Using Conservation Laws to Solve Toda Field Theories

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Abstract

We investigate the question of how the knowledge of sufficiently many local conservation laws for a model can be utilized to solve the model. We show that for models where the conservation laws can be written in one-sided forms, like $\partial Q_s = 0$, the problem can always be reduced to solving a closed system of ordinary differential equations. We investigate the $A_1$, $A_2$, and $B_2$ Toda field theories in considerable detail from this viewpoint. One of our findings is that there is in each case a transformation group intrinsic to the model. This group is built on a specific real form of the Lie algebra used to label the Toda field theory. It is the group of field transformations which leaves the conserved densities invariant.
1 Introduction

For a Hamiltonian system to be “exactly solvable” it must have a sufficient number of conservation laws. Liouville proved that if a system with $N$ degrees of freedom (i.e., with a $2N$-dimensional phase space) has $N$ independent conserved quantities with mutually vanishing Poisson brackets, then the system is integrable by quadratures[1]. This means that there exist (at least in principle) an algorithm for determining the state of system at time $t$ from the initial data at time $t_0 = 0$ (say).

There are (in two space-time dimensions) many non-linear field theories which are known to have infinitely many conserved quantities. Since there are many ways to count to infinity this fact alone is insufficient to conclude that the system is integrable; it is however a necessary condition. And, even if sufficiently many conservation laws are known, it is in general quite unclear how they can be used to construct the solution. The situation is rather the other way around, the existence and form of the conservation laws are deduced from an already known method of constructing the solution, like e.g. the inverse scattering method.

In this paper we demonstrate how knowledge of conservation laws can be explicitly used to solve e.g. the initial value problem, for a particular class of field theories.

1.1 Illustrative example

The basic idea behind the method is well illustrated by the (utterly trivial) example

\[ \bar{\partial}\partial\varphi = 0, \]  
(1)

where $\partial \equiv \partial/\partial t + \partial/\partial x \equiv \partial_z$ and $\bar{\partial} \equiv \partial/\partial t - \partial/\partial x \equiv \partial_{\bar{z}}$. Eq. (1) has infinitely many conservation laws. However, they can all be generated from the two local laws,

\[ \bar{\partial}J = 0, \quad \partial\bar{J} = 0, \]  
(2)

where $J = \partial\varphi = \varphi' + \pi$ and $\bar{J} = \bar{\partial}\varphi = \varphi' - \pi$, as the space integral of arbitrary polynomials $P$ in $J, \partial J, \partial^2 J, \ldots$, or arbitrary polynomials $\bar{P}$ in $\bar{J}, \partial\bar{J}, \partial^2\bar{J}, \ldots$. It follows from (2) and the ordinary rules of differentiation that

\[ \bar{\partial}P(J, \partial J, \partial^2 J, \ldots) = 0, \quad \partial\bar{P}(\bar{J}, \partial\bar{J}, \partial^2\bar{J}, \ldots) = 0. \]  
(3)

Thus, all information about the infinite set of conserved quantities is already contained in $J = J(x, t)$ and $\bar{J} = \bar{J}(x, t)$. This matches the fact that to solve the problem we must determine two functions, $\varphi(x, t)$ and $\pi(x, t) \equiv \dot{\varphi}(x, t)$.

At this point we should explain our notation. We do not consider a canonical formulation in light-cone coordinates, but rather at a fixed time $t$. All our conservation laws should thus be viewed as expressions in terms of the canonical fields $\varphi$ and $\pi$, with e.g. $\partial^n\varphi$ being a short-hand notation for a more complicated...
expression. The latter are found explicitly through repeated use of the equations of motion, i.e. in this example $\dot{\varphi} = \pi$, $\ddot{\varphi} = \phi''$. Here we find for $n \geq 1$

$$\partial^n \varphi = (2\partial_x)^{n-1}(\pi + \varphi'), \quad \bar{\partial}^n \varphi = (-2\partial_x)^{n-1}(\pi - \varphi'),$$

(4)

but with the non-linear equations to be considered later these substitutions become more complicated.

Now, to exploit the conservation laws to solve the initial value problem, we first determine $J(x,0) = J_0(x)$ and $\bar{J}(x,0) = \bar{J}_0(x)$ from the initial data $\varphi(x,0), \pi(x,0)$. Eq. (2) implies a simple time evolution for $J, \bar{J}$:

$$\begin{align*}
J(x,t) &= J_0(x + t), \\
\bar{J}(x,t) &= J_0(x - t).
\end{align*}$$

(5)

The fields $\varphi(x,t)$ and $\pi(x,t)$ are now found by inverting their relations to the conservation laws, i.e.

$$\begin{align*}
\pi(x,t) &= \frac{1}{2} \left[ J_0(x + t) + \bar{J}_0(x - t) \right], \\
\varphi'(x,t) &= \frac{1}{2} \left[ J_0(x + t) - \bar{J}_0(x - t) \right].
\end{align*}$$

(6)

Since this is a first order ordinary differential equation for $\varphi$ the solution is determined only modulo an integration constant $\varphi_0$. This is an ambiguity which arises because the conserved currents $J$ and $\bar{J}$ are invariant under the field transformation

$$(\varphi(x,t), \pi(x,t)) \rightarrow (\varphi(x,t) + \varphi_0, \pi(x,t)).$$

Thus, to find the complete solution we need an independent determination of say $\varphi(0,t)$. This can indeed be found by combining (8) with the equation of motion (1). Define $q(t) \equiv \varphi(0,t)$ and $p(t) \equiv \pi(0,t)$. They satisfy the equations of motion

$$\begin{align*}
\dot{q}(t) &= p(t), \\
\dot{p}(t) &= \varphi''(0,t).
\end{align*}$$

(7)

In general this would not constitute a closed system of equations, since $\varphi''(0,t)$ is a new unknown quantity. However, here it may be determined from the conservation laws. It follows from (8) that

$$\varphi''(0,t) = \frac{1}{2} \left[ J'_0(t) - \bar{J}'_0(-t) \right].$$

(8)

The right hand side of this equation is a known, and may be used to close the system (7).

The crucial points in the above analysis are (i) that we have a sufficient number of conservation laws which can be written in “one-sided” forms like (2), so that their time development are easily found, and (ii) that the relations between the fields and the conserved densities are known and can be inverted. The first condition appears to be fulfilled for all (conformal) Toda field theories. In this paper we shall explicitly consider a few of the simplest cases. The latter relations turn out to form a set of (generally non-linear) ordinary differential equations,
whose order increases with the number of field components. Thus, the integration of these equations introduces a set of undetermined integration constants, reflecting the fact that the set of conserved currents are invariant under a certain group of transformations on the fields of the model. To find the complete solution these integration constants must be determined from initial data. They play the role of angle variables in the theory. However, since there is only a (small) finite number of such undetermined quantities, most of the (infinite number of) angle variables of the model are also determined by the conserved densities. In this respect a one-sided local conservation law must contain infinitely more information than just the (globally) conserved quantities of the model.

1.2 Contents and organization of paper

The rest of this paper is organized as follows: In the next section we analyse the simplest \( (A_1) \) Toda field theory, i.e. the Liouville model, in considerable detail. We discuss in particular various methods for solving the initial value problem, but also show how the conservation of the energy-momentum tensor can be utilized in a systematic procedure for constructing the general solution. In fact, knowledge of the mechanism behind the general solution makes it easier to utilize for solving the initial value problem. We repeat this analysis in the following sections for two Toda field theories with two-component fields; in section 3 for the \( A_2 \) model and in section 4 for the \( B_2 \) model. Both of these models have one set of conservation laws in addition to conservation of the energy-momentum tensor. For \( A_2 \) the additional tensor is of spin 3, and for \( B_2 \) it is of spin 4. In both cases we show how the conservation laws can be used to derive general solutions of the equations of motion, and also indicate practical algorithms for utilizing these general solutions to solve the initial value problems.

As indicated above, each Toda field theory is labeled by the name of a classical Lie algebra. Names were given originally by Bogoyavlenski\[^2\] while giving the Lax pairs for the models. This naming scheme is connected to the method first used by Leznov and Saveliev\[^3\] to construct general solutions to Toda field theories. Here the equations of motion are enforced through a zero curvature condition for a non-abelian gauge theory constructed from the Lie algebra in question. It is not clear from this formulation whether these Lie algebras have any intrinsic connection to the Toda equations. We show that a Lie group obtained by exponentiating one of the real forms of the complex Lie algebra is an intrinsic symmetry group for the system. The symmetry in question is the group of field transformations which leave the conserved densities invariant. For the \( A_1 \) Toda field theory this group is \( SL(2, R) \), for the \( A_2 \) Toda field theory this group is \( SL(3, R) \), and for the \( B_2 \) Toda field theory this group is \( Sp(2, R) \). We conclude with a section describing the generalization to all Toda field theories.

