Electromagnetic shielding in quantum metrology

Yao Jin\textsuperscript{1} and Hongwei Yu\textsuperscript{1,2,*}

\textsuperscript{1} Institute of Physics and Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, China
\textsuperscript{2} Center for Nonlinear Science and Department of Physics, Ningbo University, Ningbo, Zhejiang 315211, China

Abstract

The dynamics of the quantum Fisher information of the parameters of the initial atomic state and atomic transition frequency is studied, in the framework of open quantum systems, for a static polarizable two-level atom coupled in the multipolar scheme to a bath of fluctuating vacuum electromagnetic fields without and with the presence of a reflecting boundary. Our results show that in the case without a boundary, the electromagnetic vacuum fluctuations always cause the quantum Fisher information of the initial parameters and thus the precision limit of parameter estimation to decrease. Remarkably, however, with the presence of a boundary, the quantum Fisher information becomes position and atomic polarization dependent, and as a result, it may be enhanced as compared to that in the case without a boundary and may even be shielded from the influence of the vacuum fluctuations in certain circumstances as if it were a closed system.

PACS numbers: 06.20.-f, 03.65.Yz, 03.65.Ta

\textsuperscript{*} Corresponding author
I. INTRODUCTION

In estimation theory, the Cramér-Rao bound \([1, 2]\) was proposed to describe how well one can estimate a parameter from the probability distribution and the Fisher information is used to describe the precision limit. Since quantum mechanics is strongly related to probability theory in the sense that when we make quantum measurements on quantum mechanical systems, the observed outcomes follow a probability distribution, the Fisher information is readily extended to quantum regime and the inverse of the so-called quantum Fisher information (QFI) gives the lower bound of the error of the parameter estimation in quantum metrology \([1–4]\). With different models of the probe systems and different parameters to be estimated, many applications of quantum metrology have been done and some of those are of practical significance in quantum technology such as quantum frequency standards \([5]\), optimal quantum clock \([6]\), measurement of gravity accelerations \([7]\), clock synchronization \([8]\), only to name a few. The central task in quantum metrology is to improve the precision of parameter estimation. Since a larger quantum Fisher information means better precision, increasing the QFI becomes a key issue in quantum metrology. A straightforward way, in addition to increasing the QFI, to enhance the precision when the probe systems are closed is by parallel measurements. Later it was shown that the use of correlated systems such as entangled states can also improve the precision of parameter estimation\([5, 9–21]\). On the other hand, however, interaction between a system and an environment is unavoidable in reality, and the quantum decoherence induced by such interactions may decrease the QFI and destroy the quantum entanglement in the probe system exploited to improve the precision. In this regard, It has been shown that the interaction between a system and an environment usually makes the measurements noisy, which in turn degrades the estimation precision \([22–41]\).

One environment which no system can be isolated from is the vacuum that fluctuates all the time in quantum sense. In the present paper, we are interested in how the vacuum fluctuations affect the QFI regarding the estimation of parameters of the initial states of a probe system which is modeled by a neutral polarizable two-level atom interacting with
emotional vacuum fluctuations. We first examine how the decoherence caused by the vacuum fluctuations decreases the QFI as time evolves in an unbound space, then we ask what happens if the vacuum fluctuations are changed somehow, for example, by the presence of a reflecting boundary, i.e., can we protect the QFI from decreasing? We demonstrate that in an unbounded space, the QFI decreases exponentially with time. This behavior is similar to that observed in the amplitude-damping channel of decoherence derived in \cite{40}. However, with the presence of a boundary, the situation changes dramatically in certain circumstances such that the QFI may even be protected from decreasing as if it were isolated from the environment. In other words, QFI may be shielded from quantum decoherence due to electromagnetic vacuum fluctuations in some special cases. At this point, let us note that the modification of vacuum fluctuations has been demonstrated to yield many interesting quantum phenomena such as the Casimir effect \cite{42, 43}, the light-cone fluctuations when gravity is quantized \cite{44}, the Brownian motion of test particles in an electromagnetic vacuum \cite{45}, the position-dependent spontaneous decay rates and geometric phase in an electromagnetic vacuum \cite{46, 47}.

