FIBER CONES OF RATIONAL NORMAL SCROLLS ARE COHEN–MACAULAY

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Abstract. In this short paper, we show that the fiber cones of rational normal scrolls are Cohen–Macaulay. As an application, we compute their Castelnuovo–Mumford regularities and $a$-invariants, as well as the reduction number of the defining ideals of the rational normal scrolls. We also characterize the Gorensteinness of the fiber cone.

1. Introduction

In this short note, we show that the fiber cones of rational normal scrolls are Cohen–Macaulay. Let $R$ be a standard graded polynomial ring over some field $K$ and $I$ be an ideal of $R$ minimally generated by some forms $f_1, \ldots, f_s$ of the same degree. Those forms define a rational map $\phi$ whose image is a variety $X$. The bi-homogeneous coordinate ring of the graph of $\phi$ is the Rees algebra of the ideal $I$, and the homogeneous coordinate ring of the image is the fiber cone (special fiber ring) of this ideal. Here, we are mostly interested in these two types of blow-up algebras when the ideal $I$ describes the rational normal scroll.

Rational normal scrolls are typical determinantal varieties, central in the study of algebraic varieties. The study of determinantal varieties has attracted earnest attentions of algebraic geometers and commutative algebraists, partly due to the beautiful structures involved and the interesting applications to the applied mathematics and statistics; see, for instance, [5], [8], [11] and [26], to name but a few.

The study of Cohen–Macaulay property of the Rees algebra and the related fiber cone is an active research area. Most of the work is focusing on giving certain assumptions on the ground ring and the ideal; see for example, [23], [24], [27], and [28]. However, not much is known about the Cohen–Macaulayness of these blow-up algebras of the determinantal varieties. The first well-known case is probably the maximal minors of a generic matrix where the fiber cone is the Grassmannian variety. Hochster [19] in 1973 showed that the Grassmannian is Cohen–Macaulay. In 1983, Eisenbud and Huneke [13] showed that the posets defining the Rees algebra and fiber cone are wonderful, and obtained the Cohen–Macaulayness of those algebras. Recently, the first author of this paper and her coauthors proved the Cohen–Macaulayness of Rees algebras and its fibers cones of closed determinantal facet ideals in [1] and sparse determinantal ideals in [6].

All the work mentioned above came from generic matrices and their specializations. Blow-up algebras of rational normal scrolls were first studied by Conca, Herzog and Valla [7] in 1996. And they proved that the Rees algebra and the fiber cone of a balanced rational normal scroll are Cohen–Macaulay normal domains. The difficulty of determining the Cohen–Macaulayness of those blow-up algebras is partly due to the lack of necessary information regarding the defining ideals of those algebras. Fortunately, Sammartano solved this implicitization problem in full generality for the rational normal scrolls in [25] recently. And this opened the door of understanding those algebras for us.

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We need to be more precise now. In the following, let \( n_1, \ldots, n_d \) be a sequence of positive integers and \( c = \sum_{i=1}^{d} n_i \). For each \( i = 1, \ldots, d \), let \( C_i \subseteq \mathbb{P}^{c+d-1} \) be a rational normal curve of degree \( n_i \) with complementary linear spans and let \( \varphi_i : \mathbb{P}^1 \to C_i \) be the corresponding isomorphism. Whence, the related rational normal scroll is simply

\[
S_{n_1, \ldots, n_d} := \bigcup_{p \in \mathbb{P}^1} \{ \varphi_1(p), \varphi_2(p), \ldots, \varphi_d(p) \} \subseteq \mathbb{P}^{c+d-1}.
\]

It is well-known (\[17\]) that \( S_{n_1, \ldots, n_d} \) is uniquely determined by the sequence \( n_1, \ldots, n_d \) up to projective equivalence. And in suitable coordinates, the ideal \( I_{n_1, \ldots, n_d} \) of \( S_{n_1, \ldots, n_d} \) is generated by the maximal minors of the matrix

\[
X := \begin{pmatrix}
x_{1,0} & x_{1,1} & \cdots & x_{1,n_1-1} \\
x_{1,1} & x_{1,2} & \cdots & x_{1,n_1} \\
& \ddots & \ddots & \vdots \\
x_{d,0} & x_{d,1} & \cdots & x_{d,n_d-1}
\end{pmatrix}.
\]

In the following, we will write \( R = \mathbb{K}[x_{i,j} \mid 0 \leq j \leq n_i - 1] \) for the ground ring, and \( I = I_{n_1, \ldots, n_d} = I_2(X) \) for this ideal. Recall that the Rees algebra of \( I \) is \( R(I) := \oplus_{t \geq 0} I^t t^t \subseteq R[t] \), and the related fiber cone will be \( \mathcal{F}(I) = \mathcal{R}(I) \otimes_{R} \mathbb{K}[t] \subseteq \mathbb{K} \), where \( t \) is a new variable. As mentioned above, Sammartano gave the implicit equations of the defining ideals of these blow-up algebras in \[25\]. Meanwhile, in the balanced case, i.e., when \( |n_i - n_j| \leq 1 \) for all \( i, j \), Conca, Herzog, and Valla showed in \[17\] that both \( \mathcal{R}(I) \) and \( \mathcal{F}(I) \) are Cohen–Macaulay normal domains.

In this paper, we will drop the balanced assumption. We can still obtain the Cohen–Macaulayness of the fiber cone \( \mathcal{F}(I) \) in Theorem 2.1. By this, we generalize the corresponding result of Conca, Herzog, and Valla. As a quick application, we compute the Castelnuovo–Mumford regularity and \( a \)-invariant of \( \mathcal{F}(I) \) in Theorem 3.3. And the reduction number of the ideal \( I \) is provided as well. As the second application, we characterize the Gorensteinness of the fiber cone.

2. Cohen–Macaulayness of the Fiber Cones of Rational Normal Scrolls

This section is devoted to prove that the fiber cone associated with the rational normal scroll is Cohen–Macaulay. To achieve that, we associate the initial ideal of the defining ideal of the fiber cone with its Stanley–Reisner complex \( \Delta \). We will show that the ideal of the Alexander dual of this complex has linear quotients. Consequently, the original simplicial complex is shellable and Cohen–Macaulay. Hence, the fiber cone is Cohen–Macaulay as well.

To establish the expected linear-quotients property, we have to resort to the binary-tree description of the facets of the simplicial complex \( \Delta \) in \[25\]. To be more accurate, instead of just using the lexicographic order on the minimal monomial generators of the Alexander dual ideal, we need to group these generators first using the leaves of the binary trees; see Notation 2.3. The subsequent proof is also quite involved, since we have to analyze the binary trees in more detail and pin down those linear generators of the successive colon ideals.

