Decremental Optimization of Dominating Sets
Under Reachability Constraints

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Abstract
Given a dominating set, how much smaller a dominating set can we find through elementary operations? Here, we proceed by iterative vertex addition and removal while maintaining the property that the set forms a dominating set of bounded size. This can be seen as the optimization variant of the dominating set reconfiguration problem, where two dominating sets are given and the question is merely whether they can be reached one from another through elementary operations. We show that this problem is PSPACE-complete, even if the input graph is a bipartite graph, a split graph, or has bounded pathwidth. On the positive side, we give linear-time algorithms for cographs, trees and interval graphs. We also study the parameterized complexity of this problem. More precisely, we show that the problem is W[2]-hard when parameterized by the upper bound on the size of an intermediary dominating set. On the other hand, we give fixed-parameter algorithms with respect to the minimum size of a vertex cover, or $d + s$ where $d$ is the degeneracy and $s$ is the upper bound of output solution.

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1 Introduction

Recently, Combinatorial reconfiguration [12] has been extensively studied in the field of theoretical computer science. (See, e.g., surveys [10, 19].) A reconfiguration problem is generally defined as follows: we are given two feasible solutions of a combinatorial search problem, and asked to determine whether we can transform one into the other via feasible solutions so that all intermediate solutions are obtained from the previous one by applying the specified reconfiguration rule. This framework is applied to several well-studied combinatorial search problems; for example, Independent Set [3, 9, 11, 14, 15], Vertex Cover [17, 18], Dominating Set [8, 16, 18, 20], and so on.

The Dominating Set Reconfiguration problem is one of the well-studied reconfiguration problems. For a graph $G = (V, E)$, a vertex subset $D \subseteq V$ is called a dominating set of $G$ if $D$ contains at least one vertex in the closed neighborhood of each vertex in $V$. Figure 1 illustrates four dominating sets of the same graph. Suppose that we are given two dominating sets $D_0$ and $D_t$ of a graph whose cardinalities are at most a given upper bound $k$. Then the Dominating Set Reconfiguration problem asks to determine whether we can transform $D_0$ into $D_t$ via dominating sets of cardinalities at most $k$ such that all intermediate ones are obtained from the previous one by adding or removing exactly one vertex. Note that this reconfiguration rule, i.e. adding or removing exactly one vertex while keeping the cardinality constraint, is called the token addition and removal (TAR) rule. Figure 1 illustrates an example of transformation between two dominating sets $D_0$ and $D_3$ for an upper bound $k = 4$.

Combinatorial reconfiguration models “dynamic” transformations of systems, where we wish to transform the current configuration of a system into a more desirable one by a step-by-step transformation. In the current framework of combinatorial reconfiguration, we need to have in advance a target (a more desirable) configuration. However, it is sometimes hard to decide a target configuration, because there may exist exponentially many desirable configurations. Based on this situation, Ito et al. introduced the new framework of reconfiguration problems, called optimization variant [13]. In this variant, we are given a single solution as a current configuration, and asked for a more “desirable” solution reachable from the given one. This variant was introduced very recently, hence it has only been applied to Independent Set Reconfiguration to the best of our knowledge. Therefore and since Dominating Set Reconfiguration is one of the well-studied reconfiguration problems as we already said, we focus on this problem and study it under this framework.
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1.1 Our problem

In this paper, we study the optimization variant of DOMINATING SET RECONFIGURATION (denoted by OPT-DSR); to avoid the confusion, we call the original DOMINATING SET RECONFIGURATION the reachability variant (denoted by REACH-DSR). Suppose that we are given a graph $G$, two integers $k, s$, and a dominating set $D$ of $G$ whose cardinality is at most $k$; we call $k$ an upper bound and $s$ a solution size. Then OPT-DSR asks for a dominating set $D_t$ satisfying the following two conditions: (a) the cardinality of $D_t$ is at most $s$, and (b) $D_t$ can be transformed from $D$ under the TAR rule with upper bound $k$. For example, if we are given a dominating set $D_0$ in Figure 1 and two integers $k = 4$ and $s = 2$, then one of the solutions is $D_3$, because $D_3$ can be transformed from $D_0$ and $|D_3| \leq 2$ holds.

1.2 Related results

Although OPT-DSR is being introduced in this paper, some results for REACH-DSR relate to OPT-DSR in the sense that the techniques to show the computational hardness or construct an algorithm will be used in our proof for OPT-DSR. We thus list such results for REACH-DSR in the following.

There are several results for the polynomial-time solvability of REACH-DSR. Haddadan et al. [8] showed that REACH-DSR under TAR rule is PSPACE-complete for split graphs, for bipartite graphs, and for planar graphs, while linear-time solvable for interval graphs, for cographs, and for forests. REACH-DSR is also studied well from the viewpoint of fixed-parameter (in)tractability. Mouawad et al. [18] showed that REACH-DSR under TAR is W[2]-hard when parameterized by an upper bound $k$. As a positive result, Lokshtanov et al. [16] gave a fixed-parameter algorithm with respect to $k + d$ for graphs that exclude $K_{d,d}$ as a subgraph.

1.3 Our results

In this paper, we study OPT-DSR from the viewpoint of the polynomial-time (in)tractability and fixed-parameter (in)tractability.
We first study the polynomial-time solvability of OPT-DSR with respect to graph classes (See Figure 2). Specifically, we show that the problem is PSPACE-complete even for split graphs, for bipartite graphs, and for bounded pathwidth graphs, and NP-hard for planar graphs with bounded maximum degree. On the other hand, the problem is linear-time solvable for cographs, trees and interval graphs. The inclusions of these graph classes are represented in Figure 2.

We next study the fixed-parameter (in)tractability of OPT-DSR. We first focus on the following four graph parameters: the degeneracy $d$, the maximum degree $\Delta$, the pathwidth $pw$, and the vertex cover number $\tau$ (that is the size of minimum vertex cover). Figure 3(a) illustrates the relationship between these parameters, where $A \rightarrow B$ means that the parameter $A$ is bounded by some function of $B$. This relation implies that if we have a result stating that OPT-DSR is fixed-parameter tractable for $A$ then the tractability for $B$ follows, while if we have a negative (i.e. intractability) result for $B$ then it extends to $A$. From results for polynomial-time solvability, we show the PSPACE-completeness for fixed $pw$ and NP-hardness for fixed $\Delta$, and hence the problem is fixed-parameter intractable for each parameter $pw$, $\Delta$ and $d$ under $P \neq PSPACE$ or $P \neq NP$. As a positive result, we give an FPT algorithm for $\tau$, hence the problem is fixed-parameter tractable for $\tau$. We then consider two input parameters: the solution size $s$ and the upper bound $k$. (See Figure 3(c).) We show that OPT-DSR is W[2]-hard when parameterized by $k$. We note that we can assume without loss of generality that $s < k$ holds, as explained in Section 2. Therefore, it immediately implies W[2]-hardness for $s$. Most single parameters (except for $\tau$) cause a negative (intractability) result. We thus finally consider combinations of one graph parameter and one input parameter. We give an FPT algorithm with respect to $s + d$. (See Figure 3(b).) In the end, we can conclude from the discussion above that for any combination of a graph parameter $p \in \{d, \Delta, pw, \tau\}$ and an input parameter $q \in \{s, k\}$, OPT-DSR is fixed-parameter tractable when parameterized by $p + q$. Due to space limitations, proofs of statements marked with (*) have been moved to Appendix.

