_t)) \leq 0),$$

then $\mathbb{P}$-a.s., for each $t \in [0, T]$, $Y_t \leq Y'_t$.

To prove the above proposition, we need the following two lemmas of Hu and Tang [26].

Lemma 2.6. Let the function $\psi$ be defined in (2.12). Then we have

(i) For each $x \in \mathbb{R}_+$, $\psi(x, \cdot)$ is increasing on $\mathbb{R}_+$.

(ii) For $\mu \in \mathbb{R}_+$, $\psi(\cdot, \mu)$ is a positive, increasing and strictly convex function on $\mathbb{R}_+$.

(iii) For all $c > 1$ and $x, \mu \in \mathbb{R}_+$, $\psi(cx, \mu) \leq \psi(c, \mu) \psi(x, \mu)$.

(iv) For all $x_1, x_2, \mu \in \mathbb{R}_+$, $\psi(x_1 + x_2, \mu) \leq \frac{1}{2} \psi(2, \mu) [\psi(x_1, \mu) + \psi(x_2, \mu)]$. 
(v) For each $x \in \mathbb{R}$, $y \in \mathbb{R}_+$ and $\mu > 0$, $e^x y \leq e^{\frac{x^2}{2\mu}} + e^{2\mu^2} \psi(y, \mu)$.

**Lemma 2.7.** Let $(q_t)_{t \in [0,T]}$ be an $\mathbb{R}^d$-valued progressively measurable process with $|q_t| \leq \epsilon$ almost surely. For each $t \in [0,T]$, if $0 \leq \lambda < \frac{1}{2\epsilon^2(T-t)}$, then

$$
\mathbb{E}\left[ e^{\lambda \int_0^t q_s \cdot dB_s} \right] \leq \frac{1}{\sqrt{1 - 2\epsilon^2(T-t)}}.
$$

Now, we prove Proposition 2.5.

**Proof of Proposition 2.5.** We only prove the case that $g$ verifies Assumptions (UN1) and (UN2), and $d\mathbb{P} \times dt$-a.e., $1_{Y_i > Y_i'}(g(t, Y_i', Z_i') - g'(t, Y_i', Z_i')) \leq 0$. In the same way, another case can be proved.

Define $\hat{Y} := Y - Y'$ and $\hat{Z} := Z - Z'$. Then the pair $(\hat{Y}, \hat{Z})$ verifies

$$
\hat{Y}_t = \xi - \xi' + \int_t^T (g(s, Y_s, Z_s) - g'(s, Y_s', Z_s')) ds - \int_t^T \hat{Z}_s \cdot dB_s, \quad t \in [0, T].
$$

(2.14)

From the assumption of $1_{Y_i > Y_i'}(g(t, Y_i', Z_i') - g'(t, Y_i', Z_i')) \leq 0$ and Assumptions (UN1) and (UN2) of the generator $g$ together with Remark 2.4, it is not difficult to deduce that $d\mathbb{P} \times ds$ - a.e.,

$$
1_{Y_i > 0}(g(s, Y_s, Z_s) - g'(s, Y_s', Z_s')) \leq \rho(\hat{Y}_s^+) + 1_{\hat{Y}_s > 0}(\kappa(|\hat{Z}_s|)) \leq A\hat{Y}_s^+ + A1_{\hat{Y}_s > 0}\hat{Z}_s + 2A.
$$

(2.15)

For each $t \in [0, T]$ and each integer $n \geq 1$, define the following stopping times:

$$
\sigma_n := \inf \left\{ s \in [t, T] : |\hat{Y}_s| + \int_t^s |\hat{Z}_r|^2 dr \geq n \right\} \wedge T.
$$

Itô-Tanaka’s formula applied (2.14) gives

$$
\hat{Y}_t^+ = \hat{Y}_{\sigma_n} + \int_t^{\sigma_n} 1_{\hat{Y}_s > 0} (g(s, Y_s, Z_s) - g'(s, Y_s', Z_s')) ds - \int_t^{\sigma_n} 1_{\hat{Y}_s > 0} \hat{Z}_s \cdot dB_s - \int_t^{\sigma_n} dL_s, \quad t \in [0, T],
$$

where $L$ is the local time of $\hat{Y}$ at the origin. Then, in view of (2.15),

$$
e^{AT}\hat{Y}_t^+ \leq e^{AT}(\hat{Y}_{\sigma_n}^+ + 2AT) + \int_t^{\sigma_n} v_s \cdot \hat{Z}_s ds - \int_t^{\sigma_n} \hat{Z}_s \cdot dB_s, \quad t \in [0, T],
$$

(2.16)

with $v_s := A \hat{Z}_s / |\hat{Z}_s|^2 1_{|\hat{Z}_s| > 0}$ and $\tilde{Z}_s := 1_{\hat{Y}_s > 0} e^{As} \hat{Z}_s$. It is clear that $|v_s| \leq A$ and it follows from (2.16) that

$$
\forall n \geq 1, \quad \hat{Y}_t^+ \leq e^{AT} \left( e^{\int_{\sigma_n} t} v_s \cdot dB_s \hat{Y}_{\sigma_n}^+ + 2AT \right), \quad t \in [0, T].
$$

(2.17)

Furthermore, in view of Assertion (v) of Lemma 2.6, we know that for each $n \geq 1$,

$$
e^{\int_{\sigma_n} t} v_s \cdot dB_s \hat{Y}_{\sigma_n}^+ \leq e^{\frac{\sigma_n - t}{\mu}} (\int_{\sigma_n} t} v_s \cdot dB_s)^2 + e^{2\mu^2} \psi(\hat{Y}_{\sigma_n}^+, \mu_t), \quad t \in (0, T).
$$

(2.18)

And, by virtue of the assumptions of $\mu$, we can pick a (unique) $T_1 \in (0, T)$ such that $\mu_{T_1}^2 = 4A^2(T - T_1)$ and $\mu_t^2 \geq 4A^2(T - t)$ for each $t \in [T_1, T]$. It then follows from Lemma 2.7 that for all $n \geq 1$,

$$
\mathbb{E}\left[ e^{\frac{\sigma_n - t}{\mu}} (\int_{\sigma_n} t} v_s \cdot dB_s)^2 \right] \leq \frac{1}{\sqrt{1 - 2A^2(T - t)}} \leq \sqrt{2}, \quad t \in [T_1, T]
$$

and then, the family of random variables $e^{\frac{\sigma_n - t}{\mu}} (\int_{\sigma_n} t} v_s \cdot dB_s)^2$ is uniformly integrable on the interval $[T_1, T]$.

On the other hand, in view of Lemma 2.6 and the monotonicity of $\mu$, observe that for all $n \geq 1$,
\[
e^{2\mu^2 \psi} \left( \hat{Y}^+_{\sigma_n}, \mu_n \right) \leq e^{2\mu^2 \psi} \left( |Y_n| + |Y_n'|, \mu_n \right) \\
\leq \frac{e^{2\mu^2 \psi(2, \mu)} |\psi(0, \mu_n) + \psi \left( |Y_n|, \mu_n \right) |}{2}, \quad t \in [0, T].
\]

It then follows from (2.18) that for \( t \in [T_1, T] \), the family of random variables \( e^{\int_{t_n}^t e^{dB_s} \hat{Y}^+_m} \) is uniformly integrable. Consequently, in view of the fact that \( \hat{Y}^+_m = (\xi - \xi')^+ = 0 \), by letting \( n \to \infty \) in (2.17) we get that \( \hat{Y}^+_m = 2ATe^{\lambda T} \) on the time interval \([T_1, T]\), which means that \( \hat{Y}^+_m \) is a bounded process on \([T_1, T]\).

Thus, by virtue of the assumptions of Proposition 2.5 again, we can apply Theorem 2.1 in [18] to obtain \( \hat{Y}^+_m = 0 \) on the time interval \([T_1, T]\).

Now, in view of the fact \( Y^+_{nT} = 0 \) obtained above, we can respectively replace \( T_1 \) and \( T_2 \) in the above analysis with \( T_1 \) and \( T_2 \in (0, T_1) \) satisfying \( \mu_1^2 = 4A^2(1 - T_2) \) and \( \mu_2^2 = 4A^2(T_1 - t) \) for each \( t \in [T_2, T_1] \), and use the same argument to get that \( \hat{Y}^+_m = 0 \) on the time interval \([T_2, T_1]\).

Finally, repeating the above procedure, we successively obtain that \( \hat{Y}^+_m = 0 \) on the time intervals \([T_3, T_2], \ldots, [T_{m+1}, T_m], \ldots\), where \( 0 < T_{m+1} < T_m < T \) for each \( m \geq 1 \), and the limit of sequence \((T_m)_{m \geq 1}\) must exist and is denoted by \( T \). Noticing that \( \mu_{T_{m+1}}^2 = 4A^2(T_{m-1} - T_m) \) by the above construction and in view of the assumptions of \( \mu \), we can conclude that \( T = \lim_{m \to \infty} T_m = 0 \). Thus, in view of the continuity of process \( \hat{Y}^+_m \) with respect to the time variable \( t \), we have \( \hat{Y}^+_m = 0 \) on the whole time interval \([0, T]\) by sending \( m \to \infty \), which is the desired result. The proof is complete.

At the end of this subsection, we would like to mention that in subsequent four sections, we shall also establish, under the additional convexity Assumption (UN3), the uniqueness theorem and the comparison theorem for unbounded solutions of BSDEs with super-linearly growing generators and four different integrability terminal conditions. To take advantage of the convexity (concavity) condition of the generator \( g \), we use the \( \theta \)-technique developed in [13] to establish the comparison theorem for the unbounded solutions, which yields directly the uniqueness of the solution. More precisely, instead of estimating the difference between the processes \( Y \) and \( Y' \), we estimate \( Y - \theta Y' \) for each \( \theta \in (0, 1) \). We found that the uniform a priori estimate is still the key point.

3. The case of \( \exp \left( \mu \sqrt{2 \ln(1 + L)} \right) \)-integrable terminal conditions

In this section, we always assume that the generator \( g \) satisfies the following assumption, which is strictly weaker than (1.2) with \( \delta = 1/2 \) and \( \lambda = 0 \).

(A1) There exist two nonnegative constants \( \beta \geq 0 \) and \( \gamma \geq 0 \) with \( \beta + \gamma > 0 \), and an \( \mathbb{R}_+ \)-valued progressively measurable process \((\alpha_t)_{t \in [0, T]}\) such that \( dP \times dt \text{-a.e.}, \)

\[
\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(y)g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| \sqrt{\ln |y|} 1_{|y| > 1} + \gamma |z|.
\]

Based on the analysis in Section 2, we need to find a smooth function \( \phi(\cdot, \cdot) : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \phi_x(s, x) > 0, \phi_{xx}(s, x) > 0 \) and Equation (2.4) is satisfied for \( \delta = 1/2 \) and \( \delta = 0 \), i.e.,

\[
\forall (s, x) \in (0, T] \times \mathbb{R}_+, \quad -\beta \phi_x(s, x) x \sqrt{\ln x} 1_{x > 1} - \frac{\gamma^2}{2} \phi_{xx}(s, x) + \phi_x(s, x) \geq 0. \quad (3.1)
\]
We assume that $\beta, \gamma \geq 0$ with $\beta + \gamma > 0$, and choose the following function
\[
\phi(s, x) := (x + e)^{\mu_s \sqrt{2 \ln(x + e) + \nu_s}} > 0, \quad (s, x) \in [0, T] \times \mathbb{R}_+ \]
to explicitly solve the inequality (3.1), where $\mu_s, \nu_s : [0, T] \to \mathbb{R}_+$ are two strictly increasing and continuous functions with zero only at the origin, and are continuously differential on the time interval $(0, T]$. For each $(s, x) \in (0, T] \times \mathbb{R}_+$, a simple computation gives
\[
\phi_x(s, x) = (x + e) \frac{\mu_s + \sqrt{2 \ln(x + e)}}{(x + e) \sqrt{2 \ln(x + e)}} > 0, \quad (3.2)
\]
\[
\phi_{xx}(s, x) = \phi(s, x) \frac{\mu_s (2 \ln(x + e) + \mu_s \sqrt{2 \ln(x + e) - 1})}{(x + e)^2 \left(\sqrt{2 \ln(x + e)}\right)^3} > 0 \quad (3.3)
\]
and
\[
\phi_s(s, x) = \phi(s, x) \left(\mu_s' \sqrt{2 \ln(x + e)} + \nu_s'\right) > 0. \quad (3.4)
\]
Note that
\[
\forall x \in \mathbb{R}_+, \quad x \sqrt{\ln x 1_{x > 1}} \leq \frac{1}{\sqrt{2}} (x + e) \sqrt{2 \ln(x + e)}. \quad (3.5)
\]
Denote $v = \sqrt{2 \ln(x + e)}$. Substituting (3.2), (3.3), (3.4) and (3.5) into the left side of (3.1) yields that
\[
-\beta \phi_x(s, x) x \sqrt{\ln x 1_{x > 1}} \leq -\beta \phi(s, x) \frac{\mu_s + v}{\sqrt{2}} - \frac{\gamma^2}{2} \phi_x(s, x) + \phi_s(s, x)
\]
\[
\geq -\beta \phi(s, x) \frac{\mu_s + v}{\sqrt{2}} \frac{v (\mu_s + v)^2}{v^2 + \mu_s v - 1} + \phi(s, x)(\mu_s' v + \nu_s'), \quad (s, x) \in (0, T] \times \mathbb{R}_+.
\]
Furthermore, in view of the fact of $v \geq \sqrt{\beta}$, we know that
\[
v \frac{(\mu_s + v)^2}{v^2 + \mu_s v - 1} = (v + \mu_s) (v^2 + \mu_s v - 1) + (v + \mu_s)
\]
\[
\leq v + \mu_s + \frac{v + \mu_s}{2 (v^2 + \mu_s v)} = v + \mu_s + \frac{2}{v} \leq v + \mu_s + \sqrt{\beta},
\]
and then
\[
-\beta \phi_x(s, x) x \sqrt{\ln x 1_{x > 1}} \leq -\beta \phi(s, x) \left\{ \left( \mu_s' - \frac{\gamma^2}{2 \mu_s} - \frac{\sqrt{\beta}}{2} \right) v + \left[ \nu_s' - \frac{\sqrt{\beta}}{2} \beta \mu_s - \frac{\gamma^2}{2} \left( 1 + \frac{\sqrt{\beta}}{\mu_s} \right) \right] \right\}, \quad (s, x) \in (0, T] \times \mathbb{R}_+.
\]
Thus, (3.1) holds if the functions $\mu_s, \nu_s \in [0, T]$ satisfies
\[
\mu_s' = \frac{\gamma^2}{2 \mu_s} + \frac{\sqrt{\beta}}{2}, \quad s \in (0, T]; \quad \mu_0 = 0 \quad (3.6)
\]
and
\[
\nu_s' = \frac{\sqrt{\beta}}{2} \beta \mu_s + \frac{\gamma^2}{2} \left( 1 + \frac{\sqrt{\beta}}{\mu_s} \right), \quad s \in (0, T]. \quad (3.7)
\]
It is not very hard to verify that for each $\beta, \gamma \geq 0$ with $\beta + \gamma > 0$, ODE (3.6) has a unique strictly increasing and continuous solution, denoted by $\mu_{\beta, \gamma}(\cdot)$; i.e.,
\[
\mu_{\beta, \gamma}(0) = 0 \quad \text{and} \quad \mu_{\beta, \gamma}'(s) = \frac{\gamma^2}{2 \mu_{\beta, \gamma}(s)} + \frac{\sqrt{\beta}}{2}, \quad s \in (0, T]. \quad (3.8)
\]
Furthermore, by integrating on both sides of the last equation, we have
\[ \mu_{\beta,\gamma}(T) - \mu_{\beta,\gamma}(t) = \frac{\gamma^2}{2} \int_t^T \frac{1}{\mu_{\beta,\gamma}(s)} \, ds + \frac{\sqrt{2}}{2} \beta(T - t), \quad t \in (0, T]. \]

Letting \( t \to 0^+ \) in the last equation yields that the integral of \( 1/\mu_{\beta,\gamma}(s) \) on \([0, T]\) is well defined, and then eq. (3.7) admits a unique solution valued zero at the origin, denoted by \( \nu_{\beta,\gamma}(\cdot) \), that is
\[ \nu_{\beta,\gamma}(s) = \int_0^s \left[ \frac{\sqrt{2}}{2} \beta \mu_{\beta,\gamma}(r) + \frac{\gamma^2}{2} \left( 1 + \frac{\sqrt{2}}{\mu_{\beta,\gamma}(r)} \right) \right] \, dr, \quad s \in [0, T]. \]  
(3.9)

Remark 3.1. In general, \( \mu_{\beta,\gamma}(\cdot) \) does not have an explicit expression. However, it is easy to check that \( \mu_{\beta,\gamma}(s) = \gamma \sqrt{s} \) in the case of \( \beta = 0 \), and \( \mu_{\beta,\gamma}(s) = \frac{\sqrt{2}}{2} \beta s \) in the case of \( \gamma = 0 \).

In summary, we have

**Proposition 3.2.** For each \( \beta, \gamma \geq 0 \) with \( \beta + \gamma > 0 \), denote the function
\[ \varphi(s, x) := (x + e)^\Delta \mu_{\beta,\gamma}(s) \sqrt{2 \ln(x + e)} + \nu_{\beta,\gamma}(s), \quad (s, x) \in [0, T] \times \mathbb{R}_+ \]  
(3.10)

with \( \mu_{\beta,\gamma}(\cdot) \) and \( \nu_{\beta,\gamma}(\cdot) \) being respectively defined in (3.8) and (3.9). Then we have

(i) \( \varphi(\cdot, \cdot) \) is continuous on \([0, T] \times \mathbb{R}_+ \); And, \( \varphi(\cdot, \cdot) \in C^{1,2}([0, T] \times \mathbb{R}_+) \);

(ii) \( \varphi(\cdot, \cdot) \) satisfies the inequality (2.3) with \( \delta = 1/2, \bar{\delta} = 0 \) and \( \lambda = 0 \), i.e.,
\[ -\varphi_x(s, x) \left( \beta x \sqrt{\ln(x + 1)} + \gamma |z| \right) + \frac{1}{2} \varphi_{xx}(s, x)|z|^2 + \varphi(s, x) \geq 0, \quad (s, x, z) \in (0, T] \times \mathbb{R}_+ \times \mathbb{R}^d. \]

The following existence and uniqueness theorem is the main result of this section.

**Theorem 3.3.** Let the functions \( \psi(\cdot, \cdot) \) and \( \mu_{\beta,\gamma}(\cdot) \) be respectively defined in (2.12) and (3.8), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in \((y, z)\). If \( g \) satisfies assumptions (EX1) and (A1) with parameters \( \alpha, \beta \) and \( \gamma \), and the terminal condition satisfies
\[ \mathbb{E} \left[ \psi \left( \xi + \int_0^T \alpha_t \, dt, \mu_{\beta,\gamma}(T) \right) \right] < +\infty, \]  
(3.11)

then BSDE(\( \xi, g \)) admits a solution \((Y_t, Z_t)_{t \in [0, T]}\) such that \((\psi(\cdot), \mu_{\beta,\gamma}(\cdot))_{t \in [0, T]}\) belongs to class (D), and \( \mathbb{P}\)-a.s., for each \( t \in [0, T] \),
\[ |Y_t| \leq \psi(|Y_t|, \mu_{\beta,\gamma}(t)) \leq CE \left[ \psi \left( \xi + \int_0^T \alpha_t \, dt, \mu_{\beta,\gamma}(T) \right) \right] \mathcal{F}_t + C, \]  
(3.12)

where \( C \) is a positive constant depending only on \((\beta, \gamma, T)\).

Moreover, if \( g \) satisfies either assumptions (UN1) and (UN2) or assumption (UN3), then BSDE \((\xi, g)\) admits a unique solution \((Y_t, Z_t)_{t \in [0, T]}\) such that \((\psi(\cdot), \mu_{\beta,\gamma}(\cdot))_{t \in [0, T]}\) belongs to class (D).

In order to prove this theorem, we need the following two propositions.

