Backward parabolic Ito equations and second fundamental inequality

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Abstract

Regularity of solutions is studied for backward stochastic parabolic Ito equations. An analog of the second energy inequality and the related existence theorem are obtained for domains with boundary.

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1 Introduction

The paper studies stochastic partial differential equations (SPDEs) in a cylinder $D \times [0, T]$ with a Dirichlet boundary condition on $\partial D$, for a region $D \subseteq \mathbb{R}^n$. We investigate regularity properties of the backward equations, i.e., equations with Cauchy condition at the final time. The difference between backward and forward equations is not that important for the deterministic equations since a deterministic backward equation can be converted to a forward equation by a time change. It cannot be done so easily for stochastic equations, because we look for solutions adapted to the driving Brownian motion. It is why the backward SPDEs require special consideration. The most common approach is to consider the so-called Bismut backward equations such the diffusion term is not given a priori but needs to be found. These approach was introduced first for ordinary linear backward stochastic equations. The backward SPDEs with similar features were widely studied (see, e.g., Pardoux and Peng (1990), Hu and Peng (1991), Dokuchaev (1992), (2003), Yong and Zhou (1999), Pardoux and Rascani (1998), Ma and Yong (1999), Hu et al (2002),
Confortola (2007), and the bibliography there given). Backward parabolic SPDEs represent analogs of backward parabolic Kolmogorov equations for non-Markov Ito processes, including the case of bounded domains, so they can be used for characterization of distributions of the first exit times in non-Markovian setting, as was shown by the author (1992, 2010a). A different type of backward equations was described in Chapter 5 of Rozovskii (1990). Forward SPDEs were also widely studied (see, e.g., Alós et al (1999), Bally et al (1994), Chojnowska-Michalik and Goldys (1995), Da Prato and Tubaro (1996), Gyöngy (1998), Krylov (1999), Maslowski (1995), Pardoux (1993), Rozovskii (1990), Walsh (1986), Zhou (1992), Dokuchaev (1995), (2002), (2005), and the bibliography there given).

For linear PDEs, existence and uniqueness at different spaces is expressed traditionally via a priori estimates, when a norm of the solution is estimated via a norm of the free term. For the second order equations, there are two most important estimates based on the $L_2$-norm: so-called “the first energy inequality” or “the first fundamental inequality”, and “the second energy inequality”, or ”the second fundamental inequality” (Ladyzhenskaya (1985)). For instance, consider a boundary value problem for the heat equation

$$u'_t = u''_{xx} + \varphi, \quad \varphi = f'_x + g,$$

$$u|_{t=0} = 0, \quad u|_{\partial D} = 0, \quad (x,t) \in Q = D \times [0,1], \quad D \subset \mathbb{R}.$$  \hspace{1cm} (1.1)

Then the first fundamental inequality is the estimate

$$\|u'_x\|^2_{L_2(Q)} + \|u\|^2_{L_2(Q)} \leq \text{const} (\|f\|^2_{L_2(Q)} + \|g\|^2_{L_2(Q)}).$$

Respectively, the second fundamental inequality is the estimate

$$\|u\|^2_{L_2(Q)} + \|u'_x\|^2_{L_2(Q)} + \|u''_{xx}\|^2_{L_2(Q)} \leq \text{const} \|\varphi\|^2_{L_2(Q)}.$$

Note that the second fundamental inequality leads to existence theorem in the class of solutions such that $u''_{xx} \in L_2(Q)$, and the first fundamental inequality leads to existence theorem in the class of solutions such that $u'_x \in L_2(Q)$, i.e., with generalized distributional derivatives $u''_{xx}$ only. For the problem without boundary value condition, with smooth coefficients, and one-dimensional $x \in \mathbb{R}$, the second fundamental inequality can be derived from the first fundamental inequality; it suffices to apply the first fundamental inequality for the parabolic equation for $u'_x$. (For the vector case of $x \in \mathbb{R}^n$, it would be more difficult since $u'_x$ is a vector satisfying a system of $n$ parabolic equations). For the problems with boundary value conditions, this approach does not work even for one-dimensional case, since the boundary values on $\partial D$ for $u'_x$ are unknown a
priori. It is why the second fundamental inequality needs to be derived separately using special methods.

For forward parabolic SPDEs, analogs of the first and the second fundamental inequalities are known. These results are summarized in Lemma 3.1 below. The first fundamental inequality for forward SPDEs in bounded domains with Dirichlet boundary condition was known long time ago (see, e.g., Rozovskii (1990)). Moreover, similar results are also known for forward SPDEs of an arbitrary high order $2m \geq 2$; in this setting, the analog of "the first fundamental inequality" is an estimate for $E\|u(\cdot,t)\|_{W^m(D)}$ (Rozovskii (1991)). In addition, a priori estimates without Dirichlet conditions, i.e., in the entire space, are known for a general setting that covers both first and second fundamental inequalities (Krylov (1999)). On the other hand, "the second fundamental inequality" for the problem with boundary conditions was more difficult to obtain. Related complications were discussed in Krylov (1999), p. 237 and in Dokuchaev (2005). Kim (2004) obtained a priori estimates for forward parabolic SPDEs for special weighted norms that devaluates boundary values; for the case of $L_2$-norms, these estimates can be interpreted as analogs of "the second fundamental inequality"; they are similar to estimates $\|r_1u''_{xx}\|_{L_2(Q)} \leq \text{const} \|r_2\varphi\|_{L_2(Q)}$ for the problem (1.1), where $r_i$ are some weight functions such that $r_i(x) \to 0$ as $x$ approaching $\partial D$. For the standard non-weighted Sobolev norms, the second fundamental inequality" was obtained in Dokuchaev (2005).

For the backward parabolic equations with Dirichlet boundary conditions, an analog of the first fundamental inequality is known (Zhou (1992), Dokuchaev (1992), (2003)). In fact, the duality relationship between forward and backward equations makes it sufficient to prove the first fundamental inequality for any one type of these two types of equations. (By duality we mean equations (6.1) connecting the solutions of SPDEs (3.2) and (3.3) respectively). However, this approach does not work for the second fundamental inequality in a bounded domain $D$, since it requires to study an adjoint equation with the free term taking values in the space $(W^2_2(D))^*$ which is too wide. It was unknown if the second fundamental inequality holds in this case.

In the present paper, we study again existence, uniqueness, and a priori estimates for solutions for backward SPDEs. As was mentioned above, the first and the second fundamental inequalities for the forward SPDEs had been proved, as well as the first fundamental inequality for the backward SPDEs, so we concentrate our efforts on the remaining problem: to investigate if an analog of the second fundamental inequality holds for the backward equations. We found sufficient conditions that ensure that the second fundamental inequality and the related
existence theorem holds (Theorem 4.1). To ensure this regularity, we required additional Condition 4.1 which is a strengthened version of the standard coercivity condition (Condition 3.1). Without this new condition, the second fundamental inequality is still not established.

Some examples of applications are discussed in Section 5.

2 Definitions

2.1 Spaces and classes of functions.

Assume that we are given an open domain $D \subseteq \mathbb{R}^n$ such that either $D = \mathbb{R}^n$ or $D$ is bounded with $C^2$-smooth boundary $\partial D$. Let $T > 0$ be given, and let $Q \triangleq D \times (0, T)$.

We are given a standard complete probability space $(\Omega, \mathcal{F}, P)$ and a right-continuous filtration $\mathcal{F}_t$ of complete $\sigma$-algebras of events, $t \geq 0$. We are given also a $N$-dimensional process $w(t) = (w_1(t), \ldots, w_N(t))$ with independent components such that it is a Wiener process with respect to $\mathcal{F}_t$.

We denote by $\| \cdot \|_X$ the norm in a linear normed space $X$, and $(\cdot, \cdot)_X$ denote the scalar product in a Hilbert space $X$.

We introduce some spaces of real valued functions.

Let $G \subset \mathbb{R}^k$ be an open domain, then $W^m_q(G)$ denote the Sobolev space of functions that belong to $L^q(G)$ with the distributional derivatives up to the $m$th order, $q \geq 1$.

We denote by $| \cdot |$ the Euclidean norm in $\mathbb{R}^k$, and we denote by $\overline{G}$ the closure of a region $G \subset \mathbb{R}^k$.

Let $H^0 \triangleq L_2(D)$, and let $H^1_0 \triangleq W^1_2(D)$ be the closure in the $W^1_2(D)$-norm of the set of all smooth functions $u : D \to \mathbb{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W^2_2(D) \cap H^1$ be the space equipped with the norm of $W^2_2(D)$. The spaces $H^k$ are Hilbert spaces, and $H^k$ is a closed subspace of $W^k_2(D)$, $k = 1, 2$.

Let $H^{-1}$ be the dual space to $H^1$, with the norm $\| \cdot \|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$. $H^{-1}$ is a Hilbert space.

We will write $(u, v)_{H^0}$ for $u \in H^{-1}$ and $v \in H^1$, meaning the obvious extension of the bilinear form from $u \in H^0$ and $v \in H^1$.

We denote by $\ell_k$ the Lebesgue measure in $\mathbb{R}^k$, and we denote by $\mathcal{B}_k$ the $\sigma$-algebra of Lebesgue sets in $\mathbb{R}^k$.

We denote by $\mathcal{P}$ the completion (with respect to the measure $\ell_1 \times \mathcal{P}$) of the $\sigma$-algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to
that are progressively measurable for any \( x \) with the norm \( \| u \| \).

Let \( \xi \) be given. Then the sequence of the integrals

\[
\int_0^t (\xi_k(x,s,\omega) - \xi_m(x,s,\omega)) \, dw_j(s) \, ds
\]

converges in \( Z_t^0 \) as \( k \to \infty \), and its limit depends on \( \xi \), but does not depend on \( \{\xi_k\} \).

**Proof** follows from completeness of \( X^0 \) and from the equality

\[
E \int_0^t \|\xi_k(\cdot,s,\omega) - \xi_m(\cdot,s,\omega)\|_{H^0}^2 \, ds = \int_D dx E \left( \int_0^t (\xi_k(x,s,\omega) - \xi_m(x,s,\omega)) \, dw_j(s) \right)^2.
\]

**Definition 2.1** Let \( \xi \in X^0 \), \( t \in [0,T] \), \( j \in \{1,\ldots,N\} \), then we define \( \int_0^t \xi(x,s,\omega) \, dw_j(s) \) as the limit in \( Z_t^0 \) as \( k \to \infty \) of a sequence \( \int_0^t \xi_k(x,s,\omega) \, dw_j(s) \), where the sequence \( \{\xi_k\} \) is such as in Proposition 2.1.

