Post-Newtonian expansions for perfect fluids

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Abstract
We prove the existence of a large class of dynamical solutions to the Einstein-Euler equations that have a first post-Newtonian expansion. The results here are based on the elliptic-hyperbolic formulation of the Einstein-Euler equations used in [15], which contains a singular parameter \( \epsilon = v_T/c \), where \( v_T \) is a characteristic velocity associated with the fluid and \( c \) is the speed of light. As in [15], energy estimates on weighted Sobolev spaces are used to analyze the behavior of solutions to the Einstein-Euler equations in the limit \( \epsilon \searrow 0 \), and to demonstrate the validity of the first post-Newtonian expansion as an approximation.

1 Introduction
The Einstein-Euler equations, which govern a gravitating perfect fluid, are given by

\[
G^{ij} = \frac{8\pi G}{c^4} T^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0,
\]

where

\[
T^{ij} = (\rho + \epsilon^{-2} p) v^i v^j + p g^{ij},
\]

with \( \rho \) the fluid density, \( p \) the fluid pressure, \( v \) the fluid four-velocity normalized by \( v^i v_i = -c^2 \), \( c \) the speed of light, and \( G \) the Newtonian gravitational constant. Defining

\[
\epsilon = \frac{v_T}{c}
\]

where \( v_T \) is a typical speed associated with the fluid, the Einstein-Euler equations, upon suitable rescaling [15], can be written in the form

\[
G^{ij} = 2\epsilon^4 T^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0,
\]

(1.1)

where

\[
T^{ij} = (\rho + \epsilon^2 p) v^i v^j + p g^{ij} \quad \text{and} \quad v^i v_i = \frac{1}{\epsilon^2}.
\]

In this formulation, the fluid four-velocity \( v^i \), the fluid density \( \rho \), the fluid pressure \( p \), the metric \( g_{ij} \), and the coordinates \( (x^i) \) \( i = 1, \ldots, 4 \) are dimensionless. By assumption, the \( (x^i) \) are global Cartesian coordinates on spacetime \( M \cong \mathbb{R}^3 \times [0, T) \), where the \( (x^I) \) \( (I = 1, 2, 3) \) are spatial coordinates that cover \( \mathbb{R}^3 \), and \( t = x^4/v_T \) is a Newtonian time coordinate that covers the interval \([0, T)\). By a choice of units, we can and will set \( v_T = 1 \).

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Post-Newtonian expansions for the Einstein-Euler system refer to expansions of solutions to this system in the parameter $\epsilon$, about $\epsilon = 0$, where the lowest expansion term is governed by the Poisson-Euler equations of Newtonian gravity:

\[ \partial_t \rho + \partial_I (\rho w^I) = 0, \tag{1.2} \]
\[ \rho (\partial_t w^I + w^J \partial_J w^I) = - (\rho \partial_J \Phi + \partial_J p), \tag{1.3} \]
\[ \Delta \Phi = \rho. \tag{1.4} \]

Here $\rho$, $p$, and $w^I$ are the fluid density, pressure, and three velocity, respectively.

Formal calculational schemes for determining the post-Newtonian expansion coefficients and the equations they satisfy exist, and are in wide use by physicists [5, 9]. In fact, these post-Newtonian computational schemes are one of the most important techniques in general relativity for calculating physical quantities for the purpose of comparing theory with experiment. For example, in gravitational wave astronomy, post-Newtonian expansions are used to calculate gravitational wave forms that are emitted during gravitational collapse [5].

It is important to stress that the formal post-Newtonian expansion schemes all implicitly rely on the assumption that the expansions exist and approximate solutions to general relativity. Therefore, to establish existence of such approximations, and to answer questions about their range of validity, a different approach must be taken to the problem. In [15], we took a first step in analyzing this problem by proving the existence of a wide class of one-parameter families of solutions to the Einstein-Euler equations that converged in a suitable sense to the Poisson-Euler equations in the limit $\epsilon \downarrow 0$. We also remark that similar results were also established, using a different method, by Alan Rendall [19] for the Einstein-Vlasov equations.

In this paper, we use the results of [15] to prove the existence of a large class of solutions to the Einstein-Euler equations that can be expanded in $\epsilon$ to the first post-Newtonian order. Moreover, we demonstrate the existence of convergent expansions in $\epsilon$ for solutions to the Einstein-Euler equations. These expansions are, in general, not of the post-Newtonian type since the expansion coefficients can depend on $\epsilon$. Nevertheless, the expansions are convergent, and therefore, represent a kind of generalized post-Newtonian expansion. We note that analogous expansions for the Vlasov-Maxwell equations and Vlasov-Nordst"om equations have been rigorously analyzed in [2–4].

The difficulty in analyzing the post-Newtonian expansions arise from the fact that the limit $\epsilon \downarrow 0$ is singular. To analyze this limit, we follow the approach of [15], which requires that the metric $g_{ij}$ and the fluid velocity $v^I$ are replaced with new variables that are compatible with the limit $\epsilon \downarrow 0$. The new gravitational variable is a density $\bar{u}^{ij}$ defined via the formula

\[ g^{ij} = \frac{\epsilon}{\sqrt{- \det(Q)}} Q^{ij} \tag{1.5} \]

where

\[ Q^{ij} = \begin{pmatrix} \delta^{ij} & 0 \\ 0 & 0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} 4\bar{u}^{IJ} & 0 \\ 0 & -1 \end{pmatrix} + 4\epsilon^3 \begin{pmatrix} 0 & \bar{u}^{I4} \\ \bar{u}^{I4} & 0 \end{pmatrix} + 4\epsilon^4 \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}^{44} \end{pmatrix}. \tag{1.6} \]

From this, it not difficult to see that the density $\bar{u}^{ij}$ is equivalent to the metric $g_{ij}$ for $\epsilon > 0$, and is well defined at $\epsilon = 0$. For the fluid, a new velocity variable $w^\nu$ is defined by

\[ v^I = w^I \quad \text{and} \quad w^4 = \frac{w^4 - 1}{\epsilon}. \tag{1.7} \]

For technical reasons, we assume an isentropic equation of state

\[ p = K \rho^{(n+1)/n}, \tag{1.8} \]
where $K \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$. This allows us to use a technique of Makino [14] to regularize the fluid equations by the use of the fluid density variable

\[ \rho = \frac{1}{(4Kn(n+1)^{n})^{2n}}. \]  

(1.9)

The resulting system can be put into a symmetric hyperbolic system that is regular across the fluid-vacuum interface. In this way, it is possible to construct solutions to the Einstein-Euler equations that represent compact gravitating fluid bodies (i.e. stars) both in the Newtonian and relativistic setting [14, 18]. In the Newtonian setting, this is straightforward to see. Using (1.8) and (1.9), the Poisson-Euler equations (1.2)–(1.4) imply that

\[ \partial_{t} \alpha = -w^{I} \partial_{I} \alpha - \frac{\alpha}{2n} \partial_{I} w^{I}, \]  

(1.10)

\[ \partial_{t} w^{I} = -\frac{\alpha}{2n} \partial^{I} \alpha - w^{I} \partial_{I} w^{I} - \partial^{I} \Phi, \]  

(1.11)

\[ \Delta \Phi = \rho \left( \rho := (4Kn(n+1))^{-n\alpha^{2n}} \right), \]  

(1.12)

which is readily seen to be regular even across regions where $\alpha$ vanishes.

As discussed by Rendall [18], the type of fluid solutions obtained by the Makino method have freely falling boundaries and hence do not include static stars of finite radius, and consequently this method is far from ideal. However, in trying to understand the post-Newtonian expansions, these solutions are general enough to obtain a comprehensive understanding of the mathematical issues involved in the post-Newtonian expansions.

As in [15], our approach to the problem of post-Newtonian expansions is to use the gravitational and matter variables $\bar{u}^{ij}, w^{i}, \alpha$ along with a harmonic gauge to put the Einstein-Euler equations into a singular (non-local) symmetric hyperbolic system of the form

\[ b^{0}_{\epsilon}(\epsilon W) \partial_{t} W = \frac{1}{\epsilon} c^{I} \partial_{I} W + b^{I}(\epsilon, W) \partial_{I} W + F(\epsilon, W). \]  

(1.13)

Singular hyperbolic systems of this form have been extensively studied in the articles [6, 11, 12, 20, 21]. Especially relevant for our purposes, is the paper [21]. There, a systematic procedure for constructing rigorous expansions to singular symmetric hyperbolic systems is developed (see also [11, 12]). However, the techniques of [6, 11, 12, 20, 21] cannot be applied directly to our case. The reason for this is that the initial data for the system (1.13) must include a $1/r$ piece for the metric and cannot lie in the Sobolev space $H^{k}$. This problem was overcome in [15] by using a one parameter family $H^{k}_{\delta, \epsilon}$ of weighted Sobolev spaces that include $1/r$ type fall off for $\epsilon > 0$, and reduce to the standard Sobolev spaces $H^{k}$ in the limit $\epsilon \rightarrow 0$. We again use these weighted Sobolev spaces, this time to generalize the results of [21] so that we can apply them to the problem of generating rigorous post-Newtonian expansions.

The next theorem is the main result of this paper, and the proof can be found in section 6. The definition of the spaces $H^{k}_{\delta, \epsilon}$ and $X_{T, s, k, \delta}$ can be found in Appendices A and B.

**Theorem 1.1.** Suppose $-1 < \delta < -1/2$, $s \geq 3$, $k \geq 3 + s$, $\alpha, w^{I}, \delta^{I} \in H^{k}_{\delta-1}, \epsilon \in H^{k-2}_{\delta-2}$, $\sup \alpha < \infty$, and let $T^{M}_{0}$ be the maximal existence time (see Proposition 1.17) for solutions to the Poisson-Euler-Makino equations (1.10)–(1.12) with initial data $\alpha(0) = \alpha, w^{I}(0) = w^{I}$. Then for any $T_{0} < T$ there exists an
\[ \epsilon_0 > 0, \text{ and maps} \]

\[ \tilde{u}_i^j(t) : \tilde{u}_i^j(t) - \tilde{u}_i^j(0), \partial_t \tilde{u}_i^j(t), \partial_i \tilde{u}_i^j(t) \in X_{T_0, s, k, \delta - 1} \quad 0 < \epsilon \leq \epsilon_0, \]

\[ \alpha_\epsilon(t), \omega^i_\epsilon(t) \in X_{T_0, s, k, \delta - 1} \quad 0 < \epsilon \leq \epsilon_0, \]

\[ \partial_t^0 \Phi(t) \in X_{T_0, s, k+2, \delta} \quad \text{with} \quad \partial_t^0 \Phi(t) \in X_{T_0, s, k+1, \delta - 1}, \]

\[ \tilde{u}_i^j(t) : \tilde{u}_i^j(t) - \tilde{u}_i^j(0), \partial_t \tilde{u}_i^j(t) \in X_{T_0, s - q, k - q, \delta - 1} \quad q = 1, 2, \]

\[ \partial_t \tilde{w}_i^j(t) \in X_{T_0, s - q, k - q, \delta - 1} \quad q = 1, 2, \]

\[ \tilde{u}_i^j(t) : \tilde{u}_i^j(t) - \tilde{u}_i^j(0), \partial_t \tilde{u}_i^j(t) \in X_{T_0, s - 3, k - 3, \delta - 1} \quad (q, \epsilon) \in \mathbb{Z}_{\geq 3} \times (0, \epsilon_0], \]

\[ \partial_t \tilde{w}_i^j(t) \in X_{T_0, s - 3, k - 3, \delta - 1} \quad (q, \epsilon) \in \mathbb{Z}_{\geq 3} \times (0, \epsilon_0], \]

such that

(i) the triple \( \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \omega^i_\epsilon(t) \} \) determines, via formulas (1.3) - (1.9), a solution to the Einstein-Euler equations (1.1) in the harmonic gauge for \( 0 < \epsilon \leq \epsilon_0 \) on the spacetime region \( (x^t, t = x^4) \in D = \mathbb{R}^3 \times (0, T_0) \),

(ii) \( \partial_t \tilde{u}_i^J(t) = c^2_{\delta J} \partial_t \tilde{u}_i^J(t) = c^2_{\delta J}, \alpha_\epsilon(0) = \alpha, \) and \( \omega^i_\epsilon(0) = \omega^i \) for \( 0 < \epsilon \leq \epsilon_0 \),

(iii) \( \{ \alpha(t), \omega^i(t), 0 \} \) is the unique solution to the Poisson-Euler-Makino equations (1.10) - (1.12) with initial data \( \alpha(0) = \alpha, \omega^i(0) = \omega^i \),

(iv) for \( q = 1, 2, \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \tilde{w}_i^j(t) \} \) satisfies a linear (non-local) symmetric hyperbolic system that only depends on \( \{ \alpha(t), \omega^i(t), 0, \Phi(t) \} \) if \( q = 1 \), and \( \{ \alpha(t), \omega^i(t), 0, \Phi(t), \tilde{u}_i^j(t), \alpha(t), \tilde{w}_i^j(t) \} \) if \( q = 2 \),

(v) for \( q \in \mathbb{Z}_{\geq 3}, \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \tilde{w}_i^j(t) \} \) satisfies a linear (non-local) symmetric hyperbolic system that only depends on \( \epsilon, \{ \alpha(t), \omega^i(t), 0, \Phi(t), \tilde{u}_i^j(t), \alpha(t), \tilde{w}_i^j(t) \} \) for \( p = 1, 2 \), and \( \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \tilde{w}_i^j(t) \} \) for \( p = 3, 4, \ldots, q - 1 \),

(vi) \( \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \omega^i_\epsilon(t) \} \) and \( \{ \tilde{u}_i^j(t), \alpha_\epsilon(t), \tilde{w}_i^j(t) \} \) for \( q \in \mathbb{Z}_{\geq 3}, \) satisfy the following estimates:

\[ \| \tilde{u}_i^j(t) \|_{L^2} + \| \partial_t \tilde{u}_i^j(t) \|_{H^s} + \epsilon \| \partial_t \tilde{u}_i^j(t) \|_{H^s} + \epsilon \| \partial_t \tilde{u}_i^j(t) \|_{H^s} + \epsilon^2 \| \partial_t^2 \tilde{u}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]

\[ \| \partial_t \tilde{u}_i^j(t) \|_{H^s} = \| \partial_t \tilde{u}_i^j(t) \|_{H^s} + \| \partial_t \alpha_\epsilon(t) \|_{H^s} + \| \partial_t \tilde{w}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]

\[ \| \tilde{w}_i^j(t) \|_{L^2} + \| \partial_t \tilde{w}_i^j(t) \|_{H^s} + \epsilon \| \partial_t \tilde{w}_i^j(t) \|_{H^s} + \epsilon \| \partial_t \tilde{w}_i^j(t) \|_{H^s} + \epsilon^2 \| \partial_t \tilde{w}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]

\[ \| \partial_t \tilde{w}_i^j(t) \|_{H^s} \lesssim 1, \| \tilde{w}_i^j(t) \|_{H^s} + \| \tilde{w}_i^j(t) \|_{H^{s-1}} + \| \partial_t \tilde{w}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]

\[ \| \partial_t \tilde{w}_i^j(t) \|_{H^{s-1}} + \| \tilde{w}_i^j(t) \|_{H^s} + \| \tilde{w}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]

\[ \| \partial_t \tilde{w}_i^j(t) \|_{H^{s-1}} \lesssim 1, \]
Remark 1.2. The following \( \eta \) where (vii) of Theorem 1.1. These higher order terms will, in general, depend on \( \epsilon \) in a non-analytic fashion, and therefore, without further analysis, the relation of these expansion terms to the standard post-Newtonian expansions is not clear.

Remark 1.2.

(a) For \( q = 1, 2 \), the equations satisfied by \( \{ q \hat{w}^i, \hat{u}^i, \hat{v}^i \} \) are the ones obtained by directly substituting the expansions of Theorem 1.1 (vii) into the Einstein-Euler equations and collecting terms to order \( \epsilon^2 \), and therefore coincide with the standard first post-Newtonian expansions.

(b) The equations satisfied by \( \{ q \hat{w}^i, \hat{u}^i, \hat{v}^i \} \) for \( q \geq 3 \) can be determined from the equations satisfied by the \( \hat{W}_\epsilon \) defined in the proof of Theorem 5.1.

