Construction of Elementary Gates in Quantum Computation by Joint Measurement

Jia-Qi Jin and Gui-Lu Long

1 Key Laboratory For Quantum Information and Measurements and Department of Physics, Tsinghua University, Beijing 100084, P.R. China
2 Center for Atomic and Molecular NanoSciences, Tsinghua University, Beijing 100084, P.R. China

(Dated: April 1, 2022)

In this paper, elementary quantum gate operations, such as the phase gate, the controlled-NOT gate, the swap and the Fredkin gate are constructed using joint measurement and pairs of entangled qubit pairs. The relation between the state of the entangled pair and the joint measurement basis is discussed. Some other generalization is also discussed.

PACS numbers: 03.67.Dd, 03.67.Hk

I. INTRODUCTION

Entanglement is one important concept in quantum mechanics. In quantum teleportation, maximally entangled states are the pivotal resource. It is also the source of power in quantum computation. Usually, quantum computation starts from an initial quantum state, say, $|0\ldots0\rangle$. Quantum gate operations are unitary operations that are constructed using a finite set of basic universal quantum computing gates. At the end of a computation, a measurement is performed on the register so that the outcome is read out. Many schemes have been proposed, and considerable progress has been made over the last decade. However, it is still a daunting task to build a practical quantum computer. Though it is difficult to build complicated quantum system and to perform complex computation operations, small quantum systems are already easily available and some basic quantum operations can be performed. Recently there have been proposals to perform complicated quantum computation with small quantum systems such as entangled pairs of qubits and simple measurement. It has been shown that quantum computation can be performed by starting from a very complicated entangled states and then proceeding to the end by merely measurements. The cluster state quantum computation scheme is typical example of these schemes. Valence bond state, proposed by Ian Affleck et al. and used in condensed matter physics, has also been found to play an important role in quantum computation recently. Verstraete and Cirac proved that the cluster state quantum computation and valance-bond-state computation are equivalent.

In Ref. [18], it was shown that universal quantum computation can be carried out by using only pairs of qubits in singlet state and by performing joint two-qubit and three-qubit measurement. It has the advantage of being deterministic and using a simple initial product state. Single-qubit operation can be implemented by performing a Bell-basis-like measurement on the target qubit and another qubit from a singlet. Controlled-Z gate is implemented by performing two GHZ-like state measurements on the two qubits and three singlets system. These two unitary gate operations are the basic building blocks for more complicated gate operations. As this scheme is practically appealing, it is interesting to construct the elementary gate operations in this scheme. In this paper, we will provide the constructions of the elementary gate operations for this scheme. The paper is organized as follows. In section II, we briefly summarize the construction of basic gates of Ref. [18]. In section III, we give the results for the well-known elementary gates mentioned in [19], which are the basic ingredients for quantum algorithms in quantum computation. In section IV, we give a brief summary.

II. THE BASIC GATES

Here we briefly review the basic gate operation in the scheme in Ref. [18]. In Ref. [18], basic single-qubit local unitary operation on a particle can be implemented in a teleportation fashion. Suppose the state of particle $A$ is $|\psi\rangle$, and particles $B$ and $C$ are in an entangled state $|H\rangle$. By performing a joint measurement on qubits $A$ and $B$ in the following-basis

$$|\alpha\rangle = (U^{+}\sigma_{\alpha} \otimes I)|H\rangle, \quad (\alpha = 1, 2, 3, 4),$$

(1)
where \( \sigma_\alpha \) denote the Pauli matrices \((\sigma_4 = I)\) and

\[
|H\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{\sqrt{2}} \\
= \frac{|0+\rangle + |1-\rangle}{\sqrt{2}},
\]

(2)
is a maximally entangled state, and

\[
|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}
\]

(3)
is the eigenstates of \( \sigma_x \). The state of particle \( C \) becomes \( \sigma_\alpha |\psi\rangle \) where \( \alpha \) is the outcome of the measurement in basis \( \{|\alpha\rangle, \alpha = 1, 2, 3, 4\} \). Particle \( C \) then works as the role of particle in the following process of the quantum computation.

It is worth mentioning here that the entangled state \((H)\) can be replaced by arbitrary maximally entangled states such as the Bell-basis states, \(|\phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \) and \(|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}\). However this substitution cannot be used for the controlled-Z gate.

The controlled-Z gate,

\[
U_{cזר} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|
\]

(4)

proposed in Ref. [18] involves two GHZ-like state joint measurements as redrawn in Fig. 2 where we have given explicit labelling to the eight qubits involved.

By doing three-qubit joint measurement of particles on \(a, e, e'\) and \(b, e', d'\) in the following basis,

\[
\{\{|\alpha\rangle\}\} = \{|\beta\rangle\} = \{(\sigma_x)^i \otimes (\sigma_x)^j \otimes I \ (|000\rangle \pm |111\rangle)\},
\]

(5)

where \(i, j \in \{0, 1\}\), the controlled-Z gate is implemented between qubits \( c \) and \( d \). The singlet states of the entangled pairs between \( e \) and \( e'\), \( c \) and \( e'\), \( d \) and \( d'\) are all \(|H\rangle\).