Sometimes we will omit \( \omega \).

## 3 Review of existence theorems for forward and backward SPDEs

Let \( (x,t) \in Q \), \( \omega \in \Omega \).

Consider the functions \( b(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^{n \times n} \), \( f(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^n \), \( \lambda(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R} \), \( \beta_j(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^n \), \( \beta \bar{i}(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R} \) that are progressively measurable for any \( x \in \mathbb{R}^n \) with respect to \( \mathcal{F}_t \).
Consider differential operators defined on functions $v : D \to \mathbb{R}$

\[
\mathcal{A}v = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left( b_{ij}(x,t,\omega) v(x) \right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( f_i(x,t,\omega) v(x) \right) + \lambda(x,t,\omega) v(x),
\]

\[
B_k v = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \beta_k(x,t,\omega) v(x) \right) + \beta_k(x,t,\omega) v(x), \quad k = 1, \ldots, N.
\]

Here $b_{ij}, f_i, x_i$ are the components of $b, f,$ and $x$.

Further, consider the operators being formally adjoint to the operators $\mathcal{A}$ and $B_k$:

\[
\mathcal{A}^* v \triangleq \sum_{i,j=1}^{n} b_{ij}(x,t,\omega) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} f_i(x,t,\omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x,t,\omega)v(x),
\]

\[
B_k^* v \triangleq \frac{dv}{dx}(x) \beta_k(x,t,\omega) + \beta_k(x,t,\omega) v(x), \quad k = 1, \ldots, N. \tag{3.1}
\]

To proceed further, we assume that Conditions 3.1 and 3.2 remain in force throughout this paper.

**Condition 3.1** (Coercivity) The matrix $b = b^\top$ is symmetric, bounded, and progressively measurable with respect to $\mathcal{F}_t$ for all $x$, and there exists a constant $\delta_1 > 0$ such that

\[
y^\top b(x,t,\omega) y - \frac{1}{2} \sum_{i=1}^{N} |y^\top \beta_i(x,t,\omega)|^2 \geq \delta_1 |y|^2 \quad \forall y \in \mathbb{R}^n, \, (x,t) \in D \times [0,T], \, \omega \in \Omega.
\]

**Condition 3.2** The functions $\lambda(x,t,\omega) : \mathbb{R}^n$ and $\beta_i(x,t,\omega)$ are bounded. The functions $b(x,t,\omega) : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^{n \times n}, f(x,t,\omega) : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^n, \lambda(x,t,\omega) : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}, \beta_i(x,t,\omega)$ and $\beta_i(x,t,\omega)$ are differentiable in $x$ and bounded in $(x,t,\omega)$, and

\[
es \sup_{x,t,\omega} \left[ \left| \frac{\partial b}{\partial x}(x,t,\omega) \right| + \left| \frac{\partial f}{\partial x}(x,t,\omega) \right| + \left| \frac{\partial \beta_i}{\partial x}(x,t,\omega) \right| < +\infty, \quad i = 1, \ldots, N. \right.
\]

We introduce the set of parameters

\[
\mathcal{P}_1 \triangleq \left( n, \, D, \, T, \, \delta, \, \operatorname{ess} \sup_{x,t,\omega} \left[ |b(x,t,\omega)| + |f(x,t,\omega)| + \left| \frac{\partial b}{\partial x}(x,t,\omega) \right| + \left| \frac{\partial f}{\partial x}(x,t,\omega) \right| \right], \right.
\]

\[
\left. \operatorname{ess} \sup_{x,t,\omega,i} \left[ |\beta_i(x,t,\omega)| + |\beta_i(x,t,\omega)| + \left| \frac{\partial \beta_i}{\partial x}(x,t,\omega) \right| \right] \right). \right.
\]

**Boundary value problems for forward and backward equations**

Let $s \in [0,T)$, $\varphi \in X^{-1}$, $h_i \in X^0$, and $\Phi, \Psi \in Z_v^0$. Consider the boundary value problem in $D \times [s,T]$

\[
d_t u = (\mathcal{A} u + \varphi) dt + \sum_{i=1}^{N} (B_i u + h_i) dw_i(t), \quad t > s,
\]

\[
u|_{t=s} = \Phi, \quad u(x,t,\omega)|_{x \in \partial D} = 0. \tag{3.2}
\]
The corresponding SPDE is a forward equation. Here $u = u(x, t, \omega)$, $(x, t) \in Q$, $\omega \in \Omega$.

Inequality (3.2) means that equation (3.2) is coercive or superparabolic, in the terminology of Rozovskii (1990).

Further, let $\xi \in X^{-1}$, and $\Psi \in Z^0_T$. Consider the boundary value problem in $Q$

$$d_t p + \left( A^* p + \sum_{i=1}^N B^*_i \chi_i + \xi \right) dt = \sum_{i=1}^N \chi_i dw_i(t), \quad t < T,$$

$$p|_{t=T} = \Psi, \quad p(x, t, \omega)|_{x \in \partial D} = 0.$$

(3.3)

The corresponding SPDE is a backward equation. Here $p = p(x, t, \omega)$, $\chi_i = \chi_i(x, t, \omega)$, $(x, t) \in Q$, $\omega \in \Omega$.

The definition of solution

Definition 3.1 Let $h_i \in X^0$ and $\varphi \in X^{-1}$. We say that equations (3.2) are satisfied for $u \in Y^1$ if

$$u(\cdot, t) = \Phi + \int_s^t \left( A u(\cdot, r) + \varphi(\cdot, r) \right) dr + \sum_{i=1}^N \int_s^t (B_i u(\cdot, r) + h_i(\cdot, r)) dw_i(r)$$

(3.4)

for all $t$ such that $s < t \leq T$, and this equality is satisfied as an equality in $Z^{-1}_T$.

Definition 3.2 We say that equation (3.3) is satisfied for $p \in Y^1$, $\Psi \in Z^0_T$, $\chi_i \in X^0$ if

$$p(\cdot, t) = \Psi + \int_t^T \left( A^* p(\cdot, s) + \sum_{i=1}^N B^*_i \chi_i(\cdot, s) + \xi(\cdot, s) \right) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) dw_i(s)$$

(3.5)

for any $t \in [0, T]$. The equality here is assumed to be an equality in the space $Z^{-1}_T$.

Note that the condition on $\partial D$ is satisfied in the following sense: $u(\cdot, t, \omega) \in H^1$ and $p(\cdot, t, \omega) \in H^1$ for a.e. $t, \omega$. Further, $u, p \in Y^1$, and the value of $u(\cdot, t)$ or $p(\cdot, t)$ is uniquely defined in $Z^0_T$ given $t$, by the definitions of the corresponding spaces. The integrals with $dw_i$ in (3.4) (3.5) are defined as elements of $Z^{-1}_T$. The integrals with $ds$ are defined as elements of $Z^{-1}_T$. (Definitions 3.1, 3.2 require for 3.3 that these integral are equal to elements of $Z^0_T$ in the sense of equality in $Z^{-1}_T$).

Existence theorems and known fundamental inequalities

The following Lemma combines the first and the second fundamental inequalities and related existence result for forward SPDEs. It gives analogs of the so-called "energy inequalities", or
"the fundamental inequalities" known for deterministic parabolic equations (Ladyzhenskaya et al (1969)).

**Lemma 3.1** Let either $k = -1$ or $k = 0$. Assume that Conditions. In addition, assume that if $k = 0$, then $\beta_i(x, t, \omega) = 0$ for $x \in \partial D$, $i = 1, ..., N$ and

$$\text{ess sup}_x \sup_{(x,t) \in Q} \left| \frac{\partial^2 b}{\partial x_k \partial x_m} (x, t, \omega) \right| < +\infty.$$ 

Let $\varphi \in X^k(s, T)$, $h_i \in X^{k+1}(s, T)$, and $\Phi \in Z^{k+1}_s$. Then problem (3.2) has an unique solution $u$ in the class $Y^1(s,T)$, and the following analog of the first fundamental inequality is satisfied:

$$\|u\|_{Y^{k+2}(s,T)} \leq c \left( \|\varphi\|_{X^k(s,T)} + \|\Phi\|_{Z^{k+1}_s} + \sum_{i=1}^N \|h_i\|_{X^{k+1}(s,T)} \right), \quad (3.6)$$

where $c = c(P_1)$ is a constant that depends on $P_1$ only.

The statement of Lemma 3.1 for $k = -1$ corresponds to the first fundamental inequality; it is a special case of Theorem 3.4.1 from Rozovskii (1990). The statement for $k = 0$ corresponds to the second energy inequality; it was obtained in Dokuchaev (2005).

The following Lemma gives the first fundamental inequalities and related existence result for backward SPDEs.

**Lemma 3.2** [Dokuchaev (1992, 2010a)] For any $\xi \in X^{-1}$ and $\Psi \in Z^0_T$, there exists a pair $(p, \chi)$ such that $p \in Y^1$, $\chi = (\chi_1, ..., \chi_N)$, $\chi_i \in X^0$, and (3.3) is satisfied. This pair is uniquely defined, and the following analog of the first fundamental inequality is satisfied:

$$\|p\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq c(\|\xi\|_{X^{-1}} + \|\Psi\|_{Z^0_T}), \quad (3.7)$$

where $c = c(P_\infty) > 0$ as a constant that does not depend on $\xi$ and $\Psi$.

Therefore, only the second fundamental inequality for backward SPDEs is missed.

### 4 The main result: the second fundamental inequality for backward equations

Starting from now, we assume that the following addition conditions are satisfied.
Condition 4.1 There exists a constant $\delta > 0$ such that
\[
\sum_{i=1}^{N} y_i^T b(x, t, \omega) y_i - \frac{1}{2} \left( \sum_{i=1}^{N} y_i^T \beta_i(x, t, \omega) \right)^2 \geq \delta \sum_{i=1}^{N} |y_i|^2
\]
\[
\forall \{y_i\}_{i=1}^{N} \subset \mathbb{R}^n, (x, t) \in D \times [0, T], \omega \in \Omega.
\]
(4.1)

For an integer $M > 0$, let $\Theta_b(M)$ denote the class of all matrix functions $b$ such that all conditions imposed in Section 3 are satisfied, and there exists a set $\{t_k\}_{i=0}^{M} = \{t_k(M)\}_{i=0}^{M}$ such that $0 = t_0 < t_1 < \cdots < t_M = T$ such that $\max_k |t_k - t_{k-1}| \to 0$ as $M \to +\infty$, and that the function $b(x, t, \omega) = b(x, \omega)$ does not depend on $t$ for $t \in [t_i, t_{i+1})$. In particular, this means that $b(x, t, \cdot)$ is $\mathcal{F}_{t_i}$-measurable for all $x \in D$, $t \in [t_i, t_{i+1})$.

