TEST FUNCTIONS IN CONSTRAINED INTERPOLATION

JAMES PICKERING

ABSTRACT. We give a set of test functions for the interpolation problem on $H^\infty_1$, the constrained interpolation problem studied by Davidson, Paulsen, Raghupathi and Singh. We show that this set of test functions is minimal.

1. BACKGROUND AND INTRODUCTION

There has been increased interest in constrained interpolation problems recently. These problems have many of the unusual characteristics of harder interpolation problems (they generally require collections of kernels, and have interesting behavior in the case of matrix valued interpolation, much like interpolation on multiply-connected domains), but are simple enough that we can do calculations explicitly (the kernels are typically rational functions), and we can reuse much of the theory of interpolation on the unit disc.

The basic problem, as discussed in [DPRS07], is the following: Under what circumstances can we find a bounded, holomorphic function $f$ on the disc, with zero derivative at 0 (that is, $f \in H^\infty_1 := \{ g \in H^\infty : g'(0) = 0 \}$) for our function $f$, which takes prescribed values $w_1, \ldots, w_n$ at prescribed points $z_1, \ldots, z_n$. The solution, as given in [DPRS07], is analogous to the Nevanlinna-Pick theorem; such a function exists if and only if

$$
\left( (1 - w_i \bar{w}_j) k^s(z_j, z_i) \right)_{i,j=1}^n \geq 0 \quad \forall s \in S^2,
$$

where $S^2$ denotes the real 2-sphere, and the kernels $k^s$ are a particular class of kernels, parameterised by points $s$ on the sphere.

In this paper, we look at interpolation with test functions in $H^\infty_1$. The test function approach to interpolation originates with Agler (see [AM02] for a treatment of Agler’s approach to this subject, as well as for general

2000 Mathematics Subject Classification. Primary: 47A57; Secondary: 32C15, 46E20, 46E22, 47B32.

This paper is based on work contributing to the author’s PhD thesis, at the University of Newcastle-upon-Tyne, under the supervision of Michael Dritschel. This work is funded by the Engineering and Physical Sciences Research Council.

They actually give two, different, Nevanlinna-Pick type theorems, but the second is not relevant here.
background on the field), although we’ll be taking results and notation from Dritschel and McCullough’s paper ([DM07]) on the subject.

We give a set of test functions for $H^\infty_1$, broadly following the approach of [AHR07], via a Herglotz representation for $H^\infty_1$. These test functions turn out to be rational functions, and the set of test functions is parameterised by the sphere. We show, using techniques similar to those in [DM07], that our set of test functions is minimal. We give some indication of how these techniques could yield test functions for other types of constrained interpolation problem, although the theory appears to be less elegant in these situations.

We’ll also introduce the idea of differentiating kernels. These are a simple analogue of reproducing kernels, and whilst they’re not particularly interesting in and of themselves, they have proved to be a useful tool when working with problems of this sort.

2. Differentiating Kernels

In this paper, it’s convenient to introduce differentiating kernels. These are along much the same lines as reproducing kernels: We know by Cauchy’s integral formula that differentiation is a bounded linear functional on $H^2$, so for each $x \in \mathbb{D}$, and for each $i = 0, 1, 2, \ldots$, there exists some function $k(x^{(i)}, \cdot) \in H^2$ such that

$$f^{(i)}(x) = \langle f(\cdot), k(x^{(i)}, \cdot) \rangle.$$ 

The above argument also holds for $H^2(R)$, for any finitely connected planar domain $R$. In the case of $H^2(= H^2(\mathbb{D}))$, we can use Cauchy’s integral formula to calculate $k(x^{(i)}, y)$ explicitly. We note that complex contour integration is with respect to $dz = 2\pi i z ds$, where $s$ is normalised arc length measure (the measure on $H^2$). So, if we let $f \in H^\infty_2$, then

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_T \frac{f(z)}{(z-x)^{n+1}} dz$$

$$= \frac{n!}{2\pi i} \int_T 2\pi i z f(z) \frac{dz}{(z-x)^{n+1}}$$

$$= \int_T f(z) \frac{n! z}{(z-x)^{n+1}} ds$$

$$= \int_T f(z) \frac{z^{-1}}{(z^{-1} - x)^{n+1}} ds$$

$$= \int_T f(z) \frac{z^n}{(1 - x z)^{n+1}} ds$$

$$= \langle f(z), n! \frac{z^n}{(1 - x z)^{n+1}} \rangle_z.$$
Since $H^\infty$ is dense in $H^2$, this also holds for $f \in H^2$. We can now see that
\[ k(x^{(n)}, y) = n! \frac{y^n}{(1-x y)^{n+1}} = \frac{\partial^n}{\partial x^n} k(x, y) \]
where $k(x, y) = (1 - x y)^{-1}$ is the ordinary Szegö kernel. For brevity, we write $k_{x^{(i)}}$ for the function $k(x^{(i)}, \cdot)$.

If $M_f$ is the multiplication operator of $f$ on $H^\infty$, these differentiating kernels satisfy
\[ M_f^* k_{x^{(n)}} \frac{n!}{n!} = \sum_{i=1}^{n} \frac{f^{(i)}(x)^*}{i!} \frac{k^{(n-i)}}{(n-i)!}, \]
as
\[ \langle g, M_f^* k_{x^{(n)}} \frac{n!}{n!} \rangle = \langle M_f g, \frac{k_{x^{(n)}}}{n!} \rangle = \langle f g, \frac{k_{x^{(n)}}}{n!} \rangle = \frac{1}{n!} (fg)^{(n)}(x) = \frac{1}{n!} \sum_{i=1}^{n} (\cdot)^{(i)} \cdot f^{(i)}(x) g^{(n-i)}(x) = \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!} \frac{g^{(n-i)}(x)}{(n-i)!} = \langle g, \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!} \frac{k_{x^{(n-i)}}}{(n-i)!} \rangle. \]

3. Test Functions

3.1. Definitions. We refer the reader to [DM07] for more in depth discussion of test functions. For our purposes, we will need some basic definitions. Note that despite the similarity in name, these test functions are unrelated to their namesakes in distribution theory.

**Definition 3.1.** A set $\Psi$ of complex valued functions on a set $X$ is a set of test functions if:

1. For each $x \in X$,
\[ \sup \left\{ |\psi(x)| : \psi \in \Psi \right\} < 1. \]
2. If $F$ is a finite set with $n$ elements, then the unital algebra generated by $\Psi|_F$ (the restriction of $\Psi$ to $F$) is $n$-dimensional (that is, $\Psi$ separates points).

