Ground state solutions for quasilinear scalar field equations arising in nonlinear optics

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Abstract. In this paper, we study a class of quasilinear elliptic equations which appears in nonlinear optics. By using the mountain pass theorem together with a technique of adding one dimension of space (Hirata et al. in Topol Methods Nonlinear Anal 35:253–276, 2010; Jeanjean in Nonlinear Anal Theory Methods Appl 28:1633–1659, 1997), we prove the existence of a non-trivial weak solution for general nonlinear terms of Berestycki–Lions’ type. The existence of a radial ground state solution and a ground state solution is also established under stronger assumptions on the quasilinear term.

Mathematics Subject Classification. 35J62, 35J20, 35Q60.

Keywords. Quasilinear elliptic equation, Variational method, Monotone operator.

1. Introduction

In this paper, we study the following quasilinear elliptic problem:

\[
\begin{aligned}
- \text{div} \left\{ \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \nabla u \right\} + \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) u &= g(u) \quad \text{in } \mathbb{R}^N, \\
\lim_{|x| \to +\infty} u(x) &= 0
\end{aligned}
\]  

(1.1)

where \( N \geq 3 \). The purpose of this paper is to establish the existence of non-trivial solutions and ground state solutions of (1.1) for general nonlinear term \( g \) by using the variational method.

On the map \( \phi : [0, +\infty) \to \mathbb{R} \), we assume:

(\(\phi_1\)) \( \phi \in C([0, +\infty)) \) and there exist two constants \( 0 < \phi_0 < \phi_1 \) such that

\[ \phi_0 \leq \phi(s) \leq \phi_1 \quad \text{for all } s \in [0, +\infty); \]

(\(\phi_2\)) the map \( t \mapsto \Phi(t^2) \) is strictly convex on \( \mathbb{R} \), where \( \Phi(s) = \int_0^s \phi(\tau) \, d\tau \).
We note that the condition \((\phi 2)\) is equivalent to

\[
\text{the map } t \mapsto t\phi(t^2) \text{ is increasing on } [0, +\infty).
\]

Typical examples of \(\phi(s)\) are given by

- \(\phi(s) = K + (1 + s)^{-\alpha} \) with \(0 \leq \alpha \leq \frac{1}{2}, K > 0\) or \(\alpha > \frac{1}{2}, K \gg 1\);
- \(\phi(s) = K - (1 + s)^{-\alpha} \) with \(\alpha > 0, K \gg 1\);
- \(\phi(s) = K + (1 + s)^{-\alpha} - (1 + s)^{-\beta} \) with \(0 \leq \alpha \leq \frac{1}{2}, \beta \geq 0, K > 1\) or \(\alpha > \frac{1}{2}, \beta > 0, K \gg 1\);
- \(\phi(s) = K + (1 + s)^{-\alpha} s^\beta \) with \(\alpha \leq \beta \leq \alpha + \frac{1}{2}, K > 0\) or \(\beta \geq 0, \beta \gg 1\);
- \(\phi(s) = K + \log(1 + (1 + s)^{-\alpha}) \) with \(\alpha > 0, K \gg 1\).

In the case \(\phi(s) = 1 + \frac{1}{\sqrt{1+s}}\), the operator \(\text{div}\{\phi(|\nabla u|^2)\nabla u\}\) is exactly the sum of the Laplacian \(\Delta\) and the mean curvature operator \(\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\).

General quasilinear elliptic problems of the form:

\[-\text{div}\{\phi(|\nabla u|^2)\nabla u\} = g(u)\]

have been studied, for example, in [2,7,9,13,14,22]. Especially the assumption \((\phi 2)\) is related with so-called \(\Delta_2\)-condition in the literature.

In our problem (1.1), the quasilinear term depends not on \(|\nabla u|^2\) but on \(u^2 + |\nabla u|^2\). This particular quasilinear term appears in the study of a nonlinear optics model describing the propagation of self-trapped beam in a cylindrical optical fiber made from a self-focusing dielectric material. (See [33,34] for the derivation.) In this model, \((\phi 2)\) plays a fundamental role as well. We also refer to [19,32] for similar problems in a bounded domain.

On the nonlinearity \(g\), we require that

- \((g1)\) \(g \in C(\mathbb{R}, \mathbb{R})\) and \(g\) is odd;
- \((g2)\) there exists \(m \in (-\phi_0, +\infty)\) such that
  \[-\infty < \liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} = -m;\]
- \((g3)\) denoting by \(2^* = \frac{2N}{N-2}\), it holds
  \[-\infty \leq \limsup_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} \leq 0;\]
- \((g4)\) there exists \(\zeta > 0\) such that \(G'(\zeta) > \Phi(\frac{\zeta^2}{2})\), where \(G(s) = \int_0^s g(\tau) d\tau\).

Whenever \(m \in (-\phi_0, 0)\), instead of \((\phi 2)\), we assume

- \((\phi 2')\) the map \(t \mapsto \Phi\left(\frac{t^2}{2}\right) + \frac{m}{2} t^2\) is strictly convex on \(\mathbb{R}\).

The conditions \((g1)-(g4)\) can be seen as a variant of Berestycki–Lions’ condition [4] for the semilinear scalar field equation:

\[-\Delta u = g(u) \quad \text{in } \mathbb{R}^N.\]

(1.2)

Indeed, under additional assumptions on \(\phi(s)\), one can show that \((g3)\) and \((g4)\) are almost optimal for the existence of non-trivial solutions of (1.1). (See
Theorem 5.4 below.) An important feature is that the mass constant \( m \) in 
(g2) can be negative because of the presence of a mass term in \( \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \) by (φ1). Our assumption (g2) means that a total mass is positive so that our 
problem (1.1) is actually the positive mass case.

To state our main results, we prepare some notations. By a weak solution
of (1.1), we mean a solution which satisfies (1.1) in the distribution sense,
equivalently, a critical point of the associated functional \( I : H^1(\mathbb{R}^N) \to \mathbb{R} \)
defined by

\[
I(u) := \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \, dx - \int_{\mathbb{R}^N} G(u) \, dx.
\]

Our first result is the following one.

**Theorem 1.1.** Assume (g1)–(g4), (φ1) and (φ2) when \( m \in [0, +\infty) \), or (φ2') when \( m \in (-\phi_0, 0) \). Then there exists a non-negative non-trivial weak solution of (1.1).

Next we consider the existence of a regular solution of (1.1), that is, a solution which belongs to the class \( C^1(\mathbb{R}^N) \) (indeed \( C^{1,\sigma}(\mathbb{R}^N) \) for some \( \sigma \in (0, 1) \)). For this purpose, we impose the following slightly stronger condition on \( \phi(s) \):

(φ3) \( \phi \in C^1([0, +\infty)) \) and there exists \( C > 0 \) such that \( s|\phi'(s)| \leq C \) for all \( s \in [0, +\infty) \). Moreover

\[
\phi_0 \leq \phi(s) + 2s\phi'(s) \quad \text{for all} \quad s \in [0, +\infty).
\]

We notice that (φ3) implies (φ2) and (φ2') respectively. One can see that (φ3) is fulfilled if

- \( \phi(s) = K + (1 + s)^{-\alpha} \), with \( 0 \leq \alpha \leq \frac{1}{2} \), \( K > 0 \);
- \( \phi(s) = K - (1 + s)^{-\alpha} \), with \( \alpha \geq 0 \), \( K > 1 \).

Then we have the following result.

**Theorem 1.2.** Suppose that (φ1), (φ3) and (g1)–(g4) hold. Then there exists a positive regular solution of (1.1), namely, a solution which is of the class \( C^{1,\sigma}(\mathbb{R}^N) \) for some \( \sigma \in (0, 1) \).

Once we have the regularity of solutions in hand, we are able to apply the Pohozaev identity (see Lemma 5.1 below) to obtain the existence of a radial ground state solution, namely, a solution of (1.1) having least energy among all non-trivial radial solutions.

**Theorem 1.3.** Suppose that (φ1), (φ3) and (g1)–(g4) hold. Then there exists a radial ground state solution of (1.1), which is of the class \( C^{1,\sigma}(\mathbb{R}^N) \) for some \( \sigma \in (0, 1) \) and positive on \( \mathbb{R}^N \).

Finally, we are interested in the existence of a ground state solution without restricting ourselves to the space of radial functions. For this purpose, we need the following additional assumption on \( \phi(s) \).
\((\phi 4)\) \(\phi \in C^2([0, +\infty))\) and there exists \(C > 0\) such that \(s^2|\phi''(s)| \leq C\) for all \(s \in [0, +\infty)\). Moreover

\[3\phi'(s) + 2s|\phi''(s)| \leq 0 \quad \text{and} \quad 0 \leq \phi(s) + 5s\phi'(s) - 2s^2|\phi''(s)| \quad \text{for all} \quad s \in [0, +\infty).\]

We observe that \((\phi 4)\) requires

\[\phi'(s) \leq 0 \quad \text{for all} \quad s \in [0, +\infty).\]  

This further implies that

\[t^2\phi(t^2) \leq \Phi(t^2) \quad \text{for all} \quad t \in \mathbb{R}.\]  

Elementary (but complicated) calculations show that \((\phi 4)\) is satisfied for

\[\phi(s) = K + (1 + s)^{-\alpha}, \quad 0 \leq \alpha \leq \frac{\sqrt{57} - 7}{4}, \quad K > 0 \quad \text{or} \quad \frac{\sqrt{57} - 7}{4} < \alpha \leq \frac{1}{2}, \quad K \gg 1.\]

Under the stronger assumption \((\phi 4)\), we are able to obtain the following result, which can be seen as an extension of the result in [4] for the semilinear case \(\phi(s) \equiv 1\).

**Theorem 1.4.** Assume \((\phi 1), (\phi 3), (\phi 4)\) and \((g1)–(g4)\). Then \((1.1)\) possesses a ground state solution \(u\), namely, \(u\) has least energy among all non-trivial solutions. Moreover, \(u\) is with fixed sign on \(\mathbb{R}^N\), radially symmetric with respect to some point and of the class \(C^{1,\sigma}(\mathbb{R}^N)\) for some \(\sigma \in (0, 1)\).

Here, we briefly introduce some ideas to obtain our main results. By \((\phi 1)\) and \((g1)–(g4)\), one can see that the functional \(I\) has the mountain pass geometry. Then the existence of a non-trivial critical point of \(I\) can be shown by establishing the Palais–Smale condition. Indeed once we could have the boundedness of Palais–Smale sequences in hand, one can expect the strong convergence of Palais–Smale sequences by decomposing the nonlinear term \(g\) into two parts as in [3,4,27] and restricting ourselves to the space of radial functions. However, as is well-known, the most difficult part is to prove the boundedness of Palais–Smale sequences.

A standard strategy of constructing a bounded Palais–Smale sequence is to apply so-called monotonicity trick as in [18,31]. However in the process of obtaining the boundedness, one needs to use the Pohozaev identity, causing the necessity of additional assumption on \(\phi\) in our problem. To avoid this technical difficulty, we adapt another strategy, namely, a technique of adding one dimension of space as established in [16,17]. This approach enables us to construct a bounded Palais–Smale sequence without using the Pohozaev identity.

Even if we could obtain the existence of a bounded Palais–Smale sequence, we face another difficulty in our quasilinear problem. In general, the boundedness of a Palais–Smale sequence \(\{u_n\}\) guarantees the weak convergence to some \(u\) in \(H^1(\mathbb{R}^N)\) and \(I'(u) = 0\) from \(I'(u_n) \to 0\). Then the strong convergence can be obtained by considering the difference between \(I'(u_n)[u_n]\) and \(I'(u)[u]\). However in our problem, one cannot show \(I'(u) = 0\) a priori, because the weak convergence of \(u_n \rightharpoonup u\) in \(H^1(\mathbb{R}^N)\) provides us no information of the pointwise convergence of \(\nabla u_n(x) \to \nabla u(x)\). To overcome this
difficulty, in the case \( m \geq 0 \), we apply the theory of monotone operator [10] for the functional

\[
\Psi_0(u) := \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \, dx, \quad u \in H^1(\mathbb{R}^N)
\]

to obtain the pointwise convergence of the gradient. Then by considering \( \Psi_0(u) - \Psi_0(u_n) - \Psi'_0(u_n)[u_n - u] \), together with the convexity of \( \Phi(t^2) \), we are able to prove the strong convergence of \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) and \( I'(u) = 0 \) \textit{a posteriori}. The case \( m \in (-\phi_0, 0) \) can be treated analogously.