Recall that a rational normal scroll is uniquely determined by a sequence of positive integers \( n = (n_1, \ldots, n_d) \) with \( n_1 \leq \cdots \leq n_d \) by \[17\]. Using the terminology stated in the above introduction, we are ready to present the main result of this paper.

**Theorem 2.1.** Let \( I_{n_1, \ldots, n_d} \) be the defining ideal of the rational normal scroll \( S_{n_1, \ldots, n_d} \subseteq \mathbb{P}^{c+d-1} \) for \( c = n_1 + \cdots + n_d \). Then the fiber cone \( \mathcal{F}(I_{n_1, \ldots, n_d}) \) is Cohen–Macaulay.

We need some preparations before laying out its proof. Let us start by recalling the indispensable constructions in \[25\] for the fiber cone \( \mathcal{F}(I) \) where \( I = I_{n_1, \ldots, n_d} \). First of all, a matrix \( M = M_{n_1, \ldots, n_d} \) was introduced in \[25\] Section 2 as follows. One starts with the
first column of the $i$-th catalecticant block $X_i$ of the original matrix $X = (X_1, X_2, \ldots, X_d)$ for each $i$ increasingly in $i$, then the second column, and so on until one has used all columns except the last one for each block; when a block runs out of columns one simply skips it. The first $c - d$ columns of $M$ form the submatrix

$$
\begin{pmatrix}
    x_{i(2),0} & x_{i(2)+1,0} & \cdots & x_{d,0} & x_{i(3),1} & x_{i(3)+1,1} & \cdots & x_{d,1} & x_{i(4),2} & \cdots & \cdots & x_{d,n_d-2} \\
    x_{i(2),1} & x_{i(2)+1,1} & \cdots & x_{d,1} & x_{i(3),2} & x_{i(3)+1,2} & \cdots & x_{d,2} & x_{i(4),3} & \cdots & \cdots & x_{d,n_d-1} \\
    \end{pmatrix},
$$

where the $i(l)$ denotes the least integer $i$ such that $n_i \geq l$. The last $d$ columns of $M$ are of the last column of the $i$-th block of $X$ for each $i$, but ordered decreasingly in $i$:

$$
\begin{pmatrix}
    x_{d,n_d-1} & x_{d-1,n_d-1} & \cdots & x_{2,n_d-1} & x_{1,n_d-1} \\
    x_{d,n_d} & x_{d-1,n_d} & \cdots & x_{2,n_d} & x_{1,n} \\
\end{pmatrix}.
$$

Interested readers are invited to go through the examples in [25, Example 2.1].

Let

$$
\pi : \mathbb{K}[T_{\alpha,\beta} \mid 1 \leq \alpha < \beta \leq c] \to F(I)
$$

be the ring epimorphism determined by $T_{\alpha,\beta} \mapsto \det(M_{\alpha,\beta})$, where $M_{\alpha,\beta}$ is the $2 \times 2$ submatrix of $M$ using the $\alpha$-th and $\beta$-th columns. Then $P := \ker(\pi)$ is called the defining ideal of the fiber cone $F(I)$. Under suitable monomial ordering introduced in [25], the initial ideal in($P$) is squarefree and quadratic. One can then consider its associated Stanley–Reisner complex $\Delta$, which will be called the initial complex of the fiber cone $F(I)$ following [25].

**Proof of Theorem 2.1.** When $c < d + 4$, the fiber cone is the coordinate ring of the Grassmann variety $G(1, c - 1) \subset P(2)^{c-1}$; see [25, Remark 3.15]. Thus, the fiber cone is Cohen–Macaulay by [19].

Now, it remains to consider the case when $c \geq d + 4$. Notice that $F(I)$ being Cohen–Macaulay is equivalent to saying that the defining ideal $P$ has this property. By [18, Corollary 3.3.5], or more strongly, by [9, Corollary 2.7], it suffices to show that in($P$) is Cohen–Macaulay. But this in turn is equivalent to showing that in($P$)$\vee$, the Alexander dual, has a linear resolution by [18, Theorem 8.1.9]. Because of this, we will show in Proposition 2.6 that in($P$)$\vee$ has linear quotients. This is sufficient for our purpose by [18, Proposition 8.2.1].

In the following, we shall assume that $c \geq d + 4$. Before we really start proving the linear quotients property, we still need some preparations. Under the natural identification $T_{\alpha,\beta} \leftrightarrow (\alpha, \beta)$, the vertex set of the aforementioned simplicial complex $\Delta$ is

$$
V = \{ (\alpha, \beta) \mid 1 \leq \alpha < \beta \leq c \}.
$$

Following [25], the vertices of $\Delta$ will also be described as open intervals in the real line $\mathbb{R}$ with integral endpoints. And we will use the familiar notions of length, intersection, and containment of intervals.

As usual, if $n$ is a positive integer, $[n]$ will be the set $\{1, 2, \ldots, n\}$. Then, with respect to the above matrix $M$, for each $\alpha \in [c - d - 2]$ and each $i \in [d]$, let $\gamma_{\alpha,i}$ be the least index $\gamma \geq \alpha + 2$ such that the $\gamma$-th column of $M$ involves variables from the block $X_i$. Then, there exists some $\ell_\alpha$ with $2 \leq \ell_\alpha \leq d + 1$ such that

$$
\{ \gamma_{\alpha,1}, \ldots, \gamma_{\alpha,d} \} = \{ \alpha + 2, \ldots, \alpha + \ell_\alpha \} \cup \{ c - d + \ell_\alpha, \ldots, c \};
$$

c.f. [25, Definition 3.6]. The key observation that we shall apply is the following result.

**Lemma 2.2** ([25, Proposition 3.8]). Suppose that $c \geq d + 4$. A subset $F \subseteq V$ is a facet of $\Delta$ if and only if the Hasse diagram $T_F$ of the poset $(F, \subseteq)$ satisfies the following conditions:
(i) $T_F$ is a rooted binary tree with the root $(1, c)$;
(ii) there exists some $\alpha \in [c-d-2]$ such that the leaves of $T_F$ are the intervals
\[
\{ (\beta, \beta + 1) \mid \beta \in \{\alpha, \ldots, \alpha + \ell_{\alpha}\} \cup \{c - d + \ell_{\alpha} - 1, \ldots, c - 1\} \} ;
\]
(iii) if $\mathcal{I} \in F$ is a node of $T_F$ with only one child $\mathcal{I}_1$, then $\text{length}(\mathcal{I}) = \text{length}(\mathcal{I}_1) + 1$ and the unique unitary interval in $\mathcal{I} \setminus \mathcal{I}_1$ does not belong to $F$;
(iv) if $\mathcal{I} \in F$ is a node of $T_F$ with two children $\mathcal{I}_1, \mathcal{I}_2$, then $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ and $\text{length}(\mathcal{I}_1) + \text{length}(\mathcal{I}_2) = \text{length}(\mathcal{I})$.

