BV SOLUTIONS CONSTRUCTED BY EPSILON-NEIGHBORHOOD METHOD

MACH NGUYET MINH
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

Abstract. We study a certain class of weak solutions to rate-independent systems, which is constructed by using the local minimality in a small neighborhood of order \(\varepsilon\) and then taking the limit \(\varepsilon \to 0\). We show that the resulting solution satisfies both the weak local stability and the new energy-dissipation balance, similarly to the BV solutions constructed by vanishing viscosity introduced recently by Mielke, Rossi and Savare \cite{19,20,21}.

1. Introduction

A rate-independent system is a specific case of quasistatic systems. It is time-dependent but its behavior is slow enough that the inertial effects can be ignored and the systems are affected only by external loadings. Some specific rate-independent systems were studied by many authors including Francfort, Marigo, Larsen, Dal Maso and Lazzaroni on brittle fractures \cite{9,8,11,6}, Dal Maso, DeSimone and Solombrino on the Cam-Clay model \cite{5}, Dal Maso, DeSimone, Mora, Morini on plasticity with softening \cite{16,15,17,18} by Mielke for the study in abstract setting as well as for further references. In this work, we consider a finite-dimensional normed vector space \(X\), an evolution \(u : [0, T] \to X\) subject to a force defined by an energy functional \(\mathcal{E} : [0, T] \times X \to [0, +\infty)\) which is of class \(C^1\), and a dissipation function \(\Psi(x)\) which is convex, non-degenerate and positively 1-homogeneous. Given an initial position \(x_0 \in X\) which is a local minimizer for the functional \(x \mapsto \mathcal{E}(0, x) + \Psi(x-x_0)\), we say that \(u\) is a solution to the rate-independent system \((\mathcal{E}, \Psi, x_0)\) if \(u(0) = x_0\) and the following inclusion holds true,

\[
0 \in \partial \Psi(u(t)) + D_\varepsilon \mathcal{E}(t, u(t)) \quad \text{in } X^*, \quad \text{for a.e. } t \in (0,T),
\]

where \(X^*\) denotes the dual space of \(X\), \(\partial \Psi\) is the subdifferential of \(\Psi\) and \(D_\varepsilon \mathcal{E}\) is the differential of \(\mathcal{E}\) w.r.t. the spatial variable \(x\).

In general, strong solutions to \((1)\) may not exist \cite{30}. Hence, the question on defining some weak solutions arises naturally.

A widely-used weak solution is the energetic solution, which was first introduced by Mielke and Theil \cite{22} (see \cite{23,12,10,19} for further studies). A function \(u : [0, T] \to X\) is called an energetic solution to the rate-independent system \((\mathcal{E}, \Psi, x_0)\), if it satisfies

(i) the initial condition \(u(0) = x_0\);
(ii) the global stability that for \((t,x) \in [0,T] \times X\),
\[
\mathcal{E}(t,u(t)) \leq \mathcal{E}(t,x) + \Psi(x-u(t));
\]
(iii) the energy-dissipation balance that for all \(0 \leq t_1 \leq t_2 \leq T\),
\[
\mathcal{E}(t_2,u(t_2)) - \mathcal{E}(t_1,u(t_1)) = \int_{t_1}^{t_2} \partial \mathcal{E}(s,u(s)) \, ds - \mathcal{Diss}_\Psi(u;[t_1,t_2]).
\]

Here \(\mathcal{Diss}_\Psi\) is the usual total variation induced by \(\Psi(\cdot)\)

\[
\mathcal{Diss}_\Psi(u(t);[t_1,t_2]) := \sup \left\{ \sum_{i=1}^{N} \Psi(u(s_i) - u(s_{i-1})) \mid N \in \mathbb{N}, t_1 = s_0 < s_1 < \cdots < s_N = t_2 \right\}.
\]

Date: June 20, 2014.
1991 Mathematics Subject Classification. Primary: 49M99; Secondary: 49J20.
Key words and phrases. Rate-independent systems, BV solutions, local minimizers, energy-dissipation balance.
Notice that when the energy functional is not convex, the global minimality makes the energetic jumps of size less than $\varepsilon$ tend to zero. They showed that the modified system $(\mathcal{E}, \Psi, x_0)$ admits a solution $u_\varepsilon$. The limit $u$ of a subsequence $u_\varepsilon$ as $\varepsilon \to 0$, called BV solution, enjoys the following properties

(i) the initial condition $u(0) = x_0$;
(ii) the weak local stability that for all $t \in [0, T] \setminus J$,

\[(4) \quad -D_\varepsilon \mathcal{E}(t, u(t)) \in \partial \Psi(0);\]
(iii) the new energy-dissipation balance that for all $0 \leq t_1 \leq t_2 \leq T$,

\[(5) \quad \mathcal{E}(t_2, u(t_2)) - \mathcal{E}(t_1, u(t_1)) = \int_{t_1}^{t_2} \partial_t s - D_\varepsilon \mathcal{E}(s, u(s)) ds - D_{\varepsilon, \text{new}}(u; [t_1, t_2]).\]

Here $J$ is the jump set of $u$ on $[0, T]$

\[J := \{t \in [0, T] \mid u(\cdot) \text{ is not continuous at } t\},\]

$\partial \Psi(0)$ is the subdifferential of $\Psi$ at 0, $(\cdot, \cdot)$ is the dual pairing between $X^*$ and $X$

\[\partial \Psi(0) := \{\eta \in X^* \mid \langle \eta, v \rangle \leq \Psi(v) \forall v \in X\},\]

and the new dissipation is defined by

\[D_{\varepsilon, \text{new}}(u; [t_1, t_2]) := D_{\varepsilon, \Psi}(u; [t_1, t_2]) + \sum_{t \in \mathcal{J}(t_1, t_2)} (\Delta_{\text{new}}(t; u(t^-(t)), u(t)) + \Delta_{\text{new}}(t; u(t), u(t^+(t)))) + \Delta_{\text{new}}(t_1; u(t_1), u(t_1^+)) + \Delta_{\text{new}}(t_2; u(t_2), u(t_2^+)) - \sum_{t \in \mathcal{J}(t_1, t_2)} (\Psi(u(t) - u(t^-)) + \Psi(u(t^+) - u(t))) - \Psi(u(t_1^+) - u(t_1)) - \Psi(u(t_2) - u(t_2^+)),\]

where $\Delta_{\text{new}}(t; a, b)$ depends also on the energy functional $\mathcal{E}$, the dissipation $\Psi$ and the viscous norm $\| \cdot \|$

\[\Delta_{\text{new}}(t; a, b) = \inf \left\{ \int_0^T \left( |\Psi(\gamma(s))| + \|\dot{\gamma}(s)\| \cdot \inf_{z \in \partial \Psi(0)} \|D_\varepsilon \mathcal{E}(t, \gamma(s)) + z\|_\ast \right) ds \mid \gamma \in AC([0, 1]; X), \gamma(0) = a, \gamma(1) = b \right\}.\]

Here the dual norm of $\| \cdot \|$ is defined by $\| \eta \|_\ast = \sup_{v \in X \setminus \{0\}} \|\eta(v)\|_\ast$ for all $\eta \in X^*$.

The new energy-dissipation balance is a deeply insight observation, which contains the information at the jump points. Indeed, it was shown in [20, 21] that if the BV solution $u$ jumps at time $t$, there exists an absolutely continuous path $\gamma : [0, 1] \to X$, which called an optimal transition between $u(t^-)$ and $u(t^+)$, such that

(i) $\gamma(0) = u(t^-)$, $\gamma(1) = u(t^+)$, and there exists $s \in [0, 1]$ such that $\gamma(s) = u(t)$;
(ii) for all $s \in [0, 1]$, $-D_\varepsilon \mathcal{E}(s, \gamma(s))$ stays outside the set $\partial \Psi(0)$ (if $\gamma$ is of viscous type), or on the boundary of $\partial \Psi(0)$ (if $\gamma$ is of sliding type);
(iii) $\dot{\mathcal{E}}(t, u(t^-)) - \dot{\mathcal{E}}(t, u(t^+)) = \int_0^1 \left( |\Psi(\dot{\gamma}(s))| + \|\dot{\gamma}(s)\| \cdot \inf_{z \in \partial \Psi(0)} \|D_\varepsilon \mathcal{E}(t, \gamma(s)) + z\|_\ast \right) ds$.

As we can see from the definition, BV solutions constructed by vanishing viscosity depend also on the viscosity. Usually, the viscosity arises naturally from physical models.

To deal with local minimizers but with a totally different approach, Larsen [11] proposed the $\varepsilon$-stability solution in the context of fracture mechanics. The idea is to choose minimizers among all $\varepsilon$-accessible states w.r.t. the discretized solution at previous time-step. A state $v$ is called $\varepsilon$-accessible w.r.t. state $z$ if the total energy at $v$ is lower than the total energy at $z$, and there is a continuous path connecting $z$ to $v$ such that along this path the total energy never increases by more than $\varepsilon$. By this way, the limit $u(t)$ when passing from discrete to continuous time satisfies the so-called $\varepsilon$-stability: $u(t)$ is $\varepsilon$-stable at every time $t$, i.e. there is no $\varepsilon$-accessible state w.r.t. $u(t)$. A similar version of optimal transition is obtained at jump points: if solution jumps at time $t$, there exists a continuous path connecting $u(t^-)$ to $u(t^+)$ such that along this path, the total energy increases no more than $\varepsilon$. The energy-dissipation upper bound is proved for fixed $\varepsilon > 0$. The energy-dissipation equality is obtained if the solution has only jumps of size less than $\varepsilon$. 
In this work, we shall discuss one more way to deal with local minimizers. The idea is more likely to the viscosity method of Mielke-Rossi-Savaré in [19, 20, 21], but instead of adding a small viscosity into the dissipation, we consider the minimization problem (2) in a small neighborhood of order $\varepsilon$. Passing from discrete to continuous time, we obtain a limit $x^\varepsilon(\cdot)$. Then taking $\varepsilon \to 0$, we get a solution $u(\cdot)$. The epsilon-neighborhood approach was first suggested in [14, Section 6] for one-dimensional case when $\varepsilon$ is chosen proportional to the square root of the time-step and the weak local stability was then obtained in [7].

