From omnitigs to macrotigs: a linear-time algorithm for safe walks – common to all closed arc-coverings of a directed graph

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Abstract
A partial solution to a problem is called safe if it appears in all solutions to the problem. Motivated by the genome assembly problem in bioinformatics, Tomescu and Medvedev (RECOMB 2016) posed the question of finding the safe walks present in all closed arc-covering walks, and gave a characterization of them (omnitigs). An $O(nm)$-time algorithm enumerating all maximal omnitigs on a directed graph with $n$ nodes and $m$ arcs was given by Cairo et al. (ACM Trans. Algorithms 2019), along with a family of graphs where the total length of maximal omnitigs is $\Theta(nm)$.

In this paper we describe an $O(m)$-time algorithm to identify all maximal omnitigs, thanks to the discovery of a family of walks (macrotigs) with the property that all the non-trivial omnitigs are univocal extensions of subwalks of a macrotig. This has several consequences: (i) A linear output-sensitive algorithm enumerating all maximal omnitigs, that avoids to pay $\Theta(nm)$ when the output is smaller, whose existence was open. (ii) A compact representation of all maximal omnitigs, which allows, e.g., for $O(m)$-time computation of various statistics on them. (iii) A powerful tool for finding safe walks for related covering problems.

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1 Introduction

Graph theory is abundant in notions capturing how resilient a graph is, for example articulation points and bridges, $k$-connectivity, and minimum cuts. When referring to a particular problem on graphs, more advanced notions include $d$-traversals (sets of nodes or edges intersecting every solution to the problem in at least $d$ elements [5]) and $d$-blockers, or most vital nodes or edges (sets of nodes or edges whose removal deteriorates the optimum solution to the problem by at least $d$ [18, 2]).

An even more fundamental notion is the one of persistency: a node or an edge is called persistent if it appears in all solutions to the problem. Persistency has been studied for
problems such as maximum independent set \cite{11}, maximum bipartite matching \cite{7}, assignment and transportation problems \cite{4}. When solutions are not just sets of nodes or edges, a natural way to generalize the notion of persistency is through the notion of safety, introduced in \cite{16,15}. A partial solution to a problem is called safe if it appears in all solutions to the problem. For example, if the solution is a certain type of walk in a graph, then a safe partial solution is a sub-walk of all such solution walks.

Safety not only captures resilience with respect to a graph problem, but is also a common sense approach to deal with problems admitting multiple solutions in practice. In fact, many real-world problems are only approximately modeled by a mathematical formulation, and as such may admit multiple solutions, with no way of telling which one is the correct one. A well-studied approach to deal with this is to enumerate all solutions, or only the first $k$-best solutions, and then “one can apply more sophisticated quality criteria, wait for data to become available to choose among them, or present them all to human decision-makers” \cite{9}. However, even in case of efficient enumeration algorithms, this approach does not scale to large numbers of solutions. Safe partial solutions are a way to tackle this obstacle since they are likely part of the correct real-world solution. For example, in bioinformatics safe partial solutions have been shown to be relevant for several problems \cite{12,14,17,5}.

In this paper we study the safe version of a fundamental graph problem. Suppose that the problem is to find one closed arc-covering walk of a graph (from this point onwards graphs are always directed), where arc-covering means that it passes through each arc at least once. This is a generalization of the closed Eulerian walk problem, since we now pass through each arc an arbitrary non-zero number of times. While finding a solution for this problem is trivial, finding the safe walks for it is not. The search of these safe walks was proposed in \cite{16,15}, motivated by the genome assembly problem, with promising results in practice. However, this problem has a long history in bioinformatics: so far, most practical genome assemblers assemble only those paths whose internal nodes have in-degree and out-degree equal to one \cite{13}, which are a sub-class of the safe walks from this paper.

A characterization of these safe walks (omnitigs, see Definition \ref{def:omnitig}) was also given in \cite{16,15}, together with a polynomial-time algorithm outputting all maximal omnitigs, based on an exhaustive search (an omnitig is maximal if it is not a sub-walk of another omnitig). Later, \cite{3} proved that the length of all maximal omnitigs of any graph with $n$ nodes and $m$ arcs is $O(nm)$, and proposed an $O(nm)$-time algorithm enumerating all maximal omnitigs. This was also proven to be optimal, in the sense that it constructed families of graphs where the total length of all maximal omnitigs is $\Theta(nm)$.

However, it remained open whether it is necessary to pay $O(nm)$ even when the total length of the maximal omnitigs is smaller. This question is important also from a practical bioinformatics point of view. Indeed, the real-world graphs on which such algorithm is to be run have the number of nodes and arcs in the order of hundreds of thousands, and yet the total length of the maximal omnitigs is linear in the size of the graph \cite{3}. In this paper we solve this problem, obtaining the best possible result for it: a linear-time output sensitive algorithm for finding all maximal omnitigs.

\begin{theorem}
Given a strongly connected graph $G$, it is possible to enumerate all maximal omnitigs of $G$ in time linear in their total length.
\end{theorem}

This result is based on the following ingredients:

- Delaying the computation of the univocal prefix and suffix of a maximal omnitig to a postprocessing phase.
- The observation that the remaining non-trivial ‘core’ omnitigs are sub-walks of a collection of walks (macrotigs) of total length $O(m)$. 

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\textbf{Theorem 1.} \textit{Given a strongly connected graph $G$, it is possible to enumerate all maximal omnitigs of $G$ in time linear in their total length.}
A linear preprocessing that simplifies the input graph by removing univocal parts. The remaining structure of the graph (macronodes) allows for an incremental construction of macrotigs.

A linear preprocessing that reduces the graph to have constant degree.

A recent result [10] showing that it is possible to preprocess the graph in linear time to answer in $O(1)$-time strong connectivity queries under single arc removal.

Novel insights on the structure of omnitigs, in particular a property allowing us to check whether an omnitig can be extended to the right with $O(1)$ such strong connectivity queries.

As opposed to the strategy adopted in [3], macrotigs can now guide the search for maximal omnitigs, allowing us to break the $O(nm)$ barrier. In fact, we obtain the following stronger result.

**Theorem 2.** Given a strongly connected graph $G$ with $m$ arcs, there exists a collection of walks of $O(m)$ total length (macrotigs) such that every maximal omnitig is either the univocal extension of a subwalk of a macrotig, or the univocal extension of a single arc not appearing in a macrotig.

Moreover, the collection of macrotigs, the endpoints of the said subwalks, and the list of the arcs not appearing in macrotigs, can all be computed in $O(m)$ total time.

