NOTES ON FROBENIUS STABLE DIRECT IMAGES

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ABSTRACT. In this note, we prove the coherence of Frobenius stable direct images in a new case. We also show a generation theorem regarding to it. Furthermore, we prove a corresponding theorem in characteristic zero.

1. Introduction

Let $X$ be a normal projective variety over an algebraically closed field of positive characteristic, let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$, and let $M$ be a Cartier divisor on $X$. In [12] (cf. [11]), Schwede introduced the subspace

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \subseteq H^0(X, \mathcal{O}_X(M)),$$

which is defined as the stable image of the trace maps of iterated Frobenius morphisms. This notion was relativized by Hacon and Xu [8] to establish the three dimensional minimal model program in positive characteristic (Definition 2.1). For a morphism $f : X \to Y$ to a variety $Y$, they define the subsheaf

$$S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \subseteq f_*\mathcal{O}_X(M)$$

by a way similar to that of $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M))$. From the definition, we cannot not see whether or not the sheaf $S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))$ is coherent. The coherence is proved in [8] Proposition 2.15], under the assumption that $M - (K_X + \Delta)$ is $f$-ample. In this note, we show the coherence of the sheaf under a weaker assumption:

**Theorem 1.1.** Let the base field be an $F$-finite field. Let $X$ be a normal projective variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $i(K_X + \Delta)$ is Cartier for an integer $i > 0$ not divisible by $p$. Let $f : X \to Y$ be a morphism to a projective variety $Y$ of dimension $n$. Let $M$ be a Cartier divisor on $X$. If $M - (K_X + \Delta)$ is relatively semi-ample over an open subset $V \subset Y$, then

$$\text{Im} \left( f_*\phi_{(X,\Delta)}^{(e)}(M) \right) |_V = S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))|_V$$

for $e$ large and divisible enough. In particular, $S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))$ is coherent over $V$.

We also prove a generation theorem on $S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))$.

**Theorem 1.2.** With the notation of Theorem 1.1, if $M - (K_X + \Delta)$ is nef and $f$-semi-ample, then the sheaf

$$\left( S^0_{f_*}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \right) \otimes \mathcal{L}^n \otimes \mathcal{A}$$

is generated by its global sections for ample line bundles $\mathcal{L}$ and $\mathcal{A}$ with $|\mathcal{L}|$ free.
For a related result, see [14, Theorem 1.11]. Furthermore, we show a corresponding theorem in characteristic zero to Theorem 1.2.

**Theorem 1.3.** Let the base field be a field of characteristic zero. Let $X$ be a normal projective variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to Y$ be a morphism to a projective variety $Y$ of dimension $n$. Let $M$ be a Cartier divisor on $X$. If $M - (K_X + \Delta)$ is nef and $f$-semi-ample, then

$$f_* (\mathcal{J}(X, \Delta)(M)) \otimes \mathcal{L}^n \otimes \mathcal{A} \quad \text{and} \quad f_* (\mathcal{J}_{\text{NLIC}}(X, \Delta)(M)) \otimes \mathcal{L}^n \otimes \mathcal{A}$$

are generated by its global sections for ample line bundles $\mathcal{L}$ and $\mathcal{A}$ with $|\mathcal{L}|$ free. In particular, if, furthermore, $(X, \Delta)$ is log canonical, then $f_* \mathcal{O}_X(M) \otimes \mathcal{L}^n \otimes \mathcal{A}$ is generated by its global sections.

Here, $\mathcal{J}(X, \Delta)$ (resp. $\mathcal{J}_{\text{NLIC}}(X, \Delta)$) is the multiplier ideal (resp. the non-lc ideal sheaf) associated to $(X, \Delta)$ (Definition 2.2). Theorem 1.3 should be compared with [6, Corollary 1.7]. Also, for several results related to Theorem 1.3, see [5, Section 9].

