Kullback-Leibler Distance Based Generalized Grey Target Decision Method With Index and Weight Both Containing Mixed Attribute Values

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ABSTRACT
This paper proposes a generalized grey target decision method (GGTDM) with index and weight both containing mixed attribute values based on Kullback-Leibler (K-L) distance. The proposed approach builds the weight function converting the mixed attribute-based weights into the certain number-based weights and takes the comprehensive weighted K-L distance as the decision-making basis (DMB). The proposed approach conducts its task in the following steps. First, all indices of alternatives are converted into binary connection numbers. Second, the two-tuple (determinacy, uncertainty) numbers originated from index binary connection numbers are obtained. Third, the two-tuple (determinacy, uncertainty) numbers of target center are calculated. Following that the certain number-based weights are obtained by the weight function. Then the comprehensive weighted K-L distance of each alternative and its target center is calculated. And the final decision making is based on the value of comprehensive weighted K-L distance with which the smaller the better. A case study illustrates the proposed approach with its effectiveness of converting the uncertain weights into the certain weights and the accurate results comparing with other decision-making methods.

INDEX TERMS
Kullback-Leibler distance, mixed attributes, generalized grey target decision method, binary connection number, weight function.

I. INTRODUCTION
The decision-making basis (DMB) of grey target decision method (GTDM) is referred to as the target center distance (TCD), which is the distance of each alternative and its target center. In certain number-based GTDM, Euclidean distance and Mahalanobis distance are often applied to obtain the TCDs [1], [2]. However, the mixed attribute-based GTDM obtains the TCD in different ways: at first, the conventional Euclidean distance-based method was reported [3]–[5]. Then the equivalent methods including cobweb area and correlation coefficient appeared [6], [7]. Besides, the proximity-based method, entropy-based method and Gini-Simpson index-based method were also investigated, as is named as generalized grey target decision method(GGTD) [8]–[10]. The GGTDM differs from the conventional one in the calculation process, but obeys the same principle [8], [9], [11]–[13]. Given that uncertainty originated from the uncertain number, an effective tool is required to measure it in the mixed attribute-based GTDM. The entropy is often used to measure the uncertainty, and entropy theory has many forms. As an important form, cross-entropy has been used widely. Ioannis and George applied cross-entropy to pattern recognition with intuitive fuzzy information and achieved desirable results [14]. Li and Wu adopted intuitionistic fuzzy cross-entropy to solve preference problem on alternatives [15]. Xia and Xu applied cross-entropy to group decision making under intuitionistic fuzzy environment [16]. Smieja and Geiger used cross-entropy theory to clustering under information restriction [17]. Tang et al. proposed an optimization algorithm based on cross-entropy [18]. While the kullback-Leibler (K-L) distance derived from cross-entropy can well reflect the degree of nearness between the two vectors, which has the
meaning of distance [19]. Therefore, the K-L distance can be applied to GGTDM for mixed attributes.

The GGTDM with index and weight both containing mixed attribute values has been investigated. However, the previous research mainly adopts the comprehensive weighted proximity(CWP) and comprehensive weighted Gini-Simpson index(CWGSI) as the DMBs [12], [20]. Accordingly, a few of methods converting uncertain weights into certain weights were presented. At first, the proximity-based method [12] and the module-based method [21] were proposed to obtain the certain weights. Later, different weight functions were also built to covert uncertain weights into certain weights [10], [22]. However, this work proposes a K-L distance-based method involving mixed attribute weight values. The decision process is as follows. First, all alternative indices are transformed into binary connection numbers and converted into two-tuple (determinacy, uncertainty) numbers. Second, the two-tuple (determinacy, uncertainty) numbers of target center under all attributes are obtained. Third, the certain number-based weights are determined by the weight function. Then the comprehensive weighted K-L distance of each alternative and its target center is calculated.

The final decision making is based on the value of comprehensive weighted K-L distance with which the smaller the better.