**Proposition 3.4.** Let the functions \( \psi(x, \mu), \mu_{\beta,\gamma}(s) \) and \( \varphi(s, x) \) be respectively defined on (2.12), (3.8) and (3.10). Then, there exists a universal constant \( K > 0 \) depending only on \((\beta, \gamma, T)\) such that
\[ \forall (s, x) \in [0, T] \times \mathbb{R}_+, \quad \psi(x, \mu_{\beta,\gamma}(s)) \leq \varphi(s, x) \leq K \psi(x, \mu_{\beta,\gamma}(s)) + K. \]  
(3.13)
Proof. The first inequality in (3.13) is obvious. We now prove the second inequality. In fact,

\[
\frac{\varphi(s, x)}{\psi(x, \mu_{\beta, \gamma}(s)) + 1} = \frac{(x + e) \exp \left( \frac{\mu_{\beta, \gamma}(s)}{x} \sqrt{2 \ln(1 + e)} + \nu_{\beta, \gamma}(s) \right)}{x \exp \left( \frac{\mu_{\beta, \gamma}(s)}{x} \sqrt{2 \ln(1 + x)} \right) + 1} \leq \frac{x + e}{x} \exp \left( \nu_{\beta, \gamma}(T) \left( \sqrt{2 \ln(1 + e)} - \sqrt{2 \ln(1 + x)} \right) + \nu_{\beta, \gamma}(T) \right) =: H_1(x; \beta, \gamma, T), (s, x) \in [0, T] \times [1, +\infty).
\]

And, in the case of \( x \in [0, 1], \)

\[
\frac{\varphi(s, x)}{\psi(x, \mu_{\beta, \gamma}(s)) + 1} \leq (1 + e) \exp \left( \nu_{\beta, \gamma}(T) \sqrt{2 \ln(1 + e)} + \nu_{\beta, \gamma}(T) \right) =: H_2(\beta, \gamma, T), s \in [0, T].
\]

Hence, for all \( x \in \mathbb{R}_+, \) we have

\[
\frac{\varphi(s, x)}{\psi(x, \mu_{\beta, \gamma}(s)) + 1} \leq H_1(x; \beta, \gamma, T)1_{x \geq 1} + H_2(\beta, \gamma, T)1_{0 \leq x < 1}, s \in [0, T]. \tag{3.14}
\]

Thus, in view of (3.14) and the fact that the function \( H_1(x; \beta, \gamma, T) \) is continuous on \([1, +\infty)\) and tends to \( \exp(\nu_{\beta, \gamma}(T)) \) as \( x \to +\infty, \) the second inequality in (3.13) follows immediately. The proof is complete. \( \square \)

**Proposition 3.5.** Let the functions \( \psi(\cdot, \cdot) \) and \( \mu_{\beta, \gamma}(\cdot) \) be respectively defined in (2.12) and (3.8), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in \((y, z), \).

If \( g \) satisfies assumption (A1) with parameters \( \alpha, \beta \) and \( \gamma, |\xi| + \int_0^T \alpha_t dt \) is a bounded random variable, and \((Y_t, Z_t)_{t \in [0, T]}\) is a solution of BSDE\((\xi, g)\) such that \( Y \) is bounded, then \( \mathbb{P}-a.s., \) for each \( t \in [0, T], \) we have

\[
|Y_t| \leq \psi(|Y_t|, \mu_{\beta, \gamma}(t)) \leq C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T \alpha_t dt, \mu_{\beta, \gamma}(T) \right) \right| \mathcal{F}_t] + C, \tag{3.15}
\]

where \( C \) is a positive constant depending only on \((\beta, \gamma, T).\)

**Proof.** Define

\[
\bar{Y}_t := |Y_t| + \int_0^t \alpha_s ds \quad \text{and} \quad \bar{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T].
\]

It follows from Itô-Tanaka’s formula that

\[
\bar{Y}_t = \bar{Y}_T + \int_0^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - \alpha_s) ds - \int_0^T \bar{Z}_s \cdot dB_s - \int_0^T dL_s, \quad t \in [0, T],
\]

where \( L_\cdot \) denotes the local time of \( Y \) at the origin. Now, we apply Itô-Tanaka’s formula to the process \( \varphi(s, \bar{Y}_s), \) where the function \( \varphi(\cdot, \cdot) \) is defined in (3.10), to derive, in view of assumption (A1),

\[
d\varphi(s, \bar{Y}_s) = \varphi_x(s, \bar{Y}_s) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + \alpha_s) ds + \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s + \varphi_x(s, \bar{Y}_s) dL_s + \frac{1}{2} \varphi_{xx}(s, \bar{Y}_s) [Z_s]^2 ds + \varphi_s(s, \bar{Y}_s) ds \geq -\varphi_x(s, \bar{Y}_s) \left( |\bar{Y}_s| \sqrt{\ln |\bar{Y}_s|} 1_{|\bar{Y}_s| > 1} + \gamma |Z_s| \right) + \frac{1}{2} \varphi_{xx}(s, \bar{Y}_s) |Z_s|^2 + \varphi_s(s, \bar{Y}_s) \right] ds + \varphi_s(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s, \quad s \in [0, T].
\]

Furthermore, in view of the fact that \( |Y_s| \sqrt{\ln |Y_s|} 1_{|Y_s| > 1} \leq \bar{Y}_s \sqrt{\ln \bar{Y}_s} 1_{\bar{Y}_s > 1}, \) thanks to Proposition 3.2 we obtain that

\[
d\varphi(s, \bar{Y}_s) \geq \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s, \quad s \in (0, T]. \tag{3.16}
\]
Let us consider, for each integer \( n \geq 1 \) and each \( t \in (0, T) \), the following stopping time

\[
\tau_n^t := \inf \left\{ s \in [t, T]: \int_t^s [\varphi_z(r, \bar{Y}_r)]^2 |\bar{Z}_r|^2 dr \geq n \right\} \wedge T.
\]

It follows from the definition of \( \tau_n^t \) and the inequality (3.16) that for each \( n \geq 1 \),

\[
\varphi(t, \bar{Y}_t) \leq \mathbb{E} \left[ \varphi(\tau_n^t, \bar{Y}_{\tau_n^t}) | \mathcal{F}_t \right], \quad t \in (0, T).
\]

By Proposition 3.4, there exists a constant \( K > 0 \) depending only on \((\beta, \gamma, T)\) such that

\[
\psi(\bar{Y}_t, \mu_{\beta, \gamma}(t)) \leq \varphi(t, \bar{Y}_t) \leq \mathbb{E} \left[ \varphi(\tau_n^t, \bar{Y}_{\tau_n^t}) | \mathcal{F}_t \right] \leq K \mathbb{E} \left[ \psi(\bar{Y}_{\tau_n^t}, \mu_{\beta, \gamma}(\tau_n^t)) | \mathcal{F}_t \right] + K, \quad t \in (0, T).
\]

And, by virtue of Lemma 2.6 we have that for each \( n \geq 1 \),

\[
|Y_t| \leq \psi(|Y_t|, \mu_{\beta, \gamma}(t)) \leq K \mathbb{E} \left[ \psi \left( |Y_{\tau_n^t}| + \int_0^{\tau_n^t} \alpha_s ds, \mu_{\beta, \gamma}(\tau_n^t) \right) | \mathcal{F}_t \right] + K, \quad t \in (0, T),
\]

from which the desired inequality (3.15) follows for \( t \in (0, T) \) by sending \( n \) to infinity. Finally, in view of the continuity of \( Y \) and the martingale in the right side of (3.15) with respect to the time variable \( t \), the inequality (3.15) holds still true for \( t = 0 \). The proof is complete.

\[ \square \]

**Remark 3.6.** From the above proof, it is not hard to check that in Proposition 3.5, if the \(|\xi|\) and \(|Y_t|\) are respectively replaced with \(\xi^+\) and \(Y_t^+\), and Assumption (A1) is replaced with the following one

(A1') There exist two nonnegative constants \( \beta \geq 0 \) and \( \gamma \geq 0 \) with \( \beta + \gamma > 0 \), and an \( \mathbb{R}_+ \)-valued progressively measurable process \( (\alpha_t)_{t \in [0, T]} \) such that \( d\mathbb{P} \times dt \)-a.e.,

\[
\forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| \sqrt{\ln |y|} \mathbf{1}_{|y| > 1} + \gamma |z|.
\]

then Proposition 3.5 still holds. For this, one only needs to respectively use \(Y^+, 1_{Y > 0}\) and \(\frac{1}{2}L\) instead of \(|Y|, \text{sgn}(Y)\) and \(L\) in the above proof.

Now, we prove Theorem 3.3.

**Proof of Theorem 3.3.** For each pair of positive integers \( n, p \geq 1 \), let \( \xi^{n,p} \) and \( g^{n,p} \) be defined in (2.1), and \( (Y^{n,p}, \bar{Z}^{n,p}) \) be the minimal (maximal) bounded solution of (2.2). It is easy to verify that the generator \( g^{n,p} \) satisfies Assumption (A1) with \( \alpha \) being replaced with \( \alpha \wedge (n \vee p) \), where \( n \vee p \) denotes the maximum of \( n \) and \( p \). Then, applying Proposition 3.5 to BSDE (2.2) yields the existence of a constant \( C > 0 \) depending only on \((\beta, \gamma, T)\) such that \( \mathbb{P}\)-a.s., for each pair of \( n, p \geq 1 \),

\[
|Y^{n,p}_t| \leq \psi(|Y^{n,p}_t|, \mu_{\beta, \gamma}(t)) \leq C \mathbb{E} \left[ \psi \left( |\xi^{n,p}| + \int_0^T [\alpha_t \wedge (n \vee p)] dt, \mu_{\beta, \gamma}(T) \right) | \mathcal{F}_t \right] + C \leq C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T \alpha_t dt, \mu_{\beta, \gamma}(T) \right) | \mathcal{F}_t \right] + C =: X_t, \quad t \in [0, T].
\]

Thus, in view of Assumption (3.11), we have found an \( \mathbb{R}_+ \)-valued, progressively measurable and continuous process \( (X_t)_{t \in [0, T]} \) such that

\[
d\mathbb{P} \times dt \text{-a.e., } \forall n, p \geq 1, \quad |Y^{n,p}| \leq X_t.
\]
Now, we can apply Proposition 2.1 to obtain the existence of a progressively measurable process \((Z_t)_{t\in[0,T]}\) such that \((Y := \inf_{t} \sup_{p} Y^{n,p} , Z)\) is a solution to BSDE \((\xi, g)\).

Furthermore, sending \(n\) and \(p\) to infinity in (3.17) yields (3.12), and then \((\psi([Y_t, \mu_{\beta,\gamma}(t)])_{t\in[0,T]}\) belongs to class (D). The existence part is then proved.

Finally, the uniqueness of the desired solution under Assumptions (UN1) and (UN2) is a direct consequence of Proposition 2.5. And, the uniqueness under Assumption (UN3) is a consequence of the following Proposition 3.7. The proof is complete.

**Proposition 3.7.** Let the functions \(\psi(\cdot, \cdot)\) and \(\mu_{\beta,\gamma}(\cdot)\) be respectively defined in (2.12) and (3.8), \(\xi\) and \(\xi'\) be two terminal conditions, \(g\) and \(g'\) be two generators which are continuous in the variables \((y, z)\), and \((Y, Z_t)_{t\in[0,T]}\) and \((Y', Z'_t)_{t\in[0,T]}\) be respectively a solution to BSDE \((\xi, g)\) and BSDE \((\xi', g')\) such that both \((\psi([Y_t, \mu_{\beta,\gamma}(t)])_{t\in[0,T]}\) and \((\psi([Y'_t, \mu_{\beta,\gamma}(t)])_{t\in[0,T]}\) belong to class (D).

Assume that \(P\)-a.s., \(\xi \leq \xi'\). If \(g\) (resp. \(g'\)) satisfies Assumptions (UN3) and (A1) with parameters \((\alpha, \beta, \gamma)\) such that \(\psi\left(\int_0^T \alpha_t dt, \mu_{\beta,\gamma}(T)\right) \in L^1\), and \(dP \times dt\)-a.e.,

\[
g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \quad (\text{resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)),
\]

then \(P\)-a.s., \(Y_t \leq Y'_t\) for all \(t \in [0, T]\).

**Proof.** We first consider the case that the generator \(g\) is convex in the state variables \((y, z)\), satisfies Assumption (A1) with parameters \((\alpha, \beta, \gamma)\) such that \(\psi\left(\int_0^T \alpha_t dt, \mu_{\beta,\gamma}(T)\right) \in L^1\), and \(dP \times dt\)-a.e.,

\[
g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t).
\]

To use the convexity condition of the generator \(g\), we use the \(\theta\)-technique developed in for example [13]. For each fixed \(\theta \in (0, 1)\), define

\[
\Delta^\theta U := \frac{Y - \theta Y'}{1 - \theta} \quad \text{and} \quad \Delta^\theta V := \frac{Z - \theta Z'}{1 - \theta}.
\]

Then the pair \((\Delta^\theta U, \Delta^\theta V)\) satisfies the following BSDE:

\[
\Delta^\theta U_t = \Delta^\theta U_T + \int_t^T \Delta^\theta g(s, \Delta^\theta U_s, \Delta^\theta V_s) ds - \int_t^T \Delta^\theta V_s \cdot dB_s, \quad t \in [0, T],
\]

where \(dP \times ds\)-a.e., for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\),

\[
\Delta^\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ g(s, (1 - \theta)y + \theta Y'_s, (1 - \theta)z + \theta Z'_s) - \theta g(s, Y'_s, Z'_s) \right] + \frac{\theta}{1 - \theta} \left[ g(s, Y'_s, Z'_s) - g(s, Y'_s, Z'_s) \right].
\]

It follows from the assumptions that \(dP \times ds\)-a.e.,

\[
\forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad \Delta^\theta g(s, y, z) \leq g(s, y, z) \leq \alpha_s + \beta|y| + \gamma|z|, \quad (3.22)
\]

which means that the generator \(\Delta^\theta g\) satisfies Assumption (A1') defined in Remark 3.6.

On the other hand, since both processes \((\psi([Y_t, \mu_{\beta,\gamma}(t)])_{t\in[0,T]}\) and \((\psi([Y'_t, \mu_{\beta,\gamma}(t)])_{t\in[0,T]}\) belong to class (D), and \(\psi\left(\int_0^T \alpha_t dt, \mu_{\beta,\gamma}(T)\right) \in L^1\), by virtue of Lemma 2.6, we conclude that the process

\[
\psi\left(\Delta^\theta U + \int_0^t \alpha_s ds, \mu_{\beta,\gamma}(t)\right), \quad t \in [0, T]
\]
belongs to class (D) for each $\theta \in (0, 1)$. Thus, for BSDE (3.20), by virtue of (3.22), Remark 3.6 and the proof of Proposition 3.5, we derive that there exists a constant $C > 0$ depending on $(\beta, \gamma, T)$ such that

$$(\Delta^\theta U_t)^+ \leq \psi \left( (\Delta^\theta U_t)^+ , \mu_{\beta, \gamma}(t) \right) \leq C \mathbb{E} \left[ \psi \left( (\Delta^\theta U_T)^+ + \int_0^T \alpha_s ds, \mu_{\beta, \gamma}(T) \right) \bigg| \mathcal{F}_t \right] + C, \quad t \in [0, T].$$

(3.23)

Furthermore, since

$$(\Delta^\theta U_T)^+ = \frac{(\xi - \theta \xi')^+}{1 - \theta} = \frac{(\xi - \theta \xi + \theta (\xi - \xi'))^+}{1 - \theta} \leq \xi^+,$$

(3.24)

we have from (3.23) that

$$(Y_t - \theta Y'_t)^+ \leq (1 - \theta) \left( C \mathbb{E} \left[ \psi \left( \xi^+ + \int_0^T \alpha_s ds, \mu_{\beta, \gamma}(T) \right) \bigg| \mathcal{F}_t \right] + C \right), \quad t \in [0, T].$$

Thus, the desired conclusion follows in the limit as $\theta \to 1$.

For the case that the generator $g$ is concave in the state variables $(y, z)$, we need to respectively use the $\theta Y - Y'$ and $\theta Z - Z'$ to replace $Y - \theta Y'$ and $Z - \theta Z'$ in (3.19). In this case, the generator $\Delta^\theta g$ in (3.21) should be replaced with

$$\Delta^\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ \theta g(s, Y_s, Z_s) - g(s, (1 - \theta)y + \theta Y_s, (1 - \theta)z + \theta Z_s) \right] + \frac{1}{1 - \theta} \left[ g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s) \right].$$

Since $g$ is concave in $(y, z)$, we have $dP \times ds - a.e.$,

$$\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad g(s, (1 - \theta)y + \theta Y_s, (1 - \theta)z + \theta Z_s) \geq \theta g(s, Y_s, Z_s) + (1 - \theta)g(t, -y, -z),$$

and then, (3.22) can be replaced by

$$\forall (y, z) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad \Delta^\theta g(s, y, z) \leq -g(s, -y, -z) \leq \alpha_s + \beta |y| |\ln |y||1_{|y|>1} + \gamma |z|,$$

which means that the generator $\Delta^\theta g$ still satisfies Assumption (A1'). Consequently, (3.23) still holds. Moreover, we use

$$(\Delta^\theta U_T)^+ = \frac{(\theta \xi - \xi')^+}{1 - \theta} = \frac{|\theta \xi - \xi + (\xi - \xi')|^+}{1 - \theta} \leq (-\xi)^+ = \xi^-,$$

instead of (3.24), and it follows from (3.23) that

$$(\theta Y_t - Y'_t)^+ \leq (1 - \theta) \left( C \mathbb{E} \left[ \psi \left( \xi^- + \int_0^T \alpha_s ds, \mu_{\beta, \gamma}(T) \right) \bigg| \mathcal{F}_t \right] + C \right), \quad t \in [0, T].$$

Thus, the desired conclusion follows in the limit as $\theta \to 1$.

Finally, in the same way as above, we can prove the desired conclusion under assumptions that the generator $g'$ satisfies Assumptions (UN3) and (A1) with parameters $(\alpha, \beta, \gamma)$ such that $\psi \left( \int_0^T \alpha_t dt, \mu_{\beta, \gamma}(T) \right) \in L^1$, and $dP \times dt-a.e., \quad g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)$. The proof is then complete.

\[\square\]

**Example 3.8.** Let $\beta > 0$, $\gamma > 0$ and $k \geq 0$. For each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$, define

$$g_1(\omega, t, y, z) := |B_t(\omega)| - ke^\beta + \beta |y| |\ln |y||1_{|y|\leq e^{\frac{1}{2k}}} + \frac{\beta}{\sqrt{2e}}|1_{|y|> e^{\frac{1}{2k}}} + \gamma \left( |z| - k \sqrt{|z|} \right)$$
and 
\[ g_2(\omega, t, y, z) := |B_t(\omega)| + ke^{-y} + \beta|y|\sqrt{\ln|y|}1_{|y| \geq 1} + \gamma|z|. \]

It is easy to verify that the generator \( g_1 \) satisfies Assumptions (EX1), (A1), (UN1) and (UN2), and the generator \( g_2 \) satisfies Assumptions (EX1), (A1) and (UN3).

4. The case of \( L \exp (\mu (\ln (1 + L))^{p}) \)-integrability terminal condition with \( p \in (1/2, 1) \)

In this section, we always assume that the generator \( g \) satisfies the following assumption, which is strictly weaker than (1.2) with \( \lambda \in (0, 1/2) \) and \( \delta = \lambda + 1/2 \).

