Frobenius and Mashke type Theorems for Doi-Hopf modules and entwined modules revisited: a unified approach

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Abstract

We study when induction functors (and their adjoints) between categories of Doi-Hopf modules and, more generally, entwined modules are separable, resp. Frobenius. We present a unified approach, leading to new proofs of old results by the authors, as well as to some new ones. Also our methods provide a categorical explanation for the relationship between separability and Frobenius properties.

0 Introduction

Let \( H \) be a Hopf algebra, \( A \) an \( H \)-comodule algebra, and \( C \) an \( H \)-module coalgebra. Doi [17] and Koppinen [21] independently introduced unifying Hopf modules, nowadays usually called Doi-Koppinen-Hopf modules, or Doi-Hopf modules. These are at the same time \( A \)-modules, and \( C \)-comodules, with a certain compatibility relation. Modules, comodules, graded modules, relative Hopf modules, dimodules and Yetter-Drinfel’d modules are all special cases of Doi-Hopf modules. Properties of Doi-Hopf modules (with applications in all the above special cases) have been studied extensively in the literature. In [12], a Maschke type theorem is given, telling when the functor \( F \) forgetting the \( C \)-coaction reflects the splitness of an exact sequence, while in [13], it is studied when this functor is a Frobenius functor, this means that its right adjoint \( \bullet \otimes C \) is at the same time a left adjoint.

The two problems look very different at first sight, but the results obtained in [12] and [13] indicate a relationship between them. The main result of [12] tells us that we have a Maschke Theorem for the functor \( F \) if \( C \) is finitely generated projective and there exists an \( A \)-bimodule \( C \)-colinear map \( A \otimes C \to C^* \otimes A \) satisfying a certain normalizing condition. In [13], we have seen that \( F \) is Frobenius if \( C \) is finitely generated and projective and \( A \otimes C \) and \( C^* \otimes A \) are isomorphic as \( A \)-bimodules and \( C \)-comodules. This isomorphism can be described using a so-called \( H \)-integral, this is an element in \( A \otimes C \) satisfying a certain centralizing condition. The same \( H \)-integrals appear also when one studies Maschke Theorems for \( G \), the right adjoint of \( F \) (see

∗EPSRC Advanced Research Fellow
†Research supported by the bilateral project “Hopf algebras and co-Galois theory” of the Flemish and Romanian governments
‡Research supported by the bilateral project “New computational, geometric and algebraic methods applied to quantum groups and differential operators” of the Flemish and Chinese governments
This connection was not well understood at the time when [12] and [13] were written. The aim of this paper is to give a satisfactory explanation; in fact we will present a unified approach to both problems, and solve them at the same time. We will then apply the same technique for proving new Frobenius type properties: we will study when the other forgetful functor forgetting the \( A \)-action is Frobenius, and when a smash product \( A \#_R B \) is a Frobenius extension of \( A \) and \( B \). Let us first give a brief overview of new results obtained after [12] and [13].

1) In [10] and [11] the notion of separable functor (see [23]) is used to reprove (and generalize) the Maschke Theorem of [12]. In fact separable functors are functors for which a “functorial” type of Maschke Theorem holds. A key result due to Rafael [25] and del Río [26] tells us when a functor having a left (resp. right) adjoint is separable: the unit (resp. the counit) of the adjunction needs a splitting (resp. a cosplitting).

2) Entwined modules introduced in [3] in the context of noncommutative geometry generalize Doi-Hopf modules. The most interesting examples of entwined modules turn out to be special cases of Doi-Hopf modules, but, on the other hand, the formalism for entwined modules is more transparent than the one for Doi-Hopf modules. Many results for Doi-Hopf modules can be generalized to entwined modules, see e.g. [3], where the results of [12] and [13] are generalized to the entwined case.

3) In [8], we look at separable and Frobenius algebras from the point of view of nonlinear equations; also here we have a connection between the two notions: both separable and Frobenius algebras can be described using normalized solutions of the so-called FS-equation. But the normalizing condition is different in the two cases.

Let \( F : C \to D \) be a covariant functor having a right adjoint \( G \). From Rafael’s Theorem, it follows that the separability of \( F \) and \( G \) is determined by the natural transformations in \( V = \text{Nat}(GF,1_C) \) and \( W = \text{Nat}(1_D,FG) \). In the case where \( F \) is the functor forgetting the coaction, \( V \) and \( W \) are computed in [10]. In fact \( V \) and \( W \) can also be used to decide when \( G \) is a left adjoint of \( F \). This is what we will do in Section 3: we will find new characterizations for \((F,G)\) to be a Frobenius pair, and we will recover the results in [13] and [3]. In Section 4, we will apply the same technique to decide when the other forgetful functor is Frobenius, and in Section 5, we will study when the smash product of two algebras \( A \) and \( B \) is a Frobenius extension of \( A \) and \( B \). This results in necessary and sufficient conditions for the Drinfel’d double of a finite dimensional Hopf algebra \( H \) (which is a special case of the smash product (see [14])) to be a Frobenius or separable over \( H \).

We begin with a short section about separable functors and Frobenius pair of functors. We will explain our approach in the most classical situation: we consider a ring extension \( R \to S \), and consider the restriction of scalars functor. We derive the (classical) conditions for an extension to be separable (i.e. the restriction of scalars functor is separable), split (i.e. the induction functor is separable), and Frobenius (i.e. restriction of scalars and induction functors form a Frobenius pair). We present the results in such a way that they can be extended to more general situations in the subsequent Sections.

Let us remark at this point that the relationship between Frobenius extensions and separable extensions is an old problem in the literature. A classical result, due to Eilenberg and Nakayama, tells us that, over a field \( k \), any separable algebra is Frobenius. Several generalizations of this property exist; conversely, one can give necessary and sufficient conditions for a Frobenius extension to be separable (see [19 Corollary 4.1]). For more results and a history of this problem, we refer to [1], [20] and [18].

We use the formalism of entwined modules, as this turns out to be more elegant and more general than that of the Doi-Hopf modules; several left-right conventions are possible and there exists a dictionary between them. In [12] and [13], we have worked with right-left Doi-Hopf modules; here we will work in the right-right case, mainly because the formulae then look more natural.

Throughout this paper, \( k \) is a commutative ring. We use the Sweedler-Heyneman notation for comultiplications and coactions. For the comultiplication \( \Delta \) on a coalgebra \( C \), we write
\[
\Delta(c) = c_{(1)} \otimes c_{(2)}.
\]
For a right $C$-coaction $\rho'$ and a left $C$-coaction $\rho^l$ on a $k$-module $N$, we write
\[ \rho'(n) = n_{[0]} \otimes n_{[1]}, \quad \rho^l(n) = n_{[-1]} \otimes n_{[0]} . \]
We omit the summation symbol $\sum$.

1 Separable functors and Frobenius pairs of functors

Let $F : C \to D$ be a covariant functor. Recall [23] that $F$ is called a separable functor if the natural transformation
\[ F : \text{Hom}_C(\bullet, \bullet) \to \text{Hom}_D(F(\bullet), (\bullet)) , \]
induced by $F$ splits. From [23] and [39], we recall the following characterisation in the case $F$ has an adjoint.

**Proposition 1.1** Let $G : D \to C$ be a right adjoint of $F$. Let $\eta : 1_C \to GF$ and $\varepsilon : FG \to 1_D$ be the unit and counit of the adjunction. Then
1) $F$ is separable if and only if there exists $\nu \in V = \text{Nat}(GF, 1_C)$ such that $\nu \circ \eta = 1_C$, the identity natural transformation on $C$.
2) $G$ is separable if and only if there exists $\zeta \in W = \text{Nat}(1_D, FG)$ such that $\varepsilon \circ \zeta = 1_D$, the identity natural transformation on $C$.

The separability of $F$ implies a Maschke type Theorem for $F$: if a morphism $f \in C$ is such that $F(f)$ has a one-sided inverse in $D$, then $f$ has a one-sided inverse in $C$.

A pair of adjoint functors $(F, G)$ is called a Frobenius pair if $G$ is not only a right adjoint, but also a left adjoint of $F$. The following result can be found in any book on category theory: $G$ is a left adjoint of $F$ if and only if there exist natural transformations $\nu \in V = \text{Nat}(GF, 1_C)$ and $\zeta \in W = \text{Nat}(1_D, FG)$ such that
\begin{align*}
F(\nu_M) \circ \zeta_{F(M)} &= I_{F(M)}, \\
\nu_{G(N)} \circ G(\zeta_N) &= I_{G(N)},
\end{align*}
for all $M \in C$, $N \in D$. In order to decide whether $F$ or $G$ is separable, or whether $(F, G)$ is a Frobenius pair, one has to investigate the natural transformations $V = \text{Nat}(GF, 1_C)$ and $W = \text{Nat}(1_D, FG)$. It often happens that the natural transformations in $V$ and $W$ are determined by single maps. In this Section we illustrate this in a classical situation and recover well-known results. In the coming Sections more general situations are considered.

Let $i : R \to S$ be a ring homomorphism, and let $F = \bullet \otimes_R S : \mathcal{M}_R \to \mathcal{M}_S$ be the induction functor. The restriction of scalars functor $G : \mathcal{M}_S \to \mathcal{M}_R$ is a right adjoint of $F$. The unit and counit of the adjunction are
\[ \forall M \in \mathcal{M}_R, \quad \eta_M : M \to M \otimes_R S, \quad \eta_M(m) = m \otimes 1, \]
\[ \forall N \in \mathcal{M}_S, \quad \varepsilon_N : N \otimes_R S \to N, \quad \varepsilon_N(n \otimes s) = ns. \]
Let us describe $V$ and $W$. Given $\nu : GF \to 1_{\mathcal{M}_R}$ in $V$, it is not hard to prove that $\nabla = \nu_R : S \to R$ is left and right $R$-linear. Conversely, given an $R$-bimodule map $\nabla : S \to R$, a natural transformation $\nu \in V$ can be constructed by
\[ \forall M \in \mathcal{M}_R, \quad \nu_M(m \otimes s) = m\nabla(s). \]
Thus we have
\[ V \cong V_1 = \text{Hom}_{R,R}(S, R) . \]
Now let ζ : 1_{M_S} → FG be in W. Then \( e = \sum e^1 \otimes e^2 = \zeta_S(1) \in S \otimes_R S \) satisfies
\[
\sum s e^1 \otimes e^2 = \sum e^1 \otimes e^2 s,
\]
for all \( s \in S \). Conversely if \( e \) satisfies (4), then we can recover \( \zeta \)
\[
\forall N \in M_S, \quad \zeta_N : N \to N \otimes_R S, \quad \zeta_N(n) = ne^1 \otimes e^2.
\]
In the sequel, we omit the summation symbol, and write \( e = e^1 \otimes e^2 \), where it is understood implicitly that we have a summation. So we have
\[
W \cong W_1 = \{ e = e^1 \otimes e^2 \in S \otimes_R S \mid se^1 \otimes e^2 = e^1 \otimes e^2 s, \text{ for all } s \in S \}.
\]
Combining all these data, we obtain the following result (cf. [23] for 1) and 2) and [8] for 3))

**Theorem 1.2** Let \( i : R \to S \) be a ringhomomorphism, \( F \) the induction functor, and \( G \) the restriction of scalars functor.
1) \( F \) is separable if and only if there exists a conditional expectation, that is \( \nabla \in V_1 \) such that \( \nabla(1) = 1 \), i.e. \( S/R \) is a split extension.
2) \( G \) is separable if and only if there exists a separability idempotent, that is \( e \in W_1 \) such that \( e^1 e^2 = 1 \), i.e. \( S/R \) is a separable extension.
3) \((F,G)\) is a Frobenius pair if and only if there exist \( \nabla \in V_1 \) and \( e \in W_1 \) such that
\[
\nabla(e^1)e^2 = e^1\nabla(e^2) = 1.
\]

Theorem 1.2 2) explains the terminology for separable functors. Theorem 1.2 3) implies the following

**Corollary 1.3** We use the same notation as in Theorem 1.2. If \((F,G)\) is a Frobenius pair, then \( S \) is finitely generated and projective as a (right) \( R \)-module.

