Fractional Kinetic Equations

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Abstract—We develop the idea of non-Markovian CTRW (continuous time random walk) approximation to the evolution of interacting particle systems, which leads to a general class of fractional kinetic measure-valued evolutions with variable order. We prove the well-posedness of the resulting new equations and present a probabilistic formula for their solutions. Though our method are quite general, for simplicity we treat in detail only the fractional versions of the interacting diffusions. The paper can be considered as a development of the ideas from the works of Belavkin and Maslov devoted to Markovian (quantum and classical) systems of interacting particles.

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1. INTRODUCTION

The derivation of kinetic equations for the interacting particles as the dynamic law of large numbers for Markovian models of interacting particles was seemingly initiated in [1] and further developed by numerous authors in a variety of forms, see, e.g., [2]–[5]. In this paper we aim to develop these approach for semi-Markovian systems of interacting particles modeled as continuous time random walks (CTRW) with non-exponential waiting times. The resulting new kinetic equations turn out to be fractional in time of variable position-dependent fractional order. All general information on fractional calculus that we are using can be found in the books [6]–[8].

As was stressed in [9], the standard diffusion process governed by the equation

\[ \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} a(x) \frac{\partial^2 u}{\partial x^2}(t, x) \]  

(1.1)

with a positive continuous function \( a(x) \), can be obtained as the limit of appropriate random walks. These prelimiting random walks can be quite different. For example, approximating random walks in continuous time can be chosen as jump type processes with the generators

\[ L_h f(x) = \frac{1}{2h^2} [f(x + h\sqrt{a(x)}) + f(x - h\sqrt{a(x)}) - 2f(x)] \]

or as jump type processes with the generators

\[ \tilde{L}_h f(x) = \frac{a(x)}{2h^2} [f(x + h) + f(x - h) - 2f(x)]. \]

Indeed, the operators \( L_h \) and \( \tilde{L}_h \) tend to \((1/2)a(x)(d^2/dx^2)\) as \( h \to 0 \), as well as the jump-type processes generated by operators \( L_h \) and \( \tilde{L}_h \) converge to the diffusion generated by the operator

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In the first approximation, the diffusion coefficient $a(x)$ is responsible for the size of jumps (with constant intensity), and in the second approximation, it is responsible for the intensity of jumps (with constant sizes). The “rough” diffusion limit does not feel the difference. The situation changes, if we model jump-type approximations via CTRW with non-exponential waiting times. If we make the diffusion coefficient responsible for the size of jumps and take waiting times from the domain of attraction of an $\alpha$-stable law with a constant intensity $a$, then the standard scaling would lead (in the limit of small jumps and large intensities) to the most standard fractional diffusion equation

$$D_{0+}^{\alpha} u(t, x) = \frac{1}{2} a(x) \left[ \frac{\partial^2 u}{\partial x^2}(t, x) \right],$$

where $D_{0+}^{\alpha}$ is the so-called Caputo–Dzerbasyan fractional derivative.

If we will use the CTRW approximations with fixed jump sizes and use $a(x)$ to distinguish intensities at different points, then (as will be shown) we get in the limit the equation with a variable position-dependent fractional derivative:

$$D_{0+}^{\alpha(x)} u(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x).$$

Of course, for any decomposition of the diffusion coefficient in a product $b(x)c(x)$ with positive functions $b(x)$ and $c(x)$, one can use $b(x)$ as a function controlling the intensity of jumps and $c(x)$ as a function controlling the spread of the jumps. In this scenario we get in the limit the equation

$$D_{0+}^{ab(x)} u(t, x) = \frac{1}{2} c(x) \frac{\partial^2 u}{\partial x^2}(t, x).$$

In [9] the rigorous derivation of equations of type (1.4) was given. In the present paper, following the same idea of the separation of diffusion coefficients in parts standing for times and sizes of jumps, we derive the fractional kinetic equations of variable order describing the scale limits of CTRW approximating systems of interacting diffusions.

Kinetic equations are measure-valued equations expressing the dynamic law of large number (LLN) limit of Markovian interacting particle systems when the number of particles tends to infinity. The resulting nonlinear measure-valued evolutions can be interpreted probabilistically as nonlinear Markov processes, see [10]. In case of a finite state space, the set of probability measures coincides with a simplex $\Sigma_n$ of sequences of nonnegative numbers $x = (x_1, \ldots, x_n)$ such that $\sum_j x_j = 1$, where $n$ can be any natural number. In a discrete case, the kinetic equations, describing the dynamic law of large number (LLN) limit of Markovian interacting particle systems when the number of particles tends to infinity, are ODEs of the form

$$\dot{x} = xQ(x) \quad \iff \quad \dot{x}_i = \sum_k x_k Q_{ki}(x) \quad \text{for all } i,$$

where $Q$ is a stochastic or Kolmogorov $Q$-matrix (that is, it has non-negative non-diagonal elements and the elements on each row sum up to zero) depending Lipschitz continuously on $x$.

The evolution of functions $F(t, x) = F(X(t, x))$, where $X(t, x)$ denotes the solution of equation (1.5) with the initial condition $x$, satisfies the equation

$$\frac{\partial}{\partial t} F(t, x) = (x, Q(x) \nabla F(t, x)) = \sum_{k,i} x_k Q_{ki}(x) \frac{\partial F(t, x)}{\partial x_i}.$$ (1.6)

More generally (see [10]), for a system of mean-field interacting particles given by a family of operators $A_\mu$, which are generators of Markov processes in some Euclidean space $\mathbb{R}^d$, depending on probability measures $\mu$ on $\mathbb{R}^d$ as parameters and having a common core, the natural scaling limit of such system, as the number of particles tends to infinity (dynamic LLN), is described by the kinetic equations in the weak form

$$(f, \mu_t) = (A_\mu f, \mu_t),$$

where $f$ is an arbitrary function from the common core of the operators $A_\mu$. 

MATHEMATICAL NOTES Vol. 112 No. 4 2022
The corresponding generalization of equation (1.6) is the differential equation in variational derivatives:

\[
\frac{\partial F(t, \mu)}{\partial t} = \int_{\mathbb{R}^d} \left( A_\mu \frac{\delta F(t, \mu)}{\delta \mu(\cdot)} \right) (z) \mu(dz). \tag{1.8}
\]

One can distinguish the cases of evolutions preserving the number of particles and changing the number of particles. The second case arises from pure jump processes. It was treated in [11]. In this paper we will deal with the first case.

**Remark 1.** The proof of the convergence result in [11] has a gap. Here we use a slightly different method (which can be used also to correct the arguments of [11]) based on the ideas from [9], [12].

