TOWARDS NONCOMMUTATIVE QUANTIZATION OF GRAVITY

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Abstract

We propose a mathematical structure, based on a noncommutative geometry, which combines essential aspects of general relativity with those of quantum mechanics, and leads to correct “limiting cases” of both these physical theories. We (algebraically) quantize a groupoid constructed on space-time rather than space-time itself. Both space and time emerge in the transition process to the commutative case. Our approach clearly suggests that quantum gravitational observables should be looked for among correlations of distant phenomena rather than among local effects. A toy model is computed (based on a finite group) which predicts the value of “cosmological constants” (in the quantum sector) which vanish when going to the standard space-time physics.
1 Introduction

There are many theoretical indications that at the fundamental level, i.e. below Planck’s scale, the manifold structure of space-time breaks down (there are so many hints scattered in the literature that it is difficult to give a list of references; for a review see [1]) and that, in particular, time loses its ordinary meaning (see for instance [2, 3, 4]). The problem is that when we give up the manifold structure so many possibilities are open, none of them being more natural than others, that we are left only with our subjective preferences. It seems that the situation started to change with the discovery of noncommutative geometry (see [15] and works cited therein). One could claim that the noncommutative generalization of the usual geometry is “natural” in the following sense. As it is well known the manifold structure can be defined in terms of the algebra of smooth functions $C^\infty(M)$ on a set $M$ (this definition is equivalent to the more standard one in terms of charts and atlas). The algebra $C^\infty(M)$ is of course commutative, and to drop the commutativity assumption seems to be a natural step to undertake. Therefore, we take any (associative) algebra and try to see what does happen if we proceed, as closely as possible, along the lines established in the usual differential geometry. It turns out that the change is radical, but surprisingly many standard methods can be adapted to the new conceptual context. In this way, many spaces usually regarded as highly pathological (e.g. non-Hausdorff, non-measurable) can effectively be investigated with the help of noncommutative methods. Since general algebras are often too difficult to deal with one looks for their suitable representation, and it turns out that such a representation is provided by the algebra of operators in a Hilbert space. And here the chain of our motivations closes up. The idea of noncommutativity appeared first in physics as the noncommutativity of quantum mechanical observables represented by operators in a Hilbert space.

Another attractive thing, from our point of view, about noncommutative spaces is their “global character”. In a differentiable manifold $M$ the existence of points is equivalent to the existence of smooth functions which vanish at these points; algebraically such functions form maximal ideals in the algebra $C^\infty(M)$ of all smooth functions. In noncommutative algebras, in general, there are no maximal ideals, and the concept of point is replaced by that of pure state which is also familiar from quantum mechanics. In spite of this, a true dynamics can be done on noncommutative spaces, for instance
in terms of derivations of a given noncommutative algebra. It is then evident that when one uses such spaces to model physical processes at the fundamental level the usual idea of space-time is replaced by something drastically different but still workable. Moreover, since each noncommutative algebra has a commutative subalgebra, called its center, the possibility is always open to go, by restricting to this subalgebra, to the standard commutative geometry.

The above attractive features of noncommutative geometries, and — last but not least — tangible successes in putting into the noncommutative framework the standard model of fundamental interactions [3], have motivated several attempts at creating a conceptual basis for the noncommutative quantum theory of gravity (see, for instance, [4], [5], [6], [7], [8], [9], [10]). All these attempts explicitly or tacitly assume that it is the geometry of space-time which should be made noncommutative (for review see [13]). In [14] we have proposed a scheme for a noncommutative quantization of gravity the main strategy of which consists in starting — from the very beginning — with an abstract noncommutative space and obtaining from it the usual space-time geometry via the correspondence with the classical case. To describe our idea, let us mention that there is the standard method of obtaining a noncommutative space from a given commutative one(see [15, p. 99-102]. Roughly speaking, a given commutative space, for instance a manifold, should be presented as the quotient of a groupoid $G$ by an equivalence relation (which can be given as the action of a group on $G$), and then one should apply the standard method of constructing a $C^*$-algebra $\mathcal{A}$ on $G$. This $C^*$-algebra is in general noncommutative and is a basis for our noncommutative geometry. In the case of space-time $M$, one should notice that $M$ can be given as the quotient $M = E/\text{SO}(3,1)$ where $E$ is the total space of the fibre bundle of frames over $M$. The point is that, by taking the Cartesian product $E \times \text{SO}(3,1)$, we obtain a groupoid, and the above method can be directly applied. This will be described in section 2. With one important proviso. We shall start, right from the beginning, with the groupoid $G = E \times \Gamma$, where $E$ is a suitable space and $\Gamma$ a suitable group, forgetting about space-time $M$, our aim being to obtain space-time when going from our noncommutative geometry to the commutative case.

Another important remark. If we assume that $E$ is a smooth manifold and $\Gamma$ a Lie group, then the noncommutative space corresponding to the $C^*$-algebra $\mathcal{A}$ will be strongly Morita equivalent to the smooth manifold.
\[ M = E/\Gamma. \] Since in noncommutative geometry, strong Morita equivalence plays the role of isomorphism (for definition see [16, p. 140] or [17, p. 40], it would seem that we have gained nothing with our construction. But this is not so. Strong Morita equivalence, being a “noncommutative isomorphism”, does not know about points and their neighbourhoods, and consequently the noncommutative space based on the algebra \( A \) is equivalent to a smooth manifold “modulo local properties”; it truly deserves the name of noncommutative manifold. In this sense, it is a strong generalization of the usual manifold concept, and can be used in physics to model nonlocal processes at the fundamental level.

Our generalization can go even further if we give up the assumption that \( E \) is a smooth manifold. For instance, we could think of it as of a generalized fibre bundle over a space-time with singularities such that the fibres over “singular points” need not be diffeomorphic to the typical fibre. Even such fibres are admitted which are reduced to the single point. In [18] it has been shown that in such cases the groupoid \( G \) is quite a regular space, and our construction can proceed essentially with no changes. However, as the result we obtain a noncommutative space which is no longer strongly Morita equivalent to a manifold. Although this case seems to be more mathematically interesting and more promising from the point of view of physics (we dealt with it in [14]), in the present paper we shall be concerned with the case where all spaces involved are assumed to be smooth manifolds. Our motivation for doing so is that in the present paper we want to introduce our approach to quantizing gravity as simply as possible, and going beyond the manifold category would provoke many questions which at the introductory stage would make things more complicated rather than smoothing them out.

