On the isentropic compressible Navier-Stokes equation

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Abstract
In this article, we consider the compressible Navier-Stokes equation with density dependent viscosity coefficients. We focus on the case where those coefficients vanish on vacuum. We prove the stability of weak solutions for periodic domain $\Omega = T^N$ as well as the whole space $\Omega = \mathbb{R}^N$, when $N = 2$ and $N = 3$. The pressure is given by $p = \rho^\gamma$, and our result holds for any $\gamma > 1$. In particular, we prove the stability of weak solutions of the Saint-Venant model for shallow water.

1 Introduction
This paper is devoted to the Cauchy problem of the compressible Navier-Stokes equation with viscosity coefficients vanishing on vacuum. Let $\rho(t,x)$ and $u(t,x)$ denote the density and the velocity of an isentropic compressible viscous fluid (as usual, $\rho$ is a non-negative function and $u$ is a vector valued function, both defined on a subset $\Omega$ of $\mathbb{R}^N$). Then, the Navier-Stokes equation for isentropic compressible viscous fluids reads (see [LL59]):

$$\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla \cdot p - \text{div} (h D(u)) - \nabla (g \text{div} u) &= 0
\end{align*}$$

(1)

where $p(\rho) = \rho^\gamma$, $\gamma > 1$, denotes the pressure, $D(u) = \frac{1}{2} \left[ \nabla u + \nabla u^T \right]$ is the strain tensor and $h$ and $g$ are the two Lamé viscosity coefficients (depending on the density $\rho$) satisfying

$$h > 0 \quad h + Ng \geq 0$$

(2)
(h is sometime called the shear viscosity of the fluid, while g is usually referred to as the second viscosity coefficient). One of the major difficulty of compressible fluid mechanics is to deal with vacuum. The problem of existence of global solution in time for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukhin [KS77], and for discontinuous one, but still with densities away from zero, by Serre [Ser86] and Hoff [Hof87]. Those results have been generalized to higher dimensions by Matsumura and Nishida [MN79] for smooth data close to equilibrium and by Hoff [Hof95b], [Hof95a] in the case of discontinuous data.

Concerning large initial data, Lions showed in [Lio98] the global existence of weak solutions for $\gamma \geq \frac{3}{2}$ for $N = 2$ and $\gamma \geq \frac{9}{5}$ for $N = 3$. This result has been extended later by Feireisl, Novotny, and Petzeltova to the range $\gamma > \frac{3}{2}$ in [FNP01], and very recently by Feireisl to the full system of the Navier-Stokes equations involving the energy equation [Fei04]. Other results provide the full range $\gamma > 1$ under symmetries assumptions on the initial datum (see for instance Jiang and Zhang [JZ03]). All those results do not require to be far from the vacuum. However they rely strongly on the assumption that the viscosity coefficients are bounded below by a positive constant. This non physical assumption allows to get some estimates on the gradient of the velocity field.

The main difficulty when dealing with vanishing viscosity coefficients on vacuum is that the velocity cannot even be defined when the density vanishes. The first result handling this difficulty is due to Bresch, Desjardins and Lin [BDL03]. They showed the $L^1$ stability of weak solutions for the following Korteweg’s system of equations:

$$\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0 \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla_x p - \nu \text{div} (\rho D(u)) &= \kappa \rho \nabla \Delta \rho.
\end{align*}$$

The result was later improved by Bresch and Desjardins in [BD03] to include the case of vanishing capillarity ($\kappa = 0$), but with an additional quadratic friction term $\rho|u| u$ (see also [BD02]). The key point in those papers is to show that the structure of the diffusion term provides some regularity for the density thanks to a new entropy inequality. However, those estimates are not enough to treat the case without capillarity and friction effects $\kappa = 0$ and $r = 0$ (which corresponds to equation (1) with $h(\rho) = \rho$ and $g(\rho) = 0$).

The main difficulty, to prove the stability of the solutions of (1), is to pass to the limit in the term $\rho(u \otimes u)$ (which requires the strong convergence of $\sqrt{\rho} u$). Note that this is easy when the viscosity coefficients are bounded below by a positive constant. On the other hand, the new bounds on the
gradient of the density make the control of the pressure term far simpler than in the case of constant viscosity coefficients.

Our result is in the same spirit as the one of Bresch, Desjardins and Lin and makes use of the same entropy inequality, first discovered by Bresch and Desjardins in [BD02] for the particular case where \( h(\rho) = \rho \) and \( g(\rho) = 0 \). We actually use a slightly more general estimate, which holds for any viscosity coefficients \( h(\rho), g(\rho) \) satisfying the relation:

\[
g(\rho) = \rho h'(\rho) - h(\rho). \tag{4}
\]

This estimate first appeared in a Note by Bresch and Desjardins [BD04] in the context of Korteweg systems of equations. However, we will see that the capillary term is by no means necessary to the derivation of the crucial estimates which thus hold for the compressible Navier-Stokes system [10].

Our main contribution is to show the \( L^1 \) stability of weak solutions of (1) under some conditions on the viscosity coefficients (including (4)) but without any additional regularizing terms. The interest of our result lie primarily in the fact that our conditions allow for viscosity coefficients that vanish on the vacuum set. It includes the case \( h(\rho) = \rho, g(\rho) = 0 \) (when \( N = 2 \) and \( \gamma = 2 \), we recover the Saint Venant model for Shallow water), but our conditions on \( h \) and \( g \) will exclude the case of constant viscosity \( h(\rho) = \mu, g(\rho) = \xi \). Indeed, it is readily seen that (4) implies that \( g(\rho) = \xi = -\mu \), and thus \( \mu + \xi = 0 \). In this border line case we thus lose all informations on the derivatives of \( u \). It is worth pointing out that while we can gain regularity on the density with this new estimate, we have to loose regularity on the velocity (on the vacuum set).

Note that the main difficulty will be to establish the compactness of \( \sqrt{\rho} u \) in \( L^2 \) strong, and the key ingredient to achieve this is an additional estimate which bounds \( \sqrt{\rho} u \) in \( L^\infty(0,T;L^{2+2\alpha}(\Omega)) \) for some small \( \alpha > 0 \) (the usual entropy estimate only gives a bound in \( L^\infty(0,T;L^2(\Omega)) \)).

For the sake of simplicity we will consider the case \( \Omega = \mathbb{R}^N \) and the case of bounded domain with periodic boundary conditions, namely \( \Omega = T^N \). For the same reason we consider only power pressure laws although the result could be extend to non monotonic pressure law of the form of [Fei02]. Note that the result holds for any power \( \gamma > 1 \) under appropriate assumptions on \( h \) and \( g \). Classically, \( L^1 \) stability is considered as the main step to prove the existence of weak solutions. To obtain the existence of weak solutions, one is thus left with the technical task of constructing a sequence of approximated solutions verifying the a priori estimates. Although this final step is in most cases quite standard, we point out that in this particular situation it
seems highly non trivial because of the complexity of the additional entropy inequality.

In the next section, we state the assumptions on the viscosity coefficients, define precisely the notion of “weak solutions” and state our main results. In Section 3 we recall the well known physical energy inequality and state the key estimates. The proof of Theorem 2.1 is detailed in Section 4. For the sake of completeness, we give in Section 5 the proof of the entropy inequality of Bresch and Desjardins in the context of compressible Navier-Stokes equation.

