Prismatic cohomology and $p$-adic homotopy theory

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Abstract
Historically, it was known by the work of Artin and Mazur that the $\ell$-adic homotopy type of a smooth complex variety with good reduction mod $p$ can be recovered from the reduction mod $p$, where $\ell$ is not $p$. This short note removes this last constraint, with an observation about the recent theory of prismatic cohomology developed by Bhatt and Scholze. In particular, by applying a functor of Mandell, we see that the étale comparison theorem in the prismatic theory reproduces the $p$-adic homotopy type for a smooth complex variety with good reduction mod $p$.

Keywords Prismatic cohomology · $p$-adic homotopy theory · $E$-infinity algebras

Introduction
The classical work of Artin and Mazur [2] shows that, for a smooth, proper complex variety $X$ with good reduction mod $p$, the mod $p$ reduction recovers the $\ell$-adic homotopy type of $X$, where $\ell$ is a prime not equal to $p$. Artin and Mazur prove this by constructing a notion of an étale homotopy type using a Čech-like construction with étale coverings, and showing that it is a pro-finite homotopy type.

The aim of this paper is to show that the constraint on the prime $\ell$ can be removed, and that the $p$-adic homotopy type is accessible from $p$-adic information. This is accomplished by utilizing a theorem of Mandell, which says that the $p$-adic homotopy type of a space is obtained from the $E_\infty$-algebra structure of its mod $p$ singular cochains, in conjunction with the étale comparison theorem from the prismatic cohomology theory of Bhatt and Scholze [7]. The étale comparison theorem does not use the reduction mod $p$ alone, but an integral model over a suitable $p$-adically complete ring of integers.

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Prismatic cohomology is defined with reference to a base prism \((A, I)\), such that the resulting cohomology groups are modules over \(A\) (see Sect. 2.2 for preliminaries for prismatic cohomology). The notation \(R\Gamma(X, \Delta_X/A)\) will denote the cochains of the prismatic cohomology of \(X\) over the base prism \((A, I)\). In particular, this paper will fix one particular prism for its setting. First, let \(\mathbb{C}_p\) denote the \(p\)-completion of an algebraic closure of \(\mathbb{Q}_p\) and let \(\mathcal{O}_{\mathbb{C}_p}\) denote its ring of integers. We can then take the tilt, \(\mathcal{O}_{\mathbb{C}_p}^\flat\), which is defined as the inverse limit of \((\phi \to \mathcal{O}_{\mathbb{C}_p}/p\phi \to \mathcal{O}_{\mathbb{C}_p}/p)\) where \(\phi\) denotes the Frobenius. Then we can take the Witt vectors of the tilt to obtain our \(\delta\)-ring \(A = A_{\text{inf}} := W(\mathcal{O}_{\mathbb{C}_p}^\flat)\). This ring has a natural map \(\theta: A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}\) and our prism is then the pair \((A_{\text{inf}}, \ker(\theta))\).

The work of Bhatt, Morrow, and Scholze in [5] yields the following corollary of the étale comparison theorem in [4] (Lecture IX, Theorem 5.1).

**Theorem 0.1** (Bhatt-Morrow-Scholze) Let \(\mathcal{X}\) denote a smooth, proper scheme over \(\mathcal{O}_{\mathbb{C}_p}\) and let \(X\) denote its fiber over \(\mathbb{C}_p\). Then there is a quasi-isomorphism

\[
(R\Gamma(\mathcal{X}, \Delta_{\mathcal{X}/A_{\text{inf}}}) \otimes^L_{A_{\text{inf}}} \mathcal{C}_p^\phi)^{\phi = 1} \simeq R\Gamma_\text{ét}(X, \mathbb{F}_p)
\]

where the notation \((-)^{\phi = 1}\) denotes the the homotopy fixed points of the lift of Frobenius \(\phi\) on the prismatic complex over the prism \((A_{\text{inf}}, \ker(A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}))\) (i.e., the mapping co-cone of the chain map \(\phi - \text{Id}\) in the derived category (so \(\text{Cone}(\phi - \text{Id})[-1]\)).

In particular, identify \(\mathbb{C}_p\) with \(\mathbb{C}\) and let \(X \subset \mathbb{P}^n\) be a smooth projective variety over \(\mathbb{C}\) that arises as the generic fiber of a smooth, proper scheme \(X_{\mathbb{Z}}\) over \(\mathbb{Z}\), e.g., one whose defining homogeneous polynomials have coefficients in \(\mathbb{Z}\) and whose Jacobian matrix has rank \(n + 1 - \dim(X)\) modulo \(p\). Then the above quasi-isomorphism holds after taking \(\mathcal{X}\) as the fiber of \(X_{\mathbb{Z}}\) over \(\mathcal{O}_{\mathbb{C}_p}\).

In fact, the proof of the above theorem proves more: there are natural \(E_\infty\)-\(\mathbb{F}_p\)-algebra structures on the cochains above, such that the quasi-isomorphism is a map of \(E_\infty\)-\(\mathbb{F}_p\)-algebras. The natural \(E_\infty\)-\(\mathbb{F}_p\)-algebra structures on the cochains arise via Godement resolutions. The theorem above then states one can recover the full \(E_\infty\)-algebra structure on the étale \(\mathbb{F}_p\)-cochains of a smooth proper complex variety with good reduction mod \(p\), by natural constructions applied to the prismatic complex over the prism \((A_{\text{inf}}, \ker(\theta))\). By suggestions of the referee, we note that although Bhatt, Morrow, and Scholze state their theorem in full generality for formal schemes \(\mathcal{X}\), for our purposes it suffices to weaken the hypothesis to smooth and proper schemes; the statement of the theorem does not change otherwise.

On the other hand, we have the following theorem of Mandell [16] (see also [16], Remark 5.1).

**Theorem 0.2** (Mandell) Let \(\mathcal{S}\) denote the homotopy category of connected \(p\)-complete nilpotent spaces of finite \(p\)-type, and let \(h\mathcal{E}\) denote the homotopy category of \(E_\infty\)-\(\mathbb{F}_p\)-algebras. The singular cochain functor \(C_{\text{sing}}^*(-, \mathbb{F}_p)\) induces a contravariant equivalence from \(\mathcal{S}\) to a full subcategory of \(h\mathcal{E}\). The quasi-inverse on the subcategory is given by \(\mathcal{U}\), the right derived functor of the functor \(A \mapsto \text{Hom}_\mathcal{E}(A, C^*(\Delta[\bullet], \mathbb{F}_p))\).
where $\Delta[\bullet]$ denotes the standard simplex as a simplicial set. Moreover, there is the following adjunction

$$[X, \mathbb{U}A] \cong [A, C^*_\text{sing}(X, \mathbb{F}_p)]$$

where $[-, -]$ denotes morphisms in the respective homotopy category. Moreover, for $X$ connected and of finite $p$-type, the natural map $X \to \mathbb{U}C^*_\text{sing}(X, \mathbb{F}_p)$ via the adjunction is naturally isomorphic to $p$-completion in the sense of Bousfield-Kan.

There is also the analog of the above theorem ([16], Proposition A.2, A.3) for the singular cochain functor with coefficients in $\mathbb{F}_p$.

**Theorem 0.3** (Mandell) Let $hE$ denote the homotopy category of $E_\infty$-$\mathbb{F}_p$-algebras. The right derived functor $\mathbb{U}$ of the functor $A \mapsto \text{Hom}_E(A, C^*(\Delta[\bullet], \mathbb{F}_p))$ from the category of $E_\infty$-$\mathbb{F}_p$-algebras to simplicial sets is right adjoint to the right derived functor of the singular cochain functor $C^*_\text{sing}(-, \mathbb{F}_p)$, such that there is a natural isomorphism $LX \to \mathbb{U}C^*_\text{sing}(X, \mathbb{F}_p)$ in the homotopy category for $X$ connected, $p$-complete, nilpotent, and of finite $p$-type, where $LX$ denotes the free loop space of $X$.

In other words, taking coefficients in $\mathbb{F}_p$ recovers the free loop space of the $p$-adic homotopy type. An immediate consequence of the theorems above is the following main theorem of the paper:

**Theorem 0.4** Let $X$ be a smooth projective variety over $\mathbb{C}$ that arises as the generic fiber of a smooth proper scheme $X_\mathbb{Z}$ over $\mathbb{Z}$. Assume further that $X$ is nilpotent. Let $\mathcal{X}$ denote the fiber of $X_\mathbb{Z}$ over $\mathcal{O}_{\mathbb{C}_p}$. Then $\mathbb{U}(R\Gamma(\mathcal{X}, \Delta\mathcal{X}/\mathcal{A}_\text{inf}) \otimes_{\mathcal{A}_\text{inf}}^L \mathbb{C}_p^0)^{\phi=1}$ is the free loop space of the $p$-completion of the complex variety $X$. Similarly, $\mathbb{U}(R\Gamma(\mathcal{X}, \Delta\mathcal{X}/\mathcal{A}_\text{inf}) \otimes_{\mathcal{A}_\text{inf}}^L \mathbb{C}_p^0)^{\phi=1} \otimes_{\mathbb{F}_p}^L \mathbb{F}_p^p)$ is the $p$-completion of the complex variety $X$.

