Directed Branched Polymers near an Attractive Line

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Abstract

We study the adsorption-desorption phase transition of directed branched polymer in $d+1$ dimensions in contact with a line by mapping it to a $d$ dimensional hard core lattice gas at negative activity. We solve the model exactly in $1+1$ dimensions, and calculate the crossover exponent related to fraction of monomers adsorbed at the critical point of surface transition, and we also determine the density profile of the polymer in different phases. We also obtain the value of crossover exponent in $2+1$ dimensions and give the scaling function of the sticking fraction for $1+1$ and $2+1$ dimensional directed branched polymer.

Linear and branched polymers, near an attractive surface, undergo an adsorption-desorption transition, which has important applications in areas ranging from technology such as in lubrication, adhesion, surface protection to biology [1, 2]. For example, adsorbed polymers are used for surface-modification of medical implants [3]. There have been several theoretical studies of the behavior of a polymer near a surface [2, 4, 5, 6, 7, 8]. Especially, effect of surface for idealized polymer (with no self-exclusion), modeled by random walks has been studied extensively. There are many exact results known for Gaussian random walks in presence of a surface [9, 10]. In comparison, linear polymer with self exclusion and branched polymer are less well studied. For a self-avoiding walk (SAW) in the vicinity of a surface the exact critical exponents are known from conformal

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field theory [11]. Directed polymer chain adsorption, modeled by a directed SAW is one of the few solvable models of surface effects in 2 and 3 dimensions [6]. For directed walks, self exclusion is automatic, and nontrivial effects of excluded volume interaction are not seen. For branched polymers (modeled by lattice animals), a relation between the exponent characterizing the number of animals, in presence of surface, and in the bulk is known from a simple argument given by De’Bell et. al [12]. In this paper we solve directed branched polymer (DBP) in 2 dimensions and 3 dimensions in presence of a 1d line exactly. Introducing a preferred direction makes the system analytically more tractable. Similar results for 2 dimensions have also been obtained by Rensburg et. al. [13] independently.

The enumeration of directed site animals in \( d + 1 \) dimensions is related to hard-core lattice gas (HCLG) at negative activity with repulsive interactions in \( d \) dimensions and the Yang-Lee edge problem in \( d \) dimensions [14, 15, 16, 17]. In this paper we give the mapping of a \( d + 1 \) dimensional directed branched polymer in presence of a line to a \( d \) dimensional HCLG with repulsive interactions.

The plan of the paper is as follows. In section 1 we will define the model of directed branched polymer and the quantities of interest. Using the above mentioned correspondence we give the mapping of a \( d + 1 \) dimensional directed branched polymer in presence of a line to a \( d \) dimensional HCLG with repulsive interactions in Section 2. In Section 3, for \( 1 + 1 \) \( d \) DBP in presence of a 1d penetrable line we solve the model exactly. Section 4 deals with the DBP in \( 1 + 1 \) \( d \) in presence of a impenetrable line. For DBP in \( 1 + 1 \) dimensions we show that the behavior at the transition point for penetrable and impenetrable wall is the same, and not just the crossover exponent but even the density profile is the same. This implies that for \( 1 + 1 \) dimensions, for impenetrable surface, at the phase transition point the decrease in entropy is exactly compensated by the increase in internal energy. This seems to be a special property of polymers in 2 dimensions. Even for linear polymers the exponent for both cases is the same and hence it is believed that for a linear polymer in 2 dimensions in presence of a impenetrable surface the phase transition point correspond to point where surface effects vanish completely and system behaves like bulk [8]. Here we are able to show it explicitly for directed branched polymers. In Section 5 using Baxter’s solution of hard hexagon gas we study 2 + 1 dimensional DBP in presence
of line and calculate the crossover exponent and sticking fraction for the directed branched polymer exactly. The scaling function of sticking fraction is a function of two intensive thermodynamic variables. We have derived its exact form in 1 + 1 dimension and 2 + 1 dimension. There are very few such exact nontrivial scaling functions of more than one thermodynamic variable known [18]. We also get the large \( w \) expansion of the sticking fraction as a power series in \( 1/\sqrt{w} \).

1 The Model

A directed branched polymer or a directed animal on a lattice, rooted at the origin is a connected cluster such that any site of the animal can be reached from the root by a walk which never goes opposite to the preferred direction. For example, on a square lattice drawn tilted at 45\(^\circ\) in Fig. 1, a directed site animal or a directed branched polymer \( \mathcal{A} \) rooted at the origin is a set of occupied sites including origin, such that for each occupied site \((x, t)\) other than the origin, at least one of the two sites \((x-1, t-1)\) and \((x+1, t-1)\) is also occupied. The number of sites in \( \mathcal{A} \) will be denoted by \( s = |\mathcal{A}| \). We define \( n(x|\mathcal{A}) \) as the number of sites of \( \mathcal{A} \) having the transverse coordinate \( x \). We study the DBP in presence of 1d line parallel to the preferred direction. This is positioned along the main diagonal of the lattice (Fig. 1). We will consider only polymers rooted at the surface in this paper.