---

Many of these ideas are also covered in [AM02], although our notation is largely taken from [DM07].
**Definition 3.2.** For any collection of test functions $\Psi$, we define a set of positive kernels

$$K_\Psi := \left\{ k : x \times x \to \mathbb{C} \mid (1 - \psi(x)\overline{\psi(y)})k(x, y) \geq 0 \forall \psi \in \Psi \right\}.$$

From this, we define a normed space $H^\infty(K_\Psi)$. We say a function $f : X \to \mathbb{C}$ is in $H^\infty(K_\Psi)$ with $\|f\|_{H^\infty(K_\Psi)} \leq 1$ if

$$(1 - f(x)f(y))k(x, y) \geq 0 \forall k \in K_\Psi.$$ 

Since we have defined the unit ball of $H^\infty(K_\Psi)$, by extension we have defined the whole of $H^\infty(K_\Psi)$, and given its norm.

### 3.2. Zero Norm Probability Measures.

We start out by finding a set of test functions for $H^\infty_1$. Since test functions have norm 1 or less, the Möbius transform $m : z \to \frac{1 + z}{1 - z}$ (which takes the unit disc to the right half plane), takes test functions to functions with positive real part. Since $H^\infty_1 \subseteq H^\infty$, our test functions must have a Herglotz representation (see Theorem 1.1.19 of [AHR07]), so if $\psi$ is a test function, $f(z) = \frac{1 + \psi(z)}{1 - \psi(z)}$, and $f(0) > 0$\footnote{This condition isn’t as problematic as it looks}, then

$$f(z) = \int_T \frac{w + z}{w - z} d\mu(w)$$

for some positive measure $\mu$. For our test function to be in $H^\infty_1$, we also need that $\psi'(0) = 0$. We can see that since

$$f'(z) = \psi'(z) m'(\psi(z))$$

and $m' \neq 0$, we have that $\psi'(0) = 0$ if and only if $f'(0) = 0$.

Now,

$$f'(z) = \int_T \left( \frac{d}{dz} \frac{w + z}{w - z} \right) d\mu(w) = \int_T \frac{2w}{(w - z)^2} d\mu(w)$$

so

$$f'(0) = \int_T \frac{2}{w} d\mu(w) = 2 \int_T \overline{w} d\mu(w) = 2 \int_T w d\mu(w).$$

so if $\mu$ is a probability measure (which it will later be convenient to assume it is), then the condition that $\psi'(0) = 0$ is equivalent to the condition that $\mathbb{E}(\mu) = 0$, so $\mu$ has zero-mean. It should be noted here that requiring $\mu$ to be a probability measure is equivalent to requiring that $f(0) = 1$, or equivalently still, that $\psi'(0) = 0$.

We have proved the following:

**Theorem 3.3.** The analytic function $\psi$ has $\|\psi\|_\infty \leq 1$, $\psi(0) = 0$, and $\psi'(0) = 0$ if and only if the corresponding measure is a zero-mean probability measure.
3.3. Extreme Directions. We’ll be using a lot of techniques and definitions from [AHR07]. In that paper, they used the convention that if $X$ was a “real function space” in some sense, then $X^0$ is the set of all real functions in $X$ corresponding to holomorphic functions. Here, we’ll use the convention that $X^1$ is the set of all real functions in $X$ corresponding to holomorphic functions with zero derivative at 0, in ways that should be fairly clear.

If $L^2_\mathbb{R}(\mathbb{T})$ is defined in the usual way, then $L^2_\mathbb{R}(\mathbb{T})$ is the set of all functions in $L^2_\mathbb{R}(\mathbb{T})$ which are the real part of an analytic function with $f'(0) = 0$. It is easy to see that $(L^2_\mathbb{R}(\mathbb{T}))^\perp = \text{span} \{\text{Im} z, \text{Re} z\}$, so $L^2_\mathbb{R}(\mathbb{T})$ has co-dimension 2 in $L^2_\mathbb{R}(\mathbb{T})$. If $M_\mathbb{R}(\mathbb{T})$ is the space of finite regular real Borel measures on $\mathbb{T}$, and $C_\mathbb{R}(\mathbb{T})$ is the space of real continuous functions on $\mathbb{T}$ ($M_\mathbb{R}(\mathbb{T})$ is the dual of $C_\mathbb{R}(\mathbb{T})$, under the weak-* and uniform topologies, respectively), then as in [AHR07]:

$$M^1_\mathbb{R}(\mathbb{T}) = \{\text{Im} z, \text{Re} z\}^\perp$$

and

$$C^1_\mathbb{R}(\mathbb{T})^\perp = \text{span} \{\text{Im} z \, ds, \text{Re} z \, ds\}.$$

We also need to make use of extreme directions. We say a vector $x$ in a cone $C$ is an extreme direction in $C$ if, to have $x = x_1 + x_2$ for some $x_1, x_2 \in C$ we need $x_1 = tx$ and $x_2 = sx$ for some $s, t \geq 0$.

This allows us to formulate the following:

**Theorem 3.4.** Let $E = \{\mu \in M^1_\mathbb{R}(\mathbb{T}) : \mu \geq 0\}$. If $\mu$ is an extreme direction in $E$, then $\mu$ is supported at three or fewer points on $\mathbb{T}$.

**Proof.** Suppose $\mu$ is supported on four or more points in $\mathbb{T}$, and divide the support of $\mathbb{T}$ into four parts, $\Delta_1$ to $\Delta_4$. Let $\mu_i = \chi_{\Delta_i} \mu$, where $\chi_{\Delta}$ is the indicator function on $\Delta$. Let $\mathcal{M} = \text{span}\{\mu_1, \ldots, \mu_4\}$. The dimension of $\mathcal{M}$ is 4, and since $M^1_\mathbb{R}(\mathbb{T})$ has co-dimension 2 in $M_\mathbb{R}(\mathbb{T})$, we must have that

$$\dim (\mathcal{M} \cap M^1_\mathbb{R}(\mathbb{T})) \geq 2.$$

Therefore, there must exist a $\nu \in \mathcal{M} \cap M^1_\mathbb{R}(\mathbb{T})$ which is linearly independent of $\mu$. Since a measure $\alpha_1 \mu_1 + \cdots + \alpha_4 \mu_4 \in \mathcal{M}$ is positive whenever $\alpha_1, \ldots, \alpha_4$ are all positive, we can choose an $\epsilon > 0$ small enough that $\mu \pm \epsilon \nu \geq 0$. Therefore, $\frac{1}{2}(\mu \pm \epsilon \nu) \in E$, but

$$\mu = \frac{1}{2}(\mu + \epsilon \nu) + \frac{1}{2}(\mu - \epsilon \nu)$$

so $\mu$ is not an extreme direction in $E$. \qed

We can combine this with Theorem 3.3 on the facing page that $\mathbb{E}(\mu) = 0$ for $\mu \in E$.