The original statement of the theorem in an earlier version of this paper involved the free loop space of the Bousfield-Kan $p$-completion of the complex variety $X$, and the Sullivan $p$-completion of the complex variety $X$ respectively. We would like to thank the referee for kindly reminding the author that all spaces of interest in this paper are of finite $p$-type, and so the natural map of $E_\infty$-$\mathbb{F}_p$-algebras $C^*_\text{sing}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^L \mathbb{F}_p \to C^*_\text{sing}(X, \mathbb{F}_p)$ is an equivalence. This implies for nilpotent spaces that the Bousfield-Kan $p$-completion and the Sullivan $p$-completion coincide (see [8], [25]). Thus we will freely use the term $p$-completion without ambiguity.

The main theorem then states, for a smooth variety over $\mathbb{C}$ with good reduction mod $p$, natural constructions applied to its prismatic complex over $\mathcal{A}_\text{inf}$ yield its $p$-adic homotopy type. We also wish to state that the theorem requires a choice of identification of $\mathbb{C}$ with $\mathbb{C}_p$, but that the result is independent of the choice. See the remark after Corollary 2.1.

The outline of the paper is as follows: in Sect. 1 we show that the classical Artin comparison theorem refines to an equivalence of $E_\infty$-algebras. In Sect. 2 we discuss...
the proof of the étale comparison theorem of Bhatt–Morrow–Scholze to see the equivalence of $E_{\infty}$-algebras between prismatic cohomology and étale cohomology. We then conclude by applying Mandell’s theorem.

**Notation and conventions.** All rings are commutative with unit. All varieties over $\mathbb{C}$ are assumed connected in the analytic topology. We use $k$ to denote the ground field, usually $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$. All $E_{\infty}$-$k$-algebras are chain complexes of $k$-modules with an action of a fixed $E_{\infty}$-operad $\mathcal{E}_k$ in $\text{Ch}(k\text{-mod})$, where $\mathcal{E}_k$ has a fixed map of operads $\mathcal{E}_k \to Z_k$ to the Eilenberg-Zilber operad $Z_k$. We refer the proof that such a map of operads always exists to [15]. Often we will omit the subscript $k$ if it is clear in context.

By a quasi-isomorphism of $E_{\infty}$-algebras, we mean a morphism of $E_{\infty}$-algebras that induces a quasi-isomorphism on the underlying chain complexes. A complex variety with good reduction mod $p$ is a variety that admits an integral model over $\text{Spec}(\mathbb{Z})$ such that the reduction mod $p$ is regular.

**1 Étale cochains and $E_{\infty}$-Artin comparison**

In this mainly expository section, we review some classical theorems regarding the $E_{\infty}$-algebra structure on étale cochains. We emphasize there are no original theorems proven in this section; many of the arguments can be found in, for example [9, 19, 20], with the main conceptual ideas originating from Godement [10]. The aim of this section is to cover the following well known theorem.

**Theorem 1.1** Let $X$ be a smooth complex variety. There is a quasi-isomorphism of $E_{\infty}$-algebras between the singular $\mathbb{F}_p$-cochains $C^*_\text{sing}(X(\mathbb{C}), \mathbb{F}_p)$ and the étale $\mathbb{F}_p$-cochains $R\Gamma_{\text{et}}(X, \mathbb{F}_p)$.

Here is an outline of the section: first, we compare the étale site of $X$ over $\mathbb{C}$ with the site of analytic open sets on its underlying complex manifold $X(\mathbb{C})$, by passing to the site of local homeomorphisms mapping to $X(\mathbb{C})$. We analyze the site of local homeomorphisms and show it has enough points. We then use the Godement resolution on all three sites to obtain $E_{\infty}$-algebras on their sheaf cohomologies; the quasi-isomorphism of the underlying complexes is omitted.

We have the following theorem from SGA IV (XII-4) [1]; we provide a translation of part of the proof, with some details provided using lemmas from the Stacks Project [24].
Theorem 1.2 Let $X$ be a smooth complex variety. There is a zig-zag of sites

$$
\begin{array}{c}
\delta & \leftarrow & X_{\text{cl}} & \xrightarrow{\varepsilon} & X_{\text{ét}}
\end{array}
$$

where $X(\mathbb{C})$ denotes the site of analytic open sets on the underlying complex manifold of $X$, $X_{\text{cl}}$ denotes the site of local homeomorphisms $U \to X(\mathbb{C})$, and $X_{\text{ét}}$ denotes the étale site of $X$.

**Proof** An object of $X_{\text{cl}}$ is a continuous map of topological spaces $f : U \to X(\mathbb{C})$ such that for every point $x \in U$, there is a neighborhood $U_x$ such that the restriction of $f$ to $U_x$ is a homeomorphism onto an open neighborhood around $f(x)$; that is, $f$ is a local homeomorphism. Since inclusions of open sets in $X(\mathbb{C})$ are local homeomorphisms, we obtain a morphism of sites $\delta : X_{\text{cl}} \to X(\mathbb{C})$ by the continuous functor $U \mapsto (U \hookrightarrow X(\mathbb{C}))$; this continuous functor is the inclusion of categories $X(\mathbb{C}) \subset X_{\text{cl}}$.

On the other hand, let $f : X' \to X$ be étale. Then the induced map on the underlying smooth manifolds $f(\mathbb{C}) : X'(\mathbb{C}) \to X(\mathbb{C})$ is a local isomorphism, by the Jacobian criterion and implicit function theorem. The functor $X' \mapsto X'(\mathbb{C})$ then induces the morphism of sites $\varepsilon : X_{\text{cl}} \to X_{\text{ét}}$.

Lemma 1.1 The functor $\delta_*$ sends surjective maps of sheaves of sets to surjective maps of sheaves of sets. Moreover, it is an equivalence of the associated topoi, and reflects injections and surjections (i.e., $\delta_* f$ is an injection (resp. surjection) implies $f$ is an injection (resp. surjection)).

**Proof** For each local homeomorphism $f : U \to X(\mathbb{C})$, for each $x \in U$, there is a neighborhood homeomorphic to an open neighborhood around $f(x)$ in $X(\mathbb{C})$. Thus, we can cover $U$ by open sets that are homeomorphic to open sets in $X(\mathbb{C})$; that is, there exists a family of open sets $\{U_i \hookrightarrow X(\mathbb{C})\}$ such that we have a commutative diagram of local homeomorphisms

$$
\begin{array}{c}
U_i & \leftarrow & U \\
& \searrow & \downarrow \\
& & X(\mathbb{C})
\end{array}
$$

and where $U$ is the union of the images of the maps from $U_i$. The above geometric argument immediately implies that the hypotheses of ([24], Tag 04D5, Lemma 7.41.2) are satisfied, and so $\delta_*$ sends surjective maps of sheaves to surjective maps of sheaves. Similarly, the hypotheses of ([24], Tag 04D5, Lemma 7.41.4) are also satisfied, so $\delta_*$ reflects injections and surjections.

Lastly, to show $\delta_*$ is an equivalence of topoi, notice that the inclusion functor $X(\mathbb{C}) \hookrightarrow X_{\text{cl}}$ is cocontinuous, and that the hypotheses of ([24], Tag 039Z, Lemma 7.29.1) are likewise satisfied by the above geometric argument. The morphism of topoi $g : \text{Sh}(X(\mathbb{C})) \to \text{Sh}(X_{\text{cl}})$ associated to the inclusion as a cocontinuous functor is then an equivalence, with the adjunction mappings $g^{-1}g_* \mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to g_*g^{-1}\mathcal{G}$ being isomorphisms. It follows from the definition of induced morphism of topoi from a cocontinuous functor that $\delta_* = g^{-1}$; in fact, since $g^{-1}$ is left adjoint to $g_*$, this shows $\delta_*$ is right exact as well. \qed
Remark The functor $\delta^{-1}$ is not the same as the induced functor $g_*$ in the argument above; the map of topoi $(\delta^{-1}, \delta_*)$ is induced from the inclusion as a continuous functor, whereas the map of topoi $(g^{-1}, g_*)$ is induced from the inclusion as a cocontinuous functor.

