Fluctuations at Phase Transitions

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Abstract

We use the closed time-path formalism to calculate fluctuations at phase transitions, both in and out of equilibrium. Specifically, we consider the creation of vortices by fluctuations, of relevance to the early universe and to $^4$He superfluidity.

1 Introduction

It has become increasingly important to understand the nature of the field fluctuations that are a consequence of the phase transitions that occurred in the very early universe. On the one hand, fluctuations at the GUT era are assumed to have seeded the large-scale structure visible today. On the other hand, fluctuations at the slightly later electroweak transition are most likely the basis of baryogenesis.

In this talk I shall consider three aspects of fluctuations. The first is combinatorical, establishing a path-integral in terms of which fluctuations can be evaluated. The second is concerned with the initial-value problem, exemplified by both equilibrium and non-equilibrium processes for a real scalar field. The third provides a non-trivial application, the creation by fluctuations of (global) vortices in a symmetry-broken $U(1)$ theory of a complex scalar field.

This application is highly relevant, on two counts. Firstly, while there is no unambiguous mechanism for large-scale structure formation in the universe, arguably one of the least artificial (originally proposed by Kibble [1]) is to attribute it to the spontaneous creation of cosmic strings (vortices) by fluctuations at the GUT transitions. [Primordial magnetic fields are created by field fluctuations in a related way. See Enqvist, these proceedings and elsewhere [2].]

Secondly, very recent experiments [3] show that vortices are naturally generated in superfluid $^4$He on quenching it through its critical density. In the Ginzburg-Landau theory of superfluidity the vortices of $^4$He are the counterparts of global U(1) cosmic strings (albeit non-relativistic). These results have been argued [4] as providing direct

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support for the creation of cosmic strings in the early universe. We have some observations on this.

This highly enjoyable meeting arrived a little too early for some of the work presented here to be completed and my conclusions are rather more qualitative than I would have liked. A fuller discussion will be given elsewhere, as indicated in the text.

2 Combinatorics of Fluctuations.

To be concrete, we begin with the simplest possible theory, that of a real scalar field \( \phi \), with order parameter \( \langle \phi \rangle \). A natural measure of the fluctuations of \( \phi(t, \vec{x}) \) is given by the field averages

\[
\phi_v(t) = \frac{1}{v} \int_v d\vec{x} \phi(t, \vec{x})
\]

over fixed volumes \( v = O(L^3) \). [The symbol \( v \) denotes both the position of the volume in question and its size]. The system as a whole is taken to have a large volume \( V = O(L^3) \gg v \), in which \( V \) is ultimately taken infinite. Since, at any time, the field \( \phi \) is correlated over some length \( \xi \), it is sufficient to take \( l \geq \xi \).

To simplify the notation, assume that the possible configurations \( \Phi_j(\vec{x}) \) of \( \phi \) in the volume \( V \) are denumerable (e.g. we impose periodic boundary conditions). Particular values of \( \phi_v \) then single out particular \( \Phi_j \).

In describing the quantum theory of \( \phi \), it is convenient to adopt the field representation. Suppose, at some initial time \( t = t_{in} \), the system, described by the state functional \( |\Phi_k, t_{in}\rangle \), is an eigenstate of \( \hat{\phi} \), eigenvalue \( \Phi_k \). That is,

\[
\hat{\phi}|\Phi_k, t_{in}\rangle = \Phi_k(\vec{x})|\Phi_k, t_{in}\rangle.
\]

At a later time \( t > t_{in} \), the state has evolved to

\[
|\Phi_k, t\rangle = \sum_j c_{kj}|\Phi_j, t\rangle
\]

a superposition of field eigenstates. The probability that the measurement of \( \hat{\phi} \) in \( |\Phi_k, t\rangle \) gives \( \Phi_j \) is \( |c_{kj}|^2 \), most conveniently written in terms of path integrals as

\[
|c_{kj}|^2 = \int_{\Phi_1(\vec{x})=\Phi_j(\vec{x})} \int_{\Phi_2(\vec{x})=\Phi_k(\vec{x})} D\phi_1 \exp\{iS[\phi_1]\} D\phi_2 \exp\{-iS[\phi_2]\}.
\]

The first term in (2.4) is \( c_{kj} \), the second \( c_{kj}^* \), where \( S[\phi] = \int d^3x \mathcal{L}(\phi) \) denotes the classical action that determines the evolution of the system.

The probability \( p(\phi_v(t) = \tilde{\phi}) \) that the coarse-grained field \( \phi_v(t) \) takes some chosen value \( \tilde{\phi} \) is obtained by checking which \( \Phi_j(\vec{x}) \) have their average as \( \tilde{\phi} \), and multiplying by the probability for finding such a \( \Phi_j \). That is,

\[
p(\phi_v(t) = \tilde{\phi}) = \sum_j |c_{kj}|^2 \delta \left( \frac{1}{v} \int_v d^3x \, \Phi_j(\vec{x}) - \tilde{\phi} \right)
\]

\[
= \sum_j \int_{\phi_1(t_{in})=\phi_2(t_{in})=\Phi_j} D\phi_1 D\phi_2 \, \delta \left( \frac{1}{v} \int_v d^3x \phi_1(t, \vec{x}) - \tilde{\phi} \right) \exp\{iS[\phi_1] - iS[\phi_2]\}
\]

\[
= \int_{\phi_1(t_{in})=\phi_2(t_{in})=\Phi_k} D\phi_1 D\phi_2 \, \delta \left( \frac{1}{v} \int_v d^3x \phi_1(t, \vec{x}) - \tilde{\phi} \right) \exp\{iS[\phi_1] - iS[\phi_2]\}.
\]

\(^1\)I have taken advantage of the few weeks since the meeting to develop some of the points a little further. Conclusions are unaltered.
It might seem that the upper bound of the field integrals in (2.6) should be $\phi_1(t, \vec{x}) = \phi_2(t, \vec{x})$, but we can extend the time contour to any final time value $t_{fin} > t$ without changing the integral. We shall take $t_{fin}$ infinite.

The expression (2.6) is written more simply as the closed-time path integral of Schwinger and Keldysh

$$p(\phi_v(t) - \bar{\phi}) = \int C D\phi \delta \left( \frac{1}{v} \int_v d^3 \bar{x}' \phi_1(t, \bar{x}') - \bar{\phi} \right) \exp \{ iS_{ctp} \}$$

(2.7)

where $C = C_1 \oplus C_2$ is the closed-time path (see below) in which $C_1$ runs from $t_{in}$ to $t_{fin}$ and $C_2$ runs backward from $t_{fin}$ to $t_{in}$, infinitesimally beneath $C_1$.

The relevant action $S_{ctp}[\phi]$, in the presence of a source $j(t, \vec{x})$, is described in terms of $L(\phi)$ by

$$S_{ctp}[\phi, j] = \int_c dtd\vec{x} \left( L(\phi) + j(t)\phi(t) \right)$$

$$= \int_{-\infty}^{\infty} dtd\vec{x} \left( L(\phi_1) - L(\phi_2) + j_1(t)\phi_1(t) + j_2(t)\phi_2(t) \right).$$

(2.8)

(2.9)

Spatial labels have been suppressed. The doublets are defined to be

$$\phi_a(t) = \begin{cases} 
\phi(t) & t \in C_1 \text{ if } a = 1. \\
\phi(t - i\epsilon) & t - i\epsilon \in C_2 \text{ if } a = 2.
\end{cases}$$

(2.10)

and

$$j_a(t) = \begin{cases} 
j(t) & t \in C_1 \text{ if } a = 1. \\
-j(t - i\epsilon) & t - i\epsilon \in C_2 \text{ if } a = 2.
\end{cases}$$

(2.11)

Note that, notionally, we have the second leg $C_2$ of our curve running back below the first.

