On the quantum equivalence of an antisymmetric field with spontaneous Lorentz violation

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We consider a minimal model of rank-2 antisymmetric field with spontaneous Lorentz violation, and obtain a classically equivalent Lagrangian consisting of vector field. The 1-loop effective actions of both theories have been derived, and compared to check their quantum equivalence. We find that the spontaneous Lorentz violating terms disturb the structure of effective action in each of these theories. In flat spacetime, it has been shown that the difference of two effective actions in consideration does not vanish for a given choice of vacuum expectation value of antisymmetric field. However, their quantum equivalence still holds because there is no field dependence in effective action, and therefore they cancel after normalization.

I. INTRODUCTION

Antisymmetric tensor fields appear in all superstring theories and are especially relevant for studies in the low-energy limit [1, 2]. They have been studied in the past in several contexts, including strong-weak coupling duality and phase transitions [3–11].

A study relevant to the present work was carried out by Altschul et al. [12], where spontaneous Lorentz violation with various rank-2 antisymmetric field models minimally and non minimally coupled to gravity was investigated. A remarkable feature of that study is the presence of distinctive physical features with phenomenological implications for tests of Lorentz violation, even with relatively simple antisymmetric field models with a gauge invariant kinetic term. Such interesting phenomenological possibilities have been a strong motivation for various works on spontaneous Lorentz violation (SLV) [13–21].

A particularly simple but interesting model is that of a rank-2 antisymmetric field minimally coupled to gravity, with the simplest choice of spontaneously Lorentz violating potential. Its classical equivalence was considered in Ref. [12] in terms of an equivalent Lagrangian consisting of a vector field $A_\mu$ coupled to auxiliary field $B_{\mu\nu}$ in Minkowski spacetime. However, checking the quantum equivalence of such classically equivalent theories is not straightforward, in flat as well as curved spacetime.

Quantum equivalence in the context of massive rank-2 and rank-3 antisymmetric fields in curved spacetime, without SLV, was first studied by Buchbinder et al. [22] and later confirmed in Ref. [23]. The proof of quantum equivalence in Ref. [22] was based on the zeta-function representation of functional determinants of $p$-form Laplacians appearing in the 1-loop effective action, and identities satisfied by zeta-functions for massless case [24–26]. Quantum equivalence results from these identities generalized to the massive case. In flat spacetime though, the proof is trivial as operators appearing in the effective action reduce to d’Alembertian operators due to vanishing commutators of covariant derivatives and equivalence follows by taking into account the independent components of each field.

On the contrary, in case of the minimal model of rank-2 antisymmetric tensor field with SLV mentioned above and a classically equivalent vector theory, we find that the simple structure of operators breaks down due to the presence of SLV terms. As a result, the difference of their effective actions does not vanish in Minkowski spacetime, contrary to the case without SLV. However, this does not threaten quantum equivalence due to a lack of field dependence in the effective actions, which will therefore cancel after normalization [27].

In curved spacetime, making a conclusive statement about quantum equivalence is a highly nontrivial task for the following reasons. First, comparing effective actions as in Ref. [23] is a difficult mathematical problem, because, to the best of our knowledge, kernels for operators involved in this problem have not yet been found in literature. Second, the formal arguments made in Ref. [22] do not apply to the present case due to the non-trivial structure of operators appearing in effective actions.

We use the quantization method developed in Ref. [28] to calculate the effective actions in curved spacetime using the DeWitt-Vilkovisky’s covariant effective action approach [29–33] and St"uckelberg procedure [34, 35]. The organization of this paper is as follows. Section II contains a review of the antisymmetric field Lagrangian in consideration, and a

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derivation of classically equivalent Lagrangian. In section III, we calculate the effective action for the two classically equivalent theories. Section IV deals with checking their quantum equivalence and problems therein.

