The shift map on Floer trajectory spaces

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Abstract
In this article we give a uniform proof why the shift map on Floer homology trajectory spaces is scale smooth. This proof works for various Floer homologies, periodic, Lagrangian, Hyperkähler, elliptic or parabolic, and uses Hilbert space valued Sobolev theory.

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1 Introduction
In Morse and Floer homology gradient flow lines play a crucial role. These locally lie in spaces of maps $\mathbb{R} \to S$ of the real line into a vector space. The
shift map $\Psi : \mathbb{R} \times \text{Map}(\mathbb{R}, S) \to \text{Map}(\mathbb{R}, S)$ is defined by $(\tau, v) \mapsto \tau_* v$, where $(\tau_* v)(t) = v(\tau + t)$ for $t \in \mathbb{R}$. After endowing the mapping space with some topology the shift map has terrible properties.

By differentiating the shift map with respect to the first variable one loses a derivative. Moreover, the shift map is merely continuous in the compact open topology, but not in the norm topology. In the last two decades this led Hofer, Wysocki, and Zehnder [HWZ17] to the discovery and exploration of a new notion of smoothness in infinite dimensions called *scale smoothness* or *sc-smoothness*. See also the article by Fabert et al [FFGW16] for the crucial importance of the new notion in various fields of symplectic topology or the article by Hofer [Hof17] for a survey.

For Morse homology the vector space $S$ is finite dimensional. However, for Floer homology the vector space $S$ will be infinite dimensional.

Floer theory arises in many features. In the study of periodic orbits of Hamiltonian systems one looks at a closed string version of Floer homology, namely, periodic Floer homology [Flo88b, Flo89]. In this case the vector space consists of loops in a finite dimensional symplectic vector space $S$. The study of gradient flow lines is based on the study of an elliptic pde on the cylinder.

In the study of Lagrangian intersection points one looks at an open string analogon, namely Floer homology with Lagrangian boundary conditions [Flo88a]. In this case the vector space $S$ consists of paths in a symplectic vector space which start and end in a Lagrangian subspace. The study of gradient flow lines is based on the study of an elliptic pde on the strip.

The second author established Morse homology for the heat flow [Web13b, Web13a] which led to the study of a parabolic pde on the cylinder. This was an essential ingredient in the joint proof with Dietmar Salamon [SW06] of the famous Viterbo isomorphism [Vit98].

Motivated by Donaldson-Thomas gauge theory in higher dimensions [DT98] Hohloch, Noetzel, and Salamon [HNS09] discovered a hyperkähler version of Floer homology which leads to dynamics in higher dimensional time, see as well Ginzburg and Hein [GH12]. In this setup the vector space $S$ consists of maps from a three-dimensional closed manifold into hyperkähler space.

Although the shift map has terrible properties in the first variable, it is quite innocent in the second variable, namely, it is linear. People familiar with Sobolev theory [AF03] know what a pain products are. Thanks to linearity in the second variable this difficulty is absent.

It is generally believed that the moduli space of (unparametrized) gradient flow lines, namely gradient trajectories modulo shift, can be interpreted as the zero set of a section from an sc-manifold into an sc-bundle over it. If the gradient flow lines are allowed to be broken, then the sc-manifold has to be replaced by an M-polyfold. In the Morse case this is explained by Albers and Wysocki [AW13]. See as well Wehrheim [Weh12] for the case of periodic Floer homology.

A crucial ingredient to construct this sc-manifold is the scale smoothness of the shift map. In view of the various Floer homologies defined on different spaces displaying elliptic and parabolic features one might worry that this property has to be proved for each Floer theory individually. The purpose of this paper is
to give a uniform treatment of the scale smoothness of the shift map which is applicable for all the above mentioned Floer homologies. This crucially uses the linearity of the shift map in the second variable.

In order to formulate this uniform treatment we use Hilbert space valued Sobolev spaces. This idea is not new in Floer homology, but has successfully been used by Robbin and Salamon [RS95] in the treatment of the spectral flow. Moreover, this gives interesting connections between Floer homology and interpolation theory. For a detailed treatment of interpolation theory see e.g. [Tri78].

The philosophy behind this comes from scale Morse homology, namely, a still to be developed Morse homology on scale manifolds. Scale Morse homology not only gives a unified treatment of various topics in Floer homology like gluing, spectral flow, and exponential decay [AF13], but due to its non-local character scale Morse homology leads to so far unexplored applications to delay equations, as discussed by the first author jointly with Peter Albers and Felix Schlenk [AFS18b, AFS18a].

This paper is organized as follows. In Section 2 we explain that the shift map is continuous in the compact open topology, but fails to be continuous in the operator topology. The strange behavior of the shift map was one of the main reasons for Hofer, Wysocki, and Zehnder to introduce scale smoothness whose definition we recall in Section 4. In order to explain scale smooth one needs the notion of a scale structure which we recall in Section 3. In Section 3 we also introduce the examples of scale structures which are relevant in Floer homology. The examples use Hilbert valued Sobolev theory. The proof that these examples actually satisfy the axioms of a scale structure is carried out in Section 8. The importance of having a scale structure is that this guarantees that the chain rule holds. We recall the chain rule in Section 5. In Section 6 we give a uniform proof that the shift map is sc-smooth for the trajectory spaces relevant in the various types of Floer homologies. The reason why the trajectory spaces which we introduce in Section 3 is explained in Section 7.

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2 Shift map on loop spaces

Throughout we identify the unit circle $S^1$ with the quotient space $\mathbb{R}/\mathbb{Z}$. To indicate that a function $v : \mathbb{R} \to \mathbb{R}$ has the property of being 1-periodic, that is $v(t + 1) = v(t)$ for every $t \in \mathbb{R}$, we use the notation $v : S^1 \to \mathbb{R}$.

As a warmup we discuss in this section the shift map on the loop space $H = L^2(S^1, \mathbb{R})$. For $\tau \in S^1$ define the shift map

$$\Psi_\tau : H \to H, \quad v \mapsto \tau_* v$$
where \((\tau v)(t) := v(t + \tau)\). Observe that \(\Psi_\tau\) is linear and an isometry

\[
\|\Psi_\tau v\|_H = \|v\|_H, \quad \tau \in \mathbb{S}^1, v \in H.
\]

**Lemma 2.1** (Continuity in compact open topology). As \(\tau\) goes to zero, \(\Psi_\tau\) converges to \(\Psi_0 = \text{id}\) in the **compact open topology**, i.e. for each \(v \in H\) it holds that

\[
\lim_{\tau \to 0} \|\Psi_\tau(v) - v\|_H = 0.
\]

**Proof.** Since \(C^\infty(\mathbb{S}^1, \mathbb{R})\) is dense in \(L^2(\mathbb{S}^1, \mathbb{R})\) there is a sequence \(v_\nu \in C^\infty(\mathbb{S}^1, \mathbb{R})\) such that \(v_\nu \to v\) in \(L^2\). Choose \(\varepsilon > 0\). Because \(v_\nu\) converges to \(v\) in \(L^2\), there is \(\nu_0 \in \mathbb{N}\) such that

\[
\|v_\nu - v\|_H \leq \varepsilon/3 \quad (2.1)
\]

whenever \(\nu \geq \nu_0\). Since \(v_\nu\) is smooth and uniformly continuous there is a time \(\tau_0 > 0\) such that for all \(\tau \in [0, \tau_0]\) it holds that

\[
|v_\nu(t + \tau) - v_\nu(t)| \leq \varepsilon/3, \quad t \in \mathbb{S}^1.
\]

We estimate for \(0 \leq \tau \leq \tau_0\)

\[
\|\Psi_\tau(v_\nu) - v_\nu\|_H = \sqrt{\int_0^1 |v_\nu(t + \tau) - v_\nu(t)|^2 \, dt}
\]

\[
\leq \sqrt{\int_0^1 (\varepsilon/3)^2 \, dt}
\]

\[
= \varepsilon/3. \quad (2.2)
\]

Combining (2.1) and (2.2) we estimate

\[
\|\Psi_\tau(v) - v\|_H \leq \|\Psi_\tau(v) - \Psi_\tau(v_\nu)\|_H + \|\Psi_\tau(v_\nu) - v_\nu\|_H + \|v_\nu - v\|_H
\]

\[
= \|v - v_\nu\|_H + \|\Psi_\tau(v_\nu) - v_\nu\|_H + \|v_\nu - v\|_H
\]

\[
= \varepsilon
\]

where the second step uses that \(\Psi_\tau\) is linear and an isometry. \(\square\)

**Lemma 2.2** (Discontinuity in norm topology). As \(\tau\) goes to zero, \(\Psi_\tau\) does not converge to \(\Psi_0 = \text{id}\) in the **norm topology**. More precisely, for each \(0 < \tau \leq 1/2\) there is an element \(v_\tau \in H\) of norm 1 and with the property that \(\|\Psi_\tau(v_\tau) - v_\tau\|_H = \sqrt{2} > 0\).

**Proof.** As illustrated by Figure 1 we define

\[
v_\tau(t) := \begin{cases} 
0 & , t \in (0, 1 - \frac{1}{2}\tau), \\
\sqrt{2/\tau} & , t \in [1 - \frac{1}{2}, 1], 
\end{cases}
\]

and compute
Figure 1: The function $\Psi_\tau(v_\tau) - v_\tau$

$$\|v_\tau\|_H = \sqrt{\int_0^1 |v_\tau(t)|^2 \, dt} = \sqrt{\frac{2}{\tau^2}} = 1.$$  

Note that

$$(\Psi_\tau(v_\tau) - v_\tau)(t) := \begin{cases} \sqrt{2/\tau}, & t \in [1 - \frac{3}{2}\tau, 1 - \tau], \\ -\sqrt{2/\tau}, & t \in [1 - \frac{5}{2}, 1], \\ 0, & \text{else.} \end{cases}$$

Hence

$$|\Psi_\tau(v_\tau) - v_\tau|^2 := \begin{cases} 2/\tau, & t \in [1 - \frac{3}{2}\tau, 1 - \tau] \cup [1 - \frac{5}{2}, 1], \\ 0, & \text{else.} \end{cases}$$

We calculate

$$\|\Psi_\tau(v_\tau) - v_\tau\|_H = \sqrt{\int_0^1 |(\Psi_\tau(v_\tau) - v_\tau)(t)|^2 \, dt} = \sqrt{\frac{2}{\tau} (\tau/2 + \tau/2)} = \sqrt{2}.$$

\[\square\]

### 3 Scale structures

Scale structures were introduced by Hofer, Wysocki, and Zehnder [HWZ07, Hof17, HWZ17]. We first recall its definition and then we discuss the main examples relevant in Morse and Floer homology. That these examples satisfy the conditions of a scale structure is proved in Section 8. The examples for Floer homology use Hilbert space valued Sobolev theory motivated by the paper of Robbin and Salamon on the spectral flow [RS95].

