Point pattern analysis and classification on compact two-point homogeneous spaces evolving time

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Abstract
This paper introduces a new modeling framework for the statistical analysis of point patterns on a manifold $\mathbb{M}_d$, defined by a connected and compact two-point homogeneous space, including the special case of the sphere. The presented approach is based on temporal Cox processes driven by a $L^2(\mathbb{M}_d)$-valued log-intensity. Different aggregation schemes on the manifold of the spatiotemporal point-referenced data are implemented in terms of the time-varying discrete Jacobi polynomial transform of the log-risk process. The $n$-dimensional microscale point pattern evolution in time at different manifold spatial scales is then characterized from such a transform. The simulation study undertaken illustrates the construction of spherical point process models displaying aggregation at low Legendre polynomial transform frequencies (large scale), while regularity is observed at high frequencies (small scale). $K$-function analysis supports these results under temporal short, intermediate and long range dependence of the log-risk process.

Keywords Connected and compact two-point homogeneous spaces · Cox processes · Discrete Jacobi polynomial transform · $K$-function · $\mathbb{M}_d$-supported random fields · Point pattern analysis · Statistical distances

1 Introduction
Several statistical approaches arise for processing spatial areally-aggregated or/and misalignment data in several environmental disciplines requiring, for example, the application of Geophysical, Ecological and Epidemiological models. The approach presented in this paper goes beyond the Euclidean setting, analyzing count models on a manifold defined by a connected and compact two-point homogeneous space. Under spatial isotropy we consider weighted aggregation schemes adapted to the geometry of the manifold, in terms of the elements of the Jacobi polynomial basis [see Theorems 4 and 5 in Ma and Malysarenko (2020), and Marinucci and Peccati (2011), for the special case of the sphere]. The application of harmonic analysis in this more general context leads to the characterization of the evolution of point patterns at different spatial scales in the manifold.

Markov random field (MRF) models, particularly, Conditional Autoregressive (CAR) models have been widely applied to represent the dynamics of the log-intensity process, interpreted as a log-risk process in the context of double stochastic Poisson processes, also named Cox processes (see Besag et al. 1991). In disease mapping, areal disease counts have been usually analyzed under this Markovian log-risk process framework (see, e.g., Ugarte et al. 2009, 2010, 2012). Particularly, different parametric, semiparametric and nonparametric statistical approaches have been adopted in the estimation of deterministic and random intensities (see Baddeley et al. 2006; Diggle et al. 2010; Goncalves and Gamerman 2018; Guan 2006), and the references therein). In point pattern analysis, special attention has been paid to functional summary statistics like the nearest neighbour, empty space, and $K$ functions (see, e.g., Diggle 2013; Illian et al. 2008). Recently,
LASSO estimation based on spherical autoregressive processes has been proposed in Caponera et al. (2021), beyond the Euclidean setting. Alternatively, in the functional data analysis (FDA) framework, conditional autoregressive Hilbertian process (CARH process) models were considered by Cugliari (2011, 2013) and Guillan (2002), developing projection estimation methods for prediction. In Ruiz-Medina et al. (2014), an Autoregressive Hilbertian process (ARH(1) process) framework was adopted to represent the dynamics of the spatiotemporal log-risk process. This framework has also been adopted in Torres-Signes et al. (2021), for COVID-19 mortality prediction by applying multivariate curve regression and machine learning. As an alternative, to analyze the spatial interaction between log-risk curves at different regions, in Frı´as et al. (2022), a Spatial Autoregressive Hilbertian process (SARH(1) process) based modeling was applied. Recently, wavelet-based projection methods are implemented in Torres-Signes et al. (2021), to developing an infinite-dimensional spatial multi-resolution point pattern analysis, based on spatiotemporal Log-Gaussian Cox processes in the Euclidean setting. The present paper goes beyond this Euclidean setting. At each spatial resolution level on the manifold, defined in terms of time-varying discrete Jacobi transform, temporal point pattern analysis is achieved from the latent random intensity process in time, and its higher order moments. In the particular case of the sphere, suitable log-intensity models can be found in Caponera and Marinucci (2021), where spherical functional autoregressive (SPHAR) processes are introduced, and their asymptotically analysis is derived. Additionally, spherical functional autoregressive-moving average (SPHARMA) processes are considered in Caponera (2021), extending SPHAR processes, for suitable approximation of isotropic and stationary sphere-cross-time random fields. Here, functional spectral analysis tools are applied, and Wold-like decomposition results are derived.

A growing interest on spherical point processes, and its functional summary statistics is observed in recent contributions (see, e.g., Moller and Rubak 2016; Robeson et al. 2014). In this paper, our interest relies on point patterns analysis in compact two-point homogeneous spaces evolving time. The framework of temporal Cox processes driven by log-intensities, evaluated in the space $L^2(\mathbb{M}_d, dv)$ of square integrable functions on a compact two-point homogeneous space $\mathbb{M}_d$ is then considered. Particularly, $\mathbb{M}_d$ is a manifold with $d$ denoting its topological dimension, and $dv$ denotes its measure, induced by the probabilistic invariant measure on the connected component of the group of isometries of $\mathbb{M}_d$. The associated infinite-dimensional $n$-order product density is identified with the infinite product of temporal $n$-order product densities. A spatial multi-scale analysis of the point process evolution is achieved from these temporal $n$-order product densities, and the usual functional summary statistics constructed from them.

