SOME VARIATIONS
ON MAXWELL’S EQUATIONS

Giorgio A. Ascoli
Krasnow Institute for Advanced Study,
George Mason University,
Fairfax, VA 22030, USA
ascoli@gmu.edu

Gerald A. Goldin
Departments of Mathematics and Physics
Rutgers University
New Brunswick, NJ 08903, USA
gagoldin@dimacs.rutgers.edu

Abstract In the first sections of this article, we discuss two variations on Maxwell’s equations that have been introduced in earlier work—a class of nonlinear Maxwell theories with well-defined Galilean limits (and correspondingly generalized Yang-Mills equations), and a linear modification motivated by the coupling of the electromagnetic potential with a certain nonlinear Schrödinger equation. In the final section, revisiting an old idea of Lorentz, we write Maxwell’s equations for a theory in which the electrostatic force of repulsion between like charges differs fundamentally in magnitude from the electrostatic force of attraction between unlike charges. We elaborate on Lorentz’ description by means of electric and magnetic field strengths, whose governing equations separate into two fully relativistic Maxwell systems—one describing ordinary electromagnetism, and the other describing a universally attractive or repulsive long-range force. If such a force cannot be ruled out a priori by known physical principles, its magnitude should be determined or bounded experimentally. Were it to exist, interesting possibilities go beyond Lorentz’ early conjecture of a relation to (Newtonian) gravity.

It is a pleasure to dedicate this paper to Gérard Emch, whose skeptical perspective helps motivate those who know him to the pursuit of deeper scientific understandings.
1. Introduction

Maxwell’s equations are among the most beautiful in physics, unifying the forces of electricity and magnetism in a classical field theory that explains electromagnetic waves [1]. Some well-known, profoundly-motivated variations on Maxwell’s equations have included the Born-Infeld theory (a nonlinear but Lorentz-covariant modification, that introduces an effective upper bound to the electric field strength), and the Yang-Mills equations (introducing non-Abelian gauge potentials) [2, 3]. These ideas go back many decades, and have deeply influenced the development of theoretical physics. Indeed, there has been a recent resurgence of interest in non-Abelian Born-Infeld Lagrangians [4], which turn out to have important application in string theory and related subjects [5, 6] [7, 8]. More recently, variations of Maxwell’s equations have been considered as ”test theories,” with respect to which observations in astrophysics can provide upper bounds to deviations from the usual equations or laws of physics [9, 10]. We nevertheless seek to approach the idea of modifying Maxwell’s equations in new ways with appropriate humility. None of the variations considered in this article is ad hoc. Rather, each occurred in answer to a specific question in fundamental physics.

Sections 2 and 3 review two such modifications considered by the second author in recent years. The first of these, proposed in joint work with Vladimir Shtelen [11, 12], is a class of Galilean nonlinear Maxwell theories, together with non-Abelian versions that generalize the Yang-Mills equations and the non-Abelian Born-Infeld equations. These possibilities arise in answer to the question of whether and how Maxwell’s equations for the four fields $E$, $B$, $D$ and $H$ can survive unchanged in the Galilean limit of $c \to \infty$, a feature that is present in neither the usual, linear Maxwell theory, nor the usual Born-Infeld theory. The second variation is a simple, linear modification, that can be associated with a change over time in some “constants” of electromagnetism. This possibility comes up in answer to the question of how to write gauge-invariant expressions for the electromagnetic field strengths $E$ and $B$, when Maxwell’s equations are coupled with a natural and very general family of non-linear Schrödinger time-evolutions in quantum mechanics [13]. Such a family of nonlinear Schrödinger equations was developed and studied in a series of articles, in joint work with Heinz-Dietrich Doebner and Peter Nattermann [14, 15, 16]. In the general nonlinear equation, a certain “frictional” term originally proposed by Kostin [17] is the one requiring a change in Maxwell’s equations for the field strengths, after nonlinear gauge transformations are taken into account.
Finally, in Section 4, we take up a different question. In a 1900 article that seems to be not very well-known today, Lorentz explored the idea that (Newtonian) gravity could be explained if the electrostatic force of repulsion between like charges were smaller in absolute magnitude than the electrostatic force of attraction between unlike charges \[18\]. Setting aside Lorentz’ conjectured relation to gravity, we want to reopen the possibility of a difference in magnitude between these forces. There is then a straightforward and elegant description of the situation by means of Maxwell’s equations, that was partially written down in Lorentz’ original article. Under the given hypothesis, one may introduce new electric and magnetic fields whose governing equations separate into two fully relativistic Maxwell systems—one describing ordinary electromagnetism, and the other describing an overall attractive or repulsive long-range force that couples to an “absolute charge.” While the latter force might conceivably have something to do with gravity, it is more plausible to regard it as an extremely small, but theoretically possible, correction to ordinary electromagnetism. Such a correction could, in principle, be time-dependent, and serve as a further “test theory” for astrophysical measurements. Whether attractive or repulsive, a modified electromagnetism could be important in modeling the early universe. Unless a known physical principle rules out such a force \textit{a priori}, its magnitude should be regarded as an experimental question. Well-known nonlinear and non-Abelian generalizations of Maxwell’s equations, and their unification with weak interactions, could then equally well be constructed from the new equations, opening up interesting possibilities.

In the remainder of this section we establish notation, summarize some elementary, familiar background material, and make a few relevant remarks. All of our discussions pertain to \((3 + 1)\)-dimensional space-time.

Let us write Maxwell’s equations in SI units, as follows \[1\]:

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, & \nabla \cdot \mathbf{D} &= \rho, \quad (1)
\end{align*}
\]

where \(\mathbf{E}(\mathbf{x}, t)\) is the electric field, \(\mathbf{D}(\mathbf{x}, t)\) the electric displacement, \(\mathbf{B}(\mathbf{x}, t)\) the magnetic induction, and \(\mathbf{H}(\mathbf{x}, t)\) the magnetic field; \(\rho(\mathbf{x}, t)\) is the charge density, and \(\mathbf{j}(\mathbf{x}, t)\) the electric current density.

The first pair of Eqs. \(1\) imply that we can write \(\mathbf{E}\) and \(\mathbf{B}\) in terms of potentials \((\Phi, \mathbf{A})\),

\[
\begin{align*}
\mathbf{B} &= \nabla \times \mathbf{A}, & \mathbf{E} &= \frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi. \quad (2)
\end{align*}
\]
The choice of $\Phi$ and $A$ is not unique. For an arbitrary smooth function $\Theta(x,t)$, new potentials $A'$ and $\Phi'$ that are obtained from the gauge transformation

$$A' = A + \nabla \Theta, \quad \Phi' = \Phi - \frac{\partial \Theta}{\partial t}$$

(3)

give just the same fields $E$ and $B$. Thus $E$ and $B$ are said to be gauge invariant. The condition $\nabla \cdot B = 0$ expresses the nonexistence in nature of magnetic monopoles, which in this article we do not consider changing. Our choice of SI units here is motivated by the desire to avoid incorporating the speed of light $c$ into the definitions of any of the fields, as we shall later be interested in considering the $c \to \infty$ limit.

From the second pair of Eqs. (1), there follows immediately the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0,$$

(4)

expressing conservation of electric charge.

Thus far, the system is underdetermined. To complete Maxwell’s equations in the presence of matter, one introduces constitutive equations relating the fields $D, H$ to the fields $E, B$. The usual system of Maxwell equations in vacuo is obtained using linear constitutive equations,

$$D = \varepsilon_0 E, \quad H = \frac{1}{\mu_0} B,$$

(5)

where $\varepsilon_0 \mu_0 = 1/c^2$. However, we shall shortly be considering a certain class of nonlinear constitutive equations.

One may take the point of view, given the absence of magnetic monopoles, that the only physically detectable fields are $E$ and $B$. These are defined operationally via the observed Lorentz force $F$ on a small “test particle” with electric charge $q$ moving with velocity $v$:

$$F = qE + qv \times B.$$

(6)

The fields $H$ and $D$ can then be regarded as unobservable constructs used to describe, by way of the latter two Maxwell equations and the constitutive equations, how the observable fields are produced by charges and currents.

