ALMOST PERIODIC SZEＧÖ COCYCLES WITH UNIFORMLY POSITIVE LYAPUNOV EXPONENTS

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Abstract. We exhibit examples of almost periodic Verblunsky coefficients for which Herman’s subharmonicity argument applies and yields that the associated Lyapunov exponents are uniformly bounded away from zero.

1. Introduction

Suppose that \((\Omega, \mu)\) is a probability measure space and \(T : \Omega \to \Omega\) is ergodic with respect to \(\mu\). A measurable map \(A : \Omega \to \text{GL}(2, \mathbb{C})\) gives rise to a so-called cocycle, which is a map from \(\Omega \times C^2\) to itself given by \((\omega, v) \mapsto (T\omega, A(\omega)v)\). This map is usually denoted by the same symbol. When studying the iterates of the cocycle, the following matrices describe the dynamics of the second component:

\[ A_n(\omega) = A(T^{n-1}\omega) \cdots A(\omega). \]

Assuming \(\log \|A\| \in L^1(\mu)\) and

\[ \inf_{n \geq 1} \frac{1}{n} \int_{\Omega} \log \|A_n(\omega)\| \, d\mu(\omega) > -\infty, \]

then, by Kingman’s subadditive ergodic theorem, the following limit exists,

\[ \gamma = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega} \log \|A_n(\omega)\| \, d\mu(\omega), \]

and we have

\[ \gamma = \lim_{n \to \infty} \frac{1}{n} \log \|A_n(\omega)\| \]

for \(\mu\)-almost every \(\omega \in \Omega\). The number \(\gamma\) is called the Lyapunov exponent of \(A\).

We will be interested in the particular case of Szeügen cocycles, which arise as follows. Denote the open unit disk in \(\mathbb{C}\) by \(\mathbb{D}\). For a measurable function \(f : \Omega \to \mathbb{D}\) with

\[ \int_{\Omega} \log(1 - |f(\omega)|) \, d\mu(\omega) > -\infty \]

and \(z \in \partial\mathbb{D}\), the cocycle \(A^z : \Omega \to \mathbb{U}(1, 1)\) is given by

\[ A^z(\omega) = \left(1 - |f(\omega)|^2\right)^{-1/2} \begin{pmatrix} z & -\overline{f(\omega)} \\ -f(\omega)z & 1 \end{pmatrix}. \]

The Lyapunov exponent of \(A^z\) will be denoted by \(\gamma(z)\). The complex numbers \(\alpha_n(\omega) = f(T^n\omega), n \geq 0\), appearing in \(A^z(T^n\omega)\) are called Verblunsky coefficients.

Szeügen cocycles play a central role in the analysis of orthogonal polynomials on the unit circle with ergodic Verblunsky coefficients; compare Simon’s recent monograph

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(see in particular [6] Section 10.5) for more information on Lyapunov exponents of Szegö cocycles. One of the major themes of [5, 6] is to work out in detail the close analogy between the spectral analysis of Jacobi matrices, or more specifically discrete one-dimensional Schrödinger operators, and that of CMV matrices. Indeed, a large portion of the second part, [6], is devoted to carrying over results and methods from the Schrödinger and Jacobi setting to the OPUC setting.

Sometimes the transition is straightforward and sometimes it is not. As discussed in the remarks and historical notes at the end of [6, Section 10.16], one of the results that Simon did not manage to carry over is Herman’s result on uniformly positive Lyapunov exponents for a certain class of almost periodic Schrödinger cocycles [4] (see also [2, Section 10.2]), which is proved by a beautiful subharmonicity argument and which is at the heart of many of the recent, far more technical, advances in the area of Schrödinger operators with almost periodic potentials; see Bourgain’s book [1] and references therein.

Our goal in this note is to present one-parameter families of almost periodic Szegö cocycles for which we prove uniformly positive Lyapunov exponents using Herman’s argument for an explicit region of parameter values.

2. Examples with Uniformly Positive Lyapunov Exponents

Consider the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ equipped with Lebesgue measure and $\mathbb{Z}_2$ equipped with the probability measure that assigns the weight $\frac{1}{2}$ to each of $0$ and $1$. Let $\Omega = \mathbb{T} \times \mathbb{Z}_2$ be the product space and $\mu$ the product measure. Fix some irrational $\alpha \in \mathbb{T}$. The transformation $T : \Omega \to \Omega$ is given by $T(\theta, j) = (\theta + \alpha, j + 1)$. It is readily verified that $T$ is ergodic with respect to $\mu$.

