ABSTRACT. For a Hopf algebra $B$ with bijective antipode, we show that the Heisenberg double $\mathcal{H}(B^*)$ is a braided commutative Yetter–Drinfeld module algebra over the Drinfeld double $\mathcal{D}(B)$. The braiding structure allows generalizing $\mathcal{H}(B^*) \cong B^\text{cop} \bowtie B$ to “Heisenberg $n$-tuples” and “chains” $\ldots \bowtie B^\text{cop} \bowtie B \bowtie B^\text{cop} \bowtie B \bowtie \ldots$, all of which are Yetter–Drinfeld $\mathcal{D}(B)$-module algebras. For $B$ a particular Taft Hopf algebra at a $2p$th root of unity, the construction is adapted to yield Yetter–Drinfeld module algebras over the $2p^3$-dimensional quantum group $\mathbb{U}_q\mathfrak{s}\ell(2)$.

1. INTRODUCTION

We establish the properties of $\mathcal{H}(B^*)$ — the Heisenberg double of a (dual) Hopf algebra — relating it to two popular structures: Yetter–Drinfeld modules (over the Drinfeld double $\mathcal{D}(B)$) and braiding. In fact, we construct new examples of Yetter–Drinfeld module algebras, some of which are in addition braided commutative.

Heisenberg doubles [1, 2, 3, 4] have been the subject of some attention, notably in relation to Hopf algebroid constructions [5, 6, 7] (the basic observation being that $\mathcal{H}(B^*)$ is a Hopf algebroid over $B^*$ [5]) and also from various other standpoints [8, 9, 10, 11, 12, 13].

We show that they are a rich source of Yetter–Drinfeld $\mathcal{D}(B)$-module algebras: $\mathcal{H}(B^*)$ is a Yetter–Drinfeld module algebra over the Drinfeld double $\mathcal{D}(B)$; it is, moreover, braided commutative. Reinterpreting the construction of $\mathcal{H}(B^*)$ in terms of the braiding in the Yetter–Drinfeld category then allows generalizing Heisenberg doubles to “$n$-tuples,” or “Heisenberg chains” (cf. [18]), which are all Yetter–Drinfeld $\mathcal{D}(B)$-module algebras.

In Sec. 2, we establish that $\mathcal{H}(B^*)$ is a Yetter–Drinfeld $\mathcal{D}(B)$-module algebra, and in Sec. 3 that it is braided ($\mathcal{D}(B)$-) commutative [19]; there, $B$ denotes a Hopf algebra with bijective antipode. In Sec. 4 where we construct Yetter–Drinfeld module algebras for a quantum $\mathfrak{s}\ell(2)$ at an even root of unity [20, 21, 22, 12, 13], $B$ becomes a particular Taft Hopf algebra.

For the left and right regular actions of a Hopf algebra $B$ on $B^*$, we use the respective notation $b \mapsto \beta = \langle \beta'', b \rangle \beta'$ and $\beta \rightarrow b = \langle \beta', b \rangle \beta''$, where $\beta \in B^*$ and $b \in B$ (and $\langle , , \rangle$ is

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1 The “true,” underlying motivation (deriving from [12, 13, 14, 15, 16, 17]) of our interest in $\mathcal{H}(B^*)$ is entirely left out here.

2 A slight mockery of the statistical-mechanics meaning of a “Heisenberg chain” may give way to a genuine, and deep, relation in the context of the previous footnote.
the evaluation). The left and right regular actions of $B^*$ on $B$ are $\beta \cdot b = \langle \beta, b' \rangle b'$ and $b \cdot \beta = \langle \beta, b' \rangle b''$. We assume the precedence $ab \cdot \beta = (ab) \cdot \beta$, $\alpha \beta \cdot a = (\alpha \beta) \cdot a$, and so on. For a Hopf algebra $H$ and a left $H$-comodule $U$, we write the coaction $\delta : U \to H \otimes U$ as $\delta(u) = u_{(-1)} \otimes u_{(0)}$; then $\langle \varepsilon, u_{(-1)} \rangle u_{(0)} = u$ and $u_{(-1)} \otimes u_{(0)} = u_{(-1)} \otimes u_{(0)}$.

2. $\mathcal{H}(B^*)$ as a Yetter–Drinfeld $\mathcal{D}(B)$-module algebra

The purpose of this section is to show that $\mathcal{H}(B^*)$ is a Yetter–Drinfeld $\mathcal{D}(B)$-module algebra. The key ingredients are the $\mathcal{D}(B)$-comodule algebra structure from [4], which we recall in 2.1.1 and the $\mathcal{D}(B)$-module algebra structure from [17], which we recall in 2.1.2. The claim then follows by direct computation.

2.1. The Heisenberg double $\mathcal{H}(B^*)$. The Heisenberg double $\mathcal{H}(B^*)$ is the smash product $B^* \# B$ with respect to the left regular action of $B$ on $B^*$, which means that the composition in $\mathcal{H}(B^*)$ is given by

$$\langle \alpha \# a \rangle (\beta \# b) = \alpha(d' \cdot \beta) \# d'' b, \quad \alpha, \beta \in B^*, \quad a, b \in B.$$  \hspace{1cm} (2.1)

2.1.1. We recall from [4] that $\mathcal{H}(B^*)$ can also be obtained by twisting the product on the Drinfeld double $\mathcal{D}(B)$ (see Appendix A) as follows. Let $\eta : \mathcal{D}(B) \otimes \mathcal{D}(B) \to k$ be given by

$$\eta(\mu \otimes m, v \otimes n) = \langle \mu, 1 \rangle \langle \varepsilon, n \rangle \langle v, m \rangle$$

and let $\cdot : \mathcal{D}(B) \otimes \mathcal{D}(B) \to \mathcal{D}(B)$ be defined as

$$M \cdot N = M' N' \eta(M'', N''), \quad M, N \in \mathcal{D}(B).$$

A simple calculation shows that $\cdot$ coincides with the product in 2.1:

$$(\mu \otimes m) \cdot (v \otimes n) = \mu(m' \cdot v) \otimes m'' n, \quad \mu, v \in B^*, \quad m, n \in B.$$  \hspace{1cm} (2.1)

From this construction of $\mathcal{H}(B^*)$, it readily follows [4] that the coproduct of $\mathcal{D}(B)$, viewed as a map

$$\delta : \mathcal{H}(B^*) \to \mathcal{D}(B) \otimes \mathcal{H}(B^*)$$

$$\beta \# b \mapsto (\beta'' \otimes b') \otimes (\beta' \# b''),$$  \hspace{1cm} (2.2)

makes $\mathcal{H}(B^*)$ into a left $\mathcal{D}(B)$-comodule algebra (i.e., $\delta$ is an algebra morphism).