*Proof*. For all \( s \in S \), we have \( s = se^1\nabla(e^2) = e^1\nabla(e^2 s) \), hence \( \{e^1, \nabla(e^2 \bullet)\} \) is a dual basis for \( S \) as a right \( R \)-module. \( \Box \)

We have a similar property if \( G \) is separable. For the proof we refer to [24].

**Proposition 1.4** With the same notation as in Theorem 1.2, if \( S \) is an algebra over a commutative ring \( R \), \( S \) is projective as an \( R \)-module and \( G \) is separable, then \( S \) is finitely generated as an \( R \)-module.

Using other descriptions of \( V \) and \( W \), we find other criteria for \( F \) and \( G \) to be separable or for \((F,G)\) to be a Frobenius pair. Let \( \text{Hom}_R(S,R) \) be the set of right \( R \)-module homomorphisms from \( S \) to \( R \). \( \text{Hom}_R(S,R) \) is an \((R,S)\)-bimodule:
\[
(rfs)(t) = rf(ts),
\]
for all \( f \in \text{Hom}_R(S,R), r \in R \) and \( s,t \in S \).

**Proposition 1.5** Let \( i : R \to S \) be a ringhomomorphism and use the notation introduced above. Then
\[
V = \text{Nat}(GF, 1_C) \cong V_2 = \text{Hom}_{R,S}(S, \text{Hom}_{R,R}(S,R)).
\]
Proof. Define \( \alpha_1 : V_1 \to V_2 \) as follows: for \( \nu \in V_1 \), let \( \alpha_1(\nu) = \overline{\phi} : S \to \text{Hom}_R(S, R) \) be given by
\[
\overline{\phi}(s)(t) = \nu(st).
\]
Given \( \overline{\phi} \in V_2 \), put
\[
\alpha_{-1}^{-1}(\overline{\phi}) = \overline{\phi}(1).
\]
We invite the reader to verify that \( \alpha_1 \) and \( \alpha_{-1}^{-1} \) are well-defined and that they are inverses of each other. \( \square \)

**Proposition 1.6** Let \( i : R \to S \) be a ring homomorphism and assume that \( S \) is finitely generated and projective as a right \( R \)-module. Then, with the notation introduced above,
\[
W = \text{Nat}(1_{D}, FG) \cong W_2 = \text{Hom}_R(S, \text{Hom}_R(S, R), S) = \text{Hom}_R(S, R).
\]

Proof. Let \( \{s_i, \sigma_i \mid i = 1, \cdots, m \} \) be a finite dual basis of \( S \) as a right \( R \)-module. Then for all \( s \in S \) and \( f \in \text{Hom}_R(S, R) \),
\[
s = \sum_i s_i \sigma_i(s) \quad \text{and} \quad f = \sum_i f(s_i) \sigma_i.
\]
Define \( \beta_1 : W_1 \to W_2 \) by \( \beta_1(e) = \phi \), with
\[
\phi(f) = f(e^1)e^2,
\]
for all \( f \in \text{Hom}_R(S, R) \). To show that \( \phi \) is a left \( R \)-linear and right \( S \)-linear map, take any \( r \in R \), \( s \in S \) and compute
\[
\phi(fs) = f(s^1)e^2 = f(e^1)e^2s = \phi(f)s,
\]
\[
\phi(rf) = \sum rf(e^1)e^2 = r\phi(f).
\]
Conversely, for \( \phi \in W_2 \) define
\[
\beta_{-1}^{-1}(\phi) = e = \sum_i s_i \otimes \phi(\sigma_i).
\]
Then for all \( s \in S \)
\[
\sum_i s_i \otimes \phi(\sigma_i)s = \sum_i s_i \otimes \phi(\sigma_is) = \sum_{i,j} s_i \otimes (\phi(\sigma_i(s)s_j)\sigma_j)
\]
\[
= \sum_{i,j} s_i \otimes (\phi(\sigma_i(s)s_j) \otimes \phi(\sigma_j))
\]
\[
= \sum_{i,j} s_i \sigma_i(s)s_j \otimes \phi(\sigma_j) = \sum_j \sum_i s_j \otimes \phi(\sigma_i(s)s_j)
\]
\[
= \sum_j \sum_i s_j \sigma_i(s)s_j \otimes \sigma_j = \sum_j s_j \otimes \sigma_j
\]
\[
i.e., e \in W_1. \text{ Finally, } \beta_1 \text{ and } \beta_{-1}^{-1} \text{ are inverses of each other since}
\]
\[
\beta_1(\beta_{-1}^{-1}(\phi))(f) = \beta_1(\sum_i s_i \otimes \phi(\sigma_i))(f) = \sum_i f(s_i)\phi(\sigma_i) = \sum_i \phi(f(s_i)\sigma_i) = \phi(f),
\]
\[
\beta_{-1}^{-1}(\beta_1(e)) = \sum_i s_i \otimes \beta_1(e)(\sigma_i) = \sum_i s_i \otimes \sigma_i(e^1)e^2 = \sum_i s_i \sigma_i(e^1) \otimes e^2 = e.
\]
\( \square \)
Corollary 1.3, we know that

**Proof.** The result is a translation of Theorem 1.2 in terms of

Let $\phi$ be a commutative ring, $A$ a (flat) $k$-coalgebra, and $\psi: C \otimes A \rightarrow A \otimes C$ a $k$-linear map. We use the following notation, inspired by the Sweedler- Heyneman notation:

$$\psi(c \otimes a) = a_\psi \otimes c_\psi.$$ If the map $\psi$ occurs more than once in the same expression, we also use $\Psi$ or $\Psi'$ as summation indices, i.e.,

$$\psi(c \otimes a) = a_\psi \otimes c_\psi = a_{\psi'} \otimes c_{\psi'}.$$ (A,C,\psi) is called a (right-right) entwining structure if the following conditions are satisfied for all $a \in A$ and $c \in C$,

\begin{align*}
(ab)_\psi \otimes c_\psi &= a_\psi b_\psi \otimes c_\psi, \\
\varepsilon_C(c_\psi)_\psi a_\psi &= \varepsilon_C(c) a, \\
(a_\psi \otimes \Delta_C(c_\psi) &= a_{\psi'} \otimes c_{(1)}^{\psi'} \otimes c_{(2)}, \\
1_\psi \otimes c_\psi &= 1 \otimes c.
\end{align*}

A $k$-module $M$ together with a right $A$-action and a right $C$-coaction satisfying the compatibility relation

$$p'(ma) = m_0|_0 a_\psi \otimes m_1^{\psi} [1]$$

is called an entwined module. The category of entwined modules and $A$-linear $C$-colinear maps is denoted by $C = \mathcal{M}(\psi)^A$. An important class of examples comes from Doi-Koppinen-Hopf structures. A (right-right) Doi-Koppinen-Hopf structure consists of a triple $(H,A,C)$, where $H$ is a $k$-bialgebra, $A$ a right $H$-comodule algebra, and $C$ a right $H$-module coalgebra. Consider the map $\psi: C \otimes A \rightarrow A \otimes C$ given by

$$\psi(c \otimes a) = a_{[0]} \otimes ca_{[1]}.$$
Then \((A, C, \psi)\) is an entwining structure, and the compatibility relation (11) takes the form

\[
\rho'(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]},
\]

A \(k\)-module with an \(A\)-action and a \(C\)-coaction satisfying (12) is called a \(\text{Doi-Koppinen-Hopf module}\) or a \(\text{Doi-Hopf module}\). Doi-Koppinen-Hopf modules were introduced independently by Doi in [17] and Koppinen in [21]. Properties of these modules were studied extensively during the last decade, see e.g. [10], [11], [12], [13], [14], [15]. Another class of entwining structures is related to coalgebra Galois extensions, see [8] for details. Entwining structures were introduced in [7]. Many properties of Doi-Hopf modules can be generalized to entwined modules (see e.g. [3], [4]). Although the most studied examples of entwined modules (graded modules, Yetter-Drinfel’d modules, dimodules, Hopf modules) are special cases of Doi-Hopf modules, their properties can be formulated more elegantly in the language of entwined modules.

The functor \(F : C = \mathcal{M}(\psi)^C_A \to \mathcal{M}_A\) forgetting the \(C\)-coaction has a right adjoint \(G = \bullet \otimes C\). The structure on \(G(M) = M \otimes C\) is given by the formulae

\[
\rho'(m \otimes c) = m \otimes c_{(1)} \otimes c_{(2)},
\]

\[
(m \otimes c)a = ma_{\psi} \otimes c_{\psi}.
\]

For later use, we list the unit and counit natural transformations describing the adjunction,

\[
\rho : 1_C \to GF \quad \text{and} \quad \varepsilon : FG \to 1_{\mathcal{M}_A},
\]

\[
\rho_M : M \to M \otimes C, \quad \rho_M(m) = \sum m_{[0]} \otimes m_{[1]},
\]

\[
\varepsilon_N = I_N \otimes \varepsilon_C : N \otimes C \to N.
\]

In particular, \(A \otimes C \in \mathcal{M}(\psi)^C_A\). \(A \otimes C\) is also a left \(A\)-module, the left \(A\)-action is given by \(a(b \otimes c) = ab \otimes c\). This makes \(A \otimes C\) into an object of \(A\mathcal{M}(\psi)^C_A\), the category of entwined modules with an additional left \(A\)-action that is right \(A\)-linear and right \(C\)-colinear.

