On a family of coupled diffusions that can never change their initial order

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Abstract
We introduce a real-valued family of interacting diffusions where their paths can meet but cannot cross each other in a way that would alter their initial order. Any given interacting pair is a solution to coupled stochastic differential equations with time-dependent coefficients satisfying certain regularity conditions with respect to each other. These coefficients explicitly determine whether these processes bounce away from each other or stick to one another if/when their paths collide. When all interacting diffusions in the system follow a martingale behaviour, and if all these paths ultimately come into collision, we show that the system reaches a random steady-state with zero fluctuation thereafter. We prove that in a special case when certain paths abide to a deterministic trend, the system reduces down to the topology of captive diffusions. We also show that square-root diffusions form a subclass of the proposed family of processes. Applications include order-driven interacting particle systems in physics, adhesive microbial dynamics in biology and risk-bounded quadratic optimization solutions in control theory.

Keywords: coupled processes, captive dynamics, stochastic domains, interacting systems

(Some figures may appear in colour only in the online journal)

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1. Introduction

This paper introduces a family of stochastic processes that maintain their initial order throughout their lifetime. In other words, by virtue of order-preservation, these random processes display a fundamental feature: they are able to collide with but never trespass each other’s randomly evolving paths, hence act as pairwise stochastic boundaries to one another. Our framework can help address a wide range of questions in physical systems that include two random objects of equivalent speed and/or velocity (moving simultaneously in the same direction), which would either bounce away from each other, or stick to one another, at time of collision, depending on their underlying parameters (e.g. mass, magnetism, shape, spin). Some possible applications may surface in the study of approximating elastic collisions in particle physics (e.g. behaviour of atoms in thermal agitation under black-body radiation), or modelling pathogenic mechanisms of microbial infections in mathematical biology, or pricing financial instruments whose maximum-minimum payoffs are capped by each other’s evolution.

For the majority of this work, order will simply refer to any two points on a totally-ordered set that can be compared with each other via the binary relation ‘⩽’ (later in the paper, we shall relax this to a partial-order on the convex cone of positive semi-definite matrices). In this sense, for points in time \( t \in [0, T] = \mathbb{T} \) with \( T < \infty \), the attribute of order-preservation in \( a_t \leq b_t \) is satisfied when \( a_t \) is bound to remain smaller or equal to \( b_t \) for the duration of its lifetime, unable to break free from its initial order with respect to \( b_t \) until \( T \). For what follows, we shall choose the totally-ordered set to be \( \mathbb{R} \), where the points refer to the values of \( \mathbb{R} \)-valued diffusions \( X_t^{(i)} \in \mathbb{T} \) for \( i \in I = \{1, \ldots, n\} \) where \( n \geq 2 \) and \( n \in \mathbb{N}_+ \). More specifically, we shall guarantee that

\[
P \left( X_t^{(1)} \leq \ldots \leq X_t^{(i)} \leq \ldots \leq X_t^{(n)} \left| X_0^{(1)} \leq \ldots \leq X_0^{(i)} \leq \ldots \leq X_0^{(n)} \right. \right) = 1 \quad \forall t \in \mathbb{T},
\]

(1)

where \( P(.) \) is a probability measure. Accordingly, every pair \( \{X_t^{(i)}, X_t^{(i+1)}\} \in \mathbb{T} \) in the system manifests either bouncy (reflective) or sticky (absorbing) behaviour if/when the two paths collide.

The study of random processes with reflective or absorbing dynamics within restricted domains has attracted significant attention in the literature due to their wide applicability in physics, engineering, economics and biology—see [1–11] amongst many more, including random walkers with Kardar–Parisi–Zhang fluctuations driven by the underlying geometry (see [12]). Essentially, the topological boundaries control the space of all possible paths that a stochastic entity may follow—e.g. a particle or a microbe. In most cases however, the domains that restrict the space of all possible paths are pre-determined from the outset, and hence, the architecture of these domains are usually not randomly-formed. As such, the boundaries that restrict the evolution of a stochastic phenomenon typically arise from the exogenously-assigned deterministic topologies that represent the relevant underlying space, which may certainly be a desired framework depending on the type of application. On the other hand, one of the blooming areas in natural sciences, in which stochastic boundaries arise randomly and endogenously, is random matrix theory, where non-colliding eigenvalue dynamics tend to appear in the specific form of Dyson’s Brownian-motion, which can be formed via harmonic Doob transforms in Weyl chambers [13–17]. In our proposed framework, we diverge from the construction of random matrix theory, and allow a high level of flexibility in defining systems where stochastic diffusions act as stochastic boundaries to each other—particles
can collide and in turn encapsulate a considerably large family of dynamics involving absorption and reflection properties, paving way to the investigation of random paths that evolve within self-induced topologies displaying highly versatile stochastic geometries that satisfy order-preservation. Therefore, in relation with the existing literature, we may still speak of topological boundaries that control the space of all possible paths that a stochastic entity may follow, but this time the boundaries are naturally and endogenously moulded in time as a result of the randomness of the agents involved. Given the mathematical generality of our approach, we can simulate systems where each particle may behave very differently, while still respecting their order within the system to sustain a degree of unity. To the best of our knowledge, there is no literature that studies this aspect as presented in this paper. In addition, we prove that captive diffusions of [18], that evolve within deterministic boundaries, form a special subclass of the processes we introduce in this paper when certain pairs of diffusion coefficients are set to zero (this will be clear to readers later on).

We note that our framework can be applied in random particle systems that interact with each other based on their order (see [18–21]), risk-controlled quadratic optimization problems (see, [22]), and sticky microbial dynamics (see, [23–25]), amongst others. We leave a more detailed account of these applications for future and focus on establishing the mathematical groundwork in this paper. This work is organised as follows: In section 2, we present our main results together with simulations for demonstration. In section 2.1, we generate Hermitian-valued processes that respect their initial Loewner-order, and produce order-preserving efficient frontiers in quadratic optimization. Section 3 concludes. Section 5 is the appendix, where we offer a generalization of our framework.

2. Main Results

We choose a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$, where all filtrations are right-continuous and complete with $\mathcal{F}_\infty = \mathcal{F}$. We let $\Omega = \mathcal{C}(\mathbb{T} \times \mathbb{R})$ be the space of continuous paths and $X^{(i)} : \mathcal{C}(\mathbb{T} \times \mathbb{R}) \to \mathbb{R}$ map a continuous-time process $\{X^{(i)}_t\}_{t \in \mathbb{T}}$ for $i \in \mathcal{I}$. We denote $\mathcal{M}(\mathbb{T} \times \mathbb{R}) \subset \mathcal{C}(\mathbb{T} \times \mathbb{R})$ as the space of continuous $\mathbb{R}$-valued $(\mathbb{P}, \{\mathcal{F}_t\})$-martingales. We collect $\{X^{(i)}_t\}_{t \in \mathbb{T}}$ in an $\mathbb{R}^n$-valued vector process $\{X_t\}_{t \in \mathbb{T}}$ as follows:

$$X_t = \left[ X^{(1)}_t, \ldots, X^{(i)}_t, \ldots, X^{(n)}_t \right]^\top.$$

Accordingly, if we have $n = 2$, we write $X_t = \left[ X^{(1)}_t, X^{(2)}_t \right]^\top$. In addition, we introduce the following notation that will be useful throughout this paper:

$$X_i[y; y] = \left[ x^{(1)}_y, \ldots, y^{(n)}_y \right]^\top,$$

(2)

to specify that the $i$th coordinate of $X_t$ takes the value $y \in \mathbb{R}$. If $i = 1$ or $i = n$, then (2) should be understood accordingly, where we have

$$X_i[1; y] = \left[ y, \ldots, x^{(n)}_y \right]^\top \quad \text{and} \quad X_i[n; y] = \left[ x^{(1)}_y, \ldots, y \right]^\top.$$

We are now in the position to introduce the main object of this work.

**Definition 2.1.** Let each $\{X^{(i)}_t\}_{t \in \mathbb{T}} \subset \mathcal{C}(\mathbb{R} \times \mathbb{T})$ for $i \in \mathcal{I}$ be a solution to a stochastic differential equation (SDE) governed by

$$X^{(i)}_t = x^{(i)}_0 + \int_0^t \mu^{(i)}(s, X_s) \, ds + \int_0^t \sigma^{(i)}(s, X_s) \, dM^{(i)}_s,$$

(3)
given that $X_0^{(i)} = x_0^{(i)}$, where $\mu^{(i)} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ and $\sigma^{(i)} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ are measurable continuous maps, and $\{M_t^{(i)}\}_{t \in \mathbb{T}} \in \mathcal{M} (\mathbb{R} \times \mathbb{T})$. The coordinates of $\{X_t\}_{t \in \mathbb{T}}$ form an order-preserving coupled system if

(a) $x_0^{(i)} \leq x_0^{(i+1)}$,

(b) $\mu^{(i)} \left( t, X_t \left[ i; X_t^{(i+1)} \right] \right) \leq \mu^{(i+1)} \left( t, X_t \left[ i+1; X_t^{(i)} \right] \right)$,

(c) $\sigma^{(i)} \left( t, X_t \left[ i; X_t^{(i+1)} \right] \right) = \sigma^{(i+1)} \left( t, X_t \left[ i+1; X_t^{(i)} \right] \right) = 0$,

for any $t \in \mathbb{T}$ when $X_t^{(i)} = X_t^{(i+1)}$ for every $i = 1, \ldots, n-1$.