The organization of our material is the following. In section 2 we construct the Hilbert space for our approach to quantum gravity. Section 3 summarizes those aspects of noncommutative geometry which are necessary to formulate a noncommutative version of general relativity. Its quantization scheme is presented in section 4, and the transitions to the usual space-time geometry, on the one hand, and to the standard quantum mechanics, on the other hand, are discussed in section 5. In section 6, we check the consistency of our scheme by computing a simple model in which the groupoid \( G \) is a Cartesian product of a 3-dimensional Minkowski space-time and the group \( \Gamma \) is a finite group \( D_4 \). In section 7, we argue that observable quantum gravitational phenomena should be looked for, as strongly suggested by our scheme,
among correlations between distant measurements rather than among “lo-
cal phenomena”. Section 8 summarizes the paper. Some overlaps with the
material presented in [14] were indispensable not only to make the present
paper self-consistent, but also because our aim in preparing it was to more
carefully discuss physical aspects of our approach than it had been possible
in [14] in which the mathematical foundations were the primary objective.

2 Groupoid of Fundamental Symmetries

In this Section we construct the Hilbert space for quantum gravity. We start
from the direct product $G = E \times \Gamma$ where $E$ is an $n$-dimensional smooth
manifold and $\Gamma$ a Lie group acting on $E$ (to the right). Elements of $\Gamma$ are
“fundamental symmetries” of our theory. Heuristically, we could think of $E$
as of the total space of the fibre bundle of frames over space-time $M$, and of
$\Gamma$ as of its structural group (a connected component of the Lorentz group).
However, we insist upon starting just from a manifold $E$ and a group $\Gamma$ of
“fundamental symmetries”, our aim being to deduce space-time $M$ from our
model via the correspondence principle with macroscopic physics. In the
present paper we leave the group $\Gamma$ unspecified. The correct choice of $\Gamma$
is left for the future development of the proposed model, and it should be
made on physical grounds. Moreover, it can turn out that our scheme is too
narrow to incorporate all required physics; in such a case the scheme could be
enlarged by substituting for $\Gamma$ a supergroup or a quantum group (and suitably
modifying the model; preliminary analysis shows that it is possible).

Our next step will be to regard $G$ as a groupoid. Roughly speaking,
groupoid differs from group by the fact that not all its elements can be com-
posed with each other (composition can be done only within certain subsets
of the groupoid). For the precise definition of groupoid see, for instance, [19];
here we shall give a less formal description of $G$ as a groupoid.

Of course, $G$ is a set of pairs $\gamma = (p, q)$ where $q = pg$, $p, q \in E$, $g \in \Gamma$ (one
can also write $\gamma = (p, g)$, $g \in \Gamma$). We can think of such a pair as of an arrow
starting at $p$ and ending at $q$. This arrow can be interpreted as a fundamental
symmetry operation (the name “fundamental symmetry” can be attributed,
by only slight abuse of language, to both elements of $\Gamma$ and elements of $G$).
For $G$ to be a groupoid two its subsets should be distinguished, namely: $G^{(0)}$
– the subset of all elements of $G$ of the from $(p, e)$, $p \in E$, where $e$ is a neutral
element of \( \Gamma \), i.e. the subset of all loops (the loop being an arrow beginning
and ending at \( p \)); and \( G^{(2)} \) – the subset of all these elements of \( G \) which can
be composed with each other; viewed as arrows two elements \( \gamma_1, \gamma_2 \in G \)
can be composed with each other \( \gamma = \gamma_1 \circ \gamma_2 \), if the end of \( \gamma_2 \) coincides with
the beginning of \( \gamma_1 \). To formally express properties of the composition one
introduces two following mappings: the source mapping
\[ s : G \rightarrow G^{(0)} \]
defined by
\[ s(p, q) = p, \]
and the range mapping
\[ r : G \rightarrow G^{(0)} \]
defined by
\[ r(p, q) = q. \]
Then, of course,
\[ G^{(2)} = \{(\gamma_1, \gamma_2) \in G \times G : s(\gamma_1) = r(\gamma_2)\}, \]
and some natural conditions are satisfied, for instance
\[ s(\gamma_1 \circ \gamma_2) = s(\gamma_2) \]
and
\[ r(\gamma_1 \circ \gamma_2) = r(\gamma_1) \]
for every \( \gamma_1, \gamma_2 \in G^{(2)} \). These conditions can be easily read from the diagram
presenting the composition \( \gamma = \gamma_1 \circ \gamma_2 \) in the form of arrows (see [13], pp.
99-100). We should also notice that each \( \gamma \in G \) has the two-sided inverse \( \gamma^{-1} \)
such that \( \gamma \gamma^{-1} = r(\gamma) \) and \( \gamma^{-1} \gamma = s(\gamma) \) (for short we omit the composition
symbol \( \circ \)). \( G \) has a natural structure of a fibred space with the fibres \( G_p = \{p\} \times \Gamma, p \in E \).

The groupoid \( G \) with the above structure is also called the direct product
of \( E \) and \( \Gamma \), and denoted by \( G = E \rtimes \Gamma \). Groupoid \( G \) is said to be smooth
if \( G \) and \( G^{(0)} \) carry differentiable structures such that the mappings \( s \) and \( r \)
are submersions, and the composition mapping \( \circ : G^{(2)} \rightarrow G \) and the natural
inclusion mapping \( i : G^{(0)} \rightarrow G \) are smooth. In our case, \( G \) is evidently a
smooth groupoid.
Now, our strategy is the following. First, we shall try to construct, basing on $G$, a noncommutative differential geometry which would allow us to introduce generalized (noncommutative) Einstein’s field equation. To this end we define the algebra $A = C^\infty_c(G, \mathbb{C})$ of smooth compactly supported complex-valued functions on $G$ with the convolution

$$(a * b)(\gamma) := \int_{G_p} a(\gamma_1)b(\gamma_2),$$

as multiplication, where $a, b \in A$, and $\gamma = \gamma_1\gamma_2$, $\gamma, \gamma_1, \gamma_2 \in G_p$, $p \in E$. If $\Gamma$ is an abelian group the convolution is commutative, if $\Gamma$ is non-abelian group the convolution is noncommutative giving rise to a noncommutative geometry (which we shall construct in the next Section). $A$ is also an involutive algebra with involution defined as $a^*(\gamma) = a(\gamma^{-1})$.

Basing on the geometry determined by the algebra $A$ we shall first define generalized Einstein’s equation in the operator form for derivations of $A$, and then, on each fibre $G_q$, define square integrable functions equipped with the suitable Hilbert space structure. The direct sum $\mathcal{H} = \bigoplus_{q \in E} L^2(G_q)$ will serve us as a state space of our quantum mechanics. The modulus squared $|\psi|^2$ of the “wave function” $\psi \in L^2(G_q)$ is the probability density of the “fundamental symmetry” $\gamma \in G$ to occur.