2.1.2. Simultaneously, $\mathcal{H}(B^*)$ is a $\mathcal{D}(B)$-module algebra, i.e.,

$$M \triangleright (AC) = (M' \triangleright A)(M'' \triangleright C)$$

for all $M \in \mathcal{D}(B)$ and $A, C \in \mathcal{H}(B^*)$, under the $\mathcal{D}(B)$ action defined in [17].
(2.4) \((\mu \otimes m) \triangleright (\alpha \# a) = \mu^m (m' \rightarrow \alpha) S^e \mu^l (\mu^l) \# (m' a S (m'')) \rightarrow S^e \mu^l (\mu^l))\),
\[\mu \otimes m \in D(B), \quad \alpha \# a \in \mathcal{H}(B^e).\]

Evidently, the right-hand side here factors into the actions of \(B^e\) and \(B\):
\[(\varepsilon \otimes m) \triangleright (\alpha \# a) = (\varepsilon \otimes 1) \triangleright ((\varepsilon \otimes m) \triangleright (\alpha \# a)),\]
where
\[(\varepsilon \otimes m) \triangleright (\alpha \# a) = (m' \rightarrow \alpha) \# (m'' a S (m''))\]
and
\[(\mu \otimes 1) \triangleright (\alpha \# a) = \mu^m \alpha S^e \mu^l \# (a \rightarrow S^e \mu^l (\mu^l)).\]
This allows verifying that (2.4) is indeed an action of \(D(B)\) independently of the argument in [17]; it suffices to show that the actions of \(B^e\) and \(B\) taken in the “reverse” order combine in accordance with the Drinfeld double multiplication, i.e., to show that
(2.5) \[(\varepsilon \otimes m) \triangleright ((\mu \otimes 1) \triangleright (\alpha \# a)) = ((\varepsilon \otimes m) (\mu \otimes 1)) \triangleright (\alpha \# a)\]
\[= (m' \rightarrow \mu \rightarrow S^e (m'')) \otimes m'' \triangleright (\alpha \# a).\]
We do this in B.1

The \(D(B)\)-module algebra property was shown in [17], but a somewhat less bulky proof can be given by considering the actions of \(\mu \otimes 1\) and \(\varepsilon \otimes m\) separately. The routine calculations are in B.2

2.2. Theorem. \(\mathcal{H}(B^e)\) is a (left–left) Yetter–Drinfeld \(D(B)\)-module algebra.

By this we mean a left module algebra and a left comodule algebra with the Yetter–Drinfeld compatibility condition
(2.6) \[(M' \triangleright A)_{(-1)} M'' \otimes (M' \triangleright A)_{(0)} = M' A_{(-1)} \otimes (M'' \triangleright A)_{(0)}\]
for all \(M \in D(B)\) and \(A \in \mathcal{H}(B^e)\). (For Yetter–Drinfeld modules, see [23, 24, 25, 26, 27, 19].) Condition (2.6) has to be shown for the \(D(B)\) action and coaction in (2.4) and (2.2).

2.2.1. Proof of 2.2. To simplify the calculation leading to (2.6), we again use that the action of \(\mu \otimes m \in D(B)\) factors through the actions of \(\mu \otimes 1\) and \(\varepsilon \otimes m\).

First, for \(M = \varepsilon \otimes m\), we evaluate the left-hand side of (2.6) as
\[(\varepsilon \otimes m') \triangleright (\alpha \# a)_{(-1)} (\varepsilon \otimes m'') \otimes ((\varepsilon \otimes m') \triangleright (\alpha \# a))_{(0)}\]
\[= ((m'^1 \rightarrow \alpha')(m'^2 a S (m'^3))(\varepsilon \otimes m'^4)) \otimes ((m'^1 \rightarrow \alpha')(m'^2 a S (m'^3))''\]
\[= ((m'^1 \rightarrow \alpha')(m'^2 a S (m'^3))' m'^4) \otimes (\alpha' \# (m'^2 a S (m'^3))'')\]
\[= ((m'^1 \rightarrow \alpha'') \otimes (m'^2 d S (m'^5)) m'^6) \otimes (\alpha' \# (m'^3 d') S (m'^4))\]
\[= ((m'^1 \rightarrow \alpha'') \otimes (m'^2 d') \otimes (\alpha' \# (m'^3 d') S (m'^4))\]

YETTER–DRINFELD STRUCTURES ON HEISENBERG DOUBLES AND CHAINS 3
but the right-hand side is given by
\[
((\varepsilon \otimes m')(\alpha'' \otimes d')) \otimes ((\varepsilon \otimes m'') \triangleright (\alpha' \neq \alpha''))
\]
\[
= ((m^{(1)} \rightarrow \alpha'' \rightarrow S^{-1}(m^{(3)}) \otimes m^{(2)}d') \otimes ((m^{(4)} \rightarrow \alpha') \neq m^{(5)}d'S(m^{(6)})))
\]
\[
= ((m^{(1)} \rightarrow \alpha'' \otimes m^{(2)}d') \otimes ((m^{(4)}S^{-1}(m^{(3)}) \rightarrow \alpha') \neq m^{(5)}d'S(m^{(6)})))
\]
(because \(\alpha' \otimes (\alpha'' \rightarrow m) = (m \rightarrow \alpha') \otimes \alpha''\), which is the same as the left-hand side.

Second, for \(M = \mu \otimes 1\), using the \(\mathcal{D}(B)\)-identity
\[
(\varepsilon \otimes (a \rightarrow S^{-1}(\mu'')))((\mu' \otimes 1) = \mu'' \otimes (S^{-1}(\mu') \rightarrow a),
\]
we evaluate the left-hand side of (2.7) as
\[
((\mu'' \otimes 1) \triangleright (\alpha \neq a))_{(\cdot)}((\mu' \otimes 1) \otimes ((\mu'' \otimes 1) \triangleright (\alpha \neq a))_{(0)}
\]
\[
= ((\mu^{(4)} \alpha S^{-1}(\mu^{(3)}))'' \otimes (a \rightarrow S^{-1}(\mu^{(2)}))(\mu^{(1)} \otimes 1))
\]
\[
\otimes ((\mu^{(4)} \alpha S^{-1}(\mu^{(3)}))' \neq (a \rightarrow S^{-1}(\mu^{(2)}))''
\]
\[
= ((\mu^{(6)} \alpha'' S^{-1}(\mu^{(3)}) \otimes (a \rightarrow S^{-1}(\mu^{(2)})))(\mu^{(1)} \otimes 1)) \otimes (\mu^{(5)} \alpha' S^{-1}(\mu^{(4)}) \neq a'')
\]
\[\tag{2.7}
= ((\mu^{(6)} \alpha'' S^{-1}(\mu^{(3)}) \mu^{(2)} \otimes (S^{-1}(\mu^{(1)}) \rightarrow a')) \otimes (\mu^{(5)} \alpha' S^{-1}(\mu^{(4)}) \neq a'')
\]
\[
= ((\mu^{(6)} \alpha'' \otimes a') \otimes (\mu^{(3)} \alpha' S^{-1}(\mu^{(2)}) \neq (a'' \rightarrow S^{-1}(\mu^{(1)})))
\]
\[
= ((\mu'' \otimes 1)(\alpha'' \otimes d')) \otimes ((\mu' \otimes 1) \triangleright (\alpha' \neq a''),
\]
which is the right-hand side.

3. \(\mathcal{H}(B^a)\) AS A BRAIDED COMMUTATIVE ALGEBRA

The category of Yetter–Drinfeld modules is well known to be braided, with the braiding \(c_{U,V} : U \otimes V \rightarrow V \otimes U\) given by
\[
c_{U,V} : u \otimes v \mapsto (u_{(\cdot)} \triangleright v) \otimes u_{(0)}.
\]
The inverse is \(c_{U,V}^{-1} : v \otimes u \mapsto u_{(0)} \otimes S^{-1}(u_{(\cdot)}) \triangleright v\).

3.1. Definition. A left \(H\)-module and left \(H\)-comodule algebra \(X\) is said to be braided commutative [7] (or \(H\)-commutative [19, 28]) if
\[
yx = (y_{(\cdot)} \triangleright x)y_{(0)}
\]
for all \(x, y \in X\).

