On lattices of convex sets in \( \mathbb{R}^n \)

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To the memory of Ivan Rival

Abstract. Properties of several sorts of lattices of convex subsets of \( \mathbb{R}^n \) are examined. The lattice of convex sets containing the origin turns out, for \( n > 1 \), to satisfy a set of identities strictly between those of the lattice of all convex subsets of \( \mathbb{R}^n \) and the lattice of all convex subsets of \( \mathbb{R}^{n-1} \). The lattices of arbitrary, of open bounded, and of compact convex sets in \( \mathbb{R}^n \) all satisfy the same identities, but the last of these is join-semidistributive, while for \( n > 1 \) the first two are not. The lattice of relatively convex subsets of a fixed set \( S \subseteq \mathbb{R}^n \) satisfies some, but in general not all of the identities of the lattice of "genuine" convex subsets of \( \mathbb{R}^n \).

1. Notation, conventions, remarks.

For \( S \) a subset of \( \mathbb{R}^n \), the convex hull of \( S \) will be denoted

\[
\text{c.h.}(S) = \{ \sum_{i=1}^{m} \lambda_i p_i \mid m \geq 1, \ p_i \in S, \ \lambda_i \in [0,1], \ \sum \lambda_i = 1 \}.
\]

When \( S \) is written as a list of elements "{ ... }" we will generally simplify "c.h.( { ... } )" to "c.h. (...)".

\( \text{Conv}(\mathbb{R}^n) \) will denote the lattice of all convex subsets of \( \mathbb{R}^n \); its lattice operations are

\[
x \wedge y = x \cap y, \quad x \vee y = \text{c.h.}(x \cup y).
\]

In any lattice, if a finite family of elements \( y_i \) has been specified, then an expression such as \( \bigwedge_i y_i \) will denote the meet over the full range of the index \( i \), and similarly for joins. Likewise, if we write something like \( \bigwedge_{j \neq i} y_j \) where \( i \) has been quantified outside this expression, then the meet will be over all values of \( j \) in the indexing family other than \( i \). If \( L \) is any lattice and \( x \) an element of \( L \), or, more generally, of an overlattice of \( L \), we define the sublattice

\[
L_{\geq x} = \{ y \in L \mid y \geq x \}.
\]

In particular, \( \text{Conv}(\mathbb{R}^n)_{\geq \{0\}} \) is the lattice of those convex subsets of \( \mathbb{R}^n \) that contain the origin.

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If $A$ is a class of lattices, $V(A)$ will denote the variety of lattices generated by $A$, that is, the class of lattices satisfying all identities (in the binary operations $\land$ and $\lor$) that hold in all lattices in $A$. Again, for $A$ given as a list, we will abbreviate $V(\{\ldots\})$ to $V(\ldots)$. (Most often $A$ will be a singleton $\{L\}$, so that we will write $V(L).$)

Given sets $x$ and $y$, we will write $x - y$ for their set-theoretic difference, $\{p \mid p \in x, p \notin y\}$.

Though I am not an expert either in lattice theory or in convex sets, I know more about the former subject than the latter; hence, I may more often state facts known to every worker in convex sets than those known to every lattice-theorist. I hope this note will nevertheless be of interest to people in both fields. I have no present plans of carrying these investigations further; others are welcome to do so.

Since obtaining the main results of this paper, I have learned that many of them were already in the literature, and have added references; thus, this is now a hybrid research/survey paper. I am grateful to Kira Adaricheva, J. B. Nation, Marina Semenova, Fred Wehrung, and the referee, for corrections, information on the literature, and many other helpful comments.

Whereas this note looks at conditions satisfied universally in various lattices of convex sets, the papers [1], [19], and others cited there study sufficient conditions for lattices to be embeddable in such lattices, in other words, existential properties of such lattices. (This note includes one result of that type, in §13.)

2. $n$-Distributivity.

The varieties of lattices we will be examining in the first few sections are those listed in

**Lemma 1.** Each lattice in the sequence $\text{Conv}(\mathbb{R}^0)_{\geq \{0\}}, \text{Conv}(\mathbb{R}^0), \text{Conv}(\mathbb{R}^1)_{\geq \{0\}}, \ldots, \text{Conv}(\mathbb{R}^n)_{\geq \{0\}}, \text{Conv}(\mathbb{R}^n), \ldots$ is embeddable in the next. Hence

$$V(\text{Conv}(\mathbb{R}^0)_{\geq \{0\}}) \subseteq V(\text{Conv}(\mathbb{R}^0)) \subseteq V(\text{Conv}(\mathbb{R}^1)_{\geq \{0\}}) \subseteq V(\text{Conv}(\mathbb{R}^1)) \subseteq \cdots \subseteq V(\text{Conv}(\mathbb{R}^n)) \subseteq \ldots .$$

**Proof.** On the one hand, $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ is a sublattice of $\text{Conv}(\mathbb{R}^n)$; on the other, one can embed $\text{Conv}(\mathbb{R}^n)$ in $\text{Conv}(\mathbb{R}^{n+1})_{\geq \{0\}}$ by sending each $x \in \text{Conv}(\mathbb{R}^n)$ to the cone $c.h.(\{0\} \cup \{(p_1, \ldots, p_n, 1) \mid (p_1, \ldots, p_n) \in x\})$. The inclusions follow from these embeddings. □

Clearly $\text{Conv}(\mathbb{R}^0)_{\geq \{0\}}$ is a trivial lattice with unique element $\{0\}$; on the other hand, $\text{Conv}(\mathbb{R}^0)$ is a two-element lattice, so $V(\text{Conv}(\mathbb{R}^0))$ is the variety of distributive lattices. Hence the first inclusion is strict. We shall see below that the next inclusion is an equality, while all subsequent inclusions are again strict.

Let us set up notation for some lattice identities.
Definition 2 (after Huhn [11]). For each positive integer n, we shall denote by $D_n(x, y_1, \ldots, y_{n+1})$ (for “n-distributivity”) the lattice-relation in $n+2$ arguments $x, y_1, \ldots, y_{n+1}$,

$$x \land (\bigvee_i y_i) = \bigvee_i (x \land \bigvee_{j \neq i} y_j). \quad (5)$$

We shall say that a lattice $L$ satisfies “the identity $D_n$” if (5) holds for all $x, y_1, \ldots, y_{n+1} \in L$.

Thus, $D_1$ is the ordinary distributivity identity.

The use of the subscript $n$ for an identity in $n+2$ variables which is symmetric in $n+1$ of these may seem confusing; a useful mnemonic is that $D_n$ is the identity that allows one to “reduce meets of $x$ with larger joins to meets of $x$ with $n$-fold joins”. An additional occasion for confusion will arise when we see that the first of these identities to be satisfied by Conv($\mathbb{R}^n$) is not $D_n$, but $D_{n+1}$. This will be a consequence of the fact that an $n$-dimensional simplex has $n+1$ vertices.

Note that the left-hand side of (5) is $\geq$ the right-hand side for any family of elements of any lattice, since each of the $n+1$ terms in the outer join on the right is majorized by the left-hand side. So to verify any instance of (5) it suffices to prove “$\leq$”.

The pioneering work on identities satisfied by lattices of convex sets was done by A. P. Huhn [10], [11]. The results in this and the next three sections will extend Huhn’s by approximately doubling both the family of lattices and the family of identities considered, and formalizing some general techniques. Huhn’s results will be recovered along with our new ones.

The key to Huhn’s and our results on $D_n$ (and some related identities) is the following standard result in the theory of convex sets. Strictly speaking, it is the first sentence below that is Carathéodory’s theorem, while the second is a well-known refinement thereof [6, p.431, line 4]. Intuitively, that second sentence says that from an arbitrary point $p_0 \in S \subseteq \mathbb{R}^n$, we can “see” any other point of ch(S) against a background of (or embedded in) some $(n-1)$-simplex with vertices in $S$.

**(Carathéodory’s Theorem).** If $S$ is a subset of $\mathbb{R}^n$, then each element $q \in$ ch(S) belongs to ch($p_0, \ldots, p_n$) for some $n+1$ points $p_0, \ldots, p_n \in S$. Moreover, $p_0$ can be taken to be any pre-specified element of $S$. □

In each paragraph of the next lemma, it is the first assertion that is due to Huhn.

Lemma 3 (cf. Huhn [11]). For every natural number $n$ and $(n+3)$-tuple of convex sets $x, y_1, \ldots, y_{n+2} \in$ Conv($\mathbb{R}^n$), the relation $D_{n+1}(x, y_1, \ldots, y_{n+2})$ holds. Moreover, if $n > 0$ and (at least) some two of the sets $y_1, \ldots, y_{n+1}$ have nonempty intersection, then $D_n(x, y_1, \ldots, y_{n+1})$ holds.

Hence Conv($\mathbb{R}^n$) satisfies the identity $D_{n+1}$, and for $n > 0$, Conv($\mathbb{R}^n$)$_{\geq\{0\}}$ satisfies the identity $D_n$.

**Proof.** We shall prove the assertions of the first paragraph, which clearly imply those of the second.

To get the first assertion, let $x, y_1, \ldots, y_{n+2} \in$ Conv($\mathbb{R}^n$), and let $p$ be a point of the convex set described by the left-hand side of (5). Then $p$ belongs to both $x$
we just showed equivalent to \( D \) the right-hand side of \( D \), hence it belongs to one of the terms on the right-hand side of \([\text{Eq}]\), hence to their join, as required.

To prove the second assertion, let \( x, y_1, \ldots, y_{n+1} \in \text{Conv}(\mathbb{R}^n) \), and suppose that two of \( y_1, \ldots, y_{n+1} \) have a point \( p_0 \) in common. Given \( p \) in the left-hand side of the desired instance of \([\text{Eq}]\), we see from the last sentence of our statement of Carathéodory’s Theorem that \( p \) will belong to the convex hull of \( p_0 \) and some \( n \) other points of \( \bigcup_i y_i \). The latter points will lie in the union of some \( n \) of the \( y \)'s, and since those \( n \) of our \( n+1 \) sets leave out only one, they do not leave out both of the sets known to contain \( p_0 \). So that family of \( n \ y \)'s contains the \( n+1 \) points whose convex hull is known to contain \( p \), and the conclusion follows as before. \( \square \)

Note that for all \( p \in \mathbb{R}^n \), we have \( \text{Conv}(\mathbb{R}^n)_{\geq \{p\}} \cong \text{Conv}(\mathbb{R}^n)_{\geq \{0\}} \); moreover, if \( p \in S \subseteq \mathbb{R}^n \), then \( \text{Conv}(\mathbb{R}^n)_{\geq S} \subseteq \text{Conv}(\mathbb{R}^n)_{\geq \{p\}} \). Thus, any identities proved for the lattice \( \text{Conv}(\mathbb{R}^n)_{\geq \{0\}} \) will hold in \( \text{Conv}(\mathbb{R}^n)_{\geq S} \) for every nonempty set \( S \).

One may ask whether, in the first assertion of the above lemma, we have failed to use the full strength of Carathéodory’s Theorem. That theorem says that the convex hull of any family of \( N > n+1 \) points is the union of the hulls of its \((n+1)\)-element subfamilies, hence for each such \( N \), we may deduce an identity like \( D_{n+1} \), but with the join of \( N \) rather than \( n+2 \) convex sets \( y_i \) on the left, and expressions involving all the \((n+1)\)-fold subjoins thereof on the right. Our \( D_{n+1} \) is the case \( N = n+2 \).

But in fact, the identities so obtained are all equivalent to \( D_{n+1} \). To derive them from it, note first that if in \( D_{n+1} \) we substitute for \( y_{n+2} \) the expression \( y_{n+2} \vee y_{n+3} \), then the left-hand side becomes \( x \wedge (\bigvee_{i=1}^{n+3} y_i) \), while on the right, some of the joins involve \( n+1 \) of the \( y \)'s and others involve \( n+2 \). If we again apply \( D_{n+1} \) to the latter joins, we get precisely the \( N = n+3 \) case of the class of identities discussed above. The identities with still larger \( N \) are gotten by repeating this argument. Conversely, one can get \( D_{n+1} \) from any of these identities by substituting for the \( y_i \) with \( i > n+2 \) repetitions of \( y_{n+2} \), and discarding from the outer join on the right-hand side joinands majorized by others.

Let us also note that \( D_{n+1} \implies D_{n+2} \). Indeed, the identity with \( n+3 \ y \)'s that we just showed equivalent to \( D_{n+1} \) has the same left-hand side as \( D_{n+2} \), while the right-hand side of \( D_{n+2} \) can be seen to lie, in an arbitrary lattice, between the two sides of that identity. In particular, the identities obtained in Lemma \( \text{B} \) are successively weaker for larger \( n \), as is reasonable in view of \([\text{Eq}]\).

The argument proving Lemma \( \text{B} \) (for simplicity let us limit ourselves to the first assertion thereof) can be formulated in a more general context, and the above observations on identities allow us to obtain a converse in that context, which we record below, though we shall not use it. Recall that a closure operator on a set \( X \) is called “finitary” (or “algebraic”) if the closure of every subset \( S \subseteq X \) is the union of the closures of the finite subsets of \( S \).

**Lemma 4** (cf. \([\text{10}]\)). Let \( \text{cl} \) be a finitary closure operator on a set \( X \), such that every singleton subset of \( X \) is closed, or more generally, such that the closure
of every singleton is join-irreducible; and let \( n \) be a positive integer. Then the lattice of closed subsets of \( X \) under \( \text{cl} \) satisfies \( D_n \) if and only if \( \text{cl} \) has the “\( n \)-Carathéodory property” that the closure of every set is the union of the closures of its \( \leq n \)-element subsets.

**Proof.** “If” is shown exactly as in the proof of Lemma 3.

Conversely, suppose the lattice of \( \text{cl} \)-closed subsets of \( X \) satisfies \( D_n \), and let \( p \in \text{cl}(S) \) for some \( S \subseteq X \). By the assumption that \( \text{cl} \) is finitary, \( p \in \text{cl}(q_1, \ldots, q_N) \) for some \( q_1, \ldots, q_N \in S \). If \( N \leq n \), we are done; if not, let us rewrite the condition \( p \in \text{cl}(q_1, \ldots, q_N) \) as a lattice relation, \( \text{cl}(p) = \text{cl}(p) \land (\bigvee_{i \in I} \text{cl}(q_i)) \). By the preceding discussion, \( D_n \) implies the identity in \( N \) variables which, when applied to the right-hand side of the above relation, turns the relation into

\[
\text{cl}(p) = \bigvee_{|I|=n} (\text{cl}(p) \land (\bigvee_{i \in I} \text{cl}(q_i))),
\]

where the subscript to the outer join means that \( I \) ranges over all \( n \)-element subsets of \( \{1, \ldots, N\} \). Now by assumption \( \text{cl}(p) \) is join-irreducible, hence it equals one of the joinands on the right, showing \( p \) to be in the closure of some set of \( n \) \( q \)'s, as required. \( \square \)

3. Tools for studying related identities.

We also appear to have used less than the full force of the middle sentence of Lemma 3 (the stronger relation holding when at least two of the \( y \)'s have nonempty intersection) in getting the second assertion of the last sentence of that lemma (the stronger identity for \( \text{Conv}((\mathbb{R}^n)_{\geq \{0\}}) \)), since the latter assertion concerns the case where not just two, but all the \( y_i \) (and also \( x \)) have a point in common. There is no evident way to take advantage of the weaker hypothesis of said middle sentence when working in the lattice \( \text{Conv}((\mathbb{R}^n)_{\geq \{0\}}) \); but might we be able to use it to prove some new identity for \( \text{Conv}(\mathbb{R}^n) \); for example, by finding a relation that holds, on the one hand, whenever a family of elements \( x, y_1, \ldots, y_{n+1} \) satisfies \( D_n \), and also, for some trivial reason, whenever \( y_1 \land y_2 = \emptyset \) ?