Finally in order to prove Theorem 1.4, as in [20], one establishes that

\[
m_0 = b, \quad (1.5)
\]

where

\[
m_0 := \inf_{u \in S} I(u), \quad S := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid I'(u) = 0 \}, \quad b := \min_{u \in P} I(u),
\]

\( P := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u \text{ satisfies the Pohozaev identity for (1.1)} \} \).

In the semilinear case (1.2), we can prove (1.5) by adapting the scaling \( \theta \mapsto u(\cdot/\theta) \) and using the variational characterization:

\[
m_0 = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}. \quad (1.6)
\]

In our problem (1.1), we cannot readily see that (1.5) holds because of the loss of scaling property. Once we could have the variational characterization (1.5) in hand, one can argue as in [1,8]. Indeed \( \phi_4 \) guarantees the convexity of some functions related with the Pohozaev identity for (1.1). This enables us to apply the generalized Polya–Szegő inequality [15] to prove that minimizing sequences for \( m_0 \) can be assumed to be radially symmetric. Then the existence of a ground state of (1.1) can be shown similarly as Theorem 1.1.

This paper is organized as follows. In Sect. 2, we investigate some basic properties of the operator \( \Psi_0 \). We prove Theorem 1.1 by applying the mountain pass theorem in Sect. 3. Section 4 is devoted to the study of regularity and positivity of weak solutions of (1.1). Theorems 1.2 and 1.3 will be shown in Sect. 5. Finally, we prove Theorem 1.4 in Sect. 6.

We conclude this introduction fixing some notations. For any \( p \geq 1 \), we denote by \( L^p(\mathbb{R}^N) \) the usual Lebesgue spaces equipped by the standard norm \( \| \cdot \|_{L^p} \). In our estimates, we will frequently denote by \( C > 0 \) fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations \( o_n(1) \) to describe a quantity which goes to zero as \( n \to +\infty \). Moreover, for any \( R > 0 \), we denote by \( B_R \) the ball of \( \mathbb{R}^N \) centered in the origin with radius \( R \).
2. Fundamental properties of the operator $\Psi_0$

In this section, we investigate fundamental properties of the operator $\Psi_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Psi_0(u) := \int_{\mathbb{R}^N} \Phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) dx, \quad u \in H^1(\mathbb{R}^N).$$

For later use, we also denote,

$$\Psi_m(u) := \Psi_0(u) + \frac{m}{2} \int_{\mathbb{R}^N} u^2 dx,$$

$$\tilde{\Psi}_m(u) := \int_{\mathbb{R}^N} \left\{ \Phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) + m \left(\frac{u^2 + |\nabla u|^2}{2}\right) \right\} dx$$

$$= \Psi_m(u) + \frac{m}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

where $m \in (-\phi_0, +\infty)$ is defined in (g2). Then one readily finds that

$$\Psi_0(u) \leq \Psi_m(u) \leq \tilde{\Psi}_m(u) \quad \text{if } m \in [0, +\infty), \quad (2.1)$$

$$\tilde{\Psi}_m(u) \leq \Psi_m(u) \leq \Psi_0(u) \quad \text{if } m \in (-\phi_0, 0). \quad (2.2)$$

Moreover from (φ1), it is standard to prove $\Psi_0, \Psi_m, \tilde{\Psi}_m \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and for $\varphi \in H^1(\mathbb{R}^N)$,

$$\Psi_m'(u)[\varphi] = \int_{\mathbb{R}^N} \left\{ \phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) (u\varphi + \nabla u \cdot \nabla \varphi) + mu\varphi \right\} dx.$$

First we recall the following convergence result, which appears in the theory of monotone operators [10, Lemma 6].

**Proposition 2.1.** Let $X$ be a finite dimensional real Hilbert space with norm $| \cdot |$ and inner product $\langle \cdot , \cdot \rangle$. Suppose that $\beta : X \to X$ is a continuous function which is strictly monotone, i.e.

$$\langle \beta(\xi) - \beta(\bar{\xi}), \xi - \bar{\xi} \rangle > 0 \quad \text{for every } \xi, \bar{\xi} \in X \text{ with } \xi \neq \bar{\xi}.$$

Let $\{\xi_n\} \subset X$ and $\xi \in X$ be such that

$$\lim_{n \to +\infty} \langle \beta(\xi_n) - \beta(\xi), \xi_n - \xi \rangle = 0.$$

Then $\{\xi_n\}$ converges to $\xi$ in $X$.

Using Proposition 2.1, we are able to obtain the following pointwise convergence result of the gradient.

**Lemma 2.2.** Assume (φ1) and (φ2) when $m \in [0, +\infty)$, or (φ2′) when $m \in (-\phi_0, 0)$. If $\{u_n\} \subset H^1(\mathbb{R}^N)$ satisfies $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $\Psi_m'(u_n)[u_n - u] \to 0$ as $n \to +\infty$, then

$$\nabla u_n(x) \to \nabla u(x) \quad \text{as } n \to +\infty \text{ a.e. in } \mathbb{R}^N.$$
Proof. First since \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \), it follows that \( \Psi_m'(u)[u_n - u] \to 0 \) as \( n \to +\infty \). Hence by the assumption, we get

\[
o_n(1) = (\Psi_m'(u_n) - \Psi_m'(u))[u_n - u]
\]

(2.3)

We distinguish the proof into the cases \( m \in [0, +\infty) \) and \( m \in (-\phi_0, 0) \).

When \( m \in [0, +\infty) \), (2.3) yields that

\[
\limsup_{n \to +\infty} (\Psi_0'(u_n) - \Psi_0'(u))[u_n - u] \leq 0.
\]

(2.4)

On the other hand, a direct computation shows that

\[
\Psi_0'(u_n) - \Psi_0'(u) = \int_{\mathbb{R}^N} \left\{ \frac{u_n^2 + |\nabla u_n|^2}{2} \right\} \nabla u_n - \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \nabla u \right\} \cdot \nabla (u_n - u) \, dx
\]

(2.5)

Putting \( X = \mathbb{R} \times \mathbb{R}^N \), \( \xi_n = (u_n(x), \nabla u_n(x)) \), \( \xi = (u(x), \nabla u(x)) \) and \( \beta(\xi) = \frac{\xi}{\sqrt{2}} \phi \left( \frac{|\xi|^2}{2} \right) \), one finds that

\[
\langle \beta(\xi) - \beta(\bar{\xi}), \xi - \bar{\xi} \rangle = \frac{1}{\sqrt{2}} \phi \left( \frac{|\xi|^2}{2} \right) |\xi - \bar{\xi}|^2 > 0 \text{ when } |\xi| = |\bar{\xi}| \text{ and } \xi \neq \bar{\xi},
\]

(2.6)

\[
\langle \beta(\xi) - \beta(\bar{\xi}), \xi - \bar{\xi} \rangle = \frac{1}{\sqrt{2}} \phi \left( \frac{|\xi|^2}{2} \right) \left( \frac{\xi}{\sqrt{2}}, \xi - \bar{\xi} \right) - \phi \left( \frac{|\xi|^2}{2} \right) \left( \frac{\bar{\xi}}{\sqrt{2}}, \xi - \bar{\xi} \right)
\]

\[
\geq \left\{ \frac{|\xi|^2}{2} \phi \left( \frac{|\xi|^2}{2} \right) - \frac{|\xi|^2}{2} \phi \left( \frac{|\xi|^2}{2} \right) \right\} (|\xi| - |\bar{\xi}|) > 0 \text{ when } |\xi| \neq |\bar{\xi}|.
\]

(2.7)

Thus from (2.4) and (2.5), we obtain

\[
0 = \lim_{n \to +\infty} \int_{\mathbb{R}^N} \langle \beta(\xi_n) - \beta(\xi), \xi_n - \xi \rangle \, dx
\]

and hence

\[
\langle \beta(\xi_n) - \beta(\xi), \xi_n - \xi \rangle \to 0 \text{ as } n \to +\infty \text{ a.e. in } \mathbb{R}^N.
\]

(2.8)

By (2.6), (2.7) and (2.8), we are able to apply Proposition 2.1 to conclude that \( \xi_n \rightharpoonup \xi \) a.e. in \( \mathbb{R}^{N+1} \) and so \( \nabla u_n(x) \rightharpoonup \nabla u(x) \) a.e. in \( \mathbb{R}^N \).

Next we consider the case \( m \in (-\phi_0, 0) \). In this case, we first find that

\[
o_n(1) = (\Psi_m'(u_n) - \Psi_m'(u))[u_n - u]
\]

(2.9)

\[
= (\Psi_m'(u_n) - \Psi_m'(u))[u_n - u] - m \int_{\mathbb{R}^N} |\nabla (u_n - u)|^2 \, dx.
\]
Putting $\beta_m(\xi) = \left( \phi \left( \frac{|\xi|^2}{2} \right) + m \right) \frac{\xi}{\sqrt{2}}$, one gets from $m < 0$ that
\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} \langle \beta_m(\xi_n) - \beta_m(\xi), \xi_n - \xi \rangle \, dx \leq 0. \tag{2.9}
\]
Moreover since $m > -\phi_0$, the function $t \mapsto \left( \phi \left( \frac{t^2}{2} \right) + m \right) \frac{t}{\sqrt{2}}$ is increasing on $[0, +\infty)$ from $(\phi'2')$. Arguing as done previously, this implies that
\[
\langle \beta_m(\xi) - \beta_m(\xi), \xi - \xi \rangle > 0 \quad \text{for any } \xi, \xi \text{ with } \xi \neq \xi. \tag{2.10}
\]
Then from (2.9) and (2.10), we conclude as above.

Finally we establish the following Brezis–Lieb type convergence result.

**Lemma 2.3.** Assume $(\phi 1)$ and $(\phi 2)$ when $m \in [0, +\infty)$, or $(\phi 2')$ when $m \in (-\phi_0, 0)$. If $\{u_n\} \subset H^1(\mathbb{R}^N)$ satisfies $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $\nabla u_n(x) \to \nabla u(x)$ a.e. in $\mathbb{R}^N$ and
\[
\limsup_{n \to +\infty} \Psi_m(u_n) \leq \Psi_m(u).
\]
Then, up to a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$.

**Proof.** Again, we distinguish the cases $m \in [0, +\infty)$ and $m \in (-\phi_0, 0)$. First we suppose that $m \in [0, +\infty)$. One finds that
\[
\Psi_m(u_n) - \Psi_m(u) - \Psi_m(u_n - u)
= \Psi_0(u_n) - \Psi_0(u) - \Psi_0(u_n - u) + m \int_{\mathbb{R}^N} u(u_n - u) \, dx. \tag{2.11}
\]
Now setting
\[
f_n(x) := \left( \frac{u_n(x)^2 + |\nabla u_n(x)|^2}{2} \right)^{\frac{1}{2}}, \quad f(x) := \left( \frac{u(x)^2 + |\nabla u(x)|^2}{2} \right)^{\frac{1}{2}},
\]
we may assume that $f_n(x) \to f(x)$ a.e. in $\mathbb{R}^N$. Moreover putting $j(s) = \Phi(s^2)$, one knows from $(\phi 2)$ that $j(s)$ is continuous, convex on $\mathbb{R}$ and $j(0) = 0$. Then by the Brezis–Lieb lemma [5, Theorem 2 and Examples (b)], it follows that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |j(f_n) - j(f) - j(f_n - f)| \, dx = 0
\]
and hence
\[
\lim_{n \to +\infty} \{ \Psi_0(u_n) - \Psi_0(u) - \Psi_0(u_n - u) \} = 0. \tag{2.12}
\]
From (2.11), (2.12), $\Psi_m \geq 0$ and by the assumptions of this lemma, we get
\[
0 \leq \liminf_{n \to +\infty} \Psi_m(u_n - u) \leq \limsup_{n \to +\infty} \Psi_m(u_n - u)
= \limsup_{n \to +\infty} \left\{ \Psi_m(u_n) - \Psi_m(u) + o_n(1) \right\} \leq 0,
\]
from which we conclude
\[
\lim_{n \to +\infty} \Psi_m(u_n - u) = 0.
\]
Since, from (2.1), \(0 \leq \Psi_0(u_n - u) \leq \Psi_m(u_n - u)\) when \(m \in [0, +\infty)\), this also yields that
\[
\lim_{n \to +\infty} \Psi_0(u_n - u) = 0.
\]
Then from (\(\phi_1\)), we conclude that \(u_n - u \to 0\) in \(H^1(\mathbb{R}^N)\), as desired.

