Longitudinal mode-coupling instabilities of proton bunches in the CERN Super Proton Synchrotron

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In this paper, we study single-bunch instabilities observed in the CERN Super Proton Synchrotron (SPS). According to the linearized Vlasov theory, radial or azimuthal mode-coupling instabilities result from a coupling of bunch-oscillation modes, which belong to either the same or adjacent azimuthal modes, respectively. We show that both instability mechanisms exist in the SPS by applying the Oide-Yokoya approach to compute van Kampen modes for the realistic longitudinal impedance model of the SPS. The results agree with macroparticle simulations and are consistent with beam measurements. In particular, we see that the uncontrolled longitudinal emittance blow-up of single bunches observed before the recent impedance reduction campaign (2018-2021) is due to the radial mode-coupling instability. Unexpectedly, this instability is as strong as the azimuthal mode-coupling instability, which is possible in the SPS for other combinations of bunch length and intensity. We also demonstrate the significant role of rf nonlinearity and potential-well distortion in determining these instability thresholds. Finally, we discuss the effect of the recent impedance reduction campaign on beam stability in single- and double-rf configurations.

I. INTRODUCTION

Longitudinal single-bunch instability is a possible performance limitation in many synchrotrons and its mechanism is a subject of various studies since long time \[1\]–\[10\]. The standard approach to evaluate beam stability is based on a solution of the linearized Vlasov equation for a small initial perturbation of a stationary distribution function. To simplify the analysis, the modification of a stationary potential well by self-induced fields, called potential-well distortion (PWD), is often neglected. The only possible mechanism of longitudinal single-bunch instability without PWD, a coupling of different azimuthal modes, was proposed by Sacherer \[1\].

Another type of instability can be caused by asymmetry of the potential well due to PWD, resulting in a coupling of two radial modes within one azimuthal mode \[8\]–\[9\]. An explicit condition required for this instability to occur was found for the double-waterbag model \[10\]. For an impedance model consisting of one broad-band resonator with frequency \[f_r = \omega_r/2\pi\], the instability thresholds computed with and without PWD are similar for \[\omega_0 \sigma \gtrsim 0.4\], where \[\sigma\] is the rms bunch length, \[8\]. This result was also confirmed in calculations based on the orthogonal polynomial expansion \[14\]. The azimuthal mode-coupling was also found in the self-consistent analysis of electron bunches for \[\omega_0 \sigma \approx \pi\] \[13\].

Similar to electron bunches, the thresholds of the single-bunch instability for proton bunches are often computed neglecting bunch asymmetry due to PWD and rf nonlinearity, as for example, in \[11\]–\[17\], and thus only azimuthal mode-coupling instability was found. To our best knowledge, for proton bunches, so far a radial mode-coupling instability was not observed in measurements nor in calculations.

In the SPS, the longitudinal instability of single proton bunches occurs during the acceleration ramp. The attempts to cure this instability by reducing the voltage in a single rf system and thus increasing the synchrotron frequency spread for a constant longitudinal emittance were not successful. Instead, a higher rf voltage was more beneficial \[18\]. In operation, this instability is cured by the application of a higher-harmonic (HH) rf system. Due to strong frequency dependence of the SPS impedance \[19\] (Fig. \[1\]), the observed instability was mainly studied in macroparticle simulations using the code B\textsc{Load}D \[20\]. The latest results of simulations through the ramp are consistent with measurements and the agreement has been improved with the refined impedance model \[21\].

In the present work, the mechanism of the SPS single-bunch instability is studied using code ME\textsc{Load}Y \[22\] which is able to find in a fully self-consistent way the numerical solutions of the semianalytic matrix equations derived from the Vlasov equation for the full SPS impedance model. We show that the previously observed instability during the ramp was due to the coupling of multiple radial modes within one azimuthal mode. For a specific set of bunch parameters, we find an instability caused by the coupling of neighboring azimuthal modes for which Landau damping is lost. The main results are confirmed by macroparticle simulations with B\textsc{Load}D and are consistent with previous measurements \[18\].

The paper is organized as follows. In Sec. \[1\] we briefly discuss the main definitions and semianalytical methods to evaluate single-bunch instabilities. Two possible instability mechanisms in the SPS and a comparison of calculations with measurements are presented in Sec. \[1\]. We consider different instability mitigation measures in Sec. \[IV\] and, finally, present the main conclusions.
II. MAIN EQUATIONS AND DEFINITIONS

The longitudinal motion of a particle in a synchrotron can be described in terms of its energy and phase deviations, $\Delta E$ and $\phi$, relative to the synchronous particle with the energy $E_0$. For beam stability analysis it is convenient to introduce another set of variables, the energy $\mathcal{E}$ and the phase $\psi$ of synchrotron oscillations

$$\mathcal{E} = \frac{\Delta^2}{2\omega_{s0}^2} + \left| U_1(\phi) \right|,$$

$$\psi = \text{sgn}(\eta \Delta E) \frac{\omega_{s0}(\mathcal{E})}{\sqrt{2\omega_{s0}}} \int_{\phi_{\text{max}}(\mathcal{E})}^{\phi} \frac{d\phi'}{\sqrt{\mathcal{E} - U_1(\phi')}}. \quad (2)$$

Here $\eta = 1/\gamma_0^2 - 1/\gamma^2$ is the phase slip factor, $\gamma$ is the relativistic Lorentz factor, $\gamma_0$ is the Lorentz factor at transition energy, $f_{s0} = \omega_{s0}/2\pi$ is the frequency of small-amplitude synchrotron oscillations in a bare single-rf system, $\omega_{s}(\mathcal{E})$ is the synchrotron frequency as a function of $\mathcal{E}$, and $\phi_{\text{max}}(\mathcal{E})$ is the maximum phase of the particle with synchrotron oscillation energy $\mathcal{E} = U_1(\phi_{\text{max}}(\mathcal{E}))$. The total potential includes contributions from both the rf system and the beam-induced fields

$$U_1(\phi) = \frac{1}{V_1 \cos \phi_{s0}} \int_{\Delta \phi}^{\phi} \left[ V_{\text{rf}}(\phi') + V_{\text{ind}}(\phi') - \delta E_0/q \right] d\phi'.$$