II. QUANTUM FISHER INFORMATION AND DYNAMICAL EVOLUTION OF A TWO-LEVEL ATOM COUPLED WITH VACUUM FLUCTUATIONS

For a given quantum state $\rho(X)$ parametrized by an unknown parameter $X$, the unknown parameter can be inferred from a set of measurements, usually modelled mathematically by a set of positive-operator valued measures, on the state. After the optimization of the measurements and the estimator, a precision limit of the unknown parameter estimation is obtained \cite{4}

$$Var(X) \geq \frac{1}{NF_X} ,$$

where $N$ represents the repeated times and $F_X$ denotes the quantum Fisher information of parameter $X$ given by

$$F_X = \text{Tr} (\rho(X)L^2) .$$

Here $L$ is the so-called symmetric logarithmic derivative Hermitian operator, which satisfies the equation $\partial_X \rho(X) = \frac{1}{2}\{\rho(X),L\}$ with $\{\}$ standing for the anti-commutator. In the
orthonormal basis \( \rho(X) = \sum_{i=1}^{N} p_i |\psi_i\rangle \langle \psi_i| \), \( L \) can be solved for and the quantum Fisher information can be written as [4]

\[
F_X = 2 \sum_{m,n} \frac{|\langle \psi_m | \partial_X \rho | \psi_n \rangle|^2}{p_m + p_n}.
\] (3)

For a two-level system, the state of the system can be expressed in the Bloch sphere representation as

\[
\rho = \frac{1}{2} (I + \omega \cdot \sigma),
\] (4)

where \( \omega = (\omega_1, \omega_2, \omega_3) \) is the Bloch vector and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) denotes the Pauli matrices. As a result, \( F_X \) can be expressed in a simple form [40]

\[
F_X = \begin{cases} 
|\partial_X \omega|^2 + \frac{|\omega \cdot \partial_X \omega|^2}{|\omega|^2}, & |\omega| < 1, \\
|\partial_X \omega|^2, & |\omega| = 1.
\end{cases}
\] (5)

Now let us first calculate the QFI for the initial state’s parameter estimation for an arbitrary state of the two-level atom

\[
|\psi\rangle = \cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle,
\] (6)

where \( \theta \) and \( \phi \) correspond to the weight parameter and phase parameter, \( |+\rangle, |-\rangle \) denote the excited state and ground state of the atom respectively, and the Bloch vector of the state can be represented as \( \omega = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). So the quantum Fisher information of \( \theta \) and \( \phi \) can be easily calculated as \( F_\theta = 1 \) and \( F_\phi = \sin^2 \theta \). Taking the atom as a closed system, whose evolution is governed by the Hamiltonian \( H_a = \frac{1}{2} \hbar \omega_0 \sigma_3 \), with \( \omega_0 \) denoting the transition frequency, one can easily show that the Bloch vector of the state with time \( \tau \) becomes \( \omega = (\sin \theta \cos(\phi + \omega_0 \tau), \sin \theta \sin(\phi + \omega_0 \tau), \cos \theta) \), and the quantum Fisher information remains \( F_\theta = 1 \) and \( F_\phi = \sin^2 \theta \), which is time independent. Thus the unitary evolution does not change the quantum Fisher information of initial parameters.

However, if the interaction with an environment is considered, the story may be different and the influence of the environment will in general be encoded in the atomic state with time. This is just what we are going to discuss next, and we consider how the QFI changes when the interaction with electromagnetic vacuum fluctuations is taken into account. For
this purpose, let us study a static polarizable two-level atom interacting with fluctuating electromagnetic fields in vacuum, and in this case, the total Hamiltonian of the coupled system can be written as \( H = H_s + H_f + H' \), where \( H_s \), which is given before, is the Hamiltonian of the atom. \( H_f \) denotes the Hamiltonian of the free electromagnetic field and its explicit expression is not required here. The Hamiltonian that describes the interaction between the atom and the electromagnetic field can be written in the multipolar coupling scheme as

\[
H' (\tau) = -e \mathbf{r} \cdot \mathbf{E}(x(\tau)),
\]

where \( e \) is the electron electric charge, \( e \mathbf{r} \) is the atomic electric dipole moment, and \( \mathbf{E}(x) \) denotes the electric field strength. Let us note that, since both \( \mathbf{r}(\tau) \) and \( \mathbf{E}(x) \) are not world vectors, the interaction Hamiltonian \( H' \) is ambiguous when we deal with atoms in motion. However, a manifestly coordinate invariant generalization of \( H' \) can be given \cite{48}:

\[
H' (\tau) = -e r^\mu(\tau) F_{\mu\nu}(x(\tau)) u^\nu(\tau),
\]

where \( F_{\mu\nu} \) is the field strength, \( r^\mu(\tau) \) is a four-vector and its temporal component in the frame of the atom vanishes and its spatial components in the same frame are given by \( \mathbf{r}(\tau) \), and \( u^\nu \) is the four velocity of the atom. Since we have \( u^\nu(\tau) = (1, 0, 0, 0) \) in the frame of the atom, this extended interaction Hamiltonian reduces to that given by Eq. (7) in the reference frame of the atom. So we choose to work in this reference frame. Notice that we assume that the atom is static, as a result, the laboratory frame is equivalent to the frame of the atom.