All these equations are derived as the natural scaling limits of some random walks on the space of the configurations of many-particle systems.

When the standard random walks are extended to more general CTRWs (continuous time random walks), characterized by the property that the random times between jumps are not exponential, but with tail probabilities decreasing as a power function, their limits turn to non-Markovian processes described by the fractional evolutions.

For instance, one can write the Caputo–Dzherbashyan fractional derivative of some fixed order \(\alpha\) instead of the usual one in (1.7), as was done, e.g., in [13]. However, taking into account the natural possibility of different waiting times for jumps from different states (as for usual Markovian approximation), one would obtain an equation of a more complicated type, with the fractional derivatives of position-dependent order. In [14] this derivation was performed for the case of a discrete state space, that is for a nonlinear Markov chain described by equation (1.5), leading to the fractional generalization of equation (1.6) of the form

\[
D_{t^{-\mu}}^{(x, \alpha)} F(s, x) = (x, Q(x)\nabla F(s, x)), \tag{1.9}
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is the vector describing the power laws of the waiting times in various states \(\{1, \ldots, n\}\) and \(D_{t^{-\mu}}^{(x, \alpha)}\) is the right Caputo–Dzherbashyan fractional derivative of order \((x, \alpha)\) depending on the position \(x\) and acting on the time variable \(t\).

As we will show, the corresponding limiting equation for the mean-field interacting particle systems in \(\mathbb{R}^d\) preserving the number of particles writes down as the equation

\[
D_{t^{-\mu}}^{(x, \alpha, \mu)} F(s, \mu) = \int_{\mathbb{R}^d} \left( A_\mu \frac{\delta F(s, \mu)}{\delta \mu(\cdot)} \right) (z) \mu(dz), \tag{1.10}
\]

where \(A_\mu\) is the family of generators of Feller processes in \(\mathbb{R}^d\), depending on \(\mu \in \mathcal{M}(\mathbb{R}^d)\). We also supply the well-posedness of this equation and the probabilistic formula for the solutions. For transparency we will deal only with the case of diffusion operators \(A_\mu\).

The content of the paper is as follows. In the next section we recall preliminary material from the theory of kinetic equations. In Sec. 3 we construct the CTRW approximations to the systems of mean-field interacting particles for the case of non-exponential waiting times between jumps. In Sec. 4 we formulate our main results on the convergence of CTRW approximations to the new kinetic equations with fractional variable order. We also supply a probabilistic formula for the solutions of these equations. In Secs. 5–7 we give the proves of the main results. In additional section 8 we recall (in a convenient for us form) the jump-type approximations to stable generators.

The bold letters \(\mathbf{E}\) and \(\mathbf{P}\) will be used to denote expectation and probability. If not mentioned otherwise, the norm on functions \(\|f\|\) denotes the usual sup-norm.

For a locally compact space \(X\), we denote by \(C(X)\) the Banach space of bounded continuous functions on \(X\), by \(C_\infty(X)\) its subspace of functions vanishing at infinity, and by \(\mathcal{M}(X)\) the space of bounded Borel measures on \(X\). For \(f \in C(X)\) and \(\mu \in \mathcal{M}(X)\), the usual pairing is given by the integration:

\[
(f, \mu) = \int f(x) \mu(dx).
\]
For a locally compact space $X$ let us denote by $C^k(\mathcal{M}(X))$ the space of functions $F$ on $\mathcal{M}(X)$ such that the variational derivatives $\delta^lF(Y)/\delta Y(x_1)\ldots\delta Y(x_l)$ are well defined for all $l \leq k$, $Y \in \mathcal{M}(X)$ and represent a continuous mapping of $(k+1)$ variables (measures are equipped with their weak topology) and $\delta^l F(Y)/(\delta Y(\cdot)\ldots\delta Y(\cdot))$ belong to $C_\infty(X^l)$ uniformly for $Y$ from bounded sets.

We will work mostly with the case $X = \mathbb{R}^d$. Then we will denote by $C^k_\infty(X)$ the subspace of $C_\infty(X)$ consisting of the functions that have derivatives up to order $k$ that belong to $C_\infty(X)$. We will denote by $C^{1,k}(\mathcal{M}(X))$ the subspace of $C^1(\mathcal{M}(X))$ consisting of the functions $F$ such that the variational derivatives $\delta^lF(Y)/\delta Y(\cdot)$ belong to $C^k_\infty(X)$ uniformly for $Y$ from bounded sets.

By $C^{2,1}\times(\mathcal{M}(X))$ we denote the subspace of $C^2(\mathcal{M}(X))$ consisting of the functions $F$ such that the derivatives

$$\frac{\partial}{\partial x} \frac{\delta^2F(Y)}{\delta Y(x)\delta Y(y)}, \quad \frac{\partial}{\partial y} \frac{\delta^2F(Y)}{\delta Y(x)\delta Y(y)}, \quad \frac{\partial^2}{\partial x \partial y} \frac{\delta^2F(Y)}{\delta Y(x)\delta Y(y)}$$

are well defined and belong to $C_\infty(X^2)$.

2. PRELIMINARIES: KINETIC EQUATIONS

Let us recall some basic notations and facts from the standard measure-valued-limits of interacting processes.

For simplicity let us take the state space of one particle as the Euclidean space $X = \mathbb{R}^d$. Denoting by $X_0$ a one-point space and by $X^j$ the powers $X \times \cdots \times X$ ($j$ times), we denote by $\mathcal{S}$ their disjoint union $\mathcal{S} = \bigcup_{j=0}^\infty X^j$, which stands for the state space of a random number of similar particles. We denote the elements of $\mathcal{S}$ by bold letters, say $\mathbf{x}$, $\mathbf{y}$.

Let us denote by $S\mathcal{S}$ (or $SX^k$, respectively) the quotient space of $\mathcal{S}$ (or $X^k$, respectively) obtained by factorization with respect to all permutations.

A key role in the theory of measure-valued limits of interacting particle systems is played by the scaled inclusion $S\mathcal{S}$ to $\mathcal{M}(X)$ given by

$$\mathbf{x} = (x_1, \ldots, x_N) \mapsto \frac{\delta x_1 + \cdots + \delta x_N}{N} = \delta \mathbf{x}_N, \quad \text{(2.1)}$$

which defines a bijection between $S\mathcal{S}$ and the set $\mathcal{M}_N^+(X)$ of normalized finite sums of Dirac’s $\delta$-measures.

Remark 2. Let us stress that we are using here a non-conventional notation: $\delta \mathbf{x} = \delta x_1 + \cdots + \delta x_N$, which is convenient for our purposes.