As noted in Ref. [11], we then actually have a more general class of linear constitutive equations,

$$\begin{bmatrix} D \\ H \end{bmatrix} = \begin{bmatrix} \varepsilon_0 & \lambda \\ -\lambda & 1/\mu_0 \end{bmatrix} \begin{bmatrix} E \\ B \end{bmatrix},$$

(7)
which—when combined with Eqs. (1)—lead for all values of the real parameter $\lambda$ to the same set of equations for the observable fields $E$ and $B$. Thus the choice $\lambda = 0$ resulting in Eqs. (5) is arbitrary.

Also relevant to the forthcoming discussion is the well-known minimal coupling of the electromagnetic potentials $(\Phi, A)$ with Schrödinger’s equation for the (complex-valued) wave function $\psi(x, t)$ of a single quantum-mechanical particle having charge $q$,

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} (-i\hbar \nabla - qA)^2 \psi + q\Phi \psi. \quad (8)$$

Local $U(1)$ gauge transformations act on $\psi$ according to the formula,

$$\psi'(x, t) = e^{i\theta(x, t)} \psi(x, t), \quad (9)$$

and it is easily checked that if $\psi$ obeys Eq. (8), then $\psi'$ obeys a gauge-transformed equation of the same form, with new electromagnetic potentials given by Eqs. (3) in which $\Theta(x, t) = (\hbar/q) \theta(x, t)$.

The gauge-invariant fields $E$ and $B$, which exert the electric and magnetic forces on the charged quantum particle, are obtained from $\Phi$ and $A$ using Eqs. (2), and satisfy Eqs. (1). The gauge-invariant probability and probability flux densities for the particle are given, respectively, by

$$\rho^g = \bar{\psi} \psi, \quad J^g = \frac{\hbar}{2\imath m} [\bar{\psi} \nabla \psi - (\nabla \bar{\psi}) \psi] - \frac{q}{m} \bar{\psi} \psi A; \quad (10)$$

these also obey an equation of continuity.

Let us remark on the fact that Eqs. (1) respect the Lorentz transformations of special relativity, while Schrödinger’s equation respects Galilean transformations. The minimal coupling of Eq. (8) is compatible with these facts because Eqs. (1) also respect Galilean transformations. It is the linear constitutive equations that impose Lorentz symmetry on the usual Maxwell equations, breaking the Galilean symmetry (see below).

We reproduce Lorentz transformations here in SI units for completeness. Let the subscript $\parallel$ indicate the component of a vector in the direction of the velocity $v$ of an inertial frame of reference, let the subscript $\perp$ indicate the component perpendicular to $v$, and let $v = |v|$. Then with

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad (11)$$
the space-time transformation under the Lorentz boost is
\[
x'_{\parallel} = \gamma (x_{\parallel} - vt), \quad x'_{\perp} = x_{\perp}, \quad t' = \gamma \left( t - \frac{v \cdot x}{c^2} \right); \quad (12)
\]
the field transformations are
\[
B'_{\parallel} = B_{\parallel}, \quad B'_{\perp} = \gamma (B - \frac{1}{c^2} v \times E)_{\perp}, \quad E'_{\parallel} = E_{\parallel}, \quad E'_{\perp} = \gamma (E + v \times B)_{\perp},
\]
\[
H'_{\parallel} = H_{\parallel}, \quad H'_{\perp} = \gamma (H - v \times D)_{\perp}, \quad (13)
\]
and the electric current and charge density transformations are
\[
j'_{\parallel} = \gamma (j_{\parallel} - \rho v), \quad j'_{\perp} = j_{\perp}, \quad \rho' = \gamma (\rho - \frac{v \cdot j}{c^2}). \quad (14)
\]
The corresponding electromagnetic potential transformations in SI units are
\[
\Phi' = \gamma (\Phi - v \cdot A), \quad A' = \gamma (A - \frac{v}{c^2} \Phi). \quad (15)
\]
Under Lorentz transformation, the following combinations of the fields are then invariant:
\[
I_1 = B^2 - \frac{1}{c^2} E^2, \quad I_2 = B \cdot E;
\]
\[
I_3 = D^2 - \frac{1}{c^2} H^2, \quad I_4 = H \cdot D; \quad (16)
\]
\[
I_5 = B \cdot H - E \cdot D, \quad I_6 = B \cdot D + \frac{1}{c^2} E \cdot H.
\]
The Born-Infeld Lagrangian as a function of these invariants is
\[
\mathcal{L} = 1 - R, \quad R = \frac{b^2}{\mu_0 c^2} \sqrt{1 + \frac{c^2}{b^2} I_1 - \frac{c^2}{b^2} I_2^2}. \quad (17)
\]
Next let us write the above in covariant notation. Define \( x^\mu = (ct, \mathbf{x}) \), \( \mu = 0, 1, 2, 3 \), and \( x_\mu = g_{\mu\nu} x^\nu = (ct, -\mathbf{x}) \), where the metric tensor \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and where summation over repeated Lorentz indices is understood; for example, \( x_\mu x^\mu = c^2 t^2 - \mathbf{x}^2 \). We further define \( \partial_\mu \equiv \partial/\partial x^\mu = [(1/c) \partial/\partial t, \nabla] \); and we shall use the antisymmetric Levi-Civita tensor \( \varepsilon^{\alpha\beta\mu\nu} \),
with \( \varepsilon^{0123} = 1 \). Then the usual relativistic tensor fields \( F_{\alpha\beta} \) and \( \mathcal{F}^{\alpha\beta} \), constructed from the fields \( E \) and \( B \), are

\[
F_{\alpha\beta} = \begin{bmatrix}
0 & (1/c)E_1 & (1/c)E_2 & (1/c)E_3 \\
-(1/c)E_1 & 0 & -B_3 & B_2 \\
-(1/c)E_2 & B_3 & 0 & -B_1 \\
-(1/c)E_3 & -B_2 & B_1 & 0
\end{bmatrix}, \quad (18)
\]

\[
\mathcal{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} = \begin{bmatrix}
0 & -B_1 & -B_2 & -B_3 \\
B_1 & 0 & (1/c)E_3 & -(1/c)E_2 \\
B_2 & -(1/c)E_3 & 0 & (1/c)E_1 \\
B_3 & (1/c)E_2 & -(1/c)E_1 & 0
\end{bmatrix},
\]

with

\[
F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}, \quad \mathcal{F}_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} \mathcal{F}^{\mu\nu}. \quad (19)
\]

Likewise,

\[
G^{\alpha\beta} = \begin{bmatrix}
0 & -cD_1 & -cD_2 & -cD_3 \\
cD_1 & 0 & -H_3 & H_2 \\
cD_2 & H_3 & 0 & -H_1 \\
cD_3 & -H_2 & H_1 & 0
\end{bmatrix}, \quad G_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} G^{\mu\nu}, \quad (20)
\]

and so forth. Maxwell’s equations (1) then become

\[
\partial_\alpha \mathcal{F}^{\alpha\beta} = 0, \quad \partial_\alpha G^{\alpha\beta} = j^\beta, \quad (21)
\]

with \( j^\beta = (c\rho, j) \). With \( A_\mu = (\Phi, -A) \), we have from the first of Eqs. (21),

\[
\mathcal{F}^{\alpha\beta} = \varepsilon^{\alpha\beta\mu\nu} \partial_\mu A_\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

The first two invariants of Eqs. (16), that enter Eq. (17) for the Born-Infeld Lagrangian density, are now written

\[
I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad I_2 = -\frac{c}{4} F_{\mu\nu} \mathcal{F}^{\mu\nu}. \quad (23)
\]

Note that our strategy, following Refs. [11] and [12], has been to postpone writing constitutive equations for as long as possible. These now relate \( D, H \) to \( E, B \); or, equivalently, they relate \( G^{\alpha\beta} \) to \( \mathcal{F}^{\alpha\beta} \). The general form for Lorentz-invariant constitutive equations is given by [19]

\[
D = MB + \frac{1}{c^2} NE, \quad H = NB - ME, \quad (24)
\]
where $M$ and $N$ are functions of the Lorentz invariants in Eqs. (16), or
\[ B = R D + \frac{1}{c^2} Q H, \quad E = Q D - R H, \] (25)
where $Q$ and $R$ are likewise functions of the invariants. The linear constitutive equations (5) correspond to the choices $M = 0$, $N = 1/\mu_0$, in which case $\varepsilon_0 = 1/\mu_0 c^2$. Other choices lead to nonlinear relativistic field equations, such as Born-Infeld or Euler-Kockel electrodynamics. It is natural in Eqs. (24) to take $M$ and $N$ to be functions of just the first two invariants $I_1$ and $I_2$ (which depend only on $E$ and $B$), or in the inverted Eqs. (25) to take $R$ and $Q$ to be functions of just the invariants $I_3$ and $I_4$ (which depend only on $D$ and $H$).