The crucial point is to make the noncommutative geometry based on the algebra $A = C^\infty_c(G, \mathbb{C})$ and the Hilbert space $\mathcal{H} = \bigoplus_{q \in E} L^2(G_q)$ to collaborate with each other. This will be achieved in the following way. First, after solving the generalized Einstein equation, we complete the algebra $A$ to a $C^*$-algebra, and then we find a representation of this algebra in the Hilbert space $\mathcal{H}$. Now, we can develop the theory of quantum gravity by following either the standard formalism of bounded operators on Hilbert space or the $C^*$-algebra approach. We shall describe all stages of the above scheme in the following Sections.

### 3 Noncommutative Geometry of the Groupoid

As it was demonstrated by Koszul [20], and later on extensively used by others, the differential geometry on a manifold $M$ can be done in terms of the algebra $C^\infty(M)$ of smooth functions on $M$ and the $C^\infty(M)$-modules of
smooth sections of smooth vector bundles over $M$. The main idea of generalizing the standard differential geometry is to replace the commutative algebra $C^\infty(M)$ by any, non necessarily commutative, associative algebra. In this way, one obtains a vast generalization of the traditional geometry but, unfortunately, the generalization is not unique: at several crucial points one can proceed in various directions, thus obtaining different versions of noncommutative differential geometry (see [21]). Happily enough, if we choose the derivation based version of differential geometry on the smooth groupoid $G = E \times \Gamma$, the generalization is practically unique. The structure of $G$ turns out to be simple enough to exclude unnecessary complications and at the same time rich enough to guarantee interesting results. The derivation based calculus has been developed in many works (for instance [23]-[17]). In the rest of this section we shall follow [27].

Derivation of the algebra $\mathcal{A}$ is defined to be a linear transformation (endomorphism) $v : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule

$$v(ab) = v(a)b + bv(a),$$

$a, b \in \mathcal{A}$. The set of all derivations of $\mathcal{A}$ is denoted by $\text{Der}\mathcal{A}$. It is a Lie algebra with respect to the bracket operation $[u, v] = uv - vu$, $u, v \in \text{Der}\mathcal{A}$. In the case of the algebra $C^\infty(M)$, $\text{Der}(C^\infty(M)$ is a $C^\infty(M)$-module, and it corresponds to all vector fields on $M$. In the case of a noncommutative algebra $\mathcal{A}$, $\text{Der}\mathcal{A}$ is not, in general, an $\mathcal{A}$-module but only a $\mathcal{Z}(\mathcal{A})$-module, where $\mathcal{Z}(\mathcal{A})$ denotes the center of $\mathcal{A}$ (i.e., the set of all elements of $\mathcal{A}$ which commute with all elements of $\mathcal{A}$). $\text{Der}\mathcal{A}$ can be thought of as a noncommutative counterpart of vectors fields. It should be emphasized that in the framework of noncommutative geometry, “vector fields” are, in general, global objects and, consequently, they cannot be said to consist of vectors.

The pair $(\mathcal{A}, V)$, where $V$ is a $\mathcal{Z}(\mathcal{A})$-submodule of $\text{Der}\mathcal{A}$, is called differential algebra. In our case $\mathcal{A} = C^\infty_c(G, \mathbb{C})$, and as $V$ we choose those derivations of $\mathcal{A}$ which are naturally adapted to the structure of $G = E \times \Gamma$ (as a direct product), i.e. all those $v \in \text{Der}\mathcal{A}$ which can be presented in the form $v = v_E + v_\Gamma$ where $v_E$ is the ”component” of $v$ parallel to $E$, and $v_\Gamma$ the ”component” of $v$ parallel to $\Gamma$. More formally, $v \in \text{Der}\mathcal{A}$ is said to be parallel to $E$ if, for any $\alpha \in C^\infty(\Gamma)$, $v(\alpha \circ pr_\Gamma) = 0$ where $pr_\Gamma$ is the obvious projection. The set of all derivations of $\mathcal{A}$ parallel to $E$ is denoted by $\text{Der}_E\mathcal{A}$. 
And analogously for derivations parallel to $\Gamma$, denoted by $\text{Der}_\Gamma \mathcal{A}$. Therefore,

$$V = \text{Der}_E \mathcal{A} \bigoplus \text{Der}_\Gamma \mathcal{A}.$$ 

To proceed further, we must introduce a metric, i.e. a $\mathcal{Z}(\mathcal{A})$-bilinear non-degenerate symmetric mapping $g : V \times V \to \mathcal{A}$. We chose the metric

$$g = pr^*_E g_E + pr^*_\Gamma g_\Gamma \quad (1)$$

where $g_E$ and $g_\Gamma$ are metrics on $E$ and $\Gamma$, respectively. The above choice of both $V$ and $g$ is the simplest and the most natural one (it is naturally adapted to the product structure of $G = E \times \Gamma$) but, if necessary, we could try other choices as well.

Now, we define the mapping $\Phi_g : V \to V^*$, where $V^*$, the dual of $V$, is the set of $\mathcal{Z}(\mathcal{A})$-homomorphism from $V$ to $\mathcal{A}$, by

$$\Phi_g(u)(v) = g(u, v),$$

$u, v \in V$. The mappings $\Phi_g$ and $\Phi^{-1}_g$ play the role analogous to that of lowering and raising indices in the standard tensorial calculus. The set $V^+$ such that $\Phi^{-1}_g(V^+) = V$ is the set of “invertible forms”. In our case, all forms are invertible, i.e. $V^+ = V^*$.

Now, we define the preconnection $\nabla^* : V \times V \to V^*$ with the help of the usual Koszul formula

$$(\nabla^*_u v)(x) = \frac{1}{2} \left[ u(g(v, x)) + v(g(u, x)) - x(g(u, v)) - v(x, [u, v]) + g(v, [u, x]) - g([u, v], x) \right],$$

for $u, v, x \in V$, and the linear connection $\nabla : V \times V \to V$ by

$$\nabla_u v = \Phi^{-1}_g(\nabla^*_u v).$$

The curvature of this connection is the operator $R : V^3 \to V$ defined by

$$R(u, x)y = \nabla_u \nabla_x y - \nabla_x \nabla_u y - \nabla_{[u, x]} y.$$ 

Since $V$ is a free $\mathcal{Z}(\mathcal{A})$-module we can chose a basis in it and, for any linear operator $T : V \to V$ define the trace of $T$ in the usual way, $\text{tr}T =$
\[ \sum_{i=1}^{k} T_i \] and, consequently, for any fixed pair \( x, y \in V \), the family of operators \( R_{xy} : V \to V \) by
\[ R_{xy}(u) = R(u, x)y. \]