3.2. Theorem. \(\mathcal{H}(B^a)\) is braided commutative with respect to the braiding associated with the Yetter–Drinfeld module structure.
3.2.1. **Remarks.**

1. The braided/\(H\)-commutativity property may be compared with “quantum commutativity” [29]. We recall that for a *quasitriangular* Hopf algebra \(H\), its module algebra \(X\) is called quantum commutative if

\[
yx = (R^2 \triangleright x)(R^1 \triangleright y) \equiv (R_{21} \triangleright (x \otimes y)), \quad x, y \in X,
\]

where \(R = R^{(1)} \otimes R^{(2)} \in H \otimes H\) is the universal \(R\)-matrix (and the dot denotes the multiplication in \(X\)). A minor source of confusion is that this useful property (see, e.g., [29, 5, 6]) is sometimes also referred to as \(H\)-commutativity [29]. For a Yetter–Drinfeld module algebra \(X\) over a quasitriangular \(H\), the properties in (3.1) and (3.2) are different (for example, a “quantum commutative” analogue of Theorem 3.2 does not hold for \(H(B^\ast)\)). We therefore consistently speak of (3.1) as of “braided commutativity” (this term is also used in [30] in related contexts, although in more than one).

2. The two properties, Eqs. (3.1) and (3.2), are “morally” similar, however. To see this, recall that a Yetter–Drinfeld \(H\)-module is the same thing as a \(D(H)\)-module, the \(D(H)\) action on a left–left Yetter–Drinfeld module \(X\) being defined as

\[
(p \otimes h) \triangleright x = \langle S_x^{-1}(p), (h \triangleright x)_{(-1)} \rangle (h \triangleright x)_{(0)}, \quad p \in H^\ast, \quad h \in H, \quad x \in X.
\]

Let then

\[
\mathcal{R} = \sum_A (\varepsilon \otimes e_A) \otimes (e_A \otimes 1) \in D(H) \otimes D(H)
\]

be the universal \(R\)-matrix for the double. It follows that

\[
\cdot (\mathcal{R}^{-1} \triangleright (x \otimes y)) = \langle \varepsilon \otimes S(e_A) \rangle (e_A \otimes 1) \triangleright y)
\]

\[
= \langle e_A, S_x^{-1}(y_{(-1)}) \rangle (S(e_A) \triangleright x)y_{(0)} = (y_{(-1)} \triangleright x)y_{(0)}
\]

for all \(x, y \in X\), and therefore the braided commutativity property can be equivalently stated in the form

\[
yx = \cdot (\mathcal{R}^{-1} \triangleright (x \otimes y))
\]

similar to Eq. (3.2) (the occurrence of \(\mathcal{R}^{-1}\) instead of \(R_{21}\) may be attributed to our choice of left–left Yetter–Drinfeld modules).

3.2.2. **Proof of 3.2.** We evaluate the right-hand side of (3.1) for \(X = H(B^\ast)\) as

\[
((\beta \# b)_{(-1)} \triangleright (\alpha \# a))((\beta \# b)_{(0)}
\]

\[
= ((\beta'' \otimes b') \triangleright (\alpha \# a))(\beta' \# b''
\]

\[
= (\beta'') (b^{(1)} \triangleright \alpha)S_x^{-1}(\beta^{(3)}) \# (b^{(2)}aS(b^{(3)})(-S_x^{-1}(\beta^{(2)}))) (\beta^{(1)} \# b^{(4)}
\]

\[
= (\beta'') (b^{(1)} \triangleright \alpha)S_x^{-1} (\beta^{(3)}) (b^{(2)}aS(b^{(3)})(-S_x^{-1}(\beta^{(2)}))) (\beta^{(1)} \# b^{(4)}
\]

\[
\# (b^{(2)}aS(b^{(3)})(-S_x^{-1}(\beta^{(2)}))) b^{(4)}
\]
\[
\begin{align*}
&= (\beta^{(4)}(b^{(1)} - \alpha)S^{*-1}(\beta^{(3)})(b^{(2)}aS(b^{(3)}))^\prime \cdash - S^{*-1}(\beta^{(2)}) - \beta^{(1)}) \\
&\quad \# (b^{(2)}aS(b^{(3)}))^\prime b^{(4)} \\
&\cong \beta^{(5)}(b^{(1)} - \alpha)S^{*-1}(\beta^{(4)})\beta^{(1)}<S^{*-1}(\beta^{(3)})\beta^{(2)}, (b^{(2)}aS(b^{(3)}))^\prime > \\
&\quad \# (b^{(2)}aS(b^{(3)}))^\prime b^{(4)} \\
&= \beta^{(3)}(b^{(1)} - \alpha)S^{*-1}(\beta^{(2)})\beta^{(1)} \# b^{(2)}aS(b^{(3)})b^{(4)} \\
&= \beta(b^{(1)} - \alpha) \# b^{(2)}a = (\beta \# b)(\alpha \# a),
\end{align*}
\]

where in \(\cong\) we used that \((a - \alpha) - \beta = \beta'\langle \alpha \beta'', a \rangle\).

### 3.3. Braided products

We now somewhat generalize the observation leading to 3.2. We first recall the definition of a braided product, then see when braided commutativity is hereditary under taking a braided product, and verify the corresponding condition for \(B^\text{cop}\) and \(B\); their braided product, which is therefore a braided commutative Yetter–Drinfeld module algebra, actually coincides with \(\mathcal{H}(B^\ast)\).

#### 3.3.1. If \(H\) is a Hopf algebra and \(X\) and \(Y\) two (left–left) Yetter–Drinfeld module algebras, their braided product \(X \bowtie Y\) is defined as the tensor product with the composition

\[
(x \bowtie y)(v \bowtie u) = x(y_{(-1)} \triangleright v) \bowtie y_{(0)} u, \quad x, v \in X, \quad y, u \in Y.
\]

This is a Yetter–Drinfeld module algebra. (Indeed, the associativity of (3.3) is ensured by \(Y\) being a comodule algebra and \(X\) being a module algebra. As a tensor product of Yetter–Drinfeld modules, \(X \bowtie Y\) is a Yetter–Drinfeld module under the diagonal action (via iterated coproduct) and codiagonal coaction of \(H\). By the Yetter–Drinfeld axiom for \(Y\) and the module algebra properties of \(X\) and \(Y\), moreover, \(X \bowtie Y\) is a module algebra; the routine verification is given in B.3 for completeness. That \(X \bowtie Y\) is a comodule algebra follows from the comodule algebra properties of \(X\) and \(Y\) and the Yetter–Drinfeld axiom for \(Y\); this is also recalled in B.3.)

#### 3.3.2. We say that two Yetter–Drinfeld modules \(X\) and \(Y\) are braided symmetric if

\[
c_{Y,X} = c_{X,Y}^{-1}
\]

(note that both sides here are maps \(Y \otimes X \to X \otimes Y\), that is,

\[
(y_{(-1)} \triangleright x) \otimes y_{(0)} = x_{(0)} \otimes (S^{-1}(x_{(-1)}) \triangleright y).
\]

#### 3.3.3. Lemma. Let \(X\) and \(Y\) be braided symmetric Yetter–Drinfeld modules, each of which is a braided commutative Yetter–Drinfeld module algebra. Then their braided product \(X \bowtie Y\) is also braided commutative.

