On the Covering Number of $U_3(q)$
Michael Epstein

Abstract
The covering number, $\sigma(G)$, of a finite, noncyclic group $G$ is the least positive integer $n$ such that $G$ is the union of $n$ proper subgroups. Here we investigate the covering numbers of the projective special unitary groups $U_3(q)$, give upper and lower bounds for $\sigma(U_3(q))$ when $q \geq 7$, and show that $\sigma(U_3(q))$ is asymptotic to $q^6/3$ as $q \to \infty$.

1 Introduction
A collection $\mathcal{C}$ of proper subgroups of a group $G$ such that $G = \bigcup \mathcal{C}$ is called a cover of $G$. Any group with a finite, noncyclic homomorphic image admits a finite cover. The covering number, $\sigma(G)$, of a group $G$ which admits a finite cover is defined to be the least positive integer $n$ such that $G$ has a cover consisting of $n$ subgroups, and a cover of this size is called a minimal cover of $G$. Following [4], we adopt the convention that $\sigma(G) = \infty$ for any group $G$ which does not admit a finite cover.

A number of results on covering numbers of finite groups can be found in [1], including results on the covering numbers of finite nilpotent and supersolvable groups. In the same paper it is conjectured that the covering number of a finite, noncyclic, solvable group is one more than the order of a suitable chief factor. This conjecture was proven by Tomkinson in [21]. Specifically he proved:

Theorem 1 (Tomkinson). If $G$ is a finite, noncyclic, solvable group, then $\sigma(G) = q + 1$, where $q$ is the order of the smallest chief factor of $G$ with more than one complement.

In light of this result, attention has shifted to investigating the covering numbers of nonsolvable groups, particularly simple and almost simple groups. Many results can be found in [1] [2] [3] [9] [11] [13] [16] [8]. In this work, we consider the covering numbers of projective special unitary groups. In light of the fact that $U_2(q)$ is known to be isomorphic to the projective special linear group $L_2(q)$, it follows from the results in [3] that $\sigma(U_2(q)) = \frac{1}{2}q(q+1)$ if $q$ is even and $\sigma(U_2(q)) = \frac{1}{2}q(q+1)+1$ if $q$ is odd, with a few exceptions for small $q$ (which are handled individually in the same paper). Consequently, we will consider the covering number of $U_3(q)$. The covering number of $U_3(q)$ is known for $q \leq 5$: $U_3(2) \cong 3^2 : Q_8$ has the Klein 4-group as a homomorphic image, and so has covering number 3, and the covering numbers of $U_3(3)$, $U_3(4)$, and $U_3(5)$ were determined to be 64, 1745, and 176 respectively in [10] (and independently in [8]). In this article we investigate the covering number of $U_3(q)$ for $q \geq 7$. The main result of this paper is the following theorem:

Theorem 2. Let $q \geq 7$ be a prime power. Then

$$k(q) + q^3(q+1)^2(q-1)/3 \leq \sigma(U_3(q)) \leq q^4 + q^2 + 1 - m(q) + q^3(q+1)^2(q-1)/3$$

where

$$k(q) = \begin{cases} 
1, & \text{if } q \text{ is a power of } 3 \\
1 + q^3, & \text{otherwise},
\end{cases}$$

and

$$m(q) = \begin{cases} 
q^4/2, & \text{if } q \text{ is a power of } 2 \\
q^3 + 2q^2 - 2q - 1, & \text{if } q \text{ is a power of } 3 \\
q^3/p, & \text{if } q \text{ is a power of a prime } p \text{ such that } p = 3n + 1 \geq 5.
\end{cases}$$

In particular,

$$1 + q^3(q+1)^2(q-1)/3 \leq \sigma(U_3(q)) \leq q^4 - q^3 - q^2 + 2q + 2 + q^3(q+1)^2(q-1)/3.$$ 

An immediate consequence of this is that $\sigma(U_3(q)) \sim \frac{q^6}{3}$ as $q \to \infty$. We also prove:

Theorem 3. $\sigma(SU_n(q)) = \sigma(U_n(q))$.

In particular, this shows that the bounds given in Theorem 2 for $\sigma(U_3(q))$ are valid for $\sigma(SU_3(q))$ as well.

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The contents of this article are based on a portion of the author’s dissertation [8], submitted in partial fulfillment of the requirements for the degree of doctor of philosophy at Florida Atlantic University.
2 Preliminaries

We assume the reader is familiar with basic group theory. Throughout this article we use the notation and terminology of [B]. We also use the notation of [F] for simple groups and subgroup structures. The reader is assumed to have some understanding of projective geometry over finite fields. We refer the reader to [H] as a reference, but recall a few basic facts here. Let \( q \) be a prime power. We denote the field of order \( q \) by \( \mathbb{F}_q \), and write \( \mathbb{F}_q^* \) for the set of nonzero elements of \( \mathbb{F}_q \). If \( n \geq 2 \) is a positive integer, we denote the point of \( PG(n-1, q) \) which corresponds to a nonzero vector \( x \in \mathbb{F}_q^n \) by \([x]\). If \( S \subseteq \mathbb{F}_q^n \), we write \([S] = \{ [x] : x \in S \setminus \{0\} \} \).

Every invertible linear map \( T : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \) induces a collineation \( \tau \) of \( PG(n-1, q) \), given by \( \tau([x]) = [T(x)] \). The map which sends each invertible linear transformation to its induced collineation is a homomorphism from the general linear group \( GL_n(q) \) into the automorphism group of \( PG(n-1, q) \) whose kernel is the center of \( GL_n(q) \), which consists of the scalar transformations in \( GL_n(q) \).

For \( \alpha \in \mathbb{F}_q^* \), let \( \alpha^q = \alpha^q \). If \( n \) is a positive integer, a conjugate symmetric sesquilinear form on \( \mathbb{F}_q^n \) is a function \( f : \mathbb{F}_q^n \times \mathbb{F}_q^n \rightarrow \mathbb{F}_q \), such that for all \( x, y, z \in \mathbb{F}_q^n \) and \( \alpha, \beta \in \mathbb{F}_q^* \), \( f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z) \) and \( f(y, x) = \overline{f(x, y)} \). We say that the vectors \( x \) and \( y \) are orthogonal if \( f(x, y) = 0 \). For \( S \subseteq \mathbb{F}_q^n \), we define \( S^\perp = \{ x \in \mathbb{F}_q^n : f(x, s) = 0 \text{ for all } s \in S \} \). We note that this is a subspace of \( \mathbb{F}_q^n \) for any subset \( S \). The form \( f \) is degenerate if there is a nonzero vector \( x \) such \( \{ x \}^\perp = \mathbb{F}_q^n \). We call a vector \( x \) isotropic if \( x \in \{ x \}^\perp \).