The Ricci curvature is \( \text{ric}(x, y) = \text{tr} R_{xy} \). Finally, by putting \( \text{ric}(x, y) = g(R(x), y) \) we obtain the Ricci operator \( R : V \to V \). (\( R \) is the adjoint operator of the \( Z(\mathcal{A}) \)-bilinear form \( \text{ric} : V \times V \to Z(\mathcal{A}) \)). This allows us to define the generalized Einstein equation in the operator form
\[ R - \frac{1}{2\alpha} r I + \Lambda I = \kappa T \tag{2} \]

where \( \alpha = \text{tr} I \), \( r = \text{tr} R \), \( \Lambda \) and \( \kappa \) are constants related to the cosmological constant and Einstein’s gravitational constant, respectively, and \( T \) a suitably generalized energy-momentum operator. Since it could be expected that “at the fundamental level” there is only “pure non-commutative geometry” we shall assume that \( T = 0 \), but for the sake of generality we shall keep \( \Lambda \) in the equation (if necessary we can always put \( \Lambda = 0 \)). Therefore, generalized Einstein’s equation assumes the form
\[ G = 0 \tag{3} \]

where \( G := R + 2\Lambda I \). It can be easily seen the set that \( \ker G := \{ v \in V : G(v) = 0 \} \) is a \( Z(\mathcal{A}) \)-submodule of \( V \). The differential algebra \( (\mathcal{A}, \ker G) \), where \( \mathcal{A} = C^\infty_c(G, \mathbb{C}) \) will be called Einstein algebra (or Einstein pair). Such an algebra can be regarded as a solution of the generalized Einstein equation (strictly speaking only \( \ker G \) is determined by this equation).

For \( \Lambda \neq 0 \), eq. (3) assumes the form
\[ R(v) = -2\Lambda I(v) \tag{4} \]

If the metric \( g \) on the \( Z(\mathcal{A}) \)-module \( V \) is given a priori, this equation can be regarded as the eigenvalue equation for the Ricci operator \( R \). In the noncommutative framework metric cannot be given independently of the module of derivations of a given algebra, and the noncommutative Einstein equation should determine both the metric and the module of derivations on which the metric is defined (it is worthwhile to notice that Madore [13] argues that there is essentially unique metric associated with each noncumulative differential calculus). If we remember that derivations can be regarded as
counterparts of vector fields, and consequently as responsible for dynamics
(“motions”) of the system, we could draw the following analogy with the
standard case. Just like in the usual general relativity it is impossible first
to specify the distribution and motions of matter and then from this to
compute the structure of space-time (see, e.g., [28, p. 84]), similarly, in our
case, motion (derivations) and geometry (metric) are so closely dynamically
linked with each other that they can only be determined simultaneously.

In this way, we have obtained the noncommutative version of general
relativity (not yet quantum gravity theory). To solve eq. (3) or eq. (2) is a
difficult task, but we can show that it has many solutions. Indeed, let \((M, g)\)
be a solution of the usual Einstein’s equation. We construct the orthonormal
frame bundle \(\pi : OM \to M\) over \(M\) with \(\text{SO}(3, 1)\) as its structural group,
and form the groupoid \(G = OM \rtimes \text{SO}(3, 1)\). For the pair \((C^\infty(G, C), \ker G)\),
which is a solution of eq. (3) or (4), there exists the pair \((C^\infty(M), \ker \tilde{G})\)
where \(\tilde{G}\) is the usual Einstein tensor written in the operator form, i.e. with
one index up and one index down, such that the algebras \(C^\infty_c(G, C)\) and
\(C^\infty(M)\) are strongly Morita equivalent. Strong Morita equivalence plays the
role of isomorphism for noncommutative algebras \([15, 16]\). The fact that
the algebras \(C^\infty_c(G, C)\) and \(C^\infty(M)\) are strongly Morita equivalent means
that, from the point of view of the noncommutative algebra, they contain
the same information (let us notice, however, that \(C^\infty_c(G, C)\) ignores local
properties of \(C^\infty(M)\)). We have shown, therefore, that for every solution
of the usual Einstein equation there exists the solution of the generalized
Einstein equation such that both solutions are Morita equivalent. Of course,
not all solutions of the generalized Einstein equation are generated in this
way.

4 Quantization of Noncommutative General
Relativity

In the present Section \((\mathcal{A}, \ker G)\) is an Einstein algebra. We remind that \(\mathcal{A}\)
is an involutive algebra with the involution defined as \(a^*(\gamma) = (\gamma^{-1})^\ast\), \(a \in \mathcal{A}, \gamma \in G\),
and convolution \((a * b)(\gamma) = \text{int}_{G_\gamma} a(\gamma_1)b(\gamma_2)\) as multiplication.
Now, our aim is to extend \(\mathcal{A}\) to a \(C^*\)-algebra and quantize it with the help
of the standard algebraic method (see, for instance, [29]).
By applying the theorem proved by Connes [30] to our case, we learn that the involutive algebra \( \mathcal{A} = C^\infty_c(G, \mathbb{C}) \), for each \( q \in G(0) \), has the representation \( \pi_q \) in a Hilbert space \( \mathcal{H} = L^2(G_q) \)

\[
\pi_q : \mathcal{A} \to \mathcal{B}(\mathcal{H}),
\]

where \( \mathcal{B}(\mathcal{H}) \) denotes the algebra of bounded operators on \( \mathcal{H} \), given by

\[
(\pi_q(a)\psi)(\gamma) = \int_{G_q} a(\gamma_1)\psi(\gamma^{-1}_1\gamma),
\]

\( \gamma = \gamma_1 \circ \gamma_2, \gamma, \gamma_1, \gamma_2 \in G_q, \psi \in L^2(G_q), a \in \mathcal{A} \), and that the completion of \( \mathcal{A} \) with respect to the norm

\[
\| a \| = \sup_{q \in G(0)} \| \pi_q(a) \|
\]
is a \( C^* \)-algebra. We shall denote it by \( \mathcal{E} \) and call \( \textit{Einstein C}^* \)-algebra.

Now, we can formulate postulates of our noncommutative theory of quantum gravity.

\textit{Postulate 1.} A quantum gravitational system is represented by an \( \textit{Einstein C}^* \)-algebra \( \mathcal{E} \), and its observables by Hermitian elements of \( \mathcal{E} \) (the set of all Hermitian elements of \( \mathcal{E} \) will be denoted by \( \mathcal{E}_H \)).