II. CLASSICAL ACTION

We consider the minimal model of a rank-2 antisymmetric tensor field, $B_{\mu\nu}$, with the simplest choice of spontaneously Lorentz violating potential [12],

$$\mathcal{L} = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{2} \lambda \left( B_{\mu\nu} B^{\mu\nu} - b_{\mu\nu} b^{\mu\nu} \right)^2. \quad (1)$$

$\lambda$ here is a massless coefficient. The first term in Eq. (1) is the gauge invariant kinetic term, where,

$$H_{\mu\nu\lambda} \equiv \nabla_\mu B_{\nu\lambda} + \nabla_\lambda B_{\mu\nu} + \nabla_\nu B_{\lambda\mu}, \quad (2)$$

and the second term is responsible for spontaneous Lorentz violation, giving rise to a non-zero vacuum expectation value,

$$\langle B_{\mu\nu} \rangle = b_{\mu\nu}. \quad (3)$$

$b_{\mu\nu}$ is also an antisymmetric tensor, which in general may not have a simple structure, but it is possible to transform to a special observer frame in which $b_{\mu\nu}$ has a block-diagonal form with its components being real numbers, provided that $b_{\mu\nu} b^{\mu\nu}$ is nonzero [12].

It is clear from Eq. (1) that the potential contains self-interaction terms for $B_{\mu\nu}$. Although it would be interesting to investigate quantum corrections in such a theory, it is out of scope of the current work. For the present study we are interested in the quantum properties of this theory with up to quadratic order terms in $B_{\mu\nu}$, and thus it is relevant to consider fluctuations of $B_{\mu\nu}$ around its vacuum expectation value so that all higher order terms, including self-interaction terms can be ignored. We define the fluctuations $\tilde{B}_{\mu\nu}$ as,

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - b_{\mu\nu}. \quad (4)$$

Substituting Eq. (4) in Eq. (1) and neglecting higher order terms and constants, yields,

$$\mathcal{L} = -\frac{1}{12} \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - 2 \lambda \left( b_{\mu\nu} \tilde{B}^{\mu\nu} \right)^2. \quad (5)$$

where $\tilde{H}_{\mu\nu\lambda}$ is now defined in terms of fluctuations $\tilde{B}_{\mu\nu}$. For convenience, we define

$$b_{\mu\nu} = bn_{\mu\nu}, \quad (6)$$

where $n_{\mu\nu}$ is an antisymmetric tensor satisfying $n_{\mu\nu} n^{\mu\nu} = 1$ so that,

$$b_{\mu\nu} b^{\mu\nu} = b^2. \quad (7)$$

Using Eq. (6), Lagrangian (5) can be written in a convenient form,

$$\mathcal{L} = -\frac{1}{12} \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - \frac{1}{4} \alpha^2 \left( n_{\mu\nu} \tilde{B}^{\mu\nu} \right)^2. \quad (8)$$

where $\alpha \equiv 8 \lambda b^2$ is now a massive coefficient.

Our intention is to check the quantum equivalence of theory (8) with a classically equivalent vector theory. Classical equivalence here means equivalence at the level of Lagrangian, that is, one Lagrangian can be obtained from other and vice versa, after manipulations. This interpretation is in-line with Refs. [12] and [22]. An equivalent Lagrangian can be obtained by introducing a vector field $A_\mu$ along with the field strength and its dual defined as,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad (9)$$

such that, Lagrangian (8) is equivalent to [12],

$$\mathcal{L} = \frac{1}{2} \tilde{B}_{\mu\nu} F^{\mu\nu} - \frac{1}{2} A^{\mu} A_\mu - \frac{1}{4} \alpha^2 \left( n_{\mu\nu} \tilde{B}^{\mu\nu} \right)^2. \quad (10)$$
In order to get rid of $\tilde{B}_{\mu\nu}$ it is handy to make use of projections of a tensor along and transverse to $n_{\mu\nu}$,

$$
T_{||\mu\nu} = n_{\rho\sigma} T^{\rho\sigma} n_{\mu\nu},
$$

$$
T_{\perp\mu\nu} = T_{\mu\nu} - T_{||\mu\nu},
$$

(11)

Substituting Eq. (11) in Eq. (10), the Lagrangian density becomes,

$$
\mathcal{L} = \frac{1}{2} \tilde{B}_{\perp\mu\nu} \mathcal{F}^{\mu\nu} + \frac{1}{2} \tilde{B}_{||\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{2} A^\mu A_\mu - \frac{1}{4} \alpha^2 \tilde{B}_{||\mu\nu} \tilde{B}^{\mu\nu}.
$$