The interest of the extended family of Cox processes analyzed here relies on well-known examples of compact two-point homogeneous spaces like the sphere $\mathbb{S}_d \subset \mathbb{R}^{d+1}$, and the projective spaces over different algebras (see Sect. 2 in Ma and Malyarenko (2020), for more details). Recent advances on modeling, analysis and simulation of Gaussian spherical isotropic random fields, including random fields obeying a fractional stochastic partial differential equation on the sphere, can be exploited in our more general $L^2(\mathbb{M}_d, dv)$-valued Gaussian log-risk process framework (see Alegrı´a and Cuevas-Pacheco 2020; Anh et al. 2018; Emery and Porcu 2019; Cleanthous et al. 2020, 2021; Leonenko et al. 2021), among others. Particularly, Anh et al. (2018), and Leonenko et al. (2021), focalize on Cosmic Microwave Background (CMB) evolution modeling and data analysis. The approach presented here can contribute to this modeling framework to approximate the distribution of CMB hot and cold spots.

In point pattern analysis on a $d$-dimensional manifold $\mathbb{M}_d$, embedded into $\mathbb{R}^{d+1}$, one can apply the isometric identification of $(\mathbb{S}_d, d_{\mathbb{S}_d})$ with $(\mathbb{M}_d, d_{\mathbb{M}_d})$ via the identity $d_{\mathbb{S}_d}(x_1, x_2) = \arccos(\langle x_1^T x_2 \rangle)$ for $x_1, x_2 \in \mathbb{S}_d$.

This geodesic distance $d_{\mathbb{M}_d}$ is involved in the definition of functional summary statistics characterizing the aggregation, regularity or inhibition of the point pattern. In particular, point pattern classification is achieved in terms of this geodesic distance. This paper presents a new manifold spatial-scale-dependent point pattern classification analysis over time, via time-varying discrete Jacobi transform, achieved in terms of different statistical distances. $K$ function analysis is also performed describing the cumulative counting properties of pair correlation function in time through different spatial scales. In the simulation study undertaken on the sphere, temporal short, intermediate and long range dependence models are tested, the statistical distance based methods implemented reflect a departure from complete randomness of the point pattern at coarser (large) scales in the manifold (low frequencies of the time-varying discrete Legendre polynomial transform). While their small scale (high-frequency) behavior shows regularity, $K$-function based analysis supports the same classification results, independently of the underlying dependence range of the log-intensity. At coarser spatial scales, stronger departure from point pattern regularity is observed when long-range dependence log-intensity models are tested. As mentioned above, the approach presented in this paper then provides a framework to detect non-uniformity of the spherical distribution of CMB hot and
cold spots, since these deviations from uniformity are usually geographically described in terms of clustering, girdling or ring structures (see, e.g., Khan and Saha 2021; Sadr and Movahed 2021).

The outline of the paper is the following. Preliminaries on connected and compact two-point homogeneous spaces are given in Sect. 2. The new class of Cox processes analyzed in a metric space framework is introduced in Sect. 3. The proposed statistical distance based classification methodology through spatial scales, involving n-order product density, is formulated in Sect. 4. K function is also explicitly computed from the time-varying discrete Jacobi transform of the second-order structure of the $L^2(M_d)$-valued temporal log-intensity. The results of the simulation study undertaken are displayed in Sect. 5. Some final remarks and discussion can be found in Sect. 6 to ending the paper.

2 Preliminaries

Let $\{X_t(\cdot), \ t \in T \subseteq \mathbb{R}\}$ be an infinite-dimensional random process such that, for each $t \in T \subseteq \mathbb{R}$, almost surely $\log(X_t) \in L^2(M_d)$, and $E[\log(X_t)] = 0$, with $\log(X_t)$ having characteristic functional

$$f_{\log(X_t)}(h) = \int_{L^2(M_d)} \exp\left(i(h \cdot \log(X_t))|L^2(M_d)|\right) \mu_{\log(X_t)}(dh \log(X_t))$$

$$= \exp\left(-\frac{\langle R_0(h), h \rangle_{L^2(M_d)}}{2}\right), \ h \in L^2(M_d),$$

(1)

where $R_0 = E[\log(X_t) \otimes \log(X_t)] \in L^1(L^2(M_d))$ denotes the covariance operator of $\log(X_t)$, and $L^1(L^2(M_d))$ is the space of trace or nuclear operators on $L^2(M_d)$. Here, $\mu_{\log(X_t)}$ is the induced Gaussian measure by $\log(X_t)$ on $(L^2(M_d), B(L^2(M_d)))$, with $B(L^2(M_d))$ being the $\sigma$-algebra generated by all cylindrical subsets of $L^2(M_d)$. In the subsequent development, we will also assume that, for any $t, s \in T$,

$$E[\log(X_t)(z) \log(X_s)(y)] = r_{t-s}(d_{M_d}(z, y))$$

$$= \overline{r}(d_{M_d}(z, y), t-s), \ z, y \in M_d,$$

(2)

i.e., stationarity in time and isotropy over $M_d$ in the weak sense are assumed. Note that the covariance operator $R_{t-s}$ with kernel $r_{t-s}(\cdot, \cdot)$ is a nuclear operator, and its kernel $r_{t-s}(d_{M_d}(z, y))$ is assumed to be continuous.

