1. Introduction

A search for traversable wormhole solutions to the gravitational field equations with realistic matter has been for long, and is still remaining to be, one of the most intriguing challenges in gravitational studies. One of the attractive features of wormholes is their ability to support electric or magnetic “charge without charge” \( \equiv 0 \) by letting the lines of force thread from one spatial asymptotic to another.

As is widely known, traversable wormholes can only exist with exotic matter sources, more precisely, if the energy-momentum tensor (EMT) of the matter source of gravity violates the local and averaged null energy condition (NEC) \( T_{\mu\nu}k^\mu k^\nu \geq 0, \ k^\mu k^\mu = 0 \) \( \equiv 0 \). It is known, for instance, that nonlinear electrodynamics with any Lagrangian of the form \( L(\mathcal{F}) = F_{\mu\nu}F^{\mu\nu} \) coupled to general relativity cannot produce a static, spherically symmetric wormhole metric \( \equiv 0 \). Though, an effective wormhole geometry for electromagnetic wave propagation can appear as a result of the electromagnetic field nonlinearity \( \equiv 0 \).

Scalar fields are able to provide good examples matter needed for wormholes: on the one hand, in many particular models they do exhibit exotic properties, on the other, many exact solutions are known for gravity with scalar sources. We will consider some examples of charged wormhole solutions in the presence of massless scalar fields.

Let us begin with the action for a general (Bergmann-Wagoner) class of scalar-tensor theories (STT), where gravity is characterized by the metric \( g_{\mu\nu} \) and the scalar field \( \phi \) in the presence of the electromagnetic field \( F_{\mu\nu} \) as the only matter source:

\[
S = \int d^4x\sqrt{-g}\{f(\phi)R[g] + h(\phi)g^{\mu\nu}\phi_\mu\phi_\nu - F_{\mu\nu}F^{\mu\nu}\}. \tag{1}
\]

Here \( R[g] \) is the scalar curvature, \( g = |\det(g_{\mu\nu})| \), \( f \) and \( h \) are certain functions of \( \phi \), varying from theory to theory. Exact static, spherically symmetric solutions for this system are well known \( \equiv 0 \), but their qualitative behaviour is rather diverse and depends on the nature of the functions \( f \) and \( h \).

Wormholes form one of the generic classes of solutions in theories where the kinetic term in \( \equiv 0 \) (more precisely, if \( f(\phi) \), defined in \( \equiv 0 \), is negative). A particular case of this kind of wormholes, namely, wormholes with a “ghost” massless minimally coupled scalar field in general relativity [Eq. (1)], \( f(\phi) \equiv 1, h(\phi) \equiv -1 \) was considered by H. Ellis \( \equiv 0 \).

The energy conditions, NEC in particular, are, however, violated as well by “less exotic” sources, such as the so-called nonminimally coupled scalar fields in general relativity, represented by the action \( \equiv 0 \) with the functions

\[
f(\phi) = 1 - \xi \phi^2, \quad \xi = \text{const}; \quad h(\phi) \equiv 1. \tag{2}
\]

Scalar-vacuum (with \( F_{\mu\nu} = 0 \)) static, spherically symmetric wormhole solutions were found in such a theory in Ref. \( \equiv 0 \) (and were recently discussed in Ref. \( \equiv 0 \)) for conformal coupling, \( \xi = 1/6 \), and in Ref. \( \equiv 0 \) for any \( \xi > 0 \). The easiness of violating the energy conditions, so evident due to the appearance of wormhole solutions, even made Barcelo and Visser discuss a “restricted domain of application of the energy conditions” \( \equiv 0 \). We recently proved \( \equiv 0 \) that all these scalar-vacuum wormhole solutions are unstable under spherically symmetric perturbations. The instability turns out to be of catastrophic nature: the increment of perturbation growth has no upper bound, hence, within linear perturbation theory, such a wormhole, if once formed, should decay immediately and instantaneously. A full dynamical solution (yet to be found) would probably show a finite but still enormous decay rate.

