ON $L^1$ LIMIT SOLUTIONS IN IMPULSIVE CONTROL

MONICA MOCCA
Dipartimento di Matematica “Tullio Levi-Civita”,
Università di Padova
Via Trieste, 63, Padova 35121, Italy

CATERINA SARTORI
Dipartimento di Matematica “Tullio Levi-Civita”,
Università di Padova
Via Trieste, 63, Padova 35121, Italy

Abstract. We consider a nonlinear control system depending on two controls $u$ and $v$, with dynamics affine in the (unbounded) derivative of $u$, and $v$ appearing initially only in the drift term. Recently, motivated by applications to optimization problems lacking coercivity, [1] proposed a notion of generalized solution $x$ for this system, called limit solution, associated to measurable $u$ and $v$, and with $u$ of possibly unbounded variation in $[0,T]$. As shown in [1], when $u$ and $x$ have bounded variation, such a solution (called in this case BV simple limit solution) coincides with the most used graph completion solution (see e.g. [6]). This correspondence has been extended in [24] to BV$_{loc}$ inputs $u$ and trajectories (with bounded variation just on any $[0,t]$ with $t < T$). Starting with an example of optimal control where the minimum does not exist in the class of limit solutions, we propose a notion of extended limit solution $x$, for which such a minimum exists. As a first result, we prove that extended and original limit solutions coincide in the special cases of BV and BV$_{loc}$ inputs $u$ (and solutions). Then we consider dynamics where the ordinary control $v$ also appears in the non-drift terms. For the associated system we prove that, in the BV case, extended limit solutions coincide with graph completion solutions.

1. Introduction

We consider a control system of the form

\begin{align}
\dot{x}(t) &= g_0(x(t),u(t),v(t)) + \sum_{i=1}^{m} g_i(x(t),u(t))\dot{u}_i(t) \quad \text{a.e. } t \in [0,T],
\end{align}

\begin{align}
x(0) &= x_0, \\ u(0) &= \bar{u}_0,
\end{align}

where $x \in \mathbb{R}^n$, $(u(t),v(t)) \in U \times V$ and $U$, $V$ are compact sets. System (1) is a so-called impulsive control system, where a solution $x$ can be provided by the usual

1991 Mathematics Subject Classification. Primary: 49N25, 93C10; Secondary: 93C15, 49J15.

Key words and phrases. Impulsive control systems. Generalized solutions. Pointwisely defined measurable solutions. Non commutative control systems. Impulsive optimal control problems.

This research is partially supported by the INdAM-GNAMPA Project 2017 "Optimal impulsive control: higher order necessary conditions and gap phenomena"; and by the Padova University grant PRAT 2015 "Control of dynamics with reactive constraints".
Carathéodory solution only if $u$ is an absolutely continuous control. For less regular $u$, several concepts of impulsive solution have been introduced in the literature, either for commutative systems, where the Lie brackets $[(e_i, g_i), (e_j, g_j)]=0$ for all $i,j=1,\ldots,m$ (see e.g. [9]), or assuming $u$ (and $x$) to be functions of bounded variation, when the Lie Algebra is non trivial. These solutions are described by different authors in fairly equivalent ways, and we will refer to them as graph completion solutions, since they are obtained by completing the graph of $u$ (see e.g. [8], [20], [25], [19], [27], [3], [14], [16]). In the less studied non commutative case with measurable controls $u$ of unbounded variation, let us mention [10], [18], and the definition of limit solution due to [1]. In the special case of $BV$ simple limit solutions, in which $u$ and $x$ are of bounded variation, in [1] the authors showed that any limit solution is a graph completion solution and vice-versa (see Definitions 3.1, 5.3, 5.4). This is an important result, since, on the one hand, graph completion solutions have a simple explicit representation formula, not available for general limit solutions. On the other hand, it proves that (pointwise defined) graph completion solutions are well-posed, in the sense that they coincide with all and only pointwise limits of classical solutions. In [24] we extended such a result to a case of unbounded variation, by introducing graph completion solutions associated to $BV_{loc}$ inputs $u$ (and trajectories) and we proved that they coincide with a special subset of simple limit solutions, the $BV_{loc}$ simple limit solutions (see Definition 3.2).

In this paper we analyse the concept of limit solution and, starting from an example in optimal control for which the infimum over limit solutions is not a minimum, we introduce a notion of extended limit solution, where such a minimum does exist. As a first result, in Theorem 4.3 we prove that this new definition coincides with the original one in the special cases of of BV simple or BV$_{loc}$ simple limit solutions (see Definitions 3.1, 3.2). As a consequence, all the results available for these two classes of limit solutions are still valid for their extended counterpart.

Furthermore, we investigate control systems of the form

$$
\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t), v(t)) \dot{u}_i(t) \quad \text{a.e. } t \in [0, T],
$$

where all the $g_0, g_1, \ldots, g_m$ depend on the control $v$. The definition of limit solution for (3) was left as an open problem in [1]. Indeed, our notion of extended limit solution can be adapted to this case, allowing us to show, in Theorem 5.2, that extended BV simple limit solution and graph completion solutions to (3), (2) coincide. This result extends to system (3) the analogous [1, Thm. 4.2] regarding (1). As remarked in [1], already when $u$ (and $x$) has bounded variation, the dependence of $g_1, \ldots, g_m$ on $v$ is much more critical than just the $v$-dependence of $g_0$, in that a simultaneous jump of $u$ and $v$ makes the determination of the corresponding jump of $x$ quite delicate.

The precise definitions of limit solution and extended limit solution will be given in Sections 3, 4. Here we just point out that the notion of limit solution involves a control $v$ which is measurable, while the control $u$ and the corresponding solution $x$ are pointwisely defined and belong to the set $L^1$ of the everywhere defined integrable functions. Let us describe a special case of extended limit solution. An extended
simple limit solution $x$ to (1), (2) associated to $(u, v)$, is the pointwise limit of a sequence of classical trajectories $(x_k)$ to (1), (2), corresponding to controls $(u_k, v_k)$ with $u_k$ absolutely continuous and pointwisely converging to $u$ and $v_k \to v$ in $L^1$-norm (see Definition 4.1). We recall that a simple limit solution $x$ is instead defined in [1] as the pointwise limit of a sequence of classical trajectories associated to controls $(u_k, v)$ with $u_k$ as above and $v$ fixed (see Definition 3.1). Our extension is motivated by the observation that in optimal control problems minimizing sequences $(x_k, u_k, v_k)$ with absolutely continuous inputs $u_k$, might converge to a map which is not a limit solution. Precisely, in Example 1 we have that the infimum value of an optimal control problem over limit solutions and extended limit solutions is the same, but it is a minimum only within the larger class of extended limit solutions. The two infima may be actually different, as shown in Example 2.

The need of considering generalized solutions to (1) or (3) and (2), associated to discontinuous $u$ comes, for instance, from optimal control, where, in absence of coercivity assumptions, it is reasonable to expect the existence of optimal solutions only in some enlarged class. The impulsive control theory, studied since the 50s, received in the last years a renewed attention because of the increasing number of applications in different fields, from Lagrangian mechanics with moving constraints [7], [6], or impactively blockable degrees of freedom [28], [13], to alternative models for hybrid systems [4], [12], [17], [15], just to give some examples. These applications set new problems also from the theoretical point of view, in particular since they lead to consider control systems nonlinear in the state variable like (1) or (3), and various types of constraints.

The paper is organized as follows. We end this section with some notation and the precise assumptions. In Section 2 we present two examples that motivate the notions of extended limit solutions, which we propose in Section 4. Section 3, is devoted to recall the original concepts of limit solution due to [1] and the recent definition of $BV_{loc}$ limit solution introduced in [24]. In Theorem 4.3 of Section 4 we prove that original and extended BVS limit solutions and $BV_{loc}$ limit solutions, respectively, coincide. In Section 5 we introduce the $v$-dependent control system (3) and in Theorem 5.2 we establish that a map $x$ is an extended BVS limit solution to (3), (2) if and only if it is a graph completion solution.

