Embeddings of quivers in derived categories of surfaces

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January 20, 2015

Abstract
In this article we give some restrictions on the finite-dimensional algebras occurring as endomorphism algebras of strong, but not necessarily full, exceptional collections on smooth projective surfaces. In particular, these results can be applied to path algebras of quivers.

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1 Introduction
There is a remarkable connection between the representation theory of finite-dimensional algebras and algebraic geometry, given by the theory of semi-orthogonal decompositions and exceptional collections. When applicable, this allows one to identify the bounded derived category of a smooth projective variety with the bounded derived category of a directed quiver with relations. At present

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however, it is not at all clear which triangulated categories admit full exceptional collections.

From now on let $k$ denote an algebraically closed field of characteristic 0. Consider a triangulated category of the form $D^b(\text{mod } / A)$, for some finite-dimensional $k$-algebra $A$. In a recent preprint, Orlov [16] showed that for any such $A$ of finite global dimension, there exists an admissible embedding

$$D^b(\text{mod } / A) \hookrightarrow D^b(\text{coh} / X), \quad (1.1)$$

for some smooth projective variety $X$. In fact, $X$ is constructed in such a way that it admits a full exceptional collection. So in particular every finite-dimensional algebra can be embedded into a triangulated category with a full exceptional collection. This result is already implicitly contained in Iyama’s beautiful paper [12], which shows that there always exists a quasi-hereditary algebra $\Delta$ and an idempotent $e$ such that $A = e\Delta e$. The conclusion follows from the fact that quasi-hereditary algebras always have full exceptional collections.

Somewhat surprisingly, not every finite-dimensional algebra of finite global dimension admits a full exceptional collection itself, so it is interesting to know more about these embeddings. In this note we focus on the geometric embedding (1.1). Orlov’s construction typically embeds $A$ into a very high-dimensional variety, so it seems worthwhile to wonder about the minimal dimension of such $X$.

Now consider the triangulated categories $D^b(\text{coh} / X)$, for $X$ smooth and projective. The existence of a full exceptional collection in such categories is a very hard question in general, though many explicit constructions are available in the literature. The dimension 1 case is understood, since a result by Okawa asserts that the derived category of a curve does not admit semi-orthogonal decompositions, save for $\mathbb{P}^1$, see lemma 3.2. Even for surfaces however, the question is open, and one is interested in knowing how complicated such a category can get. Some interesting results in the case of rational surfaces (which are conjecturally the only ones with full exceptional collections) can be found in [9, 10].

With this motivation, it is natural to ask: when does a basic finite-dimensional algebra $A = kQ/I$ allow for an admissible embedding

$$D^b(\text{mod } / A) \hookrightarrow D^b(\text{coh} / S), \quad (1.2)$$

for some smooth, projective surface $S$. It is this question that we will consider in this note, and using the Euler form we prove the following theorem.

**Theorem 1.1.** (see lemmas 5.1 and 5.2 below) Let $S$ denote a smooth projective surface and let $E = (E_1, \ldots, E_n)$ be a strong exceptional collection in $D^b(\text{coh} / S)$. Then $\text{rk}(\chi_{\text{End}_S(E)}^+) \leq 2$ and $\chi_{\text{End}_S(E)}^+$ does not admit a 3-dimensional negative definite subspace.

We also discuss some other approaches that do not appear to give effective constraints. The last section is devoted to giving explicit constructions and outlining some of the issues that remain.

**Acknowledgements** We would like to thank Louis de Thanhoffer de Volcsey and Michel Van den Bergh for stimulating discussions on these questions. Also,
we would like to thank Nathan Prabhu–Naik and Markus Perling for answering questions regarding toric geometry, and the latter for his freely available software TiltingSheaves [17].

Both authors were supported by a Ph.D. fellowship of the Research Foundation—Flanders (FWO).

2 Preliminaries

Let $T$ denote a $k$-linear triangulated category, where $k$ denotes an algebraically closed field of characteristic 0. Based on [16] we recall a couple of definitions and lemmas that we will require later on. Let $N$ denote a full triangulated subcategory of $T$.

**Definition 2.1.** A full embedding $i : N \hookrightarrow T$ is left (respectively right) admissible if there is a left (respectively right) adjoint functor $q : T \to N$ to $i$. It is admissible if $i$ is both left and right admissible.

Recall that $N^\perp$ denotes the right orthogonal to $N$: it is the full subcategory of $T$ consisting of objects $M$ such that $\text{Hom}_T(N, M) = 0$. The left orthogonal is defined similarly and is denoted $^\perp N$.

