Resonant radiation shed by dispersive shock waves

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(Dated: May 22, 2014)

We show that dispersive shock waves resulting from the nonlinearity overbalancing a weak leading-order dispersion can emit resonant radiation owing to higher-order dispersive contributions. We analyze such phenomenon for the defocusing nonlinear Schrödinger equation, giving criteria for calculating the radiated frequency based on the estimate of the shock velocity, revealing also a diversity of possible scenarios depending on the order and magnitude of the dispersive corrections.

PACS numbers: 42.65.ky, 42.65.Re, 52.35.Tc

Dispersive shock waves (DSWs) are expanding regions filled with fast oscillations that stem from the dispersive regularized formulation of classical shock waves (SWs). Originally introduced in collisionless plasmas\textsuperscript{[1]} and water waves\textsuperscript{[2]}, it is only recently that DSWs have been the focus of intense multidisciplinary efforts that have established their universal role in atom condensates\textsuperscript{[3, 4]}, light pulse (temporal)\textsuperscript{[5]} and beam (spatial)\textsuperscript{[6]} propagation, oceanography\textsuperscript{[7]}, magma flow\textsuperscript{[10]}, and granular\textsuperscript{[11]} or disordered materials\textsuperscript{[12]}. The dynamics of DSW is understood in connection with studies of supercontinuum generation\textsuperscript{[15–25]}, is a relatively well understood phenomenon (RR) due to a specific phase-matching with linear waves.

The emission of RR, usually considered for solitons\textsuperscript{[16–19]} and RR wavenumber\textsuperscript{[33]}, respectively, the radiation is resonantly amplified thanks to a well defined velocity of the shock front. While we expect this mechanism to be universal for several DSW-bearing models when HOD corrections become effective, we formulate our approach for temporal pulse propagation ruled by the dNLSE\textsuperscript{[28]}, where our results are important in view of generating a different type of supercontinuum pumped in the normal dispersion regime\textsuperscript{[29]}. They have also immediate impact to unveil the underlying mechanism of recent observations of RR produced by non-soliton pulses\textsuperscript{[30]}. We start from the dNLSE in semiclassical form\textsuperscript{[4]}, with dispersion at all orders (sum over $n$ implicitly assumes $n \geq 2$

\begin{equation}
iz\partial_t \psi + d(\partial_t)^n \psi + |\psi|^2 \psi = 0,
\end{equation}

where the link with real-world distance and retarded time (in capital) is $Z = z\sqrt{L_n L_d} = tT_0$, where $L_n = (\gamma P)^{-1}$ and $L_d = T_0^2/\partial^2 k$ are the nonlinear and dispersive length, respectively, associated with input pulse width $T_0$ and peak power $P$ ($\gamma$ is the nonlinear coefficient). The dispersive operator $d(\partial_t)$ has terms which are weighted, without loss of generality, by growing powers of the small parameter $\varepsilon = \sqrt{L_n/L_d} \ll 1$ and coefficients $\beta_n = \partial^2 k/\sqrt{(L_n)^2 (\partial^2 k)^n}$ (note that $\beta_2 = 1$), $\partial^2 k$ being $n$-order real-world dispersion arising from usual Taylor expansion of $k(\omega)$ (further details on supplemental material\textsuperscript{[31]}). We assume an input pump $\psi_0 = \psi(t, z = 0)$ with central frequency $\omega_p = 0$\textsuperscript{[32]}, and denote as $V_\omega$ the “velocity” of the SW produced by $\psi_0$ via wave-breaking (here, $V_\omega = dt/dz$ is the reciprocal of the velocity as usually defined for soliton RR\textsuperscript{[14]}), and as $d(\omega)$ the Fourier transform of $d(\partial_t)$. Linear waves $\exp(ik(\omega)z - i\omega t)$ are resonantly amplified when their wavenumber in the SW moving frame, which reads as $k(\omega) = \frac{1}{\varepsilon}[d(\omega t) - V_\omega(\varepsilon \omega)]$ equals the pump wavenumber $k_p = k(\omega_p = 0) = 0$. Denoting also as $k_{nl}$ the difference between the nonlinear contributions to the pump and RR wavenumber\textsuperscript{[33]}, respectively, the radiation is resonantly amplified at frequency $\omega = \omega_{RR}$ which solves the equation

\begin{equation}
\sum_n \frac{\beta_n}{n!}(\varepsilon \omega)^n - V_\omega(\varepsilon \omega) = \varepsilon k_{nl}.
\end{equation}

At variance with solitons of the fNLSE where $V_\omega(\varepsilon \omega) = 0$\textsuperscript{[16–19]}, DSWs possess non-zero velocity $V_\omega$, which must be carefully evaluated, having great impact on the determination of $\omega_{RR}$.

