Dynamical sampling and systems from iterative actions of operators

Akram Aldroubi and Armenak Petrosyan

Abstract In this chapter, we review some of the recent developments and prove new results concerning frames and Bessel systems generated by iterations of the form \( \{A^n g : g \in G, n = 0, 1, 2, \ldots \} \), where \( A \) is a bounded linear operator on a separable complex Hilbert space \( H \) and \( G \) is a countable set of vectors in \( H \). The system of iterations mentioned above was motivated from the so called dynamical sampling problem. In dynamical sampling, an unknown function \( f \) and its future states \( A^n f \) are coarsely sampled at each time level \( n, 0 \leq n < L \), where \( A \) is an evolution operator that drives the system. The goal is to recover \( f \) from these space-time samples.

1 Introduction

The typical dynamical sampling problem is finding spatial positions \( X = \{ x_i \in \mathbb{R}^d : i \in I \} \) that allow the reconstruction of an unknown function \( f \in \mathcal{H} \subset L^2 (\mathbb{R}^d) \) from samples of the function at spatial positions \( x_i \in X \) and subsequent samples of the functions \( A^n f, n = 0, \ldots, L \), where \( A \) is an evolution operator and \( n \) represents time. For example, \( f \) can be the temperature at time \( n = 0 \), \( A \) the heat evolution operator, and \( A^n f \) the temperature at time \( n \). The problem is then to find spatial sampling positions \( X \subset \mathbb{R}^d \), and end time \( L \), that allow the determination of the initial temperature \( f \) from samples \( \{ f |_X, (Af) |_X, \ldots, (A^L f) |_X \} \). For the heat evolution operator, the problem has been considered by Vetterli and Lu \cite{21, 20} and inspired our research in dynamical sampling, see e.g., \cite{3, 4, 5, 6}.

Akram Aldroubi
Vanderbilt University, Nashville, TN 37240, USA e-mail: akram.aldroubi@vanderbilt.edu

Armenak Petrosyan
Vanderbilt University, Nashville, TN 37240, USA e-mail: armenak.petrosyan@vanderbilt.edu
1.1 The dynamical sampling problem

Let $\mathcal{H}$ be a separable (complex) Hilbert space, $f \in \mathcal{H}$ be an unknown vector and $f_n \in \mathcal{H}$ be the state of the system at time $n$. We assume

$$f_0 = f, \quad f_n = A f_{n-1} = A^n f$$

where $A$ is a known bounded operator on $\mathcal{H}$. Given the measurements

$$\langle A^n f, g \rangle$$

for $0 \leq n < L(g), \quad g \in G$ (1)

where $G$ is a countable subset of $\mathcal{H}$ and $L : G \rightarrow \mathbb{N} \cup \{\infty\}$ is a function, the dynamical sampling problem is to recover the vector $f \in \mathcal{H}$ from the measurements (1). It is important that the recovery of $f$ be robust to noise. Thus, we also require that the sampling data allow the recovery of $f$ in a stable way. Equivalently, for any $f \in \mathcal{H}$, the samples must satisfy the stability condition

$$C_1 \|f\|^2 \leq \sum_{g \in G} \sum_{n=0}^{L(g)} |\langle A^n f, g \rangle|^2 \leq C_2 \|f\|^2,$$

for some $C_1, C_2 > 0$ absolute constants.

A related problem for band-limited signals in $\mathbb{R}^2$ (i.e., the Paley Wiener spaces $PW_\sigma$) with time varying sampling locations corresponding to trajectories but time-independent function can be found in [16].

1.2 Dynamical sampling for diagonalizable operators in $l^2(\mathbb{N})$

When the Hilbert space is $\mathcal{H} = l^2(\mathbb{N})$, and when the operator $A$ is equivalent to a diagonal matrix $D$, i.e., $A^* = B^{-1}DB$ where $D = \sum_j \lambda_j P_j$ is an infinite diagonal matrix, then a characterization of the set of sampling $I \subset \mathbb{N}$ such that any $f \in \mathcal{H}$ can be recovered from the data $Y = \{f(i), (Af)(i), \ldots, (A^l f)(i) : i \in I\}$ is obtained in [6].

The results are stated in terms of vectors $b_i$ that are the columns of $B$ corresponding to the sampling positions $i \in I$, the projections $P_i$ that are diagonal infinite matrices whose non-zero diagonals are all ones and correspond to the projection on the eigenspace of $D$ associated to $\lambda_j$, and the smallest integers $l_i$ such that the sets $\{b_i, Db_i, \ldots, D^l b_i\}$ are minimal [4].

