Temporal Matching

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Abstract

A link stream is a sequence of pairs of the form \((t, \{u, v\})\), where \(t \in \mathbb{N}\) represents a time instant and \(u \neq v\). Given an integer \(\gamma\), the \(\gamma\)-edge between vertices \(u\) and \(v\), starting at time \(t\), is the set of temporally consecutive edges defined by \(\{(t', \{u, v\}) \mid t' \in [t, t + \gamma - 1]\}\). We introduce the notion of temporal matching of a link stream to be an independent \(\gamma\)-edge set belonging to the link stream. Unexpectedly, the problem of computing a temporal matching of maximum size turns out to be \(NP\)-difficult. We show a kernelization algorithm parameterized by the solution size for the problem. As a byproduct we also depict a 2-approximation algorithm.

Both our 2-approximation and kernelization algorithms are implemented and confronted to link streams collected from real world graph data. We observe that finding temporal matchings is non trivial when sampling our data from such a perspective as: managing peer-working when any pair of peers \(X\) and \(Y\) are to collaborate over a period of one month, at an average rate of at least two exchanges every week. We furthermore design a link stream generating process by mimicking the behaviour of a random moving group of particles under natural simulation, and confront our algorithms to these generated instances of link streams. On tangent areas of our sampling method, the kernelization algorithm leads to an upper bound which nearly meets the lower bound given by the temporal matching computed by the 2-approximation algorithm. All the implementations are open source.

Key words: graph, parameterized algorithm, link stream, open source code

\textsuperscript{*} Part of the results reported in this paper were presented at CTW’18. Links to the source code and the GUI of the link stream generator:
https://github.com/antoinedimitriroux/Temporal-Matching-in-Link-Streams
https://antoinedimitriroux.github.io
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1 Introduction

The problem of finding a maximum matching – a maximum independent edge set – in graph theory is a fundamental, well studied, question. It can be solved in polynomial time by the notorious Edmonds algorithm [6]. On the theoretical side, Edmonds result plays a primary role in combinatorial optimization. This is not only because it made a major historical impact in pointing out the polytope structure of a graph problem, but also because this result has marked the beginning of a long, fruitful, list of matching algorithms all having polynomial worst case time complexity. We can cite in this sense the remarkable works reported in [2,13] where connections between MATCHING and matrix multiplication are exploited in order to sparkle new computational perspectives. In [1] a variant of matching where accents are put on fairness is investigated, and a very nice structural analysis results in a linear time procedure for finding popular matchings (albeit it must be under some natural conditions, the fact that a linear time algorithm exists for this fair version of matching is a very nice algorithmic result). Surprisingly, while being a well-known and fundamentally polynomial algorithmic problem, MATCHING has in the recent years attracted research interest in the parameterized areas of algorithmic as well [7,10]. Here, the overall effort has been put in reducing the polynomial time complexity to linear time, by means of factorising bits of the time complexity to depend on another parameter of the input instance rather than its size.

On the practical side, MATCHING is a convenient formalism to approach task management problems. For instance, in a bipartite graph where one vertex set represents chores and the other vertex set represents executors, each having the ability to execute a (different) subset of chores, MATCHING models the question of maximising the number of chores that can be executed. This problem has been intensively investigated under the setting of streaming inputs, where unpredictable arrivals of executors must be affected to chores in real time, see e.g. [11,16]. In this topic, a very clever randomized solution is given in [5]. More generally, in an arbitrary graph representing compatible coworkers, MATCHING models the concern of maximising the overall workload when work must be done by compatible pairs. This problem has recently been investigated from a heuristics point of view [4], as well as under a quantitative comparison of the greedy approach on large input [17].

From the perspective of mining data collected from human activities, however, the input graph should be taken under the light of the time dimension: graph edges are time stamped edges. They come ordered by the time instants where they are recorded. We call this kind of data a link stream, in the sense of [9,15]. The most natural illustration of such thing is web logs, where any single line of log includes a field under time format. For instance, let us consider a platform where (ideally) specialists can tell in advance the time intervals – discretized
by days, from 1 to 365 – where they are available for refereeing papers in 2018. Besides, the reviewers also communicate, via keywords, the research domains they are competent in refereeing. Let us consider that a manuscript must be processed by two reviewers over at least $\gamma = 90$ ninety consecutive time instants – in our case, days – before a decision can be made. By human resource limitations, a referee will only process a given maximum number of manuscripts at a time. Under these conditions, given a – potentially infinite – number of submitted manuscripts, how many decisions can be made in 2018? Our paper addresses this kind of question under the following formalism of $\gamma$-matching. Unluckily, we will see that answering to such a question is computationally difficult.

A link stream $L$ is a triple $L = (T, V, E)$ where $E$ is a sequence of pairs of the form $(t, \{u, v\})$, with $\{u, v\} \in \binom{V}{2}$ being an edge in the sense of classical loopless undirected simple graphs, and $t \in T \subseteq \mathbb{N}$ an integer representing a discretized time instant. If every pair $(t, \{u, v\})$ in $L$ satisfies $t = t_0$ for some fixed $t_0$, then we say that link stream $L$ is a graph, as per classical graph theory. Given an integer $\gamma$, a time instant $t$, and two distinct vertices $u$ and $v$, we define the $\gamma$-edge between $u$ and $v$ starting at time $t$ as the set $\{(t', \{u, v\}) \mid t' \in [t, t + \gamma - 1]\}$. We say that a $\gamma$-edge $\Gamma$ contains the pair $(t, u)$, for $t \in T$ and $u \in V$, if there exists a vertex $v \in V$ such that $(t, \{u, v\}) \in \Gamma$. Two $\gamma$-edges are independent if there is no pair $(t, u)$, called temporal vertex, that is contained in both of them. Finally, a $\gamma$-matching of link stream $L$ is a set of pairwise independent $\gamma$-edges where each $\gamma$-edge contains exclusively edges from $L$. We consider the problem of computing a maximum $\gamma$-matching of an input link stream, that we call $\gamma$-MATCHING. When $\gamma = 1$, this problem can be solved by a slight extension of previously mentioned Edmonds algorithm [6].