---

4Remember that we can associate a harmonic function to a measure $\mu \in M_\mathbb{R}(\mathbb{T})$ via the Poisson kernel.
If $\mu$ is supported at one point of $T$, then it is clearly impossible to have $E(\mu) = 0$.

If $\mu$ is supported at two points, then $0$ must be in the convex hull of these two points (the line between them), and so the two points must lie opposite each other on the circle, and both have equal weight (if $\mu$ is to be a probability measure, this weight must be $\frac{1}{2}$).

If $\mu$ is supported at three points, then these points must be such that $0$ is in the interior of their convex hull (that is, the interior of the triangle they form; if $0$ lies on one of the lines of the triangle, then $\mu$ will only be supported on the two points at either end of the line, so this is the degenerate case we had before). We can also see that if $\mu$ is a probability measure, then its weights are uniquely determined by the points it supports – the weights are precisely the barycentric co-ordinates of 0, with respect to the three points of the triangle.

We can now note that, if a measure $\mu$ in $E$ is supported on three or fewer points, it is uniquely determined by those points, up to multiplication by a scalar. In particular, if $\mu = t_1\mu_1 + t_2\mu_2$, for some $\mu_1, \mu_2 \in E$, then $\mu_1$ and $\mu_2$ must be supported on a subset of the support of $\mu$, so must be scalar multiples of $\mu$, so we have characterised the extreme directions in $E$.

Note that we can rescale any non-zero measure in $E$ to a probability measure.

### 3.4. Some Topology

This is a convenient time to talk about the “space” of test functions. We define a set $\widetilde{\Theta}$, containing two types of element, as shown in Figure 3.1:

- diameters of the circle
- triangles, with vertices on the circumference of the circle, and the centre of the circle in their interior

To topologise this set, we say that a sequence of triangles converges to a triangle if its points converge, and if we have a sequence of triangles where the centre of the circle seems to converge to one of the lines, then we say the sequence converges to the diameter, as in Figure 3.2 on the facing page. Essentially, the third point on the triangle disappears, which makes sense,
as in the case of the zero-mean probability measures above, the weight of the third point would tend to zero— in fact, if we identify points in \( \hat{\Theta} \) with zero-mean probability measures on the circle, as in the discussion following Theorem 3.4, we can see that this topology corresponds exactly to the weak-*-topology on \( M_R(\mathbb{T}) \).

When we're dealing with test functions (we haven't defined what these are yet, but this is the most convenient place for this discussion; it may make sense to re-read this paragraph later), we can safely identify two test functions if one is a constant, unimodular multiple of the other. If \( \mu \) is a zero-mean probability measure supported on \( n \) points, then the test function \( \psi_\mu \) induced by \( \mu \) will be an \( n \)-to-one Blaschke product, with \( \psi_\mu'(0) = 0, \psi_\mu(0) = 0 \) and \( \psi_\mu(w) = 1 \) precisely when \( w \) is in the support of \( \mu \). Conversely, if we have such a function \( \psi \), then there is a corresponding zero-mean probability measure \( \mu_\psi \), with support \( \psi^{-1}(\\{1\\}) \). If we have a test function \( \psi_\mu \), corresponding to a \( \mu \in \hat{\Theta} \), then \( \tilde{\psi} = \psi_\mu(-1)\psi_\mu \) is a two- or three-to-one inner function of the type required, and since \( \tilde{\psi}(-1) = 1 \), the corresponding measure \( \tilde{\mu} \) is supported at \(-1\).

If we define an equivalence relation \( \sim \) on \( \hat{\Theta} \) by

\[
\mu_1 \sim \mu_2 \iff \psi_{\mu_1} = \lambda \psi_{\mu_2} \text{ for some } ||\lambda|| = 1
\]

then we can define \( \Theta := \hat{\Theta}/\sim \). By the above reasoning, \( \Theta \) can be thought of as the set of all triangles or diameters in \( \hat{\Theta} \) with a vertex at \(-1\) (i.e, on the leftmost point of the circle). When viewed in this sense, there is only one diameter in \( \Theta \).

It's interesting to note that \( \Theta \) is homeomorphic to \( S^2 \). We show (using a certain amount of hand-waving) that \( \Theta \) is homeomorphic to \( \mathbb{C} \cup \{\infty\} \). First, we set the diameter (in \( \Theta \)) as \( \infty \). This leaves the set of all triangles in \( \Theta \) with a vertex at \(-1\) (i.e, on the leftmost point of the circle). When viewed in this sense, there is only one diameter in \( \Theta \).

\[\text{Figure 3.2. Convergence in } \hat{\Theta}\]
3.5. The Test Functions. If we take the set of probability measures in $M_1^R(T)$, this is a subset of $E$, and in fact corresponds to $E_\rho$, in the sense of Lemma 3.4 of [AHR07], where $\rho(\mu) = \mu(T)$. We know, therefore, that $E_\rho$ is convex, and we showed before that its extreme points are given by $\hat{\Theta}$. $E_\rho$ is compact by the Banach-Alaoglu theorem, so we can apply the Krein-Milman theorem, and we see that for any $\mu \subseteq E_\rho$, there exists a probability measure $\nu_\mu$ on $\hat{\Theta}$ such that

$$\mu = \int_{\hat{\Theta}} \delta d\nu_\mu(\delta)$$

Now, probability measures in $M_1^R(T)$ correspond to analytic functions $f$ on $\mathbb{D}$ with positive real part, $f'(0) = 0$, and $f(0) = 1$, using the Herglotz representation theorem. We define

$$h_\delta(z) := \int_T \frac{w + z}{w - z} d\delta(w)$$

and

$$\psi_\delta = \frac{h_\delta - 1}{h_\delta + 1}$$

$^5$We are, confusingly but unavoidably, talking about integrating a measure-valued function (that’s $\delta$), with respect to a measure (that’s $\nu_\mu$). Also, the points in the space that $\nu_\mu$ integrates over (that’s $\hat{\Theta}$) are, themselves, measures (they’re measures on $T$)
for all \( \vartheta \in \hat{\Theta} \). Using the reasoning above, we can write
\[
 f(z) = \int_{\hat{\Theta}} \frac{w + z}{w - z} d\mu(w) \\
 = \int_{\hat{\Theta}} \frac{w + z}{w - z} \left( \int_{\hat{\Theta}} (d\vartheta(w)) d\nu(\vartheta) \right) \\
 = \int_{\hat{\Theta}} \left[ \int_{\hat{\Theta}} \frac{w + z}{w - z} d\vartheta(w) \right] d\nu(\vartheta) \\
 = \int_{\hat{\Theta}} h_\vartheta(z) d\nu(\vartheta).
\]