**Theorem 1.** Let $A^* = B^{-1}DB$, and let $\{b_i : i \in I\}$ be the column vectors of $B$ whose indices belong to $I$. Let $l_i$ be the smallest integers such that the set $\{b_i, Db_i, \ldots, D^{l_i} b_i\}$ is minimal. Then any vector $f \in l^2(\mathbb{N})$ can be recovered from the samples

$$Y = \{f(i), Af(i), \ldots, A^l f(i) : i \in I\}$$
if and only if for each $j$, $\{P_j(b_i) : i \in I\}$ is complete in the range $E_j$ of $P_j$.

Although Theorem 1 characterizes the sets $I \subset \mathbb{N}$ such that recovery of any $f \in l^2(\mathbb{N})$ is possible, it does not provide conditions for stable recovery, i.e., recovery that is robust to noise. Results on the stable recovery are obtained for the case when $I$ is finite [4]. Stable recovery is also obtained when $\mathcal{H} = l^2(\mathbb{Z})$, $A$ is a convolution operator, and $I$ is a union of uniform grids [6]. For shift-invariant spaces, and union of uniform grids, stable recovery results can be found in [1]. Obviously, recovery and stable recovery are equivalent in finite dimensional spaces [5]. In [27] the case when the locations of the sampling positions are allowed to change is considered.

1.2.1 Connections with other fields and applications

The dynamical sampling problem has similarity with wavelets [7, 11, 12, 17, 22, 24, 25, 31]. In dynamical sampling an operator $A$ is applied iteratively to the function $f$ producing the functions $f_n = A^n f$. $f_n$ is then, typically, sampled coarsely at each level $n$. Thus, $f$ cannot be recovered from samples at any single time-level. But, similarly to the wavelet transform, the combined data at all time levels is required to reproduce $f$. However, unlike the wavelet transform, there is a single operator $A$ instead of two complementary operators $L$ (the lowpass operator) and $H$ (the high pass operator). Moreover, $A$ is imposed by the constraints of the problem, rather than designed, as in the case of $L$ and $H$ in wavelet theory. Finally, in dynamical sampling, the spatial-sampling grids is not required to be regular.

In inverse problems, given an operator $B$ that represents a physical process, the goal is to recover a function $f$ from the observation $Bf$. Deconvolution or de-blur are prototypical examples. When $B$ is not bounded below, the problem is considered ill-posed (see e.g., [23]). The dynamical sampling problem can be viewed as an inverse problem when the operator $B$ is the result of applying the operators $S_{X_0}, S_{X_1}A, S_{X_2}A^2, \ldots, S_{X_L}A^L$, where $S_{X_l}$ is the sampling operator at time $l$ on the set $X_l$, i.e., $B_X = [S_{X_0}, S_{X_1}A, S_{X_2}A^2, \ldots, S_{X_L}A^L]^T$. However, unlike the typical inverse problem, in dynamical sampling the goal is to find conditions on $L$, $\{X_i : i = 0, \ldots, L\}$, and $A$, such that $B_X$ is injective, well conditioned, etc.

The dynamical sampling problem has connections and applications to other areas of mathematics including, Banach algebras, $C^*$-algebras, spectral theory of normal operators, and frame theory [2, 8, 10, 13, 14, 15, 26, 32].

Dynamical sampling has potential applications in plenacoustic sampling, on-chip sensing, data center temperature sensing, neuron-imaging, and satellite remote sensing, and more generally to Wireless Sensor Networks (WSN). In wireless sensor networks, measurement devices are distributed to gather information about a physical quantity to be monitored, such as temperature, pressure, or pollution [13, 21, 26, 20, 30]. The goal is to exploit the evolutionary structure and the placement of sensors to reconstruct an unknown field. When it is not possible to place sampling devices at the desired locations (e.g., when there are not enough devices), then the desired information field can be recovered by placing the sensors elsewhere and taking advantage of the evolution process to recover the signals at
the relevant locations. Even when the placement of sensors is not constrained, if the cost of a sensor is expensive relative to the cost of activating the sensor, then the relevant information may be recovered with fewer sensors placed judiciously and activated frequently. Super resolution is another application when a evolutionary process acts on the signal of interest.

1.3 Contribution

In this chapter, we further develop the case of iterative systems generated by the iterative actions of normal operators which was studied in [3, 4]. This is done in Section 2.3. In Section 2.4 we study the case of general iterative systems generated by the iterative actions of operators that are not necessarily normal.

2 Frame and Bessel properties of systems from iterative actions of operators

In this section we review some results from [3, 4] on the iterative actions of normal operators, prove some new results for this case and generalize several results to the case where the operators are not necessary normal.

2.1 Equivalent formulation of the dynamical sampling problem

Using the fact that \( \langle Af, g \rangle = \langle f, A^* g \rangle \), we get the following equivalent formulation of the dynamical sampling problem described in Section 1.1.