To facilitate comparisons of the approach taken in this paper with previous studies, we define the following \( \epsilon \)-independent quantities:

\[
\hat{h}^{ij} = (4\hat{u}^{ij} - 2\eta_{k\ell}\hat{h}^{k\ell}\eta^{ij}) \quad q = 1, 2,
\]

where \( \eta_{ij} = \text{diag}(1, 1, 1, -1) \). Then a straightforward calculation, using statement (vii) of Theorem 1.1 and formulas (1.5)-(1.6), shows that the metric \( g_{ij} \) can be expanded as follows

\[
g_{44} = -\frac{1}{\epsilon^2} - 2\Phi - \epsilon h^{44} - \epsilon^2 \left( 3\Phi^2 + 2\hat{h}^{44} \right) + O(\epsilon^3),
\]

\[
g_{4l} = \epsilon^2 \hat{h}^{4l} + \epsilon^3 \hat{h}^{4l} + O(\epsilon^4),
\]

and

\[
g_{4l} = \delta_{4l} - 2\epsilon^2 \delta_{4l} \Phi - \epsilon^3 \hat{h}^{4l} \Phi - \epsilon^4 \left( \left( \Phi^2 + 2\hat{h}^{4l} \right) + O(\epsilon^5).
\]

It is worthwhile to note that higher order expansions in \( \epsilon \) can be generated for the metric \( g_{ij} \) using part (vii) of Theorem 1.1. These higher order terms will, in general, depend on \( \epsilon \) in a non-analytic fashion, and therefore, without further analysis, the relation of these expansion terms to the standard post-Newtonian expansions is not clear.

2 Einstein-Euler equations

In this section, we quickly review the formulation of the Einstein-Euler equation used in [15] to analyze the limit as \( \epsilon \to 0 \).
2.1 Reduced Einstein equations

As discussed in the introduction, we use a symmetric tensor density \( \bar{u}^{ij} \) instead of the metric \( g^{ij} \), which for \( \epsilon > 0 \) completely determines the metric via the formula

\[
(\bar{g}^{ij}) = \frac{1}{\sqrt{\bar{g}}} \left( \begin{array}{cc} \bar{g}^{IJ} & \epsilon \bar{g}^{I4} \\ \epsilon \bar{g}^{4J} & \epsilon^2 \bar{g}^{44} \end{array} \right),
\]

where

\[
\bar{g}^{ij} := \eta^{ij} + 4 \epsilon^2 \bar{u}^{ij}, \quad |\bar{g}| := -\text{det}(\bar{g}^{ij}),
\]

and

\[
\eta^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

To fix the gauge, we let

\[
\bar{\partial}_k = \begin{cases} \partial_I & \text{if } k = I \\ \epsilon \partial_4 & \text{if } k = 4 \end{cases},
\]

and demand that

\[
\bar{\partial}_i \bar{u}^{ij} = 0.
\]

For \( \epsilon > 0 \), this condition is easily seen to be equivalent to the harmonic gauge

\[
\partial_k \bar{g}^{kj} = 0.
\]

Here \( \bar{g}^{ij} = \sqrt{-\text{det}(g_{k\ell})} g^{ij} \) is the metric density in the coordinates \( (x^i) \).

Next, defining

\[
\begin{align*}
\bar{u}^{ij} &:= \epsilon \bar{u}^{ij}, \\
\bar{u}^{kij} &:= \bar{\partial}_k \bar{u}^{ij}, \\
\bar{u}^{ij} &:= (\bar{u}^{i4}, \bar{u}^{4j}, \bar{u}^{ij})^T, \\
(\bar{g}^{ij}) &:= (\bar{g}^{ij})^{-1}, \\
A^{ij} &:= 2 \left( \frac{1}{2} \bar{g}^{klt} \bar{g}^{mkn} - \bar{g}^{kml} \bar{g}^{tn} \right) (\bar{g}^{ip} \bar{g}^{jq} - \frac{1}{2} \bar{g}^{ij} \bar{g}^{pq}) \bar{\partial}_p \bar{u}^{kl} \bar{\partial}_q \bar{u}^{mn}, \\
B^{ij} &:= 4 \bar{g}^{klt} (2 \bar{g}^{n(t} \bar{\partial}_m \bar{u}^{u)}k) \bar{\partial}_n \bar{u}^{km} - \frac{1}{2} \bar{g}^{ij} \bar{\partial}_m \bar{u}^{kn} \bar{\partial}_n \bar{u}^{m\ell} - \bar{g}^{mn} \bar{\partial}_m \bar{u}^{ik} \bar{\partial}_n \bar{u}^{j\ell}),
\end{align*}
\]

and

\[
C^{ij} := 4 (\bar{\partial}_k \bar{u}^{ij} \bar{\partial}_l \bar{u}^{kl} - \bar{\partial}_k \bar{u}^{i\ell} \bar{\partial}_l \bar{u}^{k\ell}),
\]

the Einstein equations \( G^{ij} = 2 \epsilon^4 T^{ij} \), in the harmonic gauge, can be written in first order form as

\[
A^i(\epsilon \bar{u}) \bar{\partial}_i \bar{u}^{ij} = \frac{1}{\epsilon} C^i(\bar{\partial}_j \bar{u}^{ij} + A^j(\bar{\partial}_j \bar{u}^{ij} + \bar{F}_\partial^{ij}(\bar{u}) + \epsilon \bar{F}_\epsilon^{ij}(\bar{u}, \epsilon \bar{u}) - \frac{1}{\epsilon} (T^{ij}, 0, 0) T),
\]

(2.13)
In [15], we also showed that if we use the fluid variables (2.21), and choose initial data that satisfies (2.23), we will refer to the gauge fixed Einstein equation (2.13) as the reduced Einstein equations. Because of the matrix inversion (2.20) used to define the inverse density $\tilde{\eta}_{ij}$, the reduced Einstein equations will be well defined provided $\epsilon u \in \mathcal{V} = \{ (r^{ij}) \in \mathbb{M}_{4 \times 4} | \det(\eta^{ij} + 4r^{ij}) > 0 \}$.

### 2.2 Euler equations

In [15], we also showed that if we use the fluid variables (2.21), and choose initial data that satisfies (2.23), the Euler equations $\nabla_i T^{ij} = 0$ are equivalent to the system

$$a^4 \partial_i w = a^4 \partial_j w + b,$$  

(2.26)
where

\[ v^I = \bar{v}^l, \quad \bar{v}^4 = \frac{\epsilon^4}{\epsilon}, \]

\[ \bar{g}^{ij} = \frac{1}{\sqrt{|g|}} \bar{g}^{ij}, \quad (\bar{g}^{ij}) = (\bar{g}^{ij})^{-1}, \]

\[ h = \left( 1 + \frac{1}{4n(n+1)}(\epsilon \alpha)^2 \right), \quad q = \frac{1}{2nh^2}, \]

\[ L^j_i = \delta^j_i + \epsilon^2 \bar{v}^i \bar{v}^j, \quad \bar{v}^j = \bar{g}^{ij} \bar{v}^i, \]

\[ M_{ij} = \bar{g}^{ij} + 2\epsilon^2 \bar{v}^i \bar{v}^j, \]

\[ \bar{\Gamma}^k_{ij} = \epsilon^2 \left( \bar{g}^{km} (2\bar{g}_{i\ell} \bar{g}_{j\ell} - \bar{g}_{ij} \bar{g}_{\ell\ell}) \bar{\partial}_m \bar{u}^{\ell p} + 2(\bar{g}_{i\ell} \delta^k_{(i} \bar{\partial}_j) \bar{u}^{\ell p} - 2\bar{g}_{(i} \bar{\partial}_j) \bar{u}^{k\ell}) \right), \]

\[ a^4 = \left( \epsilon^2 (1 + \epsilon \omega^4) \begin{pmatrix} \epsilon q L^j_i & \epsilon q L^j_k \\ \epsilon q L^j_i & M_{ij}(1 + \epsilon \omega^4) \end{pmatrix} \right), \]

\[ a^I = \left( -h^2 w^I - q L^j_i - q L^j_k - M_{ij} w^I \right), \]

and

\[ b = \left( -q L^j_i \bar{\Gamma}^k_{ij} \bar{v}^k \bar{v}^l \right). \]

We also note that

\[ a^4 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} + \hat{a}^4(\epsilon u, \epsilon w), \]

\[ a^I = \begin{pmatrix} -w^I \delta_l^I & -\frac{\epsilon \omega^4}{\epsilon n} \delta_l^I \\ -\frac{\epsilon \omega^4}{\epsilon n} \delta_l^I & -\delta_{ij} w^I \end{pmatrix} + w^I \hat{a}(\epsilon u, \epsilon w) + \alpha \hat{a}^I(\epsilon u, \epsilon w), \]

and

\[ b = \left( -\eta^{im} (2\eta_{tp}\eta_{sp} + \eta_{tp}) u_{mp}^p - 2(\eta_{tp}\delta_{tp} u_{mp}^p - 2\eta_{tp} u_{mp}^p) \right) + \left( \alpha \hat{b}_1(\epsilon u, \epsilon w) \cdot u_k \right), \]

where \( \{ \hat{a}^4, \hat{a}, \hat{a}^I, \hat{b}_1, \hat{b}_2 \} \) are analytic in all their variables provided that \( \epsilon u \in \mathcal{V} \), \( \{ \hat{a}^4, \hat{a}, \hat{a}^I \} \) are symmetric, and \( \hat{a}^4(0,0) = 0, \hat{a}^I(0,0) = 0, \hat{a}(0,0) = 0, \hat{b}_1(0,0) = 0, \) and \( \hat{b}_2(0,0) = 0. \)

3 Uniform existence and the zeroth order equations

The combined systems (2.13) and (2.26) can be written as

\[ b^0(\epsilon V, \epsilon^2 U) \partial_t V = \frac{1}{\epsilon} \epsilon^4 \partial_t V + b^I(\epsilon U, \epsilon V, \epsilon^2 U) \partial_t V + f_0(V, \epsilon U, \epsilon V, \epsilon^2 U) + \epsilon f_1(V, \epsilon U, \epsilon V, \epsilon^2 U) + \frac{1}{\epsilon} g(V), \]
where

\[ U = (0, 0, \bar{u}^i, 0, 0)^T, \quad \bar{u}^i = u^i|_{t=0}, \quad (3.2) \]

\[ V = (u^i_4, u^i_5, \delta u^i, \alpha, u^i)^T, \quad \delta u^i = u^i - \epsilon \bar{u}^i, \quad (3.3) \]

\[ b^0(eV, e^2U) = \left( A^i_0(\epsilon u) D^0(\epsilon u, \epsilon w) \right), \quad (3.4) \]

\[ c^i = \left( C^i_0 \right), \quad (3.5) \]

\[ b^i(V, eU, eV, e^2U) = \left( A^i_0(\epsilon u) D^i(\epsilon u, \epsilon w) \right), \quad (3.6) \]

\[ f_0(V, eU, eV, e^2U) = \left( F^i_0(\epsilon u) - S^i_0(\epsilon u, \epsilon w) \right) \frac{b(\epsilon u, \epsilon w, \epsilon w)}{b(\epsilon u, \epsilon w, \epsilon w)}, \quad (3.7) \]

\[ f_1(V, eU, eV, e^2U) = \left( F^i_1(\epsilon u, \epsilon w) - S^i_1(\epsilon u, \epsilon w) \right) \frac{0}{0}, \quad (3.8) \]

and

\[ g(V) = (-\delta^i_j \delta^j_0 \rho(\alpha), 0, \ldots, 0)^T. \quad (3.9) \]

For initial data, we will often use the following notation: given a function \( z \) that depends on time \( t \), we define

\[ z_o = z|_{t=0}. \]

In addition to solving these equations, we must also solve constraint equations on the initial data to get a full solution to the Einstein-Euler equations. Letting

\[ G^{ij} = \bar{g}^{ik} \overline{\partial}_k \bar{u}^{ij} + \epsilon^2 (A^{ij} + B^{ij} + C^{ij}) + \bar{g}^{ij} \overline{\partial}_k \bar{u}^{kl} - 2 \overline{\partial}_k \overline{\partial}_l \bar{u}^{(ij)} \bar{g}^{kl}, \quad (3.10) \]

and defining

\[ C^j = \epsilon^{-1}(G^{4j} - T^{4j}), \quad C^4 = G^{44} - T^{44}, \quad C^4 \quad \text{and} \quad \mathcal{H}^j = \overline{\partial}_j \bar{u}^{ij}, \]

the constraint equations to be solved on the initial hypersurface \( S_0 = \{ (x^l, 0) \mid (x^l) \in \mathbb{R}^3 \} \) are:

\[ C^j = 0 \quad \text{(gravitational constraint equations)}, \quad (3.11) \]

\[ \mathcal{H}^j = 0 \quad \text{(harmonic gauge condition)}, \quad (3.12) \]

and

\[ \mathcal{N} = 0 \quad \text{(fluid velocity normalization)}. \quad (3.13) \]

To fix a region on which the system where both the evolution (3.1) and constraint equations (3.11)-(3.13) are well defined, we note from (2.14), (2.36), and the invertibility of the Lorentz metric \( \eta^{ij} \) that there exists a constant \( K_0 > 0 \) such that

\[ -\det(\eta^{ij} + 4\epsilon u^{ij}) > 1/16, \quad 1 + \epsilon w^4 > 1/16, \quad (3.14) \]

\[ A^i_0(\epsilon u) \geq \frac{1}{16} I, \quad a^i_0(\epsilon u, \epsilon w) \geq \frac{1}{16} I, \quad (3.15) \]

and

\[ |A^i_0(\epsilon u)| \leq 16, \quad |a^i_0(\epsilon u, \epsilon w)| \leq 16 \quad (3.16) \]

for all \( |u| \leq 2K_0, |\epsilon w| \leq 2K_0, |\epsilon a| \leq 2K_0 \). The choice of the bounds 1/16 and 16 is somewhat arbitrary, and they can be replaced by any number of the form 1/M and M for any \( M > 1 \) without changing any of the arguments presented in the following sections. However, since we are interested in the limit \( \epsilon \downarrow 0 \), we lose nothing by assuming \( M = 16 \).
3.1 Newtonian initial data

In [15], we proved the following theorem, based on previous work by Lottermoser [13], concerning the existence of \( \epsilon \)-analytic solutions to the constraints (3.11)-(3.13). Before we state the theorem, we note from (1.9), (1.8), and the weighted multiplication inequality (see [15] Lemma A.8) that if \( \alpha \in H^k_0 \) \((\delta \leq 0, k > 3/2)\) then \( \rho, \phi \in H^k_0 \).

**Proposition 3.1.** Suppose \(-1 < \delta < 0, k > 3/2 + 1, R > 0\) and \((\hat{\rho}, \hat{\phi}, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ}) \in (H^k_{k-1})^2 \times H^k_{k-2} \times H^k_{k-1} \times B_R(H^k_0)\). Then there exists an \( \epsilon_0 > 0 \), an open neighborhood \( U \) of \((\hat{\rho}, \hat{\phi}, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ})\), and analytic maps \((\epsilon_0, \epsilon_0) \times U \rightarrow H^k_{k-1} \) and \((\epsilon, \rho, \phi, w, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ}) \rightarrow \phi, \epsilon \times U \rightarrow H^k_0 \) and \((\epsilon, \rho, \phi, w, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ}) \rightarrow w^I\) such that for each \((\rho, \phi, w^I, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ}) \in U\), \((\epsilon, \rho, \phi, w^I, \hat{\omega}^I, \hat{\delta}_3^{IJ}, \hat{\delta}^{IJ}) \) is a solution to the three constraints

\[
\mathcal{C}^I = 0, \quad \mathcal{H}^I = 0, \quad \text{and} \quad \mathcal{N} = 0, \tag{3.17}
\]

where

\[
(\bar{u}^I) = \begin{pmatrix} \epsilon_3^{IJ} & \epsilon_\omega^I \\ \epsilon_\omega^I & \phi \end{pmatrix}, \tag{3.18}
\]

\[
(\partial_k \bar{u}^I) = \begin{pmatrix} \delta_3^{IJ} & -\partial_k \Lambda_3^{KJ} \\ -\partial_k \Lambda_3^{KJ} & -\partial_k \omega^K \end{pmatrix}, \tag{3.19}
\]

and

\[
w^I = \frac{1}{\epsilon} + \frac{-\epsilon \hat{g}_{4J} w^J - \sqrt{\epsilon^2 (\hat{g}_{4J} w^J)^2 - \hat{g}_{44} (\epsilon^2 g_{4J} w^I w^J + 1)}}{\epsilon \hat{g}_{44}}. \tag{3.20}
\]

Moreover, if we let \( \phi_0 = \phi\big|_{\epsilon=0}, w^I_0 = w^I\big|_{\epsilon=0}, \quad \text{and} \quad w_0^i = w^I|_{\epsilon=0}\), then \( \phi_0, w_0^I, \) and \( w_0^i \) satisfy the equations

\[
\Delta \phi_0 = \rho, \quad \Delta w_0^I = -\partial_k \Lambda_3^{KJ} + \rho w^I, \quad \text{and} \quad w_0^i = 0,
\]

respectively.

In section 5 we show that the analytic dependence of the initial data on \( \epsilon \) implies that there exists a corresponding convergent expansion in \( \epsilon \) for the solution generated from the initial data.

3.2 Uniform existence

To prove local existence of solutions to (3.1) on a uniform time interval independent of \( \epsilon \), we use a non-local symmetric hyperbolic version of (3.1). This system is essentially the one used in [15] to derive uniform existence, convergence, and error estimates for the limit \( \epsilon \searrow 0 \) of solutions to (3.1). However, we employ a few refinements that can be used to simplify the proof in [15], and will also be useful for analyzing the higher order expansions in \( \epsilon \).