This takes account of the quantum uncertainty. However, in general, the initial field $\Phi(t_{in}, \vec{x})$ is only specified statistically. Let $P_{in}[\Phi_k]$ be the probability that, at time $t = t_{in}$, the field takes configuration $\Phi_k(\vec{x})$. Then

$$p(\phi_v(t) - \bar{\phi}) = \sum_k P_{in}[\Phi_k] \int_{\phi_1=\phi_2=\Phi_k} D\phi_1 D\phi_2 \delta \left( \frac{1}{v} \int_v d^3 \bar{x}' \phi_1(t, \bar{x}') - \bar{\phi} \right) \exp \{ iS_{ctp} \}$$

$$= \int D\Phi P_{in}[\Phi] \int_{\phi_1=\phi_2=\Phi} D\phi_1 D\phi_2 \delta \left( \frac{1}{v} \int_v d^3 \bar{x}' \phi_1(t, \bar{x}') - \bar{\phi} \right) \exp \{ iS_{ctp} \}. \quad (2.12)$$

where we have now relaxed the (artificial) denumerability of the initial state. The generality of (2.12) is usually too much for us. To be tractable, simple initial conditions are required. The most convenient assumption is that $P_{in}[\Phi]$ is Boltzmann distributed as

$$P_{in}[\Phi] = N \exp \{ -\beta_{in} H_{in}[\Phi] \} \quad (2.13)$$
at some initial temperature $T_{in} = \beta_{in}^{-1}$. This permits us to introduce an initial Lagrangian density $L_{in}(\phi)$ in terms of which $P_{in}$ can be expressed as the Euclidean-time path integral (the diagonal density-matrix element)

$$P_{in}[\Phi] = \int_{\phi_3(t_{in}, \vec{x}) = \Phi(\vec{x})} D\phi_3 \exp\{-S_{in}[\phi_3]\} \tag{2.14}$$

where $S_{in}[\phi] = \int dx L_{in}(\phi(x))$ is independent of $T_{in}$. The time contour for the integral (2.14) is $C_3$ (see above), running from time $t_{in}$ to $t_{in} - i\beta_{in}$, over which $\phi_3$ is periodic. We stress that $L_{in}(\phi)$ exists only to parametrise the initial conditions and, in principle, has nothing to do with the $L(\phi)$ of later times. However, in practice we have in mind a situation in which $L_{in}$ and $L$ of the closed-time path have the same form, but in which the parameters have a time-dependence. The other obvious initial condition, in which the field $\phi$ is localised at a particular value, in effect, can also be taken as a particular choice of Boltzmann distribution, as we shall see. [In principle we can invert (2.14) to deduce $S_{in}$ from $P_{in}$, whether $P_{in}$ describes a Boltzmann distribution or not. In practice, it is all but impossible.]

With this proviso, $p$ of (2.12) can be written as

$$p(\phi_v(t) - \tilde{\phi}) = \int_C D\phi_1 D\phi_2 D\phi_3 \delta \left( \frac{1}{\nu} \int_v d^3 \vec{x}' \phi_1(t, \vec{x}') - \tilde{\phi} \right) \exp\{iS_C\} \tag{2.15}$$

where $C$ is the contour $C_1 \oplus C_2 \oplus C_3$ and $S_C = S_{in}$ on $C_3$.

Let $I(\vec{x})$ be the window function (indicator function) for the volume $v$ (i.e. $I(\vec{x}) = 1, \vec{x} \in v; I(\vec{x}) = 0, \vec{x} \notin v$). On using an exponential representation of the $\delta$-function (2.15), rewritten as

$$p(\phi_v(t) = \tilde{\phi}) = \int d\alpha \int_C D\phi_1 D\phi_2 D\phi_3 \exp\{iS_C[\phi] - \int_v d\vec{x}\alpha(\tilde{\phi} - \phi_1(t, \vec{x}))\} \tag{2.16}$$

can be further recast as

$$p(\phi_v(t) = \tilde{\phi}) = \int d\alpha \exp\{-i\alpha v \tilde{\phi}\} Z[\alpha I, 0, 0] \tag{2.17}$$

$Z[j_1, j_2, j_3]$, defined by

$$Z[j_1, j_2, j_3] = \int D\phi_1 D\phi_2 D\phi_3 \exp\{iS_C[\phi_1, j_a]\}. \tag{2.18}$$

is the generating functional for Green functions in the presence of sources on all three contours (the extension of $S_{cap}[\phi_a, j_a]$ of (2.8) to include $C_3$). In (2.17) $j_1 = \alpha I(\vec{x})$ is a source coupled to the field on $C_1$. The boundary conditions are now $\phi_1(t_{in}) = \phi_3(t_{in} - i\beta_{in}), \phi_1(t_{fin}) = \phi_2(t_{fin})$

This splitting into triplets is merely a matter of formal definition. The reason for doing this is that it manages to encode the initial conditions into an expression (2.18) that looks familiar, and is hence amenable to our usual tricks. For the circumstances that we shall consider here, a Gaussian approximation to the fluctuations is a useful first step. Adopting the notation

$$\langle F[\phi] \rangle = \int_C D\phi_1 D\phi_2 D\phi_3 F[\phi_1] \exp\{iS_C[\phi, j = 0]\} \tag{2.19}$$

for all $F[\phi]$, it follows from (2.17) that

$$p(\phi_v(t) = \tilde{\phi}) = \int d\alpha \exp\{-i\alpha v \tilde{\phi}\} \langle \exp\{i\alpha v \phi_v\} \rangle \tag{2.20}$$
The Gaussian approximation is obtained by curtailing the cumulant expansion
\[
\langle \exp\{i\alpha v\phi_v\}\rangle = \exp\{i\alpha v\langle \phi_v \rangle - \frac{1}{2}\alpha^2 v^2[\langle \phi_v \phi_v \rangle - \langle \phi_v \rangle^2] + O(\alpha^3)\}
\] (2.21)
at \(O(\alpha^2)\). Performing the \(\alpha\) integration gives
\[
p(\phi_v(t) = \bar{\phi}) = N \exp\{-\bar{\phi}^2/2\langle \phi_v \phi_v \rangle\}
\] (2.22)
where, for simplicity, we have assumed that \(\langle \phi_v \rangle = 0\). Not surprisingly, the variance in \(\bar{\phi}\) is given by the coarse-grained two-point correlation function.

More generally, we might wish to calculate the probability \(p(f_v[\phi] = \bar{f})\), where \(f_v[\phi]\) is some other coarse-grained functional of \(\phi\). In the same approximation
\[
p(f_v[\phi] = \bar{f}) = N \exp\{-\bar{f}^2/2\langle f_v f_v \rangle\}
\] (2.23)
again assuming \(\langle f_v \rangle = 0\) for simplicity.