By replacing the singlet state \(|H\rangle\) by other entangled states, it is also possible to construct the gate. The detailed combination of the states used for each pair and the appropriate basis for the joint measurement are given in Table 4. The main conclusion of this table is that the measuring-basis is determined by the state of the \(c e'\) pair, the states of \(c c'\) and \(d d'\) pairs are irrelevant. For example, as in Fig. 3 assuming the state of the two original qubits is

\[
|\psi\rangle_{ab} = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle,
\]

(6)

and the states of the \((e e')\), \((c e')\) and \((d d')\) pairs denoted by \(|\varphi_1\rangle\), \(|\varphi_2\rangle\) and \(|\varphi_3\rangle\) respectively, are in the Bell-basis states,

\[
|\varphi_1\rangle = |\varphi_2\rangle = |\varphi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\]

(7)

If the measuring bases of Eq. 3 were used, half of the information in \((c_0, c_1, c_2, c_3)\) would be lost. For instance, if \(|\alpha\rangle = |\beta\rangle = |000\rangle + |111\rangle\), then the wave function of \(c\) and \(d\) would become \(c_0|00\rangle + c_3|11\rangle\), which have lost state information in this process due to the form of entangled state of \(e\) and \(e'\). To see this more clearly, we write the state of the eight-qubit system of Fig. 4 as \(|\Phi\rangle\),

\[
|\Phi\rangle = |\Phi_1\rangle + |\Phi_2\rangle = |\psi\rangle \otimes |H\rangle_{ee'} \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle,
\]

(8)

where

\[
|\Phi_1\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|0_e, 0_{e'}\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle),
\]

(9)

\[
|\Phi_2\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|1_e, -e'\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle).
\]

(10)

If the entangled state between \(e\) and \(e'\) is the \(|H\rangle\) state, then the three-qubit measurement projections \(0_b0_c0_d|\Phi_1\rangle\), \(0_b0_c0_d|\Phi_2\rangle\), \(1_b1_c1_d|\Phi_1\rangle\), \(1_b1_c1_d|\Phi_2\rangle\) are all none-zero. However if we change the entangled state of \(e\) and \(e'\) to state \(\frac{1}{\sqrt{2}}(|0_e, 0_{e'}\rangle + |1_e, 1_{e'}\rangle)\), and turn \(|\Phi_1\rangle, |\Phi_2\rangle\) into

\[
|\Phi_1\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|0_e, 0_{e'}\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle),
\]

(11)

\[
|\Phi_2\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|1_e, 1_{e'}\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle),
\]

(12)

then two out of four projections become zero, and this leads to loss of components in the operation.
TABLE I: Measuring-basis for the controlled-Z gate

| \(|\varphi\rangle_{ee'}\) | \(|\varphi\rangle_{dd'}\) | \(|\varphi\rangle_{dd'}\) | \(|\alpha\rangle_{ac'e'}\) | \(|\beta\rangle_{bc'e'}\) |
|---|---|---|---|---|
| \(|H\rangle\) | \(|\varphi^\pm\rangle, |\psi^\pm\rangle, |H\rangle\) | \(|\varphi^\pm\rangle, |\psi^\pm\rangle, |H\rangle\) | \((\sigma_x)^0 \otimes (\sigma_z)^0 \otimes I\) | \((\sigma_z)^0 \otimes (\sigma_x)^0 \otimes I\) |
| \(|\varphi^\pm\rangle, |\psi^\pm\rangle\) | \(|\varphi^\pm\rangle, |\psi^\pm\rangle, |H\rangle\) | \(|\varphi^\pm\rangle, |\psi^\pm\rangle, |H\rangle\) | \((\sigma_x)^0 \otimes (\sigma_z)^0 \otimes I(000)\) | \((\sigma_z)^0 \otimes (\sigma_x)^0 \otimes I(000)\) |

In this case, if we change the measuring-basis turn \(|\alpha\rangle\) into

\[
|\alpha\rangle = \{ (\sigma_x)^i \otimes (\sigma_y)^j \otimes I(0a + e0c') \pm |1a - e1c'\} \}
\]

where \((i, j \in \{0, 1\})\) and keep the state of \(e\) and \(e'\) as \(|\varphi\rangle \) or \(|\psi\rangle\), then the controlled-Z gate will be also accomplished. This can be clearly seen in Table I.

We see the form of the entangled state between \(e\) and \(e'\) is crucial in determining the measuring-basis for implementing controlled-Z gate. Without \((e, e')\) in Fig. 4, it is impossible to implement the controlled-Z gate.

III. CONSTRUCTION OF ELEMENTARY QUANTUM GATES

A. Generalized Controlled-Z Gates

The controlled-Z can be implemented with more number of singlets and multi-qubit joint measurement. As in Fig. 5, the wave function of \(a\) and \(b\) before joint measurement is \(|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle\), then the wave function of \(c\) and \(d\) after joint measurement is \(|\psi\rangle = c_0|0e\rangle + c_1|1e\rangle + c_2|0e'\rangle + c_3|1e'\rangle\) or \(-1^{c_3} c_3|1e'\rangle\). Therefore, the number of particles in a joint measurement must be odd. This generalization is mathematically interesting, and it is also of practical interest because in bulk quantum system this may be the real case.

