DERIVED FUNCTOR MODULES, DUAL PAIRS AND $\mathcal{U}(g)^K$-ACTIONS

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Abstract. Derived functors (or Zuckerman functors) play a very important role in the study of unitary representations of real reductive groups. These functors are usually applied on highest weight modules in the so-called good range and the theory is well-understood. On the other hand, there were several studies on the irreducibility and unitarizability, in which derived functors are applied to singular modules. See Enright et al. (Acta. Math. 1985), for example. In this article, we apply derived functors to certain modules arising from the formalism of local theta lifting, and investigate the irreducible sub-quotients of resulting modules. The key technique is to understand $\mathcal{U}(g)^K$ actions in the setting of a see-saw pair. Our results strongly suggest that derived functor constructions are compatible with local theta lifting.

1. Introduction

In this paper, we will study some small representations obtained by applying derived functors on certain local theta lifts. The main objective of this paper is to show that the resulting representations are still theta lifts. This project is motivated by [24].

1.1. For a Harish-Chandra pair $(g,K)$, we denote by $\mathcal{C}(g,K)$ the category of $(g,K)$-modules (not necessarily admissible). For a real reductive group $G$, $(g,K)$ is a Harish-Chandra pair, where $g = \text{Lie}(G)_C$ and $K$ is a maximal compact subgroup of $G$. Let $M$ be a subgroup of $K$. Zuckerman functor $\Gamma_{g,M}^K : \mathcal{C}(g,K) \rightarrow \mathcal{C}(g,M)$ is the right adjoint functor of the forgetful functor $F_{g,M}^K : \mathcal{C}(g,K) \rightarrow \mathcal{C}(g,M)$. This functor is left exact and usually send a module in $\mathcal{C}(g,M)$ to zero. On the other hand, its derived functors $R^j\Gamma_{g,M}^K$ construct interesting objects in $\mathcal{C}(g,K)$.

Using these derived functors, one can transfer representations between different real forms of a complex reductive group as follows: Let $g$ be the complex Lie algebra of a complex reductive group $G_C$. Let $\sigma_1$ and $\sigma_2$ be two commuting involutions on $g$. They define two Harish-Chandra pairs $(g, K_i)$ and the corresponding real forms $G_i \subset G_C$ such that $\mathfrak{k}_i := \text{Lie}(K_i)_C = g^{\sigma_i}$. Therefore, every Lie algebra in the following diagram is a

2010 Mathematics Subject Classification. 22E46, 22E47.

Key words and phrases. local theta lifts, Zuckerman functor, derived functor module, singular unitary representation.
symmetric subalgebra of the Lie algebra above it.

\[
\begin{array}{c}
\mathfrak{g} \\
\mathfrak{e}_1 \\
\mathfrak{e}_2 \\
\mathfrak{e}_1 \cap \mathfrak{e}_2
\end{array}
\]

Let \( M := K_1 \cap K_2 \). We define functor \( \Gamma^j \) by composing the forgetful functor and derived functors for each non-negative integer \( j \):

\[
\Gamma^j : \mathcal{C}(\mathfrak{g}, K_1) \xrightarrow{\mathcal{F}^{g,M}_{g,K_1}} \mathcal{C}(\mathfrak{g}, M) \xrightarrow{\mathcal{R}^{g,M}_{g,K_2}} \mathcal{C}(\mathfrak{g}, K_2).
\]

We call \( \Gamma^j \) the transfer functor, which transfers \((\mathfrak{g}, K_1)\)-modules into \((\mathfrak{g}, K_2)\)-module.

1.2. Let \( G_\mathbb{C} \) be a classical complex group. Let \( G_1 \) and \( G_2 \) be subgroups of \( G_\mathbb{C} \) with commuting Cartan involutions as in Section 1.1. Let \((G_1, G'_1)\) and \((G_2, G'_2)\) be two real reductive dual pairs such that \( G'_1 \) and \( G'_2 \) are real forms of a classical group \( G'_\mathbb{C} \). We always denote by \( \tilde{G} \) certain double covering of \( G \). Let \( \theta_i : \mathcal{R}(\mathfrak{g}', \tilde{K}'_i; \mathcal{Y}_i) \to \mathcal{R}(\mathfrak{g}, \tilde{K}_i; \mathcal{Y}_i) \) be the theta lifting map from \( \tilde{G}'_i \) to \( \tilde{G}_i \), where \( \mathcal{R}(\mathfrak{g}, \tilde{K}_i) \) is the set of infinitesimal equivalent classes of admissible irreducible \((\mathfrak{g}, \tilde{K}_i)\)-modules in the domain of theta lifting (c.f. Section 2) and \( \mathcal{Y}_i \) are fixed Fock-models of oscillator representations. Set \( K_i := \tilde{K}_i \), \( M = K_1 \cap K_2 \) and \( M := \tilde{K}_1 \cap \tilde{K}_2 = \tilde{M} \). We have

\[
(1) \quad \Gamma^j := R^j \Gamma^g_{g,M} \circ \mathcal{F}^{g,M}_{g,K_1} : \mathcal{C}(\mathfrak{g}, \tilde{K}_1) \longrightarrow \mathcal{C}(\mathfrak{g}, \tilde{M}) \longrightarrow \mathcal{C}(\mathfrak{g}, \tilde{K}_2).
\]

We will exhibit some relationships between the transfer functors \( \Gamma^j \) and theta lifting maps \( \theta_i \). An optimistic guess is that there exists certain operation "?" filling the gap and make the diagram (2) commute. Here the inclusions in (2) identify an infinitesimal equivalent class of Harish-Chandra modules with an element in it.

\[
\begin{array}{ccc}
\mathcal{R}(\mathfrak{g}', \tilde{K}'_1; \mathcal{Y}_1) & \xrightarrow{\theta_1} & \mathcal{R}(\mathfrak{g}, \tilde{K}_1; \mathcal{Y}_1) \\
\downarrow & & \downarrow \Gamma^j \\
\mathcal{R}(\mathfrak{g}', \tilde{K}'_2; \mathcal{Y}_2) & \xrightarrow{\theta_2} & \mathcal{R}(\mathfrak{g}, \tilde{K}_2; \mathcal{Y}_2)
\end{array}
\]

However, the actual relationships are more subtle. For example, Theorem 5.3 shows that \( \Gamma^j \theta_1(\rho) \) for \( \rho \in \mathcal{R}(\mathfrak{g}', \tilde{K}'_1; \mathcal{Y}_1) \) could be reducible and its irreducible components are theta lifts from different real reductive dual pairs.

1.3. We fix a non-trivial unitary character of \( \mathbb{R} \). Then we could fix the Fock-models \( \mathcal{Y}_i \) and the corresponding oscillator representations in (2) (see Section 2.3). We denote by \( \Theta_i \) the full theta lifting map of pair \((G_i, G'_i)\) (see Section 2.3). Our main theorem is the following.