Theorem 1.3 does not hold in positive characteristic. Indeed, Moret-Bailly [10] constructed a semi-stable fibration $g : S \to \mathbb{P}^1$ of characteristic $p > 0$, where $S$ is a smooth projective surface, such that $g_* \omega_S \otimes \mathcal{O}(2) \cong \mathcal{O}(-1) \oplus \mathcal{O}(p)$ (see also [13, Proposition 3.16]).

The above theorems are proved in Section 3. In Section 4, we consider a question that generalizes Fujita’s freeness conjecture.

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## 2. Terminologies and definitions

In this section, we define some terminologies and notions.

Let $k$ be a field of characteristic $p \geq 0$. By a *variety* we mean an integral separated scheme of finite type over $k$.

Let $X$ be a normal variety and let $\Delta = \sum_{i=1}^d \delta_i \Delta_i$ be a $\mathbb{Q}$-Weil divisor on $X$, where each $\Delta_i$ is a prime divisor. We define the *round down* $\lfloor \Delta \rfloor$ (resp. *round up* $\lceil \Delta \rceil$) of $\Delta$ by $[\Delta] := \sum_{i=1}^d \lfloor \delta_i \rfloor \Delta_i$ (resp. $[\Delta] := \sum_{i=1}^d \lceil \delta_i \rceil \Delta_i$). Also, we use the following notation:

$$\{\Delta\} := \Delta - [\Delta]; \quad \Delta^{-1} := \sum_{\delta_i = 1} \Delta_i; \quad \Delta^{>1} := \sum_{\delta_i > 1} \Delta_i; \quad \Delta^{<1} := \sum_{\delta_i < 1} \Delta_i.$$

Assume that $p > 0$. Let $X$ be a variety. Let $F_X^e : X \to X$ denote the $e$-times iterated absolute Frobenius morphism of $X$.

Let $k$ be an $F$-field field, i.e., a field of characteristic $p > 0$ with $[k : k^p] < +\infty$. Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $i(K_X + \Delta)$ is Cartier for an integer $i > 0$ not divisible by $p$. Let $M$ be a Cartier divisor on $X$. Let $f : X \to Y$ be a projective morphism to a variety $Y$. We define
Let $e_0$ be the smallest positive integer such that $i|(p^{e_0} - 1)$. For each $e \geq 1$ with $e_0|e$, applying $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(M))$ to the composite of

$$\mathcal{O}_X \xrightarrow{F_{X}^{e}} F_{X}^{e} \mathcal{O}_X \rightarrow F_{X}^{e} \mathcal{O}_X((p^{e} - 1)\Delta),$$

we obtain the morphism

$$\phi_{(X, \Delta)}^{(e)}(M) : F_{X}^{e} \mathcal{O}_X((1 - p^{e})(K_{X} + \Delta) + p^{e}M) \rightarrow \mathcal{O}_X(M)$$

by the Grothendieck duality. Pushing this forward by $f$, we get

$$f_{\ast} \phi_{(X, \Delta)}^{(e)}(M) : F_{Y}^{e} f_{\ast} \mathcal{O}_X((1 - p^{e})(K_{X} + \Delta) + p^{e}M) \rightarrow f_{\ast} \mathcal{O}_X(M).$$

Note that $f_{\ast} F_{X}^{e} = F_{Y}^{e} f_{\ast}$. By the construction, we see that $f_{\ast} \phi_{(X, \Delta)}^{(e)}(M)$ factors through $f_{\ast} \phi_{(X, \Delta)}^{(e)}(M)$ for $e \geq e' \geq 1$ with $e_0|e$ and $e_0|e'$, so

$$\text{Im} \left( f_{\ast} \phi_{(X, \Delta)}^{(e)}(M) \right) \subseteq \text{Im} \left( f_{\ast} \phi_{(X, \Delta)}^{(e')} (M) \right).$$

**Definition 2.1.** With the above notation, we define

$$S^{0}_{\ast} f_{\ast}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M)) := \bigcap_{e \geq 1, e_0|e} \text{Im} \left( f_{\ast} \phi_{(X, \Delta)}^{(e)}(M) \right) \subseteq f_{\ast} \mathcal{O}_X(M).$$

We cannot see from the definition whether or not $S^{0}_{\ast} f_{\ast}(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))$ is coherent.

