STUDIES ON THE UV TO IR EVOLUTION OF GAUGE THEORIES AND QUASICONFORMAL BEHAVIOR

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We describe recent results from our studies of the UV to IR evolution of asymptotically free vectorial gauge theories and quasiconformal behavior. These include higher-loop calculations of the IR zero of the beta function and of the anomalous dimension of the fermion bilinear. Effects of scheme-dependence of higher-loop results are assessed in detail. Applications to models with dynamical electroweak symmetry breaking are discussed.

Keywords: UV to IR evolution, IR fixed point, quasiconformal behavior

1. Introduction

The evolution of an asymptotically free gauge theory from the ultraviolet (UV) to the infrared (IR) is of fundamental field-theoretic interest. In this SCGT12 talk we report on our calculations in\textsuperscript{[13]} with T. A. Ryttov of higher-loop corrections to this UV to IR evolution and on some new results (submitted shortly after the SCGT12 workshop in\textsuperscript{[13]}). The dependence of the gauge coupling $g(\mu)$ on the Euclidean momentum scale, $\mu$, is determined by the $\beta$ function $\beta \equiv \beta_\alpha \equiv d\alpha/dt$, where $t = \ln \mu$ and $\alpha(\mu) = g(\mu)^2/(4\pi)$ (and $\mu$ will often be suppressed in the notation). This function has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell,$$

where $a = \alpha/(4\pi)$, $\ell$ denotes the number of loops involved in the calculation of the coefficient $b_\ell$, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. The first two coefficients in $\beta$, $b_1$ and $b_2$, are scheme-independent and were calculated in\textsuperscript{[13]}\textsuperscript{[19]} The $b_\ell$ have been calculated up to 4-loop order in the $\overline{\text{MS}}$ scheme in\textsuperscript{[13]} The $n$-loop $n\ell$ beta function, denoted $\beta_{n\ell}$, is given by the RHS of Eq. (1) with the upper bound on the sum set equal to $n$ instead of $\infty$.

We consider the UV to IR evolution of an asymptotically free vectorial gauge theory with gauge group $G$ and $N_f$ massless fermions transforming according to a representation $R$ of $G$. There are two possibilities for this evolution: (i) there may
not be any IR zero in $\beta$, so that as $\mu$ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region (which is the case for QCD); (ii) $\beta$ may have an IR zero at a certain value denoted $\alpha_{IR}$, so that as $\mu$ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region (which is the case for QCD); (ii) $\beta$ may have an IR zero at a certain value denoted $\alpha_{IR}$, so that as $\mu$ decreases, $\alpha(\mu)$ increases from 0 toward $\alpha_{IR}$. In this class of theories, there are two further generic possibilities, namely $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$, where $\alpha_{cr}$ is the critical minimal value of $\alpha$ (depending on $R$) for spontaneous chiral symmetry breaking (S$\chi$SB). If $\alpha_{IR} < \alpha_{cr}$, then the zero of $\beta$ at $\alpha_{IR}$ is an exact IR fixed point (IRFP) of the renormalization group; as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions.\footnote{7,10} If $\beta$ has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as $\mu$ decreases through a scale denoted $\Lambda$, $\alpha(\mu)$ exceeds $\alpha_{cr}$ and spontaneous chiral symmetry breaking occurs, so that the fermions gain dynamical masses $\sim \Lambda$. In this case, in the low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and the $\beta$ function becomes that of a pure gauge theory, which has no (perturbative) IR zero. Hence, if $\beta$ has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approximate IRFP. If $\alpha_{IR} > \alpha_{cr}$, the effect of the approximate IRFP at $\alpha_{IR}$ depends on how close it is to $\alpha_{cr}$. The desire to understand better both quantum chromodynamics (QCD) and the properties of this IR zero have motivated calculations of higher-loop terms in $\beta$\footnote{8,9} and higher-loop corrections to the 2-loop result for the IR zero\footnote{11,12}. We denote the IR zero of the $n$-loop beta function, $\beta_{n\ell}$, as $\alpha_{IR,n\ell}$. The need to go beyond the 2-loop result for $\alpha_{IR}$ in studies of quasiconformal theories is evident from the fact that $\alpha_{cr} \sim O(1)$ and one is interested in values of $\alpha_{IR}$ near to $\alpha_{cr}$. Although the coefficients in $\beta$ at $\ell \geq 3$ loop order are scheme-dependent, the results give a measure of accuracy of the 2-loop calculation of the IR zero, and similarly with the value of the anomalous dimension of the fermion bilinear (discussed below).