**Proposition 1.** 1. Any \( f \in \mathcal{H} \) can be recovered from \( \{ \langle A^n f, g \rangle \}_{g \in G, 0 \leq n < L(g)} \) if and only if the system \( \{ (A^*)^n g \}_{g \in G, 0 \leq n < L(g)} \) is complete in \( \mathcal{H} \).
2. Any \( f \in \mathcal{H} \) can be recovered from \( \{ \langle A^n f, g \rangle \}_{g \in G, 0 \leq n < L(g)} \) in a stable way if and only if the system \( \{ (A^*)^n g \}_{g \in G, 0 \leq n < L(g)} \) is a frame in \( \mathcal{H} \).

Because of this equivalence, we drop the * and we investigate systems of iterations of the form \( \{ A^n g \}_{g \in G, 0 \leq n < L(g)} \), where \( A \) is a bounded operator on the Hilbert
space $\mathcal{H}$, $G$ is a subset of $\mathcal{H}$, and $L$ is a function from $G$ to the extended set of integers $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. The goal is then to find conditions on $A$, $G$ and $L$ so that \{\$g\in G, 0 \leq \nu < L(g)$\} is complete, Bessel, a basis, Riesz basis, frame, etc. In the remainder of this chapter, we only study the case where $L(g) = \infty$, for each $g \in G$.

### 2.2 Normal operators

Theorem 1 as well as most of the results in \cite{4} have been generalized to the case of normal operators in general Hilbert spaces (see e.g., \cite{10} Ch. IX, theorem 10.16, and \cite{9}). Since we will use this theorem again in this work, we state a version of this landmark theorem and give an example to clarify its meaning. In essence, the spectral theorem of normal operators is a way of diagonalizing any normal operator in a separable complex Hilbert space. Using an appropriate unitary transformation $U$, a normal operator $A$ is equivalent to a multiplication operator $UAU^{-1} = N_\mu f = z f$ where $f$ is a vector valued function on $\mathbb{C}$, and $N_\mu f(z) = z f(z)$ for every $z \in \mathbb{C}$. Specifically,

**Theorem 2 (Spectral theorem with multiplicity).** For any normal operator $A$ on $\mathcal{H}$ there are mutually singular compactly supported Borel measures $\mu_j$, $1 \leq j \leq \infty$ on $\mathbb{C}$, such that $A$ is equivalent to the operator

$$N^{(\infty)}_\mu \oplus N^{(1)}_{\mu_1} \oplus N^{(2)}_{\mu_2} \oplus \cdots$$

i.e. there exists a unitary operator $U$

$$U : \mathcal{H} \to \tilde{\mathcal{H}} = (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

such that

$$UAU^{-1} = N_\mu = N^{(\infty)}_{\mu_\infty} \oplus N^{(1)}_{\mu_1} \oplus N^{(2)}_{\mu_2} \oplus \cdots \quad (2)$$

Moreover, if $M$ is another normal operator with corresponding measures $\nu_\infty, \nu_1, \nu_2, \ldots$ then $M$ is equivalent to $A$ if and only if $|\nu_j| = |\mu_j|$, $j = 1, \ldots, \infty$ (are mutually absolutely continuous).

Since the measures $\{\mu_j : j \in \mathbb{N}^*\}$ are mutually singular, we can define the measure $\mu = \sum_j \mu_j$ on $\mathbb{C}$. A function $\tilde{g} \in \tilde{\mathcal{H}}$ is a vector of the form $(\tilde{g}_j)_{j \in \mathbb{N}^*}$, where $\tilde{g}_j$ is the restriction of $\tilde{g}$ to $(L^2(\mu_j))^{(j)}$.

**Remark 1.** Note that for every $1 \leq j < \infty$, $\tilde{g}_j(z)$ is a finite dimensional vector in $L^2\{1, \ldots, j\}$ and for $j = \infty$, $\tilde{g}_\infty(z)$ is a vector in $L^2(\mathbb{N})$. In order to simplify notation, we define $\Omega_j$ to be the set $\{1, \ldots, j\}$ and $\Omega_\infty$ to be the set $\mathbb{N}$. Note that $L^2(\Omega_j) \cong \mathbb{C}^j$, for $j \in \mathbb{N}$, and $L^2(\Omega_\infty) = L^2(\mathbb{N})$. For $j = 0$ we define $L^2(\Omega_0)$ to be the trivial space $\{0\}$. 


An example to clarify the use of the Theorem above is given below.