Next we assume that \(m \in (-\phi_0, 0)\). First one observes that
\[
\Psi_m(u_n) - \Psi_m(u) - \Psi_m(u_n - u) = \tilde{\Psi}_m(u_n) - \tilde{\Psi}_m(u) - \tilde{\Psi}_m(u_n - u) - m \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u_n - u) \, dx.
\]
Letting \(j_m(s) = \Phi(s^2) + ms^2\), we have from (\(\phi_2'\)) that \(s \mapsto j_m(s)\) is convex. Then we can apply the Brezis–Lieb lemma to obtain
\[
\lim_{n \to +\infty} \{ \tilde{\Psi}_m(u_n) - \tilde{\Psi}_m(u) - \tilde{\Psi}_m(u_n - u) \} = 0.
\]
In a similar argument as the case \(m \in [0, +\infty)\), it follows that
\[
\lim_{n \to +\infty} \Psi_m(u_n - u) = 0.
\]

3. Variational setting and existence of a non-negative non-trivial solution

In this section, we perform a variational setting of (1.1) and prove the existence of a non-negative non-trivial solution of (1.1).

First as in [3,4,27], we decompose the nonlinear term \(g\) as follows. Let \(s_0 := \min\{s \in [\zeta, +\infty) : g(s) = 0\}\), \(s_0 = +\infty\) if \(g(s) \neq 0\) for any \(s \geq \zeta\) and \(\tilde{g} : \mathbb{R} \to \mathbb{R}\) be the continuous function such that
\[
\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0], \\ 0 & \text{on } \mathbb{R} \setminus [0, s_0] \end{cases} \quad \text{for } s \geq 0.
\]
(3.1)
For \(s < 0\), \(\tilde{g}\) is defined by \(\tilde{g}(s) = -g(-s)\). As we will see in Proposition 3.1, any weak solution \(u \in H^1(\mathbb{R}^N)\) of (1.1) with \(\tilde{g}\) in the place of \(g\) satisfies \(|u| \leq s_0\). Thus hereafter we may replace \(g\) by \(\tilde{g}\), so that \(g\) fulfills (g1), (g2), (g4) and a stronger condition
\[
\lim_{s \to \pm \infty} \frac{|g(s)|}{|s|^{2^* - 1}} = 0.
\]
(3.2)
Next we put
\[
g_1(s) := \begin{cases} (g(s) + ms)^+ & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases}
\]
Moreover from (g2), (3.2) and (3.7), for suitable $g_2(s)$, one has
\[
\lim_{s \to 0} \frac{g_1(s)}{s} = 0, \tag{3.3}
\]
and
\[
\lim_{s \to \pm \infty} \frac{g_1(s)}{|s|^{2^* - 1}} = 0 \tag{3.4}
\]
and
\[
g_2(s) \geq ms \quad \text{for } s \geq 0. \tag{3.5}
\]

We set $G_i(s) = \int_0^s g_i(\tau) \, d\tau, \ i = 1, 2$. Observing that $G_1(s) = 0$ for $s < 0$, we also deduce that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[
0 \leq g_1(s) \leq \varepsilon |s| + C_\varepsilon |s|^{2^* - 1} \quad \text{for } s \in \mathbb{R}, \tag{3.6}
\]
and
\[
0 \leq G_1(s) \leq \frac{\varepsilon}{2} s^2 + \frac{C_\varepsilon}{2^*} |s|^{2^*} \quad \text{for } s \in \mathbb{R}. \tag{3.7}
\]
Moreover from (g2), (3.2) and (3.7), for suitable $c_1, c_2 > 0$, it holds that
\[
\frac{ms^2}{2} \leq G_2(s) = G_2(|s|) \leq G_1(|s|) + |G(|s|)| \leq c_1 |s|^2 + c_2 |s|^{2^*} \quad \text{for } s \in \mathbb{R}. \tag{3.8}
\]

Under these preparations, we define the functional
\[
I(u) := \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \, dx + \int_{\mathbb{R}^N} G_2(u) \, dx - \int_{\mathbb{R}^N} G_1(u) \, dx.
\]
Then from (\phi1), (3.7) and (3.8), $I$ is well-defined and $C^1$ on $H^1(\mathbb{R}^N)$.

**Proposition 3.1.** Assume (g1)–(g3) and (\phi1). If $u \in H^1(\mathbb{R}^N)$ is a critical point of $I$, then $u$ is a weak solution of (1.1).

**Proof.** It suffices to show that $0 \leq u \leq s_0$. Letting $u^- := \min\{0, u\}$, we have $u^- \in H^1(\mathbb{R}^N)$ and hence $I'(u)[u^-] = 0$. We denote $\Omega := \{x \in \mathbb{R}^N; \ u(x) < 0\}$. Then from (\phi1), (3.5) and the fact $g_1(s) = 0$ for $s < 0$ from (3.1), one finds that
\[
\phi_0 ||\nabla u||_{L^2(\Omega)}^2 + (\phi_0 + m)||u||_{L^2(\Omega)}^2 \\
\leq \int_{\Omega} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \left( u^2 + |\nabla u|^2 \right) \, dx + \int_{\Omega} g_2(u) u \, dx \\
= \int_{\Omega} g_1(u) u \, dx = 0,
\]
by which, since $\phi_0 + m > 0$, we deduce that $u^- = 0$ on $\mathbb{R}^N$.

Next we put $\Omega_0 := \{x \in \mathbb{R}^N; \ u(x) > s_0\}$ and assume that $\Omega_0 \neq \emptyset$. Since $g(u) = 0$ on $\Omega_0$, it follows that
\[
\phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) u - g(u) \geq 0 \quad \text{on } \Omega_0.
\]
This implies that \( u \in H^1(\mathbb{R}^N) \) is a weak solution of the differential inequality:
\[
\text{div} A(u, \nabla u) \geq 0 \quad \text{in } \Omega_0,
\]
where \( A(u, p) = \phi \left( \frac{u^2 + |p|^2}{2} \right) p \) for \((u, p) \in \mathbb{R} \times \mathbb{R}^N\). From (\(\phi 1\)), it follows that
\[
A(u, p) \cdot p = \phi \left( \frac{u^2 + |p|^2}{2} \right) |p|^2 \geq \phi_0 |p|^2,
\]
and hence we can apply the weak maximum principle [28, Theorem 3.2.1] to conclude that
\[
\max_{\Omega_0} u(x) = \max_{\partial \Omega_0} u(x).
\]
This is a contradiction to the definition of \( \Omega_0 \). Thus it holds that \( \Omega_0 = \emptyset \) and \( u(x) \leq s_0 \) a.e. in \( \mathbb{R}^N \). This completes the proof. \( \square \)

Hereafter, we work on the function space:
\[
H_+^1(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) \mid u \text{ is radial} \}
\]
and set \( ||u||^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \). Next we establish the mountain pass geometry.

**Lemma 3.2.** Assume (g1)–(g4) and (\(\phi 1\)). Then the functional \( I : H_+^1(\mathbb{R}^N) \to \mathbb{R} \) has the mountain pass geometry, i.e.

(i) there exist \( \alpha, \rho > 0 \) such that \( I(u) \geq \alpha \) for \( ||u|| = \rho \);

(ii) there exists \( z \in H_+^1(\mathbb{R}^N) \) with \( ||z|| > \rho \) such that \( I(z) < 0 \).

**Proof.** (i) From (\(\phi 1\)), (3.7) and (3.8), taking \( \varepsilon \in (0, \phi_0 + m) \), we have
\[
I(u) \geq \frac{\phi_0}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx + \int_{\mathbb{R}^N} G_2(u) \, dx - \int_{\mathbb{R}^N} G_1(u) \, dx
\]
\[
\geq \frac{\phi_0}{2} ||\nabla u||^2_{L^2} + \frac{\phi_0 + m - \varepsilon}{2} ||u||^2_{L^2} - \frac{C_2}{2} \int_{\mathbb{R}^N} |u|^{2^*_N} \, dx \text{ for } u \in H_+^1(\mathbb{R}^N).
\]

Then, by the Sobolev inequality, there exist \( \alpha, \rho > 0 \) such that \( I(u) \geq \alpha \) for \( ||u|| = \rho \).

(ii) Following [4], for any \( R > 0 \), we set \( w_R \in H_+^1(\mathbb{R}^N) \) so that
\[
w_R(x) = \begin{cases} 
\zeta & \text{if } |x| \leq R, \\
\zeta(R + 1 - |x|) & \text{if } R \leq |x| \leq R + 1, \\
0 & \text{if } |x| \geq R + 1,
\end{cases}
\]
where \( \zeta \) is defined in (g4). Using the monotonicity of the map \( t \mapsto \Phi(t^2) \) on \([0, +\infty)\), we have
\[
I(w_R) = \int_{\mathbb{R}^N} \Phi \left( \frac{w_R^2 + |\nabla w_R|^2}{2} \right) \, dx + \int_{\mathbb{R}^N} G_2(w_R) \, dx - \int_{\mathbb{R}^N} G_1(w_R) \, dx
\]
\[
= \int_{B_R} \Phi \left( \frac{\zeta^2}{2} \right) \, dx + \int_{B_{R+1} \setminus B_R} \Phi \left( \frac{w_R^2 + |\nabla w_R|^2}{2} \right) \, dx
\]
\[
+ \int_{B_R} G_2(\zeta) \, dx + \int_{B_{R+1} \setminus B_R} G_2(w_R) \, dx
\]
\[
- \int_{B_R} G_1(\zeta) \, dx - \int_{B_{R+1}\setminus B_R} G_1(w_R) \, dx \\
\leq \int_{B_R} \left\{ \Phi\left(\frac{\zeta^2}{2}\right) + G_2(\zeta) - G_1(\zeta) \right\} \, dx \\
+ \int_{B_{R+1}\setminus B_R} \left\{ \Phi(\zeta^2) + G_2(w_R) - G_1(w_R) \right\} \, dx \\
\leq C \left\{ \Phi\left(\frac{\zeta^2}{2}\right) - G(\zeta) \right\} R^N + C \max_{s \in [0, \zeta]} |\Phi(\zeta^2) - G(s)| R^{N-1}.
\]

By (g4), we can find sufficiently large \( R > 0 \) so that \( I(w_R) < 0 \) and \( \|w_R\| > \rho \). Putting \( z = w_R \), we finish the proof. \( \square \)

By Lemma 3.2, denoting \( \Gamma := \{ \gamma \in C([0, 1], H^1_r(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0 \} \), we infer that \( \Gamma \) is non-empty and
\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) \geq \alpha > 0.
\]

Now, following [16,17], we define the functional \( J : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R} \) as
\[
J(\theta, u) = I(u(e^{-\theta} \cdot)) = e^{N\theta} \int_{\mathbb{R}^N} \Phi\left(\frac{u^2 + e^{-2\theta} |\nabla u|^2}{2}\right) \, dx + e^{N\theta} \int_{\mathbb{R}^N} G_2(u) \, dx - e^{N\theta} \int_{\mathbb{R}^N} G_1(u) \, dx.
\]
With similar arguments of Lemma 3.2, \( J \) also has the mountain pass geometry and we can define its mountain pass level as
\[
\tilde{c} := \inf_{(\theta, \gamma) \in \Sigma \times \Gamma} \max_{t \in [0, 1]} J(\theta(t), \gamma(t)),
\]
where
\[
\Sigma := \{ \theta \in C([0, 1], \mathbb{R}) : \theta(0) = \theta(1) = 0 \}.
\]
Arguing as in [16, Lemma 4.1], we derive the following.

**Lemma 3.3.** The mountain pass levels of \( I \) and \( J \) coincide, namely \( c = \tilde{c} \).

Now, as a immediate consequence of Ekeland’s variational principle, we have the result below, whose proof follows as in [17, Lemma 2.3].

