Post inflationary evolution of inflation-produced large-scale magnetic fields using a generalised cosmological Ohm’s law and both standard and modified Maxwell’s equations

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Abstract

In most of the literature on evolution of cosmological magnetic fields, it is found that large-scale magnetic fields evolve as \( B^2 \propto a^{-4} \) (adiabatic magnetic decay) where \( a \) is the cosmological scale factor and \( B \) is the cosmological magnetic field. This rapid decay has been considered as the main obstacle against magnetic fields produced during the inflationary epoch from surviving until today and seeding the observed fields. However, recent reports of first ever detection of intergalactic fields, with strengths around \( 10^{-6} \) G are a mystery [1–4]. One possible explanation is that large-scale magnetic fields could have been superadiabatically amplified in their evolutionary history. Superadiabatic amplification may mean that there is an actual increase in the strength of the magnetic field or that magnetic decay-rates are slower than the standard adiabatic magnetic decay rate [5]. This can be demonstrated if we use the generalised cosmological Ohm’s law and both standard and modified Maxwell field equations; this is the goal of this study.
I. INTRODUCTION

Magnetic fields are ubiquitous in the Universe. Current known generation mechanisms are unable to account for the scope and the breadth of what is observed. The origin of some of these fields is unclear. Several known astrophysical and cosmological mechanisms for producing the galactic magnetic fields have been devised and theories of primordial magnetic fields generation have been widely studied. The proposed mechanisms include magnetogenesis during the inflationary epoch \[6\][7]. However, these proposed mechanisms are not without challenges as will be discussed in the following paragraphs.

In this paper we examine cosmological magnetic fields generated during inflation. It is noteworthy that gravitational waves are also generated during inflation \[8\] as a counterpart. The energy transfer between the two is of interest and has been studied in \[9\]. In present study we will assume that no gravitational waves are generated. In principle, inflation can generate magnetic fields on all scales although non-standard physics may have to be invoked to achieve non-minimal coupling of the electromagnetic field \[10\] to either matter or spatial curvature. In some cases, one may need a mechanism for breaking the conformal-invariance \[11\]. In order to examine if there is a link between the magnetic fields produced during inflation and those observed in the present Universe, it is crucial to understand how cosmological magnetic fields evolve during the epochs dominated by radiation (\(RD\)) , matter-radiation equality and by matter (\(MD\)) and assuming a spatially flat Friedmann Lemaître Robertson Walker (\(FLRW\)) Universe where the epochs are fluids.

In describing how cosmological magnetic fields evolve in the history of the Universe expansion, the expansion is usually broken into different intervals, namely inflation (for magnetogenesis and sometimes evolution of cosmological magnetic fields), \(RD\) and \(MD\) epochs mentioned earlier. Cosmological magnetic fields evolution is analysed separately in these epochs without analysing their evolution during the transition period of matter-radiation equality epoch. The matter-radiation equality epoch is a significant period in the evolution of cosmological magnetic fields as it lasted for thousands of years, about 23000 years \[12\] to be precise which is much longer than the duration of the inflation or reheating epoch. In analysing these epochs we employ the fluid approach because in order for us to probe the evolution of magnetic fields during the matter-radiation equality epoch we use the multi-fluid formalism which stems from using the single-fluid formalism in analysing the evolution of magnetic fields first during the \(RD\) epoch and then during the \(MD\) epoch (before analysing the evolution of magnetic fields during the \(MD\) epoch we do so during the matter-radiation equality epoch first after analysis during the \(RD\) epoch).

In most of the literature on evolution of cosmological magnetic fields, it is found that large-scale magnetic fields evolve as \(B^2 \propto a^{-4}\) (adiabatic magnetic decay) for the flat Friedman model. This rapid decay has been considered as the main obstacle against magnetic fields produced during the inflationary epoch from surviving until today and seeding the observed fields. Conventional magnetic fields in spatially flat \(FLRW\) Universes decay adiabatically.
throughout the evolution of these models and on all scales. Inflation naturally achieves superhorizon correlations, hence it can easily produce primordial fields. Magnetic fields generated just after or way after inflation are too small in scale. Cosmological magnetic fields produced during inflation decay adiabatically as soon as they cross outside the Hubble horizon. The result are astrophysically irrelevant magnetic fields today [13].

As already mentioned, recent detection of possible intergalactic fields, with strengths around $10^{-6}G$ demand an explanation. Superadiabatic amplification may be an explanation. Superadiabatic amplification can be demonstrated if we use the generalised cosmological Ohm’s law and both standard and modified Maxwell field equations as we will show in this paper; this is the goal of this paper. But first let us consider modified Maxwell’s field equations for the reasons that will be given in the following paragraphs.

A homogeneous and isotropic Universe with a uniformly distributed net charge cannot be described by standard Maxwell’s equations, because this requires that the electromagnetic field tensor in such a Universe must vanish everywhere. Standard Maxwell’s equations always fail for a closed Universe with a non-zero net charge regardless of the spacetime symmetry and the charge distribution. Modifications of systems of equations seem necessary if one is to characterise the superadiabatic modes of magnetic fields. Such changes can be implemented via the ohm’s law and Maxwell’s equation. In particular, a generalised cosmological Ohm’s law and non-trivial modifications of Maxwell’s equations are required. A Proca-type equation which contains a photon mass term is a type of modified Maxwell’s equations. These electromagnetic field equations can naturally arise from spontaneous symmetry breaking or the Higgs mechanism in quantum field theory (QFT), where photons acquire a mass by devouring massless Goldstone bosons. However, when the symmetry is restored, photons loose their mass again and the problems mentioned above reappear. The second type of modification is where an electromagnetic field potential vector is coupled to the spacetime curvature tensor. This type of electromagnetic field equations return to Maxwell’s equations in a flat or Ricci-flat spacetime and don’t introduce a new dimensional parameter. It was shown in [14] that modifications of Maxwell’s equations have no discernible impact on existing experiments and observations. The impact of a curvature coupled to field equations is, in theory testable, in astrophysical environments where the mass density is high or the gravity of electromagnetic radiations plays a dominant role in the dynamics of the system such as in the interior of neutron stars and during the early Universe.

A photon will have an effective, time-dependent mass due to the additional terms $RA^2$ or $R_{ab}A^aA^b$ or $RA^2+R_{ab}A^aA^b$ (where $R$ is the curvature scalar, $R_{ab}$ is the Ricci tensor, $A^a$ is the four vector electromagnetic potential and $a$ and $b$ are spacetime indices) . This is undesired as charge conservation is broken. Nevertheless, expected effects (due to the additional terms) which contradict present-day observations or experiments are not observed. Due to these terms we have $m_\gamma \sim R_{1/2}$ as the mass of the photon and $R^{1/2} \sim H$, where also $H$ is the expansion rate of the Universe. Today the photon mass would be $m_\gamma \sim H_{today} \sim 10^{-33}eV$, which is well below the present limits to the photon mass, that is, $m_\gamma < 3 \times 10^{-27}eV$ [15].
Though charge non-conservation would only manifest itself on scales of the horizon or larger, this has no observable consequences. The terms $RA^2 + R_{ab}A^aA^b$, $RA^2$ and $R_{ab}A^aA^b$ vanish in vacuum and as a consequence they cannot affect the propagation of photons outside massive bodies. For a $RD$ Universe, these terms introduce corrections of order $\frac{H^2}{T^2} \sim \frac{T^2}{m_{pl}}$ where $T$ is temperature and $m_{pl}$ is the Planck mass. These corrections are negligible for temperatures where the evolution of the Universe is relatively well understood. Therefore, these terms cannot spoil successful predictions made using the standard Maxwell equations [6]. Therefore, due to the brief explanation above on the terms, we will consider them.

As a result of all this discussion, it seems that the present-day amplitude of magnetic fields arising from inflationary magnetogenesis can actually be much larger than what has been claimed in most previous studies.

The paper is organised as follows: In Section II we introduce the Faraday tensor and the Bianchi identity, we discuss the generalised cosmological Ohm’s law, we discuss the matter, electromagnetic and coupling actions and the derivation of modified Maxwell tensors from the total action of the fluid action, the Maxwell action, the Coulomb action and the action due to the coupling of the terms $RA^2 + R_{ab}A^aA^b$ or $RA^2$ or $R_{ab}A^aA^b$ where $R$ is the curvature scalar, $R_{ab}$ is the Ricci tensor and $A^a$ is the four vector electromagnetic potential using the single fluid formalism. In Section III we discuss the action principle for a two-conducting fluid approximation and the derivation of the modified Maxwell tensors from the same total action except that for this case we are considering two fluids hence, using the two (multi)-conducting fluid formalism. In Section IV we use the modified Maxwell tensors derived using both the single and multi-conducting fluid formalisms to study the evolution of cosmological magnetic fields during the $RD$, matter-radiation equality and $MD$ epochs. We also discuss large-scale superadiabatic magnetic amplification during these epochs. In section V we analyse the role of initial conditions in achieving superadiabatic amplification during the mentioned epochs above. In section VI we summarise our findings for the epochs of $RD$, matter-radiation equality and $MD$. We discuss and summarise everything in this paper in Section V. For the spacetime indices we have $a, b = 0, 1, 2, 3$.

