On the estimation of parameter and stress-strength reliability for unit-Lindley distribution

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Abstract

The unit-Lindley distribution was recently introduced in the literature as a viable alternative to the Beta and the Kumaraswamy distributions with support in \((0, 1)\). This distribution enjoys many virtuous properties over the named distributions. In this article, we address the issue of parameter estimation from a Bayesian perspective and study relative performance of different estimators through extensive simulation studies. Significant emphasis is given to the estimation of stress-strength reliability employing classical as well as Bayesian approach.

keywords

unit-Lindley distribution, stress-strength reliability, conjugate prior, Metropolis-hastings algorithm, Weighted gamma distribution, Kummer’s U-function.

1 Introduction

The importance and relevance of probability models in the support \((0, 1)\) is well established in applied statistics to model certain characteristics such as, scores of some ability tests, different indices and rates, which lie in the interval \((0, 1)\). The Unit Lindley distribution was introduced by Mazucheli et al.\textsuperscript{(2018)} as a new distribution in the unit interval \((0, 1)\) with many interesting features. Its probability density function (p.d.f) and cumulative distribution function (c.d.f) is given respectively by:

\[
f(x \mid \theta) = \frac{\theta^2}{1 + \theta} (1 - x)^{-3} \exp\left(\frac{-\theta x}{1 - x}\right), \quad 0 < x < 1, \theta > 0. \quad (1.1)
\]

\[
F(x \mid \theta) = 1 - \left(1 - \frac{\theta x}{(1 + \theta)(x - 1)}\right) \exp\left(\frac{-\theta x}{1 - x}\right), \quad 0 < x < 1, \theta > 0. \quad (1.2)
\]

This is a one parameter unimodal distribution with many advantageous characteristics like closed form expressions for cumulative distribution function (c.d.f), quantile function and simple expression for lower order moments that the other distributions defined in the interval \((0, 1)\) do not posses. Mazucheli et al. \textsuperscript{(2018)} justified the utility of this distribution as a viable competitor of the existing distributions especially the beta and
Kumaraswamy distributions by providing theoretical results and practical application with illustrations.

Stress-strength reliability is an important probabilistic measure used regularly in the field of quality engineering, medical statistics, econometrics, and other branches of applied statistics. If $X$ stands for the strength of a system or component which is subjected to a stress $Y$, then the stress-strength reliability $R$ is defined as the probability that $Y$ is less than $X$, that is, stress is less than the strength. In our case, both $X$ and $Y$ are proper fractions, like per unit rate of any event of interest, proportion of some categories, relative grades of any inspection or test etc. In all such situations, it may be of interest to monitor the probability that one is smaller or larger than the other given by the following expression when $X$ and $Y$ are independent.

$$R = P(Y < X) = \int_0^1 f_X(x \mid \theta_1) F_Y(x \mid \theta_2) \, dx$$

Stress-strength reliability estimation has attracted many authors (For details see Krishna et al.(2017) and references therein). However, this problem for distributions with support $(0, 1)$ is rare in the literature. The only work we came across is for Kumaraswamy distribution, discussed in detail by Nadar et al.(2014). Throughout the rest of the text we write $L(\theta)$ and $UL(\theta)$ to denote Lindley and unit-Lindley distribution with the parameter $\theta$.

Rest of this article is organized as follows: In Section 2, we discuss the problem of estimating $\theta$. Exact expression for the Bayes estimator under both conjugate as well as flat prior is obtained in terms of Kummer’s confluent Hypergeometric function. In Section 3, main results for maximum likelihood estimation, uniformly minimum variance unbiased estimator (UMVUE) and Bayes estimation for stress-strength reliability $R = P(Y < X)$ under independent set-up are developed. Both these sections provide computational algorithms. The next section is devoted to simulation set up, results and interpretations to investigate the performance of the proposed estimators. We conclude with some discussion and directions for future extension in section 5. An appendix of relevant tables is provided at the end.

## 2 Estimation of $\theta$

In this section, we discuss maximum likelihood estimation of $\theta$ (Mazucheli et al., 2018) and introduce estimation of $\theta$ under Bayesian paradigm. We find the conjugate-prior family for $\theta$ and derive closed form expression of posterior as a weighted Gamma distribution. A computational algorithm for sampling from posterior distribution is outlined.