It is known that for a nondegenerate conjugate symmetric sesquilinear form \( f \) on \( \mathbb{F}_q^n \), there is an orthonormal basis \( \{ b_1, \ldots, b_n \} \) for \( \mathbb{F}_q^n \) such that for all \( 1 \leq i, j \leq n \), \( f(b_i, b_j) = 1 \) and \( f(b_i, b_j) = 0 \) if \( i \neq j \). In this basis, for all \( \alpha_i, \beta_i \in \mathbb{F}_q^* \), \( 1 \leq i \leq n \),

\[
f \left( \sum_{i=1}^n \alpha_i b_i, \sum_{i=1}^n \beta_i b_i \right) = \sum_{i=1}^n \alpha_i \overline{\beta_i}.
\]

Conversely, given any basis \( \{ b_1, \ldots, b_n \} \), this formula defines a conjugate symmetric sesquilinear form on \( \mathbb{F}_q^n \). One consequence of this is that any two conjugate symmetric sesquilinear forms on \( \mathbb{F}_q^n \) are equivalent, differing only by a change of basis.

An isometry of \( f \) is an invertible linear map \( T \in GL_n(q^2) \) such that \( f(T(x), T(y)) = f(x, y) \) for all \( x, y \in \mathbb{F}_q^n \). Under composition these form a group called the general unitary group \( GU_n(q) \). The special unitary group \( SU_n(q) \) is the normal subgroup consisting of the isometries with determinant 1. We may then define the projective general unitary group \( PGU_n(q) \) and the projective special unitary group \( U_n(q) \) (often denoted by \( PSU_n(q) \) or \( PSU_n(q^2) \)) as the images of \( GU_n(q) \) and \( SU_n(q) \) under the homomorphism described above. The group \( U_n(q) \) has order \( q^2 n(n-1) \) \( \prod_{i=2}^{n} (q^i - (-1)^i) \) (which is \( q^3 (q^3 + 1) (q^2 - 1) / \text{gcd}(3, q + 1) \) for \( n = 3 \)) and is known to be simple except when \( (n, q) \in \{ (2, 2), (2, 3), (3, 2) \} \).

A polarity of a projective geometry is an inclusion reversing permutation of the subspaces which has order two. A nondegenerate conjugate symmetric sesquilinear form \( f \) on \( \mathbb{F}_q^n \) gives rise a polarity \( \perp \) of \( PG(n-1, q^2) \), given by \( [W]^\perp = [W^\perp] \) for any subspace \( W \) of \( \mathbb{F}_q^n \). We call the polarity induced in this manner by a conjugate symmetric sesquilinear form a unitary polarity. A point \( P \) of the projective space is said to be an absolute point of the polarity \( \perp \) if \( P \in P^\perp \). It is easily seen that if \( \perp \) is the unitary polarity corresponding to the form \( f \), then a point \( P = [x] \) of \( PG(n-1, q^2) \) is an absolute point of \( \perp \) if and only \( x \) is a nonzero isotropic vector with respect to the form \( f \). In particular, \( PG(2, q^2) \) has \( q^3 + 1 \) absolute points of the unitary polarity \( \perp \), and any line \( \ell \) of \( PG(2, q^2) \) has 1 or \( q + 1 \) absolute points, according to whether \( \ell^\perp \) is an absolute point or not. We note that the action of \( U_3(q) \) on the absolute points of \( \perp \) in \( PG(2, q^2) \) is doubly transitive, and the action on the \( q^2 (q^2 - q + 1) \) nonabsolute points is transitive. We say a triangle \( \Delta \) in a projective plane is self-polar with respect to a polarity \( \perp \) if the polarity interchanges each vertex of the triangle with its opposite side. Note that the vertices of such a triangle are necessarily nonabsolute.

For the remainder of this article we assume that \( p \) is a prime, \( \alpha \) is a positive integer, \( q = p^\alpha \), \( f \) is a conjugate symmetric sesquilinear form on \( \mathbb{F}_{p^{2\alpha}}^3 \), and \( \perp \) is the corresponding polarity on \( PG(2, q^2) \).

3 Proof of Theorem 2

We note that for the purposes of determining a covering number of a finite group one need only consider covers consisting of maximal subgroups, and therefore we first consider the maximal subgroups of $U_3(q)$. The subgroup structure of $U_3(q)$ was investigated by H. H. Mitchell [20] and R. W. Hartley [11] for $q$ odd or even respectively. They prove:

**Theorem 4** (Mitchell). Let $H$ be a subgroup of $U_3(q)$, $q$ odd. Then, $H$ is a subgroup of the stabilizer of a point and a line, a triangle, or an imaginary triangle, (items 1–4 below) or $H$ is one of the following subgroups:

1. the stabilizer of an absolute point, with order $\frac{q^3(q + 1)(q - 1)}{\gcd(3, q + 1)}$.
2. the stabilizer of a nonabsolute point, with order $\frac{q(q + 1)^2(q - 1)}{\gcd(3, q + 1)}$.
3. the stabilizer of a triangle (in $\text{PG}(2, q^2)$), with order $\frac{6(q + 1)^2}{\gcd(3, q + 1)}$.
4. the stabilizer of an imaginary triangle (i.e. a triangle in $\text{PG}(2, q^6)$ but not $\text{PG}(2, q^2)$), with order $\frac{3(q^2 - q + 1)}{\gcd(3, q + 1)}$.
5. the stabilizer of a conic, with order $q(q + 1)(q - 1)$.
6. $U_3(q_0)$, if $q = q_0^k$ with $k$ odd.
7. $\text{PGU}_3(q_0)$, if $q = q_0^k$, $k$ is odd, and 3 divides both $k$ and $q_0 + 1$.
8. the Hessian groups of order 216 (if 9 divides $q + 1$), 72 and 36 (if 3 divides $q + 1$).
9. a group of order 168, if $q = 5^k$ with $k$ odd.
10. a group of order 360, if 5 is a square in $\mathbb{F}_q$ and $\mathbb{F}_q$ does not contain a primitive third root of unity.
11. a group of order 720, if $q = 5^k$ with $k$ odd.
12. a group of order 2520, if $q = 5^k$ with $k$ odd.