As shown in Fig. 6, the entangled state of particles \(d, e, f\) is \(|\psi\rangle\) and the entangled states of \(g\) and \(m, h, n, i\) and \(p\) are all \(|\psi\rangle\). By making joint measurement on particle groups \((a, d, g)\), \((b, e, h)\) and \((c, f, i)\) in the following measuring-basis,

\[
|\alpha\rangle = \{ (\sigma_x)^i \otimes (\sigma_y)^j \otimes I(0a + d0g) \pm |1a - d1g\}$
\]

\[
|\beta\rangle = \{ (\sigma_x)^i \otimes (\sigma_y)^j \otimes I(0b0h) \pm |1b1h\}$
\]

\[
|\gamma\rangle = \{ (\sigma_x)^i \otimes (\sigma_y)^j \otimes I(0c + f0i) \pm |1c - f1i\}$
\]

where \((i, j \in \{0, 1\})\), the following triple-qubit controlled-Z gate

\[
U_{c泽} = |000\rangle \langle 000| + |100\rangle \langle 100| + |001\rangle \langle 001|
+ |101\rangle \langle 101| + |010\rangle \langle 010| + |111\rangle \langle 111|
- |011\rangle \langle 011| - |110\rangle \langle 110|
\]

can be realized on the three resulting qubits \(m, n\) and \(p\).

B. One-qubit Quantum Gates

1. Phase Gate

By joint measurement, the simple one-qubit phase gate \(U_{phase} = (1 0 \ 0\ i)\) can be implemented by choosing the following complete bases for joint measurement on qubit \(A\) and \(B\) as shown in Fig. 4

\[
|\alpha\rangle = \{ |00\rangle + i|11\rangle |00\rangle - i|11\rangle |01\rangle + |10\rangle |01\rangle - i|10\rangle \}
\]

where the entangled state of particle \(B\) and \(C\) is \(|\psi\rangle\). After the joint measurement, a modified phase gate is applied to the resulting particle \(C\), and after an additional single qubit recovery operation in the form of a Pauli matrix or the identity operator, the phase gate is completed, and this is summarized in Table I.
TABLE II: Construction of phase gate. The initial state is \(a|0\rangle + b|1\rangle\).

| \(|\alpha\rangle\) | Qubit C state after measurement | Recovery operation |
|-----------------|--------------------------------|-------------------|
| \((|00\rangle + i|11\rangle)/\sqrt{2}\) | \(a|0\rangle - ib|1\rangle\) | \(\sigma_x\) |
| \((|00\rangle - i|11\rangle)/\sqrt{2}\) | \(a|0\rangle + ib|1\rangle\) | \(I\) |
| \((|01\rangle + i|10\rangle)/\sqrt{2}\) | \(|1\rangle - ib|0\rangle\) | \(\sigma_x\sigma_z\) |
| \((|01\rangle - i|10\rangle)/\sqrt{2}\) | \(|1\rangle + ib|0\rangle\) | \(\sigma_x\) |

2. \(\pi/8\) Gate.

The \(\pi/8\) gate, which has the unitary matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & i^{\pi/4}
\end{pmatrix}
\]

can be implemented by choosing the following measuring-basis as \(|\alpha\rangle\) = \(\{|00\rangle + e^{-i\pi/4}|11\rangle\}/\sqrt{2}, \{|00\rangle + e^{i\pi/4}|11\rangle\}/\sqrt{2},\), \((|01\rangle + e^{-i\pi/4}|10\rangle\)/\sqrt{2}, \((|01\rangle + e^{i\pi/4}|10\rangle\)/\sqrt{2}\}. The corresponding states after the joint measurement and the extra operations for the corresponding measured results are given explicitly in Table III

TABLE III: Construction of \(\pi/8\) gate. The initial state is \(a|0\rangle + b|1\rangle\)

| \(|\alpha\rangle\) | Qubit C state after measurement | Recovery Operation |
|-----------------|--------------------------------|-------------------|
| \((|00\rangle + e^{-i\pi/4}|11\rangle)/\sqrt{2}\) | \(a|0\rangle + e^{i\pi/4}|b\rangle\) | \(I\) |
| \((|00\rangle + e^{i\pi/4}|11\rangle)/\sqrt{2}\) | \(a|0\rangle + e^{-i\pi/4}|b\rangle\) | \(\sigma_x\) |
| \((|01\rangle + e^{-i\pi/4}|10\rangle)/\sqrt{2}\) | \(|1\rangle + e^{i\pi/4}|b\rangle\) | \(\sigma_x\) |
| \((|01\rangle + e^{i\pi/4}|10\rangle)/\sqrt{2}\) | \(|1\rangle + e^{-i\pi/4}|b\rangle\) | \(\sigma_x\sigma_z\) |

C. Controlled-phase Gate

Controlled-phase gate is important in constructing quantum algorithms such as the Grover algorithm\(^{20, 21}\) and in state initialization\(^{22, 23}\). The usual controlled-phase gate has been experimentally tested using NMR technique\(^{24}\), and cavity-QED realization of this gate has been proposed\(^{25}\). In the joint measurement scheme, by combining the phase gate and controlled-Z gate, we can implement controlled-phase gate

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & i
\end{pmatrix}
\]