One relation with these properties can be obtained by taking the meet of each side of \( D_n \) with \( y_1 \land y_2 \). Unfortunately, this turns out to be the trivial identity: both sides simplify to \( x \land y_1 \land y_2 \) in any lattice. However, we can circumvent this by first taking the join of both sides of \( D_n \) with a new indeterminate \( z \), and only then taking meets with \( y_1 \land y_2 \).

We shall in fact see that this provides what Lemma 3 failed to: an identity holding in \( \text{Conv}(\mathbb{R}^n) \) but not in \( \text{Conv}((\mathbb{R}^{n+1})_{\geq \{0\}}) \). However, the verification of an example showing the failure of this identity in the larger lattice (and of a similar example we will need later) is messy if done entirely “by hand”; so we shall establish in this section some general criteria for certain sorts of inequalities to be strict.

The last assertions of the next two lemmas clearly imply, for \( \text{Conv}(\mathbb{R}^n) \) and \( \text{Conv}((\mathbb{R}^{n+1})_{\geq \{0\}}) \) respectively, that if we are given expressions \( u, v \) and \( w \) in some lattice indeterminates such that the inequality \( u \geq v \) is known to hold identically in our lattice, then we can write down another inequality holding identically.
among expressions in a slightly larger set of indeterminates, for which equality will hold whenever either $u$ equals $v$, or $w$ equals the least lattice-element (i.e., $\emptyset$, respectively $\{0\}$), but which will fail in all other cases. This is what is logically called for by the program sketched above. However the earlier parts of these lemmas give some simpler inequalities for which the same is true if the hypotheses hold “in a sufficiently strong way”, and it will turn out that in the applications where we need to show failure of an identity, we will be able to use these simpler formulas.

In the statements of these lemmas, for $p, q \in \mathbb{R}^n$, “the ray drawn from $q$ through $p$” will mean $\{\lambda p + (1 - \lambda)q \mid 0 \leq \lambda < \infty\}$, even in the degenerate case $p = q$, where this set is the singleton $\{p\}$.

**Lemma 5.** Let $n$ be any natural number, and let $u \geq v$ and $w$ be three elements of $\text{Conv}(\mathbb{R}^n)$. Then

(i) The following conditions are equivalent:
   (a) There exists $z \in \text{Conv}(\mathbb{R}^n)$ such that $(u \lor z) \land w > (v \lor z) \land w$.
   (b) There exist a point $p \in u$ and a point $q \in w$ such that the ray drawn from $q$ through $p$ contains no point of $v$.

(ii) The following conditions are equivalent:
   (a) There exist $z, z' \in \text{Conv}(\mathbb{R}^n)$ such that $((u \land z') \lor z) \land w > ((v \land z') \lor z) \land w$.
   (b) $u > v$, and $w$ is nonempty.

**Proof.** In view of the hypothesis $u \geq v$, the relation “$\geq$” always holds in the inequalities of (i)(a) and (ii)(a), so in each case, strict inequality is equivalent to the existence of a point $q$ belonging to the left-hand side but not to the right-hand side.

Suppose, first, that (i)(a) holds for some $z$; thus we get a point $q$ belonging to $u \lor z$ and to $w$ but not to $v \lor z$. Note that the latter condition implies that $q$ does not lie in $z$. If $q$ lies in $u$, then (i)(b) is satisfied with $p = q$, so assume $q \notin u$. Hence, being in the convex hull of $u$ and $z$ but in neither set, $q$ must lie on the line-segment $c.h.(p, r)$ for some $p \in u, r \in z$:

|       | $p$  | $q$   | $r$   |
|-------|------|-------|-------|
|       | $\in u$ |       |       |
|       |       | $(u \lor z) \land w$ |       |
|       |       | $\notin v \lor z$ |       |
|       |       |       | $\in z$ |

Now if a point $s \in v$ lay on the ray drawn from $q$ through $p$, we would have $q \in c.h.(s, r) \subseteq v \lor z$, contradicting our assumption that $q \notin v \lor z$. This proves (b).

Conversely, if we are given $p$ and $q$ as in (i)(b), take $z = \{2q - p\}$:

|       | $p$   | $q$   | $2q - p$ |
|-------|-------|-------|----------|
|       | $\in u$ |       | $\in w$ |
|       |       |       | $\in z$ |

Thus $q \in (u \lor z) \land w$, but we claim that $q \notin v \lor z$. Indeed, if $q$ belonged to this set, then $v$ would have to meet the ray from the unique point $2q - p$ of $z$ through
The following conditions are equivalent:

and

does contradict our choice of through linear combination of 0 and but not the right-hand side of the inequality of (a), as required.

Note that (i)(b), and hence (i)(a), holds whenever and are nonempty and is empty.

Now in the situation of (ii)(b), if we take any and let , then and are respectively nonempty and empty, so applying the preceding observation with these two sets in the roles of and , we get (ii)(a). The reverse implication is trivial.

The result we shall prove for is similar. Indeed, in the lemma below, part (i) is exactly as in the preceding lemma; but (ii) becomes two statements, (ii) and (iii), the former having the same “(a)” as in (ii) above but a stronger “(b)”, the latter a weaker “(a)” and the same “(b)” as above. Note also the restriction on (only needed for (iii)).

Lemma 6. Let and let and be three elements of . Then

(i) The following conditions are equivalent:

(a) There exists such that \((u \lor z) \land w \succ (v \lor z) \land w\).

(b) There exist a point and a point such that the ray drawn from through contains no point of .

(ii) The following conditions are equivalent:

(a) There exist and such that 
\[((u \lor z') \land w) \succ ((v \lor z') \land w)\].

(b) There exist a point and a point such that the ray drawn from through contains no point of \(v \land \text{c.h.}(0,p)\).

(iii) The following conditions are equivalent:

(a) There exist , , such that 
\[((u \land z'') \lor z') \land w \succ (((v \land z'') \lor z') \land z) \land w\].

(b) \(u \succ v\), and \(w \neq \{0\}\).

Proof. (i)(a) \(\Longrightarrow\) (i)(b) holds by the preceding lemma. In proving the reverse implication, we cannot set \(z = \{2q - p\}\) as we did there, so let \(z = \text{c.h.}(0,2q - p)\). As before, we have \(q \in (u \lor z) \land w\) and need to show \(q \notin v \lor z\). If the contrary were true, then \(q\) would be a convex linear combination of \(0, 2q - p\), and a point \(r \in v\). This can be rewritten as a convex linear combination of and with a convex linear combination of 0 and \(r\); but the latter combination would also be a point of \(v\), and, as in the previous proof, would lie on the ray drawn from through \(p\), contradicting our choice of \(p\) and \(q\) as in (b).

Turning to (ii), if (ii)(a) holds then we can apply (i)(a) \(\Longrightarrow\) (i)(b) with \(u \land z'\) and \(v \land z'\) in place of \(u\) and \(v\), and the resulting \(p\) and \(q\) will satisfy (ii)(b) (since \(v \land \text{c.h.}(0,p) \subseteq v \land z'\)). Inversely, if (ii)(b) holds, take \(z' = \text{c.h.}(0,p)\), and apply (i)(b) \(\Longrightarrow\) (i)(a) with \(u \land z'\) and \(v \land z'\) in place of \(u\) and \(v\).
In statement (iii), it is clear that (a) implies (b). To prove the converse, let us assume (iii)(b), and consider two cases, according to whether the stronger statement (ii)(b) holds. If it does, we get the inequality of (ii)(a), from which we can immediately get that of (iii)(a) by choosing $z''$ to “have no effect” (e.g., by taking them to be $u$ and $v$ respectively). On the other hand, if (ii)(b) fails while (iii)(b) holds, it is easy to see that all elements of $w$ and $u-v$ must lie on a common line $x$ through $0$. In that case, we want to use $z'''$ and $z''$ to “perturb” $u$ and $v$, so that the modified $u$ has points off the line $x$, while being careful to preserve the property that $u$ is strictly larger than $v$. To do this, we begin by taking any point $p \in u-v$, letting $z''' = c.h.(0,p)$, and noting that the intersections of $u$ and $v$ with this segment are still distinct. Now taking any point $r$ not on the line $x$ (it is for this that we need $n > 1$), and letting $z'' = c.h.(0,r)$, we see that $(u \land z''') \lor z''$ and $(v \lor z''') \lor z''$ remain distinct, and that their difference now has points off $x$. Hence (ii)(b) holds with these sets in the roles of $u$ and $v$, and the implication (ii)(b) $\iff$ (ii)(a) gives the $z$ and $z'$ needed for (iii)(a). \qed

One can get criteria similar to those of the preceding lemmas for other conditions. At the trivial end, given $u \in \mathrm{Conv}(\mathbb{R}^n)$, a condition for $u$ to be nonempty is that there exist a $z$ such that $u \lor z > z$, and likewise the condition for at least one of two elements $u$ and $v$ to be nonempty is that their join have this property. The exercise below offers, for the diversion of the interested reader, some less trivial cases. (We will not use the results of this exercise.)

**Exercise 7.** (i) Find an inequality in elements $u, v$ and one or more additional lattice variables $z, \ldots$, which holds identically in lattices, and which, for any $u, v \in \mathrm{Conv}(\mathbb{R}^n)$, is strict for some values of the additional variables if and only if both $u$ and $v$ are nonempty.

(ii) Find an inequality in elements $u, v, w, x$ and additional variables which holds in any lattice when $u \geq v$, $w \geq x$, and which, for any such $u, v, w, x \in \mathrm{Conv}(\mathbb{R}^n)$, is strict for some values of the additional variables if and only if $u > v$ and $w > x$.

(iii) Same as (ii), but with “$u > v$ and $w > x$” replaced by “$u > v$ or $w > x$”.

(iv)-(vi) Like (i)-(iii), but for $\mathrm{Conv}(\mathbb{R}^n) \geq (0)$, and with “nonzero” in place of “nonempty” in (i).

(vii) In Lemma 6 conditions (i)(a), (ii)(a) and (iii)(a) involved 1, 2 and 4’s respectively; thus the condition that there exist three elements $z, z', z'' \in \mathrm{Conv}(\mathbb{R}^n)_{\geq (0)}$ such that

\[
(((u \lor z'') \land z') \lor z) \land w > (((v \lor z'') \land z') \lor z) \land w
\]

was skipped. Show by example that this condition is not equivalent to (iii)(b).

(viii) Suppose $u \geq v$ and $w$ are elements of $\mathrm{Conv}(\mathbb{R}^n)$, and consider the conditions dual to (i)(a)-(iii)(a) of Lemma 6 and to the “skipped” condition:

(a) There exists $z \in \mathrm{Conv}(\mathbb{R}^n)$ such that $(u \land z) \lor w > (v \land z) \lor w$.

(a') There exist $z, z' \in \mathrm{Conv}(\mathbb{R}^n)$ such that $((u \lor z') \land z) \lor w > ((v \lor z') \land z) \lor w$.

(a'') There exist $z, z', z'' \in \mathrm{Conv}(\mathbb{R}^n)$ such that

\[
(((u \lor z'') \lor z') \land z) \lor w > (((v \lor z'') \lor z') \land z) \lor w.
\]
4. Identities distinguishing our chain of lattices.

Given \( n > 0 \), let us write \(((D_n) \vee z) \land y_1 \land y_2\) for the equation in \( n+3 \) variables \( x, y_1, \ldots, y_{n+1}, z \) obtained by applying the operator \(((\_ \vee z) \land y_1 \land y_2)\) to both sides of the relation \( D_n \). We can now prove

**Theorem 8.** For each positive integer \( n \), \( \text{Conv}(\mathbb{R}^n) \) satisfies \(((D_n) \vee z) \land y_1 \land y_2\) but not \( D_n \), while \( \text{Conv}(\mathbb{R}^n)_{>0} \) satisfies \( D_n \) but, if \( n > 1 \), not \(((D_{n-1}) \vee z) \land y_1 \land y_2\).

Hence every inclusion in \( \mathbb{R} \) is strict except the second. Equality holds at that step.

**Proof.** To see that equality holds at the second inclusion of \( \mathbb{R} \), note that both \( \text{Conv}(\mathbb{R}^1) \) and \( \text{Conv}(\mathbb{R}^1)_{>0} \) are nontrivial lattices, which by Lemma 3 satisfy \( D_1 \), the distributive identity, and that the variety of distributive lattices is known to have no proper nontrivial subvarieties.

The positive assertions of the first paragraph of the theorem follow from Lemma 3 combined, in the case of the first of these results, with the observations of the first two paragraphs of (a) (Those observations are equivalent to the contrapositive of the easy implications (i)(a) \( \implies \) (ii)(a) \( \implies \) (ii)(b) of Lemma 3). It remains to give examples showing the negative assertions.

To see that \( \text{Conv}(\mathbb{R}^n) \) does not satisfy \( D_n \), let \( q_1, \ldots, q_{n+1} \) be the vertices of an \( n \)-simplex in \( \mathbb{R}^n \) and \( p \) an interior point of this simplex, and take for the \( y_i \) and \( x \) the singletons \( \{q_i\} \) and \( \{p\} \) respectively. Then we see that the left-hand side of \( D_n(x, y_1, \ldots, y_{n+1}) \) gives \( x \), while all the joinands on the right are empty, hence so is the right-hand side itself.

To show that \( \text{Conv}(\mathbb{R}^n)_{>0} \) does not satisfy \(((D_{n-1}) \vee z) \land y_1 \land y_2\) when \( n > 1 \), let \( P \subseteq \mathbb{R}^n \) be a hyperplane not passing through 0. The idea will be to mimic the preceding example within \( P \), then replace the resulting singleton sets with the line-segments connecting them with 0, slightly enlarge \( y_1 \), so that it has nonzero intersection with \( y_2 \), and finally apply part (i) of Lemma 3.

So let \( q_1, \ldots, q_n \) be the vertices of an \((n-1)\)-simplex in \( P \), and \( p \) a point in the relative interior of that simplex, and let \( x \) and the \( y_i \) be the line segments \( \text{c.h.}(0, p) \) and \( \text{c.h.}(0, q_i) \) respectively, except for \( y_1 \), which we take to be \( \text{c.h.}(0, q_1, q_2/2) \). Note that all of these convex sets lie in the closed half-space \( H \) bounded by \( P \) and containing 0; hence the intersection with \( P \) of any lattice expression in these convex sets can be computed as the corresponding lattice expression in their intersections with \( P \). We see that these intersections are a configuration of the form given in the preceding example, except that the dimension is lower by 1 (note that
the point “$q_2/2$” in our definition of $y_1$, not lying in $P$, does not affect the intersection $y_1 \cap P$. Hence when we evaluate the two sides of $D_{n-1}$ at these elements, the left-hand side intersects $P$ in the point $p$, and is thus the whole line-segment $x$, while the right-hand side does not meet $P$, hence is a proper subsegment of $x$.