### 2.1 Classical approach

#### 2.1.1 MLE

Let $X = (X_1, X_2, \ldots, X_m)$ be a random sample of size $m$ from $UL(\theta)$ with p.d.f. (1.1). Then, for realized $X := \mathbf{x} = (x_1, \ldots, x_n)$, the log-likelihood function of $\theta$ can be written
as:
\[
\mathcal{L}(\theta \mid \mathbf{x}) = \prod_{i=1}^{m} (1 - x_i)^{-3} \left( \frac{\theta^2}{1 + \theta} \right)^m \exp(-\theta t(\mathbf{x})) \tag{2.3}
\]
where \( t(\mathbf{x}) = \frac{\sum_{i=1}^{m} x_i}{1 - x_i} \), which gives:
\[
\hat{\theta}_{ML} = \frac{1}{2t(\mathbf{x})} \left[ m - t(\mathbf{x}) + \sqrt{t(\mathbf{x})^2 + 6mt(\mathbf{x}) + m^2} \right]. \tag{2.4}
\]

Mazucheli et al. (2018) provides the following asymptotic distribution of \( \hat{\theta}_{ML} \):
\[
\sqrt{m} (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}(0, \sigma^2(\theta))
\]
where, \( \sigma^2(\theta) = \frac{\theta^2(1 + \theta)^2}{m(\theta^2 + 4\theta + 2)} \) and thus constructed asymptotic confidence interval for \( \theta \). Table 1 of the same paper reveals that, performance of maximum likelihood estimator of \( \theta \) is satisfactory and equivalent to method of moments estimator while for small and moderate sample sizes bias corrected estimator performs better. However, due to its desirable properties such as invariance, consistency, asymptotic normality etc., \( \hat{\theta}_{ML} \) should still be preferred for further inferential work on complex parametric functions.

2.1.2 UMVUE

For unit-Lindley distribution, \( t(\mathbf{x}) \) is a complete sufficient statistic and is distributed as \( W = \sum_{i=1}^{m} W_i \), where \( W_i \sim L(\theta) \) for \( i = 1, 2, \ldots, m \) and are independent. From Zakerzadeh and Dolati (2009), the p.d.f. of \( W \) is as follows:
\[
f_W(w) = \left( \frac{\theta^2}{1 + \theta} \right)^m \sum_{k=0}^{m} \binom{m}{k} w^{2m-k-1} \exp(-\theta w) \frac{1}{\Gamma(2m-k)} \text{ for } w > 0 \tag{2.5}
\]
Since unbiased estimator of \( \theta \) is not known, direct construction of UMVUE using Lehmann-Scheffe’s theorem is not available. Mazucheli et al. (2018) indicates construction of UMVUE by bias correction of the estimator in (2.4). Maiti and Mukherjee (2018) used empirical cdf to construct UMVUE of cdf and pdf of Lindley distribution. Similar routine can be followed for unit-Lindley distribution.

2.2 Bayesian approach

In Mazucheli et al. (2018), the authors mentioned about Bayesian estimation of the parameter \( \theta \) but to the best of our knowledge no such work has so far been reported. As mentioned above in this paper, we consider the problem of Bayesian estimation of \( \theta \) with improper and conjugate priors. Prior sensitivity is also assessed through extensive simulation studies for motivating practitioners to apply the proposed method safely.

Ignoring the data-dependent part in (2.3), we rewrite the likelihood function as follows:
\[
\mathcal{L}(\theta \mid \mathbf{x}) \propto \left( \frac{\theta^2}{1 + \theta} \right)^m \exp(-\theta t(\mathbf{x})) \tag{2.6}
\]
2.2.1 Conjugate Prior

In section 2.3 of Mazucheli et al. (2018) the concerned distribution is shown to be a member of one-parameter exponential family which facilitates one to construct family of conjugate priors (see Robert, 2007). Accordingly the constructed family of conjugate priors for the unit-Lindley distribution is the following:

\[ \pi(\theta) \propto \theta^{p-1} \exp\left(-\theta \alpha\right) \frac{\left(1+\theta\right)^{\beta}}{(1+\theta)^{m+\beta}} \text{ for } \theta > 0 \]  

(2. 7)

where, \( \alpha, p > 0 \) and \( \beta \geq 0 \).

The above density can easily be identified as a weighted gamma distribution for which the normalizing constant can be seen as the confluent hypergeometric function of the second kind (Abramowitz and Stegun, 1972) popularly known as Kummer’s function. It has a convenient Maclaurine’s series expansion. Henceforth density in (2.7) will be denoted by \( WG(\alpha, \beta, p) \). Taking product of the likelihood function in (2.6) with the prior in (2.7), as follows:

\[ \pi(\theta|x) \propto \theta^{2m+p-1} \exp\left[-\theta \{\alpha + t(x)\}\right] \frac{\left(1+\theta\right)^{m+\beta}}{(1+\theta)^{m+\beta}} \text{ for } \theta > 0 \]  

(2. 8)

