Quantum Hall States in Rapidly Rotating Two-Component Bose Gases

Shunsuke Furukawa and Masahito Ueda

Department of Physics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

(Dated: May 1, 2014)

We investigate strongly correlated phases of two-component (or pseudo-spin-1/2) Bose gases under rapid rotation through exact diagonalization on a torus geometry. In the case of pseudo-spin-independent contact interactions, we find the formation of gapped spin-singlet states at the filling factors \( \nu = k/3 + k/3 \) (\( k/3 \) filling for each component) with integer \( k \). We present numerical evidences that the gapped state with \( k = 2 \) is well described as a non-Abelian spin-singlet (NASS) state, in which excitations feature non-Abelian statistics. Furthermore, we find the phase transition from the product of composite fermion states to the NASS state by changing the ratio of the intercomponent to intracomponent interactions.

PACS numbers: 03.75.Mn, 05.30.Jp, 73.43.Cd

Under rapid rotation, ultracold gases of bosonic atoms are predicted to enter a highly correlated regime, analogous to quantum Hall systems \( \nu \). This regime is reached as the number of vortices, \( N_\nu \), in a Bose-Einstein condensate becomes comparable with the number of atoms, \( N \). The relevant control parameter is the filling factor \( \nu = N/N_\nu \). For scalar bosons, it is predicted that the vortex lattice melts for \( \nu \lesssim 6 \) \cite{9,10} and that a series of gapped uncondensed states appear at various integer and fractional \( \nu \). In particular, the ground states at \( \nu = k/2 \) (with \( k = 1, 2, 3, \ldots \)) have large overlaps \cite{3,4} with the Read-Rezayi states \cite{3}, whose excitations feature non-Abelian statistics for \( k \geq 2 \). Other quantum Hall states such as composite fermion states \cite{6} have also been discussed.

Given a rich diversity of strongly correlated physics in the scalar case, it is natural to ask what happens in two-component Bose gases, such as those made up of two hyperfine spin states of the same atoms \cite{7,8}. For intermediate rotation frequencies, a variety of vortex lattices have been shown to appear \cite{3,7}. For a rapid rotation, two-component systems would offer an ideal situation in which to study the roles of (pseudo-)spin degrees of freedom non-Abelian anyonic excitations \cite{15,16}. We present numerical evidences that the gapped state with \( k = 2 \) is well described as the \( SU(3)_2 \) state. We also discuss a phase transition that occurs at a particular value of the coupling ratio \( g_{\uparrow\downarrow}/g \). We note that NASS states of bosons have also been discussed in spin-1 bosons \cite{17} and scalar bosons on optical lattices \cite{18}. The two-component case studied here offers the simplest realistic setting which allows a detailed numerical identification of a NASS state.

We consider a system of a Bose gas having two spin states (labeled by \( \alpha = \uparrow, \downarrow \)) and rapidly rotating in a harmonic trap with the cylindrical symmetry about the \( z \) axis. We denote the axial and radial trap frequencies as \( \omega_\parallel \) and \( \omega_\perp \), respectively. In the rotating frame of reference, a system of neutral particles is mathematically equivalent to that of charged particles subject to a uniform magnetic field. In analogy with the quantum Hall problem, we introduce the “magnetic” length \( \ell = \sqrt{\hbar/(2M\Omega)} \), where \( M \) is the particle’s mass and \( \Omega \) is the rotation frequency.

As \( \Omega \) increases toward \( \omega_\parallel \), the particle density decreases due to the centrifugal spreading of the gas. For \( \Omega \approx \omega_\parallel \), we may assume that the mean interaction energy per particle (roughly proportional to the coupling constants times the particle density) is much smaller than single-particle energy-level spacings, \( \hbar\omega_\parallel \). The single-particle states can then be restricted to the ground state of the axial confinement and to the lowest Landau level (LLL) in the transverse directions \( l \parallel \). The LLL state with angular momentum \( L_z = m\hbar \geq 0 \) in the \( xy \)
plane is represented as \( u_{\alpha}(z) \propto z^m \exp[-|z|^2/(4\ell^2)] \) with \( z = x + iy \). Within this restricted subspace, we consider the interaction Hamiltonian consisting of intracomponent and intercomponent contact interactions:

\[
H_{\text{int}} = \sum_{\alpha=\uparrow,\downarrow} \sum_{i<j} N_{\alpha} g_{\alpha} \delta(z_{\alpha}^i - z_{\alpha}^j) + g_{1\uparrow} \sum_{i=1}^{N_{\uparrow}} \sum_{j=1}^{N_{\downarrow}} \delta(z_{\uparrow}^i - z_{\downarrow}^j),
\]

where \( N_{\alpha} \) is the number of particles in the state \( \alpha \) and \( z_{\alpha}^\alpha \)'s are the positions of such particles. The effective coupling constants in the two-dimensional plane are given by \( g_{\alpha} = a_{\alpha} \frac{\sqrt{8\pi\hbar^2\omega/|M|}}{m_{\alpha}} \) and \( g_{1\uparrow} = a_{1\uparrow} \frac{\sqrt{8\pi\hbar^2\omega/|M|}}{m_{1\uparrow}} \), where \( a_{\alpha} \) and \( a_{1\uparrow} \) are the \( s \)-wave scattering lengths between like and unlike bosons, respectively. For simplicity, we assume \( g_{\alpha} = g_{1\uparrow} (\equiv g) \) in the following.

To study bulk properties, it is useful to work on closed uniform surfaces such as a sphere and a torus, which have no edge. Here, we perform calculations on a periodic rectangular geometry (a torus) \([1, 20]\) of sides \( L_x \) and \( L_y \), which contains \( N_V = L_x L_y/(2\pi\ell^2) \) vortices for each component. This geometry can describe the central region of a trapped atomic gas where the particle density is approximately uniform. There are \( N_V \) LLL states for each component. We introduce the filling factor as \( \nu = N/N_V \), where \( N = N_{\uparrow} + N_{\downarrow} \) denotes the total particle number. We classify all states by the pseudomomentum \( K = (K_x, K_y) = 2\pi\hbar(m_x/L_x, m_y/L_y) \) with integer \( m_x \) and \( m_y \). At \( \nu = p/q \) with \( p \) and \( q \) being coprime, all energy eigenstates possess a trivial center-of-mass degeneracy of \( q \), and the minimal Brillouin zone (without this degeneracy) consists of \( \bar{N}^2 \) points, where \( \bar{N} \) is the largest common divisor of \( N \) and \( N_V \) \([21]\). As in Ref. \([3]\), we focus on non-negative values of \( K_x \) and \( K_y \) up to the Brillouin zone boundary, since states at \((\pm K_x, \pm K_y)\) are degenerate by symmetry. Henceforth, we take the units in which \( \hbar = 1 \) and \( \ell \equiv 1 \).

We first focus on the case of pseudo-spin-independent interactions \( g_{1\uparrow} = g \), where the system possesses an \( SU(2) \) symmetry. For this case, a mean field theory \([1]\) predicted that each component forms a rectangular vortex lattice with the aspect ratio of the unit cell given by \( \sqrt{3} \). In analogy with the scalar case \([3]\), we expect to find a phase transition from the vortex lattice to gapped uncondensed states as the filling factor \( \nu \) decreases. We therefore set the aspect ratio of the torus to be compatible with this lattice, such that this lattice can be detected if it appears. For \( N_V = 6 \), for example, we set \( L_x/L_y = 3/(2\sqrt{3}) \), with which the system can host \( 3 \times 2 \) units of the rectangular vortex lattice, with the longer side of the rectangular unit in the \( y \) direction.

To search for uncondensed ground states, we introduce the “charge” gap

\[
\Delta(N) = E\left(\frac{N}{2} + 1, \frac{N}{2}\right) + E\left(\frac{N}{2} - 1, \frac{N}{2}\right) - 2E\left(\frac{N}{2}, \frac{N}{2}\right),
\]

for even integer \( N (\geq 4) \), in analogy with the studies of fermionic Hubbard models. Here, \( E(N_1, N_2) \) is the ground-state energy for fixed particle numbers, \( N_1 \) and \( N_2 \). The results for the \( SU(2) \)-symmetric case \( g_{1\uparrow} = g \) are presented in Fig. \([1]\). We find upward spikes at \( \nu = k/3 \) with integer \( k \), which indicate the appearance of gapped uncondensed states at these filling factors.

