A canonical curve of genus 17

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Abstract

We compute equations for a Hurwitz curve of genus 17 and we conclude that the canonical ideal of any Hurwitz curve of genus 14 or 17 is generated by quadrics.

1 Introduction

The curve alluded in the title is one of the 2 Hurwitz curves of genus 17 and the aim of the paper is to compute equations for its canonical embedding in \( \mathbb{P}^{16} \). Great deal of the work is about finding the quadrics \( I_2 \) in the curve’s homogeneous ideal \( I \) (see Theorem 9.1) and the rest is to verify that these quadrics generate \( I \). The results of this paper were originally meant to be part of [11] however some of the computations involved proved to be very tough and we could not reach any conclusion until long afterwards. So, to compute the space of quadrics in \( I \) we follow the method used in [11] for the case of the Hurwitz curves of genus 14; that is, we study the action of the curve’s automorphism group \( G \) on the space \( H^0(X, K_X^{\otimes d}) \) of global sections of tensor powers of the canonical sheaf (see §5). Then using a matrix representation of \( G \) for this action one tries to compute distinguished subspaces of \( I_2 \) whose sum is \( I_2 \), see §6, §7 and §8. We did not know the matrix representation of \( G \) for the action on the space \( H^0(X, K_X) \) until Vahid Dabbaghian implemented in his package Repsn of GAP [5] an algorithm to compute complex matrix representations of finite groups that worked for \( G \) (see [1]). Then the computations with some of the polynomials that we obtained turned out to be very slow and we were unable to work out the spaces corresponding to §7 and §8.

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until we used ParGap [3] to perform some parallel calculations to find the conjugating matrices (34) and (40). We did not need such matrices in [11] in the genus 14 case because the eigenspaces involved were 1-dimensional.

That I is generated by quadrics would follow at once from Petri’s Analysis of the Canonical Ideal (see for instance [13], [1], [12] or [14]) if one knew that the curve is not trigonal and although this is the case for a general curve of genus $g > 4$ a proof is required for this specific curve (we also check the genus 14 case). We used the quadrics to test the non-trigonality of the curve, see Theorem 10.1 for the genus 17 Hurwitz curve and Theorem 11.1 for the genus 14 case. Having the quadrics in the ideal available to do calculations this seemed to be a quick way to know the answer, however, this also turned out to be a very slow and long computation (at least with the algorithms that we used and I did not try to parallelize it, the tables in §11, §11 are meant to help speed up verification) and a simpler method would be desirable. Perhaps using a different interpretation of the curve one could apply tools similar to those used to determine the gonality of modular curves (see for instance [7], [8]). Nevertheless using the quadrics in the ideal to check the trigonality of modular curves is also among those tools (see §2 in [7] or §2.2 in [16]).

One can convince oneself from the calculation of the quadrics that, it is possible to obtain equations for the remaining genus 17 and genus 14 Hurwitz curves by applying appropriate elements of the absolute Galois group $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ to the coefficients of the equations of the curves considered here. In the genus 14 case, this had already been observed in [18]. It follows that the canonical ideal of any genus 14 or genus 17 Hurwitz curve is generated by quadrics.

Very little is known about the genus 17 Hurwitz curves and the information available to us about their automorphism group comes from [2] and [17]. One could highlight the fact that there is only one Hurwitz group $G$ of genus 17 and that it can be realized as the automorphism group of 2 non isomorphic genus 17 curves $X_1$ and $X_2$ (see Section 4 for the definition of $X_i$). Unlike the Hurwitz groups of genus 3, 7 and 14 our group $G$ is not simple and has a normal subgroup $N$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. One has that $G/N$ is $PSL(2, \mathbb{F}_7)$ and $X_1/N \cong X_2/N$ is the Hurwitz curve of genus 3. This of course plays a role in our calculation, although in a very subtle way, it helped us when we tried to find a point of the curve in §6 which was crucial to find some of the equations.

We remark that it is not known a way to represent a general curve of
genus \( g \) by means of polynomial equations, except for small values of \( g \) (see [6] Chap.6); and that although for some genera it is possible to give equations of special curves, it appears that for the genus 14 and genus 17 cases the Hurwitz curves are the only ones so far for which equations defining them can be computed.

The problem of computing equations for the Hurwitz curves of genus 14 and 17 had been an open problem for quite a while, see for instance the Problem 7 in [15] and §8 in [10]. Unfortunately most of the polynomials that we found have coefficients that are impossible to display here because of their size, however, producing these equations with GAP o ParGap is rather quick even in the genus 17 case (using the formulae in §9) if one has at hand the conjugating matrices (34) and (40) which can be obtained from me. For doing calculations with them it would be better to use a 64-bit version of GAP or ParGap and to have available upto 8 gigabytes of RAM memory.

### 2 Notation

We keep our notation as in [11]. Given a curve \( X \) and \( h \in \text{Aut}(X) \) with fixed point set \( \text{fix}(h) = \{p_1, \ldots, p_u\} \) then

\[
d_X h = (dh_{p_1}, \ldots, dh_{p_u}),
\]

where \( dh_{p_i} \) is the eigenvalue of \( h \) acting on \( K_{X,p_i} \), where \( K_{X,p_i} \) is the fibre of the canonical line bundle of \( X \) at \( p_i \).

If \( G \) is a finite group then \( W_1, \ldots, W_s \) denote its complex irreducible representations and \( \chi_1, \ldots, \chi_s \) denote the corresponding characters. Given a \( \mathbb{C}G \)-module \( M \) we write \( M_{W_i} \) for the sum of all \( \mathbb{C}G \)-submodules of \( M \) isomorphic to \( W_i \). The projection \( M \rightarrow M_{W_i} \) is denoted by \( \pi_{W_i} \). For \( h \in G \), \( \rho(h)_{\nu_j} \) denotes the projection from \( M \) to the \( \nu_j \)-eigenspace of \( h \) in \( M \). Throught this paper we denote by \( G \) to the Hurwitz group of genus 17.

