Order parameters and color-flavor center symmetry in QCD

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Common lore suggests that N-color QCD with massive quarks has no useful order parameters which can be non-trivial at zero baryon density. However, such order parameters do exist when there are nt quark flavors with a common mass and d \equiv \gcd(nt, N) > 1. These theories have a Zd color-flavor center symmetry arising from intertwined color center transformations and cyclic flavor permutations. The symmetry realization depends on the temperature, baryon chemical potential and value of nt/N, with implications for conformal window studies and dense quark matter.

Introduction. Defining order parameters in QCD is notoriously subtle. In pure SU(N) Yang-Mills (YM) theory, the simplest non-trivial order parameter is the expectation value of a line operator:
\[ \langle \text{tr} \Omega \rangle = \langle \text{tr} \mathcal{P} e^{i \int_0^L dx_1 A_1} \rangle, \]

when the x1 dimension is compactified with circumference L. If x1 is regarded as Euclidean time, then the gauge theory functional integral with periodic boundary conditions calculates the thermal partition function with temperature T = 1/L. The thermal expectation value (1) is the Polyakov loop confinement order parameter for temperature T conditions calculates the thermal partition function with

The key observation is that if
\[ U = \text{diag}(\nu, \nu^2, \ldots, \nu^{n_t-1}), \quad \nu = e^{2\pi i/m}, \]

or
\[ U = \text{diag}(\nu^{1/2}, \nu^{3/2}, \ldots, \nu^{n_t-1/2}). \]

With the BCs in (4), the finite L flavor symmetry is reduced to G_L = U(1)^{n_t-1} × U(1)\backslash N_3 × U(1)Q ⊂ G. The key observation is that if
\[ d \equiv \gcd(nt, N) > 1, \]

then the circle-compactified theory, with either boundary condition (4), also remains invariant under an intertwined Zd ⊂ ZN × Zp×m color-flavor center (CFC) symmetry, generated by the combination of a center transformation with phase ωN/d = e^{2\pi i/d} and a Zd cyclic flavor permutation. To see this note that, given either choice (4), a Zd center transformation effectively permutes the eigenvalues of U. Combining the center transformation with the opposite cyclic flavor permutation (which is part
of the \( U(n_t) \) flavor symmetry) leaves the boundary condition invariant.

CFC symmetry intertwines center and flavor transformations and so has both local and extended order parameters. Examples of CFC order parameters include Polyakov loops such as (1) with winding numbers which are non-zero mod \( d \), as well as \( Z_{n_t} \) Fourier transforms of fermion bilinears, \( \mathcal{O}_\Gamma^{(p)} = \sum_{a=1}^{n_t} \nu^{-ap} \bar{q}_a \Gamma q_a \), where \( \Gamma \) is an arbitrary Dirac matrix and \( p \) mod \( d \neq 0 \). The action of the \( Z_d \) CFC symmetry is

\[
\text{tr} \Gamma^p \rightarrow \omega^{np/d} \text{tr} \Gamma^p, \quad \mathcal{O}_\Gamma^{(p)} \rightarrow \nu^{np/d} \mathcal{O}_\Gamma^{(p)}. \tag{6}
\]

Other related choices of boundary conditions, and generalizations to multiple compactified directions are discussed in our Supplemental Materials.

**Center symmetry and confinement.** Consider the Polyakov loop connected correlator in QCD compactified on \( x_1 \) with circumference \( L \),

\[
\langle \text{tr} \Omega(x) \text{tr} \Omega^\dagger(0) \rangle_{\text{conn}} \equiv e^{-F(r)}, \quad r = |x| \tag{7}
\]

Suppose there is a non-zero lower bound \( E \) on the energy of states that can contribute to the correlator, so \( F(r) \sim Er \) as \( r \to \infty \). When \( n_t = 0 \), the theory has a \( Z_N \) center symmetry. If the ground state is \( Z_N \) invariant, then no intermediate state created by a local operator acting on the vacuum can contribute to the correlator. All contributions to the correlator (7) must involve flux tubes which wrap the compactified dimension, so that \( E = L\sigma \) with \( \sigma \) the string tension.

On the other hand, if center symmetry is broken, explicitly or spontaneously, then intermediate states created by local operators can also contribute to the correlator (7). The minimal energy \( E \) need not grow with \( L \). This is interpreted as a signal of string breaking. It is tempting to conclude that there is a tight link between unbroken center symmetry and confinement of static test quarks by unbreakable flux tubes.