If $\alpha_{IR}$ is only slightly greater than $\alpha_{cr}$, then, as $\alpha(\mu)$ approaches $\alpha_{IR}$, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale $\mu$, i.e., there is approximately scale-invariant (equivalently, dilatation-invariant, slow-running, or “walking”) behavior.\footnote{13} For these theories, this is equivalent to quasiconformal behavior.\footnote{14} The spontaneous chiral symmetry breaking and attendant fermion mass generation at $\Lambda$ spontaneously break the approximate dilatation symmetry, plausibly leading to a resultant light Nambu-Goldstone boson, the dilaton.\footnote{13} The dilaton is not massless, because $\beta$ is not exactly zero for $\alpha(\mu) \neq \alpha_{IR}$. Studies of gauge-singlet bound states in confining quasiconformal gauge theories, using Schwinger-Dyson and Bethe-Salpeter equations, find a significant reduction in the scalar mass divided by the vector meson mass.\footnote{15} Eventually, lattice gauge measurements may be able to determine the mass of a dilaton in a theory of this type. The quasiconformal behavior associated with an approximate IRFP has been an ingredient of DEWSB models since the 1980s, as a means of enhancing Standard-Model (SM) fermion masses while keeping neutral flavor-changing current (FCNC) processes sufficiently suppressed.\footnote{13}
2. Basic Properties of $\beta$

Since $b_1 = (1/3)(11C_A - 4N_fT_f)\beta_{2\ell}$ the asymptotic freedom of the theory requires $N_f < N_{f,b_{1z}}$, where $N_{f,b_{1z}} = 11C_A/(4T_f)$. (The subscript $b_{1z}$ stands for “$b_1$ zero”.) Since $\beta_{2\ell} = -[\alpha^2/(2\pi)](b_1 + b_2a)$, this function has an IR zero at

$$\alpha_{I R, 2\ell} = \frac{-4\pi b_1}{b_2},$$

which is physical for $b_2 < 0$. Now $b_2$ decreases linearly as a function of $N_f$; for small $N_f$, $b_2 > 0$, but as $N_f$ increases through the value

$$N_{f,b_{2z}} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)},$$

$b_2$ reverses its sign and becomes negative. Since $N_{f,b_{2z}} < N_{f,b_{1z}}$, there is always an interval of $N_f < N_{f,b_{1z}}$ for which $\beta$ has an IR zero, namely the interval

$$I : N_{f,b_{2z}} < N_f < N_{f,b_{1z}}.$$  \hspace{1cm} (4)

If $R$ is the fundamental (fund.) representation of SU($N$), then

$$I : \frac{34N^3}{13N^2 - 3} < N_f < \frac{11N}{2}, \ R = \text{fund.} $$ \hspace{1cm} (5)

For example, for $N = 2$ and $N = 3$, these intervals are $5.55 < N_f < 11$ and $8.05 < N_f < 16.5$, respectively. As $N \to \infty$, the interval $I$ is $2.62N < N_f < 5.5N$. Here and below, when an expression is given for $N_f$ that formally evaluates to a non-integral real value $\nu$, it is understood implicitly that one infers an appropriate integral value of $N_f$ from it.

If $N_f \in I$ is near to $N_{f,b_{1z}}$, so that $\alpha_{I R, 2\ell}$ is small, then the theory evolves from the UV to the IR without any spontaneous chiral symmetry breaking. As $N_f$ decreases in $I$, $\alpha_{I R, 2\ell}$ increases and, as $N_f$ decreases through a critical value denoted $N_{f,cr}$, the IR zero of $\beta$ increases through $\alpha_{cr}$, so that the UV to IR evolution leads to $S\chi$SB. For $N_f$ near lower end of $I$, $b_2 \to 0$ and $\alpha_{I R, 2\ell}$ is too large for the 2-loop calculation to be reliable.

3. Calculations of Higher-Loop Corrections to UV to IR Evolution

In this section we discuss our calculations of higher-loop corrections to the UV to IR evolution of a gauge theory and, in particular, higher-loop calculations of the IR zero of the beta function. We first recall our 3-loop calculation of the IR zero of $\beta$ in the $\overline{MS}$ scheme. The 3-loop coefficient in $\beta$, $b_3$, is a quadratic function of $N_f$ and vanishes, with sign reversal, at two values of $N_f$, denoted $N_{f,b_{3z,1}}$ and $N_{f,b_{3z,2}}$. Now $b_3$ is positive for small $N_f$ and vanishes first at $N_{f,b_{1z,1}}$, which is smaller than $N_{f,b_{2z}}$, the left endpoint of the interval $I$. Furthermore, $N_{f,b_{3z,2}} > N_{f,b_{1z}}$, the right endpoint of $I$. For example, for $N = 2$, $N_{f,b_{3z,1}} = 3.99 < N_{f,b_{2z}} = 5.55$ and $N_{f,b_{3z,2}} = 27.6 > N_{f,b_{1z}} = 11$, while for $N = 3$, $N_{f,b_{3z,1}} = 5.84 < N_{f,b_{2z}} = 8.05$ and $N_{f,b_{3z,2}} = 40.6 > N_{f,b_{1z}} = 16.5$. Hence, $b_3 < 0$ for $N_f \in I$, the interval of interest
for the IR zero of $\beta$. At this 3-loop level, $\beta_{3\ell} = -[a^2/(2\pi)](b_1 + b_2a + b_3a^2)$, so $\beta$ vanishes away from $\alpha = 0$ at two values. In terms of $\alpha$, these are

$$\alpha = \frac{2\pi}{b_3}( -b_2 \pm \sqrt{b_2^2 - 4b_1b_3} ) \quad (6)$$

Since $b_2 < 0$ in $I$ and the $\overline{MS}$ $b_3 < 0$ in $I$ also, this can be expressed in terms of positive quantities as

$$\alpha = \frac{2\pi}{|b_3|}( -|b_2| \pm \sqrt{b_2^2 + 4b_1|b_3|} ) \quad (7)$$

One of these solutions is negative and hence is unphysical; the other is manifestly positive, and is $\alpha_{IR,3\ell}$. Using this result, we proved in\cite{3} that in the $\overline{MS}$ scheme, $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$.