2 Notations and main result

Let $\Omega$ denote a subset of $\mathbb{R}^N$. We assume that $\Omega$ is either the whole space $\mathbb{R}^N$ or a bounded domain with periodic boundary conditions ($\Omega = T^N$). For the sake of simplicity, we will take $D(u) = \nabla u$, though the full strain tensor could be considered without any additional difficulty. This leads to the following system of equations:

\[ \partial_t \rho + \text{div} (\rho u) = 0 \]  
\[ \partial_t (\rho u) + \nabla (\rho u \otimes u) + \nabla \rho \gamma - \text{div} (h(\rho) \nabla u) - \nabla (g(\rho) \text{div} u) = 0, \]  

with initial conditions

\[ \rho|_{t=0} = \rho_o \geq 0, \quad \rho u|_{t=0} = m_o. \]  

Before introducing the notion of weak solution, let us state the assumptions we make on the viscosity coefficients.

**Conditions on $h(\rho)$ and $g(\rho)$:**

First we consider $f(\rho), g(\rho)$ verifying:

\[ g(\rho) = \rho h'(\rho) - h(\rho). \]

As stated in the introduction, this structure constraint is fundamental to get more regularity on the density. Moreover, we assume that there exists a positive constant $\nu \in (0, 1)$ such that

\[ h'(\rho) \geq \nu, \quad h(0) \geq 0 \]
\[ |g'(\rho)| \leq \frac{1}{\nu} h'(\rho) \]
\[ \nu h(\rho) \leq h(\rho) + Ng(\rho) \leq \frac{1}{\nu} h(\rho). \]
When $\gamma \geq 3$ and $N = 3$, we also require that

$$\liminf_{\rho \to \infty} \frac{h(\rho)}{\rho^{\gamma/3 + \varepsilon}} > 0,$$

for some small $\varepsilon > 0$. Let us make some remarks about those assumptions.

**Remark 2.1** The functions

$$h(\rho) = \rho, \quad g(\rho) = 0$$

satisfy (8-11). In fact, any linear combination of $\rho^k$ with $k \geq 1$ is an admissible function for $h(\rho)$.

**Remark 2.2** The lower estimate in (11) is trivial when $g \geq 0$, while the upper estimate is trivial when $g \leq 0$. Together they yield:

$$|g(\rho)| \leq C_\nu h(\rho) \quad \forall \rho > 0.$$

This inequality and (10) will be necessary to pass to the limit in the term

$$\nabla (g(\rho_n) \text{div} u_n).$$

**Remark 2.3** Condition (9) makes the proof simpler, but is not optimal. However, condition (11) is necessary to control the viscosity term and together with (8), it yields

$$N - 1 + \nu \leq \frac{h'(\rho)}{h(\rho)} \leq \frac{N - 1 + 1/\nu}{N \rho}, \quad \text{for all } \rho > 0,$$

and so

$$\begin{cases}
C \rho^{(N-1)/N+\nu/N} \leq h(\rho) \leq C \rho^{(N-1)/N+1/(N\nu)}, & \rho \geq 1 \\
C \rho^{(N-1)/N+1/(N\nu)} \leq h(\rho) \leq C \rho^{(N-1)/N+\nu/N}, & \rho \leq 1
\end{cases}$$

(13)

In particular, we must have $h(0) = 0$. Moreover, this shows that if we do not assume (9), the “best” $h(\rho)$ we can take is $h(\rho) = \rho^{(N-1)/N+\nu/N}$. This is actually enough to prove the stability of weak solutions for all $\gamma$ when $N = 2$ and for $\gamma < 3/2$ when $N = 3$. However, if we assume $h(\rho) \sim C \rho^{2/3 + \nu}$ for small $\rho$ and $h(\rho) \sim C \rho$ for large $\rho$, then we can take any $\gamma \in (1, 3)$ when $N = 3$. 

5
**Notion of weak solutions**

We say that \((\rho, u)\) is a weak solution of (5-6) on \(\Omega \times [0, T]\), with initial conditions (7) if

\[
\rho \in L^\infty(0, T; L^1(\Omega) \cap L^\gamma(\Omega)), \\
\sqrt{\rho} \in L^\infty(0, T; H^1(\Omega)), \\
\sqrt{\rho} u \in L^\infty(0, T; (L^2(\Omega))^N), \\
h(\rho) \nabla u \in L^2(0, T; (W^{-1,1}_{\text{loc}}(\Omega))^{N \times N}), \\
g(\rho) \text{div} u \in L^2(0, T; W^{-1,1}_{\text{loc}}(\Omega)),
\]

with \(\rho \geq 0\) and \((\rho, \sqrt{\rho} u)\) satisfying

\[
\begin{cases}
\partial_t \rho + \text{div}(\sqrt{\rho} \sqrt{\rho} u) = 0 \\
\rho(0, x) = \rho_0(x)
\end{cases}
\]

in \(\mathcal{D}'\),

and if the following equality holds for all \(\varphi(t, x)\) smooth test function with compact support such that \(\varphi(T, \cdot) = 0\):

\[
\int_\Omega m_o \cdot \varphi(0, \cdot) \, dx + \int_0^T \int_\Omega \sqrt{\rho}(\sqrt{\rho} u) \partial_t \varphi + \sqrt{\rho} u \otimes \sqrt{\rho} u : \nabla \varphi \, dx dt \\
+ \int_0^T \int_\Omega \rho \text{div} \varphi \, dx - \langle h(\rho) \nabla u, \nabla \varphi \rangle - \langle g(\rho)(\text{div} u), (\text{div} \varphi) \rangle = 0, \tag{14}
\]

where the diffusion terms make sense when written as

\[
\langle h(\rho) \nabla u, \nabla \varphi \rangle =
= -\int \frac{h(\rho)}{\sqrt{\rho}} (\sqrt{\rho} u_j \partial_i \varphi_j) \, dx dt - \int (\sqrt{\rho} u_j) 2h'(\rho) \partial_i \sqrt{\rho} \partial_i \varphi_j \, dx dt,
\]

and

\[
\langle g(\rho)(\text{div} u), (\text{div} \varphi) \rangle =
= -\int \frac{g(\rho)}{\sqrt{\rho}} (\sqrt{\rho} u_i \partial_i \varphi_j) \, dx dt - \int (\sqrt{\rho} u_i) 2g'(\rho) \partial_i \sqrt{\rho} \partial_j \varphi_j \, dx dt.
\]

In particular, the fact that the diffusion term \(h(\rho) \nabla u\) (and \(g(\rho) \text{div} u\)) lies in \(L^2(0, T; (W^{-1,1}_{\text{loc}}(\Omega))^{N \times N})\) will follow from the fact that

\[
h'(\rho) \sqrt{\rho} \in L^\infty(0, T; L^2_{\text{loc}}(\Omega)), \quad \text{and} \quad h(\rho)/\sqrt{\rho} \in L^\infty(0, T; L^2_{\text{loc}}(\Omega)),
\]

and similar conditions on \(g(\rho)\). This will be provided by assumptions (10), (9) and (13).

**Main result:**

The main result of this paper is the following:
**Theorem 2.1** Assume that $\gamma > 1$ and that $h(\rho)$ and $g(\rho)$ are two $C^2$ functions of $\rho$ satisfying conditions (8)-(11) (together with (12) if $\gamma \geq 3$ and $N = 3$). Let $(\rho_n, u_n)_{n \in \mathbb{N}}$ be a sequence of weak solutions of (5-6) satisfying entropy inequalities (18), (21) and (25), with initial data

$$\rho_n|_{t=0} = \rho^n_o(x) \quad \text{and} \quad \rho_n u_n|_{t=0} = m^n_o(x) = \rho^n_o(x) u^n_o(x),$$

where $\rho^n_o$ and $u^n_o$ are such that

$$\rho^n_o \geq 0, \quad \rho^n_o \rightarrow \rho_o \text{ in } L^1(\Omega), \quad \rho^n_o u^n_o \rightarrow \rho_o u_o \text{ in } L^1(\Omega),$$

and satisfy the following bounds (with $C$ constant independent on $n$):

$$\int_{\Omega} \rho^n_o |u^n_o|^2 \, dx + \frac{1}{\gamma - 1} \rho^n_o ^\gamma \, dx < C, \quad \int_{\Omega} \frac{1}{\rho^n_o} |\nabla h(\rho^n_o)|^2 \, dx < C,$$

and

$$\int_{\Omega} \rho^n_o |u^n_o|^{2+\delta} \, dx < C,$$

for some small $\delta > 0$.