The usual approach to writing nonlinear Maxwell theories is to begin with the Lagrangian $L$, which can be written as a function of the invariants. The constitutive equations then follow from the Euler-Lagrange equations. But not all Lorentz-covariant theories are Lagrangian, and the approach via constitutive equations is more general.

In covariant form, Eqs. (24) become
\[ G^{\mu\nu} = NF^{\mu\nu} + cM F^{\mu\nu} \equiv M_1 \frac{\partial I_1}{\partial F^{\mu\nu}} + M_2 \frac{\partial I_2}{\partial F^{\mu\nu}}, \] (26)
where $M_1$ and $M_2$ are likewise functions of the Lorentz invariants.

Now we are ready to discuss the three variations to which this article is devoted.

2. Maxwell equations having Galilean limits

When Maxwell’s equations are written for the four fields $E$, $B$, $D$, and $H$ as in Eqs. (1), the system is underdetermined. In the SI units we are using, these equations are independent of the speed of light $c$. Furthermore, the corresponding equations for the primed fields defined by Eqs. (13) and the primed currents defined by Eqs. (14), when written in the primed coordinates defined by Eqs. (12), are unchanged from Eqs. (1)—even though the Lorentz transformations given in Eqs. (12)-(14) are parameterized explicitly by $c$. Since the invariance of Eqs. (1) holds for every value of $c$, we should not be at all surprised that in the limit $c \to \infty$, these equations also respect the resulting Galilean transformations,
\[ x' = x - vt, \quad t' = t, \]
\[ B' = B, \quad E' = E + v \times B, \]
\[ H' = H - v \times D, \quad D' = D, \]
\[ j' = j - \rho v, \quad \rho' = \rho. \]  

(27)

Now, the value of \( c \) does appear in the constitutive equations. Thus the choice between Lorentz or Galilei symmetry, or the selection of a particular, finite value of \( c \) under which the Lorentz symmetry holds, resides entirely in the constitutive equations.

For the linear constitutive equations (5) or (7), the speed of light is specified by \( c = (\varepsilon_0 \mu_0)^{-1/2} \). In this case, taking a Galilean limit requires that some aspect of Maxwell’s equations be sacrificed, as discussed in detail by Le Bellac and Levy-Leblond [20].

The constitutive equations that select Galilean symmetry, when combined with Eqs. (1), are

\[ D = \hat{M} B, \quad H = \hat{N} B - \hat{M} E, \]  

(28)

or, equivalently

\[ B = \hat{R} D, \quad E = \hat{Q} D - \hat{R} H, \]  

(29)

where \( \hat{M} \) and \( \hat{N} \), \( \hat{Q} \) and \( \hat{R} \) are arbitrary functions of Galilean invariants,

\[ \hat{I}_1 = B^2, \quad \hat{I}_2 = B \cdot E; \]
\[ \hat{I}_3 = D^2, \quad \hat{I}_4 = H \cdot D; \]  

(30)

\[ \hat{I}_5 = B \cdot H - E \cdot D, \quad \hat{I}_6 = B \cdot D. \]

These constitutive equations and field invariants are respectively the formal limits as \( c \to \infty \) of their Lorentz invariant counterparts.

As discussed in Ref. [20], however, taking the mathematical step of letting \( c \to \infty \) is not precisely the same thing as imposing the low velocity condition \( v/c << 1 \) on a class of physical systems governed by the dynamical equations with Lorentz symmetry. For instance, when \( E \) and \( B \) are held fixed, the limit \( c \to \infty \) does not allow the “electric limit” of Ref. [20], although \( v/c << 1 \) is compatible with it.

Letting \( \hat{M} \) be a constant in Eqs. (28) requires (since \( \nabla \cdot B = 0 \)) that the charge density \( \rho \equiv 0 \). Hence the answer to the question of a consistent Galilean electrodynamics, retaining Maxwell’s equations, the continuity equation, and the Lorentz force, is a class of essentially nonlinear theories, that can arise as the \( c \to \infty \) limit of a class of essentially nonlinear Lorentz-covariant theories. Indeed, Le Bellac and Levy-LeBlond emphasize...
(always assuming linear constitutive equations) the mutual incompatibility of Galilean invariance, the continuity equation with non-zero values, and magnetic forces between electric currents.

With nonlinear constitutive equations, these features are no longer incompatible. Nontrivial choices of $\hat{M}$ and $\hat{N}$ in Eqs. (28), or $\hat{R}$ and $\hat{Q}$ in Eqs. (29), combined with Maxwell’s equations, yield fully consistent Galilean versions of electrodynamics.

For example, Ref. [11] proposes to set

$$\hat{Q} = \frac{1}{\varepsilon} \hat{R} = \alpha + 2\alpha^2\varepsilon \frac{\mathbf{H} \cdot \mathbf{D}}{|\mathbf{D}|^2},$$

which are homogeneous functions of the field strengths. This can be shown to lead to an interesting, albeit non-Lagrangian, theory.

Writing the Lagrangian for a nonlinear relativistic theory as

$$\mathcal{L} = \mathcal{L}(I_1, I_2),$$

a short calculation in Ref. [12] demonstrates from the Euler-Lagrange equations that

$$N = 2 \frac{\partial \mathcal{L}}{\partial I_1}, \quad M = -\frac{\partial \mathcal{L}}{\partial I_2}.$$

Therefore the necessary compatibility condition for the constitutive equations to describe such a Lagrangian theory is given by

$$2 \frac{\partial M}{\partial I_1} + \frac{\partial N}{\partial I_2} = 0.$$ (34)

In the Galilean limit, we would take $\mathcal{L} = \mathcal{L}(\hat{I}_1, \hat{I}_2)$, and argue similarly.

