A STOCHASTIC ALGORITHM FOR DETERMINISTIC MULTISTAGE OPTIMIZATION PROBLEMS

MARIANNE AKIAN*, JEAN-PHILIPPE CHANCELIER†, AND BENOÎT TRAN†

Abstract. Several attempts to dampen the curse of dimensionality problem of the Dynamic Programming approach for solving multistage optimization problems have been investigated. One popular way to address this issue is the Stochastic Dual Dynamic Programming method (SDDP) introduced by Perreira and Pinto in 1991 for Markov Decision Processes. Assuming that the value function is convex (for a minimization problem), one builds a non-decreasing sequence of lower (or outer) convex approximations of the value function. These convex approximations are constructed as a supremum of affine cuts.

On continuous time deterministic optimal control problems, assuming that the value function is semiconvex, Zheng Qu, inspired by the work of McEneaney, introduced in 2013 a stochastic max-plus scheme that builds upper (or inner) non-increasing approximations of the value function.

In this note, we build a common framework for both the SDDP and a discrete time version of Zheng Qu’s algorithm to solve deterministic multistage optimization problems. Our algorithm generates monotone approximations of the value functions as a pointwise supremum, or infimum, of basic (affine or quadratic for example) functions which are randomly selected. We give sufficient conditions on the way basic functions are selected in order to ensure almost sure convergence of the approximations to the value function on a set of interest.

Key words. Deterministic multistage optimization problem, min-plus algebra, tropical algebra, Stochastic Dual Dynamic Programming, Dynamic Programming.

Introduction. Throughout this paper we aim to solve an optimization problem involving a dynamic system in discrete time. Informally, given a time $t$ and a state $x_t$, one can apply a control $u_t$ and the next state is given by the dynamic $f_t$, that is $x_{t+1} = f_t(x_t, u_t)$. Then one wants to minimize the sum of costs $c_t(x_t, u_t)$ induced by our controls starting from a given state $x_0$ and a given time horizon $T$. Furthermore, one can add some final restrictions on the states at time $T$ which will be modeled by a function $\psi$ only depending on the final state $x_T$. As in [1] we will call such optimization problems, multistage (optimization) problems:

\[
\min_{x=(x_0,...,x_T), u=(u_0,...,u_{T-1})} \sum_{t=0}^{T-1} c_t(x_t, u_t) + \psi(x_T)
\]

(0.1)

\[
\text{s.t. } \begin{cases} 
 x_0 \in X \text{ is given,} \\
 \forall t \in [0, T-1], \ x_{t+1} = f_t(x_t, u_t).
\end{cases}
\]

One can solve the multistage problem (0.1) by Dynamic Programming as introduced by Richard Bellman around 1950 [1, 5]. This method breaks the multistage problem (0.1) into $T$ sub-problems that one can solve by backward recursion over the time $t \in [0, T]$. More precisely, denoting by $X$ the set of states and given some operators $B_t : \mathbb{R}^X \to \mathbb{R}^X$ from the set of functionals that may take infinite values to itself, one can show (see for example [3]) that solving problem (0.1) is equivalent to solving the following system of sub-problems:
\[
\begin{cases}
V_T = \psi \\
\forall t \in [0, T - 1], \ V_t : x \in X \mapsto B_t(V_{t+1})(x) \in \mathbb{R}.
\end{cases}
\]

We will call each operator $B_t$ the Bellman operator at time $t$ and each equation in (0.2) will be called the Bellman equation at time $t$. Lastly, the functions $V_t$ defined in (0.2) will be called the (Bellman) value function at time $t$. Note that by solving the system (0.2) we mean that we want to compute the value function $V_0$ at point $x_0$, that is $V_0(x_0)$. We will state several assumptions on these operators in section 1 under which we will devise an algorithm to solve the system of Bellman equations 0.2 (also called the Dynamic Programming formulation of the multistage problem).

Let us stress on the fact that although we want to solve the multistage problem 0.1, we will mostly focus on its (equivalent) Dynamic Programming formulation given by (0.2).

One issue of the Dynamic Programming approach to solve multistage problems is the so-called curse of dimensionality [2], that is, grid-based methods to compute the value functions have a complexity exponential in the dimension of the state space. One popular algorithm (see [8, 9, 10, 14, 19, 20]) that aims to dampen the curse of dimensionality is the Stochastic Dual Dynamic Programming algorithm (or SDDP for short) introduced by Pereira and Pinto in 1991. Assuming that the cost functions $c_t$ are convex and the dynamics $f_t$ are linear, the value functions defined in the Dynamic Programming formulation (0.2) are convex [8]. The SDDP algorithm aims to build lower (or outer) approximations of the value functions as a supremum of affine functions and thus, doesn’t rely on a discretisation of the state space in order to compute (approximations of) the value functions. In the aforementioned references, this approach is used to solve stochastic multistage problems, however in this article we will restrict our study to deterministic multistage problems, that is, the above formulation (0.1). Still, the SDDP algorithm can be applied to our framework. One of the main drawback of the SDDP algorithm is the lack of an efficient stopping criterion: it builds lower approximations of the value functions but upper (or inner) approximations are built through a Monte-Carlo scheme that is costly.

During her thesis [15], Zheng Qu devised an algorithm [16] which builds upper approximations of the value functions in an infinite horizon and continuous time framework where the set of controls is both discrete and continuous. This work was inspired by the work of McEneaney [12, 13] using techniques coming from tropical algebra or also called min-plus techniques. Assume that for each fixed discrete control the cost functions are convex quadratic and the dynamics are linear. If the set of discrete controls is finite, then exploiting the min-plus linearity of the Bellman operators, one can show that the value functions can be computed as a finite pointwise infimum of convex quadratic functions:

\[ V_t = \inf_{\phi_t \in F_t} \phi_t, \]

where $F_t$ is a finite set of convex quadratic forms. Moreover, in this framework, the elements of $F_t$ can be explicitly computed through the Discrete Algebraic Riccati Equation (DARE, [11]). Thus an approximation scheme that computes non-decreasing subsets $(F^k_t)_{k \in \mathbb{N}}$ of $F_t$ yields an algorithm that converges after a finite number of
improvements

\[ V_t^k := \inf_{\phi_t \in F_t^k} \phi_t \approx \inf_{\phi_t \in F_t} \phi_t = V_t. \]

However the size of the set of functions \( F_t \) that need to be computed is growing exponentially with \( t \). Informally, in order to address this issue, Qu introduced a probabilistic scheme that adds to \( F_t^k \) the best (given the current approximations) element of \( F_t \) at some point drawn on the unit sphere (assuming the space of states to be Euclidean).

Our work aims to build a general algorithm that encompasses both a deterministic version of the SDDP algorithm and an adaptation of Qu’s work to a discrete time and finite horizon framework.

This paper is divided in 3 sections. In the first section we make several assumptions on the Bellman operators \( B_t \) and define an algorithm which builds approximations of the value functions as a pointwise optimum (i.e. either a pointwise infimum or a pointwise supremum) of basic functions in order to solve the associated Dynamic Programming formulation (0.2) of the multistage problem (0.1). At each iteration, the so-called basic function that is added to the current approximation will have to satisfy two key properties at a point randomly drawn, namely, tightness and validity. A key feature of our algorithm is that it can yield either upper or lower approximations, for example:

- If the basic functions are affine, then approximating the value functions by a pointwise supremum of affine functions will yield the SDDP algorithm.
- If the basic functions are quadratic convex, then approximating the value functions by a pointwise infimum of convex quadratic functions will yield an adaptation of Qu’s algorithm.

In the following section we study the convergence of the approximations of the value functions generated by our algorithm at a given time \( t \in [0, T] \). Under the previous assumptions our approximating sequence converges almost surely (over the draws) to the value function on a set of interest (that will be specified).

Finally on the last section we will specify our algorithm to the two special cases mentioned in the first section. The convergence result of section 2 specified to these two cases will be new for (an adaptation of) Qu’s algorithm and will be the same as in the literature for the SDDP algorithm. It will be a step toward addressing the issue of computing efficient upper approximations for the SDDP algorithm and opens another way to devise algorithms for a broader class of multistage problems.

1. Notations and definitions.

Notations 1.1.

- Denote by \( X := \mathbb{R}^n \) the set of states endowed with its euclidean structure and its Borel \( \sigma \)-algebra.
- Denote by \( T \) a finite integer that we’ll call the horizon.
- Denote by \( \text{opt} \) an operation that is either the pointwise infimum or the pointwise supremum of functions which we will call the pointwise optimum. Note that once a choice of which operation is associated with \( \text{opt} \), it remains the same for the remainder of this article.
- Denote by \( \overline{\mathbb{R}} \) the extended reals endowed with the operations \( +\infty - \infty = -\infty + \infty = +\infty \).
- For every \( t \in [0, T] \), fix \( F_t \) and \( \mathbb{F}_t \) two subsets of \( (\overline{\mathbb{R}})^X \) the set of functionals on \( X \) such that \( F_t \subset \mathbb{F}_t \).
Fig. 1. The lower approximations $V^{-}_t$ will be built as a supremum of basic functions (here affine functions) that will always be below $V_t$. Likewise, the upper approximations $V^{+}_t$ will be built as an infimum of some other basic functions (here quadratic functions) that will always be above $V_t$.

— We will say that a functional $\phi$ is a basic function if it’s an element of $F_t$ for some $t \in [0,T]$.
— For every set $X \subset X$, denote by $\delta_X$ the function equal to 0 on $X$ and $+\infty$ elsewhere.
— For every $t \in [0,T]$ and every set of basic function $F_t \subset F_t$ we denote by $V_{F_t}$ its pointwise optimum, $V_{F_t} := \text{opt}_{\phi \in F_t} \phi$, i.e.

$$V_{F_t} : X \rightarrow \mathbb{R}$$
$$x \mapsto \text{opt} \{ \phi(x) | \phi \in F_t \} .$$

— Denote by $(B_t)_{t \in [0,T-1]}$ a sequence of $T$ operators from $\mathbb{R}^X$ to $\mathbb{R}^X$, that we will call the Bellman operators.
— Fix a functional $\psi : X \rightarrow \mathbb{R}$. We define a sequence of functions $(V_t)_{t \in [0,T]}$, called the value functions, by the system of Bellman equations:

$$\begin{cases}
V_T = \psi \\
\forall t \in [0,T-1], V_t : x \in X \mapsto B_t(V_{t+1})(x) \in \mathbb{R}.
\end{cases}$$

We first make several assumptions on the structure of problem (1.2). Those assumptions will be satisfied in the examples of section 3. Informally, we want some regularities on the Bellman operators so as to propagate, backward in time, good behaviours of the value function at the final time $t = T$ to the value function at the initial time $t = 0$. Moreover, at each time $t$, we ask that the basic functions that build our approximations are such that their pointwise optimum share a common regularity.

Assumptions 1.2 (Structural assumptions).
— **Common regularity:** for every $t \in [0,T]$, there exist a common (local) modulus of continuity for every $\phi \in F_t$, i.e. for every $x \in X$, there exist $\omega_x : \mathbb{R}_+ \cup \{+\infty\} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ which is increasing, equal to 0 and continuous at 0 such that for every $x \in X$ and for every $x' \in X$ we have that

$$|f(x) - f(x')| \leq \omega_x (|x - x'|) .$$
— **Final condition:** the value function at time $T$ is a pointwise optimum for some given subset $F_T$ of $F$, as in (1.1), that is $V_T := V_{F_T}$.

— **Stability by the Bellman operators:** for every $t \in [0, T - 1]$, if $\phi \in F_{t+1}$ then $B_t(\phi)$ belongs to $F_t$.

— **Stability by pointwise optimum:** for every $t \in [0, T]$, if $F_t \subset F$, then $V_{F_T} \in F_t$.

— **Order preserving operators:** the operators $B_t$, $0 \leq t \leq T - 1$ are order preserving, i.e. if $\phi, \varphi \in F_{t+1}$ are such that $\phi \leq \varphi$, then $B_t(\phi) \leq B_t(\varphi)$.

— **Existence of the value functions:** there exists a solution $(V_t)_{t \in [0, T]}$ to the Bellman equations (0.2) that never takes the value $- \infty$ and which is not equal to $+ \infty$.