From the representation of Theorem 2 one can compute in only $O(m)$ time several statistics relevant in bioinformatics, such as the average length of all maximal omnitigs [6], or how many omnitigs go through a given arc or path. One can also filter out maximal omnitigs according to some criteria, before enumerating them explicitly.

While the results in this paper are a definitive answer to the problem of safe walks for closed arc-covering walks, we believe that the structural results given here form the basis for solutions to related problems. For example, [13, 14] also considered the problem of enumerating the safe walks common to all closed node-covering walks (that is, passing through every node at least once) of a graph. Another example is [1], which defined a walk to be safe if, for any collection of closed walks that together cover all the arcs (or nodes) of the graph, the safe walk appears in some closed walk in the collection. In all these cases, safe walks are exactly the omnitigs satisfying some additional “reachability-like” conditions. We deem feasible, and leave as future work, the fact that macrotigs can be used to solve also these and other related problems more efficiently.

This paper is structured as follows. We begin in Section 2 by introducing the necessary notation and some preliminary results about omnitigs. In Section 3 we introduce three transformations, which simplify the input graph while maintaining its maximal omnitigs and organize the graph into macronodes. We show in Section 4 how to compute segments of macrotigs inside a macronode, which we call microtigs. In Section 5 we show how to combine microtigs from different macronodes to obtain macrotigs. Finally, in Section 6 we give the linear algorithm that scans macrotigs to identify all maximal omnitigs.

# 2 Preliminaries

## 2.1 Notation

In this paper, a graph is a tuple $G = (V, E, t, h)$, where $V$ is a finite set of nodes, $E$ is a finite set of arcs, and $t, h: E \rightarrow V$ assign to each arc $e \in E$ its tail node $t(e) \in V$ and its head node $h(e) \in V$. Parallel edges and self-loops are allowed. For an arbitrary arc $e \in E(G)$, we
denote $G \setminus e = (V, E \setminus \{e\}, t, h)$. The reverse graph $G^R$ of $G$ is obtained by reversing the direction of every arc, that is $G^R = (V, E, h, t)$. In the rest of this paper, we assume that a strongly connected graph $G = (V, E, t, h)$ is given, with $|V| = n$ and $|E| = m \geq n$.

A walk on $G$ is a sequence $W = (v_0, e_1, v_1, e_2, \ldots, v_{\ell-1}, e_\ell, v_\ell)$. A walk $W$ is called a split arc if $t(e) = h(e) = v$. A walk $W$ is called closed if it is non-empty and $t(W) = h(W)$, otherwise it is open. Juxtaposition $WW'$ denotes the concatenation of walks $W$ and $W'$, where $h(W) = t(W')$ is implicitly assumed.

A walk $W = (v_0, v_1, \ldots, v_\ell)$ is called a path when the nodes $v_0, v_1, \ldots, v_\ell$ are all distinct, with the exception that $v_\ell = v_0$ is allowed (in which case we have either a closed or an empty path). Subwalks of open walks are defined in the standard manner. For a closed walk $W = e_0 \ldots e_{\ell-1}$, we say that $W' = e'_0 \ldots e'_j$ is a subwalk of $W$ if there exists $i \in \{0, \ldots, \ell - 1\}$ such that for every $k \in \{0, \ldots, j\}$ it holds that $e'_k = e_{(i+k) \mod \ell}$.

Capital letters, such as $W$ or $P$, denote walks or paths, lowercase letters as $u$ and $v$ denote nodes, while letters $e, f, g, b$ denote arcs. For $v \in V$, $d^-(v)$ and $d^+(v)$ denote the in- and out-degree of $v$, respectively. The nodes and arcs of the strongly connected graph $G$ can be classified as follows:

1. A node $v$ is a join node if $d^-(v) > 1$, and a join-free node otherwise. An arc $f$ is called a join arc if $h(f)$ is a join node, and a join-free arc otherwise.
2. A node $v$ is a split node if $d^+(v) > 1$, and a split-free node otherwise. An arc $g$ is called a split arc if $t(g)$ is a split node, and a split-free arc otherwise.
3. A node or arc is called bivalent if it is both join and split, and is called bivincual if it is both split-free and join-free.

A walk $W$ is split-free (resp., join-free) if all its arcs are split-free (resp., join-free). Moreover, two arcs $e$ and $e'$ are said to be sibling arcs if $h(e) = h(e')$ or $t(e) = t(e')$ holds.

Given a walk $W$, its univocal extension $U(W)$ is defined as $W^-WW^+$, where $W^-$ is the longest join-free path to $h(W)$ and $W^+$ is the longest split-free path from $h(W)$ (observe that they are uniquely defined).

A walk $W$ is called arc covering if every arc of $G$ appears in $W$ at least once. A closed arc-covering walk exists if and only if the graph is strongly connected. In this paper we are interested in the (safe) walks that are subwalks of all closed arc-covering walks. The following characterization was proved in [16].

**Theorem 3 ([16]).** Let $G$ be a strongly connected graph different from a closed path. Then a walk $W$ is a subwalk of all closed arc-covering walks of $G$ if and only if $W$ is an omnitig.
Definition 4 (Omnitig [16]). Let $W = e_0 \ldots e_\ell$ be a walk in a graph $G$. We say that $W$ is an omnitig if and only if for all $1 \leq i \leq j \leq \ell$, there is no non-empty path from $t(e_j)$ to $h(e_{i-1})$ with first arc different from $e_j$ and last arc different from $e_{i-1}$ (we call such path forbidden).

See Figure 1 for an example of a walk which is not an omnitig.

Observe that $W$ is an omnitig in $G$ if and only if $W^R$ is an omnitig in $G^R$. Moreover, any subwalk of an omnitig is an omnitig. For every arc $e$, its univocal extension $U(e)$ is an omnitig. However, there can be longer walks that are omnitigs. A walk $W$ satisfying a property $P$ is right-maximal (resp., left-maximal) if there is no walk $W'$ (resp., $eW$) satisfying property $P$.

A walk is satisfying a property $P$ is maximal if it is both left- and right-maximal w.r.t. $P$.

Notice that if $G$ is a closed path, then every walk of $G$ is an omnitig. As such, it is relevant to find the maximal omnitigs of $G$ only when $G$ is different from a closed path. Thus, in the rest of this paper our strongly connected graph $G$ is considered to be different from a closed path, even when we do not mention it explicitly.

2.2 Preliminary properties

Consider the following result from [3].

Theorem 5 ([3]). There exists a unique left-maximal omnitig $Wg$, ending with a given split arc $g$. By symmetry, there exists a unique right-maximal omnitig $fW$, starting with a given join arc $f$.

The following properties derived from [3] offer a more operational characterization of omnitigs starting with a join arc $f$.