Set $\Theta_b \doteq \cup_{M>0} \Theta_b(M)$.

The following Condition 4.2 is rather technical.

Condition 4.2 The matrix $b$ is such that all conditions imposed in Section 3 are satisfied, and that there exists a sequence $\{b^{(M)}\}_{M=1}^{+\infty} \subset \Theta_b$ such at least one of the following conditions is satisfied:

(i) $\|b^{(M)} - b\|_{W_1^\infty} \to 0$ as $M \to +\infty$.

(ii) Condition 4.1 is satisfied for $b$ replaced by $b^{(M)}$, with the same $\delta_1 > 0$ for all $M$, and $\|b^{(M)}(\cdot, t, \omega) - b(\cdot, t, \omega)\|_{W_1^\infty(D)} \to 0$ for a.e. $(t, \omega)$ as $i \to +\infty$.

We denote by $\tilde{\Theta}_b$ the class of all functions such $b$ that Condition 4.2 is satisfied.

To proceed further, we assume that Conditions 3.1-3.2 remain in force starting from here and up to the end of this paper, as well as the previously formulated conditions.

Let $\mathcal{P} = \{P_1, \delta_1\}$.

Theorem 4.1 For any $\xi \in X^0$ and $\Psi \in Z_1^T$, there exists a pair $(p, \chi)$, such that $p \in Y^2$, $\chi = (\chi_1, \ldots, \chi_N)$, $\chi_i \in X^1$ and (3.3) is satisfied. This pair is uniquely defined, and the following analog of the second fundamental inequality holds:
\[
\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \leq c(\|\xi\|_{X^0} + \|\Psi\|_{Z_1^T}),
\]
(4.2)
where $c > 0$ is a constant that depends only on $\mathcal{P}$.

Repeat that estimate (4.2) represents an analog of the second fundamental inequality.
On the strengthened coercivity condition

Let us discuss the properties Condition 4.1 and compare it with Condition 3.1. First, let us note that it can happen that Condition 3.1 holds but Condition 4.1 does not hold. It can be seen from the following example.

Example 4.1 Assume that \( n = 2, N = 2 \),

\[
\beta_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad b \equiv \frac{1}{2} (\beta_1 \beta_1^T + \beta_2 \beta_2^T) + 0.01I_2 = 0.51I_2,
\]

where \( I_2 \) is the unit matrix in \( \mathbb{R}^{2 \times 2} \). Obviously, Condition 3.1 holds. However, Condition 4.1 does not hold for this \( b \); to see this, it suffices to take \( y_1 = \beta_1 \) and \( y_2 = \beta_2 \).

Remark 4.1 Assume that the estimate in Condition 3.1 holds with \( \delta = 0 \) only (i.e., the forward equation is dissipative, in the terms of Rozovskii (1990)). This important model covers Kolmogorov type equations for conditional densities of non-Markov Ito processes (see Rozovsky (1990), Chapter 6, and Dokuchaev (1995)). If we approximate the operator \( \mathcal{A} \) by the operator \( \mathcal{A} + \epsilon \Delta \), where \( \Delta \) is the Laplacian, then Condition 3.1 holds for the new operator for arbitrarily small \( \epsilon > 0 \). This approximation of a dissipative equation by a coercive one is a useful tool for investigation of dissipative equations and distributions of non-Markov Ito processes. Example 4.1 shows that, unfortunately, general dissipative equations cannot be approximated by equations such that Condition 4.1 holds.

The following theorems clarify the relations between Conditions 4.1 and 3.1.

Theorem 4.2 If Condition 4.1 holds then Condition 3.1 holds.

Let us give some useful criterions of validity of Condition 4.1.

Theorem 4.3 If \( n = 1 \) and Condition 3.1 holds, then Condition 4.1 holds.

Theorem 4.4 Condition 4.1 holds if there exist \( N_0 \in \{1,...,N\} \) and \( \delta_2 > 0 \) such that \( \beta_i \equiv 0 \) for \( i > N_0 \) and

\[
y^\top b(x,t,\omega) y - \frac{N_0}{2} |y^\top \beta_i(x,t,\omega)|^2 \geq \delta_2 |y|^2 \quad \forall y \in \mathbb{R}^n, \; (x,t) \in D \times [0,T], \; \omega \in \Omega, \; i = 1,...,N_0.
\]

Corollary 4.1 If \( N = 1 \) and Condition 3.1 holds then Condition 4.1 holds.

10
5 Some applications

So far, the main application is the representation theorem for functionals of non-Markov processes and their first exit times from bounded domains. These functionals are represented via solutions of backward parabolic Itô equations. The previously known results about regularity of the solution of the backward SPDE for $p$ were insufficient for the case of domains with boundary, and the representation result was never before obtained for this case. It was done only using the additional regularity in the form of the second fundamental inequality given in Theorem 4.1 (Dokuchaev (2010b)). Therefore, this regularity result opens ways to systematics of first exit times of non-Markov processes.

In addition, a priori estimates obtained above helps to establish we can show that the solution of (3.3) is robust with respect to small in $L_\infty$ norm disturbances of the coefficients.

Consider two problems (3.3), with coefficients

$$(b, f, \lambda, \xi, \beta_i, \bar{\beta}_i, \Psi) = (b^{(k)}, f^{(k)}, \lambda^{(k)}, \xi^{(k)}, \beta_i^{(k)}, \bar{\beta}_i^{(k)}, \Psi^{(k)}), \quad k = 1, 2,$$

such that Conditions 3.1-3.2 and 4.1-4.2 are satisfied for both sets of functions. Let $\mathcal{P}^{(k)}$ be the corresponding sets of parameters. Let $(p^{(k)}, \chi_1^{(k)}, ..., \chi_N^{(k)})$ be the corresponding solutions of problem (3.3), $k = 1, 2$.

**Theorem 5.1** There exists a constant $c = c(\mathcal{P}^{(1)}, \|u^{(2)}\|_{Y^2})$ such that

$$\|p^{(1)} - p^{(2)}\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i^{(1)} - \chi_i^{(2)}\|_{X^1} \leq cM,$$

where

$$M \triangleq \text{ess sup}_{x,t,\omega} \left( |b_1^{(1)}(x,t,\omega) - b_2^{(2)}(x,t,\omega)| + |f_1^{(1)}(x,t,\omega) - f_2^{(2)}(x,t,\omega)| ight)
+ |\lambda_1^{(1)}(x,t,\omega) - \lambda_2^{(2)}(x,t,\omega)| + \sum_{i=1}^{N} |\beta_i^{(1)}(x,t,\omega) - \beta_i^{(2)}(x,t,\omega)|
+ \sum_{i=1}^{N} |\bar{\beta}_i^{(1)}(x,t,\omega) - \bar{\beta}_i^{(2)}(x,t,\omega)|
+ \|\xi^{(1)} - \xi^{(2)}\|_{X^0} + \|\Psi^{(1)} - \Psi^{(2)}\|_{Z^0}.$$

Note that the first fundamental inequality can help to establish robustness only with respect to deviations of $b$ that are small together with their derivatives in $x$, and this restriction is necessary even for robustness in $X^0$. Theorem 5.1 establishes robustness in $Y^2$ for the disturbances of the coefficients that are small in $L_\infty$-norm only. For instance, if $b$ is replaced for $b + \xi$, where
ess sup_{x,t,\omega} |\xi(x, t, \omega)| \leq \varepsilon \) for a small \( \varepsilon > 0 \), then Theorem 5.1 ensures that the corresponding solution of (3.3) is close in \( Y^2 \) to the original one.

The rest part of the paper is devoted to the proofs of results given above.

6 Auxiliary facts for backward equations

In this section, we collect some facts that will be used for the proof of Theorem 4.1. Lemmas 6.1–6.3 given below were obtained in Dokuchaev (2010a), where the their proof can be found.

6.1 Decomposition of operators \( L \) and \( M_i \)

Introduce operators \( L(s, T) : X^{-1}(s, T) \to Y^1(s, T) \), \( M_i(s, T) : X^0(s, T) \to Y^1(s, T) \), and \( L(s, T) : Z^0_s \to Y^1(s, T) \), such that

\[
u = L(s, T)\varphi + L(s, T)\Phi + \sum_{i=1}^{N} M_i(s, T)h_i,
\]

where \( u \) is the solution in \( Y^1(s, T) \) of problem (3.2). These operators are linear and continuous; it follows immediately from Lemma 3.1. We will denote by \( L, M_i, \) and \( L \), the operators \( L(0, T), M_i(0, T), \) and \( L(0, T), \) correspondingly.

For \( t \in [0, T] \), define operators \( \delta_t : C([0, T]; Z^k) \to Z^k \) such that \( \delta_t u = u(\cdot, t) \).

**Lemma 6.1** In the notations of Lemma 3.2, the following duality equation is satisfied:

\[
p = L^*\xi + (\delta_T L)^*\Psi, \quad \chi_i = M_i^*\xi + (\delta_T M_i)^*\Psi, \quad p(\cdot, 0) = L^*\xi + (\delta_T L)^*\Psi,
\]

where \( L^* : X^{-1} \to X^1, M_i^* : X^{-1} \to X^0, (\delta_T L)^* : Z^0_0 \to X^1, (\delta_T M_i)^* : Z^0_0 \to X^0, \) and \( (\delta_T L)^* : Z^0_T \to Z^0_0, \) are the operators that are adjoint to the operators \( L : X^{-1} \to X^1, M_i : X^0 \to X^1, \delta_T M_i : X^{-1} \to Z^0_T, \delta_T M_i : X^0 \to Z^0_T, \) and \( \delta_T L : Z^0_0 \to Z^0_T, \) respectively.

Our method of proof of fundamental inequalities is based on decomposition of the operators to superpositions of simpler operators.