For the special case $r_{t-s}(\cdot, \cdot) = r_s(- t, \cdot)$, the following series expansion is obtained from Theorems 4 and 5 in Ma and Malyarenko (2020):

$$\log(X_t)(z) = \sum_{n=0}^{\infty} V_n(t)P_n^{(x, \beta)}(\cos(d_{M_d}(z, U))), \ z \in M_d, \ t \in \mathbb{R},$$

(3)

where $P_n^{(x, \beta)}$ is a Jacobi polynomial of degree $n$ depending on parameter vector $(x, \beta)$ (see, e.g., Andrews et al. 1999).

Here, $\{V_n(t), n \in \mathbb{N}_0\}$ is a sequence of independent stationary random processes on $T \subseteq \mathbb{R}$, satisfying $E[V_n(t)] = 0$ and $E[V_n(t_1)V_n(t_2)] = a_n^2 b_n(t_1 - t_2)$, $n \in \mathbb{N}_0$. The random variable $U$ is uniformly distributed on $M_d$, and is independent of $\{V_n(t), n \in \mathbb{N}_0\}$, and $\sum_{n=0}^{\infty} b_n(0)P_n^{(x, \beta)}(1)$ converges. Also,

$$\text{cov}\left(V_n(t)P_n^{(x, \beta)}(\cos(d_{M_d}(z, U))), V_n(t)P_n^{(x, \beta)}(\cos(d_{M_d}(z, U)))\right) = 0,$$

for $m \neq n$, $z \in M_d$, and $t \in T$.

3 Cox processes family

Let now consider the measure $dv(z)$ induced on the homogeneous space $M_d = G/K$, by the probabilistic invariant measure on $G$, with $G$ being the connected component of the group of isometries of $M_d$, and $K$ be the stationary subgroup of a fixed point $o \in M_d$. As before, $H = L^2(M_d, dv(z))$.

Our spatiotemporal count data model $\{N_t(\cdot), t \in T \subseteq \mathbb{R}\}$ characterizes the behavior of the temporal family $Y = \{Y_t, t \in T \subseteq \mathbb{R}\}$ of finite point sets of $M_d$, randomly arising at different times in the interval family $\{[0, t], t \in T\}$. Specifically, for every $t \in T$ and any Borel set $A \subseteq M_d$, $N_t(A)$ denotes the number of points in the pattern $Y_t$, falling in the region $A \subseteq M_d$, randomly arising in the interval $[0, t]$. Here, we consider the $\sigma$-algebra $\mathcal{F}$ generated by the events $\{N_t(A) = n\}$ indicating that $n$ points in $Y_t$ are falling in a region $A \subseteq M_d$, at some specific times in $[0, t]$, for any Borel set $A \subseteq M_d$, interval $[0, t]$, and integer $n \in \mathbb{N}$.

Assume that $\{N_t(\cdot), t \in T\}$ defines a spatiotemporal Cox process with random log-intensity $\log(X_t)$, whose infinite-dimensional micro-scale behavior of the random point pattern is then characterized by its $n$-order product density $p_{n_1, \ldots, n_k}^{(x)}(z_1, \ldots, z_n)$, with

$$p_{n_1, \ldots, n_k}^{(x)}(z_1, \ldots, z_n) dv^{(m)}(z_1, \ldots, z_n) dt_1, \ldots, dt_n$$

indicating the probability that $Y_t$ has a point in each one of $n$ infinitesimally small regions on $M_d$ around $z_1, \ldots, z_n$, of surface measure $dv(z_1) \cdots dv(z_n)$, over the infinitesimal time intervals around $t_1, \ldots, t_n$, of length $dt_1, \ldots, dt_n$. Under
the modeling framework introduced in Sect. 2, from Eq. (3), for any $t_1, \ldots, t_n \in \mathbb{R}$, one can compute $\rho_{(n)}^{(n)}(z_1, \ldots, z_n)$ as follows:

$$\rho_{(n)}^{(n)}(z_1, \ldots, z_n) = E \left[ \prod_{i=1}^n \exp(X_i(z_i)) \right]$$

$$= E \left[ \exp \left( \sum_{i=1}^n X_i(z_i) \right) \right]$$

$$= [\rho]^n \exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{q=0}^{n} b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right).$$

