Uncertainty and Certainty Relations for Successive Projective Measurements of a Qubit in Terms of Tsallis’ Entropies

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Abstract We study uncertainty and certainty relations for two successive measurements of two-dimensional observables. Uncertainties in successive measurement are considered within the following two scenarios. In the first scenario, the second measurement is performed on the quantum state generated after the first measurement with completely erased information. In the second scenario, the second measurement is performed on the post-first-measurement state conditioned on the actual measurement outcome. Induced quantum uncertainties are characterized by means of the Tsallis entropies. For two successive projective measurement of a qubit, we obtain minimal and maximal values of related entropic measures of induced uncertainties. Some conclusions found in the second scenario are extended to arbitrary finite dimensionality. In particular, a connection with mutual unbiasedness is emphasized.

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1 Introduction

The Heisenberg uncertainty principle[1] is one of the fundamentals of quantum theory. Despite of its wide popularity, there is no general consensus over the scope and validity.[2–3] It is typically said that measuring some observable will inevitably disturb the system, whence the context for further observations is raised. There are many ways to quantify uncertainties as well as few scenarios of measuring observables. The first form of explicit mathematical formulation has been given for the position and momentum by Kennard.[4] For this pair, a product of the standard deviations cannot be less than $\hbar/2$. For any pair of observables, this direction was realized by Robertson.[5] However, the traditional approach deals with quantum uncertainties raised in two different experiments with the same pre-measurement state. So, this approach does not reveal many details about a disturbance of the system due to performed measurements. Rather, Heisenberg’s initial reasons are better formulated in terms of noise and disturbance.[6] As was discussed in Refs. [7–8], studies of quantum uncertainties in the results of successive measurements have received less attention than they deserve.

Much attention to uncertainty relations is stimulated by a progress in using quantum systems as informational recourses.[9] Hence, formulations in information-theoretic terms including entropies are of interest. Traditional uncertainty relations for quantum measurements deal with probability distributions calculated for one and the same input state. This treatment was studied in entropic terms[10–12] and recently by means of majorization technique.[13–15] In quantum information processing, other situations are rather typical. Our subsequent manipulations deal with an output state of the latter stage. In the case of two successive measurements, the following two scenarios are typically addressed.[16–17] In the first scenario, the second measurement is performed on the quantum state generated after the first measurement with completely erased information. In the second scenario, the second measurement is performed on the post-first-measurement state conditioned on the actual measurement outcome. Thus, the scenarios are related to a realistic situation, when subsequent actions deal with an output state of the latter stage.

In the present work, we study uncertainties in successive measurements with the use of Tsallis entropies. We also focus on maximal possible values of related entropic measures. Indeed, certainty relations for successive measurements seem to be not considered previously. For successive measurements of a pair of qubit observables, we obtain uncertainty and certainty entropic bounds. The paper is organized as follows. In Sec. 2, the background material is reviewed. Here, definitions of the used entropic functions are recalled. In Sec. 3, we generally discuss quantum uncertainties induced by successive measurements of a pair of observables. Two scenarios of such measurements and related $\alpha$-entropic measures are introduced. In Sec. 4, we derive tight uncertainty and certainty relations for a qubit within the first scenario. The conditions for equality are obtained and discussed. Tight lower and upper bounds on the conditional $\alpha$-entropies within...
the second scenario are presented in Sec. 5. We also observe a connection of the equality conditions for upper bounds with mutual unbiasedness. This observation remains valid in arbitrary finite dimensions. In Sec. 6, we conclude the paper with a summary of results.

2 Preliminaries

To quantify uncertainties in generated probability distributions, we will use Tsallis’ entropies. Let discrete random variable $X$ take values on a finite set $\Omega_X$ of cardinality $#\Omega_X$. The Tsallis entropy of degree $\alpha > 0 \neq 1$ is defined by\cite{18}

$$H_\alpha(X) := \frac{1}{1-\alpha} \left( \sum_{x \in \Omega_X} p(x)^\alpha - 1 \right). \quad (1)$$

With other factor, this function was examined by Havrda and Charvát\cite{19} and later by Daróczy.\cite{20} In statistical physics, the entropy (1) is extensively used due to Tsallis.\cite{18} For other multidisciplinary applications, see the book\cite{21} and references therein. The Rényi entropies\cite{22} form another especially important family of one-parametric extensions of the Shannon entropy. For $\alpha > 0 \neq 1$, Rényi’s $\alpha$-entropy can be expressed via Tsallis’ entropy as

$$R_\alpha(X) = \frac{1}{1-\alpha} \ln(1 + (1-\alpha)H_\alpha(X)). \quad (2)$$

Despite of a direct relation, the entropies (1) and (2) differ in essential properties. As a rule, formulation in terms of one of the two entropies cannot immediately be recast for other. Such a situation takes place for entropic uncertainty relations too. In Ref.\cite{23}, uncertainty and certainty relations for the Pauli observables were derived in terms of the Rényi entropies. Using Rényi’s entropies, uncertainty relations for successive projective measurements were studied in Ref.\cite{17}. Properties and applications of both the types of entropies in quantum theory are considered in the book.\cite{24}

