Gravitational collapse in the AdS background and the black hole formation

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We study the time evolution of the Misner-Sharp mass and the apparent horizon for gravitational collapse of a massless scalar field in AdS$_5$ space-time for both cases of narrow and broad waves by numerically solving the Einstein’s equations coupled to a massless scalar field. It turns out that the Misner-Sharp mass is everywhere constant except for a rapid change across a thin shell, being characterized by stationary point of the density profile. By studying the evolution of the apparent horizon, signaling the formation of a black hole, at different times we see how asymptotically an event horizon forms. The dependence of the thermalization time on the radius of the initial black hole event horizon is also studied.

I. INTRODUCTION

There are two main motivations to study AdS space time. First, the the AdS/CFT conjecture according to which there is a correspondence between the phenomena in AdS and those in a conformal field theory [1], [2]; second, the AdS instability in the nonlinear regime [4, 7, 13, 23] which ensures that a black hole is always formed regardless of the amplitude of perturbations. In this way, the AdS/CFT correspondence was extended to include field theories at finite temperature [10]. In this correspondence a field theory in equilibrium at finite temperature is dual to an asymptotically AdS black hole. The non-equilibrium dynamics and the evolution towards thermalization in the field theory is equivalent to the evolution of the space time with the formation of a black hole horizon [3, 6, 8, 9].

Studying the thermalization in AdS/CFT needs to choose some states which are far-from-equilibrium. On the gravity side, a dynamical AdS black hole generally provides such an example. To this aim, we need to model a dynamical AdS black hole in gravity. Ashtekar and Krishnan have presented a well-defined quantity for the dynamical black hole boundary and its area law which can be applied to a non-flat background such as FRW [16] or AdS. Now, there has been some attempts to build AdS black holes [17, 18], non of them considering the dynamical perspectives of a dynamical black hole within AdS. Recall that, in the case of a dynamical black hole, the definition of mass is not trivial [19] and not unique. The black hole radiation temperature which is crucial in the CFT thermalization temperature is attributed to the apparent horizon and not to the event horizon [20, 24]. The question of the scalar field collapse in the AdS background has been recently studied in [3, 6, 8, 11, 13, 21].

We are interested in the dynamical case of the collapse of a massless scalar field coupled to gravity in the Poincare patch of the Schwarzschild-AdS space time. In section II, we present the metric and the equations of motion in addition to the details of the numerics involved. Section III is devoted to different aspects of the dynamical behavior of the model such as the mass and energy density. Both narrow and broad waves are considered. In the case of broad waves, a two stage collapse solution is found similar to [6]. In the case of narrow waves, two sets of initial conditions are discussed. Quasi-local Misner-Sharp mass for this space time is presented and its evolution is studied by plotting its behavior for different times. It is shown that the space is asymptotically AdS. It is also demonstrated that the space time is divided into three regions. In section IV, we plot the evolution of metric functions to show how the black hole forms. By evolving the equations for longer times we study apparent horizon formation. It is shown that it approaches the AdS black hole as one expects. Different quantities of interest such as event horizon radius and black hole temperature are computed. We then conclude in section V.

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II. DYNAMICAL METRIC FOR THE ADS BACKGROUND

Thermalization of a spatially homogeneous system which starts initially out of equilibrium by a scalar source coupled to a marginal operator corresponds on the gravity side to the collapse of a massless scalar field in the AdS space time. Now, the value of the (boundary) source $\phi_0$, is given by the value of the five dimensional field at the boundary of AdS space time. We, therefore, start with the Einstein-Hilbert action in $AdS_5$ with a minimally coupled massless scalar field written as

$$S = \frac{1}{2k^2} \int d^{d+1}x \sqrt{-g} (R - 2\Lambda - 2(\partial \phi)^2).$$  \(1\)

For a spatially homogeneous system on the boundary, the metric form in the Poincar coordinates is given by

$$ds^2 = \frac{1}{u^2} (-fe^\delta dt^2 + \frac{1}{f} du^2 + dx^2),$$  \(2\)

where $f$ and $\delta$ are functions of $t$ and $u$. The AdS vacuum solution corresponds to $\delta = 0$ and $f = 1$, while for the AdS black hole we set $f = 1 - \frac{u^4}{u_0^4}$, $\delta = 0$ and $\frac{1}{u_0^4} = \pi T$ with $T$ being the black hole temperature.