However one has to be careful in choosing the measuring-basis. We first give an example. Suppose the measuring-basis in Fig 2 are

\[
\{|\alpha\rangle\} = \{(\sigma_x)^{i} \otimes (\sigma_z)^{j} \otimes I(|0\rangle + |1\rangle)\}
\]

(18)

\[
\{|\beta\rangle\} = \{(\sigma_x)^{i} \otimes (\sigma_z)^{j} \otimes I(|0\rangle + |1\rangle)\}
\]

(19)

and the entangled state of \(ee'\) is

\[
\frac{1}{\sqrt{2}}(|0_c0_{c'}\rangle + p|1_c1_{c'}\rangle),
\]

(20)

the entangled state of \(ee'\) is

\[
\frac{1}{\sqrt{2}}(|0_c0_{c'}\rangle + m|1_c1_{c'}\rangle),
\]

(21)
and the entangled state of $dd'$ is

$$\frac{1}{\sqrt{2}}(0_d0_{d'} + n|1_d1_{d'})). \quad (22)$$

The normalization condition requires that $kk^* = \tilde{k}k^* = pp^* = mm^* = nn^* = 1$. Then the joint measurement will turn the state of $c$ and $d$ into

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & npk^* & 0 & 0 \\ 0 & 0 & mk^* & 0 \\ 0 & 0 & 0 & -mnkp^*k^* \end{pmatrix}, \quad (23)$$

which cannot fulfill the controlled-phase gate obviously and when all the parameters reduce to 1 then $U$ turns to be controlled-Z gate. In this case, if we still use the entangled states for the pairs of qubits as given in Eqns. 20, 21, 22, the appropriate measuring-basis are

$$\{\ket{\alpha}\} = \{(\sigma_x)^j \otimes (\sigma_z)^k \otimes I(\ket{0} + \ket{1}) \pm \ket{0} - i\ket{1}) \sqrt{\frac{1}{2}}\},$$

$$\{\ket{\beta}\} = \{(\sigma_x)^j \otimes (\sigma_z)^k \otimes I(\ket{000} \pm \ket{111})\}. \quad (24)$$

For simplicity we take the entangled states between the $cc'$, $ee'$, $dd'$ pairs as $\frac{1}{\sqrt{2}}(\ket{00} + \ket{11})(j, k \in \{0, 1\})$. It is interesting to note that the inner two states mentioned in measuring-basis $\{\ket{\alpha}\}$, i.e. $\ket{+}$ and $\ket{0 - i\ket{1}) \sqrt{\frac{1}{2}}\}$ are not orthogonal each other, the bases in $\{\ket{\alpha}\}$ are themselves orthogonal. It is because the total Hilbert space is eight-dimensional and even though $\frac{\ket{0} + \ket{1}) \sqrt{\frac{1}{2}}\} \neq 0$, the inner two parts of $\{\ket{\alpha}\}$ (e.g. $\ket{0} + \ket{1}) \sqrt{\frac{1}{2}}\}$ are orthogonal. The only request here is that $\frac{\ket{0} + \ket{1}) \sqrt{\frac{1}{2}}\} = \frac{\ket{0} + i\ket{1}) \sqrt{\frac{1}{2}}\} = 0$. In particular, the phase $i$ comes from that in $\{\ket{\alpha}\}$ basis set.

| $j,k,\pm$ | $\{\ket{\alpha}\}$ | $\{\ket{\beta}\}$ |
|------------|-------------------|-------------------|
| (0, 0, +)  | $I \otimes \sigma_x$ | $\sigma_z \otimes I$ | $I \otimes \sigma_z$ | $\sigma_z \otimes I$ |
| (0, 0, -)  | $\sigma_x \otimes \sigma_z$ | $I \otimes \sigma_z$ | $I \otimes \sigma_z$ | $\sigma_z \otimes I$ |
| (0, 1, +)  | $U_{cz}(\sigma_x U_p \otimes I)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ |
| (0, 1, -)  | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ | $U_{cz}(\sigma_x U_p \otimes \sigma_z)$ |
| (1, 0, +)  | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ |
| (1, 0, -)  | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ | $U_{cz}(\sigma_z U_p \otimes \sigma_x)$ |
| (1, 1, -)  | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ |
| (1, 1, +)  | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ | $I \otimes \sigma_z \otimes \sigma_x$ |

The details of the construction of the controlled-phase gate are given in Table IV. For example, if the result of the joint measurement is $\ket{\alpha} = \frac{\ket{0} + \ket{1}) \sqrt{\frac{1}{2}}\} \pm \ket{0} + i\ket{1} (j, k \in \{0, 1\})$ and $\ket{\beta} = \frac{\ket{0} + \ket{1}) \sqrt{\frac{1}{2}}\} = \frac{\ket{0} + i\ket{1}) \sqrt{\frac{1}{2}}\}$ then the controlled-phase gate can be implemented by an additional operation of $U_{cz}(\sigma_z U_p \otimes I)$ as shown Fig. It should be noted that $\{\sigma_z, \sigma_x\}[\sigma_z, \sigma_y]=0, U_p$ is phase gate and $U_{cz}$ is controlled-Z gate.
D. Controlled-NOT Gate

