Localized Tachyons and RG Flows

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We study condensation of closed string tachyons living on defects, such as orbifold fixed planes and Neveu-Schwarz fivebranes. We argue that the high energy density of localized states decreases in the process of condensation of such tachyons. In some cases this means that $c_{\text{eff}}$ decreases along the flow; in others, $c_{\text{eff}}$ remains constant and the decreasing quantity is a closed string analog, $g_{\text{cl}}$, of the “boundary entropy” of D-branes. We discuss the non-supersymmetric orbifolds $\mathbb{C}/\mathbb{Z}_n$ and $\mathbb{C}^2/\mathbb{Z}_n$. In the first case tachyon condensation decreases $n$ and in some cases connects type II and type 0 vacua. In the second case non-singular orbifolds are related by tachyon condensation to both singular and non-singular ones. We verify that $g_{\text{cl}}$ decreases in flows between non-singular orbifolds. The main tools in the analysis are the structure of the chiral ring of the perturbed theory, the geometry of the resolved orbifold singularities, and the throat description of singular conformal field theories.

November 16, 2001
1. Introduction

In recent years significant progress has been achieved in string theory from an analysis of string propagation in the presence of defects such as D-branes, NS5-branes and orbifolds. The study of impurities in string theory led to important insights into fundamental issues (e.g. the AdS/CFT correspondence) and might be relevant to phenomenology (for example in the framework of “brane worlds”). In such backgrounds there are two kinds of excitations, those that are localized on the defects, and those that propagate everywhere in spacetime. In the D-brane case, the localized states are open strings ending on the branes, while the delocalized ones correspond to closed strings. For orbifolds, they are twisted and untwisted sector excitations, respectively.

In the absence of spacetime supersymmetry, string vacua typically contain tachyons, and it is of interest to analyze their condensation. This requires an off-shell formulation of the theory, and is in general not well understood. It is believed that in general tachyon condensation drastically modifies the geometry of spacetime, and even its dimensionality.

For vacua with impurities, one can consider systems in which supersymmetry is only broken by the impurity. In such situations, one generically expects to find tachyons living on the impurity (“localized tachyons”), and their condensation might be under better control than the general case. For D-branes, this was widely discussed in the past two years following the work of Sen [1]. Here we will focus on closed string defects, such as orbifolds and NS5-branes. This case was recently studied in [2] (see also [3]), where the similarity of this problem to open string tachyon condensation on unstable D-brane systems was noted. We will continue this study using techniques that can be thought of as a direct generalization of [4,5,6,7] to closed strings.

We will see that, as in the open string case, the worldsheet renormalization group (RG) provides a simple conceptual framework for understanding localized tachyon condensation. It also leads to some quantitative constraints on possible decays. A nice feature of the analysis is a uniform treatment of all localized defects in weakly coupled string theory.

A characteristic feature of the renormalization group in two dimensions is a decrease along flows of the asymptotic high energy density of states

\[ \rho(E) \sim \frac{1}{2} g \left( \frac{c_{\text{eff}}}{3E^3} \right)^{\frac{3}{2}} \exp \left[ \frac{2\pi \sqrt{c_{\text{eff}} E}}{3} \right] . \]  

(1.1)

A number of examples of this phenomenon are known:
1. Unitary compact conformal field theories. Here, $c_{\text{eff}} = c$, the Virasoro central charge; the decrease of $c$ along RG flows in this case was proven by Zamolodchikov [8].

2. All compact CFT’s, perhaps non-unitary, in which all states contribute positively to the torus partition sum. Then one has [9]

$$c_{\text{eff}} = -\frac{6E_{\text{min}}}{\pi} = c - 24h_{\text{min}}, \tag{1.2}$$

where $h_{\text{min}}$ is the lowest scaling dimension in the theory and $c_{\text{eff}}$ is believed to decrease along RG flows (this has not been proven in general; see e.g. [9, 10] for discussions). Examples where RG flows of such systems have been studied include the non-unitary minimal models, for which

$$c = 1 - \frac{6(p - p')^2}{pp'}, \quad c_{\text{eff}} = 1 - \frac{6}{pp'}; \tag{1.3}$$

RG flows decrease $p$ and/or $p’$ [11, 12], and hence $c_{\text{eff}}$.

3. On worldsheets with boundary, RG flows by boundary perturbations cannot affect the central charge; in this case, it is the open string analog of the coefficient $g$ in (1.1) (which is related to the “boundary entropy” of [13]) which decreases along flows [13, 6].

One of the main purposes of this paper is to present another class of examples of the decrease of the density of states (1.1) along RG flows. Following [2], we will study non-compact sigma-models with localized states associated with impurities that break (space-time) supersymmetry. An important class of examples are the non-compact orbifolds $\mathbb{R}^n/\Gamma$. Typically, such models contain localized (twisted sector) tachyons and one can ask what happens to the density of states (1.1) as these condense. It is natural in this case to define $\rho(E)$ as the density of localized states, which are normalizable, as opposed to the delocalized states which are only delta-function normalizable (i.e. non-normalizable in infinite volume).

It was proposed in [2] that non-compact orbifold models often flow in the IR to other orbifold models. As we review in section 2, $c_{\text{eff}}$ does not change in such flows. Therefore, we propose that in these cases the quantity that decreases along the RG flow is the coefficient $g$ in (1.1), which we denote by $g_{\text{cl}}$ to distinguish it from its open string counterpart $g_{\text{op}}$. Since we have not proven the “$g_{\text{cl}}$-conjecture,” in the rest of the paper we present a number of examples of RG flows in orbifold theories, and test the conjecture for them. In

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1 By a compact CFT, we mean one with a discrete spectrum of scaling dimensions.
order to analyze the far infrared physics reached by twisted sector tachyon perturbations, we restrict to flows that preserve $N = 2$ worldsheet supersymmetry. In these cases, the chiral ring [14] provides a powerful diagnostic. The restriction to $N = 2$ supersymmetry is not a severe limitation; as we will see, it allows for many interesting flows. The picture obtained is compatible with the “$g_{\text{cl}}$-conjecture”.

NS5-branes are another interesting class of defects that are related to orbifolds but exhibit somewhat different physics. String perturbation theory typically breaks down in the presence of coincident fivebranes due to the appearance of an infinite throat [15], but there are examples in which the system is weakly coupled (obtained by going to the Coulomb branch of the theory, where the throat is capped [16]). We show that in these systems $c_{\text{eff}}$ typically decreases along RG flows (and thus $g$ in (1.1) is superfluous).

The plan of the paper is the following. In section 2 we discuss non-compact orbifolds. We show that the infinite volume limit serves as a sort of decoupling limit between twisted and untwisted states. In particular, we show that the central charge does not change under twisted sector perturbations. We introduce $g_{\text{cl}}$ and give a general formula for it. We also show that $g_{\text{cl}}$ does not change along moduli spaces corresponding to twisted sector marginal operators.

In section 3 we discuss the special case of $\mathbb{C}/\mathbb{Z}_n$. We analyze the condensation of twisted tachyons that preserves $N = 2$ worldsheet supersymmetry, both for type 0 and type II strings, by studying the perturbed chiral rings. We find that tachyon condensation relates string theory on $\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_n$ to a direct sum of disconnected cones of the form $\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_{n_i}$, with $n_i < n$. For the type II theory, one of these cones has type II strings living on it, while the rest are type 0 vacua. We also compute $g_{\text{cl}}$ and show that this process is compatible with the $g_{\text{cl}}$-conjecture.

In section 4 we discuss the $\mathbb{C}^2/\mathbb{Z}_n$ case. Subsection 4.1 discusses the Hirzebruch-Jung theory of singularity resolution and its relation to the chiral ring in the orbifold SCFT. In subsection 4.2 we review the relation between spacetime supersymmetric orbifolds and a vacuum with NS5-branes on $\mathbb{R}^3 \times S^1$. In the rest of section 4 we use these tools to analyze the RG flows in various examples.

Section 5 contains some applications of the ideas developed here to RG flows in five-brane theories with a non-compact transverse space. We end in section 6 with a discussion of our results and further applications. Three appendices contain technical results on the computation of $g_{\text{cl}}$, gauge couplings on D-brane probes, and the structure of chiral rings for certain orbifolds of $\mathbb{C}^2$.

Condensation of localized closed string tachyons has also been studied recently in [17,18,19,20,21,22].
2. Localized closed string tachyon condensation and $g_{\text{cl}}$

A prototype of the problem we will study is the following. Consider string propagation in the spacetime
\[ \mathbb{R}^{d-1,1} \times \mathbb{R}^{10-d}/\Gamma, \tag{2.1} \]
where $\Gamma$ is a discrete subgroup of $SO(10-d)$, whose action on $\mathbb{R}^{10-d}$ has a single fixed point (say $\vec{y} = 0$, with $\vec{y} \in \mathbb{R}^{10-d}$). Twisted sector states give rise to fields localized at $\vec{y} = 0$ (which thus live in $d$ spacetime dimensions), while untwisted states propagate in the full ten dimensional spacetime. Tachyons typically appear when $\Gamma$ is not in $SU(5 - [(d+1)/2])$ (or $G_2$ or $Spin(7)$ for $d = 3, 2$), so supersymmetry is broken. Here $[x]$ denotes the integer part of $x$.

“Localized instabilities” in (2.1) correspond to twisted sector scalars which are tachyonic (or massless, with a higher order unstable potential). Such scalars will condense, and it is of interest to determine the endpoint of this process.

In the open string case, it is useful to think about this problem in terms of the worldsheet RG [4]. In that case, the instabilities are associated with open string tachyons. Condensing them corresponds to studying the theory in the presence of the relevant boundary vertex operators in the worldsheet action. The endpoint of condensation corresponds to the IR fixed point of this flow.

Similarly here, condensation of localized closed string tachyons (which correspond to relevant perturbations in the twisted sector of the worldsheet CFT) can be studied by following the RG flow of the worldsheet theory perturbed by the tachyon vertex operators. By analogy with the D-brane example, one expects to be able to define an off-shell quantity which is monotonically decreasing along the flow and at the fixed points of the RG measures the number of localized degrees of freedom.

In the open string case, this quantity is the “boundary entropy” $g$, whose decrease along RG flows was conjectured by Affleck and Ludwig [13] and proven in [3] (see also [4], and [6,23,24] for the worldsheet supersymmetric case), where it was also shown that this quantity is the off-shell action for open string excitations. One way of defining $g$, which makes manifest its interpretation as the number of boundary degrees of freedom, is via the density of open string states\(^2\) at high energy, $\rho(E \to \infty)$:
\[ \rho_{\text{open}}(E) \sim \frac{1}{2} g_{\text{op}}\left(\frac{c_{\text{eff}}}{6E^3}\right)^{\frac{1}{2}} \exp\left[2\pi \sqrt{c_{\text{eff}}E/6}\right], \tag{2.2} \]

\(^2\) In situations where there are different sectors of open strings, we have in mind the total density of states, summed over the different sectors.
where $E$ is the eigenvalue of $L_0 - (c/24)$. Since the central charge does not change under boundary RG flow, $g_{op}$ is the leading measure of the change in the number of open string degrees of freedom of the system in such flows. One can show that $g_{op}$ is related to the boundary entropy $g$ via $g_{op} = g^2$ \[13\].

For non-supersymmetric orbifolds of the form \[2.1\], one might expect that the quantity that decreases along the RG when the theory is perturbed by twisted sector relevant operators is the central charge $c$. Indeed, Zamolodchikov’s $c$-theorem \[8\] asserts that in unitary theories, the central charge decreases along RG flows. We next briefly review the argument and discuss the subtleties with it in this case.

Zamolodchikov has shown that

$$
\frac{dc}{dt} = -12|z|^4\langle \Theta(z)\Theta(0) \rangle ,
$$

where $t = 2 \ln |z|$ is the log of the RG scale, $c$ is the scale-dependent central charge

$$
c(t) = 2z^4\langle T(z)T(0) \rangle - 4z^3\bar{z}\langle T(z)\Theta(0) \rangle - 6|z|^4\langle \Theta(z)\Theta(0) \rangle ,
$$

and $\Theta = T_{z\bar{z}}$ is the trace of the stress tensor. At fixed points of the RG, $\Theta$ vanishes and $c$ reduces to the standard Virasoro central charge. More generally, one has

$$
\Theta = \beta^i \phi_i
$$

where $\beta^i$ are the $\beta$-functions for the couplings $\lambda^i$, and $\phi_i$ are the corresponding perturbations. Plugging (2.5) in (2.3) one has

$$
\frac{dc}{dt} = -12\beta^i \beta^j G_{ij} ,
$$

where

$$
G_{ij} = \langle \phi_i(z)\phi_j(0) \rangle |z|^4
$$

is the Zamolodchikov metric. In a unitary QFT, the metric $G$ is expected to be positive definite, and thus $c$ decreases along RG flows.

In \[2\] it was argued that in the non-compact orbifold theories \[2.1\] this reasoning might fail, and the central charge does not in fact change along RG flows associated with twisted sector perturbations. We next discuss how these statements can be reconciled with the $c$-theorem.
It is important that, as mentioned above, in the orbifolds (2.1) there are two kinds of excitations. Vertex operators that belong to the untwisted sector of the orbifold describe excitations that propagate in the full ten dimensional spacetime, while twisted sector states are restricted to the $d$ dimensional manifold $\vec{y} = 0$. This has important implications for the correlation functions of the theory.

Denote by $U$ general untwisted sector Virasoro primaries and by $T$ twisted ones. Consider correlation functions of the form

$$\langle U_1(z_1) \cdots U_n(z_n) T_1(w_1) \cdots T_m(w_m) \rangle.$$  \hspace{1cm} (2.8)

Such correlation functions are given by the path integral over the worldsheet fields on the target space (2.1). As is standard in QFT, the correlation functions are normalized by dividing by the partition sum. There is an important difference between the behavior of correlation functions of untwisted operators (i.e. those with $m = 0$ in (2.8)), and correlation functions which contain twisted operators ($m \neq 0$). The difference has to do with the volume dependence of the correlators, and is thus easiest to explain by replacing $R^{10-d}/\Gamma$ in (2.1) by a compact orbifold with the same singularity, e.g. $S^{10-d}/\Gamma$, where $S^{10-d}$ is a round sphere of volume $V_{10-d}$. The curvature of the sphere breaks conformal symmetry, but since one is only using $V_{10-d}$ as a regulator, this can be ignored. Also, there might be more than one fixed point of $\Gamma$ on the sphere; again, this can be ignored in the large volume limit.

We would like to analyze the dependence of the correlation functions (2.8) on $V_{10-d}$ in the large volume limit. Before dividing by the partition sum $Z$, untwisted correlation functions are proportional to the volume – the vertex operators $U_i$ describe particles that live on the entire $S^{10-d}$. Since $Z$ is also proportional to $V_{10-d}$, the normalized correlators (2.8) with $m = 0$ are independent of $V_{10-d}$,

$$\langle U_1 \cdots U_n \rangle \simeq V_{10-d}^0. \hspace{1cm} (2.9)$$

On the other hand, (connected) correlation functions involving twist fields correspond to processes localized at the fixed point $\vec{y} = 0$, and therefore the normalized correlators go for $m > 0$ like

$$\langle U_1 \cdots U_n T_1 \cdots T_m \rangle \simeq \frac{1}{V_{10-d}}. \hspace{1cm} (2.10)$$

These properties may be inferred from the explicit computation of twist operator correlation functions in [25]. Note that although one can rescale the operators $U_i, T_a$ by powers
of $V_{10-d}$, (2.9), (2.10) are meaningful, since one can fix the normalization by requiring that correlators like (2.9) scale as the same power of $V_{10-d}$ for all $n$ (and similarly for (2.10)). This is a reasonable requirement to impose if one is planning on taking the large $V_{10-d}$ limit.

Equations (2.9), (2.10) imply that the OPE’s are regular in the limit $V_{10-d} \to \infty$:

$$
U_i U_j = C_{ij}^k U_k \\
T_a T_b = D_{ab}^c T_c + B_{ab}^k U_k
$$

(2.11)

with the structure constants $C_{ij}^k$, $D_{ab}^c$ finite in the large volume limit and $B_{ab}^k \sim 1/V_{10-d}$. Interestingly, as $V_{10-d} \to \infty$, all (normalized) correlation functions of the properly normalized twist operators go to zero (2.10), while their OPE’s remain finite (2.11). It is also clear that in the presence of twisted sector perturbations, (2.9) does not imply that $c$ is decreasing. Indeed, the Zamolodchikov metric (2.7), $G_{ab} = |z|^4 \langle T_a(z) T_b(0) \rangle$, vanishes to all orders in conformal perturbation theory in the twisted couplings and thus $c$ is constant along the flows.

We now return to the question of what quantity changes along the twisted RG flows discussed above. We would like to propose that it is a closed string analog of (2.2), the high energy density of localized states. This density\footnote{In this paper, we study the density of states in conformal field theory on (2.1). In string theory on this spacetime, one has to further impose the physical state condition $L_0 = \bar{L}_0$ (see \cite{9} for a related discussion).} grows for high energy $E = L_0 + \bar{L}_0 - (c/12)$ as

$$
\rho(E) \sim \frac{1}{2} g_{cl} \left( c/3E^3 \right)^{1/4} \exp \left[ 2\pi \sqrt{cE/3} \right].
$$

(2.12)

As we just saw, the leading term in $\rho(E)$, which is governed by the central charge, is unchanged along the flow. Therefore, the leading measure of the density of states that can change, is the prefactor $g_{cl}$. Since the number of degrees of freedom is expected to decrease, and in analogy to the open string case, we conjecture that along RG flows of the orbifold CFT perturbed by twisted vertex operators one has

$$
g_{cl}(UV) > g_{cl}(IR).
$$

(2.13)

Note that equation (2.12) defines $g_{cl}$ at fixed points of the RG (as was done in \cite{13} for $g_{op}$). It would be interesting to define $g_{cl}$ throughout the RG flow, and prove (2.13). One
expects to be able to define such a quantity away from the fixed points since there should exist a classical off-shell spacetime effective action for localized states. This relies on the fact that these states are described by a non-gravitational theory; it is believed that for gravitational theories one should not be able to go off-shell (due to holography).