Now suppose that \( d = \gcd(n_t, N) > 1 \), all quarks have a common mass \( m_q \), and we engineer the existence of \( Z_d \) CFC symmetry by using the BCS (4). As seen above, CFC symmetry acts on both Polyakov loops and appropriate local operators. Intermediate states created by local operators transforming the same as \( \text{tr} \Omega \) under all unbroken symmetries can contribute to the correlator (7). For example, states created by \( \mathcal{A} \equiv \sum_{a=1}^{n_t} \nu^{-ap} \bar{q}_a \gamma_1 D_1 q_a \) and \( \mathcal{B} \equiv \sum_{a=1}^{n_t} \nu^{-ap} \bar{q}_a \gamma_1 \gamma_4 q_a \) can contribute to correlators of \( \text{Re} \text{tr} \Omega^p \) and \( \text{Im} \text{tr} \Omega^p \), respectively, even when CFC symmetry is not spontaneously broken. Consequently, the string tension as defined by the asymptotic behavior of the correlator (7) vanishes regardless of the realization of the \( Z_d \) center symmetry. Of course, the minimal masses of mesons created by operators \( \mathcal{A} \) or \( \mathcal{B} \) grow with increasing quark mass \( m_q \). Due to non-uniformity in the \( m_q \to \infty \) and \( r \to \infty \) limits, a non-zero string tension does emerge if one sends \( m_q \to \infty \) first. In summary, we see that for \( n_t > 0 \) there is no relation between the presence of a non-zero string tension and the existence, or realization, of a center symmetry intertwined with flavor.

**Conformal window.** Let \( x \equiv n_t/N \), and set to zero the common quark mass and temperature, \( m_q = T = 0 \). If \( x > \frac{11}{12} \), QCD becomes an infrared-free theory. For \( x \) below some \( x_\chi < \frac{11}{12} \), perturbation theory self-consistently implies the existence of an IR fixed point with a parametrically small coupling \( \alpha \). The value of \( x_\chi \) has been the subject of intensive lattice investigations (see, e.g., Refs. [15–24]). The existence of an IR-conformal phase can be seen most easily in the Veneziano large \( N \) limit of QCD, where \( x \) is fixed along with the 't Hooft coupling \( \lambda \equiv g^2 N \) as \( N \) increases. If \( \epsilon \equiv \frac{11}{12} - x \to 0^+ \), perturbation theory self-consistently implies the existence of an IR fixed point with a parametrically small coupling \( \alpha \). One may show that \( Z_d \) CFC symmetry is spontaneously broken in the conformal window, at least at large \( N \). The Veneziano limit is taken through a sequence of values \( x \in (x_\chi, \frac{11}{12}) \), called the conformal window, QCD flows to a non-trivial infrared (IR) fixed point without chiral symmetry breaking. The value of \( x_\chi \) has been the subject of intensive lattice investigations (see, e.g., Refs. [15–24]). The existence of an IR-conformal phase can be seen most easily in the Veneziano large \( N \) limit of QCD, where \( x \) is fixed along with the 't Hooft coupling \( \lambda \equiv g^2 N \) as \( N \) increases. If \( \epsilon \equiv \frac{11}{12} - x \to 0^+ \), perturbation theory self-consistently implies the existence of an IR fixed point with a parametrically small coupling \( \alpha \). Hence, the Polyakov loop (1) remains an order parameter for the intertwined \( Z_d \) center symmetry.

The CFC realization may be determined by computing the quantum effective potential \( V_{\text{eff}}(\Omega) \). The loop expansion is controlled by the small value of \( \lambda \) at all scales when \( \epsilon \ll 1 \), rather than the small size of \( L \) compared to the inverse strong scale \( \Lambda^{-1} \) as in the classic papers [14, 27]. Hence, the following analysis is valid for any circumference \( L \), including the \( L \to \infty \) limit of interest.

Classically, \( V_{\text{eff}}(\Omega) \) is zero. Using standard methods [14, 27], at one loop one finds \( V_{\text{eff}}(\Omega) = V_\epsilon(\Omega) + V_\lambda(\Omega) \)
with gluon and fermion contributions given by

\[ V_g(\Omega) = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left( |\text{tr} \Omega|^2 - 1 \right), \tag{8} \]

and

\[ V_{\ell}(\Omega) = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left( \text{tr} U^{-n} \text{tr} \Omega^n + \text{tr} U^n \text{tr} \Omega^{-n} \right) \]

\[ = \frac{2}{\pi^2 L^4 m_3^4} \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^4} \left( \text{tr} \Omega^{n+n} + \text{h.c.} \right). \tag{9} \]

The upper/lower sign refers to BCs (4a)/(4b). As required, \( V_{\text{eff}} \) is invariant under CFC symmetry. To determine the minima of \( V_{\text{eff}} \) note that \( V_g = O(N^2) \) while, due to our imposition of flavor-twisted BCs, \( V_{\ell} = O(N^{-2}) \).

