The Weak Mixing Angle in String Theory
and the Green-Schwarz Mechanism

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Four-dimensional strings with the standard model gauge group $SU(3) \times SU(2) \times U(1)$ give model-dependent predictions for the tree level weak mixing-angle. In the presence of an extra pseudo-anomalous gauged- $U(1)_X$, the value of the weak angle may be computed purely in terms of the charges of the massless fermions of the theory, independently of the details of the massive string sector. I present the simplest such $U(1)_X$ which leads to the canonical result $\sin^2 \theta_W = 3/8$ in the supersymmetric standard model. This is a sort of gauged Peccei-Quinn symmetry which requires the presence of just the minimal set of Higgs doublets and forbids dimension-four $B$ and $L$-violating terms. In this approach the cancellation of the $U(1)_X$ anomalies through a Green-Schwarz mechanism plays a crucial role. In a different context (that of non-string low-energy supersymmetric models) I briefly discuss whether this type of anomaly cancellation mechanism could be of phenomenological relevance close to the electroweak scale.

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1. Introduction

In trying to apply string theory to the description of the low energy phenomena, the first question which naturally appears is whether one can construct a four-dimensional (4-D) string model whose massless sector coincides with that of the standard model (SM) or its supersymmetric extension (SSM). (In fact other massless particles could be present as long as they are neutral under the SM interactions. This is the “hidden sector” of the theory.) We do not have a definite answer to this question at the moment because of technical reasons. We do not know, given an specific massless sector, how to complete it with the required supermassive objects in order to constitute a consistent, modular invariant string theory. It would be a fantastic achievement if somebody could give us rules for this “upwards” procedure. Rather, the (“downwards”) approach followed up to now is to construct four-dimensional string models whose massless sector is as close as possible either to the standard model or some of its simplest non-Abelian gauge extensions like $SU(3)^3$ or $SU(5) \times U(1)$ etc. Nobody has been able up to now to construct a theory exactly resembling the SM, the closest thing achieved being models with gauge group $SU(3) \times SU(2) \times U(1)_Y \times U(1)^n \times G$, where $G$ is some “hidden sector” gauge group not coupling to the SM particles. The massless sector of these models always contains, apart from three generations of quarks and leptons, a bunch of extra particles including extra vector-like (with respect to the SM) heavy quarks, leptons, singlets etc. From this point onwards, the string model-builders abandon the stringy methods and do their best to get rid of all the extra garbage by assuming that some of the singlet scalars in the theory get vevs and give masses (through Yukawa couplings) to the extra unwanted particles. This is not so easy as it sounds because, for each given string model, the particle content and the Yukawa coupling structure is fixed and very often the required Yukawas are not present in the model. All this leads to a lot of ambiguities in the predictions of each model.

The origin of the extra unwanted particles is clear. With our present techniques the four-dimensional strings we build have a gauge group whose rank $r$ is much bigger than the one of the SM ($r = 4$), typically $r = 16 – 22$. Normally what happens is that the extra unwanted particles are required in order to cancel the anomalies induced by the extra gauge interactions beyond the ones of the SM. Of course, one can assume, as I said above, further gauge symmetry-breaking induced by giving vevs to some scalars but then one has to abandon the realm of the string theory techniques. Furthermore, typically this is not
enough and it is difficult to avoid the existence of some residual unwanted states in the massless sector.

I believe these are merely technical problems which depend on the present state of the art of four-dimensional string model construction. It is reasonable to believe that there should be a way to complete an anomaly-free field theory into a complete modular-invariant 4-D string by the addition of appropriate towers of massive states although, most likely, not any model might be completed in this way. Perhaps some theories may require its massless sector to be slightly extended in order for them to be embeddable into a 4-D string. This could be the case e.g., of the standard model. In the meantime, it makes sense to study whether a hypothetical 4-D string string with a SM massless sector (plus, possibly, a “hidden sector”) is consistent with known phenomenological facts.