1.1. Notation. Let $E \subset \mathbb{R}^N$. Given $T > 0$, let
\[ AC([0, T], E) := \{ f : [0, T] \to E, f \text{ absolutely continuous} \}, \]
\[ BV([0, T], E) := \{ f : [0, T] \to E : Var_{[0, T]}(f) < +\infty \}, \]
where $Var_{[0, T]}(f)$ denotes the (total) variation of $f$ in $[0, T]$, and
\[ BV_{loc}([0, T], E) := \{ f \in BV([0, t], E) \forall t < T, \lim_{t \to T} Var_{[0, t]}[f] \leq +\infty \}. \]
We use $L^1([0, T], E)$ to denote the set of the everywhere defined integrable functions on $[0, T]$ with values in $E$, while $L^1([0, T], E)$ is its usual quotient space with respect to the Lebesgue measure. When no confusion on the codomain may arise, we omit it and write, for instance, $AC(T)$ in place of $AC([0, T], E)$. Let us set $\mathbb{R}_+: = [0, +\infty[$ and call modulus (of continuity) any increasing, continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$ and $\omega(r) > 0$ for every $r > 0$. 
For any control \((u, v) \in AC(T) \times L^1(T)\) with \(u(0) = \bar{u}_0\), we let
\[
x = x[\bar{x}_0, \bar{u}_0, u, v]
\]
denote the (unique) Carathéodory solution to \((1)–(2)\), defined on \([0, T]\). We will say that such \((u, v)\) and \(x\) are regular.

1.2. Assumptions. Let us recall the so–called Whitney property (see \([26]\)).

**Definition 1.1** (Whitney property). A compact subset \(U \subset \mathbb{R}^m\) has the Whitney property if there is some \(C \geq 1\) such that for every pair \((u_1, u_2) \in U \times U\), there exists an absolutely continuous path \(\tilde{u} : [0, 1] \to U\) verifying
\[
\tilde{u}(0) = u_1, \quad \tilde{u}(1) = u_2, \quad \text{Var}[\tilde{u}] \leq C|u_1 - u_2|.
\]

For instance, compact, star-shaped sets enjoy the Whitney property.

Throughout the paper we assume the following hypotheses:

**\(\text{(H0)}\)**

(i) the sets \(U \subset \mathbb{R}^m, V \subset \mathbb{R}^l\) are compact and \(U\) has the Whitney property;

(ii) the control vector field \(g_0 : \mathbb{R}^n \times U \times V \to \mathbb{R}^n\) is continuous and, moreover, \((x, u) \mapsto g_0(x, u, v)\) is locally Lipschitz on \(\mathbb{R}^n \times U\) uniformly in \(v \in V\);

(iii) for each \(i = 1, \ldots, m\) the control vector field \(g_i : \mathbb{R}^n \times U \to \mathbb{R}^n\) is locally Lipschitz continuous;

(iv) there exists \(M > 0\) such that
\[
|g_0(x, u, v)|, |g_1(x, u)|, \ldots, |g_m(x, u)| \leq M(1 + |(x, u)|),
\]
for every \((x, u, v) \in \mathbb{R}^n \times U \times V\).

2. Examples

This section is devoted to motivate, by means of two simple examples, the need of enlarging the class of limit solutions, introducing a notion of extended limit solution. Precisely, in Example 1 we exhibit an optimal control problem where the infimum value over limit solutions and extended limit solutions is the same, but the minimum is achieved only within the larger class of extended limit solutions. In Example 2 we present a minimum problem where there is a gap between the infimum over limit solutions and extended limit solutions and a gap between the infimum over regular solutions and limit solutions.

These phenomena may happen since in both examples any regular minimizing control sequence \((u_k, v_k)\) verifies \(\lim_{k \to +\infty} \text{Var}(u_k) = +\infty\).

**Example 1.** Let us consider the control system in \(\mathbb{R}^4\),
\[
\dot{x} = g_0(x) v + g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2 \quad \text{a.e. } t \in [0, 2\pi], \quad |u|, |v| \leq 1,
\]
with
\[
g_0(x) := \eta(x)(0, 0, 0, v)^T \\
g_1(x) := \eta(x)(1, 0, x_3 x_2, -x_4 x_2)^T \\
g_2(x) := \eta(x)(0, 1, -x_3 x_1, x_4 x_1)^T
\]
Let us introduce the Bolza optimization problem
\[
\inf_{(x,u,v)} J(x,u,v),
\]
where
\[
J(x,u,v) := \int_0^{2\pi} (|u(t)| + |v(t)|) \, dt + (2\pi - x_4(2\pi))^2.
\]
We now construct a minimizing sequence \((x_k, u_k, v_k)\) within the class of regular trajectory-control pairs. For every \(k\), let us set, for \(t \in [0, 2\pi]\),
\[
(u_k, v_k)(t) := \left( \frac{1}{\sqrt{k}} \cos(kt) - 1, \sin(kt) \right) \chi_{[2\pi/k, 2\pi]}(t), \quad k e^{-2\pi \sqrt{k} \chi_{[0, 2\pi/k]}(t)}.
\]
The corresponding solution \(x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k]\) is given, for \(t \in [0, 2\pi]\), by
\[
\begin{aligned}
x_{1k}(t) &= u_{1k}(t), \\
x_{2k}(t) &= u_{2k}(t), \\
x_{3k}(t) &= \chi_{[0, 2\pi/k]}(t) + e^{-\sqrt{k} (t - \frac{\sin(kt)}{k} - \frac{2\pi}{k})} \chi_{[2\pi/k, 2\pi]}(t), \\
x_{4k}(t) &= k e^{-2\pi \sqrt{k} t} \chi_{[0, 2\pi/k]}(t) + 2\pi e^{\sqrt{k} (t - 2\pi - \frac{\sin(kt)}{k} - \frac{2\pi}{k})} \chi_{[2\pi/k, 2\pi]}(t).
\end{aligned}
\]
One has that
\[
\lim_{k \to +\infty} J(x_k, u_k, v_k) = 0,
\]
so that the infimum of the cost over regular trajectory-control pairs turns out to be 0. Clearly, this is not a minimum, since the unique optimal control must be \(u \equiv 0\) and \(v = 0\) a.e., whose associated Charathéodory solution to (5) gives a cost equal to \(4\pi^2\). A minimum can be reached only over some enlarged set of generalized controls and solutions. Notice that
\[
\lim_{k \to +\infty} u_k(t) = 0 \quad \forall t \in [0, 2\pi], \quad \lim_{k \to +\infty} \|v_k - v\|_{L^1(2\pi)} = 0.
\]
Hence if we define as extended limit solution to (5) associated to the control \((u, v) = (0, 0)\) a.e., the limit function
\[
x(t) := \lim_{k \to +\infty} x_k(t) = (0, 0, 1, 0) \chi_{(t=0)}(t) + (0, 0, 0, 0) \chi_{[0, 2\pi]}(t) + (0, 0, 0, 2\pi)
\]
for \(t \in [0, 2\pi]\), we obtain
\[
J(x, 0, 0) = 0.
\]
Therefore in the class of extended limit solutions the minimum does exist (see Definition 4.1).

Let us point out that \(x\) is not a limit solution as defined in [1], because of the varying \(v_k\) (see Definition 3.1). Indeed, as already observed, the optimal control has
to be \((u, v) = (0, 0)\) a.e., but any sequence \(x_k := x[x_0, \bar{u}_0, \tilde{u}_k, 0]\) associated to an arbitrary sequence \((\tilde{u}_k)\) pointwisely converging to 0, verifies
\[
\tilde{x}_4 \equiv 0 \quad \text{for every } k,
\]
so that \(J(\tilde{x}_k, \tilde{u}_k, 0) = 4\pi^2\) for every \(k\). Thus the minimum of the above optimization problem does not exist in the class of limit solutions.

Slightly modifying the previous example and adding some constraints, we can provide a case where the infima over regular solutions, over limit solutions and over extended limit solutions are all different.

**Example 2.** Let us introduce the control system in \(\mathbb{R}^5\), obtained by adding to \((5)\) the equation
\[
\dot{x}_5(t) = |v(t)| + |u(t)| \quad \text{for a.e. } t \in [0, 2\pi],
\]
with initial and end-point conditions
\[
(x, u)(0) := (\bar{x}_0, \bar{u}_0) = ((0, 0, 1, 0, 0), (0, 0)), \quad x(2\pi) \in \mathbb{R}^4 \times \{0\}.
\]
Let us now set \(\Psi(x) := |x_3| + |2\pi - x_4|\) for any \(x \in \mathbb{R}^5\) and consider the Mayer problem
\[
\inf_{(x, u, v)} \Psi(x(2\pi)).
\]
Let us call admissible the trajectory-control pairs satisfying the constraints. Since only controls \((u, v)\) with \((u, v) = 0\) a.e. give rise to admissible trajectories, the calculations in Example 1 imply that the unique admissible regular solution \(x = x[x_0, \bar{u}_0, 0, 0]\) has \((x_3, x_4) \equiv (1, 0)\). Hence the infimum of the cost over regular solutions is equal to \(1 + 2\pi\). All admissible limit solutions \(\tilde{x}\) are pointwise limits of regular solutions \(\tilde{x}_k := x[x_0, \bar{u}_0, \tilde{u}_k, 0]\), associated to regular control sequences \((\tilde{u}_k)\) converging to \(u = 0\) (and fixed \(v = 0\)). Hence \(\tilde{x}_4 \equiv 0\) in any case, but taking \(\tilde{u}_k := u_k\) defined by \((6)\), one has \(\tilde{x}_3(2\pi) = 1\), so that the minimum in the class of limit solutions is \(\Psi(\tilde{x}(2\pi)) = 2\pi\). Finally, the extended limit solution \(x = (x_1, \ldots, x_3, x_5) = (x_1, \ldots, x_4, 0)\), where \((x_1, \ldots, x_4)\) are given by \((7)\), is associated to the control \((u, v) = (0, 0)\) a.e., verifies the constraints and has cost \(\Psi(x(2\pi)) = 0\). Therefore the minimum over extended limit solutions exists and is equal to 0.

