M-theory resolution of four-dimensional cosmological singularities via U-duality

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We consider cosmological solutions of string and M-theory compactified to four dimensions by giving a general prescription to construct four-dimensional modular cosmologies with two commuting Killing vectors from vacuum solutions. By lifting these solutions to higher dimensions we analyze the existence of cosmological singularities and find that, in the case of non-closed Friedmann-Robertson-Walker universes, curvature singularities are removed from the higher-dimensional model when only one of the extra dimensions is time-varying. By studying the moduli space of compactifications of M-theory resulting in homogeneous cosmologies in four dimensions we show that U-duality transformations map singular cosmologies into non-singular ones.

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1. Introduction and motivation

It is well known that string physics has changed the way we now look at Cosmology. In particular, string theory leads to consider multidimensional cosmological scenarios in a natural way, since superstrings can only be consistently quantized in ten dimensions. M-theory has confirmed this trend extending cosmology to the realm of the eleventh dimension (see for example [1]).

Higher dimensional cosmology is not new. Before the advent of string theory the Kaluza-Klein paradigm had already put forward the idea that four-dimensional space-time should not be something taken for granted in cosmology (a collection of the most relevant articles on the subject can be found in [2]; see also [3]). In its original version, the Kaluza-Klein program aimed to describe all four-dimensional matter fields as purely gravitational (or supergravitational) degrees of freedom in $4 + N$ dimensions. Although this idea met serious obstacles when trying to account for the existence of chiral matter in four dimensions [4], it has been partially incorporated in string theory where the presence of ten-dimensional chiral matter fermions solves the problem.

When looking at the four-dimensional physics, the extra internal dimensions leave their blueprints in the form of a plethora of massless scalar and vector fields. On general grounds, the presence of these massless scalars poses a serious problem in trying to extract realistic cosmological models from string theory. The situation is complicated by the fact that most of these fields represent flat directions of the superpotential that are not lifted by quantum corrections as long as enough supersymmetry is preserved. In many occasions, however, the study is restricted to the simplest cosmological models by looking just at the effects of the dilaton and tensor fields, for example, and ignoring the remaining moduli fields altogether [5][6][7].

The ultimate motivation of string/M-theory cosmology is of course to solve the problem of cosmological singularities, explain the initial conditions in cosmology and the dimensionality of space-time. Close to $t = 0$, quantum gravity effects become dominant and hopefully will smear the semiclassical singularity, thus opening a new window to the study

\footnote{From the M-theory perspective, on the other hand, the dilaton itself is just a moduli associated with the compactification of the eleventh dimension. The reason to separate it from other fields is that in the weakly coupled string limit of M-theory its compactification scale $g_s \ell_s$ is much smaller than the string scale $\ell_s$.}
of the early universe. Although a full quantum cosmology description of the early universe is still missing, the semiclassical analysis supported by the use of stringy symmetries has been useful in getting a flavor of the physics close to the Big-Bang singularity \([7][8]\). On the other hand, from a Kaluza-Klein perspective it is possible that the existence of the initial singularity might just be the result of integrating out the physics associated with higher dimensions \([9][10]\) and that it could be removed already in the semiclassical approximation.

In this paper we will combine these two ideas and investigate the effect of extra dimensions on the initial cosmological singularity by studying a family of Friedmann-Robertson-Walker (FRW) cosmologies coupled to a number of scalar fields which we associate with compactification moduli. We find that in the particular case of open or flat \((k = -1, 0)\) universes in which only one of the extra dimensions has a non-trivial dynamics, the Big-Bang singularity might be just an artifact of the Kaluza-Klein reduction which may be removed when going to higher dimensions. On the other hand, in the case of spatially closed solutions \((k = 1)\) the Big-Bang or the Big-Crunch singularity is postponed in the higher-dimensional model, again when only one of the extra dimensions is time-varying. It is interesting that by switching on more than one dynamical internal dimension the regularization of curvature singularities is definitively spoiled in all cases. When interpreting, in the open and flat cases, this higher dimensional cosmologies as solutions in M-theory we find that the U-duality group \(G_7\) acts on the moduli space of metrics by relating singular geometries with non-singular ones. This seems to indicate that a certain class of cosmological singularities in M-theory can be physically resolved in terms of an equivalent dual regular background.

The plan of the article is the following: in the next section we will review the properties of moduli fields arising from compactifications of \((4 + N)\)-dimensional Einstein gravity on a \(N\)-dimensional straight torus and provide a general algorithm to construct Gowdy-type cosmologies in the presence of moduli fields. We will apply in Sec. 3 this algorithm to generate modular FRW cosmologies and, after undoing the Kaluza-Klein reduction, will study the structure of singularities of the higher dimensional “parent” metric. In Sec. 4 the analysis will be focused on M-theory compactifications to four dimensions and the action of U-duality on the moduli space of solutions. Finally, conclusions will be summarized in Sec. 5.
2. Cosmologies coupled to scalar fields vs. dimensional reduction

2.1. Scalar fields from dimensional reduction

Scalar fields appear naturally in the old Kaluza-Klein program or its string/M-theory versions. Here, we will consider the dimensional reduction of a \((4+N)\)-dimensional metric on an \(N\)-dimensional straight torus \(T^N = (S^1)^N\) using the ansatz\footnote{For a generic analysis of the structure of the dimensionally reduced action, see \cite{11, 12}.}

\[
\begin{align*}
  ds_{4+N}^2 &= e^{-\frac{2}{\sqrt{3}}N \psi_i} ds_4^2 + \sum_{i=1}^N e^{4 \sqrt{3} \psi_i} (dw^i)^2. \\
  \end{align*} 
\]  

(2.1)

The metric functions \(\psi_i(x^\mu) (\mu = 0, \ldots, 3)\) should be thought of as the components of the metric in the internal \(N\)-dimensional torus. Notice that this ansatz is invariant under the permutation of the \(\psi_i\) fields.