As noted by King in [17], the groups of orders 168, 360, 720, and 2520 are respectively isomorphic to $L_3(2)$, $A_6$, $A_6.2$, and $A_7$. If $q$ is even, the maximal subgroups of $U_3(q)$ are as follows:

**Theorem 5** (Hartley). Let $q$ be even and $H$ be a maximal subgroup of $U_3(q)$. Then $H$ is one of the following subgroups:

1. the stabilizer of an absolute point, with order $\frac{q^3(q + 1)(q - 1)}{\gcd(3, q + 1)}$.
2. the stabilizer of a nonabsolute point, with order $\frac{q(q + 1)^2(q - 1)}{\gcd(3, q + 1)}$.
3. the stabilizer of a triangle (in $\text{PG}(2, q^2)$), with order $\frac{6(q + 1)^2}{\gcd(3, q + 1)}$.
4. the stabilizer of an imaginary triangle, with order $\frac{3(q^2 - q + 1)}{\gcd(3, q + 1)}$.
5. $U_3(q_0)$, if $q = q_0^k$, with $k$ an odd prime.
6. $\text{PGU}_3(q_0)$, if $q = q_0^3 = 2^{3k}$ with $k$ odd.
7. A group of order 36, if $q = 2$.

In the case that $q = 2$, the group of order 36 has structure $3^2 : 4$. Since the maximal subgroups of $U_3(q)$ are described in geometric terms, it will be convenient to work with a classification of the elements of $U_3(q)$ based on the geometric objects they fix. We will consider the following three types of elements in $U_3(q)$:

**Type 1**: Elements that fix an absolute point of $PG(2, q^2)$.

**Type 2**: Elements that fix a nonabsolute point of $PG(2, q^2)$ but do not fix an absolute point.

**Type 3**: Elements that do not fix any points of $PG(2, q^2)$.

Each element of $U_3(q)$ is of exactly one of these types, and we consider the elements of each type in turn.

### 3.1 Elements of type 1

We observe that the elements of the Sylow $p$-subgroups (where $p$ is the prime divisor of $q$) of $U_3(q)$ are of this type. The stabilizers in $U_3(q)$ of the absolute points of $PG(2, q^2)$ are the normalizers of the Sylow $p$-subgroup of $U_3(q)$. These Sylow $p$-subgroups are nonabelian groups of order $q^3$, with exponent $p$ if $p$ is odd, and exponent 4 if $p = 2$. Distinct Sylow $p$-subgroups of $U_3(q)$ intersect trivially, so there are a total of $q^6 - 1$ nonidentity elements in the Sylow $p$-subgroups of $U_3(q)$.

### 3.2 Elements of type 2

Here we will prove that an element of $U_3(q)$ which fixes a nonabsolute point of $PG(2, q^2)$ but does not fix any absolute points will fix exactly three points of $PG(2, q^2)$ which are the vertices of a self-polar triangle. From this it follows that the elements of type 2 in $U_3(q)$ can be covered using any collection $S$ of stabilizers of nonabsolute points such that $S$ contains the stabilizer of at least one vertex from each self-polar triangle.

As in [7] [22], for a monic polynomial $g(x) = x^n + \alpha_{n-1}x^{n-1} + \ldots + \alpha_1 x + \alpha_0 \in \mathbb{F}_{q^2}[x]$ with $\alpha_0 \neq 0$ we define $\tilde{g}(x) = x^{\alpha_0^{-1}}(\alpha_0 x^n + \alpha_1 x^{n-1} + \ldots + \alpha_{n-1} x + 1)$. The following properties are easily proven.

**Lemma 6.** Let $g, h \in \mathbb{F}_{q^2}[x]$ be monic polynomials with $g(0) \neq 0$ and $h(0) \neq 0$. Then,

1. $\tilde{g}h(x) = \tilde{g}(x)h(x)$, and
2. if $T \in GU_3(q)$ and $g(x)$ is the minimal polynomial for $T$, then $\tilde{g}(x) = g(x)$.

The following lemma is Lemma 2 of [7].

**Lemma 7.** Let $g(x) \in \mathbb{F}_{q^2}[x]$ be irreducible and monic of degree $n$ with $g(0) \neq 0$. If $\tilde{g}(x) = g(x)$, then $n$ is odd and every root $\xi$ of $g(x)$ satisfies $\xi^{q^n+1} = 1$.

**Theorem 8.** If $\Delta$ is a self-polar triangle in $PG(2, q^2)$, then there are elements of $U_3(q)$ which fix the three vertices of $\Delta$ and no other points of $PG(2, q^2)$. Conversely, if $\tau \in U_3(q)$ fixes a nonabsolute point but does not fix an absolute point of $PG(2, q^2)$, then $\tau$ fixes exactly three points of $PG(2, q^2)$ which form a self-polar triangle.

**Proof.** First, suppose that $\Delta = \{P_0, P_1, P_2\}$ is a self-polar triangle in $PG(2, q^2)$. Then there are vectors $e_0, e_1, e_2 \in \mathbb{F}_{q^2}^3$ with $P_i = [e_i]$ for $i = 0, 1, 2$ and such that $\{e_0, e_1, e_2\}$ is an orthonormal basis for $\mathbb{F}_{q^2}^3$. Choose any $\lambda, \mu \in \mathbb{F}_{q^2}$ satisfying $\lambda^{q^2+1} = \mu^{q^2+1} = 1$ so that no two of $\lambda, \mu, \mu^{-1}$ are equal. This can always be done since we may choose, for example, $\lambda = 1$ and $\mu$ to be any element of order $q + 1$ in $\mathbb{F}_{q^2}^*$. Let $T$ be the linear transformation from $\mathbb{F}_{q^2}^3$ to itself defined by $T(e_0) = \lambda e_0, T(e_1) = \mu e_1$, and $T(e_2) = (\lambda \mu^{-1}) e_2$. It is easy to see that $T \in SU_3(q)$ and that $T$ has no eigenvectors except for scalar multiples of the $e_i$. It follows that the collineation of $PG(2, q^2)$ induced by $T$ fixes $P_0, P_1$, and $P_2$ and fixes no other points of $PG(2, q^2)$.

Suppose now that $\tau \in U_3(q)$ fixes a nonabsolute point $P$ of $PG(2, q^2)$, but does not fix any absolute points. Let $T \in SU_3(q)$ be such that the collineation of $PG(2, q^2)$ induced by $T$ is $\tau$. Since $P$ is a nonabsolute point
of $PG(2, q^2)$ there is $e_0 \in \mathbb{F}_{q^2}^3$ such that $P = [e_0]$ and $f(e_0, e_0) = 1$. Note that $e_0$ is an eigenvector for $T$ corresponding to some eigenvalue $\lambda \in \mathbb{F}_{q^2}$ satisfying $\lambda^{q+1} = 1$, and that $\{e_0\}$ can be extended to an orthonormal basis $\{e_0, e_1, e_2\}$ for $\mathbb{F}_{q^2}^3$. Note that $\text{span}\{e_1, e_2\} = \{e_0\}^\perp$ is $T$-invariant, so in this basis the matrix for $T$ has the form

$$
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \alpha & \beta \\
0 & \gamma & \delta
\end{bmatrix}
$$

where $\lambda(\alpha\delta - \beta\gamma) = 1$. The characteristic polynomial for $T$ is $c(x) = (x - \lambda)g(x)$, where $g(x) = x^2 - (\alpha + \delta)x + \alpha\delta - \beta\gamma$ is the characteristic polynomial for the restriction $T|_{\{e_0\}^\perp}$ of $T$ to $\{e_0\}^\perp$.