The purpose of this paper is to extend these results to charged wormholes. We will show in Sec. 2 that among the electrovacuum static, spherically symmetric solutions of the theory \( \equiv 0, \equiv 0 \) there is, for any \( \xi > 0 \), a 4-parameter family of wormhole solutions. (For \( \xi = 1/6 \) this is already known from \( \equiv 0 \).) The parameters can be
identified as the mass, the electric and magnetic charges and the scalar field value at infinity. One more parameter, the scalar charge, is expressed in terms of the others. The instability of these wormholes is demonstrated in Sec. 3.

As a tool, we use a transition to the Einstein conformal frame, in which the scalar field is minimally coupled to gravity. In all the wormhole solutions, the full manifold \(M_{\text{I}}[g]\) where the theory is formulated, maps to two Einstein-frame manifolds separated by the sphere \(S_{\text{trans}}\) where \(f = 0\), and the instability develops in the neighbourhood of this sphere.

## 2. Charged wormhole solutions

### 2.1. The general static solution

The general STT action \(\mathcal{A}[g]\) is simplified by the well-known conformal mapping \[13 \]
\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} / |f(\phi)|, \]
accompanied by the scalar field transformation \(\phi \mapsto \psi\) such that
\[ \frac{d\psi}{d\phi} = \pm \sqrt{|l(\phi)|} / |f(\phi)|, \quad l(\phi) \overset{\text{def}}{=} fh + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2. \]

In terms of \(\tilde{g}_{\mu\nu}\) and \(\psi\) the action takes the form
\[ S = \int d^4x \sqrt{|\tilde{g}|} \left\{ (\text{sign} f) \left[ R[\tilde{g}] \right] \right. \]
\[ + \tilde{g}^{\mu\nu}\psi_{,\mu}\psi_{,\nu} \text{sign} l(\phi) \left. - F^{\mu\nu} F_{\mu\nu} \right\}. \]

(up to a boundary term which does not affect the field equations). Here \(R[\tilde{g}]\) is the Ricci scalar obtained from \(\tilde{g}_{\mu\nu}\), and the indices are raised and lowered using \(g_{\mu\nu}\).

The electromagnetic field Lagrangian is conformally invariant, and \(F_{\mu\nu}\) is not transformed.

The space-time \(M_{\text{I}}[g]\) with the metric \(g_{\mu\nu}\) is referred to as the \textit{Jordan conformal} frame, generally regarded to be the physical frame in STT; the \textit{Einstein conformal frame} \(M_{\text{E}}[\tilde{g}]\) with the field \(\psi\) then plays an auxiliary role. The action \(\mathcal{A}[\tilde{g}]\) corresponds to conventional general relativity if \(f > 0\), and the normal sign of scalar kinetic energy is obtained for \(l(\phi) > 0\).

The general static, spherically symmetric solution to the Einstein-Maxwell-scalar equations that follow from \(\mathcal{A}[\tilde{g}]\), was first found by Penney and in a more complete form in [12]. Let us write it in the form suggested in \(\tilde{g}\), restricting ourselves to the “normal” case \(f > 0\), \(l > 0\):
\[ ds_E^2 = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2 \]
\[ = \frac{q^{-2} dt^2}{s^2(h, u + u_1)} - \frac{q^2 s^2(h, u + u_1)}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right], \]
\[ \psi(u) = Cu + \psi_1, \]
\[ F_{01} = -F_{10} = q e^{\alpha + \gamma - 2\beta} \]
\[ = \left[ q s^2(h, u + u_1) \right]^{-1}, \]
where the subscript “E” stands for the Einstein frame; \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\) is the linear element on a unit sphere; \(q = q_e\) (the electric charge), \(C\) (the scalar charge), \(h\), \(k\) and \(\psi_1\) are real integration constants. The function \(s(k, u)\) is defined as follows:
\[ s(k, u) = \begin{cases} k^{-1} \sinh ku, & k > 0 \\ u, & k = 0 \\ k^{-1} \sin ku, & k < 0. \end{cases} \]

Here \(u\) is a convenient radial variable (it is a harmonic coordinate in the Einstein frame, \(u = 0\)). The range of \(u\) is \(0 < u < u_{\text{max}}\), where \(u = 0\) corresponds to spatial infinity, while \(u_{\text{max}}\) may be finite or infinite depending on the constants \(k\), \(h\) and \(u_1\).