We must show that

\[
((x \bowtie y)_{(-1)} \triangleright (v \bowtie u))(x \bowtie y)_{(0)} = (x \bowtie y)(v \bowtie u)
\]
for all $x, v \in X$ and $y, u \in Y$. For this, we write the condition $c_{X,Y} = c_{Y,X}^{-1}$ as
\[(x_{(-1)} \triangleright y) \otimes x_{(0)} = y_{(0)} \otimes (S^{-1}(y_{(-1)}) \triangleright x)\]
and use it to establish an auxiliary identity,

\[(3.5) \quad ((x_{(-1)} \triangleright y)_{(-1)} \triangleright x_{(0)}) \otimes (x_{(-1)} \triangleright y)_{(0)} = (y_{(0)}(x_{(-1)} \triangleright (S^{-1}(y_{(-1)}) \triangleright x))) \otimes y_{(0)}(0) = (y''_{(-1)} S^{-1}(y'_{(-1)}) \triangleright x) \otimes y_{(0)}(0) = x \otimes y.\]

The left-hand side of (3.4) is
\[
\left( (x \triangleleft y)_{(-1)} \triangleright (v \triangleleft u) \right) (x \triangleleft y)_{(0)}
\]
\[
= (x_{(-1)} y_{(-1)} \triangleright (v \triangleleft u)) (x_{(0)} \triangleleft y_{(0)})
\]
\[
= (x_{(-1)} y_{(-1)} \triangleright v) \triangleright (y''_{(-1)} y'_{(-1)} \triangleright u) (x_{(0)} \triangleleft y_{(0)})
\]
\[
= (x_{(-1)} y_{(-1)} \triangleright v) ((x_{(0)}(x_{(-1)} \triangleright (y''_{(-1)} \triangleright u)))_{(-1)} \triangleright x_{(0)}(0)) \triangleright (x_{(0)}(x_{(-1)} \triangleright (y''_{(-1)} \triangleright u))_{(0)} y_{(0)}
\]
\[
= (x_{(-1)} y_{(-1)} \triangleright v) x_{(0)} \triangleright (y''_{(-1)} \triangleright u) y_{(0)},
\]
just because of (3.5) in the last line. But the right-hand side of (3.4) is
\[
(x \triangleleft y)(v \triangleleft u) = (y_{(-1)} \triangleright v) \triangleleft y_{(0)} u
\]
\[
= (x_{(-1)} y_{(-1)} \triangleright v) x_{(0)} \triangleleft (y_{(0)}(x_{(-1)} \triangleright u)) y_{(0)}(0)
\]
because $X$ and $Y$ are both braided commutative. The two expressions coincide.

3.3.4. Remark. Because the braided symmetry condition is symmetric with respect to the two modules, we also have the braided symmetric Yetter–Drinfeld module algebra $Y \triangleright \triangleleft X$, with the product
\[
(y \triangleright x)(u \triangleright v) = y(x_{(-1)} \triangleright u) \triangleright x_{(0)} v.
\]
In addition to the multiplication inside $Y$ and inside $X$, this formula expresses the relations $xu = (x_{(-1)} \triangleright u)x_{(0)}$ satisfied in $Y \triangleright X$ by $x \in X$ and $u \in Y$. Because $c_{X,Y} = c_{Y,X}^{-1}$, these are the same relations $ux = (u_{(-1)} \triangleright x)u_{(0)}$ that we have in $X \triangleright Y$. Somewhat more formally, the isomorphism
\[
\phi : X \triangleright Y \to Y \triangleright X
\]
is given by $\phi : x \triangleright y \mapsto (x_{(-1)} \triangleright y) \triangleright x_{(0)}$. This is a module map by virtue of the Yetter–Drinfeld condition, and it is immediate to verify that $\delta(\phi(x \triangleright y)) = (\text{id} \otimes \phi)(\delta(x \triangleright y))$. That $\phi$ is an algebra map follows by calculating
\[
\phi(x \triangleright y) \phi(v \triangleright u) = ((x_{(-1)} \triangleright y) \triangleright x_{(0)})(v_{(-1)} \triangleright u) \triangleright v_{(0)}
\]
\[
= (x_{(-1)} \triangleright y)(v_{(0)}(x_{(-1)} \triangleright u) \triangleright x_{(0)} v_{(0)}
\]
\[
= (x_{(-1)} y)(x_{(0)}(x_{(-1)} \triangleright u) \triangleright x_{(0)} v_{(0)}
\]
\[ x_{(-1)} \triangleright (y_{(-1)} \triangleright u) \triangleright x_{(0)} \triangleright v_{(0)} \]

and

\[
\phi((x \triangleright y)(v \triangleright u)) = \phi(x(v_{(-1)} \triangleright v) \triangleright y_{(0)} u) \\
= (x_{(-1)}(y_{(-1)} \triangleright v)_{(-1)} \triangleright (y_{(0)} u)_{(0)}) \triangleright x_{(0)}(y_{(-1)} \triangleright v)_{(0)} \\
\overset{=}{=} x_{(-1)} \triangleright (y_{(0)} u)_{(0)} \triangleright x_{(0)}((S^{-1}(y_{(0)} u)_{(-1)} \triangleright v)_{(0)}) \\
= x_{(-1)} \triangleright (y_{(0)} u)_{(0)} \triangleright x_{(0)}((S^{-1}(y'_{(-1)} u)_{(-1)} \triangleright v)_{(0)}) \\
= x_{(-1)} \triangleright (uy)_{(0)} \triangleright x_{(0)}((S^{-1}(u)_{(-1)} \triangleright v)_{(0)}) \\
\overset{=}{=} x_{(-1)} \triangleright (y(v_{(-1)} \triangleright u)) \triangleright x_{(0)} v_{(0)},
\]

where the braided symmetry condition was used in each of the \( \overset{=}{=} \) equalities.

### 3.3.5. Multiple braided products.

Further examples of Yetter–Drinfeld module algebras are provided by multiple braided products \( X_1 \triangleright \ldots \triangleright X_N \) (of Yetter–Drinfeld \( H \)-module algebras \( X_i \)), defined as the corresponding tensor products with the diagonal action and codiagonal coaction of \( H \) and with the relations

\[
(3.6) \quad x[i] \triangleright y[j] = (x_{(-1)} \triangleright y)_{[j]} \triangleright x_{(0)} [i] \quad \text{for all} \quad i > j,
\]

where \( z[i] \in X_i \). (The inverse relations are \( x[i] \triangleright y[j] = y_{(0)} [j] \triangleright (S^{-1}(y_{(-1)} \triangleright x)[i]) \), \( i < j \).

It readily follows from the Yetter–Drinfeld module algebra axioms for each of the \( X_i \) that \( X_1 \triangleright \ldots \triangleright X_N \) is an associative algebra and, in fact, a Yetter–Drinfeld \( H \)-module algebra.