**Example 1.** Let $A$ be the $8 \times 8$ diagonal matrix

$$
A = \begin{pmatrix}
\lambda_1 I_2 & 0 & 0 \\
0 & \lambda_2 I_3 & 0 \\
0 & 0 & \lambda_3 I_3
\end{pmatrix}
$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $I_j$ denotes the $j \times j$ identity matrix. For this case, the theorem above gives: $\tilde{H} = (L^2(\mu_2))^{(2)} \oplus (L^2(\mu_3))^{(3)}$, $\mu_2 = \delta_{\lambda_1}$, $\mu_3 = \delta_{\lambda_2} + \delta_{\lambda_3}$, where $\delta_x$ is the Dirac measure at $x$. If $g = (g_1, \ldots, g_8)^T$, then $Ug = \tilde{g} = (\tilde{g}_j)$. In particular, $\tilde{g}_2(\lambda_2) = \begin{pmatrix} g_3 \\ g_4 \end{pmatrix}$, $\tilde{g}_3(\lambda_3) = \begin{pmatrix} g_6 \\ g_7 \end{pmatrix}$ and $\tilde{g}_3(z) = 0$ for $z \neq \lambda_2, \lambda_3$ (in fact for $z \neq \lambda_2, \lambda_3$, $\tilde{g}_3(z)$ can take any value since the measure $\mu_3$ is concentrated on $\{\lambda_2, \lambda_3\} \subset \mathbb{C}$). We have

$$
(Uf, Ug) = \int_C \langle \tilde{f}(z), \tilde{g}(z) \rangle d\mu(z)
$$

$$
= \int_C \langle \tilde{f}_2(z), \tilde{g}_2(z) \rangle d\mu_2(z) + \int_C \langle \tilde{f}_3(z), \tilde{g}_3(z) \rangle d\mu_3(z)
$$

$$
= \langle \tilde{f}_2(\lambda_1), \tilde{g}_2(\lambda_1) \rangle + \langle \tilde{f}_3(\lambda_2), \tilde{g}_3(\lambda_2) \rangle + \langle \tilde{f}_3(\lambda_3), \tilde{g}_3(\lambda_3) \rangle
$$

$$
= \sum_{j=1}^8 f_j \tilde{g}_j = \langle f, g \rangle.
$$

Since the measures $\mu_j$ in Theorem 1 are mutually singular, there are mutually disjoint Borel sets $\{\mathcal{E}_j\}$ such that $\mu_j$ is concentrated on $\mathcal{E}_j$ for every $1 \leq j \leq \infty$.

The function $n : \mathbb{C} \to \{1, 2, \ldots, \infty\}$ given by

$$
n(z) = \begin{cases}
    j, & z \in \mathcal{E}_j \\
    0, & \text{otherwise}
\end{cases}
$$

is called multiplicity function of the operator $A$. Thus every normal operator is uniquely determined, up to a unitary equivalence, by a pair $(n, [\mu])$ where $[\mu]$ is the class of measures mutually singular with the compactly supported Borel measure $\mu$ and $n : \mathbb{C} \to \{1, 2, \ldots, \infty\}$ is a $\mu$ measurable function.

### 2.3 Action of normal operators via infinite iterations

In this section we present results from [3] about a system of infinite iterative action $\{A^n G\}_{n \geq 0}$ of a given normal operator $A \in B(H)$ on a set of vectors $G \subset H$. Some of the results assume that $A$ is reductive, i.e., every invariant subspace $V$ for $A$ is also invariant for $A^*$. 
Theorem 3. Let $A$ be a normal operator on a Hilbert space $\mathcal{H}$, and let $G$ be a countable set of vectors in $\mathcal{H}$ such that $\{A^n g\}_{g \in G, n \geq 0}$ is complete in $\mathcal{H}$. Let $\mu_0, \mu_1, \mu_2, \ldots$ be the measures in the representation (2) of the operator $A$. Then for every $1 \leq j \leq \infty$ and $\mu_j$-a.e. $z$, the system of vectors $\{\tilde{g}_j(z)\}_{g \in G}$ is complete in $l^2(\Omega_j)$.

If in addition to being normal, $A$ is also reductive, then $\{A^n g\}_{g \in G, n \geq 0}$ being complete in $\mathcal{H}$ is equivalent to $\{\tilde{g}_j(z)\}_{g \in G}$ being complete in $l^2(\Omega_j)$ $\mu_j$-a.e. $z$ for every $1 \leq j \leq \infty$.

Although, the system of iteration $\{A^n g\}_{g \in G, n \geq 0}$ is complete, it is shown in [3] that it cannot be a basis for $\mathcal{H}$. The obstruction is that $\{A^n g\}_{g \in G, n \geq 0}$ cannot be minimal and complete at the same time.

Theorem 4. If $A$ is a normal operator on $\mathcal{H}$, then, for any set of vectors $G \subset \mathcal{H}$, the system of iterates $\{A^n g\}_{g \in G, n \geq 0}$ is not a basis for $\mathcal{H}$.

