Type II vertex operators for the $A_{n-1}^{(1)}$ face model

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Abstract

Presented is a free boson representation of the type II vertex operators for the $A_{n-1}^{(1)}$ face model. Using the bosonization, we derive some properties of the type II vertex operators, such as commutation, inversion and duality relations.

1 Introduction

Recent development on integrable lattice models is mainly based on the representation theory of the quantum affine groups [1, 2]. For example, the XXZ spin chain model has the $U_q(\hat{\mathfrak{sl}}_2)$ symmetry, which is large enough to diagonalize the XXZ Hamiltonian. Correlation functions and form factors are also given as the traces of products of vertex operators. The integral formulae of the correlation functions for the XXZ model were presented by using the free boson realization of the vertex [2, 3].

In [1] vertex operators were introduced as the intertwiners among the highest weight modules and the finite dimensional representation of quantum affine groups. There exist two kinds of vertex operators, the type I and the type II vertex operators in the terminology of [1], because of the left-right asymmetry of the coproduct of the quantum affine groups. The XXZ Hamiltonian is expressed in terms of the type I vertex operators, whereas the creation and annihilation operators are constructed in terms of the type II vertex operators [1].

The vertex operator method can be also formulated for face models [1, 3]. Bosonization of the type I vertex operators for the RSOS model [3] were given by Lukyanov and Pugai [4], in which they derived an integral representation for multi-point local state probabilities. In [8] a bosonization of the type I vertex operators was realized for the $A_{n-1}^{(1)}$ face model [3], that reduces to the RSOS model [3] when $n = 2$. In [10] a bosonization of the type II vertex operators was realized for the RSOS model. The $A_{n-1}^{(1)}$ face model has the deformed Virasoro algebra symmetry [11]. Since the Virasoro algebra has no non-trivial deformation of the coproduct, those vertex operators cannot be interpreted as intertwiners.
of the deformed Virasoro algebra. However, the type I vertex operators allow a graphical interpretation which is due to the identification between the bosonized vertex operators and the half transfer matrix for the $A_{n-1}^{(1)}$ face model [3, 8].

In this paper we present a bosonization of the type II vertex operators for the $A_{n-1}^{(1)}$ face model. Unfortunately, the type II vertex operators do not allow interpretation as intertwiners of the deformed Virasoro algebra nor graphical interpretation. Our construction is thus based on the commutation relations among the type I and II vertex operators.

The rest of the paper is organized as follows. In section 2 we formulate the problem and introduce the type II vertex operators for the $A_{n-1}^{(1)}$ face model. In section 3 we realize the type II vertex operators for the $A_{n-1}^{(1)}$ face model in terms of free bosons. We prove the commutation relations of the bosonized operators in order to show that they are bona fide vertex operators. In section 4 we prove various properties of the vertex operators such as the inversion and the duality relations.

## 2 Type II vertex operators for the $A_{n-1}^{(1)}$ face model

The present section aims to formulate the problem, thereby fixing the notation.

### 2.1 Theta functions

Throughout this paper we fix the integers $n$ and $r$ such that $r \geq n + 2$, and also fix the parameter $x$ such that $0 < x < 1$. We will use the abbreviations,

$$[v] = x^{rac{r^2}{r-x}} \Theta_{x^r}(x^2 v), \quad [v]' = x^{-\frac{r^2}{r-x}} \Theta_{x^{r-2}}(x^2 v),$$

where the Jacobi theta function is given by

$$\Theta_q(z) = (q; q)_\infty (qz^{-1}; q)_\infty (q, q)_\infty,$$

$$(z; q_1, \ldots, q_m) = \prod_{i_1, \ldots, i_m \geq 0} (1 - q_1^{i_1} \cdots q_m^{i_m}).$$

For later conveniences we also introduce the following symbols

$$r_l(v) = z^{\frac{r-1}{r-1} \frac{n-l}{2l} g_l(z^{-1})}, \quad g_l(z) = \frac{\{x^{2n+2r-l-1}z\}}{\{x^{2n-l+1}z\}} \frac{\{x^{2r-l+1}z\}}{\{x^{2r-l+1}z\}},$$

$$r^*_l(v) = z^{\frac{r-1}{r-1} \frac{n-l}{2l} g^*_l(z^{-1})}, \quad g^*_l(z) = \frac{\{x^{2n+2r-l-1}z\}}{\{x^{2n-l+1}z\}} \frac{\{x^{2r-l+1}z\}}{\{x^{2r-l+1}z\}},$$

$$\chi_l(v) = z^{\frac{(r-n)}{r-1}} \frac{\rho_l(z^{-1})}{\rho_l(z)}, \quad \rho_l(z) = \frac{\{x^{2l+1}z; x^{2n}\}_\infty (1-x^{2n+2l+1}z; x^{2n}x^2)_\infty}{\{x^{2l}z; x^{2n}x^2\}_\infty (1-x^{2n+2l+1}z; x^{2n}x^2)_\infty}$$

where $z = x^2$, $1 \leq l \leq n$ and

$$\{z\} = (z; x^{2r}, x^{2n})_\infty, \quad \{z\}' = (z; x^{2r-2}, x^{2n})_\infty.$$
In particular we define \( R(v), S(v) \) and \( \chi(v) \) by
\[
R(v) = r_1(v) = z^{-\frac{n+1}{2}} \frac{\rho(z)}{\rho^*}, \quad \rho(z) = g_1(z) = \frac{\{x^{2z}\} \{x^{2n+2r-2}\}}{\{x^{2r}\} \{x^{2n}\}}, \quad (2.8)
\]
\[
S(v) = s_1(v) = z^{-\frac{n+1}{2}} \frac{\rho^*(z)}{\rho}, \quad \rho^*(z) = g_1^*(z) = \frac{\{x^{2n-2}\} \{x^{2r}\}}{\{x^{2n+2r-2}\} \{z\}}, \quad (2.9)
\]
and
\[
\chi(v) = \chi_1(v) = z^{-\frac{n+1}{2}} \frac{(-x^{2n-1} z^{-1}, x^{2n})_\infty (-x z; x^{2n})_\infty}{(-x^{-1} z; x^{2n})_\infty (-x^{2n-1} z; x^{2n})_\infty}. \quad (2.10)
\]
These factors will appear in the commutation relations among the type I and the type II vertex operators.

The integral kernel for the type I and the type II vertex operators will be given as the products of the following elliptic functions
\[
f(v, w) = \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]}, \quad h(v) = \frac{[v - 1]}{[v + 1]}, \quad (2.11)
\]
\[
f^*(v, w) = \frac{[v - \frac{1}{2} + w]'}{[v + \frac{1}{2}]'}, \quad h^*(v) = \frac{[v + 1]'}{[v - 1]'}, \quad (2.12)
\]

2.2 The weight lattice of \( A_{n-1}^{(1)} \)

Let \( V = \mathbb{C}^n \) and \( \{\varepsilon_\mu\}_{1 \leq j \leq n} \) be the standard orthogonal basis with the inner product \( \langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu} \).

The weight lattice of \( A_{n-1}^{(1)} \) is defined as follows:
\[
P = \bigoplus_{\mu=1}^n \mathbb{Z} \varepsilon_\mu, \quad (2.13)
\]
where
\[
\varepsilon_\mu = \varepsilon_\mu - \varepsilon, \quad \varepsilon = \frac{1}{n} \sum_{\mu=1}^n \varepsilon_\mu.
\]

We denote the fundamental weights by \( \omega_\mu (1 \leq \mu \leq n - 1) \)
\[
\omega_\mu = \sum_{\nu=1}^\mu \varepsilon_\nu,
\]
and also denote the simple roots by \( \alpha_\mu (1 \leq \mu \leq n - 1) \)
\[
\alpha_\mu = \varepsilon_\mu - \varepsilon_{\mu+1} = \varepsilon_\mu - \varepsilon_{\mu+1}.
\]

For \( a \in P \) we set
\[
\alpha_{\mu\nu} = \langle a + \rho, \varepsilon_\mu - \varepsilon_\nu \rangle, \quad \rho = \sum_{\mu=1}^{n-1} \omega_\mu. \quad (2.14)
\]

2.3 The \( A_{n-1}^{(1)} \) face model

An ordered pair \( (a, b) \in P^2 \) is called admissible if \( b = a + \varepsilon_\mu \), for a certain \( \mu (1 \leq \mu \leq n) \). For \( (a, b, c, d) \in P^4 \) let \( W_I \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \) be the Boltzmann weight of the \( A_{n-1}^{(1)} \) model for the state configuration \( \begin{array}{cc} c & d \\ b & a \end{array} \) round a face. Here the four states \( a, b, c \) and \( d \) are ordered clockwise from the
The bosonization of $\Phi^{\mu}$ obeys the following commutation relations

\begin{align}
W_I \left( \begin{array}{cc}
  a + 2\bar{\varepsilon}_\mu & a + \bar{\varepsilon}_\mu \\
  a + \bar{\varepsilon}_\mu & a
\end{array} \right) | v \rangle &= R(v),
\end{align}

\begin{align}
W_I \left( \begin{array}{cc}
  a + \bar{\varepsilon}_\mu & a + \bar{\varepsilon}_\nu \\
  a + \bar{\varepsilon}_\nu & a
\end{array} \right) | v \rangle &= R(v) \frac{[v][a_{\mu \nu} - 1]}{[v - 1][a_{\mu \nu}]} (\mu \neq \nu),
\end{align}

\begin{align}
W_I \left( \begin{array}{cc}
  a + \bar{\varepsilon}_\mu & a + \bar{\varepsilon}_\nu \\
  a + \bar{\varepsilon}_\nu & a
\end{array} \right) | v \rangle &= R(v) \frac{1}{[v - 1][a_{\mu \nu}]} (\mu \neq \nu).
\end{align}

