Correlators Of The Jost Functions 
In The Sine-Gordon Model

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Abstract 
In this paper the quantum direct scattering problem is solved for the Sine-Gordon model. Correlators of the Jost functions are derived by the angular quantization method.

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1. Introduction

The Inverse Scattering Method [1], [2], [3] is the most powerful technique to solve non-linear differential equations. It consists of two essential steps. First, we have to transform initial conditions to the scattering data which are the Jost functions of the auxiliary linear problem. One can consider these functions as an analogue of proper regularized Wilson P-ordered exponents in gauge theories. In terms of Jost functions the dynamics becomes trivial. Hence, the solution of the initial equation is reduced to inverting of the scattering transform.

In the quantum case handling an equation means a reconstruction of all correlators of local fields. From this point of view the solution of the quantum direct scattering problem consists of reconstructing all correlators of the Jost functions. In the present paper this problem is solved for the Sine-Gordon model.

The approach we are dealing with in this paper [4] can be considered as a proper generalization of the Baxter transfer matrix method for statistical systems [5], [6]. Notice that an analogue of the Jost functions for lattice systems is the first kind of vertex operators introduced in the papers [3]. Correlators of the Jost functions can be expressed in terms of traces over the space of angular quantization. The latter is the scaling limit of the space where the lattice corner transfer matrix acts. The angular quantization space for the Sine-Gordon model has been constructed in [7], [8]. Here it is shown that the transcendental functions introduced in [8] are nothing but correlators of the quantum Jost functions.

The paper is organized as follows. We explain the main ideas of the construction by the elementary example of the Klein-Gordon equation [9], [10]. Then they are applied to the Sine-Gordon model.

2. The Klein-Gordon model

Let us start with a consideration of the Klein-Gordon equation

\[ (\partial_t^2 - \partial_x^2)\varphi + m^2\varphi = 0. \] (2.1)

The phase space of the model is generated by the canonical variables

\[ \pi(x) = \partial_t\varphi(x,t)|_{t=0}, \quad \varphi(x) = \varphi(x,t)|_{t=0}, \]

\[ \{\pi(x),\varphi(y)\} = \delta(x-y). \] (2.2)
We shall suppose that the functions $\pi(x), \varphi(x)$ decrease at infinity fast enough and admit decomposition in Fourier integrals:

$$
\varphi(x) = 2^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} \left( A^*(\beta)e^{-imx\sinh\beta} + A(\beta)e^{imx\sinh\beta} \right),
$$

$$
\pi(x) = 2^{-\frac{1}{2}} i m \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} \cosh\beta \left( A^*(\beta)e^{-imx\sinh\beta} - A(\beta)e^{imx\sinh\beta} \right).
$$

The solution of (2.1) satisfying the initial data (2.3) has the form

$$
\varphi(x,t) = 2^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} \left( e^{-im(x\sinh\beta - t\cosh\beta)} \right) (A^*(\beta)e^{-imx\sinh\beta} + A(\beta)e^{imx\sinh\beta}).
$$

Let us fix the point $(x_0, t_0) = 0$ and introduce the Rindler coordinates

$$
x = r \cosh \theta, \quad t = r \sinh \theta;
$$

$$
-\infty < \theta < +\infty, \quad 0 < r < +\infty,
$$

in the space-like region

$$
x > |t| > 0.
$$

For any ray $\theta = const$ we define the map

$$
Scat: \varphi(r, \theta) \to \lambda^\theta(\alpha)
$$

in the following way:

$$
\lambda^\theta(\alpha) = \lim_{a \to 0} \int_{a}^{+\infty} \frac{dr e^{imr\sinh\alpha}}{r} \left( -\frac{1}{r} \partial_\theta + im \cosh\alpha \right) \varphi(r, \theta).
$$