(4)

for every $z_i \in \mathbb{M}_d, i = 1, \ldots, n$. In particular, for any $t \in T$, and, for any $t_1, t_2 \in T$, the intensity function $\rho = \rho_0 = \rho^{(1)}(t)$, and the pair correlation function $g_{(2)}(\cos(\theta_{\alpha q}(z_1, z_2)))$, $z_1, z_2 \in \mathbb{M}_d$, respectively admit the following expressions:

$$\rho = \rho_0(z) = \exp \left( \frac{1}{2} \sum_{q=0}^{n} b_q(0) P^{(x, \beta)}(1) \right) = \prod_{q=1}^{\infty} q^q, \forall z \in \mathbb{M}_d,$n

$$g_{(2)}(\cos(\theta_{\alpha q}(z_1, z_2))) = \frac{\rho_{(2)}^{(2)}(\cos(\theta_{\alpha q}(z_1, z_2)))}{\rho^2}$$

(5)

$$= \exp \left( \sum_{n=0}^{\infty} b_n(t_1 - t_2) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_1, z_2))) \right).$$

(6)

In our subsequent spatial multi-scale temporal point pattern analysis on connected and compact two-point homogeneous spaces, we apply the identification of the $n$-order product density $\rho_{(n)}^{(n)}(z_1, \ldots, z_n)$ in equation (4) with the infinite product of temporal $n$-order product densities at different spatial resolution scales, defined from the discrete Jacobi transform, i.e.,

$$\rho_{(n)}^{(n)}(z_1, \ldots, z_n) = [\rho]^n$$

$$\exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{q=0}^{n} b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right)$$

$$= \prod_{q=0}^{\infty} \rho_q^n$$

$$\exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right).$$

(7)

Thus, for each $q \geq 1$,

$$\rho_q^n(t_1, \ldots, t_n, z_1, \ldots, z_n) = [\rho_q]^n$$

$$\times \exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right),$$

(8)

where the Fourier coefficients $\{b_q(t_i - t_j), i, j = 1, \ldots, n\}$ characterize the behavior of $n$-order product density at each spatial scale $q \geq 0$ (see, e.g., Theorem 1.2.1 in Da Prato and Zabczyk (2002), where infinite-dimensional Gaussian measures are identified with the infinite product of one-dimensional measures).

4 Point patterns classification through spherical scales

Point pattern classification is performed in this section by considering different statistical distances between $n$-order product densities at different manifold spatial scales. $K$-function is computed in terms of the time-varying discrete Jacobi transform of the second-order structure of the log-intensity or log-risk process.

We first consider the following Ibragimov contrast function, also known as Shannon-entropy-based statistical distance, to measure the departure from complete randomness, by comparing $n$-order product density (8) with the $n$-order product density of homogeneous Poisson process on $\mathbb{M}_d$ evolving time (see Sect. 5 for its implementation):

$$D_q^n(\rho_q^n, \rho_q^n) = \int_{T \times \mathbb{M}_d} \rho_q^n(t_1, \ldots, t_n, z_1, \ldots, z_n)$$

$$\times \ln \left( \frac{\rho_q^n(t_1, \ldots, t_n, z_1, \ldots, z_n)}{[\rho_q]^n} \right) dt_1 \cdots dt_n d\nu(z_1), \cdots, d\nu(z_n)$$

$$= \int_{T \times \mathbb{M}_d} \rho_q^n \exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right)$$

$$\times \exp \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_q(t_i - t_j) P^{(x, \beta)}(\cos(\theta_{\alpha q}(z_i, z_j))) \right)$$

$$\times dt_1 \cdots dt_n d\nu(z_1), \cdots, d\nu(z_n).$$

(9)

Note that negative values of $D_q^n(\rho_q^n, \rho_q^n)$ mean repulsiveness or inhibition at scale $q$, while positive values mean aggregation, and null values correspond to the regular (complete randomness) case at such a scale $q$, in the $n$-order moment sense. Ibragimov contrast function corresponds to the limiting case of a more general family of functions related to Rényi-entropy based statistical distances. Specifically, one can consider for each $q \geq 0$,
Legendre polynomial are plotted for orders $l = 1, 2, 3, 4$.

\[ D_q^h(\rho_q^{(n)}(t_1, \ldots, t_n, z_1, \ldots, z_n)) = \frac{1}{h-1} \ln \left( \int_{T^n \times \Sigma^n} \rho_q^{(n)}(t_1, \ldots, t_n, z_1, \ldots, z_n) \prod_{k=1}^{n-1} \rho_q^{(h)}(t_k) dt_1 \cdots dt_n dv(z_1) \cdots dv(z_n) \right) \]

\[ \times \left[ \frac{\rho_q^{(n)}(t_1, \ldots, t_n, z_1, \ldots, z_n)}{\rho_q^{(h)}(t_1)} \right]^{h-1} dv(z_1) \cdots dv(z_n) \]

\[ = \frac{1}{h-1} \ln \left( \int_{T^n \times \Sigma^n} \rho_q^{(n)} \exp \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(t_i - t_j) P^{(2)}_{ij}(\cos(d_{ij}(z_i, z_j))) \right) \right) \times dt_1 \cdots dt_n dv(z_1) \cdots dv(z_n). \]

where the continuous positive shape parameter $h$ characterizes the $L^h(\mathbb{T}^n \times \Sigma^n, \rho_q^{(n)}(t_1, \ldots, t_n, z_1, \ldots, z_n), dt_1 \cdots dt_n dv(z_1) \cdots dv(z_n)$ space, whose norm is involved in measuring the aggregation or inhibition level of the point pattern at the logarithmic scale.