Controlled-NOT gate is important in quantum computation. Great interests are attached to this gate. Recent experimental demonstration of this controlled-NOT gate has been reported in [26, 27]. While by joint measurement, the implementation of the controlled-NOT gate (C-NOT) is not straightforward as one might have expected. After some tedious calculation, we find that to construct C-NOT gate, one has to make sure the two highlighted parts in the following equation should be different. The swapping between $|0_{d'}\rangle$ and $|1_{d'}\rangle$ in the second highlighted part leads to the switching in C-NOT gate. The minus sign in that term can be cancelled in the calculation (see also Figs. 2, 3 for illustration)

\[
\langle \alpha|\Phi \rangle = (c_{00}|0_b\rangle + c_{01}|1_b\rangle) \otimes \frac{1}{\sqrt{2}}(|0_{c'}\rangle + |1_{c'}\rangle) \otimes \frac{1}{\sqrt{2}}|0_c\rangle \otimes \frac{1}{2}(|0_d0_{d'}\rangle + |1_d1_{d'}\rangle) \\
+ (c_{10}|0_b\rangle + c_{11}|1_b\rangle) \otimes \frac{1}{\sqrt{2}}(|0_{c'}\rangle - |1_{c'}\rangle) \otimes \frac{1}{\sqrt{2}}|1_c\rangle \otimes \frac{1}{2}(-|0_d1_{d'}\rangle + |1_d0_{d'}\rangle).
\]

(25)

We note that this difference is crucial in the process of implementing the C-NOT gate. We can fulfill this by adding an extra particle $d''$ (see Fig. 5) in the first group $\{|\alpha\rangle\}$ for joint measurement. We give some necessary ingredients below.

The entangled state of $ee'$ is

\[
\frac{1}{\sqrt{2}}(|0_e0_{e'}\rangle + |1_e1_{e'}\rangle).
\]

(26)

The entangled state of $cc'$ is

\[
\frac{1}{\sqrt{2}}(|0_c0_{c'}\rangle + |1_c1_{c'}\rangle).
\]

(27)

The entangled state of $dd'd''$ is

\[
\frac{1}{2}(|0_d0_{d'}0_{d''}\rangle + |1_d1_{d'}0_{d''}\rangle + |1_d0_{d'}1_{d''}\rangle - |0_d1_{d'}1_{d''}\rangle).
\]

(28)

The measuring-basis $\{|\alpha\rangle\}$ and $\{|\beta\rangle\}$ are

\[
\{|\alpha\rangle\} = \{(\sigma_x)^i \otimes (\sigma_z)^j \otimes (\sigma_x)^k \otimes I \\
(|0_a + e^{i\theta}0_{e'}0_{d''}\rangle \pm |1_a - e^{i\theta}1_{c'}1_{d''}\rangle)\},
\]

\[
\{|\beta\rangle\} = \{(\sigma_x)^i \otimes (\sigma_x)^j \otimes I \\
(|0_00_{e'}0_{d''}\rangle \pm |1_b1_{c'}1_{d''}\rangle)\},
\]

(29)

where $(i, j, k \in \{0, 1\}; |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle))$. We choose the measuring-basis for particles $e$ and $d'$ as $|\pm_e\rangle$ and $|0_{e'}\rangle(|1_{e'}\rangle) to avoid the loss of information which was mentioned earlier in section [11]. Of course after the joint measurement, some extra operations with the form $\sigma_\alpha$ ($\alpha = 1, 2, 3, 4; \sigma_4 = I$) need to be applied to the wave function of particles $c$ and $d$ as in other operations as well. The details of construction of C-NOT gate are given in Table [V].

E. Swap Gate

The swap gate matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
TABLE V: Construction of C-NOT gate. The rows are \( (i,j,k,\pm) \) in \( \{ |\alpha\rangle \} \) and the columns are \( (i,j,\pm) \) in \( \{ |\beta\rangle \} \). The quantity is the appropriate operation to be done after the joint measurement

| \( (0,0,0,+) \) | \( (0,0,0,-) \) | \( (0,0,1,+) \) | \( (0,0,1,-) \) | \( (0,1,0,+) \) | \( (0,1,0,-) \) | \( (0,1,1,+) \) | \( (0,1,1,-) \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( I \)       | \( I \)       | \( I \)       | \( I \)       | \( I \)       | \( I \)       | \( I \)       | \( I \)       |
| \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) | \( \otimes \sigma_z \) |
| \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) | \( \sigma_z \otimes \sigma_z \) |

In the joint measurement scheme, swap gate can be implemented similarly to the C-NOT gate. Attentions have to be paid to the four highlighted parts in the following equation, which is also illustrated in Figs 12 and 9.

\[
\langle\alpha|\Phi\rangle = (c_{00}|0_b\rangle + c_{01}|1_b\rangle) \otimes \frac{1}{2}(|0_{c'}\rangle + |1_{c'}\rangle) \otimes \frac{1}{2}(|0_{c}\rangle + |1_{c}\rangle) + (c_{10}|0_b\rangle + c_{11}|1_b\rangle) \otimes \frac{1}{2}(|0_{c'}\rangle - |1_{c'}\rangle) \otimes \frac{1}{2}(|0_{d}\rangle + |1_{d}\rangle).
\]

(30)

They should not be the same. The collapse of the quantum state of particle \( d \) to \( |0_d\rangle \) and \( |1_d\rangle \) leads to the fulfillment of swap gate. In order to make this difference, we should measure \( d \) in the first group \( \{ |\alpha\rangle \} \). The minus sign in the highlighted term \( \frac{1}{2}(|0_{c}\rangle - |1_{c}\rangle) \) can be cancelled in the calculation. The scheme is illustrated in Fig 9. Explicitly the state wave functions of the pairs and the corresponding measuring-basis sets are given respectively below.

