Time averaged total force on a dipolar sphere in an electromagnetic field

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We establish the time averaged total force on a subwavelength sized particle in a time harmonic varying field. Our analysis is not restrictive about the spatial dependence of the incident field. We discuss the addition of the radiative reaction term in the polarizability in order to correctly deal with the scattering force. As a consequence and illustration, we assess the degree of accuracy of several polarizability models previously established.

In the last years there has been an increase of interest in the manipulation of small particles by means of the light Lorenz’s force. For spheres subwavelength radius the total force due to a light wave is usually split into two parts from the use of the dipole approximation (cf. Ref. [1]): a gradient force (\( p \nabla E \)), essentially due to the particle induced dipole moment \( p \) interacting with the electric field \( E \); and a scattering and absorbing forces \( \mathbf{\dot{p}} \times \mathbf{B} \), where \( \mathbf{B} \) is the magnetic vector, \( \mathbf{\dot{p}} = \partial \mathbf{p} / \partial t \), and \( c \) is the speed of light in vacuum. It has been customary after Ref. [1] to express the gradient force \( \mathbf{F}_{\text{grad}} \) as (see e.g. Ref. [2]):

\[
\mathbf{F}_{\text{grad}} = (1/2) \alpha_0 \mathbf{\nabla} E^2 ,
\]

where \( \alpha_0 \) is the particle polarizability, satisfying the Clausius-Mossotti equation:

\[
\alpha_0 = \frac{a^3 \varepsilon - 1}{\varepsilon + 2} ,
\]

\( a \) being the particle radius and \( \varepsilon \) denoting the dielectric permittivity. On the other hand, the absorbing and scattering forces are written in the approximation of small spheres through the absorbing (\( C_{\text{abs}} \)) and scattering (\( C_{\text{scat}} \)) cross sections as:

\[
\mathbf{F} = \frac{|E|^2}{(8\pi)} [C_{\text{abs}} + C_{\text{scat}} \frac{\mathbf{k}}{k}] ,
\]

where \( \mathbf{k} \) represents the light vector (\( k = |\mathbf{k}| \)). On using the expression of these cross sections in the dipole approximation, just the first term of their Taylor expansion versus the size parameter, \( x = 2\pi a / \lambda \), is usually considered.\(^3\)

At optical frequencies involved in many experiments, however, only the time average of the electromagnetic force is observed. In this letter, we establish the form of the time averaged total force on a particle without restriction on the spatial dependence of the electromagnetic field. Further, we discuss some of its consequences. For time harmonic electromagnetic waves,\(^4\) we write \( \mathbf{E}(\mathbf{r}, t) = E_0 e^{-i\omega t} \), \( \mathbf{B}(\mathbf{r}, t) = B_0 e^{-i\omega t} \) and \( \mathbf{p}(\mathbf{r}, t) = p_0 e^{-i\omega t} \), \( \mathbf{E}_0 \), \( \mathbf{B}_0 \), and \( \mathbf{p}_0 \) being complex functions of position in space, and \( \Re \) denoting the real part. Then, the time average of the total force is:

\[
< \mathbf{F} > = \frac{1}{4T} \int_{-T/2}^{T/2} \left[ (\mathbf{p} + \mathbf{p}^*) \cdot \mathbf{\nabla} (\mathbf{E} + \mathbf{E}^*) 
+ \frac{1}{c} (\mathbf{\dot{p}} + \mathbf{\dot{p}}^*) \times (\mathbf{B} + \mathbf{B}^*) \right] dt ,
\]

where * stands for the complex conjugate. On performing the integral and using \( \mathbf{E}_0 \), \( \mathbf{B}_0 \), and \( \mathbf{p}_0 \), Eq. (4) yields for each \( i \)th Cartesian component of the averaged total force:

\[
< F^i > = (1/2) \Re \left[ \alpha_0 \partial^j (E_0^i)^* + \frac{1}{c} \varepsilon_{ijk} \partial^j (E_{0k}^* B_{0j}) \right] ,
\]

for \( i = 1, 2, 3 \), where \( \varepsilon_{ijk} \) is the Levi-Civita tensor. On using the relations \( B_0 = \frac{1}{\omega} \mathbf{\nabla} \times \mathbf{E}_0 \), \( p_0 = \alpha E_0 \), and \( \mathbf{p}_0 = -i \omega \mathbf{p}_0 \) one gets for Eq. (5):

\[
< F^i > = (1/2) \Re \left[ \alpha (E_{0j} \partial^j (E_0^i)^* + \varepsilon_{ijk} \varepsilon_{klm} E_{0j} \partial^j (E_{0m}^*) \right] .
\]

On taking into account that: \( \varepsilon_{ijk} \varepsilon_{klm} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \) one can finally express \( < F^i > \) as:

\[
< F^i > = (1/2) \Re \left[ \alpha E_{0j} \partial^j (E_0^i)^* \right] .
\]

Eq. (7) is the main result of this letter. It represents the total averaged force exerted by an arbitrary time harmonic electromagnetic field on a small particle.