### 4.1 Functional Summary statistics

In the Log-Gaussian Cox process framework, the most interesting case in equation (9) corresponds to $n = 2$, where one can alternatively compute the cumulative distribution function associated with the two-order product density, in terms of the pair correlation function (6), under stationarity in time and isotropy in space. That is, we consider the following functional summary statistics $K_t(\theta)$ under the assumption that $Y$ is fully observed:

\[ K_t(\theta) = \frac{1}{\rho^2 |T| \nu(\Sigma^2)} \mathbb{E} \left[ \sum_{(x,y) \in Y, (x,z) \in Y} 1_{d_{xy} \leq t} \otimes 1_{|x-u| \leq t} \right] \]

\[ = \frac{1}{\rho^2 |T| \nu(\Sigma^2)} \int_{T^2 \times \Sigma^2} 1_{d_{xy} \leq t}(y, z) 1_{|x-u| \leq t}(x, u) P^{(2)}_{xy}(y, z) dv(y) dv(z) ds du \]

\[ = \frac{1}{|T| \nu(\Sigma^2)} \int_{T^2 \times \Sigma^2} 1_{d_{xy} \leq t}(y, z) 1_{|x-u| \leq t}(x, u) g_{x-u}(\cos(d_{xy}, y, z)) dv(y) dv(z) ds du \]

\[ = \frac{1}{|T| \nu(\Sigma^2)} \int_{T^2 \times \Sigma^2} 1_{d_{xy} \leq t}(y, z) 1_{|x-u| \leq t}(x, u) \exp \left( \sum_{q=0}^{\infty} b_q (s-u) P^{(2)}_{xy}(\cos(d_{xy}, y, z)) \right) dv(y) dv(z) ds du. \]

Fig. 1 Legendre polynomial are plotted for orders $l = 1, 2, 3, 4$

Fig. 2 Log-intensity values on sphere for times $t = 10, 20, 30, 60, 70, 80$
Specifically, $K_q(t, \theta)$ function provides the mean number of further points within geodesic distance $\theta$ occurring in a temporal interval of length less or equal than $t$. For each spatial resolution $q$, we consider, for $0 \leq \theta \leq \pi$, and $t > 0$, the $K_q(t, \theta)$ function given by

$$K_q(t, \theta) = \frac{1}{\mu(T)} \int_{T^2 \times M^2} 1_{(d_{M^2}(x,y) \leq \theta)}(y,z) 1_{(|x-u| \leq t)}(x,u) \exp\left(\beta_q(s-u)r_{M^2}(\cos(d_{M^2}(y,z)))\right) dsdv_{\mu}(z)dsdu. \quad (12)$$

At different spatial resolution levels $q$, point pattern classification is achieved by comparing function $K_q(t, \theta)$ with $K_{Pois}(t, \theta) = 2\pi(1 - \cos(\theta))$. The last one corresponds to complete randomness. Hence, one can respectively interpret aggregation and inhibition at spatial scale $q$, when $K_q(t, \theta) - K_{Pois}(t, \theta) > 0$, and $K_q(t, \theta) - K_{Pois}(t, \theta) < 0$ almost surely in $t$ and $\theta$. The pointwise null values of this difference function $K_q(t, \theta) - K_{Pois}(t, \theta)$ correspond to complete randomness. Specifically, one can compare $K_q(t, \theta)$ and $K_{Pois}(t, \theta)$ functions in terms of the $L^p$ norm of the quotient $K_q(t, \theta)/K_{Pois}(t, \theta)$ at logarithmic scale. On the other hand, pointwise information of the difference $K_q(t, \theta) - K_{Pois}(t, \theta)$, for small and large temporal $t$ and angular $\theta$ distance arguments, respectively reflects the small-scale and large-scale behavior of $K$-function. These behaviors are affected by the dependence.

Fig. 3 Statistical distance based on Ibragimov contrast function (9) between the two-order product densities of the generated spherical Log-Gaussian Cox process ($\theta = 1$), and of spherical homogeneous Poisson process over the interval $[0, 10]$, considering Legendre scales $q = 0, 1, 2, 3, 4, 5$ (top-left), $q = 0, 1, 2, 3, \ldots, 20$ (top-right), and $q = 0, 1, 2, 3, \ldots, 30$ (bottom), reflected at the horizontal axis.
range of the log-intensity process at coarser Jacobi spatial scales. While they are almost invariant at higher resolution levels of the time-varying discrete Jacobi transform, as given in Section 5 (see Figs. 7, 8 and 9).