The usual Born-Infeld theory does not have a nontrivial Galilean limit. The Lagrangian $\mathcal{L}(I_1, I_2)$ is given by

$$\mathcal{L} = 1 - \mathcal{R}, \quad \mathcal{R} = \frac{b^2}{\mu_0 c^2} \sqrt{1 + \frac{c^2}{b^2} I_1^2 - \frac{c^2}{b^4} I_2^2},$$

which leads to the constitutive equations (24) with

$$M(I_1, I_2) = \frac{I_2}{\mu_0 b^2 \mathcal{R}}, \quad N = \frac{1}{\mu_0 \mathcal{R}}.$$ (36)

Taking $c \to \infty$, we have of course $I_1 \to \hat{I}_1 = \mathbf{B}^2$, and $I_2 \to \hat{I}_2 = \mathbf{B} \cdot \mathbf{E}$. But for large $c$, one has $\mathcal{R} \approx (c/b)[\hat{I}_1 - \hat{I}_2/b^2]^{1/2}$, whence the limits of $M$ and $N$ are both zero.
It is therefore suggested in Ref. [12] to modify the Born-Infeld Lagrangian, replacing \( R \) by
\[
\tilde{R} = \sqrt{1 + \frac{c^2}{b^2} \left[ (1 + \lambda_1 c^2) I_1 - \frac{1}{b^2} (1 + \lambda_2 c^2) I_2^2 \right]},
\]
(37)
where \( \lambda_1, \lambda_2 \) are new constants with the dimensions of \( 1/c^2 \). Now, taking \( c \to \infty \), one obtains the Galilean constitutive equations (28), with
\[
\hat{M} = \frac{\lambda_2 \hat{I}_2}{\mu_0 b \sqrt{\lambda_1 \hat{I}_1 - \lambda_2 \hat{I}_2^2/b^2}}, \quad \hat{N} = \frac{b \lambda_1}{\mu_0 \sqrt{\lambda_1 \hat{I}_1 - \lambda_2 \hat{I}_2^2/b^2}}.
\]
(38)

Similarly, generalizations of classical (non-Abelian) Yang-Mills theory are written by means of Lorentz-covariant, nonlinear constitutive equations. Again, with appropriate choices for the dynamics, the new systems can have fully consistent Galilean-covariant limits as \( c \to \infty \). In analogy with Eqs. (37)-(38), one obtains a class of generalizations of non-Abelian Born-Infeld theories that are of this type [12].

3. Modification from a nonlinear Schrödinger equation

Another variation on Maxwell’s equations occurs as a result of considering the coupling of external electromagnetic fields with nonlinear Schrödinger time-evolutions [13, 16]. First we write the class of Schrödinger equations under consideration. Refs. [14] and [15] provide extensive motivation and development, that we omit here; we mainly follow the discussion in Ref. [13].

Letting \( \psi(x,t) \) be the quantum-mechanical wave function, and define
\[
\hat{x}_\rho(x,t) = \psi(x,t) \quad \text{and} \quad \hat{y}_\rho(x,t) = (1/2i) \left[ \psi \nabla \psi - (\nabla \psi) \psi \right].
\]
(39)
In this article we shall use the notation \( \hat{x}_\rho(x,t) \) to refer to the spatial probability density for the quantum-mechanical particle, to distinguish it from the net charge density \( \rho(x,t) \) that appears in Secs. 1 and 2. Thus \( \hat{x}_\rho \) is here the expression we called \( \rho_{\text{eq}} \) in the first of Eqs. (10).

Define the real, homogeneous functionals \( R_1[\psi], \ldots, R_5[\psi] \), by
\[
R_1 = \frac{\nabla \cdot \hat{x}_\rho}{\hat{x}_\rho}, \quad R_2 = \frac{\nabla^2 \hat{x}_\rho}{\hat{x}_\rho}, \quad R_3 = \frac{\hat{x}_\rho^2}{\hat{x}_\rho^2}, \quad R_4 = \frac{\hat{x}_\rho \cdot \nabla \hat{x}_\rho}{\hat{x}_\rho^2}, \quad R_5 = \frac{(\nabla \hat{x}_\rho)^2}{\hat{x}_\rho^2}.
\]
(40)
The Laplacian in the linear Schrödinger equation (8) can be expanded with respect to this basis of functionals,

\[ \nabla^2 \psi = iR_1[\psi] + \frac{1}{2} R_2[\psi] - R_3[\psi] - \frac{1}{4} R_5[\psi], \]

(41)

so that it does not appear explicitly in the equation we shall next write down.

The general family of nonlinear Schrödinger equations takes the form,

\[ i \frac{\dot{\psi}}{\psi} = i \left[ \sum_{j=1}^{2} \nu_j R_j[\psi] + \frac{\nabla \cdot (A(x,t) \hat{\rho})}{\hat{\rho}} \right] + \sum_{j=1}^{5} \mu_j R_j[\psi] + U(x,t) + \nabla \cdot \left( A'_1(x,t) \hat{\rho} \right) \hat{\rho} + \alpha_1 \ln \hat{\rho} + \alpha_2 S, \]

(42)

where \( \dot{\psi} = \frac{\partial \psi}{\partial t} \), the coefficients \( \nu_j \) (\( j = 1, 2 \)), \( \mu_j \) (\( j = 1, \ldots, 5 \)), and \( \alpha_j \) (\( j = 1, 2 \)) are all continuously differentiable, real-valued functions of \( t \); \( S(x,t) = \arg \left[ \psi(x,t) \right] \); \( U(x,t) \) is a real-valued scalar function; and \( A, A_1, \) and \( A_2 \) are distinct real-valued, time-dependent vector fields.

Writing a class of nonlinear Schrödinger equation by adding terms of the form (40) to the usual, linear Schrödinger equation as in Ref. [15], we have

\[ \hbar \frac{\partial \psi}{\partial t} = H_0 \psi + \frac{i}{2} \hbar D R_2[\psi] \psi + \hbar \sum_{j=1}^{5} D'_j R_j[\psi] \psi, \]

(43)

where \( H_0 \psi \) is given by the right-hand side of Eq. (8), and \( D \) and the \( D'_j \) have the dimension of diffusion coefficients. Then Eq. (43) is obtained from Eq. (42) with the values,

\begin{align*}
\nu_1 &= -\frac{\hbar}{2m}, & \nu_2 &= \frac{1}{2} D, & A &= \frac{q}{2m} A, \\
\mu_1 &= D'_1, & \mu_2 &= -\frac{\hbar}{4m} + D'_2, & \mu_3 &= \frac{\hbar}{2m} + D'_3, & \mu_4 &= D'_4, & \mu_5 &= \frac{\hbar}{8m} + D'_5, \\
U(x,t) &= \frac{q}{\hbar} \Phi + \frac{q^2}{2m \hbar} A^2, & A_1 &= 0, & A_2 &= -\frac{q}{m} A, \\
\alpha_1 &= \alpha_2 = 0. \end{align*}

(44)

The motivation for the form adopted in writing Eq. (42), for the presence of the terms with \( \alpha_1, \alpha_2 \) and \( A_1 \neq 0 \), and the time-dependence of the
coefficients, is the possibility of introducing a group of nonlinear gauge transformations that leave this family of equations invariant (as a class). With \( \psi = R \exp [iS] \), these take the form \( \psi \mapsto \psi' = R' \exp [iS'] \), with

\[
R' = R, \quad S' = \Lambda S + \gamma \ln R + \theta;
\]

where \( \gamma \) and \( \Lambda \) are continuously differentiable, real-valued functions of \( t \), \( \Lambda \neq 0 \), and \( \theta \) is a continuously differentiable, real-valued function of \( x \) and \( t \). Then nonlinear gauge transformations obey the group law,

\[
(\Lambda_1, \gamma_1, \theta_1) \cdot (\Lambda_2, \gamma_2, \theta_2) = (\Lambda_1 \Lambda_2, \gamma_1 + \Lambda_1 \gamma_2, \theta_1 + \Lambda_1 \theta_2).
\]

With quantum-mechanical measurements characterized as sequences of positional measurements (at distinct times), together with the application of external fields between positional measurements [21, 22], and maintaining the standard interpretation of \( \hat{\rho} = |\psi|^2 \) as a probability density for the outcomes of positional measurements, such transformations then leave the distribution of outcomes of all measurements invariant. They are also local in space-time, and respect a separation condition for multiparticle product wave functions [16, 23]. Notice that with \( \gamma \equiv 0 \) and \( \Lambda \equiv 1 \), we recover the usual local \( U(1) \) gauge group of electromagnetism (acting linearly on \( \psi \)) as a subgroup of the larger group of nonlinear gauge transformations.

