HALL-LITTLEWOOD PLANE PARTITIONS AND KP

O FODA AND M WHEELER

Abstract. MacMahon’s classic generating function of random plane partitions, which is related to Schur polynomials, was recently extended by Vuletić to a generating function of weighted plane partitions that is related to Hall-Littlewood polynomials, \( S(t) \), and further to one related to Macdonald polynomials, \( S(t, q) \).

Using Jing’s 1-parameter deformation of charged free fermions, we obtain a Fock space derivation of the Hall-Littlewood extension. Confining the plane partitions to a finite \( s \times s \) square base, we show that the resulting generating function, \( S_{s \times s}(t) \), is an evaluation of a \( \tau \)-function of KP.

0. Introduction

In [1], Vuletić obtained 1-parameter and 2-parameter deformations of MacMahon’s generating function of random plane partitions. In this work, we limit our attention mostly to the 1-parameter result

\[
S(t) = \prod_{j=1}^{\infty} \left( \frac{1 - tz^j}{1 - z^j} \right)^j
\]

where \( t \in \mathbb{C} \) is the deformation parameter, and \( z \) is a formal parameter that keeps track of the number of boxes in a plane partition (see below). Setting \( t = 0 \), one recovers MacMahon’s generating function of random plane partitions [2], which is an evaluation of a KP \( \tau \)-function [3].

The derivation of (1) in [1] was combinatorial. The purpose of this work is two-fold. 1. To use Jing’s 1-parameter deformation of free fermions [5], to obtain a Fock space derivation of (1). 2. To show that \( S(t) \) is an evaluation of a KP \( \tau \)-function.

In section 1 we recall Jing’s 1-parameter \( t \)-deformation of charged free fermions, Heisenberg algebra, vertex operators, etc. [5]. In section 2 we introduce Young diagrams\(^1\) as labels of (left and right) state vectors in (left and right) Fock spaces and evaluate their inner products.

In section 3 we show that the action of a certain \( t \)-vertex operator on a state vector labeled by a Young diagram \( \mu \) generates a weighted sum of state vectors labeled by Young diagrams that interlace with \( \mu \). The weights are skew Hall-Littlewood functions. The derivation is elementary in the sense that it follows directly from the properties of the underlying fermions. This extends a result of Okounkov and Reshetikhin for undeformed fermions and Schur functions [6].

In section 4 we compute an expectation value of (infinitely many) vertex operators in two different ways. A. By acting on Young diagrams to generate weighted interlacing Young diagrams that stack to generate weighted plane partitions. B. By commuting the vertex operators in such a way that they act on the vacuum states as the identity and thereby obtain a product expression for the expectation

\(^1\)We use ‘Young diagrams’, or ‘diagrams’, rather than ‘partitions’ to clearly distinguish between (regular) partitions and plane partitions.
value. Equating the results of these two computations, we recover the 1-parameter result of [1].

In section 5 we show that $S(t)$ is an evaluation of a KP $\tau$-function, for arbitrary values of $t$. In section 6 we comment on the 2-parameter extension of MacMahon’s generating function that is related to Macdonald polynomials [1]. In this case, a 2-parameter $(t,q)$-deformation of charged free fermions is also available [7]. Using the corresponding $(t,q)$-vertex operators, it is straightforward to compute the corresponding expectation value and obtain the analogue of computation B above. What is beyond the scope of this work is to obtain the analogue of computation A in terms of the action of underlying fermions. In section 7 we collect a number of comments.

1. $t$-Fermions and related operators

1.1. The deformation parameter $t$. We take $t = e^{\epsilon \theta}$, where $\epsilon, \theta \in \mathbb{R}$, $\epsilon > 0$. The fact that $\epsilon > 0$ will be necessary for convergence in the intermediate steps of all derivations. However, ultimately it will be possible to take the limits $\epsilon \to 0$, and $\theta \to 2\pi/n$, $n \in \mathbb{N} \geq 2$. The limit $\epsilon \to 0$ is justified by the fact that the expressions obtained are well defined.

1.2. Charged $t$-fermions and $t$-anti-commutation relations. Consider two species of charged fermions, $\{\psi_m, \psi^*_m\}$, which satisfy the following $t$-deformed anti-commutation relations

\begin{align*}
\psi_m \psi_n + \psi_n \psi_m &= t\psi_{m+1} \psi_{n-1} + t\psi_{n+1} \psi_{m-1} \\
\psi^*_m \psi^*_n + \psi^*_n \psi^*_m &= t\psi^*_m \psi^*_{n+1} + t\psi^*_n \psi^*_{m+1} \\
\psi_m \psi^*_n + \psi^*_n \psi_m &= t\psi_{m+1} \psi^*_{n-1} + t\psi^*_{n+1} \psi_{m-1} + (1 - t)^2 \delta_{m,n}
\end{align*}

where $m, n \in \mathbb{Z}$, and $t \in \mathbb{C}$ is a deformation parameter. We refer to these as $t$-fermions. For $t = 0$, we recover the charged free fermions of KP theory [3].

1.3. Re-writing the $t$-anti-commutation relations. Given the structure of the $t$-anti-commutation relations (2–4), it will be useful in later sections to re-write them differently. Using (4) on itself, we obtain

\[
\psi_m \psi_n^* = -\psi_n^* \psi_m + t\psi_{m-1}^* \psi_{n-1} + t\psi_{m+1}^* \psi_{n+1} + (1 - t)^2 \delta_{m,n}
\]
\[
= -\psi_n^* \psi_m + t \left( -\psi_{n-1}^* \psi_{m-1} + t\psi_{m-2}^* \psi_{n-2} + t\psi_n^* \psi_m + (1 - t)^2 \delta_{m,n} \right)
\]
\[
+ t\psi_{n+1}^* \psi_{m+1} + (1 - t)^2 \delta_{m,n}
\]
\[
= t\psi_{n+1}^* \psi_{m+1} + (t^2 - 1)\psi_n^* \psi_m - t\psi_{n-1}^* \psi_{m-1}
\]
\[
+ t^2\psi_{m-2}^* \psi_{n-2} + (1 + t)(1 - t)^2 \delta_{m,n}
\]
\[
\vdots
\]
\[
(5) = t\psi_{n+1}^* \psi_{m+1} + (t^2 - 1) \sum_{j=0}^{\infty} \psi_{(n-j)}^* \psi_{(m-j)} t^j + (1 - t) \delta_{m,n}
\]

which is clearly a re-writing of (4) in such a way that a $\psi^*$ operator appears on the left and a $\psi$ on the right in each bilinear term on the right hand side of the relation.

