NORM INEQUALITIES FOR ACCRETIVE-DISSIPATIVE BLOCK MATRICES

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Abstract. Let $T = [T_{ij}] \in \mathbb{M}_{mn}(\mathbb{C})$ be accretive-dissipative, where $T_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, \ldots, m$. Let $f$ be a function that is convex and increasing on $[0, \infty)$ where $f(0) = 0$. Then

$$
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T^*_{ji}|^2 \right) \right\| \leq \left\| f \left( \frac{m^2 - m}{2} |T|^2 \right) \right\|.
$$

Also, if $f$ is concave and increasing on $[0, \infty)$ where $f(0) = 0$, then

$$
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T^*_{ji}|^2 \right) \right\| \leq \left\| f \left( \frac{m^2 - m}{2} |T|^2 \right) \right\|.
$$

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. A matrix $T \in \mathbb{M}_{mn}(\mathbb{C})$ can be partitioned as an $m \times m$ block matrix ($m \in \{2, 3, 4, \ldots\}$)

$$
T = \begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1m} \\
T_{21} & T_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
T_{m1} & T_{m2} & \cdots & T_{mm}
\end{bmatrix},
$$

where $T_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, \ldots, m$.

A matrix $T \in \mathbb{M}_{mn}(\mathbb{C})$ with Cartesian decomposition $T = \text{Re} T + i \text{Im} T$ is said to be accretive-dissipative if both $\text{Re} T$ and $\text{Im} T$ are positive semidefinite. We will represent $\text{Re} T$ and $\text{Im} T$ in our work as

$$
\text{Re} T = \hat{A} = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
A^*_{12} & A_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
A^*_{1m} & \cdots & A_{mm}
\end{bmatrix} \quad \text{and} \quad \text{Im} T = \hat{B} = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1m} \\
B^*_{12} & B_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
B^*_{1m} & \cdots & B_{mm}
\end{bmatrix},
$$

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where \( A_{ij}, B_{ij} \in \mathbb{M}_n(\mathbb{C}) \) for \( i, j = 1, 2, \ldots, m \).

A principal submatrix of a square matrix \( A \) is the matrix obtained by deleting any \( j \) rows and the corresponding \( j \) columns.

On \( \mathbb{M}_n(\mathbb{C}) \), a norm \( \| \cdot \| \) satisfying the invariance property that \( \| UAV \| = \| A \| \) for every \( A, U, V \in \mathbb{M}_n(\mathbb{C}) \) where \( U, V \) are unitary is said to be unitarily invariant.

For \( A \in \mathbb{M}_n(\mathbb{C}) \) and \( B \in \mathbb{M}_{mn}(\mathbb{C}) \), the inequality \( \| A \| \leq \| B \| \) means that

\[
\| A \oplus 0 \oplus \cdots \oplus 0 \| \leq \| B \|,
\]

where the direct sum \( A \oplus 0 \oplus \cdots \oplus 0 \) is the matrix in \( \mathbb{M}_{mn}(\mathbb{C}) \) defined by

\[
A \oplus 0 \oplus \cdots \oplus 0 = \begin{bmatrix}
A & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}.
\]

The Ky Fan \( k \)–norms \( \| \cdot \|_{(k)} \) \( (k = 1, \ldots, n) \) are the norms defined on \( \mathbb{M}_n(\mathbb{C}) \) by \( \| T \|_{(k)} = \sum_{j=1}^{k} s_j(T), k = 1, \ldots, n \), where \( s_1(T) \geq \cdots \geq s_n(T) \) are the eigenvalues of the matrix \( |T| = (T^*T)^{1/2} \) arranged in decreasing order. The Ky Fan dominance principle asserts that, for every unitarily invariant norm, we have

\[
\| A \| \leq \| B \| \iff \| A \|_{(k)} \leq \| B \|_{(k)} \text{ for } k = 1, \ldots, n. \quad (1.2)
\]

Let \( \zeta \) be the class of all functions \( f \) that are increasing and nonnegative on \( [0, \infty) \) and satisfies the condition: If \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) are two decreasing sequences of nonnegative real numbers such that \( \prod_{j=1}^{k} x_j \leq \prod_{j=1}^{k} y_j \) for \( k = 1, 2, \ldots, n \), then

\[
\prod_{j=1}^{k} f(x_j) \leq \prod_{j=1}^{k} f(y_j) \text{ for } k = 1, 2, \ldots, n.
\]

A nonnegative function \( f \) defined on \( [0, \infty) \) is called submultiplicative if \( f(mn) \leq f(m)f(n) \) whenever \( m, n \in [0, \infty) \).

In [6], [12], [15], and [16], a norm inequalities that compare \( T \) with its diagonal blocks have been given.