Remark 2. The normality assumption on $A$ is essential. For example, if $S$ is the right-shift operator on $\mathcal{H} = l^2(\mathbb{N})$, then $\{S^n e_0\}_{n \geq 0}$ is an orthonormal basis for $\mathcal{H} = l^2(\mathbb{N})$. In fact, it can be shown that, in a Hilbert space, a system of vectors $\mathcal{H}$ $\{T^n g\}_{n \geq 0}$ generated by $T \in B(\mathcal{H})$ and $g \in \mathcal{H}$ is a Riesz basis if and only if it is unitarily equivalent to the right-shift operator $S$ in $l^2(\mathbb{N})$ [19].

The fact that, for a normal operator $A$, $\{A^n g\}_{g \in G, n \geq 0}$ cannot be basis is that when it is complete, it must be redundant (since it is not minimal). But it is possible for such a sequence to be a frame. For example, the following theorem characterizes frames generated by the iterative action of diagonalizable normal operators acting on a single vector $b$ [4].

Theorem 5. Let $\Lambda = \sum_j \lambda_j P_j$, acting on $l^2(\mathbb{N})$, be such that $P_j$ have rank 1 for all $j \in \mathbb{N}$, and let $b := \{b(k)\}_{k \in \mathbb{N}} \in l^2(\mathbb{N})$. Then $\{A^l b : l = 0, 1, \ldots\}$ is a frame if and only if

\begin{enumerate}
  \item $|\lambda_k| < 1$ for all $k$.
  \item $|\lambda_k| \to 1$.
  \item $\{\lambda_k\}$ satisfies Carleson’s condition

$$\inf_n \prod_{k \neq n} \frac{|\lambda_m - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta. \quad (3)$$

for some $\delta > 0$.
  \item $b(k) = m_k \sqrt{1 - |\lambda_k|^2}$ for some sequence $\{m_k\}$ satisfying $0 < C_1 \leq |m_k| \leq C_2 < \infty$.
\end{enumerate}

In the previous theorem, the spectrum lies inside the unit disk $D_1$. Moreover, the spectrum concentrates near its boundary $S_1$. These facts can be generalized for normal operators [3].

Theorem 6. Let $A$ be a normal operator on an infinite dimensional Hilbert space $\mathcal{H}$ and $G$ a system of vectors in $\mathcal{H}$. 
1. If \( \{A^n g\}_{g \in G, n \geq 0} \) is complete in \( \mathcal{H} \) and for every \( g \in G \) the system \( \{A^n g\}_{n \geq 0} \) is Bessel in \( \mathcal{H} \), then \( \mu (D^n_f) = 0 \) and \( \mu \mid_{S_1} \) is absolutely continuous with respect to arc-length measure (Lebesgue measure) on \( S_1 \).

2. If \( |G| < \infty \) and \( \{A^n g\}_{g \in G, n \geq 0} \) satisfies the lower frame bound then, for every \( 0 < \epsilon \leq 1, \mu (D^n_f \setminus \epsilon) > 0 \), where \( D^n_f \setminus \epsilon \) is the closed disc of radius \( 1 - \epsilon \).

It can be proved that if \( \mu (D^n_f) = 0 \) and \( \mu \mid_{S_1} \) is absolutely continuous with respect to arc-length measure on \( S_1 \) then there exists a set \( G \subset \mathcal{H} \) such that \( \{A^n g\}_{g \in G, n \geq 0} \) is complete and Bessel system in \( \mathcal{H} \). Other developments on this theme can be found in [23].

**Corollary 1.** If for a normal operator \( A \in B(\mathcal{H}) \) in an infinite dimensional space \( \mathcal{H} \) the system of vectors \( \{A^n g\}_{g \in G, n \geq 0} \) with \( |G| < \infty \) is a frame, then \( A \) is unitarily equivalent to an operator \( \Lambda = \sum_j \lambda_j P_j \) where \( P_j \) are projections such that \( \dim P_j \leq |G| \). In particular, if \( |G| = 1 \), then \( \lambda_j \) satisfy conditions \( i), ii) \) in Theorem 6.

**Proof.** Define the subspace \( \tilde{V}_{\rho} \) of \( \mathcal{H} \) to be \( \tilde{V}_{\rho} = \{ \tilde{f} : \supp \tilde{f} \subseteq D_{\rho} \} \). The restriction of \( UAU^* \) to \( \tilde{V}_{\rho} \) is normal with its spectrum equal to the part of the spectrum of \( A \) inside \( D_{\rho} \). If we iterate the \( z \)-multiplication operator on the projections \( G_\rho = P_{\tilde{V}_\rho}G \) of the vectors in \( G \) we get a frame for \( \tilde{V}_{\rho} \) hence, from part 2), \( \tilde{V}_\rho \) must be finite dimensional. That implies the spectrum of \( A \) is finite inside every \( D_{\rho} \) with \( \rho < 1 \). We also know from Part (1) of Theorem 6 that \( \mu (D^n_f) = 0 \). Furthermore, from Corollary 2 below, \( \mu \mid_{S_1} = 0 \). Thus, \( UAU^* \) has the form \( \Lambda = \sum_j \lambda_j P_j \). The fact that \( \dim P_j \leq |G| \) follows from Theorem 1. The rest follows from Theorem 6. \( \square \)

### 2.4 New results for general bounded operators

This section is devoted to the study of the iterative action of general bounded operators in \( B(\mathcal{H}) \).