In this coordinate system, Einstein-Klein-Gordon equations reduce to the following set of differential equations:

$$\dot{V} = u^3 \left( -fe^\delta P \frac{u^3}{u^2} \right),$$  \(3\)

$$\dot{P} = (fe^\delta V)' ,$$  \(4\)

$$\dot{f} = \frac{4}{3} uf^2 e^{-\delta} VP,$$  \(5\)

$$\delta' = \frac{2}{3} u(V^2 + P^2),$$  \(6\)

and

$$f' = \frac{2}{3} f(V^2 + P^2) + \frac{4}{u} (f - 1),$$  \(7\)

where $P = \dot{\phi}$, $V = f^{-1}e^{\delta} \dot{\phi}$, and the derivatives with respect to $t$ and $u$ are denoted by overdot and primes. The last equation is a constraint equation. We have therefore a set of differential equations defining an initial value problem. The initial conditions and boundary conditions are presented in the following subsection.

A. Initial and boundary conditions and the solution algorithm

To solve the above equations one needs to impose initial and boundary conditions. In the case of narrow waves two different sets of initial conditions for the vacuum AdS and the AdS black hole are considered. For broad waves, however, only the vacuum AdS initial condition is considered. In the case of the AdS vacuum the source is zero in the bulk. We therefore get

$$f = 1, P = V = \delta = 0,$$  \(8\)

and for the case of the AdS black hole we set

$$f = 1 - \frac{u^4}{u_0^4}, P = V = \delta = 0.$$  \(9\)

The boundary condition can not be given at $u = 0$ due to the choice of the coordinates making the system of equations singular at $u = 0$. Therefore, we set a lower bound for the numerical calculation, i.e $u_{\text{min}} = 0.005$. Now, to solve the equation \(6\) we fix the boundary condition by

$$\delta(t, u_{\text{min}}) = 0.$$  \(10\)
The $\varphi$ value at the boundary, in the dual picture, injects energy in to the boundary field [22]. To study thermalization we require that the source vanishes effectively after some time interval. The boundary conditions for the equation [3], [4] are

$$V = -2t e^{-\alpha t^2}, P(t, \infty) = 0. \quad (11)$$

In order to solve the equations numerically, a cut off $u_{\text{max}}$ is needed. We note that there is always a region in which the wave vanishes, and it does not affect the propagation of the wave. Therefore, we restrict our equations such that $P(t, u_{\text{max}}) = 0$ for $u_{\text{max}}$ being in the region in which the field vanishes. We then have used the third order Adams-Bashforth method for time-marching and finite difference in the spatial part [6], denoting by $dt$ the time intervals and by $du$ the space intervals. The value of each function at the point $(t_i, u_j)$ is then written as $h^i_j$ in which $t_i = t_0 + idt$ and $u_j = u_{\text{min}} + jdu$. Now, for each function using the third order Adams-Bashforth method we have

$$h_{j+1}^i = h_j^i + \frac{dt}{12} (23h_j^i - 16h_{j-1}^i + 5h_{j-2}^i). \quad (12)$$

Thus, the system of differential equations are given by

$$\dot{V}_j^i = -\frac{3}{u_j} (f_j^i e^{-\delta^i_j} P_j^i) + \frac{f_j^i e^{-\delta^i_j} P_j^i - f_{j-1}^i e^{\delta^i_{j-1}} P_{j-1}^i}{du} \quad (13)$$

$$\dot{P}_j^i = \frac{(f_{j+1}^i e^{-\delta^{i+1}_j} V_{j+1}^i - f_j^i e^{-\delta^i_j} V_j^i)}{du} \quad (14)$$

$$\frac{\delta^i_j - \delta^i_{j-1}}{du} = \frac{2}{3} u_j (V_j^2 + P_j^2). \quad (15)$$

Given the initial conditions, at each time interval $t_n$, we first compute $f, V, P$ using equation [12], before going to $\delta$. With this conditions, the value $\dot{P}, \dot{V}$ and $\ddot{f}$ are computed using the finite difference method for each hyper-surface.