— **Existence of optimal sets:** for every $t \in [0, T - 1]$ and every compact set $X_t \subset \text{dom}(V_t)$, there exist a compact set $X_{t+1} \subset \text{dom}(V_{t+1})$ such that for every functional $\phi \in F_{t+1}$ we have

$$B_t(\phi + \delta X_{t+1}) \leq B_t(\phi) + \delta X_t.$$  

— **Additively $M$-subhomogeneous operators:** the operators $B_t$, $0 \leq t \leq T - 1$, are additively $M$-subhomogeneous, i.e. there exist a constant $M \geq 0$ such that for every positive constant functional $\lambda$ and every functional $\phi \in F_{t+1}$ we have that

$$B_t(\phi + \lambda) + \delta_{\text{dom}(V_t)} \leq B_t(\phi) + M \lambda + \delta_{\text{dom}(V_t)}.$$  

From a set of basic functions $F_t \subset F$, one can build its pointwise optimum $V_{F_t} = \text{opt}_{\phi \in F_t} \phi$. We build monotone approximations of the value functions as an optimum of basic functions which will be computed through a compatible selection function as defined below. We illustrate this definition in Figure 2.

**DEFINITION 1.3** (Compatible selection function). Let $t \in [0, T - 1]$ be fixed. A compatible selection function is a function $\phi^*_t$ from $2^{F_{t+1}} \times X$ to $F_t$ satisfying the following properties

— **Validity:** for all set of basic functions $F_{t+1} \subset F_{t+1}$ and every $x \in X$ we have $\phi^*_t(F_{t+1}, x) \leq B_t(V_{F_{t+1}})$ (resp. $\phi^*_t(F_{t+1}, x) \geq B_t(V_{F_{t+1}})$) when $\text{opt} = \sup$ (resp. $\text{opt} = \inf$).

— **Tightness:** for all set of basic functions $F_{t+1} \subset F_{t+1}$ and every $x \in X$ the functions $\phi^*_t(F_{t+1}, x)$ and $B_t(V_{F_{t+1}})$ coincide at point $x$, that is

$$\phi^*_t(F_{t+1}, x)(x) = B_t(V_{F_{t+1}})(x).$$  

For $t = T$, we say that $\phi^*_T : 2^{F_T} \times X \to F_T$ is a compatible selection function if function $\phi^*_T$ is valid in the sense that for every $F_T \subset F_T$ and $x \in X$, the function $\phi^*_T(F_T, x)$ remains above (resp. below) the value function at time $T$ when $\text{opt} = \inf$ (resp. $\text{opt} = \sup$). Moreover, we say that function $\phi^*_T$ is tight if it coincides with the value function at point $x$, that is for every $F_T \subset F_T$ and $x \in X$ we have

$$\phi^*_T(F_T, x)(x) = V_T(x).$$  

Note that the tightness assumption only asks for equality at the point $x$ between the functions $\phi^*_t(F_{t+1}, x)$ and $B_t(V_{F_{t+1}})$ and not necessarily everywhere. The only global property between the functions $\phi^*_t(F_{t+1}, x)$ and $B_t(V_{F_{t+1}})$ is an inequality given by the validity assumption.

In Algorithm 1.1 we will generate for every time $t$ a sequence of random points of crucial importance as they will be the ones where the selection functions will be
evaluated, given the set $F_t^k$ which characterizes the current approximation. In order to
generate those points, we will assume that we have at our disposition an Oracle
that given $T+1$ set of functions (characterizing the current approximations) computes
$T+1$ compact sets and a probability law of support equal to those compact sets. This
Oracle will have to follow the following conditions on its output.

\textbf{Assumptions 1.4 (Oracle assumptions).}

The Oracle takes as input $T + 1$ sets of functions included in $F_1 \times \ldots \times F_T$. Its
output is $T + 1$ compact sets $K_0, \ldots, K_T$ each included in $X$ and a probability measure
$\mu$ on $X^{T+1}$ (where $X = \mathbb{R}^n$ is endowed with its borelian $\sigma$-algebra) such that:

- For every $t \in [0, T]$, $K_t \subset \text{dom} (V_t)$.
- For every $t \in [0, T]$, there exist a function $g_t : \mathbb{R}_+^* \rightarrow (0, 1)$ such that for
every $\eta > 0$ and every $x \in K_t$,

$$
\mu \left( B \left( x, \eta \right) \cap K_t \right) \geq g_t \left( \eta \right).
$$

An example of such Oracle would be one that outputs $T+1$ union of $N$ singletons
in $\text{dom} (V_t)$ (for some positive integer $N$) and the product measure of $\mu_t$, $0 \leq t \leq T$
where $\mu_t$ is the uniform measure over the $N$ singletons. Then for every $t \in [0, T]$ the
constant function $g_t$ equal to $\frac{1}{n}$ satisfies Assumptions 1.4.

For every time $t \in [0, T]$, we construct a sequence of functionals $(V_t^k)_{k \in \mathbb{N}}$ of $F_t$
as follows. For every time $t \in [0, T]$ and for every $k \geq 0$, we build a subset $F_t^k$ of
the set $F_t$ and define the sequence of functionals by pointwise optimum $V_t^k := V_{F_t^k} = \text{opt}_{F_t^k} \phi$. As described here, the functionals are just byproducts of Algorithm 1.1
which only describes the way the sets $F_t^k$ are defined.

As Algorithm 1.1 was inspired by Qu’s work which uses tropical algebra tech-
niques, we will call this algorithm Tropical Dynamic Programming.

2. Almost sure convergence on the set of accumulation points.
For every approximations are computed through compatible selection functions. the Bellman operators are order preserving and that the basic functions building our Assumptions 1.4, $T + 1$ compact sets $K^0_0 \times \ldots \times K^0_T$ and a probability measure $\mu^0$ on $X^{T+1}$ of support equal to those $T + 1$ compact sets.

**Output:** For every $t \in [0, T]$, a sequence of sets $(F^k_t)_{k \in \mathbb{N}}$.

- Define for every $t \in [0, T]$, $F^0_t := \emptyset$.

  - For $k \geq 1$ do
    - Draw some points $(x^{k-1}_t)_{t \in [0, T]}$ over $K_0^{k-1} \times K_1^{k-1} \times \ldots \times K_T^{k-1}$ according to $\mu^{k-1}$, independently from previous draws at iterations $k' < k$.
    - Compute $\phi^k_t := \phi^k_T (F^k_{T-1}, x^{k-1}_T)$.
    - Define $F^k_T := F^k_{T-1} \cup \{ \phi^k_T \}$.
    - for $t$ from $T - 1$ to 0 do
      - Compute $\phi^k_t := \phi^k_t (F^k_{t+1}, x^{k-1}_t)$.
      - Define $F^k_t := F^k_{t+1} \cup \{ \phi^k_t \}$.
    - end for
    - Compute $(K^0_0 \times \ldots \times K^k_T, \mu^k) := Oracle (F^0_0, \ldots, F^k_T)$.
- end for

First, we state several simple but crucial properties of the approximation functions $(V^k_t)_{k \in \mathbb{N}}$ generated by Algorithm 1.1. They are direct consequences of the facts that the Bellman operators are order preserving and that the basic functions building our approximations are computed through compatible selection functions.

**Lemma 2.1.**
1. Let $(F^k)_{k \in \mathbb{N}}$ be a non-decreasing (for the inclusion) sequence of set of functionals on $X$. Then the sequence $(V^k_F)_{k \in \mathbb{N}}$ is monotone. More precisely, when opt = inf then $(V^k_F)_{k \in \mathbb{N}}$ is non-increasing and when opt = sup then $(V^k_F)_{k \in \mathbb{N}}$ is non-decreasing.

2. **Monotone approximations:** for every indices $k < k'$ we have that
   \[(2.1) \quad V^k_t \geq V^{k'}_t \geq V_t \quad \text{when opt = inf}\]
   and $V^k_t \leq V^{k'}_t \leq V_t$ when opt = sup.

3. For every $k \in \mathbb{N}$ and every $t \in [0, T - 1]$ we have that
   \[(2.2) \quad B_t (V^k_{t+1}) \leq V^k_t \quad \text{when opt = inf}\]
   and $B_t (V^k_{t+1}) \geq V^k_t$ when opt = sup.

4. For every $k \geq 1$, we have
   \[(2.3) \quad B_t (V^k_{t+1}) (x^{k-1}_t) = V^k_t (x^{k-1}_t) .\]

**Proof.** We prove each point successively when opt = inf, as the case opt = sup is similar.

1. Let $F^k \subset F^{k'}$ be two set of functionals. When opt = inf for every $x \in X$ we have that
   \[V_{F^{k'}} (x) := \inf_{\phi \in F^{k'}} \phi (x) \leq \inf_{\phi \in F^k} \phi (x) =: V_{F^k} (x).\]

2. By construction of Algorithm 1.1, the sequence of sets $(F^k_t)_{k \in \mathbb{N}}$ is non-decreasing, thus for every indices $k < k'$ we have that $V^k_t \geq V^{k'}_t$ when opt = inf (and $V^k_t \leq V^{k'}_t$ when opt = sup).
Now we show that $V^k_t \geq V_t$ when opt = inf, the case opt = sup is analogous. Fix $k \in \mathbb{N}$, we show this by backward recursion on $t \in [0, T]$. For $t = T$, by validity of the selection functions \textbf{Definition 1.3}, for every $\phi \in F^k_T$ we have that $\phi \geq V_T$. Thus $V^k_T = V^k_{T+1} = \inf_{\phi \in F^k_T} \phi \geq V_T$. Now, suppose that for some $t \in [0, T-1]$ we have $V^k_t \geq V_{t+1}$. Applying the Bellman operator, using the definition of the value function (0.2) and as the Bellman operators are order preserving, we get the desired result.

3. We prove the assertion by induction on $k \in \mathbb{N}$ in the case opt = inf. For $k = 0$, as $F^0_t = \emptyset$ we have $V^0_t = +\infty$. Thus the assertion is true for $k = 0$. Assume that for some $k \in \mathbb{N}$ we have

\begin{equation}
B_t (V^k_{t+1}) \leq V^k_t.
\end{equation}

By (2.1) we have that $V^k_{t+1} \leq V^k_{t+1}$. Thus, as the Bellman operators are order preserving we have that $B_t (V^k_{t+1}) \leq B_t (V^k_{t+1})$. Thus by induction hypothesis (2.4) we get

\begin{equation}
B_t (V^k_{t+1}) \leq V^k_t.
\end{equation}

Moreover as the selection function is valid, we have that:

\begin{equation}
B_t (V^k_{t+1}) \leq \phi^k_{t+1}.
\end{equation}

Finally, by construction of Algorithm 1.1 we have that $V^k_{t+1} = \inf (V^k_t, \phi^k_{t+1})$, so using (2.5) and (2.6) we deduce the desired result

\begin{equation}
B_t (V^k_{t+1}) \leq V^k_{t+1}.
\end{equation}

4. As the selection function $\phi^k_{t}$ is tight in the sense of \textbf{Definition 1.3} we have by construction of \textbf{Algorithm 1.1} that

\begin{equation}
B_t (V^k_{t+1}) (x^{k-1}) = \phi^k_t (x^{k-1}).
\end{equation}

Combining it with (2.2) (or its variant when opt = sup) and the definition of $V^k_t$, one gets the desired equality. \hfill \Box

In the following two propositions, we state that the sequences of functionals $(V^k_t)_{k \in \mathbb{N}}$ and $(B_t (V^k_{t+1}))_{k \in \mathbb{N}}$ converge uniformly on any compact included in the domain of $V_t$. The limit functional of $(V^k_t)_{k \in \mathbb{N}}$, noted $V^*_t$, will be our natural candidate to be equal to the value function $V_t$. Moreover, the convergence will be $\mu$-almost sure where (see [6, pages 257-259]) $\mu$ is the unique probability measure over the countable cartesian product $X^{T+1} \times \ldots \times X^{T+1} \times \ldots$ endowed with the product $\sigma$-algebra such that for every borelian $X_i \subset X^{T+1}, 1 \leq i \leq k$,

$$
\mu \left( X_1 \times \ldots \times X_k \times \prod_{i \geq k+1} X^{T+1} \right) = \mu^1 (X_1) \times \ldots \times \mu^k (X_k),
$$

where $(\mu^k)_{k \in \mathbb{N}}$ is the sequence of probability measures generated by \textbf{Algorithm 1.1} through the Oracle.

\textbf{PROPOSITION 2.2} (Existence of an approximating limit). Let $t \in [0, T]$ be fixed. The sequence of functionals $(V^k_t)_{k \in \mathbb{N}}$ defined as $V^k_t := V^k_{F^k_t}$ (where the sets $F^k_t$ are generated by \textbf{Algorithm 1.1}) $\mu$-a.s. converges uniformly on every compact set included in the domain of $V_t$ to a functional $V^*_t$. 

Proof. Let \( t \in [0, T] \) be fixed and let \( K_t \) be a given compact set included in the domain of \( V_t \). We denote by \( (V_t^k)_{k \in \mathbb{N}} \) a sequence of approximations generated by Algorithm 1.1. The proof relies on the Arzelà-Ascoli Theorem [18, Theorem 2.13.30 p.347]. More precisely,

First, by Assumptions 1.2 each functional in \( F_t \) have a common modulus of continuity. Thus as \( F_t^k \subset F_t \), the family of functionals \( (V_t^k)_{k \geq 0} \) is equicontinuous.