For $\varepsilon \in (0, 1)$ and $k \in \mathbb{Z} \setminus \{0\}$, we define $f : \Omega \to \mathbb{D}$ by

$$f(\theta, j) = \begin{cases} (1 - \varepsilon^2)^{1/2}e^{2\pi ik\theta} & j = 0, \\ (1 - \varepsilon^2)^{1/2}e^{-2\pi ik\theta} & j = 1. \end{cases}$$

Clearly, $f$ is measurable function from $\Omega$ to $\mathbb{D}$ and satisfies $\log[(1 - |f|^2)^{-1/2}] \in L^1(\mu)$. Thus, the Lyapunov exponents $\gamma(z), z \in \partial\mathbb{D}$ exist and we wish to bound them from below.

**Theorem 1.** For $(\Omega, \mu, T)$ as above and $f$ given by (3), we have the estimate

$$\inf_{z \in \partial\mathbb{D}} \gamma(z) \geq \log \frac{(1 - \varepsilon^2)^{1/2}}{\varepsilon}.$$  

In particular, if $\varepsilon \in (0, \frac{1}{\sqrt{2}})$, the Lyapunov exponent $\gamma(\cdot)$ is uniformly positive on $\partial\mathbb{D}$.

**Proof.** We consider the case $k > 0$; the case $k < 0$ is completely analogous. Fix any $z \in \partial\mathbb{D}$. By the definition (2) of $A^2$ and the definition (3) of $f$, we have

$$A^2(\theta, j) = \begin{cases} z & j = 0, \\ -(1 - \varepsilon^2)^{1/2}e^{2\pi ik\theta}z & j = 1. \end{cases}$$
Let us conjugate these matrices as follows (cf. [3] Equation (4.10)). Define

\[
C^z(\theta, j) = \begin{cases} 
  \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & j = 0, \\
  \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} & j = 1.
\end{cases}
\]

For \( j = 0 \), we have

\[
\varepsilon C^z(\theta, j) A^z(\theta, j) C^z(\theta, j - 1)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} z \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} z & 1 \end{pmatrix} \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} z \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} z & 1 \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} z \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} z & 1 \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}
\]

while for \( j = 1 \), we have

\[
\varepsilon C^z(\theta, j) A^z(\theta, j) C^z(\theta, j - 1)^{-1} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} z \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} z \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Thus,

(4)

\[
C^z(\theta, j) A^z(\theta, j) C^z(\theta, j - 1)^{-1} = \frac{z^{1/2}}{\varepsilon} \begin{pmatrix} -(1 - \varepsilon^2)^{1/2} e^{2\pi i k\theta} & z^j \\ z^{-j} & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k\theta} \end{pmatrix}.
\]

We have \( A^z(\theta, j) = A^z(\theta + (n - 1)\alpha, j + n - 1) \cdots A^z(\theta, j) \), which, by (4), is equal to

\[
C^z(\theta, j + n - 1)^{-1} \prod_{m=n-1}^{0} \left( \begin{pmatrix} z^{1/2} & -(1 - \varepsilon^2)^{1/2} e^{2\pi i k(\theta + m\alpha)} \\ -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k(\theta + m\alpha)} & z^{-(j+m \mod 2)} \end{pmatrix} \right) C^z(\theta, j-1).
\]

Since \( C^z(\theta, j) \) is always unitary and \( w = e^{2\pi i \theta} \) and \( z^{1/2} \) both have modulus one, we find that

\[
\| A^z(\theta, j) \| = e^{-n} \left| \prod_{m=n-1}^{0} \left( \begin{pmatrix} -(1 - \varepsilon^2)^{1/2} e^{2\pi i k(\theta + m\alpha)} \\ -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k(\theta + m\alpha)} \end{pmatrix} \right) \right|.
\]

\[
= e^{-n} \left| \prod_{m=n-1}^{0} \left( \begin{pmatrix} -(1 - \varepsilon^2)^{1/2} e^{2\pi i k(\theta + m\alpha)} \\ -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k(\theta + m\alpha)} \end{pmatrix} \right) \right|.
\]
The \( w \)-dependence of the matrix in the last expression is analytic and hence the
log of its norm is subharmonic. Therefore,

\[
\int_\Omega \log \| A^n(\theta, j) \| \, d\mu(\theta, j) = \frac{1}{2} \int_\pi \log \| A^n(\theta, 0) \| \, d\theta + \frac{1}{2} \int_\pi \log \| A^n(\theta, 1) \| \, d\theta \\
\geq n \log \left( 1 - \frac{\varepsilon^2}{2} \right).
\]

Since

\[
\gamma(z) = \lim_{n \to \infty} \frac{1}{n} \int_\Omega \log \| A^n(\theta, j) \| \, d\mu(\theta, j),
\]
the result follows.

