Topologically protected qubits as minimal Josephson junction arrays with non trivial boundary conditions: a proposal

Gerardo Cristofano\textsuperscript{1}, Vincenzo Marotta\textsuperscript{2}, Adele Naddeo\textsuperscript{3}, Giuliano Niccoli\textsuperscript{4}

Abstract

Recently a one-dimensional closed ladder of Josephson junctions has been studied \cite{1} within a twisted conformal field theory (CFT) approach \cite{2, 3} and shown to develop the phenomenon of flux fractionalization \cite{4}. That led us to predict the emergence of a topological order in such a system \cite{5}. In this letter we analyze the ground states and the topological properties of fully frustrated Josephson junction arrays (JJA) arranged in a Corbino disk geometry for a variety of boundary conditions. In particular minimal configurations of fully frustrated JJA are considered and shown to exhibit the properties needed in order to build up a solid state qubit, protected from decoherence. The stability and transformation properties of the ground states of the JJA under adiabatic magnetic flux changes are analyzed in detail in order to provide a tool for the manipulation of the proposed qubit.

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\textsuperscript{1} Dipartimento di Scienze Fisiche, Università di Napoli “Federico II” and INFN, Sezione di Napoli, Via Cintia, Compl. universitario M. Sant’Angelo, 80126 Napoli, Italy

\textsuperscript{2} Dipartimento di Scienze Fisiche, Università di Napoli “Federico II” and INFN, Sezione di Napoli, Via Cintia, Compl. universitario M. Sant’Angelo, 80126 Napoli, Italy

\textsuperscript{3} Dipartimento di Fisica “E. R. Caianiello”, Università degli Studi di Salerno and CNISM, Unità di Ricerca di Salerno, Via Salvador Allende, 84081 Baronissi (SA), Italy

\textsuperscript{4} Theoretical Physics Group, DESY, NotkeStraße 85 22603, Hamburg, Germany.
1 Introduction

Today the physical realization of a quantum computer represents a very hard task because of the limits imposed by decoherence and because interactions between qubits cannot be well controlled. A lot of solid state qubit implementations have been proposed in the last ten years; among them, systems based on Josephson junctions [6] are promising because of the well assessed fabrication technology, which could allow a high degree of scalability. But anyway the path towards the physical realization of a fully integrated quantum computer still remains a remarkable challenge. Recently topologically ordered quantum systems have been proposed as physical analogues of quantum error-correcting codes [7] starting from Kitaev seminal work [8]: all such systems share a built in protection from decoherence. The idea of Kitaev involves a protected subspace created by a topological degeneracy of the ground state: such a degeneracy is typically due to a conservation law such as the conservation of the parity of the number of particles along some long contour. The underlying idea in Kitaev work is the notion of topological order, that is a new kind of order which cannot be due to a spontaneous symmetry breakdown. In such a case the system lacks of local order parameters but displays a weak form of order, which is sensitive to the topology of the underlying two dimensional manifold. Such a concept was first introduced in order to describe the ground state of a quantum Hall fluid [9] but today it is of much more general interest [10][11]. Two features of topological order are very striking: fractionally charged quasiparticles and a ground state degeneracy depending on the topology of the underlying manifold, which is lifted by quasiparticles tunneling processes. In general a system is in a topological phase if its low-energy, long-distance effective field theory is a topological quantum field theory that is, if all of its physical correlation functions are topologically invariant up to corrections of the form $e^{-\Delta T}$ at temperature $T$ for some nonzero energy gap $\Delta$. More recently superconductors have been proposed [12] in which superconductivity arises from a topological mechanism rather than from a Ginzburg-Landau paradigm: the key feature is a mapping on an effective Chern-Simons gauge theory, which turns out to be exact in the case of JJA and frustrated JJA [13]. Non-Abelian quantum Hall states, in particular the $\nu = \frac{5}{2}$ one, appear also very promising in that the quasiparticle excitations are non-Abelian anyons [14] obeying non-Abelian braiding statistics: in such a case quantum information is codified in states with multiple quasiparticles, which show a topological degeneracy. Furthermore topological phases have been recently recognized in chiral $p$-wave superconductors [15], such as $Sr_2RuO_4$, and ultra-cold atomic gases [16]. Large and small size Josephson junction arrays of special geometry [17][18] have been proposed as well, which share the property that, in the classical limit for the local superconducting variables, the ground state is highly degenerate. The residual quantum processes within such a low energy subspace lift the classical degeneracy in favor of macroscopic coherent superpositions of classical ground states [18]. The protected degeneracy in all such systems emerges as a natural property of the lattice Chern-Simons gauge theories which describe them [18]. In general, if a physical system has topological degrees of freedom that are insensitive to local perturbations (that is noise), then information contained in those degrees of freedom would be
automatically protected against errors caused by local interactions with the environment [8]. The procedure implemented in all such realizations of protected JJA based qubits runs as follows. A quantum system is required with $2^K$ quantum states ($K$ being the number of big openings in the Josephson systems under study) which are degenerate in the absence of external perturbations and are robust against local random fluctuations. This means that the Hilbert space should contain a $2^K$-dimensional subspace characterized by the crucial property that any local operator $\hat{O}$ has only state-independent diagonal matrix elements up to vanishingly small corrections: $\langle n | \hat{O} | m \rangle = O_0 \delta_{mn} + o[\exp (-L)]$, $L$ being the system size. So a possible answer to such a highly non-trivial requirement could be a system with a protected subspace built up by a topological degeneracy of the ground state [8]. An alternative procedure could be to exhibit a low-energy effective field theory for the system under study which is a topological one and whose vacua are topologically degenerate and robust against noise. Following such an approach, we were able to predict the emergence of topological order in a fully frustrated Josephson junction ladder (JJA) with Mobius boundary conditions [5, 4] and to build up a protected subspace with $2^K$ quantum states, $K = 1$, which could be identified with the two states of a “protected” qubit [1]. The low-energy effective field theory which we set up is a twisted conformal field theory, the Twisted Model (TM) [2, 3]: it accounts very well for the topological properties of the system under study [5, 4]. This finding is crucial and relies on the well known relation between Chern-Simons gauge theories in $(2 + 1)$-dimensions and 2-dimensional CFT [19].

The aim of this letter is to analyze the ground states and the topological properties of fully frustrated Josephson junction arrays (JJA) arranged in a Corbino disk geometry for a variety of boundary conditions, employing a twisted CFT approach [2, 3] which has been successfully applied to quantum Hall systems in the presence of impurities or defects [20, 21, 22] and to the study of the phase diagram of the fully frustrated $XY$ model ($FFXY$) on a square lattice [23]. In this way we extend to a more general and more involved system the results obtained in the simplest JJA case [5, 4, 1] and exhibit the minimal configuration and properties needed in order to build up a solid state qubit, protected from decoherence. We analyze in detail the stability and transformation properties of the ground state wave functions under adiabatic magnetic flux changes in order to provide a tool for the manipulation of the proposed qubit [24].

