Averaging, Renormalization Group and criticality in Cosmology

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Abstract

Several problems in physics, in particular the averaging problem in gravity, can be described in a formalism derived from the real-space Renormalization Group (RG) methods. It is shown that the RG flow is provided by the Ricci-Hamilton equations which are thereby provided with a physical interpretation. The connection between a manifold deformation according to these equations and Thurston’s conjecture is exhibited. The significance of criticality which naturally appears in this framework is briefly discussed. This article summarizes also recent work with M. Carfora [1]. Moreover, a report on some work in progress is given and some open issues in the averaging problem pointed out.

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Any theory, a physical theory in particular, describes certain kind of objects and their environments, e.g. matter, fields, space-times. Among them there is a class of objects which are easy to study, let us call them the homogeneous objects, where what we have in mind is generally that they are highly symmetric. The difficulties arise when we begin to examine inhomogeneous objects.

In physically realistic cases one necessarily has to deal with inhomogeneous objects. Homogeneous objects are easy to analyze but they are so special that it is unlikely that a physical object of a certain kind is going to be a homogeneous one. Thus we have to resort to studying inhomogeneous objects despite the related mathematical difficulties arising. Fortunately, there exist some techniques which allow us to relate various inhomogeneous objects with homogeneous ones, in such a way that we keep control of change in their properties and phenomena we are interested in.

To analyze a general case of an inhomogeneous object there are two strategies we may adopt which we will refer to as an infinitesimal change strategy and a finite change one.

In the first case, we have in mind a situation where, in a space of objects, there is a special point representing a homogeneous object. Let us perturb it slightly to an infinitesimally close position in the tangent space of the original point. What we get is an object which is almost homogeneous in the sense that the inhomogeneities generated are infinitesimally small. This kind of object is a concern of perturbation theory and this is a powerful method in a great many cases despite its obvious limitations. In the other possible approach, the strategy of finite change, we study how to deform a generic object (at some finite distance from the special one) into a special (homogeneous) one, keeping control of its properties which change during this process.

To concretize this consideration let us concentrate on three examples.

1. 3-d geometry

The aim is here to classify all 3-d manifolds. This is a difficult and not yet achieved task. Special objects in this context are e.g. homogeneous manifolds. The problems are here related with Thurston’s 3-d geometrization conjecture [2] which claims that any closed three-manifold may be canonically decomposed into pieces such that each of the pieces admits a locally homogeneous geometry. In this case it is not clear how perturbation theory applies (strategy of infinitesimal change) since the problems encountered are that of global topology. A possible solution, or rather a program (not yet fully accomplished) for proving this conjecture was put forward by R. Hamilton. Roughly, this program rests on the application of the Ricci-Hamilton flow, namely, the idea is to choose an arbitrary metric on the three-manifold and then deform it via the Ricci-Hamilton flow equation

$$\frac{\partial g_{ab}}{\partial \eta} = \frac{2}{3} < R(\eta) > g_{ab} - 2 R_{ab}(\eta),$$  \hspace{1cm} (1)

where $< R >$ denotes the average of the scalar curvature $R$ over the manifold and $R_{ab}$ the components of the Ricci tensor, respectively, whereas $\eta$ is some deformation.
parameter. The hope is then to relate the local singularities of the flow to the manifold decomposition in Thurston’s conjecture and to show that the Ricci-Hamilton flow of the geometry away from each of the local singularities approaches that of a locally homogeneous geometry in each disconnected piece. It has been shown that for certain classes of three-metrics the Ricci-Hamilton flow converges \[3, 4, 5\].

2. Critical phenomena

Critical points (second-order phase transitions) occur in liquid-gas transitions, ferromagnetic transitions, binary alloys, etc. There are close analogies relating all these critical points, which is one of the fascinations of the subject. In the case of a ferromagnet the critical temperature \(T_c\) marks the onset of spontaneous magnetization (with no external field applied) which varies as \((T_c - T)^\beta\) from below near \(T_c\). The exponent \(\beta\) is an example of a critical exponent. One of the aims of a theory of critical phenomena is to predict the values of the exponents (see e.g. \[6\]). (There are other critical exponents characterizing other power laws near the critical point). In this example we may identify as special objects systems far from criticality which, for the case of a ferromagnet, are those in low or high temperature regimes.

