Some Characteristic Properties of Parallel \( z \)-Equidistant Ruled Surfaces

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Some characteristic properties of two ruled surfaces whose principal normal vectors are parallel along their striction curves in \( E^3 \) are examined by assuming that the distance between two central planes at suitable points is constant, \( E^3 \). In case of which two ruled surfaces are close, the relationship between the integral invariants of this ruled surfaces is computed.

1. Introduction

The basis notions about ruled surfaces in \( E^3 \) are given in [1]. Parallel \( p \)-equidistant ruled surfaces are described, and some of their characteristic properties are given in Valeontis’s article entitled ”Parallel \( p \)-Aquidistante Regelflächen” [2]. Integral invariants, shape operators and spherical indicators of Parallel \( p \)-equidistant ruled surfaces were computed by Masal and Kuruoğlu in the articles [3–6]. Mannheim curves were described in Liu and Wang’s article entitled “Mannheim partner curve in 3-Space” [7]. Some characteristic properties of Mannheim curves and Mannheim offsets of ruled surfaces were studied in [8, 9].

2. Preliminaries

Let \( \alpha : I \rightarrow E^3 \) be a differentiable curve with arc-length parameter \( s \), and \( \{u_1, u_2, u_3\} \) be the Frenet frame of \( \alpha \) at the point \( \alpha(s) \), where

\[
\begin{align*}
u_1(s) &= \alpha'(s), \\
u_2(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\
u_3(s) &= u_1(s) \wedge u_2(s).
\end{align*}
\]

The Frenet formulas of \( \alpha \) are

\[
\begin{bmatrix}
u'_1 \\
u'_2 \\
u'_3
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & 0 \\
-k_1 & 0 & k_2 \\
0 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}.
\]

If \( \alpha \) is a curve and \( x \) is a generator vector, then the ruled surface \( X(s, v) \) has the following parameter representation:

\[
X(s, v) = \alpha(s) + v x(s).
\]

Namely, a ruled surface is a surface generated by the motion of a straight line \( x \) along \( \alpha \). Furthermore, if \( \alpha \) is a closed curve, then this surfaces is called closed ruled surface. Moreover, the drall \( P_x \), the striction \( \gamma \), the apex angle \( \lambda_x \), and the pitch \( l_x \) of the closed ruled surface are defined by

\[
P_x = \frac{\det(\alpha', x, x')}{\|x'\|^2},
\]

\[
\gamma(s) = \alpha(s) - \frac{\left< x'(s), \alpha'(s) \right>}{\|x'(s)\|^2} x(s),
\]

\[
\lambda_x = \langle d, x \rangle, \quad l_x = \langle V, x \rangle.
\]
The angle of the pitches, pitches, and dralls of the closed ruled surface generated by the Frenet vectors \( u_1, u_2, \) and \( u_3 \) are
\[
\begin{align*}
\lambda_{u_1} &= \frac{\phi}{\alpha_{u_1}} k_2 d s, & l_{u_1} &= \frac{\phi}{\alpha_{u_1}} d s, \\
\lambda_{u_2} &= 0, & l_{u_2} &= 0, \\
\lambda_{u_3} &= \frac{\phi}{\alpha_{u_3}} k_1 d s, & l_{u_3} &= 0,
\end{align*}
\]
where
\[
\begin{align*}
a_1 &= \int (p_{u_1} + z u_3 + q u_3) k_2 d s, \\
a_2 &= \int (p_{u_1} + z u_3 + q u_3) k_1 d s, \\
a_3 &= \int (p_{u_1} + z u_3 + q u_3) d s,
\end{align*}
\]
see [3].

**Definition 4.** Consider two space curves \( \alpha : I \to E^3 \) and \( \beta : I \to E^3 \), where \( I \) is a real interval that has at least four continuous derivatives. If there exists a corresponding relationship between the space curves \( \alpha \) and \( \beta \) such that the principal normal lines of \( \alpha \) coincide with the binormal lines of \( \beta \) at the corresponding points of the curves, then \( \alpha \) is called as a Mannheim curve, and \( \beta \) is called as a Mannheim partner curve of \( \alpha \). The pair of \( [\alpha, \beta] \) is said to be a Mannheim pair [7].