Lemma 6. Let $fW$ be an omnitig, where $f$ is a join arc. Let $f'$ be a sibling join arc of $f$. Then, every closed path in $G$ containing $f'$ is of the form $f'WQ$.

Proof. Let $P$ be a closed path in $G$ containing $f'$. If $P$ is not of the form $f'WQ$, then the suffix of $P$ starting with the first arc that is not in $W$ is a forbidden path for $fW$.

Lemma 7. Let $fW$ be an omnitig in $G$, where $f$ is a join arc. Then, there exists an arc $g$ with $t(g) = h(W)$ and a path from $h(g)$ to $h(f)$ in $G \setminus f$.

Proof. Let $f'$ be any sibling join arc of $f$. Consider any closed path $Pf'$ in $G$, which exists since $G$ is strongly connected. Notice that $Pf'$ must start with $W$, by Lemma 6. Take the first arc $g$ on $Pf'$ after the prefix $W$. The remaining suffix of $Pf'$ from $g$ is a path from $h(g)$ to $h(f)$ in $G \setminus f$.

Lemma 8. Let $fW$ be an omnitig in $G$, where $f$ is a join arc and $W$ is a join-free path. Then $fWg$ is an omnitig if and only if $g$ is the only arc with $t(g) = h(W)$ such that there exists a path from $h(g)$ to $h(f)$ in $G \setminus f$.

Proof. By Lemma 7, at least one $g$ exists which satisfies the condition.

For the direct implication, assume that there is a path $P$ in $G \setminus f$ from $h(g')$, where $g'$ sibling of $g$ and $g' \neq g$, to $h(f)$. Then, this forbidden path $P$ (Definition 1) contradicts the fact that $fWg$ is an omnitig.

For the reverse implication, assume that $fWg$ is not an omnitig. Then take any forbidden path $P$ for $fWg$. Since $fW$ is an omnitig, $P$ must start with some $g'$ sibling of $g$, $g' \neq g$. Since $W$ is join-free, then $P$ must end in $h(f)$ and have last arc different from $f$. Therefore, $P$ is a path from $h(g')$ to $h(f)$ in $G \setminus f$. 

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Corollary 9. Let $fWg$ be an omnitig, where $f$ is a join arc, $W$ is a join-free path (possibly empty) and $g$ is a split arc. Let $g'$ be an arbitrary sibling split arc of $g$. The walk $fWg'$ is not an omnitig.

The following recent result from [10] allows us to check the condition in Lemma 8 efficiently.

Theorem 10 ([10]). Let $G$ be a strongly connected graph with $n$ nodes and $m$ arcs. It is possible to build an $O(n)$-space data structure that, after $O(m + n)$-time preprocessing, given two nodes $u$, $v$, and an arc $e$, tests in $O(1)$ worst-case time if $u$ are strongly connected $v$ in $G \setminus e$.

In this paper we will use the following consequence of Theorem 10.

Corollary 11. Let $G$ be a strongly connected graph with $n$ nodes and $m$ arcs. It is possible to build an $O(n)$-space data structure that, after $O(m + n)$-time preprocessing, given a node $v$ and an arc $e$, tests in $O(1)$ worst-case time if there is a path from $v$ to $h(e)$ in $G \setminus e$.

Proof. Since $G$ is strongly connected, there is a path $P$ from $h(e)$ to $v$ in $G$. Observe that $P$ is also a path in $G \setminus e$. Thus a path from $v$ to $h(e)$ exists in $G \setminus e$ if and only if $v$ and $h(e)$ are strongly connected in $G \setminus e$. The claim follows by Theorem 10.

3 Pre-processing

In this section, we describe three transformations of the given graph $G$. These reductions preserve its maximal omnitigs, and it is immediate to see that they and their inverses can be performed in linear time. Thanks to these transformations, we can assume without loss of generality that the given input graph has some special properties, namely, it has constant maximum degree and it contains no biunivocal nodes and no biunivocal arcs.

3.1 Constant degree

The first transformation allows us to reduce to the case in which the graph has constant out-degree (see Figure 2 for an example). This is useful to prove that the final algorithm has a linear-time output sensitive complexity.

Transformation 1. Given $G$, for every node $v$ with $d^+(v) > 2$, let $e_1, e_2, \ldots, e_k$ be the arcs out-going from $v$. Replace $v$ with the path $(v_1, e'_1, v_2, e'_2, \ldots, e'_{k-2}, v_{k-1})$, where $v_1, \ldots, v_{k-1}$ are new nodes, and $e'_1, \ldots, e'_{k-2}$ are new edges. Each arc $e_i$ with $t(e_i) = v$ in $G$ now has $t(e'_i) = t(e_i) = v_i$, except for $e_k$ which has $t(e_k) = v_{k-1}$.
By also applying the symmetric transformation, the problem is thus reduced to graphs with constant out- and in-degree. The correctness of this transformation follows from Lemma 12 which we prove in the appendix.

Lemma 12. Let $G$ be a graph and let $G'$ be the graph obtained by applying Transformation 4 to $G$. Then a walk $W$ of $G$ is a maximal omnitig of $G$ if and only if there exists a maximal omnitig $W'$ of $G'$ such that $W$ is the string obtained from $W'$ by suppressing all the arcs introduced with the transformation.

3.2 Compression

In this section we ensure structural properties which are helpful to characterize macrotigs. For these reasons, we introduce the compressed graph as follows.

Definition 13 (Compressed graph). A graph $G$ is compressed if it contains no biunivocal nodes and no biunivocal arcs.

The next transformation removes biunivocal nodes, by replacing those paths whose internal nodes are biunivocal with a single arc from the tail of the path to its head (see Figure 3 for an example).

Transformation 2. Given $G$, for every longest path $P = (v_0, e_0, \ldots, e_{\ell-1}, v_\ell)$, $\ell \geq 2$, such that $v_1, \ldots, v_{\ell-1}$ are biunivocal nodes, we remove $v_1, \ldots, v_{\ell-1}$ and their incident arcs from $G$, and we add a new arc from $v_0$ to $v_\ell$.

This transformation is widely used in the genome assembly field, and it clearly preserves the maximal omnitigs of $G$, because in any closed arc-covering walk of $G$ whenever $e_0$ appears, then it is always followed by $e_1, \ldots, e_\ell$. Notice that for every node in the graph after Transformation 2 the in-degree and the out-degree are the same as in the original graph.

The last transformation contracts the biunivocal arcs of the graph (see Figure 3 for an example).

Transformation 3. Given $G$, we contract every biunivocal arc $e$, namely we set $t(e') = t(e)$ for every out-going arc from $h(e)$ and remove the node $h(e)$.

