Non-equilibrium dynamics near a quantum multicritical point

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Abstract. We study the non-equilibrium dynamics of a quantum system close to a quantum multi-critical point (MCP) using the example of a one-dimensional spin-1/2 transverse XY spin chain. We summarize earlier results of defect generation and fidelity susceptibility for quenching through MCP and close to the MCP, respectively. For a quenching scheme which enables the system to hit the MCP along different paths, we emphasize the role of path on exponents associated with quasicritical points which appear in the scaling relations. Finally, we explicitly derive the scaling of concurrence and negativity for two spin entanglement generated following a slow quenching across the MCP and enlist the results for different quenching schemes. We explicitly show the dependence of the scaling on the quenching path and discuss the limiting situations.

1. Introduction
Recent years have seen a remarkable upsurge in the studies on the non-equilibrium dynamics of quantum phase transitions (QPTs) in quantum many body systems[1, 2, 3]. A zero temperature QPT, driven by quantum fluctuations, is associated with a fundamental change in the symmetry of the ground state of a quantum system. The possibility of experimental realization of QPT in ultracold atoms trapped in optical lattices, namely Mott insulator to superfluid transition, has also paved the way for rigorous theoretical investigations [4, 5]. A quantum critical point (QCP)[1, 2] is associated with a diverging length scale as well as a diverging time scale or the relaxation time of the quantum system arising due to a vanishing energy gap. In the vicinity of a QCP at \( \lambda = 0 \), the spatial correlation length \( \xi \) diverges as \( \xi \sim |\lambda|^{-\nu} \), while the relaxation time \( \xi_\tau \) (or the inverse of the energy gap) grows as \( \xi_\tau \sim \xi^z \), where \( \nu \) and \( z \) are the quantum critical exponents characterizing the universality class of the QPT.

Divergences in length and time scales introduce non-trivial effects in the dynamics of a quantum system near a QCP. In particular, if a parameter of the Hamiltonian is tuned slowly so as to drive the system through a QCP, the dynamics fails to be adiabatic for any finite rate of variation of the parameter. This non-adiabaticity in the response of the system eventually leads to generation of defects in the final state. A quantitative analysis of defect production in a system following a quench can be carried out using the Kibble-Zurek (KZ) theory [6, 7] extended to quantum spin chains [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The KZ analysis assumes that when a parameter of the Hamiltonian is varied in time \( t \) at a uniform rate as \( (t/\tau) \) through the QCP, where \( \tau \) is the inverse rate of quenching, the system evolves adiabatically for the entire span of time except for the impulse region close to the QCP where the relaxation time...
of the system exceeds the rate of quenching; this leads to the generation of defects in the final state of a $d$-dimensional quantum Hamiltonian which scales with $\tau$ as $1/\tau^{\nu_d/(\nu_z+1)}$ [8, 9]. The scaling has also been extended to non-linear variation of parameters [16, 20].

There has also been recent developments in theoretical investigation of QPTs from the viewpoint of quantum information theory. Quantities like concurrence [21, 22], entanglement entropy[23, 24], fidelity susceptibility[25, 26, 27, 28, 29, 30, 31, 32, 33] and geometric phases[34, 35, 36, 37] have been shown to capture the non-analyticities associated with a QCP.

Fidelity is the measure of overlap of two neighbouring ground states of a quantum Hamiltonian in the parameter space. Fidelity susceptibility is the rate of change of the ground state under an infinitesimal variation of the parameters of the Hamiltonian. For two ground states $\psi_0(\lambda)$ and $\psi_0(\lambda + \delta \lambda)$ closely spaced in the parameter space ($\delta \lambda \to 0$), fidelity ($F$) is defined as[25]

$$ F = |\langle \psi_0(\lambda) | \psi_0(\lambda + \delta \lambda) \rangle| \approx 1 - \frac{\delta \lambda^2}{2} \chi_F(\lambda) + \cdots \quad (1) $$