Further, we propose the following identity

\[
(6) \psi_m^* \psi_{(m-n)}^* + (1 - t) \sum_{j=1}^{n} \psi_{(m-j)}^* \psi_{(m+j-n)} = t\psi_{(m-1-n)}^* \psi_{(m+1)}, \quad n \geq 0
\]
We refer to the statement in (4) as $P_n$, and prove it inductively as follows. Rearranging terms in $P_n$, and relabelling of the summation parameter $j$, we obtain

$$
p^*_m\psi^*_{(m-n)} + (1-t)\sum_{j=1}^{n} \psi^*_m\psi^*_{(m-j)}\psi^*_{(m+j-n)} =
$$

$$
p^*_m\psi^*_{(m-n)} - t\psi^*_m\psi^*_{(m-1)}\psi^*_{(m+1-n)} + (1-t)\psi^*_m\psi^*_{(m-n)} +
$$

$$
\begin{cases}
\psi^*_m\psi^*_{(m-1)-(n-2)} + (1-t)\sum_{j=1}^{n-2} \psi^*_m\psi^*_{(m-1)-j}\psi^*_{(m-1)+j-(n-2)} \\
\end{cases}
$$

If $P_{(n-2)}$ holds, then the parenthesised term is equal to $t\psi^*_m\psi^*_{(m-n)}$, and

$$
p^*_m\psi^*_{(m-n)} + (1-t)\sum_{j=1}^{n} \psi^*_m\psi^*_{(m-j)}\psi^*_{(m+j-n)} =
$$

$$
p^*_m\psi^*_{(m-n)} - t\psi^*_m\psi^*_{(m-1)}\psi^*_{(m+1-n)} + \psi^*_m\psi^*_{(m-n)} = t\psi^*_m\psi^*_{(m-1-n)}\psi^*_{(m+1)}
$$

where we have used the $t$-anti-commutation relation (3) in the last line. Hence $P_n$ true if $P_{n-2}$ is true. But $P_0$ and $P_1$ follow trivially from (3). They are

$$
P_0 : \psi^*_m\psi^*_m = t\psi^*_m\psi^*_m(1+1)
P_1 : \psi^*_m\psi^*_m + (1-t)\psi^*_m\psi^*_m = t\psi^*_m\psi^*_m(1+1)
$$

and $P_n$ is true for all $n \in \mathbb{N}$. From the proof, it is clear that (6) is a re-writing of (3). An analogous result holds for (2), but we will not need it in the sequel.

1.4. $t$-Heisenberg operators. The $t$-analogues of the Heisenberg generators $h_m$ are defined in terms of the $t$-fermions as

$$
h_m = \begin{cases}
\frac{1}{(1-t)^2}\sum_{j \in \mathbb{Z}} \psi^*_j\psi^*_{(j+m)} & m \geq 1 \\
\frac{1}{(1-t)(1-t|m|)}\sum_{j \in \mathbb{Z}} \psi^*_j\psi^*_{(j+m)} & m \leq -1 
\end{cases}
$$

Using the $t$-anti-commutation relations (24), one can show that

$$
\begin{align*}
h_m\psi_j - \psi_jh_m &= \psi_{(j-m)} \\
h_m\psi^*_j - \psi^*_jh_m &= -\psi^*_{(j+m)} \\
h_mh_n - h_nh_m &= \frac{m}{1-|m|}\delta_{m+n,0}
\end{align*}
$$

where $j \in \mathbb{Z}$, and $m, n \in \mathbb{Z}\setminus\{0\}$.

1.5. $t$-Vertex operators. The $t$-analogue of the vertex operators of [6] are

$$
\begin{align*}
\Gamma_+(z, t) &= \exp \left\{ -\sum_{m=1}^{\infty} \frac{1-tm}{m}z^mh_m \right\} \\
\Gamma_-(z, t) &= \exp \left\{ -\sum_{m=1}^{\infty} \frac{1-tm}{m}z^mh_{-m} \right\}
\end{align*}
$$

They satisfy the commutation relations...
\[ \Gamma_+(z, t) \Gamma_-(z', t) = \frac{z - tz'}{z - z'} \Gamma_-(z', t) \Gamma_+(z, t) \]

1.6. \textit{t-Fermion evolution equations.} Define the generating series \( \Psi(k) = \sum_{j \in \mathbb{Z}} \psi_j k^j \), \( \Psi^*(k) = \sum_{j \in \mathbb{Z}} \psi_j^* k^j \), and \( H_{\pm}(z, t) = \sum_{n \in \mathbb{Z}} \frac{t^n}{nz} h_n \). Using the relations (8) and (9), we obtain

\[ [\Psi(k), H_+(z, t)] = \Psi(k) \sum_{n \in \mathbb{N}} \frac{1 - t^n}{nz^n} k^n = \Psi(k) \log \left( \frac{1 - \frac{t^k}{z}}{1 - \frac{1}{z}} \right) \]

\[ [\Psi^*(k), H_-(z, t)] = \Psi^*(k) \sum_{n \in \mathbb{N}} \frac{1 - t^n}{n} (zk)^n = \Psi^*(k) \log \left( \frac{1 - tzk}{1 - zk} \right) \]

These equations can be cast into the form

\[ e^{-H_+(z, t)} \Psi(k) e^{H_+(z, t)} = \Psi(k) \left( \frac{1 - \frac{t^k}{z}}{1 - \frac{1}{z}} \right) \]

\[ e^{-H_-(z, t)} \Psi^*(k) e^{H_-(z, t)} = \Psi^*(k) \left( \frac{1 - tzk}{1 - zk} \right) \]

By equating coefficients of powers in \( k \), we have the following \( t \)-fermion evolution equations

\[ \Gamma_+^{-1}(z, t) \psi_m \Gamma_+(z, t) = \psi_m + (1 - t) \sum_{j=1}^{\infty} \psi_{(m-j)z^{-j}} \]

\[ \Gamma_-(z, t) \psi_m^* \Gamma_-^{-1}(z, t) = \psi_m^* + (1 - t) \sum_{j=1}^{\infty} \psi_{(m-j)z^j} \]

where we have used the fact that \( e^{-H_+(z, t)} = \Gamma_+^{-1}(z, t) \) and \( e^{-H_-(z, t)} = \Gamma_-(z, t) \).

1.7. \textit{‘Fake’ and ‘genuine’ vacuum states.} Following [3], we introduce two types of (left and right) vacuum states. The ‘fake’ vacuum states are represented by \( \langle \Omega \rangle \) and \( | \Omega \rangle \) and satisfy an inner product normalised to \( \langle \Omega | \Omega \rangle = 1 \). The ‘genuine’ vacuum states, \( | 0 \rangle \) and \( \langle 0 | \) are generated by infinite strings of \( t \)-fermion operators acting on \( \langle \Omega \rangle \) and \( | \Omega \rangle \) as follows

\[ \langle 0 | = \langle \Omega | \cdots \psi_2 \psi_1 | \psi_0 \rangle = \langle \Omega | \prod_{j=0}^{\infty} \psi_j \]

\[ | 0 \rangle = \psi_0^* \psi_1^* \psi_2^* \cdots | \Omega \rangle = \prod_{j=0}^{\infty} \psi_j^* | \Omega \rangle \]

By convention, the energy of \( | 0 \rangle \) and \( \langle 0 | \) is set to zero [3].