Figure 3 Our results for fixed-parameter tractability, where $A \rightarrow B$ means that the parameter $A$ is bounded on some function of $B$. 
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2 Preliminaries

For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set of $G$ and edge set of $G$, respectively. For a vertex $v \in V(G)$, we let $N_G(v) = \{w \mid vw \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$; we call a vertex in $N_G(v)$ a neighbor of $v$ in $G$. For a vertex subset $S \subseteq V(G)$, we let $N_G(S) = \bigcup_{v \in S} N_G(v)$. If there is no confusion, we sometimes omit $G$ from the notation.

2.1 Optimization variant of Dominating Set Reconfiguration

For a graph $G = (V, E)$, a vertex subset $D \subseteq V$ is a dominating set of $G$ if $N[D] = V(G)$. For a dominating set $D$, we say that $u \in D$ dominates $v \in V$ if $v \in N[u]$ holds. We say that a vertex $v \in D$ has a private neighbor in $D$ if there exists a vertex $u \in N[v]$ such that $N[u] \cap D = \{v\}$. In other words, the vertex $u$ is dominated only by $v$ in $D$. Note that the private neighbor of a vertex can be itself. A dominating set is (inclusion-wise) minimal if and only if each of its vertices has a private neighbor, and minimum if and only if the cardinality is minimum among all dominating sets. Notice that any minimum dominating set is minimal.

For a graph $G$, we denote by $\gamma(G)$ the dominating number of $G$ defined as the cardinality of a minimum dominating set of $G$; if it is clear from the context that $G$ is an input graph, then we just write $\gamma$ instead of $\gamma(G)$.

Let $D$ and $D'$ be two dominating sets of $G$. We say that $D$ and $D'$ are adjacent if $|D \Delta D'| = 1$, where $D \Delta D' = (D \setminus D') \cup (D' \setminus D)$ and we denote this by $D \leftrightarrow D'$. Let us now assume that both $D$ and $D'$ are both of size at most $k$, for some given $k \geq 0$. Then, a reconfiguration sequence between $D$ and $D'$ under the TAR rule (or sometimes called a TAR-sequence) is a sequence $\langle D = D_0, D_1, \ldots, D_\ell = D' \rangle$ of dominating sets of $G$ such that:

- for each $i \in \{0, 1, \ldots, \ell\}$, $D_i$ is a dominating set of $G$ such that $|D_i| \leq k$; and
- for each $i \in \{0, 1, \ldots, \ell - 1\}$, $D_i \leftrightarrow D_{i+1}$ holds.

Considering a reconfiguration sequence under the TAR rule, we sometimes write TAR($k$) instead of TAR to emphasize the upper bound $k$ on the size of a solution. We say that $D'$ is reachable from $D$ if there exists a reconfiguration sequence between $D$ and $D'$; since a reconfiguration sequence is reversible, if $D'$ is reachable from $D$, then $D$ is also reachable from $D'$. We write $D \rightharpoonup^k D'$ (resp. $D)$ is reachable from $D$ (resp. $D'$). Then, the optimization variant of the Dominating Set Reconfiguration problem (OPT-DSR) is defined as follows:

**OPT-DSR**

**Input:** A graph $G$, two integers $k, s \geq 0$, a dominating set $D$ of $G$ whose size is at most $k$.

**Output:** A dominating set $D_t$ of $G$ such that $|D_t| \leq s$ and $D \rightharpoonup^k D_t$ if it exists; no-instance otherwise.

We denote by a 4-tuple $(G, k, s, D)$ an instance of OPT-DSR.

2.2 Observations

From the definition of OPT-DSR, we have the following observation.
Observation 1. Let \((G, k, s, D)\) be an instance of OPT-DSR. If \(k, s\) and \(|D|\) violate the inequality \(s < |D| \leq k\), then \(D\) is a solution of the instance.

Proof. By the definition of \(D\), we know \(|D| \leq k\). Therefore if the inequality is violated, we have \(|D| \leq s \leq k\) or \(|D| \leq k \leq s\). In both cases, \(|D| \leq s\) holds, and hence \(D\) is a solution. ▶

It is observed that the condition in Observation 1 can be checked in linear time. Therefore, we sometimes assume without loss of generality that \(s < |D| \leq k\) holds. Then, another observation follows.

Observation 2. Let \((G, k, s, D)\) be an instance of OPT-DSR such that \(s < |D|\) holds.

If \(D\) is minimal and \(|D| = k\) holds, then the instance has no solution.

Proof. Since \(|D| = k\), we cannot add any vertex to \(D\) without exceeding the threshold \(k\). Besides, since \(D\) is minimal, we cannot remove any vertex while maintaining the domination property. As a result, there is no dominating set \(D_t\) of size at most \(s\) reachable from \(D\) i.e. \(D \xleftarrow{k} D_t\) does not hold for any dominating set \(D_t\) such that \(|D_t| \leq s\). ▶

Again, the conditions in Observation 2 can be checked in linear time, and hence we can assume without loss of generality that \(D\) is not minimal or \(|D| < k\) holds. Suppose that \(D\) is not minimal. Then we can always obtain a dominating set of size less than \(k\) by removing some vertex without private neighbor from \(D\), that is, we have a dominating set \(D'\) with \(D \xrightarrow{k} D'\) and \(|D'| < k\). Note that \((G, k, s, D)\) has a solution if and only if \((G, k, s, D')\) does. Therefore, it suffices to consider the case where \(|D| < k\) holds. Combining it with Observation 1, we sometimes assume without loss of generality that \(s < |D| < k\) holds.

3 Polynomial-time (in)tractability

In this section, we give some results for the polynomial-time solvability of OPT-DSR with respect to graph classes. In Subsection 3.1, we show the NP-hardness for the case where the input graph has maximum degree 3, or is planar with maximum degree 4. In Subsection 3.2, we show the PSPACE-completeness for bounded pathwidth graphs, for split graphs, and for bipartite graphs. In Subsection 3.3, we give polynomial-time algorithms for cographs, trees and interval graphs.

3.1 NP-hardness for planar graphs with bounded maximum degree

To show the NP-hardness, we will use the following observation:

Observation 3. Let \(G = (V, E)\) be a graph and \(s\) be an integer. Then an instance \((G, |V|, s, V)\) of OPT-DSR is equivalent to finding a dominating set of \(G\) with size at most \(s\).

Proof. We claim that any dominating set of cardinality at most \(s\) is a solution to the instance \((G, |V|, s, V)\). Suppose that \(G\) has a dominating set of cardinality at most \(s\) and let \(D_t\) be one of such dominating sets. Then, we can transform the input dominating set \(V\) into \(D_t\) by removing vertices in \(V \setminus D_t\) one by one; the observation follows. ▶

Observation 3 implies that results for the classical DOMINATING SET problem can be applied to OPT-DSR. Recall that given a graph \(G\) and an integer \(s\), the DOMINATING SET problem consists in deciding whether \(G\) admits a dominating set of size at most \(s\). This problem is known to be NP-hard even for the case where the input graph has maximum degree 3, or is planar with maximum degree 4 [7]. We thus obtain the following theorem.
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Theorem 4. OPT-DSR is NP-hard even for the case where the input graph has maximum degree 3, or is planar with maximum degree 4.

