The Group Classification of One Class of Nonlinear Wave Equations.

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Abstract

The problem of group classification of one class of quasilinear equations of hyperbolic type with two independent variables has been solved completely.

1 Introduction

The problem of group classification of differential equations is one of the central problems of modern symmetry analysis of differential equations [9]. One of the important classes are hyperbolic equations. The problem of group classification of such equations has been discussed by many authors (see e.g.[1–8],[10–11]).

In this article we consider a classes of hyperbolic equations in $1+1$ time-space:

\begin{align*}
\text{(1.1)} & \quad u_{tx} = g(t,x)u_x + f(t,x,u), \quad g_x \neq 0, \quad f_{uu} \neq 0; \\
\text{(1.2)} & \quad u_{tx} = f(t,x,u), \quad f_{uu} \neq 0.
\end{align*}

where $u = u(t,x)$ and $g, f$ is an arbitrary nonlinear differentiable function, is an arbitrary nonlinear smooth function, which dependent variables $u$ or $u_x$. We use following notation $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_t = \frac{\partial u}{\partial t}$. The approach used in the present article is that presented in [12], being a synthesis of the standard Lie algorithm for finding symmetries and the use of canonical forms of partial differential operators obtained with the equivalence group of the
equation at hand. The conditions for a linear partial differential operator to be a symmetry operator of equation (1.1) turn out to be too general to produce a manageable system of defining equations. A solution to this apparent impasse is to invoke the equivalence group of equations (1.1), (1.2) and use it to find canonical forms for the symmetry operators. In [12] this method was applied to nonlinear heat equations.

The method is as follows. First, establish the defining system for a partial differential operator to be a symmetry operators of equations (1.1), (1.2). Then calculate the equivalence group of the equation. Here, the equivalence group is following group of invertible transformations

\[ \begin{align*} \bar{t} &= T(t, x, u), \\ \bar{x} &= X(t, x, u), \\ \bar{u} &= U(t, x, u) \end{align*} \]  

(1.3)

which transforms equations (1.1), (1.2) to equations of the same forms

\[ \begin{align*} \bar{u}_{\bar{t} \bar{x}} &= \bar{u}_{\bar{x} \bar{x}} + g(\bar{t}, \bar{x})\bar{u}_{\bar{x}} + \bar{f}(\bar{t}, \bar{x}, \bar{u}); \\ \bar{u}_{\bar{t} \bar{x}} &= \bar{f}(\bar{t}, \bar{x}, \bar{u}). \end{align*} \]  

(1.4)

(1.5)

The next step is to calculate the various canonical forms for a linear partial differential operator (LPDO) with respect to this equivalence group. This is the same as linearizing the operator (more precisely, finding a simplest form of an arbitrary LPDO which is equivalent under the equivalence transformations (1.3) to the LPDO at hand).

However, these canonical forms for LPDOs and symmetry operators on their own provide no solution to the problem we wish to solve: we do not, unlike for the cases solvable by Lie’s algorithm, automatically obtain the Lie algebra of symmetry operators in this way. Our procedure requires us to make assumptions about the nature of the Lie symmetry algebra of equations (1.1), (1.2). We take an arbitrary Lie algebra and find, using our equivalence group and the conditions for an LPDO to be a symmetry operator of equations (1.1), (1.2), canonical representations of the Lie algebra as a symmetry algebra of equations (1.1), (1.2). Different representations will give different forms for the functions \( f(t, x, u), g(t, x, u) \). At present we have no result which tells us the maximal dimension for the Lie symmetry algebra, unlike the case of ordinary differential equations. However, we find that we need only Lie algebras of dimension no greater than three.

All our arguments are local, and we do not treat global questions. Also, all functions involved in our arguments are assumed to be continuously differentiable of the appropriate order.

