Multi–matrix models without continuum limit

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Abstract

We derive the discrete linear systems associated to multi–matrix models, the corresponding discrete hierarchies and the appropriate coupling conditions. We also obtain the $W_{1+\infty}$ constraints on the partition function. We then apply to multi–matrix models the technique, developed in previous papers, of extracting hierarchies of differential equations from lattice ones without passing through a continuum limit. In a q–matrix model we find $2q$ coupled differential systems. The corresponding differential hierarchies are particular versions of the KP hierarchy. We show that the multi–matrix partition function is a $\tau$–function of these hierarchies. We discuss a few examples in the dispersionless limit.

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1 Introduction

In 1–matrix models all the information is encoded in the Jacobi matrix $Q$, a semi–infinite matrix formed by three non–vanishing diagonal lines. This matrix remains the same in form whatever the polynomial potential is. This accounts for the relatively simple structure of 1–matrix models. When we turn our attention to multi–matrix models, the situation drastically changes. First of all we have several Jacobi matrices characterizing the same model. They are semi–infinite matrices with a finite or infinite band of non–vanishing diagonal lines. What is more important, the band size depends on the order of the polynomial potentials.

As is well–known the purpose of all the matrix model analysis is to extract flow equations (w.r.t. to the perturbation parameters) for the partition function, as well as constraints known as string equations. It is believed that multi–matrix models should give rise in particular to generalized KdV-hierarchies \[1\] which in turn relate to generalized Kontsevich models \[2\], \[3\], \[4\], \[5\], \[6\], \[7\].

As will be shown below, it is not difficult to extract lattice hierarchies and lattice string equations from multi–matrix models. In fact in a q–matrix model we find $2q$ linear systems and, consequently, $2q$ lattice hierarchies. They are generalizations of the Toda lattice hierarchy and they are sometimes referred to as discrete KP hierarchies. However significant information can be extracted only from differential equations. The usual way to obtain a differential hierarchy from a lattice hierarchy is via a continuum limit. However, due to the aforementioned features of multi–matrix models, the application of the continuum limit technique has not proven as successful as in 1–matrix models, and the research in this subject has remained at a rather conjectural level. The only exception are the results obtained in \[8\], \[9\], concerning the $W$–type constraints satisfied by multi–matrix models, and the analysis of some particular cases \[10\], \[11\] (see also \[12\]).

In a previous paper \[13\] (see also \[14\]) we introduced a method to extract a differential hierarchy from a lattice one. It consists essentially in using the first flow equation to eliminate the difference operations in the remaining equations. In this way the latter become differential equations with respect to the first flow parameter (which is interpreted as the ‘space’ coordinate). This procedure can be generalized to the lattice systems specific of multi–matrix models, and this is what we do in this paper. As our main result, we associate to any q–matrix model a set of differential hierarchies of flow equations, together with the appropriate string equations. The differential hierarchies turn out to be particular versions (reductions) of the generalized KP hierarchy \[15\]. We will refer to these hierarchies as KP–type hierarchies or, when no confusion is possible, KP hierarchies for short. As we will see, the structure of multi–matrix models is much richer than it is usually believed. In particular it contains much more than the n–th KdV hierarchies.

We recall that the importance of avoiding a continuum limit lies in the fact \[14\] that the differential hierarchies and string equations so obtained describe properties of the matrix–model lattice and therefore, presumably, topological properties.

The matter is arranged as follows. Section 2 is a review of multi–matrix models and an occasion to state our notations. In section 3 we represent an arbitrary multi–matrix model by means of suitable coupled discrete linear systems, and emphasize the importance of the coupling conditions; the corresponding consistency
conditions result in (generalized) 2–dimensional Toda lattice hierarchies and string equations. In section 4 we work out in detail the $W_{1+\infty}$ constraints on the partition function for two–matrix models. In section 5 we pass from lattice to differential formalism and find the KP structure of the differential hierarchies pertinent to multi–matrix models. We also prove that the multi–matrix model partition function is in fact the $\tau$ function of the appropriate KP–type hierarchy. In section 5 we see a few explicit examples of the above differential hierarchy in the dispersionless limit and prepare the ground for future developments.

2 Multi–Matrix Models: Partition Function

The partition function of the q–matrix model is given by

$$Z_N(t, c) = \int dM_1dM_2 \ldots dM_q e^{TrU}$$

where $M_1, \ldots, M_q$ are Hermitian $N \times N$ matrices and

$$U = \sum_{\alpha=1}^q V_\alpha + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} t_{\alpha,r} M_\alpha^r \quad \alpha = 1, 2, \ldots, q.$$ 

The $p_\alpha$’s are finite or infinite positive integers.

The ordinary procedure to calculate the partition function consists of three steps [16],[17],[18]:

(i). One integrates out the angular parts such that only the integrations over the eigenvalues are left,

$$Z_N(t, c) = \text{const} \int \prod_{\alpha=1}^q \prod_{i=1}^N d\lambda_{\alpha,i} \Delta(\lambda_1)e^{U} \Delta(\lambda_q),$$

where

$$U = \sum_{\alpha=1}^q \sum_{i=1}^N V(\lambda_{\alpha,i}) + \sum_{\alpha=1}^{q-1} \sum_{i=1}^N c_{\alpha,\alpha+1}\lambda_{\alpha,i}\lambda_{\alpha+1,i},$$

and $\Delta(\lambda_1)$ and $\Delta(\lambda_q)$ are Vandermonde determinants.

(ii). One introduces the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \quad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \ldots d\lambda_q \xi_n(\lambda_1)e^{V_1(\lambda_1)+\mu+V_q(\lambda_q)}\eta_m(\lambda_q) = h_n(t, c)\delta_{nm} \quad (2.1)$$
where
\[ \mu \equiv \sum_{a=2}^{q-1} \sum_{r=1}^{\infty} t_{a,r} \lambda_a^r + \sum_{a=1}^{q-1} \alpha_{a+1} \lambda_a \lambda_{a+1}. \]

(iii). We expand the Vandermonde determinants in terms of these orthogonal polynomials and using the orthogonality relation (2.1), we can easily calculate the partition function
\[ Z_N(t, c) = \text{const} \frac{N!}{N-1} \prod_{i=0}^{N-1} h_i \]  

(2.2)

Like in 1–matrix models, knowing the partition function means knowing the coefficients \( h_n(t, c) \)’s. Like in 1–matrix models, we will show in the next section that the information concerning the latter can be encoded in suitable linear systems, which have the advantage of clearly showing the integrable character of the multi–matrix models.

To end this section let us introduce some convenient notations. For any matrix \( M \), we define
\[ (M)_{ij} = M_{ij} \frac{h_j}{h_i}, \quad \tilde{M}_{ij} = M_{ji}, \quad M_l(j) \equiv M_{j,j-l}. \]

We will say that the element \( M_l(j) \) belongs to the \( j \)-th sector. As usual we also introduce a natural gradation
\[ \deg[E_{ij}] = j - i. \]

For a given matrix \( M \), if all its non–zero elements have degrees in the interval \([a, b]\), then we will simply write: \( M \in [a, b] \).

Moreover \( M_+ \) will denote the upper triangular part of \( M \) (including the main diagonal), while \( M_- = M - M_+ \). We will write
\[ \text{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}, \quad \tilde{\text{Tr}}(M) = \sum_{i=0}^{\infty} M_{ii} \]

Analogously, if \( L \) is a pseudodifferential operator, \( L_+ \) means the purely differential part of it, while \( L_- = L - L_+ \).

3 The Coupled Discrete Linear Systems in q–Matrix Model

In order to extract the discrete linear systems that characterize our q–matrix model we have to pass through some preliminaries. First we redefine the orthogonal polynomials in the following way
\[ \Psi_n(\lambda_1) = e^{V_1(\lambda_1)} \xi_n(\lambda_1), \quad \Phi_n(\lambda_q) = e^{V_q(\lambda_q)} \eta_n(\lambda_q). \]
As usual we denote the semi–infinite column vectors with components $\Psi_0, \Psi_1, \Psi_2, \ldots$, and $\Phi_0, \Phi_1, \Phi_2, \ldots$ by $\Psi$ and $\Phi$, respectively. With these polynomials, the orthogonality relation (2.1) becomes
\[
\int \prod_{\beta=1}^{q} d\lambda_\beta \Psi_n(\lambda_1) e^\mu \Phi_m(\lambda_q) = \delta_{nm} h_n(t, c). \tag{3.1}
\]
This orthogonality relation is going to play a crucial role in our analysis. Next we introduce the following $Q$–type matrices
\[
\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Psi_n(\lambda_1) e^\mu \lambda_\alpha \Phi_m(\lambda_q) \equiv Q_{nm}(\alpha) h_m = \tilde{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, \ldots, q. \tag{3.2}
\]
Among them, $Q(1), \tilde{Q}(q)$ are Jacobi matrices: their pure upper triangular part is $I+\sum_i E_{i,i+1}$. Notice also that if we call $Q_{n+1,n}(q) = R_{n+1}$ we immediately find
\[
h_{n+1} = R_{n+1} h_n.
\]
Therefore, like in 1–matrix models, we can express the partition functions in terms of $R_n$.

Beside the above $Q$ matrices, we will need two $P$–type matrices, defined by
\[
\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Psi_n(\lambda_1) e^\mu \frac{\partial}{\partial \lambda_\alpha} \Phi_m(\lambda_q) \equiv P_{nm}(1) h_m \tag{3.3}
\]
\[
\int d\lambda_1 \ldots d\lambda_q \Psi_n(\lambda_1) e^\mu \left( \frac{\partial}{\partial \lambda_q} \Phi_m(\lambda_q) \right) \equiv P_{nm}(q) h_n \tag{3.4}
\]

3.1 The coupling conditions or string equations.

One important preliminary remark is that the matrices (3.2) we introduced above are not completely independent. More precisely all the $Q(\alpha)$’s can be expressed in terms of only one of them and one matrix $P$. Expressing the trivial fact that the integral of the total derivative of the integrand in eq.(3.1) with respect to $\lambda_\alpha, 1 \leq \alpha \leq q$ vanishes, we can easily derive the constraints or coupling conditions
\[
P(1) + c_{12}Q(2) = 0, \tag{3.5a}
\]
\[
c_{\alpha-1,\alpha}Q(\alpha - 1) + V_\alpha' + c_{\alpha,\alpha+1}Q(\alpha + 1) = 0, \quad 2 \leq \alpha \leq q - 1, \tag{3.5b}
\]
\[
c_{q-1,q}Q(q - 1) + \tilde{P}(q) = 0. \tag{3.5c}
\]
where we use the notation
\[
V_\alpha' = \sum_{r=1}^{p_n} r t_{\alpha,r} Q^{r-1}(\alpha), \quad \alpha = 1, 2, \ldots, q.
\]

These conditions play an extremely important role in the study of multi–matrix models. We will also refer to them as string equations.