It was unfortunate, and quite unexpected, that we resulted in the $NP$-difficulty of $\gamma$-MATCHING, as soon as $\gamma > 1$. We subsequently address the question of pre-processing, in polynomial time, an input instance of $\gamma$-MATCHING, in order to reduce it to an equivalent instance of smaller size, in the sense of kernelization algorithms introduced in [3]. We show that $\gamma$-MATCHING when parameterized by the solution size admits a quadratic kernel. On the way to do so, we also point out a simple way to produce a 2-approximation algorithm for $\gamma$-MATCHING.

We try to comprehend our result from a practical point of view. From this perspective we design a link stream generating process by mimicking the behaviour of a random moving group of particles, using natural simulation: velocity, friction, and random walk. The generating process helps us in unit

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1 Direct link to the GUI of the generator: [https://antoinedimitriroux.github.io](https://antoinedimitriroux.github.io)
testing our implementations on small generated inputs, as well as in stress testing our implementations on large inputs mimicking natural movements.

Both our 2-approximation and kernelization algorithms are implemented\(^2\) and confronted, not only to our generated link streams, but also, and above all, to two particular sets of link streams collected from real world graph data. In one dataset the link streams have been built by samplings over exchanges collected from the Enron emailing network \([8]\). The other dataset is built by samplings over a recording of \(2 \times 80\) minute Rollerblade touring in Paris \([14]\). The reason for us to sample over the raw data is because, therein, the (machine recorded) consecutive time stamps can happen quite instantaneously for human standards. This leaves no chance for a \(\gamma\)-edge to exist, as soon as \(\gamma > 1\).

For instance with the Enron emails, if the time stamps are discretized down to the order of seconds, there is absolutely no chance for that individuals \(X\) and \(Y\) exchange two different emails in any two consecutive time stamps – seconds, in this case – in the whole link stream: There is simply no time to read the first email and type a reply. For Enron we usually merge up the time stamps to the order of half a week. From this perspective, we sample our raw data by edge contraction over the time dimension, formally as follows. For any link stream \(L = (T, V, E)\) and \(1 < \delta < |T|\), we define a \(\delta\)-sample \(L_\delta = (T_\delta, V_\delta, E_\delta)\) as \(V_\delta = V\), \(T_\delta = \left[\frac{\min T}{\delta}, \frac{\max T}{\delta}\right]\), and

\[
E_\delta = \{(t, \{u, v\}) | \exists t’ \in T : \delta t \leq t’ \leq \delta(t + 1) \land (t’, \{u, v\}) \in E}\.
\]

After running our implementation on the two datasets, we make three observations. First, the question of \(\gamma\)-matching is non trivial as soon as we sample the raw data with “human-understandable” values of \(\delta\) and \(\gamma\). For instance, let us consider the Enron emails with \(\delta \approx \frac{1}{2}\) week. There, we observe when sampling our raw data that the number of \(\gamma\)-edges for \(\gamma\) varying from 2 to 10 is: over 500 for \(\gamma = 10\); over 1000 for \(\gamma = 6\); and over 4500 for \(\gamma = 2\). Moreover, our subsequent numerical analysis can only find \(\gamma\)-matchings of size approximately one fifth of the previous number. The Enron dataset records email exchanges during a period of approximatively 3 years. We reckon during this period that \(\binom{500}{100}\) is a decent combinatorial explosion within the Enron company, and conclude that \(\gamma\)-matching is a non trivial computational problem when sampling \(\delta \approx \frac{1}{2}\) week and \(\gamma \approx 10\). These values translate the fact that any pair of collaborators in the \(\gamma\)-matching necessarily keep exchanging emails together continuously for one month, at a rate of at least one email per week and in average at least two emails every week\(^3\). Second, on small values of

\(^2\) The source code is available at
https://github.com/antoinedimitriroux/Temporal-Matching-in-Link-Streams

\(^3\) Rigorous definition of \(\delta\)-sampling when \(\delta \approx \frac{1}{2}\) week can result to two consecutive timed edges that are at about the double amount of \(\delta\) from each other in the original input: This case happens when the first timed edge is recorded rather at
\( \gamma \), namely \( \gamma \approx 20, 30 \) with \( \delta \leq 2 \) month for Enron and \( \gamma \leq 10 \) with \( \delta \leq 20 \) minutes for Rollernet, we observe that the kernelization algorithm helps in reducing the input link stream to an equivalent instance of size approximately \( 10 - 20\% \) the size of the original input. This gives measurable evidence of performance for our preprocessing by kernelization. Third, we notice from definition that the 2-approximation algorithm produces a lower bound for \( \gamma \)-MATCHING, which is at least half the optimal value. Moreover, we stress that the kernelization algorithm gives a naive upper bound for \( \gamma \)-MATCHING by simply counting the number of \( \gamma \)-edges present in the kernel. Our observation from the numerical analysis is that for tangent areas in the datasets built with our sampling method, these upper bound and lower bound meet. We stress that this case remains marginal. However, it is worth mentioning when a kernelization algorithm provides, numerically, a proof of optimality for a greedy approximation. In particular, our kernelization runtime is under ten seconds for inputs where the input size is some hundreds thousand and the parameter is some thousands. Kernelization would never be numerically helpful if the complexity analysis hides a big function of the parameter in the Landau notation. Luckily, we use simple algorithmic processes for our kernelization.

The paper is organised as follows. We first introduce the notion of temporal matching in Section 2. In Section 3, we present our algorithmic tools in order to obtain our main result, the kernelization algorithm: it is presented in Section 4. In Section 5, we present our numerical analysis. We close the paper with concluding remarks and directions for further research.

2 Temporal matching

Unless otherwise stated, graphs in this paper are simple, undirected and loopless graphs. We denote by \( \mathbb{N} \) the set of non negative integers. Given two integers \( p \) and \( q \), we denote by \([p,q]\) the set \( \{ r \in \mathbb{N} \mid p \leq r \leq q \} \). A link stream \( L \) is a triple \((T, V, E)\) such that \( T \subseteq \mathbb{N} \) is an interval, \( V \) is a set, and \( E \subseteq T \times \binom{V}{2} \). The link stream can be seen as an extension of graphs. Indeed, a graph is a link stream where \(|T| = 1\). The elements of \( V \) are called vertices and the elements of \( E \) are called (timed) edges. A temporal vertex of \( L \) is a pair \((t,v)\) such that \( t \in T \) and \( v \in V \).