Roughly speaking, epsilon-neighborhood method is a special case of vanishing viscosity approach when viscosity term is chosen as follows

$$\Psi_0(v) := \begin{cases} 0 & \text{if } |v| \leq 1, \\ +\infty & \text{if } |v| > 1. \end{cases}$$

However, this method was not discussed in [20, 21] since the viscosity there is required to be finite (see [20, Section 2.3] and [21, Section 2.1] for further discussions).

In this article, we shall show that the BV solution constructed by epsilon-neighborhood method $u(\cdot)$ satisfies both the weak local stability and the new energy-dissipation balance, i.e. it satisfies the definition in Example 2 below, we shall give a comparison between different notions of weak solutions, i.e. energetic solutions, BV solutions constructed by vanishing viscosity, BV solutions constructed by epsilon-neighborhood method as well as the solutions constructed by the method in [7]. For a detailed discussion on the different notions of weak solutions, we refer to the papers [15, 28, 31, 25].

Acknowledgments. I am indebted to Professor Giovanni Alberti for proposing to me the problem and giving many helpful discussions. I warmly thank Professor Riccarda Rossi, Li-Chang Hung and Tran Minh-Binh for their helpful comments and remarks. I really appreciate the three referees for many enlightening and insightful remarks and helpful suggestions and corrections. This work has been partially supported by the PRIN 2008 grant “Optimal mass transportation, Geometric and Functional Inequalities and Applications” and the FPT-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation”.

2. Main results

For simplicity, we shall consider the case when $X = \mathbb{R}^d$ and the unit ball of the norm $\| \cdot \|$ which defines the neighborhood has $C^1$-boundary. In addition, we assume that the energy functional $\mathcal{E}(t, x) : [0, T] \times \mathbb{R}^d \to [0, \infty)$ satisfies the following technical assumption: there exists $\lambda = \lambda(\mathcal{E})$ such that

$$|\partial_t \mathcal{E}(s, x)| \leq \lambda \mathcal{E}(s, x) \text{ for all } (s, x) \in [0, T] \times \mathbb{R}^d. \tag{6}$$

Remark. The condition (6) was proposed in [18]. Together with Gronwall’s inequality, (6) implies that

$$\mathcal{E}(r, x) \leq \mathcal{E}(s, x) e^{\lambda |r-s|}, \quad |\partial_t \mathcal{E}(r, x)| \leq \lambda \mathcal{E}(s, x) e^{\lambda |r-s|} \tag{7}$$

for any $r, s \in [0, T]$.

Definition (Construction of discretized solutions). Let $\varepsilon > 0$, $\tau > 0$ and let $N \in \mathbb{N}$ satisfy $T \in [\tau N, \tau (N+1)]$. We define a sequence $\{x^{i,\tau}_n\}_{n=0}^N$ by $x^{0,\tau}_0 = x_0$ (initial position) and

$$x^{i,\tau}_n \in \arg\min \{\mathcal{E}(t_i, x) + \Psi(x-x^{i,\tau}_{i-1}) \mid \|x-x^{i,\tau}_{i-1}\| \leq \varepsilon \} \text{ for every } i \in \{1, \ldots, N\}. \tag{8}$$

The discretized solution $x^{\varepsilon,\tau}(\cdot)$ is then constructed by interpolation

$$x^{\varepsilon,\tau}(t) := x^{i,\tau}_{i-1} \text{ for every } t \in [t_{i-1}, t_i), i \in \{1, \ldots, N\}. \tag{9}$$

Our main result is as follows.

Theorem 1 (BV solutions constructed by epsilon-neighborhood method). Let $\mathcal{E} : [0, T] \times \mathbb{R}^d \to [0, +\infty)$ be of class $C^1$ and satisfy (7). The dissipation functional $\Psi : \mathbb{R}^d \to [0, \infty)$ is assumed to be convex, positively 1-homogeneous and satisfy $\Psi(v) > 0$ for all $v \in \mathbb{R}^d \setminus \{0\}$. Given an initial datum $x_0 \in \mathbb{R}^d$ which is a local minimizer of the functional $x \mapsto \mathcal{E}(0, x) + \Psi(x-x_0)$. Then we have the following properties.

(i) (Discretized solution) For any $\varepsilon > 0$ and $\tau > 0$, there exists a discretized solution $t \mapsto x^{\varepsilon,\tau}(\cdot)$ as described above.

(ii) (Epsilon-neighborhood solution) For any fixed $\varepsilon > 0$, there exists a subsequence $\tau_n \to 0$ such that $x^{\varepsilon,\tau_n}(\cdot)$ converges pointwise to some limit $x^\varepsilon(\cdot)$. Moreover,
• (Epsilon local stability) If \( x^\varepsilon(\cdot) \) is right-continuous at \( t \), namely \( \lim_{t' \to t^+} x^\varepsilon(t') = x^\varepsilon(t) \), then \( x^\varepsilon(t) \) satisfies the epsilon local stability
\[
\mathcal{E}(t, x^\varepsilon(t)) \leq \mathcal{E}(t, x) + \Psi(x - x^\varepsilon(t)) \quad \text{for all } \|x - x^\varepsilon(t)\| \leq \varepsilon;
\]

• (Energy-dissipation inequalities) We have \( \mathcal{D}_{\text{iss}}(x^\varepsilon; [0,T]) \leq C \) (independent of \( \varepsilon \)), \( \partial_t \mathcal{E}(\cdot, x^\varepsilon(\cdot)) \in L^1(0,T) \) and for all \( 0 \leq s \leq t \leq T \),
\[
-\mathcal{D}_{\text{iss}_{\text{new}}}(x^\varepsilon; [s,t]) \leq \mathcal{E}(t, x^\varepsilon(t)) - \mathcal{E}(s, x^\varepsilon(s)) - \int_s^t \partial_r \mathcal{E}(r, x^\varepsilon(r)) \, dr \leq -\mathcal{D}_{\text{iss}}(x^\varepsilon; [s,t]).
\]

(iii) (BV solutions constructed by epsilon-neighborhood) There exists a subsequence \( \varepsilon_n \to 0 \) such that \( x^{\varepsilon_n} \) converges pointwise to some BV function \( u \). Furthermore, the function \( u \) satisfies
• (Weak local stability) If \( t \mapsto u(t) \) is continuous at \( t \), then
\[
-\nabla_x \mathcal{E}(t, u(t)) \in \partial\Psi(0);
\]
• (New energy-dissipation balance) For all \( 0 \leq s \leq t \leq T \), one has
\[
\mathcal{E}(t, u(t)) - \mathcal{E}(s, u(s)) = \int_s^t \partial_r \mathcal{E}(r, u(r)) \, dr - \mathcal{D}_{\text{iss}_{\text{new}}}(u; [s,t]).
\]

An explicit example is given below (a detail explanation can be found in Appendix).

**Example 2.** Consider the case \( X = \mathbb{R}, \Psi(x) = |x|, x_0 = 0 \) and the energy functional
\[
\mathcal{E}(t, x) := x^2 - x^4 + 0.3x^6 + t(1 - x^2) - x + 6, \quad t \in [0, 2].
\]

(i) The strong solution is \( x(t) = 0 \) for \( t \in [0, 1) \). This solution cannot be extended continuously when \( t \geq 1 \), since it would violate the local minimality.

(ii) The energetic solution constructed by time-discretization satisfies
\[
x(t) = 0 \quad \text{if} \quad t < \frac{1}{6}, \quad x(1/6) \in \{0, \sqrt{5/3}\} \quad \text{and} \quad x(t) = \sqrt{10 + \sqrt{10 + 90t}} \quad \text{if} \quad t > \frac{1}{6}.
\]

This solution jumps at \( t = 1/6 \), from \( x = 0 \) to \( x = \sqrt{5/3} \), but this jump is not physically relevant (see Fig. 1 and Fig. 3 below). The energetic solution satisfies the energy-dissipation balance but it does not satisfy the new energy-dissipation balance.

(iii) The BV solution corresponding to the viscous dissipation \( \Psi_{x}(x) = |x| + \varepsilon x^2 \) is
\[
x(t) = 0 \quad \text{for all} \quad t \in [0, 2].
\]

When \( t \geq 1 \), this solution violates the local minimality.

(iv) The BV solution constructed by epsilon-neighborhood method is
\[
x(t) = 0 \quad \text{if} \quad t < 1 \quad \text{and} \quad x(t) = \sqrt{10 + \sqrt{10 + 90t}} \quad \text{if} \quad t > 1.
\]

This solution coincides to the strong solution up to the strong solution exists. Moreover, it jumps at \( t = 1 \) which is a physical relevant jump (see Fig. 2 and Fig. 3 below). The BV solution constructed by epsilon-neighborhood method satisfies the new energy-dissipation balance but it does not satisfy the energy-dissipation balance.

(v) The solution constructed by the method in [7] coincides to the solution in (iv).

Notice that both solutions in (iii) and (iv) satisfy the definition of BV solutions [19] [20] [21]. Weak local stability in this case is: \( |\partial_x \mathcal{E}(t, x(t))| \leq 1 \).
Figure 1. $E(t, x) + |x|$ with $t = 1/6$ in Example 2.

Figure 2. $E(t, x) + |x|$ with $t = 1$ in Example 2.
Lemma 3 (Discretized solution). For any given initial state \( x_0, \tau > 0 \) and partition \( 0 = t_0 < t_1 < \cdots < t_N \leq T \) of \([0, T]\) satisfying \( t_n - t_{n-1} = \tau \) and \( T \in [\tau N, \tau (N + 1)]\), there exists a sequence \( \{x_{i}^{\tau, n}\}_{i=0}^{N} \) such that \( x_{0}^{\tau, n} = x_0 \) and for every \( i = 1, 2, \ldots, N \), \( x_{i}^{\tau, n} \) minimizes the functional

\[
x \mapsto \mathcal{E}(t, x) + \Psi(x - x_{i-1}^{\tau, n})
\]

over \( x \in \mathbb{R}^d \) with \( \|x - x_{i-1}^{\tau, n}\| \leq \varepsilon \).