### III. ENERGY AND MASS BEHAVIOR

We study both cases with the source turned on for short ($\frac{a}{A} \leq 1, \Delta t < \frac{1}{\bar{t}}$) and long ($\frac{a}{A} \geq 1, \Delta t > \frac{1}{\bar{t}}$) durations. The limit in which $\frac{a}{A} \ll 1$ induces a narrow wave near the boundary. As stated, in the case of narrow waves two sets of initial conditions are considered, namely for the vacuum AdS and the AdS black hole. The results for both cases are presented. We choose $(a, \bar{t}) = (400, 0.5)$ for the parameters. This produces a wave with the duration $\Delta t = 0.05$. The source is then turned off and an ingoing narrow wave is produced. The energy density for this space time is defined as $f(V^2 + P^2)$. FIG.1 and FIG.2 shows energy density of the field for AdS vacuum and AdS black hole initial conditions, respectively. From the figures one understands that as the wave propagates inward it becomes more compact with the density becoming sharply peaked before the formation of the apparent horizon. Our results agree with those presented in [6]. Formation of the apparent horizon is best seen by studying the behavior of the metric function $f$ which is presented in section IV.

Now, there is also the notion of mass being crucial in our study but not uniquely defined in general relativity. However, the Misner-Sharp quasi-local mass is the one having significant importance in thermodynamics of black holes and reducing to the Newtonian concept of mass [16]. The generalized quasi-local Misner-Sharp mass for the $n$ dimensional manifold of $M^2 \approx m^2 * K^{n-2}$ with the line element

$$ds^2 = g_{AB} dy^A dy^B + R(y)^2 \gamma_{ij}(z) dz^i dz^j, \quad (17)$$

is defined in [17] as

$$m = \frac{(n-2)V_{n-2}^k}{2k^n} R^{n-3} \{-\hat{\Lambda} R^2 + k - (DR)^2\}, \quad (18)$$

where $D_A$ is the covariant derivative on $M^2$ and $V_{n-2}^k$ is the area of the $n - 2$ dimensional part and $\hat{\Lambda} = \frac{2\Lambda}{(n-1)(n-2)}$. The quasi-local Misner-Sharp mass for this space time is computed as

$$M(t, u) = \frac{1 - f}{u^4}, \quad (19)$$
FIG. 1: Density evolution with the vacuum AdS initial condition

FIG. 2: Density evolution with the AdS black hole initial condition

Time evolution of this quasi-local quantity $M(t, u)$ using equation (7) is given by

$$\frac{\partial M(t, u)}{\partial t} = \frac{\partial}{\partial t} \int \varepsilon^0 \, du = \varepsilon^\mu,$$

in which

$$\varepsilon^0 = \frac{2}{3u^3} f(V^2 + P^2),$$

and

$$\varepsilon^\mu = -\frac{4}{3u^3} f^2 e^{-\delta} PV.$$

Because the Misner-Sharp mass is constant for AdS-Schwarzschild solution, we use this quantity as a measure of how different regions behave. From FIG.3 and FIG.4 we may differentiate three different parts of the space time: (1) a
static region behind the wave; (2) a narrow transition region; (3) the vacuum AdS. Thus one finds that the in the dual description thermalization happens from small scales to large scales [9].

The limit $\frac{c}{a} \geq 1$ induces a broad wave at the boundary. If the duration of the source is long enough ($\Delta t > \frac{1}{T}$) this will lead to a two stage collapse. We choose $(a, \varepsilon) = (0.1, 0.3)$ for the parameters of the wave. By starting from negative times, the energy is transferred to the bulk by two pulses of the source. This energy density at different times is plotted in Fig. 5. Two peaks are observed in the energy density leading to two collapses.
IV. BLACK HOLE FORMATION

As stated, in dynamical settings the apparent horizon is the crucial concept to be identified by the geodesic expansion of light like geodesics. We need first to obtain the induced metric on a \( d - 1 \) dimensional hyper-surface in a \( d \) dimensional space-time given by

\[
g_{ab} = g_{ab} + \ell^a n^b + n^a \ell^b, \tag{23}
\]

where \( n^a \) and \( \ell^b \) are ingoing and outgoing geodesics normal to the \( d - 1 \) dimensional hyper-surface. Two scalar quantities called geodesic expansions are defined by

\[
\theta_\ell = \tilde{q}^{ab} \nabla_a \ell_b, \tag{24}
\]

\[
\theta_n = \tilde{q}^{ab} \nabla_a n_b. \tag{25}
\]

