Weak-form differential quadrature element method for dynamic analysis of fluid-saturated soil

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Abstract. It is formulated the weak-form of differential equations that govern the two-dimensional dynamical behavior of fluid-saturated soil, then the weak-form equations are discretized by the differential quadrature technique, and finally solved by the implicit Euler method. The proposed weak-form equations and numerical programs developed are verified through comparisons with benchmark solutions, and the convergence performance of the presented method is investigated. Numerical results show that the proposed weak-form quadrature element method not only possesses significantly higher computational efficiency, for the dynamic analysis of saturated soil, than the conventional finite element method, but it is also significantly alleviated the problem of numerical smoothness in the stress analysis.

1. Introduction

Safety assessments of soils and foundations are of great importance in most of the infrastructure engineering, especially the offshore engineering, bridge engineering and underground engineering. To carry out the mechanical analyses and engineering design, the Biot [1,2] theory and the theory of porous media (TPM) [3] have matured into the most widespread theories depicting the behaviors of fluid-saturated soils. Based on the Biot theory, several aspects of analyses have been performed, including the consolidation of foundation [4], the dynamic analysis of saturated soil [5] and the seepage prediction of underground water [6], etc. Compared with the Biot theory, the TPM complies with more rigorous continuous media mixture laws and the concept of volume fraction, and is capable of dealing with nonlinear problems [7] more naturally. As a result, the TPM is getting popular, and there had been researchers investigated the saturated soil in the framework of TPM, e.g. Breuer [8] had analyzed the dynamical response of two-dimensional foundation using the standard Galerkin finite element method (FEM); Hu et al [9] had developed the strong-form differential quadrature (DQ) method for the dynamical response analysis of axisymmetric saturated porous media [10].

The FEM has nowadays undoubtedly evolved into one of the most powerful and practical analyzing techniques in the engineering practice, and commercial finite element analysis software equipped with Biot porous medium are common. Despite the success of FEM, there still exists shortcomings of the conventional FEM, e.g. the unsatisfactory discretization efficiency, that is, the acceptable analyzing results are usually based on large number of degrees of freedom (DOFs), memory and CPU/GPU time. To improve the discretization efficiency, the DQ technique [10] has recently been known as a good candidate. The weak-form differential quadrature element method (WQEM) [11,12], which incorporates DQ technique into the conventional FEM, inherits merits of both methods. And the CPU time and memory requirements may be reduced significantly by WQEM.

In this paper, the TPM is used for dynamic analysis of two-dimensional fluid-saturated soils. The
weak-form equations governing the motions of soil skeleton, pore water together with pore pressure are obtained. With the help of DQ technique, the initial value problem is formulated and the computing programs are implemented. The implicit Euler method is used to solve the initial value problem. The computer programs are verified through the numerical comparisons, and the computational efficiency of the proposed method versus the conventional FEM is examined.

2. Formulations

2.1. Governing differential equations

Let us consider a fluid-saturated porous medium with immiscible and incompressible constituents in domain \( \Omega \). The behaviors of water and solid skeleton can be formulated by TPM [7,8] as

\[
\begin{align*}
\text{div} \left( \sigma^{\text{SE}} - p \mathbf{I} \right) + \rho \left( \mathbf{b} - \mathbf{v}^s \right) - \rho^F \mathbf{w}^F &= 0 \\
\rho^F \left( \mathbf{b} - \mathbf{v}^s - \mathbf{w}^F \right) - S_r \mathbf{w}^F - n^F \text{grad} p &= 0 \\
\text{div} \left( \mathbf{v}^s + n^F \mathbf{w}^F \right) &= 0 \\
\mathbf{v}^s - \mathbf{u}^s &= 0
\end{align*}
\]