**Lemma 2.2.** A subcategory $N$ is right admissible if and only if for every object $T \in T$, there is an exact triangle $N \to T \to M$, where $N \in N$, $M \in N^\perp$. If any of these equivalent conditions holds, then $T/N$ is equivalent to $N^\perp$.

**Definition 2.3.** The triangulated category $T$ has a semi-orthogonal decomposition

$$T = \langle N_1, \ldots, N_n \rangle$$

for full triangulated subcategories $N_i$, if $T$ has an increasing filtration

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{n-1} \subset T_n = T$$

by left admissible subcategories $T_i$ such that in $T_i$, one has $^\perp T_{i-1} = N_i \cong T_i/T_{i-1}$.

We will mostly consider special semi-orthogonal decompositions, for which the quotients are as simple as possible, namely $N_i \cong D^b(\text{mod}/k)$.

**Definition 2.4.** An object $E$ in $T$ is exceptional if

$$\text{Hom}_T(E, E[m]) \cong \begin{cases} \mathbb{Z} & m = 0 \\ 0 & m \neq 0. \end{cases}$$

A sequence of exceptional objects $(E_1, \ldots, E_l)$ is an exceptional collection if

$$\text{Hom}_T(E_j, E_i[m]) = 0 \text{ for } j > i \text{ and any } m$$

An exceptional collection $(E_1, \ldots, E_l)$ is called strong if in addition

$$\text{Hom}_T(E_i, E_j[m]) = 0 \text{ for all } i, j \text{ and } m \neq 0.$$ 

An exceptional collection $(E_1, \ldots, E_n)$ is called full if it generates $T$. 

Notice that if $T$ has a full exceptional collection, then it has a semi-orthogonal decomposition with $N_i = \langle E_i \rangle \cong \text{D}^b(\text{mod}/k)$ and $T_i = \langle E_1, \ldots, E_i \rangle$. We will denote this as

$$T = \langle E_1, \ldots, E_n \rangle.$$  \hfill(2.6)

In the following sections we will be interested in the endomorphism algebras of strong but not necessarily full exceptional collections in the bounded derived category of a smooth projective variety over $k$.

### 3 Exceptional collections and smooth projective varieties

All varieties we consider are defined over $k$ and irreducible. In this section we want to prove some basic results on exceptional collections for smooth projective varieties. Recall first the following classical result [3, 1].

**Theorem 3.1.** Assume $X$ is a smooth projective variety such that its bounded derived category admits a full and strong exceptional collection $(E_1, \ldots, E_n)$. Then the functor

$$\text{RHom}_X(\bigoplus_{i=1}^n E_i, -): \text{D}^b(\text{coh}/X) \to \text{D}^b(\text{mod}/\text{End}_X(\bigoplus_{i=1}^n E_i))$$ \hfill(3.1)

is an equivalence of triangulated categories.

Note that $\text{End}_X(\bigoplus_{i=1}^n E_i)$ is a finite-dimensional $k$-algebra of finite global dimension. Ideally, one would like to know exactly which varieties admit full and strong exceptional collections, and which are the finite-dimensional algebras arising in this way.

Let us start on the geometric side and naively look at dimension. Then there is the following theorem due to Okawa [15].

**Theorem 3.2.** The only smooth projective curve $C$ such that $\text{D}^b(\text{coh}/C)$ admits a semi-orthogonal decomposition is $\mathbb{P}^1$.

Since we are only interested in exceptional collections, we can get by using an easier argument, which is subsumed in the proof of lemma 3.7.

For dimension 2, such a nice statement is not readily available. The most powerful general result seems to be due to Hille and Perling [10].

**Theorem 3.3.** Let $X$ denote a smooth projective rational surface. Then it admits a full exceptional collection of line bundles.

