Wavelets Applied to CMB Maps: a Multiresolution Analysis for Denoising

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ABSTRACT

Analysis and denoising of Cosmic Microwave Background (CMB) maps are performed using wavelet multiresolution techniques. The method is tested on 12°.8 × 12°.8 maps with resolution resembling the experimental one expected for future high resolution space observations. Semianalytic formulae of the variance of wavelet coefficients are given for the Haar and Mexican Hat wavelet bases. Results are presented for the standard Cold Dark Matter (CDM) model. Denoising of simulated maps is carried out by removal of wavelet coefficients dominated by instrumental noise. CMB maps with a signal-to-noise, S/N ∼ 1, are denoised with an error improvement factor between 3 and 5. Moreover we have also tested how well the CMB temperature power spectrum is recovered after denoising. We are able to reconstruct the Cℓ ’s up to ℓ ∼ 1500 with errors always below 20% in cases with S/N ≥ 1.

Key words: cosmology: CMB – data analysis

1 INTRODUCTION

Future CMB space experiments will provide very detailed all-sky maps of CMB temperature anisotropies; NASA MAP Mission (Bennett et al. 1996) and the ESA Planck Mission (Mandolesi et al. 1998; Puget et al. 1998). The high sensitivity of these experiments will result in unique data to constrain fundamental cosmological parameters. Moreover, future CMB maps will allow to distinguish between competing theories of structure formation in the early universe and will provide very fruitful data on astrophysical foregrounds.

The cosmological signal in CMB maps is hampered by instrumental noise and by foreground emissions. Therefore, a necessary step in analysing CMB maps is to separate the foreground emissions from the CMB signal. Several linear and non-linear methods have already been tested on simulated data (Bouchet, Gispert & Puget 1996; Tegmark & Efstathiou 1996; Hobson et al. 1998). An alternative method can be one based on wavelets. Wavelets are known to be very efficient in dealing with problems of data compression and denoising. Development of wavelet techniques applied to signal processing has been very fast in the last ten years (see Jawerth & Sweldens 1994 for an overview). These techniques have already been applied to a variety of astrophysical problems. For example, regarding cosmology, Slezak, de Lapparent & Bijaoui (1993) have applied wavelet analysis to the detection of structures in the CfA redshift survey. They have also been introduced to study the Gaussian character of CMB maps (Pando et al. 1998, Hobson et al. 1998). A study using spherical Haar wavelets to denoise CMB maps has just appeared (Tenorio et al. 1999).

We consider small patches of the sky where a flat 2-D approach is valid. We apply wavelet multiresolution techniques, known to be computationally very fast taking only O(N) operations to reconstruct an image of N pixels. In the 2-D flat wavelet analysis a single scale and two translations are usually introduced, where the basis is generated by 4 tensor products of wavelets and scaling functions. Therefore, three detail images plus an approximation image appear at each level of resolution. Different wavelet bases are characterized by their location in space. The bases considered in this work are Mexican Hat, Haar and Daubechies. The first one is the most localized though, as opposed to the other two, it does not have a compact support. As a first approach to the application of these techniques to CMB data we only consider maps with cosmological signal plus instru-
2 CONTINUOUS WAVELET ANALYSIS

2.1 One-dimensional Transform

The Fourier transform is a powerful tool in many areas but in dealing with local behaviour shows a tremendous inefficiency. For instance, a large number of complex exponentials must be combined in order to produce a spike. The wavelet transform solves this problem, introducing a good space-frequency localization. It is conceptually sim-

ple and it constitutes a fast algorithm. Let \( \psi(x) \) be a one-dimensional function satisfying the following conditions: a) \( \int_{-\infty}^{\infty} dx \psi(x) = 0 \), b) \( \int_{-\infty}^{\infty} dx \psi^2(x) = 1 \) and c) \( C_\psi \equiv \int_{-\infty}^{\infty} dk |k|^{-1} \psi(k) < \infty \), where \( \psi(k) \) is the Fourier transform of \( \psi(x) \). So, according to condition a), the wavelet must have oscillations. Condition b) is a normalization and c) re-

presents an admissibility condition in order to reconstruct a function \( f(x) \) with the basis \( \psi \) (see equation (2) for such a synthesis).