(12)

Using the equations of motion of $\tilde{B}_{||\mu\nu}$ and $\tilde{B}_{\perp\mu\nu}$ in (12) allows us to write,

$$
\alpha^2 \mathcal{L} = \frac{1}{4} (n_{\mu\nu} F^{\mu\nu})^2 - \frac{1}{2} \alpha^2 A^\mu A_\mu.
$$

(13)

Introducing the dual of $n_{\mu\nu}$, given by $\tilde{n}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} n_{\rho\sigma}$, the classically equivalent Lagrangian in terms of $F_{\mu\nu}$ reads,

$$
\alpha^2 \mathcal{L} = \frac{1}{4} (\tilde{n}_{\mu\nu} F^{\mu\nu})^2 - \frac{1}{2} \alpha^2 A^\mu A_\mu
$$

(14)

A distinctive feature of Lagrangian (14) when compared to a generic massive vector field Lagrangian like the Proca model, is its peculiar kinetic term. Instead of all modes of $F_{\mu\nu}$ only those projected along $\tilde{n}_{\mu\nu}$ are dynamical, as pointed out in Ref. [12]. Moreover, the sign of kinetic term in (14) is opposite to that in Proca model. In the context of SLV, another noteworthy feature of Lagrangian (14) is that the potential term is not affected by $n_{\mu\nu}$ unlike other vector models with SLV, for instance the Bumblebee model. It will be observed in later sections that these features lead to an effective action that has a structure different from the corresponding effective action for Lagrangian (8).

### III. THE EFFECTIVE ACTION

The classical analysis of the previous section did not take into account the gauge symmetries of equivalent Lagrangians (8) and (14). While these Lagrangians are technically not gauge invariant, they belong to a class of theories having a softly broken gauge symmetry: the kinetic terms of Lagrangians (8) and (14) are invariant under the transformations $\tilde{B}_{\mu\nu} \rightarrow \tilde{B}_{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$ and $A_\mu \rightarrow A_\mu + \nabla_\mu \Lambda$, respectively, but the potential terms are not. A standard approach for quantization of these theories is to employ the Stückelberg procedure [34, 35].

We first consider the Lagrangian (8). The first step is to restore the softly broken gauge symmetry through the introduction of a Stückelberg field $C_\mu$ such that the Lagrangian,

$$
\mathcal{L} = -\frac{1}{12} \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - \frac{1}{4} \alpha^2 n_{\mu\nu} (\tilde{B}_{\mu\nu} + \frac{1}{\alpha} F^{\mu\nu}[C])^2,
$$

(15)

becomes gauge invariant, and reduces to original Lagrangian (8) in the gauge $C_\mu = 0$. The new Lagrangian (15) is invariant under the symmetries,

$$
\tilde{B}_{\mu\nu} \rightarrow \tilde{B}_{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu,
$$

$$
C_\mu \rightarrow C_\mu - \alpha \xi_\mu,
$$

(16)

and,

$$
C_\mu \rightarrow C_\mu + \nabla_\mu \Lambda,
$$

$$
\tilde{B}_{\mu\nu} \rightarrow \tilde{B}_{\mu\nu}.
$$

(17)

In addition to the above symmetries of fields, there exists a set of transformation of gauge parameters $\Lambda$ and $\xi_\mu$ that leaves the fields $B_{\mu\nu}$ and $C_\mu$ invariant,

$$
\xi_\mu \rightarrow \xi_\mu + \nabla_\mu \psi,
$$

$$
\Lambda \rightarrow \Lambda + \alpha \psi.
$$

(18)
We simplify the kinetic term for convenience,

$$H_{\mu\nu\lambda} \dot{H}^{\mu\nu\lambda} = -3\dot{B}_{\nu\lambda} D_2 \dot{B}^{\nu\lambda},$$

(19)

where, $D_2 \dot{B} \equiv \Box \dot{B} + \nabla_\mu \nabla^\nu \dot{B} + \nabla_\mu \nabla^\nu \dot{B}$. Hence, the desired Lagrangian to be quantized is,

$$L_2 = \frac{1}{4} \dot{B}_{\nu\lambda} D_2 \dot{B}^{\nu\lambda} - \frac{1}{4} \alpha^2 \left[ n_{\mu\nu} \left( \dot{B}^{\mu\nu} + \frac{1}{\alpha} F^\mu\nu[C] \right) \right]^2.$$

