Research Article

Initial and Boundary Value Problems for a Class of Nonlinear Metaparabolic Equations

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This paper is devoted to the initial and boundary value problems for a class of nonlinear metaparabolic equations $u_t - \beta u_{xx} - k u_{xxx} + \gamma u_{xxxx} = f(u_x), \quad x \in \Omega, \ t > 0,$ \hspace{1cm} (1)

$u(x, 0) = u_0(x), \quad x \in \Omega,$ \hspace{1cm} (2)

$u(0, t) = u(1, t) = 0, \quad u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0,$ \hspace{1cm} (3)

in a bounded domain $\Omega = (0, 1)$, where $u_0(x)$ is the initial value function defined on $\Omega$, $k > 0$ is the viscosity coefficient, $\gamma > 0$ is the interfacial energy parameter, and the nonlinear smooth function $f(s)$ satisfies the following assumptions:

\begin{equation}
\begin{cases}
(i) f(s) \leq \alpha |s|^q, \quad \alpha > 0, 1 < q < +\infty, \forall s \in R, \\
(ii) (p + 1)F(s) \geq sf(s) \text{ for some } p > 1, \quad \forall s \in R, \quad F(s) = \int_0^s f(\tau)d\tau.
\end{cases}
\end{equation}

Equation (1) is a typical higher-order metaparabolic equation [1, 2], which has extensive physical background and rich theoretical connotation. This type of equation can be regarded as the regularization of Sobolev-Galpern equation by adding a fourth-order term $u_{xxxx}$. The Sobolev-Galpern equation appear in the study of various problems of fluid mechanics, solid mechanics, and heat conduction theory [3–5]. There have been many outstanding results about the qualitative theory for Sobolev-Galpern which include the existence, nonexistence, asymptotic behavior, regularities, and other some special properties of solutions. We also refer the reader to see [6, 7] and the papers cited therein. In (1), $u$ is the concentration of one of the two phases, the fourth-order term $\gamma u_{xxxx}$ denotes the capillarity-driven surface diffusion, and the nonlinear term $f(u_x)$ is an intrinsic chemical potential. For example, differentiating (1) with respect to $x$ and taking $v = u_x, \beta = 0$, then Equation (1) reduces to the well-known viscous Cahn-Hilliard equation

\begin{equation}
v_t - v_{xxx} + v_{xxxx} = \varphi(v)_{xx}, \quad x \in \Omega, \ t > 0.
\end{equation}

Equation (5) appears in the dynamics of viscous first-order phase transitions in cooling binary solutions such as glasses, alloys, and polymer mixtures [8–10]. On the other hand, Equation (5) appears in the study of the regularization of nonclassical diffusion equations by adding a
fourth-order term \( v_{xxxx} \). There have been many outstanding results about the qualitative theory for this type of equations \([11–15]\). For example, Liu and Yin \([13]\) studied Equation (5) for \( \varphi(v) = -v + \gamma_1 v^2 + \gamma_2 v^3 \) in \( R^3 \); they proved the existence and nonexistence of global classical solutions and pointed out that the sign of \( \gamma_2 \) is crucial to the global existence of solutions. In \([14]\), Grinfeld and Novick-Cohen studied a Morse decomposition of the stationary solutions of the one-dimensional viscous Cahn-Hilliard equation by explicit energy calculations. They also proved a partial picture of the variation in the structure of the attractor \((n = 1)\) for the viscous Cahn-Hilliard equation as the mass constraint and homotopy parameter are varied. Zhao and Liu \([15]\) considered the initial boundary problem for the viscous Cahn-Hilliard Equation (5). In their paper, the optimal control under boundary condition was given, and the existence of optimal solution was proved.

Let us mention that there is an abounding literature about the initial and boundary value problems or Cauchy problem to nonlinear parabolic and hyperbolic equations. We refer the reader to the monographs \([16, 17]\) which devoted to the second-order parabolic and pseudoparabolic problems. For the fourth-order nonlinear parabolic and hyperbolic equations, there are also some results about the initial boundary value and Cauchy problems, especially on global existence/nonexistence, uniqueness/nonuniqueness, and asymptotic behavior \([18–25]\). Bakiyevich and Shadrin \([21]\) studied the Cauchy problem of the metaparabolic equation

\[
\begin{align*}
\begin{cases}
 u_t - au_{xx} - \gamma u_{xxx} + \beta u_{xxxx} = f(t, x), & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = \varphi(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
\]

where \( \alpha > 0, \beta \geq 0, \) and \( \gamma > 0 \) are constants. They proved that the solutions are expressed through the sum of convolutions of functions \( \varphi(x) \) and \( f(t, x) \) with corresponding fundamental solutions.

