Algebraic bounds for heterogeneous site percolation on directed and undirected graphs

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Abstract

We analyze site percolation on directed and undirected graphs with site-dependent open-site probabilities. We construct upper bounds on cluster susceptibilities, vertex connectivity functions, and the expected number of simple open cycles through a chosen arc; separate bounds are given on finite and infinite (di)graphs. These produce lower bounds for percolation and uniqueness transitions in infinite (di)graphs, and for the formation of a giant component in finite (di)graphs. The bounds are formulated in terms of appropriately weighted adjacency and non-backtracking (Hashimoto) matrices. It turns out to be the uniqueness criterion that is most closely associated with an asymptotically vanishing probability of forming a giant strongly-connected component on a large finite (di)graph.

1. Introduction

We are currently living in an age where many scientific and industrial applications rapidly generate large datasets. The connectivity and underlying structure of this data is of great interest. As a result, graph theory has enjoyed a resurgence, becoming a prominent tool for describing complex connections in various kinds of networks: social, biological, technological[1–3], and many others. Percolation on graphs has been used to describe internet stability [4], spread of contagious diseases [5], and computer viruses [6], related models describe market crashes [7] and viral spread in social networks [8,9]. General percolation theory methods are increasingly used in quantum information theory [10,11]. Percolation is also an important phase transition in its own right [12,13] and is well established in physics as an approach for dealing with strong disorder: quantum or classical transport [14,15], bulk properties of composite materials [16,17], diluted magnetic transitions [18], or spin glass transitions [19,20].

Recently, we suggested [21] a lower bound on the site percolation transition on an infinite graph,

\[ p_c \geq 1/\rho(H). \] (1)

Here, \( \rho(H) \) is the spectral radius of the non-backtracking (Hashimoto) matrix [22,23] \( H \) associated with the graph. This expression has been proved [24] for infinite quasi-transitive graphs, a graph-theoretic analog of translationally-invariant systems with a finite number of inequivalent sites. The bound (1) is achieved on any infinite quasi-transitive tree [25], and it also gives numerically exact percolation thresholds for several families of large random graphs, as well as for some large empirical graphs [26].

In most applications of percolation theory, one encounters large, but finite, graphs. The expectation is that the corresponding crossover retains some properties of the transition in the infinite graphs, e.g., the formation of large open clusters be unlikely if the open site probability \( p \) is well below \( p_c \). However, the bound (1) tells nothing about the structure of the percolation cluster on finite graphs, and neither it gives an algorithm for computing the location of the crossover in the case of a finite graph [27]. In particular, Eq. (1) misses the mark entirely for any finite tree where \( \rho(H) = 0 \).

In this work, we construct several spectral and algebraic bounds for transitions associated with heterogeneous site percolation on directed and undirected graphs, both finite and infinite, and analyze the continuity of these bounds for a sequence of finite digraphs weakly convergent to an infinite graph. Namely, for finite digraphs, we construct explicit upper bounds for the local in-/out-/strong-cluster susceptibilities (average size of a cluster connected to a given site), the strong connectivity function (probability that a given pair of sites belongs to the same strongly-connected cluster), and the expected number of simple cycles passing through a given arc. We also construct some analogous bounds for infinite digraphs, which result in non-trivial lower bounds for the transitions associated with divergent in-/out-cluster susceptibilities, emergence of infinite in-/out-clusters, and the strong-cluster uniqueness transition.

Our results imply that Eq. (1) and its analogue for heterogeneous site percolation on a general digraph give a universal bound for the strong-cluster uniqueness transition, below which a strongly connected infinite cluster cannot be unique. Such a bound is continuous for an increasing sequence of subgraphs if the percolation problem on the limiting digraph has a finite minimum return probability, the probability that any arc and its inverse are connected by an open non-backtracking path. Finite minimum return probability also guarantees that below this bound, the strong connectivity decays exponentially with the distance, and the expected size of a strongly connected clus-
ter scales sublinearly with the number of vertices in a digraph. In comparison, the bound (1) applies only conditionally to the percolation transition proper, e.g., for a weakly-convergent sequence of quasi-transitive digraphs of increasing size, where the number of inequivalent vertex classes remains uniformly bounded.

The remainder of this paper is organized in four sections. In Sec. 2 we define several matrices associated with heterogeneous site percolation and introduce other notations. Our main results are given in Sec. 3 which contains bounds for finite digraphs, and in Sec. 4 where infinite digraphs are discussed. Finally, in Sec. 5 we compare effectiveness of different criteria in limiting the emergence of a giant component, an open cluster which contains a finite fraction of all vertices in the digraph.

2. Definitions and notations

We consider only simple directed and undirected graphs with no loops or multiple edges. A general digraph \( D = (V, E) \) is specified by its sets of vertices (also called sites) \( V \equiv V(D) \) and edges \( E \equiv E(D) \). Each edge (bond) is a pair of vertices, \((u, v) \subseteq E\) which can be directed, \( u \rightarrow v \), or undirected, \( u \leftrightarrow v \). A directed edge \( u \rightarrow v \) is also called an arc from \( u \) to \( v \); an undirected (symmetric) edge can be represented as a pair of mutually inverted arcs, \( u \leftrightarrow v \equiv \{u \rightarrow v, v \rightarrow u\} \). A digraph with no undirected edges is an oriented graph. We will denote the set of arcs in a (di)graph \( D \) as \( \mathcal{A} \equiv \mathcal{A}(D) \). Each vertex \( v \in V \) in a digraph \( D \) is characterized by its in-degree \( \text{deg}(v) \) and out-degree \( \text{od}(v) \), the number of arcs in \( \mathcal{A}(D) \) to and from \( v \), respectively. A digraph with no directed edges is an undirected graph \( G = (V, E) \). For every vertex in an undirected graph, the degree is the number of bonds that include \( v \), \( \text{deg}(v) = \text{id}(v) = \text{od}(v) \).

We say that vertex \( u \) is connected to vertex \( v \) on a digraph \( D \), if there is a path from \( u \equiv u_0 \) to \( v \equiv u_\ell \).

\[
P \equiv \{u_0 \rightarrow u_1, u_1 \rightarrow u_2, \ldots, u_{\ell-1} \rightarrow u_\ell \} \subseteq \mathcal{A}(D). \tag{2}
\]

The path is called non-backtracking if \( u_{i-1} \neq u_{i+1}, 0 < i < \ell \), and self-avoiding (simple) if \( u_i \neq u_j \) for \( 0 \leq i, j \leq \ell \). The length of the path is the number of arcs in the set, \( \ell = |P| \). The distance from \( u \) to \( v \) on \( D \), \( d(u, v) \), is the minimum length of a path from \( u \) to \( v \). We call path (2) open if \( u_0 \neq u_\ell \), and closed otherwise. A closed path is a cycle; it can be non-backtracking or self-avoiding (simple). Connectivity on an undirected graph is a symmetric relation: we just say that vertices \( u \) and \( v \) are connected (or not). On a digraph, we say that vertices \( u \) and \( v \) are strongly connected if \( u \) is connected to \( v \) and \( v \) is connected to \( u \); \( u \) and \( v \) are weakly connected on \( D \) if they are connected on the undirected graph underlying \( D \). A ray is a semi-infinite simple path, characterized as in- or out-going according to the directionality of the constituent arcs. A strong ray is a strongly connected union of in- and out-going rays; it has the property that the intersection between the vertex sets is an infinite set.

A digraph \( D \) is called transitive iff for any two vertices \( u, v \) in \( V \equiv V(D) \) there is an automorphism (symmetry) of \( D \) mapping \( u \) onto \( v \). Digraph \( D \) is called quasi-transitive if there is a finite set of vertices \( V_0 \subset V \) such that any \( u \in V \) is taken into \( V_0 \) by some automorphism of \( D \). We say that any vertex which can be mapped onto a vertex \( u_0 \in V_0 \) is in the equivalence class of \( u_0 \). The square lattice is an example of a transitive graph; a two-dimensional lattice with \( r \) inequivalent vertex classes defines a (planar) quasi-transitive graph.

A graph \( G' = (V', E') \) is called a covering graph of \( G = (V, E) \) if there is a function \( f : V' \rightarrow V \), such that an edge \((u', v') \in E' \) is mapped to the edge \((f(u'), f(v')) \in E \), with an additional property that \( f \) be invertible in the vicinity of each vertex, i.e., for a given vertex \( u' \in V' \) and an edge \((f(u'), v) \in E \), there must be a unique edge \((u', v') \in E' \) such that \( f(v') = v \). The universal cover \( \tilde{G} \) of a connected graph \( G \) is a connected covering graph which has no cycles (a tree); it is unique, up to isomorphisms. The universal cover can be constructed as a graph with the vertex set formed by all distinct non-backtracking paths from a fixed origin \( v_0 \in V \), with an edge \((P_1, P_2) \in \tilde{E} \) if \( P_2 = P_1 \cup u, u \in E \) is a simple extension of \( P_1 \). Choosing a different origin gives an isomorphic graph. The definition of a covering digraph is similar, except the mapping function \( f \) must preserve the directionality of the edges. The covering digraph \( \tilde{D} \) of a digraph \( D \) can be constructed from that of the underlying undirected graph by labeling the directionality of the corresponding edges.

2.1. Heterogeneous site percolation

Consider a connected undirected graph \( G \). We define heterogeneous site percolation on \( G \) where each vertex \( v \in V(G) \) has an associated probability \( p_v \), \( 0 < p_v \leq 1 \). A vertex is chosen to be open with probability \( p_v \), independent from other vertices. We are focusing on a subgraph \( G' \subset G \) induced by all open vertices on \( G \). For each vertex \( v \), if \( v \) is open, let \( C(v) \subset G' \) be the connected component of \( G' \) which contains the vertex \( v \), otherwise \( C(v) = \emptyset \). If \( C(v) \) is finite, for some \( v \), we say that percolation occurs. Denote

\[
\theta_v \equiv \theta_v(G, \{p\}) = \mathbb{P}_p(|C(v)| = \infty) \tag{3}
\]

the probability that \( C(v) \) is infinite. Clearly, for any pair of vertices \( u \) and \( v \), \( \theta_v \) if \( \theta_u > 0 \).