The Toda field theories have previously been considered by many authors. The treatment we feel is most similar to ours is by Bilal and Gervais\[^4, 5\]. Apart
from additional detail, our presentation differs from theirs in viewpoint and focus. Among other interesting discussions on Toda field theories we would like to mention the references[3, 4, 5]

2 The Liouville equation

The Liouville model can be described by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} (\ddot{\varphi})(\dot{\varphi}) - \sigma e^{\alpha \varphi}. \] (9)

where \( \alpha \) and \( \sigma \) are arbitrary real parameters. One may in the classical case let \( \alpha \to 1 \) and \( \sigma \to \pm 1 \) after appropriate redefinitions of the fields. Only the case \( \sigma = 1 \) gives a Hamiltonian which is unbounded from below. The equations of motion become

\[ \partial \ddot{\varphi} = -\sigma e^{\varphi}. \] (10)

2.1 Qualitative behaviour and averaged motion

We expect the solutions of (10) to have very different behaviour when \( \sigma = 1 \) and \( \sigma = -1 \). Assume \( x \)-space to be a circle of unit circumference. Then (10) can be thought to describe a string which is wrapped around a cylinder \( S_1 \times \mathbb{R} \), parametrized by the coordinates \((x, \varphi)\), and can slide along the cylinder. Thus, \( \varphi \) measures the position of the string along the cylinder axis. There is also an external potential \( \sigma e^{\varphi} \) in this direction. To obtain a qualitative understanding of the behaviour of the system consider the average position

\[ \Phi(t) = \langle \varphi(\cdot, t) \rangle = \int_0^1 dx \varphi(x, t). \]

With the approximation \( \langle e^{\varphi(\cdot, t)} \rangle = e^{\langle \varphi(\cdot, t) \rangle} \), which is exact when \( \varphi \) is independent of \( x \), we get a closed equation of motion for \( \Phi \):

\[ \frac{d^2}{dt^2} \Phi = -\sigma e^{\Phi}. \]

This equation has the solutions

\[ e^{\Phi(t)} = \frac{2a^2}{\cosh^2[a(t - t_0)]} \] (11)

when \( \sigma = 1 \), and

\[ e^{\Phi(t)} = \frac{2a^2}{\sinh^2[a(t - t_0)]} \] (12)

when \( \sigma = -1 \). In the latter case the field \( \Phi \) always becomes singular at some finite time.
When \( \sigma = 1 \) the general (physically interesting) solution can be written as a fluctuation correction to the average motion

\[
\varphi(x, t) = \Phi(t) + \delta \varphi(x, t).
\] (13)

Since \( \Phi(t) \to -\infty \) as \( |t| \to \infty \) (which makes the exponential term in (10) vanish) it must be possible decompose \( \delta \varphi(x, t) \) into right- and left-moving waves at asymptotically early and asymptotically late times,

\[
\delta \varphi(x, t) \sim \varphi_-(x + t) + \bar{\varphi}_-(x - t) \quad \text{as} \quad t \to -\infty,
\]

\[
\delta \varphi(x, t) \sim \varphi_+(x + t) + \bar{\varphi}_+(x - t) \quad \text{as} \quad t \to \infty,
\] (14)

On a cylinder the functions \( \varphi_\pm \) and \( \bar{\varphi}_\pm \) must be periodic.

### 2.2 The solution of Liouville

The solution to (10) was already found by Liouville[9]. It can be written in the form

\[
e^{\varphi(x, t)} = \frac{8 F'(x + t) G'(x - t)}{[F(x + t) + \sigma G(x - t)]^2}
\] (15)

where \( F \) and \( G \) are arbitrary functions of their arguments. Since (13) involves two arbitrary functions it can be expected to be the most general solution. However, many choices of \( F \) or \( G \) lead to solutions that are unphysical. Also, it is not entirely trivial to find which \( F, G \) correspond to a given set of initial data, e.g. \( \varphi(x, 0) \) and \( \pi(x, 0) = \dot{\varphi}(x, 0) \). The \( x \)-independent solutions (11,12) correspond to the case when \( F(\xi) = \exp[a(\xi - t_0)] \) and \( G(\xi) = \exp[a(\xi + t_0)] \). The asymptotic decompositions (14) can easily be achieved by writing \( \sigma = 1 \)

\[
F(x + t) = e^{a(x+t)} F_1(x + t), \quad G(x - t) = e^{a(x-t)} G_1(x - t).
\] (16)

By inserting these expressions into (15) we find that

\[
e^\varphi \sim 8 e^{2at} (a F_1 + F'_1)(a G_1 + G'_1)/G_1^2 \quad \text{as} \quad t \to -\infty,
\]

\[
e^\varphi \sim 8 e^{-2at} (a F_1 + F'_1)(a G_1 + G'_1)/F_1^2 \quad \text{as} \quad t \to \infty.
\]

Thus, by choosing \( F_1 \) and \( G_1 \) as the periodic solutions to

\[
F'_1 + a F_1 = a e^{\varphi_-}, \quad G'_1 + a G_1 = a G_1^2 e^{\bar{\varphi}_-},
\] (17)

we find a simple mapping between the fields at asymptotically early and asymptotically late times:

\[
\varphi_+ = \varphi_- - 2 \log F_1, \quad \bar{\varphi}_+ = \bar{\varphi}_- + 2 \log G_1.
\] (18)
Since $F_1$ resp. $G_1$ are uniquely defined functionals of $\varphi_-$ resp. $\bar{\varphi}_-$, which can be written down explicitly by solving (17), eqs. (18) define a canonical transformation which is the classical analog of the $S$-matrix.

However, for more general given initial data at (say) time $t = 0$, it is not so clear how $F$ and $G$ should be determined in the most convenient way. One (admittedly ugly) solvable system is obtained by differentiating (15) with respect to $t$ and $x$ before setting $t = 0$,

$$\frac{F''}{F'} = \frac{2F'}{(F + \sigma G)} + \frac{1}{2}(\pi_0 + \varphi'_0), \quad \frac{G''}{G'} = \frac{2\sigma G'}{F + \sigma G} - \frac{1}{2}(\pi_0 - \varphi'_0),$$

(19)

with (15) itself as a further (consistent) condition

$$F'G' = \frac{1}{8}e^{\varphi_0}(F + \sigma G)^2.$$  

(20)

Note that $F$ and $G$ are not uniquely determined from these equations; one has the freedom of choosing 3 arbitrary initial conditions for (19,20). This is related to the fact that the representation (15) for $\varphi$ is invariant under a 3-parameter (Möbius) group of transformations on $(F,G)$. This group is generated by the transformations

$$(F,G) \rightarrow (F + a, G + \sigma a), \quad (F,G) \rightarrow c(F,G), \quad (F,G) \rightarrow \left(F^{-1}, G^{-1}\right),$$

(21)

and may be viewed as a gauge group for the system.

### 2.3 Solution by direct use of conservation laws

Liouville’s solution (15) represents a situation where the mapping from the fields $(\varphi, \pi)$ to the separated solutions $(F,G)$ is quite complicated, while the inverse mapping is simple and explicit. As indicated in the introduction, one may instead attempt a different route to the solution, utilizing the fact that the system has conserved currents with simple time development. In this case the mapping of fields to separated free wave solutions is simple and explicit, while the inverse mapping is complicated. We now investigate this second possibility.

The basic conserved densities for (10) are the light-cone components of the conformally improved energy-momentum tensor,

$$T_{zz} \equiv T_{\varphi} = (\partial \varphi)^2 - 2\partial^2 \varphi, \quad \bar{T}_{\bar{z}\bar{z}} \equiv \bar{T}_{\bar{\varphi}} = (\bar{\partial} \bar{\varphi})^2 - 2\bar{\partial}^2 \bar{\varphi}.$$  

(22)

They satisfy $\bar{\partial}T = 0$ and $\partial \bar{T} = 0$. And, as with the example of the introduction, any polynomials $\mathcal{P}(T, \partial T, \partial^2 T, \ldots)$ or $\bar{\mathcal{P}}(T, \bar{T}, \bar{T}, \ldots)$ satisfy $\bar{\partial} \mathcal{P} = \partial \bar{\mathcal{P}} = 0$. The space integral of the corresponding densities will all commute with the Hamiltonian, and thus be elements in the Lie algebra of the ‘group of the Hamiltonian’. This group has sufficiently many mutually commuting elements to make the system integrable. However, all the important information is encoded in the two
fields $T$ and $\mathcal{T}$, so these are the quantities one should utilize to find the solution.

Expressed by equal time canonical fields we find

$$T = (\pi + \varphi')^2 - 4 (\pi' + \varphi'') + 2\sigma e^\varphi, \quad \mathcal{T} = (\pi - \varphi')^2 + 4 (\pi' - \varphi'') + 2\sigma e^\varphi. \quad (23)$$

There are several ways to read and utilize these equations:

1. First they can be used to find e.g. $T(x, 0) = T_0(x)$ and $\mathcal{T}(x, 0) = \mathcal{T}_0(x)$ from the initial data, $\varphi_0(x) = \varphi(x, 0)$ and $\pi_0(x) = \pi(x, 0)$. From the conservation laws for $T$ and $\mathcal{T}$ it follows that their time evolution is given as $T(x, t) = T_0(x + t)$ and $\mathcal{T}(x, t) = \mathcal{T}_0(x - t)$.