where the fidelity susceptibility $\chi_F$ is the first non-vanishing term in the expansion of fidelity. The scaling behavior of $\chi_F$ is well established near a QCP[26, 28, 29] which has recently been generalised to the case of a quantum MCP[38]. The symmetry of the ground state of a quantum many-body system undergoes a change at QCP[1] leading to a sharp drop of fidelity. At the same time, the fidelity susceptibility ($\chi_F$) usually diverges in a power law fashion with the parameter size as $\chi_F \sim L^{2/\nu}$ [26, 27, 28, 29, 30, 31, 32, 39, 40, 41, 42, 43] where $\nu$ is the correlation length exponent associated with the QCP. In recent years a series of works on the connection between fidelity susceptibility and quantum criticality has been reported [26, 27, 28, 29, 30, 31, 32, 39, 40, 41, 42, 43]. Generation of two spin entanglement following a slow quench across a QCP has also been reported recently [44]. Recently, the two spin entanglement properties of a quantum system during time evolution after a sudden quench through a critical point have been studied [45].

In the present work we discuss the dynamics of quantum spin chains, namely transverse $XY$ model near a MCP [46]. The presence of minimum energy quasicritical points near a MCP results in strikingly different behaviors of measurable parameters, for example defect density following fast and slow quenches, fidelity susceptibility, etc. We summarize recent results on the defect generation and fidelity susceptibility in sections 2 and 3. In section 4, we study the scaling of concurrence [21] and negativity [47] following a path-dependent slow quench across a quantum MCP generalizing related work on passage through a QCP [44].

2. The model

We consider a one dimensional spin $1/2$ $XY$ model in presence of a transverse field, given by

$$ H = -\sum_j \left( J_x \sigma^x_j \sigma^x_{j+1} + J_y \sigma^y_j \sigma^y_{j+1} + h \sigma^z_j \right), \quad (2) $$

where $\sigma$’s are the Pauli spin matrices, $J_x$ and $J_y$, and $h$ are interaction strengths and the transverse field, respectively. Using Jordan-Wigner transformation the above Hamiltonian can be reduced to a direct product of $2 \times 2$ Hamiltonians in the Fourier space given by the form [48, 50, 49, 51, 19]

$$ H_k = \begin{pmatrix} h + \cos k & i\gamma \sin k \\ -i\gamma \sin k & -(h + \cos k) \end{pmatrix} \quad (3) $$

with $J = J_x + J_y$ is set equal to unity and anistropy $\gamma = J_x - J_y$; the energy spectrum is

$$ \Lambda_k = \sqrt{(h + \cos k)^2 + \gamma^2 \sin^2 k}. \quad (4) $$

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Figure 1. Phase diagram of $XY$ model in presence of transverse field. The horizontal bold line shows the anisotropic transition between ferromagnetic ordering in the $x$ direction and that in the $y$ direction, whereas, the vertical bold lines are lines of Ising transitions between paramagnetic and ferromagnetic phases. The different paths of quenching through the MCP $A$ are shown.

Figure 2. Plot of defect density as a function of $\tau$ for different values of $\alpha$. $n_{ex}$ varies $\tau^{-\alpha/6}$ for $1 < \alpha < 2$, and saturates to $n_{ex} \sim \tau^{-1/3}$ for $\alpha > \alpha_c = 2$.

Eq. 4 shows that the gap vanishes for the mode $k = 0(\pi)$ for $h = -1(+1)$ indicating ferromagnetic to paramagnetic QPT with critical exponents $\nu = z = 1$. Similarly, the horizontal bold line indicate quantum transitions between two ferromagnetically ordered phases (see Fig. 1) with the same set of critical exponents. The Ising and anistropic critical lines meet at multicritical points ($h = \pm 1, \gamma = 0$) where the critical exponents are $\nu = \nu_{mc} = 1/2, z = z_{mc} = 2$, respectively.

