Vaisman Algebroid and Doubled Structure of Gauge Symmetry in Double Field Theory

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Abstract. The Vaisman algebroid is a kind of algebroid structure. It is defined by an extension of the Courant algebroid, and physically related to the gauge symmetry in Double Field Theory (DFT), which is an effective theory of string theory. DFT has T-duality as a manifest symmetry. In this study, we focus on the “doubled structure” in the Vaisman algebroid. It is already well known that some kind of Lie algebras are obtained by the Drinfel’d double of Lie bialgebras. The Courant algebroid is obtained by the Drinfel’d double of Lie bialgebroids. We find that the Vaisman algebroid can be obtained by an analogue of the “Drinfel’d double” of Lie algebroids. We discuss the algebraic origin of the strong constraint in DFT.

1. Introduction
T-duality is a duality in superstring theory. It is important to understand T-duality to study the nature of space-time. From the viewpoint of geometry, it has been studied in the framework of Hitchin’s generalized geometry [1]. The natural algebraic structure which appears in generalized geometry is the Courant bracket. Mathematically, it is known that a Courant bracket defines a Courant algebroid (for example [2]).

Double Field Theory (DFT) is a kind of effective theory of superstring theory [3]. DFT exhibits T-duality manifestly and it has other various symmetries. For example, DFT has a gauge symmetry. It is a T-duality covariantized symmetry which originates from the diffeomorphism and the $U(1)$ gauge symmetry of the NSNS $B$-field. The bracket which describes the gauge symmetry in DFT is called the C-bracket. Since the C-bracket does not satisfy the Jacobi identity, the C-bracket defines not a Lie algebra but a metric algebroid. This structure is first presented in [4]. We call this structure Vaisman algebroid. A Vaisman algebroid can be defined as an extension of a Courant algebroid. This is because a Vaisman algebroid violates some of the axioms that a Courant algebroid should satisfy. The Vaisman algebroid is known as the pre-DFT algebroid in [5].

In this proceeding, we show that the Vaisman algebroid has some kind of a doubled structure. In other words, we show that the Vaisman algebroid is constructed by a pair of Lie algebroids. This doubled structure is closely related with the Drinfel’d double. It is known that a Courant algebroid can be obtained by the Drinfel’d double of a Lie bialgebroid [6]. Our work is the generalization of this result. The contents of this proceeding is based on the first half of our paper [7].
The DFT geometry is called “doubled geometry”. Mathematically, the doubled geometry is related to the para-Hermitian geometry [8] and Born geometry [9]. It is known that the C-bracket in DFT appears as a Vaisman bracket on the para-Hermitian geometry. In the second half of our paper [7], we constructed the double structure of the Vaisman algebra on the para-Hermitian geometry and we also discussed the algebraic origin of the strong constraint which is the physical condition in DFT.

2. Lie bialgebra
First, we introduce a Lie algebra. We define a Lie algebra $\mathfrak{g}$ on a field $K$ as a set of a vector space $V$ and a Lie bracket $[\cdot, \cdot]$. The Lie bracket $[\cdot, \cdot] : V \times V \to V$ satisfies the Jacobi identity. We can define the dual Lie bracket $[\cdot, \cdot]^*$ and the dual Lie algebra $\mathfrak{g}^*$ by considering the dual vector space $V^*$. The inner product $(\cdot, \cdot)$ is defined between $\mathfrak{g}$ and $\mathfrak{g}^*$ naturally. $(\cdot, \cdot)$ take the value on $K$.

Here, we can introduce a $p$-th exterior product of $\mathfrak{g}$: $\wedge^p \mathfrak{g} = \mathfrak{g} \wedge \mathfrak{g} \wedge \cdots \wedge \mathfrak{g}$. As an analogue of an exterior derivative $d$ in the cotangent bundle on a manifold, we can define the exterior derivative $d : \wedge^p \mathfrak{g}^* \to \wedge^{p+1} \mathfrak{g}^*$, which satisfies $d^2 = 0$. Similarly, we can define an exterior derivative $d^* : \wedge^p \mathfrak{g} \to \wedge^{p+1} \mathfrak{g}$. The Lie bracket can be extended to the case where the argument is $\wedge^p \mathfrak{g}$. This bracket is called the Schouten-Nijenhuis bracket and satisfies the following properties.