We will now show that $g(x)$ factors over $\mathbb{F}_{q^2}$ by contradiction. Suppose that $g(x)$ is irreducible. Then $g(x)$ is the minimal polynomial for $T|_{\{e_0\}^\perp}$, and $T|_{\{e_0\}^\perp}$ preserves the restriction of the form $f$ to span$\{e_1, e_2\}$, and so is an element of $GU_2(q)$. Thus by Lemma 6 $g(x) = g(x)$. Note also that $T|_{\{e_0\}^\perp}$ is invertible, so $g(0) \neq 0$. Now by Lemma 4 the degree of $g(x)$ is odd which contradicts the fact that the degree of $g(x)$ is two. Consequently $g(x)$ factors in $\mathbb{F}_{q^2}[x]$.

Let us write $g(x) = (x - \mu)(x - \nu)$, where $\mu, \nu \in \mathbb{F}_{q^2}$. Note that neither $\mu$ nor $\nu$ can be equal to $\lambda$ or else $T$ would have a 2-dimensional eigenspace and would fix an absolute point of $PG(2, q^2)$. We must also have $\mu \neq \nu$ as follows: $\mu$ is an eigenvalue for $T|_{\{e_0\}^\perp}$ so there is a nonzero vector $v \in \text{span}\{e_1, e_2\}$ such that $T(v) = \mu v$. Since $T$ fixes no absolute points, $v$ is nonisotropic. Now $T$ preserves $\{e_0\}^\perp \cap \{v\}^\perp$, which is a 1-dimensional subspace of $\mathbb{F}_{q^2}$. Let $w$ generate this subspace. Note that since $v$ is nonisotropic and $w \in \{v\}^\perp$, $w$ is not a scalar multiple of $v$. Moreover, $w$ is also an eigenvector for $T|_{\{e_0\}^\perp}$. If $T(w) = \mu w$, then $T$ would have a 2-dimensional eigenspace, which as noted above is not the case. Therefore we can conclude that $T(w) = \nu w$ and that $\nu \neq \mu$. Since $T$ fixes no absolute points, $w$ is nonisotropic. Now, $e_0, e_0, v, w$ are eigenvectors corresponding to distinct eigenvalues of $T$ and so $\{e_0, v, w\}$ is a basis for $\mathbb{F}_{q^2}^3$. Thus, $\{P, [v], [w]\}$ is a triangle in $PG(2, q^2)$, and since the vectors $e_0, v, w$ are pairwise orthogonal, this triangle is self-polar. Finally, since $T$ fixes exactly these three points of $PG(2, q^2)$ and no others, the same holds for $\tau$.

\end{proof}

3.3 Elements of type 3

Our first goal in this section is to prove that elements of type 3 in $U_3(q)$ actually exist. Since these elements fix no points of $PG(2, q^2)$, and are therefore not covered by the stabilizers in $U_3(q)$ of the points of $PG(2, q^2)$, our second goal is to find a geometric object that these elements do fix, so that these elements may be covered using the corresponding stabilizers. As will be seen below, the desired object is a self-polar triangle in $PG(2, q^0)$.

Lemma 9. Let $r$ be the largest prime divisor of $q^2 - q + 1$. If $q > 2$, then $r \geq 5$ and $\gcd(r, q^3(q-1)(q+1)^2) = 1$.

\begin{proof}
Note that $q^2 - q + 1$ is odd and $q^2 - q + 1 = (q-2)(q+1)+3$ is divisible by 3 if and only if $q+1$ is, in which case $q^2 - q + 1 \equiv 3 \pmod{2}$. Thus $q^2 - q + 1$ is not a power of three unless $q = 2$. Therefore when $q > 2$ the largest prime divisor $r$ of $q^2 - q + 1$ is at least 5. Since $r \geq 5$ and $\gcd(q^2 - q + 1, q^3(q-1)(q+1)^2) = \gcd(3, q+1)$, it follows that $\gcd(r, q^3(q-1)(q+1)^2) = 1$.
\end{proof}

Theorem 10. If $q > 2$ then $U_3(q)$ contains elements not fixing any points of $PG(2, q^2)$.

\begin{proof}
Let $r$ be as in the previous lemma. Then $r$ divides the order of $U_3(q)$, so by Cauchy’s Theorem $U_3(q)$ contains an element $\rho$ of order $r$. Since $r$ divides neither the order of the stabilizers of the absolute points of $PG(2, q^2)$, nor the order of the stabilizers of the nonabsolute points, $\rho$ is not contained in a stabilizer of either type, and therefore fixes no point of $PG(2, q^2)$.
\end{proof}

Our next goal is to prove that each element of type 3 in $U_3(q)$ is contained in a cyclic subgroup of order $(q^2 - q + 1)/\gcd(3, q+1)$ inside the pointwise stabilizer of a triangle in $PG(2, q^0)$. Let $\tau \in U_3(q)$ be an element which does not fix any points of $PG(2, q^2)$, and let $T \in SU_3(q)$ be a linear transformation whose induced collineation is $\tau$. Note that $T$ has no eigenvalues in $\mathbb{F}_{q^2}$. Consequently, the characteristic polynomial for $T$
is irreducible of degree 3, and $T$ has three distinct eigenvalues in $F_p$, which must be of the form $\alpha, \alpha q^2, \alpha q^3$ for some $\alpha$ which is in $F_p$ but not in $F_q$. We may choose a basis $B = \{b_0, b_1, b_2\}$ such that the matrix for $T$ in this basis is in rational canonical form. That is, in this basis the matrix for $T$ is

$$
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & \beta \\
0 & 1 & \gamma
\end{bmatrix},
$$