We can now deduce the failure of $((D_{n-1}) \lor z) \land y_1 \land y_2$ from the implication (i)(b) $\implies$ (i)(a) of Lemma [3] using for $u$ and $v$ respectively the left and right sides of $D_{n-1}$, and for $w$ the set $y_1 \land y_2$. To see that (i)(b) holds for these sets, note that $w$ is the line-segment c.h. $(0, q_2/2)$. Since this lies in a different line through the origin from $p$, the ray drawn from any nonzero point $q$ of that segment through $p$ does not meet the line-segment $x$ in any point other than $p$, so in particular, it does not contain any point of $v$, the right-hand side of $D_{n-1}$, which we saw was a proper subset of $x$.

We remark that by alternately inserting joins and meets with more and more variables $z^{(m)}$ into “$(D_n) \ldots \land (y_1 \land y_2)$” one can get identities that, formally, are successively stronger (though still all implied by $D_n$), so that the statements that a lattice does not satisfy these identities become successively weaker. Thus, as a stronger version of the above theorem, we could have stated that $\text{Conv}(\mathbb{R}^n)$ satisfies such identities with arbitrarily long strings of inserted terms, while $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ fails to satisfy the particular one given above. But for simplicity, I used just one identity to distinguish the properties of these lattices.

The argument at the beginning of the preceding section showing that $\text{Conv}(\mathbb{R}^n)$ satisfies $((D_n) \lor z) \land y_1 \land y_2$ also clearly shows that it satisfies the formally stronger identity

$$((D_n) \lor z) \land (\bigvee_{i,j: i \neq j} y_i \land y_j).$$

With a little additional work one can get the still stronger identity:

$$((D_n) \lor z) \land \bigwedge_i (\bigvee_{j \neq i} y_j).$$

(Idea: If $\bigwedge_i (\bigvee_{j \neq i} y_j)$ is nonempty, use a point thereof as the $p_0$ in the second sentence of Carathéodory’s Theorem.)

We remark that the fact that $\text{Conv}(\mathbb{R}^n)$ satisfies the identity $D_{n+1}$ says that for any $n+2$ convex sets $y_i$, the union of the $(n+1)$-fold joins $\bigvee_{j \neq i} y_j$ ($i = 1, \ldots, n+2$) is itself convex, while 11 says essentially that the same holds for the union of the $n$-fold joins of $n+1$ convex sets, if those unions have at least one point in common.

5. Dual $n$-distributivity.

Huhn showed not only that $\text{Conv}(\mathbb{R}^n)$ satisfies $D_{n+1}$, but also that it satisfies the dual of that identity. Let us write the dual of the relation $D_n(x, y_1, \ldots, y_{n+1})$ as

$$D_n^\text{op}(x, y_1, \ldots, y_{n+1}) : \quad x \lor (\bigwedge_i y_i) = \bigwedge_i (x \lor \bigwedge_{j \neq i} y_j).$$

Just as in $D_n$ the direction $\geq$ is automatic, so $\leq$ is automatic in $D_n^\text{op}$.

Below, we will strengthen Huhn’s result that $D_{n+1}^\text{op}$ holds identically in $\text{Conv}(\mathbb{R}^n)$ by showing that it holds in the larger lattice $\text{Conv}(\mathbb{R}^{n+1})_{\geq \{0\}}$, and will again use
the method of \[8\] to manufacture a related identity which holds in \(\text{Conv}(\mathbb{R}^n)\) but not in \(\text{Conv}(\mathbb{R}^{n+1})_{\geq 0}\). Like Huhn, we start with

**Helly’s Theorem** ([8], p.391). Let \(n \geq 0\). If a finite family of convex subsets of \(\mathbb{R}^n\) has the property that every \(n+1\) of them have nonempty intersection, then the whole family has nonempty intersection. \(\square\)

We will also use the following observation.

**Lemma 9.** Let \(V\) be any real vector space. Given a convex set \(x\) and a point \(p\) in \(V\), there exists a convex set \(w\) in \(V\) such that for every nonempty convex set \(y\), one has \(p \in x \lor y\) if and only if \(y\) has nonempty intersection with \(w\).

**Proof.** It is straightforward to check that a set \(w\) with the required property (in fact, the unique such set) is the cone consisting of the union of all rays from \(p\) which meet the central reflection of \(x\) through \(p\). (If \(p \in x\) then \(w = V\); in visualizing the contrary case, it is convenient to assume without loss of generality that \(p = 0\).) \(\square\)

We can now show that certain sorts of families of convex sets satisfy the relation \(D^{op}_n\), from which we will deduce our identities.

**Lemma 10.** Let \(n\) be a natural number, and \(x, y_1, \ldots, y_{n+1}\) be \(n+2\) elements of \(\text{Conv}(\mathbb{R}^n)\) such that \(y_1 \land \cdots \land y_{n+1} \neq \emptyset\). Then \(D^{op}_n(x, y_1, \ldots, y_{n+1})\) holds.

**Proof.** Given \(p \in \bigwedge_i (x \lor \bigwedge_{j \neq i} y_j)\) we need to show that

\[p \in x \lor (\bigwedge_i y_i)\] \hspace{1cm} (9)

Let \(w\) be the set determined by \(x\) and \(p\) as in Lemma 9. Since by 9, \(p\) lies in each of the sets \(x \lor \bigwedge_{j \neq i} y_j\), our choice of \(w\) shows that for each \(i\), \((\bigwedge_{j \neq i} y_j) \lor w\) is nonempty. These conditions together with the nonemptiness of \(y_1 \land \cdots \land y_{n+1}\) allow us to apply Helly’s Theorem to the \(n+2\) convex sets \(y_1, \ldots, y_{n+1}, w\), and conclude that \((\bigwedge_i y_i) \land w\) is nonempty, which by choice of \(w\) is equivalent to 10. \(\square\)

Now for each positive integer \(n\), let \(((D^{op}_n) \land z') \lor z) \land (\bigwedge_i y_i)\) denote the relation in variables \(x, y_1, \ldots, y_{n+1}, z, z'\) obtained by applying the operation \(((\neg) \land z') \lor z) \land (\bigwedge_i y_i)\) to both sides of 8. Then we have

**Theorem 11** (cf. Huhn [10]). For each positive integer \(n\), \(\text{Conv}(\mathbb{R}^n)\) satisfies the identity \(((D^{op}_n) \land z') \lor z) \land (\bigwedge_i y_i)\) but not \(D^{op}_n\), while \(\text{Conv}(\mathbb{R}^n)_{\geq 0}\) satisfies the identity \(D^{op}_n\) but, if \(n > 1\), not \(((D^{op}_{n-1}) \land z') \lor z) \land (\bigwedge_i y_i)\).

**Proof.** The positive assertions follow from Lemma 10. (Again, the reasoning that obtains the first identity from that lemma can be considered, formally, an application of the contrapositive of an easy direction in one of our lemmas, in this case the implication (ii)(a) \(\implies\) (iii)(b) of Lemma 8.)

To get a counterexample to \(D^{op}_n\) in \(\text{Conv}(\mathbb{R}^n)\), take for \(x\) an \(n\)-simplex in \(\mathbb{R}^n\), let \(p\) be a point outside \(x\), let \(x'\) be the central reflection of \(x\) through \(p\), and
let $y_1, \ldots, y_{n+1}$ be the $(n-1)$-faces of $x'$. Then $\bigwedge_i y_i$ is empty, so the left-hand side of (8) is just $x$, while each of the intersections $\bigwedge_{i \neq j} y_j$ is a nonempty subset of $x'$, so the right-hand side of (8) contains $p$.

As in the proof of Theorem 8, we begin the counterexample to our more elaborate identity in $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ by taking a copy of our preceding example, for the next lower dimension, in a hyperplane $P \subseteq \mathbb{R}^n$ not containing 0. Let us write $p_0$ and $x_0$, $x'_0$, $y_0$, $\ldots$, $y_{n0}$ for the point and family of convex subsets of $P$ so obtained. Let us also write $q_0$ for the vertex of $x'_0$ opposite to the face $y_{10}$.

To beef these sets up to the desired members of $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$, we now take

$$x = \text{c.h.}(x_0 \cup \{0\}), \quad y_i = \text{c.h.}(y_{i0} \cup \{0\}) \quad \text{for } i \neq 1, \quad y_1 = \text{c.h.}(y_{10} \cup \{0, q_0/2\}).$$

For the same reason as in the proof of Theorem 8, the operation of intersecting with $P$ commutes with lattice operations on these convex sets; hence when $D_{n-1}^{op}$ is evaluated at the above arguments, the left-hand side meets the plane $P$ only in the set $x_0$, while the right-hand side will also contain the point $p_0$. Now observe that $\bigwedge_i y_i$ will be the line-segment $\text{c.h.}(0, q_0/2)$. Letting $p = p_0$, $q = q_0/2$, we see that these lie on different lines through 0, and deduce from the implication (ii)(b) $\implies$ (ii)(a) of Lemma 8 that $z$ and $z'$ can be chosen so that the required inequality holds.

6. Encore!

Carathéodory’s and Helly’s Theorems are two members of a well-known triad of results on convex sets in $\mathbb{R}^n$. The third is

Radon’s Theorem (E, p.391). Let $n \geq 0$. Given a natural number $n$, and $n+2$ points $p_1, \ldots, p_{n+2}$ in $\mathbb{R}^n$, there exists a partition of $\{1, \ldots, n+2\}$ into subsets $I_1$ and $I_2$ such that $\text{c.h.}(\{p_i \mid i \in I_1\}) \cap \text{c.h.}(\{p_i \mid i \in I_2\}) \neq \emptyset$.

Can we turn this, too, into an identity for lattices of convex sets?

Yes. First let’s get rid of reference to points: Clearly an equivalent statement is “Given nonempty convex sets $y_1, \ldots, y_{n+2}$ in $\mathbb{R}^n$, there exists a partition of $\{1, \ldots, n+2\}$ into subsets $I_1$ and $I_2$ such that $(\bigvee_{i \in I_1} y_i) \land (\bigvee_{i \in I_2} y_i) \neq \emptyset$.” Next, the conclusion that such a partition exists can be condensed into the single inequality $\bigwedge_{I_1, I_2} ((\bigvee_{i \in I_1} y_i) \land (\bigvee_{i \in I_2} y_i)) \neq \emptyset$, where the outer join is over the $2^{n+2} - 2$ partitions of $\{1, \ldots, n+2\}$ into two nonempty subsets. In the proof of the theorem below, Lemma 8 will be used to turn the above implication between nonemptiness statements into a lattice relation.

In that theorem, I call an inequality $a \leq b$ that always holds an “identity”, since it can be rewritten as an equation $a = a \land b$.

**Theorem 12.** For every natural number $n$, the identity in $n+3$ variables $x, y_1, \ldots, y_{n+2}$

$$\bigwedge_i (x \lor y_i) \leq x \lor \bigvee_{I_1, I_2} ((\bigvee_{i \in I_1} y_i) \land (\bigvee_{i \in I_2} y_i)), \quad (11)$$
Proof. We shall first prove (11) in the simpler case of Conv(\(\mathbb{R}^n\)) with a (\(n+1\))-simplex \(\{x_1,\ldots,x_{n+2}\}\) in \(\mathbb{R}^{n+1}\). To show that (11) does not hold in Conv(\(\mathbb{R}^n\)) with a (\(n+1\))-simplex \(\{x_1,\ldots,x_{n+2}\}\), we must show that \(p\) belongs to the left-hand side, while \(p\) does not majorize the right-hand side of (11), as required.

Given \(x, y_1, \ldots, y_{n+2} \in \text{Conv}(\mathbb{R}^n)\) and a point \(p\) belonging to the left-hand side of (11), we must show that \(p\) also belongs to the right-hand side. Let us choose \(w\) as in Lemma 9 for the given \(x\) and \(p\). The assumption that \(p\) belongs to the left-hand side of (11) says that it belongs to each of the meetands of that expression, which by choice of \(w\) means that \(w \land y_i\) is nonempty for all \(i\). Hence Radon’s Theorem applied to those \(n+2\) sets says that for some \(I_1, I_2\) partitioning \(\{1, \ldots, n+2\}\), we have \(\emptyset \neq (\bigvee_{I_1} w \land y_i) \land (\bigvee_{I_2} w \land y_i)\). The latter set is contained in \(w \land (\bigvee_{I_1} y_i) \land (\bigvee_{I_2} y_i)\), and the statement that this is nonempty now translates back to say that \(x \lor ((\bigvee_{I_1} y_i) \land (\bigvee_{I_2} y_i))\) contains \(p\), whence \(p\) belongs to the right-hand side of (11), as required.

If we are given \(x, y_1, \ldots, y_{n+2} \in \text{Conv}(\mathbb{R}^{n+1})\), and a point \(p\) on the left-hand side of (11), we begin in the same way, translating the hypothesis and desired conclusion to the same statements about the sets \(w \land y_i\) (though the convex set \(w\) will not in general belong to Conv(\(\mathbb{R}^{n+1}\)), and so neither will these intersections). This time we apply Radon’s Theorem in \(n+1\) dimensions to the sets \(w \land y_i\) together with \(\{0\}\). In the partition given by that theorem, let us assume without loss of generality that \(\{0\}\) goes into the second join; thus we get a relation \(\emptyset \neq (\bigvee_{I_1} w \land y_i) \lor (\{0\} \lor \bigvee_{I_2} w \land y_i)\). Here the first join is contained in \(w\), hence so is the whole set, so that set is contained in the meet of \(w\) with the larger set \((\bigvee_{I_1} y_i) \lor (\{0\} \lor \bigvee_{I_2} y_i)\). Since all the \(y\)’s belong to Conv(\(\mathbb{R}^{n+1}\)), the joinands \(\{0\}\) are now redundant, so we have again shown that \((w \lor (\bigvee_{I_1} y_i) \lor (\bigvee_{I_2} y_i))\) is nonempty for some partition \(\{1, \ldots, n+2\} = I_1 \lor I_2\), which, as before, yields the desired conclusion by choice of \(w\).

To show that (11) does not hold in Conv(\(\mathbb{R}^{n+1}\)), we begin essentially as in the first counterexample in the proof of Theorem 11, letting \(x\) be an \((n+1)\)-simplex in \(\mathbb{R}^{n+1}\), \(p\) a point outside that simplex, and \(x’\) the central reflection of \(x\) through \(p\). This time, however, we let \(y_1, \ldots, y_{n+2}\) be singletons, whose unique elements are the vertices of \(x’\). As in the earlier example we find that one side of the identity in question (in this case the left-hand side of (11)) contains \(p\), while the other is simply \(x\), since all joinands in the “big join” on that side are empty; so the right-hand side does not majorize the left-hand side.

I have not tried to fill in this picture, as I did with the \(D_n\) and \(D_n^{op}\), by looking for related identities that would distinguish all terms of (4); but I expect that these exist.

Another observation, relevant to the whole development up to this point, which I have not followed up on because it occurred to me late in the preparation of this paper, is that if one defines Conv(\(\mathbb{R}^n\), cone) \(\subseteq\) Conv(\(\mathbb{R}^n\)) to be the sublattice
consisting of those elements which are unions of rays through 0, then Conv$(\mathbb{R}^{n-1})$ embeds naturally in Conv$(\mathbb{R}^n, \text{cone})$, yielding an extension of the chain of varieties of Lemma 1. It would be interesting to know whether for \( n > 1 \) the varieties so interpolated are distinct from those that precede and follow them.