As obvious the the density in (2.8) is also a member of weighted-gamma family: \( WG(\alpha + t(x), m+\beta, 2m+p) \). The marginal density of data vector \( X \) can be obtained by integrating out the posterior density w.r.t \( \theta \) which takes the following form:

\[ m(x) = \frac{\Gamma(2m+p) U(2m+p, m+p+1-\beta, \alpha+t(x))}{\Gamma p U(p, p+1-\beta, \alpha)} \]  

(2. 9)

where, \( \Gamma \) denotes the gamma function and \( U \) denotes Kummer’s U function (see, Abramowitz and Stegun, 1972). Considering squared error loss function makes the posterior mean to be the Bayes’ estimator which in this case turns out to be:

\[ E(\theta|x) = \frac{\Gamma(2m+p+1)}{\Gamma(2m+p)} \frac{U(2+p, p+2-\beta, \alpha+t(x))}{U(2m+p, m+p+1-\beta, \alpha+t(x))} \]  

(2. 10)

From the literature review it is apparent that, there is a natural tendency to use gamma prior when parameter of interest has positive support (see Krishna et al., 2017 and Nadar et al., 2014 among others). In fact, gamma prior is a particular case of the proposed frame-work when \( \beta = 0 \) and we denote this by \( G(\alpha, p) \). Even in this situation, posterior of \( \theta \) is still in weighted gamma family implying that, gamma distribution plays the role of semi-conjugate prior: \( WG(\alpha + t(x), m, 2m+p) \).

As the prior is a modification of gamma distribution with simple weights, subjective belief regarding the parameter of interest can be meaningfully captured. Use of conjugate prior, provide computational tractability and enhance transparency in updation of prior through likelihood.

In full Bayesian approach, the hyper-parameters are purely in the hands of the practitioner. As a trade-off between classical and Bayesian paradigm, in empirical Bayes’ approach, the choice of hyper-parameters are data-driven. In this approach, the maximum likelihood estimates of the hyper-parameters from the marginal distribution of the
data are plugged into the expression of posterior mean to obtain Bayes’ estimator. For more details, see Efron(2012). In our case, the marginal distribution given in (2.9) is quite complex and direct computation of maximum likelihood estimates is ruled out while numerical optimization of $m(x)$ w.r.t $\alpha, \beta, p$ provides no stable solution. Even our attempt to derive the estimates of hyper-parameters by other methods proved futile. Due to the above issues, we do not pursue this method further in this work.

2.2.2 Flat Prior

In situations where no information regarding the parameter of interest is available, one may use flat priors to express indifference. Obviously, a number of conventional choices of flat priors are in use of which the present work uses the basic one:

$$\pi(\theta) \propto 1 \quad \text{for} \quad \theta > 0 \tag{2.11}$$

The corresponding posterior is $WG(t(x), m, 2m + 1)$ which indicates that the assumed improper prior is also a semi-conjugate one. It should be noted that, even with improper prior, the corresponding posterior is proper and hence under squared error loss function, the Bayes’ estimator is given by:

$$E(\theta|x) = \frac{(2m + 1)U(2m + 2, m + 3, t(x))}{U(2m + 1, m + 2, t(x))} \tag{2.12}$$

2.2.3 Computational Methods

Posterior mean of parameter is the optimum estimator under squared error loss function. For conjugate and improper cases, mean of posterior distribution is given in (2.10) and (2.12), respectively. We denote Bayes’ estimator of $\theta$ for improper prior by $\hat{\theta}_I$ whereas $\hat{\theta}_A$ denotes the same for conjugate prior. Prior-sensitivity is investigated through simulation studies by miss specifying the hyper-parameters $\alpha, \beta, p$ as $\alpha', \beta', p'$. The corresponding estimator is denoted by $\hat{\theta}_M$.

To assess the posterior density of $\theta$, one may need to draw sample. For the priors considered in this work posterior family remains the same, weighted gamma distribution. Metropolis-Hastings algorithm is widely accepted and used for drawing sample from un-normalized posterior using MCMC. We briefly state the algorithm below for single parameter:

- **Step 1**: Initialize $\theta = \theta^{(0)}$.
- **Step 2**: Draw a candidate for $\theta^* \sim h(\theta|\theta^{(0)}, x)$, $h$ being the proposal density.
- **Step 3**: Compute the acceptance ratio using target density $\pi$ and proposal $h$ as:

$$r = \frac{\pi(\theta^*|x)}{\pi(\theta^{(0)}|x)} \frac{h(\theta^{(0)}|\theta^*)}{h(\theta^*|\theta^{(0)})}$$

- **Step 4**: Draw $u \sim \text{Uniform}(0,1)$.
- **Step 5**: Set,

$$\theta^{(1)} = \begin{cases} 
\theta^* & \text{if} \quad u < r \\
\theta^{(0)} & \text{if} \quad u > r
\end{cases}$$
**Step 6:** \( \theta^{(0)} \leftarrow \theta^{(1)} \).