A natural question that arises from the analysis in\cite{3} is how general this inequality is and, specifically, whether it also holds for other schemes. We address and answer this question here\cite{4} To do this, we observe that if a scheme had $b_3 > 0$ in $I$, then, since $b_2 \to 0$ at the lower end of $I$, it would follow that $b_2^2 - 4b_1b_3 < 0$, so this scheme would not have a physical (real, positive) $\alpha_{IR,3\ell}$ in this region. Since the existence of the IR zero in $\beta_{3\ell}$ is a scheme-independent property, one may require that an acceptable scheme should preserve this to higher-loop order. To satisfy this requirement, by the argument above, it should have $b_3 < 0$ for $N_f \in I$, as is the case in the $\overline{MS}$ scheme. We now prove that in all such schemes,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell} \quad (8)$$

To prove this, we consider the difference

$$\alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{2\pi}{|b_2b_3|}[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}] \quad (9)$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0 \quad (10)$$

This difference is equal to the nonnegative quantity $(2b_1b_3)^2$, which proves the inequality. Note that, since $b_1 > 0$ from the asymptotic freedom property, this difference vanishes if and only if $b_3 = 0$, in which case $\alpha_{IR,3\ell} = \alpha_{IR,2\ell}$. This proof holds for general $G$, $R$, and $N_f \in I$.

As noted above, $\alpha_{IR,2\ell}$ is a monotonically decreasing function of $N_f \in I$. With $b_3 < 0$ for $N_f \in I$, this monotonicity property is also true of $\alpha_{IR,3\ell}$. As $N_f$ increases from $N_{f,b2z}$ to $N_{f,b1z}$ in the interval $I$, $\alpha_{IR,3\ell}$ decreases from

$$\alpha_{IR,3\ell} = 4\pi\frac{b_1}{|b_3|} \quad \text{at } N_f = N_{f,b2z} \quad (11)$$

to zero at the upper end of the interval $I$, vanishing like

$$\alpha_{IR,3\ell} = \frac{4\pi b_1}{|b_2|} \left[ 1 - \frac{|b_3|}{|b_2|} + O(b_1^2) \right] \quad \text{as } N_f \nearrow N_{f,b1z} \quad (12)$$
For the 4-loop analysis of $\beta$, we use $b_4$, which is a cubic polynomial in $N_f$. This coefficient is positive for $N_f \in I$ for $N = 2, 3$ but is negative in part of $I$ for higher $N$. The 4-loop $\beta$ function is $\beta_4 = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2 + b_4 a^3)$, so $\beta$ has three zeros away from the origin. We determine the smallest positive real zero as $\alpha_{IR,4\ell}$.

In addition to the inequality (8), we find the following general results: (i) going from the 3-loop to 4-loop level, there is a slight change in the value of the IR zero, but this change is smaller than the decrease from the 2-loop to 3-loop level, so $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$; and (ii) fractional changes in the value of the IR zero of $\beta$ decrease in magnitude as $N_f$ increases toward $N_{f,b}^1$, and all of the values of $\alpha_{IR,n\ell}$ → 0. Our finding that the fractional change in the location of the IR zero of $\beta$ is reduced at higher-loop order agrees with the general expectation that calculating a quantity to higher order in perturbation theory should give a more stable and accurate result.

Since $\alpha_{cr} \sim O(1)$, the decrease in $\alpha_{IR}$ at higher-loop order, together with the property that $\alpha_{IR}$ increases as $N_f$ decreases, means that one must go to smaller $N_f$ for $\alpha_{IR,n\ell}$ to grow to a given size for the $n = 3$ and $n = 4$ loop level as compared with the $n = 2$ loop level. This suggests that the actual lower boundary of the IR-conformal phase could lie somewhat below an estimate obtained setting $\alpha_{IR,2\ell} = \alpha_{cr}$.