**Definition 6.1** Define operators \( K : Z^0_0 \to Y^1, Q_0 : X^{-1} \to Y^1, Q_i : X^0 \to Y^1, i = 1, ..., N, \) as the operators \( L : X^{-1} \to Y^1, M_i : X^0 \to Y^1, i = 1, ..., N, \) considered for the case when \( B_i = 0 \) for all \( i \).
By Lemma 3.1, these linear operators are continuous. It follows from the definitions that

\[ K\Phi + Q_0\eta + \sum_{i=1}^{N} Q_i h_i = V, \]

where \( \eta \in X^{-1}, \Phi \in Z_0^0, \) and \( h_i \in X^0, \) and where \( V \) is the solution of the problem

\[ d_t V = (AV + \eta) \, dt + \sum_{i=1}^{N} h_i \, dw_i(t), \]

\[ V|_{t=0} = \Phi, \quad V(x,t,\omega)|_{x \in \partial D} = 0. \quad (6.2) \]

Define the operators

\[ P \triangleq \sum_{i=1}^{N} Q_i B_i, \quad P^* \triangleq \sum_{i=1}^{N} B_i^* Q_i^*. \quad (6.3) \]

By the definitions, the operator \( P : X^1 \to X^1 \) is continuous, and \( P^* : X^{-1} \to X^{-1} \) is its adjoint operator. Hence the operator \( P^* : X^{-1} \to X^{-1} \) is continuous. Let

\[ P_0 \triangleq \delta_T \sum_{i=1}^{N} Q_i B_i, \quad P_0^* \triangleq \sum_{i=1}^{N} B_i^* (\delta_T Q_i)^*. \]

By the definitions, the operator \( P_0 : X^1 \to Z_T^0 \) is continuous, and \( P_0^* : Z_T^0 \to X^{-1} \) is its adjoint operator. Hence the operator \( P_0^* : Z_T^0 X^{-1} \) is continuous.

**Lemma 6.2** The operator \((I - P)^{-1} : X^1 \to X^1\) is continuous, and

\[ L = (I - P)^{-1} Q_0, \quad M_i = (I - P)^{-1} Q_i, \]

\[ \delta_T L = P_0 (I - P)^{-1} Q_0 + \delta_T Q_0, \quad \delta_T M_i = P_0 (I - P)^{-1} Q_i + \delta_T Q_i, \quad (6.4) \]

\( i = 1, \ldots, N. \) The operator \((I - P^*)^{-1} : X^{-1} \to X^{-1}\) is also continuous, and

\[ L^* = Q_0^* (I - P^*)^{-1}, \quad M_i^* = Q_i^* (I - P^*)^{-1}, \]

\[ (\delta_T L)^* = Q_0^* (I - P^*)^{-1} P_0^* + (\delta_T Q_0)^*, \quad (\delta_T M_i)^* = Q_i^* (I - P^*)^{-1} P_0^* + (\delta_T Q_i)^*. \quad (6.5) \]

In fact, Lemma [6.2] allows to split represent solution (3.3) via solution of much simpler problem with \( B_i \equiv 0 \) and via inverse operator \((I - P^*)^{-1}. \) It can be illustrated as the following.

**Corollary 6.1** (i) For \( \Psi = 0, \) the solution \((p, \chi_1, \ldots, \chi_N)\) of problem (3.3) can be represented as

\[ p = Q_0^* g, \quad \chi_i = Q_i^* g, \] where \( g = \xi + \sum_{i=1}^{N} B_i^* \chi_i, \) and where \( \sum_{i=1}^{N} B_i \chi_i = P^* g. \)

(ii) For general \( \Psi, \) the solution \((p, \chi_1, \ldots, \chi_N)\) of problem (3.3) can be represented as

\[ p = Q_0^* g + (\delta_T Q_0)^* \Psi, \quad \chi_i = Q_i^* g + (\delta_T Q_i)^* \Psi, \]

where \( g = \xi + \sum_{i=1}^{N} B_i^* \chi_i, \) and where \( \sum_{i=1}^{N} B_i \chi_i = P^* g + P_0^* \Psi. \) In other words, \( g = (I - P^*)^{-1} \xi + (I - P^*)^{-1} P_0^* \Psi. \)

It appears that this representation helps to establish the second fundamental inequality.
6.2 Semi-group property for backward equations

It is known that the forward SPDE is casual (or it has semigroup property): if \( u = L\varphi + \mathcal{L}\Phi \), where \( \varphi \in X^{-1}, \Phi \in Z_0^0 \), then
\[
u |_{t \in [\theta, s]} = L(\theta, s)\varphi + \mathcal{L}(\theta, s)u(\cdot, \theta).
\]

(6.6)

To proceed further, we need a similar property for the backward equations.

**Lemma 6.3** Let \( 0 \leq \theta < s < T \), and let \( p = L^*\xi, \chi_i = \mathcal{M}_i\xi \), where \( \xi \in X^{-1} \) and \( \Psi \in Z_T^0 \). Then
\[
p|_{t \in [\theta, s]} = L(\theta, s)^*\xi|_{t \in [\theta, s]} + (\delta_s L(\theta, s))^*p(\cdot, s),
\]
\[
p(\cdot, \theta) = (\delta_s L(\theta, s))^*p(\cdot, s) + L(\theta, s)^*\xi,
\]
\[
\chi_k|_{t \in [\theta, s]} = \mathcal{M}_k(\theta, s)^*\xi|_{t \in [\theta, s]} + (\delta_s \mathcal{M}_k(\theta, s))^*p(\cdot, s), \quad k = 1, \ldots, N.
\]

(6.7)

(6.8)

(6.9)

Note that this semi-group property implies causality for backward equation (which is a non-trivial fact due the presence of \( \chi \)).

6.3 A special estimate for deterministic PDEs

We use notations \( \nabla u \overset{\Delta}{=} \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right)^\top \), for functions \( u : \mathbb{R}^n \to \mathbb{R} \). In addition, we use the notation \( (u, v)_{H^0} \overset{\Delta}{=} \sum_{i=1}^n (v_i, u_i)_{H^0} \) for functions \( u, v : D \to \mathbb{R}^n \), where \( u = (u_1, ..., u_n) \) and \( v = (v_1, ..., v_n) \).

For \( u \in H^1 \), let
\[
\|u\|_{H^1(t, \omega)} \overset{\Delta}{=} (\nabla u, b(\cdot, t, \omega) \nabla u)^{1/2} = \left( \sum_{i,j=1}^n \int_D \frac{\partial u}{\partial x_i}(x)b_{ij}(x, t, \omega) \frac{\partial u}{\partial x_j}(x)dx \right)^{1/2}.
\]

(6.10)

For \( K > 0 \), introduce the operator \( A^*_K = A^* - KI \), i.e., \( A^*_K u = A^* u - Ku \).

**Lemma 6.4** Let \( \theta, \tau \in [0, T] \) be given, \( 0 \leq \theta < \tau \leq T \). Let the function \( b(x, t, \omega) = b(x, \omega) \) be constant in \( t \in [\theta, \tau] \) for a.e. \( x, \omega \). Let \( h = h(x, t, \omega) \in L_2(D \times [\theta, \tau]) \), and let \( u = u(x, t, \omega) : D \times [\theta, \tau] \times \Omega \to \mathbb{R} \) be the solution of the boundary value problem
\[
\frac{\partial u}{\partial t} + A_K u = -h, \quad t \in (\theta, \tau) \quad u(x, \tau) = 0, \quad u(x, t)|_{x \in \partial D} = 0,
\]

(6.11)

Then for any \( \varepsilon > 0, M > 0 \), there exists \( K = K(\varepsilon, M, \mathcal{P}) > 0 \) such that
\[
\sup_{t \in [\theta, \tau]} \|u(\cdot, t, \omega)\|_{H^1(t, \omega)}^2 + M \sup_{t \in [\theta, \tau]} \|u(\cdot, t, \omega)\|_{H^0}^2 \leq \frac{1 + \varepsilon}{2} \int_{\theta}^{\tau} \|h(\cdot, t, \omega)\|_{H^0}^2 dt \quad a.s.
\]

This lemma follows immediately from Theorem 1 and Corollary 1 from Dokuchaev (2008).
7 The proof of Theorem 4.1

By Lemma 6.1, it suffices to show that the operators
$L^* : X^0 \to Y^2$, $(\delta_T L)^* : Z_T^1 \to Y^2$, and
$\mathcal{M}_i^* : X^0 \to X^1$, $(\delta_T \mathcal{M}_i)^* : Z_T^1 \to X^1$, are continuous, and that their norms are less or equal
than a constant $c = c(\mathcal{P})$.

We define the operators $L^*(s,T)$, $M_i^*(s,T)$, $(\delta_T L)^*$, and $(\delta_T M_i)^*$, similarly to
$L^*$, $M_i^*$, $(\delta_T L)^*$, and $(\delta_T M_i)^*$, with time interval $[0,T]$ replaced by $[s,T]$.

We denote by $\bar{P}_T$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathcal{P}$ of the
$\sigma$-algebra of subsets of $[0,T] \times \Omega$, generated by functions that are progressively measurable with respect to
$\bar{B}_1 \times \mathcal{F}_T$. Let $\bar{X}^k \triangleq L^2([0,T] \times \Omega, \bar{P}_T, \bar{\ell}_1 \times \mathcal{P}; H^k)$.

Let $E$ be the operator of projection of $\bar{X}^1$ onto $X^1$.

Let $\xi \in X^0$, $\Psi \in Z_T^1$, and let $\bar{p}$ be the solution of the boundary value problem in $Q$

$$\frac{\partial \bar{p}}{\partial t} + A^* \bar{p} = -\xi, \quad t \leq T,$$

$$\bar{p}|_{t=T} = \Psi, \quad \bar{p}(x,t,\omega)|_{x \in \partial D} = 0. \quad (7.1)$$

By the second fundamental inequality for deterministic parabolic equations, it follows that the
solution of (7.1) is such that $\bar{p} \in \bar{X}^2 \cap C^1$, (7.2) holds and

$$\|\bar{p}\|_{\bar{X}^2} + \|\bar{p}\|_{C^1} \leq c \left( \|\xi\|_{X^0} + \|\Psi\|_{Z_T^0} \right), \quad (7.2)$$

where $c = c(\mathcal{P}) > 0$ is a constant. This fact is well known; if the function $b(x,t,\omega)$ is almost
surely continuous, then (7.2) follows Theorem IV.9.1 from Ladyzenskaya et al (1968). Since the
derivative $\partial b/\partial x$ is bounded, the condition that $b$ is continuous can be lifted. In this case, (7.2)
follows from Theorem 3.1 from Dokuchaev (2005).