1.8. \textit{Finite energy states.} Finite energy states are of the form \( \langle \Omega | \cdots \psi_{m_2} \psi_{m_1} | \psi_{m_0} \rangle \) and the form \( | \psi_{m_0}^* \psi_{m_1}^* \psi_{m_2}^* \cdots | \Omega \rangle \), where \( m_n = n \) for all \( n \geq N \), for some sufficiently large \( N \). They differ from the vacuum states by a relabelling of a finite number of \( t \)-fermions with new indices.
1.9. The annihilation $t$-fermion operators. We fix
\begin{align}
(22) \quad \langle \Omega | \ldots \psi_{(t+2)} \psi_{(t+1)} \psi_j \psi_m^* = \langle l | \psi_m^* = 0 \\
(23) \quad \psi_m \psi_l^* \psi_{(t+1)} \psi_{(t+2)} \ldots | \Omega \rangle = \psi_l | l \rangle = 0 
\end{align}
for all $m < l$. There are other annihilation operators as well. In particular,
\begin{equation}
(24) \quad \psi_m^* | l \rangle = 0, \quad \langle l | \psi_m = 0
\end{equation}
for all $m \geq l$. It is sufficient for our purposes to impose (22,23) as extra conditions and note that they are consistent with (24). (24) follows from (22,23).

2. State vectors and Young diagrams

2.1. Young diagrams label state vectors. We use Young diagrams to label (left and right) state vectors in the (left and right) Fock spaces. The empty diagram labels $\langle 0 \rangle$ and also $| 0 \rangle$. Non-empty diagrams label finite energy states. If $\mu = \{ \mu_1, \ldots, \mu_l \}$ is a diagram with $l$ non-zero parts, it labels the left and right state vectors
\begin{align}
(25) \quad \langle \mu | = \langle \Omega | \ldots \psi_{(t+1)} \psi_j \psi_{m_1} \ldots \psi_m \\
(26) \quad | \mu \rangle = \psi_{m_1}^* \ldots \psi_{m_j}^* \psi_{(t+1)}^* \ldots | \Omega \rangle
\end{align}
where the integers $\{ m_1, \ldots, m_j \}$ are given by $m_j = j - 1 - \mu_j$, for $1 \leq j \leq l$. Notice that for a Young diagram with $l$ parts, the $l$ $l$-fermions that are farthest from the corresponding fake vacuum state are relabelled.

2.2. Lemma (Inner products). Left and right state vectors are orthogonal in the sense that
\begin{equation}
(27) \quad \langle \mu | \nu \rangle = b_\mu(t) \delta_{\mu,\nu}
\end{equation}
where $b_\mu(t)$ is defined as follows. If $\mu$ has $p_j(\mu)$ parts of length $j$, where $j \geq 1$, then
\begin{equation}
(28) \quad b_\mu(t) = \prod_{j=1}^{\infty} \left( \prod_{k=1}^{p_j(\mu)} (1 - t^k) \right)
\end{equation}

2.2.1. Proof. Consider the scalar product
\begin{equation}
(29) \quad \langle \mu | \nu \rangle = \langle \Omega | \ldots \psi_{\tilde{m}_1} \ldots \psi_{\tilde{m}_j} \psi_{m_1}^* \ldots \psi_{m_j}^* \psi_l^* \ldots | \Omega \rangle
\end{equation}
From (6) and (22,23), $\langle \mu | \nu \rangle = 0$ unless $\tilde{m}_1 = m_1$, hence we set $\tilde{m}_1 = m_1$ without loss of generality. Furthermore, let $m_1, \ldots, m_s$ be nearest neighbours for $1 \leq s \leq l$. That is, assume that $m_{j+1} = m_j + 1$ for $1 \leq j \leq (s - 1)$, but fix $m_{s+1} > m_s + 1$. Commuting $\psi_{\tilde{m}_j} \psi_{m_j}^*$ using (6), we obtain
\begin{align}
(30) \quad \langle \mu | \nu \rangle & = (1 - t) \langle \Omega | \ldots \psi_{\tilde{m}_1} \ldots \psi_{\tilde{m}_j} \psi_{m_2}^* \ldots \psi_{m_1}^* \psi_l^* \ldots | \Omega \rangle \\
& + t \langle \Omega | \ldots \psi_{\tilde{m}_1} \ldots \psi_{\tilde{m}_2} \psi_{m_3}^* \psi_{m_2}^* \psi_{m_1}^* \psi_l^* \ldots | \Omega \rangle \\
& \quad \vdots \\
& = (1 - t^s) \langle \Omega | \ldots \psi_{\tilde{m}_1} \ldots \psi_{m_2}^* \ldots \psi_{m_1}^* \psi_l^* \ldots | \Omega \rangle
\end{align}
Repeating this procedure, we find that $\langle \mu | \nu \rangle = 0$ unless $\tilde{m}_j = m_j$ for all $1 \leq j \leq s$. Continuing to commute the central pair of $t$-fermions $\psi_{\tilde{m}_j} \psi_{m_j}^*$, we find
Thus we have acquired a factor of \( \prod_{k=1}^{s} (1 - t^k) \) for a set of \( s \) parts of the same length in the Young diagram \( \mu \). The required result follows inductively.

2.3. Normalization of the genuine vacuum states. From (27), we obtain the normalization \( \langle 0 | 0 \rangle = 1 \), which is non-trivial to compute directly from \( \langle \Omega | \Omega \rangle = 1 \), given the nature of the \( t \)-anti-commutators (24).

2.4. Interlacing Young diagrams. Let \( \lambda = \{ \lambda_1, \ldots, \lambda_{(l+1)} \} \) be a diagram consisting of at least \( l \) non-zero parts, that is, we allow \( \lambda_{(l+1)} = 0 \). We say that \( \lambda \) interlaces with \( \mu \), where \( \mu = \{ \mu_1, \ldots, \mu_l \} \), and write \( \lambda \succ \mu \) if \( \lambda_j \geq \mu_j \geq \lambda_{(j+1)} \), for all \( 1 \leq j \leq l \).

2.5. Interlacing state vectors. Since Young diagrams label state vectors, we can define interlacing state vectors as follows. In terms of right state vectors, if

\[
|\lambda\rangle = \psi_{n_1}^* \cdots \psi_{n_j}^* \psi_{n_{(j+1)}}^* \psi_{(j+1)}^* \psi_{(j+2)}^* \cdots |\Omega\rangle
\]

(32)

\[
|\mu\rangle = \psi_{m_1}^* \cdots \psi_{m_j}^* \psi_{m_{(j+1)}}^* \psi_{(j+2)}^* \cdots |\Omega\rangle
\]

(33)

and \( n_j \leq m_j \leq n_{(j+1)} - 1 \), for all \( 1 \leq j \leq l \), we say \( |\lambda\rangle \succ |\mu\rangle \). An obvious similar definition holds for left state vectors.