From the four-dimensional point of view, the fields \(\psi_i(x^\mu)\) are scalars. Their dynamical equations are obtained by demanding that the \((4+N)\)-dimensional metric (2.1) is a vacuum solution of the Einstein equations. Writing the Einstein-Hilbert action for (2.1) we find that

\[
S = \int d^4x \sqrt{-g^{(4)}} \left[ R^{(4)} - 2 \left( \sum_{i=1}^N \partial_\mu \psi_i \partial^\mu \psi_i + \frac{2}{3} \sum_{i<j} \partial_\mu \psi_i \partial^\mu \psi_j \right) \right],
\]

(2.2)

leading to the result that the breathing modes of the higher dimensional metric appear as mixed scalar in the four-dimensional action. To eliminate this mixing we may perform a diagonalization in field space by defining the new fields \(\varphi_i\) through the relation

\[
\psi_i = D_{ij} \varphi_j
\]

(2.3)

where \(D_{ij} \in GL(N, \mathbb{R})\) is given by

\[
D = \begin{pmatrix}
  \mu_1^{-\frac{1}{2}} & \mu_2^{-\frac{1}{2}} & \mu_3^{-\frac{1}{2}} & \cdots & \mu_{N-1}^{-\frac{1}{2}} & \mu_N^{-\frac{1}{2}} \\
  -\mu_1^{-\frac{1}{2}} & \mu_2^{-\frac{1}{2}} & \mu_3^{-\frac{1}{2}} & \cdots & \mu_{N-1}^{-\frac{1}{2}} & \mu_N^{-\frac{1}{2}} \\
  0 & -2\mu_2^{-\frac{1}{2}} & \mu_3^{-\frac{1}{2}} & \cdots & \mu_{N-1}^{-\frac{1}{2}} & \mu_N^{-\frac{1}{2}} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & \mu_{N-1}^{-\frac{1}{2}} & \mu_N^{-\frac{1}{2}} \\
  0 & 0 & 0 & \cdots & -(N-1)\mu_{N-1}^{-\frac{1}{2}} & \mu_N^{-\frac{1}{2}} \\
\end{pmatrix}
\]

(2.4)
with
\[ \mu_n = \frac{2}{3} n(n+1), \quad n = 1, \ldots, N - 1 \]
\[ \mu_N = \frac{1}{3} N(N+2), \]
which is not an \( O(N) \) transformation. The new scalar fields \( \varphi_i \) will be propagation eigenstates and there will be no classical mixing among those. Their dynamics will be governed by the action
\[ S = \int d^4x \sqrt{-g^{(4)}} \left[ R^{(4)} - 2 \sum_{i=1}^{N} \partial_\mu \varphi_i \partial^\mu \varphi_i \right]. \quad (2.5) \]

In our analysis we have assumed that all scalar fields have a geometric origin as moduli of dimensional reduction. This point of view is very much appropriate for M-theory where the dilaton is on the same footing with all other scalar as compactification moduli. Nonetheless, in those regimes of M-theory that can be described in terms of a weakly coupled string theory the dilaton field plays a privileged role as the field whose vacuum expectation value determines the string coupling constant. In this case, dimensional reduction from ten dimensions will produce again a number of scalar fields in the lower-dimensional theory. The dynamics of those moduli can be extracted again from the Kaluza-Klein ansatz
\[ ds_{10}^2 = ds_4^2 + \sum_{i=1}^{6} e^{2\sqrt{2}\sigma_i} (dw^i)^2 \quad (2.6) \]
where now the ten-dimensional metric is no longer a vacuum solution of Einstein equation but rather a solution of dilaton gravity instead. The resulting four-dimensional action in string frame is
\[ S_{\text{string}} = \int d^4x \sqrt{-g^{(4)}} e^{-2\phi} \left[ R^{(4)} + 4 \partial_\mu \phi \partial^\mu \phi - 2 \sum_{i=1}^{6} \partial_\mu \sigma_i \partial^\mu \sigma_i \right] \quad (2.7) \]
where the four-dimensional dilaton \( \phi \) is defined in terms of the ten-dimensional one as
\[ \phi = \phi^{(10)} - \frac{1}{\sqrt{2}} \sum_{i=1}^{6} \sigma_i. \quad (2.8) \]
We see from (2.7) that the kinetic term for the fields \( \sigma_i \) is diagonalized in the four-dimensional action, although it is conformally coupled to the dilaton field. This conformal coupling can be removed as usual by a conformal transformation of the metric. It is however when we re-express the four-dimensional dilaton in terms of the ten-dimensional one
via (2.8) that the mixing between the different $\sigma_i$ appears. In order to recover the result from the compactification of a vacuum solution of M-theory (eq. (2.2) with $N = 7$) we would need to lift the solution (2.6) of ten-dimensional dilaton gravity to a vacuum solution in eleven dimensions.

2.2. Four-dimensional modular cosmologies from vacuum solutions

In the following we will be interested in finding exact solutions to the field equations derived from the four-dimensional action (2.5), that we can write in the manifestly $O(N)$-invariant form as

$$ S = \int d^4x \sqrt{-g} \left[ R - 2 \partial_\mu \Phi^T \partial^{\mu} \Phi \right] \quad (2.9) $$

where we have defined

$$ \Phi \equiv \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}. $$

The matter energy-momentum tensor for (2.9) can be written as a sum of the corresponding stress-energy tensors for each scalar field

$$ T_{\mu\nu} = \sum_{i=1}^{N} T_{\mu\nu}^{(i)} = 2 \left( \partial_\mu \Phi^T \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \Phi^T \partial^\sigma \Phi \right). \quad (2.10) $$

Let us concentrate our attention on Gowdy-type cosmologies with line element

$$ ds^2 = e^{f(t,z)}(-dt^2 + dz^2) + K(t,z) \left[ e^{p(t,z)} dx^2 + e^{-p(t,z)} dy^2 \right]. \quad (2.11) $$

At first sight, the Gowdy-type coordinates seem an unnecessary complication, since we will be mostly dealing with cases where both geometries and scalar fields are homogeneous. The telling point to use them, nevertheless, is that: i) The scalar field equation are linear in the metric functions. ii) The evolution of the transversal metric functions $K(t,z)$ and $p(t,z)$ is decoupled from scalar field dynamics, the longitudinal function $f(t,z)$ being the only one influenced by the presence of the scalar fields. And iii) that due to the presence of a six-parameter isometry group $G_6$ which includes the three dimensional group $G_3$ acting simply transitively on the three-dimensional surfaces of constant curvature in FRW models,

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5 We have normalized the energy-momentum tensor in such a way that the Einstein equations are $G_{\mu\nu} = T_{\mu\nu}$. 
the above line element, which has a $G_2$ isometry group, naturally includes all three FRW geometries.