Similarly, introduce the connectivity function,

\[
\tau_{u,v} \equiv \tau_{u,v}(G, \{p\}) = \mathbb{P}_p(u \in C(v)), \tag{4}
\]

the probability that vertices \( u \) and \( v \) are in the same cluster. For a pair of vertices \( u, v \) separated by the distance \( d(u, v) \), \( \tau_{u,v} \) can be bounded by the probability that \( v \) is in a cluster of size \( d(u, v) + 1 \). Thus, in the absence of percolation, \( \tau_{u,v} \rightarrow 0 \) when \( d(u, v) \rightarrow \infty \). The reverse is not necessarily true.

Yet another measure is the local susceptibility,

\[
\chi_v \equiv \chi_v(G, \{p\}) = \mathbb{E}_{p}[|C(v)|], \tag{5}
\]

the expected cluster size connected to \( v \). Equivalently, local susceptibility can be defined as the sum of probabilities that individual vertices are in the same cluster as \( v \), i.e., in terms of connectivities,

\[
\chi_v = \sum_{u \in V} \tau_{u,v}. \tag{6}
\]
If percolation occurs (i.e., with probability \( \theta > 0 \), \( |C_\infty| = \infty \)), then clearly \( \chi_v = \infty \). The reverse is known to be true in the case of heterogeneous site percolation on quasi-transitive graphs\(^4\): \( \chi_v = \infty \) can only happen inside or on the boundary of the percolating phase.

An important question is the number of infinite clusters on \( G^c \); in particular, whether an infinite cluster is unique. For infinite quasi-transitive graphs, there are only three possibilities: (a) almost surely there are no infinite clusters; (b) there are infinitely many infinite clusters; and (c) there is only one infinite cluster\(^4\). This is not necessarily so for more general graphs. Notice that when the infinite cluster is unique, the connectivity function is bounded away from zero, \( \tau_{uv} \geq \theta u \theta_v > 0 \).

In addition, uniqueness of the infinite cluster implies divergence of the local self-avoiding cycle (SAC) susceptibility \( \chi_{\text{SAC}}(a) \), the expected number of distinct simple cycles passing through the arc \( a \) on the open subgraph \( D' \). In the case of homogeneous percolation on undirected transitive graphs, such a relation is given by Theorem 3.9 in Ref.\(^{[47]} \), attributed to O. Schramm.

When the open-site probabilities are equal for all sites of an infinite graph, \( p_v = p, \forall v \in V \) (this is homogeneous site percolation), one defines the critical probabilities \( p_t \) and \( p_r \) respectively associated with formation of an infinite cluster and divergence of site susceptibilities. There is no percolation, \( \theta_t = 0 \), for \( p < p_t \), but \( \theta_t > 0 \) for \( p > p_t \). Likewise, \( \chi_v \) is finite for \( p < p_r \) but not for \( p > p_r \). A third critical probability, \( p_u \), is associated with the number of infinite clusters. Most generally, we expect \( p_r \leq p_c \leq p_u \). For a quasi-transitive graph, one has\(^4\)

\[
0 < p_r \leq p_c \leq p_u. \tag{7}
\]

Here, \( p_u \) is the uniqueness threshold, such that there can be only one infinite cluster for \( p > p_u \), whereas for \( p < p_u \), the number of infinite clusters may be zero, or infinite. For a degree-\( r \) regular tree \( T_r \) with \( r \geq 3 \), \( p_u = 1 \), \( p_c = 1/(r-1) \), while for hypercubic lattice, \( Z^d \), \( p_u = p_r \).

### 2.2. Percolation on a general digraph

There are several notions of connectivity on a digraph, and, similarly, there are several percolation transitions associated with a digraph\(^4\). For any given configuration of open vertices on a digraph \( D \) (which induce the open digraph \( D' \)) we introduce the strongly-connected cluster which includes \( v \), \( C_{\text{str}}(v) \subseteq D' \). Similarly, one also considers an out-cluster \( C_{\text{out}}(v) \subseteq D' \) and an in-cluster \( C_{\text{in}}(v) \subseteq D' \), formed by all sites which can be reached from \( v \) moving along or opposite the arcs in \( A(D) \), respectively. Finally, there is also a weakly-connected cluster \( C_{\text{weak}}(v) \) formed on the undirected graph \( G^c \) underlying \( D' \). For each of these cluster types, we introduce the quantities analogous to those in Eqs. 3, 5, and \( \delta \), e.g., the probability \( \theta_{\text{in}}(v) \) that \( v \) is in an infinite strongly-connected cluster, the strongly-connected susceptibility \( \chi_{\text{str}}(a, v) \), the two sided (strong) connectivity \( \tau_{\text{str}}(u, v) \) which implies a path from \( u \) to \( v \) and one from \( u \) to \( v \) must both be open, and the directed connectivity \( \tau_{uv} \) from \( u \) to \( v \).

#### 2.3. Emergence of a giant component

In network theory, a percolation-like transition on a finite graph is usually associated with the emergence of a giant component, an open cluster which contains a finite fraction of all vertices in a graph. The transition is sharp and it is well understood in various ensembles of random graphs and digraphs, see, e.g., Refs.\(^{[49],[54]} \).

### 2.4. Matrices associated with a digraph

For all heterogeneous site percolation problems on a digraph with the adjacency matrix \( A \), we associate the following three matrices: weighted adjacency matrix

\[
[A_p]_{ij} = p^{1/2} A_{ij} p^{1/2} \quad \text{(no summation)} \tag{8}
\]

weighted line digraph adjacency matrix \( L_p \), and weighted Hashimoto matrix \( H_p \).

**Weighted line-digraph adjacency matrix.** For any digraph \( D \), the line digraph\(^{[55]} \) \( L \equiv L(D) \) is a digraph whose vertices correspond to the arcs of the original digraph \( VA(L) = A(D) \), and it has a directed edge \( (a \rightarrow b) \in A(L) \) between vertices \( a = i \rightarrow j \) and \( b = j' \rightarrow i' \) if \( j = j' \) (that is, \( arcs \; a \) and \( b \), taken in this order, form a directed path of length two). We denote the corresponding adjacency matrix \( L_p \), and introduce the weighted matrix \( L_p \), where an entry corresponding to the directed edge \( (a \rightarrow b) \in A(L) \) (\( a = i \rightarrow j \) and \( b = j' \rightarrow i' \) are arcs in \( D \)) has weight \( p_j \):

\[
(L_p)_{ab} = p_j \delta_{i,j'}. \quad \text{(No summation)} \tag{9}
\]

In the homogeneous case, \( p_j = p \) for all sites, and the weighted matrix has the simple form, \( L_p = pL \). Notice we used the arc set \( A \) to define the line digraph; the same definition can be used to associate a line digraph \( L(G) \) with an undirected graph \( G \).

**Weighted Hashimoto matrix.** Hashimoto, or non-backtracking, matrix \( H \) has originally been defined for counting non-backtracking cycles on graphs\(^{[40]} \). This matrix is the adjacency matrix of the oriented line graph (OLG) \( H(D) \) associated with the original (di)graph\(^{[56]} \). The OLG is defined similarly to the line digraph, except that the edges corresponding to mutually inverted pairs of arcs in the original digraph are dropped. We define the corresponding weighted matrix by analogy with Eq. (7).

\[
(H_p)_{ab} = p_j \delta_{i,j}(1 - \delta_{ij}). \quad \text{(No summation)} \tag{10}
\]

where \( a = i \rightarrow j \) and \( b = j' \rightarrow i' \) are arcs in the original digraph. Again, in the homogeneous case, \( p_j = p \) for all sites, we recover the usual Hashimoto matrix, \( H_p = pH \).

Notice that in the case of an infinite digraph, the objects \( A_p \), \( L_p \), and \( H_p \) are not matrices but operators acting in the appropriate infinite-dimensional vector spaces. For a locally-finite digraph, the action of these operators is uniquely defined, respectively, by the local rules (8), (9), and (10). For convenience we will nevertheless refer the them as “matrices”, each time specifying whether the graph is finite or infinite.
2.5. Perron-Frobenius theory

Consider a square $n \times n$ matrix $B$ with non-negative matrix elements, $B_{ij} \geq 0$, and not necessarily symmetric. The spectral radius $\rho(B) \equiv \max_{i \in [n]} |\lambda_i(B)|$ and the associated eigenvectors of such a matrix are analyzed in the Perron-Frobenius theory of non-negative matrices[57–59]. In particular, there is always an eigenvalue $\lambda_{\max} = \rho(B)$, and the corresponding left and right eigenvectors $\xi_L$ and $\xi_R$, $\xi_L B = \lambda_{\max} \xi_L$, $\xi_R B = \lambda_{\max} \xi_R$, can be chosen to have non-negative components, $\xi_{Li} \geq 0$, $\xi_{Ri} \geq 0$, although in general one could have $\rho(B) = 0$. Further, in the case where $B$ is strongly connected (as determined by a digraph with the adjacency matrix given by non-zero elements of $B$), the spectral radius is strictly positive, as are the components of $\xi_L$, $\xi_R$. For such a positive vector $\xi$, we will consider the height ratio
\[
\gamma(\xi) \equiv \max_{ij} \frac{\xi_i}{\xi_j}, \quad \gamma(\xi) \geq 1.
\] (11)

3. Finite graph bounds

3.1. Approach

The derivation of Eq. (1) in Ref. [60] relied on the mapping of the percolation thresholds between the original graph $\mathcal{G}$ and its universal cover, a tree $\mathcal{T}$ locally equivalent to $\mathcal{G}$, $p_c(\mathcal{G}) \geq p_c(\mathcal{T})$. Our approach in this work is mostly algebraic. We consider the bound (1) as the convergence radius for the infinite power series[60],
\[
M \equiv M(H) \equiv \sum_{x=1}^{\infty} p^x H^x,
\] (12)
where a matrix element of $H^x$, $[H^x]_{ij}$ gives the number of non-backtracking paths starting at site $i$ along the arc $a \equiv i \rightarrow j$, ending at the arc $b$, and visiting $s - 1$ intermediate sites. Thus, the sum $\sum_b M_{ab}$ is an upper bound on the average number of sites which can be reached starting along the arc $a$. Respectively, in an infinite graph, the convergence of the series (12) implies: with probability one any given point belongs to a finite cluster. Unfortunately, this argument does not limit giant components on a finite graph. Indeed, matrix $H$ is non-symmetric, and convergence of the series can be highly non-uniform in $s$ (see, e.g., Ref. [61]), with the norm of each term exponentially increasing as $p^r[H^r] \sim p^r[H]^r$ for $s < s_0 = O(m)$, and only starting to decrease for $p < 1/\rho(H)$ at $s \geq s_0$. Thus, formal convergence does not guarantee the low probability of finding a giant component in a large but finite graph.

