On the effect of random inhomogeneities in Kerr media modelled by a nonlinear Schrödinger equation

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Abstract

We consider the propagation of optical beams under the interplay of dispersion and Kerr nonlinearity in optical fibres with impurities distributed at random uniformly on the fibre. By using a model based on the nonlinear Schrödinger equation we clarify how such inhomogeneities affect different aspects such as the number of solitons present and the intensity of the signal. We also obtain the mean distance for the signal to dissipate to a given level.

1. Introduction

In this paper we consider the evolution of a complex electric field \( u(x, t) \) in a nonlinear Kerr medium which has constant dispersion and losses and, in addition, impurities at certain points \( x_n < x_{n+1} \), which occur randomly on the fibre. We suppose that these loss elements cause the ’input’ signal \( u(x_n, t) \) to abruptly decrease to an ’output’ value \( u(x_n, t) = e^{-\gamma_n} u(x_n, t) \), where \( e^{-\gamma_n} < 1 \) measures the dimming ratio and \( u(x_n, t) \), say, denotes the limit value from the left. Assuming the validity of the self-focusing nonlinear Schrödinger (NLS) equation (see [1]) as a model of ideal transmission of one-dimensional beams under the paraxial approximation, we find that the above situation must be described by a perturbed NLS equation which written in dimensionless units reads

\[
\begin{align*}
&i \partial_x u + \partial_{tt} u + 2 |u|^2 u \\
= &\ i \left[ -\Gamma u + \sum_n \left( e^{-\gamma_n} - 1 \right) \delta(x - x_n) u(x_n, t) \right],
\end{align*}
\]

(1)

where the Dirac-delta terms account precisely for the amplitude decrease at impurities; further, \( \Gamma \geq 0 \) is the normalized loss coefficient. To avoid extra mathematical difficulties we do not consider a compensated loss mechanism;

3 See also [2] for a good discussion on the Kerr effect and the validity of the NLS equation to model pulse propagation in nonlinear optical fibres.

this will be the subject of a future publication. We also remark that with minor changes our results may be applicable to other physically interesting systems such as Bose–Einstein condensates or propagation of optical beams in a nonlocal Kerr medium.

It appears that while the effect of continuous random noise—or white noise—on NLS solitons has been well studied in the literature (see [3–7]) far less is known as regards the effects of sudden, discrete random perturbations. We intend to clarify how these inhomogeneities—which may be relevant for long-distance fibre-optic communication systems—affect the evolution of the pulse. We remark that perturbations involving delta masses also appear related to erbium-doped amplifiers and dispersion management, see [8–10]. In such a context, the positions of the amplifiers \( x_n \) are deterministic and periodically disposed, \( x_n = nx_1 \), while the strengths are constant and negative, \( \gamma_n = -\Gamma x_1 \). Kodama and Hasegawa [11] generalize the latter ideas to a random context but, unlike us, maintain the amplifier interpretation and consider the distribution of the ’intensity’ of the signal only in the limit when both \( \Delta_n \) (here \( \Delta_n = x_n - x_{n-1} \)) and \( \gamma_n \) tend to zero. Thus while these ideas have some bearing with our work, both the physical interpretation and the mathematical model are quite different.

We begin the study of equation (1) by first analyzing the case when there are no deterministic losses: \( \Gamma = 0 \), which
from a mathematical viewpoint is simpler to understand. We show that upon performing a change of dependent variable, the resulting formula can be piecewise related to the unperturbed NLS equation. Let us recall here that the classical NLS equation

\[ i\Theta_x + \Theta_{tt} + 2|\Theta|^2\Theta = 0, \quad \Theta(0, t) = \varphi(t) \quad (2) \]

was first derived by Zakharov [12] as an equation of slowly varying wave packets of small amplitude. He showed that despite its nonlinear character the corresponding initial value problem (IVP) can be reduced to a linear problem (the Zakharov–Shabat spectral problem) by the so-called inverse scattering transform (IST)—see [13, 14] for general background on the NLS equation and the IST method. Its interest has been further underlined by the realization that it also models the evolution of the complex amplitude of an optical pulse in a nonlinear fibre [1]. Applications of the NLS equation to optical communications and photonics are nowadays standard [1, 8, 15, 16].