Just like in the open string case \[13\], one can relate \( g_{\text{cl}} \) to a “non-integer ground state degeneracy” by using modular invariance of the one loop partition sum. Indeed, consider the partition sum for localized states

\[
Z_{\text{tw}}(\tau, \bar{\tau}) = \sum_{g \neq 1} \text{Tr}_g [q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}]
\]

(2.14)

where the sum over \( g \) runs over the different twisted sectors of the orbifold (the trace is over \( \Gamma \)-invariant states), and we will take the modular parameter \( q \) to be real, \( q = \exp(-2\pi \tau_2) \).

The high energy density of states (2.12) determines the behavior of the partition sum (2.14) in the limit \( \tau_2 \to 0 \). Indeed, plugging the density of states \( \rho(E) \) (2.12) into (2.14), replacing the sum in the trace by an integral, and performing it by saddle point, one finds

\[
Z_{\text{tw}}(\tau_2 \to 0) \sim g_{\text{cl}} \exp(\pi c/6\tau_2).
\]

(2.15)

The modular transformation \( \tau_2 \to 1/\tau_2 \) relates this to the contribution of the vacuum state to the torus partition sum at large \( \tau_2 \).

Consider, for example, the orbifold with a diagonal modular invariant. The twisted partition sum in this case has the form

\[
Z_{\text{tw}} = \frac{1}{|\Gamma|} \sum_{g \neq 1} \sum_{h \in \Gamma} Z^{(h \begin{array}c \square \end{array} g)}
\]

(2.16)

where

\[
Z^{(h \begin{array}c \square \end{array} g)} = \text{Tr}_g [h q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}]
\]

(2.17)

the partition sum in the sector twisted by \( g \) with an insertion of the operator representing \( h \) on the Hilbert space. The sum over \( h \in \Gamma \) in (2.16) imposes the \( \Gamma \) invariance of the physical states. We are interested in the behavior of \( Z_{\text{tw}} \) in the limit \( \tau_2 \to 0 \). Under the modular transformation \( \tau \to -1/\tau \),

\[
Z^{(1 \begin{array}c \square \end{array} g)}(-\frac{1}{\tau}) = Z^{(g \begin{array}c \square \end{array} 1)}(\tau) = \frac{1}{|\det(1 - R(g))|^2} Z^{(g \begin{array}c \square \end{array} 1)}(\tau).
\]

(2.18)
Here $Z'$ is the partition sum with the zero modes excluded, and the inverse determinant factor comes from the zero mode integral,

$$\int d^{10-d}p \delta^{10-d}(gp - p) = \frac{1}{|\det(1 - R(g))|^2}. \quad (2.19)$$

$R(g)$ is the rotation matrix representing $g$ on the string coordinates. Taking the limit $\tau_2 \to \infty$, one finds that

$$g_{cl} = \frac{1}{|\Gamma|} \sum_{g \neq 1} \frac{1}{|\det(1 - R(g))|^2}. \quad (2.20)$$

We will evaluate this expression in particular examples below. Note that for compact orbifolds, the factor $|\det(1 - R(g))|^2$ gives the number of fixed points in the sector twisted by $g$, and the more familiar form of the modular transformation (2.18) is obtained by multiplying (2.18) by the number of fixed points. We are studying a non-compact orbifold with a single fixed point, which is the origin of the inverse determinant appearing in (2.18), (2.20).

An intriguing fact is that (2.20) is essentially the same as the $\eta$ invariant for the Dirac operator in the round metric on Lens spaces of the type $\mathbb{C}^{10-d}/\Gamma$. One possible interpretation of this fact is that twisted sector strings can be thought of as random walks constrained to begin and end at the fixed point. For high energies, the random walk explores the region far from the fixed point, and most of the entropy of such strings comes from this region. This explains the role of $S^{10-d}/\Gamma$ in evaluating the asymptotic density of states. It also clarifies why $g_{cl}$ is related to a boundary contribution to an index: it should be invariant under small deformation in the vicinity of the tip of the cone, since such deformations are not expected to influence the high energy density of states.

The complexification of $\mathbb{R}^{10-d}$ to $\mathbb{C}^{10-d}$ in this interpretation is perhaps due to the need to consider both left and right-moving degrees of freedom. It is also natural to expect that a relation between $g_{cl}$ and $\eta$ invariants can be found by combining the elliptic genus with the index theorem for manifolds with boundary.

An example of a class of small perturbations under which $g_{cl}$ should be invariant is deformations by exactly marginal twisted sector operators. These give rise to moduli spaces of CFT’s which correspond to resolved orbifold singularities. Since the asymptotic geometry of the non-compact orbifold does not depend on the moduli, one would expect $g_{cl}$ to remain constant along such moduli spaces (at least for finite distance in the space of CFT’s). We next show that this is indeed the case.
Let \( \Phi \) be a truly marginal twisted sector operator. We would like to compute \( g_{\text{cl}} \) along the moduli space labeled by \( \lambda \), the coefficient of \( \Phi \) in the worldsheet Lagrangian,

\[
\delta \mathcal{L} = \lambda \Phi(z, \bar{z}).
\] (2.21)

To this end we would like to repeat the derivation of (2.20) for finite \( \lambda \). As a first step, we will establish that (2.18) holds for all \( \lambda \). In particular, the coefficient of \( Z' \) on the r.h.s. is independent of \( \lambda \). To show that, consider a general correlation function of primary operators on the torus, in the sector labeled by \( g \) and \( h \), as in (2.17),

\[
\langle \Phi_1(z_1) \Phi_2(z_2) \cdots \Phi_n(z_n) \rangle_{g, h; \tau}. 
\] (2.22)

Under the modular transformation \( \tau \to -1/\tau \), \( z_j \to -z_j/\tau \), this correlation function transforms as follows:

\[
|\tau|^{2(h_1 + h_2 + \cdots + h_n)} \langle \Phi_1(z_1) \Phi_2(z_2) \cdots \Phi_n(z_n) \rangle_{h, g; \tau} 
\] (2.23)

where \( h_j \) are the scaling dimensions of \( \Phi_j \) (which are assumed to be left-right symmetric). In particular, if the \( \Phi_j \) all have dimension \( h_j = \bar{h}_j = 1 \), and one integrates (2.23) \( \int d^2 z_j \) over the torus, one finds that to all orders in \( \lambda \) (2.21), equation (2.18) is still valid.

In order to show that \( g_{\text{cl}} \) is independent of \( \lambda \), one has to further prove that the limit as \( \tau_2 \to \infty \) of the partition sum on the r.h.s. of (2.18) has the property that (in the sector with \( h = 1 \)), the coefficient of \( (q \bar{q})^{-\frac{c}{2}} \) is independent of \( \lambda \). Indeed, the derivative of this coefficient w.r.t. \( \lambda \) (for generic \( \lambda \)) is proportional to the vacuum expectation value \( \langle 0 | \Phi(z, \bar{z}) | 0 \rangle \), which vanishes by conformal invariance.\(^4\) Therefore, we conclude that \( g_{\text{cl}} \) is independent of twisted moduli.

We end this section with a few comments. As in the open string case, (2.13) is expected to hold when we only deform the CFT by twisted sector operators (the analogs of open strings). Deformations by untwisted operators (which are analogs of closed strings) may increase or decrease \( g_{\text{cl}} \) (and will in general change the central charge as well, as discussed

\(^4\) For systems with a continuous spectrum of scaling dimensions, the last statement is in general subtle, since expectation values of \( \Phi(z, \bar{z}) \) in states with arbitrarily small scaling dimensions can in fact contribute. Here it is important that the partition sum on the r.h.s. of (2.18) has \( g \neq 1 \), so that at \( \lambda = 0 \) the spectrum of scaling dimensions that contribute to it is discrete; the spectrum must remain discrete for all finite \( \lambda \).
above). Also, there are subtleties which are familiar from the open string case, when some of the states contributing to the trace (2.14) are spacetime fermions (which contribute with a negative sign to the torus partition sum), which will be discussed below.

We have phrased the discussion in the language of non-compact orbifolds of the form (2.1), but it should apply for more general non-compact backgrounds of string theory in which there are normalizable (localized) and delta-function normalizable (delocalized) excitations which satisfy (2.11). The discussion above should be applicable to the condensation of normalizable tachyons. Since they are localized, they presumably do not change the central charge $c$, but it is natural to conjecture that the high energy density of normalizable states decreases along RG flows. We will discuss some examples of this below.

In the rest of the paper we will discuss a few classes of examples of the flows described in this section. We will verify the validity of (2.13) and introduce some additional tools for studying such flows.

3. $\mathbb{C}/\mathbb{Z}_n$ flows

Our first class of examples corresponds to $d = 8$ and $\Gamma = \mathbb{Z}_n$ in (2.1), i.e. string theory on

$$\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_n. \quad (3.1)$$

The chiral GSO projection of type II string theory acts non-trivially on the $\mathbb{C}/\mathbb{Z}_n$ factor, eliminating tachyons in the untwisted sector. It will be useful for our purposes to start with the non-chiral theory, with a diagonal GSO projection. One can think of this as type 0 string theory on (3.1). We will later discuss the modifications associated with passing from type 0 to type II.

Note that the structure of the operator algebra (2.11) guarantees that classically (i.e. on the sphere) the delocalized untwisted sector tachyon of type 0 remains unexcited along RG flows associated with condensing only twisted sector tachyons in infinite volume, and therefore it can be ignored\footnote{Just as in the open string case, one expects that it cannot be ignored when loop corrections are taken into account.}. From a more general RG perspective, the statement is that there exist RG trajectories in which the untwisted tachyon is fine tuned to vanish.

\footnote{For an early discussion of string theory on $\mathbb{C}/\mathbb{Z}_n$, see \cite{26}.}
We will parametrize the complex plane $\mathbb{C}$ in (3.1) by the worldsheet chiral superfield (we are working in the NSR formalism)

$$\Phi = Z + \theta \psi + \bar{\theta} \bar{\psi} + \cdots$$

(3.2)

The orbifold action on the worldsheet fields is

$$\Phi \rightarrow \omega^j \Phi$$

(3.3)

where $\omega = e^{2\pi i n}$. The theory contains $n-1$ twisted sectors, labeled by $j = 1, 2, \cdots, n-1$. To describe the ground states in the different sectors it is convenient to bosonize the fermions,

$$\psi = e^{iH}; \quad \bar{\psi} = e^{i\bar{H}}.$$  

(3.4)

An interesting set of operators in the orbifold CFT is

$$X_j = \sigma_{j/n} \exp \left[ i(j/n)(H - \bar{H}) \right] ; \quad j = 1, 2, \cdots, n-1$$

(3.5)

where $\sigma_{j/n}$ is the bosonic twist $j$ operator [25]. The operator $X_j$ has worldsheet scaling dimension

$$\Delta_j = \frac{j}{2n}.$$  

(3.6)

In particular, the operators (3.5) give rise to tachyons in the spacetime theory, with masses

$$\frac{\alpha'}{4} M_j^2 = \Delta_j - \frac{1}{2} = -\frac{1}{2} \left( 1 - \frac{j}{n} \right).$$  

(3.7)

It is also going to be useful to observe that CFT on $\mathbb{C}/\mathbb{Z}_n$ is $N = (2, 2)$ superconformal, with the R-charge generator $J = \psi \psi^* = i\partial H$ (and similarly for the right-movers). The operators (3.5) have R-charge

$$R_j = \frac{j}{n}.$$  

(3.8)

Comparing (3.6) and (3.8) we see that $X_j$ are chiral operators, a fact which will be useful below.

It is not difficult to see that the ground states in all twist sectors are in fact of the form $X_j$, or $X_{n-j}^*$. Thus, in order to study the condensation of the most tachyonic state in each twisted sector, it is enough to study the $X_j$. This is the problem that will be analyzed here. There are excited tachyons in the spectrum which are not of the form (3.5). Including them in the analysis below is complicated, and we will not discuss them further.
As discussed in section 2, in order to study the condensation of the twisted tachyons \( X_j \), we would like to add the vertex operators (3.3) to the worldsheet action\(^7\) and study the resulting RG flow. Since the operators (3.3) are chiral, the corresponding deformation of the worldsheet Lagrangian is by an \( N = 2 \) superpotential,

\[
\delta \mathcal{L} = \lambda^j \int d^2 \theta X_j + \text{c.c.} \quad (3.9)
\]

The \( \lambda^j \) are \( n - 1 \) complex couplings. The perturbation (3.9) breaks both conformal invariance and \( U(1)_R \) (the trace of the stress tensor \( \Theta \) (2.3) and the divergence of \( J \) are in the same multiplet under \( N = 2 \) supersymmetry). Indeed, (3.6), (3.8), (3.9) imply that \( \lambda_j \) carries R-charge \( R(\lambda_j) = 1 - (j/n) \), and scaling dimension \( \Delta(\lambda_j) = \frac{1}{2}(1 - (j/n)) \). However, as is clear from (3.9), the perturbation preserves \( N = 2 \) supersymmetry for all \( \lambda_j \). We would like to use this fact to obtain information about the possible RG flows in the model.

A strong constraint on the RG flow corresponding to (3.9) comes from analyzing the chiral ring of the theory. Consider first the chiral ring of the undeformed theory, with all \( \lambda_j = 0 \). It has two generators,

\[
X = X_1 = \sigma_{1/n} \exp \left[ \left( \frac{i}{n} \right) (H - \bar{H}) \right] \\
Y = \frac{1}{V_2} \bar{\psi} \psi = \frac{1}{V_2} \exp \left[ i (H - \bar{H}) \right] . \quad (3.10)
\]

Note that \( Y \) is the volume form of \( \mathbb{C}/\mathbb{Z}_n \) divided by the volume of the cone, \( V_2 \) (e.g. regularized as \( S^2/\mathbb{Z}_n \) [27]). The higher chiral twisted operators \( X_j \) can be thought of as powers of \( X \), \( X_j = X^j \), and there is a relation

\[
X^n = Y . \quad (3.11)
\]

One can in fact focus on the chiral ring while eliminating the rest of the dynamics, by performing a topological twist. The details of this are not important for us here, and we will not discuss them further (see [27] for a discussion).

When one turns on the superpotential (3.9), the chiral ring gets deformed. In particular, the relation (3.11) is modified to

\[
X^n + \sum_{j=1}^{n-1} g_j(\lambda) X^j = Y \quad (3.12)
\]

\(^7\) More precisely, (3.3) is the bottom component of an \( N=2 \) chiral superfield. The worldsheet Lagrangian is perturbed by adding the top component of the superfield. For simplicity, we will also denote by \( X_j \) the superfield whose lowest component is (3.3).
where \( g_j(\lambda) \) are polynomials in the couplings \( \lambda \) (but by holomorphy, are independent of \( \lambda^* \)), with R-charge \( R(g_j) = 1 - (j/n) \). To leading order one has \( g_j = c_j \lambda_j + O(\lambda^2) \), where \( c_j \) are certain non-vanishing constants. The fact that there are higher order corrections to this relation is a standard feature that has to do with coordinate choices on the space of couplings. This too is not of interest to us here, and we will ignore the difference between \( g_j \) and \( \lambda_j \).

The flow to the infrared can now be analyzed much like similar flows in Landau-Ginzburg models [28,29]. If one tunes \( g_1, g_2, \ldots, g_{n'-1} \) to zero, and the leading non-vanishing relevant coupling is \( g_{n'} \), (3.12) describes a flow from a theory with ring relation \( X^n = Y \) in the UV to one corresponding to

\[
g_{n'} X^{n'} = Y
\]

in the infrared. Equation (3.13) looks like the chiral ring relation corresponding to the orbifold \( \mathbb{C}/\mathbb{Z}_{n'} \). We conclude that the deformed ring (3.12) describes a sequence of transitions from \( \mathbb{C}/\mathbb{Z}_{n} \) to other orbifolds of the same form but with lower \( n \). Turning on generic \( g_j \) resolves the singularity completely. This is consistent with the conclusions of [2].

It should be mentioned that the above arguments do not determine the IR limit uniquely, since they only fix the chiral ring of the IR theory, and it is possible that two different CFT’s share the same chiral ring. Nevertheless, the structure of the rings places a severe constraint on the possible endpoints of the flow, and it is very likely that the above interpretation of the IR fixed point is correct.

Following the discussion of section 2, it is of interest to compute \( g_{cl} \) for the \( \mathbb{C}/\mathbb{Z}_{n} \) orbifolds and check whether it is an increasing function of \( n \). Thus, we would like to compute

\[
Z_{tw}(\tau, \bar{\tau}) = \sum_{s=1}^{n-1} \text{Tr}_s q^{L_0 - \frac{s}{n} \overline{\vartheta}} \overline{q}^{\overline{L}_0 - \frac{s}{n}},
\]

where the trace runs over \( \mathbb{Z}_n \) invariant states in the \( s \)-twisted sector. Using standard techniques one can show that the twisted partition sum is given by

\[
Z_{tw}(\tau, \bar{\tau}) = \sum_{s=1}^{n-1} \frac{1}{2n} \sum_{t=0, \epsilon_1, \epsilon_2 = 0, \frac{1}{n} \frac{1}{n}} | \vartheta[\epsilon_1 + s/n](0|\tau)e^{\frac{s}{n}}| | \vartheta[\frac{1}{n} + t/n](0|\tau) |^2 .
\]

By using the results of [27], one can obtain additional information about the deformed chiral ring, such as the structure constants. We will not pursue this here.
Performing the modular transformation $\tau \rightarrow -1/\tau$, and comparing to the definition of $g_{\text{cl}}$ (2.12), (2.13), one finds

$$g_{\text{cl}}(n) = \frac{1}{12} \left( n - \frac{1}{n} \right),$$

(3.16) in agreement with (2.20). The sum in (3.16) is evaluated in appendix A. One finds

$$g_{\text{cl}}(n) = \frac{1}{12} \left( n - \frac{1}{n} \right),$$

(3.17) which is indeed a monotonically increasing function of $n$, in agreement with the “$g_{\text{cl}}$-conjecture” (2.13). This provides further support for the flows proposed above. Note also that (3.17) vanishes for $n = 1$, in accord with the intuition that for $\mathbb{C}/\mathbb{Z}_1 = \mathbb{C}$, there are no localized states.