**Theorem A.** Let \( G'_1 \) and \( G'_2 \) be two real forms of a classical complex Lie group \( G'_\mathbb{C} \) such that \((G_i, G'_i)\) form real reductive dual pairs. Let \( \rho_1 \) and \( \rho_2 \) be characters of \( \tilde{G}'_1 \) and \( \tilde{G}'_2 \).
respectively, \( \tau_1 \) be an irreducible \((\mathfrak{q}_2, \tilde{K}_1 \cap \tilde{K}_2)\)-module and \( \tau_2 \) be an irreducible \( \tilde{K}_2 \)-module. Suppose that:

(a) \( \rho_1|_{\mathfrak{q}'} = \rho_2|_{\mathfrak{q}'} \);
(b) \( \tau_2 \) occurs in \( \Theta_2(\rho_2) \);
(c) there exists a non-zero homomorphism \( T \in \text{Hom}_{\mathfrak{q}_2, \tilde{K}_1 \cap \tilde{K}_2}(\Theta_1(\rho_1), \tau_1) \), such that \( \tau_2 \) occurs in the image of \( \Gamma^j T : \Gamma^j \Theta_1(\rho_1) \to R^i(\Gamma^t, \tilde{K}_2)\tau_1 \).

Then the two \((\mathfrak{g}, \tilde{K}_2)\)-modules, \( \Gamma^j \Theta_1(\rho_1) \) and \( \Theta_2(\rho_2) \), have isomorphic irreducible subquotients with the common \( \tilde{K}_2 \)-type \( \tau_2 \).

Roughly speaking above theorem says that the \( \tilde{K} \)-spectrums determine the theta lifts of characters. In applications, \( \Theta_1(\rho_1) \) will be a direct sum of irreducible \((\mathfrak{q}_2, \tilde{M})\)-modules and the precise \( \tilde{K} \)-spectrums of their derived functor modules are well studied, see \([2, 15, 22, 24]\) and Section 3.

We highlight the key observation Lemma 2.8. It may have potential usage beyond the case of lifts of characters.

This paper is a part of the author’s Ph.D. Thesis.

**Notation.** In this paper, little Greek letters, for example \( \tau \), denote the infinitesimal equivalent classes of \((\mathfrak{g}, K)\)-modules and \( V_\tau \) denotes a realization of \( \tau \) on a vector space. Moreover, \( \tau \) also denotes the maps from \( \mathcal{U}(\mathfrak{g}) \) and \( K \) to \( \text{End}_C(V_\tau) \). We will not distinguish representations and their isomorphism classes.

When there is no confusion, we omit the symplectic space \( W \) in \( \text{Sp}(W) \) and denote it by \( \text{Sp} \). For any subgroup \( G < \text{Sp} \), \( \tilde{G} \) denotes the inverse image of \( G \) in the metaplectic group \( \tilde{\text{Sp}} \). We always write \( \mathfrak{g} \) for the complexification of \( \text{Lie}(G) \). Let \( K \) and \( K' \) denote the maximal compact subgroups of \( G \) and \( G' \). In this paper, a complex Lie subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) would always be stable under the fixed Cartan involution. Let \( H \) denote the maximal Lie subgroup of \( G \) such that \( \text{Lie}(H)_{\mathbb{C}} = \mathfrak{h} \). Therefore \( M := H \cap K \) is a maximal compact subgroup of \( H \). Every real reductive dual pair \((G, G')\) corresponds to a complex reductive dual pair \((G_{\mathbb{C}}, G'_{\mathbb{C}})\) by base change. We always denote by \( V^G \) the subspace of \( G \)-invariants in a \( G \)-module \( V \). In particular, \( \mathcal{U}(\mathfrak{g})^H \) is the subalgebra of \( H \)-invariants in the universal enveloping subalgebra \( \mathcal{U}(\mathfrak{g}) \). In the setting of transfer, \( H \) is always a symmetric subgroup of \( G_1 \) and \( M = K_1 \cap H = K_1 \cap K_2 \) is a maximal compact subgroup of \( H \).

## 2. Theta lifting, see-saw pair and joint \( \mathcal{U}(\mathfrak{g})^K \)-actions

### 2.1. We have following generalization of Howe’s construction of maximal quotient.

**Definition 2.2.** Let \( G \) be a real reductive group. Let \( K \) and \( H \) be subgroups of \( G \) and let \( M \) be a subgroup of \( K \cap H \). Let \( V \) be a \((\mathfrak{g}, K)\)-module and \( U \) be an irreducible admissible \((\mathfrak{h}, M)\)-module. Define

\[
\Omega_{V,U} = V/N_{V,U}, \quad \text{where} \quad N_{V,U} = \bigcap_{T \in \text{Hom}_{\mathfrak{h},\mathfrak{m}}(V,U)} \text{Ker}(T).
\]
Clearly $\Omega_{V,U}$ has joint actions of $H$ and $U(\mathfrak{g})^H$. It is known to experts (see [18, Lemma 2.III.4] or [17, Section 2.3.3]) that

$$\Omega_{V,U} \cong U \otimes U', \tag{3}$$

where $U(\mathfrak{g})^H$ acts on $U' \cong \text{Hom}_{\text{b,M}}(U, \Omega_{V,U})$.

The key property of $\Omega_{V,U}$ is the following equation:

$$\text{Hom}_{\mathfrak{g}, \text{K}}(V, U) \cong \text{Hom}_{\mathfrak{g}', \text{K}}(\Omega_{V,U}, U).$$

This equation leads to the well known see-saw pair argument due to Kudla (c.f. Section 2.5).

2.3. Now let $(G, G')$ be a real reductive dual pair in $\text{Sp}$. Let $U$ be a fixed maximal compact subgroup of $\text{Sp}$. Set $G := \tilde{\text{Sp}}, K := \tilde{\text{U}}, H := \tilde{\text{G}}, M := \tilde{\text{K}}, V := \mathcal{Y}$ the Fock model and $U := V_\rho$ an irreducible admissible $(\mathfrak{g}, \tilde{\text{K}})$-module. Set $\Theta(\rho) := U'$. Then (3) gives

$$\Omega_{\mathfrak{g}, V_\rho} \cong V_\rho \otimes \Theta(\rho).$$

Howe [9] shows that when $\Theta(\rho)$ is non-zero, it is a finite length $(\mathfrak{g}', \tilde{\text{K}}')$-module and it has a unique irreducible quotient $\theta(\rho)$. We call $\Theta(\rho)$ the full (local) theta lift of $\rho$ and $\theta(\rho)$ the (local) theta lift of $\rho$.

Denote by $\mathcal{R}(\mathfrak{g}, \tilde{\text{K}}; \mathcal{Y})$ the set of infinitesimal equivalent classes of irreducible admissible $(\mathfrak{g}, \tilde{\text{K}})$-modules such that $\Theta(\rho) \neq 0$, i.e. which could be realize as a quotient of $\mathcal{Y}$. Then $\rho \mapsto \theta(\rho)$ induces a one to one correspondence

$$\mathcal{R}(\mathfrak{g}, \tilde{\text{K}}; \mathcal{Y}) \overset{\theta}{\longrightarrow} \mathcal{R}(\mathfrak{g}', \tilde{\text{K}}'; \mathcal{Y}),$$

which is called (local) theta correspondence or Howe correspondence. The roles of $G$ and $G'$ are symmetric. By abusing notation, we also denote by $\Theta$ the lifting from $\tilde{\text{G}}'$ to $\tilde{\text{G}}$.