Some numerical values of $\alpha_{IR,n\ell}$ at the 2-loop, 3-loop, and 4-loop level for fermions in the fundamental representation, $N_f \in I$, and the illustrative groups $G = SU(2)$ and $G = SU(3)$ from [2] are given in Table 1.

| $N$ | $N_f$ | $\alpha_{IR,2\ell}$ | $\alpha_{IR,3\ell}$ | $\alpha_{IR,4\ell}$ |
|-----|-------|----------------------|----------------------|----------------------|
| 2   | 7     | 2.83                 | 1.05                 | 1.21                 |
| 2   | 8     | 1.26                 | 0.688                | 0.760                |
| 2   | 9     | 0.595                | 0.418                | 0.444                |
| 2   | 10    | 0.231                | 0.196                | 0.200                |
| 3   | 10    | 2.21                 | 0.764                | 0.815                |
| 3   | 11    | 1.23                 | 0.578                | 0.626                |
| 3   | 12    | 0.754                | 0.435                | 0.470                |
| 3   | 13    | 0.468                | 0.317                | 0.337                |
| 3   | 14    | 0.278                | 0.215                | 0.224                |
| 3   | 15    | 0.143                | 0.123                | 0.126                |
| 3   | 16    | 0.0416               | 0.0397               | 0.0398               |

Corresponding higher-loop calculations were carried out in [2] for SU($N$) gauge theories with $N_f$ fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations. The general result $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ continues to apply. The difference $\alpha_{IR,4\ell} - \alpha_{IR,3\ell}$ tends to be relatively small, but can have either sign.
For example, for \( R = \text{adjoint} \), \( N_{f,b1z} = 11/4 \) and \( N_{f,b2z} = 17/16 \) (independent of \( N \)), so the interval \( I \) where \( \beta \) has an IR zero, viz., \( N_{f,b2z} < N_f < N_{f,b1z} \), is \( 1.06 < N_f < 2.75 \). This includes only one physical, integral value, \( N_f = 2 \). For this value of \( N_f \) and \( N = 2 \), one has \( \alpha_{IR,2\ell} = 0.628 \), \( \alpha_{IR,3\ell} = 0.459 \), \( \alpha_{IR,4\ell} = 0.450 \), while for \( N = 3 \), \( \alpha_{IR,2\ell} = 0.419 \), \( \alpha_{IR,3\ell} = 0.306 \), \( \alpha_{IR,4\ell} = 0.308 \).

4. Anomalous Dimension of Fermion Bilinear

The anomalous dimension \( \gamma_m \equiv \gamma \) for the fermion bilinear operator has the series expansion

\[
\gamma = \sum_{\ell=1}^{\infty} c_\ell \alpha^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell, \tag{13}
\]

where \( \bar{c}_\ell = c_\ell/(4\pi)^\ell \) is the \( \ell \)-loop coefficient. The one-loop coefficient \( c_1 \) is scheme-independent, while the \( c_\ell \) with \( \ell \geq 2 \) are scheme-dependent. The \( c_\ell \) have been calculated up to 4-loop level in the \( \overline{\text{MS}} \) scheme.\(^{[20]} \) The first two coefficients are \( c_1 = 6C_f \) and \( c_2 = 2C_f[(3/2)C_f + (97/6)C_A - (10/3)T_f N_f] \).

It is of interest to calculate \( \gamma \) at the exact IRFP in the IR-conformal phase and at the approximate IRFP in the phase with spontaneous chiral symmetry breaking. We denote \( \gamma \) calculated to \( n \)-loop (\( n\ell \)) level as \( \gamma_{n\ell} \) and, evaluated at the IR zero of \( \beta_{n\ell} \), as \( \gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell}) \). In the IR-conformal phase, an exact calculation of \( \gamma \) evaluated at the IRFP would be an exact (scheme-independent) property of the theory, but in the broken phase, just as the IR zero of \( \beta \) is only an approximate IRFP, so also, \( \gamma \) is only approximate, describing the running of \( \bar{\psi}\psi \) and the dynamically generated fermion mass near the zero of \( \beta \), according to

\[
\Sigma(k) \sim \Lambda \left( \frac{\Lambda}{k} \right)^{2-\gamma} \tag{14}
\]

for Euclidean momenta \( k \gg \Lambda \). This, in turn, affects SM fermion masses in DEWSB theories.\(^{[13]} \) In both the conformal and the chirally broken IR phases, the upper bound \( \gamma \sim 2 \) holds.

At the 2-loop level we calculate

\[
\gamma_{IR,2\ell} = \frac{C_f(11C_A - 4T_f N_f)[455C_A^2 + 99C_A C_f + (180C_f - 248C_A)T_f N_f + 80(T_f N_f)^2]}{12(-17C_A^2 + 2(5C_A + 3C_f)T_f N_f)^2}. \tag{15}
\]

Our analytic expressions for \( \gamma_{IR,n\ell} \) at the 3-loop and 4-loop level are more complicated.\(^{[13]} \) Illustrative numerical values of \( \gamma_{IR,n\ell} \) at the 2-, 3-, and 4-loop level are given below in Table 2 and Figs. 1 and 2 for \( R = \text{fund.} \), and the illustrative values \( N = 2, 3 \). (Values in parentheses violate the upper bound \( \gamma < 2 \) and reflect the inadequacy of perturbation theory if \( \alpha \) is too large.)

