A Cayley-Hamilton trace identity for $2 \times 2$ matrices over Lie-solvable rings

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Abstract. We exhibit a Cayley-Hamilton trace identity for $2 \times 2$ matrices with entries in a ring $R$ satisfying $[[x, y], [x, z]] = 0$ and $\frac{1}{2} \in R$.

1. INTRODUCTION

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field $K$ (see [2] and [3]). In case of $\text{char}(K) = 0$, Kemer’s pioneering work (see [5]) on the $T$-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = K \langle v_1, v_2, ..., v_r, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$.

For $n \times n$ matrices over a Lie-nilpotent ring $R$ satisfying the polynomial identity

$$[[[...[[x_1, x_2], x_3], ...], x_m], x_{m+1}] = 0$$

(with $[x, y] = xy - yx$), a Cayley-Hamilton identity of degree $n^m$ (with left- or right-sided scalar coefficients) was found in [6]. Since $E$ is Lie-nilpotent with $m = 2$, the above mentioned Cayley-Hamilton identity for a matrix $A \in M_n(E)$ is of degree $n^2$.

In [1] Domokos presented a slightly modified version of this identity in which the coefficients are invariant under the conjugate action of $GL_n(K)$. For a matrix $A \in M_2(E)$ he obtained the trace identity

$$A^4 - 2\text{tr}(A)A^3 + (2\text{tr}^2(A) - \text{tr}(A^2))A^2 + \left(\frac{1}{2}\text{tr}(A)\text{tr}(A^2) + \frac{1}{2}\text{tr}(A^2)\text{tr}(A) - \text{tr}^3(A)\right)A +$$
\[ \frac{1}{4}(\text{tr}^4(A) + \text{tr}^2(A^2) - \frac{5}{2}\text{tr}^2(A)\text{tr}(A^2) + \frac{1}{2}\text{tr}(A^2)\text{tr}^2(A) - 2\text{tr}(A^3)\text{tr}(A) + 2\text{tr}(A)\text{tr}(A^3))I = 0, \]

where \( I \) is the identity matrix and \( \text{tr}(A) \) denotes the trace of \( A \). A similar identity with right coefficients also holds for \( A \). Here \( E \) can be replaced by any ring \( R \) which is Lie-nilpotent of index 2.

The identity \([x, y][x, z] = 0\) is a consequence of Lie-nilpotency of index 2 (see [4]), as is obviously \([x, y], [x, z] = 0\). The first aim of the present paper is to provide an example of an algebra satisfying \([x, y], [u, v] = 0\), but neither \([x, y][u, v] = 0\) nor \([(x, y), z] = 0\). Since the above mentioned trace identity cannot be used for matrices over such an algebra, our second purpose is to exhibit a new trace identity of the same kind (of degree 4 in \( A \)) for a matrix \( A \) in \( M_2(R) \), where \( R \) is any ring satisfying the identity

\[ [(x, y), [x, z]] = 0 \]

and \( \frac{1}{2} \in R \). We note that a ring satisfying \([x, y], [u, v] = 0\) is called Lie-solvable of index 2.

From now onward \( R \) and \( S \) are rings with 1. In Section 2 we consider the ring \( U_3^*(R) \) of upper triangular \( 3 \times 3 \) matrices with equal diagonal entries over \( R \). First we observe that \( U_3^*(R) \) is never commutative. We prove that if \( R \) is commutative then the algebra \( U_3^*(R) \) satisfies the identities \([x, y][u, v] = 0\) and \([x, y], z = 0\). However, for a non-commutative \( R \) we show that the ring \( U_3^*(R) \) never satisfies any of the identities \([x, y][u, v] = 0\) and \([(x, y), z] = 0\).

The main result in Section 2 states that if \( S \) satisfies the identities \([x, y][u, v] = 0\) and \([(x, y), z = 0\), then the matrix ring \( U_3^*(S) \) is Lie-solvable of index 2. It follows that if \( R \) is commutative, then \( U_3^*(U_3^*(R)) \) is an example of an algebra satisfying \([x, y], [u, v] = 0\), but neither \([x, y][u, v] = 0\) nor \([(x, y), z] = 0\).

Section 3 is entirely devoted to the construction of our Cayley-Hamilton trace identity.