In this case we have $|F| = c + d$.

In the following, when we say the node $(\alpha', \beta')$ is a left child of the node $(\alpha'', \beta'')$ in the binary tree $T_F$, we mean it is a child and $\alpha' = \alpha''$. The notions of right child, left sibling and right sibling can be similarly defined.

We will call the set in (3) as the leaves set of $F$ and denote it by $\mathcal{L}_a$.

Notice that it is uniquely determined by its smallest point $\alpha \in [c-d-2]$ for the given matrix $M$. The following easy fact will be needed later.

**Lemma 2.3.** For each $\alpha \in [c-d-3]$, the cardinalities $|\mathcal{L}_a \setminus \mathcal{L}_{a+1}| = |\mathcal{L}_{a+1} \setminus \mathcal{L}_a| = 1$.

**Proof.** Recall that for each such $\alpha$ and each $i \in [d]$, the integer $\gamma_{\alpha, i}$ is the least index $\gamma \geq \alpha + 2$ such that the $\gamma$-th column of $M$ involves variables from the block $X_i$. Furthermore, the union in (2) is actually disjoint.

Therefore, $\gamma_{\alpha+1, i} \geq \gamma_{\alpha, i}$. And when $\gamma_{\alpha, i} \geq (\alpha + 1) + 2$, then $\gamma_{\alpha+1, i} = \gamma_{\alpha, i}$. Now, suppose that $\alpha + 2 = \gamma_{\alpha, i}$. Then for $i \neq i_0$, we have $\gamma_{\alpha, i} = \gamma_{\alpha+1, i}$. Whence, $\mathcal{L}_{a+1} \setminus \mathcal{L}_a = \{(\gamma_{\alpha+1, i_0}, \gamma_{\alpha+1, i_0} + 1)\}$ and $\mathcal{L}_a \setminus \mathcal{L}_{a+1} = \{((\alpha, \alpha) + 1)\}$. \hfill $\Box$

It is time to introduce the expected linear ordering. Let $S = \mathbb{K}[T_v \mid v \in V]$ be the base ring. Endow it with the lexicographic order $\succ_{\text{lex}}$ with respect to the linear order on the variables $T_v$’s such that
\[
T_{v_1} > T_{v_2} \iff \text{the leftmost nonzero component of } v_1 - v_2 \text{ is negative};
\]
here, by abuse of notation, we also think of $V$ as a set of integral points on $\mathbb{Z}_+^d$. Recall that the facet $F$ of the initial complex $\Delta$ is bijectively related to the monomial
\[
\hat{T}^F := \prod_{v \in V \setminus F} T_v
\]
in the minimal monomial generating set $G((\text{in}(P))^{\nu})$ by [IS, Lemma 1.5.3]. Therefore, to impose a linear order on $G((\text{in}(P))^{\nu})$ amounts to giving a linear order of the facets. For later reference, facets with the common leaves set given by (3) will be grouped into the family $\mathcal{G}_a$.

**Notation 2.4.** The expected linear order on the facets, denoted by $\succ$, will be as follows.

(a) If two facets $F \in \mathcal{G}_a$ and $F' \in \mathcal{G}_{a'}$ respectively with $\alpha > \alpha'$, then $F \succ F'$.

(b) Next, consider the facets $F$ and $F'$ within the same group $\mathcal{G}_a$. Then $F \succ F'$ if and only if $\hat{T}^F \succ_{\text{lex}} \hat{T}^{F'}$.

**Example 2.5.** The first facet of the group $\mathcal{G}_a$ with respect to our ordering $\succ$ is the one with vertices
\[
(1, c), (2, c), \ldots, (c - 2, c)
\]
in addition to the leaves given in (3).
For example, let \( \mathbf{n} = (2, 2, 4, 4) \). Then the matrix
\[
\mathbf{M} = \begin{pmatrix}
    x_{1,0} & x_{2,0} & x_{3,0} & x_{4,0} & x_{3,1} & x_{4,1} & x_{3,2} & x_{4,2} & x_{4,3} & x_{3,3} & x_{2,1} & x_{1,1} \\
    x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & x_{3,2} & x_{4,2} & x_{3,3} & x_{4,3} & x_{4,4} & x_{3,4} & x_{2,2} & x_{1,2}
\end{pmatrix}.
\]
If we choose \( \alpha = 2 \), then the leaves set is
\[
\{ (2, 3), (3, 4), (4, 5), (5, 6), (10, 11), (11, 12) \}. 
\]
see also [25, Example 3.9]. The collection of open intervals in the Figure 1 is the first facet with respect to the given \( \alpha = 2 \) in the corresponding complex \( \Delta \).

Figure 1. The first facet in \( G_\alpha \) when \( \alpha = 2 \) and \( \mathbf{n} = (2, 2, 4, 4) \)

And the Hasse diagram of the poset as considered in Lemma 2.2 is pictured in Figure 2.

Figure 2. Binary tree of the first facet in \( G_\alpha \) when \( \alpha = 2 \) and \( \mathbf{n} = (2, 2, 4, 4) \)

Now, we are ready to show that the Alexander dual ideal \( (\mathrm{in}(P))^\vee \) has linear quotients.

**Proposition 2.6.** When \( c \geq d + 4 \), the ideal \( (\mathrm{in}(P))^\vee \) has linear quotients with respect to the given ordering \( \succ \).