We note that definition 2.1 stands fairly abstract, and to keep the setup general, we deliberately avoid specifying sufficiency conditions on the SDEs above (e.g. local Lipschitz continuity and linear growth) for the existence and uniqueness of their solutions, and instead define our processes as solutions when they exist. There is a well-established literature on existence and uniqueness of SDE solutions (see, [26, 27]), which is not the focus of our study. The next result is what gives the name of the main object of our work and what in turn proves (1).

**Proposition 2.2.** Let $\{X_t\}_{t \in \mathbb{T}}$ be an order-preserving coupled system as in definition 2.1. Then,

$$X_t^{(i)} \leq X_t^{(i+1)}, \mathbb{P}\text{-a.s.}$$

for all $t \in \mathbb{T}$ and every $i = 1, \ldots, n-1$.

**Proof.** Fix any $i$ and $i+1$ from $i = 1, \ldots, n-1$. Since $\{M_t^{(i)}\}_{t \in \mathbb{T}}$ and $\{M_t^{(i+1)}\}_{t \in \mathbb{T}}$ are in general *not* right-differentiable with respect to time $t$, both $\{X_t^{(i)}\}_{t \in \mathbb{T}}$ and $\{X_t^{(i+1)}\}_{t \in \mathbb{T}}$ are right-differentiable with respect to $t$ if

$$\sigma^{(i)}(t, \ldots) = \sigma^{(i+1)}(t, \ldots) = 0$$

at that $t \in \mathbb{T}$, given that $\mu^{(i)}$ and $\mu^{(i+1)}$ are locally bounded functions (since they are continuous). Hence, from Property 3., in definition 2.1, $\{X_t^{(i)}\}_{t \in \mathbb{T}}$ and $\{X_t^{(i+1)}\}_{t \in \mathbb{T}}$ are right-differentiable at every $X_t^{(i)} = X_t^{(i+1)}$ for any $t \in \mathbb{T}$, such that

$$\lim_{\epsilon \to 0^+} \frac{X_t^{(i)} - X_t^{(i)}}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{X_t^{(i+1)} - X_t^{(i+1)}}{\epsilon},$$

$$\lim_{\epsilon \to 0^+} \frac{X_t^{(i+1)} - X_t^{(i)}}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{X_t^{(i+1)} - X_t^{(i)}}{\epsilon},$$

given that $\{X_t^{(i)}\}_{t \in \mathbb{T}} \in \mathcal{C}(\mathbb{R} \times \mathbb{T})$ and $\{X_t^{(i+1)}\}_{t \in \mathbb{T}} \in \mathcal{C}(\mathbb{R} \times \mathbb{T})$. Thus, from (4) and (5) and Property 2., in definition 2.1, we have

$$\mu^{(i)} \left( t, X_t \left[ i; X_t^{(i+1)} \right] \right) \leq \lim_{\epsilon \to 0^+} \frac{X_t^{(i)} - X_t^{(i+1)}}{\epsilon} \leq \lim_{\epsilon \to 0^+} \frac{X_t^{(i+1)} - X_t^{(i)}}{\epsilon} = \mu^{(i+1)} \left( t, X_t \left[ i+1; X_t^{(i)} \right] \right).$$
at every $X_{t}^{(i)} = X_{t}^{(i+1)}$ for any $t \in \mathbb{T}$. Since this holds at any $t \in \mathbb{T}$ where $X_{t}^{(i)} = X_{t}^{(i+1)}$ holds and since $\{X_{t}^{(i)}\}_{t \in \mathbb{T}} \in C(\mathbb{R} \times \mathbb{T})$ and $\{X_{t}^{(i+1)}\}_{t \in \mathbb{T}} \in C(\mathbb{R} \times \mathbb{T})$ with $x_{0}^{(i)} \leq x_{0}^{(i+1)}$, applying mean-value theorem gives us $X_{t}^{(i)} \leq X_{t}^{(i+1)}$ for all $t \in \mathbb{T}$, P-a.s. Same steps can be taken for every chosen pair $i$ and $i + 1$ from $i = 1, \ldots, n - 1$, and the result follows.

The statement above essentially implies that an initially-assigned order cannot be violated thereafter, hence, the naming of these stochastic processes. This order is maintained due to the interacting (time-dependent) coefficients that satisfy the aforementioned regularity conditions with respect to each other. Many models can be constructed that belong to this family—we provide an example below.

**Example 2.3.** Set $n = 2$ and fix $x_{0}^{(1)} \leq x_{0}^{(2)}$. The following is an order-preserving coupled system:

$$
\begin{align*}
X_{t}^{(1)} &= x_{0}^{(1)} + \kappa \int_{0}^{t} \left( f_{1}^{(1)} - X_{s}^{(1)} + X_{s}^{(2)} \right) ds + \alpha \int_{0}^{t} \left( X_{s}^{(2)} - X_{s}^{(1)} \right) dW_{s}^{(1)}, \\
X_{t}^{(2)} &= x_{0}^{(2)} + \kappa \int_{0}^{t} \left( f_{1}^{(2)} - X_{s}^{(1)} + X_{s}^{(2)} \right) ds + \beta \int_{0}^{t} \left( X_{s}^{(2)} - X_{s}^{(1)} \right) dW_{s}^{(2)}
\end{align*}
$$

(6)

where $\{W_{t}^{(1)}\}_{t \in \mathbb{T}}$ and $\{W_{t}^{(2)}\}_{t \in \mathbb{T}}$ are $(\mathbb{P}, \mathcal{F}_{t})$ Brownian motions, $\{f_{i}^{(1)}\}_{t \in \mathbb{T}}$ and $\{f_{i}^{(2)}\}_{t \in \mathbb{T}}$ are deterministic $\mathbb{R}$-valued continuous functions such that $f_{i}^{(1)} \leq f_{i}^{(2)}$ for all $t \in \mathbb{T}$, and the parameters satisfy: $\kappa \in [0, \infty)$, $0 < |\alpha| < \infty$ and $0 < |\beta| < \infty$. We shall provide an extended version of this example in the appendix together with a demonstrative simulation.

Now, for parsimony let $\{W_{t}^{(1)}\}_{t \in \mathbb{T}}$ and $\{W_{t}^{(2)}\}_{t \in \mathbb{T}}$ in (6) be mutually independent and $p(x, t)$ be the probability density function for $X_{t}^{(i)} \in \mathbb{R}^{2}$ in (6). Then the Fokker–Planck equation (i.e. the Kolmogorov forward equation) is given by

$$
\frac{\partial p(x, t)}{\partial t} = - \left( \frac{\partial}{\partial x^{(1)}} \left( f_{1}^{(1)} - x^{(1)} + x^{(2)} \right) p(x, t) \right) + \frac{\partial}{\partial x^{(2)}} \left( f_{1}^{(2)} - x^{(1)} + x^{(2)} \right) p(x, t) + \frac{\alpha^{2}}{2} \frac{\partial^{2}}{(\partial x^{(1)})^{2}} \left( x^{(1)} - x^{(2)} \right)^{2} p(x, t) + \frac{\beta^{2}}{2} \frac{\partial^{2}}{(\partial x^{(2)})^{2}} \left( x^{(1)} - x^{(2)} \right)^{2} p(x, t),
$$

with given initial condition. At a time of collision when $X_{t}^{(1)} = X_{t}^{(2)} = x$, the Fokker–Planck equation reduces down to the following:

$$
\frac{\partial p(x, t)}{\partial t} = - \left( f_{1}^{(1)} \frac{\partial}{\partial x} p(x, t) + f_{1}^{(2)} \frac{\partial}{\partial x} p(x, t) \right).
$$