where, \( \sigma^{\text{SE}} \), \( \mathbf{u}^s \) and \( \mathbf{v}^s \) are the tensors of effective stress, displacement and solid skeleton velocity, respectively; \( p \), \( \mathbf{w}^F \) and \( n^F \) are the pore pressure, fluid-solid relative velocity and volume fraction of water, respectively; \( \rho^F \) denotes the partial density of fluid, that is, \( \rho^F = n^F \rho^{FR} \), in which \( \rho^{FR} \) is the real density of water; \( \rho \) is the sum of \( \rho^F \) and \( \rho^s \), and herein \( \rho^s = n^s \rho^{SR} \), with \( n^s \) and \( \rho^{SR} \) the volume fraction and real density of solid skeleton, respectively; \( \mathbf{b} \) is the body force density acting upon both constituents; \( S_r \) is the coupling coefficient, determined by \( S_r = (n^F)^2 \gamma^{FR} / k^F \), with \( \gamma^{FR} = \rho^{FR} |\mathbf{b}| \) the real specific weight of water, and \( k^F \) the Darcy permeability coefficient; \( I \) is the Kronecker tensor; \( \text{div} \) is the operator for divergence, and \( \text{grad} \) denotes the operator for gradient with respect to Cartesian coordinate \( x \).

On the boundary \( S \), the boundary conditions for the fluid-solid mixture and fluid can be described as follows

\[
\begin{align*}
\mathbf{u}^s &= \mathbf{u}^s_0, \quad \mathbf{x} \in S_a, \\
\mathbf{n} \cdot (\sigma^{\text{SE}} - p \mathbf{I}) &= \mathbf{T}, \quad \mathbf{x} \in S_T \\
p &= p_0, \quad \mathbf{x} \in S_p, \\
\mathbf{n} \cdot (n^F \mathbf{w}^F) &= \mathbf{Q}, \quad \mathbf{x} \in S_Q
\end{align*}
\]

where, \( S = S_p \cup S_0 = S_a \cup S_T \); \( S_a \cap S_T = S_p \cap S_Q = \phi \); \( \mathbf{n} \) is the unit vector normal to the boundary \( S \); \( \mathbf{u}^s_0 \) and \( p_0 \) are the prescribed solid displacement on \( S_a \) and surface pore pressure of water on \( S_p \), respectively; \( \mathbf{T} \) and \( \mathbf{Q} \) are the known surface traction on \( S_T \), and water flux on \( S_Q \), respectively.

In the initial time \( t_0 \), the state of all the unknowns is known as

\[
\mathbf{u}^s = \mathbf{u}^s_0, \quad \mathbf{v}^s = \mathbf{v}^s_0, \quad \mathbf{w}^F = \mathbf{w}^F_0, \quad t = t_0
\]

For the isotropic elastic soil skeleton with small deformation, and in the case of plane strain hypothesis,

\[
\begin{align*}
\sigma_x &= (2\mu^S + \lambda^S) \frac{\partial u_x}{\partial x} + \lambda^S \frac{\partial u_y}{\partial y} + \lambda^S \frac{\partial u_z}{\partial z}, \\
\sigma_y &= (2\mu^S + \lambda^S) \frac{\partial u_y}{\partial y} + \lambda^S \frac{\partial u_x}{\partial x} + \lambda^S \frac{\partial u_z}{\partial z}, \\
\tau_{xy} &= \mu^S \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\end{align*}
\]

where, \( \mu^S \) and \( \lambda^S \) are the Lamé constants of soil skeleton,
\[ \sigma_{SE} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}, \quad \mathbf{u}^S = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \]

2.2. Weak-form equations

Equations (1) and (2) can be transformed into an integral form, which can be summarized as

\[
\begin{split}
\int_{\Omega} \left[ \mathbf{u}^T \left( \rho F \mathbf{w}^F + \rho \mathbf{v}^S \right) + \left( \rho F \mathbf{w}^T + \beta \rho \mathbf{F} \mathbf{V}^T \mathbf{p} \right) \left( \mathbf{v}^S + \mathbf{w}^T \right) + \mathbf{v}^T \mathbf{u}^S \right] d\Omega \\
+ \int_{\Omega} \left[ \mathbf{e}^T (\sigma - \mathbf{e}^T \mathbf{p} + \mathbf{w}^T (s \mathbf{w}^F + n^T \mathbf{V} p) + \mathbf{p} \mathbf{V}^T \mathbf{v}^S + \mathbf{V}^T \mathbf{p} \mathbf{n}^T \mathbf{p} - \mathbf{v}^T \mathbf{v}^S \right] d\Omega \\
= \int_{\gamma_t} \mathbf{u}^T \mathbf{t} d\mathbf{s} - \int_{\gamma_q} \mathbf{p} \mathbf{q} d\mathbf{s}
\end{split}
\]

