We extend a lower bound on average of local energy for the Ising model with quenched randomness [J. Phys. Soc. Jpn. 76, 074711 (2007)] to asymmetric distribution. Compared to the case of symmetric distribution, our bound has a non-trivial term. Applying the attained bound to the Gaussian distribution, we obtain lower bounds on the expected value of the square of the correlation function. As a result, we show that, in the Ising model with the Gaussian random field, the spin-glass order parameter always has a finite value at any temperature, regardless of the form of other interactions.
I. INTRODUCTION

Spin-glass models describe magnetic material that interacts spatially randomly. While the mean-field theory of spin-glass models, that is, the Sherrington-Kirkpatrick, was solved rigorously by the full replica symmetry breaking solution [1][4], it is very difficult to obtain analytical results for finite-dimensional models, except on the Nishimori-line [5]. While analytical approach [6] is making little progress in two-dimensional systems, analyses for three-dimensional systems have been largely untouched except for numerical analysis.

In ferromagnetic spin models, correlation inequalities play an important role in non-perturbative analysis and give us rigorous results for unsolvable models. Correlation inequalities are also valid for the Ising model with random field. Recent study [7] proved that, based on the Fortuin-Kasteleyn-Ginibre inequality, there is no spin-glass phase in the random-field Ising model with two-body interaction for all lattice and field distribution. Therefore, it is expected that the concept of correlation inequalities plays an essential role in rigorous analysis of spin-glass models, and it is a very important problem to establish correlation inequalities for spin-glass models.

There are some previous studies on correlation inequalities in spin-glass models. Recent study [8][9] showed that the response of the quenched average of the partition function with respect to the variance is always positive, which is considered as the counterpart of the Griffiths first inequality in spin-glass models. In addition, for various bond randomness including the Gaussian distribution and the binary distribution, it is shown that the counterpart of the Griffiths second inequality holds on the Nishimori-line [10][11]. However, correlation inequalities as in ferromagnetic spin models have not been obtained in general, and rigorous analysis based on correlation inequalities has not been done at the satisfactory level for spin-glass models.

In this paper, we obtain a lower bound on the average of local energy for the Ising model with quenched randomness. Although the result of the previous study [12] was limited to symmetric distribution, we generalize it to asymmetric distribution. Furthermore, as a simple application of attained inequality, we obtain correlation inequalities for the Gaussian distribution. We show that the expected value of the square of the correlation function always has a finite lower bound at any temperature. As a consequence, we prove that the spin-glass order parameter has a finite lower bound in the Ising model with the Gaussian-random field, regardless of the form of other interactions.

The organization of the paper is as follows. In Sec. II, we define the model and obtain the lower bound on the average of local energy for the Ising model with quenched randomness. In Sec. III, attained inequality is applied when the randomness of interactions follows the Gaussian distribution. Finally, our conclusion is given in Sec. IV.

II. LOWER BOUND ON LOCAL ENERGY FOR ASYMMETRIC DISTRIBUTION OF RANDOMNESS

Following Ref. [12], we consider a generic form of the Ising model,

\[ H = - \sum_{B \subset V} J_B \sigma_B, \]

where \( V \) is the set of sites, the sum over \( B \) runs over all subsets of \( V \) among which interactions exist, and the lattice structure takes any form. The probability distribution of random interactions \( J_B \) is represented by \( P_B(J_B) \). The probability distributions can be generally different from each other, \( P_B(J_B) \neq P_B'(J_B') \), and it is also allowed that the probability distribution has no randomness, \( P_B(J_B) = \delta(J - J_B) \).

The correlation function for a set of fixed interactions \( \{J_B\} \) is given by

\[ \langle \sigma_A \rangle_{\{J_B\}} = \frac{\text{Tr} \sigma_A \exp \left( \beta \sum_{B \subset V} J_B \sigma_B \right)}{\text{Tr} \exp \left( \beta \sum_{B \subset V} J_B \sigma_B \right)}. \]

The configurational average over the distribution of randomness of interactions is written as

\[ \mathbb{E}[g(\{J_B\})] = \prod_{B \subset V} \int_{-\infty}^{\infty} dJ_B P_B(J_B) g(\{J_B\}). \]