3. Action of \( t \)-vertex operators on state vectors

3.1. Skew Hall-Littlewood functions. Following [2], \( P_{\lambda/\mu}(z,t) \), the skew Hall-Littlewood function of a single variable \( z \), indexed by the skew Young diagram \( \lambda/\mu \), is

\[
P_{\lambda/\mu}(z,t) = \begin{cases} 
\Phi_{\lambda/\mu}(t) z^{\lambda\succ\mu} & \lambda \succ \mu \\
0 & \lambda \not\succ \mu
\end{cases}
\]

(34)

where \( \Phi_{\lambda/\mu}(t) \) is a polynomial in \( t \) which we now describe. Let \( \mu \) have \( p_j(\mu) \) parts of length \( j \) and \( \lambda \) have \( p_j(\lambda) \) parts of length \( j \). Define \( J_{\lambda/\mu} \) to be the set of integers \( j \) such that \( p_j(\lambda) - p_j(\mu) = -1 \). Then \( \Phi_{\lambda/\mu}(t) \) is

\[
\Phi_{\lambda/\mu}(t) = \prod_{j \in J_{\lambda/\mu}} (1 - t^{p_j(\mu)})
\]

(35)

That is, when the number of parts \( p_j(\mu) \) of length \( j \) decreases by 1 in the transition from \( \mu \) to \( \lambda \), a factor of \( (1 - t^{p_j(\mu)}) \) is acquired.

3.2. Lemma (Action of \( t \)-vertex operators). We prove the following results

\[
\Gamma_-(z,t)|\mu\rangle = \sum_{\lambda\succ\mu} P_{\lambda/\mu}(z,t)|\lambda\rangle = \sum_{\lambda\succ\mu} \left( \prod_{j \in J_{\lambda/\mu}} (1 - t^{p_j(\mu)}) z^{\lambda\succ\mu} \right) |\lambda\rangle
\]

(36)

\[
\langle \mu| \Gamma_+(z,t) = \sum_{\lambda\succ\mu} P_{\lambda/\mu}(z^{-1},t)|\lambda\rangle = \sum_{\lambda\succ\mu} \left( \prod_{j \in J_{\lambda/\mu}} (1 - t^{p_j(\mu)}) z^{\mu\succ\lambda} \right) \langle \lambda|
\]

(37)

where the sums are over all Young diagrams \( \lambda \) which interlace with \( \mu \). This lemma is the first of two results in this work. We prove (36) in detail. The proof of (37) is analogous. We require the following identities.
3.3. **Identities.** Firstly, from (39), we can derive

\[
(38) \quad \left\{ \psi_m^* + (1-t) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} \left\{ \psi_{m+1}^* + (1-t^p) \sum_{j=1}^{\infty} \psi_{m+1-j}^* z^j \right\} = \sum_{n=0}^{\infty} \left\{ \psi_m^* \psi_{m-n}^* + (1-t) \sum_{j=1}^{n} \psi_{m-j}^* \psi_{m+j-n}^* \right\} z^n = t \sum_{n=0}^{\infty} \psi_m^* \psi_{m-1-n}^* \psi_{m+1}^* z^n
\]

Next, we use (38) to derive two more identities.

\[
(39) \quad \left\{ \psi_m^* + (1-t) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} \left\{ \psi_{m+1}^* + (1-t^p) \sum_{j=1}^{\infty} \psi_{m+1-j}^* z^j \right\} = \left\{ \psi_m^* + (1-t^p+1) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} \psi_{m+1}^*
\]

\[
(40) \quad \left\{ \psi_m^* + (1-t) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} \left\{ \psi_{m+n}^* + (1-t^p) \sum_{j=1}^{n} \psi_{m+n-j}^* z^j \right\} = \left\{ \psi_m^* + (1-t) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} \left\{ \psi_{m+n}^* + (1-t^p) \sum_{j=1}^{n-1} \psi_{m+n-j}^* z^j \right\} + t(1-t^p)z^{n-1} \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \psi_{m+1}^*
\]

where \( p \geq 1 \) may be infinite, \( n \geq 2 \), and (38) was used between the first and second lines in (39) and (40).

3.4. **Notation.** We introduce the notation

\[
(41) \quad \left\{ \psi_m^* + (1-x) \sum_{j=1}^{\infty} \psi_{m-j}^* z^j \right\} = \Psi_m^*(z, x)
\]

which allows us to write (39) through (40) in the succinct form

\[
(42) \quad \Psi_m^*(z, t) \Psi_{m+1}^*(z, t^p) = \Psi_m^*(z, t^p+1) \psi_{m+1}^*
\]
Firstly, one computes the product of sums marked diagrams of the form (32). To see that this is the case, we use (42–43) iteratively.

3.5. Proof. We are now in a position to outline the proof of (36). The proof consists of three parts. 1. We show that the action of $\Gamma_-$ on a right state vector labelled by a diagram $\mu$ generates a weighted sum over all right state vectors labelled by Young diagrams $\lambda \succ \mu$. 2. We show that $\Phi_{\lambda/\mu}(t)$, the first factor in $P_{\lambda/\mu}(z, t)$, is recovered. 3. We show that $z^{[\lambda]-|\mu|}$, the second factor in $P_{\lambda/\mu}(z, t)$, is recovered.

3.5.1. Part 1: Generating all interlacing Young diagrams. We start from expression (26) for $|\mu|$. We act on this right state vector with $\Gamma_-(z, t)$ and make repeated use of the relation (19), to obtain

\[
(44) \quad \Gamma_-(z, t) |\mu\rangle = \Gamma_-(z, t) \psi_{m_1}^* \cdots \psi_{m_l}^* \psi_{(l+1)}^* \cdots |\Omega\rangle 
\]

\[
= \left( \Gamma_\psi_{m_1}^* \Gamma_-^{-1} \right) \cdots \left( \Gamma_\psi_{m_l}^* \Gamma_-^{-1} \right) \left( \Gamma_\psi_{(l+1)}^* \Gamma_-^{-1} \right) \cdots |\Omega\rangle 
\]

\[
= \left( \psi_{m_1}^* + (1-t) \sum_{j=1}^{\infty} \psi_{m_1-j}^* z^j \right) \cdots \left( \psi_{m_l}^* + (1-t) \sum_{j=1}^{\infty} \psi_{m_l-j}^* z^j \right) 
\]

\[
\times \left( \psi_{(l+1)}^* + (1-t) \sum_{j=1}^{\infty} \psi_{(l+1)-j}^* z^j \right) \cdots |\Omega\rangle 
\]

Using (32) to truncate all but the first $(l+1)$ sums appearing in (44), we obtain

\[
(45) \quad \Gamma_-(z, t) |\mu\rangle = \frac{1}{\psi_{m_1}^*} \cdots \frac{1}{\psi_{m_l}^*} \frac{1}{\psi_{(l+1)}^*} \psi_{(l+2)}^* \cdots |\Omega\rangle 
\]

The proof requires that the first $(l+1)$ sums truncate to give exactly all Young diagrams of the form (32). To see that this is the case, we use (42–43) iteratively. Firstly, one computes the product of sums marked $n_l$ and $n_{(l+1)}$, using (43) with $p = \infty$, to obtain

\[
(46) \quad \Gamma_-(z, t) |\mu\rangle = \psi_{m_1}^* \cdots \psi_{m_l}^* \psi_{(l+1)}^* \cdots |\Omega\rangle 
\]

Notice that we have truncated the $n_{(l+1)}$ sum, so that only $t$-fermions $\psi_{n_{(l+1)}}^*$ with indices $m_l + 1 \leq n_{(l+1)} \leq l$ remain. These are precisely the permissible values for $n_{(l+1)}$ in (32) so that $\lambda \succ \mu$. 

