The semi-Markov beta-Stacy process: a Bayesian non-parametric prior for semi-Markov processes.

Andrea Arfé∗†, Stefano Peluso‡, Pietro Muliere§

December 4, 2018

Abstract

The literature on Bayesian methods for the analysis of discrete-time semi-Markov processes is sparse. In this paper, we introduce the semi-Markov beta-Stacy process, a stochastic process useful for the Bayesian non-parametric analysis of semi-Markov processes. The semi-Markov beta-Stacy process is conjugate with respect to data generated by a semi-Markov process, a property which makes it easy to obtain probabilistic forecasts. Its predictive distributions are uniquely characterized by a reinforced random walk on a system of urns.

Running head: Semi-Markov beta-Stacy.

Keywords: Bayesian nonparametric; semi-Markov; beta-Stacy; reinforcement; urn model.

1 Introduction

Discrete-time Semi-Markov processes generalize Markov chains by allowing the holding times, the times spent in each visited state, to have arbitrary distributions other than the geometric (Cinlar 1969). In this paper, we address how to perform inferences and predictions for these processes from a Bayesian non-parametric perspective.

Because of their flexibility, discrete-time semi-Markov processes are used to predict many phenomena that evolve through a sequence of discrete states. Applications include time-series and longitudinal data analysis (Bulla and Bulla 2006), survival analysis and reliability (Barbu et al. 2004, Mitchell et al. 2011), finance and actuarial sciences (Janssen and Manca 2007), and biology (Barbu and Limnios 2009).

Despite their usefulness in applications, and in contrast with their continuous-time counterparts (Phelan 1990, Bulla and Muliere 2007, Zhao and Hu 2013), the literature...
on inferential or predictive approaches for discrete-time semi-Markov process is sparse (Barbu and Limnios, 2009, Chapter 4).

The available literature focuses on processes with a finite state space. From the frequentist perspective, Satten and Sternberg (1999) and Barbu and Limnios (2009) construct non-parametric estimators of the transition probabilities or the distributions of the holding times and study their asymptotic properties. From the Bayesian perspective, specific parametric models have been used in different settings (Patwardhan et al., 1980; Schiffman et al., 2007; Masala, 2013; Mitchell et al., 2011), but no general non-parametric approach has been developed.

In the sequel we will introduce the semi-Markov beta-Stacy process, a stochastic process useful for the analysis of semi-Markov models with a finite or, extending the available literature, countably infinite state space. Our perspective is both Bayesian and non-parametric because i) the Bayesian interpretation of probability is naturally suited for representing predictive uncertainty (de Finetti, 1937; Singpurwalla, 1988), and ii) non-parametric models provide a more honest assessment of posterior uncertainty than parametric models, as the formers are less tied to potentially restrictive or arbitrary parametric assumptions which may give a false sense of posterior certainty (Müller and Mitra, 2013; Hjort et al., 2010; Phadia, 2015; Ghosal and van der Vaart, 2017).

The semi-Markov beta-Stacy process is a generalization of the beta-Stacy process of Walker and Muliere (1997). Its law represents a prior distribution for the law of a discrete-time semi-Markov process. We will show below that this prior is conjugate with respect to i) the accumulating observations generated by a single process and ii) the finite histories of other similar (i.e. exchangeable) processes. This property makes it is particularly easy to perform inferences and predictions for a semi-Markov process.

In particular, the predictive distributions associated to the semi-Markov beta-Stacy process are available in closed form. These prescribe how to perform probabilistic predictions for the next state of a semi-Markov process given its observed history.

More precisely, we will show that these predictive distributions correspond to the transition kernels of a reinforced semi-Markov process. This is a novel kind of reinforced stochastic process which can be regarded as the discrete-time analogue of the reinforced continuous-time processes of Muliere et al. (2003) and Bulla and Muliere (2007). Here, the concept of “reinforcement” coincides with that of Coppersmith and Diaconis (1986) and Pemantle (1998, 2007): a process is reinforced if, whenever it visits a state, the same becomes more likely to be visited again in the future. Thus, reinforcement corresponds to a notion of learning from the past, a central idea in the Bayesian paradigm (Muliere et al., 2000, 2003; Bulla and Muliere, 2007; Peluso et al., 2015; Arfè et al., 2018).

To gain a deeper insight into the semi-Markov beta-Stacy process, we will characterize it using a reinforced urn process, i.e. a random walk over a system of reinforced urns. In the prototypical reinforced urn process, whenever a random walk visits an urn, a ball is extracted from the same. After noting its color, the extracted ball is replaced in the originating urn together with an additional ball of the same color (so the extracted color is reinforced, i.e. made more likely to be extracted in future draws from the same urn). Then, the random walk jumps to another urn determined by the extracted color.
Similar urn-based processes are receiving increasing attention in Statistics and Machine Learning to construct and understand nonparametric prior distributions for a wide range of stochastic models (Blackwell and MacQueen, 1973; Doksum, 1974; Mauldin et al., 1992; Walker and Muliere, 1997; Muliere et al., 2000, 2003; Bulla and Muliere, 2007; Ruggiero and Walker, 2009; Fortini and Petrone, 2012; Bacallado et al., 2013; Peluso et al., 2015; Caron et al., 2017; Arfé et al., 2018).

In more detail, below we show how a reinforced semi-Markov process can be interpreted as a particular reinforced urn process. By appealing to the representation theorems of Muliere et al. (2000) and Blackwell and MacQueen (1973), we also show the following characterization: if the future of a recurrent process (i.e. a process visiting all its states infinitely often) is predicted through the transition kernels of a reinforced semi-Markov process, then it will be as if i) the process being predicted is semi-Markov and ii) a semi-Markov beta-Stacy process prior is assigned to its probability law.

Before proceeding, we introduce some notational conventions. First, for convenience, if $F$ is a non-decreasing function on the integers (adjoined with the $\sigma$-algebra of all subsets), then the symbol $F$ will also be used to represent the corresponding induced measure. Hence, for example, $F(b) - F(a) = F((a, b])$ for all $a < b$, where the interval $(a, b]$ must be interpreted as the set of all integers $x$ such that $a < x \leq b$. Second, if $x = (x_1, x_2, \ldots)$ is a finite or infinite sequence, we denote with $x_{a:b}$ either the subsequence $(x_a, \ldots, x_b)$ of length $b-a+1$ if $a \leq b$ or, with some abuse of notation, the empty sequence of length 0 if $a > b$. Third and last, we adopt the standard conventions so that empty sums and products are respectively equal to 0 and 1.

The reminder of the paper is structured as follows. In Section 2 we define discrete-time semi-Markov processes and introduce several key notations. In Section 3 we introduce the semi-Markov beta-Stacy process prior. In Section 4 we derive the corresponding posterior distributions and show that this process prior is conjugate. In Section 5 we introduce reinforced semi-Markov process and show that these correspond to the predictive distributions obtained from the semi-Markov beta-Stacy process prior. In Section 6 we characterize the semi-Markov beta-Stacy process using a system of reinforced urns. In Section 7 we illustrate several generalizations, each based on alternative urn constructions. In Section 8 we illustrate the semi-Markov beta-Stacy process prior in a simulation study. Lastly, in Section 9 we provide some concluding remarks and point to possible applications of our work.

2 Semi-Markov processes: definition and basic properties

In the sequel, let $E$ be a non-empty finite or countably infinite set, adjoined with the discrete topology $\mathcal{E}$ of all its subsets.

**Definition 2.1.** Let $P = (P^{i,j})_{i,j \in E}$ be a transition matrix on $E$ such that $P^{i,i} = 0$ for all $i \in E$ and let $F = (F^i(\cdot) : i \in E)$ be a collection of probability distribution functions with support on the set of positive integers. Fixed $l_0$ in $E$, let the stochastic process
\((L, T) = (L_n, T_n)_{n \geq 0}\) be such that \(\mathbb{P}(L_0 = i | (P, F)) = I\{i = l_0\}\) and
\[\mathbb{P}(L_{n+1} = j, T_n \leq t | L_n = i, L_{0:n-1}, T_{0:n-1}, (P, F)) = F^i(t)P^{i,j}\]

almost surely for all integers \(n \geq 0, t \geq 1,\) and all \(i, j \in E.\) Then \((L, T)\) will be called a discrete-time Markov renewal process starting at \(l_0\) with characteristic couple \((P, F).\)

Suppressing the dependence on \(l_0,\) we write \((L, T) \sim MR(P, F).\)

**Remark 2.1.** In Definition 2.1 the holding time \(T_k\) depends only on the current state \(L_k\) and not on the following state \(L_{k+1}.\) More generally, \(T_k\) may depend on both \(L_k\) and \(L_{k+1}\) \((\text{Barbu and Limnios} \ 2009).\) This can be represented by substituting the distribution \(F^i\) in Definition 2.1 with one of the form \(F^{i,j}\) and letting \(F = (F^{i,j}(\cdot): i, j \in E, i \neq j).\) Each alternative may be more or less appropriate for different applications. For simplicity, we focus on the specification of Definition 2.1. In Section 7 we will describe how to generalize our results to cover the other case.

**Definition 2.2.** If \((L, T) \sim MR(P, F),\) define \(\tau_0 = 0\) and \(\tau_{n+1} = \sum_{h=0}^{n} T_h\) for all \(n \geq 0.\) Then, the process \((S_t)_{t \geq 0}\) defined by \(S_t = L_{N(t)},\) where \(N(t) = \sum_{n=1}^{\infty} I\{\tau_n \leq t\}\) for all integers \(t \geq 0,\) is the (discrete-time) semi-Markov Process associated to \((L, T),\)
\[S = (S_t)_{t \geq 0} \sim SM(P, F)\] in symbols. The times \((\tau_n)_{n \geq 1}\) are the jump times of \(S.\)

A semi-Markov process \((S_t)_{t \geq 0}\) describes the evolution in time of some system as it goes through different discrete states. The elements of \(E\) represent the possible states. Additionally, \(S_t\) is the state occupied at time \(t, N(t)\) is the number of state changes occurred up to time \(t, \tau_n\) is the time of the \(n\)-th state change, and \(T_k\) is the length of time the system spends in its \(k\)-th state (so the system first visits its \(k + 1\)-th state at time \(\tau_k + T_k).\) These interpretations are possible because the assumption that \(P^{i,j} = 0\) for all \(i \in E\) implies that \(L_k \neq L_{k+1}\) for all \(k\) with probability 1.

