Modelling the behaviour of longitudinal shear cracks in a two-layer elastic strip

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Abstract. The study is devoted to the urgent question of forecasting the development of defects in the form of cracks in a two-layer elastic material, which affect its strength and durability. The results will allow us to simulate the occurrence and behaviour of the defect from the insufficient strength of the material, as well as to formulate the optimal operating conditions for the two-layer material. Given analytical expressions will let to see the effect of various parameters of the initial state on the development of a defect in the form of a crack of a longitudinal fracture.

1. Introduction
Bimetals and composite materials are used in various areas of industry (for example, in rocket production, aircraft construction, shipbuilding). One of the actual problems in the operation of structural elements of such materials is their deformation and destruction, which are associated with the occurrence of cracks of various types, including cracks at the interface between materials.

Thus, the urgent issue is the timely identification of defective material segments and prediction of the development of cracks of various categories. The work is devoted to studying the mechanism of the behaviour of longitudinal shear cracks.

2. Problem statement
Let us consider a bioplastic strip consisting of two isotropic bands with \( \mu_1 \) and \( \mu_2 \) elastic properties (\( \mu_i \) is the elastic modulus), which are rigidly coupled to each other. The strip contains a longitudinal shear crack at \( y = 0, |x| \leq l \). Equal in magnitude but oppositely directed stresses are applied to the edges of the crack (figure 1). The first homogeneous isotropic elastic medium occupies the region \( -\infty < x < \infty, \ 0 \leq y \leq h_1 \), and the second medium occupies the region \( -\infty < x < \infty, \ h_2 \leq y \leq 0 \). The displacements on the surface of the bands are zero. The physical meaning of this condition is that the surfaces of the bioplastic strip rely on absolutely rigid bodies without a gap. At infinity, the stress and displacement tend to zero.

The geometry of the investigated problem is shown in figure 1.

![Figure 1. The geometry of the investigated problem.](image)

\[ y \]
\[ \mu_1 \]
\[ \mu_2 \]
\[ x \]
Thus, we arrive at the following mixed singular boundary value problem.

Border conditions:

\[
|x| < \infty, \ w_1(x, h_1) = 0, \\
|x| < \infty, \ w_2(x, h_2) = 0, \\
|x| \leq l, \ (\sigma_{yz})_1(x, +0) = (\sigma_{yz})_2(x, -0) = -\sigma(x) = -\sigma(-x). \\
|x| > l, \ w_1(x, +0) = w_2(x, -0), \\
|x| > l, \ (\sigma_{yz})_1(x, +0) = (\sigma_{yz})_2(x, -0)
\]

Conditions at the ends of the crack [2]

\[
\lim_{x \to l^-} \left\{ \frac{\sqrt{2\pi(l-x)}}{2\pi} \left[ \frac{\partial w_1(x, +0)}{\partial x} - \frac{\partial w_2(x, -0)}{\partial x} \right] \right\} = -\frac{(k+1)}{\mu_i} K_{III},
\]

or

\[
\lim_{x \to l^-} \left\{ \frac{\sqrt{2\pi(l-x)}}{2\pi} \left[ \frac{\partial w_1(x, +0)}{\partial x} - \frac{\partial w_2(x, -0)}{\partial x} \right] \right\} = -\frac{(k+1)}{\mu_i} K_{III}.
\]

Conditions at infinity:

\[
|y| < h, \ |x| \to \infty \ (\sigma_{yz})_1, (\sigma_{xz})_j \to 0, \ w_j \to 0.
\]

Here \(w_j(x,y)\) \((j=1, 2)\) are the displacements in the first and in the second elastic medium; \((\sigma_{yz})_j(x,y)\) are stresses in the first and in the second elastic medium; \(K_{III}\) is the stress intensity factor at the tip of the longitudinal crack.