Now, by Lemma 2.1, the sequence of functionals \( (V_t^k)_{k \geq 0} \) is monotone. Now for every \( x \in K_t \), the set \( \{V_t^k(x)\}_{k \geq 1} \) is bounded by \( V_t(x) \), which is finite since we assumed \( K_t \subset \text{dom}(V_t) \), and \( V_t^1(x) \). Hence the set \( \{V_t^k(x)\}_{k \geq 1} \) is a bounded subset of \( \mathbb{R} \) and thus relatively compact.

By Arzelà-Ascoli Theorem, the sequence of functions \( (V_t^k)_{k \geq 1} \) is a relatively compact set of \( C(K_t, \mathbb{R}) \) for the topology of the uniform convergence, i.e. there exists a subsequence of \( (V_t^k)_{k \geq 1} \) converging uniformly to a function \( V_t^* \in C(K_t, \mathbb{R}) \).

Finally, as \( (V_t^k)_{k \geq 0} \) is a monotone sequence of functions, we conclude that the sequence \( (V_t^k)_{k \geq 0} \) converges uniformly on the compact \( K_t \) to \( V_t^* \in C(K_t, \mathbb{R}) \).

PROPOSITION 2.3. Let \( t \in [0, T - 1] \) be fixed and \( V_t^* \) be the function defined in Proposition 2.2. The sequence \( B_t (V_{t+1}^k) \) \( \mu \)-a.s. converges uniformly to the continuous function \( B_t (V_{t+1}^*) \) on every compact sets included in the domain of \( V_t \).

Proof. We will stick to the case when \( \text{opt} = \inf \) and leave the other case to the reader. Let \( K_t \) be a given compact set included in \( \text{dom}(V_t) \). First, as the sequence \( (V_{t+1}^k)_{k \in \mathbb{N}} \) is non-increasing and using the fact that the operator \( B_t \) is order preserving, the sequence \( (B_t (V_{t+1}^k))_{k \in \mathbb{N}} \) is also non-increasing. By stability of the Bellman operator \( B_t \) (see Assumptions 1.2), we have that the function \( B_t (V_{t+1}^k) \) is in \( \mathbb{R} \) for every \( k \in \mathbb{N} \) and thus the family \( \{B_t (V_{t+1}^k)\}_{k \in \mathbb{N}} \) is equicontinuous using the common regularity assumption in Assumptions 1.2. Moreover, given \( x \in K_t \), the set \( \{B_t (V_{t+1}^k(x))\}_{k \geq 1} \) is bounded by \( V_t^1(x) \) and \( V_t(x) \) which take finite values on \( \text{dom}(V_t) \). Thus, using again Arzelà-Ascoli Theorem, there exists a continuous functional \( \phi \) such that \( (B_t (V_{t+1}^k))_{k \in \mathbb{N}} \) converges uniformly to \( \phi \) on any compact included in \( \text{dom}(V_t) \).

We now show that the functional \( \phi \) is equal to \( B_t (V_{t+1}^*) \) on the given compact \( K_t \) or equivalently we show that \( \phi + \delta_{K_t} = B_t (V_{t+1}^*) + \delta_{K_t} \). As already shown in Proposition 2.2, the sequence \( (V_{t+1}^k)_{k \in \mathbb{N}} \) is lower bounded by \( V_{t+1}^* \). We thus have that \( V_{t+1}^k \geq V_{t+1}^* \), which combined with the fact that the operator \( B_t \) is order preserving, gives, for every \( k \geq 1 \), that

\[
B_t (V_{t+1}^k) \geq B_t (V_{t+1}^*).
\]

Adding, on both side of the previous inequality, the mapping \( \delta_{K_t} \), and taking the limit as \( k \) goes to infinity, we have that

\[
\phi + \delta_{K_t} \geq B_t (V_{t+1}^*) + \delta_{K_t}.
\]

For the converse inequality, by the existence of optimal sets (see Assumptions 1.2), there exists a compact set \( K_{t+1} \subset \text{dom}(V_{t+1}) \) such that

\[
(2.7) \quad B_t (V_{t+1}^* + \delta_{K_{t+1}}) \leq B_t (V_{t+1}^*) + \delta_{K_t}.
\]

By Proposition 2.2, the non-increasing sequence \( (V_{t+1}^k)_{k \in \mathbb{N}} \) converges uniformly to \( V_{t+1}^* \) on the compact set \( K_{t+1} \). Thus, for any fixed \( \epsilon > 0 \), there exists an integer
k_0 \in \mathbb{N} \text{ such that we have }

\[ V_{t+1}^k \leq V_{t+1}^k + \delta_{K_{t+1}} \leq V_{t+1}^\ast + \epsilon + \delta_{K_{t+1}}, \]

for all \( k \geq k_0 \). By Assumptions 1.2, the operator \( B_t \) is order preserving and additively M-subhomogeneous, thus we get

\[ B_t (V_{t+1}^k) \leq B_t (V_{t+1}^k + \delta_{K_{t+1}}) \leq B_t (V_{t+1}^\ast + \delta_{K_{t+1}}) + M\epsilon, \]

which, combined with (2.7) gives that

\[ B_t (V_{t+1}^k + \delta_{K_t}) \leq B_t (V_{t+1}^\ast) + M\epsilon + \delta_{K_t}, \]

for every \( k \geq k_0 \). Taking the limit when \( k \) goes to infinity we obtain that

\[ \phi + \delta_{K_t} \leq B_t (V_{t+1}^\ast) + \delta_{K_t} + M\epsilon. \]

The result is proved for all \( \epsilon > 0 \) and we have thus shown that \( \phi = B_t (V_{t+1}^\ast) \) on the compact set \( K_t \). We conclude that \( (B_t (V_{t+1}^k))_{k \in \mathbb{N}} \) converges uniformly to the functional \( B_t (V_{t+1}^\ast) \) on the compact set \( K_t \). \( \Box \)

We want to exploit that our approximations of the final cost function are exact in the sense that we have equality between \( V_T^k \) and \( V_T^\ast \) at the points drawn in Algorithm 1.1, that is, the tightness assumption of the selection function is much stronger at time \( T \) than for times \( t < T \). Thus we want to propagate the information backward in time: starting from time \( t = T \) we want to deduce information on the approximations for times \( t < T \).

In order to show that \( V_t = V_t^\ast \) on some set \( X_t \), a dissymmetry between upper and lower approximations is emphasized. We introduce the notion of optimal sets \( (X_t)_{t \in [0,T]} \) with respect to a sequence of functionals \( (\phi_t)_{t \in [0,T]} \) as a condition on the sets \( (X_t)_{t \in [0,T]} \) such that if one wants to compute the restriction of \( B_t (\phi_{t+1}) \) to \( X_t \), one only need to know \( \phi_{t+1} \) on the set \( X_{t+1} \). The Figure 3 illustrates this notion.

**Definition 2.4 (Optimal sets).** Let \( (\phi_t)_{t \in [0,T]} \) be \( T + 1 \) functionals on \( \mathcal{X} \). A sequence of sets \( (X_t)_{t \in [0,T]} \) is said to be \( (\phi_t) \)-optimal or optimal with respect to \( (\phi_t)_{t \in [0,T]} \), if for every \( t \in [0,T-1] \) we have

\[ B_t (\phi_{t+1} + \delta_{X_{t+1}}) + \delta_{X_t} = B_t (\phi_{t+1}) + \delta_{X_t}. \]

When approximating from below, the optimality of sets is only needed for the functions \( (V_t^\ast)_{t \in [0,T]} \) whereas when approximating from above, one needs the optimality of sets with respect to \( (V_t)_{t \in [0,T]} \). It seems easier to ensure the optimality of sets for \( (V_t^\ast)_{t \in [0,T]} \) than for \( (V_t)_{t \in [0,T]} \) as the functional \( V_t^\ast \) is known through the sequence \( (V_t^k)_{k \in \mathbb{N}} \) whereas the function \( V_t \) is, a priori, unknown. This fact is discussed in section 3.

**Lemma 2.5 (Unicity in Bellman Equation).** Let \( (X_t)_{t \in [0,T]} \) be a sequence of sets which is

- optimal with respect to \( (V_t)_{t \in [0,T]} \) when \( \text{opt} = \inf \),
- optimal with respect to \( (V_t^\ast)_{t \in [0,T]} \) when \( \text{opt} = \sup \). If the sequence of functionals \( (V_t^\ast)_{t \in [0,T]} \) satisfies the following modified Bellman Equations:
Fig. 3. The optimality of the sets \((X_t)_{t \in [0,T]}\) means that in order to compute the restriction of \(B_t(\phi_{t+1})\) to \(X_t\), one only needs to know the values of \(\phi_{t+1}\) on the set \(X_{t+1}\).

\[
\begin{aligned}
\left\{ \begin{array}{l}
V_T^* + \delta_{X_T} = \psi + \delta_{X_T} \\
\forall t \in [0,T-1], \ B_t (V_{t+1}^*) + \delta_{X_t} = V_t^* + \delta_{X_t},
\end{array} \right. \\
\end{aligned}
\] (2.9)

Then for every \(t \in [0,T]\) and every \(x \in X_t\) we have that \(V_t^*(x) = V_t(x)\).

**Proof.** We prove the lemma by backward recursion on the time \(t \in [0,T]\), first for the case \(\text{opt} = \inf\). For time \(t = T\), since by definition of \(V_T = \psi\) (see (0.2)) we have \(V_T^* + \delta_{X_T} = \psi + \delta_{X_T} = V_T + \delta_{X_T}\). Now, let time \(t \in [0,T-1]\) be fixed and assume that we have \(V_{t+1}^*(x) = V_{t+1}(x)\) for every \(x \in X_{t+1}\) i.e.

\[
V_{t+1}^* + \delta_{X_{t+1}} = V_{t+1} + \delta_{X_{t+1}}.
\] (2.10)

Using **Lemma 2.1**, the sequence of functions \((V^*_k)\) is lower bounded by \(V_t\). Taking the limit in \(k\), we obtain that \(V_t^* \geq V_t\), we have thus only to prove that \(V_t^* \leq V_t\) on \(X_t\), that is \(V_t^* + \delta_{X_t} \leq V_t + \delta_{X_t}\). We successively have:

\[
\begin{aligned}
&\text{(by (2.9))} & V_t^* + \delta_{X_t} &= B_t (V_{t+1}^*) + \delta_{X_t} \\
&\text{(\(B_t\) is order preserving)} & & \leq B_t (V_{t+1}^* + \delta_{X_{t+1}}) + \delta_{X_t} \\
&\text{(by induction assumption (2.10))} & & = B_t (V_{t+1} + \delta_{X_{t+1}}) + \delta_{X_t} \\
&\text{\((X_t)_{t \in [0,T]}\) is \((V_t)\)-optimal)} & & = B_t (V_{t+1}) + \delta_{X_t} \\
&\text{(by (0.2))} & & = V_t + \delta_{X_t},
\end{aligned}
\]

which concludes the proof in the case of \(\text{opt} = \inf\). Now we briefly prove the case \(\text{opt} = \sup\) by backward recursion on the time \(t \in [0,T]\). As for the case \(\text{opt} = \inf\), at time \(t = T\), one has \(V_T^* + \delta_{X_T} = V_T + \delta_{X_T}\). Now assume that for some \(t \in [0,T-1]\) one has \(V_{t+1}^* + \delta_{X_{t+1}} = V_{t+1} + \delta_{X_{t+1}}\). By **Lemma 2.1**, the sequence of functions \((V^*_k)\) is upper bounded by \(V_t\). Thus, taking the limit in \(k\) we obtain that \(V_t^* \leq V_t\) and we
only need to prove that \( V_t^* + \delta_{X_t} \geq V_t + \delta_{X_t} \). We successively have:

\[
\begin{align*}
\text{(by (0.2))} & \quad V_t + \delta_{X_t} = B_t(V_{t+1}) + \delta_{X_t} , \\
\text{(B_t is order preserving)} & \quad \leq B_t(V_{t+1} + \delta_{X_{t+1}}) + \delta_{X_t} , \\
\text{(by induction assumption (2.10))} & \quad = B_t(V^*_t + \delta_{X_{t+1}}) + \delta_{X_t} , \\
\text{((X_t)_{t\in[0,T]} is (V^*_t)-optimal)} & \quad = B_t(V^*_t) + \delta_{X_t} , \\
\text{(by (2.9))} & \quad = V^*_t + \delta_{X_t} .
\end{align*}
\]

In the general case, one cannot hope for the limit object \( V_t^* \) to be equal to the value function \( V_t \) everywhere. However, one can expect an (almost sure over the draws) equality between the two functions \( V_t \) and \( V_t^* \) on all possible cluster points of sequences \( (y_k)_{k\in\mathbb{N}} \) with \( y_k \in K_t^k \) for all \( k \in \mathbb{N} \), that is, on the set \( \limsup K_t^k \) (see [17, Definition 4.1 p. 109]).