For higher dimensional varieties, the results are not so general, and we refer to the excellent [13] for an overview. A lot of work in the area is devoted to explicitly constructing full and strong exceptional collections on specific varieties. To this end, the following two general theorems of Orlov are very useful.
**Proposition 3.4** (Orlov’s blow-up formula). Let $X$ be a smooth and projective variety. Let $p$ be a point on $X$. Let $\pi: \text{Bl}_p X \to X$ be the blow-up of $X$ in $p$, whose exceptional divisor is denoted $E$. Then

$$\mathcal{D}^b(\text{coh/Bl}_p X) \cong (\pi^*(\mathcal{D}^b(\text{coh}/X)), \mathcal{O}_E).$$

(3.2)

**Proposition 3.5** (Orlov’s projective bundle formula). Let $X$ be a smooth and projective variety. Let $E$ be a vector bundle of rank $r + 1$ on $X$. Let $\pi: \text{Proj}(E) \to X$ be the associated projective bundle. If we denote

$$\mathcal{D}^b(\text{coh}/\text{Proj}(E)) \cong (\mathcal{D}^b(\text{coh}/X)^{-r}, \ldots, \mathcal{D}^b(\text{coh}/X)_0).$$

(3.3)

which is a subcategory of $\mathcal{D}^b(\text{coh/Proj}(E))$, then

$$\mathcal{D}^b(\text{coh/Proj}(E)) \cong (\mathcal{D}^b(\text{coh}/X)^{-r}, \ldots, \mathcal{D}^b(\text{coh}/X)_0).$$

(3.4)

Let us now look at the finite-dimensional algebras side. Just like for varieties, a naive way of distinguishing finite-dimensional algebras is by global dimension, so it is natural to start here.

**Lemma 3.6.** Let $X$ denote a smooth projective variety admitting a full and strong exceptional collection $(E_1, \ldots, E_n)$ such that $\text{End}_X(\bigoplus_{i=1}^n E_i)$ is semisimple. Then $X \cong \text{Spec } k$.

**Proof.** Suppose on the contrary that $X \neq \text{Spec } k$ while $\mathcal{D}^b(\text{coh}/X)$ does have a full and strong exceptional collection $E_1, \ldots, E_n$. Then the semi-orthogonal decomposition is in fact completely orthogonal, i.e.

$$\mathcal{D}^b(\text{coh}/X) = \langle E_1 \rangle \oplus \cdots \oplus \langle E_n \rangle.$$

(3.5)

But now since $X$ is connected $\mathcal{O}_X$ is an indecomposable object in $\mathcal{D}^b(\text{coh}/X)$ so it has to lie in one of the summands, which we can assume to be $\langle E_1 \rangle$ after permutation. The same holds for the skyscraper sheaves $k_x$, for all $x \in X$.

Since one always has non-zero morphisms from $\mathcal{O}_X$ to any $k_x$, and the decomposition is completely orthogonal, all skyscraper sheaves have to belong to $\langle E_1 \rangle$ as well. From this it immediately follows that all the other components have to be zero, hence

$$\mathcal{D}^b(\text{coh}/X) \cong \langle E_1 \rangle \cong \mathcal{D}^b(\text{mod}/k) \cong \mathcal{D}^b(\text{coh}/\text{Spec } k).$$

(3.6)

Having dealt with the global dimension 0 case, we proceed to global dimension 1: path algebras of acyclic quivers.

**Proposition 3.7.** Let $X$ denote a smooth projective variety admitting a full and strong exceptional collection $(E_1, \ldots, E_n)$ such that $A = \text{End}_X(\bigoplus_{i=1}^n E_i)$ is hereditary (and not semisimple). Then $X \cong \mathbb{P}^1$ and $A \cong kK_2$, the path algebra of the Kronecker quiver.
Proof. The result for \( \mathbb{P}^1 \) is standard [2]. To see that this is the only variety with this property consider the skyscraper sheaves \( k_x \), which are indecomposable objects (or more precisely, they are point objects, see [11, §7]).

A triangle equivalence sends these to indecomposable objects of \( \text{D}^b(\text{mod}/kQ) \), which correspond to the indecomposable modules up to a twist since every object therein is formal. Now by Serre duality

\[
\text{Ext}_X^d(k_x, k_x) \cong \text{Hom}_X(k_x[d], k_x \otimes \omega_X[\dim X])^\vee
\cong \text{Hom}_X(k_x, k_x[\dim X - d])^\vee,
\]

and since a hereditary algebra is of global dimension 1, \( X \) has to be a curve.

Remember that the genus of a curve is \( g = \dim_k H^0(X, \omega_X) \). So if \( g > 1 \), and using Serre duality, an exceptional object \( E \in \text{D}^b(\text{coh}/X) \) satisfies

\[
\text{Hom}^*_X(E, E[-1]) \cong \text{Hom}^*_X(E, E \otimes \omega_X)^\vee \neq 0.
\]

If \( E \) is not the shift of a sheaf, then the last inequality requires an easy truncation argument. By comparing degrees, \( \text{D}^b(\text{coh}/X) \) only contains exceptional objects if \( X \cong \mathbb{P}^1 \).