II. BASIC THEORY

A. KULLBACK-LEIBLER DISTANCE

Definition 1: Kullback-Leibler distance [14], [19]. Let $X$ and $Y$ be two random variables with their discrete distributions denoted by $X = (x_1, x_2, \cdots, x_m)^T$ and $Y = (y_1, y_2, \cdots, y_m)^T$ respectively, where $x_j, y_j \geq 0, j = 1, 2, \ldots, m, \sum_{j=1}^{m} x_j = \sum_{j=1}^{m} y_j$, then the Kullback-Leibler distance of $X$ and $Y$ can be obtained as follows:

$$H(X, Y) = \sum_{j=1}^{m} x_j \ln \frac{x_j}{y_j}$$

where $H(X, Y)$ has following properties:

1) $H(X, Y) = \sum_{j=1}^{m} x_j \ln \frac{x_j}{y_j} \geq 0$;

2) $H(X, Y) = \sum_{j=1}^{m} x_j \ln \frac{x_j}{y_j} = 0$, only $x_j = y_j, \forall j$.

When $x_j \neq 0, y_j = 0, H(X, Y) \rightarrow \infty$. Therefore, it is necessary to improve the original K-L distance in practice. The improved version of K-L distance is as follows [14], [19]:

$$K(X, Y) = \sum_{j=1}^{m} x_j \ln \frac{x_j}{\frac{1}{2}(x_j + y_j)}$$

B. UNCERTAIN NUMBER

The objective thing could be recognized by human being in an uncertain state due to the complex, uncertain thing itself, and people’s knowledge limitation and ambiguous recognition abilities. There are some ways to solve this uncertainty, such as fuzzy mathematics, grey system theory, set-pair analysis theory and rough set theory. And the uncertain number is often used to describe the characteristics, which is opposite to deterministic real number. In practical use, the data error brought by measurement and calculation, and data missing originated from incomplete information, the original data representing the characteristics of object may be uncertain. For this reason, the definitions of interval number (including the extended n-parameter interval number) and the binary connection number which can unify the determinacy and uncertainty of an uncertain number are given.

1) INTERVAL NUMBER AND MULTI-PARAMETER INTERVAL NUMBER

Definition 2: Let $R$ be a real number domain; if $\tilde{x}$ is an interval number, then it can be represented by $[x^L, x^U]$, where $x^L$ and $x^U$ are the upper and lower limits, respectively, satisfying $0 < x^L < x^U \in R$ [23], [24]. The interval number is an uncertain number with upper and lower limits. Furthermore, if an interval number contains more than two parameters in the extension of the primary form, then it can be termed an n-parameter interval number (also called multi-parameter interval number). If $n = 3$ or 4, then it can be named as three-parameter interval number or four-parameter interval number, respectively, and expressed by $[x^L, x^M, x^U]$ or $[x^L, x^M, x^N, x^U]$ respectively, where $x^L, x^M, x^N$ and $x^U$ satisfy $0 < x^L < x^M < x^N < x^U \in R$.

If $x^L = x^1, x^U = x^n$, the n-parameter interval number can be expressed as $[x^1, x^2, \ldots, x^n]$, where $x^i$ satisfy $0 < x^1 < \ldots < x^{i-1} < x^i < \ldots < x^n \in R$.

For an n-parameter interval number, if $n = 3$ or 4, it can be read as triangular fuzzy number or trapezoidal fuzzy number, respectively, in fuzzy theory; while it can also be called triangular grey number, or trapezoidal grey number, respectively, in grey theory. In this research, it is spoken of as the n-parameter interval number or multi-parameter interval number.

2) BINARY CONNECTION NUMBER

Definition 3: Let $R$ be a real domain; $\mu + \sigma i$ is a binary connection number, where $\mu$ denotes the determinism term, $\sigma$ denotes the uncertain term and $i$ is a variable term unifying the determinacy and uncertainty of an uncertain number, $\mu, \sigma \in R$ and $i \in [-1, 1]$.