We define the analyzing wavelet as \( \Psi(x; R, b) \equiv R^{-1/2} \psi(\frac{x-b}{R}) \), dependent on two parameters: dilation \( (R) \) and translation \( (b) \). It operates as a mathematical micro-

scope of magnification \( R^{-1} \) at the space point \( b \). The wavelet coefficients associated to a one-dimensional function \( f(x) \) are:

\[
w(R, b) = \int dx f(x) \Psi(x; R, b) .
\]  

(1)

It is clear from the above definition that such coefficients represent the analyzing wavelet at \( x_o \) for a delta distribution peaked at this point, i.e. for \( f(x) = \delta(x - x_o) \). For \( R = 1 \), \( w(R, b) \) is the convolution of the function \( f \) with the analyzing wavelet \( \psi \).

The reconstruction of the function \( f \) can be achieved in the form

\[
f(x) = (2\pi C_\psi)^{-1} \int \int dR db R^{-2} w(R, b) \Psi(x; R, b) .
\]  

(2)

Examples of wavelet functions are: i) Haar, \( \psi = 1(-1) \) for \( 0 < x < 1 \) (1/2 < \( x \) < 1), ii) Mexican hat, \( \psi = \frac{1}{(\pi x^2)^{1/4}}(1-x^2)e^{-x^2/2} \).

2.2 Two-dimensional Transform

Regarding the two-dimensional case, we introduce a one-

dimensional scaling function \( \phi \) normalized in the form:

\[
\int_{-\infty}^{\infty} dx \phi(x) = 1 .
\]  

Examples of scaling functions are: i) Haar, \( \phi = 1(0) \) for \( 0 < x < 1 \) (\( x \) < \( 0 \), \( x \) > 1), ii) Mexican Hat, \( \phi = \frac{2}{(\pi x^2)^{1/4}}e^{-x^2/2} \). The analyzing scaling

\[
\Phi(x; R, b) \equiv R^{-1/2} \phi(\frac{x-b}{R}) ,
\]  

allows to define details of an image, \( f(\bar{x}) \), with respect to the tensor products

\[
\Gamma_d(\bar{x}; R, \bar{b}) \equiv \Psi(x_1; R, b_1)\Psi(x_2; R, b_2) ,
\]  

(3)

\[
\Gamma_h(\bar{x}; R, \bar{b}) \equiv \Phi(x_1; R, b_1)\Psi(x_2; R, b_2) ,
\]  

(4)

\[
\Gamma_v(\bar{x}; R, \bar{b}) \equiv \Psi(x_1; R, b_1)\Phi(x_2; R, b_2) .
\]  

(5)

The diagonal, horizontal and vertical wavelet coefficients are defined by \( (\alpha \equiv d, h, v) \)

\[
w_\alpha(R, \bar{b}) = \int d\bar{x} f(\bar{x}) \Gamma_\alpha(\bar{x}; R, \bar{b}) .
\]  

(6)

Scaling functions act as low-pass filters whereas wavelet functions single out one scale. Therefore, detail coefficients provide local information about symmetrical (diagonal) and elongated/filamentary structure (vertical and horizontal).

Let us now assume an homogeneous an isotropic random field \( f(\bar{x}) \), i.e. the correlation function \( C(r) \equiv f(\bar{x})f(\bar{x} + \bar{r}) > 0, r \equiv |\bar{r}| \), where \( <\cdots> \) denotes an average value over realizations of the field. The Fourier transform of the field \( f(\bar{k}) \) satisfies

\[
f(\bar{k})f(\bar{k}') > P(k)\delta^2(\bar{k} - \bar{k}') ,
\]  

where

\[
P(k) = \frac{2\pi}{2}\int d\bar{r} f(\bar{r})f(\bar{r}') f(\bar{r} + \bar{r}') ,
\]  

\[
\Delta_C(k) = \frac{1}{2}\int d\bar{r} C(\bar{r}) .
\]  

