Error analysis for 3D shape sensing by fiber-optic distributed sensors

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Abstract

For the shape sensing of long-range cables, we study a shape sensing method that utilizes the strain data from optical fibers installed around the cable. We estimate the shape by solving the equation of a moving frame whose coefficient contains the curvature and twist rates obtained by the strain data. The estimation error process is formulated by a stochastic differential equation, and the error variance is obtained by solving the variance equation. The obtained theoretical error variance indicates that the shape estimation error becomes maximum when the curve is straight, and the general curve’s error is smaller than that for the straight line. We validated these results by numerical simulation.

1 Introduction

Recent advances in fiber-optic-distributed sensing technology are making it possible to measure such local curve parameters as curvature and torsion in high-density locations. By integrating these local parameter values, various three-dimensional (3D) shape sensing methods have been proposed and experimental results have been reported [1–7].

Since the quantity obtained by an optical fiber is only a strain, multi-core fibers [1–5] or a multi-fiber cable [6, 7] are used to measure the curvature and the torsion. Shape sensing using multi-core fibers with a fiber Bragg grating (FBG) strain sensor has also been studied [3,4]. This method requires grating each core in the optical fibers. On the other hand, distributed strain sensing using Brillouin scattering or Rayleigh scattering can be used for shape sensing without grating the fibers beforehand. Among them, Rayleigh scattering-based optical frequency domain reflectometry (OFDR), which can measure the strain with high spatial resolution and high precision, is applied to shape sensing [1,2]. However, the measurement range is restricted to several tens of meters.

Shape sensing is performed by integrating the curvature vector twice along a curve of the fiber or the cable. Sensing error increases in proportion to the 3/2-th power of their length and depends on the shape complexity. Since the previously reported estimation errors are experimental results with respect to particular arbitrary shapes, it remains unclear how much the estimation error depends on the design conditions and curve shapes. Moreover, such experiments have been restricted to short ranges (~ several tens of meters).

In this paper, for long-range shape sensing, we studied cables with helically wound multiple fibers. The parameters measured at each point of a cable are the longitudinal strain at the center of the cable, the curvature, the bending direction, and the cable twisting rate. These four parameters can be measured by the strain data of multiple (say six) fibers at each point of the cable. The cable shape can be estimated by solving the equation of a moving frame whose coefficient contains four parameters.

For sensing error evaluation, the error process is formulated by a stochastic differential equation (SDE) from which error variance equations are derived and explicit solutions are obtained. When the curve is a straight line, the solution is simple and the dependency to design parameters is clear. We also show that the shape sensing error for an arbitrary general curve is smaller than that for a straight line. Consequently, the straight line case is the worst with respect to the shape sensing error. These analytical results were validated by simulation.

2 Equation of a curve with twist

A cable can be regarded as a curve. Let $r(s) \in$
$\mathbb{R}^3$, $0 \leq s \leq L$ be a curve in three-dimensional space, where $s$ is the arc length parameter and $L$ is the curve length. We assume that the curve has a predetermined direction in the plane perpendicular to the curve at each point on the curve. We refer to the unit vector in this direction as the $y$-normal vector. The moving frame at each point on the cable is defined as follows (Fig. 1):

- $e_x$ unit directional vector of a curve,
- $e_y$ $y$-normal vector of a curve,
- $e_z$ binormal vector of a curve ($= e_x \times e_y$),

and $e_x(s)$ is the first-order derivative of $r(s)$:

$$e_x(s) = r'(s)$$

(1)

where prime (’) denotes the derivative by $s$.

Curvature vector $k(s)$ is defined by the derivative of $e_x$:

$$k(s) = e_x'(s).$$

(2)

Since the curvature vector is perpendicular to the directional vector, it can be represented as

$$k(s) = k_y(s)e_y(s) + k_z(s)e_z(s),$$

(3)

where $k_y(s)$ and $k_z(s)$ are the elements of the curvature vector in the moving frame (Fig. 2). Then the following relation that resembles the Frenet-Serret formula holds [8,9]:

$$\frac{d}{ds} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = A \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix},$$

(4)

where

$$A = \begin{pmatrix} 0 & k_y & k_z \\ -k_y & 0 & \gamma \\ -k_z & -\gamma & 0 \end{pmatrix},$$

(5)

and $\gamma$ is the twist rate (specific angle of twist).