In [8], it has been shown that for an accretive-dissipative \( 2 \times 2 \) block matrix \( T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C}) \), we have

\[
\left\| f\left(|T_{12}|^2\right) + f\left(|T_{21}|^2\right) \right\| \leq \left\| f\left(|T|^2\right) \right\| \quad (1.3)
\]

\[
\left\| f\left(|T_{12}|^2\right) + f\left(|T_{21}|^2\right) \right\| \leq 4 \left\| f\left(|T|^2/4\right) \right\| \quad (1.4)
\]

\[
\left\| f\left(|T_{12}|^2\right) + f\left(|T_{21}|^2\right) \right\| \leq \|f^p(2|T_{11}|)|^{1/p}\|f^q(2|T_{22}|)|^{1/q} \quad (1.5)
\]

and

\[
\left\| f\left(|T_{12}|^2\right) + f\left(|T_{21}|^2\right) \right\| \leq 4 \|f^p(|T_{11}|)|^{1/p}\|f^q(|T_{22}|)|^{1/q}, \quad (1.6)
\]

where in the inequality (1.3) \( f \) is a function convex and increasing on \([0, \infty)\) with \( f(0) = 0 \), in the inequality (1.4) \( f \) is a function concave and increasing on \([0, \infty)\) with \( f(0) = 0 \), in the inequality (1.5) \( f \in \zeta \) is submultiplicative convex function with \( f(0) = 0 \) and \( p, q \in (0, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), and in the inequality (1.6) \( f \in \zeta \) is submultiplicative concave function with \( f(0) = 0 \) and \( p, q \in (0, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).
In this paper, some norm inequalities concerning with accretive-dissipative block matrices in $\mathbb{M}_{mn}(\mathbb{C})$ ($m \in \{2, 3, 4, \ldots\}$) are given. In Section 2, some unitarily invariant norm inequalities that compare the accretive-dissipative matrix $T$ to its off-diagonal blocks, where $T$ is partitioned as in (1.1) are derived. In Section 3, a unitarily invariant norm inequalities for functions $f \in \zeta$ are presented. In Section 4, some results for a $2 \times 2$ accretive-dissipative block matrices are given.

2. SOME UNITARILY INVARIANT NORM INEQUALITIES

In this section, we give some unitarily invariant norm inequalities that compare the accretive-dissipative matrix $T$ to its off-diagonal blocks, where $T$ is partitioned as in (1.1). To start our work, we will use the following lemma (see [13]).

**Lemma 2.1.** Let $A = \begin{bmatrix} X & B \\ B^* & Y \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$ be positive semidefinite. Then
\[ 2s_j(B) \leq s_j(A) \]
for $j = 1, 2, \ldots, n$.

The following lemma can be shown easily depending on the inequality (1.2).

**Lemma 2.2.** Let $X, Y \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $f$ be a function that is increasing and nonnegative on $[0, \infty)$. If $s_j(X) \leq s_j(Y)$ for $j = 1, 2, \ldots, n$, then
\[ ||f(X)|| \leq ||f(Y)||. \]

The following lemma, which is essentially due to Fan and Hoffman [5], can be concluded from Lemma 3.2 in [12] or Proposition III.5.1 in [2, p. 73].

**Lemma 2.3.** Let $T \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then
\[ s_j(\text{Re} T) \leq s_j(T) \text{ and } s_j(\text{Im} T) \leq s_j(T) \]
for $j = 1, 2, \ldots, n$.

Also, we need the following lemma (see [10, p. 149]) which is essential in our work.

**Lemma 2.4.** Let $X \in \mathbb{M}_n(\mathbb{C})$ and let $Y$ be a principal submatrix. Then
\[ s_j(Y) \leq s_j(X) \]
for $j = 1, 2, \ldots, n$.

In the following lemma, part (a) is an extension of Theorem 2.3 in [1] for $n$-tuples of operators (see also [9, Theorem 1]), a stronger version of part (b) of the lemma can be obtained by invoking an argument similar to that used in the proof of Proposition 4.1 in [14]. For various Jensen type matrix inequalities, we refer to [3] and references therein. Part (c) can be found in [11] and we can find part (d) in [4]. Henceforth, we assume that every function is continuous.

**Lemma 2.5.** Let $A_1, \ldots, A_n \in \mathbb{M}_n(\mathbb{C})$ be positive and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then
\[
(a) \quad \left\| f \left( \sum_{j=1}^n \alpha_j A_j \right) \right\| \leq \left\| \sum_{j=1}^n \alpha_j f(A_j) \right\| \quad \text{for every function } f \text{ that is convex and nonnegative on } [0, \infty),
\]
\[
(b) \quad \left\| \sum_{j=1}^n \alpha_j f(A_j) \right\| \leq \left\| f \left( \sum_{j=1}^n \alpha_j A_j \right) \right\| \quad \text{for every function } f \text{ that is concave and nonnegative on } [0, \infty).
\]
Theorem 2.6. or [7, pp. 47, 82] are essential in the proof of Lemma 2.5.