Let $(E, \|\cdot\|_E)$ be a Banach space.

**Definition 3.1.** A **scale-structure** on $E$, also called an **sc-structure** or a **Banach scale**, is a nested sequence $E =: E_0 \supset E_1 \supset E_2 \supset \ldots$ of Banach spaces meeting the following requirements:
(i) Each level includes compactly into the previous one, i.e. the linear operator given by inclusion \( E_{i+1} \hookrightarrow E_i \) is compact for each \( i \in \mathbb{N}_0 \).

(ii) The intersection \( E_\infty := \cap_{i \geq 0} E_i \) is dense in each level \( E_i \).

In this case one calls \( E \) a scale Banach space and also writes \( E = (E_i)_{i \in \mathbb{N}_0} \).

**Remark 3.2.** a) It follows from (ii) that the inclusions \( E_{i+1} \hookrightarrow E_i \) are dense for all \( i \in \mathbb{N}_0 \). b) The intersection \( E_\infty \) carries the structure of a Fréchet space.

**Definition 3.3** (Shifted scale Banach space). Given a scale Banach space \( E \) and \( m \in \mathbb{N}_0 \), then one defines the scale Banach space \( E^m \) by

\[
(E^m)_k := E_{k+m}.
\]

**Definition 3.4** (Scale direct sum). If \( E \) and \( F \) are scale Banach spaces one defines their direct sum as the scale Banach space \( E \oplus F \) whose levels are given by

\[
(E \oplus F)_k := E_k \oplus F_k.
\]

**Definition 3.5** (Scale isomorphism). A map \( I : E \to F \) between scale Banach spaces is called a scale morphism, or an sc-morphism, if the restriction to each level \( E_k \) takes values in \( F_k \) and

\[
I_k := I|_{E_k} : E_k \to F_k
\]

is linear and continuous. A scale morphism is called a scale isomorphism, or an sc-isomorphism, if its restriction \( I_k \) to each level \( E_k \) is bijective. Note that by the open mapping theorem if \( I \) is a scale isomorphism its inverse is a scale isomorphism as well. Two scale Banach spaces are called scale isomorphic if there exists a scale isomorphism between them.

**Examples**

**Example 3.6** (Finite dimension). If the Banach space \( E \) is of finite dimension, then property (ii) implies that the scale-structure is constant \( E =: E_0 = E_1 = E_2 = \ldots \).

**Remark 3.7** (Infinite dimension). In contrast, if \( E \) is infinite dimensional, then the compactness requirement in property (i) enforces strict inclusions \( E_{i+1} \subsetneq E_i \). Indeed the identity map on an infinite dimensional Banach space is never compact, because the unit ball of a Banach space is compact if and only if the Banach space is finite dimensional.

Now we introduce the main examples of scale structures relevant in this text. The proofs that these examples satisfy the requirements of a scale structure are given in Section 8.
Example 3.8 (The fractal Hilbert scales $\ell^2(f)$). Given a monotone unbounded function $f : \mathbb{N} \to (0, \infty)$, define the Hilbert space of weighted $\ell^2$-sequences by $$\ell^2_f := \left\{ x = (x_\nu)_{\nu \in \mathbb{N}} \in \ell^2 \mid \sum_{\nu=1}^{\infty} f(\nu)x^2_\nu < \infty \right\}.$$ The inner product on $\ell^2_f$ is given by $$\langle x, y \rangle_f := \sum_{\nu=1}^{\infty} f(\nu)x_\nu y_\nu.$$ We obtain an sc-structure on $H = \ell^2$ by using the Hilbert spaces $H_k := (\ell^2,f)_k := \ell^2_{f_k}, \ k \in \mathbb{N}_0.$ (3.3)

Denote by $e_i = (0, \ldots, 0, 1, 0, \ldots)$ the sequence whose members are all 0 except for member $i$ which is 1. The set $\mathcal{E}$ of all $e_i$ not only forms an orthonormal basis of the Hilbert space $H_0 = \ell^2$, but simultaneously an orthogonal basis of all spaces $H_k = \ell^2_{f_k}$. Rescaling provides an orthonormal basis $$\mathcal{E}_{f_k} := \{e_{i,f_k} \mid i \in \mathbb{N}\}, \quad e_{i,f_k} := \frac{1}{f(i)^{k/2}} e_i,$$ of $H_k$. The isometric Hilbert space isomorphism obtained by identifying the canonical orthonormal bases, namely $$\phi_k : H_0 \to H_k, \quad e_i \mapsto e_{i,f_k},$$ induces a levelwise-isometric sc-isomorphism $$\phi_k : H^0 \to H^k.$$ This means that the restriction to the $i^{th}$ level $$\phi_k|_{(H^0)_i} : (H^0)_i = H_i \to (H^k)_i = H_{k+i}$$ is an isometric Hilbert space isomorphism, and this is true for every level $i \in \mathbb{N}_0$. This explains the term fractal since as a consequence each of the Banach scales $H^j$ is self-similar to any $H^k$. The fractal scales $\ell^2(f)$ are intensively studied in interpolation theory; see e.g. [Tri78].

Two monotone unbounded functions $f, g : \mathbb{N} \to (0, \infty)$ are called equivalent if there is a constant $c > 0$ such that $\frac{1}{c}f \leq g \leq cf$. Two equivalent weight functions provide equivalent inner products on the vector space $\ell^2_f = \ell^2_g$.

The product $f \ast g$ of two monotone unbounded functions $f, g : \mathbb{N} \to (0, \infty)$ is the monotone unbounded function $f \ast g : \mathbb{N} \to (0, \infty)$ whose value at $\nu$ is the $\nu^{th}$ smallest number among the members of the two lists $(f(1), f(2), \ldots)$ and $(g(1), g(2), \ldots)$. The Banach scale associated to $f \ast g$ is the scale direct sum $$\ell^2(f \ast g) = \ell^2_f \oplus \ell^2_g.$$
Example 3.9 (Path spaces for Morse homology). Fix a monotone cutoff function \( \beta \in C^\infty(\mathbb{R}, [-1, 1]) \) with \( \beta(s) - 1 \) for \( s \leq -1 \) and \( \beta(s) = 1 \) for \( s \geq 1 \). Fix a constant \( \delta > 0 \) and, see Figure 2, define a function \( \gamma_\delta : \mathbb{R} \to \mathbb{R} \) by
\[
\gamma_\delta(s) := e^{\delta \beta(s)}.
\]

Pick a constant \( p \in (1, \infty) \). Consider the Banach spaces defined for \( k \in \mathbb{N}_0 \) by
\[
W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n) := \{ v \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) \mid \gamma_\delta v \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) \} \quad (3.4)
\]
with norm
\[
\|v\|_{W^{k,p}_\delta} := \|\gamma_\delta v\|_{W^{k,p}}.
\]
Choose a sequence \( 0 = \delta_0 < \delta_1 < \delta_2 \ldots \) and define
\[
E_k := W^{k,p}_{\delta_k}(\mathbb{R}, \mathbb{R}^n), \quad k \in \mathbb{N}_0. \quad (3.5)
\]

Definition 3.10 (Weighted Hilbert space valued Sobolev spaces). Let \( k \in \mathbb{N}_0 \), \( p \in (1, \infty) \), and \( \delta \geq 0 \). Suppose \( H \) is a separable Hilbert space and define the space \( W^{k,p}_\delta(\mathbb{R}, H) \) by (3.4) with \( \mathbb{R}^n \) replaced by \( H \). This is again a Banach space, see Appendix A.2.

Example 3.11 (Path spaces for Floer homology). Pick a constant \( p \in (1, \infty) \). For \( k \in \mathbb{N}_0 \) let \( H_k \) be as in Example 3.8 and let \( \delta_k \) be a sequence as in Example 3.9. The Banach space \( E_k \) is defined as the intersection of \( k + 1 \) Banach spaces
\[
E_k := \bigcap_{i=0}^{k} W^{i,p}_{\delta_i}(\mathbb{R}, H_{k-i}), \quad k \in \mathbb{N}_0. \quad (3.6)
\]
The norm on \( E_k \) is defined by taking the maximum of the \( k + 1 \) individual norms. This not only defines a norm, but even a complete one.
4 Scale smoothness

The notion of scale smoothness is due to Hofer, Wysocki, and Zehnder [HWZ07, Hof17, HWZ17]. An elegant way to introduce scale smoothness is via the tangent map. In particular, the chain rule, see Section 5, is nicely explained using the tangent map. We also give equivalent descriptions of scale smoothness in terms of sc-differentials. This equivalent description is useful to check scale smoothness explicitly in examples.

Let $E$ be a scale Banach space. Given an open subset $U \subset E$, then the part of $U$ in $E_k$ is denoted by $U_k := U \cap E_k$. Note that $U_k$ is open in $E_k$. In particular, one obtains a nested sequence $U = U_0 \supset U_1 \supset U_2 \ldots$.

**Definition 4.1 (Scale continuity).** Suppose that $E$ and $F$ are sc-Banach spaces and $U \subset E$ is an open subset. A map $f : U \to F$ is called **scale continuous** or of class $sc^0$ if

(i) $f$ is level preserving, i.e. the restriction of $f$ to each level $U_k$ takes values in the corresponding level $F_k$, and

(ii) the map $f|_{U_k} : U_k \to F_k$ is continuous.

In order to introduce the notion of scale differentiable or $sc^1$ we first need to introduce the notion of tangent bundle. The **tangent bundle** of a scale Banach space $E$ is defined as the scale Banach space $TE := E^1 \oplus E^0$.

If $U \subset E$ is an open subset of the scale Banach space $E$, as in Definition 3.3 one denotes by $U^m \subset E^m$ the scale of open subsets whose levels are given by $(U^m)_k := U_{m+k}$ where $k \in \mathbb{N}_0$. The tangent bundle of $U$ is the open subset of $TE$ defined by $TU := U^1 \oplus E^0 \subset TE$.

Note that the filtration of $TU$ is given by

$$(TU)_k = U_{k+1} \oplus E_k, \quad k \in \mathbb{N}_0.$$  

**Definition 4.2 (Scale differentiability).** Suppose $f : U \to F$ is $sc^0$, then $f$ is called **continuously scale differentiable** or of class $sc^1$ if for every $x \in U_1$ there is a bounded linear map $Df(x) : E_0 \to F_0$, called **sc-differential**, such that the following two conditions hold:

(i) The restriction of $f$ to $U_1$ interpreted as a map $f : U_1 \to F_0$ is required to be pointwise differentiable in the usual sense. The restriction of $Df(x)$ to $E_1$ is required to be the differential of $f : U_1 \to F_0$ in the usual sense, notation $df(x) : E_1 \to F_0$, i.e.

$$Df(x)|_{E_1} = df(x). \quad (4.7)$$
The tangent map \( T_f : TU \to TF \) defined for \((x, h) \in U^1 \oplus E^0 = TU\) by

\[
T_f(x, h) := (f(x), Df(x)h)
\]

is of class sc\(^0\).