Now if \( m_l - m_{l-1} = 1 \) \((m_l - m_{l-1} \geq 2)\), take the product of the sums marked \( n_{(l-1)} \) and \( n_l \) in (45), using (42) (using (43)) with \( p = 1 \) for the first term and \( p = \infty \) for the second. This truncates the \( n_l \) terms, so that only \( t \)-fermions \( \psi^*_n \) with indices \( m_{(l-1)} + 1 \leq n_l \leq m_l \) remain. Again, these are precisely the permissible values for \( n_l \) in (42) so that \( \lambda \succ \mu \).

Continuing this way, the truncation ultimately extends to give \( n_1 \leq m_1 \) and \( m_j + 1 \leq n_{(j+1)} \leq m_{(j+1)} \), for all \( 1 \leq j \leq l - 1 \). That is, we recover exactly all Young diagrams of the form (32) from (46), proving that \( \Gamma_-(z, t) \) generates exactly all \( \lambda \succ \mu \).

3.5.2. Part 2: Factor \( \Phi_{\lambda/\mu}(t) \) of \( P_{\lambda/\mu}(z, t) \). For \( 1 \leq a \leq b \leq l \), let \( \{m_a, \ldots, m_b\} \) be a contiguous subset of the points \( \{m_1, \ldots, m_l\} \), which describe the Young diagram \( \mu \) in (26). In other words, the points \( \{m_a, \ldots, m_b\} \) are nearest neighbours. These points will give rise to \((b - a + 1)\) parts in \( \mu \) of a certain length \( h \). Under the action of \( \Gamma_- \) the points \( \{m_a, \ldots, m_b, m_{(b+1)}\} \) are shifted to \( \{n_a, \ldots, n_b, n_{(b+1)}\} \), where we define \( m_{(i+1)} = l \) if \( b = l \). Because this shifting must produce interlacing Young diagrams \( \lambda \), we can have \( m_{(a-1)} + 1 \leq n_a \leq m_a \), where \( m_0 = -\infty \) when \( a = 1 \), and \( m_b + 1 \leq n_{(b+1)} \leq m_{(b+1)} \), but all other indices are stationary.

There are three cases to consider. 1. When \( n_a = m_a \) and \( n_{(b+1)} \neq m_b + 1 \), there will still be exactly \((b - a + 1)\) parts of length \( h \) in the new Young diagram \( \lambda \), so no weight is expected. 2. When \( n_a < m_a \) and \( n_{(b+1)} \neq m_b + 1 \), there will be exactly \((b - a)\) parts of length \( h \) in \( \lambda \), meaning \( p_h(\lambda) - p_h(\mu) = -1 \). Consequently, we expect a weight of \((1 - t^{b-a+1})\) to be acquired. 3. When \( n_{(b+1)} = m_b + 1 \), there will be at least \((b - a + 1)\) parts of length \( h \) in \( \lambda \), so no weight is expected.

Let us check that these weights are in fact recovered, by analysing a general term which arises after repeated truncation of (45). In the following we abbreviate \( \Psi^*_m(z, t) = \Psi^*_m \), for the sake of visual clarity. Using (43), we have

\[
(47) \quad \Psi^*_m(z, t) = \Psi^*_m(z, t^∞) \Psi^*_m \Psi^*_m(z, t^1) \cdots |Ω| =
\]

\[
\Psi_m(z) = \Psi_m(z, t^∞) \Psi_m(z, t^1) \cdots |Ω|
\]

Using (42) repeatedly in both the first and second terms in (47), leads to

\[
(48) \quad \Psi^*_m(z, t) = \Psi^*_m(z, t^{b-a+1}) \Psi^*_m(z, t^1) \cdots |Ω|
\]

The above calculation has split the original object into two terms. The first term features Young diagrams for which \( n_{(b+1)} \neq m_b + 1 \). Hence, it has some Young diagrams \( \lambda \) for which \( p_h(\lambda) < p_h(\mu) \). Taking the product \( \Psi^*_m(z, t^{b-a+1}) \)
using (43), we find that we get no weight for \( n_a = m_a \), whilst we acquire the factor 
\((1 - t^{b-a+1})\) for \( n_a < m_a \). This is exactly as required.

The second term features Young diagrams for which \( n_{(b+1)} = m_b + 1 \). Hence it only has Young diagrams \( \lambda \) for which \( p_h(\lambda) \geq p_h(\mu) \). Taking the product 
\( \Psi_{m(a-1)} \Psi_{m_a}(z, t^{\infty}) \) using (43), we find that we get no weight for any value of \( n_a \).
Again, this is the expected result.

Note that the factors of \((1 - t^p)\) are weights due to the motion of \( m_{(b+1)} \), which belongs to a different set of contiguous points.

### 3.5.3. Part 3: Factor \( z^{\lambda - [\mu]} \) of \( P_{\lambda/\mu}(z, t) \)

If there are \( \delta \) more boxes in the Young diagram of \( \lambda \) than the Young diagram of \( \mu \), this is because the indices of the \( t \)-fermions have been shifted downward by a net \( \delta \) units. However, from (19) we see that a net downward shift by \( \delta \) units will acquire the correct weight of \( z^\delta \).

### 3.5.4. Action of \( t \)-vertex operators on the left state vectors.

Using precisely the same steps as above, one derives the result in (37).

### 4. Hall–Littlewood weighted plane partitions

#### 4.1. Plane partitions as 3-dimensional objects.

One can think of a plane partition \( \pi \) as a set of unit cubic boxes stacked in the north-west corner of a 3-dimensional room such that the heights of columns of boxes are weakly decreasing as one moves horizontally farther from the corner [2]. A 3-dimensional view of a plane partition is shown in Figure 1.

![Figure 1. A 3-dimensional view of a plane partition.](image)

#### 4.2. Plane partitions as 2-dimensional arrays.

One can also think of a plane partition \( \pi \) as a 2-dimensional array (in the shape of a Young diagram) of non-negative integers \( \pi_{i,j} \) on a 2-dimensional grid in the south-east quadrant of the plane. The 2-dimensional base of the plane partition \( \pi \) of Figure 1 is shown in Figure 2 together with its cell coordinates.

The integers \( \pi_{i,j} \), which correspond to the column heights in the 3-dimensional view of \( \pi \), satisfy

\[
\pi_{i,j} \geq \pi_{i+1,j}, \quad \pi_{i,j} \geq \pi_{i,j+1}, \quad \lim_{i \to \infty} \pi_{i,j} = \lim_{j \to \infty} \pi_{i,j} = 0
\]

for all integers \( i, j \geq 0 \).
4.3. Slicing a plane partition diagonally. As observed in [6], when plane partitions are sliced diagonally, one obtains a sequence of interlacing Young diagrams. In other words, if for \( k \geq 1 \) we define the Young diagrams

\[
\mu_{-k} = \{\pi_{k,0}, \pi_{(k+1),1}, \pi_{(k+2),2}, \ldots\} \\
\mu_0 = \{\pi_{0,0}, \pi_{1,1}, \pi_{2,2}, \ldots\} \\
\mu_k = \{\pi_{0,k}, \pi_{1,(k+1)}, \pi_{2,(k+2)}, \ldots\}
\]

then every plane partition \( \pi \) satisfies

\[
\pi = \{\emptyset = \mu_{-m} \prec \cdots \prec \mu_{-1} \prec \mu_0 > \mu_1 > \cdots > \mu_n = \emptyset\}
\]

for some \( m, n \geq 1 \). A 2-dimensional view of the plane partition of Figure 1 together with its column heights and diagonal slices, is shown in Figure 2.