Note that the Fan (dominance and maximum) principles (see, e.g., [2, pp. 24, 93] or [7, pp. 47, 82]) are essential in the proof of Lemma 2.5.

Our first main result in this section is the following theorem.

**Theorem 2.6.** Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f$ be a function that is convex and increasing on $[0, \infty)$ where $f(0) = 0$. Then

\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\| \leq f \left( \frac{m^2 - m}{2} |T|^2 \right). \tag{2.1}
\]

**Proof.** Let $C_{ij} = \begin{bmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$, then $C_{ij}$ is a principal submatrix of $T$, it follows that $C_{ij}$ is accretive-dissipative with Cartesian decomposition $C_{ij} = \begin{bmatrix} A_{ij} & A_{ij} \\ A_{ij} & A_{jj} \end{bmatrix} + i \begin{bmatrix} B_{ii} & B_{ij} \\ B_{ij} & B_{jj} \end{bmatrix}$. Using Lemma 2.1 and Lemma 2.2, we get that

\[
\left\| f \left( (2m^2 - 2m) |A_{ij}|^2 \right) \right\| \leq f \left( \frac{m^2 - m}{2} (\text{Re} \ C_{ij})^2 \right) \tag{2.2}
\]

and

\[
\left\| f \left( (2m^2 - 2m) |B_{ij}|^2 \right) \right\| \leq f \left( \frac{m^2 - m}{2} (\text{Im} \ C_{ij})^2 \right). \tag{2.3}
\]

Also, using Lemmas 2.2 and 2.3, we have

\[
\left\| f \left( \frac{m^2 - m}{2} (\text{Re} \ C_{ij})^2 \right) \right\| \leq f \left( \frac{m^2 - m}{2} |C_{ij}|^2 \right) \tag{2.4}
\]

and

\[
\left\| f \left( \frac{m^2 - m}{2} (\text{Im} \ C_{ij})^2 \right) \right\| \leq f \left( \frac{m^2 - m}{2} |C_{ij}|^2 \right). \tag{2.5}
\]

Now,

\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\|
\]

\[
\leq \left\| f \left( \sum_{i<j} \left( |T_{ij}|^2 + |T_{ji}|^2 \right) \right) \right\| \quad \text{(by Lemma 2.5(c))}
\]

\[
= \left\| f \left( \sum_{i<j} \left( |A_{ij} + iB_{ij}|^2 + |A_{ij} - iB_{ij}|^2 \right) \right) \right\|
\]

\[
= \left\| f \left( 2 \sum_{i<j} \left( |A_{ij}|^2 + |B_{ij}|^2 \right) \right) \right\|
\]
Since $C_{ij}$ is a principal submatrix of $T$, it can be inferred from Lemmas 2.4 and 2.2 that
\[ \left\| f \left( \sum_{i<j} |C_{ij}|^2 \right) \right\| \leq \left\| f \left( \frac{m^2 - m}{2} |C_{ij}|^2 \right) \right\|. \quad (2.7) \]
Now, the result follows from the inequalities (2.6) and (2.7).

Note that the inequality (1.3) follows by taking $m = 2$ in the inequality (2.1). So, the inequality (2.1) gives a generalization to the inequality (1.3).

Applications of Theorem 2.6 will be given in the following corollaries.

**Corollary 2.7.** Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f$ be a function that is convex and increasing on $[0, \infty)$ where $f(0) = 0$. For all $p \geq 2$, we have
\[ \left\| f \left( \left( \sum_{i<j} |T_{ij}|^2 \right)^{p/2} \right) \right\| + f \left( \left( \sum_{i<j} |T_{ji}|^2 \right)^{p/2} \right) \right\| \leq \left\| f \left( \frac{m^2 - m}{2} |T|^2 \right)^{p/2} \right\|. \quad (2.8) \]
In particular, when $m = 2$, we have
\[ \left\| f (|T_{12}|^p) + f (|T_{21}|^p) \right\| \leq \left\| f (|T|^p) \right\|. \]

**Proof.** The inequality (2.8) follows by applying the inequality (2.1) to the convex function $f(p/2)$.
Corollary 2.8. Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then

$$\left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) - 2I_n \right\| \leq \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) - I_{mn} \right\| .$$

Proof. The proof follows by applying the inequality (2.1) to the function $f(t) = e^t - 1$ which is a convex function that is increasing on $[0, \infty)$ with $f(0) = 0$. $\square$

Corollary 2.9. Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then

$$\left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) \right\| \leq \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) + I_{mn} \right\| . \tag{2.9}$$