**Remark 4.3** (Unique extension and continuous differentiability). Suppose \( f : U \to F \) is of class sc\(^1\).

a) Because \( E_1 \) is dense in \( E_0 \), see Remark 3.2, the map \( Df(x) \) is uniquely determined by (4.7). However, note that the mere requirement that \( f : U^1 \to F_0 \) is pointwise differentiable does not guarantee that a bounded extension of \( df(x) : E_1 \to F_0 \) to \( E_0 \) exists. Existence of such an extension is part of the definition of sc\(^1\).

b) Because \( E_1 \) includes compactly in \( E_0 \), the usual differential \( df(x) \in \mathcal{L}(E_1, F_0) \) depends continuously on \( x \in U^1 \). In other words, the restriction

\[
f \in C^1(U^1, F_0)
\]

is not only pointwise, but even continuously, differentiable in the usual sense.

To see this suppose \( x_\nu \in U^1 \subset E_1 \) is a sequence of points converging to a point \( x \in U^1 \). Now assume by contradiction that there is a constant \( \varepsilon > 0 \) and a sequence in \( E_1 \) of unit norm \( \|h_\nu\|_{E_1} = 1 \) such that \( \|df(x_\nu)h_\nu - df(x)h_\nu\|_{F_0} \geq \varepsilon \). Now since the inclusion \( E_1 \hookrightarrow E_1 \) is compact there are subsequences, still denoted by \( x_\nu \) and \( h_\nu \), such that

\[
\lim_{\nu \to \infty} \|df(x_\nu)h_\nu - df(x)h_\nu\|_{F_0} = \lim_{\nu \to \infty} \|Df(x_\nu)h_\nu - Df(x)h_\nu\|_{F_0} = 0.
\]

Here step one uses that \( Df(p)|_{E_1} = df(p) \in \mathcal{L}(E_1, F_0) \) for \( p \in U^1 \) and step two holds by continuity of \( Df \) with respect to the compact open topology. Hence \( \varepsilon = 0 \). Contradiction.

**Remark 4.4** (Level preservation). Suppose that \( x \) lies in \( U_m \) and \( k \in \{0, \ldots, m - 1\} \). Then, firstly, the restriction \( Df(x)|_{E_k} : E_k \to F_0 \) to \( E_k \) automatically takes values in the Banach space \( F_k \) and, secondly, the linear operator

\[
Df(x)|_{E_k} : E_k \to F_k
\]

is bounded. This follows from condition (ii) of scale differentiability: Because \( x \) lies in \( U_m \) and \( k < m \), one has \( x \in U_{k+1} \). Since the tangent map is sc\(^0\) it maps

\[
(TU)_k = (U^1 \oplus E^0)_k = U_{k+1} \oplus E_k
\]

to

\[
(TF)_k = (F^1 \oplus F^0)_k = F_{k+1} \oplus F_k
\]

continuously.
Remark 4.5 (Continuity in compact-open topology). A further consequence of condition (ii) in scale differentiability is that the scale differential viewed as a map

\[ Df|_{U_{k+1} \oplus E_k} : U_{k+1} \oplus E_k \to F_k \]

is continuous in the compact-open topology: Namely, if \( h \in E_k \) and \( x_\nu \) is a sequence in \( U_{k+1} \), then

\[ \lim_{\nu \to 0} \| Df(x_\nu)h - Df(x)h \|_{F_k} = 0. \]

Lemma 4.6 (Characterization of sc\(^1\) in terms of the scale-differential \( Df \)). Assume that \( f : U \to F \) is sc\(^0\). Then \( f \) is sc\(^1\) if and only if the following conditions hold:

(i) \( f : U_1 \to F_0 \) is pointwise differentiable in the usual sense;
(ii) For every \( x \in U_1 \) the differential \( df(x) : E_1 \to F_0 \) has a continuous extension \( Df(x) : E_0 \to F_0 \);
(iii) For all \( k \in \mathbb{N}_0 \) and \( x \in U_{k+1} \), the continuous extension \( Df(x) : E_0 \to F_0 \) restricts to a continuous map

\[ Df(x)|_{E_k} : E_k \to F_k \]

such that

\[ Df|_{U_{k+1} \oplus E_k} : U_{k+1} \oplus E_k \to F_k \]

is continuous in the compact-open topology.

Proof. '⇒' Suppose \( f \) is sc\(^1\). Then statements (i) and (ii) are obvious and (iii) follows from Remarks 4.4 and 4.5.

'⇐' Suppose that \( f \) is sc\(^0\) and satisfies (i–iii). We have to show that the tangent map is sc\(^0\). We first discuss why \( Tf \) maps \( (TU)_k \) to \( (TF)_k \) for every \( k \in \mathbb{N}_0 \). Pick \( (x,h) \in (TU)_k = U_{k+1} \oplus E_k \). Since \( f \) is sc\(^0\) we have that \( f(x) \in F_{k+1} \). By (iii) we have that \( Df(x)h \in F_k \). Hence

\[ Tf(x,h) = (f(x), Df(x)h) \in F_{k+1} \oplus F_k = (TF)_k. \]

This shows that \( Tf \) maps \( (TU)_k \) to \( (TF)_k \).

We next explain why \( Tf \) as a map \( Tf|(TU)_k : (TU)_k \to (TF)_k \) is continuous. Assume \( (x_\nu, h_\nu) \in (TU)_k = U_{k+1} \oplus E_k \) is a sequence which converges to \( (x,h) \in (TU)_k \). Because \( f \) is sc\(^0\), it follows that

\[ \lim_{\nu \to \infty} f(x_\nu) = f(x). \]

Again by (iii) we have that

\[ \lim_{\nu \to \infty} Df(x_\nu)h_\nu = Df(x)h. \]

Therefore

\[ \lim_{\nu \to \infty} Tf(x_\nu, h_\nu) = \lim_{\nu \to \infty} (f(x_\nu), Df(x_\nu)h_\nu) = (f(x), Df(x)h) = Tf(x,h). \]

This proves continuity and hence the lemma holds. \( \square \)
For \( m \geq 2 \) one defines higher continuous scale differentiability \( \text{sc}^m \) recursively as follows.

**Definition 4.7** (Higher scale differentiability). An \( \text{sc}^1 \)-map \( f : U \to F \) is of class \( \text{sc}^m \) if and only if its tangent map \( Tf : TU \to TF \) is \( \text{sc}^{m-1} \). It is called \( \text{sc-smooth} \), or of class \( \text{sc}^\infty \), iff it is of class \( \text{sc}^m \) for every \( m \in \mathbb{N} \).

An \( \text{sc}^m \)-map has iterated tangent maps as follows. Recursively one defines the iterated tangent bundle as \( T^{m+1}U := T(T^mU) \). For example

\[
T^2U = T(TU) = T(U^1 \oplus E^0) \\
= (U^1 \oplus E^0)^1 \oplus (E^1 \oplus E^0) \\
= U^2 \oplus E^1 \oplus E \oplus E^0.
\]

If \( f \) is of class \( \text{sc}^m \), the iterated tangent map \( T^m f : T^mU \to T^mF \) is recursively defined as

\[
T^m f := T(T^{m-1}f).
\]

For example

\[
T^2 f : U^2 \oplus E^1 \oplus E^1 \oplus E^0 \to F^2 \oplus F^1 \oplus F^1 \oplus F^0
\]

is given by

\[
T^2 f(x, h, \hat{x}, \hat{h}) = \left(f(x), Df(x)h, Df(x)\hat{x}, D^2 f(x)(h, \hat{x}) + Df(x)\hat{h}\right).
\]

**Lemma 4.8** (Characterization of \( \text{sc}^2 \) in terms of \( \text{sc} \)-differentials). Assume that \( f : U \to F \) is \( \text{sc}^1 \). Then \( f \) is \( \text{sc}^2 \) if and only if the following conditions hold:

(i) \( f : U_2 \to F_0 \) is pointwise twice differentiable in the usual sense;

(ii) For every \( x \in U_2 \), the second differential \( d^2 f(x) : E_2 \oplus E_2 \to F_0 \) has a continuous extension \( D^2 f(x) : E_1 \oplus E_1 \to F_0 \);

(iii) For all \( k \in \mathbb{N} \), \( x \in U_{k+1} \), the continuous extension \( D^2 f(x) : E_1 \oplus E_1 \to F_0 \) restricts to a continuous bilinear map

\[
D^2 f(x)|_{E_k \oplus E_k} : E_k \oplus E_k \to F_{k-1}
\]

such that

\[
D^2 f|_{U_{k+1} \oplus E_k \oplus E_k} : U_{k+1} \oplus E_k \oplus E_k \to F_{k-1}
\]

is continuous in the compact-open topology.

**Remark 4.9** (Symmetry of second scale differentials). The second scale differential \( D^2 f(x) : E_1 \oplus E_1 \to F_0 \) is symmetric, because the usual second differential \( d^2 f(x) : E_2 \oplus E_2 \to F_0 \) is symmetric and \( E_2 \) is a dense subset of the Banach space \( E_1 \).
Proposition 4.10 (Characterizing $\text{sc}^m$ by higher $\text{sc}$-differentials $D^m f(x)$). Let $f : U \to F$ be $\text{sc}^{m-1}$. Then $f$ is $\text{sc}^m$ iff the following conditions hold:

(i) $f : U_m \to F_0$ is pointwise $m$ times differentiable in the usual sense;

(ii) For every $x \in U_m$ the $m$th differential $d^m f(x) : E_m \oplus \cdots \oplus E_m \to F_0$ has a continuous extension

$D^m f(x) : E_{m-1} \oplus \cdots \oplus E_{m-1} \to F_0$;

(iii) For all $k \geq m - 1$ and $x \in U_{k+1}$, the continuous extension $D^m f(x) : E_{m-1} \oplus \cdots \oplus E_{m-1} \to F_0$ restricts to a continuous $m$-fold multilinear map

$D^m f(x) : E_k \oplus \cdots \oplus E_k \to F_{k-(m-1)} = F_{k-m+1}$

such that

$D^m f|_{U_{k+1} \oplus E_k \oplus \cdots \oplus E_k} : U_{k+1} \oplus E_k \oplus \cdots \oplus E_k \to F_{k-m+1}$

is continuous in the compact-open topology.

The higher $\text{sc}$-differentials $D^m f(x)$ are symmetric $m$-fold multilinear maps by the argument in Remark 4.9.