The above theorem and lemma say that one can replace $X(\mathbb{C})$ with $X_{\text{cl}}$ for calculation of usual sheaf cohomology. Moreover there is the following property of the morphism $\varepsilon$, again explained in SGA IV (XII-4).

**Theorem 1.3** Let $X$ be smooth over $\mathbb{C}$. There is an equivalence of categories given by the quasi-inverse functors $\varepsilon_*$ and $\varepsilon^*$ between the category of locally constant constructible torsion sheaves on $X_{\text{et}}$, and the category of locally constant finite fiber torsion sheaves on $X_{\text{cl}}$.

The proof of the above theorem is what ultimately gives the desired quasi-isomorphism between sheaf cohomologies

$$R\Gamma(X_{\text{et}}, \mathbb{F}_p) \sim R\Gamma(X_{\text{cl}}, \mathbb{F}_p) \sim R\Gamma(X(\mathbb{C}), \mathbb{F}_p)$$

where the first isomorphism is from $\varepsilon_*$ in the theorem above, and the second is by $\delta_*$ from Theorem 1.2. The quasi-isomorphism is really the difficult part of Theorem 1.1; the rest of this section is to show that the above zig-zag of maps are maps of $E^\infty_{\mathbb{F}_p}$-algebras. First, we show how we obtain induced maps on cohomology from morphisms of sites:

**Lemma 1.2** Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of sites. The following diagram commutes:

\[
\begin{array}{ccc}
D\text{AbSh}(\mathcal{C}) & \xrightarrow{Rf_*} & D\text{AbSh}(\mathcal{C}') \\
\downarrow R\Gamma & & \downarrow R\Gamma \\
D\text{Ab} & & D\text{Ab}
\end{array}
\]

In particular, we have $R\Gamma(\mathcal{C}', Rf_*\mathcal{F}) \simeq R\Gamma(\mathcal{C}, \mathcal{F})$ for any $\mathcal{F}$ in $\text{Sh}(\mathcal{C})$.

**Proof** This follows from the following commutative diagram:

\[
\begin{array}{ccc}
\text{AbSh}(\mathcal{C}) & \xrightarrow{f_*} & \text{AbSh}(\mathcal{C}') \\
\downarrow \Gamma & & \downarrow \Gamma \\
\text{Ab} & & \text{Ab}
\end{array}
\]

We have that $f_*$ is right adjoint to an exact functor $f^{-1}$. Hence, we have the following string of natural isomorphisms:

\[
\Gamma(\mathcal{C}', f_*\mathcal{F}) := \text{Hom}_{\text{AbSh}(\mathcal{C}')}(\mathbb{Z}_{\mathcal{C}'}, f_*\mathcal{F}) \\
\cong \text{Hom}_{\text{AbSh}(\mathcal{C})}(f^{-1}(\mathbb{Z}_{\mathcal{C}'}, \mathcal{F}) \\
\cong \text{Hom}_{\text{AbSh}(\mathcal{C})}(\mathbb{Z}_{\mathcal{C}'}, \mathcal{F}) \\
= : \Gamma(\mathcal{C}, \mathcal{F}).
\]
Here, \( \mathbb{Z} \) denotes the constant sheaf that takes values in \( \mathbb{Z} \): the category \( \text{AbSh}(C) \) has a unique morphism of topoi \( (p_*, p^{-1}) \) to \( \text{Ab} \), where \( \mathbb{Z}_C = p^{-1}\mathbb{Z} \). The second isomorphism holds by uniqueness of this morphism (see SGA IV (IV-4.3) [1]). All functors in the above diagram are left exact but not right exact. Since \( f_* \) preserves injective objects ([24], Tag 015Z), it follows by ([24], Tag 015L, Lemma 13.22.1) that the natural morphism \( R(\Gamma \circ f_*) \to R\Gamma \circ Rf_* \) is an isomorphism. \( \square \)

### 1.1 Godement resolutions

Let us briefly recall the Godement construction [10], considered in a general categorical context. Here, we use the language of Rodríguez-González and Roig [21]. The Godement construction will be where all of our \( E_{\infty} \)-algebra structures arise, using a theorem of Hinich and Schechtman.

**Definition 1.1** Let \( C \) be a site. A **point of the site** \( C \) is a pair of adjoint functors \( x = (x^*, x_*) \)

\[
\begin{array}{ccc}
\text{Sh}(C) & \xrightarrow{x^*} & \text{Set} \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Set}}(x^*F, E) & \cong & \text{Hom}_{\text{Sh}(C)}(F, x_*E)
\end{array}
\]

where \( x^* \) commutes with finite limits. The site \( C \) has enough points if there exists a set \( S \) of points \( x \) such that a morphism \( f \) in \( \text{Sh}(C) \) is an isomorphism if and only if \( x^*f \) is a bijection for all \( x \in S \).

The right adjoint \( x_* \) assigns to each set \( E \) the **skyscraper sheaf** \( x_*E \) at the point \( x \). The left adjoint \( x^* \) assigns to each sheaf \( F \) the **stalk** \( F_x := x^*F \) at the point \( x \).

**Example 1.1** Consider the site \( X(\mathbb{C}) \) of (analytic) open subsets of the complex manifold \( X(\mathbb{C}) \). Take \( S \) to be the underlying set of \( X(\mathbb{C}) \). The geometric points \( x \in S \) determine points of the site by taking \( x^* \) to be the stalk functor and \( x_* \) to be the skyscraper sheaf functor. Moreover, a map of sheaves is an isomorphism if and only if it is so on the stalks. Thus the site \( X(\mathbb{C}) \) has enough points.

**Example 1.2** It is a classical fact that the étale site \( X_{\text{ét}} \) has enough points; this follows from, for example, Deligne’s criterion ([1], VI Appendix, Proposition 9.0). Isomorphisms of sheaves on the étale site can be checked stalk-wise at geometric points \( \overline{x} \), where a geometric point \( \overline{x} \) (for a given point \( x \) of the underlying topological space \( X \)) is the spectrum of a separably closed field \( k(\overline{x}) \) containing \( k(x) \), with the map \( \overline{x} \to X \) induced by the inclusion of the residue fields. It suffices to choose, for each point \( x \) of the underlying topological space \( X \), a separably closed field that contains the residue field of \( x \) (see the lecture notes of Milne [18], by the argument of Lemma 7.4 therein and Remark 7.7). This gives the set of points for the étale site. We caution that the collection of all geometric points does not form a set, but rather a proper class.

**Lemma 1.3** The site \( X_{\text{ét}} \) has enough points.
Proof Recall we have an equivalence of topoi, with the equivalence induced by an inclusion of categories, with induced functor $\delta_\ast : \text{Sh}(X_{cl}) \to \text{Sh}(X(\mathbb{C}))$ with quasi-inverse $g_\ast$. In fact, $\delta_\ast$ is left adjoint to $g_\ast$. By the example above, we know $X(\mathbb{C})$ has enough points. For each point $x = (x^\ast, x_\ast)$ of $X(\mathbb{C})$, define $y = (y^\ast, y_\ast) := (x^\ast \circ \delta_\ast, g_\ast \circ x_\ast)$. We check adjointness:

$$\text{Hom}_{\text{Set}}(y^\ast \mathcal{F}, E) = \text{Hom}_{\text{Set}}(x^\ast \circ \delta_\ast \mathcal{F}, E)$$

$$\cong \text{Hom}_{\text{Sh}(X(\mathbb{C}))}(\delta_\ast \mathcal{F}, x_\ast E)$$

$$\cong \text{Hom}_{\text{Sh}(X_{cl})}(\mathcal{F}, g_\ast \circ x_\ast E)$$

$$= \text{Hom}_{\text{Sh}(X_{cl})}(\mathcal{F}, y_\ast E)$$

where all isomorphisms follow from the adjunction pairs $(x^\ast, x_\ast)$ and $(\delta_\ast, g_\ast)$. Moreover, since $x^\ast$ and $\delta_\ast$ are both exact, their composition $y^\ast$ also commutes with finite limits.

The above argument shows that $y$ as defined is a point. We take as our set of points $S$ a set with cardinality at most the cardinality of the underlying set of the manifold $X(\mathbb{C})$, via $(x^\ast, x_\ast) \mapsto (x^\ast \circ \delta_\ast, g_\ast \circ x_\ast)$. We now show we can check isomorphisms of sheaves stalk-wise.

Assume $f$ is an isomorphism in $\text{Sh}(X_{cl})$. By exactness of $\delta_\ast$, we have that $\delta_\ast f$ is an isomorphism in $\text{Sh}(X(\mathbb{C}))$. Thus, $x^\ast \delta_\ast f$ is a bijection for all points $x$. Therefore, $y^\ast f = x^\ast \delta_\ast f$ is a bijection for all $y$.