Proof. Applying Corollary 2.8 to the Ky Fan $k$–norms, we have

$$\left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) - 2I_n \right\| \oplus 0 \oplus ... \oplus 0 \leq \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) - I_{mn} \right\|_{(k)}$$

for $k = 1, ..., mn$. Thus, for $k = 1, ..., n$, we have

$$\left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) \right\| \oplus 0 \oplus ... \oplus 0 \leq -2k$$

$$= \left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) - 2I_n \right\| \oplus 0 \oplus ... \oplus 0 \right\|_{(k)} \leq \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) - I_{mn} \right\|_{(k)}$$

and for $k = n + 1, ..., mn$, we have

$$\left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) \right\| \oplus 0 \oplus ... \oplus 0 \leq -2k$$

$$\leq \left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) \right\| \oplus 0 \oplus ... \oplus 0 \right\|_{(k)} \leq -2n$$

$$= \left\| e \left( \sum_{i \leq j} |T_{ij}|^2 \right) + e \left( \sum_{i \leq j} |T_{ji}|^2 \right) - 2I_n \right\| \oplus 0 \oplus ... \oplus 0 \right\|_{(k)} \leq \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) - I_{mn} \right\|_{(k)}$$

$$= \left\| e \left( \frac{m^2 - m}{2} |T|^2 \right) \right\|_{(k)} - k. \tag{2.11}$$
From the inequalities (2.10) and (2.11), we have
\[
\left\| \left( e^{\frac{1}{2} \sum_{i<j} |T_{ij}|^2} + e^{\frac{1}{2} \sum_{i<j} |T_{ji}|^2} \right) \oplus 0 \oplus \cdots \oplus 0 \right\|_{(k)} \\
\leq \left\| e^{\frac{1}{2} \sum_{i<j} |T_{ij}|^2} \right\|_{(k)} + k \\
= \left\| e^{\frac{1}{2} \sum_{i<j} |T_{ij}|^2} + I_{mn} \right\|_{(k)}
\]
for \( k = 1, \ldots, mn \). Now, the inequality (2.9) follows from the inequality (2.12) and the Ky Fan dominance principle.

Our second main result in this section can be stated as follows.

**Theorem 2.10.** Let \( T \in \mathbb{M}_{mn}(\mathbb{C}) \) be a partitioned accretive-dissipative matrix as given in (1.1), and let \( f \) be a function that is concave and increasing on \([0, \infty)\) where \( f(0) = 0 \). Then
\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\| \leq (2m^2 - 2m) \left\| f \left( \frac{|T|^2}{4} \right) \right\|.
\]

**Proof.** Let \( C_{ij} = \begin{bmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C}) \), then \( C_{ij} \) is a principal submatrix of \( T \), it follows that \( C_{ij} \) is accretive-dissipative with Cartesian decomposition \( C_{ij} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij}^* & A_{jj} \end{bmatrix} + i \begin{bmatrix} B_{ii} & B_{ij} \\ B_{ij}^* & B_{jj} \end{bmatrix} \). Using Lemma 2.1 and Lemma 2.2, we get that
\[
\left\| f \left( |A_{ij}|^2 \right) \right\| \leq \left\| f \left( \frac{(\text{Re } C_{ij})^2}{4} \right) \right\|
\]
and
\[
\left\| f \left( |B_{ij}|^2 \right) \right\| \leq \left\| f \left( \frac{(\text{Im } C_{ij})^2}{4} \right) \right\|.
\]
And by Lemma 2.2 and Lemma 2.3, we have
\[
\left\| f \left( \frac{\text{Re } C_{ij}}{4} \right) \right\| \leq \left\| f \left( \frac{|C_{ij}|^2}{4} \right) \right\|
\]
and
\[
\left\| f \left( \frac{\text{Im } C_{ij}}{4} \right) \right\| \leq \left\| f \left( \frac{|C_{ij}|^2}{4} \right) \right\|.
\]
Now,
\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\| \\
= \left\| f \left( \sum_{i<j} |A_{ij} + iB_{ij}|^2 \right) + f \left( \sum_{i<j} |A_{ij} - iB_{ij}|^2 \right) \right\|
\]
Corollary 2.11. Let $T \in M_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1) and let $f$ be a function that is concave and increasing on $[0, \infty)$ where $f(0) = 0$. For all $0 < p \leq 2$, we have

$$
\left\| f \left( \left( \sum_{i<j} |T_{ij}|^2 \right)^{p/2} \right) + f \left( \left( \sum_{i<j} |T_{ji}|^2 \right)^{p/2} \right) \right\| \leq (2m^2 - 2m) \left\| f \left( \frac{T^2}{4} \right) \right\|.
$$

(2.20)

In particular, when $m = 2$, we have

$$
\left\| f \left( |T_{12}|^p \right) + f \left( |T_{21}|^p \right) \right\| \leq 4 \left\| f \left( \frac{T^p}{2^p} \right) \right\|.
$$

Proof. The inequality (2.20) follows by applying the inequality (2.13) to the concave function $f(p^{p/2})$. 