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n u_n})$ converges strongly to a weak solution of (5-6) satisfying entropy inequalities (18), (21) and (25) (the density $\rho_n$ converges strongly in $C^0((0, T); L^{3/2}_{loc}(\Omega))$, $\sqrt{\rho_n u_n}$ converges strongly in $L^2(0, T; L^2_{loc}(\Omega))$ and the momentum $m_n = \rho_n u_n$ converges strongly in $L^1(0, T; L^1_{loc}(\Omega))$, for any $T > 0$).

**3 Entropy inequalities and a priori estimates**

In this section, we recall the well-known energy inequality and state the main inequalities that we will use throughout the proof of Theorem 2.1.

The usual energy inequality associated with the system of equations (5,6) can be written as:

$$\frac{d}{dt} \int \rho \frac{u^2}{2} + \frac{1}{\gamma - 1} \rho^\gamma \, dx + \int h(\rho)|\nabla u|^2 \, dx + \int g(\rho)(\text{div } u)^2 \, dx \leq 0.$$  (18)

This inequality can be established for smooth solutions of (5,6) by multiplying the momentum equation by $u$.

When $h$ and $g$ satisfies $h(\rho) + Ng(\rho) \geq 0$ and if the initial data are taken in such a way that

$$\mathcal{E}_o = \int_{\Omega} \rho_o \frac{u^2_o}{2} + \frac{1}{\gamma - 1} \rho^\gamma_o \, dx < +\infty,$$
then (18) yields:
\[
\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2(\Omega))} \leq C,
\]
\[
\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C. \tag{19}
\]
Furthermore, Hypothesis (11) gives:
\[
\|\sqrt{h(\rho)} \nabla u\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{20}
\]
Finally, integrating (5) with respect to \(x\) yields the natural \(L^1\) estimate:
\[
\|\rho\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{21}
\]
Unfortunately, it is a well-known fact that those estimates are not enough to prove the stability of the solutions of (5-6). In particular, the fact that \(\rho^\gamma\) is bounded in \(L^\infty(0,T;L^1(\Omega))\) does not imply that \(\rho^\gamma_h\) converges to \(\rho^\gamma\).

However, further estimates can be obtained by means of the following lemma (the proof of which is postponed to Section 5):

**Lemma 3.1** Assume that \(h(\rho)\) and \(g(\rho)\) are two \(C^2\) functions such that (8) holds true. Then, the following inequality holds for smooth solutions of (5-6):
\[
\frac{d}{dt} \left( \frac{1}{2} \rho |u + \nabla \varphi(\rho)|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) dx + \int \nabla \varphi(\rho) \cdot \nabla \rho^\gamma dx \leq 0, \tag{21}
\]
with \(\varphi\) such that
\[
\varphi' = \frac{h'(\rho)}{\rho}. \tag{22}
\]
This lemma is similar to the result of D. Bresch and B. Desjardin in [BD04], in which the same inequality was derived when capillary effects are taken into account.

We immediately see that since the viscosity coefficient \(h(\rho)\) is an increasing function of \(\rho\) and when the initial data satisfies
\[
\int_{\Omega} \rho_o |\nabla \varphi(\rho_o)|^2 dx < +\infty,
\]
inequality (21) yields:
\[
\frac{1}{2} \|\sqrt{\rho} \nabla \varphi(\rho)\|_{L^\infty(0,T;L^2(\Omega))} = \|h'(\rho) \nabla \sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{23}
\]
and
\[
\|\sqrt{h'(\rho)} \rho^{\gamma - 2} \nabla \rho\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{24}
\]
Under assumption (9) on $h$, those estimates will give additional control on the density $\rho$ and on the pressure $\rho^\gamma$, which will be enough to prove the stability of weak solution.

Finally, we shall make use of the following result:

**Lemma 3.2** Assume

$$h(\rho) + Ng(\rho) \geq \nu h(\rho)$$

for some $\nu \in (0, 1)$ (which is a part of (11)), and let $\delta \in (0, \nu/4)$. Then, smooth solutions of (5-6) satisfy the following inequality:

$$\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2 + \delta} \, dx + \frac{\nu}{4} \int h(\rho) |u|^\delta |\nabla u|^2 \, dx$$

$$\leq \left( \int \left( \frac{\rho^{2\gamma\delta/2}}{h(\rho)} \right)^{(2/(2-\delta))^2} \, dx \right)^{2-\delta/2} \left( \int \rho |u|^2 \, dx \right)^{\delta/2}.$$ (25)

where $|\nabla u|^2 = \sum_i \sum_j |\partial_i u_j|^2$.

This inequality is quite simple to establish and will be essential in the proof of Theorem 2.1 to prove that $\sqrt{\rho_n} u_n$ is bounded in $L^\infty(0, T; L^{2+2\alpha}(\Omega))$ (see Lemma 4.3). Note, however, that to derive further estimates from this inequality, we need to control the right hand side of (25). Inequality (18) immediately provide a bound on $\int \rho |u|^2 \, dx$, so the problem will be to control enough power of $\rho$ to get a bound on $\int \left( \frac{\rho^{2\gamma\delta/2}}{h(\rho)} \right)^{2/(2-\delta)} \, dx$. This will be achieved using (24). Of course, we also need to assume that the initial condition satisfies

$$\int \rho_0 \frac{|u_0|^{2+\delta}}{2 + \delta} \, dx < C.$$

**Proof of Lemma 3.2** Let $\delta \in (0, \nu/4)$. Multiplying (9) by $u |u|^\delta$, we get:

$$\int \rho \partial_t \frac{|u|^{2+\delta}}{2 + \delta} \, dx + \int \rho u \cdot \nabla \frac{|u|^{2+\delta}}{2 + \delta} \, dx$$

$$+ \int h(\rho) |u|^\delta (\nabla u)^2 \, dx + \delta \int h(\rho) |u|^{\delta-2} u_i u_k \partial_j u_i \partial_j u_k \, dx$$

$$+ \int g(\rho) |u|^\delta (\nabla u)^2 \, dx + \delta \int g(\rho) |u|^{\delta-2} u_k u_j \partial_i u_i \partial_j u_k \, dx$$