More specifically, let \( X \) and \( Y \) be braided symmetric Yetter–Drinfeld \( H \)-module algebras, as in **3.3.2**, and consider the “alternating” products

\[
(3.7) \quad X \triangleright Y \triangleright X \triangleright Y \triangleright \ldots,
\]

with an arbitrary number of factors (or a similar product with the leftmost \( Y \), or actually their inductive limits). We let \( X[i] \) denote the \( i \)th copy of \( X \), and similarly with \( Y[j] \). For arbitrary \( x[i] \in X[i] \) and \( y[j] \in Y[j] \), we then have the relations

\[
(3.8) \quad x[2i + 1] \triangleright y[2j] = (x_{(-1)} \triangleright y)_{[2j]} \triangleright x_{(0)}[2i + 1],
\]

which by \( (3.6) \) are satisfied for all \( i \geqslant j \); but by the braided symmetry condition, relations \( (3.8) \) hold for all \( i \) and \( j \) (replicating the relations between elements of \( X \) and elements of \( Y \) in \( X \triangleright Y \)). In \( (3.7) \), also,

\[
(3.9) \quad x[2i + 1] \triangleright v[2j + 1] = (x_{(-1)} \triangleright v)_{[2j + 1]} \triangleright x_{(0)}[2i + 1], \quad x, v \in X, \quad i > j.
\]

\[
y[2i] \triangleright u[2j] = (y_{(-1)} \triangleright u)_{[2j]} \triangleright y_{(0)}[2i], \quad y, u \in Y,
\]

(These formulas also hold for \( i = j \) if \( X \) and \( Y \) are braided commutative.)
3.4. $\mathcal{H}(B^a)$ as a braided product. Theorem 3.2 can be reinterpreted by saying that the Heisenberg double of $B^a$ is a braided product,

$$\mathcal{H}(B^a) = B^{a\text{cop}} \triangleright \triangleleft B,$$

with the braiding

$$b \otimes \beta \mapsto (b_{(-1)} \triangleright \beta) \otimes b_{(0)}, \quad b \in B, \quad \beta \in B^a,$$

where we abbreviate the action of $B$ in 2.1.2 to

$$m \triangleright (\beta \bowtie b) = (m' \triangleright \beta) \bowtie (m''bS(m'''))), \quad m \in B,$$

and further use $\triangleright$ for the restriction to $B^a$, viz., $m \triangleright \beta = m - \beta$. It is also understood that $B^{a\text{cop}}$ and $B$ are viewed as left $\mathcal{D}(B)$-comodule algebras via

$$\delta : \beta \mapsto (\beta'' \otimes 1) \otimes \beta', \quad \delta : b \mapsto (\varepsilon \otimes b') \otimes b''$$

and left $\mathcal{D}(B)$-module algebras via

$$(\mu \otimes m) \triangleright \beta = \mu'' (m - \beta) S^{-1}(\mu'), \quad (\mu \otimes m) \triangleright b = (m' \bowtie bS(m''')) - S^{-1}(\mu).$$

Both $B^{a\text{cop}}$ and $B$ are then Yetter–Drinfeld $\mathcal{D}(B)$-module algebras, and each is braided commutative.

Moreover, $B^{a\text{cop}}$ and $B$ are braided symmetric because $c_{B^{a\text{cop}}, B} = c_{B, B^{a\text{cop}}}^{-1}$, i.e.,

$$(b_{(-1)} \triangleright \beta) \otimes b_{(0)} = \beta_{(0)} \otimes (S^{-1}_{2\mathcal{D}}(\beta_{(-1)}) \triangleright b).$$

The antipode here is that of $\mathcal{D}(B)$, and therefore the right-hand side evaluates as $\beta' \otimes (S^a(\beta'') \triangleright b) = \beta' \otimes (b - S^{a^{-1}}(S^a(\beta''))) = \beta' \otimes (b - \beta'')$, which is immediately seen to coincide with the left-hand side.

Thus, the result that $\mathcal{H}(B^a) = B^{a\text{cop}} \triangleright \triangleleft B$ is a braided commutative Yetter–Drinfeld module algebra now follows from 3.3.3. (This offers a nice alternative to an unilluminating brute-force proof.)

3.5. Heisenberg $n$-tuples/chains. It follows from 3.3.4 that $\mathcal{H}(B^a)$ is also isomorphic to the braided commutative Yetter–Drinfeld module algebra $B \triangleright B^{a\text{cop}}$, with the product

$$(a \triangleright \alpha)(b \triangleright \beta) = a(b - S^{a^{-1}}(\alpha'')) \triangleright \alpha' \beta.$$

We next consider “Heisenberg $n$-tuples/chains”—the alternating products

$$\mathcal{H}_{2n} = B^{a\text{cop}} \triangleright B \triangleright B^{a\text{cop}} \triangleright B \triangleright \ldots \triangleright B,$$

$$\mathcal{H}_{2n+1} = B^{a\text{cop}} \triangleright B \triangleright B^{a\text{cop}} \triangleright B \triangleright \ldots \triangleright B \triangleright B^{a\text{cop}}.$$

As we saw in 3.3.5, the relations are then given by

$$b[2i] \beta[2j + 1] = (b' - \beta)[2j + 1] b''[2i] \quad \text{for all } i \text{ and } j$$

(where $b \in B$ and $\beta \in B^a$, $B^{a\text{cop}} \to B^{a\text{cop}}[2j + 1]$ and $B \to B[2i]$ are the morphisms onto
the respective factors, and we omit \( \bowtie \) for brevity) and
\[
\alpha[2i+1] \beta[2j+1] = (\alpha'' \beta S^{-1}(\alpha''))[2j+1] \alpha'[2i+1], \quad \alpha, \beta \in B^{\ast \text{cop}}, \quad i \geq j
\]
(\(a[2i] b[2j] = (d' b S(d''))[2j] d''[2i], \quad a, b \in B,\))

The chains with the leftmost \( B \) factor are defined entirely similarly. The obvious embeddings allow defining (one-sided or two-sided) inductive limits of alternating chains. All the chains are Yetter–Drinfeld module algebras, but those with \( \geq 3 \) tensor factors are not braided commutative in general.

### 4. Yetter–Drinfeld Module Algebras for \( \mathcal{U}_q s\ell(2) \)

In this section, we construct Yetter–Drinfeld module algebras for \( \mathcal{U}_q s\ell(2) \) at an even root of unity
\[
q = e^{i\frac{\pi}{p}}
\]
for an integer \( p \geq 2 \). \( \mathcal{U}_q s\ell(2) \) is the \( 2p^3 \)-dimensional quantum group with generators \( E, K, \) and \( F \) and the relations
\[
KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},
\]
\[
E^p = F^p = 0, \quad K^{2p} = 1
\]
and the Hopf algebra structure \( \Delta(E) = E \otimes K + 1 \otimes E, \Delta(K) = K \otimes K, \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \epsilon(E) = \epsilon(F) = 0, \epsilon(K) = 1, S(E) = -EK^{-1}, S(K) = K^{-1}, S(F) = -KF \)

In \([12, 13]\), \( \mathcal{U}_q s\ell(2) \) was arrived at as a subquotient of the Drinfeld double of a Taft Hopf algebra (a trick also used, e.g., in \([35]\) for a closely related quantum group). It turns out that not only \( \mathcal{D}(B) \) but also the pair \((\mathcal{D}(B), \mathcal{H}(B^\ast))\) can be “truncated” to a pair \((\mathcal{U}_q s\ell(2), \mathcal{H}_q s\ell(2))\) of \( 2p^3 \)-dimensional algebras, with \( \mathcal{H}_q s\ell(2) \) being a braided commutative Yetter–Drinfeld \( \mathcal{U}_q s\ell(2) \)-module algebra. This is worked out in what follows. \( \mathcal{H}_q s\ell(2) \) — a “Heisenberg counterpart” of \( \mathcal{U}_q s\ell(2) \) — appears in \([4, 22]\).