(7)

(8)
where $k \equiv |\vec{k}|$ and $P(k)$ is the power spectrum (the Fourier transform of $C(r)$). In this case we can calculate the correlation and variance of the wavelet coefficients: $C_\alpha(r; R) \equiv <w_\alpha(R; \vec{b})w_\alpha(R; \vec{b} + \vec{r})>$, $\sigma_\alpha^2(R) \equiv C_\alpha(0; R)$ and we find the following equations

$$C(0) \equiv \sigma^2 = C_{\Gamma_\alpha}^{-1} \int dR R^{-2} \sigma_\alpha^2(R) ,$$

$$C_{\Gamma_\alpha} \equiv (2\pi)^2 \int d\vec{k} k^{-2} |\tilde{\Gamma}_\alpha|^2(\vec{k}) ,$$

where $\tilde{\Gamma}_\alpha(\vec{k})$ is the Fourier transform of $R \tilde{\Gamma}_\alpha$.

On the other hand, we calculate the Fourier transform of the wavelet coefficients $w_\alpha(R; \vec{b})$ with respect to the $\vec{b}$ parameters:

$$<w_\alpha(R; \vec{k})w_{\alpha'}(R'; \vec{k}')> = w_{\alpha\alpha'}(R; R'; \vec{k} - \vec{k}') ,$$

$$w_{\alpha\alpha'} = (2\pi)^2 R^2 P(k)\tilde{\Gamma}_{\alpha'}(R\vec{k})\tilde{\Gamma}_\alpha(R'\vec{k}) ,$$

that allows us to get the detail wavelet variances as

$$\sigma_\alpha^2(R) = \int d\vec{k} P(kR^{-1})|\tilde{\Gamma}_\alpha(\vec{k})|^2 .$$

The diagonal variance corresponds to the tensor product of two one-dimensional wavelets. If $|\psi(k)|^2$ is a function strongly peaked near $k \approx 1$ then $\sigma_\alpha^2(R) \approx P(k \approx R^{-1})$, taking into account the normalization of the wavelet function, that allows an estimation of the power spectrum in terms of the diagonal component. This is what happens for the Mexican hat: $|\psi(k)|^2 \propto k^4 e^{-k^2}$, with a maximum at $k = 2^{-1/2}$, whereas the Haar wavelet is not localized in Fourier space: $|\psi(k)|^2 \propto (k/4)^{-2} \sin^4(k/4)$. We can also deduce that $C_h = C_v$ and $\sigma_h^2 = \sigma_v^2$ taking into account the symmetry of the equations. Moreover, the temperature power spectrum $P(k)$ can be obtained from the detail wavelet power spectrum $w_{\alpha\alpha}(R; R; \vec{k})$ as follows

$$P(k) = \frac{1}{C_{\Gamma_\alpha}} \int \frac{dR}{R^2} \int d\theta w_{\alpha\alpha}(R; R; k\vec{n}) ,$$

$$\vec{n} = (\cos \theta, \sin \theta) .$$

For the Haar and Mexican wavelets we can calculate:

$$|\tilde{\Gamma}_h|^2 = \frac{1}{(2\pi)^2} \left( \frac{k_1 k_2}{4} \right)^{-2} \left[ \sin \frac{k_1}{4} \sin \frac{k_2}{4} \right]^{2} ,$$

$$|\tilde{\Gamma}_h|^2 = \frac{1}{(2\pi)^2} \left( \frac{k_1 k_2}{4} \right)^{-2} \left[ \sin \frac{k_2}{4} \sin \frac{k_1}{4} \right]^{2} .$$
3 DISCRETE WAVELET ANALYSIS

3.1 One-dimensional Multiresolution Analysis

An orthonormal basis of $L^2(\mathbb{R})$ can be constructed from a wavelet $\psi$ through dyadic dilations $j$ and translations $k$

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

In addition, a scaling function $\phi$ can be defined associated to the mother wavelet $\psi$. Such a function gives rise to the so called multiresolution analysis. A multiresolution analysis of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_j$ of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$. Properties can be seen in Osgden (1997).