7. The sublattice of compact convex sets.

From Carathéodory’s Theorem, we see that

\[
\text{For any compact subset } S \subseteq \mathbb{R}^n, \quad \text{c.h.}(S) \text{ is also compact.} \tag{12}
\]

Hence the join in Conv$(\mathbb{R}^n)$ of two compact subsets is compact, hence the set of compact convex subsets of $\mathbb{R}^n$ (often called “convex bodies” in the literature, e.g., [2], [11] §3.1, [19] §12) is a sublattice Conv$(\mathbb{R}^n, \text{cpct}) \subseteq \text{Conv}(\mathbb{R}^n)$. An obvious question is how the identities of this sublattice compare with those of Conv$(\mathbb{R}^n)$; i.e., whether it satisfies any identities that the larger lattice does not. Huhn [11, proof of Lemma 3.1] answered this question in the negative, by showing that the still smaller lattice of polytopes (convex hulls of finite sets) does not. Let me give a slightly different proof of the same result. Huhn used the fact that an intersection of polytopes is a polytope, but the next result is applicable to a finitary closure operator that need not have the property that an intersection of closures of finite sets is again one. (The meaning of “finitary” was recalled in the paragraph preceding Lemma 3, the notation \( V(\ldots) \) used below was defined in §1.)

**Proposition 13.** Let $\text{cl}$ be a finitary closure operator on a set $X$, and let $L$ be the lattice of subsets of $X$ closed under $\text{cl}$. Then every lattice relation satisfied by all families of elements of $L$ that are closures of finite subsets of $X$ is an identity of $L$. In particular, if $L'$ is any sublattice of $L$ which contains all closures of finite subsets of $X$, then $V(L') = V(L)$.

**Proof.** Let us topologize the power set $2^X$ by taking as a basis of open sets the sets

\[
U(A, B) = \{ Y \in 2^X \mid A \subseteq Y \subseteq B \},
\]

where $A$ ranges over the finite subsets of $X$, and $B$ over arbitrary subsets. This is stronger than the usual power-set topology, which only uses the sets of the above form with $B$ cofinite. Thus, our topology is Hausdorff, though not in general compact; hence its restriction to $L$ is also Hausdorff, with basis of open sets given by the sets $U_L(A, B) = U(A, B) \cap L$. (Of course, $U_L(A, B)$ is nonempty only when $\text{cl}(A) \subseteq B$.)

We claim that under this topology, the lattice operations of $L$ are continuous, and the closures of finite subsets of $X$ are dense in $L$. This will imply that any lattice identity holding on that dense subset must hold on all of $L$, from which the final conclusion will clearly follow.

To see that closures of finite sets are dense, note that every nonempty basic set $U_L(A, B)$ contains the element $\text{cl}(A)$, which is such a set.

The continuity of the meet operation is also straightforward: If $x, y \in L$ are such that $x \wedge y$, i.e., $x \cap y$, lies in $U_L(A, B)$, then $U_L(A, x \cup B)$ and $U_L(A, y \cup B)$ are
neighborhoods of $x$ and $y$ respectively such that the intersection of any member of the first neighborhood and any member of the second lies in $U_L(A,B)$.

Finally, suppose $x, y \in L$ are such that $x \lor y$, i.e., $\text{cl}(x \cup y)$, lies in $U_L(A,B)$. Then the finite set $A$ is contained in $\text{cl}(x \cup y)$, so by finitariness of $\text{cl}$, there is a finite subset of $x \cup y$ whose closure contains all elements of $A$; let us write this subset as $A_x \cup A_y$, where $A_x \subseteq x$ and $A_y \subseteq y$. Then $U_L(A_x,x)$ and $U_L(A_y,y)$ will be neighborhoods of $x$ and $y$ respectively such that the join of any member of the first neighborhood and any member of the second is a member of $U_L(A,B)$. (In the power-set topology, $\lor$ is generally discontinuous; this is why we needed a different topology.)

For $X = \mathbb{R}^n$ and $\text{cl} = \text{c.h.}$, the $L$ of the above proposition is $\text{Conv}(\mathbb{R}^n)$. Since the convex hull of a finite set is compact, we can apply the last sentence of the proposition with $L' = \text{Conv}(\mathbb{R}^n, \text{cpct})$, getting the first statement of the next theorem. Taking $\text{cl} = \text{c.h.}\{\varnothing \cup -\}$ we similarly get the second.

**Theorem 14 (Huhn).** For every natural number $n$, $\mathbf{V}(\text{Conv}(\mathbb{R}^n)) = \mathbf{V}(\text{Conv}(\mathbb{R}^n, \text{cpct}))$, and $\mathbf{V}(\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}) = \mathbf{V}(\text{Conv}(\mathbb{R}^n, \text{cpct})_{\geq \{0\}})$.

However, $\text{Conv}(\mathbb{R}^n, \text{cpct})$ is known also to have interesting elementary properties not possessed by $\text{Conv}(\mathbb{R}^n)$. Let us recall that an extremal point of a convex set means a point which is not in the convex hull of any two other points of the set, and the following result ([11] p.276).

**Theorem (Minkowski).** Every compact convex subset of $\mathbb{R}^n$ is the convex hull of its set of extremal points.

Recall also that a lattice $L$ is called join semidistributive if for all $x, y_1, y_2 \in L$ one has
$$x \lor y_1 = x \lor y_2 \implies x \lor y_1 = x \lor (y_1 \land y_2).$$

**Lemma 15 (= [11] Theorem 3.4, generalizing [5] Theorem 15).** For every positive integer $n$, $\text{Conv}(\mathbb{R}^n, \text{cpct})$ is join semidistributive.

**Proof.** Every extremal point of a join $x \lor y$ must belong to $x$ or to $y$, since by definition it cannot arise as a convex combination of other points of $x \lor y$, hence if $x \lor y_1 = x \lor y_2$ as in the hypothesis of (13), extremal points of this set that do not belong to $x$ must belong to $y_1$, and likewise to $y_2$. Thus every extremal point of $x \lor y_1 = x \lor y_2$ belongs to $x \lor (y_1 \land y_2)$; hence by Minkowski’s Theorem the convex hull of the latter set, $x \lor (y_1 \land y_2)$, contains the former set. The reverse inclusion is trivial.

On the other hand, for $n \geq 2$, $\text{Conv}(\mathbb{R}^n)$ is not join semidistributive; indeed, the next result will show the failure of successively weaker properties, beginning with join-semidistributivity, as $n$ increases. Following Geyer [5], let us say that a lattice $L$ is $n$-join semidistributive for a positive integer $n$ if for all $x, y_1, \ldots, y_{n+1} \in L$ one has
$$x \lor y_1 = \cdots = x \lor y_{n+1} \implies x \lor y_1 = x \lor (\bigvee_{i,j \neq j} y_i \land y_j).$$
Thus, join semidistributivity is the $n = 1$ case. One defines \emph{n-meet semidistributivity} dually.

The $(n+4)$-element lattice of height 2, $M_{n+2}$, with least element 0, greatest element 1, and $n+2$ incomparable elements $y_1, \ldots, y_{n+2}$, is neither \emph{n-join semidistributive} nor \emph{n-meet semidistributive}, as may be seen by putting $y_{n+2}$ in the role of $x$ in \eqref{16}, and in the dual statement. We shall now see that there are several sorts of sublattices with that structure within the lattices Conv($\mathbb{R}^n$). The “open bounded” case of the next result was shown to me by D. Wasserman.

**Lemma 16** (Wasserman and Bergman). For every $n > 1$, Conv($\mathbb{R}^n$) contains copies of $M_{n+1}$ consisting of open bounded sets and copies consisting of closed unbounded sets, in both cases with least element $\emptyset$; and Conv($\mathbb{R}^n$)$_{\geq 0}$ contains copies of $M_{n+1}$ consisting of bounded sets, with least element $\{0\}$. Also, Conv($\mathbb{R}^2$)$_{\geq 0}$ (and hence Conv($\mathbb{R}^n$)$_{\geq 0}$) for all $n > 1$) contains copies of $M_c$, the height-2 lattice of continuum cardinality (and hence contains copies of its sublattices $M_m$ for all natural numbers $m$) consisting of vector subspaces, with least element $\{0\}$.

In particular, for $n > 1$ neither Conv($\mathbb{R}^n$) nor Conv($\mathbb{R}^n$)$_{\geq 0}$ is \emph{N-join} or \emph{N-meet semidistributive} for any $N$, and the sublattices of open bounded sets in Conv($\mathbb{R}^n$) and of bounded sets in Conv($\mathbb{R}^n$)$_{\geq 0}$ are not \emph{(n−1)-join} or \emph{(n−1)-meet semidistributive}.

**Proof.** To get a copy of $M_{n+1}$ consisting of open bounded sets, start with any $n$-simplex $x$, let its faces be $x_1, \ldots, x_{n+1}$, choose an interior point $p$ of $x$, and for each $i$ let $y_i$ be the interior of the $n$-simplex $\{p\} \lor x_i$ (or any open convex subset of that $n$-simplex which has the whole face $x_i$ in its closure). We see that the join of any two of the $y$’s will have in its closure two faces of $x$, hence all vertices of $x$, hence its closure must be $x$, hence being itself open and convex, it must be the interior of $x$. On the other hand, the pairwise intersections of the $y_i$ are all empty. Hence the lattice generated by these sets is isomorphic to $M_{n+1}$.

For the closed unbounded example with least element again $\emptyset$, extend each of the sets in the preceding example to an infinite cone with apex $p$, displace each of these cones away from $p$ (say by translating it by the vector from the opposite vertex of $x$ to $p$), and take their closures. Then every pairwise join is seen to be the whole of $\mathbb{R}^n$, while every pairwise meet is again empty.

For the bounded example in Conv($\mathbb{R}^n$)$_{\geq 0}$, take the first example above, assuming $p = 0$, and use as our new $y_i$ the union of the $y_i$ of that example with $\{0\}$. (Thus, 0 will be the unique boundary-point belonging to each of these sets.)

Finally, an $M_c$ in Conv($\mathbb{R}^2$)$_{\geq 0}$ is given by the set of all lines through 0. \hfill \Box

There are also cases where we can show the failure of \emph{m-join semidistributivity} in a natural lattice of convex sets, but where that lattice probably does not contain a copy of $M_{m+2}$. To get such an example in the lattice of convex open subsets of $\mathbb{R}^2$ containing 0, for any $m > 1$, take $m+2$ distinct lines through 0 and “thicken” these to open sets $x, y_1, \ldots, y_{m+1}$ of width 1. The lattice that these generate will not be $M_{m+2}$, but clearly fails to satisfy \eqref{16}. We can get such examples for bounded open sets in $\mathbb{R}^2$, though in this case they fail to have a common point: Fix
a triangle $T$, and let $x$ be the interior of any triangle lying inside $T$ and sharing one edge with $T$, but not the opposite vertex $p$. Then take $m+1$ “small narrow” triangles inside $T$ that have $p$ as a common vertex but no other point in common, and the convex hull of whose union is disjoint from $x$, and let $y_1, \ldots, y_{m+1}$ be their interiors.

The parenthetical comment in the first sentence of the proof of the Lemma shows that the shapes of the convex sets forming a copy of $M_{n+1}$ in the lattice of open bounded convex sets are not unique; but I don’t know an example where the top element of such a sublattice is not an open $n$-simplex. It would also be of interest to know whether this lattice contains copies of $M_{n+2}$, and if not, whether it is $n$-join or $n$-meet semidistributive. We shall obtain a few related results in subsequent sections.

Jónsson and Rival [13, Lemma 2.1] show that a lattice is join and meet semidistributive if and only if two auxiliary overlattices contain no isomorphic copies of any member of a certain list of 6 lattices, beginning with $M_3$. The above “small narrow triangle” construction gives, when $m = 2$, a copy of the lattice $L_4$ of their list.

Incidentally, Geyer’s concept of $n$-join semidistributivity, which we have been using, does not have any obvious relationship with Huhn’s $n$-distributivity. Although for $n = 1$ they give the conditions of join-semidistributivity and distributivity respectively, of which the latter implies the former, no such implication holds for larger $n$. For instance, the lattice $M_c$ is 2-distributive in Huhn’s sense, but it is not $N$-join semidistributive for any natural number $N$ in Geyer’s sense.

8. Open bounded sets do not satisfy additional identities.

Let us denote by $\text{Conv}(\mathbb{R}^n, \text{o.bdd.})$ the lattice of open bounded convex subsets of $\mathbb{R}^n$. We shall show that this lattice, like $\text{Conv}(\mathbb{R}^n, \text{cpt})$, satisfies the same identities as $\text{Conv}(\mathbb{R}^n)$. The idea is that compact sets can be approximated by open bounded sets containing them, from which we shall deduce that any identities of $\text{Conv}(\mathbb{R}^n, \text{o.bdd.})$ are also identities of $\text{Conv}(\mathbb{R}^n, \text{cpt})$, and so by Theorem 14 are identities of $\text{Conv}(\mathbb{R}^n)$.

To approximate compact sets by open sets, we need a different topology on $2^{\mathbb{R}^n}$ from the one used earlier; let us again describe this in a general context. If $X$ is any topological space, we may topologize $2^X$ using a basis of open sets with the same form as before,

$$U(A, B) = \{ Y \in 2^X \mid A \subseteq Y \subseteq B \},$$

but where, this time, $A$ ranges over all subsets of $X$, while $B$ is restricted to open subsets. If $X$ is Hausdorff (or even T1), we see that this family of open sets again includes those defining the power-set topology, so our topology is again Hausdorff. Note that for each $A \in 2^X$, the sets $U(B, A)$ with $A$ as first argument and with second argument containing $A$ form a neighborhood basis of $A$ in $2^X$. Thus in proving the next lemma, we shall take it as understood that to be “sufficiently close to” a
set $A$ means to contain $A$ and be contained in some specified open neighborhood $B$ of $A$ in $X$.

In the formulation of that lemma, note that the statement that a function is continuous at arguments with a given property does not simply mean that the restriction of the function to the set of such arguments is continuous, but, more, that such arguments are points of continuity of the whole function.

**Lemma 17.** If $X$ is any topological space, then the binary operation $\cup : 2^X \times 2^X \to 2^X$ is continuous in the topology described above; if $X$ is normal (i.e., if disjoint closed subsets of $X$ have disjoint open neighborhoods) then the binary operation $\cap$ is continuous at arguments given by pairs of closed sets, and if $X = \mathbb{R}^n$ with the usual topology, then the unary convex-hull operation $\text{ch} : 2^X \to 2^X$ is continuous at compact sets.

**Proof.** To show continuity of $\cup$, consider sets $A_1$, $A_2$, and an open neighborhood $B$ of $A_1 \cup A_2$ in $X$. Then we see that the union of any member of $U(A_1, B)$ and any member of $U(A_2, B)$ contains $A_1 \cup A_2$ and is contained in $B$, as required.

For the case of $\cap$, let $A_1$ and $A_2$ be closed sets, and $B$ any open neighborhood of $A_1 \cap A_2$ in $X$. Then $A_1 - B$ and $A_2 - B$ are disjoint closed sets, hence they have disjoint open neighborhoods $C_1$ and $C_2$. We see that $B \cup C_1$ and $B \cup C_2$ will be open neighborhoods of $A_1$ and $A_2$ which intersect in $B$ (by distributivity of the lattice $2^X$), and it follows that the intersection of a member of $U(A_1, B \cup C_1)$ and a member of $U(A_2, B \cup C_2)$ will belong to $U(A_1 \cap A_2, B)$, as required.