\[ \square \]
Corollary 2.12. Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then
\[
\left\| \log \left( \left( \sum_{i<j} |T_{ij}|^2 \right)^{1/2} + I_n \right) + \log \left( \left( \sum_{i<j} |T_{ji}|^2 \right)^{1/2} + I_n \right) \right\| \\
\leq (2m^2 - 2m) \| \log (|T| + 2I_{mn}) - (\log 2) I_{mn} \|.
\]

Proof. The proof follows by taking $p = 1$ and applying the inequality (2.20) to the function $f(t) = \log(t + 1)$ which is a concave and increasing function on $[0, \infty)$ and satisfies that $f(0) = 0$. \hfill \Box

The following corollary can be obtained by applying Theorems 2.6 and 2.10 to the function $f(t) = t^{p/2}$.

Corollary 2.13. Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then
\[
\left\| \left( \sum_{i<j} |T_{ij}|^2 \right)^{p/2} + \left( \sum_{i<j} |T_{ji}|^2 \right)^{p/2} \right\| \leq \left( \frac{m^2 - m}{2} \right)^{p/2} |||T|||^{p/2} \quad \text{for all } p \geq 2
\]
and
\[
\left\| \left( \sum_{i<j} |T_{ij}|^2 \right)^{p/2} + \left( \sum_{i<j} |T_{ji}|^2 \right)^{p/2} \right\| \leq \frac{m^2 - m}{2^{p-1}} |||T|||^{p} \quad \text{for all } 0 < p \leq 2.
\]

3. Unitarily Invariant Norm Inequalities Involving a Special Class of Functions

In this section, we give unitarily invariant norm inequalities including functions belongs to the class $\zeta$.

We start this section with the following lemma (see [8]).

Lemma 3.1. Let $A = \begin{bmatrix} X & B \\ B^* & Y \end{bmatrix} \in \mathbb{M}_{n^2}(\mathbb{C})$ be positive semidefinite, and let $f \in \zeta$ be submultiplicative function. If $p$ and $q$ are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\left\| f(|B|^2) \right\| \leq |||f^p(X)|||^{1/p} |||f^q(Y)|||^{1/q}.
\]

Our first result in this section is the following theorem.

Theorem 3.2. Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f \in \zeta$ be a function that is convex and submultiplicative and satisfies that $f(0) = 0$. If $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\| \leq 2\left( \frac{m^2}{m^2 - m} \right) \left( \left\| f^p \left( \sqrt{2m^2 - 2m} |T_{ii}| \right) \right\|^{1/p} \left\| f^q \left( \sqrt{2m^2 - 2m} |T_{jj}| \right) \right\|^{1/q} \right). (3.2)
\]
Proof. Since $[A_{ij}]$ and $[B_{ij}]$ are positive semidefinite matrices, and by Lemma 3.1, we have

$$
\|f \left( (2m^2 - 2m) |A_{ij}|^2 \right) \| \leq \|f^p \left( \sqrt{2m^2 - 2mA_{ii}} \right) \|^{1/p} \|f^q \left( \sqrt{2m^2 - 2mA_{jj}} \right) \|^{1/q} \tag{3.3}
$$

and

$$
\|f \left( (2m^2 - 2m) |B_{ij}|^2 \right) \| \leq \|f^p \left( \sqrt{2m^2 - 2mB_{ii}} \right) \|^{1/p} \|f^q \left( \sqrt{2m^2 - 2mB_{jj}} \right) \|^{1/q} \tag{3.4}
$$

Now,

$$
\left\| \sum_{i<j} |T_{ij}|^2 \right\| + \left\| \sum_{i<j} |T_{ji}|^2 \right\| \leq \left\| \sum_{i<j} \left( |T_{ij}|^2 + |T_{ji}|^2 \right) \right\| \quad \text{(by Lemma 2.5(c))}
$$

$$
= \left\| \sum_{i<j} \left( |A_{ij} + iB_{ij}|^2 + |A_{ij} - iB_{ij}|^2 \right) \right\|
$$

$$
= \left\| \sum_{i<j} \left( 2 \sum_{i<j} \left( |A_{ij}|^2 + |B_{ij}|^2 \right) \right) \right\|
$$

$$
\leq \frac{1}{2} \left\| \sum_{i<j} \left( 4 \sum_{i<j} \left( |A_{ij}|^2 \right) \right) \right\| + \left\| \sum_{i<j} \left( 4 \sum_{i<j} \left( |B_{ij}|^2 \right) \right) \right\| \quad \text{(by Lemma 2.5(a))}
$$

$$
\leq \frac{1}{2m^2 - m} \left( \left\| \sum_{i<j} \left( (2m^2 - 2m) |A_{ij}|^2 \right) \right\| + \left\| \sum_{i<j} \left( (2m^2 - 2m) |B_{ij}|^2 \right) \right\| \right) \quad \text{(by Lemma 2.5(a))}
$$