### 3 The automorphism group

In terms of generators and relations our group \( G \) has the following two inequivalent presentations (see [17])

\[
G = \langle P, Q \mid P^7 = Q^8 = (QP^2)^3 = (QP^3)^2 = Q^4 PQ^4 P^4 Q^4 P^2 = 1 \rangle \quad (1)
\]

\[
\cong \langle P, Q \mid P^7 = Q^8 = (QP^2)^3 = (QP^3)^2 = Q^4 PQ^4 P^2 Q^4 P^4 = 1 \rangle. \quad (2)
\]
Each presentation induces a pair of generators $U, T$ given by

$$U = P^2Q,$$
$$T = P^3Q,$$

such that $U^2 = T^3 = (T^2U)^7 = 1$ from which one deduce that $G$ is a Hurwitz group. Moreover, using an isomorphism between the two presentations one has two generating triplets $(U, T, T^2U)$, $(U', T', T'^2U')$ such that $T^2U$ and $T'^2U'$ belong to different conjugacy classes. The outer automorphism group of $G$ has order 2 and the two conjugacy classes of elements of order 7 are fixed under the action of the outer automorphisms of $G$. This gives rise to the two non isomorphic genus 17 curves $X_1$ and $X_2$ with automorphism group $G$ (see for instance Theorem 2.17 in [19]).

In this paper we have chosen the presentation (1) as our definition of $G$. Using Gap one can check that the following gives us a representation of $G$ as a permutation group

$$P = (1, 13, 2, 11, 4, 5, 8)(3, 10, 6, 14, 7, 9, 12),$$
$$Q = (1, 7, 3, 4)(2, 11, 13, 9, 6, 14, 10, 5),$$

that we use to represent some elements of $G$. Since Gap some times produces random character tables for a given group, we fix here the character table of $G$ (see Table 1) that we used in our calculations. Representatives of the conjugacy classes are shown in Table 2 together with the size of the normalizer group in $G$ of the cyclic group generated by a representative.
Table 1: Character Table of G

| χ1 | χ2 | χ3 | χ4 | χ5 | χ6 | χ7 | χ8 | χ9 | χ10 | χ11 |
|----|----|----|----|----|----|----|----|----|-----|-----|
| 1  | 3  | 3  | 6  | 7  | 7  | 7  | 8  | 14 | 21  | 21  |
| 1  | 3  | 3  | 2  | 2  | 1  | 1  | 0  | 0  | 1   | 0   |
| 1  | 1  | 1  | 0  | 0  | 1  | 1  | 0  | 0  | 1   | 0   |
| - | - | - | 0  | - | - | - | - | - | -   | -   |
| 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1   | 1   |
| 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1   | 1   |

Here $\beta_j = \zeta^j + \zeta^{2j} + \zeta^{4j}$, $\zeta = e^{2\pi i/7}$.

Table 2: Representatives of the conjugacy classes of G

| Conjugacy class | Representative | $N_G(<h>)$ |
|-----------------|---------------|------------|
| 1A              | Identity      | 1344       |
| 2A              | (4, 7)(5, 9)(8, 12)(10, 13) | 192       |
| 4A              | (4, 5, 7, 9)(8, 10, 12, 13) | 64        |
| 2B              | (2, 6)(4, 9)(5, 7)(8, 13)(10, 12)(11, 14) | 16        |
| 4B              | (1, 3)(4, 5, 7, 9)(8, 13, 12, 10)(11, 14) | 64        |
| 3A              | (2, 4, 5)(6, 7, 9)(8, 10, 11)(12, 13, 14) | 12        |
| 6A              | (1, 3)(2, 4, 5)(6, 7, 9)(8, 13, 11, 12, 10, 14) | 12        |
| 8A              | (2, 4, 14, 12, 6, 7, 11, 8)(5, 13, 9, 10) | 32        |
| 8B              | (1, 3)(2, 4, 11, 12, 6, 7, 14, 8)(5, 10)(9, 13) | 32        |
| 7A              | (1, 2, 4, 8, 13, 11, 5)(3, 6, 7, 12, 10, 14, 9) | 21        |
| 7B              | (1, 8, 5, 4, 11, 2, 13)(3, 12, 9, 7, 14, 6, 10) | 21        |

Table 3: The characteristic polynomials of $h_{|W_i}$

| $h \in 7B$ | $W_2$ | $W_3$ | $W_4$ | $W_5, W_6, W_7$ | $W_8$ | $W_9$ | $W_{10}, W_{11}$ |
|------------|-------|-------|-------|----------------|-------|-------|----------------|
|            | $\lambda - \zeta^2(\lambda - \zeta^2)(\lambda - \zeta^4)$ | $(\lambda - \zeta^2)(\lambda - \zeta^2)(\lambda - \zeta^4)$ | $(\lambda^2 - 1)/(\lambda - 1)$ | $(\lambda^2 - 1)(\lambda - 1)$ | $(\lambda^2 - 1)(\lambda - 1)$ | $(\lambda^2 - 1)^2$ | $(\lambda^2 - 1)^3$ |

Here $\zeta = e^{2\pi i/7}$. 

5
4 The action on the canonical line bundle

One can characterize the Hurwitz curves $X_1$ and $X_2$ in terms of the action of
an element of order 7 on their canonical line bundles $K_{X_j}$, $j=1,2$. Let $d_X h$
be as defined in §2.

Lemma 4.1. Let $h \in 7B$. For $j = 1, 2$ we can assume that $d_{X_j} h =
(\zeta^{3^{j-1}}, \zeta^{3^{j-1}2}, \zeta^{3^{j-1}4})$, where $\zeta = e^{2i\pi/7}$.

The proof is similar to that of Lemma 2.2 in [11].

So, from now on we denote by $X_1$ to the curve such that the action of
a representative $h_{7B}$ of the conjugacy class $7B$ gives the vector $d_{X_1} h_{7B} =
(\zeta^1, \zeta^2, \zeta^4)$ and by $X_2$ to the curve such that $d_{X_2} h_{7B} = (\zeta^3, \zeta^5, \zeta^6)$. Now
one can use [11] Theorem 1 to complete the Table 4 and then compute the multiplicities $a_i(d)$ of the representations $W_i$, $i = 1, \ldots, 11$, in $H^0(X_j, K_{X_j}^d)$
(see Tables 5 and 6) by using the Atiyah–Bott fixed-point theorem and the
character table of $G$.

| Conjugacy class of $h$ | Number of fixed points in $X_j$ | $d_{X_j}(h)$ |
|------------------------|-------------------------------|--------------|
| 2A                     | 0                             | --           |
| 4A                     | 0                             | --           |
| 2B                     | 8                             | $(-1, -1, -1, -1, -1, -1, -1)$ |
| 4B                     | 0                             | --           |
| 3A                     | 4                             | $(\omega, \omega^2, \omega^2)$ |
| 6A                     | 0                             | --           |
| 8A                     | 0                             | --           |
| 8B                     | 0                             | --           |
| 7A                     | 3                             | $(\zeta^{3^j}, \zeta^{3^j2}, \zeta^{3^j4})$ |
| 7B                     | 3                             | $(\zeta^{3^{j-1}}, \zeta^{3^{j-1}2}, \zeta^{3^{j-1}4})$ |

Here $\omega = e^{2i\pi/3}$. 