2.4. Let $V$ be an admissible $(\mathfrak{g}, K)$-module and $U$ be an irreducible $K$-module. Set $G := G$, $H := M := K$. Then $\Omega_{V,U}$ is the $U$-isotypic component of $V$. There is a well known result of Harish-Chandra [13] that irreducible constituents containing the $K$-type $U$ in the $(\mathfrak{g}, K)$-module $V$ are in one to one correspondence to irreducible constituents of the $U(\mathfrak{g})^K$-module $U' := \text{Hom}_K(U, \Omega_{V,U})$. Note that the right $U(\mathfrak{g})^K$-module, $\text{Hom}_K(V, U) = \text{Hom}_K(\Omega_{V,U}, U)$, is the dual of the finite dimensional left $U(\mathfrak{g})^K$-module $U'$. So we will only consider $\text{Hom}_K(V, U)$ in the rest of this paper.

2.5. See-saw pairs. A see-saw pair is a pair of reductive dual pairs $(G, G')$ and $(H, H')$ in $\text{Sp}$ such that $H < G$ and $H' > G'$. The relationship between these groups is given in the following diagram.

$$\begin{array}{ccc}
G & H' \\
\cup & \cup & \cup \\
H & G'
\end{array}$$

\footnote{It is called the maximal Howe quotient or big theta lift in some literatures.}
Lemma 2.6. Let \( \tau \in \mathcal{B}(\mathfrak{h}, \tilde{M}; \mathcal{Y}) \) and \( \rho \in \mathcal{B}(\mathfrak{g}', \tilde{K}'; \mathcal{Y}) \). Then
\[
\text{Hom}_{\mathfrak{h}, \tilde{M}}(\Theta(\rho), V_{\tau}) \cong \text{Hom}_{(\mathfrak{h}, \tilde{M}) \times (\mathfrak{g}', \tilde{K}')}((\mathcal{Y}, V_{\tau} \otimes \mathcal{V}_{\rho}) \cong \text{Hom}_{\mathfrak{g}', \tilde{K}'}(\Theta(\tau), V_{\rho})
\]
Here \( \text{Hom}_{\mathfrak{h}, \tilde{M}}(\Theta(\rho), V_{\tau}) \) is a right \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} \)-module and \( \text{Hom}_{\mathfrak{g}', \tilde{K}'}(\Theta(\tau), V_{\rho}) \) is a right \( \mathcal{U}(\mathfrak{h}')^{\tilde{G}'} \)-module by pre-composition. The first isomorphism in \( \mathbb{H} \) is \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} \)-equivariant and the second isomorphism is \( \mathcal{U}(\mathfrak{h}')^{\tilde{G}'} \)-equivariant.

Proof. It is clear from the following calculation.
\[
\text{Hom}_{\mathfrak{h}, \tilde{M}}(\Theta(\rho), V_{\tau}) \cong \text{Hom}_{(\mathfrak{h}, \tilde{M}) \times (\mathfrak{g}', \tilde{K}')}((\Theta(\rho) \otimes V_{\rho}, V_{\tau} \otimes V_{\rho})
\]
\[
\cong \text{Hom}_{(\mathfrak{h}, \tilde{M}) \times (\mathfrak{g}', \tilde{K}')}((\Omega(\mathcal{Y}, V_{\rho}), V_{\tau} \otimes V_{\rho})
\]
\[
\cong \text{Hom}_{(\mathfrak{h}, \tilde{M}) \times (\mathfrak{g}', \tilde{K}')}((\mathcal{Y}, V_{\tau} \otimes V_{\rho}) \cong \text{Hom}_{\mathfrak{g}', \tilde{K}'}(\Theta(\tau), V_{\rho}). \quad \square
\]

The above proof is formal and is actually valid for any category of representations and any “see-saw pairs” of mutually commuting subgroups when Schur’s Lemma holds.

2.7. In the case of local theta correspondence over real, Lemma 2.8 shows that the joint actions of \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} \) and \( \mathcal{U}(\mathfrak{h}')^{\tilde{G}'} \) on \( \mathcal{Y} \) factor through the same subalgebra of \( \text{End}_{\mathbb{C}}(\mathcal{Y}) \). Therefore \( \mathbb{H} \) links the \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} \) and \( \mathcal{U}(\mathfrak{h}')^{\tilde{G}'} \)-actions on its two sides. We would like to point out that Lee-Nishiyama-Wachi \( \mathbb{H} \) have obtained the lemma when \( \tilde{H} \) and \( \tilde{G}' \) are both compact in a study of Capelli identities. Lemma 2.8 is a generalization of the correspondence of infinitesimal characters \( \mathbb{H} \), where we set \( (H, H') := (G, G') \).

Lemma 2.8. Let \( \omega \) be an oscillator representation of \( \tilde{\text{Sp}} \) with Fock model \( \mathcal{Y} \). Let \( (G, G') \) and \( (H, H') \) form a see-saw pair in \( \text{Sp} \) such that \( H < G \) and \( G' < H' \). Then

(i) as subalgebras of \( \mathcal{U}(\mathfrak{g}) \) and \( \mathcal{U}(\mathfrak{h}') \),
\[
\mathcal{U}(\mathfrak{g})^{H_c} = \mathcal{U}(\mathfrak{g})^{\tilde{H}} = \mathcal{U}(\mathfrak{h}')^{G_c} = \mathcal{U}(\mathfrak{h}')^{\tilde{G}'}
\]
respectively;

(ii) as subalgebras of \( \text{End}_{\mathbb{C}}(\mathcal{Y}) \),
\[
\omega(\mathcal{U}(\mathfrak{g})^{\tilde{H}}) = \omega(\mathcal{U}(\mathfrak{g})^{H_c}) = \omega(\mathcal{U}(\mathfrak{h}')^{G_c}) = \omega(\mathcal{U}(\mathfrak{h}')^{\tilde{G}'}).
\]

In particular, here exist a map (may not be unique and may not be an algebra homomorphism) \( \Xi_{\mathfrak{g}, \mathfrak{h}'}: \mathcal{U}(\mathfrak{g})^{H_c} \rightarrow \mathcal{U}(\mathfrak{h}')^{G_c} \) such that \( \omega(x) = \omega(\Xi_{\mathfrak{g}, \mathfrak{h}'}(x)) \). Moreover, \( \Xi_{\mathfrak{g}, \mathfrak{h}'} \) could be defined only depending on the complex dual pair, but independent of real forms and \( \omega \).

