Radix Representation of Triangular Discrete Grid System

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Abstract. Discrete Global Grid Systems (DGGSs) are spatial references that use a hierarchical tessellation of cells to partition and address the entire globe. It provides an organizational structure that permits fast integration between multiple sources of large and variable geospatial data. Although many endeavors have been done to describe certain discrete grid systems, there still lack of a uniform mathematical framework for them. This paper simplifies the planar class I aperture 4 triangular discrete grid system into a hierarchical lattice model which is proved to be a radix system in the complex number plane. Mathematical properties of the radix system reveal the discrete grid system is equivalent to the set of complex numbers with special form. The conclusion provides a potential way to build a uniform mathematical framework of DGGS and can be used to design efficient encoding and spatial operation scheme for DGGS.

1. Introduction

Discrete Global Grid Systems (DGGSs) are new forms of multi-resolution geospatial data fusion and geoscience modelling solution for the whole globe. It subdivides the globe uniformly in a particular way to provide a hierarchical structure without gap and overlapping, and uses cell address for data manipulation instead of traditional geographic coordinates. DGGSs are more suitable for large-scale questions and structurally support handling multiple-resolution data efficiently, compared with traditional local-regional organizational structure and application pattern of geospatial data [1].

An effective way for DGGS construction is to use regular polyhedrons to recursively subdivide the sphere, and the main steps are shown in figure 1. Firstly, choose a regular polyhedron and octahedron or icosahedron is commonly used; Secondly, unfold the regular polyhedron to the plane and obtain multiple triangular faces sharing one edge or one vertex; Thirdly, recursively subdivide all triangular faces into multi-resolution grids in the plane and triangle, quadrangle and hexagon are commonly used; Finally, by projection between polyhedral surface and spherical surface, map multi-resolution grids to the sphere and construct DGGS.

The obvious choice for a triangle is to subdivide it into smaller triangles. An equilateral triangle can be divided into \( p^2 (p \in \mathbb{N}, p \geq 1) \) smaller equilateral triangles by breaking each edge into \( p \) pieces and connecting the break points with lines parallel to the triangle edges (figure 2). This is referred to as Class I or alternative subdivision [2,3]. It generates consistent and aligned planar discrete grid systems with aperture (the ratio of the areas of a planar polygon cell at two successive levels) \( p^2 \) [2]. The smallest possible aperture of Class I subdivision is 4 \((p = 2)\). The subdivision is similar to the fourfold recursive subdivision of the square grid quadtree, and therefore many algorithms developed on the square grid quadtree are portable to the triangular grid quadtree with minor modifications [3]. This subdivision approach (figure 2) was used by Goodchild and Yang[4], White et al.[5], Dutton [6],...
Song et al.[7], Zhao et al.[8]. Congruent and unaligned Class I aperture 9 \((p = 3)\) subdivision has been proposed by White et al.[5] and Song et al.[7].

Main Steps:
① choose a regular polyhedron;
② unfold the polyhedron;
③ recursive subdivision of triangular faces of the polyhedron;
④ mapping from the surface of polyhedron to the sphere.

Different types of DGGSs

Figure 1. Main steps of polyhedron based DGGS construction.

Figure 2. Aperture 4 recursive triangular subdivision

The square is the most popular cell region shape for construction of planar discrete grids. Its geometric attributes make it inappropriate on the triangular face of the octahedron or icosahedron. However, White notes that pairs of adjacent triangle faces may be combined to form a diamond or rhombus, which can be recursively subdivided as square quadtree does (figure 4)[9]. This subdivision approach yields a congruent, unaligned planar discrete grid system with aperture 4. It should be noted that, similar approach can be taken to yield a congruent, aligned planar discrete grid system with aperture 9 (figure 5).

The hexagon is consistently neighboring, which takes advantage of geospatial analysis. However, it is impossible to completely tile a sphere with hexagons. A none-hexagon polygon will be formed at each of the polyhedrons' vertexes. The number of such polygons relates to the number of polyhedrons vertexes and has nothing to do with the grid resolution. The none-hexagon polygons for octahedrons and icosahedrons are 6 squares and 12 pentagons respectively. On polyhedron's triangular faces, various hexagonal grid systems can be generated by special recursive subdivisions. The aperture 3 subdivision yields an aligned grid system, whose orientation of cell alternates with successive resolutions (figure 6). This grid system is used by Sahr et al.[3,10] and PYXIS innovation[11]. For the aperture 4 subdivision, it yields similar grid systems with fixed cell orientation (figure 7). These grid systems are used by Tong et al.[12].
Above all, as beneficial supplement and improvement for existing GIS theory, DGGS has raised wide academic concern. Many scholars have done research into methods of grid partition, encoding, operation, level index, grid data visualization, etc. However, in current research situations, the academia lacks a thorough understanding of the essence of grid systems, so grid expression, encoding and operation are short of necessary theoretical support, and it is difficult to improve relevant algorithms.

