FRACTAL DIMENSIONS IN THE GROMOV–HAUSDORFF SPACE

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Abstract. In this paper, we first show that for all four non-negative real numbers, there exists a Cantor ultrametric space whose Hausdorff dimension, packing dimension, upper box dimension, and Assouad dimension are equal to given four numbers, respectively. Next, by constructing topological embeddings of an arbitrary compact metrizable space into the Gromov–Hausdorff space using a direct sum of metrics spaces, we prove that the set of all compact metric spaces possessing prescribed topological dimension, and four dimensions explained above, and the set of all compact ultrametric spaces are path-connected and have infinite topological dimension. This observation on ultrametrics provides another proof of Qiu’s theorem stating that the ratio of the Archimedean and non-Archimedean Gromov–Hausdorff distances is unbounded.

1. Introduction

In the present paper, we mainly deal with the topological dimension \( \dim_T X \), the Hausdorff dimension \( \dim_H(X, d) \), the packing dimension \( \dim_P(X, d) \), the upper box dimension \( \dim_B(X, d) \), and the Assouad dimension \( \dim_A(X, d) \) of a metric space \( (X, d) \). The definitions of these dimensions will be presented in Section 2. The topological dimension takes values in \( \mathbb{Z} \geq -1 \cup \{\infty\} \), and the other four dimensions take values in \([0, \infty]\). For every bounded metric space \( (X, d) \), we have the following basic inequalities (see Theorem 2.8):

\[
\dim_T X \leq \dim_H(X, d) \leq \dim_P(X, d) \leq \dim_B(X, d) \leq \dim_A(X, d).
\]

1.1. Fractal dimensions. A topological space is said to be a Cantor space if it is homeomorphic to the Cantor set. A metric \( d : X^2 \rightarrow [0, \infty) \) on a set \( X \) is said to be a non-Archimedean metric or ultrametric if for all \( x, y, z \in X \) the strong triangle inequality: \( d(x, y) \leq d(x, z) \vee d(z, y) \) is satisfied, where \( \vee \) is the maximal operator on \( \mathbb{R} \).

In [12], the author proved that for all \( a, b \in [0, \infty] \) with \( a \leq b \), there exists a Cantor metric space \( (X, d) \) with \( \dim_H(X, d) = a \) and \( \dim_A(X, d) = b \). As a development of this result, we solve the problems of prescribed dimensions for the five dimensions explained above.

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We denote by $\mathcal{L}$ the set of all $(a_1, a_2, a_3, a_4) \in [0, \infty]^4$ satisfying
\[ a_1 \leq a_2 \leq a_3 \leq a_4. \]
We also denote by $\mathcal{R}$ the set of all $(l, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_{\geq 0} \cup \{\infty\}) \times [0, \infty]^4$ satisfying $l \leq a_1 \leq a_2 \leq a_3 \leq a_4$.

**Theorem 1.1.** For every $(a_1, a_2, a_3, a_4) \in \mathcal{L}$, there exists a Cantor ultrametric space $(X, d)$ such that
\[ \dim_H(X, d) = a_1, \dim_P(X, d) = a_2, \dim_B(X, d) = a_3, \dim_A(X, d) = a_4. \]

By Theorem 1.1, we obtain the following (see also Theorem 2.8):

**Theorem 1.2.** For every $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$, there exists a compact metric space $(X, d)$ such that
\[ \dim_T X = l, \dim_H(X, d) = a_1, \dim_P(X, d) = a_2, \dim_B(X, d) = a_3, \dim_A(X, d) = a_4. \]

### 1.2. Topological embeddings of the Hilbert cube

In this paper, we denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance (the definition will be presented in Section 2). The space $(\mathcal{M}, \mathcal{GH})$ is called the Gromov–Hausdorff space. By abuse of notation, we represent an element of $\mathcal{M}$ as a pair $(X, d)$ of a set $X$ and a metric $d$ rather than its isometry class.

For $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$, we denote by $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ the set of all compact metric spaces in $\mathcal{M}$ satisfying
\[ \dim_T X = l, \dim_H(X, d) = a_1, \dim_P(X, d) = a_2, \dim_B(X, d) = a_3, \dim_A(X, d) = a_4. \]

Note that Theorem 1.1 implies that $\mathcal{D}(l, a_1, a_2, a_3, a_4) \neq \emptyset$ for all $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$.

We denote by $\mathcal{U}$ the set of all compact ultrametric spaces in $\mathcal{M}$.

We also define $\mathcal{Q} = \prod_{i \in \mathbb{Z}_{\geq 0}} [0, 1]$. The space $\mathcal{Q}$ (or a space homeomorphic to $\mathcal{Q}$) is called the Hilbert cube.

In [13], the author defined $\mathcal{X}(u, v, w)$ for $(u, v, w) \in \{0, 1, 2\}^3$ as sets of all compact metric spaces satisfying or not satisfying the doubling property, the uniform disconnectedness, and the uniform perfectness, which are properties appearing in the David–Semmes theorem [5, Proposition 15.11]. The author [13] proved that for all compact metric spaces $(X, d)$ and $(Y, e)$ in $\mathcal{X}(u, v, w)$ with $\mathcal{GH}((X, d), (Y, e)) > 0$, there exist continuum many geodesics connecting $(X, d)$ and $(Y, e)$ passing through $\mathcal{X}(u, v, w)$. The construction of such geodesics induces a topological embedding of the Hilbert cube into $\mathcal{X}(u, v, w)$, and hence $\mathcal{X}(u, v, w)$ has infinite topological dimension. In the present paper, as an analogue of this result, we prove that an arbitrary compact metrizable space can be topologically embedded into $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ and $\mathcal{U}$. 

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By constructing topological embeddings of an arbitrary compact metrizable space $H$ into $D(l,a_1,a_2,a_3,a_4)$ and $U$ which maps given $n+1$ points in $H$ into given $n+1$ compact metric spaces, we prove the path-connectedness and the infinite-dimensionality of $D(l,a_1,a_2,a_3,a_4)$ and $U$. The existence of these embeddings is based on a construction of a metric on a direct sum space of metric spaces using amalgamations of metrics.

**Theorem 1.3.** Assume that $S = D(l,a_1,a_2,a_3,a_4)$ for some numbers $(l,a_1,a_2,a_3,a_4) \in \mathbb{R}$ or $S = U$. Let $n \in \mathbb{Z}_{\geq 1}$, and $H$ a compact metrizable space. Take mutually distinct $n+1$ points $\{v_i\}_{i=1}^{n+1}$ in $H$, and let $\{(X_i,d_i)\}_{i=1}^{n+1}$ be compact metric spaces in $S$ satisfying that $\mathcal{GH}((X_i,d_i),(X_j,d_j)) > 0$ for all distinct $i,j$. Then there exists a topological embedding $\Phi : H \to S$ such that $\Phi(v_i) = (X_i,d_i)$ for all $i \in \{1, \ldots, n+1\}$.

In [13, Lemma 2.18], it is shown that if $D$ is any one of $\dim_T$, $\dim_H$, $\dim_P$, $\dim_B$, $\dim_A$, then the set of all compact metric space $(X,d)$ such that $D(X,d) = \infty$ is dense in $\mathcal{M}$. Using the same method, we find that the set $D(l,a_1,a_2,a_3,a_4)$ is dense in $\mathcal{M}$.

In this paper, a topological space is said to be infinite-dimensional if its topological dimension is infinite.

As consequences of Theorem 1.3, we obtain the following corollaries.

**Corollary 1.4.** For all $(l,a_1,a_2,a_3,a_4) \in \mathbb{R}$, the set $D(l,a_1,a_2,a_3,a_4)$ is path-connected, infinite-dimensional, and dense in $\mathcal{M}$.

**Corollary 1.5.** The set $U$ of all compact ultrametric spaces is path-connected and infinite-dimensional in $\mathcal{M}$.

**Remark 1.1.** For a metric space $(X,d)$, the dilation map $\lambda \mapsto (X,\lambda d)$ ($\lambda \in [0,1]$) is a geodesic connecting the one-point metric space and $(X,d)$ (see [15] (4) in Proposition 1.4]). From this observation, it follows that $(U,\mathcal{GH})$ is path-connected. Mémoli, Smith and Wan [19] proved that $(U,\mathcal{GH})$ is a geodesic space (see [19] Theorem 7.13]), and proved that $(U,\mathcal{GH})$ is closed and nowhere dense in $\mathcal{M}$ (see [19] Proposition 4.17]). Thus, the path-connectedness of $(U,\mathcal{GH})$ have been already known. Our construction of paths is obtained as a by-product of our topological embeddings of an arbitrary compact metrizable space, and this construction is enough to obtain another proof of Qiu’s theorem.

**Question 1.6.** For every $(l,a_1,a_2,a_3,a_4) \in \mathbb{R}$, is the metric space $(D(l,a_1,a_2,a_3,a_4),\mathcal{GH})$ a geodesic space?

### 1.3. Space of compact ultrametric spaces.

We provide applications of Corollary 1.5 to the non-Archimedean Gromov–Hausdorff space.

As a non-Archimedean analogue of the Gromov–Hausdorff distance $\mathcal{GH}$ on $\mathcal{M}$, the non-Archimedean Gromov–Hausdorff distance $\mathcal{NA}$ on $\mathcal{U}$ was defined in [27] (the definition will be presented in Subsection 1.2).
The space $(\mathcal{U}, \mathcal{N}, \mathcal{A})$ is called the non-Archimedean Gromov–Hausdorff space. In this paper, for a metric space $(X, d)$, and for a subset $A$ of $X$, we represent the restricted metric $d|_A$ as the same symbol $d$ as the ambient metric $d$ until otherwise stated. From Corollary 1.5, we deduce the relationship between the spaces $(\mathcal{U}, \mathcal{G}, \mathcal{H})$ and $(\mathcal{U}, \mathcal{N}, \mathcal{A})$.

**Corollary 1.7.** Let $I_{\mathcal{U}}: (\mathcal{U}, \mathcal{G}, \mathcal{H}) \to (\mathcal{U}, \mathcal{N}, \mathcal{A})$ be the identity map of $\mathcal{U}$. Then, $I_{\mathcal{U}}$ is not continuous with respect to the topologies induced from $\mathcal{G}, \mathcal{H}$ and $\mathcal{N}, \mathcal{A}$.

**Remark 1.2.** It is known that the inverse $I_{\mathcal{U}}^{-1}: (\mathcal{U}, \mathcal{N}, \mathcal{A}) \to (\mathcal{U}, \mathcal{G}, \mathcal{H})$ is $(2^{-1})$-Lipschitz (see [24, Corollary 2.9]).

The following corollary was first proven by Qiu [24, Theorem 4.8] by constructing concrete examples. Our proof is different from Qiu’s one.