If the transverse field $h$ is varied as $t/\tau$ sweeping the system across the Ising critical points, a defect density scaling as $n \sim \tau^{-1/2}$ has been derived; this is in complete agreement with the KZ scaling noting that $\nu = z = 1$ and $d = 1$ [8, 9, 10, 11, 13]. Identical scaling relation has been found for quenching across both Ising and the anisotropic critical points by varying the interaction term [14]. On the other hand, a similar quench through the MCP produces a much slower decay defect density given by $n_{ex} \sim \tau^{-\alpha/6}$ [14, 52] in sharp contrast with the KZ prediction $n_{ex} \sim \tau^{-1/4}$. A subsequent investigation on this apparent violation of KZ scaling revealed the existence of minimum energy gap points occuring away from the MCP [52] where the excitation scales with the momentum $\Delta_k \sim k^{z_{qc}}$ thereby defining an effective dynamical exponent $z_{qc}$ which turned out to be equal to three for the present model. For a system reducible to non-interacting two-level problems, a modified KZ scaling relation of the form $n_{ex} \sim \tau^{2z_{qc}}$ was proposed for quenching across a MCP [52].
The above apparent violation of KZ mechanism led to a series of works on the non-equilibrium properties of the system near a MCP [38, 53, 54]. It was shown that the dynamical response of the system stems from the presence of quasicritical points where the energy gap is minimum [53, 54] as the energy gap is not necessarily minimum at the MCP. Deng et al, [53] used a non-linear two parameter quenching scheme $|\gamma(t)| = h(t) - 1 = |t/\tau|^\alpha sgn(t)$ so that the MCP ($h = 1, \gamma = 0$) is reached at a time $t = 0$. In the case $\alpha = 1$ (linear quenching), the defect density was exactly calculated using the Landau-Zener [56] tunneling formula and the relation $n_{ex} \sim \tau^{-1/6}$ was retrieved. It was argued that quasicritical points characterized by a sharp decrease in energy lie on the ferromagnetic side of the MCP and the system encounters them whenever the transverse field satisfies the relation $h = -k^2/2$ for modes close to the critical modes during its temporal evolution. The minimum excitation at a quasicritical point $\Delta_k \sim k^3$, as compared to $\Delta_k \sim k^2$ at the MCP. This defines the effective dynamical exponent $z_{qc} = 3$ as was reported in the reference [52]. For a generic $\alpha$ and a model with $\nu_{mc}z_{mc} \neq 1$ adiabatic perturbation theory was used to derive the scaling relation [53]

$$n_{ex} \sim \tau^{-d\nu_{qc}/(\nu_{qc}+1)}$$

where $\nu_{qc} = \nu_{mc}z_{mc}/z_{qc}$.

Mukherjee and Dutta [54] introduced a quenching scheme

$$h(t) = 1 + |\gamma(t)|^\alpha sgn(t); \quad \alpha > 0 \quad \text{and} \quad \gamma(t) = -t/\tau,$$

with time varying from $-\infty$ to $\infty$ so that the system can approach the MCP through different paths characterized by the parameter $\alpha$. It was shown that whether the system encounters quasicritical points depend crucially on the path and so do the effective critical exponents $z_{qc}$ and $\nu_{qc}$. For the present model, it was shown that the system hits quasicritical points only up to a limiting path given by $\alpha = 2$ while for $\alpha > 2$, the energy gap is minimum right at the MCP. Within the linearization approximation [55], the effective dynamical exponent was found to be $z_{qc} = 2/\alpha + 1$ leading to a scaling relation $n_{ex} \sim \tau^{-\alpha/6}$ so that for $\alpha = 1$ (linear path), we retrieve the exponent $1/6$. On the other hand, in the limit $\alpha \to 2$ the scaling saturates to $n_{ex} \sim \tau^{-1/3}$ indicating that for $\alpha > 2$, the quenching scheme essentially boils down to sweeping the system across the MCP along the gapless Ising critical line [15]. In fact, it should be reiterated that for $\alpha > 2$, system does not cross any quasi-critical point which shows that the scaling and the effective exponents are indeed path-dependent and $z_{qc} = z_{mc} = 2$. Using adiabatic perturbation theory, the generic scaling form was obtained as