$$
\leq \frac{1}{m^2 - m} \sum_{i<j} \left( \|f \left( (2m^2 - 2m) |A_{ij}|^2 \right) \| + \|f \left( (2m^2 - 2m) |B_{ij}|^2 \right) \| \right)
$$

$$
\leq \frac{1}{m^2 - m} \sum_{i<j} \left( \|f^p \left( \sqrt{2m^2 - 2mA_{ii}} \right) \|^{1/p} \|f^q \left( \sqrt{2m^2 - 2mA_{jj}} \right) \|^{1/q} + \right)
$$

$$
\left( \|f^p \left( \sqrt{2m^2 - 2mB_{ii}} \right) \|^{1/p} \|f^q \left( \sqrt{2m^2 - 2mB_{jj}} \right) \|^{1/q} \right)
$$

(by the inequalities (3.3) and (3.4))

$$
\leq \frac{1}{m^2 - m} \sum_{i<j} \left( \left( \|f^p \left( \sqrt{2m^2 - 2mA_{ii}} \right) \| + \|f^p \left( \sqrt{2m^2 - 2mB_{ii}} \right) \| \right)^{1/p} \times \right)
$$

(by Hölder’s inequality)
Theorem 3.3. The inequality (3.2) gives a generalization to the inequality (1.5). Now, the result follows from the inequalities (3.5)-(3.9).

Since the matrices $T_{ii}$ are accretive-dissipative for $i = 1, \ldots, m$, it follows from Lemmas 2.2 and 2.3 that

\[
\left\| f^p \left( \sqrt{2m^2 - 2mRe T_{ii}} \right) \right\| \leq \left\| f^p \left( \sqrt{2m^2 - 2m|T_{ii}|} \right) \right\| \tag{3.6}
\]

\[
\left\| f^p \left( \sqrt{2m^2 - 2mIm T_{ii}} \right) \right\| \leq \left\| f^p \left( \sqrt{2m^2 - 2m|T_{ii}|} \right) \right\| \tag{3.7}
\]

\[
\left\| f^q \left( \sqrt{2m^2 - 2mRe T_{ii}} \right) \right\| \leq \left\| f^q \left( \sqrt{2m^2 - 2m|T_{ii}|} \right) \right\| \tag{3.8}
\]

and

\[
\left\| f^q \left( \sqrt{2m^2 - 2mIm T_{ii}} \right) \right\| \leq \left\| f^q \left( \sqrt{2m^2 - 2m|T_{ii}|} \right) \right\| \tag{3.9}
\]

Now, the result follows from the inequalities (3.5)-(3.9).

Note that the inequality (1.5) follows by taking $m = 2$ in the inequality (3.2). So, the inequality (3.2) gives a generalization to the inequality (1.5).

**Theorem 3.3.** Let $T \in \mathbb{M}_{m \times m}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f \in \zeta$ be a function that is concave and submultiplicative and satisfying that $f(0) = 0$. If $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then

\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\| \leq 4 \sum_{i<j} \left( \left\| f^p \left( |T_{ii}| \right) \right\|^{1/p} \left\| f^q \left( |T_{jj}| \right) \right\|^{1/q} \right).
\]

**Proof.**

\[
\left\| f \left( \sum_{i<j} |T_{ij}|^2 \right) + f \left( \sum_{i<j} |T_{ji}|^2 \right) \right\|
\]

\[
= \left\| f \left( \sum_{i<j} |A_{ij} + iB_{ij}|^2 \right) + f \left( \sum_{i<j} |A_{ij} - iB_{ij}|^2 \right) \right\|
\]

\[
\leq 2 \left\| f \left( \frac{\sum_{i<j} \left( |A_{ij} + iB_{ij}|^2 + |A_{ij} - iB_{ij}|^2 \right)}{2} \right) \right\| \tag{by Lemma 2.5(b)}
\]

\[
= 2 \left\| f \left( \sum_{i<j} \left( |A_{ij}|^2 + |B_{ij}|^2 \right) \right) \right\|
\]

\[
\leq 2 \sum_{i<j} \left\| f \left( |A_{ij}|^2 + |B_{ij}|^2 \right) \right\| \tag{by Lemma 2.5(d)}
\]

\[
\leq 2 \sum_{i<j} \left\| f \left( |A_{ij}|^2 + |B_{ij}|^2 \right) \right\| \tag{by Lemma 2.5(d)}
\]