Before proceeding further, let us pause a bit to make a few comments.
• It is just these coupling conditions that lead to the famous $W_{1+\infty}$–constraints on the partition function at the discrete level [19] (see section 4).

• These conditions explicitly show that the Jacobi matrices depend on the choice of the potentials. We can immediately see that these coupling conditions completely determine the degrees of the matrices $Q(\alpha)$. Since $P(1) - V'_1$ and $P(q) - \bar{V}'_q$ are purely lower triangular matrices, a simple calculation shows that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \quad \alpha = 1, 2, \ldots, q$$

where

$$m_1 = (p_q - 1) \ldots (p_3 - 1)(p_2 - 1)$$
$$m_\alpha = (p_q - 1)(p_{q-1} - 1) \ldots (p_{\alpha+1} - 1), \quad 2 \leq \alpha \leq q - 1$$
$$m_q = 1$$

and

$$n_1 = 1$$
$$n_\alpha = (p_{\alpha-1} - 1) \ldots (p_2 - 1)(p_1 - 1), \quad 2 \leq \alpha \leq q - 1$$
$$n_q = (p_{q-1} - 1) \ldots (p_2 - 1)(p_1 - 1)$$

These equations show that, if we want a finite band structure for the $Q(\alpha)$, only a finite number of perturbations is allowed. Conversely, if we want, say, $Q(1)$ to possess the full discrete KP structure – we should let $p_\alpha \to \infty$ for some $\alpha$; then all these matrices will have infinitely many non–zero diagonal lines. So, generally speaking, these coupling conditions reveal the main difference between multi–matrix models and one–matrix models.

• Finally these conditions suggest that there may be a relation between matrix models and topological models. As an illustration of this remark, we consider a particular case – the two–matrix models. In this situation, the eqs.(3.5a, 3.5c) reduce to

$$P(1) + c_{12}Q(2) = 0, \quad c_{12}Q(1) + \bar{P}(2) = 0,$$

In particular

$$c_{12}Q_+(2) = -V'_+(Q(1)), \quad c_{12}\bar{Q}_+(1) = -V'_+ (\bar{Q}(2)) \quad (3.6)$$

Now, we have $p_1 + p_2 + 1$ series of coordinates, i.e. $p_1 + p_2 + 1$ non–vanishing diagonals of $Q(1)$ and $Q(2)$. The above equations (3.6) give $(p_1 + p_2)$ series of constraints. Therefore we are only left with one–series of independent coordinates, i.e. one can express all the elements of $Q(1)$ and $Q(2)$ in terms of the $R_i$’s (and some coupling constants). This is not surprising since all the information of the matrix model is encoded in the partition function, which only involves $R_i$’s. Finally the remaining coupling conditions provides more series of constraints, so that eventually there are no ‘local’ degrees of freedom left. This means that, if multi–matrix models admit any field theory interpretation, they should correspond to topological field theories.
3.2 The associated discrete linear systems

The derivation of the linear systems associated to our matrix model is very simple. We take the derivatives of eqs. (3.1) with respect to the time parameters $t_{\alpha,r}$, and use eqs. (3.2). We get in this way the time evolution of $\Psi$ and $\Phi$, which can be represented in two different ways:

\textbf{(*)} \textit{Discrete Linear System I:}

\begin{equation}
\begin{cases}
Q(1)\Psi(\lambda_1) = \lambda_1 \Psi(\lambda_1), \\
\frac{\partial}{\partial t_{\alpha,k}} \Psi(\lambda_1) = Q^+_k(1)\Psi(\lambda_1), \quad 1 \leq k \leq p_1, \\
\frac{\partial}{\partial \lambda} \Psi(\lambda_1) = P(1)\Psi(\lambda_1).
\end{cases}
\end{equation}

The corresponding consistency conditions are

\begin{equation}
\begin{align}
\{Q(1), \ P(1)\} &= 1 \\
\frac{\partial}{\partial t_{\alpha,k}} Q(1) &= \{Q(1), \ Q^k(\alpha)\} \\
\frac{\partial}{\partial t_{\alpha,k}} P(1) &= \{P(1), \ Q^k(\alpha)\}
\end{align}
\end{equation}

\textbf{(3.8b)} \textit{and (3.8d)} are hierarchies, referred to henceforth as discrete KP hierarchies, whose integrability and meaning will be discussed later on. The first equation \textbf{(3.8a)} is often referred to as string equation. Throughout the paper we use the latter term also as synonym of coupling conditions (see subsection 3.1).

\textbf{(**)} \textit{Discrete Linear System II:}

\begin{equation}
\begin{cases}
\bar{Q}(q)\Phi(\lambda_q) = \lambda_q \Phi(\lambda_q), \\
\frac{\partial}{\partial t_{\alpha,k}} \Phi(\lambda_q) = \bar{Q}^+_k(q)\Phi(\lambda_q), \\
\frac{\partial}{\partial \lambda} \Phi(\lambda_q) = -\bar{Q}^k(\alpha)\Phi(\lambda_q), \quad 1 \leq k \leq p_\alpha; \quad 2 \leq \alpha \leq q, \\
\frac{\partial}{\partial \lambda} \Phi(\lambda_q) = P(q)\Phi(\lambda_q).
\end{cases}
\end{equation}

with consistency conditions

\begin{equation}
\begin{align}
\{\bar{Q}(q), \ P(q)\} &= 1, \\
\frac{\partial}{\partial t_{\alpha,k}} \bar{Q}(q) &= \{\bar{Q}(q), \ \bar{Q}^k(\alpha)\} \\
\frac{\partial}{\partial t_{\alpha,k}} P(q) &= \{P(q), \ \bar{Q}^k(\alpha)\}
\end{align}
\end{equation}

\textbf{(***)} \textit{The Other Discrete Linear Systems:}
One may wonder what are the equations of motion of the matrices $Q(\alpha)$’s, with $\alpha \neq 1, q$. In the light of the coupling conditions it is not surprising to find out that we have enough information to write them down. They can actually be extracted from the eqs. (3.8a–3.8c) (or (3.10a–3.10c)) and eqs. (3.5a–3.5c). We have

$$\frac{\partial}{\partial t}\beta,k Q(\alpha) = \left[ Q^k(\beta), Q(\alpha) \right], \quad 1 \leq \beta \leq \alpha$$  (3.11a)

$$\frac{\partial}{\partial t}\beta,k Q(\alpha) = \left[ Q(\alpha), Q^k(\beta) \right], \quad \alpha \leq \beta \leq q$$  (3.11b)

where $k$ runs from 1 to $p_\beta$.

It is tedious, but straightforward to prove that all the flows we have introduced so far commute with one another.

It is natural at this point to ask whether eqs. (3.11a,3.11b) can be considered as integrability conditions of some linear systems. The answer is yes, but the relevant linear systems are a bit different from the ones we met so far. Let us start from the orthogonal relation (3.1) once again, and define two series of functions

$$\xi^{(\alpha)}_n(t, \lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \Psi_n(\lambda_1)e^{\mu_\alpha}. \quad (3.12)$$

and

$$\eta^{(\alpha)}_n(t, \lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^{q} d\lambda_\beta e^{\nu_\alpha}\Phi_m(\lambda_q). \quad (3.13)$$

where we denote

$$\mu_\alpha \equiv \sum_{\beta=2}^{\alpha-1} \sum_{r=1}^{\infty} t_{\beta,r} \lambda_{\beta}^r + \sum_{\beta=1}^{\alpha-1} c_{\beta,\beta+1} \lambda_\beta \lambda_{\beta+1}. \quad \nu_\alpha \equiv \sum_{\beta=\alpha+1}^{q-1} \sum_{r=1}^{\infty} t_{\beta,r} \lambda_{\beta}^r + \sum_{\beta=\alpha}^{q-1} c_{\beta,\beta+1} \lambda_\beta \lambda_{\beta+1}. \quad (3.14)$$

Obviously we have

$$\xi^{(1)}_n(t, \lambda_1) = \Psi_n(\lambda_1), \quad \eta^{(q)}_n(t, \lambda_q) = \Phi_m(\lambda_q).$$

but for other values of $\alpha$ one sees immediately that $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ are not polynomials anymore. However they still satisfy an orthogonality relation

$$\int d\lambda_\alpha \xi^{(\alpha)}_n(t, \lambda_\alpha)e^{V_{\alpha}(\lambda_\alpha)}\eta^{(\alpha)}_m(t, \lambda_\alpha) = \delta_{nm} h_n(t, c), \quad \forall 1 \leq \alpha \leq q. \quad (3.14)$$

The $q$ series of functions $\xi^{(\alpha)}_n(t, \lambda_\alpha)$ span $q$ spaces $\mathcal{H}_\alpha$. The $q$ series of functions $\eta^{(\alpha)}_n(t, \lambda_\alpha)$ also span $q$ spaces $\mathcal{H}_\alpha^*$’s, which are dual to $\mathcal{H}_\alpha$ via the orthogonal relations $\mathcal{H}_\alpha^* \cap \mathcal{H}_\beta = \{0\}$.\footnote{This structure bears a perhaps not accidental similarity with the structure of topological field theories.}
Now we extract the spectral equations and the time evolution of these new functions. From eqs. (3.2), we immediately see that

$$\int d\lambda \lambda_\alpha \xi_\alpha(n(t, \lambda_\alpha)) e^{V_\alpha(\lambda_\alpha)} \lambda_\alpha \eta_\alpha(n(t, \lambda_\alpha)) = Q_{nm}(\alpha) n_\lambda(t, c), \quad 1 \leq \alpha \leq q. \quad (3.15)$$

This tells us that the spectral equations are

$$\lambda_\alpha \xi_\alpha = Q(\alpha) \xi_\alpha, \quad 1 \leq \alpha \leq q. \quad (3.16)$$

$$\lambda_\alpha \eta_\alpha = \bar{Q}(\alpha) \eta_\alpha, \quad 1 \leq \alpha \leq q. \quad (3.17)$$

On the other hand, from the definitions (3.12) and (3.13), and making use of eqs. (3.7) and (3.9), we can derive the evolution equations of the functions $\xi_\alpha$ and $\eta_\alpha$, which take the following form

$$\frac{\partial}{\partial t} \xi_\alpha = Q_+^r(\beta) \xi_\alpha, \quad 1 \leq \beta < \alpha - 1, \quad (3.18)$$

$$\frac{\partial}{\partial t} \xi_\alpha = -Q_-^r(\beta) \xi_\alpha, \quad \alpha \leq \beta \leq q. \quad (3.19)$$

and

$$\frac{\partial}{\partial t} \eta_\alpha = \bar{Q}_+^r(\beta) \eta_\alpha, \quad \alpha + 1 \leq \beta \leq q, \quad (3.20)$$

$$\frac{\partial}{\partial t} \eta_\alpha = -\bar{Q}_-^r(\beta) \eta_\alpha, \quad 1 \leq \beta \leq \alpha. \quad (3.21)$$

Now equations (3.16–3.21) represent 2q discrete linear systems, two of which coincide with (3.7) and (3.9), the other (2q − 2) being the additional discrete linear systems which give rise to the compatibility conditions (3.11a, 3.11b) and to the appropriate string equations.