(A2) There exist three constants \( \beta \geq 0, \gamma > 0 \) and \( \lambda \in (0, 1/2) \), and an \( \mathbb{R}_+ \)-valued progressively measurable process \( (\alpha_t)_{t \in [0, T]} \) such that \( dP \times dt \)-a.e.,

\[ \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(g(\omega, t, y, z)) \leq \alpha_t(\omega) + \beta|y|(|\ln |y||)^{\lambda + \frac{1}{2}}1_{|y| > 1} + \gamma|z||\ln |z||^\lambda. \]

For each \( \varepsilon > 0 \), by letting \( k = 1 + \varepsilon \) in Proposition 2.2 we know that there exists a constant \( C_{\lambda, \varepsilon} > 0 \) depending only on parameters \( (\lambda, \varepsilon) \) such that (2.5) holds. Then, our objective is to search for a strictly increasing and strictly convex function \( \phi \) such that for each \( (s, x) \in [0, T] \times \mathbb{R}_+, \) with \( \beta \geq 0 \) and \( \gamma > 0 \),

\[ -\beta \phi_x(s, x)(\ln x)^{\lambda + \frac{1}{2}}1_{x > 1} - \frac{\gamma^2}{2} \left( \phi_x(s, x) \right)^2 \left( 1 + \varepsilon \right) \left( \ln \phi_x(s, x) \right)^{2\lambda} + C_{\lambda, \varepsilon} + \phi_s(s, x) \geq 0, \]

which is just (2.11) with \( \lambda \in (0, 1/2) \), \( \delta = \lambda + 1/2 \), \( \delta = 0 \), \( k = 1 + \varepsilon \) and \( C_{k, \lambda} = C_{\lambda, \varepsilon} \).

Now, let \( \varepsilon > 0, \beta \geq 0, \gamma > 0, \lambda \in (0, 1/2) \) and \( \mu_s, \nu_s : [0, T] \rightarrow \mathbb{R}_+ \) be two increasing and continuously differential functions with \( \mu_0 = \varepsilon \) and \( \nu_0 = 0 \). Define

\[ k_\varepsilon := \exp \left( \frac{1 + \varepsilon}{2\varepsilon} \right) + \left( \frac{1}{2\varepsilon} \right) + \frac{\mu_T}{\gamma} + \varepsilon. \]

We choose the following function

\[ \phi(s, x; \varepsilon) := (x + k_\varepsilon) \exp \left( \mu_s \ln (x + k_\varepsilon)^{\lambda + \frac{1}{2}} + \nu_s \right), \quad (s, x) \in [0, T] \times \mathbb{R}_+ \]

to explicitly solve the inequality (4.1). For each \( (s, x) \in [0, T] \times \mathbb{R}_+ \), a simple computation gives

\[ \phi_x(s, x; \varepsilon) = \phi(s, x; \varepsilon) \frac{\left( \lambda + \frac{1}{2} \right) \mu_s + (\ln(x + k_\varepsilon))^{\frac{\gamma}{\lambda} - \lambda}}{(x + k_\varepsilon)(\ln(x + k_\varepsilon))^{\frac{\gamma}{\lambda} - \lambda}} > 0, \]

(4.3)

\[ \phi_{xx}(s, x; \varepsilon) = \phi(s, x; \varepsilon) \frac{(\lambda + \frac{1}{2}) \mu_s \ln(x + k_\varepsilon) + (\lambda + \frac{1}{2}) \mu_s (\ln(x + k_\varepsilon))^{\lambda + \frac{1}{2}} - (\frac{1}{2} - \lambda \ln(x + k_\varepsilon))^{\frac{\gamma}{\lambda} - \lambda}}{(x + k_\varepsilon)^2(\ln(x + k_\varepsilon))^{\frac{\gamma}{\lambda} - \lambda}} > 0 \]

(4.4)

and

\[ \phi_s(s, x; \varepsilon) = \phi(s, x; \varepsilon) \left( \mu'_s \ln(x + k_\varepsilon)^{\lambda + \frac{1}{2}} + \nu'_s \right) > 0. \]

(4.5)

Furthermore, from the definition of \( k_\varepsilon \) it can be directly verified that for each \( (s, x) \in [0, T] \times \mathbb{R}_+ \),

\[ \ln(x + k_\varepsilon) + \left( \lambda + \frac{1}{2} \right) \mu_s (\ln(x + k_\varepsilon))^{\lambda + \frac{1}{2}} - \left( \frac{1}{2} - \lambda \right) \]

\[ \geq \frac{1}{1 + \varepsilon} \left[ \ln(x + k_\varepsilon) + \left( \lambda + \frac{1}{2} \right) \mu_s (\ln(x + k_\varepsilon))^{\lambda + \frac{1}{2}} \right] \]

(4.6)
\[
(\ln(x + k_e))^{\frac{1}{2} - \lambda} \leq \sqrt{\ln(x + k_e)} \leq (x + k_e)^{\varepsilon}.
\] (4.7)

It follows from (4.4) and (4.6) that
\[
\phi_{xx}(s, x; \varepsilon) \geq \phi(s, x; \varepsilon) \frac{(\frac{1}{2}) \mu_s \left( (\frac{1}{2}) \mu_s + (\ln(x + k_e))^{\frac{1}{2} - \lambda} \right)}{(1 + \varepsilon)(x + k_e)^{2(\ln(x + k_e))^{1 - 2\lambda}}} , \quad (s, x) \in [0, T] \times \mathbb{R}_+ .
\]

Then, for each \((s, x) \in [0, T] \times \mathbb{R}_+\), in view of (4.3), we have
\[
\frac{\gamma^2 (\phi_x(s, x; \varepsilon))^2}{2 \phi_{xx}(s, x; \varepsilon)} \leq \frac{\gamma^2 (1 + \varepsilon)}{2} \phi(s, x; \varepsilon) \left( 1 + \frac{(\ln(x + k_e))^{\frac{1}{2} - \lambda}}{(\frac{1}{2}) \mu_s} \right) \tag{4.8}
\]
and, in view of (4.2), (4.4) and (4.7),
\[
\left\{ \begin{array}{l}
\frac{\gamma \phi_x(s, x; \varepsilon)}{\phi_{xx}(s, x; \varepsilon)} \geq \frac{\gamma}{(\lambda + \frac{1}{2}) \mu_s} (x + k_e)(\ln(x + k_e))^{\frac{1}{2} - \lambda} \geq \frac{\gamma}{\mu_T} k_e \geq 1; \\
\frac{\gamma (1 + \varepsilon)}{\phi_{xx}(s, x; \varepsilon)} (x + k_e)(\ln(x + k_e))^{\frac{1}{2} - \lambda} \leq \frac{2\gamma (1 + \varepsilon)}{\varepsilon} (x + k_e)^{1 + \varepsilon},
\end{array} \right.
\]
which yields the following
\[
\left| \ln \frac{\gamma \phi_x(s, x; \varepsilon)}{\phi_{xx}(s, x; \varepsilon)} \right|^{2\lambda} \leq \left| \ln \frac{2\gamma (1 + \varepsilon)}{\varepsilon} \right|^{2\lambda} + (1 + \varepsilon)^{2\lambda} (\ln(x + k_e))^{2\lambda} . \tag{4.9}
\]

In the sequel, observe that
\[
\forall x \in \mathbb{R}_+ , x(\ln x)^{\frac{3}{2} + \frac{1}{2}} 1_{x > 1} \leq (x + k_e)(\ln(x + k_e))^{\frac{3}{2} + \frac{1}{2}}.
\]

We substitute (4.3), (4.8), (4.9) and (4.5) into the left side of (4.1) with \(\phi(s, x; \varepsilon)\) instead of \(\phi(s, x)\) to get
\[
-\beta \phi_x(s, x; \varepsilon) \varepsilon (\ln x)^{\frac{3}{2} + \frac{1}{2}} 1_{x > 1} - \frac{\gamma^2 (\phi_x(s, x; \varepsilon))^2}{2 \phi_{xx}(s, x; \varepsilon)} \left( 1 + \varepsilon \right) \left| \ln \frac{\gamma \phi_x(s, x; \varepsilon)}{\phi_{xx}(s, x; \varepsilon)} \right|^{2\lambda} + C_{\lambda, \varepsilon} + \phi(s, x; \varepsilon)
\geq -\beta \phi(s, x; \varepsilon) \left[ (\lambda + \frac{1}{2}) \mu_s (\ln(x + k_e))^{2\lambda} + (\ln(x + k_e))^{\frac{3}{2} + \frac{1}{2}} \right] - \frac{\gamma^2 (1 + \varepsilon)}{2} \phi(s, x; \varepsilon) \left( 1 + \frac{(\ln(x + k_e))^{\frac{1}{2} - \lambda}}{(\lambda + \frac{1}{2}) \mu_s} \right) \left( 1 + \varepsilon \right)^{2\lambda + 1} (\ln(x + k_e))^{2\lambda} + C_{\lambda, \varepsilon} \leq \phi(s, x; \varepsilon) \left( \mu'_l (\ln(x + k_e))^{\frac{3}{2} + \frac{1}{2}} + \nu'_l \right) , \quad (s, x) \in [0, T] \times \mathbb{R}_+
\]
where
\[
C_{\lambda, \varepsilon} := (1 + \varepsilon) \left| \ln \frac{2\gamma (1 + \varepsilon)}{\varepsilon} \right|^{2\lambda} + C_{\lambda, \varepsilon}.
\]
Furthermore, in view of the facts that \(\mu \geq \varepsilon = \lambda \in (0, 1/2)\), using Young’s inequality, we see that there is a constant \(C_{\beta, \lambda, \varepsilon}^1 > 0\) depending only on \((\beta, \lambda, \varepsilon)\), and a constant \(C_{\gamma, \lambda, \varepsilon}^2 > 0\) depending only on \((\gamma, \lambda, \varepsilon, \bar{C}_{\lambda, \varepsilon})\) such that for each \((s, x) \in [0, T] \times \mathbb{R}_+\),
\[
\beta (\lambda + \frac{1}{2}) (\ln(x + k_e))^{2\lambda} \leq \varepsilon (\ln(x + k_e))^{\frac{3}{2} + \frac{1}{2}} + C_{\beta, \lambda, \varepsilon}^1.
\]
and
\[ \frac{\gamma^2}{2} \left( 1 + \frac{\ln(x + k_e)}{(\lambda + \frac{1}{2}) \mu_s} \right) \left[ (1 + \varepsilon)^{2\lambda+1} (\ln(x + k_e))^{2\lambda} + C_{\lambda, \varepsilon} \right] \]
\[ \leq \left( \frac{\gamma^2 (1 + \varepsilon)^{2\lambda+2}}{(2\lambda + 1) \mu_s} + \varepsilon \right) (\ln(x + k_e))^{1 + \frac{1}{2}} + C_{\gamma, \lambda, \varepsilon} \cdot \]

Then, with the notation \( v := (\ln(x + k_e))^{1 + \frac{1}{2}} \) we have for each \( (s, x) \in [0, T] \times \mathbb{R}_+ \),
\[-\beta \phi_x(s, x; \varepsilon) (\ln x)^{1 + \frac{1}{2}} \frac{1}{2} \phi_x(s, x; \varepsilon) = \frac{\gamma^2}{2} \left( \phi_x(s, x; \varepsilon) \right)^2 \left[ (1 + \varepsilon) \left\| \frac{\gamma \phi_x(s, x; \varepsilon)}{\phi_x(s, x; \varepsilon)} \right\|^{2\lambda} + C_{\lambda, \varepsilon} \right] + \phi_s(s, x; \varepsilon) \]
\[ \geq \phi(s, x; \varepsilon) \left[ -\varepsilon \mu_s - \beta - \frac{\gamma^2 (1 + \varepsilon)^{2\lambda+2}}{(2\lambda + 1) \mu_s} - \varepsilon + \mu_s' \right] v + \left( -C_{\beta, \lambda, \varepsilon} \mu_s - C_{\gamma, \lambda, \varepsilon} + \mu_s' \right) \varepsilon. \]

Thus, (4.1) holds if the functions \( \mu_s, \nu_s \in [0, T] \) satisfies
\[ \mu'_s = \varepsilon \mu_s + \frac{\gamma^2 (1 + \varepsilon)^{2\lambda+2}}{2\lambda + 1} \mu_s + \varepsilon + \beta, \quad s \in [0, T] \]  \hspace{1cm} (4.10)
and
\[ \nu_s = C_{\beta, \lambda, \varepsilon} \int_0^s \mu_r dr + C_{\gamma, \lambda, \varepsilon} s, \quad s \in [0, T]. \]  \hspace{1cm} (4.11)

It is not very hard to verify that for each \( \beta \geq 0, \gamma > 0, \lambda \in (0, 1/2) \) and \( \varepsilon > 0 \), there exists a unique strictly increasing and continuous function with \( \varepsilon \) at the origin satisfying (4.10). We denote this unique solution by \( \mu_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \), i.e., \( \mu_{\beta, \gamma, \lambda, \varepsilon}(0) = \varepsilon \) and
\[ \mu_{\beta, \gamma, \lambda, \varepsilon}'(s) = \varepsilon \mu_{\beta, \gamma, \lambda, \varepsilon}(s) + \frac{\gamma^2 (1 + \varepsilon)^{2\lambda+2}}{2\lambda + 1} \mu_{\beta, \gamma, \lambda, \varepsilon}(s) + \varepsilon + \beta, \quad s \in [0, T]. \]  \hspace{1cm} (4.12)

We also denote, in view of (4.11),
\[ \nu_{\beta, \gamma, \lambda, \varepsilon} := C_{\beta, \lambda, \varepsilon} \int_0^s \mu_{\beta, \gamma, \lambda, \varepsilon}(r) dr + C_{\gamma, \lambda, \varepsilon} s, \quad s \in [0, T]. \]  \hspace{1cm} (4.13)

Moreover, it can be directly checked that as \( \varepsilon \to 0^+ \), the function \( \mu_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \) tends decreasingly to the unique solution \( \mu_{\beta, \gamma, \lambda}(\cdot) \) of the following ODE:
\[ \mu_{\beta, \gamma, \lambda}^0(0) = 0 \quad \text{and} \quad \left( \mu_{\beta, \gamma, \lambda}^0(s) \right)' = \frac{\gamma^2}{2\lambda + 1} \mu_{\beta, \gamma, \lambda}^0(s) + \beta, \quad s \in [0, T]. \]  \hspace{1cm} (4.14)

**Remark 4.1.** In general, \( \mu_{\beta, \gamma, \lambda}^0(\cdot) \) does not have an explicit expression. However, it is easy to check that \( \mu_{\beta, \gamma, \lambda}(s) = \frac{\gamma}{\sqrt{\lambda + \frac{1}{2}}} \sqrt{s} \) in the case of \( \beta = 0 \). In addition, when \( \lambda = 0 \), we have \( \mu_{\beta, \gamma, \lambda}^0(s) = \sqrt{2} \mu_{\beta, \gamma}(s) \), where \( \mu_{\beta, \gamma}(s) \) is defined in (3.8).

In summary, we have

**Proposition 4.2.** For each \( \beta \geq 0, \gamma > 0, \lambda \in (0, 1/2) \) and \( \varepsilon > 0 \), define the function
\[ \varphi(s, x; \varepsilon) := (x + \bar{k}_e) \exp \left( \mu_{\beta, \gamma, \lambda, \varepsilon}(s) \ln (x + \bar{k}_e) \right)^{1 + \frac{1}{2}} + \nu_{\beta, \gamma, \lambda, \varepsilon}(s), \quad (s, x) \in [0, T] \times \mathbb{R}_+ \]  \hspace{1cm} (4.15)
with \( \mu_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \) and \( \nu_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \) being respectively defined in (4.12) and (4.13), and
\[ \bar{k}_e := \exp \left( \frac{1 + \varepsilon}{2} \right) + \left( \frac{1}{2\varepsilon^2} \right)^{\frac{1}{2}} + \frac{\mu_{\beta, \gamma, \lambda, \varepsilon}(T)}{\gamma} + \varepsilon, \]
which is \( k_e \) in (4.2) with \( \mu_{\beta, \gamma, \lambda, \varepsilon}(T) \) instead of \( \mu_T \). Then we have
(i) \( \varphi(\cdot, \cdot; \varepsilon) \in C^{1,2}([0, T] \times \mathbb{R}_+) \);

(ii) \( \varphi(\cdot, \cdot; \varepsilon) \) satisfies the inequality (2.3) with \( \lambda \in (0, 1/2) \), \( \delta = \lambda + 1/2 \) and \( \bar{\delta} = 0 \). i.e., for each 
\( (s, x, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^d \), we have 
\[
-\varphi_x(s, x; \varepsilon) \left( \beta x (\ln x)^{\lambda + \frac{1}{2}} 1_{x > 1} + \gamma |z| |\ln |z||^\lambda \right) + \frac{1}{2} \varphi_{xx}(s, x; \varepsilon) |z|^2 + \varphi_x(s, x; \varepsilon) \geq 0.
\]

Now, for \( \lambda \in (0, 1/2) \), we define the function
\[
\hat{\psi}(x, \mu) := x \exp \left( \mu (\ln(1 + x))^{\lambda + \frac{1}{2}} \right), \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]

The following existence and uniqueness theorem is one main result of this section.

**Theorem 4.3.** Let the functions \( \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \), \( \bar{\mu}_{\beta, \gamma, \lambda}^0(\cdot) \) and \( \hat{\psi}(\cdot, \cdot) \) be respectively defined in (4.12), (4.14), and (4.16), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in the state variables \( (y, z) \). If \( g \) satisfies Assumptions (EX1) and (A2) with parameters \( (\alpha, \beta, \gamma, \lambda) \), and there exists a positive constant \( \mu > \bar{\mu}_{\beta, \gamma, \lambda}^0(T) \) such that
\[
\mathbb{E} \left[ \hat{\psi} \left( |\xi| + \int_0^T \alpha_t dt, \mu \right) \right] < +\infty,
\]

then \( \text{BSDE}(\xi, g) \) admits a solution \( (Y_t, Z_t)_{t \in [0, T]} \) such that for some \( \varepsilon > 0 \), \( \left( \hat{\psi}(|Y_t|, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \right)_{t \in [0, T]} \) belongs to class \( (D) \), and \( \mathbb{P}\text{-a.s.}, \)
\[
|Y_t| \leq \hat{\psi}(|Y_t|, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \leq C \mathbb{E} \left[ \hat{\psi} \left( |\xi| + \int_0^T \alpha_t dt, \mu \right) \right] \mathbb{F}_t + C, \quad t \in [0, T],
\]

where \( C \) is a positive constant depending only on \( (\beta, \gamma, \lambda, \varepsilon, T) \), and \( \varepsilon \) is the unique positive constant satisfying \( \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu \).

Moreover, if \( g \) satisfies either Assumptions (UN1) and (UN2) or Assumption (UN3), then \( \text{BSDE}(\xi, g) \) admits a unique solution \( (Y_t, Z_t)_{t \in [0, T]} \) such that \( \left( \hat{\psi}(|Y_t|, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \right)_{t \in [0, T]} \) belongs to class \( (D) \).