**Definition 5.** Let \( T \) be the unit tangent vector of the curve, \( \alpha : I \to E^3 \). If \( T \) makes a constant angle with a fixed line, then the curve \( \alpha \) is called the helix curve [1].

**Theorem 6.** The distance between corresponding points of the Mannheim partner curves in \( E^3 \) is constant [8].

**Theorem 7.** For a curve \( \alpha \) in \( E^3 \), there is a curve \( \beta \), so that \( [\alpha, \beta] \) is a Mannheim pair [8].

**Theorem 8.** Let curvature and torsion of \( \alpha \) be \( k_1 \) and \( k_2 \), respectively, and \( \alpha \) is Mannheim curve \( \iff \lambda = k_1/(k_1^2 + k_2^2) \), \( \lambda = 0 = \text{const} \). See [7].

### 3. Parallel \( z \)-Equidistant Ruled Surfaces

**Definition 9.** Let \( \alpha \) and \( \overline{\alpha} \) be two curves, and let \( \{u_1, u_2, u_3\} \) and \( \{v_1, v_2, v_3\} \) be the Frenet frames of \( \alpha \) and \( \overline{\alpha} \) at the points \( \alpha(s) \) and \( \overline{\alpha}(s) \), respectively, in \( E^3 \). If the unit principal normal vectors, \( u_3(s) \) and \( v_3(s) \), are generator vectors and \( \alpha \) and \( \overline{\alpha} \) are anchor curve, then parametric equations of the two ruled surfaces are \( S(s, v) = \alpha(s) + v u_3(s) \) and \( \overline{S}(s, v) = \overline{\alpha}(s) + v v_3(s) \) in \( E^3 \). If the generator vectors are parallel, and the distance \( p \) between polar planes in suitable points are constant, then this couple ruled surface are called \( p \)-equidistant ruled surfaces [9].

**Theorem 3.** The relationships between the angles of the pitch, pitches, and dralls of parallel \( p \)-equidistant are given as follows:
\[
\begin{align*}
\lambda_{v_1} &= \lambda_{u_1} + a_1, & p_{v_1} &= p_{u_1} = 0, \\
\lambda_{v_2} &= \lambda_{u_2} = 0, & p_{v_2} &= p_{u_2} d s/d s, \\
\lambda_{v_3} &= \lambda_{u_3} + a_2, & p_{v_3} &= p_{u_3} d s/d s, \\
& l_{v_1} = l_{u_1} = l_{u_3} = 0, & l_{v_2} &= l_{u_2},
\end{align*}
\]
where
\[
\begin{align*}
a_1 &= \int (p_{u_1} + z u_3 + q u_3) k_2 d s, \\
a_2 &= \int (p_{u_1} + z u_3 + q u_3) k_1 d s, \\
a_3 &= \int (p_{u_1} + z u_3 + q u_3) d s,
\end{align*}
\]
see [3].
Theorem 10. Let striction curves of $S$ and $S$ parallel $z$-equidistant ruled surfaces be $\gamma$ and $\overline{\gamma}$. Then, the relation between striction curves is given as follows:

$$\overline{\gamma} = \gamma + pu_1 + qu_3 + \left( k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right)$$

$$- k_2 \left( q' - \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1} u_2. \quad (12)$$

Proof. By substituting (10) into (11), we get

$$\alpha - \frac{\tilde{k}_1}{k_1 + k_2} v_2 = \gamma + pu_1 + z u_2 + qu_3. \quad (13)$$

Since vectors $v_2$ and $u_2$ are parallel vectors, then we can write

$$\alpha = \gamma + pu_1 + \left( z + \frac{\tilde{k}_1}{k_1 + k_2} \right) u_2 + qu_3. \quad (14)$$

By differentiating the last equation, we have

$$\alpha' = \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' - \left( \frac{\tilde{k}_1}{k_1 + k_2} \right) k_1 u_1$$

$$+ k_2 \left( - \frac{k_1 k_2}{k_1^2 + k_2^2} + q' + \left( z + \frac{\tilde{k}_1}{k_1 + k_2} \right) k_2 \right) \times u_2$$

$$\times \frac{\tilde{k}_1}{k_1 + k_2} \left( k_1^2 + k_2^2 \right)^{-1}. \quad (15)$$