The integration constants are related by
\[ 2k^2 \text{sign} k = 2h^2 \text{sign} h + C^2, \]
\[ s^2(h, u_1) = 1/q^2. \]
The latter condition, preserving some discrete arbitrariness of \(u_1\), provides the natural choice of the time scale \((\tilde{g}_{00} = 1)\) at spatial infinity \((u = 0)\). Without loss of generality we put \(C > 0\) and \(\psi_1 = 0\).

As usual, in addition to the electric field \(F_{01} = -F_{10}\) given by \(\overline{\tilde{g}}\), one can include a radial magnetic field \(F_{02} = -F_{20}\) with \(q_m\) as the magnetic charge. One should then understand \(q^2\) in \(\tilde{g}\) and further on as \(q^2 = q_e^2 + q_m^2\), in \(\mathcal{A}\) one should replace \(q\) with \(q_e\) and in \(\overline{\tilde{g}}\) \(1/q\) with \(q_e/q^2\). In what follows, we will bear in mind this opportunity without special mentioning.

The solution contains four essential integration constants: \(k\) or \(h\) and the charges \(q_e\), \(q_m\) and \(C\). The mass \(M\) in the Einstein frame is obtained by comparing the asymptotic of \(\tilde{g}\) at small \(u\) with the Schwarzschild metric:
\[ GM = \pm \sqrt{q^2 + h^2} \text{sign} h \]
where \(G\) is Newton’s gravitational constant. The “+” sign depends on the choice of \(u_1\) among the variants admitted by \(\overline{\tilde{g}}\).

The Reissner-Nordstrom solution of general relativity is a special case obtained herefrom by putting \(C = 0\). Then from \(\tilde{g}\) it follows \(h = k\), and the familiar form of the Reissner-Nordstrom metric is recovered after a transition to the curvature coordinates, \(-\tilde{g}_{\theta\theta} = r^2\):
\[ r = |q| s(k, u + u_1). \]

To obtain another special case \(q = 0\) (the scalar-vacuum solution), one should consider the limit \(q \to 0\) preserving the boundary condition \(\overline{\tilde{g}}\). This is only possible for \(k > h \geq 0\) and \(u_1 \to \infty\). The resulting metric is
\[ ds_E^2 = e^{-2hu} dt^2 - \frac{k^2 e^{2hu}}{\sin^2(ku)} \left[ \frac{du^2}{\sin^2(ku)} + d\Omega^2 \right]. \]

The scalar field is determined, as before, from \(\tilde{g}\), and the integration constants are related by
\[ 2k^2 = 2h^2 + C^2. \]
It should be noted that in \( \|3\|, \|4\| \) the constant \( h \) can have any sign, and for the mass \( M \) we have simply \( GM = h \).

This is the Fisher solution \([4]([4]\) in terms of the harmonic \( u \) coordinate. Its more familiar form, used, in particular, in Refs. \([3]\), \([10]\), corresponds to the coordinate \( r \) connected with \( u \) by \( r = 2k/(1 - e^{-2ku}) \), and the metric in terms of \( r \) has the form
\[
d s^2_{EC} = (1 - 2k/r)^a dr^2 - (1 - 2k/r)^{-a} [d\varphi^2 + r^2 (1 - 2k/r) d\Omega^2],
\]
where \( a = h/k \). The Schwarzschild solution is then recovered in case \( C = 0, a = 1 \).

All the corresponding Jordan-frame solutions for \( l(\phi) > 0 \) are obtained from \([3], [4]\) using the transformation \([\|3\|, \|4\|]; \( \xi \)).

### 2.2. Continued solution in the Jordan frame

Let us now turn to wormhole solutions for the nonminimal coupling \([3]\). \( \xi > 0 \). The transformation \([\|3\|, \|4\|]\) takes the form
\[
d \psi = \sqrt{1 - 6\xi^2[1 - (\xi - 6\xi^2)]},
\]
where, without loss of generality, we have chosen the plus sign before the square root. We assume that spatial infinity in the Jordan-space-time \( M_\| \) corresponds to \( |\psi| < 1/\sqrt{\xi} \), where \( f(\phi) > 0 \), so that the gravitational coupling has its normal sign.