in which, \( \beta = \frac{k}{\gamma} \); \( \mathbf{a} = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T \) is the stress vector; \( \mathbf{e} \) and \( \mathbf{\varepsilon} \) are composed of arbitrary function in \( \Omega \) as

\[
\mathbf{e} = \begin{bmatrix} \partial u_x / \partial x & \partial u_y / \partial y + \partial u_x / \partial x \\ \partial u_x / \partial x + \partial u_y / \partial y \\ \partial u_y / \partial x \end{bmatrix}, \quad \mathbf{\varepsilon} = \begin{bmatrix} \partial u_x / \partial x & \partial u_y / \partial y \\ \partial u_x / \partial x + \partial u_y / \partial y \\ \partial u_y / \partial x \end{bmatrix}
\]

\( \mathbf{V} = \begin{bmatrix} \partial / \partial x & \partial / \partial y \end{bmatrix}^T \); \( \mathbf{\tilde{w}}, \mathbf{\tilde{u}} \) and \( \mathbf{\tilde{v}} \) are composed of arbitrary functions in \( \Omega \), which have expressions

\[
\mathbf{\tilde{w}} = \begin{bmatrix} \tilde{w}_x & \tilde{w}_y \end{bmatrix}^T, \quad \mathbf{\tilde{u}} = \begin{bmatrix} \tilde{u}_x & \tilde{u}_y \end{bmatrix}^T, \quad \mathbf{\tilde{v}} = \begin{bmatrix} \tilde{v}_x & \tilde{v}_y \end{bmatrix}^T
\]

As the test functions, \( \mathbf{\tilde{u}}, \mathbf{\tilde{v}} \) and \( \mathbf{\tilde{p}} \) are arbitrary except that \( \mathbf{\tilde{v}} = \mathbf{\tilde{u}} = \mathbf{0} \) on \( \mathbf{x} \in S_u \) and \( \mathbf{\tilde{p}} = 0 \) on \( \mathbf{x} \in S_p \).

2.3. DQ approximation

The core idea of the DQ technique is that the \( m^{th} \)-order \((m \in \mathbb{Z}, m \geq 1)\) derivative with respect to \( \xi \) of function \( F(\xi, t) \) with \( C^m \) continuity, can be approximated through combination of function values at \( N \) given nodes: \( -1 = \xi_1 < \xi_2 < \cdots < \xi_N = 1 \), that is

\[
\frac{\partial^m F(\xi, t)}{\partial \xi^m} = \sum_{j=1}^{N} C_{ij}^{(m)} F(\xi_j, t)
\]

where \( C_{ij}^{(m)} \) is the combination coefficient, whose generation procedure was initially proposed by Bellman et al [13,14] and improved by Shu and Richards [10]. The algorithms for generation of \( C_{ij}^{(m)} \) have been reported by many researchers, e.g. [15]. The distribution of grid points has major influences on the approximation precision of equation (9). Among all types of grid point distribution, the expanded-Chebyshev distribution

\[
\xi_i = -\cos \left( 2(2i-1)\pi/(2N) \right) \sec \left( \pi/(2N) \right), \quad i = 1,2,\ldots,N.
\]

is an excellent approximation to the Lebesgue-optimal grid distribution. Although the present differential quadrature is of Newton-Cotes type, which is less efficient than the Gaussian type or generalized Gaussian rules [16], there is an excellent property that the combination coefficients \( C_{ij}^{(m)} \) are constant and whose generation cost is very low, even without solving a linear system.
When dealing with two-dimensional problems, equation (9) can be extended into a form of

$$\frac{\partial^{mn} F(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \sum_{i=1}^{N} \sum_{j=1}^{M} C^{(m,n)}_{ij}(t) F(\xi_i, \eta_j, t) = \mathbf{N}^{(m,n)}_{ij} \mathbf{F}(t)$$

(11)

where, $M$ and $N$ are the numbers of points projecting to $\eta$ and $\xi$ axes, respectively; $\mathbf{N}^{(m,n)}_{ij}$ may be called the shape function vector at coordinate $(\xi_i, \eta_j)$, and $\mathbf{F}(t)$ is the vector of nodal value or the DOF. For simplicity, $M$ is set with same value as $N$ (see in figure 1) in the present paper.