**Step 7:** Repeat Step 2 to Step 6 \( c \) times with burn in \( b \) for \( b \leq c \) to obtain \( c - b + 1 \) sample observations.

Several improvements of the basic MH-algorithm have been proposed and implemented in R-software. The present work uses MHadaptive package from CRAN repository (see Chivers, 2015) with non-default burn in length \( b = 5000 \) and chain-length \( c = 14999 \).

### 3 Estimation of \( \mathcal{R} \)

Let us consider independent random samples, \( X = (X_1, X_2, \ldots, X_m) \) of size \( m \) from \( \text{UL}(\theta_1) \) and \( Y = (Y_1, Y_2, \ldots, Y_n) \) of size \( n \) from \( \text{UL}(\theta_2) \) with \( m > n \) such that,

\[
\frac{n}{m} = q \in (0, 1) \quad \text{for} \quad m, n \to \infty \tag{3.13}
\]

Under this set-up, the stress strength reliability parameter can be expressed as,

\[
\mathcal{R} = \frac{\theta_2^2 (\theta_1 \theta_2^2 + 2 \theta_1^2 \theta_2 + \theta_1^3 + \theta_2^2 + 4 \theta_1 \theta_2 + 3 \theta_1^2 + \theta_2 + 3 \theta_1)}{(\theta_1 + \theta_2)^3 (1 + \theta_2) (1 + \theta_1)}
\]

\[
= g(\theta_1, \theta_2), \quad \text{say} \tag{3.14}
\]

The polynomial in the denominator of (3.14) is non-zero and \( g(\theta_1, \theta_2) \) being the ratio of two polynomials, is a continuous function over \((0, 1)\). In what follows, we will consider some classical and Bayesian methods for estimating \( \mathcal{R} \).

#### 3.1 Classical approach

**3.1.1 MLE**

In view of the functional invariance property, the MLE of \( \mathcal{R} \) is given by,

\[
\hat{\mathcal{R}}_{ML} = \frac{\hat{\theta}_2^2 (\hat{\theta}_1 \hat{\theta}_2^2 + 2 \hat{\theta}_1^2 \hat{\theta}_2 + \hat{\theta}_1^3 + \hat{\theta}_2^2 + 4 \hat{\theta}_1 \hat{\theta}_2 + 3 \hat{\theta}_1^2 + \hat{\theta}_2 + 3 \hat{\theta}_1)}{(\hat{\theta}_1 + \hat{\theta}_2)^3 (1 + \theta_2) (1 + \theta_1)}
\]

\[
= g(\hat{\theta}_1, \hat{\theta}_2) \tag{3.15}
\]

where, \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are respective MLE’s of \( \theta_1 \) and \( \theta_2 \). From section (2.1.1) we have,

\[
\sqrt{m}(\hat{\theta}_1 - \theta_1) \sim \text{AN}(0, \sigma^2(\theta_1))
\]

\[
\sqrt{n}(\hat{\theta}_2 - \theta_2) \sim \text{AN}(0, \sigma^2(\theta_2))
\]

Let us denote,

\[
\theta = (\theta_1, \theta_2) \quad \text{and} \quad \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)
\]

Clearly,

\[
\hat{\theta} \sim \text{AN}_2(\theta, \Sigma(\theta))
\]

where,

\[
\Sigma(\theta) = \begin{bmatrix}
\sigma^2(\theta_1) & 0 \\
0 & \sigma^2(\theta_2)
\end{bmatrix}
\]
As mentioned earlier, $g$ is continuous and applying $\delta$-method we get,
\[
\sqrt{m} \left( g\left( \hat{\theta}_1, \hat{\theta}_2 \right) - g(\theta_1, \theta_2) \right) \sim AN \left( 0, \sigma^2_*(\theta_1, \theta_2) \right) \tag{3.16}
\]

Here, $\sigma^2_*(\theta_1, \theta_2)$ can be obtained using the following:
\[
\sigma^2_*(\theta_1, \theta_2) = \left( \frac{\delta g(\theta_1, \theta_2)}{\delta \theta_1} \right) \left( \frac{\delta g(\theta_1, \theta_2)}{\delta \theta_2} \right) \Sigma(\theta) \left( \frac{\delta g(\theta_1, \theta_2)}{\delta \theta_1} \right) \left( \frac{\delta g(\theta_1, \theta_2)}{\delta \theta_2} \right)' \tag{3.17}
\]

We introduce the following notations to state the final expression which is quite messy in our case.