where $V_1$ is the rf voltage amplitude of the main rf system, $\delta E_0$ is the energy gain per turn of the synchronous particle with charge $q$ excluding intensity effects, $\Delta \phi$ is the synchronous phase shift due to intensity effects that satisfies the relation $\delta E_0/q = V_1 \sin \phi_{s0} = V_{\text{rf}}(\Delta \phi) + V_{\text{ind}}(\Delta \phi)$, and $\phi_{s0}$ is the synchronous phase in a bare single-rf system. Below we consider a double-rf system with a total voltage of

$$V_{\text{rf}}(\phi) = V_1 \left[ \sin (\phi + \phi_{s0}) + r \sin (n\phi + n\phi_{s0} + \Phi_n) \right],$$

where $\Phi_n$ is the relative phase offset between the main and the HH rf systems with harmonic numbers $h$ and $nh$, respectively, and $V_n = rV_1$ is the voltage amplitude of the HH rf system. For particular values of $\Phi_n$ one can define two distinct regimes: bunch-shortening mode (BSM) when both rf systems are in phase at the bunch center and bunch-length mode (BLM) when both rf systems are in counter-phase at the bunch center. In the SPS operation, they are chosen such that the contribution of the HH rf system is zero at $\phi = 0$ and it does not contribute to a shift of the synchronous phase. Thus, $\Phi_n = \pi - n\phi_{s0}$ for BSM and $\Phi_n = -n\phi_{s0}$ for BLM.

In general, the total potential $U_1$ depends on the particle distribution function, impedance model, and bunch intensity. It can be calculated using an iterative procedure [12, 23]. In this work, we consider a particle distribution of the binomial family

$$\mathcal{F}(\mathcal{E}) = \frac{1}{2\pi \omega_{s0} A_N} \left( 1 - \frac{\mathcal{E}}{\mathcal{E}_{\text{max}}} \right)^\mu,$$

with the normalization constant

$$A_N = \omega_{s0} \int_0^{\mathcal{E}_{\text{max}}} \frac{d\mathcal{E}}{\omega_{s0}(\mathcal{E})} \left( 1 - \frac{\mathcal{E}}{\mathcal{E}_{\text{max}}} \right)^\mu. \quad (6)$$

For $\mu \to \infty$, the bunch has a Gaussian line density and the corresponding bunch length $\tau_{4\sigma}$ is typically defined as four times the rms bunch length $\sigma$, i.e. $\tau_{4\sigma} = 4\sigma$. The bunch length $\tau_{4\sigma}$ is related to the full-width at half-maximum (FWHM) bunch length $\tau_{\text{FWHM}}$ as

$$\tau_{4\sigma} = \tau_{\text{FWHM}} \sqrt{2/\ln 2}. \quad (7)$$

In practice, SPS proton bunches are far from being Gaussian, and the best fit to the measured bunch profiles is for $\mu \approx 1.5$. This value will be assumed in the present work for all calculations and simulations. For easy comparison with measurements, we use Eq. (7) as a definition of the bunch length. We also define the total longitudinal emittance in units of eVs as

$$\epsilon = \int \frac{\Delta E(\phi)}{n} d\phi = \sqrt{\frac{V_1 \cos \phi_{s0} q^2 E_0}{\pi n \mu^2 h^3}} \epsilon_N, \quad (8)$$
The integral equation \([12]\) is equivalent to the equation that describes collective modes in a plasma \([20, 23]\). An initial perturbation can be expressed as a superposition of van Kampen modes, which are, in general, described by nonregular functions. To show this, one can first perform the substitution in Eq. \([12]\)

\[
C_m(\mathcal{E}, \Omega) = \sqrt{-\omega_s(\mathcal{E})} \frac{dF(\mathcal{E})}{d\mathcal{E}} m \frac{\omega_s(\mathcal{E})}{\omega_s(0)} \tilde{c}_m(\mathcal{E}, \Omega),
\]

which leads to

\[
\begin{align*}
&\left[ \frac{\Omega^2}{\omega_s^2(0)} - \frac{m^2 \omega_s^2(\mathcal{E})}{\omega_s^2(0)} \right] \tilde{C}_m(\mathcal{E}, \Omega) \\
&= -2i\zeta \sum_{m'=1}^{\infty} K_{mm'}(\mathcal{E}, \mathcal{E}') \tilde{c}_{m',\Omega}(\mathcal{E}')d\mathcal{E}'.
\end{align*}
\]

Here, the kernel \(K\) is defined as

\[
K_{mm'}(\mathcal{E}, \mathcal{E}', \Omega) = \sum_{k=\infty}^{\infty} \frac{Z_k(\Omega)/k}{hZ_0} \times m \int_{0}^{E_{\text{max}}} d\mathcal{E}' \frac{dF(\mathcal{E})}{d\mathcal{E}} I_{mk}(\mathcal{E}) I_{n'k}(\mathcal{E}') C_m(\mathcal{E}', \Omega).
\]

Introducing a set of orthonormal functions \(s_{n}(m)\),

\[
\int_{0}^{E_{\text{max}}} s_{n}^{(m)}(\mathcal{E}) s_{n'}^{(m)}(\mathcal{E}) d\mathcal{E} = \delta_{mm'},
\]

with the Kronecker delta \(\delta_{ij}\), we can decompose \(\tilde{C}_m\) and \(K_{mm'}\), similarly to \([3]\):

\[
\tilde{C}_m(\mathcal{E}, \Omega) = \sum_{n=0}^{\infty} a_n(\mathcal{E}) s_n^{(m)}(\mathcal{E}),
\]

\[
K_{mm'}(\mathcal{E}, \mathcal{E}', \Omega) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} K_{mm'}^{nn'}(\mathcal{E}) s_n^{(m)}(\mathcal{E}) s_{n'}^{(m')}\mathcal{E}'\mathcal{E}'.
\]

with the coefficients \(a_n(\mathcal{E})\) and \(K_{mm'}^{nn'}\) defined as

\[
a_n^{(m)}(\mathcal{E}) = \int_{0}^{E_{\text{max}}} \tilde{C}_m(\mathcal{E}, \Omega) s_n^{(m)}(\mathcal{E}) d\mathcal{E},
\]

and

\[
K_{mm'}^{nn'}(\mathcal{E}) = \int_{0}^{E_{\text{max}}} \int_{0}^{E_{\text{max}}} K_{mm'}(\mathcal{E}, \mathcal{E}', \Omega) s_n^{(m)}(\mathcal{E}) s_{n'}^{(m')}\mathcal{E}'d\mathcal{E}d\mathcal{E}'.
\]