Any hereditary algebra derived equivalent to \( X \) is thus derived equivalent to \( kK_2 \). It is known, see [8], that any derived equivalence between basic hereditary algebras is given by a sequence of sink or source reflections, so there is no possibility other than \( K_2 \).

For algebras of global dimension 2 and higher, we do not know of any such results, and the problem seems to get a lot more difficult. A possible way of proceeding is by dropping the fullness assumption.

4 Euler forms on smooth projective surfaces

In this section we prove two propositions giving strong restrictions on the shape of the endomorphism algebra of a strong exceptional collection on a smooth projective surface \( S \). Both these propositions concern the Euler form on the Grothendieck group of such a surface.

First let \( X \) be any smooth projective variety of dimension \( n \). Recall that the bilinear Euler form is defined as

\[
\chi: K_0(X) \times K_0(X) \to \mathbb{Z}: (A, B) \mapsto \sum_i (-1)^i \dim_k \text{Ext}_X^i(A, B)
\]

Moreover one has the natural topological filtration \( F^i \) on \( \text{D}^b(\text{coh}/X) \) [SGA6, Exposé X], where \( F^i \text{D}^b(\text{coh}/X) \) consists of the complexes of coherent sheaves on \( X \) whose cohomology sheaves have support of codimension at least \( i \).

Denote by \( S \) the Serre functor on \( \text{D}^b(\text{coh}/X) \). Recall that this is defined as

\[
S(\mathcal{E}^\bullet) = \mathcal{E}^\bullet \otimes L^\omega_X[m],
\]
where $\omega_X$ denotes the canonical line bundle on $X$. The Serre functor induces an automorphism on $K_0(X)$, which we’ll denote by the same symbol, and moreover one has
\[ \chi(X, Y) = \chi(Y, SX). \] (4.3)

By definition, $F^iK_0(X)$ is the image of the morphism induced by the inclusion $F^iD^b(\text{coh}/X) \hookrightarrow D^b(\text{coh}/X)$. Using this filtration one proves the following result of Suslin [4, lemma 3.1].

**Lemma 4.1.** The operator $(-1)^nS$ is unipotent on $K_0(X)$.

To study $\chi$ using linear algebra, we pass to the numerical Grothendieck group, which is better behaved than the usual Grothendieck group in some respects [14].

**Definition 4.2.** The numerical Grothendieck group $K_0^{\text{num}}(X)$ is the quotient of $K_0(X)$ by the subspace defined by $\chi(-, K_0(X)) = \chi(K_0(X), -) = 0$.

For smooth projective varieties this group is always free of finite rank by the Grothendieck–Riemann–Roch theorem [5], so from now on we will restrict $\chi$ to $K_0^{\text{num}}(X)$, so we can use matrices. A closer inspection of the anti-symmetrisation of the Euler form
\[ \chi^{-}(A, B) := \chi(A, B) - \chi(B, A) = \chi(A, (1 - S)B), \] (4.4)
leads to the first theorem. The following result was also proved by Louis de Thanhoffer de Volcsey with a different method [6].

**Theorem 4.3.** For a smooth projective surface $S$ one has $\text{rk}\chi^{-} \leq 2$.

**Proof.** By dévissage, we can generate $K_0(S)$ by $[\mathcal{O}_S]$ and classes $[k_s]$ of skyscrapers for $s \in S$, and structure sheaves of curves $[\mathcal{O}_C]$ for all curves $C$ on $S$, see [SGA6, proposition 0.2.6].

By quotienting out the numerically trivial part to obtain $K_0^{\text{num}}(S)$ it suffices to consider a single class $[k_s]$ as all points are numerically equivalent [7, §19.3.5], and $\text{CH}^1_{\text{num}}(S) \cong \text{NS}(S)$ is a finitely generated free abelian group of rank $\rho$ [7, example 19.3.1]. Observe that under the Chern character map, $[k_s]$ is not pure of degree 2, but it has a degree 2 component whereas $[\mathcal{O}_S]$ and $[\mathcal{O}_C]$ don’t, so the rank of $K_0^{\text{num}}(S)$ is $\rho + 2$.