(20)

Now, the gauge fixing procedure requires that a gauge condition be chosen for each of the fields $B_{\mu\nu}$ and $C_\mu$ as well as for the parameter $\xi_\mu$. An important consideration while choosing a gauge condition is to ensure that all cross terms of fields in the Lagrangian cancel out or lead to a total derivative term, so that path integral can be computed with ease. Keeping this in mind, we choose [36]

$$\chi_{\xi_\mu} = n_{\mu\nu\rho\sigma} \nabla^\mu \dot{B}^{\rho\sigma} + \alpha C_\mu.$$

(21)

It turns out that the gauge fixing action term corresponding to Eq. (21) introduces yet another soft symmetry breaking in $C_\mu$ [28], so one has to introduce another Stückelberg field $\Phi$ so that,

$$C_\mu \rightarrow C_\mu + \frac{1}{\alpha} \nabla_\mu \Phi.$$

(22)

This modifies the symmetry in Eq. (17) by an additional shift transformation,

$$\Phi \rightarrow \Phi - \alpha \Lambda.$$

(23)

From Eqs. (17) and (23), the gauge condition for $C_\mu$ can be chosen to be,

$$\chi_\Lambda = \nabla^\mu C_\mu + \alpha \Phi.$$

(24)

Similarly, for the symmetry of parameters, Eq. (13), we choose

$$\tilde{\chi}_\psi = \nabla^\mu \xi_\mu - \alpha \Lambda.$$

(25)

The total gauge fixed Lagrangian is given by

$$L_2^{GF} = \frac{1}{4} \dot{B}_{\mu\nu} D_2 \dot{B}^{\mu\nu} - \frac{1}{4} \alpha^2 \left( n_{\mu\nu} \dot{B}^{\mu\nu} \right) - \frac{1}{4} \left( n_{\mu\nu} F^\mu\nu \right)^2 - \frac{1}{2} \left( n_{\mu\nu\rho\sigma} \nabla^\mu \dot{B}^{\rho\sigma} \right)^2 - \frac{1}{2} \alpha^2 C_\mu C_\mu - \frac{1}{2} \left( \nabla^\mu \Phi \right)^2 - \frac{1}{2} \left( \nabla^\mu C_\mu \right)^2 - \frac{1}{2} \alpha^2 \Phi^2.$$

(26)

The above Lagrangian can be further simplified and cast into a familiar form,

$$L_2^{GF} = \frac{1}{4} \dot{B}_{\mu\nu} D_2 \dot{B}^{\mu\nu} - \frac{1}{4} \alpha^2 \dot{B}_{\mu\nu} n_{\rho\sigma} \dot{B}^{\rho\sigma} + \frac{1}{2} C_\mu \dot{C}_\mu - \frac{1}{2} \alpha^2 C_\mu C_\mu$$

$$+ \frac{1}{2} \Phi \Box \left( \alpha^2 - \alpha \right) \Phi,$$

(27)

where,

$$\dot{D}_2 \dot{B}^{\mu\nu} \equiv D_2 \dot{B}^{\mu\nu} + 2n_{\mu\nu} n_{\rho\sigma} n_{\alpha\gamma} \nabla^\mu \dot{B}^{\rho\sigma} \nabla_{\alpha\gamma},$$

$$\dot{D}_1 C_\mu \equiv 2n_{\mu\nu} n_{\rho\sigma} \nabla^\rho C_\sigma + \nabla^\mu \nabla_\nu C_\nu.$$  

(28)

Following the method developed in Ref. [28], the calculation of ghost determinants proceeds as follows. We rewrite $\chi_{\xi_\mu}$ as,

$$\chi_{\xi_\mu} \left[ \dot{B}^{\mu\nu}, C_\mu \right] = \chi_{\xi_\mu} \left[ \dot{B}^{\mu\nu}, C_\mu, \xi_\mu, \Lambda, \tilde{\chi}_\psi \right],$$