Subspaces $V_j$ are generated by dyadic dilations and translations of the scaling function $\phi$ (this function forms an orthonormal basis of $V_0$, $\{\phi_{o,k}(x) = \phi(x-k)\}$). Moreover each $V_j$ can be expressed as the orthogonal sum $V_j = V_{j-1} \oplus W_{j-1}$, where $W_{j-1}$ is created from wavelets $\psi_{j-1,k}$. Taking into account the properties of the scaling function, together with this last expression, we can construct approximations at increasing levels of resolution. These approximations are linear combinations of dilations and translations of a scaling function $\phi$. The difference between two consecutive approximations, i.e. the detail at the corresponding resolution level, is given by a linear combination of dilations and translations of a wavelet function $\psi$.

3.2 Two-dimensional Multiresolution Analysis

The analysis performed in this work assumes equal dilations in the 2 dimensions involved. At a fixed level of resolution, subspaces in a 2-D multiresolution analysis are the tensor products of the corresponding one-dimensional ones $V_{j+1} = V_{j+1} \circ W_{j+1}$. The 2-D basis is therefore built by the product of two scaling functions (approximation), the product of wavelet and scaling functions (horizontal and vertical details) and the product of two wavelets (diagonal details):

$$V_{j+1} = (V_j \oplus W_j) \circ (V_j \oplus W_j)$$

and

$$= (V_j \oplus V_j) \circ [(V_j \circ W_j) \oplus (W_j \circ V_j) \oplus (W_j \circ W_j)]$$

Horizontal, vertical and diagonal detail coefficients represent the variations in these directions relative to a weighted average at a lower resolution level (given by the approximation coefficients).

A discrete orthonormal basis, $\Gamma_{\alpha}(\vec{x},j,\vec{k})$, can be defined by setting $R = 2^{-j}$ and $\vec{b} = 2^{-j/2}$ in equations (3-5), then $\Gamma_{\alpha}(\vec{x},j,\vec{k}) = \delta_{\alpha\alpha'}\delta_{j,j'}\delta_{\vec{k},\vec{k'}}$, where $()$ denotes the scalar product in $L^2(\mathbb{R}^2)$. If we define the discrete wavelet coefficients associated to any detail by the equation

$$w_{\alpha}(j,\vec{k}) = \int d\vec{x} f(\vec{x}) \Gamma_{\alpha}(\vec{x},j,\vec{k}),$$

we can thus reconstruct the image with all the details

$$f(\vec{x}) = \sum_{\alpha,j,\vec{k}} w_{\alpha}(j,\vec{k}) \Gamma_{\alpha}(\vec{x},j,\vec{k}).$$

In particular, we get the following expression for the second-order moment of the image

$$\langle f^2(\vec{x}) \rangle = \sum_{\alpha,j,\vec{k}} w_{\alpha}^2(j,\vec{k}),$$

that expresses how the energy of the field is distributed locally at any scale and detail.

For a finite image, $R_{\max} \times R_{\max}$, in order to reconstruct it we must add to equation (19) an approximation $w_{\alpha}(\vec{k}) \Gamma_{\alpha}(\vec{x},\vec{k})$ with $\Gamma_{\alpha}(\vec{x},\vec{k}) \equiv \Phi(\vec{x}; R_{\max},k_1)$ $\Phi(\vec{x}; R_{\max},k_2)$ and $w_{\alpha}(\vec{k}) \equiv \int d\vec{x} f(\vec{x}) \Gamma_{\alpha}(\vec{x},\vec{k})$, representing the field at the lower resolution. If $f(\vec{x})$ represents the temperature fluctuation field then the variance is given by $\langle (\Delta T/T)^2 \rangle \sim N_p^2$, being $N_p$ the number of pixels.

The orthonormal basis that we are going to use are the standard Daubechies $N$ (Haar corresponds to $N = 1$), that has been extensively used in the literature because of their special properties: they are defined in a compact support, have increasing regularity with $N$ and vanishing moments up to order $N - 1$ (Daubechies 1988). On the contrary, the Mexican Hat wavelet is not defined in a compact support and it is not appropriate for this multiresolution analysis.