From the last equation, we have

$$\langle \alpha', u_3' \rangle = -k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p'$$

$$+ k_2 \left( - \frac{k_1 k_2}{k_1^2 + k_2^2} + q' \right) \left( z + \frac{\tilde{k}_1}{k_1 + k_2} \right) k_2 \left( k_1^2 + k_2^2 \right)^{-1}. \quad (16)$$

Since vectors $v_2$ and $u_2$ are parallel vectors, then we can write

$$\overline{\gamma} = \alpha - \frac{u_2' \cdot \alpha'}{\|u_2'\|^2} u_2. \quad (17)$$

Substituting (14) and (16) into the last equation, then we obtain

$$\overline{\gamma} = \gamma + pu_1 + qu_3 + \left( k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right)$$

$$- k_2 \left( q' - \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1} u_2. \quad (18)$$

By (11), we have the following results.

Corollary 11. The distance between central planes of $S$ and $S$ parallel $z$-equidistant ruled surfaces is

$$z = \left( k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right)$$

$$- k_2 \left( q' - \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1}. \quad (19)$$

Corollary 12. If $\{\alpha, \gamma\}$ and $\{\overline{\alpha}, \overline{\gamma}\}$ pairs are Mannheim pairs, the distance between central planes of $S$ and $S$ parallel $z$-equidistant ruled surfaces is

$$z = \lambda \left( 2 + p' - q', \frac{k_1}{k_1} \right). \quad (20)$$

Corollary 13. Let the unit tangent vectors of the striction curves of surfaces $S$ and $S$ be $T$ and $\overline{T}$, respectively. Let $\delta$ and $\overline{\delta}$ be the angles between vectors $T$ and $\overline{T}$, and their projection vectors on central plane, respectively. In this case, the distance between central planes of parallel $z$-equidistant ruled surfaces $S$ and $S$ can be obtained by the following equation (Figure 1):

$$z = \left( k_1 \left( \cos \delta \sin \beta \frac{ds_y}{ds} + p' \right) - k_2 \left( \cos \delta \cos \beta \frac{ds_y}{ds} + q' \right) \right)$$

$$\times \left( k_1^2 + k_2^2 \right)^{-1}. \quad (21)$$

Proof. By some algebraic manipulations, $T$ and $\overline{T}$ can be calculated as follows:

$$T = \left( \cos \delta \sin \beta u_1 + \sin \delta u_2 + \cos \delta \cos \beta u_3 \right) \frac{ds}{ds_y}, \quad (22)$$

$$\overline{T} = \left( \cos \overline{\delta} \sin \overline{\beta} v_1 + \sin \overline{\delta} v_2 + \cos \overline{\delta} \cos \overline{\beta} v_3 \right) \frac{ds}{ds_y}. \quad (23)$$
By differentiating (14), we have

\[
\alpha' = \left[ \cos \delta \sin \beta \frac{d s_y}{d s} + p' - \left( z + \left( \frac{k_1}{k_1 + k_2} \right) \right) k_1 \right] u_1 \\
+ \left[ \sin \gamma \frac{d s_x}{d s} + pk_1 - qk_2 + \left( \frac{k_1}{k_1 + k_2} \right) \right] \]

\times u_2 + \left[ \cos \delta \cos \beta \frac{d s_y}{d s} + q' \right.

\left. + \left( z + \frac{k_1}{k_1 + k_2} \right) k_2 \right] u_3.

(24)

From the last equation, we can write

\[
\langle \alpha', u_2' \rangle = \left( -k_1 \cos \delta \sin \beta - k_1 p' + k_2 \cos \delta \cos \beta \right)
\times \frac{d s_y}{d s} + k_2 q' + \left( z + \frac{k_1}{k_1 + k_2} \right)
\times \left( k_1^2 + k_2^2 \right).

(25)

Substituting (14) and (23) into equation \( \gamma = \alpha - (\langle u_2', \alpha' \rangle / \| u_2' \|^2) u_2 \), we get

\[
\gamma = \gamma + pu_1 + qu_3
\plus \left[ \left( k_1 \left( \cos \delta \sin \beta \frac{d s_y}{d s} + p' \right) \right.