3.2 PSPACE-completeness for several graph classes

The following is the main theorem in this subsection.

Theorem 5. OPT-DSR is PSPACE-complete even for bounded pathwidth graphs, for split graphs, and for bipartite graphs.

First, observe that OPT-DSR is in PSPACE. Indeed, when we are given a dominating set \( D \) as a solution for some instance of OPT-DSR, we can check in polynomial time whether it has size at most \( s \) or not. Furthermore, since REACH-DSR is in PSPACE, we can check in polynomial space whether it is reachable from the original dominating set \( D \). Therefore, we can conclude that OPT-DSR is in PSPACE.

In the rest of this subsection, we thus show the PSPACE-hardness for bounded pathwidth, split and bipartite graphs, respectively. To this end, we give polynomial-time reductions from the optimization variant of VERTEX COVER RECONFIGURATION (denoted by OPT-VCR). We note that all reductions are almost identical to the ones of PSPACE-hardness for REACH-DSR [8].

We now give the definition of OPT-VCR. For a graph \( G = (V, E) \), a vertex subset \( C \subseteq V \) is called a vertex cover if \( C \) contains at least one endpoint of each edge in \( E \). Suppose that we are given a graph \( G \), two integers \( k, s \geq 0 \), and a vertex cover \( C \) of \( G \) whose cardinality is at most \( k \). Then OPT-VCR asks for a vertex cover \( C' \) satisfying the following two conditions: (a) the cardinality of \( C' \) is at most \( s \), and (b) \( C' \) can be transformed from \( C \) via vertex covers of size at most \( k \) such that each intermediate one can be obtained from the previous one by adding or removing exactly one vertex. The problem is known to be PSPACE-complete even for bounded pathwidth graphs [13].

In [13], Ito et al. actually showed the PSPACE-completeness for the optimization variant of INDEPENDENT SET RECONFIGURATION. However, the result can easily be converted to OPT-VCR from the observation that any vertex cover of a graph is the complement of an independent set.

We first consider bounded pathwidth graphs. The pathwidth of a graph is defined as follows. A path decomposition of \( G \) is a sequence \( P = (X_1, X_2, \ldots, X_\ell) \), where each \( X_i \subseteq V \), for each \( i \in \{1, 2, \ldots, \ell\} \), satisfies the following properties:

(i) each vertex \( v \in V \) is contained in (at least) one bag \( X_i \);
(ii) each edge \( uv \in E \) is contained in (at least) one bag i.e. there exists \( X_i \) such that \( u, v \in X_i \);
(iii) for every three indices \( i \leq j \leq k \), \( X_i \cap X_k \subseteq X_j \).

The width of a given path decomposition is one less than the size of its largest bag, that is \( \max_{1 \leq i \leq \ell} |X_i| - 1 \). Finally, the pathwidth of \( G \), denoted by \( pw(G) \), is the minimum width of any path decomposition of \( G \). Then the following lemma completes the proof of PSPACE-completeness for bounded pathwidth graphs.

Lemma 6. OPT-DSR is PSPACE-hard even for bounded pathwidth graphs.

Proof. Our reduction follows from the classical reduction from VERTEX COVER to DOMINATING SET [7]. Let \( (G', k', s', C) \) be an instance of OPT-VCR. Let \( G \) be the graph constructed from \( G' \) as follows: for each edge \( u, w \), we add a new vertex \( v_{uw} \) and join it with both of \( u \) and \( w \) by edges (see Fig. 4). Then let \( (G, k = k', s = s', D = C) \) be the corresponding instance of OPT-DSR. This construction can clearly be done in polynomial time.

1 In [13], Ito et al. actually showed the PSPACE-completeness for the optimization variant of INDEPENDENT SET RECONFIGURATION. However, the result can easily be converted to OPT-VCR from the observation that any vertex cover of a graph is the complement of an independent set.
It remains to prove that \((G', k', s', C)\) is a yes-instance for OPT-VCR if and only if \((G, k, s, D)\) is a yes-instance for OPT-DSR.

Suppose that \((G', k', s', C)\) is a yes-instance and let \(C_1\) be a vertex cover of size at most \(s'\) reachable from \(C\) under the TAR\((k')\) rule, by a sequence \(\mathcal{R}'\). Since any vertex cover of \(G'\) is a dominating set of \(G\) and \(k = k', s = s'\), then the sequence \(\mathcal{R}'\) yields a reconfiguration sequence from \(D = C\) to \(D_1 = C_1\). Thus, \((G, k, s, D)\) is a yes-instance.

We now prove the other direction. Suppose that \((G, k, s, D)\) is a yes-instance and let \(R = (D_0, D_1, \ldots, D_t)\) be a TAR\(k\) sequence of dominating sets of \(G\) starting at \(D_0\) and reaching a dominating set \(D_t\) that satisfies \(|D_t| \leq s\). Recall that \(D\) does not contain any newly added vertex in \(V(G) \setminus V(G')\). We want a sequence \(\mathcal{R}'\) that does not touch any newly added vertex \(v_{uw}\). To this end, we proceed by eliminating them one by one from the sequence. Let \(u, w\) be a vertex, and \(v_{uw}\) be the associated newly added vertex. Whenever a dominating set \(D_i\) contains \(v_{uw}\), we instead consider the set \(D'_i = (D_i \setminus v_{uw}) \cup \{u\}\), obtainable from \(D_{i-1}\) in one step under TAR\(k\) rule by simply ignoring the addition of \(v_{uw}\) and maybe adding \(u\). It is still a dominating set since \(N_G[v_{uw}] \subseteq N_G[u]\). The resulting sequence does not touch \(v_{uw}\), hence by repeating the operation on all vertices of \(V(G) \setminus V(G')\) we obtain a sequence \(\mathcal{R}'\) that does not touch any of them. In this way, we can obtain a reconfiguration sequence of vertex covers in \(G'\) between \(C\) and \(C_1 = D_1\) as needed.

Since OPT-VCR is PSPACE-complete for bounded pathwidth graphs, the reduction above implies PSPACE-hardness on bounded pathwidth graphs.

We next consider the class of split graphs. A graph is a split graph if its vertex set can be partitioned into a clique and an independent set. Then the following lemma completes the proof of PSPACE-completeness for split graphs.

**Lemma 7 (\(^\star\)).** OPT-DSR is PSPACE-hard even for split graphs.

We finally consider the class of bipartite graphs.

**Lemma 8 (\(^\star\)).** OPT-DSR is PSPACE-hard even for bipartite graphs.

### 3.3 Linear-time algorithms

In this subsection, we show that OPT-DSR can be solved in linear time for several graph classes. To this end, we deal with the concept of a canonical dominating set. We say that a dominating set \(D_c\) is canonical if \(D_c\) is a minimum dominating set which is reachable from any dominating set \(D\) under the TAR\(|D| + 1\) rule. Then we have the following theorem.

