Fourier coefficients of the net baryon number density and their scaling properties near a phase transition

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We study the Fourier coefficients $b_k(T)$ of the net baryon number density in strongly interacting matter at finite temperature. We show that singularities in the complex chemical potential plane connected with phase transitions are reflected in the asymptotic behavior of the coefficients at large $k$. We derive the scaling properties of $b_k(T)$ near a second order phase transition in the $O(4)$ and $Z(2)$ universality classes. The impact of first order and crossover transitions is also examined. The scaling properties of $b_k(T)$ are linked to the QCD phase diagram in the temperature and complex chemical potential plane.

I. INTRODUCTION

Fourier decomposition is a useful technique for exploring characteristic features of a system. In the study of hot and dense Quantum Chromodynamics (QCD), Fourier decomposition of the grand partition function for imaginary baryonic chemical potential has been used to obtain the canonical partition function at fixed net baryon number. One thus obtains thermodynamic quantities at nonzero (real) net baryon densities, bypassing the sign problem in the calculations of the grand partition function at real baryonic chemical potentials [1–10].

Recently, the Fourier decomposition of the net baryon number density $\chi^B(T, \mu_B) \equiv \partial(p/T^4)/\partial \mu_B$

$$\text{Im}[\chi^B(T, i\theta_B)] = \sum_{k=1}^{\infty} b_k(T) \sin(k \theta_B) \quad (1)$$

at imaginary baryon chemical potential $\theta_B = \text{Im} \mu_B$ has been discussed as a tool which connects thermodynamic quantities at imaginary and real chemical potentials. Here $\mu_B = \mu_B/T$ is the reduced chemical potential and $T$ the temperature. In Ref. [11] it was shown that Eq. (1) provides a good description of the density obtained in lattice QCD simulations at imaginary $\mu_B$. Given the density as a function of the imaginary chemical potential, one may compute the Fourier coefficients $b_k$ using

$$b_k = \frac{1}{\pi} \int_0^{2\pi} d\theta_B \text{Im}\chi^B(T, i\theta_B) \sin(k \theta_B). \quad (2)$$

Note that $\text{Im}\chi^B(T, i\theta_B)$ is an odd, periodic function in $\theta_B$ with the period $2\pi$.

Based on model calculations, it was shown in [12], that the Fourier coefficients $b_k(T)$ provide a signature for the deconfinement transition. On the other hand, in Ref. [13], an ansatz for the $k$–dependence of the Fourier coefficients $b_k$ consistent with the lattice data up to $b_4$ [14] was employed to explore the possible existence of a critical point at nonzero baryon density and to study fluctuations of the net baryon number.

The integration over the imaginary chemical potential in Eq. 2 may cross a phase boundary, as indicated in Fig. 1. The nature of the phase boundary depends on the temperature and on the pion mass. For instance, at high temperatures, QCD exhibits a first order phase transition at $\theta_B = \pi$, the Roberge-Weiss (RW) transition. The latter is a consequence of the $Z(3)$ center symmetry, which is related to the confinement of quarks [15]. The

![FIG. 1. A schematic QCD phase diagram in the $T–\theta_B$ plane. The arrows indicate the integration paths in Eq. (2) for high temperatures (a), for temperatures in the transition region (b) and for low temperature (c), respectively.](image-url)
discontinuity of the baryon density at the RW transition is reflected in a power-law dependence of the Fourier coefficients, \( b_k \sim 1/k \), for large \( k \). Moreover, even if the integration path does not cross a phase boundary, as in case (c), or the phase boundary in case (b) is a crossover transition, in which case the density is analytic along the integration path, there are singularities in the complex chemical potential plane, which do affect the behavior of the Fourier coefficients.

The goal of this paper is to derive model independent results for the effect of critical singularities on the asymptotics of the Fourier coefficients\(^1\). First we study the Fourier expansion of the net baryon density in the Landau theory of the phase transitions, in order to illustrate in a transparent framework how critical singularities in the complex chemical potential plane are reflected in the Fourier coefficients. We then apply the scaling theory of phase transitions to derive the properties of the Fourier coefficients near a second-order phase transition. We also consider the influence of a first-order phase transition and a crossover on the asymptotic behavior of the \( b_k(T) \). An explicit calculation of the Fourier coefficients in a QCD-inspired effective model were presented in Ref. [16].