Using proposition 2.2, when we have $n > 2$, we can construct systems whereby $\{X_{t}^{(i)}\}_{t \in \mathbb{T}}$ and $\{X_{t}^{(i+2)}\}_{t \in \mathbb{T}}$ trail a stochastic geometry over time that dynamically constrains the evolution of $\{X_{t}^{(i+1)}\}_{t \in \mathbb{T}}$. Hence, there is a level of control over $\{X_{t}^{(i+1)}\}_{t \in \mathbb{T}}$ dictated by the random topology formed in the system.
Example 2.4. Set $n = 3$ and $x_{0}^{(1)} \leq x_{0}^{(2)} \leq x_{0}^{(3)}$. The following is an order-preserving coupled system:

\[
X_{t}^{(1)} = x_{0}^{(1)} + \int_{0}^{t} \left( f_{t}^{(1)}(s) + X_{s}^{(2)}(s) - X_{s}^{(1)}(s) \right) \, ds + \alpha \int_{0}^{t} \sin \left( X_{s}^{(2)}(s) - X_{s}^{(1)}(s) \right) \, dW_{s}^{(1)}
\]

\[
X_{t}^{(2)} = x_{0}^{(2)} + \int_{0}^{t} \left( f_{t}^{(2)}(s) + \min(X_{s}^{(2)}(s) - X_{s}^{(1)}(s), X_{s}^{(3)}(s) - X_{s}^{(2)}(s)) \right) \, ds
\]

\[
+ \xi \int_{0}^{t} \left( X_{s}^{(2)}(s) - X_{s}^{(1)}(s) \right) \left( X_{s}^{(3)}(s) - X_{s}^{(2)}(s) \right) \, dW_{s}^{(2)}
\]

\[
X_{t}^{(3)} = x_{0}^{(3)} + \int_{0}^{t} \left( f_{t}^{(3)}(s) - X_{s}^{(2)}(s) + X_{s}^{(3)}(s) \right) \, ds + \beta \int_{0}^{t} \sin \left( X_{s}^{(3)}(s) - X_{s}^{(2)}(s) \right) \, dW_{s}^{(3)}
\]  

(7)

where each $\{W_{t}^{(j)}\}_{t \in \mathbb{T}}$ is an $(\mathbb{P}, \mathcal{F}_{t})$ Brownian motion, $\{f_{t}^{(1)}\}_{t \in \mathbb{T}}, \{f_{t}^{(2)}\}_{t \in \mathbb{T}}$ and $\{f_{t}^{(3)}\}_{t \in \mathbb{T}}$ are deterministic $\mathbb{R}$-valued continuous functions such that

\[
f_{t}^{(1)} \leq f_{t}^{(2)} \leq f_{t}^{(3)}
\]

for all $t \in \mathbb{T}$, and where $0 < |\alpha| < \infty, 0 < |\xi| < \infty$ and $0 < |\beta| < \infty$.

Note that in figure 1 $\{X_{t}^{(2)}\}_{t \in \mathbb{T}}$ is sandwiched between $\{X_{t}^{(1)}\}_{t \in \mathbb{T}}$ and $\{X_{t}^{(3)}\}_{t \in \mathbb{T}}$; or in other words, $\{X_{t}^{(1)}\}_{t \in \mathbb{T}}$ and $\{X_{t}^{(3)}\}_{t \in \mathbb{T}}$ act as stochastic boundaries to $\{X_{t}^{(2)}\}_{t \in \mathbb{T}}$.

Let $\{W_{t}^{(1)}\}_{t \in \mathbb{T}}$ in (7) be mutually independent and $p(x, t)$ be the probability density function for $X_{t} \in \mathbb{R}^{3}$ in (7). We shall now provide the Kolmogorov backward equation given by

\[
- \frac{\partial p(x, t)}{\partial t} = \left( f_{t}^{(1)}(x^{(2)} - x^{(1)}) - \frac{\partial}{\partial x^{(1)}} p(x, t) + \left( f_{t}^{(3)}(x^{(2)} + x^{(3)}) - \frac{\partial}{\partial x^{(3)}} p(x, t) + \frac{\alpha^{2}}{2} \sin \left( x^{(2)} - x^{(1)} \right) \right) \frac{\partial^{2}}{\partial x^{(1)^{2}}} p(x, t)
\]

\[
+ \frac{\beta^{2}}{2} \sin \left( x^{(2)} - x^{(3)} \right) \frac{\partial^{2}}{\partial x^{(3)^{2}}} p(x, t)
\]

\[
+ \frac{\xi^{2}}{2} \left( x^{(2)} - x^{(1)} \right) \left( x^{(3)} - x^{(2)} \right) \frac{\partial^{2}}{\partial x^{(2)^{2}}} p(x, t),
\]

(8)

with given terminal condition. Note that (8) is a specific case of the Feynman–Kac formula.
Depending on $\mu^{(i)}$ and $\mu^{(i+1)}$, order-preserving coupled processes can exhibit both bouncing (reflecting) and binding (absorbing) behaviour with respect to each other over non-overlapping time frames. Since the statement below follows from proposition 2.2, we shall omit its proof.

**Corollary 2.5.** Let $\{X_i\}_{i \in T}$ be an order-preserving coupled system as in definition 2.1.

(a) If $\mu^{(i)} \left( t, X_i \left[ i; X_i^{(i+1)} \right] \right) < \mu^{(i+1)} \left( t, X_i \left[ i+1; X_i^{(i)} \right] \right)$ holds for all $t \in T$, then $\{X_i^{(i)}\}_{i \in T}$ and $\{X_i^{(i+1)}\}_{i \in T}$ reflect from each other after they collide.

(b) If $\mu^{(i)} \left( t, X_i \left[ i; X_i^{(i+1)} \right] \right) = \mu^{(i+1)} \left( t, X_i \left[ i+1; X_i^{(i)} \right] \right)$ holds for all $t \in T$, then $\{X_i^{(i)}\}_{i \in T}$ and $\{X_i^{(i+1)}\}_{i \in T}$ absorb each other after they collide.

Corollary 2.5 tells us that the system shows reflecting vs absorbing properties explicitly through the drift terms. For instance, a fully reflective structure of $\mu^{(i)}$ and $\mu^{(i+1)}$ can model elastic collisions of atoms with random trajectories.

**Example 2.6.** Using corollary 2.5, we can also model systems where the coordinates of $\{X_i\}_{i \in T}$ branch out from each other only after a given time $i \in T$, until which time their trajectories have been the same. More precisely, if we keep the following:

$$
\mu^{(i)} \left( t, X_i \left[ i; X_i^{(i+1)} \right] \right) = \mu^{(i+1)} \left( t, X_i \left[ i+1; X_i^{(i)} \right] \right) \text{ for } t \leq i
$$

$$
\mu^{(i)} \left( t, X_i \left[ i; X_i^{(i+1)} \right] \right) < \mu^{(i+1)} \left( t, X_i \left[ i+1; X_i^{(i)} \right] \right) \text{ for } t > i
$$

and $x_0^{(i)} = x_0^{(i+1)} = x_0$, then the system would demonstrate branching dynamics. As an example, in figure 2, where we set $n = 2$,

$$
X_i^{(1)} = x_0 + \int_0^t \min (t-s,0) \, ds + \int_0^t \left( X_s^{(2)} - X_s^{(1)} \right) \, dW_s^{(1)},
$$

$$
X_i^{(2)} = x_0 - \int_0^t \min (t-s,0) \, ds + \int_0^t \left( X_s^{(1)} - X_s^{(2)} \right) \, dW_s^{(2)};
$$

(9)

**Remark 2.7.** Since $\{M_i^{(i)}\}_{i \in T}$ is not necessarily Markovian for any $i \in I$, $\{X_i\}_{i \in T}$ in definition 2.1 is not necessarily Markovian. For $\{X_i\}_{i \in T}$ to be Markovian, we set $\{M_i^{(i)}\}_{i \in T} = \{W_i^{(i)}\}_{i \in T}$ for every $i \in I$ as a canonical representation using Brownian motions.

From this point onwards, we shall let $\{X_i\}_{i \in T}$ stand for a Markovian order-preserving coupled system as in remark 2.7. We are interested in transformations of $\{X_i\}_{i \in T}$ that retain the system...
to be order-preserving as in definition 2.1—this allows one to form increasingly more sophisticated systems starting from simpler models by applying monotonic function compositions in succession. For the statement below, we let $C^2_b(\mathbb{R}) \subset C(\mathbb{T} \times \mathbb{R})$ be the subspace of continuous locally bounded measurable functions that are also twice-differentiable with continuous locally bounded derivatives.

**Proposition 2.8.** Let $h \in C^2_b(\mathbb{R})$ be a strictly increasing monotonic function. Then, $h(X_i^{(t)})$ for every $t \in \mathbb{T}$ and every $i \in \mathcal{I}$ form an order-preserving coupled system.