By Martingale Representation Theorem, there exist functions $\gamma_i(\cdot,t,\cdot) \in X^0$ such that

$$\bar{p}(x,t,\omega) = E\{\bar{p}(x,t,\omega)|\mathcal{F}_0\} + \sum_{i=1}^N \int_0^T \gamma_i(x,t,s,\omega)dw_i(s). \quad (7.3)$$

Lemma 7.1 Assume that the function $\mu = (b,f,\lambda)$ is such that $\mu(x,t,\omega)$ is $\mathcal{F}_0$-measurable for
all $x \in D$. Let $\xi \in X^0$, $\Psi \in Z_T^1$, let $\bar{p}$ be the solution of (7.1), and let $\gamma_j$ be the processes
presented in (7.3). Let $p, \chi_1, \ldots, \chi_2$ be defined as

$$p \triangleq E\bar{p}, \quad \chi_i(x,s,\omega) \triangleq \gamma_i(x,s,\omega). \quad (7.4)$$
Then \( p \in Y^1, \chi_i \in X^1 \), and
\[
\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \leq c \left( \|\xi\|_{X^0} + \|\Psi\|_{Z^0_T} \right),
\]
where \( c = c(P) > 0 \) is a constant. In addition,
\[
p = Q_0^* \xi + (\delta_T Q_0)^* \Psi, \quad \chi_i = Q_i^* \xi + (\delta_T Q_i)^* \Psi.
\]

**Proof of Lemma 7.1.** By Martingale Representation Theorem, there exist functions \( \gamma_i(\cdot, t, \cdot) \in X^0, \gamma_{\xi_i}(\cdot, t, \cdot) \in X^0 \), and \( \gamma_{\Psi_i}(\cdot) \in X^1 \), such that (7.3) holds as well as
\[
\xi(x, t, \omega) = E\{\xi(x, t, \omega) | F_0\} + \sum_{i=1}^{N} \int_{0}^{T} \gamma_{\xi_i}(x, t, s, \omega)dw_i(s),
\]
\[
\Psi(x, \omega) = E\{\Psi(x, \omega) | F_0\} + \sum_{i=1}^{N} \int_{0}^{T} \gamma_{\Psi_i}(x, s, \omega)dw_i(s).
\]

Moreover, it follows that \( Dg_i(\cdot, t, \cdot) \in X^0 \), where either \( D\gamma_i = \partial\gamma_i/\partial t \) or \( D\gamma_i = A^{*}\gamma_i \), and
\[
\bar{p}(x, t, \omega) = E\{\bar{p}(x, t, \omega) | F_0\} + \sum_{i=1}^{N} \int_{0}^{T} D\gamma_i(x, t, s, \omega)dw_i(s).
\]

By (7.4), it follows that
\[
\frac{\partial\gamma_i}{\partial t}(\cdot, t, s, \omega) + A^*\gamma_i(\cdot, t, s, \omega) = -\gamma_{\xi_i}(\cdot, t, s, \omega), \quad t \in (0, T),
\]
\[
\gamma_i(x, T, s, \omega) = \gamma_{\Psi_i}(x, s, \omega), \quad \gamma_i(x, t, s, \omega)|_{x \in \partial D} = 0.
\]

Again, it follows from the second fundamental inequality for deterministic parabolic equations that
\[
\sup_{t \in [s,T]} \|\gamma_i(\cdot, t, s, \omega)\|_{H^1}^2 \leq c \left( \int_{s}^{T} \|\gamma_{\xi_i}(\cdot, t, s, \omega)\|_{H^0}^2 dt + \|\gamma_{\Psi_i}(\cdot, s, \omega)\|_{H^1}^2 \right),
\]
where \( c = c(T, n, D) > 0 \) is a constant. Hence
\[
\|\gamma_i(\cdot, s, s, \omega)\|_{H^1}^2 \leq c \left( \int_{s}^{T} \|\gamma_{\xi_i}(\cdot, t, s, \omega)\|_{H^0}^2 dt + \|\gamma_{\Psi_i}(\cdot, s, \omega)\|_{H^1}^2 \right).
\]
This estimate together with (7.2) ensures that (7.5) holds for \( p \) and \( \chi_i \) defined by (7.4).

Let us show that (7.6) holds.

Clearly,
\[
\bar{p}(x, t, \omega) = p(x, t, \omega) + \sum_{i=1}^{N} \int_{t}^{T} \gamma_i(x, t, s, \omega)dw_i(s),
\]

16
\[ \bar{p}(\cdot, t) = \int_t^T \left( \mathcal{A}^* \bar{p}(\cdot, s) + \xi(\cdot, s) \right) ds. \]

Hence

\[ p(\cdot, t) = \Psi + \int_t^T \left( \mathcal{A}^* p(\cdot, s) + \xi(\cdot, s) \right) ds \]
\[ + \sum_{i=1}^N \left( \int_t^T ds \int_s^T [\mathcal{A}^* \gamma_i(\cdot, s, r) + \gamma \xi(r, s)] dw_i(r) - \int_t^T \gamma_i(\cdot, t, s) dw_i(s) \right) \]
\[ = \Psi + \int_t^T \left( \mathcal{A}^* p(\cdot, s) + \xi(\cdot, s) \right) ds \]
\[ + \sum_{i=1}^N \int_t^T dw_i(s) \left( \int_t^s [\mathcal{A}^* \gamma_i(\cdot, r, s) + \gamma \xi(r, s)] dr - \gamma_i(\cdot, t, s) \right). \]

By (7.7),
\[ \gamma_i(\cdot, t, s) - \int_t^s [\mathcal{A}^* \gamma_i(r, s) + \gamma \xi(r, s)] dr = \gamma_i(\cdot, s, s). \]

By (7.4), we have selected \( \gamma_i(\cdot, s, s) = \chi_i(\cdot, s) \). It follows that
\[ p(\cdot, t) = \Psi + \int_t^T \left( \mathcal{A}^* p(\cdot, s) + \xi(\cdot, s) \right) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) dw_i(s). \]

Finally, we obtain (7.6) from Lemma 6.1 applied to the operators \( \mathcal{Q}_0^*, \mathcal{Q}_i^*, (\delta T \mathcal{Q}_0)^*, \) and \( (\delta T \mathcal{Q}_i)^*, \)
\[ i = 1, ..., N, \] considered as special cases of \( \mathcal{L}^*, \mathcal{M}_i^*, (\delta_T \mathcal{L}_0)^*, \) and \( (\delta_T \mathcal{M}_i)^*, \) respectively. This completes the proof of Lemma 7.1.

□

In the following proof, we will explore the following observation: if \( \lambda \) is replaced by \( \lambda^{(K)}(x, t, \omega) \triangleq \lambda(x, t, \omega) + K \), i.e., if \( \mathcal{A} \) is replaced by \( \mathcal{A}_K = \mathcal{A} v + K I \), then the solution \( u \) of the problem (3.2) has to be replaced by the process
\[ u(x, t, \omega) e^{-Kt}, \]
and the solution \( (p, \chi_1, ..., \chi_N) \) of the problem (3.3) has to be replaced by the process
\[ \left( p(x, t, \omega) e^{K(T-t)}, \chi_1(x, t, \omega) e^{K(T-t)}, ..., \chi_N(x, t, \omega) e^{K(T-t)} \right). \]
Therefore, it suffices to prove theorem for any case when \( \lambda \) is replaced for \( \lambda^{(K)}(x, t, \omega) \triangleq \lambda(x, t, \omega) + K \) with some \( K > 0 \), and this \( K \) can be taken arbitrarily large.

For linear normed spaces \( \mathcal{X} \) and \( \mathcal{Y} \), we denote by \( \|T\|_{\mathcal{X}, \mathcal{Y}} \) the norm of an operator \( T : \mathcal{X} \to \mathcal{Y} \).

**Lemma 7.2** Let \( 0 \leq s < T \), and let the function \( \mu = (b, f, \lambda) \) be such that \( \mu(x, t, \cdot) \) is \( F_{s} \)-measurable for all \( x \in D, t \in [s, T) \). Moreover, we assume that \( b(x, t, \omega) = b(x, \omega) \) does not depend on \( t \in [s, T] \). Then there exist \( K > 0 \) such that if \( \lambda \) is replaced by \( \lambda(x, t, \omega) + K \), then

\[
\|L^{*}(s, T)\|_{X_{0}(s, T)\mathcal{Y}^{2}(s, T)} + \|\delta_{T}L(s, T)\|_{\mathcal{Z}^{1}_{s}, \mathcal{Y}^{2}(s, T)} + \sum_{i=1}^{n} \|\mathcal{M}_{i}(s, T)\|_{X_{0}(s, T), X^{1}} \\
+ \sum_{i=1}^{n} \|\delta_{T}\mathcal{M}_{i}(s, T)\|_{\mathcal{Z}^{1}, X^{1}(s, T)} \leq c,
\]

where \( c \in (0, +\infty) \) depends only on \( K \) and \( \mathcal{P} \).

**Proof of Lemma 7.2.** To simplify the notations, we consider only the case when \( s = 0 \).

By (6.1) and (6.5), it suffices to show that the operator \( (I - P^{*})^{-1} : X_{0} \to X_{0} \) is continuous. For this, it suffices to show that there exist \( K > 0 \) such that if \( \lambda \) is replaced for \( \lambda(x, t, \omega) + K \) then \( \|P^{*}\|_{X_{0}, X_{0}} < 1 \).

Let \( \xi \in X_{0} \), let \( \bar{p} \) be the solution of (7.1), and let \( \gamma_{j} \) be the processes presented in (7.3) with \( \Psi = 0 \). Let \( p, \chi_{1}, ..., \chi_{2} \) be defined by (7.4) with \( \Psi = 0 \). In this case,

\[
\frac{\partial \gamma_{i}}{\partial t}(-, t, s, \omega) + A^{*}\gamma_{i}(-, t, s, \omega) = -\gamma_{i}(-, t, s, \omega), \quad t \in (0, T), \\
\gamma_{i}(x, T, s, \omega) = 0, \quad \gamma_{i}(x, t, s, \omega)|_{x \in \partial D} = 0.
\]

(7.8)

By Lemma 6.4 applied to boundary value problem (7.8), for any \( \varepsilon > 0, M > 0 \), there exists \( K = K(\varepsilon, M, \mathcal{P}) > 0 \) such that

\[
\sup_{t \in [s, T]} \|\gamma_{i}(\cdot, t, s, \omega)\|_{H_{1}(t, \omega)}^{2} + M \sup_{t \in [s, T]} \|\gamma_{i}(\cdot, t, s, \omega)\|_{H_{0}}^{2} \leq \frac{1 + \varepsilon}{2} \int_{s}^{T} \|\gamma_{i}(\cdot, t, s, \omega)\|_{H_{0}}^{2} dt \quad \text{a.s.}
\]