5 Chain rule

The following theorem was proved by Hofer, Wysocki, and Zehnder in [HWZ07]. The proof relies heavily on the compactness condition on the scale inclusions $E_{i+1} \hookrightarrow E_i$ in Definition 3.1 of a Banach scale.

Theorem 5.1 (Chain rule). Consider scale Banach spaces $E,F,G$ and open subsets $U \subset E$ and $V \subset F$. Suppose the maps $f : U \to V$ and $g : V \to G$ are of class $\text{sc}^1$. Then the composition $g \circ f : U \to G$ is of class $\text{sc}^1$ and

$T(g \circ f) = Tg \circ Tf : TU \to TG$.

Concerning the proof of the chain rule in [HWZ07, p. 849] Hofer, Wysocki, and Zehnder remark the following.

"The reader should realize that in the above proof all conditions on $\text{sc}^1$ maps have been used, i.e. it just works."

An immediate consequence of the chain rule is the following corollary.

Corollary 5.2. Under the assumptions of Theorem 5.1 suppose, in addition, that $f$ and $g$ are of class $\text{sc}^m$ where $m \in \mathbb{N}$. Then the composition $g \circ f : U \to G$ is of class $\text{sc}^m$ and its $m$-fold iterated tangent map is given by

$T^m(g \circ f) = T^m g \circ T^m f : T^m U \to T^m G$. 
6 Scale smooth actions

In this section we explain that the shift map is scale smooth. We first prove this for loop spaces \( H^k = W_{k}(\mathbb{S}^1, \mathbb{R}) \), however, basically the same proof generalizes to Morse and Floer trajectory spaces \( E^k = \bigcap_{i=0}^{k} W_{d_i}^{i,2}(\mathbb{R}, H^k_{k-i}) \). Surprisingly the growth type used to define the Floer trajectory spaces does not enter the proof. This is due to the fact that the shift map is linear in the second variable.

Loop spaces

**Theorem 6.1** (Shift map on loop spaces is scale smooth). Let \( H \) be the scale Hilbert space whose levels are given by \( H^k = W_{k}(\mathbb{S}^1, \mathbb{R}) \) and consider the map

\[
\Psi : F = \mathbb{R} \oplus H \to H, \quad (\tau, v) \mapsto \tau v,
\]

where \( \mathbb{R} \) carries the constant scale structure. Then the map \( \Psi \) is \( sc \)-smooth.

**Proof.** At the point \((\tau, v) \in F_1 \) the \( sc \)-differential evaluated on \((T_1, V_1) \in F_0 \) is

\[
D\Psi(\tau, v)(T_1, V_1) = \tau v' \cdot T_1 + \tau V_1.
\]

At the point \((\tau, v) \in F_2 \) the second \( sc \)-differential \( D^2\Psi(\tau, v) \) evaluated on a pair \((T_1, V_1), (T_2, \hat{V}_2) \) \( \in F_1 \) is given by

\[
D^2\Psi(\tau, v)((T_1, V_1), (T_2, \hat{V}_2)) = \tau v'' \cdot T_1 T_2 + \tau V_1' \cdot T_2 + \tau \hat{V}_2' \cdot T_1.
\]

By induction one shows that at \((\tau, v) \in F_k \) the \( k^{th} \) iterated \( sc \)-differential

\[
D^k\Psi(\tau, v) : F_{k-1} \oplus \cdots \oplus F_{k-1} \to H_0
\]

evaluated on \( k \) elements \((T_1, V_1), \ldots, (T_k, V_k) \in F_{k-1} \) is given by the formula

\[
D^k\Psi(\tau, v)\left((T_1, V_1), \ldots, (T_k, V_k)\right) = \tau v^{(k)} \prod_{j=1}^{k} T_j + \sum_{j=1}^{k} \tau v_{j}^{(k-1)} T_1 \cdots \hat{T}_j \cdots T_k
\]  

(6.9)

where the wide hat in \( \hat{T}_j \) means to delete that term. That the iterated \( sc \)-differentials meet the requirements of Proposition 4.10 follows from Lemma 2.1.

Floer trajectory spaces

**Theorem 6.2** (Shift map on path spaces is scale smooth). Let \( E \) be the scale Banach space of path spaces arising in Morse or Floer homology as introduced in Examples 3.9 or 3.11. Then the shift map in (6.8) with \( H \) replaced by \( E \) is \( sc \)-smooth.
Proof. As proof of Theorem 6.1. More precisely, replace Lemma 2.1
- in the Morse case by Lemma 6.3 with finite dimensional $H$
- and in the Floer case by Corollary 6.4.

It is surprising that the proof of Theorem 6.2 can be given uniformly, independent of the monotone unbounded function $f : \mathbb{N} \to (0, \infty)$. This hinges on the observation that in formula (6.9) the $V_j$’s only enter linearly – there are no products. In fact, if there would be products, the regularity would strongly depend on the growth type of $f$, which is well known from Sobolev theory. It is easy to understand why the $V_j$’s only enter linearly in the formula (6.9) of the differential: The shift map (6.8) is linear in the second variable.

Lemma 6.3 (Continuity in compact open topology). Let $k \in \mathbb{N}_0$ and pick constants $p \in (1, \infty)$ and $\delta \geq 0$. Suppose $H$ is a separable Hilbert space, i.e. $H$ is either isometric to $\ell^2$ or of finite dimension, and define $W_{\delta}^{k,p}(\mathbb{R}, H)$ by (3.4) with $\mathbb{R}^n$ replaced by $H$. Then the shift map

$$\Psi_\tau : W_{\delta}^{k,p}(\mathbb{R}, H) \to \Psi_\tau : W_{\delta}^{k,p}(\mathbb{R}, H), \quad v \mapsto \tau_* v,$$

is continuous in the compact open topology, i.e.

$$\lim_{\tau \to 0} \| \Psi_\tau(v) - v \|_{W_{\delta}^{k,p}(\mathbb{R}, H)} = 0.$$

for each $v \in W_{\delta}^{k,p}(\mathbb{R}, H)$.

Proof. Same arguments as in Lemma 2.1. 

An immediate Corollary of the lemma is the following.

Corollary 6.4. Let $k \in \mathbb{N}_0$ and pick constants $p \in (1, \infty)$ and $\delta_k \geq 0$. Let $f : \mathbb{N} \to (0, \infty)$ be a monotone unbounded function and consider the weighted Hilbert spaces $H_0 := \ell^2$ and $H_j := \ell^2_{f_j}$, for $j \in \mathbb{N}$ as in (3.3). Then the shift map on the intersection Banach space

$$\Psi_\tau : E_k = \bigcap_{i=0}^{k} W_{\delta_i}^{k,p}(\mathbb{R}, H_{k-i}) \to E_k = \bigcap_{i=0}^{k} W_{\delta_i}^{k,p}(\mathbb{R}, H_{k-i}), \quad v \mapsto \tau_* v,$$

is continuous in the compact open topology.

7 Fractal Hilbert scale structures on mapping spaces

In this section we explain how fractal scale Hilbert spaces can be used to model the targets in Floer homology. Let $N$ be a closed manifold. Fix an integer $k_0 > \frac{1}{2} \dim N$ and consider the Hilbert scale defined by

$$\text{Map}(N, \mathbb{R}^r) = \text{Map}(N, \mathbb{R}^r)_0 := W^{k_0,2}(N, \mathbb{R}^r) \supset W^{k_0+1,2}(N, \mathbb{R}^r) \supset \ldots.$$
The spectral theory of the Laplace operator $\Delta_g$ associated to a Riemannian metric $g$ on $N$ shows that this Hilbert scale is given by the fractal Hilbert scale $\ell^2.f$ associated in Example 3.8 to the weight function

$$f(\nu) = \nu^{2/\dim N}.$$  \hspace{1cm} (7.10)

Observe that $f$ only depends on the dimension of the domain $N$, it is independent of the dimension of the target; cf. [Kan11]. This phenomenon is very reminiscent of the Sobolev theory which is sensible to the dimension of the domain, but not of the target.

**Periodic boundary conditions**

We illustrate this for the Hilbert space $H = \text{Map}(S^1, \mathbb{C}) := L^2(S^1, \mathbb{C})$. Here we get away with $k_0 = 0$, because we view maps $S^1 \to \mathbb{C}$ as maps $\mathbb{R} \to \mathbb{C}$ which are 1-periodic. So there is no need to take local coordinate charts on $S^1$ and therefore we don’t need continuity of the elements of our mapping space $\text{Map}(S^1, \mathbb{C})$. The Hilbert space $H$ consists of all infinite sums of the form

$$v(t) = \sum_{\ell \in \mathbb{Z}} v_\ell e^{2\pi i \ell t}$$

whose Fourier coefficient sequences $(v_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{C}$ are square summable, that is

$$\sum_{\ell \in \mathbb{Z}} |v_\ell|^2 < \infty.$$  

For $k \in \mathbb{N}_0$ the subspace $W^{k,2}(S^1, \mathbb{C})$ of $L^2(S^1, \mathbb{C})$ consists of those $v$ for which the weighted sum $\sum_{\ell \in \mathbb{Z}} (1 + \ell^2)^k |v_\ell|^2$ is finite. Up to equivalent weight functions the space $W^{k,2}(S^1, \mathbb{C})$ coincides with the $k^{\text{th}}$ level

$$H_k := (\ell^2.f)_k = \ell^{2}f_k$$

of the scale Hilbert space $\ell^2.f$ associated in Example 3.8 to the weight function $f : \mathbb{N} \to (0,\infty)$, $\nu \mapsto \nu^2$. This is consistent with formula (7.10), because $\dim S^1 = 1$.

**Lagrangian boundary conditions**

Consider the relative mapping space

$$H = \text{Map}([0, 1], \{0, 1\}; \mathbb{C}, \mathbb{R}) := W^{1,2}([0, 1], \{0, 1\}, (\mathbb{C}, \mathbb{R}))$$

that consists of all $W^{1,2}$ paths $\gamma : [0, 1] \to \mathbb{C} = \mathbb{R} \times i\mathbb{R}$ whose initial and end points lie on the real line $\mathbb{R}$. We define a scale structure on $H$ by choosing as $k^{\text{th}}$ level the space

$$H_k := \left\{ \gamma \in W^{k+1,2}([0, 1], \mathbb{C}) \mid \gamma^{(\ell)}(0), \gamma^{(\ell)}(1) \in i^\ell \mathbb{R}, \, 0 \leq \ell \leq k \right\}.$$
Note that we not have just Lagrangian boundary conditions for the path, but also for its derivatives. This is crucial to get a fractal scale Hilbert structure. These boundary conditions also appeared in the thesis of Simčević [Sim14] in her Hardy space approach to gluing. Observe that these boundary conditions are well posed, because the $k$th derivative of such a path is still continuous by the Sobolev embedding theorem $W^{k+1,2} \hookrightarrow C^k$ on the 1-dimensional domain $[0, 1]$.