2. With $T(x, t)$ and $\mathcal{T}(x, t)$ known and $t$ fixed, eq. (23) define a set of ordinary differential equations in $x$ for $\varphi(x, t)$ and $\pi(x, t)$. They are no more complicated than (19, 20). However, the complete solution to these equations requires three initial conditions to be found, say $q(t) \equiv \varphi(0, t)$, $r(t) \equiv \varphi'(0, t)$, and $p(t) \equiv \pi(0, t)$.

3. Eqs. (23) may be viewed at fixed $x$ (say $x = 0$) as a means of expressing $\varphi''(0, t)$ and $\pi'(0, t)$ in terms of $(q, r, p)$, $T$, and $\mathcal{T}$. This may be combined with (11) to find the equations of motion for $(q, r, p)$. They become

$$\dot{q} = p,$$
$$\dot{r} = \frac{1}{2} pr + \frac{1}{8} (\mathcal{T} - T),$$
$$\dot{p} = \frac{1}{4} \left( p^2 + r^2 \right) - \frac{1}{2} \sigma e^q - \frac{1}{8} \left( \mathcal{T} + T \right). \quad (24)$$

Here it is understood $T, \mathcal{T}$ are to be evaluated at $x = 0$.

In summary, to find the fields at some fixed later time, given the initial conditions, one has to solve two sets of ordinary differential equations in succession first (24) and then (23). This is still somewhat less efficient than starting from the known general solution (15). However, the advantage of this method is its systematic use of the known conservation laws. Thus, it has potential for generalization to more complicated equations where the general solution is unknown. This method will work as long as a sufficiently large set of local conservation laws can be explicitly found.

2.4 The general solution by “separation of variables”

In the previous subsection we indicated how the initial value problem for the Liouville equation could be solved by systematic use of its known conservation
laws. The procedure involved the solution of two sets of ordinary differential equations in succession. In most cases these equations must be solved numerically. With present day technology this is in practice—and as a matter of principle—comparable to the fact that even most analytic solutions at some stage must be evaluated numerically on a computer. (In this respect we still consider the numerical solution of ordinary differential equations to be significantly different from the solution of partial differential equations, both because of the computational effort involved and the number of parameters in the general solution.)

However, if we want to compute the fields for many different times, or derive asymptotic relations like (18), there is still considerable advantage in working with the general solution (15). For this reason we here propose and test a method for constructing the general solution from the knowledge of the conservation laws. Our main motivation is of course not to rederive (15), but to investigate an ansatz which can be generalized to the more complicated Toda field theories.

It is preferable to work with the field $U = e^{-\varphi/2}$. The energy-momentum tensor (22) becomes

$$T = 4 \frac{\partial^2 U}{U}, \quad \mathcal{T} = 4 \frac{\bar{\partial}^2 U}{U}.$$  

Therefore, the dependency of $U$ on $\bar{z}$ cancels in $T$, and its dependency on $z$ cancels in $\mathcal{T}$. This suggests that it should be possible to factor out the relevant dependencies on $z$ and $\bar{z}$ explicitly. To this end we write

$$U \equiv U(z, \bar{z}) = \bar{u}(z) h(z, \bar{z}) u(z) \equiv \bar{u} h u,$$

and make the ansatz that $T$ is a functional of $u$ only, and $\mathcal{T}$ is a functional of $\bar{u}$ only. This means that the expressions (23) for $T$ resp. $\mathcal{T}$ should hold with the replacements $U \to u$ resp. $U \to \bar{u}$. These are linear 2nd order ordinary differential equations for $u$ and $\bar{u}$. Inserting (26) into (23), and using the above ansatz imposes two relations on $h$:

$$\partial u^2 \partial h = 0, \quad \bar{\partial} \bar{u}^2 \bar{\partial} h = 0.$$  

It is straightforward to integrate these equations. Define functions $H = H(z)$ and $\bar{H} = \bar{H}(\bar{z})$ such that $\partial H = u^{-2}$ and $\bar{\partial} \bar{H} = \bar{u}^{-2}$. Then we obtain the representations

$$h = \bar{\alpha}_0 + \bar{\alpha}_1 H, \quad h = \alpha_0 + \alpha_1 \bar{H},$$

where $\bar{\alpha}_i = \bar{\alpha}_i(\bar{z})$ and $\alpha_i = \alpha_i(z)$. These two representations are consistent with each other if each $\alpha_i$ is a linear combination of the functions 1 and $H$. Introduce the vectors

$$\mathbf{H} = (1, H), \quad \bar{\mathbf{H}} = (1, \bar{H}).$$

Then both the linear equations (27) are satisfied by the representation (the normalization is for later convenience)

$$h = 2^{-1/2} \left( \bar{\mathbf{H}} \cdot \mathbf{X} \cdot \mathbf{H}^t \right),$$
where $\mathbf{X}$ is a $2 \times 2$ matrix of real constants. We remind the reader that our definitions of $H$ and $\bar{H}$ imply that $u = (\partial H)^{-1/2}$ and $\bar{u} = (\bar{\partial H})^{-1/2}$. We have thus found a complete solution ansatz for $U = \bar{u} \, h \, u$, which now must be inserted into (11). This gives

$$h \, \bar{\partial} h - \bar{h} \, \partial h = \sigma \, \bar{\partial} H \, \partial H,$$

which after insertion of the representation (30) gives the condition

$$\left( \bar{H} \cdot \mathbf{X} \cdot H^t \right) \left( \bar{h} \cdot \mathbf{X} \cdot h^t \right) - \left( \bar{H} \cdot \mathbf{X} \cdot h^t \right) \left( \bar{h} \cdot \mathbf{X} \cdot H^t \right) = \sigma,$$

with vectors $h = \bar{h} = (0, 1)$. On evaluating the left hand side this reduces to

$$\det \mathbf{X} = \sigma. \quad (31)$$

With this condition on $\mathbf{X}$, the conservation laws together with the ansatz (26) have indeed lead us to the general solution of the Liouville equation:

$$e^\varphi = \frac{2 \, \partial H \, \bar{\partial} H}{\left( \bar{H} \cdot \mathbf{X} \cdot H^t \right)^2}. \quad (32)$$

Liouville’s solution (15) correspond to the case that $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix}$, $F(x + t) = H(z)$, and $G(x - t) = -\bar{H}(\bar{z})$, with $z = (t + x)/2$ and $\bar{z} = (t - x)/2$.

### 2.5 The transformation group $SL(2, \mathbb{R})$

We have found a solution parametrized by two arbitrary functions in one variable, and (assume $\sigma = 1$) a matrix $\mathbf{X} \in SL(2, \mathbb{R})$. What is the role of $\mathbf{X}$? Note that $H$ resp. $\bar{H}$ is determined from $u$ resp. $\bar{u}$, hence indirectly via (25) from $T$ resp. $\bar{T}$. Equations (25) are 2nd order equations for $u$ or $\bar{u}$, and 3rd order equations for $H$ or $\bar{H}$. Thus, the general solution involves 3 arbitrary integration constants for $H$, and 3 arbitrary integration constants for $\bar{H}$. It is, for arbitrary $T$ and $\bar{T}$, not possible to find explicit expressions for $u$ and $\bar{u}$. However, assuming that some particular solutions $u_0$ and $\bar{u}_0$ have been found, it follows by variation of parameters that the general solutions to (26) can be written as

$$u = \left( \bar{a} + \bar{b} \int u_0^{-2} \right) u_0, \quad \bar{u} = \left( a + b \int \bar{u}_0^{-2} \right) \bar{u}_0. \quad (33)$$

These results applied to the definitions of $H$ and $\bar{H}$ show that when some particular solutions $H_0$ and $\bar{H}_0$ have been found the general 3-parameter classes of solutions can be expressed explicitly as

$$H_1 = \frac{a_1 H_0 + a_2}{a_3 H_0 + a_4}, \quad \bar{H}_1 = \frac{b_1 \, H_0 + b_2}{b_3 H_0 + b_4}, \quad (34)$$
where we without loss of generality may choose \( a_1 a_4 - a_2 a_3 = b_1 b_4 - b_2 b_3 = 1 \). By inserting (34) into (32) we find that the substitutions

\[
H_0 \to H_1, \quad \bar{H}_0 \to \bar{H}_1,
\]

are equivalent to keeping \( H \) and \( \bar{H} \) fixed, and instead transforming \( X \) as

\[
X \to \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} X \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \equiv B^t X A.
\] (35)

Thus, there is an \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) group of transformations on the fields such that the energy-momentum tensor is kept invariant. This group is a product of an \( SL(2, \mathbb{R}) \) group of “gauge transformations” which leaves the physical fields unchanged, \( B^t X A = X \), and an \( SL(2, \mathbb{R}) \) group which transforms the physical fields without changing the energy-momentum tensor. For a fixed \( H \) and \( \bar{H} \) the matrix \( X \) parametrizes the equivalence class of field configurations which gives rise to the same conserved densities \( T \) and \( \bar{T} \).

The Liouville equation can be formulated as a zero curvature condition connected to the Lie algebra \( A_1 \) (which has \( sl(2, \mathbb{R}) \) as one of its real forms). The above discussion show that a real form of the same Lie algebra occur in connection with physical symmetries intrinsic to the Liouville equation.