For example, the expected value of the correlation function is given by

\[ \mathbb{E}[\langle \sigma_A \rangle_{\{J_B\}}] = \left( \prod_{B \subset V} \int_{-\infty}^{\infty} dJ_B P_B(J_B) \right) \frac{\text{Tr} \sigma_A \exp \left( \beta \sum_{B \subset V} J_B \sigma_B \right)}{\text{Tr} \exp \left( \beta \sum_{B \subset V} J_B \sigma_B \right)}. \]

Our result is the following theorem.
We note that for any even function $P$ is symmetric. Furthermore, for any even function $\beta_J$ in Eq. (13), we immediately find

$$\mathbb{E} \left[ -J_A f(J_A) \langle \sigma_A \rangle_{\{J_B\}} \right] \geq \mathbb{E} \left[ -J_A f(J_A) \tanh(\beta_J A) - J_A f(J_A)(1 - e^{-\beta_J A}) \frac{1}{\sinh(2\beta J_A)} \right].$$

We note that the right-hand side of Eq. (7) does not depend on other interactions. When the distribution function is symmetric $P_A(-J_A) = P_A(J_A)$ ($\beta_{NL} = 0$) and $f(J_A) = 1$, Eq. (7) is reduced to

$$\mathbb{E} \left[ -J_A \langle \sigma_A \rangle_{\{J_B\}} \right] \geq \mathbb{E} \left[ -J_A \tanh(\beta_J A) \right],$$

which coincides with the existing result in Ref. [12]. In this case, the intuitive explanation of the inequality is possible: the local energy is always larger than or equal to the energy in the absence of all other interactions. However, for $\beta_{NL} \neq 0$, it is difficult to find an intuitive explanation because we do not give simple physical meaning of the second term of the right-hand side of Eq. (7).

**Proof.** We define $Z(\beta, J_A)$ and $\langle \sigma_A \rangle_{J_A}$ as

$$Z(\beta, J_A) = \sum_{\{\sigma\}} \exp \left( \beta \sum_{B \subset V \setminus A} J_B \sigma_B + \beta J_A \sigma_A \right),$$

$$\langle \sigma_A \rangle_{J_A} = \frac{\sum_{\{\sigma\}} \sigma_A \exp \left( \beta \sum_{B \subset V \setminus A} J_B \sigma_B + \beta J_A \sigma_A \right)}{\sum_{\{\sigma\}} \exp \left( \beta \sum_{B \subset V \setminus A} J_B \sigma_B + \beta J_A \sigma_A \right)}.$$

We note that $\langle \sigma_A \rangle_{J_A} = \langle \sigma_A \rangle_{\{J_B\}}$ but $\langle \sigma_A \rangle_{-J_A} \neq \langle \sigma_A \rangle_{\{J_B\}}$. Then, we obtain

$$\frac{Z(\beta, J_A)}{Z(\beta, -J_A)} = \cosh(2\beta J_A) + \langle \sigma_A \rangle_{-J_A} \sinh(2\beta J_A)$$

$$= e^{\beta_{NL} J_A} + \Gamma(\beta, -J_A) \frac{\sinh(2\beta J_A)}{J_A} \geq 0,$$

$$\frac{Z(\beta, -J_A)}{Z(\beta, J_A)} = \cosh(2\beta J_A) - \langle \sigma_A \rangle_{J_A} \sinh(2\beta J_A)$$

$$= e^{-\beta_{NL} J_A} + \Gamma(\beta, J_A) \frac{\sinh(2\beta J_A)}{J_A} \geq 0,$$

where $\Gamma(\beta, J_A)$ is defined as

$$\Gamma(\beta, J_A) \equiv -J_A \langle \sigma_A \rangle_{J_A} + J_A \tanh(\beta J_A) + (1 - e^{-\beta_{NL} J_A}) \frac{J_A}{\sinh(2\beta J_A)}.$$

Since Eq. (11) is the reciprocal of Eq. (12), we obtain

$$e^{-2\beta_{NL} J_A} \Gamma(\beta, -J_A) = \frac{-e^{-\beta_{NL} J_A} \Gamma(\beta, J_A)}{e^{-\beta_{NL} J_A} + \Gamma(\beta, J_A) \frac{\sinh(2\beta J_A)}{J_A}}.$$

On the other hand, from Eq. (13), we immediately find

$$\mathbb{E} \left[ -J_A f(J_A) \langle \sigma_A \rangle_{\{J_B\}} \right] = \mathbb{E} \left[ f(J_A) \Gamma(\beta, J_A) - J_A f(J_A) \tanh(\beta J_A) - J_A f(J_A)(1 - e^{-\beta_{NL} J_A}) \frac{1}{\sinh(2\beta J_A)} \right].$$