II. THE ACTION PRINCIPLE FOR A SINGLE-CONDUCTING FLUID APPROXIMATION AND MODIFIED MAXWELL FIELD EQUATIONS

In this section an action principle is set up to derive the modified Maxwell set of field equations. The pull-back formalism will be used to set up variations of $A_a$ required to get the modified Maxwell equations where $A_a$ is the four vector potential and $a$ is a spacetime index. The modified Maxwell equations are obtained from a coupling term based on the scalar $J_T^T A_a$ where $J_T^T$ is the total current of the fluid composed of the flux current of the fluid and the plasma current from the generalised cosmological Ohm’s law. The other way is by varying $A_a$, which will appear in two pieces of the total action: one constructed from
the antisymmetric Faraday tensor $F_{ab}$, defined as \[ F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{1} \]
and satisfies a Bianchi identity
\[ \nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} = 0. \tag{2} \]

It is important to note that the formalism will account for coupling of a fluid to dynamical spacetime. The fluid action $S_M$ will have as its Lagrangian an energy functional $\Lambda$ (also known as the Master function). In the following subsection, we will briefly discuss the generalised cosmological Ohm’s law before we consider a system with a single-fluid flow where the generalised cosmological Ohm’s law will be used.

\textbf{A. A discussion on the generalised cosmological Ohm’s law}

One typically applies Ohm’s law in order to solve for the evolution of the electromagnetic fields and the plasma \[17\]. Given, $\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B})$, where $\vec{v}$ is the bulk velocity of the plasma, $\sigma$ is the conductivity of plasma that has units of an inverse length, $\vec{J}$ is current due to plasma, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field, Ohm’s law represents a shortcut to solving both standard or modified Maxwell’s field equations by providing a link between current and the electromagnetic fields. After assessing the relevant evolution and interaction timescales, Ohm’s law derives from the evolution equation of the current and reduces to the simple form above. For the derivation of the generalised relativistic Ohm’s law and the current evolution see (\[32\] and references therein).

We now give a brief overview on the derivation of the generalised Ohm’s law relevant on cosmological scales (for a detailed discussion check \[17\]). Linearising equation (2.25) in the article \[17\] for a FLRW background will not alter the main features. As a result of working in an arbitrary frame with four velocity, $u^a$, the proton-electron centre of mass current, $J_{pe}^a = 0$, is present. $J_{pe}^a = 0$ if we choose $u^a$ to be the centre of mass frame, which is very close to the baryon frame. We express the spatially projected 4-vectors with respect to a comoving basis, i.e. $J^a = a^{-1}(0, \vec{J})$ where $a$ is the scale factor, and $\vec{J}$ is the 3-current due to plasma. Then we find the following linear current evolution equation
\[ \dot{\vec{J}} + (4H + \Gamma_C + \Gamma_T) \vec{J} = -\Gamma_T \vec{J}_r + \omega_p^2 \vec{E}. \tag{3} \]
where $H$ is the Hubble parameter which comes from $\Theta = 3H + O(1)$, $\Gamma_C$ is the Coulomb rate, $\Gamma_T$ is the Thomson rate, $\omega_p$ is the plasma frequency, $\vec{J}_r$ is the photon current and $\vec{E}$ is the electric field all elaborated in \[17\]. Approximating the time derivative of the current by a characteristic timescale of the problem, $\tau$, through $\dot{J} \simeq \tau^{-1}J$, we can write a generalised cosmological Ohm’s law in a familiar form. $\tau \sim \min(L, H^{-1})$ typically for
large-scale fluctuations of physical correlation lengths $L$ larger than the Silk damping scale. Therefore, the linear cosmological Ohm’s law can be written as

$$(\eta_r + 4\eta_H + \eta_C + \eta_T)\vec{J} + \eta_T\vec{J}_r \simeq \vec{E}$$

where the different resistivities are

$$\eta_r \equiv \frac{\tau^{-1}}{\omega_p^2}, \quad \eta_H \equiv \frac{H}{\omega_p^2}, \quad \eta_C \equiv \frac{\Gamma_C}{\omega_p^2}, \quad \eta_T \equiv \frac{\Gamma_T}{\omega_p^2}$$

and have the dimensions of time. With the presence of an electric field the resistivities quantify the efficiency of generating currents. You will notice that $\eta_T \gg \eta_C$ for $a \lesssim 3 \times 10^{-6}$, well before recombination, and then $\eta_T \ll \eta_C$ until today when the resistivities are compared as functions of the scale factor in the upper panel of figure 1 in the article [17]. $\eta_H$ which is resistivity due to expansion is small. It is proportional to the Hubble parameter and overcome by much faster interaction rates. $\eta_r$ which is the characteristic evolution timescale is negligible since it is inversely proportional to the large scales we consider. This means that the time derivative in the current evolution equation can safely be neglected. This implies that equation (4) can be written in the form

$$\vec{J} + \sigma_E\eta_T\vec{J}_r \simeq \sigma_E\vec{E}$$

where $\sigma_E \equiv (\eta_C + \eta_T)^{-1}$ is the electric conductivity of the plasma with the dimensions of an inverse length.

With the above discussion, we can now consider a system with a single-fluid flow in the following subsection. We will discuss the matter, electromagnetic and coupling actions.

**B. The matter, electromagnetic, and coupling actions**

In order to examine the effect of fluid couplings one needs a formalism that explicitly expresses the coupling in terms of the different fluid parameters. The most suitable formalism for this is the convective variational formalism [19–23]. In this formalism, it can be shown that the fluid action $S_M$ has as Lagrangian the Master function $\Lambda$, which depends on the $n_X^2 = -n_X^a n_X^a$, where $n_X^a$ is the number density four current or flux whose magnitude $n_X$ is the particle number density (see [19]), $X$ is a label for a fluid and $g_{ab}$ is the metric. Ignoring boundary terms throughout, an arbitrary variation of $S_M$ with respect to the flux $n_X^a$ and the metric gives (see [16])

$$\delta S_M = \delta \left( \int_{\mathcal{M}} d^4x \sqrt{-g} \Lambda \right) = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \mu^X_a \delta n_X^a + \frac{1}{2} (\Lambda g_{ab} + n_X^a \mu_X^b) \delta g_{ab} \right],$$

where $\mathcal{M}$ is the manifold or hypersurface, $g$ is the determinant of the metric, and $\mu^X_a$ are the canonically conjugate momenta to $n_X^a$; that is, letting

$$B^X = -2 \frac{\partial \Lambda}{\partial n_X^a},$$

(6)
then
\[ \mu^X_a = g_{ab} B^X n^b_X \]  
\[ (7) \]

(see [16]).

The Maxwell action is given by
\[ S_{Max} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} F_{ab} F^{ab}, \]  
\[ (8) \]

and varying this action with respect to the four-vector potential \( A_a \) (this couples the charged fluids to the electromagnetic field and vice versa) and the metric \( g_{ab} \) results in
\[ \delta S_{Max} = \frac{1}{4\pi} \int_M d^4x \sqrt{-g} (\nabla_b F^{ba}) \delta A_b - \frac{1}{32\pi} \int_M d^4x \sqrt{-g} (F_{cd} F^{cd} g^{ab} - 4F^{ac} F^b_c) \delta g_{ab} \]  
\[ (9) \]

The minimal coupling of the Maxwell field to the charge current densities is obtained from
\[ S_C = \int_M d^4x \sqrt{-g} (J_X^a + j^a) A_a, \]  
\[ (10) \]

where \( J_X^a = e_X n_X^a \) is flux current and \( j^a \) is four current due to plasma where \( j^a = (j^0, \vec{j}) = (j^0, \vec{J}) \), \( j^0 = c\rho \) is the time part where \( c \) is the speed of light and \( \rho \) is the charge density of plasma, \( \vec{J} = \vec{j} \) is the 3-current due to plasma which will be represented by the generalised cosmological Ohm’s law in simplified form in equation (5). Varying equation (10) with respect to \( n_X^a, A_a, g_{ab}, J_X^a \) and \( j^a \) results in
\[ \delta S_C = \int_M d^4x \sqrt{-g} \left[ (J_X^a + j^a) \delta A_a + e_X A_a \delta n_X^a + A_a \delta j^a + \frac{1}{2} (J_X^a + j^a) A_a g^{bc} \delta g_{bc} \right] \]  
\[ (11) \]

For a given system, the total action will thus become
\[ \delta S = \delta S_M + \delta S_{Max} + \delta S_C \]
\[ = \int_M d^4x \sqrt{-g} \left\{ [\mu^X_a + e_X A_a] \delta n_X^a + \frac{1}{4\pi} [\nabla_b F^{ba} + 4\pi (J_X^a + j^a)] \delta A_a + \frac{1}{2} \Lambda g^{ab} \\
+ n_X^a \mu^b_X + (J_X^c + j^c) A_c g^{ab} - \frac{1}{16\pi} (F_{cd} F^{cd} g^{ab} - 4F^{ac} F^b_c) \delta g_{ab} + A_a \delta j^a \right\} \]  
\[ (12) \]

where \( J_X^a + j^a = J_{T}^a \). Now in the following sections, we will consider the case where the terms \( RA^2 + R_{ab} A^a A^b \), \( RA^2 \) and \( R_{ab} A^a A^b \) are added to the action (12) above (where \( R \) is the curvature scalar, \( R_{ab} \) is the Ricci tensor and \( A^a \) is the four vector electromagnetic potential). We first do so in the following subsection. 