We devote section 2 to the study of the nonlinear dynamics of the classical solitary waves within this regime, and we show how impurities result in the appearance of radiation and general broadening of the signal. In particular, we find that solitons may be destroyed by the action of just one impurity.

When \( \Gamma \neq 0 \), equation (1) is no longer solvable in an analytic way IST; however, we find—see section 3—that the evolution of intensity, momentum and position of the pulse can be described precisely and that, under certain natural assumptions, their average values decrease exponentially due to the ‘impurities’: concretely, we suppose that the positions and strengths of impurities are statistically independent of themselves; we also suppose that in any interval \([0, x]\) impurities are uniformly distributed (provided its number is given). Nevertheless the frequency and position of the pulse are not affected.

In section 4 we study the mean distance for the signal’s intensity to attenuate to a given level due to the impurities. In applications, this level could be a recommended threshold intensity to attenuate to a given level due to the impurities. Alternatively, \( S(x) = s + \Gamma x + \sum_{j=1}^{N(x)} y_j \) if \( x_0 \leq x < x_{m+1} \).

We now focus our attention on the main equation (3). Due to the factor \( \zeta(x) \) this is a generalized NLS equation with \( x \)-dependent coefficients and generically non-integrable. We first consider the simpler case when the loss vanishes: \( \Gamma = 0 \). It turns out that, even though the resulting equation has random discontinuous coefficients, it can be piecewise reduced to an integrable equation whereupon we show how to obtain the evolution of an initial pulse (see [17] for related considerations). The reasoning in the rest of this section is essentially independent of the sequences \( \Delta_n \equiv x_n - x_{n-1} \) and \( \gamma_n \). Nevertheless we suppose that both are the sequences of positive, independent, equally distributed random variables and that \( \Delta_j \) and \( \gamma_j \) are also independent for all \( n, m \).

Note that all these assumptions are physically well founded as they imply, say, that the knowledge of the position of a given impurity does not provide any information on the location of the remaining ones. The further assumption that \( \Delta_n \) is exponentially distributed: \( \Pr(\Delta_n \geq x) = e^{-x/\lambda} \), where \( \lambda = \langle \Delta_n \rangle^{-1} \) is a certain parameter, is natural from physical principles. It has several fruitful consequences as then there follows that the number \( N(x) \) of impurities that occur on \([0, x]\) has Poisson distribution with the parameter \( \lambda x \), and that they are uniformly distributed on the interval.

It further implies the memory-less property: the distribution of impurities on \([x, x + \Delta x]\) remains unaffected given that none was observed on \([0, x]\). By contrast, we consider here a general probability density function (PDF) \( h(y) \) of \( \gamma_n \): \( \Pr(y_0 \leq y \leq y + dy) = h(y) \ dy \).

For the sake of being specific let us consider the case when the initial data are those corresponding to the classical solitary wave pulse (or soliton), namely \( u(0, t) = 2\eta \sech(2\eta t) e^{2i\xi_0} \equiv \psi(0)(t) \), where the real parameters \( \eta \) and \( \xi \) give, up to a constant, the wave’s amplitude and the carrier velocity, respectively. Note that up to the first impurity \( \upsilon(0)(x, t) \equiv u(x, t), 0 \leq x \leq x_1 \), solves the IVP

\[ iv_0(x, t) + v_0(x, t) + 2|v_0|^2v_0 = 0, \quad \upsilon(0)(0, t) = \psi(0)(t) \quad (6) \]

This is the standard IVP for the NLS equation, and hence the solution for \( 0 \leq x \leq x_1 \) is the classical soliton

\[ u(x, t) = \upsilon(0)(x, t) = 2\eta \sech(2\eta t - 4\xi x) e^{4i\xi t + 4(i\eta - \xi^2)x}. \quad (7) \]

We adopt the convention and terminology of standard NLS theory wherein \( t \) is the space and \( s \) a temporal variable, a situation opposite to that which occurs in optics.
This requirement fixes \( \psi \) for \( \xi = 0 \), i.e.
\[ \Gamma_1 \neq 0 \]
not continuous at 0. Moreover, if \( x > x_1 \), then
\[ u(x, t) = e^{-\gamma x} \psi(0). \]