**Lemma 3.4.** Let \( \varepsilon > 0 \). Suppose that \( \eta \in \Sigma \times \Gamma \) satisfies
\[
\max_{t \in [0, 1]} J(\eta(t)) \leq \tilde{c} + \varepsilon.
\]
Then there exists \( (\theta, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N) \) such that
(i) \( \text{dist}_{\mathbb{R} \times H^1_r(\mathbb{R}^N)}((\theta, u), \eta([0, 1])) \leq 2\sqrt{\varepsilon} \);
(ii) \( J(\theta, u) \in [c - \varepsilon, c + \varepsilon] \);
(iii) \( \|DJ(\theta, u)\|_{\mathbb{R} \times H^1_r(\mathbb{R}^N)'} \leq 2\sqrt{\varepsilon} \).

Arguing as in [16, Proposition 4.2], by Lemmas 3.3 and 3.4, the following proposition holds.
Proposition 3.5. There exists a sequence \( \{(\theta_n, u_n)\} \subset \mathbb{R} \times H^1_r(\mathbb{R}^N) \) such that, as \( n \to +\infty \),

(i) \( \theta_n \to 0 \);
(ii) \( J(\theta_n, u_n) \to c \);
(iii) \( \partial_\theta J(\theta_n, u_n) \to 0 \);
(iv) \( \partial_u J(\theta_n, u_n) \to 0 \) strongly in \( (H^1_r(\mathbb{R}^N))' \).

Next we recall the Strauss compactness lemma. (See [4, Theorem A.1], [30].) It will be a fundamental tool in our arguments.

Lemma 3.6. Let \( P \) and \( Q : \mathbb{R} \to \mathbb{R} \) be two continuous functions satisfying

\[
\lim_{|s| \to +\infty} \frac{P(s)}{Q(s)} = 0.
\]

Suppose that \( \{v_n\} \) and \( v \) are measurable functions from \( \mathbb{R}^N \) to \( \mathbb{R} \) such that

\[
\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(v_n(x))| \, dx < +\infty \quad \text{and} \quad P(v_n(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.
\]

Then \( \| (P(v_n) - v) \|_{L^1(B)} \to 0 \) for any bounded Borel set \( B \).

Moreover, if we have also

\[
\lim_{s \to 0} \frac{P(s)}{Q(s)} = 0 \quad \text{and} \quad \lim_{x \to \infty} \sup_{n \in \mathbb{N}} |v_n(x)| = 0,
\]

then \( \| (P(v_n) - v) \|_{L^1(\mathbb{R}^N)} \to 0 \).

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First by Proposition 3.5, there exists a sequence \( \{(\theta_n, u_n)\} \subset \mathbb{R} \times H^1_r(\mathbb{R}^N) \) such that

\[
e^{N\theta_n} \int_{\mathbb{R}^N} \Phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) \, dx + e^{N\theta_n} \int_{\mathbb{R}^N} G_2(u_n) \, dx - e^{N\theta_n} \int_{\mathbb{R}^N} G_1(u_n) \, dx = c + o_n(1),
\tag{3.9}
\]

\[
N e^{N\theta_n} \int_{\mathbb{R}^N} \Phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) \, dx - e^{(N-2)\theta_n} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 \, dx
+ N e^{N\theta_n} \int_{\mathbb{R}^N} G_2(u_n) \, dx - N e^{N\theta_n} \int_{\mathbb{R}^N} G_1(u_n) \, dx = o_n(1),
\tag{3.10}
\]

and, for all \( \varphi \in H^1_r(\mathbb{R}^N) \),

\[
e^{N\theta_n} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) (u_n \varphi + e^{-\theta_n} \nabla u_n \cdot \nabla \varphi) \, dx + e^{N\theta_n} \int_{\mathbb{R}^N} g_2(u_n) \varphi \, dx - e^{N\theta_n} \int_{\mathbb{R}^N} g_1(u_n) \varphi \, dx = o_n(1) \| \varphi \|.
\tag{3.11}
\]
By (3.9) and (3.10), we have
\[
e^{(N-2)\theta_n} \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 \, dx = c + o_n(1). \tag{3.12}
\]

From (φ1) and since \( \theta_n \to 0 \) as \( n \to +\infty \), \( \{u_n\} \) is bounded \( D^{1,2}(\mathbb{R}^N) \), namely
\[
|\nabla u_n|_{L^2} \leq C \quad \text{for some } C > 0 \text{ and for all } n \geq 1.
\]
Moreover using (3.7), (3.8), (3.10) and (3.11), one finds that
\[
\phi_0 + \frac{m}{2} \|u_n\|_{L^2}^2 \leq \int_{\mathbb{R}^N} \Phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) \, dx + \int_{\mathbb{R}^N} G_2(u_n) \, dx
\]
\[
= \frac{e^{-2\theta_n}}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n} |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 \, dx
\]
\[
+ \int_{\mathbb{R}^N} G_1(u_n) \, dx + o_n(1)
\]
\[
\leq \frac{2\phi_1}{N} \|\nabla u_n\|_{L^2}^2 + \epsilon \|u_n\|_{L^2}^2 + C_\epsilon \|u_n\|_{L^2}^2 + o_n(1).
\]
Choosing \( \epsilon > 0 \) sufficiently small so that \( \phi_0 + m - \epsilon > 0 \) and taking in account the embedding of \( D^{1,2}(\mathbb{R}^N) \) into \( L^{2^*}(\mathbb{R}^N) \), we find that \( \{u_n\} \) is bounded also in \( L^2(\mathbb{R}^N) \) and hence
\[
\|u_n\| \leq C \quad \text{for some } C > 0 \text{ and for all } n \geq 1. \tag{3.13}
\]
This implies the existence of \( u \in H^1_+(\mathbb{R}^N) \) such that
\[
u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N) \tag{3.14}
\]
and
\[
u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \tag{3.15}
\]
Moreover by the radial lemma (see [4,30]), it follows that
\[
v_n(x) \leq C |x|^{-\frac{N-2}{2}} |\nabla u_n|_{L^2} \quad \text{for some } C > 0 \text{ independent of } n. \tag{3.16}
\]

Now from (3.3), (3.4), (3.13), (3.15) and (3.16), we are able to apply Lemma 3.6 provided that \( P(s) = g_1(s)s, Q(s) = s^2 + |s|^{2^*}, \{v_n\} = \{u_n\}, \) and \( v = g_1(u)u \). Then we deduce that
\[
\int_{\mathbb{R}^N} g_1(u_n) u_n \, dx \to \int_{\mathbb{R}^N} g_1(u) u \, dx \quad \text{as } n \to +\infty.
\]
Moreover from (3.6), one finds that \( g_1(u) \in (H^1_+(\mathbb{R}^N))' \) and hence
\[
\int_{\mathbb{R}^N} g_1(u)(u_n - u) \, dx \to 0 \quad \text{as } n \to +\infty.
\]
Thus we obtain
\[
\int_{\mathbb{R}^N} g_1(u_n)(u_n - u) \, dx \tag{3.17}
\]
\[
= \int_{\mathbb{R}^N} \{g_1(u_n)u_n - g_1(u)u\} \, dx + \int_{\mathbb{R}^N} g_1(u)(u_n - u) \, dx \to 0 \quad \text{as } n \to +\infty.
\]
Next we put
\[ h(s) := g_2(s) - ms \quad \text{for } s \geq 0 \]
and extend it as an odd function for \( s < 0 \). By (3.5), we can observe that \( h(s)/s \geq 0 \) for any \( s \neq 0 \). Thus from (g2) and (3.3), one has
\[
0 \leq \liminf_{s \to 0} \frac{h(s)}{s} \leq \limsup_{s \to 0} \frac{h(s)}{s} = \limsup_{s \to 0} \frac{g_2(s) - ms}{s} = 0
\]
and hence
\[
\lim_{s \to 0} \frac{h(s)}{s} = 0. \tag{3.18}
\]
Furthermore from (3.2) and (3.4), we also have
\[
\lim_{s \to +\infty} \frac{h(s)}{|s|^{2^* - 1}} = \lim_{s \to +\infty} \frac{g_1(s) - g(s) - ms}{|s|^{2^* - 1}} = 0, \tag{3.19}
\]
\[
\lim_{s \to -\infty} \frac{h(s)}{|s|^{2^* - 1}} = \lim_{s \to -\infty} \frac{-g_1(-s) - g(s) - ms}{|s|^{2^* - 1}} = 0. \tag{3.20}
\]
By (3.18), (3.19) and (3.20), we can apply once again Lemma 3.6 to conclude that
\[
\int_{\mathbb{R}^N} h(u_n)u_n \, dx \to \int_{\mathbb{R}^N} h(u)u \, dx \quad \text{as } n \to +\infty.
\]
Repeating previous arguments, it holds that
\[
\int_{\mathbb{R}^N} h(u_n)(u_n - u) \, dx \to 0 \quad \text{as } n \to +\infty. \tag{3.21}
\]
Now by (3.11), (3.17) and (3.21), one finds that
\[
\partial_u J(\theta_n, u_n)[u_n - u]
\]
\[
= \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n}|
abla u_n|^2}{2} \right) \left\{ u_n(u_n - u) + e^{-2\theta_n}\nabla u_n \cdot \nabla (u_n - u) \right\} \, dx
\]
\[
+ m \int_{\mathbb{R}^N} u_n(u_n - u) \, dx + \int_{\mathbb{R}^N} h(u_n)(u_n - u) \, dx - \int_{\mathbb{R}^N} g_1(u_n)(u_n - u) \, dx
\]
\[
= \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n}|
abla u_n|^2}{2} \right) \left\{ u_n(u_n - u) + e^{-2\theta_n}\nabla u_n \cdot \nabla (u_n - u) \right\} \, dx
\]
\[
+ m \int_{\mathbb{R}^N} u_n(u_n - u) \, dx + o_n(1).
\]
Thus from (3.13) and since \( \partial_u J(\theta_n, u_n) \to 0 \) in \( (H^1_r(\mathbb{R}^N))' \), we deduce that
\[
e^{N\theta_n} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + e^{-2\theta_n}|
abla u_n|^2}{2} \right) \left\{ u_n(u_n - u) + e^{-2\theta_n}\nabla u_n \cdot \nabla (u_n - u) \right\} \, dx
\]
\[
+ me^{N\theta_n} \int_{\mathbb{R}^N} u_n(u_n - u) \, dx = o_n(1).
\]
Hence denoting $v_n = u_n(e^{-\theta_n})$ and $z_n = u(e^{-\theta_n})$, we obtain

$$
\Psi'_m(v_n)[v_n - z_n] \to 0 \quad \text{as } n \to +\infty.
$$

Since $\theta_n \to 0$ as $n \to +\infty$, one has $z_n \to u$ strongly in $H^1(\mathbb{R}^N)$ because $z_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $\|z_n\| \to \|u\|$. Thus by the boundedness of \{u_n\} (and so also of \{v_n\}), we get

$$
\Psi'_m(v_n)[v_n - u] \to 0 \quad \text{as } n \to +\infty. \quad (3.22)
$$

Then from (3.14) and (3.22), one can apply Lemma 2.2 to obtain

$$
\nabla v_n(x) \to \nabla u(x) \quad \text{as } n \to +\infty \text{ a.e. in } \mathbb{R}^N. \quad (3.23)
$$

Moreover arguing as in [19, Lemma 3.2] or [32, Proposition 3.1], one has from (\phi 2) that

$$
\Psi_m(u) - \Psi_m(v_n) - \Psi'_m(v_n)[u - v_n] \\
\geq \int_{\mathbb{R}^N} \phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) \left\{ (u^2 + |\nabla u|^2)^{\frac{1}{2}} (v_n^2 + |\nabla v_n|^2)^{\frac{1}{2}} - (uv_n + \nabla u \cdot \nabla v_n) \right\} \, dx \\
+ \frac{m}{2} \int_{\mathbb{R}^N} (u - v_n)^2 \, dx \geq 0.
$$

Thus from (3.22), we infer that

$$
\limsup_{n \to +\infty} \Psi_m(v_n) \leq \Psi_m(u). \quad (3.24)
$$

From (3.14), (3.23) and (3.24), we can apply Lemma 2.3 to conclude that $v_n \to u$ strongly in $H^1_r(\mathbb{R}^N)$. Hence, using again the fact that $\theta_n \to 0$ as $n \to +\infty$, we also have $u_n \to u$ strongly in $H^1_r(\mathbb{R}^N)$. Using (3.11) again, one finds that $I'(u) = 0$ and hence $u$ a solution of (1.1) by Proposition 3.1. Finally by (3.12), passing to the limit, we can see that

$$
\frac{1}{N} \int_{\mathbb{R}^N} \phi\left(\frac{u^2 + |\nabla u|^2}{2}\right)|\nabla u|^2 \, dx = c > 0.
$$

Then from (\phi 1), it follows that $u$ is non-trivial. 

\[ \square \]

### 4. Regularity and positivity of solutions for (1.1)

In this section, we show that any solution of (1.1) has $C^{1,\sigma}$-regularity and any non-negative nontrivial solution of (1.1) is indeed positive everywhere under (\phi 1) and the slightly stronger assumption (\phi 3).