Given an integer \( \gamma \), a \( \gamma \)-edge between two vertices \( u \) and \( v \) at time \( t \), denoted \( \Gamma_\gamma(t,u,v) \), is the set \( \{(t',\{u,v\}) \mid t' \in [t,t+\gamma-1]\} \). We say that a \( \gamma \)-edge the beginning of the first \( \delta \)-slice, and the second timed edge is recorded at the end of the second \( \delta \)-slice. In practice, this case is marginal. In any case, for the duration of every \( \gamma \)-edge, the average exchange rate between the two vertices in the \( \gamma \)-edge is still (at least) once every \( \delta \).
\(\Gamma\) contains a temporal vertex \((t, v)\) if there exists a vertex \(u \in V\) such that \((t, \{u, v\}) \in \Gamma\). We say that two \(\gamma\)-edges are independent if there is no temporal vertex \((t, v)\) that is contained in both of them. A \(\gamma\)-matching \(\mathcal{M}\) of a link stream \(L\) is a set of pairwise independent \(\gamma\)-edges. We say that a \(\gamma\)-edge \(\Gamma\) is incident with a vertex \(v \in V\) if there exist a vertex \(u \in V\) and an integer \(t \in T\) such that \(\Gamma = \Gamma_\gamma(t, u, v)\). We say that an edge \(e \in E\) is in a \(\gamma\)-matching \(\mathcal{M}\) if there exists \(\Gamma \in \mathcal{M}\) such that \(e \in \Gamma\).

We focus on the following problem.

**\(\gamma\)-matching**

**Input:** A link stream \(L\) and an integer \(k\).

**Output:** A \(\gamma\)-matching of \(L\) of size \(k\) or a correct answer that such a set does not exist.

**Theorem 1** \(\gamma\)-matching is \(NP\)-difficult.

**Proof:** We prove the \(NP\)-completeness of the decision version of \(\gamma\)-matching by a reduction from 3-Sat, that is well known to be \(NP\)-complete. Let \(\varphi\) be a formula with \(n\) variables \(x_1, \ldots, x_n\) and \(m\) clauses \(C_0, \ldots, C_{m-1}\) such that each clauses is of size at most 3. Without lose of generality, we assume that a clause does not contain twice the same variable. We call \(X\) the set containing the \(n\) variables and \(C\) the set containing the \(m\) clauses.

We define the link stream \(L = (T, V, E)\) in the following way:

- \(T = \{0, (m+1)\gamma - 1\}\).
- \(V = \{x^-, x^+, x_{i\gamma}^+ \mid x \in X\} \cup \{x_{i\gamma}^-, t \in \{0, (m-1)\}\} \cup \{c\}\)
- \(E = E_{\text{var}} \cup E_{\text{cla}}\) where:
  - \(E_{\text{var}} = \{(t, \{x = x^+\}), (t, \{x = x^-\}) \mid t \in \{0, (m+1)\gamma - 1\}, x \in X\}
  - \(\cup \{(t, \{x^+, x_{i\gamma}^+\}), (t, \{x^-, x_{i\gamma}^-\}) \mid t \in \{1, m\gamma\}, x \in X, i = \left\lfloor \frac{t-1}{\gamma} \right\rfloor\}\)
  - \(E_{\text{cla}} = \{(t, \{c, x_{i\gamma}^+\}) \mid t \in \{i\gamma + 1, (i+1)\gamma\}, i \in \{0, m-1\}, x \in X, x\text{ appears positively in } C_i\}
  - \(\cup \{(t, \{c, x_{i\gamma}^-\}) \mid t \in \{i\gamma + 1, (i+1)\gamma\}, i \in \{0, m-1\}, x \in X, x\text{ appears negatively in } C_i\}\).

We depict in Figure 1 the link stream build for \(\gamma = 3\) and \(\varphi = (w \lor x \lor y) \land (w \lor x \lor z)\).

We show that there is an assignments of the variables that satisfies \(\varphi\) if and only if \(L\) contains a \(\gamma\)-matching of size \((2m+1)n + m\).
Fig. 1. The constructed linkstream $L$ when $\varphi = (w \lor x \lor y) \land (w \lor x \lor z)$ and $\gamma = 3$. Here $T = [0, 8]$ and the edge $(t, \{u, v\})$ of $L$ are depicted by an edge following the vertical line corresponding to time $t$ going from the horizontal line corresponding to $u$ to the horizontal line corresponding to $v$. For readability, the edges incident with $c$ are not drawn. Instead, we have circled the vertices that are neighbors of $c$ at each specific time.
Intuitively, the edge between \((0, \{x^-, x^+\})\) and \((0, \{x^-, x^-\})\) that is in the requested \(\gamma\)-matching determines if the variable \(x\) is set to true or false. Moreover, the size of the requested \(\gamma\)-matching will ensure that if the edge \((0, \{x^-, x^+\})\) (resp. \((0, \{x^-, x^-\})\)) is in the \(\gamma\)-matching, then every edge \((t, \{x^-, x^+\})\) (resp. \((t, \{x^-, x^-\})\)), \(t \in [0, (m+1)\gamma - 1]\) and every edge \((t, \{x^+, x^-\})\) (resp. \((t, \{x^+, x^+\})\)), \(t \in [1, m\gamma]\), \(i = \lfloor \frac{i-1}{\gamma}\rfloor\), are in the \(\gamma\)-matching as well. Finally, during the time interval \([i\gamma + 1, (i+1)\gamma]\), we will certify that the clause \(C_i\) is satisfied.