## 3 The Initial Value Problem

As we have observed, the hard work has all been neatly tucked away in the boundary conditions. The familiarity of the path integral formalism makes it easy to lose the distinction between the contour \(C_3\) (the initial condition) and \(C_1(C_2)\) that carries the dynamics. However, for our first case, thermal equilibrium, the distinction has been intentionally removed.

### 3.1 Thermal Equilibrium.

This is the most extreme case, in which there is no dynamical evolution of the system. Although an unlikely occurrence in the very early universe, thermal equilibrium has been examined in great detail, in particular through the thermal effective potential \(V(\phi)\) [7]. It is the behaviour of \(V(\phi)\) that determines the order of the phase transitions in which we are interested. Our approach, given in detail elsewhere [8], is close to that of Jona-Lasinio [9], who defined the effective potential in terms of probabilities for coarse-grained fluctuations of the type proposed in the previous section. With the probability \(p(\phi_v(t) = \bar{\phi})\) now time-independent, the contours \(C_1\) and \(C_2\) can be shrunk to nothing, leaving the familiar Euclidean-time formalism on contour \(C_3\). We stress that, since thermal equilibrium corresponds to making the choice \(P_{in}[\Phi]\) of (2.14), where \(S_{in}[\phi]\) is now the classical action \(S[\phi]\) of the theory, we have no further freedom in our initial conditions. To cite one common misconception, we cannot now use the loop-expanded effective potential following from (2.14) as a 'classical' potential in which we subsequently consider the evolution of field configurations centred on metastable minima. Such an action corresponds to imposing two incompatible boundary conditions simultaneously. However, something different, but not wholly dissimilar, may make sense. [See Weinberg [10], these proceedings].

Let us return to \(P[\Phi]\) of (2.14), corresponding to equilibrium at temperature \(T = \beta^{-1}\). (With no evolution, all suffices have been dropped). Because of the reality of the integrand we can improve upon the Gaussian approximation. The integrated probability satisfies the Chebycheff inequality
\[
p(\phi_v \geq \bar{\phi}) = \int D\Phi \exp\{-\beta H[\Phi]\} \theta \left(\frac{1}{v} \int d\vec{x} \Phi(\vec{x}) - \bar{\phi}\right)
\] (3.1)
\[
\leq \int D\Phi \exp\{-\beta H[\Phi]\} \exp\{\beta \alpha \left(\frac{1}{v} \int d\vec{x} \Phi(\vec{x}) - \bar{\phi}\right)\}
\] (3.2)
for all $\alpha \geq 0$. Define $\omega_v(\alpha)$ by
\[
\exp\{\beta v \omega_v(\alpha)\} = \int D\Phi \exp\{-\beta H[\Phi] + \beta \int j \Phi\} \quad (3.3)
\]
where $j(\vec{x}) = \alpha I(\vec{x})$, as before. Then
\[
p(\phi_v \leq \bar{\phi}) \leq \exp\{\beta v (\omega_v(\alpha) - \alpha \bar{\phi})\} \quad (3.4)
\]
an inequality minimised for $\alpha = \bar{\alpha}$,
\[
\frac{\partial \omega_v(\bar{\alpha})}{\partial \alpha} = \bar{\phi}. \quad (3.5)
\]
That is
\[
p(\phi_v \leq \bar{\phi}) \leq \exp\{-\beta v V_v(\bar{\phi})\} \quad (3.6)
\]
where $V_v(\phi)$, the Legendre transform of $\omega_v(\alpha)$, is the coarse-grained effective potential.

More specifically, when $\beta \ll m$ we identify $H[\Phi]$ with $S_3[\Phi]$, the 3-dimensional action for the light Matsubara mode, obtained by integrating over all heavy modes (see Kajantie [11], these proceedings). Thus, but for light-mode self-interactions, $V_v(\bar{\phi})$ gives the usual thermal effective potential $V(\bar{\phi})$ (the convex hull of the loop-expanded potential) in the large-$v$ limit, when the inequality (3.6) becomes an equality. We note that, if we were considering a gauge theory, rather than just this simple scalar theory, the interesting features of the effective potential (e.g. the order of the transition) are largely determined by the light gauge field modes, and hence are correctly incorporated in $V_v$.

We shall be much simpler, initially restricting ourselves to a free scalar field, mass $m$, temperature $T \gg m$. If $\tilde{I}(\vec{k})$ is the Fourier transform of $I(\vec{x})$ (normalised to $\tilde{I}(\vec{0}) = v = O(l^3)$) then
\[
\langle \phi_v \phi_v \rangle = \frac{1}{v^2} \int \frac{d^3k|\tilde{I}(\vec{k})|^2G(\vec{k})}{} \quad (3.7)
\]
where
\[
G(\vec{k}) \simeq \frac{T}{\vec{k}^2 + m^2} \quad (3.8)
\]
is the free-field thermal propagator in this regime.

What we see from this is that, as we might have anticipated, the effect of coarse-graining is, through $|\tilde{I}(\vec{k})|^2$, to impose a cut-off at $|\vec{k}| < l^{-1}$. $\langle \phi_v \phi_v \rangle$ decreases from $T/m^2 v$ at large $v$ ($\tilde{I}(\vec{k}) \to \delta(\vec{k})$) to $T/Am^2 v = mT/A$ with $A \simeq 10$, when $l = \xi = O(m^{-1})$. That is, the variance in $\phi$ is
\[
(\Delta \phi)^2 \simeq m T/A \quad (3.9)
\]
for correlation-volume fluctuations. The value of $A$ depends a little on the shape of $V$.

What matters is that it reduces $(\Delta \phi)^2$ by an order of magnitude from our first guess [12]. This provides a very useful guide as to when fluctuations are important. [We note that $V_v(\phi)$ gets steeper as $v$ diminishes].

We are interested in phase transitions, and for a real scalar theory the only possibility is in the breaking of reflection invariance $\phi \to -\phi$. Let us now consider the case when the scalar field theory has broken symmetry when cold. We approximate $H[\Phi]$ by the one-loop form
\[
H[\Phi] = \int d\vec{x} \left[ \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{2} m^2 (T) \Phi^2 + \frac{1}{4} \lambda \Phi^4 \right]. \quad (3.10)
\]
The correlation length \( \xi = m(T)^{-1} \), where \( m(T) \) is the effective mass

\[
m^2(T) = m^2 \left(1 - \frac{T^2}{T_c^2}\right)
\]

(3.11)
diverges at the critical temperature \( T_c \), at which the theory undergoes a second-order transition.

It is not possible to calculate the coarse-grained \( V_v(\phi) \) of (2.29) analytically. In the symmetric phase \((T > T_c)\) perturbation theory is appropriate and we can expand about the Gaussian to include a quartic term \( \lambda_v \phi^4 \) in the coarse-grained potential \( V_v(\phi) \). Its strength \( \lambda_v \) is, not surprisingly, given in terms of the coarse-grained four-point correlation function \( \langle \phi_v \phi_v \phi_v \phi_v \rangle \). Very near to the phase transition the Gaussian approximation breaks down, but even then it remains a useful qualitative guide.