To prove this theorem, we need the following two propositions. First, similar to Proposition 3.4, we can prove the following

**Proposition 4.4.** Let the functions \( \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \), \( \varphi(\cdot, \cdot; \varepsilon) \) and \( \hat{\psi}(\cdot, \cdot) \) be respectively defined on (4.12), (4.15) and (4.16). Then, there exists a universal constant \( K > 0 \) depending on \( (\beta, \gamma, \lambda, \varepsilon, T) \) such that
\[
\forall (s, x) \in [0, T] \times \mathbb{R}_+, \quad \hat{\psi}(x, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) \leq \varphi(s, x; \varepsilon) \leq K \hat{\psi}(x, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) + K.
\]

Similar to the proof of Proposition 3.5, we can prove the following

**Proposition 4.5.** Let the functions \( \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(\cdot) \) and \( \hat{\psi}(\cdot, \cdot) \) be respectively defined in (4.12) and (4.16), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in \( (y, z) \). If \( g \) satisfies Assumption (A2) with parameters \( (\alpha, \beta, \gamma, \lambda) \), \( |\xi| + \int_0^T \alpha_t dt \) is a bounded random variable, and \( (Y_t, Z_t)_{t \in [0, T]} \) is a solution of \( \text{BSDE}(\xi, g) \) such that \( Y \) is bounded, then for each \( \varepsilon > 0 \), \( \mathbb{P}\text{-a.s.}, \)
\[
|Y_t| \leq \hat{\psi}(|Y_t|, \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \leq C \mathbb{E} \left[ \hat{\psi} \left( |\xi| + \int_0^T \alpha_t dt, \mu_{\bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T)} \right) \right] \mathbb{F}_t + C, \quad t \in [0, T],
\]

where \( C \) is a positive constant depending only on \( (\beta, \gamma, \lambda, \varepsilon, T) \).
Proof. Define
\[ \tilde{Y}_t := |Y_t| + \int_0^t \alpha_s ds \quad \text{and} \quad \tilde{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T]. \]
Using Itô-Tanaka’s formula, we have
\[ \tilde{Y}_t = \tilde{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - \alpha_s) ds - \int_t^T \tilde{Z}_s \cdot dB_s - \int_t^T dL_s, \quad t \in [0, T], \]
with \( L \) being the local time of \( Y \) at the origin. Now, we fix \( \varepsilon > 0 \) and apply Itô-Tanaka’s formula to the process \( \tilde{\varphi}(s, \tilde{Y}_s; \varepsilon) \) (see (4.15) for the function \( \tilde{\varphi}(\cdot, \cdot; \varepsilon) \)), to derive, in view of (A2),
\[
d\tilde{\varphi}(s, \tilde{Y}_s; \varepsilon) = \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + \alpha_s) ds + \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot dB_s + \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) dL_s
+ \frac{1}{2} \tilde{\varphi}_{xx}(s, \tilde{Y}_s; \varepsilon) |Z_s|^2 ds + \varphi_x(s, \tilde{Y}_s; \varepsilon) ds
\geq \left[ - \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \left( \beta|Y_s|(|\ln|Y_s||)^{\lambda+\frac{1}{2}} 1_{|Y_s|>1} + \gamma|Z_s||\ln|Z_s||^\lambda \right) \right]
+ \frac{1}{2} \tilde{\varphi}_{xx}(s, \tilde{Y}_s; \varepsilon) |Z_s|^2 ds + \varphi_x(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot dB_s, \quad s \in [0, T].
\]
Furthermore, in view of the fact that \(|Y_s|(|\ln|Y_s||)^{\lambda+\frac{1}{2}} 1_{|Y_s|>1} \leq \tilde{Y}_s|\ln\tilde{Y}_s|^{\lambda+\frac{1}{2}} 1_{\tilde{Y}_s>1} \) and Proposition 4.2, we have
\[ d\tilde{\varphi}(s, \tilde{Y}_s; \varepsilon) \geq \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot dB_s, \quad s \in [0, T]. \tag{4.20} \]
Let us consider, for each integer \( n \geq 1 \) and each \( t \in [0, T] \), the following stopping time
\[ \tau_n^t := \inf \left\{ s \in [t, T] : \int_t^s |\varphi_x(r, Y_r; \varepsilon)|^2 |Z_r|^2 dr \geq n \right\} \wedge T. \]
It follows from the definition of \( \tau_n^t \) and the inequality (4.20) that for each \( n \geq 1 \),
\[ \tilde{\varphi}(t, \tilde{Y}_t; \varepsilon) \leq \mathbb{E} \left[ \tilde{\varphi}(\tau_n^t, \tilde{Y}_{\tau_n^t}; \varepsilon) \right] + K \mathbb{E}\left[ |Y_{\tau_n^t}| + \int_0^{\tau_n^t} \alpha_s ds, \tilde{\varphi}(\cdot, \cdot; \varepsilon)(\tau_n^t) \right] + K, \quad t \in [0, T]. \]
Thus, by Proposition 4.4, there exists a constant \( K > 0 \) depending only on \((\beta, \gamma, \lambda, \varepsilon, T)\) such that
\[ \tilde{\varphi}(\tilde{Y}_t, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \leq \tilde{\varphi}(t, \tilde{Y}_t; \varepsilon) \leq \mathbb{E} \left[ \tilde{\varphi}(\tau_n^t, \tilde{Y}_{\tau_n^t}; \varepsilon) \right] \leq K \mathbb{E}\left[ \tilde{\varphi}(\tau_n^t, \tilde{Y}_{\tau_n^t}; \varepsilon) \right] + K, \quad t \in [0, T]. \]
And, since \( \tilde{\varphi}(x, \mu) \) is increasing in \( x \), we have that for each \( n \geq 1 \),
\[ |Y_t| \leq \tilde{\varphi}(|Y_t|, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) \leq K \mathbb{E}\left[ \tilde{\varphi}(Y_{\tau_n^t} + \int_0^{\tau_n^t} \alpha_s ds, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(\tau_n^t)) \right] + K, \quad t \in [0, T], \]
from which the desired inequality (4.19) follows by sending \( n \) to infinity. The proof is completed. \( \square \)

**Remark 4.6.** From the preceding proof, we can check that Assertions of Proposition 4.5 are still true if \(|\xi|, |Y_t|\) is replaced with \((\xi^+, Y_t^+)\), and Assumption (A2) is replaced with the following one (A2’):

(A2’) There exist three constants \( \beta \geq 0, \gamma > 0 \) and \( \lambda \in (0, 1/2) \), and an \( \mathbb{R}_+ \)-valued progressively measurable process \((\alpha_t)_{t \in [0, T]}\) such that \( d\mathbb{P} \times dt \text{-a.e.,} \)
\[ \forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta|y||\ln|y||^{\lambda+\frac{1}{2}} 1_{|y|>1} + \gamma|z||\ln|z||^\lambda. \]
For this, it is sufficient to use \((Y^+, 1_{Y>0}, \frac{1}{2}L)\) instead of \((|Y|, \text{sgn}(Y), L)\) in the proof.
Now, we prove Theorem 4.3.

**Proof of Theorem 4.3.** For each pair of positive integers $n, p \geq 1$, let $\xi^{n,p}$ and $g^{n,p}$ be defined in (2.1), and $(Y^{n,p}, Z^{n,p})$ be the minimal (maximal) bounded solution of (2.2). It is easy to check that the generator $g^{n,p}$ satisfies Assumption (A2) with $\alpha$ being replaced with $\alpha \wedge (n \lor p)$.

Now, we assume that there exists a positive constant $\mu > \bar{\mu}_{\beta,\gamma,\lambda}(T)$ such that (4.17) holds. From the definitions of $\bar{\mu}_{\beta,\gamma,\lambda}(\cdot)$ and $\bar{\mu}_{\beta,\gamma,\lambda}(\cdot)$ in (4.12) and (4.14), it is not very difficult to find a (unique) positive constant $\varepsilon > 0$ satisfying $\bar{\mu}_{\beta,\gamma,\lambda}(T) = \mu$. Then, applying Proposition 4.5 with this $\varepsilon$ to BSDE (2.2) yields the existence of a $C > 0$ depending only on $(\beta, \gamma, \lambda, \varepsilon, T)$ such that $\mathbb{P}$-a.s., for all $n, p \geq 1$,

$$|Y^{n,p}_t| \leq \bar{\psi}(|Y^{n,p}_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)) \leq C \mathbb{E}\left[\psi\left(|\xi^{n,p}| + \int_0^T [\alpha_t \wedge (n \lor p)]dt, \bar{\mu}_{\beta,\gamma,\lambda}(T)\right)\bigg| \mathcal{F}_t\right] + C$$

$$\leq C \mathbb{E}\left[\psi\left(|\xi| + \int_0^T \alpha_t dt, \bar{\mu}_{\beta,\gamma,\lambda}(T)\right)\bigg| \mathcal{F}_t\right] + C$$

$$= C \mathbb{E}\left[\psi\left(|\xi| + \int_0^T \alpha_t dt, \mu\right)\bigg| \mathcal{F}_t\right] + C =: X_t, \quad t \in [0,T].$$

(4.21)

Thus, in view of (4.17), we have found an $\mathbb{R}_+$-valued, progressively measurable and continuous process $(X_t)_{t \in [0,T]}$ such that

$$d\mathbb{P} \times dt \text{-a.e.,} \quad \forall \ n, p \geq 1, \quad |Y^{n,p}_t| \leq X_t.$$

Now, we can apply Proposition 2.1 to obtain the existence of a progressively measurable process $(Z_t)_{t \in [0,T]}$ such that $(Y := \inf_n \sup_n Y^{n,p}, Z \cdot)$ is a solution to BSDE$(\xi, g)$.

Furthermore, sending $n$ and $p$ to infinity in (4.21) yields the desired inequality (4.18), and then the process $(\bar{\psi}(|Y_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)))_{t \in [0,T]}$ belongs to class (D). The existence part is then proved.

As for the uniqueness, it is easy to verify that if $(\tilde{\psi}(|Y_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)))_{t \in [0,T]}$ belongs to class (D), then both $(\psi(|Y_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)))_{t \in [0,T]}$ and $(\psi(|Y_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t) - \varepsilon))_{t \in [0,T]}$ also belong to class (D), where the function $\psi(x, \mu)$ is defined in (2.12). Then, the uniqueness of the desired solution under Assumptions (UN1) and (UN2) follows from Proposition 2.5 immediately.

Finally, the uniqueness under Assumption (UN3) is a consequence of the following Proposition 4.7.

The proof is complete. □

**Proposition 4.7.** Let the functions $\bar{\mu}_{\beta,\gamma,\lambda}(\cdot)$ and $\bar{\psi}(\cdot, \cdot)$ respectively defined in (4.12) and (4.16), $\xi$ and $\xi'$ be two terminal conditions, $g$ and $g'$ be two generators which are continuous in the variables $(y, z)$, and $(Y_t, Z_t)_{t \in [0,T]}$ and $(Y'_t, Z'_t)_{t \in [0,T]}$ be respectively a solution to BSDE$(\xi, g)$ and BSDE$(\xi', g')$ such that both $(\psi(|Y_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)))_{t \in [0,T]}$ and $(\psi(|Y'_t|, \bar{\mu}_{\beta,\gamma,\lambda}(t)))_{t \in [0,T]}$ belong to class (D) for some $\varepsilon > 0$.

Assume that $\mathbb{P}$-a.s., $\xi \leq \xi'$. If $g$ (resp. $g'$) satisfies Assumptions (UN3) and (A2) with parameters $(\alpha, \beta, \gamma, \lambda)$ such that $\bar{\psi}\left(\int_0^T \alpha_t dt, \bar{\mu}_{\beta,\gamma,\lambda}(T)\right) \in L^1$, and $d\mathbb{P} \times dt$-a.e.,

$$g(t,Y'_t, Z'_t) \leq g'(t,Y'_t, Z'_t) \quad (\text{resp.} \quad g(t,Y_t, Z_t) \leq g'(t,Y_t, Z_t)), \quad (4.22)$$

then $\mathbb{P}$-a.s., for each $t \in [0,T]$, $Y_t \leq Y'_t$.
Proof. Note that the function $\tilde{\psi}(x, \mu)$ has similar properties to the function $\psi(x, \mu)$ defined in (2.12). By a same analysis as in the proof of Proposition 3.7 and with the help of Remark 4.6 and Proposition 4.5, one can derive the desired conclusion, whose proof is omitted here. \hfill \square

Example 4.8. Let $\lambda \in (0, 1/2)$, $\beta > 0$, $\gamma > 0$ and $k \geq 0$. For $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$, define

$$
g(\omega, t, y, z) := |B_t(\omega)| + ky^2 1_{y \leq 0} + \beta |y| (\ln |y|)^{\lambda/2} 1_{|y| > 1} + \gamma |z| (\ln |z|)^{\lambda/2} 1_{|z| > 1}. $$

It is easy to verify that this generator $g$ satisfies Assumptions (EX1), (A2) and (UN3).

5. The case of $L^{(1+\mu)}$-integrability terminal condition

In this section, we always assume that the generator $g$ satisfies the following assumption, which is strictly weaker than (1.2) with $\lambda = 1/2$ and $\delta = 1$.

(A3) There exist two constants $\beta \geq 0$ and $\gamma > 0$, and an $\mathbb{R}_+$-valued progressively measurable process $(\alpha_t)_{t \in [0, T]}$ such that $d\mathbb{P} \times df$-a.e.,

$$\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(y)g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| \ln |y| 1_{|y| > 1} + \gamma |z| \sqrt{|\ln |z||}. $$

As in section 4, for $\lambda = 1/2$ and $\varepsilon > 0$, by letting $k = 1 + \varepsilon$ in Proposition 2.2 we know that there exists a constant $C_\varepsilon > 0$ depending only on $\varepsilon$ such that (2.5) holds. Then, our objective is to search for a strictly increasing and convex function $\phi$ such that for each $(s, x) \in [0, T] \times \mathbb{R}_+$, with $\beta \geq 0$ and $\gamma > 0$,

$$-\beta \varphi_x(s, x) x \ln x 1_{x > 1} - \frac{\gamma^2}{2} (\phi_x(s, x))^2 \frac{\varphi_{xx}(s, x)}{\phi_{xx}(s, x)} \left(1 + \varepsilon \right) \ln \frac{\varphi_x(s, x)}{\phi_{xx}(s, x)} + C_\varepsilon \right] + \phi_s(s, x) \geq 0, \quad (5.1)$$

which is just (2.11) with $\lambda = 1/2$, $\delta = 1$, $\bar{\delta} = 0$, $k = 1 + \varepsilon$ and $C_{k, \lambda} = C_\varepsilon$.

Now, let $\varepsilon > 0$, $\beta \geq 0$, $\gamma > 0$, and $\mu_s, \nu_s : [0, T] \to \mathbb{R}_+$ be two increasing and continuously differential functions with $\mu_0 = \varepsilon$ and $\nu_0 = 1$. We choose the following function

$$\phi(s, x) := \nu_s(1 + x)^{1+\mu_x}, \quad (s, x) \in [0, T] \times \mathbb{R}_+ \quad (5.2)$$

to explicitly solve the inequality (5.1). For each $(s, x) \in [0, T] \times \mathbb{R}_+$, a simple computation gives

$$\phi_x(s, x) = \phi(s, x) \frac{1 + \mu_s}{1 + x} > 0, \quad (5.3)$$

$$\phi_{xx}(s, x) = \phi(s, x) \frac{\mu_s(1 + \mu_s)}{(1 + x)^2} > 0 \quad (5.4)$$

and

$$\phi_s(s, x) = \phi(s, x) \left( \mu_s' \ln(1 + x) + \frac{\nu_s'}{\nu_s} \right). \quad (5.5)$$

Note that

$$\forall x \in \mathbb{R}_+, \quad x \ln x 1_{x > 1} \leq (1 + x) \ln(1 + x).$$

We substitute (5.3), (5.4) and (5.5) into the left side of (5.1) to get that for each $(s, x) \in [0, T] \times \mathbb{R}_+$,
Thus, \( \mu \) the unique solution \( \tilde{\mu} \).

In general,  

Remark 5.1. \( \beta, \gamma, \varepsilon \)  

\[
\phi(s, x) \left[ (-\beta + \mu_s) - \frac{\gamma^2(1 + \mu_s)(1 + \varepsilon)}{2\mu_s} \right] \ln(1 + x) + \left( -\gamma^2(1 + \mu_s)\tilde{C}_\varepsilon + \frac{\nu_s'}{\nu_s} \right)
\]

where  

\[
\tilde{C}_\varepsilon := \frac{(1 + \varepsilon)|\ln \gamma - \ln \varepsilon| + C_\varepsilon}{2\varepsilon}.
\]

Thus, (5.1) holds if the functions \( \mu_s, \nu_s \in [0, T] \) satisfies  

\[
\mu_s' = \beta \mu_s + \frac{\gamma^2(1 + \varepsilon)}{2\mu_s} + \frac{\gamma^2(1 + \varepsilon)}{2} + \beta, \quad s \in [0, T]
\]  

(5.6)

and  

\[
\nu_s = \exp \left( \gamma^2\tilde{C}_\varepsilon \int_0^s (1 + \mu_r)dr \right), \quad s \in [0, T].
\]  

(5.7)

It is not very hard to verify that for each \( \beta \geq 0, \gamma > 0 \) and \( \varepsilon > 0 \) there exists a unique strictly increasing and continuous function with \( \varepsilon \) at the origin satisfying (5.6). We denote this unique function by \( \tilde{\mu}_{\beta, \gamma, \varepsilon}(-) \), i.e., \( \tilde{\mu}_{\beta, \gamma, \varepsilon}(0) = \varepsilon \) and  

\[
\tilde{\mu}_{\beta, \gamma, \varepsilon}^0(s) = \beta \tilde{\mu}_{\beta, \gamma, \varepsilon}(s) + \frac{\gamma^2(1 + \varepsilon)}{2\tilde{\mu}_{\beta, \gamma, \varepsilon}(s)} + \frac{\gamma^2(1 + \varepsilon)}{2} + \beta, \quad s \in [0, T].
\]  

(5.8)

We also denote, in view of (5.7),  

\[
\tilde{\nu}_{\beta, \gamma, \varepsilon}(s) := \exp \left( \gamma^2\tilde{C}_\varepsilon \int_0^s (1 + \tilde{\mu}_{\beta, \gamma, \varepsilon}(r))dr \right), \quad s \in [0, T].
\]  

(5.9)

Furthermore, it can be directly checked that as \( \varepsilon \rightarrow 0 \), the function \( \tilde{\mu}_{\beta, \gamma, \varepsilon}^0(-) \) tends decreasingly to the unique solution \( \tilde{\mu}_{\beta, \gamma}^0(-) \) of the following ODE:  

\[
\tilde{\mu}_{\beta, \gamma}^0(0) = 0 \quad \text{and} \quad (\tilde{\mu}_{\beta, \gamma}^0)' = \beta \tilde{\mu}_{\beta, \gamma}^0 + \frac{\gamma^2}{2\tilde{\mu}_{\beta, \gamma}^0} + \frac{\gamma^2}{2} + \beta, \quad s \in (0, T].
\]  

(5.10)

Remark 5.1. In general, \( \tilde{\mu}_{\beta, \gamma}^0(-) \) does not have an explicit expression.

In summary, we have proved the following proposition.

Proposition 5.2. For each \( \beta \geq 0, \gamma > 0 \) and \( \varepsilon > 0 \), define the function  

\[
\tilde{\varphi}(s, x; \varepsilon) := \tilde{\nu}_{\beta, \gamma, \varepsilon}(s)(1 + x)^{1 + \tilde{\mu}_{\beta, \gamma, \varepsilon}(s)}, \quad (s, x) \in [0, T] \times \mathbb{R}_+
\]

with \( \tilde{\mu}_{\beta, \gamma, \varepsilon}(-) \) and \( \tilde{\nu}_{\beta, \gamma, \varepsilon}(-) \) being respectively defined in (5.8) and (5.9). Then we have

(i) \( \tilde{\varphi}(-, :, \varepsilon) \in C^{1,2}([0, T] \times \mathbb{R}_+) \);

(ii) \( \tilde{\varphi}(-, :, \varepsilon) \) satisfies the inequality in (2.3) with \( \lambda = 1/2, \delta = 1 \) and \( \delta = 0 \). i.e., for each \( (s, x, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^d \), we have  

\[
-\tilde{\varphi}_x(s, x; \varepsilon) \left( \beta x \ln x 1_{x>1} + \gamma |z| \sqrt{\ln |z|} \right) + \frac{1}{2} \tilde{\varphi}_{xx}(s, x; \varepsilon) |z|^2 + \tilde{\varphi}_x(s, x; \varepsilon) \geq 0.
\]
The following existence and uniqueness theorem is one of the main results of this section.

**Theorem 5.3.** Let the functions \( \tilde{\mu}_{\beta,\gamma,\varepsilon}(\cdot) \) and \( \tilde{\mu}_{\beta,\gamma}^0(\cdot) \) be respectively defined in (5.8) and (5.10), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in the state variables \((y,z)\). If \( g \) satisfies Assumptions (\textit{EX1}) and (\textit{A3}) with parameters \((\alpha,\beta,\gamma)\), and there exists a positive constant \( \mu > \tilde{\mu}_{\beta,\gamma,\varepsilon}^0(T) \) such that

\[
E \left[ \left( |\xi| + \int_0^T \alpha_t dt \right)^{1+\mu} \right] < +\infty, \tag{5.12}
\]

then BSDE \((\xi, g)\) admits a solution \((Y_t, Z_t)_{t \in [0,T]}\) such that for some \( \varepsilon > 0 \), \( \{Y_t^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(t)}\}_{t \in [0,T]} \) belongs to class (D), \( Z \in \mathcal{M}^p \) for some \( p > 1 \) and \( \mathbb{P}\text{-a.s.}, \)

\[
|Y_t| \leq (1 + |Y_t|)^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(t)} \leq C E \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(T)} \right] \mathcal{F}_t, \tag{5.13}
\]

where \( C \) is a positive constant depending only on \((\beta,\gamma,\varepsilon,T)\), and \( \varepsilon \) is the unique positive constant satisfying \( \tilde{\mu}_{\beta,\gamma,\varepsilon}(T) = \mu \).