**Proof.** Let \( F \) be a facet of \( \Delta \) and suppose that \( F \in G_\alpha \). We want to show that the minimal monomial generating set \( G_F := G(I_F) \) of the colon ideal
\[
I_F := \langle \hat{T}_{F'} \mid F' \succ F \rangle : \hat{T}_{F'}
\]
is linear. Notice that
\[
\langle \hat{T}_{F'} \rangle : \hat{T}_{F} = \hat{T}_{F \setminus F'} := \prod_{v \in F \setminus F'} T_v.
\]
Suppose that $F$ is the first facet within $\mathcal{G}_\alpha$ with respect to the designated linear order $\succ$. If $\alpha = c - d - 2$, then $F$ is actually the first facet of $\Delta$ and there is nothing to show in this case. Otherwise, $\alpha < c - d - 2$. Let $F'$ be the first facet of $\mathcal{G}_{\alpha+1}$. It is clear that $F' \succ F$ and

$$F \setminus F' = \mathcal{L}_\alpha \setminus \mathcal{L}_{\alpha+1} = \{ (\alpha, \alpha + 1) \}$$

by Lemma 2.3 and Example 2.5. Notice that for any facet $F''$ preceding $F$, $(\alpha, \alpha + 1) \in F \setminus F''$. Whence, the colon ideal $I_F$ is principal and linear.

Suppose instead that $F$ is not the first facet in $\mathcal{G}_\alpha$. This situation is more involved. We will break the long proof into three parts, according to the following plans.

1. Firstly, we describe the expected linear generating set $G_F$.
2. Next, we show that the variables described above really show up as some colons of monomials, and they are the only variables possessing this property.
3. Finally, we show that we don’t have minimal monomial generators in $G_F$ of higher degrees.

After showing the above three parts, it is clear that the ideal $(\text{in}(P))^\vee$ has linear quotients. At this point, one may suggest only proving the last step. But to successfully give the proof, we have to clear higher-degree generators by the linear ones.

2.1. Linear generators of the colon ideal $I_F$. We separate the vertices in $F$ into collections for each fixed facet $F \in \mathcal{G}_\alpha$. For $b \in [c - 1]$, let

$$F_b := \{ v \in F \mid v = (b, *) \}.$$  \hspace{1cm} (4)

If the cardinality $k_b := |F_b| \geq 1$, we may assume that

$$F_b = \{ (b, i_{b,1}), (b, i_{b,2}), \ldots, (b, i_{b,k_b}) \}$$

with $i_{b,1} < i_{b,2} < \ldots < i_{b,k_b}$. It is clear that $(b, i_{b,j+1})$ is the parent node of $(b, i_{b,j})$ when $1 \leq j < k_b$. Now, assume that $k_b \geq 1$, and we describe whether $T_{b,i_{b,j}} \in G_F$ for $j \in [k_b]$.

(a) If $j = k_b$, then $T_{b,i_{b,k_b}} \notin G_F$.

(b) Suppose that $2 \leq j < k_b$. If

(i) the node at $(b, i_{b,j})$ has a right sibling in $\mathcal{T}_F$, or
(ii) the node at $(b, i_{b,j})$ has two children in $\mathcal{T}_F$,

then $T_{b,i_{b,j}} \notin G_F$. Otherwise, $T_{b,i_{b,j}} \notin G_F$.

(c) Suppose that $j = 1 < k_b$.

(i) Suppose that $(b, i_{b,1})$ is a leaf in $\mathcal{T}_F$, namely that $i_{b,1} = b + 1$.

1. If $b \neq \alpha$, then $T_{b,i_{b,1}} \notin G_F$.

2. Suppose that $b = \alpha$, i.e., $(b, i_{b,1})$ is the leafmost leaf in $\mathcal{T}_F$.

- If $b = c - d - 2$, then $G_\alpha$ is the first group to be considered. Whence, we expect $T_{b,i_{b,1}} \notin G_F$.
- Otherwise, we expect $T_{b,i_{b,1}} = T_{\alpha,\alpha+1} \in G_F$ and $\alpha \neq c - d + 2$.

(ii) Otherwise, $i_{b,1} > b + 1$. Whence, we always have $T_{b,i_{b,1}} \in G_F$.

For later reference, we denote the set of the expected linear generators of $F$ with respect to $b$ described above as $LG_{F,b}$. Then in the next two subsections, we will see that the minimal generating set $G_F$ is precisely the set

$$LG_F := \bigcup_{b \in [c-1]} LG_{F,b}.$$
Another fact is that the variable $b$ is a facet of the associated complex $\Delta$. Consider $b = 1$. The subset $F_b$ consists of the vertices written on the first line. And our criterion above says that the corresponding linear generators for $b = 1$ are

$$LG_{F,1} = \{T_{1,2}, T_{1,3}, T_{1,4}, T_{1,6}, T_{1,9}\} \subseteq LG_F.$$  

Indeed, it is not difficult to see that equality holds in this special case.

2.2. Why the vertex of $F$ shows or not shows up in $G_F$? Before we proceed, notice first the following consequences of Lemma 2.2.

**Observations 2.8.** Let $F$ be a facet of the aforementioned initial complex $\Delta$. Recall that it is convenient to consider the vertex $(\alpha, \beta)$ in $\Delta$ as an open interval in the real line $\mathbb{R}$ with integral endpoints.

(i) Vertices of $F$ are non-crossing, i.e., if we consider the vertices $J_1, J_2$ of $F$ as open intervals, then $J_1 \cap J_2 = \emptyset$, or $J_1 \subseteq J_2$, or $J_2 \subseteq J_1$.

(ii) If $(\alpha, \beta) \in F$ is not the root and has no sibling, then its parent node is either $(\alpha - 1, \beta)$ or $(\alpha + 1, \beta)$. Correspondingly, $(\alpha - 1, \alpha)$ or $(\beta, \beta + 1)$ is not a leaf of $T_F$.

(iii) If as an open interval, $(\alpha, \beta)$ contains no leaf of $T_F$, then the vertex $(\alpha, \beta)$ does not belong to $F$.

(iv) If both $(\alpha, \beta)$ and $(\alpha, \beta')$ belong to $F$ with $\beta < \beta'$ such that $(\beta, \beta')$ contains no leaf of $T_F$, then $(\alpha, \beta + j) \in F$ for all $j$ with $0 < j < \beta - \beta'$. To see this, notice that for each such fixed $j$, $F$ must have at least one vertex of the form $(\beta + j, \gamma)$ or $(\alpha', \beta + j)$. In the first subcase, $\gamma \leq \beta'$ is impossible by the previous observation. But when $\gamma > \beta'$, this is also impossible, since the two open intervals $(\beta + j, \gamma)$ and $(\alpha', \beta + j)$ cross. As for the second subcase, by the previous observation, it is clear that $\alpha' < \beta$. If $\alpha < \alpha' < \beta$, then the two open intervals $(\alpha', \beta + j)$ and $(\alpha, \beta)$ cross. And if $\alpha' < \alpha$, then the two open intervals $(\alpha', \beta + j)$ and $(\alpha', \beta')$ cross. Therefore, $\alpha' = \alpha$ and $(\alpha, \beta + j)$ is a vertex of $F$, as expected.