We wish to compute the rank of the matrix $1 - S = \chi^{-}$ using this choice of basis. For this we need to know the values of $\chi^{-}(\alpha, \beta)$ for $\alpha \in \text{CH}^i(S)$ and $\beta \in \text{CH}^j(S)$, with $i, j \in \{0, 1, 2\}$. We immediately get that
\[ \chi^{-}([k_s], [k_s]) = 0 \] (4.5)
and using the presentation
\[ 0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \] (4.6)
we get that
\[
\chi^{-}([\mathcal{O}_C],[k_s]) = \chi([\mathcal{O}_S],[k_s]) - \chi([\mathcal{O}_S,-C],[k_s]) + \chi([k_s],[\mathcal{O}_S])
\]
\[
= \chi([\mathcal{O}_S],[k_s]) - \chi([k_s],[\mathcal{O}_S]) - \chi([\mathcal{O}_S],[k_s]) + \chi([k_s],[\mathcal{O}_S])
\]
\[
= 0
\]
(4.7)

For \(i = j = 1\) we have that \(C \cdot D = -\chi(\mathcal{O}_C, \mathcal{O}_D)\), hence \(\chi\) is symmetric on this part by the commutativity of the intersection product, therefore it vanishes in the antisymmetric Euler form.

The (skew-symmetric) matrix one obtains is of the form
\[
\begin{pmatrix}
0 & 0 & * \\
0 & 0 & \cdot \\
* & \cdot & 0
\end{pmatrix}
\]
(4.8)
where we order our generators as \([k_s], [\mathcal{O}_C], [\mathcal{O}_S]\), so there is a block decomposition of a \((\rho + 2) \times (\rho + 2)\)-square matrix with some unknown values, but it is of rank \(\leq 2\) regardless of the unknowns.

One can also consider the *symmetrised* Euler form
\[
\chi^+(A, B) = \chi(A, B) + \chi(B, A).
\]
(4.9)
This defines a quadratic form on \(K_0^{\text{num}}(X)\), and we can consider its signature, i.e. the tuple \((n_0, n_+, n_-)\) describing the degenerate, positive definite and negative definite part of the form. The forms that we consider are non-degenerate, hence it suffices to specify \((n_+, n_-)\).

**Theorem 4.4.** Let \(S\) be a smooth projective surface. Then the signature of \(\chi^+\) is \((\rho, 2)\).

**Proof.** We use the decomposition of \(K_0^{\text{num}}(S)\) as in the proof of lemma 4.3. The Hodge index theorem, which is a statement on structure of the Néron-Severi group \(\text{NS}(S)\), gives us that the signature is \((1, \rho - 1)\) hence via the equality \(C \cdot D = -\chi(\mathcal{O}_C, \mathcal{O}_D)\) we have inverted it.

The contribution of the remaining summands to the quadratic form is a hyperbolic plane, because \(\chi^+([k_s],[k_s]) = 0\), \(\chi^+([\mathcal{O}_S],[k_s]) = 2\) using Serre duality and \(\chi^+([\mathcal{O}_S])\) is twice the Euler characteristic of \(S\), but by base change the matrix
\[
\begin{pmatrix}
0 & 2 \\
2 & 2\chi(S)
\end{pmatrix}
\]
(4.10)
corresponds to the hyperbolic plane \(x^2 - y^2 = 0\) which has signature \((1, 1)\).

### 5 Constraints on endomorphism algebras of strong exceptional collections

We will now apply the results in the previous sections to restrict the structure of the endomorphism ring of a strong (not necessarily full) exceptional
collection on a smooth projective surface. Let $A = kQ/I$ denote a basic finite-dimensional $k$-algebra with $n$ simple modules, i.e. $Q$ has $n$ vertices. Let us also assume that $A$ has finite global dimension. In that case, there is a well-defined Euler form given by

$$\chi : K_0(A) \times K_0(A) \to \mathbb{Z}: (X, Y) \mapsto \sum (-1)^i \dim_k \text{Ext}^i_A(X, Y). \quad (5.1)$$

Since the indecomposable projective modules and the simple modules form dual bases, this bilinear form is non-degenerate.

**Corollary 5.1.** Let $S$ be a smooth projective surface. Let $(E_1, \ldots, E_n)$ be a strong exceptional collection inside $D^b(\text{coh}/S)$. Then $\text{rk}(\chi^-_{\text{End}_S(\bigoplus_{i=1}^n E_i)}) \leq 2$.

