Vortices and supercurrent in AdS Born-Infeld gravity

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Abstract

In this paper, based on the fundamental principles of AdS/CFT duality and considering the probe limit, we study the effect of Born Infeld (BI) coupling both on the vortex lattice structure as well as on the expression for the supercurrent in holographic type II superconductors whose dual description consists of an abelian Higgs model coupled to gravity in a four dimensional $AdS_4$ space time. It is observed that at low temperatures the global $U(1)$ current thus computed for the boundary theory receives a highly nontrivial contribution appearing at the leading order in the BI coupling ($b$). In the light of long wave length approximation, we note that the so called non local version of the supercurrent boils down to a local form along with some finite low temperature correction to the usual Ginzburg Landau version of the supercurrent in type II superconductors. Finally, we also compute the free energy corresponding to the triangular lattice configuration. It is observed that the free energy exhibits similar low temperature effects at the leading order in the BI coupling ($b$).

1 Introduction

High $T_c$ superconductivity has been one of the most fascinating as well as evolving areas of research in the area of theoretical physics for the past couple of decades due to its several remarkable features which distinguish it from the ordinary superconducting materials. The most notable fact about these high $T_c$ materials is that we still do not have any satisfactory microscopic theory for them. These high $T_c$ materials are usually layered strongly coupled systems whose various phenomenological features could be best studied in the presence of an external magnetic field.

In order to classify superconductors in the presence of an external magnetic field one usually defines a dimensionless parameter $\kappa$ known as the Ginzburg Landau (GL) parameter which is defined as the ratio of the penetration strength ($\delta$) of the external magnetic field ($H$) to that of the Ginzburg Landau (GL) coherence length ($\xi$). With the help of this parameter one can in fact categorize superconductors into two distinct classes: $\kappa < \frac{1}{\sqrt{2}}$ usually describes what is called a type I superconductor. On the other hand for type II superconductors one finds $\kappa > \frac{1}{\sqrt{2}}$.

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For type I superconductors there exists a critical magnetic field \((H_c)\) that separates the normal phase from the superconducting phase and as the magnetic field falls below this critical value the material jumps into a superconducting phase via a first order phase transition \([1]-[3]\). For type II superconductors, on the other hand what one can find is a mixed state which could be viewed as phase that consists of many filaments embedded with quantized magnetic flux known as vortices. Therefore one could view type II superconductors as a lattice of interacting lines with certain line tension that finally results in a complex structure of interacting vortices. Each of the vortex lattice is found to be encircled with a vortex superconducting current. The objective of our present analyses would be to compute this current at strong coupling using some alternative prescription.

It is found that high \(T_c\) superconductors are of type II in nature which could be described by interacting lines of vortices as mentioned above. Vortex lattice in a high \(T_c\) superconductors is a complex system which due to thermal fluctuations could melt even at non zero temperatures forming what is known as a vortex liquid. The most significant feature of these high \(T_c\) superconductors is the formation of this vortex liquid phase which occupies a substantial region of the phase diagram curve for \(H_{c1} < H < H_{c2}\) where \(H_{c1}\) is the lower critical magnetic field and \(H_{c2}\) is the upper critical magnetic field. The movement of each filament on the vortex liquid is constrained by the fact that each vortex line must have to terminate on the surface of the superconducting material or have to form a closed loop by themselves. Melting of vortices near the lower value of the critical field strength \((H_{c1})\) is caused due to the weakening of the mutual interaction between different vortices. As one lowers the magnetic field below \(H_{c1}\), the distance between vortices increases that eventually becomes greater than the penetration depth \((\delta)\). This ultimately causes an exponential decay of the vortex-vortex interaction and as a result the vortex state destroys \([4]-[8]\).

For ordinary type II superconductors the thermodynamically most favorable configuration is characterized by a triangular lattice system that has two lattice parameters. In addition to this there exists a superconducting current that flows along the line of the uniform condensate encircling the core of the vortex that has been already mentioned above. In the usual frame work of the Ginzburg Landau (GL) theory one can in fact express this current as a local function of the vortex solution that seems to be a very specific property of type II superconductors. On the other hand, in the presence of small length scales such as the Pippard/BCS coherence length one might have certain non locality in the form of the supercurrent \([9]-[10]\).

For the past several years the AdS/CFT correspondence \([11]-[12]\) has been found to play a very significant role in order to understand several crucial properties of type II superconductors that are believed to be strongly coupled. The dual gravitational description of such superconductors/super fluids essentially consists of an abelian Higgs model coupled to gravity in an anti de-Sitter (AdS) space \([13]-[18]\). The behavior of these so called holographic superconductors in the presence of an external magnetic field have been investigated under various circumstances \([19]-[27]\), for example the holographic computation of the London equation as well as the penetration depth was first carried out in \([19]\) where the authors have shown that holographic superconductors are basically of type II in nature. The vortex lattice structure for \(s\)- wave superconductors was first studied holographically by Maeda et al in \([20]\) where the authors have shown that for \(T < T_c\) the triangular vortex configuration is the thermodynamically most favorable one. In spite of these analyses some important issues are still left behind which we would like to peruse.
in the present paper.

The motivation of the present work is therefore the following: In most of the above mentioned analyses, people have considered a dual gravitational theory that includes a complex scalar field minimally coupled to the usual Maxwell sector. It would be therefore a natural question to ask what happens to the supercurrent constructed at the boundary of the AdS if one replaces the usual Maxwell action by some non linear higher derivative terms of the $U(1)$ gauge field. What happens to the vortex structure in the presence of such higher derivative terms. It may so happen that either of them or both of them are affected due to the presence of such non linearity in the dual description.

Most importantly it would be quite interesting to explore whether under such a circumstance one can still have a local expression for the supercurrent circulating around the core. Also it is highly plausible that such non linear correction may eventually change the local form of the usual GL current that has been known to us for a long time.

In the present paper replacing the Maxwell action by the Born Infeld (BI) action\(^1\) we explore all the above mentioned issues for holographic type II superconductors in the probe limit. In order to see the effect of nonzero condensate on the boundary current, we first solve the dynamics of the scalar field as well as the gauge field in the bulk AdS space time considering the fluctuations at the leading order. Using these solutions we finally compute the boundary current which at the first level appears to be non local in the vortex solution. In order to bring it to a local form we use so called long wave length approximation that eventually leads us to the local expression of the GL form for the supercurrent along with certain nontrivial modification term appearing at the leading order in the BI coupling ($b$). Such a nontrivial contribution to the supercurrent appears at cubic order in the spatial derivatives of the condensate which we interpret as a finite temperature effect to the usual GL current that is highly suppressed at large values of the temperature and therefore could be realized in practice only at some finite non zero temperatures. Finally we compute the free energy of the system which appears to be negative for the vortex configuration and thereby indicating the triangular lattice configuration to be the thermodynamically most favorable solution. Furthermore we note that in the case for supercurrent the free energy also receives some non trivial finite temperature correction that is suppressed at high temperatures.

The organization of the paper is the following: In Section 2 we give the details of the dynamics of the scalar as well as the gauge fields considering them as a probe on the background $AdS_4$ space time. Based on the AdS/CFT prescription, computation of the boundary current up to leading order in the BI coupling ($b$) has been performed in

\(^1\)The nonlinear version of Maxwell electrodynamics was originally constructed in [28] many decades before the proposal of string theory. Later on in the context of string theory the origin of such a term was realized during the study of the low energy effective dynamics of $Dp$ branes where the vanishing of the beta function at one loop eventually leads us to something known as the Dirac Born Infeld (DBI) action in the space time [29][31]. For small field strengths the dynamics of gauge fields on $Dp$ branes is eventually captured in the Maxwell’s equations where as on the other hand if one increases the electric as well as the magnetic field strengths then one needs to incorporate the nonlinear corrections to the Maxwell sector in the form of Born Infeld correction. From the AdS/CFT perspective Born Infeld theories have received lots of attention under various occasions, for example in [32] the authors have computed the shear viscosity ratio for a five dimensional $AdS$ black brane solution in presence of BI corrections to the usual Maxwell action. Furthermore in [33] by performing the derivative expansion, the author has studied the hydrodynamics of the dual field theory to all order in the BI parameter. In spite of all these analyses, the holographic derivation of the $U(1)$ supercurrent in presence of BI term has been missing in the literature.
Section 3. In Section 4 we holographically compute the free energy corresponding to the triangular vortex solution up to leading order in the BI coupling ($b$). Finally we conclude in Section 5.