Let $L^\text{diff}_\mu$ be a family of diffusion operators in $C_\infty(X)$ (where $X = \mathbb{R}^d$), i.e.,

$$L^\text{diff}_\mu f(x) = \frac{1}{2} \text{tr} \left( G(x) \frac{\partial^2 f}{\partial x^2} \right) + \left( b_\mu(x), \frac{\partial f}{\partial x} \right), \quad \text{(2.2)}$$

with the drift coefficient depending continuously on $\mu \in \mathcal{M}(X)$.

Let us choose some jump-type approximations $L^h_\mu$ to these operators, that is, $L^h_\mu f(x) = \frac{1}{h^2} \int [f(x + hy) - f(x)] p_\mu(x, dy)$, (2.3) with some probability kernels $p_\mu(x, dy)$, so that

$$\|L^\text{diff}_\mu f - L^h_\mu f\| \to 0 \quad \text{as} \quad h \to 0 \quad \text{(2.4)}$$

for all $f \in D$. 

MATHEMATICAL NOTES Vol. 112 No. 4 2022
The simplest such approximation is of the form
\[ L^h f(x) = \frac{1}{h^2} \sum_{i=1}^{d} [f(x + h b_i e_i) + f(x - h b_i e_i) - 2f(x)] \]
\[ + \frac{1}{h^2} \sum_{i>j} [f(x + h b_{ij} e_i + e_j) + f(x - h b_{ij} e_i + e_j) - 2f(x)], \]
(2.5)
with appropriate functions \( b_i, b_{ij} \) (depending on \( x \) and \( \mu \)), where \( \{e_i\} \) is the standard basis in \( \mathbb{R}^d \).

As our basic one-particle generators we choose operators of the form
\[ A_\mu f(x) = a(x)L^\mu_{\text{diff}} f(x) \]
(2.6)
with some bounded from above and below positive function \( a(x) \). Let us stress again that the separation of the multiplier \( a(x) \) is not intrinsic. It just shows our choice of the part of the diffusion generator that will be responsible for the waiting times of jumps in our future CTRW approximation. The corresponding approximation will be denoted by \( A^h_\mu = a(x)L^h_\mu. \)

We will work under the following conditions.

**Condition A.** The functions \( G(\cdot) \) and \( b_\mu(\cdot) \) belong to \( C^2(\mathbb{R}^d) \), and the latter with the norms bounded for bounded \( \mu \) and \( G \) is uniformly elliptic; both functions have norms bounded for bounded \( \mu \). In particular, this implies that \( D = C^2_\infty(\mathbb{R}^d) \) is a common core for all Feller semigroups generated by \( L^\mu_{\text{diff}} \).

**Condition B.** The function \( b(x) \) belongs to \( C^{2;1 \times 1}(\mathcal{M}(\mathbb{R}^d)) \) uniformly in \( x \), and \( \delta b_\mu(\cdot)/\delta \mu(y) \) belongs to \( C^1(\mathbb{R}^d) \) uniformly in \( y \).

**Condition C.** The function \( a(x) \) belongs to \( C^2(\mathbb{R}^d) \) and is bounded from above and below by positive constants.

**Remark 3.** These conditions can be relaxed in many directions. One can also include the case with \( G \) and \( a \) depending on \( \mu \). Essentially what one needs is the well-posedness of kinetic equation (1.7) with \( A = L^\mu_{\text{diff}} \) and twice continuous differentiability of the solutions with respect to initial data. Chap. 6 of [15] contains various criteria for this to hold.

To any family of one-particle differential operators \( B_\mu \) on \( C_\infty(\mathbb{R}^d) \) with coefficients depending on the mean-field \( \mu \), there corresponds an operator \( \tilde{B} \), acting on the space of functions \( C(SX^N) \) on \( N \)-particle states (for any \( N \)) by the formula
\[ \tilde{B} f(x) = \sum_{j=1}^{N} B^j_{\delta x/N} f(x), \]
(2.7)
where \( B^j_{\mu} \) denotes the operator \( B_\mu \) acting on the variable \( x_j \).

**Remark 4.** In physics this extension of \( B_\mu \) from one-particle states to the operator \( \tilde{B} \) on multi-particle states is referred to as the second quantization.

By the transformation (2.1) we can transfer the operator \( \tilde{B} \) on \( C(SX) \) to the operator \( \tilde{B} \) on \( C(\mathcal{M}_\delta^+(X)) \). If \( B = A \) is a diffusion operator above, then, by Proposition 9.2 of [10],
\[ \tilde{A} F(Y) = \int_{\mathbb{R}^d} \left( A_Y \frac{\delta F(Y)}{\delta Y(\cdot)} \right)(z) Y(dz) + O\left( \frac{1}{N} \right), \]
(2.8)
for \( x = (x_1, \ldots, x_N), Y = \delta_x/N \) and any \( F \in C^{2;1 \times 1}(\mathcal{M}(\mathbb{R}^d)) \) with \( O(1/N) \) being uniform for \( F \) from bounded subsets of \( F \in C^{2;1 \times 1}(\mathcal{M}(\mathbb{R}^d)) \).
This makes it plausible to conclude (proof can be found in [10]) that, as $N \to \infty$, the process generated by $\tilde{A}$ converges weakly to a deterministic process on measures generated by the operator on the right-hand side of (1.8), that this process is given by the solution of kinetic equation (1.7). Note that (1.7) is obtained from (1.8) by choosing $F$ to be a linear function $F(\mu) = (f, \mu)$. We aim at the extension of this result for CTRW approximations with non-exponential waiting times.

**Remark 5.** Evolution (1.8) can be used to study the behavior of various functionals of the limit process. However, for our purpose, equation (1.8) is used as an auxiliary tool that allows us to exploit the theory of operator semigroups.

### 3. CTRW MODELING OF INTERACTING PARTICLE SYSTEMS

Our objective is to obtain the dynamic LLN for interacting multi-particle systems for the case of non-exponential waiting times between jumps having a power tail distributions.

Recall that a positive random variable $\sigma$ with a probability law $P$ on $[0, \infty)$ is said to have a power tail of index $\alpha$ if

$$P(\sigma > t) \sim \frac{\kappa}{t^\alpha}$$

for large $t$, that is, the ratio of the l.h.s. and the r.h.s tends to $1$ as $t \to \infty$. Here $\kappa > 0$ is a positive constant.