**Proof.** If there is a strong exceptional collection $(E_1, \ldots, E_n)$ in the derived category of a smooth projective surface $S$, this corresponds to an admissible subcategory:

$$D^b(\text{mod}/\text{End}_S(\bigoplus_{i=1}^n E_i)) \hookrightarrow D^b(\text{coh}/S). \quad (5.2)$$

Also, the global dimension of $\text{End}_S(\bigoplus_{i=1}^n E_i)$ is finite, so in particular, the Euler form $\chi$ on $K_0^{\text{num}}(S)$ restricts to the non-degenerate Euler form on the finite-dimensional algebra $\text{End}_S(\bigoplus_{i=1}^n E_i)$. Since submatrices cannot increase in rank, the statement is clear. \(\square\)

Since signatures also behave nicely under restriction to a submatrix, we obtain:

**Corollary 5.2.** Let $S$ be a smooth projective surface. Let $(E_1, \ldots, E_n)$ be a strong exceptional collection inside $D^b(\text{coh}/S)$. Then $\chi^+_{\text{End}_S(\bigoplus_{i=1}^n E_i)}$ does not admit a 3-dimensional negative definite subspace.

**Remark 5.3.** We did not manage to find any other effective constraints. One can try to use additive invariants, which are invariants of (suitably enhanced) triangulated categories that are compatible with semi-orthogonal decompositions (hence the theory of noncommutative motives comes into play), but it turns out that for a finite-dimensional algebra of finite global dimension these only depend on the number of simple modules [18].

The most meaningful result one can obtain is by applying the Hochschild–Kostant–Rosenberg decomposition to the Hochschild homology of a smooth projective variety, but this of course depends on the variety one looks at. The results in lemmas 5.1 and 5.2 are uniform in all smooth projective surfaces.

### 6 Explicit constructions for hereditary algebras

Even in the hereditary case, i.e. $A = kQ$, it is not clear which $A$ can be embedded into a smooth projective surface and which cannot. Some questions about the structure of $Q$ one could ask are:

(Q1) Is there a bound on the number of vertices of $Q$?
(Q2) Is there a bound on the number of arrows of $Q$?

(Q3) Is there a bound on the number of composable arrows of $Q$?

(Q4) Is there a bound on the length of paths in $Q$?

(Q5) Is it possible to embed any quiver on 3 vertices?

We now address these questions. In the following proposition we provide some explicit embeddings for well known families of quivers which answer some of the above questions.

**Proposition 6.1.** Let $A = kQ$ be a basic hereditary algebra.

1. If $A$ is of finite type or tame, i.e. $Q$ is a Dynkin or Euclidean quiver, then $A$ occurs as endomorphism ring of a strong exceptional collection on some smooth projective surface if and only if

   $$Q = A_2, A_3, D_4, \tilde{A}_1 \text{ or } \tilde{A}_2.$$  

2. If $Q$ occurs in the following families of quivers:

   $K_n$: $\begin{array}{c}
   \circ & \cdots & \circ \\
   1 & \cdots & 1 \\
   n & \cdots & n
   \end{array}$  

   $S_n$: $\begin{array}{c}
   0 \\
   1 \\
   \circ \\
   \vdots \\
   n
   \end{array}$  

   then $A$ occurs as endomorphism ring of a strong exceptional collection on some smooth projective surface.

**Proof.** For part (1), it suffices to compute the matrices for the anti-symmetric Euler forms, and observe that starting from $A_4$, $D_5$ and $\tilde{A}_3$ the rank is bounded below by 4 as there will be a submatrix of rank 4 in each of these coming from the smallest cases $A_4$, $D_5$ and $\tilde{A}_3$. The exceptional types $E_6, 7, 8$ or $\tilde{E}_6, 7, 8$ have corresponding ranks $6, 6, 8, 6, 6$ and 8.

For the 5 cases that are not ruled out by this restriction on the anti-symmetric Euler form, and the infinite families in part (2), there are explicit embeddings. Let $n = 2m$, then one can embed $K_{2m}$ by considering $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, m-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $n = 2m-1$, then one can embed $K_{2m-1}$ by considering $\mathcal{O}_{\text{Bl}_1 \mathbb{P}^2}$ and $\mathcal{O}_{\text{Bl}_1 \mathbb{P}^2}(E+mF)$, on the blow-up of $\mathbb{P}^2$ in a point $p$. Here, as usual, $E$ denotes the divisor associated to the $-1$-curve and $F$ the one associated to the strict transform of any line in $\mathbb{P}^2$ through $p$. 