**Proof.** Denote $Y_t^{(i)} = h(X_t^{(i)})$ for every $i \in \mathcal{I}$ such that

$$Y_t = \left[ Y_t^{(1)}, \ldots, Y_t^{(i)}, \ldots, Y_t^{(n)} \right]^\top,$$

for every $t \in \mathbb{T}$. We need to check if all the properties in definition 2.1 are satisfied. First of all, since $h \in C^2_b(\mathbb{R})$, using Itô’s lemma, we have

$$Y_t^{(i)} = h\left(x_0^{(i)}\right) + \int_0^t \frac{\partial h}{\partial x}(x) \, dX_s^{(i)} + \frac{1}{2} \int_0^t \frac{\partial^2 h}{\partial x^2}(x) \, \sigma(x)^2 \, ds,$$

where

$$\sigma(x) = \left\{ \begin{array}{ll} 0 & 2 \leq x < 3, \\ 1 & x \geq 3 \end{array} \right.$$satisfies the condition $\sigma(x)^2 = \epsilon$ when $2 \leq x < 3$.

Proof. Denote $Y_t^{(i)} = h(X_t^{(i)})$ for every $i \in \mathcal{I}$ such that

$$Y_t = \left[ Y_t^{(1)}, \ldots, Y_t^{(i)}, \ldots, Y_t^{(n)} \right]^\top,$$

for every $t \in \mathbb{T}$. We need to check if all the properties in definition 2.1 are satisfied. First of all, since $h \in C^2_b(\mathbb{R})$, using Itô’s lemma, we have

$$Y_t^{(i)} = h\left(x_0^{(i)}\right) + \int_0^t \frac{\partial h}{\partial x}(x) \, dX_s^{(i)} + \frac{1}{2} \int_0^t \frac{\partial^2 h}{\partial x^2}(x) \, \sigma(x)^2 \, ds,$$

for all $t \in \mathbb{T}$ and for every $i \in \mathcal{I}$, where we are able to write $\hat{\mu}^{(i)}$ and $\hat{\sigma}^{(i)}$ in (11) in terms of $Y$, since $h$ is a strictly increasing monotonic function, which means it has an inverse, and since $h$ is applied to every $i \in \mathcal{I}$; hence, we can find $\hat{\mu}^{(i)}$ and $\hat{\sigma}^{(i)}$ in term of $Y$ that produces (10). In addition, having $h \in C^2_b(\mathbb{R})$, both $\hat{\mu}^{(i)}$ and $\hat{\sigma}^{(i)}$ are measurable continuous maps. Moreover, since $h$ is a strictly increasing monotonic function, $Y_0^{(i)} \leq Y_{t+1}^{(i+1)}$ and $\partial h / \partial x > 0$ for any $x$, which implies we get

$$\hat{\mu}^{(i)}(t, Y_t \left[i; Y_t^{(i+1)}\right]) \leq \hat{\mu}^{(i+1)}(t, Y_t \left[i+1; Y_t^{(i)}\right]),$$

for any $t \in \mathbb{T}$ when $Y_t^{(i)} = Y_{t+1}^{(i+1)}$ for every $i = 1, \ldots, n - 1$ using (10), and we also have

$$\hat{\sigma}^{(i)}(t, Y_t \left[i; Y_t^{(i+1)}\right]) = \hat{\sigma}^{(i+1)}(t, Y_t \left[i+1; Y_t^{(i)}\right]) = 0,$$

for any $t \in \mathbb{T}$ when $Y_t^{(i)} = Y_{t+1}^{(i+1)}$ for every $i = 1, \ldots, n - 1$ that follows directly from (10), which completes the proof. \hfill \Box

**Example 2.9.** Let $h(x) = x^3$ for any $x \in \mathbb{R}$ and consider the system given in (6) in example 2.3. Then, we reach

$$\left(X_t^{(1)}\right)^3 = \left(x_0^{(1)}\right)^3 + 3\kappa \int_0^t \left(X_s^{(1)}\right)^2 \left(f_s^{(1)} - X_s^{(1)} + X_s^{(2)}\right) \, ds$$

$$+ 3\kappa^2 \int_0^t X_s^{(1)} \left(X_s^{(2)} - X_s^{(1)}\right)^2 \, ds$$

$$+ 3\alpha \int_0^t \left(X_s^{(1)}\right)^2 \left(X_s^{(2)} - X_s^{(1)}\right) \, dW_s^{(1)}.$$
\[\left( X^{(2)}_t \right)^3 = \left( x^{(2)}_0 \right)^3 + 3\alpha \int_0^t \left( X^{(2)}_s \right)^2 (r^{(2)} - X^{(1)}_s + X^{(2)}_s) \, ds \]
\[+ 3\beta^2 \int_0^t X^{(2)}_s \left( X^{(2)}_s - X^{(1)}_s \right)^2 \, ds \]
\[+ 3\beta \int_0^t \left( X^{(2)}_s \right)^2 \left( X^{(2)}_s - X^{(1)}_s \right) \, dW^{(2)}_s\]
as an order-preserving coupled system.

**Remark 2.10.** Using proposition 2.2, we can deduce that the stochastic integration below is order-preserving that can be mapped into a non-negative random variable as follows:
\[\int_0^T X^{(i)}_s \, ds \leq \int_0^T X^{(i+k)}_s \, ds \mapsto a^{(i+k)}_T = \int_0^T \left( X^{(i+k)}_s - X^{(i)}_0 \right) \, ds \in [0, \infty),\]
for \(1 \leq k \leq n - i\), which can be used to quantify the strength of attraction between the paths \(\{X^{(i)}_s\}_{s \in \mathbb{T}}\) and \(\{X^{(i+k)}_s\}_{s \in \mathbb{T}}\)—e.g. the smaller \(a^{(i+k)}_T\) is, the stronger the particles are attracted to each other. Alternative order-preserving functionals can be employed to come up with different strength quantities.

Note that \(\{X_t\}_{t \in \mathbb{T}}\) is a continuous \((\mathbb{P}, \{\mathcal{F}^X_t\})\)-semimartingale, given that each \(\mu^{(i)}\) for \(i \in \mathbb{I}\) has locally bounded variation, where
\[\mathcal{F}^X_t = \sigma \left( \{X_s\}_{0 \leq s \leq t} \right),\]
is the \(\sigma\)-algebra such that \(\mathcal{F}^X_t \subset \mathcal{F}_t\) for every \(t \in \mathbb{T}\). The following result provides the post-collision behaviour when the coupled processes are strict martingales—i.e. when two paths meet, the paths necessarily stick to each other thereafter. For the following statements, we adopt the convention \(\inf \emptyset = \infty\).

**Proposition 2.11.** Let each coordinate of \(\{X_t\}_{t \in \mathbb{T}}\) be a \((\mathbb{P}, \{\mathcal{F}^X_t\})\)-martingale and let the random variable \(\tau^{(i,i+1)}\) be given by
\[\tau^{(i,i+1)} = \inf \{t : X^{(i)}_t = X^{(i+1)}_t\}\]
as the first collision-time of \(\{X^{(i)}_t\}_{t \in \mathbb{T}}\) vs. \(\{X^{(i+1)}_t\}_{t \in \mathbb{T}}\) if \(\tau^{(i,i+1)} \in \mathbb{T}\). Then,
\[\tau^{(i,i+1)} \leq t \leq T \Rightarrow X^{(i)}_t = X^{(i+1)}_t \forall t \in \left[\tau^{(i,i+1)}, T\right],\]
and the pair reaches a steady-state and becomes constant such that
\[X^{(i)}_t = X^{(i)}_{\tau^{(i,i+1)}} \text{ and } X^{(i+1)}_t = X^{(i+1)}_{\tau^{(i,i+1)}} \forall t \in \left[\tau^{(i,i+1)}, T\right],\]

**Proof.** Since \(\{X^{(i)}_t\}_{t \in \mathbb{T}}\) and \(\{X^{(i+1)}_t\}_{t \in \mathbb{T}}\) are \((\mathbb{P}, \{\mathcal{F}^X_t\})\)-martingales, we must have the following condition on the drift terms:
\[\mu^{(i)}(t, \ldots) = \mu^{(i+1)}(t, \ldots) = 0\]
for all \(t \in \mathbb{T}\), which means the system is governed by
\[X^{(i)}_t = x^{(i)}_0 + \int_0^t \sigma^{(i)}(s, X_s) \, dM^{(i)}_s,\]
\[X^{(i+1)}_t = x^{(i+1)}_0 + \int_0^t \sigma^{(i+1)}(s, X_s) \, dM^{(i+1)}_s.\]
From Property 3., in definition 2.1, we further have
\[ \sigma(i) \left( \tau^{(i,i+1)}, X_{r(t,i+1)} \right) = \sigma^{(i+1)} \left( \tau^{(i,i+1)}, X_{r(t,i+1)} \right) = 0, \]  
(17)
and using corollary 2.5 and (12), \( X_t^{(i)} = X_t^{(i+1)} \) must hold for all \( t \in [\tau^{(i,i+1)}, T] \), which proves (13). In addition, this means
\[ \sigma(i) (t, X_t) = \sigma^{(i+1)} (t, X_t) = 0, \forall t \in [\tau^{(i,i+1)}, T], \]  
(18)
and hence, from (16), writing
\[ X_t^{(i)} = X_{\tau^{(i,i+1)}}^{(i)} + \int_{\tau^{(i,i+1)}}^t \sigma(i) (s, X_s) \, dM_s^{(i)}, \]
\[ X_t^{(i+1)} = X_{\tau^{(i,i+1)}}^{(i+1)} + \int_{\tau^{(i,i+1)}}^t \sigma^{(i+1)} (s, X_s) \, dM_s^{(i+1)}, \]  
(19)
we have (18) and (19) imply that \( dX_t^{(i)} = dX_t^{(i+1)} = 0 \) for all \( t \in [\tau^{(i,i+1)}, T] \), which proves (14).