We let \( \rho_{tot} = \rho(0) \otimes |0\rangle\langle 0| \) be the initial total density matrix of the system. Here \( \rho(0) \) is the initial reduced density matrix of the atom, which corresponds to the atomic state in Eqs. (6), and \( |0\rangle \) is the vacuum state of the field. The evolution of the total density matrix \( \rho_{tot} \) in the proper time \( \tau \) reads as

\[
\frac{\partial \rho_{tot}(\tau)}{\partial \tau} = -\frac{i}{\hbar} [H, \rho_{tot}(\tau)].
\]

We assume that the interaction between the atom and field is weak. So, the evolution of the reduced density matrix \( \rho(\tau) \) can be written in the Kossakowski-Lindblad form \cite{49,50}

\[
\frac{\partial \rho(\tau)}{\partial \tau} = -\frac{i}{\hbar} [H_{eff}, \rho(\tau)] + \mathcal{L}[\rho(\tau)],
\]
where
\[
\mathcal{L}[\rho] = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \left[ 2 \sigma_j \rho \sigma_i - \sigma_i \sigma_j \rho - \rho \sigma_i \sigma_j \right].
\] (11)

The coefficients of the Kossakowski matrix \( a_{ij} \) can be expressed as
\[
a_{ij} = A \delta_{ij} - B \epsilon_{ijk} \delta_{k3} - A \delta_{i3} \delta_{j3},
\] (12)

with
\[
A = \frac{1}{4} \left[ G(\omega_0) + G(-\omega_0) \right], \quad B = \frac{1}{4} \left[ G(\omega_0) - G(-\omega_0) \right].
\] (13)

We define a two-point correlation function, \( G^+(x - x') \), which is related to the two-point functions of the electromagnetic fields, \( \langle 0|E_i(x)E_j(x')|0 \rangle \), as
\[
G^+(x - x') = \frac{e^2}{\hbar^2} \sum_{i,j=1}^{3} \langle +|r_i|-\rangle \langle -|r_j|+\rangle \langle 0|E_i(x)E_j(x')|0 \rangle,
\] (14)

and its Fourier and Hilbert transforms, \( G(\lambda) \) and \( K(\lambda) \), then follows
\[
G(\lambda) = \int_{-\infty}^{\infty} d\Delta \tau e^{i \lambda \Delta \tau} G^+ (\Delta \tau), \quad K(\lambda) = \frac{P}{\pi i} \int_{-\infty}^{\infty} d\omega \frac{G(\omega)}{\omega - \lambda}.
\] (15)

By absorbing the Lamb shift term, the effective Hamiltonian \( H_{\text{eff}} \) can be written as
\[
H_{\text{eff}} = \frac{1}{2} \hbar \Omega \sigma_3 = \frac{\hbar}{2} \{ \omega_0 + i \frac{A}{2} [K(-\omega_0) - K(\omega_0)] \} \sigma_3,
\] (16)

where \( \Omega \) is the effective level spacing of the atom. By applying Eq. (4) to Eq. (10), the Bloch vector with proper time \( \tau \) can be solved as:
\[
\omega_1(\tau) = \sin \theta \cos (\Omega \tau + \phi) e^{-2A\tau},
\]
\[
\omega_2(\tau) = \sin \theta \sin (\Omega \tau + \phi) e^{-2A\tau},
\]
\[
\omega_3(\tau) = \cos \theta e^{-4A\tau} - \frac{B}{A} (1 - e^{-4A\tau}).
\] (17)

### III. INFLUENCE OF VACUUM FLUCTUATIONS ON INITIAL PARAMETER ESTIMATION

Let us now examine how the vacuum fluctuations affect the quantum Fisher information and thus the precision of the initial parameter estimation. Using the following electric two-
point function [51]

\[
\langle 0 | E_i(x(\tau)) E_j(x(\tau')) | 0 \rangle_0 = \frac{\hbar c}{4\pi^2\varepsilon_0} \left( \partial_0 \partial' \delta_{ij} - \partial_i \partial'_j \right) \times \frac{1}{(x - x')^2 + (y - y')^2 + (z - z')^2 - (ct - ct' - i\varepsilon)^2},
\]

where \( \varepsilon \to +0, \partial' \) denotes the differentiation with respect to \( x' \) and the subscript 0 indicates the vacuum two-point functions in the unbounded space. Applying the trajectory of the atom

\[
t(\tau) = \tau, \quad x(\tau) = x_0, \quad y(\tau) = y_0, \quad z(\tau) = z_0,
\]

we find that the electric-field two-point functions can be written as

\[
\langle 0 | E_i(x(\tau)) E_j(x(\tau')) | 0 \rangle_0 = \frac{\hbar c \pi}{2\varepsilon_0 \omega_0^3} \delta_{ij} \left( c \Delta \tau - i\varepsilon \right)^4,
\]