We assign a fugacity \( y \) to all allowed sites of the cluster. Further, if we associate an additional energy \(-E\) with each site on the surface, each site on surface will have an additional weight and the fugacity of sites about the diagonal, denoted by \( y_0 \) is equal to \( wy \) where

\[
w = \exp(E/kT)
\]

Hence \( w > 1 \) would correspond to an attractive surface.

We define \( A(w, y) \), the grand partition function of the polymer as

\[
A(w, y) = \sum_{\mathcal{A}} y^{\mathcal{A}} w^{n_0} = \sum_{s=1}^{\infty} A_s(w)y^s
\]

where \( n_0 = n(0|\mathcal{A}) \) and \( A_s(w) \) is the partition function of the polymer made of exactly \( s \) monomers.
Figure 1: A Directed Branched Polymer of size 50, rooted on the surface.

For $w = 1$, we get the statistics of equally weighted animals and $A_s(1)$ is the number of distinct directed animals having $s$ sites with given boundary conditions. For large $s$, $A_s(w)$ varies as $\lambda^s s^\theta$, where $\theta$ is known as entropic critical exponent. Similarly, the transverse size of the polymer for large $s$ scales as $s^\nu$, where $\nu$ is the exponent which defines the transverse length scale of the polymer. These exponents take different values in desorbed, adsorbed and in the critical regions. We will use subscripts $de$, $c$ and $ad$ to represent critical exponents and other quantities in desorbed, critical and adsorbed phases of the polymer.

And the free energy per monomer of the polymer in thermodynamic limit is given by

$$F(T) = \lim_{s \to \infty} \frac{k_B T}{s} \log A_s(w) \equiv k_B T \log(y_\infty(w))$$

where $y_\infty(w)$ is the value of fugacity at which $A(w, y)$ has a singularity for a given value of $w$.

Let $\phi(x, s)$ be the value of $n(x,A)$ averaged over all configurations $A$ of size $s$. We define a generating function $\Psi(x; w, y)$ as

$$\Psi(x; w, y) = \sum_A n(x|A) w^{n_0 y^{\left|A\right|}} \equiv \sum_s \phi(x, s) A_s(w) y^s$$

(4)
There is a critical value $w_c$ of wall activity such that for $w > w_c$, $\phi(0, s)$ is proportional to $s$ for large $s$ and the transverse size is finite ($\nu_{ad} = 0$). This is the adsorbed phase, in which monomers tend to stick to the surface. $w < w_c$ corresponds to the desorbed phase of the polymer in which only a finite number of monomers stick to the surface. At $w = w_c$, the critical point of the surface transition, the number of adsorbed monomers as function of polymer size in large $s$ limit have a behavior given by

$$\phi_c(0, s) \sim s^\alpha,$$

where $\alpha$ is known as the crossover exponent of the surface transition.

In the $s \to \infty$ limit, the fraction of monomers adsorbed is like order parameter of the surface phase transition. In the constant fugacity ensemble $A(w, y)$ is the partition function with fixed $w$ and $y$ and hence the average polymer size would be given by

$$\langle s(y, w) \rangle = \frac{\sum s y^s w^{n_0}}{\sum y^s w^{n_0}} = \frac{\partial \ln A(w, y)}{\partial \ln y}$$

(6)

Similarly, the average number of monomers at the surface would be

$$\langle n_0(y, w) \rangle = \frac{\sum n_0 y^s w^{n_0}}{\sum y^s w^{n_0}} = \frac{\partial \ln A(w, y)}{\partial \ln w}$$

(7)

The sticking fraction defined as the fraction of polymer segments at the surface, represented by $C_{st}(w, y)$, would be given by

$$C_{st}(w, y) = \frac{\langle n_0(y, w) \rangle}{\langle s(y, w) \rangle}$$

(8)

In the infinite polymer limit, if we represent the value of fugacity at which $\langle s(y, w) \rangle$ diverges by $y_{\infty}(w)$ for a given $w$, then the sticking fraction is only a function of the wall activity $w$ and is given by

$$C_{st}(w) = -\frac{d\ln y_{\infty}(w)}{d\ln w}$$

(9)

This is the order parameter of the surface phase transition and is zero for $w \leq w_c$, where $w_c$ is the surface phase transition point.