**Theorem 9.** Let \(\mathcal{G}\) be a class of graphs such that any graph \(G \in \mathcal{G}\) has a canonical dominating set and we can compute it in linear time. Then OPT-DSR can be solvable in linear time on \(\mathcal{G}\).
Proof. Let \((G, k, s, D)\) be an instance of OPT-DSR, where \(G \in \mathcal{G}\). Recall that we can assume without loss of generality that \(s < |D| < k\); we can check in linear time whether the inequality is satisfied or not, and if it is violated, then we know from Observation 1 and 2 that it is a trivial instance. Since \(G \in \mathcal{G}\), \(G\) admits a canonical dominating set and we can compute in linear time an actual one. Let \(D_c\) be such a canonical dominating set. Then it follows from the definition that \(D_c\) is reachable from \(D\) under the TAR\((k)\) rule since \(k \geq |D| + 1\). Since \(D_c\) is a minimum dominating set, we can output it if \(|D_c| \leq s\) holds, and no-instance otherwise. All processes can be done in linear time, and hence the theorem follows. ◁

Haddadan et al. showed in [8] that every cographs, trees (actually, forests), and interval graphs admit a canonical dominating set. Their proofs are constructive, and hence we can find an actual canonical dominating set. It is observed that the constructions on cographs and trees can be done in linear time. The construction on interval graphs can also be done in linear time with a nontrivial adaptation by using an appropriate data structure. Therefore, we have the following linear-time solvability of OPT-DSR.

Corollary 10. OPT-DSR can be solved in linear time on cographs, trees, and interval graphs.

4 Fixed-parameter (in)tractability

In this section, we study the fixed-parameter complexity of OPT-DSR with respect to several graph parameters: the upper bound \(k\), solution size \(s\), minimum size of a vertex cover \(\tau\) and degeneracy \(d\).

More precisely, we first show that OPT-DSR is \text{W}[2]-hard when parameterized by the upper bound \(k\). To prove it, we use the idea of the reduction constructed by Mouawad et al. in [18] to show the \text{W}[2]-hardness of REACH-DSR.

Theorem 11 (*). OPT-DSR is \text{W}[2]-hard when parameterized by the upper bound \(k\).

On the other hand, we give FPT algorithms with respect to the combination of the solution size \(s\) and the degeneracy \(d\) in Subsection 4.1 and the vertex cover number \(\tau\) in Subsection 4.2.

4.1 FPT algorithm for degeneracy and solution size

The following is the main theorem in this subsection.

Theorem 12. OPT-DSR is fixed-parameter tractable when parameterized by \(d + s\).

To prove the theorem, we give an FPT algorithm with respect to \(d + s\). Note that our algorithm uses the idea of an FPT algorithm solving the reachability variant of DOMINATING SET RECONFIGURATION, developed by Lokshtanov et al. [10]. Their algorithm uses the concept of domination core: for a graph \(G\), a domination core of \(G\) is a vertex subset \(C \subseteq V(G)\) such that any vertex subset \(D \subseteq V(G)\) is a dominating set of \(G\) if and only if \(C \subseteq N_G[D]\) [6].

Suppose that we are given an instance \((G, k, s, D)\) of OPT-DSR where \(G\) is a \(d\)-degenerate graph. By Observation 2 we can assume without loss of generality that \(|D| < k\). We first check whether \(G\) has a dominating set of size at most \(s\): this can be done in \(\text{FPT}(d + s)\) time for \(d\)-degenerate graphs [1]. If \(G\) does not have it, then we can instantly conclude that this is a no-instance.
In the remainder of this subsection, we assume that $G$ has a dominating set of size at most $s$. In this case, we kernelize the instance: we shrink $G$ by removing some vertices while keeping the existence of a solution until the size of the graph only depends on $d$ and $s$. To this end, we use the concept of domination core.

Lemma 13 (Lokhtanov et al. [16]). If $G$ is a $d$-degenerate graph and $G$ has a dominating set of size at most $t$, then $G$ has a domination core of size at most $dt^d$ and we can find it in FPT($d+t$) time.

Therefore, one can compute a domination core of $G$ of size at most $d s^d$ in FPT($d+s$) time by Lemma 13. In order to shrink $G$, we use the reduction rule $R1$: if there exists two vertices $v_r, v_l \in V(G) \setminus C$ such that $N_G(v_r) \cap C \subseteq N_G(v_l) \cap C$, we remove $v_r$. We need to prove that $R1$ is “safe”, that is, we can remove $v_r$ from $G$ without changing the existence of a solution. However, if the input dominating set $D$ contains $v_r$, we cannot do it immediately. Therefore, we first remove $v_r$ from $D$.

Lemma 14 (*). Let $D$ be a dominating set such that both $|D| < k$ and $v_r \in D$ hold. Then there exists $D'$ such that $v_r \notin D'$ and $D \leftrightarrow D'$, and $D'$ can be computed in linear time.

We can now redefine $D$ as a dominating set which does not contain $v_r$. We then consider removing $v_r$ from $G$. Let $G' = G[V(G) \setminus \{v_r\}]$. The following lemma ensures that removing $v_r$ keeps the existence of a solution.

Lemma 15. Let $(G, k, s, D)$ be an instance where $v_r \notin D$. Then, $(G, k, s, D)$ has a solution if and only if $(G', k, s, D)$ has a solution.

Proof. We first prove the if direction. Suppose that $(G', k, s, D)$ has a solution $D'$. Then there exists a reconfiguration sequence $D' = \langle D = D_0, D_1, \ldots, D_\ell = D' \rangle$ of dominating sets of $G'$. It suffices to show that any dominating set $D'_i$ of $G'$ in $D'$ is also a dominating set of $G$. Since $D'_i$ is a dominating set of $G'$ and $v_r \notin C$, we have $C \subseteq V(G') \subseteq N_{G'}[D'_i]$. By the definition of domination core, we know that $D'_i$ is also a dominating set of $G$.

We then prove only-if direction. Suppose that $(G, k, s, D)$ has a solution $D_s$. Then there exists a reconfiguration sequence $D = \langle D = D_0, D_1, \ldots, D_\ell = D_s \rangle$ of dominating sets of $G$. Based on $D$, we construct another sequence $D' = \langle D = D'_0, D'_1, \ldots, D'_\ell = D'_s \rangle$ of vertex sets of $G'$, where

$$D'_i = \begin{cases} D_i \setminus \{v_r\} & (v_r \in D_i) \\ D_i & \text{(otherwise)} \end{cases}$$

for each $i \in \{0, 1, \ldots, \ell\}$. Notice that any vertex subset in $D'$ does not contain $v_r$. Our claim is that $D'_s$ is a solution of $(G', k, s, D)$. To prove it, we show the following two statements:

(i) for each $i \in \{0, 1, \ldots, \ell\}$, $D'_i$ is a dominating set of $G$ (and hence of $G'$); and
(ii) for each $i \in \{0, 1, \ldots, \ell-1\}$, $|D'_i \Delta D'_{i+1}| \leq 1$ holds i.e. we have $D'_i \leftrightarrow D'_{i+1}$.