On the other hand, the expansion (12) gives a convenient tool to study percolation. By eliminating the contribution of backtracking paths, and reducing over-counting, one can get bounds tighter than what would be possible in a similar approach based on the adjacency matrix. Now, the original problem of percolation on an undirected graph is substituted by the problem of percolation on a directed graph, the OLG whose adjacency matrix is the non-backtracking matrix $H$. The same formalism can be used to bound percolation on digraphs, a problem of high importance in network theory[11,13].

3.2. Spectral bounds on susceptibilities

In the following, we only construct bounds for the out-cluster susceptibilities. The corresponding bounds for in-cluster susceptibilities, $\chi_{in}(v)$, can be obtained by considering the transposed matrices, $A^T_p$ and $H^T_p$, respectively.

Theorem 1. Consider heterogeneous site percolation on a finite strongly-connected digraph $D$. Assume that the spectral radius of the weighted adjacency matrix satisfies $\rho(A_p) < 1$, and let $\xi_R$ be the corresponding right PF vector. The out-cluster susceptibility for an arbitrary vertex $v \in V(D)$ satisfies:
\[
\chi_{out}(v) \leq \frac{C_1(\xi_R)}{1 - \rho(A_p)}, \quad C_1(\xi) \equiv \max_{u \in V} \frac{p_1^{u/2} \xi_u}{\min_{v \in V} \xi_v / p^{\gamma(\xi)}_v} \leq \gamma(\xi). \tag{13}
\]

Proof. Consider the alternative definition \( \gamma(\xi) \equiv \max_{ij} \frac{\xi_i}{\xi_j}, \gamma(\xi) \geq 1 \) of the susceptibility $\chi$, as a sum of connectivities. Any given site $u$ is in the out-cluster of $v$ iff there is an open path leading from $v$ to $u$. The corresponding probability can be estimated using the union bound, with sum of probabilities over self-avoiding paths upper-bounded by matrix elements of powers of the matrix $A_p$. Namely, the upper bound for the susceptibility $\chi_{out}(v)$ reads:
\[
\chi_{out}(v) \leq p_1^{u/2} \sum_{x=0}^{\infty} \sum_{w \in V(D)} [(A_p)^x]_{uv} p_1^{x/2}. \tag{14}
\]

For each $u$, we replace $p_1^{u/2} \leq \xi_{Ru} / \min_{v \in V} (\xi_v / p_1^{\gamma(\xi)})$, which reduces powers of $A_p$ to those of $\rho(A_p)$, thus
\[
\chi_{out}(v) \leq [1 - \rho(A_p)]^{-1} \frac{p_1^{u/2} \xi_{Ru}}{\min_{v \in V} \xi_v / p_1^{\gamma(\xi)}} \tag{15}
\]
The uniform bound (13) is obtained by maximizing over $v$. \qed

In the case of an undirected graph, or a digraph with some undirected edges, we can try to construct a better bound by considering only non-backtracking paths, with the corresponding probabilities counted using the weighted Hashimoto matrix $H_p$. The argument is simplest when the OLG of the original graph is also strongly connected. We have

Theorem 2. Consider heterogeneous site percolation on a finite digraph $D$ with a strongly connected OLG. Assume that the spectral radius of the weighted Hashimoto matrix satisfies $\rho(H_p) < 1$, and let $\eta_R$ be the corresponding right PF vector. The local out cluster susceptibility satisfies:
\[
\chi_{out}(v) \leq p_{\max} + \frac{C_2(\eta_R)}{1 - \rho(H_p)} C_2(\eta) \equiv \max_{a,b} \sum_{u \in \eta_{a \rightarrow b}} \frac{p_1^{u/2} \eta_{u \rightarrow b}}{\min_{b \rightarrow \gamma \eta_{b \rightarrow \gamma}} \eta_b / p_1^{\gamma(\xi)}} \tag{16}
\]
The proof is analogous to that of Theorem 1 except that the sum (14) is replaced by a similar sum in terms of the weighted Hashimoto matrix, with an additional summation over all arcs leaving a chosen vertex $v$. We see that $C_2(\eta) \leq o_{\max}(\gamma(\xi))$, where $o_{\max}$ is the maximum out degree of a vertex on $D$.

One possible set of sufficient conditions for an OLG to be strongly connected is given by the following Lemma:
Lemma 3. Consider a strongly-connected digraph $\mathcal{D}$. The corresponding OLG $\mathcal{H}(\mathcal{D})$ is also strongly connected if either of the following is true: (a) $\mathcal{D}$ has no undirected edges, or (b) $\mathcal{D}$ remains strongly connected after any undirected edge $i \leftrightarrow j \in \mathcal{E}(\mathcal{D})$ is replaced by either of the two arcs $a \equiv i \rightarrow j$ or $a \equiv j \rightarrow i$.

Proof. To connect the arcs $a = i \rightarrow j$ and $b = u \rightarrow v$ from $\mathcal{A}(\mathcal{D})$, take a directed path $\mathcal{P} = \{j \rightarrow j_1 \rightarrow j_2 \rightarrow \ldots \rightarrow j_{m-1} \rightarrow j\}$ from $j$ to $u$. If $j_1 = i$ [in the case the inverse of $a$ is also in the arc set, $a = j \rightarrow i \in \mathcal{A}(\mathcal{D})$], replace the first step by a directed path from $j$ to $i$ which does not include the arc $j \rightarrow i$. If needed, do the same at the other end, and remove any backtracking portions in the resulting directed path.

In the special case of a symmetric digraph $\mathcal{D}$ corresponding to an undirected graph $\mathcal{G}$, the condition in Lemma 3 is equivalent to $\mathcal{G}$ having no leaves (degree-one vertices).

For completeness, we also establish the relation between the spectral radii of the matrices $A_p$ and $H_p$, (this is an extended version of Theorem 1 from Ref. [59]; see also Ref. [41]):

Statement 4. (a) The spectral radii of the matrices $[\xi]$ and $[\xi,\rho]$ corresponding to heterogeneous site percolation on a finite strongly-connected digraph $\mathcal{D}$ satisfy $\rho(A_p) = \rho(L_p) \geq \rho(H_p)$. (b) If $\mathcal{D}$ has no undirected edges, all three spectral radii are equal. (c) If the OLG of a finite digraph $\mathcal{D}$ is strongly connected and $\rho(H_p) = \rho(A_p)$, then $\mathcal{D}$ has no undirected edges.

Proof. Consider a left PF eigenvector of the matrix $H_p$ with non-negative components, $\eta_0 \geq 0$, where $a \equiv i \rightarrow j$ goes over all arcs in the digraph $\mathcal{D}$. Such an eigenvector can always be constructed, whether or not the corresponding graph, OLG of $\mathcal{D}$, is strongly connected. According to the definition (10), the corresponding equations are

$$A\eta_{i\rightarrow j} = p_{ij} \sum_{k \in \mathcal{C}(\mathcal{D})} \eta_{j\rightarrow k} - p_{ji} \eta_{i\rightarrow j},$$

where $\lambda \equiv \rho(H_p) \geq 0$. The last term in Eq. (17) removes the backtracking contribution; it is present if the edge $(i, j) \in \mathcal{E}(\mathcal{D})$ is an undirected edge $(i \leftrightarrow j)$. Denote $x_{ij} \equiv \sqrt{p_{ij}} \sum_{j \rightarrow i} \eta_{j\rightarrow i}$ and $y_{ij} \equiv \sqrt{p_{ji}} \sum_{j \rightarrow i} p_{ji} \eta_{j\rightarrow i} \geq 0$, where in the latter case we are only summing over the neighbors $j$ connected to $i$ by undirected edges; $y_{ij} = 0$ if there are no undirected edges incident on $i$. In Eq. (17), fix $i$, sum over $j$, and multiply by $\sqrt{p_i}$; this gives

$$Ax_i = \sqrt{p_i} A \eta_{i\rightarrow j} \sqrt{p_j} x_j - y_i,$$

where $Lx = A_p x - y$. Denote $\xi$ a left PF vector of $A_p$, such that $\xi^T A_p = \rho(A_p) \xi^T$. By assumption, the original digraph is strongly connected, and all $p_i > 0$; therefore vector $\xi$ is unique, up to a normalization factor, which can be chosen to make components of $\xi$ all positive. Multiply the derived Eq. (18) by $\xi_j$ and sum over $i \in \mathcal{V}(\mathcal{D})$; this gives

$$\lambda \xi^T x = \xi^T A_p x - \xi^T y = \rho(A_p) \xi^T x - \xi^T y \leq \rho(A_p) \xi^T x.$$  

Notice that $\xi_j > 0$ while $x_j \geq 0$ and vector $x \geq 0$; thus $\xi^T x > 0$, which proves $\rho(A_p) \geq \lambda \equiv \rho(H_p)$.

Lemma 5. Consider heterogeneous percolation on a digraph $\mathcal{D}$ with $n = |\mathcal{V}(\mathcal{D})|$ vertices. For a chosen site $v$, assume that the directed connectivity decays exponentially with the distance, i.e., there are some constants $C > 0$ and $\rho < 1$ such that $\tau_{v,u} \leq C \rho^{d(\ell,\omega)}$, while the number of sites at distance $m$ from $v$ is exponentially bounded, $[u \in \mathcal{V} : d(v,u) = m] \leq C' \Delta^n$, for some $C' > 0$ and $\Delta > 1$. Then, for any $q$ such that $x \equiv \Delta \rho^q < 1$,

$$X_{ou}(v) \leq n^{-q} C(x')^{q} \left(1 - x\right)^{-q}.$$

Proof. Consider Eq. (6) as a 1-norm of an $n$-component vector of connectivities. Eq. (19) follows immediately from the Hölder’s inequality, $||u||_q \leq ||u||_p 1^{-1/q}$, where the $q$-norm is upper bounded in terms of an infinite geometrical series with the common ratio $x$.
3.3. Spectral bounds on connectivity and SAC susceptibility

It is easy to check that whenever the height ratio of the corresponding PF vector is bounded, all connectivity functions decay exponentially with the distance if $\rho(A_{\rho}) < 1$. However, since only two vertices are involved, we can get a more general statement:

**Theorem 6.** Consider heterogeneous site percolation on a finite digraph $D$ characterized by the weighted adjacency matrix $A_{\rho}$ with the spectral radius $\rho \equiv \rho(A_{\rho}) < 1$. Then, for any pair of vertices, the directed connectivity either from $u$ to $v$ or from $v$ to $u$ is exponentially bounded,

$$\tau_{u,v} \leq (1-\rho)^{-1} \rho^{d(u,v)}, \quad \text{or} \quad \tau_{v,u} \leq (1-\rho)^{-1} \rho^{d(v,u)},$$

where $d(u,v)$ is the directed distance from $u$ to $v$.