B. van Kampen modes

Detailed derivations of Eq. \([12]\) using variables \((\mathcal{E}, \psi)\) can be found in \([25]\) and it is identical to Eq. \((41)\) therein. For a given impedance model, the single-bunch stability depends on the two dimensionless parameters, \(\zeta\) and \(\epsilon_N\), as well as on the parameters of the rf potential: \(n, \phi_{0\Omega}, r,\) and \(\Phi_n\). This fact will be applied below to understand the mechanisms of instabilities observed in the SPS.
respectively. Finally, inserting Eqs. \([18, 19]\) in Eq. \((16)\), we obtain the eigenfunctions
\[
\hat{C}_m(\mathcal{E}, \Omega) = \left\{ \frac{1}{\Omega^2 - m^2 \omega_s^2(\mathcal{E})} + \alpha(\mathcal{E}) \delta \left[ \Omega^2 - m^2 \omega_s^2(\mathcal{E}) \right] \right\} 
\times 2i\zeta \omega_s^2 \sum_{m'=1}^{\infty} \sum_{n=0}^{\infty} K^{nn'}_{mm'}(\Omega) a^m_n(\Omega) s^{(m)}(\mathcal{E}), \tag{22}
\]
where \(\mathcal{P}\) denotes the principal value of the integral, and \(\alpha(\mathcal{E}, \Omega)\) can be found from the normalization condition
\[
\sum_{m=1}^{\infty} \int_{0}^{\mathcal{E}_{\text{max}}} \hat{C}_m(\mathcal{E}, \Omega) d\mathcal{E} = 1.
\]
For \(\text{Im}\Omega \to \pm 0\), for example, one gets \(\alpha = \mp i\pi\).

Most of the stable modes are located within the incoherent spectrum \(\Omega \in m\omega_s(\mathcal{E})\) and have singular eigenfunctions due to the second term in braces in Eq. \((22)\). Unstable modes and modes for which Landau damping is lost, \(\Omega \not\in m\omega_s(\mathcal{E})\), have regular eigenfunctions as they are defined by the first terms in braces.

### C. Instability mechanisms

The eigenfunctions are related to the perturbed line density harmonics (see, e.g. in \([25]\))
\[
\tilde{\lambda}_k(\Omega) = \frac{\omega_s^2}{2h} \sum_{m=1}^{\infty} \int_{0}^{\mathcal{E}_{\text{max}}} \frac{C_m(\mathcal{E}, \Omega) I^*_m(\mathcal{E})}{\omega_s(\mathcal{E})} d\mathcal{E}. \tag{23}
\]
Multiplying Eq. \((16)\) with \(\hat{C}_m(\mathcal{E}, \Omega)\) and integrating over \(\mathcal{E}\) leads to
\[
\Omega^2 = \sum_{m=1}^{\infty} \int_{0}^{\mathcal{E}_{\text{max}}} m^2 \omega_s^2(\mathcal{E}) \left| \hat{C}_m(\mathcal{E}, \Omega) \right|^2 d\mathcal{E}
+ \frac{2i\zeta \omega_s^2 \mathcal{P}}{B(\Omega)} \sum_{m=-\infty}^{\infty} \frac{\text{Im} Z_k(\Omega)/k}{Z_0} \left| \tilde{\lambda}_k(\Omega) \right|^2
- \frac{2i\zeta \omega_s^2 \mathcal{P}}{B(\Omega)} \sum_{m=-\infty}^{\infty} \frac{\text{Re} Z_k(\Omega)/k}{Z_0} \left| \tilde{\lambda}_k(\Omega) \right|^2, \tag{24}
\]
where
\[
B(\Omega) = \sum_{m=1}^{\infty} \int_{0}^{\mathcal{E}_{\text{max}}} \left| \hat{C}_m(\mathcal{E}, \Omega) \right|^2 d\mathcal{E}.
\]
Below the instability threshold, i.e. \(\text{Im}\Omega = 0\), the contribution of the last term in Eq. \((24)\) must be zero as it is purely imaginary. Depending on the sign of the sum on the second line of Eq. \((24)\), the mode frequency can be shifted upwards or downwards with respect to the weighted incoherent frequency, i.e. first line of Eq. \((24)\). Thus, two modes which frequencies approach each other can couple leading to the mode-coupling instability. In some cases one can distinguish two types of instabilities: radial and azimuthal mode-coupling. The former usually appears when different azimuthal modes are well separated as a result of a small synchrotron frequency spread and two coupled modes belong to the same azimuthal mode. The latter mechanism requires significant frequency shift of the modes as well as significant synchrotron frequency spread, so the modes with different azimuthal mode numbers can couple. Both instability types can be found using self-consistent analysis. Neglecting PWD and rf nonlinearity only the classical azimuthal mode-coupling instability \([1]\) is possible (see Appendix \([A]\)).

In the case of instability, we obtain from Eq. \((24)\) the growth rate
\[
\text{Im}\Omega = - \frac{\zeta \omega_s^2}{B(\Omega) \text{Re} \Omega} \sum_{k=-\infty}^{\infty} \frac{\text{Re} Z_k(\Omega)/k}{Z_0} \left| \tilde{\lambda}_k(\Omega) \right|^2, \tag{25}
\]
which shows that for a smooth impedance, \(Z_{-k}(\Omega) \approx Z_k(\Omega)\), the spectrum of unstable modes must be an asymmetric function of frequency \(\pm f_0\). A similar conclusion was obtained in \([13]\), pointing out the possibility of a radial mode-coupling instability within a single azimuthal mode in the presence of PWD. Equation \((25)\) shows, however, that the asymmetry of the mode spectrum is the general property of unstable modes, irrespective of the single-bunch instability mechanism.

Below we will discuss the methods of computing the van Kampen modes for cases of practical applications.