The other forgetful functor \(G' : \mathcal{M}(\psi)^C_A \to \mathcal{M}^C\) has a left adjoint \(F' = \bullet \otimes A\). The structure on \(F'(N) = N \otimes A\) is now given by

\[
\rho'(n \otimes a) = n_{[0]} \otimes a_{\psi} \otimes n_{[1]}^{\psi},
\]

\[
(n \otimes a)b = n \otimes ab.
\]

The unit and counit of the adjunction are

\[
\mu : F'G' \to 1_C \quad \text{and} \quad \eta : 1_{\mathcal{M}^C} \to G'F',
\]

\[
\mu_M : M \otimes A \to A, \quad \mu_M(m \otimes a) = ma,
\]

\[
\eta_N : N \to N \otimes A, \quad \eta_N(n) = n \otimes 1.
\]

In particular \(G'(C) = C \otimes A \in \mathcal{M}(\psi)^C_A\). The map \(\psi : C \otimes A \to A \otimes C\) is a morphism in \(\mathcal{M}(\psi)^C_A\). \(C \otimes A\) is also a left \(C\)-comodule, the left \(C\)-coaction being induced by the comultiplication on \(C\). This coaction is right \(A\)-linear and right \(C\)-colinear, and thus \(C \otimes A\) is an object of \(C\mathcal{M}(\psi)^C_A\), the category of entwined modules together with a right \(A\)-linear right \(C\)-colinear left \(C\)-coaction.
3 The functor forgetting the coaction

Let \((A, C, \psi)\) be a right-right entwining structure, \(F : \mathcal{M}(\psi)^C_A \to \mathcal{M}_A\) the functor forgetting the coaction, and \(G = \bullet \otimes C\) its adjoint. In \([13]\) necessary and sufficient conditions for \((F, G)\) to be a Frobenius pair are given (in the Doi-Hopf case; the results were generalized to the entwining case in \([3]\)), under the additional assumption that \(C\) is projective as a \(k\)-module. In this Section we give an alternative characterization that also holds if \(C\) is not necessarily projective, and we find a new proof of the results in \([13]\) and \([3]\). The method of proof is the same as in Section \([1]\), i.e., based on explicit descriptions of \(V\) and \(W\). These descriptions can be found in \([10]\), \([11]\) and \([8]\) in various degrees of generality. To keep this paper self-contained, we give a sketch of proof. We first investigate \(V = \text{Nat}(GF, 1_C)\). Let \(V_1\) be the \(k\)-module consisting of all \(k\)-linear maps \(\theta : C \otimes C \to A\) such that

\[
\theta(c \otimes d)a = a_{\psi,\psi} \theta(c^\psi \otimes d^\psi),
\]

\[
\theta(c \otimes d(1)) \otimes d(2) = \theta(c(2) \otimes d_\psi \otimes c_1^\psi).
\]

**Proposition 3.1** The map \(\alpha : V \to V_1\) given by \(\alpha(v) = \theta\), with

\[
\theta(c \otimes d) = (I_A \otimes \varepsilon_C)(v_{A \otimes C}(1_A \otimes c \otimes d)),
\]

is an isomorphism of \(k\)-modules. The inverse \(\alpha^{-1}(\theta) = \nu\) is defined as follows: \(\nu_M : M \otimes C \to M\) is given by

\[
\nu_M(m \otimes c) = m_{[0]} \theta(m_{[1]} \otimes c).
\]

**Proof.** Consider \(\nu = v_{A \otimes C}\) and \(\nabla = v_{C \otimes A}\). Due to the naturality of \(v\) and \((7)\) there is a commutative diagram

\[
\begin{array}{ccc}
C \otimes A \otimes C & \xrightarrow{\nabla} & C \otimes A \\
\downarrow \psi \otimes I_C & & \downarrow \psi \\
A \otimes C \otimes C & \xrightarrow{\nu} & A \otimes C
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow I_A & & \downarrow I_A \\
A \otimes C & \xrightarrow{\theta} & A
\end{array}
\]

Write \(\widetilde{\lambda} = (I_A \otimes \varepsilon_C) \circ \nabla\) and \(\lambda = (\varepsilon_C \otimes I_A) \circ \nu\). Then it follows that

\[
\theta(c \otimes d) = \widetilde{\lambda}(c \otimes 1 \otimes d) = \lambda(1 \otimes c \otimes d).
\]

We have seen before that \(A \otimes C \in \mathcal{A} \mathcal{M}(\psi)^C_A\). It is easy to prove that \(GF(A \otimes C) = A \otimes C \otimes C \in \mathcal{A} \mathcal{M}(\psi)^C_A\) - the left \(A\)-action is induced by the multiplication in \(A\) - and \(\nabla\) is a morphism in \(\mathcal{A} \mathcal{M}(\psi)^C_A\). Thus \(\nu\) and \(\lambda\) are left and right \(A\)-linear, and

\[
\theta(c \otimes d)a = \lambda(1 \otimes c \otimes d)a = \lambda(a_{\psi,\psi}c^\psi \otimes d^\psi)
\]

\[
= a_{\psi,\psi} \lambda(1 \otimes c^\psi \otimes d^\psi) = a_{\psi,\psi} \theta(c^\psi \otimes d^\psi),
\]

proving \((17)\). To prove \((18)\), we first observe that \(C \otimes A, GF(C \otimes A) = C \otimes C \otimes A \in \mathcal{C} \mathcal{M}(\psi)^C_A\), the left \(C\)-coaction is induced by comultiplication in \(C\) in the first factor. Also \(\nabla\) is a morphism in \(\mathcal{C} \mathcal{M}(\psi)^C_A\), and we conclude that \(\nabla\) is left and right \(C\)-colinear. Take \(c, d \in C\), and put

\[
\nabla(c \otimes d \otimes 1) = \sum_i c_i \otimes a_i.
\]
Writing down the condition that $\nabla$ is left $C$-colinear, and then applying $\varepsilon_C$ to the second factor, we find that
\[
c_{(1)} \otimes \theta(c_{(2)} \otimes d) = \sum_i c_i \otimes a_i = \nabla(c \otimes d \otimes 1).
\] (21)

Since $\nabla$ is also right $C$-colinear,
\[
\nabla(c \otimes 1 \otimes d_{(1)}) \otimes d_{(2)} = \sum_i c_{i(1)} \otimes a_i \psi \otimes c_{i(2)}^\psi
\]
and, applying $\varepsilon_C$ to the second factor, we find
\[
\theta(c \otimes d_{(1)}) \otimes d_{(2)} = \psi(\sum_i c_i \otimes a_i),
\] (22)
and (18) follows from (21) and (22). This proves that there is a well-defined map $\alpha : V \to V_1$.

To show that the map $\alpha^{-1}$ defined by (20) is well-defined, take $\theta \in V_1$, $M \in \mathcal{C}$, and let $\nu_M$ be given by (20). It needs to be shown that $\nu_M \in \mathcal{C}$, i.e., $\nu_M$ is right $A$-linear and right $C$-colinear, and that $\nu$ is a natural transformation. The right $A$-linearity follows from (17), and the right $C$-colinearity from (18). Given any morphism $f : M \to N$ in $\mathcal{C}$, one easily checks that for all $m \in M$ and $c \in C$
\[
\nu_N(f(m) \otimes c) = f(m[0]) \theta(m[1] \otimes c) = f(m[0] \theta(m[1] \otimes c)) = f(\nu_M(m \otimes c)),
\]
i.e., $\nu$ is natural. The verification that $\alpha$ and $\alpha^{-1}$ are inverses of each other is left to the reader. \qed

Now we give a description of $W = \text{Nat}(1_{\mathcal{M}_A}, FG)$. Let
\[
W_1 = \{ z \in A \otimes C \mid az = za, \text{ for all } a \in A \},
\]
i.e., $z = \sum_i a_i \otimes c_i \in W_1$ if and only if
\[
\sum_i aa_i \otimes c_i = \sum_i a_i a^\psi \otimes c_i^\psi.
\] (23)

**Proposition 3.2** Let $(A, C, \psi)$ be a right-right entwining structure. Then there is an isomorphism of $k$-modules $\beta : W \to W_1$ given by
\[
\beta(\zeta) = \zeta_A(1).
\] (24)
The inverse of $\beta$ is $\beta^{-1}(\sum_i a_i \otimes c_i) = \zeta$, with $\zeta_N : N \to N \otimes C$ given by
\[
\zeta_N(n) = \sum_i na_i \otimes c_i.
\] (25)

*Proof.* We leave the details to the reader; the proof relies on the fact that $\zeta_A$ is left and right $A$-linear. \qed

In [10], Propositions 3.1 and 3.2 are used to determine when the functor $F$ and its adjoint $G$ are separable.

**Theorem 3.3** Let $F : \mathcal{M}(\psi)_A \to \mathcal{M}_A$ be the forgetful functor, and $G = \bullet \otimes C$ its adjoint.

$F$ is separable if and only if there exists $\theta \in V_1$ such that
\[
\theta \circ \Delta_C = \varepsilon_C.
\]

$G$ is separable if and only if there exists $z = \sum_i a_i \otimes c_i \in W_1$ such that
\[
\sum_i \varepsilon_C(c_i) a_i = 1.
\]
Proof. This follows immediately from Propositions [1.1, 3.1] and 3.2

Next we show that the fact that \((F, G)\) is a Frobenius pair is also equivalent to the existence of \(\theta \in V_1\) and \(z \in W_1\), but now satisfying different normalizing conditions.

**Theorem 3.4** Let \(F : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_A\) be the forgetful functor, and \(G = \bullet \otimes C\) its adjoint. Then \((F, G)\) is a Frobenius pair if and only if there exist \(\theta \in V_1\) and \(z = \sum_l a_l \otimes c_l \in W_1\) such that the following normalizing condition holds, for all \(d \in C\):

\[
\varepsilon_C(d)1 = \sum_l a_l \theta(c_l \otimes d)\tag{26}
\]

\[
= \sum_l a_l \psi \theta(l_d \otimes c_l).	ag{27}
\]

Proof. Suppose that \((F, G)\) is a Frobenius pair. Then there exist \(v \in V\) and \(\zeta \in W\) such that \((26)\) hold. Let \(\theta = \alpha(v) \in V_1\), and \(z = \sum_l a_l \otimes c_l = \beta(\zeta) \in W_1\). Then \((26)\) can be rewritten as

\[
v_M(\sum_l ma_l \otimes c_l) = \sum_m a_m \psi \theta(m_\psi \otimes c_l) = m,	ag{28}
\]

for all \(m \in M \in \mathcal{M}(\psi)^C_A\). Taking \(M = C \otimes A\), \(m = d \otimes 1\), one obtains \((27)\).

For all \(n \in N \in \mathcal{M}_A\) and \(c \in C\), one has

\[
v_{G(N)}(G(\psi_N)(n \otimes d)) = v_{G(N)}(\sum_l na_l \otimes c_l \otimes d)
= \sum_l (na_l \otimes c_l) \theta(c_l(2) \otimes d)
= \sum_l na_l \theta(c_l(2) \otimes d) \psi \otimes c_l(1)
\]

\[= \sum_l na_l \theta(c_l \otimes d(1) \otimes d(2)), \tag{28}
\]

and \((28)\) can be written as

\[
n \otimes d = \sum_l na_l \theta(c_l \otimes d(1)) \otimes d(2),\tag{29}
\]

for all \(n \in N \in \mathcal{M}_A\) and \(d \in C\). Taking \(N = A\) and \(n = 1\), one obtains

\[1 \otimes d = \sum_l a_l \theta(c_l \otimes d(1)) \otimes d(2).
\]

Applying \(\varepsilon_C\) to the second factor, one finds \((26)\).

Conversely, suppose that \(\theta \in V_1\) and \(z \in W_1\) satisfy \((26)\) and \((27)\). \((27)\) implies \((28)\), and \((26)\) implies \((29)\). Let \(v = \alpha^{-1}(\theta)\), \(\zeta = \beta^{-1}(z)\). Then \((13)\) hold, and \((F, G)\) is a Frobenius pair. \(\square\)

In [13] it is shown that if \((H, A, C)\) is a Doi-Hopf structure, \(A\) is faithfully flat as a \(k\)-module, and \(C\) is projective as a \(k\)-module, then \(C\) is finitely generated. The next proposition shows that, in fact, one does not need the assumption that \(C\) is projective.