![Figure 1. Two-dimensional expanded-Chebyshev distribution of grid points.](image)

With the help of equation (9), all the basic unknowns $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $p$ and their partial derivatives with respect to $x$ can be approximated in terms of $\mathbf{N}^{(m,n)}_{ij}$ and vector of DOF.

$$\frac{\partial^{mn} \mathbf{u}^i(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \frac{\partial^{mn} u^i_k(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \\ \frac{\partial^{mn} u^i_l(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \end{array} \right] = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{ax} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{ay} \end{array} \right]$$

(12)

where, $\delta_{ax}$ and $\delta_{ay}$ are the nodal value vectors of $u^i_k$ and $u^i_l$ over all the points $(\xi, \eta)$, respectively; the component arrangements of $\delta_{ax}$ and $\delta_{ay}$ are consistent with $\mathbf{N}^{(m,n)}_{ij}$. Similarly, the following can also be obtained

$$\frac{\partial^{mn} \mathbf{v}^i(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \frac{\partial^{mn} v^i_k(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \\ \frac{\partial^{mn} v^i_l(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \end{array} \right] = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{bx} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{by} \end{array} \right]$$

$$\frac{\partial^{mn} \mathbf{w}^i(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \frac{\partial^{mn} w^i_k(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \\ \frac{\partial^{mn} w^i_l(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} \end{array} \right] = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{cx} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{cy} \end{array} \right]$$

(13)

For the convenience of formulation, the above DOF vectors can be augmented,

$$\frac{\partial^{mn} \mathbf{u}^e(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{ex} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{ey} \end{array} \right], \quad \frac{\partial^{mn} \mathbf{v}^e(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{bx} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{by} \end{array} \right], \quad \frac{\partial^{mn} \mathbf{w}^e(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{cx} \\ \mathbf{N}_{ij}^{(m,n)} \delta_{cy} \end{array} \right]$$

(14)

$$\frac{\partial^{mn} \mathbf{p}(\xi, \eta, t)}{\partial \xi^m \partial \eta^n} = \left[ \begin{array}{c} \mathbf{N}_{ij}^{(m,n)} \delta_{pe} \end{array} \right]$$

(15)

in which, $\delta_e = [\delta_{ax}^T \delta_{ay}^T \delta_{bx}^T \delta_{by}^T \delta_{cx}^T \delta_{cy}^T]^T$; $0$ is a zero row vector with length $N^2$.

Since all the prime unknowns and their partial derivatives are approximated by the DQ technique in $(\xi, \eta)$ coordinate system, the isoparametric coordinate transformation may be applied to transform
coordinate \((\xi, \eta)\) into physical coordinate \((x, y)\) (see in figure 1), and finally it can be obtained that

\[
\frac{\partial^{nn} u^i(x_i, y_j, t)}{\partial x^m \partial y^n} = N_u \delta_{e}^i, \quad \frac{\partial^{nn} v^i(x_i, y_j, t)}{\partial x^m \partial y^n} = N_v \delta_{e}^i.
\]

(16)

\[
\frac{\partial^{nn} w^i(x_i, y_j, t)}{\partial x^m \partial y^n} = N_w \delta_{e}^i, \quad \frac{\partial^{nn} p(x_i, y_j, t)}{\partial x^m \partial y^n} = N_p \delta_{e}^i.
\]

(17)

According to the theory of Galerkin weighted residual value method, all the test functions are approximated with the same shape functions as the basic unknowns,

\[
\frac{\partial^{nn} u^i(x_i, y_j, t)}{\partial x^m \partial y^n} = N_u \delta_{e}^i, \quad \frac{\partial^{nn} v^i(x_i, y_j, t)}{\partial x^m \partial y^n} = N_v \delta_{e}^i.
\]

(18)

where \(\delta_{e}\) is the nodal values of all the test functions in the same arrangement as \(\delta_{e}\).