The discussion above actually needs to be refined somewhat, in a way familiar from the study of Landau-Ginzburg theories. Imagine tuning the couplings in (3.12) in such a way that the relation (3.11) is deformed to

$$(X - x_1)^{n_1}(X - x_2)^{n_2} \cdots (X - x_k)^{n_k} = Y$$

(3.18) with $n = \sum_{i=1}^{k} n_i$. The $x_i$ are functions of the couplings $g_j$. When we flow to the IR they grow, and the system is expected to split into decoupled lower rank singularities. In (3.13) we focused on a particular vacuum (at $X = 0$), but in general there are additional vacua corresponding to $X$ near other $x_i$. Since the Hilbert space of localized states of the extreme IR theory is a direct sum of the Hilbert spaces corresponding to $\mathbb{C}/\mathbb{Z}_{n_i}$, we have

$$g_{uv} = \frac{1}{12} \left( n - \frac{1}{n} \right),$$

$$g_{ir} = \frac{1}{12} \sum_{i=1}^{k} (n_i - \frac{1}{n_i}).$$

(3.19) One can check that the $g_{\text{cl}}$-conjecture (2.13) is satisfied:

$$n - \frac{1}{n} > \sum_{i} (n_i - \frac{1}{n_i})$$

(3.20)

Qualitatively, the transition in (3.19) can be thought of as a process in which a $\mathbb{C}/\mathbb{Z}_n$ cone splits into $\mathbb{C}/\mathbb{Z}_{n_i}$ cones that are decoupled from each other (see figure 1).
Figure 1. Tachyon condensation of the form in equation (3.18) leads to a set of decoupled cones.

It is natural to ask how the process of “cone splitting” in figure 1 occurs in real time. There are two complications with answering this question using the discussion above. First, we have analyzed the RG evolution of the system, which is an off-shell process in the spacetime theory. One might hope that the time evolution is similar, but this is not known to be the case in general (e.g. here time evolution breaks worldsheet $N = 2$ supersymmetry, while RG evolution preserves it).

Second, the splitting of cones depicted in figure 1 is naturally described in the coordinate $X$, which is the field space of the twist field (3.10). This coordinate can be roughly thought of as associated with winding around the tip of the cone parametrized by $Z$ (3.2). All the relevant perturbations respect the $U(1)$ rotation symmetry around the tip, as can be checked by computing the action of the rotation generators on the twist fields.

Since $X$ is essentially the vertex operator of the twisted tachyon field (see (3.10)), one can think of it as a (complex) dimension of field space in the target space theory. Figure 1 seems to suggest that it is natural to think of it as an extra complex dimension in spacetime. The fragmentation of cones in figure 1 happens in this dimension, in a way compatible with the $U(1)$ rotational symmetry in $Z$.

So far we have been discussing the diagonal $\mathfrak{C}/\mathbb{Z}_n$ theory, or type 0 string theory on (3.1). We would now like to discuss the generalization to the chirally GSO projected theory, or type II on (3.1). The latter can be obtained by gauging a chiral $\mathbb{Z}_2$ symmetry in the diagonal (type 0) theory. In flat spacetime, $\mathbb{R}^{9,1}$, one gets type II from type 0 by gauging the chiral $\mathbb{Z}_2$ symmetry $(-)^F L$. In the background (3.1), the symmetry in question acts as $(-)^F L$ on $\psi_0, \psi_1, \cdots, \psi_7$, and as

$$H \to H + n \pi; \quad \bar{H} \to \bar{H}$$

(3.21)
on $\psi, \bar{\psi}$ (see (3.4)).
In order for (3.21) to act as \((-)^F L\) in the untwisted sector (and thus eliminate untwisted sector tachyons), one must take \(n\) to be odd \([2]\). In the twisted sectors, (3.21) takes
\[ X_j \to (-)^j X_j \] (3.22)
(see (3.3)), and since the \((-1, -1)\) picture vertex operators must be odd under chiral GSO, only the states with odd \(j\) survive. In the sectors with even \(j\), \(X_j\) is projected out, and so the lowest lying state is
\[ \tilde{X}_j = \sigma_{j/n} \exp[i(j/n - 1)(H - \bar{H})] , \] (3.23)
which is nothing but the complex conjugate of \(X_{n-j}\),
\[ \tilde{X}_j = X^*_{n-j} . \] (3.24)
As a check, the mass of the lowest lying states for \(j \in 2\mathbb{Z} + 1\) is given by (3.7), while for \(j \in 2\mathbb{Z}\) (3.24) implies that
\[ \frac{\alpha'}{4} M^2_j = -\frac{1}{2} \frac{j}{n} , \] (3.25)
in agreement with the light-cone Green-Schwarz analysis of [2].

We see that the effect of the chiral GSO projection is to restrict to \(n \in 2\mathbb{Z} + 1\), and allow only couplings \(\lambda_j, g_j\) with odd \(j\) in (3.9), (3.12).

The analysis of flows is similar to the one for the diagonal theory, with a few interesting differences. Since \(n, j \in 2\mathbb{Z} + 1\) in (3.12), the analog of the deformed relation (3.18) is now:
\[ X^{n'}(X - x_1)^{n_1}(X + x_1)^{n_1}(X - x_2)^{n_2}(X + x_2)^{n_2} \cdots = Y \] (3.26)
with \(n = n' + 2\sum_i n_i\), \(n'\) an odd integer, and \(x_i \neq 0\). The vacuum structure is now more intricate than in the type 0 case. The vacuum at \(X = 0\) corresponds to a \(\mathbb{C}/\mathbb{Z}_{n'}\) singularity in type II string theory. This is similar to what happens in the diagonal theory. Here, because of chiral GSO, only flows between odd \(n\)'s occur. The vacua at \(X = \pm x_i\) are more interesting, since they break the \(\mathbb{Z}_2\) symmetry \(X \to -X\). Thus, chiral GSO (3.22), which is a discrete gauge symmetry, is spontaneously broken in these vacua. Hence, the infrared theory at vacua like \(X = x_1\) in (3.26) is type 0 string theory on \(\mathbb{C}/\mathbb{Z}_{n_1}\).

We conclude that type II string theory on \(\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_n\) is connected via tachyon condensation to a decoupled sum of a type II string theory on a lower singularity, \(\mathbb{C}/\mathbb{Z}_{n'}\),
with \( n' < n \) and odd, and a set of type 0 vacua on \( \mathcal{C}/\mathbb{Z}_n, i = 1, \ldots, k \). To check whether the \( g_{\text{cl}} \)-conjecture is satisfied, we note that:

1. It is at first sight not clear how to define \( g_{\text{cl}} \) in a theory with both (spacetime) bosons and fermions. We propose to define it via the density of bosonic states \([10]\), as in open superstring theory \([30]\).

2. The value of \( g_{\text{cl}} \) for the type II theory on (3.1) is one half of that for the diagonal (type 0) theory (3.17), since half of the states in the (NS,NS) and (R,R) sectors survive the projection. More formally, this factor of two comes about because in the diagonal theory we insert the projector

\[
\frac{1}{2}(1 + (-1)^{F_L+F_R})
\]

and both terms contribute, while in the type II theory we project with

\[
\frac{1}{4}(1 + (-1)^{F_L} + (-1)^{F_R} + (-1)^{F_L+F_R})
\]

and only the first and fourth terms contribute to the asymptotic density of states.

3. \( X = x_i \) and \( X = -x_i \) in (3.26) do not give independent vacua, since they are related by a gauge symmetry (the broken chiral GSO).

Hence, the \( g_{\text{cl}} \)-conjecture is in this case the statement that

\[
\frac{1}{2}(n - \frac{1}{n}) > \frac{1}{2}(n' - \frac{1}{n'}) + \sum_{i=1}^{k}(n_i - \frac{1}{n_i})
\]

which is indeed valid, since \( n = n' + 2 \sum_i n_i \) (see (3.26)).

The type 0 components of the infrared theory have delocalized tachyons, and thus will presumably disappear when these condense. There is always a type II component in the background (corresponding to the vacuum at \( X = 0 \)), which does not have delocalized tachyons, and in it the physics is similar to that described above: the deficit angle of the cone \( \mathcal{C}/\mathbb{Z}_n \) decreases along the flows, and eventually it decays to type II string theory on a large smooth space, \( \mathbb{R}^{9,1} \).

Another approach to the study of twisted tachyon condensation, which was discussed recently in \([2]\), is to analyze the dynamics of probe D0-branes on the orbifold \( \mathcal{C}/\mathbb{Z}_n \). At low energies this is described by a quiver gauge theory with gauge group \( U(1)^n \) and matter given in \([2]\). The Higgs branch of the moduli space of vacua of this theory is the orbifold
itself. The twisted closed string tachyon v.e.v.’s $\lambda_j$ (3.9) give rise to parameters in the open string Lagrangian.

In principle, if one knows the precise form of the D-brane Lagrangian as a function of $\{\lambda_j\}$, one can analyze the moduli space and deduce from it the geometry of the orbifold after tachyon condensation. In practice, it is difficult to determine the $\lambda$ dependence of the D-brane Lagrangian for all $\lambda$. For small $\lambda$, one can show that the leading effect of tachyon condensation is to turn on (analog of) Fayet-Iliopoulos D-terms for the $U(1)^n$ gauge theory, and to modify the gauge couplings. In appendix B we review the computation of [31] and show that the FI parameters coupling to the D-terms $\{\zeta_j\}$ are given to first order in $\lambda$ by

$$\zeta_j = C \sum_{j' = 1}^n \text{Im}\left(e^{-2\pi \imath jj'/n} \lambda_j\right), \quad (3.30)$$

where $C$ is a real constant. The gauge couplings $1/g_j^2$ are modified as follows:

$$\frac{1}{g_j^2} = \frac{1}{g_0^2}(1 + B_j)$$

$$B_j = C' \sum_{j' = 1}^n \text{Re}\left(e^{-2\pi \imath jj'/n} \lambda_{j'}\right) \quad (3.31)$$

with $C'$ a real constant. By analyzing the Higgs branch of the theory with the couplings (3.30), (3.31) turned on, one may hope to get an indication of the geometry after tachyon condensation. Of course, at small $\lambda$, one expects at best to get a small patch of the new geometry near the (deformed) tip of the cone.

The leading, small $\lambda$, effect of tachyon condensation is to break spontaneously the quiver gauge group $U(1)^n$, due to the “D-term potential” associated with $\{\zeta_j\}$ (3.30). For generic $\{\lambda_j\}$ (i.e. generic $\{\zeta_j\}$ subject to the constraint $\sum_j \zeta_j = 0$), one finds [2] that the gauge group is completely broken, and the moduli space is a smooth non-singular manifold. This is consistent with our results, since for generic $\{\lambda_j\}$, one has $n' = 1$ in (3.26), and the extreme IR limit of the theory is type II string theory on $\mathbb{R}^{7,1} \times \mathbb{C}$, together with a number of decoupled unstable type 0 vacua. The latter do not seem to be visible in the quiver analysis at small $\lambda$.

As discussed in [2], one can fine tune the $\{\zeta_j\}$ (3.30) such that $U(1)^n$ is broken to $U(1)^{n'}$ with any $n' < n$. For $n'$ odd one finds the quiver for $\mathbb{C}/\mathbb{Z}_{n'}$, and it is natural to interpret this as a small $\lambda$ indication of the flow described by (3.26). Our results in fact
show that in this case the higher order corrections in $\lambda$ do not modify the leading order predictions.

For $n'$ even the situation is different. Here there is no candidate $\mathbb{C}/\mathbb{Z}_{n'}$ CFT to serve as the endpoint of the flow, and indeed the probe gauge theory is not of quiver form. Thus, there are two possibilities. One is that there is in fact a consistent background describing type II string propagation (without bulk tachyons) on $\mathbb{C}/\mathbb{Z}_{n'}$ with $n' \in 2\mathbb{Z}$, but that this background cannot be thought of as an orbifold in the usual free field theory sense. Such backgrounds were referred to as “quasi-orbifolds” in [2]. This possibility is not supported by our analysis, which shows no evidence of exotic CFT’s at the endpoint of tachyon condensation.

A second possibility is that the flow $\mathbb{C}/\mathbb{Z}_n \rightarrow \mathbb{C}/\mathbb{Z}_{n'}$ indicated by the leading order analysis is qualitatively modified by higher order effects in $\lambda$. This possibility is consistent with our results.

There are two kinds of effects that are known to be important here (there may be other effects as well). One is higher order corrections to the potential for the quiver fields. This could change qualitative features such as the rank of the maximal unbroken gauge group, and thus the singularity structure of the orbifold after tachyon condensation. For example, it could be that the exact potential for the fields on the D-brane does not have any vacua in which $U(1)^n$ is broken to $U(1)^{n-1}$, for any choice of $\{\zeta_j\}$, unlike the leading order potential studied in [2].

Another important effect is related to the behavior of the gauge couplings on the quiver. Naively, one might expect that since the D-brane probe is studied in the weak coupling limit, $g_s \rightarrow 0$, the only way factors of the quiver can disappear is by the Higgs mechanism, as described in detail in [2]. But, one of the lessons of the work on open string tachyon condensation on non-BPS branes in the last few years is that there is another, stringy, way for D-branes to disappear, by a process which formally looks like classical confinement. In this process, the whole action on the D-brane goes to zero (see e.g. [6]).

It is possible that twisted closed string tachyon condensation leads to similar effects on the probe D-brane. In order to analyze this, one would have to compute the D-brane action as a function of $\{\lambda_j\}$, and check whether it goes to zero when $\{\lambda_j\}$ approach their IR fixed point values. This would involve computing the gauge couplings (3.31), the tensions of the

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9 There has been some debate as to the precise mechanism by which the D-branes disappear [32,33,34,35]; this is irrelevant for the present discussion.
fractional branes, and other quantities, to all orders in $\lambda$. This is beyond the scope of our analysis, but it is interesting that to first order in $\{\lambda_j\}$, at least one of the couplings $\{g_j\}$ in (3.31) is getting stronger (since $\sum_j B_j = 0$). We believe that the “classical confinement” mechanism is important in this problem.

Finally, one can ask what happens when one allows $N = 2$ supersymmetric breaking on the worldsheet. One might hope that this allows independent control over all the low energy parameters of the quiver, since there are of order $n^2$ twisted sector tachyons. However, these couplings are not free parameters – the tachyon v.e.v.’s in the IR are solutions of the closed string beta function equations, which are generically isolated points in the space of couplings. The quiver gauge couplings are functions of the tachyon v.e.v.’s and take some particular values at the IR fixed points. There is a good chance that some of the gauge couplings are shifted substantially toward strong coupling during the flow, just as in the $N=2$ preserving examples. More control over the quiver analysis (or some other way of analyzing $N=1$ flows) is needed before one can claim that “quasi-orbifold” solutions to the string equations of motion exist.

4. $\mathbb{C}^2/\mathbb{Z}_n$ flows

In section 3 we saw that $\mathbb{C}/\mathbb{Z}_n$ orbifolds exhibit an interesting pattern of infrared instabilities and decays. Higher dimensional noncompact orbifolds $\mathbb{C}^m/\Gamma$ with $m > 1$ are expected to exhibit an even richer behavior. As a first step towards their study, in this section we discuss, following [2], a class of $\mathbb{C}^2/\mathbb{Z}_n$ orbifolds where $\mathbb{Z}_n$ acts as follows:

$$\mathcal{R}(Z_1, Z_2) = (\omega Z_1, \omega^p Z_2), \quad p \in \mathbb{Z},$$

(4.1)

$\omega = \exp[2\pi i/n]$, and $0 < |p| < n$, with $p, n$ relatively prime. We will denote the orbifold (4.1) by $\mathbb{C}^2/\mathbb{Z}_{n(p)}$, as in [2]. Quotients of the type (4.1) are known as Hirzebruch-Jung singularities, and their geometry is discussed for instance in [36,37].

The orbifolds $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ and $\mathbb{C}^2/\mathbb{Z}_{n(-p)}$ are related by a change of complex structure, $Z_2 \to Z_2^*$, and are thus isomorphic. We will find it convenient to keep the complex structure fixed, and discuss the cases of positive and negative $p$ separately. The Kahler form is

$$K = dZ_1 \wedge dZ_1^* + dZ_2 \wedge dZ_2^*$$

(4.2)

The worldsheet conformal field theory has a corresponding left-moving $U(1)_R$ current

$$J(z) = \psi_1 \psi_1^* + \psi_2 \psi_2^*$$

(4.3)
and similarly for the right-movers.

As for $\mathcal{C}/\mathbb{Z}_n$, the chiral ring of the orbifold CFT will play an important role in our analysis. The building blocks out of which chiral operators in $\mathcal{C}^2/\mathbb{Z}_{n(p)}$ are constructed are the twist fields $X_j^{(1)}$, $X_j^{(2)}$ (3.3) for the two complex planes parametrized by $Z^1$, $Z^2$, and the corresponding volume forms $Y^{(1)}$, $Y^{(2)}$ (3.10). Combining the two, one finds the twisted sector chiral operators

$$X_j = X_j^{(1)} X_j^{(2)}$$

where $j = 1, 2, \cdots, n - 1$ labels the twisted sectors, and $\{x\}$ is the fractional part of $x$, $\{x\} = x - [x]$, with $[x]$ the integer part of $x$ (the largest integer $\leq x$). Note that by definition $0 \leq \{x\} < 1$. The R-charges of the chiral operators (4.4) are

$$R_j = \frac{j}{n} + \left\{ \frac{jp}{n} \right\}.$$  

(4.5)

For type II strings, one has to perform further a chiral GSO projection, which can be described in a way analogous to the $\mathcal{C}/\mathbb{Z}_n$ case (see (3.21)). It acts on the bosonized fermions $H_1$, $H_2$ (which are defined by $\psi_j = \exp(iH_j)$, as in (3.4)) as the chiral $\mathbb{Z}_2$ shift

$$H_1 \rightarrow H_1 + p\pi \; , \; \; \; H_2 \rightarrow H_2 - \pi .$$

(4.6)

In the untwisted sector of the orbifold, (4.6) must reduce to the standard $(-)^{F_L}$; this implies that $p$ must be odd, in agreement with [2]. In the twisted sectors, (4.6) projects out operators with $[jp/n] \in 2\mathbb{Z}$ and keeps those with $[jp/n] \in 2\mathbb{Z} + 1$. One can show that this definition of chiral GSO in the NSR framework gives the same spectrum as the light-cone Green-Schwarz analysis.