**Corollary 1.8.** For every $c \in [2, \infty)$, there exists compact ultrametric spaces $(X, d)$ and $(Y, e)$ such that

$$c \cdot \mathcal{G}((X, d), (Y, e)) < \mathcal{N}A((X, d), (Y, e))$$

The organization of this paper is as follows: In Section 2, we prepare and explain the basic definitions and statements on metric spaces. The definitions of the five dimensions and $\mathcal{G}, \mathcal{H}$ appearing in Section 1 are given. In Section 3, we define the $(m, \alpha)$-Cantor ultrametric space, and we give formulas to calculate dimensions of such an ultrametric space. Using that formulas, first we construct spaces stated in Theorem 1.1 in cases of $a_i \in \{0, 1, \infty\}$ for all $i \in \{1, 2, 3, 4\}$. By taking a direct sum of those spaces in specific cases, we prove Theorems 1.1 and 1.2.

In Section 4, for each $n \in \mathbb{Z}_{\geq 1}$, we construct topological embeddings of an arbitrary compact metrizable space into $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ and $\mathcal{U}$. We also discuss its applications to the non-Archimedean Gromov–Hausdorff space.

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## 2. Preliminaries

In this section, we prepare and explain the basic concepts and statements on metric spaces.

### 2.1. Generalities

For $k \in \mathbb{Z}$, we denote by $\mathbb{Z}_{\geq k}$ the set of all integers greater than or equal to $k$. For a set $S$, we denote by $\text{Card}(S)$ the cardinality of $S$. Let $(X, d)$ be a metric space. For $x \in X$ and for $r \in (0, \infty)$, we denote by $B(x, r)$ the closed ball centered at $x$ with radius $r$. For a subset $A$, we denote by $\delta_d(A)$ the diameter of $A$. 

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For two metric spaces \((X, d)\) and \((Y, e)\), we denote by \(d \times_e e\) the \(\ell^\infty\)-product metric defined by \((d \times_e e)((x, y), (u, v)) = d(x, u) + e(y, v)\). Note that \(d \times_e e\) generates the product topology of \(X \times Y\).

In this paper, we sometimes use the disjoint union \(\coprod_{i \in I} X_i\) of a non-disjoint family \(\{X_i\}_{i \in I}\). Whenever we consider the disjoint union \(\coprod_{i \in I} X_i\) of a family \(\{X_i\}_{i \in I}\) of sets (this family is not necessarily disjoint), we identify the family \(\{X_i\}_{i \in I}\) with its disjoint copy unless otherwise stated. If each \(X_i\) is a topological space, we consider that \(\coprod_{i \in I} X_i\) is equipped with the direct sum topology.

2.2. Dimensions. We explain dimensions appearing in Section 1.

2.2.1. The topological dimension. In this paper, the topological dimension means the covering dimension. For a separable metrizable space, the topological dimension is equal to the large and small inductive dimensions. We refer the readers to [11], [23], [21], and [2] for the details.

2.2.2. The Hausdorff dimension. Let \((X, d)\) be a metric space. For \(\delta \in (0, \infty)\), we denote by \(P_\delta(X, d)\) the set of all subsets of \(X\) with diameter smaller than \(\delta\). For each \(s \in [0, \infty)\), and \(\delta \in (0, \infty)\), we define the measure \(H_\delta^s\) on \((X, d)\) as

\[
H_\delta^s(A) = \inf \left\{ \sum_{i=1}^\infty \delta(A_i)^s \mid A \subset \bigcup_{i=1}^\infty A_i, \ A_i \in P_\delta(X, d) \right\}.
\]

For \(s \in (0, \infty)\) we define the \(s\)-dimensional Hausdorff measure \(H^s\) on \((X, d)\) as \(H^s(A) = \sup_{\delta \in (0, \infty)} H_\delta^s(A)\). We denote by \(\dim_H(X, d)\) the Hausdorff dimension of \((X, d)\) defined as

\[
\dim_H(X, d) = \sup\{ s \in [0, \infty) \mid H^s(X) = \infty \} = \inf\{ s \in [0, \infty) \mid H^s(X) = 0 \}.
\]

2.2.3. The packing dimension. Let \((X, d)\) be a metric space. For a subset \(A\) of \(X\), and for \(\delta \in (0, \infty)\), we denote by \(P_{a, \delta}(A)\) the set of all finite or countable sequence \(\{r_i\}_{i=1}^N (N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}\) in \((0, \delta)\) for which there exists a sequence \(\{x_i\}_{i=1}^N\) in \(A\) such that if \(i \neq j\), then we have \(B(x_i, r_i) \cap B(x_j, r_j) = \emptyset\). For \(s \in [0, \infty)\), \(\delta \in (0, \infty)\), and a subset \(A\) of \(X\), we define the quantity \(\tilde{P}_\delta^s(A)\) by

\[
\tilde{P}_\delta^s(A) = \sup \left\{ \sum_{i=1}^N r_i^s \mid \{r_i\}_{i=1}^N \in P_{a, \delta}(A) \right\}.
\]

We then define the \(s\)-dimensional pre-packing measure \(\tilde{P}_0^s\) on \((X, d)\) as \(\tilde{P}_0^s(A) = \inf_{\delta \in (0, \infty)} \tilde{P}_\delta^s\), and we define the \(s\)-dimensional packing measure \(\mathcal{P}_s\) on \((X, d)\) as

\[
\mathcal{P}_s(A) = \inf \left\{ \sum_{i=1}^\infty \tilde{P}_0^s(S_i) \mid A \subset \bigcup_{i=1}^\infty S_i \right\}.
\]
We denote by \( \dim_P(X, d) \) the **packing dimension** of \((X, d)\) defined as
\[
\dim_P(X, d) = \sup \{ s \in [0, \infty) \mid P^s(X) = \infty \} = \inf \{ s \in [0, \infty) \mid P^s(X) = 0 \}.
\]

**Remark 2.1.** Our definition of the packing measure is of *radius-type*. There is another definition of *diameter-type*. If we adopt the diameter-type definition, then the statement (3) in Theorem 3.4 does not hold true in general. For the difference between two definitions, we refer the readers to [16, Lemma 1.5.7] and [4]. These two types of the packing measure are often treated indistinguishably, usually by imposing a regularity condition such as the assumption that the whole metric space is a Euclidean space, or geodesic.

2.2.4. **Box dimensions.** For a metric space \((X, d)\), and for \(r \in (0, \infty)\), a subset \(A\) of \(X\) is said to be an *\(r\)-net* if \(X = \bigcup_{a \in A} B(a, r)\). We denote by \(N_d(X, r)\) the least cardinality of \(r\)-nets of \(X\). We define the **upper box dimension** \(\dimu(X, d)\) and the **lower box dimension** \(\diml(X, d)\) by
\[
\dimu(X, d) = \limsup_{r \to 0} \frac{\log N_d(X, r)}{-\log r},
\]
\[
\diml(X, d) = \liminf_{r \to 0} \frac{\log N_d(X, r)}{-\log r}.
\]

**Remark 2.2.** In this paper, we do not mainly treat the lower box dimension.

2.2.5. **The Assouad dimension.** For a metric space \((X, d)\), we denote by \(\dim_A(X, d)\) the **Assouad dimension** of \((X, d)\) defined by the infimum of all \(\lambda \in (0, \infty)\) for which there exists \(C \in (0, \infty)\) such that for all \(R, r \in (0, \infty)\) with \(R > r\) and for all \(x \in X\) we have
\[
N_d(B(x, R), r) \leq C \left( \frac{R}{r} \right)^\lambda.
\]
If such \(\lambda\) does not exist, we define \(\dim_A(X, d) = \infty\). We say that a metric space is **doubling** if its Assouad dimension is finite.

We show some properties of the Assouad dimension. By the definition of the Assouad dimension, we first obtain:

**Lemma 2.1.** Let \(\lambda \in (0, 1)\), and \(N \in \mathbb{Z}_{\geq 2}\). Let \((X, d)\) be a metric space. If every closed ball with radius \(r\) can be covered by at most \(N\) many closed balls with radius \(\lambda r\), then we have
\[
\dim_A(X, d) \leq \log N / \log \lambda^{-1}.
\]

**Definition 2.1.** Let \((X, d)\) be a metric space. We define a function \(\Theta_{X, d}: (0, 1) \to [1, \infty]\) by \(\Theta_{X, d}(\epsilon) = \sup_{x \in X} \sup_{R \in (0, \infty)} N_d(B(x, R), \epsilon R)\).

Combining the definitions of \(\Theta_{X, d}(\epsilon)\) and the Assouad dimension, we can verify:
Lemma 2.2. Let \((X, d)\) be a metric space. Then, the Assouad dimension \(\dim_A(X, d)\) is equal to the infimum of all \(\lambda \in (0, \infty)\) for which there exists \(C \in (0, \infty)\) such that for all \(\epsilon \in (0, 1)\) we have \(\Theta_{X, d}(\epsilon) \leq C\epsilon^{-\lambda}\).

Lemma 2.1 implies:

Lemma 2.3. For every metric space \((X, d)\), the following statements are equivalent to each other.

1. The space \((X, d)\) is doubling.
2. There exists \(\epsilon \in (0, 1)\) such that \(\Theta_{X, d}(\epsilon) < \infty\).
3. For all \(\epsilon \in (0, 1)\), we have \(\Theta_{X, d}(\epsilon) < \infty\).

The next lemma stats that, to calculate the Assouad dimension, we only need the information of the behavior of \(\Theta_{X, d}(\epsilon)\) on a restricted domain.

Lemma 2.4. Let \((X, d)\) be a metric space. Then \(\dim_A(X, d)\) is equal to the infimum of all \(\lambda \in (0, \infty)\), \(C \in (0, \infty)\) and \(\epsilon_0 \in (0, 1)\) such that for all \(\epsilon \in (0, \epsilon_0]\) we have \(\Theta_{X, d}(\epsilon) \leq C\epsilon^{-\lambda}\).

Proof. Let \(\beta\) be the infimum stated in the lemma. By the definitions of \(\beta\) and the Assouad dimension, we have \(\beta \leq \dim_A(X, d)\). To prove the opposite inequality, we take \(\lambda \in (\beta, \infty)\). Then, there exist \(C_0 \in (0, \infty)\) and \(\epsilon_0 \in (0, 1)\) such that every \(\epsilon \in (0, \epsilon_0]\) satisfies \(\Theta_{X, d}(\epsilon) < \infty\). Since \(\Theta_{X, d}(\epsilon_0) \leq C_0\epsilon_0^{-\lambda} < \infty\), Lemma 2.3 shows that the space \((X, d)\) is doubling. Thus there exists \(\sigma \in (0, \infty)\) and \(C_1 \in (0, \infty)\) such that for all \(\epsilon \in (0, 1)\) we have \(\Theta_{X, d}(\epsilon) \leq C_1\epsilon^{-\sigma}\). Put \(C = \max\{1, C_0, C_1\epsilon_0^{-\sigma}\}\). We obtain \(\Theta_{X, d}(\epsilon) \leq C\epsilon^{-\lambda}\) for all \(\epsilon \in (0, 1)\). Therefore, from Lemma 2.2 it follows that \(\dim_A(X, d) \leq \beta\). □

Lemmas 2.3 and 2.4 imply the characterization of non-doubling metric spaces:

Corollary 2.5. For every metric space \((X, d)\), the following statements are equivalent to each other.