**Proof.** The statement of the theorem follows immediately if $u$ or $v$ are not strongly connected, as in this case either or both directed connectivities are zero. If $u$ and $v$ are strongly connected, the corresponding components of the right PF vector of $A_{\rho}$ are both non-zero, $\xi_u > 0$, $\xi_v > 0$. The argument similar to that in the proof of Theorem 5 gives the bound

$$\tau_{u,v} \leq \frac{\xi_u \rho^{d(u,v)}}{\xi_v (1-\rho)}.$$ (21)

The pair with smaller prefactor proves (20). \qed

Theorem 6 can be interpreted as a bound on the strong connectivity, and an associated bound on strong-cluster susceptibility from Lemma 5.

**Corollary 7.** Consider heterogeneous site percolation on a finite digraph $D$ with $n \equiv |V(D)|$ vertices, characterized by the weighted adjacency matrix $A_{\rho}$ with the spectral radius $\rho \equiv \rho(A_{\rho}) < 1$. Then, for any pair of vertices, the probability, the probability that they are in the same strongly connected cluster is exponentially bounded,

$$\tau_{u,v} \leq (1-\rho)^{-1} \rho^{d(u,v)} \min\{d(u,v), d(v,u)\}.$$ (22)

In addition, if the underlying graph of $D$ has a maximum degree $d_{\text{max}}$, strong-cluster susceptibility satisfies,

$$x_{\text{str}}(v) \leq C n^{1-1/q}, \quad C = \frac{1}{1-\rho} \left(\frac{2}{1-x}\right)^{1/q},$$

for some $q > 0$ such that $x \equiv (d_{\text{max}}-1)\rho^q < 1$.

We now would like to prove a tighter version of Theorem 6 which involves non-backtracking paths and the weighted Hashimoto matrix $H_{\rho}$, see Eq. (10). To this end, we first quantify the strong connectedness of the OLG of the digraph $D$.

**Lemma 8.** Let $\eta$ be the right PF eigenvector of the matrix $H_{\rho}$ corresponding to the positive eigenvalue $\lambda \equiv \rho(H_{\rho}) > 0$. Consider a pair of mutually inverted arcs $a \equiv i \rightarrow j$ and $\bar{a} \equiv j \rightarrow i$, $\{a, \bar{a}\} \subset \mathcal{A}(D)$, connected by a length-$\ell$ simple path,

$$\mathcal{P} \equiv \{j_0 \rightarrow j_1, j_1 \rightarrow j_2, \ldots, j_{\ell-1} \rightarrow j_\ell, j_\ell \rightarrow j_0\},$$

where $j_0 \equiv i$, and $j_1 = j_\ell \equiv j$; we assume $\mathcal{P} \subseteq \mathcal{A}(D)$. Denote

$$P \equiv P(\mathcal{P}) \equiv \lambda^{-\ell} \prod_{i=0}^{\ell} p_{j_i}, \quad \text{and} \quad x_i \equiv \sum_{j \in A_{\rho}^{\mathcal{P}}} \eta_{j-i}.$$ (25)

Then the following inequalities are true:

$$\eta_{i-j} \geq P \eta_{j-i},$$

$$\eta_{i-j} \geq \frac{x_i p_{j_i} P}{p_j + \lambda P}.$$ (26)

**Proof.** First notice that $\eta$ satisfies the eigenvalue equation [17]. Using it repeatedly along $\mathcal{P}$, write

$$\eta_{j_0 \rightarrow j_l} \geq \frac{p_{j_l}}{\lambda} \eta_{j_l \rightarrow j_{l-1}} \geq \frac{p_{j_{l-1}} p_{j_{l-2}}}{\lambda^2} \eta_{j_{l-2} \rightarrow j_{l-3}} \geq \cdots \geq \frac{\eta_{j \rightarrow j_0}}{\lambda^\ell} \prod_{i=1}^{\ell} p_{j_i},$$

which gives Eq. (26). Second, notice that the sum in the RHS of Eq. (17) equals $x_i$. Combine Eq. (17) with a similar equation for the inverted arc $j \rightarrow i$ which contains $x_i$. Together, the two equations give (we assume $\lambda^2 \neq p_{j_i} p_{j}$)

$$\eta_{i-j} \geq \frac{p_{j_l} x_j}{\lambda^2 - p_j p_j}, \quad \eta_{j-i} \geq \frac{p_{j_{l-1}} x_j}{\lambda^2 - p_j p_j}.$$ (29)

Substitute in Eq. (26) to obtain

$$\frac{p_{j_l} x_j}{\lambda^2 - p_j p_j} \leq \frac{\lambda + p_{j_l}}{\lambda} \frac{p_{j_{l-1}} x_j}{\lambda^2 - p_j p_j},$$

where the denominator is preserved for its sign. The lower bound (27) is obtained by substituting back into (29). \qed

For every vertex $j$ of a digraph $D$ with a strongly-connected OLG, let us define the minimal return probability,

$$P_j \equiv \min_{\mathcal{A}(D)} \left(\frac{p_j}{\lambda} \min_{a \leftarrow j} \max_{a \leftarrow \bar{a}} P(\mathcal{P})\right), \quad a \in \mathcal{A}(D),$$

(31)

where the minimum is taken over all arcs leading to $j$, $\mathcal{P}$ is a non-backtracking path (24) connecting $a \equiv i \rightarrow j$ and its inverse, $\bar{a} \equiv j \rightarrow i$, and $P(\mathcal{P})$ is defined in Eq. (25). For $a \equiv i \rightarrow j$ such that the inverted arc does not exist, $\bar{a} \not\in \mathcal{A}(D)$, we should use $P = p_{j_\ell}/\lambda$, see Eq. (17). Notice that thus defined $P_j$ is a local quantity; we expect it to be bounded away from zero for (d)graphs with many short cycles.

We can now prove

**Theorem 9.** Consider heterogeneous site percolation on a finite digraph $D$ characterized by the weighted Hashimoto matrix $H_{\rho}$ with the spectral radius $\rho \equiv \rho(H_{\rho})$ such that $0 < \rho < 1$. Let the OLG of $D$ be also strongly connected, with a non-zero minimal return probability, $P_{\min} \equiv \min_{\mathcal{A}(D)} \min_{a \in \mathcal{A}(D)} P_j > 0$. Then, for any pair of vertices, the directed connectivity either from $i$ to $j$ or from $j$ to $i$ is exponentially bounded,

$$\tau_{i,j} \leq \frac{P_{\min}^{(i,j)}}{(1-\rho)} (2 P_{\min}^{1})^{1}, \quad \text{or} \quad \tau_{j,i} \leq \frac{P_{\min}^{(j,i)}}{(1-\rho)} (2 P_{\min}^{1}).$$ (32)
Proof. Start with the union bound for the directed connectivity, formulated in terms of non-backtracking paths from $i$ to $j$,
\begin{equation}
\tau_{i,j} \leq p_i p_j \sum_{m \geq d(i,j)-1} \sum_{a \in A} \sum_{j_{t+1} \in v_{j_t}} [H^m_{a \rightarrow b}]_i^j \tag{33}
\end{equation}
where we assume $i \neq j$, so that $d(i,j) \geq 1$. Construct a further upper bound by introducing the factor [cf. Eq. (27)]
\begin{equation}
\frac{\eta_b}{\eta_{\min}(j)} \geq 1, \quad \eta_{\min}(i) \equiv \min_{P_i ightarrow j \in A} \eta_{i \rightarrow j} \geq \frac{x_i p_j P_j}{p_j + \rho P_j}, \tag{34}
\end{equation}
and extending the summation over $b$ to all arcs. This replaces the matrix element $[H^m_{a \rightarrow b}]$ with $\rho^m$,
\begin{equation}
\tau_{i,j} \leq p_i p_j \sum_{m \geq d(i,j)-1} \sum_{a \in A} \rho^m \frac{\eta_b}{\eta_{\min}(j)}. \tag{35}
\end{equation}
The summation over arcs $a$ from $i$ gives $x_i$ in the numerator, see Eq. (25), while the lower bound (34) gives $x_j$ in the denominator. We obtain
\begin{equation}
\tau_{i,j} \leq p_i p_j \frac{x_i}{x_j} \sum_{m \geq d(i,j)-1} \rho^m \left( \frac{\rho}{p_j} + \frac{1}{p_j} \right). \tag{36}
\end{equation}
The uniform connectivity bound (32) is obtained after the summation using the inequalities $p_i/p_j \geq p_j \geq \rho \min p_i \leq 1$. □

This gives a conditionally stronger version of Corollary 7

**Corollary 10.** Consider heterogeneous site percolation on a finite digraph $\mathcal{D}$ characterized by the weighted Hashimoto matrix $H_p$ with the spectral radius $\rho \equiv \rho(H_p)$, $0 < \rho < 1$. Then, if $\mathcal{D}$ satisfies the conditions of Lemma 8 the probability that a pair of vertices are in the same strongly-connected open cluster is exponentially bounded,
\begin{equation}
\tau_{\text{loc}}(u,v) \leq \frac{P_{\min}(d(u,v), d(v,u))^{-1}}{(1 - \rho)} (2 P_{\min}^{-1}). \tag{36}
\end{equation}
Exponential decay of connectivity with the distance also implies a sublinear scaling of the expected size of a strongly-connected cluster (Lemma 5).