In \([22]\), Liu considered the metaparabolic equation

\[
\begin{align*}
u_t - ku_{xx} + A(u)_{xxxx} = f(x, t), & \quad 0 \leq x \leq 1, 0 \leq t \leq T < +\infty,
\end{align*}
\]

where \( A(u) = \int_0^u a(s)ds, a_0 + a_1|s|^b \leq a(s), \) and \( |a''(s)| \leq a_2|s|^b \) \((a_0, a_1, a_2, \) and \( b \) are positive constants). He proved the existence of weak solutions by using the method of continuity.

Khudaverdiiyev and Farhadova \([23]\) discussed the following fourth-order semilinear pseudoparabolic equation

\[
\begin{align*}
u_t - au_{xx} + u_{xxxx} = f(t, x, u, u_1, u_2, u_3, u_4), & \quad 0 \leq x \leq 1, 0 \leq t \leq T < +\infty,
\end{align*}
\]

where \( \alpha > 0 \) is a fixed number. They proved the existence in large theorem \((i.e., \text{true for sufficiently large values of } T)\) for generalized solution by means of Schauder stronger fixed-point principle.

In \([24]\), Zhao and Xuan studied the generalized BBM-Burgers equation

\[
u_t - au_{xx} - \nu u_{xxx} + \beta u_{xxxx} + f(u)_x = 0, \quad x \in \mathbb{R}, \ t > 0. \quad (9)
\]

They obtained the existence and convergence behavior of the global smooth solutions for Equation (9).

Philippin \([25]\) studied the following fourth-order parabolic equation

\[
u_t - k_1(t)\Delta u + k_2(t)\Delta^2 u = k_3(t)u|u|^{p-1}, \quad x \in \Omega, \ t > 0, \quad (10)
\]

where \( k_i, i = 1, 2, 3 \) are positive constants or in general positive derivable functions of \( t \). Under appropriate assumptions on the data, he proved that the solutions \( u \) cannot exist for all time, and an upper bound is derived.

Equation (1) is also closely connected with many equations \([26–29]\). For example, Yang \([26]\) considered the initial and boundary value problems of the following equation

\[
u_{tt} + \lambda u_t + au_{xxxx} = f(u_1)_x, \quad x \in (0, 1), \ t > 0. \quad (11)
\]

He studied the asymptotic property of the solution and gave some sufficient conditions of the blow-up. When the weak damping term \( u_t \) of Equation (11) is replaced by the strong damping term \( -u_{xxxx} \), we have the following fourth-order wave equation

\[
u_{tt} - 2bu_{xx} + au_{xxxx} = f(u)_x, \quad x \in (0, 1), \ t > 0. \quad (12)
\]

Chen and Lu \([27]\) studied the initial and boundary value problems of Equation (12). They proved the existence and uniqueness of the global generalized solution and global classical solution by the Galerkin method. Furthermore, Xu et al. \([28]\) considered the initial and boundary value problems and proved the global existence and nonexistence of solutions by adopting and modifying the so called concavity method under some conditions with low initial energy. Ali Khelghati and Khadijeh Baghaei \([29]\) proved that the blow up for Equation (12) occurs in finite time for arbitrary positive initial energy.

Motivated by the above researches, in the present work, we mainly study the initial and boundary value problems (1)–(3) of metaparabolic equations. Hereafter, for simplicity, we set \( \alpha = \beta = \gamma = 1 \). Especially, the appearance of the dispersion term \( u_{xxx} \) and nonlinearity \( f(u)_x \) for these problems cause some difficulties such that we cannot apply the normal Galerkin approximation, concavity, and potential methods directly; we have to invent some new skills and methods to overcome these difficulties.

Our paper is organized as follows. In Section 2, we introduce some functionals and potential wells and discuss the invariance of some sets which are needed for our work. In Sections 3 and 4, the existence and nonexistence of global weak solutions for problems (1)–(3) are proved by the Galerkin approximation and potential well and improved concavity methods at low initial energy \((f(u_0) < d)\). Especially, the threshold result between global existence and nonexistence
is obtained under certain conditions. In the last section, we investigate the finite time blow-up for certain solutions of problems (1)–(3) with high initial energy.

### 2. Preliminaries

In this section, we introduce some functionals, potential wells, and important lemmas that will be needed in this paper. Throughout this paper, the following abbreviations are used for precise statement:

\[
L^2(\Omega) = L^2, \quad W^{m,q}(\Omega) = W^{m,q},
\]

\[
H^m(\Omega) = W^{m,2}(\Omega) = H^m, \quad H^m_0(\Omega) = H^m_0,
\]

\[
D = \left \{ u \in H^2(\Omega)|u(0, t) = u(1, t) = 0, u_{xx}(0, t) = u_{xx}(1, t) = 0 \right \},
\]

\[
\|u\|_{L^2(\Omega)}^2 = \|u\|^2_{L^2(\Omega)} + \|u\|_{H^m(\Omega)}^2
\]

\[
(E(t) = E(u(t)) = \int_0^t \|u(t)\|^2_{H^1} dt + \frac{1}{2} \|u_t\|^2_{H^1} + \int_\Omega F(u_x) dx,
\]

\[
J(u) = \frac{1}{2} \|u\|^2_{H^1} + \int_\Omega F(u_x) dx,
\]

\[
I(u) = \|u_x\|^2_{H^1} + \int_\Omega f(u_x) u_x dx.
\]

