Transfer reactions between odd-odd and even-even nuclei by using IBFFM

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Spectroscopic Amplitudes (SA) in the Interacting Boson Fermion Fermion Model (IBFFM) are necessary for the computation of $0\nu\beta\beta$ decays but also for cross sections of heavy-ion reactions, in particular, Double Charge Exchange reactions for the NUMEN collaboration, if one does not want to use the closure limit. We present for the first time: i) the formalism and operators to compute in a general case the spectroscopic amplitudes in the scheme IBFFM from an even-even to odd-odd nuclei, in a way suited to be used in reaction code, i.e., extracting the contribution of each orbital; 2) the odd-odd nuclei as described by the old IBFFM are obtained for the first time with the new implementation of Machine Learning (ML) techniques for fitting the parameters, getting a more realistic description. The one body transition densities for $^{116}\text{Cd} \rightarrow ^{116}\text{In}$ and $^{116}\text{In} \rightarrow ^{116}\text{Sn}$ are part of the experimental program of the NUMEN experiment, which aims to find constraints on Neutrinoless double beta decay matrix elements.

In this article, the SA operator in the scheme of IBFFM was extended to describe odd-odd nuclei, in which the two fermions (one neutron and one proton) are coupled to the even-even core described either by IBM-1 [11–13], or by IBM-2 [14]. These extensions to odd-odd nuclei are usually referred to as the Interacting Boson-Fermion-Fermion Model, IBFFM-1 and IBFFM-2, respectively. In this article we use the neutron-proton version which we will use the shorthand notation IBFFM from here on (strictly speaking it is IBFFM-2).

The IBFFM Hamiltonian is derived from semi-microscopic arguments and subsequently diagonalized numerically. It has been introduced by Lopac and Bianco [11, 12], and it is essentially an IBFM-1 in the proton, and an IBFM-1 in the neutron plus a residual interaction as introduced by Brant and Paar [13]. The original approach was extended to include the neutron-proton degree of freedom in the description of the even-even core nucleus in Ref. [13]. In this reference Yoshida and Iachello and Yoshida wrote the operator expression for single beta decay and two neutrino double beta decay, which can be considered as a particular subset of the general one-body transition operator presented in this paper.

In this paper the aim is from one side to investigate the transition from $^{116}\text{Cd}$ to $^{116}\text{In}$, and $^{116}\text{Sn}$ to $^{116}\text{In}$ and introduce the formalism of the Spectroscopic Amplitudes (SA) between even-even and odd-odd nuclei in the IBFFM that are the key ingredient for planned publications of Double Charge Exchange (DCE) reactions for the NUMEN collaboration and the study $0\nu\beta\beta$ decay.

The spectroscopic amplitudes operator dressed as a charge exchange or as simple transition amplitudes is needed for the theoretical NUMEN project Ref. [16, 17] to be inserted into a reaction code.

For the NUMEN experiment at INFN-LNS in particular there is interest for $^{116}\text{Cd} \rightarrow ^{116}\text{In}$ and $^{116}\text{In} \rightarrow ^{116}\text{Sn}$ [18].

In this article, the SA operator in the scheme of IBFFM has been derived in a general way and then calculated for the previous reactions and the results given in the form of spectroscopic amplitudes, which later on can be applied to double charge exchange, $0\nu\beta\beta$ decay processes without closure approximation. We introduce a new method to fit the odd-odd nuclei parameters by creating a multidimensional theoretical dataset using machine learning libraries such as Scikit-learn. The operators needed to compute the transitions between odd-odd and even-even nuclei by using IBFFM depends on two factors: the similarity between the states in the initial and final nucleus, which differ by two nucleons, and the transferred pair of nucleons’ correlation. We have derived the two different transition operators’ cases, the case where the numbers of bosons are conserved between the even-even

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and odd-odd nuclei and the one where is not conserved, respectively.