$$n_{ex} \sim \tau^{-d\nu_{qc}/(\nu_{qc}+1)}$$

for $\nu_{mc}z_{mc} = 1$. For a generic model with $\nu_{mc}z_{mc} \neq 1$ Eq. (7) gets modified to

$$n_{ex} \sim \tau^{-\nu_{qc}/(\nu_{qc}+1)\nu_{mc}z_{mc}(1-\alpha)}.$$  

Eq. (7) includes an additional term when compared to the scaling form given in Eq. (5) which arises due to the linearization process.

The exponents associated with the quasi critical points namely, $z_{qc}$ and $\nu_{qc}$, can be understood using a two parameter scaling of the energy gap near the MCP given by

$$\Delta_k = k^{z_{mc}} f \left( \frac{\lambda_1}{k^{1/\nu_1}}, \frac{\lambda_2}{k^{1/\nu_2}} \right)$$
where $k = |\vec{k}|$ and in the present case $\lambda_1 = 1 - h$ and $\lambda_2 = \gamma$ with exponents $\nu_1 = 1/2$ and $\nu_2 = 1$, respectively. For the path $\lambda_1 = \lambda_2$, the quasi-critical points typically appear when $\lambda_1/k^{1/\nu_1} \simeq 1$ so that $\Delta_k \sim k^z_{mc} + 1/\nu_1 + 1/\nu_2$ yielding the quasi-critical dynamical exponent given by

$$z_{qc} = z_{mc} + \frac{1}{\nu_1} + \frac{1}{\nu_2},$$

(10)

so that $z_{qc} = 3$ for $z_{mc} = 2, \nu_1 = 1/2, \nu_2 = 1$ as argued previously. For a non-linear path $\lambda_1 = \lambda_2^2$, the exponent $\nu_1$ gets renormalized to $\alpha \nu_1$ [16] leading to the relation

$$z_{qc} = z_{mc} + \frac{1}{\alpha \nu_1} + \frac{1}{\nu_2},$$

(11)

which reduces to $z_{qc} = 2/\alpha + 1$ for the transverse $XY$ chain.

Finally, the MCP can also be approached along a path lying entirely in the paramagnetic phase using the quenching scheme $h(t) = 1 + |\gamma(t)|^\alpha, \gamma = t/\tau$ [53, 54] (path III in Fig. 1). The defect density for the transverse $XY$ chain is $n_{sp} \sim \tau^{-(d\alpha)/(2(1+\alpha))}$ which can be generalised using adiabatic perturbation theory to the form

$$n_{sp} \sim \tau^{-(d + d_0(\alpha))/\nu_{mc}},$$

(12)

Eq.(12) shows that for these special paths, there is a dimensional shift in KZ scaling relation arising due to shift in the centre of the impulse region so that the non-critical modes contribute to the defect generation [53]. For the present model, this dimensional shift $d_0(\alpha) = 2(2 - \alpha)/\alpha$ does also depend on path and vanishes in the limit of $\alpha \to 2$.

3. Fidelity susceptibility near a quantum multicritical point

![Figure 3](image)

Figure 3. Plot of fidelity susceptibility as a function of $J_x$ for the critical and multicritical points. $\chi_f$ diverges as $L$ near the QCP, while an MCP results in oscillatory $\chi_f$, with the maximum values scaling as $L^{Z}$. Inset Shows the oscillations in $\chi_f$ close to the MCP.