2.6 Revisiting the initial value problem.

The solution (32) now leads to a clean way of solving the initial value problem, with say the initial data given at \( t = 0 \).

1. From the initial data we compute the conserved densities \( T \) and \( \bar{T} \). These can be found as functions of \( x \), which at \( t = 0 \) is directly related to \( z = x/2 \) and \( \bar{z} = -x/2 \).

2. We next determine the functions \( u \) and \( \bar{u} \) by solving (25). This will most likely have to be done numerically. The equations are of 2nd order, thus we need two initial conditions for both \( u \) and \( \bar{u} \). It is convenient to pick one point (say \( x = 0 \)) on the initial data curve, and require that

\[
u(0) = \bar{u}(0) = 1, \quad \partial u(0) = \bar{\partial} \bar{u}(0) = 0.
\] (36)

3. With \( u \) and \( \bar{u} \) completely known we find \( H \) and \( \bar{H} \) by integration. It is convenient to choose the integration constants such that

\[
H(0) = \bar{H}(0) = 0.
\] (37)

It now follows from (36) and the definitions of \( H \) and \( \bar{H} \) that

\[
\partial H(0) = \bar{\partial} \bar{H}(0) = 1.
\] (38)
4. It now remains to determine the matrix $X$. By inserting (36–38) into (32) we find

\[
e^{-\varphi/2}
\bigg|_0 = 2^{-1/2} X_{11},
\]

\[
\partial \left( e^{-\varphi/2} \right) \bigg|_0 = 2^{-1/2} X_{12},
\]

\[
\bar{\partial} \left( e^{-\varphi/2} \right) \bigg|_0 = 2^{-1/2} X_{21},
\]

which together with (31) gives $X$. By computing $\bar{\partial} \partial e^{-\varphi/2} \big|_0 = 2^{-1/2} X_{22}$ we again verify that the equation of motion is consistent with (31).

This provides a complete determination of all quantities in (32), which now may be used to compute the fields at arbitrary times. The connection between the initial data and general solutions of the Liouville equation have also been considered by Papadopoulos and Spence\[10]\.

3 Solution of the $A_2$ Toda field theory

The $A_2$ Toda field theory can be defined by the Lagrangian\[14]\,

\[
\mathcal{L} = \bar{\partial} \phi_1 \partial \phi_1 + \bar{\partial} \phi_2 \partial \phi_2 + \bar{\partial} \phi_2 \partial \phi_3 - \frac{1}{2} \sigma_1 e^{2(\phi_1 - \phi_2)} - \frac{1}{2} \sigma_2 e^{2(\phi_2 - \phi_3)},
\]

(40)

where the $\sigma_i$'s are parameters which may be set to $\pm 1$ after appropriate redefinitions of the fields. Only the case $\sigma_1 = \sigma_2 = 1$ gives a Hamiltonian which is bounded from below. We shall assume this case in our further analysis. The equations of motion simplify if we introduce new fields $\varphi = \frac{2}{3}(\phi_1 + \phi_2 - 2\phi_3)$ and $\chi = \frac{2}{3}(2\phi_1 - \phi_2 - \phi_3)$, in which case they become

\[
\bar{\partial} \partial \varphi = -e^{2\varphi - \chi}, \quad \bar{\partial} \bar{\partial} \chi = -e^{-\varphi + 2\chi}.
\]

(41)

The third field degree of freedom, $\Phi = \phi_1 + \phi_2 + \phi_3$, decouples and satisfies $\bar{\partial} \partial \Phi = 0$. We note that (41) is symmetric under the interchange $\varphi \Leftrightarrow \chi$. The equations are consistent with the assumption $\varphi = \chi = \phi$, in which case they reduce to the Liouville equation.

3.1 General solution

Equations (41) allow two basic sets of conserved densities. With the notation that $\varphi_n \equiv \partial^n \varphi$, $\chi_n \equiv \partial^n \chi$, the set corresponding to the conformally improved

---

2Note that the left hand sides of eqs. (39) can all be expressed in terms of the initial data $\varphi_0$ and $\pi_0$.

3For a conventional normalization of the kinetic terms we will from here on use the definitions

$\partial \equiv (\partial/\partial t + \partial/\partial x)/\sqrt{2} \equiv \partial_z$ and $\bar{\partial} \equiv (\partial/\partial t - \partial/\partial x)/\sqrt{2} \equiv \partial_{\bar{z}}$. 

11
energy momentum tensor is
\[ T = \varphi_1^2 - \varphi_1 \chi_1 + \chi_1^2 - \varphi_2 - \chi_2, \] (42)
and the same expression for \( \overline{T} \) with the replacement \( \bar{\partial} \rightarrow \partial \). The other set is two spin 3 conserved densities\[11, 12],
\[ Q = 2\varphi_1^2 \chi_1 - 2\varphi_1 \chi_1^2 - 2\varphi_1 \varphi_2 + 2\chi_1 \chi_2 + \varphi_1 \chi_2 - \varphi_2 \chi_1 + \varphi_3 - \chi_3, \] (43)
and the same expression for \( \overline{Q} \) with the replacement \( \bar{\partial} \rightarrow \partial \). Note that \((T, \overline{T})\) is even and \((Q, \overline{Q})\) is odd under the \( \varphi \leftrightarrow \chi \) symmetry. This is seen more explicitly if they are expressed by the even \( \varphi = (\varphi + \chi) / 2 \) and odd \( \psi = \varphi - \chi \) fields. With the notation that \( \phi_n \equiv \partial^n \varphi \) and \( \psi_n \equiv \partial^n \psi \),
\[ T = \phi_1^2 - 2\phi_2 + \frac{3}{4} \psi_1^2, \quad Q = (2\phi_1^2 - \phi_2)\psi_1 - \frac{1}{2} \psi_1^3 - 3\phi_1 \psi_2 + \psi_3. \] (44)
These conserved densities satisfy \( \bar{\partial} T = \partial \overline{T} = \bar{\partial} Q = \partial \overline{Q} = 0 \). Thus, their time evolution is explicitly known. It is convenient to express them by the new variables \( U = e^{-\varphi} \) and \( V = e^{-\chi} \). With the notation that \( U_n \equiv \partial^n U \) and \( V_n \equiv \partial^n V \) they become
\[ T = \frac{U_2}{U} - \frac{V_2}{V} - \frac{U_1 V_1}{U V}, \quad Q = \frac{V_3}{V} - \frac{U_3}{U} + \frac{U_1 V_2 - U_2 V_1}{U V} + \frac{U_2 U_1}{U V^2} - \frac{V_2 V_1}{V^2} + \frac{U_1 V_1^2 - U_2^2 V_1}{U^2 V}. \] (46)
and the same equations for \( \overline{T} \) and \( \overline{Q} \) with the replacement \( \partial \rightarrow \bar{\partial} \). In solving the initial value problem these relations should be viewed as equations for the fields \( U \) and \( V \), with \((T, \overline{T})\) and \((Q, \overline{Q})\) already known quantities. We want to write them in separated form. To this end we use the conserved density (45) to eliminate \( V_2 \) resp. \( U_2 \) in (46). We find
\[ \frac{U_3}{U} - T \frac{U_1}{U} = \frac{1}{2}(\partial T - Q), \quad \frac{V_3}{V} - T \frac{V_1}{V} = \frac{1}{2}(\partial T + Q), \] (47)
which are seen to be linear equations for \( U \) and \( V \). There is a similar set of equations for \( U \) and \( V \), obtained by making the replacement \( T \rightarrow \overline{T}, Q \rightarrow \overline{Q} \), and \( \partial \rightarrow \bar{\partial} \) in (47).

As for the Liouville model we try to factor out the relevant dependencies on \( z \) and \( \bar{z} \) explicitly. We write
\[ U = \bar{u}(\bar{z}) h(z, \bar{z}) u(z), \quad V = \bar{v}(\bar{z}) k(z, \bar{z}) v(z), \] (48)
and make the ansatz that \( T, Q \) are functionals of \( u \) and \( v \) only, and that \( \overline{T}, \overline{Q} \) are functionals of \( \bar{u} \) and \( \bar{v} \) only. Thus \( u, v \) and \( \bar{u}, \bar{v} \) are determined by ordinary differential equations; those for \( u, v \) by making the replacement \( (U, V) \rightarrow (u, v) \)
in (17), and the similar ones for $\bar{u}, \bar{v}$ by making the replacement $(U, V, T, Q, \partial) \rightarrow (\bar{u}, \bar{v}, T, Q, \bar{\partial})$ in (17). We still have to determine $h$ and $k$ from equations in both $\bar{z}$ and $\bar{z}$, but these will be explicitly given, and turn out to be quite manageable.