Under the nonlinear transformation in Eq. (45), we have

\[
\hat{\rho}' = \overline{\psi'} \psi' = \hat{\rho},
\]

\[
\hat{j}' = \frac{1}{2i} \left[ \overline{\psi'} \nabla \psi' - (\nabla \overline{\psi'}) \psi' \right] = \Lambda \hat{j} + \frac{\gamma}{2} \nabla \hat{\rho} + \hat{\rho} \nabla \theta;
\]

so that \( \hat{\rho} \) (as desired) is gauge-invariant (for nonlinear as well as linear gauge transformations). Thus we shall also write \( \hat{\rho} = \rho^{\text{gi}} \) when we want to emphasize this. But \( \hat{j} \) is not gauge-invariant—one must write a new gauge-invariant current (see below), to fully generalize Eqs. (10). Moreover, if \( \psi \) satisfies Eq. (42) then \( \psi' \) likewise satisfies an equation of this form, but with new (primed) coefficients and new external fields. The coefficients transformed under \( (\gamma, \Lambda, \theta) \) are given by

\[
\nu'_1 = \frac{\nu_1}{\Lambda}, \quad \nu'_2 = -\frac{\gamma}{2\Lambda} \nu_1 + \nu_2, \quad
\mu'_1 = -\frac{\gamma}{\Lambda} \nu_1 + \mu_1, \quad \mu'_2 = \frac{\gamma^2}{2\Lambda} \nu_1 - \gamma \nu_2 - \frac{\gamma}{2} \mu_1 + \Lambda \mu_2,
\]
\[
\mu'_3 = \frac{\mu_3}{\Lambda}, \quad \mu'_4 = -\frac{\gamma}{\Lambda} \mu_3 + \mu_4, \quad \mu'_5 = \frac{\gamma^2}{4\Lambda} \mu_3 - \frac{\gamma}{2} \mu_4 + \Lambda \mu_5,
\]

\[
\alpha'_1 = \Lambda \alpha_1 - \frac{\gamma}{2} \alpha_2 + \frac{1}{2} \left( \frac{\dot{\Lambda}}{\Lambda} \gamma - \dot{\gamma} \right), \quad \alpha'_2 = \alpha_2 - \frac{\dot{\Lambda}}{\Lambda}.
\] (48)

Observe that even if one begins with \(\alpha_1 = \alpha_2 = 0\) and with time-independent coefficients \(\nu_j\) and \(\mu_j\), the fact that \(\gamma\) and \(\Lambda\) can be time-dependent requires that in Eq. (42) the \(\alpha_j\) be permitted to take nonzero values, and that all the \(\nu_j, \mu_j,\) and \(\alpha_j\) be, in general, time-dependent. However, the nonlinear gauge transformations do not mix the coefficients \(\alpha_j\) with the coefficients \(\nu_j\) and \(\mu_j\).

The nonlinear term with coefficient \(\alpha_1\) was first proposed as a modification of linear quantum mechanics by Bialynicki-Birula and Micielski [24], and the term with coefficient \(\alpha_2\) was proposed still earlier by Kostin [17]. The term with coefficient \(\nu_2 \neq 0\) was derived by Doebner and Goldin from considerations of local current algebra representations [14], which led to the generalized equation containing the nonlinear functionals \(R_j\) [15].

The external fields of Eq. (42) transformed under \((\gamma, \Lambda, \theta)\) are given in Ref. [13] by

\[
\mathcal{A}' = \mathcal{A} - \frac{\nu_1}{\Lambda} \nabla \theta,
\]

\[
\mathcal{A}'_1 = \Lambda \mathcal{A}_1 - \gamma \mathcal{A} - \frac{\gamma}{2} \mathcal{A}_2 + \left( \frac{\dot{\gamma}}{\Lambda} \nu_1 - \mu_1 + \frac{\gamma}{\Lambda} \mu_3 - \mu_4 \right) \nabla \theta,
\]

\[
\mathcal{A}'_2 = \mathcal{A}_2 - \frac{2\mu_3}{\Lambda} \nabla \theta,
\]

\[
U' = \Lambda U - \dot{\theta} + \left( \frac{\dot{\Lambda}}{\Lambda} - \alpha_2 \right) \theta + \frac{\mu_3}{\Lambda} [\nabla \theta]^2 + \frac{1}{2} \gamma \cdot \mathcal{A}_2 - 2 \mathcal{A} \cdot \nabla \theta.
\] (49)

Observe that Eqs. (48)-(49) imply both the nonzero \(\mathcal{A}_1\) and nontrivial \(\mathcal{A}_2\) values are required in Eq. (42). Even if one begins with \(\mathcal{A}_1 \equiv 0\) and \(\mathcal{A}_2 \equiv -2 \mathcal{A}\), as in the linear Schrödinger equation [cf. Eqs. (44)], the nonlinear gauge transformations compel one to introduce more general values for these fields. Nonlinear Schrödinger equations with arbitrary values of \(\mathcal{A}_2\)
were proposed by Haag and Bannier [25], while the interaction with a general external vector field $\mathbf{A}_1$ was considered in Ref. [26].

Next let us write the equations of motion described by this class of nonlinear Schrödinger equations entirely in terms of gauge-invariant quantities, as in Ref. [13]—where “gauge invariance” is interpreted with respect to the group of nonlinear gauge transformations. To start, a gauge-invariant current density $J^{gi}$ may be written

$$J^{gi} = -2\nu_1 \hat{j} - 2\nu_2 \nabla \hat{\rho} - 2\hat{\rho} \mathbf{A},$$

which evidently reduces to Eq. (10) for the linear Schrödinger equation when $\nu_1 = -\hbar/2m$, $\nu_2 = 0$, $\mathbf{A} = (q/2m)\mathbf{A}$. We thus have (again) a continuity equation for the probability density, $\partial \rho^{gi} / \partial t = - \nabla \cdot J^{gi}$. Refs. [15] and [16] provide a set of gauge-invariant parameters necessary for the desired description,

$$\tau_1 = \nu_2 - \frac{1}{2} \mu_1, \quad \tau_2 = \nu_1 \mu_2 - \nu_2 \mu_1, \quad \tau_3 = \frac{\mu_3}{\nu_1}, \quad \tau_4 = \mu_4 - \mu_1 \frac{\mu_3}{\nu_1},$$

$$\tau_5 = \nu_1 \mu_5 - \nu_2 \mu_4 + \nu_2 \frac{\mu_3}{\nu_1},$$

$$\beta_1 = \nu_1 \alpha_1 - \nu_2 \alpha_2 + \nu_2 \frac{\dot{\nu}_1}{\nu_1} - \nu_2, \quad \beta_2 = \alpha_2 - \frac{\dot{\nu}_1}{\nu_1}.$$  

When $\tau_1 \neq 0$, $\tau_4 \neq 0$, or $\beta_2 \neq 0$, time-reversal invariance is violated. When $\tau_3 \neq -1$ or $\tau_4 \neq 0$, Galilean invariance is violated.

Gauge-invariant external magnetic and electric fields are now given by the formulas [13]

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{q}{2m} \mathbf{B},$$

$$\mathbf{E} = -\nabla \hat{U} - \frac{\partial \mathbf{A}}{\partial t} - \beta_2 \mathbf{A} = \frac{q}{2m} \mathbf{E},$$

where

$$\hat{U} = -\nu_1 U - \tau_3 \mathbf{A}^2 - (\tau_4 - 2\tau_1 \tau_3) \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{A}_2 - \nu_2 \nabla \cdot \mathbf{A}_2.$$  

Thus $\hat{U}$ is to be identified with $(q/2m)\Phi$ (which may be directly checked for the linear Schrödinger equation); but the main point here is that the formula for $\mathbf{E}$ in terms of $\Phi$ and $\mathbf{A}$ has been modified from Eq. (2) to include an extra term, originating with Kostin’s nonlinearity:

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} - \beta_2 \mathbf{A}.$$
The extra term is necessary—if we leave it out of Eq. (52), \( \mathcal{E} \) fails to be gauge-invariant. We also have new gauge-invariant external vector fields,

\[
A_1^{gi} = \nu_1 A_1 + \left( \frac{2\nu_2 \mu_3}{\nu_1} - \mu_1 - \mu_4 \right) A - \nu_2 A_2, \quad A_2^{gi} = \frac{\nu_1}{2\mu_3} A_2 - A. \tag{55}
\]