Critical phenomena belong to a class of phenomena where power-law behaviour occurs, but the exponent is not a simple fraction. This is one of the many problems a theory of critical phenomena must address. It is well known that the mean-field-type-theories (van der Waals, Weiss, Landau) lead to exponents in form of rational fractions, in disagreement with experiment. It happens often that apparently different physical systems exhibit precisely the same critical exponents and this fact, paradoxical on its face value, is known as universality leading to universality classes. It has been found in scattering experiments that at the critical point large-scale fluctuations are not exponentially rare, as they are above or below \(T_c\), and large-scale (macroscopic) dynamical structures, roughly scale-free, are generated. This phenomenon occurs in several physical systems not adequately understood (distribution of earthquakes, turbulence, polymers). In each case in a wide range of scales some phenomenon varies as a power law of the scale, presumably because of a gross mismatch between the largest and smallest scales in the problem. With a solution to these problems a new way of looking at physics emerged, namely Renormalization Group (RG) theory of phase transitions and critical phenomena which constitutes one of the greatest achievements in theoretical physics of the last decade. In recognition of this K. Wilson, who put forward the RG approach, was awarded the 1982 Nobel prize in physics.

3. Gravitation theory

The aim of cosmology is to describe a realistic (lumpy) universe. Unfortunately, we do not know any solutions of Einstein equations capable of describing a lumpy universe. For this reason, when considering the kinematics and dynamics of the
universe as a whole, one usually ignores the fine-graining due to the local inhomogeneities and deals with the simpler structure of space-time geometry, which is more illuminating from the point of view of cosmology. The relation however between “background models” of the universe (usually taken to be homogeneous and isotropic) and the fine-grained (more detailed) ones, such as to give a mathematical formulation of the idea that the universe is on average homogeneous is a difficult problem (see e.g. [7]). This problem, known generally as the averaging problem, lead to the assumption that, on large scales, the universe may be described by the homogeneous and isotropic FLRW metrics, the special objects in this context.

To investigate this issue, perturbation theory does not seem to be adequate since the problem is generically non-perturbative in its nature. In practical terms the heuristical justification for using FLRW models asserts that for them to hold the matter inhomogeneities have to be averaged (or smoothed-out) and redistributed homogeneously (e.g. in the form of a perfect fluid). A series of related mathematical problems arise.

First of all, let us notice that we are using continuous functions in modeling the universe (matter density, pressure or kinematical scalars of the velocity field), assuming that they represent “volume averages” of the corresponding fine-scale quantities. Einstein equations are solved with a smooth (continuously distributed) stress tensor which implies a space-time (or ensemble) averaging of a discrete matter distribution in forms of stars, galaxies, etc, has been carried out. The results of such averaging in an inhomogeneous medium depend on scale but this scale was never explicitly agreed upon. Additional problem is that a volume average for tensors is a non-covariant quantity (unlike for scalars), so a more sophisticated definition is required. The basic and tacit assumption which underlies this whole procedure is that a smoothed-out universe and the actual, inhomogeneous one, behave identically under their own gravitation. Or, more precisely, almost identically on some scales of interest, e.g. the ones that are much greater than a characteristic scale of the local inhomogeneities and much smaller than a characteristic length of the universe model under study. This assumption is usually taken for granted, but does by no means need to be true. In particular, it should be stressed, that we face here a severe problem of the non-commutativity of averaging of the metric and calculating the Einstein tensor (non-linear in the metric). Because of this feature averaging of the Einstein equations is likely to be not an easy task. Another problem of interest is here concerned with a measure of closeness between the same physical systems (universes) containing geometrical information at different scales of description enabling us to resolve less or more details [8]. Gromov’s distance (see below) might also serve as a useful concept in this context.