Sketch of the proof (see \( \mathbb{H} \) Section 2.3.4 [17] for details). The actions of the double coverings on its Lie algebra factor through the linear group, \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} = \mathcal{U}(\mathfrak{g})^{H} \) and \( \mathcal{U}(\mathfrak{h}')^{\tilde{G}'} = \mathcal{U}(\mathfrak{h}')^{G'} \). By the classification of real reductive dual pairs, \( H \) meets all the connected components of \( H_c \). Hence, \( \mathcal{U}(\mathfrak{g})^{\tilde{H}} = \mathcal{U}(\mathfrak{g})^{H_c} \) and (i) follows.

To prove (ii), it suffices to show \( \omega(\mathcal{U}(\mathfrak{g})^{H_c}) = \omega(\mathcal{U}(\mathfrak{h}')^{G_c}) \). Let \( W_c \) be the complex symplectic space defining \( \mathfrak{sp} \) and let \( \mathfrak{c} = W_c \oplus \mathfrak{c} \) be the corresponding Heisenberg Lie algebra. Under the notation in \( \mathbb{S} \), let \( \text{End}^\circ \) be the image of \( \mathcal{U}(\mathfrak{c}) \) in \( \text{End}(\mathcal{Y}) \). Howe [8, Theorem 7] shows that
\[
\omega(\mathcal{U}(\mathfrak{g})) = (\text{End}^\circ)^{G_c} \quad \text{and} \quad \omega(\mathcal{U}(\mathfrak{h}')) = (\text{End}^\circ)^{H_c}
\]
by the classical invariant theory. Therefore
\[ \omega(\mathcal{U}(g)^{H_c}) = \omega(\mathcal{U}(g))^{H_c} = ((\text{End}^o)^{G'_c})^{H_c} = ((\text{End}^o)^{H_c})^{G'_c} = \omega(\mathcal{U}(h)^{G'_c}). \]

Note that \( \text{End}^o \) could be realized as an abstract quantum algebra which is only depending on \( W_c \). This ensures that \( \Xi_{g,h'} \) could be defined independent of real forms and oscillator representations. See Section 2.9. \( \Box \)

2.9. Remarks on Fock-models. The materials in the section is due to Kudla. See [11] Section 2 or [17] Section 2.3.4. We present a sketch here for completeness.

Let \( W_c \) be a complex symplectic space with symplectic form \( \langle \cdot, \cdot \rangle \). By the commutative diagram (5), \( \bigwedge \) has a natural filtration induced by the natural filtration on \( C \). By the commutative diagram (5), \( \Xi \) could be defined independent of real forms and oscillator representations.

Let \( W_c \) be real forms of \( G \) and \( G'_c \) in Howe's picture [8]. Howe’s result is rephrased into

\[ \Xi : \mathfrak{sp}(W_c) \ni \varepsilon \mapsto \bigwedge_2(W_c) \]

extending the natural map \( W_c \to \Omega_1(W_c) \). We extend above map to universal enveloping algebras and still call it \( \omega_c \).

Now fix a real form \( W \) of \( W_c \), i.e. \( W_c = W \otimes_R C \). A totally complex polarization of \( W_c \) with respect to \( W \) is a decomposition \( W_c = X \oplus Y \) such that \( X \) and \( Y \) are maximal isotropic \( C \)-subspaces in \( W_c \) and \( X \cap W = 0 \). The Fock-model of the oscillator representation of \( \mathfrak{sp}(W) \) associated with the central character \( \varphi \) of the Heisenberg group is given by

\[ \mathcal{F} := \Omega(W_c)/\Omega(W_c)X. \]

Here \( \mathfrak{sp}(W_c) = \mathfrak{sp}(W) \) acts on \( \mathcal{F} \) by compositing \( \omega_c \) and the left multiplication. Moreover, \( \mathcal{F} \cong C[X] \) as vector spaces and \( \Omega(W_c) \cong \text{End}^o \) in Howe’s picture [8].

Let \( (G_c, G'_c) \) be a complex dual pair in \( \mathfrak{sp}(W_c) \). Recall that \( g := \text{Lie}(G_c) \) and \( g' := \text{Lie}(G'_c) \). Howe [8]'s result is rephrased into

\[ \omega_c(\mathcal{U}(g)) = \Omega(W_c)G'c \quad \text{and} \quad \omega_c(\mathcal{U}(g')) = \Omega(W_c)G'_c. \]

The following lemma is a rephrase of the equation (2.4) in [9] and one can check it case by case according to the classification of real reductive dual pairs.

**Lemma 2.10.** Let \( G \) and \( G' \) be real forms of \( G_c \) and \( G'_c \) respectively such that \( (G, G') \) is a real reductive dual pair. Then there is a real form \( W \) of \( W_c \) such that

\[ \text{Lie}(G) = \mathfrak{sp}(W) \cap g, \quad \text{Lie}(G') = \mathfrak{sp}(W) \cap g', \]

where \( g, g', \text{Lie}(G), \text{Lie}(G') \) and \( \mathfrak{sp}(W) \) are considered to be Lie subalgebras of \( \mathfrak{sp}(W_c) \). \( \Box \)

Note that the oscillator representation \( \omega \) of \( \mathfrak{sp}(W) \) acts on the Fock model \( \mathcal{F} \) factor through \( \omega_c \). By the commutative diagram (5), \( \Xi_{g,h'}(x) \) in Lemma 2.8 could be made
independent of real forms via ω_c.

\[
\begin{array}{c}
g \oplus g' \xrightarrow{\omega_c} \text{sp}(W_c) \xrightarrow{\Omega(W_c)} \text{Lie}(G) \oplus \text{Lie}(G') \xrightarrow{\omega} \text{sp}(W) \xrightarrow{\text{left multiplication}} \text{End}_{\mathbb{C}}(\mathfrak{g})
\end{array}
\]

(5)

3. A theorem of Helgason and its consequences

The following result of Helgason [6] is crucial for us.

**Theorem 3.1.** Let G be a real reductive group such that all simple factors of g are classical Lie algebras. Let H be a symmetric subgroup of G: We suppose that there is an involution σ on Lie(G) such that H is a subgroup of G with Lie algebra Lie(G)σ and meet all the connected components of G. Let \( Z(g) = U(g)^G \) be the subalgebra of G-invariants in U(g). For a one-dimensional representation \( \rho \) of H, let \( \text{Ann}_{U(\mathfrak{h})}(\rho) \) be the annihilator ideal of \( \rho \) in \( U(\mathfrak{h}) \). Then the natural map

\[
Z(g) \longrightarrow U(g)^H/(\text{Ann}_{U(\mathfrak{h})}(\rho)U(g) \cap U(g)^H)
\]

is surjective.

Our \( Z(g) \) can be smaller than the center of \( U(g) \). For example, consider \( U(\mathfrak{so}(2n))^{O(2n)} \). This is exactly the situation we will encounter in local theta correspondence. 3.1 may not hold if g has some exceptional simple factors [7].

The above theorem is a small alteration of Helgason’s original version. Helgason [6] treats the case that H is a maximal compact subgroup of G with \( \rho \) trivial. Later Shimura extends it to non-trivial \( \rho \) (c.f. [20, Theorem 2.4]).