Then, by simple calculation, it follows that

\[
J'(u) = \frac{d}{dt} J(u) = -\|u_t(t)\|^2_{H^1} \leq 0,
\]

\[
E(t) = E(u(t)) = \int_0^t \|u(t)\|^2_{H^1} dt + J(u_0) = E(0).
\]

The corresponding “Nehari manifold” and “potential well depth” are given by

\[
N = \{ u \in D \cap W^{1,q+1}(\Omega)|J(u) = 0, u \neq 0 \}, \quad d = \inf_{u \in N} J(u).
\]

In addition, we define

\[
N^+ = \{ u \in D \cap W^{1,q+1}(\Omega)|J(u) > 0 \} \cup \{0\},
\]

\[
N^- = \{ u \in D \cap W^{1,q+1}(\Omega)|J(u) < 0 \}.
\]

To obtain the results of this paper, we also introduce so called stable and unstable sets:

\[
W = \{ u \in D \cap W^{1,q+1}(\Omega)|\|u\| > 0, J(u) < d \} \cup \{0\},
\]

\[
V = \{ u \in D \cap W^{1,q+1}(\Omega)|\|u\| < 0, J(u) < d \}.
\]

Next, we shall give the following some essential lemmas which are important to obtain the main results of this paper.

**Lemma 1.** Let \( f(s) \) satisfy (4), \( u \in D \cap W^{1,q+1}(\Omega) \), then the following hold:

1. If \( 0 < \|u_x\|_{H^1} < \gamma \), then \( u \in N^+(u \neq 0) \);
2. If \( u \in N^+ \), then \( \|u_x\|_{H^1} > \gamma \);
3. If \( u \in N, \) then \( \|u_x\|_{H^1} \geq \gamma \),

where

\[
\gamma = \frac{1}{\alpha C^{q+1}}, \quad C = \sup_{\|u\|_{H^1} = 1} \frac{\|u_x\|_{H^1}^2}{\|u\|_{H^1}^2},
\]

\((u \in D \cap W^{1,q+1}(\Omega), u \neq 0).\)

**Proof.**

1. If \( 0 < \|u_x\|_{H^1} < \gamma \), then

\[
\int_\Omega f(u_x) u_x dx \leq \int_\Omega |f(u_x) u_x| dx \leq \alpha C^{q+1} \|u_x\|_{H^1}^2 \|u_x\|_{H^1}^2 \leq \|u_x\|_{H^1}^2,
\]

which gives \( I(u) > 0 \) or \( u \in N^+(u \neq 0) \).

2. If \( u \in N^- \), then \( u \neq 0 \) and

\[
\|u_x\|^2_{H^1} < -\int_\Omega f(u_x) u_x dx \leq \alpha C^{q+1} \|u_x\|_{H^1}^2 \|u_x\|_{H^1}^2,
\]

which gives \( \|u_x\|_{H^1} > \gamma \).

3. If \( u \in N, \) then

\[
\|u_x\|^2_{H^1} = -\int_\Omega f(u_x) u_x dx \leq \alpha C^{q+1} \|u_x\|_{H^1}^2 \|u_x\|_{H^1}^2,
\]

we have \( \|u_x\|_{H^1} \geq \gamma \).

**Lemma 2.** Let \( f(s) \) satisfy (4) and \( u \in N, \) then

\[
d \geq d_0 = \frac{p-1}{2(p+1)} \gamma_0^2 = \frac{p-1}{2(p+1)} \left( \frac{1}{\alpha C^{q+1}} \right)^{2(q+1)}.
\]
Proof. For any $u \in N$, we have by Lemma 1 (3) that $\|u_{n}\|_{H^{1}}$ $\geq \gamma_{0}$. Hence, from

$$J(u) = \frac{1}{2} \|u_{n}\|_{H^{1}}^{2} + \int_{\Omega} F(u_{n}) dx$$

$$\geq \frac{1}{2} \|u_{n}\|_{H^{1}}^{2} + \frac{1}{p+1} \int_{\Omega} f(u_{n}) u_{n} dx$$

$$= \frac{p-1}{2(p+1)} \|u_{n}\|_{H^{1}}^{2} + \frac{1}{p+1} I(u)$$

$$= \frac{p-1}{2(p+1)} \|u_{n}\|_{H^{1}}^{2} \geq \frac{p-1}{2(p+1)} \gamma_{0}^{2},$$

and the definition of potential depth $d$, we get $d \geq d_{0}$.

For simplicity, we define the weak solution of (1)–(3) over the interval $\Omega \times [0, T)$, but it is to be understood that $T$ is either infinity or the limit of the existence interval.