\left. - k_2 \left( \cos \delta \cos \beta \frac{d s_y}{d s} + q' \right) \right) \right]

\times \left( k_1^2 + k_2^2 \right)^{-1} u_2.

(26)

From (11), we have

\[
z = \left( k_1 \left( \cos \delta \sin \beta \frac{d s_y}{d s} + p' \right) \right.

\left. - k_2 \left( \cos \delta \cos \beta \frac{d s_y}{d s} + q' \right) \right)

\times \left( k_1^2 + k_2^2 \right)^{-1}.

(27)

Corollary 14. If \( \{ \alpha, \gamma \} \) and \( \{ \bar{\alpha}, \bar{\gamma} \} \) pairs are Mannheim pairs, then the distance between central planes of surfaces \( S \) and \( S \) is

\[
z = \lambda \left[ \left( \cos \delta \sin \beta \frac{d s_y}{d s} - k_1 \cos \delta \cos \beta \frac{d s_y}{d s} \right) \right.

\left. + p' \right]

\times \left( k_1^2 + k_2^2 \right)^{-1}.

(28)

Theorem 15. Let \( S \) and \( \bar{S} \) be parallel \( z \)-equidistant ruled surfaces. Then, the relation between Frenet frame of \( \alpha, \{ u_1, u_2, u_3 \} \) and of \( \bar{\alpha}, \{ v_1, v_2, v_3 \} \) is given as follows:

\[
v_1 = \cos \varphi u_1 - \sin \varphi u_3,
\]

\[
v_2 = u_2,
\]

\[
v_3 = \sin \varphi u_1 + \cos \varphi u_3,
\]

where \( \varphi \) is the angle between the vector \( v_1 \) and the vector \( u_1 \).

Proof. Let \( \varphi \) be the angle between the vector \( v_1 \) and the vector \( u_1 \). In this case, we can write

\[
v_1 = \cos \varphi u_1 + \sin \varphi u_3.
\]

Since the vector \( v_2 \) is parallel to the vector \( u_2 \), we have

\[
v_2 = u_2,
\]

\[
v_3 = \sin \varphi u_1 + \cos \varphi u_3.
\]

This completes the proof of the theorem.

Theorem 16. Let \( S \) and \( \bar{S} \) be parallel \( z \)-equidistant ruled surfaces. Let \( s \) and \( \bar{s} \) be arc parameters of anchor curves of \( S \)
and \( S \), respectively. If \( k_1, k_2 \) and \( \overline{k}_1, \overline{k}_2 \) are curvatures of anchor curves of \( S \) and \( \overline{S} \), respectively, there are following equations between these curvatures:

\[
\overline{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\overline{s}},
\]

\[
\overline{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\overline{s}}.
\]

**Proof.** Since \( S \) and \( \overline{S} \) parallel \( z \)-equidistant ruled surfaces, \( u_2(s) = v_2(\overline{s}) \). Differentiating this equation related to \( s \), we have

\[
-k_1 u_1 + k_2 u_3 = (-\overline{k}_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds}.
\]

Multiplying the last equation with \( v_1 \) and \( v_3 \), we have

\[
\overline{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\overline{s}},
\]

\[
\overline{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\overline{s}}.
\]

**Theorem 17.** The relations between apex angles of closed parallel \( z \)-equidistant ruled surfaces \( S \) and \( \overline{S} \)

1. \( \lambda_{v_1} = \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1, \quad b_1 = \oint_{(\alpha)} \overline{k}_2 d\overline{s} + \oint_{(\gamma)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} \)

2. \( \lambda_{v_2} = \lambda_{u_2} = 0 \)

3. \( \lambda_{v_3} = -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2, \quad b_2 = \oint_{(\alpha)} \overline{k}_1 d\overline{s} + \oint_{(\gamma)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} \)

**Proof.** The apex angle of closed ruled surface which is generated by the unit tangent vector \( v_1 \) is \( \lambda_{v_1} = \frac{1}{\varphi} \oint \overline{k}_2 d\overline{s} \). Substituting (14) into the last equation, we get

\[
\lambda_{v_1} = \oint_{(\gamma)} \overline{k}_1 d\overline{s} + \oint_{(\alpha)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds}.
\]

