Comparing concurrence and negativity in time-dependent ionic-phononic system with fifteen dimensional density matrix

Rasim Dermez$^{1,2,3}$

$^1$Department of Physics, Afyon Kocatepe University, Afyonkarahisar 03200, Turkey
$^2$Department of Electrical and Computer Engineering, Michigan Technological University, Houghton, Michigan 49931, USA
$^3$Afyon Vocational School, Afyon Kocatepe University, Afyonkarahisar 03200, Turkey

E-mail: dermez@aku.edu.tr

Abstract. It is solved a time-dependent Hamiltonian using a unitary transformation method which $\Lambda(t)$ type is used to engineer a cascade $\Xi$ scheme interaction between the vibrational phonons and trapped three-level ion. Quantum entanglement is characterized by comparing concurrence and negativity of the time-dependent ionic-phononic system. In this quantum system, we obtain that the amount of concurrence can be tuned between 0 and 0.99 while the amount of negativity changes between 0 and 0.49.

1. Introduction

The importance of quantum entanglement has particularly increased during the last thirty years. Doubtless, quantum entanglement is a distinct quantum mechanical correlation between subsystem. Measurements on an entangled state cannot be determined by classical correlations. Entangled state has been shown by different physical setups such as photon [1]. Nonclassical correlations have been explained in the EPR paradox [2], which are related to the Bell theorem [3]. Abdel-Aty focuses on the quantum entanglement of the electronic and vibrational modes of trapped ion [4]. Thus, the quantum entanglement may be modelled physically by the states of a single and three-level trapped ions [5]. It plays in quantum optics, the physics of multilevel systems can be helpful in developing quantum information theory [6]. Furthermore, entangled pure states can be produced by lasers and trapped three-level ions [7, 8]. Qudit entangled states in second order corrections can help to answer some important questions about quantum linearity [9]. Dermez et. al. have worked out the quantum correlations of a full trapped ion interacting with two stationary laser beams [10] and the three-different pure-state entanglement in the time-dependent ionic-phononic system [11]. It considered quantum dynamics of a three-level trapped ion using the quantum entropy of the system with time-dependent Rabi frequency [12].
We propose a scheme to generate quantum entanglement among time-dependent ionic-phononic system. In addition to, we determine the degree of entanglement using the concurrence and negativity which is extended with respect to weak excitation regime [13].

2. Dynamics of time-dependent ionic-phononic system

We assume the lower levels of the three-level ion, $r$ and $g$ are degenerate and the beams are of different polarizations so that each beam can drive only one transition. There are three crucial time-dependent parameters: Rabi frequency $\Omega(t)$, harmonic trap frequency $\nu(t)$, and Lamb-Dicke (LD) parameter $\eta(t)$. The time-dependent Rabi frequencies of the corresponding transitions $e-g$ and $e-r$, where $e$ represents the excited level of the ion, are also considered to be the same and denoted by $\omega_1 = \omega_2$ (see Fig. 1). Two time-dependent laser beams are characterized as their frequencies $\omega_1$ and $\omega_2$. The parameters obtained for Figure 1 are $\omega = \omega_1 = \omega_2$ and $\Omega(t) = \Omega_1(t) = \Omega_2(t)$. The annihilation ($a$) and creation ($a^\dagger$) operators of the vibrational phonons such that $a = (\nu(t)/2\hbar)^{1/2}(x + ip/m \nu(t))$, where $\nu(t)$ is the harmonic trap frequency, $x$ is the direction along the trap axis, $m$ is the mass of the ion, and $p$ is the momentum of the center of mass of the ion.

![Figure 1](image.png)

**Figure 1.** The graphic of time dependent ionic-phononic system. It is shown that excited level ($e$), ground lever ($g$) and the Raman level ($r$) become the intermediate and upper levels of the cascade, respectively. The other parameters are taken to be $\Omega(t) = \Omega_1(t) = \Omega_2(t)$, $\omega = \omega_1 = \omega_2$, $\delta = \delta_1 = \delta_2$, $\omega_1 = \omega_{eg} - \delta_1$, and $\omega_2 = \omega_{eg} - \delta_2$.