Table 4: The fixed points and $d_{X_j}(h)$
Table 5: Decomposition of $H^0(X_1, K_{X_1}^d)$

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| $a_1(d)$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| $a_2(d)$ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| $a_3(d)$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 1 | 2 |
| $a_4(d)$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 3 | 2 | 3 | 2 | 3 |
| $a_5(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_6(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_7(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_8(d)$ | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| $a_9(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_{10}(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_{11}(d)$ | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 |

Table 6: Decomposition of $H^0(X_2, K_{X_2}^d)$

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| $a_1(d)$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| $a_2(d)$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 1 | 2 |
| $a_3(d)$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| $a_4(d)$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 3 | 2 | 3 | 2 | 3 |
| $a_5(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_6(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_7(d)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 |
| $a_8(d)$ | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $a_9(d)$ | 1 | 0 | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 3 | 5 | 4 | 5 | 5 | 6 | 6 | 5 | 7 | 6 |
| $a_{10}(d)$ | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| $a_{11}(d)$ | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |

5 The quadratic equations and the matrix representation of $G$

From Tables 5 and 6 we have

$$H^0(X_i, K_{X_i}) = W_9 \oplus W_{i+1}, \quad i=1,2,$$

and

$$H^0(X_i, K_{X_i}^{\otimes 2}) = W_4 \oplus W_{10} \oplus W_{11}. $$
Table 7: Decomposition of $S^d(W_9 + W_2)$

| d   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| $a_1(d)$ | 1  | 0  | 1  | 0  | 9  | 5  | 85 | 137| 637| 1385| 4210|
| $a_2(d)$ | 0  | 0  | 0  | 5  | 6  | 57 | 145| 594| 1565| 4716| 11616|
| $a_3(d)$ | 0  | 0  | 0  | 1  | 34 | 71 | 387| 1010| 3473| 8843| 24274|
| $a_4(d)$ | 0  | 1  | 5  | 32 | 123| 447| 1470| 4398| 12171| 31680|
| $a_5(d)$ | 0  | 0  | 1  | 13| 41 | 234| 725| 2646| 7486| 21564| 54810|
| $a_6(d)$ | 0  | 3  | 13| 80 | 306| 1187| 3783| 11619| 31765| 83234|
| $a_7(d)$ | 0  | 3  | 13| 80 | 306| 1187| 3783| 11619| 31765| 83234|

Table 8: Decomposition of $S^d(W_9 + W_3)$

| d   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| $a_1(d)$ | 1  | 0  | 1  | 0  | 9  | 5  | 85 | 137| 637| 1385| 4210|
| $a_2(d)$ | 0  | 0  | 0  | 5  | 6  | 57 | 145| 594| 1565| 4716| 11616|
| $a_3(d)$ | 0  | 1  | 0  | 4  | 6  | 57 | 145| 594| 1565| 4716| 11616|
| $a_4(d)$ | 0  | 0  | 3  | 1  | 34 | 71 | 387| 1010| 3473| 8843| 24274|
| $a_5(d)$ | 0  | 0  | 8  | 20| 118| 374| 1324| 3761| 10814| 27442|
| $a_6(d)$ | 0  | 0  | 8  | 19| 117| 367| 1317| 3743| 10796| 27391|
| $a_7(d)$ | 0  | 0  | 8  | 19| 117| 367| 1317| 3743| 10796| 27391|
| $a_8(d)$ | 0  | 1  | 5  | 32| 123| 447| 1470| 4398| 12171| 31680|
| $a_9(d)$ | 0  | 3  | 13| 80 | 306| 1187| 3783| 11619| 31765| 83234|
| $a_{10}(d)$ | 0  | 3  | 13| 80 | 306| 1187| 3783| 11619| 31765| 83234|

Table 9: Decomposition of $S^dW_9$

| d   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| 1   | 1  | 0  | 1  | 0  | 4  | 2  | 34 | 40 | 183| 332| 922|
| 2   | 0  | 0  | 0  | 2  | 3  | 22 | 53 | 184| 433| 1140| 2501|
| 3   | 0  | 0  | 0  | 2  | 3  | 22 | 53 | 184| 433| 1140| 2501|
| 4   | 0  | 2  | 0  | 18| 28 | 146| 308| 980| 2120| 5284|
| 5   | 0  | 0  | 0  | 4  | 10| 46 | 140| 408| 1047| 2612| 5922|
| 6   | 0  | 0  | 0  | 5  | 9 | 50 | 133| 417| 1030| 2643| 5878|
| 7   | 0  | 0  | 0  | 5  | 9 | 50 | 133| 417| 1030| 2643| 5878|
| 8   | 0  | 1  | 2  | 17| 48 | 165| 454| 1230| 2928| 6863|
| 9   | 0  | 1  | 0  | 7  | 20| 160| 260| 843| 2060| 5267| 11772|
| 10  | 0  | 2  | 8  | 39| 130| 431| 1200| 3200| 7740| 17928|
| 11  | 0  | 2  | 8  | 39| 130| 431| 1200| 3200| 7740| 17928|

So we assume that $X_i \subset \mathbb{P}^{16} = \mathbb{P}(V_i^*)$, where $V_i$ has the structure of the representation $W_9 \oplus W_{i+1}$. From Tables 7 and 8 we have

$$S^2(V_i) = W_1 \oplus W_4^2 \oplus W_8 \oplus W_{10}^2 \oplus W_{11}^3, \quad i=1,2.$$ (9)
Table 10: Decomposition of $S^d W_2$

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| a_1(d) | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1   |
| a_2(d) | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 0   |
| a_3(d) | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 1   |
| a_4(d) | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 3 | 1 | 4   |
| a_5(d) | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 3 | 2   |
| a_6(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_7(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_8(d) | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3   |
| a_9(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_10(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_11(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |

Table 11: Decomposition of $S^d W_3$

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| a_1(d) | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1   |
| a_2(d) | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 1   |
| a_3(d) | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 0   |
| a_4(d) | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 3 | 1 | 4   |
| a_5(d) | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 3 | 2   |
| a_6(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_7(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_8(d) | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3   |
| a_9(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_10(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| a_11(d) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |

It follows that

$$H^0(\mathbb{P}^{16}, \mathcal{I}_{X_i}(2)) = W_1 \oplus W_4^2 \oplus W_8 \oplus W_8 \oplus W_{10} \oplus W_{11}, \ i=1,2. \quad (10)$$

Where $\mathcal{I}_{X_i}$ denotes the ideal sheaf of $X_i$ in $\mathbb{P}^{16}$. From now on we will use the notation

$$I_{X_i}(2) = H^0(\mathbb{P}^{16}, \mathcal{I}_{X_i}(2)).$$

We have

$$\pi_{W_1}(I_{X_i}(2)) = \pi_{W_1}(S^2V_i), \quad (11)$$

and

$$\pi_{W_8}(I_{X_i}(2)) = \pi_{W_8}(S^2V_i). \quad (12)$$

The remaining spaces $\pi_{W_4}(I_{X_i}(2)), \pi_{W_{10}}(I_{X_i}(2))$ and $\pi_{W_{11}}(I_{X_i}(2))$ will be found in Sections 6, 7 and 8 below.
For calculations we fix an action of $G$ on $\mathbb{P}^{16}$. In $V_i$ (respectively $V_i^*$) we consider a basis $\{y_1, \ldots, y_{17}\}$ (respectively the dual basis $\{e_1, \ldots, e_{17}\}$). Sometimes we use the following subspaces of $V_i$

$$V_{i,g} = \langle y_1, \ldots, y_{14} \rangle, \quad V_{i,i+1} = \langle y_{15}, y_{16}, y_{17} \rangle. \quad \text{(13)} \quad \text{(14)}$$

With respect to these bases, the action of the generators $P$ and $Q$ on $V_i$ (respectively $V_i^*$) is given by the matrices $\text{diag}(P_{w_9}, P_{w_{i+1}})$ and $\text{diag}(Q_{w_9}, Q_{w_{i+1}})$ (respectively the inverse transpose of these matrices), where the matrices $P_{w_j}, Q_{w_j}, j = 9, 2$ are defined in formulae (15), (16), (17) and (18) (recall that $\omega = e^{2i\pi/3}$ and $\zeta = e^{2i\pi/7}$). The matrices $P_{w_1}, Q_{w_3}$ could be taken as the respective inverse transpose of $P_{w_2}, Q_{w_2}$ however we will focus only on the curve $X_1$.

**Remark 5.1.** One can identify $G$ with a subgroup of $\text{Aut}(G)$. The action of $G$ extends to an action of $\text{Aut}(G)$ on $V_1$. So the exterior automorphisms move $X_1 \subset \mathbb{P}(V_1^*)$ giving rise to another copy $X'_1 \subset \mathbb{P}(V'_1)$ of $X_1$. For instance, we can assume $\text{Aut}(G) = \langle P, Q, E \rangle$, where $E$ is such that the action on $V_1$ is given by the matrix $\text{diag}(E_{w_9}, E_{w_2})$, where $E_{w_9}$ and $E_{w_2}$ are defined in formulae (19) and (21) respectively.

**Remark 5.2.** Let $Y$ denote the Klein quartic. We have that $Y$ is isomorphic to the quotient $X_i/N$ for $i = 1, 2$, where $N$ is the normal subgroup of $G$ generated by the involutions in the class 2A. Let $f_i : X_i \mapsto Y$ be the quotient maps. Then we have that the action of $G$ induced on $Y$ from $X_1$ or $X_2$ is such that $d_Y h_{7B} = (\zeta^1, \zeta^2, \zeta^3)$ or $d_Y h_{7B} = (\zeta^3, \zeta^5, \zeta^6)$ respectively. The action induced by $f_1$ or $f_2$ gives $H^0(Y, K_Y)$ the structure of the representation $W_2$ or $W_3$ respectively. Since $f_i^* K_Y \cong K_{X_i}$, we see that $f_i$ can be recovered by composing the canonical embedding $X_i \subset \mathbb{P}(V_i^*)$ with the projection $\pi_i : \mathbb{P}(V_i^*) \dashrightarrow \mathbb{P}(V_{i,i+1}^*)$.

$$P_{w_2} = \begin{pmatrix} \zeta^3 & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & \zeta^5 \end{pmatrix} \quad \text{(15)}$$

$$Q_{w_2} = \frac{1}{\sqrt{7}} \begin{pmatrix} \zeta^4 - \zeta^6 & -1 + \zeta^3 & \zeta - \zeta^2 \\ -\zeta + \zeta^4 & \zeta^2 - \zeta^3 & -1 + \zeta^5 \\ -1 + \zeta^6 & \zeta^2 - \zeta^4 & \zeta - \zeta^5 \end{pmatrix} \quad \text{(16)}$$

10
\[ P_{W_0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \] \hspace{1cm} (17)

\[ Q_{W_0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \] \hspace{1cm} (18)

\[ E_{W_0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \] \hspace{1cm} (19)

\[ E_{W_2} = \frac{1}{\sqrt{-1}} \begin{pmatrix}
\zeta - \zeta^6 & -\zeta^2 + \zeta^6 & -\zeta^2 + \zeta^3 \\
-\zeta + \zeta^5 & -\zeta^3 + \zeta^4 & -\zeta + \zeta^3 \\
-\zeta^4 + \zeta^5 & -\zeta^4 + \zeta^6 & \zeta^2 - \zeta^5 \\
\end{pmatrix} \] \hspace{1cm} (20)

\[ E_{W_2} = \begin{pmatrix}
\zeta - \zeta^6 & -\zeta^2 + \zeta^6 & -\zeta^2 + \zeta^3 \\
-\zeta + \zeta^5 & -\zeta^3 + \zeta^4 & -\zeta + \zeta^3 \\
-\zeta^4 + \zeta^5 & -\zeta^4 + \zeta^6 & \zeta^2 - \zeta^5 \\
\end{pmatrix} \] \hspace{1cm} (21)
6 The generators of \( \pi_{W_4}(I_{X_1}(2)) \)