Furthermore, for any even function $f(J_A) \geq 0$, we find $\mathbb{E} \left[ f(J_A) \Gamma(\beta, J_A) \right] \geq 0$, because

$$\mathbb{E} \left[ f(J_A) \Gamma(\beta, J_A) \right] = \int_{-\infty}^{\infty} dJ_A P_A(J_A) f(J_A) \mathbb{E} \left[ \Gamma(\beta, J_A) \right]$$

$$= \int_{0}^{\infty} dJ_A P_A(J_A) f(J_A) \mathbb{E} \left[ \Gamma(\beta, J_A) + \exp(-2\beta_{NL} J_A) \Gamma(\beta, -J_A) \right]$$

$$\geq 0,$$

$$\int_{0}^{\infty} dJ_A P_A(J_A) f(J_A) \mathbb{E} \left[ \frac{\Gamma^2(J_A) \frac{\sinh(2\beta J_A)}{J_A}}{e^{-\beta_{NL} J_A} + \Gamma(\beta, J_A) \frac{\sinh(2\beta J_A)}{J_A}} \right] \geq 0,$$

(16)
where \( E[\cdots]' \) stands for the configurational average over randomness of other interactions than \( J_A \), and we used Eq. (14) in the third identity and used Eq. (12) in the last inequality. Thus, Eqs. (15) and (16) gives Eq. (7).

### III. APPLICATION TO GAUSSIAN SPIN-GLASS MODEL

In this section, we apply Eq. (7) to spin-glass model with the Gaussian distribution. First, we consider the special case, \( P_A(J_{0,A} - J_A) = P_A(J_{0,A} + J_A) \). Then, we obtain the following result.

**Corollary 2.** When the distribution function of randomness satisfies

\[
P_A(J_{0,A} - J_A) = P_A(J_{0,A} + J_A),
\]

for any even function \( f(J_A) \geq 0 \), the system defined above satisfies the following inequality,

\[
E \left[ (J_{0,A} - J_A) f(J_A - J_{0,A}) \langle \sigma_A \rangle_{\{J_B\}} \right] = \int_{-\infty}^{\infty} dJ_A P_A(J_{0,A} + J_A) f(J_A) E \left[ -J_A f(J_A) \langle \sigma_A \rangle_{J_{0,A} + J_A} \right]'
\]

\[
\geq E \left[ -J_A f(J_A) \tanh(\beta J_A) \right].
\]

**Proof.** If we regard \( P_A(J_{0,A} + J_A) \) as a new probability distribution \( P'_A(J_A) \), \( P'_A(J_A) \) is symmetric. Therefore, using Eq. (7) for \( \beta NL = 0 \), we prove Eq. (18).

In the following, using Eq. (18), we obtain several inequalities.

**A. Correlation inequality for Gaussian spin-glass**

Next, we consider the case where all of interactions follows the Gaussian distribution with mean \( J_{0,B} \) and variance \( \Lambda_B^2 \). Each \( J_{0,B} \) and \( \Lambda_B^2 \) can take different values. We denote the configurational average over the distribution of randomness of interactions as \( E[\cdots]_{\{J_{0,B},\Lambda_B^2\}} \). Then, we obtain the following result.

**Corollary 3.** For the expected value of the square of the correlation function, we obtain a lower bound,

\[
E \left[ \tanh^2(\beta J_A) \right]_{\{0,\Lambda_A^2\}} \leq E \left[ \langle \sigma_A \rangle_{J_B}^2 \right]_{\{J_{0,B},\Lambda_B^2\}}.
\]

We note that the left-hand side of Eq. (19) is independent of mean \( J_{0,B} \).