Letting \( \chi_R \in C^\infty_0 \) be a cutoff function that satisfies

\[
\chi_R|_{B_R} = 1, \quad 0 \leq \chi_R \leq 1, \quad \text{and} \quad \text{supp} \chi_R \subset B_{2R},
\]

we replace \( g(V) \) in (3.1) with

\[
g(V) = (-\delta_3^I \delta_3^J \chi_R \rho, 0, \ldots, 0)^T, \tag{3.21}
\]

and, following [15], we define the Newtonian potential by

\[
\Delta \Phi = \chi_R \rho. \tag{3.22}
\]

Before proceeding, we first recall the following inequalities from [15]:
where

\[ W \]

Notice that the transformation (3.29) leaves the matter variables unaffected. Consequently, we can define \( H \) via the formula

\[ H \]

Embedding the fact that uniform analyticity is preserved under compositions, we get that the map \( H \) is an isomorphism by Proposition 2.2 of [1]. Next, by assumption \( \ell \geq 3/2 \), and hence it follows that the map \( H_{\eta-1}^{\ell} \ni \alpha \mapsto \rho = (4Kn(n+1))^{-n} \alpha^{2n} \in H_{\eta-1}^{\ell} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \) by Lemma 3.2.

Moreover, the linear map \( H_{\eta-1}^{\ell} \ni \alpha \rightarrow \Delta^{-1}(\chi_{\tilde{R}} \rho(\alpha)) \) is uniformly analytic and hence it follows that the map \( H_{\eta-1}^{\ell} \ni \alpha \rightarrow \Delta^{-1}(\chi_{\tilde{R}} \rho(\alpha)) \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

Next, we recall that differentiation \( H_{\eta-1}^{\ell+2} \ni \alpha \rightarrow \partial_j \alpha \in H_{\eta-1}^{\ell+2} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \), and hence it follows that the map \( H_{\eta-1}^{\ell+2} \ni \alpha \rightarrow \partial_j \alpha \in H_{\eta-1}^{\ell+2} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

Following [15], we use the Newtonian potential to define a new combined gravitational-matter variable \( W \) via the formula

\[ W = V - d\Phi, \]

where

\[ d\Phi := (0, \delta^i_j \delta^i_j \partial_j \Phi(\alpha), 0, 0, 0). \]

Notice that the transformation (3.29) leaves the matter variables unaffected. Consequently, we can define \( W \) by

\[ W = (u_t, W_t^{ij}, \partial u_t^{ij}, \alpha, w^i)^T, \]

and treat \( \Phi \) or \( d\Phi \) as a function of \( W \). In fact, by Lemma 3.2,

\[ H_{\eta-1}^{\ell} \ni W \mapsto d\Phi \in H_{\eta-1}^{\ell} \]

Lemma 3.2. Suppose \( \epsilon_0 > 0, -1 < \eta < -1/2, \) and \( \ell > 3/2 \). Then the maps

\[ \Phi : H_{\eta-1}^{\ell} \rightarrow H_{\eta}^{\ell+2} : \alpha \mapsto \Delta^{-1}(\chi_{\tilde{R}} \rho(\alpha)) \]

and

\[ \partial_t \Phi : H_{\eta-1}^{\ell} \rightarrow H_{\eta-1}^{\ell+1} : \alpha \mapsto \partial_t \Phi(\alpha) \]

are uniformly analytic \( \square \) for \( \epsilon \in [0, \epsilon_0] \).

Proof. First we recall that for \( -1 < \eta < -1/2 \), the Laplacian

\[ \Delta : H_{\eta}^{\ell+2} \rightarrow H_{\eta}^{\ell-2} \]

is an isomorphism by Proposition 2.2 of [1]. Next, by assumption \( \ell > 3/2 \), and hence it follows that the map \( H_{\eta-1}^{\ell} \ni \alpha \mapsto \rho = (4Kn(n+1))^{-n} \alpha^{2n} \in H_{\eta-1}^{\ell} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \) by Lemma 3.2.

Moreover, the linear map \( H_{\eta-1}^{\ell} \ni u \mapsto \chi_{\tilde{R}} u \in H_{\eta-1}^{\ell} \) is clearly well defined and uniformly bounded for \( \epsilon \in [0, \epsilon_0] \). Since compositions of uniformly analytic maps are again uniformly analytic, we see that the map \( H_{\eta-1}^{\ell} \ni \alpha \rightarrow \Delta^{-1}(\chi_{\tilde{R}} \rho(\alpha)) \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

Next, we recall that differentiation \( H_{\eta-1}^{\ell+2} \ni u \rightarrow \partial_j u \in H_{\eta-1}^{\ell+2} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \), and hence it follows that the map \( H_{\eta-1}^{\ell+2} \ni u \rightarrow \partial_j u \in H_{\eta-1}^{\ell+2} \) is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

Following [15], we use the Newtonian potential to define a new combined gravitational-matter variable \( W \) via the formula

\[ W = V - d\Phi, \]

where

\[ d\Phi := (0, \delta^i_j \delta^i_j \partial_j \Phi(\alpha), 0, 0, 0). \]

Notice that the transformation (3.29) leaves the matter variables unaffected. Consequently, we can define \( W \) by

\[ W = (u_t, W_t^{ij}, \partial u_t^{ij}, \alpha, w^i)^T, \]

and treat \( \Phi \) or \( d\Phi \) as a function of \( W \). In fact, by Lemma 3.2,

\[ H_{\eta-1}^{\ell} \ni W \mapsto d\Phi \in H_{\eta-1}^{\ell} \]

\( \square \)

See Appendix A for a definition of the term uniformly analytic.
defines a uniformly analytic map for \( \epsilon \in [0, \epsilon_0] \).

To formulate the evolution equation entirely in terms of \( W \), we need the “time derivative” of the \( \Phi \) map. So we define

\[
\Phi(W, \epsilon U, \epsilon W, \epsilon^2 U) := \Delta^{-1} \left( \frac{2n\chi_1\rho^{2n-1}}{(4Kn(n+1))^n} \Pi(a^I(\epsilon u, \epsilon w)^{-1}[a^I(w, \epsilon u, \epsilon w)\partial_I w + b(u, w, \epsilon u, \epsilon w)]) \right)
\]

where \( \Pi((\alpha, w)^T) = \alpha \) is a constant projection map. By construction, \( \dot{\Phi} = \partial_t \Phi \) when evaluated on a solution of the reduced Einstein–Euler equations.

**Lemma 3.3.** Suppose \( R_1 > 0, \epsilon_0 > 0, -1 < \eta < 1/2, \) and \( \ell > 3/2 \). Then there exists an \( R_2 > 0 \) such that the maps

\[
\Phi : B_{R_1}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \times B_{R_2}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \rightarrow H^{\ell+1}_\eta : (W, U, \bar{W}, \bar{U}) \mapsto \Phi(W, U, \bar{W}, \bar{U})
\]

and

\[
\partial_t \circ \Phi : B_{R_1}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \times B_{R_2}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \rightarrow H^{\ell}_\eta : (W, U, \bar{W}, \bar{U}) \mapsto \partial_t (\Phi(W, U, \bar{W}, \bar{U}))
\]

are uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

**Proof.** Fixing \( R_1 > 0, \epsilon_0 > 0, -1 < \eta < -1/2 \) and \( \ell > 3/2 \), it follows directly from Lemmas \([\Delta.1] \) and \([\Delta.7] \) that there exists a \( R_2 > 0 \) such that the map

\[
B_{R_1}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \times B_{R_2}(H^{\ell}_{\eta-1}, x) \times B_{R_2}(H^{\ell}_{\eta}) \ni (W, U, \epsilon W, \epsilon^2 U) \mapsto \chi_1 \rho^{2n-1} \Pi(a^I(\epsilon u, \epsilon w)^{-1}[a^I(w, \epsilon u, \epsilon w)\partial_I w + b(u, w, \epsilon u, \epsilon w)]) \in H^{\ell-1}_{\eta-2}
\]

is uniformly analytic for \( \epsilon \in [0, \epsilon_0] \). The rest of the proof now follows from the same arguments used in the proof of Lemma \([3.2] \).

To fit with the above notation, we define

\[
d\dot{\Phi} = (0, \delta_{ij}^1 \delta_{ij}^1 \partial_t \Phi, 0, 0, 0)^T.
\]

Noting that

\[
b^0(\epsilon V, \epsilon^2 U) = b^0(\epsilon W, \epsilon^2 U) \quad \text{and} \quad b^I(\epsilon V, \epsilon U, \epsilon V, \epsilon^2) = b^I(\epsilon W, \epsilon U, \epsilon W, \epsilon^2 U),
\]

we write \([3.31] \) as

\[
b^0(\epsilon W, \epsilon^2 U) \partial_t W = \frac{1}{\epsilon} c^I \partial_I W + b^I(\epsilon W, \epsilon U, \epsilon W, \epsilon^2 U) \partial_I W + F_0(W, \epsilon U, \epsilon W, \epsilon^2 U) + \epsilon F_1(W, \epsilon U, \epsilon W, \epsilon^2 U),
\]

where

\[
F_0(W, \epsilon U, \epsilon W, \epsilon^2 U) = f_0(W + d\Phi(W), \epsilon U, \epsilon(W + d\Phi(W)), \epsilon^2 U)
\]

\[
- b^0(\epsilon W, \epsilon^2 W) d\Phi(W, \epsilon U, \epsilon W, \epsilon^2 U) + b^I(\epsilon W, \epsilon U, \epsilon W) \partial_I d\Phi(W)
\]

and

\[
F_1(W, \epsilon U, \epsilon W, \epsilon^2 U) = f_1(W + d\Phi(W), \epsilon U, \epsilon(W + d\Phi(W)), \epsilon^2 U).
\]

**Proposition 3.4.** Suppose \(-1 < \delta < -1/2, \epsilon_0 > 0, s \in \mathbb{N}_0, R > 0, K_1 < K_0/(2\sqrt{\rho}C_{\text{Sub}}), \tau \geq 2K_1/C_{\text{Sub}}, \bar{R} > 16\tau + R, k \geq 3 + \alpha, w_I \in H^\delta_{\delta-1}, \supp{\alpha} \subset B_{\bar{R}}, s^{1j} \in H^\delta_{\delta+1}, s^{1j} \in H^\delta_{\delta-1}. \) Let \( \bar{u}_{ij} \), \( \partial_t \bar{u}_{ij} \), and \( w_{ij} \) be the initial data constructed in Proposition \([B.1] \) which, by choosing \( \epsilon_0 \leq 1 \) small enough, satisfies

\[
\left\| (\epsilon \partial_t \bar{u}_{ij}, \partial_t \bar{u}_{ij} - \delta_{ij}^1 \partial_0^1 \Delta \rho, 0, \alpha, w_{ij})^T \right\|_{H^\delta_{\delta-1}, \epsilon} \leq K_1, \quad \text{and} \quad \| w_{ij} \|^\delta_{\delta+1} \leq \frac{K_0}{\sqrt{\epsilon_0}C_{\text{Sub}}},
\]

\[
12
\]
for all \( \epsilon \in (0, \epsilon_0] \). Then there exists a \( T > 0 \) independent of \( \epsilon \in (0, \epsilon_0] \), and maps

\[
W_\epsilon = (u^{ij}_{\epsilon, t}, W^{ij}_{\epsilon, t}, \delta u^{ij}_{\epsilon}, \alpha_\epsilon, w^i_{\epsilon})^T \in X_{T, s, k, \delta - 1} \quad 0 < \epsilon \leq \epsilon_0
\]
such that

(i) \( T_\epsilon \geq T \) for \( 0 \leq \epsilon \leq \epsilon_0 \),

(ii) \( W_\epsilon \) is the unique solution to (3.34) with initial data

\[
W_\epsilon(0) = \left( \epsilon \partial_o \tilde{u}^{ij}_{\epsilon, o}, \partial_i \tilde{u}^{ij}_{\epsilon, o} - \delta^i_4 \delta^j_4 \partial_\epsilon \Delta^{-1} \rho_o, 0, \alpha, w^i_\epsilon \right)^T,
\]

(iii)

\[
\|W_\epsilon(t)\|_{H^k_{s-1, \epsilon}} \leq 2K_1, \quad \|\partial_\epsilon W_\epsilon(t)\|_{H^{k-1}_{s-1, \epsilon}} \lesssim 1,
\]

and

\[
\max \{\|\epsilon \tilde{u}^{ij}_{\epsilon}(t)\|_{L^\infty}, \|\epsilon \alpha_\epsilon (t)\|_{L^\infty}, \|\epsilon w^i(t)\|_{L^\infty}\} < 2K_0
\]

for all \((t, \epsilon) \in [0, T] \times (0, \epsilon_0]\),

(iv) if

\[
\limsup_{t \nearrow T_\epsilon} \|W_\epsilon(t)\|_{W^{1, \infty}} < \infty,
\]

and

\[
\sup_{0 \leq t < T_\epsilon} \{\|\epsilon \tilde{u}^{ij}_{\epsilon}(t)\|_{L^\infty}, \|\epsilon \alpha_\epsilon (t)\|_{L^\infty}, \|\epsilon w^i(t)\|_{L^\infty}\} < 2K_0,
\]

then the solution \(W_\epsilon(t)\) can be uniquely extended for some time \(T^*_\epsilon > T_\epsilon\),

(v) for any time \(\tilde{T}_\epsilon\) which is strictly less than the maximal existence time and for which

\[
\sup_{0 \leq t \leq \tilde{T}_\epsilon} \{\|\epsilon \tilde{u}^{ij}_{\epsilon}(t)\|_{L^\infty}, \|\epsilon \alpha_\epsilon (t)\|_{L^\infty}, \|\epsilon w^i(t)\|_{L^\infty}\} < 2K_0
\]

holds, the support of \(\alpha_\epsilon\) satisfies

\[
\text{supp } \alpha_\epsilon(t) \subset B_{\tilde{R}_\epsilon}, \quad \forall t \in [0, \tilde{T}_\epsilon],
\]

where \(\tilde{R}_\epsilon := 16 \sup_{0 \leq t \leq \tilde{T}_\epsilon} \|w^i(t)\|_{L^\infty} + R\),

(vi) \(\text{supp } \alpha_\epsilon(t) \subset B_{\tilde{R}}\) for all \((t, \epsilon) \in [0, T] \times (0, \epsilon_0]\),

(vii) \(\partial_t \tilde{u}^{ij}_{\epsilon} = \epsilon^{-1} \tilde{u}^{ij}_{\epsilon, t}\), and \(\partial_t \tilde{u}^{ij}_{\epsilon} = W^{ij}_{\epsilon, t} + \delta^i_4 \delta^j_4 \partial_\epsilon \Phi(\alpha_\epsilon)\), where \(\tilde{u}^{ij}_{\epsilon} = \tilde{u}^{ij}_{\epsilon, o} + \epsilon^{-1} \delta u^{ij}\),

(viii) the triple \(\{\tilde{u}^{ij}_{\epsilon}, \alpha_\epsilon, w^i_{\epsilon}\}\) determines, via the formulas (1.7), (1.9), (2.1), and (2.2), a solutions to the full Einstein-Euler system (1.1) in the harmonic gauge (2.4) on the spacetime region \(D_\epsilon = \mathbb{R}^3 \times [0, T]\), and

(ix) the conclusions (vii)-(viii) continue to hold on any region of the form \(D_\epsilon = \mathbb{R}^3 \times [0, \tilde{T}_\epsilon]\) provided \(\text{supp } \alpha_\epsilon(t) \subset B_{\tilde{R}}\) for all \(0 \leq t \leq \tilde{T}_\epsilon\).


Proof. (i)-(iv): Given the initial data satisfying
\[
\left\| \left( \epsilon \partial_t \tilde{u}_i^j, \partial_j \tilde{u}_i^j - \delta_i^j \delta_i^k \partial_t \bar{\alpha}_k \partial_t \bar{w}_i^j \right) \right\|_{H^k_{\bar{t} = 1, \epsilon}} \leq K_1, \quad \text{and} \quad \| \tilde{u}_i^j \|_{H^{k+1}} \leq \frac{K_0}{\epsilon_0 C_{\text{Sob}}}
\]
for all \( \epsilon \in (0, \epsilon_0] \), it is not difficult using the inequalities (3.23) and (3.24), and Lemmas 3.2, 3.3, and A.7 to verify that \( \| W_\epsilon(0) \|_{H^k_{\bar{t} = 1, \epsilon}} \leq K_1 \), \( \| \partial_t W_\epsilon(0) \|_{H^k_{\bar{t} = 1, \epsilon}} \leq 1 \), and the evolution equation (3.34) satisfies the conditions (B.3)-(B.5). Therefore, it follows directly from Theorem B.1 that there exists a time \( T > 0 \) independent of \( \epsilon \in (0, \epsilon_0] \) such that \( \| W_\epsilon(t) \|_{H^k_{\bar{t} = 1, \epsilon}} \leq 2K_1 < 2K_0/(\sqrt{\epsilon_0} C_{\text{Sob}}) \), and \( \| \partial_t W_\epsilon(t) \|_{H^k_{\bar{t} = 1, \epsilon}} \leq 1 \) for all \( 0 \leq t \leq T \). This proves (i)-(iii). Statement (iv) also follows directly from Theorem B.1.