The power tails are invariant under taking minima, in the same way, as the exponential tails are. Namely, if $\sigma_j$, $j = 1, \ldots, d$, are independent variables with a power tail of indices $\alpha_j$ and normalizing constants $\kappa_j$, then $\sigma = \min(\sigma_1, \ldots, \sigma_d)$ is clearly a variable with a power tail of index $\alpha = \alpha_1 + \cdots + \alpha_d$ and normalizing constant $\kappa_1 \cdots \kappa_d$.

In full analogy with the case of exponential times, let us assume here that the waiting time of the particle at $x_i$ has the power tail with the index $\alpha(x_i) = \alpha x_i(x_i)$ with some fixed $\alpha \in (0, 1)$. For simplicity assume that the normalizing constant $\kappa$ equals 1. Consequently, the minimal waiting time of all $N$ points in a collection $x = (x_1, \ldots, x_N)$ will have some probability distribution $Q_x(dr)$ with a tail of the index $\alpha x A(x)$, where $A(x) = \sum_i a(x_i)$.

To simplify presentation we will make the distribution $Q_x$ more precise. Namely, we assume that these distributions have continuous densities $Q_x(r)$ such that

$$Q_x(r) = \alpha x A(x) r^{-1-\alpha x A(x)} \quad \text{for} \quad r \geq B \quad \text{and} \quad Q_{s,x}(r) \leq 1 \quad \text{for all} \quad r$$

with some $B > 0$ uniformly for all $x, \tau$.

Our approximating process with power tail waiting times controlled by the function $a(x)$ and the jumps governed by the operator $L_\tau^P$ can be described probabilistically as follows. Starting from any time and current state $x$, we wait a random waiting time $\sigma$, which has the distribution $Q_x(dr)$. When $\sigma$ rings, a particle at $x_i$ that makes a transition, is chosen according to the probability law $a(x_i)/A(x)$, and then it makes an instantaneous transition to $y$ according to the distribution $p_{\delta_x/N}(x_i, dy)$. Then this procedure repeats starting from the new state $x \setminus x_i \cup y$.

In order to derive the LLN in this case, the usual trick is to lift this non-Markovian evolution on the subsets $x = (x_1, \ldots, x_N)$ or the corresponding measures $\delta_x/N$ to the discrete time Markov chain on $(M^+_\mu, S^\tau_{\mu,s})$ on $M^+_\mu = (X) \times \mathbb{R}$ by considering the total waiting time $s$ as an additional space variable. Additionally, one has to adhere to the usual scaling of the waiting time for the jumps of CTRW (see, e.g., [16] or [17]). Thus we will consider the Markov chain $(M^\tau_{\mu,s}, S^\tau_{\mu,s})(k\tau)$ on $M^+_\mu = (X) \times \mathbb{R}$ with the jumps occurring at discrete times $k\tau$, $k \in \mathbb{N}$, such that the process at a state $(x, s)$ at time $\tau k$ jumps to $(x \setminus x_i \cup (x_i + hy), s + \tau_1^{1/\alpha x A(x)} r)$ or equivalently a state $(\delta_x/N, s)$ jumps to the state $((\delta_x - \delta_{x_i} + \delta_{x_i+hy})/N, s + \tau_1^{1/\alpha x A(x)} r)$, where first $x_i$ is chosen according to the law $a(x_i)/A(x)$, and then $y$ according to the law $p_{\delta_x/N}(x_i, dy)$, where $r$ is distributed by $Q_x(dr)$.
We have three natural small parameters in the model: $1/N$ related to the number of particles, $h^2$ related to the size of jumps, and $\tau$ related to the waiting times. In order to obtain a reasonable limit, we link them as $\tau = 1/N = h^2$. Then
\[
\tau A(x) = \left( a, \frac{\delta_x}{N} \right) = \int a(y) \left( \frac{\delta_x}{N} \right) (dy).
\]
Assuming that $\alpha \sup_x a(x) < 1$, we ensure that $\alpha \tau A(x) < 1$ for all $x$.

The transition operator of the chain $(M^\tau_{\mu,s}, S^\tau_{\mu,s})(k\tau)$ is given by
\[
U^\tau F(\mu, s) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} Q_x (dr) \sum_i p_{\mu}(x_i, dy) \frac{a(x_i)}{A(x)} F(\mu - \tau \delta x_i + \tau \delta x_i + hy, s + \tau^{1/\alpha} A(x) \tau),
\]
where $\mu = \tau \delta x = \tau \sum_j \delta x_j$.

We can define the scaled CTRW of mean-field interacting particles with power-tail waiting times controlled by the function $a(x)$ and the jumps governed by the operator $L^\mu_{\alpha}$, as the value of the first coordinate $M^\tau_{\mu,s}$ of the chain $(M^\tau_{\mu,s}, S^\tau_{\mu,s})(k\tau)$, evaluated at the random time $k\tau$ when the second coordinate (expressing the total waiting time) $S^\tau_{\mu,s}(k\tau)$ reaches $t$, that is, at the time
\[
k\tau = T^\tau_{\mu,s}(t) = \inf \{ m\tau : S^\tau_{\mu,s}(m\tau) \geq t \},
\]
so that $T^\tau_{\mu,s}$ is the inverse process to $S^\tau_{\mu,s}$.

Thus this scaled CTRW is the (non-Markovian) process
\[
\widetilde{M}^\tau_{\mu,s}(t) = M^\tau_{\mu,s}(T^\tau_{\mu,s}(t)).
\]

Our main results concern the limit of this process as $\tau \to 0$.

4. MAIN RESULTS

Recall that we will always assume that Conditions A–C hold true.

Let us see first of all what happens with the process $(M^\tau_{\mu,s}, S^\tau_{\mu,s})(k\tau)$ in the limit as $\tau = h^2 \to 0$.

Namely, we are interested in the weak limit of the chains with transitions $[U^\tau]^{t/\tau}$, where $[t/\tau]$ denotes the integer part of the number $t/\tau$ as $\tau \to 0$.

It is well known [17], [18] that if such a chain converges to a Feller process, then the generator of this limiting process can be obtained as the limit
\[
\Lambda F = \lim_{\tau \to 0} \frac{1}{\tau} (U^\tau F - F).
\]

Let us introduce the space $\widehat{C}$ of continuous functions $F(\mu, s)$ on $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}$ such that $F(\mu, s)$ belongs to $C^1(\mathbb{R})$ as a function of $s$ and to $C^2(\mathbb{R}) \cap C^1(\mathbb{R}) \cap C^1(\mathbb{R})$ as a function of $\mu$.