So the Fourier transform of the correlation functions can be calculated as

\[
G^{(0)}(\lambda) = \sum_i e^{2\left(\langle -|r_i|+\rangle\right)^2\lambda^2} \frac{2 \lambda^3}{3 \varepsilon_0 \hbar c^3} \theta(\lambda),
\]

with \( \theta(\lambda) \) being the standard step function. Let us note that here we let \( \varepsilon = 0 \) after the calculation. The coefficients of the Kossakowski matrix \( a_{ij} \) and the effective level spacing of the atom are now given by

\[
A^{(0)} = B^{(0)} = \frac{\gamma_0}{4}, \quad \Omega^{(0)} = \omega_0 + \frac{\gamma_0}{2\pi \omega_0^3} P \int_0^\infty d\omega \omega^3 \left( \frac{1}{\omega + \omega_0} - \frac{1}{\omega - \omega_0} \right).
\]

Here \( \gamma_0 = e^2\left(\langle -|r|+\rangle\right)^2 \omega_0^3/3\pi \varepsilon_0 \hbar c^3 \) denotes the spontaneous emission rate in vacuum without boundaries, As a result, the quantum Fisher information of the initial weight and phase parameter become

\[
F_{\phi}^{(0)} = \sin^2 \theta e^{-\gamma_0 \tau}
\]

and

\[
F_{\theta}^{(0)} = e^{-\gamma_0 \tau}.
\]

This shows that the QFI of both weight and phase parameters decreases exponentially with time due to the decoherence caused by the interaction between the atom and the fluctuating
vacuum and the decay rate equals the spontaneous emission rate of the atom in vacuum. Therefore, when the measurement time is larger than the relaxation time \(1/\gamma_0\), the precision of the estimation is greatly damaged. This kind behavior of the QFI is the same as that in the amplitude-damping channel described in [40] as expected.

Since vacuum fluctuation will be modified if we set a boundary in the vacuum, we may wonder how the presence of a boundary which confines the vacuum fluctuations of the field affects the quantum Fisher information of the initial parameters with time. Now we consider the case with the presence of a reflecting boundary at \(z = 0\) \(^1\). The electric two-point functions in this case can be expressed as a sum of the Minkowski vacuum term and a correction term due to the boundary:

\[
\langle E_i(x(\tau))E_j(x(\tau')) \rangle = \langle E_i(x(\tau))E_j(x(\tau')) \rangle_0 + \langle E_i(x(\tau))E_j(x(\tau')) \rangle_b ,
\]

where the \(\langle E_i(x(\tau))E_j(x(\tau')) \rangle_0\) is the two-point function in the unbounded vacuum which has already been calculated above and

\[
\langle 0|E_i(x(\tau))E_j(x(\tau'))|0 \rangle_b = -\frac{\hbar c}{4\pi^2\varepsilon_0} \left[ (\delta_{ij} - 2n_in_j) \partial_0 \partial'_0 - \partial_i \partial'_j \right] \frac{1}{(x-x')^2 + (y-y')^2 + (z+z')^2 - (ct-ct' - i\varepsilon)^2} ,
\]

gives the correction due to the boundary. Here \(n = (0,0,1)\) is the unit vector normal to the boundary. By applying the trajectory in Eq. (19), the boundary term of the electric-field two-point functions can be written as

\[
\langle 0|E_x(x(\tau))E_x(x(\tau'))|0 \rangle_b = \langle 0|E_y(x(\tau))E_y(x(\tau'))|0 \rangle_b = -\frac{\hbar c}{\pi^2\varepsilon_0} \frac{c^2\Delta\tau^2 + 4z_0^2}{[ (c \Delta\tau - i\varepsilon)^2 - 4z_0^2]^3} ,
\]

\[
\langle 0|E_z(x(\tau))E_z(x(\tau'))|0 \rangle_b = \frac{\hbar c}{\pi^2\varepsilon_0} \frac{1}{[ (c \Delta\tau - i\varepsilon)^2 - 4z_0^2]^2} .
\]

The Fourier transform of the correlation functions can be calculated as [47]:

\[
\mathcal{G}(\lambda) = \sum_i \frac{e^2|\langle -|r_i|+\rangle|^2\lambda^3}{3\pi\varepsilon_0 \hbar c^3} (1 - f_i(\lambda, z_0)) \theta(\lambda) ,
\]