Also this transformation preserves the maximal omnitigs of $G$ because every maximal omnitig which contains an endpoint of $e$, also contains $e$. Notice that after Transformation 3 the maximum in-degree and the maximum out-degree are the same as in the original graph.

Figure 3 An example of Transformation 2 (T2) applied to the path $P = (v_0, e_0, \ldots, e_\ell, v_\ell)$, where $v_1, \ldots, v_{\ell-1}$ are biunivocal nodes and $e$ is the new arc from $v_0$ to $v_\ell$. The Transformation 3 (T3) compresses biunivocal arcs.
In a compressed graph all arcs are split, join or bivalent. Moreover, in compressed graphs, the following observation holds.

**Observation 14.** Let $G$ be a compressed graph. Let $f$ and $g$ be a join and a split arc, respectively, in $G$. The following hold:

(i) if $fWg$ is a walk, then $W$ has an internal node which is bivalent;
(ii) if $gWf$ is a walk, then $gWf$ contains a bivalent arc.

![Figure 4](image_url) Let $f$ be a join arc and $g$ be a split arc. Left: if $fg$ is a walk, then $h(f) = t(g)$ is a bivalent node. Right: if $gf$ is a walk in a compressed graph, then at least one among $f$ and $g$ is a bivalent arc.

**Observation 14** allows us to prove the following lemma.

**Lemma 15.** Every maximal omnitig of a compressed graph contains both a join arc and a split arc. Moreover, it has a bivalent arc or an internal node which is bivalent.

**Proof.** Consider an omnitig $W$ composed only of split-free arcs. Notice first that $W$ is a simple path. Consider any arc $e$, with $h(e) = t(W)$ and observe that $eW$ is an omnitig, since it is contained in $U(e)$. Therefore, $W$ is not a maximal omnitig. Symmetrically, no maximal omnitig is composed only of join-free arcs. This already implies the first claim in the statement: any maximal omnitig $W$ contains at least one join arc $f$ and at least one split arc $g$. If $f = g$ then $W$ contains the bivalent arc $f$. Otherwise, either $W$ contains a subwalk of the form $fW'g$ or it contains a subwalk of the form $gW'f$, where $W'$ might be an empty walk. In the first case $W$ has an internal node which is bivalent, by Observation 14 (i). In the second case $W$ contains a bivalent arc, by Observation 14 (ii).

From this point onwards, we assume that $G$ is compressed.

Once $G$ is compressed, a partition of its nodes naturally emerges: each class of the partition is identified by a bivalent node, and all other nodes of the class are those that reach it with a split-free or a join-free path (see Figure 5a in the appendix).

**Definition 16 (Macronode).** Let $v$ be a bivalent node of $G$. Consider the following sets:

i) $R^+(v) := \{ u \in V(G) : \exists \text{ a join-free path from } v \text{ to } u \}$;
ii) $R^-(v) := \{ u \in V(G) : \exists \text{ a split-free path from } u \text{ to } v \}$.

The subgraph $M_v$ induced by $R^+(v) \cup R^-(v)$ is called the macronode centered in $v$.

**Observation 17.** The set $\{ V(M_v) : v \text{ is a bivalent node of } G \}$ is a partition of $V(G)$.

**Observation 18.** In a macronode $M_v$, $R^+(v)$ and $R^-(v)$ induce two trees with common root $v$, but oriented in opposite directions. Except for the common root, the two trees are node disjoint, all nodes in $R^-(v)$ being join nodes and all nodes in $R^+(v)$ being split nodes. Moreover, the only arcs with endpoints in two different macronodes are bivalent arcs. Notice that, given any bivalent arc $b$, the endpoints of its univocal extension $U(b)$ are bivalent nodes, i.e., are centers of two macronodes, which may be the same node, or different nodes (see Figure 5a in the appendix).
4 Structure within macronodes: microtigs

In this section we analyze how omnitigs can traverse a macronode and the degrees of freedom they have in choosing their directions within the macronode.

We start by defining central-micro omnitigs, which are the smallest omnitigs that cross the center of a macronode. Then, we define right- and left-micro omnitigs, which start from a central-micro omnitig (at the center) and proceed to the periphery of a macronode. Finally, we combine right- and left-micro omnitigs into microtigs (which are not necessarily omnitigs themselves), see Figure 5b in the appendix.

Definition 19 (Central-micro omnitig). A central-micro omnitig is any omnitig \( fg \), with \( f \) a join arc and \( g \) a split arc.

Definition 20 (Right- and left-micro omnitig). Let \( fg \) be a central-micro omnitig. An omnitig \( fgW \) (respectively, \( Wfg \)) that does not contain a bivalent arc as internal arc is called an right-micro omnitig (respectively, left-micro omnitig).

Definition 21 (Microtig). An omnitig \( W \) is called a microtig if \( W = W_1fgW_2 \), where \( W_1fg \) and \( fgW_2 \) are, respectively, a left-micro omnitig, and a right-micro omnitig.

The following properties show that microtigs are very limited in their freedom.

Lemma 22. For any join arc \( f \), there exists at most one maximal right-micro omnitig \( fgW \). For any split arc \( g \) there exists at most one maximal left-micro omnitig \( Wfg \).

Proof. We prove only the first of the two symmetric statements. A minimal counterexample is the existence of two right-micro omnitigs \( fPg \) and \( fPg' \) (with \( P \) possibly empty), with \( g \) and \( g' \) distinct sibling split arcs, which contradicts Corollary 9.

Corollary 23. For any join arc \( f \), there exists at most one central-micro omnitig \( fg^* \). Symmetrically, for any split arc \( g \), there exists at most one central-micro omnitig \( f^*g \).

Corollary 24. Every central-micro omnitig \( fg \) is contained in a unique maximal microtig.

Lemma 25. For any arc \( e \), there exists at most one maximal microtig of the form \( W_1fgW_2 \), with \( fg \) central-micro omnitig and \( e \in W_1f \), and at most one maximal microtig of the form \( W_1fgW_2 \), with \( fg \) is a central-micro omnitig and \( e \in gW_2 \).

Proof. We need to prove only the first of the two symmetric statements. Consider a central-micro omnitig \( fg \) and a microtig \( W_1fgW_2 \), with \( e \in W_1f \); as such, \( e \) is a join arc. Hence, consider the unique split-free path \( P \) from \( h(e) \) to a bivalent node \( u \). Notice that \( f \) is uniquely defined as the last arc of \( P \). By Lemma 22 and its corollaries, there exists a unique maximal microtig that contains \( fg \).

Corollary 26. For any bivalent arc \( b \), there exist at most two maximal microtigs containing \( b \), of which at most one is of the form \( bW_1 \), and at most one is of the form \( W_2b \).