### 2. A PARTICULAR LIE-SOLVABLE MATRIX ALGEBRA

Since

\[ E_{1,2}, E_{2,3} \in U_3^*(R) = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in R \right\} \]

and \( E_{1,2}E_{2,3} = E_{1,3} \neq 0 = E_{2,3}E_{1,2} \), the ring \( U_3^*(R) \) is never commutative. Any element of \( U_3^*(R) \) can be written as \( aI + X \), where \( X \) is strictly upper triangular. We note that \( XYZ = 0 \) for strictly upper triangular \( 3 \times 3 \) matrices. If \( R \) is commutative, then \( aI \) is central in \( U_3^*(R) \) (of course, also in \( M_3(R) \)), \([aI + X, bI + Y] = [X, Y] \) for all \( a, b \in R \) and so \( U_3^*(R) \) satisfies all polynomial identities in which each summand is a product of certain (possibly iterated) commutators. For example,

\[ [x, y][u, v] = 0 \text{ and } [(x, y), z] = 0 \]

are typical such identities for \( U_3^*(R) \). If \( R \) is non-commutative, say \([r, s] \neq 0\) for some \( r, s \in R \), then for \( x = rI, y = sE_{1,2}, u = E_{2,2}, v = z = E_{2,3} \in U_3^*(R) \) we have

\[ [x, y][u, v] = [(x, y), z] = [r, s]E_{1,3} \neq 0. \]

#### 2.1. Theorem. If \( S \) satisfies \([x, y][u, v] = 0 \text{ and } [(x, y), z] = 0\), then \( U_3^*(S) \) satisfies \([x, y], [u, v] = 0\).
3. MATRICES WITH COMMUTATOR ENTRIES

3.1. Proposition. Let \([a, e] = [a, f] + [b, e] = [a, g] + [c, e] + (b h - f d)\) where \(a, e\) is in \(C\) and \(\alpha E\) is central in \(C\), hence \([a, e]I\) is central in \(U_3^e(S)\) (also in \(M_3(S)\)). Thus we have \([x, y, [u, v]] = ([a, e]I + C + \alpha E_{1,3}, [a', e']I + C' + \alpha' E_{1,3}] = [C + \alpha E_{1,3}, C' + \alpha' E_{1,3}] = 0\) because of \((C + \alpha E_{1,3})(C' + \alpha' E_{1,3}) = (C' + \alpha' E_{1,3})(C + \alpha E_{1,3}) = 0\). Indeed, \(CC' = C' C = 0\) is a consequence of \(C, C' \in M_3([S, S])\) and of \([x, y][u, v] = 0\) in \(S\), and \(E_{1,3} = E_{1,3} C = C' E_{1,3} = E_{1,3} C' = 0\) follows from the fact that \(C\) and \(C'\) are strictly upper triangular.\(\square\)

2.2. Corollary. If \(R\) is commutative, then the algebra \(U_3^e(U_3^e(R))\) satisfies \([x, y, [u, v]] = 0\), but neither \([x, y][u, v] = 0\) nor \([x, y, [u, v]] = 0\). The following can be considered as the “real” \(2 \times 2\) Cayley-Hamilton trace identity.

3.1. Proposition. If \(\frac{1}{2} \in R\) and \(A = [a_{ij}] \in M_2(R)\), then
\[
A^2 - \text{tr}(A)A + \frac{1}{2} \text{tr}(A^2) - \text{tr}(A^2)I = \begin{pmatrix}
\frac{1}{2}a_{11}, a_{22} + \frac{1}{2}a_{12}, a_{21} \\
\frac{1}{2}a_{21}, a_{12} - \frac{1}{2}a_{22}, a_{11}
\end{pmatrix} - \begin{pmatrix}
a_{12}, a_{22} \\
a_{21}, a_{12}
\end{pmatrix}.
\]

Proof. A straightforward computation suffices.\(\square\)

3.2. Corollary. If \(\frac{1}{2} \in R\) and \(B = [b_{ij}] \in M_2(R)\) with \(\text{tr}(B) = 0\), then
\[
B^2 - \frac{1}{2} \text{tr}(B^2)I = \begin{pmatrix}
\frac{1}{2}b_{11}, b_{22} - b_{21}, b_{12} \\
b_{21}, b_{12} - \frac{1}{2}b_{22}, b_{11}
\end{pmatrix}.
\]

Proof. Since \(b_{22} = -b_{11}\), we have \([b_{11}, b_{22}] = 0\) and \([b_{12}, b_{22}] = -[b_{12}, b_{11}]\). Thus the formula in Proposition 3.1 immediately gives the identity for \(B.\square\)

3.3. Theorem. If \(\frac{1}{2} \in R\) and \(R\) satisfies \([x, y, [x, z]] = 0\), then
\[
\left(C^2 - \frac{1}{2} \text{tr}(C^2)I\right)^2 - \frac{1}{2} \text{tr} \left((C^2 - \frac{1}{2} \text{tr}(C^2)I)^2\right) I = 0
\]
for all \(C \in M_2(R)\) with \(\text{tr}(C) = 0\).