The Boltzmann weights (2.17) solve the Yang-Baxter equation for the face model [8]:

\begin{align}
\sum_g W_I \left( \begin{array}{cc}
  d & e \\
  c & g
\end{array} \right) W_I \left( \begin{array}{cc}
  c & g \\
  b & a
\end{array} \right) W_I \left( \begin{array}{cc}
  e & f \\
  g & a
\end{array} \right) W_I \left( \begin{array}{cc}
  d & g \\
  c & b
\end{array} \right) | v_1 - v_2 \rangle
\end{align}

\begin{align}
= \sum_g W_I \left( \begin{array}{cc}
  g & f \\
  b & a
\end{array} \right) W_I \left( \begin{array}{cc}
  d & e \\
  g & f
\end{array} \right) W_I \left( \begin{array}{cc}
  e & f \\
  g & a
\end{array} \right) W_I \left( \begin{array}{cc}
  d & g \\
  c & b
\end{array} \right) | v_1 - v_2 \rangle
\end{align}

2.4 Commutation relations

Consider the type I vertex operators satisfying the following commutation relations

\begin{align}
\Phi_{\mu_1}(v_1)\Phi_{\mu_2}(v_2) = \sum_{\varepsilon_{\mu_1} + \varepsilon_{\mu_2} = \varepsilon_{\mu_1'} + \varepsilon_{\mu_2'}} W_I \left( \begin{array}{ccc}
  k + \bar{\varepsilon}_{\mu_1} + \bar{\varepsilon}_{\mu_2} & k + \bar{\varepsilon}_{\mu_2'} & v_1 - v_2 \\
  k + \bar{\varepsilon}_{\mu_2} & k &
\end{array} \right) \Phi_{\mu_2'}(v_2)\Phi_{\mu_1'}(v_1).
\end{align}

The bosonization of $\Phi_{\mu}(v)$ is realized in [8]. See also section 3. The dual type II vertex operators should obey the following commutation relations

\begin{align}
\Phi_{\mu_1}(v_1)\Psi_{\mu_2}^*(v_2) = \chi(v_1 - v_2)\Psi_{\mu_2}^*(v_2)\Phi_{\mu_1}(v_1),
\end{align}

\begin{align}
\Psi_{\mu_1}^*(v_1)\Psi_{\mu_2}^*(v_2) = \sum_{\varepsilon_{\mu_1} + \varepsilon_{\mu_2} = \varepsilon_{\mu_1'} + \varepsilon_{\mu_2'}} W_{II}^* \left( \begin{array}{ccc}
  l + \bar{\varepsilon}_{\mu_1} + \bar{\varepsilon}_{\mu_2} & l + \bar{\varepsilon}_{\mu_2'} & v_2 - v_1 \\
  l + \bar{\varepsilon}_{\mu_2} & l &
\end{array} \right) \Psi_{\mu_2'}^*(v_2)\Psi_{\mu_1'}^*(v_1),
\end{align}
where $W_{II}^* \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) | u \right)$ are defined by

$$W_{II}^* \left( \begin{array}{cc} a + 2\bar{\varepsilon}_\mu & a + \bar{\varepsilon}_\mu \\ a + \bar{\varepsilon}_\mu & a \end{array} \right) | v \right) = S(v),$$

$$W_{II}^* \left( \begin{array}{cc} a + \bar{\varepsilon}_\mu + \bar{\varepsilon}_\nu & a + \bar{\varepsilon}_\mu \\ a + \bar{\varepsilon}_\nu & a \end{array} \right) | v \right) = S(v) \left[ \frac{v'}{\nu} \right] [a_{\mu\nu}]^{'},$$

$$W_{II}^* \left( \begin{array}{cc} a + \bar{\varepsilon}_\mu + \bar{\varepsilon}_\nu & a + \bar{\varepsilon}_\nu \\ a + \bar{\varepsilon}_\nu & a \end{array} \right) | v \right) = S(v) \left[ \frac{v}{\nu} \right] [a_{\mu\nu}]^{'}. $$

Let us summarize the properties of $W_{II}^*$. Since it is obtained from $W_I$ by replacing $r \rightarrow r - 1$ up to a common scalar function, $W_{II}^*$ also satisfies the Yang-Baxter equation:

$$\sum_g W_{II}^* \left( \begin{array}{cc} d & e \\ c & g \end{array} \right) | v_1 \right) W_{II}^* \left( \begin{array}{cc} c & g \\ b & a \end{array} \right) | v_2 \right) W_{II}^* \left( \begin{array}{cc} e & f \\ g & a \end{array} \right) | v_1 - v_2 \right) = \sum_g W_{II}^* \left( \begin{array}{cc} g & f \\ b & a \end{array} \right) | v_1 \right) W_{II}^* \left( \begin{array}{cc} d & e \\ g & f \end{array} \right) | v_2 \right) W_{II}^* \left( \begin{array}{cc} d & g \\ c & b \end{array} \right) | v_1 - v_2 \right),$$

$$\quad (2.20)$$

The first and the second inversions relations as follows [8]:

$$\sum_g W_{II}^* \left( \begin{array}{cc} c & g \\ b & a \end{array} \right) | -v \right) W_{II}^* \left( \begin{array}{cc} c & d \\ g & a \end{array} \right) | v \right) = \delta_{bd},$$

$$\quad (2.21)$$

$$\sum_g G_a^* W_{II}^* \left( \begin{array}{cc} g & b \\ d & c \end{array} \right) | n - v \right) W_{II}^* \left( \begin{array}{cc} g & d \\ b & a \end{array} \right) | v \right) = \delta_{al} G_a^* G_d^*,$$

$$\quad (2.22)$$

where

$$G_a^* = \prod_{1 \leq \mu < \nu \leq n} [a_{\mu\nu}]'.$$

The Boltzmann weights (2.20) have $\sigma$-invariant [9]:

$$W_{II}^* \left( \begin{array}{cc} \sigma(c) & \sigma(d) \\ \sigma(b) & \sigma(a) \end{array} \right) | v \right) = W_{II}^* \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) | v \right),$$

$$\quad (2.23)$$

where $\sigma$ is the diagram automorphism of $A_{n-1}^{(1)}$ defined by $\sigma(\omega_\mu) = \omega_{\mu+1}$.

In section 3 we shall realize $\Psi^*_n(v)$ satisfying (2.18) and (2.19) in terms of free bosons.

### 2.5 Fused $A_{n-1}^{(1)}$ Boltzmann weight

Let us introduce $m$-fold fused Boltzmann weights for $W_{II}^*$. See [8] concerning the fused Boltzmann weights for $W_I$. 

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Let $\Lambda = \{\lambda_1, \cdots, \lambda_m\}$ be a subset of $N = \{1, \cdots, n\}$ such that $\lambda_1 < \cdots < \lambda_m$. For $\kappa, \mu \in N$ set $\mu = \kappa$ if $\kappa \in \Lambda$, otherwise set $\mu \in \Lambda \cup \{\kappa\}$. For given $\kappa, \mu, \Lambda$ let $1 \leq \nu_1 < \cdots < \nu_m \leq n$ be such that $\xi_\mu + \xi_{\nu_1} + \cdots + \xi_{\nu_m} = \xi_\kappa + \xi_{\lambda_1} + \cdots + \xi_{\lambda_m}$.

The fusion of $W_{II}^\ast$ in the horizontal direction is constructed as follows. Let $a, b, c(= c_0), c_1, \cdots c_{m-1}, c_m(= d) \in P$ satisfy

$$c = b + \xi_\mu, \ c_{j-1} - c_j = \xi_{\lambda_j} \ (1 \leq j \leq m), \ d = a + \xi_\kappa.$$ 

Note that $b = a + \xi_{\nu_1} + \cdots + \xi_{\nu_m}$ from the definition of $\nu_j$’s. Let $\sigma \in S_m$ be a permutation of $(1, \cdots, m)$, and set

$$b_0^\sigma = b, \ b_j^\sigma = b_{j-1}^\sigma - \xi_{\nu_{\sigma(j)}} \ (1 \leq j \leq m), \ b_m^\sigma = a.$$

Then $m$-fold anti-symmetric fusion of $W_{II}^\ast$ in the horizontal direction is given as

$$W_{II}^{(1,m)} \left( \begin{array}{ccc} c & d & v \\ a & b & \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W_{II}^\ast \left( \begin{array}{ccc} c_{j-1} & c_j & v + \frac{m+1}{2} - j \\ b_{j-1}^\sigma & b_j^\sigma & \end{array} \right).$$

Note that $W_{II}^{(1,m)}$ is anti-symmetric with respect to $(\lambda_1, \cdots, \lambda_m)$.