For each \( t > 0 \), and \( \theta \in [0, \pi] \), the nearest neighbour function \( G_t(\theta) \) indicates the mean number of points at a specific temporal \( t \) and angular \( \theta \) distances to the pattern. The computation of this function requires the consideration of the intensity function \( \rho \) identified with the infinite product of uniform intensity functions \( \rho_q \) at different spatial resolution scales \( q \) in (5), which are constants under isotropy in space and stationarity in time, i.e.,

\[
G_t(\theta) = \frac{1}{\rho_t |\mathcal{V}(\mathcal{M}_d)|} \int_{(y,x)\in\mathcal{V}} 1_{\{\theta \leq \psi(y,x) \leq \theta + \delta\}} 1_{\{|u| \leq \tau\}} 1_{\{|r-s| \leq \tau\}} \rho_u(y,x) dy ds.
\]

Its empirical counterpart is given, for \( t \in \mathcal{T} \), and \( \theta \in [0, \pi] \), by

\[
\hat{G}_t(\theta) = \frac{1}{N(\mathcal{T} \times \mathcal{M}_d)} \sum_{(y,x)\in\mathcal{V}} 1_{\{\theta \leq \psi(y,x) \leq \theta + \delta\}} 1_{\{|u| \leq \tau\}} 1_{\{|r-s| \leq \tau\}}.
\]

provided that \( N(\mathcal{T} \times \mathcal{M}_d) = N_T(\mathcal{M}_d) > 0 \). Given the stationarity and isotropy of the model considered, the null values of \( D_q^{(n)}(\rho_q^{(n)}, \rho_q^{(n)}) \) for \( n = 1 \), at every scale \( q \geq 0 \), in equation (9), excludes this functional summary statistics, \( G_t \) for classification purposes. The simulation study undertaken in the next section illustrates the global characterization of the point pattern through the two-order product densities at different spatial scales, in terms of statistical distances \( D_q^{(2)}(\rho_q^{(2)}, \rho_q^{(2)}) \), and \( K \)-function analysis from equations in (9), (10) and (11), respectively. This assertion is validated by computing \( D_q^{(3)}(\rho_q^{(3)}, \rho_q^{(3)}) \), \( q \geq 0 \), involving third-order product densities.

5 Simulation

In this simulation study, we restrict our attention to the case of a Log-Gaussian Cox process on \( \mathbb{S}_2 \) over the temporal interval \([0, 10]\). For this special case, we work with the time-varying discrete Legendre transform, providing spherical large and small scale information about the log-intensity and its second-order structure by projection into the Legendre polynomials \( \{P_l\} \) (see Fig. 1 for \( l = 1, 2, 3, 4 \)).

The following parametric model is considered for the temporal covariance function of the Fourier random coefficients \( \{V_l\} \) of the log-intensity \( \log(X_t) \) in equation (3), with respect to the Legendre polynomial basis (see, e.g., Caponera and Marinucci 2021; Marinucci et al. 2020):

\[
B_l(t, s) = E[V_l(t)V_l(s)] = (1/2) \left( (l + 1)^{-2 - |r-s|} / (1 + (t-s)^2)^{\beta(l)} \right)
\]

\[
\beta(l) = ((8/10)(l+1))/((l+1)^2 + 1)^{1/2}, \quad l \geq 0, \; t, s \in \mathcal{T}.
\]

Thus, as given in Theorems 4 in Ma and Malyarenko (2020), from (15), the kernel family \( \{r_{l-s}(\cdot, \cdot), l, s \in \mathcal{T}\} \) associated with the cross-covariance operator family \( \{R_{l-s} = E[\log(X_t) \otimes \log(X_s)], \; l, s \in \mathcal{T}\} \) of the \( L^2(\mathbb{S}_2) \)-valued log-intensity is given by:

\[
r_{l-s}(x, y) = \sum_{l=0}^{\infty} B_l(t-s) \left( 2l + 1 \right) P_l(\langle x, y \rangle), \; t, s \in \mathcal{T}, \; x, y \in \mathbb{S}_2.
\]
are approximated over a spherical regular grid of 225 x 225 nodes.

Shannon-entropy based distance $D_{S}^{q}$ in (9) is approximated by $D_{S}^{q}$ at Legendre scales $q=0, \ldots, 30$, to measure the statistical distance between the two-order product densities of the generated spherical Log-Gaussian Cox process, and the spherical homogeneous Poisson process over the interval [0, 10]. The estimate $D_{S}^{q}$ is computed by applying Monte Carlo numerical integration, based on a sample of size 1000, and least-squares parametric five-degree polynomial fitting for interpolation and smoothing. Figure 3 displays three plots representing the values of $D_{S}^{q}$, for three embedded spatial scale sets, i.e., for $q$-values: $q=0, 1, 2, 3, 4, 5$ (left-hand side), $q=0, \ldots, 20$ (right-hand side) and $q=0, \ldots, 30$ (bottom side). One can observe the positive values of the computed statistical distances at Legendre scales zero to four indicating clustering, while null values are displayed from scales five to thirty. Maximum distance or aggregation level is attained at Legendre high frequencies ($q \in \{5, \ldots, 30\}$), i.e., complete randomness at small scale.