The scaling of the fidelity susceptibility and defect generated following a rapid quench near a quantum MCP has been studied recently[38]. At a QCP, $\chi_f(=\chi_F/L^d)$ varies as $L^{2/\nu - d}$ where $L$ is the system size, while away from the QCP $\chi_f$ scales as $\chi_f \sim \lambda^{d - 2}$, where $\lambda$ measures the deviation from the QCP. Studies on fidelity susceptibility near MCP in Hamiltonian (2) along a linear path parametrized by $\lambda_1 = \lambda_2 = \lambda$, shows a strikingly different behavior, with $\chi_f$ exhibiting an oscillatory variation with $\lambda$; it peaks whenever a quasi-critical point at
\( \lambda = k^{1/\nu_1}/2 = k^2/2 \) is encountered while no peak is observed right at the MCP. The maximum value of any peak scales as

\[
\chi_f \sim L^{2/\nu_{qc} - d},
\]

leading to \( \chi_f \sim L^5 \) for transverse field XY chain. For a general path \( \lambda_1 = \lambda_2^\alpha \) with \( \alpha < \alpha_c \), \( \nu_{qc} = (2/\alpha + 1)^{-1} \), hence Eq. (13) reduces to

\[
\chi_f \sim L^{4/\alpha - 1}.
\]

![Figure 4](image)

**Figure 4.** Plot of kink density following a sudden quench starting from the MCP. Sudden jumps in \( n_{ex} \) are observed whenever the system encounters a quasi-critical point.

The defect density generated following a fast quench of small amplitude \( \lambda \) starting from a QCP has been argued to satisfy the scaling relation [32]

\[
n_{ex} \sim \lambda^2 \chi_f \sim \lambda^{\nu_d}.
\]

A similar approach near MCP showed a violation of the simple relation (Eq. 15) when system in quenched suddenly into the ferromagnetic region. Further, sudden increase of kink density is observed at quasicritical points. However, the coarse grained kink density still scales with \( \lambda \) as \( n_{ex} \sim \lambda^{\nu_d} \).

4. Generation of Concurrence and Negativity following a slow quench

In this section, we study generation of two-spin entanglement in the system (2) for a slow quench through the MCP and quantify it in terms of the measures concurrence [21] and negativity[47]. These quantities or their derivatives show peaks or discontinuous jumps at the QCP signalling the singularities associated with them. We generalise the results obtained by Sengupta and Sen [44] for passage through a QCP to the present case of a MCP. We employ different linear and non-linear quenching schemes parametrized by an exponent \( \alpha \) (as discussed previously) to study the scaling behaviors of concurrence and negativity of the ground state in the final state of the system.

Sengupta and Sen [44] argued that for the passage through a QCP by a linear or non-linear variation of parameters, entanglement is generated only between spins separated by even number of lattice spacings. Their study also established the existence of a critical quench rate \( \tau_c^{-1} \) such that for a fatser quench rate \( \tau^{-1} > \tau_c^{-1} \), no such two-spin entanglement is generated. Our study reveals that the above remains valid even for quenching through a MCP; the scalings of concurrence and negativity are related to the scaling of defect density which as discussed
where the matrix elements are related to the two-spin correlation function as follows:

\[
\rho^n = \begin{pmatrix}
  a_+^n & 0 & 0 & b_1^n \\
  0 & a_0^n & b_2^n & 0 \\
  0 & b_2^* & a_0^n & 0 \\
  b_1^* & 0 & 0 & a_-^n
\end{pmatrix},
\]

\[\text{(16)}\]

with

\[
\beta_n = \int_0^\pi \frac{dk}{\pi} p_k \cos(nk),
\]

\[\text{(18)}\]