Next, we define the multiplier ideal sheaf $\mathcal{J}(X, \Delta)$ and the non-lc ideal sheaf $\mathcal{J}_{\text{NLC}}(X, \Delta)$.

**Definition 2.2.** Let the base field be a field of characteristic zero. Let $X$ be a normal projective variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor such that $K_{X} + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Z \rightarrow X$ be a resolution of $X$ with $K_{Z} + \Delta_{Z} = \pi^{\ast}(K_{X} + \Delta)$ such that $\text{Supp}(\Delta_{Z})$ is simple normal crossing.

- We define the **multiplier ideal sheaf** $\mathcal{J}(X, \Delta)$ by
  $$\mathcal{J}(X, \Delta) := \pi_{\ast} \mathcal{O}_{Z}(-\lfloor \Delta_{Z} \rfloor).$$

- (Definition 2.1) We define the **non-lc ideal sheaf** $\mathcal{J}_{\text{NLC}}(X, \Delta)$ by
  $$\mathcal{J}_{\text{NLC}}(X, \Delta) := \pi_{\ast} \mathcal{O}_{Z}(\lfloor -(\lfloor \Delta_{Z}^{\leq 1} \rfloor - \lfloor \Delta_{Z}^{> 1} \rfloor) \rfloor + \lfloor \Delta_{Z}^{> 1} \rfloor) = \pi_{\ast} \mathcal{O}_{Z}(-\lfloor \Delta_{Z} \rfloor + \Delta_{Z}^{= 1}).$$

By [3, Proposition 2.6], we see that $\mathcal{J}_{\text{NLC}}(X, \Delta)$ is independent of the choice of the resolution $\pi : Z \rightarrow X$, so $\mathcal{J}_{\text{NLC}}(X, \Delta)$ is well-defined. We see from the definition that $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ (resp. $\mathcal{J}_{\text{NLC}}(X, \Delta) = \mathcal{O}_X$) if and only if $(X, \Delta)$ is Kawamata log terminal (resp. log canonical).

3. Proofs of the theorems

We first prove Theorem 1.1.

**Proof of Theorem 1.1.** We prove (1). We use the notation in Section 2. Put $\mathcal{I}^{(e)} := \text{Im} \left( f_{\ast} \phi_{(X, \Delta)}^{(e)}(M) \right)$ for each $e \geq 1$ with $e_0|e$. We show that there is an ample line
bundle $\mathcal{L}$ on $Y$ such that $\mathcal{I}^{(e)} \otimes \mathcal{L}$ is globally generated on $Y$ for $e \gg 0$ with $e_0|e$. If this holds, then for each $e \geq e' > 0$ with $e_0|e$ and $e_0|e'$, we have

$$H^0(Y, \mathcal{I}^{(e')} \otimes \mathcal{L}) \otimes_k \mathcal{O}_Y \longrightarrow \mathcal{I}^{(e')} \otimes \mathcal{L},$$

$$H^0(Y, \mathcal{I}^{(e)} \otimes \mathcal{L}) \otimes_k \mathcal{O}_Y \longrightarrow \mathcal{I}^{(e)} \otimes \mathcal{L},$$

where $k$ is the base field and the horizontal arrows are surjective on $V$. Note that the equality follows from the fact that the space of global sections is of finite dimension. Hence, $(\mathcal{I}^{(e)} \otimes \mathcal{L})|_V = (\mathcal{I}^{(e')} \otimes \mathcal{L})|_V$, which implies that $\mathcal{I}^{(e)}|_V = \mathcal{I}^{(e')}|_V$.