By inserting the factorized expressions for the $U$ and $V$ into (17), and demanding that $T$ and $Q$ should be the above mentioned functionals of $u$ and $v$ only, we obtain equations for $h$ and $k$. They may be cast into the form

$$\frac{\partial v^2}{u} \frac{\partial u^2}{v} \partial h = 0, \quad \frac{\partial u^2}{v} \frac{\partial v^2}{u} \partial k = 0.$$  \tag{49}$$

There is another set of equations for $h$ and $k$, obtained by making the replacement $(u, v, \partial) \rightarrow (\bar{u}, \bar{v}, \bar{\partial})$ in those above. It is straightforward to integrate (19). Define functions $H = H(z)$ and $K = K(z)$ such that $\partial H = v/u^2$ and $\partial K = u/v^2$. Then

$$h = \bar{\alpha}_0 + \bar{\alpha}_1 H + \bar{\alpha}_2 \int K \partial H, \quad k = \bar{\beta}_0 + \bar{\beta}_1 K + \bar{\beta}_2 \int H \partial K,$$  \tag{50}$$

where all the coefficients may depend on $\bar{z}$: $\bar{\alpha}_0 = \bar{\alpha}_0(\bar{z})$ etc. Likewise, to solve the $\bar{z}$-dependence of $h$ and $k$ we define functions $\bar{H} = \bar{H}(\bar{z})$ and $\bar{K} = \bar{K}(\bar{z})$ such that $\bar{\partial} \bar{H} = \bar{v}/\bar{u}^2$ and $\bar{\partial} \bar{K} = \bar{u}/\bar{v}^2$. Then

$$h = \alpha_0 + \alpha_1 H + \alpha_2 \int \bar{K} \partial \bar{H}, \quad k = \beta_0 + \beta_1 K + \beta_2 \int \bar{H} \partial \bar{K},$$  \tag{51}$$

where all the coefficients may depend on $z$: $\alpha_0 = \alpha_0(z)$ etc.

Equations (50,51) are consistent with each other if each of the coefficients $\alpha_i$ is a linear combination of the functions $1, H, \int K \partial H$, and each of the coefficients $\beta_i$ is a linear combination of the functions $1, K, \int H \partial K$. Define

$$H = (1, H, \int K \partial H), \quad K = (1, K, \int H \partial K),$$  \tag{52}$$

$$\bar{H} = (1, \bar{H}, \int \bar{K} \partial \bar{H}), \quad \bar{K} = (1, \bar{K}, \int \bar{H} \partial \bar{K}).$$  \tag{53}$$

Then we have found that the $h$ and $k$ must be of the form

$$h = \bar{H} \cdot \bar{X} \cdot H^t, \quad k = \bar{K} \cdot Y \cdot K^t$$  \tag{54}$$

where $\bar{X}$ and $Y$ are $3 \times 3$ matrices of real constants.

Now inserting the ansatz $\varphi = -\log(\bar{u} \, u)$ and $\chi = -\log(\bar{v} \, v)$ into the Toda equations (11) gives the equations $h \partial \partial h - \partial h \partial h = k \partial \bar{H} \partial H$ and $k \partial \partial k - \partial k \partial k = h \partial \bar{K} \partial K$. To evaluate these expressions further we first compute

$$\partial H = (0, 1, K) \partial H \equiv h \partial H, \quad \partial K = (0, 1, H) \partial K \equiv k \partial K,$$  \tag{55}$$

and the similar barred relations. It then follows from (54) that

$$h \partial h = \left(\bar{H} \cdot \bar{X} \cdot H^t\right) \left(\bar{h} \cdot \bar{X} \cdot \bar{h}^t\right) \left(\partial \bar{H} \partial H\right),$$

$$\partial \bar{h} \partial h = \left(\bar{h} \cdot \bar{X} \cdot H^t\right) \left(\bar{H} \cdot \bar{X} \cdot \bar{h}^t\right) \left(\partial \bar{H} \partial H\right),$$
and the same with \((h, H, H, h, \bar{H}, h, X) \rightarrow (k, K, K, k, \bar{K}, \bar{k}, Y)\). We obtain the algebraic equations

\[
\begin{align*}
(\bar{H} \cdot X \cdot H^t) (\bar{h} \cdot X \cdot h^t) & - (\bar{h} \cdot X \cdot H^t) (\bar{H} \cdot X \cdot h^t) = (\bar{K} \cdot Y \cdot K^t), \\
(\bar{K} \cdot Y \cdot K^t) (\bar{k} \cdot Y \cdot k^t) & - (\bar{k} \cdot Y \cdot K^t) (\bar{K} \cdot Y \cdot k^t) = (\bar{H} \cdot X \cdot H^t).
\end{align*}
\]

(56)

(57)

Regarding the expressions as polynomials in the indeterminates \(H, K, \bar{H}, \bar{K}\) (with \(\int H \partial K = H K - \int K \partial H\)), and the corresponding barred quantities, (57) are algebraic equations for \(X\) and \(Y\). Their solution is

\[
Y = P (X^t)^{-1} P, \quad \det X = \det Y = 1, \quad (58)
\]

where \(P\) is the matrix \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

We remind the reader that our definitions of \(H, K\) and \(\bar{H}, \bar{K}\) imply that

\[
u = [\partial H (\partial K)^2]^{-1/3}, \quad v = (\partial H \partial K)^{-1/3}, \quad (59)
\]

and the corresponding barred relations. With these expressions, eq. (54), and the restrictions (58), we have found the general solution to (41). This solution is expressed in terms of the 4 arbitrary functions, \((H, K, \bar{H}, \bar{K})\) and a \(SL(3, R)\) matrix (say \(X\), with \(Y\) then been determined as the corresponding representation matrix for \(X\)).

3.2 The transformation group \(SL(3, R)\).

The form of the solution we found above strongly indicates that there is a group structure behind it. As with the Liouville equation this can be interpreted as the group of field transformations which leaves the conservation laws invariant. It turns out that this group acts linearly on the fields \(h\) and \(k\). First note that the conservation laws are related to the fields \(\varphi\) and \(\chi\) through two 3rd and two 2nd order differential equations; thus there will appear a large number of integration constants in the solution of these equations. For a proper counting we find it most safe to (mentally) express everything in terms of the canonical fields \(\varphi, \pi_{\varphi}, \chi, \pi_{\chi}\), and their spatial derivatives. The number of integrations constants then turns out to be equal to the sum of the orders (spins) of the conservation laws minus the number of fields, here \(2 \times 3 + 2 \times 2 - 2 = 8\).

Consider now our definitions (52) of \(H\) and \(K\). With \(u\) and \(v\) given there is a freedom to shift \(H\) and \(K\) by constants, and the integral \(\int K \partial H\) also contains an undetermined integration constant (its lower limit of integration). However,

\[\text{If one wants to solve the model (40) with } \sigma_1, \sigma_2 \in \{1, -1\} \text{ one should at this point multiply the right hand side of (56) by } \sigma_2 \text{ and the right hand side of (57) by } \sigma_1. \text{ These modifications can be undone by the substitutions } Y \rightarrow \sigma_2 Y \text{ and } X \rightarrow \sigma_1 X.\]
such changes are equivalent to keeping \( H \) and \( K \) fixed, and instead multiplying \( X \) and \( Y \) by appropriate matrices. To be specific, changing \( H \to H + \alpha \) is equivalent to multiplying \( X \) resp. \( Y \) from the right by

\[
e^{\alpha F_1^{(X)}} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\alpha F_1^{(Y)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix}.
\]

(60)

Changing \( K \to K + \beta \) is equivalent to multiplying \( X \) resp. \( Y \) from the right by

\[
e^{\beta F_2^{(X)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}, \quad e^{\beta F_2^{(Y)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(61)

Changing \( \int K \partial H \to \int K \partial H + \gamma \) is equivalent to multiplying \( X \) resp. \( Y \) by

\[
e^{\gamma F_3^{(X)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}, \quad e^{\gamma F_3^{(Y)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma & 0 & 1 \end{pmatrix}.
\]

(62)

There is a similar freedom to shift the corresponding barred quantities. These shifts are equivalent to keeping \( \bar{H} \) and \( \bar{K} \) unchanged, and instead multiplying \( X \) resp. \( Y \) from the left by the transpose of matrices corresponding to those above, i.e. exponentials of generators \( E_i^{(\cdot)} \) which are the transpose of the generators \( F_i^{(\cdot)} \) above.

Our equations are also (for given \( T \) and \( Q \)) invariant under scale transformations: \( u \to e^\mu u, \ v \to e^\nu v, \ h\{u,v\} \to e^\mu h\{e^\mu u,e^\nu v\}, \ k\{u,v\} \to e^\nu k\{e^\mu u,e^\nu v\} \). Such transformations are equivalent to keeping \( H \) and \( K \) fixed, and multiplying \( X \) resp. \( Y \) from the right by

\[
e^{\mu H_1^{(X)} + \nu H_2^{(X)}} = \begin{pmatrix} e^\mu & 0 & 0 \\ 0 & e^{\nu - \mu} & 0 \\ 0 & 0 & e^{-\nu} \end{pmatrix}, \quad e^{\mu H_1^{(Y)} + \nu H_2^{(Y)}} = \begin{pmatrix} e^\nu & 0 & 0 \\ 0 & e^{\mu - \nu} & 0 \\ 0 & 0 & e^{-\nu} \end{pmatrix}.
\]

(63)

There is a similar invariance under rescalings of \( \bar{u} \) and \( \bar{v} \). This is equivalent to multiplications from the left by the corresponding diagonal matrices. The two sets of 8 generators \( \{H_i^{(\cdot)}, E_j^{(\cdot)}, F_j^{(\cdot)}\} \) \( (i = 1,2, j = 1, \ldots, 3) \) are representation matrices for a basis of the complex Lie algebra \( A_2 \). Upon exponentiation of these generators with real parameters, and further composition, they generate groups of real matrices, more specifically two copies of \( SL(3, R) \) which are (inequivalent) representations of each other.