Moreover, assume that \( g = g_1 + g_2 \). If \( g_1 \) satisfies Assumptions (\textit{UN1}) and (\textit{UN2}) and \( g_2 \) satisfies Assumption (\textit{UN3}), then BSDE \((\xi, g)\) admits a unique solution \((Y_t, Z_t)_{t \in [0,T]}\) such that the process \( \{Y_t^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(t)}\}_{t \in [0,T]} \) belongs to class (D) and \( Z \in \mathcal{M}^p \) for some \( p > 1 \).

In order to prove this theorem, we need the following proposition.

**Proposition 5.4.** Let the function \( \tilde{\mu}_{\beta,\gamma,\varepsilon}(\cdot) \) be defined in (5.8), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in \((y,z)\). If \( g \) satisfies Assumption (\textit{A3}) with parameters \((\alpha,\beta,\gamma)\), \( |\xi| + \int_0^T \alpha_t dt \) be a bounded random variable, and \((Y_t, Z_t)_{t \in [0,T]}\) is a solution of BSDE \((\xi, g)\) such that \( Y \) is bounded, then for each \( \varepsilon > 0 \), \( \mathbb{P}\text{-a.s.}, \)

\[
|Y_t| \leq (1 + |Y_t|)^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(t)} \leq C E \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\tilde{\mu}_{\beta,\gamma,\varepsilon}(T)} \right] \mathcal{F}_t, \tag{5.14}
\]

where \( C \) is a positive constant depending only on \((\beta,\gamma,\varepsilon,T)\).

**Proof.** Define

\[
\tilde{Y}_t := |Y_t| + \int_0^t \alpha_s ds \quad \text{and} \quad \tilde{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0,T].
\]

It follows from Itô-Tanaka’s formula that

\[
\tilde{Y}_t = \tilde{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s,Y_s,Z_s) - \alpha_s) ds - \int_t^T \tilde{Z}_s \cdot dB_s - \int_t^T dL_s, \quad t \in [0,T],
\]

where \( L \) denotes the local time of \( Y \) at the origin. Now, we fix \( \varepsilon > 0 \) and apply Itô-Tanaka’s formula to
the process $\tilde{\varphi}(s, \tilde{Y}_s; \varepsilon)$, where the function $\tilde{\varphi}(\cdot, \cdot; \varepsilon)$ is defined in (5.11), to derive, in view of (A3),
\[
\text{d}\tilde{\varphi}(s, \tilde{Y}_s; \varepsilon) = \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + \alpha_s) \text{d}s + \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot \text{d}B_s + \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \text{d}L_s
\]
\[
+ \frac{1}{2} \tilde{\varphi}_{xx}(s, \tilde{Y}_s; \varepsilon) |Z_s|^2 \text{d}s + \tilde{\varphi}_s(s, \tilde{Y}_s; \varepsilon) \text{d}s
\]
\[
\geq \left[ -\tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \left( |Y_s| \ln |Y_s|1_{|Y_s|>1} + \gamma |Z_s| \sqrt{\ln |Z_s|} \right) 
\right]
\[
+ \frac{1}{2} \tilde{\varphi}_{xx}(s, \tilde{Y}_s; \varepsilon) |Z_s|^2 + \varphi(s, \tilde{Y}_s; \varepsilon) \right] \text{d}s + \tilde{\varphi}_s(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot \text{d}B_s, \quad s \in [0, T].
\]
Furthermore, in view of the fact $|Y_s| \ln |Y_s|1_{|Y_s|>1} \leq \tilde{Y}_s \ln \tilde{Y}_s 1_{Y_s>1}$ and Proposition 5.2, we have
\[
\text{d}\tilde{\varphi}(s, \tilde{Y}_s; \varepsilon) \geq \tilde{\varphi}_x(s, \tilde{Y}_s; \varepsilon) \tilde{Z}_s \cdot \text{d}B_s, \quad s \in [0, T].
\]
(5.15)

Let us consider, for each integer $n \geq 1$ and each $t \in [0, T]$, the following stopping time
\[
\tau^t_n := \inf \left\{ s \in [t, T] : \int_t^s \tilde{\varphi}_x(r, \tilde{Y}_r; \varepsilon) \text{d}r \geq n \right\} \wedge T.
\]
In view of inequality (5.15), we have
\[
\tilde{\varphi}(t, \tilde{Y}_t; \varepsilon) \leq \mathbb{E} \left[ \tilde{\varphi}(\tau^t_n, \tilde{Y}_{\tau^t_n}; \varepsilon) \bigg| \mathcal{F}_t \right], \quad t \in [0, T], \quad n \geq 1.
\]
Thus, we have
\[
|Y_t| \leq (1 + |Y_t|)^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)} \leq \tilde{\nu}_{\beta, \gamma, \varepsilon}(t)(1 + \tilde{Y}_t)^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)}
\]
\[
\leq \tilde{\nu}_{\beta, \gamma, \varepsilon}(T) \mathbb{E} \left[ \left( 1 + |Y_{\tau^t_n}| + \int_0^{\tau^t_n} \alpha_t \text{d}t \right)^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(T)} \bigg| \mathcal{F}_t \right], \quad t \in [0, T],
\]
which yields the desired inequality (5.14) in the limit as $n \to \infty$. The proof is completed.

**Remark 5.5.**

**Assertions of Proposition 5.4 are still true if in Proposition 5.4, $(|\xi|, |Y_t|)$ is replaced with $(\xi^+, Y_t^+)$, and Assumption (A3) is replaced with the following one (A3):**

(A3') There exist two constants $\beta \geq 0$ and $\gamma > 0$, and an $\mathbb{R}_+$-valued progressively measurable process $(\alpha_t)_{t \in [0, T]}$ such that $\text{d}\tilde{\varphi} \times \text{d}t$-a.e.,
\[
\forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| \ln |y|1_{|y|>1} + \gamma |z| \sqrt{\ln |z|}.
\]

In fact, it is sufficient to use $(Y^+, 1_{Y^+}>0, Y^+ \mathbb{1}_L)$ instead of $(|Y|, \text{sgn}(Y), L)$ in the proof.

Now, we prove Theorem 5.3.

**Proof of Theorem 5.3.** For each pair of positive integers $n, p \geq 1$, let $\xi^{n,p}$ and $g^{n,p}$ be defined in (2.1), and $(Y^{n,p}, Z^{n,p})$ be the minimal (maximal) bounded solution of (2.2). It is easy to check that the generator $g^{n,p}$ satisfies assumption (A3) with $\alpha$ being replaced with $\alpha \wedge (n \vee p)$.

Now, we assume that there exists a positive constant $\mu > \tilde{\mu}^0_{\beta, \gamma}(T)$ such that (5.12) holds. From the definitions of $\tilde{\mu}_{\beta, \gamma, \varepsilon}(\cdot)$ and $\tilde{\mu}^0_{\beta, \gamma, \varepsilon}(\cdot)$ in (5.8) and (5.10), it is not very difficult to find a (unique) positive constant $\varepsilon > 0$ satisfying $\tilde{\mu}_{\beta, \gamma, \varepsilon}(T) = \mu$. Then, applying Proposition 5.4 with this $\varepsilon$ to BSDE (2.2) yields the existence of a constant $C > 0$ depending only on $(\beta, \gamma, \varepsilon, T)$ such that for all $n, p \geq 1$ and $t \in [0, T]$,
\[ |Y_t^{n,p}| \leq (1 + |Y_t^{n,p}|)^{1+\mu} \leq C \mathbb{E} \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\mu} \right] \]
\[ \leq C \mathbb{E} \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\mu} \right] = C \mathbb{E} \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\mu} \right]. \tag{5.16} \]

Thus, in view of (5.12), we have found an \( \mathbb{R}_+ \)-valued, progressively measurable and continuous process \( (X_t)_{t \in [0,T]} \) such that
\[ d\mathbb{P} \times dt \text{-a.e., } \forall n,p \geq 1, \quad |Y_t^{n,p}| \leq X_t := C \mathbb{E} \left[ \left( 1 + |\xi| + \int_0^T \alpha_t dt \right)^{1+\mu} \right]. \]

Now, we see from Proposition 2.1 that there is a progressively measurable process \( (Z_t)_{t \in [0,T]} \) such that
\[ (Y := \inf_n \sup_n Y^{n,p}, \ Z) \text{ is a solution to BSDE } (\xi, g). \]

Furthermore, sending \( n \) and \( p \) to infinity in (5.16) yields the desired inequality (5.13), and then the process \( (|Y_t|^{1+\mu})_{t \in [0,T]} \) belongs to class (D). Thus, to complete the proof of the existence part of Theorem 5.3, it remains to show that \( Z \in \mathcal{M}^P \) for some \( p > 1 \). In view of the inequality \( \mu n, \gamma, \epsilon(\cdot) \geq \epsilon, \) we see from [10, Lemma 6.1] and (5.13) that
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^{1+\frac{\mu}{\epsilon}} \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left( 1 + |Y_t|^{1+\mu} \right)^{\frac{1+\mu}{\epsilon}} \right] \leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} X_t \right)^{\frac{1+\mu}{\epsilon}} \right] \]
\[ \leq \frac{2(1+\epsilon)}{\epsilon} \left( \mathbb{E} [X_T]^{\frac{1+\mu}{\epsilon}} \right) < +\infty, \]
which implies that \( Y \in \mathcal{S}^{1+\frac{\mu}{\epsilon}}. \)

Now, we take \( p := \frac{1+\epsilon/2}{1+\epsilon/4} > 1 \) to prove \( Z \in \mathcal{M}^P. \) First, by letting \( \lambda = 1/2, \ k = 1 \) and \( y/2 \) instead of the variable \( y \) in Proposition 2.2, and using both inequalities
\[ \sqrt{\ln y} \leq \sqrt{\ln y/2} + \sqrt{\ln 2} \]
and
\[ 2x\sqrt{\ln 2} \leq (4 \ln 2)x^2 + y^2/4, \]
we see that there is a universal constant \( K > 0 \) such that
\[ \forall x, y > 0, \quad 2x\sqrt{\ln y} \leq 2x^2(\ln x + K) + \frac{3}{4}y^2. \tag{5.17} \]

Then, we pick a sufficiently large constant \( k_\epsilon > 1 \) depending only on \( \epsilon \) such that
\[ \forall x \in \mathbb{R}_+, \quad \ln(x + k_\epsilon) \leq (x + k_\epsilon)^{\frac{\epsilon}{4}}. \tag{5.18} \]

For each \( n \geq 1, \) define the following stopping time
\[ \tau_n := \inf \left\{ s \in [0,T] : \int_0^s \left( |Y_t| + k_\epsilon \right)^2 |Z_t|^2 dr \geq n \right\} \wedge T. \]

Applying Itô-Tanaka’s formula to \( (|Y_t| + k_\epsilon)^2, \) we have
\[ (|Y_0| + k_\epsilon)^2 + \int_0^{\tau_n} |Z|^2 ds \leq (|Y_{\tau_n}| + k_\epsilon)^2 + 2 \int_0^{\tau_n} (|Y_t| + k_\epsilon) \text{sgn}(Y_t) g(s, Y_s, Z_s) ds + 2 \int_0^{\tau_n} (|Y_s| + k_\epsilon) \text{sgn}(Y_s) Z_s \cdot dB_s. \tag{5.19} \]
It follows from assumption (A3) together with (5.17) and (5.18) that
\[
2(\langle Y_s \rangle + k_\varepsilon)\text{sgn}(Y_s)g(s, Y_s, Z_s)
\leq 2\alpha_t(\langle Y_s \rangle + k_\varepsilon) + 2\beta(\langle Y_s \rangle + k_\varepsilon)^2 \ln(\langle Y_s \rangle + k_\varepsilon) + 2\gamma(\langle Y_s \rangle + k_\varepsilon)\langle Z_s \rangle \sqrt{\ln|Z_s|}
\leq 2\alpha_t(\langle Y_s \rangle + k_\varepsilon) + 2(\beta + \gamma^2)(\langle Y_s \rangle + k_\varepsilon)^2 + 2\gamma^2 K(\langle Y_s \rangle + k_\varepsilon)^2 + \frac{3}{4}|Z_s|^2.
\]
Then, coming back to (5.19), we have for each \(n \geq 1\),
\[
\int_0^\tau |Z_s|^2 ds \leq 4(1 + 2\gamma^2 K) \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^2 + 8 \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon) \int_0^T \alpha_s ds
+ 8(\beta + \gamma^2) \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^2 + 8 \int_0^\tau (\langle Y_s \rangle + k_\varepsilon)\text{sgn}(Y_s)Z_s \cdot dB_s,
\]
and then there exists a positive constant \(c_p > 0\) depending only on \((p, \beta, \gamma, K)\) such that
\[
\mathbb{E}\left[ \left( \int_0^\tau |Z_s|^2 ds \right)^{\frac{p}{2}} \right] 
\leq c_p \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^p \right] + c_p \mathbb{E}\left[ \left( \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon) \int_0^T \alpha_s ds \right)^{\frac{p}{2}} \right]
+ c_p \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^{1+\frac{p}{2}} \right] + c_p \mathbb{E}\left[ \left( \int_0^\tau (\langle Y_s \rangle + k_\varepsilon)\text{sgn}(Y_s)Z_s \cdot dB_s \right)^{\frac{p}{2}} \right].
\]
Moreover, by the BDG inequality and Young’s inequality, we can find another constant \(d_p > 0\) depending only on \((p, \beta, \gamma, K)\) such that
\[
c_p \mathbb{E}\left[ \left( \int_0^\tau (\langle Y_s \rangle + k_\varepsilon)\text{sgn}(Y_s)Z_s \cdot dB_s \right)^{\frac{p}{2}} \right] \leq d_p \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^p \right] + \frac{1}{2} \mathbb{E}\left[ \left( \int_0^\tau |Z_s|^2 ds \right)^{\frac{p}{2}} \right].
\]
Substituting the above inequality to (5.20) and using Hölder’s inequality imply that for each \(n \geq 1\),
\[
\frac{1}{2} \mathbb{E}\left[ \left( \int_0^\tau |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \leq (c_p + d_p) \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^p \right] + c_p \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^{1+\frac{p}{2}} \right]
+ c_p \left( \mathbb{E}\left[ \sup_{s \in [0, T]} (\langle Y_s \rangle + k_\varepsilon)^p \right] \right)^{\frac{1}{p}} \left( \mathbb{E}\left[ \left( \int_0^T \alpha_s ds \right)^p \right] \right)^{\frac{1}{2p}}.
\]
Thus, in view of (5.12) and the facts that \(p < 1 + \frac{2}{\alpha} < 1 + \mu\) and \(Y \in S^{1+\frac{2}{\alpha}}\), using Fatou’s Lemma, we have \(Z \in \mathcal{M}^p\). The existence part is then proved.

Finally, the uniqueness part of Theorem 5.3 is a direct consequence of the following Proposition 5.6.

The proof is complete. \(\square\)

**Proposition 5.6.** Let the function \(\tilde{\mu}_{\beta, \gamma, \varepsilon}()\) be respectively defined in (5.8), \(\xi\) and \(\xi'\) be two terminal conditions, \(g\) and \(g'\) be two generators which are continuous in the variables \((y, z)\), and \((Y_t, Z_t)_{t \in [0, T]}\) and \((Y'_t, Z'_t)_{t \in [0, T]}\) be respectively a solution to BSDE \((\xi, g)\) and BSDE \((\xi', g')\) satisfying both \((\langle Y_t \rangle^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)})_{t \in [0, T]}\) and \((\langle Y'_t \rangle^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)})_{t \in [0, T]}\) belong to class \((D)\) for some \(\varepsilon_0 > 0\) and there exists an \(X \in L^1\) such that
\[
(1 + \langle Y_t \rangle + \langle Y'_t \rangle)^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)} \leq \mathbb{E}[X|\mathcal{F}_t], \quad t \in [0, T].
\]

Assume that \(\mathbb{P}\)-a.s., \(\xi \leq \xi'\). If \(g = g_1 + g_2\) (resp. \(g' = g'_1 + g'_2\)), \(g_1\) (resp. \(g'_1\)) satisfies Assumptions (UN1) and (UN2), \(g_2\) (resp. \(g'_2\)) satisfies Assumptions (UN3) and (A3) with parameters \((\alpha, \beta, \gamma)\) such
that $\mathbb{P}$-a.s., $\int_0^T \alpha_1 \, dt < +\infty$, and $\mathbb{d}\mathbb{P} \times dt$-a.e.,

$$g(t, Y_t', Z_t') \leq g'(t, Y_t', Z_t') \quad (\text{resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)), $$

then $\mathbb{P}$-a.s., for each $t \in [0, T]$, $Y_t \leq Y_t'$.

**Proof.** We only prove the case that the generator $g = g_1 + g_2$, $g_1$ satisfies (UN1) and (UN2), $g_2$ is convex in the state variables $(y, z)$ and satisfies (A3) with parameters $(\alpha, \beta, \gamma)$, and $\mathbb{d}\mathbb{P} \times dt$-a.e.,

$$g(t, Y_t', Z_t') \leq g'(t, Y_t', Z_t').$$

The other cases can be proved in the same way.