For the final assertion, let $A$ be a compact subset of $\mathbb{R}^n$, and $B$ any open neighborhood of $\text{c.h.}(A)$. By compactness of $\text{c.h.}(A)$, there is some $\epsilon > 0$ such that the set $C$ of all points having distance $< \epsilon$ from $\text{c.h.}(A)$ is contained in $B$. This set $C$ is a convex open neighborhood of $\text{c.h.}(A)$, hence $\text{c.h.}$ carries $U(A, C)$ into $U(\text{c.h.}(A), C) \subseteq U(\text{c.h.}(A), B)$, as required.

(I played with several topologies before getting the one that made the above result – in particular, continuity of intersection – easy to prove. Some of these might be preferable for other considerations of the same sort. Under the above topology, every open set $A$ is an isolated point, since $U(A, A)$ is a singleton. If one wants to approximate open sets by larger open sets, one might prefer a weaker topology in which the conditions on $Y$ in [15] are, say, strengthened to $A \subseteq Y$, $\text{cl}(Y) \subseteq B$.)

It follows from Lemma 17 that in the topology we have defined, the lattice operations of $\text{Conv}(\mathbb{R}^n)$ are continuous at arguments belonging to $\text{Conv}(\mathbb{R}^n, \text{cpt})$. Moreover, $\text{Conv}(\mathbb{R}^n, \text{cpt})$ lies in the closure of $\text{Conv}(\mathbb{R}^n, \text{o.bdd.})$, since every compact convex set $A$ is the limit in this topology, as $\epsilon \to 0$, of the open convex set of points at distance $< \epsilon$ from $A$. Hence any lattice identities holding in $\text{Conv}(\mathbb{R}^n, \text{o.bdd.})$ also hold in $\text{Conv}(\mathbb{R}^n, \text{cpt})$. The same considerations apply to the pair of lattices $\text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq 0}$ and $\text{Conv}(\mathbb{R}^n, \text{cpt})_{\geq 0}$. In view of Theorem 13, these observations give us

**Theorem 18.** For every natural number $n$, $\mathbf{V}(\text{Conv}(\mathbb{R}^n)) = \mathbf{V}(\text{Conv}(\mathbb{R}^n, \text{o.bdd.}))$, and $\mathbf{V}(\text{Conv}(\mathbb{R}^n)_{\geq 0}) = \mathbf{V}(\text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq 0})$. \[\square\]
Let us note that there is an order-preserving bijection between the elements of Conv($\mathbb{R}^n$, o.bdd.) and the compact convex subsets of $\mathbb{R}^n$ having nonempty interior, given by the operation of topological closure, with inverse given by topological interior. This is not, however, an isomorphism between sublattices of Conv($\mathbb{R}^n$), because the class of compact convex sets with nonempty interior is not closed under intersection. (E.g., consider two adjacent closed polygons in $\mathbb{R}^2$. Nor can we get around this problem by going to a homomorphic image of Conv($\mathbb{R}^n$, cpct) where sets without interior are identified with $\emptyset$, since the join of two such sets can have nonempty interior.)

On the other hand, the set of compact convex sets which are neighborhoods of 0 is a sublattice of Conv($\mathbb{R}^n$, cpct)≥{0}, and the above correspondence gives us an isomorphism between it and Conv($\mathbb{R}^n$, o.bdd.)≥{0}; so we can apply results about Conv($\mathbb{R}^n$, cpct)≥{0} to Conv($\mathbb{R}^n$, o.bdd.)≥{0}. (More generally, any sublattice of Conv($\mathbb{R}^n$, o.bdd.) whose members have a common point $p$ is contained in Conv($\mathbb{R}^n$, o.bdd.)≥{p} ≅ Conv($\mathbb{R}^n$, o.bdd.)≥{0}, and so can be studied in the same fashion.) Thus, despite the examples of Lemma 16, we have

**Corollary 19** (to Lemma 15). Conv($\mathbb{R}^n$, o.bdd.)≥{0} is join semidistributive. Hence, every counterexample to join semidistributivity in Conv($\mathbb{R}^n$, o.bdd.) has the property that the intersection of the three sets involved is empty.

Since we are considering elementary properties in which Conv($\mathbb{R}^n$) and Conv($\mathbb{R}^n$, o.bdd.) agree or differ, we should note the obvious difference, that the former is atomistic (every element is a possibly infinite join of atoms), while the latter has no atoms. Cf. [2] and papers referred to there, in which lattices of convex sets and related structures are characterized in terms of properties of their atoms, and also [1].

We noted earlier that lattices Conv($\mathbb{R}^n$)≥S for nonempty $S$ satisfy all identities holding in Conv($\mathbb{R}^n$)≥{0}. Let us end this section by using Theorem 18 to show that for bounded $S$, the converse is also true.

**Corollary 20** (to Theorem 18 and proof of Theorem 14). For every natural number $n$ and every bounded set $S \subseteq \mathbb{R}^n$,

$$V(\text{Conv}(\mathbb{R}^n)_{\geq S}) = V(\text{Conv}(\mathbb{R}^n, \text{cpct})_{\geq S}) = V(\text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq S}) = V(\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}).$$

**Proof.** By a translation, we can assume without loss of generality that 0 ∈ $S$; thus the last of the above lattices contains all the others, so letting $f(x_1, \ldots, x_m) = g(x_1, \ldots, x_m)$ be any identity not satisfied there, it suffices to prove that it is not satisfied in any of the other lattices.

Now Theorem 18 shows that $f = g$ is not an identity of Conv($\mathbb{R}^n$, o.bdd.)≥{0}, hence we can choose open bounded convex sets $x_1, \ldots, x_m$ containing 0 which do not satisfy it. The intersection of these sets is a neighborhood of the origin, and dilating the $x_i$ by a large enough real constant, we can assume without loss of generality that this neighborhood contains $S$. Hence $f = g$ is also not an identity of Conv($\mathbb{R}^n$, o.bdd.)≥S, hence not an identity of Conv($\mathbb{R}^n$)≥S either.

The case of Conv($\mathbb{R}^n$, cpct)≥S is similar. Again take $x_1, \ldots, x_m \in \text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq \{0\}}$ not satisfying $f = g$. We saw in the proof of Theorem 14 that if we approximate
x_1, \ldots, x_m \text{ closely enough from below in the topology of that proof by compact convex subsets } y_1, \ldots, y_m, \text{ these approximating sets also fail to satisfy that identity.}

Since the intersection of the x_i is a neighborhood of 0, it contains an n-simplex with 0 in its interior; so we can take all the y_i to contain the finitely many vertices of that simplex, hence to be neighborhoods of 0. As before, we may now dilate them so that they all contain $S$, getting the required result. □

9. The possibility of surface phenomena.

The technique by which we just proved Corollary 20 can be inverted to show that if $T \subseteq \mathbb{R}^n$ is any convex set with nonempty interior, then the lattice of convex subsets contained in $T$, and its sublattices of compact convex subsets of $T$ and open bounded convex subsets of $T$, satisfy the same identities as $\text{Conv}(\mathbb{R}^n)$. Namely, given any identity not holding in $\text{Conv}(\mathbb{R}^n)$, we already know that we can find elements $x_1, \ldots, x_m$ not satisfying it in $\text{Conv}(\mathbb{R}^n, \text{cpct})$. Assuming without loss of generality that 0 lies in the interior of $T$, we can shrink $x_1, \ldots, x_m$ by a constant, so that they lie in $T$ as well. A similar argument shows that the lattice of convex subsets of $T$ which contain a specified point of the interior of $T$ satisfies the same identities as $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$.

However, if we specify two convex sets $S \subseteq T$, say with $S$ compact and $T$ open, and look at the interval $[S, T] = \{x \in \text{Conv}(\mathbb{R}^n) \mid S \subseteq x \subseteq T\}$, it is not clear whether, for some choices of $S$ and $T$, this may satisfy more identities than hold in $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$. (Picturing $S$ and $T$ as “very close”, e.g., the closed ball of radius 1 and the open ball of radius $1 + \epsilon$, explains, I hope, the title of this section.)

Let us relax our assumptions on $S$ and $T$ for a moment and look at a more extreme example. If we take for $S$ the open unit ball and for $T$ the closed unit ball in $\mathbb{R}^n$, then every set $x$ with $S \subseteq x \subseteq T$ is convex, so in this case $[S, T]$ may be identified with the lattice of all subsets of the unit sphere, which is distributive, though we have seen that $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ is not even $(n-1)$-distributive.

In the case where $S$ is compact and $T$ open, however, things cannot go that far:

**Lemma 21.** Let $S \subseteq T$ be convex subsets of $\mathbb{R}^n$. If $S$ is compact and $T$ is open and nonempty, or more generally, if some hyperplane $P \subseteq \mathbb{R}^n$ disjoint from $S$ intersects $T$ in a set with nonempty relative interior (i.e., is such that $P \cap T$ is $(n-1)$-dimensional), then $\text{V}([S, T]) \supseteq \text{V}(\text{Conv}(\mathbb{R}^{n-1}))$.

**Proof.** Clearly, a pair $S \subseteq T$ satisfying the first hypothesis satisfies the second, so let us assume the latter. Let $H$ be the closed half-space of $\mathbb{R}^n$ bounded by $P$ that contains $S$ (or if $S$ is empty, either of the closed half-spaces bounded by $P$), and consider the sublattice $[S, H \cap T] \subseteq [S, T]$. The operation of intersecting with $P$ can be seen to give a lattice homomorphism $[S, H \cap T] \rightarrow [0, P \cap T]$, and this is surjective, since it has the set-theoretic section $x \mapsto S \cap x$. Since $P \cap T$ has nonempty interior in $P$, the observation in the first paragraph of this section shows that $[0, P \cap T]$ satisfies precisely the identities of $\text{Conv}(\mathbb{R}^{n-1})$. Hence $[S, H \cap T]$,
since it maps homomorphically onto $[\emptyset, P \cap T]$, cannot satisfy any identities not satisfied by $\text{Conv}(\mathbb{R}^{n-1})$, so neither can the larger lattice $[S, T]$. □

It is not evident whether, for $S$ compact and $T$ open, $V([S,T])$ can ever be strictly smaller than $V(\text{Conv}(\mathbb{R}^n)_{\geq \{0\}})$; nor, for that matter, whether it can ever fail to be strictly smaller, if $S$ has nonempty interior and $T$ is bounded. Another interval $[S,T] \subseteq \text{Conv}(\mathbb{R}^n)$ whose identities it would be interesting to investigate is given by letting $S = \{0\}$ and $T$ be a closed half-space with 0 on its boundary. Again these identities must lie somewhere between those of $\text{Conv}(\mathbb{R}^{n-1})$ and $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$.

In this and preceding sections we have used from time to time the fact that translations and dilations preserve convexity. More generally, if $\phi$ is any projective transformation on $n$-dimensional projective space $\mathbb{P}^n \supseteq \mathbb{R}^n$, then convex subsets of $\mathbb{R}^n$ which do not meet the hyperplane that $\phi$ sends to infinity are taken by $\phi$ to convex sets. Hence if $S \subseteq T$ are such convex sets, $\phi$ induces a lattice isomorphism $[S,T] \cong [\phi(S), \phi(T)]$. This observation might be useful in classifying varieties generated by such intervals.

Another sort of sublattice of $\text{Conv}(\mathbb{R}^n)$ that it might be interesting to investigate is that of all convex sets that are carried into themselves by a given affine map; e.g., the orthogonal projection onto a specified subspace. (If that subspace is $\{0\}$, we get $\text{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ with one additional element $\emptyset$ thrown in.)

10. Dualities.

Let us recall the definition of a concept we have referred to a couple of times in passing. A closed half-space in $\mathbb{R}^n$ means a set of the form
\[
\{ p \in \mathbb{R}^n \mid f(p) \leq \lambda \},
\]
for some nonzero linear functional $f$ on $\mathbb{R}^n$ and some real number $\lambda$. It is a standard result that every closed convex subset of $\mathbb{R}^n$ is an intersection of closed half-spaces.

The half-space $\{ p \in \mathbb{R}^n \mid f(p) \leq \lambda \}$ contains the point 0 if and only if $\lambda$ is nonnegative, hence closed convex sets containing 0 can be characterized as intersections of half-spaces $\{ p \in \mathbb{R}^n \mid f(p) \leq \lambda \}$ having $\lambda \geq 0$. Such sets can, in fact, be expressed as intersections of such half-spaces with $\lambda > 0$, since a half-space $\{ p \in \mathbb{R}^n \mid f(p) \leq 0 \}$ with $\lambda = 0$ is the intersection of all the half-spaces with the same $f$ and positive $\lambda$. But a half-space $\{ p \in \mathbb{R}^n \mid f(p) \leq 1 \}$ with $\lambda$ positive can be written as $\{ p \in \mathbb{R}^n \mid \lambda^{-1}f(p) \leq 1 \}$, or, expressing the linear functional $\lambda^{-1}f$ as the dot product with some $q \in \mathbb{R}^n$, as
\[
\{ p \in \mathbb{R}^n \mid q \cdot p \leq 1 \}.
\]

Thus, for any subset $S \subseteq \mathbb{R}^n$, if we define
\[
S^* = \{ q \in \mathbb{R}^n \mid (\forall p \in S) \ q \cdot p \leq 1 \},
\]
then $S^{**}$ will be the least closed convex set containing $S \cup \{0\}$. Moreover, we see that $S^*$ will also be a closed convex subset containing 0, which uniquely determines and is determined by $S^{**}$. Thus, the operator $^*$ gives a bijection of the family of
all closed convex sets containing 0 with itself, which is easily seen to be inclusion-
reversing. (This is an example of a Galois connection; cf. [3, §5.5] for a general
development of the concept, with many examples.) Let us call two closed convex
sets containing 0 that are related in this way dual to one another. (The dual of
a convex set is sometimes called its polar set, e.g., in [14].) Examples in \( \mathbb{R}^3 \) are a
cube and an octahedron of appropriate radii centered at the origin, and similarly a
dodecahedron and an icosahedron. The unit sphere is self-dual.

The class of closed convex subspaces of \( \mathbb{R}^n \) forms a lattice (by general properties
of Galois connections), which, like \( \text{Conv}(\mathbb{R}^n) \geq \{0\} \), has intersection as its meet
operation; but the join operations do not everywhere coincide – a consequence of
the fact that, given closed sets \( u, v \in \text{Conv}(\mathbb{R}^n) \geq \{0\} \), their join in that lattice,
c.h.(\( u \cup v \)), may not be closed, so that to get their join as closed convex sets, one
must take its topological closure. For example, let \( n = 2 \), and let \( u \) be the closed
strip \( \{(x, y) \mid -1 \leq y \leq 1\} \) and \( v \) the line segment c.h.(0, (0, 2)). Then the join of
\( u \) and \( v \) in \( \text{Conv}(\mathbb{R}^2) \geq \{0\} \) is
\[
\{(x, y) \mid -1 \leq y < 2\} \cup \{(0, 2)\},
\]
while their join in the corresponding lattice of closed convex sets is \( \{(x, y) \mid -1 \leq y \leq 2\} \). In view of this difference in operations, care is needed when using our
duality on closed convex sets to deduce results about the lattice \( \text{Conv}(\mathbb{R}^n) \geq \{0\} \).