$$+ \int |u|^\delta u \cdot \nabla \rho^\gamma \, dx = 0.$$
Since
\[(\text{div}\, u)^2 = \sum_i \sum_j \partial_i u_i \partial_j u_j \leq \sum_i \sum_j \frac{1}{2} (\partial_i u_i^2 + \partial_j u_j^2) \leq N |\nabla u|^2,\]
condition (11) yields:
\[
\int \rho \partial_t |u|^{2+\delta} dx + \int \rho u \cdot \nabla |u|^{2+\delta} dx + \nu \int h(\rho)|u|^{\delta} (\nabla u)^2 dx
\]
\[
+ \int |u|^\delta u \cdot \nabla \rho \gamma dx
\]
\[
\leq \delta \int h(\rho)|u|^{\delta-2} u_i \partial_j u_i \partial_j u_k dx
\]
\[
+ \delta \int g(\rho)|u|^{\delta-2} u_k \partial_i u_i \partial_j u_j dx,
\]
and since \(\delta < \nu/4\), we deduce:
\[
\int \rho \partial_t |u|^{2+\delta} dx + \int \rho u \cdot \nabla |u|^{2+\delta} dx + \frac{\nu}{2} \int h(\rho)|u|^{\delta} (\nabla u)^2 dx
\]
\[
+ \int |u|^\delta u \cdot \nabla \rho \gamma dx \leq 0.
\]
Moreover, multiplying (5) by \(\frac{|u|^{2+\delta}}{2+\delta}\) and integrating by parts, we have
\[
\int \frac{|u|^{2+\delta}}{2+\delta} \partial_t \rho dx - \int \rho u \cdot \nabla \frac{|u|^{2+\delta}}{2+\delta} dx = 0
\]
and summing the last two inequalities, we get:
\[
\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2+\delta} dx + \frac{\nu}{2} \int h(\rho)|u|^{\delta} |\nabla u|^2 dx \leq \int |u|^{\delta} u \cdot \nabla \rho \gamma dx,
\]
It remains to bound the right hand side. We have:
\[
\left| \int |u|^{\delta} u \cdot \nabla \rho \gamma dx \right| = \left| - \int \rho \gamma |u|^{\delta} \text{div} u dx - \delta \int \rho \gamma |u|^{\delta-2} (u \cdot \nabla) u dx \right|
\]
\[
\leq (\sqrt{N} + \delta) \left| \int \rho \gamma |u|^{\delta} |\nabla u| dx \right|
\]
\[
\leq (\sqrt{N} + \delta) \left( \int h(\rho)|u|^{\delta} |\nabla u|^2 dx \right)^{1/2} \left( \int \frac{\rho^{2\gamma}}{h(\rho)}|u|^{\delta} dx \right)^{1/2}
\]
\[
\leq \frac{\nu}{4} \int h(\rho)|u|^{\delta} |\nabla u|^2 dx + C \nu \int \frac{\rho^{2\gamma}}{h(\rho)}|u|^{\delta} dx,
\]
where the last term satisfies (if $\delta \in (0, 2)$):

$$\int \frac{\rho^2}{h(\rho)} |u|^\delta \, dx \leq \left( \int \left( \frac{\rho^{2\gamma-\delta/2}}{h(\rho)} \right)^{2/(2-\delta)} \, dx \right)^{(2-\delta)/2} \left( \int \rho |u|^2 \, dx \right)^{\delta/2},$$

and the lemma follows.

We now have all the necessary tools to prove Theorem 2.1.

4 Proof of Theorem 2.1

We now present the proof of Theorem 2.1. To begin with, we need to make precise the assumptions on the initial data.

Initial data:
We recall that the initial data must satisfy (16), and (17) to make use of all the inequalities presented in the previous section:

1. $\rho^o_n$ is bounded in $L^1 \cap L^{\gamma}(\Omega)$, $\rho^o_n \geq 0$ a.e. in $\Omega$
2. $\rho^o_n |u^o|^2 = |m^o_n|^2/\rho^o_n$ is bounded in $L^1(\Omega)$
3. $\sqrt{\rho^o_n} \nabla \psi(\rho^o_n) = \nabla h(\rho^o_n)/\sqrt{\rho^o_n}$ is bounded in $L^2(\Omega)$, (26)
4. $\int \rho^o_n |u^o|^{2+\delta} \, dx < C$ for some small $\delta$.

With those assumptions, and using inequalities (18) and (21), we deduce the following estimates, which we shall use throughout the proof of Theorem 2.1:

$$||\sqrt{\rho^o_n} u^o||_{L^\infty(0,T;L^2(\Omega))} \leq C$$
$$||\rho^o_n||_{L^\infty(0,T;L^1 \cap L^{\gamma}(\Omega))} \leq C$$
$$||\sqrt{h(\rho^o_n)} \nabla u^o||_{L^2(0,T;L^2(\Omega))} \leq C$$

and

$$||h'(\rho^o_n) \nabla \sqrt{\rho^o_n}||_{L^\infty(0,T;L^2(\Omega))} \leq C$$
$$||h'(\rho^o_n) \rho^o_n^{-\delta/2} \nabla \rho^o_n||_{L^2(0,T;L^2(\Omega))} \leq C$$

In view of hypothesis on the viscosity coefficient (9), the bounds (27) and (28) yields:

$$||\sqrt{\rho^o_n} \nabla u^o||_{L^2(0,T;L^2(\Omega))} \leq C$$
$$||\nabla \sqrt{\rho^o_n}||_{L^\infty(0,T;L^2(\Omega))} \leq C$$
$$||\nabla \sqrt{h^2(\rho^o_n)}||_{L^2(0,T;L^2(\Omega))} \leq C$$

(29)
The proof of Theorem 2.1 will be divided in 6 steps. In the first two steps, we show the convergence of the density and the pressure (note that the convergence of the pressure is straightforward here). The key argument of the proof is presented in the third step: We prove that \( \sqrt{\rho_n} u_n \) is bounded in a space better than \( L^\infty(0,T;L^2(\Omega)) \). In turn, this will give the convergence of the momentum (step 4) and finally the strong convergence of \( \sqrt{\rho_n} u_n \) in \( L^2_{\text{loc}}((0,T) \times \Omega) \) (step 5). The last step addresses the convergence of the diffusion terms; It is mainly technical and of minor interest.

**Step 1: Convergence of \( \sqrt{\rho_n} \).**

**Lemma 4.1** If \( h \) satisfies (9), then:

- \( \sqrt{\rho_n} \) is bounded in \( L^\infty(0,T;H^1(\Omega)) \)
- \( \partial_t \sqrt{\rho_n} \) is bounded in \( L^2(0,T;H^{-1}(\Omega)) \).

As a consequence, up to a subsequence, \( \sqrt{\rho_n} \) converges almost everywhere and strongly in \( L^2(0,T;L^2_{\text{loc}}(\Omega)) \). We write

\[
\sqrt{\rho_n} \rightharpoonup \sqrt{\rho} \quad \text{a.e and } L^2_{\text{loc}}((0,T) \times \Omega) \text{ strong.}
\]

Moreover, \( \rho_n \) converges to \( \rho \) in \( C^0(0,T;L^{3/2}_{\text{loc}}(\Omega)) \).

**Proof.** The second estimate in (29), together with the conservation of mass \( \|\rho_n(t)\|_{L^1(\Omega)} = \|\rho_{n,0}\|_{L^1(\Omega)} \) gives the \( L^\infty(0,T;H^1(\Omega)) \) bound. Next, we notice that

\[
\partial_t \sqrt{\rho_n} = -\frac{1}{2} \sqrt{\rho_n} \text{div} u_n - u_n \cdot \nabla \sqrt{\rho_n} = \frac{1}{2} \sqrt{\rho_n} \text{div} u_n - \text{div}(u_n \sqrt{\rho_n})
\]

which yields the second estimate and, thanks to Aubin’s Lemma, gives the strong convergence in \( L^2_{\text{loc}}((0,T) \times \Omega) \).

Sobolev imbedding insures that \( \sqrt{\rho_n} \) is bounded in \( L^\infty(0,T;L^q(\Omega)) \) for \( q \in [2,\infty[ \) if \( N = 2 \) and \( q \in [2,6] \) if \( N = 3 \). In either cases we deduce that \( \rho_n \) is bounded in \( L^\infty(0,T;L^3(\Omega)) \), and therefore

\[
\rho_n u_n = \sqrt{\rho_n} \sqrt{\rho_n} u_n \text{ is bounded in } L^\infty(0,T;L^{3/2}(\Omega)).
\]
The continuity equation thus yields $\partial_t \rho_n$ bounded in $L^\infty(0, T, W^{-1,3/2}(\Omega))$. Moreover, since $\nabla \rho_n = 2\sqrt{\rho_n} \nabla \sqrt{\rho_n}$, we also have that $\nabla \rho_n$ is bounded in $L^\infty(0, T; L^{3/2}(\Omega))$, hence the compactness of $\rho_n$ in $C([0, T]; L^{3/2}_{loc}(\Omega))$.