Proof. Take \(C = [c_{ij}]\). In view of Corollary 3.2 we have
\[
C^2 - \frac{1}{2} \text{tr}(C^2)I = \begin{pmatrix}
\frac{1}{2}c_{11}, c_{22} - c_{21}, c_{12} \\
c_{21}, c_{12} - \frac{1}{2}c_{22}, c_{11}
\end{pmatrix}.
\]
Thus we have
\[ C \neq \frac{1}{2} \text{tr}(C^2)I \]
and
\[ 3.6. \text{Corollary.} \quad \text{If} \; C \neq \frac{1}{2} \text{tr}(C^2)I \]
gives that
\[ \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 - \frac{1}{2} \text{tr} \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 I = \]
\[ = \frac{1}{2} \begin{bmatrix} -[[c_{12}, c_{11}], [c_{21}, c_{11}]] & [[c_{12}, c_{11}], [c_{12}, c_{21}]] \\ [[c_{21}, c_{11}], [c_{12}, c_{21}]] & [[c_{21}, c_{11}], [c_{21}, c_{11}]] \end{bmatrix}. \]
Now we have
\[ [[c_{12}, c_{11}], [c_{21}, c_{11}]] = [[c_{11}, c_{12}], [c_{11}, c_{21}]] \]
and
\[ [[c_{21}, c_{11}], [c_{12}, c_{21}]] = -[[c_{21}, c_{11}], [c_{21}, c_{12}]]. \]
Thus each entry of the above \( 2 \times 2 \) matrix is of the form \( \pm[[x, y], [x, z]] = 0 \) and the desired identity follows. □

In Corollaries 3.4 - 3.6 we assume that \( \frac{1}{2} \in R \) and \( R \) satisfies \( [[x, y], [x, z]] = 0 \).

3.4. Corollary. If \( C \in M_2(R) \) with \( \text{tr}(C) = 0 \), then
\[ C^4 - \frac{1}{2} \text{tr}(C^2)C^2 - \frac{1}{2}C^2\text{tr}(C^2) + \frac{1}{2} \left( \text{tr}^2(C^2) - \text{tr}(C^4) \right) I = 0. \]

Proof. Clearly,
\[ \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 = C^4 - \frac{1}{2} \text{tr}(C^2)C^2 - \frac{1}{2}C^2\text{tr}(C^2) + \frac{1}{4} \text{tr}^2(C^2)I \]
and
\[ \text{tr} \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 = \text{tr}(C^4) - \frac{1}{2} \text{tr}(\text{tr}(C^2)C^2) - \frac{1}{2} \text{tr}(C^2\text{tr}(C^2)) + \frac{1}{4} \text{tr}(\text{tr}^2(C^2)I) = \]
\[ = \text{tr}(C^4) - \frac{1}{2} \text{tr}^2(C^2) - \frac{1}{2} \text{tr}^2(C^2) + \frac{1}{4} \text{tr}^2(C^2) = \text{tr}(C^4) - \frac{1}{2} \text{tr}^2(C^2). \]
Thus we have
\[ \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 - \frac{1}{2} \text{tr} \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 I = \]
\[ = C^4 - \frac{1}{2} \text{tr}(C^2)C^2 - \frac{1}{2}C^2\text{tr}(C^2) + \frac{1}{4} \text{tr}^2(C^2)I - \frac{1}{2} \left( \text{tr}(C^4) - \frac{1}{2} \text{tr}^2(C^2) \right) I = \]
\[ = C^4 - \frac{1}{2} \text{tr}(C^2)C^2 - \frac{1}{2}C^2\text{tr}(C^2) + \frac{1}{2} \left( \text{tr}^2(C^2) - \text{tr}(C^4) \right) I. \]
□

3.5. Corollary. If \( C \in M_2(R) \) with \( \text{tr}(C) = \text{tr}(C^2) = \text{tr}(C^4) = 0 \), then \( C^4 = 0 \).

3.6. Corollary. If \( A \in M_2(R) \) is arbitrary, then
\[ \left( A - \frac{1}{2} \text{tr}(A)I \right)^4 - \frac{1}{2} \text{tr} \left( A - \frac{1}{2} \text{tr}(A)I \right)^2 \left( A - \frac{1}{2} \text{tr}(A)I \right)^2 - \]
\[ = \frac{1}{2} \left( A - \frac{1}{2} \text{tr}(A)I \right)^2 \text{tr} \left( A - \frac{1}{2} \text{tr}(A)I \right)^2 + \]
\[ = \frac{1}{2} \left( \text{tr}^2 \left( A - \frac{1}{2} \text{tr}(A)I \right)^2 - \text{tr} \left( A - \frac{1}{2} \text{tr}(A)I \right)^4 \right) I = 0. \]
Proof. Take $C = A - \frac{1}{2} \text{tr}(A)I$, then $\text{tr}(C) = \text{tr}(A - \frac{1}{2} \text{tr}(A)I) = 0$ and the application of Corollary 3.4 gives the identity. $\Box$