\[
a_1 = \frac{\theta^2_2 (3 + 6\theta_1 + 3\theta^2_1 + 4\theta_2 + 4\theta_1 \theta_2 + \theta^2_2)}{(1 + \theta_1) (1 + \theta_2) (\theta_1 + \theta_2)^3}
\]

\[
a_2 = \frac{3\theta^2_2 (3\theta_1 + 3\theta^2_1 + \theta^3_1 + \theta_2 + 4\theta_1 \theta_2 + 2\theta^2_1 \theta_2 + \theta^2_1 + \theta_1 \theta^2_2)}{(1 + \theta_1) (1 + \theta_2) (\theta_1 + \theta_2)^4}
\]

\[
a_3 = \frac{\theta^2_2 (3\theta_1 + 3\theta^2_1 + \theta^3_1 + \theta_2 + 4\theta_1 \theta_2 + 2\theta^2_1 \theta_2 + \theta^2_1 + \theta_1 \theta^2_2)}{(1 + \theta_1) (1 + \theta_2) (\theta_1 + \theta_2)^3}
\]

\[
e = (a_1 - a_2 - a_3)^2
\]

\[
b_1 = \theta^2_1 (1 + \theta_1)^2
\]

\[
b_2 = \theta^2_2 (1 + \theta_2)^2
\]

\[
c_1 = 2 + 4\theta_1 + \theta^2_1
\]

\[
c_2 = q (2 + 4\theta_2 + \theta^2_2)
\]

\[
d_1 = \frac{b_1 e}{c_1}
\]

\[
d_2 = \frac{b_2 e}{c_2}
\]

With these notations, we get from (3.17)
\[
\sigma^2_*(\theta_1, \theta_2) = m (d_1 + d_2) \tag{3.18}
\]

For all practical purposes, one can substitute the corresponding MLE’s in place of $\theta_1$ and $\theta_2$ to obtain
\[
\hat{\sigma}^2_*(\theta_1, \theta_2) = \sigma^2_*(\hat{\theta}_1, \hat{\theta}_2)
\]

Thus, using asymptotic normality of $\hat{R}_{ML}$ given in (3.16) $100 (1-\alpha)\%$ confidence interval can be constructed as follows:
\[
CI(R) = \left( \hat{R}_{ML} - \tau_{\alpha/2} \frac{\hat{\sigma}^2_*(\theta_1, \theta_2)}{\sqrt{m}}, \hat{R}_{ML} + \tau_{\alpha/2} \frac{\hat{\sigma}^2_*(\theta_1, \theta_2)}{\sqrt{m}} \right)
\]

where, $\tau_{\alpha}$ denotes the upper-$\alpha$ point of standard normal distribution.

### 3.1.2 UMVUE

As in subsection 2.1.2,

\[
t(x) = W = \sum_{i=1}^{m} W_i \quad \text{where} \quad W_i \sim L(\theta_1) \quad \text{for} \quad i = 1, 2, \ldots, m.
\]

\[
t(y) = V = \sum_{i=1}^{n} V_i \quad \text{where} \quad V_i \sim L(\theta_2) \quad \text{for} \quad i = 1, 2, \ldots, n.
\]
(W, V) is jointly complete sufficient for (θ₁, θ₂) and unlike for θ in case of UL(θ) here, an unbiased estimator for R can easily be constructed as

\[ \psi(X_1, Y_1) = \begin{cases} 1 & \text{if } Y_1 < X_1 \\ 0 & \text{otherwise} \end{cases} \]

It is important to note that, \( f(x) = x/(1 - x) \) is an increasing function in \( x \in (0, 1) \). Thus, the above indicator can be restated as

\[ \phi(W_1, V_1) = \begin{cases} 1 & \text{if } V_1 < W_1 \\ 0 & \text{otherwise} \end{cases} \]

One can imitate the steps in Al-Mutairi et al.(2013) to easily find the expression for UMVUE of R. Table 1 of Al-Mutairi et al.(2013) clearly indicates that, MLE beats UMVUE in terms of MSE in case of Lindley. Moreover, the complex nature of UMVUE makes computation expensive. It is reasonable to expect a similar situation in case of unit-Lindley, we refrain from computational aspect of UMVUE.