Shimura’s version and a Weyl’s “Unitary Trick” (see for example [12]) implies above theorem. We omit the routine proof, see [17, Section 3.A] for details.

3.2. Following the argument in [25], we combine 3.1 and the see-saw pair argument (Lemma 2.6). Let \( (G, G') \) and \( (H, H') \) form a see-saw pair such that H is a symmetric subgroup of G. By the classification of real reductive dual pairs, \( G' \) is automatically a symmetric subgroup of \( H' \) and satisfies the condition in 3.1.

**Lemma 3.3.** Let \( \rho \in \mathcal{R}(g', \tilde{K}'; \mathcal{Y}) \) be a character of \( g' \) and let \( \tau \in \mathcal{R}(\mathfrak{h}, \tilde{M}; \mathcal{Y}) \). Then \( U(g)^{\tilde{H}} \) acts on \( \text{Hom}_{b, \tilde{M}}(\Theta(\rho), V_\tau) \) via a character. This character is determined by the character \( \chi_\tau \) of \( Z(\mathfrak{h}) \) acting on \( \tau \) and the annihilator ideal \( \text{Ann}_{U(\mathfrak{g}')}(\rho) \).

More precisely, there exists a map

\[
\xi: U(g)^{\tilde{H}} = U(g)^{H_c} \longrightarrow Z(\mathfrak{h}) \quad \text{such that} \quad T \circ \omega(x) = \chi_\tau(\xi(x))T
\]

for \( T \in \text{Hom}_{b, \tilde{M}}(\Theta(\rho), V_\tau) \). Moreover, \( \xi \) could be defined only depending on \( \text{Ann}_{U(\mathfrak{g}')}(\rho) \), but independent of formal real forms and \( \omega \).

**Proof.** By Lemma 2.8 for \( x \in U(g)^{\tilde{H}} \), let \( x' := \Xi_{g,h'}(x) \in U(h')^{\tilde{G}} \). We have \( \omega(x) = \omega(x') \) by the definition of \( \Xi_{g,h'} \). Choose \( z' \in Z(\mathfrak{h}') \) such that \( x' - z' = a'u' \) with \( a' \in \text{Ann}_{U(\mathfrak{g}')}(\rho) \).
and $u' \in \mathcal{U}(\mathfrak{h}')$ by Lemma 3.1. Let $z := \Xi_{\mathfrak{g}',\mathfrak{h}'}(z') \in \mathcal{Z}(\mathfrak{h})$. Therefore $\omega(z) = \omega(z')$ by Lemma 2.8 again.

For any $0 \neq T \in \text{Hom}_{(\mathfrak{h},\mathfrak{h}) \times (\mathfrak{g}',\mathfrak{k}')}((\mathfrak{g}', V_\tau \otimes V_\rho),$$T \circ \omega(x) = T \circ \omega(x') = T \circ \omega(x' - z') + T \circ \omega(z')$$= T \circ \omega(a'u') + T \circ \omega(z) = \rho(a') \circ T \circ \omega(u') + \tau(z) \circ T$$= \chi_T(z)T.$

Define $\xi(x) := z$. Now the Lemma follows from the $\mathcal{U}(\mathfrak{h})$-module isomorphism (4):

$$\text{Hom}_{(\mathfrak{h},\mathfrak{h}) \times (\mathfrak{g}',\mathfrak{k}')}((\mathfrak{g}', V_\tau \otimes V_\rho, V_\tau) \cong \text{Hom}_{(\mathfrak{h},\mathfrak{h}) \times (\mathfrak{g}',\mathfrak{k}')}((\mathfrak{g}', V_\tau \otimes V_\rho)

Since $\Xi_{\mathfrak{g},\mathfrak{h}'}$, $\Xi_{\mathfrak{g}',\mathfrak{h}}$ and $z'$ could be defined independent of real forms and $\omega$, so dose $\xi$. This finish the proof. \hfill \square

4. Proof of Theorem [A]

4.1. We recall some basic facts from the construction of derived functor modules (see for example, [23, Chapter 6]). We retain the notation in Section 1.1 where $V \in \mathcal{C}(\mathfrak{g}, \mathcal{M})$.

As $K$-modules,

$$R^j(\Gamma^g_{\mathfrak{g},\mathfrak{M}})V \bigg|_K = R^j(\Gamma^t_{\mathfrak{t},\mathfrak{M}})V.$$

Hence we also denote $R^j(\Gamma^t_{\mathfrak{t},\mathfrak{M}})$ by $\Gamma^j$.

For each $x \in \mathcal{U}(\mathfrak{g})^K$, it gives a $(\mathfrak{t}, \mathcal{M})$-map $x : V \to V$. Then $x$ acts on $\Gamma^jV$ by $\Gamma^j : \Gamma^jV \to \Gamma^jV$. Hence we have following lemma.

**Lemma 4.2.** Let $V \in \mathcal{C}(\mathfrak{g}, \mathcal{M})$ and $U \in \mathcal{C}(\mathfrak{t}, \mathcal{M})$. Let $T \in \text{Hom}_{\mathfrak{t},\mathcal{M}}(V, U)$. Suppose that $T \circ x = \chi(x)T$ for a charactor $\chi : \mathcal{U}(\mathfrak{g})^K \to \mathbb{C}$. Then

(i) $\mathcal{U}(\mathfrak{g})^K$ acts on $\Gamma^jT \in \text{Hom}_{K}(\Gamma^jV, \Gamma^jU)$ via that character $\chi$;

(ii) in particular, if $\mathcal{Z}(\mathfrak{g})$ acts on $V$ via a character, then $\mathcal{Z}(\mathfrak{g})$ acts on $\Gamma^jV$ via the same character. \hfill \square

Suppose $V$ is $\mathcal{Z}$-finite where $\mathcal{Z}$ is the center of $\mathcal{U}(\mathfrak{g})$. For example $V := \Theta_1(\rho_1)$ in the next section. By Lemma 4.2 (ii), $\Gamma^jV$ is also $\mathcal{Z}$-finite. Hence, every finite dimensional $K$-invariant subspace in $\Gamma^jV$ will generate an admissible $(\mathfrak{g}, \mathcal{K})$-module (c.f. [24]). It is also clear that $\text{Ann}_{\mathcal{U}(\mathfrak{g})}(\Gamma^jV) \supset \text{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$. So an irreducible subquotient of $\Gamma^jV$ has Gelfand-Kirillov dimension less than or equal to that of $V$.