Definition 4: Let $\bar{x}$ and $\bar{v}$ be the mean value and deviation value of the $n(n \geq 2)$ parameters of an interval number $\bar{x}$, respectively, then

$$u(\bar{x}, \bar{v}) = \mu + \sigma i = \bar{x} + vi (i \in [-1, 1])$$

is called mean value-deviation value connection number, where $\bar{x}, \bar{v}, \psi$ and $\bar{v}$ are obtained by using Eqs(4) to (7):

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$$
\[
\xi = \sqrt{\frac{1}{(n-1)} \sum_{j=1}^{n} (x_j - \bar{x})^2}
\]

(5)

\[
\psi = \max \{ |x^L - \bar{x}|, |x^U - \bar{x}| \}
\]

(6)

\[
v = \min \{ \xi, \psi \}
\]

(7)

where the mean value \( \bar{x} \) can be regarded as relatively deterministic measure of \( n(n \geq 2) \) parameters about \( x \), the standard deviation \( \xi \) and the maximum deviation \( \psi \) are relatively uncertain measure of \( n \) parameters about \( \bar{x} \) [8], [25].

Definition 5: The mutual interaction of the mean value \( \bar{x} \) and the deviation value \( v \) of the mean value-deviation value connection number \( u(\bar{x}, v) \) can be mapped to the determinacy-uncertainty space (D-U space), then \((\bar{x}, v)\) is supposed to be the micro-vector in the D-U space [23], [24].

### III. METHOD FOR CONVERTING MIXED ATTRIBUTE WEIGHTS INTO CERTAIN WEIGHTS

#### A. THE DIFFICULTY OF GENERALIZED GREY TARGET DECISION MAKING WITH INDEX AND WEIGHT BOTH CONTAINING MIXED ATTRIBUTE VALUES

The difficulty of generalized grey target decision making with index and weight both containing mixed attribute values lies in [12]; (1) different types of weight values cannot be aggregated directly with mixed index values; (2) the simple conversion of different types of data into deterministic weights without scientific and reasonable method may lead to large deviation and affect the accuracy of decision; (3) how to address the mixed index weights in a unified, simple and accurate way. Let \( w_i(t = 1, 2, \ldots, m) \) be the attribute value that could be a real number or an uncertain number. When it is an uncertain number, the \( w_i \) can be expressed as interval number or multi-parameter interval number form as follows:

\[
w_i \in \left[ \omega_i^L, \omega_i^U \right], \left[ \omega_i^L, \omega_i^M, \omega_i^U \right], \left[ \omega_i^L, \omega_i^M, \omega_i^N, \omega_i^U \right], \quad t = 1, 2, \ldots, m
\]

(8)

The following relationship is generally established:

\[
\sum_{t=1}^{m} \omega_i^L \leq 1
\]

(9)

\[
\sum_{t=1}^{m} \omega_i^U \geq 1
\]

where \( \omega_i^L \) and \( \omega_i^U \) represent the lower and upper limits of an uncertain number, respectively, and the real number is the deterioration of the interval number. For example, if the parameters \( \omega_i^L \) and \( \omega_i^U \) of an interval number are equal with each other, the uncertain number will be a real number.

#### B. STEPS OF OBTAINING THE CERTAIN WEIGHTS

1) Transformation of attribute weight values into binary connection numbers

First of all, the basic parameters of each type of weight data including mean values, standard deviations and maximum deviations can be calculated by Eqs. (4) to (7). The weight values of various types of data can be converted into binary connection numbers using Eq. (3), and thus the two-tuple(determinacy, uncertainty) numbers of weights can be obtained. And the weight vector can be written as \( (a_1, b_1, a_2, b_2, \ldots, a_m, b_m)^T \).

2) Calculation of baseline value of the weight two-tuple (determinacy, uncertainty) number

According to the weight two-tuple (determinacy, uncertainty) numbers, the maximum and minimum values of the determinacy and uncertainty of each two-tuple number can be obtained using Eq. (10).

\[
w_j = \left\{ \begin{array}{l}
\frac{\alpha_j \exp \left\{ \frac{a_j - a_{\min}}{a_{\max} - a_{\min}} \right\}}{\sqrt{\omega_j^2 + b_j^2}}, \quad b_j \neq 0 \\
\frac{\exp \left\{ \frac{a_j - a_{\min}}{a_{\max} - a_{\min}} \right\}}{\sqrt{\omega_j^2 + b_j^2}}, \quad b_j = 0 \\
\end{array} \right.
\]

(11)

where \( \alpha_j \) is the contribution ratio of deterministic term of the weight two-tuple (determinacy, uncertainty) number for calculating the weight \( w_j \), and \( \beta_j \) is the contribution ratio of uncertain term; \( b_{\min} \) is the minimum value of uncertain term.