### 3.2 Bayesian approach

Prior belief on θ₁ and θ₂ suffices need for prior on R. Thus, priors mentioned through subsection 2.2 are kept intact for both θ₁ and θ₂ independently, making way for Bayes’ estimation of R. We extend the conjugate set-up for single parameter to the case of two parameters as follows:

\[ \theta_1 \sim WG(\alpha_1, \beta_1, p_1) \]
\[ \theta_2 \sim WG(\alpha_2, \beta_2, p_2) \]

Similarly, the improper set-up can be extended for two-sample situation

\[ \pi(\theta_1) \propto 1 \quad \text{and} \quad \pi(\theta_2) \propto 1 \]

Whatever be the set-up, as previously mentioned, posteriors of both the parameters are in weighted gamma family. Posterior of the parameters under conjugate set-up:

\[ \theta_1 | x \sim WG(\alpha_1 + t(x), m + \beta_1, 2m + p_1) \]
\[ \theta_2 | y \sim WG(\alpha_2 + t(y), n + \beta_2, 2n + p_2) \]

and for improper prior,

\[ \theta_1 | x \sim WG(t(x), m, 2m + 1) \]
\[ \theta_2 | y \sim WG(t(y), n, 2n + 1) \]

As we consider both the samples and priors on the parameters to be independent, posteriors of θ₁ and θ₂ are obviously independent of each other. Given the expression in \( [3, 14] \), it is near impossible to deduct the posterior distribution of R and thus the posterior mean. So a good strategy would be to draw observations from the posterior distribution of R and computing the posterior mean based on a large sample. Use of efficient algorithm warrants closeness of the true mean with the simulation based mean, which is quite common in Bayesian approach. We present algorithm for the same below:
• **Step 1:** Make an array of size \( k \) for posterior sample of \( \theta_1 \) and \( \theta_2 \) using algorithm given in subsection 2.2.3.

• **Step 2:** For all \( i = 1, 2, ..., k \) compute \( R^{(i)} = g(\theta_1^{(i)}, \theta_2^{(i)}) \) and store in array \( R \).

• **Step 3:** Calculate mean of \( R \).

Here, \( k = 10000 \) which follows from the choice of \( b \) and \( c \) given in subsection 2.2.3 which performs satisfactory in simulation studies. As mentioned in subsection 2.2.3 for \( \theta \), We denote Bayes’ estimator of \( R \) for improper prior by \( \hat{R}_I \) whereas \( \hat{R}_A \) denotes the same for conjugate prior. Prior-sensitivity is investigated through simulation studies by miss specifying the hyper-parameters \( \alpha_1, \beta_1, p_1 \) as \( \alpha_1', \beta_1', p_1' \) and \( \alpha_2, \beta_2, p_2 \) as \( \alpha_2', \beta_2', p_2' \). The corresponding estimator is denoted by \( \hat{R}_M \).

4 Simulation study

Despite the discussion regarding performance of MLE for \( \theta \) in Mazucheli et al.(2018), we extend the experiment for comparison of the same with Bayes’ estimator under improper prior. The experiment is carried out through the following steps:

• **Step 1:** For fixed value of \( \theta \) generate a sample of size \( m \) from UL(\( \theta \)) and store in \( d \).

• **Step 2:** For \( d \), calculate \( \hat{\theta}_{ML} \) and thus \( (\hat{\theta}_{ML} - \theta), (\hat{\theta}_{ML} - \theta)^2 \) and stack them into the arrays \( Bias_{ML}, MSE_{ML} \), respectively.

• **Step 3:** For \( d \), calculate \( \hat{\theta}_I \) and thus \( (\hat{\theta}_I - \theta), (\hat{\theta}_I - \theta)^2 \) and stack them into the arrays \( Bias_I, MSE_I \), respectively.

• **Step 4:** Repeat Step 1 to Step 4 for \( N = 1000 \) times.

• **Step 5:** Take average of the elements for each array: \( Bias_{ML}, MSE_{ML}, Bias_I, MSE_I \) to get bias and MSE of \( \hat{\theta}_{ML} \), bias and MSE of \( \hat{\theta}_I \).

The results for different combinations of \( \theta \) and \( m \), are presented in Table 1. From the findings of Table 1, it is quite evident that, \( \hat{\theta}_{ML} \) performs better than \( \hat{\theta}_I \).

Simulation results reported in Table 3 regarding estimation of \( R \) are also obtained using the above steps incorporating obvious extensions. It is worth observing that unlike \( \hat{\theta}_I, \hat{R}_I \) beats \( \hat{R}_{ML} \). This may be attributed to the complex form of \( \hat{R}_{ML} \). In contrary, \( \hat{R}_I \) is less sensitive to the form of \( R \) and hence, should be preferred over MLE in the absence explicit prior knowledge.

To assess the performance of Bayes’ estimators for \( \theta \) under different different we proceed through the following steps:

• **Step 1:** Fix hyper-parameters \( \alpha, \beta, p \).

• **Step 2:** Draw \( \theta \sim WG(\alpha, \beta, p) \) using the algorithm in section 2.2.3.

• **Step 3:** Generate sample of size \( m \) from UL(\( \theta \)) and store in \( d \).
Step 4: For d, compute $\hat{\theta}_A$ with accurate prior and hence stack $(\hat{\theta}_A - \theta)$, $(\hat{\theta}_A - \theta)^2$ into $Bias_A$ and $MSE_A$, respectively.