We first use the $\theta$-technique. For each fixed $\theta \in (0, 1)$, define

$$\Delta^\theta U := \frac{Y - \theta Y'}{1 - \theta} \quad \text{and} \quad \Delta^\theta V := \frac{Z - \theta Z'}{1 - \theta}. \quad (5.22)$$

Then the pair $(\Delta^\theta U, \Delta^\theta V)$ satisfies the following BSDE:

$$\Delta^\theta U_t = \Delta^\theta U_T + \int_t^T g(s, Y_s, Z_s) - \theta g'(s, Y_s', Z_s') \, ds - \int_t^T \Delta^\theta V_s \, dB_s, \quad t \in [0, T]. \quad (5.23)$$

Denote $\Delta^\theta g_1(s) := g_1(s, Y_s, Z_s) - \theta g_1(s, Y_s', Z_s')$. Observe from the assumptions that $\mathbb{d}\mathbb{P} \times ds$ - a.e.,

$$\mathbf{1}_{\Delta^\theta U_s > 0}(g(s, Y_s, Z_s) - \theta g'(s, Y_s', Z_s')) = \mathbf{1}_{\Delta^\theta U_s > 0}[g(s, Y_s, Z_s) - \theta g(s, Y_s', Z_s') + \theta g(s, Y_s', Z_s') - \theta g'(s, Y_s', Z_s')]$$

$$\leq \mathbf{1}_{\Delta^\theta U_s > 0}[\Delta^\theta g_1(s) + g_2(s, Y_s', (1 - \theta)\Delta^\theta U_s, \theta Z_s' + (1 - \theta)\Delta^\theta V_s) - \theta g_2(s, Y_s', Z_s')]$$

$$\leq \mathbf{1}_{\Delta^\theta U_s > 0}[\Delta^\theta g_1(s) + (1 - \theta)g_2(s, \Delta^\theta U_s, \Delta^\theta V_s)]$$

$$\leq \mathbf{1}_{\Delta^\theta U_s > 0}\Delta^\theta g_1(s) + (1 - \theta)\left(\alpha_s + \beta (\Delta^\theta U_s)^+ \ln(\Delta^\theta U_s)^+ \mathbf{1}_{(\Delta^\theta U_s)^+ > 1} + \gamma|\Delta^\theta V_s|\sqrt{\ln|\Delta^\theta V_s|}\right)$$

and then

$$\mathbf{1}_{\Delta^\theta U_s > 0}\frac{g(s, Y_s, Z_s) - \theta g'(s, Y_s', Z_s')}{1 - \theta}$$

$$\leq \frac{\mathbf{1}_{\Delta^\theta U_s > 0}\Delta^\theta g_1(s)}{1 - \theta} + \alpha_s + \beta (\Delta^\theta U_s)^+ \ln(\Delta^\theta U_s)^+ \mathbf{1}_{(\Delta^\theta U_s)^+ > 1} + \gamma|\Delta^\theta V_s|\sqrt{\ln|\Delta^\theta V_s|}. \quad (5.24)$$

Now, we pick $\varepsilon = \varepsilon_0/2$. Apply Itô-Tanaka’s formula to the process $\hat{\varphi}(s, (\Delta^\theta U_s)^+; \varepsilon)$, where the function $\hat{\varphi}(\cdot, \cdot; \varepsilon)$ is defined in (5.11), and in view of (5.23), (5.24) and Proposition 5.2, we use a similar argument to that in the proof of Proposition 5.4 to derive that for each $s \in [0, T]$,

$$d\hat{\varphi}(s, (\Delta^\theta U_s)^+; \varepsilon) \geq -\hat{\varphi}_x(s, (\Delta^\theta U_s)^+; \varepsilon) \left(\frac{\mathbf{1}_{\Delta^\theta U_s > 0}\Delta^\theta g_1(s)}{1 - \theta} + \alpha_s(\omega)\right) \, ds$$

$$+ \hat{\varphi}_x(s, (\Delta^\theta U_s)^+; \varepsilon) \mathbf{1}_{\Delta^\theta U_s > 0}\Delta^\theta V_s \, dB_s. \quad (5.25)$$

For each $t \in [0, T]$ and each integer $n \geq 1$, define the following stopping time

$$\tau_n^t := \inf \left\{ s \in [t, T] : |Y_s| + |Y_s'| + \int_t^s (|Z_r|^2 + |Z_r'|^2) \, dr \geq n \right\} \wedge T.$$

By virtue of (5.25) and the definition of $\hat{\varphi}$, we deduce that for each $t \in [0, T]$,
\[
\left( (\Delta^\theta U_t^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)} \right) \leq \overline{v}_{\beta,\gamma,\epsilon}(t) \left( 1 + (\Delta^\theta U_t^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)} \right)
\]
\[
\leq \overline{v}_{\beta,\gamma,\epsilon}(T) \left( 1 + (\Delta^\theta U_T^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(\tau_n^t)} \right)
\]
\[
+ \int_t^{\tau_n^t} \overline{v}_{\beta,\gamma,\epsilon}(s) \left( 1 + \overline{\mu}_{\beta,\gamma,\epsilon}(s) \right) \left( 1 + (\Delta^\theta U_s^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \right) \left( \frac{\Delta^\theta g_1(s)}{1-\theta} + \alpha_s(\omega) \right) ds.
\]
\[
- \int_t^{\tau_n^t} \overline{v}_{\beta,\gamma,\epsilon}(s) \left( 1 + \overline{\mu}_{\beta,\gamma,\epsilon}(s) \right) \left( 1 + (\Delta^\theta U_s^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \right) \Delta^\theta V_s \cdot dB_s,
\]
and then in view of (5.22), for each \( \theta \in (0,1) \) and \( n \geq 1 \),
\[
( (Y_t - \theta Y_t^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)} ) \leq K_T \left( (1-\theta) + (Y_{\tau_n^t} - \theta Y_{\tau_n^t}^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(\tau_n^t)} \right)
\]
\[
+ \int_t^{\tau_n^t} K_s \left( (1-\theta) + (Y_s - \theta Y_s^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \left[ \Delta^\theta g_1(s) + (1-\theta)\alpha_s(\omega) \right] \right) ds.
\]
(5.26)
\[
- \int_t^{\tau_n^t} K_s \left( (1-\theta) + (Y_s - \theta Y_s^t)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(s)} (Z_s - \theta Z_s^t) \right) dV_s, \quad t \in [0,T],
\]
where \( K_s := \overline{v}_{\beta,\gamma,\epsilon}(s) \left( 1 + \overline{\mu}_{\beta,\gamma,\epsilon}(s) \right), \ s \in [0,T]. \)

Moreover, define \( \bar{U}_t := Y - Y_t^t \) and \( \bar{V}_t := Z - Z_t^t \). In view of the definition of \( \Delta^\theta g_1(s) \) and Assumptions (UN1) and (UN2) of \( g_1 \), sending \( \theta \to 1 \) in (5.26) implies that for each \( n \geq 1 \),
\[
\left( \bar{U}_t^t \right)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)} \leq K_T \left( \bar{U}_{\tau_n^t}^t \right)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(\tau_n^t)} + \int_t^{\tau_n^t} K_s \left( \bar{U}_s^t \right)^{\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \left( \rho(\bar{U}_s^t) + \kappa(\bar{V}_s^t) \right) ds.
\]
(5.27)
\[
- \int_t^{\tau_n^t} K_s \left( \bar{U}_s^t \right)^{\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \bar{V}_s \cdot d\bar{B}_s, \quad t \in [0,T].
\]
In view of Remark 2.4, since \( \kappa(\cdot) \) is a continuous function of linear growth with \( \kappa(0) = 0 \), it follows from the proof of Theorem 1 in [24] that for each \( m \geq 1 \),
\[
\forall \ x \in \mathbb{R}_+, \ \kappa(x) \leq (m + 2A)x + \kappa \left( \frac{2A}{m + 2A} \right).
\]
(5.28)

Let \( \mathbb{P}_m \) be the probability on \( (\Omega, \mathcal{F}) \) which is equivalent to \( \mathbb{P} \) defined by
\[
\frac{d\mathbb{P}_m}{d\mathbb{P}} := \exp \left\{ \left( m + 2A \right) \int_0^T \bar{V}_s 1_{\bar{V}_s > 0} \cdot d\bar{B}_s + \frac{1}{2} \left( m + 2A \right)^2 \int_0^T 1_{\bar{V}_s > 0} ds \right\}.
\]

Note that \( d\mathbb{P}_m/d\mathbb{P} \) has moments of all orders. Let \( \mathbb{E}_m[\eta|\mathcal{F}_t] \) represent the conditional mathematical expectation of the random variable \( \eta \) with respect to \( \mathcal{F}_t \) under the probability measure \( \mathbb{P}_m \). Then, in view of (5.27) and (5.28), by Girsanov’s theorem we deduce that for each \( m, n \geq 1 \),
\[
\left( \bar{U}_t^t \right)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)} \leq \kappa \left( \frac{2A}{m + 2A} \right) K_T \mathbb{E}_m \left[ \left( \bar{U}_{\tau_n^t}^t \right)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(\tau_n^t)} \right| \mathcal{F}_t \right]
\]
\[
+ K_T \mathbb{E}_m \left[ \int_t^{\tau_n^t} \left( \bar{U}_s^t \right)^{\overline{\mu}_{\beta,\gamma,\epsilon}(s)} \rho(\bar{U}_s^t) ds \right| \mathcal{F}_t \right], \quad t \in [0,T].
\]
(5.29)

On the other hand, denote
\[
p := \inf_{s \in [0,T]} \frac{1 + \overline{\mu}_{\beta,\gamma,\epsilon}(s)}{1 + \overline{\mu}_{\beta,\gamma,\epsilon}(s)} \quad \text{and} \quad P_t := \left( U_t^t \right)^{1+\overline{\mu}_{\beta,\gamma,\epsilon}(t)}, \quad t \in [0,T].
\]

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Since $\varepsilon = \varepsilon_0/2$, from the definition of $\tilde{\mu}_{\beta, \gamma, \varepsilon}(s)$ it is not hard to conclude that $p > 1$. We pick a $q \in (1, p)$. It follows from [10, Lemma 6.1] and (5.21) that

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} |P_t|^q \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left( 1 + |Y_t| + |Y'_t| \right)^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}(t)} \right]^{\frac{q}{2}} \leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \mathbb{E}[X|\mathcal{F}_t] \right)^{\frac{q}{2}} \right]
$$

$$
\leq \frac{p}{p-q} (\mathbb{E}[X])^{\frac{q}{2}} < +\infty,
$$

which means that $P \in S^q$, and then, by virtue of Hölder’s inequality,

$$
\forall \ m \geq 1, \quad \mathbb{E}_m \left[ \sup_{t \in [0,T]} |P_t| \right] \leq \mathbb{E} \left( \left( \sup_{t \in [0,T]} |P_t| \right) \frac{d\mathbb{P}_m}{d\mathbb{P}} \right) < +\infty. \quad (5.30)
$$

Thus, in view of $\tilde{U}_T^+ = 0$ and the fact that $\rho(\cdot)$ is of linear growth, we can send $n$ to infinity in (5.29) to obtain that for each $m \geq 1$,

$$
P_t \leq \kappa \left( \frac{2A}{m + 2A} \right) K_T T + K_T \mathbb{E}_m \left[ \int_t^T P_s^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \rho \left( P_s^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot) \right) ds \right], \quad t \in [0, T]. \quad (5.31)
$$

Finally, define the function

$$
\bar{\rho}(x) := \sup_{t \in [0,T]} \left[ \frac{\rho \left( x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \right)}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot)} \right], \quad x \in \mathbb{R}_+.
$$

It is clear that $\bar{\rho}(0) = 0$. Since $\rho(x)$ is a nondecreasing concave function on $\mathbb{R}_+$ with $\rho(0) = 0$, it follows from [23, Lemma 6.1] that $\rho(x)/x, x > 0$ is a non-increasing function, and then,

$$
\bar{\rho}(x) = x \sup_{t \in [0,T]} \left[ \frac{\rho \left( x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \right)}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot)} \right] = x \frac{\rho \left( x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \right)}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot)} = x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot), \quad 0 < x \leq 1,
$$

and

$$
\bar{\rho}(x) = x \sup_{t \in [0,T]} \left[ \frac{\rho \left( x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \right)}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot)} \right] = x \frac{\rho \left( x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot) \right)}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}(\cdot)} = x^{1+\mu_{\beta, \gamma, \varepsilon}}(\cdot), \quad x \geq 1.
$$

Then, $\bar{\rho}(\cdot)$ is well defined on $\mathbb{R}_+$, nondecreasing, continuous and of linear growth, and

$$
\int_{0^+} \frac{dx}{\bar{\rho}(x)} = \int_{0^+} \frac{dx}{x^{1+\tilde{\mu}_{\beta, \gamma, \varepsilon}}} = (1 + \varepsilon) \int_{0^+} \frac{du}{\rho(u)} = +\infty. \quad (5.32)
$$

Now, coming back to (5.31), we have that for each $m \geq 1$,

$$
P_t \leq \kappa \left( \frac{2A}{m + 2A} \right) K_T T + K_T \mathbb{E}_m \left[ \int_t^T \bar{\rho}(P_s) ds \right], \quad t \in [0, T]. \quad (5.33)
$$

Then, in view of (5.30), (5.32), (5.33) and the fact that

$$
\lim_{m \to \infty} \kappa \left( \frac{2A}{m + 2A} \right) = 0,
$$

applying [18, Lemma 2.1] implies that $\mathbb{P}$-a.s., for each $t \in [0, T]$, $P_t = 0$, that is, $Y_t \leq Y'_t$. The proof is complete.
Example 5.7. Let $\beta > 0$, $\gamma > 0$ and $k \geq 0$. For each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$, define
\[
    g_1(\omega, t, y, z) := -ke^y + \beta|y|\ln|y|1_{|y| \leq 1} + \gamma|z|\sqrt{\ln|z||1_{|z| \leq 1}},
\]
\[
    g_2(\omega, t, y, z) := |B_t(\omega)| + ke^{-y} + \beta|y|\ln|y|1_{|y| > 1} + \gamma|z|\sqrt{\ln|z||1_{|z| > 1}}
\]
and
\[
    g(\omega, t, y, z) := |B_t(\omega)| + k(e^{-y} - e^y) + \beta|y|\ln|y| + \gamma|z|\sqrt{\ln|z|}.
\]
We can check that $g$ satisfies Assumptions (EX1) and (A2) with parameters $(|B| + \beta/e + \beta/\sqrt{\epsilon} + 2k, \beta, \gamma)$, $g = g_1 + g_2$, $g_1$ satisfies Assumptions (UN1) and (UN2), and $g_2$ satisfies Assumptions (UN3) and (A3) with parameters $(|B| + k, \beta, \gamma)$. Then the generator $g$ satisfies all assumptions in Theorem 5.3.

Remark 5.8. In comparison with the related results of [3] and [7], weaker assumptions on both the terminal condition and the generator are required in Theorem 5.3 to ensure the existence of the unbounded solution of BSDEs. In fact, the terminal condition in Theorem 5.3 may only need to belong to $L^p$ for some $1 < p \leq 2$, whereas $p > 2$ in [3] and [7], and the generator $g$ only need to satisfy Assumptions (EX1) and (A3), whereas the condition (1.2) with $\delta = 1$ and $\lambda = 1/2$ in [3] and [7]. To ensure the uniqueness of the unbounded solution, a locally monotone condition is required in [3] and [7], which is different from the assumptions in Theorem 5.3. However, we especially mention that the generator $g$ in Example 5.7 is covered by Theorem 5.3, but is not covered by the results of [3] and [7] when $k > 0$. In addition, it is noted that a general comparison theorem for the bounded solutions of BSDEs (see Proposition 5.6) is also established in this section.

6. The case of $L^p$-integrability terminal condition with $p > 1$

In this section, we always assume that the generator $g$ satisfies the following assumption, which is strictly weaker than (1.2) with $\delta = 1$ and $\lambda > 1/2$.

(A4) There exist three constants $\beta \geq 0$, $\gamma > 0$ and $\lambda > 1/2$, and an $\mathbb{R}_+$-valued progressively measurable process $(\alpha_t)_{t \in [0, T]}$ such that $d\mathbb{P} \times dt$-a.e.,
\[
    \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(y)g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta|y|\ln|y|1_{|y| > 1} + \gamma|z|\ln|z|^{\lambda}.
\]

For each $\epsilon > 0$, by letting $k = 1 + \epsilon$ in Proposition 2.2 we know that there exists a positive constant $C_{\lambda, \epsilon} > 0$ depending only on parameters $(\lambda, \epsilon)$ such that (2.5) holds. Then, our objective is to search for a positive constant $\delta > 0$ and a strictly increasing and strictly convex function $\phi$ such that for each $(s, x) \in [0, T] \times \mathbb{R}_+$, with $\beta \geq 0$ and $\gamma > 0$,
\[
    -\frac{\gamma^2}{2} \frac{(\phi_x(s, x))^2}{\phi_{xx}(s, x) - \delta} \left( (1 + \epsilon)^{4(\lambda - 1)^+} \left| \ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta} \right|^{2\lambda} + C_{\lambda, \epsilon} \right) + \beta \phi_x(s, x)x \ln x 1_{x > 1} + \phi(s, x) \geq 0,
\]
which is just (2.11) with $\lambda > 1/2$, $\delta = 1$, $k = 1 + \epsilon$ and $C_{k, \lambda} = C_{\lambda, \epsilon}$.

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Now, let $\varepsilon > 0$, $\beta \geq 0$, $\gamma > 0$, $\lambda > 1/2$ and $\mu_0, \nu_0 : [0, T] \rightarrow \mathbb{R}_+$ be two increasing and continuously differential functions with $\mu_0 = \varepsilon$ and $\nu_0 = 0$. Define

$$\tilde{k} := e + \left(\frac{2\lambda\mu_T}{\gamma}\right)^2.$$  

(6.2)

We choose the following function

$$\phi(s, x) := (x + \tilde{k}) \exp \left(\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda} + \nu_s\right), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

to explicitly solve the inequality (6.1). For each $(s, x) \in [0, T] \times \mathbb{R}_+$, a simple computation gives

$$\phi_s(s, x) = \phi(s, x) \frac{2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}{(x + \tilde{k})^2 \ln(x + \tilde{k})} > 0,$$  

(6.3)

and

$$\phi_{ss}(s, x) = \phi(s, x) \frac{2\lambda(2\lambda - 1)\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}{(x + \tilde{k})^2 \ln(x + \tilde{k})} \geq \exp \left(\varepsilon \left(\ln(x + \tilde{k})\right)^{2\lambda} - 1\right) \frac{2\lambda(2\lambda - 1)\varepsilon \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}{(x + \tilde{k}) \ln(x + \tilde{k})},$$  

(6.4)

Furthermore, for each $(s, x) \in [0, T] \times \mathbb{R}_+$, in view of $\mu_s \geq \mu_0 = \varepsilon$, we have

$$\phi(s, x) \frac{2\lambda(2\lambda - 1)\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}{(x + \tilde{k})^2 \ln(x + \tilde{k})} \geq \delta_{\lambda, \varepsilon}, \quad (s, x) \in [0, T] \times \mathbb{R}_+.$$  

(6.5)

It then follows from (6.4) that

$$\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon} \geq \phi(s, x) \frac{2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1} \left[2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1} + 1\right]}{(x + \tilde{k})^2 \ln(x + \tilde{k})}, \quad (s, x) \in [0, T] \times \mathbb{R}_+. \quad (6.7)$$

For each $(s, x) \in [0, T] \times \mathbb{R}_+$, by (6.3) and (6.7) we have

$$\frac{\gamma^2}{2} \left(\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}\right)^2 \leq \frac{\gamma^2}{2} \phi(s, x) \left(1 + \frac{1}{2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}\right) \leq \frac{\gamma^2}{2} \phi(s, x) \left(1 + \frac{1}{2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}}\right).$$  

(6.8)

And, from the definition of $\tilde{k}$ it can be directly verified that for each $(s, x) \in [0, T] \times \mathbb{R}_+$,

$$\ln(x + \tilde{k}) + 2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda} + (2\lambda - 1) \leq \frac{1 + \varepsilon}{\varepsilon} \left[\ln(x + \tilde{k}) + 2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda}\right].$$  

(6.9)

In view of (6.2), (6.3), (6.4), (6.9) and (6.7), we have

$$\begin{cases}
\frac{\gamma \phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \geq \frac{\gamma}{2\lambda(1 + \varepsilon)\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}} \geq \frac{\gamma}{2\lambda\mu_T \sqrt{x + \tilde{k}}} \geq \frac{\gamma}{2\lambda\mu_T \sqrt{k}} \geq 1;
\end{cases}$$

$$\begin{cases}
\frac{\gamma \phi_{xx}(s, x)}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \leq \frac{\gamma}{2\lambda\mu_s \left(\ln(x + \tilde{k})\right)^{2\lambda - 1}} \leq \frac{\gamma}{2\lambda\varepsilon (x + \tilde{k})},
\end{cases}$$
which yields the following

\[
\left| \ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \right|^{2\lambda} \leq 2^{2\lambda-1} \left| \ln \frac{\gamma}{2\lambda \varepsilon} \right|^{2\lambda} + 2^{2\lambda-1} \left( \ln(x + \tilde{k}) \right)^{2\lambda}. \tag{6.10}
\]

In the sequel, observe that

\[
x \ln x_1 \geq (x + \tilde{k}) \ln(x + \tilde{k}), \quad x \in \mathbb{R}_+.
\]

We substitute (6.8), (6.10), (6.3) and (6.5) into the left side of (6.1) with \(\delta_{\lambda, \varepsilon}\) instead of \(\tilde{\delta}\) to get

\[
\begin{align*}
& -\gamma^2 \frac{\phi_x(s, x)^2}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \left[ 4(\lambda-1) + (1 + \varepsilon) \right] \left| \ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \right|^{2\lambda} + C_{\lambda, \varepsilon} \\
& - \beta \phi_x(s, x) x \ln x_1 + \phi_x(s, x) \\
& \geq -\gamma^2 \frac{\phi(s, x)}{2} \left[ 1 + \frac{1}{2\beta \mu_s} \right] \left[ 1 + \mu' \right] \left( \ln(x + \tilde{k}) \right)^{2\lambda} + C_{\lambda, \varepsilon} \\
& - \beta \phi(s, x) \left( 2\lambda \mu_s \left( \ln(x + \tilde{k}) \right)^{2\lambda} + \mu' \right), \quad (s, x) \in [0, T] \times \mathbb{R}_+
\end{align*}
\]

where

\[
k_\lambda := 2^{2(\lambda-1) + 2\lambda-1} \quad \text{and} \quad C_{\lambda, \varepsilon} := (1 + \varepsilon)k_\lambda \left| \ln \frac{\gamma}{2\lambda \varepsilon} \right|^{2\lambda} + C_{\lambda, \varepsilon}.
\]