Let us point out that when there are no constraints and the cost is continuous, by the very definition of limit solution, the infimum value over the different classes of solutions considered above is always the same. The difference between the infima, as in Example 2, is instead a generic situation in the presence of constraints, which are unavoidable in most applications. In this note we do not discuss the Lavrentiev-type gap issue, that is the occurrence of infimum gaps (see e.g. \([2]\)). Let us just observe that in several real models, as for instance the mechanical examples in \([6]\), only absolutely continuous controls \(u\) are implementable. In these cases, the no-gap requirement is mandatory.

### 3. Definitions and Preliminary Results

We start recalling the concept of limit solution, given in \([1]\) for vector fields \(g_1, \ldots, g_m\) depending on \(x\) only and extended to \((x, u)\)-dependent data in \([2]\). We will write \(L^1(T) := L^1([0, T], U)\) to denote the set of pointwisely defined Lebesgue
integrable functions with values in $U$ and set $L^1(T) := L^1([0,T], V)$, $AC(T) := AC([0,T], U)$.

**Definition 3.1 (Limit solutions).** Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in L^1(T) \times L^1(T)$ with $u(0) = \bar{u}_0$.

(1) **(Limit Solution)** A map $x$ belonging to $L^1([0,T], \mathbb{R}^n)$ is called a **limit solution** of the Cauchy problem (1)-(2) corresponding to $(u, v)$ if, for every $\tau \in [0,T]$, there is a sequence of controls $(u_k^\tau) \subset AC(T)$ such that $u_k^\tau(0) = \bar{u}_0$ and

(i) the sequence $(x_k^\tau)$ of the Carathéodory solutions $x_k^\tau := x[\bar{x}_0, \bar{u}_0, u_k^\tau, v]$ to (1)-(2) is equibounded in $[0,T]$;

(ii) $|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_{L^1([\tau,T])} \to 0$ as $k \to +\infty$.

(2) **(S limit solution)** A limit solution $x$ is called a **simple limit solution** of (1)-(2), shortly S **limit solution**, if the sequences $(u_k^\tau)$ can be chosen independently of $\tau$. In this case we write $(u_k)$ to refer to the approximating sequence.

(3) **(BVS limit solution)** An S limit solution $x$ is called a **BVS limit solution** of (1)-(2) if the approximating inputs $u_k$ have equibounded variation in $[0,T]$.

For a detailed discussion on the notion of limit solution we refer the reader to [1], [2]. Here let us just underline that, already the BVS limit solution associated to a control $(u, v) \in L^1(T) \times L^1(T)$ is not unique, unless the system is commutative. Moreover, the sets of limit solutions, S limit solutions and BVS limit solutions form a decreasing sequence of sets.

The density approach adopted in Definition 3.1 allows a unified notion of trajectory (for commutative and non commutative systems with $u$ of possibly unbounded variation), but it does not give any explicit representation formula for the solution. In fact, such a representation exists if either the control system is commutative or if there are a priori bounds on the variation of the controls $u$. In particular, in the latter case [1] proves that BVS limit solutions coincide with graph completion solutions. The graph completion approach is traditionally used to study impulsive control systems with bounded variation on $u$ (see e.g. [6] and the references therein).

It provides a nice representation formula, suitable to derive, for instance necessary and sufficient optimality conditions for several optimization problems, both in terms of Pontrjagin Maximum Principle and of Hamilton-Jacobi-Bellman equations (see e.g. [25], [19], [16] and [21], [22]). In order to have a representation formula for limit solutions associated to controls with unbounded variation, in [24] we singled out the following set of controls:

$$BV_{loc}(T) := \{ u : [0,T] \to U, \ u \in BV_{loc}(T) \},$$

for which we extended the graph completion approach. Precisely, in [24] we introduced graph completions solutions associated to these controls and proved that they coincide with the following subset of S limit solutions.
**Definition 3.2.** (BV\textsuperscript{loc}S limit solution) Let \((\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U\) and let \((u, v) \in \overline{BV}\textsuperscript{loc}(T) \times L^1(T)\) with \(u(0) = \bar{u}_0\). An S limit solution \(x\) is called a BV\textsuperscript{loc}S limit solution of (1)-(2):

(i) on \([0, T]\), if there exists a sequence of controls \((u_k)\) as in the definition of S limit solution, such that for any \(t \in [0, T]\) the approximating inputs \(u_k\) have equibounded variation on \([0, t]\);

(ii) on \([0, T]\), if, moreover, \(x\) is bounded and there exists a decreasing map \(\varepsilon\) with \(\lim_{s \to +\infty} \varepsilon(s) = 0\) and there exist two strictly increasing, diverging sequences \((\tilde{s}_j) \subset \mathbb{R}_+, (k_j) \subset \mathbb{N}, k_j \geq j\), such that, for every \(k > k_j\) there is \(\tau_k^j < T\) with \(\tau_k^j + Var_{[0, \tau_k^j]}(u_k) = \tilde{s}_j\) and

\[
|(x_k, u_k)(\tau_k^j) - (x_k, u_k)(T)| \leq \varepsilon(j).
\]

The subclass of BV\textsuperscript{loc}S limit solutions is relevant in controllability issues, like approaching a target set, and in optimization problems with endpoint constraints and certain running costs lacking coercivity (see e.g. Example 3.1 in [24], involving the Brockett nonholonomic integrator).

**Remark 1.** Condition (ii) in Definition 3.2 is an equiuniformity condition on the sequence \((x_k, u_k)\) in a neighborhood of the final time \(T\). We point out that without (8), a BV\textsuperscript{loc}S limit solution \(x\) is a BV\textsuperscript{loc} graph completion solution only on \([0, T]\). Condition (ii) guarantees the equivalence of the two concepts on the closed interval \([0, T]\) (see [24]).

To better understand condition (ii) in Definition 3.2, for any trajectory-control pair \((x, u, v)\) let us introduce the following parametrization of the graph of \((x, u)\), useful also in the sequel.

**Definition 3.3** (Arc-length parametrization). Let \((u, v) \in AC(T) \times L^1(T)\) with \(u(0) = \bar{u}_0\) and set \(x := x[\bar{x}_0, \bar{u}_0, u, v]\). We call arc-length graph-parametrization of the trajectory-control pair \((x, u, v)\), the element \((\xi, \varphi_0, \varphi, \psi, S)\) defined by

\[
\sigma(t) := \int_0^t (1 + |\dot{u}(\tau)|) d\tau \quad \forall t \in [0, T], \quad S := \sigma(T)
\]

\[
\varphi_0 := \sigma^{-1}, \quad \varphi := u \circ \varphi_0, \quad \psi := v \circ \varphi_0, \quad \xi := x \circ \varphi_0.
\]

Of course, \((\xi, \varphi, \psi) \circ \sigma = (x, u, v)\).

Notice that, given \((\xi, \varphi_0, \varphi, \psi, S)\) defined as above, \((\varphi_0, \varphi)(0) = (0, \bar{u}_0), \varphi_0(S) = T\) and \(\xi\) solves the following control system

\[
\begin{align*}
\xi'(s) &= g_0(\xi, \varphi, \psi) + \sum_{i=1}^m g_i(\xi, \varphi) \varphi_i'(s) \quad s \in [0, S[,
\xi(0) &= \bar{x}_0.
\end{align*}
\]

Here the apex `'` denotes differentiation with respect to the parameter \(s\), in order to distinguish it from the time differentiation, denoted by a dot.

---

\(^1\) Since every \(L^1\) equivalence class contains Borel measurable representatives, here and in the sequel we tacitly assume that the maps \(v\) and \(\psi\) are Borel measurable, when necessary.
Differently from the original solution $x$, which is defined on the fixed time interval $[0, T]$ and depends on an unbounded control derivative $\dot{u}$, the map $\xi$ is defined on a control-dependent interval $[0, S]$ with $S = T + \text{Var}_{[0,T]}(u) \geq T$ but with $(\varphi_0, \varphi')$ bounded valued, since $\varphi'_0 + |\varphi'| = 1$ a.e. in $[0, S]$.