**Example 2.1.** Mitchell et al. \(2011\) use a semi-Markov process with state-space \(E = \{\text{"infected"}, \text{"not infected"}\}\) to model the time changes in the Human Papilloma Virus status of patients who may go through several infection periods. Here \(S_t \in E\) is the infection status of an individual at time \(t, N(t)\) is the number changes in the infection status that an individual experienced by time \(t, \tau_k\) is the time of the \(k\)-th change in the infection status of a patient, \(T_k\) is the length of time occurring between the \(k\)-th and \(k + 1\)-th changes in infection status, and \(L_k \in E\) is the infection status of a patient after these changes for the \(k\)-th time. For example, if at time \(\tau_k\) the patient becomes infected \((L_k = \text{"infected"}),\) then \(T_k\) is the length of time before the patient will become infection-free again \((L_{k+1} = \text{"not infected"}).\)

**Example 2.2.** Barbu and Limnios \(2009\) Sections 3.4) consider a semi-Markov model to describe the operation of a textile factory. To reduce pollution, the factory waste is treated in a disposal unit before being eliminated. To avoid stopping the factory, if the disposal unit fails, waste is temporarily stored in a tank. If the disposal unit is repaired before the tank is full, the factory continues operating and the tank is immediately purged. Otherwise, the whole factory must stop and a certain time is necessary to restart it. The state space of the process is thus \(E = \{1, 2, 3\}:\) 1 represents the state
where the factory if fully operational and the tank is empty, 2 represents the state where the disposal unit is malfunctioning but the factory is still operational (i.e. the tank is not full), and 3 represents the state where the factory is stopped. Additionally, it is $P^{1,3} = P^{3,2} = 0$. The distribution the time until the next disposal unit failure (i.e. the holding time of the state 1) is $F^1(\cdot)$, the distribution of the time until a malfunctioning disposal unit is either restored or when it fully breaks down (i.e. the holding time of state 2) is $F^2(\cdot)$, while the distribution of the time required to restart the factory after the tank fills up (i.e. the holding time of the state 3) is $F^3(\cdot)$.

To highlight the relation between Semi-Markov and Markov chains, suppose $S \sim SM(P, F)$ is such that $F^i(\{t\}) = p_i (1 - p_i)^{t - 1}$ for all integers $t \geq 1$ and some $p_i \in (0, 1)$ (i.e. the holding times of the state $i \in E$ are geometrical distributed with parameter $p_i$). Then $S$ is a (homogeneous) Markov chain such that $P(S_{t+1} = j | S_{0:t-1}, S_t = i) = p_{i,j}$ for all $j \in E$, $j \neq i$ and $P(S_{t+1} = i | S_{0:t-1}, S_t = i) = 1 - p_i$ for all $t \geq 1$. Conversely, if $S$ is a Markov chain with transition matrix $(p_{i,j})_{i,j \in E}$, then $S \sim SM(P, F)$ with $P^{i,j} = p_{i,j} / (1 - p_i)$ for all $j \neq i$, $P^{i,i} = 0$, and $F^i(t) = (1 - p_i) p_{i,t}^{-1}$ for all $t \geq 1$.

Note that, since $P^{i,i} = 0$ and $F^i$ has support on the positive integers for all $i \in E$, the semi-Markov process $S$ cannot have absorbing states, i.e. states such that $S_t = i$ for all sufficiently large $t \geq 0$ with positive probability. This assumption simplifies our analysis, although it might be restrictive for some applications. The presence of an absorbing state $i$ could be allowed by letting $P^{i,i} = 1$ and $F^i(\{+\infty\}) = 1$. With additional effort, the results in the following sections could be extended to this case as well.

**Remark 2.2.** Knowing $S_{0:t}$ is equivalent to knowing the values of $N(t)$, $L_{0:N(t)}$, $\tau_{1:N(t)}$, and that $\tau_{N(t)+1} > t$. Furthermore, denote $l(t) = t - \tau_{N(t)} = \max\{k = 0, 1, \ldots, t : S_k = S_{t-k} = \cdots = S_{t-k}\}$ the time spent by $S$ in the state $S_t$ just prior to time $t$. Then knowing $S_{0:t}$ is the same as knowing the values of $N(t)$, $L_{0:N(t)}$, $T_{0:N(t)-1}$, and that $T_{N(t)} > l(t)$.

**Example 2.3.** To exemplify, note that observing $S_{0:5} = (i_0, i_0, i_1, i_2, i_2, i_2)$ for some distinct $i_0, i_1, i_2 \in E$ is equivalent to observing $N(5) = 2$, $L_0 = i_0$, $L_1 = i_1$, $L_2 = i_2$, $\tau_1 = 2$, $\tau_2 = 3$, $\tau_3 > 5$, $l(5) = 2$, $T_0 = 2$, $T_1 = 1$, and $T_2 \geq 3$, i.e. $T_2 > 2 = l(5)$.

### 3 The semi-Markov beta-Stacy process prior

From a Bayesian nonparametric perspective [Ferguson, 1973; Hjort et al., 2010; Müller and Mitra, 2013], a prior distribution on the law of a semi-Markov process $S \sim SM(P, F)$ is the law of a stochastic process whose sample paths are characteristic couples $(P, F)$ with probability 1. The semi-Markov beta-Stacy process is one such stochastic process. Our strategy to define it is to separately assign a nonparametric prior distribution to i) each holding time distribution $F^i$ and ii) the transition matrix $P$.

As a starting point, we consider the discrete-time beta-Stacy process of [Walker and Muliere, 1997], a common Bayesian nonparametric prior for time-to-event distributions (Singpurwalla, 2006; Bulla and Muliere, 2007; Rigat and Muliere, 2012; Arfé et al., 2018). The beta-Stacy process will be used as the prior for the holding time distributions $F^i$. 


Definition 3.1 (Walker and Muliere (1997)). Let $c(t)$ be a positive real number for all integer $t > 0$. Also let $F_0$ be a probability distribution function with support on the set of positive integers. A random cumulative distribution function $F$ with support on the set of positive integers is said to be a beta-Stacy process $BS(c,F_0)$ if there exists a sequence $(U_i)_{i \geq 1}$ of independent random variables such that i) $U_i \sim \text{Beta}(c(t)F_0(\{t\}),c(t)F_0((t,\infty)))$ for all integers $t \geq 1$ and ii) $F((t,\infty)) = \prod_{k=1}^{t}(1-U_k)$ for all integers $t \geq 0$.

Remark 3.1. If $F \sim BS(c,F_0)$, then $E[F(t)] = F_0(t)$ and $\text{Var}(F(t))$ is a decreasing function of $c(t)$ such that $\text{Var}(F(t)) \to 0$ as $c(t) \to +\infty$. Hence $F_0$ is the mean of the process, while $c$ controls its dispersion (Walker and Muliere 1997).

The beta-Stacy process is especially useful thanks to its conjugacy property, which implies that the posterior distribution of $F \sim BS(c,F_0)$ conditional on a sample of exact observations from $F$ is again a beta-Stacy process. The beta-Stacy process is also conjugate with respect to an observation which has been censored, i.e. whose value is only known to exceed some known constant (Kalbfleisch and Prentice 2002 Singpurwalla 2006). These properties are summarized in the following Proposition, which is a specific case of the more general Theorem 1 of Walker and Muliere (1997).

Proposition 3.1 (Walker and Muliere (1997)). If, conditionally on $F \sim BS(c,F_0)$, $T_1,\ldots,T_n$ are independently distributed according to $F$, then the posterior distribution of $F$ given $T_1,\ldots,T_n$ is $BS(c_*,F_*)$, where

$$F_*(t,\infty) = \prod_{s=1}^{t} \left[ 1 - \frac{c(s)F_0(\{s\}) + N(\{s\})}{c(s)F_0([s,\infty)) + N([s,\infty))} \right]$$

$$c_*(t) = \frac{c(t)F_0((t,\infty)) + N((t,\infty))}{F_*(t,\infty)}.$$

where $N(t) = \sum_{i=1}^{n} I(T_i \leq t)$. Instead, the posterior distributions of $F$ given $T_n > t^*$ (i.e. a censored observation), where $t^*$ is a fixed constant, is $BS(c_*,F_*),$ where now

$$F_*(t,\infty) = \prod_{s=1}^{t} \left[ 1 - \frac{c(s)F_0(\{s\})}{c(s)F_0([s,\infty)) + I\{t^* \geq s\}} \right] ,$$

$$c_*(t) = \frac{c(t)F_0((t,\infty)) + I\{t^* \geq s\}}{F_*(t,\infty)}.$$
Remark 3.2. If $P \sim \text{Dir}(m)$, then $P(A) \sim \text{Beta}(m(A), m(A^c))$ for all $A$. In particular, $E[P(A)] = m(A)/m(E)$ and $\text{Var}(P(A)) \to 0$ as $m(E) \to +\infty$, so $m(E)$ controls the dispersion of $P(\cdot)$ around its mean $m(\cdot)/m(E)$. Additionally, if $A \subseteq \mathcal{E}$ is such that $m(A) = 0$, then $P(A) \sim \text{Beta}(0, m(E))$ and so $P(A) = 0$ almost surely.