where $\beta = -\alpha q^2 + 1 - \alpha q^2 + \alpha q^4 + 1$, and $\gamma = \alpha + \alpha q^2 + \alpha q^4$. Note that $B$ is also a basis for $F_q^3$ over $F_q$, and that we can extend $T$ to a linear transformation $\hat{T}$ from $F_q^3$ to itself by $\hat{T}(\sum_{i=0}^2 \delta_i b_i) = \sum_{i=0}^2 \delta_i T(b_i)$. We may also extend the form $f$ to a conjugate symmetric sesquilinear form $\hat{f}$ on $F_q^3$ by $\hat{f}(\sum_{i=0}^2 \delta_i b_i, \sum_{j=0}^2 \epsilon_j b_j) = \sum_{i,j=0}^2 \delta_i \epsilon_j^q \hat{f}(b_i, b_j)$. It is easy to verify that $\hat{T}$ is an isometry for the form $\hat{f}$. Let us define

$$
e_0 = D^{-1}[(\alpha q^3 + q^2 - \alpha q^2 + 2q^2) b_0 + (\alpha q^2 - \alpha^q b_0 + (\alpha q^4 - \alpha^q b_2)]$$

$$
e_1 = D^{-1}[(\alpha q^4 + 2 - \alpha q^2 + 1) b_0 + (\alpha q^2 - \alpha^q b_1 + (\alpha - \alpha^q) b_2)]$$

$$
e_2 = D^{-1}[(\alpha q^2 + 1 - \alpha q^2 + 2) b_0 + (\alpha q^2 - \alpha^2 b_1 + (\alpha q^2 - \alpha) b_2],$$

where $D = (\alpha q^4 - \alpha^q)(\alpha q^4 - \alpha)(\alpha q^2 - \alpha)$. Then $e_i$ is an eigenvector for $\hat{T}$ corresponding to the eigenvector $\alpha q^2 i$ for $i = 0, 1, 2$. Evidently, $T$ fixes each point $[e_i], i = 0, 1, 2$, but fixes no other points of $PG(2, q^6)$. Moreover, $\{e_0, e_1, e_2\}$ is a linearly independent set of vectors in $F_q^3$, and so the projective points $[e_0], [e_1], [e_2]$ in $PG(2, q^6)$ are not collinear and therefore form a triangle. Let $\ell_i$ be the line of $PG(2, q^6)$ which passes through the points of $\{[e_0], [e_1], [e_2]\} \setminus \{[e_i]\}$ for $i = 0, 1, 2$. That is, $\ell_i$ is the side of the triangle $\Delta$ which does not pass through $[e_i]$, and consequently $[e_2] \perp [e_1]$. Clearly $T$ fixes each line $\ell_i$, and since any collineation of a projective plane fixes equally many points and lines, it follows that the three are the only lines of $PG(2, q^6)$ fixed by $T$. We will now show that $\Delta$ is a self-polar triangle with respect to the polarity corresponding to $f$. We will ultimately show that each of the $[e_i]$ is a nonabsolute point, but first we will show that an odd number of them are nonabsolute. Suppose that $[e_0]$ is an absolute point. Then $[e_0] \perp$ is a line fixed by $T$ which passes though $[e_0]$ and therefore must be one of $\ell_0$ or $\ell_2$. Suppose without loss of generality that $[e_0] \perp \ell_1$. Then $[e_0]$ is the unique absolute point on $\ell_1$, and $\ell_1$ passes through $[e_2]$, so we can conclude that $[e_2]$ is a nonabsolute point. Thus $[e_2] \perp$ is a line fixed by $T$ which does not pass through $[e_2]$, and consequently $[e_2] \perp \ell_2$. This forces $[e_2] \perp$ to be $\ell_0$ which contains $[e_1]$ so $[e_1]$ is an absolute point. Thus exactly one or all three of the points $[e_0], [e_1],$ and $[e_2]$ are nonabsolute points.

Now, at least one of $[e_0], [e_1],$ and $[e_2]$ is a nonabsolute point. Relabeling the points if necessary, we may assume that $[e_0]$ is a nonabsolute point. Then,

$$\hat{f}(e_0, e_0) = \hat{f}(\hat{T}(e_0), \hat{T}(e_0)) = \hat{f}(\alpha e_0, \alpha e_0) = \alpha q^2 + 1 \hat{f}(e_0, e_0).$$

Since $\hat{f}(e_0, e_0) \neq 0$, it follows that $\alpha q^2 + 1 = 1$. Thus, since $\alpha q^4 + q^2 + 1 = 1$ and $gcd(q^2 + 1, q^4 + q^2 + 1) = q^2 - q + 1$, we must have $\alpha q^2 - q + 1 = 1$. We also must have

$$\hat{f}(e_0, e_1) = \hat{f}(\hat{T}(e_0), \hat{T}(e_1)) = \hat{f}(\alpha e_0, \alpha q^2 e_1) = \alpha q^2 + 1 \hat{f}(e_0, e_1),$$

from which it follows that $(\alpha q^2 + 1 - 1) \hat{f}(e_0, e_1) = 0$. We claim that $\alpha q^2 + 1$ does not equal 1. On the contrary, suppose that $\alpha q^2 + 1 = 1$. Then $\alpha q^4 (3, q^2 + 1) = 1$, since $\alpha q^2 - q + 1 = 1$ and $gcd(q^2 + 1, q^2 - q + 1) = gcd(3, q + 1)$. But every element of $F_q^3$ of order dividing gcd $(3, q + 1)$ is in $F_q^2$, and this contradicts the fact that $\alpha \notin F_q^2$. Consequently, $\alpha q^2 + 1 \neq 1$ and we may conclude that $\hat{f}(e_0, e_1) = 0$. A similar line of reasoning shows that $\hat{f}(e_1, e_2) = \hat{f}(e_2, e_0) = 0$. Now, since $e_0$, $e_1$, and $e_2$ are linearly independent and pairwise orthogonal, at most one of $[e_0], [e_1],$ and $[e_2]$ is an absolute point. Since an even number of them are absolute points as proven above, we conclude that none of them are. Thus for $i = 0, 1, 2, [e_i] \notin [e_i] \perp$, and since $\{[e_0] \perp, [e_1] \perp, [e_2] \perp\} = \{\ell_0, \ell_1, \ell_2\},$ we can conclude that $[e_i] \perp \ell_i$ for $i = 0, 1, 2,$ i.e. that $\Delta$ is a self-polar triangle.
Let $\zeta \in \mathbb{F}_{q^3}^*$ be an element of order $q^2 - q + 1$, and define $\hat{Z}$ to be the linear transformation from $\mathbb{F}_{q^3}^*$ to $\mathbb{F}_{q^3}^*$ defined by $\hat{Z}(e_i) = \zeta^{q^i} e_i$ for $i = 0, 1, 2$. It is easy to see that $\hat{Z}$ has determinant 1 and is an isometry with respect to $f$, and moreover, that $\hat{T} \in \langle \hat{Z} \rangle$. Now the matrix for $\hat{Z}$ in the basis $\{b_0, b_1, b_2\}$ is given by

$$A^{-1} = \begin{bmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{q^2} & 0 \\ 0 & 0 & \zeta^4 \end{bmatrix} A,$$

where

$$A = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^{2q^2} \\ 1 & \alpha^4 & \alpha^{q^4} \end{bmatrix}.$$

