PERMUTATIONS CONTAINING MANY PATTERNS

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Abstract. It is shown that the maximum number of patterns that can occur in a permutation of length \(n\) is asymptotically \(2^n\). This significantly improves a previous result of Coleman.

1. Introduction

Given a sequence \(t = t_1, t_2, \ldots, t_k\) of distinct elements from some totally ordered set, there is a unique permutation \(\tau\) of \([k] = \{1, 2, \ldots, k\}\) with the property that for all \(1 \leq i, j \leq k\), \(t_i < t_j\) if and only if \(\tau(i) < \tau(j)\). We call \(\tau\) the pattern of \(t\). For example, the pattern of 5, 10, 2 written in one line notation is 231. In other words, the sequence representing \(\tau\) is obtained from \(t\) simply by replacing each element of \(t\) by its rank in \(t\).

Let \(\sigma\) be a permutation of length \(n\), written in one-line notation as \(\sigma(1)\sigma(2)\cdots\sigma(n)\), and thought of as a sequence of length \(n\). For each non-empty subset \(X\) of \([n]\) define \(\sigma_X\) to be the pattern of that subsequence of \(\sigma\) whose indices belong to \(X\). Define:

\[
P(\sigma) = \{\sigma_X : \emptyset \neq X \subseteq [n]\}.
\]

That is, \(P(\sigma)\) is the set of patterns that occur in \(\sigma\). Also define \(h(n)\) to be the maximum value of \(|P(\sigma)|\) taken over all permutations \(\sigma\) of length \(n\).

Trivially, \(h(n) \leq 2^n - 1\). Slightly more precisely, for any permutation \(\sigma\) of length \(n\):

\[
|P(\sigma)| \leq \sum_{k=1}^{n} \min\left(k!, \binom{n}{k}\right)
\]

since not more than \(k!\) patterns of length \(k\) can occur. However, the expression on the right hand side of this inequality is easily seen to be asymptotically \(2^n\). At the 2003 conference on Permutation Patterns, Herb Wilf raised the issue of determining the (asymptotic) behaviour of \(h(n)\), and exhibited a sequence of permutations which established that \(h(n)\) exceeded the \(n^{th}\) Fibonacci number. Micah Coleman then
demonstrated in [1] a sequence of permutations \( \pi_n \), for \( n \) a perfect square, for which:

\[
|P(\pi_n)| > 2^{n-2\sqrt{n}+1}.
\]

Of course this establishes that \( h(n)^{1/n} \to 2 \) (for all \( n \), not just perfect squares, using the fact that \( h(n) \) is non decreasing). However, this left open the question of whether or not \( h(n)/2^n \) tends to 1 as \( n \) tends to infinity.

In this paper, we refine the counting arguments concerning the number of patterns in \( \pi_n \), for \( n \) an even perfect square, and then extend the construction to all other values of \( n \), in order to show that \( |P(\pi_n)|/2^n \to 1 \). Indeed, we will obtain:

\[
h(n) > 2^n \left( 1 - 6\sqrt{n}2^{-\sqrt{n}/2} \right)
\]

for all positive integers \( n \).

2. The main construction

Let \( k \) be a positive integer and let \( n = 4k^2 \). Let \( s \) be the sequence:

\[
s = (2k) (4k) (6k) \cdots (4k^2)
\]

and consider the permutation \( \pi_n \) which in one line notation is defined by:

\[
\pi_n = s (s-1) (s-2) \cdots (s-2k+1).
\]

Here \( s - i \) indicates the sequence obtained by subtracting \( i \) from each element of \( s \). Generally, we will suppress the subscript on \( \pi_n \) when there is no risk of confusion. Informally, the graph of \( \pi \) is obtained by taking a standard orthogonal \( 2k \times 2k \) grid and rotating it slightly in the clockwise direction around its lower left hand corner. We associate to each subset \( X \) of (the indices of) \( \pi \) a \( 2k \times 2k \) 0-1 matrix, \( M_X \), whose 1 entries correspond to the elements of the subset. We also view \( M_X \) as being partitioned into four \( k \times k \) submatrices (called the corner submatrices) in the usual way, that is, so that they form a \( 2 \times 2 \) block decomposition of \( M_X \). We say that \( X \) (or \( M_X \)) is ample if each \( k \times k \) corner submatrix of \( M_X \) has no zero rows or zero columns. An example is shown in Figure II.