Obviously, the function $(\xi^\alpha - \xi)/(1 - \alpha)$ is concave for all $\alpha > 0$. Hence, the entropy (1) is a concave function of probability distribution. The maximal value $\ln_\alpha(#\Omega_X)$ is reached by the uniform distribution. It is often convenient to rewrite Eq. (1) as

$$H_\alpha(X) = - \sum_{x \in \Omega_X} p(x)^\alpha \ln_\alpha p(x)$$

$$= \sum_{x \in \Omega_X} p(x) \ln_\alpha \left( \frac{1}{p(x)} \right). \quad (3)$$

Here, we use the $\alpha$-logarithm defined for $\alpha > 0 \neq 1$ and $\xi > 0$ as

$$\ln_\alpha(\xi) = \frac{\xi^{1-\alpha} - 1}{1 - \alpha}. \quad (4)$$

In the limit $\alpha \to 1$, we have $\ln_\alpha(\xi) \to \ln \xi$, so that the Shannon entropy $H_1(X) = - \sum_x p(x) \ln p(x)$ is raised. For brevity, we will typically write entropic sums without set symbols such as $\Omega_X$.

The above definitions are also applied in the quantum regime. Let the state of a quantum system be described by the density matrix $\rho$. It is a positive semi-definite matrix with the unit trace. The quantum $\alpha$-entropy of $\rho$ is defined as

$$H_\alpha(\rho) := \frac{1}{1-\alpha} \left( \text{Tr}(\rho^\alpha) - 1 \right). \quad (5)$$

In the case $\alpha = 1$, we deal with the von Neumann entropy $H_1(\rho) = - \text{Tr}(\rho \ln \rho)$. For general properties on the von Neumann entropy, see Refs.\cite{25–26}. In the following, the entropies (1) and (5) will be used in studying uncertainties in successive projective measurements.

In one of the two scenarios of successive measurements, the second measurement is performed on the actual post-first-measurement state. Analyzing this scenario, conditional form of entropies will be utilized. The conditional entropies are widely used in information theory\cite{27} and in applied disciplines. The standard conditional entropy is defined as follows. Let $X$ and $Z$ be random variables. For each $z \in \Omega_Z$, we take the function

$$H_1(X|z) = - \sum_{x \in \Omega_X} p(x|z) \ln p(x|z). \quad (6)$$

By $p(x|z)$, we mean the conditional probability that $X = x$ given that $Z = z$. By Bayes’ rule, it obeys $p(x|z) = p(x,z)/p(z)$. The entropy of $X$ conditional on knowing $Z$ is defined as\cite{27}

$$H_1(X|Z) := \sum_{z \in \Omega_Z} p(z) H_1(X|z)$$

$$= - \sum_{x \in \Omega_X} \sum_{z \in \Omega_Z} p(x,z) \ln p(x|z). \quad (7)$$

For brevity, we will further write entropic sums without mentioning that $x \in \Omega_X$ and $z \in \Omega_Z$. In the context of quantum theory, the conditional entropy (7) was used in information-theoretic formulations of Bell’s theorem\cite{28} and noise-disturbance uncertainty relations.\cite{29}

In the literature, two kinds of the conditional THC entropy were considered.\cite{30–31} These forms are respectively inspired by the two expressions shown in Eq. (3). The first conditional form is defined as\cite{30}

$$H_\alpha(X|Z) := \sum_z p(z)^\alpha H_\alpha(X|z), \quad (8)$$

where

$$H_\alpha(X|z) := \frac{1}{1-\alpha} \left( \sum_z p(x|z)^\alpha - 1 \right). \quad (9)$$

The conditional entropy (8) is, up to a factor, the quantity originally introduced by Daróczy.\cite{20} For any $\alpha > 0$, the conditional entropy (8) satisfies the chain rule.\cite{20,30} The chain rule is essential in some applications of conditional entropies in quantum information science. For example,
this rule for the standard conditional entropy is used in deriving the Brauneist–Caves inequality.\cite{36} This inequality expresses an entropic version of the Bell theorem.\cite{37} The $\alpha$-entropies were applied in formulating non-locality, contextuality and non-macrorealism inequalities.\cite{38,39}

Using the particular function (9), the second form of conditional $\alpha$-entropy is written as\cite{40,41,42}

$$H_\alpha(X|Z) = \sum_{z \in \text{spec}(Z)} p(z) H_\alpha(X|z).$$

(10)

It should be noted that this form of conditional entropy does not share the chain rule of usual kind.\cite{41} Nevertheless, the conditional entropy (10) is interesting at least as an auxiliary quantity.\cite{42} Moreover, it can be used in studying the Bell theorem,\cite{43} even though the chain rule is not applicable here.