Integral (9) defining $D_{S}^{q}(\rho_{q}^{(n)}, \rho_{q}^{(n)})$ is computed for $n=3$ by applying trapezoidal rule. As expected, the classification results displayed in Fig. 3 for the case of $n=2$ are

Fig. 5 Weak-dependent case ($\theta=100$). Rényi distances $D_{R}^{q}$, $q=1, 2, 3, 4, 5$ (horizontal axis), and $h \in (1, 10)$. $q=0, 1, 2, 3, 4, 5$ (left-hand side), $q=0, \ldots, 20$ (right-hand side) and $q=0, \ldots, 30$ (bottom side).
supported in the Log-Gaussian case for \( n = 3 \), over all Legendre scales tested (see Fig. 4).

Distance \( D_{q,h}^R \) in (10) is now approximated by \( \hat{D}_{q,h}^R \), computed by applying Monte Carlo numerical integration and five degree polynomial least-squares smoothing. A similar pattern to the one displayed at the left-hand-side plot in Fig. 3 is observed for the computed estimates of Rényi-entropy based distances \( D_{q,h}^R \) of different integer and fractional orders \( h \), considering Legendre scales \( q = 1, 2, 3, 4, 5 \). Such empirical distances provide additional information about the clustering index in the spatiotemporal point pattern. Figures 5 and 6 show such distances in the respective cases of short and long range dependence in time of the log-intensity, corresponding to the values \( \theta = 100 \) and \( \theta = 1/100 \) in equation (15).

All computed statistical distances reflect the same pattern with respect to Legendre scales (horizontal axis), indicating regularity at Legendre scales larger or equal than five \( (q \geq 5) \), and clustering at Legendre scales zero to four \( (q = 0, 1, 2, 3, 4) \). Spherical scales \( q = 0 \) and \( q = 1 \) display the largest aggregation index \( CI_h = \exp(D_{q,h}^R) \), under the three dependence models \( (\theta = 1, 100, 1/100) \) for all computed statistical distances. For this particular scenario where Log-Gaussian intensities are considered, the log-intensity dependence range (reflected in parameter \( \theta \)), and
the statistical distance chosen (reflected in parameter $h$) only affect the magnitude of the distances computed at the first spherical scales. Specifically, the clustering level, measured by the clustering index $CI_h$, increases when the dependence range becomes larger at these first scales ($q = 0, 1, 2$) around the integer values $h = 1$ and $h = 2$ of parameter $h$.

Large and small scale point pattern classification is here performed from Monte Carlo estimates $\hat{K}_q$, $q = 0, \ldots, 30$, of functions $K_q$, $q = 0, \ldots, 30$, respectively. The pointwise differences $\hat{K}_q - K_{Pois}$, $q = 0, \ldots, 30$, with $K_{Pois}$ denoting as before the theoretical $K$ function of spatiotemporal spherical Poisson process, are plotted in Figs. 7, 8 and 9, respectively corresponding to the long, intermediate and short range dependence cases of the log-intensity, for Legendre scales $q = 1, 7, 13, 19, 25$. These functions are evaluated at the angular distances $\{\theta_i, i = 1, \ldots, 14\} = \{0, 0.2244, 0.4488, 0.6732, 0.8976, 1.1220, 1.3464, 1.5708, 1.7952, 2.0196, 2.2440, 2.4684, 2.6928, 2.9172, 3.1416\}$ in the interval $[0, \pi]$, and at the temporal distances $\{t_i, i = 1, \ldots, 14\} = \{0, 0.7143, 1.4286, 2.1429, 2.8571, 3.5714, 4.2857, 5.0000, 5.7143, 6.4286, 7.1429, \ldots, 7.8571, 8.5714, 9.2857, 10.0000\}$, in the interval $[0, 10]$.

Figures 7, 8 and 9 show that, for all log-intensity dependence ranges, $\hat{K}_q - K_{Pois}$ values are decreasing when the Legendre spatial scale $q$ increases, going to zero when $q$ goes to infinity. Hence, for large values of $q$, it can be observed that $\hat{K}_q$ is pointwise approximating function $K_{Pois}$ (pointwise differences less than one for any temporal $t$ and angular $\theta$ distance values), supporting again the computational results showed in Figs. 3, 4, 5 and 6. Thus, regularity of spatiotemporal point patterns at high Legendre

Fig. 7 Long-range dependence (LRD) Gaussian log-intensity. Contour plots of pointwise values of empirical difference $\hat{K}_q - K_{Pois}$, for $q = 1, 7, 13$ (top) and for $q = 19, 25$ (bottom). The generated spherical Log-Gaussian Cox process over the time interval $[0, 10]$ has Legendre Fourier coefficients having covariance function (15) with $\theta = 1/100$ (LRD).
frequencies \((q \text{ large})\) is observed, while aggregation or clustering is displayed at low Legendre frequencies \((q \text{ small})\), where bigger differences are induced by the log-intensity dependence range, i.e., larger positive pointwise discrepancies (stronger departure from regularity) are observed when the dependence range increases (see, e.g., contour plots at the top-left in Figs. 7, 8 and 9).