(vi)-(ix): By (vi) we see that \( V_\epsilon(t) = W_\epsilon(t) + d\Phi(W_\epsilon(t)) \) satisfies (3.1) for \( (t, \epsilon) \in [0, T] \times (0, \epsilon_0] \). Then the same arguments used to prove (ii) and (iii) of Proposition 6.1 in [15] can be employed to prove the statements (vii)-(ix) of this Proposition.

\[\square\]

3.3 Zeroth order equation

In order to discuss equations satisfied by the zeroth and higher order expansions, we will first introduce some notation. To begin, we define
\[
\tilde{U} = (U, U, \ldots, U), \quad \tilde{W} = (W, W, \ldots, W),
\]
\[
\tilde{X} = (X, X, \ldots, X), \quad \tilde{Y} = (Y, Y, \ldots, Y),
\]
and let
\[
\mathcal{F}_\epsilon(U, W) = \mathcal{F}_0(W, \epsilon U, \epsilon W, \epsilon^2 U) + \mathcal{F}_1(W, \epsilon U, \epsilon W, \epsilon^2 U),
\]
\[
B_\epsilon(U, W, Y) = b^\ell(W, \epsilon U, \epsilon W, \epsilon^2 U)Y, \quad \epsilon^2 U, \epsilon W)X.
\]

Proposition 3.5. Suppose \( \ell > 3/2, R > 0, -1 < \eta < -1/2 \). Then there exists an \( \epsilon_0 > 0 \) such that the maps
\[
\mathcal{F}_\epsilon : B_\epsilon (\mathcal{H}_\eta^0) \times B_\epsilon (\mathcal{H}_{\eta-1, \epsilon}^0) \longrightarrow \mathcal{H}_{\eta-1, \epsilon}^0,
\]
\[
B_\epsilon : B_\epsilon (\mathcal{H}_\eta^0) \times B_\epsilon (\mathcal{H}_{\eta-1, \epsilon}^0) \times B_\epsilon (\mathcal{H}_{\eta-1, \epsilon}^0) \longrightarrow \mathcal{H}_{\eta-1, \epsilon}^0,
\]
are uniformly analytic for \( \epsilon \in [0, \epsilon_0] \).

Proof. The proof follows directly from Lemmas 3.2, 3.3, and the fact that compositions of uniformly analytic functions are again analytic. \[\square\]
Next, we define
\[
\mathcal{F}_\epsilon(\mathbf{U},\mathbf{W}) = \frac{1}{p!} \frac{d^p}{d\epsilon^p} \mathcal{F}_\epsilon(U(\epsilon), W(\epsilon)),
\]
\[
\mathcal{B}_\epsilon(\mathbf{U},\mathbf{W},\mathbf{Y}) = \frac{1}{p!} \frac{d^p}{d\epsilon^p} \mathcal{B}_\epsilon(U(\epsilon), W(\epsilon), Y(\epsilon)),
\]
and
\[
\mathcal{B}^0_\epsilon(\mathbf{U},\mathbf{W},\mathbf{X}) = \frac{1}{p!} \frac{d^p}{d\epsilon^p} \mathcal{B}^0_\epsilon(U(\epsilon), W(\epsilon), X(\epsilon)),
\]
where
\[
U(\epsilon) = \sum_{q=0}^{p-1} \epsilon^q U, \quad W(\epsilon) = \sum_{q=0}^{p-1} \epsilon^q W, \quad X(\epsilon) = \sum_{q=0}^{p-1} \epsilon^q X, \quad \text{and} \quad Y(\epsilon) = \sum_{q=0}^{p} \epsilon^q Y.
\]

**Proposition 3.6.** Suppose \( \epsilon > 3/2, R > 0, -1 < \eta < -1/2. \) Then there exists an \( \epsilon_0 > 0 \) such that the maps
\[
\mathcal{F}_\epsilon : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^p \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-1} \right) \rightarrow H^\ell_{\eta-1,\epsilon},
\]
\[
\mathcal{B}_\epsilon : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^p \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-1} \right) \times (H^\ell_{\eta-1,\epsilon})^p \rightarrow H^\ell_{\eta-1,\epsilon},
\]
and
\[
\mathcal{B}^0_\epsilon : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^{p-3} \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-2} \right) \times (H^\ell_{\eta-1,\epsilon})^p \rightarrow H^\ell_{\eta-1,\epsilon},
\]
are uniformly analytic for \( \epsilon \in [0, \epsilon_0] \). Moreover, there exists uniformly analytic maps
\[
\mathcal{F}_{R,\epsilon} : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^p \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-1} \right) \rightarrow H^\ell_{\eta-1,\epsilon},
\]
\[
\mathcal{B}_{R,\epsilon} : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^p \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-1} \right) \times (H^\ell_{\eta-1,\epsilon})^p \rightarrow H^\ell_{\eta-1,\epsilon},
\]
and
\[
\mathcal{B}^0_{R,\epsilon} : \left( B_R(H^\ell_\eta) \times (H^\ell_\eta)^{p-3} \right) \times \left( B_R(H^\ell_{\eta-1,\epsilon}) \times (H^\ell_{\eta-1,\epsilon})^{p-2} \right) \times (H^\ell_{\eta-1,\epsilon})^p \rightarrow H^\ell_{\eta-1,\epsilon}
\]
that are linear in the variables \( \frac{1}{\epsilon^{p+1}} \left[ \mathcal{F}_\epsilon(U(\epsilon), W(\epsilon)) - \sum_{q=0}^{p} \epsilon^q F(U, W) \right] = \frac{1}{\epsilon^{p+1}} \left[ \mathcal{F}_{R,\epsilon}(U, W) \right]
\]
\[
\frac{1}{\epsilon^{p+1}} \left[ \mathcal{B}_\epsilon(U(\epsilon), W(\epsilon), Y(\epsilon)) - \sum_{q=0}^{p} \epsilon^q B(U, W, Y) \right] = \frac{1}{\epsilon^{p+1}} \left[ \mathcal{B}_{R,\epsilon}(U, W, Y) \right]
\]
and
\[
\frac{1}{\epsilon^{p+1}} \left[ \mathcal{B}^0_\epsilon(U(\epsilon), W(\epsilon), X(\epsilon)) - \sum_{q=0}^{p} \epsilon^q B^0(U, W, X) \right] = \frac{1}{\epsilon^{p+1}} \left[ \mathcal{B}^0_{R,\epsilon}(U, W, X) \right].
\]
Proof. The proof follows immediately from the Taylor expansions for $\mathcal{F}_\varepsilon$, $B_\varepsilon$, and $B_\varepsilon^0$ which are uniformly analytic by Proposition \ref{prop:uniform_analyticity}.

We note that from the definition of the above maps, it is clear that

$$
\dot{B} = b'(W)Y_1 + \dot{B} = b'(\mathbf{U}, \mathbf{W}, Y_1) \quad \text{and} \quad \dot{B}^0 = X + \dot{B}^0(\mathbf{U}, \mathbf{W}, X),
$$

where

$$
\frac{b'(W)}{b'(W)} := b'(W, 0, 0, 0) \quad \text{and} \quad \dot{B} = \dot{B}^0 = 0.
$$

With our notation fixed, we are now ready to define the zeroth order equations:

$$
\partial_t^0 W = b'(W)\partial_t W + \mathcal{F}(W) + c^l\partial_t \omega, \quad c^l\partial_t \omega = 0,
$$

$$
\partial_t^0 W(0) = W_0(0)|_{t=0}.
$$

We showed in [15] that these equation are equivalent to the Poisson-Euler equations of Newtonian gravity. To see this, we first note that the Poisson-Euler-Makino system \eqref{eq:PEM} is (non-local) symmetric hyperbolic, and thus we can use the results of Appendix \ref{app:hyperbolicity} to obtain local existence of solutions.

**Proposition 3.7.** Let $k$, $s$, $\delta$, $\alpha$, and $w$ be as in Proposition \ref{prop:local_existence}. Then there exists a maximal time $T_0^M > 0$ and a unique solution

$$
\alpha_t^0, w_t^0 \in C^0([0, T_0^M), H^k_{\delta-1}), \
\Phi_0 \in C^0([0, T_0^M), H^{k+2}_{\delta+1}), \quad \partial_t \Phi_0 \in C^0([0, T_0^M), H^{k+1}_{\delta-1})
$$

to \eqref{eq:PEM} satisfying $\alpha_0^0 = \alpha$ and $w_0^0 = w^0$. Moreover,

$$
\alpha_t^0, w_t^0 \in X_{T_0^M, \delta, \delta-1}, \quad \Phi_0 \in X_{T_0^M, \delta, \delta+1}, \quad \partial_t \Phi_0 = -\partial_t \Delta^{-1}(0 \cdot w_t^0) \in X_{T_0^M, \delta, \delta-1},
$$

and

$$
\text{supp} \alpha_t^0 \subset B_{R(t)} \quad \forall t \in [0, T_0^M),
$$

where $R(t) = R + t \sup_{0 \leq s \leq t} ||w_t^0(s)||_{L^\infty}$.

**Proof.** From the weighted calculus inequalities of Appendix \ref{app:weighted_inequalities} (see also Appendix A of [15]), the Poisson-Euler-Makino system \eqref{eq:PEM} satisfies the conditions required by Theorem \ref{thm:odex}. Therefore all of the statements except for the estimate on the support of $\alpha_0^0(t)$ follow from this theorem. To prove the estimate on the support, we note that $\alpha_t^0 \in C^1([0, T_0^M), C^0_b(\mathbb{R}))$ by the Sobolev inequality \eqref{eq:sobolev}. Therefore we can integrate the differential equation $dw_t^0/dt = w_t^0(t, x)$ to get a $C^1$ flow $\psi_t^0(x)$ that is defined for all $(t, x) \in [0, T_0) \times \mathbb{R}^3$ and satisfies $\psi_0 = 1_{\mathbb{R}^3}$. For each $x \in \mathbb{R}^3$, define $\alpha_t^0(t, \psi_t^0(x))$. The evolution equation \eqref{eq:PEM} implies that

$$
\frac{d}{dt} \alpha_t^0(t) + \frac{1}{2n} \partial_t w_t^0(t, \psi_t^0(x))\alpha_t^0(t) = 0.
$$

By assumption, $\alpha_t^0(0) = \alpha(0, x) = 0$ for all $x \in E_R := \mathbb{R}^3 \setminus B_R$, and thus

$$
\alpha_t^0(t) = \alpha(t, \psi_t^0(x)) = 0 \quad \text{for all} \quad (t, x) \in [0, T_0^M) \times E_R.
$$
by the above differential equation. Moreover,
\[ |\psi_t(x) - x| \leq \int_0^t |\partial_s \psi_s(x)| \, ds \leq \int_0^t |\hat{w}^I_t(x, \psi_s(x))| \, ds \leq t \sup_{0 \leq s \leq t} \|\hat{w}^I(s)\|_{L^\infty}, \]
and hence it follows from (3.32) that \( \text{supp} \hat{a}(t) \subset B_{R(t)} \), where \( R(t) = R + t \sup_{0 \leq s \leq t} \|\hat{w}^I(s)\|_{L^\infty}. \)

Using this local existence theorem, the next proposition follows by straightforward computation.

**Proposition 3.8.** Let \( \{a(t), \hat{w}^I(t), \Phi(t)\} \) be the solution to the Makino-Euler-Poisson equations (3.10) from Proposition 3.7, and define
\[
\hat{W}(t) = (0, -\hat{\delta}^i_4 \hat{\delta}^j_4 \Phi(t), 0, \alpha(t), \delta^i_4 \hat{w}^I(t))^T \in X_{T_0^M, s, k, \delta - 1}, \quad \text{and} \quad \hat{\omega}(t) = (\hat{\omega}_4^{ij}(t), 1, \hat{\omega}_4^{ij}(t), 0, 0, 0)^T,
\]
where
\[
\hat{\omega}_4^{ij} = \hat{\delta}^i_4 \hat{\delta}^j_4 \hat{\Phi}, \quad \text{and} \quad \hat{\omega}_4^{ij} = \hat{\partial}_1 \Delta^{-1} \left( \hat{\omega}_4^{ij}(t) \hat{\omega}_4^{ij}(t) \right) \in X_{T_0^M, s, k+1, \delta - 1}.
\]

Then \( \{\hat{W}(t), \hat{\omega}(t)\} \) defines a unique solution to the initial value problem (3.30) on the time interval \( 0 \leq t < T_0^M \).

**4 First order expansion**

By Proposition 3.1, the initial data \( \hat{u}^{ij}_o \) is analytic in \( \epsilon \) and there exists a convergent expansion in \( H^{k+1}_\delta \) for \( \hat{u}^{ij}_o \) of the form \( \hat{u}^{ij}_o = \sum_{q=0}^{\infty} \epsilon^q \hat{u}^{ij}_o \) for \( 0 \leq \epsilon \leq \epsilon_0 \). Consequently, \( \hat{U} \) can be expanded as \( \hat{U} = \sum_{q=0}^{\infty} \epsilon^q \hat{U}_q \), where \( \hat{U} = (0, 0, \hat{\omega}^{ij}_o, 0, 0)^T \). Moreover, by Lemma 3.2 and the inequality (3.24), we can expand \( \hat{W}(0) \) as
\[
\hat{W}(0) = \sum_{q=0}^{\infty} \epsilon^q \hat{W}_q \tag{4.1}
\]
with the sum converging in \( H^{k}_{\delta - 1, \epsilon} \) uniformly for \( 0 \leq \epsilon \leq \epsilon_0 \).

We define the second order remainder \( \hat{Z}_r \) by
\[
\hat{W}_r = \hat{W} + \epsilon (\hat{\omega} + \hat{\omega}_r) + \epsilon^2 \hat{Z}_r, \tag{4.2}
\]
with the first order expansion term \( \hat{W}_r \) satisfying
\[
\frac{1}{\epsilon} \hat{b}^{0}_{\omega} \partial_t \hat{W}_r = \frac{1}{\epsilon} \hat{b}^{0}_{\omega} \partial_t \hat{W} + \hat{b}^{0}_{\omega} \partial_t \hat{W}_r + \hat{b}^{0}_{\omega} \partial_t \hat{\omega} + \hat{B}^{0}(\hat{U}, \hat{W}, \hat{Y}) - \hat{B}^{0}(\hat{W}, \hat{X}) + \hat{F}(\hat{U}, \hat{W}), \tag{4.3}
\]
\[
\hat{W}_r(0) = \hat{W} - \hat{\omega}(0), \tag{4.4}
\]
where
\[
\hat{b}^{0}_{\omega} = b^{0}_{\omega}(0, 0), \quad \hat{U} = \hat{U}, \quad \hat{W} = \hat{W}, \quad \hat{X} = \partial_t \hat{W}, \quad \hat{Y} = \partial_t \hat{W},
\]
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and

\[ \frac{1}{\epsilon} W = (W, \omega + \frac{1}{\epsilon} \omega), \quad \frac{1}{\epsilon} X = (\partial_t W, \partial_t \omega). \]

Observe that

\[ b^0 = \mathbb{I}, \]

by Proposition 3.8. Substituting (4.2) in (3.34) yields

\[ \frac{1}{\epsilon} \bar{B}^0 + b^0 \partial_t W + \frac{1}{\epsilon^2} \epsilon \partial_t \bar{W} + b^0 \frac{2}{\epsilon^2} \bar{Z} = \frac{1}{\epsilon} c^t \partial_t W + c^t \partial_t \omega + \frac{1}{\epsilon} \partial_t \bar{W} + \epsilon \partial_t \bar{Z} + B + \frac{1}{\epsilon} \bar{B} \frac{2}{\epsilon^2} \bar{Z} + \mathcal{F}, \]

where

\[ b^0 = b^0(\epsilon^2 U, \epsilon W), \quad \bar{B} = B(\epsilon, U, W, \partial_t W + \epsilon (\partial_t \omega + \partial_t W)), \]

and

\[ \frac{1}{\epsilon} \bar{B}^0 = B^0(\epsilon^2 U, \epsilon W, \partial_t W + \epsilon \partial_t \omega). \]

Using (4.3)-(4.4), we then find that \( Z \) satisfies

\[ \frac{1}{\epsilon} \epsilon^2 \partial_t Z = \frac{1}{\epsilon} c^t \partial_t \bar{Z} + \frac{1}{\epsilon^2} \partial_t \bar{Z} + \frac{2}{\epsilon^2} \bar{Z}, \]

where

\[ \frac{2}{\epsilon^2} \bar{K} = \frac{b^0 - b^0(\epsilon^2 U, \epsilon W)}{\epsilon^2} \epsilon \partial_t W + \frac{1}{\epsilon^2} \left[ \left( \frac{1}{\epsilon^q} B^0 \left( \frac{q-1}{\epsilon^q} \bar{W}, \bar{X} \right) - \frac{1}{\epsilon^q} \right) + \left( B - \frac{1}{\epsilon^q} B \left( \frac{q-1}{\epsilon^q} U, \bar{W}, \bar{Y} \right) \right) + \left( \mathcal{F} - \frac{1}{\epsilon^q} \mathcal{F} \left( \frac{q-1}{\epsilon^q} U, \epsilon W \right) \right) \right], \]

and

\[ U = (U, U), \quad Y = (\partial_t W, \partial_t \omega + \partial_t W). \]

Letting

\[ \bar{X} = (\partial_t W, \partial_t \omega + \partial_t W), \]

it follows from Proposition 3.6 that

\[ \frac{2}{\epsilon^2} \bar{K} = \mathcal{L}(U, \bar{W}, \bar{X}, \bar{Y}, \bar{Z}) + \mathcal{M}(\epsilon, U, \frac{1}{\epsilon} \partial_t W, \frac{1}{\epsilon} \bar{X}, Y, \frac{2}{\epsilon^2} \bar{Z}). \]

for analytic maps \( \mathcal{L} \) and \( \mathcal{M} \) with \( \mathcal{L} \) linear in \( \bar{Z} \).