\(^1\) Let us note here that external forces may be needed to balance the Casimir-Polder force to keep a polarizable atom at a fixed position near a reflecting boundary.
with
\[
f_x(\lambda, z_0) = f_y(\lambda, z_0) = \frac{3c^3}{16\lambda z_0^3} \left[ \frac{2\lambda z_0}{c} \cos \frac{2\lambda z_0}{c} + \left( \frac{4\lambda^2 z_0^2}{c^2} - 1 \right) \sin \frac{2\lambda z_0}{c} \right],
\]
\[
f_z(\lambda, z_0) = \frac{3c^3}{8\lambda z_0^3} \left[ \frac{2\lambda}{c} \cos \frac{2\lambda z_0}{c} - \sin \frac{2\lambda z_0}{c} \right].
\]
Thus the coefficients of the Kossakowski matrix \(a_{ij}\) and the effective level spacing of the atom now become
\[
A = B = \frac{\gamma_0}{4} (1 - \sum_i \alpha_i f_i(\omega_0, z_0)),
\]
\[
\Omega = \omega_0 + \frac{\gamma_0}{2\pi\omega_0^3} P \int_0^\infty d\omega \omega^3 \left( 1 - \sum_i \alpha_i f_i(\omega_0, z_0) \right) \left( \frac{1}{\omega + \omega_0} - \frac{1}{\omega - \omega_0} \right),
\]
where \(\alpha_i = |\langle -| r_i |+ \rangle|^2/|\langle -| r |+ \rangle|^2\). Physically, \(\alpha_i\) represents the relative polarizability and they satisfy \(\sum_i \alpha_i = 1\). We let \(f(\omega_0, z_0) \equiv \sum_i \alpha_i f_i(\omega_0, z_0)\). As a result, the quantum Fisher information of the initial weight and phase parameters can be expressed as follows
\[
F_\phi = \sin^2 \theta e^{-\gamma_0(1 - f(\omega_0, z_0))\tau}
\]
and
\[
F_\theta = e^{-\gamma_0(1 - f(\omega_0, z_0))\tau}.
\]
So, the decay rate of the QFI is now modified by the factor, \(1 - f(\omega_0, z_0)\), as compared to the unbounded case. Let us now first examine the asymptotic behaviors of the QFI, that is, when the atom is placed very far from the boundary or very close to it. When \(z_0 \to \infty\), \(f_i(\omega_0, z_0) \to 0\), the unbounded case is recovered as expected. When \(z_0 \to 0\), \(f_x(\omega_0, z_0) = f_y(\omega_0, z_0) = f_z(\omega_0, z_0) = 1\), so atoms with different polarization will behave differently. When the polarization is along the \(z\)-axis, i.e., \(\alpha_x = \alpha_y = 0\), the decay rate becomes double of that in the unbounded Minkowski vacuum case, which makes the QFI decay even faster than that without the boundary. However, when the polarization is in the \(xy\) plane, i.e., \(\alpha_z = 0\), the decay rate becomes zero, which means that the QFI is totally protected from electromagnetic vacuum fluctuations for transversely polarizable atoms extremely close to the boundary as if it were isolated. For an isotropic polarization, \(\alpha_x = \alpha_y = \alpha_z = \frac{1}{3}\), the decay rate is \(\frac{2\gamma_0}{3}\), so the QFI decreases slower and the precision is enhanced by the presence of the boundary as compared to the unbounded case.
For a generic position and polarization, the QFI may be decreased, or enhanced as compared to the unbounded case. This can be seen from the fact that $f_x$ decays in an oscillatory manner from 1 to 0 with increasing atomic distance, while $f_z$ does from $-1$ to 0. In general, there exist some special positions where $\sum_i \alpha_i f_i(\omega_0, z_0) = 0$. The quantum Fisher information of the initial parameters of the atom at these positions takes the same form as that in the unbounded case and the boundary effects disappear. In regions between these special positions, we have either $\sum_i \alpha_i f_i(\omega_0, z_0) > 0$ or $< 0$. As a result, the quantum Fisher information is either enhanced or decreased as compared to that in the unbounded space. To show the properties we described above graphically, we plot, in Fig. (1), $1 - f(\omega_0, z_0)$ as a function of $z_0$ for $\alpha = (1, 0, 0), (0, 0, 1), (1/3, 1/3, 1/3)$, which corresponds to the parallel, vertical, isotropic polarization cases respectively. The oscillatory behavior displayed in

![Graphs showing $1 - f$ as a function of $z_0$ for different polarizations](image)

**FIG. 1**: $1 - f$ as a function of $z_0$ for $\alpha = (1, 0, 0), (0, 0, 1), (1/3, 1/3, 1/3)$ respectively. Here $z_0$ is in the unit of $c/\omega_0$.