Conversely, suppose $y^\ast f$ is a bijection for all $y$, i.e., that $x^\ast \delta_\ast f$ is a bijection for all $x$. Then, $\delta_\ast f$ is an isomorphism in $\text{Sh}(X(\mathbb{C}))$. However, by lemma 1.1, $\delta_\ast$ reflects injections and surjections. Thus $f$ is an isomorphism. \(\Box\)

We are now ready to discuss the Godement resolution, following [21]. In fact, we have the following proposition ([21], Proposition 3.3.1):

**Proposition 1.1** Let $D$ be a category closed under products and filtered colimits, and let $x$ be a point of the site $\mathcal{C}$. Then there is an adjoint pair $(x^\ast, x_\ast)$ of functors

$$\text{Sh}(\mathcal{C}, D) \xrightarrow{x^\ast} D,$$

$$\text{Hom}_D(x^\ast \mathcal{F}, E) \cong \text{Hom}_{\text{Sh}(\mathcal{C}, D)}(\mathcal{F}, x_\ast E)$$

Rodríguez-González and Roig prove the above proposition by noting that we have explicit formulas for $x^\ast$, using filtered colimits, and $x_\ast$, using products. We refer the formulas to their paper ([21], 3.3).

**Definition 1.2** Let $\mathcal{C}$ be a site with enough points, with $S$ as its set of points. Let $D$ be a category closed under products and filtered colimits. By abuse of notation, let $S$ denote the collection considered as a discrete category, with only identity morphisms. There is a pair $(p^\ast, p_\ast)$ of adjoint functors

$$\text{Sh}(\mathcal{C}, D) \xrightarrow{p^\ast} D^S,$$

\(\odot\) Springer
\[ \text{Hom}_{D^S}(p^* F, E) \cong \text{Hom}_{\text{Sh}(C, D)}(F, p_* E) \]
defined, for \( F \in \text{Sh}(C, D) \) and \( E = (E_x)_{x \in S} \in D^S \), by \( p^* F := (F_x)_{x \in S} \) and \( p_* E := \prod_{x \in S} x_*(E_x) \).

**Definition 1.3** Let \( C \) be a site with enough points and \( D \) a symmetric monoidal category closed under products and filtered colimits. The **Godement functor** is a functor \( G : \text{Sh}(C, D) \to \text{Sh}(C, D) \) defined as \( G = p_* p^* \) for the adjoint pair \((p^*, p_*)\) defined in definition 1.2. The **cosimplicial Godement functor** is a functor \( G^\bullet : \text{Sh}(C, D) \to \Delta \text{Sh}(C, D) \) into cosimplicial sheaves, where the \( i \)-th term is given by the \( i \)-th iterate \( G^i \).

For the purposes of this paper, we will mainly be concerned with \( D = \text{Ch}(k\text{-mod}) \) the category of cochain complexes of \( k \)-modules. In this situation, we have the following classical theorem, proven for far more general \( D \) by Rodríguez-Gonález and Roig (see [20] Proposition 3.4.5, Corollary 3.4.6, Proposition 3.4.7) though definitely known to Godement [10] albeit without modern language. See also Chataur and Cirici ([9] Definition 2.5).

**Theorem 1.4** Let \( f : C \to C' \) be a morphism of sites induced by a continuous functor, where \( C \) and \( C' \) have enough points. Let \( D = \text{Ch}(k\text{-mod}) \). Then the functors \( G^\bullet, f_\ast, \Gamma \ast \) are lax symmetric monoidal.

**Remark** The symmetric monoidal structure on \( \text{Sh}(C, D) \) is by sheafifying the presheaf whose values are tensor products object-wise. That is, given \( F \) and \( G \), we can define a presheaf by \( U \mapsto F(U) \otimes_k G(U) \), and then sheafify.

We now use some definitions following Chataur and Cirici ([9], Definition 2.4), and Mandell ([15], Proposition 5.2). See also ([24], Tag 019 H) for general Dold-Kan considerations.

**Definition 1.4** The **normalized complex functor** is the functor \( N : \Delta \text{Ch}(k\text{-mod}) \to \text{Ch}(k\text{-mod}) \) given as the composition of the cosimplicial degree-wise normalization functor and the totalization functor. Normalization of a cosimplicial object means forming a chain complex by taking the intersections of kernels of the cosimplicial maps, then restricting the alternating sum of cosimplicial maps to the intersections. The **associated complex functor** is the functor \( s : \Delta \text{Ch}(k\text{-mod}) \to \text{Ch}(k\text{-mod}) \) given by taking alternating sums of the cosimplicial maps to obtain an associated double complex, then applying totalization.

It is well known that for every cosimplicial complex \( C^\bullet \), the complexes \( NC^\bullet \) and \( sC^\bullet \) are homotopy equivalent ([26], Lemma 8.3.7, Theorem 8.3.8).

**Definition 1.5** Let \( D = \text{Ch}(k\text{-mod}) \). The **normalized Godement resolution** is the functor \( NG^\bullet : \text{Sh}(C, D) \to \text{Sh}(C, D) \) given as object-wise composition of the normalized complex functor and the cosimplicial Godement functor.

**Definition 1.6** Let \( D = \text{Ch}(k\text{-mod}) \). The **Godement resolution** is the functor \( sG^\bullet : \text{Sh}(C, D) \to \text{Sh}(C, D) \) given as object-wise composition of the associated complex functor and the cosimplicial Godement functor.
The normalized cochain functor is monoidal but not symmetric monoidal; instead we have the following theorem of Hinich and Schechtman [11] (see also Chataur and Cirici [9], Proposition 2.2, and Mandell [15], Theorem 5.5).

**Theorem 1.5** (Hinich-Schechtman) Let $A^\bullet$ be a cosimplicial $P$-algebra for an arbitrary operad $P$ in $Ch(k\text{-mod})$. Then $NA^\bullet$ is a $(P \otimes_k Z)$-algebra, for the Eilenberg-Zilber operad $Z$, functorial in $A^\bullet$ and $P$.

### 1.2 Sheaves of commutative DGAs

This subsection will discuss how to obtain an $E_\infty$-algebra structure on the cochains of a sheaf of commutative DGAs (CDGAs). An outline is as follows: the above theorem of Hinich and Schechtman says by normalizing and totalizing a cosimplicial CDGA, the resulting complex is an $E_\infty$-algebra. The Godement resolution precisely takes a sheaf of CDGAs to cosimplicial sheaves of CDGAs, and then normalizes and totalizes. Taking global sections then yields an $E_\infty$-algebra structure on the sheaf cochains.

We introduce two necessary definitions from [20] and [21] before proceeding. By the remark on the symmetric monoidal structure on $\text{Sh}(\mathcal{C}, D)$ via the sheafification functor, the following is well defined. See ([20], Remark 3.4.3).

**Definition 1.7** Let $D = Ch(k\text{-mod})$. A sheaf of operads $P$ in $D$ is an operad in $\text{Sh}(\mathcal{C}, D)$.

**Definition 1.8** Let $D = Ch(k\text{-mod})$ and let $P$ be a sheaf of operads in $D$ on a site $\mathcal{C}$. A sheaf of cochain complexes $\mathcal{F} \in \text{Sh}(\mathcal{C}, D)$ is a sheaf of $P$-algebras if it is a $P$-algebra. Equivalently, for each object $U$ in $\mathcal{C}$, there are the usual structure morphisms for an algebra over an operad over an operad $P(n)(U) \otimes_k F \otimes^n(U) \rightarrow F(U)$.

**Example 1.3** Let $O$ be a sheaf of $k$-algebras. We can view this as a sheaf of trivial commutative DGAs, which are concentrated in degree 0. Then $O$ is a sheaf of $\text{Comm}$-algebras, where $\text{Comm}$ denotes the commutative operad, which has $\text{Comm}(n) = k$ in degree 0 for all $n$, with trivial symmetric group actions. Note that $\text{Comm}$-algebras are equivalent to commutative DGAs (CDGAs) ([14] Example 2.2).

**Example 1.4** Recall that the Eilenberg-Zilber operad is acyclic ([11] Theorem 2.3, [15] Proposition 5.4), and so admits a map of operads $Z \rightarrow \text{Comm}$. There is also the fixed map of operads $E \rightarrow Z$, where $E$ is our fixed $E_\infty$-operad. Thus a sheaf $O$ of $k$-algebras is a sheaf of $E_\infty$-algebras.

**Remark** Recall that the symmetric monoidal structure on $\Delta \text{Sh}(\mathcal{C}, D)$ is induced degree-wise by the symmetric monoidal structure on $\text{Sh}(\mathcal{C}, D)$, so the definitions above make sense for cosimplicial sheaves as well.