Given a path $\gamma \in H_k$, consider the associated loop in $\mathbb{C}$ defined by doubling

$$\Gamma_\gamma(t) := \begin{cases} \gamma(2t) & , t \in [0, \frac{1}{2}], \\ \bar{\gamma}(2 - 2t) & , t \in [\frac{1}{2}, 1], \end{cases}$$

where $\bar{z} = x - iy$ denotes complex conjugation of a complex number $z = x + iy$. Note that $\Gamma_\gamma$ is indeed a loop

$$\Gamma_\gamma(1) = \bar{\gamma}(0) = \gamma(0) = \Gamma_\gamma(0)$$

where the second step holds due to the condition that initial points of our paths lie on the real line $\mathbb{R} \subset \mathbb{C}$. The real endpoint condition guarantees that the loop is also continuous at $t = \frac{1}{2}$, hence everywhere. We claim that

$$\Gamma_\gamma \in W^{k+1,2}(S^1, \mathbb{C}).$$

Because $\gamma \in W^{k+1,2}([0, 1], \mathbb{C})$, it suffices to show that $\Gamma_\gamma$ is $k$ times differentiable at the points 0 and $\frac{1}{2}$. This follows from the boundary conditions imposed in the definition of the space $H_k$.

As illustrated by Figure 3 The doubling map $\gamma \mapsto \Gamma_\gamma$ gives us an embedding

$$I : H_k \hookrightarrow W^{k+1,2}(S^1, \mathbb{C}), \quad \gamma \mapsto \Gamma_\gamma.$$ 

The elements of the image of $I$ are precisely those $W^{k+1,2}$ loops $\Gamma$ in $\mathbb{C}$ that are symmetric with respect to the real line $\mathbb{R} \subset \mathbb{C}$, more precisely

$$\Gamma(t) = \overline{\Gamma(1 - t)}, \quad t \in [0, 1].$$

(7.11)
Indeed suppose $\Gamma \in W^{k+1,2}(S^1, \mathbb{C})$ satisfies (7.11), taking its first half $\gamma_\Gamma(t) := \Gamma(t/2)$ for $t \in [0,1]$ it follows from (7.11) that $\gamma_\Gamma \in H_k$ and $I(\gamma_\Gamma) = \Gamma_{\gamma_\Gamma} = \Gamma$.

Suppose $\Gamma \in \text{im}(I)$ lies in the image of $I$, that is $\Gamma : S^1 \to \mathbb{C}$ is of class $W^{k+1,2}$ and satisfies (7.11). Writing $\Gamma$ as a Fourier series we obtain that

$$
\sum_{\ell \in \mathbb{Z}} v_{k} e^{2\pi i \ell t} = \Gamma(t) = \Gamma(1-t) = \Gamma(-t) = \sum_{\ell \in \mathbb{Z}} v_{k} e^{-2\pi i \ell t} = \sum_{\ell \in \mathbb{Z}} \bar{v}_k e^{2\pi i \ell t}.
$$

This shows that all Fourier coefficients $v_k = \bar{v}_k$ are real. In particular, up to scale isomorphism, the scale relative mapping space $H$ is scale isomorphic to the fractal Hilbert scale $\ell^2(f)$ for the weight function $f(\nu) = \nu^2$, in symbols

$$
H = W^{1,2}([0,1], (0,1), (\mathbb{C}, \mathbb{R})) \simeq \ell^2(f), \quad f(\nu) = \nu^2.
$$

The growth type of various Floer homologies

In view of the discussion before we can now list the growth type of the various Floer homologies mentioned in the introduction.

The growth types are different, however, we point out that the main result, Theorem 6.2, is independent of the growth type and therefore applies to all of the following.

| Floer homology     | Order | Mapping space    | Growth type $f(\nu)$ |
|--------------------|-------|------------------|----------------------|
| Periodic           | 1st   | loop space       | $\nu^2$              |
| Lagrangian         | 1st   | path space       | $\nu^2$              |
| Hyperkähler        | 1st   | $\text{Map}(M^3, \mathbb{R}^{2n})$ | $\nu^{2/3}$          |
| Heat flow          | 2nd   | loop space       | $\nu^4$              |

8 Banach scale structures – main examples

In this section we show that the examples of scale structures introduced in Section 3 actually satisfy the axioms of scale structures.

Fractal scale Hilbert spaces

Consider a monotone unbounded function $f : \mathbb{N} \to (0, \infty)$ and consider the weighted Hilbert spaces $H_0 := \ell^2$ and $H_k := \ell^2_{f_k}$ for $k \in \mathbb{N}$ as in Example 3.8.

Our aim is to show that the nested sequence of Hilbert spaces $\ell^2(f) = \ell^2 \supset \ell^2_{f_2} \supset \ell^2_{f_3} \ldots$ carries the structure of a scale Hilbert space, that is compact inclusions and density of $\cap_{k=0}^{\infty} \ell^2_{f_k}$ in each level $\ell^2_{f_k}$.

**Compact inclusions.** Consider the finite dimensional subspace $V_N := \{ \sum_{i=1}^{N} a_i e_i | a_i \in \mathbb{R} \} \subset \ell^2_f$, the orthogonal projection $\pi_N : \ell^2_f \to V_N$, and the
non-commutative diagram

\[ \begin{array}{ccc}
\ell^2_j & \xrightarrow{I} & \ell^2 \\
\pi_N \downarrow & & \downarrow I_N \\
V_N & & \\
\end{array} \]

By finite dimension of \( V_N \) the inclusion \( I_N \) is a compact operator. Therefore the composition \( I^N := I_N \pi_N : \ell^2_j \to \ell^2 \) is compact. The estimate

\[
\| I - I^N \|_{\mathcal{L}(\ell^2_j, \ell^2)} = \sup_{\|v\|_{\ell^2_j} = 1} \| (I - I^N)v \|_{\ell^2} \\
= \sup_{\|v\|_{\ell^2_j} = 1} \| (\id - \pi_N)v \|_{\ell^2} \\
\leq 1/f(N)
\]

shows that \( I^N \to I \), as \( N \to \infty \), in the operator norm topology. Hence \( I \) is compact by Theorem A.1.

**Density.** Let \( V = \bigcup_{N=1}^\infty V_N \) be the union of all the \( V_N \). The inclusions

\[ V \subset \bigcap_{k=0}^\infty \ell^2_{j_k} \subset \ell^2_{j_k} \]

again with density of \( V \) in \( \ell^2_{j_k} \) implies density of \( \bigcap_{k=0}^\infty \ell^2_{j_k} \) in each weighted Hilbert space \( \ell^2_{j_k} \). We proved the following theorem.

**Theorem 8.1** (The fractal Hilbert scale). *The sequence of fractal Hilbert spaces \( H_k = \ell^2_{j_k} \) defined by (3.3) forms a Banach scale.*

**Morse path spaces**

Our aim is to show that the sequence of Morse path spaces \( E_k = W^k_{\delta_k}([0,1], \mathbb{R}^n) \) introduced in Example 3.9 has the two defining properties of a Banach scale, compact inclusions and density.

**Theorem 8.2** (The Morse path Banach scale). *The sequence of Morse path spaces \( E_k = W^k_{\delta_k}([0,1], \mathbb{R}^n) \) defined by (3.5) forms a Banach scale.*

**Proof.** Density: The inclusions

\[ C^\infty_c(\mathbb{R}, \mathbb{R}^n) \subset E_\infty := \bigcap_{k=0}^\infty W^k_{\delta_k}(\mathbb{R}, \mathbb{R}^n) \subset W^k_{\delta_k}(\mathbb{R}, \mathbb{R}^n) =: E_k \]

together with density of the set of compactly supported smooth functions in the Banach space \( W^k_{\delta_k}(\mathbb{R}, \mathbb{R}^n) \) implies density of \( E_\infty \) in each level \( E_k \).

Compact inclusions: Proposition 8.3. \( \square \)
Proposition 8.3 (Compact inclusions). Suppose \( k \in \mathbb{N} \) and \( p \in (1, \infty) \). For non-negative reals \( \delta_1 > \delta_0 \) the inclusion
\[
I : W^{k,p}_{\delta_1}(\mathbb{R}) \hookrightarrow W^{k-1,p}_{\delta_0}(\mathbb{R})
\]
is a compact linear operator. The Banach spaces are defined by (3.4).

In order to prove the proposition we first prove two lemmas.

Lemma 8.4 \((k = 1, \delta_0 = 0)\). For constants \( p \in (1, \infty) \) and \( \delta > 0 \) the inclusion
\[
I : W^{1,p}_{\delta}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})
\]
is compact where the Banach space \( W^{1,p}_{\delta} \) is defined by (3.4).

Without the exponential weights the inclusion \( W^{1,p}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \) is in general not compact, as shown by the sequence \( v_\nu(t) := v(t - \nu) \) of right shifts of a given function \( v \) of positive norm, for instance a bump function \( v \).

Proof. For \( T > 0 \) consider the continuous operators given by restriction
\[
R_T : W^{1,p}_{\delta}(\mathbb{R}) \to W^{1,p}([-T,T]), \quad v \mapsto v|_{[-T,T]},
\]
and extension by zero \( E_T : L^p([-T,T]) \to L^p(\mathbb{R}) \). Since \( p > \dim [-T,T] = 1 \) the inclusion operator \( I_T : W^{1,p}([-T,T]) \hookrightarrow L^p([-T,T]) \) is compact by the Sobolev embedding theorem. Hence the composition
\[
I^T := E_T I_T R_T : W^{1,p}_{\delta}(\mathbb{R}) \to L^p(\mathbb{R})
\]
is compact. Because the set of compact linear operators is norm closed, see Theorem A.1, it suffices to show that \( I^T \to I \) in the norm topology, as \( T \to \infty \). Indeed
\[
\| I - I^T \|_{\mathcal{L}(W^{1,p}_{\delta}, L^p)} = \sup_{\|v\|_{W^{1,p}_{\delta}} = 1} \| (I - I^T)v \|_{L^p} = \sup_{\|v\|_{W^{1,p}_{\delta}} = 1} \| v|_{(-\infty,-T] \cup [T,\infty)} \|_{L^p} \leq e^{-\delta T}
\]
whenever \( T \geq 1 \). To see the final estimate observe that
\[
\| v|_{(-\infty,-T] \cup [T,\infty)} \|_{L^p} \leq \frac{1}{e^{\delta T}} \| \gamma \cdot v|_{(-\infty,-T] \cup [T,\infty)} \|_{L^p} \leq \frac{1}{e^{\delta T}} \| v|_{(-\infty,-T] \cup [T,\infty)} \|_{W^{1,p}_{\delta}} \leq \frac{1}{e^{\delta T}} \| v \|_{W^{1,p}_{\delta}} = e^{-\delta T}
\]
whenever \( \|v\|_{W^{1,p}_{\delta}} = 1 \) and where step one uses that \( T \geq 1 \).