Operators whose R-charge (4.5) is smaller than one give rise to tachyons in spacetime. One new element in the discussion compared to the $\mathcal{C}/\mathbb{Z}_n$ case, is the appearance of massless states with flat potentials, corresponding to operators (4.4) with R-charge equal to one. As is well known [38], such operators correspond to exactly marginal operators on the worldsheet, and one can study the theory as a function of their coefficients in the action (moduli). As we saw in section 2, $g_{Cl}$ is independent of the moduli, for finite changes of moduli. If a modulus is taken strictly to infinity (in the natural metric on moduli space), $g_{Cl}$ can jump discontinuously (down), since in the limit some localized states might decouple.

This behavior is familiar from the study of boundary CFT’s in D-brane physics. As a simple example, consider a collection of $n$ D0-branes on an infinite line. For any finite
separation of the D-branes (which corresponds to an expectation value of a massless open string field, or boundary modulus), $g_{\text{op}} (2.2)$ is independent of the separation, $g_{\text{op}} = n^2 g_1$, where $g_1$ is the value for a single D0-brane. If we take a limit in which the collection of D-branes splits into two groups of $n_1$ and $n_2$ D-branes ($n = n_1 + n_2$) separated by an infinite distance, $g_{\text{op}}$ decreases to $g_{\text{op}} = (n_1^2 + n_2^2) g_1$, since the states associated with open strings stretched between the two clusters of branes become infinitely massive and decouple. One expects the same general behavior for $g_{\text{cl}}$ as a function of twisted sector moduli.

In this section we will discuss the physics associated with relevant and marginal perturbations of $C^2/\mathbb{Z}_{n(p)}$ by chiral operators of the form (4.4). One of the main tools that we will use to study such flows is the classical geometry of Hirzebruch-Jung quotient singularities. As we describe in the next subsection, many properties of the $C^2/\mathbb{Z}_{n(p)}$ CFT are directly related to the geometry of Hirzebruch-Jung singularities. This relation will allow us to develop a rather detailed picture of the flows in these systems.

For the special case $p = -1$, we will use another tool to study the system. In this class of orbifolds, the worldsheet symmetry is enhanced to $N = 4$ superconformal symmetry, and the GSO projection (4.6) leads to a spacetime supersymmetric background. In this case one can use a duality of the orbifold CFT $C^2/\mathbb{Z}_{n(-1)}$ to a system of NS5-branes [39,40], which provides a nice geometric picture of the possible flows. This is described in section 4.2.

One of our main purposes below is to verify the validity of the conjecture (2.13) for the different cases. For the class of orbifolds (4.1), $g_{\text{cl}}$ is given by

$$g(n, p) = \frac{1}{n} \sum_{s=1}^{n-1} \frac{1}{[4 \sin(\pi s/n) \sin(\pi ps/n)]^2} .$$  

Below we will examine in detail the cases $p = \pm 1$, for which (see appendix A)

$$g_{\text{cl}}(n, \pm 1) = \frac{(n^2 + 11)(n^2 - 1)}{45 \cdot 16n} ;$$

and $p = \pm 3$, for which

$$g_{\text{cl}}(n, \pm 3) = \begin{cases} 
\frac{(n^4 + 210n^2 - 80n - 291)}{405 \cdot 16n} & n = 2 \text{ mod } 3 \\
\frac{(n^4 + 210n^2 + 80n - 291)}{405 \cdot 16n} & n = 1 \text{ mod } 3 .
\end{cases}$$

For general $n, p$, the function (4.7) is rather complicated (see Appendix A); it is useful to note that in the limit $n \to \infty$, $p$ fixed, it simplifies: $g_{\text{cl}} \simeq \frac{1}{720} \frac{n^3}{p^2}$.

\footnote{This holds for the type 0 case. For type II there is an additional factor of 1/2, as in section 3.}
4.1. Chiral rings, Hirzebruch-Jung geometry, and singularity resolution

In this subsection we will briefly summarize the Hirzebruch-Jung theory of singularity resolution for cyclic surface singularities and its relation to the structure of the chiral ring of the corresponding orbifold $N = 2$ SCFT. We will use some of the techniques of toric geometry without detailed explanation. Useful references on this material include [36, 37, 41, 42] as well as the treatments aimed more towards physicists in [43, 44, 45, 46, 47].

4.1.1. Hirzebruch-Jung geometry

The singular geometry corresponding to the quotient (4.1) is defined in toric geometry by a cone consisting of positive real linear combinations of two generators, $v_f \equiv v_{r+1} = e_2$ and $v_i \equiv v_0 = ne_1 - pe_2$ as shown in figure 2.

$$v_j \text{ are rational, meaning they lie in } N.$$  
$$1. \text{ The } v_j \text{ are rational, meaning they lie in } N.$$  
$$2. \text{ Each successive pair of vectors } (v_0, v_1), (v_1, v_2), \cdots (v_r, v_{r+1}) \text{ spans the lattice } N \text{ as a } \mathbb{Z}\text{-module.}$$

One implication of this structure is that there exist integers $a_j$ such that

$$a_j v_j = v_{j-1} + v_{j+1} \quad (4.10)$$

for all interior vectors $v_j$. One can show that each interior vector corresponds to an exceptional divisor $E_j \cong \mathbb{P}^1$ with self-intersection number $-a_j$ and with $E_j$ intersecting $E_{j+1}$ once.

Figure 2. Cone defining the quotient singularity $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ for $(n, p) = (5, 2)$. 

Here $e_1, e_2$ are a basis for the lattice $N = \mathbb{Z}^2$. A resolution of singularities is given by adding a set of interior vectors $v_j$, $j = 1, 2, \cdots r$ lying between $v_0$ and $v_r+1$ such that

1. The $v_j$ are rational, meaning they lie in $N$.

2. Each successive pair of vectors $(v_0, v_1), (v_1, v_2), \cdots (v_r, v_{r+1})$ spans the lattice $N$ as a $\mathbb{Z}$-module.

One implication of this structure is that there exist integers $a_j$ such that

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for all interior vectors $v_j$. One can show that each interior vector corresponds to an exceptional divisor $E_j \cong \mathbb{P}^1$ with self-intersection number $-a_j$ and with $E_j$ intersecting $E_{j+1}$ once.
We may summarize this in a resolution diagram:

![Resolution diagram](image)

**Figure 3.** Resolution diagram of \(\mathbb{C}^2/\mathbb{Z}_n(p)\). The links denote unit intersection of successive curves in the resolution, and the labels on the nodes denote their self-intersection numbers.

The combinatorics of this construction are directly captured by the continued fraction expansion of \(n/k\), where \(k = p\) for \(p > 0\), and \(k = n + p\) for \(p < 0\) (in other words, \(0 < k < n\) and \(k = p \mod n\)). The continued fraction expansion is defined as

\[
\frac{n}{k} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}} \equiv [a_1, a_2, a_3, \ldots, a_r] .
\] (4.11)

with integers \(a_j \geq 2\). This is called the Hirzebruch-Jung continued fraction of \(n/k\). There is a minimal resolution of the singularity where the \(\{a_j\}\) in (4.10) are the same as the integers in (4.11). It is minimal in the sense that no curve can be blown down while leaving a nonsingular variety.

A familiar example is the resolution of singularities for the ALE space \(\mathbb{C}^2/\mathbb{Z}_n(-1)\). In toric geometry this singularity is described by the cone generated by \(v_0 = ne_1 - (n-1)e_2\) and \(v_n = e_2\) and corresponds to an \(A_{n-1}\) singularity, \(\mathbb{C}[Y_1, Y_2, Y_3]/(Y_3^{n-1} = Y_1Y_2)\). The resolved geometry is described by a fan with \(n-1\) interior vectors as illustrated in figure 4 for the case \(n = 4\). The resolved geometry has \(n-1\) \(\mathbb{P}^1\)'s which intersect according to the Dynkin diagram of \(A_{n-1}\). This agrees with the continued fraction expansion \(n/(n-1) = [2^{n-1}]\), with powers denoting repeated entries.

![Toric fan](image)

**Figure 4.** Toric fan for the resolution of the \(\mathbb{C}^2/\mathbb{Z}_{n(-1)}\) singularity for \(n = 4\).
Some examples which will be useful later include

1. \((n, p) = (n, 1)\). The continued fraction \(n/1 = [n] \) tells us that the resolved geometry has one \(\mathbb{P}^1\) with self-intersection number \(-n\).

2. \((n, p) = (n, 3)\). There are two distinct cases depending on whether \(n\) is 1 or 2 mod 3. For \((3m+1, 3)\) we have the continued fraction expansion

\[
\frac{3m+1}{3} = [m+1, 2, 2]
\]  

while for \((3m+2, 3)\) we have

\[
\frac{3m+2}{3} = [m+1, 3]
\]  

There are three or two exceptional divisors in total, and of these one has self-intersection \(-(m+1)\).

3. \((n, p) = (n, -3)\). Again there are two cases, \(n = 3m+1, n = 3m+2\):

\[
\frac{3m+1}{3m-2} = [2^{m-1}, 4]
\]  

while

\[
\frac{3m+2}{3m-1} = [2^{m-1}, 3, 2]
\]

There are now \(m - 1\) or \(m\) exceptional divisors with self-intersection \(-2\) and one additional divisor with self-intersection \(-4\) or \(-3\) respectively. Note that the resolved geometry depends dramatically on the sign of \(p\), or equivalently on the choice of complex structure.

Given a fan corresponding to a non-singular variety, it is possible to add additional internal vectors which preserve the above conditions for a non-singular fan. This corresponds to blowing up a non-singular point, or equivalently to adding a \(\mathbb{P}^1\) with self-intersection number \(-1\). For example, the cone generated by \(v_0 = e_1\) and \(v_1 = e_2\) corresponds to \(\mathbb{C}^2\).

We can add an interior vector \(v_b = e_1 + e_2\), the resulting fan describes the blow up of \(\mathbb{C}^2\) at a point and the \(\mathbb{P}^1\) corresponding to \(v_b\) has self-intersection number \(-1\) since \(v_b = v_0 + v_1\).

Conversely, we can blow down such a curve and still find a non-singular space, or in the toric description, an interior vector \(v_k\) which satisfies \(v_k = v_{k-1} + v_{k+1}\) can always be removed from the fan while preserving the conditions for a non-singular variety.
4.1.2. CFT and resolved Hirzebruch-Jung singularities

The chiral ring of an $N = 2$ conformal field theory can be specified in terms of generators and relations. Here we will summarize an empirical relation we have discovered between the generators and relations of the ring of chiral twist operators \((4.4)\) of the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifold and the combinatorial data of the toric variety giving a minimal resolution of singularities. This empirical relation has been checked in a wide variety (including several infinite classes) of examples; in particular, it holds for the particular examples studied below. A more detailed exposition will appear elsewhere [48].

The chiral ring of the orbifold is finite dimensional; we choose a set of generators $W_0 = Y^{(2)}$, $W_{r+1} = Y^{(1)}$ with $Y^{(1)}, Y^{(2)}$ defined as in \((3.10)\) for each $C$ component of $C^2$, and $W_i$, $i = 1, \ldots, r$ constructed as products of bosonic and fermionic twist fields as in \((4.4)\). The \(\{W_i\}\) are ordered by the twist sectors. Explicit descriptions of the $W_i$ in some classes of examples are given in subsection 4.3 and in Appendix C.

There is a one-to-one correspondence between generators of the chiral ring and vectors in the toric fan which describes the minimal resolution of the corresponding Hirzebruch-Jung singularity. Furthermore, the relations between the vectors of the toric fan \((4.10)\) are reflected in the chiral ring by relations between the corresponding generators:

\[
W_i^{a_i} = W_{i-1}W_{i+1}, \quad i = 1, \ldots, r .
\]  

(4.16)

In general, these provide a subset of the full set of relations satisfied by the orbifold chiral ring generators.

Thus we are led to the following general relation between the free field orbifold CFT and the corresponding resolved Hirzebruch-Jung singularity. The orbifold CFT corresponds to a minimal resolution of the singularity. There are $r$ blowing up parameters turned on, but as in the ALE case [49], they correspond to $B$-fields and not geometric resolutions. Unlike the ALE case, the operators $W_i$ are not in general marginal; thus the above non-trivial $B$-fields give rise to isolated fixed points and not lines of CFT’s.

Non-minimal resolutions of the singularity correspond to blowing up curves with self-intersection \((-1)\). A \((-1)\) curve corresponds to an interior vector $v_i = v_{i-1} + v_{i+1}$. The corresponding element of the chiral ring can be written as the product of adjacent elements, \textit{i.e.} it is not an independent generator. Blowing up a \((-1)\) curve corresponds to perturbing by this dependent element of the chiral ring. More generally, perturbing the Lagrangian by $\lambda_i \int d^2 \theta X_i$ corresponds to blowing up various curves in the orbifold. The coupling $\lambda_i$ is related to the complexified Kahler class of this blow up. This connection will prove useful later when we discuss RG flows.
4.2. $\mathbb{C}^2/\mathbb{Z}_n(-1)$: The duality between orbifolds and fivebranes

The spacetime supersymmetric orbifold theory $\mathbb{C}^2/\mathbb{Z}_n(-1)$ is T-dual to the theory of fivebranes on a circle, in an appropriate limit \cite{39,40}. Consider type II string theory on $\mathbb{R}^{8,1} \times S^1$, with $n$ NS5-branes symmetrically arranged on the circle, which we take to have circumference $R$, and parametrize by $x^9$; and let $x^{6,7,8}$ parametrize the $\mathbb{R}^3$ transverse to the fivebranes (see figure 5). The claim is that in the limit

$$g_s \to 0 \ , \quad R/l_s \to 0 \ , \quad \text{with} \quad \frac{R}{l_s g_s} \ \text{fixed} \quad (4.17)$$

type IIB string theory in the fivebrane background is equivalent to type IIA string theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_n(-1)$ (and vice versa).

**Figure 5.** Two perturbations of a $\mathbb{Z}_{2l}$ symmetric arrangement of type IIB fivebranes on a circle, dual to type IIA string theory on $\mathbb{C}^2/\mathbb{Z}_n$, with $n = 2l$: (a) moving the fivebranes on $S^1$ is related to changing NS $B$-field fluxes through vanishing cycles on the IIA side; (b) moving them in $\mathbb{R}^3$ is dual to turning on the triplets of geometrical blow up modes of the vanishing cycles on the IIA side.

The two descriptions are related by T-duality applied to the circle parametrized by $x^9$. The string couplings are related by the standard T-duality formula

$$g_s^A = g_s^0 l_s/R \ . \quad (4.18)$$
Note that the limit (4.17) is such that \( g_s^A \) is held fixed as \( g_s^B \to 0 \). The orbifold has \( n - 1 \) hypermultiplets of moduli coming from twisted sectors; the four real parameters in each hypermultiplet consist of the NS \( B \)-field flux through one of the \( n - 1 \) vanishing cycles of the orbifold ALE space, together with a triplet of modes that blow up that cycle. The \( B \)-flux is a periodic coordinate, while the blow up modes parametrize \( \mathbb{R}^3 \). These map on the fivebrane side into the relative locations of the fivebranes on the \( S^1 \) and \( \mathbb{R}^3 \), respectively. The standard \( \mathbb{C}^2/\mathbb{Z}_n(-1) \) orbifold CFT corresponds to the point in moduli space where the fivebranes are coincident in \( \mathbb{R}^3 \) and symmetrically arranged on the \( S^1 \) (as in the top of figure 5). The \( \mathbb{Z}_n \) symmetry that cyclically permutes the fivebranes is the \( \mathbb{Z}_n \) quantum symmetry of the orbifold that multiplies the \( j \)th twisted sector by \( \omega^j \).

There are two important classes of limits in the moduli space, which are easy to see in the fivebrane description (see figure 5):

(a) Separating fivebranes along \( \mathbb{R}^3 \) leads, at infinite distance, to a direct sum of decoupled theories. For example, if \( n = 2l \), we can separate the branes into two groups while preserving a \( \mathbb{Z}_l \) symmetry (figure 5b), leading to two decoupled \( \mathbb{C}^2/\mathbb{Z}_l \) theories in the limit.

(b) Bringing groups of fivebranes together so that they coincide at point(s) in \( \mathbb{R}^3 \times S^1 \) (figure 5a) leads to enhanced gauge symmetry of the fivebrane gauge theory. This limit is a finite distance in the moduli space from the orbifold point, and results in a singular CFT.

Near-coincident NS5-branes generate a target space for perturbative worldsheet string theory which develops a throat along which the string coupling grows; the throat becomes infinitely long, and the coupling at its end diverges, in the limit where fivebranes coincide [15]. On the IIA side, this singularity of the worldsheet CFT can be understood from considerations of linear sigma models.

One can also match the structure of D-branes on the two sides. The limit (4.17) keeps fixed the mass in string units of D1-branes stretching between the NS5-branes on the IIB side; their mass scales as

\[
l_sm_w = \frac{R}{n \, l_s g_s^B} \tag{4.19}
\]

at the point in moduli space related to the orbifold. D1-branes of fractional winding are pinned to the NS5-branes they begin and end on, while D1-branes of integer winding are free to move in the \( \mathbb{R}^3 \) transverse to the NS5-branes. This is exactly the same structure obtained in IIA string theory on \( \mathbb{C}^2/\mathbb{Z}_n(-1) \) [31]. There, fractional D0-branes of the orbifold
are the W-bosons of a spontaneously broken 5+1 dimensional gauge symmetry localized on the orbifold singularity; their mass is

$$l_s m_W = \frac{1}{n g_s^A}.$$  \hspace{1cm} (4.20)

These excitations are D2-branes wrapping the vanishing cycles of the ALE space, and carrying a fractional unit $1/n$ of D0-brane charge. Combining one fractional brane of each type makes a “regular” D0-brane that can be moved off the orbifold point into the ambient ALE space.

The $U(1)^n$ gauge fields and bifundamental matter content on a regular D-brane are neatly summarized in a $\mathbb{Z}_n$ quiver diagram \[31\]. In the IIB picture, the same gauge theory describes the dynamics of a D1-brane wrapped around the circle, which intersects all $n$ NS fivebranes.