1. The space \((X, d)\) is not doubling.
2. There exists \(\epsilon \in (0, 1)\) such that \(\Theta_{X, d}(\epsilon) = \infty\).
3. For all \(\epsilon \in (0, 1)\), we have \(\Theta_{X, d}(\epsilon) = \infty\).
4. There exists a sequence \(\{\epsilon_n\}_{n \in \mathbb{Z}_{\geq 0}}\) in \((0, 1)\) such that \(\epsilon_n \to 0\) as \(n \to \infty\) and \(\epsilon_n^{-n} < \Theta_{X, d}(\epsilon_n)\) for all \(n \in \mathbb{Z}_{\geq 0}\).

Definition 2.2. For a metric space \((X, d)\), we define \(\eta_{X, d}: (0, 1) \to [0, \infty]\) by

\[
\eta_{X, d}(\epsilon) = \frac{\log \Theta_{X, d}(\epsilon)}{-\log \epsilon}.
\]

Using the function \(\eta_{X, d}\), we find a simple formula of the Assouad dimension.
Proof. Due to Corollary 2.5, if \( \dim_A(X, d) = \infty \) or \( \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) = \infty \), then \( \dim_A(X, d) = \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) \). We next consider the case of \( \dim_A(X, d) < \infty \) and \( \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) < \infty \). By the definition of \( \dim_A(X, d) \), for all \( \lambda \in (\dim_A(X, d), \infty) \), we can find \( C \in (0, \infty) \) such that for all \( \epsilon \in (0, \epsilon_0) \) we have \( \Theta_{X,d}(\epsilon) \leq C\epsilon^{-\lambda} \). Then we obtain \( \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) \leq \lambda \), and hence \( \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) \leq \dim_A(X, d) \). To prove the opposite inequality, take \( \lambda \in [0, \infty) \) with \( \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) < \lambda \). In this case, there exists \( \epsilon_0 \in (0, 1) \) such that for all \( \epsilon \in (0, \epsilon_0) \) we have \( \eta_{X,d}(\epsilon) \leq \lambda \). Thus every \( \epsilon \in (0, \epsilon_0] \) satisfies \( \Theta_{X,d}(\epsilon) \leq \epsilon^{-\lambda} \). Lemma 2.4 implies that \( \dim_A(X, d) \leq \lambda \), and hence \( \dim_A(X, d) \leq \limsup_{\epsilon \to 0} \eta_{X,d}(\epsilon) \). \( \square \)

2.3. Properties of dimensions. Let \( f: (X, d) \to (Y,e) \) be a map between metric spaces. Let \( L \in [1, \infty) \) and \( \gamma \in (0, \infty) \). We say that \( f \) is \((L, \gamma)\)-homogeneously bi-Hölder if for all \( x, y \in X \) we have

\[
L^{-1}d(x, y)^\gamma \leq e(f(x), f(y)) \leq Ld(x, y)^\gamma.
\]

If \( \gamma = 1 \), the map \( f \) is said to be \( L\)-bi-Lipschitz.

Remark 2.3. For \( \gamma > 1 \), there exists a metric space \( (X, d) \) such that the function \( d^\gamma \) defined by \( d^\gamma(x, y) = (d(x, y))^\gamma \) is also a metric (for example, an ultrametric \( d \)). Then, \(( \text{bi-})\) Hölder maps with exponent \( \gamma > 1 \) between general metric spaces are meaningful.

We refer the readers to [6], [7], and [8] for the details of the following.

Proposition 2.7. Let \( (X, d) \) be a metric space. Let \( \mathcal{D} \) stand for any one of \( \dim_H \), \( \overline{\dim_B} \), \( \dim_P \), or \( \dim_A \). Then the following hold true:

1. For every subset \( A \) of \( X \), we have \( \mathcal{D}(A, d) \leq \mathcal{D}(X, d) \).
2. Let \( A, B \) be subsets of \( X \) with \( A \cup B = X \). Then we have \( \mathcal{D}(X, d) = \max\{\mathcal{D}(A, d), \mathcal{D}(B, d)\} \).
3. Let \( (Y,e) \) be a metric space, and \( f: (X, d) \to (Y,e) \) be an \((L, \gamma)\)-homogeneously bi-Hölder map. Then, we obtain the equality \( \mathcal{D}(f(X), e) = (1/\gamma)\mathcal{D}(X, d) \).
4. If \( \mathcal{D} \) is either \( \dim_H \) or \( \dim_P \), then every countable covering \( \{S_i\}_{i \in \mathbb{Z}_{\geq 1}} \) of \( X \) satisfies \( \mathcal{D}(X, d) = \sup_{i \in \mathbb{Z}_{\geq 1}} \mathcal{D}(S_i, d) \).

Theorem 2.8. Let \( (X, d) \) be a bounded metric space. Then,

\[
\dim_H X \leq \dim_H(X, d) \leq \dim_P(X, d) \leq \overline{\dim_B}(X, d) \leq \dim_A(X, d).
\]

Proof. The inequality \( \dim_H X \leq \dim_H(X, d) \) is due to Szpilrajn [25] (see also [10]). The inequality \( \overline{\dim_B}(X, d) \leq \dim_A(X, d) \) is proven in [8 Lemma 2.4.3], or we can prove it using Proposition 2.6. The proofs of the remaining inequalities are presented in [7] Chapters 2 and 3].
which proofs are valid for not only subsets of the Euclidean spaces but also general metric spaces.

For the details of the next proposition, we refer the readers to [11], [23], [21], and [2].

**Proposition 2.9.** Let $X$ and $Y$ be separable metrizable spaces. Then we have $\dim_T(X \sqcup Y) = \max\{\dim_T(X), \dim_T(Y)\}$ and $\dim_T(X \times Y) \leq \dim_T X + \dim_T Y$.

The proof of the following is presented in [7] and [8].

**Proposition 2.10.** Let $(X,d)$ and $(Y,e)$ be metric spaces. Then the following statements hold true.

1. Let $D$ be any one of $\dim_P$, $\dim_B$, or $\dim_A$, then we have $D(X \times Y, d \times_\infty e) \leq D(X, d) + D(Y, e)$.

2. $\dim_H(X \times Y, d \times_\infty e) \leq \dim_H(X, d) + \dim_P(Y, e)$.

**Remark 2.4.** In general, we can not replace $\dim_P$ with $\dim_H$ in the right-hand side in (2) of Proposition 2.10. Indeed, for all metric spaces $(X,d)$ and $(Y,e)$, we have $\dim_H(X,d) + \dim_H(Y,e) \leq \dim_H(X \times Y, d \times_\infty e)$. For the details, we refer the readers to [7] and [8].

### 2.4. The Gromov–Hausdorff distance.

For a metric space $(Z,h)$, and for subsets $A$, $B$ of $Z$, we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of $A$ and $B$ in $Z$. For metric spaces $(X,d)$ and $(Y,e)$, the Gromov–Hausdorff distance $\mathcal{GH}((X,d),(Y,e))$ between $(X,d)$ and $(Y,e)$ is defined as the infimum of all values $\mathcal{H}(i(X),j(Y);Z,h)$, where $(Z,h)$ is a metric space, and $i: X \to Z$ and $j: Y \to Z$ are isometric embeddings. Let $f: X \to Y$ be a map between metric spaces $(X,d)$ and $(Y,e)$. We define the *distortion* $\text{dis}(f)$ of $f$ by

$$\text{dis}(f) = \sup_{x,y \in X} |d(x,y) - e(f(x), f(y))|.$$  

The following is deduced from [1, Corollary 7.3.28].

**Lemma 2.11.** Let $(X, d)$ and $(Y, e)$ be compact metric spaces, and $f: X \to Y$ be a surjective map. Then $\mathcal{GH}((X,d),(Y,e)) \leq 2\text{dis}(f)$.

For a set $X$, a map $d: X \times X \to [0, \infty)$ is said to be a *pseudo-metric* if $d$ satisfies the triangle inequality and satisfies that $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x,y \in X$. If a pseudo-metric $d$ satisfies that $d(x,y) = 0$ implies $x = y$, then $d$ is a metric. A pseudo-metric is said to be a *pseudo-ultrametric* if it satisfies the strong triangle inequality. We denote by $\text{PMet}(X)$ (resp. $\text{PUMet}(X)$) the set of all pseudo-metrics (resp. pseudo-ultrametrics) on $X$. We define a metric $\mathcal{D}_X$ on $\text{PMet}(X)$ by $\mathcal{D}_X(d,e) = \sup_{x,y \in X} |d(x,y) - e(x,y)|$. Note that $\mathcal{D}_X$ can take the value $\infty$. 

Let \( d \in \text{PMet}(X) \). We denote by \( X_{/d} \) the quotient set by the relation \( \sim_d \) defined by \( x \sim_d y \iff d(x, y) = 0 \). We also denote by \([x]_d\) the equivalence class of \( x \) by \( \sim_d \). We define a metric \([d]\) on \( X_{/d} \) by \( [d]([x]_d, [y]_d) = d(x, y) \). The metric \([d]\) is well-defined. Remark that if \( d \) is a metric, then \( (X_{/d}, [d]) \) is isometric to \( (X, d) \).

**Proposition 2.12.** Let \( d, e \in \text{PMet}(X) \), and assume that \( e \) is a metric on \( X \). Then we have \( \mathcal{G} \mathcal{H} \left( (X_{/d}, [d]), (X, e) \right) \leq 2 \cdot D_X(d, e) \).

**Proof.** Let \( p: X \to X_{/d} \) be the canonical projection. Since \( \text{dis}(p) = D_X(d, e) \), the proposition follows from Lemma 2.11. \( \square \)

Proposition 2.12 implies:

**Corollary 2.13.** Let \( T \) be a topological space all of whose finite subsets are closed. Let \( X \) be a set. If a map \( h: T \to \text{PMet}(X) \) is continuous and there exists a finite subset \( L \) of \( T \) such that \( h(t) \) is a metric for all \( t \in T \setminus L \), then the map \( F: T \to \mathcal{M} \) defined by \( F(t) = (X_{/h(t)}, [h(t)]) \) is continuous.

2.5. **Amalgamations of pseudo-metrics.** For a topological space \( X \), we denote by \( \text{Met}(X) \) (resp. \( \text{UMet}(X) \)) the set of all metrics (resp. all ultrametrics) on \( X \) generating the same topology of \( X \). For every \( n \in \mathbb{Z}_{\geq 1} \), we denote by \( \hat{n} \) the set \( \{1, \ldots, n\} \). In what follows, we consider that the set \( \hat{n} \) is always equipped with the discrete topology.

Since Proposition 3.1 in [14] treats a similar construction, we omit the proofs of the following lemmas.

**Lemma 2.14.** Let \( n \in \mathbb{Z}_{\geq 2} \). Let \( \{X_i\}_{i=1}^n \) be metrizable spaces and \( \{d_i\}_{i=1}^n \) be pseudo-metrics with \( d_i \in \text{PMet}(X_i) \). Let \( r \in \text{PMet}(\hat{n}) \) and \( p_i \in X_i \). We define a symmetric function \( D \) on \( (\prod_{i=1}^n X_i)^2 \) by

\[
D(x, y) = \begin{cases} 
  d_i(x, y) & \text{if } x, y \in X_i; \\
  d_i(x, p_i) + r(i, j) + d_j(p_j, y) & \text{if } x \in X_i \text{ and } y \in X_j.
\end{cases}
\]

Then \( D \in \text{PMet}(\prod_{i=1}^n X_i) \). Moreover, if each \( d_i \) is in \( \text{Met}(X_i) \), and if \( r \in \text{Met}(\hat{n}) \), then \( D \in \text{Met}(\prod_{i=1}^n X_i) \).