**Definition 3.** We say that $u(x, t)$ is called a weak solution of the problems (1)–(3) on the interval $\Omega \times [0, T)$. If $u \in L^{\infty}_{T}[0, T] \cap W^{1, q+1} \Omega)$ satisfies the following conditions

(i) For any $v \in D \cap W^{1, q+1} \Omega)$, such that ($u_{i, v_x} + (u_{i, v_x} + (u_{i, x}, v_x) = -(f(u_{i}), v_x).$ (26)

(ii) $u(x, 0) = u_{0}$ in $D \cap W^{1, q+1} \Omega)$.

(iii) The following energy inequality holds

$$J(u) \leq J(u_{0}),$$

for any $0 \leq t < T$.

**Lemma 4.** Let $f(s)$ satisfy (4) and $u(x, t)$ be a solution of (1)–(3) over the interval $[0, T)$. If there exists a time $t_{0} \in [0, T)$ such that $u(t_{0}) \in W_{1}$, then $u(t) \in W$ for any $t \in [t_{0}, T)$, where $T$ is either infinity or the limit of the existence interval.

**Proof.** Arguing by contradiction and considering the time continuity of $I(u)$ and $J(u)$, we suppose that there exists a time $t_{1} \in [t_{0}, T)$ such that $u(t_{1}) \in W$ for any $t \in [t_{0}, t_{1})$, but $u(t_{1}) \in \partial W$, which means that (1) $I(u(t_{1})) = d$ or (2) $I(u(t_{1})) = 0, \|u(t_{1})\|_{H^{1}} \neq 0$. By (15) and $u(t_{0}) \in W$, we have $J(u(t_{1})) \leq J(u(t_{0})) < d$. It follows that case (1) is impossible. If $I(u(t_{1})) = 0, \|u(t_{1})\|_{H^{1}} \neq 0$, then by the definition of $d$, we have $J(u(t_{1})) > d$ which contradicts $J(u(t_{1})) \leq J(u(t_{0})) < d$. The case (2) is also impossible.

**Lemma 5.** Let $f(s)$ satisfy (4) and $u(x, t)$ be a solution of (1)–(3) over the interval $[0, T)$. If there exists a time $t_{0} \in [0, T)$ such that $u(t_{0}) \in V$, then $u(t) \in V$ for any $t \in [t_{0}, T)$, where $T$ is either infinity or the limit of the existence interval.

**Proof.** The proof of Lemma 5 is similar to Lemma 4.

**Lemma 6 (see [29, 30]).** Assume that the function $\phi(t) \in C^{2}$, $\phi(t) \geq 0$ satisfies

$$\phi(t)\phi''(t) - (1 + \delta)\phi'^{2}(t) \geq 0,$$

for certain real number $\delta > 0, \phi(0) > 0$, and $\phi'(0) > 0$. Then, there exists a real number $\tilde{T}$ with $0 < \tilde{T} \leq (\phi(0))/(a\phi'(0))$ such that

$$\phi(t) \longrightarrow \infty, \text{ as } t \longrightarrow \tilde{T}^{+}.$$
Proof. Multiplying (32) by $d_j'(t)$ and summing for $j = 1, \ldots, m$, then we have

$$
(u_{mx}, u_m) + (u_{mx}, u_{mx}) + (u_{mx}, u_{mx}) + (u_{mx}, u_{mx}) = -(f(u_{mx}), u_{mx}).
$$

(33)

By a direct calculation, it follows that

$$
\int_0^t \|u_m(t')\|^2\,dt' + f(u_m) = f(u_{0m}),
$$

(34)

where

$$
J(u_m) = \frac{1}{2}\|u_{mx}\|^2 + \int_{\Omega} F(u_{mx})\,dx.
$$

(35)

Utilizing the strong convergence of $u_{0m}$ in $D \cap W^{1,q+1}([\Omega])$, we note that $J(u_m) \to J(u_0) \leq d$. Hence, we get $J(u_{0m}) < d$ for sufficiently large $m$. On the other hand, from $u_0 \in W$ and $u_{0m}(x) \to u_0(x)$ in $D \cap W^{1,q+1}([\Omega])$, it follows that $u_{0m} \in W$ for sufficiently large $m$. Similar to the proof of Lemma 4, we have that the solution $u_m$ constructed by (31) remains in $W$ for $0 \leq t < \infty$ and sufficiently large $m$.