As mentioned above, one nice feature of the fivebrane description is that it provides a simple geometric interpretation of the effect of closed string moduli. This is useful for analyzing the effects on D-branes of turning on various moduli. In particular, the relative gauge couplings of the D-brane quiver gauge theory, which on the IIA side are controlled by the NS B-field flux through the ALE vanishing cycles, correspond in the IIB description, to the separations of the branes on the $x^9$ circle. The triplets of modes that blow up the vanishing cycles to a smooth ALE space on the IIA side and correspond to Fayet-Iliopoulos D-terms on the fractional D-branes, are mapped in the IIB description to the relative positions of the branes in the $\mathbb{R}^3$ parametrized by $x^{6,7,8}$.

Fractionally wound branes become massless if fivebranes coincide (IIB), or equivalently (IIA) when the B-flux through vanishing cycles of the ALE space is turned off \[49\]; the D-brane gauge dynamics then becomes strongly coupled. This is the open string reflection of the singularity of the CFT noted above. When fivebranes are infinitely separated in $\mathbb{R}^3$, the energy of the corresponding fractional branes goes to infinity.

The asymptotic density of localized states for the supersymmetric $\mathbb{C}^2/\mathbb{Z}_n(-1)$ orbifold is one half of \[L8\], due to the GSO projection (see the discussion surrounding equations \[3.27\], \[3.28\]). It is interesting that $g_{cl}$ is not linear in $n$, $g_{cl} \sim n^3$ for large $n$. This means that it is not linear in the number of fivebranes in the dual description. Since $g_{cl}$ is constant under marginal deformations, it scales like $n^3$ even for fivebranes separated by a large but finite distance. Hence, it cannot be thought of as the energy of the branes, which would be additive for well-separated branes.
4.3. Flows

In the previous two subsections we described some tools for studying the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifold CFT’s. In this subsection we will use these tools for analyzing the RG flows in these vacua. We will discuss a few special cases, and abstract from them some patterns which we believe are more general. A more complete analysis of the flows will be postponed to another publication \[48\].

4.3.1. Example 1: $\mathbb{C}^2/\mathbb{Z}_{n(1)}$.

Our first example is the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(1)}$. As discussed in section 4.1, the continued fraction expansion is in this case trivial, $n/1 = [n]$; the chiral ring has a single generator $X = X_1$ of R-charge $2/n$, subject to the relation

$$X^n = Y^{(1)}Y^{(2)},$$

(4.21)

which is a special case of (4.16). The operator $X^j$ has R-charge $2j/n$ and dimension $j/n$; thus, the operators with $j < n/2$, which are relevant, and $X^{n/2}$ (for even $n$) which is marginal, can be added to the Lagrangian as perturbations of the superpotential

$$\delta \mathcal{L} = \lambda_j \int d^2 \theta X^j + \text{c.c.}.$$  

(4.22)

The $\{\lambda_j\}$ have R-charge $1 - \frac{2j}{n}$ and quantum $\mathbb{Z}_n$ charge $-\frac{j}{n}$. Consequently, the chiral ring relation (4.21) cannot be modified at first order in the $\{\lambda_j\}$; rather the leading modification is

$$X^n + \sum_{j+k<n} c_{jk} \lambda_j \lambda_k X^{j+k} = Y^{(1)}Y^{(2)}.$$  

(4.23)

The flows are rather similar to the $\mathbb{C}/\mathbb{Z}_n$ examples studied in section 3. At large $\lambda$ one flows to the chiral ring $X^m = Y^{(1)}Y^{(2)}$ for some $m < n$. One easily checks that $g_{cl}$, given in equation (4.8), decreases along RG flows.

Along the marginal line parametrized by $\lambda_{n/2}$, the ring relation does not change:

$$X^n (1 + c\lambda^2_{n/2} + ...) = Y^{(1)}Y^{(2)}.$$  

(4.24)

This is consistent with the expectation that $g_{cl}$ is unchanged along this marginal line.

The above discussion was in the context of type 0 string theory. The chiral GSO projection (4.6) eliminates the entire chiral ring generated by $X$. Therefore, these flows are impossible in type II string theory on $\mathbb{C}^2/\mathbb{Z}_{n(1)}$ (which in fact is spacetime supersymmetric).

This example generalizes straightforwardly to orbifolds of type $\mathbb{C}^m/\mathbb{Z}_n$ with the group action $Z_i \sim \omega Z_i$ and $m > 2$. The $g_{cl}$-conjecture in this case is verified in appendix A.
4.3.2. Example 2: $\mathbb{C}^2/\mathbb{Z}_n(-1)$.

The next example we would like to analyze is $\mathbb{C}^2/\mathbb{Z}_n(-1)$, the spacetime supersymmetric orbifold. The continued fraction is $n/(n-1) = [2^{n-1}]$ (see section 4.1); thus all $n-1$ chiral operators $X_j$ \((4.4)\) are in this case generators. The continued fraction expansion suggests \((4.16)\) that $X_j^2 = X_{j-1}X_{j+1}$, but in fact both the r.h.s. and the l.h.s. are separately zero. More generally, one has $X_iX_j = 0$ for all $i,j = 1,2,\ldots,n-1$. Note also that all operators $X_j$ survive the chiral GSO projection \((4.6)\); thus the analysis of flows is the same in type 0 and type II.

The supersymmetric orbifold of course does not contain any tachyons; the chiral operators $X_j$ correspond to massless states. As discussed above, one can still ask what happens when we perturb the Lagrangian by a marginal operator $\lambda_j \int d^2 \theta X_j$, and send the modulus $\lambda_j \to \infty$.

Geometrically, turning on $\lambda_j$ and sending it to infinity corresponds to blowing up the appropriate $\mathbb{P}^1$ in the ALE geometry to infinite size. In this limit, $\lambda_j$ can be thought of as a Lagrange multiplier imposing the constraint $X_j = 0$; the corresponding infinite size $\mathbb{P}^1$ disappears from the singular geometry. The generators $X_i$ with $i < j$ decouple from those with $i > j$; the corresponding $\mathbb{P}^1$'s do not intersect (see figure 3).

All this can be summarized as the following action on the Hirzebruch-Jung continued fraction:

$$[2^{n-1}] \to [2^{j-1}] \oplus [2^{n-j-1}] \quad (4.25)$$

where we erased the $j$th entry in the continued fraction and took into account the decoupling of the left and right parts of the fraction. Eq. \((4.25)\) describes a process in which a $\mathbb{Z}_n$ singularity is split into decoupled $\mathbb{Z}_j$ and $\mathbb{Z}_{n-j}$ singularities:

$$\mathbb{C}^2/\mathbb{Z}_n(-1) \to \mathbb{C}^2/\mathbb{Z}_j(-1) \oplus \mathbb{C}^2/\mathbb{Z}_{n-j}(-1). \quad (4.26)$$

This process is easily understood in terms of fivebranes (see section 4.2). It corresponds to a deformation that takes $n$ fivebranes arranged symmetrically on a circle (as in the top figure in figure 5) and moves $j$ of them in the $(6,7,8)$ directions such that the final configuration has two groups of $j$ and $n-j$ fivebranes, infinitely separated in $\mathbb{R}^3$. It is easy to check that, as in the previous example, $g_{\text{cl}} \ (4.8)$ decreases in the process.
The discussion above can be generalized to any flow corresponding to a marginal or relevant generator of the chiral ring in any $\mathcal{C}^2/\mathbb{Z}_{n(p)}$ CFT. Adding the generator $W_j$ with a large coefficient\(^{11}\) has the following effect on the continued fraction:

\[ [a_1, \ldots, a_r] \rightarrow [a_1, \ldots, a_j - 1] \oplus [a_{j+1}, \ldots, a_r]. \tag{4.27} \]

The flow thus splits the target space into a direct sum of the corresponding Hirzebruch-Jung singularities.

4.3.3. Example 3: $\mathcal{C}^2/\mathbb{Z}_{2\ell(-3)}$.

Our next example is $\mathcal{C}^2/\mathbb{Z}_{2\ell(-3)}$. As mentioned in section 4.1 (equations (4.14), (4.15)), the continued fraction is slightly different for the cases $n = 3m + 1$ and $n = 3m + 2$.

\[
\begin{align*}
[2, \ldots, 2, 4] & , & n = 2\ell = 3m + 1 \\
\underbrace{[2, \ldots, 2, 3, 2]}_{m-1} & , & n = 2\ell = 3m + 2 . \tag{4.28}
\end{align*}
\]

For $n = 3m + 1$, the $m$ generators of the chiral ring are $W_j = X_j$ with $j = 1, 2, \ldots, m$, with R-charges\(^{43}\) $R_j = 1 - \frac{2j}{n}$. For $n = 3m + 2$ there are $m+1$ generators, the first $m$ of which are as in the previous case, while $W_{m+1} = X_{2m+1}$, whose R-charge is $R_{2m+1} = 2 - \frac{2}{n}(2m+1)$ (see Appendix C for further details). All generators are relevant operators. Chiral GSO takes $W_j \rightarrow -W_j$ for $j = 1, \ldots, m$ and for $n = 3m + 2$, $W_{m+1} \rightarrow W_{m+1}$. Thus, the generators $W_1, \ldots, W_m$ survive the GSO projection, while $W_{m+1}$ is projected out.

One can now study various relevant and marginal perturbations of the CFT. Consider first deformations by the generators. For $n = 3m + 1$, condensing $W_j$ with $1 \leq j \leq m - 1$ leads, as in the previous example, to the flow

\[ [2^{m-1}, 4] \rightarrow [2^{j-1}] \oplus [2^{m-j-1}, 4] , \tag{4.29} \]

or equivalently:

\[ \mathcal{C}^2/\mathbb{Z}_{3m+1(-3)} \rightarrow \mathcal{C}^2/\mathbb{Z}_{j(-1)} \oplus \mathcal{C}^2/\mathbb{Z}_{3(m-j)+1(-3)} . \tag{4.30} \]

\(^{11}\) Note that unlike the situation for marginal deformations, for relevant deformations one does not need to send the coupling to infinity by hand; it grows naturally along the RG flow.
This is an example where a non-supersymmetric orbifold produces a lower rank non-supersymmetric orbifold as well as a supersymmetric one. By plugging in the explicit formulae for $g_{\text{cl}}$ (4.8), (4.9) one finds that, as expected:

$$g_{\text{cl}}(3m + 1, -3) > g_{\text{cl}}(j, -1) + g_{\text{cl}}(3(m - j) + 1, -3).$$ (4.31)

Perturbing by $W_m$ gives rise to the flow

$$[2^{m-1}, 4] \to [2^{m-1}]$$

$$\mathcal{C}^2/\mathbb{Z}_{3m+1(-3)} \to \mathcal{C}^2/\mathbb{Z}_m(-1).$$ (4.32)

Again, one can check that $g_{\text{cl}}$ decreases along the flow. A similar set of flows is obtained when one perturbs by generators of the chiral ring in the case $n = 3m + 2$. Note that all the flows described above in the type 0 context exist in the type II theory as well, since the generators by which we perturbed survive the GSO projection.

A new element in this case is that one can perturb by products of generators. For example, consider the chiral operator

$$V_j = W_j W_{j+1}$$ (4.33)

with $(n - 1)/4 \leq j \leq m - 1$. One can show that $V_j$ is a non-vanishing relevant (or marginal) operator. Adding it to the worldsheet Lagrangian corresponds in the Hirzebruch-Jung geometry to blowing up the point of intersection of the $j^{\text{th}}$ and $(j + 1)^{\text{st}} \mathbb{P}^1$’s in the minimal resolution of the singularity. In general, when one blows up such a $\mathbb{P}^1$ in a resolved Hirzebruch-Jung surface, the effect on the continued fraction is the following (see [37] p. 44),

$$[a_1, \ldots, a_r] \to [a_1, \ldots, (a_j + 1), 1, (a_{j+1} + 1), \ldots, a_r].$$ (4.34)

In the CFT this corresponds to perturbing by the operator $V_j$ (4.33) and adding it to the set of ring generators, together with the trivial ring relation (4.33). Note that the continued fractions (4.11) corresponding to the l.h.s. and r.h.s. of (4.34) are equal.

Sending the coefficient of $V_j$ in the action to infinity can now be treated in the same way as before. The continued fraction is split into two disconnected components,

$$[a_1, \ldots, (a_j + 1), 1, (a_{j+1} + 1), \ldots, a_r] \to [a_1, \ldots, (a_j + 1)] \oplus [(a_{j+1} + 1), \ldots, a_r]$$ (4.35)

and the CFT correspondingly splits into a direct sum of the appropriate orbifolds.

In the examples (4.28) considered here, perturbing by (4.33) thus has the following effect:

$$[\underbrace{2, \ldots, 2, 4}_{m-1}] \to [\underbrace{2, \ldots, 2, 3}_{j-1}] \oplus [\underbrace{3, 2, \ldots, 2, 4}_{m-2-j}]$$

$$[\underbrace{2, \ldots, 2, 3, 2}_{m-1}] \to [\underbrace{2, \ldots, 2, 3}_{j-1}] \oplus [\underbrace{3, 2, \ldots, 2, 3, 2}_{m-2-j}].$$ (4.36)
4.3.4. Example 4: $\mathbb{C}^2/\mathbb{Z}_{2\ell(3)}$.

The last set of examples that we will consider here is the case $\mathbb{C}^2/\mathbb{Z}_{2\ell(3)}$, which was studied in [2]. These authors found that under a large marginal deformation, this model exhibits the flow

$$
\mathbb{C}^2/\mathbb{Z}_{2\ell(3)} \rightarrow \mathbb{C}^2/\mathbb{Z}_{\ell(1)} \oplus \mathbb{C}^2/\mathbb{Z}_{\ell(-3)}.
$$

If true, this provides a counterexample to the $g_{\text{cl}}$-conjecture. For example, for large $\ell$ one has

$$
g_{\text{cl}}(2\ell, 3) = \frac{1}{720} \frac{(2\ell)^3}{9}
\quad g_{\text{cl}}(\ell, 1) + g_{\text{cl}}(\ell, -3) = \frac{1}{720} \left( \ell^3 + \frac{\ell^3}{9} \right)
$$

and $g_{\text{cl}}(2\ell, 3) < g_{\text{cl}}(\ell, 1) + g_{\text{cl}}(\ell, -3)$. We will next examine this model in some detail. We will argue that the flow (4.37) is not possible in CFT if we interpret the right-hand side of (4.37) as a pair of free field orbifolds.

Let $n = 2\ell = 3m + 2$ and $p = 3$; the continued fraction is particularly simple in this case,

$$
\frac{n}{3} = [m + 1, 3].
$$

The generators of the chiral ring are $W_1 = X_1$ and $W_2 = X_{m+1}$. They have R-charges

$$
R_1 = 4/n = 2/\ell \quad \text{and} \quad R_2 = (m + 2)/n,
$$

and satisfy the relations $W_1^{m+1} = W_2^3 = 0$. The chiral GSO projection takes $W_1 \rightarrow W_1, W_2 \rightarrow -W_2$. The chiral ring consists of three bands:

- **band 1**: $W_1, W_1^2, \ldots, W_1^m$
- **band 2**: $W_2, W_1W_2, \ldots, W_1^mW_2$
- **band 3**: $W_2^2, W_1W_2^2, \ldots, W_1^{m-1}W_2^2$.

Only band 2 survives the chiral GSO projection. As a side remark we note that following our previous analysis it is easy to understand the perturbations

$$
\delta \mathcal{L} = \int d^2\theta \left( \lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_1 W_2 \right).
$$

Turning on $\lambda_1$ takes

$$
[m + 1, 3] \rightarrow [3];
$$

Perturbing by $\lambda_2$ takes

$$
[m + 1, 3] \rightarrow [m + 1];
$$

35
\(\lambda_3\) takes
\[
[m + 1, 3] \to [m + 2] \oplus [4]. \tag{4.44}
\]

Due to the transformation properties of \(W_j\) under the chiral GSO projection, the flow (4.42) can only be realized in the type 0 theory, while the other two flows exist in the type II theory as well.

The marginal perturbation studied in [3] corresponds to
\[
\delta\mathcal{L} = \lambda \int d^2\theta W_2 W_1^m \tag{4.45}
\]

It does not seem to fit into the class of perturbations discussed above, but it can be brought to that form by a sequence of blow ups of \(\mathbb{P}^1\)'s with self intersection \(-1\). Such blow ups are described by equation (4.34). The first blow up gives
\[
[m + 1, 3] \to [m + 2, 1, 4] \tag{4.46}
\]

where on the second line we have written the generators in the chiral ring corresponding to the different entries in the continued fraction, including the \(-1\) curve that is being blown up (here \(w_i = W_i\)). Successive blow ups yield
\[
[m + 2, 1, 4] \to [m + 3, 1, 2, 4] \to \cdots \to [m + j, 1, 2, \ldots, 2, 4] \tag{4.47}
\]

where
\[
[W_1, W_2] \to [W_1, W_1^2 W_2, W_1 W_2^2, W_2] \to \cdots \to [W_1, W_1^{j-1} W_2, W_1^{j-2} W_2, \ldots, W_1 W_2, W_2].
\]

Geometrically, after blowing up the \(j - 1\) \(\mathbb{P}^1\)'s in (4.47) one can ask what happens when the radius of the \(\mathbb{P}^1\) corresponding to \(V = W_1^{j-1} W_2\) (say) is sent to infinity. Following our previous analysis, one would say that the geometry splits into the pair of Hirzebruch-Jung singularities whose continued fractions are
\[
[m + 1, 3] \to [m + j, 1, 2, \ldots, 2, 4] \to [m + j] \oplus [2, \ldots, 2, 4], \tag{4.48}
\]

which we recognize as the pair \(\mathfrak{F}^2/\mathbb{Z}_{m+j(1)} \oplus \mathfrak{F}^2/\mathbb{Z}_{3j-2(-3)}\). In particular, for \(j = \frac{m}{2} + 1\) one recovers the result of [3], (4.37).

In the CFT, the blow up described above is supposed to correspond to perturbing the \(\mathfrak{F}^2/\mathbb{Z}_{2\ell(3)}\) CFT by
\[
\delta\mathcal{L} = \lambda_{j-1} \int d^2\theta W_2 W_1^{j-1} \tag{4.49}
\]
with \( j = 3, 4, \cdots, \frac{m}{2} + 1 \). For \( j \leq \frac{m}{2} \) the perturbation is relevant, while for \( j = \frac{m}{2} + 1 \) it reduces to \((4.43)\) and is marginal. For large enough \( j \), the process \((4.48)\) violates the \( g_{\text{cl}} \)-conjecture. We would next like to argue that it does not occur in CFT.