The organization of this paper is the following: In Sec. I we briefly review a simple theory of the one body transitions density. In Sec. II we present the derivation of the transition operator for the IBFFM. In Sec. III we discuss the theory of the odd-odd nucleus in the implemented IBFFM with Machine Learning (ML). In Sec. IV we compute the nuclear wave functions and spectroscopic amplitudes for the transitions $^{116}$In and $^{116}$Cd to $^{116}$Sn. In our last Sec. V we present our conclusions and discuss our results, also we give the future outlook.

I. SPECTROSCOPIC AMPLITUDES

The microscopic theory for direct charge exchange process by heavy ions following the approach by Greiner [19], and Etchegoyen [20] requires the One Body Transition Densities (OBTD) that are needed in the form factors of the direct charge exchange reactions. Nuclear initial and final states can be presented in the proton-neutron formalism, the one body transition densities without the isospin indices can be represented by

\[
\text{OBTD}(AB; \lambda) = \frac{\langle J_B \| [a^+_\rho \times a^-_{\rho'}]^\lambda \| J_A \rangle}{\sqrt{(2\lambda + 1)}},
\]

where $|J_A\rangle$ represents the vector of the state of the initial nucleus and $|J_B\rangle$ the final nucleus, the subindex $\rho (\rho')$ refers to a proton(neutron), the tilde denotes a time-conjugated state $\tilde{a}_{\rho j} = (-1)^{1-m} a_{\rho' j-m}$, we define the Spectroscopic Amplitudes (SA) as the reduced matrix element $\langle J_B \| [a^+_\rho \times a^-_{\rho'}]^\lambda \| J_A \rangle$.

In the isospin formalism, the nuclear states have certain isospin, which results in a non-trivial isospin factor. The IBFFM can be mapped into the proton-neutron formalism allowing to derive the one body transition densities for the bosonic-fermionic space. In the following sections, we present the derivation of the OBTD operator for the IBFFM framework, and we study the transitions $^{116}$Sn to the $^{116}$In, and $^{116}$Cd to the $^{116}$Cd that are of interest for the NUMEN experiment.

II. TRANSITION OPERATOR

The one body transition density operator from even-even to odd-odd in the IBFFM formalism can be obtained considering the mapping from the fermionic space into the boson-fermion-fermion space of the tensor operator of order $\lambda$ given by

\[
[a^+_\rho \times \tilde{a}^-_{\rho'}]^\lambda \rightarrow [P^\dagger_{\rho'} \times \tilde{P}_{\rho''}]^\lambda = T^{(\lambda)}_{j_\rho j_\rho''},
\]

where $a^+_\rho$ and $\tilde{a}^-_{\rho'}$ are the fermion creation and annihilation operators, while $P^\dagger_{\rho'}$ and $\tilde{P}_{\rho''}$ are the single nucleon transfer operator in the IBFM [21] scheme.

The operator for one nucleon transfer which the number of bosons is conserved is given as

\[
P^\dagger_{j_\rho} = \xi_{j_\rho} a^+_\rho + \sum_{j_\rho''} \xi_{j_\rho j_\rho''} (|s^\dagger \times \tilde{d}_{j_\rho''}|^2) \times a^+_\rho (|j_\rho''\rangle),
\]

where $s^\dagger$ is the creation operator, and $s = \tilde{s}$ annihilation operator of $s$-boson respectively, and $d$ is related to the $d$-boson annihilation operator by $\tilde{d}_\mu = (-1)^{1+\mu} d_{-\mu}$.

In case the number of bosons is changed by one unit

\[
P^\dagger_{j_\rho} = \theta_{j_\rho} (s^\dagger \times \tilde{a}_{j_\rho''})(|j_\rho''\rangle) + \sum_{j_\rho''} \theta_{j_\rho j_\rho''} (d^\dagger \times \tilde{a}_{j_\rho''})(|j_\rho''\rangle),
\]

where the subindex refers to a proton or a neutron occupying the orbitals $j_\rho$ and $j_\rho''$, respectively. The coefficients $\xi_{j_\rho}, \xi_{j_\rho j_\rho''}$ and $\theta_{j_\rho}, \theta_{j_\rho j_\rho''}$ are the particle- or hole-coupling coefficients defined in Ref. [21, 22].