Let \( h_{7B} = P^2 Q^{PQ} = (1, 8, 5, 4, 11, 2, 13)(3, 12, 9, 7, 14, 6, 10) \) be the representative of the conjugacy class \( 7B \) of \( G \). From Table 3, we see that \( \rho(h_{7B})_{\zeta^2}(W_4) \) has dimension 1. So, since \( W_4 \) is an irreducible \( \mathbb{C}G \)-module, the subspace \( \rho(h_{7B})_{\zeta^2}(\pi_{W_4}(I_{X_1}(2))) \) generates \( \pi_{W_4}(I_{X_1}(2)) \) as a \( \mathbb{C}G \)-module. We shall find a point \( p \in X_1 \) such that not all \( \zeta^2 \)-eigenquadrics of \( h_{7B} \) in \( \pi_{W_4}(S^2 V_1) \) vanish on \( p \). Then the subspace \( \rho(h_{7B})_{\zeta^2}(\pi_{W_4}(I_{X_1}(2))) \) is the set of all the \( \zeta^2 \)-eigenquadrics of \( h_{7B} \) in \( \pi_{W_4}(S^2 V_1) \) that vanish on \( p \).

If \( p \) is a fixed point of \( h_{7B} \) in \( X_1 \), then it lives in a subspace \( \mathbb{P}(\rho(h_{7B})_{\zeta^6}(V_1^*)) \subset \mathbb{P}(V_1^*) \) for some \( i \). It can be checked that \( i = 6, 5 \) or 3 if \( dh_{7B} = \zeta^1, \zeta^2 \) or \( \zeta^4 \) respectively.

Now we will look for the fixed point \( p \) of \( h_{7B} \) in \( X_1 \) lying on \( \mathbb{P}(\rho(h_{7B})_{\zeta^6}(V_1^*)) \). Note that \( \rho(h_{7B})_{\zeta^6}(V_{1,9}^*) \) has dimension 2 (see Table 3) and that \( v_1(p) \) (see Remark 5.2) lies on the subspace \( \rho(h_{7B})_{\zeta^6}(V_{1,2}^*) \) which is one dimensional. Therefore we can assume that

\[
p = \alpha \cdot v_1 + \beta \cdot v_2 + v_3,
\]

where \( \alpha, \beta \in \mathbb{C} \) and \( \{v_1, v_2\} \) and \( \{v_3\} \) are bases for \( \rho(h_{7B})_{\zeta^6}(V_{1,9}^*) \) and \( \rho(h_{7B})_{\zeta^6}(V_{1,2}^*) \) respectively.

To find \( \alpha \) and \( \beta \) we will evaluate on the point \( p \) the \( \zeta^2 \)-eigenquadric \( q_{\zeta^2} \) of \( h_{7B} \) in \( \pi_{W_8}(I_{X_1}(2)) \); namely

\[
q_{\zeta^2} = \rho(h_{7B})_{\zeta^2}(\pi_{W_8}(y_i^2)).
\]

We take

\[
\begin{align*}
v_i &= \rho(h_{7B})_{\zeta^6}(e_i), \quad i = 1, 2, \\
\text{and} \\
v_3 &= \rho(h_{7B})_{\zeta^6}(e_{17}).
\end{align*}
\]

We see that

\[
q_{\zeta^2}(v_i) \neq 0, \quad i = 1, 2.
\]

We assume that \( \beta = 1 \) since we always can move the fixed point \( p \) by a \( G \)-equivariant change of coordinates \( t_\lambda : e_i \to \lambda e_i, \quad i = 1, \ldots, 14 \) and \( e_i \to e_i, \quad i = 15, 16, 17 \). So we have

\[
a^2 + 1/7(-6\xi - 4\xi^2 + 4\xi^4 - 3\xi^5 - 2\xi^8 + 3\xi^{10} + 6\xi^{11} - \xi^{13} + 2\xi^{16} + \xi^{17} - 2\xi^{19} + 2\xi^{20}) = 0.
\]
where $\xi = e^{2i\pi/21}$.

Notice that the corresponding points to each root of (26) live in projectively equivalent curves in $\mathbb{P}(V_1^*)$. One is obtained from the other by applying the exterior automorphism in Remark 5.1 followed by the equivariant translation $t_\lambda$ with suitable $\lambda$.

Now, a basis for the $\zeta^2$-eigenspace of $h_7B|_{\pi W_4(S^2V_1)}$ is given by $\{b_1, b_2, b_3\}$, where

\begin{align*}
    b_1 &= \rho(h_7B)\zeta^2(\pi_{W_4}(y_1^2)), \\
    b_2 &= \rho(h_7B)\zeta^2(\pi_{W_4}(y_1y_2)), \\
    b_3 &= y_{17}^2.
\end{align*}

Evaluating on $p$ one proves that $b_i(p) \neq 0$, $i = 1, 2, 3$. Thus the quadrics

\begin{align*}
    \vartheta_1 &= b_1(p)b_2 - b_2(p)b_1 \\
    \vartheta_2 &= b_1(p)b_3 - b_3(p)b_1
\end{align*}

form a basis for the $\zeta^2$-eigenspace of $h_7B|_{\pi W_4(I_X(2))}$.

7 The generators of $\pi_{\mathcal{W}_{10}}(I_{X_1}(2))$

We shall find linearly independent elements $\theta_1, \theta_2, \theta_3 \in \pi_{\mathcal{W}_{10}}(S^2V_1)$ such that the $G$-module generated by each one of them is isomorphic to $\mathcal{W}_{10}$ and such that $\theta_1, \theta_2, \theta_3$ are $G$-equivalent, meaning that if $\phi_{ij}$ is an isomorphism of $G$-modules from the $G$-module generated by $\theta_i$ to that generated by $\theta_j$ then $\phi_{ij}(\theta_i) = \lambda_{ij}\theta_j$ for some $\lambda_{ij} \in \mathbb{C}\backslash\{0\}$. We take $\theta_i$ to be a $\zeta^2$-eigenquadric of $h_7B|_{\pi_{\mathcal{W}_{10}}(S^2V_1)}$. Let $p$ be the fixed point of $h_7B$ we found in Section 6. If $\theta_i(p) \neq 0$, $i = 1, 2, 3$ then

\begin{align*}
    \varphi_1 &= \theta_1(p)\vartheta_2 - \vartheta_2(p)\theta_1, \\
    \varphi_2 &= \theta_1(p)\vartheta_3 - \vartheta_3(p)\theta_1
\end{align*}

generate $\pi_{\mathcal{W}_{10}}(I_{X_1}(2))$.