Next consider the fusion in the vertical direction. We use same $\kappa, \mu, \lambda_j$’s and $\nu_j$’s as before. Now we set

$$b = a + \xi_\mu, \ d_{j-1} - d_j = \xi_{\lambda_j} \ (1 \leq j \leq m), \ c = d + \xi_\kappa,$$

where $d_0 = d, d_m = a$. We have $c = b + \xi_{\nu_1} + \cdots + \xi_{\nu_m}$. For $\sigma \in S_m$ set

$$c_0^\sigma = c, \ c_j^\sigma = c_{j-1}^\sigma - \xi_{\nu_{\sigma(j)}} \ (1 \leq j \leq m), \ c_m^\sigma = b.$$

Then $m$-fold anti-symmetric fusion of $W_{II}^\ast$ in the vertical direction is given as

$$W_{II}^{(m,1)} \left( \begin{array}{ccc} c & d & v \\ b & a & \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W_{II}^\ast \left( \begin{array}{ccc} \ c_{j-1} & d_{j-1} & v - \frac{m+1}{2} + j \\ \ c_j^\sigma & d_j^\sigma & \end{array} \right).$$

Note that $W_{II}^{(m,1)}$ is anti-symmetric with respect to $(\lambda_1, \cdots, \lambda_m)$.

We further introduce the fusion of $W_{II}^\ast$ in both horizontal and vertical directions. Let $\{\kappa_j\}_{1 \leq j \leq m}$ and $\{\mu_j\}_{1 \leq j \leq m}$ be subsets of $N$ such that $\sharp \{\kappa_j\} = \sharp \{\mu_j\} = m$. Let $\{\lambda_j\}_{1 \leq j \leq m'}$ and $\{\nu_j\}_{1 \leq j \leq m'}$ be subsets of $N$ such that $\sharp \{\lambda_j\} = \sharp \{\nu_j\} = m'$. Let $a, b, c, d \in P$ satisfy

$$d = a + \sum_{j=1}^{m} \xi_{\kappa_j}, \ c = d + \sum_{j=1}^{m'} \xi_{\lambda_j}, \ b = a + \sum_{j=1}^{m} \xi_{\nu_j}, \ c = b + \sum_{j=1}^{m} \xi_{\mu_j},$$

where

$$\sum_{j=1}^{m} \xi_{\kappa_j} + \sum_{j=1}^{m'} \xi_{\lambda_j} = \sum_{j=1}^{m} \xi_{\mu_j} + \sum_{j=1}^{m'} \xi_{\nu_j}.$$

The $m \times m'$-fold fusion of $W_{II}^\ast$ is defined as the antisymmetrized product of the $m'$-fold fusion of $W_{II}^\ast$ in the horizontal direction:

$$W_{II}^{(m,m')} \left( \begin{array}{ccc} c & d & v \\ b & a & \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W_{II}^{(1,m')} \left( \begin{array}{ccc} \ c_{j-1} & d_{j-1} & v - \frac{m+1}{2} + j \\ \ c_j^\sigma & d_j^\sigma & \end{array} \right).$$
where
\[ c_0^\sigma = c, \quad c_j^\sigma = c_{j-1}^\sigma - \bar{\varepsilon}_{\nu, j} (1 \leq j \leq m), \quad c_m^\sigma = b. \]

The \( W_{II}^{(m,m')} \) can be also defined as the antisymmetrized product of the \( m \)-fold fusion of \( W_{II}^* \) in the vertical direction:
\[
W_{II}^{(m,m')} (c, d; b, a | v) = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \prod_{j=1}^{m'} W_{II}^{(m,1)} (b_j^\sigma \; b_j^\sigma; \; c_j \; v + \frac{m+1}{2} - j),
\]
where
\[ b_0^\sigma = b, \quad b_j^\sigma = b_{j-1}^\sigma - \bar{\varepsilon}_{\nu, j} (1 \leq j \leq m'), \quad b_m^\sigma = a. \]

### 2.6 Fusion of the dual type II vertex operators

Here we introduce the fusion of the dual type II vertex operator \( \Psi_\mu^*(v) \). Let \( \Lambda = \{\lambda_1, \cdots, \lambda_m\} \) be the subset of \( N = \{1, \cdots, n\} \) such that \( \lambda_1 < \cdots < \lambda_m \). When \( b = a + \bar{\varepsilon}_{\lambda_1} + \cdots + \bar{\varepsilon}_{\lambda_m} \) we define the \( m \)-fold fused type II vertex operator \( \Psi_\Lambda^{(m)} \) as follows:
\[
\Psi_\Lambda^{(m)} (v) := \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \Psi_{\lambda_1}^{(1)} (v + \frac{m-1}{2}) \Psi_{\lambda_2}^{(2)} (v + \frac{m-3}{2}) \cdots \Psi_{\lambda_m}^{(m)} (v - \frac{m-1}{2}).
\]

It is clear from the definition that the following commutation relations hold
\[
\Psi_{\Lambda_1}^{(m)} (v) \Psi_{\Lambda_2}^{(m')} (v') = \sum_d W_{II}^{(m,m')} (c, d; b, a | v' - v) \Psi_{\Lambda_2}^{(m')} (v') \Psi_{\Lambda_1}^{(m)} (v),
\]
if \( \Psi_\mu^* \)'s satisfy the commutation relations (2.18).

For the special case \( \Lambda = \{1, \cdots, m\} \) and \( \Lambda_\mu = \Lambda \setminus \{\mu\} \) we define
\[
\Psi_{\mu}^{(m-1)} (v) = \prod_{j=1}^{m-1} c_{j-1} \Psi_{\lambda_{\mu}}^{(m-1)} (v) \quad (1 \leq \mu \leq m).
\]

where
\[
c_j = x^{- \frac{r-1}{2} \left( 1 + \frac{2j-1}{2r} \right)} \frac{\left( x_2^{2r-2}; x_2^{-2r-2} \right)_\infty \left( x_2^{2r-2}; x_2^{-2r-2} \right)_{2j-3}}{\eta_j^{-1} (x^{-j})}, \quad \text{for } j = 1, \cdots, n-1.
\]

In section 3 we introduce the type II vertex operator \( \bar{\Psi}_{\mu}^{(m-1)} \) in terms of free bosons. In section 4 we shall prove \( \bar{\Psi}_{\mu}^{(m-1)} \) coincides with \( \Psi_{\mu}^{(m-1)} \) up to a constant.

### 3 Bosonization of the type II vertex operators

#### 3.1 Bosons

Let us consider the bosons \( B_{j}^{l} \) \( (1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\}) \) with the commutation relations
\[
[B_{m}^{j}, B_{m'}^{k}] = \begin{cases} \frac{m(n-1)m'_{j} [(r-1)m'_{j}]_{x} \delta_{m+m',0} \delta_{j=k}}, \quad (j = k) \\
-mx^{sgn(j-k)nm} \frac{[m]_{x} [m'_{j}]_{x} (r-1)_{x} \delta_{m+m',0}}{[nm]_{x} [rm]_{x}}, \quad (j \neq k). \end{cases}
\]
where the symbol \( [a]_x \) stands for \((x^a - x^{-a})/(x - x^{-1})\). Define \( B_m^i \) by
\[
\sum_{j=1}^{n} x^{-2jm} B_m^j = 0.
\]
Then the commutation relations (3.1) holds for all \( 1 \leq j, k \leq n \). These oscillators were introduced in [12, 13].