The transformation on integration constants we have considered above only corresponds to the multiplication of \( X \) and \( Y \) by lower triangular matrices from the right and upper triangular matrices from the left. This does not generate the full group structure, because there are additional integration constants arising when we determine \( (u,v,\bar{u},\bar{v}) \) from the conservation laws. But a change \( u \to u' \) corresponds to a multiplication of \( h \) by the factor \( u/u' \). Thus, as with the
Liouville model, we expect that the freedom of changing all integration constants are equivalent to multiplying $X$ by arbitrary elements of $SL(3,R)$ from both the left and the right. A $SL(3,R)$ subgroup of these transformations are gauge transformations which leaves the physical fields unchanged.

3.3 The initial value problem.

Let us summarize how the initial value problem can be solved using the results above:

1. From the initial data at time $t = t_0$ one computes the conserved densities $T, \bar{T}, Q, \bar{Q}$. Note that this process requires a rewriting of the quantities in terms of the canonical fields and their spatial derivatives. This can be done with the use of the equations of motion, although the expressions become rather lengthy.

2. From the knowledge of the conserved densities one determines $\{u, v, \bar{u}, \bar{v}\}$, using (47) with the appropriate replacements mentioned above. For most initial data these equations will have to be solved numerically. Note that we may choose a fixed set of initial conditions in this process.

3. With further integrations one finds the vectors $\{H, K, \bar{H}, \bar{K}\}$. We may also choose a fixed set of integration constants in this process.

4. Finally $X$ and the corresponding $Y$ are determined so that the solutions $\varphi = -\log(\bar{u} h u)$ and $\chi = -\log(\bar{v} k v)$ reproduce the initial values of the canonical fields (and a sufficient number of their spatial derivatives) at some initial point $(t_0, x_0)$.

4 Solution of $B_2$ Toda field theory

The $B_2$ Toda field theory can be defined by the Lagrangian

$$L = \bar{\partial} \phi_1 \partial \phi_1 + \bar{\partial} \phi_2 \partial \phi_2 - \frac{1}{2} \sigma_1 e^{2(\phi_1 - \phi_2)} - \sigma_2 e^{2\phi_2},$$

(64)

where the $\sigma_i$’s are parameters which may be set to ±1 after appropriate redefinitions. Only the case $\sigma_1 = \sigma_2 = 1$ gives a Hamiltonian which is bounded from below. We shall assume this case in our further analysis. The equations of motion simplify if we introduce new fields $\varphi = 2\phi_1$ and $\chi = \phi_1 + \phi_2$, in which case they become

$$\bar{\partial} \bar{\partial} \varphi = -\sigma e^{2\varphi - 2\chi}, \quad \bar{\partial} \partial \chi = -\sigma e^{-\varphi + 2\chi}.$$  

(65)

These equations have no apparent symmetry. However, the ansatz

$$\varphi = 2 \phi + \log 3, \quad \chi = \frac{3}{2} \phi + \log(3/\sqrt{2}),$$  

(66)
is consistent with the equations, and leads to the Liouville equation for $\phi$.

4.1 General solution

Equations (65) also have two sets of basic conserved densities. With the notation that $\varphi_n \equiv \partial^n \varphi$, $\chi_n \equiv \partial^n \chi$ the set corresponding to the conformally improved energy momentum tensor is

$$T = \frac{1}{2} \varphi_1^2 - \varphi_1 \chi_1 + \chi_1^2 - \frac{1}{2} \varphi_2 - \chi_2,$$

and the same expression for $\overline{T}$ with the replacement $\partial \rightarrow \overline{\partial}$. Eq. (65) admit no spin 3 conserved densities, but there is a set of spin 4 conserved densities [12],

$$Q = \varphi_4^4 - 4 \varphi_1^3 \chi_1 + 12 \varphi_1^2 \chi_1^2 - 16 \varphi_1 \chi_1^3 + 8 \chi_1^4 - 2 \varphi_1^2 \varphi_2 - 4 \varphi_1 \chi_1 \varphi_2$$

$$+ 3 \varphi_2^2 - 8 \varphi_1 \chi_2 + 24 \varphi_1 \chi_1 \chi_2 - 16 \chi_1^2 \chi_2 + 4 \varphi_2 \chi_2 + 4 \chi_2^2$$

$$+ 2 \varphi_1 \varphi_3 + 2 \chi_1 \varphi_3 - 2 \varphi_1 \chi_3 - 4 \chi_1 \chi_3 - \varphi_4 + 2 \chi_4,$$

and the same expression for $\overline{Q}$ with the replacement $\partial \rightarrow \overline{\partial}$. These conserved densities satisfy $\overline{\partial}T = \partial \overline{T} = \overline{\partial}Q = \partial \overline{Q} = 0$. If we insert the ansatz (66) into these expressions the result must be expressible in terms of the conservation laws of the Liouville equation. Indeed we find $T = \frac{5}{2} T_L$, and $Q = \frac{136}{25} T_L^2 - \frac{2}{5} \partial^2 T_L$, where $T_L$ is the energy-momentum tensor (22) of the Liouville model. This provides a check of our expressions.

It is convenient to introduce new variables $U = e^{-\varphi}$ and $V = e^{-\chi}$. We obtain, with the notation that $U_n \equiv \partial^n U$ and $V_n \equiv \partial^n V$,

$$T = \frac{1}{2} \frac{U_2}{U} + \frac{V_2}{V} - \frac{U_1 V_1}{UV},$$

and a lengthy expression for $Q$. It is possible to use (69) to eliminate all explicit reference to $V$ resp. $U$ in the expression for $Q$. We get

$$\frac{U_4}{U} - \frac{U_3 U_1}{U^2} + \frac{1}{2} \frac{U_2^2}{U^2} - 4 T \frac{U_2}{U} + 2 T \frac{U_1^2}{U^2} - 2 \partial T \frac{U_1}{U} = \frac{1}{2} Q + \overline{\partial}^2 T - 4 \overline{T}^2$$

and

$$\frac{V_4}{V} - 2 \overline{T} \frac{V_2}{V} - 2 \partial \overline{T} \frac{V_1}{V} = T^2 - \frac{1}{2} \overline{\partial}^2 T - \frac{1}{4} \overline{Q}.$$
these assumptions we find equations for $h$ and $k$. The equation for $h$ can be written in the form

$$\left( \frac{u^2}{v^2} \frac{\partial v^2}{u} \frac{\partial v^2}{\partial u} \frac{\partial u^2}{v^2} \partial h \right) h + R = 0,$$

(72)

where $R$ is rather complicated. With the notation that $h_n \equiv \partial^n h$, $u_n \equiv \partial^n u$, and $v_n \equiv \partial^n v$:

$$R = -u^2 h_1 h_3 + \frac{1}{2} u^2 h_2^2 - uu_1 h_1 h_2 + \left( 2 u_1^2 - 2 uu_2 - \frac{u^2}{v} v_2 - \frac{u}{v} u_1 v_1 \right) h_1^2.$$

(73)

However, the derivative of $R$ has a compact representation,

$$\partial R = -\left( \frac{u^2}{v^2} \frac{\partial v^2}{u} \frac{\partial v^2}{\partial u} \frac{\partial u^2}{v^2} \partial h \right) \partial h.$$

(74)

Thus, by differentiating (72) we find that $h$ satisfies a 5th order linear equation,

$$\partial \frac{u^2}{v^2} \frac{\partial v^2}{u} \frac{\partial v^2}{\partial u} \frac{\partial u^2}{v^2} \partial h = 0.$$

(75)

The equation for $k$ can be written as

$$\partial \frac{v^2}{u} \frac{\partial u^2}{v^2} \frac{\partial v^2}{u} \frac{\partial u^2}{\partial h} k = 0.$$

(76)

It is now straightforward to integrate (75) and (76). Define functions $H_1$ and $K_1$ such that $\partial H_1 = v^2/u^2$ and $\partial K_1 = u/v^2$. Then

$$h = \bar{\alpha}_0 + \bar{\alpha}_1 H_1 + \bar{\alpha}_2 H_2 + \bar{\alpha}_3 H_3 + \bar{\alpha}_4 H_4,$$

$$k = \bar{\beta}_0 + \bar{\beta}_1 K_1 + \bar{\beta}_2 K_2 + \bar{\beta}_3 K_3,$$