We first notice that

\begin{equation}
    S^2V_1 = S^2V_{1,9} \oplus S^2V_{1,2} \oplus V_{1,9} \otimes V_{1,2}.
\end{equation}

Using that (see Tables 9,10)

\begin{align*}
    S^2V_{1,9} &= W_1 \oplus W_4^2 \oplus W_8 \oplus W_{10}^2 \oplus W_{11}^2, \\
    S^2V_{1,2} &= W_4,
\end{align*}

13
one concludes that
\[ V_{1,9} \otimes V_{1,2} = W_{10} \oplus W_{11}. \] (32)

Then we define an isomorphism of $G$-modules
\[ \varrho_{W_{10}} : \pi_{W_{10}}(V_{1,9} \otimes V_{1,2}) \oplus \pi_{W_{10}}(V_{1,9} \otimes V_{1,2}) \to \pi_{W_{10}}(S^2V_{1,9}). \]

So given
\[ \theta_1 \in \pi_{W_{10}}(V_{1,9} \otimes V_{1,2}) \]
one can take
\[ \theta_2 = \varrho_{W_{10}}((\theta_1, 0)) \]
\[ \theta_3 = \varrho_{W_{10}}((0, \theta_1)). \] (33)

To construct $\varrho_{W_{10}}$ we consider the bases $B_1$ of $\pi_{W_{10}}(V_{1,9} \otimes V_{1,2})$ and $B_2$ of $\pi_{W_{10}}(S^2V_{1,9})$ given in formulae (35) and (37). Let $A, C$ be the matrix representations of $G$ corresponding to the action on $\pi_{W_{10}}(V_{1,9} \otimes V_{1,2})$ and on $\pi_{W_{10}}(S^2V_{1,9})$ with respect to these bases. If $M_0$ is a $42 \times 42$ complex matrix such that $\det S \neq 0$, where $S$ given by equation (34), then one can take $\varrho_{W_{10}}$ to be the linear map induced by $S$ with respect to the bases $B_1 \cup B_1$ and $B_2$.

\[ S = \sum_{h \in G} C_h \cdot M_0 \cdot \text{diag}(A_h, A_h)^{-1}. \] (34)

For our calculation we use $M_0 = I_{42}$ the identity matrix.

\[ B_1 = \{ k_{1,j}, k_{2,j}, k_{3,j} \mid j = 1 \ldots 7 \}; \] (35)

where
\[ k_{i,j} = \rho(\eta_B)_{i,j}(Q \cdot k_{i,2}), \quad \text{for } j \neq 2, \]
\[ k_{1,2} = \rho(\eta_B)_{1,2}(\pi_{W_{10}}(y_1 y_{15})), \]
\[ k_{i+1,2} = \rho(\eta_B)_{i+1,2}(Q \cdot k_{i,2}). \] (36)

\[ B_2 = \{ \kappa_{1,j}, \kappa_{2,j}, \kappa_{3,j}, \kappa_{4,j}, \kappa_{5,j}, \kappa_{6,j} \mid j = 1 \ldots 7 \}; \] (37)

where
\[ \kappa_{i,j} = \rho(\eta_B)_{i,j}(Q \cdot \kappa_{i,2}), \quad \text{for } j \neq 2, \]
\[ \kappa_{1,2} = \rho(\eta_B)_{1,2}(\pi_{W_{10}}(y_1 y_3)), \]
\[ \kappa_{4,2} = \rho(\eta_B)_{4,2}(Q \cdot \rho(\eta_B)_{1,2}(Q \cdot \kappa_{1,2})), \]
\[ \kappa_{i+1,2} = \rho(\eta_B)_{i+1,2}(Q \cdot \kappa_{i,2}), \quad \text{for } i = 1, 2, 4, 5. \] (38)

We take $\theta_1 = k_{1,2}$. Then
\[ \theta_2 = \sum_{i=1}^{6} s_{i+6,4} \kappa_{i,2}, \]
\[ \theta_3 = \sum_{i=1}^{6} s_{i+6,25} \kappa_{i,2}. \] (39)
8 The generators of $\pi_{w_{11}}(I_{X_1}(2))$

The situation is similar to that of $\pi_{w_{10}}(I_{X_1}(2))$. We define an isomorphism of $G$-modules

$$
\varphi_{w_{11}} : \pi_{w_{11}}(V_{1,9} \otimes V_{1,2}) \oplus \pi_{w_{11}}(V_{1,9} \otimes V_{1,8}) \to \pi_{w_{11}}(S^2V_{1,9}).
$$

which corresponds to a matrix

$$
S' = \sum_{h \in G} C'_h \cdot M'_0 \cdot \text{diag}(A'_h, A'_h)^{-1}. 
$$

(40)

where $A', C'$ are the matrix representations of $G$ corresponding to the action on $\pi_{w_{11}}(V_{1,9} \otimes V_{1,2})$ and on $\pi_{w_{11}}(S^2V_{1,9})$ with respect to fixed bases $B'_1$ and $B'_2$ respectively (see formulae (41), (43)). We use $M'_0 = I_{42}$ the identity matrix. Then $\varphi'_1$ and $\varphi'_2$ in formula (46) generate $\pi_{w_{11}}(I_{X_1}(2))$.

$$
B'_1 = \{ k'_{1,j}, k'_{2,j}, k'_{3,j} \mid j = 1 \ldots 7 \}, 
$$

(41)

where

$$
k'_{i,j} = \rho(h_{7B})_{i,j}(Q \cdot k'_{i,2}), \text{ for } j \neq 2, 
k'_{1,2} = \rho(h_{7B})_{1,2}(\pi_{w_{11}}(g_{1}g_{15})), 
k'_{i+1,2} = \rho(h_{7B})_{i,2}(Q \cdot k'_{i,2}).
$$

(42)

$$
B'_2 = \{ \kappa'_{1,j}, \kappa'_{2,j}, \kappa'_{3,j}, \kappa'_{4,j}, \kappa'_{5,j}, \kappa'_{6,j} \mid j = 1 \ldots 7 \},
$$

(43)

where

$$
\kappa'_{i,j} = \rho(h_{7B})_{i,j}(Q \cdot \kappa'_{i,2}), \text{ for } j \neq 2, 
\kappa'_{1,2} = \rho(h_{7B})_{1,2}(\pi_{w_{11}}(g_{1}g_{3})), 
\kappa'_{i+1,2} = \rho(h_{7B})_{i,2}(Q \cdot \kappa'_{i,2}) \text{ for } i = 1, 2, 4, 5.
$$