Condition (ii) in Definition 3.2 is more meaningful once we read it as an hypothesis on the graphs of the approximating sequence $(x_k, u_k)_k$. Precisely, for any trajectory-control pair $(x_k, u_k, v)$ as in Definition 3.2, let $(\xi, \varphi_{0_k}, \varphi_k, v \circ \varphi_{0_k}, S_k)$ be its arc-length graph parametrization (see Definition 3.3). Then (ii) is equivalent to:

the existence of a positive, decreasing map $\bar{\varepsilon}$ with $\lim_{k \to +\infty} \bar{\varepsilon}(s) = 0$ and of two strictly increasing, diverging sequences $(\tilde{s}_j) \subset [0, \infty) \cap \mathbb{N}$, $(k_j) \subset \mathbb{N}$, $k_j \geq j$, such that, for every $k > k_j$:

$$|(\xi_k, \varphi_k)(\tilde{s}_j) - (\xi_k, \varphi_k)(S_k)| \leq \bar{\varepsilon}(j).$$

Clearly, (11) holds true when the sequence $(\xi_k, \varphi_k)$ is uniformly convergent on $\mathbb{R}$ (by considering, for every $k$, the extension $(\xi_k, \varphi_k)(s) := (\xi_k, \varphi_k)(S_k)$ for every $s \geq S_k$).

4. Extended limit solution

Motivated by Examples 1, 2, we extend here the notions of limit solution given in [1], [24], by approximating in $L^1$ the ordinary control $v$, which in the original definitions was kept fixed. Furthermore, in Theorem 4.3 we prove that extended and original $BVS$ and $BV_{loc}S$ limit solutions, respectively, coincide. Hence the results in [1], [2] and in [24], dealing with $BVS$ and $BV_{loc}S$ limit solutions, remain unchanged in the new extended framework.

Definition 4.1 (Extended limit solutions). Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in \mathcal{L}^1(T) \times \mathcal{L}^1(T)$ with $u(0) = \bar{u}_0$.

1. (E-Limit Solution) A map $x \in \mathcal{L}^1([0,T] ; \mathbb{R}^n)$ is called an extended limit solution, of the Cauchy problem (1)-(2) corresponding to $(u, v)$ if, for every $\tau \in [0, T]$, there is a sequence of controls $(u_k^\tau, v_k^\tau) \subset AC(T) \times L^1(T)$ such that $u_k^\tau(0) = \bar{u}_0$ and

(i) the sequence $(x_k^\tau)$ of the Carathéodory solutions $x_k^\tau := x[\bar{x}_0, \bar{u}_0, u_k^\tau, v_k^\tau]$ to (1)-(2) is equibounded on $[0, T]$;

(ii) $|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau, v_k^\tau) - (x, u, v)||_{L^1(T)} \to 0$ as $k \to +\infty$.

2. (E-S Limit Solution) A limit solution $x$ is called an E-simple limit solution of (1)-(2), shortly E-S limit solution, if the sequences $(u_k^\tau, v_k^\tau)$ can be chosen independently of $\tau$. In this case we write $(u_k, v_k)$ to refer to the approximating sequence.

3. (E-BVS Limit Solution) An E-limit solution $x$ is called an E-BVS limit solution, of (1)-(2) if the approximating inputs $u_k$ have equibounded variation on $[0, T]$.

Definition 4.2 (Extended $BV_{loc}S$ limit solution). Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in BV_{loc}(T) \times \mathcal{L}^1(T)$ with $u(0) = \bar{u}_0$. An E-S limit solution $x$ is called an extended $BV_{loc}S$ limit solution, shortly E-$BV_{loc}S$ limit solution, of (1)-(2):
(i) on $[0,T[$, if there exist a sequence of controls $(u_k,v_k)$ as in the definition of an E-S limit solution, such that for any $t \in [0,T[$ the approximating inputs $u_k$ have equibounded variation on $[0,t]$;

(ii) on $[0,T]$, if, moreover, $x$ is bounded and there exists a decreasing map $\tilde{\varepsilon}$ with $\lim_{s \to +\infty} \tilde{\varepsilon}(s) = 0$ and there exist two strictly increasing, diverging sequences $(\tilde{s}_j) \subset \mathbb{R}_+$, $(k_j) \subset \mathbb{N}$, $k_j \geq j$, such that, for every $k > k_j$ there is $\tau^j_k < T$ with $\tau^j_k + \text{Var}_{[0,\tau^j_k]}(u_k) = \tilde{s}_j$ and

$$|\langle x_k, u_k \rangle(\tau^j_k) - (x_k, u_k)(T)| \leq \tilde{\varepsilon}(j).$$

Analogously to the case of limit solutions, the extended limit solution associated to a control $(u,v) \in L^1(T) \times L^1(T)$ is not unique, unless the system is commutative; moreover the sets of E-limit solutions, E-S, E-BV$_{\text{loc}}$S, and E-BVS limit solutions are a decreasing sequence of sets.

**Theorem 4.3.** Let $T > 0$, $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u,v) \in L^1(T) \times L^1(T)$ be such that $u(0) = \bar{u}_0$. Then a map $x : [0,T] \to \mathbb{R}^n$ is an E-BVS limit solution [resp. E-BV$_{\text{loc}}$S limit solution] corresponding to $(u,v)$ if and only if it is a BVS limit solution [resp. BV$_{\text{loc}}$S limit solution] corresponding to the same input.

**Proof.** The “if” part is obvious for both cases. Let us prove the “only if” part.

**Case 1:** Let $x$ be an E-BVS limit solution corresponding to $(u,v)$ and let $(x_k,v_k)$ be as in Definition 4.1, so that, in particular, there is some constant $K > 0$ such that $\text{Var}_{[0,T]}(u_k) \leq K$ for every $k$. Then, setting $\hat{x}_k := x[\bar{x}_0,\bar{u}_0,u_k,v]$, by standard estimates it follows that

$$|x_k(t)|, |\hat{x}_k(t)| \leq R'$$

with $R' := \|\bar{x}_0\| + (m+1)M(T+K)e^{(m+1)M(T+K)}$. Let us denote by $\omega$ and $L$ a modulus of continuity of $g_0$ and a Lipschitz constant (in $(x,u)$) for the vector fields $g_i$, $i = 0, \ldots, m$ when $|x| \leq R'$, respectively. Gronwall’s Lemma yields that

$$|\hat{x}_k(t) - x_k(t)| \leq \left(\int_0^t \omega(|v_k(t') - v(t')|) dt'\right) e^{(m+1)L(t + \int_0^t |\hat{u}_k(t')|) dt'}.

Since there exists a subsequence of $(v_k)$ such that $v_k(t) \to v(t)$ a.e. in $[0,T]$ and $v$, $v_k$ take values in the compact set $V$, the Dominated Convergence Theorem and the continuity of $\omega$ let us conclude that, for such a subsequence,

$$\int_0^T \omega(|v_k(t) - v(t)|) dt \to 0, \quad \text{as} \ k \to +\infty$$

so that $\lim_k |\hat{x}_k(t) - x_k(t)| = 0$ for every $t \in [0,T]$. Therefore, $\lim_k \hat{x}_k(t) = \lim_k x_k(t) = x(t)$ for any $t \in [0,T]$ and $x$ is a BVS limit solution corresponding to $(u,v)$.

**Case 2:** Let now $x$ be an E-BV$_{\text{loc}}$S, not E-BVS, limit solution and let $(u_k,v_k)$, $(x_k)$, $(k_j)$ and $(\tilde{s}_j)$ be as in Definition 4.2. For every $k$, set $V_k := \text{Var}_{[0,T]}(u_k)$ and assume that $(V_k)$ is increasing and diverging. By (i) in Definition 4.2 there exists
an increasing function \( V : [0, T[ \to \mathbb{R}_+ \) with \( V(0) = 0, \lim_{t \to T} V(t) = +\infty \) and such that, for every \( k \),

\[
Var_{[0,t]}(u_k) \leq V(t) \quad \text{for every } t \in ]0,T[.
\]

Then by the proof of Case 1 we derive that

\[
\hat{x}_k(t) := x[\bar{x}_0, \bar{u}_0, u_k, v](t) \to x(t) \quad \text{for every } t \in [0, T[.
\]