Using the formula (2.4) it is easy to show that the dynamics becomes trivial in the variables $\lambda(\alpha) \equiv \lambda^{\theta=0}(\alpha)$:

$$
\lambda^\theta(\alpha) = \lambda(\alpha - \theta).
$$

This observation allows one to solve the Klein-Gordon equation by applying the direct and inverse transformation (2.7). The transformation is nothing but Laplace one and it is invertible in the class of decreasing functions. So, the solution $\varphi(x,t)$ in the space-time region $x > 0$. It might be useful to point out that this solution method of the Klein-Gordon equation is a prototype of the inverse scattering technique for integrable equations.
Let us recall now essential steps of the quantization of the Klein-Gordon equation. The canonical Poisson structure \( \{ \alpha, \beta \} \) can be quantized by using the correspondence principle, which prescribes replacing the classical Poisson bracket with the commutator \( \frac{i}{\hbar} \). All required commutation relations follow immediately from the canonical commutators. In particular,

\[
[\lambda(\alpha_1), \lambda(\alpha_2)] = i \hbar \tanh \frac{\alpha_1 - \alpha_2}{2}.
\]

After the quantization the complex conjugation \( * \) becomes the Hermitian one \( \dagger \). The Hilbert space of the theory is generated by the basic vectors

\[
|A(\beta_1)\ldots A(\beta_n)\rangle_{in} = A(\beta_n)\ldots A(\beta_1)|\text{vac}\rangle,
\]

\[
\beta_n > \beta_{n-1} > \ldots > \beta_1.
\]

Here the vacuum state \( |\text{vac}\rangle \) is specified by the conditions:

\[
A(\beta)|\text{vac}\rangle = 0, \quad <\text{vac}|A^{\dagger}(\beta) = 0.
\]

In what follows we shall need the explicit form of the vacuum average

\[
F(\alpha_1 - \alpha_2) = <\text{vac}|\lambda(\alpha_2)\lambda(\alpha_1)|\text{vac}>.
\]

Using the formulas \((2.4),(2.7),(2.11)\) one can get the relation:

\[
F(\alpha_1 - \alpha_2) = \frac{\hbar}{4} \int_{-\pi/\epsilon}^{+\pi/\epsilon} \frac{d\beta}{2\pi} \coth \frac{\beta + \alpha_1 + i0}{2} \tanh \frac{\beta + \alpha_2}{2}.
\]

The integral \((2.13)\) is divergent. To assign a meaning to it we introduce the ultraviolet cut-off

\[
-\frac{\pi}{\epsilon} < \beta < \frac{\pi}{\epsilon}, \quad \epsilon \to 0.
\]

Then

\[
F(\alpha) = -\frac{\hbar}{2\pi}(\alpha + i\pi) \tanh \frac{\alpha}{2} + \frac{\hbar}{4\epsilon} + o(\epsilon).
\]

The reason of this divergence is simple. It indicates that the limit \( a \to 0 \) in the quantum definition \((2.7)\) demands regularization. The parameters \( a \) and \( \epsilon \) are related by the obvious relation

\[
\frac{\pi}{\epsilon} \sim \ln ma.
\]
The explicit form of the function $F(\alpha)$ makes it possible to reconstruct correlation functions of any operators. For instance, as follows from definition (2.7),

$$
\int_0^{+\infty} \int_0^{+\infty} dr_1 dr_2 e^{im(r_1 \sinh \alpha_1 + r_2 \sinh \alpha_2)} < \text{vac}|\varphi(r_1, \theta_1) \varphi(r_2, \theta_2)|\text{vac}> = (m^2 \pi \cosh \alpha_1 \cosh \alpha_2)^{-1} \left[ F(\theta_1 - \theta_2 + \alpha_1 + \alpha_2 - i\pi) + F(\theta_1 - \theta_2 - \alpha_1 - \alpha_2 + i\pi) - F(\theta_1 - \theta_2 - \alpha_1 + \alpha_2) - F(\theta_1 - \theta_2 + \alpha_1 - \alpha_2) \right], \quad \theta_1 > \theta_2 .
$$

(2.17)

Inverting the Laplace transformations (2.17) one can get the familiar expression for the $T$-ordered Green’s function $< \text{vac}|T[\varphi(x_1, t_1)\varphi(x_2, t_2)]|\text{vac}>$. Of course, it does not depend on the regularization parameter $\epsilon$ (2.14).