Define the dressed bosons \( A_m^j (1 \leq j \leq n, m \in \mathbb{Z} \setminus \{0\}) \) by
\[
A_m^j = (-1)^m \frac{[rm]_x}{[r-1]^m_x} B_m^j.
\]
The expression (3.2) for \( n = 2 \) was already given in [10]. For \( \alpha, \beta \in \mathcal{P} \) let us define the zero mode operators \( P_\alpha, Q_\beta \) with the commutation relations
\[
[iP_\alpha, Q_\beta] = \langle \alpha, \beta \rangle, \quad [P_\alpha, B_m^j] = [Q_\beta, B_m^j] = 0.
\]
We will deal with the bosonic Fock spaces \( \mathcal{F}_{l,k} \), \((l, k \in \mathcal{P})\) generated by \( B_m^j \) over the vacuum vectors \(|l, k\rangle\):
\[
\mathcal{F}_{l,k} = \mathbb{C}[\{B_m^j, B_m^{-j}, \cdots \}_{1 \leq j \leq n}]|l, k\rangle,
\]
where
\[
B_m^j |k, l\rangle = 0 \ (m > 0), \quad P_\alpha |l, k\rangle = \langle \alpha, \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k |l, k\rangle, \quad |l, k\rangle = e^{i\sqrt{\frac{r}{r-1}}Q_l - i\sqrt{\frac{r-1}{r}}Q_k} |0, 0\rangle.
\]

### 3.2 Basic Operator

Let us define the basic operators for \( j = 1, \cdots, n - 1 \)

\[
U_{-\alpha_j}(z) = \exp \left(i \sqrt{\frac{r}{r-1}} (Q_{\alpha_j} - iP_{\alpha_j} \log z) \right) : \exp \left( \sum_{m \neq 0} \frac{1}{m} (B_m^j - B_m^{j+1}) (x^j z)^{-m} \right) : , \quad (3.3)
\]

\[
U_{\omega_j}(z) = \exp \left(-i \sqrt{\frac{r}{r-1}} (Q_{\omega_j} - iP_{\omega_j} \log z) \right) : \exp \left( - \sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^{j} x^{(j-2k+1)m} B_m^k z^{-m} \right) : , \quad (3.4)
\]

and

\[
V_{-\alpha_j}(z) = \exp \left(-i \sqrt{\frac{r}{r-1}} (Q_{\alpha_j} - iP_{\alpha_j} \log z) \right) : \exp \left( - \sum_{m \neq 0} \frac{1}{m} (A_m^j - A_m^{j+1}) x^{-mj} z^{-m} \right) : , \quad (3.5)
\]

\[
V_{\omega_j}(z) = \exp \left(i \sqrt{\frac{r}{r-1}} (Q_{\omega_j} - iP_{\omega_j} \log z) \right) : \exp \left( \sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^{j} x^{(j-2k+1)m} A_m^k z^{-m} \right) : . \quad (3.6)
\]
Note that the operator $U_{-\alpha_j}(z)$ and $U_{\omega_j}(z)$ ($j = 1, \cdots, n - 1$) for $A_{n-1}^{(1)}$ face model were introduced in [3], and that the operator $V_{-\alpha_1}(z)$ and $V_{\omega_1}(z)$ for the $A_1^{(1)}$ face model were introduced in [10].

We will use the variable $v$ such that $z = x^{2v}$, and set

$$
\xi_j(v) = U_{-\alpha_j}(z), \quad \eta_j(v) = U_{\omega_j}(v), \quad \xi_j^*(v) = V_{-\alpha_j}(z), \quad \eta_j^*(v) = V_{\omega_j}(z),
$$

for $j = 1, \cdots, n - 1$. These operators satisfy the following commutation relations:

$$
\begin{align*}
\xi_j(v_1)\xi_j(v_2) &= h(v_1 - v_2)\xi_j(v_2)\xi_j(v_1), \\
\xi_j^*(v_1)\xi_j^*(v_2) &= h^*(v_1 - v_2)\xi_j^*(v_2)\xi_j^*(v_1), \\
\xi_j(v_1)\xi_{j+1}(v_2) &= -f(v_1 - v_2, 0)\xi_{j+1}(v_2)\xi_j(v_1), \\
\xi_j^*(v_1)\xi_{j+1}^*(v_2) &= -f^*(v_1 - v_2, 0)\xi_{j+1}^*(v_2)\xi_j^*(v_1), \\
[\xi_j(v_1), \xi_k(v_2)] &= [\xi_j^*(v_1), \xi_k^*(v_2)] = 0, \quad (j - k > 1), \\
\xi_j(v_1)\eta_j(v_2) &= -f(v_1 - v_2, 0)\eta_j(v_2)\xi_j(v_1), \\
\xi_j^*(v_1)\eta_j^*(v_2) &= -f^*(v_1 - v_2, 0)\eta_j^*(v_2)\xi_j^*(v_1), \\
[\xi_j(v_1), \eta_k(v_2)] &= [\xi_j^*(v_1), \eta_k^*(v_2)] = 0, \quad (j \neq k), \\
[\xi_j(v_1), \xi_k^*(v_2)] &= [\xi_j^*(v_1), \eta_k(v_2)] = [\eta_j(v_1), \xi_k^*(v_2)] = 0, \quad (j, k = 1, \cdots, n - 1), \\
\eta_j(v_1)\eta_j(v_2) &= r_j(v_1 - v_2)\eta_j(v_2)\eta_j(v_1), \quad r_1(v) = R(v), \\
\eta_j^*(v_1)\eta_j^*(v_2) &= s_j(v_2 - v_1)\eta_j^*(v_2)\eta_j^*(v_1), \quad s_1(v) = S(v), \\
\eta_j(v_1)\eta_j^*(v_2) &= \chi_j(v_1 - v_2)\eta_j^*(v_2)\eta_j(v_1), \quad \chi_1(v) = \chi(v).
\end{align*}
$$

### 3.3 Type I vertex operators

In the sequel we set

$$
\pi_\mu = \sqrt{r(r - 1)}P_{\pi_\mu}, \quad \pi_{\mu\nu} = \pi_\mu - \pi_\nu.
$$

The $\pi_{\mu\nu}$ acts on $\mathcal{F}_{l,k}$ as an integer $\langle \varepsilon_\mu - \varepsilon_\nu, r l - (r - 1)k \rangle$.

For $1 \leq \mu \leq n$ define the type I vertex operator [3] by

$$
\Phi_\mu(v_0) = \oint \prod_{j=1}^{\mu-1} \frac{dz_j}{2\pi i z_j} \eta_1(v_0)\xi_1(v_1) \cdots \xi_{\mu-1}(v_{\mu-1}) \prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \pi_{j\mu}),
$$

where $z_j = x^{2v_j}$. The integral contour for $z_j$-integration encircles the poles at $z_j = x^{1+2kr}z_{j-1}$ ($k \in \mathbb{Z}_{\geq 0}$), but not the poles at $z_j = x^{-1-2kr}z_{j-1}$ ($k \in \mathbb{Z}_{\geq 0}$).

Note that

$$
\Phi_\mu(v_0) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k+\varepsilon_\mu}.
$$

We thus denote the operator (3.20) by $\Phi_\mu^{(k+\varepsilon_\mu,k)}(v_0)$ on the bosonic Fock space $\mathcal{F}_{l,k}$. The commutation relations (3.14) hold on $\mathcal{F}_{l,k}$. 

9
For $1 \leq \mu \leq n$ define the dual type I vertex operator by
\[
\tilde{\Phi}_\mu^{(m-1)}(v_m) = \oint \prod_{j=\mu}^{m-1} \frac{dz_j}{2\pi i z_j} \eta_{m-1}(v_m) \xi_{m-1}(v_{m-1}) \cdots \xi_{\mu}(v_{\mu}) \prod_{j=\mu+1}^{m} f(v_{j-1} - v_j, \pi_{j\mu}),
\] (3.22)
where $z_j = x^{2v_j}$. The integral contour for $z_j$-integration encircles the poles at $z_j = x^{1+2kr} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$), but not the poles at $z_j = x^{-1-2kr} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$).

The operators (3.22) is an operator such that
\[
\tilde{\Phi}_\mu^{(m-1)}(v_m) : F_{l,k} \rightarrow F_{l,k+\bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_m - \varepsilon_\mu}.
\] (3.23)
In particular $\tilde{\Phi}_\mu^{(n-1)}(v_n) : F_{l,k} \rightarrow F_{l,k-\varepsilon_\mu}$ for $m = n$.

### 3.4 Type II vertex operator

In this subsection we introduce the type II vertex operator for the $A^{(1)}_{n-1}$ face model. For $1 \leq \mu \leq n$, define the dual type II vertex operators by
\[
\Psi^*_\mu(v_0) = \oint \prod_{j=1}^{\mu-1} \frac{dz_j}{2\pi i z_j} \eta^*_1(v_0) \xi^*_1(v_1) \cdots \xi^*_{\mu-1}(v_{\mu-1}) \prod_{j=1}^{\mu-1} f^*(v_{j-1} - v_j, \pi_{j\mu}),
\] (3.24)
where $z_j = x^{2v_j}$. The second equality follows from (3.11), (3.12), (3.14) and (3.15). The integral contour for $z_j$-integration encloses the poles at $z_j = x^{1+2k(r-1)} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$), but not the poles at $z_j = x^{1-2k(r-1)} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$). Note that (3.24) is an operator
\[
\Psi^*_\mu(v_0) : F_{l,k} \rightarrow F_{l+\bar{\varepsilon}_\mu,k},
\] (3.25)
so that we denote $\Psi^*_\mu(v_0)$ by $\Psi^{*_{l+\varepsilon_\mu,l}}(v_0)$ on the bosonic Fock space $F_{l,k}$.