Remarkably, this equation can be reduced again to NLS: by using both temporal and translational invariance of the NLS equation one can prove that
\[ e^{2\gamma} u(x, t) = u^{(1)}(x, t) = e^{i\int_0^t \xi x \, dx} \Theta(x - \Delta \bar{\xi} x, t - 4\bar{\xi} x_1), \]
where \( \Theta \) is the solution to the NLS equation (7) with data \( \Theta(0, t) = e^{-\gamma t} \psi(0)(t). \)

As commented, we continue this solution to the interval \( x_1 \leq x \leq x_2 \) by requiring \( u(x, t) \) to be continuous at \( x = x_1 \). This requirement fixes \( u^{(1)}(x, t) \equiv u(x, t) \), \( x_1 \leq x \leq x_2 \), to satisfy the nonlinear partial differential equation
\[ i\psi_x^{(1)} + \psi_t^{(1)} + 2 e^{-2\gamma} |\psi^{(1)}|^2 \psi^{(1)} = 0, \quad \text{with} \]
\[ \psi^{(1)}(x_1, t) = 2\eta \sech(\gamma(t - 4\bar{\xi} x_1)), \]
where \( \eta \) is the modified Bessel function of zero order.

\[ e^{\gamma \xi} u(x, t) = u^{(1)}(x, t) = e^{i\int_0^t \xi x \, dx} x_1 + \gamma_1 \Theta(x - x_1, t - 4\bar{\xi} x_1), \]
where \( \Theta(x, t) \) is distributed in an exponential way between jumps.

\[ u(x, t) \] is solved by requiring
\[ u(x, t) = e^{-\gamma x} \psi(0)(t). \]

\[ \Theta(x, t) \] satisfies the nonlinear partial differential equation
\[ i\psi_x + \psi_t + 2 e^{-2\gamma} |\psi|^2 \psi = 0, \quad \text{with} \]
\[ \psi(0) = 2\eta \sech(\gamma(t - 4\bar{\xi} x_1)). \]

Remarkably, this equation can be reduced again to NLS: by using both temporal and translational invariance of the NLS equation one can prove that
\[ e^{\gamma \xi} u(x, t) = u^{(1)}(x, t) = e^{i\int_0^t \xi x \, dx} \Theta(x - x_1, t - 4\bar{\xi} x_1), \]
where \( \Theta \) is the solution to the NLS equation (7) with data \( \Theta(0, t) = e^{-\gamma t} \psi(0)(t). \)

Thus the solution \( u(x, t) \) to equation (1) with data \( u(0, t) = \psi(0)(t) \) is given for \( x_1 \leq x \leq x_2 \) in terms of the solution \( \Theta(x, t) \) to the NLS equation (7) with data \( \Theta(0, t) = e^{-\gamma t} \psi(0)(t) \). Note that, unlike \( u(x, t) \), \( u^{(1)}(x, t) \) is not continuous at \( x = x_1 \). The determination of the specific form of the last function \( \Theta(x, t) \) requires solving a linear spectral problem. The procedure is awkward but fortunately the main features of the solution may, to a large extent, be determined avoiding these complexities. We note that the solution that evolves from data \( \Theta(0, t) = e^{-\gamma t} \psi(0)(t) \) is no longer a soliton but a complicated pulse that contains radiation, in addition to the soliton. The former component has a much weaker rate of decay than the latter; concretely, it decays as the corresponding solution for the linearized Schrödinger equation (i.e. as \( x^{-\frac{1}{2}} \), see [13]). The prediction of the exact form of the solution past the first impurity is a difficult matter; however, we may say that this pulse will contain radiation and, at most, one soliton, which will be travelling in the midst of the radiation cloud and interact with the background. Further, if \( \gamma_1 > \gamma_1 \), the arriving soliton at \( x = x_1 \)—cf. equation (7)—is destroyed by the action of the first impurity after \( x_1 \). Hence the resulting configuration for \( x > x_1 \) consists solely of radiation. To be specific, suppose that the jump PDF \( h(\cdot) \) has an exponential distribution with mean 1/\( \alpha \). Then, after the first impurity, the soliton disappears with probability bounded below by \( \Pr(\gamma_1 \geq \gamma) = e^{-\alpha \gamma} \), where \( \gamma = 1.41 \).