The transitions between even-even nuclei and odd-odd nuclei can be computed by considering the tensorial product of the transfer operator of a particle and a hole coupled to the angular momenta $\lambda$, which is the value of the spin of the final state of the odd-odd nucleus. The case where the number of bosons $N_\nu, N_\pi$ between the even-even nucleus and the odd-odd core nucleus does not change, and the core of the odd-odd nucleus has attached a proton and removed a neutron the operator is written as

\[
T^{(\lambda)}_{j_\rho j_\rho''} = \xi_{j_\rho} \xi_{j_\rho''} (\tilde{a}_{j_\rho''} - a^+_\rho) (|j_\rho''\rangle),
\]

\[
+ \sum_{j_\rho''} \xi_{j_\rho j_\rho''} (s^\dagger \times \tilde{d}_{j_\rho''})(|j_\rho''\rangle) \times (a^+_\rho - \tilde{a}^-_{\rho'})(|j_\rho''\rangle),
\]

\[
+ \sum_{j_\rho''} \xi_{j_\rho j_\rho''} (d^\dagger \times \tilde{a}_{j_\rho''})(|j_\rho''\rangle) \times (\tilde{a}_{j_\rho''} - a^+_\rho)(|j_\rho''\rangle),
\]

\[
\times (|s^\dagger \times \tilde{d}_{j_\rho''}|^2) \times (\tilde{a}_{j_\rho''} - a^+_\rho)(|j_\rho''\rangle).
\]

For reactions where the number of bosons is changed by one, and the core of the odd-odd nucleus is added one neutron and removed one proton,

\[
T^{(\lambda)}_{j_\rho j_\rho''} = \theta_{j_\rho} \theta_{j_\rho''} (s^\dagger \times \tilde{a}_{j_\rho''})(|j_\rho''\rangle) \times (\tilde{a}_{j_\rho''} - a^+_\rho)(|j_\rho''\rangle),
\]

\[
+ \sum_{j_\rho''} \theta_{j_\rho j_\rho''} (s^\dagger \times \tilde{d}_{j_\rho''})(|j_\rho''\rangle) \times (\tilde{a}^-_{\rho'} - a^+_\rho)(|j_\rho''\rangle),
\]

\[
+ \sum_{j_\rho''} \theta_{j_\rho j_\rho''} (d^\dagger \times \tilde{a}_{j_\rho''})(|j_\rho''\rangle) \times (\tilde{a}_{j_\rho''} - a^+_\rho)(|j_\rho''\rangle),
\]

\[
\times (\tilde{a}_{j_\rho''} - a^+_\rho)(|j_\rho''\rangle).
\]
We proceed to compute the matrix elements of the operators of Eq. 5 and Eq. 6. For numerical calculations the \(^{116}\)Cd towards \(^{116}\)In we require to use the operator of Eq. 5 and for the \(^{116}\)Cd toward \(^{116}\)Sn, we use the operator of Eq. 6.

In the shell model, the eigenstates can be expressed in terms of the product of the proton and neutron spaces

\[ |\Phi_{JM}^{(A)}\rangle = \alpha_{\pi} J_{\pi} \alpha_{\nu} J_{\nu}; JM \]  

(7)

The relationship between nuclear matrix elements of the shell model and the IBFFM is given by

\[ \langle \Phi_{J'\nu}^{(A')} (a_{\nu}^{\dagger} \times \hat{a}_{\nu^*}) | \Phi_{J'' \nu}^{(A)} \rangle = \langle \Phi_{J'\nu}^{(A')} | T_{J' j_{\nu}}^{(\lambda)} | \Phi_{J'' \nu}^{(A)} \rangle, \]  

(8)

where \( \Psi_{J}^{(A)} \) is the even-even nuclear wave function and \( \Psi_{J'}^{(A')} \) the odd-odd nuclear wave function in the IBFFM which we will discuss in the next section.