Since the operator $\lambda(\alpha)$ depends only on $\pi(x)$, $\varphi(x)$, $x > 0$, it would seem that we reconstructed the Green’s function only in terms of the Hilbert space $\pi_Z$ associated with the half infinite line $x > 0, t = 0$. But this is not correct. The point is that the vacuum vector $|\text{vac}>$ (2.11) does not belong to the space $\pi_Z$. Contrary to the classical situation, the quantum dynamics in the area (2.6) depends on quantum fluctuations penetrating through the light cone. Hence, the function (2.12) can be expressed only in terms of some density matrix $\hat{\rho}$ which controls these effects [10]

$$
< \text{vac}|\lambda(\alpha_2)\lambda(\alpha_1)|\text{vac}> = tr_{\pi_Z} [\hat{\rho} \lambda(\alpha_2)\lambda(\alpha_1)] .
$$

(2.18)

The reconstruction of the density matrix is based on the following property of the function $F(\alpha)$ (2.15)

$$
F(\alpha) = F(-2\pi i - \alpha) .
$$

(2.19)

Consequently,

$$
\hat{\rho} = e^{2\pi i K} .
$$

(2.20)

Here $K$ is the operator of Lorentz rotations

$$
e^{\theta K} \lambda(\alpha) e^{-\theta K} = \lambda(\alpha - \theta) ,
$$

$$
K = \frac{i}{2\hbar} \int_0^{+\infty} dr \ r [\pi(r)^2 + \partial_r \varphi(r)^2 + m^2 \varphi(r)^2] .
$$

(2.21)

It is necessary to point out that the formula

$$
< \text{vac}|O_1(x_1, t_1)...O_n(x_n, t_n)|\text{vac}> = tr_{\pi_Z} [e^{2\pi i K} O_1(x_1, t_1)...O_n(x_n, t_n)] ,
$$

(2.22)
where $O_k$ are arbitrary operators, is a universal relation. It is the condition of uniqueness
of $T$-ordered Euclidean Green’s functions. For a nontrivial quantum model a description
of the physical vacuum is a difficult problem [11]. So we shall consider the formula (2.22)
as a definition of the physical vacuum state $|\text{vac} \rangle$. It will be a crucial point for our
construction. Note that an analogous idea was used to investigate the Hawking radiation
in the Kruskal coordinates [10].

Let us now discuss the structure of the space $\pi_Z$, which will be called the space of
angular quantization [9], [10]. The general solution of the Klein-Gordon equation in the
region (2.6) has the form:

$$\varphi(r, \theta) = \int_0^{+\infty} \frac{d\kappa}{\pi} K_{i\kappa}(mr)[b_{\kappa}e^{-i\kappa \theta} + b_{\kappa}^+ e^{i\kappa \theta}] .$$

(2.23)

Here $K_{i\kappa}(mr)$ are the Macdonald functions (modified Bessel function) of imaginary order.
The function $\lambda(\alpha)$ (2.7) and the operator of Lorentz rotations $K$ (2.21) can be expressed
in terms of the oscillators $b_{\kappa}, b_{-\kappa} \equiv b_{\kappa}^+ (\kappa > 0)$

$$\lambda(\alpha + \frac{i\pi}{2}) = i \int_{-\infty}^{+\infty} d\kappa \frac{b_{\kappa}}{\sinh \pi \kappa} e^{i\kappa \alpha} ,$$

(2.24)

$$K = \frac{i}{\hbar} \int_0^{+\infty} d\kappa \frac{\kappa}{\sinh \pi \kappa} b_{-\kappa} b_{\kappa} .$$

(2.24)

As it follows from the formulas (2.9), (2.24), the oscillators $b_{\kappa}$ satisfy the commutation
relations

$$[b_{\kappa}, b_{\kappa'}] = \hbar \sinh \pi \kappa \delta(\kappa + \kappa') .$$

(2.25)

The space of angular quantization $\pi_Z$ is spanned by the vectors

$$b_{\kappa_1}...b_{\kappa_n} |0 > \in \pi_Z , \quad \kappa_j < 0 .$$

(2.26)

Here the vacuum vector $|0 > \in \pi_Z$ (not to be confused with the physical vacuum $|\text{vac} >$)
obeys the equations:

$$b_{\kappa} |0 > = 0 |b_{-\kappa} = 0 , \quad \kappa > 0 .$$

(2.27)