7
C. The field equations

We now consider

\[ S_{\phi T} = S_{\phi} + S_{\phi_0} \]  

(13)

where \( S_{\phi T} \) is the total action with coupling constants \( \phi \) and \( \phi_0 \) in \( \phi RA^2 + \phi_0 R_{ab} A^a A^b \). Varying action (13) we have

\[ \delta S_{\phi T} = \delta S_{\phi} + \delta S_{\phi_0} \]  

(14)

and writing equation (14) explicitly we have

\[ \delta S_{\phi T} = \int_{\mathcal{M}} d^4x \sqrt{-g}(2\phi RA^2 + 2\phi_0 R_{ab} A^a A^b) \delta A_a - \int_{\mathcal{M}} d^4x \sqrt{-g} \phi g^{ha} g^{fb} R_{fh} A^2 \delta g_{ab} 
+ 2 \int_{\mathcal{M}} d^4x \sqrt{-g} \phi g^{ab} A^2 \left\{ \frac{1}{2} \nabla_c [g^{cd}(\nabla_a \delta g_{db} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ba})] 
+ \frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{f^h} \right\} + \frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g} g^{ab}(\phi RA^2 + \phi_0 R_{cd} A^c A^d) \delta g_{ab} \]  

(15)

The total action then takes the form

\[ \delta S_T = \delta S_M + \delta S_{Max} + \delta S_C + \delta S_{\phi T} \]  

(16)

and writing explicitly we have

\[ \delta S_T = \int_{\mathcal{M}} d^4x \sqrt{-g} [\mu^X_a + e_X A_a] \delta n^X_a + \frac{1}{4\pi} [\nabla_b F^{ba} + 4\pi (J^a_X + j^a) + 8\pi \phi RA^a + 8\pi \phi_0 R_{cd} A^c A^d] \delta A_a 
+ \frac{1}{2} \left[ \Lambda g^{ab} + n_X^a \mu_X^b + J^c_X A_g^{ab} \right] - \frac{1}{16\pi} (F_{cd} F^{cd} g^{ab} - 4F^{ac} F_{c}^{b}) - \phi g^{ha} g^{fb} R_{fh} A^2 \right] \delta g_{ab} 
+ 2\phi g^{ab} A^2 \left[ \frac{1}{2} \nabla_c (g^{cd}(\nabla_a \delta g_{db} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ba})) + \frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{f^h} \right] + A_a \delta j^a \right\} \]

(17)

The minimal coupling of the Maxwell field to the charge current densities of fluid \( X \) has given a modification of the conjugate momentum, namely,

\[ \tilde{\mu}_a^X = \mu_a^X + e_X A_a \]  

(18)

The field equations obtained from the variation above cannot be the final form, since the term proportional to \( \delta n^X_a \) implies that the momentum \( \tilde{\mu}_a^X \) must vanish. This is essentially the condition that there be no energy present but clearly this is not viable. This happens because the components of \( \delta n^X_a \) cannot all be varied independently. A constrained variation
is needed. A set of alternative variables which does precisely that is provided by the pull-back formalism—the $X^4$ can be varied independently, where $X^A$ (where $A = (1, 2, 3)$) are coordinates of the $X$-fluid matter space. We also need to incorporate the fact that the fluid momenta have changed from $\mu_a^X$ to $\tilde{\mu}_a^X$. This means that we need to use the pull-back formalism for a single fluid approximation. For references on the pull-back approach please check [24] [25] [26] [19]. The equations of motion for a general relativistic fluid are obtained from an action principle. This will form the foundation for the variations of the fundamental fluid variables in the action principle. After using the pull-back approach [19], the equations of motion can be derived from the action principle. Therefore, a first-order variation of the fluid Lagrangian plus other variations of other Lagrangians that make up the total variation of equation (17) results in

$$\delta S_T = \int_M d^4x \sqrt{-g} \left\{ [\mu^X \alpha + e_x A_\alpha] \delta n_X^\alpha + \frac{1}{4\pi} [\nabla_b F^{ba} + 4\pi (J^a + j^a) + 8\pi \phi R A^a + 8\pi \phi_0 R^a_b A^b] \delta A_a \\
+ \frac{1}{2} \left[ (\psi \delta^a_X + n^X \mu^X) g^{cb} - \frac{1}{16\pi} (F_{cd} F^{cd} g^{ab} - 4 F^{ac} F_{cb}) - \phi g^{ab} g^{fh} R_{fh} A^2 \right] \delta g_{ab} \\
+ 2\phi g^{ab} A^2 \left[ \frac{1}{2} \nabla_c (g^{cd} (\nabla_d g_{ab} + \nabla_b g_{ad} - \nabla_a g_{bd})) + \frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{fh} \right] \\
+ \frac{1}{2} g^{ab} [\phi R A^2 + \phi_0 R_{cd} A^c A^d] \delta g_{ab} - \mathcal{F}_b^X \delta X_b + A^a \delta j^a \right\},$$

(19)

where $\mathcal{F}_b^X$ is the force density given by

$$\mathcal{F}_b^X = n^X \mathcal{W}_{ab}^X$$

(20)

where $\mathcal{W}_{ab}^X$ is defined as

$$\mathcal{W}_{ab}^X = 2\nabla_{[a} \mu_{b]}^X = \nabla_a \mu^X_b - \nabla_b \mu^X_a$$

(21)

and $\psi$ is defined to be

$$\psi \equiv \Lambda - n^X \mu^X_a$$

(22)

For $S_T$ to be an extremum, the coefficients of $\delta A_a$ of equation (19) demand

$$\nabla_b F^{ab} - 8\pi \phi R A^a - 8\pi \phi_0 R^a_b A^b = 4\pi J_T^a$$

(23)

(see [16]) including also equation (2), which are the equations of motion. For equation (2), symmetry of Christoffel symbols, $\Gamma^d_{ab} = \Gamma^d_{ba}$, enables us to substitute usual derivatives instead of covariant ones. Indeed,

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$$

(24)
means that
\[ \partial_a F_{bc} - \Gamma^d_{ab} F_{dc} - \Gamma^d_{ac} F_{bd} + \partial_b F_{ca} - \Gamma^d_{bc} F_{da} = 0 \] (25)

Rewriting equation (25), we get,
\[ \partial_a F_{bc} - \Gamma^d_{ab} F_{dc} - \Gamma^d_{ac} F_{bd} + \partial_b F_{ca} - \Gamma^d_{bc} F_{da} + \Gamma^d_{ba} F_{dc} + \partial_c F_{ab} + \Gamma^d_{ca} F_{bd} + \Gamma^d_{cb} F_{da} = 0. \] (26)

This yields
\[ \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \] (27)

where all spatial derivatives are with respect to the comoving coordinates [6]. The formalism we used above to derive the equations of motion for fluid \( X \) is for a single-fluid approximation. In the next section, we will use a two-fluid formalism to derive the equations of motion for a two-fluid system but before doing so we will discuss briefly the action principle for a two-conducting fluid approximation.

### III. THE ACTION PRINCIPLE FOR A TWO-CONDUCTING FLUID APPROXIMATION AND MODIFIED MAXWELL FIELD EQUATIONS

Now, the fluid action \( S_M \) has as its Lagrangian the master function \( \Lambda \), which depends on the \( n_X^2 = -n_X^a n_X^a \), \( n_Y^2 = -n_Y^b n_Y^b \) and the \( n_{XY}^2 = -g_{ab} n_X^a n_Y^b \), where \( n_X^a \) or \( n_Y^b \) is the number density four current or flux whose magnitude \( n_X \) or \( n_Y \) is the particle number density (see [19]), \( X \) is a label for one fluid while \( Y \) is a label for a different fluid from fluid \( X \) and \( g_{ab} \) is the metric. Ignoring boundary terms throughout, an arbitrary variation of \( S_M \) with respect to the fluxes \( n_X^a \), \( n_Y^b \) and the metric results in
\[ \delta S_M = \delta \int d^4 x \sqrt{-g} \Lambda = \int_M d^4 x \sqrt{-g} \left\{ \sum_{i = \{X,Y\}} \mu^i_a \delta n_i^a + \frac{1}{2} \Lambda g^{ab} + \sum_{i = \{X,Y\}} n_i^a \mu^i_b \delta g_{ab} \right\} \] (28)

where \( M \) is the manifold or hypersurface, \( g \) is the determinant of the metric, and \( \mu^i_a \) are the canonically conjugate momenta to \( n_i^a \); that is, letting [21]
\[ B^X = -2 \frac{\partial \Lambda}{\partial n_X^2}, \]
\[ B^Y = -2 \frac{\partial \Lambda}{\partial n_Y^2}, \]
\[ A^{XY} = A^{YX} = -\frac{\partial \Lambda}{\partial n_{XY}^2}, \] (29)

then
\[ \mu^X_a = g_{ab} \{ B^X n_Y^b + \sum_{Y \neq X} A^{XY} n_Y^b \}, \]
\[ \mu^Y_a = g_{ab} \{ B^Y n_X^b + \sum_{X \neq Y} A^{YX} n_X^b \}, \] (30)
and varying this action with respect to the vector potential $A_a$ and the metric $g_{ab}$ results in

$$
\delta S_{Max} = \frac{1}{4\pi} \int_M d^4x \sqrt{-g} \left\{ \nabla_a F^{ab} \right\} \delta A_b - \frac{1}{32\pi} \int_M d^4x \sqrt{-g} (F_{cd} F^{cd} g^{ab} - 4 F^{ac} F^b_c) \delta g_{ab}
$$

The minimal coupling of the Maxwell field to the charge current densities is obtained from

$$
S_C = \int_M d^4x \sqrt{-g} \left( \sum_{i=\{X,Y\}} J^a_i + j^a \right) A_a,
$$

where $J^a_i$ are the flux currents. Varying action (33) with respect to $n^a_i$, $A_a$, $g_{ab}$ and $j^a$ results in

$$
\delta S_C = \int_M d^4x \sqrt{-g} \sum_{i=\{X,Y\}} \left\{ [J^a_i + j^a] \delta A_a + \epsilon_i A_a \delta n^a_i + A_a \delta j^a + \frac{1}{2} [J^a_i + j^a] A_a g^{bc} \delta g_{bc} \right\}
$$