In general, in the large polymer limit, near critical value of $w$, as $w \to w_c^+$, $C_{st}(w, y)$ is expected to have scaling form
\[ C_s(w, y) = \epsilon^{1-\alpha} h((w - w_c)\epsilon^{-\alpha}) \]  

where \( \epsilon = 1 - y/y_\infty(w) \). The scaling function \( h(u) \) where \( u = (w - w_c)\epsilon^{-\alpha} \), is a function of \( w \) and \( y \) which are both intensive thermodynamic variables. As \( u \to \infty \), \( h(u) \sim u^{(1-\alpha)/\alpha} \).

2 General Results

The directed site animal enumeration (DSAE) problem in \( d + 1 \)-dimensions is related to time development of thermal relaxation of a hard core lattice gas (HCLG) with nearest neighbor exclusion on \( d \) dimensional lattice [15]. In [19], we have shown that this correspondence relates the density at a site \( i \) in steady state to sum of weights of all animals rooted at \( i \), the grand partition function of the animal. Also, the average number of sites at a given transverse distance \( x \) from the origin for a \( d + 1 \) dimensional directed animal is related to the density-density correlation function of the lattice gas in \( d \) dimensions.

Specifically, if on a \( d+1 \) dimensional body-centered hyper-cubic lattice we define weight of an animal \( A \) as the product of weights of all occupied sites, with weight corresponding to a site with \( x \) coordinate \( i \) being \( y_i \), then the DSAE problem on this \( d + 1 \) dimensional lattice gets related to time development of HCLG with nearest neighbor exclusion on a \( d \) dimensional body-centered hyper-cubic lattice with the rates which satisfy detailed balance condition corresponding to the Hamiltonian

\[ H = +\infty \sum_{<ij>} n_in_j - \sum_i (\ln z_i)n_i \]  

where \( z_i = -y_i/(1+y_i) \) and the animal number generating function is just the negative of density of HCLG with change of variables from \( z \) to \( y \). Here we have used the convention that if \( \sum_{<ij>} n_in_j = 0 \) then the corresponding term in the Hamiltonian is zero. The configurations with any pair of occupied nearest neighbor have infinite energy and do not contribute to the partition function.

The partition function is linear in all \( z_i \)'s. The linearity of the partition function in \( z_i \)'s implies that in case when the activity about \( x = 0 \) is different from that in the rest of the sample, i.e if we let the activity about \( x = 0 \) be \( z_0 \) and activity in rest of space be \( z \), then the partition function of the HCLG can be written as
\[ Z(z_0, z) = A(z) + z_0 B(z) \]  \hspace{1cm} (12)

where \( A(z) \) and \( B(z) \) are polynomials in \( z \). If \( \rho \) represents the density of HCLG when the activity about each site is the same, then the density of HCLG about the origin in the present case \( \rho_0(z_0, z) \) can be written in terms of \( \rho \) as

\[ \rho_0(z_0, z) = \frac{z_0 \rho}{\rho z_0 + z (1 - \rho)} \]  \hspace{1cm} (13)

Same observation has been made by Cardy in [20]. Correspondingly, since \( A(w, y) \) is just the negative of \( \rho_0(z_0, z) \) with \( z_0 = -wy/(1 + wy) \) and \( z = -y/(1 + y) \), we can express \( A(w, y) \) in terms of \( A(1, y) \) and this is given by

\[ A(w, y) = \frac{w(1 + y)A(1, y)}{(1 + wy) + A(1, y)(1 - w)} \]  \hspace{1cm} (14)

Moreover, the density-density correlation function of HCLG \( G(x; w, z) \) with \( w \neq 1 \) can be expressed in terms of density density correlation function when \( w = 1 \). We find that the density density correlation function is related to \( \Psi(x; w, y) \) on a hyper-cubic lattice as follows

\[ \Psi(x; w, y) = -\frac{1}{1 + y} G \left( x; w, z = \frac{-y}{1 + y} \right) \]  \hspace{1cm} (15)

from this we get

\[ \frac{\Psi(x; w, y)}{\Psi(x; 1, y)} = \frac{w(1 + y)[1 + wy - (1 - A(1, y))(1 - w)]}{[1 + wy + A(1, y)(1 - w)]^2} \]  \hspace{1cm} (16)

Since \( \rho \) is the density of the HCLG, then as discussed in [19], for \( x = 0 \), the density density correlation of HCLG is always equal to \( \rho(\rho - 1) \) for any \( d \) dimensional case and hence \( \Psi(0; 1, y) \) can be completely expressed in terms of \( A(1, y) \). Hence we get,

\[ \Psi(0; w, y) = \frac{w(1 + y)A(1, y)(1 + A(1, y))}{[1 + wy + A(1, y)(1 - w)]^2} \]  \hspace{1cm} (17)

Eq. (14-17) hold for all dimensions. Hence, in presence of 1d surface, a DBP in \( d + 1 \) dimensions rooted on the surface can be studied using the mapping to HCLG. Moreover the generating functions \( A(w, y) \) and \( \Psi(0; w, y) \) can be completely expressed in terms of animal number generating function when wall is neutral i.e, in terms of \( A(1, y) \). We will
use these results in rest of the paper to study the surface effects for DBP in 2 and 3 dimensions.