Eq. (11) indicates that the deterministic weight depends on the deterministic and uncertain parts of the two-tuple (determinacy, uncertainty) number for \( b_j \neq 0 \); otherwise \( w_j \) simply relies on the deterministic part.

In Eq. (11), \( \alpha_j \) can be determined as required. However, \( \alpha_j \) and \( \beta_j \) can also be determined by the uncertain number itself regarding the information contained therein. So, the improved function can be given as follows.

\[
w_j = \left\{ \begin{array}{l}
\frac{\alpha_j}{\sqrt{a_j^2 + b_j^2}} \exp \left\{ \frac{a_j - a_{\min}}{a_{\max} - a_{\min}} \right\}, \quad b_j \neq 0 \\
\frac{b_j}{\sqrt{a_j^2 + b_j^2}} \exp \left\{ \frac{b_{\max} - b_j}{b_{\max} - b_{\min}} \right\}, \quad b_j = 0 \\
\end{array} \right.
\]

(12)

where \( \frac{a_j}{\sqrt{a_j^2 + b_j^2}} \) and \( \frac{b_j}{\sqrt{a_j^2 + b_j^2}} \) originated from a binary connection number denote the contribution of deterministic term.
and that of the uncertain term in a mixed attribute weight respectively.

4) Index weight normalization

The index weight obtained in step (3) is unnormalized, which is different from that the sum of the index weights is one. Therefore, the normalized weights can be obtained by using Eq. (15).

IV. KULLBACK-LEIBLER DISTANCE BASED GGTDM WITH INDEX AND WEIGHT BOTH CONTAINING MIXED ATTRIBUTE VALUES

Let \( T = \{ T_1, T_2, \cdots, T_n \} \), \( Z = \{ Z_1, Z_2, \cdots, Z_m \} \) and \( W = (w_1, w_2, \cdots, w_m)^T \) be alternative set, attribute set and weight vector of index attributes, respectively, then the index value of alternative \( T_i \) under attribute \( Z_t \) is \( v_{st} (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \).

A. TRANSFORMATION OF ALTERNATIVE INDICES INTO BINARY CONNECTION NUMBERS

Different types of index values can be converted into binary connection numbers \( \mu + \sigma i \) using Eqs. (4) to (7). The form \( \mu + 0i \) means that the deterministic term is the real number itself and the uncertain term is 0. The converted index number can be expressed as \( V_{st} = \mu_{st} + \sigma_{st} i (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \).

B. DETERMINATION OF THE TARGET CENTER INDICES

The binary connection numbers \( V_{st} = \mu_{st} + \sigma_{st} i (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \) that converted from all indices can also be expressed as two-tuple numbers \( U_{st} = (\mu_{st}, \sigma_{st}) (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \). The benefit-type set and cost-type set can be denoted by \( J^+ \) and \( J^- \), respectively. Then the two-tuple (determinacy, uncertainty) numbers of target center can be obtained using Eq.(13).

\[
C_t^+ = \begin{cases} 
\{ \max \{ \mu_{st} \}, \min \{ \sigma_{st} \} \}, & U_{st} \in J^+ \\
\{ \min \{ \mu_{st} \}, \min \{ \sigma_{st} \} \}, & U_{st} \in J^- 
\end{cases},
\]

\( s = 1, 2, \cdots, n; t = 1, 2, \cdots, m \) (13)

C. NORMALIZATION OF ALL ALTERNATIVE INDICES

The index two-tuple numbers of all alternatives \( U_{st} = (\mu_{st}, \sigma_{st}) (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \) and target center indices \( C_t = ((\mu_c, \sigma_c)) (c = n + 1; t = 1, 2, \cdots, m) \) can be expressed as two-tuple (deterministic degree, uncertainty degree) numbers:

\[
a_{st} = \frac{\mu_{st}}{\mu_{st} + \sigma_{st}}, \quad b_{st} = \frac{\sigma_{st}}{\mu_{st} + \sigma_{st}}, \quad (s = 1, 2, \cdots, n; t = 1, 2, \cdots, m) \] (14)