As we shall see in Theorem 4.2 when the initial data is chosen such that \( \| \partial^2 W(0) \|_{H^k} \) remains bounded as \( \epsilon \downarrow 0 \), the \( \epsilon \) dependence can be removed from the first order expansion coefficient. This is
accomplished by replacing \([4.3] \text{--} [4.4]\) with a related, but different \(\epsilon\) independent version. To describe this system, we let
\[
\frac{1}{W} = (W_4^{ij}, W_7^{ij}, \partial_t W^{ij}, \alpha, W^{1/1_i})^T,
\]
and define projection operators by
\[
\Pi_4(\frac{1}{W}) = (u^{ij}_4) \quad \text{and} \quad \Pi_4(\frac{1}{W}) = (W^{ij}_4).
\]
Then the system that replaces \([4.3] \text{--} [4.4]\) is:
\[
\partial_t \frac{1}{W} = b^j \partial_j W_t + b^j \partial_j \omega + B(U, W, Y) - B^0(W, X) + F(U, W) + c^j \partial_j W^2,
\]
and define projection operators by
\[
\Pi_4(\frac{1}{W}) = (u^{ij}_4) \quad \text{and} \quad \Pi_4(\frac{1}{W}) = (W^{ij}_4).
\]
Existence of solutions to the initial value problem \([4.12] \text{--} [4.13]\) is covered by the following Proposition.

**Proposition 4.1.** Let \(\delta, k, s, K_1, R, \hat{R}, \) and \(\tau\) be as in Proposition \([3.4]\) \(T^M_0\) be as in Proposition \([3.7]\) and suppose \(T_0 < T^M_0\). If \(s\) and \(\tau\) are chosen so that \(s \geq 1\), and \(16\tau > \max\{32K_1, T_0 \sup_{0 \leq t \leq T_0} \sup \|\psi^j(t)\|_{L^\infty}\}\), then there exists a map
\[
\frac{1}{W} \in X_{T_0, s-1, k-1, \delta}
\]
such that \(\frac{1}{W}(t)\) is the unique solution to the initial value problem \([4.12] \text{--} [4.13]\), and \(\sup \frac{1}{W}(t) \subset B_{\hat{R}}\) for \(0 \leq t < T_0\),
where \(\rho = \frac{2}{(4Kn(n+1))^n} 0^{2n-1} 0^{1} \alpha\). Moreover, if the initial data satisfies \(c^j \partial_j \frac{1}{W}(0) = 0\), then
\[
\frac{1}{W}(t) = 0 \quad \text{for} \quad 0 \leq t < T_0, \quad \text{and} \quad \omega, \omega_4 \in X_{T_0, s-1, k-1, \delta-1}.
\]

**Proof.** By construction, we have
\[
\frac{1}{W_0} - \frac{1}{\omega}(0) \in H^{k-1}_{\delta-1}.
\]
Next, we observe that the map
\[
H^{l+1}_{\delta} \times (H^{l}_{\delta-1} \times H^{l-1}_{\delta-1}) \times H^{l-1}_{\delta-1} \ni (U, W, X, Y)
\]
\[
\longmapsto \Pi_4 (B(U, W, X) - B^0(W, X) + F(U, W)) \in H^{l-1}_{\delta-2}
\]
is analytic for \(l > 3/2 + 1\), which follows directly from the weighted estimates of Appendix \([A]\) (see also Appendix A of \([15]\)). It therefore follows that the system \([4.12] \text{--} [4.16]\) satisfies all the hypotheses of Theorem \([3.1]\). Thus, there exists a unique solution
\[
\frac{1}{W} \in X_{T_0, s-1, k-1, \delta-1}
\]
satisfying the initial value problem (4.12)-(4.13). Furthermore, from (3.28)-(4.18), it is clear that \( \omega = \partial_t \Omega \in X_{T_0, s-1, k, \delta-1} \). Note that we have used the linearity of the system (4.12)-(4.16) in \( W \) to conclude that the solution can be continued as long as the coefficients are well defined, which is the case for \( 0 \leq t \leq T_0 < T_0^M \).

By assumption, the initial data satisfies
\[
c^I \partial_t \frac{1}{2} W(0) = 0, \tag{4.20}
\]
while from Proposition 3.8 we have that
\[
0 \omega_{ij}(t) = \delta_i^I \delta_j^I \partial_t \Phi(t), \quad \text{and} \quad 0 \omega_{ij}(t) = 0, \tag{4.21}
\]
and hence
\[
0 u_{ij}^I(t) = 0, \quad 0 u_{ij}^I(t) = \delta_i^I \delta_j^I \partial_t \Phi(t), \quad \text{and} \quad 0 u_{ij}^I(t) = 0. \tag{4.22}
\]

From this it follows that \( 0 b' \) has a block diagonal structure of the form
\[
0 b' = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},
\]
and consequently
\[
0 b' \partial_t \omega = 0, \quad \Pi_i (0 b' \partial_t W) = 0, \quad \text{and} \quad \Pi_j (0 b' \partial_t W) = 0. \tag{4.23}
\]

Next, a straightforward calculation using (4.12), (4.14)-(4.15), and (4.23) shows that \( \partial_t \frac{1}{2} W(t) = 0 \) for \( 0 \leq t < T_0 \) by (4.20). By the definition of the \( c^I \), this is equivalent to (since \( \delta < 0 \))
\[
\frac{1}{2} W(t) = 0 \quad \text{and} \quad \partial^I \frac{1}{2} W(t) = 0. \tag{4.24}
\]

A short calculation using (4.12) and (4.24) then shows that
\[
\partial_i \delta_i^I \omega_{ij}^I = \omega_{ij}^I = \delta_i^I \delta_j^I \partial_t \Phi \tag{4.25}
\]
However, \( \delta_i^I \omega_{ij}^I(t) = 0 \) (see Proposition 3.1), and so integrating (4.25) yields
\[
\delta_i^I \omega_{ij}^I = \delta_i^I \delta_j^I \left( \Phi(t) - \Phi(0) \right), \tag{4.26}
\]
and
\[
\frac{1}{2} u_{ij}^I(t) = \frac{0}{0} u_{ij}^I + \delta_i^I \delta_j^I \Phi(t). \tag{4.27}
\]

Also by (4.24), we have that
\[
\frac{1}{2} u_{ij}^I(t) = \omega_{ij}^I(t) + \frac{1}{2} u_{ij}^I(t) = \delta_i^I \delta_j^I \Phi(t), \tag{4.28}
\]
while
\[
\frac{1}{2} u_{ij}^I(t) = \omega_{ij}^I(t) + \delta_i^I \delta_j^I \partial_t \Phi(t) + W_{ij}^I(t), \tag{4.29}
\]
where
\[
\Delta \frac{1}{2} \Phi = \frac{1}{2} \rho. \tag{4.30}
\]
We remark that in obtaining (4.30), we have used \( \text{supp } \hat{\rho}(t) \subset B_R \) for \( 0 \leq t < T_0 \), which follows from the definition of \( \hat{\rho} \) and Proposition 3.7.

Using (4.21), (4.22), (4.24), (4.27), (4.28), and (4.29) together, we can write (4.15) as
\[
\Delta \omega_i^4 = \partial^j \left( \partial_i \omega_i^{4j} + \delta_i^j \delta_i^k \partial_t \Phi \right).
\]

Moreover, it follows from the evolution equation (4.12) that
\[
\partial_t \omega_i^{4j} = -\partial_i \omega_i^{4j} + \partial_t \omega_i^{4j} - \delta_i^j \delta_i^k \partial_t \Phi.
\]

We also note that
\[
\omega_i^{4j} = \partial_t \Delta^{-1} (2 \rho \delta_i^j \delta_i^k \omega_i^{4j}),
\]
by Proposition 3.8, and hence
\[
\partial_t \partial_i \omega_i^{4j} = 0
\]
by (4.32). However, \( \partial_t \omega_i^{4j}(0) = 0 \) by Proposition 3.1, and thus we get from (4.34) that \( \partial_t \omega_i^{4j}(t) = 0 \).

This combined with (4.24) shows that (since \( \delta < 0 \))
\[
\omega_i^{4j}(t) = 0,
\]
and hence
\[
\omega_i^{4j}(t) = \omega_i^{4j}(t) + \delta_i^j \delta_i^k \partial_t \Phi(t).
\]

Using (4.21), (4.22), (4.24), (4.27), (4.28), and (4.29), and the evolution equation (4.12), a straightforward calculation then shows that the pair \( \{\alpha, \omega_i^{4j}\} \) satisfy
\[
\begin{align*}
\partial_t \omega_i^{4j} &= \partial_i \Delta^{-1} (2 \rho \delta_i^j \delta_i^k \omega_i^{4j}), \\
\partial_t \omega_i^{4j} &= -\partial_i \omega_i^{4j} + \partial_t \omega_i^{4j} - \delta_i^j \delta_i^k \partial_t \Phi
\end{align*}
\]
(4.32)

by (4.32). However, \( \partial_t \omega_i^{4j}(0) = 0 \) by Proposition 3.1, and thus we get from (4.34) that \( \partial_t \omega_i^{4j}(t) = 0 \).

This combined with (4.24) shows that (since \( \delta < 0 \))
\[
\omega_i^{4j}(t) = 0,
\]
and hence
\[
\omega_i^{4j}(t) = \omega_i^{4j}(t) + \delta_i^j \delta_i^k \partial_t \Phi(t).
\]

Using (4.21), (4.22), (4.27), (4.28), and (4.29), and the evolution equation (4.12), a straightforward calculation then shows that the pair \( \{\alpha, \omega_i^{4j}\} \) satisfy
\[
\begin{align*}
\partial_t \omega_i^{4j} &= \partial_i \Delta^{-1} (2 \rho \delta_i^j \delta_i^k \omega_i^{4j}), \\
\partial_t \omega_i^{4j} &= -\partial_i \omega_i^{4j} + \partial_t \omega_i^{4j} - \delta_i^j \delta_i^k \partial_t \Phi
\end{align*}
\]
(4.32)

by (4.32). However, \( \partial_t \omega_i^{4j}(0) = 0 \) by Proposition 3.1, and thus we get from (4.34) that \( \partial_t \omega_i^{4j}(t) = 0 \).

This combined with (4.24) shows that (since \( \delta < 0 \))
\[
\omega_i^{4j}(t) = 0,
\]
and hence
\[
\omega_i^{4j}(t) = \omega_i^{4j}(t) + \delta_i^j \delta_i^k \partial_t \Phi(t).
\]

Using (4.21), (4.22), (4.27), (4.28), and (4.29), and the evolution equation (4.12), a straightforward calculation then shows that the pair \( \{\alpha, \omega_i^{4j}\} \) satisfy
\[
\begin{align*}
\partial_t \omega_i^{4j} &= \partial_i \Delta^{-1} (2 \rho \delta_i^j \delta_i^k \omega_i^{4j}), \\
\partial_t \omega_i^{4j} &= -\partial_i \omega_i^{4j} + \partial_t \omega_i^{4j} - \delta_i^j \delta_i^k \partial_t \Phi
\end{align*}
\]
(4.32)

by (4.32). However, \( \partial_t \omega_i^{4j}(0) = 0 \) by Proposition 3.1, and thus we get from (4.34) that \( \partial_t \omega_i^{4j}(t) = 0 \).

This combined with (4.24) shows that (since \( \delta < 0 \))
\[
\omega_i^{4j}(t) = 0,
\]
and hence
\[
\omega_i^{4j}(t) = \omega_i^{4j}(t) + \delta_i^j \delta_i^k \partial_t \Phi(t).
\]
Proof. (i)-(ii): and by (4.39), (4.40), and Proposition 3.7.

and hence Theorem 4.2.

Let by (4.37), (4.40), (4.41), and (4.42). It then follows from (4.30) that (ii) and there exists maps $K$, $\omega$, such that the solution $W_\epsilon(t)$ $(0 < \epsilon \leq \epsilon_0)$ exists on the interval $[0, \tilde{T}_\epsilon)$, where

$$\tilde{T}_\epsilon = \min \left\{ T_0, \frac{1}{K_2} \ln \left( \frac{K_3}{\epsilon} \right) \right\},$$

and obeys the bounds

$$\sup_{0 \leq t < \tilde{T}_\epsilon} \max \{ \| \epsilon u_\epsilon(t) \|_{L^\infty}, \| \epsilon \alpha_\epsilon(t) \|, \| \epsilon w_\epsilon(t) \|_{L^\infty} \} < 2K_0,$$

$$\sup_{0 \leq t < \tilde{T}_\epsilon} \| W_\epsilon(t) \|_{W^{1,\infty}} < \infty, \quad \text{supp} \rho_\epsilon(t) \subset B_{\tilde{R}},$$

(iii) and there exists maps

$$\frac{1}{\epsilon} W_\epsilon \in X_{T_0,s-1,k-1,\delta-1} \quad 0 < \epsilon \leq \epsilon_0,$$

such that $\frac{1}{\epsilon} W_\epsilon$ is the unique solution to the initial value problem (4.13)–(4.14), and

$$\| W_\epsilon(t) - \frac{1}{\epsilon} W(t) - \epsilon \left( \frac{1}{\epsilon} \omega(t) + \frac{1}{\epsilon} \omega(t) \right) \|_{H^{k-2}} \lesssim \| W_\epsilon(t) - \frac{1}{\epsilon} W(t) - \epsilon \left( \frac{1}{\epsilon} \omega(t) + \frac{1}{\epsilon} \omega(t) \right) \|_{H^{k-2}} \lesssim e^{K_2t} \epsilon^2$$

for all $(t, \epsilon) \in [0, \tilde{T}_\epsilon) \times (0, \epsilon_0]$.

(iii) Moreover, if $W_\epsilon(0)$ satisfies $\| \partial^2 \omega \epsilon(0) \|_{H^{k-2}} \lesssim 1$ for $0 \leq \epsilon \leq \epsilon_0$, then

$$\| W_\epsilon(t) - \frac{1}{\epsilon} W(t) - \epsilon \left( \frac{1}{\epsilon} \omega(t) + \frac{1}{\epsilon} \omega(t) \right) \|_{H^{k-2}} \lesssim \| W_\epsilon(t) - \frac{1}{\epsilon} W(t) - \epsilon \left( \frac{1}{\epsilon} \omega(t) + \frac{1}{\epsilon} \omega(t) \right) \|_{H^{k-2}} \lesssim e^{K_2t} \epsilon^2$$

for all $(t, \epsilon) \in [0, \tilde{T}_\epsilon) \times (0, \epsilon_0]$, where $\frac{1}{\epsilon} W \in X_{T_0,s-1,k-1,\delta-1}$ is the unique solution to the initial value problem (4.13)–(4.14).