Various approaches have been attempted in tackling the averaging problem with various competing definitions of averaging or, sometimes, with no definition at all where the averages were introduced formally (for a review see [9] or [10]). Worth mentioning is here a macroscopic gravity theory [11]. This is an axiomatic approach whose averaging
refers to a derivation of the so-called macroscopic field theories whereby the Einstein equations are averaged in a covariant procedure using bilocal tensor-valued p-forms. This approach generalizes an averaging procedure due to Lorentz in electrodynamics in deriving Maxwell’s equations (which is a macroscopic theory of electromagnetism) by means of a space-time averaging of the Maxwell-Lorentz equations. For more details on the physical status of the macroscopic gravity theory and its comparison with other approaches we refer the reader to [12].

Now, it is the subject of this article to briefly describe another possible approach to tackle the averaging problem in gravitational theory with an application to cosmology in mind (for details we point the reader to [1]). This approach belongs to the finite change strategy methods remarked earlier. Our idea was to use RG method, theory and philosophy, interpreted broadly to include various kinds of “multi-length-scale” and “coarse-graining” phenomena and arguments. This enabled us to find an interesting and deep connection of the averaging problem with the problems of 3-d geometry, as summarized in our first example.

The usual RG transformations invoking averaging over square blocks are designed mostly having ferromagnetic systems in mind. However there are many open problems suitable for the RG methods. They belong to the most difficult problems known in physics where their difficulty can be traced to a multiplicity of scales and where the reductionist approach fails. For each new physical situation one has to “custom-make” the RG map and according to M. Fisher: A “good” renormalization group must be “apt” or appropriate for the problem at hand, and it must, in particular, “focus” properly on the critical phenomena of interest [13]. The RG can be used in various formulations. Apart from displaying the scale dependence of the fundamental laws and constants, it can be useful to eliminate unimportant variables. Usually, the mathematical expressions are similar but the interpretations can differ.

The basic physical insight on which the technique of RG is built, due to Kadanoff, was an assumption that near a critical point the system “looks the same on all length scales” (roughly), namely, he hypothesized that if the block lattice is considered, then at the critical point the block-lattice Hamiltonian may be reduced to the initial lattice Hamiltonian by scaling transformations. Then in the 70’s Wilson gave a formulation which provided a systematic way of implementing the integration over a finite fraction of degrees of freedom in a near-critical system, and quantifying the effect on the remaining variables, providing in this way all the mathematical infrastructure required to explain scaling and universality. In other words, the effect of the long wave-length fluctuations could be calculated using self-similarity properties of a critical system under scale transformations. When the effect on the long wave components of the integration over the short wavelength ones vanishes, the transformation has a fixed point which determines the universal critical singularities. If we think of a RG transformation as a transformation in a space of coupling constants of a theory, then the fixed point does not flow under RG transformation. The critical behaviour of the model can be gotten from the RG flow. The otherwise complicated flow pattern becomes particularly simple in the vicinity of
the fixed point where the linearized RG flow and scaling hold. It should be clear that the averaging problem in cosmology belongs to a multi-length-scale class of problems as it is effectively a question of how a system behaves when we make the scale successively coarser. As such it is then most naturally addressed using the RG approach, thought of as a general strategy to handle problems with multi-length-scales and enabling us to extract the long distance behaviour of the system. Hopefully choosing this path we will be able to say something about scaling (possibly universality) in the universe, which RG theory neatly explains in most cases. Indeed notice that there are scaling laws exhibited in the universe (e.g. the power law behaviour of the two-point correlation function for galaxies) which seem to be a hint that the universe might in fact be not far from criticality. For other problems in cosmology and astrophysics where critical phenomena of various types may play a role, see the review by Smolin [14]. This being the motivation, the main question then to address is how to implement the RG philosophy in gravitational physics. This is what we are going to describe now.

We can approach the problem by setting up a program for approximating cosmological space-time solutions of Einstein’s equations via the development of a procedure for smoothing sets of initial data for such space-times. In doing so we choose to handle the unwanted fluctuations of matter and space-time geometry on small scales at the level of data sets, since the time evolution of the initial data set for the Einstein equations is actually determined by the constraints which that data set has to satisfy. Thus our purpose is to extract effective dynamics capturing the global dynamics of the original space-time.