**Lemma 2.15.** Let \( n \in \mathbb{Z}_{\geq 2} \). Let \( \{X_i\}_{i=1}^n \) be ultrametrizable spaces and \( \{d_i\}_{i=1}^n \) be pseudo-ultrametrics with \( d_i \in \text{UMet}(X_i) \). Let \( r \in \text{UMet}(\hat{n}) \) and \( p_i \in X_i \). We define a symmetric function \( D \) on the set \( (\prod_{i=1}^n X_i)^2 \) by

\[
D(x, y) = \begin{cases} 
  d_i(x, y) & \text{if } x, y \in X_i; \\
  d_i(x, p_i) \lor r(i, j) \lor d_j(p_j, y) & \text{if } x \in X_i \text{ and } y \in X_j.
\end{cases}
\]

Then \( D \in \text{UMet}(\prod_{i=1}^n X_i) \). Moreover, if each \( d_i \) is in \( \text{UMet}(X_i) \), and if \( r \in \text{UMet}(\hat{n}) \), then \( D \in \text{UMet}(\prod_{i=1}^n X_i) \).
3. Prescribed dimensions

3.1. Cantor ultrametric spaces.

**Definition 3.1.** Let $\mathbb{M}$ be the set of all sequences $\mathbf{m} = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of integers with $m_i \geq 2$ for all $i \in \mathbb{Z}_{\geq 0}$. In other words, $\mathbb{M} = (\mathbb{Z}_{\geq 2})^{\mathbb{Z}_{\geq 0}}$. A map $\alpha : \mathbb{Z}_{\geq 0} \to (0, \infty)$ is said to be a shrinking sequence if $\alpha$ is strictly decreasing and converges to 0. We denote by $\mathbb{SH}$ the set of all shrinking sequences. Let $S(\mathbf{m})$ be the set of all maps $x$ from $\mathbb{Z}_{\geq 0}$ into $\mathbb{Z}_{\geq 0}$ such that $x(i) \in \{0, 1, \ldots, m_i - 1\}$ for all $i \in \mathbb{Z}_{\geq 0}$. We define a valuation $\nu : S(\mathbf{m}) \times S(\mathbf{m}) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $\nu(x, y) = \min \{n \in \mathbb{Z}_{\geq 0} \mid x(n) \neq y(n)\}$ if $x \neq y$; otherwise $\nu(x, y) = \infty$. We also define $\alpha(x, y) = \alpha(\nu(x, y))$, where we put $\alpha(\infty) = 0$. Then $\alpha$ is an ultrametric on $S(\mathbf{m})$. Notice that $(S(\mathbf{m}), \alpha)$ is a Cantor space. In this paper, the space $(S(\mathbf{m}), \alpha)$ is called the $(\mathbf{m}, \alpha)$-Cantor ultrametric space. This space is a generalization of sequentially metrized Cantor spaces defined in the author’s paper [12]. This construction of Cantor spaces has been utilized in fractal geometry (for example, [16]).

For $\alpha \in \mathbb{SH}$, and for $\mathbf{m} = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in \mathbb{M}$, and for $k \in \mathbb{Z}_{\geq 1}$, we define the $k$-shifted shrinking sequence $\alpha^{(k)}$ of $\alpha$ by $\alpha^{(k)}(n) = \alpha(n + k)$, and define $k$-shifted sequence $\mathbf{m}^{(k)} = \{m_i^{(k)}\}_{i \in \mathbb{Z}_{\geq 0}}$ of $\mathbf{m}$ by $m_i^{(k)} = m_{i+k}$.

From the definition of $k$-shifted sequences, we deduce the following lemma.

**Lemma 3.1.** Let $\mathbf{m} \in \mathbb{M}$, and let $\alpha \in \mathbb{SH}$. Let $k \in \mathbb{Z}_{\geq 0}$. Then the metric space $(S(\mathbf{m}^{(k)}), (\alpha^{(k)})_\mathbf{m})$ is isometric to the closed ball $B(x, \alpha(k))$ in $(S(\mathbf{m}), \alpha_\mathbf{m})$ for all $x \in S(\mathbf{m})$.

**Lemma 3.2.** Let $\mathbf{m} = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in \mathbb{M}$ and $\alpha \in \mathbb{SH}$. If $r \in (0, \infty)$ and $n \in \mathbb{Z}_{\geq 0}$ satisfy that $\alpha(n + 1) \leq r < \alpha(n)$, then we have

$$N_{\alpha_\mathbf{m}}(S(\mathbf{m}), r) = m_0 \cdots m_n.$$ 

Moreover, if $R, r \in (0, \infty)$ and $n, k \in \mathbb{Z}_{\geq 0}$ satisfy $R > r$, $\alpha(n + 1) \leq R < \alpha(n)$, and $\alpha(n + k + 1) \leq r < \alpha(n + k)$, then

$$N_{\alpha_\mathbf{m}}(B(x, R), r) = \begin{cases} 1 & \text{if } k = 0; \\ m_{n+1} \cdots m_{n+k} & \text{otherwise.} \end{cases}$$

**Proof.** Let $A$ be the set of all $x \in S(\mathbf{m})$ with $x(k) = 0$ for all $k \in \mathbb{Z}_{\geq n+1}$. Then $A$ is a minimal $r$-net, and hence $N_{\alpha_\mathbf{m}}(S(\mathbf{m}), r) = \text{Card}(A) = m_0 \cdots m_n$. The latter part follows from Lemma 3.1. \qed

The upper and lower box dimensions of the $(\mathbf{m}, \alpha)$-Cantor ultrametric space can be computed using $\mathbf{m}$ and $\alpha$. 
Proposition 3.3. For all \( m = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in M \) and \( \alpha \in \mathbb{S}\mathbb{H} \), we have

\[
\dim_b(S(m), \alpha) = \liminf_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n+1)},
\]

(3.1)

\[
\overline{\dim_b}(S(m), \alpha) = \limsup_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n)}.
\]

(3.2)

Proof. Take \( N \in \mathbb{Z}_{\geq 0} \) with \( \alpha(N) < 1 \). Take \( r \in (0, \alpha(N)) \), and \( n \in \mathbb{Z}_{\geq N} \) with \( \alpha(n+1) \leq r < \alpha(n) \). Then, by \( \alpha(n) < 1 \),

\[-\log r < \frac{1}{-\log \alpha(n+1)} \leq \frac{1}{-\log \alpha(n)}.
\]

From these inequalities, and Lemma \ref{lem:box_dms}, and the definitions of the lower and upper box dimensions, Proposition \ref{prop:dim_box} follows. \( \square \)

To calculate the Hausdorff dimension and the packing dimension, we use the local dimensions of measures on metric spaces. Let \( (X, d) \) be a separable metric space, and \( \mu \) be a finite Borel measure on \( X \). For every \( x \in X \), we define the upper (reps. lower) local dimension \( \dim_{\text{loc}} \mu(x) \) (resp. \( \overline{\dim}_{\text{loc}} \mu(x) \)) by

\[
\dim_{\text{loc}} \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]

(3.3)

\[
\overline{\dim}_{\text{loc}} \mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]

(3.4)

The following theorem is well-known. The proofs on the Hausdorff dimension, and the packing dimension are presented in \cite[Lemmas 2.1 and 2.2]{3}, and \cite[Corollary 3.20]{4}, respectively. The paper \cite[Corollary 2.9]{22} provides its sophisticated version. Proposition 2.3 in the book \cite{6} treats Theorem \ref{thm:dim_box} only in the Euclidean setting; however, that proof is also valid for the general setting since the so-called “5r covering lemma” holds true in general metric spaces (see \cite[Theorem 1.2]{10}).

Theorem 3.4. Let \( (X, d) \) be a separable metric space, and \( \mu \) be a finite Borel measure on \( X \). Let \( s \in [0, \infty) \). Then, we obtain:

1. If \( s \leq \dim_{\text{loc}} \mu(x) \) for all \( x \in X \) and \( \mu(X) > 0 \), then we have \( s \leq \dim_{\text{H}}(X, d) \).
2. If \( \dim_{\text{loc}} \mu(x) \leq s \) for all \( x \in X \), then we have \( \dim_{\text{H}}(X, d) \leq s \).
3. If \( s \leq \overline{\dim}_{\text{loc}} \mu(x) \) for all \( x \in X \) and \( \mu(X) > 0 \), then we have \( s \leq \overline{\dim}_{\text{P}}(X, d) \).
4. If \( \overline{\dim}_{\text{loc}} \mu(x) \leq s \) for all \( x \in X \), then we have \( \overline{\dim}_{\text{P}}(X, d) \leq s \).

Definition 3.2. Let \( m = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in M \) and \( \alpha \in \mathbb{S}\mathbb{H} \). We denote by \( \mu_{m, \alpha} \) the probability Borel measure on \( (S(m), \alpha) \) satisfying

\[
\mu_{m, \alpha}(B(x, \alpha(n+1))) = \frac{1}{m_0 \cdots m_n},
\]

where \( B(x, r) = \{y \in X : d(x, y) < r\} \) for \( x \in X \) and \( r > 0 \).
for all $x \in S(\mathbf{m})$ and $n \in \mathbb{Z}_{\geq 0}$. Note that $\mu_{\mathbf{m},\alpha}$ always exists since it is the countable product of uniform measures on $\{0, 1, \ldots, m_i - 1\}$ (see also the argument of “repeated subdivision” in [6] and [7]).

Due to the measure $\mu_{\mathbf{m},\alpha}$, we obtain the formulas to calculate the Hausdorff and packing dimensions of $(S(\mathbf{m}), \alpha)$. 

**Proposition 3.5.** For all $\mathbf{m} = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in \mathbb{M}$ and $\alpha \in \mathbb{SH}$, we have

\[
\dim_H(S(\mathbf{m}), \alpha) = \liminf_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n + 1)},
\]

\[
\dim_P(S(\mathbf{m}), \alpha) = \limsup_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n)}.
\]

**Proof.** Take $N \in \mathbb{Z}_{\geq 0}$ with $\alpha(N) < 1$. Take $r \in (0, \alpha(N))$, and $n \in \mathbb{Z}_{\geq N}$ with $\alpha(n + 1) \leq r < \alpha(n)$. Take arbitrary $x \in S(\mathbf{m})$. Then $B(x, r) = B(x, \alpha(n + 1))$, and $\mu_{\mathbf{m},\alpha}(B(x, r)) = 1/(m_0 \cdots m_n)$. By $\alpha(n) < 1$, we obtain

\[
\frac{1}{-\log \alpha(n + 1)} \leq \frac{1}{-\log r} < \frac{1}{-\log \alpha(n)}
\]

Thus, for all $x \in S(\mathbf{m})$ we obtain

\[
\dim_{loc} \mu_{\mathbf{m},\alpha}(x) = \liminf_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n + 1)},
\]

\[
\overline{\dim}_{loc} \mu_{\mathbf{m},\alpha}(x) = \limsup_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{-\log \alpha(n)}.
\]

Thus, Proposition 3.5 implies the proposition. $\square$

**Remark 3.1.** From Propositions 3.3 and 3.5 it follows that for all $\mathbf{m} \in \mathbb{M}$ and $\alpha \in \mathbb{SH}$, we have $\dim_{\mathbb{B}}(S(\mathbf{m}), \alpha) = \dim_H(S(\mathbf{m}), \alpha)$ and $\overline{\dim}_{\mathbb{B}}(S(\mathbf{m}), \alpha) = \dim_P(S(\mathbf{m}), \alpha)$.