### D. Methods to solve the linearized Vlasov equation

To solve the integral equation \((12)\), it is usually converted to the infinite system of equations
\[
\frac{\Omega^2}{\omega_s^2} a^n_m(\Omega) = \sum_{n'=0}^{\infty} \sum_{m'=1}^{\infty} M^n_{mm'}(\Omega) a^{n'}_{m'}(\Omega), \tag{26}
\]
where the matrix elements are defined as (see, e.g. \([2, 4]\))
\[
M^n_{mm'}(\Omega) = m^2 \delta_{mn} \int_{0}^{\mathcal{E}_{\text{max}}} \frac{\omega_s^2(\mathcal{E})}{\omega_s^2} s^{(n)}(\mathcal{E}) s^{(m)}(\mathcal{E}) d\mathcal{E}
- 2i\zeta K^n_{mm'}(\Omega). \tag{27}
\]
The system of equations \((26)\) becomes linear, if the dependence of the matrix elements \(M^n_{mm'}(\Omega)\) on \(\Omega\) can be neglected. This requires a rather smooth dependence of impedance on the frequency, so that \(Z_k(\Omega) \approx Z_k(0)\), which is valid for the SPS impedance where the bandwidth of all relevant resonant peaks is larger than the revolution frequency \(f_0\). Then, after truncation of the infinite sums over indices \(m, n, \) and \(k\), Eq. \((26)\) can be solved as a standard eigenvalue problem.

Two methods exist to compute the matrix elements depending on the choice of the orthonormal functions:
the orthogonal polynomial expansion, and the Oide-Yokoya method [8]. For the former method, the orthonormal functions for a particle distribution according to Eq. (3) can be constructed from the Jacobi polynomials \( P_n^{(α, β)}(x) \),

\[
\begin{align*}
    s_n^{(m)}(E) &= \sqrt{\frac{(2n + m + μ)}{Γ(n + m + μ)}} \frac{Γ(n + 1)Γ(n + m + μ)}{Γ(n + 1)Γ(n + m + μ)} \\
    &\times \sqrt{-\frac{2πω_α A_N dF(E)}{μ}} \frac{dE}{E_{max}} \left( \frac{E}{E_{max}} \right)^{m/2} \\
    &\times P_n^{(m, μ-1)} \left( 1 - \frac{2E}{E_{max}} \right),
\end{align*}
\]

where \( n \) is the radial mode number, and \( Γ(x) \) is the Gamma function. For the latter approach step-like functions [8] are used

\[
\begin{align*}
    s_n^{(m)}(E) &= \begin{cases} 
    \frac{1}{ΔE_n}, & E_n - \frac{ΔE_n}{2} < E ≤ E_n + \frac{ΔE_n}{2} \\
    0, & \text{elsewhere},
\end{cases}
\end{align*}
\]

where \( E_n \) is the \( n \)th mesh point on the energy grid, and \( ΔE_n \) is the thickness of the corresponding strip. Additionally, integration is approximated by a sum, so the matrix elements of the eigenvalue problem become

\[
M_{mm'}^{nn'}(Ω) = \frac{mω_α^2(E_n)}{ω_0^2} \frac{mω_β^2(E_n)}{ω_0^2} \frac{Δnn'}{δmm'} \\
- 2iζ√(ΔE_n ΔE_n') K_{mm'} (E_n, E_n', Ω).
\]

Both methods solve the linearized Vlasov equation are implemented in the MELODY code and allow the evaluation of single-bunch stability for arbitrary impedance models. We find that the Oide-Yokoya method has better convergence properties than the orthogonal polynomial expansion. This is thanks to a nonuniform mesh with respect to the energy of synchrotron oscillations \( E_n \), which allows to improve the resolution around ‘critical’ points. For example, in the case of instability with \( \text{Re}Ω = mω_α(E) \), one can expect a resonance with a characteristic width of

\[
δE ≈ \left| \text{Im}Ω \right| \left| \frac{dω_α(E)}{dE} \right| \left| E = E_n \right|.
\]

If \( δE ≪ E_{max} \), a very high-resolution mesh in \( E \) might be needed to find a converged solution.

### III. LONGITUDINAL SINGLE-BUNCH INSTABILITIES IN THE SPS

In this section, we will show first the results of stability analysis at the SPS flat top (450 GeV) and then through acceleration. The SPS has a double-rf system with frequencies of 200 MHz and 800 MHz \((n = 4)\). The main accelerator and beam parameters are listed in Table I.

Here, we will discuss a single-rf operation, while stability in a double-rf case will be addressed in Sec. IV.

It has been shown in previous studies that in a single-rf operation, the single-bunch intensity threshold is a noncontinuous and nonmonotonic function of the bunch length [18, 29]. Since then, the SPS impedance model has been further refined, and below we will show the simulation results based on this latest model (Fig. 1).

The beam stability in simulations is probed by observing the growing oscillations of the bunch position (dipole) and length (quadrupole). Even though the actual evolution of the bunch parameters depends on the initial seed at the moment of bunch generation, a fast-growing instability can still be detected. In Fig. 2, we show a stability map obtained from a scan in bunch intensity \( N_p \) (in steps of \( 2.5 × 10^{10} \)) and bunch length \( τ_σ \) (in steps of 50 ps) in simulations for \( 10^6 \) macroparticles. The initial bunches are matched to the total potential \( U_t \) including intensity effects and then tracked for \( 10^5 \) turns (about 600 synchrotron periods). The impedance model before the second long shutdown (LS2) shown in Fig. 1 was taken. In simulations, the maximum amplitude of the bunch-length oscillations \( Δτ_σ \) divided by the average bunch length \( τ_σ \) indicates the bunch stability.

The semianalytical results calculated using code MELODY with up to ten azimuthal modes are shown in Fig. 3. The scan was performed on a grid of \((E, ζ)\) values. The bunch length \( τ_σ \) and intensity \( N_p \) values were computed for each pair of parameters and then used to create an interpolated map of the corresponding growth rates. The impedance model is truncated at \( k_{max}f_0 ≈ 6.4 \text{ GHz} \), while the radial mesh is automatically refined depending on the bunch length (in some cases up to \( N_ζ ≈ 10^3 \) mesh points in the energy of synchrotron oscillations).