**Theorem 7.** If for an operator \( A \in B(\mathcal{H}) \) there exists a set of vectors \( G \) in \( \mathcal{H} \) such that \( \{A^n g\}_{g \in G, n \geq 0} \) is a frame in \( \mathcal{H} \) then for every \( f \in \mathcal{H}, (A^n)^m f \to 0 \) as \( n \to \infty \).

**Proof.** Suppose, for some \( \{g\}_{g \in G}, \{A^n g\}_{g \in G, n \geq 0} \) is a frame with frame bounds \( B_1 \) and \( B_2 \). Let \( f \in \mathcal{H} \). Then for any \( m \in \mathbb{Z} \) we have

\[
\sum_{g \in G} \sum_{n=0}^\infty |\langle (A^*)^m f, A^n g \rangle|^2 = \sum_{g \in G} \sum_{n=0}^\infty |\langle f, A^{n+m} g \rangle|^2 = \sum_{g \in G} \sum_{n=-m}^{\infty} |\langle f, A^n g \rangle|^2.
\]

(4)

Since \( \sum_{g \in G} \sum_{n=0}^\infty |\langle f, A^n g \rangle|^2 \leq B_2 \|f\|^2 \), we conclude that \( \sum_{n=m}^\infty \sum_{g \in G} |\langle f, A^n g \rangle|^2 \to 0 \) as \( m \to \infty \). Thus, from (4), we get that \( \sum_{g \in G} \sum_{n=0}^\infty |\langle (A^*)^m f, A^n g \rangle|^2 \to 0 \) as \( m \to \infty \). Using the lower frame inequality, we get
B₁∥(A*)m f∥ ≤ \sum_{g∈G} \sum_{n=0}^{m} |⟨(A*)^m f, A^n g⟩|^2.

Since the right side of the inequality tends to zero as \(m\) tends to infinity we get that \((A*)^m f → 0\) as \(m → ∞\).

**Corollary 2.** For any unitary operator \(A : H → H\) and any set of vectors \(G ⊂ H\), \(\{A^n g\}_{g∈G, n≥0}\) is not a frame in \(H\).

If for every \(f ∈ H\), \((A*)^n f → 0\) as \(n → ∞\), then we can get the following existence theorem of frames for \(H\) from iterations.

**Theorem 8.** If \(A\) is a contraction (i.e., \(∥A∥ ≤ 1\)), and for every \(f ∈ H\), \((A*)^n f → 0\) as \(n → ∞\), then we can get the following existence theorem of frames for \(H\) from iterations.

**Remark 3.** The system we find in this case is not very useful since the initial system \(G\) is 'too large' (it is complete in \(H\) in some cases). Moreover, the condition \(∥A∥ ≤ 1\) is not necessary for the existence of a frame with iterations. For example, we can take nilpotent operators with large operator norm for which there are frames with iterations.

**Proof.** Suppose for any \(f ∈ H\), \((A*)^n f → 0\) as \(n → ∞\) and \(∥A∥ ≤ 1\). Let \(D = (I − AA^*)^{1/2}\) and \(V = cl(DH)\). Let \(\{h\}_{h∈F}\) be an orthonormal basis for \(V\). Then

\[
\sum_{n=0}^{m} \sum_{h∈F} |f, A^n Dh|^2 = \sum_{n=0}^{m} |D(A*)^n f, h|^2 \\
= \sum_{n=0}^{m} \|D(A*)^n f\|^2 \\
= \sum_{n=0}^{m} (D(A*)^n f, (A*)^n f) \\
= \sum_{n=0}^{m} \langle (I − AA^*)(A*)^n f, (A*)^n f \rangle \\
= \|f\|^2 − \|(A*)^{m+1} f\|.
\]

Taking limits as \(m → ∞\) and using the fact that \((A*)^m f → 0\) we get from the identity above that

\[
\sum_{n=0}^{∞} \sum_{h∈F} |f, A^n Dh|^2 = \|f\|^2.
\]

Therefore the system of vectors \(G = \{g = Dh : h ∈ F\}\) is a tight frame for \(H\).