**Remark** The equivalence above in definition 1.8 follows from the existence of the sheafification functor for presheaves with values in $D = Ch(k\text{-mod})$. Part of the work of Rodríguez-Gónzalez and Roig in [20] concerns the conditions on the coefficient category $D$ for which a sheafification functor is available. For the purposes of this paper, we will actually mainly be concerned with the underlying presheaf structure than the sheaf structure per se.
Lemma 1.4 The category of Comm-algebras, equivalent to the category of commutative differential graded k-algebras (CDGAs), is closed under filtered limits and filtered colimits.

Proof Limits and colimits of cochain complexes are computed degree-wise on the underlying k-modules, with the differential defined by universal property. A CDGA is precisely a cochain complex $C^\bullet$ with a chain map $a : C^\bullet \otimes_k C^\bullet \to C^\bullet$ unital and associative in the appropriate sense, and satisfying graded commutativity. We sketch a proof of the filtered colimit case in this paragraph and omit the filtered limit case, which is identical. For filtered colimits in CDGAs, we take the colimits of the underlying cochain complexes. Let $D^\bullet$ denote the colimit of a system of CDGAs $C^\bullet_i \overset{f_{ji}}\longrightarrow C^\bullet_j$. We define the algebra map by

$$D^\bullet \otimes_k D^\bullet = \text{colim} C^\bullet_i \otimes_k \text{colim} C^\bullet_j \cong \text{colim}(C^\bullet_i \otimes_k C^\bullet_i) \overset{\text{colim} a_i}{\longrightarrow} \text{colim} C^\bullet_i = D^\bullet$$

where the center isomorphism follows from invoking the universal property of $\text{colim} (C^\bullet_i \otimes_k C^\bullet_i)$ for the system $\{C^\bullet_i \otimes_k C^\bullet_i\}$, via the maps $C^\bullet_i \otimes_k C^\bullet_i \overset{f_{ji} \otimes \text{id}}{\longrightarrow} C^\bullet_j \otimes_k C^\bullet_j$ if $i \leq j$, and $C^\bullet_i \otimes_k C^\bullet_j \overset{\text{id} \otimes f_{ij}}{\longrightarrow} C^\bullet_i \otimes_k C^\bullet_i \to \text{colim} C^\bullet_i \otimes_k C^\bullet_i$ if $i > j$; one then uses the universal property twice, and commutativity of all diagrams involved and uniqueness of the maps arising from the universal property verify the isomorphism. Graded commutativity and the Leibniz formula follow from all the terms in the equations being colimits of elements and colimits of maps respectively, and functoriality of colim.

Lemma 1.5 Let $F$ be a sheaf of Comm-algebras and let $f : C \to C'$ be a morphism of sites with enough points induced by a continuous functor $u : C' \to C$. Then $G^\bullet F$ is a cosimplicial presheaf of Comm-algebras. If $F$ is a sheaf of $P$-algebras for a fixed operad $P$, then $f_* F$ is a presheaf of $P$-algebras.

Proof Recall that $G^0 F = p_* p^* F$. By definition, $G^0$ is a product of filtered colimits of Comm-algebras, and so is a presheaf of Comm-algebras by Lemma 1.4. By definition, $f_* F(U) = F(u(U))$ for objects $U \in C$; since $F$ is a presheaf of $P$-algebras, it follows that $f_* F$ is a presheaf of $P$-algebras.

Remark By Theorem 1.4 and results of [20], we obtain an induced functor, which we also denote by $G^\bullet$, that sends sheaves of operads $\mathcal{P}$ in $\text{Sh}(\mathcal{C}, D)$ to cosimplicial sheaves of operads $G^\bullet \mathcal{P}$, which is by definition the same as a cosimplicial operad in $\text{Sh}(\mathcal{C}, D)$. Similarly, we obtain an induced functor $G^\bullet$ that sends sheaves $\mathcal{F}$ of $\mathcal{P}$-algebras to cosimplicial sheaves $G^\bullet \mathcal{F}$ which are $G^\bullet \mathcal{P}$-algebras, i.e., algebras over the cosimplicial operad $G^\bullet \mathcal{P}$.

An immediate consequence of Theorem 1.4 and Theorem 1.5 is then the following corollary.
Corollary 1.1 Let $D = \text{Ch}(k\text{-mod})$ and let $C$ be a site with enough points with a terminal object $X$. Let $F$ be a sheaf of $\text{Comm}$-algebras. Then $NG^\bullet(F)$ is a presheaf of $\mathcal{Z}$-algebras. In particular, $R\Gamma(C, F)$ is an $\mathcal{Z}$-algebra.

**Proof** By Theorem 1.4 and Lemma 1.5, $G^\bullet F$ is a cosimplicial presheaf of $\text{Comm}$-algebras. Thus on each object $U$ of $C$, we have that $NG^\bullet(F)(U)$ is the normalized complex of a cosimplicial $\text{Comm}$-algebra. By Theorem 1.5, $NG^\bullet(F)(U)$ is a $(\text{Comm} \otimes_k \mathcal{Z})$-algebra for each object $U$. Since $\text{Comm} \otimes_k \mathcal{Z} = \mathcal{Z}$, we have that this is a presheaf of $\mathcal{Z}$-algebras. Finally, we have $R\Gamma(C, F) = NG^\bullet(F)(X)$ since the Godement resolution is a resolution by flasque sheaves, which are acyclic. □

We need just one more theorem, due to Mandell ([15], Theorem 5.8). See also Chataur and Cirici ([9], Definition 2.4).

**Theorem 1.6** There is a cosimplicial normalization functor $N$ that sends cosimplicial $E_\infty$-algebras to $E_\infty$-algebras, that agrees with the normalized cochain functor $N$ on the underlying cochain complexes, and such that, for a constant cosimplicial $E_\infty$-algebra $A^\bullet$, the isomorphism of cochain complexes $A^0 \cong N(A^\bullet)$ is a morphism of $E_\infty$-algebras.

**Lemma 1.6** Let $\mathcal{O}$ be a sheaf of $k$-algebras. The cosimplicial Godement functor has an augmentation $\mathcal{O} \rightarrow G^\bullet \mathcal{O}$ which is a map of presheaves of $k$-algebras in degree 0. The existence of the augmentation is equivalent to the existence of a map of cosimplicial presheaves of $k$-algebras $\mathcal{O}^\bullet \rightarrow G^\bullet \mathcal{O}$. The composition $\mathcal{O} \rightarrow NG^\bullet \mathcal{O}$ is a map of presheaves of $E_\infty$-algebras.

**Proof** That an augmentation of a cosimplicial object is equivalent to a map from the constant cosimplicial object is by composing the augmentation map with the cosimplicial maps; see ([24] Tag 018F Lemma 14.20.2). The map $\mathcal{O}^\bullet \rightarrow G^\bullet \mathcal{O}$ of cosimplicial presheaves of $k$-algebras is then a map of cosimplicial presheaves of $\text{Comm}$-algebras, and so $E_\infty$-algebras. Applying Theorem 1.6 gives the last statement. □

The following proposition summarizes the above section in one clean statement.

**Proposition 1.2** Let $F$ be a sheaf of commutative $k$-algebras (or more generally of commutative DGAs) on a site $C$ with enough points. Then $\Gamma(C, NG^\bullet F)$ is an $E_\infty$-$k$-algebra, where $NG^\bullet F$ is the normalization of the cosimplicial Godement resolution. Thus the object $R\Gamma(C, F)$ representing sheaf cohomology has an $E_\infty$-algebra structure.

### 1.3 $E_\infty$-Artin comparison

We are now ready to prove the statement that the classical Artin comparison theorem is a quasi-isomorphism of $E_\infty$-algebras, with the corresponding $E_\infty$-algebra structures arising from Godement resolutions.

**Lemma 1.7** Let $\mathcal{O}$ and $\mathcal{O}'$ be sheaves of $k$-algebras on sites $C, C'$ respectively. Assume $C$ and $C'$ both have terminal objects. A morphism of ringed sites $(C, \mathcal{O}) \rightarrow (C', \mathcal{O}')$...
induced by a continuous functor $\mathcal{C}' \to \mathcal{C}$ induces a map of $E_\infty$-algebras $R\Gamma(\mathcal{C}', \mathcal{O}') \to R\Gamma(\mathcal{C}, \mathcal{O})$.