2 Theory in the bulk $AdS_4$

The present analyses of this paper is based on several crucial assumptions and facts. We would like to specify all of them in this section. Based on these assumptions, the goal of this section would be to provide a description regarding the dynamics of the scalar as well as the gauge fields in the bulk $AdS_4$. We start our analysis with the action which is basically the abelian Higgs model coupled to gravity in a four dimensional $AdS_4$ space-time,

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left[ R - 2\Lambda + \frac{1}{b} \left( 1 - \sqrt{1 + \frac{bf}{2}} \right) - |\nabla_\mu \Psi - iA_\mu \Psi|^2 - m^2 |\Psi|^2 \right].$$

The above action is the non linear generalization of the abelian Higgs model proposed by Gubser originally in [14], where the usual Maxwell term has been replaced by the Born Infeld (BI) term, where $b$ is the BI coupling parameter and $F = F^{\mu\nu}F_{\mu\nu}$. It could be easily checked that in the limit $b \rightarrow 0$ one recovers the usual Maxwell action.

In our analyses we consider both the scalar as well as the gauge fields in the probe limit that could be realized by taking the large charge limit ($q \rightarrow \infty$) while keeping $A_\mu$ and $\psi$ fixed. This essentially means that we ignore the back reaction of the matter fields on the background space time whereas on the other hand, the non trivial interactions between the scalar field and the gauge fields are retained. In our analyses we consider the effect of the BI term (which is noting but the higher derivative corrections of the gauge fields to the usual Maxwell action) perturbatively in the BI parameter ($b$) and the entire analyses has been performed considering the effects only in the leading order in $b$.

The background over which the analyses is performed is an asymptotically $AdS_4$ black brane solution,

$$ds^2 = -f(u)dt^2 + \frac{r^2}{u^4}f^{-1}(u)du^2 + \frac{r^2}{u^2} dx^2$$

where,

$$f(u) = \frac{r^2}{u^2}(1 - u^3).$$

Note that in these coordinates the horizon is located at $u = 1$ whereas on the other hand, the boundary of the AdS is located at $u = 0$. The temperature of the black brane is given by,

$$T = \frac{3r_+}{4\pi}$$

which is considered to be fixed for the present analyses. Therefore the boundary field theory could also be considered to be at the same temperature as that of the black brane.

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2The quantity $\Lambda(=-3/l^2)$ is the cosmological constant. In the present analyses we set $l = 1$. 

4
Following the AdS/CFT correspondence one can identify the $U(1)$ gauge field ($A_\mu$) in the bulk acting as a source to some global $U(1)$ current ($J^\mu$) in the boundary theory. Also the scalar field ($\Psi$) could be considered as a source to some dual (condensation) operator ($O$) of the boundary theory. Apart from these facts we also use the following holographic dictionary in order to carry out the analyses further.

- In order to have a finite norm of the $U(1)$ gauge field ($A_\mu$) near the horizon, the spatial component $A_i(x, u = 1)$ has to be regular and on the other hand, the temporal part should vanish ($A_t(x, u = 1) = 0$) at the horizon.
- The boundary value of $A_t(x, u = 0)$ gives the chemical potential ($\mu$) of the boundary field theory.
- The asymptotic value of $F_{xy}(x, u = 0)$ gives rise to some non zero magnetic field ($H$) at the boundary.

In the present analyses we keep the chemical potential ($\mu$) of the boundary theory to be fixed while changing the magnetic field. There exists an upper critical magnetic field ($H = H_{c2}$) above which the condensation vanishes.

In order to proceed further we first define a parameter $\varepsilon = \frac{H_{c2} - H}{H_{c2}}$, where $H_{c2}$ is the (upper) critical magnetic field strength above which the charge condensate ($\Psi$) vanishes. The parameter $\varepsilon$ could be called the deviation parameter as it essentially takes us a little away from the critical point ($H = H_{c2}$). We expand both the gauge filed as well as the scalar field in this parameter,

$$A_\mu = A_\mu^{(0)} + \varepsilon A_\mu^{(1)}(u, x) + \mathcal{O}(\varepsilon^2)$$

$$\Psi = \varepsilon^{1/2}\psi_1(u, x) + \mathcal{O}(\varepsilon^{3/2}).$$

From the above expansion the following two things could be noted,

- The above expansion is valid iff the parameter $|\varepsilon| \ll 1$ which means we are performing our analyses very close to the critical point.
- The positivity of the parameter ($\varepsilon > 0$) implies that the applied magnetic field ($H$) is always less than $H_{c2}$ and therefore in our analyses we shall always have the presence of some non trivial charge condensate that acts as an order parameter for our theory.

Note that here $A_\mu^{(0)}$ is the solution of the Maxwell’s equation when the charge condensate ($\Psi$) is zero. In the present analyses we choose the following ansatz for $A_\mu^{(0)}$,

$$A_\mu^{(0)} = (A_t^{(0)}(u), 0, 0, A_y^{(0)}(x))$$

where the spatial component of the gauge field acts as a source to some non zero magnetic field ($H_{c2}$) at the boundary.

Furthermore here $A_\mu^{(i)} (i = 1, 2, ..)$ s are the fluctuations of the $U(1)$ gauge field in the presence of the non zero charge condensate. The quantity $\psi_1$ denotes the first non trivial fluctuation in the charge condensate. Our next goal would be to provide a detail of the dynamics of both the scalar field as well as the gauge field considering only the leading order terms in $\varepsilon$. 

5
2.1 Dynamics of scalar field: vortex structure

In this part of our analysis, considering the probe limit we would like to explore the effect of BI correction on the dynamics of the scalar field. We start our analysis considering the following ansatz for the scalar field namely,

$$\Psi = \Psi(u, x, y).$$ (8)

Using the above ansatz (8) the equation for the scalar field turns out to be,

$$\nabla^\mu \nabla_\mu \Psi - i A^\mu \nabla_\mu \Psi - i \nabla_\mu (A^\mu \Psi) - A^2 \Psi - m^2 \Psi = 0. \tag{9}$$

Considering the leading order fluctuations (6) in $\Psi$ the above equation turns out to be,

$$\partial^2 u \psi_1 + \frac{f'(u)}{f(u)} \partial_u \psi_1 + \frac{r^2 A_i^{(0)2}}{u^4 f^2(u)} \psi_1 - \frac{m^2 r^2}{u^4 f(u)} \psi_1 + \frac{1}{u^2 f(u)} (\Delta - 2i H_{c2} x \partial_y - H_{c2}^2 x^2) \psi_1 = 0. \tag{10}$$

Our next goal would be to solve (10) using the separation of variable. In order to do so we choose the following ansatz

$$\psi_1(u, x) = \rho_1(u) e^{i k_y y} X(x) = \rho_1(u) \mathcal{V}(x, y). \tag{11}$$

Substituting (11) into (10) we are essentially left with the following two sets of equations,

$$\partial^2 \rho_1 + \frac{f'(u)}{f(u)} \partial_u \rho_1 + \frac{r^2 A_i^{(0)2}}{u^4 f^2(u)} \rho_1(u) - \frac{r^2 A_i^{(0)2}}{u^4 f(u)} \rho_1(u) = \frac{\rho_1(u)}{\xi^2 u^2 f(u)} \tag{12}$$

and,

$$- X''(x) + H_{c2}^2 \left( x - \frac{k_y}{H_{c2}} \right)^2 = \frac{X(x)}{\xi^2}. \tag{13}$$

Eq (13) looks exactly similar to that of the equation of motion of an one dimensional quantum harmonic oscillator in the presence of a uniform background magnetic field ($H_{c2}$). One may further note that this is the equation of motion of a harmonic oscillator with the shifted minimum whose minima has been shifted by an amount $x_0(= \frac{k_y}{H_{c2}})$ from its original equilibrium position.