In theoretical physics, martingale property has been interpreted as a stochastic analogue for the conservation law of energy (see [28–30]). In our framework, proposition 2.11 tells us that martingales can be interpreted as systems that (sequentially) reach a collection of random steady-states as the paths meet each other—i.e. if the particles collide with at least one other particle after some finite time. Surely, we can have more than two paths colliding with each other to attain their random steady-state. The proposition below gives the case of three paths meeting at the same level, which can be generalised to any number of meeting paths via following a similar logic.

**Proposition 2.12.** Let each coordinate of \( \{ X_t \}_{t \in \mathbb{T}} \) be a \( (\mathbb{P}, \{ \mathcal{F}_t \}) \)-martingale and let the random variables \( \tau^{(i,i+1)} \) and \( \tau^{(i+1,i+2)} \) be given by
\[ \tau^{(i,i+1)} = \inf \left\{ t : X_t^{(i)} = X_t^{(i+1)} \right\}, \]
\[ \tau^{(i+1,i+2)} = \inf \left\{ t : X_t^{(i+1)} = X_t^{(i+2)} \right\}, \]  
(20)
as the first collision-times of \( \{ X_t^{(i)} \}_{t \in \mathbb{T}} \) vs \( \{ X_t^{(i+1)} \}_{t \in \mathbb{T}} \) if \( \tau^{(i,i+1)} \in \mathbb{T} \), and \( \{ X_t^{(i+1)} \}_{t \in \mathbb{T}} \) vs \( \{ X_t^{(i+2)} \}_{t \in \mathbb{T}} \) if \( \tau^{(i+1,i+2)} \in \mathbb{T} \), respectively. Then,
\[ \tau^{(i,i+1)} \leq u \leq T \Rightarrow X_u^{(i)} = X_u^{(i+1)} \forall u \in [\tau^{(i,i+1)}, T], \]
\[ \tau^{(i+1,i+2)} \leq s \leq T \Rightarrow X_s^{(i+1)} = X_s^{(i+2)} \forall s \in [\tau^{(i+1,i+2)}, T], \]  
(21)
and the triplet reaches a steady-state and becomes constant such that
\[ X_u^{(i)} = X_{\tau^{(i,i+1)}}^{(i)} \forall u \in [\tau^{(i,i+1)}, T], \]
\[ X_t^{(i+1)} = X_{\tau^{(i,i+1)} \wedge \tau^{(i+1,i+2)}}^{(i+1)} \forall r \in [\tau^{(i,i+1)} \wedge \tau^{(i+1,i+2)}, T], \]
\[ X_s^{(i+2)} = X_{\tau^{(i+1,i+2)}}^{(i+2)} \forall s \in [\tau^{(i+1,i+2)}, T]. \]

**Proof.** The proof is similar to that of proposition 2.11, where we must now have
\[ \mu^{(i)} (t, ..) = \mu^{(i+1)} (t, ..) = \mu^{(i+2)} (t, ..) = 0 \]  
(22)
for all \( t \in \mathbb{T} \), since every coordinate is a \((\mathbb{P}, \{\mathcal{F}_t^X\})\)-martingale. Hence,

\[
\begin{align*}
X^{(i)}_t &= x_0^{(i)}(t) + \int_0^t \sigma^{(i)}(s, X_s) \, dM_s^{(i)}, \\
X^{(i+1)}_t &= x_0^{(i+1)}(t) + \int_0^t \sigma^{(i+1)}(s, X_s) \, dM_s^{(i+1)}, \\
X^{(i+2)}_t &= x_0^{(i+2)}(t) + \int_0^t \sigma^{(i+2)}(s, X_s) \, dM_s^{(i+2)},
\end{align*}
\]

(23)

From Property 3., in definition 2.1, we have

\[
\begin{align*}
\sigma^{(i)}(\tau^{(i,j+1)}, X_{\tau^{(i,j+1)}}) &= \sigma^{(i+1)}(\tau^{(i,j+1)}, X_{\tau^{(i,j+1)}}) = 0, \\
\sigma^{(i+1)}(\tau^{(i+1,j+2)}, X_{\tau^{(i+1,j+2)}}) &= \sigma^{(i+2)}(\tau^{(i+1,j+2)}, X_{\tau^{(i+1,j+2)}}) = 0,
\end{align*}
\]

(24)

which from (23) and (24) imply that \( X^{(i)}_u = X^{(i+1)}_u \) must hold for all \( u \in [\tau^{(i,j+1)}, T] \), and \( X^{(i+1)}_s = X^{(i+2)}_s \) must hold for all \( s \in [\tau^{(i+1,j+2)}, T] \), proving (21). This further implies

\[
\begin{align*}
\sigma^{(i)}(u, X_u) &= \sigma^{(i+1)}(u, X_u) = 0, \forall u \in [\tau^{(i,j+1)}, T], \\
\sigma^{(i+1)}(s, X_s) &= \sigma^{(i+2)}(s, X_s) = 0, \forall s \in [\tau^{(i+1,j+2)}, T],
\end{align*}
\]

(25)

Therefore, using (23), we have

\[
\begin{align*}
X^{(i)}_t &= X^{(i)}_{\tau^{(i,j+1)}} + \int_{\tau^{(i,j+1)}}^t \sigma^{(i)}(s, X_s) \, dM_s^{(i)}, \\
X^{(i+1)}_t &= X^{(i+1)}_{\tau^{(i+1,j+2)}} + \int_{\tau^{(i+1,j+2)}}^t \sigma^{(i+1)}(s, X_s) \, dM_s^{(i+1)}, \\
X^{(i+2)}_t &= X^{(i+2)}_{\tau^{(i+1,j+2)}} + \int_{\tau^{(i+1,j+2)}}^t \sigma^{(i+2)}(s, X_s) \, dM_s^{(i+2)},
\end{align*}
\]

(26)

which, by using (25), completes the proof.

\( \square \)

**Example 2.13.** Note that if we set \( \kappa = 0 \) in example 2.3, then \( \{X_t\}_{t \in \mathbb{T}} \) becomes a \((\mathbb{P}, \{\mathcal{F}_t^X\})\)-martingale.

Figure 3 demonstrates the collision-behaviour of martingales as formalised in proposition 2.11—when the paths meet, they absorb each other, and the system collapses to a (random) steady-state.

The next result connects proposition 2.2 and the family of captive diffusions of [18] [definition 2.2], where the stochastic paths of \( \{X^{(i)}_t\}_{t \in \mathbb{T}} \) and \( \{X^{(i+2)}_t\}_{t \in \mathbb{T}} \) reduce down to everywhere-right-differentiable continuous paths that form the boundaries of \( \{X^{(i+1)}_t\}_{t \in \mathbb{T}} \) for \( n > 2 \). Accordingly, we see that proposition 2.2 forms a natural generalisation of captive diffusions.

**Proposition 2.14.** Set \( n > 2 \) where

\[
\sigma^{(i)}(t, \ldots, \cdot) = \sigma^{(i+2)}(t, \ldots, \cdot) = 0
\]

(27)

for all \( t \in \mathbb{T} \). Then, \( \{X^{(i+1)}_t\}_{t \in \mathbb{T}} \) belongs to the family of captive diffusions with respect to \( \{X^{(i)}_t\}_{t \in \mathbb{T}} \) and \( \{X^{(i+2)}_t\}_{t \in \mathbb{T}} \).
Proof. The statement follows since $\sigma(i)(t,\ldots) = \sigma(i+2)(t,\ldots) = 0$ for all $t \in \mathbb{T}$ implies $\mu(i)(t,\ldots)$ and $\mu(i+2)(t,\ldots)$ compute the derivatives $dX_i(t)/dt$ and $dX_{i+2}(t)/dt$, respectively, which exist given that $\mu(i)(t,\ldots)$ and $\mu(i+2)(t,\ldots)$ are measurable continuous (locally bounded) functions. Due to continuity of $\mu(i)(t,\ldots)$ and $\mu(i+2)(t,\ldots)$, $dX_i(t)/dt$ and $dX_{i+2}(t)/dt$ are equal to their right-derivatives. Thus, Property 2., and Property 3., in definition 2.1 coincide with Property 1., and Property 2., in [18] [definition 2.2].