(v) Suppose that $(\gamma, \gamma + 1) \in V$ is not a leaf of $T_F$. Consider the smallest interval (node) in $T_F$ that contains $(\gamma, \gamma + 1)$. Then by Lemma 2.2 and the minimality, this node is either $(\gamma', \gamma + 1)$ or $(\gamma, \gamma')$ for some $\gamma'$. And correspondingly it has a unique child node $(\gamma', \gamma)$ or $(\gamma + 1, \gamma')$.

Another fact is that the variable $T_v \in G_F$ if and only if $F \setminus F' = \{v\}$ for some $F' > F$. And for this pair $F$ and $F'$, if $F \setminus F' = \{(i,j)\}$ while $F' \setminus F = \{(i',j')\}$, then

either $i' > i$ or $i' = i$ and $j' > j$.

Now, we are ready to justify the linear generating set $LG_F$ presented in the previous subsection 2.1. Correspondingly, we also have three cases.

(a) If $j = k_b$, we show that $T_{b, k_b} \notin G_F$. We don’t have to worry about the case when $b = 1$, since $(1, c)$ is the common root. Suppose now for contradiction that there is some $F'$ with $F' > F$ and $F \setminus F' = \{(b, i_b, k_b)\}$. Thus, we have two subcases.

Suppose first that $F \in G_{\alpha}$ and $F' \in G_{\alpha'}$ with $\alpha < \alpha'$. Since necessarily $(\alpha, \alpha + 1) \in \mathcal{L}_\alpha \setminus \mathcal{L}_{\alpha'} \subseteq F \setminus F'$ while $|F \setminus F'| = 1$, it follows that $b = \alpha$ and $i_b, k_b = \alpha + 1$. Since the leftmost leaf of $T_F$ is $(\alpha, \alpha + 1)$ and $|F_{\alpha}| = 1$, $F$ must contain the vertex
(α − 1, α + 1) by item [ii] of Observations 2.8. Hence, F′ must also contain this vertex. But the leftmost leaf of T_F is (α′, α′ + 1) with α′ > α, contradicting the item [iii] of Observations 2.8.

Suppose now instead that F and F′ have the same leaves set, i.e., α = α′. Then (b, i_b,b_k) is not a leaf of T_F and must have some child. The parent node of (b, i_b,b_k) in T_F must have the form (b′, i_b,b_k) for some b′ < b. Notice that the unique vertex in F′ \ F that the vertex (b, i_b,b_k) is switched to cannot take the form (b, b′) with b′ > i_b,k, since the interval (b, i_b,b_k) will cross the interval (b′, i_b,b_k). Meanwhile, this unique vertex cannot take the form (b′′, b′′′) with b′′ > b, since (b′′, i_b,b_k) will not have a legal branch in T_F; see [iii] and [iv] of Lemma 2.2. Therefore, we have a contradiction to the fact stated at the end of Observations 2.8.

Thus, T_b,i_b,b_k ∉ G_F.

(b) Suppose that 2 ≤ j < k_b.

(i) Suppose that the node (b, i_b,j) has a sibling in T_F. It is clear that the parent node of (b, i_b,j) is (b, i_b,j+1) and (b, i_b,j) is the left child in the binary tree. Whence, the sibling of (b, i_b,j) is (i_b,j, i_b,j+1). Now, let

F′ := F \ {(i_b,j−1, i_b,j+1)} ∩ {(b, i_b,j)}.

Obviously F′ \ F. And it is easy to verify that F′ is a legal facet and the node (b, i_b,j+1) has two children (b, i_b,j−1) and (i_b,j−1, i_b,j+1) in T_F. This implies that T_b,i_b,j ∈ G_F.

(ii) Suppose that the node (b, i_b,j) has two children in T_F. They have to be (b, i_b,j−1) and (i_b,j−1, i_b,j). We similarly define

F′ := F \ {(i_b,j−1, i_b,j+1)} ∩ {(b, i_b,j)},

and see that T_b,i_b,j ∈ G_F.

(iii) Suppose that the node (b, i_b,j) has only one child and has no right sibling. Whence, i_b,j±1 = i_b,j + 1 respectively and (i_b,j−1, i_b,j+1) contains no leaf in T_F by Observations 2.8(ii). Suppose for contradiction that we have some facet F′ such that F′ > F and F \ F′ = {(b, i_b,j)}. Since (b, i_b,j) is not a leaf, F and F′ have identical leaves set. Therefore (i_b,j−1, i_b,j+1) contains no leaf in T_F as well. The node (b, i_b,j) must be a grandchild of (b, i_b,j+1) in T_F by Observations 2.8(iv) and Lemma 2.2(iii) and (iv). Meanwhile, (b, i_b,j) is the only possible option to connect (b, i_b,j−1) with (b, i_b,j+1). This makes F = F′, a contradiction. Therefore, T_b,i_b,j ∉ G_F.

(c) Suppose that j = 1 < k_b.

(i) We first consider the case when (b, i_b,1) is a leaf in T_F, namely that i_b,1 = b+1. Suppose that T_b,i_b,1 ∈ G_F. Then we have F ∈ G_α and F′ ∈ G_{α′} for some α < α′ such that F \ F′ = {(b, b+1)}. It is clear then that α < α′ ≤ c − d − 2. And again, since (α, α + 1) ∈ L_α \ L_{α′} ⊆ F \ F′ while |F \ F′| = 1, it follows that b = α.

Conversely, suppose that b = α < c − d − 2 with F ∈ G_α. Notice that L_α \ L_{α+1} = {β, β + 1} for some β by Lemma 2.3 and this (β, β + 1) ∉ F. Meanwhile, since the invariant L_α defined before satisfies L_α ≥ 2, both (α, α + 1) and (α + 1, α + 2) are leaves of T_F. Thus, as k_α ≥ 2, the node (α, α + 1) has a right sibling (α + 1, i_α,2) by Lemma 2.2(iii). It is clear that any vertex in T_F strictly containing (α, α + 1) will contain its parent node (α, i_α,2), and consequently will contain (α + 1, i_α,2). Now, it is not difficult to
verify that
\[ F' := F \cup \{(\beta, \beta + 1)\} \setminus \{(\alpha, \alpha + 1)\} \in \mathcal{G}_{\alpha + 1} \]
is a well-defined facet, and therefore, \( T_{\alpha, \alpha + 1} \in \mathcal{F}_F \).