The first step in our programme is to find the conditions for a vector field to be a point symmetry of equations (1.1), (1.2). To this end we consider a vector field of the form

\[ Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \]  

(1.6)

where \( \tau, \xi, \eta \) are arbitrary, real-valued smooth functions defined in some subspace of the space \( V = X \otimes R^1 \) of the independent variables \( X = \langle t, x \rangle \) and the dependent variable \( R^1 = \langle u \rangle \). As a result, we find that the operator (1.6) generates a one-parameter symmetry group of equation (1.1) if

\[ \varphi^{tx} - \varphi^{xx} - [\tau g_t - \xi g_x]u_x - \varphi^x g - \tau f_t - \xi f_x - \eta f_u \bigg|_{utx = gux + f} = 0, \]  

(1.7)

where

\[ \begin{align*} \varphi^t &= D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi), \\ \varphi^x &= D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi), \\ \varphi^{xx} &= D_x(\varphi^x) - u_{tx}D_x(\tau) - u_{tx}D_x(\xi) \end{align*} \]  

(1.8)
and $D_t, D_x$ are operators of total differentiation in $t$ and $x$ respectively.

We then find the following results:

**Theorem 1** The symmetry group of the equation (1.1) is generated by the infinitesimal operators of the form

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + [h(t)u + r(t,x)]\partial_u,$$

(1.9)

where $\tau, \xi, \eta$ are real-valued functions that satisfy the system

$$r_{tx} + f [h - \tau_t - \xi_x] = gr_x + \tau f_t + \xi f_x + [hu + r]f_u,$$

$$h_t = \tau g + \tau g_t + \xi g_x.$$

(1.10)

**Theorem 2** The symmetry group of the equation (1.2) is generated by the infinitesimal operators of the form

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + (ku + r(t,x))\partial_u,$$

(1.11)

where $\tau, \xi, \eta$ are real-valued functions that satisfy the equation

$$r_{tx} + [k - \tau' - \xi']f = \tau f_t + \xi f_x + [ku + r]f_u.$$

**Lemma 1** The maximal equivalence group $\mathcal{E}$ of equation (1.1) reads as

(1) $\bar{t} = T(t), \bar{x} = X(x), \nu = U(t)u + Y(t,x), \ t'X'U \neq 0$;

(2) $\bar{t} = T(t), \bar{x} = X(t), \nu = \Psi(x)\Phi(t,x)u + Y(t,x), \ t'X'\Phi \neq 0,$

$$\Phi(t,x) = \exp[-\int g(t,x)dt], \ g_x \neq 0,$$

(1.12)

where $\dot{T} \neq 0,$ $D(X,U)/D(x,u) \neq 0.$

**Lemma 2** The maximal equivalence group $\mathcal{E}$ of equation (1.2) reads as

(1) $\bar{t} = T(t), \bar{x} = X(x), \nu = mu + Y(t,x),$

(2) $\bar{t} = T(t), \bar{x} = X(t), \nu = mu + Y(t,x), \ T'X'm \neq 0.$

(1.13)

where $\dot{T} \neq 0,$ $D(X,U)/D(x,u) \neq 0.$

The proof is by direct calculation using the chain rule, and we omit it here.

The first step in our method is to take a canonical form for a vector field. This is essentially the same as linearising a vector field, but we use only the transformations of the equivalence group $\mathcal{E}$ of the equations (1.1), (1.2). The reason for this is that the linearising transformations must not take us out of the classes of equations of the type given in (1.1), (1.2). We characterize the possible canonical forms for vector fields in the following result:

**Theorem 3** There are changes of variables (1.12) that reduce an operator (1.9) to one of the operators below:

$$Q = t\partial_t + x\partial_x; \ Q = \partial_t; \ Q = \partial_x + tu\partial_u;$$

$$Q = \partial_x + \epsilon u\partial_u, \ \epsilon = 0, 1; \ Q = tu\partial_u,$$

$$Q = u\partial_u, \ Q = r(t,x)\partial_u, \ r \neq 0.$$

(1.14)
Proof. The first group of transformation (1.12) reduces operator $Q$ (1.9) to the form

$$
\tilde{Q} = \tau T' \partial_t + \xi X' \partial_x + [(\tau U' + U h) u + \tau Y_t + \xi Y_x + U r] \partial_v.
$$

(1.15)

