Ramond Equations of Motion in Superstring Field Theory

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Problems with the Ramond Sector

We’d like to write a free action

\[ S = \frac{1}{2} \langle \psi, Q\psi \rangle \]

but \( \langle, \rangle \) requires picture \(-2\). Thus \( \psi \) must have picture \(-1\).

OK for NS sector, but not OK for Ramond sector.

Let’s find EOM instead
Review of NS Sector

(Open Superstring with Witten Vertex)

NS string field: $\Phi_N$, picture $-1$.

EOM:

$$0 = Q\Phi_N + M_2(\Phi_N, \Phi_N) + ...$$

$M_2$ must carry picture $+1$.

Witten inserts picture changing operator at midpoint of open string star product. This is problematic. Instead we use contour integral of picture changing operator:

![Diagram](image-url)
Compute associator of $M_2$:

\[
\frac{1}{9} \left( \begin{array}{c}
1 \\
4 \\
\hline
2 \\
3
\end{array} \right) + \begin{array}{c}
1 \\
4 \\
\hline
2 \\
3
\end{array}
\]

This is not zero!

**We need an $A_\infty$ algebra**

**EOM:**

\[
0 = Q\Phi_N + M_2(\Phi_N, \Phi_N) + M_3(\Phi_N, \Phi_N, \Phi_N) + \ldots
\]

$M_n$s satisfy $A_\infty$ relations.
$A_{\infty}$ relations require:

\[ \text{Associator of } M_2 = Q(3\text{-string product } M_3) \]

Pretend picture changing operator is BRST exact.

\[ X = [Q, \xi] \]

Then we can simply factor $Q$ out of $M_2$ associator to find $M_3$.
Oops. $X$ is not BRST exact.

At least not in the small Hilbert space.

Have to make sure that $M_3$ is in the small Hilbert space. Upshot is that we have to add to the stuff under the parentheses of $Q$ the BRST variation of

$$\frac{1}{6} \left( \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array} \right) + \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array}$$

This works! We have a solution to the $A_\infty$ relations, and therefore the EOM, out to third order.
Need equations to go along with these pictures

Signs: degree(ψ) = Grassmann parity(ψ) + 1 (Don’t ask...)

Act string products on any number of copies of state space \( \mathcal{H} \):

\[
Q\mathcal{H} = Q\mathcal{H}
\]
\[
Q(\mathcal{H} \otimes \mathcal{H}) = (Q\mathcal{H}) \otimes \mathcal{H} + \mathcal{H} \otimes (Q\mathcal{H})
\]
\[
Q(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) = (Q\mathcal{H}) \otimes \mathcal{H} \otimes \mathcal{H} + \mathcal{H} \otimes (Q\mathcal{H}) \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H} \otimes (Q\mathcal{H})
\]
\[
\vdots
\]
\[
M_2(\mathcal{H}) = 0
\]
\[
M_2(\mathcal{H} \otimes \mathcal{H}) = M_2(\mathcal{H}, \mathcal{H})
\]
\[
M_2(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) = M_2(\mathcal{H}, \mathcal{H}) \otimes \mathcal{H} + \mathcal{H} \otimes M_2(\mathcal{H}, \mathcal{H})
\]
\[
\vdots
\]
For example:

\[ m_2 = \text{Witten's open string star product} \]

Bosonic SFT axioms can be expressed:

* BRST operator is nilpotent: \[ [Q, Q] = 0 \]
* BRST operator is derivation: \[ [Q, m_2] = 0 \]
* Star product is associative: \[ [m_2, m_2] = 0 \]
More generally, and $A_\infty$ algebra is defined by a sequence of products $Q, M_2, M_3, ...$ which satisfy $A_\infty$ relations:

$$[Q, M_n] + [M_2, M_{n-1}] + ... + [M_{n-1}, M_2] + [M_n, Q] = 0$$

Or, even more simply, we can take the sum

$$M = Q + M_2 + M_3 + M_4 + ...$$

The $A_\infty$ relations imply that $M$ is nilpotent:

$$[M, M] = 0$$
Now let’s put equations to the pictures.