(ii) Next, we assume that \( (b, i_{b,1}) \) is not a leaf in \( \mathcal{T}_F \), namely that \( i_{b,1} > b + 1 \). Since \( j = 1 \), \( (b, b + 1) \) is not a leaf and \( (b, i_{b,1}) \) has only the right child \( (b + 1, i_{b,1}) \) in \( \mathcal{T}_F \). We have two subcases.

- Suppose that \( (b, i_{b,1}) \) has a right sibling in \( \mathcal{T}_F \). It has to be \( (i_{b,1}, i_{b,2}) \).
- Suppose that \( (b, i_{b,1}) \) does not have a right sibling in \( \mathcal{T}_F \). Whence, \( i_{b,2} = i_{b,1} + 1 \) and \( (i_{b,1}, i_{b,1} + 1) \) is not a leaf of \( \mathcal{T}_F \) by Observations 2.8 in \([\text{ii}]\).

In either subcase, let
\[ F' := F \cup \{(b + 1, i_{b,2})\} \setminus \{(b, i_{b,1})\}, \]
and we can see as above that \( T_{b,i_{b,1}} \in \mathcal{G}_F \). And this completes our argument in this subsection.

2.3. No minimal monomial generator of higher degrees. In this subsection, we finish the proof by showing that \( \mathcal{G}_F = \mathcal{L}_F \).

Suppose for contradiction that \( F \in \mathcal{G}_\alpha \) and \( F' \in \mathcal{G}_{\alpha'} \) with \( F' \succ F \) such that \( \langle \hat{\mathcal{T}}^{F'} \rangle : \hat{\mathcal{T}}^F \) provides a minimal monomial generator of higher degree in \( \mathcal{G}_F \setminus \mathcal{L}_F \). Whence, none of the linear generators in \( \mathcal{L}_F \) will ever divide the principal generator of \( \langle \hat{\mathcal{T}}^{F'} \rangle : \hat{\mathcal{T}}^F \). If we apply the identification \( T_{i,j} \leftrightarrow (i,j) \in \Delta \), then this is just
\[ \mathcal{L}_F \cap (F \setminus F') = \emptyset, \]

or simply, \( \mathcal{L}_F \subseteq F \cap F' \). \hspace{1cm} (5)

If \( \alpha \neq \alpha' \), let \( a \leq \alpha \) be the smallest such that \( (a, \alpha + 1) \in F \). We claim that \( (a, \alpha + 1) \in \mathcal{L}_F \). It is clear when \( a = \alpha \) since \( \alpha < \alpha' \leq c - d - 2 \). Thus, we will assume \( a < \alpha \). The parent node of \( (a, \alpha + 1) \) has the form either \( (a', \alpha + 1) \) for some \( a' < a \), or \( (a, \alpha'') \) for some \( \alpha'' > \alpha + 1 \). The first case contradicts the minimality of \( a \). Thus, we have the second case and consequently \( |F_a| \geq 2 \).

Whence, \( (a, \alpha + 1) \in \mathcal{L}_F \), still establishing the claim. Meanwhile, since \( \alpha' > \alpha \), \( (a, \alpha + 1) \) does not contain any leaf of \( \mathcal{T}_F' \). This implies that \( (a, \alpha + 1) \in F \setminus F' \), contradicting the assumption in \([\text{5}]\).

Therefore, we will assume in the following that \( \alpha = \alpha' \), i.e., \( \mathcal{T}_F \) and \( \mathcal{T}_F' \) have a common leaf set. Now, let \( b \in [c - 1] \) be the smallest such that \( F_b \neq F'_b \). Here, \( F'_b \) is a subset of \( F' \), just as \( F_b \) defined for \( F \) in \([\text{4}]\).

(a) Suppose that \( k_b = 1 \). By the minimality of \( b \) and the fact that \( \langle \hat{\mathcal{T}}^{F'} \rangle \gg \langle \hat{\mathcal{T}}^F \rangle \), we have \( (b, j) \notin F'_b \) for all \( j \) with \( b + 1 \leq j \leq i_{b,1} \). Therefore, either \( (b, i') \in F'_b \) for some \( i' > i_{b,1} \) or \( F'_b = \emptyset \).

In the first subcase, notice that the parent node of \( (b, i_{b,1}) \) in \( \mathcal{T}_F \) must take the form \( (b', i_{b,1}) \) for some \( b' < b \). By the minimality of \( b \), \( (b', i_{b,1}) \in F' \). But since \( (b, i') \) will cross \( (b', i_{b,1}) \) when considered as open intervals, this is impossible.

In the remaining subcase, one can conclude that \( F' \) contains some vertices of the form \((b', b + 1)\) and \((b', b)\) such that \((b, b + 1)\) is not a leaf of \( \mathcal{T}_F' \), by Observations 2.8 in \([\text{v}]\). Since \( b' < b \), we also have \((b', b + 1), (b', b) \in F' \) by the minimality of \( b \). Meanwhile, as \( \mathcal{T}_F \) and \( \mathcal{T}_F' \) have a common leaf set, \((b, b + 1)\) is not a leaf of \( \mathcal{T}_F \) as well. Whence, \( i_{b,1} > b + 1 \) and we have two crossing open intervals \((b', b + 1)\) and \((b, i_{b,1})\) for \( F \), which is another contradiction.
(b) When $k_b = 2$, we observe first that $(b, i_{b,1}) \in F \cap F'$. To see this, notice that if $(b, i_{b,1}) \in F \setminus F'$, then $(b, i_{b,1})$ is not a common leaf. Whence, $(b, i_{b,1}) \in LG_F$ by our description in subsection 2.1. Now, we have $(b, i_{b,1}) \in LG_F \cap (F \setminus F')$ contradicting the assumption in [\[\text{[2.1]}\].

Therefore, by the minimality of $b$ and the fact that $\overrightarrow{T_F} >_{\text{lex}} \overrightarrow{T'_F}$, we have two subcases: either $(b, i') \in F'_b$ for some $i' > i_{b,2}$ or $F'_b = \{(b, i_{b,1})\}$. One can likewise exclude the first subcase using the non-crossing argument. Whence, we only need to check with the $F'_b = \{(b, i_{b,1})\}$ case. In $T_F$, the parent node of $(b, i_{b,1})$ must take the form of $(b', i_{b,1})$ for some $b' < b$. By the minimally of $b$, $(b', i_{b,1})$ belongs to $F \cap F'$ and will consequently be the parent node of $(b, i_{b,1})$ in both $T_F$ and $T_{F'}$. But the parent node of $(b, i_{b,1})$ in $T_F$ is $(b, i_{b,1})$, still a contradiction.