We speak of “observables” in the quantum gravity regime by analogy with the standard quantum mechanics. Whether these “observables” leave traces in the macroscopic world remains to be seen (we come back to this question in Sec. 5). By the same analogy we can say that the spectrum of a Hermitian element of \( \mathcal{E} \) represents possible measurement results of this observable.

\textit{Postulate 2.} Let \( \mathcal{S} \) denote the set of all states of the algebra \( \mathcal{E} \); elements of \( \mathcal{S} \) represent states of the system and pure states of \( \mathcal{E} \) represent pure states of the system.

\textit{Postulate 3.} If \( a \in \mathcal{E}_H \) and \( \phi \in \mathcal{S} \) then \( \phi(a) \) is the expectation value of the observable \( a \) when the system is in the state \( \phi \).

We remind that \textit{states} of \( \mathcal{E} \) are defined to be positive linear functionals \( \phi \) on \( \mathcal{E} \) such that \( \| \phi \| = 1 \). Convex combinations of states are states. A state which cannot be expressed as a convex combination of other states is said to be a \textit{pure state}.

The above three postulates are in the analogy with the standard \( C^* \)-algebraic approach to quantum mechanics, the fourth postulate is a new ingredient of the noncommutative quantization of gravity.
Postulate 4. The dynamical equation of the system described by $\mathcal{E}$ is

$$i\hbar \pi_q(v(a)) = [\pi_q(a), F]$$

(5)

for every $q \in G^{(0)}$. Let us notice that here $v \in G$, and in this way generalized Einstein’s equation (3) is coupled to quantum dynamical equation (5). $F$ is a Fredholm operator, i.e. an operator $F : \mathcal{H} \to \mathcal{H}$ such that $F(\mathcal{H})$ is closed and the dimensions of its kernel and cokernel are finite. Together with the group $\Gamma$, the operator $F$ is a “free entry” of our quantization scheme. It should be specified on physical grounds. To solve eq. (3) means to find $a \in \ker G \subset \mathcal{E}$ such that $\pi_q(v(a))$ for $v \in \text{Der} \mathcal{E}$, would give the same result as $-i/\hbar [\pi_q(a), F]$ when acting on $\psi \in L^2(G_q)$.

Equation (4) is a noncommutative counterpart of the Schrödinger equation in the Heisenberg picture of the usual quantum mechanics

$$i\hbar \left( \frac{d}{dt} \hat{A}(t) \right) = [\hat{A}(t), H]$$

where $H$ is the Hamilton operator, in which state vectors are independent of time and all time dependence goes to operators. Since in the noncommutative framework the standard concept of time breaks down, the dynamics of the system is expressed in terms of derivation of the Einstein algebra. This remark could be elaborated in the following way. Let $S$ be the set of all elements of the Einstein algebra $\mathcal{E}$ satisfying eq. (3), and let us consider the set $\pi_q(S) \subset \mathcal{B}(\mathcal{H})$. Let further $(\pi_q(S))''$ be the commutant of the commutant of $\pi_q(S)$ (we remaind that the commutant of a subset $\mathcal{M}$ of an algebra $\mathcal{A}$ is defined to be the set of all elements of $\mathcal{A}$ which commute with all elements of $\mathcal{M}$). $(\pi_q(S))''$ is the smallest von Neumann algebra generated by the “space of solutions” $\pi_q(S)$ (see [17, p. 14]). We remind that the subset $\mathcal{N}$ of $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra if $\mathcal{N} = \mathcal{N}''$. It can be argued that in the context of noncommutative geometries von Neumann algebras encode dynamical aspects of the system in question. Very roughly speaking, an algebra of operators in a separable Hilbert space is a von Neumann algebra if it commutes with unitary operators in this Hilbert space and, as it is well known from the standard quantum mechanics, unitary operators are responsible for the dynamical evolution of the system. In the noncommutative context there is no obvious counterpart of the local time concept, and it is precisely von Neumann algebra that can be regarded as implementing the abstract idea of dynamics (see also [3]).
Equation (5) acts on that Hilbert space $L^2(G_q)$. This space should be regarded as a counterpart of the Hilbert space in the position representation in quantum mechanics. However, now the “position space” is more abstract: the quantity $|\psi(\gamma)|^2$ is the probability density of the “fundamental symmetry” $\gamma \in G_q$ to occur.

It might turn out that in order to make eq. (5) manageable we would have to impose on it some further conditions, for instance to assume that the triple $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module or to assume some its summability properties (see [13, pp. 288-291], [17, pp. 117-120]).

To substantiate our approach we should show that it reproduces, in a suitable limit, the usual general relativity and the usual quantum mechanics. We shall demonstrate this in the subsequent Section.

5 Transition to General Relativity and Quantum Mechanics

The “canonical way” of obtaining the commutative geometry from a noncommutative one is to restrict the corresponding noncommutative algebra $\mathcal{A}$ to its center $Z(\mathcal{A})$. In our case this restriction could be interpreted in the following manner. The noncommutative algebra $\mathcal{A}$ can be thought of as a “deformation” of a commutative algebra of “classical observables” with the Planck constant $\hbar$ as a “deformation parameter”, typically $[a, b] = i\hbar 1$, for two noncommuting elements $a$ and $b$ of $\mathcal{A}$. Since the center $Z(\mathcal{A})$ of $\mathcal{A}$ is the set of elements of $\mathcal{A}$ which commute with all elements of $\mathcal{A}$, the transition from $\mathcal{A}$ to $Z(\mathcal{A})$ can be implemented by postulating $\hbar \to 0$. This means that quantum effects are negligible and should reduce our theory to the usual theory of general relativity. We shall show below that this is indeed the case.

$Z(\mathcal{A})$ is of course a commutative algebra with convolution as multiplication (since $Z(\mathcal{A})$ is a subalgebra of $\mathcal{A}$). Let $Z(\mathcal{A})^*$ be the set of all characters of $Z(\mathcal{A})$, i.e. the set of all *-homomorphisms from $Z(\mathcal{A})$ to $\mathbb{C}$. On the strength of the Gel’fand theorem the algebra $Z(\mathcal{A})$ is isomorphic with the algebra of continuous functions on the groupoid $G$ (with the usual multiplication). This algebra is given by the Gel’fand representation

$$\rho^{Z(\mathcal{A})} : Z(\mathcal{A}) \to C^*Z(\mathcal{A})^*$$
defined by
\[ \rho Z(A)(a)(\chi) = \chi(a) \]
where \( a \in Z(A), \ \chi \in Z(A)^\# \), and \( Z(A)^\# \) can be identified with \( G \). The algebra \( \rho Z(A)(Z(A)) \) consists of continuous functions on \( G \), but since \( G \) is a smooth manifold we can assume that these functions are smooth (if necessary we can restrict this algebra to the subalgebra of smooth functions) (see [31]).