**Theorem 2.6** (Convergence of Tropical Dynamic Programming). Define \( K_t^* := \limsup K_t^k \), for every time \( t \in [0,T] \). Assume that, \( \mu \)-a.s the sets \( (K_t^*)_{t\in[0,T]} \) are (\( V_t \))-optimal when \( \text{opt} = \inf \) (resp. \( (V^*_t) \)-optimal when \( \text{opt} = \sup \)).

Then, \( \mu \)-a.s. for every \( t \in [0,T] \) the functional \( V_t^* \) defined in Proposition 2.2 is equal to the value function \( V_t \) on \( K_t^* \).

**Proof.** We will only study the case \( \text{opt} = \inf \) as the case \( \text{opt} = \sup \) is analogous. We will show that (2.9) holds with \( X_t = K_t^* \). By tightness of the selection function at time \( T \), we get that \( V_T = V_T^* \) on \( K_T^* \). Now, fix \( t \in [0,T-1] \).

— First, we show that \( B_t(V_{t+1}^*) \leq V_t^* \). By Lemma 2.1, for every \( k \in \mathbb{N} \) we have that

\[
B_t(V_{t+1}^*) \leq V_t^k .
\]

Moreover, by Lemma 2.1 we have \( V_{t+1}^* \leq V_t^k \), thus, as the operator \( B_t \) is order preserving we have

\[
B_t(V_{t+1}^*) \leq B_t(V_t^k) \leq V_t^k \xrightarrow{k \to +\infty} V_t^* \quad \text{(by Proposition 2.2)} .
\]

— Now, we will show by contradiction that \( B_t(V_{t+1}^*) \geq V_t^* \) on \( K_t^* \). Suppose that there exist \( x \in K_t^* \) such that

\[
B_t(V_{t+1}^*)(x) < V_t^*(x) .
\]

Define \( h := V_t^*(x) - B_t(V_{t+1}^*)(x) > 0 \). As illustrated in Figure 4, we will show that there is an index \( \bar{k} \) such that \( V_t^{\bar{k}+1}(x) \) will be closer to \( B_t(V_{t+1}^*)(x) \) than \( V_t^*(x) \), which will contradict the fact that the sequence \( (V_t^k(x))_{k\in\mathbb{N}} \) is non-increasing.

To this end, we state several facts:

— By equicontinuity of the sequence of functionals \( (V_t^k)_{k\in\mathbb{N}} \) there exist \( \eta > 0 \) such that for every \( y \in B(x, \eta) \), for every index \( k \in \mathbb{N} \) we have

\[
|V_t^k(y) - V_t^k(x)| \leq \frac{h}{4} .
\]

— By Lemma 2.1, for every index \( k \in \mathbb{N} \) the \( k \)-th draw at time \( t \), noted \( x_t^k \) is such that

\[
B_t(V_{t+1}^*)(x_t^k) = V_t^{k+1}(x_t^k) .
\]
If there exist $x \in K_t^*$ such that $B_t(V_{t+1}^*) (x) < V_t^*(x)$ then there is an index $\bar{k}$ such that $V_{\bar{k}+1}^*(x)$ will be closer to $B_t(V_{t+1}^*) (x)$ than $V_t^*(x)$ as the selection function is tight. This will contradict the fact that the sequence $(V_k^*(x))_{k \in \mathbb{N}}$ is non-increasing.

- By Proposition 2.3, the sequence $(B_t(V_k^*_{t+1}))_{k \in \mathbb{N}}$ converges uniformly to the continuous functional $B_t(V_{t+1}^*)$ on arbitrary compact sets included in $\text{dom}(V_t)$. Hence, it converges pointwise to the continuous function $B_t(V_{t+1}^*)$ on $\text{dom}(V_t)$. Thus, we get the following inequality: for any $y \in \text{dom}(V_t)$, there exist a rank $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ we have that

$$|B_t(V_{t+1}^*) (y) - B_t(V_k^*_{t+1}) (y)| \leq \frac{h}{4}.$$  

(2.14)

- By continuity of $B_t(V_{t+1}^*)$ at $x$, there exist $\eta' > 0$ such that $\eta' < \eta$ and for every $y \in B(x, \eta')$ we have

$$|B_t(V_{t+1}^*) (y) - B_t(V_{t+1}^*) (x)| \leq \frac{h}{4}.$$  

(2.15)

- As $x$ is in $K_t^* := \limsup_k K_t^k$, by definition of the limsup, there exist an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of points $y_{\sigma(k)}^* \in K_{\sigma(k)}^t$ such that $y_{\sigma(k)}^* \to x$. Thus, there exist a rank $k_1 \geq k_0$ such that if $k \geq k_1$ then $y_{\sigma(k)}^* \in B\left(x, \frac{\eta'}{2}\right)$ and a fortiori $B\left(y_{\sigma(k)}^*, \frac{\eta'}{2}\right) \subset B(x, \eta')$. Let $(X_{t}^k)_{k \in \mathbb{N}}$ be the sequence of random variables where for each $k \in \mathbb{N}$, $X_t^k$ is the $t$-th marginal of a random variable of probability law $\mu^k$. We have that for every $k \geq k_1$,

$$\mathbb{P}\left(X_{t}^\sigma(k) \in B(x, \eta')\right) \geq \mathbb{P}\left(X_{t}^\sigma(k) \in B\left(y_{\sigma(k)}^*, \eta'/2\right)\right) \geq \mathbb{P}\left(X_{t}^\sigma(k) \in B\left(y_{\sigma(k)}^*, \eta'/2\right) \cap K_{\sigma(k)}^t\right) = \mu^\sigma(k)\left(B\left(y_{\sigma(k)}^*, \eta'/2\right) \cap K_{\sigma(k)}^t\right) \geq g\left(\frac{\eta'}{2}\right) \quad \text{(by Assumptions 1.4)}.$$  


Hence, the probability that the subsequence never draw a point in the ball $B(x, \eta')$ after the rank $\sigma(k_1)$ is bounded from above by

$$\prod_{k \geq k_1} (1 - g(\eta'/2)) = 0.$$  

Therefore, $\mu$-almost surely, there exists an index $k_2 \geq k_1$ such that the $\sigma(k_2)$-th draw at time $t$ in Algorithm 1.1 satisfies:

(2.16) $$x_t^{\kappa} \in B(x, \eta'),$$

where $\kappa$ is a simplified notation for $\sigma(k_2)$, $\kappa := \sigma(k_2)$.

By (2.16), the state $x_t^{\kappa}$ satisfies both (2.12) and (2.15). Thus, we can conclude that $V_t^{\kappa+1}(x)$ is closer to $B_t(V_t^{\star+1})(x)$ than $V_t^{\star}(x)$ as detailed now:

$$|V_t^{\kappa+1}(x) - B_t(V_t^{\star+1})(x)| \leq |V_t^{\kappa+1}(x) - V_t^{\kappa+1}(x_t^{\kappa})|$$

$$\leq \frac{h}{4} \text{ by (2.12)}$$

$$+ |V_t^{\kappa+1}(x_t^{\kappa}) - B_t(V_t^{\kappa+1})(x_t^{\kappa})|$$

$$= 0 \text{ by (2.13)}$$

$$+ |B_t(V_t^{\kappa+1})(x_t^{\kappa}) - B_t(V_t^{\star+1})(x_t^{\kappa})|$$

$$\leq \frac{h}{4} \text{ by (2.14)}$$

$$+ |B_t(V_t^{\star+1})(x_t^{\kappa}) - B_t(V_t^{\star+1})(x)|$$

$$\leq \frac{3h}{4} \text{ by (2.15)}$$

Hence we have that

$$V_t^{\kappa+1}(x) = V_t^{\kappa+1}(x) - B_t(V_t^{\star+1})(x) + B_t(V_t^{\star+1})(x)$$

$$\leq B_t(V_t^{\star+1})(x) + \frac{3h}{4}.$$  

And finally we get

$$V_t^{\kappa+1}(x) - V_t^{\star}(x) = \left(V_t^{\kappa+1}(x) - B_t(V_t^{\star+1})(x) \right) - \left(V_t^{\star}(x) - B_t(V_t^{\star+1})(x) \right)$$

$$\leq \frac{3h}{4} - h$$

$$< 0,$$

which contradicts the fact that the sequence $(V_t^k)_{k \in \mathbb{N}}$ is non-increasing (Lemma 2.1). Hence, there is no $x \in K_t$ such that (2.11) holds. We conclude that the sequence $(V_t^\star)_{k \in \mathbb{N}}$ satisfies the modified Bellman Equation (2.9) with the sequence $(K_t^\star)_{k \in \mathbb{N}}$. The conclusion follows from the Unicity Lemma Lemma 2.5.

3. The multistage framework and examples of selection functions.
3.1. SDDP selection function: lower approximations in the linear-convex framework. We will show that our framework contains a similar framework of (the deterministic version of) the SDDP algorithm as described in [8] and yields the same result of convergence. Let $X = \mathbb{R}^n$ be a continuous state space and $U = \mathbb{R}^m$ a continuous control space. We want to solve the following problem

$$\min_{x=(x_0,\ldots,x_T), u=(u_0,\ldots,u_{T-1})} \sum_{t=0}^{T-1} c_t(x_t, u_t) + \psi(x_T)$$

\(3.1\)

s.t.

\begin{align*}
& x_0 \in X \text{ is given,} \\
& \forall t \in [0,T], \; x_t \in X_t \subset X, \\
& \forall t \in [0, T-1], \; u_t \in U_t \subset U, \\
& \forall t \in [0, T-1], \; x_{t+1} = f_t(x_t, u_t).
\end{align*}

We make similar assumptions as in [8], one can look at this article for more in-depth comments about them.

**Assumptions 3.1.** For all $t \in [0, T-1]$ we assume that:

- The set $X_t \subset X$ and $X_T \subset X$ are convex and compacts with non-empty relative interior.
- The set $U_t$ is non-empty, convex and compact.
- The dynamic $f_t : X \times U \rightarrow X$ is linear:

$$f_t(x,u) = A_t x + B_t u,$$

for some given matrices $A_t$ and $B_t$ of compatible dimensions.
- The cost function $c_t : X \times U \rightarrow \mathbb{R}$ is a proper convex lower semicontinuous (lsc) function and is a $C_t$-Lipschitz continuous function on $X_t \times U_t$.
- The final cost function $\psi : X \rightarrow \mathbb{R}$ is a proper convex lsc function and is a $C_T$-Lipschitz continuous function on $X_T$.
- Relatively Complete Recourse (RCR). For every $x \in X_t$ we have that $f_t(x, U_t) \cap \text{ri} (X_{t+1}) \neq \emptyset$.

For every time step $t \in [0, T-1]$, we define the Bellman operator $B_t$ for every functional $\phi : X \rightarrow \mathbb{R}$ by:

$$B_t(\phi) := \inf_{u \in U} c_t(\cdot, u) + \phi(f_t(\cdot, u)).$$

Moreover, for every functional $\phi : X \rightarrow \mathbb{R}$ and every $(x, u) \in X \times U$ we define

$$B_t^n(\phi)(x) := c_t(x, u) + \phi(f_t(\cdot, u)) \in \mathbb{R}.$$

The Dynamic Programming principle using the Bellman’s operators $B_t$ yields:

\begin{align*}
V_T &= \psi \\
\forall t \in [0, T-1], V_t : x \in X \mapsto B_t(V_{t+1})(x) \in \mathbb{R}.
\end{align*}

\(3.2\)

Using a generalization of Hoffman’s Lemma [4, Theorem 9] that bounds from above the distance between the image by a linear mapping of a point and a convex set, we show that the image of a Lipschitz continuous function by a Bellman operator will also be Lipschitz continuous, with an explicit (conservative) constant.
Theorem 9]) there exist a constant \( \kappa \) such that if \( x \) is a marginal of a jointly convex function. Then \( \lambda \) is a control such that \( x \rightarrow \inf_{u \in U} c_t (x, u) + \phi (f_t (x, u)) \) is a constant \( \kappa > 0 \) such that we have

\[
\inf_{u \in U} c_t (x, u) + \phi (f_t (x, u)) \leq \kappa (x, u_t) + \phi (f_t (x, u_t)) < +\infty.
\]

Moreover, by compactness of \( U \) and lower semicontinuity of \( u \rightarrow c_t (x, u) \), \( \phi \) and \( f_t \), the above minimum is attained. We have shown that dom \((B_t (\phi))\) includes \( X_t \).