3.2. Proof of Theorems 1.1 and 1.2

**Definition 3.3.** For $(a_1, a_2, a_3, a_4) \in \mathcal{L}$, we say that a metric space $(X,d)$ is of the dimensional type $(a_1, a_2, a_3, a_4)$ if we have

\[
\dim_H(X,d) = a_1, \dim_P(X,d) = a_2, \overline{\dim}_{\mathbb{B}}(X,d) = a_3, \dim_{\mathcal{A}}(X,d) = a_4.
\]

To prove Theorems 1.1 and 1.2 we construct Cantor ultrametric spaces of the dimensional type $(a_1, a_2, a_3, a_4)$ with $a_i \in \{0, 1, \infty\}$ for all $i \in \hat{4}$ using Propositions 3.3 and 3.5.

**Definition 3.4.** We denote by $\mathbb{2}$ the sequence in $\mathbb{M}$ all of whose entries are equal to 2.

Using Lemma 2.1 we find:

**Lemma 3.6.** If $\alpha \in \mathbb{SH}$ satisfies $\alpha(i + 1)/\alpha(i) \leq 2^{-1}$ for all $i \in \mathbb{Z}_{\geq 0}$, then we have $\dim_{\mathcal{A}}(S(\mathbb{2}), \alpha) \leq 1$. 

Lemma 3.7. For all \( m \in \mathbb{M} \) and \( \alpha \in \mathbb{SH} \), the space \((S(m), \alpha)\) contains a Cantor subspace of the dimensional type \((0, 0, 0, 0)\).

**Proof.** First, we can find a sequence \( \{k(i)\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( \mathbb{Z}_{\geq 0} \) satisfying that

1. \( \alpha(k(0)) < 1 \);
2. for all \( i \in \mathbb{Z}_{\geq 0} \), we have \( k(i) < k(i + 1) \);
3. if \( s, t, i \in \mathbb{Z}_{\geq 0} \) satisfy \( s + t \leq i \), then \( \alpha(k(i)) < 2^{-s} \alpha(k(t)) \).

We define a subset \( T \) of \( S(m) \) by the set of all \( x \in S(m) \) such that \( x(n) \in \{0, 1\} \) if \( n = k(i) \) for some \( i \in \mathbb{Z}_{\geq 0} \); otherwise \( x(n) = 0 \).

We define \( \beta \in \mathbb{SH} \) by \( \beta(i) = \alpha(k(i)) \). Then the space \((T, \alpha)\) is isometric to \((S(2), \beta)\). Due to Theorem 2.8, it suffices to show that \( \dim_{\alpha}(S(2), \beta) = 0 \).

Take \( R, \epsilon \in (0, 1) \). Let \( n, m, N \in \mathbb{Z}_{\geq 0} \) be integers satisfying that \( \beta(n + 1) \leq R < \beta(n) \), \( 2^{-m^2} \leq \epsilon < 2^{-(m - 1)^2} \), and \( \beta(N + 1) \leq \epsilon R < \beta(N) \). Since \( 2^{-m^2} \beta(n + 1) \leq \beta(N) \), using the property \( \beta(k) \) of \( \{k_i\}_{i \in \mathbb{Z}_{\geq 0}} \), we obtain \( N < m + n + 2 \).

Lemma 3.2 implies \( \lim sup_{x, \epsilon \to 0} \eta_{S(2), \beta}(\epsilon) = 0 \). Hence \( \lim sup_{x, \epsilon \to 0} \eta_{S(2), \beta}(\epsilon) = 0 \).

Therefore, Proposition 2.6 shows that \( \dim_{\alpha}(S(2), \beta) = 0 \). \( \square \)

Lemma 3.8. There exist \( m \in \mathbb{M} \) and \( \alpha \in \mathbb{SH} \) such that the space \((S(m), \alpha)\) is of the dimensional type \((1, 1, 1, 1)\).

**Proof.** We define \( \alpha \in \mathbb{SH} \) by \( \alpha(k) = 2^{-1} \). Then, combining Lemma 3.6, Proposition 3.5, and Theorem 2.8, we notice that the space \((S(2), \alpha)\) is of the dimension type \((1, 1, 1, 1)\) (see also [12, Lemma 8.8]). \( \square \)

Lemma 3.9. There exist \( m \in \mathbb{M} \) and \( \alpha \in \mathbb{SH} \) such that the space \((S(m), \alpha)\) is of the dimension type \((\infty, \infty, \infty, \infty)\).

**Proof.** We define \( \alpha \in \mathbb{SH} \) by \( \alpha(k) = (n + 1)^{-1} \). Then, by Proposition 3.5 and Theorem 2.8, the space \((S(2), \alpha)\) is a desired one. \( \square \)

Before proving the next lemma, we introduce two asymptotic notations of functions. Let \( f, g : \mathbb{Z}_{\geq 0} \to (0, \infty) \) be arbitrary maps. We write \( f(i) \asymp g(i) \) if there exist \( M_0, M_1 \in (0, \infty) \) and \( N \in \mathbb{Z}_{\geq 0} \) such that for all \( i \in \mathbb{Z}_{\geq 0} \) with \( N < i \), we have \( M_0 \cdot g(i) \leq f(i) \leq M_1 \cdot g(i) \). We also write \( f(i) = o(g(i)) (i \to \infty) \) if \( \lim_{i \to \infty} f(i)/g(i) = 0 \).

Lemma 3.10. There exist \( m \in \mathbb{M} \) and \( \alpha \in \mathbb{SH} \) such that the space \((S(m), \alpha)\) is of the dimensional type \((0, 1, 1, 1)\).

**Proof.** Take sequences \( \{k(i)\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( \mathbb{Z}_{\geq 1} \) and \( \{c_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \((0, 1)\) with

1. for every \( i \in \mathbb{Z}_{\geq 0} \), we have \( k(i) + 1 < k(i + 1) \);
2. for every \( i \in \mathbb{Z}_{\geq 0} \), we have \( c_i \leq 2^{-1} \);
3. if \( n \in \mathbb{Z}_{\geq 0} \) satisfies \( k(i) + 2 \leq n \leq k(i + 1) \), we have \( c_n = 2^{-1} \);
(4) the following equalities hold true:

(4.A) \[ \lim_{i \to \infty} \frac{k(i)}{- \log(c_0 \cdots c_{k(i)+1})} = 0; \]

(4.B) \[ \lim_{i \to \infty} \frac{k(i)}{k(i+1) - k(i) - 1} = 0; \]

(4.C) \[ \lim_{i \to \infty} \frac{- \log(c_0 \cdots c_{k(i)+1})}{k(i+1) - k(i) - 1} = 0. \]

For example, we define \( \{t(n)\}_{n \in \mathbb{Z}_{\geq 0}} \) by \( t(0) = 2 \) and \( t(n+1) = 2^{t(n)} \). Using Knuth’s up-arrow notation, we can represent \( t(n) = 2 \uparrow \uparrow (n+1) \). We define \( k(i) = t(5i+5) \). We also define \( \{c_n\}_{n \in \mathbb{Z}_{\geq 0}} \) by \( c_n = 1/t(5i+8) \) if \( n = k(i)+1 \) for some \( i \); otherwise \( c_n = 2^{-1} \). Since \( k(i) \sim t(5i+5) \), \( (k(i+1) - k(i) - 1) \approx t(5i+10) \), and \( - \log(c_0 \cdots c_{k(i)+1}) \approx t(5i+7) \), and since \( t(n) = o(t(n+1)) \) \( (n \to \infty) \), the conditions (3.9)–(3.10) are satisfied.

Put \( m = 2 \), i.e., \( m_i = 2 \) for all \( i \in \mathbb{Z}_{\geq 0} \). We define \( \alpha \in \mathbb{SH} \) by \( \alpha(n) = c_0 \cdots c_n \). The condition (4.A) implies

\[
\frac{\log(m_0 \cdots m_{k(i)})}{- \log \alpha(k(i)+1)} = \frac{(k(i) + 1) \log 2}{- \log(c_0 \cdots c_{k(i)+1})} \to 0 \quad (\text{as } i \to \infty).
\]

Thus \( \dim_{\mathbb{H}}(S(2), \alpha_2) = 0 \). The conditions (3), (4.B) and (4.C) imply

\[
\frac{\log(m_0 \cdots m_{k(i+1)})}{- \log \alpha(k(i)+1)} = \frac{(k(i+1) + 1) \log 2}{- \log(c_0 \cdots c_{k(i+1)})} = \frac{(k(i+1) + 1) \log 2}{- \log(c_0 \cdots c_{k(i)+1}) + (k(i+1) - k(i) - 1) \log 2} \]

\[
= \frac{k(i+2)}{k(i+1) - k(i) - 1} \log 2 + \log 2 \to 1 \quad (\text{as } i \to \infty).
\]

Hence \( 1 \leq \dim_{\mathbb{P}}(S(2), \alpha_2) \). By the condition (2), and Lemma 3.6 we have \( \dim_{\mathbb{A}}(S(2), \alpha_2) \leq 1 \). Therefore the space \( (S(2), \alpha_2) \) is of the dimensional type \((0, 1, 1, 1)\). \( \square \)

**Lemma 3.11.** There exist \( m \in \mathbb{M} \) and \( \alpha \in \mathbb{SH} \) such that the space \( (S(m), \alpha_2) \) is of the dimensional type \((0, \infty, \infty, \infty)\).

**Proof.** Take \( m = \{m_i\}_{i \in \mathbb{Z}_{\geq 0}} \in \mathbb{M} \) and a sequence \( \{c_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \((0, 1)\) satisfying that

(3.9) \[ \lim_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{- \log(c_0 \cdots c_{n+1})} = 0, \]

(3.10) \[ \lim_{n \to \infty} \frac{\log(m_0 \cdots m_n)}{- \log(c_0 \cdots c_{n})} = \infty. \]

For example, if we put \( g(n) = 2^{n^2} \), and we define \( m_i = 2^{g(2i+2)} \) and \( c_i = 2^{-g(2i+1)} \), then the conditions (3.9) and (3.10) are satisfied.
We define \( \alpha \in \mathbb{SH} \) by \( \alpha(n) = c_0 \cdots c_n \). Then, by Proposition 3.5, the space \((S(m), \alpha_2)\) is of the dimensional type \((0, \infty, \infty, \infty)\). □

We next construct a Cantor ultrametric space of the dimensional type \((0, 0, 1, 1)\). As noted in Remark 3.1, the packing dimension and the upper box dimension of the \((m, \alpha)\)-Cantor ultrametric space coincide with each other. Thus, a metric space of the dimensional type \((0, 0, 1, 1)\) is not the \((m, \alpha)\)-Cantor ultrametric space for any \(m \in \mathbb{M}\) and any \(\alpha \in \mathbb{SH}\). We realize it as a subspace of the \((m, \alpha)\)-Cantor ultrametric space using the following theorem due to Mišik–Záčik [20, Theorem 4].