**Proposition 3.5** Let \((A, C, \psi)\) be a right-right entwining structure. If \((F, G)\) is a Frobenius pair, then \(A \otimes C\) is finitely generated and projective as a left \(A\)-module.
Proof. Let θ and \( z = \sum_i a_i \otimes c_i \) be as in Theorem 3.4. Then for all \( d \in C \),

\[
1 \otimes d = \Psi(d \otimes 1) = \psi(\langle d_1 \rangle \otimes e(\langle 2 \rangle) 1) = \sum_i \psi \left( d_1 \otimes a_i \Psi \langle d_2 \rangle \otimes c_i \right)
\]

so then for all \( \theta \in A \) and \( c \in C \),

\[
\Psi(\langle 1 \rangle) \otimes \theta(\langle 2 \rangle) \otimes c = \sum_i \psi \left( d_1 \otimes a_i \Psi \langle d_2 \rangle \otimes c_i \right)
\]

\[
\Psi(\langle 1 \rangle) \otimes \theta(\langle 2 \rangle) \otimes c = \sum_i \psi \left( d_1 \otimes a_i \Psi \langle d_2 \rangle \otimes c_i \right)
\]

Write \( c_i(1) \otimes c_i(2) = \sum c_i c_i^j \otimes c_i^j \) and for all \( l, j \) consider the map

\[
\sigma_{ij} : A \otimes C \rightarrow A, \quad \sigma_{ij}(a \otimes d) = a a_i \Psi \langle d \rangle \otimes c_i^j \).
\]

Then for all \( a \in A \) and \( d \in C \),

\[
a \otimes d = \sum_{i, j} \sigma_{ij}(a \otimes d)(1 \otimes c_i^j),
\]

so \( \{ \sigma_{ij}, c_i^j \mid i = 1, \cdots, n, j = 1, \cdots, m_i \} \) is a finite dual basis for \( A \otimes C \) as a left \( A \)-module.

In some situations, one can conclude that \( C \) is finitely generated and projective as a \( k \)-module.

Corollary 3.6 Let \((A, C, \Psi)\) be a right-right entwining structure, and assume that \((F, G)\) is a Frobenius pair.  
1) If \( A \) is faithfully flat as a \( k \)-module, then \( C \) is finitely generated as a \( k \)-module.  
2) If \( A \) is commutative and faithfully flat as a \( k \)-module, then \( C \) is finitely generated projective as a \( k \)-module.  
3) If \( k \) is a field, then \( C \) is finite dimensional as a \( k \)-vector space.  
4) If \( A = k \), then \( C \) is finitely generated projective as a \( k \)-module.

Proof. 1) With notation as in Proposition 3.5, let \( M \) be the \( k \)-module generated by the \( c_i^j \). Then for all \( d \in C \),

\[
1 \otimes d = \sum_{i, j} \sigma_{ij}(1 \otimes d)(1 \otimes c_i^j) \in A \otimes M.
\]

Since \( A \) is faithfully flat, it follows that \( d \in M \), hence \( M = C \) is finitely generated.

2) From descent theory: if a \( k \)-module becomes finitely generated and projective after a faithfully flat commutative base extension, then it is itself finitely generated and projective.

3) Follows immediately from 1): since \( k \) is a field, \( A \) is faithfully flat as a \( k \)-module, and \( C \) is projective as a \( k \)-module.

4) Follows immediately from 2).
This can be checked directly. An explanation for this at first sight artificial structure is given in Section \[\text{Proposition 3.1}\]. We now give alternative descriptions for \(\nu\). As we have seen, a natural transformation \(V_k\)-module, then it turns out that \(\nu\) isomorphic to \(\nu\) as the image space, at some other we prefer \(\nu\). This can be checked directly. An explanation for this at first sight artificial structure is given in Section 5. We first show that \(\Phi\) is left \(\lambda\)-linear. It is also right \(\lambda\)-linear because

\[
\Phi(a \otimes c)(d) = \lambda_\lambda(a, d^\Psi \otimes c) = a_{\Psi}(d^\Psi \otimes c),
\]

or

\[
\Phi(a \otimes c) = \sum_i d_i^\Psi \otimes a_{\Psi}(d_i^\Psi \otimes c).
\]  \hspace{1cm} (32)

It turns out that \(\Phi\) is a morphism in \(A\). More specifically, one has

**Proposition 3.7** Let \((A, C, \Psi)\) be a right-right entwining structure. If \(C\) is a finitely generated and projective \(k\)-module, then

\[
V \cong V_1 \cong V_2 = \text{Hom}_{AA}(A \otimes C, C^\ast \otimes A).
\]

The isomorphism is \(\alpha_1 : V_1 \to V_2\), with \(\alpha_1(\Phi) = \Phi\) given by (32). The inverse of \(\alpha_1\) is

\[
\alpha_1^{-1}(\Phi)(d \otimes c) = \Phi(1 \otimes c)(d).
\]  \hspace{1cm} (33)

**Proof.** We first show that \(\Phi \in V_2\). For all \(a, b \in A\) and \(c \in C\), we have

\[
\Phi(a \otimes c)(d) = b \left( \sum_i d_i^\Psi \otimes a_{\Psi}(d_i^\Psi \otimes c) \right)
\]

proving that \(\Phi\) is left \(A\)-linear. It is also right \(A\)-linear because

\[
\Phi(a \otimes c)b = \sum_i d_i^\Psi \otimes a_{\Psi}(d_i^\Psi \otimes c)b
\]

\[
= \sum_i (d_i^\Psi \otimes b_{\Psi}a_{\Psi}(d_i^\Psi \otimes c))
\]

\[
= \sum_i d_i^\Psi \otimes b_{\Psi}a_{\Psi}(d_i^\Psi \otimes c)
\]

\[
= \sum_i d_i^\Psi \otimes (ba_{\Psi})(d_i^\Psi \otimes c)
\]

\[
= \Phi(ba \otimes c) = \Phi(b(a \otimes c)),
\]

proving that \(\Phi\) is right \(A\)-linear. Notice that the dual basis for \(C\) satisfies the following equality (the proof is left to the reader):

\[
\sum_i \Delta(d_i) \otimes d_i^\ast = \sum_{i,j} d_i \otimes d_j \otimes d_i^\ast \ast d_j^\ast.
\]  \hspace{1cm} (34)
Using this equality one computes
\[ \rho^\nu(\varphi(a \otimes c)) = \rho^\nu\left(\sum_i d_i^a \otimes a_\psi \theta(d_i^\psi \otimes c)\right) \]

\[ (31) = \sum_{i,j} d_i^a \otimes d_j^\nu \cap \left(a_\psi \theta(d_i^\psi \otimes c)\right) \otimes d_j^\nu \]

\[ (34) = \sum_i d_i^a \otimes \left(a_\psi \theta(d_i^\psi \otimes c)\right) \otimes d_i^\nu \]

\[ (37) = \sum_i d_i^a \otimes a_\psi \theta\left(d_i^\psi \otimes c\right) \otimes d_i^\nu \]

\[ (38) = \sum_i d_i^a \otimes a_\psi \theta\left(d_i^\psi \otimes c\right) \otimes c(1) \otimes c(2) \]

\[ = \varphi(a \otimes c(1)) \otimes c(2). \]

This proves that \( \varphi \) is right \( C \)-colinear. Conversely, given \( \varphi \in V_2 \), first one needs to show that \( \theta = \alpha^{-1}_1(\varphi) \in V_1 \).

It is now more convenient to work with \( \text{Hom}(C, A) \) rather than \( C^* \otimes A \). For \( f \in \text{Hom}(C, A) \), \( b, b' \in A \), \( (34) \) can be rewritten as
\[ (bfb')(c) = b_\psi f(c^\psi)b'. \] (35)

Take any \( c, d \in C, a \in A \) and compute
\[ \theta(c \otimes d)a = \left(\varphi(1 \otimes d)(c)\right)a \]

\[ (35) = \left(\varphi(1 \otimes d)(c)\right)a \]

\[ \left(\varphi \right) \text{ is right } A \text{-linear} = \left(\varphi(a) \otimes d^\nu\right)(c) \]

\[ \left(\varphi \right) \text{ is left } A \text{-linear} = \left(a_\psi \varphi(1 \otimes d^\nu)\right)(c) \]

\[ (35) = a_\psi \varphi\left(\varphi(1 \otimes d^\nu)(c^\nu)\right) \]

\[ = a_\psi \varphi\varphi(c^\nu \otimes d^\nu). \]

This proves that \( \theta \) satisfies \( (17) \). Before proving \( (18) \), we look at the right \( C \)-coaction \( \rho^\nu \) on \( f = c^* \otimes a \in \text{Hom}(C, A) \cong C^* \otimes A \). Write \( \rho^\nu(f) = f_{0[0]} \otimes f_{1[1]} \in \text{Hom}(C, A) \otimes C \). Using \( (34) \), we find, for all \( c \in C \),
\[ f_{0[0]}(c) \otimes f_{1[1]} = \sum_i \langle d_i^a \otimes c^*, c \rangle a d_i^\nu \]

\[ = \sum_i \langle d_i^a, c(1) \rangle \langle c^*, c(2) \rangle a d_i^\nu \]

\[ = \langle c^*, c(2) \rangle a d_i^\nu \]

\[ = \psi(c(1) \otimes f(c(2))). \]

This means that for all \( f \in \text{Hom}(C, A) \)
\[ f_{0[0]}(c) \otimes f_{1[1]} = \psi(c(1) \otimes f(c(2))). \] (36)

This can be used to show that \( \theta \) satisfies \( (18) \). Explicitly,
\[ \theta(c(2) \otimes d) \psi \psi c(1) = \psi(c(1) \otimes \theta(c(2) \otimes d)) \]
\((\overline{\phi})\) = \(\overline{\phi}(1 \otimes d)(c) \otimes \overline{\phi}(1 \otimes d)(1)\)

\((\overline{\phi})\) is right \(C\)-colinear

= \(\overline{\phi}(1 \otimes d(1))(c) \otimes d(2)\)

= \(\theta(c \otimes d(1)) \otimes d(2)\).

It remains to be shown that \(\alpha_1\) and \(\alpha_1^{-1}\) are inverses of each other. First take \(\theta \in V_1\). Then for all \(c, d \in C\),

\[
\left((\alpha_1^{-1} \circ \alpha_1)(\theta)\right)(d \otimes c) = \alpha_1(\theta)(1 \otimes c)(d) = \sum_l \langle d^*_l, d \rangle_1 \psi \theta(d^*_l \otimes c) = \theta(d \otimes c).
\]

Finally, for \(\overline{\phi} \in V_2, a \in A\) and \(c, d \in C\):

\[
\left((\alpha_1 \circ \alpha_1^{-1})(\overline{\phi})\right)(a \otimes c)(d) = \sum_l \langle d^*_l, d \rangle a \psi \alpha_1^{-1}(\overline{\phi})(d^*_l \otimes c) = \sum_l \langle d^*_l, d \rangle a \psi \overline{\phi}(1 \otimes c)(d^*_l) = a \psi \overline{\phi}(1 \otimes c)(d) = (a \overline{\phi}(1 \otimes c))(d) = \overline{\phi}(a \otimes c)(d).
\]

\(\square\)

Now we give an alternative description for \(W_2\).

**Proposition 3.8** Let \(C\) be finitely generated and projective as a \(k\)-module. Then

\(W \cong W_1 \cong W_2 = \text{Hom}^{kC}_{AA}(C^* \otimes A, A \otimes C)\).