For $1 \leq \mu \leq n$, define the type II vertex operators by
\[
\tilde{\Psi}^*_\mu^{(m-1)}(v_m) = \oint \prod_{j=\mu}^{m-1} \frac{dz_j}{2\pi i z_j} \eta^*_{m-1}(v_m) \xi^*_{m-1}(v_{m-1}) \cdots \xi^*_\mu(v_{\mu}) \prod_{j=\mu+1}^{m} f^*(v_{j-1} - v_j, \pi_{j\mu}),
\] (3.26)
where $z_j = x^{2v_j}$. The second equality again follows from (3.11), (3.12), (3.14) and (3.15). The integral contour for $z_j$-integration encloses the poles at $z_j = x^{1+2k(r-1)} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$), but not the poles at $z_j = x^{1-2k(r-1)} z_{j+1}$ ($k \in \mathbb{Z}_{\geq 0}$). The $\tilde{\Psi}^*_\mu^{(m-1)}(v_m)$ is an operator
\[
\tilde{\Psi}^*_\mu^{(m-1)}(v_m) : F_{l,k} \rightarrow F_{l+\bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_m - \varepsilon_\mu,k}.
\] (3.26)
3.5 Proof of commutation relations

In this subsection we shall show that (3.24) gives a bosonization of the dual type II vertex operators for
the $A_{n-1}^{(1)}$ face model. For that purpose we prove the commutation relations (2.18) and (2.19).

**Proposition 3.1** Type I and Type II vertex operators commute modulo the ‘energy functions’ $\chi_m$:

\[
\Phi_{\mu_1}(v_1)\Phi^*_{\mu_2}(v_2) = \chi(v_1 - v_2)\Phi^*_{\mu_2}(v_2)\Phi_{\mu_1}(v_1),
\]

\[
\Phi^*_{\mu_1}(v_1)\Phi_{\mu_2}(v_2) = \chi(v_1 - v_2)\Phi_{\mu_2}(v_2)\Phi^*_{\mu_1}(v_1).
\]

**Proof.** It is clear from (3.16) and (3.19). \(\square\)

**Theorem 3.2** The operators (3.24) satisfy the commutation relations (2.19).

**Proof.** The claim of the theorem is equivalent to the following equations

\[
\Psi_{\mu}(v_0)\Psi^*_{\mu}(v'_0) = S(v'_0 - v_0)\Psi^*_{\mu}(v'_0)\Phi^*_{\mu}(v_0),
\]

\[
\Psi^*_{\mu}(v_0)\Psi^*_{\mu}(v'_0) = S(v'_0 - v_0)\{\Psi^*_{\mu}(v'_0)\Psi^*_{\mu}(v_0)b^*(v'_0 - v_0, \pi_{\mu\nu}) + \Psi^*_{\mu}(v'_0)\Phi^*_{\mu}(v_0)c^*(v'_0 - v_0, \pi_{\mu\nu})\} \ (\mu \neq \nu),
\]

where

\[
b^*(v, w) = \frac{[v]\cdot[w - 1]'}{[v - 1]'[w]}, \quad c^*(v, w) = \frac{[v - w]\cdot[1]'}{[v - 1]'[w]}.
\]

Let us first prove (3.27). By using

\[
f^*(v, \pi_{\mu\nu})\eta^*_{\mu}(v'_0)\xi^*_{\mu}(v'_1)\cdots\xi^*_{\mu-1}(v'_\mu-1) = \eta^*_{\mu}(v'_0)\xi^*_{\mu}(v'_1)\cdots\xi^*_{\mu-1}(v'_\mu-1) f^*(v, \pi_{\mu\nu} + r(\delta_{\mu\nu} - \delta_{\mu\nu})),
\]

and the commutation relations (3.11), (3.12), (3.14) we have

\[
\Psi^*_{\mu}(v_0)\Psi^*_{\mu}(v'_0) = \int \prod_{j=1}^{\mu-1} \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} \eta^*_{\mu}(v'_0)\eta^*_{\mu}(v'_0)\xi^*_{\mu}(v'_1)\cdots\xi^*_{\mu-1}(v'_\mu-1)\xi^*_{\mu-1}(v'_\mu-1)
\]

\[
\times \int \prod_{j=1}^{\mu-1} f^*(v_j - v'_j - 1, 0) f^*(v_j - v'_j - 1, \pi_{\mu\nu} - 1) f^*(v_j - v'_j - 1, \pi_{\mu\nu}).
\]

Since $v_1, v'_1, \cdots, v_{\mu-1}, v'_{\mu-1}$ are integral variables we can deform an integrand without changing the value of the integral. Actually the following deformation is allowed for any function $F(v_j, v'_j)$ coupled to $\xi^*_{\mu}(v_j)\xi^*_{\mu}(v'_j)$:

\[
\int \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} \sum_{j=1}^{\mu} F(v_j, v'_j) = \int \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} \sum_{j=1}^{\mu} F(v_j, v'_j) \xi^*_{\mu}(v_j)\xi^*_{\mu}(v'_j)
\]

\[
= \frac{1}{2} \int \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} \xi^*_{\mu}(v_j)\xi^*_{\mu}(v'_j) \left\{ F(v_j, v'_j) + h^*(v'_j - v_j) F(v'_j, v_j) \right\},
\]

where we use (3.9). We thus denote $F(v_j, v'_j) \sim F'(v_j, v'_j)$ if $F(v_j, v'_j)$ and $F'(v_j, v'_j)$ satisfy

\[
F(v_j, v'_j) + h^*(v'_j - v_j) F(v'_j, v_j) = F'(v_j, v'_j) + h^*(v'_j - v_j) F'(v'_j, v_j).
\]
From (3.31), (3.18) and (3.32) we can prove (3.27) by showing

\[ f_{11}^*(v_0, v_0', v_1, v_1') \sim f_{11}^*(v_0', v_0, v_1, v_1'), \quad (3.33) \]

where

\[ f_{11}^*(v_0, v_0', v_1, v_1') = f^*(v_1 - v_0', 0) f^*(v_1 - v_0, w - 1) f^*(v_1' - v_0', w). \]

Let

\[ \tilde{f}_{11}^*(v_0, v_0', v_1, v_1') = f_{11}^*(v_0, v_0', v_1, v_1') + h^*(v_1' - v_1) f_{11}^*(v_0, v_0', v_1, v_1) \]

and set

\[ F_{11}^*(v_1) = \tilde{f}_{11}^*(v_0, v_0', v_1, v_1') - \tilde{f}_{11}^*(v_0', v_0, v_1, v_1'). \]

Since the residues at \( v_1 = v_0 - \frac{1}{2}, \ v_1 = v_0' - \frac{1}{2}, \ v_1 = v_1' - 1 \) vanish, \( F_{11}^* \) is a regular double periodic function of \( v_1 \), and hence a constant. We therefore get

\[ F_{11}^*(v_1) = F_{11}^*(v_0 + \frac{1}{2} - w) = 0, \quad (3.34) \]

which implies (3.33).

Next we prove (3.28) for \( \mu < \nu \). (We can show (3.28) for \( \mu > \nu \) in a similar manner.) When \( \mu = 1 < \nu \), we have

\[
\Psi_1^*(v_0) \Psi_\nu^*(v_0') = \oint \prod_{j=1}^{\nu-1} \frac{dz_j^*}{2\pi i z_j^*} \eta_j^*(v_0) \eta_j'(v_0') \xi_j^*(v_1') \cdots \xi_{\nu-1}^*(v_{\nu-1}') \prod_{j=1}^{\nu-1} f^*(v_j' - v_{j-1}' - 1, \pi_{j\nu}),
\]

\[
\Psi_\nu^*(v_0') \Psi_1^*(v_0) = \oint \prod_{j=1}^{\nu-1} \frac{dz_j^*}{2\pi i z_j^*} \eta_j^*(v_0') \eta_j'(v_0) \xi_j^*(v_1) \cdots \xi_{\nu-1}^*(v_{\nu-1}) \times f^*(v_1' - v_0', \pi_{1\nu} - 1) f^*(v_1' - v_0, 0) \prod_{j=2}^{\nu-1} f^*(v_j' - v_{j-1}' - 1, \pi_{j\nu}).
\]

By the same argument as we show (3.34) we obtain

\[
 f^*(v_j' - v_{j-1}', \pi_{j\nu}) = b^*(v_{j-1}' - v_{j-1}, \pi_{j\nu}) f^*(v_j' - v_{j-1}', \pi_{j\nu} + 1) f^*(v_j' - v_{j-1}, 0) + c^*(v_{j-1}' - v_{j-1}, \pi_{j\nu}) f^*(v_j' - v_{j-1}, \pi_{j\nu}). \quad (3.36)
\]

From (3.35) and (3.36) with \( j = 1 \) we have (3.28) for \( \mu = 1 < \nu \).