Set $N := M - K_X - \Delta$. Then $iN$ is Cartier. Since $N$ is relatively ample over $V$, there is $d \geq 2$ with $i|d$ and $m_0 \geq 1$ such that the natural morphism

$$f_*\mathcal{O}_X(dN) \otimes f_*\mathcal{O}_X(mN + M) \rightarrow f_*\mathcal{O}_X((d + m)N + M)$$

is surjective on $V$ for $m \geq m_0$ with $i|m$. Let $\mathcal{L}$ be an ample line bundle on $Y$ such that $\mathcal{L}$ and $f_*\mathcal{O}_X(dN) \otimes \mathcal{L}$ are globally generated. We prove that $\mathcal{I}^{(e)} \otimes \mathcal{L}^{n+1}$ is globally generated on $V$ for each $e \gg 0$ with $e_0|e$. Let $q_e$ and $r_e$ be integers such that $p^e - 1 = q_e d + r_e$ and $m_0 \leq r_e < m_0 + d$. Note that if $e_0|e$, then $i|r_e$.

Put $\mathcal{G} := \bigoplus_{m_0 \leq r < m_0 + d, i|r} f_*\mathcal{O}_X(r N + M)$. We then have the following sequence of morphisms that are surjective on $V$ for each $e \geq 1$ with $e_0|e$:

$$\mathcal{I}^{(e)} \otimes \mathcal{L}^{n+1}$$

\[ F_{Y*}^e \left( f_* \mathcal{O}_X((p^e - 1)N + M) \otimes \mathcal{L}^{p^e(n+1)} \right) \]

\[ \left( f_* \mathcal{O}_X((p^e - 1)N + M) \otimes \mathcal{L}^{p^e(n+1)} \right) \]

\[ \cong F_{Y*}^e \left( f_* \mathcal{O}_X(dN) \otimes f_* \mathcal{O}_X(r_e N + M) \otimes \mathcal{L}^{p^e(n+1-q_e)} \right) \]

\[ \cong F_{Y*}^e \left( \left( \bigoplus \mathcal{O}_Y \right) \otimes f_* \mathcal{O}_X(r_e N + M) \otimes \mathcal{L}^{p^e(n+1-q_e)} \right) \]

\[ \cong \bigoplus F_{Y*}^e \left( f_* \mathcal{O}_X(r_e N + M) \otimes \mathcal{L}^{p^e(n+1-q_e)} \right) \]

Therefore, it is enough to show that $F_{Y*}^e \left( \mathcal{G} \otimes \mathcal{L}^{p^e(n+1-q_e)} \right)$ is globally generated for $e \gg 0$. We check that the sheaf is 0-regular with respect to $\mathcal{L}$ in the sense of Castelnuovo–Mumford ([3, Theorem 1.8.5]). For each $0 < j \leq n$, we have

$$H^j \left( Y, F_{Y*}^e \left( \mathcal{G} \otimes \mathcal{L}^{p^e(n+1-q_e)} \right) \otimes \mathcal{L}^{-j} \right) = H^j \left( Y, F_{Y*}^e \left( \mathcal{G} \otimes \mathcal{L}^{p^e(n+1-j-q_e)} \right) \right)$$

Note that $F_{Y*}^e$ is finite, since $k$ is $F$-finite. By $q_e/p^e \xrightarrow{e \to +\infty} 1/d < 1$, we get $p^e(n + 1 - j - q_e) \xrightarrow{e \to +\infty} +\infty$, so our claim follows from the Serre vanishing theorem. □

Next, we show Theorem 1.2. To this end, we need Lemma 3.2. To prove the following:
Lemma 3.1 ([2] Lemma 3.4]). Let $f : X \to Y$ be a morphism between projective varieties over a field. Let $\mathcal{F}$ be a coherent sheaf on $X$. Let $D$ be an ample Cartier divisor on $X$. Then there exists an integer $m_0 \geq 1$ such that

$$f_*\mathcal{F}(mD + N)$$

is generated by its global sections for each $m \geq m_0$ and every nef Cartier divisor $N$ on $X$.