**Corollary 11.** Consider heterogeneous site percolation on a digraph $\mathcal{D}$ with $n \equiv |V(\mathcal{D})|$ vertices, characterized by the weighted Hashimoto matrix $H_p$ with the spectral radius $\rho \equiv \rho(H_p)$, $0 < \rho < 1$. Let $d_{\text{max}}$ be the maximum degree of the undirected graph underlying $\mathcal{D}$, and $q \geq 1$ satisfy the inequality $x \equiv (d_{\text{max}} - 1) \rho^q < 1$. Then, if the graph satisfies the conditions of Lemma 8 we have the following bound for strongly-connected cluster susceptibility,
\begin{equation}
\chi_{\text{loc}}(v) \leq \eta^{-1/q} \left( \frac{2 P_{\min}^{-1}}{(1 - \rho)(1 - x)} \right)^{1/q}. \tag{37}
\end{equation}
This bound guarantees that in a large digraph, a giant strongly connected component occurs with asymptotically zero probability as long as the corresponding prefactor remains bounded.

We conclude this subsection with the following universal bound on the expected number of SACs through a given arc:

**Theorem 12.** Let $H_p$ be the weighted Hashimoto matrix for heterogeneous site percolation on a finite digraph $\mathcal{D}$. Then, if the spectral radius $\rho(H_p) < 1$, the SAC susceptibility for any arc $a \in \mathcal{A}(\mathcal{D})$ is bounded,
\begin{equation}
\chi_{\text{SAC}}(a) \leq [1 - \rho(H_p)]^{-1}. \tag{38}
\end{equation}
Proof. We first consider the case of a digraph whose OLG is strongly connected. The expected number of SACs, $\chi_{\text{SAC}}$, is non-negative, the RHS of Eq. (39) can be further bounded by
\begin{equation}
[H_p^a] \leq \sum_{b \in A} [H_p^1]_a \delta_{ba} \leq [H_p^1]_a \eta_a \eta_b = \rho(H_p)^\eta, \tag{40}
\end{equation}
where there is no implicit summation over the chosen $a \in \mathcal{A}(\mathcal{D})$. Summation over $s$ gives Eq. (38).

For a general digraph $\mathcal{D}$, not necessarily connected, we notice that the bound (38) is independent of the actual values of the components of the PF vector $\eta$. A general finite digraph can be made strongly connected by introducing additional vertices and additional edges connecting these vertices to different strongly connected components of $\mathcal{D}$, with arbitrarily small probabilities $p_i = \epsilon > 0$ for these vertices to be open. The statement of the Theorem is obtained in the limit $\epsilon \to 0$. □

### 3.4. Matrix norm bounds

For digraphs where the height ratios $\gamma_h$ and $\gamma_c$ may be large, we give a weaker general bound for the local in-/out-cluster susceptibilities:

**Theorem 13.** Let the induced one-norm of the weighted Hashimoto matrix corresponding to heterogeneous site percolation on a digraph $\mathcal{D}$ satisfy $\|H_p\|_1 < 1$. Then, the in-cluster susceptibility for vertex $j$ is bounded,
\begin{equation}
\chi_{\text{in}}(j) \leq 1 + \text{id}(j)(1 - \|H_p\|_1)^{-1}, \tag{41}
\end{equation}
where $\text{id}(j)$ is the in-degree of $j$.

We note that $\|H_p\|_1$ equals to the maximum column weight of $H_p$. The analogous result for the out-cluster susceptibility is obtained by considering the transposed matrix $H_p^\top$, which gives the maximum row weight of $H_p$ (also, $\|H_p\|_1 = \|H_p^\top\|_\infty$).

**Proof of Theorem 13.** We start with a version of Eq. (13) for in-cluster susceptibility.
\begin{equation}
\chi_{\text{in}}(j) \leq p_j + \sum_{n=0}^\infty \left( (H_p^\top)^j \right)_{ia} p^i \left( H_p^\top \right)^n, \tag{42}
\end{equation}
where the first term accounts for the starting point $j$, summation over the arcs $v \equiv i \to i'$ and $u \equiv j' \to j$ is assumed, and the expression inside the norm is a vector with non-negative components labeled by the site index $i$. Statement of the Theorem is obtained with the help of the standard norm expansion, also using $p_i \leq 1$. □
Notice that Theorem 13 works for finite or infinite digraphs. This implies that in- and out-cluster site-uniform percolation threshold in an infinite (di)graph satisfy, respectively,

\[ p_c^{\text{in}} \geq p_T^{\text{in}} \geq ||H||_1^{-1}, \quad (43) \]
\[ p_c^{\text{out}} \geq p_T^{\text{out}} \geq ||H^T||_1^{-1} = ||H||_{\infty}^{-1}. \quad (44) \]

Thus, Theorem 13 gives a direct generalization of the well-known maximum-degree bound (62).

\[ p_c \geq (d_{\text{max}} - 1)^{-1}, \quad (45) \]

to heterogeneous site percolation on a digraph.

Notice that any induced matrix norm satisfies the inequality \([H]\| \leq \rho(H_p), \) so that these bounds are generally weaker than Eq. (41). Nevertheless, Example 1 below shows that the bounds (43) and (44) are tight: the finiteness of the height ratio \(g_k\) is an essential condition in Theorem 14 and Theorem 2.

**Example 1.** Consider a family of girth-L random strongly connected oriented graphs with 2L^2 vertices, parameterized by in/out degrees \(d_1 \geq d_2 > 1\) [see Fig. 2]. The graphs are constructed from 2L directed cycles of length \(L\), by adding \(D_1\) randomly placed arcs from each site of the \(i\)-th cycle to those of the cycle \((i+1)\), with the constraint that the in-degrees on the latter cycle are all the same (and equal to \(D_i+1\)). We set \(D_i = d_i - 1\) for \(i = 0, \ldots, L-1\), and \(D_i = d_i - 1\) for \(i = L, \ldots, 2L-1\), with the sites of the last cycle connecting to those of the first.

In the degree-regular case \(d_1 = d_2 = d\), the PF vectors of the Hashimoto matrix have height ratios \(\gamma_1 = \gamma_k = 1\), which implies that the in-out-cluster susceptibilities for this family of digraphs is bounded for \(p < \rho(H)^{-1} = 1/d\).

More generally, for this family of digraphs, \(\rho(A) - 1\) equals the geometrical mean of the parameters \(D_i\). In the case \(d_1 > d_2\), we get \(\rho(A) = \rho(H) = 1 + [(d_1 - 1)(d_2 - 1)]^{1/2}\), with the height ratios \(\gamma_1\) and \(\gamma_k\) diverging exponentially with \(L\). In this case the more general weaker bounds (43) and (44) apply, with \(\|H\|_1 = \|H\|_{\infty} = d_1 > d_2\). Numerically, for \(d_1 = 3\) and \(d_2 = 2\), we find \(p_{\text{out}}^{\text{out}} = 0.346 \pm 0.01\), see Fig. 2.

The bound in Theorem 13 is local and independent of the size of the graph. The following gives bounds for the in-/out-cluster susceptibilities averaged over all sites:

**Theorem 14.** Let the \(q\)-norm \((q \geq 1)\) of the weighted adjacency matrix \(A\) for heterogeneous site percolation on a digraph with \(n\) vertices satisfy \(\|A\|_0 < 1\). Then the \(q\)-th power average of the out-cluster susceptibilities satisfies

\[ \langle \chi^q \rangle_{\text{out}} = \frac{1}{n} \sum_{i=1}^{n} \chi^q_{\text{out}}(i) \leq \left(1 - \|A\|_0\right)^{-q}. \quad (46) \]

**Proof.** Write a version of the bound (14) as

\[ \chi_{\text{out}}(v) \leq \sum_{s=0}^{\infty} ||A^s||_1 e_u, \quad (47) \]

where we introduce the vector with all-one components, \(e_u = 1\), \(u = 1, \ldots, n\). Consider this expression as an element of a vector \(\chi^q\) of susceptibilities. Taking the \(q\)-norm of this vector and using the standard norm expansion, we get, in self-evident notations

\[ \|\chi^q\|_q = \left[ \frac{1}{n} \sum_{i=1}^{n} (\chi^q_{\text{out}}(i))^q \right]^{1/q} \leq \left(1 - \|A\|_0\right)^{-q/2}[\|\chi^q\|_q], \quad (48) \]

Eq. (46) immediately follows.

We note that one can also derive a version of Eq. (46) with the non-backtracking matrix. However, in the cases where the \(q\)-norm of a large matrix can be computed efficiently, \(q = 1\), \(q = 2\), and \(q = \infty\), the resulting bounds are superseded by Theorem 13; it is easy to check that \(\|H^p\|_2 = \|H\|_1\), and \(\|H^p\|_{\infty} = \|H\|_{\infty}\).

In the special case of heterogeneous site percolation on an undirected graph \(G\) where the matrix \(A_p\) is symmetric, \(A_p = A_p^T\), the two-norm of \(A_p\) equals its spectral radius, \(\|A_p\|_2 = \rho(A_p)\); in this case we obtain a stronger version of Corollary 1 for undirected graphs:

**Corollary 15.** Let the spectral radius of the weighted adjacency matrix \(A\) for heterogeneous site percolation on an undirected graph with \(n\) vertices satisfy \(\rho(A_p) < 1\). Then cluster susceptibility at any site \(i\) is bounded,

\[ \chi(i) \leq \frac{n^{1/2}}{1 - \rho(A_p)}. \quad (49) \]

This bound guarantees the absence of a giant component in a sufficiently large graph with \(\rho(A_p) < 1\). The scaling with \(n\) is not an artifact of the approximation; this is illustrated by the following
Example 2 (Percolation on a rooted tree). Consider an undirected r-generation D-ary rooted tree with
\[ n = 1 + D + \ldots + D^r = \frac{D^{r+1} - 1}{D - 1} \]
vertices. Homogeneous site percolation with probability \( p \) gives the susceptibility at the root of the tree
\[ \chi(0) = p[1 + pD + \ldots + (pD)^r] = \frac{p(pD)^{r+1} - 1}{pD - 1}. \]
In the limit \( r \to \infty \) percolation threshold is \( p_T = 1/D \), same as \( p_c \) for the infinite tree \( T_D \), \( D = D + 1 \). On the other hand, the spectral radius of the adjacency matrix
\[ \rho(A) = 2\sqrt{D}\cos(\pi/(r + 1)) \leq 2\sqrt{D}. \]
For \( p = 1/\rho(A) > D^{-1} \) (in the case \( D > 4 \)), at large \( n \), \( \chi(0) = O(n^{1/2}) \), in agreement with the bound [40].