The entangled state of \( ee' \) is

\[
\frac{1}{\sqrt{2}}(|0_{c}0_{c'}\rangle + |1_{c}1_{c'}\rangle).
\]

(31)

The entangled state of \( dd' \) is

\[
\frac{1}{\sqrt{2}}(|0_{d}0_{d'}\rangle + |1_{d}1_{d'}\rangle).
\]

(32)

The GHZ-like entangled state of \( ee'c' \) is

\[
\frac{1}{2}(|0_{c}0_{c'}0_{c'}\rangle + |1_{c}0_{c'}1_{c'}\rangle + |0_{c}1_{c'}0_{c'}\rangle - |1_{c}1_{c'}1_{c'}\rangle) = \frac{1}{\sqrt{2}}(|0_{c}0_{c'}\rangle + |1_{c}1_{c'}\rangle) |0_{c'}\rangle + \frac{1}{\sqrt{2}}(|0_{c}0_{c'}\rangle - |1_{c}1_{c'}\rangle) |1_{c'}\rangle).
\]

(33)

We can add a Hadamard operation on \( c' \) to change this entangled state to the term \( \frac{1}{\sqrt{2}}(|0_{c}0_{c'}0_{c'}\rangle + |1_{c}1_{c'}1_{c'}\rangle) \).
The measuring-basis sets are
\[
\{\left| \alpha \right\rangle \} = \{(\sigma_x)^i \otimes (\sigma_z)^j \otimes I \otimes (\sigma_x)^k \\
\left( |0_a + e_{0c} 0_{d'} \rangle \pm |1_a - e_{1c} 1_{d'} \rangle \right)\},
\]
\[
\{\left| \beta \right\rangle \} = \{(\sigma_x)^i \otimes (\sigma_x)^j \otimes I \\
\left( |0_b 0_{c'} 0_{d''} \rangle \pm |1_b 1_{c'} 1_{d''} \rangle \right)\},
\]
where \((i, j, k \in \{0, 1\})\). It is worth pointing that particles and \(c', d'\) are crucial in implementing this quantum gate. The pattern of entangled states for constructing swap gate is just the same as that for constructing the C-NOT gate. The difference is at the distinct ways for the joint measurement and the different three-particle entangled states respectively. The details of construction of swap gate are given in Table VI.

### Table VI: Construction of swap gate

The rows are \((i, j, \pm)\) in \(\{\left| \alpha \right\rangle \}\) and the columns are \((i, j, \pm)\) in \(\{\left| \beta \right\rangle \}\). The quantity is the appropriate operation to be done after the joint measurement.

|         | (0, 0, 0, +) | (0, 0, 0, −) | (0, 0, 1, +) | (0, 0, 1, −) | (0, 1, 0, +) | (0, 1, 0, −) | (0, 1, 1, +) | (0, 1, 1, −) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| (0, 0, +) | \(I\)       | \(\sigma_x \otimes I\) | \(\sigma_z \otimes I\) | \(\sigma_x \otimes \sigma_z\) | \(I \otimes \sigma_x\) | \(\sigma_z \otimes I\) | \(\sigma_z \otimes \sigma_x\) | \(\sigma_z \otimes \sigma_x\) |
| (0, 0, −) | \(I \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) | \(\sigma_z \otimes I\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) |
| (0, 1, +) | \(\sigma_z \otimes I\) | \(I \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) | \(\sigma_z \otimes I\) | \(\sigma_x \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) |
| (0, 1, −) | \(\sigma_x \otimes \sigma_z\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) | \(I \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) | \(\sigma_z \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_z\) |
| (1, 0, +) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_z\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_z \otimes I\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_z \otimes I\) |
| (1, 0, −) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_z \otimes \sigma_x\) | \(I \otimes \sigma_z\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_z \otimes I\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_z \otimes I\) |
| (1, 1, +) | \(\sigma_x \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) |
| (1, 1, −) | \(\sigma_x \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(I \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) | \(\sigma_x \otimes \sigma_x\) |

#### F. Toffoli Gate \(U_T\)

Toffoli gate is the further development of C-NOT gate. We present the details for implementing the Toffoli gate by doing joint measurement as shown in Fig. 10. Explicitly, the entangled state of \(df\) is
\[
\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),
\]
and the entangled state of \(gm, hn\) are all
\[
\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\]