In addition to the singularities associated with phase transitions, the thermodynamic potentials have thermal singularities [17]. In particular, the thermal branch points of the net baryon density are generated by the poles of the Fermi-Dirac function at complex values of the baryon chemical potential. For baryonic degrees of freedom of mass \( m \), these are located at \( \tilde{\mu}_B = \pm \tilde{m} \pm \i\pi \), where \( \tilde{m} = m/T \). The influence of the thermal singularities on the Fourier coefficients is also briefly discussed.

We present our results in the context of the expected criticality of QCD matter in the temperature and imaginary chemical potential plane (see Fig. 1). In particular, we show that at the chiral \( O(4) \) and the RW \( Z(2) \) critical points, the asymptotic Fourier coefficients exhibit a power-law behavior, which depends on the critical exponents \( \alpha \) and \( \delta \), respectively. Furthermore, in the temperature range between the \( O(4) \) critical point and the RW endpoint, an oscillation is superimposed on the power-law dependence. The frequency of the oscillation is determined by the value of the imaginary chemical potential at the critical point.

For temperatures below the \( \mu_B = 0 \) chiral critical point, the \( O(4) \) singularity is, in the chiral limit, located on the real \( \mu_B \) axis. This implies that the contribution of the critical singularity to large-order Fourier coefficients is exponentially suppressed. Therefore the critical behavior at real \( \mu_B \) does not have a conspicuous impact on the asymptotic form of the Fourier coefficients \( b_k \). In particular, the existence of a \( Z(2) \) critical point at large real \( \mu_B \) is not perceptible in the high-order Fourier components.

Moreover, for physical quark masses, the \( O(4) \) chiral transition is of the crossover type. In this case the critical singularities are located in the complex \( \tilde{\mu}_B \) plane and the \( k \) dependence of the Fourier coefficients \( b_k(T) \) corresponds to damped oscillations superimposed on a power-law dependence. The damping rate is determined by the real and the oscillation frequency by the imaginary part of the baryon chemical potential at the crossover branch point singularity. At temperatures between the chiral transition at \( \mu_B = 0 \) and the RW critical point, the imaginary part dominates and the oscillations are underdamped. Conversely, at lower temperatures the real part of \( \mu_B \) dominates and the oscillations are overdamped.

This paper is organized as follows. In section II we discuss the analytic structure of the density in Landau theory. We then derive the asymptotic behavior of \( b_k(T) \) in section III. Finally, section IV is devoted to a summary.

**II. CRITICAL SINGULARITIES IN THE COMPLEX \( \mu_B \) PLANE**

In the chiral limit of QCD, one expects to find a second order chiral critical line, belonging to the \( O(4) \) universality class, at a critical temperature \( T_c \approx 140 \text{ MeV} \) and \( \mu_B = 0 \) \( [18] \) and by continuity at small values of the baryon chemical potential. This line continues to larger \( \mu_B \) and ends in a chiral tricritical point at \((T_{TCP}, \mu_{TCP})\), assuming that such a point exists. The second-order critical points and the tricritical point correspond to singularities on the real \( \mu_B \) axis.

Given the negative curvature of the phase boundary at small \( \mu_B \) and nonzero quark masses \([19, 20]\), it is generally assumed that \( T_{TCP} < T_c \). Beyond the tricritical point, i.e., at still larger values of the baryon chemical potential, the chiral transition would be first order.

On the other hand, at temperatures \( T > T_c \), one expects a chiral critical point for purely imaginary values of the chemical potential, as shown in Fig. 1. With increasing temperature, this transition occurs at larger values of \( \theta_B \) and eventually approaches \( \theta_B = \pi \), at or close to the temperature of the Roberge-Weiss endpoint.

For nonzero quark masses, chiral symmetry is explicitly broken and the chiral critical line is replaced by a line of crossover transitions. The corresponding branch points in the complex chemical potential plane are shifted away from the real or imaginary axes to\(^2\).

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\(^1\) We note that although all singularities in the complex \( \mu_B \) plane contribute to the Fourier coefficients, their asymptotic behavior is determined only by the singularity closest to the imaginary axis. Thus, the contribution to the asymptotics from the critical singularities considered in this letter, may be “screened” by other non-analyticities, located closer to the imaginary \( \mu_B \) axis. Moreover, for singularities on the imaginary axis, the strongest singularity determines the asymptotics of the Fourier coefficients. Thus, again the contribution of a particular critical singularity may be eclipsed by other, stronger ones.

\(^2\) The singularities come in complex conjugate pairs, because the thermodynamic functions are real for real values of the chemical
\[ \mu_{br} = \pm \mu_c \pm i T \theta_c \] (see Fig. 2). The conjectured first-order chiral transition at large \( \mu_B \) then ends in a critical point \( (T_{CP}, \mu_{CP}) \), located on the real \( \mu_B \) axis. This point is also referred to as the chiral critical endpoint \[21, 22].