3 On Successive Projective Measurements in Finite Dimensions

In this section, we consider several facts concerning uncertainty and certainty relations for successive measurements in finite dimensions. Let $Z$ be an observable of some finite-level quantum system. By $\text{spec}(Z)$, we denote the spectrum of $Z$. The spectral decomposition is written as

$$Z = \sum_{z \in \text{spec}(Z)} z P_z,$$

(11)

where $P_z$ denotes the orthogonal projection on the corresponding eigenspace of $Z$. The operators $P_z$ are mutually orthogonal and satisfy the completeness relation

$$\sum_{z \in \text{spec}(Z)} P_z = 1.$$

(12)

By $I$, we denote the identity operator in the Hilbert space of studied system. Let the pre-measurement state be described by the density matrix $\rho$. The probability of each outcome $z$ is expressed as $\text{Tr}(P_z \rho)$. Due to Eq. (12), these probabilities are summarized to 1. Calculating the $\alpha$-entropy of the generated probability distribution, we will deal with the quantity

$$H_\alpha(Z; \rho) = \frac{1}{1-\alpha} \left( \sum_{z \in \text{spec}(Z)} \left( \text{Tr}(P_z \rho) \right)^\alpha - 1 \right).$$

(13)

Let $X$ be another observable with the spectral decomposition

$$X = \sum_{x \in \text{spec}(X)} x Q_x.$$

(14)

Here, the operator $Q_x$ is the orthogonal projection on the corresponding eigenspace of $X$. We consider an amount of uncertainties induced by two successive measurements of the observables, $Z$ at first and $X$ later. As was above mentioned, there are two possible scenarios of interest.

In the first scenario, the second measurement is performed on the quantum state generated after the first measurement with completely erased information. That is, the second measurement is performed with the pre-measurement state

$$E_Z(\rho) = \sum_{z \in \text{spec}(Z)} P_z \rho P_z.$$  

(15)

The linear map $E_Z$ describes the action of the projective measurement of $Z$. The right-hand side of Eq. (15) is an operator-sum representation of the map.\cite{44} The completeness relation (12) provides that the map (15) preserves the trace for all inputs. The uncertainty in the second measurement is quantified by means of the entropy $H_\alpha(X; E_Z(\rho))$. The latter is expressed similarly to Eq. (13), but with the probabilities $\text{Tr}(Q_x E_Z(\rho))$. We will characterize a total amount of uncertainty in the first scenario by the sum of the classical entropies of two generated probability distributions. Note that the post-first-measurement state (15) obeys the following property. If we have measured $Z$ in the state $E_Z(\rho)$, we again deal with probabilities $\text{Tr}(P_z \rho)$. Thus, we write the formula

$$H_\alpha(Z; \rho) + H_\alpha(X; E_Z(\rho)) = H_\alpha(E_Z(\rho)) + H_\alpha(E_X \circ E_Z(\rho)).$$

(16)

By $E_X \circ E_Z$, we mean the composition of two quantum operations.\cite{45} Further, the projectors $Q_x$ give rise to the map

$$E_X(\omega) = \sum_{x \in \text{spec}(X)} Q_x \omega Q_x,$$

(17)

where $\omega$ is a density matrix. The left-hand side of Eq. (16) is the sum of classical $\alpha$-entropies of the two probability distributions, whereas the right-hand side is the sum of quantum $\alpha$-entropies. To formulate uncertainty and certainty relations for successive measurements, we aim to have a two-sided estimate on the quantity (16).

Let us recall one of physically important properties of the von Neumann entropy related to the measurement process. In effect, projective measurements cannot decrease the von Neumann entropy (see, e.g., theorem 11.9 in Ref. [36]), that is

$$H_1(\mathcal{E}_Z(\rho)) \geq H_1(\rho).$$

(18)

In the paper,\cite{46} we extended the above property to the family of quantum unified entropies. In particular, for all $\alpha > 0$ we have\cite{47}

$$H_\alpha(\mathcal{E}_Z(\rho)) \geq H_\alpha(\rho).$$

(19)

It follows from Eqs. (16) and (19) that

$$H_\alpha(Z; \rho) + H_\alpha(X; E_Z(\rho)) \geq H_\alpha(\rho) + H_\alpha(E_Z(\rho)).$$

(20)

This inequality can be treated as an uncertainty relation expressed in terms of the quantum $\alpha$-entropies of $\rho$ and $E_Z(\rho)$. Certainty relations are formulated as upper bounds on the sum of considered entropies. At this stage, only simple bounds may be given. We merely recall that the quantum $\alpha$-entropy is not more than the $\alpha$-entropy of the completely mixed state. Thus, we obtain

$$H_\alpha(Z; \rho) + H_\alpha(X; E_Z(\rho)) \leq 2H_\alpha(\varrho).$$

(21)
where the completely mixed state \( \rho_c = 1/ \text{Tr} (1) \). We aim to obtain uncertainty and certainty relations connected with the purity \( \text{Tr} (\rho^2) \) of the input state. In the following sections, more detailed relations for two successive measurements will be formulated in the qubit case.

In another scenario of successive measurements, the second measurement is performed on the post-first-measurement state conditioned on the actual measurement outcome. Let the first measurement has given the outcome \( z \). The probability of this event is written as \( p(z) = \text{Tr} (P_z \rho) \). According to the projection postulate, the state right after the measurement is described by the projector \( P_z \). In the second measurement, therefore, the outcome \( x \) is obtained with the probability \( p(x|z) = \text{Tr} (Q_x P_z) \).