Continuing to follow Ref. [13], we are now in a position to write equations of motion obtained from Eq. (42), in a form that is manifestly gauge-invariant for the group of nonlinear gauge transformations. We use the hydrodynamical variables variables \( \rho^{gi} \) and \( \mathbf{J}^{gi}/\rho^{gi} \), where the latter has the interpretation of a gauge-invariant velocity field. We then have the interpretation of the (gauge-invariant) magnetic induction as a vorticity field,

\[
\nabla \times \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) = -2 \mathbf{B} = \frac{q}{m} \mathbf{B}. \tag{56}
\]

The dynamical equations are the equation of continuity that we wrote above,

\[
\frac{\partial \rho^{gi}}{\partial t} = -\nabla \cdot \mathbf{J}^{gi}, \tag{57}
\]

together with the equation for the changing velocity field,

\[
\frac{\partial}{\partial t} \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) = \nabla \left[ 2\tau_1 \nabla \cdot \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) + 2\tau_2 \nabla^2 \rho^{gi} + \frac{1}{2} \tau_3 \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right)^2 \right] + \nabla \left[ (2\tau_1 [1 + \tau_3] - \tau_4) \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) \cdot \nabla \rho^{gi} + 2\tau_5 \left( \frac{\nabla \rho^{gi}}{\rho^{gi}} \right)^2 \right] + \nabla \left[ 2 \frac{\nabla \cdot (A_1^{gi} \rho^{gi})}{\rho^{gi}} - 2\tau_3 A_2^{gi} \cdot \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) + 2\beta_1 \ln \rho^{gi} \right] - \beta_2 \left( \frac{\mathbf{J}^{gi}}{\rho^{gi}} \right) + \frac{q}{m} \mathbf{E}. \tag{58}
\]

In Eq. (58) we see that \( \beta_2 \), taken to be positive, has a natural interpretation as a gauge-invariant coefficient of friction—it governs the magnitude of the term in \( \partial_t (\mathbf{J}^{gi}/\rho^{gi}) \) that is proportional to \( \mathbf{J}^{gi}/\rho^{gi} \).

In the framework of the nonlinear quantum mechanics discussed here, the (gauge-invariant) expected values for the position, velocity, and acceleration
of the quantum particle (all of which are functions of \( t \)) are given respectively by the following expressions:

\[
\begin{align*}
\langle x \rangle &= \int x \rho_{gi}(x) \, dx, \\
\langle v \rangle &= \frac{\partial \langle x \rangle}{\partial t} = \int \rho_{gi} \left( \frac{J_{gi}}{\rho_{gi}} \right) \, dx = \int J_{gi}(x) \, dx, \\
\langle a \rangle &= \frac{\partial \langle v \rangle}{\partial t} = \\
&= \int \rho_{gi} \left[ \frac{1}{2} \nabla \left( \frac{J_{gi}}{\rho_{gi}} \right)^2 + \left( \frac{J_{gi}}{\rho_{gi}} \right) \times \frac{q}{m} B + \frac{\partial}{\partial t} \left( \frac{J_{gi}}{\rho_{gi}} \right) \right] \, dx.
\end{align*}
\]

In Eqs. (58)-(59), we see that the laws of force describing the interaction of the charged particle with the \( E \) and \( B \) fields are unchanged from those in linear quantum mechanics.

Now Eq. (54) gives us the variation on Maxwell’s equations that is the focus of this section. The usual equations for \( E \) and \( B \) are replaced by

\[
\begin{align*}
\nabla \times E &= -\frac{\partial B}{\partial t} - \beta_2 B, \\
\nabla \cdot B &= 0.
\end{align*}
\]

Note that the second of these equations is still consistent with the first. Let us take \( \beta_2 \) to be a constant, independent of \( t \). If \( E_0(x,t) \), \( B_0(x,t) \) satisfy the original Maxwell equations (with \( \beta_2 = 0 \)), then fields satisfying Eqs. (60) are given by

\[
E = E_0 \, e^{-\beta_2 t}, \quad B = B_0 \, e^{-\beta_2 t}.
\]

But nonlinear quantum mechanics alone does not specify the remaining two Maxwell equations. One possibility is to maintain the constitutive equations (5), with fixed coefficients \( \varepsilon_0 \) and \( \mu_0 \). Then taking \( E_0 \), \( B_0 \), \( D_0 = \varepsilon_0 E_0 \), \( H_0 = (1/\mu_0)B_0 \), \( \rho_0 \) and \( j_0 \) to satisfy the usual, linear Maxwell equations (1), we must have

\[
\begin{align*}
D &= D_0 \, e^{-\beta_2 t}, \\
H &= H_0 \, e^{-\beta_2 t},
\end{align*}
\]

and

\[
\begin{align*}
\nabla \cdot D &= \rho, \quad \text{with} \quad \rho = \rho_0 \, e^{-\beta_2 t}, \\
\nabla \times H &= \frac{\partial D}{\partial t} + \beta_2 D + j, \quad \text{with} \quad j = j_0 \, e^{-\beta_2 t}.
\end{align*}
\]
That is, with $\beta_2 > 0$, the magnitudes of all the electric charges and currents are decaying exponentially with time. Of course, the equation of continuity for $\rho$ and $j$ no longer holds, and net charge is no longer conserved. Instead, we have the equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j - \beta_2 \rho.$$ \hspace{1cm} (64)

The nonzero value for $\beta_2$, interpreted as a coefficient of friction experienced by a charged particle, has introduced a preferred universal reference frame. The corresponding Maxwell theory is no longer covariant.

Another alternative is to join the standard Maxwell equations for $D$ and $H$ with Eqs. (60) by supposing that the coefficients in the constitutive equations—the permittivity and permeability of free space—are time-dependent. With

$$\varepsilon(t) = \varepsilon_0 e^{\beta_2 t}, \quad \mu(t) = \mu_0 e^{-\beta_2 t},$$ \hspace{1cm} (65)

we may combine Eqs. (60) and (61) with the constitutive equations,

$$D = \varepsilon(t)E, \quad H = \frac{1}{\mu(t)}B.$$ \hspace{1cm} (66)

Then $D = D_0$, $H = H_0$, $\rho = \rho_0$, and $j = j_0$. We have $\varepsilon(t)\mu(t) = 1/c^2$ for all $t$. The equation of continuity holds for $\rho$ and $j$, and the net charge is conserved.

Since current systems of units define the vacuum permittivity to have a fixed numerical value, the modification embodied in Eqs. (66) requires some reexamination of the way in which we define our units of measurement for electromagnetism. We shall return briefly to the idea of time-dependent permittivity and permeability toward the end of the next section.

4. Force differences between like and unlike charges

For our final variation on Maxwell’s equations, we pursue the suggestion by Lorentz that the electrostatic force of repulsion between like charges is slightly different in magnitude from the electrostatic force of attraction between unlike charges. While Lorentz proposed this idea in 1900 as a way to arrive at a universally attractive (Newtonian) gravitational force, we want to regard it here as just a modification of classical electrodynamics. Let us write a description of the situation in terms of field strengths.
The electrostatic forces among pairs of positively-charged point particles having charges $Q_+ > 0$ and $q_+ > 0$ and/or negatively-charged point particles $Q_- < 0$ and $q_- < 0$, is described by
\[
F_\ell = \frac{1}{4\pi \varepsilon_\ell} \frac{Q_+ q_+}{r^2} \hat{r} = \frac{1}{4\pi \varepsilon_\ell} \frac{Q_- q_-}{r^2} \hat{r},
\]
\[
F_u = \frac{1}{4\pi \varepsilon_u} \frac{Q_- q_+}{r^2} \hat{r} \quad \text{or} \quad F_u = \frac{1}{4\pi \varepsilon_u} \frac{Q_+ q_-}{r^2} \hat{r},
\]
where $\hat{r}$ is the unit vector at the location of each particle, pointing away from the other, and $r$ is the distance between the pair. The subscripts $\ell$ and $u$ stand for “like” and “unlike” respectively, and $\varepsilon_\ell \neq \varepsilon_u$. If $|Q_+| = |Q_-|$ and $|q_+| = |q_-|$, then the forces $F_u$ in the second equation are equal.