(29)

which yields,

$$\chi_{\xi_\mu} = n_{\mu\nu\rho\sigma} \nabla^\mu \dot{B}^{\rho\sigma} + \alpha \nabla^\mu \psi + 2n_{\mu\nu} n_{\rho\sigma} \nabla^\mu \xi_\sigma + \nabla^\mu \nabla_\nu \psi - \alpha^2 \xi_\nu - \nabla_\nu \tilde{\chi}_\psi$$

(30)
Then, using the definition of \( Q'_{\xi_\alpha} \), we get
\[
Q'_{\xi_\alpha}(\xi_\mu) = 2n_{\mu\nu}n_{\rho\sigma} \nabla^\mu \nabla^\nu + \nabla_\nu \nabla_\alpha - \alpha^2 \delta_{\nu\alpha} = \hat{D}_1 - \alpha^2 \delta_{\nu\alpha}
\]  
(31)

A straightforward calculation leads to other non-zero components of ghost determinant,
\[
Q'_{\Lambda} = \frac{\delta \chi_\Lambda}{\delta \Lambda} = \square_x - \alpha^2 \]  
(32)

\[
\hat{Q}_{\psi} = \frac{\delta \hat{\chi}_\psi}{\delta \psi} = \square_x - \alpha^2 \]  
(33)

Using the definition of effective action obtained in [28],
\[
\exp(i\Gamma[\bar{B}, \bar{C}]) = \int \prod \mu \, dC_\mu \prod \rho \sigma \, dB_{\rho\sigma} \prod \Phi \, \det(Q'_{\Lambda}) \det(Q'_{\xi_\alpha}) (\det \hat{Q}_\psi)^{-1} \times \exp \left\{ i \left( \int dv_x \mathcal{L}_{GF}^2 \right) + (\bar{B}_{\mu\nu} - B_{\mu\nu}) \frac{\delta}{\delta B_{\mu\nu}} \Gamma[\bar{B}, \bar{C}] \right\}.
\]  
(34)

The 1-loop effective action is obtained as,
\[
\Gamma_1^{(1)} = \frac{i\hbar}{2} \left[ \ln \det(\hat{D}_2 - \alpha^2 n^{\mu\nu} n_{\rho\sigma}) - \ln \det(\hat{D}_1 - \alpha^2) + \ln \det(\square_x - \alpha^2) \right]
\]  
(35)

It is to be noted that the coefficient of \( \alpha^2 \) in the first term in Eq. (35) ensures that massive modes correspond to field components along vacuum expectation tensor \( n_{\mu\nu} \) and massless modes correspond to transverse components. An interesting observation here is the last term, which is unaffected by \( n_{\mu\nu} \). In case of no SLV, the last term causes the quantum discontinuity when going from massive to massless case [23].

To compare Eq. (35) with the effective action of classically equivalent Lagrangian, the Lagrangian in (14) is treated with the Stückelberg procedure to obtain,
\[
\tilde{\mathcal{L}}_1 = \frac{1}{4} \left( \tilde{n}_{\mu\nu} F^{\mu\nu} \right)^2 - \frac{1}{2} \alpha^2 (C_\mu + \frac{1}{\alpha} \nabla_\mu \Phi)^2
\]  
(36)

The above Lagrangian is invariant under a transformation identical to Eqs. (17) and (23),
\[
C_\mu \rightarrow C_\mu + \nabla_\mu \Lambda, \quad \Phi \rightarrow \Phi - \alpha \Lambda.
\]  
(37)

With the gauge condition Eq. (24), the gauge fixed Lagrangian reads,
\[
\tilde{\mathcal{L}}_{GF}^1 = \frac{1}{2} C_\mu D_1 C^\mu - \frac{1}{2} \alpha^2 C_\mu C^\mu + \frac{1}{2} \Phi (\square_x - \alpha^2) \Phi.
\]  
(38)

where,
\[
D_1 C_\mu = -2 \tilde{n}_{\nu\mu} \tilde{n}_{\rho\sigma} \nabla^\nu \nabla^\rho C^\sigma + \nabla_\mu \nabla_\nu C^\nu.
\]  
(39)