Then the sequence obtained by removing redundant ones from $D'$ is a reconfiguration sequence from $D$ to $D'_s$.

We first show the statement (i). Let $D_i$ be any dominating set in $D$. If $v_r \notin D_i$, then the statement clearly holds. Thus we consider the other case where $v_r \in D_i$. Since $D_i$ is a dominating set of $G$, we know $C \subseteq V(G) \subseteq N_G[D_i]$. Furthermore, since $N_G(v_r) \cap C \subseteq N_G(v_l) \cap C$, we have $C \subseteq N_G[D_i \setminus \{v_r\} \cup \{v_l\}] \subseteq N_G[D'_i]$. By the definition of domination core, $D'_i$ is a dominating set of $G$, and hence the statement (i) follows.
We then show the statement. Let $D_i$ and $D_{i+1}$ be any two consecutive dominating
sets in $\mathcal{D}$. Then, we know $|D_i \Delta D_{i+1}| = 1$. We assume without loss of
generality that $D_i \subseteq D_{i+1}$; otherwise the proof is symmetric. We prove the statement
in the following three cases:

- **Case 1**: both $v_r \notin D_i$ and $v_r \notin D_{i+1}$ hold;
- **Case 2**: either $v_r \in D_i$ or $v_r \in D_{i+1}$ holds (but not both); and
- **Case 3**: both $v_r \in D_i$ and $v_r \in D_{i+1}$ hold.

In **Case 1**, we know that $|D_i \Delta D'_{i+1}| = |D_i \Delta D_{i+1}| = 1$, and hence the statement
clearly holds. We then consider **Case 2**. In this case, since $D_i \subseteq D_{i+1}$, we observe that $v_r \notin D_i$ and
$v_r \in D_{i+1}$, and hence $\{v_r\} = D_{i+1} \setminus D_i$. Therefore, $D' \Delta D'_{i+1} = D_i \Delta (D_{i+1} \setminus \{v_r\}) \subset \{v_r\} \subset \{v_r\}$. Thus we can conclude that $|D_i \Delta D'_{i+1}| \leq 1$, and hence the statement follows. We finally deal with **Case 3**. In this case, we have $D_i \Delta D'_{i+1} = (D_i \setminus \{v_r\} \Delta (D_{i+1} \setminus \{v_r\} \cup \{v_r\})) \subset \{v_r\}$. Therefore, $|D_i \Delta D'_{i+1}| \leq |D_i \Delta D_{i+1}| = 1$ holds, and hence the statement follows.

In this way, we can conclude that $D'_i$ is a solution of $(G', k, s, d)$. This concludes the
proof.

We exhaustively apply the reduction rule $R_1$ to shrink $G$. Let $G_k$ and $D_k$ be the resulting
graph and dominating set, respectively. Then, any two vertices $u, v \in (V(G_k) \setminus C$ satisfy
$N_{G_k}(u) \cap C \neq N_{G_k}(v) \cap C$ (more precisely, $N_{G_k}(u) \cap C \neq N_{G_k}(v) \cap C$.) Then the following
lemma completes the proof of Theorem 12.

**Lemma 16.** $(G_k, k, s, D_k)$ can be solved in $\text{FPT}(d + s)$ time.

**Proof.** We first show that the size of the vertex set of $G_k$ is at most $f(d, s) = ds^d + 2^{ds^s}$. Since $|C| \leq ds^d$, it suffices to show that $|V(G_k) \setminus C| \leq 2^{ds^s}$ holds. Recall that any two vertices
$u, v \in (V(G_k) \setminus C$ satisfy $N_{G_k}(u) \cap C \neq N_{G_k}(v) \cap C$. Then since the number of combination
of vertices in $C$ is at most $2^{|C|} \leq 2^{ds^s}$, we have the desired upper bound $|V(G_k) \setminus C| \leq 2^{ds^s}$.

We now prove that $(G_k, k, s, D_k)$ can be solved in $\text{FPT}(d + s)$ time. To this end, we
construct an auxiliary graph $G_A$, where the vertex set of $G_A$ is the set of all dominating
sets of $G_k$, and any two nodes (that correspond to dominating sets of $G_k$) $D$ and $D'$ in $G_A$
are adjacent if and only if $|D \Delta D'| = 1$ holds. Let $n = |V(G_k)|$ and $m = |E(G_k)|$. Then
the number of candidate nodes in $G_A$ (vertex subsets of $G_k$) is bounded by $O(2^n)$. For
each candidate, we can check in $O(n + m)$ time if it forms a dominating set. Thus we can
construct the vertex set of $G_A$ in $O(2^n(n + m))$ time. We then construct the edge set of $G_A$.
There are at most $O(|V(G_A)|^2) = O(4^n)$ pairs of nodes in $G_A$. For each pair of nodes, we
can check in $O(n)$ time if their corresponding dominating sets differ in exactly one vertex.
Therefore we can construct the edge set of $G_A$ in $O(4^n n)$ time, and hence the total time
to construct $G_A$ is $O(4^n n + 2^n(n + m))$ time. We finally search a solution by running a
breadth-first search algorithm from $D_k$ on $G_A$ in $O(|V(G_A)| + |E(G_A)|) = O(4^n)$ time.

We can conclude that our algorithm runs in time $O(4^n n + 2^n(n + m))$ in total. Since
$n \leq f(d, s)$ and $m \leq n^2 \leq (f(d, s))^2$, this is an FPT time algorithm.

### 4.2 FPT algorithm for vertex cover number

Let $(G, k, s, D)$ be an instance of OPT-DSR. As in the previous section, we may first assume
by Observation 2 that $|D| < k$. We recall that $\tau(G)$ is the size of a minimum vertex cover of $G$. In order to lighten notations, we simply denote by $\tau$ the vertex cover number of the
input graph. Then, we have the following:

**Theorem 17.** OPT-DSR is fixed-parameter tractable when parameterized by $\tau$. 

We first establish the following fact that is going to be useful later.

\textbf{Observation 18.} If \( G \) is \( d \)-degenerate, then \( d \leq \tau \).

\textbf{Proof.} Let \( G \) be a graph, \( X \) a minimum vertex cover of \( G \) and \( H \) be any subgraph of \( G \). If \( H \) contains a vertex \( v \) outside \( X \), then \( v \) has a degree at most \( \tau \) in \( G \) and therefore in \( H \). Otherwise, \( H \) is a subgraph of \( G[X] \) and thus has at most \( \tau \) vertices. Hence all vertices of \( H \) have degree at most \( \tau \) in \( H \). Therefore, since any subgraph \( H \) of \( G \) contains a vertex of degree at most \( \tau \), \( G \) is \( \tau \)-degenerate. \hfill \blacktriangle

We are now able to get down to the proof of Theorem 17 by providing an algorithm that solves OPT-DSR and runs in time \( \text{FPT}(\tau) \). We first compute a minimum vertex cover \( X \subseteq V(G) \) of \( G \) in time \( \text{FPT}(\tau) \)). We partition the vertices of \( G \) into two components, the vertex cover \( X \) and the remaining vertices \( I \). By definition of vertex cover, no edge can have both ends outside \( X \), therefore \( I \) is an independent set. Note that if \( s \leq \tau \), then by Observation 18 we have \( d + s \leq 2\tau \), where \( d \) is the degeneracy of \( G \). In this case we are able to use the algorithm of the last section, that runs in time \( \text{FPT}(d + s) \).