Proof. (i)-(ii): Fix $T_\epsilon < \min \{ T, T_0 \}$, and let

$$C_1 = \sup_{0 \leq t \leq T_\epsilon} \| \frac{1}{\epsilon} W(t) \|_{H^{k-1}} + \sup_{0 \leq t \leq T_\epsilon} \| \partial \frac{1}{\epsilon} W(t) \|_{H^{k-1}},$$

$$C_2 = \sup_{0 \leq t \leq T_\epsilon} \| \frac{1}{\epsilon} \omega(t) \|_{H^{k-1}} + \sup_{0 \leq t \leq T_\epsilon} \| \partial \frac{1}{\epsilon} \omega(t) \|_{H^{k-1}},$$

and

$$C_3 = \| \frac{1}{\epsilon} W - \frac{1}{\epsilon} \omega(0) \|_{H^{k-1}}.$$
Since

$$\|\tilde{u}^{ij}_\delta\|_{H^{k+1}_\delta} \leq \frac{K_0}{\sqrt[6]{C_{\text{Sob}}}}$$

and \( \frac{1}{\epsilon} \) satisfies the linear equation \((4.33)\), it follows from the energy estimates derived in the proof of Theorem \([3.7]\) that there exists a constant \( K_2 = K_2(C_1, C_2, K_0/(\sqrt[6]{C_{\text{Sob}}})) \) such that

$$\|W_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} \leq e^{K_2 T} C_3 + K_2 \quad \forall (t, \epsilon) \in [0, T_\ast] \times (0, \epsilon_0].$$

Next, we observe that

\[(iii):\]

To prove statement (iii), we first observe that it follows from the evolution equation \((3.34)\) that the \( \frac{1}{\epsilon} \) satisfies an estimate of the form

$$\epsilon C_0 (\|Z_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} + \|\dot{W}_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} + C_2),$$

where

$$\|W_\epsilon(t)\|_{W^{1, \infty}_\delta} \leq C_{\text{Sob}} \left[ \epsilon^2 \|Z_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} + \epsilon \|\dot{W}_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} + C_2 \right].$$

Setting \( \frac{1}{\epsilon} \) = \( \frac{1}{\epsilon} \) satisfying the equation \( \|Z_\epsilon(t)\|_{H^{k-1}_\delta, \epsilon} \leq \epsilon C_4 \). Moreover, from the error equation \((4.38)\), it is clear that \( \frac{1}{\epsilon} \) satisfies an equation to which Theorem \([3.1]\) applies. Therefore, for any \( K_3 > \epsilon C_4 (0 \leq \epsilon \leq \epsilon_0) \) there exists constants \( K_4, K_5 \) such that \( \frac{1}{\epsilon} \) satisfies an estimate of the form

$$\|Z_\epsilon(t)\| \leq \epsilon \left( e^{K_3 t} [C_4 + K_5] - K_3 \right) \leq K_3 \quad \text{for } 0 \leq t < \bar{T},$$

where

$$\bar{T} = \min \left\{ T_\ast, \frac{1}{K_4} \ln \left( \frac{K_3 + \epsilon K_5}{\epsilon (C_4 + K_5)} \right) \right\}.$$

Statements (i) and (ii) now follow directly from Propositions \([3.4]\) and \([3.7]\) and the estimates \((4.25), (4.41) - (4.46), (4.47),\) and \((4.48)\), provided \( \epsilon_0 \) is chosen small enough.

(iii): To prove statement (iii), we first observe that it follows from the evolution equation \((4.34)\) that the condition \( \|\partial^2_t W_\epsilon(0)\|_{H^{k-1}_\delta} \leq 1 \) for \( 0 < \epsilon \leq \epsilon_0 \) is equivalent to the condition \( e^t \partial^2_t \dot{W}(0) = 0 \). Then replacing \( \frac{1}{\epsilon} \) with \( \frac{1}{\epsilon} \) in \((4.2)\) with \( \frac{1}{\epsilon} \) and \( \frac{1}{\epsilon} + Z_\epsilon(t) \), respectively, it is not difficult using Proposition \([3.1]\) to show that the new error term \( \frac{1}{\epsilon} \) will satisfy the same type of estimate as above. We emphasize that the key property used to make this replacement is that \( \frac{1}{\epsilon} \) and \( \frac{1}{\epsilon} \) satisfy \( e^t \partial^2_t \dot{W}(t) = 0 \) and \( \frac{1}{\epsilon} \in X_{T_\ast, s-1, k-1, \delta-1} \). The proof of statement (iii) now follows as we are able to replace \( \frac{1}{\epsilon} \) with \( \frac{1}{\epsilon} \) everywhere in the above estimates.

\[\square\]

5 Higher order expansions and convergence

**Theorem 5.1.** Let \( \delta, k, s, K_1, R, \bar{R}, \) and \( W_\epsilon(t) \) be as in Proposition \([3.3]\), \( \{W(t), \omega(t)\} \) as in Proposition \([3.8]\), \( T_0^M \) as in Proposition \([3.7]\), \( \dot{W}_\epsilon(t) \) and \( \bar{\tau} \) as in Theorem \([4.2]\), and suppose \( T_0 < T_0^M \). If \( s \geq 3 \), then
for $\epsilon_0$ small enough, there exists an infinite sequence of maps

$$W_\epsilon \in X_{T_0, s-2, k-2, \delta-1} \quad q \in \mathbb{Z}_{\geq 2}$$

such that

(i) each $\tilde{W}(t)$ satisfies a linear (non-local) symmetric hyperbolic system with initial data $\tilde{W}_\epsilon(0) = \tilde{W}_0$

and coefficients depending on $\epsilon$, $W$, $\omega$, $U$ for $0 \leq r \leq q$, and $W_\epsilon$ for $1 \leq r \leq q - 1$,

(ii) $\|W_\epsilon(t)\|_{H^{k-2}} + \epsilon\|\partial_t W_\epsilon(t)\|_{H^{k-3}} \lesssim \|W_\epsilon(t)\|_{H^{k-2}} + \epsilon\|\partial_t W_\epsilon\|_{H^{k-3}} \lesssim 1$

for all $(t, \epsilon, q) \in [0, T_0) \times (0, \epsilon_0) \times \mathbb{Z}_{\geq 2}$, and

(iii) $W_\epsilon(t) = 0 + W(\omega(t) + W) + \sum_{q=0}^{\infty} \epsilon^q W_\epsilon(t) \quad (t, \epsilon) \in [0, T_0) \times (0, \epsilon_0], \quad (5.1)$

where the sum converges uniformly in $C^0([0, T_0); H^{k-3}_{\delta-1, \epsilon})$ and $C^0([0, T_0); H^{k-3}_{\delta-1, \epsilon})$.

(iv) Moreover, if $s - 2 \geq p \geq 1$, and the initial data is chosen so that

$$\|\partial_t^{q+1} W_\epsilon(0)\|_{H^{k-1}_{s-1, \epsilon}} \lesssim 1 \quad q = 1, 2, \ldots, p,$$

then there exists $\epsilon$-independent maps

$$\tilde{W} \in X_{T_0, s-\epsilon, k, q, \delta-1} \quad \text{and} \quad \tilde{W}_\epsilon \in X_{T_0, s-\epsilon, k, q, \delta-1} \quad q = 1, 2, \ldots, p$$

such that

(iv,a) each $\tilde{W}$ satisfies a $\epsilon$-independent linear (non-local) symmetric hyperbolic system with coefficients depending only on $\tilde{W}$ for $0 \leq r \leq q$, $\tilde{W}$ for $0 \leq r \leq q + 1$, and $\tilde{W}$ for $0 \leq r \leq q - 1$, and

(iv,b) the terms $\tilde{W}_\epsilon$ in the sum (5.1) can be replaced by $\tilde{W} + \tilde{W}_\epsilon$ for $1 \leq q \leq p$ with the sum converging uniformly $C^0([0, T_0); H^{k-1(q+2)}_{\delta-1, \epsilon})$ and $C^0([0, T_0); H^{k-1(q+2)}_{\delta-1, \epsilon})$.

Proof. The proof of this Theorem follows from a straightforward adaptation of the proof of Theorem 3 in [21]. We will only sketch the details.

Following Schochet [21] (see also [11]), we consider the following iteration:

$$b^0(Z_\epsilon) = \frac{1}{\epsilon} e_0 \partial_t Z_\epsilon + b^0(Z_\epsilon) \partial_t s + L(Z_\epsilon) + e^m Z_\epsilon, \quad (5.2)$$

where

$$Z_1 = 0, \quad \tilde{W}_\epsilon = W + e(\omega + W) + e^m Z_\epsilon, \quad b^1(Z_\epsilon) = b^1(W, \epsilon U, \epsilon W, e^2 U),$$

$$b^0(Z_\epsilon) = b^0(e^2 U, e^m W), \quad L(Z_\epsilon) = L(U, W, X, Y, Z_\epsilon), \quad \mathcal{M}(Z_\epsilon) = \mathcal{M}(\epsilon, U, W, \epsilon U, Z_\epsilon).$$

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Using the energy estimates of Theorem 3.1 and the weighted Sobolev estimates in Appendix A (see also [15]), it is clear the arguments of Schochet can be generalized to show that
\[ \|Z(t)\|_{H^{k-2}_{\delta_{-1},e}} + \epsilon \|\partial_t^m Z(t)\|_{H^{k-3}_{\delta_{-1},e}} \lesssim 1, \] (5.4)
and
\[ \|Z(t) - Z(t)\|_{H^{k-3}_{\delta_{-1},e}} \lesssim \epsilon \|Z(t) - Z(t)\|_{H^{k-3}_{\delta_{-1},e}} \] (5.5)
for all \((t, \epsilon) \in [0, T_0] \times (0, \epsilon_0].\) Therefore by (4.1), (4.4), (5.5), and the uniqueness of solutions to the evolution equation (5.2), we see that for \(\epsilon_0\) small enough the sequence \(W(t) + \epsilon(W_\epsilon(t) + \omega(t)) + \epsilon^2 \frac{1}{m} Z_\epsilon(t)\) converges in \(C^0([0, T_0), H^{k-3}_{\delta_{-1},e})\) to \(W_\epsilon(t)\) for each \(\epsilon \in (0, \epsilon_0].\) Therefore, defining
\[ m^+ = \frac{m+1}{\epsilon^{m-1}}, \]
we have that
\[ W_\epsilon(t) = W(t) + \epsilon(W_\epsilon(t) + \omega(t)) + \sum_{q=2}^\infty \epsilon^q W_\epsilon(t) \]
with the sum converging in \(C^0([0, T_0), H^{k-3}_{\delta_{-1},e})\) for each \((\epsilon \in (0, \epsilon_0].\) Moreover, because of the inequality (5.2), it follows that the sum converges uniformly in \(C^0([0, T_0), H^{k-3})\) for \(\epsilon \in (0, \epsilon_0].\) This completes the proof of statements (i)-(iii). The proof of statement (iv) also follows easily from the arguments used in the proof of Theorem 3 in [21].

\[ \square \]

**Remark 5.2.** The equations satisfied by the \(W\) from part (iv) of Theorem 5.1 are:
\[ \partial_t W = b'(W)\partial_t W + b'(W)\partial_t \omega + B(U, W, \omega) - \partial_t \omega \]
\[ - B'(U, W, X) + \mathcal{F}(U, W) + \epsilon \partial_t \omega_{+1}, \]
\[ \epsilon \partial_t W_0 = 0, \]
\[ W(0) = \tilde{W}(0), \]
where
\[ U = (U, \ldots, \tilde{U}), \quad \tilde{W} = (0, W, \omega + W), \]
\[ X = (\partial_t W, \partial_t \omega + \partial_t W, \ldots, \partial_t \omega + \partial_t \omega), \quad \text{and} \quad \tilde{Y} = (\partial_t W, \partial_t W, \frac{1}{\epsilon} \partial_t \omega + \partial_t \omega). \]

## 6 The first post-Newtonian expansion

We are now ready to prove the main theorem that guarantees the existence of a large class of solutions to the Einstein-Euler equations that can be expanded to the first post-Newtonian order.

**Proof of Theorem 6.1.** Using the harmonic equations
\[ \epsilon \partial_t \tilde{u}^{44} = -\partial_t \tilde{u}^{44}, \quad \text{and} \quad \epsilon \partial_t \tilde{u}^{IJ} = -\partial_t \tilde{u}^{IJ}, \] (6.1)
we can write the constraint equations (3.11) as
\[ \Delta \tilde{u}^{4k} = \delta^k_4 \rho - \delta^k_4 \partial_L \partial_t \tilde{u}^{4L} + \epsilon \left[ Q_0^{4k}(\epsilon \tilde{u}^{ij}, \partial_t \tilde{u}^{ij}, \epsilon \partial_t \tilde{u}^{KL}) + Q_1^{4k}(\epsilon^2 \tilde{u}^{ij}, \epsilon \partial_t \tilde{u}^{ij} + \epsilon \partial_t \tilde{u}^{ij} + \epsilon \partial_t \tilde{u}^{ij}) + Q_2^{4k}(\epsilon^2 \tilde{u}^{ij}, \epsilon \partial_t \tilde{u}^{ij} + \epsilon \partial_t \tilde{u}^{ij} + \epsilon \partial_t \tilde{u}^{ij}) \right] \]
where $Q^{ij}_{ij}(y_1, y_2, y_3)$ is bilinear in $y_1$ and $(y_2, y_3)$. $Q^{ij}_{ij}(y_1, y_2, y_3)$ is quadratic in $y_2, y_3$, and the maps $Q^{ijk}_{ij}$ ($\nu = 0, 1, 2$) are analytic in all their variables for $\epsilon^2 \bar{u}^{ij} \in \mathcal{V}$. We can also write the $KL$-components of the reduced Einstein equations (2.13) as

$$
\partial_t^2 \bar{u}^{KL} = \frac{1}{\epsilon^2 (1 - \epsilon^2 \bar{u}^{44})} \left[ \Delta \bar{u}^{KL} + 2 \epsilon^3 \bar{u}^{14} \partial_t \partial_t \bar{u}^{KL} + \epsilon^2 \bar{u}^{1j} \partial_t \partial_t \bar{u}^{KL} + \epsilon^2 Q_0^{KL} (\epsilon^2 \bar{u}^{ij}, \partial_t \partial_t \bar{u}^{ij}, \epsilon \partial_t \bar{u}^{i}) \right]$$

$$
- \epsilon^2 \left( \rho w^K w^L + \rho \delta^{KL} \right) + \epsilon^3 Q_1^{KL} (\epsilon \bar{u}^{ij}, \epsilon^2 \bar{u}^{ij}, w, \epsilon w) \right],
$$

where $Q_0^{KL}(y_1, y_2, y_3)$ is quadratic in $(y_2, y_3)$,

$$
Q_1^{KL} = Q_2^{KL} (\epsilon \bar{u}^{ij}, \epsilon^2 \bar{u}^{ij}, w, \epsilon w) \alpha^2 + Q_3^{KL} (\epsilon \bar{u}^{ij}, \epsilon^2 \bar{u}^{ij}, w, \epsilon w) \bar{w}^I w^J,$

and all of the maps $Q^{KL}_\nu$ ($\nu = 0, 1, 2, 3$) are analytic in their arguments for $\epsilon^2 \bar{u} \in \mathcal{V}$.

We now take

$$
\left\{ \begin{array}{l}
\partial_t \bar{u}^{i} (0) = \epsilon^2 \bar{f}_i, \\
\alpha (0) = \alpha_0, w^I (0) = w^I_0, \bar{f}^I (0)
\end{array} \right.
$$

as the prescribed initial data, and solve the non-linear elliptic system

$$
\Delta \bar{u}^{ik} = \bar{\Lambda}^{ik} := \delta^k_i \rho - \delta^k_i \partial_L \epsilon \bar{u}^{KL} + \epsilon \left[ Q_0^{ik} (\epsilon \bar{u}^{ij}, \partial_t \partial_t \bar{u}^{ij}, \epsilon \partial_t \partial_t \bar{u}^{KL}) + Q_1^{ij} (\epsilon^2 \bar{u}^{ij}, \partial_t \partial_t \bar{u}^{ij}, \epsilon \partial_t \partial_t \bar{u}^{KL}) + Q_4^{ij} (\epsilon \bar{u}^{ij}, \epsilon^2 \bar{u}^{ij}, w, \epsilon w) \alpha^2 \right],
$$

$$
\Delta \bar{u}^{KL} = \bar{\Lambda}^{KL} := -2 \epsilon^3 \bar{u}^{14} \partial_t \partial_t \bar{u}^{KL} - \epsilon^2 \bar{u}^{1j} \partial_t \partial_t \bar{u}^{KL} - \epsilon^2 Q_0^{KL} (\epsilon^2 \bar{u}^{ij}, \partial_t \partial_t \bar{u}^{ij}, \epsilon \partial_t \bar{u}^{i})$$

$$
+ \epsilon^2 \left( \rho w^K w^L + \rho \delta^{KL} \right) - \epsilon^3 Q_1^{KL} (\epsilon \bar{u}^{ij}, \epsilon^2 \bar{u}^{ij}, w, \epsilon w) + \epsilon^4 (1 - \epsilon^2 \bar{u}^{14}) f_K^L,
$$

to determine the initial data $\{ \bar{u}^{ij} |_{t=0}, \partial_t \bar{u}^{ij} |_{t=0} \}$ on $S_0 = \{ (x^I, 0) | (x^I) \in \mathbb{R}^3 \}$. Note that $w^4$ is determined by the fluid velocity normalization (3.13), which can be written as

$$
w^4 = \frac{1}{\epsilon} f (\epsilon w^I, \epsilon^2 \bar{u}^{ij}),
$$

where $f(y_1, y_2)$ is analytic in a neighborhood of $(0,0)$ and $f(y) = O(|y|^2)$ as $y \to 0$.