The approach we adopt is that of a 3+1 formulation of GR. Let us assume that we have a differentiable, compact riemannian three-manifold $\Sigma$ without a boundary, to be thought of as a particular hypersurface in a 4-dim space-time. We also assume that $\Sigma$ is a manifold of bounded geometry. Such manifolds, or more precisely the corresponding riemannian structures, can then be classified according to how they can be covered by small metric balls (to be defined later). Moreover, this set of riemannian structures has some remarkable compactness properties. This is a classical result due to M. Gromov, related to the possibility of introducing a distance function which, roughly speaking, enables one to say how close riemannian manifolds are to each other. Of particular interest in this context is the fact that nearby riemannian manifolds (in the sense of Gromov distance) can be covered with metric balls arranged in similar packing configurations [16, 15].

To study local properties on a manifold we need to introduce coordinates. This is an arbitrary choice but there is a preferred system of coordinates particularly well suited to studying riemannian geometry [17]. This is the geodesic system. This coordinate system is also a natural one from the point of view of a human observer. The insight here is that by using, in an appropriate way, the geodesic system of coordinates one can also give a good description of the global properties of riemannian structures captured by the coverings of the manifold with the metric balls.

In order to define such coverings [15], let us parameterize the geodesics by arc-length,
and for any point \( p \in \Sigma \), let \( d_\Sigma(x, p) \) denote the distance function of a generic point \( x \) from the chosen one \( p \). Then for any given \( \epsilon > 0 \) it is always possible to find an ordered set of points \( \{ p_1, \cdots, p_N \} \) in \( \Sigma \), so that

the open metric balls (geodesic balls) \( B_\Sigma(p_i, \epsilon) = \{ x \in \Sigma | d_\Sigma(x, p_i) < \epsilon \}, \ i = 1, \cdots, N \),

cover \( \Sigma \); in other words the collection \( \{ p_1, \cdots, p_N \} \) is an \( \epsilon \)-net in \( \Sigma \).

the open balls \( B_\Sigma(p_i, \epsilon/2), i = 1, \cdots, N \), are disjoint, i.e. \( \{ p_1, \cdots, p_N \} \) is a minimal \( \epsilon \)-net in \( \Sigma \).

This way we obtain a discretized manifold model which is obviously simpler to deal with than the original manifold, in the sense that it is a union of geodesic balls. In particular we can carry out in this model averaging of tensors, which we cannot in curved space. Moreover, this discrete object describes the manifold well, the underlying construction being somewhat similar in spirit to that of Regge Calculus or dynamical triangulations used in Monte Carlo approaches to 4-dim Quantum Gravity. It luckily happens also that the \( \epsilon \)-nets just introduced are useful from a physical point of view. To see this let us consider the average of a scalar function \( f \) on \( \Sigma \)

\[
\langle f \rangle_\Sigma(g) = \frac{\int_\Sigma f \, d\mu_g}{\text{vol}(\Sigma, g)},
\]

where \( \text{vol}(\Sigma, g) = \int_\Sigma d\mu_g \) and \( \mu_g \) is the riemannian measure associated with the three-metric \( g \) of \( \Sigma \). If the geometry of \( \Sigma \) is not known on large scale we cannot take (2) as an operational way of defining the average of \( f \). Taking a pragmatic viewpoint and assuming that we can only experience geometry in sufficiently small neighbourhoods of a finite set of instantaneous observers, it makes much more sense to replace (2) with a suitable average based on the geometrical information available on the length scale of such observers. For simplicity, given a finite set of instantaneous observers, located at the points \( x_1, \cdots, x_N \in \Sigma \), we may assume that these regions susceptible to observation are suitably small geodesic balls of radius \( \epsilon \), scattered over the hypersurface \( \Sigma \) so as to cover it. In other words, we assume that \( \{ x_1, \cdots, x_N \} \) is a minimal \( \epsilon \)-net in \( \Sigma \). With the above in mind, we can approximate the average of \( f \) over \( \Sigma \) by averaging over euclidean balls, whose Lebesgue measure is locally weighted by Puiseux’ formula, rewriting formula (2) to leading order as