**Step 2: Convergence of the pressure**

**Lemma 4.2** The pressure $\rho_n^\gamma$ is bounded in $L^{5/3}((0, T) \times \Omega)$ when $N = 3$ and $L^r((0, T) \times \Omega)$ for all $r \in [1, 2]$ when $N = 2$. In particular, $\rho_n^\gamma$ converges to $\rho^\gamma$ strongly in $L^1_{loc}((0, T) \times \Omega)$.

**Proof.** Inequalities (29) and (27) yield $\rho_n^\gamma / 2 \in L^2(0, T; H^1(\Omega))$.

When $N = 2$, we deduce $\rho_n^\gamma / 2 \in L^2(0, T; L^q(\Omega))$ for all $q \in [2, \infty]$. So $\rho_n^\gamma$ is bounded in $L^1(0, T; L^p(\Omega)) \cap L^\infty(0, T; L^1(\Omega))$ for all $p \in [1, \infty]$, hence $\rho_n^\gamma$ is bounded in $L^r((0, T) \times \Omega)$ for all $r \in [1, 2]$.

When $N = 3$, we only get $\rho_n^\gamma / 2 \in L^2(0, T; L^6(\Omega))$, or $\rho_n^\gamma \in L^1(0, T; L^3(\Omega))$.

Since $\rho_n^\gamma$ is bounded in $L^\infty(0, T; L^1(\Omega))$, Hölder inequality gives

$$||\rho_n^\gamma||_{L^{5/3}((0, T) \times \Omega)} \leq ||\rho_n^\gamma||_{L^\infty(0, T; L^1(\Omega))}^{2/5} ||\rho_n^\gamma||_{L^1(0, T; L^3(\Omega))}^{3/5} \leq C,$$

hence $\rho_n^\gamma$ is bounded in $L^{5/3}((0, T) \times \Omega)$.

Since we already know that $\rho_n^\gamma$ converges almost everywhere to $\rho^\gamma$, those bounds yield the strong convergence of $\rho_n^\gamma$ in $L^1_{loc}((0, T) \times \Omega)$.

**Step 3: Bounds for $\sqrt{\rho_n}u_n$**

**Lemma 4.3** If $\gamma < 3$, or if $N = 3$, $\gamma \geq 3$ and (12) holds, then $\sqrt{\rho_n}u_n$ is bounded in $L^\infty(0, T; L^{2+2\alpha}(\Omega))$

for some small $\alpha > 0$.

This Lemma is really the corner stone of the stability result. As a matter of fact, at this point, the main difficulty is to prove the strong convergence of $\sqrt{\rho_n}u_n$ in $L^1(0, T; L^2_{loc}(\Omega))$. A first consequence of Lemma 4.3 is that it will be enough to prove the convergence almost everywhere. However, since we are only able to prove the convergence of the momentum $\rho_n u_n$ (see Step 4, which makes use of Lemma 4.3 as well), we need to control $\sqrt{\rho_n}u_n$ on
the vacuum set \( \{ \rho(t, x) = 0 \} \) (and prove that it converges to zero almost everywhere). And this fact also will be a consequence of Lemma 4.3 (see Step 5).

**Proof.** The proof of Lemma 4.3 relies on Lemma 3.2: for \( \delta \) small enough \((\delta \in \langle 0, \nu/4 \rangle)\), we have:

\[
\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2} dx + \frac{\nu}{2} \int h(\rho)|u|^{\delta} |\nabla u|^2 dx \leq \left( \int \left( \frac{\rho^{2\gamma-\delta/2}}{h(\rho)} \right)^{2/(2-\delta)} dx \right)^{(2-\delta)/2} \left( \int \rho|u|^2 dx \right)^{\delta/2}. \tag{30}
\]

Using (27), we deduce:

\[
\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2} dx \leq C \left( \int \left( \frac{\rho^{2\gamma-\delta/2}}{h(\rho)} \right)^{2/(2-\delta)} dx \right)^{(2-\delta)/2}.
\]

Condition (34) yields \( h(\rho) \geq \nu \rho \) and so

\[
\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2} dx \leq C \left( \int \left( \rho^{2\gamma-1-\delta/2} \right)^{2/(2-\delta)} dx \right)^{(2-\delta)/2}.
\]

Using Lemma 4.2 we readily check that the right hand side is bounded \( L^1 \) in time (for small \( \delta \)), without any condition when \( N = 2 \), and when \( N = 3 \) under the condition that

\[
2\gamma - 1 < \frac{5}{3}\gamma,
\]

which gives rise to the restriction \( \gamma < 3 \). In either cases, we deduce

\[
\frac{d}{dt} \int \rho \frac{|u|^{2+\delta}}{2+\delta} dx \leq C.
\]

and (17) gives the lemma. When \( N = 3 \) and \( \gamma \geq 3 \) we need the extra hypothesis (12) to achieve the same result.

Finally, for \( \alpha < \delta/2 \), we have:

\[
\int (\rho|u|^2)^{1+\alpha} dx \leq \left( \int \rho|u|^{2+\delta} dx \right)^{\frac{2+2\alpha}{2+\delta}} \left( \int \rho^{\delta(1+\alpha-(2+2\alpha)/(2+\delta))} dx \right)^{\frac{1}{\delta}}.
\]
with \( q = (1 - (2 + 2\alpha)/(2 + \delta))^{-1} \), so that the exponent of \( \rho \) goes to 1 when \( \alpha \) goes to zero. In particular, it is less than 3 for \( \alpha \) small enough, and since \( \rho_n \) is bounded in \( L^\infty(0,T;L^3(\Omega)) \), we deduce Lemma 4.3.

**Step 4: Convergence of the momentum**

**Lemma 4.4** Up to a subsequence, the momentum \( m_n = \rho_n u_n \) converges strongly in \( L^2(0,T;L^{1+\varepsilon}_\text{loc}(\Omega)) \) (for some positive \( \varepsilon \)) and almost everywhere to some \( m(x,t) \).

Note that we can already define \( u(x,t) = m(x,t)/\rho(x,t) \) outside the vacuum set \( \{\rho(x,t) = 0\} \), but we do not know yet whether \( m(x,t) \) is zero on the vacuum set.

**Proof.** We have

\[
\rho_n u_n = \sqrt{\rho_n} \sqrt{\rho_n} u_n,
\]

where \( \sqrt{\rho_n} \) is bounded in \( L^\infty(0,T;L^q(\Omega)) \) for \( q \in [2, +\infty[ \) if \( N = 2 \) and \( q \in [2, 6] \) if \( N = 3 \); Since \( \sqrt{\rho_n} u_n \) is bounded in \( L^\infty(0,T;L^2(\Omega)) \), we deduce that \( \rho_n u_n \) is bounded in \( L^\infty(0,T,L^q(\Omega)) \) for all \( q \in [1, 3/2] \).