At this point one has to make a choice: should we have an intermediate ”GUT” stage (e.g., $SU(5)$) or not? I personally think that one should first study the case of a hypothetical 4-D string with gauge group $SU(3) \times SU(2) \times U(1) \times G$. I see four reasons for that: 1) As usual in physics, one must start with the simplest possibility consistent with the observed facts; 2) Most of the unification achievements in GUTs are already present in strings. That is particularly the case of charge quantization and the unification of gauge coupling constants; 3) GUTs have a couple of unsolved problems which I find hard to swallow. These are the Higgs doublet-triplet splitting problem (this is, in my opinion, a fatal problem) and the wrong predictions for the first generation quark and lepton masses; 4) In order to obtain a 4-D string with a massless sector resembling a standard GUT like $SU(5)$ one has to use “higher Kac-Moody level” models. This is not only technically complicated. The chances of getting e.g. an $SU(5)$ model without other unwanted Higgses (e.g., more than one adjoint, 15-plets etc) and with some authomatic solution of the doublet-triplet splitting problem are nihil.

In spite of the above four points, there is one outstanding success of the simplest (SUSY) GUT scenarios: they naturally predict the ”canonical” value for the tree-level weak angle, $\sin^2 \theta_W = 3/8$, which leads to amazingly good agreement with data once renormalization effects are taken into account. A random $SU(3) \times SU(2) \times U(1)$ 4-D string will give also a definite prediction for the weak angle, but it will not in general coincide with the ”canonical” value, and typically it will be very different. Moreover, in order to find what is the prediction of a given string model for $\sin^2 \theta_W$, one needs to know information about the complete string model, it is not enough to know what is its massless
sector. More precisely, one needs to know the normalization of the weak hypercharge $U(1)$ generator (which is given by the tree-level coupling of the $U(1)$ gauge boson to a pair of gravitons [2]) and the Kac-Moody level $k_2$ of the $SU(2)_W$ factor. This looks a bit deceptive since then the weak angle can only be computed using information about the full specific 4-D string and in a model by model basis.

2. The Weak Mixing Angle and the Green-Schwarz Mechanism

In ref.[3] I described an exception to the fact discussed above. There is a large class of $SU(3) \times SU(2) \times U(1)_Y$ models in which one can compute the value of $\sin^2 \theta_W$ only in terms of the massless spectrum. Those are models with an additional ”pseudoanomalous” $U(1)_X$ gauge factor whose anomalies are cancelled by the f-D version [4] of the Green-Schwarz mechanism [5]. The existence of a pseudoanomalous $U(1)$s with these characteristics is extremely common in specific 4-D strings constructed in the past and may be considered indeed as a generic situation. The point here is that this mechanism in four-dimensions is sensitive to the normalization (the ”levels”) of the different gauge generators.

The 4-D Green-Schwarz mechanism allows for gauged $U(1)_X$ currents whose anomalies, as naively computed through the triangle graphs, are non-vanishing. The anomalies are in fact cancelled by assigning a non-trivial gauge transformation to an axion $\eta(x)$ present in the theory (the pseudoscalar partner of the dilaton) which couples universally to all gauge groups [4]. The quadratic gauge piece of the Lagrangian has the form

$$\frac{1}{g^2(M)} \sum_{i=1,2,3,X} k_i F_i^2 + i \eta(x) \sum_{i=1,2,3,X} k_i F_i \tilde{F}_i, \quad (2.1)$$

where $g$ is the gauge coupling constant at the string scale $M$, and $F_i$ are the gauge field strengths. The coefficients $k_i$ are the Kac-Moody levels of the corresponding gauge algebra [2]. For the case of non-Abelian groups like $SU(3)$ and $SU(2)$ those levels are integer and in practically all models constructed up to now one has $k_2 = k_3 = 1$. In the case of an Abelian group like $U(1)$-hypercharge, $k_1$ is a normalization factor (not necessarily integer) and is model dependent. Notice that the above action does not assume ‘a priori’ any GUT-like symmetry relating the different $k_i$s. Below the string scale, the coupling constants will run as usual according to their renormalization group equations [4]. The index $i$ runs over
the three gauge groups \( U(1) \otimes SU(2) \otimes SU(3) \) of the SM and the extra ‘anomalous’ gauge group \( U(1)_X \). Under a \( U(1)_X \) gauge transformation one has

\[
A_X^\mu \to A_X^\mu + \partial^\mu \theta(x) \\
\eta \to \eta - \theta(x) \delta_{GS}
\]