In the symmetry-broken phase perturbation theory breaks down, and even the Gaussian approximation is difficult. A Gaussian bound can be made by weakening the inequality (3.6) further by restricting the spatial integral to \( \vec{x} \in v \) in (3.10). Without worrying about the details the end result is the highly sensible bound [8]

\[
(\Delta \phi)^2 = \langle \phi_v \phi_v \rangle < \sigma^2 + \frac{1}{\beta v m^2 T}
\]

(3.12)

The first term on the right hand side of (3.12) describes the long-range field correlations in the symmetry-broken phase, the second the fluctuations about the minima at \( |\phi| = \sigma \). For correlation volumes \( v = O(\xi^3) = O(m(T)^{-3}) \) the free field result suggests that the inequality is improved by replacing \( T m(T) = (\beta v m^2(T))^{-1} \) by \( T m(T)/A \). When \( T m(T)/A = O(\sigma^2(T)) \) there is a significant probability that correlation-volumes of false vacuum can be created. This is more transparent if written as the constraint on the effective thermal coupling strength \( \lambda_3 \), that

\[
\lambda_3 = \lambda T/m(T) = O(1)
\]

(3.13)

This condition is seen to be just the Ginzburg criterion [13], the condition that the effective thermal coupling \( \lambda_3 \) is becoming large, and signalling that the one-loop form (3.10) is beginning to break down. [The failure of the Gaussian approximation in the symmetric phase is now seen to correspond to \( |\lambda_3| = O(1) \)].

If \( H[\Phi] \) describes a theory experiencing a first-order transition, as the EW transition is expected to do, the situation is more complicated. There has been an extensive discussion of the relative role of fluctuations versus quantum tunnelling by Gleiser, Kolb and others [14] (although our earlier caveat about incompatible boundary conditions is valid here). The consensus that thermal fluctuations are inadequate to populate the symmetry-broken vacuum state is largely confirmed by our analysis, which will be published elsewhere [15].

### 3.2 Non-Equilibrium Behaviour.

The picture that we have given above, in which correlation volumes of false vacuum are produced with significant probability in the Ginzburg regime demonstrates how large fluctuations can occur. However, it may be positively misleading. The expansion of the very early universe is so rapid that particles are separated from one another before they have undergone sufficient collisions to attain it. To exemplify the nature of out-of-equilibrium behaviour, and to offer the strongest contrast to thermal equilibrium, we
consider the case of rapid quenching of the scalar theory from an initially symmetric state at time $t = t_{\text{in}}$ to a symmetry-broken state, described by the classical action

$$S[\phi] = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \right].$$

(3.14)

More details are given by Lee (these proceedings, and work by Lee, Boyanovsky, de Vega [16]) in which a similar calculation is performed, and from which we borrow results. [Although motivated by early universe considerations, for simplicity we stay in flat spacetime. For a more realistic temporal evolution in the early universe, see de Vega [17] (these proceedings).]

At $t = t_{\text{in}}$ we would like the initial state to be symmetric, with $P_{\text{in}}$ strongly peaked about $\phi = 0$. Despite our caveats about an inability to define temperature in strongly non-equilibrium environments our earlier observations suggest that this is most simply achieved by taking a Boltzmann probability distribution $P_{\text{in}}[\Phi] = N \exp\{-\beta_{\text{in}} H_{\text{in}}[\Phi]\}$. $H_{\text{in}}[\Phi]$ is derived, as in (2.14), from an action

$$S_{\text{in}}[\phi] = \int dx \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_{\text{in}}^2 \phi^2 \right]$$

(3.15)

for some $m_{\text{in}}$. While $S_{\text{in}}$ is a phenomenological action that induces the required initial peaking, the effect is the same as if the system were in thermal equilibrium at temperature $T = T_{\text{in}} = \beta_{\text{in}}^{-1}$ for all time $t < t_{\text{in}}$. The dispersion in $\phi$ at this initial time is given by (31) as $(\Delta \phi)^2 = O(m_{\text{in}} T_{\text{in}})$ on coarse-graining, chosen as we wish. The parameter $\beta_{\text{in}}$ serves solely to fix this. For $t > t_{\text{in}}$ the evolution of the initial state is taken to be determined by $S[\phi]$ of (3.14 ). We take $t_{\text{in}} = 0$ for convenience.

In effect, we are taking the theory to be determined for all time by the action

$$S_{\text{f}}[\phi] = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2(t) \phi^2 + \frac{1}{4} \lambda(t) \phi^4 \right]$$

(3.16)

where

$$m^2(t) = \begin{cases} m_{\text{in}}^2 & t < 0 \\ -m^2 & t > 0 \end{cases}$$

(3.17)

and

$$\lambda(t) = \begin{cases} 0 & t < 0 \\ \lambda & t > 0 \end{cases}$$

(3.18)

subject to the condition that, for $t < 0$, it was in thermal equilibrium. In this extreme case it is apparent that the equilibrium thermal effective potential $V$ and its coarse-grained descendents $V_v$ have no role to play and it makes little sense to talk about the order of the transition.

We could have taken $\lambda$ constant for all time, but have not done so in order to delineate between its role in establishing the initial condition (for which it is an artefact) and its role in the subsequent dynamics, in which it is ultimately a constant of the universe. Unsurprisingly, taking interactions into account properly is hard. In practice, because of the failure of perturbation theory, it is difficult to do better than a self-consistent (Hartree) linearisation of (3.16). Despite our preamble, for our purposes it is sufficient to set $\lambda(t) = 0$ in (3.18) (whence the classical potential is an upturned parabola for $t > 0$) but only to consider the evolution of the system to times at which the field configuration would have spread from $\phi = 0$ to the point of inflexion $\phi_{\text{inf}} = O(m/\sqrt{\lambda})$. Since the characteristic growth rate is $\phi/m = e^{mt}$, this means times for which $t = O(m^{-1} \ln(1/\lambda))$. 

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The spinodal dispersion in \( \phi \) is now given by the ‘free-field’ equal time correlation function (explicitly dependent on \( t \))

\[
\langle \phi_v \phi_v \rangle = \frac{1}{v^2} \int d\bar{x} \, d\bar{x}' \, \langle \phi(t, \bar{x}) \phi(t, \bar{x}') \rangle
\]

\[= \frac{1}{v^2} \int d\bar{x} \, d\bar{x}' \, I(\bar{x}) I(\bar{x}') G(\bar{x} - \bar{x}'; t)
\]

\[= \frac{1}{v^2} \int d\bar{k} \, |I(\bar{k})|^2 G(\bar{k}; t)
\]

in terms of the Fourier transform

\[G(\bar{k}; t) = \int d\bar{x} G(\bar{x}; t) e^{-i\bar{k} \cdot \bar{x}}.
\]

At equal-time all correlation functions are the same, once the boundary condition \( G(\bar{x}; t_m) = G(\bar{x}; t_m - i\beta_m) \) has been implemented.