(44)

Taking $\theta'_1 = k'_{1,2}$ we take

$$
\theta'_2 = \sum_{i=1}^{6} s'_{i+6,4} \kappa'_{i,2}, 
\theta'_3 = \sum_{i=1}^{6} s'_{i+6,25} \kappa'_{i,2}.
$$

(45)

One checks that $\theta'_1(p)\theta'_2(p)\theta'_3(p) \neq 0$. So we define

$$
\varphi'_1 = \theta'_1(p)\theta'_2 - \theta'_2(p)\theta'_1, 
\varphi'_2 = \theta'_1(p)\theta'_3 - \theta'_3(p)\theta'_1.
$$

(46)

Although one can speed up the calculation of (44) and (46) by using representatives of cosets of a subgroup of $G$, the computation was still very slow.
9 A basis for $I_{X_1}(2)$

Now we can give formulae for the elements of a basis for the space of quadrics in the canonical ideal of $X_1$.

**Theorem 9.1.** The union of the sets of quadrics $D_i$, $i = 1, 4, 8, 10, 11$ below form a basis for $H^0(\mathbb{P}^{16}, I_{X_1}(2))$.

Let $\alpha$ be a solution of (26) and $p$ be the corresponding point. Let $q_{i^2}$ be as defined in (23). Let $\vartheta_1, \vartheta_2$ be as defined in (28). Let $\phi_{1,2}, \phi'_{1}$ and $\phi'_2$ be as defined in (29) and (46) respectively. The sets $D_i$, $i = 1, 4, 8, 10, 11$ are chosen so that each one is the homomorphic image of a basis for $W_1, W_4, W_8, W_{10} \oplus W_{10}, W_{11} \oplus W_{11}$, respectively, under some $G$-isomorphism to $\pi_{w_i}(I_{X_1}(2))$. Then we have

\[ D_1 = \{d_{1,7,1}\}, \quad \text{where} \quad d_{1,7,1} = \pi_{w_1}(y_1y_2). \]

\[ D_4 = D_{4,1} \cup D_{4,2}, \quad \text{where} \quad D_{4,i} = \{d_{4,j,i} \mid j = 1, \ldots, 6\}, \quad i = 1, 2, \]

and \[ d_{4,j,i} = \rho(h_{7B})\zeta(Q \cdot \vartheta_i). \]

\[ D_8 = \{d_{8,j,1} \mid j = 1, \ldots, 7\} \cup \{d_{8,7,2}\}, \quad \text{where} \]

\[ d_{8,j,1} = \rho(h_{7B})\zeta(Q \cdot q_{i^2}), \]

\[ d_{8,7,2} = \rho(h_{7B})1(Q^2 \cdot q_{i^2}). \]

\[ D_{10} = D_{10,1} \cup D_{10,2}, \quad \text{where} \quad D_{10,i} = \phi_{\theta_1,\varphi_1}(B_1), \]

$B_1$ is defined in formula (35) and \[ \phi_{\theta_1,\varphi_1} : \mathbb{C}G_{<\theta_1>} \to \mathbb{C}G_{<\varphi_1>} \]

is the isomorphism of $G$-modules that maps $\theta_1$ to $\varphi_i$. We write

\[ d_{10,j,k} = \phi_{\theta_1,\varphi_1}(k_{j,k}), \]

\[ d_{10,j,k+3} = \phi_{\theta_1,\varphi_2}(k_{j,k}), \quad k = 1, 2, 3. \]
One has for instance
\[
\begin{align*}
d_{10,2,1} & = \varphi_1, \\
d_{10,2,k+1} & = \rho(h_{7B})_{Q} (Q \cdot d_{10,2,k}), \ k=1,2, \\
d_{10,j,k} & = \rho(h_{7B})_{Q} (Q \cdot d_{10,2,k}), \ j=1,3,4,5,6,7, \ k=1,2,3.
\end{align*}
\]
Similarly we define
\[
D_{11} = D_{11,1} \cup D_{11,2},
\]
where
\[
D_{11,i} = \{d_{11,j,3(i-1)+k} \mid j=1,\ldots,7, \ k=1,2,3, \ i=1,2, \}
\]
and
\[
d_{11,j,3(i-1)+k} = \phi_{i,j} (L_{k,j}).
\]

10 The space of cubic forms

We conclude the genus 17 case with the following theorem.

**Theorem 10.1.** The homogeneous ideal of the canonical embedding of \(X_1\) in \(\mathbb{P}^{16}\) is generated by the quadratic forms in it.

For the proof we used the basis defined in Section 9 to show that the space of cubic forms generated by \(I_{X_1}(2)\) is \(I_{X_1}(3)\), namely \(V_1 \cdot I_1 = I_{X_1}(3)\). To do that it is enough to show that \(\pi_{w_1}(V_1 \cdot I_1) = \pi_{w_1}(I_{X_1}(3))\). So we computed the dimensions for the eigenspaces \(\rho(h_{7B})_{Q} (\pi_{w_1}(V_1 \cdot I_1))\), for some \(j\). Thus the multiplicity of the representation \(W_i \in \pi_{w_1}(V_1 \cdot I_1)\) can be deduced from the information in Table 3. In Table 12 we list the dimensions of the spaces \(\rho(h_{7B})_{Q} (\pi_{w_1}(V_1 \cdot I_1))\) and we also list bases for these spaces in order to help speed up verification since this calculation was very slow. On the other hand, from Tables 5 and 7 we have

\[
H^0(X_1, K^\otimes 3) = W_3 \oplus W_5 \oplus W_6 \oplus W_7 \oplus W_9 \oplus W_{10} \oplus W_{11},
\]
and
\[
S^3(V_1) = W_4^2 \oplus W_5^2 \oplus W_4 \oplus W_5^2 \oplus W_6^2 \oplus W_7^2 \oplus W_9 \oplus W_{10} \oplus W_{11}.
\]