First assume that \(\varphi\) is satisfiable. Let \(\psi\) be a satisfying assignment of \(\varphi\) and let \(\chi : C \rightarrow V\) be a function that, for each clause \(C_i\), \(i \in [0, m-1]\), arbitrary chooses a variable \(x \in X\), such that the assignment of \(x\) given by \(\psi\) satisfies \(C_i\), and returns \(x_i^+\) (resp. \(x_i^-\)) if \(\psi(x) = true\) (resp. \(\psi(x) = false\)). Let

\[
\mathcal{M} = \{\Gamma_i(i \cdot \gamma, x^-, x^+) \mid x \in X, \psi(x) = true, i \in [0, m]\} \\
\cup \{\Gamma_i(i \cdot \gamma, x^-, x^-) \mid x \in X, \psi(x) = false, i \in [0, m]\} \\
\cup \{\Gamma_i(i \cdot \gamma + 1, x^-, x_i^-) \mid x \in X, \psi(x) = true, i \in [1, m]\} \\
\cup \{\Gamma_i(i \cdot \gamma + 1, x^+, x_i^+) \mid x \in X, \psi(x) = false, i \in [1, m]\} \\
\cup \{\Gamma_i(i \cdot \gamma + 1, c, \chi(C_i)) \mid i \in [1, m]\}\]

One can verify that \(\mathcal{M}\) is a \(\gamma\)-matching of \(L\) of size \((2m+1)n + m\).

Assume now that \(L\) contains a \(\gamma\)-matching \(\mathcal{M}\) of size \((2m+1)n + m\). We use several claim in order to construct a satisfying assignment of \(\varphi\).

Claim 1 For each \(x \in X\), \(\mathcal{M}\) contains at most \(m + 1\) \(\gamma\)-edges incident with \(x^+\) (resp. \(x^-\)).

Proof: This result follows from the fact that \(T = [0, (m+1)\gamma - 1]\) is of size \((m+1)\gamma\) and so, cannot be divided into \(m+2\) pairwise disjoint sets of size \(\gamma\).

Claim 2 Given \(x \in X\), if \(\Gamma_i(0, x^-, x^+) \notin \mathcal{M}\) (resp. \(\Gamma_i(0, x^-, x^-) \notin \mathcal{M}\)), then \(\mathcal{M}\) contains at most \(m\) \(\gamma\)-edges incident with \(x^+\) (resp. \(x^-\)).

Proof: As \(\Gamma_i(0, x^-, x^+)\) is the only \(\gamma\)-edge of \(L\) that contains the edge \(e_x^0 = (0, \{x^-, x^+\})\), this implies that the edge \(e_x^0\) is not contained in any \(\gamma\)-edge of \(\mathcal{M}\). So the \(\gamma\)-edge of \(\mathcal{M}\) that are incident with \(x^+\) are constraint to exist in the time interval \(I = [1, (m+1)\gamma - 1]\) that is of size \((m+1)\gamma - 1\). Thus, \(I\) cannot be divided in \(m+1\) pairwise disjoint sets of size \(\gamma\). The claim follows.

Claim 3 \(\mathcal{M}\) contains exactly \(m\) \(\gamma\)-edges incident with \(c\) and contains exactly
2m + 1 \gamma\text{-edges incident with } x^+ \text{ or } x^-, \text{ for each } x \in X.

\textbf{Proof:} As } \mathcal{M} \text{ is a } \gamma\text{-matching, then for each } x \in X, \Gamma_\gamma(0, x^-, x^+) \not\subseteq \mathcal{M} \text{ or } \Gamma_\gamma(0, x^+, x^-) \not\subseteq \mathcal{M}. \text{ So Claim 1 and Claim 2 imply that for each } x \in X, \mathcal{M} \text{ contains at most } 2m + 1 \gamma\text{-edges incident with } x^+ \text{ or } x^- \text{. Moreover, by construction } \mathcal{M} \text{ can contains at most } m \gamma\text{-edges incident with } c \text{ and } L \text{ does not contains any edge of the form } (t, \{x^+, y^+\}), (t, \{x^+, y^-\}), (t, \{x^-, c\}), \text{ or } (t, \{x^-, c\}), \text{ for any } x, y \in X. \text{ Thus the budget is tight. The claim follows.} 

\square

\text{Note that by construction, if } \mathcal{M} \text{ contains a } \gamma\text{-edge incident with } x_i^{++} \text{ for some } x \in X \text{ and } i \in [0, m - 1], \text{ then this } \gamma\text{-edge has to be either } \Gamma_\gamma(i\gamma + 1, c, x_i^{++}) \text{ or } \Gamma_\gamma(i\gamma + 1, x^+, x_i^{++}). \text{ Moreover Claim 4 give us some information in the case where } \Gamma_\gamma(i\gamma + 1, x^+, x_i^{++}) \in \mathcal{M}.

\textbf{Claim 4} Given } x \in X, \text{ if } \mathcal{M} \text{ contains a } \gamma\text{-edge } \Gamma_\gamma(i\gamma + 1, x^+, x_i^{++}) \text{ (resp. } \Gamma_\gamma(i\gamma + 1, x^-, x_i^{--})\text{) for some } i \in [0, m - 1], \text{ then } \mathcal{M} \text{ contains at most } m \gamma\text{-edges incident with } x^+ \text{ (resp } x^-). \n
\textbf{Proof:} \text{ Let } x \in X \text{ and let } i \text{ be the first value such that } \Gamma_\gamma(i\gamma + 1, x^+, x_i^{++}) \in \mathcal{M}. \text{ As } \gamma \text{ does not divide } i\gamma + 1, \text{ this implies that, in the interval } [0, i\gamma], \text{ at least one edge } e_x^i = (t, \{x^-, x^+\}), t \in [0, i\gamma] \text{ is not in } \mathcal{M}. \text{ So the } \gamma\text{-edges of } \mathcal{M} \text{ that are incident with } x^+ \text{ are constraint to exist in the time interval } I = T \setminus t \text{ that is of size } (m + 1)\gamma - 1. \text{ Thus, } I \text{ cannot be divided in } m + 1 \text{ pairwise disjoint sets of size } \gamma. \text{ The claim follows.} \n
\square

\text{Let } x \in X. \text{ Using Claim 3, we know that } \mathcal{M} \text{ contains exactly } 2m + 1 \gamma\text{-edges incident with } x^+ \text{ or } x^- \text{. By the pigeonhole principle, we know that for } x^+ \text{ or } x^-, \text{ say } x^+, \mathcal{M} \text{ contains exactly } m + 1 \gamma\text{-edges incident with } x^+. \text{ By Claim 4, this implies that } \{\Gamma_\gamma(i\gamma, x^-, x^+) \mid i \in [0, m]\} \subseteq \mathcal{M}. \text{ Thus, as } \mathcal{M} \text{ is a } \gamma\text{-matching that contains } m \gamma\text{-edges incident with } x^-, \text{ this also implies that } \{\Gamma_\gamma(i\gamma + 1, x^-, x^-) \mid i \in [0, m - 1]\} \subseteq \mathcal{M}.