\( G(\bar{k}; t) \) possesses both oscillatory modes and exponentially growing (fading) modes, as solutions to

\[(\partial_t^2 + \bar{k}^2 + m^2(t))U_0(t) = 0
\]

from which it is constructed. The dominant contributions to (3.21) are the exponential modes, and if we take them only then

\[G(\bar{k}; t) \simeq \frac{\theta(m^2 - \bar{k}^2)}{2\omega_{in}(\bar{k})} \left( 1 + \frac{1}{2} \left( \frac{m^2 + m_{in}^2}{m^2 - \bar{k}^2} \right) (\cosh(2t\sqrt{m^2 - \bar{k}^2}) - 1) \right) \coth\left[ \frac{1}{2} \beta_m \omega_{in}(\bar{k}) \right]
\]

where \( \omega_{in}(\bar{k}) = \sqrt{\bar{k}^2 + m_{in}^2} \). If we consider fluctuations coarse-grained to volumes \( v = O(l^3) \) the momentum is cut off at \( |\bar{k}| = O(l^{-1}) \), whereas the \( \theta \)-function imposes a cut-off at \( |\bar{k}| = m \). Since we are only interested in \( l \geq m^{-1} \) the \( \theta \)-function is irrelevant. On converting to spherical polars in (3.22) \( k^2 G(\bar{k}; t) \) has, for large values of \( mt \), a sharp peak at \( k^2 = O(m/t) \). For this to be inside the integral, so that the probability is significant, requires \( l^2 < t/m \). Thus the field is correlated on a scale \( \xi = O(\sqrt{t/m}) \). At the relevant time \( t = m^{-1} ln(1/\lambda) \) at which domain formation rapidly slows to a halt, the domain size is then

\[\xi = O(m^{-1}(ln(1/\lambda))^{\frac{1}{2}})
\]

This is in contrast to \( \xi = O(m^{-1}/\sqrt{\lambda}) \) in the equilibrium case at the Ginzberg temperature (3.13). In practice, unless \( \lambda \) is extremely small the sizes can be comparable when prefactors are taken into account properly. Further details are given by Lee.

4 Vortex Creation from Fluctuations.

Consider a field theory invariant under a group \( G \). Suppose the effect of spontaneous symmetry-breaking is to reduce \( G \) to its subgroup \( H \). The vacuum manifold \( M \) can be identified with the coset space \( G/H \). If \( M \) has a non-trivial first homotopy group \( \Pi_1(M) \) (i.e. the embeddings of loops in \( M \)), then vortices (strings) can form [1]. If the second homotopy group \( \Pi_2(M) \) is non-trivial we can have monopoles, and if the third homotopy group \( \Pi_3(M) \) is non-trivial we can have textures. As we said earlier, the production of vortices in the early universe has the capacity to seed large-scale structure through their gravitational effects. In general, GUT vortices are assumed to arise from the breaking of
a local gauge theory. However, because of their relative simplicity, a considerable effort has gone into understanding global vortices, and it is they that we shall mainly consider here. Given that the vortices of superfluid $^4$He are global vortices, they are of more than theoretical interest.

The simplest theory permitting such vortices is the global $U(1)$ theory of a complex scalar field $\phi$, with Minkowski space-time action

$$S[\phi] = \int d^4x \left[ \frac{1}{2}(|\partial\phi|^2 + \frac{1}{2}m^2|\phi|^2 - \frac{1}{4}\lambda|\phi|^4). \right] \tag{4.1}$$

With $m^2 > 0$ the $U(1)$ symmetry is spontaneously broken, and the classical vacuum determined by $|\phi|^2 = \sigma^2 = m^2/\lambda$. The vacuum manifold $M = S^1$ has homotopy group $\Pi_1(S^1) = Z$, and vortices can exist with integer winding number $n$, Vortices with winding number $|n| > 1$ are unstable to decay into vortices with $|n| = 1$.

Consider an open surface $S$ with oriented boundary $\partial S$. The line integral

$$N_S(t) = -\frac{i}{2\pi} \int_{\partial S} d\sigma \frac{\bar{\phi} \partial^\dagger \bar{\phi}}{|\phi|^2} \tag{4.2}$$

measures the winding number of the field configuration on $S$ at time $t$. (i.e. $2\pi N_S$ is the change in phase of the field $\phi$ as it is taken once around $\partial S$).

If $N_S \neq 0$, continuity of $\phi$ requires that it vanish at some point or points of $S$. A vortex of winding number $n$ is a tube of 'false' vacuum ($\phi \approx 0$) containing a line of zeros of $\phi$ for which $N_S$ of (4.2) has value $n$ for any closed path enclosing this line. For a large loop $\partial S$, $N_S$ measures the net winding number of the vortices that pass through it. At a distance $r$ from the vortex centre $|\phi|$ approaches its vacuum value $|\phi| = |\sigma(1 - O(e^{-mr})|$. In practice it is more convenient to evaluate the related quantity

$$\bar{N}_S(t) = -\frac{i}{2\pi} \int_{\partial S} d\sigma \frac{\bar{\phi} \partial^\dagger \bar{\phi}}{|\phi|^2} \tag{4.3}$$

$$= -\frac{i}{2\pi |\sigma|^2} \int_S d^2S' (\partial\bar{\phi}^\dagger \partial\phi) \frac{\sigma}{|\phi|} \tag{4.4}$$

$$= \frac{2}{2\pi |\sigma|} \int_S \partial^S \cdot (\partial\rho \wedge \partial\chi) \tag{4.5}$$

where, in (3.5), we have used the radial/angular decomposition

$$\phi = \rho e^{i\chi} \tag{4.6}$$

of the field. For a large loop $\partial S$ the difference between $N_S$ and $\bar{N}_S$ (not integer) is vanishingly small if no vortices pass close to $\partial S$, and $\bar{N}_S$ remains a good indicator of vortex production.

On further decomposing the radial field $\rho$ as $\rho = |\sigma| + h$ for Higgs field $h$, $S$ of (4.1) becomes

$$S[h, \chi] = \int dx \left[ \frac{1}{2}(\partial h)^2 + \frac{1}{2}\sigma^2(\partial\chi)^2 - m^2 h^2 - \lambda\sigma h^3 - \frac{1}{4}\lambda h^4 \right] \tag{4.7}$$

In the Gaussian approximation the probability that $\bar{N}_S(t)$ takes the value $n$ is, from (2.23)

$$p(\bar{N}_S(t) = n) = \exp\left\{ -n^2/2\langle\bar{N}_S(t)\bar{N}_S(t)\rangle \right\} \tag{4.8}$$

On further defining the Goldstone mode $g$ by $g = \sigma\chi$, from (4.5) it follows that

$$\langle\bar{N}_S(t)\bar{N}_S(t)\rangle = \left(\frac{2}{2\pi |\sigma|^2}\right)^2 \int_S d^2S' d^2S'' \langle(\partial h' \wedge \partial g')(\partial h'' \wedge \partial g'')\rangle \tag{4.9}$$
All fields are at time $t$. The primes (doubleprimes) denote fields in the infinitesimal areas $dS', dS''$ of $S$ respectively. For economy of notation we have not made the scalar products explicit. Without loss of generality we take $S$ in the 1-2 plane, whence

$$\langle \bar{N}_S(t) N_S(t) \rangle = \left( \frac{2}{2\pi \sigma^2} \right)^2 \int dS' dS'' \langle \partial_i h' \partial_j h'' \partial_j g' \partial_j g'' - \partial_i h' \partial_j h'' \partial_j g' \partial_j g'' \rangle \quad (4.10)$$

It is convenient to refine our notation further, decomposing space-time as $x = (t, \bar{x}) = (t, \bar{x}_L, x_T)$ where $\bar{x}_L = (x_1, x_2)$ denotes the co-ordinates of $S$, and $x_T = x_3$ the transverse direction to $S$. Similarly, we separate 4-momentum $p$ as $p = (E, \bar{p}_L, p_T)$. .