With the identification of $\xi \left( \sim (T_c - T)^{-1/2} \right)$ as the correlation length that diverges near the critical point of the phase transition, Eq (13) remarkably agrees to that with the vortex lattice equation for ordinary type II superconductors obtained from the usual Ginzburg-Landau theory in the presence of an external magnetic field [10]. Considering the lowest energy ($n = 0$) solution of (13), the upper critical magnetic field could be expressed as

$$H_{c2} = \frac{1}{\xi^2} \sim \left( 1 - \frac{T}{T_c} \right) \tag{14}$$

3From Eq (10) it could be easily noticed that the particle experiences a non trivial force along the $x$ direction while on the other hand, no such constraints exists for the motion along the $y$ direction. This means that the particle behaves as a free particle along $y$ direction.

4In type II superconductors the magnetic field starts to penetrate the sample below certain lower critical magnetic field $H = H_{c1}$ and the sample could exists in a mixed phase for $H_{c1} < H < H_{c2}$. Finally as the magnetic field is increased further the sample starts to lose its superconducting property and at $H = H_{c2}$ the sample goes to a normal metallic phase via a second order phase transition [10].
which is the critical value of the magnetic field above which the order parameter vanishes. Note that this is something that we already know from the usual Ginzburg Landau (GL) theory for type II superconductors \[10\].

Following the original arguments of Abrikosov, we assume that there exist a periodic lattice which is caused due to the penetration of parallel magnetic flux lines known as vortex beams. We assume periodicity both in the $x$ and the $y$ directions. In order to do that we define two arbitrary parameters $a_x$ and $a_y$. The periodicity in $y$ direction could be expressed as,

$$
k_y = \frac{2\pi l}{a_y}, \quad l \in Z \tag{15}
$$

where the coefficient $a_y$ stands for the periodicity along $y$ direction.

The most general lowest energy ($n = 0$) solution could be expressed as a linear superposition of a set of (eigen) functions corresponding to different values of $l$,

$$
V(x, y) = \sum_{l=-\infty}^{\infty} c_l \exp \left( \frac{-i\pi a_x \xi^2 l^2}{a_y^2} \right) \cdot \exp \left( -\frac{1}{2\xi^2} \left( x - \frac{2\pi l \xi^2}{a_y} \right)^2 \right) \tag{16}
$$

where the coefficients $c_l$ could be expressed as,

$$
c_l = \exp \left( -\frac{i\pi a_x \xi^2 l^2}{a_y^2} \right) \cdot \tag{17}
$$

Finally, using elliptic theta function, the above solution \[11\] could be rewritten as,

$$
\psi_1(u, x) = \rho_1(u) e^{\frac{-x^2}{2\xi^2}} \vartheta_3(v, \tau) \tag{18}
$$

where the elliptic theta function could be formally expressed as,

$$
\vartheta_3(v, \tau) = \sum_{l=-\infty}^{\infty} q^l z^{2l} \tag{19}
$$

with,

$$
q = \exp \left( i\pi \xi^2 \frac{2\pi i - a_x}{a_y} \right) \tag{20}
$$

$$
z = \exp \left( i\pi \frac{y - ix}{a_y} \right).
$$

In this context one may define a two dimensional function $\sigma(x) (= |e^{-\frac{x^2}{2\xi^2}} \vartheta_3(v, \tau)|^2)$ that essentially represents a (triangular) vortex lattice whose fundamental region could be spanned by the two lattice vectors $e_1 = a_y \partial_y$ and $e_2 = \frac{2\pi \xi^2}{a_y} \partial_x + \frac{a_x \xi^2}{a_y} \partial_y$. We are now in a position to make the following two comments.

- The natural size of the vortex is essentially determined by the square of the correlation length ($\xi$). As it can be easily seen from \[18\], that the solution has Gaussian behavior along the $x$ direction which means that the vortex structure eventually dies out for $|x| \gg \xi$. Thus the correlation length ($\xi$) acts as a natural length scale in order to determine the size of a single vortex lattice.
The BI term has no effect on the triangular vortex solution. It could have its effect only through the radial part of the equation (12).

Before we conclude our discussion on scalar field, it is now customary to mention about the boundary behavior of the radial function $\rho_1(u)$. Near the AdS boundary the radial function has different fall offs for different choices of the mass ($m^2$) of the scalar field. Considering $m^2 = -2$, which is well above the BF bound for the present case, one may note that,

$$\rho_1(u)|_{u \to 0} = c_1u + c_2u^2. \quad (21)$$

Since here both the modes are normalisable, therefore we may set either of the coefficients to be zero. For the present analyses we set $c_1 = 0$ and therefore $c_2$ will play the role of the dual operator ($\mathcal{O}$) in boundary theory.

### 2.2 Dynamics of gauge fields

In this section we explore the dynamics of the abelian gauge field in the presence of zeroth as well as the first order fluctuations in the charge condensation. Our aim would be to solve the equations perturbatively both for the scalar field fluctuations as well as for the BI coupling ($b$). Our solutions could be considered to be valid only upto leading order in the BI coupling ($b$). To start with we note that the Maxwells equation turns out to be,

$$\nabla_\mu \left( \frac{F^\nu_\mu}{\sqrt{1 + bF^2}} \right) = j^\nu \quad (22)$$

where,

$$j^\nu = i(\Psi(D^\nu\Psi)^\dagger - \Psi^\dagger D^\nu\Psi) \quad (23)$$

with $D_\mu = \partial_\mu - iA_\mu$ as the gauge covariant derivative.

Since we are interested in solving the equations only upto leading order in the BI coupling ($b$), therefore we expand the l.h.s. of the above equation (22) upto leading order in $b$ which finally yields,

$$\nabla_\mu F^{\nu\mu} - \frac{b}{4}F^{\nu\mu}\partial_\mu F = j^\nu \left( 1 + \frac{bF}{4} \right). \quad (24)$$

Considering the perturbative expansions provided in (5) and (6), we first expand both the l.h.s. as well as the r.h.s. of (24) perturbatively in the parameter $\varepsilon$ and separate the gauge field equations both for the zeroth order as well as for the first order fluctuations in the charged condensation which could be expressed as,

$$\nabla_\mu F^{\nu\mu(0)} - \frac{b}{4}F^{\nu\mu(0)}\partial_\mu (F^{\lambda\sigma(0)}F_{\lambda\sigma}^{(0)}) = 0 \quad (25)$$

$$\nabla_\mu F^{\nu\mu(1)} - \frac{b}{4}[2F^{\nu\mu(0)}\partial_\mu (F^{\lambda\sigma(0)}F_{\lambda\sigma}^{(1)}) + F^{\nu\mu(1)}\partial_\mu (F^{\lambda\sigma(0)}F_{\lambda\sigma}^{(0)})] = \left( 1 + \frac{b}{4}F^{\lambda\sigma(0)}F_{\lambda\sigma}^{(0)} \right) j^{\nu(1)}. \quad (26)$$
Note that the r.h.s. of (26) is arising solely due to the presence of a non trivial scalar hair that essentially contributes to the fluctuations of the $U(1)$ the gauge field in the bulk. At this level one can also note that due to the presence of the non linear interactions between the gauge field and the scalar field it would be quite difficult to solve the above set of equations (25) and (26) for all order in the BI coupling ($b$). Therefore in order to proceed further we need to solve the above set of equations perturbatively in the BI coupling ($b$). This further demands that one needs to expand the $U(1)$ gauge field perturbatively in the BI coupling ($b$),