Our system has been developed for studying macroscopic quantum superposition states, or so called Schrodinger cat states, of the vibrational phonons [13]. A scheme of optical transitions is described by a total Hamiltonian. The total Hamiltonian for the time-dependent ionic-phononic quantum system is $H(t) = H_{\text{ion}}(t) + H_1(t) + H_2(t)$. The
trapped ion Hamiltonian and the interacting Hamiltonians are given as ($\hbar = 1$)

$$H_{\text{ion}}(t) = \frac{p^2}{2m} + \frac{1}{2}m\nu^2(t)x^2_{\text{ion}} + \omega_{eg}|e\rangle\langle e|$$  

$$H_1(t) = \frac{\Omega_1(t)}{2}\exp(i(k_1x_{\text{ion}} - \omega_1t)|e\rangle\langle g| + h.c.$$  

$$H_2(t) = \frac{\Omega_2(t)}{2}\exp(-i(k_2x_{\text{ion}} - \omega_2t)|e\rangle\langle r| + h.c.)$$

where both wavelength of laser beams $\lambda = \lambda_1 = \lambda_2$, are slightly red detuned from the $e - g$ transition angular frequency of the ion $\omega_{eg}$ by the same amount $\delta = \nu(t)\eta^2(t)$, so that $\omega = \omega_{eg} - \delta$. $\eta(t) = \frac{k}{\sqrt{2\nu(t)}}$ is the time-dependent LD parameter and $k = \frac{2\pi}{\lambda}$.  

Additionally, the weak excitation regime is received to satisfy $\nu(t) = \frac{\Omega(t)}{2}$. It used growing time-dependent Rabi frequency $\Omega(t) = \sin\theta(\omega_{eg}t)$ [4]. In the weak excitation regime, above optical $\Lambda$ scheme has been shown to be equivalent to a cascade $\Xi$ scheme for the vibrational phonon transitions, under a unitary transformation, characterized by the transformation matrix $U$ [13]. The time evolution of an initial state $\psi(0)$ is written as:

$$|\psi(t)\rangle = U|\psi(0)\rangle,$$

where $K(t)$ is the well-known the propagator for the cascade Hamiltonian, $\hbar = 1$, and $\exp[-\int i\tilde{H}_0(t)dt]$ are the interaction picture transformation. A given state of the vibrational phonons and the ion evolves in the $\Xi$ configuration according to the propagator $K(t)$ in the interaction picture [13]. Initial state for the ionic-phononic system becomes:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[(|g\rangle - |r\rangle) \otimes (|0\rangle + |1\rangle)].$$

With acting the propagator $K(t)$ to the initial state of the cascade [i.e., $U^\dagger|\psi(0)\rangle$], we obtain the state $|\psi_K(t)\rangle = \sum_{ip}(A_{ip}(t)|i,p\rangle)$ where $i$ stands for the electronic state (e, r, g) and $p$ stands for the vibrational quantum number (0, 1, 2, 3, 4). The Hilbert space $H = H_e \otimes H_p$ of dimension is 15. Thus, we need the fifteen probability amplitudes. Two version of the new final state can be written as

$$|\psi(t)\rangle = \sum_{ip}A_{ip}(t)|i,p\rangle,$$

$$|\psi(t)\rangle = A_{e0}(t) + A_{e1}(t) + A_{e2}(t) + A_{e3}(t) + A_{e4}(t) + A_{r0}(t) + A_{r1}(t) + A_{r2}(t) + A_{r3}(t) + A_{r4}(t) + A_{g0}(t) + A_{g1}(t) + A_{g2}(t) + A_{g3}(t) + A_{g4}(t).$$

With the fifteen probability amplitudes, we have calculated for Eq. (8) of concurrence and Eq. (11) of negativity in the next section.