For this purpose, we first observe that (1.1) can be written by the form:

$$
div A(u, \nabla u) + B(u, \nabla u) = 0, \quad (4.1)
$$

where for $(u, p) \in \mathbb{R} \times \mathbb{R}^N$,

$$
A(u, p) = \phi\left(\frac{u^2 + |p|^2}{2}\right)p, \quad A_i(u, p) = \phi\left(\frac{u^2 + |p|^2}{2}\right)p_i, \quad i = 1, \ldots, N, \\
B(u, p) = -\phi\left(\frac{u^2 + |p|^2}{2}\right)u + g(u).
$$
Then from (ϕ1), one finds that
\[
A(u, p) \cdot p = \phi\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 \geq \phi_0|p|^2, \tag{4.2}
\]
\[
|A(u, p)| \leq \phi\left(\frac{u^2 + |p|^2}{2}\right)|p| \leq \phi_1|p|,
\]
\[- \phi_1|u| + g(u) \leq B(u, p) \leq \phi_1|u| + g(u). \tag{4.3}
\]
Especially for $|u| \leq M$ and $p \in \mathbb{R}^N$, it follows from (g2) that
\[
|B(u, p)| \leq K|u| \quad \text{for some } K > 0 \text{ depending on } M. \tag{4.4}
\]
Moreover direct calculations yield that for $i, j = 1, \ldots, N$,
\[
\frac{\partial A_i}{\partial p_j}(u, p) = \phi'\left(\frac{u^2 + |p|^2}{2}\right)p_i p_j + \phi\left(\frac{u^2 + |p|^2}{2}\right)\delta_{ij},
\]
\[
\frac{\partial A_i}{\partial u}(u, p) = \phi'\left(\frac{u^2 + |p|^2}{2}\right)u_i.
\]
From (ϕ3), for $|u| \leq M$, one can see that
\[
\sum_{i,j=1}^{N} \frac{\partial A_i}{\partial p_j} \xi_i \xi_j = \phi'\left(\frac{u^2 + |p|^2}{2}\right)(p \cdot \xi)^2 + \phi\left(\frac{u^2 + |p|^2}{2}\right)|\xi|^2
\]
\[
\leq \left\{\phi'\left(\frac{u^2 + |p|^2}{2}\right)\right\} |p|^2 + \phi\left(\frac{u^2 + |p|^2}{2}\right)|\xi|^2
\]
\[
\leq \left\{2\left(\frac{u^2 + |p|^2}{2}\right)\phi'\left(\frac{u^2 + |p|^2}{2}\right)\right\} + \phi_1 \right\} |\xi|^2
\]
\[
\leq C|\xi|^2, \tag{4.5}
\]
and, for some $K > 0$ depending on $M$,
\[
\sum_{i=1}^{N} \left(\frac{|\partial A_i|}{\partial u} + |A_i|\right)(1 + |p|) + |B|
\]
\[
\leq N \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|(1 + |p|) + N\phi\left(\frac{u^2 + |p|^2}{2}\right)|p|(1 + |p|)
\]
\[
+ \phi\left(\frac{u^2 + |p|^2}{2}\right)|u| + g(u)
\]
\[
\leq N\left(\frac{u^2 + |p|^2}{2}\right)\phi'\left(\frac{u^2 + |p|^2}{2}\right)(1 + |p|) + N\phi_1(1 + |p|)^2 + \phi_1 |u| + g(u)
\]
\[
\leq K(1 + |p|)^2. \tag{4.6}
\]
Moreover if $(u, p) \in \mathbb{R} \times \mathbb{R}^N$ satisfies $\phi'\left(\frac{u^2 + |p|^2}{2}\right) < 0$, we have from (ϕ3) that
\[
\sum_{i,j=1}^{N} \frac{\partial A_i}{\partial p_j} \xi_i \xi_j = \phi'\left(\frac{u^2 + |p|^2}{2}\right)(p \cdot \xi)^2 + \phi\left(\frac{u^2 + |p|^2}{2}\right)|\xi|^2
\]
\[
\geq \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|^2|\xi|^2 + \phi\left(\frac{u^2 + |p|^2}{2}\right)|\xi|^2.
\[ \begin{align*}
\{ 2 \left( \frac{u^2 + |p|^2}{2} \right) \phi' \left( \frac{u^2 + |p|^2}{2} \right) + \phi \left( \frac{u^2 + |p|^2}{2} \right) \} |\xi|^2 \\
\geq \phi_0 |\xi|^2.
\end{align*} \]

In the case \( \phi' \left( \frac{u^2 + |p|^2}{2} \right) \geq 0 \), one easily finds that
\[ \sum_{i,j=1}^N \frac{\partial A_i}{\partial p_j} \xi_i \xi_j \geq \phi_0 |\xi|^2, \]
yielding that
\[ \sum_{i,j=1}^N \frac{\partial A_i}{\partial p_j} \xi_i \xi_j \geq \phi_0 |\xi|^2 \quad \text{for all } (u, p, \xi) \in \mathbb{R} \times \mathbb{R}_N \times \mathbb{R}_N. \quad (4.7) \]

Under these preparations, we establish the following regularity result.

**Proposition 4.1.** Assume (\( \phi_1 \)), (\( \phi_3 \)) and (g1)–(g3). Then any weak solution \( u \in H^1(\mathbb{R}^N) \) of (1.1) belongs to the class \( C^{1,\sigma}(\mathbb{R}^N) \) for some \( \sigma \in (0,1) \).

**Proof.** The proof consists of three steps.

**Step 1:** \( u \in L^q(\mathbb{R}^N) \) for any \( q > \frac{2N}{N-2} \).

This kind of property can be obtained by applying the Brezis–Kato lemma as in [4, p. 329]. However since our problem (1.1) is quasilinear, we give the proof for the sake of completeness.

For \( L > 0 \) and \( s \geq 0 \), we define
\[ \varphi := u \min \{|u|^{2s}, L^2\} \in H^1(\mathbb{R}^N). \]

Then multiplying (1.1) by \( \varphi \), one has
\[ \begin{align*}
\int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 \min \{|u|^{2s}, L^2\} \, dx \\
+ \frac{s}{2} \int_{\{x: |u(x)| \leq L\}} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 |u|^{2s-2} \, dx \\
+ \int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) u \varphi \, dx = \int_{\mathbb{R}^N} g(u) \varphi \, dx.
\end{align*} \]

(4.8)

By (g1)–(g3), for any \( \varepsilon \in (0, m + \phi_0) \), there exists \( C_\varepsilon > 0 \) such that
\[ g(s) \leq -(m - \varepsilon) s + C_\varepsilon s^{\frac{N+2}{N-2}} \quad \text{for } s \geq 0. \]

Since \( u \) and \( \varphi \) have the same sign, it holds that
\[ g(u) \varphi \leq -(m - \varepsilon) u \varphi + C_\varepsilon |u|^{\frac{4}{N-2}} u \varphi. \]

Then from (4.8) and (\( \phi_1 \)), we get
\[ \phi_0 \int_{\mathbb{R}^N} |\nabla \left( u \min\{|u|^{s}, L\} \right)|^2 \, dx \leq C \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} u \varphi \, dx. \]

(4.9)

For any \( K > 0 \), by the Hölder and the Sobolev inequalities, one has
\[ \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} u \varphi \, dx \]
Thus by Step 1, we obtain

\[
\begin{align*}
&= \int_{\{x:|u(x)|\geq K\}} |u|^{\frac{N+2}{N-2}} u \varphi \, dx + \int_{\{x:|u(x)|\leq K\}} |u|^{\frac{N+2}{N-2}} u \varphi \, dx \\
&\leq C \left( \int_{\{x:|u(x)|\geq K\}} |u|^{\frac{2N}{N-2}} \, dx \right)^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |u| \min\{|u|^{s}, L\}^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \\
&\quad + K^{\frac{N-2}{N}} \int_{\mathbb{R}^N} |u|^{2s+2} \, dx
\end{align*}
\]

Since \( u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \), it follows that

\[
\int_{\{x:|u(x)|\geq K\}} |u|^{\frac{2N}{N-2}} \, dx \to 0 \quad \text{as} \quad K \to +\infty.
\]

Thus from (4.9), choosing sufficiently large \( K \), we find that

\[
\int_{\mathbb{R}^N} |\nabla (u \min\{|u|^{s}, L\})|^{2} \, dx \leq C\|u\|^{2s+2}_{L^{2s+2}(\mathbb{R}^N)}.
\]

Letting \( L \to +\infty \), we conclude that

\[
\|\nabla (|u|^{s+1})\|_{L^2(\mathbb{R}^N)} \leq C\|u\|^{s+1}_{L^{2s+2}(\mathbb{R}^N)}. \tag{4.10}
\]

Now putting \( s_0 = 0 \) and \( s_i + 1 = (s_{i-1} + 1) \frac{N}{N-2} \) for \( i \geq 1 \), one deduces from (4.10) that

\[
\begin{align*}
\quad u \in L^{2}_{s_0=0} \Rightarrow \nabla |u| \in L^{2} \Rightarrow u \in L^{\frac{2N}{N-2}}_{s_1=\frac{N+2}{N-2}} \Rightarrow \nabla |u|^{\frac{N}{N-2}} \in L^{2} \\
&\quad \Rightarrow u \in L^{2}(\nabla^{\frac{N}{N-2}})^{2}_{s_2=(\frac{N}{N-2})^{2}-1} \Rightarrow \nabla |u|^{\frac{N}{N-2}} \in L^{2} \ldots
\end{align*}
\]

and hence \( u \in L^{q}(\mathbb{R}^N) \) for any \( q > \frac{2N}{N-2} \), as desired.

**Step 2:** \( u \in L^{\infty}_{\text{loc}}(\mathbb{R}^N) \).

By (\( \phi1 \)) and (\( g1 \))–(\( g3 \)), one has

\[
B(u, \mathbf{p}) \text{ sign } u = -\phi \left( \frac{u^2 + |\mathbf{p}|^2}{2} \right) u \text{ sign } u + g(u) \text{ sign } u
\]

\[
\leq -\left( \phi_0 + m - \varepsilon \right) |u| + C_\varepsilon |u|^{\frac{N+2}{N-2}}
\]

\[
\leq C_\varepsilon |u|^{\frac{4}{N-2}} |u|.
\]

Thus by Step 1, we obtain

\[
B(u, \mathbf{p}) \text{ sign } u \leq a(x) |u|, \quad a(x) = C_\varepsilon |u|^{\frac{4}{N-2}} \in L^{r}(\mathbb{R}^N) \text{ for any } r > \frac{N}{2}.
\tag{4.11}
\]

Then the claim follows from (4.2), (4.11) and by the Moser type iteration. (See [12] or [23, Theorem 7.1, p. 286].)
Step 3: $u \in C^{1,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$.

Once we get the $L^\infty$-boundedness, together with (4.4), (4.5), (4.6) and (4.7), we are able to apply the regularity result for quasilinear elliptic problems of the divergence form (4.1) due to [23, Chapter 4, Theorems 3.1, 5.2 and 6.2], [35] to conclude that $u \in C^{1,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$.

From (4.2), (4.3) and (4.4), we can also apply the strong maximum principle [28, Theorem 2.5.1] or the Harnack inequality [36]. Then we obtain the following positivity result.