Here \( \| \cdot \|_{H_{1}(t, \omega)} \) is defined by (6.10). Hence

\[
\int_{0}^{T} \|\gamma_{i}(\cdot, s, \omega)\|_{H_{1}(t, \omega)}^{2} ds + M \int_{0}^{T} \|\gamma_{i}(\cdot, s, s, \omega)\|_{H_{0}}^{2} ds \leq \frac{1 + \varepsilon}{2} \int_{0}^{T} ds \int_{s}^{T} \|\gamma_{i}(\cdot, t, s, \omega)\|_{H_{0}}^{2} dt.
\]

Note that

\[
E \sum_{i=1}^{N} \int_{0}^{T} dt \int_{0}^{T} \|\gamma_{i}(\cdot, t, s, \omega)\|_{H_{0}}^{2} ds \leq \|\xi\|_{X_{0}}^{2}.
\]
Hence
\[ \mathbf{E} \int_0^T \| \gamma_i(\cdot, s, s, \omega) \|_{H^1(t, \omega)}^2 ds + M \mathbf{E} \int_0^T \| \gamma_i(\cdot, s, s, \omega) \|_{H^0}^2 ds \leq \frac{1 + \varepsilon}{2} \| \xi \|_{X^0}^2. \]

By (7.6), it can be rewritten as
\[ \mathbf{E} \int_0^T \| \chi_i(\cdot, t, \omega) \|_{H^1(t, \omega)}^2 dt + M \mathbf{E} \int_0^T \| \chi_i(\cdot, t, \omega) \|_{H^0}^2 dt \leq \frac{1 + \varepsilon}{2} \| \xi \|_{X^0}^2. \] (7.9)

Remind that
\[ P^* \xi = \sum_{j=1}^N B_j^* \chi_j. \]

By Condition 4.1, there exists \( M = M(\mathcal{P}) > 0 \) such that
\[ \left\| \sum_{j=1}^N B_j^* \chi_j \right\|_{H^0}^2 \leq 2 \sum_{j=1}^N \| \chi_j \|_{H^1(t, \omega)}^2 + 2M \sum_{j=1}^N \| \chi_j \|_{H^0}^2 - 2 \delta_1 \sum_{j=1}^N \| \nabla \chi_j \|_{H^0}^2 \quad \forall t, \omega. \] (7.10)

By (7.9) and (7.10), it follows that a small enough \( \varepsilon > 0 \) and a large enough \( K > 0 \) can be found such that
\[ \| P^* \xi \|_{X^0}^2 = \left\| \sum_{i=1}^N B_i^* \chi_i \right\|_{X^0}^2 \leq c \| \xi \|_{X^0}^2 \]
for this \( K \) with some \( c = c(\mathcal{P}, K) < 1 \). Hence
\[ \| P^* \xi \|_{X^0} \leq \sqrt{c} \| \xi \|_{X^0}. \]

Therefore, we have proved that there exist \( K = K(\mathcal{P}) > 0 \) such that if \( \lambda \) is replaced for \( \lambda(x, t, \omega) + K \) then \( \| P^* \|_{X^0, X^0} < 1 \), and, therefore, the operator \((I - P^*)^{-1} : X^0 \rightarrow X^0\) is continuous. By the first equation in (7.5), it follows that the operator \( \mathcal{Q}_0^* : X^0 \rightarrow Y^2 \) is continuous. In addition, it follows from (7.6) and (7.9) that the operators \( \mathcal{Q}_i^* : X^0 \rightarrow X^1 \) are continuous. Then the proof of Lemma 7.2 for the special case of \( \Psi = 0 \) follows from the first equations for adjoint operators in (6.5).

To complete the proof of Lemma 7.2 for general \( \Psi \), By (7.5), (7.6), it follows that it suffices to show that the operators \((\delta_T Q_0)^* : Z^1_T \rightarrow Y^2 \) and \((\delta_T Q_i)^* : Z^1_T \rightarrow X^1 \) are continuous, \( i = 1, ..., N \). In addition, the upper bound of the norms of these operators depends on \( \mathcal{P} \) only. Then the proof follows from the last two equations for the adjoint operators in (6.5). This completes the proof of Lemma 7.2. □

For an integer \( M > 0 \), we denote by \( \Theta(M) \) the class of all functions \( \mu = (b, f, \lambda) \) such that all conditions imposed in Section 3 are satisfied, and that there exists a set \( \{t_i\}_{i=0}^M \) such
that $0 = t_0 < t_1 < \cdots < t_M = T$ and that the function $\mu(x, t, \cdot) = (b(x, t, \cdot), f(x, t, \cdot), \lambda(x, t, \cdot))$ is $\mathcal{F}_{t_i}$-measurable for all $x \in D$, $t \in [t_i, t_{i+1})$ and that the function $b(x, t, \omega) = b(x, \omega)$ does not depend on $t$ for $t \in [t_i, t_{i+1})$.

Let $\Theta \triangleq \cup_{M > 0} \Theta(M)$.

**Lemma 7.3** Let $(b, f, \lambda) \in \Theta(M)$ for some $M > 0$. Then there exists $K > 0$ such that if $\lambda$ is replaced by $\lambda(x, t, \omega) + K$, then

\[
\|L^*\|_{X^0,Y^1} + \sum_{i=1}^n \|M_i^*\|_{X^0,X^1} + \|\delta_T L^*\|_{Z_{1,T}^1,Y^1} + \sum_{i=1}^n \|\delta_T M_i^*\|_{Z_{1,T}^1,X^1} \leq c,
\]

where $c \in (0, +\infty)$ does not depend on $M$ and depends only on $K$ and $P$.

**Proof** of this lemma follows immediately from Lemma 6.3 and from Lemma 7.2 applied consequently for all time intervals from the definition of $\Theta(M)$ backward from terminal time.

**Corollary 7.1** Under assumption of Lemma 7.3, Theorem 4.1 holds and there exists $K > 0$ such that the operators $L^* : X^0 \to Y^2$, $M_j^* : X^0 \to X^1$, $(\delta_T L)^* : Z_{T}^1 \to Y^2$, and $(\delta_T M_i)^* : Z_{T}^1 \to X^1$, $j = 1, \ldots, N$, are continuous, and their norms do not depend on $M$.

Up to the end of this section, we assume that $\lambda$ is replaced for $\lambda(x, t, \omega) + K$ such that the conclusion of Lemma 7.3 holds.

Now we are in position to prove Theorem 4.1 for the case of $(b, f, \lambda)$ of the general kind.

Let $M = 1, 2, \ldots, M \to +\infty$. Let $\varepsilon \triangleq M^{-1}$. By Condition 4.2 there exist a subsequence of $M$ such that there exists $b_{\varepsilon} \in \Theta_b(M)$ for any $M$ with the corresponding sets $\{t_k\} = \{t_k(M)\}$, $0 = t_0 < \ldots < t_k < t_M = T$ such that $\max_k |t_k - t_{k-1}| \to 0$ as $M \to +\infty$, $b_{\varepsilon}(x, t, \omega) = b(t_k, t, \omega)$, $t \in [t_k, t_{k+1})$, and that there exists $q, r \in [1, +\infty]$ such that

\[b_{\varepsilon} \to b \quad \text{in} \quad \mathcal{W}_{q,r}^1 \quad \text{as} \quad \varepsilon \to 0.\]

Further, we introduce functions $f_{\varepsilon}$, $\lambda_{\varepsilon}$, such that

\[f_{\varepsilon}(x, t, \omega) = \mathbf{E}\{f(x, t, \omega) | \mathcal{F}_{t_k}\}, \quad \lambda_{\varepsilon}(x, t, \omega) = \mathbf{E}\{\lambda(x, t, \omega) | \mathcal{F}_{t_k}\}, \quad t \in [t_k, t_{k+1}).\]

**Proposition 7.1** Let us show that Condition 4.1 implies that:

(a) Condition 4.1 is satisfied for $b$ replaced by $b_{\varepsilon}$, with the same $\delta_1 > 0$ for all $\varepsilon$,

(b) Without a loss of generality, we can assume that $\sup_{\varepsilon > 0} \|b_{\varepsilon}\|_{\mathcal{W}_{q,r}^1} < +\infty$. 

20
Proof of Proposition 7.1. It suffices to show that Condition 4.1(i) implies (a) and that Condition 4.1(ii) implies (b).

Let us show that Condition 4.1(i) implies (a). Let \( A = A(x, t, \omega) \in \mathbb{R}^{nN \times nN} \) be the symmetric matrix that defines the quadratic form on the vectors \( Y = (y_1, \ldots, y_N) \in \mathbb{R}^{nN} \) in (4.1), and let \( A_{\varepsilon} \) be the similar matrix defined for \( b = b_{\varepsilon} \). By Condition 4.1 the minimal eigenvalue of \( A \) is positive and is separated from zero uniformly over \( \varepsilon, x, t, \omega \). By the definitions, it follows that \( \|A_{\varepsilon} - A\|_{\mathbb{W}_\infty}^n \to 0 \). Since the minimal eigenvalue of a matrix depends continuously of its coefficients, it follows that the minimal eigenvalue of \( A_{\varepsilon} \) is positive and is separated from zero uniformly over \( \varepsilon, x, t, \omega \). Hence Condition 4.1(i) implies (a).