**Proof** A morphism of ringed sites by definition has the data of a map of sheaves of rings $\mathcal{O}' \to f_*\mathcal{O}$. Viewing $f_*\mathcal{O}$ as an object in the derived category concentrated in degree 0, there is a natural chain map $f_*\mathcal{O} \to Rf_*\mathcal{O}$. By functoriality of $R\text{Hom}_{\text{PSh}}(\mathcal{C}', \ast_{\mathcal{C}'}, \ast)$ we obtain maps

$$R\text{Hom}_{\text{PSh}}(\mathcal{C}', \ast_{\mathcal{C}'}, \mathcal{O}') \to R\text{Hom}_{\text{PSh}}(\mathcal{C}', f_*\mathcal{O}) \to R\text{Hom}_{\text{PSh}}(\mathcal{C}', f_*\mathcal{O}, Rf_*\mathcal{O}) \simeq R\text{Hom}_{\text{PSh}}(\mathcal{C}, \ast_{\mathcal{C}}, \mathcal{O})$$

where the last isomorphism follows from Lemma 1.2. The composition of these maps gives our map $R\Gamma(\mathcal{C}', \mathcal{O}') \to R\Gamma(\mathcal{C}, \mathcal{O})$. The claim is that this naturally induced map is a map of $E_\infty$-algebras. Note that Theorem 1.4 also implies that the last isomorphism from Lemma 1.2 is a quasi-isomorphism of $E_\infty$-algebras, for sheaves of $k$-algebras.

First, note we already have maps of bounded below cochain complexes of sheaves of abelian groups $\mathcal{O}' \to f_*\mathcal{O} \to Rf_*\mathcal{O}$ where we view $\mathcal{O}'$ and $f_*\mathcal{O}$ as complexes concentrated in degree 0. The first map $\mathcal{O}' \to f_*\mathcal{O}$ is a map of presheaves of $k$-algebras, so of $\text{Comm}$-algebras, and so of $E_\infty$-algebras. We have that $Rf_*\mathcal{O}$ is computed as $f_*NG^\bullet\mathcal{O}$, which is a presheaf of $\mathbb{Z}$-algebras, and so of $E_\infty$-algebras, by Corollary 1.1 and Lemma 1.5. The second map is in fact the functor $f^*$ applied to the map $\mathcal{O} \to NG^\bullet\mathcal{O}$ in Lemma 1.6; since this map is a map of presheaves of $E_\infty$-algebras, applying the functor $f^*$ yields a map of presheaves of $E_\infty$-algebras. We then have the following commutative square

$$\begin{array}{ccc}
\mathcal{O}' & \simeq & NG^\bullet(\mathcal{O}') \\
\downarrow & & \downarrow^* \\
Rf_*\mathcal{O} & \simeq & NG^\bullet(Rf_*\mathcal{O})
\end{array}$$

by functoriality of the Godement resolution on the underlying complexes. The left hand vertical map is a map of presheaves of $E_\infty$-algebras, by the above discussion. The right hand vertical map is then a map of presheaves of $(E_\infty \otimes_k \mathbb{Z})$-algebras, and so a map of presheaves of $E_\infty$-algebras; this follows from the existence of maps of operads $\mathcal{E} \to \mathcal{E} \otimes_k \mathcal{E} \to \mathcal{E} \otimes_k \mathbb{Z}$. The map $\mathcal{E} \to \mathcal{E} \otimes_k \mathcal{E}$ exists using cofibrancy of an appropriate choice of $E_\infty$-operad, or for example using the Barratt-Eccles operad, which is known to have this property [3]. Taking global sections gives us a map of $E_\infty$-algebras

$$R\Gamma(\mathcal{C}', \mathcal{O}') = \Gamma(\mathcal{C}', NG^\bullet(\mathcal{O}')) \xrightarrow{\ast} \Gamma(\mathcal{C}', NG^\bullet(Rf_*\mathcal{O})) = R\Gamma(\mathcal{C}, \mathcal{O})$$

where the map $\ast$ is exactly the map described in the first paragraph above, by definition of derived global sections. \qed

The zig-zag of sites in Theorem 1.2 is what is classically used to prove the quasi-isomorphism between étale cohomology and sheaf cohomology on $X(\mathbb{C})$. The central corollary of the above discussion is then that the underlying maps of the quasi-isomorphism are maps of $E_\infty$-algebras.
Corollary 1.2 Let $X$ be a smooth complex variety. The zig-zag of sites in Theorem 1.2 gives a map of $E_\infty$-algebras between $R\Gamma_{\text{ét}}(X, \mathcal{F}_p)$ and $R\Gamma(X(\mathbb{C}), \mathcal{F}_p)$.

Recall that the singular cochains of a space $C^*_{\text{sing}}(X, k)$ have a natural $E_\infty$-$k$-algebra structure, as an algebra over the Eilenberg-Zilber operad $Z_k$; this structure arises from the failure of commutativity of the cup product on cochains with values in a general field $k$, and the Eilenberg-Zilber operad action is essentially given by the Alexander-Whitney maps ([17], Theorem 3.9). This $E_\infty$-algebra structure can be computed again using the Godement resolution. We conclude with the following (again, very classical) statement neatly explained by Petersen in [19], who also assumes the space only be cohomologically locally connected.

Theorem 1.7 Let $X$ be a locally contractible, paracompact Hausdorff space. There is a quasi-isomorphism of $E_\infty$-algebras between $R\Gamma(X, k)$ and $C^*_{\text{sing}}(X, k)$.

Petersen’s proof of the above follows the same strategy as the other proofs in this section, where one passes to the Godement resolution to see that one has a map of $E_\infty$-algebras. Of course, one uses that the constant sheaf admits a resolution by the sheaf of singular cochains, and then applies Godement to both complexes.

Corollary 1.3 An equivalence of sites $C \xrightarrow{f} C'$ induces a quasi-isomorphism of $E_\infty$-algebras $R\Gamma(C', A) \rightarrow R\Gamma(C, A)$, where $A$ denotes the constant sheaf with values in a fixed abelian group.

Proof An equivalence of sites induces an equivalence of topoi via the pushforward $f_*$ functor, which is an equivalence of the underlying categories. The map of $E_\infty$-algebras in Lemma 1.7 is a composition of the following maps:

$$R\text{Hom}_{\text{PSh}(C')}(\mathcal{C}', A) \rightarrow R\text{Hom}_{\text{PSh}(C')}(\mathcal{C}', f_*A)$$

$$\rightarrow R\text{Hom}_{\text{PSh}(C')}(\mathcal{C}', Rf_*A) \simeq R\text{Hom}_{\text{PSh}(C)}(\mathcal{C}, A).$$

Since $f$ is an equivalence of sites, we have that $f_* \simeq (f^{-1})^*$, where $f^{-1}$ denotes the quasi-inverse functor. Since we are working with constant sheaves, we have that $A \simeq (f^{-1})^*A \simeq f_*A$. Thus the first map above is an isomorphism. Similarly, since $f$ is an equivalence, we have that $f_*$ is exact, so the second map is an isomorphism.

Notice that in the proof of the above corollary, all one really needs is an equivalence of topoi.

2 Prismatic cohomology and $p$-adic homotopy theory

This section will cover preliminaries regarding prisms and prismatic cohomology. We will also briefly discuss Huber’s étale comparison theorem for adic spaces, and Bhatt–Scholze’s étale comparison theorem for prismatic cohomology.
2.1 Huber’s comparison theorem

We will briefly discuss adic spaces in the sense of Huber. We will restrict ourselves to the affinoid setting as that will be all we require. Most of the content here can be found in Huber’s work [12] and [13].

**Definition 2.1** Let $A$ be a topological ring. We say $A$ is $f$-adic or a Huber ring if it has an open subring $A_0 \subset A$ where the subspace topology is the $I$-adic topology given by some finitely generated ideal $I \subset A_0$. An affinoid ring is a pair $(A, A^+)$ where $A$ is a Huber ring and $A^+$ is an open subring of $A$ that is integrally closed in $A$ and contained in the set of power-bounded elements of $A$.

We can associate to an affinoid ring $(A, A^+)$ the space $Spa(A, A^+) = \{v|v$ is a continuous valuation of $A$ with $v(a) \leq 1$ for every $a \in A^+\}$ equipped with the topology generated by the sets $\{v \in Spa(A, A^+)|v(a) \leq v(b) \neq 0\}(a, b \in A)$.

Roughly, an adic space is a topological space that is locally isomorphic to an affinoid space $Spa(A, A^+)$ for some affinoid ring. We will not give the exact definition here; for the full definition, see [12]. For our purposes, the original statement of Theorem 0.1 in [5] (the étale comparison theorem) views the generic fiber $X$ over $\mathbb{C}_p$ of the formal scheme $X$ over $\mathcal{O}_{\mathbb{C}_p}$ as a rigid analytic variety, which then has an associated adic space with equivalent étale information.