For discrete wavelet analysis of the CMB maps we use the Matlab Wavelet Toolbox (Misiti et al. 1996). This toolbox is an extensive collection of programs for analyzing, denoising and compressing signals and 2-D images. Discrete Wavelet decomposition is performed as described above to obtain the approximation and detail coefficients of the 2-D CMB maps at several levels.

4 DENOISING OF CMB MAPS

Future CMB space experiments will provide maps with resolution scales of few arcminutes. In this work we analyze simulated maps of $12.8 \times 12.8$ square degrees with pixel size of 1.5 arcmin. Simulations are made assuming the standard CDM, $\Omega = 1$ and $H_0 = 50$ km/sec/Mpc. The maps are filtered with a 4.5 FWHM Gaussian beam to approximately reproduce the filtering scale of the High Frequency Channels of the Planck Mission. Simulated maps have a rms signal of $\Delta T/T = 3.7 \times 10^{-5}$. Gaussian noise is added to these maps at different S/N levels between 0.7 and 3. A non-uniform noise is also considered to account for the non-uniform sampling introduced in satellite observations. As an extreme case we have assigned the signal-to-noise at each pixel from a truncated (at the 2$\sigma$ level) Gaussian distribution with a mean value of 2 and a dispersion of 0.5. We use the set of Matlab Wavelet 2-D programs with the corresponding graphical interface to analyze and denoise those maps. Suitable bases of wavelets are studied. Daubechies 4
wavelets are the ones used in this analysis. No significant changes are observed when the analysis is carried out using other higher order Daubechies bases. On the other hand, the Haar system is not so efficient for denoising CMB maps since it produces reconstruction errors much larger than using high order Daubechies systems.

First of all, three wavelet decompositions are performed obtaining wavelet coefficients corresponding to the CMB original map, to the signal plus noise map and to the pure noise map. Decompositions are carried out up to the fourth resolution level. Denoising of the signal plus noise maps is based on subtraction of certain sets of coefficients affected by noise. White noise is the most common in CMB experiments. The dispersion of wavelet coefficients of that type of noise is constant as can be seen from equation 10. On the contrary CMB detail wavelet dispersions go to zero as \( R \) goes to zero. Therefore first level wavelet coefficients are dominated by noise and then, for a given signal plus noise map, it is possible to know the noise and consequently the CMB wavelet coefficient dispersions at all levels. CMB maps produced by typical experiments with a ratio between antenna and pixel size of \( \approx 3 \) will have wavelet coefficients containing the relevant information on the signal at level 3 and above. As shown below, level 3 is the critical one to perform denoising as the noise can still be at a level comparable to the signal. Figure 2 shows rms deviations and corresponding ratios for two simulations with \( S/N = 0.7 \) and \( S/N = 2 \). Detail coefficient numbering corresponds to the three directions diagonal, vertical and horizontal at the three consecutive levels, i.e., numbers 1,2,3 correspond to diagonal, vertical and horizontal coefficients at the first resolution level, 4,5,6 to the second level coefficients in the same order and

\[ \text{Figure 3. Mean value (solid line) and } 1\sigma \text{ error (dashed-dotted lines) of the absolute value of the relative errors, } \Delta C_l / C_l. \text{ Top-left panel corresponds to } S/N = 1.0, \text{ top-right to } S/N = 2, \text{ bottom-left to } S/N = 2 \text{ with non-uniform noise and bottom-right to } S/N = 3. \]
Figure 4. Absolute value of the relative errors, $\Delta C_l/C_l$, of the CMB power spectrum obtained from signal-plus-noise maps (solid lines), wavelet denoised maps (short dashed lines) and Wiener denoised maps (dashed lines). Top-left panel corresponds to $S/N = 1$ (wavelet denoised maps removing all coefficients at levels 1, 2, 3 and 3h is included as long dashed lines), top-right to $S/N = 2$, bottom-left to $S/N = 2$ with non-uniform noise and bottom-right to $S/N = 3$.