The Dirichlet process is a particular case of the beta-Stacy process. In particular, since in our setting $E$ is countable, this can be identified with a set of the form $E = \{1, 2, \ldots, k\}$ for some $k \leq +\infty$. With this identification, let $m$ be a measure on $(E, \mathcal{E})$ such that $0 < m(E) < +\infty$ and let $P \sim \text{Dir}(m)$. The probability measure $P$ is entirely determined by its distribution function $F(t) = \sum_{x \in E: x \leq t} P(\{x\})$. Following the same reasoning as in [Walker and Muliere (1997)], it can be shown that $F \sim BS(c, F_0)$, where $c(t) = m(E)$ for all integers $t > 0$ and $F_0$ is determined by $F_0(\{t\}) = m(\{t\})/m(E)$ for all integers $t$ such that $0 < t \leq k$, and $F_0(\{t\}) = 0$ for $t > k$.

Akin as the beta-Stacy process, the Dirichlet process is also conjugate, as highlighted by the following proposition. This could be proved either by representing the Dirichlet process as a specific case of the beta-Stacy process, or by appealing to Theorem 1 of [Ferguson (1973)] and the facts that $E$ is countable and $\mathcal{E}$ is its power set.

Proposition 3.2 ([Ferguson (1973)]). Suppose that $P \sim \text{Dir}(m)$ and, conditionally on $P$, $X_1, \ldots, X_n$ are independently distributed with common law $P$. Then the posterior distribution of $P$ given $X_1, \ldots, X_n$ is $\text{Dir}(m_\ast)$, where $m_\ast$ is the measure on $E$ determined by $m_\ast(\{i\}) = m(\{i\}) + \sum_{j=1}^n I\{X_j = i\}$ for all $i \in E$.

Having introduced all required elements, we are finally ready to define the semi-Markov beta-Stacy process. To do so, let $m^i(\cdot)$ be a measure on $(E, \mathcal{E})$ such that $0 < m^i(E) < +\infty$ and $m^i(\{i\}) = 0$ for all $i \in E$. Let $c^i(t)$ be a positive real number for any integer $t > 0$. Also let $F^i_0$ be a distribution function with support on the set of positive integers for all $i \in E$. Lastly, let $m = (m^i)_{i \in E}$, $c = (c^i)_{i \in E}$, and $F_0 = (F^i_0)_{i \in E}$.

Definition 3.3. A random characteristic couple $(P, F)$ has a semi-Markov beta-Stacy distribution with parameters $(m, c, F_0)$, or $(P, F) \sim \text{SMBS}(m, c, F_0)$, if:

1. $P$ and $F$ are independent;
2. the rows $P^i(\cdot)$, $i \in E$, of $P$ are independent;
3. the distributions $F^i$, $i \in E$, in $F$ are independent;
4. $P^i(\cdot)$ is a Dirichlet process with base measure $m^i$ for all $i \in E$: $P^i(\cdot) \sim \text{Dir}(m^i)$;
5. for all $i \in E$, $F^i$ is a beta-Stacy process with precision parameters $c^i$ and centering distribution $F^i_0$, $F^i \sim BS(c^i, F^i_0)$.

Note that each realization of $(P, F) \sim \text{SMBS}(m, c, F_0)$ is a valid characteristic couple, justifying the use of the law of a semi-Markov beta-Stacy process as a prior distribution for a characteristic couple $(P, F)$.

More precisely, if $(P, F) \sim \text{SMBS}(m, c, F_0)$, then with probability 1, i) $F^i(\cdot)$ is a cumulative distribution function with support on the positive integers and ii) $P$ is a
transition matrix on $E$ such that $P^{i,i} = 0$ for all $i \in E$. The first point follows directly from the properties of the beta-Stacy process. The second point instead follows because each realization of the Dirichlet process is almost surely a probability measure. This implies that $0 \leq P^{i,j} = P^i(\{j\}) \leq 1$ and $\sum_{j \in E} P^{i,j} = \sum_{j \in E} P^i(\{j\}) = P^i(E) = 1$ for all $i, j \in E$ with probability 1. Since $m^i(\{i\}) = 0$ for all $i \in E$, it must also be $P^{i,i} = 0$ almost surely by Remark 3.2.

More generally, it will be $P^i(\{j\}) = P^{i,j} = 0$ almost surely for all $j \in E$ such that $m^i(\{j\}) = 0$. In this case, each realization of a SMBS$(m, c, F_0)$ will be the law of a semi-Markov process which cannot perform transition from $i$ to $j$.

4 Posterior computations

We will now prove that the semi-Markov beta-Stacy process prior is conjugate. To do so, we will need to introduce some additional notions.

Consider a finite sequence of states $s_{0:t} = (s_0, \ldots, s_t) \in E^{t+1}$. For each $i \in E$, any maximal subsequences $s_{a:b}$ (0 $\leq a \leq b \leq t$) such that $s_c = i$ for all $a \leq c \leq b$ will be called an $i$-block of $s_{0:t}$. In particular, an $i$-block $s_{a:b}$ will be called terminal if $b = t$, non-terminal otherwise. Suppose now that $S \sim SM(P, F)$. Moreover, $N^{i,t}(l)$ will denote the number of non-terminal $i$-blocks of length $l$ present in $S_{0:t} \in E^{t+1}$. Additionally, for all $i, j \in E$, $i \neq j$, let $M^{i,j}(t) = \sum_{k=1}^t I\{S_{k-1} = i, S_k = j\}$ be the number of transitions from state $i$ to state $j$ in $S_{0:t}$.

**Remark 4.1.** As observed in Remark 2.2, knowing $S_{0:t}$ is equivalent to knowing the values of $N(t)$, $L_{0,N(t)}$-block $\tau_1$, $N(t)$, $T_{0,N(t)}$, and that $T_{N(t)} > l(t)$. This implies that the terminal block of $S_{0:t}$ is an $L_{N(t)}$-block $S_{l(t):l(t)}$ of length $x(t)$. Additionally, $S_{0:t}$ contains exactly $N(t)$ non-terminal blocks. For $k = 0, \ldots, N(t) - 1$, the $k + 1$-th of such non-terminal blocks is the $L_k$-block $S_{\tau_k, \tau_k + 1}$ of length $T_k$. Consequently, $N^{i,t}(l) = \sum_{k=0}^{N(t)-1} I\{T_k \leq l, L_k = i\}$.

**Example 4.1.** Going back to Example 2.3, the blocks of $S_{0:5} = (i_0, i_0, i_1, i_2, i_2, i_2)$ are the non-terminal $i_0$-block $S_{0:2}$ of length $T_0 = 2$, the non-terminal $i_1$-block $S_{2:2}$ of length $T_1 = 1$, and the terminal $i_2$-block $S_{3:5}$ of length $3 = l(5) + 1$. Additionally: $N^{i_0,5}(1) = 0$, $N^{i_0,5}(2) = N^{i_0,5}(1) = 1$ for all $l \geq 2$; $N^{i_1,5}(1) = N^{i_1,5}(5) = 1$ for all $l \geq 1$; $N^{i_2,5}(l) = 0$ for all $l > 0$ if $i \neq i_0, i_1$; and $M^{i_0,i_1}(5) = M^{i_1,i_2}(5) = 1$.

With these notations, we can now state the following theorem.

**Theorem 4.1.** Suppose that, conditionally on $(P, F) \sim SMBS(m, c, F_0)$, it is $S \sim SM(P, F)$. Then, the posterior distribution of $(P, F)$ conditional on $S_{0:t} = i_{0:t}$ is $SMBS(m_*, c_*, F_*)$, where:

1. For all $i \in E$, $m^i_*$ is defined by $m^i_*(\{j\}) = m^i(\{j\}) + M^{i,j}(t)$ for $j \in E$, $j \neq i$. 

8
2. For all $i \in E$, $i \neq i_t$, $F^i_t$ and $c^i_t$ are determined by letting
\[
F^i_t((u, +\infty)) = \prod_{s=1}^{u} \left[ 1 - \frac{c^i(s)F^i_0([s, +\infty)) + N^{i,t}([s, +\infty))}{c^i(s)F^i_0([s, +\infty)) + N^{i,t}([s, +\infty))} \right]
\]
for each integer $u > 0$.

3. For $i = i_t$, $F^i_t$ and $c^i_t$ are instead determined by letting
\[
F^i_t((u, +\infty)) = \prod_{s=1}^{u} \left[ 1 - \frac{c^i(s)F^i_0([s, +\infty)) + N^{i,t}([s, +\infty)) + I\{l(t) \geq s\}}{c^i(s)F^i_0([s, +\infty)) + N^{i,t}([s, +\infty)) + I\{l(t) \geq s\}} \right]
\]
for each integer $u > 0$.

Proof. To begin, note that by Remark 2.2 observing a sequence of states $S_{0:t} = i_{0:t}$ such that $N(t) = n$ is equivalent to observing $L_{0:n} = l_{0:n}$, $T_{0:n-1} = t_{0:n-1}$, and $T_n > l(t)$, where $l_{0:n}$ is the sequence of distinct states in $i_{0:t}$ (in the same order) and the times $t_{0:n-1}$ are determined uniquely by the position of the the state changes in the sequence $i_{0:t}$. Consequently, by Remark 4.1 the likelihood function associated to the observation of $S_{0:t} = i_{0:t}$ is given by
\[
\mathbb{P}(S_{0:t} = i_{0:t} | \mathbf{P}, \mathbf{F}) = \mathbb{P}(L_{0:n} = l_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > l(t) | \mathbf{P}, \mathbf{F})
\]
\[
= F^{l_0}(t_0) \left[ \prod_{k=1}^{n-1} F^{l_k}(\{t_k\}) P^{l_{k-1},l_k} \right] \cdot \left[ F^{l_n}(l(t), +\infty) \right]
\]
\[
= \left[ \prod_{i \in E} \prod_{s=1}^{t} F^i(\{s\})^{N^i(\{s\})} F^i(l(t), +\infty) \right] \left[ \prod_{i \in E} P^i(\{j\})^{M^i(\{j\})} \right]
\]

Since the likelihood can be factorized as the product of individual terms depending only on $F^i$ or $P^i(\cdot)$ for some $i$, by points 1-3 of Definition 3.3 it follows that, conditionally on $S_{0:t} = i_{0:t}$, i) $\mathbf{P}$ and $\mathbf{F}$ are independent, ii) the rows $P^i(\cdot)$, $i \in E$, of $\mathbf{P}$ are independent, and iii) the distributions $F^i$, $i \in E$, in $\mathbf{F}$ are independent.