Finally, we mention that by using similar ideas one can extend the solution to \( x > x_n \) by solving (3) with data \( \psi^{(n-1)}(x_n, t) \), where as before \( \psi^{(n)}(x, t) \) denotes the general solution \( \psi(x, t) \) restricted on \( x_{n-1} \leq x \leq x_n \). Translation invariance allows one to reduce this to the NLS equation with new data which involves a contraction factor \( e^{-\gamma(n-1)\xi} \). Eventually, this dimming of the initial signal results in a disappearance of the starting solitons into radiation, an indication that, as a result of impurities, broadening of the signal takes place. We skip the mathematical details.

3. General case with deterministic loss and impurities

When \( \Gamma > 0 \), equation (3) can be mapped into the so-called dispersion-managed NLS equation, which, unfortunately, is not solvable in an analytic way, either by using IST or by any other method. It is then remarkable that the evolution of the main physically observable functionals can be discerned in an exact way. Consider the following quantities:
\[ M(x) = \int_{-\infty}^{\infty} |u(x, t)|^2 \, dt, \]
\[ P(x) = i \int_{-\infty}^{\infty} \bar{u}(x, t) u_t(x, t) \, dt \quad \text{and} \]
\[ Q(x) = \int_{-\infty}^{\infty} t |u(x, t)|^2 \, dt, \]

This stems from the fact that the condition \( \mathbb{E}^2 \mathbb{E}(2\mathbb{E}) < 1 \) on the initial data guarantees that no solitons will be formed upon evolution \([13, 18]\). Here \( \mathbb{E}(\cdot) \) denotes the modified Bessel function of zero order and \( \mathbb{E} = \int_{-\infty}^{\infty} |\Theta(0, t)| \, dt = \pi e^{-\gamma \xi} \).
where \( M(x) \) and \( P(x) \) are the (accumulated) intensity and momentum of the signal, respectively, at a position \( x \), while \( Q(x)/M(x) = T(x) \) is the pulse position. The functional \( P(x)/M(x) = \Omega(x) \) is interpreted as the pulse-centre frequency. The singular nature of the delta terms prevents us from determining the relevant evolution by manipulating equation (1). Nevertheless, one can rely again on the decomposition \( n(x,t) = \zeta(t) \nu(x,t) \) and use equation (3). Then, the proper manipulation of the latter expressions yields that

\[
M(x) = M(0) e^{-2S(x)}, \\
P(x) = P(0) e^{-2S(x)} \quad \text{and} \\
Q(x) = [Q(0) - 2P(0)x] e^{-2S(x)}.
\]

Thus the effect of the presence of impurities results in the decomposition of (2) into a multiplicative random factor \( e^{-2S(x)} \) in both intensity and momentum. Note however that \( \Omega(x) = \Omega(0) \) and \( T(x) = [T(0) - 2\Omega(0)x] \), and hence that inhomogeneities have no effect whatsoever on position and frequency, a fact that accords with physical intuition.

It is therefore of interest to evaluate the mean amplitude decrease. We do so by first assuming that previously \( n \) defects have occurred: \( N(x) = n \). Let \( E_\nu \) denote statistical averaging and \( E(\zeta^2(x) | N(x) = n) \) be the mean value of \( \zeta^2(x) \) knowing that exactly \( n \) jumps have occurred on \([0, x]\). Note that given this information one has \( S(x) = \Gamma x + \sum_{j=1}^n \eta_j \), i.e. only the uncertainty regarding the value of the \( \eta_j \)'s remains but not that associated with the number of summands \( N(x) \). In view of the assumed statistical independence we have that the mean factorizes as

\[
E(\zeta^2(x)|N(x) = n) = \left( e^{-2\Gamma x} \prod_{j=1}^n e^{-2\eta_j} \right)
\]

\[
e^{-2\Gamma x} \prod_{j=1}^n (e^{-2\eta_j}) = e^{-2\Gamma x} Q_r^2,
\]

where \( Q_r \equiv E[\exp(-r\eta_j)] = \int_0^\infty e^{-y\eta} h(y) \, dy < 1 \) is the Laplace transform of the jump-size PDF. The mean intensity is obtained by further averaging with respect to the number of impurities:

\[
E[M(x)] = M_0 E[\zeta^2(x)]
\]

\[
= M_0 \sum_{n=0}^\infty \frac{(\lambda x)^n e^{-\lambda x}}{n!} E(\zeta^2(x) | N(x) = n)
\]

\[
= M_0 e^{-2\Gamma x + (1 - Q_r)x}, \quad (8)
\]

where \( M(0) = M_0 \) and we used that if \( \Delta_j \) has exponential distribution, i.e. if \( Pr(\Delta_j > x) = e^{-\lambda x} \) for some \( \lambda > 0 \), then \( N(x) \), the number of defects on \([0, x]\), is Poisson distributed: \( Pr(N(x) = n) = (\lambda x)^n e^{-\lambda x} / n! \). Hence we obtain that the existence of defects implies an exponential decrease in the field’s intensity and momentum at a rate \( 2\lambda(1 - Q_r) \), an effect which might result in the degradation of the bit patterns. This additional decrease is in agreement with the results of section 2 wherein it was proved that when \( \Gamma = 0 \) impurities dim the initial’s signal amplitude exponentially which eventually results in the disappearance of the solitons into radiation.

4. Mean half-life

A natural related problem of interest is determining the distance \( x \) at which \( M(x) \) dissipates from a starting value \( M_0 \) to a given level \( M_1 \), such that \( M(x) = M_1 \). For convenience we set \( M_1 = M_0 e^{-2b} \) and hence require \( S(x) = b \). This distance could be considered as a threshold value below which the signal is no longer reliable (it gives the mean half-life of the signal if \( M_0 = 2M_1 \)). In the deterministic case \( (\lambda = 0) \) this distance follows inverting \( M_1 = M_0 \exp(-2T(x)) \)

\[
x = \frac{\lambda}{2} \log \frac{M_0}{M_1}.
\]

When inhomogeneities are present, \( x \) is a random variable whose mean is not obtained by inverting equation (8)—as it might have been naively thought. Instead, we reason as follows: call \( x' \), see figure 1, the (random) distance that the generalized process \( S(x) \) with the initial value \( S(0) = s \)—cf equation (5)—covers to go beyond the level \( b \). It turns out (see the appendix) that \( X(s) \equiv E(x') \) satisfies the linear integral equation

\[
X(s) = \frac{1 - e^{-2s}}{\lambda} \left( \frac{1}{2} + \frac{e^{-b}}{s} \right) \int_0^s dy \left( \frac{1 - e^{-y}}{s} \right), \quad (9)
\]

where \( \varrho = \frac{x'}{x} \), and we recall that \( h(x) \) is the density of \( \eta \).

This equation can be solved in a closed form by the Laplace transformation. We consider again the case corresponding to a jump PDF also exponential with mean \( \sigma^{-1} = \langle \eta \rangle \), i.e. \( h(x) = \sigma e^{-\sigma x} \), \( \sigma > 0 \). If \( k = \lambda + \sigma \Gamma \), the Laplace transformation yields the solution to (9) as

\[
X(s) = \frac{\Gamma}{k} + \frac{\lambda}{k^2} (1 - e^{-s\varrho}).
\]

The mean distance for the amplitude to decrease to \( M_1 \) follows letting \( s = 0 \) and \( b = \frac{1}{2} \log \frac{M_0}{M_1} \) as

\[
E(x) = E(0) = \frac{1}{2(\Gamma + \lambda/\sigma)} \log \frac{M_0}{M_1} + \frac{\lambda}{k^2} \left[ 1 - \left( \frac{M_1}{M_0} \right)^{2} \right]. \quad (10)
\]

Note how our analysis corrects in a significant way the situation corresponding to an impurity-free medium, where \( x = \frac{1}{\sigma} \log \frac{M_0}{M_1} \)—a formula which is recovered by setting \( \lambda = 0 \). Another interesting limit is that of the vanishing deterministic loss rate \( \Gamma = 0 \). The mean attenuation distance can only be accounted to the presence of impurities and reads

\[
E(x) = \lambda + \frac{\sigma}{\sigma} \log \frac{M_0}{M_1}.
\]