Let us show that Condition 4.1(ii) implies (b). Let \( R = \sup_{x,t,\omega} \|b\|_{\mathbb{W}_\infty} \), and let \( \gamma \) be the supremum over \( x, t, \omega \) of the maximal eigenvalue of \( b(x, t, \omega) \). It suffices to show that, without a loss of generality, we can assume that

\[
\sup_{\varepsilon} \|b_{\varepsilon}\|_{\mathbb{W}_\infty^n} \leq n\gamma + 2R + 1. \tag{7.11}
\]

Suppose that (7.11) does not hold, i.e., that there exists some \( M \) such that for \( \varepsilon = M^{-1} \) and some \( t_k = t_k(M) \) there exists \( \Gamma \subset \Omega \) such that \( \Gamma \in \mathcal{F}_{t_k}, \ P(\Gamma) > 0, \)

\[
b_{\varepsilon}(\cdot, t, \omega) \equiv b_{\varepsilon}(t_k, x, \omega), \quad t \in [t_k, t_{k+1}), \quad \|b_{\varepsilon}(\cdot, t_k, \omega)\|_{\mathcal{W}_\infty^1(\Omega)} > n\gamma + 2R + 1 \quad \text{iff} \quad \omega \in \Gamma.
\]

In this case, one can replace \( b_{\varepsilon}(\cdot, t)|_{t \in [t_k, t_{k+1})} \), by

\[
\tilde{b}_{\varepsilon}(x, t, \omega) = b_{\varepsilon}(x(t, t, \omega)\mathbb{1}_{\Omega\Gamma}(\omega) + \gamma I_n\mathbb{1}_{\Gamma}(\omega), \quad t \in [t_k, t_{k+1}),
\]

where \( \mathbb{1} \) is the indicator function, and where \( I_n \) is the unit matrix in \( \mathbb{R}^n \). Obviously, Condition 4.1 is satisfied for \( \tilde{b}_{\varepsilon} \) replacing \( b_{\varepsilon} \), with the same \( \delta_1 > 0 \) for all \( \varepsilon \). In addition, we have that

\[
\|\tilde{b}(M) - b\|_{\mathcal{W}_\infty^1(\Omega)} \leq \|\tilde{b}(M)\|_{\mathcal{W}_\infty^1(\Omega)} + \|b\|_{\mathcal{W}_\infty^1(\Omega)} \leq n\gamma + R, \quad \omega \in \Gamma,
\]

\[
\|\tilde{b}(M) - b\|_{\mathcal{W}_\infty^1(\Omega)} \geq \|\tilde{b}(M)\|_{\mathcal{W}_\infty^1(\Omega)} - \|b\|_{\mathcal{W}_\infty^1(\Omega)} \geq n\gamma + 2R - R = n\gamma + R, \quad \omega \in \Gamma.
\]

It follows that Condition 4.2 holds for the new selection \( \tilde{b}_{\varepsilon} \). This completes the proof of Proposition 7.1. \( \square \)

Further, it follows from Proposition 7.1 and from the definitions that

\[
\sup_{x, t, \omega, \varepsilon} \left( |b_{\varepsilon}(x, t, \omega)| + |\frac{\partial b_{\varepsilon}}{\partial x}(x, t, \omega)| + |f_{\varepsilon}(x, t, \omega)| + |\frac{\partial f_{\varepsilon}}{\partial x}(x, t, \omega)| + |\lambda_{\varepsilon}(x, t, \omega)| \right) < +\infty. \tag{7.12}
\]

Let us consider a subsequence \( \varepsilon = \varepsilon_i \to 0 \) such that

\[
b_{\varepsilon} \to b, \quad f_{\varepsilon} \to f, \quad \lambda_{\varepsilon} \to \lambda, \quad \frac{\partial b_{\varepsilon}}{\partial x} \to \frac{\partial b}{\partial x}, \quad \frac{\partial f_{\varepsilon}}{\partial x} \to \frac{\partial f}{\partial x} \quad \text{in} \quad X^0 \quad \text{and a.e. as} \quad \varepsilon \to 0. \tag{7.13}
\]
Let \( p_\varepsilon \overset{\Delta}{=} L_\varepsilon^* \xi + (\delta_T L_\varepsilon)^* \Psi, \ \chi_{\varepsilon i} \overset{\Delta}{=} M_{\varepsilon i}^* \xi + (\delta_T M_{\varepsilon i})^* \Psi, \) and let \( p \overset{\Delta}{=} L^* \xi + (\delta_T L)^* \Psi, \ \chi_i \overset{\Delta}{=} M_i^* \xi + (\delta_T M_i)^* \Psi. \) The operators \( L_\varepsilon^* : X^{-1} \rightarrow Y^1, \) etc., are defined similarly to \( L^* : X^{-1} \rightarrow Y^1, \) etc., with substituting \((b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon).\)

By Lemma \( \ref{7.3} \) the sequences \( \{p_\varepsilon\} \) and \( \{\chi_{\varepsilon i}\} \) belong to the closed balls in the spaces \( X^2 \) and \( X^1 \) respectively with the centers at the zero and with the radius \( c(||\xi||_{X^0} + ||\Psi||_{Z_f^1}), \) where \( c = c(\mathcal{P}) > 0 \) does not depend on \( \varepsilon. \) The balls mentioned are closed, concave, and bounded. It follows that these balls are weakly closed and weakly compact in the reflexible Banach spaces \( X^2 \) and \( X^1 \) respectively. It follows that the sequences \( \{p_\varepsilon\} \) and \( \{\chi_{\varepsilon i}\} \) has subsequences with weak limits \( \bar{p} \) and \( \bar{\chi}_i, \) in the corresponding balls, i.e.,

\[
\|\bar{p}\|_{X^2} + \sum_{i=1}^{N} \|\bar{\chi}_i\|_{X^1} \leq c(||\xi||_{X^0} + ||\Psi||_{Z_f^1}).
\]

Assume that we can show that \( \bar{p}_\varepsilon \rightarrow p \) weakly in \( X^2 \) and \( \chi_{\varepsilon i} \rightarrow \chi_i \) weakly in \( X^1 \) for all \( i. \) It follows that \( \bar{p} = p \) and \( \bar{\chi}_i = \chi_i \) and

\[
\|p\|_{X^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \leq c(||\xi||_{X^0} + ||\Psi||_{Z_f^1}). \tag{7.14}
\]

It follows that

\[
\left\| \sum_{i=1}^{N} B_i^* \chi_i \right\|_{X^0} \leq c_1(||\xi||_{X^0} + ||\Psi||_{Z_f^1}),
\]

where \( c_1 = c_1(\mathcal{P}) \) is a constant. Hence \( g \overset{\Delta}{=} \xi + \sum_{i=1}^{N} B_i^* \chi_i \) is such that

\[
\|g\|_{X^0} \leq c_2(||\xi||_{X^0} + ||\Psi||_{Z_f^1}),
\]

where \( c_2 = c_2(\mathcal{P}) \) is a constant. Remind that, by Lemma \( \ref{6.2} \) and Corollary \( \ref{6.1} \)

\[
g = (I - P^*)^{-1} \xi + (I - P^*)^{-1} P_0^* \Psi, \quad p = Q_0^* g + (\delta_T Q_0)^* \Psi.
\]

By Lemma \( \ref{7.1} \) it follows that \( p \in Y^2 \) and

\[
\|p\|_{Y^2} + \left\| \sum_{i=1}^{N} B_i^* \chi_i \right\|_{X^0} \leq c_3(||g||_{X^0} + ||\Psi||_{Z_f^1}),
\]

where \( c_3 = c_3(\mathcal{P}) \). By \( \ref{7.15} \), it follows that \( p \in Y^2 \) and

\[
\|p\|_{Y^2} + \left\| \sum_{i=1}^{N} B_i^* \chi_i \right\|_{X^0} \leq c_4(||\xi||_{X^0} + ||\Psi||_{Z_f^1}),
\]

where \( c_4 = c_4(\mathcal{P}) \) is a constant. Then the proof of Theorem \( \ref{4.1} \) follows provided that the weak convergence of the sequence \( \{\chi_{\varepsilon i}\} \) to \( \chi_i \) is established.
Therefore, it suffices prove this weak convergence, i.e., it suffices to show that
\[ I_\varepsilon \mathcal{\Delta} (p_\varepsilon - p, h) \chi_0 \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \forall h \in X^0, \quad (7.15) \]
\[ J_\varepsilon \mathcal{\Delta} (\chi_\varepsilon - \chi_i, h) \chi_0 \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \forall h \in X^0, \quad i \in \{1, \ldots, N\}. \quad (7.16) \]

Let us show that \((7.15)\) holds. Set \( u_\varepsilon \Delta = L_i \varepsilon \) and \( u_\Delta = L_i h \), where the operators \( L_i \varepsilon : X^0 \to Y^1 \) are defined similarly to the operators \( L_i : X^0 \to Y^1 \) with substituting \((b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon)\).

By the definitions of the corresponding adjoint operators,
\[ I_\varepsilon \mathcal{\Delta} = (L_\varepsilon \xi - L_i \xi, h) \chi_0 + ((\delta_T L_\varepsilon)^* \Psi - (\delta_T L_i)^* \Psi, h) \chi_0 + (\xi, u_\xi - u) \chi_0 + (\Psi, u_\xi(\cdot, T) - u(\cdot, T)) \chi_0 \to 0, \quad (7.17) \]
and that \( \mathcal{A}_\varepsilon u - A u \) is represented as
\[ \mathcal{A}_\varepsilon u - A u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( [b_{ij\varepsilon} - b_{ij}] \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n \tilde{f}_\varepsilon \frac{\partial u}{\partial x_i} + \tilde{\lambda}_\varepsilon u. \]

By \((7.12)-(7.13)\), it follows that
\[ \tilde{f}_\varepsilon \to 0 \quad \text{and} \quad \tilde{\lambda}_\varepsilon \to 0 \quad \text{in} \quad X^0 \quad \text{and a.e.} \quad (7.17) \]

The function \( U_\varepsilon \mathcal{\Delta} = u_\varepsilon - u \) is the solution in \( Q \) of the boundary value problem
\[ d_t U_\varepsilon = (\mathcal{A}_\varepsilon U_\varepsilon + F_\varepsilon(u)) \ dt + \sum_{i=1}^N B_i U_\varepsilon \ dw_i(t), \]
\[ U_\varepsilon(x, 0) = 0, \quad U_\varepsilon(x, t)|_{x \in \partial D} = 0, \]
and where the linear operator \( F_\varepsilon(\cdot) \) is defined as
\[ F_\varepsilon(u) \Delta = r_\varepsilon(u) + q_\varepsilon(u), \quad r_\varepsilon(u) \Delta = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( [\tilde{b}_{ij\varepsilon} - b_{ij}] \frac{\partial u}{\partial x_j} \right), \quad q_\varepsilon(u) \Delta = \frac{\partial u}{\partial x} \tilde{f}_\varepsilon + \tilde{\lambda}_\varepsilon u. \]

Here \( \tilde{b}_{ij\varepsilon} \) are the components of the matrix \( \tilde{b}_\varepsilon \). By Lemma 3.3, it follows that
\[ \|U_\varepsilon\|_{Y^1} \leq C \|F_\varepsilon(u)\|_{X^{-1}}, \]
for a constant $C_1 = C_1(P)$. It follows that there exists a constant $C = C(P) > 0$ such that

$$|I_\varepsilon| \leq C\|U_\varepsilon\|_{Y^1}(\|\xi\|_{X^{-1}} + \|\Psi\|_{Z^T_0}) \leq C\|F_\varepsilon(u)\|_{X^{-1}}(\|\xi\|_{X^0} + \|\Psi\|_{Z^T_0}).$$

We have that

$$\|r_\varepsilon(u)\|_{X^{-1}}^2 = E \int_0^T \left\| \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(\hat{b}_{eij} - b_{ij}) \frac{\partial u}{\partial x_j} \right\|_{H^{-1}}^2 dt \leq C_1 \sum_{i,j=1}^n E \int_0^T \left\| \hat{b}_{eij} - b_{ij} \right\|_{H^0}^2 dt,$$

for a constant $C = C(n)$. The functions $b_\varepsilon$ and $b$ are bounded, hence

$$\left\| \hat{b}_{eij} - b_{ij} \right\|_{H^0} \leq C_1 \left| \frac{\partial u}{\partial x}(x, t, \omega) \right|$$

for a constant $C_1 = C_1(P)$. We have that $u \in X^1$. By the Lebesgue’s Dominated Convergence Theorem, it follows that $\left\| \hat{b}_{eij} - b_{ij} \right\|_{X^0} \rightarrow 0$. Hence $\|r_\varepsilon(u)\|_{X^{-1}} \rightarrow 0$.