3. Generalized concurrence and negativity

Quantum entanglement can be described by comparing concurrence and negativity of time-dependent ionic-phononic system. The three-level ion subsystem are associated
with a three-dimensional Hilbert space $H_i$ of a qutrit spanned by the basis states $H_i$, while the vibrational phonon subsystem is described by a five-dimensional Hilbert space $H_p$. Therefore, the initial state in the previous section evolves in the Hilbert space $(C^3 \otimes C^5 = C^{15})$. Concurrence for the ionic-phononic system $(i - p)$ is written as [14]

$$C = \sqrt{2(\text{Tr}(\rho_{i-p}) - \text{Tr}(\rho_{ion}^2))}. \quad (8)$$

The full density matrix $\rho_{i-p}$ can be represented by the basis of $|i, p\rangle$. By tracing over the phononic variable, we get a $3 \times 3$ square reduced density matrix $\rho_{ion}$ such that

$$\rho_{ion} = \text{Tr}_{\text{phonon}}(\rho_{i-p}) = \begin{pmatrix} \text{Tr}\langle e | e \rangle & \text{Tr}\langle e | r \rangle & \text{Tr}\langle e | g \rangle \\
\text{Tr}\langle r | e \rangle & \text{Tr}\langle r | r \rangle & \text{Tr}\langle r | g \rangle \\
\text{Tr}\langle g | e \rangle & \text{Tr}\langle g | r \rangle & \text{Tr}\langle g | g \rangle \end{pmatrix}, \quad (9)$$

where $|e\rangle$ is a $5 \times 5$ square matrix. The full density matrix is $\rho_{i-p} = |a\rangle\langle a|$ that $|a\rangle\langle a|$ is a $15 \times 15$ square matrix. Also, other measurement of entanglement is negativity. According to the Schmidt decomposition theorem, the pure state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_j \sqrt{\mu_j}|a_j\rangle|b_j\rangle \quad (10)$$

where $\sqrt{\mu_j}$ ($j = 1, \ldots, d$) are the Schmidt coefficients (SCs) obtained from the three eigenvalues of the reduced density matrix $\rho_{ion}$. $|a_j\rangle$ and $|b_j\rangle$ are the orthogonal basis in $H_i$ and $H_p$ respectively. The negativity of any arbitrary bipartite pure state $|\psi\rangle$ is given by [15]

$$N(|\psi\rangle) = \frac{2}{d-1} \sum_{i<j} \sqrt{\mu_i\mu_j} = \frac{2}{3-1} (\sqrt{\mu_1\mu_2} + \sqrt{\mu_1\mu_3} + \sqrt{\mu_2\mu_3}), \quad (11)$$

with $d = \text{min}(d_1 = 3, d_2 = 5)$, and $\mu_1, \mu_2, \mu_3$ are three eigenvalues.

**Figure 2.** The time evolution of the concurrence and negativity of time-dependent ionic-phononic system in the fifteen dimensional Hilbert space. The time-dependent harmonic trap frequency $\nu(t) = \frac{\sinh(\omega_{eg}t)}{2}$ and Rabi frequency $\Omega(t) = \sinh(\omega_{eg}t)$ are used with $\omega_{eg} = 5 \times 10^{14}\text{Hz}$. The system is initially prepared in the state $\psi(0) = (1/\sqrt{2})(|g\rangle - |r\rangle) \otimes (|0\rangle + \alpha|1\rangle)$ with $\alpha = 0.01$.

Concurrence and negativity are given in Figure 2 with the help of Equations (8,11). The time-dependent harmonic trap frequency $\nu(t) = \frac{\sinh(\omega_{eg}t)}{2}$ and Rabi frequency
Figure 3. The time dependence of SCs in time-dependent ionic-phononic system using $\hbar = 1$. The other parameters are same as Fig. 2

$\Omega(t) = \sinh(\omega_{eg} t)$ are used with $\omega_{eg} = 5 \times 10^{14} \text{Hz}$. Our calculations are presented in Figures 2 – 3, for the ionic-phononic system. Concurrence values (in Fig. 2, left panel) determined to be

$$C(t = 1.0 \times 10^{-17}s = 0.01fs) = 0.002,$$
$$C(t = 1.7 \times 10^{-14}s = 17fs) = 0.999,$$
$$C(t = 4.0 \times 10^{-14}s = 40fs) = 0.007. \quad (12)$$