Lemma 3.2. Let $f : X \to Y$ be a morphism between projective varieties over a field. Let $\mathcal{F}$ be a coherent sheaf on $X$. Let $D$ be a nef and $f$-semi-ample Cartier divisor on $X$. Let $A$ be an ample line bundle on $Y$. Then there exists integers $n_0 \geq 1$ and $l_0 \geq 1$ such that

$$f_*\mathcal{F}(nD) \otimes A^l$$

is generated by its global sections for all $n \geq n_0$ and $l \geq l_0$.

Proof. Since $D$ is $f$-semi-ample, there are projective morphisms $\sigma : X \to W$ and $\tau : W \to Y$ with $\tau \circ \sigma = f$ such that $mD \sim \sigma^*D'$ for an $m \geq 1$ and a nef and $\tau$-ample Cartier divisor $D'$ on $Z$. Set $\mathcal{F}' := \bigoplus_{i=0}^{m-1} \mathcal{F}(iD)$. Replacing $f : X \to Y$, $D$ and $\mathcal{F}$ by $\tau : W \to Y$, $D'$ and $\sigma^*\mathcal{F}'$, respectively, we may assume that $D$ is $f$-ample. Since $D$ is nef and $f$-ample, $D + f^*A$ is ample, where $A$ is a Cartier divisor with $\mathcal{O}_Y(A) \cong A$. Then by Lemma 3.1, there is an $m_0 \geq 1$ such that

$$f_*\mathcal{F}(nD + l|f^*A) \cong f_*\mathcal{F}(nD) \otimes A^l$$

is globally generated for each $n \geq m_0$. \hfill \Box

Proof of Theorem 1.3. We use the same notation as that of the proof of Theorem 1.1. By Theorem 1.1, it is enough to show that $\mathcal{T}^{(e)} \otimes \mathcal{L}^n \otimes \mathcal{A}$ is globally generated for $e \gg 0$ with $e_0|e$. By Lemma 3.2, there is an $l \geq 1$ such that

$$f_*\mathcal{O}_X(mN + M) \otimes \mathcal{A}^l$$

is globally generated for each $m \geq 1$ with $i|m$. We then have the following sequence of surjective morphisms for each $e \geq 1$ with $e_0|e$:

$$\mathcal{T}^{(e)} \otimes \mathcal{L}^n \otimes \mathcal{A} \leftarrow (F_{Y*}^e (f_*\mathcal{O}_X ((p^e - 1)N + M) \otimes \mathcal{A}^p)) \otimes \mathcal{L}^n$$

$$\cong (F_{Y*}^e (f_*\mathcal{O}_X ((p^e - 1)N + M) \otimes \mathcal{A}^l \otimes \mathcal{A}^{p^e - l})) \otimes \mathcal{L}^n$$

$$\cong (F_{Y*}^e (\bigoplus \mathcal{O}_Y) \otimes \mathcal{A}^{p^e - l}) \otimes \mathcal{L}^n$$

When $e \gg 0$, we see that the last sheaf is 0-regular with respect to $\mathcal{L}$ by an argument similar to the proof of Theorem 1.1 so it is globally generated, and hence so is $\mathcal{T}^{(e)} \otimes \mathcal{L}^n \otimes \mathcal{A}$. \hfill \Box