Further, the functions $\hat{f}_\varepsilon$ and $\hat{\lambda}_\varepsilon$ are bounded, hence

$$|q_\varepsilon(u)(x, t, \omega)| \leq C_2 \left( \left| \frac{\partial u}{\partial x}(x, t, \omega) \right| + |u(x, t, \omega)| \right)$$

for a constant $C_2 = C_2(P) > 0$. By the Lebesgue’s Dominated Convergence Theorem again, it follows that $\|q_\varepsilon(u)\|_{X^0} \rightarrow 0$. Therefore, we obtain that $\|U_\varepsilon(u)\|_{X^0} \rightarrow 0$. By (7.18), it follows that (7.15) holds.

Let us show that (7.16) holds. Set $v_\varepsilon \triangleq M_{1\varepsilon} h$ and $v \triangleq M_i h$, where the operators $M_{1\varepsilon} : X^0 \rightarrow Y^1$ are defined similarly to the operators $M_i : X^0 \rightarrow Y^1$ with substituting $(b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$. By the definitions of the corresponding adjoint operators,

$$J_\varepsilon = (M_{1\varepsilon}^* \xi - M_i^* \xi, h)_{X^0} + ((\delta_T M_{1\varepsilon})^* \Psi - (\delta_T M_i)^* \Psi, h)_{X^0}$$

$$= (\xi, v_\varepsilon - v)_{X^0} + \left( \Psi, v_\varepsilon(\cdot, T) - v(\cdot, T) \right)_{Z^T_0}.$$

The function $V_\varepsilon \triangleq v_\varepsilon - v$ is the solution in $Q$ of the boundary value problem

$$dt V_\varepsilon = (A_\varepsilon V_\varepsilon + F_\varepsilon(v)) dt + \sum_{i=1}^N B_i V_\varepsilon dw_i(t),$$

$$V_\varepsilon(x, 0) = 0, \quad V_\varepsilon(x, t)_{|x \in \partial D} = 0,$$

where the operator $F_\varepsilon(\cdot)$ is defined above. The remaining part of the proof of (7.16) repeats the proof of (7.15). This completes the proof of Theorem 4.1. □
8 The proof of Theorems 4.2, 4.4 and 5.1

Proof of Theorem 4.2. Assume that Condition 4.1 holds. Let
\[ S_N = \left\{ \alpha = (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{R}^N : |\alpha| = \left( \sum_{i=1}^{N} |\alpha_i|^2 \right)^{1/2} \leq 1 \right\}. \]

Let \( y \in \mathbb{R}^n \) be fixed and let \( y_i = y_i(\alpha) \triangleq \alpha_i y, \alpha \in S_N \). Let \( y_i \triangleq \alpha_i y \) and \( z_i = z_i(y) = \beta_i^T y, \)
\( z = z(y) = (z_1, \ldots, z_N)^T \). By Condition 4.1,
\[ \sum_{i=1}^{N} y_i^\top b_i y_i \geq \frac{1}{2} \left( \sum_{i=1}^{N} y_i^\top \beta_i \right)^2 + \delta_1 \sum_{i=1}^{N} |y_i|^2 \]
for all \( \alpha \in S_N, (x,t) \in D \times [0, T] \) and \( \omega \in \Omega \). Hence
\[ y^\top b y = \sum_{i=1}^{N} \alpha_i^2 y^\top b y \geq \frac{1}{2} \left( \sum_{i=1}^{N} \alpha_i y^\top \beta_i \right)^2 + \delta_1 \sum_{i=1}^{N} |\alpha_i|^2 \]
for any \( \alpha \in S_N \). Hence
\[ y^\top b y \geq \sup_{\alpha \in S_N} \frac{1}{2} \left( \alpha^\top z(y) \right)^2 + \delta_1 |y|^2 = \frac{1}{2} |z(y)|^2 + \delta_1 |y|^2. \]
On the other hand,
\[ |z(y)|^2 = \sum_{i=1}^{N} |z_i(y)|^2 = \sum_{i=1}^{N} |y^\top \beta_i|^2. \]
Hence
\[ y^\top b y \geq \frac{1}{2} \sum_{i=1}^{N} |y^\top \beta_i|^2 + \delta_1 |y|^2. \]
Hence Condition 3.1 holds with \( \delta = \delta_1 \). □

Proof of Theorem 4.3. We have that \( 2b = \gamma + R \), where \( \gamma = \sum_{i=1}^{N} \beta_i^2 \) and \( R = R(x,t,\omega) \geq 2\delta \).
Let \( D \triangleq BB^\top = \{ \beta_i \beta_j \}_{i,j=1}^{N} \), where \( B \triangleq (\beta_1, \ldots, \beta_N)^\top \). It suffices to show that there exists \( \delta_1 > 0 \) such that
\[ \gamma(x,t,\omega)I_N - D(x,t,\omega) \geq 0 \quad (8.1) \]
for all \( x,t,\omega \), where \( I_N \) is the unit matrix in \( \mathbb{R}^{N \times N} \). Let \( \lambda = \lambda(x,t,\omega) \) be the minimal eigenvalue of the matrix \( \gamma(x,t,\omega)I_N - D(x,t,\omega) \). It suffices to show that \( \lambda \geq 0 \). Let \( z = z(x,t,\omega) \) be
a corresponding eigenvector such that $|z| = |B| \neq 0$ (for the trivial case $|B| = 0$, we have immediately that $\lambda = 0$). We have that $z = cB + B'$, where $c \in [-1, 1]$ and $B' = B'(x, t, \omega)$ is a vector such that $B^\top B' = 0$. By the definitions, we have that $\gamma = |B|^2$ and

$$\lambda z = (\gamma I_N - D)z = (\gamma I_N - BB^\top)(cB + B') = \gamma(cB + B') - c|B|^2 B$$

$$= \gamma cB + \gamma B' - c\gamma B = \gamma B'.$$

Hence $\lambda(cB + B') = \gamma B'$. It follows that either $B' \neq 0$, $c = 0$, and $\lambda = \gamma \geq 0$, or $\lambda = 0$ and $B' = 0$. This completes the proof. □

**Proof of Theorem 4.4.** By Hölder inequality, we have that

$$\left(\sum_{i=1}^{N_0} y_i^\top \beta_i\right)^2 \leq N_0 \sum_{i=1}^{N_0} \left(y_i^\top \beta_i\right)^2.$$

Hence

$$\sum_{i=1}^{N} y_i^\top b y_i - \frac{1}{2} \left(\sum_{i=1}^{N} y_i^\top \beta_i\right)^2 = \sum_{i=1}^{N} y_i^\top b y_i - \frac{1}{2} \left(\sum_{i=1}^{N} y_i^\top \beta_i\right)^2 \geq \sum_{i=1}^{N} y_i^\top b y_i - \frac{N_0}{2} \sum_{i=1}^{N_0} \left(y_i^\top \beta_i\right)^2 \geq \delta_2 \sum_{i=1}^{N} |y_i|^2.$$

This completes the proof. □.

**Proof of Theorem 5.1.** Let $p_\Delta = p^{(1)} - p^{(2)}$, $\chi_i_\Delta = \chi^{(1)}_i - \chi^{(2)}_i$, and let $A^{(k)*}, B_i^{(k)*}$ be the corresponding operators (3.1), $k = 1, 2$. We have that

$$dt p + (A^{(1)*} p + \psi) dt + \sum_{i=1}^{N} B_i^{(1)} \chi_i dt + \psi = \chi_i dw_i(t), \quad t \leq T,$$

$$p(x, 0, \omega) = \Psi^{(1)}(x, \omega) - \Psi^{(2)}(x, \omega), \quad p(x, t, \omega) |_{x \in \partial D} = 0.$$

Here

$$\psi \triangleq \xi^{(1)} - \xi^{(2)} + A^{(1)*} p^{(2)} - A^{(2)*} p^{(2)} + \sum_{i=1}^{N} (B_i^{(1)*} \chi^{(2)}_i - B_i^{(2)*} \chi^{(2)}_i).$$

By Theorem 4.1, it follows that there exists a constant $C_0 = C_0(\mathcal{P}^{(1)})$ such that

$$\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \leq C_0 \left(\|\psi\|_{X^0} + \|\Psi^{(1)} - \Psi^{(2)}\|_{Z_0}\right). \quad (8.2)$$
Further, we have $\psi = \sum_{m=0,1,2} \psi_m$, where
\[
\psi_0 = \xi^{(1)} - \xi^{(2)}, \\
\psi_1 = \Delta \sum_{i,j=1}^{n} [b^{(1)}_{ij} - b^{(2)}_{ij}] \frac{\partial^2 p^{(2)}}{\partial x_i \partial x_j} + \sum_{i=1}^{n} [f^{(1)}_i - f^{(2)}_i] \frac{\partial p^{(2)}}{\partial x_i} + [\chi^{(1)} - \chi^{(2)}] p^{(2)}, \\
\psi_2 = \sum_{i=1}^{N} \left( \sum_{i=1}^{n} [\beta^{(1)}_i - \beta^{(2)}_i] \frac{\partial \chi^{(2)}_i}{\partial x_i} + [\hat{\beta}^{(1)} - \hat{\beta}^{(2)}] \chi^{(2)}_i \right).
\]

Clearly,
\[
\|\psi_0\|_{X^0} \leq M, \quad \|\psi_1\|_{X^0} \leq CM\|p^{(2)}\|_{Y^2}, \quad \|\psi_2\|_{X^0} \leq CM + \sum_{i=1}^{N} \|\chi^{(2)}\|_{X^1},
\]
where $C = C(n)$ is a constant. Finally, we obtain
\[
\|\psi\|_{X^0} \leq C_1 M (\|p^{(2)}\|_{Y^2} + \sum_{i=1}^{N} \|\chi^{(2)}\|_{X^1} + 1),
\]
where $C_1 = C_1(n)$ is a constant. By (8.2), the desired estimate follows. This completes the proof. □

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