**Figure 2.** A 2-dimensional view of the plane partition in Figure 1 and its diagonal slices. The upper (large sized) integers in the cells are the heights of the corresponding columns. The lower (small sized) integers are the \((i,j)\) cell coordinates.

**Figure 3.** A 2-dimensional view of a plane partition. \( H_m \) in each cell stand for height \( H \) and level \( m \). There are 13 level-1, 3 level-2, and 1 level-3 paths. The associated weight is \( A_\pi(t) = (1 - t)^{13}(1 - t^2)^3(1 - t^3) \).

4.4. Levels and paths. Consider the elements \( \{\pi_{i,j}\} \) that form a plane partition array. The element \( \pi_{i,j} \) with coordinates \((i, j)\) is contiguous with four surrounding elements at \((i - 1, j), (i + 1, j), (i, j - 1), (i, j + 1)\), possibly including trivial elements that have numerical value 0 (trivial elements of a plane partition that have height 0).
Following [1], we say that \( \pi_{i,j} \) has level \( l \geq 1 \) if

\[
\pi_{i,j} = \ldots = \pi_{(i+l-1),(j+l-1)} > \pi_{(i+l),(j+l)} \tag{52}
\]

With these definitions, a path of level-\( l \) is a set of contiguous level-\( l \) elements that have the same numerical value. They correspond to equal-height columns in the 3-dimensional view of \( \pi \).

We say that the plane partition \( \pi \) has \( p_j(\pi) \) level-\( j \) paths and assign it the weight \( A_\pi(t) \), given by

\[
A_\pi(t) = \prod_{j=1}^{\infty} \left( 1 - t^j \right)^{p_j(\pi)} \tag{53}
\]

An example of a larger plane partition is shown in Figure 3. In this example, there are 3 paths of column height 5. They consist of a level-1, a level-2 and a level-3 path. Equivalently, this height 5 horizontal plane (a set of contiguous paths of the same column height) has a maximal width of 3 paths, when measured diagonally.

It is straightforward to show that

\[
A_\pi(t) = b_{\mu_0}(t) \prod_{j=1}^{m} \Phi_{\mu_{(j+1)/\mu_{(j-1)}}}(t) \prod_{k=1}^{n} \Phi_{\mu_{(k-1)/\mu_{(k)}}}(t) \tag{54}
\]

where the Young diagrams \( \mu_j \) are the diagonal slices of \( \pi \), as given by (51).

4.5. A scalar product generates weighted plane partitions. We study the infinite scalar product

\[
\mathcal{S}(t) = \langle 0 | \ldots \Gamma_+(z^{-\frac{1}{2}}, t) \Gamma_+(z^{-\frac{1}{2}}, t) \Gamma_-(z^{\frac{1}{2}}, t) \Gamma_-(z^{\frac{3}{2}}, t) \ldots | 0 \rangle \tag{55}
\]

\[
= \langle 0 | \prod_{j=1}^{\infty} \Gamma_+(z^{-\frac{2j+1}{2}}, t) \prod_{k=1}^{\infty} \Gamma_-(z^{\frac{2k-1}{2}}, t) | 0 \rangle
\]

The scalar product \( \mathcal{S}(t) \) provides a generating function for plane partitions. This is seen in the following way. We insert a complete set of states \( \sum_{\mu_0} \frac{1}{b_{\mu_0}(t)} | \mu_0 \rangle \langle \mu_0 | \) between the central pair of vertex operators in (55), giving

\[
\mathcal{S}(t) = \sum_{\mu_0} \frac{1}{b_{\mu_0}(t)} \langle 0 | \prod_{j=1}^{\infty} \Gamma_+(z^{-\frac{2j+1}{2}}, t) | \mu_0 \rangle \langle \mu_0 | \prod_{k=1}^{\infty} \Gamma_-(z^{\frac{2k-1}{2}}, t) | 0 \rangle \tag{56}
\]

From (56) for right state vectors, and (57) for left state vectors, it is clear that all plane partitions of the form (51) receive a weight equal to

\[
b_{\mu_0}(t) \prod_{j=1}^{m} \Phi_{\mu_{(j+1)/\mu_{(j-1)}}}(t) z^{\frac{2j-1}{2}} | |_{\mu_{(j+1)}} - | |_{\mu_{(j-1)}} |)
\]

\[
\times \prod_{k=1}^{n} \Phi_{\mu_{(k-1)/\mu_{(k)}}}(t) z^{\frac{2k-1}{2}} | |_{\mu_{(k-1)}} - | |_{\mu_{(k)}} |)
\]

\[
= A_\pi(t) z^{\sum_j |\mu_j|} = A_\pi(t) z^{\frac{\pi}{|\pi|}}
\]

from (56). In other words,
Evaluating the right hand side of (62) using (13), we obtain

\[ S(t) = \sum_{\pi} A_{\pi}(t)z^{\pi} \]

4.6. Evaluating the scalar product directly. On the other hand, it is possible to evaluate \( S(t) \) directly. Using the commutation relation (13) repeatedly, we find

\[
S(t) = \left( \prod_{j=1}^{\infty} \frac{1 - tz^j}{1 - z^j} \right)^j \langle 0 | \prod_{j=1}^{\infty} \frac{1 - tz^j}{1 - z^j} \prod_{k=2}^{\infty} \frac{\Gamma_+ \left( z^{\frac{1}{2k-1}} \right), t^k \right| 0 \\
= \left( \prod_{j=1}^{\infty} \frac{1 - tz^j}{1 - z^j} \right)^j \langle 0 | \prod_{j=1}^{\infty} \frac{1 - tz^j}{1 - z^j} \prod_{k=2}^{\infty} \frac{\Gamma_+ \left( z^{\frac{1}{2k-1}} \right), t^k \right| 0 \\
(59) = \prod_{j=1}^{\infty} \left( \frac{1 - tz^j}{1 - z^j} \right)^j \langle 0 | 0 \rangle = \prod_{j=1}^{\infty} \left( \frac{1 - tz^j}{1 - z^j} \right)^j
\]

where we have used the fact that \( \langle 0 | \Gamma_{-} (z, t) = \langle 0 |, \) and \( \Gamma_+ (z, t) | 0 \rangle = | 0 \rangle \).