7,8,9,10,11,12 to levels 3 and 4 respectively. As it can be seen, the first two levels are entirely dominated by noise as pointed out before. Therefore, all these coefficients can be removed to reconstruct a denoised map. This is equivalent to using a hard thresholding assuming a threshold above all these coefficients. On the other side, level 4 is completely dominated by the CMB signal and is left untouched. Ratios between rms deviations of the signal and noise maps at the third resolution level are not always clearly dominated by noise or signal. Ratios of $\approx 1$ are treated with a soft thresholding technique (in practice we consider ratios in the range $0.3 - 1.5$ though changes in this interval do not significantly affect results). Soft thresholding consists of removing all coefficients with absolute values smaller than the threshold defined in terms of the noise dispersion ($\sigma_n$). Coefficients with absolute values above the defined threshold are rescaled by subtracting the threshold to the positive ones and adding it to the negative ones. To define these thresholds we use the so-called SURE thresholding technique introduced by Donoho & Johnstone 1995. This technique is based on finding an estimator of the signal that will minimize the expected loss or risk defined as the mean value of $(1/N)\sum_{i=1}^{N}(T_d - T_i)^2$, where $T_i$ is the temperature at pixel $i$ in the original signal map and $T_d$ is the estimator at pixel $i$ (temperature in the final denoised map). The minimization is finally achieved in the wavelet domain by choosing a threshold value that minimizes the risk at each wavelet level (see for instance Ogden 1997).

Results of the errors in the map reconstruction are shown in Table 1. The map error is defined as:
Performing 20 simulations (proved to be enough as results reached stable values) at each S/N level we have also calculated the \(1\sigma\) error. The error improvement achieved with the denoising technique applied goes from factors of 3 to 5 for \(S/N = 3\) to \(S/N = 0.7\).

It is also interesting to see how well the denoising method performs to reconstruct the temperature power spectrum. Mean values and \(1\sigma\) errors of the relative errors, \(|\Delta C_\ell / C_\ell|\), are shown in Figure 3 for three \(S/N\) ratios and the case of non-uniform noise considered in this work. The \(C_\ell\)'s are reconstructed from the denoised maps with \(|\Delta C_\ell / C_\ell| \leq 10\%\) up to \(l \sim 1000\) in cases \(S/N \geq 1\). This error can only be achieved up to an \(l \sim 700\) in the \(S/N = 0.7\) case. Higher order multipoles \((l \leq 1500)\) are reconstructed with \(|\Delta C_\ell / C_\ell| \leq 20\%\). Absolute relative errors and reconstructed \(C_\ell\)'s for a given map are presented in Figures 4 and 5 respectively.

In order to check the performance of the SURE thresholding technique, knowing the original maps we can find

\[
\left( \frac{\sum_{i=1}^{n_{\text{pixels}}} (T_i - T_d_i)^2}{\sum_{i=1}^{n_{\text{pixels}}} T_i^2} \right)^{1/2}.
\]  

(21)

**Figure 5.** Power spectrum obtained from signal-plus-noise maps (dashed lines), signal maps (solid lines) and denoised maps (dashed-dotted lines). Top-left \(S/N = 0.7\), top-right \(S/N = 1\), bottom-left \(S/N = 2\) and bottom-right \(S/N = 3\).

**Table 1.** Reconstruction errors vs \(S/N\).

| \(S/N\) | \% map error \(\pm 1\sigma\) |
|--------|-----------------------------|
| 0.7    | 26.3±0.4                    |
| 1.0    | 20.7±0.4                    |
| 2.0    | 13.3±0.2                    |
| 2.0 (n.u.) | 14.3±0.3                  |
| 3.0    | 10.3±0.2                    |
the optimal threshold to get a reconstructed map with a minimum error (as defined above). In $S/N = 1$ maps the optimal threshold is found to be $0.6 - 0.7 \sigma_n$. Thresholds between $0.3 - 1\sigma_n$ do not make substantial changes in the reconstructed map (see Table 2). The hard case included in that table stands for a case where all coefficients below a signal-to-noise dispersion ration $< 1.5$ are removed, leaving the others untouched. For comparison, the error obtained comparing the signal plus noise map with the original signal map is also presented in Table 2. We can see that the error reconstruction achieved with the SURE technique equals the one obtained with the optimal threshold.