### A. Landau theory

The Landau theory of phase transitions offers a transparent framework for exploring the qualitative features of the singularity structures discussed above. The Landau free energy is given by

\[ \Omega[\sigma] = \frac{1}{2} a \sigma^2 + \frac{1}{4} b \sigma^4 - h \sigma, \quad (3) \]

where \( \sigma \) is the order parameter, \( a \) and \( b \) are real functions of the thermodynamic variables (here \( T \) and \( \mu_B \)), while \( h \) is a symmetry breaking external field\[3]. The equilibrium value of the order parameter is determined by solving the gap equation,

\[ \frac{\partial \Omega}{\partial \sigma} = a \sigma + b \sigma^3 - h = 0. \quad (4) \]

By taking a further derivative with respect to \( \sigma \), we obtain the order-parameter susceptibility

\[ \chi_{\sigma} = (a + 3b \sigma^2)^{-1}, \quad (5) \]

where the order parameter takes on the value that minimizes \( \Omega \).

The solutions of the gap equation read

\begin{align*}
\sigma_1 &= \frac{f(a, b, h)}{\sqrt{2}3^{2/3} b} - \frac{1}{\sqrt{3}a}\frac{1}{f(a, b, h)}, \quad (6) \\
\sigma_2 &= \frac{(1 + i\sqrt{3}) a}{2^{2/3}3^{1/3}f(a, b, h)} - \frac{1}{2\sqrt{2}3^{2/3} b}, \quad (7) \\
\sigma_3 &= (\sigma_2)^*, \quad (8)
\end{align*}

where \( f(a, b, h) = \sqrt{9} h b^2 + \sqrt{3} \sqrt{27} h^2 b^4 + 4 a^3 b^3 \). The first solution is real, while the other two are real or complex, for real \( a, b \) and \( h \). In the limit \( h \to 0 \), where the symmetry under \( \sigma \to -\sigma \) is restored, the solutions reduce to \( \sigma = 0 \) and \( \sigma = \pm i \sqrt{a/b} \). The first one is a minimum of the effective potential for \( a > 0 \), while the other solutions correspond to two degenerate minima for \( a < 0 \). Thus, for \( a < 0 \) the expectation value of the order parameter is nonzero and the symmetry is spontaneously broken.

While for \( h = 0 \) the critical point is located at \( \sigma = 0 \) and \( a = 0 \), for nonzero \( h \) the singularities are shifted to complex values of \( \sigma \) and \( a \). These singularities are branch points where the susceptibility \( (5) \) diverges for a solution of the gap equation. For the three solutions \( (6) \)–\( (8) \), these branch points are found at \[23\]

\[ a_{br1} = -3 \left( \frac{bh^2}{4} \right)^{1/3}, \quad (9) \]

\[ a_{br2,3} = 3e^{\pm i\pi/3} \left( \frac{bh^2}{4} \right)^{1/3}. \quad (10) \]

Thus, the location of the branch points \( a_{br} \) scale with \( h^{2/3} \). We note that the first one, \( a_{br1} \), is located on an unphysical Riemann sheet in the complex \( a \)-plane, while the two complex singularities \( (10) \) are responsible for the crossover transition at real \( a \) and nonzero \( h \).

To map the singularities onto the complex chemical potential plane, we use the parameterization \( a = t' = t + \kappa(\mu_B/T_c)^2 \), where \( \kappa \) represents the curvature of the critical line, \( t = T/T_c - 1 \) is the reduced temperature and \( T_c \) the critical temperature at \( \mu_B = 0 \). Then, the location of the four branch points is given by

\[ \mu_{br} = \pm T_c \sqrt{-\frac{t + \frac{3}{2}(1 \pm \sqrt{3}i) \left( \frac{bh^2}{4} \right)^{1/3}}{\kappa}}. \quad (11) \]

We note that for \( t = 0 \), \( \mu_{br} \) scales with \( h^{1/3} \). As observed above, there are four branch points in the complex \( \mu_B \) plane. In the chiral \( (h \to 0) \) limit, the branch points approach the real \( \mu_B \) axis for \( t < 0 \) (below \( T_c \)).
and the imaginary axis for \( t > 0 \) (above \( T_c \)), as indicated in Fig. 2. In the chiral limit of QCD, the corresponding branch points make up the \( O(4) \) critical line \([17, 23, 24]\).