3. Solving the task

In the case of antiplane deformation, the nonzero components of \(\sigma_{yz}(x,y)\) and \(\sigma_{xz}(x,y)\) stress tensor are related to \(w(x,y)\) displacement as follows:

\[
\sigma_{yz}(x,y) = \mu \frac{\partial w(x,y)}{\partial y}, \quad \sigma_{xz}(x,y) = \mu \frac{\partial w(x,y)}{\partial x},
\]

and besides

\[
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = 0,
\]

Applying the cosine - Fourier transform to the Laplace equations for displacements, we obtain an ordinary differential equation:

\[
\frac{d^2 w^*(\lambda,y)}{dy^2} - \lambda^2 w^*(\lambda,y) = 0,
\]

where

\[
w^*_j(\lambda,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty w_j(x,y) \cos \lambda x dx,
\]

Solving this equation and taking into account (1), (2) and (11) using the cosine - Fourier transform inversion formula, we obtain:

First elastic medium:
\[ w_1(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\lambda) \frac{\text{sh}\lambda(h_1 - y)}{\text{sh}\lambda h_1} \cos \lambda x d\lambda, \] (12)

\[ (\sigma_{xc})_1(x, y) = -\mu_1 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda A(\lambda) \frac{\text{ch}\lambda(h_1 - y)}{\text{sh}\lambda h_1} \cos \lambda x d\lambda, \] (13)

\[ (\sigma_{xc})_2(x, y) = -\mu_2 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda A(\lambda) \frac{\text{sh}\lambda(h_1 - y)}{\text{sh}\lambda h_1} \sin \lambda x d\lambda. \] (14)

Second elastic medium:

\[ w_2(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty C(\lambda) \frac{\text{sh}\lambda(h_2 + y)}{\text{sh}\lambda h_2} \cos \lambda x d\lambda, \] (15)

\[ (\sigma_{xc})_2(x, y) = -\mu_2 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda C(\lambda) \frac{\text{ch}\lambda(h_2 + y)}{\text{sh}\lambda h_2} \cos \lambda x d\lambda, \] (16)

\[ (\sigma_{xc})_2(x, y) = -\mu_2 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda C(\lambda) \frac{\text{sh}\lambda(h_2 + y)}{\text{sh}\lambda h_2} \sin \lambda x d\lambda. \] (17)

From (12) and (15) owing to (4) we have:

\[ w_1(x, +0) - w_2(x, -0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [A(\lambda) - C(\lambda)] \cos \lambda x d\lambda = 0 \quad (|x| > l) \] (18)

From (13) and (16), owing to (3) and (5), we obtain

\[ (\sigma_{xc})_1(x, +0) - (\sigma_{xc})_2(x, -0) = \]

\[ = -\sqrt{\frac{2}{\pi}} \int_0^\infty \lambda \left[ \mu_1 A(\lambda) \frac{\text{ch}\lambda h_1}{\text{sh}\lambda h_1} + \mu_2 C(\lambda) \frac{\text{ch}\lambda h_2}{\text{sh}\lambda h_2} \right] \cos \lambda x d\lambda = 0 \] (19)

\((x \geq 0).\)

From (18) with the help of the discontinuous Dirichlet integral [2]

\[ \int_0^\infty \frac{\sin \frac{\lambda t}{\lambda}}{\lambda} \cos \lambda x d\lambda = \begin{cases} \pi / 2, & x < t, \\ \pi / 4, & x = t, \\ 0, & x > t, \end{cases} \] (20)

we find:

\[ A(\lambda) - C(\lambda) = \frac{k + 1}{\mu_1} \sqrt{\frac{2}{\pi}} \int_0^t f(t) \frac{\sin \frac{\lambda t}{\lambda}}{\lambda} dt. \] (21)

Here \( f(t) \) is a new unknown function.

If we take into account (21) in (18), owing to (20) it is easy to verify that equality (18) is identically satisfied.