It can now be seen that: i) the posterior distribution of $F^i$, $i \neq l_n$ depends only on the observed values of those $T_k$ such that $L_k = i$ and it is the same as if these value were obtained as a random sample of independent and identically distributed observations from $F^i$; ii) the same is true for the posterior distribution of $F^{l_n}$ except that $T_n$ is censored, as only $T_n > l(t)$ is known; iii) the posterior distribution of $P(l(\cdot))$ depends only on each and only those $l_k$ in the sequence $l_{0:n}$ which are preceded by the state $i$; the corresponding posterior distribution is the same as if these were a random sample from $P^i(\cdot)$. The thesis now follows from Propositions 3.1 and 3.2. \qed
Theorem 4.1 allows to compute the posterior distribution of \((P,F)\) associated to the observation of the history \(S_{0:t}\) up to time \(t\) of some semi-Markov process \(S \sim SM(P,F)\). For example, in the context of Example 2.2, the history \(S_{0:t}\) may represent the (unreplicable) history of failures in the operation of the textile factory.

In some settings, however, multiple independent semi-Markov processes \(S^1, \ldots, S^n \sim SM(P,F)\) may be observed up to fixed time points \(t^1, \ldots, t^n\), generating data \(S^1_{0:t^1}, \ldots, S^n_{0:t^n}\). For instance, in the context of Example 2.1, \(S^1_{0:t^1}, \ldots, S^n_{0:t^n}\) may represent the histories of infection status of \(n\) independent patients. In this case, the posterior distribution of \((P,F)\) is obtained by iteratively applying Theorem 4.1.

It could also be shown that Theorem 4.1 remains valid also if the the process \(S\) is observed up to some stopping time \(\tau\), so that the posterior distribution of \((P,F)\) given \(S_{0:\tau}\) is the semi-Markov beta-Stacy process obtained by applying Theorem 4.1 after substituting \(S_{0:\tau}\) for \(S_{0:t}\). Following an argument similar as those presented by Heitjan and Rubin (1991), the same result also holds if \(\tau\) is a random variable a priori independent of \(S\) and \((P,F)\).

5 Predictive laws and reinforced semi-Markov processes

We now address the problem of predicting the evolution of a process \(S \sim SM(P,F)\). Specifically, assuming \((P,F) \sim SMBS(m,c,F_0)\), we derive the one-step-ahead predictive distributions of \(S\), i.e. the conditional distributions \(P(S_{t+1} = |S_{0:t})\) for \(t \geq 0\). These play an important role in applications. For instance, in Example 2.1, they allow to predict the future infection status of an individual patient given its history of infections. Instead, in Example 2.2, they allow to quantify the future risk that the textile factory will have to stop its operations.

**Theorem 5.1.** Suppose that, conditionally on \((P,F) \sim SMBS(m,c,F_0)\), it is \(S \sim SM(P,F)\). Define for simplicity \(x(t) = l(t) + 1\) for all integers \(t \geq 0\). Then, with probability 1, \(P(S_{t+1} = |S_{0:t}) = k_t(S_{0:t}; \cdot)\) for all integers \(t \geq 0\), where, letting \(S_t = i\), \(k_t\) is the transition kernel defined as follows:

\[
k_t(S_{0:t}; i) = \frac{d^i(x(t))F_0^i((x(t), +\infty)) + N^{i,t}((x(t), +\infty))}{c^i(x(t))F_0^i([x(t), +\infty)) + N^{i,t}([x(t), +\infty))} \times \frac{m^i(\{j\}) + M^{i,j}(t)}{m^i(E) + \sum_{h \neq i} M^{i,h}(t)},
\]

for all \(j \neq i\).

**Proof.** Suppose that, conditionally on \((P,F) \sim SMBS(m,c,F_0)\), \(S \sim SM(P,F)\). To prove the thesis, observe that by Remark 2.2 it is

\[
P(S_{t+1} = j|S_{0:t}, P, F) = P(L_{N(t+1)} = j|N(t), L_{0:N(t)}, T_{0:N(t)-1}, T_{N(t)} > l(t), P, F)
\]
Thus, from Theorem 4.1, 

$$P(S_{t+1} = j|S_{0:t}, P, F) = P(T_n > x(t)|L_{0:n} = i_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > l(t), P, F)$$

$$= \frac{F^{i_n}((x(t), +\infty))}{F^{i_n}((l(t), +\infty))}.$$  

Since $F^{i_n}$ has a beta-Stacy distribution, by Theorem 4.1 conditionally on $S_{0:t}$ it is

$$F^{i_n}((x(t), +\infty)) = \prod_{k=1}^{x(t)} (1 - U_k) = (1 - U_x(t))F^{i_n}((l(t), +\infty))$$

for independent $U_1, \ldots, U_x(t)$ such that

$$U_x(t) \sim Beta(c^{i_n}(x(t))F^{i_n}_0(\{x(t)\}), c^{i_n}(x(t))F^{i_n}_1((x(t), +\infty))).$$

Thus, from Theorem 4.1

$$P(S_{t+1} = j|S_{0:t}) = E[P(S_{t+1} = j|S_{0:t}, P, F)|S_{0:t}]$$

$$= E[1 - U_x(t)|S_{0:t}]$$

$$= \frac{F^{i_n}((x(t), +\infty))}{F^{i_n}([x(t), +\infty])}$$

$$= 1 - \frac{c^{i_n}(x(t))F^{i_n}_0(\{x(t)\}) + N^{i_n,d}(\{x(t)\})}{c^{i_n}(x(t))F^{i_n}_1([x(t), +\infty]) + N^{i_n,d}([x(t), +\infty]) + I\{l(t) \geq x(t)\}}$$

$$= k_t(S_{0:t}, j)$$

as needed.

Continuing, on the event $\{N(t) = n, L_{0:n} = i_{0:n}, T_{0:n-1} = t_{0:n-1}\}$ with $j \neq i_n$, $P(S_{t+1} = j|S_{0:t}, P, F)$ equals

$$P(T_n = x(t), L_{n+1} = j|L_{0:n} = i_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > l(t), P, F) =$$

$$= \frac{F^{i_n}((x(t)))}{F^{i_n}([x(t), +\infty])}, P^{i_n,j}$$

$$= \left(1 - \frac{F^{i_n}((x(t), +\infty))}{F^{i_n}([x(t), +\infty])}\right), P^{i_n,j}$$

$$= U_x(t)P^{i_n,j}.$$ (1)

By Theorem 4.1 $U_x(t)$ and $P^{i_n,j} \sim Beta(m^{i_n}_{x}(\{j\}), m^{i_n}_{x}(E\{j\}))$ are independent given on $S_{0:t}$. The thesis now follows by taking expectations conditionally on $S_{0:t}$. \qed

By the Ionescu-Tulcea Theorem (Cinlar 2011 Theorem 4.7), the sequence of predictive distributions $k_t$ defines the law of a new stochastic process:

11
Definition 5.1. A stochastic process \( S = (S_t)_{t \geq 0} \) with state space \((E, \mathcal{E})\) is called a reinforced semi-Markov process with parameters \((m, c, F_0)\), or \( S \sim \text{RSM}(m, c, F_0) \), if \( \mathbb{P}(S_0 = l_0) = 1 \) and \( \mathbb{P}(S_{t+1} = j | S_{0:t}) = k_t(S_{0:t}; j) \) almost surely for all \( j \in E \) and \( t \geq 0 \).

With this definition, the following is a trivial corollary of Theorem 5.1.

Corollary 5.1. If, conditionally on \((P, F) \sim \text{SMBS}(m, c, F_0)\), \( S \sim \text{SM}(P, F) \), then marginally it is \( S \sim \text{RSM}(m, c, F_0) \).

Compatibly with the definition of [Coppersmith and Diaconis (1986) and Pemantle (1988, 2007)], the reinforced semi-Markov process \( S \sim \text{RSM}(m, c, F_0) \) is “reinforced” in the following sense: if \( S \) performs a transition from a state \( i \) to a state \( j \neq i \), this becomes more likely in the future. More precisely, say that \( S_t = i \) and consider the probability \( k_t(S_{0:t}, j) \) for some \( j \neq i \). By Equations (1), \( k_t(S_{0:t}, j) \) is increasing in \( M^{i,j}(t) \), i.e. the number of times that a transition from \( i \) to \( j \) already occurred by time \( t \).

6 Predictive characterization by reinforced urn processes

A sequence \((L_n)_{n \geq 1}\) of random elements of \( E \) is said to be a Pólya sequence generated by a measure \( m(\cdot) \) on \( E \) if it is the result of successive draws from a generalized Pólya urn whose initial composition is determined by \( m \).

Specifically, this is a reinforced urn \( U \) which initially contains \( m\{i\} \) balls of color \( i \in E \). Balls are repeatedly extracted from the urn and, after every draw, the extracted ball is replaced together with another ball of the same color. The color of the ball extracted at the \( n \)-th draw gives the value of \( L_n \), so \( \mathbb{P}(L_1 = i) = m\{i\}/m(E) \) and, for all \( n \geq 1 \),

\[
\mathbb{P}(L_{n+1} = i | L_{1:n}) = \frac{m\{i\} + \sum_{h=1}^{n} I\{L_h = i\}}{m(E) + n}.
\]

The seminal results of [Blackwell and MacQueen (1973)] imply that \((L_n)_{n \geq 1}\) is exchangeable and its de Finetti measure is \( \text{Dir}(m) \). In other words, there exists a random probability measure \( P \sim \text{Dir}(m) \) such that the \( L_n \) are independent and have common distribution \( P(\cdot) \), conditionally on \( P(\cdot) \).