This gives us a new “Herglotz representation”, which I’ll call a Herglotz-Agler representation:

**Theorem 3.5.** If \( f \) is an analytic function on \( D \) with positive real part, \( f'(0) = 0 \), and \( f(0) > 1 \), then there exists some positive real measure \( \nu \) on \( \hat{\Theta} \) such that
\[
 f(z) = \int_{\hat{\Theta}} h_\vartheta(z) d\nu(\vartheta).
\]

We now prove the main result, which uses terminology from [DM07]. I’ll be using \( \Psi \) to refer to the set of test functions associated with \( \Theta \), so \( \Psi := \{ \psi_\vartheta : \vartheta \in \Theta \} \):

**Theorem 3.6.** The two spaces \( H^\infty(K_\Psi) \) and \( H^\infty(D) \) are isometrically isomorphic, that is, \( H^\infty(K_\Psi) = H^\infty(D) \) and \( \| \cdot \|_{K_\Psi} = \| \cdot \|_{H^\infty(D)} \)

**Proof.** One way is simple. Since \( \Psi \subseteq H^\infty(D) \), we know that \( K_\Psi \) contains the set \( K_1^\infty \) of reproducing kernels given in [DPRS07], so if \( \zeta \in H^\infty(K_\Psi) \), and \( \| \zeta \|_{K_\Psi} \leq 1 \), then
\[
 \left( \left[ 1 - \zeta(z)\overline{\zeta(w)} \right] k(x, w) \right) \geq 0
\]
for all \( k \in K_1^\infty \). Therefore, \( \zeta \) must be in \( H^\infty(D) \), with \( \| \zeta \|_{H^\infty(D)} \leq 1 \), so \( H^\infty(K_\Psi) \subseteq H^\infty(D) \).

Now, suppose that \( \zeta \in H^\infty(D) \) and \( \| \zeta \|_{H^\infty(D)} \leq 1 \). For now, we also suppose that \( \zeta(0) = 0 \). We let
\[
 f := \frac{1 + \zeta}{1 - \zeta}
\]
so
\[
 \zeta = \frac{f - 1}{f + 1}
\]
Hence
\[
 1 - \zeta(z)\overline{\zeta(w)} = 2 \frac{f(z) + f(w)}{(f(z) + 1)(f(w) + 1)}
\]
We know that \( f \) has positive real part, \( f(0) = 1 \) and \( f'(0) = 0 \), so we can use our Herglotz-Agler representation, Theorem 3.5, and find that there is
a measure \( \nu \) on \( \widehat{\Theta} \) such that
\[
f = \int_{\Theta} h_\delta d\nu(\delta).
\]

We can then show, using the definition of \( \psi_\delta \) and (3.1), that
\[
1 - \zeta(z)\zeta(w) = \int_{\Theta} \frac{1 - \psi_\delta(z)\psi_\delta(w)}{(f(z) + 1)(1 - \psi_\delta(z))(1 - \psi_\delta(w))(f(w) + 1)} d\nu(\delta)
\]

We know that if \( \delta \in \Theta \), then \( \psi_\delta(-1)\psi_\delta \in \Psi \). If we define a positive kernel \( \Gamma : \mathbb{D} \times \mathbb{D} \to C_b(\Psi)^* \) by
\[
\Gamma(z, w)\alpha = \int_{\Theta} \frac{\alpha(\psi_\delta(-1)\psi_\delta)}{(f(z) + 1)(1 - \psi_\delta(z))(1 - \psi_\delta(w))(f(w) + 1)} d\nu(\delta)
\]

We can then see that
\[
1 - \zeta(z)\zeta(w) = \Gamma(z, w)(1 - E(z)E(w)^* )
\]

so \( \zeta \in H^\infty(\mathcal{K}_\Psi) \) and \( ||\zeta||_{\mathcal{K}_\Psi} \leq 1 \).

To see that this holds when \( \zeta(0) \neq 0 \), simply note that
\[
1 - \left( \frac{\zeta(z) - a}{1 - a\zeta(z)} \right) \left( \frac{\zeta(w) - a}{1 - a\zeta(w)} \right) = \frac{(1 - aa)(1 - \zeta(z)\zeta(w))}{(1 - a\zeta(z))(1 - a\zeta(w))}.
\]

Therefore, \( \zeta \in H^\infty(\mathcal{K}_\Psi) \), and \( ||\zeta||_{\mathcal{K}_\Psi} \leq 1 \), so \( H^\infty(\mathcal{K}_\Psi) = H^\infty_1(\mathbb{D}) \) and \( ||\cdot||_{\mathcal{K}_\Psi} = ||\cdot||_{H^\infty_1(\mathbb{D})} \), as required. \( \square \)

3.6. Minimality. As in [DM07], we show that this set of test functions is minimal, in the sense that there is no closed subset \( C \) of \( \Psi \) so that \( C \) is a set of test functions for \( H^\infty_1 \). First, we need a lemma.

**Lemma 3.7.** If, for some measure \( \mu \) on a space \( C \), and some separable Hilbert space \( H \), \( M \in B(H) \), \( f \in B(H) \otimes L^1(\mu) \), \( f(x) \geq 0 \) \( \mu \)-almost everywhere, and \( M \geq \int_C f(x) \, d\mu(x) \), then for all scalars \( \delta > 0 \) there exists some \( C' \subseteq C \) and some scalar \( N_0 > 0 \), such that \( \mu(C \setminus C') < \delta \) and \( M \geq N_0 f(x) \).