**Theorem 3.12.** Let \((X, d)\) be an infinite compact metric space. Then for every \(w \in [0, \overline{\dim}_B(X, d)]\), there exist a convergent sequence \(\{a_i\}_{i \in \mathbb{Z}_0} \) in \((S(2), \alpha_2)\) with its limit \(l \in X\) such that \(\overline{\dim}_B(\{l\} \cup \{a_i \mid i \in \mathbb{Z}_0\}, d) = w\).

**Lemma 3.13.** There exists a Cantor ultrametric space of the dimensional type \((0, 0, 1, 1)\).

**Proof.** We define \(\alpha \in \mathbb{SH}\) by \(\alpha(i) = 2^{-i}\). From Lemma 3.8 it follows that the space \((S(2), \alpha_2)\) is of the dimensional type \((1, 1, 1, 1)\). Theorem 3.12 implies that there exists a convergent sequence \(\{a_i\}_{i \in \mathbb{Z}_0}\) in \((S(2), \alpha_2)\) with its limit \(l \in S(2)\) such that \(\overline{\dim}_B(\{l\} \cup \{a_i \mid i \in \mathbb{Z}_0\}, \alpha_2) = 1\). For each \(n \in \mathbb{Z}_0\), let \(r_n\) be the distance between \(\{a_n\}\) and \(\{l\} \cup \{a_i \mid i \in \mathbb{Z}_0, i \neq n\}\). Note that \(r_n > 0\) and \(r_n \to 0\) as \(n \to \infty\). For each \(i \in \mathbb{Z}_0\), let \(K_i\) be a Cantor subspace of \(B(a_i, r_i/3)\) of the dimensional type \((0, 0, 0, 0)\) (see Lemmas 3.1 and 3.7). We define \(K = \{l\} \cup \bigcup_{i \in \mathbb{Z}_0} K_i\). Then, the space \(K\) is a Cantor space (see [26, Theorem 30.3]). Using the statement (1) in Proposition 2.7, we have \(\overline{\dim}_B(K, \alpha_2) = \overline{\dim}_P(K, \alpha_2) = 0\). Due to the statement (1) in the Proposition 2.7, we find that \(\overline{\dim}_B(K, \alpha_2) = \overline{\dim}_A(K, \alpha_2) = 1\). Therefore the space \((K, \alpha_2)\) is a Cantor ultrametric space of the dimensional type \((0, 0, 1, 1)\). □

**Lemma 3.14.** There exists a Cantor ultrametric space of the dimensional type \((0, 0, \infty, \infty)\).

**Proof.** We define \(\beta \in \mathbb{SH}\) by \(\beta(i) = (i + 1)^{-1}\). Then \((S(2), \beta_2)\) is of the dimensional type \((\infty, \infty, \infty, \infty)\) (see Lemma 3.9). By replacing the role of \((S(2), \alpha_2)\) with that of \((S(2), \beta_2)\) in the proof of Lemma 3.13 we obtain Lemma 3.14. □

**Lemma 3.15.** There exist \(m \in \mathbb{M}\) and \(\alpha \in \mathbb{SH}\) such that the space \((S(m), \alpha_2)\) is of the dimensional type \((0, 0, 0, 1)\).

**Proof.** The construction in this proof has already appeared in [12, Proposition 8.7]. We define \(\alpha \in \mathbb{SH}\) by \(\alpha(n) = 2^{-n^3}\). Let \(\theta \in \mathbb{SH}\) be the shrinking sequence such that
\[
\{\theta(n) \mid n \in \mathbb{Z}_0\} = \alpha(\mathbb{Z}_0) \cup \{2^{-k}\alpha(n) \mid n \in \mathbb{Z}_0, k = 1, \ldots, n\}.
\]
Let $\varphi: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a map satisfying that $\theta(\varphi(n)) = \alpha(n) = 2^{-n^3}$ for all $n \in \mathbb{Z}_{\geq 0}$. Let $\psi: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a map satisfying that $\varphi(\psi(n)) \leq n < \varphi(\psi(n) + 1)$ for all $n \in \mathbb{Z}_{\geq 0}$. Then, $\varphi(n) \asymp n^2$, and hence $\psi(n) \asymp \sqrt{n}$. Since $\alpha(\psi(n)) \leq \theta(n) < \alpha(\psi(n) + 1)$ for all $n \in \mathbb{Z}_{\geq 0}$, we have $-\log \theta(n) \asymp n^{3/2}$. Using Lemma 3.2, we find $N_{\theta_b}(S(2), \theta(n)) = 2^n$. Thus $\dim_B(S(2), \theta_2) = 0$. Next we estimate the Assouad dimension. By the definition of $\theta$, for all $x \in S(2)$ and $n \in \mathbb{Z}_{\geq 0}$, we have $N_{\theta_b}(B(x, 2^{-n^3}), 2^{-n} \cdot 2^{-n^3}) = 2^{n+1}$, thus $1 \leq \limsup_{t \to 0} \eta_{S(2), \theta_b}(t)$. Hence $\dim_A(S(2), \theta_2) = 1$. □

**Lemma 3.16.** There exists a Cantor ultrametric space of $(0, 0, 0, \infty)$.

**Proof.** We define $\alpha \in \mathbb{S} \mathbb{H}$ by $\alpha(n) = 2^{-n^3}$. Let $\theta \in \mathbb{S} \mathbb{H}$ be the shrinking sequence such that

$$\{ \theta(n) \mid n \in \mathbb{Z}_{\geq 0} \} = \alpha(\mathbb{Z}_{\geq 0}) \cup \left\{ \frac{k}{k+1} \alpha(n) \mid n \in \mathbb{Z}_{\geq 0}, k = 1, \ldots, n \right\}.$$

Using the same method as the proof of Lemma 3.15, we conclude that $\dim_B(S(2), \theta_4) = 0$. Next we estimate the Assouad dimension. By the definition of $\theta$, for all $x \in S(2)$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$N_{\theta_b}(B(x, 2^{-n^3}), 2^{-1} \cdot 2^{-n^3}) = 2^{n+1}.$$  

Thus $\Theta_{S(2), \theta_4}(1/2) = \infty$. According to Corollary 2.5, we conclude that $\dim_A(S(2), \theta_4) = \infty$. □

By the definition of ultrametrics and by Proposition 2.7, we obtain:

**Lemma 3.17.** Let $(a_1, a_2, a_3, a_4) \in \mathcal{L}$, and let $(X, d)$ be an ultrametric space of the dimensional type $(a_1, a_2, a_3, a_4)$. Then for every $\eta \in (0, \infty)$, the function $d^{1/\eta}$ defined by $d^{1/\eta}(x,y) = (d(x,y))^{1/\eta}$ belongs to UMet$(X)$, and the space $(X, d^{1/\eta})$ is of the dimensional type $(\eta a_1, \eta a_2, \eta a_3, \eta a_4)$.

We now prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Let $(a_1, a_2, a_3, a_4) \in \mathcal{L}$. First we handle the case of $a_i < \infty$ for all $i \in \mathcal{A}$. If $a_i = 0$ for all $i \in \mathcal{A}$, then Lemma 3.7 implies the theorem. We may assume that $a_i > 0$ for some $i$. Let $(X_1, d_1)$, $(X_2, d_2)$, $(X_3, d_3)$, and $(X_4, d_4)$ be Cantor ultrametric spaces of the dimensional type $(1, 1, 1, 1)$, $(0, 0, 1, 1)$, and $(0, 0, 0, 1)$, respectively (see Lemmas 3.8, 3.10, 3.13, and 3.15). Let $P$ be the set of all $i \in \mathcal{A}$ with $a_i > 0$. Remark that $P \neq \emptyset$. Put $Y = \bigsqcup_{i \in P} X_i$. Take $r \in \text{Met}(P)$. Applying Lemma 2.13 to $r$ and $\{d_i^{1/a_i}\}_{i \in P}$, we obtain an ultrametric $e$ on $Y$ such that $e|_{X_i^2} = d_i^{1/a_i}$ for all $i \in P$. Notice that $(Y, e)$ is a Cantor space. According to the statement (2) in Proposition 2.7 and Lemma 3.17, we conclude that $(Y, e)$ is of the dimensional type $(a_1, a_2, a_3, a_4)$. By a similar method using Lemmas 3.9, 3.11, 3.14, and
we can prove the theorem in other cases. This finishes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Let \((l, a_1, a_2, a_3, a_4) \in \mathcal{R}\). Based on Theorem 1.1, we may assume that \(l > 0\). In the case of \(l < \infty\), let \((M,e)\) be the metric space \(([0,1]^3, d_{\mathbb{R}})\), where \(d_{\mathbb{R}}\) is the Euclidean metric. In the case of \(l = \infty\), let \((M,e)\) be the metric space \((Q,u)\), where \(u \in \text{Met}(Q)\). In any case, we notice that the space \((M,e)\) is of the dimensional type \((l,l,l,l)\), where we define a metric \(u\) and define a map \(v\). Then we can find a Cantor ultrametric space \((X,d)\) of the dimensional type \((a_1,a_2,a_3,a_4)\). Put \(Y = X \sqcup M\). Let \(h\) be a metric in \(\text{Met}(Y)\) with \(h|_{X^2} = d\) and \(h|_{([0,1]^2)} = e\) (see Lemma 2.14). Since \(\dim_T M = l\), by Proposition 2.9, we conclude that \((Y,h)\) is a metric space as desired. This completes the proof of Theorem 1.2.

4. **Topological embeddings of the Hilbert cube**

4.1. **Construction of embeddings.** We refer to the construction in [13].

**Definition 4.1.** We define \(\ell : \mathbb{R} \to [0,\infty)\) by \(\ell(x) = \sqrt{2 - 2 \cos(x)}\), and define a map \(\vartheta : [0,1] \to [\pi/6, \pi/3]\) by \(\vartheta(t) = (\pi/6)(t + 1)\). For \(q \in Q\), and \(j \in \mathbb{Z}_{\geq 0}\) we define a metric \(e_{j,q} \in \text{Met}(\mathbb{Z})\) by

\[
e_{j,q}(a,b) = \begin{cases} 2^{-j-1} & \text{if } \{a,b\} = \{1,2\} \text{ or } \{2,3\}; \\ 2^{-j-1} \cdot \ell(\vartheta(q_j)) & \text{if } \{a,b\} = \{1,3\}. \end{cases}
\]

The metric space \(\mathbb{Z} = (\mathbb{Z}, e_{j,q})\) is the set of vertices of the isosceles triangle whose apex angle is \(\vartheta(q_j)\) and whose length of the legs is \(2^{-j-1}\).