For a given system, the total action will thus become

$$
\delta S = \delta S_M + \delta S_{Max} + \delta S_C
$$

Adding terms of the form $R A^2 + R_{ab} A^a A^b$ results in the total action taking the form below,

$$
\delta S_T = \delta S_M + \delta S_{Max} + \delta S_C + \delta S_{\phi T}
$$
and writing explicitly we have

$$\delta S_T = \int_M d^4x \sqrt{-g} \left\{ \sum_{i=X,Y} [\mu_i^a + e_i A_a] \delta n_i^a + \frac{1}{4\pi} [\nabla_b F^{ba} + 4\pi \sum_{i=X,Y} (J_i^a + j^a)] + 8\pi \phi R A^a + 8\pi \phi_0 R_b^a A^b \right\} \delta n^a_i + \frac{1}{2} \left[ \Lambda g^{ab} + \sum_{i=X,Y} (n_i^a \mu_i^b + (J_i^c + j^c) A_c g^{ab}) - \frac{1}{16\pi} (F_{cd} F^{cd} g^{ab} - 4 F^{ac} F_c^b) - \phi g^{ha} g^{fb} R_{fh} A^2 \right] \delta g_{ab} + 2\phi g^{ab} A^2 \left[ \frac{1}{2} \nabla_c (g^{cd} (\nabla_a \delta g_{db} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ba})) + \frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{fh} \right] + \frac{1}{2} g^{ab} \left( \phi R A^2 + \phi_0 R_{cd} A^c A^d \right) \delta g_{ab} + A_a \delta j^a \right\}$$

(37)

Similarly, as in the single-fluids approximation case, the minimal coupling of the Maxwell field to the charge current densities of the coupled fluids of $X$ and $Y$ has given a modification of the conjugate momentum, namely,

$$\tilde{\mu}_a^i = \mu_a^i + e_i A_a$$

(38)

Again and similarly the field equations obtained from the final variation above cannot be the final form, since the term proportional to $\delta n_i^a$ implies that the momentum $\tilde{\mu}_a^i$ must vanish. This is essentially the condition that there be no energy present but clearly this is not viable. This happens because the components of $\delta n_X^i$ cannot all be varied independently. A constrained variation is needed. A set of alternative variables which does precisely that is provided by the pull-back formalism-the $X_i^A$ can be varied independently. We also need to incorporate the fact that the fluid momenta have changed from $\mu_a^i$ to $\tilde{\mu}_a^i$. This means that we need to use the pull-back formalism for two-fluids approximation. For a detailed discussion the reader is referred to [19, 24–26]. The equations of motion for a general relativistic fluid are obtained from an action principle. This will form the foundation for the variations of the fundamental fluid variables in the action principle. After using the pull-back approach for a two-fluid model [19], the equations of motion can be derived from the action principle. Therefore, a first-order variation of the fluid Lagrangian plus other variations of the other
Lagrangians that make up the total variation of equation (37) results in

\[ \delta S_T = \int \mathcal{M} d^4x \sqrt{-g} \left\{ \sum_{i=\{X,Y\}} \left[ (\mu^i_a + c_i A_a) \delta n^a_i \right] + \frac{1}{4\pi} [\nabla_b F^{ba} + 4\pi \left( \sum_{i=\{X,Y\}} J^a_i + j^a \right) + 8\pi \phi R A^a \right. \\
+ 8\pi \phi_0 R_a^b A^b \delta A_a + \frac{1}{2} \left( \psi \delta_c^a + \sum_{i=\{X,Y\}} n^a_i \mu^i_c \right) g^{cb} - \frac{1}{16\pi} (F_{cd} F^{cd} g^{ab} - 4 F^{ac} F^{b}_c) \right. \\
- \phi g^{ab} g^{fh} R_{fh} A^2 \left[ \delta g_{ab} + 2 \phi g^{ab} A^2 \left[ \frac{1}{2} \nabla_c (g^{cd} (\nabla_a \delta g_{db} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ba})) + \frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{lh} \right] \right. \\
. \\
+ \frac{1}{2} [\phi R A^2 + \phi_0 R_{cd} A^c A^d] \delta g_{ab} - \sum_{i=\{X,Y\}} \mathcal{F}^{i b \delta b}_i \right\} \right. \\
\right. \] (39)

where \( \mathcal{F}^i_b \) is as defined in equation (20) except that the individual velocities are no longer parallel. \( \psi \) is given by

\[ \psi = \Lambda - \sum_{i=\{X,Y\}} n^b_i \mu^i_b. \] (40)

Similarly, by varying \( A_a \), modified Maxwell field equations are obtained as in the single-fluids approximation case. For \( S_T \) to be an extremum, the coefficients of \( \delta A_a \) of equation (39) demand

\[ \nabla_b F^{ab} - 8\pi \phi R A^a - 8\pi \phi_0 R_a^b A^b = 4\pi \left( \sum_{i=\{X,Y\}} J^a_i + j^a \right) \] (41)

(see [16]) including also equation (2), which are the equations of motion. Similarly, as in the single-fluids approximation case, for equation (2), symmetry of Christoffel symbols, \( \Gamma^d_{ab} = \Gamma^d_{ba} \), enables us to rewrite it in the form below,

\[ \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \] (42)

where all spatial derivatives are with respect to the comoving coordinates [0].

Note that, even though the particular system we concentrate on consists of only two fluids, it illustrates all new features of a general multi-fluid system. We will scrutinise the effect of adding terms of the form \( R A^2 + R_{ab} A^a A^b \) first to equations (12) and (35). The greatest step is to go from one to two fluids conceptually. Once this is done, a generalisation to a system with more degrees of freedom is straightforward. Now, in order to study the evolution of magnetic fields we need to rewrite the Maxwell tensors in terms of the magnetic flux strength. We assume that magnetic evolution evolves through \( RD \), matter-radiation equality and \( MD \) epochs. We will do this in the sections to follow. We will start with the evolution of magnetic fields during the expansion of the Universe driven by the fluid of radiation. Therefore, we will use the equations of motion from the single-fluid formalism above.
IV. MAGNETIC FIELDS EVOLUTION DURING THE RD, MATTER-RADIATION EQUALITY AND MD EPOCHS

In this section we consider the evolution of magnetic fields during the RD epoch first, then matter-radiation equality epoch and finally the MD epoch. We do so in the following subsections.

A. RD epoch

The formalism we used to derive the Maxwell tensors can handle a number of different fluids [22]. In the case we are considering, electromagnetism is incorporated in the action of the formalism, thus allowing for plasmas and their effects on the system. We consider equations (23) and (27). To study these equations we write them in terms of the \( \vec{E} \) and \( \vec{B} \) flux strength. The results will be vectors. The matrix we will use to enable us write equations (23) and (26) in terms of \( \vec{E} \) and \( \vec{B} \) flux strength is given below,

\[
\begin{bmatrix}
0 & -a^2 E_x & -a^2 E_y & -a^2 E_z \\
a^2 E_x & 0 & a^2 B_z & -a^2 B_y \\
a^2 E_y & -a^2 B_z & 0 & a^2 B_x \\
a^2 E_z & a^2 B_y & -a^2 B_x & 0
\end{bmatrix}
\]

where \( [F_{ab}] = F_{ab} [6] \). Considering equation (27), using \( F_{ab} \) and the fact that \( R_i = 6 \frac{\ddot{a}}{a^3} \) where \( a \) is the cosmic scale factor, we recast equation (27) as

\[
\frac{1}{a^2} \frac{\partial a^2 \vec{B}}{\partial \eta} + \nabla \times \vec{E} = 0
\]

in vector form. Using \( F_{ab}, R_i = \frac{\ddot{a}}{a} + \left[ \frac{\dot{a}}{a} \right]^2 \) (no sum on \( i \)) and \( R = \frac{6\ddot{a}}{a^4} \) we can recast equation (23) as

\[
\frac{1}{a^2} \frac{\partial (a^2 \vec{E})}{\partial \eta} - \nabla \times \vec{B} - \frac{n}{\eta^2 a^2} \vec{A} = 4\pi (\vec{J}_X + \vec{j})
\]

in vector form where

\[
n = \eta^2 \left[ 8\pi \left( 6\phi + \phi_0 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right) \right) \right]
\]

where \( \phi \) and \( \phi_0 \) are coupling constants and \( n \) is a constant whenever \( a(\eta) \) varies as a power of \( \eta \). In equation (44) we have \( \vec{J}_X \) which is the current due to fluid \( X \) representing the RD epoch as a fluid and \( \vec{j} \) is the current due to plasma. Therefore, we have

\[
\vec{J}_X + \vec{j} = \vec{J}_T
\]
where \( \vec{J}_T \) is the total current. Due to the overwhelming plasma effects to flux current, \( \vec{J}_X \) or when

\[
\vec{j} \gg \vec{J}_X
\]

yields

\[
\vec{J}_T \approx \vec{j}
\]

This implies that equation (44) can be rewritten as

\[
\frac{1}{a^2} \frac{\partial (a^2 \vec{E})}{\partial \eta} - \nabla \times \vec{B} - \frac{n}{\eta^2 a^2} \frac{\vec{A}}{a} = 4\pi \vec{j}
\]

in vector form. We can now study the evolution of magnetic fields in the RD epoch of the Universe using the generalised cosmological Ohm’s law where \( \vec{j} \approx \vec{J} \).