**Proof.** Define, for \( N > 0 \), and \( \varphi \in H \),
\[
C^0_N = \{ x \in C : \langle (M - Nf(x)) \varphi, \varphi \rangle \geq 0 \}
\]
\[
C_N = \{ x \in C : M \geq Nf(x) \}
\]
\[
C_0 = \bigcup_{N>0} C_N \quad C^0_0 = \bigcup_{N>0} C^0_N
\]

We can see that, for any given \( \varphi \in H \), \( C \setminus C^0_0 \) is a \( \mu \)-null set. To see this, note that for \( x \) to be in \( C \setminus C^0_0 \), we’d need to have \( \langle M\varphi, \varphi \rangle = 0 \) but \( \langle f(x)\varphi, \varphi \rangle > 0 \), however,
\[
\langle M\varphi, \varphi \rangle \geq \left( \int_C f(x) \, d\mu(x) \varphi, \varphi \right) \geq \int_{C \setminus C^0_0} \langle f(x) \varphi, \varphi \rangle \, d\mu(x) \geq 0,
\]
which is a contradiction.

If $\Phi$ is a countable dense subset of the unit ball in $H$, we can see that
\[ C_N = \bigcap_{\varphi \in \Phi} C^{\varphi}_N \]
and so
\[ C - C_0 = \bigcap_{N > 0} (C - C_N) = \bigcap_{N > 0} \left( \bigcup_{\varphi \in \Phi} C - C^{\varphi}_N \right) \]
\[ = \bigcup_{\varphi \in \Phi} \left( \bigcap_{N > 0} C - C^{\varphi}_N \right) = \bigcup_{\varphi \in \Phi} (C - C^{\varphi}_0), \]
which is a countable union of null-sets, and so is a null set.

We can now see that as $N \to 0$, $\mu(C - C_N) \to 0$, and $M \geq N f(x)$ for all $x \in C_N$, so our result is proved. \hfill \Box

**Theorem 3.8.** No proper closed subset $C$ of $\Psi$ is a set of test functions for $H^\infty_1$.

**Proof.** Suppose, towards an eventual contradiction, that $C$ is a proper closed subset of $\Psi$ and $\psi_0 = \psi_{\delta_0} \notin C$. Since $C$ is closed, its complement is open, so we can safely assume that $\delta_0$ is not a diameter, and not an equilateral triangle.

We notice that the differentiating kernels we defined in Section 2 on page 2 are rational functions, so we can extend them to the entire Riemann sphere $C \cup \{\infty\}$. If we do this, then we see that if $x \neq 0$, $k_{\delta(x)}$ has $n + 1$ poles, all at $x^{-1}$, and $k_{0(\infty)}$ has $n$ poles, all at $\infty$.

The kernels
\[ \Delta_\delta(z, w) := (1 - \psi_{\delta}(z)\psi_{\delta}(w)^*) k(w, z) \]
are positive and have rank at most three ($k$ is just the Szegö kernel). To see this, first note that $\psi_{\delta}$ has at most three zeroes, and that at least two of them must be at zero, as $\psi_{\delta}(0) = 0$ and $\psi_{\delta}'(0) = 0$. Also note that $M_\delta$, the operator of multiplication by $\psi_{\delta}$, is an isometry on $H^2$, so $1 - M_\delta^* M_\delta$ is the projection onto
\[ \mathfrak{m}_\delta := \ker M_\delta^* = \text{Span} \{k_0, k_{\delta'}, k_{a_\delta} \} \]
where $a_\delta$ is the third zero of $\delta$ (if $\delta$ has three zeroes at zero, then $a_\delta = 0''$; if $\delta$ is a diameter, then the span doesn’t include $k_{a_\delta}$).

Now,
\[ \Delta_\delta(z, w) = \left( 1 - M_\delta M_\delta^* \right) k_w, k_z := \langle P_\delta k_w, k_z \rangle = \langle P_\delta k_w, P_\delta k_z \rangle, \]

---

6If we do the calculations, we discover that these two possibilities correspond to the test functions $z^2$ and $-z^3$, which are inconvenient corner cases.
7The fact that this is homeomorphic to $\Theta$ is not relevant here, and appears to be a coincidence.
8If $\delta$ is a diameter, then it has two zeroes, otherwise it has three.
and we can think of this as being a holomorphic function in $z$, and an antiholomorphic function in $w$. If we think of the antiholomorphic function as being in the dual of $H^2$, then

$$\Delta_\delta \in H^2 \otimes (H^2)\ast \cong B(H^2).$$

More explicitly, $\Delta_\delta$ defines an operator on $H^2$ as

$$\Delta_\delta f(z) := \int_T \Delta_\delta(z, w) f(w) \, ds(w),$$

so

$$\langle \Delta_\delta f, g \rangle = \int_T \int_T \overline{g(z)} \Delta_\delta(z, w) f(w) \, ds(w) \, ds(z)$$

$$= \int_T \int_T \overline{g(z)} \langle P_{\delta k_w}, P_{\delta k_z} \rangle f(w) \, ds(w) \, ds(z)$$

$$= \int_T \int_T \langle f(w) P_{\delta k_w}, g(z) P_{\delta k_z} \rangle \, ds(w) \, ds(z)$$

$$= \left\langle \int_T f(w) P_{\delta k_w} \, ds(w), \int_T g(z) P_{\delta k_z} \, ds(z) \right\rangle_{\mathcal{M}_\delta}$$

so we have factorised $\Delta_\delta$ as $A_\delta^* A_\delta$, where $A_\delta : H^2 \to \mathcal{M}_\delta$ is given by

$$A_\delta f := \int_T f(w) P_{\delta k_w} \, ds(w).$$

We also note now, for use later, that $A_\delta^* = I_{\mathcal{M}_\delta}$, the embedding map of $\mathcal{M}_\delta$ into $H^2$, as

$$\langle A_\delta f, g \rangle = \left\langle \int_T f(w) P_{\delta k_w} \, ds(w), \overline{g} \right\rangle_{\mathcal{M}_\delta}$$

$$= \int_T f(w) \left\langle P_{\delta k_w}, \overline{g} \right\rangle_{\mathcal{M}_\delta} \, ds(w)$$

$$= \int_T f(w) \left\langle P_{\delta k_w}, I_{\mathcal{M}_\delta} g \right\rangle \, ds(w)$$

$$= \int_T f(w) \left\langle k_w, I_{\mathcal{M}_\delta} g \right\rangle \, ds(w)$$

$$= \int_T f(w) \overline{I_{\mathcal{M}_\delta} g} \, ds(w)$$

$$= \left\langle f, I_{\mathcal{M}_\delta} g \right\rangle.$$

We choose any set of four points $F = \{z_1, z_2, z_3, z_4\} \in \mathbb{D}$, and consider the classical Nevanlinna-Pick problem, of finding a contractive function $\varphi \in H^\infty$ such that $\varphi(z_i) = \psi_0(z_i)$ for all $i$. Since $\Delta_0(z, w)$ has rank at most
three, the $4 \times 4$ matrix

$$
\left( \begin{bmatrix} 1 - \psi_0(z_i)\psi_0(z_j) \\ \cdots \\ 1 - \psi_0(z_i)\psi_0(z_j) \end{bmatrix}k(z_j, z_i) \right)_{i,j=1}^4
$$

must be singular, so the problem has a unique solution, $\varphi = \psi_0$.