Spurring from the work of [Blackwell and MacQueen (1973)], other urn models have been used to characterize many other common nonparametric prior processes. For example, using models based on Pólya urns it is possible to generate Pólya trees (Mauldin et al. 1992) or the beta-Stacy process (Walker and Muliere, 1997). Fortini and Petrone (2012) provide references to other modern examples. Many of these constructions can be unified using the reinforced urn processes of [Muliere et al. (2000)], which also provide a tool to characterize general neutral-to-the-right processes (Doksum 1974).

Of particular interest to us is the following urn scheme characterizing the discrete-time beta-Stacy process. Here, let \( c(t) \) be a positive real number for all integer \( t > 0 \) and \( F_0 \) be a distribution function with support on the positive integers.

Suppose \( V_1, V_2, V_3, \ldots, V_k, \ldots \) is an infinite sequence of Pólya urns. Each urn \( V_k \) contains \( c(t)F_0\{t\} \) black balls and \( c(t)F_0((t, +\infty)) \) white balls. As before, every time a ball is extracted from an urn, it is replaced together with another ball of the same color.
Starting from $V_1$, for $k \geq 1$ sample a ball from $V_k$. If its color is white, continue sampling from $V_{k+1}$, otherwise set $T_1 = k$ and return to $V_1$ after having reinforced all visited urns. Restarting from $V_1$ and repeating the process it is possible to generate the variables $T_2$, $T_3$, $T_4$, and so on. It is possible to show that the urn $V_1$ is recurrent, i.e. it is visited infinitely often with probability 1. Consequently, this scheme generates an infinite sequence $(T_n)_{n \geq 1}$ of random variables such that

$$\mathbb{P}(T_{n+1} > t | T_{1:n}) = \prod_{s=1}^{t} \left[ 1 - \frac{c(s)F_0([s]) + N([s])}{c(s)F_0([s, +\infty)) + N([s, +\infty))} \right],$$

where $N(t) = \sum_{i=1}^{n} I\{T_i \leq t\}$ (the right-hand side is exactly $F_t$ from Proposition 3.1).

Here, Muliere et al. (2000) have shown that $(T_n)_{n \geq 1}$ is exchangeable and its de Finetti measure is the $BS(c,F_0)$ distribution. Hence, there exists a random $F(\cdot) \sim BS(c,F_0)$ such that the $T_n$ are independent and have distribution $F(\cdot)$, conditional on $F(\cdot)$.

**Definition 6.1.** For simplicity, we will say that a generalized Pólya urn $U$ like the one used above to characterize the $Dir(m)$ process is a $Dir(m)$-urn. Similarly, we say that a system $V$ of reinforced urns $V_1$, $V_2$, $V_3$, ... like the one used to characterize the $BS(c,F_0)$ process is a $BS(c,F_0)$-system.

We can now describe an urn-based characterization of the semi-Markov beta-Stacy process. To do so, associate every $i \in E$ with a $Dir(m^i)$-urn $U_i$ and a $BS(c^i,F_0^i)$-system $V_i$ made up of the urns $V_{i,1}$, $V_{i,2}$, $V_{i,3}$, and so on. Generate a sequence $\{(L_k,T_k)\}_{k \geq 0}$ as follows. Set $L_0 = l_0$. Then, for all $k \geq 0$, generate $T_k$ from $V_{L_k}$ as above, and, independently, set $L_{k+1}$ to the color of the ball extracted from $U_{L_k}$. This generative process is illustrated graphically in Figure 4.

Continuing, define a process $S = (S_t)_{t \geq 0}$ with state space $E$ as follows. Define $\tau_0 = 0$, $\tau_{n+1} = \sum_{h=0}^{n} T_h$ for all $n \geq 0$, and $N(t) = \sum_{n=1}^{+\infty} I\{\tau_n \leq t\}$ for all integers $t \geq 0$. Lastly, define the process $S = (S_t)_{t \geq 0}$ by letting $S_t = L_{N(t)}$ for all integers $t \geq 0$. It is not hard to show that $\mathbb{P}(S_{t+1} = \cdot | S_{0:t}) = k_t(S_{0:t}, \cdot)$, where the kernel $k_t$ is the same as in Theorem 5.1. This shows that $S \sim RSM(m,c,F_0)$. Clearly, any $RSM(m,c,F_0)$ process can be generated in this way.

Now, for all $i \in E$ let $v_{i,0} = -1$ and, for all integers $n \geq 1$, let $v_{i,n} = \inf\{k \geq v_{i,n-1} : L_k = i\}$ be the time of the $n$-th visit of the sequence $(L_k)_{k \geq 0}$ to the state $i$. The process $S = (S_t)_{t \geq 0}$ just introduced will be said to be recurrent if

$$\mathbb{P}\left( \bigcap_{i \in E} \bigcap_{n=1}^{+\infty} \left\{ v_{i,n} < +\infty \right\} \right) = 1. \quad (2)$$

In other words, $S$ is recurrent if it visits every state in $E$ an infinite number of times with probability 1. If $S$ is recurrent, for each $i \in E$ we can define the infinite sequence $\{(L_{i,n},T_{i,n}) = (L_{v_{i,n-1}+1},T_{v_{i,n}})\}_{n \geq 1}$. Note that $T_{i,n}$ is the (finite) length of the $n$-th $i$-block in $S$, which is immediately followed by a $L_{i,n}$-block. In other words, $T_{i,n}$ the length of time $S$ stays in $i$ during the $n$-th visit to that state, while $L_{i,n}$ is the state visited by $S$ immediately after its $n$-th visit to $i$ is over.

With these notions, we can now show the following partial converse of Corollary 5.1.
Figure 1: Graphical illustration of the reinforced urn process of Section 6. In the figure, the path of the process corresponds to the observation of \((L_0, T_0) = (1, 3), (L_1, T_1) = (3, 2),\) and \(L_2 = 2.\) Specifically, the process starts from the urn corresponding to the value \(T_0 = 1\) for the holding time of the state \(L_0 = 1.\) The \(BS(c^3, F_0^1)\)-system \(V_{11}, V_{12}, V_{13}, \ldots\) is traversed left to right until a black ball is extracted from \(V_{13},\) determining the value \(T_0 = 3.\) The process then jumps to the \(Dir(m^1)\)-urn \(U_1,\) from which a ball of color “3” is extracted. Thus, \(L_1 = 3\) and the process jumps to \(V_{31},\) the first urn of the \(BS(c^3, F_0^3)\)-system represented in the third row of the graph. The process then resumes similarly to generate the values \(T_1 = 2\) and \(L_2 = 2.\)
Theorem 6.1. Suppose $S \sim RSM(m, c, F_0)$ is recurrent. Then there exists a random characteristic couple $(P, F)$ such that:

1. conditional on $(P, F)$, $S \sim SM(P, F)$;
2. $(P, F) \sim SMBS(m, c, F_0)$.

To show this result we will make use of the following lemma:

Lemma 6.1. Suppose $S \sim RSM(m, c, F_0)$ is recurrent. Then:

1. the sequences $\{(L_{i,n}, T_{i,n})\}_{n \geq 1}$ for $i \in E$ are independent;
2. the sequences $(L_{i,n})_{n \geq 1}$ and $(T_{i,n})_{n \geq 1}$ are independent for all $i \in E$;
3. there exists a random probability measure $P^i \sim Dir(m^i)$ such that the $L_{i,n}$ are independent and have common distribution $P^i(\cdot)$, conditional on $P^i(\cdot)$;
4. there exists a random distribution $F^i(\cdot) \sim BS(c^i, F^i_0)$ such that the $T_{i,n}$ are independent and have common distribution $F^i(\cdot)$, conditional on $F^i(\cdot)$;
5. all the $P^i(\cdot)$ and $F^i(\cdot)$ are independent.

Proof of Lemma 6.1. To show points (1)-(4) it suffices to note that: i) for all $i \in E$, the sequence $(L_{i,n})_{n \geq 1}$ is generated by $Dir(m^i)$-urn $U_i$; ii) for all $i \in E$, $(T_{i,n})_{n \geq 1}$ is generated by the $BS(c^i, F^i_0)$-system $V_i$; iii) the outcomes of the urns $U_i, V_{i,1}, V_{i,2}, \ldots$, for all $i \in E$ are independent of each other. To prove (5), since $(L_{i,n})_{n \geq 1}$ and $(T_{i,n})_{n \geq 1}$ are exchangeable, by the de Finetti representation theorem $P^i(\cdot) = \mathbb{P}(L_{i,1} \in \cdot | \mathcal{L}_i)$ and $F^i(\cdot) = \mathbb{P}(T_{i,1} \in \cdot | \mathcal{T}_i)$ with probability 1, where $\mathcal{L}_i$ and $\mathcal{T}_i$ are, respectively, the tail $\sigma$-fields of $(L_{i,n})_{n \geq 1}$ and $(T_{i,n})_{n \geq 1}$ [Kallenberg 2006 Chapter 1]. The thesis now follows because all $\sigma$-fields $\mathcal{L}_i$ and $\mathcal{T}_i$, $i \in E$, are independent by (1) and (2).