By a straightforward calculation, which we omit, one can verify that the entries of this matrix are all

$$\begin{bmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{q^2} & 0 \\ 0 & 0 & \zeta^4 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^{2q^2} \\ 1 & \alpha^4 & \alpha^{q^4} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^{2q^2} \\ 1 & \alpha^4 & \alpha^{q^4} \end{bmatrix}.$$

Theorem 11. If $\tau \in U_3(q)$ is an element that does not fix any points of $PG(2, q^2)$, then

1. $\tau$ fixes exactly three points of $PG(2, q^2)$ that form a triangle $\Delta_{\tau}$,

2. there is $\sigma \in U_3(q)$ which also fixes the three vertices of $\Delta_{\tau}$ such that

   (a) $|\sigma| = (q^2 - q + 1)/\gcd(3, q + 1)$, and

   (b) $\tau \in \langle \sigma \rangle$.

In particular, this theorem shows that every element of type 3 in $U_3(q)$ is contained in a cyclic subgroup generated by an element of order $(q^2 - q + 1)/\gcd(3, q + 1)$, so in covering the elements of type 3 in $U_3(q)$ we need only consider how to cover those of order $(q^2 - q + 1)/\gcd(3, q + 1)$ or $(q^2 - q + 1)/\gcd(3, q + 1)$. Our next goal is to prove Theorem 12 which shows that the unique way to do this using maximal subgroups is by all of the stabilizers of the imaginary triangles fixed by the elements of type 3 in $U_3(q)$.

Theorem 12. If $q \geq 7$ and $\sigma \in U_3(q)$ has order $(q^2 - q + 1)/\gcd(3, q + 1)$, then $\sigma$ is contained in a unique maximal subgroup of $U_3(q)$, namely the stabilizer of the imaginary triangle $\Delta_{\sigma}$.

Proof. We note that $(q^2 - q + 1)/\gcd(3, q + 1)$ divides neither $q^2(q^2 - 1)/\gcd(3, q + 1)$, the order of the stabilizers of the absolute points of $PG(2, q^2)$, nor $(q^2 + 1)^2(q - 1)/\gcd(3, q + 1)$, the order of the stabilizers of the nonabsolute points, so $\sigma$ fixes no points of $PG(2, q^2)$ and is therefore of type 3. By Theorem 11 $\sigma$ is contained in the stabilizer $S$ of a unique imaginary triangle $\Delta_{\sigma}$. It is not immediately obvious that $S$ is a maximal subgroup of $U_3(q)$ when $q \geq 7$ considering the fact that the stabilizer of an imaginary triangle is not maximal in $U_3(5)$. However we will show both that $S$ is maximal and that $S$ is the unique maximal subgroup of $U_3(q)$ containing $\sigma$ when $q \geq 7$ by showing that $S$ is the only group from the lists of subgroups in Theorems 4 and 5 which contains $\sigma$. Thus far we have already established that $\sigma$ is not contained in the stabilizer of any point, nor in the stabilizer of any imaginary triangle other than $\Delta_{\sigma}$. To see that $\sigma$ is not contained in the stabilizer of a triangle in $PG(2, q^2)$ or the stabilizer of a conic, observe that $(q^2 - q + 1)/\gcd(3, q + 1)$ divides neither $6(q + 1)^2(q - 1)/\gcd(3, q + 1)$ nor $q(q^2 - 1)$, the order of the stabilizer of a triangle or conic respectively. Nor does $(q^2 - q + 1)/\gcd(3, q + 1)$ divide $3q^3(0)(q^2 + 1)(q^2 - 1)$ where $q = q_0^k$, $k$ is odd and $k \geq 3$, and consequently $\sigma$ is not contained in a maximal subgroup of $U_3(q)$ isomorphic to $U_3(q_0)$ or $PGU_3(q_0)$. This completes the proof if $q$ is even.

For the odd case, we must show that $\sigma$ is not contained in any of the subgroups from items 8-12 of the list in Theorem 4. We do this by showing that $|\sigma|$ does not divide any of the orders of these subgroups, which are 36, 72, 168, 216, 360, 720, and 2520. If $q \geq 88$, then $(q^2 - q + 1)/\gcd(3, q + 1) > 2520$, so we need only consider the case that $7 \leq q < 88$. In this case, $q \neq 5k$ with $k$ odd, so we need not consider the subgroups isomorphic to $A_4$ or $A_4$. If $q \geq 34$, then $(q^2 - q + 1)/\gcd(3, q + 1) > 360$, and the theorem holds in this case also. It is easily checked that $(q^2 - q + 1)/\gcd(3, q + 1)$ does not divide 36, 72, 168, 216, or 360 in any of the remaining cases, i.e. when $q \in \{7, 9, 11, 13, 17, 19, 23, 27, 29, 31\}$, which completes the proof.
Finally, we note that the stabilizers of the imaginary triangles fixed by the elements of type 3 in $U_3(q)$
are the normalizers of Sylow subgroups of $U_3(q)$ and are therefore all conjugate.

**Lemma 13.** Let $q > 2$ and $\tau \in U_3(q)$ be an element that does not fix any points of $PG(2, q^2)$. Then
the stabilizer of $\Delta_\tau$ in $U_3(q)$ is the normalizer of a Sylow $r$-subgroup of $U_3(q)$, where $r$ is the largest
prime divisor of $q^2 - q + 1$.

**Proof.** Let $S$ be the stabilizer of $\Delta_\tau$ in $U_3(q)$. Note that $S$ has order $3(q^2 - q + 1)/\gcd(3, q + 1)$ by Theorem 11
if $q$ is even, and by Theorem 12 if $q$ is odd. Therefore, $\sigma \in S$ of order $(q^2 - q + 1)/\gcd(3, q + 1)$. The index of $\langle \sigma \rangle$
in $S$ is 3, which is the minimal prime divisor of the order of $S$, so $\langle \sigma \rangle$ is normal in $S$. Now, $\langle \sigma \rangle$
contains a unique Sylow $r$-subgroup $R$ of $U_3(q)$ which is characteristic in $\langle \sigma \rangle$. Since $\langle \sigma \rangle$ is normal in $S$, it follows that
$R$ is also normal in $S$.