This paper simplifies planar triangular discrete grid system into a hierarchal lattice model which is proved to be a radix system in the complex number plane. Mathematical properties of the radix system reveal the discrete grid system is equivalent to the set of complex numbers with special form. The conclusion provides a potential way to build a uniform mathematical framework of DGGS and can be used to design efficient encoding and spatial operation scheme for DGGS.

2. Definition of planar grid system
A planar grid system consists of a set of regions that form a subdivision of the 2D plane, where each region has a single point contained in the region associated with it. Each region/point combination is a cell. Depending on the application, data objects or vectors of values may be associated with regions, points, or cells. If an application defines only the regions, the centroids of the regions form a suitable set of associated points. Conversely, if an application defines only the points, the Voronoi regions of those points form an obvious set of associated cell regions.
It is convenient to represent each cell in a grid by its centroid. The centroids of the cells at a given resolution \( n(n \geq 1) \) are a finite set of points of a lattice in the plane. The elements of the lattice are called lattice points. The Voronoi cell of a lattice point \( x_i \) in a two-dimensional lattice \( L_n \) is a set consisting of all points of \( \mathbb{R}^2 \) which are at least as close to \( x_i \) as to any other lattice point of \( L_n \). If the Voronoi cells are regular triangles, then the lattice is called a triangular lattice (figure 8). A finite, non-empty subset of a lattice, or the corresponding set of Voronoi cells, is equivalent. Throughout this paper we often refer to the lattice point and its Voronoi cell interchangeably.

3. Mathematical model of triangular lattice

According to the analysis above, Class I triangular subdivision theory can produce grid systems with aperture \( p^2 \). But in fact, great apertures lead to severe jumping of resolution between two successive levels. Small apertures have the advantage of generating more grid resolutions, thus giving applications more resolutions from which to choose. Small apertures of triangular subdivision are 4 and 9. Subdivision of aperture 4 generates aligned grid systems while aperture 9 generates unaligned grid systems. Since aligned systems are convenient for expression of lattice and formula derivation, this article takes an aperture 4 triangular grid system as an example to establish the mathematical model of lattice.

**Theorem 1** On the complex plane, letting \( \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \), \( D = \{0, \omega, \omega^2, \omega^3\} \), lattice of nth level in aperture 4 Class I triangular grid system is

\[
\mathbb{L}_n = \frac{1}{2}e_1 + f(e_1)\frac{1}{4}e_2 + f(e_1)f(e_2)\frac{1}{8}e_3 + f(e_1) \cdot f(e_2) \cdots f(e_{n-1})\frac{1}{2^n}e_n = \sum_{i=1}^{n-1} \left[ \prod_{j=1}^{i-1} f(e_j) \right] \frac{1}{2^i}e_i
\]

Among this, \( e_i \in D(i = 1, 2, \cdots, n) \), \( f(x) = \begin{cases} -1 & x = 0 \\ 1 & x \neq 0 \end{cases} \) is the direction adjustment function, \( \sum_p \) are accumulation and plus operators between sets.

**Proof.** Since the aperture of the grid system is four, according to the law that the side ratio of two successive levels is 2, one cell in the level \( i-1(i \geq 2) \) is divided into four cells in level \( i \), as shown in figure 9. The lattice of one sub-cell is identical with the father cell, but the orientations are opposite. The other three sub-cells are rotational symmetry about the father cell and orientations are identical with the father cell.

**Figure 9.** Lattice relationship between two successive levels

Take the lattice in \( \mathbb{L}_{i-1} \) as the origin and establish the complex plane, letting \( D = \{0, \omega, \omega^2, \omega^3\} \), so lattice relationship between two successive levels is (figure 10):

\[
\mathbb{L}_i = \mathbb{L}_{i-1} + \frac{1}{2}f(e_1)f(e_2) \cdots f(e_{i-1})e_i
\]
Among this, \( e_j \in D(j = 1, 2, \ldots, i) \), \( f(x) = \begin{cases} -1 & x = 0 \\ 1 & x \neq 0 \end{cases} \)