If \( x \) and \( y \) are mutually dual closed convex sets containing 0 in \( \mathbb{R}^n \), it is not
hard to see that one of them is bounded (i.e., compact) if and only if 0 is an interior
point of the other. It follows that the class of closed bounded convex sets having 0
in their interior is self-dual; moreover, we saw at the beginning of \( \S 7 \) that the join in
\( \text{Conv}(\mathbb{R}^n) \) of two compact sets is again compact, from which it follows that unlike
the lattice of all closed convex sets containing 0, this is a sublattice of \( \text{Conv}(\mathbb{R}^n) \).
It is easy to see from the equality of the second and fourth varieties in Corollary 20
that this sublattice satisfies the same identities as \( \text{Conv}(\mathbb{R}^n) \). Hence the existence
of the anti-automorphism just noted gives us

**Theorem 22.** The class of lattice identities satisfied by \( \text{Conv}(\mathbb{R}^n) \geq \{0\} \) is self-dual,
i.e., closed under interchanging all instances of \( \lor \) and \( \land \). Equivalently, the variety
\( V(\text{Conv}(\mathbb{R}^n) \geq \{0\}) \) is closed under taking dual lattices. \( \square \)

And indeed, the identities proved for \( \text{Conv}(\mathbb{R}^n) \geq \{0\} \) in Theorems \( \S \) and \( \| \) respectively are dual to one another. This is not true of the identities proved for
\( \text{Conv}(\mathbb{R}^n) \) in those theorems, as is easily verified:

**Exercise 23.** (i) For every positive integer \( n \), show by example that \( \text{Conv}(\mathbb{R}^n) \)
does not satisfy either of the identities
\[
((D^p_n) \land z) \lor y_1 \lor y_2 \quad \text{and} \quad (((D_n) \lor z') \land z) \lor (\lor_i y_i),
\]
dual to the identities proved for that lattice in Theorems \( \S \) and \( \| \) respectively.

(ii) Show that these observations together with Theorem 22 yield an alternative
way of verifying that in each of Theorems \( \S \) and \( \| \) the identities proved for
\( \text{Conv}(\mathbb{R}^n) \) are not satisfied by \( \text{Conv}(\mathbb{R}^{n+1}) \geq \{0\}) \).
Theorem 22 also implies that Conv(\(\mathbb{R}^{n+1}_{\geq 0}\)), and hence also Conv(\(\mathbb{R}^n\)), satisfies the dual of the identity of Theorem 12, which we had not previously obtained.

Just as every closed convex set is an intersection of closed half-spaces, so every open convex set is an intersection of open half-spaces,

\[ \{ p \in \mathbb{R}^n \mid f(p) < \lambda \} \]

But here the converse is not true. Indeed, every closed half-space is also an intersection of open half-spaces, so the class of intersections of open half-spaces includes both the open and the closed convex sets. If for every subset \( S \subseteq \mathbb{R}^n \) one defines

\[ S^\circ = \{ q \in \mathbb{R}^n \mid (\forall p \in S) \, q \cdot p < 1 \} \]

one gets a duality theory for the class of \( ^\circ \)-invariant sets, i.e., sets satisfying \( x^{\circ \circ} = x \), a larger class than that covered by the preceding duality. It is not hard to see that a necessary and sufficient condition for a set \( x \in \text{Conv}(\mathbb{R}^n)_{\geq 0} \) to be \( ^\circ \)-invariant is that for every point \( p \) of the boundary of \( x \) which does not belong to \( x \), there exist a supporting hyperplane of \( x \) through \( p \) containing no point of \( x \). From this we can see that the union of the open unit ball \( B \) in \( \mathbb{R}^n \) with any subset of its boundary is \( ^\circ \)-invariant; its \( ^\circ \)-dual is the union of \( B \) with the complementary subset of the boundary. On the other hand, we find that the union of an open polygonal neighborhood \( P \) of \( 0 \) in the plane with a nonempty finite (or countable) subset \( S \) of its boundary is \( ^\circ \)-invariant, since by the above characterization of \( ^\circ \)-invariant sets, each point \( q \in S \) forces the \( ^\circ \)-closure of \( P \cup S \) to contain all points \( p \) of the open edge(s) of our polygon containing or adjacent to \( q \).

Like the \(*\)-invariant sets (closed convex sets containing \( 0 \)), the \( ^\circ \)-invariant subsets of \( \mathbb{R}^n \) form a lattice by general properties of Galois connections, which has the same meet operation as Conv(\(\mathbb{R}^n\)) \( \geq 0 \), but different join operation. For instance, if \( P \) is, as above, an open polygonal neighborhood of \( 0 \) in \( \mathbb{R}^2 \), \( q \) a point of its boundary, and \( Q = \text{c.h.}(0,q) \), then writing \( \lor \) for the join operation of Conv(\(\mathbb{R}^2\)) \( \geq 0 \), we find that \( P \lor Q = P \cup Q = P \cup \{q\} \), as we saw above, this set is not \( ^\circ \)-invariant. Rather, \( (P \cup Q)^{\circ \circ} \), the join of \( P \) and \( Q \) in the lattice of \( ^\circ \)-invariant sets, is obtained by attaching to \( P \cup \{q\} \) the open edge(s) containing or adjacent to \( q \). Further, the \(*\)-invariant subsets of \( \mathbb{R}^n \) do not even form a sublattice of the \( ^\circ \)-invariant sets. To see this in \( \mathbb{R}^2 \), let

\[ u = \{(x, y) \mid |x| < 1, y \geq 1/(1-x^2) - 2\} \]
\[ v = \{(x, y) \mid |x| < 1, y \leq -1/(1-x^2) + 2\} \]

We see that the join of \( u \) and \( v \) in Conv(\(\mathbb{R}^2\)) \( \geq 0 \) is \( \{(x, y) \mid |x| < 1\} \), which is open, hence is \( ^\circ \)-invariant, hence is their join in the lattice of \( ^\circ \)-invariant sets; but it is not closed, hence is not their join in the lattice of \(*\)-invariant sets.

But again, one can apply \( ^\circ \)-duality to sublattices of Conv(\(\mathbb{R}^n\)) \( \geq 0 \) which consist of \( ^\circ \)-invariant elements, since for these the two lattice structures in question must agree. In particular, one can verify that \( ^\circ \)-duality interchanges compact and open sets, giving an anti-isomorphism between the sublattices Conv(\(\mathbb{R}^n\), cpt) \( \geq 0 \)
and \( \text{Conv}(\mathbb{R}^n, \text{open}) \geq \{0\} \). (This anti-isomorphism can also be obtained by composing the anti-isomorphism “\(^*\)” between \( \text{Conv}(\mathbb{R}^n, \text{cpt}) \geq \{0\} \) and the lattice of closed convex sets that are neighborhoods of the origin, with the isomorphism noted earlier between that lattice and \( \text{Conv}(\mathbb{R}^n, \text{open}) \geq \{0\} \).) The existence of this anti-isomorphism gives

**Corollary 24** (to Lemma 15). For every positive integer \( n \), \( \text{Conv}(\mathbb{R}^n, \text{open}) \geq \{0\} \) is meet semidistributive, that is, satisfies the dual of (13). □

This explains the fact that in Lemma 16 and the discussion that followed, though we saw that \( \text{Conv}(\mathbb{R}^n, \text{open}) \geq \{0\} \) was not join semidistributive, we found no copies of \( M_3 \) in it.

Meet-semidistributivity does not hold in any of the lattices of convex sets we have considered that are not defined so as to make those sets all have some point (such as \( 0 \)) in common; for in each of these lattices, it is easy to get examples of a set \( x \) having empty intersection with each of two sets \( y_1 \) and \( y_2 \), but nonempty intersection with their join. It also does not hold in \( \text{Conv}(\mathbb{R}^n, \text{cpt}) \geq \{0\} \) for \( n > 1 \), since that lattice embeds in \( \text{Conv}(\mathbb{R}^{n-1}, \text{cpt}) \) (cf. proof of Lemma 11), to which the above observation applies.

While on the topic of join- and meet-semidistributivity, I will note some questions and examples which are easier to state now that we have named several sublattices of \( \text{Conv}(\mathbb{R}^n) \). Kira Adaricheva (personal communication) has posed several questions of the following form: If we take one of the sublattices of \( \text{Conv}(\mathbb{R}^n) \) that we know to be join- or meet-semidistributive, and extend it by adjoining, within \( \text{Conv}(\mathbb{R}^n) \), a single “nice” outside element, is the property in question already lost, and if so, does the resulting lattice in fact contain copies of \( M_3 \)?

In many cases this does happen. For instance, the sublattice \( L \subseteq \text{Conv}(\mathbb{R}^2) \) generated by the join-semidistributive sublattice \( \text{Conv}(\mathbb{R}^2, \text{cpt}) \), and the open disk, which we shall denote \( B \), contains a copy of \( M_3 \). To describe it, let \( p_0 \), \( p_1 \), \( p_2 \) be any three distinct points on the unit circle, and for \( i = 0, 1, 2 \) define the point \( q_i = 1/5p_i + 2/5p_{i+1} + 2/5p_{i+2} \) (subscripts evaluated mod 3; I suggest making a sketch). Let \( x_i = (\text{c.h.}(q_i, p_{i+1}) \land B) \lor (\text{c.h.}(q_i, p_{i+2}) \land B) \). Then \( x_1 \), \( x_2 \), \( x_3 \) can be seen to generate a copy of \( M_3 \) in \( \text{Conv}(\mathbb{R}^2) \), resembling, though not identical to, the \( n = 2 \) case of the first example described in Lemma 10. An analogous construction gives a family of elements of \( L \) resembling the example described immediately following that lemma, showing that \( L \) is not \( m \)-join semidistributive for any \( m \).

As another example let \( L \) be the sublattice of \( \text{Conv}(\mathbb{R}^3) \geq \{0\} \) generated by \( \text{Conv}(\mathbb{R}^3, \text{o.bdd.}) \geq \{0\} \) and the cube \( C = [-1, +1]^3 \). Letting \( B \) denote the interior of \( C \), I claim that for any triangle \( T \) drawn on a face \( F \) of \( C \), \( L \) contains the union of \( B \) with the interior of \( T \) relative to \( F \). Indeed, we can find in \( \text{Conv}(\mathbb{R}^3, \text{o.bdd.}) \geq \{0\} \) an open pyramid \( P \) (with apex near \( 0 \)) meeting no face of \( C \) except \( F \), and meeting \( F \) in precisely the relative interior of \( T \). Then \( (P \land C) \lor B \) will be the desired set. If we construct three sets of this sort using as our \( T \)’s three triangles in a common face \( F \) of \( C \), whose relative interiors form a copy of the \( n = 2 \) case of the first example of Lemma 10, then the sublattice generated by
the three resulting sets \((P \land C) \lor B\) will have the form \(M_3\) (with \(B\) as its least element; cf. the last sentence of Corollary 10). The same trick can be used to get examples like those of the paragraph following Lemma 10 showing that \(L\) is not \(m\)-join semidistributive for any positive \(m\).

On the other hand, I do not know what can be said about the lattices gotten by adjoining to \(\text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq 0}\) a closed ball \(D\) about 0, respectively an open ball \(B\) not containing 0, except that the second of these lattices is not meet-semidistributive, since we can find three “fingers” in \(\text{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq 0}\) which meet \(B\) in disjoint subsets \(x, y_1, y_2\), such that \(x \land (y_1 \lor y_2)\) is nonempty.

11. Relatively convex sets.

Let \(S\) be a subset of \(\mathbb{R}^n\), in general non-convex. We shall call a subset \(x \subseteq S\) convex relative to \(S\) if \(x = \text{c.h.}(x) \cap S\); equivalently, if \(x\) is the intersection of \(S\) with some convex subset of \(\mathbb{R}^n\). Such sets \(x\) will form a lattice \(\text{RelConv}(S)\), with meet operation given, as in (2), by intersection, but join now given by

\[ x \lor y = \text{c.h.}(x \cup y) \cap S. \]  

(20)

Note that the map \(x \mapsto \text{c.h.}(x)\) gives a bijection between the relatively convex subsets of \(S\), and the subsets of \(\mathbb{R}^n\) which are convex hulls of subsets of \(S\), the inverse map being given by \(-\cap S\). Convex sets of the latter sort form a lattice with join as in (2), but with meet operation

\[ x \land y = \text{c.h.}(x \cap y \cap S). \]  

(21)

We observe that Carathéodory’s Theorem (with the “refinement” given in the final sentence), regarded as a statement about the closure operator \(\text{c.h.}(-)\) on \(\mathbb{R}^n\), entails the same properties for the closure operator \(\text{c.h.}(-) \cap S\) on \(S\). Hence the proofs of Lemma 8 and of the positive assertions of Theorem 8 immediately yield

**Proposition 25** (cf. Huhn [11] Lemma 3.2]). If \(n\) is a natural number and \(S\) a subset of \(\mathbb{R}^n\), then \(\text{RelConv}(S)\) satisfies the identity \(((D_n) \lor z) \land y_1 \land y_2\), and if \(p\) is a point of \(S\), \(\text{RelConv}(S)_{\geq \{p\}}\) satisfies the identity \(D_n\).

However, the next lemma shows that the corresponding results fail badly for the identities involving \(D_n^{op}\) obtained in Theorem 11. In this lemma, the second assertion embraces the first (plus two obvious intermediate results not stated); however I include the first assertion because both its statement and the example proving it are more transparent than for the second.

**Lemma 26.** For \(S\) a subset of \(\mathbb{R}^2\), \(\text{RelConv}(S)\) need not satisfy the identity \(D_n^{op}\) for any positive integer \(n\). In fact, for \(p\) an element of such an \(S\), \(\text{RelConv}(S)_{\geq \{p\}}\) need not satisfy the identity \(((D_n^{op}) \lor z') \lor (\bigwedge_i y_i)\).

**Proof.** To get the first assertion, let \(S\) be the set consisting of the unit circle and its center, 0. Given \(n\), let \(q_1, \ldots, q_{2n+2}\) be the successive vertices of a regular \((2n+2)\)-gon on that circle; for \(i = 1, \ldots, n+1\), let \(y_i = \{q_1, \ldots, q_{n+1}\} \setminus \{q_i\}\), and let \(x = \{q_{n+2}, \ldots, q_{2n+2}\}\). Observe that each intersection of \(n\) of the \(y\)’s consists
of one of the points \( q_i \), and that \( x \) contains the antipodal point; hence the join of \( x \) with each intersection of \( n \) \( y \)'s contains 0. On the other hand, the intersection of all the \( y \)'s is empty, hence its join with \( x \) is \( x \), which does not contain 0. So \( D_{n}^{\text{op}} \) fails for this choice of arguments.