(2.2)

where \( \delta_{GS} \) is a constant and \( \eta(x) \) is the axion field. If the coefficients \( C_i \) of the mixed \( U(1)_X - SU(3), SU(2), U(1) \) are in the ratio

\[
\frac{C_1}{k_1} = \frac{C_2}{k_2} = \frac{C_3}{k_3} = \delta_{GS},
\]

(2.3)

those mixed anomalies will be cancelled by the gauge variation of the second term in eq.(2.1). Since there may be in the spectrum extra singlet particles with \( U(1)_X \) quantum numbers but no SM gauge interactions, we will not consider here the equivalent conditions involving the \( U(1)_X \) anomaly coefficient, since those singlets can always be chosen so that that anomaly is cancelled. For the same reason we will not consider the mixed \( U(1)_X \)-gravitational anomalies. On the other hand, to be consistent, one has to impose that the mixed \( U(1)_Y - U(1)_X \) anomaly vanishes identically since it only involves standard model fermions and cannot be cancelled by a GS mechanism.

From eqs.(2.1) and (2.3) one obtains \( \frac{3}{8} \) for the tree level weak angle at the string scale

\[
sin^2 \theta_W = \frac{k_2}{k_1 + k_2} = \frac{C_2}{C_1 + C_2}.
\]

(2.4)

The above expression shows that, for each given ‘anomalous’ \( U(1)_X \), the cancellation of the anomalies through a GS mechanism gives a definite prediction for the weak angle in terms of the coefficients of the anomaly. The latter may be computed in terms of the \( U(1)_X \) charges of the massless fermions of the theory.

The above mechanism gives us an alternative to GUTs concerning the derivation of \( sin^2 \theta_W = 3/8 \). In our context, the success of that prediction would be an indication of the existence of a 4-D string with gauge group of the form

\[
SU(3) \times SU(2) \times U(1)_Y \times U(1)_X \times G
\]

(2.5)

and with mixed \( U(1)_X \) anomalies in the ratio \( C_2/(C_1 + C_2) = 3/8 \). It is not difficult to find an example of a \( U(1)_X \) giving that ratio. In fact, since we know that \( U(1)_X \) must have mixed anomalies with QCD, the natural candidates must be symmetries of
the Peccei-Quinn (PQ) type. Indeed, as shown in ref.[3], the simplest PQ symmetry in the two-Higgs non-supersymmetric standard model does the job automatically. This is a generation-independent $U(1)_X$ with charge assignments:

$$Q_X(Q_L, U_c^L, D_c^L, l_L, l_c^L, H, \bar{H}) = (0, 0, -1, 0, 1, 0)$$

in an obvious notation. One easily finds in this case

$$C_3 = \frac{-N_g}{2}; \quad C_2 = \frac{-N_g}{2}; \quad C_1 = -\frac{5}{6}N_g,$$

where $N_g$ is the number of generations. This leads to the canonical 3/8 automatically. Notice also that there are no mixed $U(1)_Y - U(1)_X$ anomalies.