Next, we have

\[
\partial_t (\rho_n u_n) = \rho_n \partial_t u_n + u_n \partial_t \rho_n = \sqrt{\rho_n} \sqrt{\rho_n} \partial_t u_n + 2 \sqrt{\rho_n} u_n \partial_t \sqrt{\rho_n}.
\]

Using Lemma 4.3 and (29), it is readily seen that the second term is bounded in \( L^\infty(0,T;L^{1+\varepsilon}(\Omega)) \) for some small \( \varepsilon > 0 \), while the first term is bounded in \( L^2(0,T,L^q(\Omega)) \) for all \( q \in [1, 3/2] \). Hence

\[
\nabla (\rho_n u_n) \text{ is bounded in } L^2(0,T;L^{1+\varepsilon}(\Omega)).
\]

In particular, we have

\[
\rho_n u_n \text{ bounded in } L^2(0,T;W^{1,1+\varepsilon}(\Omega)).
\]

It remains to show that for every compact set \( K \subset \Omega \), we have

\[
\partial_t (\rho_n u_n) \text{ is bounded in } L^{5/3}(0,T;W^{-2,3/2}(K)). \tag{31}
\]
As a matter of fact, we observe that $W^{1,3}_0(K) \subset L^{1+\varepsilon}(K)$ for small $\varepsilon$ (for $N = 2$ or 3), and therefore

$$L^{1+\varepsilon}(K) \subset W^{-1,3/2}(K) \subset W^{-2,3/2}(K),$$

so (31) together with Aubin’s Lemma, yields the compactness of $\rho_n u_n$ in $L^2(0,T;L^{1+\varepsilon}(K))$.

To prove (31), we use the momentum equation (6), first noticing from Lemma 4.2 and Lemma 4.3 that

$$\text{div} \left( \sqrt{\rho_n} u_n \otimes \sqrt{\rho_n} u_n \right) \in L^\infty(0,T;W^{-1,1+\varepsilon+\varepsilon}(K))$$

and

$$\nabla \rho_n^\gamma \in L^{5/3}(0,T;W^{-1,1+\varepsilon}(K)).$$

So we only have to check that $\nabla (h(\rho_n) \nabla u_n)$ and $\nabla (g(\rho_n) \text{div} u_n)$ are bounded in $L^\infty(0,T;W^{-2,3/2}(K))$. To that purpose, we write

$$h(\rho_n) \nabla u_n = \nabla (h(\rho_n) u_n) - u_n \nabla h(\rho_n),$$

(32)

(and similarly with $g(\rho_n)$). The second term in (32) is

$$u_n \nabla h(\rho_n) = \sqrt{\rho_n} u_n \frac{\nabla h(\rho_n)}{\sqrt{\rho_n}} = 2 \sqrt{\rho_n} u_n h'(\rho_n) \nabla \sqrt{\rho_n}$$

which is bounded in $L^\infty(0,T;L^{1+\varepsilon}(\Omega))$ thanks to (28) and Lemma 4.3. The first term in (32) can be rewritten

$$\nabla [h(\rho_n) u_n] = \nabla \left[ \frac{h(\rho_n)}{\sqrt{\rho_n}} \sqrt{\rho_n} u_n \right],$$

which is bounded in $L^\infty(0,T;W^{-1,3/2}(\Omega))$ thanks to the following lemma:

**Lemma 4.5** For all compact set $K$, $h(\rho_n)/\sqrt{\rho_n}$ and $g(\rho_n)/\sqrt{\rho_n}$ are bounded in $L^\infty(0,T;L^6(K))$.

The proof of this Lemma is a bit technical in full generality and will be postponed to Appendix A. However, note that, in the particular case $h(\rho) = \nu \rho$, we have $h(\rho_n)/\sqrt{\rho_n} = \sqrt{\rho_n}$ and Lemma 4.5 follows straightforwardly from Lemma 4.1.

We deduce that $h(\rho_n) \nabla u_n$ and $g(\rho_n) \text{div} u_n$ are bounded in

$$L^\infty(0,T;W^{-1,3/2}(K) + L^{1+\varepsilon}(K)).$$
and since $L^{1+\varepsilon}(K) \subset W^{-1,3/2}(K)$ we can conclude that $h(\rho_n) \nabla u_n$ and $g(\rho_n) \text{div} u_n$ are bounded in $L^\infty(0,T;W^{-1,3/2}(K))$, which conclude the proof of Lemma 4.4.

**Step 5: Convergence of $\sqrt{\rho_n}u_n$**

**Lemma 4.6** The quantity $\sqrt{\rho_n}u_n$ converges strongly in $L^1$ and $L^2_{\text{loc}}((0,T) \times \Omega)$ to $m/\sqrt{\rho}$ (defined to be zero when $m = 0$).

In particular, we have $m(x,t) = 0$ a.e. on $\{\rho(x,t) = 0\}$ and there exists a function $u(x,t)$ such that

$$m(x,t) = \rho(x,t)u(x,t)$$

(note that $u$ is not uniquely defined on the vacuum set $\{\rho(x,t) = 0\}$).

**Proof.** First of all, since $m_n/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Fatou’s lemma yields

$$\int \liminf \frac{m_n^2}{\rho_n} \, dx < \infty.$$ 

In particular, we have $m(x,t) = 0$ a.e. in $\{\rho(x,t) = 0\}$, and if we define $m^2/\rho$ to be 0 when $m = 0$, we have

$$\int \frac{m^2}{\rho} \, dx < \infty.$$ 

Lemma 4.3 implies that $\sqrt{\rho_n}|u_n|$ is bounded in $L^\infty(0,T;L^{2+2\alpha}(\Omega))$ for a small $\alpha$. It is thus enough to prove the convergence almost everywhere, or in $L^1_{\text{loc}}((0,T) \times \Omega)$, to prove the strong convergence in $L^2_{\text{loc}}$.

First of all, we note that in $\{\rho(x,t) \neq 0\}$, $\sqrt{\rho_n}u_n$ converges almost everywhere to $m/\sqrt{\rho}$. So, if we denote the vacuum set by

$$A = \{\rho(x,t) = 0\},$$

we deduce

$$\sqrt{\rho_n}u_n1_A \rightarrow \frac{m}{\sqrt{\rho}}1_A \quad \text{a.e.} \quad (33)$$

To control $\sqrt{\rho_n}u_n$ on the vacuum set, we introduce the set

$$B^n_M = \{\rho_n^{1/(2+\delta)}|u_n| \geq M\},$$

17
for \( M > 0 \). We then cut the \( L^1 \) norm as follows:

\[
\int |\sqrt{\rho_n} u_n - \frac{m}{\sqrt{\rho}}| \, dx \, dt = \int_{\mathcal{C}B_M^A} \cdots + \int_{\mathcal{C}B_M^A \cap A} \cdots + \int_{B_M^A} \cdots
\]

The \( L^\infty(0, T; L^2(\Omega)) \) bound and (33) gives the convergence of the first integral:

\[
\int 1_{\mathcal{C}B_M^A \cap A} |\sqrt{\rho_n} u_n - \frac{m}{\sqrt{\rho}}| \, dx \, dt \longrightarrow 0. \tag{34}
\]

Moreover, Lemma 4.3 and Tchebychev’s inequality yields

\[
|B_M^n| \leq \frac{C}{M^2},
\]

and so

\[
\int_{B_M^A} |\sqrt{\rho_n} u_n - m/\sqrt{\rho}| \leq \sqrt{|B_M^n|} \left( \int \rho_n |u_n|^2 + |m|^2/\rho \, dx \right)^{1/2} \leq \frac{C}{M}. \tag{35}
\]

Finally, on \( \mathcal{C}B_M^A \cap A \), we have

\[
|\sqrt{\rho_n} u_n| \leq M \rho_n^{1/2 - 1/(2 + \delta)} \longrightarrow 0 \quad \text{a.e. ,}
\]

since \( \rho_n \to 0 \) a.e. and \( 1/2 - 1/(2 + \delta) > 0 \). So \( 1_{\mathcal{C}B_M^A \cap A} |\sqrt{\rho_n} u_n| \) converges almost everywhere to 0. In particular, the \( L^\infty(0, T; L^2(\Omega)) \) bound yields

\[
\int 1_{B_M^A \cap A} |\sqrt{\rho_n} u_n| \, dx \, dt \longrightarrow 0.
\]