In order to prove (3.28) for \( 1 < \mu < \nu \) it is enough to show the following relation

\[
 b^*(v_0' - v_0, \pi_{\mu\nu}) f^*(v_1' - v_0, \pi_{1\nu}) f^*(v_1 - v_0, 0) f^*(v_0' - v_{\mu-1}' - v_0', 0) \\
 \times \prod_{j=2}^{\mu-1} f^*(v_j - v_{j-1}, \pi_{j\mu}) f^*(v_j - v_{j-1}', 0) f^*(v_j' - v_{j-1}', \pi_{j\nu}) \\
 \sim \prod_{j=1}^{\mu-1} f^*(v_j - v_{j-1}, \pi_{j\mu}) f^*(v_j - v_{j-1}', 0) f^*(v_j' - v_{j-1}', \pi_{j\nu}) \\
 - c^*(v_0' - v_0, \pi_{\mu\nu}) f^*(v_1' - v_0, \pi_{1\nu}) f^*(v_1 - v_0, 0) f^*(v_0' - v_{\mu-1}' - v_0', 0) \\
 \times \prod_{j=2}^{\mu-1} f^*(v_j - v_{j-1}, \pi_{j\mu}) f^*(v_j - v_{j-1}', 0) f^*(v_j' - v_{j-1}', \pi_{j\nu}). \quad (3.37)
\]
We would like to prove (3.37) by induction with respect to \( \mu \). Set \( \mu = 2 < \nu \). Then (3.37) reduces to

\[
\begin{align*}
\quad & b^* (v'_0 - v_0, \pi_{2\nu}) f^* (v'_2 - v'_1, \pi_{2\nu} + 1) f^* (v'_2 - v_1, 0) \\
\times & f^* (v_1 - v_0, \pi_{12}) f^* (v'_1 - v_0, 0) f^* (v'_1 - v_0, \pi_{1\nu}) \\
\sim & f^* (v'_2 - v'_1, \pi_{2\nu}) f^* (v_1 - v_0, \pi_{12}) f^* (v'_1 - v_0, 0) f^* (v'_1 - v_0, \pi_{1\nu}) \\
- & c^* (v'_0 - v_0, \pi_{2\nu}) f^* (v'_2 - v'_1, \pi_{2\nu}) f^* (v_1 - v_0, \pi_{12}) f^* (v'_1 - v_0, 0) f^* (v'_1 - v_0, \pi_{1\nu}).
\end{align*}
\]

\hspace{1cm}

(3.38)

Here we exchange \( v_1 \) and \( v'_1 \) in the term including \( b^* \) because they are integral variables. Owing to (3.36) with \( j = 2 \), (3.38) is equivalent to

\[
\begin{align*}
\quad & f^* (v'_2 - v'_1, \pi_{2\nu}) f^* (v'_1 - v'_0, \pi_{1\nu}) f^* (v_1 - v_0, \pi_{12}) \\
\times & \left\{ b^* (v'_0 - v_0, \pi_{2\nu}) f^* (v_1 - v_0, 0) - f^* (v_1 - v'_0, 0) \right\} \\
+ & f^* (v'_2 - v'_1, \pi_{2\nu}) f^* (v_1 - v_0, 0) \left\{ c^* (v'_0 - v_0, \pi_{2\nu}) f^* (v'_1 - v_0, \pi_{1\nu}) \right\} \\
- & b^* (v_1 - v'_1, \pi_{2\nu}) f^* (v'_1 - v_0, \pi_{1\nu}) f^* (v_1 - v_0, \pi_{1\nu}) \\
\sim & 0,
\end{align*}
\]

where we use the relation \( b^* (v'_1 - v_1, w) \sim b^* (v_1 - v'_1, w) \). By using the identities

\[
\begin{align*}
\quad & b^* (v'_0 - v_0, \pi_{2\nu}) f^* (v'_1 - v_0, 0) - f^* (v_1 - v'_0, 0) = \frac{\left[ 1 \right] [v'_0 - v_0 + v_1 - v'_1]\left[ v_0 - v_1 + \frac{1}{2}\right]}{[v'_0 - v_0 - 1][v_1 - v'_0] [v'_1 - v_0 + \frac{1}{2}] [v_0 - v_0 - \frac{1}{2}]}, \tag{3.40}
\end{align*}
\]

\[
\begin{align*}
\quad & c^* (v'_0 - v_0, \pi_{2\nu}) - c^* (v_1 - v'_1, \pi_{2\nu}) b^* (v'_0 - v_0, 0) f^* (v'_1 - v_0, 0) f^* (v'_1 - v'_0, 0) \\
= & \frac{\left[ 1 \right] [v'_0 - v_0 + v_1 - v'_1]\left[ v_0 - v_1 + \frac{1}{2} + w\right]}{[v'_0 - v_0 - 1][v_1 - v'_0] [v'_1 - v_0 - \frac{1}{2} + w][v_0 - v_0 - \frac{1}{2} + w]}, \tag{3.41}
\end{align*}
\]

we have (3.39), which implies (3.38).

Suppose \( 2 < \mu < \nu \). From the assumption of the induction

\[
\begin{align*}
\quad & b^* (v_1 - v'_1, \pi_{\mu \nu}) f^* (v'_\mu - v'_\mu - 1, 0) f^* (v'_\mu - v_\mu - 1, \pi_{\mu \nu} + 1) \\
\times & \prod_{j=2}^{\mu-1} \left( f^* (v_j - v_1 - \pi_{j\mu}) f^* (v_j - v'_1, 0) f^* (v'_j - v_{j-1}, \pi_{j\nu}) \right) \\
\sim & f^* (v_2 - v'_1, \pi_{\mu \nu}) f^* (v_2 - v_1, 0) f^* (v'_2 - v_1, \pi_{2\nu}) \\
\times & \prod_{j=3}^{\mu-1} \left( f^* (v_j - v_1 - \pi_{j\mu}) \prod_{j=3}^{\mu-1} f^* (v_j - v'_j - 0) \prod_{j=3}^{\mu-1} f^* (v'_j - v_{j-1}, \pi_{j\nu}) \right) \\
- & c^* (v_1 - v'_1, \pi_{\mu \nu}) \prod_{j=2}^{\mu-1} f^* (v_j - v_1 - \pi_{j\mu}) \prod_{j=2}^{\mu-1} f^* (v_j - v'_j - 0) \prod_{j=2}^{\mu-1} f^* (v'_j - v_{j-1}, \pi_{j\nu}),
\end{align*}
\]

we have

\[
\begin{align*}
\text{LHS of (3.37)} \quad & \sim \frac{b^* (v'_0 - v_0, \pi_{\mu \nu})}{b^* (v_1 - v'_1, \pi_{\mu \nu})} \left\{ \prod_{j=1}^{\mu-1} f^* (v_j - v_1 - \pi_{j\mu}) \prod_{j=2}^{\mu-1} f^* (v_j - v'_j - 0) \prod_{j=1}^{\mu-1} f^* (v'_j - v_{j-1}, \pi_{j\nu}) \\
- & c^* (v_1 - v'_1, \pi_{\mu \nu}) f^* (v'_1 - v_0, \pi_{1\mu}) f^* (v_1 - v_0, 0) f^* (v_1 - v'_0, \pi_{1\nu}) \\
\times & \prod_{j=2}^{\mu-1} f^* (v_j - v_1 - \pi_{j\mu}) \prod_{j=2}^{\mu-1} f^* (v_j - v'_j - 0) \prod_{j=2}^{\mu-1} f^* (v'_j - v_{j-1}, \pi_{j\nu}) \right\}. \tag{3.43}
\end{align*}
\]

Repeating the same procedure and using (3.40)(3.41) as we show (3.38), we obtain (3.37) for \( \mu < \nu \). □
4 Inversion and duality

In this section we prove various properties of the vertex operators for the $A_{n-1}^{(1)}$ face model. Besides the formulae listed in the last section we will use the following formulae of normal ordering and commutation relations among the basic operators:

\[ \eta_j^* (v_1) \xi_j^* (v_2) = \eta_j^* (v_1) \xi_j^* (v_2) : x_1^{- \frac{1}{2}} (x^{2r-1 \frac{2 \eta}{z_1}}, x^{2r-2}) : \]

\[ \xi_j^* (v_1) \eta_j^* (v_2) = : \xi_j^* (v_1) \eta_j^* (v_2) : x_1^{- \frac{1}{2}} (x^{2r-1 \frac{2 \eta}{z_1}}, x^{2r-2}) : \]

\[ \xi_j^* (v_1) \eta_k^* (v_2) = : \eta_k^* (v_2) \xi_j^* (v_1) ; \quad (j \neq k), \]

\[ \eta_j^* (v_1) \xi_k^* (v_2) = : \xi_k^* (v_2) \eta_j^* (v_1) ; \quad (j \neq k), \]

\[ \xi_j^* (v_1) \xi_{j+1}^* (v_2) = : \xi_j^* (v_1) \xi_{j+1}^* (v_2) : x_1^{- \frac{1}{2}} (x^{2r-1 \frac{2 \eta}{z_1}}, x^{2r-2}) : \]

\[ \xi_j^* (v_1) \xi_1^* (v_2) = : \xi_j^* (v_1) \xi_1^* (v_2) ; \quad (|j-k| > 1), \]

\[ \eta_j^* (v_1) \eta_j^* (v_2) = \eta_j^* (v_2) \eta_j^* (v_1) : x_1^{- \frac{1}{2}} \frac{1}{x} g^*_1 (z_2 / z_1), \]

\[ \eta_j^* (v_2) \eta_j^* (v_1) = \eta_j^* (v_1) \eta_j^* (v_2) : x_2^{- \frac{1}{2}} \frac{1}{x} g^*_1 (z_1 / z_2), \]

\[ \eta_j^* (v_1) \eta_j^* (v_2) = r^*_j (v_1 - v_2) \eta_j^* (v_2) \eta_j^* (v_1), \]

where

\[ \{ z \}_\infty = (z; x^{2n}, x^{2r}, x^2), \quad \{ z \}'_\infty = (z; x^{2n}, x^{2r-2}, x^2). \]

Do not confuse $\{ z \}, \{ z \}'$ defined in (2.7) and $\{ z \}_\infty, \{ z \}'_\infty$, respectively.