The latter is the conditional probability of outcome \( x \) given that the previous measurement of \( Z \) has resulted in \( z \). In our case, the function (9) is obviously expressed as

\[
H_\alpha (X|z) = \frac{1}{1-\alpha} \left( \sum_{x \in \text{spec}(X)} (\text{Tr} (Q_x P_z))^\alpha - 1 \right). \tag{22}
\]

It should be emphasized that this quantity does not depend on \( \rho \). For the scenario considered, an amount of uncertainties is characterized by means of the conditional entropies

\[
H_\alpha (X|Z; \rho) = \sum_{z \in \text{spec}(Z)} (\text{Tr} (P_z \rho))^\alpha H_\alpha (X|z), \tag{23}
\]

\[
\tilde{H}_\alpha (X|Z; \rho) = \sum_{z \in \text{spec}(Z)} \text{Tr} (P_z \rho) H_\alpha (X|z). \tag{24}
\]

Taking \( \alpha = 1 \), both the \( \alpha \)-entropies are reduced to the standard conditional entropy \( H_1 (X|Z; \rho) \). In Ref. [16], the latter entropy was examined as a measure of uncertainties in successive measurements. In the following, we will give minimal and maximal values of the conditional entropies (23) and (24) in the qubit case.

### 4 First-Scenario Relations for Successive Qubit Measurements

In this section, we will obtain uncertainty and certainty relations for a qubit within the first scenario. The formulation of uncertainty relations for successive measurements in terms of the Shannon entropies was given in Ref. [16]. The authors of Ref. [17] studied corresponding uncertainty relations in terms of the Rényi entropies. Although the Rényi and Tsallis entropies are closely connected, relations for one of them are not immediately applicable to other. In particular, the formula (2) does not allow to move between the Rényi-entropy and Tsallis-entropy formulations of the uncertainty principle. This fact also holds for the case of successive measurements. We will formulate uncertainty relations for two successive projective measurements in terms of Tsallis’ entropies.

Each density matrix in two dimensions can be represented in terms of its Bloch vector as

\[
\rho = \frac{1}{2} (1 + \vec{r} \cdot \vec{\sigma}). \tag{25}
\]

By \( \vec{\sigma} \), we denote the vector of the Pauli matrices \( \sigma_1, \sigma_2, \sigma_3 \). The three-dimensional vector \( \vec{r} = (r_1, r_2, r_3) \) obeys \( |\vec{r}| \leq 1 \), with equality if and only if the state is pure. By calculations, we obtain the purity

\[
\text{Tr} (\rho^2) = \frac{1 + |\vec{r}|^2}{2}. \tag{26}
\]

The language of Bloch vectors is very convenient in description of qubit states and their transformations [24, 36]. Following Ref. [17], we will also use this approach in representing projectors of the observables \( Z \) and \( X \).

Without loss of generality, we assume the observables to be non-degenerate. Indeed, in two dimensions any degenerate observable is inevitably proportional to the identity operator. We will exclude this trivial case. Further simplification is reached by rescaling eigenvalues of the observables. Working in information-theoretic terms, we mainly deal with probability distributions. In such a consideration, we can turn observables to be dimensionless. Moreover, we can further shift eigenvalues of the observables without altering the probabilities. Of course, such actions are not appropriate for more traditional approach, using the mean value and the variance. Thus, the spectral decompositions are written as

\[
Z = z_+ P_+ + z_- P_-, \tag{27}
\]

\[
X = x_+ Q_+ + x_- Q_. \tag{28}
\]

Taking the observables \( Z \) and \( X \) with eigenvalues \( \pm 1 \), we will arrive at dimensionless spin observables. Any projector describes a pure state and, herewith, is represented by means of some Bloch vector. We introduce two unit vectors \( \vec{p} \) and \( \vec{q} \) such that

\[
P_\pm = \frac{1}{2} (\mathbb{1} \pm \vec{p} \cdot \vec{\sigma}), \tag{29}
\]

\[
Q_\pm = \frac{1}{2} (\mathbb{1} \pm \vec{q} \cdot \vec{\sigma}). \tag{30}
\]

When eigenvalues of each of the observables are \( \pm 1 \), we simply have \( Z = \vec{p} \cdot \vec{\sigma} \) and \( X = \vec{q} \cdot \vec{\sigma} \). In the first measurement, we measure \( Z \) in the pre-measurement state (25).