Generically, the solution in \( M_{EC} \| \) has a naked singularity at \( u = u_{\text{max}} \), and, though its nature can change due to the transformation to \( g_{\mu\nu} \), it remains to be a singularity in \( M_\| \). An exception is the case when the solution is smoothly continued in \( M_\| \) through the sphere \( S_{\text{trans}} \ (u = \infty, \ \phi = 1/\sqrt{\xi}) \) which is singular in \( M_{EC} \| \) but regular in \( M_\| \). The infinity of the conformal factor \( 1/f \) thus compensates the zero of both \( \phi \| \) and \( g_{\phi\theta} \) simultaneously. Wormhole solutions can only be found in this case. It happens when, in accord with \([\|1\|]\),
\[
k = 2h = 2C/\sqrt{\xi} > 0, \quad u_{\text{max}} > 0,
\]
which selects a special subfamily among all solutions. We will restrict our attention to this subfamily. Note that now \( s(k, u) = (2h)^{-1} \sinh(2hu) \), \( s(h, u + u_1) = h^{-1} \sinh(hu + hu_1) \) and \( u_{\text{max}} = \infty \). According to \([\|1\|]\) and \([\|8\|]\), we have \( \psi \to \infty \) as \( \phi \to 1/\sqrt{\xi} \).

Under the condition \([\|1\|]\) the solutions with and without charge in \( M_{EC} \) are conveniently written in isotropic coordinates. Indeed, putting \( y = \tanh(hu) \), we obtain:
\[
d s^2_{EC} = \frac{(1 - y^2)^2}{y + y_1} \left[ dy^2 - h^2 \frac{(y + y_1)^2}{y_1^2 y^4} (dy^2 + y^2 d\Omega^2) \right],
\]
\[
\psi = \frac{\sqrt{6}}{2} \ln \frac{1 + y}{1 - y},
\]
\[
F_{01} = -F_{10} = \frac{g e}{h} \frac{y^2}{(y + y_1)^2},
\]
where
\[
y_1 = \tanh(hu_1) = \frac{h}{\sqrt{h^2 + q^2}}.
\]
The vacuum solution is included here as the special case \( q = 0, y_1 = 1 \). The range of \( u, \ u \in \mathbb{R}_+ \), is converted into \( y \in (0, 1) \) where \( y = 0 \) corresponds to spatial infinity and \( y = 1 \) to a naked singularity.

To proceed to the Jordan frame, let us integrate Eq. \([\|8\|]\). This gives \([\|1\|]\),
\[
\psi = -\sqrt{3/2} \ln[B(\phi)H^2(\phi)]
\]
where
\[
B(\phi) = B_0 \frac{\sqrt{1 - \eta \phi^2 - \sqrt{6}\xi \phi}}{\sqrt{1 - \eta \phi^2 + \sqrt{6}\xi \phi}}.
\]

\( B_0 = \text{const.} \), while \( H(\phi) \) is different for different \( \xi \):
\[
\begin{align*}
0 < \xi < 1/6: & \quad H(\phi) = \exp \left[ -\frac{\sqrt{1 - 6\xi}}{\sqrt{6\xi}} \arcsin \sqrt{\eta \phi} \right], \\
\xi > 1/6: & \quad H(\phi) = \left[ \sqrt{-\eta \phi + \sqrt{1 - \eta \phi^2}} \right]^{\frac{\sqrt{6\xi - 1}}{\sqrt{6\xi}}},
\end{align*}
\]
where \( \eta = \xi(1 - 6\xi) \), and \( H \equiv 1 \) for \( \xi = 1/6 \). The function \( H(\phi) \) is finite in the whole range of \( \phi \) under consideration.