Proposition 4.2. Assume $(\phi_1)$, $(\phi_3)$ and $(g_1)$–$(g_3)$. Then any non-negative non-trivial (regular) solution of (1.1) is positive on $\mathbb{R}^N$.

5. Existence of a positive solution and a radial ground state solution

In this section, we prove the existence of a positive solution and a radial ground state solution of (1.1).

Proof of Theorem 1.2. Under $(\phi_1)$ and $(\phi_3)$, we know that any non-negative non-trivial of (1.1) is of the class $C^{1,\sigma}$ and positive on $\mathbb{R}^N$ by Propositions 4.1 and 4.2. Thus the claim follows from Theorem 1.1.

In the next lemma, we show that each solution of (1.1) satisfies a Pohozaev type identity.

Lemma 5.1. Assume that $(\phi_1)$, $(\phi_3)$ and $(g_1)$–$(g_3)$. Then if $u \in H^1(\mathbb{R}^N)$ is a solution of (1.1), it satisfies the following type Pohozaev identity:

$$\int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) \right\} dx = 0.$$  

(5.1)

Proof. We argue as in [25]. By Proposition 4.1, we know that $u \in C^{1,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$. Then, since the function

$$\mathcal{L}(s, p) = \Phi \left( \frac{s^2 + |p|^2}{2} \right)$$

associated with the differential operator in (1.1) is strictly convex for all $s \in \mathbb{R}$, we can apply the Pohozaev identity due to [11] by choosing $h(x) = h_k(x) = H(x/k)x \in C^1_0(B_{2k}(0), \mathbb{R}^N)$ for $k \in \mathbb{N}$, where $H \in C^1_0(\mathbb{R}^N)$ is such that $H(x) = 1$ on $|x| \leq 1$ and $H(x) = 0$ for $|x| \geq 2$. Letting $k \to +\infty$ and taking into account that

$$\Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right), \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2$$

and $G(u) \in L^1(\mathbb{R}^N)$, we obtain (5.1) as claimed.
Next we show the existence of a radial ground state solution of (1.1). Let us define by $S_{\text{rad}}$ the set of the nontrivial radial solutions of (1.1), namely 

$$S_{\text{rad}} := \{ u \in H^1_r(\mathbb{R}^N) \setminus \{0\} ; \; I'(u) = 0 \}.$$ 

By Theorem 1.1, we know that $S_{\text{rad}} \neq \emptyset$.

**Lemma 5.2.** Assume that $(\phi 1)$, $(\phi 3)$ and $(g1)$–$(g4)$. Then it holds that

$$\inf_{u \in S_{\text{rad}}} \| u \| > 0.$$ 

**Proof.** If $u \in S_{\text{rad}}$, since $I'(u)[u] = 0$, we have

$$\int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \left( u^2 + |\nabla u|^2 \right) dx + \int_{\mathbb{R}^N} g_2(u) u dx - \int_{\mathbb{R}^N} g_1(u) u dx = 0.$$ 

Therefore, by (3.5), (3.6) and $(\phi 1)$, we have

$$\phi_0 \| \nabla u \|^2_{L^2} + (\phi_0 + m) \| u \|^2_{L^2} \leq \int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \left( u^2 + |\nabla u|^2 \right) dx + \int_{\mathbb{R}^N} g_2(u) u dx \leq \epsilon \| u \|^2_{L^2} + C_\varepsilon \| u \|^2_{L^2^*}.$$ 

by which, taking $\varepsilon > 0$ sufficiently so such that $\phi_0 + m - \varepsilon > 0$, the conclusion follows immediately. □

**Lemma 5.3.** Assume that $(\phi 1)$, $(\phi 3)$ and $(g1)$–$(g3)$. Then it follows that

$$m_{0,\text{rad}} := \inf_{u \in S_{\text{rad}}} I(u) > 0.$$ 

**Proof.** Suppose by contradiction that $m_{0,\text{rad}} = 0$. Then there exists $\{u_n\} \subset S_{\text{rad}}$ such that $I(u_n) \to 0$ as $n \to +\infty$. Since $u_n$ is a solution of (1.1), it follows by Lemma 5.1 that $u_n$ satisfies the following Pohozaev type identity:

$$\int_{\mathbb{R}^N} \Phi \left( \frac{u_n^2 + |\nabla u_n|^2}{2} \right) \left( u_n^2 + |\nabla u_n|^2 \right) dx - \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} G_2(u_n) dx - \int_{\mathbb{R}^N} G_1(u_n) dx = 0$$

and hence

$$I(u_n) = \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 dx \to 0 \quad \text{as } n \to +\infty.$$ 

By $(\phi 1)$, this implies that

$$\| \nabla u_n \|_{L^2} \to 0 \quad \text{as } n \to +\infty. \quad (5.2)$$

Moreover using (3.7), (3.8) (5.2), $(\phi 1)$ and the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, we have

$$\frac{\phi_0 + m}{2} \| u_n \|^2_{L^2} \leq \int_{\mathbb{R}^N} \Phi \left( \frac{u_n^2 + |\nabla u_n|^2}{2} \right) dx + \int_{\mathbb{R}^N} G_2(u_n) dx.$$
\[
\frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^2 + |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} G_1(u_n) \, dx \\
\leq o_n(1) + \frac{\varepsilon}{2} \|u_n\|_{L^2}^2 + \frac{C_\varepsilon}{2} \|u_n\|_{L^{2^*}}^{2^*} \\
\leq o_n(1) + \frac{\varepsilon}{2} \|u_n\|_{L^2}^2.
\]

Choosing sufficiently small \(\varepsilon > 0\), together with (4.2), we find that \(\|u_n\| \to 0\) reaching a contradiction with Lemma 5.2. \(\square\)

By Lemma 5.3, we are ready to prove the existence of a radial ground state solution of (1.1).

**Proof of Theorem 1.3.** Let \(\{u_n\} \subset S_{\text{rad}}\) be a minimizing sequence such that \(I'(u_n) = 0\) and \(I(u_n) \to m_{0,\text{rad}}\) as \(n \to +\infty\). Repeating the arguments of the proof of Theorem 1.1, we can prove that \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^N)\). This implies the existence of \(\bar{u} \in H^1(\mathbb{R}^N)\) such that \(u_n \rightharpoonup \bar{u}\) in \(H^1(\mathbb{R}^N)\). Arguing as in the proof of Theorem 1.1, we deduce that \(u_n \to \bar{u}\) strongly in \(H^1(\mathbb{R}^N)\), and therefore \(\bar{u}\) satisfies

\[ I(\bar{u}) = m_{0,\text{rad}} = \min_{u \in S_{\text{rad}}} I(u), \]

namely, \(\bar{u}\) is a radial ground state solution of (1.1). The regularity and the positivity of \(\bar{u}\) follow by Propositions 4.1 and 4.2. \(\square\)

Finally by applying the Pohozaev identity (5.1), we establish the following non-existence result, which indicates (g3) and (g4) are almost optimal.

**Theorem 5.4.** Assume (\(\phi_1\)), (\(\phi_3\)) and \(\Phi(t^2) \geq t^2\phi(t^2)\) on \(\mathbb{R}\). Then (1.1) has no non-trivial regular solution if one of the following conditions holds:

(i) \(g(s) = -ms + |s|^{p-1}s\), with \(m \in (-\phi_0, +\infty)\) and \(p \geq \frac{N+2}{N-2}\);

(ii) \(G(s) \leq \frac{\phi_0}{2} s^2\), for all \(s \in \mathbb{R}\).

**Proof.** (i) Let \(u\) be a solution of (1.1). From \(I'(u)[u] = 0\), it holds that

\[
\int_{\mathbb{R}^N} \left\{ \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \left( u^2 + |\nabla u|^2 \right) + mu^2 \right\} \, dx = \int_{\mathbb{R}^N} |u|^{p+1} \, dx.
\]

Combining this equation with (5.1), one finds that

\[
\int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \left( \frac{u^2 + |\nabla u|^2}{2} \right) \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \right\} dx \\
+ \frac{1}{N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) u^2 + \frac{m}{N} u^2 \right\} \, dx \\
= \left( \frac{1}{p+1} - \frac{N-2}{2N} \right) \int_{\mathbb{R}^N} |u|^{p+1} \, dx. \tag{5.3}
\]

Since \(p \geq \frac{N+2}{N-2}\), r.h.s. of (5.3) is non-positive. On the other hand, we have from (\(\phi_1\)) and by the assumption \(\Phi(t^2) \geq t^2\phi(t^2)\) that

\[
\text{l.h.s. of (5.3)} \geq \frac{\phi_0 + m}{N} \int_{\mathbb{R}^N} u^2 \, dx.
\]
Since $0 < m + \phi_0$, l.h.s. of (5.3) is positive if $u \neq 0$. This is a contradiction and hence $u \equiv 0$.

(ii) Let $u$ be a solution of (1.1). Using (5.1) again, one finds that

$$
\int_{\mathbb{R}^N} \left\{ \Phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \frac{\phi_0}{2} u^2 \right\} \, dx
$$

$$
= \int_{\mathbb{R}^N} \left( G(u) - \frac{\phi_0}{2} u^2 \right) \, dx. \tag{5.4}
$$

By the assumption, r.h.s. of (5.4) is non-positive, while $(\phi_1)$ and the fact that $\Phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) \geq u^2$ yield that

$$
\Phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \frac{\phi_0}{2} u^2
$$

$$
\geq \left( \frac{u^2 + |\nabla u|^2}{2} \right) \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \frac{\phi_0}{2} u^2
$$

$$
= \frac{N - 2}{2N} \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 + \frac{u^2}{2} \left( \phi\left( \frac{u^2 + |\nabla u|^2}{2} \right) - \phi_0 \right)
$$

$$
\geq \frac{(N - 2)\phi_0}{2N} |\nabla u|^2 \text{ a.e. in } \mathbb{R}^N,
$$

showing that l.h.s. of (5.4) is positive if $u \neq 0$. This is a contradiction again. \hfill \Box

**Remark 5.5.** From Theorem 5.4 (ii), we see that a natural assumption for the existence seems to be

$$
G(\zeta) > \frac{\phi_0}{2} \zeta^2 \text{ for some } \zeta > 0
$$

instead of ($g_4$). However for the moment, we don’t know whether the functional $I$ has the mountain pass geometry as in Lemma 3.2 under this slightly weaker assumption.

### 6. Existence of a ground state solution

In this section, we show the existence of a ground state solution of (1.1) under more stronger assumption $(\phi_4)$.

For this purpose, we begin with the following lemma.

**Lemma 6.1.** Assume $(\phi_1)$, $(\phi_3)$ and $(\phi_4)$.

(i) $J_1(s, b) := \phi\left( \frac{s^2 + b^2}{2} \right) b^2$ is increasing and convex with respect to $b$ for all $(s, b) \in \mathbb{R} \times \mathbb{R}_+$.

(ii) $J_2(s, b) := \Phi\left( \frac{s^2 + b^2}{2} \right) - \frac{1}{N} \phi\left( \frac{s^2 + b^2}{2} \right) b^2$ is increasing and convex with respect to $b$ for all $(s, b) \in \mathbb{R} \times \mathbb{R}_+$.