Next we define \(\Omega = (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \sqcup \{\infty\}\). To simplify our description, for \(o \in \mathbb{Z}\) and \(i \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}\), we represent an element of \(\Omega\) as \(o_i = (o,i) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) if \(i \neq \infty\); otherwise, \(1_\infty = 2_\infty = 3_\infty = \infty\). For \(q \in Q\), we define a metric \(u[q]\) on \(\Omega\) by

\[
u[q](o_i, o_{i'}) = \begin{cases} e_{i,q}(o,o') & \text{if } i = j, \\ 2^{-i} & \text{if } o_i = \infty \text{ and } o_{i'} \neq \infty, \end{cases}
\]

and we also define an ultrametric \(v[q]\) on \(\Omega\) by

\[
u[q](o_i, o_{i'}) = \begin{cases} e_{i,q}(o,o') & \text{if } i = j, \\ \max\{2^{-i}, 2^{-j}\} & \text{if } i \neq j, \\ 2^{-i} & \text{if } o_i = \infty \text{ and } o_{i'} \neq \infty. \end{cases}
\]

These constructions have already appeared in [12] as *telescope spaces*. Note that \(u[q]\) and \(v[q]\) generate the same topology, which makes \(\Omega\) a
countable compact metrizable space possessing the unique accumulation point \( \infty \).

**Lemma 4.1.** For all \( q \in \mathbb{Q} \), the spaces \( (\Omega, u[q]) \) and \( (\Omega, v[q]) \) are of the dimensional type \( (0,0,0,0) \).

**Proof.** For each \( o \in \mathbb{Z} \), we define \( A_o = \{ o_i \mid i \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \} \). Note that \( \Omega = A_1 \cup A_2 \cup A_3 \). By the definition of \( u[q] \), each \( A_o \) is isometric to the space \( \{0\} \cup \{2^{-i} \mid i \in \mathbb{Z}_{\geq 0}\} \) equipped with the Euclidean metric. According to [18, Theorem 14] or [9, Corollary 7], the Assouad dimension of \( \{0\} \cup \{2^{-i} \mid i \in \mathbb{Z}_{\geq 0}\} \) is 0. Using the statement (2) in Proposition 2.7, we find that \( \dim_{\Omega}(\Omega, u[q]) = 0 \). Since \( (\Omega, v[q]) \) is bi-Lipschitz equivalent to \( (\Omega, u[q]) \) (see [12, Proposition 3.6]), by (3) in Proposition 2.7, we have \( \dim_{\Omega}(\Omega, v[q]) = 0 \). \( \square \)

The proofs of the metric parts of the following two propositions are presented in [13] Propositions 4.5 and 4.6. The ultrametric parts can be proven by the same method.

**Proposition 4.2.** The maps \( U : \mathbb{Q} \to \text{PMet}(\Omega) \) and \( V : \mathbb{Q} \to \text{PUMet}(\Omega) \) defined by \( U(q) = u[q] \) and \( V(q) = v[q] \) are continuous.

**Proposition 4.3.** Let \( q, r \in \mathbb{Q} \), and \( K, L \in (0, \infty) \). Assume that the spaces \( (\Omega, K \cdot u[q]) \) and \( (\Omega, L \cdot u[r]) \), or the spaces \( (\Omega, K \cdot v[q]) \) and \( (\Omega, L \cdot v[r]) \) are isometric to each other. Then we have \( q = r \).

The following proposition occupies the main part of the proof of Theorem 1.3.

**Proposition 4.4.** Let \( (l, a_1, a_2, a_3, a_4) \in \mathcal{R}, n \in \mathbb{Z}_{\geq 1} \) and \( m \in \mathbb{Z}_{\geq 2} \). Let \( H \) be a compact metrizable space and \( \{v_i\}_{i=1}^{n+1} \) be \( n + 1 \) points in \( H \). Put \( H^\times = H \setminus \{ v_i \mid i = 1, \ldots, n + 1 \} \). Let \( \{(X_i, d_i)\}_{i=1}^{n+1} \) be a sequence of compact metric spaces satisfying that \( \mathcal{G} \mathcal{H}((X_i, d_i), (X_j, d_j)) > 0 \) for all distinct \( i, j \). Then there exists a continuous map \( F : H^\times \times \hat{n} \to \mathcal{D}(l, a_1, a_2, a_3, a_4) \) such that

1. for all \( i, j \in \hat{n} + 1 \) and \( j \in \hat{n} \) we have \( F(v_i, j) = (X_i, d_i) \);
2. for all \( (s, i), (t, j) \in H^\times \times \hat{n} \) with \( (s, i) \neq (t, j) \), we have \( F(s, i) \neq F(t, j) \).

**Proof.** Since the space \( H^\times \times \hat{n} \) is compact and metrizable, there exists a topological embedding \( \tau : H^\times \times \hat{n} \to \mathbb{Q} \) (this is the Urysohn Metrization Theorem, see [17]). Since every metrizable space is perfectly normal (see [23, Proposition 4.18]), for each \( i \in \hat{n} + 1 \), there exists a continuous function \( \zeta_i : H \to [0, 1] \) such that \( \zeta_i^{-1}(0) = \{ v_k \mid k \neq i \} \) and \( \zeta_i^{-1}(1) = \{ v_i \} \), and there exists a continuous function \( \xi : H \to [0, 1] \) with \( \xi^{-1}(0) = \{ v_i \mid i = 1, \ldots, n + 1 \} \). Due to Lemma 3.7, we can take a Cantor ultrametric space \( (P, v) \) of the dimensional type \( (0,0,0,0) \). For each \( s \in H \), and for each \( i \in \hat{n} + 1 \), we define \( (Y_i, h_{i,s}) = (X_i \times P, d_i \times \infty (\xi(s) \cdot v)) \),
and we define \( Y_{n+2} = \Omega \). For all \( i \in \mathbb{N} + 2, s \in H, \) and \( k \in \hat{m} \), we also define a metric \( E_{i,s,k} \) on \( Y_i \) by

\[
E_{i,s,k}(x, y) = \begin{cases} 
\zeta(s) \cdot h_{i,s}(x, y) & \text{if } i \neq n + 2; \\
\xi(s) \cdot u[\tau(s, k)](x, y) & \text{if } i = n + 2.
\end{cases}
\]

Put \( Z = \prod_{i=1}^{n+2} Y_i \), and take \( r \in \text{Met}(n+2) \) and \( p_i \in Y_i \). For each \( (s, k) \in H \times \hat{m} \), we define a symmetric function \( D_{s,k} \) on \( Z \) by

\[
D_{s,k}(x, y) = \begin{cases} 
E_{i,s,k}(x, y) & \text{if } x, y \in Y_i; \\
E_{i,s,k}(x, p_i) + \xi(s) r(i, j) + E_{j,s,k}(p_j, y) & \text{if } x \in Y_i \text{ and } y \in Y_j.
\end{cases}
\]

Lemma \( \text{2.14} \) implies that \( D_{s,k} \) is a pseudo-metric on \( Z \) for all \( (s, k) \in H \times \hat{m} \). We notice that \( D_{s,k} \) is a metric if and only if \( s \in H^\times \). We also notice that for all \( i \in \mathbb{N} + 1 \) and \( k \in \hat{m} \), the quotient metric space \( \left( Z / D_{s,k}, [D_{v,k}] \right) \) is isometric to \( (X_i, d_i) \). Combining Propositions \( \text{2.7, 2.9, and 2.10} \) and Lemma \( \text{4.1} \), we conclude that \( (Z, D_{s,k}) \in \mathcal{D}(l, a_1, a_2, a_3, a_4) \) for all \( (s, k) \in H^\times \times \hat{m} \).

We define a map \( F : H \times \hat{m} \to \mathcal{D}(l, a_1, a_2, a_3, a_4) \) by

\[
F(s, k) = \begin{cases} 
(X_i, d_i) & \text{if } s = v_i \text{ for some } i \in n + 1; \\
(Z, D_{s,k}) & \text{otherwise}.
\end{cases}
\]

Then the condition (1) is satisfied. By the definition of \( D_{s,k} \), and by Proposition \( \text{1.2} \), the map \( W : H \times \hat{m} \to \text{PMet}(Z) \) defined by \( W(s, k) = D_{s,k} \) is continuous. Therefore, Corollary \( \text{2.13} \) guarantees the continuity of the map \( F \).

Next we prove the condition (2). For a metric space \( (S, h) \), we denote by \( \overline{\mathcal{C}I}(S, h) \) the closure of the set of all isolated point of \( (S, h) \). Note that if metric spaces \( (S, h) \) and \( (S', h') \) are isometric to each other, then so are \( \overline{\mathcal{C}I}(S, h) \) and \( \overline{\mathcal{C}I}(S', h') \). Since \( (P, v) \) has no isolated points, so does \( Y_i \) for all \( i \in \mathbb{N} + 1 \). Then, by the definitions of \( \Omega \), and \( D_{s,k} \), the space \( \overline{\mathcal{C}I}(Z, D_{s,k}) \) is isometric to \( (\Omega, \xi(s) \cdot u[\tau(s, k)]) \) for all \( (s, k) \in H^\times \times \hat{m} \). Since \( \overline{\mathcal{C}I} \) is isometrically invariant, and since \( \tau \) is injective, Proposition \( \text{4.3} \) implies that the condition (2) is satisfied. This finishes the proof. \( \square \)

Similarly to Proposition \( \text{4.4} \), we obtain its analogue for ultrametrics.

**Proposition 4.5.** Fix \( n \in \mathbb{Z}_{\geq 0} \). Let \( H \) be a compact metrizable space and \( \{v_i\}_{i=1}^{n+1} = n+1 \) points in \( H \). Put \( H^\times = H \setminus \{v_i \mid i = 1, \ldots, n+1\} \). Let \( m \in \mathbb{Z}_{\geq 2} \). Let \( \{(X_i, d_i)\}_{i=1}^{n+1} \) be a sequence of compact ultrametric spaces with \( \mathcal{G}H((X_i, d_i), (X_j, d_j)) > 0 \) for all distinct \( i, j \). Then there exists a continuous map \( F : H \times \hat{m} \to \mathcal{U} \) such that

1. For all \( i \in \mathbb{N} + 1 \) and \( j \in \hat{m} \) we have \( F(v_i, j) = (X_i, d_i) \);
(2) for all \((s,i),(t,j) \in H^\times \times \hat{m}\) with \((s,i) \neq (t,j)\), we have \(F(s,i) \neq F(t,j)\).

Proof. By replacing the metric \(u[\tau(s,k)]\) with the ultrametric \(v[\tau(s,k)]\), and replacing the symbol \(\text{“}+\text{”}\) with the symbol \(\text{“}\lor\text{”}\) in the definition of \(D_{s,k}\) in the proof of Proposition 4.4, and using Lemma 2.15 instead of Lemma 2.14 we obtain the proof of Proposition 4.5. □

We give the proof of Theorem 1.3

Proof of Theorem 1.3. Let \(n \in \mathbb{Z}_{\geq 1}\), and \(H\) a compact metrizable space. Take mutually distinct \(n+1\) points \(\{v_i\}_{i=1}^{n+1}\) in \(H\), and let \(\{(X_i,d_i)\}_{i=1}^{n+1}\) be a sequence in \(S\) such that \(\mathcal{GH}((X_i,d_i),(X_j,d_j)) > 0\) for all distinct \(i,j\).