4.3. We retain the notation in Section 1.3.

**Proof of Theorem 4.4.** We can fix a complex dual pair $(G_c, G'_c)$ sitting in $\text{Sp}(W_c)$ such that $G_i \subset G_c$ and $G'_i \subset G'_c$ with commuting Cartan involutions.\footnote{These Cartan involutions of $G_i$ and $G'_i$ can be obtained by restricting a pair of commuting Cartan involutions of $\text{Sp}(W_c)$}

Let $\mathfrak{h} := \mathfrak{t}_2$ and $H$ be the maximal subgroup in $G_1$ with Lie algebra $\mathfrak{h} \cap \text{Lie}(G_1)$. Now $H$ is the member of a dual pair $(H, H')$ and $M := H \cap K_2 = K_1 \cap K_2$ is a maximal compact subgroup of $H$ by the classification of real reductive dual pairs.
Let $V_{\tau_1}$ be a $(\mathfrak{t}_2, \widetilde{M})$-module of type $\tau_1$. Let

$$0 \neq T \in \text{Hom}_{\mathfrak{t}_2, \tilde{M}}(\Theta_1(\rho_1), V_{\tau_1})$$

be the map in assumption (c). Since $\tau_2$ occur in the image of $\Gamma^j T$, we fix an irreducible $\tilde{K}_2$-submodule $U_{\tau_2}$ of type $\tau_2$ in $\Gamma^j \Theta(\rho_1)$ such that $(\Gamma^j T)(U_{\tau_2}) \neq 0$. Let

$$\mathcal{V} := \mathcal{U}(\mathfrak{g}) U_{\tau_2} \subset \Gamma^j \Theta(\rho_1)$$

be the admissible $(\mathfrak{g}, \tilde{K}_2)$-submodule of $\Gamma^j V$ generated by $U_{\tau_2}$. Let $\mathcal{V}$ be the image of $U_{\tau_2}$ under $\Gamma^j T$. We have a right $\mathcal{U}(\mathfrak{g})\tilde{K}_2$-module homomorphism

$$\text{Hom}_{\tilde{K}_2}(\Gamma^j \Theta(\rho_1), \Gamma^j V_{\tau_1}) \longrightarrow \text{Hom}_{\tilde{K}_2}(\mathcal{V}, \mathcal{V})$$

by pre-compose the restriction and post-compose the projection. Denote by $\mathcal{F}$ the image of $\Gamma^j T$ under above homomorphism.

Notice that, as subalgebras in $\mathcal{U}(\mathfrak{sp})$, $\mathcal{U}(\mathfrak{g})\tilde{H} = \mathcal{U}(\mathfrak{g})H = \mathcal{U}(\mathfrak{g})K_2 = \mathcal{U}(\mathfrak{g})\tilde{K}_2$ and $\mathcal{Z}(\mathfrak{h}) = \mathcal{U}(\mathfrak{t}_2)^\tilde{H} = \mathcal{U}(\mathfrak{t}_2)^H = \mathcal{U}(\mathfrak{t}_2)^{K_2} = \mathcal{Z}(\mathfrak{t}_2)$.

Since $V_{\tau_1}$ is an irreducible $(\mathfrak{t}_2, \tilde{M})$-module and $\mathcal{V}$ is an irreducible $\tilde{K}_2$-submodule, $\mathcal{Z}(\mathfrak{t}_2)$ act on them by characters. On the other hand, $\mathcal{V}$ is a submodule in $\Gamma^j V_{\tau_1}$, we conclude that $\mathcal{Z}(\mathfrak{t}_2)$ acts on $V_{\tau_1}$, $\mathcal{V}$ and $\Gamma^j V_{\tau_1}$ by the same character (c.f. Lemma 4.2 (ii)).

The assumption (a) implies $\text{Ann}_{\mathcal{U}(\mathfrak{g})\rho_1} = \text{Ann}_{\mathcal{U}(\mathfrak{g})\rho_2}$. Now $\mathcal{U}(\mathfrak{g})\tilde{H} = \mathcal{U}(\mathfrak{g})\tilde{K}_2$ act on $\text{Hom}_{\mathfrak{t}_2, \tilde{M}}(\Theta_1(\rho_1), V_{\tau_1})$ and $\text{Hom}_{\tilde{K}_2}(\Theta_2(\rho_2), \mathcal{V})$ via the same character by Lemma 3.3. In particular, $\mathcal{U}(\mathfrak{g})\tilde{K}_2$ acts on $c\mathcal{F}$ via this character by Lemma 4.2 (i).

Hence $\mathcal{V}$ has an irreducible quotient containing $U_{\tau_2}$ and it is isomorphic to the irreducible subquotient of $\Theta_2(\rho_2)$ containing $\tau_2$ by the discussion in Section 2.4. This finished the proof.

It is known to experts that $\Theta_2(\rho_2)$ is $\tilde{K}_2$-multiplicity free for any character $\rho_2$, i.e. $\dim \text{Hom}_{\tilde{M}}(\Theta_2(\rho_2), \tau_2) \leq 1$ for every $\tilde{K}_2$-type $\tau_2$ (see for example, [26] or [17] Section 2.3.6). On the other hand, the multiplicity of $\tau_2$ in $\Gamma^j \Theta(\rho_1)$ could be greater than one. For all examples in Section 5, it is the $\tilde{K}_2$-isotypic component of several copies of a $\tilde{K}_2$-multiplicity free irreducible $(\mathfrak{g}, \tilde{K}_2)$-module.

5. Examples

In this section, we give two types of examples. In these examples, the full theta liftings are already irreducible. So we replace $\Theta$ by $\theta$ when we apply A.

5.1. A decomposition of the derived functor module. We retain notations in Section 1.1. Suppose the $(\mathfrak{g}, M)$-module $V$ is a direct sum of irreducible unitarizable $(\mathfrak{t}, M)$-modules, i.e.

$$V = \bigoplus_{l \in L} V_l,$$

where the Harish-Chandra pair $(\mathfrak{t}, M)$ comes from some real reductive group $H$ and $V_l$ are irreducible unitarizable $(\mathfrak{t}, M)$-modules. In fact, all examples in the next sections are in
this case and they are typical examples of discretely decomposable modules in the sense of Kobayashi [10].

Following Wallach-Zhu [24], we have a decomposition of the \((\mathfrak{g}, K)\)-module \(\Gamma^j V\) by Vogan-Zuckerman’s theory [21]:

\[
\Gamma^j V = \bigoplus W \Gamma_W V.
\]

We describe the decomposition briefly. See [24] or [17, Section 2.4.2] for the details of the construction. Here we fix a decomposition of \(\bigwedge \mathfrak{k}/\mathfrak{m}\) into irreducible \(M\)-submodules and \(W\) runs over all its irreducible components. For each \(V_i\) let \(\hat{K}_i\) be the set of \(K\)-type \(\gamma\) such that \(V_i\) and \(\gamma\) have the same infinitesimal characters and central characters. Then the \((\mathfrak{g}, K)\)-module

\[
\Gamma_W V := \bigoplus_{l \in L} \bigoplus_{\gamma \in \hat{K}_i} \text{Hom}_M(W, \bigoplus_{j} V_j \otimes \gamma^\ast) \otimes \gamma
\]

as \(K\)-module, where \(\Gamma_W V_i\) are \(K\)-submodules of \(\Gamma^j V_i\). Most of the terms \(\Gamma_W V\) in (6) are zero, one can determine the non-zero terms by Vogan-Zuckerman’s theory [21].

