Correction to Friis Noise Factors

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Abstract
The signal-to-noise ratio of a multistage cascade network is often estimated using Friis formulas for noise factors (or the noise figures in decibel). In this letter, the correct formulas to calculate the stage-wise noise factors and the total noise factor in terms of the stage-wise noise factors of an $n$-stage cascade network are derived. A comparison of our derived formulas for noise factors with Friis’s formulas is presented. Contrary to Friis’s total noise factor in terms of the stage-wise noise factors, we define the actual total noise factor of the $n$-stage cascade network as the product of its stage-wise noise factors.

1 Introduction
Cascade systems are widely used in various electrical and electronics engineering domains such as telecommunications and signal processing [1–7], circuits [8–16], networks and systems [17–20], solid-state devices [21–29], and so on. To extract the actual signal component at the output of these systems, it is critical to calculate its total noise factor ($F_T$). Friis’s formulas for noise are most commonly used to calculate the total noise factor of $n$-stage cascade networks [1–29].

In this letter, we briefly discuss the existing theory to calculate the noise factors of a cascade structure and the well-known Friis formulas for the stage-wise and the total noise factors of a cascade structure. We then derive the correct formulas for the stage-wise noise factor and the total noise factor in terms of the stage-wise noise factors of $n$-stage cascade networks. Our derived formulas are then compared with Friis’s formulas, with a discussion on the correction required to Friis noise factors.

2 Theory and Discussion
Fig. 1 shows the block diagram of an $n$-stage cascade network. From the block diagram, $S_i$ is the input signal to the network from a source, $N_i$ is the noise from the input source, $M_x$ is the gain of the $x$-th stage, $F_x$ is the noise factor at the $x$-th stage, $N_{a(x)}$ is the added noise at the output of the $x$-th stage, $\text{SNR}_i(x)$ is the input signal-to-noise ratio (SNR) of the $x$-th stage, $\text{SNR}_o(x)$ is the output SNR of the $x$-th stage, $\text{SNR}_i$ is the input SNR of the network, $\text{SNR}_o$ is the output SNR of the network, $S_o$ is the output signal of the network, and $N_o$ is the output noise of the network.

2.1 Existing Noise Theory and Friis Noise Factors
The total noise factor of the $n$-stage cascade network ($F_T$) is defined as the ratio of input SNR to output SNR [1,15,30]. Solving for $F_T$ of the $n$-stage cascade network shown in Fig. 1, we get $F_T$ as given by equation (1).
2.2 Correct Formulas for Noise Factors

For cascade networks, the output of the previous stage will be the input to the next stage. Therefore, the stage-wise noise factor must be equal to the ratio of SNR at the input of the stage to SNR at the output of the corresponding stage [1, 15, 30]. From the block diagram shown in Fig. 1, the noise factor at the x-th stage is,

$$F_x = \frac{\text{SNR}_i(x)}{\text{SNR}_o(x)} = \frac{\text{SNR}_{i(x-1)}}{\text{SNR}_{o(x)}}.$$  (5)

Solving equation (5) for the 1-st stage of the network, we obtain the 1-st stage noise factor ($F_1$) same as that of Friis's 1-st stage noise factor ($F_{1\text{Friis}}$). However, for higher stages ($x \geq 2$), the x-th stage noise factor will not be equal to the corresponding Friis's stage-wise noise factor. For illustration, considering the noise factor at the 2-nd stage of the network,

$$F_2 = \frac{\text{SNR}_i(2)}{\text{SNR}_o(2)} = \frac{\text{SNR}_i(1)}{\text{SNR}_o(2)} = \frac{\left(\frac{S_i}{N_i}\right)}{\left(\frac{S_i}{N_i} + \frac{N_{a(1)}}{N_i}M_1 + \frac{N_{a(2)}}{N_i}M_2 + \frac{N_{a(3)}}{N_i}M_3 + \cdots M_x\right)}.$$  (6)

From equation (6), the stage-wise noise factor may also be defined as the ratio of the total noise at the output of the stage ($N_o(x)$) to the total noise at its input ($N_i(x)$) multiplied by the stage gain ($M_x$). This definition does not agree with Friis’s stage-wise noise factors. A comparison of the stage-wise noise factor for the 2-nd stage of the cascade network is shown in equation (7).