We have also performed these calculations for \( G = \text{SU}(N) \) and higher fermion representations \( R \). In general, we find that, for a given \( N, R, \) and \( N_f, \) the values of \( \gamma_{IR,n\ell} \) calculated to 3-loop and 4-loop order are smaller than the 2-loop value. The value of these higher-loop calculations is evident from the figures. A necessary
Table 2. Values of $\gamma_{IR,n\ell}$ for an SU($N$) gauge theory with $N_f$ fermions in the fundamental representation, for $N = 2, 3$ and $N_f \in I$.

| $N$ | $N_f$ | $\gamma_{IR,2\ell}$ | $\gamma_{IR,3\ell}$ | $\gamma_{IR,4\ell}$ |
|-----|-------|-----------------|-----------------|-----------------|
| 2   | 7     | (2.67)          | 0.457           | 0.0325          |
| 2   | 8     | 0.752           | 0.272           | 0.204           |
| 2   | 9     | 0.275           | 0.161           | 0.157           |
| 2   | 10    | 0.0910          | 0.0738          | 0.0748          |
| 3   | 10    | (4.19)          | 0.647           | 0.156           |
| 3   | 11    | 1.61            | 0.439           | 0.250           |
| 3   | 12    | 0.773           | 0.312           | 0.253           |
| 3   | 13    | 0.404           | 0.220           | 0.210           |
| 3   | 14    | 0.212           | 0.146           | 0.147           |
| 3   | 15    | 0.0997          | 0.0826          | 0.0836          |
| 3   | 16    | 0.0272          | 0.0258          | 0.0259          |

Fig. 1. $\gamma_{IR}$ for SU(2) and fermions in the fundamental representation. From top to bottom, the curves show $\gamma_{IR,2\ell}$, $\gamma_{IR,3\ell}$, and $\gamma_{IR,4\ell}$.

condition for a perturbative calculation to be reliable is that higher-order contributions do not modify the result too much. One sees from the tables and figures that, especially for smaller $N_f$, there is a substantial decrease in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ when one goes from 2-loop to 3-loop order, but for a reasonable range of $N_f$, the 3-loop and 4-loop results are close to each other. Thus, our higher-loop calculations of $\alpha_{IR}$
and γ allow us to probe the theory reliably down to smaller values of \( N_f \) and thus stronger couplings.

It is useful to give a comparison of our calculations with lattice measurements. The theory with SU(3), \( R = \text{fund.} \), and \( N_f = 12 \) has been the subject of intensive lattice study, so we focus on this for the comparison. For this theory we calculate (see Table 2) \( \gamma_{IR,2\ell} = 0.77 \), \( \gamma_{IR,3\ell} = 0.31 \), and \( \gamma_{IR,4\ell} = 0.25 \) (to two significant figures). Lattice results include \( \gamma = 0.414 \pm 0.016 \) \( \gamma \sim 0.35 \) \( 0.2 \leq \gamma \leq 0.4 \) \( \gamma = 0.4 - 0.5 \) and \( \gamma = 0.27 \pm 0.03 \) Here the 2-loop value of \( \gamma \) is larger than, and the 3-loop and 4-loop values are closer to, these lattice measurements. This illustrates how higher-loop calculations of \( \gamma \) can improve agreement with lattice measurements. Note that there is not yet a consensus among lattice groups as to whether this theory has an IR phase with chiral symmetry or spontaneous chiral symmetry breaking. Similar comparisons can be given for other values of \( G, R \) and \( N_f \). In particular, for SU(3), \( R = \text{fund.} \), \( N_f = 10 \), one group obtains \( \gamma_{IR} \sim 1 \) consistent with the idea that \( \gamma_{IR} \sim 1 \) at the lower end of the IR-conformal phase. This region is difficult to probe perturbatively because of the strongly coupled nature of the physics.

An interesting property of the values of \( \alpha_{IR,n\ell} \) and \( \gamma_{IR,n\ell} \) in the case where \( R = \text{fund.} \) is that these are similar in theories with different values of \( N \) and \( N_f \), provided that these theories have the same or similar values of the ratio \( N_f/N \). This can be understood as a result of a rapid approach to the ’t Hooft-Veneziano limit.
$N \to \infty$, $N_f \to \infty$ with $N_f/N$ fixed.

5. Supersymmetric Gauge Theory

It is valuable to carry out a similar analysis in an asymptotically free $N = 1$ supersymmetric gauge theory with vectorial chiral superfield content $\Phi_i$, $\Phi_i$, $i = 1, ..., N_f$ in the $R$, $\bar{R}$ reps., respectively. An appeal of this analysis is that exact results on the IR properties of the theory are known. Thus, one can compare results from higher-loop perturbative calculations with exact results, in particular, for $N_f$, $c_r$. The coefficients of $\beta$ and $\gamma$ have been calculated up to 3-loop order. The one-loop coefficient is $b_{1,s} = 3C_A - 2T_f N_f$ (Here and below, the subscript $s$, for “supersymmetric”, is appended to distinguish these quantities from those for the nonsupersymmetric theory). Asymptotic freedom requires that $N_f < N_{f, \text{brz}, s}$, where $N_{f, \text{brz}, s} = 3C_A/(2T_f)$. The 2-loop coefficient of $\beta$ is $b_{2,s} = 6C_A^2 - 4T_f N_f (C_A + 2C_f)$, which is positive for small $N_f$, but vanishes with sign reversal as $N_f$ increases through the value

$$N_{f, \text{brz}, s} = \frac{3C_A^2}{2T_f (C_A + 2C_f)},$$

which is smaller than $N_{f, \text{brz}, s}$. Thus, for this theory, there is again an interval $I_s$ in which $\beta_{2t}$ has an IR zero, namely

$$I_s : \quad N_{f, \text{brz}, s} < N_f < N_{f, \text{brz}, s}.$$

For $R = \text{fund.}$ this interval $I_s$ is $3N^3/(2N^2 - 1) < N_f < 3N$. In general, this IR zero of $\beta_{2t}$ occurs at

$$\alpha_{IR, 2t} = \frac{2\pi (3C_A - 2T_f N_f)}{2T_f N_f (C_A + 2C_f) - 3C_A^2}.$$