From (19) it follows

\[ kA(\lambda) \text{ch}\lambda h_1 + C(\lambda) \text{ch}\lambda h_2 = 0. \] (22)

From the system of linear algebraic equations (21) and (22) we find:
Thus, the solution of the above-stated boundary value problem (1) - (8) is reduced to finding one unknown function $f(t)$, where $t \in [0, l]$. It can be proved that $f(x) \in K_{1/2}[-l, l]$, that is:

$$
(\sigma_\infty)_1(x, y) = -\frac{2}{\pi} \int_0^l \int_0^\infty \frac{c\lambda(h_1 - y)}{shh_1} \sin \lambda t \cos \lambda x \, d\lambda \, dt -
- \frac{2k}{\pi} \int_0^l \int_0^\infty \frac{c\lambda h_2 - c\lambda h_1}{kcth\lambda h_1 + c\lambda h_2} \frac{c\lambda(h_1 - y)}{shh_1} \sin \lambda t \cos \lambda x \, d\lambda \, dt
$$

(28)

From (28), having satisfied (3), we arrive at a generalized singular integral equation of the first kind:

$$
\sigma(x) = \frac{1}{2h \chi h^2} \left[ \int_t^\infty \frac{\pi x}{2h} f(t) dt + \int_t^{\pi t} \frac{\pi x}{2h} f(t) dt \right] + \frac{1}{\pi} \int_{-t}^t f(t) K(x, t) \, dt,
$$

$$
K(x, t) = k \int_0^{\infty} \frac{c\lambda h_2 - c\lambda h_1}{kcth\lambda h_1 + c\lambda h_2} c\lambda h_1 \sin \lambda t \cos \lambda x \, d\lambda
$$

(30)

In (29), the first integral is understood in the sense of the main value.

In (29), $\sigma(x)$ free term is assumed to be a continuous function defined on $[-l, l]$ interval. From (30) it can be seen that $K(x, t)$ kernel is also continuous on this segment.

Singular integral equations of the first kind, as a rule, have the form:
where $K(x,t)$ kernel and $\gamma(x)$ free term are the continuous functions of their arguments defined on $[-l,l]$ interval.

Comparing (31) with (29), we are convinced that in order to apply the available numerical and approximate methods suitable for the solution (31) to (29), additional research should be conducted.

We shall introduce the notation:

$$f_1(x) = K_{l/2} [-l,l[.$$ (29)

As $\sigma(x) \in C[-l,l]$, moreover $\sigma(-x) = -\sigma(x)$, which follows from (25) and (29), $f(x) \in K_{l/2} [-l,l]$, then:

$$f_1(x) = f(x) \in K_{l/2} [-l,l[.$$ (32)

(29) taking into account (32) can be written in the following form:

$$p(x) = \frac{1}{4h^2} \int_0^l f_1(t) \left( \frac{2t}{2h} \right) \frac{1}{\left( \frac{\pi}{2} / \frac{2h}{} \right) - \left( \frac{\pi}{2} / \frac{2h}{} \right)} dt +$$

$$+ \frac{1}{\pi} \int_{-l}^l f_1(t) K_1(x,t) dt,$$

$$K_1(x,t) = \frac{k}{2h} \int_0^\infty \left[ \frac{1}{k} \left( \frac{\pi}{2} - \frac{\pi}{2} \alpha^2 \right) \frac{\pi}{2} \cos \alpha x \cdot d \alpha \right.$$ (33)

Singular equation owing to [1] can be presented as follows:

$$f(x) = - \frac{1}{h \cdot ch^2 \left( \frac{\pi}{2} \alpha^2 \right)} \sqrt{\left[ \frac{\pi}{2} - \frac{\pi}{2} \alpha^2 \right] \left( \frac{\pi}{2} l - \frac{\pi}{2} x \right)}$$

$$\times \int_0^l \sqrt{\left[ \frac{\pi}{2} - \frac{\pi}{2} \alpha^2 \right] \left( \frac{\pi}{2} l - \frac{\pi}{2} x \right)} \left[ p(\xi) - \frac{1}{2\pi} \int_0^l f(t) K(\xi,t) dt \right] d\xi.$$ (33)

Here $K(\xi,t)$ is determined by the formula (30) (instead of $x$ in (30), we should write $\xi$).

We shall consider two cases.