In general, one can see that the stable solutions (purple) correspond to BLonD simulations where bunch length stays constant (the oscillation amplitude is below the noise level \( Δτ_σ/τ_σ < 0.04 \)), while \( τ_σ \) grows for most of the unstable solutions with nonnegligible growth rates. Both methods show that there is an unstable ‘island’ approximately for intensities above \( 1.5 × 10^{11} \) and the bunch length in the range of 1.5 to 1.9 ns.

In addition, in Fig. 3 the circles indicate the cases with a nonmonotonic behavior of the synchrotron frequency amplitude. As this happens, the rate of growth changes and a stable region appears.

### TABLE I. The SPS parameters at the flat-top energy [30].

| Parameter                     | Units   | Value  |
|-------------------------------|---------|--------|
| Circumference, \( C \)        | m       | 6911.554 |
| Beam energy, \( E_0 \)        | GeV     | 450    |
| Transition Lorentz factor, \( γ_{tr} \) |            | 17.951 |
| Main harmonic number, \( h \) |         | 4620   |
| Main rf frequency, \( f_t \)  | MHz     | 200.394 |
| Main rf voltage amplitude, \( V_1 \) | MV     | 7.2     |
as a function of the energy of synchrotron oscillations. In the past, it was shown that coupling of radial modes within a single azimuthal can occur in this case [9]. We observe that the instability often is correlated with the presence of $\frac{d\omega}{dE} = 0$ inside the bunch, however, it is not always the case. This can be understood from the fact that for a particular set of bunch parameters the growth rate can be very small, and it is difficult to detect in simulations or to properly resolve in semianalytical calculations. Still, the nonmonotonic behavior of the synchrotron frequency distribution could be a useful indication that we can expect instability.

We also show in Fig. [3] with a solid white line, the threshold of loss of Landau damping (LLD) computed for a dipole mode ($m = 1$) in the same way as in [25]. The LLD threshold is a monotonic function, thus it increases for a larger bunch length. However, the actual instability threshold is significantly higher. Even though Landau damping is already lost and the coherent modes are outside the incoherent band, their amplitudes do not grow in simulations without external noise. In operation, the excitation of these modes, for example, due to some noise in the rf system, is suppressed by the phase loop. Thus, we conclude that LLD does not trigger the observed instability of a single-bunch in the SPS. Two other instability mechanisms will be discussed in the following subsection.

### A. Radial mode-coupling instability

To clarify the instability mechanism inside an unstable ‘island’ we plot van Kampen modes as a function of intensity for constant energy of synchrotron oscillations $\mathcal{E}_{\text{max}} = 0.57$ (beam parameters according to the red line in Fig. [2]) in Fig. [4]. The real part of the first six azimuthal modes (top plot) does not overlap for the entire intensity range. It means that only coupling of radial modes of the same azimuthal mode is possible for these parameters. The lowest threshold intensity is for the third azimuthal mode (green curve on the bottom plot of Fig. [4]), while for higher intensity other azimuthal modes are also unstable. In that case, we can observe ‘microwave-like’ instability as the bunch line density will be modulated by a mixture of several unstable modes.

The synchrotron frequency distribution inside the unstable ‘island’ is strongly affected by PWD and the initial rf nonlinearity plays an important role (see Fig. [5]). At the threshold of instability (orange curve) $d\omega(\mathcal{E})/d\mathcal{E}$ approaches zero and at the same time $d^2\omega(\mathcal{E})/d\mathcal{E}^2 = 0$. This shape can be achieved for double-rf operation in BSM for $n > 2$, while here it is a result of PWD. For even higher intensity the overlap of synchrotron frequencies for different $\mathcal{E}$ values becomes more significant.

In this example, instability occurs due to the coupling of two modes inside the incoherent synchrotron frequency spread. The unstable modes of the first four azimuthal modes move below the minimum incoherent frequency and remain unstable for higher intensities. The instability of higher-order azimuthal modes is suppressed above a certain intensity. For example, a zoomed region around the 5th azimuthal mode is shown in Fig. [6]. At the inten-
FIG. 4. Real (top) and imaginary (bottom) parts of van Kampen modes (light blue lines and dots) as a function of bunch intensity for $\mathcal{E}_{\text{max}} = 0.57$. Unstable modes of the first six azimuthal modes are plotted as colored lines. Beam parameters correspond to those indicated by the red line in Fig. 3.

sity of about $3.1 \times 10^{11}$ the modes are again decoupled, which results in suppression of this instability.

For completeness, the results of analysis when rf nonlinearity and PWD are neglected are also performed for the same $\mathcal{E}_{\text{max}} = 0.57$. As discussed in Sec. II C, only azimuthal mode-coupling instability is possible in this case. Its threshold is almost a factor of 5 higher than the one obtained in self-consistent calculations (Fig. 7). It means that the self-consistent analysis is crucial to predict the instability threshold for relatively long bunches.

**B. Azimuthal mode-coupling instability**

A different mechanism of instability can be observed for shorter bunches. Figure 8 shows the real part of van Kampen modes of quadrupole and sextupole azimuthal modes as a function of intensity for $\mathcal{E}_{\text{max}} = 0.12$ (black curve in Fig. 3). One can see a quadrupole mode for which Landau damping is lost. It is moving towards the frequency band of radial modes with azimuthal mode number $m = 3$. At the same time, a sextupole mode of LLD type moves down towards the radial modes with $m = 2$ (Fig. 9). This mode could emerge due to a strong PWD, which leads to the formation of a local minimum of the synchrotron frequency as a function of the synchrotron oscillation energy (blue curve in Fig. 10). Note also that rf nonlinearity does not play a significant role, as itself it results in a synchrotron frequency spread that is negligible in comparison to the one created by PWD (black curve in Fig. 10). At the intensity of $1.37 \times 10^{11}$, those two modes are coupled and thus result in azimuthal mode-coupling instability. The growth rate of the instability increases while the coupled modes remain outside the incoherent frequency bands ($N_p \lesssim 1.4 \times 10^{11}$). Then,

FIG. 5. Synchrotron frequency as a function of synchrotron-oscillation energy for different intensities and $\mathcal{E}_{\text{max}} = 0.57$. Solid lines are for single-rf potential, while dashed lines are for a parabolic-rf potential. The black curve illustrates the case without PWD.