**Proof.** (i) A direct calculation shows that

$$
\frac{\partial J_1}{\partial b} = 2\phi\left( \frac{s^2 + b^2}{2} \right) b + \phi'\left( \frac{s^2 + b^2}{2} \right) b^3,
$$
\[
\frac{\partial^2 J_1}{\partial b^2} = 2\phi\left(\frac{s^2 + b^2}{2}\right) + 5\phi'\left(\frac{s^2 + b^2}{2}\right)b^2 + \phi''\left(\frac{s^2 + b^2}{2}\right)b^4.
\]

Since \(\phi' \leq 0\) by (1.3), we have from (\(\phi 1\)) and (\(\phi 3\)) that
\[
\frac{\partial J_1}{\partial b} \geq 2b \left\{ \phi\left(\frac{s^2 + b^2}{2}\right) + \frac{5}{2}\phi'\left(\frac{s^2 + b^2}{2}\right) \right\} \\
\geq 2b \left\{ \frac{\phi_0}{2} + \frac{1}{2}\phi\left(\frac{s^2 + b^2}{2}\right) + \frac{5}{2}\phi'\left(\frac{s^2 + b^2}{2}\right) \right\} \\
\geq 2\phi_0 b > 0.
\]

Moreover from (\(\phi 4\)), we deduce the following pointwise estimate:
\[
\frac{1}{2} \frac{\partial^2 J_1}{\partial b^2} \\
\geq \begin{cases} 
\phi\left(\frac{s^2 + b^2}{2}\right) + \frac{5}{2}\phi'\left(\frac{s^2 + b^2}{2}\right) \\
\phi\left(\frac{s^2 + b^2}{2}\right) + \frac{5}{2}\phi'\left(\frac{s^2 + b^2}{2}\right) + 2\left(\frac{s^2 + b^2}{2}\right)\phi''\left(\frac{s^2 + b^2}{2}\right) \end{cases} \\
\text{if } \phi'' \geq 0, \\
\text{if } \phi'' \leq 0.
\]

(ii) First from \(\phi' \leq 0\) by (1.3), we observe that
\[
\frac{\partial J_2}{\partial b} = \frac{N - 2}{N} \phi\left(\frac{s^2 + b^2}{2}\right)b - \frac{1}{N} \phi'\left(\frac{s^2 + b^2}{2}\right)b^3 \geq \frac{N - 2}{N} \phi_0 b > 0.
\]

Next, by a simple computation, one has
\[
\frac{\partial^2 J_2}{\partial b^2} = \frac{N - 2}{N} \phi\left(\frac{s^2 + b^2}{2}\right) + \frac{N - 5}{N} \phi'\left(\frac{s^2 + b^2}{2}\right)b^2 - \frac{1}{N} \phi''\left(\frac{s^2 + b^2}{2}\right)b^4 \\
= \frac{N - 2}{N} \left\{ \phi\left(\frac{s^2 + b^2}{2}\right) + \phi'\left(\frac{s^2 + b^2}{2}\right)b^2 \right\} \\
- \frac{b^2}{N} \left\{ 3\phi'\left(\frac{s^2 + b^2}{2}\right) + \phi''\left(\frac{s^2 + b^2}{2}\right)b^2 \right\} \\
\geq \frac{N - 2}{N} \left\{ \phi\left(\frac{s^2 + b^2}{2}\right) + 2\left(\frac{s^2 + b^2}{2}\right)\phi'\left(\frac{s^2 + b^2}{2}\right) \right\} \\
- \frac{b^2}{N} \left\{ 3\phi'\left(\frac{s^2 + b^2}{2}\right) + 2\left(\frac{s^2 + b^2}{2}\right)\phi''\left(\frac{s^2 + b^2}{2}\right) \right\}.
\]

Then by (\(\phi 3\)) and (\(\phi 4\)), it holds that \(\frac{\partial^2 J_2}{\partial b^2} \geq 0\). \(\square\)

Since \(J_2(s, b) > J_2(s, 0) = \Phi\left(\frac{s^2}{2}\right)\) for all \(b > 0\) by Lemma 6.1, we find that for any \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\),
\[
\int_{\mathbb{R}^N} \left\{ \Phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) - \frac{1}{N} \phi\left(\frac{u^2 + |\nabla u|^2}{2}\right)|\nabla u|^2 - \Phi\left(\frac{u^2}{2}\right) \right\} \, dx > 0.
\]

(6.1)

Next for \(u \in H^1(\mathbb{R}^N)\), we define
\[
P(u) := \int_{\mathbb{R}^N} \left\{ \Phi\left(\frac{u^2 + |\nabla u|^2}{2}\right) - \frac{1}{N} \phi\left(\frac{u^2 + |\nabla u|^2}{2}\right)|\nabla u|^2 - G(u) \right\} \, dx,
\]
\[
\mathcal{P} := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : P(u) = 0 \}.\]
From (φ1), (φ3) and (g1)–(g3), \( P \) is a \( C^1 \)-functional on \( H^1(\mathbb{R}^N) \). The equation \( P(u) = 0 \) is exactly the Pohozaev identity as established in Lemma 5.1. Especially by Theorem 1.2, it follows that \( P \neq 0 \). The next result shows that the set \( P \) is actually a \( C^1 \)-manifold.

**Proposition 6.2.** Assume (φ1), (φ3), (φ4) and (g1)–(g4). Then the set \( P \) is a co-dimension one manifold, bounded away from zero. Moreover \( P \) is a natural constraint for the functional \( I \).

**Proof.** The proof consists of four steps.

**Step 1:** \( P \) is bounded away from zero.

For this purpose, we put \( \varepsilon = \frac{N-2}{2N}(\phi_0 + m) \) and \( c_0 = \min \left\{ \frac{\phi_0 + m}{2}, \phi_0 \right\} \).

Then by (3.7), (3.8), (φ1) and (1.4), one has

\[
P(u) = \frac{N-2}{N} \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \, dx \\
+ \frac{2}{N} \int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \right\} \, dx \\
+ \int_{\mathbb{R}^N} G_2(u) \, dx - \int_{\mathbb{R}^N} G_1(u) \, dx \\
\geq \frac{N-2}{2N} \phi_0 \| u \|^2 \\
+ \frac{2}{N} \int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \left( \frac{u^2 + |\nabla u|^2}{2} \right) \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \right\} \, dx \\
+ \frac{m}{2} \| u \|^2 - \frac{\varepsilon}{2} \| u \|^2 - C \| u \|^{2*} \\
\geq \frac{N-2}{2N} \phi_0 \| \nabla u \|^2 + \frac{N-2}{4N} \phi_0 \, \| u \|^2 - C \| u \|^{2*} \\
\geq \frac{N-2}{2N} c_0 \| u \|^2 - C \| u \|^{2*}
\]

for some \( C > 0 \). This implies that there exists \( \delta > 0 \) such that \( P(u) > 0 \) for any \( u \in H^1(\mathbb{R}^N) \) with \( 0 < \| u \| < \delta \) and hence \( P \) is bounded away from zero.

**Step 2:** if \( P'(u) = 0 \), then \( u \) satisfies another Pohozaev type identity \( \tilde{P}(u) = 0 \), where

\[
\tilde{P}(u) := \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) \, dx - \frac{2N-2}{N^2} \int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 \, dx \\
+ \frac{1}{N^2} \int_{\mathbb{R}^N} \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 \, dx - \int_{\mathbb{R}^N} G(u) \, dx.
\tag{6.2}
\]

We note that \( \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 \in L^1(\mathbb{R}^N) \) because \( s|\phi'(s)| \leq C \) for \( s \in [0, +\infty) \) by (φ3).

By a direct calculation, \( P'(u) = 0 \) implies that \( u \) is a weak solution of the following elliptic equation of the divergence form:

\[
\text{div} \, \tilde{A}(u, \nabla u) + \tilde{B}(u, \nabla u) = 0,
\tag{6.3}
\]
where for \((u, p) \in \mathbb{R} \times \mathbb{R}^N\), \(\tilde{A} = (\tilde{A}_i)_{i=1,\ldots,N}\),
\[
\tilde{A}(u, p) = \frac{N-2}{N} \phi\left(\frac{u^2 + |p|^2}{2}\right)p - \frac{1}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 p,
\]
\[
\tilde{B}(u, p) = -\phi\left(\frac{u^2 + |p|^2}{2}\right)u + \frac{1}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 u + g(u).
\]

If we could establish that \(u \in C^1(\mathbb{R}^N)\), then we are able to apply the generalized Pohozaev identity due to [11] as in Lemma 5.1, completing the proof of Step 2. Thus it remains to show that any weak solution of (6.3) belongs to the class \(C^1(\mathbb{R}^N)\) as in Proposition 4.1.

To this aim, we investigate uniform ellipticity of the operator \(\text{div} \tilde{A} + \tilde{B}\). First from \(\phi' \leq 0\), (φ1) and (φ3), one finds that
\[
\tilde{A}(u, p) \cdot p = \frac{N-2}{N} \phi\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 - \frac{1}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|^4 \geq \frac{N-2}{N} \phi_0|p|^2,
\]
\[
|\tilde{A}(u, p)| \leq \frac{N-2}{N} \phi_1|p| + \frac{2}{N} C|p|,
\]
\[
|\tilde{B}(u, p)| \leq K|u| \quad \text{for } |u| \leq M.
\]

Next, by a direct computation, we have
\[
\sum_{i,j=1}^{N} \frac{\partial \tilde{A}_i}{\partial p_j} \xi_i \xi_j = \frac{N-4}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)(p \cdot \xi)^2 - \frac{1}{N} \phi''\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 (p \cdot \xi)^2
\]
\[
+ \frac{N-2}{N} \phi\left(\frac{u^2 + |p|^2}{2}\right)|\xi|^2 - \frac{1}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 |\xi|^2,
\]
\[
\frac{\partial \tilde{A}_i}{\partial u}(u, p) = \frac{N-2}{N} \phi'\left(\frac{u^2 + |p|^2}{2}\right)p_i u - \frac{1}{N} \phi''\left(\frac{u^2 + |p|^2}{2}\right)|p|^2 p_i u.
\]

By using the assumptions \(s|\phi'(s)| \leq C\), \(s^2|\phi''(s)| \leq C\) and (φ1), one gets
\[
\sum_{i,j=1}^{N} \frac{\partial \tilde{A}_i}{\partial p_j} \xi_i \xi_j \leq C|\xi|^2.
\]

Moreover applying the Young inequality, we also have
\[
\sum_{i=1}^{N} \left| \frac{\partial \tilde{A}_i}{\partial u}(u, p) \right|
\]
\[
\leq (N-2) \left| \phi'\left(\frac{u^2 + |p|^2}{2}\right)\right||p||u| + \left| \phi''\left(\frac{u^2 + |p|^2}{2}\right)\right||p|^3|u|
\]
\[
\leq (N-2) \left(\frac{u^2 + |p|^2}{2}\right) \left| \phi'\left(\frac{u^2 + |p|^2}{2}\right)\right| + \left| \phi''\left(\frac{u^2 + |p|^2}{2}\right)\right| \left(\frac{3}{4}|p|^4 + \frac{1}{4}|u|^4\right)
\]
\[
\leq (N-2) \left(\frac{u^2 + |p|^2}{2}\right) \left| \phi'\left(\frac{u^2 + |p|^2}{2}\right)\right| + 3 \left(\frac{u^2 + |p|^2}{2}\right)^2 \left| \phi''\left(\frac{u^2 + |p|^2}{2}\right)\right|,
\]
from which we conclude that
\[
\sum_{i=1}^{N} \left( \left| \frac{\partial \tilde{A}_i}{\partial u} \right| + |\tilde{A}_i| \right)(1 + |p|) + |\tilde{B}|
\]
\[
\leq C(1 + |p|^2) + K|u| \leq C(1 + |p|^2) \quad \text{for } |u| \leq M.
\]

Moreover by \(\phi' \leq 0\) by (1.3), (\(\phi 3\)) and (\(\phi 4\)), it follows that
\[
\sum_{i,j=1}^{N} \frac{\partial A_i}{\partial p_j} \xi_i \xi_j
\]
\[
= \frac{1}{N} \left\{ (N - 2)\phi \left( \frac{u^2 + |p|^2}{2} \right) |\xi|^2 + (N - 1)\phi' \left( \frac{u^2 + |p|^2}{2} \right) (p \cdot \xi)^2 
- \phi' \left( \frac{u^2 + |p|^2}{2} \right) |p|^2 |\xi|^2 \right\}
\[
- \frac{1}{N} \left\{ 3\phi' \left( \frac{u^2 + |p|^2}{2} \right) + \phi'' \left( \frac{u^2 + |p|^2}{2} \right) |p|^2 \right\} (p \cdot \xi)^2
\geq \frac{N - 2}{N} \left\{ \phi \left( \frac{u^2 + |p|^2}{2} \right) + 2 \left( \frac{u^2 + |p|^2}{2} \right) \phi' \left( \frac{u^2 + |p|^2}{2} \right) \right\} |\xi|^2
\[
- \frac{1}{N} \left\{ 3\phi' \left( \frac{u^2 + |p|^2}{2} \right) + 2 \left( \frac{u^2 + |p|^2}{2} \right) \phi'' \left( \frac{u^2 + |p|^2}{2} \right) \right\} (p \cdot \xi)^2
\geq \frac{N - 2}{N} \phi_0 |\xi|^2.
\]

Finally using \(\phi' \leq 0\), (\(\phi 1\)) and (g1)–(g3), we obtain
\[
\tilde{B}(u, p) \text{ sign } u \leq - (\phi_0 + m - \varepsilon)|u| + C_{\varepsilon}|u|^\frac{N+2}{\varepsilon-2} \leq C_{\varepsilon}|u|^\frac{N}{\sigma-2}|u|.
\]

Under these preparations, we are able to apply the regularity result as in Proposition 4.1 to obtain \(u \in C^{1,\sigma}(\mathbb{R}^N)\) for some \(\sigma \in (0, 1)\).