Finally we notice that, thanks to the generalized systems of orthogonal functions just introduced, we can represent all the linear systems in a unified form as follows

$$\begin{cases}
Q(\alpha) \xi(\lambda_\alpha) = \lambda_\alpha \xi(\lambda_\alpha), \\
\frac{\partial}{\partial t} \xi(\lambda_\alpha) = Q^r_+(\beta) \xi(\lambda_\alpha), \quad 1 \leq \beta < \alpha, \\
\frac{\partial}{\partial t} \xi(\lambda_\alpha) = -Q^r_-(\alpha) \xi(\lambda_\alpha), \quad \alpha \leq \beta \leq q.
\end{cases} \quad (3.22)$$

and

$$\begin{cases}
\bar{Q}(\alpha) \eta(\lambda_\alpha) = \lambda_\alpha \eta(\lambda_\alpha), \\
\frac{\partial}{\partial t} \eta(\lambda_\alpha) = \bar{Q}_+^r(\beta) \eta(\lambda_\alpha), \quad \alpha + 1 \leq \beta \leq q, \\
\frac{\partial}{\partial t} \eta(\lambda_\alpha) = -\bar{Q}_-^r(\alpha) \eta(\lambda_\alpha), \quad 1 \leq \beta \leq \alpha.
\end{cases} \quad (3.23)$$

One could phrase the above results by saying that introducing the functions $\xi_\alpha^{(n)}(t, \lambda_\alpha)$'s and $\eta_\alpha^{(n)}(t, \lambda_\alpha)$ can be considered as a formal reductions of the multi–matrix models to 1–matrix models, since we are left with the orthogonal relations
which are similar to the ones in 1–matrix models. The major difference comes
from the fact that $\xi^{(\alpha)}_a(t, \lambda_\alpha)$'s and $\eta^{(\alpha)}_a(t, \lambda_\alpha)$ are very complicated functions rather
than the polynomials.$^3$

A comment is in order concerning the plethora of linear systems we found. First
we notice that, due to the coupling conditions (3.7), any $Q(\alpha)$ can be expressed in
terms of $Q(1)$ and $P(1)$ or, alternatively, of $Q(q)$ and $P(q)$. This means that the
linear system $I$ or the linear system $II$ can be considered as the fundamental ones.
We already noticed that the equations of motion for any $Q(\alpha)$ can be derived from
the flow equations of these two systems. In other words these equations of motion
admit 2q different linearizations. In the following we will mostly be dealing with the
systems $I$ and $II$. However one cannot exclude that a linear system different from $I$
and $II$ might be more convenient for specific purposes.

3.2.1 Coordinatization of the $Q$ and $P$ matrices

Henceforth we will use the following coordinatization of the Jacobi matrices

$$Q(1) = I_+ + \sum_{i}^{m_1} \sum_{l=0}^{a_l(i)} a_l(i) E_{i,i-l}, \quad Q(q) = I_+ + \sum_{i}^{n_q} \sum_{l=0}^{b_l(i)} b_l(i) E_{i,i-l} \quad (3.24)$$

and for the supplementary matrices

$$Q(\alpha) = \sum_{i}^{m_\alpha} \sum_{l=-n_\alpha}^{r_\alpha(i)} T^{(\alpha)}_l(i) E_{i,i-l}, \quad 2 \leq \alpha \leq q - 1. \quad (3.25)$$

One should never forget that

$$\begin{cases}
  a_l(i) = 0 & l > i \\
  b_l(i) = 0 & l > i \\
  T^{(\alpha)}_l(i) = 0 & l > i
\end{cases} \quad (3.26)$$

Let us start from the coordinatization (3.24). We can see immediately that

$$\left( Q_+(1) \right)_{ij} = \delta_{j,i+1} + a_0(i) \delta_{i,j}, \quad \left( Q_-(q) \right)_{ij} = R_l \delta_{j,i-1}$$

Therefore we can write down the $t_{1,1}$– and $t_{q,1}$–flows explicitly

$$\frac{\partial}{\partial t_{1,1}} a_l(j) = a_{l+1}(j+1) - a_{l+1}(j) + a_l(j) \left( a_0(j) - a_0(j-l) \right) \quad (3.27a)$$
$$\frac{\partial}{\partial t_{q,1}} a_l(j) = R_{j-l+1} a_{l+1}(j) - R_j a_{l-1}(j-1) \quad (3.27b)$$
$$\frac{\partial}{\partial t_{1,1}} b_l(j) = R_{j-l+1} b_{l+1}(j) - R_j b_{l-1}(j-1) \quad (3.27c)$$
$$\frac{\partial}{\partial t_{q,1}} b_l(j) = b_{l+1}(j+1) - b_{l+1}(j) + b_l(j) \left( b_0(j) - b_0(j-l) \right) \quad (3.27d)$$
$$\frac{\partial}{\partial t_{1,1}} T^{(\alpha)}_l(j) = T^{(\alpha)}_{l+1}(j+1) - T^{(\alpha)}_{l+1}(j) + T^{(\alpha)}_l(j) \left( a_0(j) - a_0(j-l) \right) \quad (3.27e)$$
$$\frac{\partial}{\partial t_{q,1}} T^{(\alpha)}_l(j) = R_{j-l+1} T^{(\alpha)}_{l+1}(j) - R_j T^{(\alpha)}_{l-1}(j-1) \quad (3.27f)$$

$^3$Does this mean that multi–matrix models can be realized as 1-matrix models if we choose suitable
non–polynomial interactions?
We will also need the following coordinatization of \( P(1) \) and \( P(q) \)

\[
P(1) = \sum_{i} \sum_{t=-\infty}^{s_{1}} P_{i}^{(1)}(t) e_{i,i-l}
\]

\[
P(q) = \sum_{i} \sum_{t=-\infty}^{s_{q}} P_{i}^{(q)}(t) e_{i,i-l}
\]

where \( s_{1} \) and \( s_{q} \) are suitable positive (possibly infinite) integers.

### 3.3 Flow equations for the partition function.

As a byproduct of the above results we can easily get the flow equations for the partition function

\[
\frac{\partial}{\partial t_{\alpha,r}} \ln Z_{N}(t,c) = \text{Tr} \left( Q_{r}(\alpha) \right),
\]

\[
1 \leq r \leq p_{\alpha}; \quad 1 \leq \alpha \leq q
\]

Using the consistency conditions, we can rewrite these equations in the following form

\[
\frac{\partial^{2}}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_{N}(t,c) = \left( Q_{r}(\alpha) \right)_{N,N-1},
\]

\[
1 \leq r \leq p_{\alpha}; \quad 1 \leq \alpha \leq q
\]

or equivalently

\[
\frac{\partial^{2}}{\partial t_{q,1} \partial t_{\alpha,r}} \ln Z_{N}(t,c) = \left( \bar{Q}_{r}(\alpha) \right)_{N,N-1},
\]

\[
1 \leq r \leq p_{\alpha}; \quad 1 \leq \alpha \leq q
\]

We end this subsection with one more remark. The information concerning multi–matrix models is encoded in the discrete hierarchy (3.11a–3.11b). Once we solve these equations, we can reconstruct the partition functions of the matrix models: they will automatically satisfy (3.30).

### 3.4 The hidden 2-dimensional Toda lattice

In subsection 3.2 we have shown that multi–matrix models are equivalent to certain discrete linear systems subject to some constraints. If we ignore the coupling conditions, the discrete linear systems lead to discrete hierarchies. In fact we will show that these discrete or lattice hierarchies, in the \( q = 2 \) case, are nothing but 2-dimensional Toda lattice hierarchies [21][22], and in the generic \( q \) case, they are 2-dimensional Toda lattices with additional flows.

Let us start from the first flow equations written above. In particular, from eq. (3.27a), we have

\[
\frac{\partial}{\partial t_{q,1}} a_{0}(j) = R_{j+1} - R_{j}
\]
while, from (3.30), one can show that
\[ \frac{\partial}{\partial t_{1,1}} R_j = R_j(a_0(j) - a_0(j - 1)) \] (3.34)

Combining these two equations we obtain the following 2-dimensional Toda lattice equation
\[ \frac{\partial^2}{\partial t_{1,1} \partial t_{q,1}} \ln R_j = R_{j+1} - 2R_j + R_{j-1} \] (3.35)

In fact in terms of the coordinates \( \phi_j = \ln h_j \), the above equation takes the familiar form
\[ \frac{\partial^2}{\partial t_{1,1} \partial t_{q,1}} \phi_j = e^{\phi_{j+1} - \phi_j} - e^{\phi_{j} - \phi_{j-1}} \] (3.36)

We could have gotten the same result from
\[ \frac{\partial}{\partial t_{1,1}} b_0(j) = R_{j+1} - R_j \] (3.37)

and from
\[ \frac{\partial}{\partial t_{q,1}} R_j = R_j(b_0(j) - b_0(j - 1)) \] (3.38)

the latter being obtained again from (3.30).

(3.36) is nothing but the two–dimensional Toda lattice equation. In the \( q = 2 \) case (i.e. 2–matrix model) the other time flows generate the complete 2–dimensional Toda lattice hierarchy [21]. From the analysis in subsection 3.2, we know that the system is restricted by the coupling conditions, therefore we can say that 2–matrix models are nothing but constrained 2–dimensional Toda lattices. The \( q \)–matrix models with \( q > 2 \) contains not only the Toda lattice hierarchy, but also additional series of flows.

It is perhaps interesting to remark that, from eqs. (3.34) and (3.36), the parameters \( t_{1,1} \) and \( t_{q,1} \) could be interpreted as space coordinates while the other parameters are really time flow parameters. Therefore the multi–matrix models hierarchies could underlie a 2+1 D theory.

### 3.5 Integrability

The discrete linear systems ensuing from multi–matrix models are integrable as is guaranteed by the fact that all the flows commute with one another. Most integrable systems we know possess a bi–hamiltonian structure with Hamiltonians in involutions. For the linear systems of multi–matrix models we are discussing here we can define a bi–hamiltonian structure, i.e. two compatible Poisson brackets.