Using the oscillator representation of the space $\pi_Z$ one can check the formula (2.18)
directly. The calculations have been done for the more general case of the Sine-Gordon
model in the work [8].
At the end of this section let us discuss the property of the two point average

\[ f(\alpha_1 - \alpha_2) = \langle 0|\lambda(\alpha_2)\lambda(\alpha_1)|0 \rangle . \]  

(2.28)

One can get an explicit form for this function:

\[ f(\alpha) = \hbar \pi \psi\left(\frac{1}{2} + \frac{i\alpha}{2\pi}\right) + \text{const} . \]  

(2.29)

Here \( \psi(z) \) is the logarithmic derivative of the gamma function. The function \( f(\alpha) \) satisfies the conditions:

a. Due to the commutation relation (2.9), it obeys the functional equation

\[ f(\alpha) - f(-\alpha) = i \hbar \tanh \frac{\alpha}{2} . \]  

(2.30)

b. Due to the vacuum definitions (2.27), \( f(\alpha) \) is an analytical function in the half plane \( \Im \alpha \leq 0 \).

c. If \( \Re \alpha \to \pm \infty \), then \( \partial_\alpha f(\alpha) \to 0 \).

The last requirement has a simple physical meaning. Indeed, from (2.7),(2.8) it follows that it is equivalent to the boundary conditions

\[ \partial_\theta \varphi(r, \theta)|0 \rangle \to 0, \quad \theta \to -\infty, \]

\[ < 0|\partial_\theta \varphi(r, \theta) \to 0, \quad \theta \to +\infty, \]  

(2.31)

which can be considered as conditions of absence of the flow through the light cone.

It is easy to see that the requirements (a-c) uniquely determine the explicit form of \( f(\alpha) \) (2.29) and the whole structure of the space \( \pi_Z \). They admit natural generalization for the Sine-Gordon model and can be used as the basis for a reconstruction of the space of angular quantization \[8\] .

3. The Sine-Gordon model

The simplest integrable generalization of the Klein-Gordon equation is the Sine-Gordon model

\[ (\partial_t^2 - \partial_x^2)\varphi + \frac{m^2}{b} \sin b\varphi = 0, \]  

(3.1)

where \( b \) is the interaction constant. The equation (3.1) can be represented as the zero-curvature condition \[2,3\] :

\[ [\partial_\mu - \mathcal{A}_\mu, \partial_\nu - \mathcal{A}_\nu] = 0 . \]  

(3.2)
It provides the classical integrability of (3.1). We shall consider the solution of the Sine-Gordon equation in the region (2.6). Then, in the Rindler coordinates (2.5) the connection $A_\mu$ has the form:

\[ A_r = \frac{1}{4i} b \partial_\alpha \varphi_3 + 2m \cosh(\theta + \alpha) \sin \frac{b \varphi}{2} \sigma_1 + 2m \sinh(\theta + \alpha) \cos \frac{b \varphi}{2} \sigma_2, \]

\[ A_\theta = \frac{r}{4i} b \partial_r \varphi_3 + 2m \sinh(\theta + \alpha) \sin \frac{b \varphi}{2} \sigma_1 + 2m \cosh(\theta + \alpha) \cos \frac{b \varphi}{2} \sigma_2. \] (3.3)

Here $\sigma_k$ ($k = 1, 2, 3$) are Pauli matrices and $\alpha$ is a spectral parameter.

The phase space of the model is generated by the canonical variables (2.2). We shall discuss the decreasing boundary conditions

\[ \pi(r) = \frac{1}{r} \partial_\theta \varphi(r, \theta) |_{\theta=0} \to 0, \]

\[ \varphi(r) = \varphi(r, \theta) |_{\theta=0} \to \frac{\pi l}{b}, \quad r \to +\infty. \] (3.4)

Contrary to the free massive model, the phase space contains an infinite number of components specified by the integer $l$. To simplify the discussion we will assume that

\[ l = 0 \pmod{4}. \] (3.5)

For any solution $\varphi(r, \theta)$ of the Sine-Gordon equation in the region (2.6) we can construct the following matrix

\[ T^\theta(\alpha) = \lim_{a \to 0} e^{\frac{imR}{\sqrt{2}} \sinh \alpha \sigma_3} U T \exp[\int_a^R dr A_r(r, \theta, \alpha)]. \] (3.6)