In Eq.(14), \( a_{st} \) and \( b_{st} \) denote the deterministic degree and uncertain degree under the same attribute in the normalized binary connection number. Then the alternative vector of two-tuple (deterministic degree, uncertainty degree) numbers can be given as \( (a_{s1}, b_{s1}), (a_{s2}, b_{s2}), \ldots, (a_{sm}, b_{sm}) \). The values \( a_{st} \) and \( b_{st} \) in a two-tuple (determinacy, uncertainty) number should be normalized as they are incomparable under different attributes. The normalized equation is as follows.

\[
a'_{st} = \frac{a_{st}}{\sum_{s=1}^{n} a_{st}}, \quad b'_{st} = \frac{b_{st}}{\sum_{s=1}^{n} b_{st}}, \quad s = 1 \ldots n; \quad t = 1 \ldots m \] (15)

where \( a'_{st} \) and \( b'_{st} \) are the normalized deterministic and uncertain terms of the two-tuple number, respectively.

D. CALCULATION OF COMPREHENSIVE WEIGHTED KULLBACK-LEIBLER DISTANCE

Having obtained the vector constituted by two-tuple numbers of alternatives and target center, then the comprehensive weighted K-L distance can be obtained.

Definition 6: Comprehensive weighted Kullback-Leibler distance (CWKL). Suppose that \( S = ((x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m))^T \) and \( E = ((p_1, q_1), (p_2, q_2), \ldots, (p_m, q_m))^T \) are two vectors of two-tuple numbers, and let \( W = (w_1, w_2, \cdots, w_m)^T \) be the weight vector, then the comprehensive weighted Kullback-Leibler distance of \( S \) and \( E \) can be obtained.

\[
H_W(S, E) = \sum_{j=1}^{m} w_j \left( x_j \left| \ln \frac{x_j}{p_j} \right| + y_j \left| \ln \frac{y_j}{q_j} \right| \right) \] (16)

The properties of \( H_W(S, E) \) are as follows:

1) \( H_W(S, E) \geq 0 \);
2) \( H_W(S, E) = 0 \), when and only when, \( S = E \), which means \( x_j = p_j, y_j = q_j, \forall j \);
3) When \( x_j = p_j = 0 \) or \( y_j = q_j = 0 \), then by definition \( x_j \ln \frac{x_j}{p_j} = 0 \) or \( y_j \ln \frac{y_j}{q_j} = 0 \).

The situations \( x_j \neq 0 \), \( p_j = 0 \) and \( y_j \neq 0 \), \( q_j = 0 \) may occur in Eq.(16). In order to avoid this, the modified equation can be given as follows.

\[
K_W(S, E) = \sum_{j=1}^{m} w_j \left( x_j \left| \ln \frac{x_j}{p_j + \frac{x_j}{2}} \right| + y_j \left| \ln \frac{y_j}{q_j + \frac{y_j}{2}} \right| \right) \] (17)

E. DECISION-MAKING STEPS

The algorithm of GGTDM is as follows.

1) All alternative indices and weight values are converted into binary connection numbers by Eqs.(4) to (7), and the results can be transformed into two-tuple (determinacy, uncertainty) numbers.
2) The two-tuple (determinacy, uncertainty) numbers of target center under all attributes can be obtained by Eq.(13).
3) The two-tuple (determinacy, uncertainty) numbers of all indices and target center can be normalized by Eq.(14), then the two-tuple (deterministic degree, uncertainty degree) numbers of all attributes can also be normalized by Eq.(15).
4) The deterministic weights can be obtained by Eq.(10) and Eq.(12).
V. CASE STUDY

A. DESCRIPTION OF DECISION PROBLEM

To evaluate a weapon system, six indices including mobility (km h$^{-1}$), reliability, accuracy (km), maintainability, warhead payload (kg) and price (10$^6$ g) denoted by $Z_1$ to $Z_6$ are considered. Among these attributes $Z_3$ and $Z_6$ are cost-type indices and the others are benefit-type indices. The attribute weights are given as $W = (0.1, [0.14, 0.18], [0.16, 0.18, 0.2], [0.12, 0.16], [0.18, 0.2, 0.22], [0.18, 0.2, 0.22, 0.24])$, and there are four feasible alternatives denoted by $T_1$ to $T_4$. The data is shown in Table 1 [26].