The first term is the mean time for the first jump at \( x_1 \) to happen; the logarithmic correction corresponds to the mean time to go beyond the level \( b \) after the first jump. Actually, this rate rules the mean dissipation distance whenever \( M_0 \gg M_1 \) and \( \lambda/\sigma \gg \Gamma \). In figure 2 we perform a plot of this function. Note, by contrast, how the distance implied inverting equation (8), namely

\[
E(M(x)) = M_0 \exp \left[ -2x \left( \Gamma + \frac{\lambda}{\sigma+2} \right) \right], \quad (11)
\]

and therefore

\[
x = 2 \left( \Gamma + \frac{\lambda}{\sigma+2} \right)^{-1} \log \frac{M_0}{M_1}, \quad (12)
\]

deviates from the correct result, equation (10), and fails to capture the sharp behaviour occurring for \( M_1 \approx M_0 \). The error increases as \( \Gamma \) decreases.
Figure 2. Mean distance in terms of $M_0/M_i$ for $\lambda = 2.0$, $\sigma = 3.0$ while $\Gamma = 0.5$ (red line) and $\Gamma = 0.05$ (blue line) as follows from (10). Note how in the latter case $\bar{X}(0)$ jumps an amount $\langle \Delta_x \rangle = 0.5$ right after the origin. The green and magenta curves represent the (incorrect) mean distances implied by equation (12) with the above parameters.

5. Conclusions

We have analysed how the existence of randomly distributed impurities affects the evolution of an optical pulse in a nonlinear Kerr medium with constant dispersion and loss $\Gamma$. We suppose that the unperturbed situation is described by the NLS equation. When the deterministic loss vanishes, it is shown by changing the dependent variable that the resulting equation can still be piecewise related to the unperturbed NLS equation. The effect of impurities in the nonlinear propagation is pinpointed. In particular we address the issue of how they affect the initial solitons and the possibility of dissipating them into radiation. In the general, non-solvable $\Gamma \neq 0$ case we show that while impurities do not influence the frequency and position of the signal, they induce an exponential decrease of the main physical observables’ intensity and momentum and hence a general degradation. We also determine the mean half-life or mean distance for the signal to dissipate to a given threshold value. We find that this distance satisfies a certain integral equation. Its analysis shows that impurities result in a general degradation. We also determine the mean dissipation distance. To overcome these effects the addition of amplifiers is in order.

Appendix

We sketch the derivation of the integral equation (9) (see [19] for similar ideas in a financial context with $\Gamma = 0$). With $S(0) = s$ there are three possibilities for the future evolution. If the first jump satisfies $x_1 > b$, then $S$ reaches the level $b$ at $x = \varrho$. If this is not the case and if the jump at $x_1$ satisfies $s + \Gamma x_1 + \gamma_1 > b$, then the process goes past $b$ at $x = x_1$. Otherwise the process still remains within $[0, b)$ at $x = x_1$ and starts afresh with an initial value $S(x_1) = s + \Gamma x_1 + \gamma_1 < b$ (hence the process will exit $[0, b)$ at $x_1 + x_1^{s+\Gamma x_1+\gamma_1}$). Upon appropriate rearrangement this reasoning leads to

$$x' = \varrho \theta(x_1 - \varrho) + x_1 \theta(q - x_1) + x_1^{s+\Gamma x_1+\gamma_1} \theta(b - s - \Gamma x_1 - \gamma_1) \theta(q - x_1).$$

(A.1)

Averaging this relationship yields first

$$E(\varrho \theta(x_1 - \varrho) + x_1 \theta(q - x_1)) = \int_0^\varrho e^{-\lambda l} dl = \frac{1 - e^{-\lambda q}}{\lambda}.$$ 

Further by taking an average in the last term of equation (A.1) and conditioning in an appropriate way one can prove that

$$E(x_1^{s+\Gamma x_1+\gamma_1} \theta(b - s - \Gamma x_1 - \gamma_1) \theta(q - x_1)) = \frac{1}{\Gamma} \int_0^{b-s} dl e^{\frac{s-(b-s)}{\Gamma}} \int_0^l dy X(y + b - l) h(y).$$

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