At large \( N \), the minima of \( V_{\text{eff}} \) are entirely determined by the gluonic contribution \( V_g \), which favors coinciding eigenvalues, \( \Omega \propto 1 \). Consequently, when \( \epsilon = \frac{12}{\pi^2} x < 1 \) the CFC symmetry is spontaneously broken at any \( L \). On the other hand, at the pure Yang-Mill point, \( x = 0 \), center symmetry is certainly expected to be unbroken at large \( L \), and standard large \( N \) counting arguments imply that the intertwined center symmetry should remain unbroken for sufficiently small \( x \). Hence, there must be at least one transition at some \( x = x_{\text{CFC}} \) where the realization of the CFC symmetry changes. This point may or may not coincide with the point \( x_{\chi} \) where the chiral symmetry realization changes. Logically possible phase diagrams are sketched in Fig. 1.

Introducing a non-zero quark mass or temperature gives a richer phase structure. With \( \epsilon \ll 1 \) and small quark mass, the theory develops a new strong scale, \( \Lambda_m \sim m_q e^{-75/(8\epsilon^2)} \). As \( L \) increases and becomes comparable to \( \Lambda_m^{-1} \) we expect a \( Z_3 \) center-restoring phase transition. We also expect a CFC-restoring phase transition at a non-zero temperature \( T_c \sim 1/L \) when \( m_q = 0 \), similar to the large \( N \) deconfinement transition in \( N = 4 \) super-Yang-Mills theory on \( S^3 \times S^1 \) [28, 29].

Now consider \( N = 3 \) and massless quarks. If the \( n_t = 15 \) IR fixed point is weakly coupled, as widely believed, then our above calculation applies and \( Z_3 \) center symmetry is spontaneously broken at \( n_t = 15 \). At \( n_t = 3 \), lattice calculations [6] with boundary conditions (4a) are consistent with unbroken \( Z_3 \) center symmetry when \( L \Lambda > 1 \). So for integer values of \( x = n_{\text{CFC}} / 3 \), there must be a minimal value \( 2 \leq x_{\text{CFC}} \leq 5 \) where the \( Z_3 \) CFC symmetry first becomes spontaneously broken.

**Dense quark matter.** Consider the phase diagram of QCD with \( N = n_t = 3 \) and a common quark mass \( m_q \), as a function of the \( U(1)_Q \) chemical potential \( \mu \) and temperature \( T \). Previously known symmetry principles only suggest the existence of a curve \( T(\mu) \) in the \( (T, \mu) \) plane below which lies a superfluid phase with spontaneously broken \( U(1)_Q \) symmetry, leading to a hypothesis of continuity of quark matter and hadronic nuclear matter [30]. Consideration of CFC symmetry implies the existence of additional phase structure when QCD is compactified with CFC-preserving BCs on a spatial circle large compared to other spatial scales.

First, consider the small \( T, \mu \) regime. Here lattice studies [6] imply that \( \langle \text{tr} \Omega \rangle = 0 \) at large \( L \). Next, consider high temperatures, \( T \gg \max(\Lambda, \mu) \). Here, the dynamics on spatial scales large compared to \( (g^2 T)^{-1} \) are described by pure 3D YM theory [27] which confines, so \( \langle \text{tr} \Omega \rangle = 0 \) at high \( T \). We expect this high-temperature region to be smoothly connected to the region near \( T = \mu = 0 \). However, as we next discuss the CFC symmetry realization behaves non-trivially when \( T \rightarrow 0 \) with \( \mu \gg \max(\Lambda, m) \). A simple phase diagram consistent with our results is sketched in Fig. 2.

High density QCD, \( \mu \gg \Lambda \), is believed to be in a "color-flavor-locked" (CFL) color-superconducting phase [31] when \( T < T_{\text{CFL}} \). The phase transition temperature \( T_{\text{CFL}} \) is comparable to the superconducting gap, \( T_{\text{CFL}} \sim \Delta \sim g^{5} e^{-\left(3\pi^2/\sqrt{2}\right)/g} \) Electric and magnetic gluons develop Debye and Meissner static screening masses, respectively, both of order \( g \mu \) in the CFL phase [32, 33]. For \( T_{\text{CFL}} < T \ll g \mu \), low frequency magnetic fluctuations experience Landau damping. Consequently, for \( T \lesssim g \mu \) the relevant gauge coupling is small, \( (g(\mu)) \ll 1 \), and cold dense quark matter is weakly coupled.