What is the fate of the extra $U(1)_X$ interaction? The structure of the GS mechanism forces this gauge boson to become massive by swallowing the axion field as its longitudinal component [4]. This is more clearly seen in the dual formulation of the axion field in terms of a two index antisymmetric tensor $B_{\mu\nu}$. The field strength of this tensor $H_{\mu\nu\rho}$ (which contains the standard gauge Chern-Simons term) is related to the axion field by

$$\partial_\mu \eta(x) = \epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}.$$  

In this equivalent formulation the anomaly cancellation mechanism requires a one-loop counterterm in the Lagrangian of the form $M^2 \epsilon_{\mu\nu\rho\sigma} B^{\mu\nu} F^{\rho\sigma}$. After the duality transformation this term becomes $M^2 \partial_\mu \eta A_X^\mu$ in terms of the axion. This is nothing but a typical Higgs mechanism term which gives a mass $\simeq M$ to the gauge boson $A_X$. In string theory, the role of radial mode in the Higgs mechanism is played by the dilaton field.

The above $U(1)_X$ symmetry gives the canonical result for the non-supersymmetric standard model but it fails to do so in the supersymmetric standard model. This is due to the contribution of the higgsinos to the mixed anomalies. Indeed, it was found in ref.[3] that there is no flavour-independent $U(1)_X$ which would give the canonical 3/8 in the supersymmetric case (as long as we stick to the non-singlet particle content of the SSM). Thus some of the assumptions concerning the $U(1)_X$ assignments has to be abandoned, the simplest of them being flavour-independence in the quark sector.

The simplest supersymmetric $SU(3) \times SU(2) \times U(1)_Y \times U(1)_X$ model giving rise to the canonical value is the following. The $Q_X$ charges of the chiral multiplets of e.g., the second and third generations are as in eq.(2.6) whereas those e.g., for the first generation are slightly changed:

$$Q_X(q_L, u_c^L, d_c^L, l_L, e_c^L) = (0, 0, 0, -1, 0)$$  

(2.8)
i.e., the $d$ quark has charge zero instead of $-1$. This modest change of assignments in the symmetry already gives rise to the canonical $3/8$. Indeed, leaving free the number of Higgs pairs $N_D$, one obtains for the anomaly coefficients:

$$C_3 = -1 ; \quad C_2 = -\frac{3}{2} + \frac{1}{2} N_D; \quad C_1 = -\frac{13}{6} + \frac{1}{2} N_D. \quad (2.9)$$

For the minimal number of Higgs pairs $N_D = 1$ one obtains $C_2/(C_1 + C_2) = 3/8$ and $k_3 = k_2$. It is also easy to check that the mixed $Y - Q^2_X$ anomalies vanish identically. It is remarkable that this $U(1)_X$ symmetry may only be gauged and give $\sin^2 \theta_W = 3/8$ if in addition the minimal set of Higgs fields is present. This correlation between the mixing angle and the presence of the minimal set of Higgs fields is very attractive.

It is interesting to examine the structure of the dim=$4$ and $5$ operators allowed by this type of symmetry. All dimension=$4$ terms violating B or L are forbidden. Indeed, $Q_X$ forbids couplings of the $UDD, QDL$ and $LLE$ type (a coupling involving the first generation $udd$ is allowed by $Q_X$ but is forbidden by Fermi statistics). Dimension five operators of the type $QQQL$ (which can mediate proton decay once appropriately dressed) are also forbidden by the gauge symmetry. A dim=$5$ operator of the type $(ucde)$ involving only right-handed particles is, on the other hand, allowed, but the experimental constraints on this operator are considerably weaker than those for $QQQL$ (this is due to the fact that the wino does not couple directly to right-handed objects). Concerning the usual dim=$4$ Yukawa terms which give masses to quarks and leptons, all of them are allowed except the ones involving the right-handed down quark, $d^c_L$. Indeed, all couplings of the type $(Q_i d^c_L H_i)$, for $i = 1, 2, 3$ vanish. Thus, as long as $Q_X$ is unbroken the down-quark would remain massless. On the other hand, as we argued above, the $U(1)_X$ symmetry is generically spontaneously broken slightly below the Planck mass. Due to supersymmetry, the Green-Schwarz mechanism comes along with a dilaton-dependent Fayet-Iliopoulos term [4] associated to $U(1)_X$. Usually there are singlet chiral superfields $X_i$ with non-vanishing $Q_X$ charges in the spectrum which are required to cancel the $Q^3_X$ and gravitational anomalies. Some of these singlets are forced by the $U(1)_X$ $D^2$-term in the scalar potential to get a non-vanishing vev. This breaks the $U(1)_X$ symmetry spontaneously. Then, dim=$5$ superpotential terms of the type $Q d^c_i H X_i$ can generate the desired d-quark mass once the $X_i$-vev is inserted. On the other hand, one has to check this is not happening with the B- and L-violating dim=$4$ couplings since they could be also regenerated by this mechanism.
It is important to realize that, even though the pseudoanomalous $U(1)_X$ is spontaneously broken, the fact that the value of the weak angle is given by eq.(2.4) remains true. Let me also remark that an alternative to pseudoanomalous gauge $U(1)_X$ symmetries is provided by local $U(1)_R$ anomalous $R$-symmetries often present in string models. This possibility is discussed in ref. [3].