The matrix that arranges the moving frame vectors as columns

$$R = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix} = \begin{pmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{pmatrix},$$

(6)

becomes a three-dimensional rotation matrix, i.e., $R \in SO(3)$, and follows the following equation:

$$\frac{dR}{ds} = R A^T.$$

(7)

### 3 Parameter estimation from fiber strain data

#### 3.1 Fiber layout

When an optical fiber is placed on a cable’s surface with a layout angle of $\varphi$ (Fig. 3), its strain is represented as [6]:

$$\epsilon_t = a(\epsilon_c + \langle k, \rho \rangle) + b\rho\gamma,$$

(8)

where $a = \sin^2 \varphi - \nu \cos^2 \varphi$, $b = \sin \varphi \cos \varphi$, and $c$ is the strain of the cable’s central axis, $\nu$ is its Poisson ratio, $\rho$ is the distance from its center to the fiber, $k = (k_y, k_z)$ and $\rho = (\rho \cos \theta, \rho \sin \theta)$ are the curvature vector and the fiber’s position vector in the $y-z$ plane, respectively, $\theta$ is the angle between the fiber and $y$ axis (Fig. 3), and $\langle \cdot, \cdot \rangle$ stands for the inner product.

Since there are four unknown parameters, $\epsilon_c, k_y, k_z$, and $\gamma$, at least four fibers are necessary. The strains of the $M (\geq 4)$ fibers are represented as

$$\epsilon_t = Bp,$$

(11)

where

$$\epsilon_t = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_M \end{pmatrix}, \quad p = \begin{pmatrix} \epsilon_c \\ k_y \\ k_z \\ \gamma \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_1 & a_1\rho_{1,y} & a_1\rho_{1,z} & b_1\rho_1 \\ a_2 & a_2\rho_{2,y} & a_2\rho_{2,z} & b_2\rho_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_M & a_M\rho_{M,y} & a_M\rho_{M,z} & b_M\rho_M \end{pmatrix}.$$
If a fiber is parallel to the cable ($\varphi = \pi/2$), then the $b$ of Eq. (10) becomes zero and $\gamma$ cannot be measured by Eq. (8); hence, not all of the fibers should be placed parallel to the cable. In addition, if all the fibers have the same layout angle $\varphi$, then $\varepsilon_c$ and $\gamma$ cannot be separated. Considering these, $M$ optical fibers are placed on the cable’s surface with layout angles:

$$\varphi_m = \left\{ \begin{array}{ll}
\varphi, & 1 \leq m \leq M/2 \\
\pi - \varphi, & M/2 + 1 \leq m \leq M
\end{array} \right. \quad (13) $$

Moreover, the fibers are evenly spaced apart for $1 \leq m \leq M/2$ and $M/2 + 1 \leq m \leq M$. Then $B^T B$ becomes a diagonal matrix, which simplifies the error analysis that is described below. An example of fiber placement for $M = 6$ is shown in Fig. 4.

![Cable and optical fibers](image)

**Fig. 4: Cable and optical fibers**

### 3.2 Local parameter estimation

Let $\hat{\varepsilon}_f$ be the strain data of the $M$ fibers containing measurement noise. Since the measurement errors of the $M$ fiber strains are independent and identically distributed, the parameter estimates can be obtained by the least squares method:

$$\hat{p} = B^+ \hat{\varepsilon}_f, \quad (14)$$

where $B^+ = (B^T B)^{-1} B^T$ is the generalized inverse matrix of $B$. The true values of the strains and the parameters also satisfy

$$p = B^+ \epsilon_f. \quad (15)$$

Hence, the relation between measurement error $\Delta \epsilon_f = \hat{\varepsilon}_f - \epsilon_f$ and parameter estimation error $\Delta p = \hat{p} - p$ is given by

$$\Delta p = B^+ \Delta \epsilon_f. \quad (16)$$

Let $\sigma^2$ be the variance of the strain measurement error. Then the covariance of the parameter estimation errors is given by

$$C_p = E(\Delta p \Delta p^T) = (B^T B)^{-1} \sigma^2 \\
= \frac{\sigma^2}{M} \text{diag} \left[ \frac{1}{a^2}, \frac{2}{a^2 \rho^2}, \frac{2}{a \rho^2}, \frac{1}{b \rho^2} \right]. \quad (17)$$

where $E(\cdot)$ and $\text{diag}[\cdot]$ respectively stand for the expectation and diagonal matrices. From this expression, we find that the parameter estimation errors become independent.