$$A_{\mu}^{(m)} = A^{(m)(0)} + bA^{(m)(1)} + \mathcal{O}(b^2).$$

In the above expansion (27) we have used two different indices in order to incorporate the effect of fluctuations of two different kinds. The index ($m$) correspond to the fluctuation in the order parameter ($\Psi$). In other words, various terms corresponding to different values of ($m$) stands for different terms in the perturbative $\varepsilon$ expansion of (5). On the other hand, the indices $b^n (n = 0, 1, 2, \ldots)$ stand for perturbations at different levels in the BI coupling ($b$). Using the expansion (27) one can further express the field strength tensor ($F^{\mu\nu}$) as,

$$F^{\mu\nu(m)} = F^{\mu\nu(m)(0)} + bF^{\mu\nu(m)(1)} + \mathcal{O}(b^2).$$

With the above machinery in hand, we are now in a position to solve the equations (25) and (26) perturbatively in the BI parameter ($b$). Let us first consider (25). Using (28) one can in fact show that it leads to the following set of equations,

$$\nabla_\mu F^{\mu\nu(0)}(b^{(0)}) = 0$$

$$\nabla_\mu F^{\nu\mu(0)}(b^{(1)}) - \frac{1}{4} F^{\nu\mu(0)(b^{(0)})} \partial_\mu (F_{\lambda\sigma(0)}(b^{(0)}) F^{\lambda\sigma(0)}(b^{(0)})) = 0$$

whose solutions could be expressed as,

$$A_{t}^{(0)}(u) = \mu(1-u) \left[ 1 - \frac{b}{10r_+^2} (\mu r_+^2 - H_{c2}^2) \right] + \mathcal{O}(b^2)$$

$$A_{y}^{(0)}(x) = H_{c2}x$$

where, $\zeta(u) = u(1 + u + u^2 + u^3)$.

As a next step, our job is to solve (26) which governs the dynamics of the gauge fluctuations at the first order level. In order to do that we first split Eq (26) into different components which could be enumerated as follows,

$$L_t A_{t}^{(1)} - \frac{b u^2 f(u)}{4} \left[ 2 \partial_u A_{t}^{(0)} \partial_u (F^{\lambda\sigma(0)}(b^{(1)}) F_{\lambda\sigma(1)}) + \partial_u A_{t}^{(1)} \partial_u F^{(0)} \right] = \frac{2r_+^2 A_{t}^{(0)}}{u^2} \left| \psi_1 \right|^2 \left( 1 + \frac{b}{4} F^{(0)} \right)$$

$$L_x A_{x}^{(1)} + \frac{b}{4} \left( 2 H_{c2} \partial_y (F^{\lambda\sigma(0)}(b^{(1)}) F_{\lambda\sigma(1)}) + u^2 f(u) F_{xu}^{(1)} \partial_u F^{(0)} \right) = -\frac{r_+^2}{u^2} \left( 1 + \frac{b}{4} F^{(0)} \right) j_{x}^{(1)}$$

$$L_y A_{y}^{(1)} - \frac{b}{4} \left( 2 H_{c2} \partial_x (F^{\lambda\sigma(0)}(b^{(1)}) - u^2 f(u) F_{yu}^{(1)} \partial_u F^{(0)} \right) = -\frac{r_+^2}{u^2} \left( 1 + \frac{b}{4} F^{(0)} \right) j_{y}^{(1)}$$
where, \( L_t = u^2 f(u) \partial_u^2 + \Delta \) and \( L_s = \partial_u(u^2 f(u) \partial_u) + \Delta \) are the differential operators and \( F^{(0)} = F^{\lambda \sigma}(0) \). In order to arrive at the above set of equations we choose a particular gauge \( A_u = 0 \) and also exploit the residual gauge symmetry \( A_i^{(1)} \rightarrow A_i^{(1)} - \partial_i \varpi(x) \) to fix the gauge \( \partial_x A_x^{(1)} + \partial_y A_y^{(1)} = 0 \).

We are now in a position to solve these equations perturbatively in the BI parameter \((b)\). As a first step one may note that the solution of the radial equation \((12)\) could be expressed as a perturbation in the BI coupling \((b)\) as,

\[
\rho_1(u) = \rho_1^{(b(0))} + b \rho_1^{(b(1))} + \mathcal{O}(b^2). \tag{36}
\]

With the above prescriptions \((28)\) and \((36)\) in hand we are now in a position to solve the above set of equations \((33)-(35)\) order by order as a perturbation in the BI coupling \((b)\). Let us first note down the equations at the at the zeroth order level which turns out to be,

\[
\begin{align*}
L_t A_i^{(1)(b(0))} &= \frac{2r_+^2 + \rho_1^{2(b(0))}}{u^2} A_i^{(0)(b(0))} \sigma(x) \tag{37} \\
L_s A_x^{(1)(b(0))} &= \frac{r_+^2 \rho_1^{2(b(0))}}{u^2} \epsilon_{xz} \sigma(x) \tag{38} \\
L_s A_y^{(1)(b(0))} &= \frac{r_+^2 \rho_1^{2(b(0))}}{u^2} \epsilon_{yz} \sigma(x) \tag{39}
\end{align*}
\]

where, \( \sigma(x) = (= |e^{-\frac{x^2}{4u}} \partial_3 (v, \tau)|^2) \) corresponds to the triangular vortex solution in the \((x, y)\) plane and \( j_i^{(1)} = \rho_1^{2} \epsilon_{ij} \partial_j \sigma(x) \) is the current due to fluctuations in the scalar field\(^5\).

Note that here \( \epsilon_{ij} \) is an anti symmetric tensor with \( \epsilon_{xy} = -\epsilon_{yx} = 1 \).

At this stage it is quite evident that the above set of equations \((37)-(39)\) essentially correspond to a set of inhomogeneous differential equations with a source term on the r.h.s. of it. Therefore in general the solutions to these equations could be expressed in terms of Green’s functions that satisfy suitable boundary conditions near the boundary of the AdS. The solutions could be expressed as,

\[
\begin{align*}
A_i^{(1)(b(0))} &= -2r_+^2 \int_0^1 du' \rho_1^{2(b(0))} \frac{(u')}{u^2} A_i^{(0)(b(0))}(u') \int d\mathbf{x}' G_i(u, u'; \mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') \\
A_i^{(1)(b(0))} &= a_i(x) - 2r_+^2 \epsilon_{ij} \int_0^1 du' \rho_1^{2(b(0))} \frac{(u')}{u^2} \int d\mathbf{x}' G_s(u, u'; \mathbf{x}, \mathbf{x}') \partial_j \sigma(\mathbf{x}'). \tag{40}
\end{align*}
\]

Here \( a_i(x) \) is the homogeneous part of the solution of \((38)-(39)\) which is the only term that contributes to a uniform magnetic field \((H_{c2} = \epsilon_{ij} \partial_i a_j)\) at the boundary of the AdS. On the other hand, \( G_i(u, u'; \mathbf{x}, \mathbf{x}') \) and \( G_s(u, u'; \mathbf{x}, \mathbf{x}') \) are the Green’s functions corresponding to the above set of equations \((37)-(39)\) that obey the following differential equations,

\[
\begin{align*}
L_t G_i(u, u'; \mathbf{x}, \mathbf{x}') &= -\delta(u - u') \delta(\mathbf{x} - \mathbf{x}') \\
L_s G_s(u, u'; \mathbf{x}, \mathbf{x}') &= -\delta(u - u') \delta(\mathbf{x} - \mathbf{x}') \tag{41}
\end{align*}
\]

\(^5\)Here \( i (= x, y) \) denotes the spatial coordinates.
along with the following (Dirichlet) boundary conditions near the boundary of the AdS namely,

\[ G_t(u, u'; x, x')|_{u=0} = G_t(u, u'; x, x')|_{u=1} = 0 \]
\[ G_s(u, u'; x, x')|_{u=0} = u^2 f(u) \partial_u G_s(u, u'; x, x')|_{u=1} = 0. \]  
(42)

The above boundary conditions eventually clarifies the following two things. Firstly, we are working with a fixed chemical potential \( \mu \) at the boundary which is reflected in the fact that any non trivial fluctuation of \( A_t \) eventually vanishes near the boundary of the AdS. Secondly, we have a uniform magnetic field \( H_{c2} \) at the boundary since any correction appearing to the spatial component of the gauge field due to the presence of the non trivial fluctuations in the scalar profile eventually dies out at the boundary.