(i) $[a, b]_S = -(\cdot)^{(p-1)q-1}[b, a]_S$.
(ii) $[a, b \wedge c]_S = [a, b]_S \wedge c + (\cdot)^{(p-1)q}b \wedge [a, c]_S$.
(iii) $(-)^{(p-1)q-1}([a, b, c]_S + \cdot (\cdot)^{(q-1)p-1}b, [c, a]_S) + (\cdot)^{(q-1)p-1}c, [a, b]_S) = 0$.
(iv) The bracket of an element $\wedge^p \mathfrak{g}$ and an element in $\wedge^0 \mathfrak{g} = K$ is 0.

Where $a \in \wedge^p \mathfrak{g}$, $b \in \wedge^q \mathfrak{g}$, $c \in \wedge^r \mathfrak{g}$. The Schouten-Nijenhuis bracket defines a Gerstenhaber algebra.

Next, we examine the algebraic structure between $\mathfrak{g}$ and $\mathfrak{g}^*$. If we consider the Lie bracket $[\cdot, \cdot]$ as a bilinear map $\mu : \wedge^2 \mathfrak{g} \to \mathfrak{g}$, we can define the co-bracket $\delta : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ similarly. A dual Lie bracket can be defined by $\mu^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$. This $\mu^*$ is an adjoint of $\delta$ which means the relation $(x, \mu^* \xi) = (\mu^* x, \xi)$ holds. Then, $\mu^*$ corresponds to the co-bracket. The co-bracket satisfies the co-Jacobi identity. This is equivalent to the Jacobi identity for the dual Lie bracket $[\cdot, \cdot]^*$. Even more, the co-bracket satisfies the following 1-cocycle condition:

$$\delta([x, y]) = \text{ad}_x^2 \delta(y) - \text{ad}_y^2 \delta(x), \quad x, y \in \mathfrak{g}.$$ (1)

Then, the set $(\mathfrak{g}, \mu, \delta)$ is called Lie bialgebra. The same Lie bialgebra can be defined as $(\mathfrak{g}^*, \mu^*, \delta^*)$.

3. From Lie algebra to Lie algebroid
A Lie algebroid is a structure defined as a generalization of Lie algebras, this is defined by a vector bundle $E$ on the manifold $M$, an anchor map $\rho : E \to TM$ and a Lie algebroid bracket $[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying the Jacobi identity. The following two expressions are required for $\rho$ and $[\cdot, \cdot]_E$:

$$[X, fY]_E = (\rho(X) \cdot f)Y + f[X, Y]_E, \quad (2)$$

$$\rho([X, Y]_E) = [\rho(X), \rho(Y)]. \quad (3)$$

Here, $f \in C^\infty(M)$, and $[\cdot, \cdot]$ is the Lie bracket on $\Gamma(TM)$. (2) is the Leibniz rule of the Lie algebroid bracket. (3) is the homomorphism of $\rho$ with respect to $[\cdot, \cdot]_E$. Considering dual vector bundle $E^*$ for $E$, a dual Lie algebroid $(E^*, \rho^*, [\cdot, \cdot]_E^*)$ can also be defined on the same manifold $M$. The inner product $(\cdot, \cdot)$ can be defined naturally between $E$ and $E^*$.
We can define the multi-vectors $\Gamma(\wedge^n E)$ and the multi-forms $\Gamma(\wedge^n E^*)$. We define the exterior derivative $d$ for $\Gamma(\wedge^n E^*)$. The exterior derivative $d$ is defined as follows [10]:

$$
\begin{align*}
\frac{d\xi}{\partial x_i} &= \sum_{i=1}^{p+1} (-)^{i+1} \rho(X_i) \cdot (\xi(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})) \\
&+ \sum_{i<j} (-)^{i+j} \xi([X_i, X_j]_E, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),
\end{align*}
$$

where $\xi \in \Gamma(\wedge^p E^*)$, $X_i \in \Gamma(E)$. The notation $\hat{X}_i$ stands for that the term is omitted in the expression. Similarly, a Lie derivative $\mathcal{L}_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^p E^*)$ by $X \in \Gamma(E)$ is defined by

$$
\mathcal{L}_X(\xi)(Y_1, \ldots, Y_p) = \rho(X) \cdot (\xi(Y_1, \ldots, Y_p)) - \sum_{i=1}^{p} \xi(Y_1, \ldots, [X, Y_i], \ldots, Y_p).$$

where $Y_1, \ldots, Y_p \in \Gamma(E)$, $\xi \in \Gamma(\wedge^p E^*)$. The interior product $\iota_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-1} E^*)$ by $X \in \Gamma(E)$ is defined by

$$
\iota_X(\xi(Y_1, \ldots, Y_{p-1})) = \xi(X, Y_1, \ldots, Y_{p-1}),$$

where $Y_1, \ldots, Y_{p-1} \in \Gamma(E)$, $\xi \in \Gamma(\wedge^p E^*)$. We also define the exterior derivative $d_*$, Lie derivative and the interior product $\iota_*$ for $\Gamma(\wedge^* E)$.

For $X \in \Gamma(\wedge^{p+1} E), Y \in \Gamma(\wedge^{p+1} E)$, the Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\mathcal{S}}$ is defined as an extension of the Lie algebroid bracket to multi vectors (or forms). $[\cdot, \cdot]_{\mathcal{S}}$ has the following properties.

(i) $[X, Y]_{\mathcal{S}} = -(-)^{p|Y}[Y, X]_{\mathcal{S}}$.
(ii) $[X, f]_{\mathcal{S}} = \rho(X) \cdot f$ for $X \in \Gamma(E)$.
(iii) For $X \in \Gamma(\wedge^{p+1} E)$, the bracket $[X, \cdot]_{\mathcal{S}}$ acts on $\Gamma(\wedge^q E)$ as a degree-$p$ derivation.

If the derivation condition

$$
d_*[X, Y]_{\mathcal{S}} = [d_* X, Y]_{\mathcal{S}} + [X, d_* Y]_{\mathcal{S}}
$$

is satisfied between these dual Lie algebroids, this Lie algebroid pair becomes a Lie bialgebroid [10]. A Lie bialgebroid is a generalization of Lie bialgebras. The derivation condition corresponds to the 1-cocycle condition (1) of Lie bialgebra. It is known that we obtain a Courant algebroid structure by the Drinfel’d double of a Lie bialgebroid [6].

4. Doubled structure of Vaisman algebroid

A Vaisman algebroid is defined by a vector bundle $\mathcal{V}$ on a manifold $M$, an anchor map $\rho : \mathcal{V} \to TM$, a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and a Vaisman bracket $[\cdot, \cdot]_{\mathcal{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$. The Vaisman algebroid $(\mathcal{V}, \rho, (\cdot, \cdot), [\cdot, \cdot]_{\mathcal{V}})$ satisfies the following two axioms.

**Axiom V1.** $[e_1, f e_2]_{\mathcal{V}} = f [e_1, e_2]_{\mathcal{V}} + (\rho(e_1) \cdot f)e_2 - (e_1, e_2)Df$.

**Axiom V2.** $\rho(e_1) \cdot (e_2, e_3) = ([e_1, e_2]_{\mathcal{V}} + D(e_1, e_2), e_3) + (e_2, [e_1, e_3]_{\mathcal{V}} + D(e_1, e_3))$.

Here $e_1, e_2, e_3 \in \Gamma(\mathcal{V})$ and $D$ is a differential operator over $f \in C^\infty(M)$.