\[
\delta_{e} = \left[ \begin{array}{cccc} \delta_{e}^x & \delta_{e}^y & \delta_{e}^r \end{array} \right]^T
\]

(19)

2.4. Weak-form DQ discretization

Applying equations (16) and (18) to equation (6), and using the condition that \(\delta_{e}\) is arbitrary in each sub-domain \(\Omega\), it can be finally obtained the weak-form differential quadrature element (WQE) equations

\[
C_{e} \delta_{e} + K_{e} \delta_{e} = R_{e}
\]

(20)

where,

\[
C_{e} = \int_{\Omega} \left[ N_u^T \left( \rho^e N_u + \rho N_s \right) + \left( \rho^e N_u + \beta \rho^e \nabla N_p \right)^T \left( N_e + N_u \right) + N_e^T N_u \right] d\Omega
\]

(21)

\[
K_{e} = \int_{\Omega} \left( G N_u \right)^T D G N_u - \left( r_1 \frac{\partial N_u}{\partial x} + r_2 \frac{\partial N_u}{\partial y} \right) N_p + N_p^T \left( s_{\nu} N_u + n^\nu \nabla N_p \right)
\]

\[
+ N_p^T \left( r_1 \frac{\partial N_u}{\partial x} + r_2 \frac{\partial N_u}{\partial y} \right) + \beta \left( \frac{\partial N_p^T}{\partial x} + \frac{\partial N_p^T}{\partial y} \right) - N_p^T N_e \right] d\Omega
\]

(22)

\[
R_{e} = \int_{\Omega} \left[ N_u^T \rho b + N_u^T \rho^e b + \left( \frac{\partial N_p}{\partial x} + \frac{\partial N_p}{\partial y} \right)^T \rho^e \beta b \right] d\Omega + \int_{s_r} N_e^T T dS - \int_{s_q} N_e^T Q dS
\]

(23)

Herein, \(r_1 = [1 \ 0]\), \(r_2 = [0 \ 1]\); \(G\) is a differential operator for strain; \(D\) is a elasticity constants matrix. \(G\) and \(D\) have expressions as follows
It can be observed that equations (21), (22) and (23) cannot be evaluated through numerical quadrature such as the Gauss-Legendre quadrature for the moment, because the integrand there can only be acquired in the two-dimensional expanded-Chebyshev points. To cope with this, [17] and [15] have already presented solutions to this problem by using the Lagrange interpolation. As a result, the Gauss-Legendre quadrature can be used in the present numerical program, where only the inner and constant sum of points of each element may be used. This is different from the high efficient procedures of the conventional FEM, whose quadrature point sampling spans many elements, and the quadrature points number in each element varies. After assembly of equation (20) and other procedures of the conventional FEM the first-order initial value problem is solved by the implicit Euler method step by step.

3. Numerical verification
According to the above formulations, the numerical programs are implemented using Julia language on a PC, which configures: OS Windows 10 64 bit, RAM 16 GB and CPU intel i7 8700k. A full-band sparse matrix storage is used for the assembly of element matrices, and the gaussian factorisation method is used for solution of linear equation system.

3.1. Numerical example 1
There is a half-space foundation consisting of saturated-soil. And its top surface is subjected to a uniform pressure \( q(t) = 3000[1 - \cos(75 \ t)] \). In this case, the three-dimensional problem degrades into a one-dimensional one. And de Boer et al [20] had contributed the analytical solutions to this problem. In this numerical example, the uniform mesh shown in figure 2 is used to represent the one-dimensional case approximately, where 50 four-node elements with side length 0.2 m are used; the time step length is fixed 0.1 ms; all the physical parameters are also listed in figure 2.

![Figure 2. Physical, geometric and meshing parameters together with boundary conditions.](image)

![Figure 3. Verification for solutions of pore pressure.](image)

Figure 3 illustrates the pore pressure time-history at depth from ground top 0.4 m, 1.0 m and 6.0 m. There an excellent agreement of pore pressure results is observed, which to some extent verifies the present formulations and numerical program.
3.2. Numerical example 2
The same physical parameters as numerical example 1 are used in this example, except that the Darcy permeability coefficient herein is modified as $k^F = 0.0001$ m/s. A square fluid-saturated soil block with side length 10 m, shown in figure 4, is analyzed, where four uniform five-order WQEs are used to discretized the whole block, and the detailed displacement, water flux and stress boundary conditions are marked in the figure. A fixed time length 0.01 s is used in the implicit Euler method.

