The conjugacy class graphs of non-abelian 3-groups

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INTRODUCTION

The conjugacy class graph has been introduced by Bertram et al. in 1990 [1] and many researchers are interested in studying the relation of conjugacy class graph with groups. Casolo and Dolfi [2] experimented on the diameter of conjugacy class graph for finite groups in 1996. In 2013, Moradipour et al. [3] investigated the conjugacy class graph of some metacyclic 2-groups where the chromatic number and the clique number of these graphs are found to be similar. Meanwhile, in the same year, Omar et al. [4] constructed the conjugacy class graph using the probability of metacyclic 2-groups. Later, Omar et al. [5] explored the conjugacy class graphs for the probability of symmetric groups. In this research, the conjugacy class graphs on non-abelian 3-groups [6] are constructed based on two classifications of these groups, which are:

\[ G = \langle x, y : x^3 = y^3 = (xy)^2 = 1, z = [x, y], zx = xz, xy = yx \rangle \]

\[ H = \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle \]

Basic definitions in group theory

Some basic definitions in group theory used in this research such as the center of a group, the conjugate element, and the conjugacy class of an element of a group, are given in this section.

Basic definitions in graph theory

Some basic definitions in graph theory are used in this research including the conjugacy class graph, the complete graph, the chromatic number, the dominating number and the diameter of a graph, are given in this section.

Definition 1 [7] The center \( Z(G) \) of a group \( G \) is the set of elements that commute with all elements of \( G \) where \( Z(G) = \{ a \in G : ag = ga, \forall g \in G \} \).

Definition 2 [1] If \( G \) is a group and \( a \in G \), then the conjugate of \( a \) is any element in \( G \) of the form \( gag^{-1} \) where \( g \in G \).

Definition 3 [8] Let \( a, b \in G \). We say \( a \) and \( b \) are conjugated in \( G \) or \( b \) is a conjugate of \( a \) if \( x^{-1}ax = b \) for some \( x \) in \( G \). The conjugacy class of \( a \) is the set \( cl(a) = \{ x^{-1}ax | x \in G \} \).

Basic definitions in graph theory

Some basic definitions in graph theory are used in this research including the conjugacy class graph, the complete graph, the chromatic number, the dominating number and the diameter of a graph, are given in this section.

Definition 4 [9] Let \( G \) be a group and \( Z(G) \) is the center of \( G \). A conjugacy class graph of \( G \) is a graph when its vertices are the non-central conjugacy classes of \( G \), and two distinct vertices are connected if their cardinalities are not co-prime. A conjugacy class graph of \( G \) is denoted as \( \Gamma_G \).
Definition 5 [10] A complete graph is a simple graph in which every pair of distinct vertices is adjacent. The complete graph with \( n \) vertices is denoted as \( K_n \).

Definition 6 [11] A chromatic number of a graph \( \Gamma_G \), denoted as \( \chi(\Gamma_G) \), is a proper colouring or simply a colouring of the vertices of \( G \) in such a way that adjacent vertices have distinct colours. It is also the minimal number of colours in a (vertex) colouring of \( G \).

Definition 7 [12] A set \( S \) of vertices of \( G \) is the dominating set of \( G \) if every vertex in \( V(G) - S \) is adjacent to some vertices in \( S \). A dominating number of \( G \), denoted as \( \gamma(\Gamma_G) \), is the cardinality of a minimum dominating set.

Definition 8 [10] The diameter of a connected graph \( \Gamma \), denoted as \( \text{diam} (\Gamma) \), is the maximum possible distance between any two vertices.

RESULTS AND DISCUSSION

This section presents the results of the conjugacy class graphs for non-abelian 3-groups of order 27 along with some properties of the graphs, given in the following theorems and propositions.

The conjugacy class graphs for \( G \) and their properties

In this section, the results for the conjugacy class graph of \( G \) are shown as in the following theorem and propositions.

Theorem 1 Let \( G =< x, y : x^3 = y^3 = z^3 = 1, z = [x, y], zx = xz, zy = yz > \) be a non-abelian 3-group of order 27. Then, the conjugacy class graph of \( G \) is the complete graph with eight vertices, namely as \( \Gamma_G^{CC} = K_8 \).

Proof The elements of \( G \) are as in the following:

\[
G = \{1, x^2yxy^2, x^2y^2xy, x, y, x^2, y, xy, x^2y, y^2, xy, x^2y^2, y^2x, x^2xy, y^2x^2, xy^2, x^2xy^2, y^2x^2, x^2y^2x, y^2x^2y, x^2xy^2, y^2x^2y^2, y_2x, x^2y_2, y^2x_2, x^2y_2x, y^2x_2y, x^2y_2y, y^2x_2y^2\}.
\]

From the set of conjugate elements, the center of \( G \) is \( Z(G) = \{1, x^2yxy, x^2y^2xy\} \). Then, by using Definition 3, the cardinalities of the non-central conjugacy classes of \( G \) are as in the following:

1. \( |cl(1)| = 1 \)
2. \( |cl(x^2yxy^2)| = x^2yxy^2 \)
3. \( |cl(x^2y^2xy)| = x^2y^2xy \)
4. \( |cl(x)| = [x, y, y^2x, y_2] \)
5. \( |cl(y)| = [y, y_2, x_2y, x^2y] \)
6. \( |cl(xy)| = [xy, yx, x^2y^2] \)
7. \( |cl(x^2y^2)| = [x^2y^2, y^2x, x^2y_2] \)
8. \( |cl(x^2y^2x)| = [x^2y^2x, y^2x^2, x^2y_2x, y^2x^2y] \)
9. \( |cl(x^2y^2y)| = [x^2y^2y, y^2x^2y, x^2y_2y, y^2x^2y^2] \)
10. \( |cl(xy^2)| = [xy^2, yx^2, y_2x, x_2y] \)
11. \( |cl(x^2y^2x)| = [x^2y^2x, x^2y^2 xy, y^2x^2y] \)

From the set of conjugate elements, the center of \( G \) is \( Z(G) = \{1, x^2yxy, x^2y^2xy\} \). Then, by using Definition 3, the cardinalities of the non-central conjugacy classes of \( G \) are as in the following:

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2. \( |cl(x^2yxy^2)| = x^2yxy^2 \)
3. \( |cl(x^2y^2xy)| = x^2y^2xy \)
4. \( |cl(x)| = [x, y, y^2x, y_2] \)
5. \( |cl(y)| = [y, y_2, x_2y, x^2y] \)
6. \( |cl(xy)| = [xy, yx, x^2y^2] \)
7. \( |cl(x^2y^2)| = [x^2y^2, y^2x, x^2y_2] \)
8. \( |cl(x^2y^2x)| = [x^2y^2x, y^2x^2, x^2y_2x, y^2x^2y] \)

The set of non-central conjugacy classes of \( G \), which is the set of vertices of the conjugacy class graph of \( G \) that have the same cardinalities, which is three. Then, all the vertices are connected with each other and a complete graph with eight vertices has been formed as illustrated in the following Fig. 1.

![Fig. 1 The conjugacy class graph of \( G \).](image)

Proposition 1 Let \( G =< x, y : x^3 = y^3 = z^3 = 1, z = [x, y], zx = xz, zy = yz > \) be a non-abelian 3-group of order 27. Then, the chromatic number of the commuting graph of \( G \) is eight, namely as \( \chi(\Gamma_G^{CC}) = 8 \).

Proof By referring to the Fig. 1, the minimum number of colours needed to color all the vertices in a way that all the adjacent vertices have different colours is eight. Then, by using Definition 6, the chromatic number of the conjugacy class graph of \( G \) is eight as illustrated in the Fig. 2.

![Fig. 2 The colouring of the conjugacy class graph of \( G \).](image)

Proposition 2 Let \( G =< x, y : x^3 = y^3 = z^3 = 1, z = [x, y], zx = xz, zy = yz > \) be a non-abelian 3-group of order 27. Then, the dominating number of the conjugacy class graph of \( G \) is one, namely as \( \gamma(\Gamma_G^{CC}) = 1 \).

Proof By referring to Figure 1, the minimum number of vertices needed to ensure all the vertices are adjacent is one. Then, by using Definition 7, the dominating number of the conjugacy class graph of \( G \) is one.

Proposition 3 Let \( G =< x, y : x^3 = y^3 = z^3 = 1, z = [x, y], zx = xz, zy = yz > \) be a non-abelian 3-group of order 27. Then, the diameter of the conjugacy class graph of \( G \) is one, namely as \( \text{diam}(\Gamma_G^{CC}) = 1 \).
**Proof** By referring to Fig. 1, the longest path between any two distinct vertices is one. By using Definition 8, the diameter of conjugacy class graph of $G$ is one.

By using similar method, the results for $H$ are obtained as stated in the following theorem and propositions.

**The conjugacy class graph and its properties of the graph for $H$**

This section is on the results for the conjugacy class graph and its properties for the graph of $H$. The proofs are similar as in the previous section.

**Theorem 2** Let $H = \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle$ be a non-abelian 3-group of order 27. Then, the conjugacy class graph of $H$ is the complete graph with eight vertices, namely $\Gamma^c(\mathcal{C}_H) = K_8$.

![Fig. 3](image1.png)

**Fig. 3** The conjugacy class graph of $H$.

**Proposition 4** Let $H = \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle$ be a non-abelian 3-group of order 27. Then, the chromatic number of the commuting graph of $H$ is eight, namely $\chi(\Gamma^c(\mathcal{C}_H)) = 8$.

![Fig. 4](image2.png)

**Fig. 4** The colouring of the conjugacy class graph of $H$.

**Proposition 5** Let $H = \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle$ be a non-abelian 3-group of order 27. Then, the dominating number of the conjugacy class graph of $H$ is one, namely $\gamma(\Gamma^c(\mathcal{C}_H)) = 1$.

**Proposition 6** Let $H = \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle$ be a non-abelian 3-group of order 27. Then, the diameter of the conjugacy class graph of $H$ is one, namely $\diam(\Gamma^c(\mathcal{C}_H)) = 1$.

**CONCLUSION**

The conjugacy class graphs of $G$ and $H$ are turned out to be complete graphs with eight vertices. In addition, the properties of the conjugacy class graphs for $G$ and $H$, which are the diameter is found to be one, the chromatic number is found to be eight, and the dominating number is found to be one.

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