### III. DESCRIPTION OF ODD-ODD NUCLEI

In the IBFFM, odd-odd nuclei are described in terms of a system of \( N_{\pi} \) proton bosons \( \{s_{\pi} \text{ and } d_{\pi}\} \) and \( N_{\nu} \) neutron bosons \( \{s_{\nu} \text{ and } d_{\nu}\} \) coupled to a single proton \( \langle j_{\pi} \rangle \) and a single neutron \( \langle j_{\nu} \rangle \). The proton and neutron orbitals are those of the active major shell for protons and neutrons, respectively.

The Hamiltonian of the IBFFM for the odd-odd nuclei can be written as

\[ H = H^B + H^F + V^B_\pi + H^F_\pi + V^B_\nu + V^F_\nu + V_{res} \]  

(9)

\[ H^B = \sum_{j_{\pi}} \epsilon_{j_{\pi}} \hat{n}_{j_{\pi}}, \]  

(10)

where \( \epsilon_{j_{\pi}} \) is the quasi-particle energy of the extra nucleon, and \( \hat{n} \) is the number operator. The quasi-particle energies \( \epsilon_{j_{\pi}} \) are obtained within the BCS approximation with a gap \( \Delta = 12 / \sqrt{A} \), where \( A \) is the mass number of the even-even core nucleus. In the BCS, the quasi-particle energy is related to the single-particle level \( \epsilon^{sp} \), the occupation probabilities \( v_{j} \), and the fermi level \( \lambda \) as follows:

\[ \epsilon_{j} = \sqrt{(\epsilon^{sp}_{j} - \lambda)^2 + \Delta^2}, \]  

\[ v_{j}^2 = \frac{1}{2} \left( 1 - \frac{\epsilon^{sp}_{j} - \lambda}{\epsilon_{j}} \right), \]  

\[ u_{j}^2 = 1 - v_{j}^2. \]  

(11)

\( V^B_\pi \) and \( V^B_\nu \) describe the core-particle coupling of the odd proton and odd neutron in the IBFM-2 model \cite{21, 22} as the sum of a quadrupole term \((\Gamma_{\rho})\), an exchange term \((\Lambda_{\rho})\) and a monopole term \((\Lambda_{\rho})\)

\[ V^B_\pi = \Gamma_{\rho} Q^{(2)}_{\rho} + q^{(2)}_\pi + \Lambda_{\rho} F_{\rho} + \Lambda_{\rho} n_{d_{\rho}} \cdot \hat{n}_{\rho} \]  

(12)

where \( \rho' \neq \rho \) and \( \rho, \rho' = \nu, \pi \). The first term in Eq. 12 is a quadrupole-quadrupole interaction with

\[ q^{(2)}_\rho = \sum_{j_{\nu}, j_{\nu}'} (u_{j_{\nu}} u_{j_{\nu}'} - v_{j_{\nu}} v_{j_{\nu}'}) Q_{j_{\nu}, j_{\nu}'} (a_{j_{\nu}}^{\dagger} \times a_{j_{\nu}'}^{\dagger})^{(2)} \]  

\[ Q^{(2)}_\rho = (s_{\rho}^{\dagger} \times \hat{d}_{\rho} + d_{\rho}^{\dagger} \times \hat{s}_{\rho})^{(2)} + \chi_{\rho}' (d_{\rho}^{\dagger} \times \hat{d}_{\rho})^{(2)}. \]  

(13)

The second term is the exchange interaction

\[ F_{\rho, \rho'} = - \sum_{j_{\nu}, j_{\nu}'} \beta_{j_{\nu} j_{\nu}'} \tilde{\beta}_{j_{\nu} j_{\nu}'} \]  

\[ Q_{j_{\nu}, j_{\nu}'} = \langle \rho || Y^{(2)} || \rho' \rangle. \]  

(15)

The last term is the monopole-monopole interaction with

\[ n_{d_{\rho}} = \sum_{m} d_{\rho, m}^{\dagger} d_{\rho, m}, \]  

\[ \hat{n}_{\rho} = \sum_{j_{\nu}, m} \hat{a}_{j_{\nu}, m}^{\dagger} \hat{a}_{j_{\nu}, m} \]  