FIG. 6. Zoomed version of in Fig. 4 (top plot). Blue lines represent van Kampen modes, while the unstable mode is shown in purple. The minimum and maximum incoherent synchrotron frequencies of the 5th azimuthal mode plotted as dashed and dotted lines, respectively.

FIG. 4. Real (top) and imaginary (bottom) parts of van Kampen modes (light blue lines and dots) as a function of bunch intensity for $\mathcal{E}_{\text{max}} = 0.57$. The minimum and maximum incoherent synchrotron frequencies of the 5th azimuthal mode plotted as dashed and dotted lines, respectively.
the growth rate is slightly suppressed and weakly depends on intensity once the coupled modes move inside the sextupole synchrotron frequency band. This is very different to the azimuthal-mode coupling instability predicted by Sacherer \[1\] for which the growth rate is a strong function of intensity. Since this instability involves the LLD mode of the sextupole synchrotron frequency band, which emerges below the minimum synchrotron frequency, it is very sensitive to the particle distribution. As the distribution modifies after start of the instability, the modes can be decoupled again leading to beam stabilization.

Equation (24) can be applied to understand why these modes move in different directions as a function of intensity. Figure 11 shows that the power spectrum of the quadrupole mode (blue curve) has a larger overlap with positive values of Im\(Z_k/k\) (Fig. 1). Thus, its frequency shift will be positive as the cumulative sum \(\sum_k \text{Im} Z_k/k \overline{\tilde{\lambda}_k}^2\) is positive. The opposite situation is true for the LLD mode with \(m = 3\), for which the maximum power spectrum is localized at the higher frequencies where Im\(Z_k/k\) is mostly negative. In this case, the mode is moving downwards in frequency as the intensity increases.

We also find that the LLD mode above the maximum incoherent frequency of the second azimuthal mode is dominated by the zeroth radial mode (dashed curve in Fig. 11), which is defined by Eqs. (23) and (28). On the contrary, the LLD mode below the third azimuthal mode is actually a superposition of a few hundred radial modes, which makes it more difficult to find a converging solution using the orthogonal polynomial expansion. To see this precisely, the computed expansion coefficients \(a_{m}^n\) from Eqs. (20) and (28) are shown in Fig. 12.

There are no LLD modes above maximum incoherent frequency for \(m > 2\). This can be understood from the fact that their power spectra are localized at higher frequencies, and the modes sample mostly the negative values of Im\(Z_k/k\). Thus, the modes are pushed inside the incoherent frequency band and remain Landau-damped.

Although the threshold of azimuthal mode-coupling instability is clearly defined \(N_p \approx 1.37 \times 10^{14}\), one can observe the modes with growth rates Im\(\Omega/\omega_0\) \(\sim 10^{-4}\) for lower intensities. In this case, \(\omega_s(E)\) has a local minimum (Fig. 10) and, therefore, radial mode-coupling instability can appear. However, these unstable modes are localized at small values of \(E\), which makes it difficult to find converged solutions as was discussed in Sec. II D. Additionally, the spectra of these modes are localized at high frequencies and they sample a rather small residual part of the real impedance (Fig. 1), which supports expectations of very small growth rates.

Above a certain intensity of about \(\sim 1.8 \times 10^{11}\), we also observe another unstable mode (blue curve in Fig. 3). It involves van Kampen modes, which are inside the synchrotron frequency spread leading to the radial mode-coupling instability. However, a strong PWD leads to an overlap of several azimuthal synchrotron frequency...
bands as the minimum synchrotron frequency is rather low (Fig. 10). The eigenvector of the unstable mode has resonant behavior for several azimuthal modes where \( m\omega_s(E) = \text{Re}\Omega \), which can be seen in the longitudinal phase space (Fig. 13). This can be called a ‘mixed’ mode-coupling instability, which has features of both radial and azimuthal mode-coupling instabilities. For even higher intensities, the PWD leads to formation of the second local minimum of the potential well and the minimum synchrotron frequency approaches zero. In that case, there are two distinct branches for the possible solutions that start from two different fixed points [8, 14]. In our calculations we show the results for the branch with the lowest minimum of the potential well and, in most cases, we observe an extremely fast instability with a growth time of a few synchrotron periods. Since the synchrotron frequency bands for different azimuthal modes significantly overlap, it is difficult to distinguish, which modes are coupled for a particular set of bunch parameters.

Figure 14 shows the results of calculations where rf nonlinearity and PWD are neglected, similarly to Fig. 7 in the previous subsection. In that case, we again find the coupling of the quadrupole and sextupole azimuthal modes, while the instability threshold is about 25% higher in comparison to the results of the self-consistent analysis. A similar dependence of the mode frequencies as a function of intensity can be achieved by using a single broad-band resonator impedance model with the resonant frequency at about 1.4 GHz. Since the zero-intensity bunch length is \( \tau_{\sigma} \approx 0.75 \text{ ns} \), we get \( \omega_r\sigma \approx 1.7 \).
As was shown in the original work of Oide and Yokoya [8], the results of self-consistent calculations are close to those obtained neglecting PWD and rf nonlinearity \( \omega, \sigma > 0.4 \).

In the recent publication [17], the instability analysis was performed for the SPS parameters and the azimuthal mode-coupling instability was found assuming a single broad-band resonator impedance with the resonant frequency of 1 GHz. Additionally, a ten times smaller rf frequency compared to actual rf frequency was used to exclude the contribution of rf nonlinearity. The computed instability threshold was slightly higher than the one obtained in simulations. Our self-consistent analysis, instead, finds the radial mode-coupling instability. Since \( \omega, \sigma \approx 4.3 \), two types of semianalytical calculations give similar results as expected. Nevertheless, the growth rates as a function of intensity could be very different.