But before doing so we will rescale the electromagnetic fields, currents and the electromagnetic four vector potential. We can do this because of conformal invariance. The equations that result from this are \([17]\)

\[
\tilde{E} \equiv a^2 \vec{E}, \quad \tilde{B} \equiv a^2 \vec{B}, \quad \tilde{J} \equiv a^3 \vec{J}, \quad \tilde{J}_r \equiv a^3 \vec{J}_r, \quad \tilde{A} \equiv a\vec{A}
\]

We now rewrite equation (49) in the form below,

\[
\frac{\partial (a^2 \vec{E})}{\partial \eta} - \nabla \times \vec{B} - \frac{n}{\eta^2 a^2} \frac{\vec{A}}{a} = 4\pi a^3 \vec{j}
\]

in vector form. Simplifying yields,

\[
\frac{\partial \tilde{E}}{\partial \eta} - \nabla \times \tilde{B} - \frac{na\vec{A}}{\eta^2 a^2} = 4\pi \vec{j}
\]

We then take the curl of equation (51) and this yields,

\[
\frac{\partial \nabla \times \tilde{E}}{\partial \eta} - \nabla \times \nabla \times \tilde{B} - \frac{n}{\eta^2 a^2} \nabla \times \vec{A} = 4\pi \nabla \times \tilde{j}
\]

where \( \tilde{\nabla} = a\nabla \). But equation (43) can be written in the form below,

\[
\frac{\partial a^2 \tilde{B}}{\partial \eta} + \nabla \times a^2 \vec{E} = 0
\]

and simplifying,

\[
\frac{\partial \tilde{B}}{\partial \eta} + \nabla \times \tilde{E} = 0.
\]

Using equation (54) and the identities

\[
\nabla \times \nabla \times \tilde{B} = \nabla (\nabla \cdot \tilde{B}) - \nabla^2 \tilde{B}
\]

\[
= a^2 \nabla (\nabla \cdot \tilde{B}) - \nabla^2 \tilde{B}
\]

\[
= 0 - \nabla^2 \tilde{B}
\]

\[
= -\nabla^2 \tilde{B}
\]
(since $\nabla a^2 = 0$) and
\[ \tilde{\nabla} \times \tilde{A} = \tilde{B} \] (56)
in equation (52) results in
\[ \frac{\partial}{\partial \eta} \left( -\frac{\partial \tilde{B}}{\partial \eta} \right) + \nabla^2 \tilde{B} - \frac{n}{\eta^2 a^2} \tilde{B} = \frac{4\pi}{a} \nabla \times \tilde{J} \] (57)
which can be rewritten in the form below,
\[ \frac{\partial^2 \tilde{B}}{\partial \eta^2} - \nabla^2 \tilde{B} + \frac{n}{\eta^2 a^2} \tilde{B} = -\frac{4\pi}{a} \nabla \times \tilde{J}. \] (58)
Now, equation (5) in rescaled form is
\[ \tilde{J} = \hat{\sigma} \tilde{E} - \eta T \hat{\sigma} \tilde{J} \] (59)
where
\[ \hat{\sigma} = a\sigma. \] (60)
Then from equation (58) we have
\[ \nabla \times \tilde{J} = \hat{\sigma} \nabla \times \tilde{E} - \frac{\eta \tau \hat{\sigma}}{a} \nabla \times \tilde{J}_r. \] (61)
With equation (54), equation (61) can be rewritten in the form below,
\[ \nabla \times \tilde{J} = -\hat{\sigma} \frac{\partial \tilde{B}}{\partial \eta} - \frac{\eta \tau \hat{\sigma}}{a} \nabla \times \tilde{J}_r. \] (62)
Using equation (62), equation (58) can be rewritten in the form
\[ \frac{\partial^2 \tilde{B}}{\partial \eta^2} - \nabla^2 \tilde{B} + \frac{n}{\eta^2 a^2} \tilde{B} = -\frac{4\pi}{a} \left( -\hat{\sigma} \frac{\partial \tilde{B}}{\partial \eta} - \frac{\eta \tau \hat{\sigma}}{a} \nabla \times \tilde{J}_r \right). \] (63)
Rewriting equation (63) in a more suitable form results in
\[ \tilde{B}'' - \nabla^2 \tilde{B} + \frac{n}{\eta^2 a^2} \tilde{B} = -\frac{4\pi \hat{\sigma}}{a^2} \nabla \times \tilde{J}_r \] (64)
where primes denote derivatives with respect to conformal time. In $[6] \frac{1}{\eta a} \sim H$. But $H \sim R^{\frac{1}{2}}$, $m_\gamma \sim R^{\frac{1}{2}}$, [6] where $m_\gamma$ is the photon mass. This implies that $\frac{1}{\eta a} \sim m_\gamma$. Therefore, equation (64) will become
\[ \tilde{B}'' - \nabla^2 \tilde{B} + m_\gamma^2 \tilde{B} = -\frac{4\pi \hat{\sigma}}{a^2} \nabla \times \tilde{J}_r \] (65)
On comoving scales larger than $\hat{\sigma}$ one can conclude that the first three (3) terms on the left hand side ($LHS$) can be neglected with respect to the fourth term (check condition of
coupling constants $\phi_0$ and $\phi$ just after equation (74). This is equivalent to dropping $\vec{J}$ with respect to $\sigma_E \vec{E}$ in equation (5). This implies that equation (61) becomes

$$\frac{\eta T}{a} \nabla \times \ddot{\vec{r}} = \dot{\sigma} \nabla \times \vec{E}.$$  \hfill (66)

Then using equation (66) in equation (65) we have

$$\dddot{\vec{B}} - \nabla^2 \vec{B} + m^2 n \vec{B} - \frac{4\pi \dot{\sigma}}{a} \vec{B}' = \frac{4\pi \dot{\sigma}}{a} \nabla \times \vec{E}.$$  \hfill (67)

Using equation (54) in equation (67) yields,

$$\dddot{\vec{B}} - \nabla^2 \vec{B} + m^2 n \vec{B} - \frac{4\pi \dot{\sigma}}{a} \vec{B}' = -\frac{4\pi \dot{\sigma}}{a} \vec{B}'.$$  \hfill (68)

This simplifies to

$$\dddot{\vec{B}} - \nabla^2 \vec{B} + m^2 n \vec{B} = 0.$$  \hfill (69)

To solve equation (69) we introduce the harmonic splitting of $\vec{B} = \sum_k \vec{B}_k Q^k$ where $\vec{B}_k$ is the $k^{th}$ magnetic mode and $k$ is the associated comoving eigenvalue. $Q^k$ are pure-vector harmonics that satisfy the conditions $Q'^k = 0 = \nabla Q^k$ and the other version of the Laplace-Beltrami equation, that is

$$\nabla^2 Q^k = -k^2 Q^k.$$  \hfill (70)

Applying the above decomposition to equation (69), the harmonics decouple, and the wave formula of the $k^{th}$ magnetic mode assumes the form

$$\dddot{\vec{B}}_k + k^2 \vec{B}_k + m^2 n \vec{B}_k = 0.$$  \hfill (71)

We consider the case where $n \neq 0$. Rewriting equation (71) in a more suitable form we have

$$\dddot{\vec{B}}_k + (k^2 + m^2 n) \vec{B}_k = 0.$$  \hfill (72)

Solving the above equation yields (note that $|\vec{B}_k| = B_k$)

$$B_k = \left[ C_1 \cos(k^2 + m^2 n) \eta + C_2 \sin(k^2 + m^2 n) \eta \right] \left[ \frac{a_0}{a} \right]^2.$$  \hfill (73)

where $C_1$ and $C_2$ are constants and the equation is written for the actual magnetic field ($B = \frac{\dot{\vec{B}}}{a^2}$). This equation is similar to the equation that applies to inflationary magnetic fields as they cross the horizon during the de Sitter era. But we are considering the $RD$ epoch. This equation then applies to inflation-produced magnetic fields outside the Hubble horizon but still near the horizon during the $RD$ epoch (or matter-radiation equality epoch or $MD$ epoch as we will find later on when analysing these epochs). This equation is a differential equation that accepts an oscillatory solution where $(k^2 + m^2 n) \eta = \frac{\lambda_0}{\lambda_k}$.
Relative to the Hubble horizon ($\lambda_H = \frac{1}{H}$) where $H$ is the Hubble radius the ratio ($\frac{\lambda_H}{\lambda_k}$) measures the physical size of the magnetic mode ($\lambda = \frac{\phi}{k}$) \[13\]. So when the inflation-produced magnetic fields near the horizon initially are well outside the Hubble radius, that is for $\frac{\lambda_H}{\lambda_k} \ll 1$, meaning for $(k^2 + m^2)\eta \ll 1$ in conformal-time terms, a simple Taylor expansion reduces the above equation (73) to the power law

$$a^2B_k = C_1 + C_2(k^2 + m^2)\eta$$

(74)

where $a = a(\eta)$. $k$, $\phi_0$ and $\phi$ are constants whose values are much much less than 1 (that is $k \ll 1$, $\phi_0 \ll 1$ and $\phi \ll 1$) and therefore, the condition $(k^2 + m^2)\eta \ll 1$ is achieved. Note that the subscript 0 on $\phi_0$ doesn’t indicate initial time or today. It is just zero in order to differentiate $\phi_0$ from $\phi$.