The appearance of a gap at \( \nu = 1/3 + 1/3 \) can be naturally interpreted from the formation of the Halperin \((221)\) state \([22, 23]\), which is an Abelian spin-singlet state. On a disc geometry, its wave function is written as

\[
\tilde{\Psi}^{221} = \prod_{i<j} (z_i^\uparrow - z_j^\uparrow)^2 \prod_{i<j} (z_i^\downarrow - z_j^\downarrow)^2 \prod_{i,j} (z_i^\uparrow - z_j^\downarrow).
\]

Here, as in Ref. \([1]\), a tilde on the wave function indicates that it has to be multiplied by the usual Gaussian factors for the \( xy \) plane. The contact interactions in Eq. \([1]\) vanish for this wave function, and therefore Eq. \([1]\) is an exact zero-energy ground state for arbitrary \( g_{1\uparrow} \geq 0 \) and \( g \geq 0 \). On a torus, there appear triply-degenerate zero-energy ground states due to the center-of-mass degeneracy. Performing exact diagonalization up to \( N_V = 15 \) and \( N_1 = N_\downarrow = 5 \) with \( g_{1\uparrow} = g \), we find a stable energy gap of magnitude \( \sim 0.035g \) above these zero-energy states.

We now investigate the origins of the gapped states at \( \nu = k/3 + k/3 \) with \( k \geq 2 \). Precisely at these filling factors, Ardonne and Schoutens \([15]\) proposed non-Abelian extensions of the Halperin \((221)\) state, termed the \( SU(3)_k \) states \([24]\). On a disc, their wave functions are written as

\[
\tilde{\Psi}^{SU(3)}_k = \mathcal{S}_{\text{group}} \prod_{\text{group}} \tilde{\Psi}^{221}.
\]

In this construction, the \( N = rk + rk \) bosons are first partitioned into \( k \) groups, each with \( r \) particles in each spin state \( \uparrow, \downarrow \). For each group we write a Halperin \( \tilde{\Psi}^{221} \)
factor, and then such factors are multiplied together. Finally, we apply the symmetrization operation $S_{\text{group}}$ over all different ways of dividing the particles into $k$ groups. Our motivation to consider the states \( \{ 4 \} \) stems from the analogy with the scalar case; their single-component counterparts, Read-Reazayi [SU(2)$_k$] states,\cite{Read90} give good approximations to the ground states of the scalar Bose gas.\cite{Senthil97} The wave function \( \{ 4 \} \) is a unique zero-energy eigenstate of a Hamiltonian consisting of a \((k + 1)\)-body interaction \( \{ 16 \} \):

\[
H_k = \sum_{i_1 < \cdots < i_{k+1}} \delta(z_{i_1} - z_{i_2}) \cdots \delta(z_{i_k} - z_{i_{k+1}}),
\]

where the indices $i_1, \ldots, i_{k+1}$ run over all the particles of both components. On a torus, this Hamiltonian leads to a ground-state degeneracy of \((k + 1)(k + 2)/2\). The appearance of this non-trivial degeneracy is related to an underlying topological order, and can be used as an indicator of the corresponding topological phase.

To discuss the possibility of SU(3)$_2$ state at $\nu = 2/3 + 2/3$, we present the energy spectra versus the pseudomomentum $K$ in Fig. 2. At $K = 0$, we further decompose the Hilbert space using quantum numbers $p_\pi = \pm 1$ and $p_{\pi\uparrow\downarrow} = \pm 1$ for the $\pi$ spatial rotation $R_\pi$ and the interchange of two components, $P_{\pi\uparrow\downarrow}$, respectively. In addition to the equal-population case $N_\uparrow = N_\downarrow = N/2$, we also present data for the minimally imbalanced case $N_\uparrow = N/2 + 1$ and $N_\downarrow = N/2 - 1$ by cross symbols for $N = 8$ and 12; equal-population states which are not degenerate with crosses are identified as spin singlets. We find that for all the system sizes examined in Fig. 2, the two lowest-energy states are spin singlets in the sector with $K = 0$, $p_\pi = +1$, and $p_{\pi\uparrow\downarrow} = +1$. Diamonds, representing $\psi_3$, indicate the lowest-energy state in the sector with $K = 0$, $p_\pi = -1$, and $p_{\pi\uparrow\downarrow} = -1$. Circles indicate other eigenstates in the equal-population case $N_\uparrow = N_\downarrow = N/2$. Crosses show eigenstates for the minimally imbalanced case $N_\uparrow = N/2 + 1$ and $N_\downarrow = N/2 - 1$. Only the two lowest energies are displayed in each sector.