$$F_2^{\text{Cor}} = 1 + \frac{N_{a(2)}}{N_iM_1} = \frac{N_{a(2)}}{N_iM_2} = F_2^{\text{Cor}}.$$  (7)
Therefore, the correct generalized formula for the stage-wise noise factor at the $x$-th stage ($F_{x}^{\text{Cor}}$) must be,

$$F_{x}^{\text{Cor}} = 1 + \frac{N_{n(x)}}{N_{i(x)}M_{x}} = 1 + \frac{N_{n(x)}}{N_{i(x)}M_{x}} \cdot \frac{N_{a(x)}}{N_{i(x)}M_{x}}.$$

Comparing equations (2) and (8), the stage-wise noise factors will be ‘equal to one’ if the stage-wise added noises are ‘equal to zero’. Thus, if $N_{n(x)} = 0$, then, $F_{x} = F_{x}^{\text{Cor}} = F_{x}^{\text{Friis}} = 1$. However, if there is a stage-wise added noise that is ‘greater than zero’ and ‘equal at all the stages’, then, a bar chart comparing the relative values of the stage-wise noise factors calculated using Friis formula (equation (2)) and our formula (equation (8)) for up to the 6-th stage is shown in Fig. 2. Here, firstly it is observed that the 1-st stage noise factors calculated using Friis and our formulas are equal. Whereas, for higher stages, the noise factors calculated using Friis formula are greater than the corresponding stage-wise noise factors calculated using our formula. Secondly, the Friis stage-wise noise factor values remain the same for all stages if the stage-wise added noises are equal. However, our formula suggests that if all the stage-wise added noises are equal and greater than zero, then the stage-wise noise factors reduce with the stage number. This is because, as the stage number increases, the total noise at its input also increases. Thus, if $\forall x = \{1, 2, \ldots, n\} \exists N_{n(x)} > 0 \Rightarrow \{N_{n(1)} = N_{n(2)} = \ldots = N_{n(n)}\}$, then, (i) $F_{x}^{\text{Friis}} = F_{x}^{\text{Cor}}$ for $x = 1$ and $F_{x}^{\text{Friis}} > F_{x}^{\text{Cor}}$ for $x \geq 2$; (ii) $F_{1}^{\text{Cor}} = F_{2}^{\text{Friis}} = \ldots = F_{n}^{\text{Friis}}$, whereas, $F_{1}^{\text{Cor}} > F_{2}^{\text{Cor}} > \ldots > F_{n}^{\text{Cor}}$. Therefore, our formula for the stage-wise noise factor is a correction to Friis’s formula.

Moreover, the correct generalized formula for the stage-wise noise factor at the $x$-th stage in terms of the stage-wise noise factors of the previous stages is written as,

$$F_{x}^{\text{Cor}} = 1 + \frac{N_{n(x)}}{N_{i(x)} \prod_{j=1}^{x} M_{j} \prod_{k=1}^{x-1} F_{k}^{\text{Cor}}}.$$  

Rearranging equation (9) and substituting it in equation (1) we get,

$$F_{T} = 1 + \frac{N_{n(1)}}{N_{i(1)} M_{1}} + \frac{N_{n(2)}}{N_{i(2)} M_{1} M_{2}} = \prod_{i=1}^{n} F_{x}^{\text{Cor}} = (F_{1}^{\text{Cor}} - 1) F_{1}^{\text{Cor}} + \ldots + \frac{N_{n(n)}}{N_{i(n)} M_{2} \ldots M_{n}} = (F_{1}^{\text{Cor}} - 1) \prod_{x=1}^{n-1} F_{x}^{\text{Cor}}.$$  

Therefore, the correct total noise factor of the $n$-stage cascade network ($F_{T}^{\text{Cor}}$) in terms of the stage-wise noise factors ($F_{x}^{\text{Cor}}$) is given by equation (11), which is not equal to equation (4).

$$F_{T}^{\text{Cor}} = \prod_{x=1}^{n} F_{x}^{\text{Cor}} \neq (F_{T}^{\text{Friis}})$$  

From equation (11), the actual total noise factor of the $n$-stage cascade network is defined as the product of the stage-wise noise factors.

3 Conclusion

We conclude that our derived formulas for the stage-wise noise factor for stages $x \geq 2$ and the total noise factor in terms of the stage-wise noise factors of the
$n$-stage cascade network are a correction to Friis’s formulas.

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