Figure 5 shows the vertical displacement at the two top-surface corners, $p_1$ and $p_2$ (see in figure 4). It can be seen in figure 5 that in the very beginning of the loading the rise of $p_1$ and the sink of $p_2$ almost coincide with each other due to the low-permeability and incompressibility of the saturated soil, while the rise and sink diverge when more time elapses, due to the seepage from the drainage surface. The good agreement of results generated by the WQEM and FEM of [7] again verifies the work of the present paper.

4. Numerical performances

4.1. Numerical convergence
Keeping all the physical and geometric parameters unchanged except the Darcy permeability coefficient $k^F = 0.01$ m/s, the saturated-soil block analyzed by section 3.2 is re-analyzed. There are two factors, the time step-length and the mesh size, controlling the convergence of analysis results. To investigate these factors, the vertical displacement of the central point $p_3$ (see in figure 4) in time $t = 0.1$ s is computed. Considering three types of uniform quadrilateral mesh 64x64, 16x16 and 6x6, which respectively denote element sizes 10/64 m, 10/16 m and 10/6 m, figure 6 shows the convergence course of vertical displacement at $p_3$ as the number of fixed time-length increases. It can be seen, from figure 6, clearly that the vertical displacement converges stably for soil block with different meshing densities.
Figure 6. Influence of time-integration on convergence of vertical displacement at $p_3$.

From figure 6, it can be judged that a fixed time step-length 0.2 ms may provide time integration with enough precision. As a result, figure 7 is further presented, setting the time-step length fixed 0.2 ms, to investigate the convergence efficiency of WQEM. Herein, the soil-block is meshed uniformly. The WQE model is discretized by only four WQEs, whose system DOF is adjusted through changing the number of expanded-Chebyshev grid points each element. In the FE model, the system DOF is adjusted by varying the meshing density. It is observed from figure 7 that there is much less requirement of system DOF for the WQEM than the conventional FEM to pursue a certain accuracy goal. The CPU time spent by the FEM is 2857.960 s for an accuracy of 0.01 mm, while it takes 58.5% less CPU time in WQEM for the same accuracy.

4.2. Stress smoothness

A foundation pit shown in figure 8 is analyzed, whose physical parameters are the same as those used in section 3.1. The geometry information is also illustrated in figure 8. As the structure is vertically symmetric, half structure is considered and meshed with two types of mesh grids (see in figure 9). In mesh 0 and mesh 1, there are 16 six-order WQEs and 736 conventional FEs used, respectively; the half structure is discretized into a system with DOFs 4209 in WQE model and 5349 in FE model, respectively. A fixed time step-length 0.5 ms is used in the implicit time integration.

Figure 8. Dimensions and boundary conditions of a foundation pit.

Figure 9. Mesh grid for (a) mesh0: higher order QEM model, (b) mesh 1: conventional FEM model.
Figure 10. Effective stress $\sigma_x$, using (a) six-order QEM and (b) conventional FEM.

Figure 11. Effective stress $\tau_{xy}$, using (a) six-order QEM and (b) conventional FEM.

The effective stresses responses, $\sigma_x$ and $\tau_{xy}$, are captured by the two models in the moment when time reaches 0.1 s, and are showed in figures 10 and 11. In the figures, no numerical smoothness technique is applied to the nodes connecting neighboring elements. And it can be observed that a serious discontinuity of stress occurs in the FEM model, while it is not a problem in the QEM model. This also demonstrates why it is usually required of numerical smoothness in some of the commercial FE software.

5. Conclusions
Taking advantage of the DQ technique, a WQE for the dynamic analyses of two-dimensional fluid-saturated soil is developed. The formulations and computer programs are verified through two comparison examples with the previous results. Convergence analyses for both time and space discretization are performed, from which it is found that the proposed WQEM is able to converge at much less cost of system DOFs than the conventional FEM; in the stress analyses it is demonstrated that the presented WQEM dramatically alleviates the smoothness problem of the conventional displacement-based FEM.

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