Thus, from (4) and

$$
d > J(u_m) = \frac{1}{2}\|u_{mx}\|^2 + \int_{\Omega} F(u_{mx})\,dx
$$

$$
\geq \frac{1}{2}\|u_{mx}\|^2 + \frac{1}{p+1}\int_{\Omega} f(u_{mx})u_{mx}\,dx
$$

$$
\geq \frac{p-1}{2(p+1)}\|u_{mx}\|^2 + \frac{1}{p+1}J(u_m) \geq 0,
$$

we obtain

$$
\int_0^t \|u_m(t')\|^2\,dt' < d, \quad \forall t < \infty,
$$

$$
\|u_{mx}\|^2 < \frac{2(p+1)}{p-1}d, \quad \forall t < \infty,
$$

$$
\|u_{mx}\|^2_{q+1} \leq C_s^2\|u_{mx}\|^2_{q+1} < C_s^2\frac{2(p+1)}{p-1}d, \quad \forall t < \infty,
$$

$$
\|f(u_{mx})\|_{r} \leq \alpha^r\|u_{mx}\|_{r+1}^{(r+1)/2}
$$

$$
< \alpha^r C_s^{r+1}\left(\frac{2(p+1)}{p-1}d\right)^{(r+1)/2}, \quad \forall t < \infty,
$$

(37)

where $r = (q+1)/q, 1/(q+1) + (1/r) = 1$. Therefore, there exist a subsequence of $\{u_m\}$ which from now on will be also denoted by $\{u_m\}$ such that as $m \to \infty$

$$
\lim_{m \to \infty} J(u_m) = \int_{\Omega} F(u_{mx})\,dx = \int_{\Omega} F(u_0)\,dx .
$$

(46)

Then, making use of Fatou’s Lemma and (34), (46), we deduce

$$
\lim_{m \to \infty} J(u_m) = \int_{\Omega} F(u_0)\,dx .
$$

(47)

which yields (27). Thus, we obtain that $u$ is a global weak solution of problems (1)–(3). Finally, making use of Lemma 4 again, we get $u(t) \in W$ for $0 \leq t < \infty$. 

$$
u_{mx} \to u_t \text{ in } L^{\infty}([0, \infty); D \cap W^{1,q+1}([\Omega])),
$$

(38)

$$
u_m \to u \text{ a.e. } Q = \Omega \times (0, \infty),
$$

(39)

$$
u_{mt} \to u_t \text{ in } L^2([0, \infty); H^1_0([\Omega])),
$$

(40)

and

$$
f(u_{mx}) \to \chi \text{ in } L^{\infty}([0, \infty); L^r([\Omega])),
$$

(41)

$$
u_{mx} \to u_t \text{ in } L^{\infty}([0, \infty); L^{q+1}([\Omega])),
$$

(42)
4. Finite Time Blow-up of Solutions with $f(u_0) < d_0$

In this section, we consider the finite time blow up of solutions with $E(0) = f(u_0) < d_0$ for the problems (1)–(3).

**Theorem 8.** Let $f$ satisfy (4), and $u_0 \in D \cap W^{1,q+1}(\Omega)$. Assume that $I(u_0) < 0$ and $E(0) = f(u_0) < d_p^*$, where $d_p^*$ is defined in Lemma 2, then the weak solution $u(t)$ of problems (1)–(3) blow-up in finite time.

**Proof.** Let $u(t)$ be any weak solution of the problems (1)–(3) with $I(u_0) < 0$ and $E(0) = f(u_0) < d_0$, $T$ be the maximal existence time of $u(t)$. Next, we will prove $T < \infty$. Arguing by contradiction, we suppose $T = +\infty$. We define the function $\Psi : [0, T_1] \rightarrow \mathbb{R}^+$ by

$$\Psi(t) = \int_0^t ||u(\tau)||_{H}^2 \, d\tau + (T_1 - t)||u_0||_{H}^2 + b(t + T_0)^2,$$  \hspace{1cm} (48)

where $b$, $T_0$, and $T_1$ are positive constants to be chosen later. By simple calculation, we have

$$\Psi'(t) = \int_0^t \int_\Omega u(\tau)u_\tau(\tau) \, dx \, d\tau$$

$$+ 2 \int_0^t \int_\Omega u_\tau(\tau)u_\tau(\tau) \, dx \, d\tau + 2b(t + T_0),$$  \hspace{1cm} (49)

$$\Psi''(t) = 2 \int_\Omega u(t)u_\tau(t) \, dx + 2 \int_\Omega u_\tau(t)u_\tau(t) \, dx + 2b.$$  \hspace{1cm} (50)

By (1), we obtain

$$\Psi''(t) = 2 \int_\Omega u[u_{xx} - u_{xxxx} + f(u)], \, dx + 2b$$

$$= -2\|u_s\|_{H}^2 - 2 \int_\Omega u_xf(u_s), \, dx + 2b.$$  \hspace{1cm} (51)

Therefore, we can get

$$\Psi(t)\Psi''(t) - \frac{p + 3}{4} \Psi'(t)^2$$

$$= \Psi(t) \left[-2\|u_s\|_{H}^2 - 2 \int_\Omega u_xf(u_s), \, dx + 2b\right]$$

$$- (p + 3) \left[\int_0^t \int_\Omega u(\tau)u_\tau(\tau) \, dx \, d\tauight]$$

$$+ \left[\int_0^t \int_\Omega u_\tau(\tau)u_\tau(\tau) \, dx \, d\tau + b(t + T_0)^2\right]^2$$  \hspace{1cm} (52)

where

$$\eta(t) = \left[\int_0^t ||u(\tau)||_{H}^2 \, d\tau + b(t + T_0)^2\right]$$

$$- \left[\int_0^t u(\tau)u_s(\tau) \, dx \, d\tau - \int_0^t u_s(\tau)u_\tau(\tau) \, dx \, d\tau\right]$$

$$+ b(t + T_0)^2.$$  \hspace{1cm} (53)