Specifically, there is a notion of étale morphisms between adic spaces (Definition 1.6.5 in [13]) and a notion of étale site for an adic space (see Sect. 2.1 in [13]). Similarly, there is a notion of étale morphisms between rigid analytic varieties, and a notion of étale site for rigid analytic varieties (see Section 0 in [13]). Finally, Huber defines a functor $r$ that associates to a rigid analytic variety $X$ an associated adic space $r(X)$ (1.1.11 in [13]). This gives Huber the following étale comparison theorem (Proposition 2.1.4 in [13]).

**Proposition 2.1** The functor $r : X \mapsto r(X)$ induces a morphism of sites $r(X)_{\text{ét}} \rightarrow X_{\text{ét}}$ between the étale site of the rigid analytic variety $X$ and the étale site of its associated adic space $r(X)$. The morphism of sites induces an equivalence of topoi.

By Corollary 1.3 we see that the equivalence of topoi yields for free a map of $E_\infty$-algebras for the cohomology of constant sheaves.

2.2 Prismatic cohomology

Prismatic cohomology arises as the sheaf cohomology of a sheaf of rings $\mathcal{O}_\Delta$ on an appropriate site. The underlying category of the site is a category of objects, called prisms, with morphisms to the variety of interest. Informally, a prism is a special type of characteristic 0 ring that has the data of a lift of Frobenius; they are pairs $(A, I)$, where $A$ is a $\delta$-ring, i.e., a ring equipped with the data of a set map $\delta : A \rightarrow A$ such that the map $\phi(x) = x^p + p\delta(x)$ reduces mod $p$ to the Frobenius endomorphism, and $I$ is an invertible ideal. An illustrative example to keep in mind is $(\mathbb{Z}_p, (p))$ where $\mathbb{Z}_p$ denotes the $p$-adic integers, and $\delta(x) = \frac{x^p}{p}$, which is well defined as $\mathbb{Z}_p$ is $p$-torsion free. Morally, prisms are mixed characteristic thickenings of characteristic $p$ varieties.
whose infinitesimal neighborhoods record a lift of the Frobenius endomorphism. We formally define them below, starting with the definition of a $\delta$-ring.

**Definition 2.2** A $\delta$-ring is a ring $A$ equipped with a set map $\delta : A \to A$ such that $\delta(0) = \delta(1) = 0$, and satisfying the two identities:

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y)$$

and

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}.$$ 

Given a $\delta$, one can define a map $\phi : A \to A$ by $\phi(x) = x^p + p\delta(x)$. By the identities in the definition above, $\phi$ is a ring homomorphism. Moreover, $\phi$ reduces mod $p$ to the Frobenius homomorphism. We call $\phi$ a lift of Frobenius. A morphism of $\delta$-rings is a map of rings which commute with the respective $\delta$ maps.

We now briefly review the notion of derived completion.

**Definition 2.3** Let $A$ be a ring with finitely generated ideal $I$. An $A$-complex $M \in D(A)$ is derived $I$-complete if for each $f \in I$, the natural map $M \to R\text{lim}(M \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^n))$ is a quasi-isomorphism, where we view $M$ as a $\mathbb{Z}[x]$-module where $x$ acts by multiplication by $f$.

If an $A$-module $M$ is classically $I$-complete, then it is derived $I$-complete.

**Definition 2.4** A prism is a pair $(A, I)$ where $A$ is a $\delta$-ring and $I$ defines a Cartier divisor on $\text{Spec}(A)$ such that $A$ is derived $(p, I)$-complete and $p \in I + \phi(I)A$.

All prisms used seriously in this paper will be equipped with principal ideals $I = (d)$ for some choice of generator $d$, and will all be classically $(p, I)$-complete. We include the definitions as they are, for full generality. Morphisms of prisms $(A, I) \to (B, J)$ are maps of $\delta$-rings that send the ideal $I$ into $J$. In fact, it is a small lemma that any map of prisms forces $J = IB$.

**Definition 2.5** A prism is perfect if the lift of Frobenius $\phi$ is an isomorphism. A ring $R$ is (integral) perfectoid if $R = A/I$ for a perfect prism $(A, I)$.

The above notion of perfectoid differs slightly from the notion of perfectoid in Fontaine’s sense [22]. For the relation between the two notions, see Lemma 3.20 in [5]; in context for the étale comparison theorem later, the two notions will be compatible via this lemma. We use the notion of perfectoid above in this paper.

We now fix a base prism $(A, I)$. We now assume all formal schemes are equipped with the $p$-adic topology. Recall that ordinary schemes in characteristic $p$ can be considered $p$-adic formal schemes, since $p = 0$.

**Definition 2.6** Fix a smooth formal scheme $\mathcal{X}$ over $A/I$. The prismatic site of $\mathcal{X}$ is a category, where objects are diagrams
Prismatic cohomology and $p$-adic...

$$\text{Spf}(B/IB) \longrightarrow \text{Spf}(B)$$

$$\downarrow$$

$$\text{Spf}(A/I) \longrightarrow \text{Spf}(A)$$

where the pairs $(B, IB)$ are prisms over $(A, I)$. We equip the category with the flat topology. The structure sheaf $\mathcal{O}_{\Delta}$ sends a prism $(B, IB)$ to $B$.

There is a map of topoi $\nu : \text{Sh}(X/A)_{\Delta} \rightarrow \text{Sh}(X_{\text{ét}})$ (see [7] Construction 4.4) that localizes prismatic cohomology on étale coverings, given by sending the above diagram to étale coverings $\text{Spf}(B/IB) \rightarrow X$. The prismatic complex $\mathcal{H}_{\Delta}X/A$ is defined as $R\nu_{*}\mathcal{O}_{\Delta} \in D(X_{\text{ét}})$. This is a complex of étale sheaves of $A$-modules, and in fact a commutative algebra object in the derived category. That is, it can be viewed as a presheaf of $E_{\infty}$-algebras.

**Definition 2.7** We define **prismatic cohomology** $H_{\Delta}(X/A)$ over the base prism $(A, I)$ to be the hypercohomology of the complex of étale sheaves $\mathcal{H}_{\Delta}X/A$.

We finally end our review with one last definition.

**Definition 2.8** The **tilt** of a ring $R$ is the inverse limit $\lim_{\leftarrow} \phi R/p$ where $\phi$ denotes the Frobenius $x \mapsto x^p$.

If $(A, I)$ is a perfect prism, then there is a natural isomorphism $A \simeq W((A/I)^{\phi})$. Thus, if $R = A/I$ is perfectoid with perfect prism $(A, I)$ then its tilt $R^{\flat}$ is also perfectoid with perfect prism $(W(R^{\phi}), (p)) = (A, (p))$. Indeed, there is an equivalence of categories between the category of perfectoid rings, and the category of perfect prisms.

We are now ready to relate the result of Bhatt-Morrow-Scholze to the $p$-adic homotopy theory of the complex variety $X$. We simply apply $- \otimes_{\mathbb{F}_p}^{L} \mathbb{F}_p$ to the Bhatt-Morrow-Scholze result and apply Mandell’s functor $\mathbb{U}$. Notice that since $\mathbb{F}_p$ is a field extension over $\mathbb{F}_p$, it is faithfully flat. Thus, the functor $- \otimes_{\mathbb{F}_p}^{L} \mathbb{F}_p$ is equal to the functor $- \otimes_{\mathbb{F}_p} \mathbb{F}_p$ on the underlying complexes in the derived category.

**Theorem 0.1** can be obtained by first proving the following theorem ([7], Theorem 9.1) for affine opens of $X$, then gluing the result to pass from local to global. We note that the fraction field of $\mathcal{O}_{\mathbb{C}_p}^{\phi}$ is $\mathcal{O}_{\mathbb{C}_p}^{\phi}[1/d] = \mathbb{C}_p$ where $d$ is the element that generates the kernel of the map $W(\mathcal{O}_{\mathbb{C}_p}^{\phi}) \rightarrow \mathcal{O}_{\mathbb{C}_p}$. Since $R\Gamma(X, \mathcal{H}_{\Delta}X/A)$ takes values in $A = A_{\text{inf}}$-modules, tensoring with the fraction field is equivalent (locally) to inverting $d$ and taking mod $p$.

Recall that there is a short exact sequence (the Artin-Schreier-Witt short exact sequence) of étale sheaves:

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{G}_a \xrightarrow{\phi-1d} \mathbb{G}_a \rightarrow 0.$$
which will be used in the argument below. The point is that prismatic cohomology is acyclic (as a complex of sheaves) for perfectoid rings, and is given by the Witt vectors of their tilt in degree 0. Moreover, the étale theory of perfectoid rings is equivalent to their tilts.

We are now ready to briefly discuss the proof of the étale comparison theorem for prismatic cohomology. We include it here to show that the comparison is one of $E_\infty$-algebras as explicitly as possible.