#### 4.1. \( \mathcal{D}(B) \) and \( \mathcal{H}(B^\ast) \) for the Taft Hopf algebra \( B \)

##### 4.1.1. The Taft Hopf algebra \( B \)

Let
\[
B = \text{Span}(E^m k^n), \quad 0 \leq m \leq p - 1, \quad 0 \leq n \leq 4p - 1,
\]
be the \( 4p^2 \)-dimensional Hopf algebra generated by \( E \) and \( k \) with the relations

---

In an “applied” context (see, e.g., \([14, 31, 32]\)), this quantum group first appeared in \([12, 13]\); subsequently, it gradually transpired (with the final picture having emerged from \([33]\)) that that was just a continuation of a series of previous (re)discoveries \([20, 21, 22]\) (also see \([34]\)). The ribbon and (somewhat stretching the definition) factorizable structures of \( \mathcal{U}_q s\ell(2) \) were worked out in \([12]\).
The Hopf-algebra structure

\[ (4.1) \quad kE = qE_k, \quad E^p = 0, \quad k^{4p} = 1, \]

and with the comultiplication, counit, and antipode given by

\[ (4.2) \quad \Delta(E) = 1 \otimes E + E \otimes k^2, \quad \Delta(k) = k \otimes k, \]
\[ \epsilon(E) = 0, \quad \epsilon(k) = 1, \]
\[ S(E) = -Ek^{-2}, \quad S(k) = k^{-1}. \]

Dual elements \( F, z \in B^* \) are introduced as

\[ \langle F, E^m k^n \rangle = \delta_{m,1} \frac{q^{-n}}{q-q^{-1}}, \quad \langle z, E^m k^n \rangle = \delta_{m,0} q^{-n/2}. \]

Then \([12]\)

\[ B^* = \text{Span}(F^a z^b), \quad 0 \leq a \leq p - 1, \quad 0 \leq b \leq 4p - 1. \]

**4.1.2. The Drinfeld double** \( D(B) \). Straightforward calculation shows \([12]\) that the Drinfeld double \( D(B) \) is the Hopf algebra generated by \( E, F, k, \) and \( z \) with the relations given by

i) relations \((4.1)\) in \( B \),

ii) the relations

\[ zF = qFz, \quad F^p = 0, \quad z^{4p} = 1 \]

in \( B^* \), and

iii) the cross-relations

\[ kz = zk, \quad kFk^{-1} = q^{-1}F, \quad zEz^{-1} = q^{-1}E, \quad [E, F] = \frac{k^2 - z^2}{q-q^{-1}}. \]

The Hopf-algebra structure \((\Delta_D, \epsilon_D, S_D)\) of \( D(B) \) is given by \((4.2)\) and

\[ \Delta_D(F) = z^2 \otimes F + F \otimes 1, \quad \Delta_D(z) = z \otimes z, \quad \epsilon_D(F) = 0, \quad \epsilon_D(z) = 1, \]
\[ S_D(F) = -z^{-2}F, \quad S_D(z) = z^{-1}. \]

**4.1.3. The Heisenberg double** \( \mathcal{H}(B^*) \). For the above \( B \), \( \mathcal{H}(B^*) \) is spanned by

\[ (4.3) \quad F^a z^b \# E^c k^d, \quad a, c = 0, \ldots, p - 1, \quad b, d \in \mathbb{Z}/(4p\mathbb{Z}), \]

where \( z^{4p} = 1, k^{4p} = 1, F^p = 0, \) and \( E^p = 0. \) Then the product in \((2.1)\) becomes \([17]\)

\[ (4.4) \quad (F^r z^s \# E^m k^n)(F^a z^b \# E^c k^d) \]
\[ = \sum_{u \geq 0} q^{-\frac{1}{2}u(u-1)} \binom{m}{u} \frac{[u]!}{[u]!} q^{-\frac{1}{2}b(n+2s)+a(s-n)+u(2c-a-b+m-s)} \]
\[ \times F^{a+r-u} z^{b+s} \# E^{m+c-u} k^{n+d+2u}. \]

A convenient basis in \( \mathcal{H}(B^*) \) can be chosen as \((z, z, \lambda, \partial)\), where \( z \) is understood as \( z \neq 1 \) and

\[ z = -(q-q^{-1})\epsilon \# E k^{-2}, \]
\[ \lambda = \kappa \neq k, \]
\[ \partial = (q - q^{-1})F \neq 1. \]

The relations in \( \mathcal{H}(B^\ast) \) then become \( \kappa z = q^{-1}z\kappa, \kappa \lambda = q^2 \lambda \kappa, \kappa \partial = q \partial \kappa, \partial^4 p = 1, \) and

\[ \lambda^4 p = 1, \quad z^p = 0, \quad \partial^p = 0, \]
\[ \lambda z = z \lambda, \quad \lambda \partial = \partial \lambda, \]
\[ \partial z = (q - q^{-1})1 + q^{-2}z \partial. \]

4.2. The \((\overline{U}_q \mathfrak{sl}(2), \overline{H}_q \mathfrak{sl}(2))\) pair.

4.2.1. From \( D(B) \) to \( \overline{U}_q \mathfrak{sl}(2) \). The “truncation” whereby \( D(B) \) yields \( \overline{U}_q \mathfrak{sl}(2) \) consists of two steps: first, taking the quotient

\[ (4.5) \quad \overline{D}(B) = D(B)/(\kappa k - 1) \]

by the Hopf ideal generated by the central element \( \kappa \otimes k - \epsilon \otimes 1 \) and, second, identifying \( \overline{U}_q \mathfrak{sl}(2) \) as the subalgebra in \( \overline{D}(B) \) spanned by \( F^\ell E^m k^{2n} \) (tensor product omitted) with \( \ell, m = 0, \ldots, p - 1, n = 0, \ldots, 2p - 1 \). It then follows that \( \overline{U}_q \mathfrak{sl}(2) \) is a Hopf algebra — the one described at the beginning of this section, where \( K = k^2 \).

4.2.2. From \( \mathcal{H}(B^\ast) \) to \( \overline{H}_q \mathfrak{sl}(2) \). In \( \mathcal{H}(B^\ast) \), dually, we take a subalgebra and then a quotient \([17]\). In the basis chosen above, the subalgebra (which is also a \( \overline{U}_q \mathfrak{sl}(2) \) submodule) is the one generated by \( z, \partial, \lambda \). Its quotient by \( \lambda^2 p = 1 \) gives a \( 2p^3 \)-dimensional algebra \( \overline{H}_q \mathfrak{sl}(2) \) — the “Heisenberg counterpart” of \( \overline{U}_q \mathfrak{sl}(2) \) \([17]\).

As an associative algebra,

\[ \overline{H}_q \mathfrak{sl}(2) = \mathbb{C}[z, \partial] \otimes (\mathbb{C}[\lambda]/(\lambda^{2p} - 1)), \]

with the \( p^2 \)-dimensional algebra

\[ \mathbb{C}[z, \partial] = \mathbb{C}[z, \partial]/(z^p, \partial^p, \partial z - (q - q^{-1}) - q^{-2}z \partial). \]

The \( \overline{U}_q \mathfrak{sl}(2) \) action on \( \overline{H}_q \mathfrak{sl}(2) \) follows from \((2.4)\) as

\[ E \triangleright \lambda^n = q^{-\frac{n}{2}} \left[ \frac{n}{2} \right] \lambda^n z, \quad k^2 \triangleright \lambda^n = q^{-n} \lambda, \quad F \triangleright \lambda^n = -q^2 \left[ \frac{n}{2} \right] \lambda^n \partial, \]
\[ E \triangleright z^n = -q^n [n] z^{n+1}, \quad k^2 \triangleright z^n = q^{2n} z^n, \quad F \triangleright z^n = [n] q^{1-n} z^{n+1}, \]
\[ E \triangleright \partial^n = q^{1-n} [n] \partial^{n+1}, \quad k^2 \triangleright \partial^n = q^{-2n} \partial^n, \quad F \triangleright \partial^n = -q^n [n] \partial^{n+1}. \]