In typical gauge-dependent language, CFL superconductivity is driven by an expectation value for diquark operators, \( \langle q_{i}^{a} C_{\gamma_{5} q_{j}}^{b} \rangle \propto \epsilon^{a b K} \epsilon_{ij K} \) [31]. The uncontracted flavor indices on the "condensate" might lead one to think that flavor permutation symmetry is broken, automatically implying accompanying spontaneous breaking of the CFC symmetry [7] when \( x_2 \) is compactified with BCs [4]. But this gauge-dependent language is misleading. The true gauge-invariant order parameters for spontaneous breaking of chiral and \( U(1)_Q \) symmetries, schematically \( \langle q_{i} C_{\gamma_{5} q_{j}} \rangle \rangle \) and \( \langle (q C_{\gamma_{5} q}) \rangle \rangle \), are \( SU(n_f) \) singlets [31]. So the development of CFL superconductivity does not, ipso facto, imply spontaneous breaking of CFC.
under the CFC symmetry, as required. To examine the realization of CFC symmetry, we work in the simplifying limit $m_q \ll \mu$ and focus on the regime $\mu L \gg 1$. If $T L \gg 1$, then the sum (11) is dominated by the $k = \pm \frac{1}{2}$, $n = 1$ terms, giving

$$V_l(\Omega) = \frac{\pm 2T e^{-n_l^2 L T}}{n_l \pi L^2} [\mu \sin(n_l \mu L) + \pi T \cos(n_l \mu L)]$$

\[ \times (\text{tr} \Omega^{n_l} + \text{h.c.}) + \text{(holonomy-independent)} \] (12)

up to exponentially small corrections. The $e^{-\pi T L n_i \mu}$ factor arises from the lowest fermionic Matsubara frequency and our twisted boundary conditions. Alternatively, if $TL \to 0$ then the prefactor in (12) becomes $\pm(n_i^2 \pi^2 L^3)^{-1}$. In either regime of $TL$, neglecting subdominant contributions, $V_{eff}(\Omega) \propto \text{Re} \text{tr} \Omega^n$ with an amplitude which oscillates as a function of $n_l \mu L$.

For $n_l = N = 3$, extrema of $V_l(\Omega)$ fall into four categories: (a) one center-symmetric extremum at $\Omega = \text{diag}(1, e^{2\pi i/3}, e^{4\pi i/3})$, where the $SU(3)$ gauge symmetry is “broken” down to $U(1)^2$ with the holonomy playing the role of an adjoint Higgs field; (b) three center-broken extrema with “residual” $SU(3)$ gauge group $U(1)^2$ where $\Omega = \text{diag}(e^{(2k-1)\pi i/3}, e^{2\pi i/3}, e^{e^{(2k+1)\pi i/3}}, k = 0, 1, 2$; (c) nine center-broken “$SU(2) \times U(1)$” extrema at $\Omega = \text{diag}(e^{(2k+1)\pi i/3}, e^{2(2k+1)\pi i/3})$ with $k \mod 6 = 2, 3$ or 4; (d) three center-broken “$SU(3)$” extrema, $\Omega = \text{diag}(e^{e^{2\pi i/3}}, e^{2\pi i/3}, e^{3e^{2\pi i/3}}, k = 0, 1, 2$. These “$SU(3)$” extrema are also minima of $V_\theta$.

The form (12) implies that the the locations of the minima of $V_{eff}(\Omega)$ oscillate as a function of $\mu L$, as illustrated in the contour plots in Fig. 3 for two nearby values of $\mu L$. (Each plot shows a fundamental domain of the Weyl group of $SU(3)$, which acts by permuting the eigenvalues of $\Omega$.) There are quantum oscillations in the phase structure of cold dense QCD on a circle, with the minima of $V_{eff}$ cycling through two inequivalent sets of local minima as $\mu L$ varies. These come in two groups within which $V_l(\Omega)$ is degenerate. One group consists of the center-symmetric and three $SU(3)$ extrema. The other consists of the six $SU(2) \times U(1)$ extrema with $\Omega = \text{diag}(e^{e^{2\pi i/3}}, e^{2e^{2\pi i/3}}, e^{2e^{2\pi i/3}})$ with $k \mod 6 = 2$ and 4. (The remaining six extrema are always saddle points for $TL \gg 1$.)

At $T = 0$ there are quantum phase transitions when the minimum energy state switches from one set of extremum to another, with associated jumps in the ground state degeneracy. (Similar behavior in other circle-compactified theories has been seen in Refs. [34, 35].) There are an infinite number of phase transitions in the cold dense limit as $L$ increases and successive energy bands pass through the value of the chemical potential, with an accumulation point at $L = \infty$. Borrowing a term from the condensed-matter literature [36, 37], each point in the $(T, \mu)$ phase diagram for QCD where this phenomenon occurs can be called a multi-phase point [38]. As we discuss below, this

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![Contour plots](imageurl)

**FIG. 3.** (Color online.) Contour plots of $V_l$ for $N = n_l = 3$, with BCs (4a), as a function of $\theta_1, \theta_2$ for two nearby values of $\mu L$ with $T/\mu = 10^{-3}$, illustrating the quantum oscillations described in the text. Darker colors indicate lower values of $L^4V_{eff}$. Regions outside the triangle shown are gauge-equivalent to points within the triangle. The center-symmetric point $(\theta_1, \theta_2, \theta_3) = (0, 2\pi/3, 4\pi/3)$ lies at the center of the triangle while the corners are the coinciding eigenvalue points $(0, 0, 0)$ and $(2\pi/3, 2\pi/3, 2\pi/3)$. Dots denote critical points of $V_l$. Results with BCs (4b) are similar.
behavior is expected in a finite area domain of the \((T, \mu)\) phase diagram, so in fact we find a multi-phase region.