To get an example where all the given sets have a common element \( p \), and where, moreover, the indicated weaker identity fails, let us take for \( S \) the same set as in the above example, with the addition of one arbitrary point \( r \) outside the circle, at distance \( \geq 2 \) from 0. This time, let \( q_1, \ldots, q_{2n+6} \) be the successive vertices of a regular \((2n+6)\)-gon on \( S \), placed so that \( q_{n+2} \) lies on the line connecting 0 with the external point \( r \). For \( i = 1, \ldots, n+1 \), let \( y_i = \{ q_1, \ldots, q_{n+3} \} - \{ q_i \} \), and let \( x = \{ q_{n+3}, \ldots, q_{2n+4} \} \). We note that the set of subscripts of the \( q \)'s occurring in each of these sets lies in an interval of length \( < n+3 \), so the absence of the point 0 does not contradict convexity of these sets relative to \( S \). Taking \( p = q_{n+3} \) we see that all of these sets belong to \( \text{RelConv}(S) \geq \{ p \} \).

As in the previous example, the intersection of any \( n \) of the \( y \)'s contains one of \( q_1, \ldots, q_{n+1} \), and \( x \) contains its antipodal point, so that the join of \( x \) with that intersection contains 0, hence so does the intersection of all these joins, i.e., the value of the right-hand side of \( D_{n}^{\text{op}} \) at these arguments. On the other hand, the intersection of all the \( y \)'s is \( \{ q_{n+2}, q_{n+3} \} \), and the union of this set with \( x \) still has all subscripts lying in an interval of length \( < n+3 \), so the join of those two sets, the left-hand side of \( D_{n}^{\text{op}} \), does not contain 0. Hence if we take \( z' = \{ q_{n+3}, 0 \} \), the intersection of the right-hand side of \( D_{n}^{\text{op}} \) with \( z' \) is \( z' = \{ q_{n+3}, 0 \} \), while intersection of the left-hand side with \( z' \) is \( \{ q_{n+3} \} \). If, finally, we let \( z = \{ q_{n+3}, r \} \) and take the joins of this element with those two intersections, we see that in the first case the resulting set contains the point \( q_{n+2} \), since we assumed this to lie on the line-segment from 0 to \( r \), while in the second, it does not. Since \( q_{n+2} \in \bigwedge_i y_i \), the two sets remain distinct on intersecting with \( \bigwedge_i y_i \), showing the failure of
\[
( (D_{n}^{\text{op}} \land z') \lor z ) \land (\bigwedge_i y_i).
\]

The above example in \( \mathbb{R}^2 \) is also an example in \( \mathbb{R}^n \) for any \( n \geq 2 \). For completeness, we should also consider dimensions \( n = 0 \) and 1. If we look at the chain of varieties corresponding to that of Lemma 1 but with each \( \text{V}(\text{Conv}(\mathbb{R}^n)) \) replaced by the variety generated by all the lattices \( \text{RelConv}(S) \) for subsets \( S \subseteq \mathbb{R}^n \), and each \( \text{V}(\text{Conv}(\mathbb{R}^n) \geq \{ 0 \}) \) by the variety generated by lattices \( \text{RelConv}(S) \geq \{ p \} \), then we see that the first term of this chain is still the trivial variety and the next two still satisfy the distributive identity (in the last case, by the final assertion of Proposition 25 for \( n = 1 \)). Since the distributive identity implies the identities of every nontrivial variety, we conclude that allowing lattices of relatively convex sets has not enlarged the varieties we get at these three steps. We have just shown the contrary from the fifth step on; this leaves only the fourth step, i.e., the relation between \( \text{V}(\text{Conv}(\mathbb{R}^1)) \) and the variety generated by all lattices of form \( \text{RelConv}(S) \) with \( S \subseteq \mathbb{R}^1 \). Here again, it turns out that we have equality. This follows from our observation that \( \text{RelConv}(S) \) is isomorphic to the lattice of convex hulls of subsets of \( S \), together with
Lemma 27. Let $S$ be a subset of $\mathbb{R}^1$. Then the convex hulls of subsets of $S$ form a sublattice of $\text{Conv}(\mathbb{R}^1)$.

Proof. We have noted, for arbitrary $n$ and $S \subseteq \mathbb{R}^n$, that the convex hulls of subsets of $S$ are closed under the join operation of the full lattices of convex sets, so we need only show them closed under meets, i.e., intersections, when $n = 1$. This comes down to showing that if $p \in \text{c.h.}(x) \cap \text{c.h.}(y)$, where $x$ and $y$ are relatively convex subsets of $S$, then $p \in \text{c.h.}(x \cap y)$. The case where $p \in S$ is immediate, so assume the contrary. The fact that $p \in \text{c.h.}(x)$ then means that $q_1 < p < q_2$ for some $q_1, q_2 \in x$; similarly, $q_3 < p < q_4$ for some $q_3, q_4 \in y$. From the relative convexity of $x$ and $y$ and the order-relations of these elements, we now see that $\max(q_1, q_3) \in x \cap y$ and $\min(q_2, q_4) \in x \cap y$. Hence $p \in \text{c.h.}(\max(q_1, q_3), \min(q_2, q_4)) \subseteq \text{c.h.}(x \cap y)$. □

For further results on $\text{V(\text{Conv}(\mathbb{R}^1))}$ and its subvarieties, see [17]. Incidentally, the lattice $\text{RelConv}(S)$, for $S$ the set used in first part of the proof of Lemma 26 (consisting of the unit circle and its center) shows that none of the identities $((D_{n}^{op}) \wedge z') \vee z \wedge (\bigwedge_{i} y_{i})$ implies any of the identities $D_{n}^{op}$. Indeed, we showed that $\text{RelConv}(S)$ satisfies none of the latter identities; let us now show that (unlike the lattice considered in the second part of that proof) it satisfies all of the former. It suffices to verify the strongest of these, the case $n = 1$. Writing $S = C \cup \{0\}$, where $C$ is the unit circle, we see that the map $- \cap C : \text{RelConv}(S) \to \text{RelConv}(C) = 2^{C}$ is a homomorphism; so if two lattice expressions in $x, y_{1}, y_{2}, z, z'$ are identically equal in distributive lattices, their values in $\text{RelConv}(S)$ will always agree except, perhaps, as to whether they contain 0. We now consider separately the cases $0 \in y_{1} \wedge y_{2}$ and $0 \notin y_{1} \wedge y_{2}$. In the former case, the two sides of $D_{1}^{op}(x, y_{1}, y_{2})$ agree in containing 0, hence are equal, so a fortiori the two sides of $((D_{1}^{op}) \wedge z') \vee z \wedge (y_{1} \wedge y_{2})$ are equal. In the latter case, neither side of the latter relation can contain 0, hence again they are equal.

12. The snowflake.

Let us look at a particularly neat example of a lattice of relatively convex subsets of a set $S$.

Let $p_{1}, p_{2}, p_{3}, -p_{1}, -p_{2}, -p_{3}$ be the successive vertices of a regular hexagon in $\mathbb{R}^2$ centered at 0. Let

$$S_{1} = \text{c.h.}(p_{1}, -p_{1}), \quad S_{2} = \text{c.h.}(p_{2}, -p_{2}), \quad S_{3} = \text{c.h.}(p_{3}, -p_{3}),$$

$$S = S_{1} \cup S_{2} \cup S_{3},$$

and let $L$ be the sublattice of $\text{RelConv}(S)_{\geq 0}$ generated by the three line-segments $S_{1}$, $S_{2}$ and $S_{3}$. (In view of the form of $S$, I think of this example as “the snowflake”.)

Every element of $L$ will clearly be centrally symmetric and topologically closed; hence every such element has the form

$$\lambda_{1} S_{1} \cup \lambda_{2} S_{2} \cup \lambda_{3} S_{3} \quad (\lambda_{1}, \lambda_{2}, \lambda_{3} \in [0, 1]). \quad (22)$$
The join of $S_1$ and $S_2$ in this lattice must have the form $S_1 \cup S_2 \cup \lambda S_3$ for some $\lambda \in (0, 1)$. (Elementary geometry shows that $\lambda = 1/2$; but we don’t need to know this now, and will get it from a general formula soon.) Intersecting this join with $S_3$, we see that $\lambda S_3 \in L$. By symmetry we also have $\lambda S_1, \lambda S_2 \in L$, and we see that these together generate a proper sublattice of $L$ isomorphic to the whole lattice. In particular, $L$ has infinite descending chains of elements, e.g., $S_1 > \lambda S_1 > \lambda^2 S_1 > \ldots$.

Let’s figure out how to calculate in $L$. The first thing we should find are the conditions on $\lambda_1$, $\lambda_2$ and $\lambda_3$ for a set (22) to be relatively convex. Calculation shows that for $\lambda_1$, $\lambda_2$ and $\lambda_3$ nonzero, the points $\lambda_1 p_1$, $\lambda_2 p_2$ and $\lambda_3 p_3$ are collinear if and only if $\lambda_2^{-1} = \lambda_1^{-1} + \lambda_3^{-1}$. (In verifying this, the key relation is $p_2 = p_1 + p_3$.) Hence we see that one necessary condition for the convexity of (22) is $\lambda_2^{-1} \leq \lambda_1^{-1} + \lambda_3^{-1}$; by symmetry, the remaining conditions are $\lambda_1^{-1} \leq \lambda_2^{-1} + \lambda_3^{-1}$ and $\lambda_3^{-1} \leq \lambda_1^{-1} + \lambda_2^{-1}$. It is easy to verify that if the $\lambda_i$ are also allowed to be zero, and we write $0^{-1} = \infty$ and consider $\infty$ greater than all real numbers, then these three conditions continue to be necessary and sufficient for convexity.

Let us therefore index elements (22) by the three parameters $\lambda_1^{-1}$, $\lambda_2^{-1}$, $\lambda_3^{-1}$, defining

$$[a_1, a_2, a_3] = a_1^{-1} S_1 \cup a_2^{-1} S_2 \cup a_3^{-1} S_3 \quad (a_1, a_2, a_3 \in [1, \infty]),$$

so that the lattice of centrally symmetric elements of $\text{RelConv}(S)_{\geq (0)}$ consists of the sets $[a_1, a_2, a_3]$ with

$$a_1 \leq a_2 + a_3, \quad a_2 \leq a_1 + a_3, \quad a_3 \leq a_1 + a_2.$$

Note that the ordering of this lattice is by reverse componentwise comparison of expressions $[a_1, a_2, a_3]$, and that $S_1$, $S_2$, $S_3$ are the elements $[1, \infty, \infty]$, $[\infty, 1, \infty]$, $[\infty, \infty, 1]$. Lattice-theoretic meet is clearly given by componentwise supremum, while the lattice-theoretic join of two elements is gotten by first taking their componentwise infimum, which represents their set-theoretic union, then getting its relative convex hull by reducing the largest entry to the sum of the other two if it exceeds this. So, for instance, $S_1 \cup S_2 = [1, \infty, \infty] \vee [\infty, 1, \infty]$ is gotten by forming the componentwise infimum, $[1, 1, \infty]$, and then decreasing the last component to the sum of the first two, getting $[1, 1, 2]$ (confirming the value $\lambda = 1/2$ in our earlier description of this element). We can now calculate, e.g., the meet (componentwise supremum) of this with $S_3 = [\infty, \infty, 1]$, namely $[\infty, \infty, 2]$; and take the join of this meet with $S_2 = [\infty, 1, \infty]$ by forming the componentwise infimum $[\infty, 1, 2]$, and adjusting the first component as above, getting $[3, 1, 2]$.

Note that the only way these lattice operations yield components in their values that did not occur as components in their arguments is by addition; hence, as $L$ was defined to be generated by $S_1$, $S_2$ and $S_3$, all finite components $a_i$ that occur in the expressions (23) for elements of $L$ are positive integers. It is not hard to verify that all positive integers indeed occur, and that the elements of $L$ are all the elements (23) satisfying (24) with $a_1, a_2, a_3 \in \{1, 2, 3, \ldots, \infty\}$. So we have a very arithmetic description of this lattice.
Though we have seen that $L$ contains an infinite descending chain, it is interesting to note that the sublattice generated by any finite set of elements, none of which have any infinite components, is finite; for the lattice operations will not produce, in any position, entries larger than the corresponding entries of their arguments.

It would be of interest to examine more general lattices of the form $\text{RelConv}(S)_{\geq\{0\}}$ for sets $S$ which are unions of finitely many line-segments (or rays) through the origin in $\mathbb{R}^n$. In this situation, the conditions for convexity are always given by linear inequalities in the $\lambda_i^{-1}$; let me sketch why.

First, some general observations. Suppose $p_1, \ldots, p_m$ $(m \geq 3)$ are a minimal linearly dependent family of vectors in $\mathbb{R}^n$; thus they satisfy a nontrivial linear relation $\sum c_i p_i = 0$, unique up to scalars. For which families of positive real numbers $\lambda_1, \ldots, \lambda_m$ will the points $\lambda_1 p_1, \ldots, \lambda_m p_m$ lie in an $(m-1)$-dimensional affine subspace of their $m$-dimensional span? This will hold if and only if the linear relation satisfied by these modified elements is an affine relation; i.e., has coefficients summing to 0. Those coefficients are $c_i \lambda_i^{-1}$, so the condition is

$$\sum c_i \lambda_i^{-1} = 0.$$  

In this situation, can the affine relation among the $\lambda_i p_i$ be written as an expression for one of them, say $\lambda_j p_j$, as a convex linear combination of the others? One sees that this is so if and only if the coefficient $c_j$ is opposite in sign to all the other $c_i$; in that situation, let us rewrite the expression satisfied by the $\lambda_i^{-1}$ as

$$\lambda_j^{-1} = -c_j^{-1} \sum_{i \neq j} c_i \lambda_i^{-1}.$$  

Note that the abovementioned condition on signs can be satisfied by at most one $j$, and can be looked at as saying that the ray determined by the corresponding vector $p_j$ is in the convex hull of the rays determined by the other $p_i$. When it is satisfied, one finds that a union of intervals $\bigcup_i [0, \lambda_i p_i]$ is relatively convex in the union of the rays determined by the $p_i$ if and only if $\lambda_j$ has at least the value given by the above formula, i.e., if and only if

$$\lambda_j^{-1} \leq -c_j^{-1} \sum_{i \neq j} c_i \lambda_i^{-1}. \quad (25)$$  

If the $c_i$ do not consist of one of one sign and the rest of the opposite sign, then none of the rays determined by the $p_i$ is in the convex hull of the rest, and every set $\bigcup_j [0, \lambda_j p_j]$ $(\lambda_1, \ldots, \lambda_m > 0)$ is relatively convex. (An example of this situation is given for $n = 3$ by letting $p_1, \ldots, p_4$ be the vertices of a convex quadrilateral lying in a plane not containing 0.)

Let us now drop the condition that the $p_i$ are minimal among linearly dependent families, assuming only that none of them is a nonnegative multiple of another (i.e., that they determine distinct rays), and let $S$ denote the union of the rays through 0 that they determine. Then one can show that a union of intervals $\bigcup_j [0, \lambda_j p_j]$ is relatively convex in $S$ if and only if (25) holds for each minimal linearly dependent family of $\geq 3$ of the $p_i$ whose unique linear relation has exactly one coefficient $c_j$ of different sign from the rest. (The reduction to the minimal-linearly-independent-family case can be gotten by a recursive application
of Carathéodory’s Theorem, with \( p_0 = 0 \), within subspaces spanned by successively smaller subsets of \( \{ p_1, \ldots, p_n \} \).