In Theorem 1 we considered functions given by simple exponentials. Since we
obtained explicit terms which bound the Lyapunov exponents uniformly from be-
low, it is possible to add small perturbations to the function and retain uniform
positivity of the Lyapunov exponents. For example, given an integer \( k \geq 1 \) and \( \lambda, a_{-k}, \ldots, a_{k-1} \in \mathbb{C} \), we set

\[
f_\lambda(\theta, j) = \begin{cases} 
(1 - \varepsilon^2)^{1/2} \left( e^{2\pi ik\theta} + \lambda \sum_{l=-k}^{k-1} a_1 e^{2\pi il\theta} \right) & j = 0, \\
(1 - \varepsilon^2)^{1/2} \left( e^{-2\pi ik\theta} + \lambda \sum_{l=-k}^{k-1} a_1 e^{-2\pi il\theta} \right) & j = 1.
\end{cases}
\]

Since we need \( f_\lambda \) to take values in \( \mathbb{D} \), we have to impose an upper bound on the
admissible values of \( \lambda \). Clearly, once \( \varepsilon \in (0, 1) \), \( k \geq 1 \) and the numbers \( a_i \in \mathbb{C} \) are
chosen, there is \( \lambda_0 > 0 \) such that for \( \lambda \) with modulus bounded by \( \lambda_0 \), the range of
\( f_\lambda \) is contained in \( \mathbb{D} \).

Theorem 2. Let \((\Omega, \mu, T)\) be as above. For every \( \varepsilon \in (0, 1) \), \( k \in \mathbb{Z}_+ \), and \( \{a_i\}_{l=-k}^{k-1} \subset \mathbb{C} \), there is \( \lambda_1 > 0 \) such that for every \( \lambda \) with \( |\lambda| < \lambda_1 \), there is \( \gamma_- > 0 \)
for which the Lyapunov exponent \( \gamma(\cdot) \) associated with \( f_\lambda \) given by (5) satisfies

\[
\inf_{\varepsilon \in \partial \mathbb{D}} \gamma(z) \geq \gamma_-.
\]

Proof. The smallness condition \( |\lambda| < \lambda_1 \) needs to address two issues. First, the
range of the function \( f_\lambda \) must be contained in \( \mathbb{D} \), so we need \( \lambda_1 \leq \lambda_0 \). Second, the
explicit strictly positive uniform lower bound obtained in the proof of Theorem 1
for the case \( \lambda = 0 \) changes continuously once the perturbation is turned on. Thus, it
remains strictly positive for \( |\lambda| \) small enough. Notice that the degree requirements
for the perturbation in (5) are such that the subharmonicity argument from the
proof of Theorem 1 goes through without any changes.

3. Discussion

In the previous section we proved a uniform lower bound for the Lyapunov
exponents associated with strongly coupled almost periodic sequences of Verblunsky
coefficients. A few remarks are in order.

The Verblunsky coefficients take values in the open unit disk and the unit circle
has to be regarded as the analogue of infinity in the Schrödinger case. Thus, just
as the coupling constant is sent to infinity in the application of Herman’s argument
in the Schrödinger case, the coupling constant is sent to one in our study. Notice
that we need a rather uniform convergence to the unit circle, whereas one may have
zeros in the Schrödinger case. In particular, while Herman’s argument applies to
all non-constant trigonometric polynomials in the Schrödinger case, we only treat small perturbations of simple exponentials.

Another limitation of our proof is that it requires the consideration of the product $T \times \mathbb{Z}_2$. It would be nicer to have genuine quasi-periodic examples, that is, generated by minimal translations on a finite-dimensional torus. Our attempts to produce such examples have run into trouble with analyticity issues. It would be of interest to produce quasi-periodic examples or to demonstrate why Herman’s argument cannot work for any of them.

As explained by Simon in [6, Theorem 12.6.1], as soon as one knows that $\gamma(z)$ is positive for (Lebesgue almost) every $z \in \mathbb{D}$, one can immediately deduce that for $\mu$-almost all elements of $\Omega$, Lebesgue almost all Aleksandrov measures associated with the sequence of Verblunsky coefficients in question are pure point. This is applicable to our examples for $\varepsilon \in (0, \sqrt{2})$ and $|\lambda|$ small enough.

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