Remark 2.15. Using proposition 2.14, the system reduces down to the topology of captive diffusions if odd-numbered paths $i \in I \cap \mathcal{O}$ (where $\mathcal{O}$ is the set of odd integers) are deterministic paths. Such a construct is relevant for modelling stochastic particles evolving within impenetrable tunnels.

2.1. Loewner-order preserving coupled processes

In a multivariate setting, the framework naturally lends itself to Hermitian-valued coupled stochastic processes that cannot change their initially-assigned Loewner-order. More precisely, we can produce matrix-valued Hermitian processes $\{H_t(i)\}_{t \in \mathbb{T}}$ for $i \in I$ on the space of positive semi-definite Hermitian matrices $\mathbb{H}_{m \times m}$ for some $m \in \mathbb{N}_+$ that maintain their initially given order on the induced convex cone, i.e.

$$P \left( H_t^{(1)} \preceq \ldots \preceq H_t^{(i)} \preceq \ldots \preceq H_t^{(n)} \right| H_0^{(1)} \preceq \ldots \preceq H_0^{(i)} \preceq \ldots \preceq H_0^{(n)} ) = 1 \quad \forall t \in \mathbb{T},$$

where ‘$\preceq$’ is the Loewner-order; a partial-order on $\mathbb{H}_{m \times m}$ such that

$$H_t^{(i)} \preceq H_t^{(i+1)} \Rightarrow Z_t^{(i+1)} = H_t^{(i+1)} - H_t^{(i)}$$

is a positive semi-definite matrix. In doing so, we can select a collection of $m$ order-preserving coupled systems from definition 2.1 as follows:

$$\left\{ \left[ X_t^{(i,1)}, X_t^{(i+1,1)} \right], \ldots, \left[ X_t^{(i,n)}, X_t^{(i+1,n)} \right] \right\}_{t \in \mathbb{T}}, \quad \forall i \in I,$$
unitarily diagonalized, and hence, we can produce Hermitian matrices as follows:

\[
\begin{bmatrix}
X_t^{(i,1)}, X_t^{(i+1,1)} \\
\vdots \\
X_t^{(i,j)}, X_t^{(i+1,j)} \\
\vdots \\
X_t^{(i,m)}, X_t^{(i+1,m)}
\end{bmatrix}
\mapsto
\begin{bmatrix}
\Lambda_t^{(i)} = \\
\Lambda_t^{(i+1)} =
\end{bmatrix}
\begin{bmatrix}
X_t^{(i,1)}, \ldots, 0, \ldots, 0 \\
\vdots \\
0, \ldots, X_t^{(j,1)}, \ldots, 0 \\
\vdots \\
0, \ldots, 0, X_t^{(i,1)}
\end{bmatrix}
\]

to model diagonal-eigenvalue matrix processes \(\{(\Lambda_t^{(i)}, \Lambda_t^{(i+1)})\}_{t \in \mathbb{T}}\), where we have \(x_t^{(i,j)} \leq x_t^{(i+1,j)}\) for any fixed \(j = 1, \ldots, m\), given that the initial conditions are \(X_0^{(i,j)} = x_t^{(i,j)}\) and \(X_0^{(i+1,j)} = x_t^{(i+1,j)}\) for every \(i \in \{1, \ldots, n-1\}\).

We set \(X_t^{(i,j)} = c_t^{(i,j)} \geq 0\) for all \(t \in \mathbb{T}\) and \(j = 1, \ldots, m\) to ensure every eigenvalue in the system is non-negative. From spectral decomposition theorem, every element in \(\mathbb{H}_{+}^{m \times m}\) can be unitarily diagonalized, and hence, we can produce Hermitian matrices as follows:

\[
\begin{align*}
H_t^{(i)} &= V_t \Lambda_t^{(i)} V_t^*, \\
H_t^{(i+1)} &= V_t \Lambda_t^{(i+1)} V_t^*,
\end{align*}
\]
given that \(\{V_t\}_{t \in \mathbb{T}}\) is a unitary matrix process and \(\{V_t^*\}_{t \in \mathbb{T}}\) is its conjugate transpose. Therefore, we have

\[
\Lambda_t^{(i+1)} - \Lambda_t^{(i)} \cong \Lambda_t^{(i,i+1)} \in \mathbb{H}_{+}^{m \times m}
\Rightarrow H_t^{(i)} \preceq H_t^{(i+1)} \quad \forall t \in \mathbb{T}, \text{P-a.s.,}
\]
for every \(i \in \{1, \ldots, n-1\}\), which follows since \(\Lambda_t^{(i,i+1)}\) as defined above is symmetric with all eigenvalues non-negative due to \(X_t^{(i,j)} \leq X_t^{(i+1,j)}\) for all \(t \in \mathbb{T}\), \(\text{P-a.s.}\) for any fixed \(j = 1, \ldots, m\) from proposition 2.2.

Such Hermitian systems can be used in solving quadratic optimization problems that can generate order-preserving efficient-frontiers where the controller is interested in producing truncated distributions of optimal solutions dictated by covariance structures bounded in the Loewner sense (see [22] for details) – note that elements in \(\mathbb{H}_{+}^{m \times m} \cap \mathbb{R}^{m \times m}\) are covariance matrices. Extending [22], we can now solve optimization problems of the form:

\[
\begin{align*}
\min_{w_t} & \quad \gamma^* w_t^T H_t^{(i+1)} w_t - \beta_t^T w_t \\
\text{s.t.} & \quad Aw_t \preceq b \\
& \quad Cw_t = d \\
& \quad H_t^{(i)} \preceq H_t^{(i+1)} \preceq H_t^{(i+2)}
\end{align*}
\]

for all \(t \in \mathbb{T}\), where \(\gamma \in \mathbb{R}_+, A \in \mathbb{R}^{k \times n}, b \in \mathbb{R}^k, C \in \mathbb{R}^{l \times m}, d \in \mathbb{R}^l, \beta \in \mathbb{R}^m\) and

\[
H_t^{(i)}, H_t^{(i+1)}, H_t^{(i+2)} \in \mathbb{H}_+^{m \times m} \cap \mathbb{R}^{m \times m}
\]
for some $k, l \in \mathbb{N}_+$—here $w_t \in \mathbb{R}^m$ is the control variable. For our next result, we define the following maps:

\[
\begin{align*}
\gamma^{*H(0)}(t) & = \arg\min_{\gamma} \gamma^\top H^{(0)}_t w_t - \beta^{\top} w_t \text{ s.t. } A w_t \leq b \text{ and } C w_t = d, \\
\gamma^{*H(i+1)}(t) & = \arg\min_{\gamma} \gamma^\top H^{(i+1)}_t w_t - \beta^{\top} w_t \text{ s.t. } A w_t \leq b \text{ and } C w_t = d, \\
\gamma^{*H(i+2)}(t) & = \arg\min_{\gamma} \gamma^\top H^{(i+2)}_t w_t - \beta^{\top} w_t \text{ s.t. } A w_t \leq b \text{ and } C w_t = d,
\end{align*}
\]

for every $t \in T$ and $\gamma \in \gamma$ and generate the following maps from the optimal solutions in (32)–(34) given by

\[
\begin{align*}
G^{H(0)}_t & \leftarrow \left\{ \left( \gamma^{*H(0)}(t) \right)^\top H^{(0)}_t \gamma^{*H(0)}(t) \right\} : \forall \gamma \in \gamma, \\
G^{H(i+1)}_t & \leftarrow \left\{ \left( \gamma^{*H(i+1)}(t) \right)^\top H^{(i+1)}_t \gamma^{*H(i+1)}(t) \right\} : \forall \gamma \in \gamma, \\
G^{H(i+2)}_t & \leftarrow \left\{ \left( \gamma^{*H(i+2)}(t) \right)^\top H^{(i+2)}_t \gamma^{*H(i+2)}(t) \right\} : \forall \gamma \in \gamma.
\end{align*}
\]

We are interested in possible order-preserving relationships between the efficient-frontier curves $G^{H(0)}_t$, $G^{H(i+1)}_t$ and $G^{H(i+2)}_t$ for any $t \in T$. Accordingly, we introduce the following order relationship across these curves.

**Definition 2.16.** Let $\preceq^R$ be a binary relation such that

\[
G^{H(i)}_t \preceq^R G^{H(i+1)}_t \preceq^R G^{H(i+2)}_t
\]

if and only if

\[
\gamma^{*H(0)}(t)^\top H^{(0)}_t \gamma^{*H(0)}(t) \leq \gamma^{*H(i+1)}(t)^\top H^{(i+1)}_t \gamma^{*H(i+1)}(t) \leq \gamma^{*H(i+2)}(t)^\top H^{(i+2)}_t \gamma^{*H(i+2)}(t),
\]

for all $\gamma \in \gamma$.