Fig. (1) is related to the stationary modes of the fields as a result of the superposition of the propagating incident and reflected modes. The oscillations of both the tangential and normal components of the field lead to the different oscillatory behaviors of the decay rates of both the atoms polarizable in the directions parallel and vertical to the planar boundary.
IV. EFFECTS OF VACUUM FLUCTUATIONS ON ATOMIC FREQUENCY ESTIMATION

As we have demonstrated, the presence of a reflecting boundary may protect the estimation precision of the initial parameters from the influence of the environment in certain circumstances. We may also wonder how it affects other parameters’ estimation such as the frequency estimation. Now, our unknown parameter becomes the atomic frequency $\omega_0$. By using Eq. (5) and Eq. (17), the quantum Fisher information of $\omega_0$ in the unbounded case and the bounded case can be calculated, after dropping the terms which are of higher order in terms of the fine structure constant in the effective level spacing of the atom, to the first order as

\[
F_{\omega_0}^{(0)} \approx \sin^2 \theta e^{-\gamma_0 \tau} \tau^2 + 16 \left( \frac{\partial \gamma_0}{\partial \omega_0} \right)^2 \left[ 1 + \frac{\sin^2 \theta}{4} + \cos \theta (\cos \theta + 2) e^{-\gamma_0 \tau} \right] e^{-\gamma_0 \tau} \tau^2 \quad (36)
\]

and

\[
F_{\omega_0} \approx \sin^2 \theta e^{-\gamma_0 \tau (1-f(\omega_0, z_0))} \tau^2 + 16 \left\{ \frac{\partial [\gamma_0 (1-f(\omega_0, z_0))]}{\partial \omega_0} \right\}^2 \times \left[ 1 + \frac{\sin^2 \theta}{4} + \cos \theta (\cos \theta + 2) e^{-\gamma_0 \tau (1-f(\omega_0, z_0))} \right] \times e^{-\gamma_0 \tau (1-f(\omega_0, z_0))} \tau^2 . \quad (37)
\]

So, the maximum quantum Fisher information is obtained when $\theta = \frac{\pi}{2}$, which is given, after dropping the higher order term, by

\[
F_{\omega_0}^{(0)} \approx e^{-\gamma_0 \tau} \tau^2 \quad (38)
\]

and

\[
F_{\omega_0} \approx e^{-\gamma_0 \tau (1-f(\omega_0, z_0))} \tau^2 . \quad (39)
\]

Applying the equation $\frac{\partial F_{\omega_0}}{\partial \tau} = 0$, we find that the maxima of $F_{\omega_0}^{(0)}$ and $F_{\omega_0}$, $\frac{4}{\gamma_0} e^{-2}$ and $\frac{4}{\gamma_0 (1-f(\omega_0, z_0))} e^{-2}$, can be reached at $\tau^{(0)} = \frac{2}{\gamma_0}$ and $\tau = \frac{2}{\gamma_0 (1-f(\omega_0, z_0))}$. When $z_0 \to \infty$, $f_x(\omega_0, z_0) \to 0$, the bounded vacuum case reduces to the unbounded case as expected. When $z_0 \to 0$, $f_x(\omega_0, z_0) = f_y(\omega_0, z_0) = -f_z(\omega_0, z_0) = 1$, thus different polarization directions lead to different results. When the polarization direction is along the $z$ axis, i.e., $\alpha_x = \alpha_y = 0$, $\alpha_z = 1,$ and $2.$
the optimal measurement time and the maximal quantum Fisher information in the bounded vacuum case become \( \tau = \frac{1}{\gamma_0} \) and \( F_{\omega_0} = \frac{1}{\gamma_0^2} e^{-2} \). The optimal measurement time is half of that in the unbounded vacuum case and the maximal quantum Fisher information is a quarter of that in the unbounded case. When the polarization direction is in the \( xy \) plane, i.e., \( \alpha_z = 0 \), both the optimal measurement time and the maximal quantum Fisher information in the bounded case approach to infinity, and the quantum Fisher information with time \( \tau \) approaches to \( \tau^2 \), which is consistent with the quantum Fisher information in unitary evolution \( \omega = (\cos(\phi + \omega_0 \tau), \sin(\phi + \omega_0 \tau), 0) \). When polarization direction is isotropic, \( \alpha_x = \alpha_y = \alpha_z = \frac{1}{3} \), the optimal measurement time and the maximal quantum Fisher information in the bounded vacuum case become \( \tau = \frac{3}{\gamma_0} \) and \( F_{\omega_0} = \frac{9}{\gamma_0^2} e^{-2} \), which are both larger than those in unbounded vacuum case. For an arbitrary polarization, in regions where \( f(\omega_0, z_0) > 0 \), both the optimal measurement time and the maximal quantum Fisher information are enhanced as compared to the unbounded vacuum case, while in regions where \( f(\omega_0, z_0) < 0 \), they are depressed. Let us note that in estimation of a transition frequency, the total time \( T \) of probing is also a resource. For a given \( T \gg \frac{1}{\gamma_0} \), we can improve the precision by accomplishing a sequence of measurements [23]. We let \( N = \frac{T}{\tau} \) denote the repeated times of the measurements. Then, according to Eq. (1), the uncertainty of the atomic frequency satisfies