We may therefore assume \( \tau < s \). In the remainder of the proof, we assume that the graph \( G \) has no isolated vertex since an isolated vertex must belong to any dominating set of \( G \). We now prove that \( (G, k, s, D) \) is always a yes-instance i.e. there exists a dominating set of size at most \( \tau \) that is reacheable from \( D \) under the TAR(\( k \)) rule.

We associate to every vertex \( v \in X \setminus D \) a special neighbor among its neighbors that dominate it (which can be either in \( X \) or \( I \)) i.e. we pick arbitrarily a vertex \( v \in N_G[v] \cap D \). We denote this special neighbor \( t(v) \). Let \( T \) be the set of special neighbors i.e. \( T := \{ t(v) \mid v \in X \setminus D \} \). This corresponds to the set of vertices that are used to dominate the vertices in \( X \) that do not belong to \( D \). Note that \( |T| \leq \tau \).

We are now able to describe the algorithm we use to output \( D_t \), the target dominating set. It consists in exhaustively applying the following rules on the vertices of \( I \) that belong to the current dominating set:

(i) if there is a vertex \( v \) in \( I \) but not in \( T \) that is already dominated by another vertex, then we remove \( v \) from the dominating set; and

(ii) if there is a vertex \( v \) in \( I \) but not in \( T \) that is dominated only by itself, then we add any one of its neighbors \( u \in X \) to the dominating set, and then remove \( v \). The vertex \( u \) does not need a special neighbor anymore, since it now belongs to the dominating set. We thus update the set \( T \) by only keeping the special neighbors \( t(w) \) of vertices \( w \) that are still in \( X \setminus D \).

We first prove that these two rules are safe i.e. we do not break the domination property at any step. Since Rule (i) removes a vertex \( v \) that is not required to dominate itself or another vertex \( u \in X \) (because it has not been chosen in \( T \)), we can safely remove it. In Rule (ii) after adding a neighbor of \( v \) to the dominating set, \( v \) is not required to dominate itself anymore. Since \( v \) is not in \( T \), we can now apply Rule (i) which is safe.

Recall that \( |D| < k \). Then, each dominating set obtained after applying one of these rules is of size at most \( k \) since Rule (i) only removes vertices and Rule (ii) consists in an addition immediately followed by a removal.

Now, let \( D_t \) be the dominating set obtained once we cannot apply Rule (i) and Rule (ii) anymore (see Figure 5 for an example). All remaining vertices in \( I \cap D_t \) now belong to \( T \). By definition of \( T \), each vertex in \( X \setminus D_t \) has (exactly) one neighbor in \( T \) (but they are not necessarily distinct). Therefore, \( |I \cap D_t| \leq |X \setminus D_t| \). As a result, \( |D_t| = |X \setminus D_t| + |I \cap D_t| \leq |X \setminus D_t| + |X \setminus D_t| = |X| = \tau \). Since \( \tau < s \), the size of \( D_t \) is at most \( s \), as desired.
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![Diagrams of reconfiguration sequences](image)

**Figure 5** Reconfiguration sequence from the original dominating set $D = I$ to the target one $D_t = \{b, v_1, v_3\}$. $D_1$ is obtained from $D_0$ by applying Rule (ii) and $D_2$ (resp. $D_3$) obtained from $D_1$ (resp. $D_2$) by applying Rule (i). The special neighbor of a vertex $v \in X \setminus D$ is the one pointed by its outgoing edge.

Finally, we focus on the complexity of this algorithm. As we already said, we first compute a minimum vertex cover $X$ of $G$ in time $\text{FPT}(\tau)$. If $s \leq \tau$, we run the FPT algorithm of Section 4.1. Otherwise, we compute the set $T$ and we run the subroutine that exhaustively applies the two aforementioned rules. Since these rules only apply to vertices in $I$ and whenever one is applied, exactly one vertex in $I$ is removed (and none is added), these rules are applied at most $|I \setminus D|$ times. Therefore, the subroutine runs in polynomial time and produces the desired dominating set $D_t$. As a result, this algorithm is FPT with respect to $\tau$. This concludes the proof.

## 5 Conclusion

In this paper, we have studied a new variant of combinatorial reconfiguration recently introduced by Ito et al. in [13] and we have applied it to the well-studied Dominating Set Reconfiguration problem. We have tackled this problem from a complexity perspective with respect to some graph parameters or graph classes. More precisely, we have shown that $\text{OPT-DSR}$ is PSPACE-complete, even when restricted to bounded pathwidth graphs, split graphs or bipartite graphs. On the other hand, we have shown that the problem is linear-time tractable on cographs, trees, and interval graphs. These results highlight the frontier between hardness and tractability since the problem is PSPACE-hard for bipartite graphs but linear for trees. We have also studied the problem from a parameterized complexity viewpoint and we have showed that $\text{OPT-DSR}$ is fixed-parameter tractable when parameterized by the minimum size of a vertex cover or by the degeneracy and the size of the desired dominating set.
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Omitted proof for Lemma 7

Proof. We again give a polynomial-time reduction from OPT-VCR. We extend the idea developed for the NP-hardness proof of DOMINATING SET problem on split graphs [2].

Let \((G', k', s', C)\) be an instance of OPT-VCR, where \(V(G') = \{v_1, v_2, \ldots, v_n\}\) and \(E(G') = \{e_1, e_2, \ldots, e_m\}\). We construct the corresponding split graph \(G\) as follows (see also Figure 6). Let \(V(G) = A \cup B\), where \(A = V(G')\) and \(B = \{w_1, w_2, \ldots, w_m\}\); the vertex \(w_i \in B\) corresponds to the edge \(e_i \in E(G')\). We join all pairs of vertices in \(A\) so that \(A\) forms a clique in \(G\). In addition, for each edge \(e_i = v_p v_q\) in \(E(G')\), we join \(w_i \in B\) with each of \(v_p\) and \(v_q\). Let \(G\) be the resulting graph, and let \((G', k = k', s = s', D = C)\) be the corresponding instance of OPT-DSR (we will prove later that \(D\) is a dominating set of \(G\)). Clearly, this instance can be constructed in polynomial time. It remains to prove that \((G', k', s', C)\) is a yes-instance if and only if \((G, k, s, D)\) is a yes-instance.