Eq. \( [24]\) is valid for \( \phi < 1/\sqrt{\xi} \), and the Jordan-frame metric \( g_{\mu\nu} = \tilde{g}_{\mu\nu}/f \) under the condition \([\|1\|]\) can be written in terms of the coordinate \( y \) as follows:
\[
d s^2_j = \frac{BH^2}{1 - \xi \phi^2} \left[ \frac{(1 + y)^2}{(y + y_1)^2} y_1^2 dy^2 - h^2 \frac{(y + y_1)^2}{y_1^2 y^4} (dy^2 + y^2 d\Omega^2) \right],
\]
where \( y \) can be expressed in terms of \( \phi \):
\[
y = \frac{1 - BH^2}{1 + BH^2}.
\]

The metric is thus actually expressed in terms of the scalar field \( \phi \) used as a coordinate. The isotropic coordinate \( y \) conveniently shortens the expression \([\|2\|]\) and makes it easy to see that the metric, originally built for \( \phi < 1/\sqrt{\xi} \) \((y < 1)\), is smoothly continued across the surface \( S_{\text{trans}} \ (\phi = 1/\sqrt{\xi}, \ y = 1) \). Indeed, in a close neighbourhood of \( S_{\text{trans}} \), for \( \phi = (\phi - \delta)/\sqrt{\xi} \) with \( \delta \ll 1 \) one has
\[
B \approx B_0 \delta/(12\xi), \quad 1 - \xi \phi^2 \approx 2\delta
\]
whence
\[
\frac{BH^2}{1 - \xi \phi^2} \bigg|_{y=1} = \frac{B_0 H^2}{24\xi} \bigg|_{\phi=1/\sqrt{\xi}}.
\]

\(^{1}\)We have changed the notations as compared with \([\|1\|]\), in particular, we have replaced \( \Phi_2 \to \sqrt{6\phi} \), \( H \to 1/H \) and \( F^2 \to 1/B \), to avoid imaginary \( F \) at \( \phi > 1/\sqrt{\xi} \).
It is easily shown that this ratio is not only finite on \( \mathcal{S}_{\text{trans}} \) but also smoothly changes across it, so that Eq. (27) comprises an analytic continuation of the metric, obtained from (\[3\]) in case (\[19\]) by the transformation (\[3\]), (\[4\]), beyond \( \mathcal{S}_{\text{trans}} \). The coordinate \( y \) covers the whole manifold \( M_1[g] \), and it is now possible to study the properties of the system as a whole.

Before doing that, let us note that the new region \( \phi > 1/\sqrt{x} \) \( (y > 1) \) in \( M_1 \) can also be obtained by the same transformation (\[3\]), (\[4\]) from a certain Einstein frame. An essential difference from the previous solution is that, since \( f(\phi) \) is now negative, (\[4\]) leads to the Einstein equations with a reversed sign of the electromagnetic energy-momentum tensor. As a result, the solution in this second Einstein-frame manifold \( M_{E'} \) will have the same form (\[6\])–(\[9\]), but with the replacement

\[
s(h, u + u_1) \rightarrow h' - 1 \cosh(h' u + h' u_1),
\]

where \( h' > 0 \), and the relation (\[11\]) is replaced by \( 2k'^2 = 2h'^2 + C'^2 \) where \( k' > 0 \).

The solution in \( M_{E'} \) is also regularized by the factor \( 1/f \) on \( \mathcal{S}_{\text{trans}} \), and the integration constants in it satisfy the condition \( k' = 2h' \), similar to (\[14\]). Other integration constants are adjusted as well, in particular, the charges \( q_c \) and \( q_m \) are the same on both sides of \( \mathcal{S}_{\text{trans}} \), providing the continuity of the electromagnetic field.

### 2.3. Wormhole solutions

Let us begin with the simplest case \( \xi = 1/6 \) (conformal coupling). Then instead of (\[24\])–(\[26\]) one can write for \( \phi < \sqrt{6} \)

\[
\phi = \sqrt{6} \tanh[(\psi + \psi_0)/\sqrt{6}], \quad \psi_0 = \text{const},
\]

where \( \psi = Cu \) and due to (\[19\]) \( C = h/\sqrt{6} \). The Jordan-frame solution in terms of the isotropic coordinate \( y \) takes the form (\[3\]), (\[4\])

\[
\begin{aligned}
\text{ds}^2 &= \frac{(1 + y_0 y)^2}{1 - y_0^2} \left[ \frac{y_1^2 dt^2}{(y + y_1)^2} - h^2 \frac{(y + y_1)^2}{y_1 y y^2} (dy^2 + y^2 d\Omega^2) \right], \\
\phi &= \sqrt{6} \frac{y + y_0}{1 + y_0 y_1},
\end{aligned}
\]

where \( y_0 = \tanh(\psi_0/\sqrt{6}) \) and \( y_1 \in (0, 1) \); the expressions for \( F_{\mu\nu} \) are evident.