The letter is organized as follows.

In Section 2 we briefly discuss all the relevant phenomenology of 2-dimensional fully frustrated JJA with an emphasis on the underlying physical model, the $FFXY$ model on a square lattice [25]. The analysis will focus on some models which share the same $U(1) \otimes Z_2$ degenerate ground state [26] and are believed to be in the same universality class.

In Section 3 we recall some aspects of the $m$-reduction procedure, in particular we show how the $m = 2, p = 0$ case gives rise to the required $U(1) \otimes Z_2$ mixed symmetry
[23] of the *FFXY* model. Then we focus on the discrete version of such a procedure. In such a framework we give the whole primary field content of the theory on the plane.

In Section 4 we properly define a JJA on the torus topology, which is equivalent to the Corbino disk geometry with coinciding boundary, and introduce the magnetic translation operators. We build up the ground state wave functions in order to provide a physical identification of the characters of our CFT, the TM, as the components of the “center of charge” for such wave functions. Then we describe the general topological structure of the vacua on the torus corresponding to all the possible boundary conditions.

In Section 5 we study the stability and transformation properties of the four ground states of the JJA, arranged in the closed geometry under adiabatic magnetic flux changes through the central hole of the Corbino disk. That allows us to identify the two states of a possible flux qubit [27], protected from decoherence, and to provide a tool for its manipulation.

In Section 6 some comments and outlooks are given while in the Appendix the explicit expressions of the TM characters on the torus are presented in detail.

### 2 Fully frustrated Josephson junction arrays

Josephson junction arrays (JJA) are prototype systems to investigate a variety of phase transitions induced by thermal or quantum fluctuations [28]. If the superconducting islands are of submicron size, quantum fluctuations play a crucial role and drive the zero temperature Superconductor-Insulator (SI) phase transition [29]. There are two energy scales in JJA: the Josephson energy $J$, which is associated to the tunneling of Cooper pairs between neighboring islands, and the charging energy $U$, which is the energy needed in order to add an extra Cooper pair on a neutral island. A finite $U$ is responsible of quantum fluctuations of the phases $\{\varphi\}$ of the superconducting order parameter on each island. In the regime $J \gg U$, named the classical case, the fluctuations of the phases are small, the system is globally coherent and superconducting. Conversely, in the opposite regime $J \ll U$, strong quantum phase fluctuations prevent the array from reaching long-range phase coherence and make it a Mott insulator. Magnetic frustration can be introduced in a JJA by applying a magnetic field transversal to the plane of the array [30][31]; it can be defined as $f = \frac{\Phi}{\Phi_0}$, where $\Phi$ is the magnetic flux threading each plaquette and $\Phi_0 = \frac{hc}{2e}$ is the superconducting flux quantum. The external magnetic field induces vortices in the array and, in the case of rational frustration $f = \frac{p}{q}$, the ground state shows a checkerboard configuration of vortices on a $q \times q$ elementary supercell. In the case of full frustration, i.e. $f = \frac{1}{2}$, of interest to us here, there are two degenerate ground states built of a vortex lattice with a $2 \times 2$ elementary supercell; the corresponding current flows either clockwise or anticlockwise in each plaquette, giving rise to a chiral phase configuration. Quantum fluctuations affect the superconducting regime but do not
destroy the checkerboard structure of the ground state \[32\]. All the relevant physics of the JJA is captured by the quantum phase model (QPM), whose Hamiltonian is:

\[
H_{\text{QPM}} = \sum_{i,j} (n_i - n_x) U_{ij} (n_j - n_x) - J \sum_{(ij)} \cos (\varphi_i - \varphi_j - A_{ij}),
\]

(1)

where \(n_i\) and \(\varphi_i\) are canonically conjugated variables defined on the sites and satisfying the commutation relations \([\varphi_i, n_j] = 2 \epsilon_i \delta_{ij}\), the second sum is over nearest neighbors, \(J > 0\) is the Josephson energy, the matrix \(U_{ij} = 4 \epsilon_i \epsilon_j \) describes the Coulomb interaction (\(C_{ij}\) is the capacitance matrix), the external voltage \(V_x\) enters through the induced charge \(n_x\) and fixes the average charge on each island and \(A_{ij} = \frac{2}{\hbar c} \int_i^j A \cdot dl\) is the line integral along the bond between adjacent sites \(i\) and \(j\). We consider the case where the bond variables \(A_{ij}\) are fixed, uniformly quenched, out of equilibrium with the site variables and satisfy the condition \(\sum_p A_{ij} = 2 \pi f\); here the sum is over each set of bonds of an elementary plaquette and \(f\) is the strength of frustration. In the following we will focus on the limit \(J \gg U\), where the JJA in the presence of an external magnetic field transversal to the lattice plane is a physical realization of the fully frustrated \(XY\) model on the square lattice. Let us then focus on such a model, which is described by the action:

\[
H = -J \sum_{(ij)} \cos (\varphi_i - \varphi_j - A_{ij}).
\]

(2)

We assume that the local magnetic field in Eq. (2) is equal to the uniform applied field; such an approximation is more valid the smaller is the sample size \(L\) compared with the transverse penetration depth \(\lambda_\perp\). In the case under study, \(f = \frac{1}{2}\), such a model has a continuous \(U(1)\) symmetry associated with the rotation of spins and an extra discrete \(Z_2\) symmetry, as it has been shown analyzing the degeneracy of the ground state \[25\]. Choosing the Landau gauge, such that the vector potential vanishes on all horizontal bonds and on alternating vertical bonds, we get a lattice where each plaquette displays one antiferromagnetic and three ferromagnetic bonds. Such a choice corresponds to switching the sign of the interaction term in Eq. (2) and is closely related to the presence of two ground states with opposite chiralities, the first one invariant under shifts by two lattice spacings and the second one invariant under shifts by one lattice spacing. The action (2) can be cast into a form where both the \(U(1)\) and \(Z_2\) symmetries are manifest, through the Villain approximation \[33\]. In this way the spin-wave and the vortex contributions can be separated. Furthermore, by integrating out the spin waves, the resulting vortex contribution can be rewritten as a fractionally charged Coulomb gas (CG) defined on the dual lattice \[34\]:

\[
H = -J \sum_{r,r'} (m (r) + f) G (r, r') (m (r') + f).
\]

(3)

Here we have \(\lim_{|r-r'| \to \infty} G (r, r') = \log |r - r'| + \frac{1}{2} \pi\) and the neutrality condition \(\sum_r (m (r) + f) = \sum_r n (r) = 0\) must be satisfied. It is now evident that the ground state for \(f = \frac{1}{2}\) consists
of an alternating lattice of logarithmically interacting $\pm \frac{1}{2}$ charges and is doubly degenerate. Such a model exhibits two possible phase transitions, an Ising and a vortex-unbinding one [35], and their relative order has been deeply studied in the literature [25].