The isomorphism \(\beta_1: W_1 \rightarrow W_2\) is given by \(\beta_1(z) = \phi\) with

\[\phi(c^* \otimes a) = \sum_l a_l a \psi \otimes \langle c^*, c_{l(2)} \rangle c^*_{l(1)}\],

and the inverse of \(\beta_1\) is given by

\[\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1)\]. (37)

**Proof.** We have to show that \(\beta_1(z) = \phi\) is left and right \(A\)-linear and right \(C\)-colinear. For all \(c^* \in C^*\) and \(a, b \in A\),

\[\Phi(c^* \otimes ab) = \sum_l a_l a \psi(b \otimes \langle c^*, c_{l(2)} \rangle c^*_{l(1)})\]

\[
= \sum_l a_l a \psi \otimes \langle c^*, c_{l(2)} \rangle c^*_{l(1)}b
= (\sum_l a_l a \psi \otimes \langle c^*, c_{l(2)} \rangle c^*_{l(1)})b
= \Phi(c^* \otimes a)b,
\]
same procedure to determine when $a \in z$

Take $\varepsilon$

Conversely, let $\Phi$ be right $A$-linear. The proof of left $A$-linearity goes as follows:

$$
\Phi(b(c^* \otimes a)) = \sum_i \Phi\left( (c^*, d_i^\Psi) d_i^* \otimes b \otimes a \right)
= \sum_i \langle c^*, d_i^\Psi \rangle a_i b \otimes (d_i^*, c_{l(2)}) c_{l(1)}^\Psi
= \sum_i \langle c^*, d_i^\Psi \rangle a_i b a \otimes (d_i^*, c_{l(2)}) c_{l(1)}^\Psi
= \sum_i \langle c^*, d_i^\Psi \rangle a_i b a \otimes (c_{l(2)}) c_{l(1)}^\Psi
= \sum_i \langle c^*, (c_{l(2)}) a_i b a \otimes (c_{l(2)}) c_{l(1)}^\Psi
= b \Phi(c^* \otimes a).
$$

Next one needs to show that $\Phi$ is right $C$-colinear. Using (33), one finds

$$
\overline{\Phi}(c^* \otimes a) \otimes (c^* \otimes a) = \sum_i \overline{\Phi}(d_i^* \otimes c^* \otimes a) \otimes d_i^\Psi
= \sum_i a_i a \otimes (d_i^*, c_{l(2)}) c_{l(1)}^\Psi \otimes d_i^\Psi
= \sum_i a_i a \otimes (c_{l(2)}, c_{l(3)}) c_{l(1)} \otimes d_i^\Psi
= \sum_i a_i a \otimes (c_{l(2)}, c_{l(3)}) c_{l(1)} \otimes c_{l(2)}^\Psi
= \sum_i a_i a \otimes (c_{l(2)}, c_{l(3)}) c_{l(1)} \otimes c_{l(2)}^\Psi
= \rho^*(\overline{\Phi}(c^* \otimes a)).
$$

Conversely, let $\phi \in W_2$ and put $z = \phi(\varepsilon \otimes 1) = \sum_i a_i \otimes c_i$. Using (33), we see that $a(\varepsilon \otimes 1) = (\varepsilon \otimes 1) a$, for all $a \in A$, hence $az = a\phi(\varepsilon \otimes 1) = \phi(a(\varepsilon \otimes 1)) = \phi(\varepsilon \otimes 1)a = a$, and $z \in W_1$.

Take $z = \sum_i a_i \otimes c_i \in W_1$. Then

$$
\beta_{i-1}(\beta_1(z)) = \sum_i a_i 1_{\Psi} \otimes (\varepsilon, c_{l(2)}) c_{l(1)}^\Psi = z.
$$

Finally, take $\phi \in W_2$, and write $\beta_{i-1}(\phi) = \phi(\varepsilon \otimes 1) = \sum_i a_i \otimes c_i$. $C^* \otimes A$ and $A \otimes C$ are right $C$-comodules and left $C^*$-modules. Since $\phi$ is right $A$-linear, right $C$-colinear and left $C^*$-linear,

$$
\phi(c^* \otimes a) = \phi(c^* \otimes 1) a = (c^* \cdot \phi(\varepsilon \otimes 1)) a = (c^* \cdot (\sum_i a_i \otimes c_i)) a
= \langle c^*, c_{l(2)} \rangle \sum_i a_i a \otimes c_{l(1)}^\Psi = \beta_1(z)(c^* \otimes a)
$$

and it follows that $\phi = \beta_1(z) = \beta_1(\beta_{i-1}(\phi))$, as required.

\[\square\]

Suppose that $C$ is finitely generated and projective as a $k$-module. From Proposition 3.7, it follows that $F$ is separable if and only if there exists a map $\phi \in V = \text{Hom}_k(A \otimes C, C^* \otimes A)$ such that $\phi(1 \otimes c_{l(2)}) c_{l(1)} = \varepsilon(c) 1$, for all $c \in C$. In the Doi-Hopf case, this implies the Maschke Theorem in [12]. Now we apply the same procedure to determine when $(F, G)$ is a Frobenius pair.
Theorem 3.9 Consider an entwining structure \((A, C, \psi)\), and assume that \(C\) is finitely generated projective as a \(k\)-module. Let \(F : M(\psi)^C_A \to M_A\) be the functor forgetting the \(C\)-coaction, and \(G = \bullet \otimes C\) be its right adjoint. Then the following statements are equivalent:

1) \((F, G)\) is a Frobenius pair.

2) There exist \(z = \sum a_i \otimes c_i \in W_1\) and \(\theta \in V_1\) such that the maps

\[
\phi : C^* \otimes A \to A \otimes C \quad \text{and} \quad \overline{\phi} : A \otimes C \to C^* \otimes A,
\]

given by

\[
\phi(c^* \otimes a) = \sum_i a_i \psi(c^*, c_i(2))c_i^{\psi}(c_i(1)), \quad (38)
\]

\[
\overline{\phi}(a \otimes c) = \sum_i d_i^* \otimes a_i \theta(d_i^{\psi} \otimes c), \quad (39)
\]

are inverses of each other.

3) \(C^* \otimes A\) and \(A \otimes C\) are isomorphic as objects in \(A M(\psi)^C_A\).

Proof. 1) \(\Rightarrow\) 2). Let \(z \in W_1\) and \(\theta \in V_1\) be as in Theorem 3.4. Then \(\phi = \beta_1(z)\) and \(\overline{\phi} = \alpha_1(\theta)\) are morphisms in \(A M(\psi)^C_A\), and

\[
\overline{\phi}(\varepsilon \otimes 1) = \overline{\phi}(z) = \sum_{i,l} d_i^* \otimes a_i \psi(\theta(d_i^{\psi} \otimes c_l)) = \sum_{i,l} d_i^* \otimes \varepsilon(d_l)1 = \varepsilon \otimes 1. \quad (27)
\]

The fact that \(\phi\) and \(\overline{\phi}\) are right \(A\)-linear and left \(C^*\)-linear implies that \(\overline{\phi} \circ \phi = I_{C^* \otimes A}\). Similarly, for all \(c \in C\),

\[
\phi(\overline{\phi}(1 \otimes c)) = \phi(\sum_i d_i^* \otimes \theta(d_i \otimes c)) = \sum_{i,l} a_i \theta(d_i \otimes c) \psi(\theta(d_i^{\psi} \otimes c_l))c_i^{\psi}(c_l(1)) = \sum_{i,l} a_i \theta(c_l(2) \otimes c) \psi(c_l(1))c_i^{\psi}(c_l(2)) = \sum_{i,l} a_i \theta(c_{l(1)} \otimes c_{l(2)})(c_i(1)) \otimes c_{l(2)} = \varepsilon(c_i(1))1 \otimes c_{l(2)} = 1 \otimes c. \quad (26)
\]

Since \(\phi\) and \(\overline{\phi}\) are left \(A\)-linear, \(\phi \circ \overline{\phi} = I_{A \otimes C}\).

2) \(\Rightarrow\) 3). Obvious, since \(\phi\) and \(\overline{\phi}\) are in \(A M(\psi)^C_A\).

3) \(\Rightarrow\) 1). Let \(\phi : C^* \otimes A \to A \otimes C\) be the connecting isomorphism, and put \(z = \phi(\varepsilon \otimes 1) = \sum_l a_l \otimes c_l \in W_1\), \(\theta = \alpha_i^{-1}(\phi^{-1}) \in V_1\). Applying (32) and (37), one finds

\[
\varepsilon \otimes 1 = \phi^{-1}(\phi(\varepsilon \otimes 1)) = \sum_i d_i^* \otimes a_i \psi(\theta(d_i^{\psi} \otimes c_l)).
\]

Evaluating this equality at \(c \in C\), one obtains (27). For all \(c \in C\),

\[
1 \otimes c = \phi(\phi^{-1}(1 \otimes c)) = \sum_l a_l \theta(c_l \otimes c_{l(1)}) \otimes c_{l(2)}.
\]

Applying \(\varepsilon\) to the second factor, one finds (26). Theorem 3.4 implies that \((F, G)\) is a Frobenius pair. \(\Box\)
Remark 3.10 Recently M. Takeuchi observed that entwined modules can be viewed as comodules over certain corings. This observation has been exploited in [5] to derive some properties of coring counterparts of functors $F$ and $G$. It is quite clear that the procedure applied in Section 1 to extension and restriction of scalars can be adapted to functors associated to corings, leading to a generalization of the results in this Section. This will be the subject of a future publication.

4 The functor forgetting the $A$-action

Again, let $(A, C, \psi)$ be a right-right entwining structure. The functor $G' : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}^C$ forgetting the $A$-action has a left adjoint $F'$. The unit $\mu$ and the counit $\eta$ of the adjunction are given at the end of Section 2.

Lemma 4.1 Let $M \in \mathcal{M}(\psi)_A^C$, $N \in \mathcal{M}(\psi)_A^C$. Then $F'G'(M) \in \mathcal{M}(\psi)_A^C$ and $G'F' \in \mathcal{M}(\psi)_A^C$. The left structures are given by

$$a(m \otimes b) = am \otimes b \quad \text{and} \quad p^l(n \otimes b) = \sum n[1] \otimes n[0] \otimes b,$$

for all $a, b \in A$, $m \in M$, $n \in N$. Furthermore $\mu_M$ is left $A$-linear, and $\nu_N$ is left $C$-colinear.

Now write $V' = \text{Nat}(G'F', 1_{\mathcal{M}(\psi)}), W' = \text{Nat}(1_C, F'G')$. Following the philosophy of the previous Sections, we give more explicit descriptions of $V'$ and $W'$. We do not give detailed proofs, however, since the arguments are dual to the ones in the previous Section. Let

$$V'_1 = \{ \hat{\vartheta} \in (C \otimes A)^s \mid \hat{\vartheta}(c(1) \otimes a) = \hat{\vartheta}(c(2) \otimes a)\}, \quad \text{for all } c \in C, a \in A \}. \quad (40)$$

Proposition 4.2 The map $\alpha : V' \rightarrow V'_1$, $\alpha(v') = \varepsilon \circ \nu_C$ is an isomorphism.