For the crossover transition, one can define a pseudocritical temperature at the maximum of the order parameter susceptibility \((5)\). The solution of \( \partial \chi_\sigma / \partial a = 0 \) is, for the first solution of the gap equation \((6)\), \( a = 3(b h^2 / 16)^{1/3} \), while the corresponding pseudocritical temperature at \( \mu_B = 0 \) is given by

\[
T_{pc} = T_c \left[ 1 + 3 \left( \frac{b h^3}{16} \right)^{1/3} \right] \tag{12}
\]

Using \((12)\) in \((11)\), one finds the location of the singularities at the pseudocritical temperature\(^4\)

\[
\mu_{br}(T = T_{pc}) = \pm T_c \sqrt{\frac{3 b (1 - \sqrt{2} \pm \sqrt{3} i)}{\sqrt{32} \kappa}} h^{1/3} \tag{13}
\]

The trajectories of the crossover branch point in the first quadrant of the complex \( \mu_B \)-plane are shown in Fig. 2 for \( h = 0.1, 0.5 \) and 1. The circles and triangles indicate the locations of the branch point for fixed values of \( T/T_{pc} \).

By adding a \( \delta^6 \) term to the Landau free energy \((3)\), one can explore the singularities associated with a critical point for \( h = 0 \) and a critical point for \( h \neq 0 \). In the latter case, the two complex conjugate branch points of the crossover transition merge on the real axis \([23]\) at \( T = T_{CP} \). The possible existence of such a branch point, which in QCD would correspond to a critical point belonging to the \( Z(2) \) universality class, is under intensive scrutiny in theoretical and experimental studies of nucleus-nucleus collisions \([21, 24, 26]\).

In lattice calculations it is found that the RW endpoint is a triple point for large and small quark masses \([27]\). In the latter case, the chiral transition at imaginary \( \mu_B \) is first order close to the RW endpoint, and may remain first order up to \( \mu_B = 0 \) and beyond, i.e., also for real values of \( \mu_B \). Recent lattice results suggest that for physical quark masses the RW endpoint is a second-order critical point belonging to the \( Z(2) \) universality class \([28]\). Thus, depending on the value of the quark mass, the chiral transition at imaginary \( \mu_B \), i.e., in the temperature range \( T_{pc} \leq T \leq T_{RW} \), can be first order, of the crossover type or partly first order, partly crossover with a critical point. In the following we explore the characteristics of the Fourier coefficients for these possibilities.

### B. Scaling theory

This discussion can be generalized to properly account for critical fluctuations using the scaling theory of phase transitions. The singular part of the free energy is expressed in terms of the universal scaling function \( f_f(z) \)

\[
f^{\text{sing}}(t, h) \sim h^{1+1/\delta} f_f(z), \tag{14}
\]

where \( z \equiv t'/h^{1/\beta} \) is the scaling variable.

The scaling function exhibits universal branch points in the complex \( z \) plane at \( z = z_{br} \) and \( z = z_{br}^* \). For the corresponding singularities in the complex \( \mu_B \) plane, we find, using the parametrization for \( t' \) introduced above,

\[
\mu_{br} = \pm T_c \left[ \frac{1}{\kappa}(z_{br} - z_0) \right]^{1/2} h^{1/2\delta}, \tag{15}
\]

where \( z_0 = (T/T_c - 1)/h^{1/\beta} \). The two remaining singular points are obtained by replacing \( z_{br} \) by \( z_{br}^* \) in \((15)\). The behavior of the singular parts of the free energy and the baryon density close to the branch point is obtained by analyzing \((14)\).

We note that singularities of the thermodynamic functions influence the asymptotic behavior of the corresponding Fourier series obtained at imaginary values of the chemical potential \([29, 30]\). By appropriately deforming the integration contour in the complex \( \mu_B \) plane, one can isolate the contribution of each individual singularity to the Fourier coefficients. In the following, we determine the asymptotic behavior of the Fourier series stemming from the singularities associated with phase transitions by using the universal properties of the free energy.

### III. ASYMPTOTICS OF FOURIER COEFFICIENTS

For the discussion in this section, the Riemann-Lebesgue (RL) lemma \([30]\) is of central importance. Applied to Eq. \((2)\), it states that the Fourier coefficients \( b_k \to 0 \) for \( k \to \infty \), provided the integral of the density \( \chi_1^B \) over the same interval exists. In the appendix, the Riemann-Lebesgue lemma is proven for the more restricted case of a differentiable function.