Let $f(t, z) = f(t, z)_{\text{vac}}$ such that (2.11) is a solution of the Einstein vacuum equations. In this case the functions $p(t, z)$ and $K(t, z)$ satisfy the following conditions

$$\frac{\partial}{\partial t}[K(t, z)\dot{p}(t, z)] - \frac{\partial}{\partial z}[K(t, z)p'(t, z)] = 0$$

and

$$\ddot{K}(t, z) - K''(t, z) = 0.$$

The idea now is to solve Einstein equations with the energy-momentum tensor (2.10). As it happens with a single scalar field (see for example [14]), the transverse part characterized by the metric functions $K(t, z)$ and $p(t, z)$ will be left unchanged. On the other hand, the longitudinal function $f(t, z)_{\text{vac}}$ is replaced by

$$f(t, z)_{\text{vac}} \rightarrow f(t, z)_{\text{vac}} + f(t, z)_{\text{sc}}$$

and the equations for $f(t, z)_{\text{sc}}$ can be written from the Einstein equations as

$$\dot{f}(t, z)_{\text{sc}} = \frac{2K}{K^2 - K'^2} \left( K'T_{tz} - \dot{K}T_{tt} \right)$$

$$f'(t, z)_{\text{sc}} = \frac{2K}{K^2 - K'^2} \left( K'T_{tt} - \dot{K}T_{tz} \right)$$

where $T_{\mu\nu}$ are the components of the energy-momentum tensor. Substituting (2.10) we finally get

$$\dot{f}(t, z)_{\text{sc}} = \frac{2K}{K^2 - K'^2} \left[ 2K' \sum_{i=1}^{N} \phi_i \dot{\phi}_i - K \left( \sum_{i=1}^{N} \phi_i^2 + \sum_{i=1}^{N} \phi_i'^2 \right) \right]$$

$$f'(t, z)_{\text{sc}} = \frac{2K}{K^2 - K'^2} \left[ K' \left( \sum_{i=1}^{N} \phi_i^2 + \sum_{i=1}^{N} \phi_i'^2 \right) - 2\dot{K} \sum_{i=1}^{N} \phi_i \phi_i' \right].$$

The structure of this expression (sum over each scalar field) is the result of the fact that Einstein equations are linear in the energy-momentum tensor and the energy-momentum tensor itself is a sum of the contributions from each scalar field. Expressions (2.12) are invariant under the global $O(N)$ symmetry of (2.5), as one expects, since a rotation of the fields by an element of this group does not modify the geometry. In addition, the scalar fields $\varphi_i$ must satisfy the wave equation

$$\frac{\partial}{\partial t}[K(t, z)\dot{\varphi}_i(t, z)] - \frac{\partial}{\partial z}[K(t, z)\varphi'_i(t, z)] = 0, \quad i = 1, \ldots, N. \quad (2.13)$$
Eqs. (2.12) can be used to generalize the algorithm of generation of string cosmologies given in ref. [6]. Taking $\varphi_i = \sigma_i$ ($i = 1, \ldots, 6$) and $\varphi_7 = \phi$, the four dimensional dilaton, we generate exact solutions of the Einstein equations with seven scalar fields that, after the conformal transformation by $e^{2\phi}$, will provide us with solutions to the equations derived from the string theory action (2.7). After this, we can further transform the resulting four-dimensional metric by $O(2, 2; \mathbb{R})$ [6] or $SL(2, \mathbb{R})$ [15] to generate other fields, the moduli $\sigma_i$ remaining invariant under these transformations.

In addition to this, we can use this generation technique to directly construct four-dimensional modular cosmologies representing toroidal compactifications of cosmological solutions of M-theory. In the following, we will particularize our analysis to a certain class of these cosmological solutions that render homogeneous cosmologies in four dimensions and use them to study how four-dimensional physics can be regularized in eleven dimensions. This family of solutions will also be useful to study the moduli space of M-theory on $\mathbb{T}^7$.

3. FRW cosmologies with moduli fields

We will now apply what we have learnt to the construction of rolling moduli cosmologies in four dimensions [16]. We start with a vacuum solution and apply the algorithm (2.12) with a collection of $N$ time-dependent scalar fields of the form

$$\varphi_i(t) = q_i \varphi_0(t), \quad i = 1, \ldots, N$$

with $q_i$ a $N$-tuple of real numbers and $\varphi_0(t)$ a particular homogeneous solution to the wave equation (2.13). The numbers $\{q_i\}$ play now the role of coordinates in the moduli space of solutions. From (2.12) we get the following equations for $f(t, z)_{sc}$

$$\dot{f}(t, z)_{sc} = \left(\sum_{i=1}^{N} q_i^2\right) \frac{2\dot{K} \dot{\varphi}_0}{K' - K}$$

$$f'(t, z)_{sc} = \left(\sum_{i=1}^{N} q_i^2\right) \frac{2K' \dot{\varphi}_0}{K' - K}^2.$$  

Notice that in the family of solutions under consideration, the $O(N)$ global symmetry of the Einstein-Klein-Gordon action (2.5) acts linearly on the $q_i$’s and that the numerical prefactor in (3.2) is just the $O(N)$-invariant metric on the moduli space.