The $FFXY$ model has been studied analyzing other models which have the same $U(1) \otimes Z_2$ degenerate ground state and are believed to be in the same universality class. In particular it can be reformulated in terms of a system of two coupled $XY$ models with a symmetry breaking term [36]:

$$H = A \left[ \sum_{i=1,2} \sum_{\langle r,r' \rangle} \cos (\varphi^{(i)}(r) - \varphi^{(i)}(r')) \right] + h \sum_r \cos 2 (\varphi^{(1)}(r) - \varphi^{(2)}(r)).$$

(4)

The limit $h \to 0$ corresponds to a full decoupling of the fields $\varphi^{(i)}$, $i = 1, 2$, so giving rise to a CFT with central charge $c = 2$ which describes two independent classical $XY$ models (in the continuum limit). In the $h \to \infty$ limit the two phases $\varphi^{(i)}$, $i = 1, 2$ are locked [36], i.e. $\varphi^{(1)}(r) - \varphi^{(2)}(r) = \pi j$, $j = 1, 2$; as a consequence the model gains a symmetry $U(1) \otimes Z_2$ and its Hamiltonian renormalizes towards a model described by:

$$H = H (h \to \infty) = A \sum_{\langle r,r' \rangle} (1 + s_r s_{r'}) \cos (\varphi^{(1)}(r) - \varphi^{(1)}(r')).$$

(5)

where $\varphi^{(1)}(r)$ and $s_r = \cos \pi j = \pm 1$ are planar and Ising spins respectively. In this way a model is obtained which is consistent with the required symmetry, the $XY$-Ising one, and whose Hamiltonian has the general form [37]:

$$H_{XY-I} = \sum_{\langle r,r' \rangle} \left[ A \left( 1 + s_r s_{r'} \right) \cos \left( \varphi^{(1)}(r) - \varphi^{(1)}(r') \right) + C s_r s_{r'} \right].$$

(6)

In the presence of such a symmetry we can use the $m$-reduction technique [3] which has been successfully applied to a quantum Hall fluid in [2, 3, 20, 21] and to a fully frustrated Josephson junction ladder with Mobius boundary conditions in [5, 4, 1].

Let us now have a look at the full spectrum of excitations of the $FFXY$ model: vortices, domain walls, kinks and antikinks. Vortices are point-like defects such that the phase rotates by $\pm 2\pi$ in going around them [35]. A domain wall is a topological excitation of the double degenerate ground state and it can be defined as a line of links, each one separating two plaquettes with the same chirality. So through a domain wall the alternating structure (the checkerboard pattern) of the ground state is lost. Kinks and antikinks are excitations which live on the domain walls and are described by fractional vortices with $+1/2$ and $-1/2$ topological charge. As we will see, domain wall excitations can be generated from the ground state by closing the lattice. This is our strategy: the JJA under study can be closed and arranged in a Corbino disk geometry, which is the relevant geometry for the implementation of a protected qubit. Imposing the coincidence
between the internal and the external edge we obtain the discretized analogue of a torus. In this way we generate a variety of boundary conditions which we can classify by noticing that, for the ground state, topologically inequivalent circumstances arise for even or odd number of plaquettes along the cycles of the torus. In the even case the end plaquettes on the opposite sides of the lattice have opposite chirality, while in the odd case they have the same chirality. So the ground state on a lattice maps into the ground state on the torus only if the torus has an even number of plaquettes along the two cycles. On the other hand a straight domain wall is generated along any cycle of the torus which corresponds to opposite sides of the lattice separated by an odd number of plaquettes; all the possible cases are illustrated in Figs. 1,...,4 in the so-called minimal configurations. Such a behavior has to be taken into account by opportune boundary conditions on the fields \( \varphi^{(i)} \), \( i = 1, 2 \) at the edges of the finite lattice. These non trivial boundary conditions naturally arise when we implement the \( m \)-reduction procedure in the discrete case. So the closed geometry gives rise to non-trivial topological properties, which appear deeply related to the twofold degeneracy of the ground state and are the source of protection from external perturbations.

The next step is to show how our \( m \)-reduction procedure will give rise in a natural way to such non trivial topological properties for the closed JJA. The first step will be to implement the \( m \)-reduction in the discrete case and then to perform the continuum limit first on the plane and then on the torus. In this way, by using the powerful techniques of the CFT, we will be able to propose a qubit device protected from decoherence.

### 3 The \( m \)-reduction procedure for the JJA: a summary

Here we briefly summarize the main results of our theory, the TM, for the fully frustrated JJA [23]. We first construct the bosonic theory on the plane and show that it gives rise to a system with the required \( U(1) \otimes \mathbb{Z}_2 \) symmetry. That allows us to describe the JJA excitations in terms of the primary fields \( V_\alpha (z) \). Then we implement a discrete version of the \( m \)-reduction procedure in order to clarify the central role played by the closed geometry and by the non-trivial boundary conditions in the description of the excitations spectrum. The topological structure of the vacua on the torus will be outlined in Section 4 together with the corresponding physical configurations of the JJA, the detailed expressions of the conformal blocks being reported in the Appendix.

Let us focus on the \( m \)-reduction procedure [3] for the special \( m = 2 \) case (see Ref. [2] for the general case), since we are interested in a system with \( U(1) \otimes \mathbb{Z}_2 \) symmetry and choose the “bosonic” theory [23], which well adapts to the description of a system with Cooper pairs of electric charge \( 2e \) in the presence of non-trivial boundary conditions [20], i.e. a fully frustrated JJA. As a result of the 2-reduction procedure [2][3] we get a \( c = 2 \) orbifold CFT, the TM, whose fields have well defined transformation properties under the
discrete $Z_2$ (twist) group, which is a symmetry of the TM. Its primary fields content can be expressed in terms of a $Z_2$-invariant scalar field $X(z)$, given by

$$X(z) = \frac{1}{2} \left( Q^{(1)}(z) + Q^{(2)}(z) \right),$$

(7)

describing the continuous phase sector of the theory, and a twisted field

$$\phi(z) = \frac{1}{2} \left( Q^{(1)}(z) - Q^{(2)}(z) \right),$$

(8)

which satisfies the twisted boundary conditions $\phi(e^{i\pi z}) = -\phi(z)$ [2]. The whole TM theory decomposes into a tensor product of two CFTs, a twisted invariant one with $c = 3/2$ and the remaining $c = 1/2$ one realized by a Majorana fermion in the twisted sector. In the $c = 3/2$ sub-theory the primary fields are composite vertex operators $V(z) = U_{X}^{\alpha_i}(z) \psi(z)$ or $V_{qh}(z) = U_{X}^{\alpha_i}(z) \sigma(z)$, where

$$U_{X}^{\alpha_i}(z) = \frac{1}{\sqrt{z}} : e^{i\alpha_l X(z)} :$$

(9)

is the vertex of the continuous sector with $\alpha_l = \frac{l}{2}$, $l = 1, \ldots, 4$ for the $SU(2)$ Cooper pairing symmetry used here. Regarding the other component, the highest weight state in the isospin sector, it can be classified by the two chiral operators:

$$\psi(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\sqrt{2} \phi(z)} : + : e^{i\sqrt{2} \phi(-z)} : \right), \quad \overline{\psi}(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\sqrt{2} \phi(z)} : - : e^{i\sqrt{2} \phi(-z)} : \right);$$