Moreover, the function \( t \mapsto x_{i}^{\tau, n}(t) \) defined by the interpolation \( x_{i}^{\tau, n}(t) = x_{i-1}^{\tau, n} \) if \( t \in [t_{i-1}, t_{i}] \), \( i \in \{1, \ldots, N\} \) satisfies the following energy estimates.

(i) (Discrete bound) For any \( n \in \{1, \ldots, N\} \) we have

\[
\mathcal{E}(t_{n}, x_{n}^{\tau}) \leq \mathcal{E}(0, x_0) e^{\lambda t_{n}} \quad \text{and} \quad \mathcal{E}(0, x_{n}^{\tau}) \leq \mathcal{E}(0, x_0) e^{2\lambda t_{n}};
\]

(ii) (Integral bound) For all \( 0 \leq s \leq t \leq T \), it holds that \( \mathcal{R}(x^{\tau}; [s, t]) < \infty \), \( \partial_{t} \mathcal{E}(\cdot, x^{\tau}(\cdot)) \in L^1(0, T) \) and

\[
\mathcal{E}(t, x^{\tau}(t)) - \mathcal{E}(s, x^{\tau}(s)) \leq \int_{s}^{t} \partial_{t} \mathcal{E}(r, x^{\tau}(r)) \, dr - \mathcal{R}(x^{\tau}; [s, t]).
\]

Proof. Since \( x \mapsto \mathcal{E}(t_{n}, x) + \Psi(x - x_{i-1}^{\tau, n}) \) is continuous, this functional has a minimizer \( x_{i}^{\tau, n} \) in the compact set \( \|x - x_{i-1}^{\tau, n}\| \leq \varepsilon \). The energy estimates can be proved similarly for energetic solutions (see e.g. [10]). A detailed proof can be found in the Appendix.

Lemma 4 (Epsilon-neighborhood solution \( x^\varepsilon \)). Given any initial datum \( x_0 \in \mathbb{R}^d \) such that \( x_0 \) is a local minimizer of the functional \( x \mapsto \mathcal{E}(0, x) + \Psi(x - x_0) \). Let \( x^{\tau, n} \) be as in Lemma 3. There exists a subsequence \( \tau_n \to 0 \) such that \( x^{\tau_n, n}(t) \to x^\varepsilon(t) \) for all \( t \in [0, T] \). Moreover, the epsilon-neighborhood solution \( x^\varepsilon(\cdot) \) satisfies the following properties:
(i) (Epsilon local stability) If \( x^\varepsilon(\cdot) \) is right-continuous at \( t \), namely \( \lim_{t' \to t^+} x^\varepsilon(t') = x^\varepsilon(t) \), then \( x^\varepsilon(t) \) satisfies the epsilon local stability

\[
\mathcal{E}(t, x^\varepsilon(t)) \leq \mathcal{E}(t, x) + \Psi(x - x^\varepsilon(t)) \quad \text{for all } \|x - x^\varepsilon(t)\| \leq \varepsilon;
\]

(ii) (Energy-dissipation inequalities) We have \( \mathcal{Diss}_\Psi(x^\varepsilon; [0, T]) \leq C \) (independent of \( \varepsilon \)), \( \partial_t \mathcal{E}(\cdot, x^\varepsilon(\cdot)) \in L^1(0, T) \) and for all \( 0 \leq s \leq t \leq T \),

\[
-\mathcal{Diss}_{new}(x^\varepsilon; [s, t]) \leq \mathcal{E}(t, x^\varepsilon(t)) - \mathcal{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t \mathcal{E}(r, x^\varepsilon(r)) \, dr \leq -\mathcal{Diss}_\Psi(x^\varepsilon; [s, t]).
\]

**Proof.** **Step 1. Existence.** By the Integral bound in Lemma 3, the fact that \( \mathcal{E} \) is non-negative, and condition (7), we have

\[
\mathcal{Diss}_\Psi(x^\varepsilon, \tau; [0, T]) \leq \mathcal{E}(0, x_0) - \mathcal{E}(T, x^\varepsilon(T)) + \int_0^T \partial_t \mathcal{E}(r, x^\varepsilon(T)) \, dr
\]

\[
\leq \mathcal{E}(0, x_0) + \sum_{i=1}^{N+1} \int_{t_i-1}^{t_i} \lambda \mathcal{E}(t_i-1, x^\varepsilon(t_i-1)) e^{\lambda(r-t_1)} \, dr.
\]

Here we denote \( T \) by \( t_{N+1} \). Using the Discrete bound in Lemma 3 we get

\[
\mathcal{Diss}_\Psi(x^\varepsilon, \tau; [0, T]) \leq \mathcal{E}(0, x_0) + \int_0^T \lambda \mathcal{E}(0, x_0) e^{\lambda r} \, dr = \mathcal{E}(0, x_0) e^{\lambda T}.
\]

Thus, \( \{x^\varepsilon, \tau\} \) has uniformly bounded variation and it is uniformly bounded. Therefore, applying Helly’s selection principle [22, 27], we can find a subsequence \( \tau_n \to 0 \) and a BV function \( x^\varepsilon(\cdot) \) such that \( x^\varepsilon, \tau_n(t) \to x^\varepsilon(t) \) as \( n \to \infty \) for all \( t \in [0, T] \).

**Step 2. A consequence of the right-continuity.** Let us denote by \( \{t^\varepsilon_n\}_{i=0}^{N_n} \) the partition corresponding to \( \tau_n \) and assume that \( t \in [t^\varepsilon_{n-1}, t^\varepsilon_n] \). It is obvious that

\[
x^\varepsilon_{i-1} = x^\varepsilon_{i-1}(t) \to x^\varepsilon(t)
\]

as \( n \to \infty \). Now we show that if \( x^\varepsilon(\cdot) \) is right-continuous at \( t \), then

\[
x^\varepsilon_i = x^\varepsilon_{i-1}(t^\varepsilon_i) \to x^\varepsilon(t).
\]

Let \( t' > t \). Thanks to the Integral bound in Lemma 3 we have

\[
\mathcal{E}(t', x^\varepsilon_{i-1}(t')) - \mathcal{E}(t, x^\varepsilon_{i-1}(t)) + \mathcal{Diss}_\Psi(x^\varepsilon_{i-1}; [t, t']) \leq \int_t^{t'} \partial_t \mathcal{E}(r, x^\varepsilon_{i-1}(r)) \, dr \leq C|t' - t|.
\]

Here the last inequality is due to the continuity of \( \partial_t \mathcal{E} \) and the fact that \( x^\varepsilon, \tau_n \) is bounded on \([0, T]\). For \( n \) large enough, we have \( t < t^\varepsilon_n < t' \). Therefore,

\[
\Psi(x^\varepsilon_{i-1} - x^\varepsilon_{i-1}) \leq \mathcal{Diss}_\Psi(x^\varepsilon_{i-1}; [t, t'])).
\]

Moreover, when \( n \to \infty \), we get

\[
x^\varepsilon_{i-1}(t) \to x^\varepsilon(t) \text{ and } x^\varepsilon_{i-1}(t') \to x^\varepsilon(t').
\]

Thus it follows from the above integral bound that

\[
\mathcal{E}(t', x^\varepsilon(t')) - \mathcal{E}(t, x^\varepsilon(t)) + \limsup_{n \to \infty} \Psi(x^\varepsilon_{i-1} - x^\varepsilon_{i-1}) \leq C|t' - t|.
\]

Notice that the inequality above holds for all \( t' > t \). Hence, we can take \( t' \to t \) and use the assumption \( x^\varepsilon(t^\varepsilon) = x^\varepsilon(t) \) to obtain

\[
\limsup_{n \to \infty} \Psi(x^\varepsilon_{i-1} - x^\varepsilon_{i-1}) \leq 0.
\]

Since \( x^\varepsilon_{i-1} \to x^\varepsilon(t) \), we can conclude that \( x^\varepsilon_{i-1} \to x^\varepsilon(t) \) as \( n \to \infty \).

**Step 3. Stability.** We show that for all \( t \in [0, T] \), if \( x^\varepsilon(\cdot) \) is right-continuous at \( t \), then

\[
\mathcal{E}(t, x^\varepsilon(t)) \leq \mathcal{E}(t, z) + \Psi(z - x^\varepsilon(t)) \quad \text{for all } \|z - x^\varepsilon(t)\| \leq \varepsilon.
\]

To this end, we first prove the result for \( z \in \mathbb{R}^d \) with \( \|z - x^\varepsilon(t)\| < \varepsilon \). Since \( \lim_{n \to \infty} x^\varepsilon, \tau_n(t) = x^\varepsilon(t) \), we get

\[
\|z - x^\varepsilon, \tau_n(t)\| < \varepsilon
\]
for $n$ large enough. We shall follow the notations in Step 2. The fact that $t \in [t^n_{i-1}, t^n_i)$ yields $x^{\epsilon, \tau_n}(t) = x^{\epsilon, \tau_n}_i$.

From the definition of $x^{\epsilon, \tau_n}_i$ and condition $\|z - x^{\epsilon, \tau_n}_i\| < \varepsilon$, we obtain

$$\mathcal{E}(t^n_i, x^{\epsilon, \tau_n}_i) + \Psi(x^{\epsilon, \tau_n}_i - x^{\epsilon, \tau_n}_i) \leq \mathcal{E}(t^n_i, z) + \Psi(z - x^{\epsilon, \tau_n}_i).$$

Taking the limit as $n \to \infty$ and using the fact that both $x^{\epsilon, \tau_n}_i$ and $x^{\epsilon, \tau_n}_i$ converge to $x^\epsilon(t)$ (see Step 2), we have

$$\mathcal{E}(t, x^\epsilon(t)) \leq \mathcal{E}(t, z) + \Psi(z - x^\epsilon(t)) \text{ for all } \|z - x^\epsilon(t)\| < \varepsilon. \quad (9)$$

Now for any $z$ satisfying $\|z - x^\epsilon(t)\| = \varepsilon$, we can choose a sequence $z_n$ converging to $z$ such that $\|z_n - x^\epsilon(t)\| < \varepsilon$. Applying (9) for $z_n$, we get

$$\mathcal{E}(t, x^\epsilon(t)) \leq \mathcal{E}(t, z_n) + \Psi(z_n - x^\epsilon(t)) \quad \text{ (10)}$$

Note that the mapping $y \mapsto \mathcal{E}(t, y) + \Psi(y - x^\epsilon(t))$ is continuous, taking the limit in (10), we obtain the result also for $\|z - x^\epsilon(t)\| = \varepsilon$.