Now, if we assume that $C$ is a set of test functions for $H_1^\infty$, then by Theorem 2.3 of [DM07] there must be a positive kernel $\Gamma : F \times F \to C(C)^*$ such that

$$
1 - \psi_0(z_i)\psi_0(z_j) = \Gamma(z_i, z_j)\left(1 - E(z_i)E(z_j)^*\right).
$$

Indeed, by Theorem 2.2 of [DM07], this kernel must extend to the whole of $D \times D$. We can rewrite this, in our case, by saying that there exists a measure $\mu$ on $C$, and functions $h_l(z, \cdot) \in L^2(\mu)$, for $l = 1, \ldots, 4$, such that

$$
(3.2) \quad 1 - \psi_0(z)\psi_0(w)^* = \int \sum_{i=1}^4 h_l(z, \psi)h_l(w, \psi)^*(1 - \psi(z)\psi(w)^*)d\mu(\psi).
$$

Multiplying this equation by $k(z, w)$ gives

$$
\Delta_0(z, w) = \int \sum_{i=1}^4 h_l(z, \psi)\Delta_\psi(z, w)h_l(w, \psi)^*d\mu(\psi).
$$

Since $\Delta_\psi$ is a positive kernel and a positive operator, when seen as an operator on $H^2$, as above, we can say that for all $l$,

$$
\Delta_0(z, w) \geq \int \sum_{i=1}^4 h_l(z, \psi)\Delta_\psi(z, w)h_l(w, \psi)^*d\mu(\psi).
$$

We now know, by Lemma [3.7 on page 10], that for any $\delta > 0$, there is a set $C'$, and a constant $c_0 > 0$ such that $\mu(C - C') < \delta$, and

$$
\Delta_0(z, w) \geq c_0h_l(z, \psi)\Delta_\psi(z, w)h_l(w, \psi)^*
$$

for all $\psi \in C'$.

If we use our factorisation of $\Delta_\psi$ from above, we see that

$$
A_0^*A_0 \geq c_0h_l(z, \psi)A_\psi^*A_\psi h_l(w, \psi)^*
$$

and so we can apply Douglas’ Lemma ([Dou66]), to see that the range of $h_l(\cdot, \psi)A_\psi^*$ is contained in the range of $A_0^*$, therefore, there exist constants $c_1, \ldots, c_9$ so that

$$
\begin{align*}
(3.3) \quad h_l(\cdot, \psi)k_0 & = c_1k_0 + c_2k_0 + c_3k_0 \\
(3.4) \quad h_l(\cdot, \psi)k_{0'} & = c_4k_0 + c_5k_0 + c_6k_0 \\
(3.5) \quad h_l(\cdot, \psi)k_{\epsilon_0} & = c_7k_0 + c_8k_0 + c_9k_0.
\end{align*}
$$

By letting $\delta$ go to zero, we see that these equations must hold for $\mu$-almost-all $\psi \in \Psi$.

Equation (3.3) tells us that $h_l(\cdot, \psi) = c_1k_0 + c_2k_0 + c_3k_0$, as $k_0$ is constant. We can also see that $h_l(\cdot, \psi)$ must extend meromorphically to the Riemann sphere, as these kernels do so. Equation (3.4) tells us that $c_2 = 0$, as otherwise
the left hand side of the equation has a triple pole at $\infty$, but the right hand side has at most only a double pole.

We consider equation (3.5) in three cases. Firstly, if $\psi$ has only two zeroes, there is no equation (3.5), so $h_1(\cdot, \psi) = c_1 + c_3 k_{a_0}$.

If $a_\psi \neq 0$, then

$$h_1(y, \psi)k_{a_\psi}(y) = c_7 k_0(y) + c_8 k_0'(y) + c_9 k_{a_0}(y)$$

$$\left[ c_1 + c_3 \frac{1}{1 - \alpha_0 y} \right] \frac{1}{1 - \alpha_\psi y} = c_7 + c_8 y + c_9 \frac{1}{1 - \alpha_0 y}$$

$$c_1 + c_3 \frac{1}{1 - \alpha_0 y} = c_7 - c_7 \alpha_\psi y + c_8 y - c_8 \alpha_\psi y^2 + \frac{c_9 - c_9 \alpha_\psi}{1 - \alpha_0 y}$$

$$c_1 - c_1 \alpha_0 y + c_3 = \begin{cases} c_7 - c_7 \alpha_\psi y - c_7 \alpha_0 y + c_7 \alpha_\psi \alpha_0 y^2 \\ + c_8 y - c_8 \alpha_\psi y^2 - c_8 \alpha_0 y^2 + c_8 \alpha_\psi \alpha_0 y^3 \end{cases} + c_9 - c_9 \alpha_\psi y$$

Looking at the $y^3$ coefficient tells us that $c_8 = 0$, looking at the $y^2$ coefficient then tells us that $c_7 = 0$, and so comparing the constant and $y$ coefficients, we see that

$$\begin{cases} c_1 + c_3 & = c_9 \\ c_1 \alpha_0 & = c_9 \alpha_\psi \\ c_1 & = c_9 \left( \frac{\alpha_\psi}{\alpha_0} \right) \\ c_3 & = c_9 \left( 1 - \frac{\alpha_\psi}{\alpha_0} \right) \end{cases}$$

and so

$$h_1(y, \psi) = c_9 \left( \frac{\alpha_\psi}{\alpha_0} + \frac{1}{1 - \alpha_\psi y} \right) \frac{1}{1 - \alpha_0 y} = c_9 \frac{1 - \alpha_\psi y}{1 - \alpha_0 y}$$