1. Let $h_1 = h_2 = h$. Then $K(\xi,t) = 0$. In this case from (33) it follows that:

$$f(x) = - \frac{1}{h \cdot ch^2 \left( \frac{\pi}{2} \alpha^2 \right)} \sqrt{\left[ \frac{\pi}{2} - \frac{\pi}{2} \alpha^2 \right] \left( \frac{\pi}{2} l - \frac{\pi}{2} x \right)}$$
\[ \times \int_{0}^{l} \sqrt{\frac{\pi}{2h} \frac{l - \pi}{2h} \frac{\varpi}{2h} \varpi} \sigma (\varpi) \text{ch}^{2} \frac{\pi}{2h} \varpi \cdot d\varpi. \]

From where, owing to (25) we shall have:

\[ f_{0}(x) = - \frac{\varpi}{2h h \cdot \text{ch}^{2}(\frac{\varpi x}{2h})} \sqrt{\frac{\pi}{2h} \frac{l - \varpi}{2h} \frac{\varpi}{2h} x} \times \]

\[ \times \int_{0}^{l} \sqrt{\frac{\pi}{2h} \frac{l - \varpi}{2h} \frac{\varpi}{2h} \varpi} \sigma (\varpi) \text{ch}^{2} \frac{\pi}{2h} \varpi \cdot d\varpi. \]

With the help of (26) and (34) we shall find \( K_{III} \) stress intensity factor:

\[ K_{III} = \frac{1}{\text{ch} \frac{\pi l}{2h}} \sqrt{\frac{2}{\text{ch} \frac{\pi l}{2h}}} \int_{0}^{l} \sigma (\varpi) \text{ch}^{2} \frac{\pi}{2h} \varpi \cdot d\varpi. \]

We shall give \( K_{III} \) stress intensity factor. Let

\[ \sigma (\varpi) \text{ch}^{2} \frac{\pi}{2h} \varpi = \sigma_{0} \equiv \text{const}. \]

Then we will find from (35):

\[ K_{III} = \sigma_{0} \sqrt{2h \cdot \text{ch} \frac{\pi l}{2h}}, \]

where by \( h \rightarrow +\infty \) we have:

\[ K_{III} = \sigma_{0} \sqrt{\pi l}. \]

If \( h \rightarrow +0 \) from (36) it follows that:

\[ K_{III} = \sigma_{0} \lim_{h \rightarrow +0} \sqrt{2h \cdot \text{ch} (\frac{\pi l}{2h})} = 0, \]

as it could be expected.

**Consequence 1.**

If \( h_{1} = h_{2} = h \), \( K_{III} \) stress intensity factor does not depend on \( k \), where \( k = \mu_{1}/\mu_{2} \). The same phenomenon is found in work [3].

Let \( h \rightarrow +\infty \), and \( h_{2} = \text{const} \). Then from (29) we get Fredholm integral equation of the second kind [2]:

\[ \frac{2}{\pi} \int_{0}^{l} \frac{\sigma (\tau)}{\sqrt{\tau^{2} - x^{2}}} d\tau = \psi (x) + \int_{0}^{l} \psi(t)K_{\varphi} (x, t) dt, \]

\[ K_{\varphi} (x, t) = k \int_{0}^{\infty} \frac{\lambda t e^{-\lambda h_{2}}}{\text{ch} \lambda h_{2} + k \cdot s h \lambda h_{2}} J_{0} (\lambda t) J_{0} (\lambda x) d\lambda, \]
\[ \psi(x) = \frac{2}{\pi} \int_{x}^{l} \frac{f(\tau)}{\sqrt{\tau^2 - x^2}} d\tau, \quad 0 \leq x \leq l, \quad f(\tau) \in K_{1/2} \] 

Here \( J_0(u) \) is Bessel function of the first kind of zero order, \( \psi(x) \) function belongs to the class of \( C[0,l] \) continuous functions.

From (26) we shall find \( K_{III} \) stress intensity factor [2]:

\[ K_{III} = \sqrt{\pi l} \psi(l), \]

as \( \psi(x) = f_0(l)/l \).

**Consequence 2.** If \( h_1 \neq h_2 \), \( K_{III} \) stress intensity factor depends on \( k \), where \( k = \mu_1/\mu_2 \).

The results obtained are important when choosing materials at the design stage [4-7].

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