Let us be more explicit. Following [23], let us consider system I and forget, for the time being, the coupling conditions. We simplify the notation by writing \( Q \) instead of \( Q(1) \). Bearing in mind the coordinatization (3.24), we notice that all the functions \( a_l(j) \) can be represented as
\[ f_X(Q) \equiv \text{Tr}(QX) \equiv \langle QX \rangle 

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by means of a suitable upper triangular matrix $X$. We introduce two Poisson brackets as follows

$$\{ f_X, f_Y \}_1(Q) = \langle Q[X,Y] \rangle \quad (3.39)$$

$$\{ f_X, f_Y \}_2(Q) = \langle (XQ)_+YQ \rangle - \langle QX_+QY \rangle + \langle QD_I_+Y \rangle - \langle QYI_+D \rangle \quad (3.40)$$

where

$$D = \sum_{j=0}^{\infty} \sum_{l=0}^{j} [X,Q]_{ll} E_{j+1,j}$$

In terms of the coordinates we have

$$\{ a_m(i), a_n(j) \}_1 = a_{m+n}(j)\delta_{i,j-n} - a_{m+n}(i)\delta_{j,i-m}$$

and

$$\{ a_m(i), a_n(j) \}_2 = a_{m+n+1}(j)\delta_{i,j-n-1} - a_{m+n+1}(i)\delta_{i,j+m+1}$$

$$+ \sum_{l=0}^{m} \left( a_l(i)a_{n+m-l}(j)\delta_{i,j-n+l} - a_{n+m-l}(i)a_l(j)\delta_{i,j+m-l} \right)$$

$$+ a_m(i)a_n(j) \sum_{l=j-n}^{j} (\delta_{il} - \delta_{i-m,l})$$

Introducing the Hamiltonians $H_k \equiv H_k(1) \equiv \frac{1}{k} \text{Tr}(Q^k)$, one can verify that the two Poisson brackets are compatible, i.e.

$$\{ H_{k+1}, f_X \}_1(Q) = \{ H_k, f_X \}_2(Q),$$

that they are in involution and that they generate the flows with respect to $t_{1,k}$.

With minor modifications we could do the same for system II. For example, in the 2–matrix model we can find another couple of compatible Poisson brackets and another set of Hamiltonians $H_k(2)$ in involutions. As we already noticed the flows generated by the two series of Hamiltonians commute. However we are not able to decide whether the $H_k(1)$’s commute with the $H_k(2)$’s. If not, this would be an example of integrable systems with non–commuting hamiltonians.

4 $W_{1+\infty}$ Constraints in Two–Matrix Models

It was shown in [13] that the path integral of multi–matrix models has a large symmetry that can be translated into algebraic constraints on the partition function. It is instructive to see how this large symmetry is inherited by the corresponding linear systems. We will limit ourselves here to two–matrix models. Starting from the string equations and the discrete hierarchies we found in the previous section, we will prove that the partition function of the two–matrix model satisfies $W_{1+\infty}$–constraints. To this end it is actually convenient to consider a much more general model than the one studied so far, which is characterized by the most general polynomial coupling
between the two matrices. The reason is that this enlarged model possesses a larger symmetry than the original one. By using the ensuing algebraic structure our goal will be reached in a much easier way. Eventually, by reduction, we will find the result appropriate to the original 2–matrix model.

4.1 The generalized coupling conditions

Let us introduce a more general polynomial interaction term into the partition function

\[ Z_N(t, C) = \text{const} \int \prod_{\alpha=1}^{2} \prod_{i=1}^{N} d\lambda_{\alpha,i} \Delta(\lambda_1) \Delta(\lambda_2) e^U, \quad (4.1) \]

\[ U = \sum_{k=1}^{\infty} t_{1,k} \sum_{i=1}^{N} \lambda_{1,i}^k + \sum_{r=1}^{\infty} t_{2,r} \sum_{i=1}^{N} \lambda_{2,i}^r + \sum_{a,b \geq 1} C_{ab} \sum_{i=1}^{N} \lambda_{1,i}^a \lambda_{2,i}^b. \quad (4.2) \]

In order to recover our original two–matrix model we have to simply set \( C_{ab} = 0 \) except for \( C_{11} = c_{12} \), where the latter is the coupling of section 2.

We follow the same procedure we followed in section 2 after integrating out the angular part, and introduce the orthogonal polynomials

\[ \xi_n(\lambda_1, t) = \lambda_1^n + \cdots; \quad \eta_n(\lambda_2, t) = \lambda_2^n + \cdots. \]

which satisfy the improved orthogonal relation

\[ \int d\lambda_1 d\lambda_2 \xi_n(\lambda_1, t) e^U \eta_m(\lambda_2, t) = h_n \delta_{nm}. \quad (4.3) \]

where

\[ U(\lambda_1, \lambda_2) = V_1(\lambda_1) + V_2(\lambda_2) + \sum_{a,b \geq 1} C_{ab} \lambda_1^a \lambda_2^b, \]

\[ V_1(\lambda_1) = \sum_{k=1}^{\infty} t_{1,k} \lambda_1^k, \quad V_2(\lambda_2) = \sum_{r=1}^{\infty} t_{2,r} \lambda_2^r. \]

Next we define the \( Q \) matrices

\[ \int d\lambda_1 d\lambda_2 \xi_n(\lambda_1, t) e^U \lambda_\alpha \eta_m(\lambda_2, t) = Q_{nm}(\alpha) h_m = \bar{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, 2 \quad (4.4) \]

and their conjugates

\[ P_\circ(1) \xi = \frac{\partial}{\partial \lambda_1} \xi, \quad P_\circ(2) \eta = \frac{\partial}{\partial \lambda_2} \eta. \]

We appended the label \( \circ \) in order to distinguish these operators from the previously defined \( P(1) \) and \( P(2) \).

As in section 3 we can get the spectral equations

\[ \lambda_1 \xi = Q(1) \xi, \quad \lambda_2 \eta = \bar{Q}(2) \eta, \quad (4.5) \]

\(^8\)At times, eqs. (4.19) and (4.25), we will use a compact notation in which \( C_{0,b} \equiv t_{2,b} \) and \( C_{a,0} \equiv t_{1,a} \).
and the equations of motion of the polynomials, in the form of two coupled linear systems, which coincide with the systems I and II of the previous section when \( q = 2 \). The only differences can be found when the coupling conditions are involved. Noting the spectral equations, one can easily see that the coupling conditions become

\[
P_2(1) + V_1'(1) + \sum_{a,b \geq 1} aC_{ab}Q^{a-1}(1)Q^b(2) = 0, \quad \text{(4.6a)}
\]

\[
\tilde{P}_2(2) + V_2'(2) + \sum_{a,b \geq 1} bC_{ab}Q^a(1)Q^{b-1}(2) = 0. \quad \text{(4.6b)}
\]

The time evolution of the partition function is given by eq. (3.30), with \( \alpha = 1 \) and 2. We can also write down the coupling dependence of the partition function

\[
\frac{\partial}{\partial C_{ab}} \ln Z_N(t, C) = \text{Tr} \left( Q^a(1)Q^b(2) \right), \quad \forall a, b \geq 1. \quad \text{(4.7)}
\]

This is all we need for the derivation of the constraints.

## 4.2 Virasoro constraints

The \( W_{1+\infty} \)-algebraic constraints contain a particularly important subset, the Virasoro constraints. We begin with them. Since the discussion is valid simultaneously for both linear systems, we temporarily omit the system indices and consider a general Jacobi matrix \( Q \) and its conjugate \( P \)

\[
Q_{ij} = \delta_{j,i+1} + S_j\delta_{i,j} + L_j\delta_{j,i-1} + \ldots, \quad [Q, P] = 1. \quad \text{(4.8)}
\]

As was shown in [20], the string equation completely determines the matrix \( P \). Proceeding as in [20], we can express the string equation as a series of equations

\[
\text{Tr} \left( Q^{n+1}(1) \left( P_2(1) + V_1'(1) + \sum_{a,b \geq 1} aC_{ab}Q^{a-1}(1)Q^b(2) \right) \right) = 0, \quad n \geq -1 \quad \text{(4.9)}
\]

Now using the relation between partition function and \( Q \) matrix, these are equivalent to

\[
(L_n^{[\text{l}]}(1) + T_n^{[\text{l}]}(1)) Z_N(t; C) = 0, \quad n \geq -1. \quad \text{(4.10)}
\]

where

\[
L_n^{[\text{l}]}(1) = \sum_{k=2}^{\infty} kt_{1,k} \frac{\partial}{\partial t_{1,1}} + Nt_{1,1},
\]

\[
V_1'(1) = \sum_{k=1}^{\infty} kt_{1,k}Q^{k-1}(1),
\]

\[
V_1''(1) = \sum_{k=2}^{\infty} k(k-1)t_{1,k}Q^{k-2}(1),
\]

etc.
\[
\mathcal{L}_0^{[1]}(1) = \sum_{k=1}^{\infty} k t_{1,k} \frac{\partial}{\partial t_{1,k}} + \frac{1}{2} N(N + 1), \tag{4.11}
\]
\[
\mathcal{L}_n^{[1]}(1) = \sum_{k=1}^{\infty} k t_{1,k} \frac{\partial}{\partial t_{1,k+n}} + (N + \frac{n+1}{2}) \frac{\partial}{\partial t_{1,n}} + \frac{1}{2} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,n-k}}, \quad n \geq 1.
\]

and

\[
T_{-1}^{[1]}(1) \equiv \sum_{a \geq 2, b \geq 1} a C_{ab} \frac{\partial}{\partial C_{a-1,b}} + \sum_{b \geq 1} C_{1b} \frac{\partial}{\partial t_{2,b}}, \tag{4.12a}
\]
\[
T_{n}^{[1]}(1) \equiv \sum_{a,b \geq 1} a C_{ab} \frac{\partial}{\partial C_{a+n,b}}, \quad n \geq 0, \tag{4.12b}
\]

These operators satisfy the Virasoro algebras

\[
[L_n^{[1]}(1), L_m^{[1]}(1)] = (n-m) L_{n+m}^{[1]}(1), \quad n, m \geq -1, \tag{4.13a}
\]
\[
[L_n^{[1]}(1), T_m^{[1]}(1)] = 0, \quad n, m \geq -1, \tag{4.13b}
\]
\[
[T_n^{[1]}(1), T_m^{[1]}(1)] = (n-m) T_{n+m}^{[1]}(1), \quad n, m \geq -1. \tag{4.13c}
\]

In order to recover the result we are interested in for our original two–matrix model we should simply set, in eqs. (4.10, 4.12a, 4.12b), \(C_{ab} = 0\) except for \(C_{11} = c_{12}\), where the latter is the coupling of section 2.