(16)

The residual interaction between the odd-proton and odd-neutron is defined as \cite{13, 14}

\[ V_{res} = H_{\delta} + H_{\sigma \delta} + H_{\sigma \sigma} + H_{T}, \]  

(17)

with

\[ H_{\delta} = 4 \pi V_{\delta} \delta(\bar{r}_{\pi} - \bar{r}_{\nu}) \delta(r_{\pi} - R_{0}) \delta(r_{\nu} - R_{0}), \]  

\[ H_{\sigma \delta} = 4 \pi V_{\sigma \delta} \delta(\bar{r}_{\pi} - \bar{r}_{\nu})(\bar{\sigma}_{\pi} \cdot \bar{\sigma}_{\nu}), \]  

\[ \times \delta(r_{\pi} - R_{0}) \delta(r_{\nu} - R_{0}), \]  

\[ H_{\sigma \sigma} = - \sqrt{3} V_{\sigma \sigma} \bar{\sigma}_{\pi} \cdot \bar{\sigma}_{\nu}, \]  

\[ H_{T} = V_{T} \left[ 3(\bar{\sigma}_{\pi} \cdot \bar{r}_{\pi})(\bar{\sigma}_{\nu} \cdot \bar{r}_{\nu}) - \bar{\sigma}_{\pi} \cdot \bar{\sigma}_{\nu} \right], \]  

(18)

where \( \bar{r}_{\pi \nu} = \bar{r}_{\pi} - \bar{r}_{\nu} \) and \( R_{0} = 1.2A^{1/3}\text{fm} \). The matrix elements of the residual interaction are calculated in the quasi-particle basis which is related to the particle basis by \cite{14}

\[ \langle j_{\nu}', j_{\pi}' ; J | V_{res} | j_{\nu}, j_{\pi}; J \rangle = \]  

\[ = (u_{j_{\nu}'}, v_{j_{\pi}'}, u_{j_{\nu}} v_{j_{\pi}}) \times (j_{\nu}', j_{\pi}') \right| \left( j_{\nu}, j_{\pi}; J \right) \]  

\[ - (u_{j_{\nu}'}, v_{j_{\pi}'}, u_{j_{\nu}} v_{j_{\pi}}) \times (j_{\nu}', j_{\pi}') \right| \left( j_{\nu}, j_{\pi}; J \right) \]  

\[ \sum_{J} (2J + 1) \left\{ J_{\nu}' J_{\pi}' J_{\nu} J_{\pi} J \right\} \times (j_{\nu}', j_{\pi}'; J | V_{RES} | j_{\nu}, j_{\pi}; J') \]  

(19)

where \( v_{j}^2 = 1 - u_{j}^2 \) denotes the occupation probability.
IV. NUMERICAL RESULTS

In the context of the IBFFM, the odd-odd nucleus is described by coupling a proton and a neutron to its core (even-even nucleus). Thus the first step is to construct the core nucleus. It is described by the IBM-2 Hamiltonian \( H^B \) \[26\], which only depends on the neutron- and proton-boson degrees of freedom, the expression of the boson Hamiltonian \( H^B \) is taken as in Ref. [14].

To construct the odd-odd wave function of \(^{116}\)In, we consider \(^{116}\)Cd as the core nucleus. The parameters for the core nucleus are taken from the literature [25]. They are shown in Table I. The spectrum of \(^{116}\)Cd obtained in our calculation is presented in Fig. 1 compared to the available experimental data [27]. We can see that the agreement is quite good [28].

The second step is to construct the two odd-even associated nuclei, i.e. the core nucleus plus an extra neutron and the core nucleus plus an extra proton. It allows us to get a reliable set of parameters that we use to construct the odd-odd wave function. In this case, the odd-even nuclei are described in the context of the IBFM-2 \[22–24\] where the degrees of freedom of the extra nucleon are taken into account. In the IBFM-2, the Hamiltonian is given by

\[
H = H^B + H^F + V_{BF}^\rho, \tag{20}
\]

where \( H^B \) is the boson Hamiltonian that describes the core nucleus, the label \( \rho \) refers to the \( \pi \) (extra proton) or \( \nu \) (extra neutron) is added in the even-even core to form the odd-even nucleus.