Furthermore, in view of \(2\lambda > 1\), by Young’s inequality it is not very hard to verify that there exists a constant \(C_{\beta, \lambda, \varepsilon} > 0\) depending only on \((\beta, \lambda, \varepsilon)\) such that for each \((s, x) \in [0, T] \times \mathbb{R}_+\),

\[
\beta \ln(x + \tilde{k}) \leq \varepsilon \left( \ln(x + \tilde{k}) \right)^{2\lambda} + C_{\beta, \lambda, \varepsilon}
\]

Then, for each \((s, x) \in [0, T] \times \mathbb{R}_+\), we have

\[
\begin{align*}
& -\gamma^2 \frac{\phi_x(s, x)^2}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \left[ 4(\lambda-1) + (1 + \varepsilon) \right] \left| \ln \frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta_{\lambda, \varepsilon}} \right|^{2\lambda} + C_{\lambda, \varepsilon} \\
& - \beta \phi_x(s, x) x \ln x_1 + \phi_x(s, x) \\
& \geq \phi(s, x) \left\{ \left[ -\gamma^2 \frac{(1 + \varepsilon)k_\lambda}{2} \left( 1 + \frac{1}{2\beta \mu_s} \right) - 2\beta \lambda \mu_s - \varepsilon + \mu'_s \right] \left( \ln(x + \tilde{k}) \right)^{2\lambda} \\
& + \left[ -\gamma^2 \frac{1}{2\beta \mu_s} \right] C_{\lambda, \varepsilon} - \tilde{C}_{\beta, \lambda, \varepsilon} + \mu'_s \right\}.
\end{align*}
\]

Thus, (6.1) holds if the functions \(\mu_s, \nu_s \in [0, T]\) satisfies

\[
\mu'_s = \gamma^2 \frac{(1 + \varepsilon)k_\lambda}{2 \lambda} \left( 1 + \frac{1}{2\beta \mu_s} \right) + 2\beta \lambda \mu_s + \varepsilon \quad s \in [0, T] \tag{6.11}
\]

and

\[
\nu_s = \gamma^2 \frac{1}{4\lambda} C_{\lambda, \varepsilon} \int_0^s \frac{1}{\mu_r} dr + \left( \frac{\gamma^2}{2} \tilde{C}_{\lambda, \varepsilon} + \tilde{C}_{\beta, \lambda, \varepsilon} \right) s, \quad s \in [0, T]. \tag{6.12}
\]
It is not very hard to verify that for each \( \beta \geq 0, \gamma > 0, \lambda > 1/2 \) and \( \varepsilon > 0 \), there exists a unique strictly increasing and continuous function with \( \varepsilon \) at the origin satisfying (6.11). We denote this unique solution by \( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(\cdot) \), i.e., \( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(0) = \varepsilon \) and
\[
\bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}'(s) = \frac{\gamma^2(1 + \varepsilon)k_\lambda}{2} \left( 1 + \frac{1}{2\lambda \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(s)} \right) + 2\beta \lambda \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(s) + \varepsilon, \quad s \in [0, T].
\]
(6.13)

We also denote, in view of (6.12),
\[
\bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(s) := \frac{\gamma^2}{4\lambda} C_{\lambda,\varepsilon} \int_0^s \frac{1}{\bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(r)} \, dr + \left( \frac{\gamma^2}{2} C_{\lambda,\varepsilon} + \bar{C}_{\lambda,\varepsilon} \right) s, \quad s \in [0, T].
\]
(6.14)

Moreover, it can be directly checked that as \( \varepsilon \to 0^+ \), the function \( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(\cdot) \) tends decreasingly to the unique solution \( \bar{\mu}^0_{\beta,\gamma,\lambda}(\cdot) \) of the following ODE:
\[
\bar{\mu}^0_{\beta,\gamma,\lambda}(0) = 0 \quad \text{and} \quad (\bar{\mu}^0_{\beta,\gamma,\lambda}(s))' = \frac{\gamma^2 k_\lambda}{2} \left( 1 + \frac{1}{2 \lambda \bar{\mu}^0_{\beta,\gamma,\lambda}(s)} \right) + 2 \beta \lambda \bar{\mu}^0_{\beta,\gamma,\lambda}(s), \quad s \in (0, T].
\]
(6.15)

In summary, we have proved the following proposition.

**Proposition 6.1.** For each \( \beta \geq 0, \gamma > 0, \lambda > 1/2 \) and \( \varepsilon > 0 \), define the function
\[
\bar{\varphi}(s, x; \varepsilon) := (x + \bar{k}) \exp \left( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(s) \left( \ln \left( x + \bar{k} \right) \right)^{2\lambda} + \bar{\varphi}_{\beta,\gamma,\lambda,\varepsilon}(s) \right), \quad (s, x) \in [0, T] \times \mathbb{R}_+
\]
with \( \bar{\varphi}_{\beta,\gamma,\lambda,\varepsilon}(\cdot) \) and \( \bar{\varphi}_{\beta,\gamma,\lambda,\varepsilon}(\cdot) \) being respectively defined in (6.13) and (6.14), and
\[
\bar{k} := e + \left( \frac{2 \lambda \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(T)}{\gamma} \right)^2,
\]
which is \( \bar{k} \) in (6.2) with \( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(T) \) instead of \( \mu_T \). Then we have

(i) \( \bar{\varphi}(\cdot, \cdot; \varepsilon) \in C^{1,2}([0, T] \times \mathbb{R}_+) \);

(ii) There exists a positive constant \( \delta_{\lambda,\varepsilon} > 0 \) depending only on \( \lambda, \varepsilon \) such that \( \bar{\varphi}(\cdot, \cdot; \varepsilon) \) satisfies the inequality (2.4), i.e., for each \( (s, x, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^d \), we have
\[
-\bar{\varphi}_x(s, x; \varepsilon) \left( \beta_x \ln x 1_{x>1} + \gamma |z||\ln |z||^\lambda \right) + \frac{1}{2} (\bar{\varphi}_{xx}(s, x; \varepsilon) - \delta_{\lambda,\varepsilon}) |z|^2 + \bar{\varphi}_z(s, x; \varepsilon) \geq 0.
\]

Now, for \( \lambda > 1/2 \), we define the function
\[
\tilde{\psi}(x, \mu) := x \exp \left( \mu (\ln(1 + x))^{2\lambda} \right), \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]
(6.17)

The following existence and uniqueness theorem is one main result of this section.

**Theorem 6.2.** Let the functions \( \bar{\beta}_{\beta,\gamma,\lambda,\varepsilon}(\cdot), \bar{\mu}^0_{\beta,\gamma,\lambda}(\cdot) \) and \( \bar{\psi}(\cdot, \cdot) \) be respectively defined in (6.13), (6.15), and (6.17), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in the state variables \( (y, z) \). If \( g \) satisfies Assumptions (EX1) and (A4) with parameters \( (\alpha, \beta, \gamma, \lambda) \), and there exists a positive constant \( \mu > \bar{\mu}^0_{\beta,\gamma,\lambda}(T) \) such that
\[
\mathbb{E} \left[ \bar{\psi} \left( |\xi| + \int_0^T \alpha_t dt, \mu \right) \right] < +\infty,
\]
(6.18)
then BSDE($\xi, g$) admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that \( \left( \tilde{\psi} \left( |Y_t| + \int_0^t \alpha_s \, ds, \, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t) \right) \right)_{t \in [0,T]} \) belongs to class (D) for some $\varepsilon > 0$ and $Z \in M^2$, where $\varepsilon$ is the unique positive constant satisfying $\tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$. And, there exists a constant $\delta_{\varepsilon} > 0$ depending only on $(\lambda, \varepsilon)$ such that $P$-a.s., for each $t \in [0,T]$,

$$|Y_t| \leq \tilde{\psi}(|Y_t| + \int_0^t \alpha_s \, ds, \, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)) + \frac{\delta_{\varepsilon}}{2} \mathbb{E} \left[ \int_t^T |Z_s|^2 \, ds \mid \mathcal{F}_t \right]$$

(6.19)

where $C > 0$ is a positive constant depending only on $(\beta, \gamma, \lambda, \varepsilon, T)$.

Moreover, if the inequality (6.18) holds for some $\mu > 2q3^{2\lambda-1}\bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T)$ with $q > 1$, then BSDE $(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that the process

$$\left( \tilde{\psi} \left( |Y_t| + \int_0^t \alpha_s \, ds, \, 2q3^{2\lambda-1}\bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t) \right) \right)_{t \in [0,T]}$$

belongs to class (D), where $\varepsilon$ is the unique positive constant satisfying $2q3^{2\lambda-1}\bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$. And, if the generator $g$ further satisfies Assumption (UN3), then the solution satisfying the preceding condition is also unique.

In order to prove this theorem, we need the following two propositions.

**Proposition 6.3.** Let functions $\tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)$, $\tilde{\psi}(s, x; \varepsilon)$ and $\tilde{\psi}(x, \mu)$ be respectively defined on (6.13), (6.16) and (6.17). Then, there exists a universal constant $K > 0$ depending on $(\beta, \gamma, \lambda, \varepsilon, T)$ such that

$$\forall \ (s, x) \in [0,T] \times \mathbb{R}_+, \quad \tilde{\psi}(x, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) \leq \tilde{\psi}(s, x; \varepsilon) \leq K \tilde{\psi}(x, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) + K.$$  (6.20)

**Proof.** The first inequality in (6.20) is obvious. We now prove the second inequality. In fact,

$$\frac{\tilde{\psi}(s, x; \varepsilon)}{\tilde{\psi}(x, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) + 1} = \frac{(x + \tilde{k}) \exp \left( \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s) \left( \ln \left( x + \tilde{k} \right) \right)^{2\lambda} + \tilde{\nu}_{\beta, \gamma, \lambda, \varepsilon}(s) \right)}{x \exp \left( \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s) \left( \ln (1 + x) \right)^{2\lambda} + 1 \right)} + 1 \leq \frac{x + \tilde{k}}{x} \exp \left( \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) \left( \left( \ln \left( x + \tilde{k} \right) \right)^{2\lambda} - \left( \ln (1 + x) \right)^{2\lambda} \right) + \tilde{\nu}_{\beta, \gamma, \lambda, \varepsilon}(T) \right)$$

$$=: H_1(x; \beta, \gamma, \lambda, \varepsilon, T), \quad (s, x) \in [0,T] \times [1, +\infty),$$

with $\tilde{k}$ being defined in Proposition 6.1. And, in the case of $x \in [0, 1]$,

$$\frac{\tilde{\psi}(s, x; \varepsilon)}{\tilde{\psi}(x, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) + 1} \leq (1 + \tilde{k}) \exp \left( \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) \left( \ln \left( 1 + \tilde{k} \right) \right)^{2\lambda} + \tilde{\nu}_{\beta, \gamma, \lambda, \varepsilon}(T) \right)$$

$$=: H_2(\beta, \gamma, \lambda, \varepsilon, T), \quad s \in [0,T].$$

Hence, for all $x \in \mathbb{R}_+$, we have

$$\frac{\tilde{\psi}(s, x; \varepsilon)}{\tilde{\psi}(x, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(s)) + 1} \leq H_1(x; \beta, \gamma, \lambda, \varepsilon, T)1_{x \geq 1} + H_2(\beta, \gamma, \lambda, \varepsilon, T)1_{0 < x < 1}, \quad s \in [0,T].$$  (6.21)

Thus, in view of (6.21) and the fact that the function $H_1(x; \beta, \gamma, \lambda, \varepsilon, T)$ is continuous on $[1, +\infty)$ and tends to $\exp(\tilde{\nu}_{\beta, \gamma, \lambda, \varepsilon}(T))$ as $x \to +\infty$, the second inequality in (6.20) follows immediately. \qed
Proposition 6.4. Let functions $\tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(\cdot)$ and $\tilde{\psi}(\cdot,\cdot)$ be respectively defined in (6.13) and (6.17), $\xi$ be a terminal condition and $g$ be a generator which is continuous in $(y,z)$. If $g$ satisfies Assumption (A4) with parameters $(\alpha, \beta, \gamma, \lambda)$, $|\xi| + \int_0^T \alpha_t \, dt$ is a bounded random variable, and $(Y_t, Z_t)_{t \in [0,T]}$ is a solution of BSDE $(\xi, g)$ such that $Y_t$ is bounded, then for each $\varepsilon > 0$, there exists a constant $\delta_{\lambda,\varepsilon} > 0$ depending only on $(\lambda, \varepsilon)$ such that $\mathbb{P}$-a.s., for each $t \in [0,T]$, we have

$$|Y_t| \leq \tilde{\psi}(|Y_t| + \int_0^t \alpha_s \, ds, \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(t)) + \frac{\delta_{\lambda,\varepsilon}}{2} \mathbb{E} \left[ \int_t^T |Z_s|^2 \, ds \middle| F_t \right]$$

$$\leq C \mathbb{E} \left[ \tilde{\psi} \left( |\xi| + \int_0^T \alpha_t \, dt, \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(T) \right) \right] + C,$$

where $C$ is a positive constant depending only on $(\beta, \gamma, \lambda, \varepsilon, T)$.

Proof. Define

$$\tilde{Y}_t := |Y_t| + \int_0^t \alpha_s \, ds \quad \text{and} \quad \tilde{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T].$$

Using Itô-Tanaka’s formula, we have

$$\tilde{Y}_t = \tilde{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - \alpha_s) \, ds - \int_t^T \tilde{Z}_s \cdot dB_s - \int_t^T dL_s, \quad t \in [0, T],$$

where $L_t$ is the local time of $Y_t$ at the origin. Now, we fix $\varepsilon > 0$ and apply Itô-Tanaka’s formula to the process $\tilde{\psi}(s, \tilde{Y}_t; \varepsilon)$, where the function $\tilde{\psi}(s, \cdot; \varepsilon)$ is defined in (6.16), to derive, in view of (A4),

$$d\tilde{\psi}(s, \tilde{Y}_t; \varepsilon) = \tilde{\psi}_x(s, \tilde{Y}_t; \varepsilon) \left( -\text{sgn}(Y_s)g(s, Y_s, Z_s) + \alpha_s \right) \, ds + \tilde{\psi}_x(s, \tilde{Y}_t; \varepsilon) \tilde{Z}_s \cdot dB_s + \tilde{\psi}_x(s, \tilde{Y}_t; \varepsilon) dL_s + \frac{1}{2} \tilde{\psi}_{xx}(s, \tilde{Y}_t; \varepsilon) |Z_s|^2 \, ds + \tilde{\psi}_s(s, \tilde{Y}_t; \varepsilon) ds$$

$$\geq \left[ -\tilde{\psi}_x(s, \tilde{Y}_t; \varepsilon) (\beta|Y_s| \ln |Y_s|) \cdot 1_{|Y_s| > 1} + \gamma|Z_s| \cdot \ln |Z_s| \cdot \lambda \right]$$

$$+ \frac{1}{2} \tilde{\psi}_{xx}(s, \tilde{Y}_t; \varepsilon) |Z_s|^2 + \tilde{\psi}_s(s, \tilde{Y}_t; \varepsilon) \tilde{Z}_s \cdot dB_s, \quad s \in [0, T].$$

Furthermore, from the inequality $|Y_s| \ln |Y_s| \cdot 1_{|Y_s| > 1} \leq \tilde{Y}_s \ln \tilde{Y}_s \cdot 1_{\tilde{Y}_s > 1}$ and Proposition 6.1, we see that there is a constant $\delta_{\lambda,\varepsilon} > 0$ depending only on $(\lambda, \varepsilon)$ such that

$$d\tilde{\psi}(s, \tilde{Y}_t; \varepsilon) \geq \frac{\delta_{\lambda,\varepsilon}}{2} |Z_s|^2 \, ds + \tilde{\psi}_s(s, \tilde{Y}_t; \varepsilon) \tilde{Z}_s \cdot dB_s, \quad s \in [0, T].$$

(6.23)

Let us consider, for each integer $n \geq 1$ and each $t \in [0, T]$, the following stopping time

$$\tau_n^t := \inf \left\{ s \in [t, T] : \int_t^s \left[ \tilde{\psi}_x(r, \tilde{Y}_r; \varepsilon) \right]^2 \tilde{Z}_r^2 \, dr \geq n \right\} \wedge T.$$

In view of the inequality (6.23), we have that for each $n \geq 1$,

$$\tilde{\psi}(t, \tilde{Y}_t; \varepsilon) + \frac{\delta_{\lambda,\varepsilon}}{2} \mathbb{E} \left[ \int_t^{\tau_n^t} |Z_s|^2 \, ds \middle| F_t \right] \leq \mathbb{E} \left[ \tilde{\psi}(\tau_n^t, \tilde{Y}_{\tau_n^t}; \varepsilon) \middle| F_t \right], \quad t \in [0, T].$$

Thus, by Proposition 6.3, there exists a constant $K > 0$ depending only on $(\beta, \gamma, \lambda, \varepsilon, T)$ such that

$$\tilde{\psi}(\tilde{Y}_t, \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(t)) \leq \tilde{\psi}(t, \tilde{Y}_t; \varepsilon) \leq K \mathbb{E} \left[ \tilde{\psi}(\tau_n^t, \tilde{Y}_{\tau_n^t}; \varepsilon) \middle| F_t \right] \leq K \mathbb{E} \left[ \tilde{\psi}(\tilde{Y}_{\tau_n^t}, \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(\tau_n^t)) \middle| F_t \right] + K, \quad t \in [0, T].$$

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And, since $\tilde{\psi}(x, \mu)$ is increasing in $x$, we have that for each $n \geq 1$,

$$|Y_t| \leq \tilde{\psi}\left(|Y_t| + \int_0^t \alpha_s \, ds, \tilde{\mu}_{\beta, \gamma, \lambda, \epsilon}(t)\right) + \frac{\delta_{\lambda, \epsilon}}{2} \mathbb{E}\left[\int_t^T |Z_s| \, ds \bigg| F_t\right]$$

$$\leq K \mathbb{E}\left[\tilde{\psi}\left(|Y_t| + \int_0^t \alpha_s \, ds, \tilde{\mu}_{\beta, \gamma, \lambda, \epsilon}(t)\right) \bigg| F_t\right] + K, \quad t \in [0, T],$$

which gives the desired inequality (6.22) in the limit as $n$ to infinity. The proof is completed. \(\square\)

**Remark 6.5.** Assertions of Proposition 6.4 are still true if $(|\xi|, |Y_t|)$ is replaced with $(\xi^+, Y_t^+)$, and Assumption (A4) is replaced with the following one (A4'):

(A4') There exist three constants $\beta \geq 0$, $\gamma > 0$ and $\lambda \in (0, 1/2)$, and an $\mathbb{R}_+$-valued progressively measurable process $(\alpha_t)_{t \in [0, T]}$ such that $d\mathbb{P} \times dt$-a.e.,

$$\forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| + \gamma |z|,$$

To show this, it is sufficient to use $(Y^+, 1_{Y > 0}, \frac{1}{2} L)$ instead of $(|Y|, \text{sgn}(Y), L)$ in the proof.

Now, we prove Theorem 6.2.

**Proof of Theorem 6.2.** For each pair of positive integers $n, p \geq 1$, let $\xi^{n,p}$ and $g^{n,p}$ be defined in (2.1), and $(Y^{n,p}, Z^{n,p})$ be the minimal (maximal) bounded solution of (2.2). It is easy to check that the generator $g^{n,p}$ satisfies Assumption (A4) with $\alpha$ being replaced with $\alpha \wedge (n \vee p)$.

