On self-dual Yang–Mills fields in eight and seven dimensions

E.K. Loginov
Physics Department, Ivanovo State University
Ermaka St. 39, Ivanovo, 153025, Russia

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Abstract
The self-duality equations for gauge fields in pseudoeuclidean spaces of eight and seven dimensions are considered. Some new classes of solutions of the equations are found.

1 Introduction
In 1983 Corrigan et al. [1] have proposed a generalization of the self-dual Yang–Mills equations in dimension $d > 4$:

$$f_{mnp}F^{ps} = \lambda F_{mn},$$

(1)

where the numerical tensor $f_{mnp}$ is completely antisymmetric and $\lambda = const$ is a non-zero eigenvalue. By the Bianchi identity $D[pF_{mn}] = 0$, it follows that any solution of (1) is a solution of the Yang–Mills equations $D_m F_{mn} = 0$. Some of these solutions have found in [2].

The many-dimensional Yang–Mills equations appear in the low-energy effective theory of the heterotic string [3] and in the many-dimensional theories of supergravity [4]. In addition, there is a hope that Higgs fields and supersymmetry can be understood through dimensional reduction from $d > 4$ dimensions down to $d = 4$ [5].

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The paper is organized as follows. Section 2 contains well-known facts about Cayley-Dickson algebras and connected with them Lie algebras. Sections 3 contains the main results.

2 Cayley-Dickson algebra

Let us recall that the algebra $A$ satisfying the identities

$$x^2y = x(xy), \quad yx^2 = (yx)x$$

is called alternative. It is obvious that any associative algebra is alternative. The most important example of nonassociative alternative algebra is Cayley-Dickson algebra. Let us recall its construction (see [6]).

Let $A$ be an algebra with an involution $x \rightarrow \bar{x}$ over a field $F$ of characteristic $\neq 2$. Given a nonzero $\alpha \in F$ we define a multiplication on the vector space $(A, \alpha) = A \oplus Ae$ by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - \alpha \bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2).$$

This makes $(A, \alpha)$ an algebra over $F$. It is clear that $A$ is isomorphically embedded into $(A, \alpha)$ and $\dim(A, \alpha) = 2\dim A$. Let $e = (0, 1)$. Then $e^2 = -\alpha$ end $(A, \alpha) = A \oplus Ae$. Given any $z = x + ye$ in $(A, \alpha)$ we suppose $\bar{z} = \bar{x} - ye$. Then the mapping $z \rightarrow \bar{z}$ is an involution in $(A, \alpha)$.

Starting with the base field $F$ the Cayley-Dickson construction leads to the following tower of alternative algebras:

1) $F$, the base field.
2) $\mathbb{C}(\alpha) = (F, \alpha)$, a field if $x^2 + \alpha$ is the irreducible polynomial over $F$; otherwise, $\mathbb{C}(\alpha) \simeq F \oplus F$.
3) $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta)$, a generalized quaternion algebra. This algebra is associative but not commutative.
4) $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma)$, a Cayley-Dickson algebra. Since this algebra is not associative the Cayley-Dickson construction ends here.

The algebras in 1) – 4) are called composition. Any of them has the non-degenerate quadratic form (norm) $n(x) = x\bar{x}$, such that $n(xy) = n(x)n(y)$. In particular, over the field $\mathbb{R}$ of real numbers, the above construction gives 3 split algebras (e.g., if $\alpha = \beta = \gamma = -1$) and 4 division algebras (if $\alpha = \beta = \gamma = 1$): the fields of real $\mathbb{R}$ and complex $\mathbb{C}$ numbers, the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, taken with the Euclidean norm $n(x)$. 

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Note also that any simple nonassociative alternative algebra is isomorphic to Cayley-Dickson algebra $O(\alpha, \beta, \gamma)$.