**Proof.** For the Gaussian distribution with mean \( J_{0,B} \) and variance \( \Lambda_B^2 \), and \( f(J_A) = 1 \), Eq. (18) is reduced to

\[
E \left[ (J_{0,A} - J_A) f(J_A - J_{0,A}) \langle \sigma_A \rangle_{\{J_B\}} \right]_{\{J_{0,B},\Lambda_B^2\}} = E \left[ -J_A f(J_A) \langle \sigma_A \rangle_{J_{0,A} + J_{0,B}} \right]_{\{0,\Lambda_B^2\}}
\]

\[
\geq E \left[ -J_A f(J_A) \tanh(\beta J_A) \right]_{\{0,\Lambda_B^2\}}.
\]

Furthermore, using integration by parts, we obtain Eq. (19)\[\square\]

A similar calculation is possible for higher order terms. Taking \( f(J_A) = J_A^2 \) in Eq. (18), we obtain

\[
E \left[ -(J_A - J_{0,A})^3 \langle \sigma_A \rangle_{\{J_B\}} \right]_{\{J_{0,B},\Lambda_B^2\}} = E \left[ -J_A^3 \langle \sigma_A \rangle_{J_{0,A} + J_{0,B}} \right]_{\{0,\Lambda_B^2\}} \geq E \left[ -J_A^3 \tanh(\beta J_A) \right]_{\{0,\Lambda_B^2\}}
\]

Using integration by parts and Eq. (19), we get a lower bound on the expected value of the fourth power of the correlation function,

\[
E \left[ \tanh^4(\beta J_A) \right]_{\{0,\Lambda_A^2\}} \leq E \left[ \langle \sigma_A \rangle_{J_B}^4 \right]_{\{J_{0,B},\Lambda_B^2\}}.
\]

Therefore, it is expected that the following relation holds for any natural number \( k \),

\[
E \left[ \tanh^{2k}(\beta J_A) \right]_{\{0,\Lambda_A^2\}} \leq E \left[ \langle \sigma_A \rangle_{J_B}^{2k} \right]_{\{J_{0,B},\Lambda_B^2\}}.
\]

However, we have not obtained a general proof or counter example.
B. Lower bound on spin-glass order-parameter in Gaussian random-field Ising model

Finally, we show that the spin-glass order-parameter in the Ising model with the Gaussian random-field always takes a finite value at any temperature, regardless of the form of other interactions.

We consider the case where a random-field \{h_i\} is independently applied to all sites and \{h_i\} follows the Gaussian distribution with mean \(J_0\) and variance \(\Lambda^2\). The Hamiltonian is given by

\[
H = - \sum_{B \subset V} J_B \sigma_B \\
= - \sum_{B \subset V \setminus \{h_i\}} J_B \sigma_B - \sum_{i=1}^{N} h_i \sigma_i,
\]

where interaction \(J_B\) other than \{h_i\} takes any form. Then, Eq. (19) is reduced to

\[
E \left[ \tanh^2(\beta h_i) \right]_{\{h_i, \Lambda^2\}} \leq E \left[ \langle \sigma_i \rangle^2_{\{J_B\}} \right]_{\{J_0, \Lambda^2\}}.
\]

Furthermore, because the same inequality holds for all sites, we obtain the following result.

**Corollary 4.** For the spin-glass order-parameter \(q\),

\[
q = \frac{1}{N} \sum_i E \left[ \langle \sigma_i \rangle^2_{\{J_B\}} \right]_{\{J_0, \Lambda^2\}},
\]

the system (24) satisfies the following inequality,

\[
E \left[ \tanh^2(\beta h_i) \right]_{\{h_i, \Lambda^2\}} \leq q.
\]

Thus, when the Gaussian random field is applied, the spin-glass order-parameter has generally a non-zero lower bound. In ferromagnetic models, the ferromagnetic order parameter, that is, the magnetization, has a finite value when a magnetic field is applied. Equation (27) implies that a similar phenomenon occurs in the Ising model with the Gaussian random field. This is a natural consequence, but the existence of a finite lower bound is not obvious.

In addition, we note that Eq. (27) does not mean that there is a spin-glass phase in the Ising model with the Gaussian random field.

### IV. CONCLUSIONS

We have obtained the lower bound on the local energy for the Ising model with quenched randomness. We emphasize that obtained inequality (7) is independent of other interactions. Our result is a natural generalization of Ref. [12] where symmetric distribution was considered.

Applying obtained inequality to the Gaussian spin-glass model, we find that the expected value of the square of the correlation function always has a finite lower bound at any temperature. As a consequence, the spin-glass order-parameter in the Ising model with the Gaussian random field always takes a finite value at any temperature, which is a natural but not obvious result.

It is an interesting question whether a similar inequality as Eq. (19) holds for general distribution function of random interactions or not. Our proof relied on the property of the Gaussian distribution, and we have not found a proof for other distribution.

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