Using the Schwarz and Young inequalities, we have

$$\left(\int_0^t (u(\tau), u_s(\tau)) d\tau\right)^2 \leq \int_0^t ||u(\tau)||_{H}^2 \, dx \int_0^t ||u_s(\tau)||_{H}^2 \, dx,$$  \hspace{1cm} (54)

$$\left(\int_0^t (u(\tau), u_\tau(\tau)) d\tau\right)^2 \leq \int_0^t ||u(\tau)||_{H}^2 \, dx \int_0^t ||u_\tau(\tau)||_{H}^2 \, dx,$$  \hspace{1cm} (55)

Inserting (53)–(55) into (52), we have

$$\eta(t) \geq 0, \quad t \in [0, T_1].$$  \hspace{1cm} (56)

Thus,

$$\Psi''(t)\Psi(t) - \frac{p + 3}{4} \Psi'(t)^2 \geq \Psi(t)\xi(t),$$  \hspace{1cm} (57)

where

$$\xi(t) = -2\|u_s\|_{H}^2 - 2 \int_\Omega u_xf(u_s), \, dx + 2b$$

$$- (p + 3) \left[\int_\Omega ||u(\tau)||_{H}^2 \, dx \, d\tau + b\right]$$

$$\geq - (p + 3) \left[\int_\Omega ||u(\tau)||_{H}^2 \, dx \, d\tau - (p + 1)b\right]$$

$$- 2(p + 1) \int_\Omega F(u_s), \, dx - 2||u_s||_{H}^2.$$  \hspace{1cm} (58)

From $I(u_0) < 0$, $E(0) = f(u_0) < d_0$, and Lemma 5, we have $I(u) < 0$ for all $t \in [0, \infty)$. Hence, by Lemma 1, it follows that
\[ \|u_0\|_{H^1}^2 > \gamma_0^2 = ((2p+1)/(p-1))d_0 = (1/(aC^{p+1}))^{2/(p-1)}. \] Thus, we have from (58) that
\[ \xi(t) > 2(p+1)(d_0 - J(u_0)) - (p+1)b. \] (59)

We choose \( b \) small enough such that \( b \leq 2(d_0 - J(u_0)) \), then we have
\[ \Psi''(t)\Psi(t) - \frac{b + 3}{4} \Psi'(t)^2 \geq \Psi(t)\xi(t) > 0, \] (60)
for all \( t \in [0, T_1) \). From \( I(u) < 0 \) for all \( t \in [0, T_1) \) and (50), we get \( \Psi''(t) > 0 \). Hence, we have \( \Psi'(t) > \Psi'(0) = 2bT_0 > 0 \) for all \( t \in (0, T_1) \).

From what has been discussed above, using Lemma 6 and letting \( \delta = (p-1)/4 \), we can obtain that there exists a finite time \( \tilde{T} > 0 \) such that
\[ \lim_{t \to \tilde{T}} \Psi(t) = +\infty, \] (61)
or
\[ \lim_{t \to \tilde{T}} \left( \int_0^t \|u(r)\|_{H^1}^2 dr + (T_1 - t)\|u_0\|_{H^1}^2 + b(t + T_0)^2 \right) = +\infty, \] (62)
where
\[ \tilde{T} \leq \frac{\Psi(0)}{y\Psi'(0)} = \frac{2[T_1\|u_0\|_{H^1}^2 + bT_0^2]}{(p - 1)bT_0}. \] (63)
which contradicts \( \tilde{T} = +\infty \). Hence, the desired assertion immediately follows.

From the discussed above in Sections 3 and 4, a threshold result of global existence and nonexistence of solutions for problems (1)–(3) has been obtained as follows.

**Corollary 9.** Let \( f \) satisfy (4), and \( u_0 \in D \cap W^{1,q+1}(\Omega) \). Assume that \( E(0) = J(u_0) < d_0 \). Then, problems (1)–(3) admits a global weak solution \( I(u_0) \geq 0 \) (includes \( u_0 = 0 \)); Problems (1)–(3) does not admit any global solution provided \( I(u_0) < 0 \).

**5. Finite Time Blow-up of Solutions with High Initial Energy**

In this section, we shall state and prove the finite time blow-up result with high initial energy for the problems (1)–(3).