Proof. Details are left to the reader. Given $\hat{\vartheta} \in V'$, for $N \in \mathcal{M}^C$, the natural map $\nu'_N : N \otimes A \rightarrow N$

$$\nu'_N(n \otimes a) = \sum \hat{\vartheta}(n[1] \otimes a)n[0].$$

For any $k$-linear map $e : C \rightarrow A \otimes A$ and $c \in C$, we use the notation $e(c) = e^1(c) \otimes e^2(c)$ (summation understood). Let $W'_1$ be the $k$-submodule of $\text{Hom}(C, A \otimes A)$ consisting of maps $e$ satisfying

$$e^1(c(1)) \otimes e^2(c(1)) \otimes c(2) = e^1(c(2)) \otimes e^2(c(2)) \otimes c^{wv}, \quad (41)$$

$$e^1(c) \otimes e^2(c)a = a e^1(c)\otimes e^2(c). \quad (42)$$

Proposition 4.3 The map $\beta : W' \rightarrow W'_1$ given by

$$\beta(\zeta') = (\varepsilon \otimes I_A \otimes I_A) \circ \zeta'_1 \circ (\eta_A \otimes I_C)$$

$$= (I_A \otimes \varepsilon \otimes I_A) \circ \zeta'_2 \circ (I_C \otimes \eta_A)$$

is an isomorphism. Given $e \in W'_1$, $\zeta' = \beta^{-1}(e)$ is recovered from $e$ as follows: for $M \in \mathcal{M}(\psi)_A^C$,

$$\zeta'_M(m) = \sum m[0]e^1(m[1]) \otimes e^2(m[1]).$$
Proof. We show that $\beta$ is well-defined, leaving other details to the reader. Consider a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{I_C \otimes \eta_A} & C \otimes A \\
& \searrow \psi \swarrow & \psi \otimes I_A \\
C & \xrightarrow{\eta_A \otimes I_C} & A \otimes C \\
\end{array}
\]

The map $\lambda = \zeta'_C \otimes (I_C \otimes \eta_A)$ is left and right C-colinear. Write $\lambda(c) = \sum c_i \otimes a_i \otimes d'_i$. Then

\[
c_{(1)} \otimes \lambda(c_{(2)}) = \sum_i c_{i(1)} \otimes c_{i(2)} \otimes a_i \otimes d'_i.
\]

Applying $\varepsilon$ to the second factor, one finds

\[
c_{(1)} \otimes e(c_{(2)}) = \lambda(c).
\]

The right C-colinearity of $\lambda$ implies that

\[
\lambda(c_{(1)}) \otimes c_{(2)} = \sum_i a_{i\psi} \otimes d_{i\psi} \otimes c_i^{\psi}\psi
\]

and hence proves (41). To prove (42) note that $\lambda = \zeta'_A \otimes C \circ (\eta_A \otimes I_C)$ is left and right A-linear, hence

\[
e^1(c) \otimes e^2(c)a = (I_A \otimes \varepsilon \otimes I_A)(\zeta'_A \otimes C((1 \otimes c)a))
\]

Proposition 4.4 Let $(A, C, \psi)$ be a right-right entwining structure.

1) $F' = \bullet \otimes A : \mathcal{M}^C \to \mathcal{M}(\psi)_A^C$ is separable if and only if there exists $\vartheta \in V'_1$ such that for all $c \in C$,

\[
\vartheta(c \otimes 1) = \varepsilon(c).
\]

2) $G' : \mathcal{M}(\psi)_A^C \to \mathcal{M}^C$ is separable if and only if there exists $e \in W'_1$ such that for all $c \in C$,

\[
e^1(c)e^2(c) = \varepsilon(c)1.
\]

3) $(F', G')$ is a Frobenius pair if and only if there exist $\vartheta \in V'_1$ and $e \in W'_1$ such that

\[
\varepsilon(c)1 = \vartheta(c_{(1)} \otimes e^1(c_{(2)}))e^2(c_{(2)}) = \vartheta(c_{(1)} \otimes e^2(c_{(2)}))e^1(c_{(2)})\psi.
\]
Proof. We only prove 3). If \((F', G')\) is a Frobenius pair, then there exist \(v' \in V'\) and \(\zeta' \in W'\) such that (1) and (2) hold. Take \(\varnothing \in V'\) and \(e \in W'\) corresponding to \(v'\) and \(\zeta'\) and write down (1) applied to \(n \odot 1\) with \(n \in N \in M^C\),
\[
n \odot 1 = ((n'_N \otimes I_A) \circ (\zeta'_N \otimes I_A))(n \odot 1) = \varnothing(n_{[1]} \otimes e^1(n_{[2]}))n_{[0]} \otimes e^1(n_{[2]})).
\]  
(47)

Taking \(N = C\), \(n = c\), and applying \(\varepsilon_C\) to the first factor, one obtains (45). Conversely, if \(\varnothing \in V'\) and \(e \in W'\) satisfy (45), then (47) is satisfied for all \(N \in M^C\), and (1) follows since \(V_N \otimes I_A\) and \(\zeta'_N \otimes I_A\) are right \(A\)-linear. Now write down (2) applied to \(m \in M \in M(\psi)\)
\[
m = (V'_{G(M \otimes A)} \circ G'((\zeta'_M))(m) = \theta(m^w_{[1]} \otimes e^2(m_{[2]}))m_{[0]} \otimes e^1(m_{[2]}))\psi.
\]  
(48)

Take \(M = C \otimes A\), \(m = c \otimes 1\), and apply \(\varepsilon_C\) to the first factor. This gives (46). Conversely, if \(\varnothing \in V'\) and \(e \in W'\) satisfy (47), then application of (46) to the second and third factors in \(m_{[0]} \otimes m_{[1]} \otimes 1\), and then \(\varepsilon_C\) to the second factor shows that (48) holds for all \(M \in M(\psi)\). Finally, note that (48) is equivalent to (2). \(\square\)

Inspired by the results in the previous Section, we ask the following question: assuming \((F', G')\) is a Frobenius pair, when is \(A\) finitely generated projective as a \(k\)-module. We give a partial answer in the next Proposition. We assume that \(\psi\) is bijective (cf. [2, Section 6]). In the Doi-Hopf case, this is true if the underlying Hopf algebra \(H\) has a twisted antipode. The inverse of \(\psi\) is then given by the formula
\[
\psi^{-1}(a \otimes c) = c \overline{\delta}(a(1)) \otimes a(0).
\]

Proposition 4.5 Let \((A, C, \psi)\) be a right-right entwining structure. With notation as above, assume that \((F', G')\) is a Frobenius pair. If there exists \(c \in C\) such that \(\varepsilon(c) = 1\), and if \(\psi\) is invertible, with inverse \(\varphi = \psi^{-1} : A \otimes C \to C \otimes A\), then \(A\) is finitely generated and projective as a \(k\)-module.

Proof. Observe first that \((A, C, \varphi)\) is a left-left entwining structure. This means that (7-10) hold, but with \(A\) and \(C\) replaced by \(A^{\text{op}}\) and \(C^{\text{cop}}\). In particular,
\[
\varepsilon(\varphi)a = \varepsilon(c)a,  \quad (49)
\]
\[
a \otimes \Delta(\varphi) = a \otimes \varphi \otimes \varphi.  \quad (50)
\]

This can be seen as follows: rewrite (8) and (9) as commutative diagrams, reverse the arrows, and replace \(\psi\) by \(\varphi\). Then we have (43) and (50) in diagram form. Now fix \(c \in C\) such that \(\varepsilon(c) = 1\). Then for all \(a \in A\),
\[
a = \varepsilon(c)a = \varepsilon(c)a = \varnothing((c^0(1) \otimes e^1((c^0)_{(2)}))e^2((c^0)_{(2)}))a = \varepsilon(c_1 \otimes e^1(c_{(2)}))e^2(c_{(2)})a_{\varphi} = \varepsilon(c_1 \otimes e^1(c_{(2)}))e^2(c_{(2)})a_{\varphi} = \varepsilon(c_1 \otimes e^1(c_{(2)}))e^2(c_{(2)}).
\]

Write \((I \otimes e)\Delta(c) = \sum_{i=1}^m c_i \otimes b_i \otimes a_i \in C \otimes A \otimes A\). For \(i = 1, \ldots, m\), define \(a_i^* \in A^*\) by
\[
\langle a_i^*, a \rangle = \varnothing(c^0_i \otimes a_{\varphi}b_i).
\]

Then \(\{a_i, a_i^* \mid i = 1, \ldots, m\}\) is a finite dual basis of \(A\) as a \(k\)-module. \(\square\)

From now on we assume that \(A\) is finitely generated and projective with finite dual basis \(\{a_i, a_i^* \mid i = 1, \ldots, m\}\). The proof of the next Lemma is straightforward, and therefore left to the reader.

\[19\]
**Lemma 4.6** Let \((A, C, \psi)\) be a right-right entwining structure, and assume that \(A\) is finitely generated and projective as a \(k\)-module. Then \(A^* \otimes C \in \mathcal{CM} (\psi)_A^k\). The structure is given by the formulae

\[
(a^* \otimes c)b = \langle a^*, b_\psi a_i \rangle a_i^* \otimes c^\psi_i, \\
\rho'(a^* \otimes c) = a^* \otimes c_{(1)} \otimes c_{(2)}, \\
\rho^l(a^* \otimes c) = \langle a^*, a_\psi \rangle c_{(1)}^\psi \otimes a_i^* \otimes c_{(2)}^\psi.
\]

We now give alternative descriptions of \(V'\) and \(W'\).

**Proposition 4.7** Let \((A, C, \psi)\) be a right-right entwining structure, and assume that \(A\) is finitely generated and projective as a \(k\)-module. Then there is an isomorphism

\[
\beta_1 : W'_1 \rightarrow W'_2 = \text{Hom}_{kA}^C (A^* \otimes C, C \otimes A),
\]

\[
\beta_1(e) = \Omega, \quad \text{with} \quad \Omega(a^* \otimes c) = \langle a^*, e^1 (c_{(2)})_\psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)}).
\]

The inverse of \(\beta_1\) is given by \(\beta_1^{-1}(\Omega) = e\) with

\[
e(c) = \sum_i a_i \otimes (e_C \otimes I_A) \Omega(a^* \otimes c).
\]