\text{For each variable } x \in X, \text{ we set } x \text{ to } \text{true} \text{ (resp. } \text{false}) \text{ if } \Gamma_\gamma(0, x^-, x^+) \in \mathcal{M} \text{ (resp. } \Gamma_\gamma(0, x^-, x^-) \in \mathcal{M}). \text{ Let } \varphi \text{ be the so obtained assignment. Let } i \in [0, m - 1]. \text{ We know that there exists } x \in X \text{ such that either } \Gamma_\gamma(i\gamma + 1, c, x_i^{++}) \in \mathcal{M} \text{ or } \Gamma_\gamma(i\gamma + 1, c, x_i^{--}) \in \mathcal{M}. \text{ Let fix this } x \in X \text{ and assume that } \Gamma_\gamma(i\gamma + 1, c, x_i^{++}) \in \mathcal{M}, \text{ meaning that } x \text{ appears positively in } C_i. \text{ This implies that } \Gamma_\gamma(i\gamma + 1, x^+, x^{++}) \not\subseteq \mathcal{M}, \text{ so that } \{\Gamma_\gamma(i'\gamma, x^-, x^+) \mid i' \in [0, m]\} \subseteq \mathcal{M}. \text{ Thus, } x \text{ is set to } \text{true} \text{ by } \varphi \text{ and } x \text{ satisfies } C_i. \text{ This conclude the proof.} \n
\square
3 Approximation algorithm

In classical graph theory, it is folklore that any maximal matching is also a 2-approximation of a maximum matching. Fortunately enough, it is roughly the same situation with link streams. Precisely, in this section, we adopt the greedy approach—finding a maximal $\gamma$-matching—in order to provide a 2-approximation algorithm for $\gamma$-matching.

Let $L = (T, V, E)$ be a link stream. Let $\mathcal{P}$ be the set of every $\gamma$-edges of $L$. Let $\preceq$ be an arbitrary total ordering on the elements of $\mathcal{P}$ such that given for any two elements of $\mathcal{P}$, $\Gamma_1 = \Gamma_\gamma(t_1, u_1, v_1)$ and $\Gamma_2 = \Gamma_\gamma(t_2, u_2, v_2)$ such that $t_1 < t_2$, we have $\Gamma_1 \preceq \Gamma_2$.

Let $\mathcal{A}$ the following greedy algorithm. The algorithm starts with $M = \emptyset$, $Q = \mathcal{P}$, and a function $\rho : V \times T \to \{0, 1\}$ such that for each $(t, v) \in T \times V$, $\rho(t, v) = 0$. The purpose of $\rho$ is to keep track of the temporal vertices that are contained in a $\gamma$-edge of $M$. As long as $Q$ is not empty, the algorithm selects $\Gamma$, the $\gamma$-edge of $Q$ that is minimum for $\preceq$, and removes it from $Q$. Let $K$ be the set of the $2\gamma$ temporal vertices that are contained in $\Gamma$. If, for each $(t, v) \in K$, $\rho(t, v) = 0$, then the algorithm adds $\Gamma$ to $M$, otherwise it does nothing at this step. For each $(t, v) \in K$, it sets $\rho(t, v)$ to 1 and repeats.

If $Q = \emptyset$, it returns $M$.

As $\mathcal{P}$ can be determined in a sorted way in time $O(m)$, this algorithm runs in time $O(n\tau + m)$, where $\tau = |T|$, $n = |V|$, $m = |E|$, and where $\gamma$ is a constant hidden in the $O$.

Given a $\gamma$-matching $\mathcal{M}$, we define the bottom temporal vertices of $\mathcal{M}$, denoted $\text{bot}(\mathcal{M})$, as the set $\{(t + \gamma - 1, u), (t + \gamma - 1, v) \mid \Gamma_\gamma(t, u, v) \in \mathcal{M}\}$. Lemma 1 shows the crucial role of the bottom temporal vertices of the matchings returned by $\mathcal{A}$.

**Lemma 1** Let $\gamma$ be a positive integer, let $L$ be a link stream, and let $\mathcal{M}$ be a $\gamma$-matching returned by $\mathcal{A}$ when applied to $L$. If $\mathcal{M}'$ is a $\gamma$-matching of $L$, then every $\gamma$-edge of $\mathcal{M}'$ contains, at least, one temporal vertex of $\text{bot}(\mathcal{M})$.

**Proof:** First, note that any $\gamma$-edge of $\mathcal{M}$ contains two temporal vertices of $\text{bot}(\mathcal{M})$, and so, at least one. Let $\Gamma'$ be a $\gamma$-edge of $\mathcal{M}'$ that is not in $\mathcal{M}$. Let $\mathcal{M}'^* \subseteq \mathcal{M}$ be the set of every $\gamma$-edge $\Gamma'^*$ of $\mathcal{M}$ such that there exists a temporal vertex $(t, v)$ that is contained in both $\Gamma'$ and $\Gamma'^*$. Assume that $\Gamma = \Gamma_\gamma(t, u, v)$. If there exists $\Gamma'^* \in \mathcal{M}$ such that $\Gamma'^* = \Gamma_\gamma(t', u, v')$ and $t' \leq t$, then we have that $(t' + \gamma - 1, u) \in \text{bot}(\mathcal{M})$ is contained in $\Gamma$. Otherwise, we have that for each $\Gamma'^* \in \mathcal{M}'^*$ such that $\Gamma'^* = \Gamma_\gamma(t', u', v')$, $t' > t$. This is not possible by construction of $\mathcal{A}$. This concludes the proof. \qed
Lemma 1 plays a cornerstone role in the proof of subsequent Theorem 2. As a byproduct, we also obtain the following result.