4. Bounds for infinite digraphs and digraph sequences

4.1. Bounds for infinite digraphs

In the majority of graph theory applications, one is interested in perhaps large but nevertheless finite graphs, considering the percolation transition as an idealization of a finite-graph crossover which gets sharper as \( n \to \infty \). It would be conceptually easiest to define an infinite limit by considering a sequence of increasing induced subgraphs, \( \mathcal{V}(\mathcal{G}_i) \subset \mathcal{V}(\mathcal{G}_{i+1}) \), such that every vertex in the infinite graph \( \mathcal{G} \) is eventually covered. However, this is not necessarily a good idea. In addition to regular \( D \)-dimensional lattices (e.g., \( \mathbb{Z}^D \)), or graphs which can be embedded in \( \mathbb{Z}^D \) with finite distortions, there are many important graph families which are, in effect, infinite-dimensional. Such graphs are characterized by a non-zero Cheeger constant, \( \chi(0) = \frac{|\mathcal{G}'|}{\min(\mathcal{G}', \mathcal{G} \setminus \mathcal{G}')}. \)

4.2. Bounds for infinite digraphs

In the denser region, the spectral radius of the adjacency matrix is connecting the subgraph \( \mathcal{G} \) where \( \mathcal{G} \) of increasing induced subgraphs, \( \mathcal{G} \). For \( 0 < i < L \), choose \( D_i = d_i - 1 \), for \( L \leq i < 2L \) (last ring connects to first), \( D_i = d_i - 1 \). Shown is an actual sample with \( L = 4 \), \( d_1 = 3 \), \( d_2 = 2 \). For such graphs, at large \( L \), in/out-cluster percolation is determined by that in the denser region, \( p^{(out)}_i \geq 1/d_i \) assuming \( d_1 > d_2 \), see text.
graph and its any cover can be counted with the help of the Hashimoto matrix.

To construct a similar bound for the case of heterogeneous site percolation on a locally-finite digraph, we similarly associate growth with any non-negative matrix,

$$
\text{gr } H = \max_{a} \inf_{m \to \infty} \left\{ \lambda > 0 : \liminf_{m \to \infty} \lambda^{-m} \sum_{b} |H^m_{ab}| = 0 \right\}. \quad (54)
$$

It is a simple exercise to check that on a strongly-connected digraph the infimum is independent of the chosen arc a. We prove the following generalization of the bound [1]:

**Theorem 16.** Consider heterogeneous site percolation on a strongly connected locally-finite digraph $G$. Let on-site probabilities be such that the growth of the corresponding weighted Hashimoto matrix satisfy $\text{gr } H_p < 1$. Then, with probability one, any out cluster on the open subgraph is finite.

The proof is a variant of the first half of the proof of Theorem 6.2 in Ref. [64].

**Proof.** Suppose an open out-cluster for a chosen vertex $v \in V$ is infinite. Then, there should be a simple path from $v$ to infinity, entirely constructed from open vertices. Then there is also a directed SAW from some arc $a \in \mathcal{A}$ leaving $v$ to infinity, which implies that for any $m > 0$, the number of points reachable from $a$ in $m$ steps should be at least one. Then, the probability $\theta_{out}(v) \geq 0$ to have an infinite out-cluster starting from $v$, we have the following uniform bound for any $m > 0$,

$$
\rho_p(d, m) \max_{a \in \mathcal{A}} \left\{ \sum_{b \in \mathcal{A}} |H^m_{ab}| \geq \theta_{out}(v). \right\} \quad (55)
$$

By assumption $\text{gr } H_p < 1$, thus $\lim_{m \to \infty} \lambda^{-m} \sum_{b} |H^m_{ab}| = 0$ for any arc $a \in \mathcal{A}$. That is, there should be an increasing sequence $m_i, i = 1, 2, \ldots$, such that the corresponding limit is zero. Then, Eq. [55] gives $\theta_{out}(v) = 0$. QED

Similarly, to get a bound on the transition associated with divergent susceptibility, we need to ensure the convergence of the series similar to that in Eq. [15]. To this end, we introduce the uniformly bounded growth:

$$
\overline{\text{gr }} H \equiv \inf_{m \to \infty} \left\{ \lambda > 0 : \limsup_{m \to \infty} \lambda^{-m} \sup_{u} \sum_{v} |H^m_{uv}| = 0 \right\}. \quad (56)
$$

In particular, if $H$ is a Hashimoto matrix of a tree, then $\rho_T = 1/\overline{\text{gr }} H$. More generally, we have

**Theorem 17.** Consider heterogeneous site percolation on a bounded-degree digraph $\mathcal{D}$, characterized by the weighted Hashimoto matrix $H_p$. If $\overline{\text{gr }} H_p < 1$, then there exists a constant $C > 0$ such that for any $v \in \mathcal{V}(\mathcal{D}), \chi_{out}(v) \leq 1 + C o(d(v)).$

**Proof.** The definition [56] implies that for any $\lambda > \overline{\text{gr }} H_p$ and any $\epsilon > 0$, there exists $m_0$ such that for all $m > m_0$,

$$
\lambda^{-m} \sup_{a \in \mathcal{A}(\mathcal{D})} \left\{ \sum_{b \in \mathcal{A}(\mathcal{D})} |H^m_{ab}| \right\} < \epsilon. \quad (57)
$$

Take $\epsilon = 1$ and $\lambda = (1 + \sqrt[p]{H_p})/2 < 1$, and define a constant

$$
A \equiv \sum_{m=0}^{m_0} \sup_{a} \sum_{b} |H^m_{ab}|. \quad (58)
$$

The supremum is finite for any finite $m$ by the assumption of a finite maximum degree. Now, use our usual bound for the out-cluster susceptibility, in terms of all non-backtracking paths from $v \in \mathcal{V}(\mathcal{D})$, counted using the powers of $H_p$. Summation over $m$ gives the stated bound with $C = A + \lambda^{m+1}/(1-\lambda)$. □

In comparison, the connectivity is bounded in terms of the $\hat{\mathcal{P}}(\mathcal{A})$ spectral radius $\rho_{\mathcal{P}}(H_p)$:

**Theorem 18.** Consider heterogeneous site percolation on a locally-finite digraph $\mathcal{D}$ characterized by the weighted Hashimoto matrix $H_p$. Then, if $\rho_{\mathcal{P}}(H_p) < 1$, there exists a base $\rho' < 1$ and a constant $C \geq (1-\rho')^{-1}$ such that the directed connectivity between arcs $u$ and $v$ decays exponentially with the distance $d(u, v)$,

$$
\tau_{uv} \leq C(\rho')^d(u,v). \quad (59)
$$

**Proof.** Start with our usual bound in terms of the weighted Hashimoto matrix,

$$
\tau_{uv} \leq \sum_{m \geq 0}(H^m_{uv})'; \quad (60)
$$

the series converges since $\rho < 1$ by assumption. Notice that in $\hat{\mathcal{P}}(\mathcal{A})$, the spectral radius of $H_p$ can be defined as the limit,

$$
\rho = \lim_{m \to \infty} ||H^m_{uv}||^{1/m}, \quad (61)
$$

where $||H^m_{uv}||^{1/m} \geq \rho$ for every $m > 0$. Then, for any $\epsilon > 0$, there exists $m_0$ such that $||H^m_{uv}||^{1/m} < \rho + \epsilon$ for all $m \geq m_0$. Choose $\epsilon = (1-\rho)/2$, define $\rho' = \rho + \epsilon = (1+\rho)/2$, and also the constant $B = \max_{0 \leq m < m_0}||H^m_{uv}||/(1-\rho'), B \geq 1$. The statement of the theorem is satisfied with $C = (1-\rho')^{-1}B$. □

The spectral radius $\rho_{\mathcal{P}}(H_p)$ satisfies the following bounds:

**Statement 19.** Consider heterogeneous site percolation on a locally-finite digraph $\mathcal{D}$ characterized by the weighted Hashimoto matrix $H_p$, and the corresponding problem on the universal cover $\hat{\mathcal{D}}$ characterized by the matrix $\hat{H}_p$. The following bounds are true:

$$
\rho_{\mathcal{P}}(H_p) \leq \rho_{\mathcal{P}}(H) \leq \overline{\text{gr }} H_p. \quad (62)
$$

**Proof.** For any $\lambda \neq 0$ and $u \in \mathcal{A}$, define the vectors $\xi^{(m)}(u)$ with components $\xi^{(m)}(u) \equiv \lambda^{-m} |H^m_{uv}|$, and $\xi(u) \equiv \sum_{m \geq 0} \xi^{(m)}(u)$. The parameter $\lambda \neq 0$ is outside of the spectrum of $H_p$, iff the series $\sum_{m \geq 0} H^m_{uv} / \lambda^m$ define a bounded operator on $\hat{\mathcal{P}}(\mathcal{A})$ [cf. Eq. [52]]. Equivalently, since individual arcs form a basis of $\hat{\mathcal{P}}(\mathcal{A})$, for any $u \in \mathcal{A}$, the vector $\xi(u)$ should have a finite norm $||\xi(u)||_2$. Since the matrix elements $(H_p)_{uv} \geq 0$, the sum in Eq. [56] is the 1-norm of $\xi^{(m)}(u)$. On the other hand, for any $\lambda > \overline{\text{gr }} H_p$, we can write

$$
||\xi(u)||_2 \leq \sum_{m \geq 0} ||\xi^{(m)}(u)||_2 \leq \sum_{m \geq 0} ||\xi^{(m)}(u)||_1, \quad (63)
$$
where the rightmost series is convergent since it is asymptotically majored by the sum of $(\overline{\rho} H_p / \lambda)^m$; this proves the upper bound. To prove the lower bound, consider a similarly defined vector $\xi(u)$ on the universal cover; components $\xi_i(u)$ are the sums of the non-negative components of $\xi(\alpha)$ corresponding to different non-backtracking paths from $u$ to $v$; we thus have $|\xi(u)|_2 \leq |\xi(\alpha)|_2$. Thus, a point outside the spectrum $\sigma(\overline{H}_p)$ is also outside the spectrum $\sigma(H_p)$. \qed

Further, in the case of homogeneous percolation, the spectral radius $\rho(\overline{H}_p) = [br \overline{D}]^{1/2}$ is exactly the tree’s point spectral radius. Also, for any quasi-transitive digraph $D$, we have $br D \leq gr H \leq \overline{gr} H$. The corresponding value can be found as the spectral radius $\rho(H')$ for a finite graph.