Next we almost follow the same procedure in order to solve the equations (33)-(35) for the leading order in the BI coupling \( (b) \). Let us first note down the equations corresponding to leading order in the \( b \) which turns out to be,

\[
L_t A_t^{(1)(b(1))} - \frac{u^2 f(u)}{4} \left[ 2 \partial_u A_t^{(b(0))} \partial_u (F^{\lambda \sigma (b)(0)} F_{\lambda \sigma}^{(1)(b(0))}) + \partial_u A_t^{(1)(b(0))} \partial_u F^{(b)(0)} \right] = \frac{2r_+^2}{u^2} J(u) \sigma(x)
\]
\[
L_x A_x^{(1)(b(1))} + \frac{1}{4} \left[ 2H_{c2} \partial_y (F^{\lambda \sigma (b)(0)} F_{\lambda \sigma}^{(1)(b(0))}) + u^2 f(u) F_{xu}^{(1)(b(0))} \partial_u F^{(b)(0)} \right] = -\frac{r_+^2}{u^2} I(u) \epsilon^y \partial_y \sigma
\]
\[
L_y A_y^{(1)(b(1))} - \frac{1}{4} \left[ 2H_{c2} \partial_x (F^{\lambda \sigma (b)(0)} F_{\lambda \sigma}^{(1)(b(0))}) - u^2 f(u) F_{yu}^{(1)(b(0))} \partial_u F^{(b)(0)} \right] = -\frac{r_+^2}{u^2} I(u) \epsilon^x \partial_x \sigma
\]  
(43)

where \( J(u) \) and \( I(u) \) are some radial functions which could be expressed as,

\[ J(u) = \frac{1}{4} A_t^{(0)(b(0))} F^{(0)(b(0))} \rho_1^{2(b(0))} + A_t^{(0)(b(1))} \rho_1^{2(b(0))} + 2 \rho_1^{(b(1))} \rho_1^{(b(0))} A_t^{(0)(b(0))} \]
\[ I(u) = \frac{\rho_1^{2(b(0))}}{4} F^{(0)(b(0))} + 2 \rho_1^{(b(0))} \rho_1^{(b(1))}. \]  
(44)

These are again set of inhomogeneous differential equations whose solutions could be expressed in terms of Green’s functions namely,

\[ A_t^{(1)(b(1))} = -\int_0^1 du' \int dx' \mathcal{P}_t(u', x') \mathcal{G}_t(u, u'; x, x') \]
\[ A_i^{(1)(b(1))} = -\int_0^1 du' \int dx' \mathcal{Q}_i(u', x') \mathcal{G}_s(u, u'; x, x') \]  
(45)

where \( \mathcal{P}_i(u, x) \) and \( \mathcal{Q}_i(u, x) \) are some nontrivial functions that act as a source for the above set of equations (43). These functions essentially carry the information of how the triangular vortex that is arising below certain critical temperature \( (T < T_c) \) could in principle be responsible for giving rise to some non trivial fluctuations in the gauge fields in the presence of non linear (BI) corrections to the usual Maxwell sector. The explicit form of these functions could be expressed as,

\[ \mathcal{P}_i = \frac{2r_+^2}{u^2} J(u) \sigma(x) + \frac{u^2 f(u)}{4} \left[ 2 \partial_u A_t^{(0)(b(0))} \partial_u (F^{\lambda \sigma (0)(b(0))} F_{\lambda \sigma}^{(1)(b(0))}) + \partial_u A_t^{(1)(b(0))} \partial_u F^{(b)(0)} \right] \]
\[ \mathcal{Q}_i = -\epsilon^i_\jmath \partial_\jmath \mathcal{R}(u, x) \]  
(46)
with,
\[
\Re(u, x) = \frac{r^2}{u^2} I(u) \sigma(x) + \frac{Hr^2}{2} F^{\lambda(0)}(\beta^{(0)}) F^{(1)}(\beta^{(0)})
+ \frac{r^2 u^2 f(u)}{4} \partial_u F^{(0)}(\beta^{(0)}) \int_0^1 du' \rho^2(u') \frac{2(\beta^{(0)})}{u'^2} \partial_u \int d^2 x' G_s(u, u'^{\prime}; x, x'^{\prime}) \sigma(x').
\]

(47)

With the above solutions in hand and using the AdS/CFT prescription we are now in a position to compute the Global \((U(1))\) current \((J_\mu)\) for the boundary theory. Our goal would be to express this current as a *local* function of the vortex solution \(\sigma(x)\).

### 3 Holographic Ginzburg-Landau current

In this section, based on the gauge/gravity duality we would like to compute the global \(U(1)\) current \((J_\mu)\) corresponding to the boundary theory of the four dimensional anti-de Sitter (AdS\(_4\)) space time. The goal of the present analyses is twofold. Firstly, we would like to explore the effect of non trivial fluctuations of the scalar hair on the boundary current itself. To see this effect we need to compute the current corresponding to the first order in the gauge fluctuations. This would eventually enable us to write down the current as a nontrivial function of the vortex solution \(\sigma(x)\). Secondly, and most importantly, we would like to see whether we can indeed express this current as a *local* function of the vortex solution in the presence of the BI correction to the usual Maxwell action. Furthermore it would be also interesting to explore whether the BI corrections can at all affect the local form of the current at least in the leading order in the BI coupling \((b)\). In this sense it would be a *holographic* derivation of the familiar Ginzburg-Landau (GL) current in presence of some non linear corrections to the abelian Higgs model originally proposed in \([14]\). Since the boundary theory is living on a \((2 + 1)\) dimensional flat (Minkowski) back ground, therefore from this analyses one might infer that whether there could be any non trivial modification of the usual GL current that might appear for the real life superconducting materials at some finite temperature \((T < T_c)\).

From the AdS/CFT dictionary, it is well known that the \(U(1)\) gauge field in the bulk acts as the source to some global \(U(1)\) current defined at the boundary of the AdS. Following the AdS/CFT prescription \([15]\), this current could be computed considering the variation in the action due to the corresponding change in the background electromagnetic field \((A_\mu)\) namely,

\[
\langle J^\mu \rangle = \lim_{u \to 0} \frac{\delta S^{(os)}}{\delta A^\mu} = \lim_{u \to 0} \frac{\sqrt{-g} F^{\mu u}}{\sqrt{1 + b E}}.
\]

(48)

where we have set \(16\pi G_4 = 1\).