![FIG. 2: Energy spectra versus the pseudomomentum $K$ at the filling factor $\nu = 2/3 + 2/3$ for the SU(2)$_2$-symmetric case $g_{\pi\uparrow\downarrow} = g$. The ground-state energy is subtracted from the spectrum. For $N_\pi = 6$ and 9, the same aspect ratios as in Fig. 1 are used; for $N_\pi = 12$, we set $L_x/L_y = 4/(3\sqrt{3})$. Upward (\$\Delta$) and downward (\$\uparrow\downarrow$) triangles, representing $\psi_1$ and $\psi_2$ respectively, indicate the two lowest-energy states in the sector with $K = 0$, $p_\pi = +1$, and $p_{\pi\uparrow\downarrow} = +1$. Diamonds, representing $\psi_3$, indicate the lowest-energy state in the sector with $K = 0$, $p_\pi = -1$, and $p_{\pi\uparrow\downarrow} = -1$. Circles indicate other eigenstates in the equal-population case $N_\uparrow = N_\downarrow = N/2$. Crosses show eigenstates for the minimally imbalanced case $N_\uparrow = N/2 + 1$ and $N_\downarrow = N/2 - 1$. Only the two lowest energies are displayed in each sector.](image)

![FIG. 3: (a,b) Energy spectra versus the aspect ratio $L_x/L_y$, at the filling factor $\nu = 2/3 + 2/3$ in the SU(2)$_2$-symmetric case $g_{\pi\uparrow\downarrow} = g$. The same symbols as in Fig. 1 are used. Solid curves are guides for the eyes for the states $\psi_{\pi\uparrow\downarrow}$. (c,d) Squared overlaps of $\psi_{\pi\uparrow\downarrow}$ with the SU(3)$_2$ states, plotted against the aspect ratio.](image)
larger $g_{\uparrow \downarrow}/g(\gtrsim 0.6)$, we find that the energy of $\psi_3$ goes up, while $\psi_1$ and $\psi_2$ remain the lowest two states, leading to a total quasi-degeneracy of 6 as expected for the $SU(3)_2$ state. These results point to a phase transition from the direct product of composite fermion states to the $SU(3)_2$ state at $g_{\uparrow \downarrow}/g \approx 0.6$. Essentially the same behavior is found for a larger system size ($N_V = 9$ and $N = 12$; not shown), although the quasi-degeneracy of $\psi_1$ and $\psi_2$ around $g_{\uparrow \downarrow}/g = 1$ is less evident than in Fig. 4(a).

The occurrence of the phase transition can be further supported by the results of the squared overlaps in Fig. 4(b). For small $g_{\uparrow \downarrow}/g$, the states $\psi_{1,2,3}$ continue to have large overlaps with those in the decoupled case $g_{\uparrow \downarrow} = 0$. For $g_{\uparrow \downarrow}/g(\gtrsim 0.6)$, $\psi_{1,2}$ have larger overlaps with the $SU(3)_2$ states than with the states at $g_{\uparrow \downarrow} = 0$. It remains unclear whether a phase transition occurs directly between the two quantum Hall states or any intermediate phase exists.

In summary, we have studied quantum Hall states in rapidly rotating two-component Bose gases. We have presented numerical evidences that a NASS state appears at $\nu = 2/3 + 2/3$. Non-Abelian anyonic excitations in the NASS state can carry spins $\frac{1}{2}$ and $\frac{3}{2}$, and spin-selective operations would potentially offer better probe and control of such excitations than in scalar Bose gases. Furthermore, we have demonstrated that changing the ratio $g_{\uparrow \downarrow}/g$ drives a phase transition from the product of separate quantum Hall states to a spin-singlet quantum Hall state. Systematic appearance of spikes at $\nu = k_3/k_1 + k_3/k_1$ with integer $k$ in Fig. 4(b) suggests that $SU(3)_k$ with $k \geq 3$ might also appear for the realistic contact interactions [1]. More precise characterizations of the ground states at these filling factors deserve further studies.