The following Theorem is a generalization of Theorem 3.9 from Ref. 17, attributed to O. Schramm.

**Theorem 20.** Consider heterogeneous site percolation on an infinite locally finite digraph $D$, with site probabilities bounded from above, $p_v \leq \rho_m < 1$, $v \in V(D)$. Then, if strong percolation occurs, and a strongly connected infinite open cluster is unique with probability one, the SAC susceptibility is unbounded,

$$\sup_{u \in \mathcal{A}} \chi_{SAC}(a) = \infty. \quad (64)$$

**Proof.** First, uniqueness of the infinite strongly-connected open cluster $K \subseteq D$ implies that, with probability one, $K$ is one-ended: it cannot be separated into two or more strongly connected infinite components by removing any finite set of vertices. Indeed, otherwise, we would have a non-unique infinite cluster with a finite probability, which contradicts the assumption. Second, with probability one, $K$ contains two disjoint strong rays. Indeed, let us assume that not to be the case. Then, according to Menger’s theorem, for any $v \in V(D)$, the open subgraph would have an infinite number of single-vertex cut sets separating $v$ from infinity or from infinity. This would imply $\theta_{uv}(v) = 0$, counter to the assumption. The one-endedness of $K$ implies that outside of any finite ball, the two strong rays must remain strongly connected with each other. This means that with probability one, for some $a \in \mathcal{A}$, the open strongly-connected cluster $K$ contains an infinite number of simple cycles passing through $a$, which implies Eq. (64). \qed

Notice that an upper bound on probabilities $p_v$ is an essential condition. This eliminates the case of a digraph with a ray whose vertices all have $p_v = 1$.

### 4.2. Bounds for sequences of finite digraphs

In Sections 3.2 and 3.3 we constructed several upper bounds for susceptibility and connectivity in heterogeneous percolation on finite digraphs, formulated in terms of the spectral radius of the corresponding weighted Hashimoto matrix $H_p$ and the associated PF vectors $\eta_L, \eta_R$. In contrast, bounds in Section 4.1 are formulated directly on infinite digraphs. We would like to see the correspondence between these bounds for weakly convergent digraph sequences.

Given an infinite graph $G(V, E)$, we say that a sequence of graphs $G^{(t)}, t = 1, 2, \ldots$, (weakly) converges to $G$ near the origin $v_0 \in V$, if for any $R > 0$ there is $t_0$ such that the radius-$R$ vicinity of $v_0$ on $G$ for every $t \geq t_0$ is isomorphic to a subgraph of $G^{(t)}$. For heterogeneous site percolation, we require $p_v$ to match on corresponding sites. For digraphs, we also require the bond directions to match (while using undirected distance to define the ball). In the following, when discussing a sequence of digraphs, objects referring to the digraph $D^{(t)}$ are denoted with the corresponding superscript, e.g., the weighted Hashimoto matrix $H_p^{(t)}$ and its right PF vector $\eta^{(t)} \in \ell^2(\mathcal{A}^{(t)})$. We will also use the same notation for the corresponding vector mapped to $\mathcal{A}$ under the isomorphism map, $\eta^{(t)} \in \ell^2(\mathcal{A})$, dropping any arcs not in $\mathcal{A}$, and adding zeros for arcs not in $\mathcal{A}^{(t)}$.

We first compare the bounds for the transition associated with emergence of infinite cluster and divergent out-cluster susceptibilities. The bound in Theorem 2 is formulated in terms of the spectral radius $\rho(H_p)$ of the Hashimoto matrix and a pre-actor $C_2(\eta_R)$, while bounds in Theorems 16 and 17 are formulated in terms of the growth $gr H$ and uniformly-bounded growth $\overline{gr} H$. A sufficient condition for continuity between these bounds is given by the following:

**Statement 21.** Consider heterogeneous site percolation on an infinite digraph $D$, characterized by the weighted Hashimoto matrix $H_p$, and a sequence of finite digraphs $D^{(t)}$ with strongly connected OLGs, converging to $D$ around some origin. Then, if the right PF vectors of $H_p^{(t)}$ have uniformly bounded height ratios, $\gamma(\eta^{(t)}R) \leq M$, the following limit exists, and

$$\lim_{t \to \infty} \rho(H_p^{(t)}) = gr(H_p) = \overline{gr}(H_p). \quad (65)$$

**Proof.** For each $D^{(t)}$, the corresponding universal cover $\overline{D}^{(t)}$ is a quasi-transitive tree; a lift $\overline{\eta}_R^{(t)}$ of the PF vector $\eta_R^{(t)}$ is the eigenvector of $H_p^{(t)}$ with all positive components. For a given $u \in \mathcal{A}$ and $t$ large enough, the conditions guarantee that the radius-$m$ vicinity of $u$ on $D$ is entirely within the subgraph of $D^{(t)}$ isomorphic with that of $D$; the same is true for the universal covers. We can therefore construct the upper and lower bounds on the sum in Eq. (56) in terms of the right PF vectors $\eta^{(t)}$, or, using the assumed uniform bound on the height ratios,

$$\frac{1}{\gamma(\eta^{(t)}R)} ||\mu(\eta^{(t)}R)||^m \leq \sum_v ||H_p^{(t)}||_v \leq \gamma(\eta^{(t)}R)\mu(\eta^{(t)}R)||^m, \quad (66)$$

$$\frac{1}{M} ||\mu(\eta^{(t)}R)||^m \leq \sum_v ||H_p^{(t)}||_v \leq M||\mu(\eta^{(t)}R)||^m. \quad (67)$$

Now, let us choose a subsequence with spectral radii converging to $\liminf_{t \to \infty} \rho(H_p^{(t)})$. Using only these digraphs in the upper bound (67), the definition (56) implies

$$\overline{gr} H_p = \liminf_{t \to \infty} \rho(H_p^{(t)}).$$

The same calculation can be repeated for the lower bound, with a subsequence of graphs whose spectral radii converge to the corresponding superior limit; we get

$$\limsup_{t \to \infty} \rho(H_p^{(t)}) \leq gr(H_p) \leq \overline{gr}(H_p) \leq \liminf_{t \to \infty} \rho(H_p^{(t)}). \quad (68)$$
This implies that the limit exists and satisfies Eq. \(65\). \qed

Similarly, for a finite digraph, Corollary \(10\) gives strong connectivity exponentially decaying with the distance for \(\rho(H_p) < 1\), as long as the corresponding OLG is locally strongly connected. If we consider a sequence of such finite digraphs converging to an infinite digraph around some origin, we expect exponential decay of strong connectivity for \(\lim_{t\to\infty} \rho(H_p^t) < 1\), with a bounded prefactor. On the other hand, Theorem \(18\) gives exponential decay of directed connectivity with the distance on an infinite digraph with \(\rho(H_p) < 1\), without an explicit bound on the prefactor. The following gives partial correspondence between these results for undirected graphs:

**Statement 22.** Consider heterogeneous site percolation on a locally-finitely infinite connected graph \(G\), characterized by the weighted Hashimoto matrix \(H_p\), and a sequence of finite graphs \(G^t\) converging to \(G\) around some origin. If the ratios of the components of the right PF vectors \(\eta^t\) of \(H_p^t\) corresponding to each arc and its inverse are uniformly bounded by a fixed \(M \geq 1\),

\[
M^{-1} \leq \frac{\eta^t(a)}{\eta^t(b)} \leq M, \quad \text{then} \quad \lim_{t\to\infty} \rho(H_p^t) \geq \rho_p(H_p). \tag{69}
\]

**Proof.** Define the parity operator as in Ref. \([63]\), \(P_{ab} = \delta_{a,b}\), to connect each arc \(a\) with its inverse \(\bar{a}\); we have \(PH_p, P = H_p\). The condition of the theorem allows to use the components of \(P\eta^t\) as a lower or an upper bound on those of \(\eta^t\). It also guarantees that the OLGs of the graphs \(G^t\) are strongly connected. For any \(m \geq 0\) and \(v \in A\), there exists \(t_0\) such that for all \(t > t_0\), the radius-\(m\) vicinity of \(v\) will be isomorphic to a subgraph of \(G^t\). If we introduce the vector \(e_v\) with the only non-zero component (equal to one) at the arc \(v\), we can write for \(t > t_0\),

\[
\|H_p^m e_v\|_2^2 = e_v^T PH_p^m PH_p e_v = e_v^T P(H_p)^m P(H_p)^m e_v, \tag{70}
\]

where we used the same notation for the corresponding vector under the isomorphism map. We can now use the PF vector \(\eta^t\) to construct the upper bound,

\[
\|H_p^m e_v\|_2^2 \leq e_v^T P(H_p)^m P(H_p)^m \eta^t(1) \frac{1}{\eta^t} \leq M^2 \frac{\eta^t(1)}{\eta^t}, \quad \text{where we used the identity } |P \eta^t|_v \leq M \eta^t(1) \text{ twice. This shows that for any } v \in A \text{ and any } \lambda > \lim_{t\to\infty} \rho(H_p^t), \text{ the two-norm of } H_p^m e_v \text{ converges to zero, i.e., the operator } (\lambda I - H_p)^{-1} \text{ is bounded in } l^2(A), \text{ thus } \rho_p(H_p) \leq \lim_{t\to\infty} \rho(H_p^t). \tag{71}
\]

We note that for an increasing sequence of subgraphs \(G\), \(\rho(H_p^t) \leq \rho_p(H_p)\) in a non-decreasing bounded function of \(t\). Also, the condition on the PF vector can be guaranteed by Lemma \(8\). This implies

**Corollary 23.** Consider heterogeneous site percolation on a locally-finitely infinite graph \(G\) whose OLG is locally strongly connected, i.e., for every arc \(a \in A(G)\), there is a non-backtracking path of length at most \(t\) from \(a\) to \(\hat{a}\). Assume that the site probabilities are bounded from below: \(p_v > p_{\min} > 0\). Consider any increasing sequence of finite subgraphs \(G^t \subset G^{t+1} \subset G\), convergent to \(G\) around some origin. Then,

\[
\lim_{t\to\infty} \rho(H_p^t) = \rho_p(H_p). \tag{72}
\]

Our final result is a bound on the SAC susceptibility for a weakly convergent sequence of finite digraphs, and an associated bound for the transition associated with the number of strongly-connected infinite clusters. We notice that the SAC susceptibility counts only finite-length cycles. This implies, most generally:

**Theorem 24.** Consider heterogeneous site percolation on an infinite locally-finitely infinite digraph \(D\), characterized by the weighted Hashimoto matrix \(H_p\), and an increasing sequence of finite subgraphs \(D^t \subset D^{t+1} \subset D\) converging to \(D\) around some origin. Consider \(\rho_0 \equiv \lim_{t\to\infty} \rho(H_p^t)\). Then, if \(\rho_0 < 1\), the SAC susceptibility at any arc \(a \in A\) is bounded,

\[
\chi_{\text{SAC}}(a) \leq (1 - \rho_0)^{-1}. \tag{73}
\]

Notice that this bound does not include limitations as in Corollary \(22\). Generally, for an increasing sequence,

\[
\rho_0 \equiv \lim_{t\to\infty} \rho(H_p^t) \leq \rho_p(H_p). \tag{74}
\]

It may well happen that \(\rho_0 < \rho_p(H_p)\), as in the case where \(D\) is a tree, cf. Example \(8\).