3.7. Theorem. If $\frac{1}{2} \in R$ and $R$ is a Lie-solvable ring satisfying $[[x, y], [x, z]] = 0$, then for all $A \in M_2(R)$ we have

$$A^4 - \frac{1}{2} A^2 \text{tr}(A)A - \frac{1}{2} \text{Atr}(A)A^2 - \frac{1}{2} A^3 \text{tr}(A) - \frac{1}{2} \text{tr}(A)A^2 + \frac{1}{2} A^2 \text{tr}^2(A) + \frac{1}{2} \text{tr}^2(A)A^2 -$$

$$\frac{1}{2} A^2 \text{tr}(A^2) - \frac{1}{2} \text{tr}(A^2)A^2 + \frac{1}{4} \text{Atr}(A)\text{tr}(A)A + \frac{1}{4} \text{tr}(A)\text{Atr}(A)A +$$

$$\frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{Atr}^2(A)A - \frac{1}{4} \text{tr}(A)\text{Atr}^2(A) - \frac{1}{4} \text{tr}^2(A)\text{Atr}(A) +$$

$$\frac{1}{4} \text{tr}(A)\text{Atr}^2(A) + \frac{1}{4} \text{tr}(A^2)\text{Atr}(A) - \frac{1}{4} \text{Atr}^3(A) - \frac{1}{4} \text{tr}^3(A)A +$$

$$\frac{1}{4} \text{Atr}(A)\text{tr}(A^2) + \frac{1}{4} \text{tr}(A^2)\text{tr}(A)A - \frac{1}{2} \text{tr}(A^2)\text{tr}^2(A)I + \frac{1}{2} \text{tr}(A^2)\text{tr}^2(A)I +$$

$$\frac{1}{2} \text{tr}^2(A^2)I + \frac{1}{4} \text{tr}(A^2)\text{tr}(A)A + \frac{1}{4} \text{tr}(A)\text{tr}(A^2)I + \frac{1}{4} \text{tr}(A^3)\text{tr}(A)I + \frac{1}{4} \text{tr}(A)\text{tr}(A^3)I -$$

$$\frac{1}{8} \text{tr}(A)\text{tr}(A)\text{tr}(A)I - \frac{1}{8} \text{tr}(A)\text{tr}(A)\text{tr}(A)I - \frac{1}{8} \text{tr}(A)\text{tr}(A)\text{tr}(A)I -$$

$$\frac{1}{2} \text{tr}^4(A)I - \frac{1}{2} \text{tr}(A^4)I = 0.$$

Proof. Take $C = A - \frac{1}{2} \text{tr}(A)I$, then $\text{tr}(C) = \text{tr}(A - \frac{1}{2} \text{tr}(A)I) = 0$. We have

$$C^2 - \frac{1}{2} \text{tr}(C^2)I = (A - \frac{1}{2} \text{tr}(A)I)^2 - \frac{1}{2} \text{tr}\left( (A - \frac{1}{2} \text{tr}(A)I)^2 \right) I =$$

$$A^2 - \frac{1}{2} \text{tr}(A^2)A - \frac{1}{2} \text{tr}(A)A - \frac{1}{2} \text{Atr}(A)A - \frac{1}{2} \text{Atr}(A)A - \frac{1}{2} \text{tr}(A)A - \frac{1}{2} \text{Atr}(A)A -$$

$$\frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) +$$

$$\frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) +$$

and

$$\left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 = \left( A^2 - \frac{1}{2} \text{tr}(A)A - \frac{1}{2} \text{Atr}(A)A + \frac{1}{2} \text{tr}(A)I - \frac{1}{2} \text{tr}(A^2)I \right)^2 =$$

$$A^4 - \frac{1}{2} A^2 \text{tr}(A)A - \frac{1}{2} A^3 \text{tr}(A) - \frac{1}{2} A^2 \text{tr}^2(A) - \frac{1}{2} \text{tr}(A)A^3 +$$

$$\frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) +$$

$$\frac{1}{2} \text{tr}(A^2)A^2 - \frac{1}{4} \text{tr}(A^2)A^2 - \frac{1}{4} \text{tr}(A^2)A^2 - \frac{1}{4} \text{tr}(A^2)A^2 -$$