### 4 Shape estimation from measured strain data

Let $\hat{p}_s = (\hat{\varepsilon}_f(s), \hat{\kappa}_y(s), \hat{\kappa}_z(s), \hat{\gamma}(s))^T$ be the parameter estimates obtained by (14) at position $s$ and write the coefficient matrix substituted by the parameter estimates as

$$\hat{A}_s = \begin{pmatrix}
0 & \hat{\kappa}_y(s) & \hat{\kappa}_z(s) \\
-\hat{\kappa}_y(s) & 0 & \hat{\gamma}(s) \\
-\hat{\kappa}_z(s) & -\hat{\gamma}(s) & 0
\end{pmatrix}. \quad (18)$$

The estimate of the moving frame, $\hat{R}_s = [\hat{e}_x(s) \ \hat{e}_y(s) \ \hat{e}_z(s)]$, can be obtained by solving

$$\frac{d}{ds} \hat{R}_s = (1 + \hat{\varepsilon}_c(s)) \hat{R}_s \hat{A}_s^T, \quad (20)$$

with initial value $\hat{R}_0 = R_0$. The first column of $\hat{R}_s$ is $\hat{e}_x(s)$, the estimate of the directional vector. Position estimate $\hat{r}(s)$ and $\hat{e}_x(s)$ have the following relation,

$$\frac{d}{ds} \hat{r}(s) = (1 + \hat{\varepsilon}_c(s)) \hat{e}_x(s), \quad (21)$$

whose integration is used to obtain the following position estimate:

$$\hat{r}(s) = r(0) + \int_0^s (1 + \hat{\varepsilon}_c(u)) \hat{e}_x(u) du. \quad (22)$$

In actual situations, since the parameter measurement values are given at the sampling points on the optical fibers, the differential equation is discretized on the sampling points and the estimates are obtained by solving the approximate difference equation.

### 5 Error analysis of shape sensing

#### 5.1 Position estimation error

The strain measurement errors of a fiber at the sampling points are independent of each other. We assume that sampling period $\Delta s$ is sufficiently small and approximate strain measurement errors $\Delta \epsilon_f(s) = \hat{\varepsilon}_f(s) - \epsilon_f(s)$ as a white Gaussian process, characterized by

$$E[\Delta \epsilon_f(s_1) \Delta \epsilon_f^T(s_2)] = \sigma^2 I_M \delta(s_1 - s_2) \Delta s, \quad (23)$$

where $I_M$ stands for the $M$-dimensional identity matrix and $\delta(\cdot)$ is the Dirac delta function. Then parameter estimation error $\Delta p_s = \hat{p}(s) - p(s)$ becomes a white Gaussian process satisfying

$$E[\Delta p_s \Delta p_s^T] = C_p \delta(s_1 - s_2) \Delta s, \quad (24)$$

where $C_p$ is a covariance matrix defined by (17). In the same manner, the error of coefficient matrix $\Delta A_s =$
\(A(p(s)) - A(p(s))\) becomes a white Gaussian process satisfying
\[
E[\Delta A_s, \Delta A_{s_2}^T] = C_{A,s} \delta(s_1 - s_2) \Delta s,
\]
where
\[
C_{A,s} = E[A(\Delta p)A^T(\Delta p)]
\]
\[
= E \left( \begin{array}{cccc} 
\Delta k_1^2 + \Delta k_2^2 & \Delta k_1 \Delta k_2 - \Delta k_1 \Delta k_2 & \Delta k_1 \Delta k_2 & \Delta k_2 \Delta k_2 \\
\Delta k_1 \Delta k_2 & \Delta k_1^2 + \Delta k_2^2 & \Delta k_1 \Delta k_2 & \Delta k_2 \Delta k_2 \\
\Delta k_1 \Delta k_2 & \Delta k_1 \Delta k_2 & \Delta k_1^2 + \Delta k_2^2 & \Delta k_2 \Delta k_2 \\
\Delta k_2 \Delta k_2 & \Delta k_2 \Delta k_2 & \Delta k_2 \Delta k_2 & \Delta k_2^2 + \Delta k_2^2 
\end{array} \right).
\]

The elements of \(C_{A,s}\) are calculated by (17), and we have
\[
C_{A,s} = \frac{\sigma^2}{M^2} \text{diag} \left[ \frac{4}{a^2}, \frac{2}{a^2 + \frac{1}{b^2}}, \frac{2}{a^2} + \frac{1}{b^2} \right].
\]