**Proof of Theorem 24.** The sequence \(\rho(H_p^t)\) is non-decreasing. By assumption, it is also bounded, thus the limit exists. For any \(s\), the cycles of length \(s\) are contained in a finite vicinity of the original arc \(a\); the corresponding contribution does not exceed \(\rho_0\) (see the proof of Theorem \(12\)). Summation over \(s\) gives the bound \(73\). \qed

Combined with Theorem \(20\), this gives the following Corollary (its weaker version previously appeared as a conjecture in Ref. \([66]\)).

**Corollary 25.** Consider heterogeneous site percolation on an infinite, locally-finitely infinite digraph \(D\), characterized by the weighted Hashimoto matrix \(H_p\), with on-site probabilities \(p_v \leq p_{\max} < 1\). For an increasing sequence of finite subgraphs \(D^t \subset D^{t+1} \subset D\) weakly converging to \(D\), let \(\rho_0 \equiv \lim_{t\to\infty} \rho(H_p^t)\). Then, if \(\rho_0 < 1\), the percolating cluster cannot be unique.

If \(\rho_0 < 1\) and yet an infinite strongly-connected cluster exists on the induced subgraph \(D^t\), the system is expected to be below any transition associated with the number of percolating clusters. While such a transition is usually called “uniqueness” transition, we note that one-endedness of the infinite cluster is necessary but not sufficient to have a unique percolating cluster. An example could be any graph with a finite number of ends.

**Example 3.** A degree-\(d\) infinite tree \(T_d\) can be obtained as a limit around its root of the following graph sequences: \(a\) a sequence of its subgraphs, \(t\)-generation trees \(T_d^t\); \(b\) sequence of...
graphs obtained from $T_r^{(0)}$ by pairing degree-one vertices arbitrarily to form degree-two vertices; (c) a sequence of $d$-regular graphs obtained from $T_r^{(0)}$ by joining the leaves in pairs and replacing any resulting pair of edges connected by a degree-two vertex with a single edge. In the case (a), $p(H^{(0)}) = 0$ for any $t$; in the case (b), $\lim_{t \to \infty} p(H^{(0)}) = (d-1)^{1/2} = \sqrt{\text{gr} T_d}/2$, with the components of the PF vectors falling exponentially away from the center. In the case (c), $p(H^{(0)}) = d - 1 = \sqrt{\text{gr} T_d}$, and the PF vectors have all equal components. The sequence of subgraphs (a) correctly reproduces (the absence of) the uniqueness transition, while the sequences of degree-regular graphs (c) give the percolation transition at $p_t = 1/(d - 1)$.

Example 4. Consider a $(d, d)$-regular locally-planar hyperbolic graph $G_{d, d}$, with $d$ identical $d$-sided plaquettes meeting at every vertex. This graph can be obtained as a limit of (a) an increasing sequence of radius-$t$ subgraphs, with $\lim_{t \to \infty} p(H^{(0)}) = \rho_c(\text{gr})$, or (b) a sequence of $d$-regular graphs [67] whose spectral radii coincide with $\sqrt{\text{gr} H} = 2d - 1$. The spectral radius of the adjacency matrix satisfies the following bounds based on the Cheeger constant [68, 69].

$$2\sqrt{d - 1} \leq \rho_c(A) \leq 2\sqrt{d};$$

(75) these result in

$$\sqrt{d - 1} \leq \rho_c(H) \leq 1 + \sqrt{d}.$$  

(76) For site percolation on $G_d$, we recover the maximum degree bound for percolation transition, $p_s = p_t \geq 1/(d - 1)$, and get the bound $p_c \geq 1/\sqrt{d - 1}$ for the uniqueness transition.

Example 5. For a $D$-dimensional hypercubic lattice, $Z^D$, we have $\rho_c(H) = \sqrt{\text{gr} \, H} = 2D - 1 = d - 1$. There is only one percolation transition, $p_s = p_c = p_t$. Our bounds for these transitions coincide and recover the maximum degree bound.

Example 6. Consider strongly-connected cluster percolation on the oriented graphs in Example 4. For any finite $L$, a strongly-connected cluster can be formed. Numerically, we get $\rho_c^{(s)}(H) \approx 0.53$, see Fig. 3. This is a reasonable value since directed-cluster percolation in both regions is necessary to form a large strong cluster. However, the weak limit of this graph sequence is an infinite oriented tree. For such a tree, any strongly-connected cluster is limited to one site: strong-cluster percolation never happens. Respectively, the limiting spectral radius in Corollary 23 is $\rho_0 = 0$.

Example 7. For an integer $d_0 > 2$, consider site percolation on a graph $G$ constructed from a well-connected core, a large random $d_0$-regular graph $G_0$ with $n_0$ vertices, by connecting $r$ additional otherwise disjoint edges to each vertex in $V(G_0)$. With $n_0$ large and $r$ bounded, formation of a giant component on $G$ is governed by the corresponding transition on $G_0$. Spectral radii of the Hashimoto and the adjacency matrices of $G$ are $\rho(H) = d_0 - 1$ and $\rho(A) = \left[d_0 + \left(4r + d_0^2\right)^{1/2}/2\right]$. We see that the bound in Corollary 17 becomes increasingly loose as $r$ is increased in the region $r > d_0^2/4$.

5. Discussion

We constructed several bounds for heterogeneous percolation on general graphs, directed or undirected. We obtained explicit expressions for finite and infinite (di)graphs, and, in two cases, analyzed the continuity of the bounds in the infinite-graph limit. Most bounds are obtained from the non-backtracking path expansion, and formulated in terms of appropriately weighted Hashimoto matrices $H_p$. While in several cases stronger bounds (e.g., in terms of self-avoiding paths, or using modified Hashimoto matrix as in Ref. [70]) may be readily available, one main advantage of the results presented here is that spectral radii and norms can be calculated efficiently.

For a general infinite undirected graph, there are three transitions usually associated with percolation: divergence of the cluster susceptibility, formation of an infinite cluster, and the uniqueness transition. Bounds for all three transitions are formulated in terms of the weighted Hashimoto matrix, see Theorems [17, 18] and Corollary 25. These bounds also apply for directed- or strong-cluster percolation on digraphs. In addition, the condition $\rho_c(H_p) < 1$ guarantees that connectivity on an infinite digraph decays exponentially with the distance.

In practical network theory applications, more important is the transition associated with the formation of a giant component. Several simple criteria for the emergence of a giant component that are commonly used in network theory rely on the degree distribution of a graph. First is the lower bound for percolation on an arbitrary graph in terms of the maximum degree $d_0$, see Eq. (45). While it is universally applicable, the issue with this inequality is that it tends to give very low bounds on graphs with wide degree distribution. Our Theorem [17] gives a generalization of this bound to heterogeneous site percolation on arbitrary digraphs. Second is the Molloy-Reed criterion [9, 10, 71, 72] which gives the percolation threshold on random graphs in terms of the two first moments of degree
distribution. It has been recently generalized to giant in-/out-clusters on random digraphs\textsuperscript{[7]}. While these formulas are asymptotically exact in random graph ensembles\textsuperscript{[2]}, there is no guarantee: on actual networks the Molloy-Reed criterion can substantially overestimate or underestimate the threshold\textsuperscript{[7]}.

The spectral radius has also been used to study percolation. On large dense graphs, under mild conditions, the critical probability where a giant cluster emerges is very close to the inverse spectral radius of the adjacency matrix\textsuperscript{[53]}. Our Corollary \textsuperscript{[7]} gives a related strict bound for emergence of a giant strongly-connected component in heterogeneous site percolation on an arbitrary digraph (a slightly stronger bound specific for undirected graphs is given in Corollary \textsuperscript{[5]}). One substantial advantage of these bounds is their universal applicability. On the other hand, bounds in terms of \(\rho(A_p)\) can become loose in certain carefully designed networks, see Example \textsuperscript{[7]}.

In comparison, a bound in terms of the spectral radius of the (weighted) Hashimoto matrix does not change upon the addition of leaves or finite trees. It is also asymptotically exact for tree-like graphs with few short cycles, large random graphs being the most important example. While such a bound does indeed limit the percolation transition on highly-uniform (e.g., quasi-transitive) (di)graphs, most generally the condition \(\rho(H_p) < 1\) is a bound on the strong-cluster uniqueness transition. On an infinite digraph, such a bound is constructed as the limit of the spectral radii for an increasing sequence of subgraphs, which recovers the absence of the uniqueness transition on an arbitrary infinite tree. Further, for \(\rho(H_p) < 1\), on an infinite digraph, connectivity decays exponentially with the distance. On a finite digraph, with \(\rho(H_p) < 1\), strong connectivity also decays exponentially with an additional local strong connectivity condition needed to limit the prefactor in the bound. Such an exponential decay also implies sublinear scaling of the expected size of the largest cluster, which in turn guarantees that a giant strongly-connected component containing a non-zero fraction of all vertices emerges with an asymptotically zero probability.

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