(10)

which correspond to two $c = 1/2$ Majorana fermions with Ramond (invariant under the $Z_2$ twist) or Neveu-Schwartz ($Z_2$ twisted) boundary conditions [2][3] in a fermionized version of the theory. The Ramond fields are the degrees of freedom which survive after the tunnelling and the parity symmetry, which exchanges the two Ising fermions, is broken. Besides the fields appearing in eq. (10), there are the $\sigma(z)$ fields, also called the twist fields, which appear in the quasi-hole primary fields $V_{qh}(z)$. The twist fields have non local properties and decide also for the non trivial properties of the vacuum state, which in fact can be twisted or not in our formalism. Indeed the whole TM theory decomposes into a tensor product of two CFTs, a twisted invariant one with $c = 3/2$ (the Moore-Read (MR) theory with symmetry $U(1) \otimes Z_2$) and the remaining $c = 1/2$ one realized by a Majorana fermion in the twisted sector. Such a factorization can be unambiguously pointed out on the torus topology as we will show in the following [3]. But now let us focus on a discrete version of the procedure just outlined in order to clarify the role of the closed geometry and of the non-trivial boundary conditions in the description of the full spectrum of excitations of the model [23]. To such an extent let $(-L/2, 0)$, $(L/2, 0)$, $(L/2, L)$, $(-L/2, L)$ be the corners of the square lattice $L$ and assume that the fields $\varphi^{(i)}$, $i = 1, 2$ satisfy the following boundary conditions:

$$\varphi^{(1)}(r) = \varphi^{(2)}(r) \quad \text{for } r \in L \cap x,$$

(11)
where \(x\) is the \(x\) axis. The above boundary conditions allow us to consider the two fields \(\varphi^{(1)}\) and \(\varphi^{(2)}\) on the square lattice \(L\) as the folding of a single field \(Q\), defined on the lattice \(L_0\) with corners \((-L/2, -L), (L/2, -L), (L/2, L), (-L/2, L)\). More precisely we define the field \(Q\) as:

\[
Q(r) = \begin{cases} 
\varphi^{(1)}(r) & \text{for } r \in L \cap L_0, \\
\varphi^{(2)}(-r) & \text{for } r \in (-L) \cap L_0.
\end{cases}
\]

(12)

We can implement now a discrete version of the \(m\)-reduction procedure \((m = 2)\) by defining the fields:

\[
\mathcal{X}(r) = \frac{1}{2} (Q(r) + Q(-r)),
\]

(13)

\[
\Phi(r) = \frac{1}{2} (Q(r) - Q(-r)),
\]

(14)

where \(r \in L_0\). The resemblance with the continuum version of the procedure is evident and the fields \(\mathcal{X}\) and \(\Phi\) are symmetric and antisymmetric with respect to the action of the generator \(g : r \to -r\) of the discrete group \(Z_2\). The Hamiltonian in Eq. (4) can be rewritten in terms of these fields and, for \(\hbar = 0\), it becomes:

\[
H = 2A \sum_{\langle r, r' \rangle \in L} \cos \left( \mathcal{X}(r) - \mathcal{X}(r') \right) \cos \left( \Phi(r) - \Phi(r') \right),
\]

(15)

which in the continuum limit corresponds to the action of our TM model:

\[
\mathcal{A} = \int \left[ \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} (\partial \Phi)^2 \right] d^2 x.
\]

(16)

It is worth pointing out that the fields \(\mathcal{X}\) and \(\Phi\) are scalar fields and so the chiral fields defined by Eqs. (7), (8) can be seen as their chiral components. Moreover the group \(Z_2\) is a discrete symmetry group, indeed both \(H\) and \(\mathcal{A}\) are invariant under its action.

4 JJA on the torus topology: magnetic translations and wave functions

The non perturbative ground state wave functions of the JJA system in the torus topology represent coherent states of Cooper pairs on the torus. It can be inferred that such ground states can be expressed as correlation functions of the primary fields describing the elementary particles (Cooper pairs) of JJA. In the following we will explicitly show that the characters of the theory are in one to one correspondence with such ground states. In particular they describe the components of the “center of charge” for the corresponding ground state.
wave functions [38], encoding all the topological properties of the JJA system. To such an extent let us define for a single Cooper pair on a torus $a \times b$ an effective mean-field Hamiltonian of the kind $H(x, y) = H_0(x, y) + V(x, y)$, where $H_0(x, y) = \left[-i\hbar \nabla - 2e \mathcal{A}/c \right]^2/2m$ is the Hamiltonian in the presence of an uniform magnetic field and $V(x, y)$ is a mean-field scalar potential such that $V(x, y) = V(x + a, y) = V(x, y + b)$. It is now possible to define the magnetic translations operators $\tilde{S} = e^{i\theta_x a/\hbar}$ and $\tilde{T} = e^{i\theta_y b/\hbar}$ along the two cycles $A$ (i.e. the real axis $x$) and $B$ (i.e. the imaginary axis $y$) of the torus respectively, where:

$$\theta_x = \pi x - \frac{2c}{\mathcal{A}} By = -i\hbar \partial_x, \quad \theta_y = \pi y + \frac{2c}{\mathcal{A}} Bx = -i\hbar \partial_y + \frac{2c}{\mathcal{A}} Bx$$ (17)

and the gauge choice $\mathcal{A}(x, y) = (-By, 0)$ has been made. They satisfy the relations:

$$(\tilde{S}, \mathcal{H}(x, y)) = (\tilde{T}, \mathcal{H}(x, y)) = 0, \quad \tilde{S}\tilde{T} = e^{2\pi i\Phi_{ab}/\Phi_0} \tilde{T}\tilde{S},$$ (18)

where $\Phi_{ab}$ is the magnetic flux threading the torus surface, and their action on the wave functions can be defined as:

$$\tilde{S}\varphi(x, y) = \varphi(x + a, y), \quad \tilde{T}\varphi(x, y) = e^{2\pi i Bax/\Phi_0} \varphi(x, y + b).$$ (19)