\[
\langle f \rangle_\Sigma(g) \approx \langle f \rangle_\epsilon \equiv \frac{\sum_i \left[ f_i + \frac{\Delta f_i - R_i f_i}{2(n+2)} \epsilon^2 \right]}{\sum_i \left[ 1 - \frac{R_i}{6(n+2)} \epsilon^2 \right]},
\]

where \( f \equiv f(x_i) \), \( \Delta \) is the Laplacian operator relative to the manifold, \( R_i \equiv R(x_i) \) is the scalar curvature at the center of the ball and \( n = \dim \Sigma \). This formula (3) can then be taken as a suitable scale dependent approximation to \( \langle f \rangle_\Sigma(g) \).

There are certain problems here which need to be clarified. Obviously, there are “unwanted” details affecting this averaging, the immediate one being that associated
with use of a collection of geodesic balls. The important question to ask is what happens to the average $< f >_\epsilon$ when we change the length scale represented by the radius of the balls. Clearly, on scales big enough the average should not be sensitive to the details of the underlying geometry since homogeneity and isotropy prevail. This is the reason why averaging in constant curvature spaces is well defined since there one can move the balls freely and deform them but by so doing no new geometric details, which measure the inhomogeneities, will be felt in the average values. A natural question to ask is then how the geometry, i.e. curvature inhomogeneities, should depend on scale so that the average $< f >_\epsilon$ over the balls is scale independent, or equivalently, how do we have to deform the geometry in order to achieve the scaling limit when the size of the balls no longer matters?

To this end, let us consider the average $< f >_{\epsilon_0+\eta}$, with $\eta$ a positive number, $\eta/\epsilon_0 \ll 1$. Upon expanding $< f >_{\epsilon_0+\eta}$ in $\eta$ we get to leading order

$$
\epsilon_0 \frac{d}{d\eta} < f >_{\epsilon_0+\eta} \big|_{\eta/\epsilon_0=0} = \frac{1}{n+2} < \Delta f >_{\epsilon_0} + \frac{1}{3(n+2)} [< R >_{\epsilon_0} < f >_{\epsilon_0} - < Rf >_{\epsilon_0}],
$$

where $< f >_{\epsilon_0}$ is the average of $f$ over the set of $N$ instantaneous observers (with similar expressions for $< R >_{\epsilon_0}$, $< Rf >_{\epsilon_0}$ and $< \Delta f >_{\epsilon_0}$). It is evident from the above formula that on a curved, inhomogeneous, manifold the curvature necessarily enters in the process of averaging. Namely, note the following feature of (4): if $R = \text{const}$ the second term on the r.h.s. of (4) disappears. In these circumstances, if $f$ is related to geometry, one cannot speak of averaging without taking into account the backreaction of the curvature.

Since, in general, sources are coupled to the gravitational field, we ought to treat the full dynamical system, i.e. the cosmological fluid and the geometry of $\Sigma$, in the RG procedure of blocking. In this setting the blocking is to be understood as passing from an $\epsilon_m$-net to an $\epsilon_{m+1}$-net, given by a collection of geodesic balls with radius $\epsilon_m = (m+1)\epsilon_0$ for $m = 0, 1, 2, \cdots$, thought of as à la Kadanoff blocking, applied here to the geometry itself. In the RG treatment of the lattice models in statistical mechanics, blocking engenders a change (flow) of the coupling constants in the corresponding Hamiltonian, according to the RG equations. We wish to imitate the above picture in gravity and thus blocking of the manifold, in our setting, should give rise to a deformation of geometry. Consequently one can see here that it is not possible to provide a consistent averaging procedure of the sources without taking into account the backreaction of the geometry.