Substituting (9) into the last equation, we get

\[
\lambda_{v_1} = \oint_{(\alpha)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} + \oint_{(\gamma)} \overline{k}_1 d\overline{s}.
\]

Impending,

\[
b_1 = \oint_{(\alpha)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} + \oint_{(\gamma)} \overline{k}_1 d\overline{s}.
\]

Substituting (33) into the last equation, we have

\[
\lambda_{v_1} = \sin \varphi \oint_{(\alpha)} k_1 ds + \cos \varphi \oint_{(\gamma)} k_2 ds + b_1.
\]

From (5), we can write

\[
\lambda_{v_2} = \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1.
\]

From (5), the apex angles of closed ruled surfaces which are generated with vector \( v_2 \) and \( u_3 \) are

\[
\lambda_{v_2} = \lambda_{u_2} = 0.
\]

The apex angle of closed ruled surface which is generated with vector \( v_3 \) is \( \lambda_{v_3} = \oint_{(\alpha)} \overline{k}_1 d\overline{s} \). Substituting (14) into the last equation, we get

\[
\lambda_{v_3} = \oint_{(\gamma)} \overline{k}_1 d\overline{s} + \oint_{(\alpha)} \overline{k}_1 d\overline{s}.
\]

Substituting (9) into the last equation, we have

\[
\lambda_{v_3} = \oint_{(\alpha)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} + \oint_{(\gamma)} \overline{k}_1 d\overline{s}.
\]

Impending,

\[
b_2 = \oint_{(\alpha)} (\overline{k}_1 - k_1 v_1 + \overline{k}_2 v_3) \frac{d\overline{s}}{ds} + \oint_{(\gamma)} \overline{k}_1 d\overline{s}.
\]

\[
\lambda_{v_3} = \oint_{(\alpha)} \overline{k}_1 d\overline{s} + b_2.
\]
If \( \alpha \) and \( \gamma \) pairs are Mannheim pairs, then the relations between apex angles of closed parallel \( z \)-equidistant ruled surfaces \( S \) and \( \overline{S} \) are given as follows:

\[
\begin{align*}
(1) \quad \lambda_{v_1} &= \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_2} + b_1, \quad b_1 = \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s} + \oint_{(\gamma - (k_1 + k_2)^2)u_2} \kappa_1 d\bar{s}, \\
(2) \quad \lambda_{v_2} &= \lambda_{u_2}, \\
(3) \quad \lambda_{v_3} &= -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2, \quad b_2 = \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s} + \oint_{(\gamma - (k_1 + k_2)^2)u_2} \kappa_1 d\bar{s}.
\end{align*}
\]

**Theorem 19.** If we specially take helix curve instead of anchor curve of closed parallel \( z \)-equidistant ruled surface, there is relation between \( l_{u_1} \) and \( l_{v_1} \) as follows:

\[
l_{v_1} = \left( \cos \frac{k_1}{k_1} - \sin \frac{k_2}{k_2} \right) l_{u_1} + b_3,
\]

\[
b_3 = \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s} + \oint_{(\gamma - (k_1 + k_2)^2)u_2} \kappa_1 d\bar{s}.
\]

**Proof.** From (5), the pitch of closed ruled surface which is generated vector \( v_1 \) is \( l_{v_1} = \oint_{(a)} d\bar{s} \). Substituting (14) into the last equation, we get

\[
l_{v_1} = \oint_{(a)} d\bar{s} + \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s},
\]

Substituting (9) into the last equation, we get

\[
l_{v_1} = \oint_{(a)} d\bar{s} + \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s},
\]

Impeding,

\[
b_3 = \oint_{(pu_1 + (z + \frac{1}{k_1} + \frac{1}{k_2}))u_2 + qu_1} \kappa_1 d\bar{s} + \oint_{(\gamma - (k_1 + k_2)^2)u_2} \kappa_1 d\bar{s},
\]

Substituting (32) into this last equation, we have

\[
l_{v_1} = \oint_{(a)} \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) d\bar{s} + b_3.
\]

Since anchor curve is helix curve, then curvatures are constant. In this situation, we have

\[
l_{v_1} = \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) \oint_{(a)} d\bar{s} + b_3.
\]