\( \square \)
Lemma 8.5 (General $k$, $\delta_0 = 0$). Given $k \in \mathbb{N}$ and reals $p \in (1, \infty)$ and $\delta > 0$, then the inclusion
\[ I : W^{k,p}_\delta(\mathbb{R}) \hookrightarrow W^{k-1,p}(\mathbb{R}) \]
is compact. The Banach spaces $W^{k,p}_\delta$ are defined by (3.4).

Proof. The lemma follows by induction on $k$. For $k = 1$ the assertion is true by Lemma 8.4. To prove the induction step $k \Rightarrow k+1$ suppose the lemma holds true for $k$. Let $v_\nu$ be a sequence in the unit ball of $W^{k-1,p}_\delta(\mathbb{R})$. Hence both $v_\nu$ and its derivative $\dot{v}_\nu$ lie in the unit ball of $W^{k,p}_\delta(\mathbb{R})$. By induction hypothesis there exist elements $v,w \in W^{k-1,p}_\delta(\mathbb{R})$ and a subsequence, still denoted by $v_\nu$, such that as $\nu \to \infty$ it holds that
\[ v_\nu \overset{W^{k-1,p}}{\longrightarrow} v \quad \text{and} \quad \dot{v}_\nu \overset{W^{k-1,p}}{\longrightarrow} w. \]

Note that $w$ is equal to the weak derivative $\dot{v}$. Indeed the definition of the weak derivative provides the first of the identities
\begin{align*}
\int_{\mathbb{R}} \phi \dot{v} &= - \int_{\mathbb{R}} \dot{\phi} v = - \lim_{\nu \to \infty} \int_{\mathbb{R}} \dot{\phi} v_\nu = \lim_{\nu \to \infty} \int_{\mathbb{R}} \phi \dot{v}_\nu = \int_{\mathbb{R}} \phi w \quad (8.12)
\end{align*}
which hold true for every test function $\phi \in C_0^\infty(\mathbb{R})$. In particular, the element $v$ lies in $W^{k,p}$ and $v_\nu \to v$ in $W^{k,p}$.

\[ \square \]

Proof of Proposition 8.3. Consider the isomorphisms
\[ T : W^{k,p}_{\delta_1} \to W^{k,p}_{\delta_1 - \delta_0}, \quad S : W^{k-1,p}_{\delta_0} \to W^{k-1,p}, \quad v \mapsto \gamma_{\delta_0} \cdot v, \]
which both act by multiplication by the weight function $\gamma_{\delta_0}$. The assertion of the proposition – compactness of the inclusion $I : W^{k,p}_{\delta_1}(\mathbb{R}) \hookrightarrow W^{k-1,p}_{\delta_0}(\mathbb{R})$ – follows since the diagram commutes
\[ W^{k,p}_{\delta_1 - \delta_0} \overset{I}{\hookrightarrow} W^{k-1,p} \]
\[ T \uparrow \cong S \]
\[ W^{k,p}_{\delta_1} \overset{I}{\hookrightarrow} W^{k-1,p}_{\delta_0} \quad (8.13) \]
and the upper inclusion is compact by Lemma 8.5.

\[ \square \]

Floer path spaces

Our aim is to show that the sequence of Floer path spaces $E_k$ introduced in Example 3.11 has the two defining properties of a Banach scale, compact inclusions and density. Hence from now on let $f : \mathbb{N} \to (0, \infty)$ be a monotone unbounded function and consider the weighted Hilbert spaces $H_0 := \ell^2$ and $H_k := \ell^2_{f_k}$ for $k \in \mathbb{N}$ as in (3.3).
**Theorem 8.6** (The Floer path Banach scale). *The sequence of Floer path spaces $E_k$ defined by (3.6) forms a Banach scale.*

**Proof.** Density: Consider the dense subset $V$ of $\ell^2$ that consists of all finite sums

$$V := \left\{ \sum_{i=1}^{N} a_i e_i \mid N \in \mathbb{N}, a_1, \ldots, a_N \in \mathbb{R} \right\} \subset \ell^2.$$ 

The inclusions $V \subset \bigcap_{k=0}^{\infty} \ell^2_{f^k} \subset \ell^2_{f^k}$ together with density of $V$ in $\ell^2_{f^k}$ implies density of $V$ in each weighted Hilbert space $\ell^2_{f^k}$. The inclusions

$$C^\infty_c(\mathbb{R}, V) \subset E_\infty := \bigcap_{k=0}^{\infty} E_k \subset E_k$$

together with density of the set $C^\infty_c(\mathbb{R}, V)$ in the Banach space $E_k$ implies density of $E_\infty$ in each level $E_k$.

Compact inclusions: Proposition 8.7.

**Proposition 8.7** (Compact inclusions). *Suppose $k \in \mathbb{N}$ and $p \in (1, \infty)$. For non-negative reals $\delta_k > \delta_{k-1}$ the inclusion*

$$I : E_k = \bigcap_{i=0}^{k} W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i}) \to \bigcap_{i=0}^{k-1} W^{i,p}_{\delta_{k-1}}(\mathbb{R}, H_{k-1-i}) = E_{k-1}$$

*is a compact linear operator.*

In order to prove the proposition we first prove two lemmas.

**Lemma 8.8** ($k = 1, \delta_0 = 0$). *Pick reals $p > 1$ and $\delta > 0 = \delta_0$. Then the inclusion*

$$I : E_1 = W^{1,p}_{\delta}(\mathbb{R}, H_0) \cap L^{p}_0(\mathbb{R}, H_1) \to L^p(\mathbb{R}, H_0) = E_0$$

*is a compact linear operator.*

Recall from Example 3.11 that the norm on an intersection of Banach spaces is the maximum of the individual norms.

**Proof of Lemma 8.8** ($k = 1$). Denote by $e_i = (0, \ldots, 0, 1, 0, \ldots)$ the sequence whose members are all 0 except for member $i$ which is 1. The set of all $e_i$ not only form an orthonormal basis of the Hilbert space $H_0 = \ell^2$, but simultaneously an orthogonal basis of $H_1 = \ell^2_f$, although not of unit length any more.

For $N \in \mathbb{N}$ consider the subspace $V_N = \text{span} \{ e_1, \ldots, e_N \} \subset H_0$ of finite dimension and the corresponding orthogonal projection $\pi_N : H_0 = \ell^2 \to V_N$. Its restriction $\pi_N|_{\ell^2_f} : H_1 = \ell^2_f \to V_N$ is also an orthogonal projection.

Define a linear projection by

$$\Pi_N : W^{1,p}_{\delta}(\mathbb{R}, H_0) \cap L^{p}_0(\mathbb{R}, H_1) \to W^{1,p}_{\delta}(\mathbb{R}, V_N)$$

$$v \mapsto \pi_N \circ v$$
The linear operator given by inclusion and denoted by
\[ I_N : W^{1,p}_\delta(\mathbb{R}, V_N) \hookrightarrow L^p(\mathbb{R}, V_N), \quad v \mapsto v, \]
is compact by Lemma 8.4 for \( I = I_N \). The inclusion \( j_N : V_N \hookrightarrow \ell^2 = H_0 \) induces the inclusion
\[ J_N : L^p(\mathbb{R}, V_N) \hookrightarrow L^p(\mathbb{R}, H_0), \quad v \mapsto j_N \circ v. \]
The inclusion given by composition of bounded linear operators
\[ I_N = J_N I_N \Pi_N : E_1 = W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1) \to L^p(\mathbb{R}, H_0) = E_0 \]
is compact since \( I_N \) is compact.

To see that \( I_N \) converges to \( I \) in the norm topology observe that
\[
\| I - I_N \|_{\mathcal{L}(E_1, E_0)} = \sup_{\| v \|_{E_1} = 1} \| (I - I_N)v \|_{E_0} = \sup_{\| v \|_{E_1} = 1} \| (\text{Id} - \pi_N)v \|_{L^p(\mathbb{R}, H_0)}.
\]
Observe that
\[
\| (\text{Id} - \pi_N)v \|_{L^p(\mathbb{R}, H_0)} = \left( \int_{-\infty}^{\infty} \left( \| (\text{Id} - \pi_N)v(s) \|_{H_0}^p \right)^{1/p} \right)^{1/p} \leq \frac{1}{f(N)} \| v \|_{L^p_\delta(\mathbb{R}, H_1)} \leq \frac{1}{f(N)} \| v \|_{W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1)}.
\]
This proves that \( \| I - I_N \|_{\mathcal{L}(E_1, E_0)} \leq 1/f(N) \). By unboundedness of \( f \) the sequence of compact operators \( I_N \) converges to \( I \) in norm. Thus the limit \( I \) is compact, too, by Theorem A.1.

In view of the fractal structure, i.e. all levels are self-similar (isometrically isomorphic), an immediate Corollary to Lemma 8.8 is the following.

**Corollary 8.9.** Pick reals \( p > 1 \) and \( \delta > 0 = \delta_0 \). Then each of the inclusions
\[ I : W^{1,p}_\delta(\mathbb{R}, H_k) \cap L^p_\delta(\mathbb{R}, H_{k+1}) \to L^p(\mathbb{R}, H_k), \quad k \in \mathbb{N}_0, \]
is a compact linear operator.

**Lemma 8.10** \((k \in \mathbb{N}, \delta_0 = 0)\). For \( p \in (1, \infty) \) and \( \delta > 0 \) the inclusion
\[ I : \bigcap_{i=0}^{k} W^{1,p}_\delta(\mathbb{R}, H_{k-i}) \to \bigcap_{i=0}^{k-1} W^{i,p}(\mathbb{R}, H_{k-1-i}) \]
is a compact linear operator. The Banach spaces \( W^{1,p}_\delta \) are defined by (3.4).
Proof. Lemma 8.10 follows from Lemma 8.8 \((k = 1)\) by induction similarly as in the Morse case (where Lemma 8.5 followed from Lemma 8.4). In order to illustrate the adjustments one has to do, we show how the case \(k = 2\) follows from the case \(k = 1\) which is Lemma 8.8.