**Lemma 1.** Assume that $F \in \widehat{C}$. If $\mu = \tau \delta x$ converges weakly to some measure, which we will also denote by $\mu$ with some abuse of notation, as $\tau \to 0$, then
\[
\Lambda F(\mu, s) = \lim_{\tau \to 0} \frac{1}{\tau} (U^\tau F - F)(\mu, s)
\]
exists and
\[
\Lambda F(\mu, s) = \alpha(a, \mu) \int_0^\infty \frac{F(\mu, s + r) - F(\mu, s)}{r^{1+\alpha(a, \mu)}} dr + \frac{1}{a(\mu)} \int_{\mathbb{R}^d} a(z) \left( \int_{\mathbb{R}^d} \frac{\delta F(\mu, z)}{\delta \mu(\cdot)} (z) \mu(dz) \right).
\]

**Theorem 1.** (i) The operator (4.2) generates a contraction semigroup in the space of continuous functions $F(\mu, x)$ such that $F(\mu, \cdot) \in C^1(\mathbb{R}^d)$ uniformly on bounded subsets of $\mathcal{M}(\mathbb{R}^d)$ with an invariant core $\widehat{C}$. It specifies a Markov process $(M_{\mu,s}(t), S_{\mu,s}(t))$ in $\mathcal{M}(\mathbb{R}) \times \mathbb{R}$ such that...
• the first coordinate does not depend on \( s \) (and thus can be denoted shortly \( M_\mu(t) \)), is deterministic, and solves the kinetic equation
\[
\frac{d}{dt}(f, \mu_t) = \frac{a(\cdot)}{(a, \mu)} (L^\text{diff}_\mu f, \mu_t);
\] (4.3)

• the second coordinate is a time nonhomogeneous stable-like subordinator generated by the time dependent family of generators
\[
\Lambda^t_s g(s) = \alpha(a, \mu_s) \int_0^\infty \frac{g(s + r) - g(s)}{r^{1+\alpha(a, \mu_s)}} dr,
\] (4.4)

this subordinator has a smooth transition probability density \( G(u; \mu, s; S) \) (for the transition from \( s \) to \( S \) during the time interval \([0, u]\)).

(ii) The discrete time Markov chains \((M^\tau_\mu, S^\tau_\mu)\) given by (3.2) converge weakly to the Markov process \((M_\mu, S_\mu)\), that is the corresponding finite-dimensional distributions converge. Moreover, the convergence is uniform on compact subsets of time, so that, in particular, for any continuous bounded function \( F \),
\[
\lim_{\tau \to 0, k \tau \to t} \mathbb{E} F((M^\tau_\mu, S^\tau_\mu)(k\tau)) = \mathbb{E} F((M_\mu, S_\mu)(t))
\] (4.5)
uniformly for \( t \) from any compact set.

Remark 6. An important recent contribution [19] allows one to identify a proper topology on the space of probability measures that allows one to identify appropriate spaces of functions \( F(\mu, x) \), where the Markov semigroups of the type given above become strongly continuous.

Finally we can formulate our main result.

Theorem 2. (i) The marginal distributions of the process
\[
\tilde{M}_{\mu,s}(t) = M_\mu(T_{\mu,s}(t)),
\] (4.6)
where
\[
T_{\mu,s}(t) = \inf\{u : S_{\mu,s}(u) \geq t\}
\]
is the inverse process to \( S_{\mu,s} \), can be expressed explicitly as
\[
\mathbb{E}[F(\tilde{M}_{\mu,s}(t))] = \int_{0}^{\infty} du \int_{0}^{t} dS(t - S)^{-\alpha(a, M_\mu(u))} G(u; \mu, s; S) F(M_\mu(u)).
\] (4.7)

Its cut-off version writes down as
\[
\mathbb{E} \left[ F(\tilde{M}_{\mu,s}(t)) 1 \left( T_t \in \left[ \frac{1}{K}, K \right] \right) \right] = \int_{1/K}^{K} du \int_{0}^{t} dS(t - S)^{-\alpha(a, M_\mu(u))} G(u; \mu, s; S) F(M_\mu(u)).
\] (4.8)

(ii) The marginal distributions of the scaled CTRW of mean-field interacting particles (3.3) converge to the marginal distributions of the process (4.6), i.e.,
\[
\lim_{\tau \to 0} \mathbb{E} F(\tilde{M}^{\tau}_{\mu,s}(t)) = \mathbb{E} F(\tilde{M}_{\mu,s}(t))
\] (4.9)
for a bounded continuous functions \( F(\mu) \).
(iii) For any smooth function $F(\cdot) \in C^{2,1}(\mathcal{M}(\mathbb{R}^d)) \cap C^{1,2}(\mathcal{M}(\mathbb{R}^d))$ (with continuous bounded variational derivative), the evolution of averages $F(\mu, s) = EF(\hat{M}_{\mu,s}(t))$ satisfies the mixed fractional kinetic equation

$$D_t^{\alpha(a,\mu)} F(\mu, s) = \frac{1}{(a, \mu)} \int_{\mathbb{R}^d} a(z) \left( L_{\mu}^{\text{df}} \frac{\delta F}{\delta \mu(z)} \right) (z) \mu(dz), \quad s \in [0, t],$$

(4.10)

with the terminal condition $F(\mu, t) = F(\mu)$, where the right fractional derivative acting on the variable $s \leq t$ of $F(\mu, s)$ is defined as

$$D_t^{\alpha(a,\mu)} g(s) = -\alpha(a, \mu) \int_0^{t-s} \frac{g(s+y) - g(s)}{y^{1+\alpha(a,\mu)}} dy - \alpha(a, \mu)(g(t) - g(s)) \int_{t-s}^\infty \frac{dy}{y^{1+\alpha(a,\mu)}}. \quad (4.11)$$

5. PROOF OF LEMMA 1

We have

$$\frac{1}{\tau} (U_t F - F)(\mu, s)$$

$$= \frac{1}{\tau} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q_x(dr)$$

$$\times \sum_i p(x_i, dy) \frac{a(x_i)}{A(x)} \left[ F(\mu - \tau \delta x_i + \tau \delta x_i + \sqrt{\tau} y, s + \tau^{1/\alpha(a,\mu)} r) - F(\mu, s) \right]$$

$$= \frac{1}{\tau} \int_{\mathbb{R}^d} Q_x(dr) \left[ F(\mu, s + \tau^{1/\alpha(a,\mu)} r) - F(\mu, s) \right]$$

$$+ \frac{1}{\tau} \int_{\mathbb{R}^d} \sum_i p(x_i, dy) \frac{a(x_i)}{A(x)} \left[ F(\mu - \tau \delta x_i + \tau \delta x_i + \sqrt{\tau} y, s) - F(\mu, s) \right] + R, \quad (5.1)$$

where the error term is

$$R = \frac{1}{\tau} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q_x(dr) \sum_i p(x_i, dy) \frac{a(x_i)}{A(x)} \left[ g_{i,y}(\mu, s + \tau^{1/\alpha A(x)} r) - g_{i,y}(\mu, s) \right].$$