Thus, we have

\[
A_t \quad \phi (f_t (x, u_t)) = B_t (\phi) (x) \quad \text{is convex on} \quad X_t \quad \text{as a marginal of a jointly convex function.}
\]

We finally show that the function \( B_t (\phi) \) is Lipschitz on \( X_t \) with a constant \( L_t > 0 \) detailed below. Let \( x \) be in \( X_t \) and note \( u_x \) an optimal control at \( x \), that is \( u_x \) satisfies

\[
B_t^{u_x} (\phi) (x) = B_t (\phi) (x).
\]

Note that for every \( x' \in X_t \), by the RCR assumption, there exist a control \( u \) such that \( f_x (u) := A_t y + B_t u \) is an element of \( \text{ri} X_{t+1} \). Thus, by Hoffman’s Lemma (see [4, Theorem 9]) there exist a constant \( \gamma > 0 \) such that

\[
\min_{u \in U_t} c_t (x, u) + \phi (f_t (x, u)) \leq \gamma \, \text{dist} (f_t (x, u), X_{t+1}).
\]

We want to bound from above \( \min_{u \in U_t} c_t (x, u) + \phi (f_t (x, u)) \) by a constant times \( \|x-x'\| \). As \( (A_t x) \in B_t u_x - X_{t+1} \), by triangle inequality we have that

\[
\text{dist} (B_t u_x - (A_t x'), X_{t+1}) \leq \text{dist} (A_t x' + B_t u_x, A_t x + B_t u_x) = \text{dist} (A_t x', A_t x) \leq \|A_t x' - A_t x\|
\]

(3.4) \[
\text{dist} (f_t (x, u), X_{t+1}) \leq \lambda_{\max} (A_t^T A_t)^{1/2} \|x - x'\|.
\]

Setting \( \kappa_1 := \gamma \lambda_{\max} (A_t^T A_t)^{1/2} \), by (3.3) and (3.4) we have shown that

\[
\inf_{u \in U_t, f_x (u) \in X_{t+1} \} \|x - u\| \leq \kappa_1 \|x - x'\|.
\]

Similarly, denoting \( u_{x'} \) a control such that \( B_t^{u_{x'}} (\phi) (x') = B_t (\phi) (x') \), there exists a constant \( \kappa_2 > 0 \) such that we have

\[
\inf_{u \in U_t, f_x (u) \in X_{t+1} \} \|x - u\| \leq \kappa_2 \|x - x'\|.
\]

Now for every \( u \) such that \( f_x (u) \in X_{t+1} \), as \( c_t \) is \( C_1 \)-Lipschitz continuous, \( \phi \) is \( L_{t+1} \)-Lipschitz continuous and \( f_t \) is linear, we have that

\[
B_t (\phi) (x') \leq B_t (\phi) (x) + B_t^{u} (\phi) (x') - B_t (\phi) (x) = B_t (\phi) (x) + c_t (x', u) + \phi (f_t (x', u)) - c_t (x, u_x) - \phi (f_t (x, u_x)) \leq B_t (\phi) (x) + c_t (\|x - x'\| + \|u_x - u\|) + L_{t+1} (\lambda_{\max} (A_t^T A_t) \|x - x'\| + \lambda_{\max} (B_t^T B_t) \|u - u_x\|).
\]
So, taking the infimum over the $u$ such that $f_{x'}(u) \in X_{t+1}$, setting $\kappa := \max(\kappa_1, \kappa_2)$ and $L_t := C_t(1+\kappa) + L_{t+1}(\lambda_{\max}(A_t^T A_t) + \lambda_{\max}(B_t^T B_t) \kappa)$ one has by (3.6) that

$$
B_t(\phi)(x') - B_t(\phi)(x) \leq L_t \|x - x'\|.
$$

(3.7)

Similarly, one can show that

$$
B_t(\phi)(x) - B_t(\phi)(x') \leq L_t \|x - x'\|.
$$

(3.8)

The result follows from both (3.7) and (3.8).

As lower semicontinuous proper convex functions can be approximated by a supremum of affine functions, for every $t \in [0, T]$ we define $F_t^{SDDP}$ to be the set of affine functions $\phi : x \in X \mapsto \langle a, x \rangle + b \in \mathbb{R}$, $a \in X$ and $b \in \mathbb{R}$ with $\|a\|_2 \leq L_t$. Moreover we'll denote by $F_t^{SDDP}$ the set of convex functionals $\phi : X \mapsto \mathbb{R}$ which are $L_t$-Lipschitz continuous on $X_t$ and proper.

**Proposition 3.3.** Under Assumptions 3.1, the problem (3.1) and the Bellman operators defined in (3.2) satisfy the structural assumptions Assumptions 1.2.

**Proof.**

— **Common regularity.** By construction, for all $t \in [0, T]$, every element of $F_t^{SDDP}$ is $L_t$-Lipschitz continuous.

— **Final condition.** As $\psi$ is convex proper and $L_T$-Lipschitz continuous on $X_T$, it is a countable (as $\mathbb{R}^n$ is separable) supremum of $L_T$-Lipschitz affine functions.

— **Stability by the Bellman operators.** This has been shown in Proposition 3.2.

— **Stability by pointwise optimum.** Recall that we are here on the case $\text{opt} = \sup$. Fix $t \in [0, T]$ and let $F \subset F_t^{SDDP}$ be a set of affine $L_t$-Lipschitz continuous functionals. For every $x, x' \in \text{dom}(V_t)$, we have that

$$
|V_F(x) - V_F(x')| = \sup_{\phi \in F} \phi(x) - \sup_{\phi' \in F} \phi'(x') \leq \sup_{\phi \in F} |\phi(x) - \phi(x')| \leq L_t \|x - x'\|.
$$

Thus the function $V_F$ is $L_t$-Lipschitz continuous. As a supremum of affine functions is convex, $V_F$ is convex and finite valued. We have shown that $V_F$ is an element of $F_t^{SDDP}$.

— **Order preserving operators.** Let $\phi_1$ and $\phi_2$ be two functionals over $X$ such that $\phi_1 \leq \phi_2$ i.e. for every $x \in X$ we have $\phi_1(x) \leq \phi_2(x)$. We want to show that $B_t(\phi_1) \leq B_t(\phi_2)$. Let $x \in X$, we have:

$$
B_t(\phi_1)(x) = \inf_{u \in U} x^T C_t x + u^T D_t u + \phi_1(f_t(x, u))
\leq \inf_{u \in U} x^T C_t x + u^T D_t u + \phi_2(f_t(x, u))
= B_t(\phi_2)(x).
$$

— **Existence of the value functions.** By backward recursion on the time step $t \in [0, T]$ and by Proposition 3.2, for every time step $t \in [0, T]$ the function $V_t$ defined by the Dynamic Programming equation (3.2) is convex and $L_t$-Lipschitz continuous on $X_t$.

— **Existence of optimal sets.** Fix an arbitrary element $\phi \in F_t^{SDDP}$. We will show that for every compact set $K_t \subset \text{dom}(V_t)$, there exist a compact set $K_{t+1} \subset \text{dom}(V_{t+1})$ such that

$$
B_t(\phi + \delta_{K_{t+1}}) + \delta_{K_t} = B_t(\phi) + \delta_{K_t}.
$$

(3.9)
which will imply the desired result. Now equation (3.9) is equivalent to the fact that for every state $x_t \in K_t$, there exist a control $u_t \in U_t$ such that

\[
\begin{cases}
    f_t (x_t, u_t) \in K_{t+1} \\
    u_t \in \arg \min_{u \in U_t} c_t (x_t, u) + \phi (f_t (x_t, u))
\end{cases}
\]

Now as the function $B : u \in U_t \mapsto B(u) \in \mathbb{R}$ is lower semicontinuous proper and convex (as the sum of two convex functions) we have that $B$ attains its minimum on the compact $U_t$ and that its set of minimizers, that we will note $U^*_t$, is bounded. As it is also a closed set of $\mathbb{R}^n$, $U^*_t$ is a compact set of $\mathbb{R}^n$. Thus setting

\[
K_{t+1} := f_t (X_t, U^*_t)
\]

we have that equation (3.9) is satisfied and that $K_{t+1}$ is compact as $f_t$ is continuous and by compactness of $X_t$ and $U^*_t$. Lastly, by the RCR Assumption, we have that $K_{t+1} \subset \text{dom} (V_{t+1})$.

---

**Additively $M$-subhomogeneous operators.** We will show that $B_t$ is additively homogeneous, that is $M = 1$. Let $\lambda : X \rightarrow \mathbb{R}$ be a positive constant function, we will identify $\lambda$ with its value. Let $\phi$ be a functional over $X$. For any $x \in X$ we have:

\[
B_t (\lambda + \phi) (x) = \inf_{u \in U} c_t (x, u) + (\lambda + \phi) (f_t (x, u))
\]

\[
= \inf_{u \in U} c_t (x, u) + \lambda + \phi (f_t (x, u))
\]

\[
= \lambda + \inf_{u \in U} c_t (x, u) + \phi (f_t (x, u))
\]

\[
= \lambda + B_t (\phi) (x).
\]

Note that the constraint $f_t (x', u)$ restricts the current approximation at time $t+1$ to the set $X_{t+1}$ included in the domain of the true value function $V_{t+1}$.

Now we define a compatible selection function. Let $t \in [0, T - 1]$, for any $F \subset F^{\text{SDDP}}_t$ and $x \in X$, define the following optimization problem

\[
\min_{x' \in X} \left[ c_t (x', u) + \sup_{\phi \in F} (f_t (x', u)) \right] \quad \text{s.t.} \quad x' = x, \ f_t (x', u) \in X_{t+1}.
\]

If we denote by $b$ its optimal value and by $a$, a Lagrange multiplier of the constraint $x' - x = 0$ at the optimum, then we define

\[
\phi^{\text{SDDP}}_t (F, x) := \langle a, x' - x \rangle + b.
\]

Finally, at time $t = T$, for any $F \in F^{\text{SDDP}}_t$ and $x \in X$ we define

\[
\phi^{\text{SDDP}}_T (F, x) := \langle a, x' - x \rangle + V_T (x),
\]

where $a \in \partial V_T(x)$.

**Proposition 3.4.** For every time $t \in [0, T]$, the function $\phi^{\text{SDDP}}_t$ is a compatible selection function in the sense of Definition 1.3.
Proof. Fix $t \in [0, T - 1]$, $F \subset F_{t, \text{SDDP}}^T$ and $x \in X$. We have $B_t(V_F)(x) = b$ by definition of $B_t$, thus the function $\phi_t^{\text{SDDP}}(F, x)$ is tight and it is valid as $a$ is a subgradient of the convex function $B_t(V_F)$ at $x$.

For $t = T$, the selection function $\phi_T^{\text{SDDP}}$ is tight and valid by convexity of $V_T$. \(\square\)

If we want to apply the convergence result from Theorem 2.6, as we approximate from below the value functions (opt = sup) then one has to make the draws according to some sets $K^k_0$ such that the sets $K^*_0 := \limsup_{k \in \mathbb{N}} K^k_0$ are $V^*_t$ optimal. As done in the litterature of the Stochastic Dual Dynamic Programming algorithm (see for example [8], [20] or [14]), one can study the case when the draws are made along the optimal trajectories of the current approximations.

More precisely, fix $k \in \mathbb{N}$ we define a sequence $(x^k_0, x^k_1, \ldots, x^k_T)$ by

$$
\begin{cases}
  x^k_0 := x_0 \\
  \forall t \in [0, T - 1], \ x^k_{t+1} := f_t \left( x^k_t, u^k_t \right),
\end{cases}
$$

where $u^k_t \in \arg\min_u \mathcal{B}_t^k \left( V_{x^k_t}^k \right) (x^k_t)$. We say that such a sequence $(x^k_0, x^k_1, \ldots, x^k_T)$ is an optimal trajectory for the $k$-th approximations starting from $x_0$.

**Proposition 3.5 (Optimality of trajectories).** For every $k \in \mathbb{N}$ define a sequence of singletons:

$$
\begin{cases}
  K^k_0 := \{ x_0 \} \\
  \forall t \in [0, T - 1], \ K^k_{t+1} := \{ x^k_{t+1} \},
\end{cases}
$$

where $(x^k_0, x^k_1, \ldots, x^k_T)$ is an optimal trajectory for the $k$-th approximations starting from $x_0$. Lastly, for every $t \in [0, T]$ define $K^*_t := \limsup_k K^k_t$.