We shall denote the algebra of these functions by \( G^\infty \). As it is well known, there is the bijection
\[ Z(A)^\# \rightarrow \text{Spec} Z(A), \]
where \( \text{Spec} Z(A) \) denotes the set of maximal ideals of \( Z(A) \), given by
\[ \chi \mapsto \ker \chi, \]
\( \chi \in Z(A)^\# \). Each maximal ideal \( \ker \chi \) determines a point of \( G \) (such a point is given by the set of functions belonging to \( G^\infty \) vanishing at this point). We remember that points of \( G \) are “fundamental symmetries” of our theory. In this way, the full geometry of the groupoid is given by the pair \( (G, G^\infty) \).

Since the prototype of our groupoid was the Cartesian product \( G = E \times \Gamma \) (see Introduction) where \( E \) was supposed to be the total space of the frame bundle over space-time \( M \), we recover \( M \) by forming, first, the quotient \( E = G/\Gamma \), and, second, \( M = E/\Gamma \) (or \( M = (E/\Gamma)/\Gamma \), see below for the detailed construction). Generalized Einstein’s equation (2) “projected down” in this way to space-time \( M \) gives the usual Einstein equation of general relativity.

Now, we shall discuss the transition from our noncommutative theory to the usual quantum mechanics. To do this, we assume that the gravitational field is weak so that quantum gravity effects can be neglected. This means that in dynamical equation (3) the assumption that \( v \in \ker G \) can be omitted, i.e. generalized Einstein’s equation is decoupled from ordinary quantum effects.

Equation (3) is defined on the Hilbert space \( \bigoplus_{q \in E} L^2(G_q) \). We want to “project it down” to the more usual Hilbert space \( L^2(M) \). We do that, essentially, as above, by forming the “double quotient” \( (E/\Gamma)/\Gamma \). However, more practical way is the following.

Let \( (p_1, g_1), \ (p_2, g_2) \in G, \ p_1, p_2 \in E, \ g_1, g_2 \in \Gamma \). We define the equivalence relation
\[ (p_1, g_1) \sim (p_2, g_2) \iff \exists g \in \Gamma \ p_2 = p_1 g. \]
Then we consider only those “wave functions” \( \tilde{\psi} \) which have the following invariance property
\[
\tilde{\psi}(p_1, g_1) = \tilde{\psi}(p_2, g_2)
\]
(\( \tilde{\psi} \) is constant on equivalence classes of \( \sim \)). It can be easily seen that equation (5) restricted to functions \( \tilde{\psi} \) is essentially the Schrödinger equation of quantum mechanics in its Heisenberg picture provided that the Fredholm operator \( F \) is correctly chosen to reproduce the Hamiltonian of the system. This, of course, had to be expected. In our noncommutative quantum gravity theory there is no concepts of points and time instants (space and time are somehow hidden in the subalgebra of \( Z(A) \)); the standard concept of space-time appears only in the transition process to the standard physics. No wonder that we obtain the Heisenberg picture in which state vectors are time independent and all time dependence goes to operators.

6 A Simple Example

In this Section we shall analyse a simple model of our scheme to quantize gravity. The basis of this model is the groupoid \( G = E \times D_4 \), where \( E \) is the total space of the frame bundle over a three-dimensional Minkowski spacetime \( M^3 \) and \( D_4 \) is a group consisting of 4 rotations by the angle \( \pi/2 \) and 4 reflections with respect to two directions crossing each other at the origin. If \( r \) denotes rotation and \( s \) reflection, the following relations are assumed to be satisfied
\[
r^4 = 1, \quad s^2 = 1, \quad srs = r^{-1}.
\]
\( D_4 \) is a finite noncommutative subgroup of the group \( SU(2) \). \( D_4 \) acts on \( E \) (to the right) on the plane \((x, y)\) leaving the \( t \)-axis fixed.

For every vector field \( X \in \mathcal{X}(M^3) \) on the Minkowski space-time \( M^3 \), there exists its lifting to \( \tilde{X} \in \mathcal{X}(G) \) to \( G \). It can be easily seen that the vector field \( \tilde{X} \) is constant on the fibres (of \( G \)) parallel to \( M^3 \). All such fields form a \( Z(A) \)-submodule \( V_E \) of the \( Z(A) \)-module \( \text{Der}(A) \), where \( A = C^\infty(G) \). Analogously, we have a \( Z(A) \)-submodule \( V_{D_4} \) of the \( Z(A) \)-module \( \text{Der}(A) \). Our simple model will be based on the differential algebra \((A, V)\), where \( V = V_E \oplus V_{D_4}, V \subset \text{Der}(A) \). Accordingly, the algebra \( C^\infty(G) \) can be “decomposed” into the algebras \( C^\infty(E) \) and \( C[D_4] \), where \( C^\infty(E) = \iota_E^*(C^\infty(G)) \) and \( C[D_4] = \iota_{D_4}^*(C^\infty(G)) \), \( \iota_E \) and \( \iota_{D_4} \) being natural embeddings of \( E \) and \( D_4 \) into \( G \), respectively.
In agreement with formula (1) (Sec. 3), we assume a metric on the \( \mathcal{Z}(A) \)-module \( V \) of the form

\[
g = pr_E^* \circ \pi_1^* \eta + pr_{D_4}^* g_{D_4},
\]

where \( \eta \) and \( g_{D_4} \) are the Minkowski metric on \( M^3 \) and a metric on the group \( D_4 \), respectively; \( pr_E \) and \( pr_{D_4} \) are obvious projections from the groupoid, and \( \pi_1 \) is the canonical projection from \( E \) to \( M^3 \).

Since the “parallel geometry” (geometry based on \( V_E \)) is rather obvious (see below), we shall focus on the “vertical geometry” (geometry of \( D_4 \)). As it is well known, the group algebra \( \mathbb{C}[D_4] \) can be constructed in the following way

\[
\mathbb{C}[D_4] = \{ \sum_{i=1}^8 c_i A_i : c_i \in \mathbb{C}, A_i \in D_4, i = 1, \ldots, 8 \}
\]

(8 is the rank of \( D_4 \)), with the usual addition and convolution as multiplication. For any finite group \( \Gamma \) there exists an isomorphism

\[
T : \mathbb{C}[\Gamma] \to \Pi_{i=1}^{k} M_{n_i}(\mathbb{C}),
\]

where \( M_{n_i}(\mathbb{C}) \) are \( n_i \times n_i \) matrices and \( i \) runs over all irreducible representations of \( \Gamma \), such that \( T(\varphi \ast \psi) = T(\varphi) \cdot T(\psi) \) with asterisk denoting convolution and dot the usual matrix multiplication. The group \( D_4 \) has 4 irreducible representations of rank 1, and 1 irreducible representation of rank 2 \[34\]. Therefore, in our case, the above isomorphism assumes the form

\[
T : \mathbb{C}[D_4] \to \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})
\]

given by

\[
T = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho^1),
\]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are the rank 1 irreducible representations of \( D_4 \), and \( \rho^1 \) is the rank 2 irreducible representation of \( D_4 \).