### 4.3 Higher rank constraints

In the previous section we obtained the Virasoro constraints, which may be referred to as rank 2 tensorial constraints. In the following we deal with higher rank constraints. In order to get the spin–3 operators, we introduce the following notations

\[
\mathcal{V} \equiv \sum_{a,b \geq 1} C_{ab} Q_a^{(1)} Q_b^{(2)},
\]
\[
\mathcal{V}' \equiv \sum_{a,b \geq 1} a C_{ab} Q_a^{a-1} Q_b^{(2)},
\]
\[
: \mathcal{V}^2 : \equiv \sum_{a,b,a',b' \geq 1} C_{ab} C_{a'b'} Q^{a+a'} Q^{b+b'} (1)^{a} (1)^{b} (1)^{a'} (1)^{b'}.
\]

From the trivial relation

\[
\int d\lambda_1 d\lambda_2 \frac{\partial^2}{\partial \lambda_1^2} \left( \xi_n(\lambda_1, t) e^{U} \eta_m(\lambda_2, t) \right) = 0,
\]

we get the following identity

\[
- P_2^{[1]}(1) + V''(1) + V'^2(1) + 2V'(1) V' + : V'^2 : + V'' = 0, \tag{4.14}
\]

We claim that multiplying (4.14) by \(Q^{n+2}(1)\) from the left and taking the trace, we obtain the rank–3 constraints

\[
(\mathcal{L}_n^{[2]}(1) - T_n^{[2]}(1)) Z_N(t; C) = 0, \quad n \geq -2. \tag{4.15}
\]
with the definitions

\[
L_{-2}^{[2]}(1) \equiv \sum_{l_1+l_2=3} l_1 l_2 t_{1,l_1} t_{1,l_2} \frac{\partial}{\partial t_{1,l_1+l_2}} + \sum_{l=3}^{\infty} l t_{1,l} \sum_{k=1}^{l-3} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,l-k}} + 2N \sum_{l=3}^{\infty} l t_{1,l} \frac{\partial}{\partial t_{1,l}} + N t_{1,2} + \frac{1}{2} t_{1,1}^2, \quad (4.16a)
\]

\[
L_{-1}^{[2]}(1) \equiv \sum_{l_1+l_2=1} l_1 l_2 t_{1,l_1} t_{1,l_2} \frac{\partial}{\partial t_{1,l_1+l_2}} + \sum_{l=2}^{\infty} l t_{1,l} \sum_{k=1}^{l-2} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,l-k}} + 2N \sum_{l=2}^{\infty} l t_{1,l} \frac{\partial}{\partial t_{1,l}} + N^2 t_{1,1} + \frac{1}{3} N(N+1)(N+2), \quad (4.16b)
\]

\[
L_{0}^{[2]}(1) \equiv \sum_{l_1+l_2=1} l_1 l_2 t_{1,l_1} t_{1,l_2} \frac{\partial}{\partial t_{1,l_1+l_2}} + \sum_{l=1}^{\infty} l t_{1,l} \sum_{k=1}^{l-1} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,l-k}} + \frac{1}{2(n+3)} \sum_{l=1}^{n-2} \sum_{k=1}^{n-l-1} (n-l+2) \frac{\partial^3}{\partial t_{1,l} \partial t_{1,k} \partial t_{1,n-l-k}} + (N^2 + (n+1)N + \frac{1}{6}(n+1)(n+2)) \frac{\partial}{\partial t_{1,n}} + (2N + n + 2) L_{-1}^{[1]}(1) \quad n \geq 1. \quad (4.16d)
\]

and

\[
T_{-2}^{[2]}(1) \equiv \sum_{a+a'=3, b, b' \geq 1} a a' C_{ab} C_{a'b'} \frac{\partial}{\partial C_{a+a'-2, b+b'}} + \sum_{a, b \geq 1} a(a-1) C_{ab} \frac{\partial}{\partial C_{a-2, b}} + \sum_{b, b' \geq 1} C_{1b} C_{1b'} \frac{\partial}{\partial t_{1, b+b'}} + 2 \sum_{b \geq 1} C_{2b} \frac{\partial}{\partial t_{2, b}}, \quad (4.17a)
\]

\[
T_{n}^{[2]}(1) \equiv \sum_{a, a', b, b' \geq 1} a a' C_{ab} C_{a'b'} \frac{\partial}{\partial C_{a+a'+n, b+b'}} + \sum_{a, b \geq 1} a(a-1) C_{ab} \frac{\partial}{\partial C_{a+n, b}}, \quad n \geq -1. \quad (4.17b)
\]

They satisfy the following algebra

\[
[L_{n}^{[2]}(1), L_{m}^{[1]}(1)] = (n-2m)L_{n+m}^{[2]}(1) + m(m+1)L_{n+m}^{[1]}(1) \quad (4.18a)
\]

\[
[L_{n}^{[2]}(1), L_{m}^{[2]}(1)] = 2(n-m)L_{n+m}^{[3]}(1) - (n-m)(n+m+3)L_{n+m}^{[2]}(1) \quad (4.18b)
\]

\[
[T_{n}^{[2]}(1), T_{m}^{[1]}(1)] = (n-2m)T_{n+m}^{[2]}(1) - m(m+1)T_{n+m}^{[1]}(1) \quad (4.18c)
\]

\[
[T_{n}^{[2]}(1), T_{m}^{[2]}(1)] = 2(n-m)T_{n+m}^{[3]}(1) + (n-m)(n+m+3)T_{n+m}^{[2]}(1) \quad (4.18d)
\]
where we have introduced the spin–4 operator
\[
T_n^{[3]}(1) \equiv \sum_{a_1, b_1, a_2, b_2, a_3, b_3 \geq 1} \frac{a_1 a_2 a_3 C_{a_1 b_1} C_{a_2 b_2} C_{a_3 b_3}}{\partial C_{a_1+a_2+a_3+n,b_1+b_2+b_3}}\]
\[
+ \sum_{a_1, b_1, a_2, b_2 \geq 1} \frac{3}{2} a_1 a_2 (a_1 + a_2 - 2) C_{a_1 b_1} C_{a_2 b_2} \frac{\partial}{\partial C_{a_1+a_2+n,b_1+b_2}}\]
\[
+ \sum_{a,b \geq 1} a(a-1)(a-2) C_{a b} \frac{\partial}{\partial C_{a+n,b}}, \quad n \geq -3.
\] (4.19)

As for \(L_n^{[3]}(1)\), it is a very complicated expression, which will not be written down here. One can use eq. (4.18b) as its definition.

Now let us prove our claim—the constraints (4.15). At first, we notice that
\[
\text{Tr}(P_n^2(1)) = 0,
\]
So taking the trace of eq.(4.14), we can easily rewrite it as follows
\[
(L_n^{[2]}(1) - T_n^{[2]}(1)) Z_N(t; C) = 0,
\] (4.20)
which is a particular case of eqs.(4.15), when \(n = -2\). Then, the algebras (4.18a) and (4.18c) guarantee that all the other constraints in eqs.(4.15) are true as well. On the other hand from (4.18b) and (4.18d), together with (4.15), we obtain the rank 4 operator constraints,
\[
(L_n^{[3]}(1) + T_n^{[3]}(1)) Z_N(t; C) = 0, \quad n \geq -3.
\] (4.21)

We see that a larger algebraic structure helps us simplifying the calculations drastically. Therefore it is evident that we should keep introducing higher and higher tensor operators in order to close the algebra generated by \(\{L_n^{[1]}(1) + T_n^{[1]}(1)\}, n \geq -1; (L_n^{[2]}(1) - T_n^{[2]}(1)), n \geq -2\). For example, from eqs.(4.18b) and (4.18d), we obtain the rank–4 operators \(L_n^{[3]}(1)\) and \(T_n^{[3]}(1)\). In turn the commutators of these operators with \(L_n^{[2]}(1)\) and \(T_n^{[2]}(1)\) generate the rank–5 operators and so on. In this way we must introduce the whole sequence of operators
\[
T_n^{[r]}(1) \equiv \sum_{a_1, b_1, \ldots, a_r, b_r \geq 1} a_1 a_2 \ldots a_r C_{a_1 b_1} \ldots C_{a_r b_r} \frac{\partial}{\partial C_{a_1+\ldots+a_r+n,b_1+\ldots+b_r}}\]
+ lower orders

and the corresponding \(L_n^{[r]}(1)\). All these generators form a closed \(W_{1+\infty}\)-algebra
\[
[T_n^{[r]}(1), T_m^{[s]}(1)] = (sn - rm) T_n^{[r+s-1]}(1) + \ldots,\] (4.22a)
\[
[L_n^{[r]}(1), L_m^{[s]}(1)] = (sn - rm) L_n^{[r+s-1]}(1) + \ldots,\] (4.22b)
\[
[T_n^{[r]}(1), L_m^{[s]}(1)] = 0,\] (4.22c)
for \(r, s \geq 1; n \geq -r, m \geq -s\). Here dots denote lower than \(r+s-1\) rank operators. Since we have already shown that a combination of the rank–2 and rank–3 operators
annihilate the partition function, eqs. (4.10) and (4.13), then, due to the above $W_{1+\infty}$ algebra \footnote{In fact we have two $W_{1+\infty}$ algebras, one formed by the $T$ generators and the other by the $\mathcal{L}$ generators, but since for the constraints the relevant algebra is the direct sum $T_n^\text{[1]} + \mathcal{L}_n^\text{[1]}$, we keep speaking about a unique algebra.}, the same combination of higher rank generators will annihilate the partition function, that is to say
\[
(L_n^\text{[r]}(1) - (-1)^r T_n^\text{[r]}(1))Z_N(t; C) = 0, \quad r \geq 1; \quad n \geq -r. \tag{4.23}
\]