### 5.2. Transfer of unitary highest weight modules

To have unitary highest weight modules, the pair \((\mathfrak{g}, K)\) should be Hermitian symmetric. We will study three families of examples where \(\mathfrak{g}\) has root system of type \(A, C\) and \(D\) respectively.

| Type | \(G\) | \(G'_{p,q}\) | \(H\) | stable range | \(j(p,q)\) |
|------|-------|--------|------|-------------|----------|
| \(A\) | \(U(n, n)\) | \(U(p, q)\) | \(U(r, s) \times U(s, r)\) | \(n \geq p + q\) | \(rs - (r - p)(s - q)\) |
| \(C\) | \(\text{Sp}(2n, \mathbb{R})\) | \(\text{O}(p, q)\) | \(U(r, s)\) | \(n \geq p + q\) | \(2(rs - (r - p)(s - q))\) |
| \(D\) | \(\text{O}^\ast(2n)\) | \(\text{Sp}(p, q)\) | \(U(r, s)\) | \(n \geq 2(p + q)\) | \(rs - (r - 2p)(s - 2q)\) |

**Table 1.** Transfer of unitary highest weight modules: (Here \(r + s = n\))

See Table 1 for notation. We fix a real form \(G \subset G_C\) with Cartan involution \(\sigma_1\). We set \(G_1 := G\). Fix an involution \(\sigma_2\) commuting with \(\sigma_1\) such that \(H = G_2^{\sigma_2} \cap G_1\). Let \(G_2\) be the real form of \(G_C\) such that \(\sigma_2\) is its Cartan involution. Let \(K_i = G_1^{\sigma_i}, M = K_1 \cap K_2\) and define \(\Gamma^j\) as in (1).

In this setting, we have \(\tilde{G} = \tilde{G}_1 \cong \tilde{G}_2\). So \(\Gamma^j\) can be thought as an operation which transfers representations of \(\tilde{G}\).

Let \(\theta^{p,q}\) be the theta lifting map from \(\tilde{G}'_{p,q}\) to \(\tilde{G}\). In the dual pair \((G, G'_{p,q})\), the double covering \(\tilde{G}'_{p,q}\) split: \(\tilde{G}'_{p,q} \cong \mathbb{Z}/2\mathbb{Z} \times G'_{p,q}\). So there is a canonical genuine character \(\zeta\) of \(\tilde{G}'_{p,q}\) whose restriction on \(G'_{p,q}\) part is trivial. By twisting with \(\zeta\), we identify \(G'_{p,q}\)-module with \(\tilde{G}'_{p,q}\)-module. We abuse notation and denote \(\theta^{p,q}(\rho \otimes \zeta)\) by \(\theta^{p,q}(\rho)\).

We consider characters of \(G'_{p,q}\) whose restriction on its Lie algebra is trivial. In type \(A\) and type \(D\), trivial representation is the only one. In type \(C\), \(G'_{p,q} = \text{O}(p, q)\). Let \(1^{\xi, \eta}\) be the character of \(\text{O}(p, q)\) such that \(1^{\xi, \eta}|_{\text{O}(p) \times \text{O}(q)} = \det^{\xi}_{\text{O}(p)} \otimes \det^{\eta}_{\text{O}(q)}\). When \(p, q \neq 0\), i.e. \(\text{O}(p, q)\) is non-compact, all of the four characters of \(\text{O}(p, q)\) are represented by \(1^{\xi, \eta}\) with
ξ, η ∈ Z/2Z. When one of p, q is zero, i.e. O(p, q) is compact, there are two characters: the trivial representation and the determinant.

**Theorem 5.3.** Fix positive integers n, m, r, s such that r + s = n. Let (G, G'_{m,0}) be in the stable range with G'_{m,0} the smaller member. For type A and type D, let ρ and ρ_{p,q} be trivial representations; for type C, let ρ = det^{ε}_{O(m)} and ρ_{p,q} = 1^{ε,m} where ξ ≡ ε − (s − q) (mod 2) and η ≡ ε − (r − p) (mod 2). Then

\[
\Gamma^j \theta^{m,0}(\rho) = \bigoplus_{j = j(p,q)} \theta^{p,q}(\rho_{p,q}),
\]

Here 0 ≤ p ≤ r, 0 ≤ q ≤ s in type A and type C; 0 ≤ 2p ≤ r, 0 ≤ 2q ≤ s in type D.

4. The module θ^{m,0}(\rho) is a singular unitary highest weight module. All such modules are classified in [3] and they are obtained from theta lifting for classical groups. So θ^{m,0}(\rho) ⊆ Γ^j \theta^{m,0}(\rho) is a cohomological operation constructing \( \tilde{G} \)-modules from these relatively well understood modules. This type of construction is studied extensively in [2][4][5][22][24]. Frajria [5] studied Γ^j \theta^{m,0}(\rho) at the first non-vanishing degree. He expects these modules could be fit in the dual pair correspondence. Later, Wallach and Zhu made a precise conjecture [24, Conjecture 5.1] for type C and they show that (7) holds on K-spectrum level. In these works, the irreducibility and unitarizability of the resulting modules is the main concern. Here the unitarizability of Γ^j \theta^{m,0}(\rho) could follow from Li’s result [14] on the unitarizability of stable range theta lifts by 5.3. Moreover Γ^j \theta^{m,0}(\rho) could be reducible when there are (precisely) two pairs \((p_1, q_1)\) and \((p_2, q_2)\) such that \(j = j(p_1, q_1) = j(p_2, q_2)\).

If we consider the derived functor Γ^j in all degrees together, there is a simple formula:

\[
\bigoplus_{j \in \mathbb{N}} \Gamma^j \theta^{m,0}(\rho) \cong \bigoplus_{p+q=m} \theta^{p,q}(\rho_{p,q}).
\]

It would be nice if one could explain above formula in terms of some Euler characteristic formula in the Grothendieck group by adding some ±-signs. However, we have no idea how to do it yet.

**Sketch of the proof of 5.4.** Let \(H'\) be the centralizer of \(H\) in the real symplectic group containing the pair \((G, G'_{m,0})\). Now \((H, H')\) is a compact dual pair, the Fock space \(\mathcal{Y}\) is already decomposing into a direct sum of irreducible unitarizable \((\mathfrak{h}, \tilde{M})\)-modules. So it is easy to see that \(\theta^{m,0}(\rho)\) has the same property (see [24] or [17, Lemma 49]).