To handle the convergence at \( t = T \), we use part (ii) of the definition of E-BV\text{loc}S limit solution. Let us introduce, for every \( k \), the arc-length graph parametrizations \((\xi_k, \varphi_{0_k}, \varphi_k, v_k \circ \varphi_{0_k}, T + V_k)\) and \((\hat{\xi}_k, \varphi_{0_k}, \varphi_k, v \circ \varphi_{0_k}, T + V_k)\) of \((x_k, u_k, v_k)\) and \((\hat{x}_k, u_k, v)\), respectively (see Definition 3.3). Let us suppose that these arc-length graph parametrizations are extended to \([T + V_k, +\infty[\) by the constant value assumed at \( T + V_k \). By assumption, there exists a constant \( R > 0 \) such that

\[
\sup_{s \in \mathbb{R}_+} |\xi_k(s)| = \sup_{t \in [0,T]} |x_k(t)| \leq R \quad \text{for every } k
\]

and, recalling that \( \varphi_{0_k}(s) + |\varphi_k'(s)| \leq 1 \) a.e., standard estimates imply that for any \( j \) there is some \( R_j > 0 \) such that

\[
\sup_{s \in [0,\bar{\xi}_j]} |\dot{\xi}_k(s)| \leq R_j \quad \text{for every } k.
\]

Let \( \omega_j \) and \( L_j \) be a modulus of continuity of \( g_0 \) and a Lipschitz constant (in \((x,u)\)) of the vector fields \( g_i, i = 0, \ldots, m \) for \( |x| \leq \max\{R, R_j\} \), respectively. Gronwall’s Lemma yields, for every \( k \),

\[
\sup_{t \in [0, \bar{\xi}_j]} |\dot{x}_k(t) - x_k(t)| = \sup_{[0, \bar{\xi}_j]} |\dot{\xi}_k(s) - \xi_k(s)| \leq
\]

\[
\int_0^{\bar{\xi}_j} \omega_j(|(v_k - v) \circ \varphi_{0_k}(r)|) |\varphi_{0_k}(r)| dr \cdot e^{(m+1)L_j \int_0^{\bar{\xi}_j} (\varphi_{0_k}(r) + |\varphi_k(r)|) dr} \leq
\]

\[
\int_0^T \omega_j(|v_k(t) - v(t)|) dt \cdot e^{(m+1)L_j \bar{\xi}_j} =: \varepsilon^2_j(k)
\]

with \( \varepsilon^2_j(k) \leq \varepsilon^2_{j+1}(k) \). Passing to a suitable subsequence of \( (v_k) \), still denoted by \((v_k)\), as in (15) we have that, for every fixed \( j \), \( \lim_k \varepsilon^2_j(k) = 0 \). Now we can construct a sequence \( (k^1_j) \), with \( k^1_j \geq k_j \), such that

\[
\varepsilon^2_j(k) \leq 1/j \quad \text{for all } k \geq k^1_j.
\]

In particular, this implies that, for some \( \hat{R} > 0 \),

\[
\sup_{[0, \bar{\xi}_j]} |\dot{x}_k| \leq \hat{R} \quad \forall k \geq k^1_j.
\]

Since \( \lim_k V_k = +\infty \), we need to modify the sequence \((\hat{x}_k, \hat{u}_k)\) using the Whitney property. Precisely, we set \( \tau^j_k := \tau^j_{k^1_j} \) and

\[
\hat{u}_j := \hat{u}_{k^1_j}(t) \chi_{[0,\tau^j_k]}(t) + \hat{u}_j \left( \frac{t - \tau^j_k}{\tau^j_{k^1_j} - \tau^j_k} \right) \chi_{[\tau^j_{k^1_j}, T]},
\]

\[
\hat{x}_j := x[\bar{x}_0, \bar{u}_0, \hat{u}_j, v],
\]
where $\tilde{u}_j \in AC(1)$ joins $\hat{u}_{k_j}^1(\tau^j) = \varphi_j(s_j)$ to $u(T)$ and $Var_{[0,1]} u_j \leq C|\varphi(s_j) - u(T)|$. We have $\ddot{x}_j(\tau^j) = \hat{x}_{k_j}^1(\tau^j)$, and by standard estimates it follows that $\sup_{t \in [0,T]} |\ddot{x}_j(t)| \leq \bar{R}$ for some $\bar{R} > 0$, and

$$|\ddot{x}_j(T) - \ddot{x}_j(\tau^j)| \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$  

Hence by (17), (8) and (17) we get

$$|\ddot{x}_j(T) - x(T)| \leq |\ddot{x}_j(T) - \ddot{x}_j(\tau^j)| + |\ddot{x}_{k_j}^1(\tau^j) - x_{k_j}^1(\tau^j)| +$$

$$|x_{k_j}^1(\tau^j) - x_{k_j}^1(T)| + |x_j^1(T) - x(T)| \leq$$

$$|\ddot{x}_j(T) - \ddot{x}_j(\tau^j)| + \frac{1}{j} + \tilde{\varepsilon}(j) + |x_j^1(T) - x(T)|.$$  

The r.h.s. of (20) approaches 0 since by (19) its first term goes to 0 and, being $x$ an E-BV loc limit solution, the last term approaches 0 too. Therefore, renaming the index $j$ in the sequence $(\ddot{x}_j, \tilde{u}_j)$ by $k$, it is not difficult to prove that the sequence $(\ddot{x}_k, \tilde{u}_k)$ verifies statements (i) and (ii) and, by (20), also (ii) of Definition 4.1.  

5. A further extension

For $u$ with bounded variation, the graph completion technique has been extended since the 90s to control systems of the form

$$(21) \quad \dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t), v(t)) \dot{u}_i(t) \quad \text{a.e. } t \in [0,T],$$

$$(22) \quad x(0) = \bar{x}_0, \quad u(0) = \bar{u}_0,$$

where the dependence on the ordinary control $v$ appears also in the coefficients $g_1, \ldots, g_m$ of the control derivatives $\dot{u}_i$. This notion has been applied to several problems (see [20], [19], [16] and the references therein). As mentioned in [1], this kind of equation is relevant in mechanical applications, for instance, when $u$ is a shape parameter and $v$ is a control representing an external force or torque and in min-max control problems where the adjoint equations may contain a $v$-dependent term multiplied by an unbounded control, like in (21) (see e.g. [5]). In this section we adapt the notion of extended BVS limit solution introduced in Definition 4.1 to (21), (22) and in Theorem 5.2 below we prove the one-to-one correspondence between such limit solutions and graph completion solutions to (21), (22). In this way we extend the result of [1, Thm. 4.2], where the same assertion is proved for $g_1, \ldots, g_m$ independent of $v$.

Throughout this section we assume that for every $i = 0, \ldots, m$, the control vector field $g_i : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$ is continuous, $(x, u) \mapsto g_i(x, u, v)$ is locally Lipschitz on $\mathbb{R}^n \times U$ uniformly in $v \in V$ and there exists $M > 0$ such that

$$|g_i(x, u, v)| \leq M(1 + |(x, u)|) \quad \forall (x, u, v) \in \mathbb{R}^n \times U \times V.$$  

The notion of extended BVS limit solution to (21), (22) that we are going to introduce coincides with the Definition 4.1, 3., for $g_1, \ldots, g_m$ not depending on $v$, but
the presence of the ordinary control in the $g_i$ for $i = 1, \ldots, m$ requires to take into account the interplay between $u$ and $v$. We distinguish the two situations ($v$ just in the drift or $v$ ‘everywhere’) by considering the more general control system

$$
(23) \quad \dot{x}(t) = g_0(x(t), u(t), v_1(t)) + \sum_{i=1}^m g_i(x(t), u(t), v_2(t)) \dot{u}_i(t) \quad \text{a.e. } t \in [0,T],
$$

with $v := (v_1, v_2)$ taking values in $V \times V$. For simplicity, we use the same notation of Definition 4.1 and still denote by $x[\bar{x}_0, \bar{u}_0, u, v]$ a regular solution to (23), (22) associated to $(u, v) = (u, v_1, v_2) \in AC(T) \times L^1(T) \times L^1(T)$.

**Definition 5.1 (Extended BV Solution).** Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) = (u, v_1, v_2) \in L^1(0,T) \times L^1(0,T) \times L^1(0,T)$ with $u(0) = \bar{u}_0$.

A map $x \in \mathcal{L}^1([0,T],\mathbb{R}^n)$ is called an extended BV limit solution, shortly E-BVS limit solution, of the Cauchy problem (23)-(22) corresponding to $(u, v)$ if there is a sequence of controls $(u_k, v_k) = (u_k, v_{1k}, v_{2k}) \in AC(T) \times L^1(T) \times L^1(T)$ such that $u_k(0) = \bar{u}_0$, the approximating inputs $u_k$ have equibounded variation on $[0,T]$ and

(i) the sequence $(x_k)$ of the Carathéodory solutions $x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k]$ to (23)-(22) verifies for every $\tau \in [0,T],$

$$
|(x_k, u_k)(\tau) - (x, u)(\tau)| + \|(x_k, u_k, v_k) - (x, u, v)\|_{L^1(\tau)} \rightarrow 0 \quad \text{as } k \rightarrow +\infty;
$$

(ii) there is some $\psi_2 \in L^1(\mathbb{R}_+, V)$ such that, setting $\sigma_k(t) := t + \text{Var}_{[0,t]}(u_k)$, $V_k := \text{Var}_{[0,T]}(u_k)$, one has $\|(v_{2k} \circ \sigma_k)^{-1} - \psi_2\|_{L^1(\mathbb{R}_+)} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$

**Theorem 5.2.** A map $x : [0,T] \rightarrow \mathbb{R}^n$ is a E-BVS-limit solution to (23)-(22) associated to $(u, v) \in BV(T) \times L^1(T) \times L^1(T)$ with $u(0) = \bar{u}_0$ if and only if it is a graph completion solution to (23), (22) associated to the same control.