Introduce the electric field $E_+$ exerting force on positive charges, and the electric field $E_-$ exerting force on negative charges; also the displacement fields $D_+$ and $D_-$ produced (respectively) by positive and negative charges. In ordinary electromagnetism, $E = E_+ = E_-$, while $D = D_+ + D_-$. Evidently we must keep track separately of the density $\rho_+ \geq 0$ of positive charge and the density $\rho_- \leq 0$ of negative charge. From Eqs. (67), the force $F_+$ experienced by the positively charged particle with charge $q_+$ in the presence of a composite having charges $Q_+$ and $Q_-$ at a distance $r$ is given by
\[
F_+ = q_+ E_+ = q_+ \left( \frac{1}{4\pi \varepsilon_\ell} \frac{Q_+}{r^2} + \frac{1}{4\pi \varepsilon_u} \frac{Q_-}{r^2} \right) \hat{r},
\]
and similarly for a negatively charge particle,
\[
F_- = q_- E_- = q_- \left( \frac{1}{4\pi \varepsilon_u} \frac{Q_+}{r^2} + \frac{1}{4\pi \varepsilon_\ell} \frac{Q_-}{r^2} \right) \hat{r}.
\]

There is a net electrostatic force between neutral composites that is attractive if $\varepsilon_u < \varepsilon_\ell$, and repulsive if $\varepsilon_u > \varepsilon_\ell$. The situation thus far is described by the Maxwell equation
\[
\nabla \cdot \begin{bmatrix} D_+ \\ D_- \end{bmatrix} = \begin{bmatrix} \rho_+ \\ \rho_- \end{bmatrix},
\]
and the constitutive equation
\[
\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \begin{bmatrix} 1/\varepsilon_\ell & 1/\varepsilon_u \\ 1/\varepsilon_u & 1/\varepsilon_\ell \end{bmatrix} \begin{bmatrix} D_+ \\ D_- \end{bmatrix}.
\]
Similarly, introduce $\mathbf{B}_+$ and $\mathbf{B}_-$ as the magnetic inductions that exert velocity-dependent forces on positive and negative moving charges $q_+$ and $q_-$ (respectively); so that the total forces are given (respectively) by

$$F_+ = q_+ \mathbf{E}_+ + q_+ \mathbf{v} \times \mathbf{B}_+ , \quad F_- = q_- \mathbf{E}_- + q_- \mathbf{v} \times \mathbf{B}_- .$$

(72)

Evidently, we must also keep track separately of the current of positive charge $\mathbf{j}_+$ and the current of negative charge $\mathbf{j}_-$. We introduce $\mathbf{H}_+$ and $\mathbf{H}_-$ as the magnetic fields produced (respectively) by electric currents $\mathbf{j}_+$ and $\mathbf{j}_-$, and (respectively) by changing displacement fields $\mathbf{D}_+$ and $\mathbf{D}_-$. Then we obtain, consistent with Lorentz covariance and the absence of magnetic monopoles, the additional Maxwell equations,

$$\nabla \times \begin{bmatrix} \mathbf{H}_+ \\ \mathbf{H}_- \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} \mathbf{D}_+ \\ \mathbf{D}_- \end{bmatrix} + \begin{bmatrix} \mathbf{j}_+ \\ \mathbf{j}_- \end{bmatrix},$$

$$\nabla \times \begin{bmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{bmatrix} = -\frac{\partial}{\partial t} \begin{bmatrix} \mathbf{B}_+ \\ \mathbf{B}_- \end{bmatrix} , \quad \nabla \cdot \begin{bmatrix} \mathbf{B}_+ \\ \mathbf{B}_- \end{bmatrix} = 0 ,$$

(73)

with the constitutive equation

$$\begin{bmatrix} \mathbf{B}_+ \\ \mathbf{B}_- \end{bmatrix} = \begin{bmatrix} \mu_\ell & \mu_u \\ \mu_u & \mu_\ell \end{bmatrix} \begin{bmatrix} \mathbf{H}_+ \\ \mathbf{H}_- \end{bmatrix} ,$$

(74)

where

$$\varepsilon_\ell \mu_\ell = \varepsilon_u \mu_u = \frac{1}{c^2} .$$

(75)

From Eq. (70) and the first of Eqs. (73), we have separate continuity equations for $\rho_+$, $\mathbf{j}_+$ and $\rho_-$, $\mathbf{j}_-$,

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho_+ \\ \rho_- \end{bmatrix} + \nabla \cdot \begin{bmatrix} \mathbf{j}_+ \\ \mathbf{j}_- \end{bmatrix} = 0 .$$

(76)

Thus far we have written in Eqs. (70) and (73) a doubled set of Maxwell equations, indexed by $(+, -)$, that are coupled by the matrix constitutive equations (71) and (74). To recover the theory as a perturbation of ordinary classical electrodynamics, define $\rho = \rho_+ + \rho_-$ and $\mathbf{D} = \mathbf{D}_+ + \mathbf{D}_-$; so that $\rho$ is the net charge density, and $\nabla \cdot \mathbf{D} = \rho$. Defining $\mathbf{E} = \frac{1}{2} (\mathbf{E}_+ + \mathbf{E}_-)$, we recover the constitutive equation $\mathbf{E} = (1/\varepsilon_0) \mathbf{D}$ by setting

$$\frac{1}{\varepsilon_0} = \frac{1}{2} \left( \frac{1}{\varepsilon_\ell} + \frac{1}{\varepsilon_u} \right) .$$

(77)
Similarly, defining \( j = j_+ + j_- \) and \( H = H_+ + H_- \), we have the usual Maxwell equation \( \nabla \times H = \partial D / \partial t + j \). Letting \( B = \frac{1}{2}(B_+ + B_-) \), we also have the Maxwell equations \( \nabla \times E = -\partial B / \partial t \) and \( \nabla \cdot B = 0 \), and we recover the constitutive equation \( B = \mu_0 H \) by setting
\[
\mu_0 = \frac{1}{2} (\mu_\ell + \mu_u). \tag{78}
\]
Then it follows from Eqs. (75), (77), and (78) that \( \varepsilon_0 \mu_0 = \frac{1}{c^2} \). In short, we still have the four fields \( E, B, D, \) and \( H \), and they still satisfy the usual Maxwell equations incorporating the net charge density and net electric current density, with coefficients \( \varepsilon_0 \) and \( \mu_0 = 1/\varepsilon_0 c^2 \).

But we also have an additional set of fields, density, and current,
\[
\tilde{D} = D_+ - D_- \quad \text{and} \quad \tilde{E} = \frac{1}{2} (E_+ - E_-),
\]
\[
\tilde{H} = H_+ - H_- \quad \text{and} \quad \tilde{B} = \frac{1}{2} (B_+ - B_-). \tag{79}
\]
These also obey Maxwell’s equations; but with new constants in their constitutive equations, \( \tilde{\varepsilon} \) and \( \tilde{\mu} \), that are given by
\[
\frac{1}{\tilde{\varepsilon}} = \frac{1}{2} \left( \frac{1}{\varepsilon_\ell} - \frac{1}{\varepsilon_u} \right) \quad \text{and} \quad \tilde{\mu} = \frac{1}{2} (\mu_\ell - \mu_u). \tag{80}
\]
In ordinary electromagnetism, \( \tilde{\varepsilon} \) is infinite and \( \tilde{\mu} \) is zero—so that, although \( \tilde{\rho}, \tilde{j}, \tilde{D}, \) and \( \tilde{H} \) are defined and nontrivial, \( \tilde{E} \equiv 0 \) and \( \tilde{B} \equiv 0 \).