In the adsorbed regime the number of monomers in direct contact with the wall is proportional to $s$ and $\nu_{ad} = 0$. This implies that the scaling form of $\phi(x, s)$ in the adsorbed regime would be

$$\phi_{ad}(x, s) \sim \frac{s}{\xi d} g(|x|/\xi)$$

where $\xi = (w - w_c)^2$ is the characteristic length scale in the system. Since we are away from the critical regime, $\xi$ is well behaved and never diverges for finite $w$. Also $\xi$ is independent of the size $s$ of the polymers. The normalization of scaling function $g(r)$ is chosen such that

$$\int_{-\infty}^{\infty} d^d x g(|x|) = 1$$

$A_s(w)\phi(x, s)$ is the coefficient of $y^s$ in the expansion of $\Psi(x; w, y)$. In the adsorbed regime, $A_s(w) \sim (y_\infty(w))^{-s}$ for large $s$ and behavior of $\phi(x, s)$ is given by Eq. (18), hence $\Psi(x; w, y)$ will have a scaling form

$$\Psi(x; w, y) \sim \frac{\epsilon^2}{\xi d} g(|x|/\xi)$$

where $\epsilon = 1 - y/y_\infty(w)$.

Since the scaling function $g(|x|/\xi)$ has no $y$ dependence, hence the scaling function of $G(x; w, z)$ would also be just $g(|x|/\xi)$ for $w > w_c$.

3 Two dimensional Directed Branched Polymer in presence of 1-d penetrable surface

For a penetrable surface, since the configurations spanning through the surface are allowed, there is no loss of entropy per monomer to take into account (Fig 1). Hence, $w = 1$ corresponds to a zero gain in free energy per monomer of the surface. We find that the value of $y$ at which $A(1, y)$ diverges, the only value of $w$ which makes $A(w, y)$ also divergent is $w = 1$. This implies that $w_c = 1$ for a DBP in any dimension in presence of a 1d line as long as $A(1, y)$ gets singular at finite value of $y$. Then the polymer has bulk behavior at the critical point. At $w = 1$, i.e for directed branched polymer in bulk, we
have shown in an earlier paper [19] by scaling arguments and dimensional analysis φ(x, s) has a scaling form

$$\phi_c(x, s) \sim s^{1-\nu_c} f(|x|^\epsilon^{\nu_c})$$  \hspace{1cm} (21)

This implies \(\phi_c(0, s) \sim s^{1-\nu_c}\), and the crossover exponent \(\alpha\) is exactly given by

$$\alpha = 1 - d\nu_c \equiv 1 - \theta$$  \hspace{1cm} (22)

where \(\nu_c\) is the transverse correlation exponent of a \(d+1\) dimensional DA in bulk, which is equal to the correlation length exponent for a \(d\) dimensional HCLG with nearest neighbor exclusion.

As we go to higher dimensions even though entropy loss and energy gain balances each other at \(w = 1\), the polymer might start binding to a line only at wall activity greater than 1. For directed branched polymers, when \(A(1, y)\) has no divergence, \(w = 1\) is not the critical point of the surface transition. Instead it is given by

$$w_c = \frac{1 + 1/A(1, y_c)}{1 - y_c/A(1, y_c)}$$  \hspace{1cm} (23)

where \(y_c\) is the large polymer limit fugacity value of the polymer with neutral wall i.e., when \(w = 1\).

As an example, on a Bethe lattice with co-ordination number 3 the function \(A(1, y)\) is

$$A_B(1, y) = \frac{1 - \sqrt{1 - 4y}}{2y}$$  \hspace{1cm} (24)

and \(y_c = 1/4\). At \(y = 1/4\) the function \(A_B(1, y) = 2\), and substituting in Eq.23 we get \(w_c = 12/7\), which is greater than 1.