Let $G_h(t, \bar{x}' - \bar{x}'') = \langle h(t, \bar{x}) h(t, \bar{x}'') \rangle$, $G_g(t, \bar{x}' - \bar{x}'') = \langle g(t, \bar{x}) g(t, \bar{x}'') \rangle$ be the Higgs field and Goldstone mode correlation functions respectively. As a first step we ignore correlations between Higgs and Goldstone fields. That is, we retain only the disconnected parts of $\langle \bar{N}_S N_S \rangle$. Eqn. (4.10) then simplifies to

$$\langle \bar{N}_S(t) N_S(t) \rangle = \left( \frac{2}{2\pi \sigma^2} \right)^2 \int dS' dS'' \langle \partial_i h' \partial_i h'' \rangle \langle \partial_j g' \partial_j g'' \rangle$$

which can be written as

$$\langle \bar{N}_S(t) N_S(t) \rangle = \left( \frac{2}{2\pi \sigma^2} \right)^2 \int d^3 p^i d^3 p'' G_h(\bar{p}', t) G_g(\bar{p}'', t) |\bar{I}(\vec{p}_L'')|^2 |(\vec{p}'')^2 - (\vec{p}_L'')^2|^2$$

(4.12)

In (4.12) $\bar{I}(\vec{p}_L)$ is the Fourier transform of the window function $I(\bar{x}_L)$ of the surface $S$ (i.e. $I(\bar{x}_L) = 1$ if $\bar{x}_L \subset S$, otherwise zero). The $G(\vec{p}, t)$ are defined as in (3.22).

We coarse-grain in the transverse and longitudinal directions by imposing a cut-off in three-momenta at $|p_i| < \Lambda = l^{-1}$, for some $l$, as before. Thus $\bar{N}_S$ is now understood as the average value over a closed set of correlation-volume 'beads' through which $\partial S$ runs like a necklace. However, we leave the notation unchanged.

For large loops $\partial S$, $\bar{I}(\vec{q}_L) \simeq \delta(\vec{q}_L)$, enabling us to write

$$\langle \bar{N}_S(t) N_S(t) \rangle = \left( \frac{2}{2\pi \sigma^2} \right)^2 \int dq_T d^3 p \ G_h(\vec{p} + \vec{q}_T, t) G_g(\vec{p}, t) \int d^2 q_L |\bar{I}(\vec{q}_L)|^2 |(\vec{p}_L)^2 (\vec{q}_L)^2 - (\vec{p}_L \cdot \vec{q}_L)^2|^2$$

(4.13)

By $\vec{p} + \vec{q}_T$ we mean $(\vec{p}_L, p_T + q_T)$. The dependence on the contour $\partial S$ is contained in the final integral

$$\mathcal{J} = \int d^2 q_L \left| \bar{I}(\vec{q}_L) \right|^2 |(\vec{p}_L)^2 (\vec{q}_L)^2 - (\vec{p}_L \cdot \vec{q}_L)^2|^2$$

(4.14)

$$\simeq \pi p_L^2 \int^{\Lambda} dq_L q_L^2 |\bar{I}(q_L)|^2$$

(4.15)

If this is evaluated for a circular loop of radius $R$, we find

$$\mathcal{J} = p_L^2 O(2\pi R/l)$$

(4.16)

as we might have anticipated. The rms winding number behaves with path length as $\Delta n \propto \mathcal{J}^{\frac{1}{2}} = O(L^{\frac{3}{2}})$, where $L$ is the number of steps of length $l$. This is consistent with the coarse-grained field phases of different volumes $v$ being randomly distributed.

The final step is to relate the magnitude of the fluctuations in winding number $N_S$ to the magnitude of the radial (Higgs) field fluctuations and angular (Goldstone) field fluctuations. To see this requires a specific choice of initial conditions, and we repeat those of the previous section, thermal equilibrium and instantaneous quenching.
4.1 Thermal Equilibrium.

To estimate the thermal fluctuations in $\bar{N}_S$ in equilibrium we (a) neglect the positivity of $\rho$ and the Jacobean from the non-linear transformation (4.6) and (b) the non-singlevaluedness of $\chi$. While valid for small fluctuations around the global minima this can only be approximate for large fluctuations. With this proviso, at temperature $T$ the equilibrium propagators are time-independent, read off from (3.7) as

$$G_h(\vec{p}) = \frac{T}{\vec{p}^2 + m_H^2(T)} \quad (4.17)$$

$$G_g(\vec{p}) = \frac{T}{\vec{p}^2} \quad (4.18)$$

where $m_H(T) = \sqrt{2}m(T)$ is the effective Higgs mass at temperature $T$. If we take $\bar{p}_L^2 = 2\bar{p}^2/3$ in the integral then, up to numerical factors, we can approximate (3.10) as

$$\langle \bar{N}_S \bar{N}_S \rangle \simeq J \left( \frac{2}{2\pi \sigma^2} \right) m(T) T \int_{|\vec{p}|<m(T)} d\vec{p} G_h(\vec{p}) \quad (4.19)$$

where we have coarse-grained to the Higgs correlation length $\xi = m_H^{-1}(T)$. [We have further assumed that the $q_T$ integration can be approximated by setting $q_T$ to zero in the integrand. Qualitatively this is a reasonable simplification]. The integral in (4.19) is essentially the integral (3.21). The end result is that, after substitution,

$$\langle \bar{N}_S \bar{N}_S \rangle = O \left( \left( \frac{m_H(T)^2}{\sigma^2(T)} \right)^2 \right) O(L) \quad (4.20)$$

where $L$ is the length of the path in units of $\xi$. Equivalently, on using our previous results for equilibrium

$$\langle \bar{N}_S \bar{N}_S \rangle = O \left( \left( \frac{\langle h_v h_v \rangle}{\sigma^2(T)} \right)^2 \right) O(L) \quad (4.21)$$

One power of $\langle h_v h_v \rangle$ comes from $G_h$, the other from residual factors. That is, the phase fluctuations are scaled by the Higgs fluctuations. In the Ginzburg regime, when correlation-volumes of the Higgs field can fluctuate to the false vacuum with significant probability, the fluctuations in field phase on the same distance scale are of order unity and, from the $O(L)$ term, are distributed randomly. [It is unclear whether the inclusion of connected correlation functions - linking the Higgs and Goldstone fields - would change the results qualitatively in this regime, and is under examination. The inclusion of four-point correlation functions in the real scalar field calculations had no dramatic effect, although the circumstances were somewhat different.] This ability not to just produce ‘beads’ of false vacuum, but to string them as vortices, is the Kibble mechanism referred to earlier [1], a common starting-point for numerical calculations [18].