The small residual gluon contribution to \(V_{\text{eff}}\) favors configurations with clumped holonomy eigenvalues, lowering the energy of \(SU(3)\) extrema relative to the center-symmetric point. Hence, we expect that all genuine minima of \(V_{\text{eff}}\) in this multi-phase region are associated with broken CFC symmetry, with \(\langle \text{tr} \Omega \rangle \neq 0\) \[39\].

Putting everything together, we conclude that there must be some curve \(T = T_{\text{CFC}}(\mu)\) below which the CFC symmetry is spontaneously broken with oscillatory multi-phase behavior. We lack a definitive calculation of \(V_{\text{eff}}(\Omega)\) valid for \(T > T_{\text{CFL}}\), but we expect that \(T_{\text{CFC}}(\mu)\) is \(\mathcal{O}(g\mu)\), greatly exceeding \(T_{\text{CFL}}\) at large \(\mu\). The \(T_{\text{CFC}}\) curve must end at some point \(\mu_c\) on the \(T = 0\) axis. The simplest hypothesis is that \(\mu_c\) coincides with \(\mu_n \sim \Lambda\), the critical chemical potential needed to produce pressurless nuclear matter at \(T = 0\), as illustrated in Fig. 2.

**Conclusions.** We have shown that there are well-defined and non-trivial order parameters for quantum and thermal phase transitions in QCD, compactified on a circle, provided \(\text{gcd}(n_q, N) > 1\) with quarks having a common mass \(m_q\). This is a consequence of the existence of color-flavor center symmetry, and has interesting implications for the phase structure of QCD as a function \(n_q/N, \mu, T, \text{and } m_q\).

There are many worthwhile extensions of our observations. Consideration of CFC symmetry may be helpful in studies of QCD behavior near the lower edge of the conformal window [40]. For applications to dense QCD, an explicit calculation of the one-loop gluon contribution to \(V_{\text{eff}}(\Omega)\) in the hard dense loop approximation would give a better estimate for the CFC symmetry restoration temperature \(T_{\text{CFC}}(\mu)\). The role of explicit \(SU(3)_{\text{V}}\) symmetry breaking should be explored. Finally, it would be interesting to study local order parameters for CFC symmetry. At \(\mu = 0\) these order parameters violate \(SU(n_q)_{\text{V}}\) symmetry, and hence must vanish as one takes the \(R^4\) limit by the Vafa-Witten theorem [1]. But this theorem does not apply at finite \(\mu\).