A see-saw pair argument gives

\[
\theta^{m,0}(\rho)|_{\mathfrak{h}, \tilde{M}} = \bigoplus_{\mu \in \mathcal{R}(H'; H)} n_{\mu} L(\mu).
\]

Here \(L(\mu)\) is the theta lift of the \(\tilde{H}'\)-module \(\mu\); \(n_{\mu} = \dim \text{Hom}_{\tilde{G}}(\mu, \rho)\) is the multiplicity of \(L(\mu)\) occur in \(\theta^{m,0}(\rho)\). In fact, the decomposition is multiplicity free, since \(G'_{m,0}\) is a symmetric subgroup of \(H'\).
Note that $L(\mu)$ is irreducible and unitarizable. We apply the decomposition in Section 5.1: as $(\mathfrak{g}, \tilde{K}_2)$-module,

$$\Gamma^j \theta^{m,0}(\rho) = \bigoplus_{\mathcal{W}} \Gamma_{\mathcal{W}} \theta^{m,0}(\rho).$$

We apply Vogan-Zuckerman’s theory [21] to calculate $\Gamma^j L(\mu)$ case by case, see [24] and [17] Section 3.5.1 for details. The calculation shows that there are $\tilde{M}$-submodules $\mathcal{W}_{p,q}$ with multiplicity one in $\wedge^{(p,q)} \mathfrak{h}/\mathfrak{m}$ such that as $\tilde{K}_2$-module,

$$(8) \quad \Gamma_{\mathcal{W}_{p,q}} \theta^{m,0}(\rho) \cong \theta^{p,q}(\rho_{p,q}).$$

Moreover, $\Gamma_{\mathcal{W}} \theta^{m,0}(\rho) = 0$ for other $\mathcal{W}$.

Setting $G'_1 := G'_{m,0}$ and $G'_2 := G'_{p,q}$, we conclude that (8) also holds as $(\mathfrak{g}, \tilde{K}_2)$-module by \(\Box\). This completes the proof.

5.5. Transfer of singular unitary representations. In this section we consider another type of examples in which singular unitary representations are transferred between different real forms.

$$\begin{array}{|c|c|c|c|c|} \hline \text{Type} & G_{p,q} & G' & \text{stable range} & j_0 \\ \hline A & U(p, q) & U(n_1, n_2) & p, q \geq n_1 + n_2 & (n_1 + n_2)r \\ C & \text{Sp}(p, q) & \text{O}^*(2n) & p, q \geq n & 2nr \\ D & \text{O}(p, q) & \text{Sp}(2n, \mathbb{R}) & \text{max}\{p, q\} \geq 2n & nr \\ \hline \end{array}$$

Table 2. Transfer of singular unitary representations

See Table 2 for notation. We fix a real form $G_1 \subset G_{\mathbb{C}}$ with Cartan involution $\sigma_1$ such that $G_1 \cong G_{p,q}$. Now fix an involution $\sigma_2$ commuting with $\sigma_1$ such that $H := G_{p,q}^c \cap G_1 \cong G_{p,r} \times G_{0, q-r}$. Let $G_2$ be the real form of $G_{\mathbb{C}}$ such that $\sigma_2$ is its Cartan involution. It is clear that $G_2 \cong G_{p+r,q-r}$. We define functor $\Gamma^j$ by (1).

In type $A$ and type $D$, we will assume $p+q$ is even. Then the double covering $\tilde{G}'$ is split, i.e. $\tilde{G}' \cong G' \times \mathbb{Z}/2\mathbb{Z}$ with respect to the dual pair $(G_{p,q}, G')$. Fix the genuine character $\varsigma$ of $\tilde{G}'$ which is nontrivial on $\mathbb{Z}/2\mathbb{Z}$ and trivial on $G'$. We again identify genuine $\tilde{G}'$-modules with $G'$-modules via twisting of $\varsigma$. In particular, the trivial $G'$-module 1 corresponds to $\varsigma$. We denote by $\theta_{p,q}$ the theta lifting map from $\tilde{G}'$ to $\tilde{G}_{p,q}$.

Note that $\theta_{p,q}(1)$ is not a highest weight module except for the pairs $(\text{O}(2, q), \text{Sp}(2, \mathbb{R}))$. The latter situation is studied in [4] Section 8.

**Theorem 5.6.** Fix positive integers $p, q, n, r$ ($n_1, n_2$ for type $A$) such that $p+q$ is even in type $A$ and type $D$. We assume that $(G_{p,q}, G')$ is in the stable range with $G'$ the smaller member and $r < q$. Let $\theta_{p,q}(1)$ be the theta lift of the trivial representation of $G'$.

(i) If $(G_{p+r,q-r}, G')$ is outside the stable range, $\Gamma^j \theta_{p,q}(1) = 0$ for every $j$.

(ii) If $(G_{p+r,q-r}, G')$ is in the stable range,

$$\Gamma^j \theta_{p,q}(1) = \begin{cases} \theta_{p+r,q-r}(1) & \text{when } j = j_0 \\ 0 & \text{when } j < j_0, \end{cases}$$
where $j_0$ is defined in Table 3. If $j > j_0$, $\Gamma^j \theta_{p,q}(1)$ is a direct sum of several copies of $\theta_{p+r,q-r}(1)$.

Sketch of the proof. See [17, Section 3.5.2] for the details. Here $\theta_{p,q}(1)$ is again a direct sum of irreducible unitarizable $(\mathfrak{h}, \tilde{M})$-modules by the argument in [15]. We apply the decomposition in Section 5.1. For $\Gamma^j \theta_{p,q}(1) = 0$ as $\tilde{K}_2$-module already; For there is a unique $\tilde{M}$-type $W_0 \subset \bigwedge \mathfrak{h}$ such that $\Gamma_{W_0} V_l$ is non-zero for some $l$. It first occurs in $\bigwedge \mathfrak{h}/m$ with multiplicity 1 and occurs in $\bigwedge \mathfrak{h}/m$ with some multiplicities for $j > j_0$. One calculates that

$$\Gamma_{W_0} \theta_{p,q}(1) = \theta_{p+r,q-r}(1)$$

as $\tilde{K}_2$-module. Setting $G_1 = G_2 = G'$ and applying $A$, we finish the proof. □

5.7. The above theorem generalize the results of type $D$ in [16] and the proof here is conceptually simpler. By the same argument in [16], we extend the theorem to theta lifts of unitary highest weight modules:

**Corollary 5.8.** Fix integers $p, q, r, s, n$ ($n_1, n_2$ for type $A$) such that $p + q + s$ is even in type $A$ and type $D$. We retain notations in Table 3 and assume that $(G_{p,q+s}, G')$ is in the stable range. Let $\mu$ be a genuine $\tilde{G}_{s,0}$-module (finite dimensional since $\tilde{G}_{s,0}$ is compact). Let $L(\mu)$ be the unitary highest weight $\tilde{G}'$-module lifted from $\mu$.

(i) If $(G_{p+r,q+s-r}, G')$ is not in the stable range, $\Gamma^j \theta_{p,q}(L(\mu)) = 0$ for all $j \in \mathbb{N}$.

(ii) If $(G_{p+r,q+s-r}, G')$ is in the stable range,

$$\Gamma^j \theta_{p,q}(L(\mu)) = \begin{cases} \theta_{p+r,q-r}(L(\mu)) & \text{when } j = j_0 \\ 0 & \text{when } j < j_0 \end{cases}$$

where $j_0$ is defined in Table 3. If $j > j_0$, $\Gamma^j \theta_{p,q}(L(\mu))$ is a direct sum of several copies of $\theta_{p+r,q-r}(L(\mu))$.

5.9. In view of 5.3, 5.6 and 5.8, we speculate that theta correspondence and derived functors would be compatible upto Langlands packets or Arthur packets.

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