The process of wave-breaking ruled by Eq.\textsuperscript{[11]} can be described by applying the Madelung transformation $\psi = \sqrt{\rho} \exp(iS/\varepsilon)$. At leading-order in $\varepsilon$, we obtain a quasi-linear hydrodynamic reduction, with $\rho = |\psi|^2$ and
\( u = -S_l \) (chirp) equivalent density and velocity of the flow, which can be further cast in the form

\[
\partial_t \rho + \partial_x \left[ \sum_n \frac{\beta_n \rho u^{n-1}}{(n-1)!} \right] = 0, \\
\partial_t (\rho u) + \partial_x \left[ \sum_n \frac{\beta_n \rho u^n + \frac{1}{2} \rho^2}{(n-1)!} \right] = 0,
\]

of a conservation law \( \partial_t q + \partial_x f(q) = 0 \) for mass and momentum, with \( q = (\rho, \rho u) \). This system can be diagonalized to yield \( \partial_t x^+ \pm V^\pm \partial_x x^+ = 0 \), by introducing the evolutions \( V = \sum_n \beta_n u^{n-1}/(n-1)! \) \pm \sqrt{\rho \sum_n \beta_n u^{n-2}/(n-2)!} \) and the Riemann invariants.

Equations (3-4), as far as HOD is such that they remain hyperbolic, admit weak solutions in the form of classical SWs, i.e., traveling discontinuity from left \((\rho_l, u_l)\) to right \((\rho_r, u_r)\) values, whose velocity \( V_c \) can be found from the so-called Rankine-Hugoniot (RH) condition \( V_c(q_l - q_r) = [f(q_l) - f(q_r)] \). In the 2 \times 2 case, the RH equations fix both \( V_c \) and the admissible value of one of the parameters of the jump, e.g., \( u_r \) given \( \rho_r, \rho_l, u_l \). For instance, when no HOD is effective \((\beta_2 = 1)\), an admissible right-going shock which satisfies the entropy condition \( \rho_l > \rho_r \), can be obtained with

\[
w_r = u_l - (\rho_l - \rho_r) \sqrt{\frac{\rho_r + \rho_l}{2\rho_l \rho_r}} V_c = u_l + \rho_l \sqrt{\frac{\rho_r + \rho_l}{2\rho_l \rho_r}}.
\]

This result can be generalized for HOD, thanks to Eqs. (3-4). For instance, if \( \beta_3 \neq 0 \), the SW velocity becomes

\[
V_c = \frac{\beta_3 (\rho_l u_l - \rho_r u_r) + \beta_3 (\rho_l u_l^2 - \rho_r u_r^2)/2}{(\rho_l - \rho_r)},
\]

where \( u_r \) is obtained as the real root of the cubic equation

\[
\beta_3 (u_l - u_r)^2 (u_l + u_r) + 2\beta_3 (u_l - u_r)^2 = g(\rho_l, \rho_r),
\]

where \( g(\rho_l, \rho_r) \equiv (\rho_l - \rho_r)^2 (\rho_l + \rho_r)/(\rho_l \rho_r) \) (see supplemental for more details).

Second-order dispersion, however, is known to regularize classical SWs by replacing the jump with an expanding fan filled with oscillations (i.e., a DSW) characterized by classical SWs by replacing the jump with an expanding wave of a conservation law

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\]

and fourth-order (4-HOD) dispersion.

In particular, when 3-HOD is effective we find a crossover from a perturbative regime \((|\beta_3| \lesssim 0.5)\) where the DSW leading edge turns out to be responsible for the RR, to a regime where the 3-HOD is strong enough \((|\beta_3| \sim 1)\) to modify the shock formation, leading to enhanced RR generated by a traveling front which is approximated with a classical SW. To show this and verify that Eq. (2) is able to predict the RR frequency in both regimes, we consider first a step initial value that allows us to calculate the velocity analytically, taking \( \beta_3 < 0 \) without loss of generality. Specifically, we consider the evolution of an initial jump from the state \( \rho_l, u_l = 0 \) to the state \( \rho_r, u_r = 2(\sqrt{\rho_l} - \sqrt{\rho_r}) \), which is such to maintain constant \( r^- (z, t) \) upon evolution (only \( r^+ (z, t) \) varies). In this case, the modulated wavetrain produced upon evolution [see Fig. 1(a,c)] in the limit \( \beta_3 = 0 \) is described by a rarefaction wave of the Whitham modulation equations for the unperturbed dNLSE [4]. Following the approach of Ref. [4], one can calculate the edge velocities of the fan. What is relevant for the RR is the leading-edge velocity, which we find to be \( V_f = \sqrt{\rho_l} + u_r = 2\sqrt{\rho_l} - \sqrt{\rho_r} \). Given a gray soliton on unchirped background A, \( \psi = A[w \exp(i\theta)/\epsilon] \) with \( \theta = w/(t-Az) \), \( w^2 = 1 - v^2 \), \( V_f \) turns out to coincide with the soliton velocity \( V_{sol} = AV = \sqrt{\rho_{min}} \), with natural position \( A = \sqrt{\rho_l}, v = (2\sqrt{\rho_l} - \sqrt{\rho_r})/\sqrt{\rho_l} \), and the dip velocity \( \rho_{min} = (2\sqrt{\rho_l} - \sqrt{\rho_l})^2 \). We emphasize, however, that the equivalence of the leading edge with a gray soliton holds only locally since the DSW is strictly speaking a modulated nonlinear wave. In this regime, if we account for \( k_{nl} = k_{sol} = k_{RR} = -\rho_l \) arising from the soliton \( k_{sol} = \rho_l / \epsilon \) and the cross-induced contribution \( k_{RR} = 2\rho_l / \epsilon \) to the RR, Eq. (2) explicitly reads as