The \(1 + 1\) \(d\) DA gets mapped to a \(1d\) HCLG. For \(1 + 1\) dimensional DAs in bulk in [19] we have derived the exact expressions of \(A(1, y)\) and \(\Psi(x; 1, y)\). Using them and Eq. 14-16 we get the expressions for \(A(w, y)\) and \(\Psi(x; w, y)\) as follows

$$A(w, y) = \frac{2wy(1 + y)}{(1 - y - wy - 3wy^2) + (1 + wy)(1 - 3y)(1 + y)}$$  \hspace{1cm} (25)

The connected density-density correlation function of the corresponding gas is simple exponential and hence the generating function \(\Psi(x; w, y)\) has a form
\[ \Psi(x; w, y) = K(w, y)\exp(-b(y)|x|) \]  

(26)

where it is straightforward to calculate \( K(w, y) \) and \( b(y) \), and we get

\[ K(w, y) = \frac{2wy(1-3y)(1+wy)(1-y+\sqrt{(1-3y)(1+y)})}{[(1-3y)(1+wy)\sqrt{1+y+(1-y-wy-3wy^2)^2}]} \]  

(27)

and

\[ b(y) = \log(\sqrt{1+y} + \sqrt{1-3y}) - \log(\sqrt{1+y} - \sqrt{1-3y}) \]  

(28)

The generating functions \( A(w, y) \) and \( \Psi(x; w, y) \) have branch cut at \( y = 1/3 \). For \( w = 1 \), they also have a pole singularity at \( y = 1/3 \). Hence, clearly the phase transition from desorbed to adsorbed phase occurs at \( w = 1 \), i.e \( w_c = 1 \). For \( w \leq 1 \), \( y_\infty(w) = 1/3 \) and for \( w > 1 \) it is given by the real positive solution of

\[ w - y - w(3+2w)y^2 - 3w^2y^3 = 0 \]  

(29)

Near the critical point, for \( w = 1 + \delta \), to leading order we get \( y_\infty(w) \) to be

\[ y_\infty(w) = \frac{1}{3} - \frac{\delta^2}{16} + \text{higher order terms} \]  

(30)

The sticking fraction \( C_{st}(w, y) \) can also be exactly calculated and we get it to be

\[ C_{st}(w, y) = \left[ \frac{y(1-w)}{1+y} + \frac{1+wy}{\sqrt{(1+y)(1-3y)}} \right]^{-1} \]  

(31)

From this, near the critical point, we get the scaling form of \( C_{st}(w, y) \) to be

\[ C_{st}(w, y) = \sqrt{\epsilon} \ h(u) \]  

(32)

where \( \epsilon = 1 - y/y_\infty(w) \) and \( u = \epsilon^{-1/2}\delta \) and we get

\[ h(u) = \sqrt{\frac{3}{2}} \left[ 1 + \frac{9u^2}{48} \right]^{\frac{1}{2}} \]  

(33)

This gives the order parameter \( C_{st}(w) \) near the critical point to be proportional to \( \frac{3\delta^4}{8} \).

For large values of \( w \), expanding \( C_{st}(w) \) in powers of \( (1/w) \) we get
be mapped to the HCLG in 1-d with fugacity 0 for all sites lying along the negative axis.

detrable surface, about the diagonal (Fig 2). From the exact gene rating function

Two dimensional Directed Branched Polymer in presence of a solid wall along the growth direction can take place at a non trivial value of adsorption activity.

Figure 2: Directed Branched Polymer on a Square lattice in presence of a 1d impenetrable line about the diagonal.

\[ C_{st}(w) \sim \frac{1}{2} - \frac{3\sqrt{2}}{16w} - \frac{3}{16w} - \ldots \]  
\hspace{1cm} (34)

For \( w \to \infty \) it approaches 1/2, the maximum possible fraction that can stick to wall, as expected.

4 Two dimensional Directed Branched Polymer in presence of 1-d impenetrable surface

In the presence of an impenetrable surface, because of loss in entropy per monomer on the wall, the transition from desorbed to adsorbed phase takes place at a non trivial value of adsorption activity.

Here we study a DBP in 1 + 1 dimension on a square lattice, in presence of an impenetrable surface, about the diagonal (Fig 2). From the exact generating function \( A(1, y) \), \( A(w, y) \) and \( \Psi(x; w, y) \), it is straightforward to determine the critical value of \( w \) and the sticking fraction and the density profile in the desorbed, critical and adsorbed phase of the system. The 1 + 1 d case in presence of a solid wall along the growth direction can be mapped to the HCLG in 1-d with fugacity 0 for all sites lying along the negative axis. Making use of this mapping we get
\[ A(1, y) = \frac{(1 - y - \sqrt{1 - 2y - 3y^2})}{2y} \] (35)

and \( A(w, y) \) is easy to get by substituting in Eq. 14. This result can also be obtained using heap method. For an alternate treatment see [13].