The next step to consider is then a deformation of geometry. How do we achieve this? Let us formally write that the effect of the renormalization induced by the blocking (and appropriate rescaling, see [1] for details) can be symbolized as a non-linear operation acting on a metric $g^{(m)}$ so as to produce $g^{(m+1)}$, i.e.

$$
g^{(m+1)} = \mathcal{R}(g^{(m)}).
$$

Since it is natural in this setting to consider the metric as a running coupling constant, depending on the cut-off, while thinking of the average $< f >_\epsilon$ as a functional of the
metric, we can equivalently interpret (4) as obtained by considering the variation of \( < f >_\epsilon \) under a suitable smooth deformation of the metric, rather than deforming the (euclidean) radius of the balls \( \{ B(x_i, \epsilon) \} \). We can equivalently rewrite the second term on the r.h.s. of (4) as

\[
< R >_\epsilon < f >_\epsilon - < Rf >_\epsilon = -D < f >_\epsilon \frac{\partial g_{ab}}{\partial \eta},
\]

where \( D < f >_\epsilon \cdot \partial g_{ab}/\partial \eta \) denotes a formal linearization of the functional \( < f >_\epsilon \) in the direction of the symmetric 2-tensor \( \partial g_{ab}/\partial \eta \), and where

\[
\frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} < R(\eta) > g_{ab}(\eta) - 2R_{ab}(\eta),
\]

\( R_{ab}(\eta) \) being the components of the Ricci tensor and \( < R(\eta) > \) is the average scalar curvature given by

\[
< R(\eta) >= \frac{1}{\text{vol} \Sigma} \int_{\Sigma} R(\eta)d\mu_\eta.
\]

The metric flow (7) is known as the Ricci-Hamilton flow studied in connection with the quasi-parabolic flows on manifolds. Its use in cosmology advocated earlier in [18] was quite ad hoc. The Ricci-Hamilton flow is always solvable for sufficiently small \( \eta \) and has a number of useful properties, namely, it preserves the volume (due to the normalization chosen), and any symmetries of the original metric are preserved along the flow, and the limiting metric \( \bar{g}_{ab} = \lim_{\eta \to \infty} g_{ab}(\eta) \) (if attained) has constant sectional curvature.

Smoothing-out matter sources, as described by a set of instantaneous observers, means thus eliminating from the distribution of such sources on \( \Sigma \) all fluctuations on scales smaller than the cut-off distance \( \epsilon \), leaving an effective probability distribution of fluctuations for the remaining degrees of freedom. The underlying RG philosophy tells us that this effective distribution has the same properties as the original one at distances much larger than \( \epsilon \) (i.e. for fluctuations with wavelengths much larger than \( \epsilon \)). Since we are adopting the Hamiltonian point of view, the field \( f \) characterizing matter sources, as described by the instantaneous observers on \( \Sigma \) is given by

\[
f \equiv \alpha \rho + \alpha^i J_i,
\]

where \( \rho \) is the matter density, \( J \) momentum density and \( \alpha, \alpha^i \) stand for the lapse function and shift vector, respectively, whose assignment on \( \Sigma \) specifies the observers in question. The independent parameters in the effective Hamiltonian \( H^{(m)} \) are then the metric and the second fundamental form \( K \) (whose renormalization is generated by the linearization of (F)), whereas \( \rho^{(m)} \) and \( J^{(m)} \) are at each stage of the renormalization connected to \( g^{(m)} \) and \( K^{(m)} \) by the constraints that hold at each stage (they fix the effective Hamiltonians which otherwise are undetermined up to a constant factor [I]). These requirements imply that the full effective Hamiltonian (matter and geometry) takes on the standard
ADM form pertaining to gravity interacting with a barotropic fluid at each stage of
renormalization. This way the renormalization of matter fields is intrinsically tied with
the renormalization (deformation) of the metric.