The 3-loop coefficient $b_{3,s}$ is positive for small $N_f$ and vanishes at two values of $N_f$, denoted $N_{f, \text{brz}, 1,s}$ and $N_{f, \text{brz}, 2,s}$. As before, we find that $N_{f, \text{brz}, 1,s} < N_{f, \text{brz}, s}$ and $N_{f, \text{brz}, 2,s} > N_{f, \text{brz}, s}$, so $b_{3,s} < 0$ for $N_f \in I_s$. Since $b_{3,s} < 0$ for $N_f \in I$, we find, by the same type of proof as given above, that for any $G$, $R$, and $N_f \in I_s$

$$\alpha_{IR, 3t} < \alpha_{IR, 2t}.$$

For fixed $N$, we find that $\alpha_{IR, n}$ increases monotonically with decreasing $N_f$ at both the 2-loop and 3-loop level.

Next, we analyze the anomalous dimension $\gamma$ of the superfield operator product $\Phi \Phi$ containing the term $\theta \bar{\psi} \psi$. In a conformally invariant $d$-dimensional field theory (whether supersymmetric or not), unitarity yields a lower bound on the dimension $D_O$ of a spin-0 operator $O$ (other than the identity), namely, $D_O \geq (d - 2)/2$, where $d$ = spacetime dim.; so $D_O \geq 1$ here. In the nonsupersymmetric theory, with $\text{dim}(\bar{\psi} \psi) = 3 - \gamma$, this constraint is $D_{\bar{\psi} \psi} = 3 - \gamma > 1$, so $\gamma < 2$. In the supersymmetric theory, with $\text{dim}(\theta) = -1/2$ and $\text{dim}(\psi \bar{\psi}) = 3 - \gamma$, the constraint $D_{\Phi \Phi} = -1 + 3 - \gamma > 1$, so $\gamma < 1$. 
At the 2-loop level, we find
\[
\gamma_{IR,2\ell,s} = \frac{C_f(3C_A - 2T_f N_f)(2T_f N_f - C_A)(2T_f N_f - 3C_A + 6C_f)}{[2(C_A + 2C_f)T_f N_f - 3C_A^2]^2}.
\] (20)

We have also calculated \(\gamma_{IR,3\ell,s}\) and find that, as in the nonsupersymmetric case, \(\gamma_{IR,3\ell} < \gamma_{IR,2\ell}\). Let us focus on the case \(R = \text{fund}\), for which \(N_{f,bz,s} = 3N\) and \(N_{f,cr,s} = (3/2)N\). One perturbative estimate of \(N_{f,cr}\) can be obtained by assuming that the upper bound \(\gamma \leq 1\) is saturated as \(N \rightarrow N_{f,cr,s}\). Solving this, we obtain that for \(N = 2\), \(N_{f,cr,s,est.} = 4.24\), a factor of 1.41 larger than exact \(N_{f,cr,s} = 3\), while for \(N = 3\) \(N_{f,cr,s,est.} = 6.15\) a factor of 1.37 larger than exact \(N_{f,cr,s} = 4.5\). As \(N \rightarrow \infty\), this procedure yields \(N_{f,cr,s,est.} \rightarrow 2N\), a factor of (4/3) larger than the exact result, \(N_{f,cr,s} = (3/2)N\).

This comparison for the \(N = 1\) supersymmetric gauge theory suggests that the perturbative calculation slightly overestimates the value of \(N_{f,cr}\), i.e., slightly underestimates the size of the IR-conformal phase, similar to what we found for the nonsupersymmetric theory.

6. Scheme-Dependence in Calculation of IR Fixed Point

Since the coefficients in \(\beta\) at the level of \(n \geq 3\) loops are scheme-dependent, so is the resultant value of \(\alpha_{IR,n\ell}\). It is important to assess quantitatively the uncertainty in the analysis of the UV to IR evolution due to this scheme dependence. A way to do this is to perform scheme transformations and determine how much of a change there is in \(\alpha_{IR,n\ell}\).