\[
|\Delta \omega_0| \geq \frac{1}{\sqrt{NF_Q}} = \frac{1}{\sqrt{TF_Q/\tau}} = \frac{1}{\sqrt{T\tau e^{-\gamma_0 \tau(1 - f(\omega_0, z_0))}}} \tag{40}
\]

Applying \( \frac{\partial |\Delta \omega_0|}{\partial \tau} = 0 \), we obtain the minimum uncertainty of \( \omega_0 \), i.e., \( |\Delta \omega_0|_{min} = \sqrt{\frac{\gamma_0(1 - f(\omega_0, z_0))}{T}} \), and the optimal sequence measurement time \( \tau = \frac{1}{\gamma_0(1 - f(\omega_0, z_0))} \). As a result, the optimal repeated times of measurements in a given total time of probing to maximize the precision is \( T \gamma_0 (1 - f(\omega_0, z_0)) \). In regions where \( f(\omega_0, z_0) > 0 \), the precision is enhanced and the measurement times we need to obtain the maximum precision is less than that in the unbounded case. In regions where \( f(\omega_0, z_0) < 0 \), the precision is degraded and the measurement times become more than that in the unbounded case. It could also be possible to further improve this precision by the use of an entangled probe system following [23], the detailed analysis of which is left as future work. Here, we give a very brief comment on the
issue. Take \( N \) maximally entangled atoms in vacuum for example and let \( \rho_{tot} = \rho(0) \otimes |0\rangle\langle 0| \) be the initial total density matrix of the system, where \( \rho(0) \) is the initial reduced density matrix of the \( N \) maximally entangled atoms. Then we can use Eq. (9) to study the evolution of the total state. After obtaining the state of the atoms in time, we can in principle calculate the quantum Fisher information using Eq. (3). Because of the indirect interaction between atoms caused by the field, it is not easy to find an analytical result of the reduced density matrix of the \( N \) atoms in time \( \tau \). But in the special case when all the coefficients of the dissipative part in the evolution equation of the reduced density matrix vanish, all the atoms behave like closed systems and the Heisenberg precision limit is protected from deterioration. This condition can be approximately fulfilled if atoms are transversely polarizable and very close to the boundary.

V. CONCLUSION

In summary, we studied the dynamics of the quantum Fisher information for the atomic parameter estimation for a static polarizable two-level atom coupled in the multipolar scheme to a bath of fluctuating vacuum electromagnetic fields without and with the presence of a reflecting boundary. When we estimate the parameters of initial atomic state, we find that, in the case without a boundary, the electromagnetic vacuum fluctuations always cause the quantum Fisher information of the initial parameters and thus the precision limit of parameter estimation to decrease. However, with the presence of a boundary, the quantum Fisher information becomes position and atomic polarization dependent, and thus the precision of the initial parameter estimation may be decreased, enhanced or remain unchanged as compared to the case without a boundary depending on the position and polarization. When the atom is extremely close to the boundary and is transversely polarizable, the quantum Fisher information may even be shielded from the influence of the vacuum fluctuations and remains constant with time as if it were a closed system. For the estimation of the atomic frequency, there exist a maximum quantum Fisher information and optimal measurement time, which can also both be enhanced or decreased as compared to the case without a
boundary.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grants No. 11435006, and No. 11375092, and the Specialized Research Fund for the Doctoral Program of Higher Education under Grant No. 20124306110001

[1] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976).
[2] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North-Holland, Amsterdam, 1982).
[3] M. Hübner, Phys. Lett. A 163, 239 (1992); 179, 226 (1993).
[4] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994).
[5] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A 54, R4649(R) (1996).
[6] V. Bužek, R. Derka, and S. Massar, Phys. Rev. Lett. 82, 2207 (1999).
[7] A. Peters, K. Y. Chung, and S. Chu, Nature (London) 400, 849 (1999).
[8] R. Jozsa, D. S. Abrams, J. P. Dowling, and C. P. Williams, Phys. Rev. Lett. 85, 2010 (2000).
[9] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A 33, 4033 (1986).
[10] J. P. Dowling, Phys. Rev. A 57, 4736 (1998).
[11] P. Kok, S. L. Braunstein, and J. P. Dowling, J. Opt. B Quantum Semiclass. 6, S811 (2004).
[12] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[13] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. 96, 010401 (2006).
[14] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Phys. Rev. Lett. 98, 090401 (2007).
[15] S. M. Roy and S. L. Braunstein, Phys. Rev. Lett. 100, 220501 (2008).
[16] S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, Phys. Rev. Lett.
101, 040403 (2008).