The invariance of long distance properties of the matter distribution, under simulta-
neous change of the cut-off (blocking) and deformation of $g$, can be expressed as a
differential equation for the effective Hamiltonian $H(\rho, J)$ (actually for its partition
function), namely,

$$[-\epsilon \frac{\partial}{\partial \epsilon} + \beta_{ab}(g) \frac{\partial}{\partial g_{ab}}] \sum_{\rho, J} \exp[-H(\rho, J)] = 0,$$

(10)

where $\beta_{ab}(g) \equiv \epsilon \frac{\partial}{\partial \epsilon} g_{ab}$ is the $\beta$-function associated with (10). It can be shown that in
order that (10) is satisfied the $\beta$-function has to be given by (7), where the parameter $\eta$
is the logarithmic change of the cut-off length $\epsilon$. We have thus arrived at the conclusion
that in order to reach a fixed point of (11) the geometry has to be deformed according to
the Ricci-Hamilton flow (7).

There are sets of metrics flowing under (7) to respective limit points. All those limit
points, either fixed or not, have their own basins of attractions. There are 9 such basins,
corresponding to the classes of homogeneous geometries that can be used to model the
locally inhomogeneous geometries on closed three-manifolds (cf. Thurston’s conjecture).
It is here where Hamilton’s program, which is an analytical approach to Thurston’s
geometrization conjecture, enters the stage. This program, roughly speaking, amounts
to saying that any three-manifold can be decomposed into pieces on which the Ricci-
Hamilton flow is global and thereby each of those pieces can be smoothly deformed into
a locally homogeneous three-manifold. Hamilton noticed that there are patterns in the
kind of singularities that may develop in the regions connecting the smooth-able pieces
[4, 5].

The above analysis of the averaging problem in cosmology can be seen as a physi-
cally non-trivial application of the Hamilton-Thurston geometrization programme. It is
striking that motivations coming from geometry and physics, namely, the construction
of cosmological models out of a local gravitational theory, go hand in hand in such a
way.

Considering further the Ricci-Hamilton evolution of some model geometries, one can
give characterization of the critical fixed points associated with the Ricci-Hamilton flow
(7) [1]. Since in this cases we are in the presence of significant finite-size effects, the
existence of the critical points is accompanied by the effective reduction of dimensionality
(crossover), and the stable phases, under the renormalization generated by (7), are sep-
"arated by a critical surface. It is tempting to speculate that, a striking feature of these
topological crossover phenomena, associated with the renormalization of the cosmologi-
cal matter distribution, is that their pattern resembles the linear sheet-like (sponge-like)
structure in the distribution of galaxies on large scales. It happens that if the initial
data set $(\Sigma, g, K, \rho, J)$ for the real universe is close to criticality then the corresponding
averaged model exhibits a tendency to such topological crossovers in various regions.
Filament-like and sheet-like structures would emerge, and the overall situation would be the one where such structures appear altogether with regions of high homogeneity and isotropy, in some sort of hierarchy. This situation is akin to that of a ferromagnet near its critical temperature, whereby we have islands of spins up and down in some sort of nested pattern. It is fair to say that an ultimate solution to the averaging problem is in this picture connected to the understanding of the critical surface in the space of riemannian metrics. Understanding this surface is however equivalent, roughly speaking, to proving Thurston’s conjecture which is still one of the grandest topics of research in mathematics.

In this way the above analysis makes accessible in cosmology too the whole subject of critical phenomena. They are manifested macroscopically since phase transitions are collective phenomena. To gain a thorough understanding of their relevance in structure formation (possibly pattern formation as well) and clustering in the universe will however require further research to be done. It is worthwhile to notice that the notion of criticality is of particular importance in non-perturbative formulations of Quantum Gravity where the problem of the classical limit seems to be exactly that of critical phenomena, namely, the theory has to be tuned to a critical point to achieve this limit (see e.g. [19] for a review). In this sense a classical space-time emerges in a limit somewhat similar to a thermodynamical limit. Perhaps this suggests a deep connection between space-time, thermodynamics and criticality. The numerical results on black hole gravitational collapse (e.g. [21]) seem to point in this direction. We venture to suggest that the critical exponent found in these simulations, close to 0.37, for the black hole mass is a deep property of the gravitational field equations. It thus seems likely that critical phenomena found in gravitational collapse could be described using RG approach, treating Einstein’s equations as generating a RG flow on the space of initial data (for an attempt in this direction see [21]). Independently, we notice that the above connection is already at work in the evaporation of black holes.

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