Now for $\Phi_{ab} = M\Phi_0$ (i.e. when the magnetic flux $\Phi_{ab}$ is an integer number $M$ of flux quanta $\Phi_0 = \frac{hc}{2e}$) the condition $[\tilde{S}, \tilde{T}] = 0$ holds and we can simultaneously diagonalize the operators $H(x, y)$, $\tilde{S}$, $\tilde{T}$. By introducing adimensional coordinates on the torus, Eqs. (19) can be rewritten as:

$$\tilde{S}\varphi(\omega) = \varphi(\omega + 1), \quad \tilde{T}\varphi(\omega) = e^{2\pi i Mx/\Phi_0} \varphi(\omega + \tau),$$ (20)

where $\omega = x + \tau y$, $x \in [0, 1]$, $y \in [0, 1]$. One can look for eigenfunctions $\varphi(\omega) = e^{i\pi My^2\tau} f(\omega)$ and define magnetic translation operators $S_\alpha, T_\alpha$ acting only on the corresponding holomorphic part $f(\omega)$:

$$S_\alpha f(\omega) = f(\omega + \alpha), \quad T_\alpha f(\omega) = e^{i\pi M(a^2\tau + 2\alpha\omega)} f(\omega + \alpha\tau).$$ (21)

In this way Eqs. (20) become:

$$\tilde{S}\varphi(\omega) = e^{i\pi My^2\tau} S_1 f(\omega), \quad \tilde{T}\varphi(\omega) = e^{i\pi My^2\tau} T_1 f(\omega).$$ (22)

It is straightforward to show that the ground state wave function for $M$ Cooper pairs is a coherent state given by:

$$\psi_0(\omega_1, \ldots, \omega_M) = e^{i\pi M\tau \sum_{i=1}^M y_i^2} f_0(\omega_1, \ldots, \omega_M),$$ (23)

$$f_0(\omega_1, \ldots, \omega_M) = \prod_{i<j=1}^M \left[ \frac{\theta_1(\omega_{ij}, \tau)}{\theta_1^* (\omega_{ij}, \tau)} \right]^4 \chi_0(\omega_i | \tau),$$ (24)

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where \( \omega = \sum_{i=1}^{M} \omega_i \) is the “center of charge” variable and the non local functions \( \chi_\alpha(\omega|\tau) \) are the characters of our theory (the TM), whose analytical expressions are reported in the Appendix. In fact it can be shown that such characters are eigenfunctions of the following generalized magnetic translations operators:

\[
S_\alpha = \prod_{i=1}^{M} S_{\alpha/M}^i, \quad T_\alpha = \prod_{i=1}^{M} T_{\alpha/M}^i, 
\]

where \( S_{\alpha/M}^i \) and \( T_{\alpha/M}^i \) are the magnetic translation operators for the single Cooper pair. In particular, by recalling the explicit expressions of the conformal blocks (48)-(52) and (44)-(47) in the four topological sectors [3], given in the Appendix, and the transformation properties of the Theta functions [39], we get (for the step \( \alpha = 1 \), see also Eq. (22)):

\[
T \tilde{\chi}_\alpha^+(w_c|\tau) = \tilde{\chi}_\alpha^+(w_c|\tau), \quad T \tilde{\chi}_\beta^+(w_c|\tau) = \tilde{\chi}_\beta^+(w_c|\tau), 
\]

for the \( P - P \) sector,

\[
T \tilde{\chi}_\alpha^-(w_c|\tau) = i\tilde{\chi}_\alpha^-(w_c|\tau), \quad T \tilde{\chi}_\beta^-(w_c|\tau) = -i\tilde{\chi}_\beta^-(w_c|\tau), 
\]

for the \( P - A \) sector, where we defined \( \tilde{\chi}_\alpha^\pm(w_c|\tau) = \tilde{\chi}_\alpha(w_c|\tau) \pm i\tilde{\chi}_\alpha(w_c|\tau) \),

\[
T \chi_{(0)}^+(w_c|\tau) = \chi_{(0)}^+(w_c|\tau), \quad T \chi_{(1)}^+(w_c|\tau) = \chi_{(1)}^+(w_c|\tau), 
\]

for the \( A - P \) sector, and

\[
T \chi_{(0)}^-(w_c|\tau) = i\chi_{(0)}^-(w_c|\tau), \quad T \chi_{(1)}^-(w_c|\tau) = i\chi_{(1)}^-(w_c|\tau), 
\]

for the \( A - A \) sector respectively. From such a picture it is evident then how the degeneracy of the non perturbative ground states is closely related to the number of primary states and that our characters represent highly non local functions: all the topological properties of our system are codified in such functions.

Let us now focus on the general topological structure of the vacua on the torus and on their corresponding physical JJA realization in relation to the various boundary conditions imposed by closing the square lattice. Let \( A \) be the cycle of the torus which surrounds the hole of the Corbino disk and \( B \) the cycle in the radial direction. On a pure topological ground we expect for the torus a doubling of the ground state degeneracy, which can be seen at the level of the conformal blocks (characters) of our TM (see Appendix). Indeed we get for periodic boundary conditions (i.e. an even number of plaquettes) along the \( A \)-cycle an untwisted sector, \( P - P \) and \( P - A \), characterized by periodic \( (P) \) and twisted \( (A) \) boundary conditions along the \( B \)-cycle respectively, described by the four conformal blocks (48)-(51); likewise we get for twisted boundary conditions (i.e. an odd number of plaquettes) along the \( A \)-cycle a twisted sector, \( A - P \) and \( A - A \), characterized by periodic and twisted boundary conditions along the \( B \)-cycle respectively, described by the four conformal blocks (44)-(47). The four different topological sectors just obtained can
be put in correspondence with four minimal configurations for the closed fully frustrated JJA as follows.

Regarding the untwisted sector, in Fig. 1 we show the two vacua corresponding to the \( P - P \) sector, characterized by a JJA with \( N = 4 \) plaquettes along the circular direction and \( N = 2 \) plaquettes along the radial direction, 4 and 2 being the minimum number of plaquettes respectively needed in order to fulfill the even-even boundary conditions. Then in Fig. 2 the two vacua in the \( P - A \) sector, implemented by a JJA with even-odd boundary conditions along the two cycles respectively, are sketched.

Regarding the twisted sector, we get for the \( A - P \) and the \( A - A \) sectors the configurations shown in Figs. 3 and 4 respectively, characterized by the required odd-even and

Figure 1: The two vacua in the \( P - P \) sector.

Figure 2: The two vacua in the \( P - A \) sector. A straight domain wall is generated along the \( A \)-cycle.
Figure 3: The two vacua in the $A - P$ sector. A straight domain wall is generated along the $B-$cycle.

Figure 4: The two vacua in the $A - A$ sector. A straight domain wall is generated along both the $A-$cycle and the $B-$cycle.

odd-odd boundary conditions.