Negativity values (in Fig. 2, right panel) determined to be

$$N(t = 1.0 \times 10^{-17}s = 0.01fs) = 0.001,$$
$$N(t = 1.7 \times 10^{-14}s = 17fs) = 0.49,$$
$$N(t = 4.0 \times 10^{-14}s = 40fs) = 0.003. \quad (13)$$

These curves in Fig. 2 both resemble the Gaussian profile. This state ($C=0.999, N=0.49$) is maximally entangled state at a specific time of $17fs = 1.7 \times 10^{-14}s$, in the middle of these curves (see left and right panels in Fig. 2). For instance, the system is completely disentangled at $t = 0.01fs$ and $t = 40fs$, in the begin and in the end of these curves, where the system reaches the separable state. Figure 3 shows the time dependence of Schmidt coefficients (SCs) which are the three eigenvalues of the reduced density matrix in Eq. (9). Therefore, normalization of the quantum mechanical is equal to $\mu_1 + \mu_2 + \mu_3 = 1$.

4. Conclusion

We have theoretically studied the quantum entanglement process as a result of a time-dependent interaction of a three-level trapped ion with two laser beams in the $\Lambda$ scheme using a unitary transformation. Measurements of entanglement, concurrence and
negativity are too difficult to understand because of complicated structure of the ionic-phononic system and infinite-dimensional Hilbert space. In this circumstances, the system of exact solution seem foggy. Afterwards, we restrict the Hilbert space that the new final state in the Eq. (6) can be easily understood. In this way, the fog has lifted around ionic-phononic system.

Our suggestions exhibit an explicit expression in time-dependent fifteen-dimensional Hilbert space. We have seen the hidden entanglement into structure of the ionic-phononic system. They are clearly calculated concurrence and negativity in order to the ionic-phononic system. Therefore, it is obtained that the new final state \(|\psi(t)\rangle\) is superposition of fifteen function. So the new final function can be a robust entangled state.

In our approach, time evolution of concurrence and negativity obtained high amount of entanglement. In the Equations (12, 13), we have found the degree of entanglement created in the system are rather big values, \(C(17fs) = 0.999\) and \(N(17fs) = 0.49\). Entangled states as demonstrated through first order corrections by time-dependent coupling parameters and optimum entanglement can be beneficial for researchers in quantum computation and quantum optics. We hope such a practical availability of entangled states can be helpful for the ongoing efforts in ion-phonon based quantum information where a high degree of entanglement can be useful for practical quantum computation applications.

5. Acknowledgments

This work has been supported by Afyon Kocatepe University 07-FENED.09 project. I thank to T. Dermez and C. Bulutay for their patience and also I would like to thank Prof. Dr. O. E. Mustecaplioglu and D. Güney for their inspired communications.

6. References

[1] Tittel W and Weihs G 2001, Quantum Inf. Comput. 2,3
[2] Einstein A, Podolsky B and Rosen N 1935, Phys. Rev. 47, 777.
[3] Bell J S 1964, Physics 1.
[4] Abdel-Aty M 2006, Optics Communications 266 225-230.
[5] Wang Z-J and Chen F 2007, Chin. Phys. Lett. 24 1570.
[6] Can M A, Cakir O, Klyachko A, Shumousky A 2004, J. Op. B:Quantum Semiclass. Opt. 6 513.
[7] Dermez R and Müstecaplıoğlu Ö E 2009, Phys. Scr. 71 015304.
[8] Dermez R and Özen S 2010, Eur. Phys. J. D. 57 431.
[9] Dermez R and Özen S 2012, Phys. Scr. 85 055009.
[10] Dermez R and Khalek S A 2011, J. Russ. Laser Res. 32 287.
[11] Dermez R and Khalek S A, Kara K, Deveci B and Gunaydin G N 2012, J. Russ. Laser Res. 33 42.
[12] Dermez R, Deveci B and Güney D Ö 2013, Eur. Phys. J. D. 67 120.
[13] Müstecaplıoğlu Ö E 2003, Phys. Rev. A 68 023811.
[14] Hill S and Wooteers W K 1997, Phys. Rev. Lett. 78 5022.
[15] Lee S, Chi D P, Oh S D and Kim J 2003, Phys. Rev. A 68, 062304.