In the same way from the second linear system we can derive another copy of $W_{1+\infty}$ algebra as well as another version of the $W_{1+\infty}$ constraints
\[
(L_n^\text{[r]}(2) - (-1)^r T_n^\text{[r]}(2))Z_N(t; C) = 0, \quad r \geq 1. \tag{4.24}
\]
where $L_n^\text{[r]}(2)$ can be obtained from $L_n^\text{[r]}(1)$ by simply replacing the $t_{1,k}$ parameters with the $t_{2,r}$ ones. However, as for the $T_n^\text{[r]}(2)$ generators, we have

\[
T_n^\text{[1]}(2) = \sum_{a,b \geq 1} bC_{ab} \frac{\partial}{\partial C_{a,b+n}}, \quad n \geq -1
\]
\[
T_n^\text{[2]}(2) = \sum_{a,b,a',b' \geq 1} bb' C_{ab} C_{a' b'} \frac{\partial}{\partial C_{a+a',b+b'+n}}
+ \sum_{a,b \geq 1} b(b-1)C_{ab} \frac{\partial}{\partial C_{a,b+n}}, \quad n \geq -2, \tag{4.25a}
\]
\[
T_n^\text{[3]}(2) = \sum_{a_1,b_1,a_2,b_2,a_3,b_3 \geq 1} b_1 b_2 b_3 C_{a_1 b_1} C_{a_2 b_2} C_{a_3 b_3} \frac{\partial}{\partial C_{a_1+a_2+a_3,b_1+b_2+b_3+n}}
+ \sum_{a_1,b_1,a_2,b_2 \geq 1} \frac{3}{2} b_1 b_2 (b_1 + b_2 - 2) C_{a_1 b_1} C_{a_2 b_2} \frac{\partial}{\partial C_{a_1+a_2,b_1+b_2+n}}
+ \sum_{a,b \geq 1} b(b-1)(b-2) C_{ab} \frac{\partial}{\partial C_{a,b+n}}, \quad n \geq -3, \tag{4.25b}
\]
\[
T_n^\text{[r]}(2) = \sum_{a_1,b_1,...,a_r,b_r \geq 1} b_1 b_2 ... b_r C_{a_1 b_1} ... C_{a_r b_r} \frac{\partial}{\partial C_{a_1+...+a_r,b_1+...+b_r+n}}
+ \text{lower order terms}, \tag{4.25c}
\]

with the following algebra, which is isomorphic to (4.22a, 4.22b)
\[
[T_n^\text{[r]}(2), T_m^\text{[s]}(2)] = (sn - rm) T_{n+m}^\text{[r+s-1]}(2) + \ldots, \tag{4.26a}
\]
\[
[L_n^\text{[r]}(2), L_m^\text{[s]}(2)] = (sn - rm) L_{n+m}^\text{[r+s-1]}(2) + \ldots, \tag{4.26b}
\]
\[
[T_n^\text{[r]}(2), L_m^\text{[s]}(2)] = 0, \tag{4.26c}
\]

for $r, s \geq 1; n \geq -r, m \geq -s$.

In conclusion, we see that, in the case of a general interaction, there are two isomorphic $W_{1+\infty}$ algebra constraints. The closed algebra formed by these two pieces of $W_{1+\infty}$ algebras gives the complete constraints (one should notice that they do not form a direct product).
4.4 $W_{1+\infty}$ constraints

From eq. (4.23), it is easy to see that if we return to our original model, i.e. we set all the $C_{ab}, a, b \geq 1$ equal to zero, and only keep $C_{11} = c_{12} = c \neq 0$, we have

$$T_n^{[r]}(1)Z_N(t; C) = c^r \text{Tr}(Q^r(1)Q^r(2))Z_N(t; C) \quad (4.27a)$$

$$T_n^{[r]}(2)Z_N(t; C) = c^r \text{Tr}(Q^r(1)Q^r(2))Z_N(t; C). \quad (4.27b)$$

Substituting these into eqs. (4.23) and (4.24), we get the following constraints

$$W_n^{[r]}Z_N(t, C) = 0, \quad r \geq 0; \quad n \geq -r,$$

(4.28)

where

$$W_n^{[r]} = (-c)^n L_n^{[r]}(1) - L_{-n}^{[r+n]}(2). \quad (4.29)$$

form a $W_{1+\infty}$ algebra. One can explicitly check that this result coincides with the one in ref. [19].

A further reduction is possible. Suppose we set $t_{2,k} = 0, k > q$, then from the above equation, we have

$$L_{-n}^{[r]}(1) + c^r \text{Tr}(Q^r(2)) = 0 \quad (4.30)$$

Substituting it into the other constraints, we can get other $W_{1+\infty}$ constraints, which are only expressed in terms of $t_{2,k}$ with $1 \leq k \leq q$ (beside all the $t_{1,k}$'s, of course). The operators form a subalgebra of the $W_{1+\infty}$ in eq. (4.28–4.29).

5 Differential Hierarchies of Multi–Matrix Models.

So far we have shown that multi–matrix models can be represented by means of coupled discrete linear systems, whose consistency conditions give rise to discrete KP hierarchies and string equations. In this section, we will use the method introduced in [13] to transform the discrete linear systems into equivalent differential systems whose consistency conditions are purely differential hierarchies. In the most general case they coincide with the generalized KP hierarchy. In this elaboration no continuum limit is involved.

As anticipated in the introduction the clue to the construction are the $t_{1,1}$ and $t_{q,1}$ flows. On the one hand, it is just the first flow equations of $\Psi$ and $\Phi$ that enable us to eliminate the difference operations from the RHS of the other flow equations and to recast the discrete linear systems into a purely differential form. On the other hand, using them, we can express the flows of the coordinates of the $j$–th sector as functions of the coordinates in the same sector. This remarkable property is far from obvious a priori.

(*) Differential linear system I.

We apply the procedure outlined above to the discrete linear systems derived in the previous section. Since for q–matrix models we derived $2q$ discrete linear
systems, when we pass to the differential language we expect to find 2q differential linear systems. We start with the DLS (3.7). Using the eqs. (3.34) and (3.33), we have

\[ a_0(j - 1) = a_0(j) - (\ln R_j)' \]
\[ a_0(j - 2) = a_0(j) - (\ln R_j)' - \left( \ln \left[ R_j - a_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1} \partial t_{q,1}} \right] \right)' \]

where for any function \( f(t) \), we write

\[ f' \equiv \frac{\partial f}{\partial t_{1,1}} \equiv \partial f, \quad \dot{f} \equiv \frac{\partial f}{\partial t_{q,1}} \equiv \tilde{\partial} f \]

In general, defining

\[ R_{j+r} \equiv F_r^+(j), \quad a_0(j + r) \equiv G_r^+(j) \]
\[ R_{j-r} \equiv F_r^-(j), \quad a_0(j - r) \equiv G_r^-(j) \]

we obtain the recursion relation

\[ F_{r+1}^+(j) = F_r^+(j) + \hat{G}_r^+(j) \] (5.1a)
\[ G_{r+1}^+(j) = G_r^+(j) + \left( \ln |F_r^+(j) + \hat{G}_r^+(j)| \right)' \] (5.1b)
\[ F_{r-1}^-(j) = F_r^-(j) - \hat{G}_r^-(j) + \frac{\partial^2 \ln F_r^-}{\partial t_{1,1} \partial t_{q,1}} \] (5.1c)
\[ G_{r-1}^-(j) = G_r^-(j) - (\ln F_r^-)' \] (5.1d)

These results guarantee: 1) that all the \( a_0(i) \)'s and \( R_i \)'s (\( i \neq j \)) can be expressed in terms of \( a_0(j) \) and \( R_j \) and their \( t_{1,1} \) and \( t_{q,1} \)-derivatives; 2) that, substituting these results into eqs. (3.27a–3.27f), we can recursively express all \( a_l(i) \)'s and \( T_l^{(a)}(i) \)'s (\( i \neq j \)) as functions of the coordinates in the \( j \)-th sector.

Now let us consider the \( t_{1,1} \)- and \( t_{q,1} \)-flows of \( \Psi_j \). From eqs. (5.4), we can write down their explicit forms

\[ \frac{\partial}{\partial t_{1,1}} \Psi_j = \Psi_{j+1} + a_0(j) \Psi_j \] (5.2)
\[ \frac{\partial}{\partial t_{q,1}} \Psi_j = -R_j \Psi_{j-1} \] (5.3)

which lead to the following equalities

\[ \Psi_{j+1} = \hat{A}_j \Psi_j, \quad \Psi_{j-1} = \hat{B}_{j-1} \Psi_j \] (5.4)

where

\[ \hat{A}_j \equiv \partial - a_0(j) = -\tilde{\partial}^{-1} R_{j+1} \]
\[ \hat{B}_j \equiv \partial^{-1} \sum_{i=0}^{\infty} (a_0(j) \partial^{-1})^i = -\frac{1}{R_{j+1}} \tilde{\partial} \]
It is easy to see that
\[ \hat{A}_j \hat{B}_j = \hat{B}_j \hat{A}_j = 1 \quad \forall j \geq 1 \]

Using eq.(5.4) we can rewrite the spectral equation in (3.7) as a purely differential equation
\[ L_j(1) \Psi_j = \lambda_1 \Psi_j \quad (5.5) \]

where
\[ L_j(1) = \partial + \sum_{l=1}^{j,m_1} a_l(j) \frac{1}{\partial - a_0(j - l - 1)} \cdot \frac{1}{\partial - a_0(j - l)} \cdot \frac{1}{\partial - a_0(j - l + 1)} \]
\[ \equiv \partial + \sum_{l=1}^{\infty} u_l(j) \partial^{-l} \quad (5.6) \]

where \([j, m_1]\) denotes the least between \(j\) and \(m_1\). All the functions \(u_l(j)\) are functions of the coordinates in \(j\)-th sector only.

\[ u_1(j) = a_1(j), \]
\[ u_2(j) = a_2(j) + a_1(j) \left( a_0(j) - (\ln R_j) \right)' \]
\[ u_3(j) = a_3(j) + a_2(j) \left[ 2a_0(j) - 2(\ln R_j)' - \left( \ln \left[ R_j - a_0(j) + \frac{\partial^2 \ln R_j}{\partial \ell_{1,1} \partial q_{1,1}} \right] \right)' \right] \]
\[ + a_1(j) \left( \left( a_0(j) - (\ln R_j)' \right)^2 - a_0'(j) + (\ln R_j)'' \right) \]

and so on. \(L_j(1)\) is an operator of KP type. Actually it is in general a reduction of the KP operator
\[ L_{KP} = \partial + \sum_{l=1}^{\infty} w_l(j) \partial^{-l} \]

where \(w_l\) are generic (i.e. unrestricted) coordinates. We notice that due to the limitation in the first two summations in (5.6), the operators \(L_j(1)\) do not have in general a universal form in terms of the coordinates \(a_l(j)\). In fact they change with \(j\) as long as \(j < m_1\), while the form of \(L_j(1)\) is independent of \(j\) if \(j \geq m_1\). We will comment upon this point later on.