#### A. First order transition at imaginary \( \mu_B \)

Let us first consider the case when the density has a discontinuity at \( \theta_B = \pi \). This corresponds to the RW phase transition in QCD at high temperature, as indicated by case (a) in Fig. 1. At temperatures below \( T_{RW} \) the density vanishes at \( \theta_B = \pi \), while above \( T_{RW} \) it is non-zero, \( \text{Im} \chi_1^B(T > T_{RW}, \theta_B = i \pi) \neq 0 \). Since \( \chi_1^B(T, i \theta_B) \) is odd in \( \theta_B \) and periodic under \( \theta_B \to \theta_B + 2\pi \), this implies that the density must be discontinuous in this point. Using the periodicity and the symmetry of

\(^4\) As indicated in Fig. 2, \( \mu_{br}(T = T_{pc}) \) scales with \( h^{1+2\delta} \), which with mean-field exponents, \( \beta = 1/2 \) and \( \delta = 3 \), yields \( h^{1/3} \).
the integrand in (2), one finds after integration by parts,
\[ b_k = \frac{\alpha}{\pi} \int_0^\pi d\theta_B \left[ \text{Im} \chi^B_1(T, i\theta_B) \right] \sin(k \theta_B) \]
\[ = 2 \frac{(-1)^{k-1}}{\pi k} \chi^B_1(T, i\theta_B = i\pi) + \frac{2}{\pi k} \int_0^\pi dk \theta_B \left[ \text{Re} \chi^B_2(T, i\theta_B) \right] \cos(k \theta_B). \]  
(16)

The expression in parentheses in the last line vanishes for \( k \to \infty \) due to the RL lemma, discussed in the appendix. Hence, for \( T > T_{\text{RW}} \), the asymptotic behavior of the \( b_k \) is \( b_k \sim (-1)^{k-1}/k \), and the prefactor\(^5\) is determined by the density at \( \theta_B = \pi \).

The case of a first order transition at some intermediate point, \( 0 < \theta_c < \pi \), on the imaginary chemical potential axis can be handled analogously. As mentioned in Section II.A, this case may correspond to QCD for light or heavy quark masses, where the RW endpoint is expected to be a triple point \[27\]. Note, that in this case the density at \( \theta_B = \pi \) vanishes. On either side of the transition, the density is a smooth function given by \( f(\theta_B) \) and \( g(\theta_B) \), respectively. Thus,
\[ \text{Im} \chi^B_1(T, i\theta_B) = f(\theta_B) \Theta(\theta_c - \theta_B) + g(\theta_B) \Theta(\theta_B - \theta_c), \]  
(17)
where \( \Theta \) denotes the Heaviside step function. After integration by parts, the Fourier coefficients read
\[ b_k = \frac{\alpha}{\pi} \left( \int_0^{\theta_c} d\theta_B f(\theta_B) \sin(k \theta_B) \right) + \frac{\alpha}{\pi} \int_0^{\pi} d\theta_B g(\theta_B) \sin(k \theta_B) \]
\[ = \frac{2 \cos(k \theta_c)}{\pi k} (g(\theta_c) - f(\theta_c)) + \frac{2}{\pi k} \Delta b_k, \]  
(18)
where
\[ \Delta b_k = \int_0^{\theta_c} d\theta_B f(\theta_B) \cos(k \theta_B) \]
\[ + \int_0^{\pi} d\theta_B g(\theta_B) \cos(k \theta_B). \]  
(19)

For \( k \to \infty \), the term in Eq. (19) is subleading compared to the first one, owing to the RL lemma. Hence, in the case of a first order transition, the asymptotic form of the Fourier coefficients is \( \sim \cos(k \theta_c)/k \), with amplitude and frequency determined by the jump in density and the location of the transition, respectively.

At this point, two remarks are called for. First, \( b_k \sim k^{-(n+1)} \) is expected for a discontinuity in the \( n \)-th

dervative of the density \( [30] \). The phase of the oscillations depends on whether the discontinuity appears in an even or odd derivative. A relevant example is a second-order transition, which in the mean-field approximation is associated with a discontinuity in the derivative of the density at the critical point. Thus, such a transition at imaginary \( \mu_B \) yields high-order Fourier coefficients of the form \( [16] b_k \sim \sin(k \theta_c)/k^2 \).