We now consider the case where $n = 0$. This case is for standard electromagnetism. Equation (71) will reduce to

$$\tilde{B}''_k + k^2B_k = 0$$

(75)

Solving the equation yields

$$a^2B_k = C_3 \cos k\eta + C_4 \sin k\eta$$

(76)

where $C_3$ and $C_4$ are constants. This equation is similar to the equation that applies to inflationary magnetic fields as they cross the horizon during the de Sitter era \[13\]. But we are considering the $RD$ epoch. This equation then applies to inflation-produced magnetic fields outside the Hubble horizon but still near the horizon during the $RD$ epoch (or matter-radiation equality epoch or $MD$ epoch as we will find later on when analysing these epochs). This equation is a differential equation that accepts an oscillatory solution where $k\eta = \frac{\lambda_H}{\lambda_k}$ \[13\]. Relative to the Hubble horizon ($\lambda_H = \frac{1}{H}$) where $H$ is the Hubble radius the ratio ($\frac{\lambda_H}{\lambda_k}$) measures the physical size of the magnetic mode ($\lambda = \frac{\phi}{k}$) \[13\]. So when the inflation-produced magnetic fields near the horizon initially are well outside the Hubble radius, that is for $\frac{\lambda_H}{\lambda_k} \ll 1$, meaning for $k\eta \ll 1$ in conformal-time terms, a simple Taylor expansion reduces the above equation (77) to the power law

$$a^2B_k = C_3 + C_4k\eta$$

(77)

with $a = a(\eta)$.

When we consider the case where we add terms of the forms $RA^2$ and $R_{ab}A^aA^b$ in the action (12), the result is the same form or similar equations to equations (74) and (77). The only difference is that for the third term in equation (64) we have for the constant $n$,

$$n = n_0 = \eta^28\pi6\phi\frac{\dot{a}}{a}$$

(78)

for the case $RA^2$ and

$$n = n_1 = \eta^28\pi\phi_0\left[\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)\right]$$

(79)
for the case $R_{ab}A^aA^b$ and we obtain the same form or similar equations to equations (74) and (77). We now consider evolution of magnetic fields in the matter-radiation equality epoch.

**B. Matter-radiation equality epoch**

It is shown in [22] that it is possible to have a cosmological epoch where there is a relative flow of radiation with respect to the matter, but out of which the expansion becomes isotropic and the relative flow dissipates. This transit of matter-radiation equality epoch puts forward a Bianchi $I$ behaviour with a spacelike privileged direction. This two-fluid nature of the problem introduces several terms that are not present in the one-fluid case. We introduce the so-called cross-constituent coupling, which occurs when the $EoS$ has terms containing both fluid densities. It is an equilibrium property and hence, non-dissipative.

We then assume a general relativistic two-fluid model for a coupled system of matter (non-zero rest mass) and radiation (zero rest mass). The two fluids are allowed to interpenetrate and exhibit a relative flow with respect to each other (, implying, in general, an anisotropic Universe). The initial conditions used are such that the massless fluid flux dominates early on so that the situation is effectively that of a single-fluid and one has the usual $FLRW$ spacetime. This two-fluid model introduced is valid for applications in cosmology.

The model that will be developed is the simplest set-up of what might be envisioned for the transition itself, for we have not taken into account, for example, the non-conservation of the photon number through its coupling with luminous matter, matter flows with more than one constituent or relative flows at arbitrary angles. One would not be too surprised if comparisons to observational data indicated the need for a more elaborate model. Here also, we have assumed the matter fluid to have only one flux-component. This assumption might not be reasonable; however, it is largely a scale-dependent statement.

The formalism developed can in principle handle a number of different fluids other than just two. In the case we are considering, electromagnetism is incorporated in the action, thus allowing for plasmas and their effects on the system. Now the two-fluid cosmology we will consider has a combination of matter with mass $m^Y = m$, and radiation, which means that $m^X = 0$. As mentioned above we assume that there is cross-constituent coupling and zero entrainment [22]. We have particle flux of matter given by $n_Y = n$ and ignoring dissipation we have for the entropy flux of the system $n_X = s$, and the bulk of this is due to radiation. We set $\mu^X = T$, which is the temperature.

We now investigate how a cross-constituent term can come about. We consider the usual way of combining a (non-relativistic) gas and radiation in the energy density and pressure:

$$\rho = mn + \frac{3}{2}nT + \alpha T^4, \quad (80)$$

$$p = nT + \frac{1}{3}\alpha T^4, \quad (81)$$
where $\alpha$ is constant. Taking as our fundamental thermodynamic variables $n$ and $s$ will result in $T = \frac{\partial p}{\partial s}$ which is a function of both. Therefore, the ideal gas contribution will generate a cross-constituent coupling and a measure of the interactions are the cross-constituent couplings defined as

$$C_{YX} \equiv \frac{\partial \ln \mu^Y}{\partial \ln n_X} = \frac{\mu^X_n}{\mu^Y_n} C_{XY}. \quad (82)$$

The $C_{YX}$ represents a key channel through which the two fluids see each other especially when the entrainment is zero.

We now consider equation (41) with the aim of finding out how cosmological magnetic fields evolve during the transit of matter-radiation equality epoch. Rewriting it in more suitable form (tensor form still) we have

$$\nabla_b F^{ab} - 8\pi \phi R^a - 8\pi \phi_0 R^a_b A^b = 4\pi (J^a_X + J^a_Y + j^a). \quad (83)$$

Now using $F_{ab}$, the fact that $R_{ii} = \ddot{a} a + \left[\dot{a} \frac{a}{\dot{a}}\right]^2$ (no sum on $i$) and $R = 6 \ddot{a} a a$ we can recast equation (83) as

$$\frac{1}{a^2} \partial \left\{ a^2 \vec{E} \right\} - \nabla \times \vec{B} - \frac{n \vec{A}}{\eta^2 a^2} = 4\pi (\vec{J}_X + \vec{J}_Y + \vec{j}) \quad (84)$$

in vector form where $\frac{n}{\eta^2}$ is equation (45).

$$\vec{J}_X + \vec{J}_Y + \vec{j} = \vec{J}_T \quad (85)$$

where $\vec{J}_T$ is the total current. Due to the overwhelming plasma effects to the total of flux currents, $\vec{J}_X + \vec{J}_Y$ or when

$$\vec{j} \gg \vec{J}_X + \vec{J}_Y \quad (86)$$

yields

$$\vec{J}_T \approx \vec{j} \quad (87)$$

This implies that equation (84) can be rewritten as

$$\frac{1}{a^2} \partial \left\{ a^2 \vec{E} \right\} - \nabla \times \vec{B} - \frac{n \vec{A}}{\eta^2 a^2} = 4\pi \vec{j} \quad (88)$$

in vector form.

We can see that equation (88) above is the same as equation (49). From this point onwards [from equation (49)], we can use the same analysis as we did during the RD epoch. We will obtain the same equations and hence we will have the same conclusions as in the analysis of the RD epoch. We now consider the MD epoch in the next subsection.
C. *MD* epoch

The modelling or analysis of magnetic fields evolution during the *MD* epoch is exactly the same as that during the *RD* epoch up to equation (43).

Then using $F_{ab}, R_i^i = \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2$ (no sum on $i$) and $R = 6\frac{\ddot{a}}{a}$ we can recast equation (23) as

$$\frac{1}{a^2} \frac{\partial (a^2 \vec{E})}{\partial \eta} - \nabla \times \vec{B} - \frac{n}{\eta^2 a^2} \vec{A} = 4\pi \vec{J_Y} + \vec{j}$$  \hspace{1cm} (89)

in vector form where $\frac{n}{\eta^2}$ is equation (45). In equation (89) we have $\vec{J_Y}$ which is the current due to the flux of fluid $Y$ representing the *MD* epoch as a fluid and $\vec{j}$ is the current due to plasma. Therefore, we have

$$\vec{J_Y} + \vec{j} = \vec{J_T}$$  \hspace{1cm} (90)

where $\vec{J_T}$ is the total current. Due to the overwhelming plasma effects to flux current, $\vec{J_Y}$ or when

$$\vec{j} \gg \vec{J_Y}$$  \hspace{1cm} (91)

yields

$$\vec{J_T} \approx \vec{j}$$  \hspace{1cm} (92)

This implies that equation (89) can be written as

$$\frac{1}{a^2} \frac{\partial (a^2 \vec{E})}{\partial \eta} - \nabla \times \vec{B} - \frac{n}{\eta^2 a^2} \vec{A} = 4\pi \vec{j}$$  \hspace{1cm} (93)

in vector form.

We can see that equation (93) above is the same as equation (49). From this point onwards [from equation (49)], we can use the same analysis as we did during the subsection of the *RD* epoch. We will obtain the same equations and hence, the same conclusions as in the analysis of the subsection of the *RD* epoch. We now consider large-scale superadiabatic amplification of magnetic fields in the next subsection.

D. Large-scale superadiabatic magnetic amplification

Considering the arguments (check modifications in reference [29] in [13] in [27]) after equation (7) in [13] up to section 3.2 in [13], large-scale (causally disconnected) magnetic fields evolve in line with the power-law equations (74) and (77) from the time they are well outside the Hubble radius during inflation or during the *RD* or matter-radiation equality transition or *MD* epoch until the time of their re-entry later in the *MD* epoch (or dust
Equations (74) and (77) are no longer valid once the magnetic fields are back inside the horizon. From then onwards, the ideal-MHD limit applies, the magnetic flux remains conserved and the magnetic field decays adiabatically \( (i.e., B \propto a^{-2}) \).

We will now consider the evolution of superhorizon-sized cosmological magnetic fields after inflation \((i.e., at the beginning of RD or matter-radiation equality or MD epoch)\). After evaluating the two integration constants on the RHS of equations (74) and (77), the equation recasts into the form below [28],

\[
B = \left[ B_0 - \eta_0(2a_0H_0B_0 + B'_0) \right] \left( \frac{a_0}{a} \right)^2 + \eta_0(2a_0H_0B_0 + B'_0) \left( \frac{a_0}{a} \right)^2 \left( \frac{\eta}{\eta_0} \right) \tag{94}
\]

where the relation \( H = \frac{a'}{a^2} \) was used for the Hubble parameter. The above equation (94) monitors the linear evolution of superhorizon-sized magnetic fields on spatially flat \( FLRW \) backgrounds. \( w = w(t) \) meaning that the barotropic index of the matter is not necessarily constant but it can vary with time. Hence, equation (94) applies continously throughout the lifetime of the Universe \[from the epoch of RD to MD or matter-radiation equality to MD or beginning of MD to later during the MD (or dust era)]\, provided the cosmological expansion is entirely smooth and the matter can always be treated as a single barotropic medium. Under this condition, equation (94) also monitors the magnetic evolution through the matter-radiation equality transition epoch.