**Theorem 10.** Let \( f \) satisfy (4), and \( u_0 \in D \cap W^{1,q+1}(\Omega) \). Assume that
\[ 0 \leq J(u_0) \leq \frac{(p-1)\lambda_1}{2(p+1)(1 + \lambda_1)}\|u_0\|_{H^1}^2, \] (64)
where \( \lambda_1 \) is the optimal constant satisfying the Poincaré inequality \( \|u_0\|_{H^1}^2 \geq \lambda_1\|u_0\|_{L^2}^2 \), then \( u(t) \) is a global weak solution of the problems (1)–(3).

**Proof.** Arguing by contradiction, we suppose that \( u(t) \) is a global weak solution of the problems (1)–(3). Considering that
\[ \int_0^t u_i(r)dr = u(t) - u_0, \quad \forall t \in (0, \infty), \] (65)
so we have
\[ \int_0^t \|u_i(r)\|^2 H^1 dr = \|u(t) - u_0\|^2 H^1, \] (66)
From (16), (66) and Hölder’s inequality, we obtain
\[ \|u(t)\|^2 H^1 \leq \|u_0\|^2 H^1 + \int_0^t \|u_i(r)\|^2 H^1 dr \leq \|u_0\|^2 H^1 + t^{1/2} (\int_0^t \|u_i(r)\|^2 H^1 dr)^{1/2} \leq \|u_0\|^2 H^1 + t^{1/2} (J(u_0) - J(u(t)))^{1/2}. \] (67)
Since assume that \( u(t) \) is a global weak solution of the problems (1)–(3), we get \( J(u(t)) \geq 0 \) for all \( t \in [0, \infty) \). Otherwise, there exists a time \( t_0 \in (0, \infty) \) such that \( J(u(t_0)) < 0 \). Hence, from
\[ 0 > J(u(t_0)) = \frac{1}{2} \int_{\Omega} F(u(t_0))dx \geq \frac{1}{2} \|u_0\|^2 H^1 + \frac{1}{p + 1} \int_{\Omega} f(u_0)|u_0|^p dx \geq \frac{p - 1}{2(p + 1)} \|u_0\|^2 H^1 + \frac{1}{p + 1} (J(u(t_0))), \] (68)
we have \( I(u(t_0)) < 0 \) and \( J(u(t_0)) < 0 \) which implies that \( u(t_0) \in V \). Therefore, by the results of Theorem 8, we obtain that \( u(x, t; t_0, u(t_0)) \) blows up in finite time, which is a contradiction. Thus, we have
\[ J(u_0) \geq J(u(t)) \geq 0, \quad \forall t \in (0, \infty). \] (69)

Next, combining (67) and (69), we get
\[ \|u(t)\|^2 H^1 \leq \|u_0\|^2 H^1 + t^{1/2} (J(u_0) - J(u(t)))^{1/2} \leq \|u_0\|^2 H^1 + t^{1/2} (J(u(t_0)))^{1/2}, \] (70)
for all \( t \in (0, \infty) \).
On the other hand, multiplying $u$ on two sides of Equation (1) and integrating by parts, we have

$$
(u_t, u) + (u_{xt}, u_x) + (u_x, u_x) + (u_{xxx}, u_x) = -\langle f(u), u \rangle_x.
$$

(71)

The Poincaré inequality gives $\|u_x\|_2^2 \geq \lambda_1 \|u\|_2^2$, where $\lambda_1$ is the first eigenvalue of the problem

$$
\begin{cases}
    w_{xx} + \lambda w = 0, \text{ in } \Omega,
    \\
    w = 0, \text{ on } \partial \Omega.
\end{cases}
$$

(72)

Thus, we have

$$
\|u\|_{H^1}^2 = \|u\|_2^2 + \|u_x\|_2^2 \leq \frac{1 + \lambda_1}{\lambda_1} \|u\|_2^2 \leq \frac{1 + \lambda_1}{\lambda_1} \|u_x\|_{H^1}^2.
$$

(73)

By the combination of (4), (73), and Sobolev's inequality, we can get that

$$
\frac{d}{dt} \left( \frac{1}{2} \|u\|_{H^1}^2 \right) = -\langle u_x, f(u) \rangle_x \\
\quad \geq -\|u_x\|_{H^1}^2 - (p + 1) \int_0^1 F(u_x)dx \\
\quad = \frac{p-1}{2} \|u_x\|_{H^1}^2 - (p + 1)F(u_t) \\
\quad \geq \frac{(p-1)\lambda_1}{2(1+\lambda_1)} \|u_x\|_{H^1}^2 - (p + 1)F(u_t).
$$

(74)

Since $\langle d/dt \rangle F(u(t)) \leq 0$, for $\forall k > 0$, we have

$$
\frac{d}{dt} \left( \frac{1}{2} \|u\|_{H^1}^2 - kF(u(t)) \right) \\
\quad \geq \frac{d}{dt} \left( \frac{1}{2} \|u\|_{H^1}^2 \right) - \frac{(p-1)\lambda_1}{2(1+\lambda_1)} \|u_x\|_{H^1}^2 - (p + 1)F(u_t) \\
\quad = \frac{(p-1)\lambda_1}{1+\lambda_1} \left( \frac{1}{2} \|u_x\|_{H^1}^2 - \frac{p+1}{p-1}(1+\lambda_1) \|u_x\|_{H^1}^2 \right) F(u(t)) \\
\quad \geq \frac{(p-1)\lambda_1}{1+\lambda_1} \left( \frac{1}{2} \|u_x\|_{H^1}^2 \right) F(u(t)) - \frac{p+1}{p-1}(1+\lambda_1) \|u_x\|_{H^1}^2 F(u(t)).
$$