**Corollary 1** \( A \) is a 2-approximation of the \( \gamma \)-matching problem.

**Proof:** Let \( L \) be the input link stream. Let \( M \) be a solution returned by the algorithm \( A \) when applied to \( L \), and let \( M' \) be a \( \gamma \)-matching of \( L \). As \( |\text{bot}(M)| = 2|M| \), two \( \gamma \)-edges of \( M' \) cannot contain the same temporal vertex, and, by Lemma 1, every \( \gamma \)-edge of \( M' \) contains at least one element of \( \text{bot}(M) \), we obtained that \( |M'| \leq 2|M| \).

\( \square \)

4 Kernelization algorithm

We now show a kernelization algorithm for \( \gamma \)-matching by a direct pruning process based on Lemma 1. The main idea is as follows. First, we compute the set \( S \) of all bottom temporal vertices of a \( \gamma \)-matching produced by previously defined algorithm \( A \). Then, we prune the original instance by only keeping edges that belong to a \( \gamma \)-edge incident to a temporal vertex of \( S \). More precisely, we prove the following result.

**Theorem 2** There exists a polynomial-time algorithm that for each instance \((L,k)\), either correctly determines if \( L \) contains a \( \gamma \)-matching of size \( k \), or returns an equivalence instance \((L',k')\) such that the number of edges of \( L' \) is \( 2(k-1)(2k-1)\gamma^2 \).

**Proof:** Let \( L = (T,V,E) \) be a link stream and \( k \) be an integer. We first run the algorithm \( A \) on \( L \). Let \( M \) be the \( \gamma \)-matching outputed by the algorithm and let \( \ell = |M| \). If \( \ell \geq k \), then we already have a solution and then return a true instance. If \( \ell < \frac{k}{2} \), then, by Corollary 1, we know that the instance does not contains a \( \gamma \)-matching of size \( k \), and then we return a false instance. We now assume that \( \frac{k}{2} \leq \ell < k \).

Lemma 1 justifies that we are now focusing on the temporal vertices of \( \text{bot}(M) \) in order to find the requested kernel. We construct a set \( \mathcal{P} \) of \( \gamma \)-edges and we show that any edge \( e \), that is not in a \( \gamma \)-edge of \( \mathcal{P} \), is useless when looking for a \( \gamma \)-matching of size \( k \). For each \((t,u) \in \text{bot}(M)\), and for each \( t' \) such that \( \max(0, t - \gamma + 1) \leq t' \leq t \), we consider the set \( S(t', u) \) of every \( \gamma \)-edge, existing in \( L \), with the form \( \Gamma_\gamma(t', u, v) \) with \( v \in V \). If the set \( S(t', u) \) is of size at most \( 2k - 1 \), we add every element of \( S(t', u) \) to \( \mathcal{P} \). Otherwise, we select \( 2k - 1 \) elements of \( S(t', u) \) that we add to \( \mathcal{P} \). In both cases, we denote by \( S'(t', u) \) the set of elements of \( S(t', u) \) that we have added to \( \mathcal{P} \). This finish the construction of \( \mathcal{P} \). As \( |\text{bot}(M)| = 2\ell \) and for each element of \( \text{bot}(M) \) we have added at most \( (2k-1)\gamma \) \( \gamma \)-edges to \( \mathcal{P} \), we have that \( |\mathcal{P}| \leq 2\ell(2k-1)\gamma \leq 2(k-1)(2k-1)\gamma \).
We now prove that if $L$ contains a $\gamma$-matching $M'$ of size $k$, then it also contains a $\gamma$-matching $M''$ of size $k$ such that $M'' \subseteq P$. Let $M'$ be a $\gamma$-matching of $L$ of size $k$ such that $p = |M' \setminus P|$ is minimum. We have to prove that $p = 0$. Assume that $p \geq 1$. Let $\Gamma$ be a $\gamma$-edge in $M' \setminus P$. Let $(t,u)$ be a temporal vertex of $\text{bot}(M)$ that is contained in $\Gamma$. We know by Lemma 1 that this temporal vertex exists. Assume that $\Gamma \in S(t', u)$ and so $|S'(t', u)| = 2k - 1$. Let $N_S(t', u)$ be the set of vertices $w$ of $V \setminus \{u\}$ such that a $\gamma$-edge of $S'(t', u)$ is incident to $w$. As $M' \setminus \{\Gamma\}$ is of size $k - 1$, the $\gamma$-edges that it contains can be incident to at most $2k - 2$ vertices. This means that there exists $w \in N_S(t', u)$ such that no $\gamma$-edge of $M' \setminus \{\Gamma\}$ is incident to $w$. Thus $(M' \setminus \{\Gamma\}) \cup \{\Gamma_{\gamma}(t', u, w)\}$ is a $\gamma$-matching of size $k$. As $\Gamma \notin P$ and $\Gamma_{\gamma}(t', u, v) \in P$, this contradicts the fact that $p$ is minimum.

We now can define the link stream $L' = (T,V,E')$ such that $E' = \{e \in E : \exists \Gamma \in P : e \in \Gamma\}$. As $|P| \leq 2(k - 1)(2k - 1)^2$ and every element of $P$ is a $\gamma$-edge, we have that $|E'| \leq 2(k - 1)(2k - 1)^2 \gamma^2$. The theorem follows. \hfill \Box

5 Experimental result

For easy diffusion, both our 2-approximation and kernelization algorithms are implemented in Java and JavaScript\footnote{The source code is available at \url{https://github.com/antoinedimitriroux/Temporal-Matching-in-Link-Streams}}. Experiments are run on a standard laptop clocking at 3.1 Ghz with DDR3 16 Go memory.

5.1 Dataset

We carried out our experiments on two main types of datasets: those that are randomly generated\footnote{Direct link to the GUI of the generator: \url{https://antoinedimitriroux.github.io}} and those that are collected from human activities.