Second, we note that the calculation of the Fourier series for the derivative of a discontinuous function must be done with caution. By differentiating, e.g., the Fourier expansion of the density \( (17) \) (for real values of the chemical potential), one finds
\[ \chi_2^B = \frac{\partial \chi_1^B}{\partial \mu_B} = \sum_{k=1}^{\infty} c_k \cosh(k \hat{\mu}_B), \]  
(20)
where the Fourier coefficients \( c_k = k b_k \) are non-zero in the limit\(^6\) \( k \to \infty \), since \( b_k \sim k^{-1} \). On the other hand, according to the RL lemma, the Fourier coefficients of a piecewise smooth function vanish asymptotically. This discrepancy is a consequence of the fact that the term-by-term derivative of the Fourier series of an odd function \( f(x) \) with period \( 2\pi \) reproduces that of its derivative \( f'(x) \) if and only if the original function is continuous \( [30] \).

B. Second order transition

We now consider the effect of a second-order phase transition. Here our goal is to extract the leading contribution to the Fourier coefficients \( b_k \), induced by the critical singularity.

In analogy to the mean-field treatment presented in Sect. II, we parameterize the second-order \( O(4) \) critical line near \( \mu_B = 0 \) in terms of the cruvature of the phase boundary \( \kappa \),
\[ T_c(\mu_B) = T_c - \kappa \hat{\mu}_B^2. \]  
(21)
Using (14) and the scaling relation \( \alpha = 2 - \beta(\delta + 1) \), one finds that, in the chiral limit \( (h \to 0) \), the singular parts of the free energy and the baryon density, on the imaginary \( \hat{\mu}_B \) axis, are given by
\[ f^{\text{sing}} \sim -|t - \kappa \theta_B^2|^{2-\alpha}, \]
\[ \text{Im} \chi^B_1 \sim |t - \kappa \theta_B^2|^{1-\alpha} \theta_B, \]  
(22)
where \( t \) is the reduced temperature introduced in Sect. II. Thus, at a given temperature \( T \geq T_c \), the critical point is located at \( \theta_c = \sqrt{T/\kappa} \). The singular contribution to

\(^5\) To be more precise, the prefactor is the left-sided limit of the density, \( \lim_{\theta_B \to \pi} \text{Im} \chi_1(T, i\theta_B) \).

\(^6\) For the \( n \)-th derivative of the density \( (n > 1) \), the corresponding Fourier coefficients diverge, \( c_k^{(n)} \sim k^{n-1} \).
the Fourier coefficients for \( T > T_c \) is obtained using (2) and (22),

\[
\begin{align*}
 b_k \sim & \ A^+ \int_0^{\theta_c} d\theta_B (\theta_B^2 - \theta_c^2)^{\phi} \theta_B \sin (k \theta_B) \\
 & + A^- \int_{\theta_c}^{\pi} d\theta_B (\theta_B^2 - \theta_c^2)^{\phi} \theta_B \sin (k \theta_B) \\
 & \sim \Gamma(1 + \phi) (\theta_c/k)^{1+\phi} \left[ A^+ \sin (k \theta_c - (1 + \phi) \pi/2) \\
 & + A^- \sin (k \theta_c + (1 + \phi) \pi/2) \right],
\end{align*}
\]

with the universal ratio \( R_\pm = A^+/A^- \). In the \( O(4) \) universality class [32], \( R_\pm \approx 1.85 \) and \( \alpha \approx -0.21 \).

At the \( O(4) \) critical point \( (T = T_c, \theta_c = 0) \), the singularity is weaker, the term in (24) vanishes and the asymptotic form of the Fourier coefficients is a pure power law

\[
b_k \sim k^{2\alpha - 4} = k^{-4.42}. \tag{25}
\]

The contribution of the Roberge-Weiss endpoint is evaluated analogously. At intermediate quark masses, this is a critical endpoint at \( \theta_B = \pi \), belonging to the \( Z(2) \) universality class of the Ising model in three dimensions [27, 28]. Hence, the asymptotics of the Fourier coefficients at \( T = T_{RW} \) is determined by the \( Z(2) \) critical exponents.

Since the RW transition line is a symmetry axis for reflections of the phase diagram (cf. Fig. 1) approaching the singularity at constant \( T \) corresponds to probing the system in the Ising h, i.e., external field, direction. Consequently, the singular part of the density is expected to be characterized by the critical exponent \( \delta \),

\[
r \sim (\pi - \theta_B)^{1/\delta}. \tag{26}
\]

In this case, the leading singular contribution to the Fourier coefficients is given by

\[
\begin{align*}
 b_k \sim & \int_0^{\pi} d\theta_B (\pi - \theta_B)^{1/\delta} \sin(k \theta_B) \\
 & \sim (-1)^{k-1} \frac{1}{k^{1+1/\delta}}.
\end{align*}
\]

Thus, at \( T = T_{RW} \), the power law decay of the Fourier coefficients is indeed determined by the critical exponent \( \delta \). We note that this result is obtained also by setting \( \theta_c = \pi \) and \( \phi = 1/\delta \) in (23). The fact that the RW endpoint is located at \( \theta_B = \pi \), gives rise to the alternating phase of the Fourier coefficients, as for the first-order RW transition in (16). The mean-field case, where \( \delta = 3 \), has been discussed in [16].