**Proof.** We first prove that \(\beta_1\) is well-defined.

a) \(\beta_1(e) = \Omega\) is right \(A\)-linear: for all \(a^* \in A^*\), \(c \in C\) and \(b \in A\), we have

\[
\Omega((a^* \otimes c)b) = \sum_i \langle a^*, b_\psi a_i \rangle \Omega(a_i^* \otimes c^\psi_i) \\
= \sum_i \langle a^*, b_\psi a_i \rangle \langle a_i^*, e^1 ((c^\psi)_2(2))_\psi \rangle (c^\psi)^{(1)}_2 \otimes e^2 ((c^\psi)_2) \\
= \langle a^*, b_\psi e^1 ((c^\psi)_2(2))_\psi \rangle (c^\psi)^{(1)}_2 \otimes e^2 ((c^\psi)_2) \\
= \langle a^*, b_\psi e^1 ((c^\psi)_2(2))_\psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)}) \\
= \langle a^*, (b_\psi e^1 ((c^\psi)_2(2)))_\psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)}) \\
= \langle a^*, e^1 (c_{(2)})_\psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)})b \\
= \Omega(a^* \otimes c)b.
\]

b) \(\beta_1(e) = \Omega\) is right \(C\)-linear: for all \(a^* \in A^*\) and \(c \in C\), we have

\[
\rho'(\Omega(a^* \otimes c)) = \langle a^*, e^1 (c_{(2)})_\psi \rangle \rho'(c_{(1)}^\psi \otimes e^2 (c_{(2)})) \\
= \langle a^*, e^1 (c_{(2)})_\psi \rangle (c_{(1)}^\psi)^{(1)}_1 \otimes e^2 (c_{(2)})_\psi \otimes (c_{(1)}^\psi)^{(2)}_2 \\
= \langle a^*, e^1 (c_{(3)})_\psi \psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)})_\psi \otimes c_{(2)}^\psi \\
= \langle a^*, e^1 (c_{(2)})_\psi \psi \rangle c_{(1)}^\psi \otimes e^2 (c_{(2)})_\psi \otimes c_{(2)}^\psi \\
= \Omega(a^* \otimes c_{(1)}) \otimes c_{(2)}.
\]

c) \(\beta_1(e) = \Omega\) is left \(C\)-linear: for all \(a^* \in A^*\) and \(c \in C\), we have

\[
\rho^l(\Omega(a^* \otimes c)) = \langle a^*, e^1 (c_{(2)})_\psi \rangle (c_{(1)}^\psi)^{(1)}_1 \otimes (c_{(1)}^\psi)^{(2)}_2 \otimes e^2 (c_{(2)}).
\]
The proof that $\beta_1^{-1}(\Omega) = e$ satisfies (41) and (42) is left to the reader. The maps $\beta_1$ and $\beta_1^{-1}$ are inverses of each other since

$$
\beta_1^{-1}(\beta(e))(c) = \sum_i a_i \otimes (e_C \otimes I_A) (a_i^* \otimes a_i) \otimes e^2(c(2))
$$

$$
\beta_1^{-1}(\beta_1(\omega))(a^* \otimes c) = \langle a^*, (a_i)_C \rangle c^y(1) \otimes (e_C \otimes I_A) \Omega (a_i^* \otimes c(2))
$$

At the last step, we used that for all $c \in C$ and $a \in A$,

$$(I_C \otimes e_C \otimes I_A) p^i(c \otimes a) = c \otimes a.$$ 

$\square$

**Proposition 4.8** Let $(A, C, \psi)$ be a right-right entwining structure If $A$ is a finitely generated projective $k$-module, then the map

$$\alpha_1 : V'_1 \rightarrow V'_2 = \text{Hom}^C_{kA}(C \otimes A, A^* \otimes C),$$

defined by $\alpha_1(\vartheta) = \overline{\Omega}$, with

$$\overline{\Omega}(c \otimes a) = \langle \vartheta, c(1) \otimes a \psi a_i \rangle a_i^* \otimes c^y(2)$$

is an isomorphism. The inverse of $\alpha_1$ is given by $\alpha_1^{-1}(\overline{\Omega}) = \vartheta$ with

$$\vartheta(c \otimes a) = \overline{\Omega}(c \otimes a), 1_A \otimes e_C).$$

**Proof.** We first show that $\alpha_1$ is well-defined. Take $\vartheta \in V'_1$, and let $\alpha_1(\vartheta) = \overline{\Omega}$.

a) $\overline{\Omega}$ is right $A$-linear since for all $a, b \in A$ and $c \in C$,

$$\overline{\Omega}(c \otimes a) b = \sum_{i,j} \langle \vartheta, c(1) \otimes a \psi a_i \rangle \langle a_i^*, b \psi a_j \rangle a_j^* \otimes c^y(2)$$

$$\overline{\Omega}(c \otimes ab) = \sum_j \langle \vartheta, c(1) \otimes (ab) \psi a_j \rangle a_j^* \otimes c^y(2)$$

$\square$
b) $\overline{\Omega}$ is right $C$-colinear since for all $a \in A$ and $c \in C$,

$$\rho^l(\overline{\Omega}(c \otimes a)) = \vartheta(c_{(1)} \otimes a_{\varphi} a_i) a_i^* \otimes (c_{(2)}^\varphi)_{(1)} \otimes (c_{(2)}^\varphi)_{(2)}$$

$$= \vartheta(c_{(1)} \otimes a_{\varphi} a_i) a_i^* \otimes c_{(2)}^\varphi \otimes c_{(3)}^\varphi$$

$$= \overline{\Omega}(c_{(1)} \otimes a_{\varphi}) \otimes c_{(2)}^\varphi.$$  

c) $\overline{\Omega}$ is left $C$-colinear since for all $a \in A$ and $c \in C$,

$$p^l(\overline{\Omega}(c \otimes a)) = \sum_{i,j} \vartheta(c_{(1)} \otimes a_{\varphi} a_i) (a_i^*, a_{i \varphi}) (c_{(2)}^\varphi)_{(1)} \otimes a_j^* \otimes (c_{(2)}^\varphi)_{(2)}$$

$$= \sum_{i,j} \vartheta(c_{(1)} \otimes a_{\varphi} a_i) (a_i^*, a_{i \varphi}) c_{(2)}^\varphi \otimes a_j^* \otimes c_{(3)}^\varphi$$

$$= \sum_j \vartheta(c_{(1)} \otimes (a_{\varphi} a_j)') c_{(2)}^\varphi \otimes a_j^* \otimes c_{(3)}^\varphi$$

$$= \sum_j \vartheta(c_{(2)} \otimes a_{\varphi} a_j) c_{(1)} \otimes a_j^* \otimes c_{(3)}^\varphi$$

$$= c_{(1)} \otimes \overline{\Omega}(c_{(2)} \otimes a).$$

Conversely, given $\overline{\Omega}$, we have to show that $\alpha_1^{-1}(\overline{\Omega}) = \vartheta$ satisfies (14). Take any $c \otimes a \in C \otimes A$ and write $\overline{\Omega}(c \otimes a) = \sum_i b_i^* \otimes d_i \in A^* \otimes C$. Since $\overline{\Omega}$ is right and left $C$-colinear, we have

$$\overline{\Omega}(c_{(1)} \otimes a_{\varphi}) \otimes c_{(2)}^\varphi = \sum_i b_i^* \otimes d_{i(1)} \otimes d_{i(2)}$$

$$c_{(1)} \otimes \overline{\Omega}(c_{(2)} \otimes a) = \sum_i \langle b_i^*, a_{i \varphi} \rangle d_i^\varphi \otimes a_i^* \otimes d_{i(2)}.$$  

Therefore we can compute

$$\vartheta(c_{(2)} \otimes a) c_{(1)} = \langle \overline{\Omega}(c_{(2)} \otimes a), 1_A \otimes \varepsilon \rangle c_{(1)}$$

$$= \sum_i \langle b_i^*, a_{i \varphi} \rangle \langle a_i^*, 1 \rangle \langle \varepsilon, d_{i(2)}^\varphi \rangle d_i^\varphi$$

$$= \sum_i \langle b_i^* 1_{\varphi} \rangle d_i^\varphi$$

$$= \sum_i \langle b_i^* 1 \rangle d_i$$

$$= \langle \overline{\Omega}(c_{(1)} \otimes a_{\varphi}), 1_A \otimes \varepsilon_C \rangle c_{(2)}^\varphi$$

$$= \vartheta(c_{(1)} \otimes a_{\varphi}) c_{(2)}^\varphi.$$  

Thus (14) follows. Finally, we show that $\alpha_1$ and $\alpha_1^{-1}$ are inverses of each other.

$$\alpha_1^{-1}(\alpha_1(\vartheta))(c \otimes a) = \langle \vartheta(c_{(1)} \otimes a_{\varphi} a_i) a_i^* \otimes c_{(2)}^\varphi, 1_A \otimes \varepsilon_C \rangle = \vartheta(c_{(1)} \otimes a a_i) (a_i^*, 1_A) = \vartheta(c \otimes a).$$

We know that $\alpha_1(\alpha_1^{-1}(\overline{\Omega}))$ is right $A$-linear. Hence suffices it to show that

$$\alpha_1(\alpha_1^{-1}(\overline{\Omega}))(c \otimes 1) = c \otimes 1,$$
for all $c \in C$. From (51), we compute
\[
\langle (a^* \otimes c)b, 1_A \otimes \varepsilon_C \rangle = \langle a^*, b \rangle \varepsilon(c) = \langle a^* \otimes c, b \otimes \varepsilon_C \rangle.
\]

Now write \(\overline{\Omega}(c \otimes 1) = \sum_r a_i^* \otimes c_r\) and compute
\[
\begin{align*}
\alpha_1(\alpha_1^{-1}(\overline{\Omega}))(c \otimes 1) &= \langle \overline{\Omega}(c(1) \otimes a_i), 1_A \otimes \varepsilon_C \rangle a_i^* \otimes c(2) \\
&= \langle \overline{\Omega}(c(1) \otimes 1), a_i \otimes \varepsilon_C \rangle a_i^* \otimes c(2) \\
&= \overline{\Omega}(c \otimes 1)[a_i \otimes \varepsilon_C]a_i^* \otimes \overline{\Omega}(c \otimes 1)[1] \\
&= \sum_r (a_i^*, a_i) \langle \varepsilon, c_r(1) \rangle a_i^* \otimes c_r(2) \\
&= \sum_r a_i^* \otimes c_r = \overline{\Omega}(c \otimes 1).
\end{align*}
\]

\(\square\)

**Theorem 4.9** Let \((A, C, \psi)\) be a right-right entwining structure, and assume that \(A\) is finitely generated and projective as a \(k\)-module. With notation as above, we have the following properties:
1) \(F'\) is separable if and only if there exists \(\overline{\Omega} \in V'_2\) such that for all \(c \in C\),
\[
\langle \overline{\Omega}(c \otimes 1), 1_A \otimes \varepsilon_C \rangle = \varepsilon_C(c).
\]
2) \(G'\) is separable if and only if there exists \(\Omega \in W'_2\) such that for all \(c \in C\),
\[
\sum_i (\varepsilon_C \otimes I_A) \Omega (a_i^* \otimes c) = \varepsilon_C(c) 1.
\]
3) The following assertions are equivalent:
   a) \((F', G')\) is a Frobenius pair.
   b) There exist \(e \in W'_1\), \(\hat{\Theta} \in V'_1\) such that \(\Omega = \beta_1(e)\) and \(\overline{\Omega} = \alpha_1(\hat{\Theta})\) are inverses of each other.
   c) \(A^* \otimes C\) and \(C \otimes A\) are isomorphic objects in \(\mathcal{C} \mathcal{M}(\psi)^C_A\).