The entangled state of \(i, i', i'', p\) is
\[
\frac{\sqrt{3}}{4}(|0_i 0_p 0_{i'} 0_{i''}\rangle + |1_i 1_p 0_{i'} 0_{i''}\rangle + |0_i 0_p 1_{i'} 0_{i''}\rangle \\
- |1_i 1_p 1_{i'} 0_{i''}\rangle + |0_i 0_p 0_{i'} 1_{i''}\rangle + |1_i 1_p 0_{i'} 1_{i''}\rangle \\
- |0_i 1_p 1_{i'} 1_{i''}\rangle + |1_i 0_p 1_{i'} 1_{i''}\rangle).
\]
The joint measurement basis sets are

\[ \{ \alpha \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes I \otimes (\sigma_x)^k \}
\]

\[ (|0_a + d 0_{i'} 0_{i''} 0_g \rangle \pm |1_a - d 1_{i'} 1_{i''} 1_g \rangle), \]

\[ \{ \beta \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes I \otimes (\sigma_x)^k \}
\]

\[ (|0_b 0_c 0_d 0_{i''} \rangle \pm |1_b 1_c 1_d 1_{i''} \rangle), \]

\[ \{ \gamma \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes I \}
\]

\[ (|0_e + f 0_i \rangle \pm |1_e - f 1_i \rangle), \]

(39) (40) (41)

where \((i, j, k, w \in \{0, 1\}; |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)).\) We note that the extra particles \(i\) and \(i''\) are necessary in this process. Some extra operations should be applied to the wave function of particles \(m, n, p\) after joint measurement too. For instance, as shown in Fig. 10, if the results of measuring-basis sets \(a\) re \(|\rangle = |0_a - d 0_{i'} 1_g \rangle - |1_a + d 1_{i'} 0_g \rangle, |\beta \rangle = |0_b 1_c 0_d 0_{i''} \rangle - |1_b 0_c 1_d 1_{i''} \rangle\) and \(|\gamma \rangle = |1_c - f 0_i \rangle - |0_c + f 1_i \rangle\), then an additional operation \(\sigma_x \otimes \sigma_z \otimes \sigma_x\) on particles \(m, n, p\) is necessary.

G. Fredkin (controlled-swap) Gate

Fredkin gate is the development of swap gate in a three-qubit system. The details of the implementation of this quantum gate is given in Fig. 11. Here the entangled state of \(d, e, f\) is

\[ \frac{1}{\sqrt{2}}(|000 \rangle + |111 \rangle). \]

(42)

The entangled state of \(g, m\) is

\[ \frac{1}{\sqrt{2}}(|00 \rangle + |11 \rangle). \]

(43)

The entangled state of \(i, i', i'', p\) is

\[ \sqrt{\frac{2}{4}}(|0_0 0_{i'} 0_{i''} \rangle + |1_1 0_{i'} 0_{i''} \rangle + |0_1 0_{i'} 0_{i''} \rangle
\]

\[ - |1_1 0_{i'} 0_{i''} \rangle + |0_0 0_{i'} 1_{i''} \rangle - |1_1 0_{i'} 1_{i''} \rangle
\]

\[ - |0_1 0_{i'} 1_{i''} \rangle + |1_1 0_{i'} 1_{i''} \rangle). \]

(44)

The entangled state of \(h, h', h'', n\):

\[ \sqrt{\frac{2}{4}}(|0_0 0_{h'} 0_{h''} \rangle + |0_0 1_{h'} 0_{h''} \rangle + |0_0 0_{h'} 0_{h''} \rangle
\]

\[ - |1_1 1_{h'} 0_{h''} \rangle + |1_1 0_{h'} 1_{h''} \rangle + |1_1 1_{h'} 1_{h''} \rangle
\]

\[ + |0_0 0_{h'} 1_{h''} \rangle + |1_1 0_{h'} 1_{h''} \rangle). \]

(45)

The measuring-basis sets are

\[ \{ \alpha \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes (\sigma_x)^k \otimes I \otimes (\sigma_x)^w \}
\]

\[ (|0_a + d 0_{i'} 0_{i''} 0_g \rangle \pm |1_a - d 1_{i'} 1_{i''} 1_g \rangle), \]

(46)

\[ \{ \beta \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes I \otimes (\sigma_x)^k \}
\]

\[ (|0_b 0_{i'} 0_{i''} \rangle \pm |1_b 1_{i'} 1_{i''} \rangle), \]

(47)

\[ \{ \gamma \} = \{ (\sigma_x)^i \otimes (\sigma_z)^j \otimes (\sigma_x)^k \otimes I \}
\]

\[ (|0_c + f 0_{i'} 0_{i''} \rangle \pm |1_c - f 1_{i'} 1_{i''} \rangle), \]

(48)

where \((i, j, k, w \in \{0, 1\})\). It is worth noting the extra particles \(h', h'', i', i''\) are crucial to implement the Fredkin gate.
IV. SUMMARY

In this paper, we have constructed explicitly the elementary gates for the measurement-based quantum computation, including the generalized controlled-Z gate, the phase gate and $\frac{\pi}{8}$ gate, the swap gate, the C-NOT gate, the Fredkin gate and Toffoli gate. We have studied the relation between the form of the measuring-basis and the entangled states of the pairs. It is found that the they have an exquisite relation among them. In some cases, some parts in the joint measurement basis are necessarily distinct for different qubits in a group so that they can implement the desired quantum gate. It is interesting to mention that as shown in Ref. [18], the teleportation-based quantum computation and the VBS quantum computation scheme are equivalent. It is also equivalent to the cluster-state quantum computation. In addition, we note that the patterns of entangled states in the process of constructing C-NOT gate and swap gate are the same and they only differ at the ways of the joint measurement. For these matters, the topological construction of quantum computation [28, 29] is worthwhile for further study at this point. In all these schemes, entanglement is the core of quantum computation. In the cluster-state quantum computation scheme, the entanglement is condensed into the initial state, while in the teleportation-based quantum computation as studied in this paper, the entanglement in injected into the quantum computing system by using entangled pairs of particles.