For \( T < T_c \), the branch point singularities are located on the real \( \mu_B \) axis [17], at \( \mu_B = \pm \mu_c \). The corresponding singular contribution to the Fourier coefficients \( b_k \) is given by

\[
b_k \sim \text{Im} \left[ \int_0^{\pi} d\theta_B (-t + \kappa \theta_B^2)^{\phi} \theta_B e^{i \kappa \theta_B} \right] \\
\sim \Gamma(1 + \phi) (\mu_c/k)^{1+\phi} \sin[(1 + \phi) \pi] e^{-k \mu_c}, \tag{28}
\]

where again \( \phi = 1 - \alpha \) and \( \mu_c = \mu_c/T = \sqrt{-t/\kappa} \). Thus, the asymptotic contribution of a critical point on the real \( \mu_B \) axis is a decreasing exponential superimposed on a power law\(^8\). The location of the critical point determines the range of the exponential, while the critical exponent is reflected in the power law. Thus, the corresponding contribution to the asymptotic form of \( b_k \), is in general, strongly suppressed.

The contribution of a possible critical endpoint of the \( Z(2) \) universality class [33] and a first-order transition at real values of the baryon chemical potential to the asymptotics of the Fourier coefficients would also be exponentially suppressed.

We note that for temperatures below the chiral transition at \( \mu_B = 0 \), this is the case also for the part of the Fourier coefficients emanating from the regular part of the net baryon density. The contribution of the thermal singularities, located at \( \mu_B = \pm \bar{\mu}_c \pm i \pi \), is of the form \( b_k \sim (-1)^{k-1} e^{-k m/\sqrt{N}/T} \), where \( \bar{\mu}_c \approx m_N/T \approx 6 \), since the nucleon is the lightest baryonic degree of freedom in this temperature range. Consequently, for \( T < T_c \), it is difficult to extract information on the singularity structure in the complex \( \mu_B \) plane from high-order Fourier coefficients.

C. Crossover

The leading contribution of the crossover singularities (15) in the scaling free energy (14), located at \( \mu_{bc} = T(\pm \bar{\mu}_c \pm i \theta_c) \), is obtained by a judicious choice

\[^7\text{We note that in the mean-field approximation } A^+ = 0 \text{ and } \phi = 1. \text{ Thus, in this case the singular contribution to the Fourier coefficients is } b_k \sim \sin(k \theta_c)/k^2.\]

\[^8\text{In the mean-field approximation } (\alpha = 0) \text{ the singular contribution to } b_k \text{ from a critical point on the real axis vanishes.}\]
of the integration contour in (2),
\begin{align*}
b_k &\sim e^{-k\hat{\mu}_c} \Gamma(1 + \phi) \left(\frac{\pi}{T}\right)^{\phi+1} \\
&\left[ A^+ \sin \left( k \theta_c - (\phi + 1) \left(\frac{\pi}{2} - \varphi_{br}\right) \right) \\
&+ A^- \sin \left( k \theta_c + (\phi + 1) \left(\frac{\pi}{2} + \varphi_{br}\right) \right) \right],
\end{align*}
where \( A^+ \) and \( A^- \) are the amplitudes of the singularity above and below the crossover branch cut and \( \phi \) is the corresponding exponent. Moreover, \( r = (\hat{\mu}_c^2 + \theta_c^2)^{1/2} \) and \( \varphi_{br} = \arctan(\hat{\mu}_c/\theta_c) \) is the phase of the crossover branchpoint in the first quadrant. Consequently, the asymptotic behavior of the Fourier coefficients \( b_k \) is that of a damped oscillator superimposed on a power-law dependence on \( k \).

For temperatures between the chiral and RW transitions, the imaginary part dominates and the oscillations prevail, while at low temperatures the real part dominates, and the Fourier coefficients are strongly damped with increasing \( k \). In the chiral limit, (29) reduces to (23) or (28), for \( t > 0 \) and \( t < 0 \) respectively. The corresponding mean-field behaviour in a QCD-like effective model was discussed in Ref. [16].