The density-density correlation of the corresponding gas is exponential and hence the generating function \( \Psi(x; w, y) \) still has a form given by Eq. 26, but \( K(w, y) \) and \( b(y) \) are now given by

\[
K(w) = \frac{wy(1 + wy)(\sqrt{1 + y} + \sqrt{1 - 3y})}{(1 + y)(1 - w^2 y^2)\sqrt{1 - 3y} + (1 - y - (4 - w)wy^2 - w^2 y^3)\sqrt{1 + y}}
\] (36)

and

\[
b(y) = \log(\sqrt{1 + y} + \sqrt{1 - 3y}) - \log(\sqrt{1 + y} - \sqrt{1 - 3y})
\] (37)

The generating functions \( A(w, y) \) and \( \Psi(x; w, y) \) have a branch cut at \( y = 1/3 \). At \( w = 1 \), \( A(1, y) \) has no divergence and \( y_c = 1/3 \). Substituting in Eq. (23), we get \( w_c = 3 \).

This value is greater than the value for 1 + 1 d DBP with a penetrable surface. This is expected, since the tendency of polymer to grow away from the surface is more when the surface is impenetrable and hence only when the surface gets sufficiently attractive, the polymer starts sticking to it. For \( w > 3 \), the closest singularity to the origin occurs at

\[ y_s = \frac{\sqrt{4w - 3} - 1}{2w} \] (38)

For \( w \leq 3 \) the branch cut singularity 1/3 dominates and hence \( y_\infty(w) \), the infinite polymer limit fugacity value is equal to 1/3 for \( w \leq 3 \). Whereas for \( w > 3 \), \( y_\infty(w) = y_s \).

Free energy is a constant and the order parameter, \( C_{st}(w) \) is zero for \( w < 3 \).

We get the sticking fraction, \( C_{st}(w, y) \) to be

\[
C_{st}(w, y) = \frac{1 - 2y - 3y^2 + (-1 + y + 2y^2)\sqrt{(1 + y)(1 - 3y)}}{y[-2y + (w + 2y - wy)\sqrt{(1 + y)(1 - 3y)} + w(-1 + 2y + 2y^2)]}
\] (39)

Near the critical point for \( w = 3 + \delta \) and \( y = y_s(1 - \epsilon) \), we get the same scaling form for \( C_{st}(w, y) \) as given by Eq. 32, with the scaling function \( h(u) \) to be

\[
h(u) = \frac{2}{\sqrt{3}} \left[ 1 + \frac{u^2}{27} \right]^{\frac{1}{2}}
\] (40)
Hence $C_{st}(w)$ is proportional to $\frac{2\delta}{w}$ near the critical point and approaches $1/2$ as $w \to \infty$. This is plotted in Fig 3 along with the sticking fraction for the penetrable case. The qualitative behavior in both cases is just the same, the main difference being the shift of the transition point from 1 to 3 and the initial slope. For large value of $w$ it is easy to expand $C_{st}(w)$ in powers of $1/w$. It should be noted that large $w$ expansion of $C_{st}(w)$ will involve powers of $w^{-1/2}$ in this case as well.

Using the exact equations for generating function for $\Psi(x; w, y)$ we translate these results to the constant number ensemble and we get the function $\phi(x, s)$ in three regions giving the spread of sites as a function of distance from the wall. Here we give these calculations for the impenetrable case only because the qualitative behavior in both impenetrable and penetrable case is exactly same for $1+1$ dimensional system.

In the desorbed phase($w < 3$), expanding near $y_c = 1/3$ as $y = y_c e^{-\epsilon}$, we get the scaling form for $\Psi(x; w, y)$ to be

$$
\Psi(x; w, \epsilon) = c(w)\exp(-x\sqrt{3\epsilon})
$$

where, $c(w)$ is a $w$ dependent constant and is $\frac{3(3+w)}{2(3-w)^2}$.

To obtain $\phi(x, s)$ for large $s$, we need to determine the coefficient of $y^s$ in the series expansion of $\Psi(x; w, \epsilon)$ i.e,

$$
\Psi(x; w, \epsilon) \equiv \sum_s \phi(x, s)A_s(w) y^s = c(w) \sum_{k=0}^{\infty} \frac{(-\sqrt{3x})^k}{\Gamma[k + 1]} (1 - 3y)^{\frac{k}{2}}
$$
\[
\phi(x, s) = c(w) \sum_{s=0}^{\infty} (3y)^s \sum_{k=0}^{\infty} \frac{(-\sqrt{3}x)^k}{\Gamma[k+1] \Gamma[s+1] \Gamma[-k/2]} \Gamma[s-k/2] \tag{43}
\]

For fixed \(k\) and large \(s\),

\[
\frac{\Gamma[s-k/2]}{\Gamma[s+1]} \to s^{-1-k/2} \tag{44}
\]

Hence the leading singular behavior of \(\phi(x, s)A_s(w)\) in the desorbed phase is given by

\[
\phi(x, s)A_s = \frac{3^s c(w)}{s} \sum_{k=0}^{\infty} \frac{(-\sqrt{3}x/\sqrt{s})^k}{\Gamma[k+1] \Gamma[-k/2]} \Gamma[s-k/2] \tag{45}
\]