\[
\frac{\beta_3}{6} (\epsilon \omega)^3 + \frac{\beta_3^2}{2} (\epsilon \omega)^2 - V_f (\epsilon \omega) + \rho_l = 0.
\]

Real solutions \( \omega = \omega_{RR} \) of Eq. (4), with \( V_f = V_i = 2\sqrt{\rho_l} - \sqrt{\rho_r} \) correctly predicts the RR as long as \( |\beta_3| \lesssim 0.5 \), as shown by the dNLSE simulation in Fig. 1. The DSW displayed in Fig. 1(a) clearly exhibits a spectral RR peak besides spectral shoulders due to the oscillating

![Image](image_url)
front, as shown by the spectral evolution in Fig. 1b) and the output spectrum in Fig. 1d. Perfect agreement is found between the RR peak obtained in the numerics and the prediction [dashed vertical line in Fig. 1b,d] from Eq. (7) with velocity $V_s = V_l$ characteristic of the integrable limit ($\beta_3 = 0$, snapshots in Fig. 1f). Indeed, in this regime, the DSW leading edge is nearly unaffected by 3-HOD, whereas using the velocity $V_c$ [Eq. (3)] of the equivalent classical SW [reported for comparison in Fig. 1c]] would miss the correct estimate of $\omega_{RR}$. We also point out that $k_{nl}$ represents a small correction, so $\omega_{RR}$ can be safely approximated by dropping the last term in Eq. (7) to yield $\omega_{RR} = \frac{1}{2\beta_3}(-\beta_2 \mp \sqrt{\beta_2^2 + 8V_s\beta_3/3})$,

or $\omega_{RR} = -\frac{3\beta_2}{2\beta_3}$ for $\beta_3V_s \rightarrow 0$.

When $|\beta_3|$ grows larger, the aperture of the shock fan reduces, until eventually the DSW resembles a single traveling front, i.e. a classical SW \cite{35}. In this regime, we find that Eq. (7) still gives the correct frequency $\omega_{RR}$ provided that $V_s$ is taken as the Rankine-Hugoniot velocity $V_c$ of the equivalent classical SW calculated for $\beta_3 \neq 0$ [Eq. (8)]. This case is illustrated in Fig. 2 for $\beta_3 = -1$. The RR becomes clearly visible in the temporal evolution [Fig. 2a,c)], and is sufficiently strong to generate $-\omega_{RR}$ via four-wave mixing [Fig. 2b,d)]. Perfect agreement between the numerics and the value predicted from Eq. (7), once we set $V_s = V_c$, is found also in this case.

The behaviors of step initial data are basically recovered for pulse waveforms that are more manageable in experiments. Figure 3 shows the transition from the perturbative [Fig. 3a)] to the non-perturbative [Fig. 3b)] regime, for an input gaussian pulse $\psi_0 = \nu + (1 - \nu) \exp(-t^2)$ with background to peak density ratio $\nu^2 = 0.09$. As shown in Fig. 3a), for relatively small $\beta_3$, two asymmetric DSWs emerge from wave-breaking points on the two pulse edges, which occur at different distances due to broken symmetry in time caused by 3-HOD. Phase-matching is achieved only for the DSW traveling with $V_s > 0$. The corresponding $\omega_{RR}$ can be obtained from Eq. (7) provided we set $V_s = V_l$, with the DSW leading edge velocity being (following the discussion of Fig. 1) $V_l = \sqrt{\rho_{min} + u_l}$, where the minimum density and the correction $u_l$ due to the local non-zero chirp are evaluated numerically after wave-breaking as shown in Fig. 3c). Also in this case, a larger $|\beta_3|$ results in a narrower fan, until eventually a simple front is left which strongly radiates, as shown in Fig. 3b). In this regime, a good approximation of the front velocity is obtained by the approximating classical SW in Eq. (8). In both the regimes shown in Fig. 3a) and (b), Eq. (7) provides a good description of the RR frequency observed in the numerics [see Fig. 3d)]. Notice also that, for symmetry reasons, sign reversal of 3-HOD ($\beta_3 > 0$) simply results into RR with opposite frequency, generated by the DSW with opposite velocity ($V_s < 0$, left DSW).