Alternately, if $a_\psi = 0$, then equation (3.5) becomes

$$h_1(y, \psi) y^2 = c_7 + c_8 y + c_9 \frac{1}{1 - \alpha_0 y}$$

$$c_1 y^2 + c_3 \frac{y^2}{1 - \alpha_0 y} = c_7 + c_8 y + c_9 \frac{1}{1 - \alpha_0 y}$$

$$c_1 y^2 - c_1 \alpha_0 y^3 + c_3 y^2 = c_7 + c_7 \alpha_0 y + c_8 y + c_8 \alpha_0 y^2 + c_9$$

Looking at the $y^3$ coefficient tells us that $c_1 \alpha_0 = 0$, so $c_1 = 0$. The $y^2$ terms then tell us that $c_3 = c_8 \alpha_0$. We then see that

$$h_1(y, \psi) = c_3 \frac{1}{1 - \alpha_0 y} = c_3 \frac{1 - \alpha_\psi y}{1 - \alpha_0 y}$$

as before.
Combining these consequences of equations (3.3)-(3.5) with (3.2) gives a more explicit realisation than the one in (3.2); that is,

\[
1 - \psi_0(z)\psi_0(w)^* = \\
\sum_{l=1}^{4} \left( \alpha_l + \frac{\beta_l}{1 - a_0 z} \right) \left( \overline{\alpha_l} + \frac{\overline{\beta_l}}{1 - \overline{a_0 w}} \right) (1 - \psi_0(z)\psi_0(w)^*) \\
+ \int_{C \setminus \{\infty\}} c(\psi) \left( \frac{1 - a_\psi z}{1 - a_0 z} \right) \left( \frac{1 - a_\psi w}{1 - a_0 w} \right) (1 - \psi(z)\psi(w)^*) d\mu(\psi)
\]

for some positive \(c \in L^1(\mu)\) and some \(\alpha_1, \beta_2, \ldots, a_4, \beta_4 \in \mathbb{C}\). We know that the \(\psi_s\) are Blaschke products, and we know their roots, so we can write this even more explicitly as

\[
1 - z^2\overline{w}^2 \frac{z - a_0}{1 - a_0 z} \frac{\overline{w} - \overline{a_0}}{1 - \overline{a_0 w}} = \\
\sum_{l=1}^{4} \left( \alpha_l (1 - \overline{a_0 z}) + \beta_l \right) \left( \overline{\alpha_l} (1 - a_0 \overline{w}) + \overline{\beta_l} \right) (1 - z^2\overline{w}^2) \\
+ \int_{C \setminus \{\infty\}} c(\psi) \left( (1 - \overline{a_0 z}) (1 - a_\psi \overline{w}) - z^2\overline{w}^2 (z - a_\psi) (\overline{w} - \overline{a_\psi}) \right) d\mu(\psi)
\]

If we multiply both sides by \((1 - \overline{a_0 z}) (1 - a_0 \overline{w})\) we get

\[
(1 - \overline{a_0 z}) (1 - a_0 \overline{w}) - z^2\overline{w}^2 (z - a_0) (\overline{w} - \overline{a_0}) = \\
\sum_{l=1}^{4} \left( \alpha_l (1 - \overline{a_0 z}) + \beta_l \right) \left( \overline{\alpha_l} (1 - a_0 \overline{w}) + \overline{\beta_l} \right) (1 - z^2\overline{w}^2) \\
+ \int_{C \setminus \{\infty\}} c(\psi) \left( (1 - \overline{a_0 z}) (1 - a_\psi \overline{w}) - z^2\overline{w}^2 (z - a_\psi) (\overline{w} - \overline{a_\psi}) \right) d\mu(\psi)
\]

which we can expand to get

\[
1 - \overline{a_0 z} - a_0 \overline{w} + |a_0|^2 z\overline{w} - z^2\overline{w}^2 + a_0 z^2\overline{w}^2 + a_0^{-2} z^2 \overline{w}^2 - |a_0|^2 z^2\overline{w}^2 = \\
\sum_{l=1}^{4} \left( |\alpha_l| + |\beta_l| \right)^2 \\
-(\alpha_l + \beta_l) |\alpha_l| a_0 \overline{w} + (\alpha_l + \beta_l) |\alpha_l| a_0 z^2 \overline{w}^2 \\
-(\overline{\alpha_l} + \overline{\beta_l}) |\overline{\alpha_l}| a_0 \overline{w} + (\overline{\alpha_l} + \overline{\beta_l}) |\alpha_l| a_0 z^2 \overline{w}^2 \\
+ |\alpha_l|^2 |a_0|^2 z\overline{w} - |\alpha_l|^2 |a_0|^2 z^2 \overline{w}^2 \\
+ \int_{C \setminus \{\infty\}} c(\psi) \left( 1 - \overline{a_\psi z} - a_\psi \overline{w} + |a_\psi|^2 z\overline{w} - z^2 \overline{w}^2 + a_\psi z^2 \overline{w}^2 + a_\psi^{-2} z^2 \overline{w}^2 - |a_\psi|^2 z^2 \overline{w}^2 \right) d\mu(\psi)
\]
To get our contradiction, we look at the $\overline{w}$, $z^3\overline{w}$ and $z^2\overline{w}$ coefficients of this equation, i.e,

\[(3.6)\quad a_0 = \sum_{l=1}^{4} (\alpha_l + \beta_l)\overline{a}_0 + \int_{C\backslash[0\infty]} c(\psi)a_\psi \, d\mu(\psi)\]

\[(3.7)\quad 1 = \sum_{l=1}^{4} |\alpha_l|^2 |a_0|^2 + \int_{C\backslash[0\infty]} c(\psi)\, d\mu(\psi)\]

\[(3.8)\quad |a_0|^2 = \sum_{l=1}^{4} |\alpha_l + \beta_l|^2 + \int_{C\backslash[0\infty]} c(\psi) |a_\psi|^2 .\]

We can easily see that (3.6) implies

\[|a_0|^2 = \left| \sum_{l=1}^{4} (\alpha_l + \beta_l)\overline{a}_0 + \int_{C\backslash[0\infty]} c(\psi)a_\psi \, d\mu(\psi) \right|^2\]

and if we define a Hilbert space $H = \mathbb{C}^4 \oplus L^2(\mu)$, then we can rewrite this as

\[(3.9)\quad |a_0|^2 = \left| \left( \frac{(a_0, \overline{a}_0)}{c(\psi)^{1/2}} \right) \left( \frac{(\overline{a}_0 + \beta_0)}{c(\psi)^{1/2} \overline{a}_0} \right) \right|^2\]

and apply the Cauchy-Schwarz inequality, so

\[|a_0|^2 \leq \left( \sum_{l=1}^{4} |\alpha_l|^2 |a_0|^2 + \int_{C\backslash[0\infty]} c(\psi)\, d\mu(\psi) \right) \left( \sum_{l=1}^{4} |\alpha_l + \beta_l|^2 + \int_{C\backslash[0\infty]} c(\psi) |a_\psi|^2 \right)\]

\[= 1 \text{ by (3.7)} = |a_0|^2 \text{ by (3.8)} = |a_0|^2 .\]

Further, the two vectors in (3.9) are linearly independent, as $\overline{a}_0$ is the complex conjugate of the third root of $\psi$, which is different for each $\psi$, so the inequality is strict, which is a contradiction.\(\Box\)

Remark 3.9. Clearly, this proof is not very good. It would seem more natural to use the fact that the Herglotz-Agler representation from Theorem 3.5 was parameterised by extreme measures, although there is no obvious way to do this.