The generated probability distribution is written as

\[
p(z \pm 1) = \frac{1 \pm \vec{p} \cdot \vec{r}}{2}. \tag{31}
\]

In the considered scenario, information contained in the qubit after the first measurement is completely erased. By calculations, we now have

\[
\hat{E}_Z (\rho) = P_+ \rho P_+ + P_- \rho P_- = \frac{1}{2} (1 + (\vec{p} \cdot \vec{r}) \vec{p} \cdot \vec{\sigma}), \tag{32}
\]

with the Bloch vector \( (\vec{p} \cdot \vec{r}) \vec{p} \). Here, the map turns the Bloch vector into its projection on \( \vec{p} \). The density matrix

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\footnote{In the non-degenerate case.}
(32) describes the pre-measurement state of the measurement of $X$. Similarly to Eq. (31), we write generated probabilities in the form

$$p(x = \pm 1) = \frac{1 \pm (\vec{q} \cdot \vec{p})(\vec{p} \cdot \vec{r})}{2}. \quad (33)$$

To avoid bulky expressions, we put the function of positive variable with the parameter $\alpha > 0$,

$$\eta_\alpha(\xi) := \frac{\xi^\alpha - \xi}{1 - \alpha} = -\xi^\alpha \ln \xi,$$

including $\eta_1(\xi) = -\xi \ln \xi$. Due to the space isotropy, we can choose the frame of references in such a way that $\vec{p} = \vec{e}_3$. Then the entropic quantity (16) is written as

$$\sum_{m=\pm 1} \eta_\alpha\left(\frac{1 + m|\vec{p} \cdot \vec{r}|}{2}\right) + \sum_{n=\pm 1} \eta_\alpha\left(\frac{1 + n|\vec{q} \cdot \vec{p}||\vec{p} \cdot \vec{r}|}{2}\right) = \sum_{m=\pm 1} \eta_\alpha\left(\frac{1 + m\eta_3|\vec{p} \cdot \vec{r}|}{2}\right). \quad (35)$$

For brevity, we denote $\mu = (\vec{q} \cdot \vec{p}) \in [-1; +1]$. To obtain purity-based uncertainty and certainty relations, we will search the minimum and the maximum of Eq. (35) under the restriction that the Bloch vector length $|\vec{r}|$ is fixed. Due to Eq. (26), the purity of a quantum state then remains unchanged. The following statement takes place.

**Proposition 1** Let the length $|\vec{r}|$ of the Bloch vector of $\rho$ be fixed. For all $\alpha > 0$, the sum of $\alpha$-entropies for successive measurements of non-degenerate observables $Z$ and $X$ is bounded from below as

$$H_\alpha(Z; \rho) + H_\alpha(X; \mathcal{E}_Z(\rho)) \geq \sum_{m=\pm 1} \eta_\alpha\left(\frac{1 + m|\vec{r}|}{2}\right) + \sum_{n=\pm 1} \eta_\alpha\left(\frac{1 + n\mu|\vec{r}|}{2}\right). \quad (36)$$

The equality in Eq. (36) holds if and only if $\rho$ commutes with $Z$, i.e., $\rho Z = Z \rho$. For all $\alpha > 0$, the sum of $\alpha$-entropies is bounded from above as

$$H_\alpha(Z; \rho) + H_\alpha(X; \mathcal{E}_Z(\rho)) \leq 2 \ln_\alpha(2). \quad (37)$$

The equality in Eq. (37) holds if and only if $\text{Tr}(Z \rho) = \text{Tr}(Z) / 2$.

**Proof** For the given $|\vec{r}|$, the component $r_3 = (\vec{p} \cdot \vec{r})$ obeys $-|\vec{r}| \leq r_3 \leq +|\vec{r}|$. Since the right-hand side of Eq. (35) is an even function of $r_3$, we can restrict our consideration to the interval $r_3 \in [0; |\vec{r}|]$. For all $\alpha > 0$, the function (34) is concave. Thus, the right-hand side of Eq. (35) is concave with respect to $r_3$. A concave function reaches the minimal value at one or more least points of the interval. To minimize the function (35), we should therefore compare its values for $r_3 = 0$ and for $r_3 = |\vec{r}|$. The latter actually leads to the minimum, whence the claim (36) is proved. Indeed, substituting $r_3 = 0$ leads to the uniform distribution and, therefore, to the maximal value $\ln_\alpha(2)$ of each of two entropies. The last comment justifies the claim (37).

We shall now prove conditions for the equality. Substituting $r_3 = \pm |\vec{r}|$, the right-hand side of Eq. (35) is equal to the right-hand side of Eq. (36). Some inspection shows that deviating $r_3$ from the points $\pm |\vec{r}|$ will certainly increase the term (35). Thus, these points are the only case when the inequality (36) is saturated. In this case, we have $\vec{r} \parallel \vec{p}$, whence the density matrix is diagonal with respect to the common eigenbasis of the projectors $P_\pm$. Hence, the operators $\rho$ and $Z$ commute. To saturate the inequality (37), the two entropies must reach the maximal value. The only case is $r_3 = 0$, when the probability distributions $(1 \pm r_3)/2$ and $(1 \pm \mu r_3)/2$ are both uniform. Since

$$\text{Tr} ((\vec{p} \cdot \vec{r})((\vec{p} \cdot \vec{r}))) = 2(\vec{p} \cdot \vec{r})^2, \quad (38)$$

for $\vec{r} \perp \vec{p}$ we obtain $\text{Tr}(P_\pm \rho) = 1/2$. Combining the latter with (27) gives $\text{Tr}(Z \rho) = \text{Tr}(Z)/2$.