For simplicity, we consider the case of no strain at the cable’s central axis, i.e., \(\epsilon_c(s) \equiv 0\). The position estimation error of the cable follows the differential equation:
\[
\frac{d\Delta r_s}{ds} = \Delta e_{x,s},
\]
and the estimation error of moving frame \(\Delta R_s = \hat{R}_s - R_s\) approximates follows the differential equation:
\[
\frac{d}{ds} \Delta R_s = \Delta R_s A_{s}^T + R_s \Delta A_{s}^T,
\]
which is driven by white Gaussian process \(\Delta A_s\). Although (29) is stochastic, it is linear and the noise term does not depend on state variable \(\Delta R_s\); hence the covariance equations can be obtained in a closed form. We define the covariances of the estimation errors by
\[
\sigma_r^2(s) = E[\Delta r^T_1 \Delta r_2],
\]
\[
C_{R,r}(s) = E[\Delta R^T_1 \Delta r_2],
\]
\[
C_{e_x,r}(s) = E[\Delta e^T_{x,s} \Delta r_2] = [C_{R,r}(s)]_{1st \text{ element}},
\]
\[
C_r(s) = E[\Delta R^T_1 \Delta R_2],
\]
\[
C_{R,e_x} = E[\Delta R^T_1 \Delta e_{x,s}] = [C_r(s)]_{1st \text{ column}}.
\]

Then they satisfy the following equations:
\[
\frac{d\sigma_r^2}{ds} = 2 C_{e_x,r},
\]
\[
\frac{d}{ds} C_{R,r} = A_s C_{R,r} + C_{R,e_x},
\]
\[
\frac{d}{ds} C_R = A_s C_R + C_R A_{s}^T + C_A \Delta s,
\]
by solving which with the initial values of zeros, we have
\[
\sigma_r^2(s) = 2 \int_0^s C_{e_x,r}(s_1) ds_1,
\]
\[
C_{e_x,r}(s) = \int_0^s R_{s_1} C_{R,e_x}(s_1) ds_1,
\]
\[
C_{R,e_x}(s) = R_{s}^T \int_0^s R_{s_1} C_R \Delta s_1 e_{x,s} ds_1 e_{x,s}. \quad (40)
\]

Substituting (39) and (40) into (38) yields the expression of the variance of the position estimation error:
\[
\sigma_r^2(s) = 2 \int_0^s \int_0^{s_1} e_{x,s_1}^T R_{s_1} C_{R,s_1} e_{x,s_2} ds_1 ds_2 ds_1. \quad (41)
\]

### 5.2 Estimation error for a straight line

When a curve is a straight line, which is achieved by setting the curvature and torsion to zero, frame \(R_s\) becomes an identity matrix and the integrand of (41) becomes
\[
e_{x,s_1}^T R_{s_1} C_{R,s_1} e_{x,s_2} = \frac{4\sigma^2 s}{M^2 a^2} \quad (42)
\]
from which we get
\[
\sigma_r^2(s) = \frac{4\sigma^2 s^3}{3M a^2 \rho} \quad (43)
\]
In this straight line case, the standard deviation of the position estimation error becomes
\[
\sigma_r(s) = \sqrt{\frac{4s^3 \Delta s}{3M a^2 \rho}} \quad (44)
\]
which is simple and the dependency to design parameters is clear.

### 5.3 Estimation errors for general curves

Since \(C_A\), which is included in the integrand of (41), is positive definite, we can define an inner product for two arbitrary vectors \(u, v \in \mathbb{R}^3\) as
\[
\langle u, v \rangle_{C_A} = u^T C_A v \quad (45)
\]
and write norm \(\| \cdot \|_{C_A}\) that corresponds to the above inner product as
\[
\| u \|_{C_A} = \sqrt{\langle u, u \rangle_{C_A}}. \quad (46)
\]
Then the following Schwartz’s inequality holds:
\[
\| \langle u, v \rangle_{C_A} \| \leq \| u \|_{C_A} \| v \|_{C_A}. \quad (47)
\]
Moreover, if we denote the maximum eigenvalue of \(C_A\) by \(\lambda_{\max}\), then it holds that
\[
\| u \|_{C_A}^2 = \langle u, C_A u \rangle \leq \lambda_{\max} \| u \|^2. \quad (48)
\]
Since the same inequality applies to \(v\),
\[
\| \langle u, v \rangle_{C_A} \| \leq \lambda_{\max} \| u \| \| v \| \quad (49)
\]
holds. By making substitutions \(u = R_{s_1} e_{x,s_1}\) and \(v = R_{s_2} e_{x,s_2}\), and noting \(\| u \| = \| e_{x,s_1} \| = 1\) and \(\| v \| = \| e_{x,s_2} \| = 1\) because \(R_{s_1}\) is a rotation matrix, we have
\[
\| e_{x,s_1}^T R_{s_1} C_{R,s_1} R_{s_2} e_{x,s_2} \| \leq \lambda_{\max}. \quad (50)
\]
Substituting this inequality into (41) and integrating it with respect to \(s\) yields
\[
\sigma_r^2(s) = \frac{1}{3} \lambda_{\max} s^3. \quad (51)
\]
The maximum eigenvalue of $C_A$ is given by (27):