Definition 2.16 essentially tells us that $G^{H(i)}_t$, $G^{H(i+1)}_t$ and $G^{H(i+2)}_t$ preserve a particular type of order at that given $t \in T$, and $G^{H(i+1)}_t$ is sandwiched in the sense of (35). We shall prove a result that provides us with sufficient (but not strictly necessary) conditions to guarantee the existence of efficient frontiers that abide the order in (35).

**Proposition 2.17.** Keep the setup of proposition 2.2 and (31). If the conditions

\[
\begin{align*}
\beta^{\top} \left( \gamma^{*H(0)}(t) - \gamma^{*H(i+1)}(t) \right) & \leq 0, \\
\beta^{\top} \left( \gamma^{*H(i+2)}(t) - \gamma^{*H(i+1)}(t) \right) & \geq 0
\end{align*}
\]

are satisfied for all $\gamma \in \gamma$, then the following holds:

\[
G^{H(i)}_t \preceq^R G^{H(i+1)}_t \preceq^R G^{H(i+2)}_t \text{ a.s.}
\]
**Proof.** From (31), denote each objective function as follows:

\[
\begin{align*}
    f^{H^{(i+1)}}(w_t) &= \gamma + w_t^T H_t^{(i+1)} w_t - \beta_t^T w_t \\
    f^{H^{(i+1)}}(w_t) &= \gamma + w_t^T H_t^{(i+1)} w_t - \beta_t^T w_t \\
    f^{H^{(i+2)}}(w_t) &= \gamma + w_t^T H_t^{(i+2)} w_t - \beta_t^T w_t
\end{align*}
\]

Each objective function produces a convex surface in \(\mathbb{R}^{m+1}\) with respect to \(w_t\), where the minimums are given by \(w_t^* H^{(i)}(\gamma), w_t^* H^{(i+1)}(\gamma)\) and \(w_t^* H^{(i+2)}(\gamma)\), respectively, such that \(A w_t \preceq b\) and \(C w_t = d\). From proposition 2.2 and the construction in (30), we know that

\[
H_t^{(i)} \preceq H_t^{(i+1)} \preceq H_t^{(i+2)}
\]

for all \(t \in T\), \(\mathbb{P}\)-a.s. We thus have the following:

\[
\begin{align*}
    (H_t^{(i+1)} - H_t^{(i)}) &\in \mathbb{H}^{m \times m}_+ \\
    (H_t^{(i+2)} - H_t^{(i+1)}) &\in \mathbb{H}^{m \times m}_+ \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

which implies that the following must hold:

\[
\begin{align*}
    w_t^T H_t^{(i)} w_t &\leq w_t^T H_t^{(i+1)} w_t \quad \mathbb{P}\text{-a.s.}, \\
    w_t^T H_t^{(i+2)} w_t &\geq w_t^T H_t^{(i+1)} w_t \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

for any \(w_t \in \mathbb{R}^m\). Therefore,

\[
f^{H^{(i)}}(w_t) \leq f^{H^{(i+1)}}(w_t) \leq f^{H^{(i+2)}}(w_t) \quad \mathbb{P}\text{-a.s.}
\]

for any \(w_t \in \mathbb{R}^m\), which means that over the same feasible region generated by \(A w_t \preceq b\) and \(C w_t = d\), we must have

\[
f^{H^{(i)}}(w_t^* H^{(i)}) \leq f^{H^{(i+1)}}(w_t^* H^{(i+1)}) \leq f^{H^{(i+2)}}(w_t^* H^{(i+2)}) \quad \mathbb{P}\text{-a.s.}
\]

(40)

When we write (40) explicitly and add \(\beta_t^T w_t^* H^{(i+1)}(\gamma)\) on every side, we get

\[
\begin{align*}
    w_t^* H^{(i)}(\gamma)^T H_t^{(i)}(w_t^* H^{(i)}) - \beta_t^T (w_t^* H^{(i)} - w_t^* H^{(i+1)}(\gamma)) \\
    \leq w_t^* H^{(i+1)}(\gamma)^T H_t^{(i+1)}(w_t^* H^{(i+2)}(\gamma)) \\
    \leq w_t^* H^{(i+2)}(\gamma)^T H_t^{(i+2)}(w_t^* H^{(i+3)}(\gamma)) \\
    \geq w_t^* H^{(i+1)}(\gamma)^T H_t^{(i+1)}(w_t^* H^{(i+2)}(\gamma))
\end{align*}
\]

\(\mathbb{P}\)-a.s. for all \(\gamma \in \gamma\). Since we operate in \(\mathbb{H}^{m \times m}_+\), the following hold:

\[
\begin{align*}
    w_t^* H^{(i)}(\gamma)^T H_t^{(i)}(w_t^* H^{(i)}(\gamma)) &\geq 0, \\
    w_t^* H^{(i+1)}(\gamma)^T H_t^{(i+1)}(w_t^* H^{(i+1)}(\gamma)) &\geq 0, \\
    w_t^* H^{(i+2)}(\gamma)^T H_t^{(i+2)}(w_t^* H^{(i+2)}(\gamma)) &\geq 0
\end{align*}
\]

for any \(\gamma \in \gamma\). Using the positivity properties above, we have

\[
\begin{align*}
    w_t^* H^{(i)}(\gamma)^T H_t^{(i)}(w_t^* H^{(i)}(\gamma)) &\leq w_t^* H^{(i+1)}(\gamma)^T H_t^{(i+1)}(w_t^* H^{(i+1)}(\gamma))
\end{align*}
\]

(41)
Efficient-frontiers are models to quantify risk-return trade-offs in decision-making (see [31, 32]) and the relation in (38) provides a particular way to order risk via definition (2.16). For demonstration, figure 4. displays an example where the efficient-frontiers satisfy the order given in (38).

We believe such Loewner-order preserving matrix constructs are relevant in the fields of control theory, operations research and economics, where one can produce stochastic covariance processes that evolve between pairs of other stochastic covariance processes in the Loewner-sense, which in turn generate order-preserving efficient-frontier dynamics as in (38).

3. Conclusion

The main objective of this paper has been to establish a theoretical framework for representing random systems where paths of stochastic processes can collide but cannot cross each other, by virtue of preserving their initially given order over their lifetime. The main objective is geared towards building a robust but an adaptable mathematical foundation, rather than discussing in detail the practical applications which are introduced to spur new ideas but are left for future research. The proposed setup includes multiple stochastic paths as solutions to SDEs defined through their drift and diffusion coefficients interacting with each other. The particular nature of these cross-communicating coefficients that satisfy certain regularity conditions forms the basis of how these paths draw in space such stochastic trajectories that in turn manifest as stochastic boundaries to one another. This construction includes cases of potential collisions that would either cause particles to bounce away from each other, or to bind to one another, depending on the coupling force characterised through their drift components. We show that, if all interacting particles follow a martingale behaviour, the system ultimately reaches a random steady-state with zero fluctuation at the time of final collision, forcing all paths to necessarily remain attached to one another thereafter. We also conclude that the manifestation of the captive diffusion topology, as introduced in [18], boils down to being a specific case of our proposed framework. Furthermore, we prove that order-preserving coupled systems are invariant under monotonic transformations, which enables one to transition from simple models to
more sophisticated, if not more realistic, representations of certain physical dynamics. One of the key discussions in the paper revolves around the framework’s high degree of flexibility in allowing stochastic paths to endogenously sculpt the geometry of their own system, which brings a level of self-induced dynamic control to the system through order-preservation. Due to the versatility of our framework, supported by numerical simulations to inspire imagination, we envisage its use in many areas such as particle physics, microbiology and control theory as mentioned in the paper (e.g. the modelling, calibration and simulation of interacting particle systems, where collisions may exhibit either reflection or absorption dynamics against each other).

**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

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**Appendix**

We shall also show how we can extend the framework by introducing a near-collision setup via controlling the distance between any two stochastic paths \( \{X^{(i)}_t\}_{t \in \mathbb{T}} \) and \( \{X^{(i+1)}_{t}\}_{t \in \mathbb{T}} \) to remain greater than or equal to some constant \( \Delta^{(i)} \in \mathbb{R}_+ \). To be precise, we shall provide a generalization that allows these processes to get only \( \Delta^{(i)} \)-ball near each other such that

\[
P(X^{(i+1)}_t - X^{(i)}_t \geq \Delta^{(i)} : \forall i \in \mathcal{I} \setminus \{n\} \bigg| X^{(i+1)}_0 - X^{(i)}_0 \geq \Delta^{(i)} : \forall i \in \mathcal{I} \setminus \{n\}) = 1 \quad \forall t \in \mathcal{T}.
\]

(43)

We believe such an extension is meaningful for reaching a wider avenue of applications. In order to achieve this, we augment definition 2.1 as follows.