\section*{Step 4. Energy-dissipation inequalities.}

By the Integral bound in Lemma 3 we have for all $0 \leq s \leq t \leq T$,

$$\mathcal{E}(t, x^\epsilon(t)) - \mathcal{E}(s, x^\epsilon(s)) \leq \int_s^t \partial_t \mathcal{E}(r, x^\epsilon(r)) dr - \mathcal{D}_{iss}(x^\epsilon; [s, t]).$$

Since $x^{\epsilon, \tau_n}(r) \to x^\epsilon(r)$ for all $r \in [0, T]$, we have

$$\mathcal{E}(t, x^{\epsilon, \tau_n}(t)) - \mathcal{E}(s, x^{\epsilon, \tau_n}(s)) \to \mathcal{E}(t, x^\epsilon(t)) - \mathcal{E}(s, x^\epsilon(s))$$

and

$$\int_s^t \partial_t \mathcal{E}(r, x^{\epsilon, \tau_n}(r)) dr \to \int_s^t \partial_t \mathcal{E}(r, x^\epsilon(r)) dr$$

as $n \to \infty$. Moreover, one has

$$\liminf_{n \to \infty} \mathcal{D}_{iss}(x^{\epsilon, \tau_n}; [s, t]) \geq \mathcal{D}_{iss}(x^\epsilon; [s, t]).$$

Thus we can derive one energy-dissipation inequality

$$\mathcal{E}(t, x^\epsilon(t)) - \mathcal{E}(s, x^\epsilon(s)) \leq \int_s^t \partial_t \mathcal{E}(r, x^\epsilon(r)) dr - \mathcal{D}_{iss}(x^\epsilon; [s, t]).$$

We shall use Lemma 5 to obtain the other energy-dissipation inequality,

$$\mathcal{E}(t, x^\epsilon(t)) - \mathcal{E}(s, x^\epsilon(s)) \geq \int_s^t \partial_t \mathcal{E}(r, x^\epsilon(r)) dr - \mathcal{D}_{iss_{new}}(x^\epsilon; [s, t]).$$

To apply Lemma 5 it is sufficient to verify that $-\nabla_x \mathcal{E}(t, x^\epsilon(t)) \in \partial \Psi(0)$ for a.e. $t \in (0, T)$. Indeed, for every $t \in [0, T]$ such that $x^\epsilon(t)$ is right-continuous at $t$, we have proved in Step 3 the $\varepsilon$-stability

$$\mathcal{E}(t, x^\epsilon(t)) \leq \mathcal{E}(t, x) + \Psi(x - x^\epsilon(t)) \quad \text{ for all } \|x - x^\epsilon(t)\| \leq \varepsilon.$$

For every $x$ satisfying $\|x - x^\epsilon(t)\| \leq \varepsilon$ and for every $s \in [0, 1]$, denote by $z = x^\epsilon(t) + s(x - x^\epsilon(t))$. Clearly, $\|z - x^\epsilon(t)\| \leq \varepsilon$. Thus,

$$\mathcal{E}(t, x^\epsilon(t)) \leq \mathcal{E}(t, z) + \Psi(z - x^\epsilon(t)).$$

This inequality is equivalent to

$$\frac{\mathcal{E}(t, x^\epsilon(t)) - \mathcal{E}(t, x^\epsilon(t) + s(x - x^\epsilon(t)))}{s} \leq \Psi(x - x^\epsilon(t)).$$

Taking $s \to 0^+$ and notice that $\mathcal{E}$ is of class $C^1$, we obtain that

$$(-\nabla_x \mathcal{E}(t, x^\epsilon(t)), x - x^\epsilon(t)) \leq \Psi(x - x^\epsilon(t)) \quad \text{ for all } \|x - x^\epsilon(t)\| \leq \varepsilon.$$

Now for every $y \in \mathbb{R}^A \setminus \{0\}$, applying the inequality above for $\tilde{y} = x^\epsilon(t) + \varepsilon y/\|y\|$, we get

$$(-\nabla_x \mathcal{E}(t, x^\epsilon(t)), \tilde{y}) \leq \Psi(y).$$

Hence, $-\nabla_x \mathcal{E}(t, x^\epsilon(t)) \in \partial \Psi(0)$ whenever $x^\epsilon(t)$ is right-continuous at $t$.

On the other hand, since $x^\epsilon(t)$ is a BV function, it is continuous except at most countably many points. Thus, we can conclude that $-\nabla_x \mathcal{E}(t, x^\epsilon(t)) \in \partial \Psi(0)$ for a.e. $t \in (0, T)$. \qed
Lemma 5 (Lower bound of the new energy-dissipation balance). For any BV function $u : [0, T] \to \mathbb{R}^d$, energy functional $\mathcal{E} \in C^1([0, T] \times \mathbb{R}^d)$ and dissipation functional $\Psi$ which is convex and positively 1-homogeneous, if $-\nabla x \mathcal{E}(t, u(t)) \in \partial \Psi(0)$ for a.e. $t \in (0, T)$, it holds that
\[
\mathcal{E}(t_1, u(t_1)) - \mathcal{E}(t_0, u(t_0)) \geq \int_{t_0}^{t_1} \partial_t \mathcal{E}(s, u(s)) ds - \mathcal{Diss}_{\text{new}}(u; [t_0, t_1]), \quad \text{for all } 0 \leq t_0 < t_1 \leq T.
\]

This result is due to Mielke, Rossi and Savaré (see [20, Proposition 4.2] for finite-dimensional space and [21, Theorem 3.11] for infinite-dimensional space). For the readers’ convenience, a proof of Lemma 5 is included in Appendix.

4. BV solutions constructed by epsilon-neighborhood method

Lemma 6 (Limit of epsilon-neighborhood solutions). Given an initial datum $x_0 \in \mathbb{R}^d$ which is a local minimizer of the functional $x \mapsto \mathcal{E}(0, x) + \Psi(x - x_0)$. Let $x^\varepsilon$ be as in Lemma 4. There exists a subsequence $\varepsilon_n \to 0$ and a BV function $u$ such that $x^\varepsilon(t) \to u(t)$ for all $t \in [0, T]$. Moreover, the function $u$ satisfies the following properties

- (Weak local stability) If $t \mapsto u(t)$ is continuous at $t$, then $-\nabla x \mathcal{E}(t, u(t)) \in \partial \Psi(0)$;
- (New energy-dissipation balance) For all $0 \leq s \leq t \leq T$, one has
\[
\mathcal{E}(t, u(t)) - \mathcal{E}(s, u(s)) = \int_s^t \partial \mathcal{E}(r, u(r)) dr - \mathcal{Diss}_{\text{new}}(u; [s, t]).
\]

Proof. **Step 1. Existence.** Since $\mathcal{Diss}_\Psi(x^\varepsilon; [0, T]) \leq C$ independent of $\varepsilon$, by Helly’s selection principle we can find a subsequence $\varepsilon_n \to 0$ and a BV function $u$ such that $x^\varepsilon(t) \to u(t)$ as $n \to \infty$ for all $t \in [0, T]$.

**Step 2. Stability.** Let $A := \{ t \in [0, T] \mid x^\varepsilon(t) \text{ is right continuous at } t \}$.

Then $[0, T] \setminus A$ is at most countable. Moreover, for $t \in A$, by Lemma 4 we get
\[
\mathcal{E}(t, x^\varepsilon(t)) \leq \mathcal{E}(t, z) + \Psi(z - x^\varepsilon(t)) \text{ for all } \| z - x^\varepsilon(t) \| \leq \varepsilon_n
\]
for all $n \geq 1$. For $t \in A$ and $n \geq 1$,
\[
\langle -\nabla x \mathcal{E}(t, x^\varepsilon(t)), z \rangle \leq \Psi(z) \text{ for all } z \in \mathbb{R}^d,
\]
can be shown in a similar manner as in Step 4, Lemma 4. Taking $n \to \infty$, we obtain
\[
\langle -\nabla x \mathcal{E}(t, u(t)), z \rangle \leq \Psi(z) \text{ for all } z \in \mathbb{R}^d, \text{ for all } t \in A.
\]
By continuity, we immediately have $-\nabla x \mathcal{E}(t, u(t)) \in \partial \Psi(0)$ provided that $u$ is continuous at $t$.

**Step 3. New energy-dissipation balance.** By means of a similar proof of the energy inequalities in Lemma 4, we have
\[
-\mathcal{Diss}_{\text{new}}(u; [s, t]) \leq \mathcal{E}(t, u(t)) - \mathcal{E}(s, u(s)) - \int_s^t \partial \mathcal{E}(r, u(r)) dr \leq -\mathcal{Diss}(u; [s, t]).
\]
The second inequality is a consequence of the corresponding inequality of $x^\varepsilon$ in Lemma 4 and Fatou’s lemma, while the first inequality follows from Lemma 3.

Notice that if the solution $t \mapsto u(t)$ is continuous on $[a, b] \subseteq [0, T]$, then $\mathcal{Diss}(u; [a, b]) = \mathcal{Diss}_{\text{new}}(u; [a, b])$. Thus, we have immediately the energy-dissipation balance
\[
\mathcal{E}(b, u(b)) - \mathcal{E}(a, u(a)) - \int_a^b \partial \mathcal{E}(r, u(r)) dr = -\mathcal{Diss}(u; [a, b]) = -\mathcal{Diss}_{\text{new}}(u; [a, b]).
\]

Therefore, it remains to consider jump points. More precisely, we need to show that if $u$ jumps at $t \in (0, T)$, namely $u(t^-) \neq u(t^+)$, then
\[
\mathcal{E}(t, u(t^+)) - \mathcal{E}(t, u(t^-)) = \Delta_{\text{new}}(t, u(t^+), u(t)) - \Delta_{\text{new}}(t, u(t), u(t^+)).
\]
This fact follows from Lemma 5, 7 and 8.

□
To prove the upper bound, we start by showing that the discretized solution $x^{ε, r}$ is "almost" an optimal transition.