The goal of the present paper is to compute the above current \((48)\) for the leading order in the gauge fluctuations. Keeping terms upto leading order in \(b\) and considering only the spatial components of the current we finally have,

\[
\langle J_i \rangle = \left[ F^{(1)}(\beta^{(0)}) + b F^{(1)}(\beta^{(1)}) \right]_{u=0} + \mathcal{O}(b^2).
\]

(49)
where we have re-scaled the current by a factor of $\varepsilon r_+$. Substituting the above set of solutions (40) and (45) into (49) we finally obtain,

$$\langle J_i \rangle = \varepsilon_i \partial_j \Theta(x)$$  \hspace{1cm} (50)

where,

$$\Theta(x) = r_+^2 \int_0^1 du' \frac{\rho_1^{2(b(0))}(u')}{u'^2} \partial_u \int dx' G_s(u, u'; x, x')\sigma(x')|_{u=0}$$

$$-b \int_0^1 du' \partial_u \int dx' R(u', x')G_s(u, u'; x, x')|_{u=0} + O(b^2).$$  \hspace{1cm} (51)

Eq (50) is the exact expression for the boundary current considering the leading order corrections due to the BI coupling ($b$). At this point of our analyses, one can immediately make the following comments looking at the expression (50) above. These are the following.

- The current that has been computed so far is a highly non trivial function of the gauge fields, the vortex function and their derivatives.

- Secondly, and most importantly, the boundary current ($J_i$) is a non local function of the vortex solution $\sigma(x)$ in the sense that in order to evaluate the above current one needs to integrate the vortex function ($\sigma(x)$) over a region around some point ($x$) where vortex is localized. In other words in order to compute the current we need the information from a finite definite region around $x$ for which the vortex function ($\sigma(x)$) is non zero.\(^6\)

The goal of the present analyses is to remove the above non locality and express the current as a local function of the vortex solution $\sigma(x)$. In order to remove the above non locality we take the following steps. As a first step, we explicitly decompose the full Green’s functions into following two pieces,

$$G_t(u, u'; x, x') = \sum_{\alpha} \vartheta_{\alpha}(u)\vartheta_{\alpha}^\dagger(u')\tilde{G}_t(x - x', \alpha)$$

$$G_s(u, u'; x, x') = \sum_{\lambda} \zeta_{\lambda}(u)\zeta_{\lambda}^\dagger(u')\tilde{G}_s(x - x', \lambda)$$  \hspace{1cm} (52)

where, $\vartheta_{\alpha}(u)$ and $\zeta_{\lambda}(u)$ are the radial functions that satisfy the following eigen value equations namely\(^7\).

$$\mathcal{L}_t \vartheta_{\alpha}(u) = \alpha \vartheta_{\alpha}(u); \hspace{1cm} \sum_{\alpha} \vartheta_{\alpha}(u)\vartheta_{\alpha}^\dagger(u') = \delta(u - u'); \hspace{1cm} \langle \vartheta_{\alpha}|\vartheta_{\alpha}' \rangle = \delta_{\alpha\alpha'}$$

$$\mathcal{L}_s \zeta_{\lambda}(u) = \lambda \zeta_{\lambda}(u); \hspace{1cm} \sum_{\lambda} \zeta_{\lambda}(u)\zeta_{\lambda}^\dagger(u') = \delta(u - u'); \hspace{1cm} \langle \zeta_{\lambda}|\zeta_{\lambda'} \rangle = \delta_{\lambda\lambda'}.$$  \hspace{1cm} (53)

\(^6\)The size of this region is determined by the coherence length length $\xi$ and is typically of the order ($\sim \xi^2$).

\(^7\)Following the boundary conditions (42), the value of these radial functions near the boundary of the AdS could be set as, $\vartheta_{\alpha}(0) = 0$ and $\zeta_{\lambda}(0) = 0$. 

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where, $\mathcal{L}_t = -u^2 f(u) \partial_u^2$ and $\mathcal{L}_s = -\partial_u (u^2 f(u) \partial_u)$ are the differential operators that solely depend on the radial coordinate ($u$).

Furthermore note that here $\tilde{\mathcal{G}}_t(x - x', \alpha)$ and $\tilde{\mathcal{G}}_s(x - x', \lambda)$ are the Green’s functions defined on the two dimensional $(x, y)$ plane over which the condensate forms. Following the definitions (41) and (53), it is indeed quite trivial to show that these two dimensional Green’s functions satisfy the differential equation of the following form,

$$(\Delta - k^2) \tilde{\mathcal{G}}(x, k^2) = -\delta (x)$$  \hspace{1cm} (54)

for any real positive value of $k^2$. The solution of (54) could be expressed in terms of modified Bessel function namely,

$$\tilde{\mathcal{G}}(x, k^2) = \frac{1}{2\pi} K_0(|k|)$$  \hspace{1cm} (55)

which satisfies the boundary condition $\lim_{|x| \to \infty} |\tilde{\mathcal{G}}(x)| < \infty$.

With the above machinery in hand, we are now in a position to remove the non locality in the supercurrent ($J_i$) (Eq. (50)) computed at the boundary of the AdS. Note that we have two length scales in our theory. One is the length scale $\frac{1}{\sqrt{\lambda}}$ (or $\frac{1}{\sqrt{\alpha}}$) over which the Green’s function $\mathcal{G}_s(x - x', \lambda)$ (or $\mathcal{G}_t(x - x', \alpha)$) varies\(^8\) and the other one is the correlation length ($\xi$) which essentially determines the size of the vortex and thereby define a scale over which the vortex could exist. In order to remove the non localities associated with (50) we assume that the scale over which the Green’s function fluctuates is quite small (i.e; $\lambda$ (or $\alpha$) is quite large) compared to that of the correlation length ($\xi$) over which the vortex fluctuates i.e; $\frac{1}{\sqrt{\lambda}}$ (or $\frac{1}{\sqrt{\alpha}}$) $\ll \xi$. In other words, with the above assumption in mind we can take the condensate to be almost uniform over the scale $\frac{1}{\sqrt{\lambda}}$ (or $\frac{1}{\sqrt{\alpha}}$) which eventually results in the following mathematical identities,

$$\int dx' \tilde{\mathcal{G}}_t(x - x', \lambda) \sigma(x') = \frac{\sigma(x)}{\alpha} + \mathcal{O}(\frac{1}{\alpha^{3/2}})$$

$$\int dx' \tilde{\mathcal{G}}_s(x - x', \lambda) \sigma(x') = \frac{\sigma(x)}{\lambda} + \mathcal{O}(\frac{1}{\lambda^{3/2}}).$$  \hspace{1cm} (56)

where we have used (54) in order to arrive at the above identity\(^9\). Also we have ignored all the sub leading terms in the Taylor expansion of $\sigma(x')$ about the point $x' = x$ since they are highly suppressed compared with the leading term in the large $\lambda$ (or $\alpha$) limit.

The whole idea described above could be termed as the long wave length approximation that eventually leads to the following local expression for the supercurrent namely,

$$\langle J_i \rangle = D \epsilon^i_j \partial_j \sigma(x)$$  \hspace{1cm} (57)

where the coefficient $D$ is given by\(^{10}\)

$$D = D_1 + D_2^1 + D_2^{2(1)} + D_2^{2(2)} \Delta + D_2^{2(3)}.$$  \hspace{1cm} (58)

\(^8\)One can check that as we move away from the origin, the special Bessel function of the kind $K_0(|k|)$ has a sharp fall off that is determined by the factor $\frac{1}{\xi^2}$ which essentially measures the width of the curve about the origin. Therefore as the value of $k$ increases the width decreases which results in a faster fall off of the function.

\(^9\)In order to arrive at the above identity (56) one needs to consider the integral version of (54).

\(^{10}\)See Appendix for detailed expressions of individual coefficients.
From the above expression we note that it is the fourth term appearing with the coefficient $D^2(2)$ that dramatically modify the local form of the current and leads to a new physics beyond the scope of the usual GL theory of ordinary type II superconductors. Our next aim would be to extract all the relevant physical information associated with this coefficient.