A similar analysis can be performed for $^4He$, treated as a quantum-mechanical system of non-relativistic bosonic point-particles with pair-wise potentials $V(\vec{x}_i - \vec{x}_j)$. In thermal equilibrium it is well-known [18] that the grand canonical partition function for such a system is identical to the partition function

$$Z = \int D\Phi \ exp\{-S[\Phi]\} \quad (4.22)$$
of a complex non-relativistic quantum field $\Phi$. The action in Euclidean space-time (with Euclidean time $0 \leq \tau \leq \beta$) for particles of mass $m$ and (effective) chemical potential $\mu$ is

$$S[\Phi] = \int d^4x \Phi^\dagger(x) \left( -\frac{1}{2m} \nabla^2 - \mu + \frac{\partial}{\partial \tau} \right) \Phi(x) + \frac{1}{2} \int d^4x \int d^4x' |\Phi(x)|^2 V(x-x') |\Phi(x')|^2$$

(4.23)

$V(x) = \delta(t)V(\vec{x})$ is the time-instantaneous generalisation of the two-body potential introduced earlier.

Equation (4.23) is exact. If we adopt a local approximation for the two-body forces that enables us to replace the final term in (4.23) by $\lambda |\Phi|^4$, and integrate out the $\tau$-dependent modes, the end result is a three-dimensional Landau-Ginzburg theory with effective Hamiltonian

$$H[\Phi] = \int d\vec{x} \left[ \frac{1}{2m} |\nabla \Phi|^2 - \mu(T)|\Phi|^2 + \frac{1}{2} \lambda |\Phi|^4 \right].$$

(4.24)

like that of (3.10). At this level of approximation, in which Euclidean time has disappeared, the distinction between relativistic and non-relativistic theories has also disappeared (except for the details of $\mu(T)$). But for a slight change of definition, repeating our analysis will give a variance $(\Delta n)^2$ in winding number (along $\partial S$) proportional to the path length and to the radial field fluctuation. In thermal equilibrium there is, therefore, a true identity between the vortex fluctuations of a relativistic scalar field of the type that could occur in the early universe and those of superfluid $^4$He. In the Ginzburg regime, when a random phase distribution is relevant, the vortex density will be large.

As a further example of scalar fields in thermal equilibrium we observed earlier that if $\Pi_2(M)$ is non-trivial we can have monopoles. In a cosmological context they are an embarrassment [1], since they would most likely dominate the energy of the universe if produced at the GUT phase transition and allowed to survive. As a final generalisation of our result (4.21) for winding number we note that it is straightforward to repeat our analysis for the production of global monopoles. Consider the simplest case of a vector triplet of scalar fields in a broken global $O(3)$ theory. The variance in the monopole charge $Q_v$ in a volume $v$ is, in the same approximation as the above, a product of three two-point correlation functions. The end result is [20] that, in thermal equilibrium at temperature $T$, for coupling constant $\lambda$ and effective Higgs mass $m(T)$,

$$\langle Q_v Q_v \rangle = O((\lambda T/m(T))^3)O((m(T)^3 v)^{2/3})$$

(4.25)

where $m^3 v$ is the volume $v$ measured in terms of correlation volumes, correlation length $\xi = m(T)^{-1}$. In the Ginzburg regime this is understood as the field taking random directions in field space in different correlation volumes.

### 4.2 Out-Of-Equilibrium Behaviour.

Our discussion above has shown that, in thermal equilibrium close to the phase transition, field fluctuations are large enough to produce local topological charge (winding number, monopole charge) at high density, as anticipated by Kibble. However, this is of little use unless the future evolution of the system is able to freeze this charge in. Let us return to relativistic vortices. The random distribution of field phases shown above leads to macroscopic vortex fluctuations, crossing the system volume $V$. Numerical simulations
may take an initial configuration of vortices based on random phases and then, switching to zero (or low) temperature solve for their classical evolution. This freezing in of defects goes beyond anything that we have said so far. As with the real scalar field of the previous section it is simplest to adopt a quenched approximation in which the closed time-path action for \( t > 0 \) is \( S \) of (4.1). In the light of our previous comments one of the more interesting choices would be to take the initial state as one of thermal equilibrium in the Ginzburg regime with its large fluctuations and watch to see if the vortices freeze. However, a simpler, and probably more relevant, choice is to move instantaneously from a symmetric to a spontaneously-broken theory, rather as in (3.16), but with the differences outlined below.

Again we assume an initial symmetric state, centred at \( \phi = 0 \). Our first guess might be to take field configurations Boltzmann-distributed as

\[
P(\Phi) = \frac{N}{\exp \left\{ -\frac{\beta}{2} H(\Phi) \right\}}
\]

where

\[
H(\Phi) = \int d^4x \left\{ \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} m^2 \phi^2 \right\}
\]

(4.26)

the \( U(1) \) generalisation of (2.38). However, when (4.22) is decomposed into radial and angular fields as

\[
S_{in}[\rho, \chi] = \int d^4x \left\{ \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} \rho^2 (\partial \chi)^2 - \frac{1}{2} m^2 \rho^2 \right\}
\]

(4.27)

the coupling between them (the second term) cannot be ignored since \( \langle \rho \rangle = 0 \). In the absence of any compelling reason (initial thermal equilibrium in the symmetric phase is hardly likely in the universe, although it would be appropriate for superfluid \(^4\)He) we may as well make a choice of initial condition that is solvable. In this spirit, we take

\[
S_{in}[\rho, \chi] = \int d^4x \left\{ \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} \sigma^2 (\partial \chi)^2 - \frac{1}{2} m^2 \rho^2 \right\}
\]

(4.28)

where \( \sigma^2 = m^2 / \lambda \) determines the vacuum in the subsequent symmetry-broken phase.

This \( S_{in} \), which characterises the initial probabilities, is to be contrasted to the action which determines the subsequent spinodal evolution of the field. A similar problem arises for times \( t > t_m \). Our first thought, the quadratic (in \( \phi \)) action with downturned potential

\[
S[\rho, \chi] = \int d^4x \left\{ \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} \rho^2 (\partial \chi)^2 + \frac{1}{2} m^2 \rho^2 \right\}
\]

(4.29)

with \( m_H = \sqrt{2} m \), involves interactions between the Higgs and Goldstone fields after the radial/angular field decomposition. In the spirit of the Gaussian approximation we replace it by

\[
S[\rho, \chi] = \int d^4x \left\{ \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} \sigma^2 (\partial \chi)^2 + \frac{1}{2} m^2 \rho^2 \right\}
\]

(4.30)

in which the decoupling between \( \rho \) and \( \chi \) is enforced by putting \( \rho^2 = \sigma^2 \) in the second term of (4.29) The identity of the massless \( \chi \) action in (4.28) and (4.30) guarantees that it remains in thermal equilibrium at all times. Although \( S \) of (4.30) is artificial, it is realistic in that we would not expect the \( \chi \) modes to be dominantly exponential, but oscillatory. This is in contrast to the quenched behaviour of the radial Higgs field \( \rho \) as it falls off the hill, with its exponentially growing long-wavelength fluctuations. The Goldstone field, always being massless, possesses the IR singularity for small \( \vec{p} \),

\[
G_\chi(\vec{p}, t) \approx \frac{T_m}{\vec{p}^2}
\]

(4.31)
With this $1/p^2$ effectively cancelling the $p^2$ in $J$, $\langle \bar{N}_S(t)\bar{N}_S(t) \rangle$ behaves as

$$\langle \bar{N}_S(t)\bar{N}_S(t) \rangle \simeq J \left( \frac{2}{2\pi\sigma^2} \right) mT_{in} \int_{|\vec{p}|<m} d\vec{p} G_\rho(\vec{p},t)$$

$$= O(L) \left( \frac{2}{2\pi\sigma^2} \right) mT_{in} \langle \rho_v(t)\rho_v(t) \rangle$$

(4.33)

where $L$ is the path length in units of $l$ which, now, we take to be $\xi$ of (3.25).