The above discussion may be summarized as follows: the apparent success of the canonical prediction $\sin^2\theta_W = 3/8$ may be evidence not for a GUT-type symmetry but for the existence of a gauged $U(1)_X$ symmetry of the Peccei-Quinn type whose anomaly is cancelled by a Green-Schwarz mechanism. Of course, the outstanding problem of finding an specific 4-D string with these properties remains. It must be emphasized though that the presence of pseudoanomalous $U(1)$s in string models is quite generic.

A final comment is in order. There is a known method to construct string models with the canonical values for the gauge coupling constant normalizations. This may be achieved starting with a $(2,2)$-type compactification (leading to an $E_6 \times E_8$ gauge group) and then assuming there are additional gauge backgrounds (Wilson lines) further breaking $E_6$ to the standard model or some extension. If the Wilson lines are associated to a particular type of isometries of the compactifying variety (isometries leaving no fixed points), the canonical ($E_6$-like) relationships between gauge coupling constants are preserved. This is the Hosotani-Witten mechanism. Indeed there is nothing wrong with this method, but, in my opinion, it gives a rather dissapointing answer to the question, why $\sin^2\theta_W = 3/8$? The answer to this question within this point of view would be something like this: because the underlying theory has a $(2,2)$ structure with gauge backgrounds associated to isometries leaving no fixed points. I find this answer i) rather technical and unphysical: why nature should prefer isometries without fixed points rather than isometries with them? ii) relying on superheavy dynamics iii) antropocentric; it is more concerned with our model building limitations than with the actual physical dynamics and iv) quite steryle, since in this context the coupling normalization is completely unrelated to other properties of the theory like anomaly structure, existence of just the minimal set of Higgses etc. One should add to these (just aesthetical) arguments the difficulties appearing in $E_6$ string-based models in order to obtain consistent phenomenology. The presence of too many light chiral multiplets makes the gauge couplings to explode in its running up to the Planck scale.
3. A Green-Schwarz Mechanism Close to the Weak Scale?

Let us change of subject and consider now low-energy supersymmetric extensions of the standard model (forget about strings in this section). The Green-Schwarz mechanism seems to be independent of string theory. One can conceive the existence of axion-like fields $\eta_j$ with couplings of the form

$$i \sum_{j=1,2,3,X} a_j \eta_j F_j \tilde{F}_j$$

(3.1)

to the standard model groups and a pseudoanomalous $U(1)_X$. Here the $a_j$ are possible group-dependent constant coefficients. Under a $U(1)_X$ gauge transformation these axions would transform like $\eta_j \rightarrow \eta_j - \theta(x) \delta_j$, where $\theta(x)$ is the gauge function and $\delta_j$ are constants. The mixed anomalies will be cancelled as long as the constants involved are related with the triangle anomaly coefficients $C_j$ by

$$C_j = a_j \delta_j$$

(3.2)

This is telling us that we can have an extra anomalous $U(1)_X$s added to the SUSY standard model assuming there is a low energy GS-mechanism at work. (Notice that, unlike the case of string theory, there is no reason to set equal normalization coefficients for the $F^2$ and the $\tilde{F}F$ terms. Thus imposing cancellation of anomalies does not give us any information about the coupling constant normalizations nor, e.g., the weak mixing angle).