The single-particle energies \( \epsilon^{sp} \) are obtained by solving the Schrödinger equation with the phenomenological Woods-Saxon potential based upon the sum of a spin-independent central potential, a spin-orbit potential, and the Coulomb potential [29],

\[
V_{os}(r) = V_o(r) + V_{so}(r)\hat{l} \cdot \hat{s} + V_c(r), \tag{21}
\]

where \( V_o(r) \) is the spin-independent central potential:

\[
V_o(r) = V_o f_o(r), \tag{22}
\]

with a fermi shape

\[
f_o(r) = \frac{1}{1 + \exp(10^2 - R_0/a_0)} \tag{23}
\]

\( V_{so}(r) \) is the spin-orbit potential:

\[
V_{so}(r) = V_{so} \frac{1}{r} \frac{df_{so}(r)}{dr}, \tag{24}
\]

with

\[
f_{so}(r) = \frac{1}{1 + \exp(10^2 - R_{so}/a_0)} \tag{25}
\]

and \( V_c(r) \) is the Coulomb potential for protons based upon the Coulomb potential of a sphere of radius \( R_c \):

\[
V_c(r) = \frac{Ze^2}{r} \text{ for } r \geq R_c, \tag{26}
\]

and

\[
V_c(r) = \frac{Ze^2}{2R_c} \left[ \frac{3}{2} - \frac{r^2}{2R_c^2} \right] \text{ for } r \leq R_c. \tag{27}
\]

The radii \( R_0, R_{so}, \) and \( R_c \) are usually expressed as:

\[
R_i = r_i A^{1/3}. \tag{28}
\]

since the average proton-neutron potential is stronger than the average neutron-neutron (or proton-proton) potential, thus in a nuclei with a neutron excess, the
protons will feel a stronger potential than the neutrons, for this reason we take

\[ V_{op} = V_0 + \frac{N - Z}{A} V_1 \]  

for protons, \( (29) \) and

\[ V_{on} = V_0 - \frac{N - Z}{A} V_1 \]  

for neutrons. \( (30) \)

The single particle energies are obtained by solving the Woods-Saxon with a typical set of parameters with \( V_0 = -53 \) MeV, \( V_1 = -30 \) MeV and \( V_0 = 22 \) MeV for the strengths, and \( r_0 = r_{ws} = 1.29 \) fm and \( a_0 = a_{ws} = 0.70 \) fm for the geometry. For the Coulomb term the radius is a little smaller with \( r_c = 1.20 \) fm.

The quasi-particle energies and occupation probabilities were obtained solving the BCS equations with the single particle energies calculated and reported in the Table II. The value of \( \lambda \) is constrained to the conservation of the number of particles as follows:

\[ 2N_\rho = \sum_j v_j^2 (2j_\rho + 1). \]  

\( (31) \)

| Nucleus | \( N_\nu \) | \( N_\pi \) | \( \epsilon_d \) | \( \kappa \) | \( \chi_\nu \) | \( \chi_\pi \) | \( \xi_1 \) | \( \xi_2 \) | \( c^{(1)}_\nu \) | \( c^{(2)}_\nu \) | \( v_0^\nu \) | \( v_2^\nu \) | \( v_0^\pi \) | \( v_2^\pi \) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \(^{116}\text{Sn}\) | 8 | 0 | 1.32 | -0.5 | -0.22 | -0.07 | -0.06 | 0.04 |
| \(^{116}\text{Cd}\) | 7 | 1 | 0.85 | -0.27 | -0.58 | 0.24 | -0.18 | -0.18 | -0.15 | -0.15 | -0.06 |

TABLE II. The single particle energies, quasi-particle energies and occupation probabilities used in the OBTD for \(^{116}\text{Sn}\) to \(^{116}\text{In}\). For the neutron in the 50-82 shell as a hole and for the proton 28-50 shell as a hole.