First we assume that \(S = \mathcal{D}(l,a_1,a_2,a_3,a_4)\) for some \((l,a_1,a_2,a_3,a_4)\) in \(\mathcal{R}\). Put \(m = n+2\). Let \(F : H^\times \times \hat{m} \to \mathcal{D}(a_1,a_2,a_3,a_4)\) be a map stated in Proposition 4.4. For all \(i \in \hat{m}+1\) and \(j \in \hat{m}\), we define \(S(i,j) = \{s \in H^\times \mid F(s,j) = (X_i,d_i)\}\). According to the conditions (1) and (2) in Proposition 4.3 for all \(i \in \hat{m}+1\) the set \(\bigcup_{j=1}^{m} S(i,j)\) is empty or a singleton. Thus, by \(m = n+2\), we obtain \(\hat{m} \setminus \bigcup_{i=1}^{n+1} \{j \in \hat{m} \mid \text{Card}(S(i,j)) > 0\} \neq \emptyset\), and we can take \(k\) from this set. Then \(\{(X_i,d_i) \mid i = 1,\ldots,n+1\} \cap F(H^\times \times \{k\}) = \emptyset\). Therefore, the function \(\Phi : \mathcal{D}(a_1,a_2,a_3,a_4) \to \mathcal{D}(l,a_1,a_2,a_3,a_4)\) defined by \(\Phi(s) = F(s,k)\) is injective, and hence \(\Phi\) is a topological embedding since \(H\) is compact. This completes the proof for \(S = \mathcal{D}(l,a_1,a_2,a_3,a_4)\).

Theorem 1.3 for \(S = \mathcal{U}\) is obtained by the same method using Proposition 4.3 instead of Proposition 4.4. This finishes the proof of Theorem 1.3. □

We next show Corollaries 1.4 and 1.5.

Proof of Corollary 1.4. Applying Theorem 1.3 to \(H = [0,1]\) and \(H = \mathbb{Q}\), it is shown that \(\mathcal{D}(l,a_1,a_2,a_3,a_4)\) is path-connected and infinite-dimensional, respectively. By the same method as Lemmas 2.17 and 2.18 using the property (2) in Proposition 2.7, we conclude that \(\mathcal{D}(l,a_1,a_2,a_3,a_4)\) is dense in \(\mathcal{M}\). □

Proof of Corollary 1.5. Similarly to the proof of Corollary 1.4, applying Theorem 1.3 to \(H = [0,1]\) and \(H = \mathbb{Q}\), it is shown that \(\mathcal{U}\) is path-connected and infinite-dimensional, respectively. □

Remark 4.1. Mémoli, Smith and Wan [19] introduced the \(p\)-metrics as follows: For \(p \in [1,\infty]\) and \(a,b \in [0,\infty]\), we define \(a+b = (a^p + b^p)^{1/p}\) if \(p \neq \infty\); otherwise \(\max\{a,b\}\). A metric \(d\) on a set \(X\) is a \(p\)-metric if \(d(x,y) \leq d(x,z) \oplus d(z,y)\) for all \(x,y,z \in X\). We denote by \(\mathcal{M}_p\) the set of all compact \(p\)-metric spaces in \(\mathcal{M}\). By replacing the symbol \(\text{“}+\text{”}\) with \(\text{“}\oplus\text{”}\), we can prove \(p\)-metric analogues of Lemma 2.14, Proposition 4.4, and Theorem 1.3. From these arguments, we can deduce the
path-connectedness and the infinite-dimensionality of \((\mathcal{M}_p, \mathcal{GH})\) for all \(p \in [1, \infty]\) (compare with [19, Theorem 7.11]). Since 1-metrics and \(\infty\)-metrics are identical with ordinal metrics and ultrametrics, respectively, the cases of \(p = 1, \infty\) are contained in Theorem 1.3.

4.2. The non-Archimedean Gromov–Hausdorff space. Let \((X, d)\) and \((Y, e)\) be ultrametric spaces. We define the non-Archimedean Gromov–Hausdorff distance \(\mathcal{NA}((X, d), (Y, e))\) between \((X, d)\) and \((Y, e)\) as the infimum of all \(\mathcal{HD}(i(X), j(Y); Z, h)\), where \((Z, h)\) is an ultrametric space, and \(i \colon X \rightarrow Z\) and \(j \colon Y \rightarrow Z\) are isometric embeddings.

The proof of the following proposition is presented in [27].

**Proposition 4.6.** The non-Archimedean Gromov–Hausdorff distance \(\mathcal{NA}\) is an ultrametric on \(\mathcal{U}\).

We shall prove Corollaries 1.7 and 1.8.

**Proof of Corollary 1.7.** By Corollary 1.5, the space \((\mathcal{U}, \mathcal{GH})\) is path-connected; however \((\mathcal{U}, \mathcal{NA})\) is not so since \(\mathcal{NA}\) is an ultrametric (Proposition 4.6). Thus \(I_{\mathcal{U}}\) is not continuous. Note that we can use the infinite-dimensionality instead of the path-connectedness of \((\mathcal{U}, \mathcal{GH})\) since all ultrametric spaces have zero topological dimension. □

**Proof of Corollary 1.8.** For the sake of contradiction, suppose that there exists \(c \in [2, \infty)\) such that for all \((X, d), (Y, e) \in \mathcal{NA}\) we have
\[
\mathcal{NA}((X, d), (Y, e)) \leq c \cdot \mathcal{GH}((X, d), (Y, e)).
\]
Then \(I_{\mathcal{U}}\) is continuous. This contradicts Corollary 1.7. □

Moreover, we can obtain stronger versions of Corollaries 1.7 and 1.8.

Before proving these statements, we prepare two propositions on \(\mathcal{NA}\).

The following is [24, (3B) in Theorem 4.2].

**Proposition 4.7.** Let \((X, d)\) and \((Y, e)\) be bounded ultrametric spaces. If \(\delta_d(X) \neq \delta_e(Y)\), then \(\mathcal{NA}((X, d), (Y, e)) = \max\{\delta_d(X), \delta_e(Y)\}\).

By Proposition 4.7, we obtain:

**Proposition 4.8.** Let \((X, d)\) be a compact ultrametric space, and \(\epsilon \in (0, \infty)\). Put \(d_\epsilon = (1 + \epsilon)d\). Then \(d_\epsilon\) is an ultrametric and we have
\[
\begin{align*}
\mathcal{GH}((X, d), (X, d_\epsilon)) &\leq \epsilon \delta_d(X); \\
\mathcal{NA}((X, d), (X, d_\epsilon)) &\leq (1 + \epsilon)\delta_d(X).
\end{align*}
\]

Proposition 4.8 implies stronger versions of Corollaries 1.7 and 1.8.

**Corollary 4.9.** The map \(I_{\mathcal{U}} \colon (\mathcal{U}, \mathcal{GH}) \rightarrow (\mathcal{U}, \mathcal{NA})\) is not continuous at any point in \(\mathcal{M}\) except the one-point metric space.

**Corollary 4.10.** For every \(c \in [2, \infty)\), and for every compact ultrametric space \((X, d)\) containing at least two points, there exists a compact ultrametric space \((Y, e)\) such that
\[
c \cdot \mathcal{GH}((X, d), (Y, e)) < \mathcal{NA}((X, d), (Y, e)).
\]
Moreover, \((Y,e)\) can be chosen as a metric space homeomorphic to \((X,d)\). 

**Remark 4.2.** Corollaries [1.3, 1.7, 4.9, and 4.10] can be considered as partial answers of Question 2.18 in [24] asking the relations between \((U,\mathcal{G}H)\) and \((U,\mathcal{N},A)\).

**References**

1. D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.
2. M. G. Charalambous, *Dimension theory: A selection of theorems and counterexamples*, Atlantis Studies in Mathematics, vol. 7, Springer, Cham, 2019.
3. C. D. Cutler, *Connecting ergodicity and dimension in dynamical systems*, Ergodic Theory Dynam. Systems **10** (1990), no. 3, 451–462.
4. , *The density theorem and Hausdorff inequality for packing measure in general metric spaces*, Illinois J. Math. **39** (1995), no. 4, 676–694.
5. G. David and S. Semmes, *Fractured fractals and broken dreams: Self-similar geometry through metric and measure*, Oxford Lecture Ser. Math. Appl., vol. 7, Oxford Univ. Press, 1997.
6. K. Falconer, *Techniques in fractal geometry*, Wiley, Chichester, 1997.
7. , *Fractal geometry: mathematical foundations and applications*, 3rd ed., John Wiley & Sons, 2004.
8. J. M. Fraser, *Assouad dimension and fractal geometry*, Tracts in Mathematics Series, vol. 222, Cambridge University Press, 2020.
9. I. García, K. Hare, and F. Mendivil, *Assouad dimensions of complementary sets*, Proc. Roy. Soc. Edinburgh Sect. A **148** (2018), no. 3, 517–540.
10. J. Heinonen, *Lectures on analysis on metric spaces*, Springer-Verlag, New York, 2001.
11. W. Hurewicz and H. Wallman, *Dimension theory*, revised ed., Princeton Univ. Press, 1948.
12. Y. Ishiki, *Quasi-symmetric invariant properties of Cantor metric spaces*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 6, 2681–2721.
13. , *Branching geodesics of the Gromov–Hausdorff distance*, (2021), preprint, arXiv:2108.06970.
14. , *On dense subsets in spaces of metrics*, (2021), preprint arXiv:2104.12450, to appear in Colloq. Math.
15. A. O. Ivanov and A. A. Tuzhilin, *Isometry group of Gromov-Hausdorff space*, Mat. Vesnik **71** (2019), no. 1-2, 123–154.
16. H. J. Joyce, *Packing measures, packing dimensions, and the existence of sets of positive finite measure*, University of London, University College London (United Kingdom), 1995, PhD Thesis.
17. J. L. Kelly, *General topology*, Springer-Verlag New York, 1975.
18. D. G. Larman, *A new theory of dimension*, Proceedings of the London Mathematical Society **3** (1967), no. 1, 178–192.
19. F. Mémoli, Z. Smith, and Z. Wan, *Gromov-Hausdorff distances on p-metric spaces and ultrametric spaces*, arXiv preprint arXiv:1912.00564 (2019).
20. L. Mišú and T. Záčik, *On some properties of the metric dimension*, Comment. Math. Univ. Carol. **31** (1990), no. 4, 781–791.
21. J. Nagata, *Modern dimension theory*, revised ed., Sigma Series in Pure Mathematics, vol. 2, Heldermann Verlag, Berlin, 1983.
22. L. Olsen, *A multifractal formalism*, Adv. Math. **116** (1995), no. 1, 82–196.
23. A. R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press, 1975.
24. D. Qiu, *Geometry of non-Archimedean Gromov-Hausdorff distance*, \( p \)-Adic Numb. Ultr. Anal. Appl. 1 (2009), no. 4, 317–337.
25. E. Szpilrajn, *La dimension et la mesure*, Fund. Math. 28 (1937), no. 1, 81–89.
26. S. Willard, *General topology*, Dover Publications, 2004; originally published by the Addison-Wesley Publishing Company in 1970.
27. I. Zarichnyi, *Gromov-Hausdorff ultrametric*, arXiv:math/0511436v1., 2005.

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