Consider first the case where \( \lambda_{j-1} \) is a relevant perturbation. In order to analyze the perturbation \((4.49)\) one can proceed as indicated in \((4.48)\). One first blows up by a small amount \( j - 2 \) cycles, and by a much larger amount the cycle corresponding to \((4.49)\), and then flows to the IR on the worldsheet (thereby sending \( \lambda_{j-1} \) and the radius of the cycle to infinity). In order to focus on the perturbation \((4.49)\) one would like to send the radii of the other \( j - 2 \) cycles to zero.

The \( j - 2 \) blowing up parameters correspond to coefficients in the Lagrangian of \( W_1 W_2, W_1^2 W_2, \cdots, W_1^{j-2} W_2 \). These operators are relevant, and thus their coefficients in the Lagrangian can really be thought of as a series of (energy) scales \( \mu_1, \cdots, \mu_{j-2} \). The coupling \( \lambda_{j-1} \) also gives rise to a scale, \( \mu_{j-1} \), and one can study the situation in which \( \mu_{j-1} \gg \mu_k, k = 1, 2, \cdots, j - 2 \). The original problem, of analyzing \((4.43)\) corresponds to sending \( \mu_k \to 0 \).

From the point of view of the geometric analysis \((4.48)\), the limit \( \mu_k \to 0 \) is singular. For the transition

\[
\mathbb{C}^2/\mathbb{Z}_{2l(3)} \to \mathbb{C}^2/\mathbb{Z}_{m+j(1)} \oplus \mathbb{C}^2/\mathbb{Z}_{3j-2(-3)} \tag{4.50}
\]

to occur, it was important that the parameters \( \mu_k \) are kept finite. But if the \( \mu_k \) are finite, as the system evolves to the IR it will eventually probe the region \( E \ll \mu_k \), where the geometry is no longer described by the r.h.s.’s of equations \((4.48)\), \((4.50)\). If we try to take the scales \( \mu_k \) to zero, the geometric analysis seems to suggest that we find a singular CFT. Therefore, it is not clear to us that there is a limit in which the CFT realizes the flow \((4.50)\).

We next turn to the case where \( \lambda_{j-1} \) is a marginal operator, \((4.45)\). The discussion here is similar to the above, except \( \lambda \) is now a dimensionless parameter. In order to arrive at the flow \((4.37)\) by using \((4.48)\) one again has to blow up \( m/2 \) \( \mathbb{P}^1 \)'s which correspond in the CFT to the relevant operators \( W_2, W_1 W_2, \cdots, W_1^{j-2} W_2 \). This introduces \( m/2 \) scales \( \mu_k, k = 1, 2, \cdots, m/2 \). If one attempts to send the modulus \( \lambda \to \infty \) while sending the scales \( \mu_k \to 0 \), one again seems to find a singular CFT, while if one keeps the scales \( \mu_k \) finite, the RG flow eventually makes them very large and the analysis that led to \((4.37)\) breaks down.
One could also analyze this example in the following way. We first consider the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. This theory has a twisted marginal operator; perturbing by it gives an Eguchi-Hanson space, $T^*\mathbb{P}^1$, with the modulus $\lambda$ controlling the size of the $\mathbb{P}^1$. Choose complex coordinates $x_\pm$ on the northern and southern hemispheres of $\mathbb{P}^1$. Cotangent vectors are parametrized by $p_+dx_+ = p_-dx_-$, so the transition functions are

$$
\begin{align*}
x_+ &= 1/x_- \\
p_+ &= -p_-x_-^2.
\end{align*}
$$

The coordinates $(Z^1, Z^2)$ of $\mathbb{C}^2/\mathbb{Z}_2$ are related to those of the resolved space $T^*\mathbb{P}^1$ via

$$
\begin{align*}
x_+ &= Z_1/Z_2 & p_+ &= Z_2^2 \\
x_- &= Z_2/Z_1 & p_- &= -Z_1^2.
\end{align*}
$$

We can then act with $\mathbb{Z}_\ell$ on the resolved space in the way induced by this map from the action of $\mathbb{Z}_{2\ell}(3)$ on $\mathbb{C}^2$. Since $\mathbb{C}^2/\mathbb{Z}_{2\ell} = (\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_\ell$ (for $\ell \in 2\mathbb{Z} + 1$), the resulting space is the same as one would get by quotienting by the full $\mathbb{Z}_{2\ell}$, and then resolving. In the limit of large radius of the $\mathbb{P}^1$ one finds in this way two “daughter” singularities at the north and south poles of the $\mathbb{P}^1$ which look locally like $\mathbb{C}^2/\mathbb{Z}_\ell(1)$ and $\mathbb{C}^2/\mathbb{Z}_\ell(-3)$. Assuming that these singularities are described by standard free field orbifolds leads to (4.37).

As mentioned above, we believe that in fact the daughter singularities do not correspond to standard orbifold CFT’s. Indeed, our results on the relation of the Hirzebruch-Jung resolution to the chiral rings of the orbifolds strongly suggests the following picture.

The daughter conformal field theories $D_i$, $i = 1, 2$ are linear sigma models with target space metric which is the flat metric on $\mathbb{C}^2/\mathbb{Z}_\ell(r_i)$, with $r_1 = 1$, $r_2 = -3$. However to define the theories one must also specify the $B$-fields. It is likely that the $B$-fields for the daughter conformal field theories $D_i$ differ from those of the standard free field orbifolds $\mathbb{C}^2/\mathbb{Z}_\ell(r_i)$. In particular, if the $B$-fields are such that one or both of the daughters corresponds to a singular CFT, the decoupling between the North and South poles might break down due to the appearance of a throat [15]. In addition, the strong coupling region in the throat would invalidate the CFT analysis.

From the discussion of section 4.1 it is in fact natural to expect that the marginal deformation (4.45) leads to a singular CFT. The reason is that as we have seen in section 4.1.2, to reach a non-singular CFT one needs to blow up the $\mathbb{P}^1$’s (4.48), but in the flow (4.45) these parameters are tuned to zero.
So where is the decoupled theory on the r.h.s. of (4.37) in the space of 2d field theories? It is obtained by adding half a unit of $B$ flux on the $\frac{m}{2} - 1$ collapsed cycles at the north and south poles of $T^*\mathbb{P}^1$, and then sending $\lambda$ to infinity keeping the $B$ flux fixed. The limiting theory cannot be reached from any finite point on the marginal line by an RG flow, for the reasons given below equation (4.50).

It would be very interesting to test this picture more thoroughly. Some tools which might prove useful in doing this include the gauged linear sigma model of [50] and the elliptic genus [51].

To summarize, we believe that the geometrical transition (4.37) might not be possible in CFT, in agreement with the $g_{ct}$-conjecture.

4.3.5. The general structure

While most of the discussion in this subsection was done in special cases, it is clear that the structure generalizes for all $p$. We will leave a detailed description of the general structure to another publication [48] and restrict here to a few brief comments.

The chiral ring, which always contains $n - 1$ twisted sector operators, splits in general into $|p|$ bands, labeled by $[|p|j/n]$. For large $n$ and small positive $p$, the structure is similar to that described above for the cases $p = 1$, and $p = 3$. There is a single generator of the ring in the first band of the chiral ring, with the rest of the operators in this band being powers of this generator. Similarly, in higher bands there is at most one new generator per band.

For small negative $p$, the structure is like that described above for $p = -1$, and $p = -3$. All the operators in the first band are generators, and there are sometimes generators in higher bands as well. The structure of the chiral ring in general is best described in terms of toric geometry (section 4.1).

5. Fivebranes and Liouville flows

As described in section 4, twisted sector RG flows in $\mathbb{C}^2/\mathbb{Z}_n$ orbifold models are related to the dynamics of NS5-branes on a small transverse circle. The physics of fivebranes on a large (or non-compact) transverse space is rather different, and thus provides additional examples of the phenomena explored in this paper. In this section we briefly discuss these examples.
We start with a discussion of the supersymmetric system of $n$ parallel NS5-branes. We take the fivebranes to be localized on an $\mathbb{R}^4$ labeled by $(x^6, x^7, x^8, x^9)$, and extended in the remaining $5 + 1$ directions. This system has no infrared instabilities (tachyons), but just as in section 4 we can study its behavior as a function of the moduli, which are the locations of the fivebranes in $\mathbb{R}^4$.

As a representative example, consider a system of $n$ coincident fivebranes, and turn on moduli that correspond to moving $n - n'$ fivebranes to different locations in $\mathbb{R}^4$, which are then sent to infinity. Following the discussion of the previous sections, one expects the number of degrees of freedom localized on the fivebranes to decrease in the process.

Systems of coincident fivebranes are in general outside of the realm of applicability of our analysis. Indeed, it is well known [15] that they develop a throat with a linear dilaton along it, and the string coupling diverges as one approaches the fivebranes. The high energy density of states of the system is nevertheless known, from thermodynamic considerations. It grows as

$$\rho(E) \sim E^\alpha \exp(\beta_H E)$$

where $\beta_H = 2\pi \sqrt{n\alpha'}$, and $\alpha$ is a known constant [52]. Thus, the spectrum of states exhibits Hagedorn growth, with a Hagedorn temperature determined by the number of fivebranes. It should be emphasized that most of the states contributing to $\rho(E)$ are expected to be non-perturbative. Also, $E$ here is the energy in spacetime, while in previous formulae, such as (2.12), it was energy on the worldsheet.

What happens when the fivebranes are separated? For finite separations, it is clear that at energies much higher than the mass of the W-bosons of the broken gauge symmetry on the fivebranes [12], the system is insensitive to the existence of the separation, and the density of states is the same as at the origin of the Coulomb branch (5.1). However, if we first send $n - n'$ fivebranes to infinity (and thus take the corresponding $m_W \to \infty$), the effective number of fivebranes decreases from $n$ to $n'$, and the corresponding density of states is now given by (5.1) but with a smaller $\beta_H = 2\pi \sqrt{n'\alpha'}$. Thus, we conclude that it is indeed true for fivebranes that “flows” in which some moduli are sent to infinity decrease the number of states of the system, as one would expect.

In order to connect to the discussion of the previous sections, we would like to construct a fivebrane configuration which can be described by weakly coupled string theory. It is

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12 We are discussing the type IIB fivebrane case, for concreteness. A similar analysis holds for the IIA case.
known how to do that by going to the Coulomb branch of the fivebrane theory. It was shown in [39,16,53] that the near-horizon geometry of a system of $n$ fivebranes symmetrically arranged on a circle of radius $r_0$ in $\mathbb{R}^4$ is
\[
\mathbb{R}^{5,1} \times \left( \frac{SL(2)_n}{U(1)} \times \frac{SU(2)_n}{U(1)} \right) / \mathbb{Z}_n
\] (5.2)
The radius of the circle on which the fivebranes are arranged, $r_0$, determines the string coupling at the tip of the cigar $SL(2)/U(1)$. As $r_0 \to \infty$, the string coupling at the tip of the cigar goes to zero.

The geometry (5.2) provides a holographically dual description of the dynamics of the fivebranes, also known as Little String Theory [54,16]. Normalizable states in the geometry (5.2) correspond to states in LST; non-normalizable states give rise to off-shell observables. Actually, in the geometry (5.2) one finds three classes of vertex operators:

(1) Normalizable states living near the tip of the cigar, corresponding to principal discrete series representations of $SL(2)$.

(2) Delta function normalizable states living in the bulk of the cigar (5.2). These correspond to the principal continuous series representations of $SL(2)$. They form a continuum above a finite energy gap, and are in a sense intermediate between states bound to the fivebranes and states that propagate in the full ten dimensional space-time far from the branes.

(3) Non-normalizable vertex operators, whose wavefunctions are supported in the weak coupling region far from the tip of the cigar. As mentioned above, these correspond to off-shell observables.

Since string theory in the background (5.2) can be made arbitrarily weakly coupled by tuning the string coupling at the tip of the cigar, or equivalently the radius of the circle on which the fivebranes are placed, one can study this system using our techniques.

To evaluate the density of localized states, one simply computes the torus partition sum in the background (5.2). It is not difficult to show that the high energy density of states grows like (1.1) with
\[
c_{\text{eff}} = 6 - \frac{6}{n}
\] (5.3)
This follows from the fact that the cigar CFT has $c_{\text{eff}} = 3$ (this is analogous to the fact that Liouville CFT has $c_{\text{eff}} = 1$ for all values of the central charge [4]), while the $SU(2)_n/U(1)$ CFT is an $N = 2$ minimal model with $c_{\text{eff}} = c = 3 - (6/n)$.

Note also that $c_{\text{eff}}$ is additive for products of CFT’s, and that the $\mathbb{Z}_n$ orbifold in (5.2) cannot change it (it only influences the prefactor $g$).
Marginal flows that separate the fivebranes in the plane containing the circle which defines the original configuration are particularly simple to study using this approach. The $SU(2)/U(1)$ factor in the background (5.2) can be represented by an $N = 2$ Landau-Ginzburg system for a superfield $\chi$, with a superpotential

$$W = \chi^n$$

(5.4)

The cigar CFT $SL(2)/U(1)$ is equivalent [16] to $N = 2$ Liouville, a chiral superfield $\Phi$ with the superpotential

$$W = \mu e^{-\frac{1}{2}\Phi}$$

(5.5)

where $Q^2 = 2/n$ is the linear dilaton slope in the Liouville direction $\phi = \text{Re} \Phi$, and $\mu$ determines the string coupling at the tip of the cigar (and thus is related to the radius of the circle in the original fivebrane geometry). The moduli space of fivebranes is explored by deforming the superpotential (5.4), (5.5) to

$$W = \chi^n + \mu e^{-\frac{1}{2}\Phi} + \sum_{j=2}^{n-1} \lambda_j \chi^{n-j} e^{-\frac{Q}{2} j \Phi}$$

(5.6)

Note that all terms in (5.6) are invariant under the $\mathbb{Z}_n$ symmetry in (5.2), which acts as

$$\chi \to e^{\frac{2\pi i}{n}} \chi; \quad \Phi \to \Phi - 2\pi i Q; \quad \theta \to e^{2\pi i} \theta; \quad \bar{\theta} \to \bar{\theta}$$

(5.7)

The parameters $\lambda_j$ in (5.6) determine the locations of the fivebranes. It is important to emphasize that unlike the formally similar situation studied in section 3 (see e.g. equation (3.9)), here $\lambda_j$ are exactly marginal perturbations, reflecting the fact that parallel fivebranes do not exert a net force on each other.

The main features of the moduli space can be read directly from (5.6), as in [28,29]. For finite $\{\lambda_j\}$, the large $\chi$ behavior of the superpotential is dominated by the $\chi^n$ term. This implies that the high energy density of states is governed by (5.3) for all finite $\{\lambda_j\}$. If some or all of the $\{\lambda_j\}$ are taken to infinity, the asymptotic behavior of the potential changes, and $c_{\text{eff}}$ decreases accordingly. For example, if $\lambda_{n-n'}$ is sent to infinity ($1 \leq n' \leq n - 2$) while keeping all other $\lambda_j = 0$, the system decays into one with $n$ replaced by $n'$. This is the Landau-Ginzburg description of the flow in which $n-n'$ fivebranes are sent to infinity.

A few comments about the preceding discussion are in order here:
In analyzing the behavior of (5.6), we focused on the dynamics of \( \chi \), since in systems of \((N = 2)\) matter coupled to \((N = 2)\) Liouville, the RG flow of the matter system is exhibited by the full system as evolution as a function of \( \phi (= \text{Re } \Phi) \) [10].

The behavior of (5.6) as a function of the moduli \( \{\lambda_j\} \) illustrates a phenomenon we encountered in section 4. In the discussion of \( \mathbb{C}^2/\mathbb{Z}_n \) regular orbifolds we argued that the high energy density of states \( (c_{\text{eff}} \text{ and } g_{\text{cl}}) \) does not change along moduli spaces associated with localized perturbations, but \textit{can} decrease (in that case it was only \( g_{\text{cl}} \) that changed) if we go an infinite distance in moduli space. The fivebrane system (5.6) provides an explicit demonstration of this: for finite values of the moduli \( \{\lambda_j\} \), the high energy density of states remains unchanged, while when some of the \( \{\lambda_j\} \) become infinite, \( c_{\text{eff}} \) decreases.

It was seen in section 4 that for \( n \) fivebranes arranged in two widely separated groups of \( n_1 \) and \( n_2 \) branes, \( g_{\text{cl}} \) was \textit{not} approximately equal to \( g_{\text{cl}}(n_1) + g_{\text{cl}}(n_2) \). While this was found for fivebranes in the regular orbifold limit (4.17), we see the same phenomenon here. In the unperturbed system (5.6) with \( \{\lambda_j = 0\} \), the density of perturbative localized states is independent of \( \mu \), which determines the radius \( r_0 \) of the circle that the \( n \) fivebranes live on. What changes as \( r_0 \to \infty \) is the coupling between these states (which goes to zero). It is natural that the same happens for \( \mathbb{C}^2/\mathbb{Z}_n \) orbifolds: as we tune moduli, \( g_{\text{cl}} \) does not change, but as the moduli are sent to infinity the couplings of some of the states to others go to zero.

As mentioned in the beginning of this section, the physics of fivebranes in a large transverse space is quite different from that of the regular orbifolds \( \mathbb{C}^2/\mathbb{Z}_n \), which correspond to fivebranes on a small transverse circle. In particular, while for \( \mathbb{C}^2/\mathbb{Z}_n \) one has \( c_{\text{eff}} = 6 \), for fivebranes it is given by (5.3) \textit{(i.e. it is smaller)}. A related fact is that in flows between regular orbifolds, \( c_{\text{eff}} \) remains fixed, and \( g_{\text{cl}} \) is changing, while flows between fivebrane systems typically involve a change in \( c_{\text{eff}} \). One can ask why all this is not in contradiction with the "\( g_{\text{cl}} \)-conjecture", which presumably implies that both \( c_{\text{eff}} \) and \( g_{\text{cl}} \) are constant along marginal deformations – after all, \( \mathbb{C}^2/\mathbb{Z}_n \) and (5.2) are related by adjusting moduli. A possible answer is that to go from \( \mathbb{C}^2/\mathbb{Z}_n \) to (5.2) one has to: (a) change the value of an untwisted modulus (the size of the circle the fivebranes live on); (b) go an infinite distance in moduli space. Both of these things are in general expected to lead to large changes in the high energy density of states.