(77)

where all the coefficients $\bar{\alpha}_i$, $\bar{\beta}_i$ may depend on $\bar{z}$: $\bar{\alpha}_0 = \bar{\alpha}_0(\bar{z})$ etc. The functions $H_i$ and $K_i$ are given by the integrals

$$H_2 = \int \partial H_1 \int \partial K_1 = \int K_1 \partial H_1,$$

$$H_3 = \int \partial H_1 \int \partial K_1 \int \partial K_1 = \frac{1}{2} \int K_1^2 \partial H_1,$$

$$H_4 = \int \partial H_1 \int \partial K_1 \int \partial K_1 \int \partial H_1 = H_1 H_3 - \frac{1}{2} H_2^2,$$

$$K_2 = \int \partial K_1 \int \partial H_1 = \int H_1 \partial K_1 = H_1 K_1 - H_2,$$

$$K_3 = \int \partial K_1 \int \partial H_1 \int \partial K_1 = H_2 K_1 - 2 H_3.$$

(78)
Here the various identities among the integrals are derived by (repeated) partial integrations, and depends on certain assumptions about the undetermined integration constants (i.e., the lower integration limits). This is the maximum set of relations which may be derived between the integrals without making specific assumptions about the functions \( u \) and \( v \). It is also the minimum set required for the rest of the solution to go through. However, the identities still allow us to freely change the outer integration limits in the definitions of \( H_2 \) and \( H_3 \), and to shift \( H_1 \) and \( K_1 \) by independent constants. Note that we have the relation \( 2 \, H_1 - 2 \, H_3 \, H_1 + H_2^2 = 0 \). With the additional constraint that \( 2 \, \bar{a}_4 \bar{a}_0 - 2 \, \bar{a}_3 \bar{a}_1 + \bar{a}_2^2 = 0 \) the expression (77) for \( h \) is also a solution of (72), which in fact was the equation we first solved to determine \( h \).

The functions \( h \) and \( k \) must also satisfy differential equations in \( z \), the same as (75) and (76) with the replacements \((u, v, \partial) \rightarrow (\bar{u}, \bar{v}, \bar{\partial})\). This means that we also have the representation

\[
\begin{align*}
  h &= \alpha_0 + \alpha_1 \, H_1 + \alpha_2 \, H_2 + \alpha_3 \, H_3 + \alpha_4 \, H_4, \\
  k &= \beta_0 + \beta_1 \, K_1 + \beta_2 \, K_2 + \beta_3 \, K_3,
\end{align*}
\]

(79)

where the \( \bar{H}_i \)'s and \( \bar{K}_i \)'s are given by the barred version of (78), and the coefficients \( \alpha_i, \beta_i \) may depend on \( z \): \( \alpha_0 = \alpha_0(z) \) etc. Equations (72), (77), (79) are consistent with each other if each of the coefficients \( \alpha_i \) is a linear combination of the functions \( 1, H_1, H_2, H_3, H_4 \), and each of the coefficients \( \beta_i \) is a linear combination of the functions \( 1, K_1, K_2, K_3 \). Define (with coefficients choosen for later convenience)

\[
\begin{align*}
  H &= (1, H_1, 2^{1/2} \, H_2, 2 \, H_3, 2 \, H_4), \\
  \bar{H} &= (1, \bar{H}_1, 2^{1/2} \, \bar{H}_2, 2 \, \bar{H}_3, 2 \, \bar{H}_4), \\
  K &= (1, K_1, K_2, K_3), \\
  \bar{K} &= (1, \bar{K}_1, \bar{K}_2, \bar{K}_3).
\end{align*}
\]

(80)

Then all the linear equations are satisfied by the representations

\[
\begin{align*}
  h &= \bar{H} \cdot X \cdot H^t, \\
  k &= \bar{K} \cdot Y \cdot K^t,
\end{align*}
\]

(82)

where \( X \) is a \( 5 \times 5 \) matrix of real constants and \( Y \) is a \( 4 \times 4 \) matrix of real constants. However, not all the representations (82) will satisfy the original Toda equations (83). To check our ansatz we first compute

\[
\begin{align*}
  \partial H &= (0, 1, 2^{1/2} \, K_1, K_2^2, H_1 \, K_1^2 - 2 \, H_2 \, K_1 + 2 \, H_3) \, \partial H_1 \equiv h \, \partial H_1, \\
  \partial K &= (0, 1, H_1, H_2) \, \partial K_1 \equiv h \, \partial K_1,
\end{align*}
\]

(83)

and the corresponding barred quantities. Inserting \( \varphi = - \log(\bar{u} \, h \, u) \) and \( \chi = - \log(\bar{v} \, k \, v) \) into (83) gives the equations \( h \, \partial \varphi - \bar{h} \, \partial \bar{\varphi} = k^2 \, \partial \bar{H}_1 \, \partial H_1 \) and \( k \, \partial \chi - \bar{k} \, \partial \bar{\chi} = h \, \partial \bar{K}_1 \, \partial K_1 \). These are purely algebraic conditions\(^5\).

\[
\begin{align*}
  \left( \bar{H} \cdot X \cdot H^t \right) (\bar{h} \cdot X \cdot h^t) - (\bar{h} \cdot X \cdot H^t) \left( \bar{H} \cdot X \cdot h^t \right) &= \left( \bar{K} \cdot Y \cdot K^t \right)^2, \\
  \left( \bar{K} \cdot Y \cdot K^t \right) (\bar{k} \cdot Y \cdot k^t) - (\bar{k} \cdot Y \cdot K^t) \left( \bar{K} \cdot Y \cdot k^t \right) &= \left( \bar{H} \cdot X \cdot H^t \right).
\end{align*}
\]

(84)

If one wants to solve the model (83) with \( \sigma_1, \sigma_2 \in \{1, -1\} \) one should at this point multiply the right hand side of (84) by \( \sigma_1 \) and the right hand side of (85) by \( \sigma_2 \).
Regarding the expressions as polynomials in the indeterminates $H_1, H_2, H_3, K_1$ (and the corresponding barred quantities) these are sets of algebraic equations for the elements of $X$ and $Y$.

### 4.2 The transformation group $Sp(2, R)$

The equations (84-85) are nonlinear and involve many $(5^2 + 4^2 = 41)$ variables. They are not easy to solve by brute force. However, with the ansatz that $X$ and $Y$ are diagonal a two-parameter class of solutions is straightforward to find:

$$X = \text{diag} \left( e^\mu, e^{2\nu - \mu}, 1, e^{\mu - 2\nu}, e^{-\mu} \right), \quad Y = \text{diag} \left( e^{\nu}, e^{\mu - \nu}, e^{\nu - \mu}, e^{-\nu} \right). \quad (86)$$

The two parameters have their origin in the symmetry that our equations for $H_1, H_2, H_3$ and $K_1$ were not fully specified. We are free to shift $H_1 \rightarrow H_1 + \alpha$. This is equivalent to keeping $H$ and $K$ unchanged, and instead multiplying $X$ resp. $Y$ from the right by diagonal matrices,

$$\exp \left( \mu H_1^{(X)} + \nu H_2^{(X)} \right) = \text{diag} \left( e^\mu, e^{2\nu - \mu}, 1, e^{\mu - 2\nu}, e^{-\mu} \right),$$

$$\exp \left( \mu H_1^{(Y)} + \nu H_2^{(Y)} \right) = \text{diag} \left( e^{\nu}, e^{\mu - \nu}, e^{\nu - \mu}, e^{-\nu} \right). \quad (87)$$

There is a similar invariance under rescalings of $\bar{u}$ and $\bar{v}$. This is equivalent to multiplications from the left by the corresponding diagonal matrices.

From the solutions (86) we can easily find many more, since the lower integration limits in the expressions (78) for $H_1, H_2, H_3$ and $K_1$ were not fully specified. We are free to shift $H_1 \rightarrow H_1 + \alpha$. This is equivalent to keeping $H$ and $K$ unchanged, and instead multiplying $X$ resp. $Y$ from the right by matrices

$$e^\alpha F_1^{(X)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \alpha & 1
\end{pmatrix}, \quad e^\alpha F_1^{(Y)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \alpha & 1
\end{pmatrix}$$

We are free to shift $K_1 \rightarrow K_1 + \beta$. This is equivalent to keeping $H$ and $K$ unchanged, and instead multiplying $X$ resp. $Y$ from the right by matrices

$$e^\beta F_2^{(X)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & \beta \sqrt{2} & 1 & 0 & 0 \\
0 & \beta^2 \sqrt{2} & 1 & 0 & 0 \\
0 & 0 & 0 & \beta \sqrt{2} & 1
\end{pmatrix}, \quad e^\beta F_2^{(Y)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \beta & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \beta & 1
\end{pmatrix}$$

We are free to shift $H_2 \rightarrow H_2 + \gamma$. This is equivalent to keeping $H$ and $K$ unchanged, and instead multiplying $X$ resp. $Y$ from the right by matrices

$$e^\gamma F_3^{(X)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\gamma \sqrt{2} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-\gamma^2 & 0 & -\gamma \sqrt{2} & 0 & 1
\end{pmatrix}, \quad e^\gamma F_3^{(Y)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-\gamma & 0 & 1 & 0 & 0 \\
0 & \gamma & 0 & 1
\end{pmatrix}$$
We are free to shift $H_3 \rightarrow H_3 - \delta/2$. This is equivalent to keeping $H$ and $K$ unchanged, and instead multiplying $X$ resp. $Y$ from the right by matrices

$$e^{\delta F_i^{(X)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & 0 & 1 \end{pmatrix}, \quad e^{\delta F_i^{(Y)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \delta & 0 & 0 \end{pmatrix}$$