Let $A$ be Cayley-Dickson algebra and $x \in A$. Denote by $R_x$ and $L_x$ the operators of right and left multiplication in $A$

$$R_x : a \to ax, \quad L_x : a \to xa.$$ 

It follows from (2) that

$$R_{ab} - R_a R_b = [R_a, L_b] = [L_a, R_b] = L_{ba} - L_a L_b.$$  \hspace{1cm} (3)

Consider the Lie algebra $L(A)$ generated by all operators $R_x$ and $L_x$ in $A$. Choose in $L(A)$ the subspaces $R(A), S(A)$, and $D(A)$ generated by the operators $R_x, S_x = R_x + 2L_x$, and $2D_{x,y} = [S_x, S_y] + S_{[x,y]}$ respectively. Using (3), it is easy to prove that

$$3[R_x, R_y] = D_{x,y} + S_{[x,y]}, \hspace{1cm} (4)$$

$$[R_x, S_y] = R_{[x,y]}, \hspace{1cm} (5)$$

$$[R_x, D_{y,z}] = R_{[x,y,z]}, \hspace{1cm} (6)$$

$$[S_x, S_y] = D_{x,y} - S_{[x,y]}, \hspace{1cm} (7)$$

$$[S_x, D_{y,z}] = S_{[x,y,z]}, \hspace{1cm} (8)$$

$$[D_{x,y}, D_{z,t}] = D_{[x,z,t],y} + D_{x,[y,z,t]}, \hspace{1cm} (9)$$

where $[x, y, z] = [x, [y, z]] - [y, [z, x]] - [z, [x, y]]$. It follows from (4)–(9) that the algebra $L(A)$ is decomposed in the direct sum

$$L(A) = R(A) \oplus S(A) \oplus D(A)$$

of the Lie subalgebras $D(A), D(A) \oplus S(A)$ and the vector space $R(A)$.

In particular, if $A$ is a real division algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to the compact Lie algebras $g_2$ and $so(7)$ respectively. If $A$ is a real split algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to noncompact Lie algebras $g'_2$ and $so(3, 4)$.

### 3 Solutions of the self-duality equations

Let $A$ be a real linear space equipped with a nondenerate symmetric metric $g$ of signature $(8, 0)$ or $(4, 4)$. Choose the basis $\{1, e_1, \ldots, e_7\}$ in $A$ such that

$$g = \text{diag}(1, -\alpha, -\beta, \alpha \beta, -\gamma, \alpha \gamma, \beta \gamma, -\alpha \beta \gamma), \hspace{1cm} (10)$$