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So far we have discussed the description of the supersymmetric system of near-coincident parallel fivebranes. As for non-singular orbifolds (2.1), one expects to find a rich set of non-supersymmetric fivebrane configurations with localized tachyons, whose condensation has similar effects to those described in previous sections. In particular, we argued in section 4 that it should be possible to flow from regular orbifolds to geometries with throats.

We will not attempt a comprehensive discussion of non-supersymmetric throat geometries here, but will give one class of constructions where tachyon condensation seems to connect non-supersymmetric CHS type geometries to supersymmetric ones.

Consider the background

\[
\mathbb{R}^{5,1} \times \left( \frac{SL(2)^n \times SU(2)^n}{U(1) \times U(1)} \right) / \mathbb{Z}_{n'}
\]  

(5.8)

where we assume that the integers \(n, n'\) are related via:

\[
n = n'(2l + 1); \quad l \in \mathbb{Z}
\]  

(5.9)

The unperturbed superpotential describing this system is

\[
W = \chi^{n'(2l+1)} + \mu e^{-\frac{1}{2\lambda} \Phi}
\]  

(5.10)

The new phenomenon that occurs here compared to (5.6) is that there exist chiral relevant operators that survive the (chiral) GSO projection. They correspond to deformations of the superpotential of the form

\[
\delta W = \sum_{j=0}^{l-1} \lambda_j \chi^{n'(2j+1)}
\]  

(5.11)

The analysis of the perturbed system is elementary in this case, since it only involves a relevant deformation of the \(N = 2\) minimal model. One finds a cascade of the form discussed in sections 3,4, with a stable endpoint corresponding to \(\delta W = \lambda_0 \chi^{n'}\). As \(\lambda_0 \to \infty\) in the IR, the system approaches that describing \(n'\) NS5-branes arranged on a circle; this configuration is stable and does not decay further.

This provides an example of a flow from a non-supersymmetric to a supersymmetric fivebrane system. It is presumably easy to construct many other such flows.

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6. Discussion

The two main results we have found are the following. First, we have defined a quantity $g_{cl}$ in theories containing localized closed string tachyons which appears to decrease along RG flows that leave $c_{\text{eff}}$ unchanged. Second, we have shown that $N = 2$ world-sheet supersymmetry and its associated chiral ring is a powerful tool for studying tachyon condensation in one class of these theories, namely non-supersymmetric orbifolds with localized tachyons. In particular, as in supersymmetric orbifolds, the world-sheet CFT captures all the geometrical information of the resolved geometry. This is perhaps not surprising since the CFT is non-singular and thus provides in some sense a smoothed out version of the geometry, but it is striking how direct the correspondence is.

There are a number of points which should be better understood in what we have done. We conjecture that $g_{cl}$ decreases along RG flow and have checked this in many examples, but we have no proof of this result. A proof of the analogous result in open string theory required an off-shell definition of $g_{\text{op}}$ and the techniques of BSFT [6]. It would be very interesting to see if similar methods could be used to define an off-shell version of $g_{cl}$ and a corresponding string field theory of the localized closed string states. The possibility that $g_{cl}$ should be related to an action for localized closed string states also has clear implications for recent work on closed string tachyons and black hole entropy [55].

We also noted earlier that the formula for $g_{cl}$ in orbifold CFT is mathematically the same as the formula for the $\eta$ invariant for a quotient of $\mathbb{C}^d$. If this connection can be made more precise it would help to explain some of the properties of $g_{cl}$. For example it would provide an alternative explanation of the fact that $g_{cl}$ does not change under localized perturbations that do not change the geometry at infinity. The combinatorics of the Hirzebruch-Jung resolution also encode the surgery data for constructing the link of the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ singularity. We expect that there are several interesting relations of $g_{cl}$ to 3-manifold invariants and Seiberg-Witten theory.

The relation between geometry and the chiral ring also needs to be understood in more detail. In supersymmetric orbifolds, the chiral ring can be identified with the (quantum) cohomology ring of the resolved geometry. In the non-supersymmetric theories we are considering, the chiral ring clearly cannot be identified directly with the ordinary cohomology of the resolution, because the grading given by the $U(1)_R$ charge is not integer, and also because some of the relations are incompatible with ordinary cohomology.
The chiral ring clearly defines some interesting generalization of cohomology. In the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifold the number of generators in the cohomology defined by the chiral ring is the length of the continued fraction $n/p$ (or $n/(n+p)$ for $p < 0$) defined in section 4. The appropriate $K$-theory lattice is expected to be of rank $n$. It would be interesting to learn if there is some generalization of the McKay correspondence.

A related issue, raised in [3,2], involves the fate of the fractional branes that carry these $K$-theory charges, and twisted sector RR gauge fields that measure those charges, after the process of tachyon condensation. Similar issues arise in studies of tachyon condensation in open string theory; it may be that some of these questions can be addressed more fruitfully in the present context.

Indeed, an interesting issue is the interplay of open and closed string dynamics in the presence of a closed string tachyon. A concrete step in this direction is the calculation (in appendix B) of the leading corrections to the open string dynamics due to an expectation value of a twisted sector closed string tachyon. One may imagine that at finite string coupling, the presence of D-branes may significantly modify the course of tachyon condensation; since the tachyon condensate has the effect of blowing up a cycle, and D-brane tension tries to shrink a cycle, there should be interesting mechanisms for stabilizing cycles due to the competition between these two effects.

This work also suggests a number of other directions for future research. It would be interesting to generalize our results to higher dimensional orbifolds and to orbifolds of the heterotic string. It would also be interesting to study other systems with localized closed string tachyons. These include orbifolds of $AdS_3$ [3], and "fluxbrane" solutions in string theory [56,57,58,59,17,60,61,19,20,21].

Acknowledgements: We would like to thank A. Adams, J. Distler, M. Douglas, E. Diaconescu, D. Friedan, J. Maldacena, D. Morrison, J. Polchinski, N. Seiberg, E. Silverstein, E. Witten and A. B. Zamolodchikov for useful discussions. The support and hospitality of the Aspen Center for Physics during the initial stages of this work is gratefully appreciated. D.K. thanks the Rutgers NHETC for hospitality during part of this work. In addition, the work of D.K. and E.M. is supported in part by DOE grant DE-FG02-90ER40560; J.H. is supported in part by NSF grant PHY-9901194; and the work of G.M. is supported in part by DOE grant DE-FG02-96ER40949.
Appendix A. Evaluation of $g_{\text{cl}}$ for some orbifolds

In this appendix we evaluate certain sums that are needed for computing $g_{\text{cl}}$ for some of the orbifold CFT’s discussed in the text. In particular, we focus on the case of $\mathbb{C}^m/\mathbb{Z}_n$ orbifolds where $\mathbb{Z}_n$ rotates the superfields by $g \cdot Z^i \sim \omega^p_i Z^i$ (with $\omega = e^{2\pi i/n}$) and we take the diagonal modular invariant (and spin structure sum). The partition sum of the twisted sector states, with states weighted by one is

$$Z_{\text{tw}}(\tau, \bar{\tau}) = \frac{1}{2n} \sum_{t=0}^{2n-1} \prod_{i=1}^{m} \left| \theta^t \left( \frac{1}{2} + p_{si}/n \right)(0|\tau) \right|^2 .$$  \hspace{1cm} (A.1)

Making a modular transformation to $\tau' = -1/\tau$, the leading terms in the $q'$ expansion comes from the terms with $t = \epsilon_2 = 0$. These give

$$g(p_1, \ldots, p_m; n) = \frac{1}{n} \sum_{s=1}^{n-1} \prod_{i=1}^{m} \left( \frac{2 \sin \pi p_{si}/n}{2 \sin \pi s/n} \right)^2 .$$  \hspace{1cm} (A.2)

Curiously, these expressions turn out to be rational functions of $n$. In fact, they can be evaluated explicitly by considering contour integrals of the function

$$f(z) = \cot(\pi n z) \prod_{i=1}^{m} \frac{1}{\sin^2(\pi p_i z)}$$  \hspace{1cm} (A.3)

We will illustrate the method in some special cases.

In all cases we consider contour integrals around the contour given by $C_1 - C_2 - C_3 + C_4$ where $C_1$ runs along $1 - \epsilon + iy$, $-\Lambda \leq y \leq \Lambda$, $C_2$ runs along $x + i\Lambda$, $-\epsilon \leq x \leq 1 - \epsilon$, $C_3$ runs along $-\epsilon + iy$, $-\Lambda \leq y \leq \Lambda$, and $C_4$ runs along $x - i\Lambda$, $-\epsilon \leq x \leq 1 - \epsilon$. Here $\epsilon < 1/n$.

Let us first consider the case $p_1 = \cdots = p_m = 1$. Simple contour integration gives:

$$g_m(n) := g(1,1,\ldots,1; n) = \frac{1}{n} \sum_{s=1}^{n-1} \frac{1}{(2 \sin \pi s/n)^{2m}} = -\text{Res}_{x=0} \frac{\cot nx}{(2 \sin x)^{2m}} .$$  \hspace{1cm} (A.4)

With this explicit formula we can prove that $g_m(n)$ is a strictly increasing function of $n$ for $n \geq 1$ and fixed $m$, and thus verify the $g_{\text{cl}}$-conjecture for these special $C^m/\mathbb{Z}_n$ orbifolds as follows. From (A.4) it follows that $g_m(n)$ is of the form $P_m(n)/n$ for some polynomial $P_m$ of order $2m$. The polynomial $P_m(n)$ has a root for $n = 1$. Physically, this is because $\mathbb{C}^m/\mathbb{Z}_n$ has no localized states for $n = 1$. Mathematically, it follows since

$$-2m \cot x (\sin x)^{-2m} = \frac{d}{dx} (\sin x)^{-2m} .$$

We now use the expansions

$$\cot x = \frac{1}{x} \left( 1 - \sum_{j=1}^{\infty} \frac{2^{2j} |B_{2j}| x^{2j}}{(2j)!} \right)$$  \hspace{1cm} (A.5)
\[
\frac{1}{\sin^2 x} = \frac{1}{x^2} \left( 1 + \sum_{j=1}^{\infty} \frac{2^{2j}|B_{2j}|}{(2j)!} (2j-1)x^{2j} \right)
\]  
(A.6)

and subtract the expression in (A.4) from its value at \( n = 1 \) to write:

\[
2^{2m} P_m(n) = \text{Res}_{x=0} \left\{ \frac{1}{x^{2m+1}} \left( \sum_{j=1}^{\infty} \frac{|B_{2j}|}{(2j)!} n^{2j-1}(2x)^{2j} \right) \left( 1 + \sum_{j=1}^{\infty} \frac{(2j-1)|B_{2j}|}{(2j)!} (2x)^{2j} \right)^m \right\} .
\]  
(A.7)

In particular we have

\[
g_1(n) = \frac{1}{12} \left( n - \frac{1}{n} \right)
\]  
(A.8)

\[
g_2(n) = \frac{(n^2 + 11)(n^2 - 1)}{45 \cdot 16n} .
\]

In general we can note that all the coefficients in the series expansions in \( x \) which appear in (A.7) are positive. Thus

\[
g_m(n) = (n - \frac{1}{n})(A_{2m-2}n^{2m-2} + \cdots + A_0),
\]  
(A.9)

where \( A_i \) are positive rational numbers. It follows that \( g_m(n) \) is strictly increasing for \( n \geq 1 \).

In the case when the \( p_i \) are not all equal we must work harder. We will illustrate the method for the case \( m = 2 \) studied in this paper.

For the case \( p = \pm 3 \) used in the text the method used below gives

\[
g(1, 3; n) = \frac{(n^4 + 210n^2 - 80n - 291)}{405 \cdot 16n} 
\begin{align*}
&= \frac{(n^4 + 210n^2 + 80n - 291)}{405 \cdot 16n} & n = 1 \mod 3, \\
&= \frac{(n^4 + 210n^2 - 80n - 291)}{405 \cdot 16n} & n = 2 \mod 3.
\end{align*}
\]  
(A.10)

The answer for \( g(1, p; n) \) for general \( n, p \) is expressed in terms of the continued fraction expansion \( n/p = [a_1, \ldots, a_r] \). Define

\[
\frac{q_i}{q_{i+1}} = [a_i, a_{i+1}, \ldots, a_r] = a_i - \frac{q_{i+2}}{q_{i+1}}
\]  
(A.11)

where \( n = q_1 > q_2 > q_3 > \cdots > q_r = a_r > q_{r+1} = 1 \).

Then the general formula is

\[
g(1, p; n) = n \left[ \sum_{s=1}^{r} H\left( \frac{q_s}{q_{s+1}} \right) + \sum_{s=1}^{r-1} K\left( \frac{q_s}{q_{s+1}} \right) \right]
\]  
(A.12)
Here $H\left(\frac{n}{p}\right)$ is defined for fractions in lowest terms with

$$H\left(\frac{n}{p}\right) = \frac{1}{720n^2p^2} \left(n^4 + 5n^2p^2 - 3p^4 + 5n^2 - 5p^2 - 3\right)$$  \hspace{1cm} (A.13)

while $K\left(\frac{n}{p}\right)$ is defined for fractions in lowest terms with $p > 1$ as

$$K\left(\frac{n}{p}\right) := \frac{1}{np} \left[f(q_2, q_3) + f(q_3, q_4) + \cdots + f(q_{r-1}, q_r) + f(q_r, 1)\right]$$

$$f(j, k) := -\frac{1}{360jk} (j^4 - 5j^2k^2 + k^4 + 3)$$  \hspace{1cm} (A.14)

Let us briefly indicate the proof of this result. Evaluating residues we get:

$$0 = \text{Res}_{z=0} \left(\frac{\cot(\pi nz)}{\sin^2(\pi z) \sin^2(\pi pz)}\right) + \frac{1}{\pi} g(1, p; n) + \sum_{t=1}^{p-1} \text{Res}_{z=t/p} \left(\frac{\cot(\pi nz)}{\sin^2(\pi z) \sin^2(\pi pz)}\right)$$

From this one derives the recursion relation

$$\frac{1}{q_1} g(1, q_2; q_1) = \frac{1}{q_2} g(1, q_3; q_2) + H\left(\frac{q_1}{q_2}\right) + \frac{1}{q_1q_2} \kappa\left(\frac{q_1}{q_2}\right)$$  \hspace{1cm} (A.15)

where

$$\kappa\left(\frac{n}{p}\right) := \frac{1}{8p} \sum_{t=1}^{p-1} \frac{\cot(\pi t/p) \cot(\pi nt/p)}{\sin^2(\pi t/p)}$$  \hspace{1cm} (A.17)

This function can, in turn, be evaluated using residues and recursion relations. By evaluating residues of the function

$$\cot \pi z \cot(\pi nz) \cot(\pi pz) / \sin^2(\pi z)$$

along the same contour as above we produce a nice reciprocity formula

$$\kappa\left(\frac{n}{p}\right) + \kappa\left(\frac{p}{n}\right) + f(n, p) = 0,$$  \hspace{1cm} (A.18)

where

$$f(n, p) = -\frac{1}{360} \left(n^4 - 5n^2p^2 + p^4 + 3\right).$$  \hspace{1cm} (A.19)
Equation (A.18) is only valid for $p > 1, n > 1$ and $n, p$ relatively prime. Now note that
\[ \kappa\left(\frac{n}{p}\right) = \kappa\left(\frac{q_2}{q_2}\right) = -\kappa\left(\frac{q_2}{q_2}\right). \]
Using this we can evaluate
\[ \kappa\left(\frac{n}{p}\right) = -\kappa\left(\frac{q_3}{q_2}\right) \]
\[ = \kappa\left(\frac{q_3}{q_2}\right) + f(q_2, q_3) \]
\[ = -\kappa\left(\frac{q_3}{q_2}\right) + f(q_2, q_3) \]
\[ = \kappa\left(\frac{q_3}{q_2}\right) + f(q_3, q_4) + f(q_2, q_3) \]
\[ = \ldots \]
(A.20)
The process terminates when we get to
\[ \kappa\left(\frac{q_{r-1}}{q_r}\right) = -\kappa\left(\frac{1}{q_r}\right) = f(q_r, 1) \]
(A.21)
This completes the proof.

Appendix B. Tachyon contributions to scalar potentials and gauge couplings

B.1. Results

In this appendix we generalize the computation of [31] of the couplings of twist fields to the scalars and gauge fields in a D-brane probe.

The results are as follows: In the $\mathbb{C}/\mathbb{Z}_n$ orbifold, with $n$ odd, the v.e.v.’s of the twist fields $\lambda_j$ of the $j^{th}$ twisted sector couple to the scalars in the D-brane probe Lagrangian as
\[ -8\pi \sum_{k=1}^{n} \zeta_k D_k \]
(B.1)
where
\[ \zeta_k = \text{Im}\left( \sum_{j \text{ odd}} \lambda_j e^{-2\pi i \frac{j}{n}} \right) \]
(B.2)
and
\[ D_k = |Z_{k+1, k}|^2 - |Z_{k, k-1}|^2 \]
(B.3)
while the coupling to the gauge fields is determined from
\[ \frac{1}{4g_0^2} \sum_{k=1}^{n} (1 + B_k) F^{(k)}_{\mu\nu} F^{(k), \mu\nu} \]
(B.4)
with
\[ B_k = 4\pi \text{Re}\left(\sum_{j=1}^{n-1} \lambda_j e^{-2\pi i \frac{jk}{n}}\right) \] (B.5)

Let us now consider higher-dimensional orbifolds. The computation can be repeated for each superfield, and need only be done for chiral or antichiral operators, for which the fermionic twistfield is of the form \( e^{i\phi(H-H^\dagger)} \) or \( e^{i(\phi-1)(H-H^\dagger)} \), respectively. Here \( \phi \) is the twist of the superfield under the orbifold action. In particular for the \( \mathbb{C}^2/\mathbb{Z}_n(p) \) orbifold the results for the gauge couplings, (B.4), (B.5) remain unchanged.