The coaction \( \delta : \overline{H}_q \mathfrak{sl}(2) \to \overline{U}_q \mathfrak{sl}(2) \otimes \overline{H}_q \mathfrak{sl}(2) \) follows from \((2.2)\) as

\[ \lambda \mapsto 1 \otimes \lambda, \]
\[ z^n \mapsto \sum_{s=0}^{m} (-1)^s q^s(1-s) \left( q - q^{-1} \right)^m \left[ \begin{array}{c} m \\ s \end{array} \right] E^s k^{-2m} \otimes z^{m-s}, \]
\[ \partial^m \mapsto \sum_{s=0}^{m} q^{s(m-s)} (q - q^{-1})^s \left[ \frac{m}{s} \right] F^s k^{-2(m-s)} \otimes \partial^{m-s}. \]

In particular, \( z \mapsto k^{-2} \otimes z - (q - q^{-1}) Ek^{-2} \otimes 1 \) and \( \partial \mapsto k^{-2} \otimes \partial + (q - q^{-1}) F \otimes 1. \)

4.2.3. With the \( \overline{\mathcal{U}}_{q}\mathfrak{sl}(2) \) action and coaction given above, \( \overline{\mathcal{H}}_{q}\mathfrak{sl}(2) \) is a braided commutative Yetter–Drinfeld \( \mathcal{U}_{q}\mathfrak{sl}(2) \)-module algebra.

Hence, in particular, \( C_q[z, \partial] \) is also a braided commutative Yetter–Drinfeld \( \mathcal{U}_{q}\mathfrak{sl}(2) \)-module algebra.\(^4\)

4.3. Heisenberg "chains." The Heisenberg \( n \)-tuples/chains defined in 3.5 can also be "truncated" similarly to how we passed from \( \mathcal{H}(\mathcal{B}^n) \) to \( \mathcal{U}_{q}\mathfrak{sl}(2) \). An additional possibility here is to drop the coinvariant \( \lambda \) altogether, which leaves us with the “truly Heisenberg” Yetter–Drinfeld \( \mathcal{U}_{q}\mathfrak{sl}(2) \)-module algebras

\[
\begin{align*}
\mathcal{H}_2 &= C_q^a[z_1] \otimes C_q^b[z_2] = C_q[z_2, \partial_1], \\
\mathcal{H}_{2n} &= C_q^a[z_1] \otimes C_q^a[z_2] \otimes \cdots \otimes C_q^a[\partial_{2n-1}] \otimes C_q^b[z_{2n}], \\
\mathcal{H}_{2n+1} &= C_q^a[z_1] \otimes C_q^a[z_2] \otimes \cdots \otimes C_q^a[\partial_{2n-1}] \otimes C_q^b[z_{2n}] \otimes C_q^a[\partial_{2n+1}]
\end{align*}
\]

(or their infinite versions), where \( C_q^a[z] = C[\partial]/\partial^p \) and \( C_q^b[z] = C[z]/z^p \), with the braiding inherited from 3.5 which amounts to using the relations

\[ \partial_i z_j = q - q^{-1} + q^{-2} z_j \partial_i \]

for all (odd) \( i \) and (even) \( j \), and

\[ z_i z_j = q^{-2} z_j z_i + (1 - q^{-2}) z_i^2, \]

\[ \partial_i \partial_j = q^{2} \partial_j \partial_i + (1 - q^{2}) \partial_i^2, \quad i \geq j \]

(and \( z_i^p = 0 \) and \( \partial_i^p = 0 \); our relations may be interestingly compared with those in para-Grassmann algebras in 3.6).

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Appendix A. Drinfeld Double

We recall that the Drinfeld double of \( B \), denoted by \( \mathcal{D}(B) \), is \( B^* \otimes B \) as a vector space, endowed with the structure of a quasitriangular Hopf algebra given as follows. The co-algebra structure is that of \( B^* \otimes B \), the algebra structure is given by

\[ (\mu \otimes m)(\nu \otimes n) = \mu(m' - \nu - S^{-1}(m'')) \otimes m'n \]

\(^4\)We also recall that \( C_q[z, \partial] \) is in fact \( \text{Mat}_p(\mathbb{C}) \) 16.
for all $\mu, \nu \in B^s$ and $m, n \in B$, the antipode is given by

(A.2) \[ S_D(\mu \otimes m) = (\varepsilon \otimes S(m))(S^s(\mu) \otimes 1) = (S(m^\prime\prime) \rightarrow S^s(\mu) \rightarrow m^\prime) \otimes S(m^\prime\prime), \]

and the universal $R$-matrix is

(A.3) \[ R = \sum_i (\varepsilon \otimes e_i) \otimes (e^i \otimes 1), \]

where $\{e_i\}$ is a basis of $B$ and $\{e^i\}$ its dual basis in $B^s$.

**APPENDIX B. STANDARD CALCULATIONS**

**B.1. Proof of the action in (2.4).** To show that (2.4) defines an action of $\mathcal{D}(B)$, we verify (2.5) by first evaluating its right-hand side:

\[ \left((m^\prime \rightarrow \mu \rightarrow S^s(\mu^{\prime\prime})) \otimes m^{\prime\prime}\right) \triangleright (\alpha \neq a) \]

\[ = \langle \mu^\prime, S^s(\mu^{\prime\prime}) \rangle \langle \mu^{\prime\prime}, m^{(1)} \rangle (\mu^{\prime\prime} \otimes 1) \triangleright \left((m^{(2)} \rightarrow \alpha) \# m^{(3)} \alpha S(m^{(4)})\right) \]

\[ = \langle \mu^{(1)}, S^s(\mu^{(5)}) \rangle \langle \mu^{(5)}, m^{(1)} \rangle \mu^{(4)}(m^{(2)} \rightarrow \alpha)S^s(\mu^{(3)}) \# (m^{(3)} \alpha S(m^{(4)}) \rightarrow S^s(\mu^{(2)})) \]

\[ = \langle \mu^{(1)}, S^s(\mu^{(7)}) \rangle \langle \mu^{(5)}, m^{(1)} \rangle \mu^{(4)}(m^{(2)} \rightarrow \alpha)S^s(\mu^{(3)}) \# m^{(4)} \alpha d''S(m^{(5)}) \]

\[ \times \langle \mu^{(2)}, m^{(6)} S^{s-1}(d'', S^{s-1}(m^{(3)})) \rangle \]

\[ = \langle \mu^{(1)}, S^s(\mu^{(3)}) \rangle \langle \mu^{(4)}, m^{(1)} \rangle \mu^{(3)}(m^{(2)} \rightarrow \alpha)S^s(\mu^{(2)} \# m^{(4)} d''S(m^{(5)}) \rangle \]

\[ = \langle m^{(1)} \rightarrow S^s(\mu^{(3)}) \rangle \# m^{(2)}(a \rightarrow S^s(\mu^{(1)}))S(m^{(3)}), \]

which is the same as the left-hand side:

\[ (\varepsilon \otimes m) \triangleright ((\mu \otimes 1) \triangleright (\alpha \neq a)) = (\varepsilon \otimes m) \triangleright (\mu^{\prime\prime} \alpha S^s(\mu^{\prime\prime}) \# (a \rightarrow S^s(\mu^{\prime\prime}))) \]

\[ = (m^\prime \rightarrow m^\prime \alpha S^s(\mu^{\prime\prime}) \# (a \rightarrow S^s(\mu^{\prime\prime}))) \]

\[ = (m^\prime \rightarrow (m^\prime \alpha S^s(\mu^{\prime\prime}))) \# (m^\prime \alpha S^s(\mu^{\prime\prime})))S(m^{\prime\prime}). \]

**B.2. Proof of the $\mathcal{D}(B)$-module algebra property for $\mathcal{H}(B^s)$.** To show (2.3) for the action in (2.4), we do this for $M = \varepsilon \otimes m$ and $M = \mu \otimes 1$ separately.