In the generic case, the resulting family of metrics will always have strong curvature singularities at some values of the time coordinate $t$. In what follows we will study what happens to these singularities from a higher-dimensional point of view, when we undo the Kaluza-Klein reduction.
3.1. Open FRW

Let us start with the following solution to the vacuum Einstein equations in four dimensions

\[ ds^2_{\text{vac}} = (\sinh 2t)^{-\frac{1}{2}} (\cosh 4t - \cosh 4z)^{\frac{1}{4}}(-dt^2 + dz^2) + \frac{1}{2} \sinh 2t \sinh 2z \left( \tanh z \, dx^2 + \cotanh z \, dy^2 \right) \]

(3.3)

and “dress” it with the homogeneous scalar fields (3.1) taking

\[ \varphi_0(t) = \frac{\sqrt{3}}{2} \log \tanh t. \]

The solution coupled to the \( N \) scalar fields is

\[ ds^2 = (\sinh 2t)^{\frac{1}{4} (3\lambda - 1)} (\cosh 4t - \cosh 4z)^{\frac{1}{4} (1-\lambda)}(-dt^2 + dz^2) + \frac{1}{2} \sinh 2t \sinh 2z \left( \tanh z \, dx^2 + \cotanh z \, dy^2 \right), \]

(3.4)

where we have defined \( \lambda \) as the \( O(N) \)-invariant quadratic form

\[ \lambda \equiv \sum_{i=1}^{N} q_i^2. \]

The interesting feature of these solutions is that the dynamics of the scalar fields is only relevant at early times, saturating to a constant when \( t \to \infty \). One would expect this sort of behavior in “realistic” models for the modular fields in string/M-theory cosmology. The scalar fields are supposed to play an important role in the evolution during the early epoch, but are expected to be frozen at a certain vacuum expectation value after supersymmetry breaking. From that time on, the dynamics of the universe is dominated by matter or radiation.

The case with \( \lambda = 1 \) is especially interesting, since we recover an open FRW universe

\[ ds^2_{\lambda=1} = \sinh 2t (-dt^2 + dz^2) + \frac{1}{2} \sinh 2t \sinh 2z \left( \tanh z \, dx^2 + \cotanh z \, dy^2 \right) \]

\[ \varphi_i(t) = \frac{\sqrt{3}}{2} q_i \log \tanh t, \quad \text{with} \quad \sum_{i=1}^{N} q_i^2 = 1. \]

(3.6)

It is straightforward to check that the metric (3.6) has a cosmological singularity at \( t = 0 \) where the curvature scalar blows up. Actually, the physical properties of the singularity are independent of the particular values \( q_i \) take, as long as the condition (3.3) with \( \lambda = 1 \)
is satisfied. This is in accordance with the requirement of $O(N)$ global invariance of the theory.

Let us now “re-construct” the $(4+N)$-dimensional theory from which (3.6) is being obtained by dimensional reduction. Here we have to keep in mind that the scalar fields associated with the scale factors of the internal torus are related to a linear combination of the original fields $\varphi_i(t)$, as shown above. This means that the higher dimensional metric will be

$$ds_{4+N}^2 = 2(\sinh t)^{-1} \sum_{i=1}^{N} p_i (\cosh t)^{1+N} \sum_{i=1}^{N} p_i (-dt^2 + dz^2 + \sinh^2 z dx^2 + \cosh^2 z dy^2) + \sum_{i=1}^{N} \tanh^{2p_i} t (dw^i)^2,$$

(3.7)

where we have introduced the new constants $p_i = D_{ij}q_j$, with $D_{ij}$ the matrix (2.4). We can now rewrite the condition $\lambda = 1$ in terms of the new moduli space coordinates $\{p_i\}$ as

$$\lambda \equiv \sum_{i=1}^{N} p_i^2 + \frac{2}{3} \sum_{i<j} p_i p_j = 1$$

(3.8)

The new parameters just provide a non-orthogonal system of coordinates in moduli space. Actually we see from (3.7) that the $p_i$’s can be thought of as a kind of Kasner exponents in the $N$-dimensional internal torus when $t \to 0$.

The metric (3.7) with $\lambda = 1$ is a vacuum solution of Einstein equations in $4+N$ dimensions (in fact it is such for any $\lambda$). Naively, the singularity seems to be at $t = 0$, as it was in the original four-dimensional space-time. However, it may happen that this apparent singularity is just the result of choosing a singular coordinate system. To clarify this, we can evaluate the square of the Riemann curvature tensor $R_{abcd}R^{abcd}$, which is non-vanishing, in the $t \to 0$ limit

$$R_{abcd}R^{abcd} \sim C(p_i)t^{2(S-3)} + O[t^{2(S-2)}]$$

(3.9)

where we have written $S = \sum_{i=1}^{N} p_i$ and $C(p_i)$ is defined by

$$C(p_i) = \frac{3}{16}(S-1)^2(S^2-2S+5) + \sum_{i=1}^{N} p_i^2[3 + p_i^2 + (S-3)(p_i + S)] + \sum_{i<j} p_i p_j$$

First, let us notice that $S \leq \sqrt{3\lambda} < 3$ so the leading term in (3.9) will always diverge at $t \to 0$ limit. Thus, in general, we will have curvature singularities in $4+N$ dimensions in
this limit. However, we find the surprising result that whenever \( p_i = 1, \) \( p_j = 0 \) \( (j \neq i) \) \( C(p_i) = 0 \) and actually the curvature invariant is regular for all times and given by\(^6\)

\[
\mathcal{R}_{abcd}\mathcal{R}^{abcd} = \frac{18}{\cosh^8 t}.
\]

(3.10)

This corresponds to having only one dynamically non-trivial cycle whose scale factor evolve with time. In the four dimensional metric, on the other hand, due to the non-diagonal relation between \( \{p_i\} \) and \( \{q_i\} \), in general all scalar fields will have a non-trivial dynamics. Note, however, that the dimensionally reduced model (3.4) is generally inhomogeneous for generic \( \lambda \), the regular higher-dimensional solution falling into a homogeneous and isotropic class. One might wonder whether higher-dimensional regularity and lower-dimensional isotropy are physically linked somehow.