| Neutron case | orbit | spec. (MeV) | qspec.(MeV) | \( v_j^2 \) |
|-------------|--------|-------------|-------------|-------------|
| n | \( i_j \) | \( \epsilon_j^p \) | \( \epsilon_j^{qpe} \) | \( v_j^2 \) |
| 2d5/2 | 2.7090 | 2.1430 | 0.0735 |
| 1g7/2 | 1.5300 | 1.2940 | 0.2489 |
| 3s1/2 | 0.3730 | 1.2286 | 0.7064 |
| 1h11/2 | 0.0000 | 1.4237 | 0.8091 |
| 2d3/2 | -0.2890 | 1.6184 | 0.8612 |

| Proton case | orbit | spec. (MeV) | qspec.(MeV) | \( v_j^2 \) |
|-------------|--------|-------------|-------------|-------------|
| n | \( i_j \) | \( \epsilon_j^p \) | \( \epsilon_j^{qpe} \) | \( v_j^2 \) |
| 2p1/2 | 0.2610 | 3.8569 | 0.9789 |
| 2p3/2 | 1.8900 | 2.3440 | 0.9405 |
| 1f5/2 | 2.3530 | 1.9485 | 0.9110 |
| 1g9/2 | 0.0000 | 4.1075 | 0.9814 |

The odd-odd \(^{116}\text{In}\) nucleus uses the quasi-particle energies obtained from the single-particle energies for the odd-neutron nucleus \(^{115}\text{Cd}\) and the odd-proton nucleus \(^{117}\text{In}\). Both odd-even nuclei have the same even-even core nucleus \(^{116}\text{Cd}\). The core-particle coupling parameters reported in Table II were obtained by fitting the experimental data for \(^{115}\text{Cd}\) and \(^{117}\text{In}\).

For the odd-odd nucleus \(^{116}\text{In}\), the tensor and Surface Delta Interaction (SDI) play an essential role. In this work, the parameters are obtained by minimization of the Mean Square Error (MSE) on the test set given by MSE = \( \frac{1}{N} \sum_{i,j} (f(x) - c_i)^2 \) where \( f(x) \) are our theoretical calculations and \( c_i \) are the experimental values \( [27] \). We are interested in studying the low-lying spectra of the odd-odd nucleus. We consider the first three levels of the \(^{116}\text{In}\), and this procedure may be applied to many levels as are needed. It is computed the Mean Square Error for different values of the parameter of Tensor and SDI. We have seen that the MSE is very sensitive to the parameters, as shown in Fig. 4.

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The theoretical energy surface of the ground state 1+ of \(^{116}\text{In}\) is depicted in Fig. 3. We found that the MSE is very sensitive to the parameters, as we see in Fig. 4. The minimization procedure consists of the creation of an uninterpolated contour surface with a given set of parameters over a grid of dimensions at least of dimension 11×11. After that, we generate an interpolated function of a grid dimension 30×30, which is used to determine the best parameters to fit the experimental data.
The fitting method is given by the Nelder-Mead simplex algorithm to find the minimum of a function of one or more variables. The library used is the SciPy that uses NumPy, the Machine Learning Python library used for scientific and technical computing. We obtain the minimum of the surface at the point with SDI 0.1500 and Tensor 0.200, with min error 6.53e-05. The result is depicted in the Fig. 5. The spectrum of the $^{116}$In is presented in the Fig. 6. This method also can be used for the fitting parameters for the odd-even nuclei, and even-even nuclei.

In this work, the parameters are fitted to reproduce the experimental level energies and depicted in the Figs. 6 and are reported in Table V.

In the following section, we compute the spectroscopic amplitudes by using the previous nuclear wave functions.

### A. Spectroscopic Amplitudes in IBFFM

The model space of transition is given by the model space of the odd-odd nucleus. The model is given by five active neutron orbitals $2d_5/2$, $1g_7/2$, $3s_{1/2}$, $2d_3/2$ and $1h_{11/2}$, and by four active proton orbitals $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$ and $1g_{9/2}$.