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1. C. Vafa and E. Witten, Nucl. Phys. B234, 173 (1984).
2. H. Kouno, Y. Sakai, T. Makiyama, K. Tokunaga, T. Sasaki, and M. Yahiro, J. Phys. G39, 085010 (2012).
3. Y. Sakai, H. Kouno, T. Sasaki, and M. Yahiro, Phys. Lett. B718, 130 (2012), arXiv:1204.0228 [hep-ph].
4. H. Kouno, T. Makiyama, T. Sasaki, Y. Sakai, and M. Yahiro, J. Phys. G40, 095003 (2013), arXiv:1301.4013 [hep-ph].
5. H. Kouno, T. Misumi, K. Kashiwa, T. Makiyama, T. Sasaki, and M. Yahiro, Phys. Rev. D88, 016002 (2013), arXiv:1304.3274 [hep-ph].
6. T. Iritani, E. Itou, and T. Misumi, JHEP 11, 159 (2015), arXiv:1508.07132 [hep-lat].
7. H. Kouno, K. Kashiwa, J. Takahashi, T. Misumi, and M. Yahiro, Phys. Rev. D93, 056009 (2016), arXiv:1504.07585 [hep-ph].
8. T. Hirakida, H. Kouno, J. Takahashi, and M. Yahiro, Phys. Rev. D94, 014011 (2016), arXiv:1604.02977 [hep-lat].
9. T. Hirakida, J. Sugano, H. Kouno, J. Takahashi, and M. Yahiro, (2017), arXiv:1705.00665 [hep-lat].
10. R. A. Briceno, Z. Davoudi, T. C. Luu, and M. J. Savage, Phys. Rev. D90, 074509 (2014), arXiv:1311.7686 [hep-lat].
11. A. Cherman, T. Schäfer, and M. Ünsal, Phys. Rev. Lett. 117, 081601 (2016), arXiv:1604.06108 [hep-th].
12. Y. Liu, E. Shuryak, and I. Zahed, Phys. Rev. D94, 105013 (2016), arXiv:1606.02996 [hep-ph].
13. A. Cherman, S. Sen, M. L. Wagman, and L. G. Yaffe, Phys. Rev. D95, 074512 (2017), arXiv:1612.00403 [hep-lat].
14. N. Weiss, Phys. Rev. D24, 475 (1981).
15. Y. Aoki et al. (LatKMI), Phys. Rev. D89, 111502 (2014), arXiv:1403.5000 [hep-lat].
16. E. Rinaldi (LSD), in Sakata Memorial KMI Workshop on Origin of Mass and Strong Coupling Gauge Theories (SCGT15) Nagoya, Japan, March 3-6, 2015 (2015) arXiv:1510.06771 [hep-lat].
17. T. Appelquist et al., Phys. Rev. D93, 114514 (2016), arXiv:1601.04027 [hep-lat].
18. Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B681, 353 (2009), arXiv:0907.4562 [hep-lat].
19. Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-i. Nagai, H. Ohki, A. Shibata, K. Yamawaki, and T. Yamazaki (LatKMI), Phys. Rev. D87, 094511 (2013), arXiv:1302.6859 [hep-lat].
20. T. Appelquist et al. (LSD), Phys. Rev. D90, 114502 (2014), arXiv:1405.4752 [hep-lat].
21. T. Appelquist et al., (2012), arXiv:1204.6000 [hep-ph].
22. Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B703, 348 (2011), arXiv:1104.3124 [hep-lat].
23. Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-i. Nagai, H. Ohki, A. Shibata, K. Yamawaki, and T. Yamazaki, Phys. Rev. D86, 054506 (2012), arXiv:1207.3060 [hep-lat].
24. Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-i. Nagai, H. Ohki, E. Rinaldi, A. Shibata, K. Yamawaki, and
T. Schäfer and F. Wilczek, Phys. Rev. Lett. 111, 162001 (2013), arXiv:1305.6006 [hep-lat].

[25] W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974).

[26] T. Banks and A. Zaks, Nucl. Phys. B196, 189 (1982).

[27] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[28] E. Witten, Adv. Theor. Math. Phys. 44 (2003), arXiv:hep-th/0301025 [hep-th].

[29] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, Lie theory and its applications in physics. Proceedings, 5th International Workshop, Varna, Bulgaria, June 16-22, 2003, Adv. Theor. Math. Phys. 8, 603 (2004), [161(2003)], arXiv:hep-th/0301025 [hep-th].

[30] T. Schäfer and F. Wilczek, Phys. Rev. Lett. 82, 3956 (1999), arXiv:hep-ph/9811473 [hep-ph].

[31] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schäfer, Rev. Mod. Phys. 80, 1455 (2008), arXiv:0709.4635 [hep-ph].

[32] D. T. Son and M. A. Stephanov, Phys. Rev. D61, 074012 (2000), arXiv:hep-ph/9910491 [hep-ph].

[33] D. H. Rischke, Phys. Rev. D62, 054017 (2000), arXiv:nucl-th/0003063 [nucl-th].

[34] A. S. Vshivtsev, M. A. Vdovichenko, and K. G. Klimenko, J. Exp. Theor. Phys. 87, 229 (1998), [Zh. Eksp. Teor. Fiz.114,418(1998)].

[35] T. Kanazawa, M. Únsal, and N. Yamamoto, (2017), arXiv:1703.06411 [hep-th].

[36] M. E. Fisher and W. Selke, Phys. Rev. Lett. 44, 1502 (1980).

[37] M. E. Fisher and W. Selke, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 302, 1 (1981).

[38] Unlike the spin models discussed in Refs. [36, 37], our multi-phase points involve no fine-tuning, as they are obtained as limits of one of our two symmetry-preserving choices of compactification.

[39] Higher order calculations of $V_t$ are needed to confirm that the center-broken $SU(3)$ minima are favored, as an $O(g^2)$ two-loop fermion contribution could overwhelm the one-loop gluonic contribution when it is suppressed by Meissner screening.

[40] D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, Phys. Rev. D80, 125005 (2009), arXiv:0905.4752 [hep-lat].

[41] A. Gonzalez-Arroyo, R. Narayanan, and H. Neuberger, Phys. Lett. B631, 133 (2005), arXiv:hep-lat/0509074 [hep-lat].