From the expressions of the individual coefficients $D_i$ we note that they basically contain an infinite sum over the eigenvalues $\lambda^{-1}$ (or $\alpha^{-1}$). Our next aim would be to consider the most dominant term in this series. In other words, we shall replace $\lambda$ (or $\alpha$) in the above series by $\lambda_{\text{min}}$ (or $\alpha_{\text{min}}$) which could be termed as the large $\lambda$ (or $\alpha$) approximation. Using this approximation, the most dominant contribution to the current could be expressed as,

$$\langle J_i \rangle = Z_1 \epsilon^j_i \partial_j \sigma(x) + \frac{9bH^2Z_2}{16\pi^2T^2\lambda_{\text{min}}^2} \epsilon^j_i \partial_j \Delta \sigma(x) + O(b^2)$$  \hspace{1cm} (59)$$

where the coefficients $Z_1$ and $Z_2$ are given by,

$$Z_1 = A_1 + A_2^{(1)} + A_2^{(2)}$$
$$Z_2 = -\zeta'_\lambda(0) \int_0^1 u^4 \zeta'_\lambda(u') \zeta'_\lambda^{-1}(u') du' \int_0^1 du'' \rho_1^2(0) \frac{u''}{u'^2} \zeta'_\lambda(u'').$$ \hspace{1cm} (60)$$

Eq (59) gives the final expression for the supercurrent as a local function of the vortex solution $\sigma(x)$ in the presence of the BI corrections to the usual Maxwell sector of the abelian Higgs model proposed in \[14\]. There are few comments that we would like to make at this stage. First of all we note that in the presence of non linear corrections to the usual Maxwell sector of the model, the local form of the boundary current gets modified dramatically. This modification appears with several parameters. Considering the boundary behavior of the radial function $\rho(u)$ we note that the coefficient $Z_2$ yields a finite value. Secondly and most importantly, considering (3), (4) and (53) and noting the fact that $u(= r+/r)$ is a dimensionless parameter, one can see that $\lambda \sim T^2$ where $T$ is the temperature of the system. Therefore the modification term appearing in (59) is a term that eventually goes as ($\sim \frac{1}{T^6}$) compared with the original term and thereby highly suppressed at large values of the temperature. In other words we may say that the above correction appearing in (59) is nothing but the finite temperature correction to the usual GL current for type II superconductors in $(2 + 1)$ dimensions which could be tested only at some finite non zero temperatures.

We are now in a position to summarize the outcome of the entire discussion made so far which could be stated as follows: The AdS/CFT duality inspires us to consider the following finite temperature correction to the usual GL current for ordinary type II superconductors namely,

$$j_i|_{\text{new}} \rightarrow j_i|_{\text{GL}} + \frac{\#}{T^6} \epsilon^j_i \partial_j \Delta \sigma(x)$$ \hspace{1cm} (61)$$

where the first term correspond to the usual GL current familiar to the condensed matter community. The remaining one is a temperature dependent (correction) term that appears

\[11\] Here $\Delta = \partial_x^2 + \partial_y^2$ is the usual Laplacian operator.
\[12\] See Appendix for details.
as a cubic derivative of the vortex function $\sigma(x)$. According to gauge/gravity duality the origin of such a non trivial low temperature correction would be encoded in certain dual gravitational description of the system where one needs to modify the usual abelian sector by certain appropriate correction terms.

4 Free energy

We would like to conclude our analyses of the present paper by computing the free energy for the dual theory living in the boundary of the $AdS_4$. In this section, considering the large $\lambda$ approximation, our goal is to study the effect of BI corrections on the free energy of the system in the presence of (triangular) vortex lattice. In order to carry out our analyses we consider that the scalar condensation is confined within a compact region of volume $V$ whose size is much bigger than the size of a single unit cell of the triangular lattice.

The free energy of the system could be computed from the knowledge of the onshell action evaluated with appropriate counter terms namely,

$$ F = -S_{(os)}.$$ (62)

The full onshell action consists of two parts. Let us first consider the onshell action corresponding to the scalar field ($\Psi$). Using the equation of motion (9), one can in fact show that the onshell action for the scalar field turns out to be,

$$ S_{\psi} |_{(os)} = -\frac{1}{2} \int_{\partial M} d\Sigma_{\mu} \sqrt{-g} (\nabla^\mu - iA^\mu) |\Psi|^2.$$ (63)

where $d\Sigma_{\mu}$ is the volume measured at the boundary of the $AdS_4$.

In order to evaluate the above integral one needs to take into account different choices for the hyper surface ($\partial M$) corresponding to different values of $\mu$ which are the following:

- For $\mu = t$, the above integral vanishes if we take our field configuration to be stationary at the past and future space like surfaces.

- Considering the radial behavior (21), the above integral (63) also vanishes for $\mu = u$.

- Finally for $\mu = i$, at large values of the spatial coordinates, the above integral will vanish due to the Gaussian behavior of the vortex solution (See Eq (18)).

Therefore from the above discussion we conclude that the scalar field does not directly contribute to the free energy of the system. Thus we are only left with the onshell action corresponding to the $U(1)$ gauge sector. Since we are interested to calculate the free energy in the presence of the vortex solution, therefore our next aim would be to compute the onshell action corresponding to the fluctuations in the $U(1)$ gauge field. This is how the scalar condensation could indirectly affect the free energy of the system through its interaction with the gauge fields.

In order to proceed further we first consider the following perturbative expansion for the onshell action in the parameter $\varepsilon$ namely,

$$ S_{(os)} = S_{(os)}^{(0)} + \varepsilon S_{(os)}^{(1)} + \varepsilon^2 S_{(os)}^{(2)} + O(\varepsilon^3).$$ (64)
Let us now consider the action corresponding to the first order in the gauge fluctuations which turns out to be,

\[ S^{(1)}_{(os)} = -\frac{1}{2} \int d^4x \sqrt{-g} F^{\nu\mu(0)} F^{(1)}_{\nu\mu} \left(1 - \frac{b}{4} F^{(0)}\right) \]

\[ = -\int_{\partial M} d\Sigma u \sqrt{-g} F^{u\mu(0)} \left(1 - \frac{b}{4} F^{(0)}\right) A^{(1)}_\mu |_{u=0}. \quad (65) \]

Since our boundary theory has been kept at a fixed chemical potential (∝µ), therefore the above integral yields a vanishing contribution to the free energy of the system.

At this stage of our analyses it is now quite reasonable to expect that the first nontrivial contribution to the free energy of the system might arise at the quadratic level in the gauge fluctuations. The onshell action at the quadratic level in the gauge fluctuations turns out to be,

\[ S^{(2)}_{(os)} = -\frac{1}{2} \int d^4x \sqrt{-g} F^{\nu\mu(0)} F^{(2)}_{\nu\mu} \left(1 - \frac{b}{4} F^{(0)}\right) \]

\[ -\frac{1}{4} \int d^4x \sqrt{-g} \left[ F^{\nu\mu(1)} F^{(1)}_{\nu\mu} - \frac{b}{4} F^{\nu\mu(1)} F^{(1)}_{\nu\mu} F^{(0)} - \frac{b}{2} F^{\nu\mu(0)} F^{(1)}_{\nu\mu} F^{\lambda\sigma(0)} F^{(1)}_{\lambda\sigma}\right]. \quad (66) \]

After using the equation of motion (26) it takes the following form,

\[ S^{(2)}_{(os)} = -\int_{\partial M} d\Sigma u \sqrt{-g} F^{u\mu(0)} \left(1 - \frac{b}{4} F^{(0)}\right) A^{(2)}_\mu |_{u=0} + \frac{r_0}{2} \int_{\partial M} d\Sigma u \langle J^1 \rangle a_i(x) \quad (67) \]

where in order to arrive at the above relation we have used the following orthogonality condition namely,

\[ \int_M d^4x \sqrt{-g} A^{(1)}_\mu j^{(1)} j^{(1)} = 0. \quad (68) \]

Following our previous arguments we note that the first term on the r.h.s. of (67) vanishes identically. Finally using (50), the expression for the free energy turns out to be,

\[ F = -\frac{\varepsilon_r^2 H_{c2}}{2} \int_{\mathbb{R}^2} dx \Theta(x). \quad (69) \]

Note that this is again a non local expression in the vortex function σ(x). Following our previous arguments namely the large λ approximation, we may convert the above equation (69) into a local function of the vortex solution σ(x) averaged over the finite volume V on which the condensate forms. This eventually leads us to the following local expression for the free energy,

\[ F/V = -\frac{\varepsilon_r^2 H_{c2}}{2} \left(Z_1 \overline{\sigma(x)} + \frac{9bH_{c2}^2 Z_2}{16\pi^2 T^2 \lambda_{min}^2} \overline{\Delta \sigma(x)}\right) + \mathcal{O}(b^2) \quad (70) \]

where the bar indicates the average value of the condensate over the region under consideration.