The result (36) is a Tsallis-entropy family of uncertainty relations for successive projective measurements of a qubit. The pre-measurement density matrix $\rho$ describes a mixed state of the purity (26). For the given $|\vec{r}|$, this purity is constant. In two dimensions, a geometrical description in terms of the Bloch vector is more convenient. The equality in Eq. (36) takes place, if and only if $\vec{r} \parallel \vec{p}$. Among states of the fixed purity, minimal uncertainties are revealed by states, whose Bloch vector is collinear to the Bloch vector associated with the projectors on the eigenspaces of $Z$. In terms of operators, this condition implies the commutativity of $Z$ and $\rho$. Further, the density matrix is then a fixed point of $\mathcal{E}_Z$, i.e., $\mathcal{E}_Z(\rho) = \rho$ for $\vec{r} \parallel \vec{p}$.

We further note that the lower bound (20) is typically not saturated for $\vec{r} \parallel \vec{p}$. Indeed, the first entropic sum in the quantity (36) actually becomes equal to the quantum $\alpha$-entropy of $\rho$. This is not the case for the second one. In general, the second entropic sum in the formula (36) is strictly larger than the quantum $\alpha$-entropy of $\mathcal{E}_Z(\rho)$. The equality takes place only for $\mu = \pm 1$ that is equivalent to $\vec{q} \parallel \vec{p}$. In such a situation, the operators $\rho$, $Z$, and $X$ are all diagonal in the same common eigenbasis. Then the picture becomes purely classical in character.

As we see, uncertainties in considered successive measurements are minimized for $\vec{r} \parallel \vec{p}$. It is natural to expect that the condition $\vec{r} \perp \vec{p}$ will lead to an opposite case of maximal uncertainties. In effect, this case actually leads to the maximal values to both the entropies of the left-hand side of Eq. (37). Then the quantum operation $\mathcal{E}_Z$ maps the input $\rho$ into the completely mixed state. Here, we have $\mathcal{E}_X \circ \mathcal{E}_Z(\rho) = \mathcal{E}_Z(\rho) = \varrho$. Thus, the upper bound (21) is actually saturated. When the two eigenvalues of $Z$ are symmetric with respect to 0, as for spin observables, the condition $\vec{r} \perp \vec{p}$ implies that the mean value of $Z$ in the state $\rho$ is zero, i.e., $\text{Tr}(Z \rho) = 0$. Indeed, in such a case we deal with the traceless observable.
5 Bounds for the Second Scenario of Successive Qubit Measurements

In this section, we will obtain uncertainty and certainty relations for a qubit within the second scenario. Here, the second measurement is performed on the post-first-measurement state conditioned on the actual measurement outcome. Using the conditional Rényi entropy, corresponding uncertainty relations were derived in Ref. [17]. To quantify uncertainties, we will use the conditional Tsallis entropies (8) and (10). They cannot be related immediately to the conditional Rényi entropy. So, a formulation in terms the conditional entropies (8) and (10) is of own interest. It turns out that the entropy (8) rather gives a more sensitive measure than (10). Entropic certainty bounds for successive measurements seem to be not studied in the literature.

We first measure the observable $Z$ in the state (25), for which the probabilities of outcomes is calculated according to Eq. (31). If the first measurement has given the outcome $m$, then the post-first-measurement state is described by the projector $P_m$ with the Bloch vector $m\vec{p}$. Then we perform the measurement of $X$ with generating the probability distribution

$$p(x = n | z = m) = \frac{1 + n(\vec{q} \cdot m\vec{p})}{2}, \quad (39)$$

where $m, n = \pm 1$. These quantities are conditional probabilities used in Eq. (22). For both $m = \pm 1$, the entropic function (22) is expressed as

$$H_\alpha(X|z = m) = H_\alpha(\mathcal{E}_X(P_m)) = \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right). \quad (40)$$

Hence, the two conditional Tsallis entropies are represented as

$$H_\alpha(X|Z; \rho) = \sum_{m = \pm 1} \left(\frac{1 + mr_3}{2}\right) \alpha \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right), \quad (41)$$

$$\tilde{H}_\alpha(X|Z; \rho) = \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right). \quad (42)$$

Thus, the second form of conditional $\alpha$-entropies does not depend on the input state $\rho$. This entropic quantity is completely given by taking the observables $Z$ and $X$. Except for $\alpha = 1$, the first conditional entropy (41) depends on the state $\rho$ and both the observables. The right-hand side of Eq. (41) gives a general expression of the entropy. Let us examine an interval, in which this quantity ranges as a function of inputs of the fixed purity.

**Proposition 2** Let the length $|\vec{r}|$ of the Bloch vector of $\rho$ be fixed. For $\alpha \in (0; 1)$, the conditional $\alpha$-entropy for successive measurements of non-degenerate observables $Z$ and $X$ obeys

$$(1 + (1 - \alpha)H_\alpha(\rho)) \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right) \leq H_\alpha(X|Z; \rho), \quad (43)$$

$$H_\alpha(X|Z; \rho) \leq 2^{1-\alpha} \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right). \quad (44)$$

The lower bound (43) is reached if and only if $\rho$ commutes with $Z$, i.e., $\rho Z = Z \rho$. The upper bound (44) is reached if and only if $\text{Tr}(Z \rho) = \text{Tr}(Z)/2$.