\[
\leq 2 \sum_{i<j} \left\| f \left( |A_{ij}|^2 \right) + f \left( |B_{ij}|^2 \right) \right\| \tag{by Lemma 2.5(d)}
\]
\[ \leq 2 \sum_{i<j} \left( \| f(A_{ij})^2 \| + \| f(|B_{ij}|^2) \| \right) \]
\[ \leq 2 \sum_{i<j} \left( \| f^p(A_{ii}) \|^{1/p} \| f^q(A_{jj}) \|^{1/q} + \| f^p(B_{ii}) \|^{1/p} \| f^q(B_{jj}) \|^{1/q} \right) \]
(by Lemma 3.1)
\[ \leq 2 \sum_{i<j} \left( \| f^p(A_{ii}) \| \| f^q(A_{jj}) \| \right)^{1/p} \left( \| f^q(B_{ii}) \| \| f^q(B_{jj}) \| \right)^{1/q} \]
(by Hölder’s inequality)
\[ \leq 2 \sum_{i\neq j} \left( \| f^p(T_{ii}) \| \| f^q(T_{jj}) \| \right)^{1/p} \left( \| f^q(T_{jj}) \| \| f^q(T_{jj}) \| \right)^{1/q} \]
(by Lemmas 2.2 and 2.3)
\[ \leq 4 \sum_{i<j} \left( \| f^p(|T_{ii}|) \| \| f^q(|T_{jj}|) \| \right)^{1/q} . \]

\[ \Box \]

Note that the inequality (1.6) follows by taking \( m = 2 \) in the inequality (3.10). So, the inequality (3.10) gives a generalization to the inequality (1.6).

4. Some results for 2 × 2 block matrices

In this section, our results consider the case when \( T \) partitioned as in (1.1) with \( m = 2 \). Our first result in this section is the following theorem.

**Theorem 4.1.** Let \( T \in \mathbb{M}_{2n}(\mathbb{C}) \) be a partitioned accretive-dissipative matrix as given in (1.1), and let \( f \) be a function that is increasing on \([0, \infty)\) with \( f(0) = 0 \) such that \( f(\sqrt{t}) \) is convex. Then
\[ \| f(|T_{12}| + |T_{21}^*|) + f(||T_{12}| - |T_{21}^*||) \| \leq \| f\left(\sqrt{2} |T|\right)\| . \]

**Proof.** Let \( g(t) = f(\sqrt{t}), t \in [0, \infty) \). Then \( g \) is an increasing convex function on \([0, \infty)\).

Since \( T = A+iB \) with \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \) and \( B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} \) are positive semidefinite, it follows from Lemma 2.1 and Lemma 2.2 that
\[ \| g\left(8 |A_{12}|^2\right)\| \leq \| g\left(2 (Re T)^2\right)\| \text{ and } \| g\left(8 |B_{12}|^2\right)\| \leq \| g\left(2 (Im T)^2\right)\| . \]

Also, using Lemmas 2.2 and 2.3, we have
\[ \| g\left(2 (Re T)^2\right)\| \leq \| g\left(2 |T|^2\right)\| \text{ and } \| g\left(2 (Im T)^2\right)\| \leq \| g\left(2 |T|^2\right)\| . \]

Now,
\[ \| f(|T_{12}| + |T_{21}^*|) + f(||T_{12}| - |T_{21}^*||) \|
\[ = \| g\left(\left(|T_{12}| + |T_{21}^*|\right)^2\right) + g\left(||T_{12}| - |T_{21}^*||^2\right)\|
\[ \leq \| g\left(\left(|T_{12}| + |T_{21}^*|\right)^2 + ||T_{12}| - |T_{21}^*||^2\right)\| \] (by Lemma 2.5(c))
\[ = \| g\left(2 |T_{12}|^2 + 2 |T_{21}^*|^2\right)\|
\[ = \| g\left(4 |A_{12}|^2 + 4 |B_{12}|^2\right)\|
\[ \leq \frac{1}{2} \| g\left(8 |A_{12}|^2\right) + g\left(8 |B_{12}|^2\right)\| \] (by Lemma 2.5(a))
\[ \leq \frac{1}{2} \left\| g \left( 8 |A_{12}|^2 \right) \right\| + \frac{1}{2} \left\| g \left( 8 |B_{12}|^2 \right) \right\| \]
\[ \leq \frac{1}{2} \left\| g \left( 2 (\text{Re} \, T)^2 \right) \right\| + \frac{1}{2} \left\| g \left( 2 (\text{Im} \, T)^2 \right) \right\| \quad \text{(by the inequalities (4.2))} \]
\[ \leq \left\| g \left( 2 |T|^2 \right) \right\| \quad \text{(by the inequalities (4.3))} \]
\[ = \left\| f \left( \sqrt{2} |T| \right) \right\|. \]

The following example asserts that the convexity of the function \( f(\sqrt{t}) \) given in Theorem 4.1 is essential and cannot be replaced by \( f(t) \) to be convex.