To the reader familiar with the prehistory of SUSY model-building this possibility looks very interesting. Indeed, the first SUSY versions of the standard model by Fayet [8] included an extra $U(1)$ in order to do both the SUSY-breaking and the $SU(2) \times U(1)_Y$-breaking. This $U(1)$ was typically an anomalous symmetry and this is one of the reasons why this type of models were not pursued further. With the introduction of the GS-mechanism at low energies it seems one could in principle resurect these models.

Although this possibility is very exciting, it is not obvious that it can work in practice. Couplings like those in eq.(3.1) are non-renormalizable. For the axion fields to have canonical dimension=1 one has to divide by some mass scale $M$. In the case of strings $M$ is nothing but the string scale, a well motivated scale, but in our case it has to be some new mass scale of unknown origin. Thus, for energies above $M$ some new physics must appear. A low-energy GS-mechanism must necessarily be an effective mechanism induced by some underlying new physics.
I think an old model by Weinberg (2.4) may give an example of what new physics could be involved. The model is a SUSY standard model enlarged by an extra $U(1)_X$ symmetry. All quarks and leptons have charge = 1 under this symmetry whereas the Higgs superfields have charges = $-2$. With this particle content this model would have $U(1)_X$ anomalies. Weinberg found a simple extension of the model (2.4) which is anomaly-free for three quark-lepton generations. He added chiral superfields transforming under $SU(3) \times SU(2) \times U(1)_Y \times U(1)_X$ like

$$O = (8, 1, 0, -2) \quad (3.3)$$
$$T = (1, 3, 0, -2) \quad (3.4)$$
$$E_i = (1, 1, 1, -2) \text{, } i = 1, 2 \quad (3.5)$$
$$\bar{E}_i = (1, 1, -1, -2) \text{, } i = 1, 2 \quad (3.6)$$

Notice that all extra fields have $U(1)_X$-charge = $-2$. Imagine we now introduce a singlet field with quantum numbers $X = (1, 1, 0, +4)$ and couplings to the above chiral multiplets as follows:

$$\lambda_O(XOO) + \lambda_T(XTT) + \lambda_{ij}(XE_i\bar{E}_j) \text{, } i, j = 1, 2 \quad (3.7)$$

Let us further assume that the singlet $X$ gets a non-vanishing vev, $<X> \simeq M$. Then, due to the couplings in (3.7), all the extra particles will become massive. Let us now take the (formal) limit $\lambda_O, \lambda_T, \lambda_{ij} \to \infty$, keeping $<X> \simeq M$ fixed. In this limit, since the extra particles have disappeared from the low energy spectrum, one would again recover the $U(1)_X$ anomalies we originally had. The situation now is quite analogous to the heavy top limit considered by D’Hooker and Fahri [7] for the SM. In their case they showed that extra terms are generated in the Lagrangian when taking the $m_{top} \to \infty$ limit. This extra terms cancel the low energy anomalies. I would expect something analogous (although not identical, since the group structure and transformation properties of the massive fields are different) going on in our supersymmetric model. In the large Yukawa coupling limit terms like those in eq.(3.1) would appear in which a single axion $\eta$ (associated to the phase of the field $X$) would be operative. In this limit an effective GS-mechanism would be at work.

If the above example is generic it is not clear whether a low-energy GS mechanism would be of any use. It will be just an effective mechanism, a particular limit of some
underlying model including extra particles. But, on the other hand, in the underlying model, anomalies will be cancelled in the usual way and the problems of the SUSY models with an extra $U(1)$ will again reappear in this complete theory. In particular, the models discussed in ref.\(^{(2,4)}\) had problems with the existence of charge- and colour-breaking supersymmetric minima and also with too small gaugino masses. The possible uses of a low-energy GS-mechanism for SUSY phenomenology do not look particularly bright if the above argumentation is correct.
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