For the purpose of this appendix, it is useful to adopt a somewhat different convention for the perturbations than is used in the main text. The orbifolds \( \mathbb{C}^2/\mathbb{Z}_n(p) \) and \( \mathbb{C}^2/\mathbb{Z}_n(-p) \) are isomorphic as CFT’s, with (as discussed at the beginning of section 4) the isomorphism being the change of complex structure \( \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^* \). One could then consider (say) only \( p \) positive, but allow perturbations that are chiral with respect to either one or the other of the two R-symmetries, (4.3) or

\[ J = \psi_1\psi_1^* - \psi_2\psi_2^*. \] (B.6)

We denote fields chiral with respect to (4.3) as being of type \((c_1,c_2)\), and fields chiral with respect to (B.6) as being of type \((c_1,a_2)\).

The scalar couplings depend on whether the tachyon v.e.v. is of \((c_1,c_2)\) type (or its \((a_1,a_2)\) complex conjugate) or of \((c_1,a_2)\) type (or its \((a_1,c_2)\) complex conjugate). The coupling to \((c_1,c_2)\)-type tachyons is of the form (B.1) where

\[ \zeta_k = \text{Im}\left(\sum_{(c_1,c_2)-\text{type}} T_\phi e^{-2\pi ik\phi}\right) \] (B.7)

and

\[ D_k = |Z_{k+1,k}^{(1)}|^2 - |Z_{k,k-1}^{(1)}|^2 + |Z_{k+p,k}^{(2)}|^2 - |Z_{k,k-p}^{(2)}|^2 \] (B.8)

In equation (B.7) \( \phi \) is the twist of the field \( Z^{(1)} \).

Similarly, the coupling to \((c_1,a_2)\)-type tachyons is of the form (B.1) where

\[ \zeta_k = \text{Im}\left(\sum_{(c_1,a_2)-\text{type}} T_\phi e^{-2\pi ik\phi}\right) \] (B.9)

and

\[ D_k = |Z_{k+1,k}^{(1)}|^2 - |Z_{k,k-1}^{(1)}|^2 - |Z_{k+p,k}^{(2)}|^2 + |Z_{k,k-p}^{(2)}|^2 \] (B.10)
B.2. Computation

We need to compute disk amplitudes with one twistfield at the center and two vertex operators on the boundary. It suffices to consider the case of one twisted $N = 2$ scalar superfield $Z$ in a twisted sector with $g \cdot Z = e^{2\pi i \phi} Z$. Here $\phi = \left\{ \frac{2j}{n} \right\}$ for the $j^{th}$ twisted sector of a $\mathbb{Z}_n$ orbifold when the generator of the orbifold group acts as $Z \rightarrow \omega^p Z$.

For the coupling to twisted scalars we require the vertex operators at momentum $k$:

\[
\begin{align*}
\bar{Z}(k) \otimes V(x; k) \\
Z(k) \otimes \bar{V}(x; k)
\end{align*}
\]

(B.11)

Here $Z(k)$ is a matrix valued spacetime field for the spacetime coordinate $Z$. The field $\bar{Z}$ is the Hermitian conjugate. These fields are valued in the Chan-Paton representation of the orbifold group. $x$ is a coordinate along the boundary of the upper half-plane or the unit disk. The orbifold projection is given by $\gamma Z \gamma^{-1} = \omega^p Z$. Thus (B.11) are invariant operators. In this paper we are only considering the regular representation of the orbifold group, appropriate to a D-brane probe, so the nonzero entries of $Z(k)$ are $Z_{r+p,r}$, $1 \leq r \leq n$.

Inclusion of the twistfield follows an important rule of boundary conformal field theory, first stated in [31]. In boundary conformal field theory the twistfield vertex operator $V_{\phi}(z, \bar{z})$ is not mutually local with respect to the scalar vertex operators $V(x; k)$ and $\bar{V}(x; k)$. Thus, amplitudes with a twistfield in the disk together with scalar vertex operators on the boundary are not single valued as functions of $x$. This difficulty may be corrected by accompanying the insertion of the twistfield with an insertion of the Chan-Paton group action in the combination:

\[\gamma^{-1}_{\phi} \otimes V_{\phi}\] (B.12)

where $\gamma_{\phi} Z \gamma^{-1}_{\phi} = e^{2\pi i \phi} Z$. The order of Chan-Paton matrices reflects the order of operators relative to the intersection of the cut with the boundary of the disk. With this insertion, the correlators of boundary vertex operators (in a given order) (B.11) are single-valued in the presence of the twistfield. In particular, the ordering of the boundary operators relative to the cut is irrelevant.

Similarly, the gauge fields have vertex operators $A_{\mu}(k) \otimes V_{\mu}(x; k)$ where $\gamma_{\phi} A_{\mu}(k) \gamma^{-1}_{\phi} = A_{\mu}(k)$ in the Chan-Paton representation.
The amplitude connecting the twistfield to the scalars is determined by computing the amplitude

\[
T\phi(k_1) \left[ \text{Tr}\gamma^{-1}_{\phi} \bar{Z}(k_2)Z(k_3) \int dx_2dx_3 \langle V_\phi(z,\bar{z}; \frac{1}{2}k_1)V(x_2;k_2)\bar{V}(x_3;k_3) \rangle + \\
+ \text{Tr}\gamma^{-1}_{\phi} Z(k_3)\bar{Z}(k_2) \int dx_2dx_3 \langle V_\phi(z,\bar{z}; \frac{1}{2}k_1)\bar{V}(x_2;k_3)V(x_3;k_2) \rangle \right]
\]  

Here we are using the upper-half-plane model of the disk. We partially fix the Mobius invariance by putting the twistfield at \( z = i \) and then we integrate over \(-\infty < x_2 < x_3 < \infty \). The momenta satisfy \( k_1 + k_2 + k_3 = 0 \).

Similarly, the coupling to the gauge fields is

\[
T\phi(k_1) \left[ \text{Tr}\gamma^{-1}_{\phi} A_\mu(k_2)A_\nu(k_3) \int dx_2dx_3 \langle V_\phi(z,\bar{z}; \frac{1}{2}k_1)V^\mu(x_2;k_2)V^\nu(x_3;k_3) \rangle + \\
+ \text{Tr}\gamma^{-1}_{\phi} A_\nu(k_3)A_\mu(k_2) \int dx_2dx_3 \langle V_\phi(z,\bar{z}; \frac{1}{2}k_1)V^\nu(x_2;k_3)V^\mu(x_3;k_2) \rangle \right]
\]  

In order to proceed with the computation we begin by noting that, although a priori one must include both orderings the two lines in (B.13)-(B.14) make equal contributions (due to mutual locality of (B.11) and (B.12)).

We are interested in the coupling of the twist fields \( T_\phi(k) \) at zero momentum. In nonsupersymmetric orbifolds this means that this coupling is an off-shell coupling in those sectors where \( T_\phi \) is massive or tachyonic. We will take the fields \( Z(k), A_\mu(k) \) to be on-shell, so that \( k_2^2 = k_3^2 = 0 \). Let \( \delta := k_2 \cdot k_3 = \frac{1}{2}k_1^2 \). We will compute the vertex operator correlators in the flat metric for arbitrary \( \delta \) and then analytically continue \( \delta \to 0 \). This defines our off-shell continuation. Technically, it is important to keep the momenta nonvanishing in the computation because many terms have canceling zeroes and poles in the \( \delta \to 0 \) limit.

Let us now describe the vertex operators. The gauge field vertex operators will be in the 0-picture:

\[
\epsilon_\mu V^\mu(x;k) = \epsilon_\mu \left( \partial Y^\mu - ik \cdot \psi^\mu \right) e^{ik \cdot Y(x)}
\]  

where \( Y^\mu \) are the untwisted directions along the brane probe and \( \psi^\mu \) are the \( N = 1 \) superconformal partners. These satisfy Neumann boundary conditions.

\[\text{We suppress all the ghost factors in the text below.}\]
The scalar vertex operators are (in the 0-picture):

\[ V(x; k) = (\partial Z - ik \cdot \psi e^{iH})e^{ik \cdot Y}(x) \]
\[ \bar{V}(x; k) = (\partial \bar{Z} - ik \cdot \psi e^{-iH})e^{ik \cdot Y}(x) \]  
(B.16)

If the orbifold group acts in the \(X^8 - X^9\) plane then the complex scalar is \(Z = \frac{1}{\sqrt{2}}(X^8 + iX^9)\) and the superconformal partner is bosonized via \(\psi = \frac{1}{\sqrt{2}}(\psi^8 + i\psi^9) = e^{iH}\). The relative phases in (B.16), which are crucial, are most readily determined by acting with the \(N = 1\) supercurrent which is proportional to

\[ G^- + G^+ = \partial Ze^{-iH} + \partial \bar{Z}e^{iH} + \partial Y^\mu \psi^\mu \]  
(B.17)

where

\[ G^- = \partial Ze^{-iH} + \cdots \]
\[ G^+ = \partial \bar{Z}e^{iH} + \cdots \]
\[ \bar{G}^- = \bar{\partial}Ze^{-i\bar{H}} + \cdots \]
\[ \bar{G}^+ = \bar{\partial}\bar{Z}e^{i\bar{H}} + \cdots \]  
(B.18)

The fields \(Z\) satisfy Dirichlet boundary conditions (hence \(B\)-type supersymmetry) and hence

\[ \tilde{\psi}(\bar{z}) = -\psi(\bar{z}). \]  
(B.19)

In terms of the bosonized field \(H\):

\[ \bar{H}(\bar{z}) = H(\bar{z}) + \pi. \]  
(B.20)

Finally, the twistfield vertex operator will be taken in the \((-1)\) picture

\[ V_\phi(z, \bar{z}; k) := \sigma_\phi(z, \bar{z})e^{i\phi(H - \bar{H})}e^{ik \cdot Y}(z, \bar{z}) \]  
(B.21)

for \(C\)-type twistfields and

\[ V_\phi(z, \bar{z}; k) := \sigma_\phi(z, \bar{z})e^{i(\phi - 1)(H - \bar{H})}e^{ik \cdot Y}(z, \bar{z}) \]  
(B.22)

for \(A\)-type twistfields. Here \(\sigma_\phi\) is the bosonic twistfield and we take the convention of \(\frac{1}{2}5\) that

\[ \partial Z(z)\sigma_\phi(w, \bar{w}) \sim (z - w)^{-1 - \phi} \tau(w, \bar{w}) \]  
(B.23)

---

\(^{15}\) We use the notation \(Z\) for the matrix field, the conformal field theory scalar field, and the conformal field theory \(N = 2\) superfield. Nevertheless, we hope no confusion will arise.
where $\tau$ is a single-valued field.

The main twistfield correlator we need is

$$
\langle \sigma_\phi(z, \bar{z}) \partial Z(x_2) \partial \bar{Z}(x_3) \rangle_H
$$

where $0 \leq \phi < 1$. Recall that we will put $z = i$. We choose branch cuts so that

$$
w = e^{i\theta} = \frac{x - i}{x + i}
$$

(B.25)

has argument $0 \leq \theta < 2\pi$. Similarly

$$
\langle e^{i\phi(H - \bar{H})} k_2 \cdot \psi e^{iH}(x_2) k_3 \cdot \psi e^{-iH}(x_3) \rangle
$$

is

$$
\langle e^{i\phi(H - \bar{H})} \rangle = -\frac{\delta}{x_2^2 x_3^2} \frac{\phi + (1 - \phi) x_2 - z}{x_3 - z} \phi
$$

(B.26)

We are assuming the cocycles are such that all fermions anticommute, so $e^{iH}$ anticommutes with $\psi$.

The amplitude (B.14) for gauge fields $A^r(k_2) = \epsilon_2$ and $A^r(k_3) = \epsilon_3$ and all other gauge fields zero is

$$
2 \langle V_\phi \rangle e^{i\Phi(\delta)} \delta^{r', r} e^{-2\pi ir\phi} \left[ \epsilon_2 \cdot \epsilon_3 \delta - \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \right] \frac{\Gamma(\frac{1}{2}(\delta + 1)) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} \delta + 1)}
$$

(B.27)

where $e^{i\Phi(\delta)} \to 1$ as $\delta \to 0$, and $\langle V_\phi \rangle$ is the one-point function of the twistfield on the disk. We have used the BRST conditions on the polarizations. The integral is most easily done by fixing $x_2 = 0$ and integrating $x_3$ along the real axis.

The amplitude (B.14) for scalar fields is most easily done by mapping the amplitude to the integral around the boundary of the unit disk using (B.25). The contour integral can be deformed to an integral around a cut in the $w$ plane along $[0, 1]$ using:

$$
J(a, b) := \oint d\xi \xi^a (1 - \xi)^b
$$

$$
= \int_0^{2\pi} i\xi d\theta e^{i\theta} (1 - e^{i\theta})^b
$$

(B.28)

$$
= (e^{2\pi i a} - e^{2\pi i b}) \Gamma(a + 1) \Gamma(b + 1) \frac{\Gamma(a + b + 2)}{\Gamma(a + b + 1)}
$$

(here $\xi = e^{i\theta}$ in the integrand).
The result is expressed in terms of several gamma functions. Taking the limit \( \delta \to 0 \) results in the amplitude

\[
\pm 4\pi i \langle V_\phi \rangle \text{Tr}(\gamma^{-1}_\phi \bar{Z}Z)(1 - e^{-2\pi i \phi})
\]  

where we take the + sign for \( C_\phi \) type twistfields and the - sign for \( A_\phi \)-type twistfields. Absorbing \( \langle V_\phi \rangle \) into the normalization of the twistfield \( T_\phi(k) \) leads to the D-brane probe couplings quoted above.

**Appendix C. Chiral rings for the marginal deformation**

\[ \Phi^2 / \mathbb{Z}_{12k+2(-3)} \to \Phi^2 / \mathbb{Z}_{6k+1(-2)} \oplus \Phi^2 / \mathbb{Z}_{6k+1(3k-1)} \]

In this appendix we discuss the chiral ring of

\[ \Phi^2 / \mathbb{Z}_{2\ell(-3)}; \ \ell = 6k + 1 \]  

which was discussed in section 4. We also describe the effect on the chiral ring of a certain marginal perturbation, which is a special case of those described in equation (4.36). We discuss the type 0 version of the theory.

The chiral ring of the model contains the operators

\[
\sigma^{(1)}_j e^{i\frac{4j}{2\ell}(H_1 - \bar{H}_1)} \sigma^{(2)}_j e^{i(1 - \{\frac{3j}{2\ell}\})(H_2 - \bar{H}_2)}
\]

where \( j = 1, 2, \cdots, 2\ell - 1; \{\theta\} \) is the fractional part of \( \theta \). The R-charges of the operators (C.2) are

\[
R_j = \frac{j}{2\ell} + 1 - \{\frac{3j}{2\ell}\}.
\]

Equation (C.3) describes three bands:

I : \( j = 1, 2, \cdots, 4k \); \quad R_j = 1 - \frac{j}{\ell}

II : \( j = 4k + 1, \cdots, 8k + 1 \); \quad R_j = 2 - \frac{j}{\ell}

III : \( j = 8k + 2, \cdots, 12k + 1 \); \quad R_j = 3 - \frac{j}{\ell}

The generators of the ring are all operators in the first band, \( X_j \) with \( j = 1, 2, \cdots, 4k \), as well as the last operator in the second band, \( X_{8k+1} \). One can check that they satisfy the relations (4.16) implied by the appropriate continued fraction, \([2^{4k-1}, 3, 2]\), (4.15).
The operator with \( j = \ell = 6k + 1 \) in band II is marginal, and one can ask what happens when it is turned on with a large coefficient. As shown in section 4, in this limit the CFT is expected to decompose into the decoupled factors

\[
\mathcal{C}^2/\mathbb{Z}_{2\ell(-3)} \rightarrow \mathcal{C}^2/\mathbb{Z}_{\ell(-2)} \oplus \mathcal{C}^2/\mathbb{Z}_{\ell(3k-1)},
\]

(C.5)
corresponding to the decomposition of the continued fraction

\[
[2^{4k-1}, 3, 2] \rightarrow [2^{3k-1}, 3] \oplus [3, 2^{k-2}, 3, 2].
\]

(C.6)

We next show that the chiral ring of the r.h.s. in (C.5) agrees with (C.4). Consider first the factor \( \mathcal{C}^2/\mathbb{Z}_{\ell(-2)} \) in (C.5). The chiral operators are similar to (C.2) and have R-charges

\[
\tilde{R}_j = \frac{j}{\ell} + 1 - \left\{ \frac{2j}{\ell} \right\}; \quad j = 1, 2, \ldots, \ell - 1.
\]

(C.7)

They thus arrange into two bands:

\[
\tilde{I} : j = 1, 2, \ldots, 3k; \quad \tilde{R}_j = 1 - \frac{j}{\ell}
\]

\[
\tilde{II} : j = 3k + 1, \ldots, 6k; \quad \tilde{R}_j = 2 - \frac{j}{\ell}
\]

(C.8)

One can check that the chiral operators (C.8) fill out part of the original list (C.4):

\[
\tilde{R}_j = R_j \quad \text{for } j = 1, 2, \ldots, 3k
\]

\[
\tilde{R}_j = R_{\ell+j} \quad \text{for } j = 3k + 1, 2, \ldots, 6k
\]

(C.9)

Thus, the \( \mathcal{C}^2/\mathbb{Z}_{\ell(-2)} \) factor provides the first \((\ell - 1)/2\) operators in band I and the last \((\ell - 1)/2\) operators in band III in (C.4).

Moving on to the second factor in (C.5), \( \mathcal{C}^2/\mathbb{Z}_{\ell(3k-1)} \), one finds chiral operators with R-charges

\[
R'_j = \frac{j}{\ell} + \left\{ \frac{3k-1}{\ell} j \right\}; \quad j = 1, 2, \ldots, \ell - 1.
\]

(C.10)

One can show that (C.10) precisely completes (C.8) to cover all of (C.4) (except for the marginal operator \( R_\ell \), which is added to the Lagrangian with a large coefficient and decouples):

\[
R'_{2j-1} = R_{3k+j}
\]

\[
R'_{2j} = R_{6k+1+j}
\]

(C.11)

where \( j = 1, 2, \ldots, 3k \). In verifying (C.11) it is useful to note that on the first line, for \( j = 1, 2, \ldots, k \), \( R_{3k+j} \) belongs to band I in (C.4), while for \( j = k + 1, \ldots, 3k \) it is in band II. Similarly, on the second line of (C.11), \( j = 1, 2, \ldots, 2k \) gives operators in band II in (C.4), while \( j = 2k + 1, \ldots, 3k \) gives operators in band III.
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