4.7. Equating the two results. Equating the above results, we have a new derivation of the 1-parameter extension of MacMahon’s generating function, first obtained in [1]

\[
(60) \sum_{\pi} A_{\pi}(t)z^{\pi} = \prod_{j=1}^{\infty} \left( \frac{1 - tz^j}{1 - z^j} \right)^j
\]

4.8. Weighted plane partitions with restricted levels. From the form of \( A_{\pi}(t) \) in (59), it is clear that by setting the deformation parameter to a primitive \( n \)-th root of unity, \( t = e^{2\pi i/n} \), \( n \in \mathbb{N} \geq 2 \), \( S(t) \) in (58–59) is the generating function of weighted plane partitions with paths of maximally level \( (n - 1) \).

5. Connection with the KP integrable hierarchy

5.1. Finite scalar products. We consider \( S_{s \times s}(t) \), the generating function of plane partitions of the same type as those generated by \( S(t) \), but now the plane partitions are confined to an \( s \times s \) square base (with no conditions on the column heights). \( S_{s \times s}(t) \) is obtained from the finite scalar product

\[
S_{s \times s}(t) = \langle 0 | \prod_{j=1}^{s} \frac{\Gamma_+ \left( z^{\frac{1}{2j-1}} \right), t^j \right| 0 \prod_{k=1}^{s} \frac{\Gamma_\left( z^{\frac{1}{2k-1}} \right), t^k \right| 0 \\
(61) = \langle 0 | \prod_{j=1}^{s} \frac{\Gamma_+ \left( z^{\frac{1}{2j-1}} \right), t^j \right| 0 \prod_{k=1}^{s} \frac{\Gamma_\left( z^{\frac{1}{2k-1}} \right), t^k \right| 0 \\
(61) = \prod_{j=1}^{s} \frac{1 - tu_jv_j}{1 - u_jv_j} \]

Next, we introduce dependence on two finite sets of parameters \( \{u_1, \ldots, u_s\} \) and \( \{v_1, \ldots, v_s\} \) by considering the more general finite scalar product

\[
S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s | 0) = \prod_{j=1}^{s} \frac{1 - tu_jv_j}{1 - u_jv_j} \]

We note that \( S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t) \) is the restriction of a function of two infinite sets of parameters \( u = \{u_1, u_2, \ldots\} \) and \( v = \{v_1, v_2, \ldots\} \), obtained by setting \( u_t = v_t = 0 \), for all \( t > s \), and we show that this function is a
KP $\tau$-function. More precisely, we show that there is an extension $A(u, v; t)$ of $S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)$ that is a $\tau$-function of KP in the power sum variables $x_m = \frac{1}{m} \sum_{j=1}^{\infty} u_j^m$.

5.2. An extension that is a non-trivial KP $\tau$-function. From chapter III, equation (4.7) in [2]

\begin{equation}
S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t) = \sum_{\lambda \mid l(\lambda) \leq s} s_{\lambda}(u_1, \ldots, u_s) S_{\lambda}(v_1, \ldots, v_s; t)
\end{equation}

where $s_{\lambda}(u_1, \ldots, u_s)$ is a Schur function

\begin{equation}
s_{\lambda}(u_1, \ldots, u_s) = \det \left(h_{\lambda_i - i + j}(u_1, \ldots, u_s)\right)
\end{equation}

\begin{equation}
\sum_{m=0}^{\infty} h_m(u_1, \ldots, u_s) z^m = \prod_{j=1}^{s} \frac{1}{1 - u_j z}
\end{equation}

and

\begin{equation}
S_{\lambda}(v_1, \ldots, v_s; t) = \det \left(q_{\lambda_i - i + j}(v_1, \ldots, v_s; t)\right)
\end{equation}

\begin{equation}
\sum_{m=0}^{\infty} q_m(v_1, \ldots, v_s; t) z^m = \prod_{j=1}^{s} \frac{1 - tv_j z}{1 - v_j z}
\end{equation}

Now consider

\begin{equation}
A(u, v; t) = \sum_{\lambda \mid l(\lambda) \leq s} s_{\lambda}(u_1, u_2, \ldots) S_{\lambda}(v_1, v_2, \ldots; t)
\end{equation}

where the sets $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ are now infinite, but the sum over the partitions $\lambda$ is still restricted, $\{\lambda \mid l(\lambda) \leq s\}$. $S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)$ is the restriction of $A(u, v; t)$ obtained by setting $u_t = v_t = 0$ for all $t > s$. Letting $\frac{1}{m} \sum_{j=1}^{\infty} u_j^m = x_m$ and $\frac{1}{m} \sum_{j=1}^{\infty} v_j^m = y_m$, we can write

\begin{equation}
A(u, v; t) = \sum_{\lambda \mid l(\lambda) \leq s} \chi_{\lambda}(x_1, x_2, \ldots) \chi_{\lambda}(\hat{y}_1, \hat{y}_2, \ldots)
\end{equation}

where $\hat{y}_m = (1 - t^m)y_m$, and $\chi_{\lambda}(x_1, x_2, \ldots)$ is the character polynomial

\begin{equation}
\chi_{\lambda}(x_1, x_2, \ldots) = \det \left(p_{\lambda_i - i + j}(x_1, x_2, \ldots)\right)
\end{equation}

\begin{equation}
\sum_{m=0}^{\infty} p_m(x_1, x_2, \ldots) z^m = \exp \left(\sum_{m=1}^{\infty} x_m z^m\right)
\end{equation}

The coefficients $\chi_{\lambda}(\hat{y}_1, \hat{y}_2, \ldots)$ are of the right form to satisfy the Plücker relations non-trivially [3], hence $A(u, v; t)$ is a non-trivial KP $\tau$-function in the variables $\{x_1, x_2, \ldots\}$ and $S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)$ is the restriction obtained by setting $u_t = v_t = 0$ for all $t > s$, in the power sums $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$.

5.3. Another extension that is a trivial KP $\tau$-function. It is possible to write $S_{s \times s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)$ as a restriction of another function $B(u, v)$, as follows.
\[
S_{s	imes s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t) = \prod_{i,j=1}^{s} \frac{1 - tu_i v_j}{1 - u_i v_j}
\]
\[
= \exp \left\{ \sum_{i,j=1}^{s} \left( \log(1 - tu_i v_j) - \log(1 - u_i v_j) \right) \right\}
\]
\[
= \exp \left\{ \sum_{k=1}^{\infty} \sum_{i,j=1}^{s} \frac{(1 - t^k)(u_i v_j)^k}{k} \right\}
\]

(73)

Now consider

\[
B(u, v) = \exp \left\{ \sum_{k=1}^{\infty} \sum_{i,j=1}^{s} \frac{(1 - t^k)(u_i v_j)^k}{k} \right\}
\]

(74)

where the sets \{u_1, u_2, \ldots\} and \{v_1, v_2, \ldots\} are now infinite. \(S_{s	imes s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)\) is the restriction of \(B(u, v; t)\) obtained by setting \(u_t = v_t = 0\) for all \(t > s\). Letting \(\frac{1}{s} \sum_{i=1}^{\infty} u_i^k = x_k\) and \(\frac{1}{s} \sum_{j=1}^{\infty} v_j^k = y_k\), we can write

\[
B(u, v) = \exp \left\{ \sum_{k=1}^{\infty} k(1 - t^k)x_k y_k \right\}
\]

(75)

which is a trivial KP \(\tau\)-function, since it is the exponential of a linear function in \(\{x_1, x_2, \ldots\}\), so it satisfies any nonlinear partial differential equation in the KP hierarchy trivially.