In our paper, we have shown that a Vaisman algebroid can be obtained by a double of a pair of Lie algebroids. In the following, only the brief flow of proofs is introduced. The details can be found in [7].
First, we show that the Axioms V1 and V2 are satisfied only by the properties of a Lie algebroid. In the following, we consider \( \mathcal{V} = E \oplus E^* \) and \( X \in \Gamma(E), \xi \in \Gamma(E^*) \). The Vaisman bracket is defined by

\[
[e_1, e_2]_V = [X_1, X_2]_E + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - d_*(e_1, e_2)_- + [\xi_1, \xi_2]_{E^*} + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + d(e_1, e_2)_- ,
\]

(8)

where

\[
(e_1, e_2)_\pm = \frac{1}{2} (\langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle).
\]

We employ \((\cdot, \cdot)_+\) as the non-degenerate symmetric bilinear form \((\cdot, \cdot)\).

Axiom V1 is the Leibniz rule for the Vaisman bracket. If we calculate the left hand side of Axiom V1, we obtain

\[
[e_1, e_2]_V = [X_1, fX_2]_V + [X_1, f\xi_2]_V + [\xi_1, fX_2]_V + [\xi_1, f\xi_2]_V,
\]

(10)

Terms in the right hand side of (10) becomes

\[
[X_1, f\xi_2]_V = f[X_1, \xi_2]_V + (\rho(X_1) \cdot f)\xi_2 - \frac{1}{2} \mathcal{D}g(\xi_2, X_1),
\]

\[
[\xi_1, fX_2]_V = f[\xi_1, X_2]_V + (\rho_*(\xi_1) \cdot f)X_2 - \frac{1}{2} \mathcal{D}f(\xi_1, X_2),
\]

\[
[X_1, fX_2]_V = f[X_1, X_2]_V + (\rho(X_1) \cdot f)X_2,
\]

\[
[\xi_1, f\xi_2]_V = f[\xi_1, \xi_2]_V + (\rho_*(\xi_1) \cdot f)\xi_2.
\]

(11)

If we add all the terms in the right hand side of (10) by using (11), we obtain

\[
[e_1, f e_2]_V = f[e_1, e_2]_V + (\rho(e_1) \cdot f)e_2 - (e_1, e_2)_+ \mathcal{D}f.
\]

(12)

Therefore, Axiom V1 is satisfied.

Axiom V2 is the compatibility condition between \((\cdot, \cdot)\) and \(\mathcal{D}\). Due to the property of \((\cdot, \cdot)\), the left hand side of Axiom V2 is written as follows:

\[
\rho_V(e) \cdot (e_1, e_2)_+ = ([e, e_1]_V, e_2)_+ + (e_1, [e, e_2]_V)_+ + \frac{1}{2} \rho_V(e_1) \cdot (e, e_2)_+ + \frac{1}{2} \rho_V(e_2) \cdot (e, e_1)_+.
\]

(13)

If we use the following relation

\[
\frac{1}{2} \rho_V(e_1) \cdot (e, e_2)_+ = (\mathcal{D}(e, e_2), e_1)_+ ,
\]

\[
\frac{1}{2} \rho_V(e_2) \cdot (e, e_1)_+ = (\mathcal{D}(e, e_1), e_2)_+ ,
\]

then, we obtain the following result:

\[
\rho(e_1) \cdot (e_2, e_3)_+ = ([e_1, e_2]_V + \mathcal{D}(e_1, e_2)_+, e_3)_+ + (e_2, [e_1, e_3]_V + \mathcal{D}(e_1, e_3)_+) .
\]

(14)

Therefore Axiom V2 holds.