The original Einstein-frame solution corresponds to \( y < 1, \phi = 0 \) is spatial infinity while the sphere \( y = 1 \) is \( \mathcal{S}_{\text{trans}} \), where the solution (\[32\]), (\[33\]) is manifestly regular. The region \( y > 1 \) is an analytic continuation of the solution in \( M_1[g] \) to \( \phi > \sqrt{6} \) and corresponds to another Einstein-frame solution described above.

The properties of the solution at \( y > 1 \) depend on the constant \( y_0 \) which characterizes the \( \phi \) field at spatial infinity. Namely, if \( y_0 < 0 \), then the solution has a naked singularity at \( y = -1/y_0 > 1 \). If \( y_0 = 0 \), we obtain a black hole with electromagnetic and scalar charges (\[13\] \[3\] \[4\] \[7\]); introducing \( r = h(y + y_1)/(y_1 y) \), we obtain

\[
\begin{aligned}
ds^2 &= (1 - m/r)^2 dt^2 - (1 - m/r)^{-2} dr^2 - r^2 d\Omega^2, \\
\phi &= C/(r - m)
\end{aligned}
\]

where \( m = GM/\sqrt{h^2 + q^2}, C = \sqrt{h} \). On the horizon, \( r = m \), despite \( \phi \rightarrow \infty \), the energy-momentum tensor of the scalar field is finite. This solution (mainly its neutral special case \( q = 0 \)) was repeatedly discussed as an interesting counterexample of the well-known no-hair theorems; its instability under spherically symmetric perturbations has been proved in Ref. (\[13\]).

Lastly, if \( y_0 > 0 \), then \( y \) ranges from 0 to \( \infty \), and \( y = \infty \) is another flat spatial infinity. This is the sought-for wormhole solution, parametrized by the four constants \( h, q_c, q_m \) and \( y_0 \). The position and radius of the wormhole neck (minimum of \( r^2 = -g_{\theta\theta} \)) are given by

\[
\begin{aligned}
y_{\text{neck}} &= \sqrt{y_1} \sqrt{y_0}, \\
r_{\text{neck}} &= \frac{h(1 + \sqrt{y_0 y_1})}{y_1 \sqrt{1 - y_0}}.
\end{aligned}
\]

For \( \xi \neq 1/6 \) the analytical relations are much more complicated, but the qualitative behaviour of the solution can be described rather easily.

In case \( \xi > 1/6 \), for any \( B_0 \), with growing \( \phi \) the quantity \( B^2 H^{-4} \) eventually reaches the value 1, where \( g_{\theta\theta} \rightarrow \infty \), i.e., we arrive at another spatial asymptotic, and it is straightforward to verify that this infinity is flat. In other words, we obtain again a static wormhole.

In case \( \xi < 1/6 \) everything depends on \( B_0 \). If

\[
B_0 < B_0^* = \exp(-\pi \sqrt{\frac{1 - 6\xi}{6\xi}}),
\]

the situation is the same as for \( \xi > 1/6 \), i.e., a wormhole. If \( B_0 > B_0^* \), then, while \( g_{\theta\theta} \) is still finite, \( \phi \) reaches the value \( 1/\sqrt{\xi} = 1/\sqrt{\xi(1 - 6\xi)} \), the location of a curvature singularity (\[14\]). So we have a naked singularity instead of a wormhole. Lastly, for \( B_0 = B_0^* \), the maximum value of \( \phi \) is again \( 1/\sqrt{\xi} \), but now it is non-flat spatial infinity.