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. We first prove the statement on $\mathcal{J}_{\text{NLC}}(X, \Delta)$. One can easily check that we may assume that the base field is an algebraically closed field. Let $\pi : Z \to X$ be a resolution of $(X, \Delta)$ with $K_Z + \Delta_Z = \pi^*(K_X + \Delta)$ such that $\text{Supp}(\Delta_Z)$ is simple normal crossing. Put $\Delta' := \{\Delta_Z\} + \Delta_Z^{\leq 1}$. Then each coefficient in $\Delta'$ is at
most one and \(\text{Supp}(\Delta')\) is simple normal crossing. Set \(M' := \pi^*M - [\Delta_Z] + \Delta_Z^{-1}\). Then
\[
M' - (K_Z + \Delta') = \pi^*M - [\Delta_Z] + \Delta_Z^{-1} - K_Z - \{\Delta_Z\} - \Delta_Z^{-1} = \pi^*M - K_Z - \Delta_Z = \pi^*(M - (K_X + \Delta)),
\]
so \(M' - (K_Z + \Delta')\) is nef and \(g\)-semi-ample, where \(g := f \circ \pi : Z \to Y\). We also have \(\pi_*\mathcal{O}_Z(M') \cong \mathcal{J}_{\text{NLC}}(X, \Delta)(M)\) by the projection formula. Let \(L\) (resp. \(A\)) be a Cartier divisor on \(Y\) such that \(\mathcal{O}_Y(L) \cong \mathcal{L}\) (resp. \(\mathcal{O}_Y(A) \cong \mathcal{A}\)). Put \(L^{(i)} := (n-i)L + \frac{1}{2}A\) for each \(0 < i \leq n\). Then each \(L^{(i)}\) is ample, and \(M' - (K_Z + \Delta') + g^*L^{(i)}\) is semi-ample. Indeed, since \(M' - (K_Z + \Delta')\) is nef and \(g\)-semi-ample, there are projective morphisms \(\sigma : X \to W\) and \(\tau : W \to Y\) with \(\tau \circ \sigma = g\) such that \(M' - (K_Z + \Delta') \sim_Q \sigma^*N\) for a nef and \(\tau\)-ample \(\mathbb{Q}\)-Cartier divisor \(N\), and then \(N + \tau^*L^{(i)}\) is ample, so \(M' - (K_Z + \Delta') + g^*L^{(i)}\) is semi-ample. Therefore, we can find an effective \(\mathbb{Q}\)-divisor \(F^{(i)} \sim_Q M' - (K_Z + \Delta') + g^*L^{(i)}\) such that the support of \(\Delta^{(i)} := \Delta' + F^{(i)}\) is simple normal crossing and that each coefficient in \(\Delta^{(i)}\) is at most one. Then
\[
M' + g^*((n - i)L + A) - (K_Z + \Delta^{(i)}) = M' + g^*((n - i)L + A) - (K_Z + \Delta') - F^{(i)} \\
\sim_Q M' + g^*((n - i)L + A) - (K_Z + \Delta') - M' + K_Z + \Delta' - g^*L^{(i)} = g^*\left((n - i)L + A - (n - i)L - \frac{1}{2}A\right) = g^*\left(\frac{1}{2}A\right),
\]
so we can apply [1] Theorem 3.2 or [3] Theorem 6.3, from which we obtain that
\[
0 = H^i(Y, g_*\mathcal{O}_Z(M' - g^*((n - i)L + A))) \\
\cong H^i(Y, f_*(\mathcal{J}_{\text{NLC}}(X, \Delta)(M - f^*((n - i)L + A)))) \\
\cong H^i(Y, f_*(\mathcal{J}_{\text{NLC}}(X, \Delta)(M))) \otimes \mathcal{L}^{n-i} \otimes \mathcal{A}
\]
for each \(0 < i \leq n\). This implies that \(f_*(\mathcal{J}_{\text{NLC}}(X, \Delta)(M))) \otimes \mathcal{L}^{n} \otimes \mathcal{A}\) is 0-regular with respect to \(\mathcal{L}\) in the sense of Castelnuovo–Mumford, and hence it is globally generated ([3] Theorem 1.8.5]).