**Step 3:** \(P\) is a co-dimension one manifold.

For this purpose, we argue as in [29, Lemma 1.4] and suppose by contradiction that there exists \(u \in \mathcal{P}\) such that \(P'(u) = 0\). Then by using two Pohozaev type identities (5.1) and (6.2), we obtain
\[
\int_{\mathbb{R}^N} \left\{ (N - 2)\phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 \right\} dx = 0.
\]

Since \(\phi' \leq 0\), it follows by (\(\phi 1\)) that
\[
0 \geq \int_{\mathbb{R}^N} \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 dx - 0 = (N - 2) \int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 dx
\geq (N - 2) \phi_0 \int_{\mathbb{R}^N} |\nabla u|^2 dx,
\]
yielding that \(u \equiv 0\). This contradicts to Step 1 and hence \(P'(u) \neq 0\) for any \(u \in \mathcal{P}\).

**Step 4:** \(P\) is a natural constraint for \(I\).

Again, we follow the argument in [29, Theorem 1.6] (see also [26]). Let \(u \in \mathcal{P}\) be a critical point of the functional \(I|_{\mathcal{P}}\). By Step 3, we are able to apply the method of Lagrange multiplier to obtain the existence of \(\mu \in \mathbb{R}\) such that
\[
I'(u) = \mu P'(u).
\]
As a consequence, together with Step 2, \( u \) satisfies the following identity:

\[
P(u) = \mu \tilde{P}(u) = \mu P(u) - \frac{\mu}{N^2} \int_{\mathbb{R}^n} \left\{ (N - 2) \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 \right\} \, dx
\]

Since \( P(u) = 0 \), this yields that

\[
\mu \int_{\mathbb{R}^n} \left\{ (N - 2) \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \phi' \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^4 \right\} \, dx = 0.
\]

However as we can see by the proof of Step 3, this is possible only if \( \mu = 0 \). This completes the proof. \( \square \)

Next, let us denote \( m_0 \) by the ground state energy level:

\[
m_0 := \inf_{u \in S} I(u), \quad \text{where } S := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} ; I'(u) = 0 \}.
\]

By Theorem 1.1, it holds that \( S \neq \emptyset \). Moreover since the proofs of Lemmas 5.2 and 5.3 do not rely on the radial symmetry, one can see that \( m_0 > 0 \).

**Lemma 6.3.** Let

\[
b := \inf \{ I(u) ; u \in \mathcal{P} \}.
\]

We have that \( 0 < b \leq m_0 \). Moreover if \( b \) is attained, then it holds that \( m_0 = b \).

**Proof.** First by Proposition 6.2, we find that

\[
I(u) = \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 \, dx \geq \frac{\phi_0}{N} \|u\|_{L^2}^2 > 0,
\]

for any \( u \in \mathcal{P} \), and hence \( b > 0 \).

If \( I'(u) = 0 \) and \( u \neq 0 \), Proposition 4.1 and Lemma 5.1 yield that \( u \in \mathcal{P} \) and thus

\[
b \leq I(u) \quad \text{for any } u \in S.
\]

This implies that \( b \leq m_0 \).

On the other hand if \( b \) is attained, then there exists \( u \in \mathcal{P} \) such that it is a minimizer of \( I|_{\mathcal{P}} \). Then by Proposition 6.2, we have that \( I'(u) = 0 \) and \( u \neq 0 \), from which one concludes that \( m_0 \leq I(u) = b \). Thus we obtain \( m_0 = b \), as claimed. \( \square \)

**Lemma 6.4.** Assume \((\phi 1), (\phi 3), (\phi 4)\) and \((g1)-(g4)\). For any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) which satisfies \( \int_{\mathbb{R}^N} G(u) - \Phi(u^2) \, dx > 0 \), there exists \( \theta_0 > 0 \) such that \( u_{\theta_0}(\cdot) = u(\cdot/\theta_0) \in \mathcal{P} \). If further \( P(u) \leq 0 \), then it holds that \( 0 < \theta_0 \leq 1 \).

**Proof.** First we define a \( C^1 \)-function \( f(\theta) \) on \([0, +\infty)\) by

\[
f(\theta) := I(u_{\theta}) = \theta^N \int_{\mathbb{R}^N} \Phi \left( \frac{u^2 + \theta^{-2} |\nabla u|^2}{2} \right) \, dx - \theta^N \int_{\mathbb{R}^N} G(u) \, dx.
\]

Since \( \int_{\mathbb{R}^N} G(u) - \Phi(u^2) \, dx > 0 \), it follows that \( f(\theta) \to -\infty \) as \( \theta \to +\infty \). Moreover \((\phi 1)\) and \((g1)-(g3)\) imply that \( f(\theta) > 0 \) for sufficiently small \( \theta > 0 \). Thus there exists \( \theta_0 > 0 \) such that \( f'(\theta_0) = 0 \). Then by a direct calculation, one finds that \( P(u_{\theta_0}) = 0 \).
Next we suppose that \( P(u) \leq 0 \). Then from (6.1), we have
\[
0 < \int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 - \Phi \left( \frac{u^2}{2} \right) \right\} \, dx
\]
\[
\leq \int_{\mathbb{R}^N} G(u) - \Phi \left( \frac{u^2}{2} \right) \, dx,
\]
from which we obtain the existence of \( \theta_0 > 0 \) so that \( P(u_{\theta_0}) = 0 \). Now since \( P(u_{\theta_0}) = 0 \) and \( P(u) \leq 0 \), one finds that
\[
\int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + \theta_0^{-2} |\nabla u|^2}{2} \right) - \frac{1}{N} \phi \left( \frac{u^2 + \theta_0^{-2} |\nabla u|^2}{2} \right) \theta_0^{-2} |\nabla u|^2 \right\} \, dx = \int_{\mathbb{R}^N} G(u) \, dx, \tag{6.4}
\]
\[
\int_{\mathbb{R}^N} \left\{ \Phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) - \frac{1}{N} \phi \left( \frac{u^2 + |\nabla u|^2}{2} \right) |\nabla u|^2 \right\} \, dx \leq \int_{\mathbb{R}^N} G(u) \, dx. \tag{6.5}
\]
From (6.4) and (6.5), it follows that
\[
\int_{\mathbb{R}^N} J_2(u, \theta_0^{-1} |\nabla u|) \, dx \geq \int_{\mathbb{R}^N} J_2(u, |\nabla u|) \, dx.
\]
Then by Lemma 6.1 (ii), we conclude that \( \theta_0 \leq 1 \). \( \square \)

Under these preparations, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We argue as in [1,8]. By Lemma 6.3, it suffices to show that there exists \( u_0 \in \mathcal{P} \) such that
\[
I(u_0) = b = \min_{u \in \mathcal{P}} I(u).
\]
Let \( \{u_n\} \subset \mathcal{P} \) be a minimizing sequence for \( b \). We may assume that \( u_n \geq 0 \) because \( P(|u_n|) = P(u_n) = 0 \) and \( I(|u_n|) = I(u_n) \).

For all \( n \geq 1 \), let \( u_n^* \) be the Schwarz symmetrization of \( u_n \). Then, by Lemma 6.1, we are able to apply the generalized Polya–Szegö inequality (see [15, Proposition 3.11]) to obtain
\[
\int_{\mathbb{R}^N} J_i(u_n^*, |\nabla u_n^*|) \, dx \leq \int_{\mathbb{R}^N} J_i(u_n, |\nabla u_n|) \, dx, \quad \text{for } i = 1, 2. \tag{6.6}
\]
Applying (6.6) for \( i = 2 \), since \( \int_{\mathbb{R}^N} G(u_n^*) \, dx = \int_{\mathbb{R}^N} G(u_n) \, dx \), we find that \( P(u_n) = 0 \) implies \( P(u_n^*) \leq 0 \). Thus by Lemma 6.4, there exists \( 0 < \theta_n \leq 1 \) such that \( v_n := (u_n^*)_{\theta_n} = u_n^*(\cdot/\theta_n) \in \mathcal{P} \).

Now applying the generalized Polya–Szegö inequality (6.6) for \( i = 1 \), we find that
\[
b + o_n(1) = I(u_n) = \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{u_n^{*2} + |\nabla u_n|^2}{2} \right) |\nabla u_n|^2 \, dx
\]
\[
\geq \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{(u_n^*)^{2} + |\nabla u_n^*|^2}{2} \right) |\nabla u_n^*|^2 \, dx
\]
\[
= \frac{\theta_n^{-N+2}}{N} \int_{\mathbb{R}^N} \phi \left( \frac{v_n^{*2} + \theta_n^2 |\nabla v_n|^2}{2} \right) |\nabla v_n|^2 \, dx.
\]
Since \( v_n \in \mathcal{P} \), \( \phi' \leq 0 \) by (1.3) and \( \theta_n \leq 1 \), one obtains
\[
b + o_n(1) \geq \frac{1}{N} \int_{\mathbb{R}^N} \phi \left( \frac{v_n^2 + |\nabla v_n|^2}{2} \right) |\nabla v_n|^2 \, dx = I(v_n) \geq b,
\]
yielding that \( \{v_n\} \) is also a minimizing sequence for \( b \). By the radial symmetry of \( v_n \), we can argue as in the proof of Theorem 1.1 to prove that \( v_n \rightharpoonup u_0 \) in \( H^1(\mathbb{R}^N) \) for some \( u_0 \in \mathcal{P} \). This implies that
\[
I(u_0) = b = m_0 = \min_{u \in S} I(u), \quad \text{(6.7)}
\]
as claimed. Moreover since \( v_n \in H^1_1(\mathbb{R}^N) \) and \( v_n \geq 0 \), it follows that \( u_0 \) is radially symmetric with respect to the origin (up to translation) and non-negative. Then by Propositions 4.1 and 4.2, we have \( u_0 \in C^{1,\sigma}(\mathbb{R}^N) \) for some \( \sigma \in (0,1) \) and positive on \( \mathbb{R}^N \), finishing the proof. \( \square \)

**Remark 6.5.** By the variational characterization (6.7), the oddness of \( g \) in (g1) and Proposition 4.2, we can see that any ground state solution \( w \) of (1.1) has fixed sign on \( \mathbb{R}^N \).

One also expects that any ground state solution \( w \) of (1.1) is radially symmetric with respect to some point as in [1,8]. But for the moment, we are not able to prove it. Actually for this purpose, we first need that the function \( J_1(s,b) \) defined in Lemma 6.1 is strictly convex with respect to \( b \), which follows by assuming that
\[
0 < \phi(s) + 5s\phi'(s) - 2s^2|\phi''(s)| \quad \text{for all } s \in [0, +\infty).
\]
Then one can apply the case of equality for the generalized Polya–Szegö inequality to \( J_1 \) ([15, Theorem 2.11 and Corollary 2.12]), showing that \( w = w^* \) a.e. in \( \mathbb{R}^N \) provided that
\[
\mathcal{L}\{x \in \mathbb{R}^N ; 0 < w^*(x) < \text{ess sup } w \quad \text{and} \quad \nabla w^*(x) = 0\} = 0. \quad \text{(6.8)}
\]
(See also [6, Theorem 1.1] and [21, Corollary 2.33].) Especially if \( w \) is analytic, then (6.8) can be established. We also note that one cannot apply the symmetry result due to [24] because of the loss of a variational characterization like (1.6) in our problem.

**Acknowledgements**

The first author is supported by PRIN 2017JPCAPN *Qualitative and quantitative aspects of nonlinear PDEs*. The second author is supported by JSPS Grant-in-Aid for Scientific Research (C) (No. 18K03383).

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Received: 13 April 2020.
Accepted: 4 March 2021.