(c) Assume now that $k_b \geq 3$. If $(b, i_{b,j}) \in F'_b$ for all $1 \leq j \leq k_b - 1$, then we can argue as in the previous case [\[\text{[b]}\]. Thus, we have some $j$ with $1 \leq j \leq k_b - 1$ such that $(b, i_{b,j}) \notin F'_b$. Let $j_0$ be the smallest with this property. Notice that

- we always have $(b, i_{b,1}) \in F'$ as in [\[\text{[b]}\], and
- for $j$ with $2 \leq j \leq k_b - 1$, if $(b, i_{b,j})$ has a sibling or two children, then $(b, i_{b,j}) \in LG_F \subset F'$ by [\[\text{[5]}\].

Thus, $2 \leq j_0 \leq k_b - 1$ and $(b, i_{b,j_0})$ has neither a right child nor a right sibling in $T_F$.

As a matter of fact, $(b, i_{b,j})$ has no right sibling in $T_F$ for $j_0 \leq j \leq k_b - 1$. To see this, suppose for contradiction that there exists some $j_1 > j_0$ such that $(b, i_{b,j_1})$ has a right sibling. Let $j_1$ be the smallest. Therefore, for all $j$ with $j_0 \leq j < j_1$, $(b, i_{b,j})$ has no right sibling. Notice that $(b, i_{b,j_0-1})$ also has no right sibling. Therefore, $(i_{b,j_0-1}, i_{b,j_1})$ contains no leaf of $T_F$ and $i_{b,j} = i_{b,j_0-1} + j - (j_0 - 1)$ for $j_0 \leq j \leq j_1$ by Observations 2.8 [\[\text{[iii]}\] since $F, F' \in G_a$, it follows that $(i_{b,j_0-1}, i_{b,j_1})$ contains no leaf of $T_F$. Meanwhile, as $(b, i_{b,j_1})$ has a right sibling in $T_F$, $(b, i_{b,j_1}) \in LG_F \subset F'$ as well. Since we have assumed that $(b, i_{b,j_0-1}) \in F'$, this forces $(b, i_{b,j}) \in F'$ for $j_0 - 1 < j \leq j_1$ by Observations 2.8 [\[\text{[iv]}\] contradicting the choice of $j_0$.

Now, $(b, i_{b,j})$ has no right sibling in $T_F$ for $j_0 \leq j \leq k_b - 1$. It follows from Observations 2.8 [\[\text{[ii]}\] that the interval $(i_{b,j_0-1}, i_{b,k_b})$ has no leaf of $T_F$, and

$$i_{b,j} = i_{b,j_0-1} + j - (j_0 - 1) \text{ for } j_0 - 1 \leq j \leq k_b.$$  \hspace{1cm} (6)

Furthermore, as $(b, i_{b,j_0-1}) \in F'$ while $(b, i_{b,j_0}) \notin F'$, we have

$$\{ (b, i_{b,j}) \mid 1 \leq j \leq j_0 - 1 \} \subseteq F'_b.$$  \hspace{1cm} (7)

by Observations 2.8 [\[\text{[iv]}\].

So far, $(b, i_{b,1}) \in F \cap F'$ and $(b, i_{b,j}) \in F'_b$ for all $j = 2, 3, \ldots, j_0 - 1$ by the minimality of $j_0$. Therefore, we have

$$\{ (b, i_{b,j}) \mid 1 \leq j \leq j_0 - 1 \} \subseteq F'_b.$$  \hspace{1cm} (8)

If the containment in [\[\text{[8]}\] is strict, then we have some $(b, i') \in F'_b \setminus F_b$. Since $F' \succ F$, $i' > i_{b,j_0}$. And by [\[\text{[6]}\] with [\[\text{[7]}\], we must have $i' > i_{b,k_b}$. If $b = 1$, then $(b, i_{b,k_b})$ is the common root $(1, c)$. The existence of such an $i'$ is impossible. Therefore, $b > 1$ and the parent node of $(b, i_{b,k_b})$ in $T_F$ must have the form $(b', i_{b,k_b})$ for some $b' < b$. By the minimality of $b$, $(b', i_{b,k_b}) \in F'$. Whence, we will arrive at a contradiction due to crossing. Therefore, the containment in [\[\text{[8]}\] is actually an equality:

$$\{ (b, i_{b,j}) \mid 1 \leq j \leq j_0 - 1 \} = F'_b.$$
Now, we consider the parent node of \((b, i_{k,j_{0}-1})\) in \(\mathcal{T}_P\), and get a similar contradiction due to crossing. And this completes the proof in this subsection.

In summary, the ideal \((\text{in}(P))^{\vee}\) has linear quotients, as expected. \(\square\)

**Corollary 2.9.** When \(c \geq d + 4\), the initial complex \(\Delta\) is shellable.

**Proof.** It follows from [18] Proposition 8.2.5 and Proposition 2.6. \(\square\)

3. **APPLICATIONS OF THE COHEN–MACAULAY PROPERTY**

In this final section, \(I_{n_1,\ldots,n_d}\) is still the defining ideal of the rational normal scroll \(S_{n_1,\ldots,n_d} \subseteq \mathbb{P}^{c+d-1}\) for \(c = n_1 + \cdots + n_d\).

As the first application to our main result Theorem 2.1, we compute the Castelnuovo–Mumford regularity of the fiber cone \(\mathcal{F}(I_{n_1,\ldots,n_d})\) in Theorem 3.4. Its argument depends on the Cohen–Macaulayness established in section 2 and the regularity result of balanced cases in [21]. Again, by the Cohen–Macaulayness, the regularity of the fiber cone can be related to its \(a\)-invariant, as well as the reduction number of the ideal \(I_{n_1,\ldots,n_d}\).

It is worth mentioning that, [12, Section 20.5] provides a concise introduction to the notion of Castelnuovo–Mumford regularity, as well as some historical notes. And the \(a\)-invariant was introduced by Goto and Watanabe in [15] Definition 3.1.4. A nice thing is that, for any standard graded Cohen–Macaulay algebra \(A\) over \(\mathbb{K}\), one has

\[
a(A) = \text{reg}(A) - \text{dim}(A),
\]

in view of the equivalent definition of regularity in [22] Definitions 1 and 3. As for the reduction number of an ideal and some related topics, one may wish to consult [20] Section 8.2. Since these three invariants have been widely studied in the commutative algebra literature, we will not bother repeating their definitions here.