In this connection, Ref. [5] establishes the average force on an object represented by a set of dipoles when the electromagnetic field is a plane wave. We notice that in this case Eq. (7) reduces to just Eq. (3), in agreement with Ref. [5]. However, as we shall illustrate next, Eq. (7) permits to apply the couple dipole method (CDM) to more complex configurations like that of a small particle in front of a dielectric surface, under arbitrary illumination (see Ref. [6] for a discussion on the CDM for large particles). Also, the absence of the magnetic field in Eq. (7) eases the computations.

Conversely, when Eq. (2) for the polarizability is introduced into Eq. (7), one obtains for the \( i \)th component of the time averaged optical force:
\[
<F^i> = (1/2)a_0 \Re \left[ E_0 \partial^i (E_0)^* \right]
\]
\[
= (1/4)a_0 \Re \left[ \partial^i |E_0|^2 \right] = (1/4)a_0 \Re \left[ \partial^i |E_0|^2 \right]
(8)
\]
which is just the gradient force. Notice the factor \((1/4)\) (see e.g. Ref. [7]) instead of which the factor \((1/2)\) for non-averaged fields often appears in the literature (see for example Refs. [2,8,9]). In agreement with the remarks of Ref. [10], now the scattering force, Eq. (3), vanishes and thus, \(<F_\perp>\) reduces to the gradient force. Therefore, \(a_0\) must be replaced from its static expression (2) by the addition of a damping term. This was done by Draine,\(^{10}\) who with the help of the optical theorem, obtained:

\[
\alpha = a_0/(1 - (2/3)ik^3a_0).
(9)
\]
The existence of the imaginary term for \(\alpha\) in Eq. (9) is essential to derive the correct value for the averaged total force due to a time varying field.

As an illustration, let the field that illuminates the particle be the beam whose electric vector is:

\[
E_x = \exp(-x^2/2) \exp(i(kz - \omega t)), \quad E_y = 0, \quad E_z = 0
(10)
\]
On using Eqs. (2) and (10) in Eq. (7), we find:

\[
<F_x> = -(a_0/2)x \exp(-x^2) \quad (11a)
\]
\[
<F_z> = 0. \quad (11b)
\]

On the other hand, if the correct polarizability, Eq. (9), is introduced with Eq. (10) into Eq. (7), the total force is then expressed as:

\[
<F_x> = (1/2)\Re \left[ -k^2 \alpha x \exp(-x^2) \right]
\]
\[
= -(a_0/2)k \exp(-x^2) \left[ 1 + (4/9)k^3a_0^2 \right] \quad (12a)
\]
\[
<F_z> = (1/2)k \exp(-x^2) \Re \left[ -i\alpha \right]
\]
\[
= \exp(-x^2)k^4a_0^2/3 \left[ 1 + (4/9)k^3a_0^2 \right]. \quad (12b)
\]
For a particle with a radius \(a \ll \lambda\), e.g. \(a = 10\) nm, at wavelength \(\lambda = 632.8\) nm and \(\epsilon = 2.25\), the factor \((1 + (4/9)k^3a_0^2)\) is very close to one (notice in passing that the expression used in Ref. [11] for \(\alpha\) makes this factor unity). We thus see that in contrast to Eqs. (11), the correct form for the polarizability, Eq. (9), leads to a total force given by Eqs. (12a) and (12b), which can be associated to the gradient and scattering components, namely, to the time average of Eq. (1) and Eq. (3) with \(C_{abs} = 0\), respectively.

In the case of an absorbing sphere, the dielectric constant becomes complex and so is \(a_0\). Then, Eqs. (12) with \(a \ll \lambda\) become:

\[
<F_x> = -(1/2)\Re \left[ a_0 \alpha x \exp(-x^2) \right] \quad (13a)
\]
\[
<F_z> = \frac{\exp(-x^2)k^4|a_0|^2}{3} + \frac{k \exp(-x^2)3m|a_0|}{2}. \quad (13b)
\]
The imaginary part of \(a_0\) does not contribute to the component \(<F_x>\), namely, to the gradient force Eq. (13a). On the other hand the absorbing and scattering force, Eq. (13b), exactly coincides with the expression obtained from Eq. (3).

FIG. 1. a) Relative difference between the force computed by the exact Mie calculation and by the dipole approximation: CM-RR (full line), LAK (thick line), DB (dashed line). The sphere is of glass (\(\epsilon = 2.25\)), illuminated by an incident propagating plane wave (\(\lambda = 600\) nm). b) Same as Fig. 1a for a silver sphere (\(\lambda = 400\) nm, \(\epsilon = -4 + i0.7\)).