B. DECISION PROCESS

1) Calculation of the parameters of binary connection numbers for all alternative indices

The parameters of binary connection numbers of all alternatives can be calculated by the data shown in Table 1 using Eqs. (4) to (7). The results are shown in Table 2.

2) Translation of all index values into binary connection numbers

All index values can be transformed into binary connection numbers using Eqs. (3) to (7) based on the data shown in Table 2. The results are shown in Table 3.

3) Obtain the two-tuple (determinacy, uncertainty) numbers of target center

The two-tuple (determinacy, uncertainty) numbers of target center are calculated as $((55.5, 0.5), (0.7, 0.1), (1.8, 0), (0.9, 0.1), (540, 0), (4.7, 0.5))$ using Eq. (13).

4) Normalization of the two-tuple (determinacy, uncertainty) numbers of all alternatives and target center

The two-tuple (determinacy, uncertainty) numbers of all alternative indices can be obtained from Table 3, the results are shown in Table 4.

The two-tuple numbers of alternative indices and target center indices can be normalized using Eq. (15), and the results are shown in Table 5.

5) Determination of the weight of each index attribute

Given the mixed attribute weights $W = (0.1, [0.14, 0.18], [0.16, 0.18, 0.2], [0.12, 0.16], [0.18, 0.2, 0.22], [0.18, 0.2, 0.22, 0.24])$, the parameters of all index weights can be calculated by Eqs. (3) to (7), the results are shown in Table 6.

Next, the weight values can be expressed as binary connection numbers, the results are shown in Table 7.

The maximum and minimum values of the determined terms in weight two-tuple (determinacy, uncertainty) numbers are calculated as 0.21 and 0.1 using Eq. (10); and the maximum and minimum values of the uncertain term of that are 0.0258 and 0.02 respectively. Then the certain weights can be calculated as (1.0000, 2.0492, 2.3570, 1.8085, 2.7402, 2.8199) using Eq. (16).
Finally, the original weights are normalized as \( W_F = (0.0783, 0.1604, 0.1845, 0.1416, 0.2145, 0.2207) \).

6) Obtain the comprehensive weighted K-L distance between each alternative and target center

Having obtained the certain weight vector \( W_F = (0.0783, 0.1604, 0.1845, 0.1416, 0.2145, 0.2207) \), the comprehensive weighted K-L distances of all alternatives and target center can be determined as \( I_{CWL} = (0.1210, 0.1679, 0.0681, 0.0269) \). The ranking result of alternatives is \( T_4 \succ T_3 \succ T_1 \succ T_2 \).

### C. COMPARISON AND ANALYSIS

1) COMPARISON OF METHODS FOR OBTAINING THE CERTAIN WEIGHTS

To verify the weight function method transforming mixed attribute weights into certain weights, the proximity-based method discussed in [12] and the module-based method presented in [21] are used to make a comparison. Given the weight vector \( W = (0.1, 0.14, 0.18, 0.16, 0.18, 0.16, 0.18, 0.2, 0.12, 0.16, 0.2, 0.22, 0.18, 0.2, 0.22, 0.24) \), the certain weights can be calculated as \( W_P = (0.101, 0.162, 0.182, 0.141, 0.202, 0.212) \) by proximity-based method. While using the module-based method, the certain weights can be obtained as \( W_M = (0.1004, 0.1618, 0.1818, 0.1419, 0.2017, 0.2124) \) with the same mixed attribute weights. The certain weights of all attributes determined by different methods are shown in Table 8 and Figure 1. Seen from Table 8, the certain weights determined by three methods have some differences under all attributes. It should be noted that the result by proximity-based method is almost the same as that by the module-based method. But the result determined by the weight function method is obviously different from the results calculated by the other two methods. Seen from Figure 1, the two curves representing the results that determined by the proximity-based method and that by the module-based method almost overlap with each other, while the curve denoting the result that obtained by the weight function method can be discriminated easily.