This work is supported by the National Fundamental Research Program, Grant No. 001CB309308, China National Natural Science Foundation, Grant No. 60073009, 10325521, the SRFDP program of Education Ministry of China.

[1] A. Barenco et al., Phys. Rev. A52, 3457 (1995); A. Barenco, Proc. R. Soc. London A449, 679 (1995).
[2] M. A. Nielsen and I. Chuang, Phys. Rev. A79, 321 (1997).
[3] D. Gottesman and I. Chuang, Nature 402, 390 (1999).
[4] E. Knill, R. Laflamme and G. Milburn, Nature 409, 46 (2001).
[5] M. Nielsen, Phys. Lett. A 308, 96 (2003).
[6] D. Leung, quant-ph/0111122 and quant-ph/0310189.
[7] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001); Quant. Inf. Comp. 6, 443 (2002).
[8] H. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001).
[9] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki Commun. Math. Phys. 115, 477-528 (1988).
[10] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki Phys. Rev. Lett. 59, 799-802 (1987).
[11] M. Fannes, B. Nachtergaele and R. F. Werner, Commun. Math. Phys. 114, 443 (1992).
[12] G.-M. Zhang and S.-Q. Shen Phys. Rev. Lett. 87, 157201 (2001).
[13] M. Nakamura and S. Todo Phys. Rev. Lett. 89, 077204 (2002).
[14] T. Koretsune and M. Ogata Phys. Rev. Lett. 89, 116401 (2002).
[15] G.-M. Zhang, H. Hu, and L. Yu Phys. Rev. Lett. 91, 067201 (2003).
[16] F. Verstraete, M. Popp and J. I. Cirac Phys. Rev. Lett. 92, 027901 (2004).
[17] F. Verstraete and M. A. Martin-Delgado, J. I. Cirac Phys. Rev. Lett. 92, 087201 (2004).
[18] F. Verstraete and J. I. Cirac, quant-ph/0311130.
[19] M. A. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press (2000).
[20] L. K. Grover, Phys. Rev. Lett. 79, 325 (1997).
[21] G. L. Long, Phys. Rev. A64, 022307 (2001).
[22] G. L. Long and Y. Sun, Phys. Rev. A64, 014303 (2001).
[23] K. Maruyama and P. L. Knight, Phys. Rev. A67, 032303 (2003).
[24] J. F. Zhang, W. Z. Liu, Z. W. Deng, Z. H. Lu and G. L. Long quant-ph/0406209.
[25] Y. F. Xiao, X. M. Lin, J. Gao, Y. Yang, Z. F. Han and G. C. Guo quant-ph/0408033.
[26] J. L. O’Brien, G. J. Pryde, A. G. White, T. C. Ralph and D. Branning Nature 426, 264 (2003).
[27] S. Gasparoni, J. W. Pan, P. Walther, T. Rudolph and A. Zeilinger, quant-ph/0404107.
[28] A. Marzuoli and M. Rasetti Phys. Lett. A 306, 79-87 (2002).
[29] A. Marzuoli and M. Rasetti, quant-ph/0407119.
FIG. 1: Implementation of a single-qubit unitary operation using a pair of state in the $|H\rangle$ entangled state.

FIG. 2: Implementation of the controlled-phase gate in Ref. [18].

FIG. 3: The implementation of the controlled-phase gate using different pairs of entangled states.

FIG. 4: Without the crucial particles $e$ and $e'$, the controlled-phase gate cannot be implemented.
FIG. 5: Implementation of the controlled-phase gate using $n$ pairs of singlet.

FIG. 6: Implementation of triple-qubit controlled-phase gate by making joint measurement on particle groups $(a, d, g)$ $(b, e, h)$ and $(c, f, i)$.

FIG. 7: An additional operation of $U_{cz} \sigma_z U_p \otimes I$ on particles $c$ and $d$ for implementing controlled-phase gate if the result of the joint measurement is $|\alpha\rangle = |0\rangle_a (|00\rangle + |11\rangle)_{c,d'} + |1\rangle_a (|00\rangle - |11\rangle)_{c,d'}$ and $|\beta\rangle = |0\rangle_b |0\rangle_{c,d'} + |1\rangle_b |0\rangle_{c,d'}$. It should be noted that $\{|\bar{\alpha}\rangle\}$ is the same as in Table II and $\{|\bar{\beta}\rangle\} = (|\sigma_x\rangle^i \otimes |\sigma_x\rangle^j \otimes I(|00\rangle \pm |11\rangle))$. Besides, all the entangled pairs in this figure are in the form $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$. 
FIG. 8: Implementation of C-NOT gate by joint measurement using singlets and three-particle entanglement.

FIG. 9: Implementation of swap gate by joint measurement using singlets and GHZ-like state. Note that the pattern of entangled states in this process is the same as that in Fig. 8.

FIG. 10: Implementation of Toffoli gate by joint measurement.

FIG. 11: Implementation of Fredkin gate by joint measurement.