Noting that \(p_k\) is invariant under \(k \rightarrow \pi - k\) and hence \(\beta_n = 0\) for odd \(n\), it can be readily shown from Eq. (17) that \(b_1^n = \langle \sigma_i^- \sigma_{i+n}^+ \rangle = 0\) for all \(n\) as the correlations between two fermionic creation or annihilation operators vanish. The transformations \(\sigma_i^\pm \rightarrow (-1)^n \sigma_i^\mp\), \(\sigma_i^x \rightarrow (-1)^n \sigma_i^x\) and \(\sigma_i^z \rightarrow \sigma_i^z\) leave the Hamiltonian (2) (the \(Z_2\) symmetry) and as a consequence \(p_k\) is also invariant. As a consequence, \(b_2^n = \langle \sigma_i^- \sigma_{i+n}^- \rangle\) vanishes for odd \(n\) so that the two-spin entanglement will not be generated for spins separated by odd lattice spacing [13]. This can also be deduced from the fact that the spin chain forms even and odd sublattice structure as explained in Ref. [13].

Correlation functions (19) can be evaluated by expressing the correlators as polynomials of pair correlators of Majorana Fermions [13, 48]. Using Eqs. (17), (18) and (19), we obtain the matrix elements of (16) as

\[
\begin{align*}
\langle \sigma_i^+ \sigma_{i+2}^- \rangle &= \beta_2 \langle \sigma_i^\pm \rangle, \\
\langle \sigma_i^+ \sigma_{i+4}^- \rangle &= (\beta_4 \langle \sigma_i^\pm \rangle + 2\beta_2^2 \langle \sigma_i^+ \sigma_{i+2}^- \rangle), \\
\langle \sigma_i^+ \sigma_{i+6}^- \rangle &= [\beta_6 \langle \sigma_i^+ \sigma_{i+2}^- \rangle + 4\beta_2(\beta_4 - \beta_2^2 + 4\beta_4 \langle \sigma_i^\pm \rangle)] \times \left[\langle \sigma_i^\pm \rangle \{\langle \sigma_i^+ \sigma_{i+2}^- \rangle - 4(\beta_2^2 + \beta_4^2)\} - 16\beta_2^2 \beta_4^2\right],
\end{align*}
\]

\[\text{(20)}\]

With \(b_1^n = 0\), the matrix \(\rho^n\) reduces to the form

\[
\rho^n = \begin{pmatrix}
  A & 0 & 0 & 0 \\
  0 & B & C & 0 \\
  0 & C & G & 0 \\
  0 & 0 & 0 & D
\end{pmatrix},
\]
For a matrix of this form, substituting the elements from Eq. (17) the expressions for concurrence and negativity simply become [44, 21]

\[
\mathcal{C}^n = \max\{0, 2(|b_0^n| - \sqrt{a_0^n a_0^n})\}, \\
\mathcal{N}^n = \max\{0, \frac{1}{2} \left[ a_+^n + a_-^n - \sqrt{(a_+^n - a_-^n)^2 + 4|b_0^n|^2} \right] \}.
\]

The scalings of concurrence and negativity of spins at site \( i \) and \( i + n \) therefore depends on the scalings of \( a_0^n, a_0^\pm \) and \( b_0^n \) as shown below for \( n = 2 \):

\[
a_+^2 = \frac{1}{4} ((1 \pm \sigma_i^z)(1 \pm \sigma_{i+2}^z)), \\
= \frac{1}{4} \left[ 1 \pm 2 \langle \sigma_i^z \rangle + \langle \sigma_i^- \sigma_{i+2}^z \rangle \right], \\
= \frac{1}{4} \left[ 1 \pm (2 - 4\beta_0) + (1 - 2\beta_0)^2 - 4\beta_2^2 \right],
\]

which simplifies to

\[
a_+^2 = (1 - \beta_0)^2 - \beta_2^2, \\
a_-^2 = \beta_0^2 - \beta_2^2.
\]