A scheme transformation (ST) is a map between \(a\) and \(a'\) or equivalently, \(a\) and \(a'\), where \(a = \alpha/(4\pi)\), which can be expressed as
\[
a = a' f(a')
\] (21)
with \(f(0) = 1\) to keep the UV properties unchanged. We write
\[
f(a') = 1 + \sum_{s=1}^{s_{max}} k_s(a')^s = 1 + \sum_{s=1}^{s_{max}} \tilde{k}_s(a')^s,
\] (22)
where the \(k_s\) are constants, \(\tilde{k}_s = k_s/(4\pi)^s\), and \(s_{max}\) may be finite or infinite. Hence, the Jacobian
\[
J = \frac{da}{da'} = \frac{d\alpha}{d\alpha'}
\] (23)
satisfies \(J = 1\) at \(a = a' = 0\). We have
\[
\beta_{\alpha'} = \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha} \frac{d\alpha}{dt} = J^{-1} \beta_{\alpha}.
\] (24)
This has the expansion
\[
\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \tilde{b}'_\ell (a')^\ell,
\] (25)
where $\bar{b}' = b'/(4\pi)^\ell$.

Using these two equivalent expressions for $\beta_\alpha'$, one can solve for the $b'_{\ell}$ in terms of the $b_\ell$ and $k_s$. This yields the well-known result that $b'_1 = b_1$ and $b'_2 = b_2$. To assess the scheme-dependence of an IRFP, we have calculated the relations between the $b'_{\ell}$ and $b_\ell$ for higher $\ell$ values. For example, for $\ell = 3, 4, 5$, we obtain

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 ,$$

(26)

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

(27)

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$

$$+ (4k_1^4 - 11k_1^2 k_2 + 6k_1^3 + 4k_2^2 - 3k_4) b_1 .$$

(28)

To be physically acceptable, a ST must satisfy several conditions, $C_i$:

- $C_1$: the ST must map a real positive $\alpha$ to a real positive $\alpha'$, since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex $\alpha'$ would violate the unitarity of the theory.
- $C_2$: the ST should not map a moderate value of $\alpha$, for which perturbative calculations may be reliable, to an excessively large value of $\alpha'$ where perturbative calculations are inapplicable.
- $C_3$: $J$ should not vanish in the region of $\alpha$ and $\alpha'$ of interest, or else there would be a pole in the relation between $\beta_\alpha$ and $\beta_\alpha'$.
- $C_4$: The existence of an IR zero of $\beta$ is a scheme-independent property, depending (in an asymptotically free theory) only on the condition that $b_2 < 0$. Hence, a ST should satisfy the condition that $\beta_\alpha$ has an IR zero if and only if $\beta_\alpha'$ has an IR zero.

These four conditions can always be satisfied by scheme transformations near a UV fixed point, and hence in applications to perturbative QCD calculation, since $\alpha$ is small, and one can choose the $k_s$ to be small also, so $\alpha' \approx \alpha$. However, these conditions C1-C4 are not automatically satisfied, and are a significant constraint, on a scheme transformation applied in the vicinity of an IRFP, where $\alpha$ may be $O(1)$. For example, consider the scheme transformation

$$\alpha = \tanh(\alpha')$$

(29)

with inverse

$$\alpha' = \frac{1}{2} \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) .$$

(30)

If $\alpha << 1$, as at a UVFP, this is acceptable, but as $\alpha$ approaches 1 from below it maps a moderate value of $\alpha$ to an arbitrarily large $\alpha'$ and hence fails condition C2, and if $\alpha$ exceeds 1, even if by a small amount, then it fails conditions C1 and C4, since it maps a real positive $\alpha$ to a complex $\alpha'$.
We have studied the scheme dependence of the IR zero of $\beta$ in $\overline{\text{MS}}$ using several scheme transformations; e.g., the ST (depending on a parameter $r$)

$$S_{sh,r} : \quad a = \frac{\sinh(ra')}{r}$$

(31)

Since $\sinh(ra')/r$ is an even fn. of $r$, we take $r > 0$ with no loss of generality. This transformation has the inverse

$$a' = \frac{1}{r} \ln \left[ ra + \sqrt{1 + (ra)^2} \right]$$

(32)

and the Jacobian $J = \cosh(ra')$. For this ST,

$$f(a') = \frac{\sinh(ra')}{ra'}.$$  

(33)

This $f(a')$ has a series expansion with $k_s = 0$ for odd $s$ and, for even $s$,

$$k_2 = \frac{r^2}{6}, \quad k_4 = \frac{r^4}{120}, \quad k_6 = \frac{r^6}{5040}, \quad k_8 = \frac{r^8}{362880},$$

(34)

etc. for higher $s$. Substituting these results for $k_s$ into the equations for the $b'_\ell$, we obtain

$$b'_3 = b_3 - \frac{r^2 b_1}{6}, \quad b'_4 = b_4,$$

(35)

$$b'_5 = b_5 + \frac{r^2 b_3}{6} + \frac{31 r^4 b_1}{360},$$

(36)

and so forth for higher $\ell$.