**Theorem 20.** Let \( \{u_1, u_2, u_3\} \) and \( \{v_1, v_2, v_3\} \) be Frenet frame of anchor curves of \( S \) and \( \overline{S} \) closed parallel \( z \)-equidistant ruled surfaces. Then, the relations between dralls of ruled surfaces \( S \) and \( \overline{S} \) are given as follows:

\[
\begin{align*}
P_{v_1} &= P_{u_1} = 0, \\
P_{v_2} &= P_{u_2} \left( \cos \varphi + \sin \varphi \frac{k_1}{k_2} \right) \frac{d\bar{s}}{d\bar{s}}, \\
P_{v_3} &= P_{u_3} \left( \cos \varphi \frac{k_1}{k_2} + \sin \varphi \frac{k_2}{k_2} \right) \frac{d\bar{s}}{d\bar{s}}.
\end{align*}
\]

**Proof.** From (5), the drall of closed ruled surface which is generated vector \( v_1 \) is

\[
P_{v_1} = 0.
\]

From (5), the drall of closed ruled surface which is generated vector \( v_2 \) is

\[
P_{v_2} = \frac{k_2}{k_1 + k_2}.
\]

Substituting (32) and (33) into the last equation, we have

\[
P_{v_1} = \left( \sin \varphi k_1 + \cos \varphi k_2 \right) \frac{d\bar{s}}{d\bar{s}} \\
\times \left[ \left( \cos \varphi k_1 - \sin \varphi k_2 \right) \frac{d\bar{s}}{d\bar{s}} \right]^2 + \left( \sin \varphi k_1 + \cos \varphi k_2 \right) \left( \frac{d\bar{s}}{d\bar{s}} \right)^{-1},
\]

\[
P_{v_2} = \cos \varphi k_1 + \sin \varphi \frac{k_2}{k_2} \frac{k_2}{k_2} \frac{d\bar{s}}{d\bar{s}}.
\]

From (5),

\[
P_{v_2} = P_{u_2} \left( \cos \varphi + \sin \frac{k_1}{k_2} \right) \frac{d\bar{s}}{d\bar{s}}.
\]
The draft of closed ruled surface which is generated vector $v_3$ is $P_{v_3} = 1/k_2$,

$$P_{v_3} = \frac{k_2}{k_2}$$  \hspace{1cm} (58)

Substituting (5) and (33) into this last equation, we have

$$P_{v_3} = P_{u_3} \frac{k_2}{(\sin \varphi k_1 + \cos \varphi k_2)} ds.$$ \hspace{1cm} (59)

References

[1] H. H. Hacısalıoğlu, *Differential Geometry*, vol. 2, Faculty of Science, Ankara University, 1994.

[2] I. E. Valeontis, “Parallel-$p$-äquidistante Regelflächen,” *Manuscripta Mathematica*, vol. 54, no. 4, pp. 391–404, 1986.

[3] M. Masal, *A new some characteristic properties parallel p-equidistante ruled surfaces [M.S. thesis]*, Samsun, Turkey, 1994.

[4] M. Masal and N. Kuruoğlu, “Some characteristic properties of the parallel $p$-equidistant ruled surfaces in the Euclidean space,” *Pure Applied Mathematica Sciences*, vol. 49-50, 1999.

[5] M. Masal and N. Kuruoğlu, “Some characteristic properties of the shape operators of parallel $p$-equidistant rule surfaces,” *Bulletin of Pure & Applied Sciences E*, vol. 19, no. 2, pp. 361–364, 2000.

[6] M. Masal and N. Kuruoğlu, “Some characteristic properties of the spherical indicatrices leading curves of parallel $p$-equidistant rule surfaces,” *Bulletin of Pure & Applied Sciences E*, vol. 19, no. 2, pp. 405–410, 2000.

[7] H. Liu and F. Wang, “Mannheim partner curves in $3$-space,” *Journal of Geometry*, vol. 88, no. 1-2, pp. 120–126, 2008.

[8] K. Orbay and E. Kasap, “On mannheim partner curves in $E^3$,” *International Journal of Physical Sciences*, vol. 4, no. 5, pp. 261–264, 2009.

[9] K. Orbay, E. Kasap, and I. Aydemir, “Mannheim offsets of ruled surfaces,” *Mathematical Problems in Engineering*, vol. 2009, Article ID 160917, 9 pages, 2009.
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