[17] H. F. Hofmann, Phys. Rev. A 79, 033822 (2009).

[18] J. Estèe, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature (London) 455, 1216 (2008).

[19] P. Hyllus, L. Pezzé, and A. Smerzi, Phys. Rev. Lett. 105, 120501 (2010).

[20] L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).

[21] L. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Phys. Rev. A 85, 022322 (2012), G. Toth, Phys. Rev. A 85, 022322 (2012)

[22] M. Rosenkranz and D. Jaksch, Phys. Rev. A 79, 022103 (2009).

[23] S. F. Huelga, C. Macchiavello, T. Pellizzari, and A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. 79, 3865 (1997).

[24] D. Ulam-Orgikh and M. Kitagawa, Phys. Rev. A 64, 052106 (2001).

[25] M. Sasaki, M. Ban, and S. M. Barnett, Phys. Rev. A 66, 022308 (2002).

[26] A. Shaji and C. M. Caves, Phys. Rev. A 76, 032111 (2007).

[27] A. Monras and M. G. A. Paris, Phys. Rev. Lett. 98, 160401 (2007).

[28] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guta, Nat. Commun. 3, 1063 (2012).

[29] R. Demkowicz-Dobrzański, U. Dorner, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. A 80, 013825 (2009).

[30] T.-W. Lee, S. D. Huver, H. Lee, L. Kaplan, S. B. McCracken, C. Min, D. B. Uskov, C. F. Wildfeuer, G. Veronis, and J. P. Dowling, Phys. Rev. A 80, 063803 (2009).

[31] U. Dorner, R. Demkowicz-Dobrzański, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. 102, 040403 (2009).

[32] Y. Watanabe, T. Sagawa, and M. Ueda, Phys. Rev. Lett. 104, 020401 (2010).

[33] S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, Phys. Rev. A 83, 021804 (2011).

[34] J. Kołodyński, and R. Demkowicz-Dobrzański, Phys. Rev. A 82, 053804 (2010).

[35] M. Kacprowicz, R. Demkowicz-Dobrzański, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Nature Photonics 4, 357 (2010).

[36] M. G. A. Genoni, S. Olivares, and M. G. Paris, Phys. Rev. Lett. 106, 153603 (2011).
[37] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 109, 233601 (2012).
[38] B. M. Escher, R. L. de MatosFilho, and L. Davidovich, Nature Physics 7, 406 (2011).
[39] J. Ma, Y. Huang, X. Wang, and C. Sun, Phys. Rev. A 84, 022302 (2011).
[40] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori, Phys. Rev. A 87, 022337 (2013).
[41] R. Chaves, J. B. Brask, M. Markiewicz, J. Kołodyński, and A. Acín, Phys. Rev. Lett. 111, 120401 (2013).
[42] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).
[43] H. B. G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948).
[44] H. Yu and L. H. Ford, Phys. Rev. D 60, 084023 (1999); H. Yu and L. H. Ford, Phys. Lett. B 496, 107 (2000); H. Yu and P. X. Wu, Phys. Rev. D 68, 084019 (2003); H. Yu, N. F. Svaiter and L. H. Ford, Phys. Rev. D 80, 124019 (2009).
[45] H. Yu and L. H. Ford, Phys. Rev. D 70, 065009 (2004); H. Yu and J. Chen, Phys. Rev. D 70, 125006 (2004); H. Yu, X. Fu and P. Wu, J. Phys. A: Math. Theor. 41, 335402 (2008); H. Yu, J. Chen and P. Wu, JHEP 02, 058 (2006).
[46] Z. Zhu, H. Yu and S. Lu, Phys. Rev. D 73, 107501 (2006).
[47] H. Yu and J. Hu, Phys. Rev. A 86, 064103 (2012).
[48] S. Takagi, Prog. Theor. Phys. Suppl. 88, 1, (1986).
[49] V. Gorini, A. Kossakowski, and E. C. G. Surdeshan, J. Math. Phys. 17, 821 (1976); G. Lindblad, Commun. Math. Phys. 48, 119 (1976).
[50] F. Benatti, R. Floreanini and M. Piani, Phys. Rev. Lett. 91, 070402 (2003).
[51] W. Greiner, J. Reinhardt, Field Quantization (Springer, 1996).