During previous instability studies [18], macroparticle simulations were compared with dedicated measurements, where bunches with different intensities and approximately constant longitudinal emittance were injected into the SPS and accelerated following the cycle shown in Fig. [15]. In general, the results of simulations through the ramp are very close to the observations if the latest impedance model is applied [21]. However, the impact of the transfer function of the measurement system on the bunch profiles was not yet taken into account. In Fig. [16] the corrected data for bunch lengths and intensities is presented. Bunches with intensity below \( 1.2 \times 10^{11} \) remain stable for the whole cycle, which is consistent with predictions from simulations (Fig. [2]) and semianalytical analysis (Fig. [3]). There is a ‘mild’ instability for the intensity range \( 1.2 - 1.8 \times 10^{11} \) already during the acceleration. Note that the bunch length as a function of intensity approximately follows the line for the total longitudinal emittance of 0.33 eVs up to \( N_p \approx 1.8 \times 10^{11} \) (red dashed line). Bunches with even higher intensities suffer from a strong instability, which leads to the uncontrolled longitudinal emittance blowup. The presence of instability during the ramp is also consistent with simulations for the full acceleration cycle [18, 21]. Still, stability maps computed for the SPS flat-top energy are not sufficient to fully explain the observations.

To understand the instability during the acceleration we apply the eigenvalue analysis using matrix equations [29], which is implemented in the code MELODY. The main advantage of this approach is that we can obtain a snapshot of beam stability at any time during the cycle. In simulations, however, this is difficult since the energy changes every turn during acceleration, and only a very fast instability can be properly observed during the limited time of the small change in beam energy. Note
that in the stability analysis one also has to include the space-charge contribution to the total impedance model. At the injection energy of 26 GeV, the computed \( \text{Im} Z/k \) is about \(-1 \text{ \Omega}\) [18], but it drops by an order of magnitude at 100 GeV and thus can be neglected at higher energies.

As was discussed in Sec. II A, the single-bunch stability for a given impedance model depends only on few dimensionless parameters. In the single-rf case these are \( \zeta \), \( \epsilon_N \), and \( \varphi_0 \). Thanks to the systematic analysis performed by Gadioux [32], we have found that stability maps at intermediate energies through the cycle have similar nonmonotonic behavior than the one at the flat top with the unstable ‘island’. In Fig. 17 as an example, the map of bunch intensity versus the dimensionless emittance obtained from calculations with code MELODY is shown for \( \varphi_0 = 2.59 \). It illustrates the beam stability during acceleration from 200 to 300 GeV since \( \varphi_0 \) and \( V_1 \) are constant (see Fig. 15). For longitudinal emittance of 0.33 eVs, the bunches with an intensity above \( 1.2 \times 10^{11} \) will enter the unstable ‘island’ where the radial mode-coupling instability occurs. Depending on the time spent in that region, a fast enough instability will be able to develop. For even higher intensities, the bunches will become unstable even earlier in the ramp. Thus, we can conclude that the instabilities observed in the dedicated measurements during the ramp are indeed caused by the coupling of the radial modes.

IV. MITIGATION OF THE SPS INSTABILITIES

In this section, we discuss the ways applied to improve beam stability in the SPS. We saw above that the type of single-bunch instability and its threshold strongly depend on the synchrotron frequency distribution \( \omega_s(\mathcal{E}) \).

This function can be significantly modified by the PWD, determined in turn by the SPS impedance model, and a HH rf system. The two stabilization methods based on this effect are used in the SPS and considered below.

A. Impact of the SPS impedance reduction

As an injector for the Large Hadron Collider (LHC), the SPS has been majorly upgraded during the last long shutdown (LS2) to be able to produce the multi-bunch beam required for the High Luminosity (HL)-LHC [33–35]. In particular, an impedance reduction campaign [35] has been performed, involving shielding of the specific type of vacuum flanges, 200 MHz rf system upgrade, and damping of its higher-order mode around 630 MHz by a factor of three. A significant increase of instability thresholds for single-bunch and LHC-type beams in the double-rf system was expected from simulations and already observed for available intensities [36]. However, stability of a single bunch in a single-rf configuration could be degraded for some bunch parameters [29]. Indeed, the top plot in Fig. 18 demonstrates that the stable region on the left-hand side of the unstable ‘island’ now disappeared.

This effect can be again understood from the new shapes of \( \omega_s(\mathcal{E}) \) shown in Fig. 19. Bunches with an intensity of \( 2 \times 10^{11} \) and a length of 1.3 ns were stable before impedance reduction because \( \omega_s(\mathcal{E}) \) was a monotonic function (blue). After LS2 an overlap of synchrotron frequencies for different \( \mathcal{E} \) (orange) occurs, which results in the radial mode-coupling instability.
Bunch length $\tau_{4\sigma}$ (ns)

Particles per bunch $N_p \times 10^{11}$

max $\left(\Delta \tau_{4\sigma}/\tau_{4\sigma}\right)$

FIG. 18. Longitudinal stability map obtained from BLoD simulations for the SPS impedance model after LS2 (orange curve in Fig. 1) and single-rf configuration. Background colors indicate the maximum relative oscillation amplitude of the bunch length. The SPS parameters are according to Table I.

Pre-LS2, single rf
Post-LS2, single rf
Post-LS2, BSM
Post-LS2, BLM

Synchrotron frequency $\omega/\omega_0$

0.0 0.1 0.2 0.3 0.4 0.5

Energy of synchrotron oscillations $E$

FIG. 19. Synchrotron frequency distributions for different impedance models and rf configurations at the intensity $N_p = 2 \times 10^{11}$ and the bunch length $\tau_{4\sigma} = 1.3$ ns.

B. High-harmonic rf system

Double-rf operation in the SPS significantly increases the stability of all beams. The 800 MHz voltage is always applied in BSM, while there is no improvement of stability in BLM. The latter is practically limited by the extremely accurate programming of the relative phase $\Phi_4$ in Eq. (4) between the 200 MHz and 800 MHz rf systems during the entire cycle [37]. Additionally, a strong PWD, especially due to the resistive part of the SPS impedance, also introduces a shift of the synchronous phase $\Delta \phi_4$, such that the relative phase $\Phi_4$ computed without intensity effects is not optimal anymore. This leads to a nonmonotonic synchrotron frequency as a function of $E$ (Fig. 19 red) and radial mode-coupling instability can occur. To suppress this instability for given bunch parameters, the relative phase $\Phi_4$ must be precisely tuned to reestablish $\omega_s(E)$ as a monotonic function. If the intensity is very low, or the resistive impedance is negligible, the instability can be avoided, which is consistent with observations in another CERN synchrotron, the PS Booster (PSB) [38]. It is also known, that HH cavities operating in BLM allow raising the instability threshold of electron bunches (e.g., [13]). In particular, passive rf cavities can be used with a purely beam-induced rf voltage. To obtain a flat bunch profile, the amplitude and phase of the rf voltage are controlled by tuning the cavity center-frequency. In this case, a weak radial mode-coupling instability due to nonmonotonic synchrotron frequency distribution is suppressed by the synchrotron radiation damping.