Since \(\Gamma[-k/2]\) has poles when \(k\) is an even integer, only odd terms contribute to the sum. It is easy to sum the resulting series, giving \(\phi(x, s)\) for large \(s\) in the desorbed phase to be

\[
\phi_{de}(x, s) = \frac{3}{2} x \exp \left( -\frac{3x^2}{4s} \right) \tag{46}
\]

For \(w = 3\), \(c(w)\) is singular and we have to keep terms till first order in \(\epsilon\) in the expansion (for \(w < 3\) the constant term dominates) and we get

\[
\Psi(x; 3, \epsilon) = \frac{1}{\epsilon} \exp(-x\sqrt{3}\epsilon) \tag{47}
\]

Again, just as in the desorbed phase expanding \(\Psi(x; 3, \epsilon)\) in powers of \(y^s\), the average number of sites at a distance \(x\), i.e \(\phi(x, s)\) for the critical region for large \(s\) is

\[
\phi(x, s) = \frac{\sqrt{3\pi s}}{2} \text{erfc} \left( \frac{\sqrt{3}x}{2\sqrt{s}} \right) \tag{48}
\]

Hence we see that at \(w = 3\) not just the crossover exponent \(\alpha\) is equal to \(1/2\), but even the scaling form of \(\phi(x, s)\) is same as that of a \((1+1)\) dimensional DA in bulk [19] and hence same as that of the penetrable wall at the critical point. This unusual result can be understood as coming from exact cancellation of decrease in entropy and increase in internal energy at the critical point. Also note that the value of exponent \(\alpha = 1/2\) for DBP is equal to the estimates of \(\alpha\) for branched polymers [21] and linear polymers [22] in 2 dimensions. Infact for adsorption of an undirected \(d\) dimensional branched polymer to a \(d-1\) dimensional surface, the crossover exponent \(\alpha\) is conjectured to be \(1/2\) in all spatial dimensions [23].
For $w > 3$, the behavior of the generating function is dominated by the singularity given by Eq. (38). For $w \gg 3$, $y_s \approx 1/\sqrt{w}$ and we get the large $s$ behavior of $\phi(x, s)$ to be

$$\phi(x, s) = s \exp(-x)$$  

(49)

i.e, most of the sites stick to the origin as expected.

Similarly, expanding $A(w, y)$ about $y_c$ and then going to constant number $(s)$ ensemble, we get $A_s(w)$ for large $s$ as $A_s(w) \sim \frac{\sqrt{3}}{2\sqrt{\pi}} c(w) 3^s s^{-\frac{3}{2}}$ in the desorbed regime. Hence the number of animals in presence of the 1d impenetrable wall i.e $A_s(1)$ for large $s$ are $A_s(1) \sim \frac{\sqrt{3}}{2\sqrt{\pi}} 3^s s^{-\frac{3}{2}}$. This gives $\theta_{de}$ to be $3/2$. This is consistent with the result derived for lattice trees by De’Bell et al [12]. Also we get at the critical point $w = 3$, $A_s(3) \sim \frac{2}{3\sqrt{\pi}} 3^s s^{-\frac{1}{2}}$, implying $\theta_c$ to be $1/2$. For $w \gg 3$, $A_s(w) \sim (\sqrt{w})^s$, giving $\theta_{ad} = 0$.

The function $\phi(x, s)$ gives the density profile of the polymer as a function of distance from the surface. Since the configurations are very different in two phases as shown schematically in Fig 2 hence $\phi(x, s)$ is very different in three regions. In desorbed phase it peaks away from the surface at a distance of the order of the average transverse diameter of the polymer in the large $s$ limit. Whereas at the critical point it peaks at the surface (Fig 4).
5 Three dimensional Directed Branched Polymer in presence of an attractive line

In $2 + 1$ dimensions, a DBP on a simple cubic lattice with nearest and next nearest neighbor connections gets mapped to the hard hexagon gas model in 2 dimensions at negative activity in the disordered regime, which was solved by Baxter [24]. He obtained the equation for the average density of the gas. It was shown by Joyce, there is an algebraic equation in $z$ (activity of the gas) and $\rho$ (density of the gas) [25].