Case \(k = 1 \Rightarrow k = 2\): Pick a sequence \(v_\nu\) in the unit ball of the space

\[ W_\delta^{2,p}(\mathbb{R}, H_0) \cap W_\delta^{1,p}(\mathbb{R}, H_1) \cap L_\delta^p(\mathbb{R}, H_2). \]

So \(v_\nu\) is a sequence in the unit ball of

\[ W_\delta^{1,p}(\mathbb{R}, H_1) \cap L_\delta^p(\mathbb{R}, H_2). \]

Hence by Corollary 8.9 for \(k = 1\) there is a subsequence, still denoted by \(v_\nu\), and an element \(v \in L^p(\mathbb{R}, H_1)\) such that

\[ v_\nu \to v \quad \text{in} \quad L^p(\mathbb{R}, H_1). \]

Moreover, the weak derivatives \(\dot{v}_\nu\) form a sequence in the unit ball of

\[ W_\delta^{1,p}(\mathbb{R}, H_0) \cap L_\delta^p(\mathbb{R}, H_1). \]

Hence by Lemma 8.8 there is a subsequence, still denoted by \(v_\nu\), and an element \(w \in L^p(\mathbb{R}, H_0)\) such that

\[ \dot{v}_\nu \to w \quad \text{in} \quad L^p(\mathbb{R}, H_0). \]

Similarly as in (8.12) one gets \(w = \dot{v}\). Hence \(v\) is in \(W^{1,p}(\mathbb{R}, H_0) \cap L^p(\mathbb{R}, H_1)\) and \(v_\nu \to v\) in \(W^{1,p}(\mathbb{R}, H_0) \cap L^p(\mathbb{R}, H_1)\). This shows that the inclusion

\[ W_\delta^{2,p}(\mathbb{R}, H_0) \cap W_\delta^{1,p}(\mathbb{R}, H_1) \cap L_\delta^p(\mathbb{R}, H_2) \hookrightarrow W^{1,p}(\mathbb{R}, H_1) \cap L^p(\mathbb{R}, H_2) \]

is a compact linear operator.

Case \(k \Rightarrow k + 1\): Follows along similar lines as \(k = 1 \Rightarrow k = 2\). \qed

Proof of Proposition 8.7. As in the Morse case, Lemma 8.10 implies Proposition 8.3 in view of the commutative diagram (8.13). \qed

A Background from Functional Analysis

A.1 Compact operators

A useful fact to prove compactness of a linear operator is that the space of compact operators is closed in the space of bounded linear operators with respect to the operator norm topology. We use this fact heavily in Section 8. For the readers convenience we recall in this section the proof of this well known fact.

Suppose \(E\) and \(F\) are Banach spaces. Let \(\mathcal{L}(E, F)\) be the Banach space of bounded linear operators \(T : E \to F\) whose operator norm defined by

\[ ||T|| = ||T||_{\mathcal{L}(E, F)} := \sup_{||x||_E = 1} ||Tx||_F \]
is finite. An operator $T \in \mathcal{L}(E, F)$ is called **compact** if the image under $T$ of any bounded sequence $x_\nu \in E$ admits a convergent subsequence. Since $T$ is linear it suffices to show this for sequences in the unit ball of $E$.

**Theorem A.1.** Let $T_\nu \in \mathcal{L}(E, F)$ be a sequence of compact linear operators which converges in the operator topology to $T \in \mathcal{L}(E, F)$. Then $T$ is compact.

**Proof.** Let $x_k \in E$ be a sequence in the unit ball of $E$. Because each $T_\nu$ is compact, by a diagonal argument there is a subsequence $(x_{k_j})_j$ such that each image sequence $(T_\nu x_{k_j})_j$ converges in $F$.

We claim that $(Tx_{k_j})_j$ is a Cauchy sequence in $F$. In order to see this pick $\varepsilon > 0$. Because $T_\nu \to T$ in the operator topology, there is $\nu_0 \in \mathbb{N}$ such that $\|T - T_{\nu_0}\| \leq \varepsilon/3$. Because the sequence $(T_{\nu_0} x_{k_j})_j$ converges in $F$, it is a Cauchy sequence. In particular, there is $j_0 \in \mathbb{N}$ such that for every $j_1, j_2 \geq j_0$ we have

$$\|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\|_F \leq \varepsilon/3.$$

We estimate

$$\|Tx_{k_{j_1}} - Tx_{k_{j_2}}\|_F$$

$$\leq \|Tx_{k_{j_1}} - T_{\nu_0} x_{k_{j_1}}\|_F + \|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\|_F + \|T_{\nu_0} x_{k_{j_2}} - T x_{k_{j_2}}\|_F$$

$$\leq \|T - T_{\nu_0}\| \cdot \|x_{k_{j_1}}\|_E + \|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\|_F + \|T - T_{\nu_0}\| \cdot \|x_{k_{j_2}}\|_E$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \cdot 1 = \varepsilon.$$

This shows that $Tx_{k_j}$ is a Cauchy sequence in $F$. Since $F$ is a Banach space each Cauchy sequence converges. Hence the linear operator $T$ is compact. \qed

### A.2 Hilbert space valued Sobolev theory

In this appendix we recall Sobolev theory for Hilbert space valued functions in the separable case. For a more general treatment, which as well treats non-separable Banach spaces, see e.g. [Coh93, Eva98, Kre15, Yos95]. In particular, we recall that Hilbert valued Sobolev spaces are complete and therefore Banach spaces.

Suppose $H$ is a separable Hilbert space, i.e. $H$ is isometrically isomorphic to $\ell^2$ or of finite dimension, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Throughout $H$ is endowed with the **Borel $\sigma$-algebra** $\mathcal{B} = \mathcal{B}(H)$, i.e. the smallest $\sigma$-algebra that contains the open sets. Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ be the Borel $\sigma$-algebra on the real line and $\mathcal{A} = \mathcal{A}(\mathbb{R})$ be the Lebesgue $\sigma$-algebra. The elements of a $\sigma$-algebra are called **measurable sets**. Recall that a map is called measurable if pre-images of measurable sets are measurable.

**The Banach space $L^1(\mathbb{R}, H)$**

We need the following theorem of Pettis which makes use of the fact that our Hilbert space is separable.
Theorem A.2 (Pettis [Pet38]). Consider a Hilbert space valued function \( f : \mathbb{R} \to H \). The following assertions are equivalent.

1) Every function \( \langle f, x \rangle : (\mathbb{R}, A) \to (\mathbb{R}, B) \), where \( x \in H \), is measurable.

2) The map \( f : (\mathbb{R}, A) \to (H, B) \) is measurable.

Remark A.3. That 2) implies 1) follows from two facts in measure theory. Firstly, continuous maps are measurable and, secondly, compositions of measurable maps are measurable.

Remark A.4. By the same reasoning as in Remark A.3 if \( f : \mathbb{R} \to H \) meets one, hence both, conditions in Theorem A.2 it follows that the map \( \|f\| : (\mathbb{R}, A) \to (\mathbb{R}, B) \) is measurable.

Definition A.5. A Hilbert space valued function \( f : \mathbb{R} \to H \) is called integrable if it satisfies the following two conditions.

(i) Every function \( \langle f, x \rangle : (\mathbb{R}, A) \to (\mathbb{R}, B) \), where \( x \in H \), is measurable.

(ii) The integral \( \int_{\mathbb{R}} \|f(t)\| \, dt < \infty \) is finite.

By \( \mathcal{L}^1(\mathbb{R}, H) \) we denote the set of integrable Hilbert space valued functions.

Proposition A.6. The set \( \mathcal{L}^1(\mathbb{R}, H) \) is a real vector space.

Proof. Suppose \( f, g \in \mathcal{L}^1(\mathbb{R}, H) \) and \( \lambda, \mu \in \mathbb{R} \) we need to show that \( \lambda f + \mu g \) satisfies (i) and (ii) in the definition. Part (i) follows from the fact that the space of measurable functions \( (\mathbb{R}, A) \to (\mathbb{R}, B) \) is a vector space. To prove part (ii) we observe that by part (i) combined with Pettis' Theorem A.2, 1) \( \Rightarrow \) 2), and Remark A.4 the map \( \|\lambda f + \mu g\| : (\mathbb{R}, A) \to (\mathbb{R}, B) \) is measurable. Therefore the integral

\[
\int_{\mathbb{R}} \|\lambda f + \mu g\| \leq |\lambda| \int_{\mathbb{R}} \|f\| + |\mu| \int_{\mathbb{R}} \|g\| < \infty
\]

is finite. This proves part (ii), hence the proposition. \( \square \)

Proposition A.7. The vector space \( \mathcal{L}^1(\mathbb{R}, H) \) is complete with respect to the semi-norm defined by

\[
\|f\|_1 := \|f\|_{\mathcal{L}^1(\mathbb{R}, H)} := \int_{\mathbb{R}} \|f(t)\| \, dt.
\]

Proof. Fix a Cauchy sequence \( f_\nu \in \mathcal{L}^1(\mathbb{R}, H) \). Motivated by the real valued case, see e.g. Rudin [Rud87, Ch. 3] or [Sal16, Thm. 4.9], pick a subsequence, still denoted by \( f_\nu \), such that each difference \( \|f_\nu - f_{\nu+1}\|_1 \) is less than \( 2^{-\nu} \). That is

\[
\|f_\nu - f_{\nu+1}\|_{\mathcal{L}^1(\mathbb{R}, H)} = \int_{\mathbb{R}} \|f_\nu(t) - f_{\nu+1}(t)\| \, dt \leq 2^{-\nu}, \quad \nu \in \mathbb{N}.
\]
**Claim 1.** The infinite sum of the $g_\nu$'s is finite outside a null set. More precisely, there is a function $g : \mathbb{R} \to [0, \infty)$ and a Lebesgue null set $N \subset \mathbb{R}$ such that

$$g = \sum_{\nu=1}^{\infty} g_\nu, \quad \text{on } \mathbb{R} \setminus N.$$  

*Proof of Claim 1.* Setting $G_n := \sum_{\nu=1}^{n} g_\nu$ we obtain a pointwise monotone sequence $G_n \leq G_{n+1}$ since the $g_\nu$ are non-negative. Define $G : \mathbb{R} \to [0, \infty]$ by

$$G(t) := \sum_{\nu=1}^{\infty} g_\nu(t), \quad \text{for } t \in \mathbb{R}.$$  

The Lebesgue monotone convergence theorem, see e.g. [Sal16, Thm. 1.37], asserts that the function $G$ is measurable and provides the first step in

$$\int_{\mathbb{R}} G(t) \, dt = \lim_{n \to \infty} \int_{\mathbb{R}} G_n(t) \, dt = \lim_{n \to \infty} \int_{\mathbb{R}} \sum_{\nu=1}^{n} g_\nu(t) \, dt = \sum_{\nu=1}^{\infty} \int_{\mathbb{R}} g_\nu(t) \, dt \leq 1.$$  

The inequality uses (A.14). By finiteness of the integral $G$ can only take on an infinite value on a set $N$ of measure zero. Consequently the function defined by

$$g(t) := \begin{cases} G(t) & , t \in \mathbb{R} \setminus N, \\ 0 & , t \in N, \end{cases}$$

is pointwise finite. It is also measurable.  