**Lemma 7** (Approximate optimal transition). For the discretized solution $x^{ε, r}$, if we write $x_j := x^{ε, r}(t_j)$, it holds that

$$\langle -∇_x \mathcal{E}(t_i, x_i), x_i - x_{i-1} \rangle = \Psi(x_i - x_{i-1}) + \min_{\eta \in \partial \Psi(0)} \|\eta + ∇_x \mathcal{E}(t_i, x_i)\|_s \cdot \|x_i - x_{i-1}\|.$$  

Consequently, if $δ ≥ ε + |t - t_i|$ and $v : [a, b] → \mathbb{R}^d$ is the linear curve connecting $x_{i-1}$ and $x_i$, namely

$$v(s) = x_i - 1 + \frac{s - a}{b - a}(x_i - x_{i-1}),$$

there exists $g(δ)$ such that $g(δ) → 0$ as $δ → 0$ and

$$\mathcal{E}(t, x_{i-1}) - \mathcal{E}(t, x_i) ≥ \int_a^b \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \|\eta + ∇_x \mathcal{E}(t, v(s))\|_s \cdot \|\dot{v}(s)\| ds - (b - a)g(δ)\|x_i - x_{i-1}\|.$$  

**Proof.** The proof is trivial when $x_i = x_{i-1}$. Hence, we shall assume that $x_i \neq x_{i-1}$.

**Step 1.** Denote by $m(z) := \|z - x_{i-1}\|$ and $h(z) := \mathcal{E}(t_i, z) + \Psi(z - x_{i-1})$. Recall that $x_i$ is a minimizer for

$$\inf_{m(z) ≤ ε} h(z).$$

Denote by $c := \|x_i - x_{i-1}\|$. Since $c ≤ ε$, we can consider $x_i$ as a minimizer for

$$\inf_{m(z) = c} h(z).$$

Here $(x_i - x_{i-1})^T$ stands for the transpose of $(x_i - x_{i-1})$. By Lagrange multiplier, there exists $λ ∈ \mathbb{R}$ such that $λ∇m(x_i) ∈ ∂h(x_i)$, or equivalently

$$λ\left(\frac{(x_i - x_{i-1})^T}{\|x_i - x_{i-1}\|}\right) = -∇_x \mathcal{E}(t_i, x_i) ∈ ∂h(x_i - x_{i-1}).$$

The inclusion above implies two following conditions

i. For all $z ∈ \mathbb{R}^d$, it holds that $\langle -∇_x \mathcal{E}(t_i, x_i) + λ\left(\frac{(x_i - x_{i-1})^T}{\|x_i - x_{i-1}\|}\right), z \rangle ≤ Ψ(z).$

ii. $\langle -∇_x \mathcal{E}(t_i, x_i) + λ\left(\frac{(x_i - x_{i-1})^T}{\|x_i - x_{i-1}\|}\right), x_i - x_{i-1} \rangle = Ψ(x_i - x_{i-1}).$

**Step 2.** Since the function $h_1(s) = h(x_{i-1} + s(x_i - x_{i-1}))$ satisfies $h_1(s) ≥ h_1(1)$ for all $s ∈ [0, 1]$, it follows that

$$\mathcal{E}(t_i, x_{i-1} + s(x_i - x_{i-1})) + sΨ(x_i - x_{i-1}) ≥ \mathcal{E}(t_i, x_i) + Ψ(x_i - x_{i-1}).$$

The above inequality can be rewritten as

$$\frac{\mathcal{E}(t_i, x_i + (s - 1)(x_i - x_{i-1})) - \mathcal{E}(t_i, x_i)}{s - 1} + Ψ(x_i - x_{i-1}) ≤ 0.$$

Since $\mathcal{E}$ is of class $C^1$, we can conclude that

$$\langle ∇_x \mathcal{E}(t_i, x_i), x_i - x_{i-1} \rangle + Ψ(x_i - x_{i-1}) ≤ 0.$$  

In addition, [12] and Condition ii) in Step 1 give $λ ≤ 0$. Moreover, for all $η ∈ ∂Ψ(0)$ we have $-Ψ(x_i - x_{i-1}) ≤ \langle -η, x_i - x_{i-1} \rangle$. Thus, condition ii) implies

$$-λ\left(\frac{(x_i - x_{i-1})^T}{\|x_i - x_{i-1}\|}\right) = \langle -∇_x \mathcal{E}(t_i, x_i), x_i - x_{i-1} \rangle - Ψ(x_i - x_{i-1}) ≤ \langle -∇_x \mathcal{E}(t_i, x_i) - η, x_i - x_{i-1} \rangle ≤ -\| -∇_x \mathcal{E}(t_i, x_i) - η\|_s \cdot \|x_i - x_{i-1}\|.$$  

Thanks to Condition i) in Step 1, $η_0 ∈ ∂Ψ(0)$, where $η_0$ is chosen so that $η_0 = -∇_x \mathcal{E}(t_i, x_i) - k(x_i - x_{i-1})^T$ and $k = -\|x_i - x_{i-1}\| ≥ 0$. Moreover, the above two inequalities becomes equalities with such choice of $η_0$. Thus, we can write

$$-λ\left(\frac{(x_i - x_{i-1})^T}{\|x_i - x_{i-1}\|}\right) = \min_{η ∈ ∂Ψ(0)} \|η + ∇_x \mathcal{E}(t_i, x_i)\|_s \cdot \|x_i - x_{i-1}\|.$$


Hence, we obtain that
\[
\langle -\nabla_x \mathcal{E}(t, x_i), x_i - x_{i-1} \rangle = \Psi(x_i - x_{i-1}) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, x_i) \|_* \cdot \| x_i - x_{i-1} \|.
\]

**Step 3.** Consequently, using \(|t - t_i| \leq \delta, \|x_{i-1} - x_i\| \leq \varepsilon \leq \delta\) and the fact that \(\nabla_x \mathcal{E}(\cdot, \cdot)\) is continuous on compact sets, there exists \(g(\delta)\) such that \(g(\delta) \to 0\) when \(\delta \to 0\) and
\[
\langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle \geq \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| - g(\delta) \| \dot{v}(s) \|
\]
for every \(s \in [a, b] \). Therefore,
\[
\mathcal{E}(t, x_{i-1}) - \mathcal{E}(t, x_i) = \int_a^b \langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle \, ds \\
\geq \int_a^b \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| \, ds - (b - a)g(\delta)\| x_i - x_{i-1} \|.
\]

Now we are in the position to prove the new energy-dissipation upper bound at jumps.

**Lemma 8** (Upper bound). Let \(u \) be the function as in Lemma 7. If \(u(t^-) \neq u(t)\), then
\[
\Delta_{\text{new}}(t, u(t^-), u(t)) \leq \mathcal{E}(t, u(t^-)) - \mathcal{E}(t, u(t)).
\]

**Proof.** Let \(0 < \tau < \varepsilon < \delta < 1\). By the definition of the discretized solution \(x^{\varepsilon,\tau}\), for every \(t \in (0, T)\) we have
\[
x^{\varepsilon,\tau}(t - \delta) = x^{\varepsilon,\tau}(t_i) \quad \text{and} \quad x^{\varepsilon,\tau}(t) = x^{\varepsilon,\tau}(t_{i+k})
\]
for \(t_i, t_{i+k} \in [t - 2\delta, t + \delta]\).

We can construct an absolutely continuous function \(v : [0, 1] \to \mathbb{R}^d\) by linearly interpolating the following \((k + 3)\) points:
\[
u(t^-), x^{\varepsilon,\tau}(t - \delta), x^{\varepsilon,\tau}(t_i), x^{\varepsilon,\tau}(t_{i+1}), \ldots, x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t), u(t).
\]

More precisely, we define
\[
\begin{align*}
z_0 &= u(t^-), \\
z_1 &= x^{\varepsilon,\tau}(t - \delta), \\
z_2 &= x^{\varepsilon,\tau}(t_{i+1}), \\
&\vdots \\
z_{k+1} &= x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t), \\
z_{k+2} &= u(t),
\end{align*}
\]
and denote \(r := 1/(k + 2)\) and
\[
v(s) = z_j + \frac{s - jr}{r}(z_{j+1} - z_j) \quad \text{when} \quad s \in [jr, (j + 1)r], \quad j = 0, 1, \ldots, k + 1.
\]

By the definition of the new dissipation, we have
\[
\Delta_{\text{new}}(t, u(t^-), u(t)) \leq \int_0^1 \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| \, ds \\
= \sum_{j=0}^{k+1} \int_{jr}^{(j+1)r} \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| \, ds.
\]

When \(j = 0\) and \(j = k + 1\), we estimate
\[
\int_{jr}^{(j+1)r} \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| \leq C \int_{jr}^{(j+1)r} \| \dot{v}(s) \| \, ds = C \| z_{j+1} - z_j \|.
\]

When \(j = 1, 2, \ldots, k\), Lemma 7 yields the following equation
\[
\int_{jr}^{(j+1)r} \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \| \, ds \leq \mathcal{E}(t, x^{\varepsilon,\tau}(t_{i+j-1})) - \mathcal{E}(t, x^{\varepsilon,\tau}(t_{i+j})) + rg(\delta) \cdot \| x^{\varepsilon,\tau}(t_{i+j}) - x^{\varepsilon,\tau}(t_{i+j-1}) \|,
\]
where \( g(\delta) \to 0 \) as \( \delta \to 0 \). Taking the sum over \( j = 0, 1, \ldots, k+1 \) and using the bound \( \|x - \tau; [0,T]\| \leq C \) (independent of \( \varepsilon \) and \( \tau \)), we find that

\[
\Delta_{\text{new}}(t, u(t^-), u(t)) \leq \int_0^1 \Psi(\dot{v}(s)) + \min_{\eta \in \partial \Phi(0)} \|\eta + \nabla x \mathcal{E}(t, v(s))\|_{\ast} \cdot \|\dot{v}(s)\| \, ds \\
\leq \mathcal{E}(t, x^\varepsilon(t) - \delta) - \mathcal{E}(t, x^\varepsilon(t)) + Cg(\delta) + C\|u(t^-) - x^\varepsilon(t) - \delta\| + C\|x^\varepsilon(t) - u(t)\|.
\]

Taking the limit \( \tau \to 0 \), then \( \varepsilon \to 0 \), then \( \delta \to 0 \), we conclude that

\[
\Delta_{\text{new}}(t, u(t^-), u(t)) \leq \mathcal{E}(t, u(t^-)) - \mathcal{E}(t, u(t)).
\]

This finishes the proof. \( \square \)

5. APPENDIX: TECHNICAL PROOFS

5.1. Example [2] First of all, it is easy to verify that \( \mathcal{E}(t, x) : [0, 2] \times \mathbb{R} \to [0, +\infty) \) is \( C^1 \) and satisfies condition [8]. Moreover, \( x_0 = 0 \) is a local minimizer for the functional \( x \mapsto \mathcal{E}(0, x) + |x| \).