Since we’re pretending that $X$ is BRST exact, the product $M_2$ in the EOM is BRST exact:

$$M_2 = [Q, \mu_2]$$
$$m_2 = [\eta, \mu_2]$$

$\mu_2$ is the same as $M_2$ with the replacement $X \rightarrow \xi$. Again, $m_2$ is the ordinary star product.

Note

$$[\eta, M_2] = -[Q, m_2] = 0$$
Now let’s derive the 3-product $M_3$. (Bear with me.) Pretending $X$ is BRST exact, we can pull $Q$ out of third $A_\infty$ relation:

$$0 = 2[Q, M_3] + [M_2, M_2]$$
$$= [Q, 2M_3 − [M_2, \mu_2]]$$

Therefore

$$M_3 = \frac{1}{2}([Q, \mu_3] + [M_2, \mu_2])$$

$[Q, \mu_3]$ is the extra term needed to make sure $M_3$ is in small Hilbert space. Defining $m_3 = [\eta, \mu_3]$ we must have

$$0 = [\eta, M_3] = [Q, m_3] + [M_2, m_2] = [Q, m_3 − [m_2, \mu_2]]$$

so

$$m_3 = [m_2, \mu_2]$$

Note $[\eta, m_3] = 0$. Surrounding $m_3$ with $\xi$ defines $\mu_3$, and therefore the product that we want!
This is how it works at all orders: Defining

\[ M(t) = Q + tM_2 + t^2M_3 + t^3M_4 + \ldots \]
\[ \mu(t) = \mu_2 + t\mu_3 + t^2\mu_4 + \ldots \]
\[ m(t) = m_2 + tm_3 + t^2m_4 + \ldots \]

The products are defined by solution of the equations

\[ \frac{d}{dt} M(t) = [M(t), \mu(t)] \]
\[ \frac{d}{dt} m(t) = [m(t), \mu(t)] \]
\[ [\eta, \mu(t)] = m(t) \]

Finally, we have the EOM for the NS open superstring!
Ramond Equations of Motion
(Open Superstring with Witten Vertex)

Include Ramond string field $\psi_R$ with picture $-1/2$.

EOM:

$$0 = Q\phi_N + M_2(\phi_N, \phi_N) + m_2(\psi_R, \psi_R) + ...$$

$$0 = Q\psi_R + M_2(\psi_R, \phi_N) + M_2(\phi_N, \psi_R) + ...$$
Define composite string field $\tilde{\Phi} = \Phi_N + \Psi_R$. EOM can be written

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + ...$$

with

$$\tilde{M}_2(N, N) = M_2(N, N),$$
$$\tilde{M}_2(N, R) = M_2(N, R),$$
$$\tilde{M}_2(R, N) = M_2(R, N),$$
$$\tilde{M}_2(R, R) = m_2(R, R)$$

Want $\tilde{M}_3$

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + \tilde{M}_3(\tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) + ...$$

so that third $A_\infty$ relation is obeyed:

$$2[Q, \tilde{M}_3] + [\tilde{M}_2, \tilde{M}_2] = 0$$
Again we just want to pull a $Q$ out of the associator to find the 3-product.