[42] P. Kovtun, M. Únsal, and L. G. Yaffe, JHEP 06, 019 (2007), arXiv:hep-th/0702021 [HEP-TH].

**Supplemental Material**

### Multiple compactified dimensions

Suppose multiple dimensions are compactified, so that the theory lives on $\mathbb{R}^{D-k} \times T^k$. If one weakly gauges the $SU(n_1) V$ flavor symmetry, then quarks become bifundamentals under $SU(N) \times SU(n_1)$ and there is a $(Z_d)^k$ center symmetry (see, e.g., Refs. [41, 42]). Charged operators are Wilson loops wrapping non-trivial cycles of $T^k$ with non-zero winding numbers mod $d$. In the limit of vanishing $SU(n_1)$ gauge coupling, where fluctuations in the $SU(n_1)$ gauge field become negligible, it is possible to preserve a single diagonal $\mathbb{Z}_d$ subgroup of $(\mathbb{Z}_d)^k$ by intertwining it with the $\mathbb{Z}_d$ subgroup of cyclic flavor permutations. This is achieved by setting $q(x_i + L_i) = U q(x_i)$, with $\{x_i\}$ parametrizing $T^k$, and $U$ given by one of the choices (4).

If $n_1$ and $N$ have multiple common divisors, then one can choose BCs which preserve different embeddings of $\mathbb{Z}_d$ (or a chosen subgroup of $\mathbb{Z}_d$) within $(\mathbb{Z}_N)^k \ltimes SU(n_1) \nu$. As an example, suppose $N = n_1 = 4$, with two compactified directions. Instead of a common boundary condition for both directions, one could choose differing flavor-twisted boundary conditions (3) for the two compact directions, with

$$U_1 \equiv \text{diag}(1,-1,1,-1), \quad U_2 \equiv \text{diag}(1,1,-1,-1).$$ (13)

Eigenvalues of these $U_k$ are transposed under the action of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the $\mathbb{Z}_4 \times \mathbb{Z}_4$ center symmetry. These transpositions can be compensated by flavor permutations with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the $(SU(4) \nu$ flavor symmetry, so these boundary conditions produce a compactified theory with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ CFC symmetry in which each $\mathbb{Z}_2$ factor affects only a single compact dimension.

### Holonomy effective potential on $\mathbb{R}^2 \times T^2$

Consider a 2-torus $T^2 = S_{L_x}^1 \times S_{L_y}^1$ with $L$ and $\beta \equiv 1/T$ regarded as spatial $x_1$ and thermal $x_4$ circle sizes, respectively. Assign quarks twisted boundary conditions (4) in the $x_1$ direction, and thermal boundary conditions with a $U(1)_Q$ chemical potential in $x_4$,

$$q(x_4 = \beta) = -e^{-\beta \mu} q(x_4 = 0),$$ (14a)

$$\bar{q}(x_4 = \beta) = -e^{\beta \mu} \bar{q}(x_4 = 0).$$ (14b)

Assume a constant spatial holonomy $\Omega$, with eigenvalues $\{e^{i\theta_a}\}$. In $A_1 = 0$ gauge, the holonomy appears as additional phases in the $x_1$ boundary condition,

$$q(x_1 = L)_{aA} = e^{i\theta_a} \nu^A q(x_1 = 0)_{aA},$$ (15a)

$$\bar{q}(x_1 = L)_{aA} = e^{-i\theta_a} \nu^{-A} \bar{q}(x_1 = 0)_{aA},$$ (15b)

where $\mu = 1, \ldots, N$ is a color index and $A$ is a flavor index running from 0 to $n_1-1$ for BC (4a), or a half-integer $\frac{1}{2}, \ldots, n_1-\frac{1}{2}$ for BC (4b). The usual quark action, $S_F = \int d^4 x \bar{q}(\gamma^\mu \partial_\mu + m_q) q$, leads to a free energy density

$$F = -\frac{1}{\beta V_N} \ln \det (\gamma^\mu \partial_\mu + m_q)$$ (16)

$$= -\frac{1}{\beta L} \int d^2 p_{\perp} \sum_{a,n,k} \ln \det (i\gamma^\mu p_{n,k}(\Omega) + m_q),$$

where $p_{n,k}(\Omega) \equiv (p_{a,n}(\Omega), \bar{p}_{\perp}, \bar{p}^{(k)})$ are momenta consistent with the above boundary conditions. The thermal (KMS) conditions (14) imply $\bar{p}_{\perp}^{(k)} = 2\pi k T + i\mu$ with $k \in \mathbb{Z} + \frac{1}{2}$. The flavor twisted conditions (15) imply $\bar{p}^{(n)}_{a,n} = \theta_a/L + 2\pi n/(n_1 L)$ with $n \in \mathbb{Z}$ for BC (4a), or...
\[ n \in \mathbb{Z} + \frac{1}{2} \] for BC (4b). Performing the Dirac determinant reduces the free energy density to

\[ F = -\frac{2}{\beta L} \int \frac{d^2p_\perp}{(2\pi)^2} \sum_{a,n,k} \ln(p^2_{\mu} + m^2_q), \tag{17} \]

where the indices labeling the quantized momentum components are suppressed.