Here

$$g_{i,y}(\mu, s) = F(\mu - \tau \delta x_i + \tau \delta x_i + \sqrt{\tau} y, s) - F(\mu, s).$$

Assuming that $\tau \delta x$ converges to some measure, which we also denote by $\mu$, as $\tau \to 0$, we can conclude by (8.3) that the first term in (5.1) converges as $\tau \to 0$ to

$$\alpha(a, \mu) \int_0^\infty \frac{F(\mu, s + r) - F(\mu, s)}{r^{1+\alpha(a,\mu)}} dr,$$

whenever $F$ is continuously differentiable in $s$. By (2.4) and the definition of the variational derivative, the second term in (5.1) converges as $\tau \to 0$ to

$$\sum_i \frac{a(x_i)}{A(x)} \left( L_{\mu}^{\text{df}} \frac{\delta F}{\delta \mu(x_i)} \right) = \frac{1}{(a, \mu)} \int_{\mathbb{R}^d} a(z) \left( L_{\mu}^{\text{df}} \frac{\delta F}{\delta \mu(z)} \right) (z) \mu(dz).$$

To estimate the term $R$ we note that if $F$ has a bounded derivative in $t$ and a bounded variational derivative in $\mu$, then $g(x)$ is uniformly bounded by $\tau$ and the derivative $|\partial g/\partial s|$ is uniformly bounded. Hence by (8.3) it follows that $R \to 0$ as $\tau \to 0$, implying (4.2).
6. PROOF OF THEOREM 1

(i) By Theorem 6.8.4 in [15] under Conditions A–C equation (1.7) with $A_{\mu} = a(\cdot)L_{\mu}^{\text{diff}}$ is well-posed in the set of probability measures, and moreover, the solutions are twice continuously differentiable with respect to the initial data. This implies that the resolving operators for the corresponding equation (1.8) preserve the space $C^{2,1,1}((\mathcal{M}(\mathbb{R}^d)) \cap C^{1,2}((\mathcal{M}(\mathbb{R}^d))$. As can be seen from the proof of this result, it still hold true if one changes $L_{\mu}^{\text{diff}}$ by a common multiplier $1/(a, \mu)$, and thus the same holds for equation (4.3). Therefore the second term of operator (4.2) generates a deterministic process $M_{\mu}(t)$ solving (4.3). Once $\mu_t$ are well defined, the operator (4.4) has the standard form of a stable-like subordinator (with a time dependent index of stability), implying the rest of statement (i). The last statement is a well known result for stable-like processes, see, e.g., [17].

(ii) The convergence follows directly from Lemma 1, statement (i), and the standard result (see, e.g., [18]) stating that the convergence (4.1) for $F$ from a core of the limiting process implies the weak convergence of the corresponding discrete Markov chains to this limiting process, which is uniform on compact subsets of time. More precisely, the result from [18] proves this fact for Markov chains in locally compact spaces, but the proof is seen to be valid for Markov chains with values in the space of probability measures on $\mathbb{R}^d$.

7. PROOF OF THEOREM 2

(i) If $M_{\mu}(s)$ is a continuous dynamic system in some metric space (starting in $\mu$ at time zero) and $\Sigma$ a positive random variable with density $g(\sigma)$, then for any $K > 0$ and a continuous bounded function $F(\mu)$,

$$
E \left[F(M_{\mu}(\Sigma))1\left(\Sigma \in \left[\frac{1}{K}, K\right]\right)\right] = \int_{1/K}^{K} F(M_{\mu}(s))g(s) \, ds. \quad (7.1)
$$

If $V_{v}(s)$ is an increasing Markov process (starting in $v$ at time zero) with time dependent generators $A_{M_{\mu}(s)}$ depending on $M_{\mu}(s)$ and with the transition density $G(s; \mu, v; V)$ (from $v$ at time zero to $V$ at time $s$), then the random vector $(M_{\mu}(s), V_{v}(u))$ has the distribution

$$
\phi_{\mu,v}(s, u; M, V) \, dM \, dV = \delta(M_{\mu}(s) - M)G(u; \mu, v; V) \, dV. \quad (7.2)
$$

Consequently,

$$
\frac{\partial}{\partial u} \phi_{\mu,v}(s, u; M, V) = \delta(M_{\mu}(s) - M)A_{M_{\mu}(u)}^{*}G(u; \mu, v; V) \, dV, \quad (7.3)
$$

where $A^{*}$ acts on the variable $V$.

Let the process $V_{v}(t)$ be strictly increasing a.s., so that its generalized inverse (or hitting times) process

$$
Z_{t} = \sup\{u \geq 0 : V_{v}(u) \leq t\} = \inf\{u \geq 0 : V_{v}(u) > t\}, \quad t > v,
$$

is continuous. Since $(Z_{t} \leq z) = (V_{v}(z) \geq t) = (V_{v}(z) > t)$ a.s., for the density $g_{\mu,v}(s, t; M, z)$ of the pair $(M_{\mu}(s), Z_{t})$, $s > 0, t > v$, the following formula holds:

$$
g_{\mu,v}(s, t; M, z) = \frac{\partial}{\partial z} \int_{t}^{\infty} \phi_{\mu,v}(s, z; M, V) \, dV = \delta(M_{\mu}(s) - M) \int_{t}^{\infty} A_{M_{\mu}(u)}^{*}G(z; \mu, v; V) \, dV. \quad (7.4)
$$

From (7.4) and (7.1) it follows that

$$
E \left[F(M_{\mu}(Z_{t}))1\left(Z_{t} \in \left[\frac{1}{K}, K\right]\right)\right] = \int_{1/K}^{K} du \int_{t}^{\infty} dV A_{M_{\mu}(u)}^{*}G(u; \mu, v; V)F(M_{\mu}(u)) \quad (7.5)
$$

and therefore

$$
E \left[F(M_{\mu}(Z_{t}))1\left(Z_{t} \in \left[\frac{1}{K}, K\right]\right)\right].
$$
\[
\int_{1/K}^{K} ds \int_{0}^{\infty} dV A_{\mu, (s, \theta \geq t)}(V) G(s; \mu, v; V) F(M_\mu(s)),
\]
where \( \theta \geq t \) is the indicator function of the interval \([t, \infty)\).