**Part I. Energetic solution via time-discretization.**

**Step 1.** Fix a time step \( \tau > 0 \). To find the discretized solution \( x^\tau(t) \), it suffices to calculate \( x_i := x^\tau(t_i) \) where \( 0 = t_0 < \cdots < t_N \leq 1 \) and \( t_i - t_{i-1} = \tau \) for all \( i = 1, 2, \ldots, N \). Here \( N \in \mathbb{N} \) satisfies \( i \in \{\tau N, \tau(N+1)\} \).

We have \( x_0 = 0 \) and for all \( i = 1, 2, \ldots, N \), \( x_i \) is a minimizer of the functional

\[
x \in \mathbb{R} \mapsto \mathcal{E}(t_i, x) + |x - x_{i-1}|.
\]

**Step 2.** Let us fix \( t \in [0, 2] \) and consider the functional

\[
F(x) := \mathcal{E}(t, x) + |x| = x^2 - x^4 + 0.3 x^6 + t(1 - x^2) - x + |x| + 6, \quad x \in \mathbb{R}.
\]

It is readily seen that

- When \( t \leq 1 \), \( F(x) \) has two local minimizers (see Fig. 1)

\[
x = 0 \quad \text{and} \quad x = y(t) := \frac{\sqrt{10 + \sqrt{10 + 90t} - 10}}{3}.
\]

Moreover,

\[
F(y(t)) - F(0) = \frac{1}{243}(10 + \sqrt{10 + 90t})(8 - 18t - \sqrt{10 + 90t}),
\]

which is positive if \( t < 1/6 \) and negative if \( t > 1/6 \). Hence \( F \) has a unique global minimizer \( x = 0 \) if \( 0 \leq t < 1/6 \), and then \( F \) has a unique global minimizer at \( x = y(t) \) if \( 1/6 < t < 1 \).

- When \( t > 1 \), \( F(x) \) has a unique local (also global) minimizer at \( x = y(t) \).

**Step 3.** By induction, we can show that if \( t_{i_0} < 1/6 \leq t_{i_0+1} \), then \( x_i = 0 \) for all \( i = 1, 2, \ldots, i_0 \), and either \( x_{i_0+1} = y(t_{i_0+1}) \), or \( x_{i_0+1} = 0 \) and \( x_{i_0+2} = y(t_{i_0+2}) \).

Next, we show that if \( i_{i-1} \geq 1/6 \) and \( x_{i-1} = y(t_{i-1}) > 0 \), then \( x_i = y(t_i) \). Recall that \( x_i \) is a global minimizer for the functional

\[
x \in \mathbb{R} \mapsto F_i(x) := \mathcal{E}(t_i, x) + |x - x_{i-1}| = x^2 - x^4 + 0.3 x^6 + t_i(1 - x^2) - x + |x - x_{i-1}| + 6.
\]

By using the triangle inequality \( -x + |x - x_{i-1}| \geq -x_{i-1} \) and the same analysis of \( F \), we can conclude that \( x_i = y(t_i) \).

Taking the limit as \( \tau \to 0 \), we obtain the energetic solution

\[
x(t) = 0 \quad \text{if} \quad t \in [0, 1/6), \quad x(1/6) \in \{0, \sqrt{5/3}\}, \quad x(t) = y(t) \quad \text{if} \quad t \in [1/6, 2].
\]

**Step 4.** Finally, we show that the energetic solution does not satisfies the new energy-dissipation balance. It suffices to show that at the jump point \( t = 1/6 \),

\[
\mathcal{E}(t, x(t^+)) - \mathcal{E}(t, x(t^-)) > -\Delta_{\text{new}}(t, x(t^-), x(t^+)).
\]

Indeed, a direct computation gives us that at \( t = 1/6 \),

\[
\mathcal{E}(t, x(t^+)) - \mathcal{E}(t, x(t^-)) = \mathcal{E}(1/6, \sqrt{5/3}) - \mathcal{E}(1/6, 0) = -\sqrt{5}/3.
\]

On the other hand, at \( t = 1/6 \) we have

\[
\Delta_{\text{new}}(t, x(t^-), x(t^+)) = \int_0^{\sqrt{5/3}} \max \left\{ 1, \frac{2}{3}y - 4y^3 + 1.8y^5 - 1 \right\} \, dy = \frac{185}{486} + \sqrt{5}/3.
\]
Thus,
\[ \mathcal{E}(t, x(t^+)) - \mathcal{E}(t, x(t^-)) > -\Delta_{neq}(t, x(t^-), x(t^+)) \] at \( t = 1/6 \).

**Part II. BV solution constructed by the viscous dissipation.**

\( \Psi_\varepsilon(x) = |x| + \varepsilon x^2 \). We construct the BV solution via vanishing viscosity with the viscous term \( \varepsilon x^2 \) by the method used in [20].

Let us briefly recall the construction of the BV solution. Given \( \varepsilon > 0 \) and \( \tau > 0 \). We denote by \( \tau = \varepsilon / \tau \).

Let \( 0 = t_0 < \cdots < t_N \leq T \) be a partition of \([0, T]\) satisfying \( t_i - t_{i-1} = \tau \) for every \( i \in \{1, \ldots, N\} \) and \( T - t_N < \tau \). The discretized problem is to find a sequence \( \{x^{\tau}_{\varepsilon}\}_{i=1}^N \) such that \( x^{\tau}_{\varepsilon} = 0 \) and \( x^{\tau}_{\varepsilon} \) is a global minimizer for the functional

\[
x \in \mathbb{R} \mapsto \{ \mathcal{E}(t_i, x) + |x - x^{\tau}_{\varepsilon}| + \varepsilon|x - x^{\tau}_{\varepsilon}|^2 \}
\]

for every \( i = 1, 2, \ldots, N \) and \( \varepsilon = \varepsilon / \tau \). Then using interpolation and passing to the pointwise limit as \( \tau \to 0, \varepsilon \to 0 \) and \( \varepsilon = \varepsilon / \tau \to \infty \), we obtain the BV solution.

Now coming back to our example, for \( t \in (0, 2) \), we consider the function

\[
F(x) := \mathcal{E}(t, x) + |x|e^2 = t + (1 + e - t) x^2 - x^4 + 0.3 x^6 - x + |x| + 6, \quad x \in \mathbb{R}.
\]

If \( e \) is large enough (such that \( 1 + e - t \geq 1 \)), one has

\[
F(x) \geq t + x^2 - x^4 + 0.3 x^6 + 6 = t + \frac{1}{6} e^2 + \left( \sqrt{\frac{5}{6}} x - \sqrt{\frac{3}{10}} x^3 \right)^2 + 6 \geq t + 6 = F(0).
\]

Thus \( F \) has a unique global minimizer at \( x = 0 \). Therefore, the discretized sequence \( \{x_i^{\tau}_{\varepsilon}\} \) is identically equal to 0 and so is the BV solution.

**Part III. BV solution constructed by epsilon-neighborhood method.**

**Step 1.** Let \( \varepsilon > 0 \) and \( \tau > 0 \) be small. Let us compute \( x_i := x^{\tau}_{\varepsilon}(t_i) \), where \( t_i = i/N \) for \( i = 0, 1, \ldots, N \).

Here \( N \in \mathbb{N} \) with \( 1 \in (\tau N, \tau (N + 1)) \).

By definition, \( x_0 = 0 \) and \( x_1 \) is a minimizer for the functional

\[
F_1(x) := \mathcal{E}(t_1, x) + |x - x_{i-1}| = x^2 - x^4 + 0.3 x^6 + t_i(1 - x^2) - x + |x - x_{i-1}| + 6
\]

over \( x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon] \). In particular, if \( x_{i-1} = 0 \), then \( x_i \) is a minimizer for

\[
\tilde{F}_1(x) := x^2 - x^4 + 0.3 x^6 + t_i(1 - x^2) - x + |x| + 6
\]

over \( x \in [-\varepsilon, \varepsilon] \).

Recall that if \( t_i < 1 \), \( \tilde{F}_1(x) \) has two local minimizers at \( x = 0 \) and

\[
x = y(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} > 1.
\]

Choose \( \varepsilon < 1 \), then \( x = 0 \) is the unique minimizer for \( \tilde{F}_1(x) \) on \( x \in [-\varepsilon, \varepsilon] \). Thus, we can conclude that \( x_i = 0 \) whenever \( t_i < 1 \).

**Step 2.** Assume that \( t_i \in [1, 2] \). We prove that \( x_i \leq y(t_i) \) for all \( i \) by contradiction. Indeed, by induction we can assume that \( x_{i-1} \leq y(t_{i-1}) \). Suppose that \( x_i > y(t_i) \). Since \( x_{i-1} \leq y(t_{i-1}) < y(t_i) < x_i \leq x_{i-1} + \varepsilon \), there exists an \( \alpha \in (y(t_i), x_i) \subset [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon] \). Using the fact that the function \( x \mapsto g_i(x) = x^2 - x^4 + 0.3 x^6 + t_i(1 - x^2) + 6 \) is strictly increasing in the interval \( [y(t_i), \infty) \) and the triangle inequality \( f(x) = -x + |x - x_{i-1}| \geq -x_{i-1} \), we have

\[
F_i(x_i) = x_i^2 - x_i^4 + 0.3 x_i^6 + t_i(1 - x_i^2) - x_i + |x_i - x_{i-1}| + 6 > a^2 - a^4 + 0.3 a^6 + t_i(1 - a^2) - x_{i-1} + 6 = F_i(a).
\]

This contradicts to the assumption that \( x_i \) is a minimizer for \( F_i(x) \) over \( x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon] \). Thus, we must have \( x_i \leq y(t_i) \).