If $\sigma \cdot \xi \neq 0$, then, the function $T, X, U$ (1.12) is a solutions of the first-order PDE

$$
\tau T' = T, \quad \xi X' = X, \quad \tau U' + h U = 0,
$$

and the function $Y$ is a solution of the equation

$$
\tau Y_t + \xi Y_x + U r = 0,
$$

we find that the operator $\tilde{Q}$ (1.15) takes the form

$$
\tilde{Q} = \bar{t} \partial_{\bar{t}} + \bar{x} \partial_{\bar{x}}.
$$

If $\tau \neq 0$, a $\xi = 0$, choosing in (1.12) $T, U, Y$ a particular solution of PDE

$$
\tau T' = 1, \quad \tau U' + h U = 0 (U \neq 0), \quad \tau Y_t + U r = 0,
$$

we transform (1.9) to become

$$
\tilde{Q} = \partial_{\bar{t}}.
$$

If $\tau = 0$, $\xi \neq 0$, and

- $h' \neq 0$ we get the operator $\tilde{Q} = \partial_{\bar{x}} + \bar{v} \partial_{\bar{v}}$.
- $h' = 0$, we get the operator $\tilde{Q} = \partial_{\bar{x}} + \epsilon v \partial_{\bar{v}}$, where $\epsilon = 0$ or $\epsilon = 1$.

If $\tau = \xi = 0$, that we get the following case:

$$
\tilde{Q} = \bar{v} \partial_{\bar{v}}, \quad \tilde{Q} = v \partial_{\bar{v}}, \quad \tilde{Q} = r(f, \bar{x}) \partial_{\bar{v}}.
$$

The theorem is proved.

**Theorem 4** There are changes of variables (??) that reduce an operator (1.11) to one of the operators below:

$$
Q = \partial_t + \partial_x + \epsilon u \partial_u \quad (\epsilon = 0, 1);
Q = \partial_t + \epsilon u \partial_u \quad (\epsilon = 0, 1);
Q = u \partial_u, \quad Q = g(t, x) \partial_u \quad (g \neq 0).
$$

**Theorem 5** There are three equations of the form given in (1.1) which admit local one-parameter symmetry groups generated by the canonical forms given in Theorem 3. They are described by the following list, where $\langle Q \rangle$ denotes the algebra generated by the operator $Q$ and we define the equation of the form (1.1) by the form of the functions $g(t, x)$ and $f(t, x, u)$:

$$
A_1 = \langle t \partial_t + x \partial_x \rangle : g = t^{-1} \tilde{g}(\omega), \quad f = t^{-2} f(u, \omega), \quad \omega = tx^{-1}, \quad \tilde{g}_\omega \neq 0, \quad f_{uu} \neq 0;
$$

$$
A_2 = \langle \partial_t \rangle : g = \tilde{g}(x), \quad f = \tilde{f}(x, u), \quad \tilde{g}' \neq 0, \quad \tilde{f}_{uu} \neq 0;
$$

$$
A_3 = \langle \partial_x + tu \partial_u \rangle : g = x + \tilde{g}(t), \quad f = e^{tx} \tilde{f}(t, \omega), \quad \omega = e^{-tx} u, \quad \tilde{f}_{\omega} \neq 0.
$$
Having established the one-dimensional Lie point symmetry algebras of equation (1.1) we deal with the semi-simple Lie algebras. In fact, we show that no semi-simple Lie algebra has a representation in terms of the given vector fields. We prove this result for the two real simple Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$.

**Theorem 6** The real simple Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$ do not have any realizations as symmetry algebras of equation (1.1).