If it is not actually infinite, the magnitude of \( \tilde{\varepsilon} \) is presumably very large compared with that of \( \varepsilon_0 \); while if it is not zero, the magnitude of \( \tilde{\mu} \) is small compared with that of \( \mu_0 \). One verifies straightforwardly that \( \tilde{\epsilon} \tilde{\mu} = 1/c^2 \); but the new constants may be of either sign—both positive (for a net attractive force between neutral composites), or both negative (for a net repulsive force). We thus have obtained a pair of fully decoupled Maxwell systems, consistent with the equations set down in Lorentz’ original article.

An idealized composite point particle, having positive charge \( q_+ \geq 0 \) and negative charge \( q_- \leq 0 \), may be equivalently described as having net charge \( q = q_+ + q_- \) and absolute charge \( \tilde{q} = q_+ - q_- \geq 0 \). Such a particle, moving with velocity \( v \), experiences according to Eqs. (72) the total force
\[
F = F_+ + F_- = q (E + v \times B) + \tilde{q} (\tilde{E} + v \times \tilde{B}). \tag{81}
\]
Thus the new fields $\tilde{E}$, $\tilde{B}$, $\tilde{D}$, and $\tilde{H}$ couple to the absolute charge (which is always positive), and the absolute current; while the usual fields $E$, $B$, $D$, and $H$ still exist in this framework and couple to the net charge and the net current. One should no longer automatically take positive charge flowing to the right to be indistinguishable mathematically or physically from the same amount of negative charge flowing to the left. The net currents $\mathbf{j}$ are the same in these two cases, but the absolute currents $\tilde{\mathbf{j}}$ are equal and opposite.

Let us close this section with some comments and speculations, many of them rather obvious, about such a “doubled electromagnetism” theory.

In his original article [18], Lorentz took the new force to be attractive. He sought to identify the absolute charge with mass (and consequently the absolute current with momentum), and to calculate whether the precession of the perihelion of Mercury’s orbit could then be understood as due to the (very small) magnetic force that would originate from the absolute current. He concluded that the resulting force would be too weak to explain the astronomical observations, and of course this line of thinking was superseded by the success of Einstein’s general relativity.

However, we want to entertain the idea of a modified electrodynamics that does not identify the extra fields with gravity, and that treats absolute charge not as mass but as an additional property of matter. Of course, this does not preclude the possibility that existing measurements of gravitational forces have erroneously incorporated a small extra electrostatic force (attractive or repulsive). It would seem to be an especially interesting conjecture that the new force is repulsive. In any case, we appear to have an additional parameter with which to fit cosmological models, and an additional “test theory” for study through observations in astrophysics.

Since we have a new set of fields obeying Maxwell’s equations, we would also need to have a new type of electromagnetic wave (coupling weakly with the absolute charge), a new type of photon, and a new quantum electrodynamics (see below).

Now the absolute electric charge of a system must be at least equal to the net electric charge, and at least equal to the sum of the absolute charges of the system’s components. However, it could in principle be greater. The absolute charge of a nucleon, for example, might be the sum of the absolute values of the charges of its constituent quarks; but one could also conjecture additional, unobserved positive and negative charges in equal measure, contributing to a larger value of the overall absolute charge. While net charge is quantized in fixed units, it is plausible but not necessary that absolute charge be similarly quantized. Thus, it does not appear to be inconsistent to take the absolute
charge to be proportional to the mass, as Lorentz implicitly did.

However, modern particle physics offers no fundamental theoretical reason to make such an assumption. If we make reference only to constituent quarks and leptons, the absolute charge of a proton (comprised of two up quarks and one down quark) is $5/3$; that of a neutron (comprised of one up quark and two down quarks) is $4/3$; and that of an electron (taken to be fundamental) is $1$. Then the absolute charge of a proton together with an electron is $8/3$, double that of the neutron, while the respective masses are very close to equal. Under these assumptions, the absolute charge per gram of electrically neutral matter comprised of heavier elements is macroscopically different from that of matter comprised of lighter elements, and their accelerations under the Earth’s absolute electric field would be different in magnitude. Since such differences are not observed, we should take the empirically-determined magnitude of any new inverse-square-law force of “absolute electromagnetism” to be small compared with Newtonian gravity—making it extremely small in comparison with ordinary electromagnetism.

Still, one may conjecture that physical “constants” are not actually constant, but change as the universe ages. We discussed in Sec. 3 the possibility of the vacuum permittivity and permeability changing exponentially with $t$, producing a modification in Maxwell’s equations and a universal frictional force that breaks covariance. One may instead take these to be fixed at $\varepsilon_0$ and $\mu_0$, but entertain the possibility that $\varepsilon_u$ and $\varepsilon_l$ are changing, and that they have not always been as close as they are today. For example, we could have

$$\varepsilon(t) = \varepsilon_0 e^{\beta t}, \quad \mu(t) = \mu_0 e^{-\beta t},$$

and modify the Maxwell equations for the perturbing fields $\tilde{E}$, $\tilde{B}$, $\tilde{D}$, and $\tilde{H}$ as in Sec. 3.

Furthermore, should there be regions of space-time containing plasmas of electrons and positrons, these might contribute proportionally more to absolute charge than to gravitational mass. Such speculations leave open some possibilities for observable effects in astrophysics, even for a small force.

While net charge and absolute charge are both conserved when there is no particle creation or annihilation, it seems clear that absolute charge is not conserved by fundamental particle processes. [Of course, at the time of Lorentz’ paper, mass, positive charge, and negative charge would all have been taken as separately conserved.] As long as we stay with Lorentz’ idea that absolute charge is proportional to mass, then (as mass is transformed into energy during fundamental particle processes) we have a conservation
law. But if we take absolute charge to be an independent quantity with which the new electromagnetic fields couple, it becomes an unlikely, speculative possibility that absolute charge transforms into something previously unknown during annihilation processes, so as to maintain a conservation law. More likely, one should take the new $U(1)$ gauge symmetry to be broken outside the classical domain that is governed by Maxwell’s equations, requiring a different quantum electrodynamics for absolute electromagnetism.

We have not discussed the question of absolute charge from virtual particle-antiparticle pairs, or vacuum polarization. It appears that, unlike the situation for net charge, it should be possible to have a coherent superposition of quantum states having different absolute charges. For example, the neutral pion, written as a linear combination of up and down quark-antiquark pairs, $(\bar{u}u - \bar{d}d)/\sqrt{2}$, would combine states of absolute charges $4/3$ and $2/3$ (in units of the electron’s charge).

If Lorentz’ conjecture breaks no known physical principle, then the question of a discrepancy in magnitude between the electrostatic forces between like and unlike charges is purely an empirical one, and the best we can do in principle is to establish an experimental upper bound to this discrepancy (or, equivalently, to the ratio $\varepsilon_0/\bar{\varepsilon}$). For example, it is clear that his conjecture continues to respect the charge conjugation invariance of electromagnetism. However, it should be noted that current physics tends to assume the equality in magnitude between these forces. Thus the present, official definition of the coulomb is effectively as a unit of net charge, while the value of $\varepsilon_0$ is not measured but fixed by definition. Just as we have come to distinguish (theoretically) gravitational mass from inertial mass, and to regard their proportionality as a question to be determined by experiment, so may it be necessary to distinguish (theoretically) $\varepsilon_\ell$ from $\varepsilon_u$, and to regard their closeness as having a value to be bounded by experiment.

Possible further development of a “doubled electromagnetism” theory could entail its involvement in electroweak unification or in the standard model, its nonlinear modifications (as discussed for Maxwell’s equations in Sec. 2 of this article), its coupling with linear and nonlinear Schrödinger quantum mechanics (as discussed for Maxwell fields in Sec. 3 of this article), its non-Abelian generalizations, and its quantum electrodynamics.

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