Here

\[ U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}. \]

If the functions $\pi(r), \varphi(r)$ tend to their limiting values (3.4) fast enough, the standard arguments [2],[3] allow one to prove that the matrix $T^\theta(\alpha)$ exists and satisfies the conditions:

1. The dynamical equation

\[ T^\theta(\alpha) = T(\alpha - \theta). \] (3.7)

2. The matrix $T(\alpha)$ has the following analytical structure

\[ T(\alpha) = \begin{bmatrix} i T_+(\alpha) & i T_-(\alpha) \\ T_+(\alpha + i\pi) & -T_-(\alpha + i\pi) \end{bmatrix}. \] (3.8)
Here \( T_a(\alpha) \) \((a = \pm)\) are analytical functions in the strip \( 0 \leq \Im \alpha \leq \pi \). Hereinafter we shall call them the Jost functions.

3. For real \( \alpha \) the Jost functions satisfy the conjugation condition

\[
i T^*_a(\alpha) = C^{ab}T_b(\alpha + i\pi).
\]

(3.9)

Here we use the notation \( C^{ab} = C_{ab} = \delta_{a+b} \).

4. The determinant of the matrix \( T(\alpha) \) is equal to one:

\[
C^{ab}T_a(\alpha + i\pi)T_b(\alpha) = i.
\]

(3.10)

We restrict the class of initial data by the following additional requirements:

5. The function \( T_+(\alpha) \) does not have zeros in the strip of analyticity \( 0 \leq \Im \alpha \leq \pi \).

6. If \( \Re \alpha \to \pm\infty \), \( 0 \leq \Im \alpha \leq \pi \) then

\[
[T_+(\alpha)]^{-1} = o(\alpha^{-\frac{1}{2}}e^{\frac{|\Re \alpha|}{\alpha}}).
\]

(3.11)

It is necessary to point out that the analytical condition 2 can be considered as a constraint on asymptotic behavior at \( r \sim R \to +\infty \) of the initial functions \( \pi(r), \varphi(r) \) (3.4). At the same time the requirement 6 determines their permissible behavior in other asymptotic area \( r \sim a \to 0 \). Let us also note that the additional requirements 5,6 considered as constraints on the initial data are compatible with the dynamical equation (3.7).

Using the properties (1-6) of the Jost functions it is easy to prove the dispersion relation:

\[
\frac{T_-(\alpha)}{T_+(\alpha)} = \int_{-\infty}^{+\infty} \frac{d\delta}{2\pi} \frac{[T_+(\delta)T_+(\delta + i\pi)]^{-1}}{\sinh(\delta - \alpha - i\theta)}.
\]

(3.12)

Introduce the function

\[
\lambda(\alpha) = -i \ln T_+(\alpha).
\]

(3.13)

From the above, the formula (3.6) determines the transformation

\[
\text{Scat} : \varphi(r, \theta) \to \lambda^\theta(\alpha) = \lambda(\alpha - \theta).
\]

(3.14)

It generalizes the transformation (2.7) for the Klein-Gordon model and in the weak-coupling limit \( b \to 0 \)

\[
\lambda^{SG}(\alpha) \to \frac{b}{4}\lambda^{KG}(\alpha) + \text{const.}
\]

(3.15)
Note that the solution of the Sine-Gordon equation in the region \( x > |t| > 0 \) is reduced to applying the direct and inverse transformation \( \text{Scat} \) \((3.14)\).

Let us describe now the canonical Poisson structure \((2.2)\) in terms of the Jost functions. Using the familiar method of calculation \([3]\), one can show that the functions \( T_a(\alpha) \) form a quadratic Poisson algebra:

\[
\{T_a(\alpha_1), T_b(\alpha_2)\} = r_{ab}^{cd}(\alpha_1 - \alpha_2)T_d(\alpha_2)T_c(\alpha_1), \quad -\pi < \Im (\alpha_1 - \alpha_2) < \pi .
\]