Since we defined \( m/\sqrt{\rho} \) to be 0 on \( A \), we also have

\[
1_{\mathcal{C}B_M^A} \frac{m}{\sqrt{\rho}}(x, t) = 0 \text{ a.e. } \forall n
\]

hence

\[
\int 1_{\mathcal{C}B_M^A \cap A} \sqrt{\rho_n} u_n - \frac{m}{\sqrt{\rho}} \, dx \, dt \longrightarrow 0. \tag{36}
\]

Putting (34), (35) and (36) together, we deduce

\[
\limsup_{n \to \infty} \int |\sqrt{\rho_n} u_n - \frac{m}{\sqrt{\rho}}| \, dx \, dt \leq \frac{C}{M}
\]

for all \( M > 0 \), and so \( \sqrt{\rho_n} u_n \) converges to \( \frac{m}{\sqrt{\rho}} \) in \( L^1((0, T) \times \Omega) \) strong. The lemma follows.
Step 6: Convergence of the diffusion terms

Lemma 4.7 We have

\[ h(\rho_n)\nabla u_n \to h(\rho)\nabla u \text{ in } \mathcal{D}' \]

and

\[ g(\rho_n)\text{div} u_n \to g(\rho)\text{div} u \text{ in } \mathcal{D}' \]

Proof. Let \( \phi \) be a test function, then

\[
\int h(\rho_n)\nabla u_n \phi \, dx \, dt = -\int h(\rho_n)u_n\nabla \phi \, dx \, dt + \int u_n \nabla h(\rho_n)\phi \, dx \, dt
\]

\[
= -\int \frac{h(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n\nabla \phi \, dx \, dt + \int \sqrt{\rho_n}u_n \frac{h'(\rho_n)}{\sqrt{\rho_n}}\nabla \rho_n \phi \, dx \, dt
\]

Thanks to Lemma 4.5, we know that \( \frac{h(\rho_n)}{\sqrt{\rho_n}} \) is bounded in \( L^\infty(0, T; L^6_{\text{loc}}(\Omega)) \). Moreover, since \( h(\rho_n)/\sqrt{\rho_n} \leq \nu, \sqrt{\rho_n} \), this term converges almost everywhere to \( h(\rho)/\sqrt{\rho} \) (defined to be zero on the vacuum set). Therefore, it converges strongly in \( L^2_{\text{loc}}((0, T) \times \Omega) \); This is enough to prove the convergence of the first term.

Next, we note that

\[
\frac{h'(\rho_n)}{\sqrt{\rho_n}}\nabla \rho_n = \nabla \psi(\rho_n)
\]

with \( \psi'(\rho) = h'(\rho)/\sqrt{\rho} = \sqrt{\rho} \psi'(\rho) \). Since

\[
\int |\nabla \psi(\rho)|^2 \, dx = \int \rho |\nabla \varphi(\rho)|^2 \, dx,
\]

we have that \( \nabla \psi(\rho_n) \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \). Moreover, \[13\] yields

\[ h'(\rho) \leq C\rho^{-1/2+\nu/3} \text{ when } \rho \leq 1 \]

and so

\[ \psi(\rho) \leq C\rho^{\nu/3} \text{ when } \rho \leq 1. \]

Therefore, an argument similar to the proof of Lemma 4.5 shows that \( \psi(\rho_n) \) is bounded in \( L^\infty(0, T; L^6_{\text{loc}}(\Omega)) \). Since it converges almost everywhere (\( \psi \) is a continuous function), it converges strongly in \( L^2_{\text{loc}}((0, T) \times \Omega) \). It follows that

\[ \nabla \psi(\rho_n) \rightharpoonup \nabla \psi(\rho) \text{ in } L^2_{\text{loc}}((0, T) \times \Omega)-\text{weak} \]

A similar argument holds for \( g(\rho_n)\text{div} u_n \) using the fact that \( |g(\rho)| \leq Ch(\rho) \) and \( |g'(\rho)| \leq Ch'(\rho) \).
5 Proof of Lemma 3.1

We conclude this paper by giving the proof of the estimate (21). To that purpose, we have to evaluate

\[
\frac{d}{dt} \int \left[ \frac{1}{2} \rho |u|^2 + \rho u \cdot \nabla \varphi(\rho) + \frac{1}{2} \rho |\nabla \varphi(\rho)|^2 \right] \, dx + \frac{d}{dt} \int \frac{1}{\gamma - 1} \rho^\gamma \, dx.
\]

**Step 1:** First of all, we recall the usual entropy equality:

\[
\frac{d}{dt} \int \left[ \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right] \, dx = - \int h(\rho) |\nabla u|^2 \, dx - \int g(\rho) |\text{div } u|^2 \, dx
\]

**Step 2:** Next, (5) gives

\[
\int \rho \frac{\partial}{\partial t} \left[ \frac{1}{2} |\nabla \varphi(\rho)|^2 \right] \, dx - \int \left[ \frac{1}{2} |\nabla \varphi(\rho)|^2 \right] \text{div } \rho u \, dx \\
= - \int \rho \nabla u : \nabla \varphi(\rho) \otimes \nabla \varphi(\rho) \, dx + \int \rho^2 \varphi'(\rho) \Delta \varphi(\rho) \text{div } u \, dx \\
+ \int \rho |\nabla \varphi(\rho)|^2 \text{div } u \, dx
\]

and so

\[
\frac{d}{dt} \int \rho \left[ \frac{1}{2} |\nabla \varphi(\rho)|^2 \right] \, dx = - \int \rho \nabla u : \nabla \varphi(\rho) \otimes \nabla \varphi(\rho) \, dx \\
+ \int \rho^2 \varphi'(\rho) \Delta \varphi(\rho) \text{div } u \, dx \\
+ \int \rho |\nabla \varphi(\rho)|^2 \text{div } u \, dx \quad (37)
\]

**Step 3:** It remains to evaluate the derivative of the cross-product:

\[
\frac{d}{dt} \int \rho u \cdot \nabla \varphi(\rho) \, dx = \int \nabla \varphi(\rho) \cdot \partial_t (\rho u) \, dx + \int \rho u \cdot \partial_t \nabla \varphi(\rho) \, dx \\
= \int \nabla \varphi(\rho) \cdot \partial_t (\rho u) \, dx - \int \text{div } (\rho u) \varphi'(\rho) \partial_t \rho \, dx \\
= \int \nabla \varphi(\rho) \cdot \partial_t (\rho u) \, dx + \int (\text{div } (\rho u))^2 \varphi'(\rho) \, dx \quad (38)
\]
Multiplying (6) by $\nabla \varphi(\rho)$, we get:

$$
\int \nabla \varphi(\rho) \cdot \partial_t(\rho u) \, dx
= - \int (h(\rho) + g(\rho)) \Delta \varphi(\rho) \text{div } u \, dx + \int \nabla u : \nabla \varphi(\rho) \otimes \nabla h(\rho) \, dx
- \int \nabla \varphi(\rho) \cdot \nabla h(\rho) \text{div } u \, dx - \int \nabla \varphi(\rho) \cdot \nabla \rho^\gamma \, dx
- \int \nabla \varphi(\rho) \text{div } (\rho u \otimes u) \, dx,
$$

where we used the fact that

$$
\int \nabla (g(\rho) \text{div } u) \cdot \nabla \varphi(\rho) \, dx = - \int g(\rho) \Delta \varphi(\rho) \text{div } u \, dx
$$

and

$$
\int \text{div } (h(\rho) \nabla u) \cdot \nabla \varphi(\rho) \, dx
= \int \partial_j (h(\rho) \partial_j u_i) \partial_i \varphi(\rho) \, dx
= \int \partial_t (h(\rho) \partial_j u_i) \partial_j \varphi(\rho) \, dx
= \int \partial_t h(\rho) \partial_j u_i \partial_j \varphi(\rho) \, dx - \int \partial_t u_i \partial_j h(\rho) \partial_j \varphi(\rho) \, dx
- \int \partial_i u_i h(\rho) \partial_j \varphi(\rho) \, dx
= \int \nabla u : \nabla h(\rho) \otimes \nabla \varphi(\rho) \, dx - \int \nabla h(\rho) \cdot \nabla \varphi(\rho) \text{div } u \, dx
- \int h(\rho) \Delta \varphi(\rho) \text{div } u \, dx.
$$