Without repeating all the steps of [13], the rules to pass from lattice to purely differential language are:

\((i)\). The Jacobi matrices are mapped into KP–type operators, i.e.

\[ \begin{cases} 
(Q(\alpha) \Psi)_j \Rightarrow L_j(\alpha) \Psi_j, & \alpha = 1, 2, \ldots, q \\
(P(1) \Psi)_j \Rightarrow M_j(1) \Psi_j.
\end{cases} \quad (5.8) \]
where
\[
L(\alpha)_j = T_0(\alpha)(j) + \sum_{l=1}^{[j,m_\alpha]} T_{-l}(\alpha)(j) \hat{A}_{j+l-1} \hat{A}_{j+l-2} \cdots \hat{A}_j + \sum_{l=1}^{[j,m_\alpha]} T_{l}(\alpha)(j) \hat{B}_{j-l} \hat{B}_{j-l+1} \cdots \hat{B}_{j-1}
\]

\[M_j(1) = P_0(1)(j) + \sum_{l=1}^{\infty} P_{-l}(1)(j) \hat{A}_{j+l-1} \hat{A}_{j+l-2} \cdots \hat{A}_j + \sum_{l=1}^{[j,s_1]} P_{l}(1)(j) \hat{B}_{j-l} \hat{B}_{j-l+1} \cdots \hat{B}_{j-1}
\]

We remark again the non-universality of such expression except for \( j \) large enough.

(ii). The lower triangular part of a Jacobi matrix (or its powers) maps to the pure integration part of the KP operator (or its powers);

(iii). The upper triangular part together with the main diagonal line of the Jacobi matrix (or its powers) correspond to the purely differential part of the KP operator (or its powers).

(iv). The residue of the KP operators have a particularly simple form,
\[
\text{res} \partial (L_j(1)) = a_1(j) = Q_{j,j-1}(1).
\]

More generally, we can obtain
\[
\text{res} \partial (L_j^r(1)) = Q_{j,j-1}(1).
\]

and
\[
\text{res} \partial (L_j(\alpha)) = T_1(\alpha)(j) = Q_{j,j-1}(\alpha).
\]

as well as
\[
\text{res} \partial (L_j^r(\alpha)) = \left( Q_{j,j-1}(\alpha) \right)^r_{j,j-1}(\alpha).
\]

Expanding all the above operators in powers of \( \partial \), the coefficients can be expressed as functions of the coordinates in the \( j \)-th sector. Collecting all the results we get a linear system
\[
\begin{cases}
L_j(1)\Psi_j = \lambda_1 \Psi_j \\
\frac{\partial}{\partial t_{1,r}} \Psi_j = \left( L_j^r(1) \right)_+ \Psi_j \\
\frac{\partial}{\partial t_{n,r}} \Psi_j = -\left( L_j^r(\alpha) \right)_- \Psi_j, \quad \alpha = 2, 3, \ldots, q \\
M_j(1)\Psi_j = \frac{\partial}{\partial \lambda_1} \Psi_j
\end{cases}
\]

The consistency conditions are
\[
\frac{\partial}{\partial t_{\beta r}} L_j(\alpha) = [(L_j^r(\beta))_+, L_j(\alpha)], \quad 1 \leq \beta \leq \alpha
\]
\[
\frac{\partial}{\partial t_{\beta r}} L_j(\alpha) = [L_j(\alpha), (L_j^r(\beta))_-], \quad \alpha \leq \beta \leq n
\]

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We do not drop here the label $j$ in the above equations in order to stress the fact that, although the hierarchy is expressed in terms of the coordinates of the $j$–th sector only, $L_j$ depends on $j$, as we have already remarked.

The linear system (5.13) is exactly the same as the one obtained in [15] in a completely different context, i.e. by trying to generalize the KP hierarchy.

We remark that if we impose the condition
\[ a_l(j) = 0, \quad \forall \ l \geq 2 \]
then the second expression of (5.6) gives the so–called two–bosonic representation of the KP hierarchy. Motivated by this fact, we refer to the full expression of eq.(5.6) as the multi–bosonic representation of the KP hierarchy.

(**) Differential linear system II.

In the above analysis, we considered $t_{1,1}$ as the space coordinate, and $\partial$ as the basic derivative. However, as we already pointed out, it is also possible to consider $t_{q,1}$ as the space coordinate and $\tilde{\partial}$ as the basic derivative. In this way we would get another linear system, in which $\Phi$ plays the role of Baker–Akhiezer function. In the following we discuss this system. As before we have
\[
\Phi_{j+1} = \hat{C}_j \Phi_j, \quad \Phi_{j-1} = \hat{D}_{j-1} \Phi_j
\]
(5.16)

where
\[
\hat{C}_j \equiv \tilde{\partial} - b_0(j) = -\partial^{-1} R_{j+1}
\]
\[
\hat{D}_j \equiv \tilde{\partial}^{-1} \sum_{l=0}^{\infty} (b_0(j)\tilde{\partial}^{-1})^l = -\frac{1}{R_{j+1}} \partial
\]

The linear system is
\[
\mathcal{L}_j(q) \Phi_j = \lambda_q \Phi_j \quad (5.17a)
\]
\[
\frac{\partial}{\partial t_{q,r}} \Phi_j = \left( \mathcal{L}_j(q) \right)_+ \Phi_j \quad (5.17b)
\]
\[
\frac{\partial}{\partial t_{\alpha,r}} \Phi_j = -\left( \mathcal{L}_j(\alpha) \right)_- \Phi_j, \quad \alpha = 1, 2, \ldots, q - 1 \quad (5.17c)
\]
\[
\tilde{M}_j \Phi_j = \frac{\partial}{\partial \lambda_q} \Phi_j \quad (5.17d)
\]

Now the KP operator takes the following form
\[
\mathcal{L}_j(q) = \tilde{\partial} + \sum_{l=1}^{j} b_l(j) \hat{D}_{j-l} \hat{D}_{j-l+1} \cdots \hat{D}_{j-1}
\]
\[
= \tilde{\partial} + \sum_{l=1}^{j} b_l(j) \frac{1}{\tilde{\partial} - b_0(j - l)} \frac{1}{\tilde{\partial} - b_0(j - l + 1)} \cdots \frac{1}{\tilde{\partial} - b_0(j - 1) - 1}
\]
\[
\equiv \tilde{\partial} + \sum_{l=1}^{\infty} v_l(j) \tilde{\partial}^{-l} \quad (5.18)
\]
All the functions \( v_l(j) \) are only functions of the coordinates in the \( j \)-th sector.

\[
v_1(j) = b_1(j),
\]

\[
v_2(j) = b_2(j) + b_1(j) \left( b_0(j) - \tilde{\partial}(\ln R_j) \right)
\]

\[
v_3(j) = b_3(j) + b_2(j) \left[ 2b_0(j) - 2\tilde{\partial}(\ln R_j) - \tilde{\partial} \left( \ln \left[ R_j - \partial b_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1} \partial t_{q,1}} \right] \right) \right]
\]

\[
+ b_1(j) \left( [b_0(j) - \tilde{\partial}(\ln R_j)]^2 - \tilde{\partial} b_0(j) + \tilde{\partial}^2 \ln R_j \right)
\]

and so on. Using the rules to go from the lattice to the differential language, we obtain

\[
\left( \tilde{Q}(\alpha) \Phi \right)_j \implies \tilde{L}_j(\alpha) \Phi_j, \quad \alpha = 1, 2, \ldots, q \quad (5.20a)
\]

\[
\left( P(q) \Phi \right)_j \implies \tilde{M}_j(q) \Phi_j. \quad (5.20b)
\]

with new differential operators

\[
\tilde{L}(\alpha)_j = T_0^{(\alpha)}(j) + \sum_{l=1}^{\infty} T_l^{(\alpha)}(j) \hat{C}_{j+l-1} \hat{C}_{j+l-2} \cdots \hat{C}_j
\]

\[
+ \sum_{l=1}^{j} T_l^{(\alpha)}(j) \hat{D}_{j-l} \hat{D}_{j-l+1} \cdots \hat{D}_{j-1} \quad (5.21a)
\]

\[
\tilde{M}_j(q) = P_0^{(q)}(j) + \sum_{l=1}^{\infty} P_l^{(q)}(j) \hat{C}_{j+l-1} \hat{C}_{j+l-2} \cdots \hat{C}_j
\]

\[
+ \sum_{l=1}^{s_q} P_l^{(q)}(j) \hat{D}_{j-l} \hat{D}_{j-l+1} \cdots \hat{D}_{j-1} \quad (5.21b)
\]

Therefore we can write down another linear system

\[
\begin{cases}
\tilde{L}_j(q) \Phi_j = \lambda_q \Phi_j \\
\frac{\partial}{\partial r_{\alpha,r}} \Phi_j = \left( \tilde{L}_j^{(\alpha)}(q) \right)_+ \Phi_j \\
\frac{\partial}{\partial r_{\alpha,r}} \Phi_j = - \left( \tilde{L}_j^{(\alpha)}(q) \right)_- \Phi_j, \quad \alpha = 1, 2, \ldots, q - 1 \\
M_j(1) \Phi_j = \frac{\partial}{\partial \lambda_1} \Phi_j
\end{cases}
\]

whose consistency conditions also give the hierarchy \((5.14-5.15)\).

We could also start from the general form of the discrete linear systems \((3.22)\) and \((3.23)\), translate them into the differential language by using the rules we listed before. Once again they lead to the hierarchy \((5.14-5.15)\). We will spare the reader the details.

We end this section with a remark on the flows in \((5.14)\) and \((5.15)\). They all commute with one another, since their lattice versions do. So all the differentiable hierarchies \((5.14)\) and \((5.15)\) are integrable. This is a very remarkable result. We know in fact that attempts at enlarging the KP hierarchy in a very general way have led to incompatible KP flows \[(24)\]. Therefore we may say that the multi–matrix models give a natural realization of commuting KP flows.
5.1 The coupling conditions

Our final task is to reexpress the coupling conditions (3.5a–3.5b) in the differential language. This can be done very easily by replacing matrices by their operator versions. The only problem we should care about is the matrix $\overline{P}(q)$, since its differential version is not $\overline{M}(q)$. We may simply denote its differential version by $M(q)$, then the coupling conditions are

\begin{align}
M(1) + c_{1,2}L(2) &= 0, \quad (5.23a) \\
c_{\alpha-1,\alpha}L(\alpha-1) + V'(\alpha) + c_{\alpha,\alpha+1}L(\alpha+1) &= 0, \quad 2 \leq \alpha \leq q-1; \quad (5.23b) \\
c_{q-1,q}L(q-1) + M(q) &= 0. \quad (5.23c)
\end{align}

where for simplicity we dropped the label $j$, denoting the sector. Of course

$$V'(\alpha) = \sum_{r=1}^{p_{\alpha}} r t_{\alpha,r} L_{\alpha,r}^{r-1}(\alpha)$$

Some of these equations are also met in [15], so they are characteristic of the generalized KP hierarchy. But not all of them appear simultaneously in the generalized KP hierarchy. More precisely, if we are in the $t_{1,1}$–picture, that is $t_{1,1}$ is assumed to be the ‘space’ variable, then in the generalized KP hierarchy there naturally appear only (5.23a) and (5.23b); therefore in this picture (5.23c) is a constraint on the generalized KP hierarchy. Vice versa, in the $t_{q,1}$–picture (5.23c) and (5.23b) appear naturally in the generalized KP hierarchy, while (5.23a) plays the role of constraint.