In the next Section the relevant two states of the qubit will be found and will lead us to propose the configuration shown in Fig. 3 as the minimal one required to build a reliable qubit device protected from decoherence.
5 Stability and transformation properties of the ground states: the physical qubit

In this Section we will study the stability and transformation properties of the four ground states of the JJA arranged in the Corbino disk geometry under an adiabatic elementary flux change \((\pm \frac{hc}{2e})\) through the central hole. Because of the finite energy gap to fractionally charged excitation states (in complete analogy with the presence of a gap separating the ground state from higher energy states in the Laughlin Hall fluid [40]), such an adiabatic transformation is believed to leave the system in a ground state which can be different from the original one, due to the occurrence of the ground state degeneracy. This analysis will lead us to the identification of the two states of a possible protected qubit and to the definition of a tool for its manipulation.

We use the results provided by our TM model in Section 4 and in the Appendix in order to analyze such properties by standard conformal techniques. As we showed in the previous Section, in the torus topology the characters of the theory are in one to one correspondence with the ground states and any one of them describes the component of the “center of charge” for the corresponding ground state wave function. On a pure topological ground the torus shows a doubling of the ground state degeneracy, which can be seen at the level of the conformal blocks (characters) of our TM. Indeed, as outlined in Section 4, we get a rich structure for the vacua corresponding to different boundary conditions at the ends of the square lattice. That gives rise to an untwisted sector, \(P - P\) and \(P - A\), described by the four conformal blocks (48)-(51), and a twisted sector, \(A - P\) and \(A - A\), described by the four conformal blocks (44)-(47) respectively. Now we are going to extract from such vacua the two states of the “protected” qubit.

Let us notice that the ground state wave functions of the twisted and untwisted sectors of the TM are characterized by different monodromy properties along the \(A\)-cycle. In particular the characters of the untwisted sector are single-valued functions along the \(A\)-cycle while the characters of the twisted sector pick up a common \((-1)\) phase factor along the \(A\)-cycle. Such phase factors can be interpreted as Bohm-Aharonov phases generated while a Cooper pair is taken along the \(A\)-cycle. The above observation evidences a strong difference between the two inequivalent topological sectors, the untwisted and the twisted one respectively, on the torus. Indeed in the twisted sector the ground state wave functions show a non trivial behavior implying the trapping of a half flux quantum \((\frac{1}{2} \frac{hc}{2e})\) in the hole of the Corbino disk. Instead in the untwisted sector, due to the single-valued ground state wave functions, only integer numbers of flux quanta can be attached to the hole. It is worth pointing out the central role played by the isospin (or neutral) component of the TM in producing the discussed non trivial monodromy properties. To this end let us recall that the TM is a \(c = 2\) CFT, composed by a \(c = 1\) charged and a \(c = 1\) isospin CFT components, as it is well evidenced by the characters decompositions given in Section 4. Furthermore the transport of a Cooper pair along the \(A\)-cycle can be implemented by a simultaneous and identical translation \(\Delta w_c = \Delta w_n = 2\) of the charged and the
isospin variables. The *charged* characters have trivial monodromy with respect to this transformation, being:

\[ K_l(w_c + 2|\tau) = K_l(w_c|\tau), \quad l = 0, \ldots, 3, \quad (30) \]

while the *isospin* contribution is the one responsible for the non trivial monodromy of the complete ground state wave functions:

\[ \chi_{0,\frac{1}{2}}(2|\tau) = \chi_{0,\frac{1}{2}}(0|\tau), \quad \chi_{\frac{1}{2}}(2|\tau) = (-1)\chi_{\frac{1}{2}}(0|\tau) \quad (31) \]

and the same is true for the characters \( \tilde{\chi}_{\beta} \). Observe that the change in sign in the last relation of Eq. (31) shows the presence in the spectrum of excitations carrying fractionalized charge quanta. More precisely the presence in the isospin component of one twist-field (with conformal dimension \( \Delta = 1/16 \)) characterizes all the conformal blocks of the twisted sector and accounts for the trapping of a half flux quantum in the hole of the Corbino disk.

We are now ready to study the stability and transformation properties of the ground state wave functions when a magnetic flux change takes place through the central hole of the closed JJA. The above analysis shows that at the level of the wave functions it has the effect to change the monodromy along the \( A \)-cycle due to the corresponding change in the Bohm-Aharonov phase. Such a modification can be implemented on the center of charge component of the wave function, i.e. the characters, with a well defined transformation. In the case of the *charged* component this analysis has been brought out already in [41] for the quantum Hall effect. Let us adapt here the results for the *charged* component of our TM. On a pure physical ground the fact that we are considering a magnetic flux change, which is on one side integer in the flux quantum (one flux quantum change \( \pm \frac{hc}{2e} \)) and on the other side adiabatic suggests both that the monodromy properties do not change and that the system remains in a degenerate ground state. Such a physical picture is in fact confirmed for the *charged* component of our TM; indeed, the flux change is implemented on the *charged* characters by the transformation \( T_{1/2}^c \):

\[ T_{1/2}^c K_l(w_c|\tau) \equiv e^{(\frac{1}{2})^2 \pi \tau + \frac{2i\pi w_c}{\tau}} K_l \left( w_c + \frac{\tau}{2} |\tau \right) = K_{l+1} (w_c|\tau), \quad l = 0, \ldots, 3. \quad (32) \]

Let us notice that the *charged* component wave functions realize a flip process \( l \rightarrow l + 1 \) under one magnetic flux quantum change.

However the analysis for the complete TM, with *charged* and *isospin* components, is more involved, due to the non trivial interplay between *charged* and *isospin* components summarized in the so-called \( m \)-ality parity rule, which characterizes the gluing condition for the *charged* and *isospin* excitations. The main point being the compatibility between such parity rule and the transformation of the complete characters of the TM under the insertion of a magnetic flux quantum through the hole of the closed JJA, which reads as:

\[ T_{1/2} f(w_n|w_c|\tau) = e^{2i\pi (\alpha^2 \tau + \alpha (w_n + w_c))} f(w_n + \alpha \tau|w_c + \alpha \tau|\tau) \bigg|_{\alpha = 1/2}, \quad (33) \]
where \( f(w_n = 0|w_c|\tau) \) stays for any character of the TM. More in detail the full list of character transformations is as follows.