**Claim 2.** Outside the null set $N$ from Claim 1, that is for $t \in \mathbb{R} \setminus N$, the sequence $f_\nu(t)$ is Cauchy in $H$.

*Proof of Claim 2.* Given $t \in \mathbb{R} \setminus N$, pick $\varepsilon > 0$. Because $G(t) = \sum_{\nu=1}^{\infty} g_\nu(t) < \infty$ is finite, there is an index $\nu_0 = \nu_0(\varepsilon)$ such that $\sum_{\nu=\nu_0}^{\infty} g_\nu(t) < \varepsilon$. Pick $\nu_2 \geq \nu_1 \geq \nu_0$. We estimate

$$\| (f_{\nu_1} - f_{\nu_2}) (t) \| \leq \sum_{\nu=\nu_1}^{\nu_2-1} \| (f_{\nu+1} - f_{\nu}) (t) \| \leq \sum_{\nu=\nu_1}^{\nu_2-1} \| (f_{\nu+1} - f_{\nu}) (t) \| g_\nu(t) \leq \sum_{\nu=\nu_0}^{\infty} g_\nu(t) < \varepsilon.$$  

This proves Claim 2.
Because $H$ is complete it follows from Claim 2 that for all $t \in \mathbb{R} \setminus N$ the limit $\lim_{\nu \to \infty} f_{\nu}(t) \in H$ exists. We obtain a function $f : \mathbb{R} \to H$ by defining

$$f(t) := \begin{cases} 
\lim_{\nu \to \infty} f_{\nu}(t), & t \in \mathbb{R} \setminus N, \\
0, & t \in N.
\end{cases}$$

By Theorem A.2 of Pettis measurability of $f : \mathbb{R} \to H$ is equivalent to the following claim.

**Claim 3.** (Measurability of $f : \mathbb{R} \to H$) The function $\varphi_x := \langle f, x \rangle : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable $\forall x \in H$.

**Proof of Claim 3.** Define the function $\varphi_{x,\nu} : \mathbb{R} \to \mathbb{R}$ by $t \mapsto \langle x, f_{\nu}(t) \rangle$. Note that for every $t \in \mathbb{R} \setminus N$ one has $\lim_{\nu \to \infty} \varphi_{x,\nu}(t) = \varphi_x(t)$, because $f_{\nu}(t) \to f(t)$ by definition of $f$. Therefore $\varphi_x$ is up to the set $N$ of measure zero the pointwise limit of a sequence of measurable functions and hence itself measurable. □

**Claim 4.** (Convergence) $\lim_{\nu \to \infty} \|f - f_{\nu}\|_{L^1(\mathbb{R}, H)} = 0$.

**Proof of Claim 4.** Given $\varepsilon > 0$, choose $\nu$ such that $1/2^{\nu-1} < \varepsilon$. Using Fatou’s Lemma to obtain the first inequality we estimate

$$\|f - f_{\nu}\|_{L^1(\mathbb{R}, H)} = \int_{\mathbb{R}} \|f_{\nu}(t) - f(t)\| \, dt = \int_{\mathbb{R}} \liminf_{k \to \infty} \|f_{\nu}(t) - f_k(t)\| \, dt \leq \liminf_{k \to \infty} \int_{\mathbb{R}} \|f_{\nu}(t) - f_k(t)\| \, dt \leq \sum_{j=0}^{k-1} \int_{\mathbb{R}} \|f_j(t) - f_{j+1}(t)\| \, dt \leq \liminf_{k \to \infty} \sum_{j=\nu}^{k-1} \int_{\mathbb{R}} \|f_j(t) - f_{j+1}(t)\| \, dt \leq \sum_{j=\nu}^{\infty} \frac{1}{2^j} = \frac{1}{2^{\nu-1}} < \varepsilon.$$ 

This proves Claim 4. □

By Claim 4 the limit $f$ is in $L^1(\mathbb{R}, H)$ and $f_{\nu} \to f$ in $L^1(\mathbb{R}, H)$. This proves Proposition A.7.

On $L^1(\mathbb{R}, H)$ consider the equivalence relation $f \sim g$ if the two maps are equal outside a set of measure zero. On the quotient space

$$L^1(\mathbb{R}, H) := L^1(\mathbb{R}, H) / \sim$$

the semi-norm $\|\cdot\|_1$ is a norm. Hence $L^1(\mathbb{R}, H)$ is a Banach space by Proposition A.7. By abuse of notation we denote the elements of $L^1(\mathbb{R}, H)$ still by $f$. 

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The Banach spaces $L^p(\mathbb{R}, H)$

Similarly for $p \in (1, \infty)$ one calls a Hilbert space valued function $f : \mathbb{R} \to H$ \textbf{$p$-integrable} if it satisfies (i) in Definition A.5 and (ii) is replaced by finiteness of the $p$-semi-norm

$$
\|f\|_p := \|f\|_{L^p(\mathbb{R}, H)} := \left( \int_{\mathbb{R}} \|f(t)\|^p \, dt \right)^{\frac{1}{p}}.
$$

Let $L^p(\mathbb{R}, H)$ be the set of all $p$-integrable functions $f : \mathbb{R} \to H$. As in Propositions A.6 and A.7 one shows that $L^p(\mathbb{R}, H)$ is a vector space which is complete with respect to the $p$-semi-norm. On the quotient space

$$
L^p(\mathbb{R}, H) := L^p(\mathbb{R}, H)/ \sim
$$

the semi-norm $\|\cdot\|_p$ is a norm. Hence $L^p(\mathbb{R}, H)$ is a Banach space. Again we denote the elements $[f]$ of $L^p(\mathbb{R}, H)$ still by $f$.

The Sobolev space $W^{1,p}(\mathbb{R}, H)$

Fix $p \in [1, \infty)$ and let $W^{1,p}(\mathbb{R}, H)$ be the vector space of all $f \in L^p(\mathbb{R}, H)$ for which there exists an element $v \in L^p(\mathbb{R}, H)$ such that

$$
\int_{\mathbb{R}} \langle f(t), \dot{\varphi}(t) \rangle \, dt = - \int_{\mathbb{R}} \langle v(t), \varphi(t) \rangle \, dt
$$

for every $\varphi \in C^\infty_c(\mathbb{R}, H)$. If such a map $v$ exists, then it is unique and called the \textbf{weak derivative} of $f$. We denote $v$ by the symbol $\dot{f}$ or $f'$. The vector space $W^{1,p}(\mathbb{R}, H)$ is endowed with the norm

$$
\|f\|_{W^{1,p}} := \|f\|_p + \|\dot{f}\|_p.
$$

\textbf{Proposition A.8.} The space $W^{1,p}(\mathbb{R}, H)$ is a Banach space.

\textit{Proof.} Let $f_\nu$ by a Cauchy sequence in $W^{1,p}(\mathbb{R}, H)$. Hence $f_\nu$ forms a Cauchy sequence in $L^p(\mathbb{R}, H)$ as well as the weak derivatives $\dot{f_\nu}$. By completeness of $L^p(\mathbb{R}, H)$ there are elements $f, v \in L^p(\mathbb{R}, H)$ such that

$$
f_\nu \xrightarrow{L^p} f, \quad \dot{f_\nu} \xrightarrow{L^p} v.
$$

In view of Lemma A.9 we compute

$$
\int_{\mathbb{R}} \langle f(t), \dot{\varphi}(t) \rangle \, dt = \lim_{\nu \to \infty} \int_{\mathbb{R}} \langle f_\nu(t), \dot{\varphi}(t) \rangle \, dt
$$

$$
= - \lim_{\nu \to \infty} \int_{\mathbb{R}} \langle \dot{f_\nu}(t), \varphi(t) \rangle \, dt
$$

$$
= - \int_{\mathbb{R}} \langle v(t), \varphi(t) \rangle \, dt.
$$

This shows that $v$ is the weak derivative of $f$. Hence $f \in W^{1,p}(\mathbb{R}, H)$ and $f_\nu \to f$ in $W^{1,p}(\mathbb{R}, H)$. This shows completeness and proves Proposition A.8. \qed
Lemma A.9. Let $p \in [1, \infty)$. Let $f_\nu \in L^p(\mathbb{R}, H)$ be a sequence that converges to an element $f \in L^p(\mathbb{R}, H)$ and $\varphi \in C_0^\infty(\mathbb{R}, H)$ is of compact support. Then

$$\int_\mathbb{R} \langle f, \varphi \rangle \, dt = \lim_{\nu \to \infty} \int_\mathbb{R} \langle f_\nu, \varphi \rangle \, dt.$$ 

Proof. The support of $\varphi$ is contained in $[-T, T]$ for $T > 0$ sufficiently large. Moreover, there is a constant $c > 0$ such that $\|\varphi(t)\| \leq c$ for every $t \in \mathbb{R}$. Let $q$ be such that $1/p + 1/q = 1$. We estimate

$$\left| \int_\mathbb{R} (f(t) - f_\nu(t), \varphi(t)) \, dt \right| = \left| \int_{-T}^{T} (f(t) - f_\nu(t), \varphi(t)) \, dt \right|$$

$$\leq \int_{-T}^{T} \|f(t) - f_\nu(t)\| \cdot \|\varphi(t)\| \, dt$$

$$\leq c \int_{-T}^{T} 1 \cdot \|f(t) - f_\nu(t)\| \, dt$$

$$\leq c(2T)^{1/q} \left( \int_{-T}^{T} \|f(t) - f_\nu(t)\|^p \, dt \right)^{1/p}$$

$$\leq c(2T)^{1/q} \|f - f_\nu\|_{L^p(\mathbb{R}, H)}.$$ 

The first inequality uses the Cauchy-Schwarz inequality in the Hilbert space $H$. The third inequality uses Hölder for real valued functions. □

The Sobolev spaces $W^{k,p}(\mathbb{R}, H)$

Recursively, for $k \in \mathbb{N}$, we define $W^{k+1,p}(\mathbb{R}, H)$ to be the space of all functions $f \in W^{1,p}(\mathbb{R}, H)$ whose weak derivative $\dot{f}$ lies in $W^{k,p}(\mathbb{R}, H)$. The vector space $W^{k+1,p}(\mathbb{R}, H)$ is endowed with the norm $\|f\|_{k+1,p} := \|f\|_p + \|\dot{f}\|_{k,p}$. Using the argument in the proof of Proposition A.8 inductively we obtain that $W^{k+1,p}(\mathbb{R}, H)$ is a Banach space.

Proposition A.10. The space $W^{k,p}(\mathbb{R}, H)$ is a Banach space whenever $k \in \mathbb{N}_0$ and $p \in [1, \infty)$.

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