Moreover, if we choose \( \varepsilon < 1/2 \), it holds that \( x_i \geq x_{i-1} \). Indeed, since \( g_i(x) \) decreases in \( [-1/2, y(t_i)] \) and \( f(x) \) strictly decreases when \( x < x_{i-1} \), for every \( z \in [-1/2, x_{i-1}) \)

\[
F_i(z) = g_i(z) + f(z) > g_i(x_{i-1}) + f(x_{i-1}) = F_i(x_{i-1}).
\]

For the determination of \( x_{i-1} \), we have the following cases.

- \( x_{i-1} \leq y(t_i) - \varepsilon \). Observe that \( y(t) \) strictly increases in \( t \). We can choose \( \tau \) small enough (in this case \( \tau \leq \varepsilon \)) so that \( y(t_i) - y(t_{i-1}) < \varepsilon \). Thus, \( x_{i-1} < y(t_{i-1}) \). Since \( f(x) = x_{i-1} \) for \( x \geq x_{i-1} \) and \( g_i(x) \) decreases in the interval \([x_{i-1}, y(t_i)]\), the function \( F_i(x) = g_i(x) + f(x) \) decreases in the interval \([x_{i-1}, y(t_i)]\). Thus, \( x_i = x_{i-1} + \varepsilon \).
• For the case when $x_{i-1} \in [y(t_i) - \varepsilon, y(t_{i-1})]$, $y(t_i)$ is the unique minimizer of $F_i(x)$ in the interval $[x_{i-1}, x_{i-1} + \varepsilon]$. Thus, $x_i = y(t_i)$.

**Step 3.** Taking the largest $k$ and the smallest $m$ such that $x_k = 0$ and $x_m = y(t_m)$. The number of steps $L$ to move from $x_k$ to $x_m$ is the integer part of $\frac{\log_\varepsilon(\tau)}{\varepsilon}$. Since $\varepsilon$ is fixed, this value is bounded from above by a constant $C = \frac{2}{\varepsilon^2} + 1$. Hence,

$$t_m = t_k + L\tau \leq t_k + C\tau.$$  

Taking $\tau \to 0$, we have $t_m \approx t_k \approx 1$. Thus, for $\varepsilon < \frac{1}{2}$, the BV solution constructed by epsilon-neighborhood method is $x(t) = x^\varepsilon(t) = 0$ if $t \in [0, 1)$ and $x(t) = x^\varepsilon(t) = y(t)$ if $t \in (1, 2]$. At $t = 1$, $x(t)$ can take values either 0 or $y(1)$.

**Step 4.** We show that the BV solution constructed by epsilon-neighborhood does not satisfy the energy-dissipation balance. At the jump point $t = 1$, one has

$$-|x(t^-) - x(t^+)| = -\frac{2\sqrt{5}}{3} > \mathcal{E}(t, x(t^+)) - \mathcal{E}(t, x(t^-)) = \frac{400}{243} - \frac{2\sqrt{20}}{3}.$$  

**Part IV. The solution constructed by the method in [7].** Let us briefly recall the method used in [7]. Let $N \in \mathbb{N}$ be the numbers of time step. The neighborhood is chosen equal to the usual time-step, i.e. $\varepsilon = \tau = \frac{t}{N}$. Take $t_0 = 0$ and $x_0 = 0$. For $j = 1, \ldots, N$, $x_j$ and $t_j$ are defined as follows.

- $x_j$ minimizes $\{\mathcal{E}(t_{j-1}, x) + |x - x_{j-1}|\}$ among all states $x$ such that $|x - x_{j-1}| \leq \varepsilon$.
- $t_j = t_{j-1} + \tau - |x_j - x_{j-1}|$.

By the same argument as in Part III, Step 1, we deduce that $x_{i+1} = 0$ and $t_{i+1} = t_i + \tau$ if $N < T$ and $t_i < 1$.

Now assume that $t_i \in [1, 2]$. Argue as in Part III, Step 2, we have $x_{i+1} \in [x_i, y(t_i)]$ and

- If $x_i \in [0, y(t_i) - \varepsilon)$: $x_{i+1} = x_i + \varepsilon$ and $t_{i+1} = t_i$.
- If $x_i \in [y(t_i) - \varepsilon, y(t_{i-1})]$: $x_{i+1} = y(t_i)$ and $t_{i+1} = t_i + \tau - |y(t_i) - x_i|$.

Taking $\tau$ to 0, we obtain the solution $x(t) = 0$ if $t < 1$, $x(t) = y(t)$ if $t \geq 1$.

### 5.2. Proof of the energy estimate in Lemma [3]

#### Step 1.
By the minimality of $x^{\varepsilon, \tau}_n$ at time $t_n$, we have

$$\mathcal{E}(t_n, x^{\varepsilon, \tau}_n) + \Psi(x^{\varepsilon, \tau}_n - x^{\varepsilon, \tau}_{n-1}) \leq \mathcal{E}(t_n, x^{\varepsilon, \tau}_{n-1}) = \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) + \int_{t_{n-1}}^{t_n} \partial_i \mathcal{E}(t, x^{\varepsilon, \tau}_n)\, dt.$$  

It follows from the assumption [7] that

$$\partial_i \mathcal{E}(t, x^{\varepsilon, \tau}_n) \leq \lambda \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) e^{\lambda(t - t_{n-1})}$$

for all $t \in [t_{n-1}, t_n]$.

Applying Gronwall’s inequality we obtain

$$\mathcal{E}(t_n, x^{\varepsilon, \tau}_n) \leq \mathcal{E}(t_n, x^{\varepsilon, \tau}_{n-1}) + \int_{t_{n-1}}^{t_n} \lambda \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) e^{\lambda(t - t_{n-1})} dt + \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) = \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) + \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) e^{\lambda(t - t_{n-1})}.$$  

By induction,

$$\mathcal{E}(t_n, x^{\varepsilon, \tau}_n) \leq \mathcal{E}(t_{n-1}, x^{\varepsilon, \tau}_{n-1}) e^{\lambda(t_{n-1} - t_{n-1})} \leq \mathcal{E}(t_{n-2}, x^{\varepsilon, \tau}_{n-2}) e^{\lambda(t_{n-2} - t_{n-2})} e^{\lambda(t_{n-1} - t_{n-1})} \leq \cdots \leq \mathcal{E}(0, x_0) e^{\lambda(t_{n-1} - t_0)} e^{\lambda(t_{n-2} - t_1)} \cdots e^{\lambda(t_{n-1} - t_{n-1})}.$$  

Finally, by [7] again, we have

$$\mathcal{E}(0, x^{\varepsilon, \tau}_n) \leq \mathcal{E}(t_n, x^{\varepsilon, \tau}_n) e^{\lambda t_n} \leq \mathcal{E}(0, x_0) e^{2\lambda t_n}.$$  

#### Step 2.
Now we prove the integral bound. Assume that $t_{i-1} < s \leq t_i < t_{i+1} < \cdots < t_j \leq t < t_{j+1}$, where $\{t_n\}$ is the partition corresponding to $x^{\varepsilon, \tau}$. We start by writing

$$\mathcal{E}(t, x^{\varepsilon, \tau}(t)) = \mathcal{E}(s, x^{\varepsilon, \tau}(s)) = \mathcal{E}(t, x^{\varepsilon, \tau}(t)) - \mathcal{E}(t, x^{\varepsilon, \tau}(t_i)) + \cdots + \mathcal{E}(t, x^{\varepsilon, \tau}(t_j)) - \mathcal{E}(t, x^{\varepsilon, \tau}(t_{j-1})) + \mathcal{E}(t, x^{\varepsilon, \tau}(t_i)) - \mathcal{E}(s, x^{\varepsilon, \tau}(s)).$$
By the minimality of \( x_k := x^{ε,τ}(t_k) \) at time \( t_k \), we have
\[
\mathcal{E}(t_k, x_k) - \mathcal{E}(t_{k-1}, x_{k-1}) \leq \mathcal{E}(t_k, x_{k-1}) - \Psi(x_k - x_{k-1}) = \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(r, x_{k-1}) \, dr - \Psi(x_k - x_{k-1}).
\]

Taking the sum for all \( k \) from \( i + 1 \) to \( j \) and using \( x^{ε,τ}(r) = x_{k-1} \) for all \( r \in [t_{k-1}, t_k) \), we get
\[
\sum_{k=i+1}^{j} [\mathcal{E}(t_k, x_k) - \mathcal{E}(t_{k-1}, x_{k-1})] \leq \sum_{k=i+1}^{j} \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(r, x^{ε,τ}(r)) \, dr - \sum_{k=i+1}^{j} \Psi(x_k - x_{k-1}).
\]

Moreover, since \( t_{i-1} < s \leq t_i \) and \( t_j < t < t_{j+1} \), we can write
\[
\mathcal{E}(t, x^{ε,τ}(t)) - \mathcal{E}(s, x^{ε,τ}(s)) = \mathcal{E}(t, x_1) - \mathcal{E}(s, x_{i-1}) \leq \mathcal{E}(t, x_1) - \Psi(x_1 - x_{i-1}) - \mathcal{E}(s, x_{i-1})
\]
\[
= \int_s^t \partial_t \mathcal{E}(r, x^{ε,τ}(r)) \, dr - \Psi(x_1 - x^{ε,τ}(s)).
\]

Thus, it follows from (13), (14), (16) and (15) that
\[
\mathcal{E}(t, x^{ε,τ}(t)) - \mathcal{E}(s, x^{ε,τ}(s)) \leq \int_s^t \partial_t \mathcal{E}(r, x^{ε,τ}(r)) \, dr - \Psi(x_1 - x^{ε,τ}(s)) \leq \int_s^t \partial_t \mathcal{E}(r, x^{ε,τ}(r)) \, dr - \Psi(x_1 - x^{ε,τ}(s)).
\]