\( w \) is believed to maintain constant value during prolonged periods in the lifetime of the Universe. For example, the matter-radiation equality epoch which is more prolonged than the inflation or reheating epochs. The cosmological scale factor and the conformal time are related by

\[
a = a_0 \left( \frac{\eta}{\eta_0} \right)^\frac{2}{1+3\omega} \tag{95}
\]

as long as \( w \) remains invariant where \( w = -\frac{1}{3} \) and the zero suffix indicates a given initial time. Using the above equation (95), it is easy to show that \( H = \frac{a'}{a^2} = \frac{2}{(1+3\omega)a\eta} \) and then recast equation (94) into

\[
B = -\left[ \left( \frac{4}{1+3\omega} - 1 \right) B_0 + \eta_0B'_0 \right] \left( \frac{a_0}{a} \right)^2 + \left( \frac{4B_0}{1+3\omega} + \eta_0B'_0 \right) \left( \frac{a_0}{a} \right)^\frac{3(1-\omega)}{2} \tag{96}
\]

This equation monitors the linear evolution of superhorizon-sized magnetic fields on spatially flat \( FLRW \) backgrounds filled with a single barotropic medium. The difference with equation (94) is that here the barotropic index of the matter has been treated as a constant. As a result, equation (96) does not apply constantly throughout the evolution of the Universe, but only to periods during which \( w = \text{constant} \neq -\frac{1}{3} \) \( (e.g., \text{matter-radiation equality epoch and the } RD \text{ epoch}) \). This means that equation (96) is a special case of equation (94).
Well outside the Hubble radius during either the \textit{RD} epoch or matter-radiation equality transition epoch or \textit{MD} epoch, large-scale magnetic fields on spatially flat \textit{FLRW} backgrounds obey the above equations (94) and (96) because they contain modes with decay rates slower than the adiabatic. These slowly-decaying magnetic modes depend on their associated coefficients. When the coefficients are of roughly the same order of magnitude, the slowly decaying modes quickly take over and dictate the subsequent evolution of the magnetic field. This means that the initial conditions at the beginning of the post-inflationary evolution of magnetic fields, the nature of the transitions and the beginning and end of the reheating epoch or \textit{RD} epoch if we are considering inflation-produced magnetic fields to be outside the horizon but near it during the \textit{RD} epoch or \textit{MD} epoch respectively are very important. For the matter-radiation equality epoch the conditions at the beginning of the post-inflationary evolution of magnetic fields, the nature of the transitions, the beginning and the end of the \textit{RD} epoch would determine the initial conditions for matter-radiation equality epoch and these too are very important and that is when inflation-produced magnetic fields are outside the Hubble horizon but near it during this transition epoch. We now analyse the role of the initial conditions in the following section.

\textbf{V. THE ROLE OF INITIAL CONDITIONS}

The initial conditions are decided by the field’s behaviour in the de Sitter phase and by the nature of the transitions (epoch in the case of matter-radiation equality transition epoch) to the reheating or \textit{RD} or \textit{MD} epochs and the moment when inflation-produced magnetic fields cross into the matter-radiation equality transition epoch from the \textit{RD} epoch or from the equality epoch to \textit{MD} epoch. We will discuss one typical and matching initial-condition scenario [for a detailed discussion please check ([13] and [32])]. The scenario we will consider in the first subsection is the first case. We will start with Scenario \textit{AI} then Scenario \textit{AII} in the second subsection.

\textbf{A. Scenario \textit{AI}}

In this Scenario, \( w \) undergoes an abrupt change from \( w^- \) before the transition to \( w^+ \) afterwards (with \( w^+ \neq w^- \))\textsuperscript{33}.

The usual inflationary magnetogenesis scenarios demand that the magnetic field decays adiabatically throughout the de Sitter era (\( i.e., B \propto a^{-2} \)). On the other hand, the usual non-conventional mechanisms of primordial magnetic generation amplify their magnetic fields superadiabatically during inflation (\( i.e., B \propto a^{-m} \) with \( 0 \leq m < 2 \))\textsuperscript{34,35}. Now let us assume that all along the de Sitter phase the magnetic field obeys the law

\[ B = B_0 \left( \frac{a_0}{a} \right)^m = B_0 \left( \frac{\eta}{\eta_0} \right)^m \]  \hspace{1cm} (97)
where $0 \leq m \leq 2$ and the zero suffix indicates the beginning of the exponential expansion ($w = -1$). Differentiating equation (97) with respect to the conformal time gives $B' = \frac{mB}{\eta}$, which implies that,
\begin{equation}
\eta_* B_*' = mB_*
\end{equation}

at the end of inflation proper. During reheating $w$ changes from $w_*^- = -1$ to $w_*^+ = 0$. Then throughout reheating superhorizon-sized magnetic fields evolve as
\begin{equation}
B = B_*^+ \left( \frac{a_*^+}{a} \right)^2
\end{equation}
with $a \geq a_*^+$. Due to the arguments between equations (17) and (18) in [13], constraint equation (98) translates into
\begin{equation}
\eta_*^+ B_*^+' = -mB_*^+
\end{equation}
The above sets the initial conditions for the evolution of the magnetic fields where we have equation (99). Therefore, magnetic fields drop as $B \propto a^{-2}$ and the magnetic fields decay adiabatically throughout reheating.

Let us consider magnetic evolution in the subsequent epoch of $RD$. Following equation (99) and keeping in mind that $a \propto \eta^2$ during reheating, we deduce that $B \propto \eta^{-4}$ throughout that period. Then,
\begin{equation}
\eta_*^- B_*^- = -4B_*^-
\end{equation}
just before the transition to the epoch of $RD$. During that time, $w$ changes from $w_*^- = 0$ to $w_*^+ = \frac{1}{3}$ and equation (96) reads
\begin{equation}
B = -(B_*^+ + \eta_*^+ B_*^+) \left( \frac{a_*^+}{a} \right)^2 + (2B_*^+ + \eta_*^+ B_*^+) \left( \frac{a_*^+}{a} \right)
\end{equation}
with $a \geq a_*^+$. Due to the arguments between equations (21) and (22) in [13], constraint equation (101) recasts into
\begin{equation}
\eta_*^+ B_*^+' = -4B_*^+
\end{equation}
and sets the initial conditions for the magnetic evolution in the $RD$ epoch. Substituting the above into the RHS of equation (102) results in
\begin{equation}
B = 3B_*^+ \left( \frac{a_*^+}{a} \right)^2 - 2B_*^+ \left( \frac{a_*^+}{a} \right)
\end{equation}
where $a \geq a_*^+$. As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., $B \propto a^{-1}$) all along the $RD$ epoch.
We now consider evolution of magnetic fields during the matter-radiation equality epoch. We don’t pick up from equation (102), instead we continue from equation (103) and assume that the evolution of magnetic fields during the RD epoch are monitored by equation (99). With equation (103) as the initial conditions, magnetic fields decay adiabatically all along the RD epoch.

We find that $\eta^* B^r_\ast = -2B^r_\ast$ prior to the equilibrium time, since $a \propto \eta$ when $w = \frac{1}{3}$. This sets the initial conditions for the magnetic evolution in the matter-radiation equality epoch. For $w = \frac{1}{9}$ (check [31] an upcoming paper for a detailed discussion on the chosen value of $w$), we have

$$B = -(2B^r_\ast + \eta^* B^r_\ast \left( \frac{a^-}{a} \right)^2 + (3B^r_\ast + \eta^* B^r_\ast) \left( \frac{a^-}{a} \right)^3$$

(105)

where $a \geq a^-$. Substituting the initial condition into the RHS of equation (105) results in

$$B = B^r_\ast \left( \frac{a^-}{a} \right)$$

(106)

where $a \geq a^-$. Therefore, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., $B \propto a^{-1}$) all along the transit of the matter-radiation equality epoch.

We now consider the evolution of magnetic fields during the MD epoch. Due to the arguments between equations (23) and (24) in [13] the constraint $\eta^* B^r_\ast = -2B^r_\ast$ recasts into

$$\eta^+ B^r_\ast = -2B^r_\ast$$

(107)

at the beginning of the MD epoch. During the time of matter-radiation equality epoch $w$ changes from $w^- = \frac{1}{3}$ to $w^+ = 0$ and equation (96) reads

$$B = -(3B^r_\ast + \eta^+ B^r_\ast \left( \frac{a^+}{a} \right)^2 + (4B^r_\ast + \eta^+ B^r_\ast) \left( \frac{a^+}{a} \right)^3$$

(108)

where $a \geq a^+$. Substituting equation (107) results in

$$B = -B^r_\ast \left( \frac{a^+}{a} \right)^2 + 2B^r_\ast \left( \frac{a^+}{a} \right)^{\frac{3}{2}}$$

(109)

where $a \geq a^+$. As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., $B \propto a^{-\frac{3}{2}}$) during the dust era as well though not all along up to the end of the dust era. Sometime later in the dust era inflation-produced magnetic fields exit the horizon because by then $\eta$ becomes $\eta \gg 1$. This implies that equations (74) and (77) will no longer be valid and hence magnetic fields exit the horizon and go back to adiabatic decay until today.
B. Scenario AII

This is just a continuation from Scenario AI. Prior to equilibrium of the matter-radiation equality epoch, we find \( \eta^* B^*_{s} = -B_{s} \) since \( a \propto \eta \) when \( w = \frac{1}{3} \) for the RD epoch. We are following equation (104). Inserting the condition above in equation (105) results in

\[
B = -B_{s} \left( \frac{a_{s}}{a} \right)^{2} + 2B_{s} \left( \frac{a_{s}}{a} \right) ^{2}\tag{110}
\]

where \( a \geq a_{s} \). As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., \( B \propto a^{-1} \)) all along the matter-radiation equality epoch.