For $\alpha \in (1; \infty)$, the conditional $\alpha$-entropy for successive measurements of non-degenerate observables $Z$ and $X$ obeys

$$2^{1-\alpha} \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right) \leq H_\alpha(X|Z; \rho), \quad (45)$$

$$H_\alpha(X|Z; \rho) \leq (1 + (1 - \alpha)H_\alpha(\rho)) \sum_{n = \pm 1} \eta_\alpha \left(\frac{1 + n\mu}{2}\right). \quad (46)$$

The lower bound (45) is reached if and only if $\text{Tr}(Z \rho) = \text{Tr}(Z)/2$. The upper bound (46) is reached if and only if $\rho Z = Z \rho$.

**Proof** We need only find those least values that exactly bound the first sum in the right-hand side of Eq. (41). For brevity, we define the function

$$g_\alpha(r_3) := \left(\frac{1 + r_3}{2}\right)^\alpha + \left(\frac{1 - r_3}{2}\right)^\alpha. \quad (47)$$

As this function is even, the aim is to find its minimum and maximum on the interval $r_3 \in [0; |\vec{r}|]$.

For $\alpha \in (0; 1)$, the function $r_3 \mapsto g_\alpha(r_3)$ is concave. So, we have $g_\alpha(r_3) \leq 2^{1-\alpha} = g_0(0)$ by using Jensen’s inequality with the weights 1/2. Hence, the claim (44) follows. Due to concavity, the minimum is reached for one of two least points of the interval $r_3 \in [0; |\vec{r}|]$. So, the desired minimum is

$$g_\alpha(|\vec{r}|) = \left(\frac{1 + |\vec{r}|}{2}\right)^\alpha + \left(\frac{1 - |\vec{r}|}{2}\right)^\alpha = 1 + (1 - \alpha)H_\alpha(\rho), \quad (48)$$

as the eigenvalues of $\rho$ are $(1 \pm |\vec{r}|)/2$. Combining this with Eq. (41) finally gives Eq. (43). Focusing on the conditions for equality, $r_3 = |\vec{r}|$ is equivalent to $\vec{r} \parallel \vec{p}$, and $r_3 = 0$ is equivalent to $\vec{r} \perp \vec{p}$. The former implies $\rho Z = Z \rho$, the latter implies $\text{Tr}(Z \rho) = \text{Tr}(Z)/2$.

For $\alpha \in (1; \infty)$, the function $r_3 \mapsto g_\alpha(r_3)$ is convex. Now, we have $g_\alpha(r_3) \geq 2^{1-\alpha} = g_0(0)$ by using Jensen’s inequality with the weights 1/2. By convexity, the maximum is reached for one of two least points of the interval $r_3 \in [0; |\vec{r}|]$. In this case, the right-hand side of Eq. (48) represents the desired maximum. The conditions for equality are treated similarly.

□

The formulas (43) and (45) describe the minimal values of the conditional $\alpha$-entropy (41) for the states of fixed purity. The bounds (44) and (46) present the maximal values. Unlike Eq. (41), the inequalities (43)–(46) do not depend on a direction of the Bloch vector. For the given observables $Z$ and $X$, the mentioned results involve the constant factor equal to Eq. (40). For the case $\alpha \in (0; 1)$, the conditional entropy is minimized or maximized under the same conditions as the sum of two $\alpha$-entropies in the first scenario (see Proposition 1 above). For $\alpha \in (1; \infty)$,
the conditions for the equality are swapped. For $\alpha = 1$, both the forms of conditional entropies give

$$H_1(X|Z; \rho) = \sum_{n=\pm 1} \eta n \left( \frac{1 + n \mu}{2} \right) \leq \ln 2.$$  \hspace{1cm} (49)

Using the standard conditional entropy, this formula expresses an upper bound for the second scenario of two successive measurements. Overall, we can say the following. The conditional $\alpha$-entropy (41) seems to be more appropriate, since it depends also on the pre-measurement state. This state is not involved by the conditional $\alpha$-entropy (42) including the standard case (49). However, such a property is two-dimensional in character. In more dimensions, entropic functions of the form (22) will generally depend on the label $z$. Hence, the $\alpha$-entropy (24) will be dependent on the pre-measurement state.