If
\[
A_\mu \phi(V) = \alpha(a, \mu) \int_{0}^{\infty} \frac{\phi(V + w) - \phi(V)}{w^{1+\alpha(a, \mu)}} dw,
\]
then
\[
A_\mu \theta \geq t(V) = \begin{cases} 
\alpha(a, \mu) \int_{t-V}^{\infty} \frac{dw}{w^{1+\alpha(a, \mu)}} = (t-V)^{-\alpha(a, \mu)}, & t > V; \\
0, & t \leq V
\end{cases}
\]

Therefore,
\[
E\left[ F(M(\mu)(Z_t)) \mathbf{1}\left( Z_t \in \left[ \frac{1}{K}, K \right] \right) \right] = \int_{1/K}^{K} du \int_{0}^{t} dV (t-V)^{-\alpha(a, \mu)} G(u; \mu, v; V) F(M_\mu(u)),
\]
which is (4.8) in slightly different notations. Passing to the limit as \( K \to \infty \) in (4.8) yields (4.7).

(ii) Step 1. As a starting point, we make some preliminary calculations for the subordinated Markov chains in discrete times.

Let \( M(k \tau) \) be an adapted process (with values in some metric space) on a stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) and let \( \Sigma \) be a stopping time with values in \( \{k \tau\}, k \in \mathbb{N} \). Let the random variables \( M(k \tau) \mathbf{1}(\Sigma = k \tau) \) have distributions \( g_{m \tau}(dM, k \tau) \). Then, for any \( K > 0 \) and a continuous bounded function \( F(M) \),
\[
E\left[ F(M(S)) \mathbf{1}\left( \Sigma \in \left[ \frac{1}{K}, K \right] \right) \right] = \sum_{k \tau \in [1/K, K]} \int g_{k \tau}(dM, k \tau) F(M).
\]

Let \( M_{\mu}(m \tau) \) be a Markov chain (with values in some metric space) with a transition operator
\[
U^\tau_M F(\mu) = \int F(M) Q^\tau(\mu, dM).
\]
Moreover, let the pair \((M_{\mu}(m \tau), S_{\mu,s}(m \tau))\) (with \( S \in \mathbb{R} \)) also form a Markov chain with the transition operator \( U^\tau \) of the form
\[
U^\tau F(\mu, v) = \int F(M, S) P^\tau(\mu, s; S, dM) dS = \int F(M, S) Q^\tau(\mu, dM) G^\tau(\mu, s; S) dS
\]
with some density \( G^\tau(\mu, s; S) \).

Then the distribution of the random vector \((M_{\mu}(k \tau), S_{\mu,s}(k \tau), S_{\mu,s}((k-1) \tau))\) is
\[
\int Q^\tau(R, dM) G^\tau(R, S; W) P_{(k-1) \tau}(\mu, s; S, dR) dS
\]
(integration carried out over the variable \( R \)), because
\[
E\left[ F(M_{\mu}(k \tau), S_{\mu,s}(k \tau), S_{\mu,s}((k-1) \tau)) \right] = \int E(F(M_{\mu}(k \tau), S_{\mu,s}(k \tau), S)) | M_{\mu}((k-1) \tau) = R, S_{\mu,s}((k-1) \tau) = S)
\times P_{(k-1) \tau}(\mu, s; S, dR) dS
= \int F(M, W, S) Q^\tau(R, dM) G^\tau(R, S; W) dW P_{(k-1) \tau}(\mu, s; S, dR) dS.
\]
Let the coordinate \( S_{\mu,s}(k\tau) \) be strictly increasing, and let
\[
Z^*_t = \sup\{m\tau: S_{\mu,s}(m\tau) \leq t\} = \inf\{m\tau: S_{\mu,s}(m\tau) > t\}, \quad t > s,
\]
be its (generalized) inverse (or hitting times) process.
Then
\[
\begin{align*}
E\left[F\left(M_{\mu}\left(Z^*_t\right)\right)1\left(Z^*_t \in \left[\frac{1}{K}, K\right]\right)\right] &= E \sum_{k\tau \in [1/K, K]} F\left(M_{\mu}(k\tau)\right)1\left(Z^*_t = k\tau\right) \\
&= E \sum_{k\tau \in [1/K, K]} F\left(M_{\mu}(k\tau)\right)1\left(S_{\mu,s}\left((k-1)\tau\right) < t \leq S_{\mu,s}(k\tau)\right) \\
&= \int F(M) \sum_{k\tau \in [1/K, K]} \int 1(S < t \leq W) Q_{\tau}(R, dM) G_{\tau}(R, S; W) dW \\
& \quad \times P_{(k-1)\tau}(\mu; s, S, dR) dS.
\end{align*}
\]
This can be rewritten as
\[
E\left[F\left(M_{\mu}\left(Z^*_t\right)\right)1\left(Z^*_t \in \left[\frac{1}{K}, K\right]\right)\right] = \int \sum_{k\tau \in [1/K, K]} (U^T F_S)(R, S) P_{(k-1)\tau}(\mu; s, dR) dS, \quad (7.11)
\]
where \( F_S(R, W) = F(R)1(S < t \leq W) \).

**Step 2.** In our case, \( U^T \) is given by (3.2) and, therefore, for \( R = \tau \sum_{j} \delta z_j \), (7.11) turns to the equation
\[
\begin{align*}
E\left[F\left(M_{\mu}\left(Z^*_t\right)\right)1\left(Z^*_t \in \left[\frac{1}{K}, K\right]\right)\right] &= \int \sum_{k\tau \in [1/K, K]} (U^T F_S)(R, S) P_{(k-1)\tau}(\mu; s, dR) dS, \\
&= \int_{R^d} \int_{0}^{t} \sum_{k\tau \in [1/K, K]} \int_{R^d} Q_{\tau}(dR) \sum_{i} p_{\mu}(z_i, dy) \frac{a(z_i)}{A(z)} F(R - \tau \delta z_i + \tau \delta z_i + \sqrt{\tau}y) \\
& \quad \times 1(S \leq t \leq S + \tau^{1/\alpha A(z)} \tau) P_{(k-1)\tau}(\mu; s, dR) dS.
\end{align*}
\]
We need to show that these expectations converge towards (4.8). By the density argument it is sufficient to show the convergence for smooth \( F \) only. One of the key observations is that the limit will be the same if we change \( F(R - \tau \delta z_i + \tau \delta z_i + h_y) \) to just \( F(R) \) as the difference will be small as compared to the latter limit. Therefore, due to the relation
\[
\sum_{i} \int p_{\mu}(z_i, dy) \frac{a(z_i)}{A(z)} = 1
\]
we need to show the convergence of the expressions
\[
\int_{R^d} \int_{0}^{t} \sum_{k\tau \in [1/K, K]} \int_{R^d} Q_{\tau}(dR) F(R)1(S \leq t \leq S + \tau^{1/\alpha A(z)} \tau) P_{(k-1)\tau}(\mu; s, dR) dS
\]
towards (4.8).