There is a similar freedom to shift the corresponding barred quantities. These shifts are equivalent to keeping $\bar{H}$ and $\bar{K}$ unchanged, and instead multiplying $X$ resp. $Y$ from the left by the transpose of matrices corresponding to those above, i.e. exponentials of generators $E_i^{(\cdot)}$ which are the transpose of the generators $F_i^{(\cdot)}$ above. The two sets of 10 generators $\{H_i^{(\cdot)}, E_j^{(\cdot)}, F_j^{(\cdot)}\}$ ($i = 1, 2, j = 1, \ldots, 4$) are representation matrices for a basis of the complex Lie algebra $B_2$. Upon exponentiation of these generators with real parameters, and further composition, they generate groups of real matrices. More specifically, with a symplectic form $\epsilon$ and a metric form $\eta$,

$$\epsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (88)$$

the 4-dimensional representation generate real symplectic matrices $S \in Sp(2, R)$ ($S^t \epsilon S = \epsilon$), and the 5-dimensional representation generate pseudo-orthogonal matrices $O \in O(3, 2)$ ($O^t \eta O = \eta$). It is now straightforward to verify by direct insertion that a solution to (84,85) is obtained by choosing any $Y \in Sp(2, R)$ (as defined above) and taking $X$ to be the corresponding $SO(3, 2)$ representation matrix. The precise connection between corresponding $X$ and $Y$’s is implicitly defined by the connection between the 4- and 5-dimensional generators listed in the appendix. A more direct relation between $X$ and $Y$ is defined by (85), which moreover is an explicit expression for the function $h$. Eq. (85) show that $Y$ and $-Y$ leads to the same solution for $X$, which means that $Sp(2, R)$ is a double covering of $SO(3, 2)$. Thus the transformation group in this case is $Sp(2, R)$ (or rather $Sp(2, R) \times Sp(2, R)$ if we include the gauge transformations).

The initial value problem can now be solved by the same procedure as described for the $A_2$ Toda field theory. The practical implementation of this procedure only becomes more cumbersome.

5 Concluding remarks

In this paper we have in detail discussed the explicit solutions of the three simplest Toda field theories. From this a general picture emerges, which seems to be as
follows: For each Lie algebra \( A_n \) \((n \geq 1)\), \( B_n \) \((n \geq 2)\), \( C_n \) \((n \geq 3)\), \( D_n \) \((n \geq 3)\), \( E_n \) \((n = 6, 7, 8)\), \( F_4 \), and \( G_2 \) there exists a Toda field theory. The number of independent field components in the model is equal to the index \( n \). Thus, to solve the model by the method used in this paper we have to find \( 2n \) independent conservation laws which can be written in one-sided forms,

\[
\partial Q_s = 0, \quad \bar{\partial} Q_s = 0.
\]  

Such conservation laws can be found in all the cases we have considered. The conserved densities always occur in parity related pairs \((Q_s, \bar{Q}_s)\). For \( A_n \) there is one independent density \( Q_s \) for each spin \( s = 2, 3, \ldots, n + 1 \). For \( B_n \) and \( C_n \) there is one independent density \( Q_s \) for each spin \( s = 2, 4, \ldots, 2n \). For \( D_n \) there is one independent density \( Q_s \) for each spin \( s = 2, 4, \ldots, 2n - 2 \), and an additional one of spin \( s = n \). For \( E_6 \) there is one independent density for each spin \( s \in \{2, 5, 6, 8, 9, 12\} \). For \( E_7 \) there is one independent density for each spin \( s \in \{2, 6, 8, 10, 12, 14, 18\} \). For \( E_8 \) there is one independent density for each spin \( s \in \{2, 8, 12, 14, 18, 20, 24, 30\} \). For \( F_4 \) there is one independent density for each spin \( s \in \{2, 6, 8, 12\} \). For \( G_2 \) there is one independent density for each spin \( s \in \{2, 6\} \). Counting shows that the number of independent conserved densities is always equal to the number of field components.

The fields in the model are related to a conserved density by a differential relation of the same order as its spin. By considering the canonical degrees of freedom one finds that the number \( N \) of integration constants which occur when one attempts to determine the canonical fields from the conserved densities is equal to the sum of the spins (counting both \( Q_s \) and \( \bar{Q}_s \)), minus the number of field components. This number is \( N = (n + 1)^2 - 1 \) for \( A_n \), \( N = (2n + 1)n \) for \( B_n \) and \( C_n \), \( N = (2n - 1)n \) for \( D_n \), and \( N = (78, 133, 248, 52, 14) \) for \( (E_6, E_7, E_8, F_4, G_2) \). These numbers are precisely equal to the dimensions of the Lie algebras which label the models. Thus there is a \( N \)-dimensional manifold of field configurations which lead to the same conserved densities, and a continuous transformation group acting on this manifold. Our experience from the previous sections is that the manifold may be identified with the transformation group, and that the group has as its Lie algebra one of the real forms of the (complex) Lie algebra which labels the Toda field theory. The freedom of choosing integration constants is probably such that the transformation group will consist of one simply connected component.

How useful are the solutions we have found? If the purpose is to determine the classical fields on a dense grid of points covering a finite regular region of space-time, it may in fact be difficult to beat a direct numerical solution of the partial differential equations in efficiency (although probably in accuracy). The computational effort must in any case be proportional to the number of grid points, and the proportionality factor is rather small for a direct numerical method. However, if we are interested in finding the fields along a line or on small
set of points only, the use of our solutions would be vastly more efficient than a direct numerical method. The same is true if the interest is in how asymptotic field configurations at \( t \to -\infty \) develop into asymptotic field configurations at \( t \to \infty \). Finally there is the problem of understanding the quantized versions of these models. For this any improved understanding of their analytic structure is of potential use.

A The complex Lie algebra \( B_2 \).

The complex Lie algebra \( B_2 \) can be represented by \( 5 \times 5 \) or \( 4 \times 4 \) matrices. Let \( \text{diag}^{n\uparrow} (\cdot, \ldots, \cdot) \) denote a matrix whose only non-zero entries are on the \( n \)'th superdiagonal, and \( \text{diag}^{n\downarrow} (\cdot, \ldots, \cdot) \) denote a matrix whose only non-zero entries are on the \( n \)'th subdiagonal. Then the transformations found in section 4.2 correspond to a 5-dimensional matrix representation of a basis of \( B_2 \),

\[
\begin{align*}
H_1^{(5)} &= \text{diag} (1, -1, 0, 1, -1), & H_2^{(5)} &= \text{diag} (0, 2, 0, -2, 0), \\
E_1^{(5)} &= \text{diag}^{1\uparrow} (1, 0, 0, 1), & E_2^{(5)} &= \text{diag}^{1\uparrow} (0, \sqrt{2}, \sqrt{2}, 0), \\
E_3^{(5)} &= \text{diag}^{2\uparrow} (\sqrt{2}, 0, -\sqrt{2}), & E_4^{(5)} &= \text{diag}^{3\uparrow} (-1, -1), \\
F_1^{(5)} &= \text{diag}^{1\downarrow} (1, 0, 0, 1), & F_2^{(5)} &= \text{diag}^{1\downarrow} (0, \sqrt{2}, \sqrt{2}, 0), \\
F_3^{(5)} &= \text{diag}^{2\downarrow} (\sqrt{2}, 0, -\sqrt{2}), & F_4^{(5)} &= \text{diag}^{3\downarrow} (-1, -1),
\end{align*}
\]

and a corresponding 4-dimensional representation,

\[
\begin{align*}
H_1^{(4)} &= \text{diag} (0, 1, -1, 0), & H_2^{(4)} &= \text{diag} (1, -1, 1, -1), \\
E_1^{(4)} &= \text{diag}^{1\uparrow} (0, 1, 0), & E_2^{(4)} &= \text{diag}^{1\uparrow} (1, 0, 1), \\
E_3^{(4)} &= \text{diag}^{2\uparrow} (-1, 1), & E_4^{(4)} &= \text{diag}^{3\uparrow} (1), \\
F_1^{(4)} &= \text{diag}^{1\downarrow} (0, 1, 0), & F_2^{(4)} &= \text{diag}^{1\downarrow} (1, 0, 1), \\
F_3^{(4)} &= \text{diag}^{2\downarrow} (-1, 1), & F_4^{(4)} &= \text{diag}^{3\downarrow} (1).
\end{align*}
\]

These generators are given in the Chevalley basis. Thus they are normalized so that \([E_i, F_j] = \delta_{ij} H_i\) for \( i, j \in \{1, 2\} \). Not all relative signs are determined by the conventions of a Chevalley basis. Here we have chosen signs so that \([E_1, E_2] = E_3\) and \([E_2, E_3] = 2E_4\).

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