---

This is a slight oversimplification. If $c(\psi)$ is non-zero at exactly one point $\psi$ (and that $\psi$ is singular with respect to $\mu$), then this argument doesn’t hold. However, a simple calculation by equating coefficients (which is omitted) shows that if the vectors are linearly dependent, so $c(\psi)$ is non-zero at exactly one point, then that point must be $\psi_0 \not\in C$, which is also a contradiction.
The fact that the minimum set of test functions is parameterised by the sphere is interesting, as the set of kernels given in [DPRS07] is also parameterised by the sphere, and conjectured to be minimal (that paper contains some partial results, towards this aim).

Similarly, Abrahamse gave (in [Abr79]) a set of kernels corresponding to interpolation on a multiply connected ($n$-holed) domain, parameterised by the $n$-torus, and conjectured that this set of kernels was minimal (there are some partial results in this direction in [BC96]; in [DM07] and [Pic08], the authors give sets of test functions for $n$-holed domains, which are also parameterised by the $n$-torus, and minimal $^{10}$.

It’s not clear whether there is some sort of duality between minimal sets of test functions and minimal sets of kernels. A possible counterexample to such a duality is the bidisc; it’s well known (see for example [AM02]) that only two test functions are needed for the bidisc, whereas in [MP02], the authors conjecture that infinitely many kernels are required.

3.7. Generalisation. It’s worth briefly discussing how these ideas could be generalised to other spaces. In [Rag08], the author looks at spaces of the form $^{11}C + BH^{\infty}$. These spaces are a natural generalisation of the space $H^{\infty}$, and the author provides a generalisation of the Nevanlinna-Pick theorem from [DPRS07].

We used the Herglotz representation trick to turn a linear equation ($f'(0) = 0$) into a constraint on probability measures (that they have zero mean), found the extreme points of the set of constrained probability measures, and then used those extreme probability measures to generate our test functions.

Suppose we want to apply these techniques to $C + BH^{\infty}$. It’s fairly clear that we can come up with a set of linear equations that all functions in $C + BH^{\infty}$ must satisfy (functions must be constant at zeroes of $B$, and have a prescribed number of zero derivatives at repeated zeroes of $B$). It should also be fairly easy to turn this set of linear equations into a set of constraints on probability measures.

The difficulty comes when we calculate extreme points. Theorem 3.4 should work well enough, so if we have $n$ independent equations, we can say that extreme measures are supported on at most $2n + 1$ points. However, the reasoning that followed Theorem 3.4 (which showed precisely which

---

$^{10}$ The main aim of [Pic08] was not to investigate test functions, although Note 2.10 of [Pic08] gives the set of test functions, and the results of Section 5.2 of [Pic08] are broadly analogous to the results used in the proof of Theorem 3.8 in this paper, and in Proposition 5.3 of [DM07].

$^{11}$ Here, $B$ is a Blaschke product

$^{12}$ The author only generalises the first form here; a generalisation of the second form is given in [BBT08].
such measures would give functions in $H^\infty$) relied on a geometric interpretation of the probability constraint, which doesn’t obviously generalise to other types of constraint.

If we can calculate the extreme measures, these should generate test functions in precisely the same way as above. However, the proof that the test functions we’ve found are minimal is heavily dependent – perhaps overly dependent – on explicit calculations using the test functions.

References

[Abr79] Marine B. Abrahamse, The Pick interpolation theorem for finitely connected domains, Michigan Math. J. 26 (1979), no. 2, 195–203.

[AHR07] Jim Agler, John Harland, and Benjamin Raphael, Classical function theory, operator dilation theory, and machine computations on multiply connected domains, Memoirs of the American Mathematical Society, AMS Bookstore, 2007.

[AM02] Jim Agler and John E. McCarthy, Pick interpolation and Hilbert function spaces, Graduate Studies in Mathematics, AMS, 2002.

[BBT08] Joseph A. Ball, Vladimir Bolotnikov, and Sanne Ter Horst, A constrained Nevanlinna-Pick interpolation problem for matrix-valued functions, ArXiv: 0809:2345, September 2008.

[BC96] Joseph A. Ball and Kevin F. Clancey, Reproducing kernels for Hardy spaces on multiply connected domains, Integral Equations and Operator Theory 25 (1996), 35–57.

[DM07] Michael A. Dritschel and Scott McCullough, Test functions, kernels, realizations and interpolation, Operator Theory, Structured Matrices and Dilations: Tiberiu Constantinescu Memorial Volume, pp. 153–179, Theta Foundation, Bucharest, 2007.

[Dou66] Ronald G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, Proceedings of the American Mathematical Society 17 (1966), no. 2, 413–415.

[DPRS07] Kenneth R. Davidson, Vern Paulsen, Mrinal Raghupathi, and Dinesh Singh, A constrained Nevanlinna-Pick interpolation problem, ArXiv:0711.2032, Nov 2007.

[MP02] Scott McCullough and Vern Paulsen, $\mathcal{C}$-envelopes and interpolation theory, Indiana University Mathematics Journal 51 (2002), no. 2, 479–505.

[Pic08] James Pickering, Counterexamples to rational dilation on symmetric multiply connected domains, Complex Analysis and Operator Theory (2008), Online edition, yet to appear in print.

[Rag08] Mrinal Raghupathi, Nevanlinna-Pick interpolation for $\mathcal{C} + BH^\infty$, ArXiv:0803.1278, 2008.

James Pickering, Department of Mathematics, University of Newcastle-upon-Tyne, Newcastle-upon-Tyne, NE8 3JT, United Kingdom

E-mail address: james.pickering@ncl.ac.uk

URL: http://www.jamespic.me.uk