We apply this $S_{sh,r}$ ST to the $\beta$ function in the $\overline{\text{MS}}$ scheme, calculated up to $\ell = 4$ loop level. For $N_f \in I$ where $\beta_{2\ell}$ has an IR zero, we then calculate the resultant IR zeros in $\beta_{\alpha'}$ at the 3-loop and 4-loop order and compare the values with those in the $\overline{\text{MS}}$ scheme. We list some numerical results for illustrative values of $r$ and for $N = 2, 3$ below. We denote the IR zero of $\beta_{\alpha'}$ at the $n$-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$. For example, for $N = 3, N_f = 10$, $\alpha_{IR,2\ell} = 2.21$, and:

$$\alpha'_{IR,3\ell,\overline{\text{MS}}} = 0.764, \quad \alpha'_{IR,3\ell,r=3} = 0.762, \quad \alpha'_{IR,3\ell,r=6} = 0.754,$$

(37)

$$\alpha'_{IR,3\ell,r=9} = 0.742, \quad \alpha'_{IR,3\ell,r=4\pi} = 0.723$$

(38)

$$\alpha_{IR,4\ell,\overline{\text{MS}}} = 0.815, \quad \alpha'_{IR,4\ell,r=3} = 0.812, \quad \alpha'_{IR,4\ell,r=6} = 0.802,$$

(39)

$$\alpha'_{IR,4\ell,r=9} = 0.786, \quad \alpha'_{IR,4\ell,r=4\pi} = 0.762.$$  

(40)

In general, the effect of scheme dependence tends to be reduced (i) for a given $N$ and $N_f$, as one calculates to higher-loop order, and (ii) for a given $N$, as $N_f \rightarrow N_f, {\text{Max}}$, so that the value of $\alpha_{IR} \rightarrow 0$. These results provide a quantitative measure of scheme dependence of the location of an IR zero of $\beta$.

Since the $\beta$ function coefficients $b_\ell$ with $\ell \geq 3$ are scheme-dependent, there should exist a ST that renders these coefficients equal to zero (i.e., maps to the 't
Hooft scheme). We constructed an explicit ST that can does this at a UVFP. This necessarily has $s_{\text{max}} = \infty$. For simplicity, we set $k_1 = 0$. Our solutions for the first few $k_s$ are

$$k_2 = \frac{b_3}{b_1}, \quad k_3 = \frac{b_4}{2b_1},$$

(41)

$$k_4 = \frac{b_5}{3b_1} - \frac{b_2b_4}{6b_1^2} + \frac{5b_3^2}{3b_1^3},$$

(42)

$$k_5 = \frac{b_6}{4b_1} - \frac{b_2b_5}{6b_1^2} + \frac{2b_3b_4}{b_1} + \frac{b_5^2b_4}{12b_1^3} - \frac{b_2b_4^2}{12b_1^3},$$

(43)

and so forth for higher $s$.

7. Application to Models of Dynamical Electroweak Symmetry Breaking

Models with dynamical EWSB have been of interest as one way to avoid the hierarchy (fine-tuning) problem with the Standard Model, while supersymmetry provides another way to do this. DEWSB models use an asymptotically free vectorial gauge interaction with a certain set of fermions subject to this gauge interaction, which becomes strong at the TeV scale, producing bilinear condensates of these fermions. These models involve further ingredients to give masses to SM fermions. The running mass $m_f(p)$ of a SM fermion of generation $i$ is constant up to the scale where it is dynamically generated, and has a power-law decay above this scale. Quasiconformal behavior in these models resulting from an approximate IRFP is important as a means to produce a substantial $\gamma_m$ and hence enhance SM fermion masses.

Since the quasiconformal property depends on having an approximate IRFP slightly greater than $\alpha_{\text{cr}}$, the higher-loop calculations of the UV to IR evolution described here are valuable not only for their intrinsic field-theoretic interest, but also as applied to the construction of such models. Studies of reasonably UV-complete DEWSB models showed that approximate residual generational symmetries suppress FCNC effects but it is challenging to reproduce all features of SM fermion masses, such as $m_t >> m_b$, etc.

Concerning dilatons resulting from quasiconformal DEWSB models, we note that such models need not have any color-nonsinglet fermions subject to the new gauge interaction. The minimal SM-nonsinglet fermion content consists of one SU(2)$_L$ doublet with corresponding right-handed SU(2)$_L$ singlets, all of which are color-singlets. DEWSB models of this type can exhibit quasiconformal behavior. Hence, in contrast to one-family DEWSB models, in these minimal models, the resultant dilaton would be comprised purely of color-singlet constituents. This difference is recognized in some of the phenomenological papers on dilatons. The boson with mass $\sim 125$ GeV discovered at the LHC by ATLAS and CMS is consistent with being the SM Higgs. An important question is to determine how well it can,
alternatively, be modelled as a dilaton resulting from a quasi-conformal DEWSB model. Further experimental and theoretical work should settle this question.

8. Conclusions

In conclusion, understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest. Our higher-loop calculations give new information on this UV to IR flow and on the determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$. It is valuable to compare our higher-order calculations of $\gamma_{IR,n\ell}$ with lattice measurements. The higher-loop study of the UV to IR flow for supersymmetric gauge theories yields further insights. We have carried out a study of scheme-dependence in higher-loop calculations, noting that scheme transformations are subject to constraints that are easily satisfied at a UVFP but are quite restrictive at IRFP. In addition to their intrinsic interest, quasi-conformal gauge theories are relevant to models of dynamical EWSB, plausibly yielding a light pseudo-Nambu-Goldstone boson, the dilaton.

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