The set of derivations \( \text{Der}(\mathbb{C}[D_4]) \) of the algebra \( \mathbb{C}[D_4] \) is isomorphic with \( \text{Der}(M_2(\mathbb{C})) \). It can be shown, by the straightforward computation, that if \( (e_\alpha), \alpha = 1, \ldots, n^2 - 1 \), is a basis of the Lie algebra \( \text{su}(n) \) of the Lie group \( \text{SU}(n) \), and if \( c^\gamma_{\alpha\beta}, \alpha, \beta, \gamma = 1, \ldots, n^2 - 1 \), are structure constants of \( \text{su}(n) \) with respect to the basis \( (e_\alpha) \), then \( c^\gamma_{\alpha\beta} \) are also structure constants of the
Lie algebra $\text{Der}(M_n(\mathbb{C})$ with respect to the basis $(\text{ad}e_\alpha)$, $\alpha = 1, \ldots, n^2 - 1$. In our case, we choose the following basis for $\mathfrak{su}(2)$

$$e_1 = \text{ad} \frac{i}{2} \sigma_1, \quad e_2 = \text{ad} \frac{i}{2} \sigma_2, \quad e_3 = \text{ad} \frac{i}{2} \sigma_3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices, and the structure constants assume the simple form $c_{\alpha\beta}^\gamma = \epsilon_{\alpha\beta\gamma}$, i.e. they are equal 1 for even permutations, and 0 otherwise (see, for instance [33, p. 183]).

Now, we chose the metric $g_{D_4} : V_{D_4} \times V_{D_4} \to \mathbb{Z}(\mathbb{C}[D_4])$ defined by

$$(g_{D_4})_{11} = kI, \quad (g_{D_4})_{ij} = \delta_{ij}I, \text{ if } i \neq 1, j \neq 1,$$

where $k \in \mathbb{R}$, and develop differential geometry as in Sec. 4. As we shall see below, even such a trivial deviation from the “Euclidean metric” gives us interesting insights into the nature of the problem at hand.

Straightforward computations give the following non-vanishing components of the Ricci operator

$$R_1^1 = -\frac{k}{2},$$

$$R_2^2 = R_3^3 = \frac{k}{2} - 1.$$

It can be easily seen that, in this case, the Einstein equation $R_{D_4}(w) = 0$ is satisfied only for $w = 0$, $w \in V_{D_4}$. To find non-trivial solutions we should try the Einstein equation with the cosmological constant. For $w = w^iv_i$ it can be written in the form

$$(R_{D_4} + 2\Lambda I)w^iv_i = 0.$$

This is the eigenvalue equation for the operator $R_{D_4}$. The eigenvalues $\lambda = 2\Lambda$ can be easily found. To find them we should distinguish two cases.

Case 1: $k \neq 1$. The eigenvalues are: $\Lambda_1 = \frac{k}{4}$ and $\Lambda_2 = \frac{1}{2} - \frac{k}{4}$.

The eigenvectors corresponding to the eigenvalue $\Lambda_1$ are of the form $w = tv_i = t\text{ad} \frac{1}{2} \sigma_1$, where $t \in \mathbb{C}$. Therefore, for $\Lambda_1$ the solution of the “vertical part” of Einstein equation (3) is

$$\ker(G_{D_4}) = \{tv_1 : t \in \mathbb{C}\}$$
where $G_{D_4} = R_{D_4} + 2\Lambda I$. And correspondingly for the eigenvalue $\Lambda_2$ one has the solution

$$\ker(G_{D_4}) = \{rv_2 + sv_3 : r, s \in \mathbb{C}\}.$$

It is a remarkable fact that our simple model requires, for its consistency, the existence of two “cosmological constants” in the quantum sector of the model (i.e., in its “vertical geometry”), and predicts their values. These “cosmological constants” are eigenvalues of the Ricci operator $R_{D_4}$ (up to a constant factor). Of course, because of the “toy” character of the model, these values are rather symbolic.

**Case 2:** $k = 1$. There is only one eigenvalue $\Lambda = \frac{1}{4}$, and correspondingly one has

$$\ker(G_{D_4}) = \{tv_1 : s \in \mathbb{C}\}.$$

Now, we must return to the “parallel geometry”. We shall consider the algebra $A_E = \pi_2^*(C^\infty(E))$ where $\pi_2$ is the natural projection from $G = E \times D_4$ to $E$. Since in our case the frame bundle $\pi_1 : E \to M^3$ is a trivial bundle, there is the natural embedding $\iota_k : M \to E$, $k \in \Gamma$, where $\Gamma$ is the structural group of this bundle, i.e. the Lorentz group. There is also another natural embedding $j_h : E \to G$, $h \in D_4$. With the help of these two embeddings we “push forward” the basis $(\partial_t, \partial_x, \partial_y)$ in $M^3$ to the basis $(\bar{\partial}_t, \bar{\partial}_x, \bar{\partial}_y)$ in $G$. In this way, we obtain the $Z(A)$-module of derivations

$$V_E = \{a^1\bar{\partial}_t + a^2\bar{\partial}_x + a^3\bar{\partial}_z : a^1, a^2, a^3 \in A_E\}.$$

We equip this module with the metric $\bar{\eta}$ lifted from the Minkowski metric $\eta$ on $M^3$, i.e. $\bar{\eta} = \tau^*\eta$ where $\tau = \pi_1 \circ \pi_2$. It can be easily seen that in fact $\bar{\eta}$ is also Minkowski metric. Indeed, let $\bar{X}, \bar{Y} \in V_E$, then $\bar{\eta}(\bar{X}, \bar{Y}) = (\tau^*\eta)(\bar{x}, \bar{y}) = \eta(\tau_*\bar{x}, \tau_*\bar{y}) = \eta(X, Y)$. Therefore, we have

$$R_E = 0.$$

Now, we can consider the algebra $A = C^\infty(E \times D_4)$ together with the $Z(A)$-module

$$V = V_E \oplus V_{D_4} = \{a^\alpha\bar{\partial}_a + \beta^i\bar{v}_i : a^\alpha, \beta^i \in A, \ a = 0, 1, 2; \ i = 1, 2, 3\}.$$

We should now see how the generalized Einstein equation interacts with the quantum dynamical equation (5). As an example let us consider case
1 when, in the “vertical geometry”, the metric coefficient $k \neq 1$. Since we postulate that the derivation $v$ in the left hand side of eq. (5) should be a solution of generalized Einstein equation, eq. (5) splits into two equations

$$i\hbar \pi_q((\alpha^a \partial_a + t \text{ad}_{\frac{i}{2} \sigma_1})(a)) = [\pi_q(a), F],$$

and

$$i\hbar \pi_q((\alpha^a \partial_a + r \text{ad}_{\frac{i}{2} \sigma_2} + s \text{ad}_{\frac{i}{2} \sigma_3})(a)) = [\pi_q, F].$$

When a Fredholm operator $F$ is given, these equations should be solved for $a \in \mathcal{A}$.