FIG. 2. a) Relative difference between the component of the force perpendicular to the incident wave vector obtained from CDM and by the dipole approximation: CM-RR (full line), LAK (thick line), DB (dashed line). The sphere is of glass (\(\epsilon = 2.25\)) illuminated by an incident evanescent wave (\(\lambda = 600\) nm). b) Same as Fig. 2a for a silver sphere (\(\lambda = 400\) nm, \(\epsilon = -4 + i0.7\)).

We next illustrate the above arguments with some numerical calculations that permit us to assess the degree of accuracy of several polarizability models previously established. We first compare the relative difference between the force obtained from the exact Mie calculation and the most usual polarizabilities models, namely, those of Lakhtakia\(^{12}\) (LAK), Dungey and Bohren\(^{13}\) (DB), and the Clausius-Mossotti relation with the radiative reaction term (CM-RR)\(^{10}\) previously discussed, versus the radius \(a\) of a sphere illuminated by a plane propagating wave in free space (Fig. 1a and 1b). Secondly, when this sphere is illuminated by an evanescent wave created by total internal reflection on a dielectric surface, the component of the force perpendicular to the incident wave vector (Figs. 2a and 2b) is compared with the result derived...
from the CDM (as discussed in Ref. [6]). All curves are represented up to $a = \lambda/10$. The relative difference (%) plotted is defined as: $100 \times (F_{\text{ref}} - F_{\text{pol.}})/F_{\text{ref}}$ where $\text{pol.}$ denotes the force obtained from the corresponding method used for the polarizability (among LAK, DB, CM-RR) and $\text{ref}$ stands for the force derived from the Mie calculation when the incident wave is propagating, and from the CDM when the incident wave is evanescent.

We first consider a dielectric wave (glass, $\epsilon = 2.25$) illuminated at $\lambda = 600nm$ (Figs. 1a and 2a). We observe that, for an incident propagating wave (Fig. 1a), the result from the CM-RR relation is better than that of DB, and this, in turn, is better than the result from LAK. The force over a dielectric particle given by the exact Mie calculation is: $F = C_{\text{scat}}(1 - \cos \theta)|E|^2/(8\pi)$, and that obtained from the dipole approximation is: $F = (1/2)|E|^2 \Re[-i\alpha]$. When the DB model is used, then $\alpha = (3/2)ia_1/k^3$ where $a_1$ is the first Mie coefficient, hence, $4\pi \Re[-i\alpha]$ is the scattering cross section for an electric dipole. However, when Eq. (9) for the CM-RR is employed, $4\pi \Re[-i\alpha]$ constitutes only the first term of the Taylor expansion of the scattering cross section versus the size parameter $x$. This is why $C_{\text{scat}}$ is underestimated when it is calculated from the CM-RR model. Therefore, the DB model should be better. However, in both cases the factor $\cos \theta$ has not been taken into account in the dipole approximation and, thus, both results overestimate the force. Hence, this factor $\cos \theta$ makes a balance making the CM-RR result closer to the Mie’s solution. In the case of an incident evanescent wave (Fig. 2a), DB and CM-RR results are very close together; this is due to the fact that the real parts of both polarizabilities are very close to each other. We see that LAK result, as with a propagating wave, is far from the correct result.

As a second example, we consider a metallic sphere (silver) illuminated at $\lambda = 400nm$ ($\epsilon = -4 + i0.7$). We now observe that for an incident propagating wave (Fig. 1b), the DB model yields the best result. The force can exactly be written as $F = (C_{\text{ext}} + C_{\text{scat}} \cos \theta)|E|^2/(8\pi)$. We notice that now $C_{\text{scat}} \cos \theta$ is of sixth order in $x$ in comparison with $C_{\text{ext}}$. Since $C_{\text{ext}} \propto \Re[a_1]$ in the electric dipole limit, the DB formulation appears as the best here. Also, for incident evanescent waves (Fig. 2b), DB gives the most accurate solution. However, for a metallic sphere, the relative permittivity much depends on the wavelength used, hence, it is now difficult to establish a generalization of these results. We have checked, notwithstanding, that for a gold or silver sphere in free space in the visible, DB is often the best.

In summary, we have established the average total force on a little particle in a time harmonic varying field of arbitrary form, and thus clarify its use in the interpretation of experiments, as well as in some previous theoretical works. For instance, we see that Eq. (7) is not just the gradient force as stated in some previous work (see e.g. Ref. [14]). Also, this general expression shows the importance of the radiative reaction term in the polarizability of the sphere put forward by other authors. Its derivation makes no assumptions about the surrounding environment. It is just necessary to know both the electric field and its derivative at the position of the sphere, and thus it allows an easy handling of illuminating evanescent fields. An immediate important consequence is that it permits to assess the adequacy of several polarizability models.

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