### 3.2. Closed FRW

We now pass to discuss a four-dimensional metric with closed spatial sections. The vacuum seed metric is obtained from (3.3) by replacing hyperbolic functions by their trigonometric counterparts. We again couple the space-time to \( N \) scalar fields of the form (3.1) now with

\[
\varphi_0(t) = \frac{\sqrt{3}}{2} \log \tan t.
\]

Imposing as before \( \lambda = 1 \), we are left with the family of four-dimensional metrics

\[
\begin{align*}
\text{ds}^2_{\lambda=1} &= \sin 2t (-dt^2 + dz^2) + \frac{1}{2} \sin 2t \sin 2z \left( \tan z dx^2 + \cotan z dy^2 \right) \\
\varphi_i(t) &= \frac{\sqrt{3}}{2} q_i \log \tan t, \quad \text{with} \quad \sum_{i=1}^{N} q_i^2 = 1.
\end{align*}
\]

(3.11)

The coordinate transformation from the standard closed FRW metric to (3.11) are given in the Appendix of ref. [14], the coordinates \( z, x \) and \( y \) used here being just Euler angles for \( S^3 \). In spite of the similarities with the flat case, the structure of cosmological singularities is now richer. Studying the scalar curvature \( \mathcal{R} \) we find that the metric (3.11) has curvature

\[^6\] Actually, it is easy to realize that, provided this curvature invariant is finite, so are all other scalars constructed from contractions of any number of Riemann tensors. In our case this follows from the fact that \( \mathcal{R}^{1}_{a;1b} \) is finite when \( t \to 0 \), where the index 1 denotes the dynamical internal dimension \( w^1 \).
singularities at $t = \ell\frac{\pi}{2}$ with $\ell$ being an integer. We have a bouncing universe that evolves from a Big-Bang singularity at $t = \ell\pi$ into a Big-Crunch at $t = (2\ell + 1)\frac{\pi}{2}$.

In going to $4 + N$ dimensions, we get

$$ds_{4+N}^2 = 2\left(\sin t\right)^{1-\sum_{i=1}^N p_i} (\cos t)^{1+\sum_{i=1}^N p_i} (\cos dt^2 + \sin^2 z\, dx^2 + \cos^2 z\, dy^2)$$

$$+ \sum_{i=1}^N \tan^2 p_i (dw^i)^2,$$

where the same definition for the $p_i$ as in Sec. 3.1, satisfying the condition (3.8), has been used. From inspection of (3.12) we might be tempted to infer that the higher dimensional theory will be singular again when $t = \ell\frac{\pi}{2}$. Computing the curvature invariant when $t \to 0^+$ we get eq. (3.9), as in the previous case. However if we now compute the same invariant close to the Big-Crunch singularity ($t \to \frac{\pi}{2}^-$) we find

$$\mathcal{R}_{abcd}\mathcal{R}^{abcd} \sim C(-p_i)t^{-2(S+3)} + \mathcal{O}[t^{-2(S-2)}].$$

The situation is somewhat different as compared to the open case. There we found that when the condition $\lambda = 1$ is saturated by a single $p_i = 1$, the initial Big-Bang singularity disappears altogether by going to higher dimensions. Here we find that we do not get rid of all curvature singularities, but only of “half” of them. The resulting geometry in this case in only singular when $t = \frac{\pi}{2}(2\ell + 1)$, whereas those at $t = \pi\ell$ are smeared in $4 + N$ dimensions. The curvature invariant now is given by (3.10) with the hyperbolic cosine replaced by a trigonometric one. The reverse situation happens when we take $p_i = -1$ and $p_j = 0 \,(j \neq 0)$: we avoid singularities located at $t = \frac{\pi}{2}(2\ell + 1)$, the geometry being the same as for the previous case but now with $t \to t + \frac{\pi}{2}$.

In ref. [10] a similar situation was noticed for a closed FRW cosmology that could be resolved into a five-dimensional black hole interior with just one curvature singularity in the past or in the future. The difference with the model analyzed here is that in the case at hand our higher-dimensional universe still has a finite life, although it is doubled with respect to the one of the four-dimensional geometry.
3.3. Flat FRW

Finally, let us briefly analyze the case of cosmological models with flat spatial sections. The four-dimensional metric and scalar fields system is given by

\[
ds^2 = 2t (-dt^2 + dz^2 + dx^2 + dy^2)
\]
\[
\varphi_i(t) = \frac{\sqrt{3}}{2} q_i \log t, \quad \text{with} \quad \sum_{i=1}^{N} q_i^2 = 1,
\]

which is singular when \( t = 0 \). The higher-dimensional version of this metric according to the ansatz (2.1) is

\[
ds^{2}_{4+N} = 2t^{1-\sum_{i=1}^{N} p_i} (-dt^2 + dz^2 + dx^2 + dz^2) + \sum_{i=1}^{N} t^{2p_i} (dw^i)^2
\]

We can transform this solution into a standard Kasner form by rewriting it in co-moving time coordinates (and re-scaling spatial coordinates) to give

\[
ds^{2}_{4+N} = -dT^2 + T^{2\alpha_0} (dX^2 + dY^2 + dZ^2) + \sum_{i=1}^{N} T^{2\alpha_i} (dW^i)^2,
\]

where

\[
\alpha_0 = \frac{1 - \sum_{j=1}^{N} p_j}{3 - \sum_{j=1}^{N} p_j}, \quad \alpha_i = \frac{2p_i}{3 - \sum_{j=1}^{N} p_j} \quad (i = 1, \ldots, N),
\]

which can be easily shown to satisfy the Kasner equalities

\[
3\alpha_0 + \sum_{i=1}^{N} \alpha_i = 1, \quad 3\alpha_0^2 + \sum_{i=1}^{N} \alpha_i^2 = 1 - \frac{6(\lambda - 1)}{(3 - \sum_{i=1}^{N} p_i)^2} = 1.
\]

Here, we have implemented \( \lambda = 1 \) to get the second condition.