### 3. Stability analysis

#### 3.1. Perturbation equations

Consider small (linear) spherically symmetric perturbations of the above wormholes. It is helpful to work separately in each of the two Einstein-frame manifolds \( M_E \) and \( M_{E'} \), perturbing the metric quantities \( \alpha, \beta, \gamma \) in (\[3\]) and the field \( \psi \), replacing

\[
\psi(u) \rightarrow \psi(u, t) = \psi(u) + \delta \psi(u, t)
\]

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and similarly for other quantities; the same is done for their counterparts in $\mathcal{M}_E'$. Due to spherical symmetry, the only dynamical degree of freedom is the scalar field, obeying the equation $\Box \psi = 0$, while other perturbations must be expressed in terms of $\delta \psi$ and its derivatives via the Einstein equations. The perturbed scalar equation has the form

$$e^{-\gamma + \alpha + 2\beta} \psi - (e^{\gamma - \alpha + 2\beta} \psi_u)_u = 0. \quad (38)$$

where the dot stands for $\partial/\partial t$ and the subscript $u$ for the radial coordinate derivative $\partial/\partial x^1$. One can notice that Eq. (38) decouples from perturbations other than $\delta \psi$ if one chooses the frame of reference and the coordinates in the perturbed space-time (the gauge for short) so that

$$\delta \alpha = 2 \delta \beta + \delta \gamma. \quad (39)$$

The relation $\alpha = 2 \beta + \gamma$ thus holds for both the static background written as in (7), (6) and the perturbations. The unperturbed part of Eq. (38) reads $\psi_{uu} = 0$ and is satisfied by (7), while for $\delta \psi$ we obtain the wave equation

$$e^{4\beta(u)}(\delta \psi')' \omega - \delta \psi_{uu} = 0. \quad (40)$$

The static nature of the background solution makes it possible to separate the variables,

$$\delta \psi = \Phi(u) e^{\omega t}, \quad (41)$$

and to reduce the stability problem to a boundary-value problem for $\Phi(u)$. Namely, if there exists a nontrivial solution to (41) with $\omega^2 < 0$, satisfying some physically reasonable boundary conditions, then the static background system is unstable since the perturbations can exponentially grow with $t$. Otherwise it is stable in the linear approximation.

Suppose $-\omega^2 = \Omega^2$, $\Omega > 0$. The equation that follows directly from (41),

$$\Phi_{uu} - \Omega^2 e^{4\beta(u)} \Phi = 0, \quad (42)$$

is converted to the normal Liouville (Schrödinger-like) form

$$d^2 Y/dx^2 - (\Omega^2 + V(x)) Y(x) = 0,$$

$$V(x) = e^{-4\beta}(\beta_{uu} - \beta_u^2). \quad (43)$$

by the transformation

$$\Phi(u) = Y(x) e^{-\beta}, \quad x = -\int e^{2\beta(u)} du. \quad (44)$$

Eq. (42) makes it possible to use the experience of quantum mechanics (QM): $\Omega^2$ here corresponds to $-E$ in the Schrödinger equation. In other words, the presence of “negative energy levels” $E = -\Omega^2 < 0$ for the potential $V(x)$ indicates the instability of our system.

The variable $x$ behaves as follows at small and large $u$:

- $u \to 0$ (spatial infinity): $x \approx e^\beta \approx 1/u$.

For the potential $V(x)$ one finds:

$$V(x) \approx 2h/x^3 \quad (x \to \infty \text{ spatial asymptotic}),$$

$$V(x) \approx -1/(4x^2) \quad (x \to 0 \text{ the sphere } S_{\text{trans}}). \quad (45)$$

Thus we have a quadratic potential well at $S_{\text{trans}}$, which is placed at $x = 0$ by choosing the proper value of the arbitrary constant in the definition of $x$ in Eq. (43).

The same form of Eq. (43) is obtained for the Einstein frame $\mathcal{M}_E'$, but with another potential $V(\phi)$ due to the slightly different form of the solution in this “anti-gravitational” region. One easily finds, however, that the asymptotics of the potential at $x \to 0$ and $x \to \infty$ are again given by (43), though with $h$ replaced by some $h' > 0$ which is, in general, not equal to $h$.

It makes sense to change $x \to -x$ in $\mathcal{M}_E'$, which does not affect Eq. (43) but makes it possible to unify the perturbation equations for the two parts of $\mathcal{M}_J$, the space-time of the Jordan-frame. We thus obtain Eq. (43) with $x \in \mathbb{R}$ and a certain function $V(x)$, vanishing at large $|x|$ and providing a potential well of the form $V \approx 1/(4x^2)$ near $x = 0$.