Before proving the theorem, let us briefly describe the graph completion approach and give the precise definition of graph completion solution to (23), (2). For more details we refer the interested reader to [20] and the references therein.

For $L > 0$ and $S > 0$, let $\mathcal{U}_L(S)$ denote the subset of $L$-Lipschitz maps

$$(\varphi_0, \varphi) : [0,S] \rightarrow \mathbb{R}_+ \times U,$$

such that $\varphi_0(0) = 0$, and $\varphi'_0(s) \geq 0$, $\varphi_0(s) + |\varphi'(s)| \leq L$ for almost every $s \in [0,S]$. We set $L^1(S) := L^1([0,S], V)$.

We call space-time controls the elements $(\varphi_0, \varphi, \psi, S) = (\varphi_0, \varphi, \psi_1, \psi_2)$ with $S > 0$ and $(\varphi_0, \varphi, \psi_1, \psi_2) \in \bigcup_{S>0} \mathcal{U}_L(S) \times L^1(S) \times L^1(S)$. Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$. We denote by $\Gamma(\bar{u}_0)$ the subset of space-time controls verifying $(\varphi_0, \varphi)(0) = (0, \bar{u}_0)$ and $\varphi_0(S) = T$. The space-time control system is defined by

$$
(24) \quad \begin{cases}
\xi'(s) = g_0(\xi, \varphi, \psi_1)\varphi_0'(s) + \sum_{i=1}^m g_i(\xi, \varphi, \psi_2)\varphi_i'(s) \quad \text{for a.e. } s \in [0,S], \\
\xi(0) = \bar{x}_0
\end{cases}
$$

and we use $\xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$ to denote its solution. Notice that by just identifying regular controls $u$ and trajectories $x$ with their graphs and considering a time parametrization $t = \varphi_0(s)$, (21) can be embedded in the space-time system (24).
However, when a space-time control has \( t = \varphi_0(s) = \text{const} \) for \( s \in I := [s_1, s_2] \), the pair \((\xi, \varphi)\) describes on \( I \) the ‘instantaneous evolution’ at time \( t \) of the system; this is a way to define generalised controls and trajectories for the original control system in the extended, space-time setting. Now any space-time trajectory-control pair gives rise to a \textit{set-valued} notion of generalized solution \( x(t) := \xi \circ \varphi_0^{-1}(t) \) to (21), associated to a control \((u, v)\) with \((u, v)(t) \in (\varphi, \psi) \circ \varphi_0^{-1}(t)\); following [1], a (univalued) concept of graph completion solution is then obtained by the choice of a suitable selection.

Since the space-time control system (24) is rate-independent, without loss of generality we consider just controls verifying
\[ \varphi_0'(s) + |\varphi'(s)| = 1 \quad \text{for a.e. } s \in [0, S]. \]

\( \Gamma_f(\bar{u}_0) \) will denote the subset of such controls, to which we will refer to as \textit{feasible space-time controls}.

**Definition 5.3.** Let \((u, v) = (u_1, v_1, v_2) \in BV(T) \times L^1(T) \times L^1(T)\) and \( u(0) = \bar{u}_0 \in U \). We say that a space-time control \((\varphi_0, \varphi, \psi, S) \in \Gamma_f(\bar{u}_0)\) is a \textit{graph completion} of \((u, v)\) if
\[ \forall t \in [0, T], \exists s \in [0, S] \text{ such that } (\varphi_0, \varphi, \psi)(s) = (t, u(t), v(t)). \]

We call a \textit{clock} any strictly increasing, surjective function \( \sigma : [0, T] \to [0, S] \) such that
\[ (\varphi_0, \varphi)(\sigma(t)) = (t, u(t)) \quad \text{for every } t \in [0, T]. \]

**Definition 5.4.** Given a control \((u, v) \in BV(T) \times L^1(T) \times L^1(T)\) with \( u(0) = \bar{u}_0 \), let \((\varphi_0, \varphi, \psi, S)\) be a graph-completion of \((u, v)\) and let \( \sigma \) be a clock. Set \( \xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi] \). A map
\[ x : [0, T] \to \mathbb{R}^n, \quad x(t) := \xi \circ \sigma(t) \quad \forall t \in [0, T], \]
is called a \textit{graph completion solution} to (23), (2).

**Proof of Theorem 5.2.** Let \((u, v) \in BV(T) \times L^1(T) \times L^1(T)\) and \( u(0) = \bar{u}_0 \in U \). We begin by showing that a graph completion solution \( x \) to (23), (22) associated to \((u, v)\) is an E-BVS limit solution. By Definitions 5.3 and 5.4, there exist a feasible space-time control \((\varphi_0, \varphi, \psi, S) \in \Gamma(\bar{u}_0)\) and a surjective, strictly increasing function \( \sigma : [0, T] \to [0, S] \) such that, setting \( \xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi] \), one has
\[ (\xi, \varphi_0, \varphi, \psi) \circ \sigma(t) = (x(t), t, u(t), v(t)) \quad \forall t \in [0, T]. \]

By [1, Thm. 5.1] as revisited in [24, Thm. 4.2], there exists a sequence \( (\sigma_k) \) of absolutely continuous, strictly increasing maps \( \sigma_k : [0, T] \to [0, S] \), such that
\begin{itemize}
  \item[(i)] \( \sigma_k(0) = 0, \sigma_k(T) = S \), and
  \item[(26)] \( \sigma_k(t) \geq 1 \text{ for a.e. } t \in [0, T], \lim_{k \to +\infty} \sigma_k(t) = \sigma(t) \forall t \in [0, T]; \)
\end{itemize}

(ii) the maps \( \varphi_0 := \sigma_k^{-1} : [0, S] \to [0, T] \) are strictly increasing, 1-Lipschitz continuous, surjective and converge uniformly to \( \varphi_0 \) in \( [0, S] \).
We are going to show that the sequences \((u_k, v_k)\) and \((x_k)\) defined by
\[
    u_k := \varphi \circ \sigma_k, \quad v_k := \psi \circ \sigma_k, \quad x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k],
\]
verify all the requirements of Definition 5.1, so proving that \(x\) is a E-BVS limit solution of (21), (22) associated to \((u, v)\).

In view of definition (25), the pointwise convergence of \(u_k\) to \(u\) follows from the continuity of \(\varphi\). Moreover, the sequence \((u_k)\) has equibounded variation, since \(\text{Var}_{[0,T]}(u_k) = \text{Var}_{[0,S]}(\varphi)\) for every \(k\). In order to show that \(\lim_{k \to +\infty} \|v_k - v\|_{L^1(T)} = 0\), take an arbitrary \(\varepsilon > 0\) and consider a bounded, continuous map \(\tilde{\psi} : [0,S] \to \mathbb{R}^2\) such that
\[
    \int_0^S |\tilde{\psi}(s) - \psi(s)| \, ds < \varepsilon,
\]
(such \(\tilde{\psi}\) exists by well known density results). Hence
\[
    \int_0^T |v_k(t) - v(t)| \, dt = \int_0^T |\psi(\sigma_k(t)) - \tilde{\psi}(\sigma_k(t))| \, dt \leq \int_0^T |\psi(\sigma_k(t)) - \bar{\psi}(\sigma(t))| \, dt + \int_0^T |\bar{\psi}(\sigma(t)) - \bar{\psi}(\sigma(t))| \, dt \leq \int_0^T \bar{\psi}(\sigma(t)) \, ds + \int_0^T |\bar{\psi}(\sigma_k(t)) - \bar{\psi}(\sigma(t))| \, dt.
\]
where the last inequality follows from the properties of \(\sigma\) and \(\sigma_k\). Now the first and the third integrals in the r.h.s., by the (continuous) change of variable \(s = \sigma_k(t)\) and the discontinuous one \(s = \sigma(t)\) (see e.g. [11]) respectively, are both less than \(\varepsilon\), while the second integral tends to 0 by the Dominated Convergence Theorem, since \(\bar{\psi}\) is bounded and continuous. By the arbitrariness of \(\varepsilon > 0\), this concludes the proof that \(\lim_{k \to +\infty} \|v_k - v\|_{L^1(T)} = 0\). Since
\[
    v_{2_k} \circ \sigma_k^{-1} = \psi_2 \circ \sigma_k \circ \sigma_k^{-1} \equiv \psi_2,
\]
the condition \(\|(v_{2_k} \circ \sigma_k^{-1} - \psi_2) \chi_{[0,T] \times V}\|_{L^1(\mathbb{R}^+)} \to 0\) as \(k \to +\infty\) is trivially satisfied. It remains to show that \(x\) is the pointwise limit of \((x_k)\). To this aim, let us set \(\xi_k := \xi[\bar{x}_0, \bar{u}_0, \varphi_0_k, \varphi, \psi]\). By the continuity of the input-output map associated to the control system (21) (see [20, Thm. 4.1]) we derive that \((\xi_k)\) converges uniformly to \(\xi\) on \([0, S]\). Since \(x_k = \xi_k \circ \sigma_k\) on \([0, T]\), we finally obtain that, for every \(t \in [0, T]\), one has
\[
    \lim_{k \to +\infty} |x_k(t) - x(t)| = \lim_{k \to +\infty} |\xi_k(\sigma_k(s)) - \xi(\sigma(t))| = 0.
\]
Hence \(x\) is a E-BVS limit solution.