Step 5: For d, compute $\hat{\theta}_M$ with miss specified prior and hence stack $(\hat{\theta}_M - \theta)$, $(\hat{\theta}_M - \theta)^2$ into $Bias_M$ and $MSE_M$, respectively.

Step 6: For d, compute $\hat{\theta}_I$ with flat prior and hence stack $(\hat{\theta}_I - \theta)$, $(\hat{\theta}_I - \theta)^2$ into $Bias_I$ and $MSE_I$, respectively.

Step 7: Repeat Step 2 to Step 6 $N = 1000$ times.

Step 8: Take average of the elements for each array: $Bias_A$, $MSE_A$, $Bias_M$, $MSE_M$, $Bias_I$, $MSE_I$ to get bias and MSE of $\hat{\theta}_A$, bias and MSE of $\hat{\theta}_M$ and bias and MSE of $\hat{\theta}_I$.

As expected $\hat{\theta}_A$ performs the best whereas $\hat{\theta}_M$ beats $\hat{\theta}_I$ in most cases.

Simulation results reported in Table 4 regarding estimation of $\mathcal{R}$ are also obtained using the above steps incorporating obvious extensions. It is worth observing that unlike $\hat{\theta}_I$, $\hat{\mathcal{R}}_I$ beats $\hat{\mathcal{R}}_M$. This is due to the joint effect of miss specification for both the parameters. Needless to mention, $\hat{\mathcal{R}}_A$ performs best.

5 Concluding Remarks

Bayes estimation of the unit-Lindley distribution is thoroughly investigated. Satisfactory results are obtained. The direction of future study may include development of bivariate unit-Lindley distribution to account for dependence which is quite common in many practical applications and related inferential issues should follow. Incorporation of dependence will necessitate a fresh look into the structure of $\mathcal{R}$ and its estimation.

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**Appendix: Tables**

| $m$ | $\theta$ | $\hat{\theta}_{ML}$ | $\hat{\theta}_I$ |
|-----|----------|----------------------|-------------------|
| 20  | 9.00     | 0.445(4.377)         | 0.872(5.374)      |
|     | 2.33     | 0.097(0.225)         | 0.186(0.271)      |
|     | 1.50     | 0.049(0.083)         | 0.101(0.097)      |
|     | 4.00     | 0.166(0.743)         | 0.336(0.901)      |
|     | 0.43     | 0.011(0.006)         | 0.023(0.006)      |
| 40  | 9.00     | 0.204(1.915)         | 0.411(2.136)      |
|     | 2.33     | 0.041(0.098)         | 0.085(0.108)      |
|     | 1.50     | 0.023(0.038)         | 0.052(0.041)      |
|     | 4.00     | 0.080(0.335)         | 0.163(0.372)      |
|     | 0.43     | 0.006(0.003)         | 0.012(0.003)      |
| 60  | 9.00     | 0.153(1.230)         | 0.290(1.330)      |
|     | 2.33     | 0.033(0.067)         | 0.062(0.071)      |
|     | 1.50     | 0.014(0.023)         | 0.031(0.025)      |
|     | 4.00     | 0.062(0.209)         | 0.116(0.226)      |
|     | 0.43     | 0.005(0.002)         | 0.008(0.002)      |
| $m$ | $(\alpha, \beta, p)$ | $(\alpha', \beta', p')$ | $\theta_A$ | $\theta_M$ | $\theta_I$ |
|-----|---------------------|---------------------|----------|----------|----------|
| 20  | (1,1,1)             | (3.1,1.9,2)         | 331.324(345.607) | -154.985(367.185) | 795.456(718.155) |
|     | (2.3,4,7,3)        | (1.2,3.6,4)         | 251.977(137.109)  | 684.887(228.899)   | 553.082(227.156) |
|     | (6.5,2,4)           | (6,4,5)             | 116.380(58.499)   | 69.801(58.227)     | 456.371(103.095) |
| 40  | (1,1,1)             | (3.1,1.9,2)         | 185.712(135.623)  | -87.022(187.999)   | 384.747(276.104) |
|     | (2.3,4,7,3)        | (1.2,3.6,4)         | 119.179(70.325)   | 337.114(84.453)    | 375.501(126.903) |
|     | (6.5,2,4)           | (6,4,5)             | 55.452(30.878)    | 24.091(31.667)     | 214.942(41.407)  |
| 60  | (1,1,1)             | (3.1,1.9,2)         | 89.539(98.716)    | -121.197(116.965)  | 193.227(118.203) |
|     | (2.3,4,7,3)        | (1.2,3.6,4)         | 65.139(52.178)    | 181.075(56.888)    | 211.592(61.833)  |
|     | (6.5,2,4)           | (6,4,5)             | 82.809(19.451)    | 13.223(19.755)     | 165.456(28.34)   |