The statement on \(\mathcal{J}(X, \Delta)\) can be proved by an argument similar to the above, by putting \(\Delta' := \{\Delta_Z\}\) and \(M' := \pi^*M - [\Delta_Z]\).

\[Q \square\]

4. Question

In this section, we consider the following question:

**Question 4.1**. Let the base field be an algebraically closed field of characteristic zero. Let \(X\) be a normal projective variety and let \(\Delta\) be an effective \(\mathbb{Q}\)-Weil divisor on \(X\) such that \((X, \Delta)\) is log canonical. Let \(M\) be a Cartier divisor on \(X\). Let \(f : X \to Y\) be a surjective morphism to a smooth projective variety \(Y\) of dimension \(n\). If \(M - (K_X + \Delta)\) is nef and \(f\)-semi-ample, then is
\[
f_*\mathcal{O}_X(M) \otimes \mathcal{L}^{n+1}
\]
generated by its global sections for an ample line bundle \(\mathcal{L}\) on \(Y\)?
This question is a generalization of Fujita's freeness conjecture.

In positive characteristic, Question 4.1 has been already answered negatively, even if we employ \( S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \) instead of \( f_* \mathcal{O}_X(M) \). Indeed, Gu–Zhang–Zhang [7] constructed a smooth projective surface \( S \) on which there is an ample Cartier divisor \( A \) such that

\[
f_* \mathcal{O}_S(K_S) \otimes A^3 \cong S^0 f_*(\sigma(S, 0) \otimes \mathcal{O}_S(K_S)) \otimes A^3 \cong \omega_S \otimes A^3
\]

is not globally generated, where \( f = \text{id} : S \to S \).

We answer affirmatively Question 4.1 when \( Y \) is a smooth projective curve.

**Theorem 4.2.** Let the base field be an algebraically closed field of characteristic \( p \geq 0 \). Let \( X \) be a normal projective variety and let \( \Delta \) be a \( \mathbb{Q} \)-Weil divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Let \( f : X \to Y \) be a morphism to a smooth projective curve \( Y \). Let \( M \) be a Cartier divisor on \( X \) such that \( M - (K_X + \Delta) \) is nef and \( f \)-semi-ample, and let \( \mathcal{L} \) be an ample line bundle on \( Y \).

1. **Suppose that** \( p = 0 \). **Then**

\[
f_*(\mathcal{J}_{\text{NLC}}(X, \Delta)(M)) \otimes \mathcal{L}^l
\]

is generated by its global sections for \( l \geq 2 \).

2. **Suppose that** \( p > 0 \) and \( i(K_X + \Delta) \) is Cartier for an integer \( i > 0 \) not divisible by \( p \). **Then**

\[
(S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))) \otimes \mathcal{L}^l
\]

is generated by its global sections for \( l \geq 2 \).

**Proof.** First, we prove (1). Take a closed point \( y \in Y \). It is enough to show that

\[
H^1(Y, f_*(\mathcal{J}_{\text{NLC}}(X, \Delta)(M)) \otimes \mathcal{L}^l(-y)) = 0.
\]

This follows from the proof of Theorem 1.3, since \( \mathcal{L}^l(-y) \) is ample.

Next, we show (2). By the same argument as the above, it is enough to show that

\[
H^1(Y, (S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))) \otimes \mathcal{A}) = 0
\]

for an ample line bundle \( \mathcal{A} \) on \( Y \). By Theorem 1.1 and the proof of Theorem 1.2 we have the surjective morphism

\[
\bigoplus F_{Y*} \mathcal{A}^{l-1} \to (S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(M))) \otimes \mathcal{A}
\]

for an \( l \geq 1 \) and each \( e \) large and divisible enough. Hence, it suffices to prove that

\[
H^1(Y, F_{Y*} \mathcal{A}^{l-1}) = 0,
\]

but this follows from an argument similar to that of the proof of Theorem 1.1. \( \square \)

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