Since $\tilde{M}_2$ is different depending on the number of $R$ states being multiplied, we have to do this separately for the 8 possible ways three NS and $R$ states can multiply. Upshot:

$$
\begin{align*}
\tilde{M}_3(N, N, N) &= M_3(N, N, N) \\
\tilde{M}_3(N, N, R) &= M_3(N, N, R) \\
\tilde{M}_3(N, R, N) &= M_3(N, R, N) \\
\tilde{M}_3(R, N, N) &= M_3(R, N, N) \\
\tilde{M}_3(N, R, R) &= m_2(\mu_2(N, R), R) - \mu_2(N, m_2(R, R)) \\
\tilde{M}_3(R, N, R) &= m_2(\mu_2(R, N), R) + m_2(R, \mu_2(N, R)) \\
\tilde{M}_3(R, R, N) &= -\mu_2(m_2(R, R), N) + m_2(R, \mu_2(R, N)) \\
\tilde{M}_3(R, R, R) &= -\mu_2(m_2(R, R), R) - \mu_2(R, m_2(R, R))
\end{align*}
$$

Whew!
Note $\tilde{M}_3(R, R, R) \neq 0$.

What about $\tilde{M}_4(R, R, R, R)$? Must vanish by ghost and picture counting.

EOM is precisely cubic in the Ramond string field.
Finding all ways NS and R states multiply seems like a pain

**Key technical idea: Ramond number:**

\[
\text{Ramond number} = \text{Number of Ramond inputs} - \text{Number of Ramond outputs}
\]

Denote

\[ b_n|_N \]

Can decompose products into components of definite Ramond number:

\[ b_n = b_n|-1 + b_n|_0 + b_n|_1 + \ldots + b_n|_n \]
We can write the composite 2-product:

$$\tilde{M}_2 = M_2|_0 + m_2|_2$$

We can write the composite 3-product:

$$\tilde{M}_3 = M_3|_0 + m'_3|_2$$

with

$$m'_3|_2 = [m_2|_2, \mu_2|_0]$$

Ramond number restriction of products in commutator automatically takes care of different NS and R multiplications

All composite products have components at Ramond number 0 and 2:

$$\tilde{M}_n = M_n|_0 + m'_n|_2$$

Products of 4 or more Ramond states vanish.
This is how it works at all orders: Defining

\[ M(t) = Q + tM_2|_0 + t^2 M_3|_0 + t^3 M_4|_0 + \ldots \]
\[ \mu(t) = \mu_2|_0 + t\mu_3|_0 + t^2 \mu_4|_0 + \ldots \]
\[ m(t) = m_2|_0 + tm_3|_0 + t^2 m_4|_0 + \ldots \]
\[ m'(t) = m_2|_2 + tm'_3|_2 + t^2 m'_4|_2 + \ldots \]

The products are defined by solution of the equations

\[
\frac{d}{dt} M(t) = [M(t), \mu(t)] \\
\frac{d}{dt} m(t) = [m(t), \mu(t)] \\
\frac{d}{dt} m'(t) = [m'(t), \mu(t)] \\
[\eta, \mu(t)] = m(t)
\]

Finally, we have the NS+R equations of motion for the NS open superstring!
Solving the differential equations gives a set of recursive equations for the products. The recursion is solved by following the diagram:
The story for the NS+R equations of motion of the heterotic string is similar but more intricate. In the end you solve for a bunch of products by following a recursion illustrated by the diagram:
The story for the NS-NS+R-NS+NS-R +R-R equations of motion for the type II closed superstring is similar but even more complicated. You find the 2-string products by following the diagram:
Then the 3-string products by following the diagram:
Then the 4-string products by following the diagram:

And so on.
Supersymmetry

Easiest to describe SUSY in another set of field variables $\Phi \rightarrow \Phi$ where the equations of motion take the form

$$0 = (Q - \eta)\Phi + \Phi^2$$

(See Okawa’s talk)

SUSY transformation takes the form:

$$\delta \Phi = q \psi_R + [\psi_R, q \xi \phi_N]$$
$$\delta \psi_R = q \chi \phi_N + q \xi (\psi_R)^2$$

Now we can ask whether SFT solutions are supersymmetric. For example, can translate reference BPS D-brane with analytic solution

$$\phi_N = \psi_{tv} - \Sigma \psi_{tv} \overline{\psi}$$

SUSY invariance is easy to check.
Thank you!