**Definition 4.1.** Let \( \{X^{(i)}_t\}_{t \in \mathbb{T}} \subseteq C(\mathbb{R} \times \mathbb{T}) \) for \( i \in \mathcal{I} \) be a solution to a stochastic differential equations (SDEs) governed by

\[
X^{(i)}_t = x^{(i)}_0 + \int_0^t \mu^{(i)}(s,X_t) \, ds + \int_0^t \sigma^{(i)}(s,X_t) \, dM^{(i)}_s,
\]

given that \( X^{(i)}_0 = x^{(i)}_0 \), where \( \mu^{(i)} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R} \) and \( \sigma^{(i)} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n \) are measurable continuous maps, and \( \{M^{(i)}_t\}_{t \in \mathbb{T}} \subseteq \mathcal{M}(\mathbb{R} \times \mathbb{T}) \). The coordinates of \( \{X_t\}_{t \in \mathbb{T}} \) form a \( \Delta \)-buffered order-preserving coupled system if

(a) \( x^{(i+1)}_0 - x^{(i)}_0 \geq \Delta^{(i)} \) for some \( \Delta^{(i)} \geq 0 \),

(b) \( \mu^{(i)}(t, X_t, i; X^{(i+1)}_t - \Delta^{(i)}) \leq \mu^{(i+1)}(t, X_t, i+1; X^{(i+1)}_t + \Delta^{(i)}) \),

(c) \( \sigma^{(i)}(t, X_t, i; X^{(i+1)}_t - \Delta^{(i)}) = \sigma^{(i+1)}(t, X_t, i+1; X^{(i+1)}_t + \Delta^{(i)}) = 0 \),

for any \( t \in \mathbb{T} \) when \( X^{(i+1)}_t - X^{(i)}_t = \Delta^{(i)} \) for every \( i \in \{1, \ldots, n-1\} \).
Figure 5. Sample paths for (44) with $\kappa = 1$.

Note that we recover definition 2.1 when we set $\Delta^{(i)} = 0$ for every $i \in I$. We also have the following result that extends proposition 2.2.

**Proposition 4.2.** Let $\{X_i\}_{i \in I}$ be a $\Delta$-buffered order-preserving coupled system as in definition 4.1. Then,

$$X_t^{(i+1)} - X_t^{(i)} \geq \Delta^{(i)}, \quad \mathbb{P}\text{-a.s.,}$$

for all $t \in \mathbb{T}$ and every $i \in \{1, \ldots, n-1\}$.

We omit the proof of proposition 4.2 due to its similarity to that of proposition 2.2, whereby right-derivatives exist whenever $X_t^{(i+1)} - X_t^{(i)} = \Delta^{(i)}$ holds for any $t \in \mathbb{T}$ and $i \in \{1, \ldots, n-1\}$. Accordingly, we can extend all the examples and results we presented in section 2.

**Remark 4.3.** From proposition 4.2, we also see that

$$X_t^{(i+k)} - X_t^{(i)} \geq \sum_{j=i}^{i+k-1} \Delta^{(j)}, \quad \mathbb{P}\text{-a.s.,}$$

for $1 \leq k \leq n-i$.

**Example 4.4.** Set $n = 2$ and fix $x_0^{(2)} - x_0^{(1)} \geq \Delta^{(1)} \geq 0$. The following is a $\Delta$-buffered order-preserving coupled system:

$$X_t^{(1)} = x_0^{(1)} + \kappa \int_0^t \left( f_t^{(1)} - X_s^{(1)} + X_s^{(2)} - \Delta^{(1)} \right) ds + \alpha \int_0^t \left( X_s^{(2)} - X_s^{(1)} - \Delta^{(1)} \right) dW_t^{(1)},$$

$$X_t^{(2)} = x_0^{(2)} + \kappa \int_0^t \left( f_t^{(2)} - X_s^{(1)} + X_s^{(2)} - \Delta^{(1)} \right) ds + \beta \int_0^t \left( X_s^{(2)} - X_s^{(1)} - \Delta^{(1)} \right) dW_t^{(2)},$$

(44)

where $\{W_t^{(1)}\}_{t \in \mathbb{T}}$ and $\{W_t^{(2)}\}_{t \in \mathbb{T}}$ are $(\mathbb{P}, \{\mathcal{F}_t\})$ Brownian motions, $\{f_t^{(1)}\}_{t \in \mathbb{T}}$ and $\{f_t^{(2)}\}_{t \in \mathbb{T}}$ are deterministic $\mathbb{R}$-valued functions such that $f_t^{(1)} \leq f_t^{(2)}$ for all $t \in \mathbb{T}$ and the parameters satisfy: $\kappa \in [0, \infty)$, $0 < |\alpha| < \infty$ and $0 < |\beta| < \infty$. See figure 5 for demonstrative sample paths.

This model extends example 2.3 through the additional term $\Delta^{(1)}$. 

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We shall not present the extended versions of all the results from section 2 for \( \Delta \)-buffered order-preserving coupled systems, and leave them to the interested reader, but as an example, the following result generalises proposition 2.11. In order to avoid repetition, we omit its proof.

**Proposition 4.5.** Let each coordinate of \( \{X_i\}_{i \in \mathbb{T}} \) be a \((\mathbb{P}, \{\mathcal{F}_i^X\})\)-martingale and let the random variable \( \tau^{(i,i+1)} \) be given by

\[
\tau^{(i,i+1)} = \inf \left\{ t : X_i^{(i+1)} - X_i^{(i)} = \Delta^{(i)} \right\}
\]

(45)

as the first time \( \{X_i^{(i)}\}_{i \in \mathbb{T}} \) vs \( \{X_i^{(i+1)}\}_{i \in \mathbb{T}} \) has a distance of \( \Delta^{(i)} \) if \( \tau^{(i,i+1)} \in \mathbb{T} \). Then,

\[
\tau^{(i,i+1)} \leq t \leq T \Rightarrow X_i^{(i+1)} - X_i^{(i)} = \Delta^{(i)} \forall t \in \left[ \tau^{(i,i+1)}, T \right].
\]

(46)

and the pair reaches their steady-state such that

\[
X_i^{(i)} = X_i^{(i)} \text{ and } X_i^{(i+1)} = X_i^{(i+1)} \forall t \in \left[ \tau^{(i,i+1)}, T \right].
\]

(47)

We believe \( \Delta \)-buffered order-preserving coupled processes may find applications in modelling electromagnetic fields with charged stochastic particles that repel each other at the \( \Delta \)-level. We leave this application for future research.

**A. \( \Delta \)-buffered square-root diffusions**

Finally, we shall show that square-root diffusions—also known as Cox–Ingersoll–Ross processes—form a subclass of order-preserving coupled systems, and then use this connection to naturally extend square-root diffusions as part of \( \Delta \)-buffered order-preserving coupled systems. A (standard) square-root diffusion, which we shall denote as \( \{X_i^{(2)}\}_{i \in \mathbb{T}} \), is governed by the following SDE:

\[
X_i^{(2)} = x_0^{(2)} + \kappa \int_0^t \left( \theta - X_i^{(2)} \right) ds + \beta \int_0^t \sqrt{X_i^{(2)}} dW_s,
\]

for all \( t \in \mathbb{T} \), where \( x_0^{(2)} \geq 0, \kappa, \theta \in (0, \infty) \) and \( \beta \in \mathbb{R} \setminus \{0\} \). Note that if we set \( n = 2 \), where \( X_i^{(1)} = 0 \) for all \( t \in \mathbb{T} \), then \( \{X_i^{(1)}, X_i^{(2)}\}^{\top}_{i \in \mathbb{T}} \) forms an order-preserving coupled system as in definition 2.1 that is reflective as in corollary 2.5. Accordingly, we can extend this model to what we call \( \Delta \)-Buffered Square-Root Diffusions using definition 4.1 by setting

\[
X_i^{(1)} = 0
\]

\[
X_i^{(2)} = x_0^{(2)} + \kappa \int_0^t \left( \theta - X_i^{(2)} + \Delta \right) ds + \beta \int_0^t \sqrt{\left( X_i^{(2)} - \Delta \right)} dW_s,
\]

for all \( t \in \mathbb{T} \), where \( x_0^{(2)} \geq \Delta \geq 0, \kappa, \theta \in (0, \infty) \) and \( \beta \in \mathbb{R} \setminus \{0\} \). As such, \( \{X_i^{(2)}\}_{i \in \mathbb{T}} \) can never go below the \( \Delta \)-level.

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