Corollary 27. For any non-bivalent arc \( e \), there exists at most one maximal microtig containing \( e \).

Maximal left- and right-micro omnitig can be constructed by repeatedly applying Lemma 8 (see Algorithm 1 in the appendix), where each iteration takes only constant time thanks to Corollary 11 and the assumption of constant degree. They are then joined to form maximal microtigs (see Algorithm 3 in the appendix).
Macronodes are connected with each other by bivalent arcs. We need to distinguish two different classes of bivalent arcs (recall Observation 18): those that connect a macronode with itself (self-bivalent) and those that connect two different macronodes (cross-bivalent), see Figure 5a in the appendix.

Definition 28 (Self-bivalent and cross-bivalent arcs). A bivalent arc \( b \) is called a self-bivalent arc if \( U(b) \) goes from a bivalent node to itself. Otherwise it is called a cross-bivalent arc.

A macrotig is obtained by connecting together those microtigs from different macronodes which overlap on a cross-bivalent arc (see Figure 5c in the appendix).

Definition 29 (Macrotig). Let \( W \) be any walk. \( W \) is called a macrotig if

1. \( W \) is a microtig, or
2. by writing \( W = W_0b_1W_1b_2\ldots b_{k-1}W_{k-1}b_kW_k \), where \( b_1, \ldots, b_k \) are all the internal bivalent arcs of \( W \), the following conditions hold:
   a. the arcs \( b_1, \ldots, b_k \) are all cross-bivalent arcs, and
   b. \( W_0b_1, b_1W_1b_2, \ldots, b_{k-1}W_{k-1}b_k, b_kW_k \) are all microtigs.

Notice that, by Definition 29, given any two macrotigs of the form \( W_1b \) and \( bW_2 \), where \( b \) is a cross-bivalent arc, also \( W_1bW_2 \) is a macrotig.

The following properties, proven in the appendix, are helpful to illustrate the relationship between omnitigs and macrotigs.

Lemma 30. Let \( e \) be a join or a split arc. No omnitig can traverse \( e \) twice.

Lemma 31. Let \( u \) be a bivalent node. No omnitig contains \( u \) twice as an internal node.

The following lemma shows that macrotigs are a super-class of those omnitigs that start with a join arc and end with a split arc. As we will show in Section 6, this is without loss of generality thanks to the structure of macronodes: any omnitig can be obtained from this restricted class of omnitigs, if necessary, by the addition of split-free and join-free paths at the endpoints, until the center of a macronode is reached (univocal extension).

Lemma 32. Let \( fWg \) be an omnitig where \( f \) is a join arc and \( g \) is a split arc. Then, \( fWg \) is a macrotig.

Proof. By Observation 14, \( fWg \) contains an internal bivalent node. We prove the statement by induction on the length of \( W \).

Case 1: \( W \) contains no internal bivalent arcs.

Since \( fWg \) contains a bivalent node, it is of the form \( fWg = W_1f'g'W_2 \), with \( h(f') = t(g') = u \) bivalent node. Notice that \( W_1f'g'W_2 \) is an microtig and thus it is a macrotig, by definition.

Case 2: \( fWg \) contains an internal bivalent arc \( b \), i.e. \( fWg = W_1bW_2 \), with \( W_1, W_2 \) non empty.

By induction, \( W_1b \) and \( bW_2 \) are macrotigs and both contain a bivalent node as internal node. Suppose \( b \) is a self-bivalent arc, then both \( W_1b \) and \( bW_2 \) would contain the same bivalent node \( u \) as internal node, contradicting Lemma 31. Thus, \( b \) is a cross-bivalent arc and \( W_1bW_2 \) is also a macrotig, by definition.
We now show that macrotigs do not close into cycles, and they do not split nor join, so they organize themselves into few maximal macrotigs which are almost arc-disjoint paths.

To show that they do not form cycles, we need the following definition and lemma from [3].

Definition 33 ([3]). For any two distinct non-sibling split arcs \( g, g' \), write \( g \lhd g' \) if there exists an omnitig \( gPg' \) where \( P \) is split-free.

Lemma 34 ([3]). The relation \( \lhd \) is acyclic.

Lemma 35. A macrotig \( W \) does not contain an arc \( e \) twice, unless \( e \) is a self-bivalent arc.

Proof. Consider \( g \) and \( g' \) two closest split arcs along \( W \) which are not self-bivalent arcs: that is \( gPg' \) subwalk of \( W \), with \( P \) a split-free path. Notice that \( g \) and \( g' \) are not siblings, otherwise \( g \) is a self-bivalent arc, by Observation 14.

If \( t(g') \) is not a bivalent node, then \( P \) is empty. In this case, \( g \) is a join-free path so \( gg' \) is an omnitig; as such, \( g \lhd g' \). Otherwise, if \( t(g') \) is a bivalent node, then \( gPg' \) is a left-micro omnitig and so it is an omnitig; as such, again, \( g \lhd g' \). Suppose a split arc, which is not self-bivalent, is traversed twice. Since there are no internal self-bivalent arcs, this would result in a cycle in the relation \( \lhd \). Since the relation \( \lhd \) is acyclic, by Lemma 34, no such arc is traversed twice. The same reasoning applies, symmetrically, for join arcs which are not self-bivalent.

Corollary 36. Every macrotig is a subwalk of a maximal macrotig.

Lemma 37. For any macrotig \( W \) there exists a unique maximal macrotig containing \( W \).

Proof. The existence of a maximal macrotig containing \( W \) is guaranteed by Corollary 36. Notice that there exist only two ways to extend a macrotig \( W_0b_1W_1b_2\ldots b_{k-1}W_{k-1}b_kW_k \) to its right: either by extending to the right the microtig \( b_kW_k \), which can be done in at most one way, by Corollary 24, or, if \( b_kW_k \) ends with a cross-bivalent arc \( b_k+1 \) (as such the microtig is right-maximal, by definition), by adding another microtig \( b_{k+1}W_{k+1} \), which also can be done in at most one way by Corollary 26. Hence, there is a unique way to maximally extend a macrotig to its right and by symmetry, there is a unique way to maximally extend it to the left. These extensions can be combined to obtain a maximal macrotig containing \( W \).

Corollary 38. For any self-bivalent arc \( e \), there are at most two maximal macrotigs containing \( e \) (at most one of the form \( eW \) and at most one of the form \( We \)).

Proof. Recall that, by Corollary 26, each self-bivalent arc is contained in at most two microtigs (at most one of the form \( eW \) and at most one of the form \( We \)), which can each be extended to maximal microtigs (in opposite directions).

Corollary 39. For any arc \( e \) that is not a self-bivalent arc, there is at most one maximal macrotig containing \( e \).