With this expression for the metric we can easily compute the curvature invariant, with the result

\[
R_{abcd}R^{abcd} = 4 t^4 \left[ 6\alpha_0^2(1 - \alpha_0) - 3\alpha_0^4 + \sum_{i=1}^{N} \alpha_i^2(\alpha_i - 1)^2 + \sum_{i<j}^{N} \alpha_i^2\alpha_j^2 \right]
\]

Using the definition of the Kasner exponents in terms of the original \( p_i \) it is possible to check that the only case in which we will have a regular geometry at \( t = 0 \) will again occur if
\( p_i = 1 \) with all other \( p_j \) \((j \neq i)\) vanishing. In this case the resulting metric is flat. Actually it can be seen to be \((4 + N)\)-dimensional Minkowski space-time in Rindler coordinates. The non-trivial topology of the internal dimensions will hinder global identification of the manifold with static Minkowski solution.

### 4. M-theory connections

Let us particularize the study of the cosmological models of the previous section to the case with \( N = 7 \) in which they can be interpreted as four-dimensional cosmologies arising from a compactification of M-theory on \( T^7 = (S^1)^7 \) with vanishing three-form. In describing the moduli space of cosmological solutions coupled to scalar fields of the form (3.1), we have two different possibilities. In the four-dimensional theory it seems natural to take the coordinates \( q_i \) \((i = 1, \ldots, 7)\) which characterize the seven independent scalar fields coupled to gravity. Due to the global \( O(7) \) symmetry of the low-energy action, there is a natural choice for the moduli space metric

\[
I_4 = \sum_{i=1}^{7} q_i^2.
\]

The resulting geometry will only depend on \( q_i \) through the quadratic form \( I_4 \) as seen in equation (3.2).

On the other hand, in eleven dimensional language it seems more appropriate to change coordinates in moduli space and use instead of \( \{q_i\} \) the new parameters \( \{p_i\} \) which determine the scale factors of the internal compactified dimensions and are related to the original coordinates by a non-orthogonal linear transformation, \( p_i = D_{ij} q_j \). If we now express the moduli space metric in the new coordinates, we find

\[
I_{11} = \sum_{i=1}^{7} p_i^2 + \frac{2}{3} \sum_{i<j}^{7} p_i p_j.
\]  

(4.1)

\footnote{In ref. [9] an inverse procedure was evoked by dimensionally reducing from a five-dimensional flat model to four dimensions to argue that the cosmological singularity could be an artifact of dimensional reduction.}

\footnote{Since we are just performing a change of coordinates, \( I_4 = I_{11} \). We will use however different notation to indicate the coordinates used to write the quadratic form.}
Incidentally, this moduli space metric is the same as the one obtained in \cite{8} (see also \cite{17}) for the compactifications of M-theory on a seven-torus with \(A_{MNP} = 0\) using group theoretical considerations. The \(O(7)\) global symmetry will act linearly on the \(p_i\)'s leaving \(I_{11}\) invariant. It is however important to notice that the eleven-dimensional geometry do transform under \(O(7)\), in contrast to the four-dimensional one that was a singlet under the action of this group. In eleven dimensions the only transformations that leave invariant the space-time metric are permutations of the \(p_i\)'s, which generates the Weyl subgroup of the mapping class group of the seven dimensional torus, \(SL(7, \mathbb{Z})\).

We will consider the family of M-theory metrics obtained by taking \(N = 7\) in (3.7). These line elements are vacuum solution of eleven-dimensional supergravity everywhere in the submanifold of the moduli space defined by \(I_{11} = 1\). We can ask now about the action on this metric of \(G_7 \equiv \text{Weyl}[E_7(7)(\mathbb{Z})]\), the subgroup of U-duality transformations preserving the straight torus \(T^7 = (S^1)^7\) and the vanishing of the three-form \(\mathcal{I}\) \cite{17}, which is generated by permutations of the \(p_i\)'s and the so-called 2/5 transformation

\[
(p_1, \ldots, p_7) \rightarrow \left( p_1 - \frac{2s}{3}, p_2 - \frac{2s}{3}, p_3 - \frac{2s}{3}, p_3 + \frac{s}{3}, \ldots, p_7 + \frac{s}{3} \right)
\]

with \(s = p_1 + p_2 + p_3\). It is easy to show that \(G_7\) leaves invariant the bilinear form \(I_{11}\) and it is thus a discrete finite subgroup of \(O(7)\) connecting physically equivalent M-theory vacua.

In the following we will restrict our analysis to the open FRW metric of Sec. 3.1 although most of our results can be extended to the flat case as well. Acting with the elements of \(G_7\) on the different solutions characterized by \(\{p_i\}\) we can look how sensitive M-theory physics is to the semiclassical geometry. The most striking fact we find is that the U-duality group maps solutions with a Big-bang initial singularity into geometries that are regular for all values of the cosmic time. For example, the metric

\[
ds_{11}^2 = 2 \left( \sinh t \cosh^2 t \right)^{\frac{2}{3}} \left( -dt^2 + dz^2 + \sinh^2 z \, dx^2 + \cosh^2 z \, dy^2 \right) + (\tanh t)^{\frac{2}{3}} \sum_{i=1}^{5} (dw^i)^2 + (\tanh t)^{-\frac{2}{3}} \sum_{i=6}^{7} (dw^i)^2
\]

which is singular at \(t = 0\) (\(\mathcal{R}_{abcd} \mathcal{R}^{abcd} \sim t^{-16/3}\)) can be mapped by \(G_7\) into one of the non-singular M-theory cosmologies using the following sequence of permutations and 2/5 transformations

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 2 & -2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 2 & 1 & 1 & 1 & 1 \\
-2 & -3 & -3 & 3 & 3 & 3 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 2 & 1 & 1 & 1 & 1 \\
-2 & -3 & -3 & 3 & 3 & 3 & 3
\end{pmatrix}
\rightarrow
(1, 0, 0, 0, 0, 0, 0).
\]
We have concluded that using an U-duality transformation, which is an exact symmetry of M-theory, we can transform a singular background into a non-singular one. The obvious bottom line seems to be that, at least in the low-energy limit, M-theory physics is insensitive to a certain class of cosmological singularities.