Proof of Theorem 6.1. Take $P^i(\cdot)$ and $F^i(\cdot)$ for $i \in E$ as given by Lemma 6.1. Define $(P, F)$ by letting $P = (P^i({\{j}\}))_{i,j \in E}$ (note that $P^i({i}) = 0$ almost surely since $m^i({i}) = 0$) and $F = \{F^i(\cdot) : i \in E\}$. To prove the thesis it suffices to show that, conditional on $(P, F)$, $\{(L_k, T_k)\}_{k \geq 0}$ is a Markov renewal process with characteristic couple $(P, F)$. To do so, note that $\mathbb{P}(L_0 = l_0|(P, F)) = 1$ by definition. Moreover, on the event $\{L_n = i, v_{i,k} = n\}$, $k \leq n$, it is

\[
\mathbb{P}(L_{n+1} = j, T_n \leq t|L_{0:n}, T_{0:n-1}, (P, F)) = \mathbb{P}(L_{i,k} = j, T_{i,k} \leq t|(P, F)) = P^i({j})F^i(t).
\]

This concludes the proof.
7 Generalizations of the semi-Markov beta-Stacy process

As anticipated in Remark 2.1, here we illustrate how the semi-Markov beta-Stacy process can be generalized to the setting where the distribution of the holding time $T_k$ is assumed to depend on both $L_k$ and $L_{k+1}$:

$$\mathbb{P}(L_{k+1} = j, T_k \leq t | L_k = i, L_{0:k-1}, T_{0:k-1}, (P, F)) = F^{i,j}(t) P^{i,j}$$  \hspace{1cm} (3)$$

for all $i, j \in E$ and $k \geq 0$, where now $F = (F^{i,j} : i, j \in E, i \neq j)$, while $P = (P^{i,j} : i, j \in E)$ as in Definition 2.1. This corresponds to the assumption that the process $\{(L_k, T_k)\}_{k \geq 0}$ evolves by first deciding which state $L_{k+1} \sim P^{L_k}(\cdot)$ will be visited after leaving the current state $L_k$, and only then decide how much time $T_k \sim F^{L_k,L_{k+1}}(\cdot)$ to spend in the current state $L_k$. Compared to the formulation of Definition 2.1, the present one may be more appropriate in some applications (Barbu and Limnios, 2009).

In this new setting, the definition of the semi-Markov beta-Stacy process can be extended in two ways based on different prior assumptions. These generalizations and the process of Definition 3.3 are all characterized by similar reinforced urn models. These uniquely determine the predictive distributions associated to each process.

7.1 A first non-conjugate generalization

The most natural approach consists in defining $c = (c^{i,j} : i, j \in E, i \neq j)$, $F_0 = (F_0^{i,j} : i, j \in E, i \neq j)$ and then substituting the symbols $c^i$ and $F^i$ with $c^{i,j}$ and $F^{i,j}$ in points 3 and 5 of Definition 3.3 (all other points remaining unchanged).

Despite its simplicity, this approach leads to a generalization of the semi-Markov beta-Stacy process which does not retain all the properties shown in the previous section. In particular, the natural generalization of Theorem 4.1 does not hold, as now the process in not necessarily conjugate.

This lack of conjugacy is evident from the structure of the likelihood function of $(P, F)$ for data $S_{0:t} = i_{0:t}$, whose general form when $N(t) = n$ is

$$\mathbb{P}(S_{0:t} = i_{0:t} | P, F) = \mathbb{P}(L_{0:n} = l_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > t(t) | P, F)$$

$$= \prod_{i,j \in E, i \neq j} \prod_{s=1}^{t} F^{i,j}(\{s\})^{N^{i,j}(\{s\})}$$

$$\cdot \sum_{l_{n+1} \in E, l_{n+1} \neq l_n} P^{l_n}(\{l_{n+1}\}) F^{l_n,l_{n+1}}([l(t), +\infty))$$

$$\cdot \prod_{i,j \in E, i \neq j} P^i(\{j\})^{M^{i,j}(t)},$$

16
where \( N^{i,j,l} \) is the number of non-terminal \( i \)-blocks of length \( \leq l \) which are immediately followed by a \( j \)-block in \( S_{0,t} \) (note that \( N^{i,j,l}(l) = \sum_{j \in E} N^{i,j,l}(l) \)).

Here, if \( l(t) = 0 \), i.e. \( i_t \neq i_{t-1} \), the second term in the square brackets is equal to 1 (since \( F^{i,j}((0, +\infty)) = 1 \) for all \( i \) and \( j \)). In this case, the posterior distribution of \((P,F)\) given the observation of the event \( \{S_{0,t} = i_{0:t}\} = \{L_{0:n} = l_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > l(t)\} \) is again a semi-Markov beta-Stacy process, call it \( SMBS(l_{0:n}, t_{0:n-1}) \), whose parameters can be obtained from obvious analogues of points 1-3 of Theorem 6.1.

On the other hand, if \( l(t) > 0 \), i.e. \( i_t = i_{t-1} \), it can be shown that the posterior distribution of \((P,F)\) given the observation of the event \( \{S_{0,t} = i_{0:t}\} = \{L_{0:n} = l_{0:n}, T_{0:n-1} = t_{0:n-1}, T_n > l(t)\} \) is the mixture of semi-Markov beta-Stacy processes \( SMBS((l_{0:n}, L_{n+1}), (t_{0:n-1}, T_n)) \), where the mixing measure is the distribution of \((L_{n+1}, T_n)\) given \( L_{0:n} = l_{0:n}, T_{0:n-1} = t_{0:n-1}, \) and \( T_n > l(t) \).

Although the posterior distribution associated to the generalized semi-Markov beta-Stacy process is not immediately available, it is still possible to characterize its associated predictive distributions using a new reinforced urn process.

Specifically, associate every state \( i \in E \) with a \( Dir(m^i) \)-urn \( U_i \) and every pair \((i,j)\) \( \in E \times E, i \neq j \), with the \( BS(c^{i,j}, F^{i,j}) \)-system of urns \( V_{i,j,1}, V_{i,j,2}, V_{i,j,3} \), and so on. Generate a sequence \( \{\{L_k, T_k\}\}_{k \geq 0} \) as follows. First, set \( L_0 = 0 \). Then, for all \( k \geq 0 \), generate \( L_{k+1} \) from \( U_{L_k} \) and, independently, \( T_k \) from \( V_{L_k,L_{k+1}} \). Lastly, denote with \( S = (S_t)_{t \geq 0} \) the process with state space \( E \) induced by \( \{(L_k, T_k)\}_{k \geq 0} \) as in Section 6.

Additionally, for all \((i,j)\) \( \in E \times E, i \neq j \), let \( v_{i,j,0} = -1 \) and, for all integers \( n \geq 1 \), let \( v_{i,j,n} = \inf\{k > v_{i,j,n-1} : L_k = i\} \) be the time the sequence \( (L_k)_{k \geq 0} \) performs its \( n \)-th transition from the state \( i \) to the state \( j \). Note that \( v_{i,n} \), the time of the \( n \)-th visit to \( i \), is related to the \( v_{i,j,n}, j \neq i \), by \( v_{i,n} = \min\{v_{i,j,k} : j \neq i, k \leq n, v_{i,j,k} > v_{i,n-1}\} \). We will consider the following strengthening of the recurrence condition of Equation 2.

\[
P\left( \bigcap_{(i,j) \in E \times E, n=1}^{+\infty} \bigcap_{i \neq j} \{v_{i,j,n} < +\infty\} \right) = 1. \tag{4}
\]

This not only implies that \((S_t)_{t \geq 0} \) is recurrent, but also that it performs every allowable transition an infinite number of times with probability 1. Hence, the sequences \( (L_{i,k})_{k \geq 0} = (L_{i,v_{i,k+1}})_{k \geq 0} \) and \( (T_{i,j,k})_{k \geq 0} = (T_{v_{i,j,k}})_{k \geq 0} \) are infinite with probability 1.

Importantly, under the condition of Equation 4, Theorem 6.1 still holds. In fact, proceeding as in the proof of Lemma 6.1 under condition 4 it can be shown that: i) the arrays of random variables \( \{(L_{i,n}, T_{i,j,n} : j \neq i)\}_{n \geq 0}, i \in E, \) are independent of each other; ii) the sequences \( (L_{i,n})_{n \geq 0}, (T_{i,j,n})_{n \geq 0}, i \neq j, \) are all independent of each other; iii) for all \( i \in E, \) the \( L_{i,n} \) are independent and identically distributed as \( P^i(\cdot) \) for some random \( P^i(\cdot) \sim Dir(m^i); \) iv) for all \( i \neq j, \) the \( T_{i,j,n} \) are independent and identically distributed as \( F^{i,j}(\cdot) \) for some random \( F^{i,j}(\cdot) \sim BS(c^{i,j}, F_{0}^{i,j}); \) and v) all the \( P^i(\cdot) \) and \( F^{i,j}(\cdot), i \neq j, \) are independent of each other.

Consequently, letting \( P = \{P^i(\{\cdot\})\}_{i,j \in E} \) and \( F = \{F^{i,j}(\cdot) : i, j \in E, i \neq j\} \), it is \( P(L_0 = l_0 | (P, F)) = 1. \) Moreover, on the event \( \{L_n = i, v_{i,k} = v_{i,j,h} = n\} \) with
of \( P \)

\[ \mathbb{P}(L_{n+1} = j, T_n \leq t | L_0, T_{0:n-1}, (P, F)) = \mathbb{P}(L_{i,k} = j, T_{i,j,h} \leq t | (P, F)) = P^i(\{j\}) F^{i,j}(t), \]

as desired.

7.2 An alternative conjugate generalization

To arrive at an alternative generalization, we consider the following approach. First, note that Equation 3 can be equivalently expressed as

\[ \mathbb{P}(L_{k+1} = j, T_k = t | L_k = i, (L_{0:k-1}, (P, F))) = F^i(\{j\}) P_t^{i,j} \tag{5} \]

where \( F^i(t) = \sum_{j \neq i} F^{i,j}(t) P_t^{i,j} \) and \( P_t^{i,j} = F^{i,j}(t) P^{i,j} / F^i(t) \), where now \( P = (P_t^{i,j} = P_t^j(\{j\}) : i, j \in E, t \geq 1) \), while \( F = (F^i(\cdot) : i \in E) \) as in Definition 2.1. In this formulation, the process \( \{(L_k, T_k)\}_{k \geq 0} \) evolves by first deciding the time \( T_k \sim F^{L_k}(\cdot) \) to spend in the current state \( L_k \) subsequently deciding the next state \( L_{k+1} \sim P^{L_k}_{L_k}(\cdot) \).