We shall now consider the entropic quantity (40). It reaches its maximal value $\ln_\alpha(2)$ for $\vec{p} \perp \vec{q}$, when $\mu = 0$. Here, we have arrived at an interesting observation. For the given input $\rho$, both the measures (41) and (42) are maximized, when the eigenbases of $Z$ and $X$ are mutually unbiased. The condition $\vec{p} \perp \vec{q}$ actually implies that $|\langle z|x \rangle| = 1/\sqrt{2}$ for all labels $z$ and $x$. For instance, this property takes place for eigenbases of any two of the three Pauli matrices. Such eigenstates are indistinguishable in the following sense. The detection of a particular basis state reveals no information about the state, which was prepared in another basis. Indeed, two possible outcomes are then equiprobable. As is known, this property is used in the BB84 protocol of quantum key distribution.[48]

Thus, formulation of certainty relations for successive measurement again emphasizes a role of mutual unbiasedness. The concept of mutually unbiased bases is naturally posed in arbitrary finite dimensions. If two bases are mutually unbiased, then the overlaps between any basis state in one basis and all basis states in the other are the same. Mutually unbiased bases have found use in many questions of quantum information. They also connected with important mathematical problems (see the review[39] and references therein). Some of the above conclusions can be extended to an arbitrary dimensionality. The generalization is formulated as follows.

**Proposition 3** Let $Z$ and $X$ be two non-degenerate $d$-dimensional observables. For each density $d \times d$-matrix $\rho$, the conditional entropies of successive measurements of $Z$ and $X$ obey

$$H_\alpha(X|Z; \rho) \leq \text{Tr}(\mathcal{E}_Z(\rho)^\alpha) \ln_\alpha(d), \hspace{1cm} (50)$$

$$\tilde{H}_\alpha(X|Z; \rho) \leq \ln_\alpha(d). \hspace{1cm} (51)$$

If the eigenbases of $Z$ and $X$ are mutually unbiased, then the equality is reached in both Eqs. (50) and (51). For strictly positive $\rho$, mutual unbiasedness of the eigenbases is the necessary condition for equality.

**Proof** By $\{|z\rangle\}$ and $\{|x\rangle\}$, we respectively denote eigenbases of the non-degenerate observables $Z$ and $X$. Recall that the Tsallis $\alpha$-entropy is not larger than $\ln_\alpha(d)$, where $d$ is the number of outcomes. For each eigenvalue $z$, we then have

$$H_\alpha(X|z) \leq \ln_\alpha(d).$$  \hspace{1cm} (52)

In the case considered, we clearly have $\sum_z p(z)^\alpha = \text{Tr}(\mathcal{E}_Z(\rho)^\alpha)$. Combining the latter with Eq. (52), provides the claims (50) and (51). We shall now proceed to the conditions for equality. The maximal entropic value $\ln_\alpha(d)$ is reached only for the uniform distribution, when the probabilities are all $1/d$. To saturate Eq. (52) with the given $z$, we should therefore have

$$\forall x \in \text{spec}(X) : |\langle x|z \rangle| = \frac{1}{\sqrt{d}}.$$  \hspace{1cm} (53)

When $\rho$ is strictly positive, $p(z) \neq 0$ for all $z \in \text{spec}(Z)$. To reach the equality in both Eqs. (50) and (51), the condition (53) should be provided for all $z \in \text{spec}(Z)$. Then the two eigenbases are mutually unbiased. $\square$

Although the BB84 scheme of quantum cryptography is primarily important, other protocols have been studied in Ref. [40]. Some of them are based on mutually unbiased bases (see, e.g., Refs. [41–42]). We have seen above that the conditional entropies for a pair of successive projective measurements are maximized just in the case of mutual unbiasedness. Entropic uncertainty relations for several mutually unbiased bases were studied in Refs. [43–46]. It would be interesting to consider main questions of this section for many mutually unbiased bases. Such investigations may give an additional perspective of possible use of mutually unbiased bases in quantum information processing. It could be a theme of separate investigation.

**6 Conclusion**

We have studied Tsallis-entropy uncertainty and certainty relations for two subsequent measurements of a qubit. Despite of very wide prevalence of the Heisenberg principle, there is no general consensus over its scope and validity. The following claim is commonly accepted. It is not possible to assign jointly exact values for two or more incompatible observables. There are several ways to fit this claim as a quantitative statement. Heisenberg’s original argument is adequately formulated in terms of noise and disturbance.[6] Using the two scenarios of successive measurements, we are able to fit the question of measuring uncertainties in a different way. Such an approach may be more significant in the sense of its potential applications in studying protocols of quantum information. Indeed, subsequent manipulations with qubits rather deal with an output state of the latter stage. Uncertainty relations for two successive measurements were already examined in terms of the standard entropies[16] and the Rényi
entropies. At the same time, certainty relations for successive measurements seem to be not addressed in the literature.

The following two scenarios of successive measurements were considered. In the first scenario, a subsequent measurement is performed on the quantum state generated after the previous stage with completely erased information. In the second scenario, a subsequent measurement is performed on the post-measurement state conditioned on the actual measurement outcome. For both the scenarios, we derived uncertainty and certainty bounds on α-entropic functions related to successive measurements of a pair of qubit observables. The conditions for equality in these bounds were obtained as well. Some of found results in the second scenario were extended to arbitrary finite dimensionality. They are connected with a frequently used property of mutually unbiased bases. In effect, the detection of a particular basis state reveals no information about the state, which was prepared in another basis. It would be interesting to study uncertainty and certainty relations for successive measurements in several mutually unbiased bases.

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