(75)

Taking $k = ((p+1)(1+\lambda_1))/((p-1)\lambda_1)$ in (75) and $G(t) = 1/2 \|u\|_{H^1}^2 - (((p+1)(1+\lambda_1))/((p-1)\lambda_1))F(u(t))$, then we have

$$
\frac{d}{dt} G(t) \geq \frac{(p-1)\lambda_1}{1+\lambda_1} G(t).
$$

(76)

Integrating the inequality (76) from 0 to $t$, we see

$$
G(t) \geq e^{((p-1)\lambda_1)/(1+\lambda_1)} G(0), \quad t \in [0, \infty).
$$

(77)

which means that

$$
\|u\|_{H^1}^2 \geq \frac{2(p+1)\lambda_1}{(p-1)(1+\lambda_1)} F(u(t)) + 2e^{((p-1)\lambda_1)/(1+\lambda_1)^2} G(0), \quad t \in [0, \infty).
$$

(78)

From the assumption condition (64), we have $G(0) > 0$. Hence, we get from (69) and (78) that $\|u\|_{H^1}^2 \geq 2 e^{((p-1)\lambda_1)/(1+\lambda_1)^2} G(0)$, i.e.,

$$
\|u\|_{H^1}^2 \geq [2G(0)]^{1/2} e^{((p-1)\lambda_1)/(2(1+\lambda_1)^2)} t, \quad t \in [0, \infty).
$$

(79)

From the combination of (70) and (79), we have

$$
[2G(0)]^{1/2} e^{((p-1)\lambda_1)/(2(1+\lambda_1)^2)} t \leq \|u\|_{H^1}^2 + t^{1/2} (J(u_0))^{1/2}.
$$

(80)

Clearly, the above inequality cannot hold for $t$ large enough, this means that the solution $u$ of problems (1)–(3) cannot exist all time.

Furthermore, by (80) and $e^{((p-1)\lambda_1)/(2(1+\lambda_1)^2)} t \geq ((p-1)\lambda_1)/(2(1+\lambda_1)) t$, we can obtain the inequality

$$
\sqrt{2G(0)}(p-1)\lambda_1 \frac{1}{2} \|u\|_{H^1}^2 \geq \frac{1}{2} \|u\|_{H^1}^2 - (J(u_0))^{1/2} t^{1/2} \|u\|_{H^1}^2 \leq 0,
$$

(81)

which implies that there exists a finite time $T > 0$ such that

$$
\lim_{t \to T^-} \|u\|_{H^1}^2 = +\infty,
$$

(82)

and $T^{1/2}$ is the largest root of the following equation

$$
\sqrt{2G(0)}(p-1)\lambda_1 \frac{1}{2} t^2 - (J(u_0))^{1/2} t - \|u\|_{H^1}^2 = 0.
$$

(83)

This completes the proof.

### 6. Conclusion and Future Work

In our work, we mainly study the qualitative properties of the solutions for the initial and boundary value problems (1)–(3). It is well known that Equation (1) is a typical higher-order metaparabolic equation, which has extensive practical background and rich theoretical notation. For example, the solutions $u$ of (1) can be used to denote the concentration of one of the two phases, the fourth-order term $uy_{xxxx}$ presents the capillarity-driven surface diffusion, and the nonlinear term $f(u_x)$ is an intrinsic chemical potential. Especially, the interaction between the dispersion term $u_{xx}$ and nonlinearity $f(u_x)$ of these problems cause some difficulties such that we cannot apply the normal Galerkin approximation, concavity, and potential methods directly. Considering the above situation, at low initial energy level, we first prove the existence of global weak solutions for these problems by the Galerkin approximation and potential well methods and obtain the finite time blow-up result by the potential well and improved concavity skills. In addition, we establish the
finite time blow-up result for certain solutions with high initial energy. However, as far as we know, there is little information on the long-time behavior of global solutions for above problems. Whether the global solutions will exhibit a long-time dynamic behavior at low initial energy? Do both problems (1)–(3) have the global solutions and asymptotic property at high initial energy level? These questions are all opening, and we are now working on these problems. On the other hand, we note that the fractional partial differential equations have been applied in various areas of science, and their related theoretical results and applications have been investigated by some authors (see [31–33] and the references therein). The study of their qualitative properties is one of the hot topics. Do the conclusions of present paper also hold for therein). The study of their qualitative properties is one of the hot topics. Do the conclusions of present paper also hold for

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no competing interests.

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