Then the sequence of sets $(K^*_t)_{t \in [0, T]}$ is optimal with respect to $(V^*_t)_{t \in [0, T]}$ in the sense of Definition 2.4.

**Proof.**

First we show that for every $k \in \mathbb{N}$ the sets $(K^k_t)_{t \in [0, T]}$ are $V^k_t$-optimal.

Let $k \in \mathbb{N}$ and $t \in [0, T - 1]$ be fixed. We want to show that

$$
B_t \left( V^k_{t+1} + \delta_{K^k_{t+1}} \right) + \delta_{K^k_t} = B_t \left( V^k_{t+1} \right) + \delta_{K^k_t},
$$

which is equivalent to prove that for every $x^k_t \in K^k_t$ we have

\begin{equation}
B_t \left( V^k_{t+1} + \delta_{K^k_{t+1}} \right) (x^k_t) = B_t \left( V^k_{t+1} \right) (x^k_t).
\end{equation}

Now using the definition of the Bellman Operators (0.2), the equation (3.11) is satisfied if, and only if, there exist a control $u^k_t \in U$ such that

\begin{equation}
\begin{cases}
  u^k_t \in \arg\min_{u \in U} \mathcal{B}_t^k (u) \quad \text{where} \quad \mathcal{B}_t^k (u) := c_t \left( x^k_t, u \right) + V^k_{t+1} \left( f_t \left( x^k_t, u \right) \right) \\
  f_t \left( x^k_t, u^k_t \right) \in K^k_{t+1}.
\end{cases}
\end{equation}

Which is true by construction of the sets $(K^k_t)_{t \in [0, T]}$.

Now we can deduce that the sets $(K^*_t)_{t \in [0, T]}$ are $V^*_t$-optimal. First, as for (3.12), the compatibility relation

$$
B_t \left( V^*_{t+1} + \delta_{K^*_t} \right) + \delta_{K^*_t} = B_t \left( V^*_{t+1} \right) + \delta_{K^*_t},
$$
is satisfied if, and only if, for every \( x^*_t \in K^*_t \) there exist a control \( u^*_t \in U_t \) such that

\[
\begin{cases}
  u^*_t \in \arg\min_u B^*_t (u) & \text{with } B^*_t (u) = c_t (x^*_t, u) + V^*_{t+1} (f_t (x^*_t, u)) \\
  f_t (x^*_t, u^*_t) \in K^*_{t+1}.
\end{cases}
\]  

(3.13)

For every \( k \in \mathbb{N} \) denote by \( u^*_k \) the control defining the set \( K^*_{t+1} \). As \( U_t \) is compact, one can extract from the sequence of controls \( (u^*_k)_{k \in \mathbb{N}} \) a converging subsequence to a control \( u^*_t \) that we will also note \( (u^*_k)_{k \in \mathbb{N}} \). Let \( x^*_t \) be an element of \( K^*_t \), by definition of \( K^*_t := \limsup_k K^*_t \), one can find a sequence of points \( x^*_k \in K^*_t \) such that one of its subsequence converges to \( x^*_t \). Thus, extracting subsequences if needed, one can consider that we have simultaneously

\[
\begin{cases}
  x^*_k \to x^*_t \quad \text{as } k \to +\infty \\
  u^*_k \to u^*_t \quad \text{as } k \to +\infty
\end{cases}
\]

Which implies, by continuity of \( f_t \), that the sequence \( (f_t (x^*_t, u^*_k)) \) converges to a point in \( K^*_{t+1} \).

Let \( k, k' \in \mathbb{N} \) be two indices, as for every \( u \in U_t \) the function \( x \mapsto c_t (x^*_t, u) \) is \( C_t \)-Lipschitz continuous, \( V^*_{t+1} \in \mathbb{R}^{+}_{t+1} \) is \( L_{t+1} \) Lipschitz and for every \( u \in U_t \) the linear function \( x \mapsto f_t (x, u) \) is \( \lambda^{1/2}_t \) \( (A^T_t A_t) \)-Lipschitz continuous, so for every \( u \in U_t \) we have that

\[
\sup_{u \in U_t} |B^*_k (u) - B^*_k' (u)| \leq \left(C_t + L_{t+1} \lambda^{1/2}_t \|A^T_t A_t\| \right) \|x^*_k - x^*_k'\| \to 0 \quad \text{as } k \to +\infty.
\]

Thus we have shown that the sequence of functions \( (B^*_k)_{k \in \mathbb{N}} \) converges uniformly on the compact \( U_t \). Moreover as it converges pointwise to \( B^*_t \), the sequence \( (B^*_k)_{k \in \mathbb{N}} \) converges uniformly to \( B^*_t \) on \( U_t \). Finally, as for every \( k \in \mathbb{N} \), \( u^*_k \in \arg\min_{u \in U_t} B^*_k (u) \) we have that \( u^*_t \in \arg\min_{u \in U_t} B^*_t (u) \). \( \square \)

As the structural assumptions Assumptions 1.2 are satisfied, as the functions \( \phi_t^{\text{SDDP}} \), \( 0 \leq t \leq T \) are compatible selections and the sets \( (K^*_t)_{t \in [0, T)} \) are \( V^*_t \)-optimal (case opt = sup) by Theorem 2.6 we have the following convergence result, which is analogous to the ones in the literature.

**Theorem 3.6 (Lower (outer) approximations of the value functions).** Under Assumptions 3.1, for every \( t \in [0, T] \), denote by \( (V^*_t)_{k \in \mathbb{N}} \) the sequence of functionals generated by Tropical Dynamic Programming with the selection function \( \phi_t^{\text{SDDP}} \) and the draws made uniformly over the sets \( K^*_t \) defined in Proposition 3.5.

Then the sequence \( (V^*_t)_{k \in \mathbb{N}} \) is nondecreasing, bounded from above by \( V_t \), and converges uniformly to \( V^*_t \) on every compact set included in \( \text{dom}(V_t) \). Moreover, almost surely over the draws, \( V^*_t = V_t \) on \( \limsup_{k \in \mathbb{N}} K^*_t \).

**3.2. A min-plus selection function: upper approximations in the linear-quadratic framework with both continuous and discrete controls.**

In this example, the cost functions will be quadratic without mixing terms, in the sense of the definition below. We will explain at the end of the section why we don’t lose generality (if we increase the dimension of the state space by 1) when studying
such specific quadratic forms instead of general ones. In particular this allows us to restrict our study the (compact) unit sphere as explained below. We will denote by \( S_n \) the set of \( n \times n \) symmetric real matrices.

**Definition 3.7 (Pure quadratic form).** We say that a functional \( q : X \to \mathbb{R} \) is a pure quadratic form if there exist a symmetric matrix \( M \in S_n \) such that for every \( x \in X \) we have

\[
q(x) = x^T M x.
\]

Similarly, a functional \( q : X \times U \to \mathbb{R} \) is a pure quadratic form if there exist two symmetric matrices \( M_1 \in S_n \) and \( M_2 \in S_m \) such that for every \( x \in X \) we have

\[
q(x, u) = x^T M_1 x + u^T M_2 u.
\]

Let \( X = \mathbb{R}^n \) be a continuous state space (endowed with its euclidean structure), \( U = \mathbb{R}^m \) a continuous control space and \( V \) a finite set of discrete (or switching) controls. We want to solve the following optimization problem

\[
\min_{(x_0, \dots, x_{T-1}, u_0, \dots, u_{T-1})} \sum_{t=0}^{T-1} c_t(x_t, u_t) + \psi(x_T)
\]

s.t.

\[
\begin{align*}
x_0 & \in X \\
\forall t \in [0, T-1], \quad x_{t+1} &= f^v_t(x_t, u_t).
\end{align*}
\]

**Assumptions 3.8.** Let \( t \in [0, T - 1] \) and \( v \in V \) be arbitrary.

- The dynamic \( f^v_t : X \times U \to X \) is linear:

\[
f^v_t(x, u) = A^v_t x + B^v_t u,
\]

for some given matrices \( A^v_t \) and \( B^v_t \) of compatible dimensions.

- The cost function \( c_t^v : X \times U \to \mathbb{R} \) is a pure convex quadratic form,

\[
c_t^v(x, u) = x^T C^v_t x + u^T D^v_t u,
\]

where the matrix \( C^v_t \) is symmetric semidefinite positive and the matrix \( D^v_t \) is symmetric definite positive.

- The final cost function \( \psi := \inf_{i \in I} \psi_i \) is a finite infimum of pure convex quadratic form, of matrix \( M_i \in S_n \) with \( i \in I \) a finite set, such that there exists a constant \( \alpha_T \geq 0 \) satisfying for every \( i \in I \)

\[
0 \preceq M_i \preceq \alpha_T I.
\]

- The maximal eigenvalue of the symmetric semidefinite matrix \( (A^v_t)^T A^v_t \) is less than 1:

\[
\lambda_{\max} \left( (A^v_t)^T A^v_t \right) < 1.
\]

One can write the Dynamic Programming principle for (3.14):

\[
\begin{cases}
V_T = \psi \\
\forall t \in [0, T - 1], V_t : x \in X \mapsto \inf_{v \in V} \inf_{u \in U} c_t^v(x, u) + \phi(f^v_t(x, u)).
\end{cases}
\]

The following result is crucial in order to study this example: the value functions are 2-homogeneous, allowing us to restrict their study to the unit sphere.
Proposition 3.9. For every time step \( t \in [0, T] \), the value function \( V_t \), solution of (3.15) is 2-homogeneous, i.e. for every \( x \in X \) and every \( \lambda \in \mathbb{R} \) we have

\[
V_t(\lambda x) = \lambda^2 V_t(x).
\]

Proof. We proceed by backward recursion on the time step \( t \in [0, T] \). For \( t = T \) it is true by Assumptions 3.8. Assume that it is true for some \( t \in [1, T] \). Fix \( \lambda \in \mathbb{R} \), then by definition of \( V_{t-1} \), for every \( x \in X \) we have

\[
V_{t-1}(\lambda x) = \min_{v \in V} \min_{u \in U^c_v} c_{t-1}^v(\lambda x, u) + V_t(f_{t-1}^v(\lambda x, u)),
\]

which yields the result by 2-homogeneity of \( x \mapsto c_{t-1}^v(x, u) \), linearity of \( f_{t-1}^v \) and 2-homogeneity of \( V_t \).

Thus, in order to compute \( V_t \), one only needs to know its values on the unit (euclidean) sphere \( S \) as for every non-zero \( x \in X \),

\[
V_t(x) = \|x\|_2^2 V_t(\|x\|_2).
\]

We will study a dynamic programming formulation that exploits this fact. For every time \( t \in [0, T-1] \) we define the Bellman operator with fixed switching control \( B_t^v \) for every functional \( \phi : X \to \mathbb{R} \) by:

\[
B_t^v(\phi) := \inf_{u \in U} c_t^v(\cdot, u) + \|f_t^v(\cdot, u)\|_2^2 \phi \left( \frac{f_t^v(\cdot, u)}{\|f_t^v(\cdot, u)\|} \right).
\]

For every time \( t \in [0, T-1] \) we define the Bellman operator \( B_t \) for every functional \( \phi : X \to \mathbb{R} \) by:

\[
B_t(\phi) := \inf_{v \in V} B_t^v(\phi).
\]

Then one can rewrite the Dynamic Programming equation (3.15) as

\[
\begin{align*}
V_T &= \psi \\
\forall t \in [0, T-1], V_t &= B_t(V_{t+1}).
\end{align*}
\]

The point of writing the Dynamic Programming equation this way is to be able to compute the image by the Bellman operator of a functional by only knowing this functional on the unit (euclidean) sphere \( S \). This will ensure that the unit sphere \( S \) is \( V_t \)-optimal in the sense of Definition 2.4.

Now in order to apply the Tropical Dynamic Programming algorithm to (3.16) we need to check Assumptions 1.2. Under Assumptions 3.8, there exist an interval in the cone of symmetric semidefinite matrices which is stable by every Bellman operator \( B_t \) in the sense of the proposition below. We will consider the Loewner order on the cone of (real) symmetric semidefinite matrices, i.e. for every couple of matrices of symmetric matrices \( (M_1, M_2) \) we say that \( M_1 \preceq M_2 \) if, and only if, \( M_2 - M_1 \) is semidefinite positive. Moreover we’ll identify a pure quadratic form with its symmetric matrix, thus when we write an infimum over symmetric matrices, we mean the pointwise infimum over their associated pure quadratic forms.