The operation in BSM is much less sensitive to the relative phase between rf systems and can help to suppress the harmful impact of PWD (Fig. 19 green). This rf configuration is routinely used in the SPS to deliver high-intensity bunches to the AWAKE experiment [39]. For example, for voltage ratio of $r = 0.1$ a significant improvement in beam stability, in comparison to a single-rf case (Fig. 20). Typically, the instability threshold for short bunches gets higher for a larger ratio of rf voltages $r$, however, special attention is necessary for long bunches. These bunches may have a nonmonotonic synchrotron distribution for $r > 1/n$ and $n > 2$ even at zero intensity. The instability mechanism in this configuration is under study.
V. CONCLUSION

We have demonstrated the existence of two different mechanisms of longitudinal single-bunch instability in the SPS: radial and azimuthal mode-coupling. The results of fully self-consistent semianalytic calculations done using the code MELODY are able to explain all relevant observations in the SPS, including nonmonotonic dependence of thresholds on intensity and the presence of the unstable ‘island’. The theoretical predictions also agree well with the performed macroparticle simulations.

Previously, radial-mode coupling instability was obtained for very short, electron, bunches, when potential-well distortion plays an important role but rf nonlinearity can be neglected. We have shown that neglecting rf nonlinearity for long proton bunches leads to underestimation of the real instability threshold up to a factor of five. With intensity increase, radial mode-coupling can appear simultaneously for several azimuthal modes and then has a signature of microwave instability.

Another instability type, azimuthal mode-coupling, is also possible in the SPS, but for shorter bunches. In this case, the corresponding modes lose Landau damping (moving outside the incoherent frequency bands) and can become coupled above a certain beam intensity. This instability has a similar threshold but seems to be significantly weaker than the one obtained previously using non-self-consistent analysis (without PWD and rf nonlinearity). It can be suppressed by a small change in the particle distribution or even increase of intensity. Additionally, we also observed a ‘mixed’ mode-coupling instability, which phase-space perturbation involves several azimuthal modes, and it has higher growth rates.

Longitudinal instabilities in the SPS are cured by operating in the bunch-shortening mode of the double-rf configuration, as it removes the nonmonotonic behavior of the synchrotron frequency as a function of synchrotron oscillation energy at the bunch center. Thus, for example, applying the 4th harmonic voltage at 10% of the main-rf voltage improves the stability of short bunches by almost a factor of three, especially after the recent impedance reduction campaign. Longer bunches should be treated with caution due to nonmonotonic behavior of synchrotron frequency distribution occurring with 4th harmonic rf system.

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Appendix A: Azimuthal mode-coupling instability

The azimuthal mode-coupling instability mechanism was proposed by Sacherer in [1]. Here, we demonstrate that this is the only instability mechanism when rf nonlinearity and PWD are neglected. In this case, the functions \( I_{mk} \) according to Eq. (14) can be evaluated analytically. They become

\[
I_{mk}(\mathcal{E}) = i^n J_m \left( \frac{k \sqrt{2\mathcal{E}}}{\hbar} \right)
\]

where, \( J_m(x) \) is the Bessel function of the first kind and the order \( m \). Then, the nominator of the integrand of the spectral harmonics \( \tilde{\lambda}_k(\Omega) \) from Eq. (23) becomes

\[
\sum_{m=1}^{\infty} C_m(\mathcal{E}, \Omega) I_{mk}(\mathcal{E}) = \sum_{p=1}^{\infty} (-1)^p \left\{ \text{Re}[C_{2p}(\mathcal{E}, \Omega)] J_{2p} \left( \frac{k \sqrt{2\mathcal{E}}}{\hbar} \right) - \text{Im}[C_{2p-1}(\mathcal{E}, \Omega)] J_{2p-1} \left( \frac{k \sqrt{2\mathcal{E}}}{\hbar} \right) \right\} + i \sum_{p=1}^{\infty} (-1)^p \left\{ \text{Im}[C_{2p}(\mathcal{E}, \Omega)] J_{2p} \left( \frac{k \sqrt{2\mathcal{E}}}{\hbar} \right) + \text{Re}[C_{2p-1}(\mathcal{E}, \Omega)] J_{2p-1} \left( \frac{k \sqrt{2\mathcal{E}}}{\hbar} \right) \right\}.
\]

Based on the Bessel function property for the integer argument \( J_m(-x) = (-1)^m J_m(x) \), \( \left| \tilde{\lambda}_k(\Omega) \right|^2 \) is a symmetric function of \( k \) if either \( \text{Re}[C_m(\mathcal{E}, \Omega)] = 0 \) or \( \text{Im}[C_m(\mathcal{E}, \Omega)] = 0 \) for all azimuthal modes \( m \). Thus, \( \text{Im}\Omega = 0 \) according to Eq. (25) and the mode is stable. The instability requires that for at least one pair of azimuthal modes \( m \) and \( m + 1 \) the eigenfunctions \( \text{Re}[C_m(\mathcal{E}, \Omega)] \neq 0 \) and \( \text{Im}[C_m(\mathcal{E}, \Omega)] \neq 0 \). Then, at least one element of the sum in the integrand is non-zero and \( \left| \tilde{\lambda}_k(\Omega) \right|^2 \) is asymmetric function of \( k \), which is necessary condition for the instability. If only single azimuthal mode \( m \) has nonzero eigenfunctions \( \text{Re}[C_m(\mathcal{E}, \Omega)] \) and \( \text{Im}[C_m(\mathcal{E}, \Omega)] \), the symmetry of \( \left| \tilde{\lambda}_k(\Omega) \right|^2 \) is preserved and bunch remains stable.

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