Thus we may conclude that multi–matrix models correspond to the generalized KP hierarchy subject to certain constraints.

5.2 The partition function as $\tau$–function

In the generalized KP hierarchy we can define the following $\tau$–function

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln \tau = \text{res}_{\vartheta} L'(\alpha), \quad \forall \ \alpha, r$$

(5.24)

where $L$ is the generalized KP operator. We want to prove that the partition function of multi–matrix models is a $\tau$–function of the $\alpha$–th generalized KP hierarchy. To achieve this we only need to prove that eqs. (3.31), once translated into the differential language, have the same form as eqs. (5.24). This can be easily done, if we note that eq. (5.12) is valid for any $1 \leq \alpha \leq q$, and any positive integers $r, j$. Choosing in particular $j = N$, we have

$$\left(Q^r(\alpha)\right)_{N,N-1} = \text{res}_{\vartheta} \left(L_N(\alpha)\right)^r$$

(5.25)

This equality together with eq. (3.31) gives

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_N(t, c) = \text{res}_{\vartheta} \left(L_N(\alpha)\right)^r$$

(5.26)

This is nothing but (5.24). Therefore we can conclude that the partition functions of multi–matrix models are exactly $\tau$–functions of the generalized KP hierarchy.
6 Examples and Conclusion

In the previous sections we were able to extract differential hierarchies from multi-matrix models, without passing through a continuum limit but by simply studying the properties of the matrix model lattice. These are KP type hierarchies of which the original partition function is a $\tau$–function. To determine the initial conditions for this $\tau$–function we dug out additional equations: the coupling conditions or string equations.

What we have worked out so far is a very general scheme which allows for a vast number of possibilities, i.e. physical models, the physical meaning being attached to the coordinates we have introduced. The analysis of all the different possibilities is out of the question. Our present purpose is much more modest: we want to single out significant subclasses of models. To do this we can play with two tools: the first is the type of potentials introduced in the original multi–matrix path–integral; the second tool, once the potentials are fixed and the corresponding differential linear systems are worked out, is to study the various possible reductions of the latter.

To start with we want to split all the possibilities we are faced with in two broad classes or alternatives. We recall what we repeatedly pointed out in the last section: the differential hierarchies obtained from the lattice are in general not universal in that they depend on $j$, i.e. on the sector. In particular, the flows of the logarithm of the partition function depend in general on the sector. For the differential hierarchies coming from a given lattice hierarchy we have therefore two possibilities, which are connected with specific choices of the numbers $p_\alpha$ characterizing the order of the potential $V_\alpha$:

– the non–universal alternative: the differential hierarchies coming from the same lattice hierarchy are all different in different sectors. In this case one can hardly consider these hierarchies an intrinsic property of the lattice. Presumably the only way (if any) to treat this case is via a continuum limit.

– the universal alternative: the differential hierarchies coming from the same lattice hierarchy become isomorphic, i.e. truly independent of $j$, for $j$ large enough. No doubt, the $j$–independent hierarchies, referred to henceforth as the relevant universal hierarchy, express intrinsic properties of the lattice, and they can presumably be realized in terms of topological field theories. Needless to say, we are interested in this case.

Let us see, as an example, the two–matrix model case. We have two parameters: $p_1$ and $p_2$. On the basis of the discussion in subsection 3.1, we see that the lattice system $I$ gives rise to a universal hierarchy only if $p_2$ is finite, while the lattice system $II$ does the same only if $p_1$ is finite. Generalizing this discussion to other multi–matrix models is straightforward.

This distinction between universal and non–universal alternative is perhaps difficult to grasp at first reading. However it has a rather natural meaning: we recall once again that the information concerning the multi–matrix model is contained in the partition function; the flows of the latter depend on the sector $j = N$ (see subsection 5.2); in the universal alternative the potentials are simple enough as to allow us to retrieve all the information contained in the lattice if we choose a sufficiently large (but finite) $N$, while of course only partial information is retrieved if $N$ is too small; in the non–universal case the potentials are too complicated for us to retrieve all the
To end this paper we would like to give a few more explicit examples of the differential systems of the last section. We will consider the differential system \( I \) in multi-matrix models; not in full generality however, only in the dispersionless limit \( 1 \). This limit is in any case very interesting. It represents the genus 0 approximation and it might characterize the full hierarchy. So let us concentrate on the system \( I \) of the previous section and, in particular, on the differential operator \( L_j(1) \), eq.(5.6), with the following specifications: \( p_1 = \infty \) and \( p_2 \) finite, so that \( L_j(1) \) satisfies a universal hierarchy for large enough \( j \). For this reason we will drop in the following the label \( j \).

The technique to obtain the dispersionless hierarchy is very simple once we know the full hierarchy: in the flow equations we simply discard terms containing more than one first order derivative or higher order derivatives. A simple way to obtain it without previous knowledge is to replace \( \partial \) with a variable \( p \) which is Poisson conjugate to \( t_1 \),

\[
\{p, t_1\} = 1
\]

and fully exploit the integrable structure. Therefore we start with the replacement

\[
L(1) \implies \mathcal{L} = p + \sum_{l=1}^{\nu} a_l \frac{1}{(p - S)^l} \equiv p + \sum_{l=1}^{\infty} u_l p^l
\]  

(6.1)

where

\[
u_{l+1} = \sum_{k=0}^{l} \binom{l}{k} a_{k+1} S^{l-k}, \quad a_l = 0 \quad \text{for} \quad l > \nu
\]

In (6.1) we set \( a_0 = S \) in analogy with [13]. At this point, in order to find the flows of these coordinates, we could simply use the consistency conditions of the system \( I \) where commutators are replaced by corresponding Poisson brackets. A better way is to use the Poisson algebra. The first Poisson bracket is given by

\[
\{a_1(x), S(y)\}_1 = \delta'(x - y) \\
\{a_i(x), a_j(y)\}_1 = \left[(i + j - 2)a_{i+j-2}\partial + (j - 1)a'_{i+j-2}\right]\delta(x - y), \quad 2 \leq i, j \leq \nu
\]  

(6.2)

while all the other brackets vanish. The second Poisson bracket is

\ [
\{S(x), S(y)\}_2 = \frac{\nu+1}{\nu} \delta'(x - y) \\
\{S(x), a_1(y)\}_2 = \partial S\delta(x - y) \\
\{S(x), a_j(y)\}_2 = \frac{\nu-j+1}{\nu} \partial a_{j-1}\delta(x - y) \\
\{a_1(x), a_1(y)\}_2 = \left[a_1\partial + \partial a_1\right]\delta(x - y)
\]

(6.3)

We advise the reader not to rely too much in this regard on the analogy with 1-matrix models. The analogy between 1-matrix models and multi-matrix models is only partial. One-matrix models are all to be classified in the universal alternative, as their Jacobi matrix has at most three non-vanishing diagonal lines. The dependence of equations and differential operators on the sector is instead present in 1-matrix models as well, although in a quite unconspicuous way as it appears only in the zeroth sector while everything takes the universal form starting from the first sector on.
In these equations \(i, j \geq 2\). Next we write down the Hamiltonians
\[
\mathcal{H}_r = \frac{1}{r} \mathcal{L}_r^{(-1)}
\] (6.4)
The RHS denotes the coefficient of \(p^{-1}\) in \(\mathcal{L}^r\). The two Poisson brackets are compatible with respect to these Hamiltonians. Using this fact we can easily write down the flows in terms of
\[
F_r = \frac{\delta \mathcal{H}_r}{\delta S}, \quad G_r = \frac{\delta \mathcal{H}_r}{\delta a_1}, \quad E_r = \frac{\delta \mathcal{H}_r}{\delta a_2}, \ldots
\]
and recursion relations for the latter. For example, in the case of three fields, i.e. \(\nu = 2\), we obtain
\[
\begin{align*}
\frac{\partial S}{\partial t_r} &= G'_{r+1} \\
\frac{\partial a_1}{\partial t_r} &= F'_{r+1} \\
\frac{\partial a_2}{\partial t_r} &= 2a_2 E'_{r+1} + a'_2 E_{r+1}
\end{align*}
\] (6.5)
The recursion relations are
\[
\begin{align*}
G'_{r+1} &= \frac{3}{2} F'_r + (S G'_r)' + \frac{1}{2} (a_1 E'_r)' \\
F'_{r+1} &= S F'_r + 2a_1 G'_r + a'_1 G_r + 3a_2 E'_r + 2a'_2 E_r \\
2a_2 E'_{r+1} + a'_2 E_{r+1} &= \frac{1}{2} a_1 F'_r + 3a_2 G'_r + a'_2 G_r \\
& \quad + (2a_2 S - \frac{1}{2} a_1^2) E'_r + (a_2 S - \frac{1}{4} a_1^2)' E_r
\end{align*}
\] (6.6)
the initial conditions being \(F_1 = E_1 = G_1 = 0\).

From this result we can learn something important. In [13], [14] we obtained the KdV hierarchy reducing the NLS hierarchy. The latter is generated as the consistency condition of system I for a KP operator of the type \(L(1)\) with only two non–vanishing coordinates \(a_0 = R\) and \(S\). The reduction to the KdV hierarchy is obtained by setting \(S = 0\). Therefore it would seem not unmotivated to expect that setting \(S = 0\) in (6.5) one finds the dispersionless 3–KdV (or Boussinesq) hierarchy. However an explicit calculation proves that this is not the case. In other words this approach to find the n–KdV hierarchies as reductions of the differential systems ensuing from multi–matrix models is too naive. A way to obtain the n–KdV hierarchy could be as follows: reduce \(L(1)\) in just the same way we reduce the KP operator down to the n–KdV.
operator; if the order of the $V_2$ potential (in two–matrix model) is large enough, we can obtain all the flows we wish of the $n$–KdV hierarchy.

In any case it is evident that the reduction to the $n$–KdV hierarchies is a very drastic one: most of the generalized KP structure gets lost in this passage. Taught by the 1–matrix model example, [14], we think a lot of information, in other words of topological models, is contained in the generalized KP structure as it appears in multi–matrix models. We intend to deal elsewhere with this part of the analysis of multi–matrix models.

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