The equation given by Joyce is quartic in $z$ and 12th order in $\rho$. For convenience we will just rewrite it here [25]

$$ \rho(1 - \rho)^{11} - (1 - \rho)^5 P_1(\rho) z + \rho^2 (1 - \rho)^2 P_2(\rho) z^2 - \rho^5 P_1(\rho) z^3 + \rho^{11} (1 - \rho) z^4 = 0 \quad (50) $$

where

$$ P_1(\rho) = (1 - 13\rho + 66\rho^2 - 165\rho^3 + 220\rho^4 - 165\rho^5 + 77\rho^6 - 22\rho^7) $$
$$ P_2(\rho) = (1 - 13\rho + 63\rho^2 - 125\rho^3 + 6\rho^4 + 401\rho^5 - 689\rho^6 + 476\rho^7 - 119\rho^8) $$

The density $\rho$ of the HCLG is just the negative of $A(1, y)$ and $z = -y/(1 + y)$. It is straightforward to get an algebraic equation in $A(1, y)$ as a function of $y$ [26]. As $A(w, y)$ is a simple rational function of $A(1, y)$, $y$ and $w$ (see Eq.14), substituting $\rho$ in terms of $A(w, y)$, the grand partition function of the 2 + 1 dimensional DA in presence of an one dimensional line about the main diagonal of the lattice, we get a 12th order polynomial equation in $A(w, y)$, where the coefficients are functions of $w$ and $y$. Explicit writing down the equation is rather tedious and is omitted. Since $A(1, y)$ becomes singular for $y = y_c = 2/(9 + 5\sqrt{5})$, hence in presence of 1-dimensional line the polymer will undergo a desorption-adsorption transition at $w = 1$. For $w \leq 1$ the dominant singularity will be $y_c$ and $y_\infty(w) = y_c$. For $w > 1$, at $y_\infty(w)$, $A(w, y)$ tends to infinity and at this point the coefficient of highest order term must be zero. Since we have a 12th order equation in $A(w, y)$ hence by equating the coefficient of the 12th order term to zero, we get a polynomial equation in $y$ and $w$ ($Q(y, w) = 0$) whose smallest positive real root would be $y_\infty(w)$. This polynomial is 12th order in $w$. But we can find the root numerically. The free-energy is just $\log(y_\infty(w))$ and hence can be evaluated numerically.
In this case the expressions of $A(w, y)$ and other generating functions are rather complicated and hence it is difficult to go to the constant size ensemble. But at critical point system behaves like bulk and since $\theta = 5/6$, by hyper-scaling arguments $\nu_c = 5/12$ which implies that the crossover exponent $\alpha = 1/6$ (Eq. 22). And by Eq. 10, the sticking fraction $C_{st}(w, y) \sim (1 - y/y_\infty(w))^5/6$ as $w \to 1^+$ asymptotically.

By solving $Q(y, w) = 0$ we get $y_\infty(w)$ as a function of $w$. Near the critical point for $w = 1 + \delta$, to leading order we get

$$y_\infty(w) = \frac{c}{y - y_c} \left(1 - \epsilon \right)$$  

(51)

where $c = 5(5\gamma)^3$ with $\gamma = (13\sqrt{5} - 25)/50$.

In the large polymer limit, for $y$ very close to $y_c$, $A(1, y)$ has a scaling form

$$A(1, y) = a_0 \left(1 - \frac{y}{y_c}\right)^{\frac{5}{6}} \left[1 + a_1 \left(1 - \frac{y}{y_c}\right)^{\frac{5}{6}} + \ldots\right]$$  

(52)

where $a_0 = (\sqrt{5}\gamma^{1/6})^{-1}$ [26].

Hence taking $y = y_\infty(w)(1 - \epsilon)$ and $w = 1 + \delta$ we get the scaling function of $C_{st}(w, y) = \epsilon^{5/6} h(u)$ to be

$$h(u) = \frac{6a_0}{1 + y_c} (1 + cu^6)^{\frac{5}{6}} - 6cu^5$$  

(53)

where $u = \delta \epsilon^{-1/6}$. The scaling function $h(u)$ is a function of $w$ and $y$, which are both thermodynamic variables.

For large $w$, expanding in power of $1/w$ we get

$$y_\infty(w) \sim \frac{1}{\sqrt{6}w} - \frac{1}{4\sqrt{2}} - \frac{11}{16}\sqrt{\frac{3}{2}} \left(\frac{1 \sqrt{3}}{w}\right)^{\frac{3}{2}} - \ldots$$  

(54)

and

$$C_{st}(w) \sim \frac{1}{2} - \frac{1}{4\sqrt{2}} - \frac{9}{4\sqrt{2w}} - \ldots$$  

(55)

As $w \to \infty$, $C_{st}(w)$ approaches $1/2$, the maximum possible fraction of adsorption. It is like order parameter of the surface transition. It is plotted in Fig 3. As is clear from the scaling function, the sticking fraction increases much slowly than in the $1 + 1$ $d$ case. This is expected as there the polymer in $d + 1$ dimensions was getting adsorbed at a $d$
dimensional surface whereas here a polymer in $d + 1$ dimensions is getting adsorbed on a $d - 1$ dimensional surface.

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