**Proof:** First, $\mathfrak{so}(3)$. The commutation relations are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$ 

One of these operators may be taken to be in one of the canonical forms given in Theorem 3. We do the calculation for the first canonical form and take $e_1 = t\partial_t + x\partial_x$. Then we take $e_2$ and $e_3$ in general form:

$$e_2 = \lambda e_1 + \lambda_1 \partial_t + \lambda_2 \partial_x + (h(t)u + r(t,x))\partial_u$$
$$e_3 = \mu e_1 + \mu_1 \partial_t + \mu_2 \partial_x + (g(t)u + s(t,x))\partial_u.$$

We may set $\lambda = \mu = 0$ since $e_1$ commutes with itself (this is equivalent to replacing $e_2$ and $e_3$ by $e_2 - \lambda e_1$ and $e_3 - \mu e_1$). Thus we take

$$e_2 = \lambda_1 \partial_t + \lambda_2 \partial_x + (h(t)u + r(t,x))\partial_u$$
$$e_3 = \mu_1 \partial_t + \mu_2 \partial_x + (g(t)u + s(t,x))\partial_u.$$

Then $[e_1, e_2] = e_3$ and $[e_3, e_1] = e_2$ give us

$$\mu_1 \partial_t + \mu_2 \partial_x + (g(t)u + s(t,x))\partial_u = -\lambda_1 \partial_t - \lambda_2 \partial_x + (th'(t)u + tr_1(t,x) + xr_x(t,x))\partial_u$$
$$\mu_1 \partial_t - \mu_2 \partial_x - (g(t)u + s(t,x))\partial_u = -\lambda_1 \partial_t - \lambda_2 \partial_x + (tg'(t)u + ts_1(t,x) + xs_x(t,x))\partial_u$$

From this we see that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$ so that

$$e_2 = (h(t)u + r(t,x))\partial_u, \quad e_3 = (g(t)u + s(t,x))\partial_u$$

which gives $[e_2, e_3] = 0 \neq e_1$. All the other calculations lead to the same result for both $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$.

The theorem is proved.

From this theorem we obtain the following:

**Note 1** In the class of operators (1.1) there are no realizations of any real semi-simple Lie algebras;

**Note 2** There are no equations (1.1) which has algebras of invariance, which are isomorphic by real semi-simple algebras, or conclude those algebras as subalgebras.
Every solvable Lie algebra \( g \) has, as is well-known, a composition series

\[
g = g_n \triangleright g_{n-1} \triangleright \cdots \triangleright g_0 = \{0\}
\]

where each \( g_i \) is an ideal of codimension one in \( g_{i+1} \) for \( i = 0, \ldots, n - 1 \). This structure allows us to perform an inductive construction of the realizations of solvable Lie algebras as vector fields: having constructed realizations for solvable Lie algebras of dimension \( n \), we may construct a realization for any solvable Lie algebra \( g_{n+1} \) of dimension \( n + 1 \) by adding the appropriate \((n+1)\)-st element to a solvable Lie algebra \( g_n \) of dimension \( n \) in the composition series of \( g_{n+1} \).

In Theorem 7 we gave all realizations of one-dimensional Lie algebras which give symmetries of equation (1.1); there are three inequivalent such realizations. We then note that there are two inequivalent solvable Lie algebras of dimension two with generators \( e_1, e_2 \):

\[
A_{2,1} = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = 0
\]

\[
A_{2,2} = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_2,
\]

**Theorem 7** Any \( A_{2,2} \)-invariant equation of the type (1.1) is equivalent to one of the following inequivalent equations (where we give the realization of \( A_{2,2} \) and the forms of the functions \( f \) and \( g \)).

\[
A^1_{2,2} = \langle t \partial_t + x \partial_x, t^2 \partial_t + x^2 \partial_x + mt \partial_u \rangle \quad (m \in R) : \quad
g = [mt + (k - m)x]t^{-1}(t - x)^{-1}, \quad k \neq 0,
\]

\[
f = |t - x|^{m-2}|x|^{-m} \bar{f}(\omega),
\]

\[
\omega = u|t - x|^{-m}|x|^{m}, \quad \bar{f}_{\omega \omega} \neq 0;
\]

\[
A^2_{2,2} = \langle t \partial_t + x \partial_x, t^2 \partial_t + mt \partial_u \rangle \quad (m \in R) : \quad
g = t^{-2}[kx + mt], \quad k \neq 0,
\]

\[
f = |t|^{m-2}|x|^{-m} \bar{f}(\omega),
\]

\[
\omega = |t|^{-m}|x|^{m}u, \quad \bar{f}_{\omega \omega} \neq 0;
\]

\[
A^3_{2,2} = \langle t \partial_t + x \partial_x, x^2 \partial_x + tu \partial_u \rangle :
\]

\[
g = (tx)^{-1}(mx - t) \quad (m \in R), \quad f = x^{-2}e^{-tx^{-1}} \bar{f}(\omega),
\]

\[
\omega = ue^{tx^{-1}}, \quad \bar{f}_{\omega \omega} \neq 0.
\]

In Theorem 7 we gave all realizations of two-dimensional Lie algebras which give symmetries of equation (1.1); there are three inequivalent such realizations.