In agreement with the discussion of Sec. 5, in order to obtain classical case we must restrict the algebra $\mathcal{A} = C^\infty(G)$ to the algebra $\mathcal{A}_{\text{proj}}$. In such a case, the “vertical geometry” projects to zero, and we are left with the ordinary Minkowski space-time $M^3$ and the corresponding Einstein field equation $R = 0$ (this effect, in the considered model, is trivial since the “parallel geometry” has been obtained by lifting the Minkowski geometry to the groupoid $G$). Let us notice that, even in this toy model, we have an interesting result: in the noncommutative regime the kind of cosmological constants appear (as eigenvalues of the Ricci operator for the “quantum sector”) which vanish if we go to the classical case.

### 7 Observables and Their Eigenvalues

In postulate 2 of our quantization scheme we have identified quantum gravity observables with the Hermitian elements of the algebra $\mathcal{A}$ by following strict analogy with the $C^*$-algebraic quantization of the usual quantum mechanics. However, from the experimental point of view we are interested only in those observables which leave some traces in the macroscopic world and thus have chances to be detected. As we have seen in the preceding section, such observables must belong to $\mathcal{A}_{\text{proj}}$. Let $a$ be such an observable, and let the system be in a state $\psi$ which, in order “to be reached” by a macroscopic observer, must be $\Gamma$-invariant (see the preceding section). Measuring an observable quantity corresponding to $a$ when the system is in a $\Gamma$-invariant state $\psi \in L^2(G_q)$ means to act with $a$ upon $\psi$. The measurement will give as its result the eigenvalue $r_q$ as determined by the eigenvalue equation

$$\pi_q(a)\psi = r_q \psi$$  \hspace{1cm} (6)
where, for simplicity we consider a non-degenerate case. Taking into account the form of the representation $\pi_q$ (eq. (4)) the above equation is equivalent to

$$\int_{G_q} a(\gamma_1)\psi(\gamma_1^{-1}\gamma) = r_q\psi(\gamma).$$

From the $\Gamma$-invariance of $\psi$ it follows that $\psi$ is constant on $G_q$; therefore, we can write

$$\psi(\gamma_1^{-1}\gamma)\int_{G_q} a(\gamma_1) = r_q\psi(\gamma)$$

and consequently

$$r_q = \int_{G_q} a(\gamma_1).$$

We have proved the following fact:

Lemma. If $\psi \in L^2(G_q)$ is $\Gamma$-invariant and if it is an eigenfunction of $a \in \mathcal{E}_H$, the eigenvalue of $a$ is $r_q = \int_{G_q} a(\gamma_1)$.

This is a nice conclusion. Let us notice that the result $r_q$ of a measurement is a measure in the mathematical sense (in this case we deal with the Haar measure on the group $\Gamma$). But we can go even further. Let us define the “total phase space” of our system

$$L^2(G) := \bigoplus_{q \in G^{(0)}} L^2(G_q),$$

with the operator

$$\pi(a) := (\pi_q(a))_{q \in G^{(0)}}$$

acting on it. Now, eigenvalue equation (3) can be naturally written as

$$\pi(a)\psi = r\psi$$

where $r$ is a function on $G$; since, however, $\psi$ is $\Gamma$-invariant $r$ can be interpreted as the function on space-time $M$

$$r : M \to \mathbb{R}$$

defined by

$$r(x) = r_q = \int_{G_q} a(\gamma_1)$$

with $x$ being a point in $M$ to which the “frame” $q$ is attached. The measurement result is not a “naked number”, but a value of a function at a given
point \(x \in M\) the domain of which is the entire space-time \(M\). As the consequence of this, if the measurement is performed at a point \(x \in M\), its result is correlated with the result of another measurement performed on another component of the same system even if it is situated at a very distant point \(y \in M\). This also suggests that typically quantum gravitational phenomena should be looked for among correlations between distant measurements rather than among “local phenomena”. This could be regarded as a relic of the pre-Planckian era in which “everything was global”. It would be interesting to examine nonlocal phenomena of quantum physics detected by the Aspect type experiments in the light of the above remarks.

8 Concluding Remarks

In the present paper we have proposed a mathematical structure which combines essential aspects of general relativity (its main geometric elements) with those of quantum physics (algebra of operators of a Hilbert space), and leads to correct “special cases” of standard general relativity and standard quantum mechanics. In our opinion, this is the main result of our approach. It is rooted in the fact that we have performed the quantization of a groupoid (over a space-time) rather than of space-time itself.

Although we do not think that our approach is a full theory of quantum gravity (it is a scheme rather than a theory since two of its important elements, the group \(\Gamma\) and the Fredholm operator \(F\), remain unspecified), it gives some hints of what the future theory could be like. It clearly suggests that measurable effects of quantum gravity should be looked for among correlations of distant phenomena rather than among local effects (Sec. 7). It is also a remarkable fact that a simple model based on the groupoid \(G = E \times D_4\) (Sc. 6) predicts the value of “cosmological constants” which, after projecting down to space-time, vanish.

To choose the correct group \(\Gamma\) it is important to experiment with various possibilities. In [14] we have computed an example where \(\Gamma\) is any finite group. A generalization to the case when \(\Gamma\) is compact would not be difficult. From the physical point of view the most interesting are cases with \(\Gamma\) non-compact but then, unfortunately, difficulties increase dramatically. We hope, however, that they are only of the technical nature. For instance, as the work by Fell [36] demonstrates, the group algebra of \(\text{SL}(2,\mathbb{C})\) (which physically is
very interesting) can be expressed in terms of algebras of operator fields on a locally compact Hausdorff space, which in turn is isomorphic with the algebra of norm-continuous functions on a certain parameter space.

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