Following equation (106) and keeping in mind that \( a \propto \eta^{\frac{3}{2}} \) during the equality epoch we deduce that \( B \propto \eta^{-\frac{3}{2}} \) throughout that period. Then,

\[
\eta^* B_{s} = -\frac{3}{2} B_{s}^+\tag{111}
\]

at the start of the MD epoch. Inserting equation (111) in equation (108) results in

\[
B = -\frac{3}{2} \left( \frac{a_{s}^+}{a} \right)^{2} + 5 \left( \frac{a_{s}^+}{a} \right)^{3/2}\tag{112}
\]

where \( a \geq a_{s}^+ \). As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., \( B \propto a^{-\frac{3}{2}} \)) during the dust era as well.

We now consider evolution of magnetic fields during the MD epoch again. We use the approach in [13]. Due to the arguments between equations (23) and (24) in [13] the constraint \( \eta^* B_{s} = -B_{s} \) recasts into

\[
\eta^* B_{s} = -B_{s}^+\tag{113}
\]

at the beginning of the MD epoch. During the time of the matter-radiation equality epoch \( w \) changes from \( w_{s} = \frac{1}{3} \) to \( w_{s}^+ = 0 \) and equation (96) reads as equation (108). Substituting equation (113) in equation (108) results in

\[
B = -2B_{s}^+ \left( \frac{a_{s}^+}{a} \right)^{2} + 3B_{s}^+ \left( \frac{a_{s}^+}{a} \right)^{3/2}\tag{114}
\]

where \( a \geq a_{s}^+ \). As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., \( B \propto a^{-\frac{3}{2}} \)) during the dust era as well.

Now assuming adiabatic decay during the matter-radiation equality epoch, we will deduce that \( B \propto \eta^{-3} \) throughout that period. Then,

\[
\eta^* B_{s} = -3B_{s}^+\tag{115}
\]
at the beginning of the $MD$ epoch. Inserting equation (115) in equation (108) results in

$$B = B^+_s \left( \frac{a^+_s}{a} \right)^{\frac{3}{2}}$$  \hspace{1cm} (116)$$

where $a \geq a^+_s$. As a result, superhorizon-sized magnetic fields are superadiabatically amplified (i.e., $B \propto a^{-\frac{3}{2}}$) during the dust era as well again.

VI. SUMMARY OF FINDINGS DURING THE EPOCHS OF $RD$, MATTER-RADIATION EQUALITY AND $MD$

After examining equations (94) and (96) we notice that the first of the two magnetic modes on the RHS decay adiabatically. However, the rate of the second mode is not a priori fixed but depends on the equation of state ($EoS$) of the cosmic medium. The relation between the cosmological scale factor and the conformal time is determined by the latter. In particular, as long as $w=$constant$> -\frac{1}{3}$ the second mode on the RHS of equation (96) decays at a rate slower than the adiabatic. The same behaviour can also be seen in equation (94). Hence, when dealing with conventional matter, superhorizon-sized magnetic fields on spatially flat $FLRW$ backgrounds are superadiabatically amplified provided the initial conditions allow the second modes in equations (94) and (96) to survive and dominate. The initial conditions of the post-inflationary magnetic evolution are determined by the field’s behaviour at the beginning of the post-inflationary Universe and the nature of the

![Figure 1](image-url)

**FIG. 1.** This figure shows the typical behaviour of magnetic fields during each epoch of $RD$ or matter-radiation equality or $MD$. $B$ is the magnetic field while $\eta$ is the conformal time on the figure. The continous line denoted by $B_{k0}$ is for superadiabatic amplification while the discontinous line denoted by $B_k$ is for adiabatic decay. Superadiabatic amplification may mean increase in strength of the magnetic field or slower magnetic decay rates than the standard magnetic decay rate; in cosmology the latter is usually the case and the figure depicts that...
transitions to the reheating epoch or RD epoch or the MD epoch. Some of the initial condition scenarios are again discussed in detail in [13].

Therefore, depending on the initial conditions, conventional large-scale magnetic fields can be superadiabatically amplified throughout the post-inflationary evolution of a flat FLRW Universe (from the epoch of RD to MD or matter-radiation to MD or MD to later during MD epoch or dust era when magnetic fields cross the horizon and go back to adiabatic decay until today). This amplification occurs at the beginning of the RD or matter-radiation equality or MD epoch until the second horizon crossing. Once inside the horizon together with inflation-produced magnetic fields that were well outside the Hubble radius during the de Sitter phase, they decay adiabatically leading to the magnitude of $10^{-33}G$ or more for cosmological magnetic fields instead of $10^{-53}G$ today due to adiabatic decay throughout the post-inflationary evolution of the Universe. This shows that by appealing to causality, one can increase the final strength of conventional inflationary magnetic fields by roughly 20 or more orders of magnitude.

Similar equations to equations (73) and (75) were found in [29] and it was found in the mentioned article that conventional large-scale magnetic fields are superadiabatically amplified throughout the post-inflationary Universe.

VII. DISCUSSIONS AND CONCLUSIONS

We have examined (i) the single-fluid and multi-fluid approximations and used them to derive the modified Maxwell tensors, (ii) the generalised cosmological Ohm’s law valid on cosmological scales and used it in both standard and modified electromagnetism, (iii) the terms $RA^2$, $R_{ab}A^aA^b$ and $RA^2 + R_{ab}A^aA^b$ and their significance in the evolution of cosmological magnetic fields in the post-inflationary Universe, (iv) the post-inflationary evolution of cosmological magnetic fields using the generalised cosmological Ohm’s law, the flux currents of both radiation and matter fluids and the modified Maxwell tensors derived from both single-fluid and multi-fluid approaches. The evolution of the inflation-produced magnetic fields is examined during the RD, matter-radiation equality epoch and the MD epoch with initial conditions starting during the inflationary era. We find that cosmological magnetic fields are superadiabatically amplified throughout the post-inflationary Universe (i.e., from the epoch of RD to MD or from matter-radiation equality to MD or from the beginning of MD itself to later during the epoch of MD when the magnetic fields exit the horizon going back to adiabatic decay) when we use the generalised cosmological Ohm’s law and both standard and modified Maxwell field equations. This confirms that superadiabatic amplification of large-scale magnetic fields generated during inflation is possible in flat FLRW spacetime (we were working in flat FLRW spacetime). This is so in both cases where we used modified Maxwell’s equations, fluid flux currents and the generalised cosmological Ohm’s law and the case where we used standard Maxwell’s equations, fluid flux currents and
again the generalised cosmological Ohm’s law. We have also shown that the single-fluid and multi-fluid approaches are resourceful tools for investigating the evolution of cosmological magnetic fields throughout the post-inflationary Universe especially during the relatively explored epoch of the matter-radiation equality transit where we employed the multi-fluid approach.

Adiabatic decay of magnetic fields on flat FLRW spacetime translates into magnetic strengths below $10^{-50}G$ today [5] . As far as we know currently, such fields can never seed the galactic dynamo as mentioned earlier or can never affect the dynamics of our Universe. Therefore, our goal in this paper was to show that superadiabatic amplification is possible on flat FLRW spacetime which translates into magnetic strengths above $10^{-50}$ today.

The consequences of the modified Maxwell’s equations were investigated in [14]. For reasonable parameters it was shown that modification does not affect existing experiments and observations. Nevertheless, it is argued that, the field equations with a curvature-coupled term can be testable in astrophysical environments where the mass density is high or the gravity of electromagnetic radiation plays a dominant role in dynamics, e.g., the interior of neutron stars and the early Universe.

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The electrical properties of a medium are reflected in the generalised cosmological Ohm’s law valid on cosmological scales only not superhorizon scales. Consequently, to eliminate a superhorizon-sized electric field, requires the presence of currents coherent on the same scales. Given that causality forbids the existence of such currents, the electric field will not vanish, unless it is fragmented into individually causally connected parts. Nevertheless, even if we assume that the electric field has somehow been fragmented and eliminated by the local currents valid on cosmological scales, its superhorizon-sized magnetic counterpart is unaware to speak of that for as long as it remains causally disconnected.

Calculating the integration constants of equation (75) gives 

$$C_1 = [B_0 - \eta_0(2a_0H_0B_0 + B_0')]a_0^2$$

and 

$$C_2 = \frac{n(2a_0H_0B_0 + B_0')}k\eta_0a_0^2$$

. Given that \(k\eta_0 \ll 1\) on super-Hubble scales, we deduce that 

$$C_2 \gg C_1$$

(unless \(2a_0H_0B_0 + B_0' = 0\)) . All these explain why one should not a priori discard the second mode of equation (75) before evaluating the integration constants first.

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. Given that \(k\eta_0 \ll 1\) on super-Hubble scales, we deduce that 

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(unless \(2a_0H_0B_0 + B_0' = 0\)) . All these explain why one should not a priori discard the second mode of equation (75) before evaluating the integration constants first.

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