where \( \alpha, \beta, \gamma = \pm 1 \). Define the multiplication
\[
e_i e_j = -g_{ij} + c_{ij}^k e_k,
\]
where the structural constants \( c_{ijk} = g_{ks} c_{ij}^s \) are completely antisymmetric and different from 0 only if
\[
c_{123} = c_{145} = c_{167} = c_{246} = c_{374} = c_{365} = 1.
\]
The multiplication (11) transforms \( A \) into a linear algebra. It can easily be checked that the algebra \( A \simeq O(\alpha, \beta, \gamma) \). In the basic \( \{1, e_1, \ldots, e_7\} \) the operators
\[
R_{e_i} = e_{i0} + \frac{1}{2} c_{ij}^k e_{jk}, \quad L_{e_i} = e_{i0} - \frac{1}{2} c_{ij}^k e_{jk},
\]
where \( e_{ij} \) are generators of Lie algebra \( \mathcal{L}(A) \) satisfying the switching relations
\[
[e_{mn}, e_{ps}] = g_{mp} e_{ns} - g_{ms} e_{np} - g_{np} e_{ms} + g_{ns} e_{mp}.
\]
\[(12)\]
Now, let \( H \) and \( G \) be matrix Lie groups constructed by the Lie algebras \( D(A) \) and \( D(A) \oplus S(A) \) respectively. In the space \( A \) equipped with a metric (10) we define the completely antisymmetric \( H \)-invariant tensor \( h_{ijkl} \)
\[
h_{ijkl} = g_{ij} g_{jk} - g_{ik} g_{jl} + c_{ijk} c_{kl}^m,
\]
\[
h_{ij00} = 0,
\]
and the completely antisymmetric \( G \)-invariant tensor \( f_{mnps} \)
\[
f_{ijkl} = h_{ijkl}, \quad f_{ij00} = c_{ijk},
\]
where \( i, j, k, l \neq 0 \). The tensors \( f_{mnps} \) and \( h_{mnps} \) satisfy the identities (cp. [7])
\[
f_{mnrij} f^{pqst} = \delta^{(p}_{m} \delta^{s}_{n} \delta^{t}_{r} ) - \delta^{(p}_{m} \delta^{s}_{r} \delta^{t}_{n} ) + f_{mn}^{(ps}_{r} \delta^{t)}),
\]
\[
h_{mnrij} h^{pqst} = \delta^{(p}_{m} \delta^{s}_{n} \delta^{t}_{r} ) - \delta^{(p}_{m} \delta^{s}_{r} \delta^{t}_{n} ) + h_{mn}^{(ps}_{r} \delta^{t)} - c_{mnrc}^{(p}_{s} \delta^{t}_{r} )\]
\[
f_{mnij} f_{ps}^{ij} = 6(g_{mp} g_{ns} - g_{ms} g_{np}) + 4f_{mnps},
\]
\[
h_{mnij} h_{ps}^{ij} = 4(g_{mp} g_{ns} - g_{ms} g_{np}) + 2h_{mnps}.
\]
\[(13)-(16)\]
where \( (pst) \) is cyclic sum for the indexes. Define the projectors \( \tilde{f}_{mnps} \) and \( \tilde{h}_{mnps} \) of \( \mathcal{L}(A) \) onto the subspaces \( D(A) \oplus S(A) \) and \( D(A) \) respectively by
\[
\tilde{f}_{mnps} = \frac{1}{8} (3g_{mp} g_{ns} - 3g_{ms} g_{np} - f_{mnps}),
\]
\[
\tilde{h}_{mnps} = \frac{1}{6} (2g_{mp} g_{ns} - 2g_{ms} g_{np} - h_{mnps}).
\]
It follows from (15)–(16) that
\[ f_{mnij} \tilde{f}_{ps}^{ij} = -2 \tilde{f}_{mnps}, \quad (17) \]
\[ h_{mnij} \tilde{h}_{ps}^{ij} = -2 \tilde{h}_{mnps}. \quad (18) \]
It is obvious that the elements
\[ \tilde{f}_{mn} = \tilde{f}_{mn}^{ij} e_{ij}, \quad (19) \]
\[ \tilde{h}_{mn} = \tilde{h}_{mn}^{ij} e_{ij}, \quad (20) \]
generate the subspaces \( D(A) \oplus S(A) \) and \( D(A) \) respectively. Using the identities (12)–(14), we get the switching relations
\[ [\tilde{f}_{mn}, \tilde{f}_{ps}] = \frac{3}{4} (\tilde{f}_{m[p} g_{s|n} - \tilde{f}_{n[p} g_{s|m}) - \frac{1}{8} (f_{mn}^{k[p} \tilde{f}_{s]k} - f_{ps}^{k[m} \tilde{f}_{n]k}), \quad (21) \]
\[ [\tilde{h}_{mn}, \tilde{h}_{ps}] = \frac{2}{3} (\tilde{h}_{m[p} g_{s|n} - \tilde{h}_{n[p} g_{s|m}) - \frac{1}{6} (h_{mn}^{k[p} \tilde{h}_{s]k} - h_{ps}^{k[m} \tilde{h}_{n]k}). \quad (22) \]
Now we can find solutions of (1). We choose the ansatzs (cp. [2])
\[ A_m(x) = \frac{4}{3} \frac{\tilde{f}_{mi} x^i}{\lambda^2 + x_k x^k}; \]
\[ B_m(x) = \frac{3}{2} \frac{\tilde{h}_{mi} x^i}{\lambda^2 + x_k x^k}. \]
Using the switching relations (21)–(22), we get
\[ F_{mn}(x) = -\frac{4}{9} (6 \lambda^2 + 3 x_i x^i) \tilde{f}_{mn} + \frac{8 \tilde{f}_{mi} \tilde{f}_{sj} x^i x^j}{(\lambda^2 + x_k x^k)^2}; \]
\[ G_{mn}(x) = -\frac{3}{2} \frac{(2 \lambda^2 + x_i x^i) \tilde{h}_{mn} + 3 \tilde{h}_{mi} \tilde{h}_{sj} x^i x^j}{(\lambda^2 + x_k x^k)^2}. \]
It follows from (17)–(20) that the tensors \( F_{mn} \) and \( G_{mn} \) are self-dual. If the metric (10) is Euclidean, then we have the well-known solutions of equations (1) (see. [2]). If the metric (10) is pseudoeuclidean, then we have new solutions.
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