**Step 3.** The key point for the argument is that, by Theorem 1 the sums
\[
\tau \int_{R^d} \int_{0}^{t} \sum_{k\tau \in [1/K, K]} \Omega(R, S) P_{(k-1)\tau}(\mu; s, dR) dS, \quad (7.14)
\]
approximate uniformly the Riemannian sums for the integral
\[
\int_{0}^{t} dS \int_{[1/K, K]} du \Omega(M_{\mu}(u), S) G(u; \mu, s; S)
\]
and thus converge to this integral for bounded continuous functions \( \Omega \).
It follows that we can slightly reduce the domain of integration in (7.13). Namely, the limit of these expressions will be the same as for the expressions

\[
\int_{\mathbb{R}^+} Q_\tau(dr) F(R) \mathbf{1}(S \leq t \leq S + \tau^{1/\alpha} A(\tau) r) \times P_{(k-1)\tau}(\mu, s; S, dR) dS. \tag{7.16}
\]

In fact the difference of these expressions with (7.13) is bounded in magnitude by

\[
\|F\| \int_{\mathbb{R}^+} \sum_{k \in [1/K, K]} P_{(k-1)\tau}(\mu, s; S, dR) dS,
\]

which, according to (7.15) is of order $\tau^{1/\alpha} A(\tau) r^{-1}$, which tends to zero as $\tau \to 0$, because $\alpha A(\tau) < 1$.

Now let us rewrite expressions (7.16) as

\[
\int_{\mathbb{R}^+} Q_\tau(dr) [\theta_{\geq t}(S + \tau^{1/\alpha} A(\tau) r) - \theta_{\geq t}(S)] \times P_{(k-1)\tau}(\mu, s; S, dR) dS. \tag{7.17}
\]

The next key observation is that

\[
\theta_{\geq t}(S + \tau^{1/\alpha} A(\tau) r) - \theta_{\geq t}(S) = 0
\]

for $S < t - B\tau^{1/\alpha} A(\tau)$ and $r < B$. Therefore, by (8.1)

\[
\int_{\mathbb{R}^+} Q_\tau(dr) [\theta_{\geq t}(S + \tau^{1/\alpha} A(\tau) r) - \theta_{\geq t}(S)] = (\alpha A(\tau))^{-1} \int_{t-S}^{\infty} \theta_{\geq t}(S + r)r^{-1-\alpha A(\tau)} dr
\]

\[
= (\alpha A(\tau))^{-1} \int_{t-S}^{\infty} r^{-1-\alpha A(\tau)} dr = (t-S)^{-\alpha A(\tau)}
\]

for $S < t - B\tau^{1/\alpha} A(\tau)$.

Therefore expression (7.17) rewrites as

\[
\int_{\mathbb{R}^+} Q_\tau(dr) F(R)(t-S)^{-\alpha A(\tau)} P_{(k-1)\tau}(\mu, s; S, dR) dS. \tag{7.18}
\]

Noticing that changing the integration over $[0, t - B\tau^{1/\alpha} A(\tau)]$ back to $[0, t]$ does not spoil the limit and using (7.15) shows that expressions (7.18) converge to (4.8), as was claimed.

(iii) Standard probabilistic arguments (using Dynkin’s martingale) show that any solution to equation (4.10) has the probabilistic representation $F(\mu, s) = \mathbb{E}F(\tilde{M}_{\mu, s}(t))$. On the other hand, formula (4.7) allows one to check that this function does solve problem (4.10). These arguments are almost identical to the those used in [9] in a slightly different situation, and we omit them here.

8. APPENDIX: CONVERGENCE RATES FOR THE STANDARD CTRW

For convenient referencing, we recall here the basic scheme of jump-type approximations to stable generators.

**Proposition 1.** Let $p(y)$ be a probability density on $\mathbb{R}_+$ such that $p(y) = y^{-1-\alpha}$ for $y \geq B$ with some $\alpha \in (0, 1)$ and $B > 0$. Then

(i) for any bounded measurable $f$ having support on $[Bh, \infty)$,

\[
h^{-\alpha} \int_0^\infty f(hy)p(y) dy = \int_0^\infty \frac{f(y) dy}{y^{1+\alpha}}; \tag{8.1}
\]
(ii) for any continuous \( f \) on \( \mathbb{R}_+ \) such that \( f(0) = f(\infty) = 0 \) and \( f \) is Lipschitz at zero so that \( |f(y)| \leq Ly \) for \( y \in [0, Bh] \) and some constant \( L \), it follows that
\[
\left| h^{-\alpha} \int_0^\infty f(hy)p(y) \, dy - \int_0^\infty f(y) \, dy \right| \leq C_B h^{1-\alpha},
\]
where
\[
C_B = \frac{B^{1-\alpha}}{1-\alpha} + \int_0^B yp(y) \, dy.
\]

**Proof.** (i) It follows from the conditions on the support of the function \( f \) that we can replace \( p(y) \) by \( y^{-1-\alpha} \) and then make the change of variable \( hy = y' \) which completes the proof.

(ii) Let \( f \) have support on \( [0, Bh] \). Then the estimates
\[
\left| h^{-\alpha} \int_0^\infty f(hy) \, dy \right| = \left| h^{-\alpha} \int_0^B f(hy)p(y) \, dy \right| \leq h^{1-\alpha} L \int_0^B yp(y) \, dy,
\]
\[
\left| \int_0^\infty \frac{f(y) \, dy}{y^{1+\alpha}} \right| = \left| \int_0^{Bh} \frac{f(y) \, dy}{y^{1+\alpha}} \right| \leq L \int_0^{Bh} \frac{dy}{y^\alpha} = \frac{B^{1-\alpha}}{1-\alpha} Lh^{1-\alpha},
\]

imply (8.2).

In particular, setting \( \tau = h^\alpha \), it follows that
\[
\left| \tau^{-1} \int_0^\infty (f(x + \tau^{1/\alpha} y) - f(x))p(y) \, dy - \int_0^\infty \frac{f(x + y) - f(x)}{y^{1+\alpha}} \, dy \right| \leq C_B L\tau^{(1/\alpha)-1},
\]
where \( L \) is the sup of the derivative of \( f \) near \( x \).

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