On the other hand, suppose that $\nu \in U_3(q)$ normalizes $R$. Notice that $R$ is cyclic, and let $\rho$ be a generator
for $R$. Then for some integer $k$ and each vertex $P$ of $\Delta_\tau$, $\rho^k(P) = \nu^k(P) = \nu(P)$. So $\rho$ fixes each of the
points $\nu(P)$, where $P \in \Delta_\tau$, but the only points of $PG(2, q^2)$ fixed by $\rho$ are the vertices of $\Delta_\tau$, so it follows
that $\nu$ permutes these and thus $\nu \in S$.

It follows from the previous lemma that the stabilizers of the imaginary triangles fixed by the elements
of type 3 in $U_3(q)$ are self-normalizing and form a single conjugacy class of size $q^3(q + 1)^2(q - 1)/3$.

### 3.4 Bounds on the covering number

We are nearly ready to establish the upper and lower bounds on $\sigma(U_3(q))$ given in Theorem 2. We will need
to make use of some known results on polarity graphs, as defined below.

**Definition 14.** Let $\Pi$ be a projective plane and $\perp$ be a polarity on $\Pi$. The *polarity graph* associated to the
pair $(\Pi, \perp)$, is a graph $\mathcal{G}$ whose vertices are the points of $\Pi$, where there is an edge from $P$ to $Q$ if and only
if $P \in Q^\perp$.

We note that the triangles in the polarity graph $\mathcal{G}$ correspond to the self-polar triangles in $\Pi$. We will
need the following result, which is a combination of Theorems 1.3, 4.7, and Remark 4.6 of [19].

**Theorem 15.** Let $\perp$ be a unitary polarity on $PG(2, q^2)$, and let $\mathcal{G}$ be the corresponding polarity graph. Then
there exists a set $S$ of $m(q)$ nonabsolute points of $PG(2, q^2)$ such that the subgraph of $\mathcal{G}$ induced by $S$ is
triangle-free, where

$$m(q) = \begin{cases} q^4/2, & \text{if } q \text{ is a power of } 2 \\ q^3 + 2q^2 - 2q - 1, & \text{if } q \text{ is a power of } 3 \\ kq^4/p, & \text{if } q \text{ is a power of a prime } p \text{ such that } p = 3k \pm 1 \geq 5. \end{cases}$$

With this result on triangle-free subsets in unitary polarity graphs, we can now prove the upper and lower bounds on $\sigma(U_3(q))$ for $q \geq 7$ given in Theorem 2.

Let $S$ be a set as in the previous theorem, and let $S'$ be the set of all nonabsolute points of $PG(2, q^2)$
which are not in $S$. Note that $|S'| = q^4 - q^3 + q^2 - m(q)$. Since the subgraph of $\mathcal{G}$ induced by $S$ is triangle-free,
every self-polar triangle in $PG(2, q^2)$ has at least one vertex in $S'$. By Lemma 8 every element of $U_3(q)$
which fixes a nonabsolute point of $PG(2, q^2)$ but not an absolute point is contained in the stabilizer in $U_3(q)$
of at least one point of $S'$. Consequently the set $\mathcal{C}$ consisting of

1. the $q^3 + 1$ stabilizers of the absolute points of $PG(2, q^2)$,
2. the $q^4 - q^3 + q^2 - m(q)$ stabilizers of the nonabsolute points from $S'$, and
3. the stabilizers of the $q^3(q + 1)^2(q - 1)/3$ imaginary triangles fixed by the elements of type 3 in $U_3(q)$
is a cover of $U_3(q)$ with $|\mathcal{C}| = q^4 + q^2 + 1 - m(q) + q^3(q + 1)^2(q - 1)/3$. Note that for $q \geq 7$, $m(q) \geq q^3 + 2q^2 - 2q - 1$,
so $|\mathcal{C}| \leq q^4 - q^3 - q^2 + 2q + 2 + q^3(q + 1)^2(q - 1)/3$. This establishes the upper bound given in Theorem 2.
To justify the lower bound, let $C$ be a minimal cover for $U_3(q)$. We may without loss of generality assume that $C$ consists of maximal subgroups of $U_3(q)$. Note that by Theorem 12 and the comments following the proof of Lemma 13, $C$ must contain the stabilizers of all $q^3(q + 1)^2(q - 1)/3$ imaginary triangles fixed by the elements of $U_3(q)$ that fix no point of $PG(2, q^2)$. Since these subgroups do not contain, for example, any of the involutions in $U_3(q)$, we must have $|C| > q^3(q + 1)^2(q - 1)/3$.

For the remainder of the proof we will assume that the prime divisor $p$ of $q$ is not equal to 3. Let $\Omega$ be the set of nonidentity elements contained in the union of all of the Sylow $p$-subgroups of $U_3(q)$. We will partition the maximal subgroups of $U_3(q)$ into sets $X_i$. Specifically, let $X_1$ be the set of stabilizers in $U_3(q)$ of the absolute points of $PG(2, q^2)$, $X_2$ be the set of stabilizers of the nonabsolute points, $X_3$ be the set consisting of the stabilizers of the triangles in $PG(2, q^2)$, $X_4$ be the set of the stabilizers of the imaginary triangles fixed by the elements of type 3 in $U_3(q)$, and $X_5$ be the set of subgroups of $U_3(q)$ isomorphic to $U_3(q_0)$ or $PGU_3(q_0)$ where $q = q_0^k$ with $k$ odd and $k \geq 3$. If $p$ is odd let $X_6$ be the set of stabilizers of conics in $U_3(q)$ and $X_7$ be the set of maximal subgroups of $U_3(q)$ which are isomorphic to any of $L_3(2)$, $A_6$, $A_6^2$, $A_7$, or the Hessian groups of order 216, 72, or 36. If $p = 2$, set $X_6 = X_7 = \emptyset$. Finally, for $1 \leq i \leq 7$ let $x_i = |X_i \cap C|$ and let $a_i = \max_{H \in X_i} |H \cap \Omega|$ if $X_i \neq \emptyset$, and $a_i = 0$ otherwise. By Theorems 4 and 5 $\bigcup_{i=1}^{7} X_i$ contains all of the maximal subgroups of $U_3(q)$.