This work was supported by a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Quantum Phenomena” (No. 22103005), KAKENHI 22340114, a Global COE Program “the Physical Science Frontier”, and the Photon Frontier Network Program, from MEXT of Japan.

Note added. During the preparation of this Rapid Communication, we became aware of an independent work by Graß et al. [20], where similar numerical evidences for the NASS states are presented.

[1] For a review, see N. R. Cooper, Adv. Phys. 57, 539 (2008).
[2] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, Phys. Rev. Lett. 80, 2245 (1998).
[3] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. 87, 20403 (2001).
[4] N. Regnault and Th. Jolicoeur, Phys. Rev. B 76, 235324 (2007).
[5] N. Read and E. H. Rezayi, Phys. Rev. B 59, 8084 (1999).
[6] N. Regnault and Th. Jolicoeur, Phys. Rev. Lett. 91, 130402 (2003); C.-C. Chang, N. Regnault, Th. Jolicoeur, and J. K. Jain, Phys. Rev. A 72, 013611 (2005).
[7] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 81, 1539 (1998).
[8] V. Schweikhard, I. Coddington, P. Engels, S. Tung, and E. A. Cornell, Phys. Rev. Lett. 93, 210403 (2004).
[9] E. J. Mueller and T.-L. Ho, Phys. Rev. Lett. 88, 180403 (2002).
[10] K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. Lett. 91, 150401 (2003); Int. J. Mod. Phys. B 19, 1835 (2005).
[11] V. Schweikhard, I. Coddington, P. Engels, V. P. Moggendorff, and E. A. Cornell, Phys. Rev. Lett. 92, 040404 (2004).
[12] A. G. Morris and D. L. Feder, Phys. Rev. Lett. 99, 240401 (2007).
[13] M. Roncaglia, M. Rizzi, and J. Dalibard, Scientific Reports 1, 43 (2011).
[14] Y.-J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto, and I. B. Spielman, Nature 462, 628 (2009).
[15] E. Ardonne and K. Schoutens, Phys. Rev. Lett. 82, 5096 (1999).
[16] E. Ardonne, N. Read, E. Rezayi, K. Schoutens, Nucl. Phys. B607 549 (2001).
[17] J. W. Reijnders, F. J. M. van Lankvelt, K. Schoutens, and N. Read, Phys. Rev. Lett. 89, 120401 (2002); Phys. Rev. A 69, 023612 (2004).
[18] L. Hormozi, G. Möller, and S. H. Simon, Phys. Rev. Lett. 108, 256809 (2012).
[19] These are obtained by multiplying the factor $\sqrt{M\omega_{\parallel}/(2\pi\hbar)}$ to the coupling constants $g^{(3D)}_{\alpha} = 4\pi\hbar^2a_{\alpha}/M$ and $g^{(3D)}_{\alpha\beta} = 4\pi\hbar^2a_{\alpha\beta}/M$ for the three-dimensional contact interactions. This factor arises from the restriction to the ground state of the axial confinement.

[20] D. Yoshioka, Phys. Rev. B 29, 6833 (1984).
[21] F. D. M. Haldane, Phys. Rev. Lett. 55, 2095 (1985).
[22] B. Halperin, Helv. Phys. Acta 56, 75 (1983).
[23] B. Paredes, P. Zoller, and J. I. Cirac, Phys. Rev. A 66, 033609 (2002).
[24] This is the $n = 2$ case of $SU(n+1)_k$ states proposed for $n$-component systems in Ref. [17].
[25] For $N = 4$, the overlap is perfect (exactly 1) for any aspect ratio $L_x/L_y$, due to a limited Hilbert space.
[26] T. Graß, B. Juliá-Díaz, N. Barberán, and M. Lewenstein, Phys. Rev. A 86, 021603 (R) (2012).