We conclude that it is possible to write \(S_{s	imes s}(u_1, \ldots, u_s, v_1, \ldots, v_s; t)\) as a restriction of either \(A(u, v)\) or \(B(u, v)\). The advantage of the former is that it offers a connection with a non-trivial solution of KP, in contrast with the latter which satisfies KP trivially.

6. \((t,q)\)-Operators and Macdonald Weighted Plane Partitions

6.1. \((t,q)\)-Fermions. The t-fermions are a special case of more general \((t,q)\)-fermions that depend on two deformation parameters \(t\) and \(q\). We obtain the latter as the components of the fermion generating functions

\[
\sum_{j \in \mathbb{Z}} \psi_j z^j = \exp \left\{ \sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^m \right\}
\]
\[
\times \exp \left\{ -\sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^{-m} \right\}
\]
\[
(76)
\]

and

\[
\sum_{j \in \mathbb{Z}} \psi_j^* z^j = \exp \left\{ -\sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^m \right\}
\]
\[
\times \exp \left\{ \sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^{-m} \right\}
\]

(77)

where the \((t,q)\)-Heisenberg generators \(h_m\) satisfy

\[
[h_m, h_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}
\]

(78)
The $t$-fermions and all related operators are obtained by setting $q = 0$. From the $(t, q)$-fermion generating functions, one deduces the $(t, q)$-(half-)vertex operators

\begin{align}
\Gamma_+(z, t, q) & = \exp \left( - \sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^{-m} \right) \\
\Gamma_-(z, t, q) & = \exp \left( - \sum_{m=1}^{\infty} \frac{1 - t^m}{m(1 - q^m)} h_m z^m \right)
\end{align}

which satisfy the commutation relation

\begin{align}
\Gamma_+(z, t, q) \Gamma_-(z', t, q) & = \prod_{n=0}^{\infty} \frac{1 - q^n \frac{t z'}{z}}{1 - q^n \frac{t z}{z'}} \Gamma_-(z', t, q) \Gamma_+(z, t, q) \\
& = \left( \frac{t z'}{z}, q \right)_\infty \Gamma_-(z', t, q) \Gamma_+(z, t, q)
\end{align}

6.2. Generating function for Macdonald weighted plane partitions. Consider the scalar product

\begin{align}
S(t, q) & = \langle 0 | \ldots \Gamma_+(z^{-\frac{3}{2}}, t, q) \Gamma_+(z^{-\frac{1}{2}}, t, q) \Gamma_-(z^\frac{1}{2}, t, q) \Gamma_-(z^\frac{3}{2}, t, q) \ldots | 0 \rangle \\
& = \langle 0 \rangle \prod_{j=1}^{\infty} \Gamma_+(z^{-\frac{2j+1}{2}}, t, q) \prod_{k=1}^{\infty} \Gamma_-(z^{\frac{2k-1}{2}}, t, q) | 0 \rangle
\end{align}

Using (81), one easily obtains

\begin{align}
S(t, q) & = \prod_{n=1}^{\infty} \left( \frac{\left( t z^n, q \right)_\infty}{\left( z^n, q \right)_\infty} \right)^n
\end{align}

which is the right-hand-side of the following equation for the generating function of 2-parameter weighted plane partitions derived combinatorially in [1]

\begin{align}
\sum_{\pi} F_\pi(q, t) z^{\left| \pi \right|} = \prod_{n=1}^{\infty} \left( \frac{\left( t z^n, q \right)_\infty}{\left( z^n, q \right)_\infty} \right)^n
\end{align}

What is missing is a Fock space derivation of the left-hand-side of (84) which requires a proof of the following statements

\begin{align}
\Gamma_-(z, t, q) | \mu \rangle & = \sum_{\lambda > \mu} P_{\lambda/\mu}(z, t, q) | \lambda \rangle \\
\langle \mu | \Gamma_+(z, t, q) & = \sum_{\lambda > \mu} P_{\lambda/\mu}(z^{-1}, t, q) \langle \lambda |
\end{align}

where $P_{\lambda/\mu}(z, t, q)$ is the skew Macdonald polynomial of a single variable $z$ [2].

7. Comments

We obtained a Fock space derivation of the 1-parameter $t$-deformation of MacMahon’s plane partition generating function in [1], starting from the underlying $t$-fermions of [5]. For $t = e^{2\pi i/n}$, we obtained generating functions of weighted plane partitions with horizontal paths that are maximally $n - 1$ rim hooks wide. When
formulated on a finite $s \times s$ square base, we showed that these generating functions are evaluations of KP $\tau$-functions.

Our proofs are based on the action of underlying fermions on vector states. This complicates the analysis, because we start from the anti-commutation relations and we derive the action of vertex operators (which are exponentials in bilinears in the fermions) on Fock space state vectors. It is possible to bypass these complications and work entirely in terms of the action of the Heisenberg generators as in [8]. This simplifies the treatment, as one starts by requiring the vertex operator actions that one wants, such as \[ S_3 \] at the expense of losing contact with the underlying fermions.

In [9], a connection between the $t$-deformation of MacMahon’s generating function was discussed also on the basis of Jing’s $t$-fermions. There it was pointed out, amongst other results, that for $t \in \mathbb{R}$ the expression \[ 60 \] is the topological string partition function on a conifold and a geometric interpretation of the parameter $t$ (called $Q$ in [9]) was obtained. The derivation of \[ 60 \] in [9] was based on commuting the vertex operators in the scalar product. Further, $t$-deformed free fermions and results that are related to those discussed in this work (and in [9]) were also obtained in [10].

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References

[1] M Vuletić, A generalization of MacMahon’s formula, \url{http://arxiv.org/abs/0707.0532}
[2] I G Macdonald, Symmetric functions and Hall polynomials, Oxford University Press, 1995.
[3] T Miwa, M Jimbo and E Date, Solitons, Cambridge University Press, 2000.
[4] M Vuletić, \url{http://arxiv.org/abs/math-ph/0702068} Int Math Res Notices IMRN (2007) Article 043.
[5] N Jing, J Math. Phys. 36 (12) (1995) 7073-7080.
[6] A Okounkov and N Reshetikhin, Amer Math Soc 16 (2003) 581–603. \url{math.CO/0107056}
[7] N Jing, J of Algebraic Combinatorics 3 (1994) 291-305.
[8] Th Lam, \url{http://arxiv.org/abs/math/0507341} Math Res Lett 13 (2006) 377–392.
[9] P Sulkowski, Deformed boson-fermion correspondence, $Q$-bosons, and topological strings on the conifold, \url{http://arxiv.org/abs/0808.2327}.
[10] N Tsilevich, Quantum inverse scattering method for the $q$-boson model and symmetric functions, \url{http://arxiv.org/abs/math/0512389}.

Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia.

E-mail address: foda, mwheeler@ms.unimelb.edu.au