**Theorem 9.** If \(dim H = ∞\), \(|G| < ∞\), and \(\{A^n g\}_{g∈G, n≥0}\) satisfies the lower frame bound, then \(∥A∥ ≥ 1\).
Proof. Suppose $\|A\| < 1$. Since $\{g\} \in G$ is finite and $\dim(\mathcal{H}) = \infty$, for any fixed $N$ there exists a vector $f \in \mathcal{H}$ with $\|f\| = 1$ such that $\langle A^n g, f \rangle = 0$, for every $g \in G$ and $0 \leq n \leq N$. Then

$$
\sum_{g \in G} \sum_{n \geq 0} |\langle A^n g, f \rangle|^2
= \sum_{g \in G} \sum_{n = N}^{\infty} |\langle A^n g, f \rangle|^2
\leq \sum_{g \in G} \|g\| \sum_{n = N}^{\infty} \|A\|^{2n} \to 0
$$
as $N \to \infty$ hence the lower frame bound cannot hold. \hfill \Box

**Corollary 3.** Let $\{A^n g\} \in G, n \geq 0$ with $|G| < \infty$ satisfies the lower frame bound. Then for any co-invariant subspace $\mathcal{V} \subset \mathcal{H}$ of $A$ with $\|P_{\mathcal{V}} A P_{\mathcal{V}}\| < 1$ we have that $\dim(\mathcal{V}) < \infty$.

*Proof.* $\mathcal{V}$ is co-invariant for $A$ is equivalent to

$$
P_{\mathcal{V}} A = P_{\mathcal{V}} A P_{\mathcal{V}}.
$$

It follows that $P_{\mathcal{V}} A^n = P_{\mathcal{V}} A^n P_{\mathcal{V}}$. Hence, if $\{A^n g\} \in G, n \geq 0$ satisfies the lower frame inequality in $\mathcal{H}$, then $\{(P_{\mathcal{V}} A P_{\mathcal{V}})^n g\} \in G, n \geq 0$ also satisfies the lower frame inequality for $\mathcal{V}$ and hence from the previous theorem if $\dim(\mathcal{V}) = \infty$, then $\|P_{\mathcal{V}} A P_{\mathcal{V}}\| \geq 1$. \hfill \Box

### 3 Related work and concluding remarks

There are several features that are particular to the present work: In the system of iterations $\{A^n g\} \in G, 0 \leq n \leq L(g)$ that we considered in this chapter, we let $L(g) = \infty$ for all $g \in G$. This setting implies strong constraints on the spectrum of $A$ when we further require that the system is a Bessel system, a frame, etc. Since in finite dimensional spaces every finite spanning set is a frame, and since for fixed $g$, if $K > \dim(\mathcal{H})$, then the set $\{(A^n g)\} \in G, 0 \leq n \leq K$ is always linearly dependent, it does not make sense to let $L(g) > \dim(\mathcal{H}) + 1$. In fact, the finite dimensional problem has first been studied in which $L(g)$ is a constant for all $g \in G$ and is as small as possible in some sense.

Acknowledgements This work has been partially supported by NSF/DMS grant 1322099. Akram Aldroubi would like to thank Charlotte Avant and Barbara Corley for their attendance to the comfort and entertainment during the preparation of this manuscript.