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For the family $S_n$, by Orlov’s blow-up formula lemma 3.4 we obtain a semi-orthogonal decomposition

$$D^b(\text{coh}/\text{Bl}_1\mathbb{P}^2) = \langle \pi^* (D^b(\text{coh}/\mathbb{P}^2)), \mathcal{O}_E \rangle$$ (6.2)

where $\pi: \text{Bl}_1\mathbb{P}^2 \rightarrow \mathbb{P}^2$ is the blow-up morphism, and $\mathcal{O}_E$ is the structure sheaf of the exceptional divisor. The blow-up locus is denoted $p$. Consider the exceptional line bundle $\mathcal{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$, then one checks by adjunction

$$\text{Hom}_{D^b(\text{coh}/\text{Bl}_1\mathbb{P}^2)}(\pi^* (\mathcal{O}_{\mathbb{P}^2}), \mathcal{O}_E) \cong \text{Hom}_{D^b(\text{coh}/\mathbb{P}^2)}(\mathcal{O}_{\mathbb{P}^2}, k_p)$$ (6.3)

that the exceptional pair $(\pi^* (\mathcal{O}_{\mathbb{P}^2}), \mathcal{O}_E$ has endomorphism ring $kS_1$. Using the blow-up formula inductively, this gives a realisation of $kS_n$ using $\text{Bl}_n\mathbb{P}^2$.

By the identifications $A_2 = S_1$, $A_3 = S_2$ (using reflection), $A_4 = S_3$ (using reflection) and $K_2 = \tilde{A}_1$ the only remaining quiver is $\tilde{A}_2$, and for this one we use some elementary toric geometry. The variety $\text{Bl}_2\mathbb{P}^2$ can be represented by the fan

![Fan Diagram](6.4)

As basis for $\text{Pic}(\text{Bl}_2\mathbb{P}^2)$, we choose the first three torus-invariant divisors $D_1$, $D_2$ and $D_3$. It is then not hard to see that

$$(\mathcal{O}_{\text{Bl}_2\mathbb{P}^2}, \mathcal{O}_{\text{Bl}_2\mathbb{P}^2}(D_1 - D_3), \mathcal{O}(D_1))$$ (6.5)

has the desired structure.

**Remark 6.2.** Of course, there are a lot of alternatives to the above embeddings. The Kronecker quivers $K_n$ for example can also be embedded using $\mathcal{O}(E)$ and $\mathcal{O}(E + nF)$ on $\text{F}_n = \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, the $n$th Hirzebruch surface.

With this proposition, (Q1) and (Q2) clearly have a negative answer. For (Q3) the answer is also no, since one can always reflect the $S_n$-quiver in some non-zero vertex. The questions (Q4) and (Q5) are more subtle.

Let us say a quiver $Q'$ is **forbidden** if the rank of $\chi^-$ is strictly greater than 2.

**Proposition 6.3.** If a quiver $Q$ contains a forbidden quiver $Q'$ as a full sub-quiver, then it cannot be embedded into a smooth projective surface.

**Proof.** The fullness ensures that the $\chi^-$ matrix of $Q'$ occurs as a block in that of $Q$ (for the basis of simples for example), so $\text{rk}(\chi^-_Q) > 2$ and the quiver cannot be embedded. □

The question about path length can be partially answered by plugging $A_4$ into this proposition, as we know from lemma 5.1 that $A_4$ cannot be embedded into the derived category of a surface. Observe that $A_4$ does satisfy the condition on the negative definite subspaces for $\chi^+$, as in lemma 5.2.
However, if $A_4$ occurs as a subquiver of $Q$ not satisfying the condition of the proposition, it is not completely clear what happens in general. The following example shows that in some cases one can have an embedding.

**Example 6.4.** Consider the following quiver:

\[ Q: \quad \begin{array}{ccc}
\circ & \longrightarrow & \circ \\
\circ & \longrightarrow & \circ \\
\end{array} \quad (6.6) \]

It has $\text{rk}(\chi^-) = 2$ but obviously contains $A_4$ as a subquiver, in a way so as not to satisfy the conditions of the lemma. In fact, it can be embedded into $\text{Bl}_2 \mathbb{P}^2$ by extending the strong exceptional collection we already had for $A_2$. In terms of the fan mentioned in (6.4), the collection

\[ (\mathcal{O}_{\text{Bl}_2 \mathbb{P}^2}, \mathcal{O}_{\text{Bl}_2 \mathbb{P}^2}(D_1 - D_3), \mathcal{O}_{\text{Bl}_2 \mathbb{P}^2}(D_1), \mathcal{O}_{\text{Bl}_2 \mathbb{P}^2}(D_2)) \quad (6.7) \]

can be checked to be a strong exceptional collection with the right endomorphism ring, after reflecting in the vertex corresponding to $\mathcal{O}_{\text{Bl}_2 \mathbb{P}^2}$.