We also emphasize two important points: (i) RR occurs also in the limit of vanishing background $\nu = 0$, as shown in Fig. 3a), allowing us to conclude that a bright pulse does not need to be a soliton (as in the fNLSE, $\beta_2 = -1$) to radiate. Importantly, experimental evidence for such RR scenario was reported very recently in fiber optics \cite{35}, without explaining the underlying mechanism, which our theory individuates in the shock formation. Indeed, the physical parameters used in Fig. 1 of Ref. \cite{35}, i.e. power $P = 600$ W, pulse duration $T_0 = 1$
ps, nonlinear coefficient $\gamma = 2.5 \text{ W}^{-1} \text{ km}^{-1}$, dispersion $\beta_2 k = 7.5 \text{ ps}^2/\text{km}$, $\beta_3 k = 0.2 \text{ ps}^3/\text{km}$, gives normalized parameters $\varepsilon \simeq 0.07$ and $\beta_3 \simeq 0.37$, typical of the wave-breaking regime ($\varepsilon \ll 1$) with perturbative 3-HOD. Since $\beta_3 > 0$, the radiating shock turns out to be the one on the leading edge ($t < 0$), and its velocity $V_t = -0.75$, inserted in Eq. (2), gives a negative [opposite of Fig. 4(a)] frequency detuning $\Delta f_{RR} = \omega_{RR} T_0^2 / 2\pi \simeq 13 \text{ THz}$, in excellent agreement with the value reported in Ref. [37]. (ii) a limitation exists (regardless of $\nu$) on the value of $|\beta_3|$ to observe RR, since large 3-HOD feature a qualitative different wave-breaking mechanism, as shown in Fig. 4(b) for $\beta_3 = -2$. While the non-radiating (left) DSW simply develop at shorter $z$ without qualitative changes, on the right ($t > 0$) the pulse undergoes a different catastrophe, reminiscent of the fNLSE. Indeed in this case the eigenvelocities $V^\pm = \beta_2 u + \beta_3 u^2 / 2 \pm \sqrt{\rho(\beta_2 + \beta_3 u)}$ become locally complex conjugate where $u > 0$, implying that Eqs. (4) show a mixed hyperbolic-elliptic behavior reminiscent of the so-called transsonic flow [37].

A completely different scenario occurs when the dispersive correction is due to 4-HOD. In this case, the shock formation can compete with a different breaking mechanism, namely modulational instability (MI). The latter, characteristic of the fNLSE, is known to extend to the defocusing regime $\beta_2 = 1$, whenever $\beta_4 < 0$ [37] (see also supplemental material [31]). Moreover, the problem is symmetric in time and the shocks from both edges of a pulse radiate. Overall four RR frequencies result from Eq. (2), which is now fourth-order: the frequency pair $\omega_{RR1} - \omega_{RR2}$ ($\omega_{RR1,RR2} > 0$) and the opposite pair $-\omega_{RR1} = \omega_{RR2}$, induced by the shock with positive ($V_t > 0$) and negative velocity ($V_t < 0$), respectively. Since the MI has a spectral narrow bandwidth that turns out to lie exactly in between the two frequencies $\omega_{RR1,RR2}$ (arising from shocks on opposite edges), it serves as a seed for the RR. This is shown in Fig. 5: the pair of twin-band RR starts to grow, triggered by MI, even during the process of pulse edge steepening ($z < 1$), while becoming prominent as the DSWs form and travel with definite velocities (here $V_t = \pm 0.77$). The four-band RR from the spectral evolution in Fig. 5(b) fits well the prediction from Eq. (2) (dashed lines), while coexistence of the two wave-breaking phenomena (MI and DSW) is clearly visible in the output snapshot in Fig. 5(a).

In summary, we have demonstrated that higher-order dispersive corrections force DSWs to radiate, following different observable cross-over scenarios depending on the nature and magnitude of such corrections. Funding from MIUR (grant PRIN 2009P3K72Z) is gratefully acknowledged.

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[31] See Supplemental Material at [URL will be inserted by publisher] for more technical details on the normalization of Eq. (1), evaluation of Rankine-Hugoniot velocity, and MI analysis in the presence of 4-HOD.
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