On the other hand, \( a_0 \) is given by

\[
a_0^2 = \frac{1}{4} \langle (1 \pm \sigma_i^z)(1 \mp \sigma_{i+2}^z) \rangle, \\
= \frac{1}{4} \left[ 1 - \langle \sigma_i^z \sigma_{i+2}^z \rangle \right], \\
= \beta_0 (1 - \beta_0) - \beta_2^2.
\]

while

\[
b_2^2 = b_2^2 = \langle \sigma_i^- \sigma_{i+2}^+ \rangle = \beta_2 (1 - 2\beta_0)
\]

From Eq. (18), it is clearly seen that \( \beta_n \) scales in the same way for all \( n \). Retaining the highest order terms of Eqs. (24),(25) and (26), it can be shown that concurrence \( \mathcal{C}^n \sim \beta_n \) whereas \( \mathcal{N}^n \sim \beta_0^2 \). On the other hand, the defect density \( n_{ex} \sim \beta_n \) as seen from Eq. (18) thus establishing a direct relation with the scalings of concurrence and negativity.

By employing various quenching schemes to drive the system through the MCP, we get scaling relations of concurrence and negativity as tabulated in Table (1) (also see Fig. (1)). Scheme (1) summarises results for a non-linear queching of the transverse field through the Ising critical point as reported in reference [44] while schemes (2),(3) [53] and (4) [54] denote quenching across

| S. no. | Quenching Scheme | Defect density \( n_d \) | \( \mathcal{C}^n \) | \( \mathcal{N}^n \) |
|-------|-----------------|-----------------|-----|-----|
| 1.    | \( h(t) = h(t) = \frac{t}{T} \), \( \gamma(t) = \frac{t}{T} \) | \( \sqrt{\frac{a}{\tau}} \) | \( \sqrt{\frac{a}{\tau}} \) | \( \frac{a}{\tau} \) |
| 2.    | \( h(t) - 1 = \frac{t}{T}, \gamma(t) = \frac{t}{T} \) | \( \frac{1}{\sqrt{\tau}} \) | \( \frac{1}{\sqrt{\tau}} \) | \( \frac{1}{\sqrt{\tau}} \) |
| 3.    | \( h(t) - 1 = \frac{t}{T} \), \( \gamma(t) = \frac{t}{T} \) | \( \frac{a}{\tau} \) \( \frac{a}{\tau} \) | \( \frac{a}{\tau} \) \( \frac{a}{\tau} \) |
| 4.    | \( h(t) - 1 = \frac{t}{T} \), \( \gamma(t) = \frac{t}{T} \) | \( \frac{2}{\tau} \) | \( \frac{2}{\tau} \) | \( \frac{2}{\tau} \) |
| 5.    | \( h = 1, \gamma = \frac{1}{T} \) | \( \frac{1}{\tau} \) | \( \frac{1}{\tau} \) | \( \frac{1}{\tau} \) |

**Table 1.** Comparative study of different quenching schemes.
the MCP where quasicritical points play a dominant role. Finally, in the scheme (5) [15] the system swept along a gapless Ising transition line across the MCP. In the scheme 4, which we have introduced in section 2., the quasicritical exponents and consequently the scaling of C_n and \lambda_n depend on path up to the limting path \alpha = 2 beyond which scaling relations for scheme (4) converges to those for scheme (5) as shown in the case of defect density. From Eq. (21), we note the existence of a critical quench rate when |b_2| equals \sqrt{a_n^2 + a_n^{-2}} above which \lambda_n = \lambda_n^c = 0 and no two-spin entanglement is generated. Interestingly, the critical rate does not depend on the path.

5. Conclusion
We have summarized recent results on the non-equilibrium dynamics of a quantum spin chain, mainly the defect density following a slow quench across a MCP and the scaling of fidelity susceptibility close to it. In the final section, we have calculated the scaling of the concurrence and negativity for a multicritical slow quantum quench. We show that scaling of both the afore mentioned quantities with the rate of quenching depends crucially on path for a path dependent quenching scheme discussed here. The scaling exponent saturates to a constant value beyond a limiting path when the system does no longer encounter any quasicritical point.

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