It is also possible (but less trivial) to find an example of a quiver that satisfies lemma 5.1 but violates lemma 5.2.

**Example 6.5.** Consider an acyclic quiver on 5 vertices $v_1, \ldots, v_n$, whose $\chi$ is given by the matrix

\[ \begin{pmatrix}
1 & 2 & 4 & 3 & 0 \\
0 & 1 & 4 & 5 & 2 \\
0 & 0 & 1 & 4 & 4 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (6.8) \]

It is straightforward to check that $\text{rk} \chi^- = 2$, but $\chi^+$ has a negative-definite subspace of dimension 3.

The reason for posing question (Q5) is that any skew-symmetric $3 \times 3$-matrix has rank $\leq 2$, and it also cannot have a 3-dimensional negative definite subspace, since one can always look at the projective indecomposables yielding a nonzero positive definite subspace. Also, since every quiver on 2 vertices can be embedded, these provide a natural next step.

Such a quiver can be presented as

\[ \quad \begin{array}{ccc}
\circ & \longrightarrow & \circ \\
\circ & \longrightarrow & \circ \\
\end{array} \quad (6.9) \]

Again, we do not know what happens in general. A fertile testing ground for these matters is toric geometry, where one can use computer algebra to construct exceptional collections of line bundles.

Recall that all smooth projective toric surfaces are iterated blow-ups of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_r$ for $r \geq 2$. The following proposition gives some families of 3-vertex
quivers that can be realized in $\text{Bl}_3\mathbb{P}^2$. We have not found any other quiver in any other smooth toric surface. From now on $a, b$ and $c$ will denote the respective number of arrows as in (6.9).

The toric surface $X = \text{Bl}_3\mathbb{P}^2$ can be represented by the fan

$$
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
$$

(6.10)

and as a basis for $\text{Pic}(X)$ we choose the torus-invariant divisors $D_1, \ldots, D_4$.

Since we are only looking at line bundles one can always assume that this collection is of the form $(\mathcal{O}_X, \mathcal{O}_X(D), \mathcal{O}_X(E))$ for two divisors $D$ and $E$. For given $D$ and $E$ it is easy to check whether they form an exceptional collection and what the associated quiver is by computing the (higher) cohomology of $D, -D, E, -E, D - E$ and $E - D$.

In the following lemmas we give the required constructions without writing down the straightforward computations. Remark that the constructions are far from unique: reflections allow for symmetries in (6.9).

**Proposition 6.6.** There exist divisors $D$ and $E$ on $X$ such that the associated exceptional collection $(\mathcal{O}_X, \mathcal{O}_X(D), \mathcal{O}_X(E))$ has $a = c \geq 0$ and $b = 0$.

**Proof.** If $a = c = 2n$ is even one takes

$$
D := D_2 + nD_3 + (n - 1)D_4,
$$

$$
E := D_1 + D_2 + (n - 1)D_3 + (n - 1)D_4
$$

(6.11)

whereas if $a = c = 2n + 1$ is odd one takes

$$
D := D_2 + nD_3 + nD_4,
$$

$$
E := D_1 + D_2 + nD_3 + (n - 1)D_4.
$$

(6.12)

\hfill \Box

**Proposition 6.7.** There exist divisors $D$ and $E$ on $X$ such that the associated exceptional collection $(\mathcal{O}_X, \mathcal{O}_X(D), \mathcal{O}_X(E))$ has $a = b = 1$ and $c \geq 0$.

**Proof.** If $c = 2n$ is even one takes

$$
D := D_2 + nD_3 + nD_4
$$

$$
E := D_4,
$$

(6.13)

whereas if $c = 2n + 1$ is odd one takes

$$
D := D_2 + (n + 1)D_3 + nD_4,
$$

$$
E := D_3.
$$

(6.14)

\hfill \Box
By considering “random” smooth projective toric surfaces and sufficiently big hypercubes in $\text{Pic}(X)$ we have not found other configurations without relations. These experiments were performed with a modified version of [17]. Hence we are led to believe that these constructions exhaust all the configurations, and that they all occur as soon as $\text{rkPic}(X) \geq 5$. If one allows relations, only one new class appears, where $a \geq 1$, $b = 2$ and $c = 0$. 
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