**Proposition 1.** The number of ample matrices is greater than

\[
2^n \left( 1 - \frac{4\sqrt{n}}{2\sqrt{n}/2} \right).
\]

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\(^1\)We have adjusted the notation slightly from that of [1] — what was there called \( \pi_k \) we are calling \( \pi_{k^2} \) so that the subscript is equal to the length of the permutation.
Figure 1. The graph of the permutation \( \pi_{64} \), an ample subset of its elements indicated by filled circles, together with the corresponding matrix divided into its corner submatrices.

**Proof.** Recall that \( n = 4k^2 \). Suppose that we sample an \( n \times n \) 0-1 matrix uniformly at random from among all \( n \times n \) 0-1 matrices. The probability that any particular row or column sum of one of the corner submatrices is 0 is \( 1/2^k \). There are \( 8k \) such sums which must all be non zero in order for the matrix to be ample. However, the probability that any of them are 0 is less than \( 8k/2^k \). So, the probability that all are non zero is greater than

\[
1 - \frac{8k}{2^k} = 1 - \frac{4\sqrt{n}}{2\sqrt{n}/2},
\]

which is equivalent to the stated result. \( \square \)

**Proposition 2.** Let \( X \) and \( Y \) be ample sets. Then \( \pi_X = \pi_Y \) implies \( X = Y \).

**Proof.** We must show that, if \( X \) is ample, then it can be reconstructed from just the permutation \( \pi_X \). Since \( X \) is ample, the column sum of both the top half and bottom half of each column of \( M_X \) is non zero. Therefore, there are \( 2k - 1 \) descents in \( \pi_X \), corresponding to the transitions between columns of \( M_X \). Thus, we can associate the elements of \( \pi_X \) with their correct columns. However, this argument applies equally well to the rows of \( M_X \) — as is most easily seen by considering \( \pi^{-1} \). Determining the row and column that represents each element of \( \pi_X \) is exactly the same as reconstructing \( X \). \( \square \)

Combining these two results we have:
**Theorem 3.** If $n$ is an even perfect square, then

$$h(n) > 2^n \left(1 - \frac{4\sqrt{n}}{2\sqrt{n/2}}\right).$$

We will refer to the second term inside the parentheses above as the *correction term* for this estimate.

### 3. Refinements

It is easy to extend the above arguments to give lower bounds on $h(n)$ that are valid for *all* values of $n$. We can do this by using the basic construction of the previous section, and adding some extra elements in appropriate places to construct permutations $\pi_n$ of length $n$ that contain many patterns.

First suppose that $n = 4k^2 + l$ where $0 < l < 2k$. Take the grid associated to the permutation $\pi_{4k^2}$ and add a (partial) column on the right hand side at the bottom containing not more than $k$ elements, and, if necessary, a partial row on top at the right hand side, also not containing more than $k$ elements, so that the total number of elements added is $l$. As before, rotate this grid slightly, and view the result as the graph of a permutation, $\pi_n$. An example is shown in Figure 2. Call the elements of this permutation arising from the original grid defining $\pi_{4k^2}$ the *main elements*, and the remaining elements the *extra elements*. Define a subset of the indices of $\pi_n$ to be *ample* if its intersection with the main elements would be ample for $\pi_{4k^2}$.

![Figure 2](image.png)

**Figure 2.** The graph of the permutation $\pi_{70}$, together with the matrix associated with a particular ample subset of its elements indicated by filled circles.
Proposition 4. Let $X$ and $Y$ be ample sets. Then $\pi_X = \pi_Y$ implies $X = Y$.