Artificially generated link streams, and stress test:
In order to generate random sets of link stream instances, we adopt the more realistic point of view of random geometric graphs, rather than the classically theoretic Erdős-Rényi model, as follows. Let $S$ be a 2D Euclidian space. We define a particle as a point in space $S$. Every particle is given along with a radius representing the maximum communication distance it can have with another particle. Thus, the particle together with its range define a disk in

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\url{https://antoinedimitriroux.github.io}
Fig. 2. Stress test on generated input, with $\gamma = 5$. Interestingly, for some input with a large number of timed edges, but no $\gamma$-edges, the runtime can be instantaneous (left figure). For this reason, it is much more interesting to examine the runtime as a function of the number of $\gamma$-edges of the input (right figure). For easy eye-comparison, we tried to parametrise the generator in a way that most of the generated instances have roughly the same number of timed edges and $\gamma$-edges. For instance, most instances with less than 100000 timed edges also have roughly the same number of $\gamma$-edges. However, the situation is more random for instances having between 100000 and 200000 timed edges.

Let $\mathcal{S}$ be the space of all particle sets. Let $\mathcal{P}$ be a set of particles, given along with the same value of radius. We partition set $\mathcal{P}$ into $n$ parts, $\mathcal{P} = P_1 \cup P_2 \cup \cdots \cup P_n$, of roughly equal size. We will construct a link stream $L = (T, V, E)$ as follows. Let $V = \{P_1, P_2, \ldots, P_n\}$. At time zero, let $E_0 = \{(0,\{P_i, P_j\}) | \exists a \in P_i \land b \in P_j, \text{the distance between } a \text{ and } b \text{ is less than their radius}\}$. In other words, if there are at least two particles under communication range $a \in P_i, b \in P_j$ of different groups $P_i \neq P_j$ (at time zero), then, we consider there is a (zero-timed) edge between $P_i$ and $P_j$. Every particle has a velocity that is defined as follows. First, the velocity of a particle at time $t$ is a fraction of the velocity at time $t - 1$ of that particle (friction). Second, all velocity vectors are subject to a small random additional factor (random walk factor modeling the wind condition). Finally, we truncate every velocity vector in order to insure that the norm of the vector is lower than a given maximum particle speed (physical limits). We then let the system evolve during a given laps of time, that we also refer to as $T$. At every time instant $t \in T$, we define, similarly as before, the $t$-timed edge set $E_t = \{(t,\{P_i, P_j\}) | \exists a \in P_i \land b \in P_j, \text{the distance between } a \text{ and } b \text{ is less than their radius}\}$. Finally, we define our link stream as $E = \bigcup_{t \in T} E_t$.

Roughly, increasing any of the three parameters which are defined by the particle radius, the cardinality of $\mathcal{P}$, and the maximum particle speed, results in the same effect on the generated link stream, that is, to produce a dense link stream. The generated data allows us to unit test our code, and especially to verify that approximation and kernelization runtime is sound on large input. For instance, with inputs containing hundreds of thousand timed edges, our runtime is under ten seconds, cf. Fig. 2.
Real world datasets:
We also confront the implementations of our algorithms to two particular link streams collected from real world graph data. In one data set the link stream has been built by analysing a recording of $2 \times 80$ minute Rollerblade touring in Paris [14]. The other data set is built from emailing information collected from the Enron company [8].

We noticed with our raw datasets that the instants where some timed edge is present can be very sparse, especially with the Enron dataset, leaving no chance for a $\gamma$-edge to exist as soon as $\gamma > 1$. We will, for this reason, sample our raw datasets by the following process. For any link stream $L = (T, V, E)$ and for any $1 < \delta < |T|$, we define a $\delta$-sample $L_\delta = (T_\delta, V_\delta, E_\delta)$ as $V_\delta = V$, $T_\delta = [\min \frac{T}{\delta}, \max \frac{T}{\delta}]$, and

$$E_\delta = \{(t, \{u, v\}) \mid \exists t' \in T : \delta t \leq t' \leq \delta (t+1) \land (t', \{u, v\}) \in E\}.$$ 

Needless to say, our sampling process breaks down the number of timed edges in the dataset. We ensure that the process does not result in trivial inputs. Luckily, for sensible values of $\delta$, e.g. $\frac{1}{2}$ week for Enron or 15 minutes for Rollernet, there is still a large number of timed edges after $\delta$-sampling, cf. Fig. 3. We give in the subsequent Fig. 4 the runtime of our algorithm on random pieces of the two Enron and Rollernet datasets, where we observe that our runtime is instantaneous.

5.2 Hypothesis

We theorise three hypotheses. For each hypothesis we run experiment on the above described datasets, and expose our results in the next Subsection 5.3. We also confirm and negate our hypothesis therein. We discuss and conclude our numerical analysis in the subsequent and last Subsection 5.4 of current Section 5.

Hypothesis 1. Consistence of the formalism:
We would like to verify that $\gamma$-MATCHING is non trivial on human values for $\delta$ and $\gamma$. For instance, we suppose when mining emails that collaborating during a month at a rate of at least two emails per week (round trip) is sensibly human values. When mining proximity records of $2 \times 80$ minute Rollerblade touring Paris, we consider that collaborating during 80 minutes at a rate of one visit every quarter hour (water/snack supplying) is sensibly human values.

Hypothesis 2. Kernelization quality:
Our second hypothesis is that solving $\gamma$-MATCHING can benefit from the kernelization algorithm described in Theorem 2, at least on well-chosen intervals
Fig. 3. Enron dataset (left) and Rollernet dataset (right): remaining timed edges after $\delta$-sampling. For instance, with $\delta = \frac{1}{2}$ week, the number of timed edges after $\delta$-sampling Enron is over ten thousand. With $\delta = 15$ minutes, the number of timed edges after $\delta$-sampling Rollernet is also over ten thousand. We conclude that for sensible values of $\delta$, our sampling does not break down the input to a trivial instance.