Proposition 3.10 (Existence of a stable interval). Fix \( t \in [0, T-1] \). Under Assumptions 3.8, there exists a constant \( \alpha_t > 0 \) such that for every \( \beta \geq \alpha_t \) we have:

\[
0 \preceq M \preceq \beta I \Rightarrow 0 \preceq B_t(M) \preceq \beta I.
\]
Proof. We want to show that if $M \succeq 0$ then $B_t(M) \succeq 0$. First, as in Proposition 3.3 one can show that the Bellman operator $B_t$ is order preserving. Therefore, if $M \succeq 0$ then $B_t(M) \succeq B_t(0)$. Hence it is enough to show that $B_t(0) \succeq 0$. But by formula (A.2), we have that $B_t(0) = \min_{v \in V} C_t^v \succeq 0$ (by Assumptions 3.8) hence the result follows.

Now, as $B_t$ is order preserving, if $M \succeq \beta I$ then $B_t(M) \succeq B_t(\beta I)$. Hence it suffices to find $\beta > 0$ such that $B_t(\beta I) \succeq \beta I$. By formula (A.2) we have:

\begin{equation}
B_t^v (\beta I) \succeq \beta I \iff \beta (A_t^v)^T \left( I + \beta B_t^v (D_t^v)^{-1} (B_t^v)^T \right)^{-1} A_t^v + C_t^v \succeq \beta I.
\end{equation}

Now, using propositions Proposition B.1, Proposition B.2 and using the notations introduced in those propositions, finding $\beta > 0$ satisfying equation (3.18) is equivalent to find $\beta > 0$ such that

$$\frac{\beta \lambda_{\max} \left( (A_t^v)^T A_t^v \right)}{\lambda_{\min} \left( I + \beta B_t^v (D_t^v)^{-1} (B_t^v)^T \right)} + \lambda_{\max} \left( C_t^v \right) \leq \beta,$$

Noting that as $\lambda_{\min} \left( I + \beta B_t^v (D_t^v)^{-1} (B_t^v)^T \right) \geq 1$, it suffices to find $\beta \geq 0$ such that

$$\beta \lambda_{\max} \left( (A_t^v)^T A_t^v \right) + \lambda_{\max} (C_t^v) \leq \beta,$$

which, under Assumptions 3.8, is equivalent to

\begin{equation}
\beta \geq \frac{\lambda_{\max} (C_t^v)}{1 - \lambda_{\max} \left( (A_t^v)^T A_t^v \right)}.
\end{equation}

Finally, by setting

$$\alpha_t := \max_{v \in V} \frac{\lambda_{\max} (C_t^v)}{1 - \lambda_{\max} \left( (A_t^v)^T A_t^v \right)} > 0,$$

every $\beta \geq \alpha_t$ satisfies (3.19) which concludes the proof.

For every time step $t \in [0, T]$, we define

\begin{equation}
\beta_t := \max_{s \in [t, T]} \alpha_s,
\end{equation}

the basic functions $F_{t}^{Qu}$ will be pure quadratic convex forms bounded in the Loewner sense by 0 and $\beta_t I$,

$$F_{t}^{Qu} := \left\{ \phi : x \in X \mapsto x^T M x \in \mathbb{R} \mid M \in \mathbb{S}_n, \ 0 \preceq M \preceq \beta_t I \right\},$$

and we define the following class of functions which will be stable by pointwise infimum of elements in $F_{t}^{Qu}$,

\begin{equation}
\mathcal{F}_{t}^{Qu} := \left\{ V_F \mid F \subset F_{t}^{Qu} \right\}.
\end{equation}

Proposition 3.11. Under Assumptions 3.8, the Problem (3.14) and the Bellman operators defined in (3.15) satisfy the structural assumptions given in Assumptions 1.2.
Proof.
— **Common regularity.** We will show that every element of $F_{t}^{Qu}$ is $2\beta$-Lipschitz continuous on $S$. Let $F = \{ \phi_i \}_{i \in N} \subset F_{t}^{Qu}$ with $N \subset N$ and $\phi_i \in F_{t}^{Qu}$ with associated symmetric matrix $M_i$. Fix $x, y \in S$, we have

$$\|V_F(x) - V_F(y)\| = \| \inf_{i \in N} x^T M_i x - \inf_{i \in N} y^T M_i y \|$$

$$\leq \max_{i \in N} \| x^T M_i x - y^T M_i y \|$$

$$\leq \max_{i \in N} \| x^T M_i (x - y) + y^T M_i (x - y) \|$$

($M^T = M$)

(Cauchy-Schwarz)

$$\leq \| x + y \| \cdot \max_{i \in N} \| M_i (x - y) \|$$

$$\leq \| x + y \| \cdot \max_{i \in N} \| M_i \| \cdot \| x - y \|$$

$$\leq \beta_i \| x + y \| \cdot \| x - y \|$$

$$\leq 2 \beta_i \| x - y \|_2.$$ 

— **Final condition.** By Assumptions 3.8, the final cost function $\psi$ is an element of $F_{t}^{Qu}$.

— **Stability by the Bellman operators.** Let $\phi$ be in $F_{t}^{Qu}$. By construction, the symmetric matrix $M$ associated with the pure quadratic form $\phi$ is such that $0 \preceq M \preceq \beta_{t+1}I$. By (3.20) we have $0 \preceq t \preceq \beta_{t+1}I$, thus $0 \preceq M \preceq \beta_t I$ and by Proposition 3.10 we have that

$$0 \preceq B_t (M) \preceq \beta_t I.$$ 

We have shown that if $\phi$ is in $F_{t+1}^{Qu}$, then $B_t (\phi)$ is in $F_{t}^{Qu}$.

— **Stability by pointwise optimum.** This is true by construction of $F_{t}^{Qu}$ (3.21).

— **Order preserving operators.** Proceed as in Proposition 3.3.

— **Existence of the value functions.** This is a consequence of (A.1) and the fact that for every functionals $\phi_1, \phi_2 : X \to \mathbb{R}$ we have that

$$B_t (\inf (\phi_1, \phi_2)) = \inf (B_t (\phi_1), B_t (\phi_2)).$$

— **Existence of optimal sets.** For every time step $t \in [0, T-1]$, every compact $X_t \subset \text{dom}(V_t) (= X)$ and every functional $\phi \in F_{t}^{Qu}$ the unit sphere $S$ satisfies

$$B_t (\phi + \delta_S) = B_t (\phi),$$

which implies the desired inequality.

— **Additively $M$-subhomogeneous operators.** Set $M \geq 0$ to be

$$M := \inf_{(u,v) \in \bar{U} \times \bar{V}} \sup_{x \in S} \| f_t^u (x, u) \|^2.$$ 

Then $M$ is finite as for every $(u,v) \in \bar{U} \times \bar{V}$ the function $f_t^v (\cdot, u)$ is continuous over the compact $S$. Let $\lambda : X \to \mathbb{R}$ be a positive constant function, we will
identify \( \lambda \) with its value. Let \( \phi \) be an element of \( F^Q_{t+1} \). For any \( x \in S \) we have:

\[
B_t (\lambda + \phi) (x) = \inf_{(u,v) \in U \times V} c_t^v (x, u) + \| f_t^u (x, u) \|^2 (\lambda + \phi) \left( \frac{f_t^u (x, u)}{\| f_t^u (x, u) \|} \right)
\leq \lambda M + B_t (\phi) (x).
\]

Thus \( B_t \) is additively \( M \)-subhomogeneous.

We now define the selection functions \( \phi_t^Q \). At time \( t \in [0, T - 1] \) we define for any \( F \subset F^Q_t \) and \( x \in X \)

\[
\phi_t^Q(F, x) \in B_t \left( \arg \min_{\phi \in F} B_t (\phi) (x) \right).
\]

At time \( t = T \) we define for any \( F \subset F^Q_T \) and \( x \in X \) we set

\[
\phi_T^Q(F, x) = \arg \min_{\psi_t} \psi_t(x).
\]

**Proposition 3.12.** For every time \( t \in [0, T] \), the function \( \phi_t^Q \) is a compatible selection function in the sense of Definition 1.3.

**Proof.** Fix \( t = T \). The function \( \phi_T^Q \) is tight and valid as \( V_T = \psi \). Now fix \( t \in [0, T - 1] \). Let \( F \subset F^Q_t \) and \( x \in X \) be arbitrary. We have

\[
B_t (V_T) (x) = B_t \left( \inf_{\phi \in F} \phi \right) (x)
\]

\[
= \inf_{(u,v) \in U \times V} c_t^v (x, u) + \| f_t^u (x, u) \|^2 \inf_{\phi \in F} \phi \left( \frac{f_t^u (x, u)}{\| f_t^u (x, u) \|} \right)
\]

\[
= \inf_{\phi \in F} \inf_{(u,v) \in U \times V} c_t^v (x, u) + \| f_t^u (x, u) \|^2 \phi \left( \frac{f_t^u (x, u)}{\| f_t^u (x, u) \|} \right)
\]

\[
= \inf_{\phi \in F} B_t (\phi) (x)
\]

\[
= \phi_t^Q(F, x) (x).
\]

Thus \( \phi_t^Q \) is tight. By similar arguments we have for every \( x' \in X \) that

\[
B_t (V_T) (x') = \inf_{\phi \in F} B_t (\phi) (x') \geq \phi_t^Q(F, x) (x').
\]

As the structural assumptions, Assumptions 1.2 are satisfied, as the functions \( \phi_t^Q \), \( 0 \leq t \leq T \) are compatible selections and the unit sphere \( S \) is \( V_t \)-optimal (case \( \text{opt} = \inf \)) we have by Theorem 2.6 the following convergence result

**Theorem 3.13 (Upper (inner) approximations of the value functions).** Under Assumptions 3.8, for every \( t \in [0, T] \), denote by \( (V^k_t)_{k \in \mathbb{N}} \) the sequence of functionals generated by Tropical Dynamic Programming with the selection function \( \phi_t^Q \) and the draws made uniformly over the sphere \( K_t := S \).

Then the sequence \( (V^k_t)_{k \in \mathbb{N}} \) is nonincreasing, bounded from below by \( V_t \) and converges uniformly to \( V^*_t \) on \( S \). Moreover, almost surely over the draws, \( V^*_t = V_t \) on \( S \).
Finally, we briefly explain why, by adding another dimension to the state variable, there’s no loss of generality induced by the choice of studying pure quadratic forms in (3.14) instead of quadratic forms, nor is there one for studying linear dynamics instead of affine dynamics. First, we define the operator $\theta$ that maps any functional $\phi : \mathbb{X} \to \mathbb{R}$ to a 2-homogeneous functional $\phi^\theta$

$$\theta : \mathbb{R}^\mathbb{X} \to \mathbb{R}^{\mathbb{X} \times \mathbb{R}}$$

$$\phi \mapsto \phi^\theta : (x, y) \mapsto y^2 \phi \left( \frac{x}{y} \right) \text{ if } y \neq 0, \ 0 \text{ otherwise.}$$

When $\phi$ is affine, abusively denote by $\theta$ again the operator that maps any affine functional $\phi$ to the linear functional $\phi^\theta$,

$$\theta : \mathbb{R}^\mathbb{X} \to \mathbb{R}^{\mathbb{X} \times \mathbb{R}}$$

$$\phi \mapsto \phi^\theta : (x, y) \mapsto y \phi \left( \frac{x}{y} \right) \text{ if } y \neq 0, \ 0 \text{ otherwise.}$$

Now consider $(B_t)_{t \in [0, T-1]}$ the Bellman operators associated to an analogous optimization problem as (3.14) but where we allow the costs and final cost functions to be general quadratic forms and the dynamics to be affine. Furthermore, one can turn this non-homogeneous optimization problem into a homogeneous problem by applying $\theta$ to the costs functions, final cost function and dynamics of the previous optimization problem. We note by $(B^\theta_t)_{t \in [0, T-1]}$ the Bellman operators of the homogenized optimization problem. Then one can deduce $B_t$ from $B^\theta_t\theta$, meaning that if one can solve the (homogenized) optimization problem (3.14) where the costs are pure quadratics forms and the dynamics linear, then one can solve the (non-homogeneous) problem where the costs are quadratic forms and the dynamics are affine.