Proof. If \( e \) is cross-bivalent, by Corollary 26, then \( e \) is contained in at most two microtigs (at most one of the form \( eW \) and at most one of the form \( We \)), which, if both present, can be joined into a macrotig. Otherwise, by Corollary 27, \( e \) is contained in at most one maximal microtig. In either case, it can be extended, in both directions, to a unique maximal macrotig.

Corollary 40. The total size of the maximal macrotigs of \( G \) is \( O(m) \).

To find the maximal macrotigs, it is sufficient to join together microtigs overlapping on cross-bivalent arcs (see Algorithm 4 in the appendix).
6 Maximal omnitig enumeration

Once macro tigs have been found, Lemma 42 guarantees that it is sufficient to identify subwalks of macro tigs which are also omnitigs.

To this end, we need to prove a stronger version of Lemma 8 stated in Lemma 42 and whose proof requires the following lemma.

Lemma 41. Let \( fW \) be an omnitig, where \( f \) is a join arc. Let \( P \) be a path in \( G \), from \( h(W) = t(P) \) to a node in \( W \), such that the last arc of \( P \) is not an arc of \( fW \). Then no internal node of \( P \) is a node of \( W \).

Proof. Consider \( P_W \) the longest suffix of \( P \), such that no internal node of \( P_W \) is a node of \( W \). If \( P_W = P \), the lemma trivially holds. Let \( W = (u_0, e_1, u_1, e_2, \ldots, e_k, u_k) \). Let \( u_i = t(P_W) \) and \( u_j = h(P_W) \). If \( i \geq j \), then \( P_W \) is a forbidden path for \( fW \), a contradiction. Hence, assume \( i < j < k \). Let \( f'WQ \) be a closed path, as in Lemma 5. Consider the walk \( Z = P_{W(e_{j+1} \ldots e_kQ)} \). Notice that \( e_{i+1} \notin Z \) and \( f \notin Z \). Therefore \( Z \) can transformed in a forbidden path, from \( u_i \) to \( h(f) \), for \( fW \).

Lemma 42. Let \( fW \) be an omnitig in \( G \), where \( f \) is a join arc. Then \( fWg \) is an omnitig if and only if \( g \) is the only arc with \( t(g) = h(W) \) such that there exists a path from \( h(g) \) to \( h(f) \) in \( G \setminus f \).

Proof. By Lemma 4, at least one \( g \) exists which satisfies the condition. Assume \( g \) is a split arc, otherwise the statement trivially holds.

First, assume that there is a \( g' \) sibling split arc of \( g \) and a path \( P \) from \( h(g) \) to \( h(f) \) in \( G \setminus f \). We prove that there exists a forbidden path for \( fWg \). Let \( P_W \) be the prefix of \( P \) ending in the first occurrence of a node in \( W \) (i.e., no node of \( P_W \) belongs to \( W \), except for \( h(P_W) \)). Notice that \( g'P_W \) is a forbidden path for the omnitig \( fWg \) (it is possible, but not necessary, that \( h(P_W) = h(f) \)).

Second, take any forbidden path \( P \) for the omnitig \( fWg \). We prove that there exists a \( g' \) sibling split arc of \( g \) and a path from \( h(g) \) to \( h(f) \) in \( G \setminus f \). Notice that \( t(P) = h(W) = t(g) \), otherwise \( P \) would be a forbidden path for \( fW \). As such, \( P \) starts with a split arc \( g' \neq g \) and, by Lemma 41, \( P \) does not contain \( f \). Thus, the suffix of \( P \) from \( h(g') \) is a path in \( G \setminus f \) from \( h(g') \) to \( h(f) \).

Lemma 42 is exploited to check if a given omnitig can be extended to the right (see Algorithm 5 in the appendix). This will be used to grow omnitigs along a macro tig.

To identify maximal omnitigs along macro tigs (see Figure 5d in the appendix), one can scan each macro tig with two pointers, a left one always on a join arc, and a right one always on a split arc (see Algorithm 6 in the appendix). Both pointers move from left to right such that the subwalk between them is always an omnitig. The subwalk is grown to the right by moving the right pointer as long as it remains an omnitig (checked with Algorithm 5). When growing to the right is no longer possible, the omnitig is shrunk from the left by moving the left pointer. This two-pointer technique avoids testing those segments whose univocal extension is neither a left- nor a right-maximal omnitig, and for this reason runs in linear time.

To get all maximal omnitigs, it is sufficient to compute the univocal extensions of maximal omnitigs along a macro tig as well as bivalent arcs which do not belong to a macro tig.
7 Conclusions and future work

The main contribution of this paper is a linear-time algorithm to identify all maximal omnitigs of a graph, which is made possible by the discovery of macrotigs. Indeed, even if there are families of graphs where the sum of the lengths of all maximal omnitigs is \(\Theta(nm)\), non-trivial maximal omnitigs are always subwalks of macrotigs (apart from a join-free prefix and a split-free suffix), which have \(O(m)\) total length.

This fact has several consequences. First, this property allows the linear-time output-sensitive explicit enumeration of all maximal omnitigs, which remained an open problem in [3]. In addition, macrotigs act as a compact data structure that allows for various operations on maximal omnitigs, without listing them explicitly. For example, by pre-computing univocal extensions from any node, one can compute in \(O(m)\) total time various statistics about maximal omnitigs (such as minimum, maximum, and average length). One can also filter out subfamilies of them, for example those of length smaller than a given value.

Although this paper closes most of the questions about finding those walks which are safe for closed arc-covering walks, there are many variants of this problem for which the existence of a fast algorithm is still open. For example, one can consider closed node-covering walks (as in [16, 15]), or arc-covering collections of multiple closed walks (as in [1]). We deem feasible that the structure of macrotigs can provide useful insights for these variants as well, and we leave this as subject of further research.

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From omnitigs to macrotigs: a linear-time algorithm for safe walks

To prove the correctness of Transformation 1, we proceed as follows. Let $c_e(G)$ be the graph obtained from $G$ by contracting an arc $e$ (contracting $e$ means that we remove $e$ and identify its endpoints). For every walk $W$ of $G$, we denote by $c_e(W)$ the walk of $c_e(G)$, obtained from $W$ by removing every occurrence of $e$ (here we regard walks as sequences of arcs). In the following, we regard $c_e$ as a surjective function from the family of walks of $G$ to the family of walks of $c_e(G)$.

▶ **Observation 43.** When $e$ is a split-free or join-free arc, then $c_e$ is a bijection when restricted to the closed (arc-covering) walks, or to the open walks of $G$ whose first and last arc are different than $e$.