The region in which the expansions \( \theta_\ell \) and \( \theta_n \) are negative is called the trapped region and its boundary is the apparent horizon. Now, in our case the \( f = 0 \) hypersurface is the boundary of the trapped region, or the apparent horizon, where the expansion of the outgoing geodesics is zero. However, to calculate numerically, we set a lower bound for \( f \). The behavior of the metric function \( f \) for the narrow wave is shown in FIG.6 and FIG.7. As the wave propagates inward its potential well increases until the black hole forms. For the vacuum AdS initial condition the collapse occurs at \( u \sim 1.1 \). If we define the apparent horizon formation time \( t_A \) such that \( f \) is approximately less than 0.02 then we find \( t_A \sim 2.3 \). Given the initial condition of the AdS black hole, the collapse occurs at \( u \sim 0.9 \) and \( t_A \sim 1.7 \). Therefore, in the presence of an initial black hole with the same parameters of the boundary source, the formation time of the apparent horizon decreases. The question of how the presence of a black hole may affect the thermalization is studied in the last section.

![FIG. 6: The evolution for \( f \) in the vacuum AdS](image)

The case of broad waves is shown in FIG.8 where we see the formation of a two phase black hole. The first collapse happens at \( u \sim 0.7 \) and the second one at \( u \sim 0.5 \).

A. The horizon behavior

The collapse of a wave in an AdS-Schwarzschild black hole, as we will show, ends asymptotically into an AdS-Schwarzschild black hole again, irrespective of the wave amplitude. During the collapse the apparent horizon is defined by the boundary of the region where \( f(t, u) = 0 \) vanishes. This is done numerically by setting the amplitude of the boundary source to \( \epsilon = 0.5 \). FIG.9 shows the behavior of the apparent horizon, approaching asymptotically the event horizon. This is a typical behavior independent of the initial black hole radius and of amplitude \( \epsilon \), as may be seen from FIG.10 and FIG.11. Note that the radius of the resulting black hole and the time of its formation decreases as the amplitude of the wave increases. The temperature of the black hole is given by \( T = \frac{1}{\pi u_0} \), where the \( u_0 \) defines the apparent horizon radius at late time \([24]\). Therefore, the temperature of the black hole increases as the amplitude
increases. We set the parameters of the source as $\epsilon = 0.5$. This time we change the initial black hole radius. This behavior is shown in . This allows us to conclude that decreasing the initial black hole radius, decreases the time apparent horizon forms.

B. Black hole thermalization

We define the thermalization time as the time in which $f$ reaches a minimum value of 0.01. In FIG.12 we have depicted the dependence of the thermalization time $t_T$ on the radius of the initial black hole event horizon, for the same parameters of the wave packet. In the dual description, this corresponds to the dependence of the thermalization time by starting from thermal state at different temperatures. In this coordinate the mass of the black hole is given by $M = \frac{1}{u_{0T}}$. Therefore, the thermalization time increases with the decrease of the black hole mass.

V. CONCLUSION

We have studied the time evolution of the Misner-Sharp mass and the evolution of the apparent horizon for the gravitational collapse of a massless scalar field in the five dimensional Schwarzschild-AdS space-time. The Misner-
Sharp mass at different times turns out to be almost constant except for the narrow transition region identifying three different regions of the space-time. The stationary point in the transfer region is related to the minimum of the density as shown in FIG.1 and FIG.2. The formation time of the apparent horizon decreases as the source amplitude increases. We have also shown the effect of initial AdS black hole radius on the apparent horizon formation time: a decrease in the initial AdS black hole radius decreases the formation time of the apparent horizon.

The presence of an initial black hole affects the thermalization time. Using the thermalization definition given in [6],
FIG. 12: The thermalization time for different initial black hole radius interpreted in terms of Wilson loops, we have found that the increase of the radius of the initial black hole increases the thermalization time.

VI. ACKNOWLEDGMENT

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