Proof. As before, we must describe how to reconstruct $X$ from $\pi_X$. However, we can identify the extra elements (and hence the main elements) in $\pi_X$. If there are any belonging to the new partial column, then they are exactly the elements following the $(2k)^{th}$ descent, while those belonging to the new partial row, if such exist, are exactly those lying above the maximum element of the first $k$ columns. Since the main elements form an ample subset of $\pi_{4k^2}$ we can use the preceding result to identify their values. Once the values of the main elements are known, so are the values of the extra elements. \hfill \Box

Therefore, for such $n$, 
\[ h(n) \geq |P(\pi_n)| > 2^{4k^2} \left(1 - \frac{8k}{2^k}\right) 2^t. \]

Certainly $k \leq \sqrt{n}/2$, but also $(2k + 1/2)^2 > n$ so $k > (\sqrt{n} - 1/2)/2$. Applying these estimates we obtain:
\[ h(n) > 2^n \left(1 - \frac{2^{9/4} \sqrt{n}}{2^{\sqrt{n}/2}}\right). \]

This differs from our previous estimate by a factor of $2^{1/4}$ in the correction term.

For $n = 4k^2 + 2k$, we switch to a grid consisting of $2k + 1$ columns of size $2k$ and define $\pi_n$ appropriately. As in the previous section, we define the four corner submatrices, except now those on the right hand side of the matrix are $k \times (k + 1)$ instead of $k \times k$. The probability of a subset of the matrix not being ample is not as much as:
\[ \frac{2(2k+1)}{2^k} + \frac{2k}{2^k} + \frac{2k}{2^{k+1}} = \frac{7k+2}{2^k}. \]

Using the same bounds as before (which still apply) plus trivial estimates for $k \leq 2$ it is easy to check that the bound
\[ h(n) > 2^n \left(1 - \frac{2^{9/4} \sqrt{n}}{2^{\sqrt{n}/2}}\right) \]

still applies in this case. We can proceed from this point with the half-row/half-column construction again (possibly at a penalty of another factor of $2^{1/4}$ in the correction term) as far as $n = (2k+1)^2$. At this point we pause for a detailed re-evaluation. In a $(2k+1) \times (2k+1)$ grid, divided into corner submatrices of sizes $k \times k$, $k \times (k+1)$, $(k+1) \times k$
and \((k + 1) \times (k + 1)\), the probability that a subset is not ample is less than:

\[
\frac{2k}{2^k} + 2 \left( \frac{k}{2^{k+1}} + \frac{k + 1}{2^k} \right) + \frac{2(k + 1)}{2^{k+1}} = \frac{6k + 3}{2^k}.
\]

Since \(k = (\sqrt{n} - 1)/2\), this equals

\[
\frac{(3\sqrt{2})\sqrt{n}}{2\sqrt{n/2}}.
\]

We can pursue these constructions through to the next even perfect square, and, allowing for a further penalty of \(\sqrt{2}\) in the correction term (which we leave to the reader to verify is generous), obtain:

**Theorem 5.** For all positive integers \(n\),

\[
h(n) > 2^n \left(1 - \frac{6\sqrt{n}}{2\sqrt{n/2}}\right).
\]

4. Conclusions

It would be interesting to know just how close to \(2^n\) the value of \(h(n)\) actually is. A more careful analysis of the various steps in moving from one square grid to the next might well provide a small improvement in the constant factor of the correction term of our estimate. Similarly, an analysis of conditions weaker than ample which none the less would allow for a reconstruction result might actually improve the asymptotic form of the correction term. However, the simplicity of the main construction (for \(n = 4k^2\)) and of the proof that ample subsets can be reconstructed from their patterns, together with the lack of any great need for more precise estimates of \(h(n)\) somewhat dampens our enthusiasm for further investigations in that direction. Of perhaps greater interest would be to investigate the distribution of the statistic \(|P(\pi)|\) as \(\pi\) ranges over permutations of length \(n\).

We would like to thank Herb Wilf for having posed such an interesting problem!

**References**

[1] Micah Coleman. An answer to a question by Wilf on packing distinct patterns in a permutation. *Electron. J. Combin.*, 11(1):Note 8, 4 pp. (electronic), 2004.
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