$$\lambda_{\text{max}} = \frac{\sigma_r^2}{Ma^2} \max \left\{ \frac{4}{a^2}, \frac{2}{a^2}, \frac{1}{b^2} \right\}.$$  \hfill (52)

Hence, when $2b^2 \leq a^2$, it holds that

$$\lambda_{\text{max}} = \frac{4\sigma_r^2}{Ma^2a^2},$$  \hfill (53)

and we have

$$\sigma_r(s) \leq \sqrt{\frac{4s^4A\sigma_r}{3Ma^2a^2}}.$$  \hfill (54)

which implies that the error for the case of a straight line becomes the upper bound of the estimation error for general curves.

6 Evaluation by simulation

We simulated the estimation of the 3D shapes of the cables from the strain data of the optical fibers. In all cases, the cable lengths are $L = 5$ km, and the cable radii are $\rho = 2.5$ cm, and six optical fibers are installed spirally around the cable. The layout angles of the fibers are $\varphi_m = \pi/3$ ($m = 1, 2, 3$), $2\pi/3$ ($m = 4, 5, 6$), and three of the fibers are evenly spaced apart. The standard deviations of the strain measurement errors are assumed to be $\sigma_r = 10 \mu e$ and $1 \mu e$.

The simulation results of the four cases are shown in Figs. 5–8. In each case, the estimation results of ten trials are plotted. Actual and estimated shapes are plotted in (a), and estimation errors with the theoretical values are plotted in (b), where $\sigma_r$ and $\sigma_r$ for straight lines are given by (41) and (44), respectively.

Figure 5 is the case of a twisted arch. Since the curve is close to a straight line, $\sigma_r$ is almost the same as $\sigma_r$ for a straight line. Fig. 6 is the case of a helix. Here $\sigma_r$ is much smaller than that for a straight line.

Figure 7 is the case of a random curve, which is generated by randomly giving curvature and torsion. Even in such a complex curve case, $\sigma_r$ is smaller than that for a straight line and the estimation errors are distributed around it.

Figure 8 is the case of another random curve. But the strain measurement errors are $1 \mu e$, which is one tenth of Fig. 7. As might be expected, the estimation errors become one tenth of those of Fig. 7.

As observed above, the shape estimation errors for any complex curves are distributed around the theoretically obtained $\sigma_r$. The value of $\sigma_r$ is maximum when curve is a straight line. When the cable length is $5$ km and the strain measurement error is $1 \mu e$, the estimation errors is less than about $3$ m.

7 Conclusions

For the shape sensing of long-range cables, we studied a sensing method utilizing the strain data of optical fibers installed around the cable. The shape is estimated by solving the equation of a moving frame whose coefficient contains curvature and twist rates obtained by the strain data.

By formulating the estimation error as a stochastic differential equation, we derived error variance equations. The obtained theoretical error variance indicates that the shape estimation error becomes maximum when the curve is straight, and the error for the general curve is smaller than that for the straight line. The expression of the error variance for a straight line is simple and clarifies the dependence of the error on design parameters. These values were evaluated by simulation. We also observed in our simulations that the shape sensing error is about $3$ m when the cable length is $5$ km and the strain measurement error is $1 \mu e$. Due to the recent advances in optical fiber sensing technology, accuracy of about $1 \mu e$ has been achieved for long-range measurements, and we expect that accurate shape sensing will become possible with our proposed method.

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Fig. 5: Shape sensing for a twisted arch: $L = 5$ km, $\sigma_\epsilon = 10 \mu\epsilon$.

Fig. 6: Shape sensing for a helix: $L = 5$ km, $\sigma_\epsilon = 10 \mu\epsilon$.

Fig. 7: Shape sensing for a random curve: $L = 5$ km, $\sigma_\epsilon = 10 \mu\epsilon$.

Fig. 8: Shape sensing for another random curve: $L = 5$ km, $\sigma_\epsilon = 1 \mu\epsilon$. 
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