4.1 Inversion relations

In this subsection we prove the following two theorems:
**Theorem 4.1** (Inversion identity) As \( v' \to v - \frac{n}{z} \),
\[
\sum_{\mu=1}^{n} \frac{\bar{\Psi}_\mu^{(n)}(v) \Psi_\mu(v') \prod_{j=1}^{n} \left[ \pi_{j\mu} \right]'^{-1}}{1 - (x^n z')/z'}, \tag{4.11}
\]
\[
\sum_{\mu=1}^{n} \frac{\Psi_\mu(v) \bar{\Psi}_\mu^{(n-1)}(v') \prod_{j=1}^{n} \left[ \pi_{j\mu} \right]'^{-1}}{1 - (x^n z')/z'}. \tag{4.12}
\]

where
\[
g'_n = (-1)^{n-1} \frac{x^{-n(n-1)/2z}}{(x^{2r-2}; x^{2r-2})_{\infty}^{n-1} (x^{-2r}; x^{2r-2})_{\infty}^{n-1} (x^{2r-2}; x^{2r-2})_{\infty}^{n-1} (x^{2r-2}; x^{2r-2})_{\infty}^{n-1} (x^{2r-2}; x^{2r-2})_{\infty}^{n-1}}. \tag{4.13}
\]

**Theorem 4.2** As \( v' \to v + \frac{n}{z} \), the product of vertex operators behaves like
\[
\bar{\Psi}_\mu^{(n-1)}(v) \Psi_\nu(v') = \delta_{\mu\nu} \frac{g_n}{1 - z'(x^n z)} \prod_{j=1}^{n} \left[ \frac{1 - \pi_{j\mu}}{1 - \pi_{j\nu}} \right], \tag{4.14}
\]

where
\[
g_n = (-1)^{n-1} x^{-n(n-1)/2z} \left( \frac{x^{2r}; x^{2r-2})_{\infty}^{n} (x^{2r}; x^{2r-2})_{\infty}^{n} (x^{2r}; x^{2r-2})_{\infty}^{n} (x^{2r}; x^{2r-2})_{\infty}^{n} (x^{2r}; x^{2r-2})_{\infty}^{n} \right). \tag{4.15}
\]

Let us begin from the following Lemma:

**Lemma 4.3** For \( 1 \leq \mu \leq m \) we have
\[
\bar{\Psi}_\mu^{(m-1)}(v_m) \Psi_\mu^*(v_0) = -r_{m-1}^{*}(v' - v)^{-1} \sum_{\nu=1}^{m} \Psi_\nu^*(v') \bar{\Psi}_\nu^{(m-1)}(v) \left[ \frac{v - v' - m}{1 - \pi_{\nu\mu} + \frac{m}{z}} \right] \prod_{j=1}^{m} \left[ \frac{1 - \pi_{j\mu}}{1 - \pi_{j\nu}} \right]. \tag{4.16}
\]

**Proof.** From the commutation relations (3.11), (3.12), (3.14), (3.15) and (4.10),
\[
\Psi_\mu^{(m-1)}(v_m) \Psi_\mu^*(v_0) = \left( -1 \right)^{m-1} \int \prod_{j=1}^{m-1} \frac{d\zeta_j}{2\pi i \zeta_j} \eta_{m-1}^* (v_{m-1}) \xi^*_{m-1}(v_{m-1}) \cdots \xi^*_1(v_1) \eta_1^*(v_0)
\]
\[
\times \prod_{j=1}^{m} f^*(v_{j-1} - v_j, 1 - \pi_{j\mu}), \tag{4.17}
\]
\[
\Psi_\nu^*(v_0) \Psi_\mu^{(m-1)}(v_m) = \int \prod_{j=1}^{m-1} \frac{d\zeta_j}{2\pi i \zeta_j} \eta_{m-1}^* (v_{m-1}) \xi^*_{m-1}(v_{m-1}) \cdots \xi^*_1(v_1) \eta_1^*(v_0)
\]
\[
\times r_{m-1}^{*}(v_0 - v_m) \prod_{j=1}^{m} f^*(v_{j-1} - v_j, \pi_{j\nu}). \tag{4.18}
\]

In order to show (4.16) let us consider the following elliptic function
\[
F^*(u) = \frac{\left[ v_m - v_0 - \frac{m}{z} - 1 - \pi_{\mu\mu} + u \right]}{1 + \pi_{\mu\nu} - \frac{m}{z}} \prod_{j=1}^{m} \left[ \frac{v_{j-1} - v_j - \frac{m}{z} - \pi_{j\mu} + u}{\pi_{j\nu}} \right].
\]

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Set the sum of all residues of $F^*$ in the period to be zero. Then we obtain

$$
[v_m - v_0 - \frac{m}{2}] \prod_{j=1}^{m} \left[ \frac{v_j - v_j + \frac{1}{2} - \pi_j + \pi_j}{[\pi_j - \pi_j - 1]} \right]^
u \prod_{j \neq \nu} \left[ \frac{v_j - v_j - \frac{1}{2} - \pi_j + \pi_j}{[\pi_j - \pi_j']} \right]^
u,
$$

which implies (4.16) because of (4.17) and (4.18). \(\Box\)

**Lemma 4.4** For $1 \leq m \leq n - 1$

$$
\eta_{m-1}^* \left( v + \frac{1}{2} \right) \xi_{m-1}^* (v) \cdots \xi_1^* \left( v - \frac{m - 2}{2} \right) \eta_1^* \left( v - \frac{m - 1}{2} \right)
$$

$$
x \mapsto \frac{(x^{2r-2}, x^{2r-2})_\infty}{(x^{-2}, x^{2r-2})_\infty} g_{m-1}(x^{-m}) \times \eta_m^*(v).
$$

For $m = n$, as $v_n \to v + \frac{1}{2}$, $v_{n-1} \to v$, \ldots, $v_0 \to v - \frac{n-1}{2}$,

$$
\eta_{n-1}^*(v_n) \xi_{n-1}^* (v_{n-1}) \cdots \xi_1^* \left( v_1 \right) \eta_1^* \left( v_0 \right)
$$

$$
x \mapsto \frac{(x^{2r-2}, x^{2r-2})_\infty}{(x^{-2}, x^{2r-2})_\infty} g_{n-1} \left( \frac{2n}{z_n} \right) \times \eta_n^*(v).
$$

**Proof.** The claim for $1 \leq m \leq n - 1$ follows from (4.3–4.10). For $m = n$ note that $g_{n-1}^*(z)$ has a pole at $z = x^{-n}$. We obtain (4.21) by taking the limit $v_j = v - \frac{n-1}{2} - j$ ($0 \leq j \leq n$). \(\Box\)

**Lemma 4.5** For $1 \leq m \leq n - 1$, the following relations hold:

$$
\sum_{\mu=1}^{m} \Psi_{\mu}^*(v) \left( v + \frac{1}{2} \right) \Psi_{\mu}^* \left( v - \frac{m - 1}{2} \right) \prod_{j=1}^{m} \left[ \pi_{jmu} \right]^{-1} = c_m^{-1} \Psi_{m+1}^*(v),
$$

where $c_m$ is defined in (2.31). For $m = n$, we have

$$
\sum_{\mu=1}^{n} \Psi_{\mu}^{(n-1)}(v) \Psi_{\mu}^* (v') \prod_{j=1}^{n} \left[ \pi_{jmu} \right]^{-1} = \frac{g_n^*}{1 - (x^n z^*)^{-2}}.
$$

where $g_n^*$ is defined in (4.13).

**Proof.** Let $v_0 = v - \frac{m-1}{2}$ and $v_m = v + \frac{1}{2}$. From (4.17) we have

$$
\sum_{\mu=1}^{m} \Psi_{\mu}^{(m-1)}(v + \frac{1}{2}) \Psi_{\mu}^* (v - \frac{m - 1}{2}) \prod_{j=1}^{m} \left[ \pi_{jmu} \right]^{-1}
$$

$$
= \sum_{\mu=1}^{m} (-1)^{m-1} \int_{C_{\mu}} \prod_{j=1}^{m-1} \frac{dz_j}{2\pi i z_j} \eta_{m-1}^*(v_m) \xi_{m-1}^* (v_{m-1}) \cdots \xi_1^* \left( v_1 \right) \eta_1^* \left( v_0 \right) F^*_\mu (v_1, \cdots, v_{m-1}),
$$

where

$$
F^*_\mu (v_1, \cdots, v_{m-1}) = \prod_{1 \leq j \leq m} \frac{f^* (v_{j-1} - v_j, 1 - \pi_{jmu})}{\left[ \pi_{jmu} \right]^{-1}}.
$$
The contour $C_\mu$ is chosen as
\[
+\{\vert z_j \vert = x^{-m+j+1}(\vert z \vert - j\varepsilon)\} \\
+\{\vert z_j - x^{-m+j-1}z \vert = \varepsilon\} - \{\vert z_j - x^{-m+j+3}z \vert = \varepsilon\} \quad (1 \leq j \leq \mu - 1), \\
+\{\vert z_j \vert = x^{-m+j+1}(\vert z \vert + (m-j)\varepsilon)\} \\
+\{\vert z_j - x^{-m+j-1}z \vert = \varepsilon\} - \{\vert z_j - x^{-m+j+3}z \vert = \varepsilon\} \quad (\mu \leq j \leq m-1)
\]

for a small number $\varepsilon > 0$. Here the signs of the integral paths represent the directions. The plus sign refers to an anti-clockwise contour, and the minus sign refers to a clockwise contour.