In the case of the open \((k = -1)\) solution, however, the asymptotic form of the metrics at large times is to a great extent left invariant by the action of \(G_7\). The general feature of all solutions \((3.7)\) when \(t \to \infty\) is that the open three dimensional space-time inflates while the internal torus reach a constant volume, namely

\[
\frac{\text{Vol}_{3D}}{\text{Vol}_{T^7}} \sim \tau^3,
\]

with \(\tau\) the co-moving time. Physically, what we have is a cosmological solution that evolves from an eleven-dimensional regime in which all scale factors are of the same size into an asymptotic state with describes a “large” four-dimensional expanding universe and a “small” static seven-dimensional torus (cf. [18][19]).

In the case of the flat solution, on the other hand, the large-time behavior of the metric is not universal. As we saw in Sec. 3.3, in this case we can rewrite the metric in the Kasner form \((3.15)\) with exponents given by \((3.16)\). The U-duality group \(G_7\) acts on these exponents in a rather complicated way through the transformation of the moduli space coordinates \(\{p_i\}\) (cf. [8]). Since the sum \(\sum_{i=1}^{7} p_i\) is not invariant under the 2/5 transformation, the asymptotic form of the metrics \((3.14)\) or \((3.15)\) will be sensitive to rotations by elements of the U-duality group. In addition to this, the internal manifold will not remain “small” in general. We will have the usual behavior of any Kasner universe, with at least one contracting direction and a number of expanding ones.

The main characteristic of the non-singular M-theory cosmologies that we have obtained is that in all cases there is only one time-varying coordinate, while the remaining ones are constant during the evolution of the Universe. In trying to make a string theory interpretation of such solutions it seems natural to identify the dynamical coordinate with the eleventh dimension associated with the dilaton field. If we do so and perform a dimensional reduction of the open \((k = -1)\) regular eleven-dimensional solution down to ten dimensions, the resulting metric and dilaton in string frame are

\[
d s^2_{\text{string}} = \sinh 2t \left( -dt^2 + dz^2 + \sinh^2 z \, dx^2 + \cosh^2 z \, dy^2 \right) + \tanh t \sum_{i=1}^{6} (dw^i)^2,
\]

\[
\phi(t) = \frac{3}{2} \log \tanh t
\]

\[9\] The appropriate Kaluza-Klein ansatz in this case is \(ds^2_{11} = e^{-\frac{2}{3} \phi} ds^2_{10} + e^{\frac{4}{3} \phi} dw^2\).
which corresponds to a string background with three open spatial dimensions and another six compactified on a torus whose dynamics is characterized by a global breathing mode (a situation extensively studied in the Kaluza-Klein and string cosmology literature \cite{20,7,16}). The geometry is singular when $t \to 0$ but, on the other, at large times we find again an inflating three-dimensional space together with a frozen internal torus, now also with a constant dilaton. Notice, however, that M-theory U-duality maps these kind of singular string cosmologies into the regular ones that are obtained by identifying the dilaton field with one of the static circles. These transformations involve the interchange of the dilaton with other moduli fields and thus are intrinsically non-perturbative from the point of view of string theory.

Incidentally, let us remark that because of the invariance of $I_{11} \equiv \lambda$ under $O(7)$, the family of solutions (3.4) is stable under the U-duality group which thus maps solutions into solutions (cf. \cite{8}). This is what one should expect, since $G_7$ is a symmetry of M-theory on a straight torus with a vanishing three-form.

5. Concluding remarks

In this paper we have been concerned with four-dimensional cosmologies that arise from the compactification of string/M-theory on a straight torus. We have given a general algorithm to generate these modular cosmologies in four dimensions with two commuting isometries. Using this technique we have also constructed FRW metrics and observed that, although they are singular in four dimensions, in the open ($k = -1$) and flat ($k = 0$) cases their lifted higher-dimensional vacuum images have a regular curvature invariant $R_{abcd}R^{abcd}$. In the case of closed FRW cosmologies ($k = 1$), singularities are not removed but the lifetime of the higher-dimensional universe (i.e. the time elapsed from the Big-Bang to the Big-Crunch) is twice that of their four-dimensional versions.

It is important to stress, however, that in general the homogeneous character of the scale factors of the extra dimensions derived from (3.4) does not ensure the homogeneity of the reduced four-dimensional cosmology. Given the absence of a singularity-free higher-dimensional “parent” cosmology for the four-dimensional inhomogeneous solutions, it is remarkable that the regularity condition for the space-time in higher dimensions translates itself into isotropy of the four-dimensional solution. Thus, the initially isotropic universe
could be naturally chosen by the requirement of regularity of the higher-dimensional space-time.

We have also studied four-dimensional modular cosmologies related to toroidal compactifications of string/M-theory. In the case of M-theory on a straight seven-torus with vanishing three form we have found open and flat singular metrics that can be rotated into regular ones by the elements of the U-duality group $G_7$, thus indicating that the physics of a certain class of singular universe can be described in terms of a dual regular geometry. In the case of homogeneous open universes, however, the large time asymptotic behavior of the metric is insensitive to the action of $G_7$. In all cases we find an inflating three-dimensional space with $k = -1$ and an internal torus whose size asymptotically reaches a constant value. It is remarkable that U-duality is able to relate geometries with such a different behavior near $t = 0$ without modifying the dynamics of the universe at large times. Incidentally, this stabilization of the internal dimensions is produced by the presence of the positive spatial curvature and it is absent in the case of the flat solutions.

Because of the $O(7)$ invariance of the dimensionally reduced cosmologies, all the eleven-dimensional metrics labeled by $\{p_i\}$ and the same value of $I_{11}$ will produce exactly the same geometry in four-dimensions, although it will be coupled to scalar fields with different amplitudes in each case. In particular, using the classical symmetry of the four dimensional action any solution in our family with $I_4 = 1$ can be rotated into one with a regular higher-dimensional “parent” metric. However, since M-theory in this background is not invariant under the full group $O(7)$, one expects that quantum effects will break this global symmetry down to the U-duality group $G_7$ which preserves the lattice of charges. The situation is completely analogous to the classical invariance of the four-dimensional effective action under $O(6, 6; \mathbb{R})$ [13], a symmetry that is broken down to the T-duality group $O(6, 6; \mathbb{Z})$ by string effects.