From the above expression (70), we note that the free energy for the vortex configuration is negative. This suggests that the (triangular) lattice configuration is more stable than the normal phase with the vanishing order parameter (Ψ = 0). Furthermore we note that due to the presence of the BI term in the dual gravitational description, the corresponding free energy of the boundary theory also receives a nontrivial finite temperature correction that we have already found during the computation of the supercurrent in the previous section.
5 Summary and final remarks

Based on the AdS$_4$/CFT$_3$ correspondence, the goal of the present analyses was to compute the global $U(1)$ current for the boundary theory in the presence of Born Infeld (BI) corrections to the usual Maxwell action of the abelian Higgs model proposed in [14]. The outcome of the entire analysis could be summarized as follows: First of all it is confirmed that using the AdS/CFT prescription even in the presence of nonlinear terms in the bulk theory one can arrive at the familiar Ginzburg Landau (GL) expression for the supercurrent that is circulating around the core of the vortex in type II superconductors. It is observed that the presence of such non linear terms in the dual gravitational description dramatically affects the local form of the current at the leading order in the BI coupling ($b$). Such a non trivial effect comes with a cubic derivative term that essentially corresponds to a finite temperature correction to the standard form of the GL current in type II superconductors. Since the modification of the above kind goes inversely as the sixth power of the temperature therefore it is highly suppressed at the large values of the temperature and could be effective only at some finite low temperatures. We also compute the free energy of the system that comes out to be negative for the triangular lattice configuration indicating the vortex structure to be the most stable thermodynamic phase. Due to the presence of the BI coupling we also find some corresponding finite temperature correction to the free energy of the system at the leading order in the coupling parameter ($b$).

It should be noted that in the above analyses we have removed the non locality in the expression for the boundary current by virtue of a strong assumption known as the long wave length approximation which may not be a good approximation in the presence of small length scales such as the Pippard/BCS coherence lengths where one can not ignore the fluctuation of the electromagnetic signal in the superconductor. Therefore in that case one needs to find a way to deal with those non local terms appearing in the expression of the current. Finally, it would be also an interesting exercise to carry out the above analysis in the presence of the back reaction.

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Appendix

A Calculation of $\mathcal{D}$

The appendix consists of all the details of the coefficients that appear in Eq. (58) in the expression for the coefficient $\mathcal{D}$. Also it contains the expression for the corresponding quantities in the large $\lambda$ (or $\alpha$) limit.

First Term:

$$\mathcal{D}_1 = \pi^2 r^2 \sum_{\lambda} \zeta_{\lambda}(0) \frac{\lambda}{\lambda} \int_0^1 du' \frac{u'}{u^2} P_1^{2(b(0))}(u') \zeta_{\lambda}^*(u').$$

(71)
Second Term:
\[ D_2^1 = -br_+^2 \sum_{\lambda} \frac{\zeta_\lambda'(0)}{\lambda} \int_0^1 \frac{du'}{u'^2} \mathcal{I}(u') \zeta_\lambda^\dagger(u'). \] (72)

Third Term:
\[ D_2^{2(1)} = 2b\mu H c_2 \sum_{\lambda, \alpha} \frac{\zeta_\lambda'(0)}{\lambda \alpha} \int_0^1 u'^4 \frac{\partial \varphi_{\alpha}(u')}{\alpha} \zeta_\lambda^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} A_t^{(0)(b(0))}(u'') \varphi_{\alpha}^\dagger(u''). \] (73)

Fourth Term:
\[ D_2^{2(2)} = -\frac{b H c_2}{r_+^2} \sum_{\lambda, \alpha} \frac{\zeta_\lambda'(0)}{\lambda \alpha} \int_0^1 u'^4 \zeta_\lambda(u') \zeta_\lambda^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} \zeta_\lambda^\dagger(u''). \] (74)

Fifth Term:
\[ D_2^{2(3)} = -\frac{br_+^2}{4} \sum_{\lambda, \alpha} \frac{\zeta_\lambda'(0)}{\lambda \alpha} \int_0^1 u'^4 f(u') \partial_{u'} \mathcal{F}^{(0)(b(0))}(u') \zeta_\lambda(u') \zeta_\lambda^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} \zeta_\lambda^\dagger(u''). \] (75)

Large \( \lambda \) approximation: Replacing \( \lambda \) (or \( \alpha \)) in the above series by \( \lambda_{\text{min}} \) (or \( \alpha_{\text{min}} \)) we arrive at the following expressions for the corresponding coefficients \( D_i \) written above.

First Term:
\[ A_1 = r_+^2 \frac{\zeta_{\lambda_{\text{min}}}'(0)}{\lambda_{\text{min}}} \int_0^1 \frac{du'}{u'^2} \frac{\rho_1^{2(0)}(u')}{u'^2} \zeta_{\lambda_{\text{min}}}^\dagger(u'). \] (76)

Second Term:
\[ A_2^1 = -br_+^2 \frac{\zeta_{\lambda_{\text{min}}}'(0)}{\lambda_{\text{min}}} \int_0^1 \frac{du'}{u'^2} \mathcal{I}(u') \zeta_{\lambda_{\text{min}}}^\dagger(u'). \] (77)

Third Term:
\[ A_2^{2(1)} = 2b\mu H c_2 \frac{\zeta_{\lambda_{\text{min}}}'(0)}{\lambda_{\text{min}}} \int_0^1 u'^4 \frac{\partial \varphi_{\alpha_{\text{min}}}(u')}{\alpha_{\text{min}}} \zeta_{\lambda_{\text{min}}}^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} A_t^{(0)(b(0))}(u'') \varphi_{\alpha_{\text{min}}}^\dagger(u''). \] (78)

Fourth Term:
\[ A_2^{2(2)} = -\frac{b H c_2}{r_+^2} \frac{\zeta_{\lambda_{\text{min}}}'(0)}{\lambda_{\text{min}}^2} \int_0^1 u'^4 \zeta_{\lambda_{\text{min}}}'(u') \zeta_{\lambda_{\text{min}}}^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} \zeta_{\lambda_{\text{min}}}^\dagger(u''). \] (79)

Fifth Term:
\[ A_2^{2(3)} = -\frac{br_+^2}{4} \frac{\zeta_{\lambda_{\text{min}}}'(0)}{\lambda_{\text{min}}^2} \int_0^1 u'^4 f(u') \partial_{u'} \mathcal{F}^{(0)(b(0))}(u') \zeta_{\lambda_{\text{min}}}'(u') \zeta_{\lambda_{\text{min}}}^\dagger(u') du' \int_0^1 du'' \frac{\rho_1^{2(0)}(u'')}{u''^2} \zeta_{\lambda_{\text{min}}}^\dagger(u''). \] (80)
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