5.3. Proof of Lemma 5

Proof. Applying the chain rule formula for \( \mathcal{E} \in C^1 \) and \( u \in BV \) (see [2]), we get
\[
\mathcal{E}(t_1, u(t_1)) - \mathcal{E}(t_0, u(t_0)) = \int_{t_0}^{t_1} \partial_t \mathcal{E}(s, u(s)) \, ds + \int_{t_0}^{t_1} \langle \nabla_x \mathcal{E}(s, u(s)), u'_o(s) \rangle \, ds
\]
\[
+ \sum_{t \in J \cap (t_0, t_1)} [\mathcal{E}(t, u(t)) - \mathcal{E}(t, u(t^-))] + \sum_{t \in J \cap (t_0, t_1)} [\mathcal{E}(t, u(t^+)) - \mathcal{E}(t, u(t))]
\]
\[
+ \mathcal{E}(t_0, u(t_0)) - \mathcal{E}(t_0, u(t_0)) + \mathcal{E}(t_1, u(t_1)) - \mathcal{E}(t_1, u(t_1^-)).
\]

The fact that \( -\nabla_x \mathcal{E}(t, u(t)) \in \partial \Psi(0) \) whenever \( u(t) \) is continuous at \( t \) yields
\[
\int_{t_0}^{t_1} \langle \nabla_x \mathcal{E}(s, u(s)), u'_o(s) \rangle \, ds \geq -\int_{t_0}^{t_1} \Psi(u'_o(s)) \, ds.
\]

Notice that
\[
\int_{t_0}^{t_1} \Psi(u'_o(s)) \, ds = \Psiiss(u; [t_0, t_1]) - \sum_{t \in J \cap (t_0, t_1)} \Psi(u(t) - u(t^-)) - \sum_{t \in J \cap (t_0, t_1)} \Psi(u(t^+) - u(t))
\]
\[
- \Psi(u(t_0^+) - u(t_0)) - \Psi(u(t_1) - u(t_1^-)).
\]

Moreover, for every absolutely continuous curve \( v \in AC([0, 1]; \mathbb{R}^d) \) such that \( v(0) = u(t^-), v(1) = u(t) \) we have
\[
|\mathcal{E}(t, u(t)) - \mathcal{E}(t, u(t^-))| = \int_0^1 \langle \nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle \, ds.
\]

For any \( \eta \in \partial \Psi(0) \), it holds that \( \langle \eta, v \rangle \leq \Psi(v) \) for all \( v \in \mathbb{R}^d \). Thus, at every point \( s \in [0, 1] \) for which the derivative \( \dot{v}(s) \) exists, we can write
\[
-\Psi(\dot{v}(s)) \leq \langle -\eta, \dot{v}(s) \rangle.
\]

Hence,
\[
\langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle = \Psi(\dot{v}(s)) - \Psi(\dot{v}(s)) + \langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle
\]
\[
\leq \Psi(\dot{v}(s)) + \langle -\eta, \dot{v}(s) \rangle + \langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle
\]
\[
\leq \Psi(\dot{v}(s)) + \| -\eta - \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \|.
\]
The inequality above holds for every \( \eta \in \partial \Psi(0) \). Thus, we obtain
\[
\langle -\nabla_x \mathcal{E}(t, v(s)), \dot{v}(s) \rangle \leq \Psi(\dot{v}(s)) + \inf_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \|.
\]
Therefore, for any absolutely continuous curve \( v \) in \( AC([0,1]; \mathbb{R}^d) \) satisfying \( v(0) = u(t^\ell), v(1) = u(t) \), it holds that
\[
|\mathcal{E}(t, u(t)) - \mathcal{E}(t, u(t^-))| \leq \int_0^1 \Psi(\dot{v}(s)) + \inf_{\eta \in \partial \Psi(0)} \| \eta + \nabla_x \mathcal{E}(t, v(s)) \|_* \cdot \| \dot{v}(s) \|.
\]
By the definition of \( \Delta_{\text{new}}(t, u(t^-), u(t)) \), we can conclude that
\[
|\mathcal{E}(t, u(t)) - \mathcal{E}(t, u(t^-))| \leq \Delta_{\text{new}}(t, u(t^-), u(t)).
\]
Similarly, we also get
\[
|\mathcal{E}(t, u(t^\ell)) - \mathcal{E}(t, u(t))| \leq \Delta_{\text{new}}(t, u(t), u(t^\ell)).
\]
Thus, it follows from (17), (18), (19) and (20) that
\[
\mathcal{E}(t_1, u(t_1)) - \mathcal{E}(t_0, u(t_0)) \geq \int_{t_0}^{t_1} \partial_t \mathcal{E}(s, u(s)) \, ds - \mathcal{D} \text{iss}(u; [t_0, t_1])
\]
\[
+ \sum_{t \in J \cap (t_0, t_1)} \Psi(u(t^-) - u(t)) + \sum_{t \in J \cup (t_0, t_1)} \Psi(u(t) - u(t^\ell))
\]
\[
+ \Psi(u(t_0) - u(t_0^-)) + \Psi(u(t_1^\ell) - u(t_1))
\]
\[
- \sum_{t \in J \cap (t_0, t_1)} \Delta_{\text{new}}(t, u(t^-), u(t)) - \sum_{t \in J \cup (t_0, t_1)} \Delta_{\text{new}}(t, u(t), u(t^\ell))
\]
\[
- \Delta_{\text{new}}(t_0, u(t_0), u(t_0^\ell)) - \Delta_{\text{new}}(t_1, u(t_1^-), u(t_1))
\]
\[
= \int_{t_0}^{t_1} \partial_t \mathcal{E}(s, u(s)) \, ds - \mathcal{D} \text{iss}_{\text{new}}(u; [t_0, t_1]).
\]
This ends the proof of Lemma 5. \( \square \)

**References**

[1] G. Alberti and A. DeSimone, *Quasistatic evolution of sessile drops and contact angle hysteresis*, Arch. Rational Mech. Anal., 202 (2011), pp. 295–348.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon Press, 2000.

[3] G. Dal Maso, A. DeSimone, M. G. Mora, and M. Morini, *Globally stable quasistatic evolution in plasticity with softening*, Netw. Heterog. Media, 3 (2008), pp. 567–614.

[4] ———, *A vanishing viscosity approach to quasistatic evolution in plasticity with softening*, Arch. Ration. Mech. Anal., 189 (2008), pp. 469–544.

[5] G. Dal Maso, A. DeSimone, and F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling*, Cal. Var. and PDE., 40 (2008), pp. 125–181.

[6] G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257–290.

[7] M. Efendiev and A. Mielke, *On the rate-independent limit of systems with dry friction and small viscosity*, J. Convex Analysis, 13 (2006), pp. 151–167.

[8] G. Francfort and C. J. Larsen, *Existence and convergence for quasistatic evolution in brittle fracture*, Comm. Pure Appl. Math., 56 (2003), pp. 1465–1500.

[9] G. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.

[10] G. Francfort and A. Mielke, *Existence results for a class of rate-independent material models with nonconvex elastic energies*, J. Reine Angew. Math., 595 (2006), pp. 55–91.

[11] C. J. Larsen, *Epsilon-stable quasistatic brittle fracture evolution*, Comm. Pure Appl. Math., 63 (2010), pp. 630–654.

[12] A. Mainik and A. Mielke, *Existence results for energetic models for rate-independent systems*, Calc. Var. PDE., 22 (2005), pp. 73–99.

[13] A. Mielke, *Finite elastoplasticity, Lie groups and geodesics on \( SL(d) \)*, In P. Newton, A. Weinstein, and P. Holmes editors, Geometry, Dynamics, and Mechanics, Springer-Verlag, 2003, pp. 61–90.

[14] ———, *Energetic formulation of multiplicative elastoplasticity using dissipation distances*, Cont. Mech. Thermodynamics, 15 (2003), pp. 351–382.

[15] ———, *Evolution of rate-independent systems*. Handbook of Differential Equations, Evolutionary equations, Elsevier B. V., 2 (2005), pp. 461–559.
A mathematical framework for generalized standard materials in the rate-independent case, in Multifield problems in Fluid and Solid Mechanics, vol. Series Lecture Notes in Applied and Computational Mechanics, Springer, 2006.

Modeling and analysis of rate-independent processes, 2007. Lipschitz Lectures, University of Bonn.

Differential, energetic and metric formulations for rate-independent processes, 2008. Lecture Notes of C.I.M.E. Summer School on Nonlinear PDEs and Applications, Cetraro.

A. Mielke, R. Rossi, and G. Savaré, Modeling solutions with jumps for rate-independent systems on metric spaces, Discrete Contin. Dyn. Syst., 2 (2010), pp. 585–615.

BV solutions and viscosity approximations of rate-independent systems, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 36–80.

Balanced Viscosity (BV) solutions to infinite-dimensional rate-independent systems, Submitted Paper, 2013.

A. Mielke and F. Theil, A mathematical model for rate-independent phase transformations with hysteresis, vol. Models of Continuum Mechanics in Analysis and Engineering, Shaker Ver., Aachen, 1999.

On rate-independent hysteresis models, NoDEA Nonlinear Differential Equations Appl., 11 (2004), pp. 151–189.

A. Mielke, F. Theil, and V. Levitas, A variational formulation of rate-independent phase transformations using an extremum principle, Arch. Rational Mech. Anal., 162 (2002), pp. 137–177.

M. N. Minh, Weak solutions to rate-independent systems: Existence and Regularity, PhD Thesis, 2012.

S. Müller, Variational models for microstructure and phase transitions, In Calculus of Variations and Geometric Evolution Problems, Cetraro, 1999, pp. 85–210. Springer, Berlin, 1999.

I. P. Natanson, Theory of Functions of a Real Variable, Frederick Ungar, New York, 1965.

M. Negri, A comparative analysis on variational models for quasi-static brittle crack propagation, Adv. Calc. Var. 3 (2010), pp. 149–212.

F. Schmid and A. Mielke, Vortex pinning in superconductivity as a rate-independent process, Europ. J. Appl. Math., 2005.

U. Stefanelli, A variational characterization of rate-independent evolution, Math. Nach., 282 (2009), pp. 1492–1512.

R. Rossi and G. Savaré, A characterization of energetic and BV solutions to one-dimensional rate-independent systems, Discrete Contin. Dyn. Syst. Ser. S 6 (2013), pp. 167–191.

E-mail address: mach@mail.dm.unipi.it