Let us now show that an E-BVS limit solution \(x\) to (23), (22) associated to \((u, v)\) is a graph completion solution. By Definition 5.1, there exist \(\psi_2 \in L^1(T)\) and a sequence \((u_k, v_k) \subset AC(T) \times L^1(T) \times L^1(T)\) with \(u_k(0) = \bar{u}_0\) and \(V_k := \text{Var}(u_k) \leq K\) for some \(K > 0\) such that, setting
\[
    \sigma_k(t) := t + \text{Var}_{[0,t]}(u_k) \quad (\leq S := T + K)
\]
\[
    \bar{\psi}(\sigma(t)) \equiv \psi_2,
\]
where the last inequality follows from the properties of \(\sigma\) and \(\sigma_k\). Now the first and
and \( x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k] \), one has
\[
\lim_{k \to +\infty} (x_k(t), u_k(t)) = (x(t), u(t)) \quad \text{for any } t \in [0, T],
\]
(28)
\[
\lim_{k \to +\infty} \int_0^T |v_k(t) - v(t)| \, dt = 0,
\]
\[
\lim_{k \to +\infty} \int_\mathbb{R}_+ |v_{2k} \circ \sigma_k^{-1}(s) - \psi_2(s)| \chi_{[0, T+V_k]} \, ds = 0.
\]

Arguing as in the proof of Theorem 4.3, Case 1, one can prove that it is possible to assume, without loss of generality, that \( v_{1k} = v_1 \) for every \( k \). Let \( \varphi_{0_k} : [0, S] \to [0, T] \) be the 1-Lipschitz continuous, increasing function such that
\[
\varphi_{0_k} := \sigma_k^{-1} \quad \text{on } [0, T + V_k], \quad \text{and} \quad \varphi_{0_k}(s) = T \quad \text{for all } s \in [T + V_k, S].
\]
Set \( \varphi_k := u_k \circ \varphi_{0_k} \). Then the sequence of space-time controls \( (\varphi_{0_k}, \varphi_k) \) is 1-Lipschitz continuous on \([0, S]\) and satisfies \( \varphi_{0_k}'(s) + |\varphi_k'(s)| = 1 \) for a.e. \( s \in [0, T + V_k] \) (and \( \varphi_{0_k}'(s) + |\varphi_k'(s)| = 0 \) for \( s > T + V_k \)). Therefore by Ascoli-Arzelà’s Theorem, taking if necessary a subsequence, still denoted by \( (\varphi_{0_k}, \varphi_k) \), it converges uniformly to a Lipschitz continuous function \( (\varphi_0, \varphi) \) such that \( \varphi_0'(s) + |\varphi'(s)| \leq 1 \) for \( s \in [0, S] \). Let us observe that \( (\varphi_0, \varphi) \) is a graph completion of \( u \), possibly not feasible (namely, not verifying the equality \( \varphi_0'(s) + |\varphi'(s)| = 1 \) a.e.). Indeed, for every \( t \in [0, T] \), there exist a subsequence \( (\sigma_{k'}(t)) \) and \( \sigma(t) \in [0, S] \) such that \( \lim_{k'} \sigma_{k'}(t) = \sigma(t) \).

Therefore, by the uniform convergence of \( (\varphi_{0_k}, \varphi_k) \) it follows that
\[
(\varphi_0, \varphi) \circ \sigma(t) = \lim_{k' \to +\infty} (\varphi_{0_k}, \varphi_{k'}) \circ \sigma_{k'}(t) = (t, u(t)).
\]

Set
\[
\psi_1 := v \circ \varphi_0, \quad \psi := (\psi_1, \psi_2),
\]
where \( \psi_2 \) is the same as in (28) and define the solution \( \xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi] \) associated to the space-time control \( (\varphi_0, \varphi, \psi, S) \). Moreover, let \( \psi_k := (v_1 \circ \varphi_{0_k}, v_2 \circ \varphi_{0_k}) \) and \( \xi_k := \xi[\bar{x}_0, \bar{u}_0, \varphi_{0_k}, \varphi_k, \psi_k] \). Clearly, \( x_k = \xi_k \circ \sigma_k \).

In order to prove that \( x \) is a graph completion solution, let us first verify that \( x = \xi \circ \sigma \). To this aim, we observe that this is true as soon as there exists a subsequence of \( (\xi_k) \) uniformly converging in \([0, S]\) to \( \xi \). In this case indeed, for every \( t \in [0, T] \), the pointwise convergence of \( \sigma_{k'}(t) \) to \( \sigma(t) \) implies that
\[
x(t) = \lim_{k' \to +\infty} x_{k'}(t) = \lim_{k'} \xi_{k'} \circ \sigma_{k'}(t) = \xi \circ \sigma(t).
\]

At this point, if we introduce the change of variable
\[
\eta(s) := \int_0^s \left[ \varphi'_0(r) + |\varphi'(r)| \right] \, dr \quad \forall s \in [0, S], \quad \tilde{V} := \eta(S) - T,
\]
denote by \( s(\cdot) : [0, T + \tilde{V}] \to [0, S] \) its the strictly increasing right-inverse, define the feasible space-time control
\[
(\tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}, \tilde{S}) := (\varphi_0 \circ s, \varphi \circ s, \psi \circ s, T + \tilde{V}),
\]
and the clock \( \tilde{\sigma} := \eta \circ \sigma \), we can easily obtain that \( x \) is a graph completion solution, since
\[
x = \xi \circ \sigma = \tilde{\xi} \circ \tilde{\sigma} \quad (\tilde{\xi} := \xi[\bar{x}_0, \bar{u}_0, \tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}]).
\]
To conclude the proof it remains to show that, eventually for a subsequence, one has
\[
\lim_{k \to +\infty} \sup_{s \in [0,S]} |\xi_k(s) - \xi(s)| = 0. \tag{29}
\]
Since both the derivatives \((\varphi'_0, \varphi')\), \((\varphi'_0, \varphi')\) are bounded, by standard estimates it follows that
\[
\sup_{s \in [0,S]} |\xi(s)|, \sup_{s \in [0,S]} |\xi_k(s)| \leq M := (|\bar{x}_0| + (m + 1)MS)e^{(m+1)MS}.
\]
Let us denote by \(\omega\) a modulus of continuity of \(g_0(x, u, \cdot), \ldots, g_m(x, u, \cdot)\) by \(\tilde{L}\) a Lipschitz constant of \(g_0, \ldots, g_m\) in \((x, u)\) uniformly w.r.t. \(v\), and by \(\bar{M}\) an upper bound for all the vector fields \(g_i, i = 0, \ldots, m\), in the compact set \(B_n(0, M) \times U \times V\). After some calculations, setting
\[
f_k(s) := \left| \int_0^s [g_0(\xi(r), \varphi(r), v_1 \circ \varphi_0(r))[\varphi'_0(r) - \varphi'_0(r)] + \right.
\]
\[
\left. \sum_{i=1}^m g_i(\xi(r), \varphi(r), \psi_2(r))[\varphi'_i(r) - \varphi'_i(r)] dr \right|,
\]
and
\[
\rho_{1k} := \int_0^s \omega(|v_1 \circ \varphi_0(k) - v_1 \circ \varphi_0(r)| \varphi'_0(k) dr,
\]
\[
\rho_{2k} := \sum_{i=1}^m \int_0^s \omega(|v_2 \circ \varphi_0(k) - \psi_2(r)| \varphi'_i(k) dr,
\]
by the Gronwall’s Lemma we get to
\[
|\xi_k(s) - \xi(s)| \leq e^{(m+1)Ls} \left( \sup_{s \in [0,S]} f_k(s) + \rho_{1k} + \rho_{2k} \right). \tag{30}
\]
The uniform convergence of \((\varphi_0, \varphi_k)\) to \((\varphi_0, \varphi)\) on \([0, S]\) implies that the maps \((\varphi'_0, \varphi'_k)\) tend to \((\varphi'_0, \varphi')\) in the weak* topology of \(L^\infty([0,S], \mathbb{R}^{1+m})\), so that \(f_k(s)\) tends to 0 as \(k \to +\infty\) for every \(s \in [0, S]\). The uniform convergence to 0 of the \(f_k's\) now follows from Ascoli-Arzelà Theorem, for the \(f_k's\) are equibounded and equi-Lipschitzian. By \((28)\) and the inequality \(|\varphi'_i| \leq 1\) a.e., we derive that \(\lim_{k \to +\infty} \rho_{2k} = 0\).