In the untwisted sector, we have that the two ground state wave functions of the \( P - A \) sector decouple, being

\[
T_{1/2}\tilde{\chi}^-_{(0)}(0|w_c|\tau) = 0, \quad T_{1/2}\tilde{\chi}^-_{(1)}(0|w_c|\tau) = 0. \tag{34}
\]

Concerning the \( P - P \) sector, we have:

\[
T_{1/2}\tilde{\chi}^+_{\alpha}(0|w_c|\tau) = 0 \tag{35}
\]

and

\[
T_{1/2}\tilde{\chi}^+_{\beta}(0|w_c|\tau) = \tilde{\chi}^+_{\gamma}(0|w_c|\tau). \tag{36}
\]

Such transformations show the instability of the \( P - P \) sector under the insertion of a flux quantum through the hole of the closed JJA. More precisely the state \( \tilde{\chi}^+_{\alpha}(0|w_c|\tau) \) decouples while the state \( \tilde{\chi}^+_{\beta}(0|w_c|\tau) \) gets excited to the state with a kink-antikink configuration \( \tilde{\chi}^+_{\gamma}(0|w_c|\tau) \) (see Eq. (52) and comments afterwards).

Furthermore in the twisted sector, we have that the two ground state wave functions of the \( A - A \) sector decouple, being

\[
T_{1/2}\tilde{\chi}^-_{(0)}(0|w_c|\tau) = 0, \quad T_{1/2}\tilde{\chi}^-_{(1)}(0|w_c|\tau) = 0. \tag{37}
\]

Concerning the \( A - P \) sector, we have that the two ground state wave functions transform as:

\[
T_{1/2}\tilde{\chi}^+_{(0)}(0|w_c|\tau) = \chi^+_{(1)}(0|w_c|\tau), \quad T_{1/2}\tilde{\chi}^+_{(1)}(0|w_c|\tau) = \chi^+_{(0)}(0|w_c|\tau). \tag{38}
\]

Concluding, the \( A - P \) sector only is stable under the insertion of a magnetic flux quantum through the central hole, with the two ground states flipping one into the other under an adiabatic flux change of \( \pm \frac{hc}{2e} \). That allows us to make the following identifications:

\[
|0\rangle \sim \chi^+_{(0)}(0|w_c|\tau), \quad |1\rangle \sim \chi^+_{(1)}(0|w_c|\tau), \tag{39}
\]

in terms of the ground states center of charge wave functions (characters). Then \(|0\rangle\) and \(|1\rangle\) are the two ground states of the closed JJA characterized by an odd number of plaquettes along the \( A \)-cycle and an even number of plaquettes along the \( B \)-cycle. If we choose a closed path along the \( A \)-cycle, we can define a topological invariant as the size invariant sum over all its plaquettes \( \sum_p \chi_p \), which results equal to \(-1\) and \(+1\) for \(|0\rangle\) and \(|1\rangle\) respectively.

The results just obtained lead us to propose the JJA closed in a Corbino disk geometry, characterized by an odd number of plaquettes along circular direction and an even number
of plaquettes along radial direction, as our protected qubit. In fact the two ground states \(|0\rangle\) and \(|1\rangle\) work as the two logical states of the qubit and the required one qubit operations:

\[ |0\rangle \rightarrow |1\rangle, \ |1\rangle \rightarrow |0\rangle, \]

are simply implemented by insertion of a flux quantum \((\pm \frac{\Phi_0}{2e})\) through the central hole. That provides a tool for the control and the manipulation of our device, which is indeed a “flux” qubit [27].

In Fig. 3 we showed the minimal configuration for such a device, that is a closed fully frustrated JJA with \(N = 3\) plaquettes along the circular direction and \(N = 2\) plaquettes along the radial direction, 3 and 2 being the minimum odd and even number of plaquettes respectively needed in order to fulfill all the above requests.

Given the two degenerate ground states (39) and the operation mode defined in (38) it should be possible to prepare the qubit in a definite state and to realize all elementary one-qubit operations. We showed in particular how to implement the \textit{NOT} operation, the Hadamard gate will be implemented in a future publication. All that is accomplished by an adiabatic flux change of \(\pm \frac{\Phi_0}{2e}\) through the central hole of the Corbino disk.

6 Conclusions and outlooks

The ground states and the topological properties of fully frustrated Josephson Junction arrays (JJA) arranged in a Corbino disk geometry for a variety of boundary conditions have been investigated in detail, employing a twisted CFT approach [2, 3]. In this way we built up a low-energy effective field theory which is a topological one, due to the well known relation with Chern-Simons gauge theories in \((2 + 1)\)-dimensions [19]. The vacua obtained have been classified in four different topological sectors, according to the different boundary conditions imposed at the ends of the square lattice, and put in correspondence with four minimal configurations for the closed fully frustrated JJA. Then a careful analysis of the stability and transformation properties of the conformal blocks corresponding to such vacua allowed us to propose a solid state qubit, protected from decoherence, whose operation mode is based only on adiabatic magnetic flux changes through the central hole of the Corbino disk.

Josephson junction arrays have been fabricated within the trilayer \(Nb/Al – AlO_x/Nb\) or \(Pb/Sn/Pb\) technology as well as all aluminum technology and experimentally investigated [42], but in such a case the application of an external transverse magnetic field is needed in order to fulfill the requirement of full frustration and that could be another source of decoherence. It is now possible to avoid such a problem by realizing arrays with a built-in frustration. In fact high-\(T_c\) Josephson junction arrays have been recently proposed [43], which support degenerate spontaneous current states in zero magnetic field due to the presence of plaquettes containing an odd number of \(\pi\)-junctions [44]. Such
unconventional junctions can be realized because of the \textit{d}-wave symmetry of high-\textit{T}_c superconductors \cite{45}, which produces a $\pi$-shift in the phase of the wave function on one side of the junction. Furthermore $\pi$-junctions can be obtained also with superconducting-ferromagnetic-superconducting (SFS) \cite{46} and superconducting-insulator-ferromagnetic-superconducting (SIFS) \cite{47} structures. One can also achieve the same effect using a barrier which effectively flips the spin of a tunneling electron, i. e. when the barrier is made of a ferromagnetic insulator \cite{48}, of a carbon nanotube \cite{49} or of a quantum dot \cite{50} created by gating a semiconducting nanowire. In this way it is possible to avoid the external frustration bias but, in any case, external magnetic fields are needed for control and read-out operations. So in principle an experimental setup for the realization of our protected qubit can be easily conceived, whose basis is a fully frustrated JJA arranged in a Corbino disk geometry with an odd number of plaquettes along the inner hole and an even number of them along the radial direction.

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\section*{Appendix: the TM on the torus}

In the following we summarize the whole primary field content of our theory, the TM, on the torus topology \cite{3}.