Using the weighted multiplication inequality (see [15], Lemma A.8) and Lemma A.7, it is straightforward to verify that there exists an $\epsilon_0 > 0$ such that $\Lambda^{ij}$ (see (6.3)-(6.4)) defines an analytic map

$$
(\epsilon, \frac{1}{\epsilon}, 0, w^I, \bar{f}^I, \bar{u}^{ij}) \in (-\epsilon_0, \epsilon_0) \times H^{k-1}_{\delta-1} \times H^{k}_{\delta-1} \times H^{k}_{\delta-2} \times H^{k}_{\delta-2} \times H^{k}_{\delta-2} \longrightarrow \Lambda^{ij} \in H^{k-2}_{\delta-2},
$$

where

$$
\Lambda^{4i} = \delta^k_i \rho + O(\epsilon) \quad \text{and} \quad \Lambda^{KL} = 0(\epsilon^2) \quad \text{as} \quad \epsilon \searrow 0.
$$

Writing (6.3)-(6.4) as

$$
\bar{u}^{ij} = \Delta^{-1} \Lambda (\epsilon, \frac{1}{\epsilon}, 0, w^I, \bar{f}^I, \bar{u}^{ij}),
$$

it follows from (6.6) and the invertibility of the Laplacian $\Delta : H^{k}_{\delta} \to H^{k-2}_{\delta-2}$ that we can use the analytic version of the implicit function theorem [8] to conclude that there exists an open neighborhood $U$ of any point in $H^{k}_{\delta-1} \times H^{k}_{\delta-1} \times H^{k}_{\delta-2} \times H^{k-2}_{\delta-2} \times H^{k}_{\delta-2}$, and analytic maps

$$
(\epsilon, \frac{1}{\epsilon}, 0, w^I, \bar{f}^I) \in (-\epsilon_0, \epsilon_0) \times U \longrightarrow \bar{u}^{ij} \in H^{k}_{\delta}
$$

that solve equations (6.3)-(6.4). Moreover, it follows from (6.6) that

$$
\| \bar{u}^{KL}(0) \|_{H^{k}_{\delta-1}} \lesssim \epsilon^2 \quad \forall \epsilon \in [0, \epsilon_0],
$$

(6.7)
and hence
\[ \| \partial_t^2 \tilde{u}_x^{KL}(0) \|_{H_{3-\ell}^k} \lesssim \epsilon^2 \quad \forall \epsilon \in [0, \epsilon_0]. \] (6.8)

Also, we note that by construction
\[ \| \partial_t \tilde{u}_x^{KL}(0) \|_{H_{3-1}^{k-1}} \lesssim \epsilon^2 \quad \forall \epsilon \in [0, \epsilon_0]. \] (6.9)

Differentiating the harmonic conditions (6.1) with respect to \( t \), and using (3.25), (3.29), and (3.31). The proof of Theorem 1.1, now follows directly from Theorem 5.1, and the estimates (6.17)-(6.19).

Using (6.1), the Euler equations (2.26) can be written as
\[ \partial_t w = [a^4(\epsilon^2 \tilde{u}^{ij}, \epsilon w)]^{-1} \left( a^4(w, \epsilon^2 \tilde{u}^{ij}, \epsilon w) \partial_t w + b_0(\partial_t \tilde{u}^{ij}, \epsilon \partial_t \tilde{u}^{IJ}) + b_1(\epsilon w, \epsilon^2 \tilde{u}^{ij}, \epsilon \partial_t \tilde{u}^{IJ}, \epsilon^2 \partial_t \tilde{u}^{IJ}) \right), \] (6.12)

where the maps \( a^4, a^t, b_0, b_1 \) are analytic in all their arguments for \( \epsilon \tilde{u} \in \mathcal{V} \), and \( a^4(0,0) = I \), \( a^t(0,0,0,0) = 0 \), \( b_0(y_1,y_2) \) is linear, and \( b^4(y_1,y_2,y_3,y_4,y_5,y_6,y_7) \) is linear in \( (y_4,y_5,y_6,y_7) \) and satisfies \( b^4(0,0,0,4,5,6,7) = 0 \). Then differentiating (6.1), (6.2), and (6.12) with respect to \( t \) while using (6.7)-(6.11) shows that
\[ \| \partial_t^p \tilde{u}_x^{KL}(0) \|_{H_{3-\ell}^k} \lesssim 1 \quad p = 3, 4, \] (6.13)
\[ \| \partial_t^p \alpha_x(0) \|_{H_{3-1}^{k-1}} \lesssim 1 \quad p = 0, \ldots, 3, \] (6.14)
\[ \| \partial_t^p w_x(0) \|_{H_{3-1}^{k-1}} \lesssim 1 \quad p = 0, \ldots, 3, \] (6.15)

and
\[ \| \epsilon \partial_t^4 \tilde{u}^{4J} \|_{H_{3-3}^{k-4}} \lesssim 1 \] (6.16)

for all \( \epsilon \in [0, \epsilon_0] \). We then find from the definition of \( W_\epsilon \), the estimates (6.7)-(6.11), and (6.13)-(6.16), that
\[ \| \partial_t^3 W_\epsilon(0) \|_{H_{3-1}^{k-1}} \lesssim 1 \quad \text{for } p = 0, 1, 2, 3 \text{ and } 0 \leq \epsilon \leq \epsilon_0. \] (6.17)

Next, we observe that
\[ \| \tilde{u}_x^{ij}(t) \|_{L_\ell^2} = \| \tilde{u}_x^{ij}(0) + \epsilon^{-1} \delta \tilde{u}_x^{ij}(t) \|_{L_\ell^2} \lesssim \| \tilde{u}_x^{ij}(0) \|_{L_\ell^2} + \frac{1}{\epsilon} \| \delta \tilde{u}_x^{ij}(t) \|_{L_{3-1,\epsilon}^2}, \] (6.18)

by (3.26) and (3.27), while for any \( 0 \leq \ell \leq k \),
\[ \| V_\epsilon(t) \|_{H_{3-1,\epsilon}^0} \lesssim \| V_\epsilon(t) \|_{H_{3-1,\epsilon}^0} = \| W_\epsilon(t) + d\Phi(W_\epsilon(t)) \|_{H_{3-1,\epsilon}^0} \lesssim \| W_\epsilon(t) \|_{H_{3-1,\epsilon}^0} \] (6.19)

by (3.25), (3.29), and (3.31). The proof of Theorem 1.1 now follows directly from Theorem 5.1 and the estimates (6.17)-(6.19).
7 Discussion

In this article, we have established the existence of a large class of dynamical solutions to the Einstein-Euler equations that have a first post-Newtonian expansion. Although this is an improvement over existing rigorous results [15, 19], which only cover the Newtonian limit situation (i.e. the “zeroth” post-Newtonian expansion), the results of this paper are almost certainly not optimal. In general, one expects that with a suitable gauge choice, it should be possible to generate post-Newtonian expansions to at least the $2.5$ post-Newtonian order after which there are indications that the post-Newtonian expansions will break down. For a lucid discussion of this phenomenon see [17].

As remarked in [17], the choice of harmonic gauge may be the reason for not being able to reach the $2.5$ post-Newtonian order. At the formal level, there exist other gauges that perform better than the harmonic gauge for the post-Newtonian expansions. However, it remains to be seen if these other gauges are compatible with the singular hyperbolic energy estimates that are guaranteed to arise in the dynamical setting. We are presently investigating this problem.

From the proof of Theorem 1.1 and the paper [15], it is clear that conditions of the form

$$\|\partial_t^p W_\epsilon(0)\|_{H^{k-1}} \lesssim 1 \text{ as } \epsilon \searrow 0$$

(7.1)

on the initial data play a crucial role in generating the post-Newtonian expansions. This leads to the question of what happens when one considers initial data that does not satisfy (7.1) for any $p \in \mathbb{Z} \geq 0$. In [16], we address this question for the situation where

$$\limsup_{\epsilon \searrow 0} \|\partial_t W_\epsilon(0)\|_{H^{k-1}} = \infty.$$ 

There we find that a Newtonian description is still appropriate for the motion of the matter, but the gravitational field no longer vanishes in the limit $\epsilon \searrow 0$. Instead, there exists high frequency gravitational radiation that is not small at the $\epsilon^0$ order, and this will necessarily affect the higher order expansions.

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A Weighted calculus inequalities

In this section, we prove additional weighted calculus inequalities that are similar in spirit to those in Appendix A of [15]. We first recall from [15] the definition of the weighted Sobolev spaces. Let $V$ be a finite dimensional vector space with inner product $(\cdot|\cdot)$ and corresponding norm $|\cdot|$. For $u \in L^p_{\text{loc}}(\mathbb{R}^n, V)$, $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$, and $\epsilon \in \mathbb{R}_{\geq 0}$, the weighted $L^p$ norm of $u$ is defined by

$$\|u\|_{L^p_{\delta, \epsilon}} := \begin{cases} 
\|\sigma_\epsilon^{-\delta-n/p} u\|_{L^p} & \text{if } 1 \leq p < \infty \\
\|\sigma_\epsilon^{-\delta} u\|_{L^\infty} & \text{if } p = \infty
\end{cases}$$

(A.1)

where $\sigma_\epsilon(x) := \sqrt{1 + \frac{1}{4} |\epsilon x|^2}$. The weighted Sobolev norms are then defined by

$$\|u\|_{W^{k,p}_{\delta, \epsilon}} := \begin{cases} 
\left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p_{\delta-|\alpha|, \epsilon}} \right)^{1/p} & \text{if } 1 \leq p < \infty \\
\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty_{\delta-|\alpha|, \epsilon}} & \text{if } p = \infty
\end{cases}$$

(A.2)
where \( k \in \mathbb{N}_0, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) is a multi-index and \( D^n = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \). Here
\[
\partial_1 = \frac{\partial}{\partial x^1}
\]
where \((x^1, \ldots, x^n)\) are the standard Cartesian coordinates on \( \mathbb{R}^n \). The weighted Sobolev spaces are then defined as
\[
W^{k,p}_{\delta,\epsilon} = \{ u \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, V) \mid \|u\|_{W^{k,p}_{\delta,\epsilon}} < \infty \}.
\]
We note that \( W^{k,p}_{\delta,\epsilon} \) are the standard Sobolev spaces, and for \( \epsilon > 0 \) the \( W^{k,p}_{\delta,\epsilon} \) are equivalent to the radially weighted Sobolev spaces \([1, 7]\). For \( p = 2 \), we use the alternate notation \( H^{k,2}_{\delta,\epsilon} := W^{k,2}_{\delta,\epsilon} \). The spaces \( L^2_{\delta,\epsilon} \) and \( H^k_{\delta,\epsilon} \) are Hilbert spaces with inner products

\[
\langle u | v \rangle_{L^2_{\delta,\epsilon}} := \int_{\mathbb{R}^n} (u(x)\sigma_{\epsilon}^{\delta - n} d^n x, \tag{A.3}
\]

and

\[
\langle u | v \rangle_{H^k_{\delta,\epsilon}} := \sum_{|\alpha| \leq k} \| D^\alpha u | D^\alpha v \|_{L^2_{\delta,\epsilon}}, \tag{A.4}
\]

respectively. When \( \epsilon = 1 \), we will also use the notation \( W^{k,p}_{\delta} = W^{k,p}_{\delta,1} \) and \( H^k_{\delta} = H^k_{\delta,1} \).

**Lemma A.1.** Suppose \( \epsilon_0 > 0, \delta_1 \geq \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\} \), then
\[
\|uv\|_{L^\infty_{\delta,\epsilon}} \lesssim \|u\|_{L^\infty_{\delta,\epsilon}} \|v\|_{H^k_{\delta,\epsilon}} + (\|Du\|_{H^{k-1}_{\delta,\epsilon}} + \epsilon \|u\|_{L^2_{\delta,\epsilon}}) \|v\|_{L^\infty_{\delta,\epsilon}},
\]

for all \( \epsilon \in [0, \epsilon_0], u \in L^\infty_{\delta,\epsilon} \cap H^k_{\delta,\epsilon}, \) and \( v \in L^\infty_{\delta,\epsilon} \cap H^k_{\delta,\epsilon} \).

**Proof.** This follows directly from the inequality
\[
\|uv\|_{H^k} \lesssim \|u\|_{L^\infty} \|v\|_{H^k} + \|Du\|_{H^{k-1}} \|v\|_{L^\infty}
\]
and Lemma A.4 of [15]. \( \square \)

**Lemma A.2.** Suppose \( \epsilon_0 > 0, \delta \leq 0, -n/2 \leq \lambda \leq -n/2 + 1, \lambda \geq \delta, k > n/2, \) and \( f \in C^k_e(\mathbb{R} L \times \mathbb{R}^N, \mathbb{M}^{M \times M}) \) with \( f(0,0) = 0 \). Then there exists a polynomial \( p(y_1, y_2, y_3) \) such that
\[
\|f(u, w)v\|_{H^k_{\delta,\epsilon}} \lesssim \|f\|_{C^k_e(\mathbb{R} L \times \mathbb{R}^N, \mathbb{M}^{M \times M})} \|u\|_{H^k_{\delta,\epsilon}} \|w\|_{H^k_{\delta,\epsilon}} \|v\|_{H^k_{\delta,\epsilon}},
\]

for all \( \epsilon \in [0, \epsilon_0], u \in H^k_{\delta,\epsilon} \) and \( w, v \in H^k_{\delta,\epsilon} \).

**Proof.** Since \( \delta \leq \lambda \), it follows from Lemma A.1 that
\[
\|f(u, w)v\|_{H^k_{\delta,\epsilon}} \lesssim \|f(u, w)\|_{L^\infty} \|v\|_{H^k_{\delta,\epsilon}} + (\|D(f(u, w))\|_{H^{k-1}_{\delta,\epsilon}} + \epsilon \|f(u, w)\|_{H^k_{\delta,\epsilon}}) \|v\|_{L^\infty_{\delta,\epsilon}}.
\]

Using Lemma A.9 of [15], we can write the above inequality as
\[
\|f(u, w)v\|_{H^k_{\delta,\epsilon}} \lesssim \|f(u, w)\|_{L^\infty} \|v\|_{H^k_{\delta,\epsilon}} + \|f\|_{C^k_e} \left[ 1 + \left( \|u\|_{L^\infty} + \|w\|_{L^\infty} \right)^{k-1} \right] (\|Du\|_{H^{k-1}_{\delta,\epsilon}} + \epsilon \|u\|_{H^k_{\delta,\epsilon}} + \|w\|_{H^{k-1}_{\delta,\epsilon}}) \|v\|_{L^\infty_{\delta,\epsilon}}. \tag{A.5}
\]

But \( k > n/2 \) and \( \lambda \leq \delta \leq 0 \) implies that
\[
\|u\|_{L^\infty} \lesssim \|u\|_{H^k_{\delta}}, \quad \|w\|_{L^\infty} \lesssim \|w\|_{H^k_{\delta}}, \quad \|v\|_{L^\infty} \lesssim \|v\|_{L^\infty_{\delta,\epsilon}} \lesssim \|v\|_{H^k_{\delta,\epsilon}}, \tag{A.6}
\]
29
and

\[ \| u \|_{H^k_L} \lesssim \| u \|_{H^k_{\lambda, \epsilon}} \]  \tag{A.7} 

by equation A.24 and Lemma A.7 of [15], while

\[ \| Du \|_{H^{k-1}_{\lambda, \epsilon}} + \epsilon \| u \|_{L^2_{\lambda, \epsilon}} \lesssim \| u \|_{H^k_{\lambda, \epsilon}} \]  \tag{A.8} 

follows from Lemma A.11 of [15] since \(-n/2 \leq \lambda \leq -n/2 + 1\). The proof now follows directly from the inequalities \(A.5\)-\(A.8\).

**Lemma A.3.** Suppose \(\epsilon_0 > 0, \delta \leq 0, -n/2 \leq \lambda \leq -n/2 + 1, \lambda \geq \delta, k > n/2 + 1\), and \(f \in C^k_b(\mathbb{R}^L \times \mathbb{R}^N, M^{M \times M})\) with \(f(0,0) = 0\). Then there exists a polynomial \(p(y_1, y_2)\) such that

\[ \| [D^\alpha, f(u, w)]v \|_{L^2_{\lambda, \epsilon}} \lesssim \| f \|_{C^k_b}(\| u \|_{H^k_{\lambda}}, \| w \|_{H^k_{\lambda}})(\| u \|_{H^k_{\lambda}} + \| w \|_{H^k_{\lambda}})\| v \|_{H^{k-1}_{\lambda, \epsilon}} \]

for all \(\epsilon \in [0, \epsilon_0]\), \(1 \leq |\alpha| \leq k\), \(u \in H^k_{\lambda, \epsilon}\), \(w \in H^k_{\delta, \epsilon}\), and \(v \in H^{k-1}_{\delta, \epsilon}\).

**Proof.** The proof follows directly from Lemma A.9 of [15] and the inequalities \(A.5\)-\(A.8\).

**Lemma A.4.** Suppose \(\epsilon_0 > 0, \delta \leq 0, -n/2 \leq \lambda \leq -n/2 + 1, \lambda \geq \delta, k > n/2\). Then there exists a constant \(C > 0\) such that

\[ \| u_1 u_2 \|_{H^k_L} \leq C \| u_1 \|_{H^k_L} \| u_2 \|_{H^k_L}, \]

\[ \| u_1 v_1 \|_{H^k_{\lambda, \epsilon}} \leq C \| u_1 \|_{H^k_{\lambda, \epsilon}} \| v_1 \|_{H^k_{\lambda, \epsilon}}, \]

and

\[ \| v_1 v_2 \|_{H^k_{\lambda, \epsilon}} \leq C \| v_1 \|_{H^k_{\lambda, \epsilon}} \| v_2 \|_{H^k_{\lambda, \epsilon}} \]

for all \(u_1, u_2 \in H^k_L, v_1, v_2 \in H^k_{\delta, \epsilon}\), and \(\epsilon \in [0, \epsilon_0]\).