The matrix \( r_{ab}^{cd}(\alpha) \) has the following nontrivial elements:

\[
\begin{align*}
r_{++}^{++} &= r_{--}^{--} = \frac{b^2}{16} \tanh \frac{\alpha}{2}, \\
r_{+-}^{+-} &= r_{-+}^{-+} = -\frac{b^2}{16} \text{P.V.} \coth \frac{\alpha}{2}, \\
r_{+-}^{-+} &= r_{-+}^{+-} = \frac{b^2}{16} \text{P.V.} \frac{2}{\sinh \alpha}.
\end{align*}
\]

These formulas are equivalent to the simple Poisson bracket for the function \( \lambda(\alpha) \) \((3.13)\):

\[
\{\lambda(\alpha_1), \lambda(\alpha_2)\} = -\frac{b^2}{16} \tanh \frac{\alpha_1 - \alpha_2}{2}, \quad -\pi < \Im (\alpha_1 - \alpha_2) < \pi .
\]

Hence we have exactly the same Poisson structure as for the Klein-Gordon equation.

The quantization technique for the Poisson algebra \((3.16)\) has been developed in \([8]\). In principle, it is a natural generalization of ideas considered in Sec. 2. Here we only note that the quantum operators \( T_a(\alpha) \) satisfy the condition \((3.10)\) and the commutation relation:

\[
T_a(\alpha_1)T_b(\alpha_2) = R_{ab}^{cd}(\alpha_1 - \alpha_2)T_d(\alpha_2)T_c(\alpha_1).
\]

The matrix \( R_{ab}^{cd}(\alpha) \) reads explicitly

\[
\begin{align*}
R_{++}^{++} &= R_{--}^{--} = R(\alpha), \\
R_{+-}^{+-} &= R_{-+}^{-+} = -R(\alpha) \frac{\sinh \alpha \nu}{\sinh(i\pi + \alpha) \nu}, \\
R_{+-}^{-+} &= R_{-+}^{+-} = R(\alpha) \frac{\sinh i\pi \nu}{\sinh(i\pi + \alpha) \nu}.
\end{align*}
\]

Here the function \( R(\alpha) \) is represented by:

\[
R(\alpha) = \frac{\Gamma(\nu)\Gamma(1 - \frac{i\alpha \nu}{\pi})}{\Gamma(\nu - \frac{i\alpha \nu}{\pi})} \prod_{p=1}^{\infty} \frac{R_p'(-\alpha)R_p'(i\pi + \alpha)}{R_p'(0)R_p'(i\pi)},
\]
\[
R'_{\nu}(\alpha) = \frac{\Gamma(2\nu + \frac{i\alpha}{\pi})\Gamma(1 + 2\nu + \frac{i\alpha}{\pi})}{\Gamma((2\nu + 1)\nu + \frac{i\alpha}{\pi})\Gamma(1 + (2\nu - 1)\nu + \frac{i\alpha}{\pi})},
\]
and the parameter \(\nu\) is connected with the constant of interaction \(b\) in the following way:

\[
\nu = 1 - \frac{b^2\hbar}{8\pi}. \tag{3.22}
\]

One can point out that quantization of the dispersion relation (3.12) leads to the free field representation of the algebra (3.19).

As already discussed in the Sec. 2 the vacuum correlation functions

\[
<\text{vac}|T_{\theta_{\nu}}^{\alpha_{n}}(\alpha_{n})...T_{\theta_{\nu}}^{\alpha_{1}}(\alpha_{1})|\text{vac}> \equiv \mathcal{F}_{\alpha_{1}...\alpha_{n}}(\alpha_{1} - \theta_{1},...\alpha_{n} - \theta_{n})
\]
can be expressed in terms of traces over the space of angular quantization \(\pi\). Using the notation of [8] one can write them in the following form:

\[
\mathcal{F}_{\alpha_{1}...\alpha_{n}}(\alpha_{1},...\alpha_{n}) = tr_{\pi} \left[ e^{2\pi iK'_{\nu}(\alpha_{n})...K'_{\nu}(\alpha_{1})} \right]. \tag{3.23}
\]

In [8] the integral representation for these function has been constructed. Here we present explicitly only the simplest nontrivial one.