IV. CONCLUSION

In this paper, we have demonstrated that the asymptotic behavior of the Fourier coefficients of the density at imaginary chemical potential is, for temperatures above the chiral transition at \( \mu_B = 0 \), governed by the singularities associated with the phase transition or crossover in the chemical potential plane. The asymptotic behavior in the different regimes is summarized in Table I. Our results indicate that the singularity structure in the complex chemical potential plane is reflected in the asymptotic behavior of the \( b_k \). Indeed, a calculation, within a chiral effective model [16], demonstrates that in the chiral limit the location of the critical point, \( \theta_c \), can for \( T > T_c \) be extracted from a fit to \( b_k \) with the expected asymptotic form. Although the exponential damping of \( b_k \) leads to numerical difficulties when the singularity is located far from the imaginary axis, \( \mu_c \gg T \), a study of the dependence of the Fourier coefficients on the pion mass may provide information on the nature of the chiral phase transition, as discussed in Ref. [16].

In this work, the Fourier decomposition was applied to the net baryon density on the imaginary \( \mu_B \) axis, with applications to QCD matter in mind. Given that the results presented in this letter are model independent, we expect them to be generic and thus relevant also for other systems with phase transitions or transitions of the crossover type. However, as noted above, the critical contribution to the asymptotics of the Fourier coefficients may be eclipsed by other non-analyticities, located closer to the imaginary \( \mu_B \) axis.

In order to apply this approach to LQCD, it would be useful to extend our discussion to systems of finite volume, where the analytic structure in the complex chemical potential plane is different because the cuts and branch points are replaced by Yang-Lee zeros and edge singularities [34], respectively. In practice, the calculation of Fourier coefficients at large order in LQCD is certainly numerically very challenging. Nevertheless, since the location of the singularities depend on the volume as well as on the pion mass, a combined study of the dependence of high-order Fourier coefficients on the volume as well as on the pion mass may provide some insight into the phase structure of QCD matter.

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Appendix: Large \( k \) limit

In this appendix we derive a relation for the asymptotic form of the Fourier coefficients \( b_k \), which is a special case of the Riemann-Lebesgue lemma [30]. For an odd, differentiable function, \( f(x) \), the magnitude of the Fourier coefficients for large \( k \) decrease as \( \sim 1/k \) or faster. Integrating by parts, one finds
\begin{align*}
|b_k| &= \left| \int_0^\pi dx f(x) \sin(kx) \right| \\
&= \left| \frac{f(0) + (-1)^{k-1} f(\pi)}{k} + \frac{1}{k} \int_0^\pi dx f'(x) \cos(kx) \right| \\
&\leq \frac{1}{k} \left( |f(0)| + |f(\pi)| + \int_0^\pi dx |f'(x)| \right),
\end{align*}
where the expression in parantheses is finite, provided the integral of \( f' \) is absolutely convergent. An analogous result is obtained for the Fourier coefficients of even functions. In Sec. III, the asymptotic behavior of the Fourier coefficients is determined with the help of the RL lemma.
TABLE I. Summary of the asymptotic behavior of the Fourier coefficients of the net baryon density, $b_k$, for different classes of the transition. Here $\mu_c$ and $\theta_c$ are the real and imaginary parts of the reduced baryon chemical potential at the branch point singularity, $\mu_{br}/T$. Moreover, $T_{RW}$ is the Roberge-Weiss temperature, $h$ is the strength of the external symmetry-breaking field while $\alpha$ and $\delta$ are critical exponents.

| Temperature range | $T < T_c$ | $T = T_c$ | $T_c < T < T_{RW}$ | $T = T_{RW}$ | $T > T_{RW}$ |
|-------------------|------------|----------|---------------------|-------------|------------|
| $h = 0$ | $e^{-k\mu_c}/k^{2-\alpha}$ | $1/k^{4-2\alpha}$ | $\sin(k\theta_c - \alpha \pi/2) + R_+ \sin(k\theta_c + \alpha \pi/2)$ | $(-1)^{k+1}k^{1+1/\delta}$ | $(-1)^{k+1}k^{1+1/\delta}$ Im$\chi_B^0(\theta_B = \pi)$ |
| $h \neq 0$ | $e^{-k\mu_c}(\sin(k\theta_c - \alpha \pi/2) + R_\pm \sin(k\theta_c + \alpha \pi/2))$ | $k^{2-\alpha}$ | | | |

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