Set the sum of all residues in the period we have to be zero. Then we have
\[
\sum_{\mu=1}^{m} F_{\mu}'(v_1, \cdots, v_{m-1}) = 0. \quad (4.26)
\]

Note that the RHS of $(4.24)$ vanishes from $(4.26)$ if the exchange of the order of the sum and the integral is permitted. In the neighbourhood of the contour $C_\mu$, the poles of the integrand of $(4.24)$ are those of $F_\mu$, and located at $z_j = x^{-m+j+1}z$. For $\mu \geq 2$, changing the contour for $z_j$ into $\{\vert z \vert = x^{-m+2}(\vert z \vert + (m-1)\varepsilon)\} + \{\vert z - x^{-m+j-1}z \vert = \varepsilon\} - \{\vert z - x^{-m+j+3}z \vert = \varepsilon\}$, the integrals are taken on a contour common to all $\mu$. Then the RHS of $(4.24)$ reduces to its residue at $z_j = x^{-m+2}z$:
\[
\sum_{\mu=2}^{m} (-1)^{m-1} \oint_{C'_{\mu}} \prod_{j=2}^{m-1} \frac{dz_j}{2\pi i z_j} \eta_{m-1}^*(v_m) \xi_{m-1}^*(v_{m-1}) \cdots \xi_1^*(v_1) \eta_1^*(v_0) \\
\times \text{Res}_{v_1=v_0} E_{\mu}'(v_1, \cdots, v_{m-1}) \frac{dz_1}{z_1} \\
= B \sum_{\mu=2}^{m} (-1)^{m-2} \oint_{C'_{\mu}} \prod_{j=2}^{m-1} \frac{dz_j}{2\pi i z_j} \eta_{m-1}^*(v_m) \xi_{m-1}^*(v_{m-1}) \cdots \xi_1^*(v_1) \eta_1^*(v_0) \\
\times F_{\mu}'(v_2, \cdots, v_{m-1}),
\]

where $v_0 = v - \frac{m-1}{2}$, $v_1 = v - \frac{m-2}{2}$ and
\[
F_{\mu}'(v_2, \cdots, v_{m-1}) = \prod_{2 \leq j \leq m \atop j \neq \mu} \frac{f^*(v_{j-1} - v_j, 1 - \pi_{j\mu})}{[\pi_{j\mu}]'},
\]

\[
B = -\text{Res}_{v=0} \frac{1}{[0]^'} \frac{dz}{z} = \frac{1}{(x^{2r-2}; x^{2r-2})_\infty},
\]

and the contour $C'_\mu$ is given by $(4.27)$ with $j = 2$. Repeating this procedure, we have
\[
B^{m-1} \eta_{m-1}^*(v_m) \xi_{m-1}^*(v_{m-1}) \cdots \xi_1^*(v_1) \eta_1^*(v_0).
\]

Thus Lemma $(4.22)$ follows from Lemma $(4.4)$.

For $m = n$, keep $(4.22)$ in mind and repeat the similar procedure. Then we obtain $(4.23)$. □

Proof of Theorem $(4.4)$ Lemma $(4.3)$ for $m = n$ implies $(4.11)$. You can also prove $(4.12)$ in a similar way. □
Proof of Theorem 4.2. Set $v_n = v, v_0 = v'$ and suppose $\mu > \nu$. From (4.1–4.10) the product $\tilde{\Psi}^{(n-1)}(v)\Psi^*(v')$ is regular at $v' = v + \frac{n}{2}$, which implies the claim of the theorem for $\mu > \nu$. The case $\mu < \nu$ is similar.

Suppose $\mu = \nu$. Then we have
\[
\bar{\Psi}^{(n-1)}(v)\Psi^*(v') = (-1)^{n-1} \prod_{j=1}^{n-1} \frac{dz_j}{2\pi i z_j} \eta^*_{n-1}(v_n) \xi^*_{n-1}(v_{n-1}) \cdots \xi^*_{1}(v_{1}) \eta^*_{1}(v_0) \prod_{j=1}^{n} f^*(v_{j-1} - v_j, 1 - \pi_{j\mu}).
\]
(4.27)

As $v' \to v + \frac{\nu}{2}$, the contour is pinched. The limit is calculated by successively taking the residues at $v_j = v_{j-1} - \frac{1}{2}$ for $1 \leq j \leq \mu - 1$, and $v_j = v_{j+1} + \frac{1}{2}$ for $\mu \leq j \leq n - 1$. As $z' = z_0 \to z_1 \to x^2 z_2 \to \cdots \to x^n z_n = x^n z$ the operator part behaves like,
\[
\bar{\eta}^*_{n-1}(v_n) \xi^*_{n-1}(v_{n-1}) \cdots \xi^*_{1}(v_{1}) \eta^*_{1}(v_0) \to x^{\frac{1}{2}} x^{-\frac{1}{2}} \bar{g}_{n-1}(x) \left( \frac{x^{2r}; x^{2r-2}}{x^{2r-2}; x^{2r-2}} \right) \prod_{j=0}^{n-1} \frac{1}{1 - z_j/(x z_{j+1})},
\]
which implies the claim of the theorem. $\square$

4.2 Duality relations

In this subsection we prove the following theorem:

Theorem 4.6 (Duality) For $1 \leq \mu \leq m \leq n$, the duality relation is given by
\[
\tilde{\Psi}^{(m-1)}(v) = \prod_{1 \leq k \leq \mu} [\pi_{j,k}]^{m-1} \Psi^{(m-1)}(v).
\]
(4.28)

Proof. Define $\tilde{\Psi}^{(m)}(v)$ by
\[
\tilde{\Psi}^{(m)}(v) = \prod_{1 \leq k \leq \mu} [\pi_{j,k}]^{m-1} \cdot \tilde{\Psi}^{(m)}(v).
\]
Let us prove that
\[
\tilde{\Psi}^{(m)}(v) = \Psi^{(m)}(v)
\]
(4.29)
by induction with respect to $m$. The case $m = 1$ is trivially true. First we consider the case $\mu = m + 1$.

By performing the cofactor expansion for (4.30) we have
\[
\tilde{\Psi}^{(m)}_{m+1}(v) = c_m \sum_{\mu=1}^{m} \prod_{j=1}^{m} \frac{[\pi_{j\mu}]^{m-1} \tilde{\Psi}^{(m-1)}_{m+1}(v + \frac{1}{2}) \Psi^*(v - \frac{m-1}{2})}{\Psi^*(v - \frac{m-1}{2})}.
\]

Because of the assumption of the induction and Lemma 4.5 we obtain that $\tilde{\Psi}^{(m)}_{m+1}(v) = \Psi^{(m)}_{m+1}(v)$. Since $\tilde{\Psi}^{(m)}_{m+1}(v)$ satisfy the same commutation relation (4.16) as $\tilde{\Psi}^{(m)}_{m+1}(v)$, taking $\mu = m + 1$ and calculating
\[(\tilde{\Psi}^{(m)}_{m+1}(v) - \tilde{\Psi}^{(m)}_{m+1}(v'))\Psi^*_m(v'),\text{ we have}\]
\[0 = \sum_{\nu=1}^{m} \Psi^*_\nu(v')(\tilde{\Psi}^{(m)}_\nu(v) - \tilde{\Psi}^{(m)}_\nu(v)) \frac{(v - v' - \frac{m+1}{2} - 1 - \pi_{\mu \nu})}{[v - v' - \frac{m+1}{2}]'} \prod_{1 \leq \kappa \leq m+1; \kappa \neq \nu} \frac{[1 - \pi_{\kappa \mu}]}{[\pi_{\kappa \nu}]} \tag{4.30}\]

Multiplying \(\tilde{\Psi}^{(n-1)}_m(v' - \frac{n}{2})\) from the left and applying Theorem 12, we get

\[0 = \tilde{\Psi}^{(m)}_m(v) - \tilde{\Psi}^{(m)}_m(v)\]

Applying this to (4.30) and multiplying \(\tilde{\Psi}^{(n-1)}_{m-1}(v' - \frac{n}{2})\) from the left, we get

\[0 = \tilde{\Psi}^{(m)}_{m-1}(v) - \tilde{\Psi}^{(m)}_{m-1}(v)\]

Repeating this procedure we have (4.29). \(\square\)

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