By a time-change, we get
\[
\int_0^S |v_1 \circ \varphi_0(k) - v_1 \circ \varphi_0(r)| \varphi'_0(r) dr = \int_0^T |v_1(t) - v_1 \circ \varphi_0 \circ \sigma_k(t)| dt.
\]
Hence, if we show that
\[
\lim_{k \to +\infty} \int_0^S |v_1 \circ \varphi_0(k) - v_1 \circ \varphi_0(r)| \varphi'_0(r) dr = 0, \tag{31}
\]
then there exists a subsequence of \((v_1 - v_1 \circ \varphi_0 \circ \sigma_k)\) converging to 0 a.e. on \([0, T]\), and by the Dominated Convergence Theorem we obtain that, for such subsequence,
\[
\rho_{1k} = \int_0^T \omega(|v(t) - v \circ \varphi_0 \circ \sigma_k(t)|) dt \to 0 \quad \text{as} \quad k \to +\infty, \tag{32}
\]
so concluding the proof of \((29)\).
Since $|\varphi'_{0_k}| \leq 1$, when $v_1$ is a continuous function (31) holds true owing to the uniform continuity of $v_1$ and to the uniform convergence of $\varphi_{0_k}$ to $\varphi_0$ on $[0,S]$. For $v_1 \in L^1(T)$, $\forall \varepsilon > 0$ there exists, by density, $\tilde{v}_1 \in C_c([0,T], \mathbb{R}^l)$ such that 
\[ \int_0^T |\tilde{v}_1(t) - v_1(t)| \, dt \leq \varepsilon. \]

Hence we get 
\[ \int_0^S |v_1 \circ \varphi_{0_k}(s) - v_1 \circ \varphi_0(s)|\varphi'_{0_k}(s) \, ds \leq \int_0^S |v_1 \circ \varphi_{0_k}(s) - \tilde{v}_1 \circ \varphi_{0_k}(s)|\varphi'_{0_k}(s) \, ds + \int_0^S |	ilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)|\varphi'_0(s) \, ds. \]

Performing the change of variable $t = \varphi_{0_k}(s)$, the first integral on the r.h.s. is smaller than $\varepsilon$, while the second one converges to 0 because $\tilde{v}_1$ is continuous. For the third integral on the r.h.s., taking into account that $v_1$ and $\tilde{v}_1$ are bounded maps, by the weak* convergence of $\varphi_{0_k}$ to $\varphi'_0$ we derive that 
\[ \int_0^S |\tilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)|\varphi'_0(s) \, ds \to \int_0^S |\tilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)|\varphi'_0(s) \, ds \]

as $k \to +\infty$, and the last term is smaller than $\varepsilon$ by the change of variable $t = \varphi_0(s)$. By the arbitrariness of $\varepsilon > 0$ this concludes the proof of (31). \[\square\]

**References**

[1] M.S. Aronna, F. Rampazzo, $L^1$ limit solutions for control systems, *J. Differential Equations*, 258 no. 3 (2015), 954–979.

[2] M.S. Aronna, M. Motta and F. Rampazzo, Infimum gaps for limit solutions, *Set-Valued Var. Anal.*, 23 no. 1 (2015), 3–22.

[3] A. Arutyunov, D. Karamzin and F. Pereira, On a generalization of the impulsive control concept: controlling system jumps, *Discrete Contin. Dyn. Syst.*, 29 no. 2 (2011), 403–415.

[4] J.-P. Aubin, *Impulse Differential Equations and Hybrid Systems: A Viability Approach*. Lecture Notes. University of California, Berkeley, 2000

[5] E. Barron, H. Ishii, The Bellman equation for minimizing the maximum cost, *Nonlinear Anal.*, 13 no. 9 (1989), 1067–1090.

[6] A. Bressan, B. Piccoli, *Introduction to the mathematical theory of control*. AIMS Series on Applied Mathematics, 2. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.

[7] A. Bressan, F. Rampazzo, Moving constraints as stabilizing controls in classical mechanics. *Arch. Ration. Mech. Anal.*, 196 no. 1 (2010), 97–14.

[8] A. Bressan, F. Rampazzo, On differential systems with vector-valued impulsive controls. *Boll. Un. Mat. Ital. B (7) 2 no. 3* (1988), 641–656.

[9] A. Bressan, F. Rampazzo, Impulsive control systems with commutative vector fields. *J. Optim. Theory Appl.*, 71 no. 1 (1991), 67–83.

[10] A. Bressan, F. Rampazzo, Impulsive control systems without commutativity assumptions. *J. Optim. Theory Appl.*, 81 no. 3 (1994), 435–457.

[11] N. Falkner, G. Teschl, On the substitution rule for Lebesgue-Stieltjes integrals. *Expo. Math.*, 30 no. 4 (2012), 412–418.

[12] S. L. Fraga, R. Gomes and F.L. Pereira, An impulsive framework for the control of hybrid systems, in *Proc. 46 IEEE Conf. Decision Control*, (2007) 54445449.

[13] E. Goncharova, M. Staritsyn, Optimization of Measure-Driven Hybrid Systems, *J. Optim. Theory Appl.*, 153 (2012), 139–156

[14] M. Guerra, A. Sarychev, Fréchet generalized trajectories and minimizers for variational problems of low coercivity, *J. Dyn. Control Syst.*, 21, no. 3 (2015), 351–377.
[15] W. Haddad, V. Chellaboina and S. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton University Press, Princeton, 2006.

[16] D.Y. Karamzin, V.A. de Oliveira, F.L. Pereira and G.N. Silva, On the properness of an impulsive control extension of dynamic optimization problems, *ESAIM Control Optim. Calc. Var.*, 21, no. 3 (2015), 857–875.

[17] A. Kurzhanski, P. Tochilin, Impulse Controls in Models of Hybrid Systems, *Differential Equations*, 45 no. 5 (2009), 731–742.

[18] T. Lyons, Z. Qian, *System control and rough paths*. Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2002.

[19] M. Miller, E. Y. Rubinovich, *Impulsive control in continuous and discrete-continuous systems*. Kluwer Academic/Plenum Publishers, New York, 2003.

[20] M. Motta, F. Rampazzo, Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential Integral Equations*, 8 no. 2 (1995), 269–288.

[21] M. Motta, F. Rampazzo, Dynamic programming for nonlinear systems driven by ordinary and impulsive controls. *SIAM J. Control Optim.*, 34 no. 1, (1996), 199–225.

[22] M. Motta, C. Sartori, Uniqueness results for boundary value problems arising from finite fuel and other singular and unbounded stochastic control problems, *Discrete Contin. Dyn. Syst.*, 21 no. 2 (2008), 513–535.

[23] M. Motta, C. Sartori, On asymptotic exit-time control problems lacking coercivity. *ESAIM Control Optim. Calc. Var.*, 20 no. 4 (2014), 957–982.

[24] M. Motta, C. Sartori, Unbounded variation and solutions of impulsive control systems, preprint, http://arxiv.org/abs/1705.01724

[25] G. Silva, R. Vinter, Measure driven differential inclusions, *J. Math. Anal. Appl.*, 202 no. 3 (1996), 727–746.

[26] H. Whitney, Functions differentiable on the boundaries of regions, *Ann. of Math.*, 35 no. 3 (1934), 482–485.

[27] P. Wołęnski, S. Żabić, A sampling method and approximation results for impulsive systems, *SIAM J. Control Optim.*, 46 no. 3 (2007), 983–998.

[28] K. Yunt, Modelling of Mechanical Blocking, *Recent Researches in Circuits, Systems, Mechanics and Transportation Systems*, (2011), 123–128.

E-mail address: motta@math.unipd.it
E-mail address: sartori@math.unipd.it