▶ **Lemma 44.** Let $e$ be a join-free arc of $G$. A walk $W'$ of $c_e(G)$ is an omnitig of $c_e(G)$ if and only if there exists an omnitig $W$ of $G$ such that $W' = c_e(W)$.

**Proof.** Consider the shortest walk $W$ of $G$ such that $W' = c_e(W)$. Notice that the first and last arc of $W$ are different than $e$. Moreover, $W'$ is an omnitig of $c_e(G)$ if $W$ is an omnitig.
(a) Classification of nodes and arcs: join-free (blue), split-free (red), bivalent nodes (black), self-bivalent arcs (purple), cross-bivalent arcs (green).

(b) Maximal microtigs.

(c) Maximal macrotigs.

(d) Maximal non-trivial omnitigs (obtained from macrotigs). Univocal extensions are dotted.

Figure 5 A strongly connected and compressed graph, its macronodes (gray areas), classification of nodes and arcs, microtigs, macrotigs, and omnitigs.
of $G$. Indeed, for every circular covering $C$ of $G$ it holds that $C$ avoids $\bar{W}$ iff $c_e(C)$ avoids $W'$. ▶

**Corollary 45.** Let $e$ be a join-free arc of $G$. A walk $W'$ of $c_e(G)$ is a maximal omnitig of $c_e(G)$ if and only if there exists a maximal omnitig $W$ of $G$ such that $W' = c_e(W)$. 

**Proof.** Let $W$ be a maximal omnitig of $G$. Then $c_e(W)$ is an omnitig of $c_e(G)$ by Lemma 44. Moreover, if $W'$ was an omnitig of $c_e(G)$ strictly containing $c_e(W)$, then there would exist an omnitig $\bar{W}$ of $G$ such that $W' = c_e(\bar{W})$, by Lemma 44. Clearly, $\bar{W}$ would contain $W$ and contradict its maximality. Therefore, $c_e(W)$ is a maximal omnitig of $c_e(G)$.

For the converse, let $W'$ be a maximal omnitig of $c_e(G)$. Let $W$ be the shortest and unique minimal walk of $G$ such that $W' = c_e(W)$. By Lemma 44, $W$ is an omnitig of $G$. Let $\bar{W}$ be any maximal omnitig of $G$ containing $W'$. We claim that $c_e(\bar{W}) = W' = c_e(W)$, which concludes the proof. If not, then $c_e(\bar{W})$ would strictly contain $W'$ and contradict its maximality since also $c_e(W)$ would be an omnitig of $c_e(G)$ by Lemma 44. ▶

**Lemma 12.** Let $G$ be a graph and let $G'$ be the graph obtained by applying Transformation 1 to $G$. Then a walk $W$ of $G$ is a maximal omnitig of $G$ if and only if there exists a maximal omnitig $W'$ of $G'$ such that $W$ is the string obtained from $W'$ by suppressing all the arcs introduced with the transformation.

**Proof.** Notice that $G$ is obtained by applying $c_e$ to each arc $e$ introduced by Transformation 1, that is, to each arc of $G'$ that is not an arc of $G$. Notice that $W$ is the string obtained from $W'$ by suppressing all the arcs introduced with the transformation if and only if $W$ is obtained from $W'$ by contracting each arc $e$ introduced by Transformation 1. Apply Corollary 45. ▶

**Lemma 30.** Let $e$ be a join or a split arc. No omnitig can traverse $e$ twice.

**Proof.** By symmetry, we only consider the case of two sibling split arcs $g$ and $g'$. Since prefixes and suffixes of omnitigs are omnitigs, then a minimal violating omnitig would be of the form $gZg$, with $g \not\in Z$. Since $G$ is strongly connected, then there exists a simple cycle $C$ of $G$ with $g' \in C$ and with $g'$ as its first arc. Notice that $g \not\in C$, since $C$ is simple. Consider then the first node $u$ shared by both $C$ and $Z$, and let $e$ be the arc of $C$ with $h(e) = u$. Clearly, $e \not\in Z$; in addition, $e \not\not\in Z$, since $C$ is a path. Let $C_u$ represent the prefix of $C$ ending in $u$. Therefore, $C_u$ is a forbidden path for the omnitig $gZg$, since it starts from $t(g) = t(g')$, with $g' \not\not\in Z$, and it ends in $u$ with $e \not\not\in Z$. ▶

**Lemma 31.** Let $u$ be a bivalent node. No omnitig contains $u$ twice as an internal node.

**Proof.** Suppose for a contradiction, there exist an omnitig $W$ that contains $u$ twice as internal node. Since $u$ is an internal node of $W$, we can distinguish the case in which an omnitig contains twice a central-micro omnitig that traverses $u$, and the case in which an omnitig contains both the central-micro omnitigs that traverse $u$. In the first case, let $fg$ be the central-micro omnitig of an omnitig $W$ that traverses $u$. Notice that $f$ is a join arc contained twice in $W$, contradicting Lemma 30. In the latter case, let $f_1g_1$ and $f_2g_2$ the two central-micro omnitigs that traverse $u$, with $f_1 \neq f_2$ and $g_1 \neq g_2$. Consider $W$ to be a minimal violating omnitig of the form $f_1g_1Wf_2g_2$. Notice that $u \not\in W$, by minimality; hence $g_1Wf_2$ is a forbidden path, contradicting $W$ being an omnitig. ▶
Algorithm pseudo-code and analysis

B.1 Microtigs

To describe the algorithm to construct all maximal microtigs in the graph (Algorithm 3), we first introduce two additional procedures (Algorithms 1 and 2).

Algorithm 1

Function RightExtension(G, f, W)

1. **Input**: The compressed graph G, fW omnitig with W join-free.
2. **Returns**: The unique arc e such that fWe is an omnitig, if it exists. Otherwise, nil.

```plaintext
S ← \{ e ∈ E(G) | t(e) = h(W) and there is a path from h(e) to h(f) in G \setminus f \}
if there is exactly one arc e ∈ S then return e
return nil
```

Lemma 46. Algorithm 1 is correct and it can be implemented to run in constant time, after preprocessing the graph in \( O(m + n) \) time.

Proof. Recall Lemma 8 and Corollary 11 and the assumption of constant degree for every node, by Transformation 1.

Algorithm 2

Function MaximalRightMicroOmnitig(G, f, g)

1. **Input**: The compressed graph G, fg omnitig with f join arc and g split arc.
2. **Returns**: The path W such that fgW is a maximal right-micro omnitig.

```plaintext
W ← empty path
while True do
  if fgW ends with a bivalent arc then return W
  e ← RightExtension(G, f, R)
  if e = nil then return W
  W ← We
```

Lemma 47. Algorithm 2 is correct and it can be implemented to run in time linear in its output, after preprocessing the graph in \( O(m + n) \) time.