\[
F_{ab}(\alpha_{1}, \alpha_{2}) = C_{ab} \frac{e^{\frac{\alpha_{1} - \alpha_{2}}{2} - i\pi}}{4\nu \cos \pi \nu} \frac{\sinh \nu(\alpha_{1} - \alpha_{2} + i\pi)}{\cosh \frac{\alpha_{1} - \alpha_{2}}{2}} \frac{G'_{\nu}(\alpha_{1} - \alpha_{2})}{G'_{\nu}(\nu)}, \tag{3.24}
\]

where \(G'_{\nu}(\alpha)\) is given by

\[
G'_{\nu}(\alpha) = \exp \left[ \int_{0}^{+\infty} dt \frac{\sinh t(1 - \frac{i\alpha}{\pi})}{t} \frac{\sinh t(1 - \nu)}{\sinh \frac{t}{\nu}} \right]. \tag{3.25}
\]

So far we have considered only region \(x > |t| > 0\). Let us discuss now another space-like region in Minkowski space

\[
x < -|t| < 0. \tag{3.26}
\]

The coordinates in (3.26) can be chosen as (2.5), where \(\theta = -\pi i + \zeta, -\infty < \zeta < \infty\). One can introduce the Jost functions \(Q_{\zeta}^a(\alpha)\) in an analogous way as for the region \(x > |t| > 0:\)

\[
Q_{\zeta}(\alpha) = Q(\alpha - \zeta) = \lim_{a \to 0} \lim_{R \to +\infty} \hat{P} \exp \left[ \int_{-\infty}^{a} dr A_{r,-i\pi + \zeta}(\alpha) \right] U^{-1} e^{i\frac{imR}{2} \sinh \alpha \sigma_{3}}. \tag{3.27}
\]
and
\[ Q(\alpha) = \begin{pmatrix} Q^+(\alpha) & iQ^+(\alpha + i\pi) \\ Q^-(\alpha) & -iQ^-(\alpha + i\pi) \end{pmatrix}. \] (3.28)

Then standard arguments \[10\] show that vacuum correlation functions of the operators \( T^q_a(\alpha) \) and \( Q^b_\xi(\alpha) \) can be expressed in terms of (3.23):

\[ <\text{vac}|T^n_{a_n}(\alpha_n)\ldots T^1_{a_1}(\alpha_1)Q^b_{\xi_1}(\alpha_{m+1})\ldots Q^b_{\xi_m}(\alpha_{n+m})|\text{vac}> = C^{b_1c_1}\ldots C^{b_mc_m} \times \]
\[ \times \mathcal{F}_{a_1\ldots a_n,c_1\ldots c_m}(\alpha_1 - \theta_1, \ldots, \alpha_n - \theta_n, i\pi + \alpha_{n+1} - \zeta_1, \ldots, i\pi + \alpha_{n+m} - \zeta_m). \] (3.29)

Particularly, the functions
\[ \mathcal{F}_{a_1\ldots a_n,c_1\ldots c_m}(\alpha_1, \ldots, \alpha_n, i\pi + \alpha_n, \ldots, i\pi + \alpha_1) \]
describe the vacuum correlators of the elements of the monodromy matrix
\[ M(\alpha) = T(\alpha)Q(\alpha) \] (3.30)
in the Sine-Gordon model. The technique developed in the work \[8\] also provides calculations of matrix elements of arbitrary combination of the Jost functions for soliton asymptotic states. In the notation of \[8\] they are expressed in terms of functions:
\[ \text{tr}_{\pi\zeta}\left[ e^{2\pi iK}Z'_{a_n}(\alpha_n)\ldots Z'_{a_1}(\alpha_1)Z_{b_1}(\beta_1)\ldots Z_{b_k}(\beta_k) \right], \] (3.31)
which admit integral representations as well as (3.23).

4. Conclusion

The quantum direct scattering problem for the Sine-Gordon model admits a complete solution. Now we should try to solve the inverse scattering problem, which is equivalent to determining correlation functions of local operators. At present there is only an indirect way to do this: The asymptotic of functions (3.31) at \( \text{Re } \alpha_k \to \pm \infty \) defines the form-factors of all local operators in the theory \[8\]. Knowing the form-factors one can reconstruct the correlation functions in the familiar manner \[12, 13, 14\]. Hopefully, there is a more effective and direct way to derive physical correlation functions in terms of only correlators of the Jost functions (3.29). From this point of view, it will be very useful to analyze the free fermion point \( b^2\hbar = 4\pi \) \[13\].

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