A comparison of wavelet techniques with Wiener filter (see for instance Press et al. 1994) has also been performed. In relation to map reconstruction the error affecting the Wiener reconstructed maps is comparable to the error for the wavelet reconstructed maps, in all cases. However, in order to apply Wiener filter previous knowledge of signal power spectrum is required. Reconstructed and residual maps using both, wavelets and Wiener filter, are shown in Figure 6. Regarding the $C_\ell$s, performance of Wiener filter is clearly worse than Wavelets for $\ell = 1000 - 1500$, as can be seen in Figure 4. For example, for a $S/N = 1$ the $C_\ell$s are recovered using Wiener filter with an error between 20% and 70% for $\ell$s between 1000 and 1500 being this error smaller by a factor of 2-4 for Wavelet reconstruction. The error is also clearly larger for Wiener reconstruction than for Wavelet reconstruction, up to $\ell \sim 2000$ in cases with $S/N > 1$.

We have checked for non-Gaussian features possibly introduced by the non-linearity of the soft thresholding used in the wavelet methods applied for denoising. Distributions of Skewness and Kurtosis have been obtained for the original signal maps as well as for the denoised ones. No significant differences can be appreciated between both distributions. However this method could not be good enough to detect non-Gaussian features. As recently claimed by Hobson et al. (1998), the analysis of the distribution of wavelet coefficients is one of the most efficient methods to detect them. We have performed a similar analysis using the Daubechies 4 multiresolution wavelet coefficients. These coefficients are Gaussian distributed in the case of a temperature Gaussian random field. The application of soft thresholding to the wavelet coefficients at a certain level clearly changes the Gaussian distribution by removing all coefficients whose absolute values are below the imposed threshold and shifting the remaining ones by that threshold. As an example, in the previous case $S/N = 1$ the kurtosis of the diagonal level 3 distribution changes from $3.3 \pm 0.1$ to $34 \pm 10$ ! (notice that the change strongly depends on the threshold imposed). This result is not surprising as any non-linear method used for denoising or foreground separation will introduce non-Gaussianity at different levels in the reconstructed map. Fortunately there are two ways of overcoming the question of determining the Gaussianity of the CMB signal. One way would be to check the Gaussian character of the data before applying denoising to maps affected by Gaussian noise. We have checked this by looking at the multiresolution wavelet coefficient distributions in the case of $S/N = 1$. The addition of white noise didn’t change the mean value and error bar of the kurtosis. The second way would be applying a linear denoising method. We have used a simple one consisting in removing all detail coefficients at levels with signal-to-noise dispersion ratio $< 1.5$ (notice that 1.5 corresponds the upper value of the threshold interval where soft thresholding was applied). This method is equivalent to applying hard thresholding with a threshold above all the coefficients. The errors of the reconstructed map and its corresponding $C_\ell$s increase slightly compared to the SURE thresholding method (see table 2and top-left panel of figure 4). The same hard thresholding linear method will give even better results using 2-D Wavelets with two scales of dilation (Sanz et al. 1999) instead of the one-scale multiresolution techniques, since the former works with many more resolution levels being therefore more selective in removing the coefficients.

### Table 2. Reconstruction errors vs threshold, $S/N=1$.

| Threshold | % map error |
|-----------|-------------|
| hard      | 23.5        |
| $1.5 \sigma_n$ | 21.7        |
| $1.0 \sigma_n$ | 20.7        |
| $0.7 \sigma_n$ | 20.5        |
| $0.6 \sigma_n$ | 20.6        |
| $0.5 \sigma_n$ | 20.7        |
| $0.4 \sigma_n$ | 21.0        |
| $0.3 \sigma_n$ | 21.3        |
| signal+noise | 100.0       |

Figure 6. $12^\circ.8 \times 12^\circ.8$ maps of the cosmological signal (top left), signal plus noise with $S/N = 1$ (top right), denoised map using a soft thresholding as explained in the text (middle left) and residual map obtained from the CMB signal map minus the denoised one (middle right). For comparison a denoised map using Wiener filter is presented in the bottom left panel together with the residuals in the bottom right panel.