Proof. Every iteration of the while loop increases the output by one arc and takes constant time, by Lemma 46.

Algorithm 3

Function MaximalMicroOmnitig(G, f, g)

1. **Input**: The compressed graph G, fg omnitig.
2. **Returns**: The path W such that fgW is a maximal microtig.

```plaintext
if \exists \text{ central-micro omnitig} fg of the graph then
  \text{compute a maximal microtig and take linear time in its size, by Definition 21 and Lemma 47}
\end{algorithm}
```

Lemma 48. Algorithm 3 is correct and can be implemented to run in linear time.

Proof. Since every node of the graph has constant degree, the if check runs a number of times linear in the size of the graph. Checking the condition takes constant time, by Lemma 46 in addition, the condition is true for every central-micro omnitig fg of the graph. The then block computes a maximal microtig and takes linear time in its size, by Definition 21 and Lemma 47. By Corollary 24 and Corollary 27 we find every microtig in linear total time.
Algorithm 3: Function AllMaximalMicrotigs

1. Function AllMaximalMicrotigs(G)
   2. Input: The compressed graph G.
   3. Returns: All the maximal microtigs in G.
   4. S ← ∅
   5. foreach bivalent node u in G do
   6.     foreach join arc f with h(f) = u do
   7.         foreach split arc g with t(g) = u do
   8.             if g = RightExtension(G, f, ∅) then
   9.                 W1fgW2 to S
   10. return S

Algorithm 4: Function AllMaximalMacrotigs.

1. Function AllMaximalMacrotigs(G)
   2. Input: The compressed graph G.
   3. Returns: All the maximal macrotigs in G.
   4. S ← AllMaximalMicrotigs(G)
   5. while there exists W1b ∈ S and bW2 ∈ S with b cross-bivalent arc and non-empty W1, W2 do
   6.     remove W1b and bW2 from S
   7.     add W1bW2 to S
   8. return S

Lemma 49. Algorithm 4 is correct and it can be implemented to run in linear time.

Proof. Recall Lemma 37 and Corollaries 38 and 40.

B.3 Maximal omnitigs

To describe the algorithm that identifies all maximal omnitigs (Algorithm 6), we first introduce an auxiliary procedure (Algorithm 5).

Lemma 50. Algorithm 5 is correct and can be implemented to run in constant time, after preprocessing the graph in O(m + n) time.
Algorithm 5 Function IsOmnitigRightExtension

Function IsOmnitigRightExtension(G, f, g)

Input : The compressed graph G. A join arc f and a split arc g such that there exists a walk fWg where fW is an omnitig.

Returns : Whether fWg is also an omnitig.

1. \( S \leftarrow \{ g' \in E(G) \mid t(g') = t(g) \text{ and there is a path from } h(g') \text{ to } h(f) \text{ in } G \setminus f \} \)

2. return True if \( S = \{ g \} \) and False otherwise

Proof. Recall Corollary 11 and Lemma 42, and the fact that every node in G has constant degree.

Maximal omnitigs are identified with a two-pointer scan of maximal macrotigs (Algorithm 6), while also adding univocal extensions and trivial omnitigs (of the form \( U(b) \) for a bivalent arc \( b \)).

Algorithm 6 Computing all maximal omnitigs

Input : The compressed graph G.

Outputs : All maximal omnitigs of G.

1. \( B \leftarrow \{ b \text{ bivalent arc } \mid b \text{ does not occur in any } W \in \text{AllMaximalMacrotigs}(G) \} \)

2. foreach \( b \in B \) do output \( U(b) \)

3. foreach \( f^*Xg^* \in \text{AllMaximalMacrotigs}(G) \) do

   \[ \text{With the notation } X[f..g], \text{ we refer to the subwalk of } f^*Wg^* \text{ starting with the occurrence of } f \text{ in } f^*X \text{ (unique by Lemma 35) and ending with the occurrence of } g \text{ in } Xg^* \text{ (unique by Lemma 35).} \]

4. \( f \leftarrow f^* \)

5. \( g \leftarrow \text{nil} \)

6. \( g' \leftarrow \text{first split arc in } Xg^* \)

7. while \( g' \neq \text{nil} \) do

   \[ \text{Grow } X[f..g] \text{ to the right as long as possible} \]

8. \( g \leftarrow g' \)

9. \( g' \leftarrow \text{next split arc in } Xg^* \text{ after } g \)

10. \( X[f..g] \text{ cannot be grown to the right anymore} \)

11. output \( U(X[f..g]) \)

12. while \( g' \neq \text{nil and not IsOmnitigRightExtension}(f, g') \) do

   \[ \text{Shrink } X[f..g] \text{ from the left until it can be grown to the right again} \]

13. \( i \leftarrow \text{index of next join arc in } f^*X \text{ after } f \)

Lemma 51. Algorithm 6 is correct and runs in time linear in the total size of the graph and of its output.
Proof. Recall that, by Lemma 15, any maximal omnitig of $G$ contains both a join arc and a split arc. Hence, it can be written either in the form $U(b)$, where $b$ is a bivalent arc, or in the form $U(fWg)$, where $f$ is a join arc and $g$ is a split arc.

In the first case, that is, if $U(b)$ is a maximal omnitig, then $b$ does not occur in any macrotig (otherwise, it can be extended using at least one microtig that contains $b$). Any such omnitig is returned in Line 2.

In the second case, by Lemma 32, $fWg$ is a macrotig, contained in a maximal macrotig, by Corollary 36. By Lemma 49, Algorithm 4 returns every maximal macrotig in linear time.

It remains to prove that the external while cycle, in Algorithm 6, outputs all the maximal omnitigs of the form $U(fWg)$ where $fWg$ is contained in a maximal macrotig $f^*Xg^*$.

At the beginning of the first iteration, $W = U(X[f..g'])$ is left-maximal since $f = f^*$. The first internal while cycle, in Algorithm 6, ensures that $W = U(X[f..g])$ is also right-maximal, at which point it is printed in output. Then, the second internal while cycle, in Algorithm 6, ensures that $W = U(X[f..g'])$ is a left-maximal omnitig, and the external cycle repeats.

To prove the running time bound, observe that each iteration of the foreach cycle takes time linear in the total size of the maximal macrotig $X$ and of its output, and that the total size of all maximal macrotigs is linear, by Corollary 40.

Observation 52. Maximal omnitigs of the form $U(b)$ can be uniquely reconstructed from the bivalent arc $b$ in time linear in their length. Given the set of maximal macrotigs, also omnitigs of the form $U(X[f..g])$ can be uniquely reconstructed from the arcs $f$ and $g$, in time linear in their length.