**Step 4:** When $\varphi$, $h$ and $g$ satisfies (8) and (22), then (37) and (38) yields

$$
\frac{d}{dt} \left\{ \int \rho u \cdot \nabla \varphi(\rho) + \rho \frac{[\nabla \varphi(\rho)]^2}{2} \, dx \right\} + \int \nabla \varphi(\rho) \cdot \nabla p \, dx
= - \int \nabla \varphi(\rho) \text{div } (\rho u \otimes u) \, dx + \int \varphi'(\rho)(\text{div } (\rho u))^2 \, dx.
$$
Finally, we have

\[- \int \nabla \varphi(\rho) \text{div} (\rho u \otimes u) \, dx + \int \varphi'(\rho)(\text{div} (\rho u))^2 \, dx \]

\[= \int -\varphi'(\rho) u \cdot \nabla \rho \text{div} (\rho u) - \varphi'(\rho) \nabla \rho (\rho u \cdot \nabla u) + \varphi'(\rho)(\text{div} \rho u)^2 \, dx \]

\[= \int \rho \varphi'(\rho) \text{div} u \text{div} (\rho u) - \rho \varphi'(\rho) \nabla \rho (u \cdot \nabla u) \, dx \]

\[= \int \rho^2 \varphi'(\rho)(\text{div} u)^2 + \rho \varphi'(\rho) u \cdot \nabla \rho \text{div} u - \rho \varphi'(\rho) \nabla \rho (u \cdot \nabla u) \, dx \]

so using (22) and (8), we get

\[- \int \nabla \varphi(\rho) \text{div} (\rho u \otimes u) \, dx + \int \varphi'(\rho)(\text{div} (\rho u))^2 \, dx \]

\[= \int \rho h'(\rho)(\text{div} u)^2 + \nabla (h(\rho)) \cdot u \text{div} u - \nabla (h(\rho))(u \cdot \nabla u) \, dx \]

\[= \int \rho h'(\rho)(\text{div} u)^2 - h(\rho)(\text{div} u)^2 - h(\rho) u \cdot \nabla \text{div} u \, dx \]

\[+ \int h(\rho) \partial_i u_j \partial_j u_i + h(\rho) u \cdot \nabla \text{div} u \, dx \]

\[= \int (\rho h' - h)(\text{div} u)^2 + h(\rho) \partial_i u_j \partial_j u_i \, dx \]

\[= \int g(\rho) \, dx + \int h(\rho) \partial_i u_j \partial_j u_i \, dx \]

which yields

\[\frac{d}{dt} \left\{ \int \rho u \cdot \nabla \varphi(\rho) + \rho \frac{|\nabla \varphi(\rho)|^2}{2} \, dx \right\} + \int \nabla \varphi(\rho) \cdot \nabla \rho \, dx \]

\[\leq \int g(\rho)(\text{div} u)^2 \, dx + \int h(\rho)|\nabla u|^2 \, dx, \]

and the proof is complete.

**A Proof of Lemma 4.5**

We shall only prove the result for $h(\rho_n)/\sqrt{n}$. Using the fact that

\[|g(\rho)| \leq Ch(\rho), \quad \text{and} \quad |g'(\rho)| \leq Ch'(\rho) \quad \text{for all} \; \rho,\]

22
a similar proof follows for $q(\rho_n)/\sqrt{\rho_n}$.

Note that in view of (13), we have

$$\frac{h(\rho)}{\sqrt{\rho}} \leq C\rho^\nu$$

so we only need to control $\frac{h(\rho_n)}{\sqrt{\rho_n}}$ for large $\rho_n$. This will be achieved differently depending on the dimension.

When $N = 2$, the fact that $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and Sobolev’s inequalities implies that $\rho_n$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,\infty[$. Moreover, in view of (13), we have

$$\frac{h(\rho)}{\sqrt{\rho}} \leq \left\{ \begin{array}{ll}
C\rho^{1/\nu} & \text{if } \rho \geq 1 \\
C\rho^\nu & \text{if } \rho \leq 1
\end{array} \right.$$

So there exists $q_0 > 1$ such that $\frac{h(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for all $q > q_0$. In particular, $\frac{h(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^p(K))$ for all $p \in [1,\infty[$ for any compact set $K$.

When $N = 3$, we note that

$$\nabla \left( \frac{h(\rho)}{\sqrt{\rho}} \right) = 2h'(\rho)\nabla \sqrt{\rho} - \frac{h(\rho)}{2\rho^{3/2}} \nabla \rho,$$

and since conditions (8) and (11) yields

$$h'(\rho)\rho = g(\rho) + h(\rho) \geq \frac{3g(\rho) + h(\rho)}{3} \geq \frac{\nu}{3} h(\rho),$$

we have

$$|\nabla \left( \frac{h(\rho)}{\sqrt{\rho}} \right)| \leq C|h'(\rho)\nabla \sqrt{\rho}|.$$ 

So inequality (23) yields

$$||\nabla \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right)||_{L^\infty(0,T;L^2(\Omega))} \leq C$$  \hspace{1cm} (39)

When $\Omega = \mathbb{R}^3$, Sobolev’s inequalities implies that $\frac{h(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^6(\Omega))$. When $\Omega$ is a subset of $\mathbb{R}^3$, we note that (13) gives

$$\frac{h(\rho)}{\sqrt{\rho}} \leq \left\{ \begin{array}{ll}
C\rho^{1/6+3/\nu} & \text{if } \rho \geq 1 \\
C\rho^{1/6+\nu/3} & \text{if } \rho \leq 1
\end{array} \right..$$
So there exists a constant \( s \leq 1 \) such that
\[
\left( \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right)^s - 1 \right) + \in L^\infty(0,T;L^2(\Omega))
\]

Moreover
\[
\left| \nabla \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right)^s 1_{\rho_n \geq 1} \right| = \left| \nabla \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right)^{s-1} \nabla \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right) 1_{\rho_n \geq 1} \right|
\leq \left| \nabla \left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right) \right| \in L^\infty(0,T;L^2(\Omega)),
\]
using the fact that \( s - 1 \leq 0 \). It follows that \( (h(\rho_n)/\sqrt{\rho_n})^1 1_{\rho_n \geq 1} \) is bounded in \( L^\infty(0,T;H^1(\Omega)) \) which in turn gives
\[
\left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right)^{s_1} 1_{\rho_n \geq 1} \in L^\infty(0,T;L^2(\Omega)),
\]
for all \( s_1 \in (s,3s) \). As long as \( 3s \leq 1 \), we can repeat this argument with \( 3s \) instead of \( s \). Eventually, this will lead to
\[
\left( \frac{h(\rho_n)}{\sqrt{\rho_n}} \right) 1_{\rho_n \geq 1} \in L^\infty(0,T;L^2(\Omega)),
\]
which, together with (39) implies that \( (h(\rho_n)/\sqrt{\rho_n}) 1_{\rho_n \geq 1} \) is bounded in \( L^\infty(0,T;L^6(\Omega)) \).

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