**Example 4.2.** Consider \( T = A + iB = \begin{bmatrix} 1 + i & 2i \\ 0 & 1 + i \end{bmatrix} \). Then \( A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) are positive semidefinite matrices. Take \( f(t) = t \), then for the spectral norm \( \left\| \cdot \right\| \), the right hand side of the inequality (4.1) equals 4 and the left hand side of the inequality (4.1) equals \( \sqrt{8 + 4\sqrt{3}} \), but \( 4 \not\leq \sqrt{8 + 4\sqrt{3}} \).

The following corollary gives an application on Theorem 4.1.

**Corollary 4.3.** Let \( T \in \mathbb{M}_{2n}(\mathbb{C}) \) be a partitioned accretive-dissipative matrix as given in (1.1). Then
\[ \left\| \left( ||T_{12}| + |T_{21}^*| \right) + ||T_{12} - |T_{21}^*| \right\| \leq 2^{p/2} \left\| \left| T \right|^p \right\| \quad \text{for all } p \geq 2. \]

**Proof.** The proof follows by applying the inequality (4.1) to the function \( f(t) = t^p, p \geq 2. \)

Our second result in this section is given in the following theorem.

**Theorem 4.4.** Let \( T \in \mathbb{M}_{2n}(\mathbb{C}) \) be a partitioned accretive-dissipative matrix as given in (1.1), and let \( f \) be a function that is increasing on \([0, \infty)\) with \( f(0) = 0 \) such that \( f(\sqrt{t}) \) is concave. Then
\[ \left\| f \left( ||T_{12}| + |T_{21}^*| \right) + f \left( ||T_{12} - |T_{21}^*| \right) \right\| \leq 4 \left\| f \left( \left| T \right|^2 \right) \right\|. \quad (4.4) \]

**Proof.** Let \( g(t) = f(\sqrt{t}), t \in [0, \infty) \). Then \( g \) is a concave and increasing function on \([0, \infty)\).

Since \( T = A + iB \) with \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \) and \( B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} \) are positive semidefinite, using Lemma 2.1 and Lemma 2.2, we get that
\[ \left\| g \left( 2 |A_{12}|^2 \right) \right\| \leq \left\| g \left( \frac{\text{Re} \, T)^2}{2} \right) \right\| \quad \text{and} \quad \left\| g \left( 2 |B_{12}|^2 \right) \right\| \leq \left\| g \left( \frac{\text{Im} \, T)^2}{2} \right) \right\|. \quad (4.5) \]

Also, using Lemmas 2.2 and 2.3, we have
\[ \left\| g \left( \frac{\text{Re} \, T^2}{2} \right) \right\| \leq \left\| g \left( \frac{|T|^2}{2} \right) \right\| \quad \text{and} \quad \left\| g \left( \frac{\text{Im} \, T^2}{2} \right) \right\| \leq \left\| g \left( \frac{|T|^2}{2} \right) \right\|. \quad (4.6) \]

Now,
\[ \left\| f \left( ||T_{12}| + |T_{21}^*| \right) + f \left( ||T_{12} - |T_{21}^*| \right) \right\| \]
\[ = \left\| g \left( (||T_{12}| + |T_{21}^*|)^2 \right) + g \left( ||T_{12} - |T_{21}^*| \right)^2) \right\|. \]
\[ \begin{align*}
\leq & 2 \left\| g \left( \frac{(|T_{12}| + |T_{21}|)^2 + |T_{12}| - |T_{21}|)}{2} \right) \right\| \quad \text{(by Lemma 2.5(b))} \\
= & 2 \left\| g \left( |T_{12}|^2 + |T_{21}|^2 \right) \right\| \\
= & 2 \left\| g \left( 2 |A_{12}|^2 + 2 |B_{12}|^2 \right) \right\| \\
\leq & 2 \left( \left\| g \left( 2 |A_{12}|^2 \right) \right\| + \left\| g \left( 2 |B_{12}|^2 \right) \right\| \right) \\
\leq & 2 \left( \left\| g \left( \left( \frac{\text{Re} T}{2} \right)^2 \right) \right\| + \left\| g \left( \left( \frac{\text{Im} T}{2} \right)^2 \right) \right\| \right) \quad \text{(by the inequalities (4.5))} \\
\leq & 4 \left\| g \left( \frac{|T|^2}{2} \right) \right\| \quad \text{(by the inequalities (4.6))} \\
= & 4 \left\| f \left( \frac{|T|}{\sqrt{2}} \right) \right\|. 
\end{align*} \]

We conclude this paper by the following corollary.

**Corollary 4.5.** Let \( T \in \mathbb{M}_{2n}(\mathbb{C}) \) be partitioned accretive-dissipative matrix as given in (1.1). Then
\[
\| (|T_{12}| + |T_{21}|)^p + |T_{12}| - |T_{21}| \|^p \leq 2^{2-p/2} \| |T|^p \| \quad \text{for all } 0 < p \leq 2.
\]

**Proof.** The proof follows by applying the inequality (4.4) to the function \( f(t) = t^p, 0 < p \leq 2 \). \( \square \)

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