Finally, it would be interesting to check whether our results can be extended to compactifications of M-theory on a generic seven-torus with a non-vanishing three form. From our analysis it is clear at least that there will be solutions of this kind that can be transformed into non-singular universes by U-duality, as can be seen by acting on the regular metric with the Borel generators of $E_{7(7)}(\mathbb{Z})$ [12].
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References

[1] H. Lü, S. Mukherji, C.N. Pope and K.-W. Xu, Phys. Rev. D55 (1997) 7926 (hep-th/9610107);
A. Lukas, B.A. Ovrut and D. Waldram, Nucl. Phys. B495 (1997) 365 (hep-th/9610238);
A. Lukas and B. Ovrut, Phys. Lett. B437 (1998) 291 (hep-th/9709030);
N. Kaloper, I. Kogan and K.A. Olive, Phys. Rev. D57 (1998) 7340 (hep-th/9711027);
K. Benakli, Int. J. Mod. Phys. D8 (1999) 153 (hep-th/9804096);
A. Lukas, B.A. Ovrut and D. Waldram, Cosmological solutions of Horava-Witten theory, hep-th/9806022;
H.S. Reall, Phys. Rev. D59 (1999) 103506 (hep-th/9809195);
M. Maggiore and A. Riotto, D-branes and cosmology, hep-th/9811089.

[2] T. Appelquist, A. Chodos and P.G.O. Freund, Modern Kaluza-Klein Theories, Addison Wesley 1987.

[3] J.M. Overduin and P.S. Wesson, Phys. Rept. 283 (1997) 303 (hep-th/9805018).

[4] E. Witten, Nucl. Phys. B186 (1981) 412; Fermion quantum numbers in Kaluza-Klein theory, Proceedings of the Shelter Island II conference, M.I.T. Press, 1985.

[5] E.J. Copelan, A. Lahiri and D. Wands, Phys. Rev. D50 (1994) 4868 (hep-th/9406216);
N.A. Batakis and A.A. Kehagias, Nucl. Phys. B449 (1995) 248 (hep-th/9502007);
N.A. Batakis, Phys. Lett. B353 (1995) 450 (hep-th/9503142); Phys. Lett. B353 (1995) 39 (hep-th/9504057);
J.D. Barrow and K.E. Kunze, Phys. Rev. D55 (1997) 623 (hep-th/9608045); Phys. Rev. D56 (1997) 741 (hep-th/9701085);
J.D. Barrow and M.P. Dabrowski, Phys. Rev. D55 (1997) 630 (hep-th/9608136).

[6] A. Feinstein, R. Lazkoz and M.A. Vázquez-Mozo, Phys. Rev. D56 (1997) 5166 (hep-th/9704173).

[7] E. Alvarez, Phys. Rev. D31 (1985) 418;
R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1989) 191;
K.A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; Mod. Phys. Lett. A6 (1991) 3397 (hep-th/9110004);
M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B296 (1992) 51 (hep-th/9209052);
M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1997) 317 (hep-th/9211021);
N. Kaloper R. Madden and K.A. Olive, *Nucl. Phys.* B452 (1995) 677 (hep-th/9506027);
M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys.* B949 (1997) 315 (hep-th/9611039);
R. Brandenberger, R. Easther and J. Maia, *JHEP* 08 (1998) 007 (gr-qc/9806111);
D.A. Easson and R. Brandenberger, *Nonsingular dilaton cosmology in the string frame*, (hep-th/9905175).

[8] T. Banks, W. Fischler and L. Motl, *JHEP* 01 (1999) 019 (hep-th/9811194).

[9] G.W. Gibbons and P.K. Townsend, *Nucl. Phys.* B282 (1987) 610;
G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Class. Quant. Grav.* 12 (1995) 297 (hep-th/9410073).

[10] F. Larsen and F. Wilczek, *Phys. Rev.* D55 (1997) 4591 (hep-th/9610252).

[11] H. Lü and C.N. Pope, *Nucl. Phys.* B465 (1996) 127 (hep-th/9512012).

[12] N.A. Obers and B. Pioline, *U-duality and M-theory*, hep-th/9809039, to appear in Physics Reports.

[13] J. Maharana and J.H. Schwarz, *Nucl. Phys.* B390 (1993) 3 (hep-th/9207016).

[14] M. Carmeli, C. Charach and A. Feinstein, *Annals Phys.* 150 (1983) 392.

[15] D. Clancy, A. Feinstein, J. Lidsey and R. Tavakol, *Inhomogeneous Einstein-Rosen string cosmologies*, (gr-qc/9901062).

[16] M. Mueller, *Nucl. Phys.* B337 (1990) 37;
G. Veneziano, *Phys. Lett.* B265 (1991) 287;
A.A. Tseytlin and C. Vafa, *Nucl. Phys.* B372 (1992) 443 (hep-th/9109048);
A.A. Tseytlin, *Phys. Lett.* B334 (1994) 315 (hep-th/9404191)

[17] O. Aharony, *Nucl. Phys.* B476 (1996) 470 (hep-th/9604103);
S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *Nucl. Phys.* B509 (1998) 122 (hep-th/9707217).

[18] A. Chodos and S. Detweiler, *Phys. Rev.* D21 (1980) 2167.

[19] J. Demaret and J.-L. Hanquin, *Phys. Rev.* D31 (1985) 258;
J. Demaret, J.-L. Hanquin, M. Henneaux and P. Spindel, *Nucl. Phys.* B252 (1985) 538.

[20] P.G.O. Freund, *Nucl. Phys.* B209 (1982) 146;
D. Sahdev, *Phys. Lett.* B137 (1984) 155;
S. Randjbar-Daemi, A. Salam and J. Straathdee, *Phys. Lett.* B135 (1984) 388