Fig. 4. Enron dataset (left) and Rollernet dataset (right): runtime of kernelization algorithm in function of the number of timed edges, with $\gamma = 2$. The (rawly recorded) runtime is instantaneous, hence, probably subject to many noises. We rather refer to Fig. 2 for evaluating performance. Each dot in current Fig. 4 is obtained by first truncating the raw input with varying maximum value of time instants, then, $\delta$-sampling the previously obtained link streams with $\delta = 100$. The only observation we make with this figure is that, on real world dataset, our combined runtime for both the 2-approximation and the kernelization algorithms is very likely instantaneous.
of $\delta$ and $\gamma$.

**Hypothesis 3. Approximation quality:**
We stress that MATCHING in a classical graph is polynomial. Unfortunately, $\gamma$-MATCHING in a link stream is $NP$-difficult. However, in practice, a lot of $NP$-complete problems are not difficult on datasets arising from human activities. What’s more, some such problems can be solved near-optimally by simple algorithms such as by a random or greedy approach, or a mix of both approaches, even on arbitrary inputs. A notorious example is COLORING [12]. Accordingly, our last hypothesis is that, in practice, finding an optimal $\gamma$-matching need not to be difficult. Moreover, we hypothesise that the greedy 2-approximation described in Lemma 1 can produce near-optimal $\gamma$-matching on real world dataset, as well as artificial datasets that mimic real word datasets.

5.3 Result

Both our 2-approximation and kernelization algorithms are implemented and confronted to on the above mentioned datasets. Essentially, our experiments confirm the first two hypotheses. They somewhat negate the last hypothesis.

**Confirmation of Hypothesis 1. Consistence of the formalism:**
Results are given in Fig. 5. We observe on Enron dataset with $\delta \approx \frac{1}{2}$ week that, after $\delta$-sampling, the number of $\gamma$-edges for $\gamma$ varying from 2 to 10 is: over 500 for $\gamma = 10$; over 1000 for $\gamma = 6$; and over 4500 for $\gamma = 2$. Moreover, we will see in the next paragraph that our numerical analysis can only find $\gamma$-matchings of size approximately one fifth of the previous number. The Enron dataset records email exchanges during a period of approximatively 3 years. We reckon during this period that $\binom{500}{100}$ is a decent combinatorial explosion within the Enron company, and conclude that Hypothesis 1 is sound, that is, $\gamma$-MATCHING is non trivial for decent values, namely $\delta \approx \frac{1}{2}$ week and $\gamma = 10$ on Enron. Indeed, these values translate the fact that any pair of collaborators in the $\gamma$-matching necessarily keep exchanging emails together continuously for one month, at a rate of at least one email per week and in average at least two emails every week. As a side note, we remark that our last statement is mathematically cautious: Rigorous definition of $\delta$-sampling when $\delta \approx \frac{1}{2}$ week can result to two consecutive timed edges that are at about the double amount of $\delta$ from each other in the original input. This case happens when the first timed edge is recorded rather at the beginning of the first $\delta$-slice, and the second timed edge is recorded at the end of the second $\delta$-slice. We stress that, in practice, this case is marginal. Basically, up to our knowledge after random checking in the dataset with non trivial values of $\delta$, this case never occurs. In any case, for the duration of every $\gamma$-edge, the average exchange rate between the two vertices in the $\gamma$-edge is still (at least) once every $\delta$. 

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Fig. 5. Enron dataset (left) and Rollernet dataset (right): number of $\gamma$-edges after $\delta$-sampling, for varying values of $\delta$ and $\gamma$.

Fig. 6. Enron dataset (left) and Rollernet dataset (right): ratio obtained by dividing the number of $\gamma$-edges in the kernelization output by the number of $\gamma$-edges in the kernelization input (which is obtained after $\delta$-sampling). Here, the darker is the better.
Confirmation of Hypothesis 2. Kernelization quality:
Results are given in Fig. 6. We observe that on well chosen intervals of $\Delta$ and $\gamma$, kernelization reduces the input size down to under twenty per cent. This is particularly true for $\gamma \approx 20,30$ with $\Delta \leq 2$ month on Enron; and $\gamma \leq 10$ with $\Delta \leq 20$ minutes for Rollernet. We observe on these values that the kernelization algorithm reduces the input link stream to an equivalent instance of size smaller than twenty per cent the size of the original input, and sometimes under ten per cent.

General failure, with some marginal positive results, of Hypothesis 3. Approximation quality:
Results are given in Fig. 7. We notice from definition that the 2-approximation algorithm produces a lower bound for $\gamma$-MATCHING, which is at least half the optimal value. Moreover, we stress that the kernelization algorithm gives a naive upper bound for $\gamma$-MATCHING by simply counting the number of $\gamma$-edges present in the kernel. We observe for tangent areas in Fig. 7 that these two upper and lower bounds meet. This means that the 2-approximation outputs an optimal solution for $\gamma$-MATCHING on these areas. However, we observe that for most parts of our dataset, Hypothesis 3 is not confirmed.
5.4 Discussion

Our experiment results point out several questions. While they allow us to safely conclude that:

- preprocessing an instance of $\gamma$-MATCHING by a greedy process, and then kernelization as described in Theorem 2, is sound;
- the preprocessing is instantaneous on real world input;
- the preprocessing is robust versus stress testing on large inputs using common laptop: below ten seconds on input of hundreds thousand timed edges;

they also testify that works still need to be done for further investigating $\gamma$-MATCHING, especially that:

- the optimisation problem is $NP$-difficult;
- the 2-approximation algorithm only finds optimal values for very marginal samplings of Enron and Rollernet datasets;
- the kernelization algorithm helps in reducing the input down to $10 - 20\%$ for interesting samplings of Enron and Rollernet datasets, but we do not know how then to find a $\gamma$-matching of the kernel that is better than the output of the 2-approximation;
- in particular, we do not know if the approximation factor can be improved.

6 Conclusion and perspectives

We introduce the notion of temporal matching in a link stream. Unexpectedly, the problem of computing a temporal matching of maximum size, called $\gamma$-MATCHING, turns out to be $NP$-difficult. We then show a kernelization algorithm for $\gamma$-MATCHING parameterized by the size of the solution. Our process produces quadratic kernels. On the way to obtaining the kernelization algorithm, we also provide a 2-approximation algorithm for $\gamma$-MATCHING. We believe that the same techniques extend to a large class of hitting set problems in link streams.

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