From this perspective, Definition 3.3 can be generalized by letting \( m = (m_t^i : i \in E, t \geq 1) \) be a family of measures on \( E \) and then supposing that the \( P_t^i(\cdot) \) are independent \( \text{Dir}(m_t^i) \) processes on \( E \) for all \( i \in E \) and \( t \geq 1 \) (all other assumptions remaining as is).

Contrary to the previous case, this generalization of the semi-Markov beta-Stacy process is easily seen to be conjugate. In fact, an immediate generalization of Theorem 4.1 can be obtained by noting that the likelihood function of \((P, F)\) for data \( S_{0:t} = i_{0:t} \) such that \( N(t) = n \) now takes the form

\[ \mathbb{P}(S_{0:t} = i_{0:t} | P, F) = \prod_{i \in E} \prod_{s=1}^{t} \left[ F^i(\{s\})^{N_{i,s}^i}(\{s\}) F^i((l(t), +\infty)) | i = i_n \right] \times \left[ \prod_{i \in E} \prod_{s=1}^{t} m_t^i(\{j\})^{N_{i,j,s}^i}(\{s\}) \right] \]

(note that \( M_t^{i,j}(t) = \sum_{s=1}^{t} N_{i,j,s}^i(\{s\}) \) for all \( i \neq j \)). Thus, the posterior distribution of \((P, F)\) given \( S_{0:t} = i_{0:t} \) is a \( \text{SMBS}(m_s, c_s, F_s) \), where \( c_s \) and \( F_s \) are defined as in Theorem 4.1 while \( m_s = (m_{i,s}^i : i \in E, s \geq 1) \) is obtained by letting \( m_{i,s}^i(\cdot) = m_{i,s}^i(\cdot) + N_{i,j,s}^i(\{s\}) \) for all \( i \in E \) and \( s \geq 1 \).

It should be clear by now that this generalization can also be characterized by another reinforced urn process. Specifically, associate every \( i \in E \) with a \( BS(c_s, F_s^0) \)-system \( V_i \) as in Section 6 and every couple \((i, t) \in E \times \{1, 2, 3 \ldots \} \) with a \( \text{Dir}(m_t^i) \)-urn \( U_{i,t} \). Suppose that \( \{(L_k, T_k)\}_{k \geq 0} \) is generated first by letting \( L_0 = 0 \) and then by iteratively generating \( T_k \) from \( V_{L_k} \) and \( L_{k+1} \) from \( U_{L_k, T_k} \) for all \( k \geq 0 \). In this set up, a generalization of Theorem 6.1 can be shown to hold under an appropriate strengthening of the recurrence condition of Equation 2. In particular, it suffices to require that every urn is visited infinitely often with probability one.
8 Simulation study

To illustrate the semi-Markov beta-Stacy process in action, we conducted a simulation study based on the textile factory scenario of Example 2.2.

8.1 Description of the simulation study

Following Barbu and Limnios (2009, Sections 4.3), we generated a single realization \( s_{0:1,000} \) from the semi-Markov process \( (S_t)_{t \geq 0} \) describing the day-by-day status of the factory from day 0 to day 1,000. The law of this process was determined by assuming that: i) \( S_0 = 1 \) (so the factory begins fully functional); ii) the transition matrix is

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0.95 & 0 & 0.05 \\
1 & 0 & 0
\end{bmatrix};
\]

(6)

iii) \( F^1(\cdot) \) is the geometric distribution \( F^1(\{t\}) = p(1-p)^{t-1}, t \geq 1, \) with parameter \( p = 0.8; \) iv) \( F^2(\cdot) \) is the first-type discrete Weibull distribution \( F^2(t) = 1 - q^k, t \geq 1, \) of Nakagawa and Osaki (1975) with parameters \( q = 0.3 \) and \( k = 0.5 \) (when \( k = 1, \) this distribution reduces to the geometric distribution with parameter \( 1 - q); \) v) \( F^3(\cdot) \) is the first-type discrete Weibull distribution with parameters \( q = 0.6 \) and \( k = 0.9 \). The observed sequence \( s_{0:1,000} \) was considered as data to perform posterior inferences.

8.2 Prior specification

We assign a semi-Markov beta-Stacy prior distribution \( SMBS(\mathbf{m}, \mathbf{c}, \mathbf{F}_0) \) to the data-generating characteristic couple \( (\mathbf{P}, \mathbf{F}) \). We consider the measures \( m^1(\cdot), m^2(\cdot), \) and \( m^3(\cdot) \) on \( E = \{1, 2, 3\} \) determined by the conditions \( m^i(\{1, 2, 3\}) = m^1(\{2\}) = m^2(\{1\}) = m^2(\{3\}) = m^3(\{1\}) = 1 \) for all \( i \in E \) (in particular, this implies that both \( P^{2,1} \) and \( P^{2,3} \) are marginally uniformly distributed over \((0,1))\). For all \( i = 1, 2, 3, \) \( F_0^i(\cdot) \) will be the geometric distribution with parameter \( p = 0.3 \) (a prior assumption clearly incompatible with the data-generating mechanism). For all \( i \in E, \) we consider \( c^i(t) = c \) for all \( t \geq 1 \) and some constant \( c > 0, \) successively considering the values \( c = 0.1, 1, \) and 10.

8.3 Posterior distributions

Figure 2 shows the plots of the posterior mean of \( F^2(\cdot), \) together with a sample of 500 samples from the corresponding distribution. Posterior distributions were obtained from Theorem 4.1 using data \( s_{0:M} \) with \( M = 0 \) (so the posterior coincides with the prior), \( M = 100, \) or \( M = 1000 \) (so whole simulated path is used). For comparison, the figure also reports the data-generating distribution of the holding-times of the state 2, i.e. of the time elapsed until either the tank is repaired or the factory has to stop after a failure.

Figure 2 highlights how the posterior distribution obtained from the semi-Markov beta-Stacy prior is able to recover the underlying data-generating distribution by flexibly adapting to the observations, even when these deviate from prior assumptions. This
Figure 2: Plot of the posterior distribution of $F^2(\cdot)$ for the semi-Markov process priors of Section 8. Results are shown for different values of: i) the prior concentration parameter $c$, which specifies the weight assigned to the prior centering distributions $F^2_0(\cdot)$; ii) the length $N$ of the observation period during which data $S_{0:N}$ is collected (if $N = 0$, the posterior distribution coincides with the prior). Blue lines: values of the true data-generating distribution $F^2(\cdot)$ (see Section 8.1). Black lines: posterior mean of $F^2(\cdot)$. Orange lines: graph of 500 samples from the posterior distribution of $F^2(\cdot)$.

is true both for data reflecting a short ($M = 100$) or long ($M = 1,000$) period of observation. The figure also highlights the impact of the concentration parameters $c$. As this increases, the dispersion of the distribution of $F^2(\cdot)$ around its mean decreases.

8.4 Predictive distributions

Figure 3 reports the estimates of the predictive distributions $P_h(j) = \mathbb{P}(S_{1,000+h} = j | S_{0:1,000} = s_{0:1,000})$ obtained from the semi-Markov beta-Stacy prior with $c = 1$ for all $h = 1, \ldots, 100$ and all $j = 1, 2, 3$. These were obtained by simulating $10^5$ future paths $(S_{1,000+h})_{h=1,\ldots,100}$ conditional on the past observation of $S_{0:1,000} = s_{0:1,000}$ by sampling from the reinforced semi-Markov kernels of Corollary 5.1. Then, $P_h(j)$ was estimated as the proportion of simulations in which $S_{1,000+h} = j$. 

20
Figure 3: Plot of the predictive probabilities \( P_h(j) = \mathbb{P}(S_{1,000+h} = j | S_{0:1,000} = s_{0:1,000}) \) obtained from the semi-Markov beta-Stacy process of Section 8 with \( c = 1 \) for all \( h = 1, \ldots, 100 \). The value \( P_h(j) \) is the probability that the factory will be in state \( j = 1, 2, 3 \) after \( h \) days in the future given its past history \( S_{0:1000} \). The black, red, and blue lines are, respectively, the values of \( P_h(1), P_h(2), \) and \( P_h(3) \). The dashed lines represent the limiting distribution of the underlying data-generating semi-Markov process.

Figure 3 shows how the the \( P_h(j) \) adapt over time as \( h \) increases for all \( j = 1, 2, 3 \), whose values stabilize in the long run. Specifically, for large \( h \) the vector \( (P_h(1), P_h(2), P_h(3)) \) remain close to the limiting distribution \( (\nu_1, \nu_2, \nu_3) \) of the data-generating semi-Markov process. This is obtained from Proposition 3.9 of Barbu and Limnios (2009) as \( \nu_j = e_j m_j / \sum_{i=1}^{3} e_i m_i \), where \( (e_1, e_2, e_3) = (1/0.05, 1/0.05, 0.05/0.05) \) is the equilibrium distribution of the transition matrix \( P \) in Equation 6, while \( m_j = \sum_{t=0}^{+\infty} (1 - F_j(t)) \) is the expected sojourn time in the state \( j \).

9 Concluding remarks

In this paper we introduced the semi-Markov beta-Stacy process, a Bayesian nonparametric process prior for semi-Markov models, and some related generalizations. Each
was characterized from a predictive perspective by “piecing together” different reinforced urn models characterizing simpler processes.

This approach is conceptually valuable, as it provides a fresh strategy for the specification of Bayesian nonparametric models for the prediction of complex processes. Importantly, as previously noted by Muliere et al. (2003), reinforced stochastic processes can be used to perform predictions from a Bayesian nonparametric perspective without requiring knowledge of difficultly obtained aspects of the prior or posterior distributions (Ghosal and van der Vaart, 2017).