To obtain a UV-safe quantity and focus on the effects of the holonomy, we subtract the holonomy-independent infinite volume limit,

\[ \Delta F(\Omega, L) = F(\Omega, L) - F(\Omega, L \to \infty) = \frac{2}{\beta} \sum_k \int_0^\infty \frac{dz}{z} \int \frac{d^2p_\perp}{(2\pi)^2} \left( \frac{1}{L} \sum_{a,n} - \int \frac{dp_\parallel}{2\pi} \right) e^{-z(p^2_{\mu} + m^2_q)}. \tag{18} \]

The sum-integral difference can be evaluated using Poisson summation, yielding

\[ \Delta F = \frac{1}{4\pi^{3/2}} \beta \int_0^\infty \frac{dz}{z^{5/2}} e^{-zm^2_q} \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{-z(2\pi k / \beta + i\mu)^2} \times \sum_{n \neq 0} e^{-n^2L^2/(4z)} \sum_{a,A} e^{i(\theta_a + 2\pi A/n)}n \]

\[ = \frac{1}{\pi L^3 \beta} \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m_k} (1 + nLm_k) e^{-nLm_k}, \tag{19} \]

where the effective mass \( m_k \) of each fermion mode is given by

\[ m_k^2 \equiv (2\pi k T + i\mu)^2 + m^2_q. \tag{20} \]

The result (19) is the holonomy dependent part of the fermion contribution to the effective potential \( V_{\text{eff}}(\Omega) \).

The form (11) in the main text follows from evaluating the traces of the flavor holonomy \( \mathcal{U} \). At leading order in \( m_q/\mu \), one may perform the sum over \( k \) and obtain

\[ \Delta F = \frac{T}{n_l^3 \pi L^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \left( \frac{\pi n L \sin(n \mu L)}{\coth(\pi n TL)} + n \mu L \sin(n \mu L) + \cos(n \mu L) \right). \tag{21} \]

**Zero temperature limit.** When \( TL \to 0 \), the holonomy-dependent free energy (21) reduces to

\[ \Delta F = \frac{1}{n_l^3 \pi^2 L^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \left( \frac{\pi n L \sin(n \mu L)}{\coth(\pi n TL)} + n \mu L \sin(n \mu L) + 2 \cos(n \mu L) \right). \tag{22} \]

In this \( TL = 0 \) limit, both the shape and sign of the potential depends on \( \mu L \). However, the subsequent terms in the sum over \( n \) rapidly decrease as \( 1/n^4 \). The extrema of the potential at \( TL = 0 \) are in the same locations as the extrema at \( TL \gg 1 \).

One finds essentially the same quantum oscillations in the minima as a function of \( \mu L \) at \( TL = 0 \) as in our large \( TL \) expressions in the main text. For completeness, snapshots of the behavior at \( TL = 0 \), for \( N = n_l = 3 \) and several nearby values of \( \mu L \), are shown in Fig. 4. The figure highlights one qualitative difference between the \( TL = 0 \) and \( TL \gg 1 \) results. As noted in the main text, for \( TL \gg 1 \), six of the center-breaking extrema are always saddlepoints of the effective potential, regardless of the value of \( \mu L \). But at \( TL = 0 \) these extrema also turn as minima of the potential as \( \mu L \) varies. Hence, unsurprisingly, there is non-uniformity between the \( T \to 0 \) and \( L \to \infty \) limits. If one views the spatial compactification purely as a device used to probe the behavior of the system, then it is natural to choose \( L \) large compared to all other physical length scales. This motivated our emphasis on the large \( TL \) regime in the main text.
FIG. 4. (Color online.) Contour plots of $L^4V_{\text{eff}}(\Omega)$ as a function of $\theta_1$ and $\theta_2$, for $N = n_f = 3$ and $TL = 0$, with BCs (4a). Shown are four nearby values of $\mu L$, illustrating the existence of quantum oscillations as a function of $\mu L$. Darker colors indicate lower values of $L^4V_{\text{eff}}$, and regions outside the triangle shown are gauge-equivalent to points within the triangle. Dots denote critical points of $V_f$. Results with BCs (4b) are similar.

\[
\begin{align*}
\mu L &= 10.0 \\
\mu L &= 10.6 \\
\mu L &= 11.2 \\
\mu L &= 11.8
\end{align*}
\]