As before, the fluctuations in winding number vary as the square root of the step number, a consequence of random phase fluctuations, with scale set by the fluctuations of the Higgs field at the cessation of domain growth. However, with $\rho$ positive there is now only one side of the hill to roll down. The non-vanishing of $\langle \rho_v \rangle$ prevents the Higgs two-point correlation function being identical to that of the scalar field in (3.21) and (3.24). However, with $\langle \rho_v \rangle$ itself showing exponential growth the effect may be qualitatively the same. Details will be given elsewhere [21]. How these domains then aggregate to produce the true vacuum is beyond our calculations, as is the subsequent evolution of the vortex network. What is interesting is the similarity between the equilibrium result (4.21) and (4.33). Whether the vortices are produced in equilibrium or strongly out of equilibrium the Kibble conjecture seems substantially correct.

Global monopoles can be treated in the same way. If the Goldstone modes are allowed to stay in equilibrium the fluctuations in monopole charge retain the equilibrium volume dependence of (4.25), scaled by the Higgs fluctuations. This is not necessarily the case for $^4He$. The form (4.23) for the $^4He \Phi$-field partition function relied on thermal equilibrium for the particle system to be identical to a field theory. We do not know to what extent a non-equilibrium system of point particles can be put into correspondence with non-equilibrium field theory. However, if it were possible to take a real-time continuation of (4.23) as a basis for non-equilibrium calculations there would be no difficulty in principle in adopting the same approximations, although the difference between non-relativistic and relativistic thermal field theories would now have to be addressed.

### 4.3 Local Vortices

Our experience of continuous symmetries in particle physics has been that all such symmetries have been made local by the presence of gauge fields. In that sense the global vortices that we have been discussing are unnatural.

The local gauge extension of the $U(1)$ theory (4.1) has action

$$S[\phi, A] = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |\partial_\mu \phi - ieA_\mu|^2 + m^2|\phi|^2 - \frac{1}{4}\lambda|\phi|^4 \right].$$

(4.34)

(changing factors of $\frac{1}{2}$ for convenience). This still permits relativistic vortices, the simplest candidates for local cosmic strings. [Whereas the non-relativistic global strings are the vortices of superfluids, the non-relativistic counterparts of local strings are the vortices of superconductors]. For $e^2/\lambda \ll 1$ we have a Type-II theory and the strings are approximately global, and our previous results should apply.

The gauge-invariant expression (4.2) for winding number is still valid, but it is more convenient to define it through the gauge field $A_\mu$ as the line integral

$$N_S(t) = \frac{e}{2\pi} \int_{\partial S} dl A$$

(4.35)
As before, field fluctuations will create local winding number. In the Gaussian approximation its variance is

\[ \langle N_S(t)N_S(t) \rangle = \left( \frac{e}{2\pi} \right)^2 \int_{\partial S} dl_i' \int_{\partial S} dl_j'' \langle A_i(t, \vec{x}') A_j(t, \vec{x}'') \rangle \] (4.36)

where \( \vec{x}', \vec{x}'' \) denote the positions of the line increments on \( \partial S \). In one sense this is significantly simpler than its scalar counterpart (4.10) since we do not have to worry about disconnected and connected parts. The difficulty lies in the more complicated correlation function. There is no intrinsic problem in calculating \( \langle N_S N_S \rangle \) in thermal equilibrium. At the same level of approximation as before \( \langle N_S(t)N_S(t) \rangle = O(e^2LT) \), up to infrared logarithms. This shows the usual \( O(L) \) behaviour and, in step lengths \( \xi = m_v^{-1} = (e\sigma(T))^{-1} \), \( \langle N_S(t)N_S(t) \rangle \) shows large steps in winding number in the Ginzburg regime. Further, since the winding number is essentially the magnetic flux a similar calculation can be performed for estimating primordial magnetic fields [2]. However, as of this moment the work is not complete, but we hope to give the results later. This seems a good place to stop.

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References

[1] T. W. B. Kibble, J. Phys. A9, 1387 (1976).

[2] K. Enqvist, Magnetic fields of electroweak origin, these proceedings.

K. Enqvist and P. Oleson, Phys. Lett. B319, 178 (1993)

[3] P. C. Hendry, N. S. Lawson, R. A. M. Lee, P. V. E. McClintoch and C. D. H. Williams, Nature 368, 315 (1994)

[4] W. H. Zurek, Nature 368, 292 (1994)

[5] G. W. Semenoff and N. Weiss, Phys. Rev. D31 (1985) 689; Phys. Rev. D31 (1985) 699.

[6] I. D. Lawrie, Phys. Rev. D40 (1989) 3330; J. Phys. A25 (1992) 2493.

[7] D. A. Kirzhnits and A. D. Linde, Phys. Lett. 42B, 471 (1972). L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974); S. Weinberg, Phys. Rev. D9, 3357 (1974).

[8] M. B. Hindmarsh and R. J. Rivers, Nucl. Phys. B417, 506 (1994).

[9] G. Jona-Lasinio, in Scaling and Self-Similarity in Physics ed. J. Frolich, Progress in Physics, Vol.7, 11 (Birkhauser Press, 1983).

[10] E. Weinberg, Bubble nucleation rates, these proceedings.
[11] K. Kajantie, *Three dimensional physics and the EW phase transition*, these proceedings.

[12] M. Dine, R. G. Leigh, P. Huet, A. Linde and D. Linde, Phys. Rev. *D46*, 550 (1992).

[13] V. I, Ginzburg, Fiz. Tverd. Tela 2; 2031 (1960); [Sov. Phys. Solid State 2; 1826 (1961)].

[14] M. Gleiser, Phys. Rev. *D42*, 3350 (1990), M. Gleiser, E. W. Kolb and R. Watkins, Nucl. Phys. *B364*, 411 (1991).

[15] L. Bettencourt and R. J. Rivers, Imperial College preprint, in preparation.

[16] D. Boyanovsky and H. J. de Vega, Phys. Rev *D47*, 2343 (1993); D. Boyanovsky, Da-Shin Lee and A. Singh, Phys. Rev. *D48*, 800 (1993).

[17] H. J. de Vega, *Quantum non-equilibrium evolution and the phase transition in the early universe*, these proceedings.

[18] T. Vachasparti and A. Vilenkin, Phys. Rev. *D10*, 2036 (1984).

[19] F. Wiegel, *Introduction to Path-Integral Methods in Physics and Polymer Science* (World Scientific, Singapore, 1986).

[20] A. Gill, Imperial College preprint, in preparation.

[21] T. S. Evans and R. J. Rivers, Imperial College preprint, in preparation.