### References

1. Roza Aceska, Akram Aldroubi, Jacqueline Davis, and Armenak Petrosyan, *Dynamical sampling in shift invariant spaces*, Commutative and Noncommutative Harmonic Analysis and Applications (Azita Mayeli, Alex Iosevich, Palle E. T. Jorgensen, and Gestur Olafsson, eds.), Contemp. Math., vol. 603, Amer. Math. Soc., Providence, RI, 2013, pp. 139–148.
2. Akram Aldroubi, Anatoly Baskakov, and Ilya Krishtal, *Slanted matrices, Banach frames, and sampling*, J. Funct. Anal. **255** (2008), no. 7, 1667–1691. MR 2442078 (2010a:46059)
3. Akram Aldroubi, Carlos Cabrelli, Ahmet F. Çakmak, Ursula Molter, and Petrosyan Armenak, *Iterative actions of normal operators*, arXiv:1602.04527 (2016).
4. Akram Aldroubi, Carlos Cabrelli, Ursula Molter, and Sui Tang, *Dynamical sampling*, Appl. Comput. Harmon. Anal. (in press, 2016), ArXiv:1409.8333.
5. Akram Aldroubi, Jacqueline Davis, and Ilya Krishtal, *Dynamical sampling; time-space trade-off*, Appl. Comput. Harmon. Anal. **34** (2013), no. 3, 495–503. MR 3027915
6. **Exact reconstruction of signals in evolutionary systems via spatiotemporal trade-off**, J. Fourier Anal. Appl. **21** (2015), 11–31.
7. Ola Bratteli and Palle Jorgensen, *Wavelets through a looking glass*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2002, The world of the spectrum. MR 1913212 (2003i:42001)
8. Peter G. Casazza, Gitta Kutyniok, and Shidong Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. **25** (2008), no. 1, 114–132. MR 2419707 (2009d:42094)
9. John B. Conway, *Subnormal operators*, Research Notes in Mathematics, vol. 51, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1981. MR 634507 (83i:47030)
10. **A course in functional analysis**, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713
11. Brad Currey and Azita Mayeli, *Gabor fields and wavelet sets for the Heisenberg group*, Monatsh. Math. **162** (2011), no. 2, 119–142. MR 2769882 (2012d:42069)
12. Ingrid Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR 1162107
13. Brendan Farrell and Thomas Strohmer, *Inverse-closedness of a Banach algebra of integral operators on the Heisenberg group*, J. Operator Theory **64** (2010), no. 1, 189–205. MR 2669435
14. Karlheinz Gröchenig, *Localization of frames, Banach frames, and the invertibility of the frame operator*, J. Fourier Anal. Appl. **10** (2004), no. 2, 105–132. MR 2054304 (2005f:42096)
15. Karlheinz Gröchenig and Michael Leinert, *Wiener's lemma for twisted convolution and Gabor frames*, J. Amer. Math. Soc. **17** (2004), no. 1, 1–18 (electronic). MR 2015328 (2004m:42037)
16. Karlheinz Gröchenig, José Luis Romero, Jayakrishnan Unnikrishnan, and Martin Vetterli, *On minimal trajectories for mobile sampling of bandlimited fields*, Appl. Comput. Harmon. Anal. **39** (2015), no. 3, 487–510. MR 3398946
17. Eugenio Hernández and Guido Weiss, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996, With a foreword by Yves Meyer. MR 1408902 (97i:42015)
18. Ali Hormati, Olivier Roy, Yue M. Lu, and Martin Vetterli, *Distributed sampling of signals linked by sparse filtering: Theory and applications*, Signal Processing, IEEE Transactions on **58** (2010), no. 3, 1095 –1109.
19. Illia Karabash, *Unpublished notes*, Private Communication (2016).
20. Yu-Ming Lu and M. Vetterli, *Spatial super-resolution of a diffusion field by temporal oversampling in sensor networks*, Acoustics, Speech and Signal Processing, 2009. ICASSP 2009. IEEE International Conference on, april 2009, pp. 2249–2252.
21. Yue M. Lu, Pier-Luigi Dragotti, and Martin Vetterli, *Localization of diffusive sources using spatiotemporal measurements*, Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on, Sept 2011, pp. 1072–1076.
22. Stéphane Mallat, *A wavelet tour of signal processing*, Academic Press Inc., San Diego, CA, 1998. MR 1614527 (99m:94012)
23. Zuhair M. Nashed, *Inverse problems, moment problems, signal processing: un menage a trois*, Mathematics in science and technology, World Sci. Publ., Hackensack, NJ, 2011, pp. 2–19. MR 2883419
24. Gestur Ölfsson and Darrin Speegle, *Wavelets, wavelet sets, and linear actions on \( \mathbb{R}^n \)*, Wavelets, frames and operator theory, Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 253–281. MR 2066833 (2005h:42075)
25. Isaac Z. Pesenson, *Multiresolution analysis on compact Riemannian manifolds*, Multiscale analysis and nonlinear dynamics, Rev. Nonlinear Dyn. Complex., Wiley-VCH, Weinheim, 2013, pp. 65–82. MR 3221687
26. ______, *Sampling, splines and frames on compact manifolds*, GEM Int. J. Geomath. 6 (2015), no. 1, 43–81. MR 3322489
27. A. Petrosyan, *Dynamical sampling with moving devices*, Proc. of the Yerevan State Univ., Phys. and Math. Sci. (2015), no. 1, 31–35.
28. Frederich Philipp, *Unpublished notes*, Private Communication (2016).
29. Juri Ranieri, Amina Chebira, Yue M. Lu, and Martin Vetterli, *Sampling and reconstructing diffusion fields with localized sources*, Acoustics, Speech and Signal Processing (ICASSP), 2011 IEEE International Conference on, May 2011, pp. 4016–4019.
30. Günter Reise, Gerald Matz, and Karlheinz Gröchenig, *Distributed field reconstruction in wireless sensor networks based on hybrid shift-invariant spaces*, IEEE Trans. Signal Process. 60 (2012), no. 10, 5426–5439. MR 2979004
31. Gilbert Strang and Truong Nguyen, *Wavelets and filter banks*, Wellesley-Cambridge Press, Wellesley, MA, 1996. MR 1411910 (98b:94003)
32. Qiyu Sun, *Frames in spaces with finite rate of innovation*, Adv. Comput. Math. 28 (2008), no. 4, 301–329. MR 2390281 (2009c:42093)