It is straightforward to check that the 1-loop effective action is,
\[
\Gamma_1^{(1)} = \frac{i\hbar}{2} \left[ \ln \det(D_1 - \alpha^2) - \ln \det(\square_x - \alpha^2) \right].
\]  
(40)

Similar to Eq. (35), the scalar term is unaffected by \( n_{\mu\nu} \) and the operator \( D_1 \) possesses a non-trivial structure. The expression for \( D_1 \) has a striking resemblance to that of \( \hat{D}_1 \), which has opposite sign in the first term and \( n_{\mu\nu} \) instead of \( \tilde{n}_{\mu\nu} \). Particularly interesting is the fact that this difference is, by design, built into the equivalent Lagrangian (14) and is apparent even before, in Eq. (26), where the kinetic part of Stückelberg field \(-\frac{i}{4} n_{\mu\nu} F^{\mu\nu})^2 \) has a sign opposite to that of Eq. (14).
IV. QUANTUM EQUIVALENCE IN FLAT SPACETIME

To compare Eqs. (35) and (40), we define the difference in 1-loop effective actions given by,

$$\Delta \Gamma = \Gamma^{(1)} - \Gamma^{(1)}_1$$

$$= \frac{i}{2} \left[ \ln \det(\hat{D}_2 - \alpha^2 n^{\mu\nu} n_{\mu\nu}) - \ln \det(\hat{D}_1 - \alpha^2) - \ln \det(D_1 - \alpha^2) \right]$$

$$+ 2 \ln \det(\Box_{\alpha} - \alpha^2).$$  \hspace{1cm} (41)

In contrast, the corresponding difference in 1-loop effective action in the case of massive antisymmetric and vector fields, with mass $m$, with no spontaneous Lorentz violation is given by [22],

$$\Delta \Gamma' = \frac{i}{2} \left[ \ln \det(\Box_2 - m^2) - \ln \det(\Box_1 - m^2) + 2 \ln \det(\Box_x - m^2) \right],$$  \hspace{1cm} (42)

where,

$$\Box_2 B_{\mu\nu} = \Box_x B_{\mu\nu} - [\nabla^\rho, \nabla_\nu] B_{\rho\mu} - [\nabla^\mu, \nabla_\rho] B_{\rho\nu},$$

$$\Box_1 C_{\mu} = \Box_x C_{\nu} - [\nabla^\nu, \nabla_\mu] C^\mu.$$  \hspace{1cm} (43)

This comparison between cases with and without SLV is quite insightful, because it helps in understanding how SLV disturbs the structure of effective action. In the later case, the operator for St"uckelberg vector field and that for vector field of equivalent Lagrangian are equal, while in the former case they are not, as was noted earlier. Moreover, operators in Eq. (41) do not contain the commutator terms due to presence of $n_{\mu\nu}$, and hence do not simplify in flat spacetime unlike their counterparts in Eq. (42).

In flat spacetime, it can be explicitly checked that Eq. (42) vanishes, taking into account the number of independent components of respective fields, because the commutators in Eq. (43) vanish and hence the operators $\Box_2$, $\Box_1$, and $\Box_x$ are identical. Inferring quantum equivalence is thus trivial. However, this is clearly not the case in Eq. (41) due to the non-trivial structure of operators $\hat{D}_2$ and $\hat{D}_1$. This can be demonstrated in a rather simple example when a special choice of tensor $n_{\mu\nu}$ is considered. It can be shown that in Minkowski spacetime, $n_{\mu\nu}$ can be chosen to have a special form

$$n_{\mu\nu} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & -b & 0 \end{pmatrix},$$  \hspace{1cm} (44)

where $a$ and $b$ are real numbers, provided atleast one of the quantities $x_1 = -2(a^2 - b^2)$ and $x_2 = 4ab$ are non-zero [12]. For simplicity, we may choose $b = 0$. Further, the constraint $n_{\mu\nu} n^{\mu\nu} = 1$ implies that $a = 1/\sqrt{2}$. Therefore, the only non-zero components of $n_{\mu\nu}$ are $n_{10} = 1/\sqrt{2}$ and $n_{01} = -1/\sqrt{2}$. For the dual tensor $\tilde{n}_{\mu\nu}$, the non-zero components are $\tilde{n}_{32} = -1/\sqrt{2}$ and $\tilde{n}_{23} = 1/\sqrt{2}$. Substituting in Eq. (28), one obtains, for the non-zero components of $n_{\mu\nu}$ and $\tilde{n}_{\mu\nu}$,