and we obtain that

$$\text{tr}\left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 = \text{tr}(A^4) - \frac{1}{2} \text{tr}(A^2)\text{tr}(A)A - \frac{1}{2} \text{tr}(A^3)\text{tr}(A) +$$
\[
\frac{1}{2} \text{tr}(A^2)\text{tr}^2(A) - \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) - \frac{1}{2} \text{tr}(A)\text{tr}(A^3) + \\
\frac{1}{4} \text{tr}(A)\text{tr}(\text{tr}(A)A) + \frac{1}{4} \text{tr}(A)\text{tr}(A^2)\text{tr}(A) - \\
\frac{1}{4} \text{tr}(A)\text{tr}(\text{tr}(A)A) + \frac{1}{4} \text{tr}(A)\text{tr}(A^2)\text{tr}(A) + \frac{1}{4} \text{tr}(\text{tr}(A^2)A) + \\
\frac{1}{4} \text{tr}(\text{tr}(A^2)A) + \frac{1}{4} \text{tr}(A)\text{tr}(A^2)\text{tr}(A) + \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) - \\
\frac{1}{4} \text{tr}^3(A)\text{tr}(A) - \frac{1}{4} \text{tr}^2(A)\text{tr}(A)\text{tr}(A) + \\
\frac{1}{2} \text{tr}^4(A) - \frac{1}{2} \text{tr}^2(A)\text{tr}(A^2) - \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) + \\
\frac{1}{4} \text{tr}(A^2)\text{tr}(A)\text{tr}(A) + \frac{1}{4} \text{tr}(A^2)\text{tr}(A)\text{tr}(A) - \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) + \frac{1}{2} \text{tr}^2(A^2) = \\
= \text{tr}(A^4) - \frac{1}{2} \text{tr}(A^2\text{tr}(A)A) - \frac{1}{2} \text{tr}(\text{tr}(A)A^2) - \\
\frac{1}{2} \text{tr}(A^3)\text{tr}(A) - \frac{1}{2} \text{tr}(A)\text{tr}(A^3) + \frac{1}{4} \text{tr}(A)\text{tr}(\text{tr}(A)A) + \frac{1}{4} \text{tr}(\text{tr}(A^2)A) + \\
\frac{1}{4} \text{tr}(\text{tr}(A^2)A) + \frac{1}{4} \text{tr}(A)\text{tr}(A^2)\text{tr}(A) + \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) + \frac{1}{2} \text{tr}(A^2)\text{tr}(A^2) - \\
\frac{1}{2} \text{tr}^2(A^2) - \frac{1}{2} \text{tr}^4(A).
\]

Now the calculation of
\[
\left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 - \frac{1}{2} \text{tr} \left( \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 \right) I
\]
and the application of Theorem 3.3 yield the identity. □

**Question.** Throughout this section we have used the identity \([x, y, [x, z]] = 0\). We do not know whether this identity implies the “seemingly” stronger identity \([[x, y], [u, v]] = 0\) which plays an important role in Section 2.

Starting with a matrix \(C \in M_2(R)\) such that \(\text{tr}(C) = 0\), define the sequence \((C_k)_{k \geq 0}\) by the following recursion: \(C_0 = C\) and
\[
C_{k+1} = C_k^2 - \frac{1}{2} \text{tr}(C_k^2)I.
\]
Clearly, \(\text{tr}(C_k) = 0\) for all \(k \geq 0\) and \(C_k\) is a trace polynomial expression of \(C\). In view of Corollary 3.2, the entries of \(C_1\) are of the form \([x_1, x_2]\). The repeated application of Corollary 3.2 (as it can be seen in the proof of Theorem 3.3) and a straightforward induction show that the (four) entries of \(C_k\) are all of the form \([x_1, x_2, ..., x_2^k]_{\text{solv}}\), where \([x_1, x_2]_{\text{solv}} = [x_1, x_2]\) and for \(i \geq 1\) we take the Lie brackets as
\[
[x_1, x_2, ..., x_2^{i+1}]_{\text{solv}} = [[x_1, x_2, ..., x_2^i]_{\text{solv}}, [x_2^{i+1}, x_2^{i+2}, ..., x_2^{i+2}]_{\text{solv}}].
\]
If \(R\) satisfies the general identity
\[
[x_1, x_2, ..., x_2^k]_{\text{solv}} = 0
\]
of Lie solvability, then \(C_k = 0\), whence we can derive a trace identity for \(C\). Thus the substitution \(C = A - \frac{1}{2} \text{tr}(A)I\) gives a trace identity for an arbitrary \(A \in M_2(R)\).
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