On the torus \cite{3} the TM primary fields are described in terms of the conformal blocks of the $Z_2$-invariant $c = \frac{3}{2}$ sub-theory and of the non invariant $c = \frac{1}{2}$ Ising model, so reflecting the decomposition on the plane outlined in Section 3. The characters $\tilde{\chi}_0(0|\tau)$, $\tilde{\chi}_\frac{1}{2}(0|\tau)$, $\tilde{\chi}_{\frac{1}{16}}(0|\tau)$ express the primary fields content of the Ising model \cite{51} with Neveu-Schwartz ($Z_2$ twisted) boundary conditions \cite{3}, while

\begin{align}
\chi_{c=3/2}^{(0)}(0|w_c|\tau) &= \chi_0(0|\tau) K_0(w_c|\tau) + \chi_{\frac{1}{2}}(0|\tau) K_2(w_c|\tau), \\
\chi_{c=3/2}^{(1)}(0|w_c|\tau) &= \chi_{\frac{1}{16}}(0|\tau) (K_1(w_c|\tau) + K_3(w_c|\tau)), \\
\chi_{c=3/2}^{(2)}(0|w_c|\tau) &= \chi_{\frac{1}{2}}(0|\tau) K_0(w_c|\tau) + \chi_0(0|\tau) K_2(w_c|\tau)
\end{align}

represent those of the $Z_2$-invariant $c = \frac{3}{2}$ CFT. They are given in terms of a “charged” $K_\alpha(w_c|\tau)$ contribution:

\begin{equation}
K_{2l+i}(w|\tau) = \frac{1}{\eta(\tau)} \Theta \left[ \begin{array}{c} \frac{2l+i}{4} \\ 0 \end{array} \right] (2w|4\tau), \quad \text{with } l = 0, 1 \text{ and } i = 0, 1,
\end{equation}
and a “isospin” one $\chi_\beta(0|\tau)$, (the conformal blocks of the Ising Model), where $w_c = \frac{1}{2\pi i} \ln z_c$ is the torus variable of the “charged” component while the corresponding argument of the isospin block is $w_n = 0$ everywhere.

If we now turn to the whole $c = 2$ theory, the characters of the twisted sector are given by:

$$\chi^+_\alpha(0|w_c|\tau) = \chi^+_\alpha(0|\tau) \left( \chi_0 + \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_0 + K_2 \right)(w_c|\tau), \quad (44)$$

$$\chi^+_\beta(0|w_c|\tau) = \chi^+_\beta(0|\tau) \left( \chi_0 + \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_1 + K_3 \right)(w_c|\tau), \quad (45)$$

for the $A - P$ sector and by:

$$\chi^-_\alpha(0|w_c|\tau) = \chi^-_\alpha(0|\tau) \left( \chi_0 - \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_0 - K_2 \right)(w_c|\tau), \quad (46)$$

$$\chi^-_\beta(0|w_c|\tau) = \chi^-_\beta(0|\tau) \left( \chi_0 - \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_1 + K_3 \right)(w_c|\tau), \quad (47)$$

for the $A - A$ one. Furthermore the characters of the untwisted sector are [3]:

$$\bar{\chi}^-_\alpha(0|w_c|\tau) = \left( \bar{\chi}_0 \chi_0 - \bar{\chi}_{\frac{1}{2}} \chi_{\frac{1}{2}} \right)(0|\tau)K_0(w_c|\tau) + \left( \bar{\chi}_0 \chi_{\frac{1}{2}} - \bar{\chi}_{\frac{1}{2}} \chi_0 \right)(0|\tau)K_2(w_c|\tau), \quad (48)$$

$$\bar{\chi}^-_\beta(0|w_c|\tau) = \left( \bar{\chi}_0 \chi_{\frac{1}{2}} - \bar{\chi}_{\frac{1}{2}} \chi_0 \right)(0|\tau)K_0(w_c|\tau) + \left( \bar{\chi}_0 \chi_0 - \bar{\chi}_{\frac{1}{2}} \chi_{\frac{1}{2}} \right)(0|\tau)K_2(w_c|\tau), \quad (49)$$

for the $P - A$ sector while for the $P - P$ sector we have:

$$\bar{\chi}^+_\alpha(0|w_c|\tau) = \frac{1}{2} \left( \bar{\chi}_0 - \bar{\chi}_{\frac{1}{2}} \right)(0|\tau) \left( \chi_0 - \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_0 - K_2 \right)(w_c|\tau), \quad (50)$$

$$\bar{\chi}^+_\beta(0|w_c|\tau) = \frac{1}{2} \left( \bar{\chi}_0 + \bar{\chi}_{\frac{1}{2}} \right)(0|\tau) \left( \chi_0 + \chi_{\frac{1}{2}} \right)(0|\tau) \left( K_0 + K_2 \right)(w_c|\tau), \quad (51)$$

and

$$\bar{\chi}^+_\gamma(0|w_c|\tau) = \bar{\chi}^+_\alpha(0|\tau)\chi^+_\alpha(0|\tau) \left( K_1 + K_3 \right)(w_c|\tau). \quad (52)$$

Let us point out that the above factorization expresses the parity selection rule ($m$-ality), which gives a gluing condition for the “charged” and “isospin” excitations.

Furthermore, in the $P - P$ sector, unlike for the other sectors, modular invariance constraint requires the presence of three different characters. The isospin operator content of the character $\bar{\chi}^+_\gamma(0|w_c|\tau)$ clearly evidences its peculiarity with respect to the other states of the periodic (even ladder) case. Indeed it is characterized by two twist fields ($\Delta = 1/16$) in the isospin components. The occurrence of the double twist in the state described by $\bar{\chi}^+_\gamma(0|w_c|\tau)$ is simply the reason why such a state is a periodic state. Indeed, being an isospin twist field the representation in the continuum limit of a kink (or equivalently a
half flux quantum trapping), the double twist corresponds to a double half flux quantum trapping, i.e. one flux quantum, typical of the periodic configuration. Indeed the state described by $\tilde{\chi}^\gamma_+ (0|w_c|\tau)$ embeds in the continuum limit a kink-antikink excitation, i.e. it represents an excited state in the $P-P$ sector. In this way, as it happens for all the other sectors, the $P-P$ sector is left with just two degenerate ground states ($\tilde{\chi}^\gamma_+(0|w_c|\tau)$ and $\tilde{\chi}^\gamma_+ (0|w_c|\tau)$) and, as expected on a pure topological base, the ground state degeneracy in the torus topology is the double of that of the disk.

Finally the partition function of the TM model on the torus has the following factorized form (see [3]):

$$Z(w_c|\tau) = Z^{MR}(w_c|\tau)Z_T(\tau),$$

in terms of the Moore-Read partition function $Z^{MR} (c = 3/2)$ and of the Ising partition function $Z_T (c = 1/2)$. They have the following expression:

$$Z^{MR}(w_c|\tau) = \left|\chi_0^{MC}(0|w_c|\tau)\right|^2 + \left|\chi_1^{MC}(0|w_c|\tau)\right|^2 + \left|\chi_2^{MC}(0|w_c|\tau)\right|^2,$$

$$Z_T(\tau) = \left|\tilde{\chi}_0(\tau)\right|^2 + \left|\tilde{\chi}_1(\tau)\right|^2 + \left|\tilde{\chi}_2(\tau)\right|^2.$$  (55)

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