We first prove the only-if direction. Suppose that \((G', k', s', C)\) is a yes-instance. Then, there exists a vertex cover \(C_1\) of size at most \(s'\) reachable from \(C\) under the TAR\((k')\) rule. Since \(k' = k\) and both problems employ the same reconfiguration rule, it suffices to prove that any vertex cover of \(G'\) is a dominating set of \(G\). Since \(C \subseteq V(G') = A\) and \(A\) is a clique, all vertices in \(A \setminus C\) are dominated by the vertices in \(C\). Thus, consider a vertex \(w_i \in B\), which corresponds to the edge \(e_i = v_p v_q\) in \(E(G')\). Then, since \(C\) is a vertex cover of \(G'\), at least one of \(v_p\) and \(v_q\) must be contained in \(C\). This means that \(w_i\) is dominated by the endpoint \(v_p\) or \(v_q\) in \(G\). Therefore, each vertex cover in the reconfiguration sequence between \(C\) and \(C_1\) is a dominating set of \(G\) (including \(D = C\) and \(D_t = C_t\)) and thus, \((G, k, s, D)\) is a yes-instance.

We now focus on the if direction. Suppose that \((G, k, s, D)\) is a yes-instance. Then, there exists a dominating set \(D_t\) of \(G\) of size at most \(s\) reachable under the TAR\((k)\) rule by a sequence \(R = \{D_0, D_1, \ldots, D_t\}\), with \(D = D_0\). Recall that \(D = C\) and thus \(D\) is a vertex cover of \(G'\). We want to produce a sequence of dominating sets that are subsets of \(A\). To this end, in the same spirit as in the previous proof, we eliminate the vertices of \(B\) one by one. If a \(D_t\) contains a vertex \(w_j\) associated to the edge \(v_k v_j\), then we replace \(D_t\) by \(D'_t = (D_t \setminus w_j) \cup \{v_k\}\), which is also a dominating set and is reachable in one step from \(D_{t-1}\). Thus, the resulting sequence does not touch \(w_j\), and by repeating the operation to all vertices of \(B\), we obtain the wanted TAR\((k)\) sequence \(R'\) of subsets of \(A\). Observe that any dominating set \(D\) of \(G\) such that \(D \subseteq A = V(G')\) forms a vertex cover of \(G'\), because each vertex \(w_i \in B\) is dominated by at least one vertex in \(D \subseteq V(G')\). Therefore, \((G', s', k', C)\) is
a yes-instance. The conclusion follows.

B Omitted proof of Lemma 8

Proof. We give a polynomial-time reduction from OPT-DSR on split graphs to the same problem restricted to bipartite graphs. The same idea is used in the NP-hardness proof of DOMINATING SET problem on bipartite graphs [2].

Let \((G', k', s', D')\) be an instance of OPT-DSR, where \(G'\) is a split graph. Then \(V(G')\) can be partitioned into two subsets \(A\) and \(B\) which form a clique and an independent set in \(G'\), respectively. Furthermore, by the reduction given in the proof of Lemma 7, the problem on split graph remains PSPACE-complete even if the given dominating set \(D'\) consists of vertices only in \(A\). We thus assume that \(D' \subseteq A\) holds.

We now construct the corresponding bipartite graph \(G\), as follows. First, we delete any edge joining two vertices in \(A\) so that \(A\) forms an independent set. Then, we add a new edge consisting of two new vertices \(x\) and \(y\), and join \(y\) with each vertex in \(A\). The resulting graph \(G\) is bipartite (see Fig. 6 for an example). Let \(D = D' \cup \{y\}\), \(k = k' + 1\) and \(s = s' + 1\). Then we obtain the corresponding OPT-DSR instance \((G, k, s, D)\) where \(G\) is bipartite (here again, we will prove later that \(D\) is dominating set of \(G\)). Clearly, this instance can be constructed in polynomial time. We then prove that \((G, k', s', D')\) is a yes-instance if and only if \((G, k, s, D)\) is a yes-instance.

We first prove the only-if direction. Suppose that there exists a dominating set \(D'_1\) of \(G'\) such that \(D' \xrightarrow{\leq k'} D'_1\) and \(|D'_1| \leq s'\). Consider any dominating set \(D''\) of \(G'\). Then, \(B \subseteq N_G[D'']\) holds because \(B \subseteq N_G[D']\) and we have deleted only the edges which have both endpoints in \(A\). Since \(N_G[y] = A \cup \{x\}\), we can conclude that \(D'' \cup \{y\}\) is a dominating set of \(G\). Furthermore, \(|D'_1 \cup \{y\}| \leq s' + 1 = s\). Thus there exists a dominating set \(D_s\) of \(G\) such that \(D \xrightarrow{\leq k} D_s\) and \(|D_s| \leq s\), as desired.

We then prove the if direction. Suppose that there exists a dominating set \(D_t\) of \(G\), of size at most \(s\) and reachable from \(D\) by a TAR\((k)\) sequence \(\mathcal{R} = \langle D_0, D_1, \ldots, D_t \rangle\), with \(D = D_0\). Recall that \(D = D' \cup \{y\}\), and notice that any dominating set of \(G\) contains at least one of \(x\) and \(y\). Since \(N_G[x] \subseteq N_G[y]\), we can assume that \(D_s\) contains \(y\). Therefore, we can also assume that \(y\) is contained in every dominating set of the reconfiguration sequence. Recall that the assumption \(D' \subseteq A\) holds. As in the previous proof, we can produce an equivalent sequence \(\mathcal{R}'\) that does not touch any vertex of \(B\). Again, if a dominating set \(D_t\) touches a vertex \(w_j\) associated to the edge \(v_k\), we replace \(D_t\) by \(D'_t = (D_t \setminus w_j) \cup v_k\). We repeat the operation for all \(w_j\) and obtain the wanted sequence.

Consider any dominating set \(D\) of \(G\) in such a reconfiguration sequence. Since \(y \in D\), we have \(|D \cap V(G')| \leq k - 1 = k'\). Furthermore, since \(D \cap V(G') \subseteq A\) and \(A\) forms a clique in \(G'\), we have \(A \subseteq N_G[D \cap V(G')]\). Since there is no edge joining \(y\) and a vertex in \(B\), each vertex in \(B\) is dominated by some vertex in \(D \cap V(G')\). Therefore, \(D \cap V(G')\) is a dominating set of \(G'\) with cardinality at most \(k'\), and hence there exists a dominating set \(D'_t\) of \(G'\) such that \(D' \xrightarrow{\leq k'} D'_t\) and \(|D'_t| \leq s'\).

C Omitted proof of Theorem 11

Proof. We give an FPT-reduction from the (original) DOMINATING SET problem that is \(W[2]\)-hard when parameterized by its natural parameter \(k\).