Table 3: Bias (mean-squared error) (in order of $10^{-4}$) of different estimators for $R$.

| $(m, n)$ | $(\theta_1, \theta_2)$ | $R$ | $R_{ML}$ | $R_I$ | CI        |
|--------|------------------------|-----|----------|------|-----------|
| (9,0.04,4.00) | 0.285             | 55.731(45.359) | 92.353(44.959) | (0.192,0.389) |
| (2.3,3.4,0.00) | 0.659             | -18.103(47.751) | -47.524(46.538) | (0.448,0.867) |
| (1.50,1.00,0.00) | 0.366             | -21.088(56.039) | 4.405(54.099) | (0.240,0.489) |
| (4.00,0.67,0.00) | 0.084             | 15.438(7.490) | 48.785(8.000) | (0.049,0.122) |
| (0.43,1.00,0.00) | 0.771             | -66.636(36.076) | -110.326(36.224) | (0.545,0.983) |
| (9,0.04,4.00,0.00) | 0.285             | 51.471(33.882) | 102.787(34.420) | (0.215,0.365) |
| (2.3,3.4,0.00,0.00) | 0.659             | -18.103(47.751) | -47.524(46.538) | (0.448,0.867) |
| (1.50,1.00,0.00,0.00) | 0.366             | -21.088(56.039) | 4.405(54.099) | (0.240,0.489) |
| (4.00,0.67,0.00,0.00) | 0.084             | 15.438(7.490) | 48.785(8.000) | (0.049,0.122) |
| (0.43,1.00,0.00,0.00) | 0.771             | -66.636(36.076) | -110.326(36.224) | (0.545,0.983) |
| (9,0.04,4.00,0.00,0.00) | 0.285             | 3.809(21.539) | 23.897(21.362) | (0.216,0.354) |
| (2.3,3.4,0.00,0.00,0.00) | 0.659             | 11.378(26.327) | -4.726(25.879) | (0.512,0.809) |
| (1.50,1.00,0.00,0.00,0.00) | 0.366             | 50.071(31.136) | 63.263(30.793) | (0.282,0.461) |
| (4.00,0.67,0.00,0.00,0.00) | 0.084             | 20.052(3.769) | 37.273(3.963) | (0.060,0.112) |
| (0.43,1.00,0.00,0.00,0.00) | 0.771             | -24.496(18.560) | -47.450(18.608) | (0.612,0.924) |
Table 4: Bias (mean-squared error) (in order of 10^{-5}) of Bayes’ estimators for \( R \) with different priors.

| \( (m,n) \) | (\( \alpha_1, \beta_1, p_1 \)) | (\( \alpha_2, \beta_2, p_2 \)) | (\( \alpha'_1, \beta'_1, p'_1 \)) | (\( \alpha'_2, \beta'_2, p'_2 \)) | \( \hat{R}_A \) | \( \hat{R}_M \) | \( \hat{R}_I \) |
|-------------|----------------------------|----------------------------|-------------------|-------------------|----------------|----------------|----------------|
| (1,1,1)     | (1,1,1)                   | (3.1,1.9,2)               | (2.2,3.2,2)       | 95.127(268.202)  | 62.538(230.162)| 33.884(269.368)|
| (20,20)     | (2.3,4.7,3)               | (5.2,3.6,4)               | (7.3,5.0,3)       | -371.875(426.978)| -3086.691(519.702)| 92.753(417.404)|
|             | (6.5,2.4)                 | (6,4.5)                  | (7.1,5.3,1)       | -257.403(432.304)| -2303.736(467.789)| -119.287(446.023)|
| (40,20)     | (2.3,4.7,3)               | (5.2,3.6,4)               | (7.3,5.0,3)       | -33.006(211.139) | -1765.020(250.514)| -44.443(217.053)|
|             | (6.5,2.4)                 | (6,4.5)                  | (7.1,5.3,1)       | 104.657(221.095) | -1479.458(240.938)| -144.490(236.988)|
| (60,60)     | (1,1,1)                   | (1,1,1)                  | (3.1,1.9,2)       | 31.100(96.994)   | -15.983(92.742) | -117.199(98.549)|
|             | (2.3,4.7,3)               | (5.2,3.6,4)               | (7.3,5.0,3)       | 13.014(154.847)  | -1246.592(183.749)| 69.951(156.650)|
|             | (6.5,2.4)                 | (6,4.5)                  | (7.1,5.3,1)       | -258.978(138.277) | -836.536(159.317)| -185.032(146.759)|