**Proof.** The proof follows immediately from Lemma [A.1] and the inequalities \(A.6\)-\(A.8\).

We now recall the definition of analytic maps between Banach spaces.

**Definition A.5.** Suppose \(Y\) and \(Z\) are Banach spaces, \(U \subseteq Y\) is an open set, and \(\mathcal{L}_j(Y,Z)\) is the space of continuous, \(j\)-multilinear maps from \(Y\) to \(Z\) with norm

\[ \| F \|_{\mathcal{L}_j(Y,Z)} = \sup \{ \| F(u_1, u_2, \ldots, u_j) \|_{Z} \mid u_j \in U \text{ and } \sup\{\| u_1 \|_{Y}, \| u_2 \|_{Y}, \ldots, \| u_3 \|_{Y} \} \leq 1\}. \]

Then a map \(f : U \to Z\) is **analytic in \(U\)**, if for each \(u_0 \in U\) there exists a \(\rho > 0\), and a sequence of maps multilinear maps \(f_j \in \mathcal{L}_j(Y,Z)\) such that

\[ \sum_{j=0}^{\infty} \| f_j \|_{\mathcal{L}_j(Y,Z)} \rho^j < \infty, \]

and

\[ f(u) = \sum_{j=0}^{\infty} f_j(u - u_0, \ldots, u - u_0) \]  \tag{A.9} 

for all \(u \in U\) satisfying \(\| u - u_0 \|_Y < \rho\). The set of all analytic functions in \(U\) will be denoted \(C^\omega(U, Z)\).

In addition to analytic maps, we will need analytic maps that are uniformly analytic on the \(H^k_{\lambda, \epsilon}\) spaces as \(\epsilon\) varies.
**Definition A.6.** Suppose $R > 0$, $Y$, $Z$ are Banach spaces, and $V \subset Y$ is open. Then a sequence a maps $f_\epsilon : B_{R}(H^{k_1}_\epsilon) \times V \to H^{k_2}_\epsilon \times Z$ will be called uniformly analytic for $\epsilon \in [0, \epsilon_0]$, if

(i) $f_\epsilon \in C^\omega(B_{R}(H^{k_1}_\epsilon) \times V; H^{k_2}_\epsilon \times Z)$ for $0 \leq \epsilon \leq \epsilon_0$, and

(ii) for each $v_0 \in V$ there exists constants $\rho, c_j > 0$, and a sequence of maps multilinear maps $f_j \in \mathcal{L}_j(H^{k_1}_{\epsilon_1} \times Y, H^{k_2}_{\epsilon_2} \times Z)$ such that

$$
\| f_j \|_{\mathcal{L}_j(H^{k_1}_{\epsilon_1} \times Y, H^{k_2}_{\epsilon_2} \times Z)} \leq c_j \quad 0 \leq \epsilon \leq \epsilon_0,
$$

and

$$
\sum_{j=0}^\infty c_j(\rho + R)^j < \infty,
$$

and

$$
f_\epsilon(u, v) = \sum_{j=0}^\infty f_j(u, v - v_0, \ldots, u, v - v_0) \quad 0 \leq \epsilon \leq \epsilon_0
$$

for all $(u, v) \in H^{k_1}_{\epsilon_1} \times V$ satisfying $\| u \|_{H^{k_1}_{\epsilon_1}} < R$, and $\| v - v_0 \|_V < \rho$.

The next lemma shows how to construct a particular class of uniformly analytic functions.

**Lemma A.7.** Suppose $\epsilon_0 > 0$, $\delta \leq 0$, $-n/2 \leq \lambda \leq -n/2 + 1$, $k > n/2$, $F \in C^\omega(B_{R_1}(\mathbb{R}) \times B_{R_2}(\mathbb{R}), \mathbb{R})$, $F(\cdot, 0) = 0$, and $C$ is the $\epsilon$ independent constant from Lemma A.4. Then for $0 \leq \epsilon \leq \epsilon_0$,

$$
F(u, v) = \sum_{p=0}^\infty \sum_{q=1}^\infty \frac{1}{q!p!} (\partial^p \partial^q F)(0, 0) u^p v^q
$$

defines a function of class $C^\omega(B_{R_1}(H^k_{\lambda}) \times B_{R_2}(H^k_{\delta, \epsilon}), H^k_{\delta, \epsilon})$ where $R_1 = R_1/C$ and $R_2 = R_2/C$.

**Proof.** Using Lemma A.4, the proof follows from a slight modification of the proof of Proposition 3.6 from [10].

We note that the above Lemma can be easily generalized to maps $f \in C^\omega(B_{R}(\mathbb{R}^N) \times B_{R}(\mathbb{R}^M), \mathcal{M}_M \times \mathcal{M})$.

## B Symmetric hyperbolic equations

The hyperbolic equations that we will consider are of the form

$$
b^0(\epsilon u_\epsilon, \epsilon w_\epsilon, \epsilon v_\epsilon) \partial_1 v_\epsilon = \frac{1}{\epsilon} \epsilon^j \partial_j v_\epsilon + b^j(\epsilon, u_\epsilon, w_\epsilon, v_\epsilon) \partial_j v_\epsilon + \gamma F(\epsilon, u_\epsilon, w_\epsilon, v_\epsilon),
$$

$$
v_\epsilon |_{t=0} = v_0,
$$

where

(i) the maps $u_\epsilon = u_\epsilon(x)$ and $w_\epsilon = w_\epsilon(t, x)$ are $\mathbb{R}^L$ and $\mathbb{R}^N$ valued, respectively, while the map $v_\epsilon = v_\epsilon(t, x)$ is $\mathbb{R}^M$-valued,

(ii) $F$ is a (possibly non-local) map satisfying

$$
\| F(\epsilon, u, w_1, v_1) - F(\epsilon, u, w_2, v_2) \|_{H^k_{\delta, \epsilon}} \leq_{\rho, \epsilon_0, k, \epsilon} \| w_1 - w_2 \|_{H^k_{\delta, \epsilon}} + \| v_1 - v_2 \|_{H^k_{\delta, \epsilon}}
$$

for all $\epsilon \in [0, \epsilon_0]$, $u \in B_{\rho}(H^k_{\lambda})$, $w_1, w_2, v_1, v_2 \in B_{\rho}(H^k_{\delta, \epsilon})$, and

$$
\| F(\epsilon, u, w, v) \|_{H^{k}_{\delta, \epsilon}} \leq p(\| u \|_{H^k_{\lambda}}, \| w \|_{H^{k}_{\delta, \epsilon}}, \| v \|_{H^{k}_{\delta, \epsilon}})(\| w \|_{H^{k}_{\delta, \epsilon}} + \| v \|_{H^{k}_{\delta, \epsilon}})
$$

for all $\epsilon \in [0, \epsilon]$, $u \in H^k_{\lambda}$, and $w, v \in H^k_{\delta, \epsilon}$.
Theorem B.1. Suppose \( \epsilon \) such that for all \( \epsilon \in (0, \epsilon_0) \),

(iv) \( b^0 \) and \( b^j \) are symmetric,

(v) the \( c^j \) are constant symmetric matrices, and

(vi) there exists a constant \( \omega > 0 \) such that

\[
 b^0(\xi_1, \xi_2, \xi_3) \geq \omega \| M \times M \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^k \times \mathbb{R}^M \times \mathbb{R}^M.
\]  

(B.5)

Let \([n/2]\) denote the largest integer with \([n/2] \leq n/2\), \( k_0 = [n/2] + 2 \), and

\[
 X_{T,s,k,\delta} = \bigcap_{t=0}^{s+1} C^k([0,T), \mathbb{R}^{k-\ell}).
\]

Theorem B.1. Suppose \( \epsilon_0 > 0 \), \( T > 0 \), \( s \in \mathbb{N} \), \( k = k_0 + s \), \( \delta \leq 0 \), \( -n/2 \leq \lambda \leq -n/2 + 1 \),

\[
 v_\epsilon \in H^k_\delta, \quad u_\epsilon \in H^k_\lambda, \quad w_\epsilon \in X_{T,s,k,\delta}, \quad 0 < \epsilon \leq \epsilon_0,
\]

and

\[
 \| v_\epsilon \|_{H^k_\delta} \leq C_1, \quad \| w_\epsilon(t) \|_{H^k_\delta} + \| \partial_t w_\epsilon(t) \|_{H^{k-1}_\delta} \leq C_2, \quad \| u_\epsilon \|_{H^k_\delta} \leq C_3,
\]

for constants \( C_1, C_2, C_3 \), independent of \( (t, \epsilon) \in [0,T) \times (0,\epsilon_0] \). Then there exists a polynomial \( p(y_1, y_2, y_3) \) and maps

\[
 v_\epsilon \in X_{T_\epsilon,s,k,\delta} \quad 0 < \epsilon \leq \epsilon_0,
\]

such that for all \( \epsilon \in (0, \epsilon_0] \)

(i) \( v_\epsilon(t, x) \) is the unique solution in \( L^\infty((0,T^\epsilon), H^k_\delta) \cap \text{Lip}((0,T^\epsilon), H^{k-1}_\delta) \) to the initial value problem (B.1)–(B.2),

(ii) if \( \limsup_{t \to T^\epsilon} \| v_\epsilon \|_{W^{1,\infty}} < \infty \), then the solution \( v_\epsilon \) can be extended (uniquely) for time \( T^\epsilon_\epsilon \in [T^\epsilon, T) \),

(iii) for any constant \( K_1 > C_1 \),

\[
 \| v_\epsilon(t) \|_{H^k_\delta} \leq \exp(K_2(1+\gamma)t) \left[ \| v_\epsilon(0) \|_{H^k_\delta} + \frac{\gamma C_2}{K_2(1+\gamma)} \right] - \frac{\gamma C_2}{K_2(1+\gamma)} \leq K_1
\]

for all \( (t, \epsilon) \in [0,\tilde{T}) \times (0,\epsilon_0] \), where

\[
 K_2 := p(C_3, C_2, K_1),
\]

and

\[
 T_\epsilon \geq \tilde{T} := \min \left\{ T, \frac{1}{K_2(1+\gamma)} \ln \left( \frac{K_1 K_2(1+\gamma) + \gamma C_2}{C_1 K_2(1+\gamma) + \gamma C_2} \right) \right\},
\]

(iv) \( \epsilon \| v_\epsilon(t) \|_{H^k_\delta} \lesssim 1 \) for all \( (t, \epsilon) \in [0,\tilde{T}) \times (0,\epsilon_0] \),

(iv) and if \( \| c^j \partial_\alpha v_\epsilon \|_{H^{k-1}_\delta} \lesssim \epsilon \), then \( \| \partial_t v_\epsilon(t) \|_{H^{k-1}_\delta} \lesssim 1 \) for all \( (t, \epsilon) \in [0,\tilde{T}) \times (0,\epsilon_0] \).
Proof. We will only prove statements (iii)-(v) as (i)-(ii) follow from a slight modification of arguments in Appendix B of [15].

Let \( v^\alpha = D^\alpha v, b^\alpha = b^0(\epsilon u, \epsilon w, \epsilon v), b^\beta = b^j(\epsilon, u, w, v), \) and \( F = F(\epsilon, u, w, v). \) Then from the evolution equation \((B.1)\), we find that

\[
\partial_t v^\alpha = (b^0)^{-1} \left[ \frac{1}{\epsilon} \epsilon^j \partial_j v^\alpha + b^\beta \partial_j v^\alpha + \gamma F \right].
\]

(B.6)

Differentiating this yields

\[
b^0_\epsilon \partial_t v^\alpha = \frac{1}{\epsilon} \epsilon^j \partial_j v^\alpha + b^\beta \partial_j v^\alpha + f^\alpha,
\]

where

\[
f^\alpha = b^0_\epsilon \left[ D^\alpha, (b^0)^{-1} (\epsilon^{-1} \epsilon^j + b^j) \right] \partial_j v^\alpha + \gamma b^0_\epsilon D^\alpha \left( (b^0)^{-1} F \right).
\]

(B.8)

Energy estimates (see Lemma 7.1 in [15]) then show that

\[
\frac{d}{dt} \|v^\alpha\|_{0,\delta}^2 \lesssim \left( \|\text{div} b\|_{L^\infty} + \|\epsilon^j \epsilon^\beta\|_{L^\infty} \right) \|v^\alpha\|_{L^2_{\delta,\epsilon}}^2 + \|f^\alpha\|_{L^2_{\delta,\epsilon}} \|v^\alpha\|_{L^2_{\delta,\epsilon}}
\]

(B.9)

where \( \text{div} b = \partial_t b^0_\epsilon + \partial_j b^\beta, \epsilon^j = (\epsilon, \ldots, \epsilon^\alpha), \) \( \epsilon = (b^1, \ldots, b^\beta), \) and

\[
\|\cdot\|_{k,\delta,\epsilon} := \sum_{|\alpha| \leq k} \left( D^\alpha (-) \right) |b^0_\epsilon D^\alpha (-) |.
\]

(B.10)

Since \( b^0_\epsilon = b^0(\epsilon u, \epsilon w, \epsilon v), \) it follows from Lemma A.3 that

\[
\|[D^\alpha, (b^0)^{-1} (\epsilon^{-1} \epsilon^j + b^j)] \|_{L^2_{\delta,\epsilon}} \lesssim p(\|u\|_{H^k}, \|w\|_{H^k}, \|v\|_{H^k}) \left( \|u\|_{H^k} + \|w\|_{H^k} + \|v\|_{H^k} \right) \|v\|_{H^k}
\]

(B.11)

for some polynomial \( p(y_1, y_2, y_3). \) Using this estimate along with (B.4) and Lemma A.2, we find that

\[
\|f^\alpha\|_{L^2_{\delta,\epsilon}} \lesssim p(\|u\|_{H^k}, \|w\|_{H^k}, \|v\|_{H^k}) \left( \|u\|_{H^k} + \|w\|_{H^k} \right) \|v\|_{H^k}
\]

(B.12)

for some polynomial \( p(y_1, y_2, y_3). \) Combining the two estimates (B.9) and (B.13), and summing over \( \alpha \) (0 \( \leq |\alpha| \leq k)) yields

\[
\frac{d}{dt} \|v\|_{k,\delta,\epsilon} \lesssim p \left( \|u\|_{H^k}, \|w\|_{H^k}, \|v\|_{H^k} \right) (1 + \gamma) \|v\|_{H^k}.
\]

(B.14)

But

\[
\omega \|v\|_{H^k} \leq \|v\|_{k,\delta,\epsilon} \leq \|b^0\|_{C^0} \|v\|_{H^k},
\]

by (B.5), and so it follows from (B.14) and Gronwall’s inequality that for any constant \( K_1 > C_1, \) if we let \( K_2 = p(C_3, C_2, K_1), \) then

\[
\|v(t)\|_{H^k} \leq \exp \left( K_2 (1 + \gamma) t \right) \left[ \|v(0)\|_{H^k} + \frac{\gamma C_2}{K_2(1 + \gamma)} \right] - \frac{\gamma C_2}{K_2(1 + \gamma)}
\]

for all \( t \) such that \( \|v(t)\|_{H^k} \leq K_1. \) But \( \|v\|_{W^{1,\infty}} \lesssim \|v\|_{H^k}, \) by Lemma A.7 of [15], and hence, by the continuation principle (ii), we see that

\[
T_\epsilon \geq \tilde{T} := \min \left\{ T, \frac{1}{K_2(1 + \gamma)} \ln \left( \frac{K_1 K_2 (1 + \gamma) + \gamma C_2}{C_1 K_2 (1 + \gamma) + C_2} \right) \right\}
\]

(B.16)

\(^2\) The only real difference is the proof of the convergence of the Galerkin approximations. For the non-local problem, one can use the global compact imbedding \( H^{k} \subset H^{k}_\epsilon (k > \ell, \delta < \eta) \) to obtain convergence instead of the local compact \( H^{k}(B_R) \subset H(B_R) \) \((k > \ell)\) imbedding used in [15].
Next, differentiating (B.1) with respect to $t$, it is clear that $\partial_t v_\epsilon$ satisfies a linear equation of the same structure as (B.1), and therefore the same estimates used to derive (B.15) also show that there exists constants $K_2$, $K_3$ such that

$$\|\partial_t v_\epsilon(t)\|_{H^{k,\delta}_{\bar{\epsilon}}} \leq e^{K_1 t} \|\partial_t v_\epsilon(0)\|_{H^{k-1,\delta}_{\bar{\epsilon}}} + K_3 \quad \forall (t, \epsilon) \in [0, T_\ast] \times (0, \epsilon_0]. \quad (B.17)$$

The proof now follows from the estimates (B.15)-(B.17).

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