Let \((G', k')\) be an instance of DOMINATING SET, where \(|V(G')| = n'\) and \(V(G') = \{v_1, v_2, \ldots, v_{n'}\}\). Then we construct the corresponding instance \((G, k, s, D)\) of OPT-DSR, as
follows. We first describe the construction of $G$. Let $G_0$ be the graph obtained by adding a universal vertex $v_0$ to $G'$, and $G_1, G_2, \ldots, G_k'$ be $k'$ copies of $G_0$. The vertex set of $G$ consists of $\bigcup_{j \in \{0, 1, \ldots, k'\}} V(G_j)$. For any $j \in \{1, 2, \ldots, k'\}$ and $i \in \{0, 1, \ldots, n'\}$, we use $v_{j,i}$ to denote the vertex in $G_j$ corresponding to $v_i$ in $G_0$. Then, for each vertex $v_i$ in $G_0$ except for $v_0$, we connect $v_i$ by new edges to all vertices in $N_{G_j}[v_{j,i}]$ in each $j \in \{1, 2, \ldots, k'\}$; formally, the edge set of $G$ consists of $\bigcup_{j \in \{0, 1, \ldots, k'\}} E(G_j) \cup \bigcup_{i \in \{1, 2, \ldots, n'\}} \bigcup_{j \in \{1, 2, \ldots, k'\}} \{v_{i,w} \mid w \in N_{G_j}[v_{j,i}]\}$. This completes the construction of $G$; see Figure 7 for an example of this reduction. However, for readability purposes, we do not draw all the edges between the vertices in $G'$ and those of $G_j$, for $j \in \{1, 2\}$. The only such drawn edges are the dotted ones (in gray) that are incident to the vertices $v_1$ and $v_3$. We set $k = 2k' + 1$, $s = k'$, and $D = \{v_{j,0} \mid j \in \{0, 1, \ldots, k'\}\}$; notice that $D$ has $k' + 1$ vertices. In this way, we constructed the corresponding instance $(G, k, s, D)$. Then our claim is that $(G', k')$ is a yes-instance if and only if $(G, k, s, D)$ has a solution.

We first prove the only-if direction. Suppose that $(G', k')$ is a yes-instance, hence there exists a dominating set $D'$ of $G'$ of size at most $k'$. Then by the construction of $G$, we know that $D'$ is also a dominating set of $G$ (if we identify the vertices of $G'$ with those of $G_0$).

Thus it suffices to show that $D \leftrightarrow D'$, since $D'$ has at most $s = k'$ vertices. We first add vertices in $D'$ to $D$ one by one; this transformation can be done under TAR($k$) since $|D \cup D'| \leq (k' + 1) + k' = k$. We then remove vertices in $D$ one by one. In this way, we can transform $D$ into $D'$ under TAR($k$), and hence $(G, k, s, D)$ has a solution $D'$.

We then prove the if direction. Suppose that $(G, k, s, D)$ has a solution $D'$. We know that $D'$ has at most $s = k'$ vertices. Then, since $G$ has $k' + 1$ copies $G_0, G_1, \ldots, G_k'$, there exists a copy $G_j \in \{G_0, G_1, \ldots, G_k'\}$ such that $V(G_j) \cap D' = \emptyset$. We know that $j \neq 0$ because all neighbors of $v_0$ are in $V(G_0)$, hence $D'$ contains at least one vertex in $V(G_0)$. For any $p \in \{1, 2, \ldots, k'\} \setminus \{j\}$, there is no edge joining a vertex in $V(G_j)$ and a vertex in $V(G_p)$. Therefore, for any vertex $v_{j,i}$ in $G_j$, a vertex $u \in D'$ which dominates $v_{j,i}$ is contained in $(V(G_0) \cap D') \setminus \{v_0\}$. Then, by the construction of $G$, $u$ also dominates the corresponding vertex $v_i$ in $G_0$. Thus, we know that $D'' = (V(G_0) \cap D') \setminus \{v_0\}$ is a dominating set of $G'$. Since $|D''| \leq |D'| \leq s = k'$ holds, $D''$ is a desired dominating set of $G'$.

\subsection*{D Omitted proof of Lemma 14}

\textbf{Proof.} We first consider the case where $v_i \in D$. In this case, we simply remove $v_r$ from $D$; let $D'$ be the resulting vertex subset. It is clear that $D \leftrightarrow D'$, and hence it suffices to show that $D'$ is a dominating set of $G$. We know that $C \subseteq N_G[D]$ holds by the definition of a domination core. Then since $N_G(v_r) \cap C \subseteq N_G(v_i) \cap C$ and $v_i \in D$ hold, we have

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\node[draw, circle] (v0) at (0,0) {$v_0$};
\node[draw, circle] (v1) at (1,1) {$v_1$};
\node[draw, circle] (v2) at (2,0) {$v_2$};
\node[draw, circle] (v3) at (1,-1) {$v_3$};
\node[draw, circle] (v4) at (0,-2) {$v_4$};
\node[draw, circle] (v5) at (2,-2) {$v_5$};
\draw (v0) -- (v1); \draw (v0) -- (v2); \draw (v0) -- (v3); \draw (v1) -- (v2); \draw (v1) -- (v4); \draw (v2) -- (v5); \draw (v3) -- (v4); \draw (v3) -- (v5);
\end{tikzpicture}
\caption{$(G', k' = 2)$}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\node[draw, circle] (v1_1) at (0,0) {$v_{1,1}$};
\node[draw, circle] (v1_0) at (0,-1) {$v_{1,0}$};
\node[draw, circle] (v2_0) at (1,-1) {$v_{2,0}$};
\node[draw, circle] (v2_1) at (1,0) {$v_{2,1}$};
\node[draw, circle] (v3_1) at (1,-2) {$v_{3,1}$};
\node[draw, circle] (v3_0) at (0,-2) {$v_{3,0}$};
\node[draw, circle] (v4_1) at (2,-1) {$v_{4,1}$};
\node[draw, circle] (v4_0) at (2,-2) {$v_{4,0}$};
\node[draw, circle] (v5_1) at (2,0) {$v_{5,1}$};
\node[draw, circle] (v5_0) at (2,1) {$v_{5,0}$};
\node[draw, circle] (v6_1) at (2,-2) {$v_{6,1}$};
\node[draw, circle] (v6_0) at (2,1) {$v_{6,0}$};
\draw (v1_1) -- (v1_0); \draw (v1_1) -- (v2_0); \draw (v1_1) -- (v3_0); \draw (v1_1) -- (v4_0); \draw (v1_1) -- (v5_0); \draw (v1_1) -- (v6_0); \draw (v2_0) -- (v2_1); \draw (v2_0) -- (v3_0); \draw (v2_0) -- (v4_0); \draw (v2_0) -- (v5_0); \draw (v2_0) -- (v6_0); \draw (v3_0) -- (v3_1); \draw (v3_0) -- (v4_0); \draw (v3_0) -- (v5_0); \draw (v3_0) -- (v6_0); \draw (v4_0) -- (v4_1); \draw (v4_0) -- (v5_0); \draw (v4_0) -- (v6_0); \draw (v5_0) -- (v5_1); \draw (v5_0) -- (v6_0); \draw (v6_0) -- (v6_1);
\end{tikzpicture}
\caption{$(G, k = 5, s = 2, D)$}
\end{subfigure}
\caption{Reduction for Theorem 11 with $D' = \{v_1, v_3\}$ and $D = \{v_0, v_{1,0}, v_{2,0}\}$.}
\end{figure}
\( C \subseteq N_G[D \setminus \{v_r\}] = N_G[D'] \). Thus \( D' \) is a dominating set of \( G \).

We then consider the remaining case where \( v_l \notin D \). In this case, we can add \( v_l \) to \( D \) since \( |D| < k \). Then the resulting dominating set contains \( v_l \), and we can remove \( v_r \) as discussed above.

\[ \text{\footnotesize $\blacktriangle$} \]