As an example of the sort of lattice one gets, let us drop the central-symmetry condition from our snowflake construction, writing \(-p_1, -p_2, -p_3\) as \( p_4, p_5, p_6\), and considering general sets

\[
\lambda_1 \text{ c.h.} (0, p_1) \cup \ldots \cup \lambda_6 \text{ c.h.} (0, p_6) \quad (\lambda_1, \ldots, \lambda_6 \in (0, \infty])
\]

Then setting \( a_i = \lambda_i^{-1} \), we get six conditions for relative convexity, namely

\[
a_2 \leq a_1 + a_3, \quad a_3 \leq a_2 + a_4, \quad \ldots, \quad a_1 \leq a_6 + a_2,
\]

each corresponding to the fact that one of our six rays lies in the cone spanned by its two immediate neighbors.

Our snowflake example showed that a lattice \( \text{RelConv}(S)_{\geq \{0\}} \) could contain a 3-generator sublattice with an infinite descending chain. Let me sketch an example with an infinite ascending chain. In \( \mathbb{R}^2 \), let

\[
p_1 = (0, 3), \quad p_2 = (1, 2), \quad p_3 = (2, 1), \quad p_4 = (3, 0).
\]

(Or for a more abstract description, take any four points of \( \mathbb{R}^n \) in arithmetic progression, on a line not passing through the origin.) Define

\[
S_i = \text{c.h.} (0, p_i) \quad \text{and} \quad S = S_1 \cup S_2 \cup S_3 \cup S_4.
\]

Now let \( L \) be the sublattice of \( \text{RelConv}(S)_{\geq \{0\}} \) generated by

\[
x_1 = S_1 \cup (S_2/2), \quad y = S_2 \cup S_3, \quad x_2 = (S_3/2) \cup S_4.
\]

It is easy to see from a sketch that, starting with \( x_1 \land y \), if we alternately apply \((- \lor x_2) \land y\) and \((- \lor x_1) \land y\), we obtain an infinite ascending chain of subsets.

Is the existence of infinite chains in sublattices of lattices \( \text{RelConv}(S) \) generated by few elements limited to cases where \( S \neq \mathbb{R}^n \), or does it also occur in lattices \( \text{Conv}(\mathbb{R}^n) \)? To get a large part of the answer without any computation, recall that \( \text{RelConv}(S) \) is isomorphic to the lattice of convex hulls in \( \mathbb{R}^n \) of subsets of \( S \). This lattice, which we shall here denote \( L_S \), is a subset but not a sublattice of \( \text{Conv}(\mathbb{R}^n) \); however, in cases like those considered above, where \( S \) is a finite union of convex sets, \( S = S_1 \cup \cdots \cup S_m \), we can write the operations of this lattice as “polynomial operations” in those of \( \text{Conv}(\mathbb{R}^n) \). Namely, temporarily writing \( \land_S \) and \( \lor_S \) for the operations of \( L_S \), and \( \land \) and \( \lor \) for those of \( \text{Conv}(\mathbb{R}^n) \), we get, for \( x, y \in L_S \),

\[
x \lor_S y = x \lor y, \quad x \land_S y = \lor_S (x \land y \land S_i). \quad (26)
\]

Hence if the sublattice of \( L_S \) generated by elements \( y_1, \ldots, y_k \) has an infinite ascending or descending chain (or any other specified join-sublattice), so will the sublattice of \( \text{Conv}(\mathbb{R}^n) \) generated by \( y_1, \ldots, y_k, S_1, \ldots, S_m \). In the case of our “snowflake lattice”, the \( y \)’s and the \( S \)’s happen to be the same, so we immediately conclude that the sublattice of \( \text{Conv}(\mathbb{R}^2)_{\geq \{0\}} \) generated by those three elements has an infinite descending chain. From our example with an ascending chain, the best conclusion this general argument gives is that the sublattice of \( \text{Conv}(\mathbb{R}^2)_{\geq \{0\}} \) generated by the six elements \( S_1, S_2, S_3, S_4, S_2/2 \) and \( S_3/2 \) has an infinite
ascending chain. Nevertheless, a little diagram-drawing shows that in this case, the sublattice of \( \text{Conv}(\mathbb{R}^2) \geq \{0\} \) generated by the three elements \( \text{c.h.}(x_1) \), \( \text{c.h.}(x_2) \) and \( \text{c.h.}(y) \) shows essentially the same behavior as our lattice of relatively convex subsets.

I do not know any examples of 3-generator lattices of convex sets (relative or absolute) that have both infinite ascending and descending chains, or that have infinite antichains. On the other hand, it is not hard to show that the 4-generator sublattice of \( \text{Conv}(\mathbb{R}^2) \) generated by the diameters of a regular octagon has all three.

13. Notes on related work on relatively convex sets, and some further observations.

The lattices \( \text{RelConv}(S) \) are examples of what are known as convex geometries; for the definition, and results on these, see [7], [1].

Huhn [11] looked briefly at lattices of relatively convex sets determined by finite sets \( S \subseteq \mathbb{R}^n \) for the purpose of “approximating” \( \text{Conv}(\mathbb{R}^n) \) by finite lattices, and for the same purpose he considered in [10, §2] the dual construction, namely the lattice of those subsets of \( \mathbb{R}^n \) which can be represented as intersections of members of a given finite set of closed half-spaces. Not surprisingly in view of the dual natures of these two sorts of relativization, he found that lattices of the latter sort satisfied the identities of the form \( D_n \) that he had obtained in the nonrelativized lattice, but not those of the form \( D_n^{op} \) (cf. Proposition 25 and Lemma 26 above).

In [19] it is shown that for every finite lattice \( L \) which is “lower bounded” (a strengthening of join semidistributive), there exist an \( n \) and a finite subset \( S \subseteq \mathbb{Q}^n \) such that \( L \) is embeddable in the sublattice of \( \text{Conv}(\mathbb{R}^n) \) generated by \( \{ \{p\} \mid p \in S \} \). Clearly, such an embedding will send all members of \( L \) to convex polytopes; if we let \( S' \) denote the union of the vertex-sets of this finite set of polytopes, it is not hard to see that we get an embedding of \( L \) in \( \text{RelConv}(S') \). Whether embeddability in the lattice of relatively convex sets of a finite subset of \( \mathbb{R}^n \) holds not only for lower bounded lattices, but for all join-semidistributive lattices, is an open question [19, Problem 1], cf. [11, Problem 3]. It is also shown in [19] that every lattice can be embedded in the lattice of convex subsets of some infinite-dimensional vector space (over an arbitrary totally ordered division ring).

In particular, such lattices need not satisfy any nontrivial lattice identities. In contrast, the lattice of subspaces of any vector space satisfies the modular identity and others.

Here is another embedding result using not necessarily finite-dimensional vector spaces, although it may be seen that the connection with convexity is somewhat artificial, based on the fact that subspaces are in particular convex sets; it is essentially a result on lattices of “relative subspaces”. (Since most of the results in this note concerned subspaces of \( \mathbb{R}^n \), we only gave our general definitions for that case, but we shall use them here without that restriction.)
Lemma 28. Let $V$ be a real vector space and $B$ a basis of $V$. For each pair of distinct elements $a,b \in B$, let $S_{a,b}$ be the subspace of $V$ spanned by $a - b$. Define the set

$$S = \bigcup_{a,b} S_{a,b} \subseteq V.$$ 

Then the following lattices are isomorphic:

(a) The lattice $\text{Equiv}(B)$ of all equivalence relations on $B$ (ordered by inclusion).

(b) The lattice $\text{RelSubSp}(S)$ consisting of all sets of the form $S \cap U$ where $U$ is a subspace of $V$. (Here meet is given by intersection; join by taking the subspace spanned by the union of the sets in question, and intersecting this with $S$.)

(b') The lattice of all subspaces of $V$ spanned by subsets of $S$. (The join of two such subspaces $W_1$, $W_2$ is $W_1 + W_2$, the meet is the span of $W_1 \cap W_2 \cap S$.)

(c) The sublattice of $\text{RelConv}(S)$ consisting of all elements thereof that are unions of subspaces $S_{a,b}$.

(c') The lattice of all subsets of $V$ which are convex hulls of unions of subspaces $S_{a,b}$. (Join as in $\text{Conv}(\mathbb{R}^n)$, while the meet of $x_1$, $x_2$ is c.h.$(x_1 \cap x_2 \cap S)$).

Proof. That the sets described in (b), (b') and (c'), ordered by inclusion, form lattices, with meet and join as described, is immediate.

Note that the convex hull of a union of subspaces of $V$ is the sum of those subspaces, and that the intersection of $S$ with a subspace is always a union of certain of the $S_{a,b}$. From this it is easily seen that (b) and (c) are not merely, as asserted, isomorphic, but equal, and likewise (b') and (c'). It is also clear that (b) is isomorphic to (b'), via the “span of” map in one direction and the operator $- \cap S$ in the other. So these four lattices are isomorphic; to complete the proof we shall describe an isomorphism between the lattice $\text{Equiv}(B)$ of (a) and the lattice of (b').

Given an equivalence relation $R \in \text{Equiv}(B)$, let $\phi(R)$ be the subspace of $V$ spanned by all elements $a - b$ with $(a,b) \in R$, which by definition belongs to (b'), while given a subspace $W \subseteq V$ spanned by a subset of $S$, let $\psi(W) = \{(a,b) \mid a - b \in W\}$, which it is easy to check is an equivalence relation on $B$. The maps $\phi$ and $\psi$ are clearly isotone, and from the definition of (b'), we see that $\phi \psi$ is the identity function thereof; moreover, for any $R \in \text{Equiv}(B)$ it is clear that $\psi(\phi(R)) \geq R$, so it remains to prove the reverse inequality.

So suppose that $(a,b) \in \psi(\phi(R))$, i.e., that $a - b \in \phi(R)$. From the definition of $\phi(R)$ it is easy to see that for each $R$-equivalence class $C \subseteq B$, the sum of the coefficients of all members of $C$ in any element of $\phi(R)$ is zero. But the only way this can hold for $a - b$ is if $a$ and $b$ are in the same equivalence class, i.e., $(a,b) \in R$, as required to complete our proof. \qed

Pudlák and Tůma [18] have shown that every finite lattice $L$ embeds in $\text{Equiv}(X)$ for some finite set $X$. Hence by the above lemma, for every such $L$ one can find an $n$ and a subset $S \subseteq \mathbb{R}^n$ such that $L$ is embeddable in $\text{RelConv}(S)$. It would be interesting to know whether this same conclusion can be proved without using the deep result of [18]. So far as is known, the embedding of $L$ in a lattice $\text{Equiv}(X)$
may require a set $X$ whose cardinality is enormous compared with that of $L$ (see [14], [12] for some improvements on the bound of [13]), but it is plausible that one could do better with embeddings in lattices of relatively convex sets.

Our final remark will concern the observations at the end of the preceding section on the form of the lattice operations of $\text{RelConv}(S)$ when $S$ is a finite union of convex sets $S_1 \cup \cdots \cup S_m$. Let us put these in a more general context. (Readers allergic to category theory may ignore this discussion.)

Let $\text{Lattice}$ denote the category of all lattices, objects of which we will here write $L = (|L|, \lor, \land)$, distinguishing between the lattice $L$ and its underlying set $|L|$. For $m$ a natural number, let $\text{Lattice}^{m-\text{pt}}$ denote the category of lattices with $m$ distinguished elements, i.e., of systems $(|L|, \lor, \land, S_1, \ldots, S_m)$ such that $(|L|, \lor, \land)$ is a lattice and $S_1, \ldots, S_m \in |L|$, and where a morphism between such systems means a lattice homomorphism which respects the ordered $m$-tuple of distinguished elements. Let us, finally, write $\text{Lattice}^{\text{int}}$ for the category of objects $(|L|, \lor, \land, \text{int})$, where $(|L|, \lor, \land)$ is a lattice, and $\text{int}$ is an “interior operator” (the dual of a closure operator), that is, a map $|L| \to |L|$ satisfying

$$\text{int}(x) \leq x, \quad x \leq y \implies \text{int}(x) \leq \text{int}(y), \quad \text{int}(\text{int}(x)) = \text{int}(x).$$

(Here by “$u \leq v$” we of course mean $u = u \land v$, equivalently, $u \lor v = v$.) The morphisms of $\text{Lattice}^{\text{int}}$ will be the lattice homomorphisms respecting this additional operation.

We can define a functor $\text{Lattice}^{m-\text{pt}} \to \text{Lattice}^{\text{int}}$ taking each object $(|L|, \lor, \land, S_1, \ldots, S_m)$ to the object $(|L|, \lor, \land, \text{int}(S_i))$, where the interior operator is defined by

$$\text{int}(S_i)(x) = \lor_i (x \land S_i),$$

and another functor $\text{Lattice}^{\text{int}} \to \text{Lattice}$, taking each object $(|L|, \lor, \land, \text{int})$ to the object $(|L|^{\text{int}}, \lor, \land^{\text{int}})$, where

$$|L|^{\text{int}} = \{ x \in |L| \mid x = \text{int}(x) \} \quad \text{and} \quad x \land^{\text{int}} y = \text{int}(x \land y).$$

(It is not hard to verify that the join operation of $L$ carries $|L|^{\text{int}}$ into itself.)

We now see that if we take $L = \text{Conv}(\mathbb{R}^n)$, and let $S_1, \ldots, S_m$ be any $m$ elements of this lattice, then the composite of the above two functors, applied to $(|L|, \lor, \land, S_1, \ldots, S_m)$, gives precisely the lattice we named $L_S \cong \text{RelConv}(S)$, for $S = S_1 \cup \cdots \cup S_m$.

The constructions given by the above functor $\text{Lattice}^{\text{int}} \to \text{Lattice}$, and its dual, with a closure operator replacing the interior operator, are well-known, if not in this functorial form. I do not know whether the construction $\text{Lattice}^{m-\text{pt}} \to \text{Lattice}^{\text{int}}$ (and its variant with lattices replaced by complete lattices and the specified finite family by an arbitrary family) has been considered.

Stepping back a little further, we may observe that the lattice $\text{Conv}(\mathbb{R}^n)$ arises as the fixed set of the closure operator c.h.(-) on the lattice $2^{\mathbb{R}^n}$ of subsets of $\mathbb{R}^n$, so that the construction of the mutually isomorphic lattices $L_S$ and $\text{RelConv}(S)$ can be seen as arising from the interaction of the closure operator c.h.(-) and the interior operator $\land \cap S$ on $2^{\mathbb{R}^n}$. Again, this situation can be made into a general construction.
The reader familiar with the concept of representable algebra-valued functors (§ Chapter 9 or § §1, §8) will be happy to observe that all the functors of the above discussion are representable.

14. A question.

The referee has pointed out that some properties of the lattice of convex sets are known to change if the base field \( \mathbb{R} \) is replaced by another ordered field (e.g., \( \mathbb{Q} \)), but that the arguments of §§1-6 look as though they should work over any ordered field; perhaps even any ordered division ring. I have the same feeling, but as an amateur in the area, I will leave this question to others. (I do not know whether the theorems of Helly, Carathéodory and Radon hold in that context, nor how much of what I have justified as geometrically evident may rely on properties of the reals.)

A straightforward generalization of these results could not, of course, extend to §7 on compact convex sets, since over an ordered field which is not locally compact, the only nonempty compact convex sets are the singletons, which do not form a lattice. One might be able to prove results like those of that section with “compact” replaced by “closed and bounded”, but new proofs would be needed, since the theorem of Minkowski we used there is not true in that context. Later sections depend to varying degrees on that one.

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