By equation (1):
\[
\mathbb{L}_{i-1} = \mathbb{L}_{i-2} + \frac{1}{2^{i-1}} f(e_1)f(e_2) \cdots f(e_{i-2})e_{i-1}
\]
\[
\mathbb{L}_{i-2} = \mathbb{L}_{i-3} + \frac{1}{2^{i-2}} f(e_1)f(e_2) \cdots f(e_{i-3})e_{i-2}
\]
\[
\vdots
\]
\[
\mathbb{L}_2 = \mathbb{L}_1 + \frac{1}{4} f(e_1)e_2
\]

So the expression of \( \mathbb{L}_n \) is
\[
\mathbb{L}_n = \frac{1}{2} e_1 + \frac{1}{4} e_2 + \frac{1}{8} e_3 + \frac{1}{16} e_4 + \frac{1}{32} e_5 + \cdots + \frac{1}{2^n} e_n
\]

Theorem 1 reveals, aperture 4 triangular lattice is essentially identical to a Radix System or a Positional Number System such as binary, octonary and decimalism, and it is a ‘number’ with specific form on the complex field. The system of this positional number system is 2 and the digit set is \( D \). For \( x \in \mathbb{L}_n \), its specific form is:
\[
x = \frac{1}{2} e_1 \pm \frac{1}{4} e_2 \pm \frac{1}{8} e_3 \pm \cdots \pm \frac{1}{2^n} e_n (e_1, e_2, \ldots, e_n \in D)
\]

If the base of power is omitted, exponents like decimalism, \( x \) is denoted by \( x = .e_1e_2 \cdots e_n \). The radix points show that the lattice is more and more intensive when the subdivision level is ingrown. Express elements in \( \mathbb{L}_1 \sim \mathbb{L}_5 \) using points with different sizes and colours, their distribution on the complex plane is shown as figure 11.

**Theorem 2** The aperture 4 triangular grid system \( \mathbb{L}_n (n \geq 1) \) has the following properties:
- \( \mathbb{L}_n \) has inclusion relation, that is \( \mathbb{L}_1 \subset \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_n \);
- The number of lattice in level \( i \) is \( 4^i \).
Proof. Since \( L_i = L_{i-1} + \frac{1}{2^i} f(e_1)f(e_2) \ldots f(e_{i-1})e_i \), that is to say, lattices in level \( i (i \geq 2) \) take lattices in level \( i - 1 \) as centre, \( \frac{1}{2^i} \) as radius and then expand three elements in the directions of nonzero elements of \( D \), so \( L_{i-1} \) must be proper subset of \( L_i \). Combined with recursive relations, it is concluded that \( L_1 \subset L_2 \subset \ldots \subset L_n \).

Letting the lattice count of level \( i (i \geq 2) \) is \( a_i \), it is obvious that \( a_i = 4a_{i-1} \), Combined with recursive relations, and \( a_1 = 4 \), so \( a_i = 4^i \).

4. Conclusion
This paper employs lattice as the equivalent object of grids and reveals the mathematical essence of the planar Class I aperture 4 triangular grid system. Although only one grid system is discussed in this paper, the proof processes of relevant theorems have shown that multi-resolution regular grids are self-replication of the finite lattice set, which has nothing to do with the shape of grids. Thus, this research thought can be expanded to other regular grid systems. And as an effective tool of formal description, it perfects the theoretical basis of DGGS and provides theoretical support for design and analysis of algorithms.

References
[1] Open Geospatial Consortium 2015 OGC Discrete Global Grid System (DGGS) Core Standard https://portal.opengeospatial.org/files/66643
[2] Kenner H 1976 Geodesic math and how to use it (California: University of California Press) p172
[3] Sahr K, White D and Kimerling A 2003 Cartography and Geographic Information Science, 30 (2) p121–134
[4] Goodchild M and Yang S 1992 Graphical Models and Image Processing 54(1) p31–44
[5] White D, Kimerling A, Sahr K, and Song L 1998 International Journal of Geographical Information Science 12 p805–27
[6] Dutton G 1999 A hierarchical coordinate system for geoprocessing and cartography. Germany: Springer-Verlag. p231
[7] Song L, Kimerling A and Sahr K 2002 Discrete global grids: A web book (Santa Barbara: University of California)
[8] Zhao X, Hou M and Bai J 2007 Spatial digital modeling of the global discrete grids (Beijing: SinoMaps Press) p46
[9] White D 2000. Environmental Monitoring and Assessment 64(1) p93–103
[10] Sahr K 2008 Environment and Urban Systems 32 174–187
[11] PYXIS Innovation 2016 Pyxis public wiki http://www.pyxisinnovation.com/pyxwiki/index.php?title=Main_Page
[12] Tong X, Ben J, Wan Y, Zhang Y and Pei T 2013 International Journal of Geographical Information Science 27(5) p898–921

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