The nuclear states of the $^{116}$In system are in our model restricted to be $J^P = 0^+, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$, and $9^+$, within the first 40 excited states.

### TABLE IV. The parameters of the proton-neutron residual interaction.

| Parameter | Value (in MeV) |
|-----------|----------------|
| $V_{116}$In | -0.1500. |
| $V_{\sigma\sigma}$ | 0.0 |
| $V_T$ | 0.200 |

### TABLE V. Theoretical energy Levels obtained in the calculation in comparison with the available experimental data for odd-odd nucleus $^{116}$In [27].

| Spin $^+$ | Spin $^+$ | $E_{[KeV]}$ | $E_{[KeV]}$ |
|-----------|-----------|-------------|-------------|
| 1$^+$ | 1$^+$ | 0.0 | 0.0 |
| 5$^+$ | 5$^+$ | 127.267 | 120.4 |
| 4$^+$ | 4$^+$ | 223.330 | 219.8 |
| 2$^+$ | 2$^+$ | 272.966 | 183.4 |
| 4$^+, 5^+$ | 4$^+$ | 313.476 | 297.5 |
| 4$^+$ | 4$^+$ | 425.930 | 468.1 |
| 4$^+, 5^+$ | 5$^+$ | 460.0 | 336.6 |
| 3$^+$ | 3$^+$ | 508.241 | 238.2 |
calculated excitation energies (in MeV). We have decided to use the calculated rather than experimental energies because some of the experimentally known states may not correspond. The odd-odd nuclei’s parities positives are chosen because we are interested in the odd-odd nucleus’s ground state. This fact affects the selection rules of the available orbitals of the transition operator for neutrons and protons. The neutron and proton model space chosen in the IBFFM depends on the active orbitals. It was derived the odd-even and odd-odd parameters of neutron and proton for particle and hole coupling, we have considered the first 40 excited states. The numerical results of the spectroscopic amplitudes are exhibited in the Table VI for various cases of interest. We denote the spectroscopic amplitudes needed to calculate the spectroscopic amplitudes for the transitions $^{116}\text{Cd} \rightarrow ^{116}\text{In}$ and $^{116}\text{Sn} \rightarrow ^{116}\text{In}$ as SA1 and SA2, respectively. The squared of the spectroscopic amplitudes are denoted as SA$^2$ and SA$^2$ respectively. The IBM and IBFFM wave functions obtained after diagonalization were used. The double charge exchange without closure. For spins of intermediate states higher than 6 MeV, the SA tends to zero. To summarize, the spectroscopic amplitudes of the $^{116}\text{Cd}$, $^{116}\text{Sn}$ to $^{116}\text{In}$ nuclei are investigated within the IBFFM approach. The even-even boson-core Hamiltonian, and essential building blocks of the particle-boson coupling Hamiltonians, i.e., single-particle energies and odd particles’ occupation probabilities. A few coupling constants for the boson-fermion Hamiltonians and the residual neutron-proton interaction remain the only free parameters of the model. They are determined by using an implemented method of fitting using Machine learning libraries, which allowed computing more realistic nuclear wave functions and presented

V. SUMMARY AND CONCLUDING REMARKS

It was presented for the first time the explicit operator of the IBFFM for calculating one body transition densities in terms of active orbitals. It was derived the odd-even and odd-odd parameters by using an implemented method of fitting using Machine learning libraries, which allowed computing more realistic nuclear wave functions and presented
of the corresponding Hamiltonians for the parent and daughter nuclei. For the first time, the parameters of the odd-odd $^{116}\text{In}$ are presented. This is attributed to the combination of various factors adopted in the theoretical procedure, such as the chosen boson-fermion coupling constants, residual neutron-proton interaction, and underlying microscopic inputs. The spectroscopic amplitudes converge to zero after a certain energetic value of the odd-odd nuclei’s intermediate states, as shown in Fig. 7. It has been correlations of SA between

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