Now, since the stabilizer of each absolute point contains a unique Sylow $p$-subgroup, every subgroup in $X_1$ has exactly $q^3 - 1$ elements of $\Omega$. The stabilizer of a nonabsolute point $P$ has exactly $q^2 - 1$ elements of $\Omega$, namely the (nonidentity) elations in $U_3(q)$ whose centers are on $P^1$. Consequently $a_1 = q^3 - 1$ and $a_2 = q^3 - 1$. The stabilizer of a triangle contains no elements of $\Omega$ if $p \geq 5$, in which case $a_3 = 0$. If $p = 2$, $0 \leq a_3 < 6(q + 1)^2/\gcd(3, q + 1)$ and $q \geq 8$. In both cases $a_3 < q^3 - 1$. Since we are assuming that $p \neq 3$, the stabilizer of an imaginary triangle contains no elements of $\Omega$ and $a_4 = 0$. A subgroup of $U_3(q)$ isomorphic to $U_3(q_0)$ or $PGU_3(q_0)$ has $q_0^6 - 1$ elements of $\Omega$, but if $q = q_0^k$ with $k \geq 3$, then $q_0^6 - 1 < q^3 - 1$, and we conclude that $a_5 < q^3 - 1$. If $p = 2$ then $a_6 = a_7 = 0$. So suppose that $p \geq 5$. The stabilizer of a conic has order $q(q^2 - 1)$, so $a_6 \leq (q - 1)(q^2 - 1) < q^3 - 1$. Now since $q \geq 7$, $q^3 - 1 \geq 342$, and the only subgroup in $X_7$ with more than 342 elements of order $p$ is $A_7$ in the case that $p = 5$ which contains 504 elements of order 5. But if $p = 5$ and $q \geq 7$, then in fact $q \geq 25$. So $504 < 15624 \leq q^3 - 1$ in this case. Consequently, we conclude that in all cases, $a_i < q^3 - 1 = a_1$ for $2 \leq i \leq 7$.

Since $C$ is contained in $\bigcup_{i=1}^{7} X_i$, $C$ covers the $q^6 - 1$ elements of $\Omega$, and $a_4 = 0$, we must have

$$\sum_{i=1}^{7} a_i x_i \geq q^6 - 1.$$ 

Furthermore, since $a_i < q^3 - 1 = a_1$ for $2 \leq i \leq 7$, it follows that

$$(q^3 - 1) \sum_{i=1}^{7} x_i \geq \sum_{i=1}^{7} a_i x_i \geq q^6 - 1.$$ 

Therefore $\sum_{i=1}^{7} x_i \geq q^3 + 1$, and since $x_4 = q^3(q + 1)^2(q - 1)/3$ as noted earlier in the proof, we conclude that

$$\sigma(G) = |C| = \sum_{i=1}^{7} x_i \geq q^3 + 1 + q^3(q + 1)^2(q - 1)/3.$$ 

### 4 Proof of Theorem 3

We conclude with a proof of Theorem 3 which states that $\sigma(SU_n(q)) = \sigma(U_n(q))$. We will make use of the following lemmas:
**Lemma 16.** Let $G$ be a finite noncyclic group and $N$ be a normal subgroup of $G$. Then $\sigma(G) = \sigma(G/N)$ if and only if there is a minimal cover of $G$ which consists of subgroups containing $N$. In particular, if $N$ is contained in every maximal subgroup of $G$, then $\sigma(G) = \sigma(G/N)$.

**Proof.** Note that the inequality $\sigma(G) \leq \sigma(G/N)$ holds for any normal subgroup $N$ of $G$ (this is Lemma 2 of [4]), and let $\eta$ be the canonical map from $G$ onto $G/N$. First suppose that there exists a minimal cover $C$ of $G$ consisting of subgroups which contain $N$, and let $C' = \{\eta(H) : H \in C\}$. Then $C'$ is a cover of $G/N$ with $|C'| = |C| = \sigma(G)$. So $\sigma(G/N) \leq |C'| = \sigma(G)$, and we conclude that $\sigma(G) = \sigma(G/N)$.

Now suppose that there does not exist a minimal cover of $G$ consisting only of subgroups which contain $N$. If $G/N$ is cyclic, then by convention $SU(G/N) = \infty$, and so it follows that $\sigma(G) < \sigma(G/N)$. If not, let $C$ be a minimal cover of $G/N$, and let $C' = \{\eta^{-1}(H) : H \in C\}$. Then $C'$ is a cover of $G$ and $|C'| = |C| = \sigma(G/N)$. But since every member of $C'$ contains $N$, $C'$ cannot be a minimal cover for $G$ and so $\sigma(G) < \sigma(G/N)$.

Finally, if $N$ is contained in every maximal subgroup of $G$, choose any minimal cover of $G$ consisting of maximal subgroups. Then every member of $C$ contains $N$, so $\sigma(G) = \sigma(G/N)$ by the first part of the theorem.

**Lemma 17.** Let $H$ be a maximal subgroup of a group $G$. Then $H$ contains the center or commutator subgroup of $G$.

**Proof.** If $H$ does not contain the center $Z$ of $G$, then $H \leq HZ = G$, and $G/H = Hz/H \cong Z/(H \cap Z)$ is abelian, from which it follows that $H$ contains the commutator subgroup of $G$.

**Lemma 18.** Let $G$ be a finite nontrivial perfect group and $Z$ be the center of $G$. Then $\sigma(G) = \sigma(G/Z)$.

**Proof.** Since $G$ is perfect, no maximal subgroup contains the commutator subgroup of $G$, and therefore by the previous lemma, every maximal subgroup of $G$ contains the center of $G$. The desired result now follows from Lemma [16].

To see that $\sigma(SU_n(q)) = \sigma(U_n(q))$, first note that $SU_2(2) \cong U_2(2) \cong S_3$, so $\sigma(SU_2(2)) = \sigma(U_2(2)) = 4$. In the case $n = 2$ and $q = 3$, $SU_2(3) \cong SL_2(3)$ and $U_2(3) \cong L_2(3)$, so it follows from the results in [3] that $\sigma(SU_2(3)) = \sigma(U_2(3)) = 5$. Note that $U_3(2)$ has the Klein 4-group as a homomorphic image. Therefore $SU_3(2)$ does as well, and so $\sigma(SU_3(2)) = \sigma(U_3(2)) = 3$. So we may assume that $(n, q) \notin \{(2, 2), (2, 3), (3, 2)\}$ in which case $SU_n(q)$ is perfect, so that $\sigma(SU_n(q)) = \sigma(U_n(q))$ by Lemma [18].

We note that Lemma [18] can be applied to other families of (usually) quasisimple groups. For example, applying it to the symplectic groups $Sp_{2m}(q)$ yields $\sigma(Sp_{2m}(q)) = \sigma(S_{2m}(q))$, which holds even in the cases $(m, q) \in \{(1, 2), (1, 3), (2, 2)\}$ where $Sp_{2m}(q)$ is not perfect, since $Sp_2(2) \cong S_2(2)$, $Sp_4(2) \cong S_4(2)$, $Sp_2(3) \cong SL_2(3)$, $Sp_3(3) \cong L_2(3)$, and $\sigma(SL_2(3)) = \sigma(L_2(3)) = 5$.

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