**Theorem 8** There are two equations of the form given in (1.2) which admit local one-parameter symmetry groups generated by the canonical forms given in Theorem 4. They are described by the following list, where \( \langle Q \rangle \) denotes the algebra generated by the operator \( Q \) and we define the equation of the form (1.3) by the form of the functions \( f(t, x, u) \):

\[
A^1_1 = \langle \partial_t + \partial_x + e u \partial_u \rangle \quad (e = 0, 1) : \quad f = e^{ct} \bar{f}(\theta, \omega), \quad \theta = t - x, \quad \omega = e^{-ct}u; \quad \bar{f}_{\omega \omega} \neq 0;
\]

\[
A^2_1 = \langle \partial_t + e u \partial_u \rangle \quad (e = 0, 1) : \quad f = e^{ct} \bar{f}(x, \omega), \quad \omega = e^{-ct}u, \quad \bar{f}_{\omega \omega} \neq 0.
\]
The results of the group classification of the equation (1.2) has been done in following table. The forms of the functions $f$ determining the corresponding invariant equations (1.2) are given as follows:

| Number | Function $f$ | Symmetry operator | Algebra |
|--------|--------------|-------------------|---------|
| 1      | $e^{t}f(\omega)$, $\omega = u e^{-t}, f_{\omega} \neq 0$ | $\partial_t + u \partial_u, \partial_x$ | $A_{2,1}$ |
| 2      | $e^{t+x}f(\omega)$, $\omega = u e^{-t-x}, f_{\omega} \neq 0$ | $\partial_t + u \partial_u, \partial_x + u \partial_u$ | $A_{2,1}$ |
| 3      | $(t-x)^{-3}f(\omega)$, $\omega = (t-x)u, f_{\omega} \neq 0$ | $-t \partial_t - x \partial_x + u \partial_u, \partial_t + \partial_x$ | $A_{2,2}$ |
| 4      | $x^{-1}f(\omega)$, $\omega = x^{-1}u, f_{\omega} \neq 0$ | $-t \partial_t - x \partial_x - u \partial_u, \partial_t$ | $A_{2,2}$ |
| 5      | $(t-x)^{-2}f(u)$, $f_{uu} \neq 0$ | $\partial_t + \partial_x, t \partial_t + x \partial_x, t^2 \partial_t + x^2 \partial_x$ | $sl(2, \mathbb{R})$ |
| 6      | $e^{x^{-1}u}$ | $-t \partial_t + x \partial_u, \partial_t, x \partial_x + u \partial_u$ | $A_{2,2} \oplus A_1$ |
| 7      | $\lambda|x|^{-m-1}|u|^{m+1}$, $\lambda \neq 0, m \neq 0, -1 - 2$ | $\partial_t, t \partial_t - \frac{1}{m} u \partial_u, x \partial_x + \frac{m+1}{m} u \partial_u$ | $A_{2,2} \oplus A_1$ |
| 8      | $f(u), f_{uu} \neq 0$ | $\partial_t, \partial_x, -t \partial_t - x \partial_x$ | $A_{3,6}$ |
| 9      | $\lambda|u|^{n+1}$, $\lambda \neq 0, n \neq 0, -1$ | $t \partial_t - \frac{1}{n} u \partial_u, x \partial_x - \frac{n}{n} u \partial_u, \partial_t, \partial_x$ | $A_{2,2} \oplus A_{2,2}$ |

So the problem of group classification of one class of quasilinear equations of hyperbolic type with two independent variables has been solved completely.

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