The semi-Markov beta-Stacy may be amenable to more generalization than the ones considered here by modifying its underlying reinforced urn process. First, each extracted ball may be reinforced by a fixed or random amount of multiple balls of the same or different colors, akin as in Muliere et al. (2006). This could allow a finer control of the level of uncertainty attached to the urns’ initial composition, i.e. to the centering distribution of the prior (Arfé et al., 2018).

Second, a form of dependence across different components of the prior may be introduced by reinforcing urns other than the one from which a ball was extracted. This form of interaction among urns could lead to interesting models in which observations provide indirect information about distributions that have not generated them directly (Paganoni and Secchi, 2004; Muliere et al., 2005).

From a more applied perspective, we are investigating different ways to exploit the semi-Markov beta-Stacy process in more complex Bayesian non-parametric models based on semi-Markov processes. In particular, we are implementing a regression model in which the distribution of the holding times and the transition matrices depend on a vector of covariates. As in Arfé et al. (2018), this is done by letting the initial composition of the urns be a function of both the covariates and some additional parameters, which are then assigned their own prior distribution. Such model could be used for the analysis of multi-stage diseases in medical studies (Barbu et al., 2004; Mitchell et al., 2011).

Additionally, we are applying the semi-Markov beta-Stacy process to perform inference and predictions in Hidden Semi-Markov Models (HSMMs), in which the sequence of visited states is observed only indirectly (Barbu and Limnios, 2009, Chapter 6). As a specific application, we are developing a novel approach for changepoint analysis in which the state of a semi-Markov process represents the latent regimen of a time series (Smith, 1975; Muliere and Scarsini, 1985; Ko et al., 2015; Peluso et al., 2018).

Acknowledgments

AA would like to thank Sarah Craver for her helpful suggestions.

References

Arfé, A., Peluso, P. and Muliere, P. (2018) Reinforced urns and the subdistribution beta-Stacy process prior for competing risks analysis. Scandinavian Journal of Statistics, Forthcoming.
Bacallado, S., Favaro, S. and Trippa, L. (2013) Bayesian nonparametric analysis of reversible Markov chains. *The Annals of Statistics*, 870–896.

Barbu, V., Boussemart, M. and Limnios, N. (2004) Discrete-time semi-markov model for reliability and survival analysis. *Communications in Statistics-Theory and Methods*, **33**, 2833–2868.

Barbu, V. and Limnios, N. (2009) *Semi-Markov Chains and Hidden Semi-Markov Models toward Applications: Their Use in Reliability and DNA Analysis*. Lecture Notes in Statistics. Springer New York.

Blackwell, D. and MacQueen, J. B. (1973) Ferguson distributions via Polya urn schemes. *Ann. Statist.*, **1**, 353–355.

Bulla, J. and Bulla, I. (2006) Stylized facts of financial time series and hidden semi-Markov models. *Computational Statistics & Data Analysis*, **51**, 2192–2209.

Bulla, P. and Muliere, P. (2007) Bayesian nonparametric estimation for reinforced markov renewal processes. *Statistical Inference for Stochastic Processes*, **10**, 283–303.

Caron, F., Neiswanger, W., Wood, F., Doucet, A. and Davy, M. (2017) Generalized Polya urn for time-varying pitman-yor processes. *Journal of Machine Learning Research*, **18**, 1–32.

Çinlar, E. (2011) *Probability and Stochastics*. New York: Springer.

Çinlar, E. (1969) Markov renewal theory. *Advances in Applied Probability*, **1**, 123–187.

Coppersmith, D. and Diaconis, P. (1986) Random walk with reinforcement. unpublished manuscript.

de Finetti, B. (1937) La prévision: ses lois logiques, ses sources subjectives. *Annales de l'institut Henri Poincaré*, **7**, 1–68. English translation, “Foresight: its Logical Laws, Its Subjective Sources”, in H. E. Kyburg and H. E. Smokler (editors), Studies in Subjective Probability. New York: Wiley, 1964.

Doksum, K. (1974) Tailfree and neutral random probabilities and their posterior distributions. *The Annals of Probability*, 183–201.

Ferguson, T. S. (1973) A Bayesian analysis of some nonparametric problems. *The Annals of Statistics*, **1**, 209–230.

Fortini, S. and Petrone, S. (2012) Predictive construction of priors in Bayesian nonparametrics. *Brazilian Journal of Probability and Statistics*, **26**, 423–449.

Ghosal, S. and van der Vaart, A. (2017) *Fundamentals of Nonparametric Bayesian Inference*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.
Heitjan, D. F. and Rubin, D. B. (1991) Ignorability and coarse data. *The Annals of Statistics, 19*, 2244–2253.

Hjort, N. L., Holmes, C., Müller, P. and Walker, S. G. (2010) *Bayesian nonparametrics*. Cambridge, UK: Cambridge University Press.

Janssen, J. and Manca, R. (2007) *Semi-Markov Risk Models for Finance, Insurance and Reliability*. Springer US.

Kalbfleisch, J. D. and Prentice, R. L. (2002) *The statistical analysis of failure time data*. Hoboken, New Jersey: John Wiley & Sons, 2nd edition edn.

Kallenberg, O. (2006) *Probabilistic symmetries and invariance principles*. Springer Science & Business Media.

Ko, S. I., Chong, T. T., Ghosh, P. et al. (2015) Dirichlet process hidden Markov multiple change-point model. *Bayesian Analysis, 10*, 275–296.

Masala, G. (2013) Hurricane lifespan modeling through a semi-Markov parametric approach. *Journal of Forecasting, 32*, 369–384.

Mauldin, R. D., Sudderth, W. D. and Williams, S. (1992) Polya trees and random distributions. *The Annals of Statistics, 20*, 1203–1221.

Mitchell, C., Hudgens, M., King, C., Cu-Uvin, S., Lo, Y., Rompalo, A., Sobel, J. and Smith, J. (2011) Discrete-time semi-Markov modeling of human papillomavirus persistence. *Statistics in medicine, 30*, 2160–2170.

Muliere, P., Paganoni, A. M. and Secchi, P. (2006) A randomly reinforced urn. *Journal of Statistical Planning and Inference, 136*, 1853–1874.

Muliere, P. and Scarsini, M. (1985) Change-point problems: A and Bayesian nonparametric approach. *Aplikace Matematiky, 30*, 397–402.

Muliere, P., Secchi, P. and Walker, S. (2000) Urn schemes and reinforced random walks. *Stochastic Processes and their Applications, 88*, 59–78.

— (2005) Partially exchangeable processes indexed by the vertices of a k-tree constructed via reinforcement. *Stochastic processes and their applications, 115*, 661–677.

Muliere, P., Secchi, P. and Walker, S. G. (2003) Reinforced random processes in continuous time. *Stochastic Processes and their Applications, 104*, 117–130.

Müller, P. and Mitra, R. (2013) Bayesian nonparametric inference—why and how. *Bayesian Analysis, 8*, 269–302.

Nakagawa, T. and Osaki, S. (1975) The discrete weibull distribution. *IEEE Transactions on Reliability, 24*, 300–301.
Paganoni, A. M. and Secchi, P. (2004) Interacting reinforced-urn systems. *Advances in applied probability, 36*, 791–804.

Patwardhan, A. S., Kulkarni, R. B. and Tocher, D. (1980) A semi-Markov model for characterizing recurrence of great earthquakes. *Bulletin of the seismological society of America, 70*, 323–347.

Peluso, S., Chib, S., Mira, A. et al. (2018) Semiparametric multivariate and multiple change-point modeling. *Bayesian Analysis*.

Peluso, S., Mira, A. and Muliere, P. (2015) Reinforced urn processes for credit risk models. *Journal of Econometrics, 184*, 1–12.

Pemantle, R. (1988) *Random processes with reinforcement*. Ph.D. thesis, Massachusetts Institute of Technology.

—— (2007) A survey of random processes with reinforcement. *Probabability Surveys, 4*, 1–79.

Phadia, E. G. (2015) *Prior processes and their applications*. New York: Springer.

Phelan, M. J. (1990) Bayes estimation from a Markov renewal process. *The Annals of Statistics, 18*, 603–616.

Rigat, F. and Muliere, P. (2012) Nonparametric survival regression using the beta-Stacy process. *Journal of Statistical Planning and Inference, 142*, 2688–2700.

Ruggiero, M. and Walker, S. G. (2009) Bayesian nonparametric construction of the Fleming-Viot process with fertility selection. *Statistica Sinica, 707–720*.

Satten, G. A. and Sternberg, M. R. (1999) Fitting semi-Markov models to interval-censored data with unknown initiation times. *Biometrics, 55*, 507–513.

Schiffman, M., Castle, P. E., Maucort-Boulch, D., Wheeler, C. M., of Undetermined Significance/Low-Grade Squamous Intraepithelial Lesions Triage Study) Group, A. A. S. C. and Plummer, M. (2007) A 2-year prospective study of human papillomavirus persistence among women with a cytological diagnosis of atypical squamous cells of undetermined significance or low-grade squamous intraepithelial lesion. *The Journal of infectious diseases, 195*, 1582–1589.

Singpurwalla, N. D. (1988) Foundational issues in reliability and risk analysis. *SIAM Review, 30*, 264–282.

—— (2006) *Reliability and risk: a Bayesian perspective*. Chichester, England: John Wiley & Sons.

Smith, A. (1975) A Bayesian approach to inference about a change-point in a sequence of random variables. *Biometrika, 62*, 407–416.
Walker, S. and Muliere, P. (1997) Beta-Stacy processes and a generalization of the pólya-urn scheme. *The Annals of Statistics*, **25**, 1762–1780.

Zhao, L. and Hu, X. J. (2013) Estimation with right-censored observations under a semi-Markov model. *Canadian Journal of Statistics*, **41**, 237–256.