The structures of the algebroid are different from those in a Courant algebroid which appears in the generalized geometry. This is because the three axioms that Courant algebroids should
satisfy are not included in this algebroid (Axiom C1, C2, C4 in our paper [7]). Axiom C1 is a deformed Jacobi identity. The Jacobiator is calculated as follows:

\[
[[e_1, e_2]_V, e_3]_V + \text{c.p.} = DT(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p.})
\]

where

\[
T(e_1, e_2, e_3) = \frac{1}{3} \left( ([e_1, e_2]_V, e_3)_+ + \text{c.p.} \right),
\]

\[
J_1 = \iota_{\xi_1_1}(\mathcal{L}_{\xi_1_1}^d d(\xi_2) _V - \mathcal{L}_{\xi_2} d(\xi_1) _V + \iota_{\xi_2_1}(\mathcal{L}_{\xi_2}^d d(\xi_1) _V - \mathcal{L}_{\xi_1} d(\xi_2) _V + \mathcal{L}_{\xi_1}^d d(\xi_2) _V - \mathcal{L}_{\xi_2} d(\xi_1) _V),
\]

\[
J_2 = \left( \mathcal{L}_{d^a(e_1, e_2)_-} \xi_3 + [d(e_1, e_2)_-, \xi_3]_V - \left( \mathcal{L}_{d^a(e_1, e_2)_-} X_3 + [d(e_1, e_2)_-, X_3]_V \right) \right). 
\]

Since the term \((J_1 + J_2 + \text{c.p.})\) remains, Axiom C1 does not hold under the current conditions. Here, c.p. is the cyclic permutation.

Axiom C2 means the homomorphism of \(\rho\) with respect to \([\cdot, \cdot]_V\), this axiom corresponds to the relation (3). By subtracting the right hand side from the left hand side of (3) for the Vaisman bracket \([\cdot, \cdot]_V\), we obtain

\[
\rho([e_1, e_2]_V) \cdot f - [\rho(e_1), \rho(e_2)] f
= -\langle \xi_1, (\mathcal{L}_{d^a} X_2 - [X_2, d^a f]_E) \rangle + \langle \xi_2, (\mathcal{L}_{d^a} X_1 - [X_1, d^a f]_E) \rangle + \frac{1}{2} (\rho \rho^* + \rho^* \rho) d_0 (\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle) f,
\]

where \(d_0\) is the usual exterior derivative for \(\Gamma(T^* M)\). We have used the relation, \(d = \rho^* d_0, d^a = \rho^* d_0\). Axiom C2 holds when this right hand side in (19) vanishes. However, the result is generically non-zero.

If we calculate the left hand side of Axiom C4, we obtain

\[
(D f, D g)_+ = \frac{1}{2} (\rho \rho^* + \rho^* \rho) (d_0 f) g.
\]

Axiom C4 holds when the right hand side vanished. However, the result is generically non-zero.

If a derivation condition (7) is imposed between the dual Lie algebroids and the condition of Lie bialgebroid holds, the above Axioms C1, C2, and C4 are satisfied. Therefore, imposing the derivation condition to the Vaisman algebroid constructed in this way gives a Courant algebroid of [6] (see figure 1). Conversely, we can say that a Vaisman algebroid can be composed of two Lie algebroids, not of Lie bialgebroids. According to the DFT context, this derivation condition corresponds to the strong constraint [7].
5. Conclusion and discussions
In this proceeding, we consider the Vaisman algebroid as the gauge symmetry algebra in DFT. We have shown that the Vaisman algebroid can be obtained by the double of Lie algebroids. From the physical viewpoint, the double structure of the gauge symmetry in DFT originates from the symmetry between the Kaluza-Klein modes and winding modes of strings. Mathematically, it was understood as an analogue of the Drinfel’d double for Lie algebroids. In our paper, we discuss the Vaisman algebroid based on a doubled structure which appears naturally on a para-Hermitian manifold. Based on $\Gamma(E)$ and $\Gamma(E^*)$ in section 2 in this proceeding, a concrete construction of the exterior algebras in DFT was also performed in our paper.

The algebroid structures discussed in this proceeding provide the local structure in the para-Hermitian manifold. The problem of finding the corresponding groupoid structure describing the global nature of the symmetry is called coquecigrue problem (first appearance in [11]). Examining the groupoid structure corresponding for the Vaisman algebroid in DFT is an interesting problem [12].

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