**Proposition 3.14.** Define for every $t \in [0, T-1]$, two operators

$$B_t : \mathbb{R}^\mathbb{X} \to \mathbb{R}^\mathbb{X}$$

$$\phi \mapsto x \mapsto \min_{u \in \mathbb{U}} c_t (x, u) + \phi \left( f_t (x, u) \right),$$

$$B^\theta_t : \mathbb{R}^{\mathbb{X} \times \mathbb{R}} \to \mathbb{R}^{\mathbb{X} \times \mathbb{R}}$$

$$\phi \mapsto (x, y) \mapsto \min_{u \in \mathbb{U}} c_t^\theta (x, y, u) + \phi \left( f_t^\theta (x, y, u) \right).$$

For every functional $\phi : \mathbb{X} \to \mathbb{R}$, for every $x \in \mathbb{X}$ and $y \in \mathbb{R}^*$ we have that

$$B^\theta_t (\theta \phi) (x, y) = \theta B_t (\phi) (x, y) \quad \text{and} \quad B_t (\phi) (x) = B^\theta_t (\theta \phi) (x, 1).$$

**Proof.** First, remark that if the first equality holds, then as for every $x \in \mathbb{X}$ one has $\theta B_t (\phi) (x, 1) = B_t (\phi) (x)$, setting $y = 1$ one gets the second equality. Now fix a
Fig. 5. Illustration of the multistage (two stages here) optimization problem studied in Example 3.15. On the right we have the final cost function (infimum of $\psi_1$ and $\psi_2$) and on the left we have the image of $\psi_1$ and $\psi_2$ by the Bellman operator $B$. The image of $\psi_2$ is strictly greater than the image of $\psi_1$. At the final step $t = 1$, the best function at the point $-1$ is $\psi_2$. The image by the $k$-th optimal dynamic of $x_0 = -2$ is $x_1 = -1$.

functional $\phi : X \rightarrow \mathbb{R}$, $x \in X$ and $y \in \mathbb{R}^*$, we have that

$$B_t^\theta(\theta \phi)(x, y) = \min_{u \in \mathcal{U}} c_t^\theta(x, y, u) + \phi^\theta(f_t^\theta(x, y, u))$$

$$= \min_{u \in \mathcal{U}} y^2 c_t \left( \frac{x}{y}, \frac{u}{y} \right) + \phi \left( y c_t \left( \frac{x}{y}, \frac{u}{y} \right) \right)$$

$$= y^2 \min_{u' \in \mathcal{U}} c_t \left( \frac{x}{y}, u' \right) + \phi \left( c_t \left( \frac{x}{y}, u' \right) \right)$$

$$= y^2 B_t(\phi) \left( \frac{x}{y} \right)$$

$$= \theta B_t(\phi)(x).$$

Example 3.15. We show an example where approximating from above fails to converge when the points are drawn accordingly to optimal trajectories for the current approximations (as done in subsection 3.1 where we approximate from below). As shown by Proposition 3.14 there’s no loss of generality in considering the framework of this section but with non-homogeneous functions.

We consider a (non-homogeneous) problem with only two time steps, that is $T = 1$ and $t \in \{0, 1\}$ such that

- The state space $X$ and the control space $U$ are equal to $\mathbb{R}$.
- The linear dynamic is $f(x, u) = x + u$.
- The quadratic cost is $c(x, u) = x^2 + u^2$.
- The final cost function is the infimum between two quadratics, $\psi_1(x) = (x + 2)^2 + 1$ and $\psi_2(x) = x^2$ i.e.

$$\psi = \inf (\psi_1, \psi_2).$$

Here the Bellman operator $B$ of this multistage optimization problem is defined
for every $\phi : X \to \mathbb{R}$ and every $x \in X$ by
\[ B(\phi)(x) = \min_{u \in U} x^2 + u^2 + \phi(x + u) = x^2 + \min_{u \in U} u^2 + \phi(x + u). \]

For the case where $\phi_{a,b}(\cdot) = (\cdot + a)^2 + b$ with $a, b \in \mathbb{R}$ one has for every $x \in \mathbb{R}$
\[ (3.24) \quad B(\phi_{a,b})(x) = \frac{3}{2} x^2 + ax + b. \]

Fix $x_0 = x_0^k = -2$ for every $k \in \mathbb{N}$. As described in Algorithm 1.1, the approximations of the value functions $V_1$ and $V_0$ are initialized to $+\infty$. Thus every control $u \in U$ is optimal in the sense that $u \in \arg\min_{u' \in U} x^2 + (u')^2 + \phi(x + u')$. Hence if we set $x_0^1 := -1 = f(x_0, 1)$ then $(x_0, x_0^1)$ is an optimal trajectory as described in Proposition 3.5.

We deduce from (3.24) the following facts, illustrated in Figure 5.
1. The image of $\psi_2$ is strictly greater than the image of $\psi_1$ by the Bellman operator $B$, i.e.
   \[ B(\psi_2)(-2) > B(\psi_1)(-2). \]
2. The image by the $k$-th optimal dynamic of $-2$ is $-1$, i.e. setting $u_0^k := \arg\min_{u' \in U} (-2)^2 + (u')^2 + V_1^k(-2 + u')$ (the arg min is here a singleton) one has
   \[ f(-2, u_0^k) = -1. \]
3. At the final step $t = 1$, the best function at the point $-1$ is $\psi_2$, i.e.
   \[ \psi(-1) = \inf(\psi_1(-1), \psi_2(-2)) = \psi_2(-2). \]

From those three facts, one can deduce starting $x_0 = -2$ and $x_1 = -1$, the optimal trajectory for the current approximations will always be sent to $x_1 = -1$. But, as shown in the proof of Proposition 3.12 one can show that the image by $B$ of an infimum is the infimum of the images by $B$:
\[ V_0(-2) = B(\inf(\psi_1, \psi_2))(-2) = \inf(B(\psi_1)(-2), B(\psi_1)(-2)). \]

Thus for every $k \in \mathbb{N}$,
\[ V_0(-2) = B(\psi_1)(-2) < B(\psi_1)(-2) = V_0^k(-2). \]

**Conclusion.** In this article we have devised an algorithm, Tropical Dynamic Programming, that encompasses both a discrete time version of Qu’s work and the SDDP algorithm in the deterministic case. We have shown in the last section that Tropical Dynamic Programming can be applied to two natural frameworks: one for Qu’s adaptation and one for SDDP. In the case where both framework intersects, one could apply Tropical Dynamic Programming with the selection functions $\phi_{Qu}^t$ and get non-increasing upper approximations of the value function. Simultaneously, by applying Tropical Dynamic Programming with the selection function $\phi_{SDDP}^t$, one would get non-decreasing lower approximations of the value function. Moreover, we have shown that the upper approximations are, almost surely, asymptotically equal to the value function on the whole space of states $X$ and that the lower approximations are, almost surely, asymptotically equal to the value function on a set of interest.

Thus, in those particular cases we get converging bounds for $V_0(x_0)$, which is the value of the multistage optimization problem (0.1), along with asymptotically exact minimizing policies. In those cases, we have shown a possible way to address the issue.
of computing efficient upper bounds when running the SDDP algorithm by simultaneously running another algorithm (namely TDP with Qu’s selection functions). This claim has yet to be tested numerically: the results presented here act as a safeguard and are not proofs of efficiency.

In future works, we would like to extend the current framework to risk-averse multistage stochastic optimization problems, explicitly give a way to generate deterministic converging upper and lower bounds and to provide numerical experiments.

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Appendix A. Algebraic Riccati Equation. This section gives complementary results for subsection 3.2. We use the same framework and notations introduced in subsection 3.2.

Proposition A.1. Fix a discrete control \( v \in \mathcal{V} \) and a time step \( t \in [0, T - 1] \).

The operator \( B_t^v \) restricted to the pure quadratic forms (identified with \( \mathbb{S}_n \) the space of the symmetric semidefinite positive matrices) is given by the discrete time algebraic Riccati equation

\[
B_t^v : \mathbb{S}_n \rightarrow \mathbb{S}_n^+ \\
M \mapsto C_t^v + (A_t^v)^T M A_t^v \\
- (A_t^v)^T M B_t^v \left( D_t^v + B_t^v M (B_t^v)^T \right)^{-1} (B_t^v)^T M A_t^v.
\]  

We can rewrite the discrete time algebraic Riccati equation Appendix B (A.1).

For every \( M \in \mathbb{S}_n^+ \) we have:

\[
B_t^v(M) = (A_t^v)^T M \left( I + B_t^v (D_t^v)^{-1} (B_t^v)^T M \right)^{-1} A_t^v + C_t^v.
\]  

Proof.

First note that if \( M \) is symmetric, then \( B_t^v(M) \) is also symmetric. Let \( t \in \{T - 1, T - 2, \ldots, 0\} \) and \( M \in \mathbb{S}_n^+ \). Let \( x \in \mathbb{X} \), we have:

\[
B_t^v(M)(x) = \inf_{u \in \mathcal{U}} c_t(x, u) + \| f_t^v(x, u) \|^2_2 \left( \frac{f_t^v(x, u)}{\| f_t^v(x, u) \|_2} \right)^T M \frac{f_t^v(x, u)}{\| f_t^v(x, u) \|_2} = \inf_{u \in \mathcal{U}} x^T C_t^v x + u^T D_t^v u + f_t^v(x, u)^T M f_t^v(x, u)
\]

(\ref{eq:algebraic Riccati equation})

\[
B_t^v(M)(x) = x^T C_t^v x + \inf_{u \in \mathcal{U}} u^T D_t^v u + f_t^v(x, u)^T M f_t^v(x, u).
\]

As \( u \mapsto f_t(x, u) \) is linear, \( D_t^v \geq 0 \) and \( M \geq 0 \) we have that \( g : u \in \mathcal{U} \mapsto u^T D_t^v u + f_t^v(x, u)^T M f_t^v(x, u) \in \mathbb{R} \) is convex, hence is minimal when \( \nabla g(u) = 0 \) i.e. for \( u \in \mathcal{U} \) such that :

\[
\left( D_t^v + (B_t^v)^T M B_t^v \right) u + (B_t^v)^T M (A_t^v) x = 0.
\]  

(A.4)

Now we will show that \( D_t^v + (B_t^v)^T M B_t^v \) is invertible. As \( M \) is symmetric semidefinite positive, for every \( u \in \mathcal{U} \) we have :

\[
u^T \left( D_t^v + (B_t^v)^T M B_t^v \right) u = u^T D_t^v u + (B_t^v u)^T M (B_t^v u) \geq 0
\]

\[
g \geq 0
\]

\[
0 > 0
\]

We have shown that the symmetric matrix \( D_t^v + (B_t^v)^T M B_t^v \) is definite positive and thus invertible. We conclude that equation (A.4) is equivalent to:

\[
u = - \left( D_t^v + (B_t^v)^T M B_t^v \right)^{-1} (B_t^v)^T M (A_t^v) x.
\]  

(A.5)
Finally replacing (A.5) in equation (A.3) we get after simplifications that

\[ B^v_t(M)(x) = x^T [C^v_t + (A^v_t)^T MA^v_t \]
\[ - (A^v_t)^T MB^v_t \left( D^v_t + (B^v_t)^T MB^v_t \right)^{-1} (B^v_t)^T MA^v_t]x, \]

which gives (A.1).
— See [11, Proposition 12.1.1 page 271].

\section*{Appendix B. Smallest and greatest eigenvalues.}
Here we recall some formulas on the lowest and greatest eigenvalues of a matrix.

\begin{proposition}
\label{prop:B1}
Let \( A \) and \( B \) two symmetric real matrices. Denote the smallest eigenvalue of a symmetric real matrix \( M \) by \( \lambda_{\text{min}}(M) \) (every eigenvalue of \( M \) is real) and by \( \lambda_{\text{max}}(M) \) its greatest eigenvalue. We have the following inequalities:
\begin{enumerate}
\item \( \lambda_{\text{min}}(A + B) \geq \lambda_{\text{min}}(A) + \lambda_{\text{min}}(B) \),
\item \( \lambda_{\text{max}}(A + B) \leq \lambda_{\text{max}}(A) + \lambda_{\text{max}}(B) \).
\end{enumerate}
\end{proposition}

\begin{proposition}
\label{prop:B2}
Let \( A \) be a real matrix and \( B \) be a symmetric real matrix. Denote by \( \lambda_{\text{min}}(M) \) the smallest eigenvalue of a symmetric real matrix \( M \) (every eigenvalue of \( M \) is real) and by \( \lambda_{\text{max}}(M) \) its greatest eigenvalue. We have the following inequalities:
\begin{enumerate}
\item \( \lambda_{\text{min}}(A^T BA) \geq \lambda_{\text{min}}(A^T A) \lambda_{\text{min}}(B) \),
\item \( \lambda_{\text{max}}(A^T BA) \leq \lambda_{\text{max}}(A^T A) \lambda_{\text{max}}(B) \).
\end{enumerate}
Moreover if \( A \) and \( B \) are symmetric definite positive matrices, then we have:
\begin{enumerate}
\item \( \lambda_{\text{min}}(AB) \geq \lambda_{\text{min}}(A) \lambda_{\text{min}}(B) \),
\item \( \lambda_{\text{max}}(AB) \leq \lambda_{\text{max}}(A) \lambda_{\text{max}}(B) \).
\end{enumerate}
\end{proposition}

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