However, before stating the concrete result and displaying its proof, we still need to include three important facts here for completeness.

**Lemma 3.1** ([3, Proposition 7.43]). Let \(M\) be a graded Cohen–Macaulay module over the polynomial ring \(\mathbb{K}[x_1,\ldots,x_n]\), and \(P_M(t)\) the numerator Laurent polynomial of the Hilbert series of \(M\). Then \(\text{reg}(M) = \deg(P_M(t))\).

**Lemma 3.2** ([10, Proposition 6.6] or [14, Proposition 1.2]). Let \(I \subseteq R = \mathbb{K}[x_1,\ldots,x_N]\) be a homogeneous ideal that is generated in one degree, say \(d\). Assume that the fiber cone \(\mathcal{F}(I)\) is Cohen–Macaulay. Then each minimal reduction of \(I\) is generated by \(\dim(\mathcal{F}(I))\) homogeneous polynomials of degree \(d\), and \(I\) has the reduction number \(r(I) = \text{reg}(\mathcal{F}(I))\).

Recall that the rational normal scroll \(S_{n_1,\ldots,n_d}\) is balanced precisely when \(|n_i - n_j| \leq 1\) for all \(i, j\). Whence, we can rearrange the columns of the matrix \(X\) in the equation (11) and rewrite it as the Hankel matrix \(H_{2,c,d}\) for \(c = n_1 + \cdots + n_d\). And here is the last piece that we need for the regularity result.

**Lemma 3.3** ([21, Theorem 3.1]). Let \(I_2(H_{2,c,d})\) be the defining ideal of the rational normal scroll \(S_{n_1,\ldots,n_d} \subseteq \mathbb{P}^{c+d-1}\) in the balanced case. Then, the Castelnuovo–Mumford regularity of the fiber cone \(\mathcal{F}(I_2(H_{2,c,d}))\) is

\[
\text{reg}(\mathcal{F}(I_2(H_{2,c,d}))) = \begin{cases} 
\left\lfloor (c + d - 1)/2 \right\rfloor, & \text{if } 4 + d \leq c, \\
\left\lfloor c - 3 \right\rfloor, & \text{if } 2 < c \leq 4 + d.
\end{cases}
\]

It is time to describe the first result of this section.

**Theorem 3.4.** Let \(I_{n_1,\ldots,n_d}\) be the defining ideal of the rational normal scroll \(S_{n_1,\ldots,n_d} \subseteq \mathbb{P}^{c+d-1}\) for \(c = n_1 + \cdots + n_d\).
(i) The Castelnuovo–Mumford regularity of the fiber cone is given by
\[
\text{reg}(F(I_{n_1,\ldots,n_d})) = \begin{cases} 
    \lceil (c + d - 1)/2 \rceil, & \text{if } 4 + d \leq c, \\
    c - 3, & \text{if } 2 < c < 4 + d.
\end{cases}
\]

(ii) The reduction number of \(I_{n_1,\ldots,n_d}\) is given by
\[
\text{r}(I_{n_1,\ldots,n_d}) = \begin{cases} 
    \lceil (c + d - 1)/2 \rceil, & \text{if } 4 + d \leq c, \\
    c - 3, & \text{if } 2 < c < 4 + d.
\end{cases}
\]

(iii) The \(a\)-invariant of the fiber cone is given by
\[
\text{a}(F(I_{n_1,\ldots,n_d})) = \begin{cases} 
    \lceil (c + d - 1)/2 \rceil - c - d, & \text{if } 4 + d \leq c, \\
    -c, & \text{if } 2 < c < 4 + d.
\end{cases}
\]

Proof. Notice that the fiber cone \(F(I_{n_1,\ldots,n_d})\) is Cohen–Macaulay by Theorem 2.1. Thus, the item (ii) follows directly from the item (i) and Lemma 3.2. And similarly, the item (iii) follows from the item (i) Equation (9) and the fact that the fiber cones of the ideal \(I_{n_1,\ldots,n_d}\) by [25, Proposition 3.1, Corollary 3.10].

As for the item (i), just as observed in the proof of [25, Theorem 3.13], the Hilbert function of the fiber cone of the rational normal scroll \(S_{n_1,\ldots,n_d}\) depends only on \(c\) and \(d\) by [2, Theorem 3.7]. Whence, the fiber cones of the ideal \(I_{n_1,\ldots,n_d}\) above and the ideal \(I_2(H_{2,c,d})\) in [21, Section 3] have the same Laurent polynomial. Now, by the Cohen–Macaulayness of \(F(I_{n_1,\ldots,n_d})\) established in Theorem 2.1 these fiber cones have the same Castelnuovo–Mumford regularity by Lemma 3.3.

Now, it remains to apply Lemma 3.3. \(\square\)

As the second application to our main result Theorem 2.1 we characterize the Gorensteinness of the fiber cone. The corresponding numerical condition in the balanced case is the following.

Lemma 3.5 ([21, Proposition 2.10]). Let \(I_2(H_{2,c,d})\) be the defining ideal of the rational normal scroll \(S_{n_1,\ldots,n_d} \subseteq \mathbb{P}^{c+d-1}\) in the balanced case. Then, the fiber cone \(F(I_2(H_{2,c,d}))\) is Gorenstein if and only if \(c \in \{2, 3, 2+d, 3+d, 4+d\}\).

And here is the last result that this paper wants to describe.

Theorem 3.6. Let \(I_{n_1,\ldots,n_d}\) be the defining ideal of the rational normal scroll \(S_{n_1,\ldots,n_d} \subseteq \mathbb{P}^{c+d-1}\) for \(c = n_1 + \cdots + n_d\). Then, the fiber cone \(F(I_{n_1,\ldots,n_d})\) is Gorenstein if and only if \(c \in \{2, 3, 2+d, 3+d, 4+d\}\).

Proof. The fiber cone \(F(I_{n_1,\ldots,n_d})\) is Cohen–Macaulay by Theorem 2.1. And since it is isomorphic to the toric ring \(\mathbb{K}[I_{n_1,\ldots,n_d}]\), it is an integral domain. As mentioned in the proof of Theorem 3.4, the fiber cones \(F(I_{n_1,\ldots,n_d})\) and \(F(I_2(H_{2,c,d}))\) have identical Hilbert series. Thus, by both (b) and (c) of [24, Corollary 4.4.6], \(F(I_{n_1,\ldots,n_d})\) is Gorenstein, if and only \(F(I_2(H_{2,c,d}))\) is so. Now, it suffices to apply Lemma 3.5. \(\square\)

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12




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