$$D_1 C^2 = \partial_1^2 C^2 - \partial_2^2 C^2 + 2 \partial_1 \partial_2 C^3 + \partial_1^2 \partial_2 C^1,$$

$$D_1 C^3 = -\partial_2^2 C^3 + \partial_1^2 C^3 + 2 \partial_1 \partial_2 C^2 + \partial_1^2 C^1,$$

$$\hat{\Delta}_1 C^{0} = \partial_2^2 C^{0} + \partial_1^2 C^{0} + \partial_0 \partial_1 C^3,$$

$$\hat{\Delta}_1 C^{1} = \partial_2^2 C^{1} + \partial_1^2 C^{1} + \partial_0 \partial_1 C^3,$$

$$\hat{\Delta}_1 C^{2} = \partial_2^2 C^{2} + \partial_1^2 C^{2} + \partial_0 \partial_1 C^3,$$

$$\hat{\Delta}_1 C^{3} = \partial_2^2 C^{3} + \partial_1^2 C^{3} + \partial_0 \partial_1 C^3.$$  \hspace{1cm} (45)

where, $j = 2, 3$ and $i = 0, 1$. The remaining components of operators $\hat{D}_2$, $\hat{D}_1$ and $D_1$ are given by,

$$\hat{\Delta}_2 B^{jk} = \Box_x \hat{B}^{jk} + \partial_{\rho} \hat{B}^{jk} \partial_{\sigma} + \partial_{\rho} \hat{B}^{jk} \partial_{\sigma}, \hspace{1cm} B^{jk} \neq B^{10},$$

$$\hat{\Delta}_1 C^{l} = \partial^l \partial_{\sigma} C^{\sigma}, \hspace{1cm} l = 2, 3,$$

$$D_1 C^{k} = \partial_{\sigma} \partial_{\tau} C^{\sigma\tau}, \hspace{1cm} k = 0, 1.$$  \hspace{1cm} (46)

Eqs. (45) and (46) show explicitly that $\Delta \Gamma$ in Eq. (41) does not vanish. But this does not imply quantum inequivalence of theories (35) and (40). The reason is, functional determinants in Eq. (41) do not have field dependence and can only contribute as infinite (regularization-dependent) constants. So, they will cancel upon normalization, hence preserving the quantum equivalence.
V. SUMMARY

We derived the Lagrangian for a vector field $C_{\mu}$ which is classically equivalent to a rank-2 antisymmetric tensor field with a spontaneously Lorentz violating potential, by extending the calculations carried out in Ref. [12]. We computed the 1-loop effective action for both theories, and found that the operators have complicated structures due to the presence of vacuum expectation tensor $\eta_{\mu\nu}$. In flat spacetime, we explicitly checked for a simple choice of $\eta_{\mu\nu}$ that although the difference of effective actions, $\Delta \Gamma$, does not vanish, their quantum equivalence still holds once normalization of functional determinants are taken into account. This confirms the fact that two free field theories which are classically equivalent, must also be quantum equivalent.

In curved spacetime, however, it is difficult to make a precise statement because an explicit comparison of operators is not possible unless one uses a regularization scheme to find an appropriate expression for operators in $\Delta \Gamma$, as done in Refs. [23] and [22]. A good starting point for answering this question would be to explicitly write the heat kernel for these operators. Unfortunately, we could not find in literature a suitable heat kernel for operators encountered in the present problem.

In conclusion, it can be stated that inferring quantum equivalence from calculation of effective action is not always trivial, in the sense that physical arguments like normalization must be considered. This may serve as a motivation for future studies on quantum equivalence. An interesting problem is that of Bumblebee model, whose equivalence has although been checked in specific contexts [37], an effective action analysis has not yet been performed. It must also be pointed out that calculating heat kernel for the present problem would be useful while studying gravitational corrections in cosmological contexts.

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