5 DISCUSSION AND CONCLUSIONS

A wavelet multiresolution technique has been presented and used to analyse and denoise CMB maps. This method has been proved to be one of the best to reconstruct observed CMB maps as well as power spectra by removing a significant percentage of the noise. The analysis has been carried out assuming a uniform Gaussian noise as would be expected in a small sky patch, e.g. $12^\circ.8 \times 12^\circ.8$, observed by satellite scans. Analysis of whole sky CMB maps using wavelets will be performed in a future work. Since these data will be affected by non-uniform noise, the use of wavelet techniques to localize map features will be highly suitable.

A semi-analytical calculation of the variance of the wavelet coefficients has been presented. The behaviour of the variance of the detail coefficients is given for a standard CDM model in the case of Haar and Mexican Hat bases. The acoustic peaks can be noticed in the wavelet coefficient variance represented in Figure 1. Moreover, these peaks are better defined for the Mexican Hat wavelet system since these wavelets are more localized than the Haar ones.

Denoising of CMB maps has been carried out by using a signal-independent prescription, the SURE thresholding...
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method. The results are model independent depending only on the observed data. However, a good knowledge of the noise affecting the observed CMB maps is required. For a typical case of $S/N \sim 1$ the high order detail coefficients are dominated by the signal, whereas the lowest ones are noise dominated. This behaviour is due to the expected dependence of the temperature power spectrum, $C_l \propto l^{-2}$. The applied wavelet method is able to reconstruct maps with an error improvement factor between 3 and 5 and the CMB power spectrum of the denoised maps carries relative errors below 20% up to $l \sim 1500$ for $S/N \geq 1$. We have also checked that SURE thresholding methods are providing thresholds in agreement with the optimal ones.

For comparison Wiener filter has also been applied to the simulations considered in this paper. This method reconstructs CMB maps after denoising with errors comparable to the Wavelet method we propose, as shown in Figure 6. However, the $C_l$s of the denoised maps obtained applying Wiener filter have relative errors larger than a factor of 2 than the relative errors of the $C_l$s obtained from the wavelet reconstructed maps in the range $l = 1000 - 1700$. In addition we have applied a Maximum Entropy Method (MEM) to the maps used in this work, with the definition of entropy given by Hobson & Lasenby (1998). This method provides reconstruction errors at the same level as multiresolution wavelet methods. However, the later are easier (not requiring iterative processes) and faster ($O(N)$) to apply than MEM.

A possible handicap of denoising methods based on soft thresholding of wavelet coefficients as well as other non-linear methods are the non-Gaussian features introduced in the reconstructed map. However one can still detect the possible intrinsic non-Gaussianity of the CMB signal by studying it in the signal plus noise map using the wavelet coefficient distribution. Moreover a valid reconstruction can be obtained by applying a “hard” thresholding linear method as discussed in the text.

In a different work, we are studying the case of using a wavelet method based on two scales of dilation (Sanz et al. 1999). Though this method has the advantage of keeping information on two different scales, for the purpose of denoising both methods give comparable results. The linear hard thresholding method is expected to perform better for 2-D wavelets than for multiresolution ones as the former works with many more resolution levels.

Summarizing, the main advantages of the wavelet method are: to provide local information of the contribution from different scales, to be computationally very fast $O(N)$, absence of tunning parameters and the most important, the good performance on denoising CMB maps. The best reconstruction is achieved using soft thresholding techniques. Concerning the Gaussianity of the signal one can apply the suggested linear method for denoising. Moreover, the soft thresholding technique will provide a good reference map and power spectrum for the signal, that can be used to check the quality of other reconstructions based on linear methods. Wavelets are also expected to be a very valuable tool to analyse future CMB maps as those that will be provided by future missions like MAP and Planck.
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