First, the right-hand side of (2.3) with $M = \varepsilon \otimes m$ is

\[ \left((\varepsilon \otimes m^\prime) \triangleright (\alpha \neq a)\right) \left((\varepsilon \otimes m^{\prime\prime}) \triangleright (\beta \neq b)\right) \]

\[ = ((m^{(1)} \rightarrow \alpha) \# m^{(2)} \alpha S(m^{(3)}))(m^{(4)} \rightarrow \beta) \# m^{(5)} bS(m^{(6)})) \]

\[ = (m^{(1)} \rightarrow \alpha) \left(((m^{(2)} \alpha S(m^{(3)}))b(m^{(4)} \rightarrow \beta) \# (m^{(3)} \alpha S(m^{(4)}))b(m^{(5)} bS(m^{(6)})) \]

\[ = (m^{(1)} \rightarrow \alpha)m^{(2)} \alpha d''S(m^{(4)}))b(m^{(5)} bS(m^{(6)})) \]

\[ = (m^{(1)} \rightarrow \alpha)(m^{(2)} \alpha d''S(m^{(4)})), \]
which is the left-hand side \((\varepsilon \otimes m) \triangleright (\alpha(d \rightarrow \beta) \# d''b)\).

Second, the left-hand side of (2.3) with \(M = \mu \otimes 1\) is

\[
(\mu \otimes 1) \triangleright (\alpha(d \rightarrow \beta) \# d''b) \\
= \mu'' \alpha(d \rightarrow \beta)S^{a-1}((\mu'') \# (d''b) - S^{a-1}(\mu')) \\
= \mu'' \alpha(d \rightarrow \beta)S^{a-1}(\mu^{(4)}) \# (d'' - S^{a-1}(\mu^{(1)}))(b - S^{a-1}(\mu^{(1)})).
\]

But the right-hand side of (2.3) evaluates the same:

\[
((\mu'' \otimes 1) \triangleright (\alpha \# a)) \triangleright ((\mu' \otimes 1) \triangleright (\beta \# b)) \\
= (\mu''(\alpha S^{a-1}(\mu^{(5)})) \# (a - S^{a-1}(\mu^{(4)})))(\mu^{(3)}\beta S^{a-1}(\mu^{(2)})) \# (b - S^{a-1}(\mu^{(1)})) \\
= \mu^{(6)} \alpha S^{a-1}(\mu^{(5)})(a - S^{a-1}(\mu^{(4)})) \triangleright \mu^{(3)}\beta S^{a-1}(\mu^{(2)}) \# a''(b - S^{a-1}(\mu^{(1)})) \\
= \mu^{(6)} \alpha S^{a-1}(\mu^{(5)})\triangleright (a - S^{a-1}(\mu^{(4)})) - (\mu^{(3)}\beta S^{a-1}(\mu^{(2)})) \# a''(b - S^{a-1}(\mu^{(1)})) \\
= \mu^{(6)} \alpha S^{a-1}(\mu^{(5)})\triangleright (a - S^{a-1}(\mu^{(4)})) - (\mu^{(3)}\beta S^{a-1}(\mu^{(2)})) \# a''(b - S^{a-1}(\mu^{(1)})) \\
= \beta'' S^{a-1}(\mu^{(2)}) \cdot d'' \triangleright \alpha \beta' S^{a-1}(\mu^{(3)}) \# a''(b - S^{a-1}(\mu^{(1)})) \\
= \beta'' \cdot d'' \triangleright \mu^{(4)} \alpha \beta' S^{a-1}(\mu^{(3)}) \# a''(b - S^{a-1}(\mu^{(1)})) \\
= \mu^{(4)} \alpha (d \rightarrow \beta)S^{a-1}(\mu^{(3)}) \# (d'' - S^{a-1}(\mu^{(2)}))(b - S^{a-1}(\mu^{(1)})).
\]

### B.3. Standard checks for braided products

Here, we give the standard calculations establishing the module algebra and comodule algebra properties for the product defined in (3.3).

The module algebra property follows by calculating

\[
(h \triangleright (x \triangleright y))(h'' \triangleright (v \triangleright u)) = ((h^{(1)} \triangleright x) \triangleright (h^{(3)} \triangleright y))(h^{(5)} \triangleright v) \triangleright (h^{(4)} \triangleright u) \\
= (h^{(1)} \triangleright x)((h^{(3)} \triangleright y)(x) \triangleright (h^{(5)} \triangleright v) \triangleright y) = (h^{(4)} \triangleright u) \\
= (h^{(1)} \triangleright x)(h^{(2)} y(x) \triangleright y \triangleright v) \triangleright (h^{(3)} \triangleright y)(h^{(4)} \triangleright u) \\
= h \triangleright (x(y \triangleright v) \triangleright y \triangleright u) = h \triangleright ((x \triangleright y)(v \triangleright u)).
\]

To verify the comodule algebra property \(\delta((x \triangleright y)(v \triangleright u)) = \delta(x \triangleright y)\delta(v \triangleright u)\), we calculate the left-hand side using that \(X\) and \(Y\) are comodule algebras and that \(Y\) is Yetter–Drinfeld:

\[
\delta((x \triangleright y)(v \triangleright u)) = \delta(x(y \triangleright v) \triangleright y \triangleright u)
\]
\[
\begin{align*}
&= (x(y_{(-1)} \triangleright v))^{-1}_{(-1)} (y(0)u)^{-1}_{(-1)} \otimes (x(y_{(-1)} \triangleright v))_{(0)} \triangleright (y(0)^{u}_{(0)}u)_{(0)} \\
&= x_{(-1)}^\prime (y_{(-1)} \triangleright v)_{(-1)} y(0)_{(-1)} u_{(-1)} \otimes x(0) (y_{(-1)} \triangleright v)_{(0)} \triangleright y(0)_{(0)} u_{(0)} \\
&= x_{(-1)}^\prime y_{(-1)}^\prime v_{(-1)}^\prime u_{(-1)} \otimes (x(0) (y_{(-1)} \triangleright v)_{(0)} \triangleright y(0)_{(0)} u_{(0)}) \\
&= x_{(-1)}^\prime y_{(-1)}^\prime v_{(-1)}^\prime u_{(-1)} \otimes (x(0) (y_{(-1)} \triangleright v)_{(0)} \triangleright y(0)_{(0)} u_{(0)}),
\end{align*}
\]

which is the same as the right-hand side by another use of the comodule axiom for \( Y \):

\[
\delta(x \otimes y) \delta(v \otimes u) = (x_{(-1)} y_{(-1)} \otimes (x(0) \triangleright y_{(0)})) (v_{(-1)}^\prime u_{(-1)}^\prime \otimes (v_{(0)} \triangleright u_{(0)})) \\
= (x_{(-1)} y_{(-1)}^\prime v_{(-1)}^\prime u_{(-1)}^\prime) \otimes (x(0) (y_{(-1)} \triangleright v)_{(0)} \triangleright y(0)_{(0)} u_{(0)}) \\
= (x_{(-1)} y_{(-1)}^\prime v_{(-1)}^\prime u_{(-1)}^\prime) \otimes (x(0) (y_{(-1)} \triangleright v)_{(0)} \triangleright y(0)_{(0)} u_{(0)}).
\]

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