Summarizing, for small arguments \(t\) and \(\theta\) of \(\tilde{K}_q - K_{\text{Pois}}\) functions, more pronounced differences are observed through Legendre scales, while a regular behavior is observed for large values of \(t\) and \(\theta\), i.e., null values of \(\tilde{K}_q - K_{\text{Pois}}\) functions for every \(q > 1\). For \(q = 1\), positive pointwise discrepancies between \(K\)-functions hold for all values analyzed of \(t\) and \(\theta\). One can also observe that for this \(q = 1\) value the effect of the dependence range of the Gaussian log-intensity is stronger, increasing positive discrepancies between \(K\) functions compared, for all arguments \(t\) and \(\theta\). For the rest of scales \((q \geq 2)\), the effect of the dependence range is more pronounced at small values of \(t\) and \(\theta\).

6 Final comments

Under stationarity in time and isotropy in space, the present paper performs a statistical analysis of point patterns on a connected and compact two-point homogeneous space. Specifically, this analysis is based on Cox processes whose log-intensity (log-risk process) is Gaussian or belongs to the class of second-order mean-square continuous elliptically contoured random fields on a manifold (see, e.g., Ma and Malyarenko 2020). In the Gaussian case, a countable family of independent stationary centered Gaussian processes defines the time-varying discrete Jacobi
polynomial transform of the log-intensity spatiotemporal random field. The \( n \)-order product density then admits an expression in terms of the infinite-product of \( n \)-order product densities corresponding to different Jacobi polynomial scales.

The simulation study undertaken is based on Monte Carlo numerical integration and least-squares parametric polynomial curve fitting, allowing the implementation of the proposed point pattern analysis, based on empirical statistical distances, and \( K \)-functions, in terms of the time-varying discrete Legendre polynomial transform. By exploiting the isometry properties with the sphere, the numerical results derived in this simulation study are extended to the case of Log-Gaussian Cox processes on a connected and compact two point homogeneous space evolving time. Thus, one can conclude for the wider introduced family of Log-Gaussian Cox processes, the regular behavior (complete randomness) of the point process at small scale in the manifold (high-frequency behavior of the discrete Jacobi transform). While aggregation or clustering is displayed at large (coarser) scales in the manifold (low-frequency behavior of the discrete Jacobi transform). This low- and high-frequency analysis in the domain of the Jacobi transform is achieved by measuring the statistical distance between the \( n \)-order product densities of the analyzed point process, at different Jacobi polynomial scales, and the \( n \)-order product density of homogeneous Poisson process on the manifold over a time interval. Different statistical distances are tested within the Shannon- and Rényi-entropy based distances. The last ones providing a micro-scale aggregation (or clustering) index of the point pattern depending on Jacobi scale. The effect of the temporal dependence range of the log-risk process is more pronounced at low frequencies of discrete Jacobi polynomial transform. Particularly, the Rényi-based micro-scale aggregation index increases when Fig. 9 Short-range dependence in the Gaussian log-intensity. Contour plots of pointwise values of empirical difference \( K_q - K_{\text{Pois}} \), for \( q = 1, 7, 13 \) (top) and for \( q = 19, 25 \) (bottom). The generated spherical Log-Gaussian Cox process over the time interval \([0, 10]\) has Legendre Fourier coefficients having covariance function (15) with \( \theta = 100 \).
the temporal dependence range of the log-intensity increases at low Jacobi frequencies. While the effect of the temporal dependence range asymptotically disappears at high frequencies of the discrete Jacobi polynomial transform. The analysis of point patterns in terms of the associated countable family of $K$-functions, arising from discrete Jacobi polynomial transform, also supports the conclusions of the micro-scale analysis based on statistical distances between $n$-order product densities. Particularly, at low discrete frequencies (large scale), stronger differences between complete randomness and scale-dependent $K$-functions of the analyzed point pattern are observed.

The statistical methodology proposed for analysis and multi-scale classification of point patterns on a manifold over time, in the context of connected and compact two-point homogeneous spaces, within the framework of Cox processes, will be extended to the case of multifractal spherical log-risk processes in a subsequent paper (see, e.g., Leonenko et al. 2021). Finally, we remark that the presented approach is applicable to further families of point processes, including the family of determinantal point process on a manifold evolving time (see, e.g., Möller et al. 2018; Möller and Rubak 2016, for the spatial spherical case).

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Declaration

Conflict of interest The authors declare that they have no conflict of interest.

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