**Theorem 2.1** (Bhatt-Scholze) Let $\mathcal{X} = \text{Spf}(S)$ be a formal affine scheme over a perfectoid ring $R$ corresponding to a perfect prism $(A, (d))$. There is a canonical quasi-isomorphism of $E_\infty$-algebras (with the $E_\infty$-algebra structure induced by the Godement resolution for étale sheaves)

$$R\Gamma_{\text{et}}(\text{Spec}(S[1/p]), \mathbb{Z}/p^n) \simeq (\Delta_{S/A}[1/d]/p^n)^{\phi=1}$$

for each $n \geq 1$.

**Very rough sketch of proof.** Bhatt and Scholze first prove the two functors

$$\text{Spf}(S) \leftrightarrow R\Gamma_{\text{et}}(\text{Spec}(S[1/p]), \mathbb{Z}/p^n)$$

are sheaves in the arc topology on affine formal schemes. Étale cohomology satisfies descent with respect to the arc topology ([6], Theorem 5.4). Given this, they assume $S$ is perfectoid, using that affine perfectoids are a basis for the arc topology ([7], Lemma 8.8). Since $S$ is perfectoid, they can then identify $(\Delta_{S/A}[1/d]/p^n)^{\phi=1}$ with $R\Gamma_{\text{et}}(\text{Spec}(S^\flat[1/d]), \mathbb{Z}/p^n)$ where $S^\flat$ is the tilt of $S$, described as follows. As $S$ is perfectoid, the complex of sheaves $\Delta_{S/A}[1/d]/p^n$ is concentrated in degree 0 and given by $W(S^\flat)[1/d]/p^n$. By the Artin-Schreier-Witt sequence, taking Frobenius fixed points yields $R\Gamma_{\text{et}}(\text{Spec}(S^\flat[1/d]), \mathbb{Z}/p^n)$, after viewing the Witt vectors as group schemes (which is why we get cohomology on $S^\flat[1/d]$). Moreover, as $S$ is perfectoid, the space $\text{Spa}(S[1/p], S)$ is perfectoid in the sense of Fontaine, by Lemma 3.20 in [5]. By Huber’s comparison theorem ([13], Proposition 2.1.4 and Corollary 3.2.2), there is an equivalence of étale sites between $\text{Spec}(S[1/p])$ and the étale site of $\text{Spa}(S[1/p], S).$ The analogous equivalence holds for $S^\flat[1/d].$ Moreover, we have that the tilt of the perfectoid space $\text{Spa}(S[1/p], S)$ is the perfectoid space $\text{Spa}(S^\flat[1/d], S^\flat).$ By Scholze ([22], Theorem 1.11), there is an equivalence of étale sites induced by the tilting functor for perfectoid spaces. Thus we have an equivalence of étale sites between $S[1/p]$ and $S^\flat[1/d].$

The equivalence of étale sites by Scholze is then the root of the $E_\infty$-algebra comparison as in Sect. 1, again using that the étale site has enough points, and passing these points through via the equivalences. □
Remark The above lemma is a comparison theorem on affine opens. To recover the full theorem as stated, one glues affines together in such a way that the isomorphisms are compatible. In the original context of [4] (Lecture IX, Theorem 5.1), the hypotheses of smooth and proper are used to impose a finiteness condition on the complexes involved. Then, under this finiteness condition, they utilize a semicontinuity theorem to obtain a dimension inequality between de Rham cohomology of the special fiber and étale cohomology of the generic fiber. In fact, the hypothesis of proper is unnecessary for our theorem.

2.3 Proof of main theorem

We are now ready to prove the main Theorem 0.4. There is a slight nuance kindly pointed out to me by Mark De Cataldo: there are many non-canonical identifications of $\mathbb{C}$ with $\mathbb{C}_p$. For a complex variety $X = X_\mathbb{C}$ that admits a model over $\text{Spec}(\mathbb{Z})$, to apply the work of Bhatt and Scholze we must consider the model over the ring of integers $\mathcal{O}_{\mathbb{C}_p}$. We then have a model $X_{\mathbb{C}_p}$ over $\mathbb{C}_p$. Fixing an identification of $\mathbb{C}$ with $\mathbb{C}_p$ yields $X_\mathbb{C}$ as the pullback of $X_{\mathbb{C}_p}$ under the field isomorphism.

Lemma 2.1 Let $X = X_\mathbb{C}$ denote the smooth complex variety over $\mathbb{C}$ with an integral model over $\mathbb{Z}$, and let $Y = X_{\mathbb{C}_p}$ denote the model over $\mathbb{C}_p$. Choose a field isomorphism $\mathbb{C}_p \to \mathbb{C}$ so that we have the following fiber product diagram:

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \rightarrow & \text{Spec}(\mathbb{C}_p)
\end{array}
$$

We then have an equivalence of sites between $X_{\text{ét}}$ and $Y_{\text{ét}}$.

Proof Using the map $X \to Y$ induced by the field isomorphism, one can pull back étale coverings over $Y$ to étale coverings over $X$. Base-changing along this map produces a continuous functor between sites. By base-changing along the map induced by the inverse field isomorphism, one obtains the inverse continuous functor. To prove that the composition of base-changes with respect to the induced maps is naturally isomorphic to the identity, we present the following diagram:

$$
\begin{array}{ccc}
(U \times_X Y) \times_Y X & \rightarrow & U \times_X Y \\
\downarrow & & \downarrow \\
X & \rightarrow & Y & \rightarrow & X
\end{array}
$$

where each inner square is a pullback diagram and $U \to X$ is étale. This implies the larger square with the four corners is a pullback diagram. However, the composition of horizontal arrows on the bottom is the identity, since it is the composition of the maps induced by the field isomorphisms. Thus the top left corner is naturally isomorphic to the top right corner. □

Corollary 2.1 Fix a field isomorphism $\mathbb{C} \to \mathbb{C}_p$. We have a quasi-isomorphism of $E_\infty$-algebras $R\Gamma(X_{\mathbb{C}_p}, \mathbb{F}_p) \simeq R\Gamma(X_\mathbb{C}, \mathbb{F}_p)$. 

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Remark Suppose $X$ only admits a model over $\mathcal{O}_{\mathbb{C}_p}$ without admitting a model over $\text{Spec}(\mathbb{Z})$. One can produce a complex variety by using any field isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$. However, it is entirely possible that two different field isomorphisms could yield two non-isomorphic complex varieties. The two varieties produced would then be related by an action of the Galois group $\text{Gal}(\mathbb{C}/\mathbb{Q})$. However, there are explicit examples due to Serre [23], of complex varieties that differ by a Galois group action and whose fundamental groups are not isomorphic; these examples are not even homeomorphic. This checks out, as these latter automorphisms are not even continuous.

However, the argument above of Lemma 2.1 shows that if two complex varieties differ by a field automorphism, their étale sites are nonetheless equivalent. So they have isomorphic étale cohomology and étale fundamental groups (which are the profinite completions of their ordinary fundamental groups). Moreover, they have isomorphic $p$-adic (étale) homotopy types. Thus, the output of the main theorem being “the” $p$-adic homotopy type of a complex variety still makes sense.

We now prove the main theorem.

Proof of Theorem 0.4 Fix an identification of $\mathbb{C}$ with $\mathbb{C}_p$. The sheaf cohomology of any sheaf on the étale site of $X$ inherits an $E_\infty$-algebra structure by Godement considerations, as in Sect. 1. By Bhatt–Morrow–Scholze 0.1 and Theorem 2.1, we have that

$$(\Gamma(X, \Delta\chi_{\text{Art}}) \otimes_{\text{Art}} \mathbb{C}_p^b, \phi=1) \simeq \Gamma_{\text{ét}}(X, \mathbb{F}_p)$$

is a quasi-isomorphism of $E_\infty$-$\mathbb{F}_p$-algebras. By the entirety of Sect. 1, the right hand side is quasi-isomorphic to $\Gamma(X(\mathbb{C}), \mathbb{F}_p)$ as $E_\infty$-$\mathbb{F}_p$-algebras. By Theorem 1.7, we again obtain a quasi-isomorphism of $E_\infty$-$\mathbb{F}_p$-algebras with $C^*_\text{sing}(X(\mathbb{C}), \mathbb{F}_p)$. By Mandell 0.3, applying the functor $U$ to the left hand side then yields the free loop space of the $p$-completion of $X(\mathbb{C})$. 

On the other hand, we can apply $-\otimes_{\mathbb{F}_p} \mathbb{F}_p$ to the equation above. Again by Mandell’s Theorem 0.2, applying the functor $\overline{U}$ to the resulting expression gives the $p$-completion of $X(\mathbb{C})$.

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