**Proof.** We only prove a) \(\Rightarrow\) b) in 3). First we show that \(\Omega\) is a left inverse of \(\overline{\Omega}\). Since \(\Omega \circ \overline{\Omega}\) is right \(A\)-linear, it suffices to show that
\[
\begin{align*}
\Omega(\overline{\Omega}(c \otimes 1)) &= \sum_i \hat{\Theta}(c(1) \otimes a_i) \Omega(a_i^* \otimes c(2)) \\
&= \sum_i \hat{\Theta}(c(1) \otimes a_i) (a_i^* e^1(c(3))_\psi)c_{(2)}^\psi \otimes e^2(c(3)) \\
&= \hat{\Theta}(c(1) \otimes e^1(c(3))_\psi)c_{(2)}^\psi \otimes e^2(c(3)) \\
&= \hat{\Theta}(c(2) \otimes e^1(c(3))_\psi)c_{(1)}^\psi \otimes e^2(c(3)) \\
&= e \otimes 1.
\end{align*}
\]

To show that \(\Omega\) is a right inverse of \(\overline{\Omega}\) we use that \(\overline{\Omega} \circ \Omega\) is right \(C\)-colinear and conclude that it suffices to show that for all \(c \in C\) and \(a^* \in A^*\),
\[
(L_A^* \otimes \varepsilon_C)(\overline{\Omega}(\Omega(a^* \otimes c))) = \varepsilon_C(c)a^*.
\]
Both sides of the equation are in \( A^* \), so the proof is completed if we show that both sides are equal when evaluated at an arbitrary \( a \in A \). Observe that

\[
\overline{\Omega}(\Omega(a^* \otimes c)) = \sum_i (a^*, e^1(c(2)_\psi) \partial((c(1)_1)^{(i)} \otimes e^2(c(2)_\psi)a_i) a^* \otimes (c(1)_2))
\]

hence

\[
(I_{A^*} \otimes \varepsilon_C)(\overline{\Omega}(\Omega(a^* \otimes c)))(a) = (a^*, e^1(c(2)_\psi) \partial((c(1)_1)^{(i)} \otimes e^2(c(2)))
\]

\[\tag{42}\]

\[
= (a^*, (a \psi e^1(c(2)_\psi)) \partial((c(1)_1)^{(i)} \otimes e^2(c(2))))
\]

\[\tag{7}\]

\[
= (a^*, a \psi e^1(c(2)_\psi)) \partial((c(1)_1)^{(i)} \otimes e^2((c(2))))
\]

\[\tag{9}\]

\[
= (a^*, a \psi e^1(c(2)_\psi)) \partial((c(1)_1)^{(i)} \otimes e^2((c(2))))
\]

\[\tag{16}\]

\[
= (a^*, a \psi) \varepsilon((c(2))) = (a^*, a) \varepsilon(c),
\]

as required. \( \square \)

5 The smash product

Let \((B, A, R)\) be a factorization structure (sometimes also called a smash or twisted tensor product structure, cf. \([27]\), [22], pp. 299-300). This means that \( A \) and \( B \) are \( k \)-algebras and that \( R : A \otimes B \to B \otimes A \) is a \( k \)-linear map such that for all \( a, c \in A, b, d \in B \),

\[
R(ac \otimes b) = b_{Rr} \otimes ac, \quad R(ad \otimes b) = b_{Rd} \otimes ac, \quad R(a \otimes 1_B) = 1_B \otimes a, \quad R(1_A \otimes b) = b \otimes 1_A.
\]

We use the notation \( R(a \otimes b) = b_{R} \otimes a_{R}. B_{R}A \) is the \( k \)-module \( B \otimes A \) with new multiplication

\[
(b \# a)(d \# c) = bd_{R}a_{R}.c.
\]

\( B_{R}A \) is an associative algebra with unit \( 1_{B_{R}}1_{A} \) if and only if \([24]\, [27]\) hold. In this Section we want to examine when \( B_{R}A/A \) and \( B_{R}A/B \) are separable or Frobenius. This will be a direct application of the results in the second part of Section \( [1] \).

In Section \( [1] \), take \( R = A, S = B_{R}A. \) For \( \nabla \in V_{1} = \operatorname{Hom}_{R,R}(S, R), \) define \( \kappa : B \to A \) by

\[
\kappa(b) = \nabla(b \# 1).
\]

Then \( \nabla \) can be recovered form \( \kappa, \) since \( \nabla(b \# a) = \kappa(b) a. \) Furthermore

\[
a \kappa(b) = \nabla(b \# 1) = \nabla(b_{R}a_{R}) = \kappa(b_{R}a_{R})
\]

and we find that

\[
V \cong V_{1} \cong V_{3} = \{ \kappa : B \to A \mid a \kappa(b) = \kappa(b_{R}a_{R}) \}.
\]

Now we simplify the description of \( W \cong W_{1} \subset (B_{R}A) \otimes_{A} (B_{R}A). \) Note that there is a \( k \)-module isomorphism

\[
\gamma : (B_{R}A) \otimes_{A} (B_{R}A) \to B \otimes B \otimes A,
\]

where
defined by
\[
\gamma((b\#a) \otimes (d\#c)) = b \otimes d_R \otimes a_{Rc}, \\
\gamma^{-1}(b \otimes d \otimes c) = (b\#1) \otimes (d\#c).
\]

Let \(W_3 = \gamma(W_1) \subset B \otimes B \otimes A\). Take \(e = b^1 \otimes b^2 \otimes a^2 \in B \otimes B \otimes A\) (summation implicitly understood). Then \(e \in W_3\) if and only if (4) holds, for all \(s = b^1\) and \(s = 1\#a\) with \(b \in B\) and \(a \in A\), if and only if
\[
bb^1 \otimes b^2 \otimes a^2 = b^1 \otimes b^2 b_R \otimes a_{R}^2, \\
(b^1)_R \otimes (b^2)_r \otimes a_{R\#a}^2 = b^1 \otimes b^2 \otimes a_{\#}^2a,
\]
for all \(a \in A\), \(b \in B\). This implies isomorphisms
\[
W \cong W_3 \cong W_2 = \{e = b^1 \otimes b^2 \otimes a^2 \in B \otimes B \otimes A \mid (59) \text{ and } (60) \text{ hold}\}.
\]

Using these descriptions of \(V\) and \(W\), we find immediately that Theorem 1.2 takes the following form.

**Theorem 5.1** Let \((B,A,R)\) be a factorization structure over a commutative ring \(k\).
1) \(B\#_RA/A\) is separable (i.e. the restriction of scalars functor \(G: \mathcal{M}_{B\#_RA} \rightarrow \mathcal{M}_A\) is separable) if and only if there exists \(e = b^1 \otimes b^2 \otimes a^2 \in W_3\) such that
\[
b^1 b^2 \otimes a^2 = 1_B \otimes 1_A \in B \otimes A.
\]
2) \(B\#_RA/A\) is split (i.e. the induction functor \(F: \mathcal{M}_A \rightarrow \mathcal{M}_{B\#_RA}\) is separable) if and only if there exists \(\kappa \in V_3\) such that
\[
\kappa(1_B) = 1_A.
\]
3) \(B\#_RA/A\) is Frobenius (i.e. \((F,G)\) is Frobenius pair) if and only if there exist \(\kappa \in V_3\), \(e \in W_3\) such that
\[
(b^2)^R \otimes \kappa(b^1)_R \otimes a^2 = b^1 \otimes \kappa(b^2)_R \otimes a^2 = 1_B \otimes 1_A.
\]

Theorem 1.7 can be reformulated in the same style. Notice that
\[
\text{Hom}_R(S,R) = \text{Hom}_A(B\#_RA,A) \cong \text{Hom}_R(B,A).
\]

\(\text{Hom}_R(B,A)\) has the following \((A,B\#_RA)\)-bimodule structure (cf. (5)):
\[
(cf(b\#a))(d) = cf(db)a,
\]
for all \(a,c \in C\) and \(b,d \in B\). From Proposition 1.3 we deduce that
\[
V \cong V_2 \cong V_4 = \text{Hom}_{A,B\#_RA}(B\#_RA, \text{Hom}(B,A)).
\]

If \(B\) is finitely generated and projective as a \(k\)-module, then we find using Proposition 1.6
\[
W \cong W_2 \cong W_4 = \text{Hom}_{A,B\#_RA}(\text{Hom}(B,A), B\#_RA).
\]

Theorem 1.7 now takes the following form:
Theorem 5.2 Let \((B,A,R)\) be a factorization structure over a commutative ring \(k\), and assume that \(B\) is finitely generated and projective as a \(k\)-module. Let \(\{b_i, b_i^* \mid i = 1, \ldots, m\}\) be a finite dual basis for \(B\).

1) \(B\#_RA/A\) is separable if and only if there exists an \((A, B\#_RA)\)-bimodule map \(\phi : \text{Hom}(B, A) \cong B^* \otimes A \to B\#_RA\) such that

\[
\sum_i (b_i \# 1) \phi(b_i^* \# 1) = 1_B \otimes 1_A.
\]

2) \(B\#_RA/A\) is split if and only if there exists an \((A, B\#_RA)\)-bimodule map \(\bar{\phi} : B\#_RA \to \text{Hom}(B, A)\) such that

\[
\bar{\phi}(1_B \# 1_A)(1_B) = 1_A.
\]

3) \(B\#_RA/A\) is Frobenius if and only if \(B^* \otimes A\) and \(B\#_RA\) are isomorphic as \((A, B\#_RA)\)-bimodules. This is also equivalent to the existence of \(\kappa \in V_3, e = b^1 \otimes b^2 \otimes a^2 \in W_3\) such that the maps

\[
\phi : \text{Hom}(B, A) \to B\#_RA, \quad \phi(f) = f(b^1)b^2#a^2
\]

and

\[
\bar{\phi} : B\#_RA \to \text{Hom}(B, A), \quad \bar{\phi}(b#a)(d) = \kappa(bd_R)a_R
\]

are inverses of each other.

The same method can be applied to the extension \(B\#_RA/B\). There are two ways to proceed: as above, but applying the left-handed version of Theorem 1.7 (left and right separable (resp. Frobenius) extension coincide). Another possibility is to use “op”-arguments. If \(R : A \otimes B \to B \otimes A\) makes \((B, A, R)\) into a factorization structure, then

\[
\tilde{R} : B^{op} \otimes A^{op} \to A^{op} \otimes B^{op}
\]

makes \((A^{op}, B^{op}, \tilde{R})\) into a factorization structure. It is not hard to see that there is an algebra isomorphism

\[
(A^{op}\#_RB^{op})^{op} \cong B\#_RA.
\]

Using the left-right symmetry again, we find that \(B\#_RA/B\) is Frobenius if and only if \((A^{op}\#_RB^{op})^{op}/B\) is Frobenius if and only if \((A^{op}\#_RB^{op})/B^{op}\) is Frobenius, and we can apply Theorems 5.1 and 5.2. We invite the reader to write down explicit results.

Our final aim is to link the results in this Section to the ones in Section 3 at least in the case of finitely generated, projective \(B\). Let \((A, C, \psi)\) be a right-right entwining structure, with \(C\) finitely generated and projective, and put \(B = (C^*)^{op}\). Let \(\{c_i, c_i^* \mid i = 1, \ldots, n\}\) be a dual basis for \(C\). There is a bijective correspondence between right-right entwining structures \((A, C, \psi)\) and smash product structures \((C^{op}, A, R)\). \(R\) and \(\psi\) can be recovered from each other using the formulae

\[
R(a \otimes c^*) = \sum_i \langle c^*, c_i^* \rangle c_i^* \otimes a, \quad \psi(c \otimes a) = \sum_i \langle c_i^* \rangle_R c_i \otimes a_R.
\]

Moreover, there are isomorphisms of categories

\[
\mathcal{M}(\psi)^C_A \cong \mathcal{M}_{B^{op}A} \quad \text{and} \quad \mathcal{M}(\psi)^C_A \cong \mathcal{M}_{B^{op}A}.
\]

In particular, \(B\#_RA\) can be made into an object of \(\mathcal{M}(\psi)^C_A\), and this explains the structure on \(C^* \otimes A\) used in Section 3. Combining Theorems 3.9 and 5.2, we find that the forgetful functor \(\mathcal{M}(\psi)^C_A \to \mathcal{M}_A\) and its adjoint form a Frobenius pair if and only if \(C^* \otimes A\) and \(A \otimes C\) are isomorphic as \((A, (C^*)^{op}\#_RA)\)-bimodules, if and only if the extension \((C^*)^{op}\#_RA/A\) is Frobenius.
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