![FIGURE 1. Comparison of certain weights determined by different methods.](image-url)
TABLE 8. Comparison of certain weights determined by different methods.

| Method                  | $Z_1$   | $Z_2$   | $Z_3$   | $Z_4$   | $Z_5$   | $Z_6$   |
|-------------------------|---------|---------|---------|---------|---------|---------|
| Weight function method  | 0.0783  | 0.1604  | 0.1845  | 0.1416  | 0.2145  | 0.2207  |
| Proximity-based method  | 0.1010  | 0.1620  | 0.1820  | 0.1410  | 0.2020  | 0.2120  |
| Module-based method     | 0.1004  | 0.1618  | 0.1818  | 0.1419  | 0.2017  | 0.2124  |

TABLE 9. Comparison of the decision-making results by different methods.

| $T_i$ | Proposed method | Proximity-based method-1 | Proximity-based method-2 |
|-------|-----------------|--------------------------|--------------------------|
|       | $I_{CW1}$ | Rank | $I_{CWP1}$ | Rank | $I_{CWP2}$ | Rank |
| $T_1$ | 0.1210 | 3    | 0.2681 | 3    | 0.2695 | 3    |
| $T_2$ | 0.1679 | 4    | 0.2894 | 4    | 0.2928 | 4    |
| $T_3$ | 0.0681 | 2    | 0.2405 | 2    | 0.2354 | 2    |
| $T_4$ | 0.0269 | 1    | 0.2021 | 1    | 0.2023 | 1    |

FIGURE 2. Comparison of the results by different methods.

2) COMPARISON OF DECISION-MAKING RESULTS

The comprehensive weighted proximity-based method is adopted to make a comparison with the proposed approach. In previous research, the CWP based GGTDM with index and weight both containing mixed attribute values has two ways to fulfill the decision-making task: one way is to obtain the certain weights by proximity-based method and adopt the CWP as the DMB [12]; the other is to arrive at the certain weights by module-based method and also take the CWP as the DMB [21]. The former way could be called proximity-based method-1, while the latter could be referred to as proximity-based method-2.

Having obtained the certain weights $W_P = (0.101, 0.162, 0.182, 0.141, 0.202, 0.212)$ by the proximity-based method, the comprehensive weighted proximity vector can be calculated as $I_{CWP1} = (0.2681, 0.2894, 0.2405, 0.2021)$. According to the principle the smaller the proximity the better the alternative, the alternatives ranking by proximity-based method-1 is: $T_3 > T_1 > T_2$. Given the certain weights $W_M = (0.1004, 0.1618, 0.1818, 0.1419, 0.2017, 0.2124)$ determined by module-based method, the comprehensive weighted proximity vector can be obtained as $I_{CWP2} = (0.2695, 0.2928, 0.2354, 0.2023)$ by proximity-based method-2. Thus the alternatives are ranked as: $T_4 > T_3 > T_1 > T_2$. It is obvious that the decision-making result determined by the proposed approach is completely in accordance with the results that by proximity-based method-1 and proximity-based method-2. The decision-making results are listed in Table 9 and shown in Figure 2.

VI. CONCLUSION

This work proposes a novel GGTDM with index and weight both involving mixed attribute values. The novelty of this paper lies in two aspects: the improved K-L distance is adopted as the DMB, and the weight function is built to obtain the certain weights. The principle of the proposed method is that the comprehensive weighted K-L distance reflecting the uncertain measure between the two vectors is applied to fulfill the decision making. Furthermore, the weight function is constructed to transform mixed attribute weights into deterministic weights, which considers the information of mixed attribute values. The proposed method is verified in ranking alternatives effectiveness and accuracy compared with other methods. In the future, more suitable DMB of GGTDM and mixed attribute weights determining method should be investigated to support the decision making accurately and effectively.

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