Now, we assume that there exists a positive constant $\mu > \tilde{\mu}_{\beta, \gamma, \lambda}(T)$ such that (6.18) holds. From the definitions of $\tilde{\mu}_{\beta, \gamma, \lambda}(\cdot)$ and $\tilde{\mu}_{\beta, \gamma, \lambda}(\cdot)$ in (6.13) and (6.15), it is not very difficult to find a (unique) constant $\varepsilon > 0$ satisfying $\tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$. Then, applying Proposition 6.4 with this $\varepsilon$ to BSDE (2.2) yields that there exists a constant $\delta_{\lambda, \varepsilon} > 0$ depending only on $(\lambda, \varepsilon)$ such that $\mathbb{P}$-a.s., for all $n, p \geq 1$,

$$|Y_t^{n,p}| \leq \tilde{\psi}\left(|Y_t| + \int_0^t [\alpha_s \wedge (n \vee p)] \, ds, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)\right) + \frac{\delta_{\lambda, \epsilon}}{2} \mathbb{E}\left[\int_t^T |Z_s^{n,p}| \, ds \bigg| F_t\right]$$

$$\leq C \mathbb{E}\left[\tilde{\psi}\left(|Y_t^{n,p}| + \int_0^T [\alpha_t \wedge (n \vee p)] \, dt, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T)\right) \bigg| F_t\right] + C$$

(6.24)

$$\leq C \mathbb{E}\left[\tilde{\psi}\left(|\xi| + \int_0^T \alpha_t \, dt, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T)\right) \bigg| F_t\right] + C =: X_t, \quad t \in [0, T],$$

where $C > 0$ is a positive constant depending only on $(\beta, \gamma, \lambda, \varepsilon, T)$. Thus, in view of (6.24), there is an $\mathbb{R}_+$-valued, progressively measurable and continuous process $(X_t)_{t \in [0, T]}$ such that

$$d\mathbb{P} \times dt$$-a.e., $\forall n, p \geq 1, \quad |Y^{n,p}| \leq X.$$

Now, we can apply Proposition 2.1 to obtain the existence of a progressively measurable process $(Z_t)_{t \in [0, T]}$ such that $(Y := \inf_n \sup_n Y^{n,p}, \ Z_t)$ is a solution to BSDE$(\xi, g)$.

Furthermore, sending $n$ and $p$ to infinity in (6.24) yields the desired inequality (6.19), and then the process $\left(\tilde{\psi}\left(|Y_t| + \int_0^t \alpha_s \, ds, \tilde{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t)\right)\right)_{t \in [0, T]}$ belongs to class (D).
In the sequel, we assume that (6.18) holds for some \( \mu > 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda}(T) \) with \( q > 1 \). Then, there exists a unique constant \( \varepsilon > 0 \) such that \( 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda}(T) = \mu \). In view of the definition of \( \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(-) \) and by virtue of the analysis from (6.1) to (6.15), it is not very difficult to verify that Proposition 6.1 still holds when the functions \( \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(-) \) in (6.14), (6.16) and the definition of \( \tilde{k} \) are all replaced with \( 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(-) \), and then BSDE\( (\xi, g) \) admits a solution \((Y_t, Z_t)_{t \in [0,T]}\) such that the process
\[
\left( \tilde{\psi} \left( Y_t \right) + \int_0^t \alpha_sds, 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(t) \right)_{t \in [0,T]}
\]
belong to class (D).

Finally, we suppose further that the generator \( g \) satisfies Assumption (UN3). The desired uniqueness is a direct consequence of the following Proposition 6.6. The proof is complete.

**Proposition 6.6.** Let the functions \( \tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(-) \) and \( \tilde{\psi}(-,\cdot) \) be respectively defined in (6.13) and (6.17), \( \xi \) and \( \xi' \) be two terminal conditions, \( g \) and \( g' \) be two generators which are continuous in the variables \((y,z)\), and \((Y_t, Z_t)_{t \in [0,T]}\) and \((Y'_t, Z'_t)_{t \in [0,T]}\) be respectively a solution to BSDE\( (\xi, g) \) and BSDE\( (\xi', g') \) such that for some \( \varepsilon > 0 \) and \( q > 1 \), both
\[
\left( \tilde{\psi} \left( Y_t \right) + \int_0^t \alpha_sds, 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(t) \right)_{t \in [0,T]}
\]
and
\[
\left( \tilde{\psi} \left( Y'_t \right) + \int_0^t \alpha_sds, 2q3^{2\lambda-1}\tilde{\mu}_{\beta,\gamma,\lambda,\varepsilon}(t) \right)_{t \in [0,T]}
\]
belong to class (D). Assume that \( \mathbb{P}\text{-a.s.}, \xi \leq \xi' \). If \( g \) (resp. \( g' \)) satisfies Assumptions (UN3) and (A4) with parameters \((\alpha, \beta, \gamma, \lambda)\), and \( d\mathbb{P} \times dt\text{-a.e.}, \)
\[
g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \quad (\text{resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)), \tag{6.25}
\]
then \( \mathbb{P}\text{-a.s.}, \) for each \( t \in [0, T] \), \( Y_t \leq Y'_t \).

**Proof.** We first consider the case that the generator \( g \) is convex in the state variables \((y,z)\), satisfies Assumption (A4) with parameters \((\alpha, \beta, \gamma, \lambda)\), and \( d\mathbb{P} \times dt\text{-a.e.}, \)
\[
g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t),
\]
In order to utilize the convexity condition of the generator \( g \), we use the \( \theta \)-technique developed in for example [13]. For each fixed \( \theta \in (0, 1) \), define
\[
\Delta^\theta U := \frac{Y - \theta Y'}{1 - \theta} \quad \text{and} \quad \Delta^\theta V := \frac{Z - \theta Z'}{1 - \theta}.
\tag{6.26}
\]
Then the pair \((\Delta^\theta U, \Delta^\theta V)\) satisfies the following BSDE:
\[
\Delta^\theta U_t = \Delta^\theta U_T + \int_t^T \Delta^\theta g(s, \Delta^\theta U_s, \Delta^\theta V_s)ds - \int_t^T \Delta^\theta V_s \cdot dB_s, \quad t \in [0, T],
\tag{6.27}
\]
where \( d\mathbb{P} \times ds \text{-a.e.}, \) for each \((y,z) \in \mathbb{R} \times \mathbb{R}^d, \)
\[
\Delta^\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ g(s, (1 - \theta)y + \theta Y'_s, (1 - \theta)z + \theta Z'_s) - \theta g(s, Y'_s, Z'_s) \right]
\tag{6.28}
+ \frac{\theta}{1 - \theta} \left[ g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s) \right].
\]
It follows from the assumptions that \( d\mathbb{P} \times ds \ - a.e. \),

\[
\forall \ (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad \Delta^\theta g(s, y, z) \leq g(s, y, z) \leq \alpha + \beta|y| \ln |y| 1_{|y| > 1} + \gamma|z| \ln |z|^{\lambda},
\]

(6.29)

which means that the generator \( \Delta^\theta g \) satisfies Assumption \((A4')\) defined in Remark 6.5.

On the other hand, note that for each \( x, y, z \geq 0, \ c > 0 \) and \( \lambda > 1/2 \), we have

\[
1 + cx + cy \leq (1 + c)(1 + x)(1 + y)
\]

and

\[
(x + y + z)^{2\lambda} \leq 3^{2\lambda - 1} (x^{2\lambda} + y^{2\lambda} + z^{2\lambda}).
\]

It follows that for each \( x, y \geq 0, \ \theta \in (0, 1) \) and \( \lambda > 1/2 \),

\[
\left[ \ln \left( 1 + \frac{(x - \theta y)^+}{1 - \theta} \right) \right]^{2\lambda} \leq 3^{2\lambda - 1} \left[ \ln \left( 1 + \frac{1}{1 - \theta} \right) \right]^{2\lambda} + (\ln (1 + x))^{2\lambda} + (\ln (1 + y))^{2\lambda}.
\]

Furthermore, note that for each \( \mu > 0, \ \lambda > 1/2, \ p > 1 \) and \( x > 0 \), we have

\[
\exp \left( \mu (\ln (1 + x))^{2\lambda} \right) \geq x^p.
\]

By virtue of the last two inequalities, Young’s inequality, the definition of \( \tilde{\psi} \) and the assumptions of Proposition 6.6, it is not very hard to verify that the process

\[
\tilde{\psi} \left( (\Delta^\theta U_t)^+ + \int_0^t \alpha_s ds, \ \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t) \right), \quad t \in [0, T]
\]

belongs to class \((D)\) for each \( \theta \in (0, 1) \). Thus, for BSDE \((6.27)\), by virtue of \((6.29)\), Remark 6.5 and the proof of Proposition 6.4, we derive that there exists a \( C > 0 \) depending on \((\beta, \gamma, \lambda, \varepsilon, T)\) such that

\[
(\Delta^\theta U_t)^+ \leq \tilde{\psi} \left( (\Delta^\theta U_t)^+, \ \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(t) \right) \leq CE \tilde{\psi} \left( (\Delta^\theta U_T)^+ + \int_0^T \alpha_s ds, \ \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) \right) | F_t | + C, \quad t \in [0, T].
\]

(6.30)

Furthermore, in view of the fact that

\[
(\Delta^\theta U_T)^+ = \frac{(\xi - \theta \xi')^+}{1 - \theta} = \frac{(\xi - \theta \xi + \theta (\xi - \xi'))^+}{1 - \theta} \leq \xi^+,
\]

(6.31)

it follows from \((6.30)\) that for each \( \theta \in (0, 1) \),

\[
(Y_t - \theta Y_t')^+ \leq (1 - \theta) \left( CE \tilde{\psi} \left( \xi^+ + \int_0^T \alpha_s ds, \ \bar{\mu}_{\beta, \gamma, \lambda, \varepsilon}(T) \right) | F_t | + C \right), \quad t \in [0, T].
\]

Thus, the desired conclusion follows by sending \( \theta \to 1 \) in above inequality.

For the case that the generator \( g \) is concave in the state variables \((y, z)\), we need to respectively use the \( \theta Y \ - Y' \) and \( \theta Z \ - Z' \) to replace \( Y \ - \theta Y' \) and \( Z \ - \theta Z' \) in \((6.26)\). In this case, the generator \( \Delta^\theta g \) in \((6.28)\) should be replaced with

\[
\Delta^\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ \theta g(s, Y_s, Z_s) - g(s, (1 - \theta)y + \theta Y_s, (1 - \theta)z + \theta Z_s) \right] + \frac{1}{1 - \theta} \left[ g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s) \right].
\]

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Since \( g \) is concave in \((y, z)\), we have \( d\mathbb{P} \times ds - a.e. \)

\[
\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad g(s, -(1 - \theta)y + \theta Y_s, -(1 - \theta)z + \theta Z_s) \geq \theta g(s, Y_s, Z_s) + (1 - \theta)g(t, -y, -z),
\]

and then, (6.29) can be replaced by

\[
\forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad \Delta^\theta g(s, y, z) \leq -g(s, -y, -z) \leq \alpha_s + \beta |y| \ln |y| 1_{|y| > 1} + \gamma |z| \ln |z| ^\lambda,
\]

which means that the generator \( \Delta^\theta g \) still satisfies Assumption \( (A4') \). Consequently, (6.30) still holds. Moreover, we use

\[
(\Delta^\theta U_T) ^+ = \frac{(\theta \xi - \xi )^+}{1 - \theta} = \frac{[\theta \xi - \xi + (\xi - \xi')]^+}{1 - \theta} \leq (-\xi)^+ = \xi^-,
\]

instead of (6.31), and it follows from (6.30) that for each \( \theta \in (0, 1) \),

\[
(\theta Y_t - Y_t') ^+ \leq (1 - \theta) \left( C \mathbb{E} \left[ \tilde{\psi} \left( \xi^- + \int_0^T \alpha_s ds, \tilde{\mu}_{\beta, \gamma, \lambda, c}(T) \right) \bigg| \mathcal{F}_t \right] + C \right), \quad t \in [0, T].
\]

Thus, the desired conclusion follows by sending \( \theta \to 1 \) in above inequality.

Finally, in the same way as above, we can prove the desired conclusion under assumptions that the generator \( g' \) satisfies \( (UN3) \) and \( (A4) \) with parameters \( (\alpha, \beta, \gamma) \), and \( d\mathbb{P} \times dt-a.e. \), \( g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t) \). The proof is then complete. \( \square \)

**Example 6.7.** Let \( \lambda > 1/2, \beta > 0, \gamma > 0 \) and \( k \geq 0 \). For \((\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d\), define

\[
g(\omega, t, y, z) := |B_t(\omega)| + ky^2 1_{y \leq 0} + \beta |y| \ln |y| 1_{|y| > 1} + \gamma |z| \ln |z| ^\lambda 1_{|z| > 1}.
\]

It is easy to verify that this generator \( g \) satisfies Assumptions \( (EX1), (A4) \) and \( (UN3) \).

### 7. Conclusion

The objective of this section is to unify the existence and uniqueness results obtained in previous four sections, namely, Theorems 3.3, 4.3, 5.3 and 6.2. First, we unify Assumptions \( (A1), (A2), (A3) \) and \( (A4) \) into the following one:

**\( (EX2) \)** There exist three constants \( \beta \geq 0, \gamma > 0 \) and \( \lambda \in [0, +\infty) \), and an \( \mathbb{R}_+ \)-valued progressively measurable process \((\alpha_t)_{t \in [0, T]}\) such that \( d\mathbb{P} \times dt-a.e. \),

\[
\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(y) g(\omega, t, y, z) \leq \alpha_t(\omega) + \beta |y| \ln |y| ^{\lambda} 1_{|y| > 1} + \gamma |z| \ln |z| ^\lambda.
\]

For each \( \lambda \in [0, +\infty) \), we define the function

\[
\psi(x, \mu; \lambda) := x \exp \left( \mu (\ln(1 + x)) ^{(\lambda+\frac{1}{2})\lambda} \right), \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]  

(7.1)

It can be easily verified that

(i) if \( \lambda \in [0, 1/2) \), then for each \( p > 1 \),

\[
x \ln(1 + x) < \psi(x, \mu; \lambda) = x \exp \left( \mu (\ln(1 + x)) ^{\lambda+\frac{1}{2}} \right) < x^p, \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+;
\]
(ii) if \( \lambda = 1/2 \), then
\[
x^{1+\mu} < \psi(x, \mu; \lambda) = x(1+x)^\mu < (1+x)^{1+\mu}, \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+;
\]

(iii) if \( \lambda \in (1/2, +\infty) \), then for each \( p > 1 \) and \( \epsilon \in (0, 1) \),
\[
x^p < \psi(x, \mu; \lambda) = x \exp \left( \mu (\ln(1+x))^{2\lambda} \right) < \exp(x^\epsilon), \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]
Moreover, for each \( \beta \geq 0, \gamma > 0, \) and \( \epsilon > 0 \), let \( \mu_{\beta, \gamma, \lambda, \epsilon}(\cdot) \) with \( \lambda > 0 \) be the unique solution of the following ODE: \( \mu(0) = \epsilon \) and
\[
\mu'(s) = \begin{cases} 
\frac{\gamma^2}{2} + \frac{1}{2\lambda + 1} + \beta + \epsilon, & \lambda \in (0, 1/2); \\
\beta \mu_s + \frac{\gamma^2(1+\epsilon)}{2} \left( 1 + \frac{1}{\mu_s} \right) + \beta, & \lambda = 1/2; \quad \text{for } s \in [0, T], \quad (7.2) \\
2\beta \lambda \mu_s + \frac{\gamma^2(1+\epsilon) k_\lambda}{4\lambda} \left( 1 + \frac{1}{\mu_s} \right) + \epsilon, & \lambda \in (1/2, +\infty)
\end{cases}
\]
and \( \mu_{\beta, \gamma, \lambda}^0(\cdot) \) with \( \lambda \geq 0 \) be the unique solution of the following ODE: \( \mu(0) = 0 \) and
\[
\mu'(s) = \begin{cases} 
\frac{\gamma^2}{2\lambda + 1} \mu_s + \beta, & \lambda \in [0, 1/2); \\
\beta \mu_s + \frac{\gamma^2}{2} \left( 1 + \frac{1}{\mu_s} \right) + \beta, & \lambda = 1/2; \quad \text{for } s \in (0, T] \quad (7.3) \\
2\beta \lambda \mu_s + \frac{\gamma^2 k_\lambda}{4\lambda} \left( 1 + \frac{1}{\mu_s} \right), & \lambda \in (1/2, +\infty)
\end{cases}
\]
with
\[ k_\lambda := 2^{2(\lambda-1)^+ + 2\lambda - 1}. \]

It is not difficult to check that as \( \epsilon \to 0^+ \), \( \mu_{\beta, \gamma, \lambda, \epsilon}(\cdot) \) tends decreasingly to \( \mu_{\beta, \gamma, \lambda}^0(\cdot) \) on \([0, T]\).

In view of Remark 4.1, Theorems 3.3, 4.3, 5.3 and 6.2 can be unified into the following one.

**Theorem 7.1.** Let the functions \( \psi(x, \mu; \lambda), \mu_{\beta, \gamma, \lambda, \epsilon}(\cdot) \) and \( \mu_{\beta, \gamma, \lambda}^0(\cdot) \) be respectively defined in (7.1), (7.2) and (7.3), \( \xi \) be a terminal condition and \( g \) be a generator which is continuous in the state variables \((y, z)\) and satisfies Assumptions (EX1) and (EX2) with parameters \((\alpha, \beta, \gamma)\). We have

(i) If \( \lambda = 0 \) and the terminal condition satisfies
\[
\psi \left( |\xi| + \int_0^T \alpha_r dt, \mu_{\beta, \gamma, \lambda}^0(T); \lambda \right) \in L^1,
\]
then BSDE \((\xi, g)\) admits a solution \((Y_t, Z_t)_{t \in [0, T]}\) such that \( \psi \left( |Y_T|, \mu_{\beta, \gamma, \lambda}(T); \lambda \right) \) belongs to class \((D)\).

Moreover, if \( g \) also satisfies either Assumptions (UN1) and (UN2) or Assumption (UN3), then the solution satisfying the preceding condition is unique.

(ii) If \( \lambda = (0, 1/2) \) and there exists a constant \( \mu > \mu_{\beta, \gamma, \lambda}(T) \) such that
\[
\psi \left( |\xi| + \int_0^T \alpha_r dt, \mu; \lambda \right) \in L^1,
\]
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then BSDE$(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that for some $\varepsilon > 0$ satisfying $\mu_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$, $\psi \left( Y_t, \mu_{\beta, \gamma, \lambda, \varepsilon}(t); \lambda \right)$ belongs to class $(D)$.

Moreover, if $g$ also satisfies either Assumptions \textbf{(UN1)} and \textbf{(UN2)} or Assumption \textbf{(UN3)}, then the solution satisfying the preceding condition is unique.

(iii) If $\lambda = 1/2$ and there exists a constant $\mu > \mu_{\beta, \gamma, \lambda}^0(T)$ such that the condition (7.5) holds, then BSDE$(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that for some $\varepsilon > 0$ satisfying $\mu_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$, $\psi \left( Y_t, \mu_{\beta, \gamma, \lambda, \varepsilon}(t); \lambda \right)$ belongs to class $(D)$ and $Z \in \mathcal{M}^p$ for some $p > 1$.

Moreover, if $g = g_1 + g_2$, $g_1$ satisfies Assumptions \textbf{(UN1)} and \textbf{(UN2)} and $g_2$ satisfies Assumption \textbf{(UN3)}, then the solution satisfying the preceding condition is unique.

(iv) If $\lambda > 1/2$ and there exists a constant $\mu > \mu_{\beta, \gamma, \lambda}^0(T)$ such that the condition (7.5) holds, then BSDE $(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that for some $\varepsilon > 0$ satisfying $\mu_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$, $\psi \left( Y_t + \int_0^t \alpha_s ds, \mu_{\beta, \gamma, \lambda, \varepsilon}(t); \lambda \right)$ belongs to class $(D)$ and $Z \in \mathcal{M}^2$.

Moreover, if the condition (7.5) holds for some $\mu > 2q3^{2\lambda-1}\mu_{\beta, \gamma, \lambda}^0(T)$ with $q > 1$, then BSDE$(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that the process
\[
\left( \psi \left( Y_t + \int_0^t \alpha_s ds, 2q3^{2\lambda-1}\mu_{\beta, \gamma, \lambda, \varepsilon}(t); \lambda \right) \right)_{t \in [0,T]}
\]
belongs to class $(D)$, where $\varepsilon$ is the unique positive constant satisfying $2q3^{2\lambda-1}\mu_{\beta, \gamma, \lambda, \varepsilon}(T) = \mu$. And, if $g$ further satisfies Assumption \textbf{(UN3)}, then the solution satisfying the preceding condition is also unique.

Finally, we would like to mention that the integrability conditions (7.4) and (7.5) are believed to be reasonably weakest possible for the existence of the solution, but it remains to be proved. To the best of our knowledge, all conclusions in Theorem 7.1 can not be obtained from any existing results.

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