The boundary conditions at both spatial asymptotics are obtained from the requirement that the perturbations should possess finite energy. This requirement upon the perturbed EMT leads to the condition $x Y \to 0$ as $x \to \pm \infty$. Meanwhile, the asymptotic form of any solution of (43) with $\Omega > 0$ at large $|x|$ is

$$Y \approx C_1 e^{\Omega|x|} + C_2 e^{-\Omega|x|}, \quad C_{1,2} = \text{const.} \quad (46)$$

Therefore an admissible solution is the one with $C_1 = 0$, with only a decaying exponential. Actually, the conditions at both infinities are that $Y \to 0$, i.e., coincide with the boundary conditions for the one-dimensional wave function under the same potential in QM.

As is evident from QM (see, e.g., (9)), a potential well of the form $V \approx 1/(4x^2)$ always possesses negative energy levels, $E = -\Omega^2 < 0$; moreover, the absolute value of $\Omega$ has no upper bound. The latter statement can be proved, e.g., by comparing Eq. (44) with our $V(x)$ and with rectangular potentials $V \geq V$ for which $Y(x)$ and $E$ are easily found; one can then use the fact that $E_{\text{min}}[V] < E_{\text{min}}[\tilde{V}]$ where $E_{\text{min}}$ is the lowest energy level (ground state) for a given potential.

Recalling that $\Omega$ is the perturbation growth increment, we can conclude that our wormholes decay instantaneously within linear perturbation theory. Non-perturbative analysis would probably smooth out this infinite decay rate.

The behaviour of the perturbations near $S_{\text{trans}}$ ($x = 0$) of interest. The asymptotic form of the solution to (43) at small $x$ is

$$Y = \sqrt{|x|}(c_1 + c_2 \ln |x|), \quad c_{1,2} = \text{const}, \quad (47)$$

therefore the perturbation $\delta \psi \sim \Phi \sim Y/\sqrt{|x|}$ behaves as $c_1 + c_2 \ln |x|$, i.e., generically blows up at $x = 0$ but at the same rate as $\psi$ itself, so that the perturbation
scheme still works. Furthermore, with \( \delta \phi \) it is easy to find that the perturbation \( \delta \phi \) behaves at small \( x \) as \( x(c_1 + c_2 \ln|x|) \), so that \( \delta \phi(0) = 0 \). In other words, the perturbation as a function of time rapidly grows around \( S_{\text{trans}} \) due to the instability but vanishes on this sphere itself.

4. Concluding remarks

Our perturbation analysis proves the instability of both neutral and charged wormhole solutions for any \( \xi > 0 \). This conclusion can be extended to all solutions continued through the sphere \( S_{\text{trans}} \), where the original function \( f(\phi) \) in the action \([1]\) vanishes. This actually means that the effective gravitational constant, proportional to \( f^{-1} \), blows up and changes its sign. The violent instability occurs due to a negative pole of the perturbational effective potential \( V(x) \) on \( S_{\text{trans}} \). Comparing the present results with those of our previous paper \([11]\), we see that neither the particular value of the coupling constant \( \xi \) nor the presence of matter (the electromagnetic field in our case) change the situation.

The instability of black holes with a conformal scalar field, found long ago in Ref. \([18]\), is another example of such a phenomenon. A similar instability was pointed out by Starobinsky \([20]\) for cosmological models with conformally coupled scalar fields.

It is quite plausible that instabilities of this kind are a common feature of STT solutions with conformal continuations, for which the transformation \([3]\) maps the Einstein-frame manifold \( M_E(g) \) to only a part of the whole Jordan-frame manifold \( M_J[g] \). Solutions containing such continuations always exist in STT in which the function \( f(\phi) \) has at least one simple zero \([21]\), irrespective of the particular form of \( f(\phi) \). Wormholes solutions turn out to be generic among the conformally continued solutions, but there also exist other kinds of configurations \([21]\). One can anticipate that all of them are unstable since the cause of instability is the transition itself rather than the wormhole nature of the solutions.

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