Then
\[
H^0(\mathbb{P}^{16}, I_{X_1}(3)) = W_2^2 \oplus W_4 \oplus W_5 \oplus W_6 \oplus W_7 \oplus W_9 \oplus W_{10} \oplus W_{11}.
\]
Table 12: Generators for $I_{X_1}(3)$

| $W_{i,j}$ | $\rho(h_B)^e_i(\pi_W(V_1,I_{X_1}(2)))$ | $\rho(h_B)^e_i(\pi_W(V_1,I_{X_1}(2)))$ |
|-----------|---------------------------------|---------------------------------|
| $W_{1,-}$ | $0$                             | $\pi_{W_1} \left( \begin{array}{c} [y_1]_c \cdot d_1, \ldots, [y_{11}]_c \cdot d_1, d_2, d_3, d_4, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{2,2}$ | $4$                             | $\pi_{W_2} \left( \begin{array}{c} [y_1]_c \cdot d_2, \ldots, [y_{11}]_c \cdot d_2, d_3, d_4, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{3,3}$ | $4$                             | $\pi_{W_3} \left( \begin{array}{c} [y_1]_c \cdot d_3, \ldots, [y_{11}]_c \cdot d_3, d_4, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{4,2}$ | $1$                             | $\pi_{W_4} \left( \begin{array}{c} [y_1]_c \cdot d_4, \ldots, [y_{11}]_c \cdot d_4, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{5,2}$ | $7$                             | $\pi_{W_5} \left( \begin{array}{c} [y_1]_c \cdot d_5, \ldots, [y_{11}]_c \cdot d_5, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{6,2}$ | $7$                             | $\pi_{W_6} \left( \begin{array}{c} [y_1]_c \cdot d_6, \ldots, [y_{11}]_c \cdot d_6, d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{7,2}$ | $7$                             | $\pi_{W_7} \left( \begin{array}{c} [y_1]_c \cdot d_7, \ldots, [y_{11}]_c \cdot d_7, d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{8,2}$ | $5$                             | $\pi_{W_8} \left( \begin{array}{c} [y_1]_c \cdot d_8, \ldots, [y_{11}]_c \cdot d_8, d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{9,2}$ | $24$                            | $\pi_{W_9} \left( \begin{array}{c} [y_1]_c \cdot d_9, \ldots, [y_{11}]_c \cdot d_9, d_{11}, d_{12} \end{array} \right)$ |
| $W_{10,2}$| $36$                            | $\pi_{W_{10}} \left( \begin{array}{c} [y_1]_c \cdot d_{10}, \ldots, [y_{11}]_c \cdot d_{10}, d_{11}, d_{12} \end{array} \right)$ |
| $W_{11,2}$| $36$                            | $\pi_{W_{11}} \left( \begin{array}{c} [y_1]_c \cdot d_{11}, \ldots, [y_{11}]_c \cdot d_{11}, d_{12} \end{array} \right)$ |

Here we write $[y_i]_c = \rho(h_B)^e_i(y_i)$ and $\{D_v\}_V = \{d_{i,j} \in D_v \mid j \equiv v \mod 7\}$.
11 The genus 14 case

Let $X_1$ be the Hurwitz curve of genus 14 considered in [11]. Using the quadrics in the Theorem 2.5 of [11] one also proves the following.

**Theorem 11.1.** The homogeneous ideal of the canonical embedding of $X_1$ in $\mathbb{P}^{13}$ is generated quadrics.

We proceed exactly as in the proof of Theorem 10.1. In this case we used the Table 13 below, the Table 3 in [11] and the decomposition of $H^0(\mathbb{P}^{13}, I_{X_1}(3))$ into a sum of irreducible representations of $PSL(2, \mathbb{F}_{13})$. That is (use Tables 5 and 8 in [11]),

$$H^0(\mathbb{P}^{13}, I_{X_1}(3)) = W^5_2 \oplus W^5_3 \oplus W^6_6 \oplus W^5_5 \oplus W^5_8 \oplus W^4_8 \oplus W^8_9. \quad (53)$$

When computing the quadrics in Theorem 2.5 of [11] one should have into account two misprints: the first in the definition of $\epsilon$ (before formula (9)) where $\epsilon_2$ should be equal to $\rho(h_{7A})\zeta_1(\epsilon_3)$ and the second in the definition of $q$ (also before formula (9)) where $q = \rho(h_{7A})\zeta_5(\pi_{w_6}(y_2^4))$ should be $q = \rho(h_{7A})\zeta_5(\pi_{w_6}(y_2^4))$. 

Table 13: Generators for $I_{X_1}(3)$ in the genus 14 case.

| $W_{i,j}$ | $\rho(h_{13A})_{ij}(\pi_{W_k}(V_1 \cdot I_{X_1}(2)))$ | $\rho(h_{13A})_{ij}(\pi_{W_k}(V_1 \cdot I_{X_1}(2)))$ |
|-----------|---------------------------------------------------|---------------------------------------------------|
| $W_{1,-}$ | 0                                                 | $\pi_{W_2}\left(\{\left[y_1\right],\left[d_{5,12,1}\right],\left[y_1\right],\left[d_{6,1,1}\right]\}\right)$ |
| $W_{2,2}$ | 5                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{7,1,1}\right],\left[y_1\right],\left[d_{6,1,1}\right],\left[y_1\right],\left[d_{6,1,3}\right]\right\}\right)$ |
| $W_{3,1}$ | 5                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{7,13,1}\right],\left[y_1\right],\left[d_{6,13,1}\right],\left[y_1\right],\left[d_{6,13,2}\right]\right\}\right)$ |
| $W_{4,1}$ | 5                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{6,1,1}\right],\left[y_1\right],\left[d_{6,11,1}\right],\left[y_1\right],\left[d_{6,12,1}\right]\right\}\right)$ |
| $W_{5,1}$ | 6                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{5,11,1}\right],\left[y_1\right],\left[d_{5,10,1}\right],\left[y_1\right],\left[d_{5,12,1}\right],\left[y_1\right],\left[d_{5,13,1}\right],\left[y_1\right],\left[d_{5,14,1}\right]\right\}\right)$ |
| $W_{6,1}$ | 5                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{6,11,1}\right],\left[y_1\right],\left[d_{6,11,1}\right],\left[y_1\right],\left[d_{6,13,1}\right],\left[y_1\right],\left[d_{6,14,1}\right]\right\}\right)$ |
| $W_{7,1}$ | 5                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{5,12,1}\right],\left[y_1\right],\left[d_{5,13,1}\right],\left[y_1\right],\left[d_{5,14,1}\right]\right\}\right)$ |
| $W_{8,1}$ | 4                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{5,12,1}\right],\left[y_1\right],\left[d_{5,13,1}\right],\left[y_1\right],\left[d_{5,14,1}\right]\right\}\right)$ |
| $W_{9,1}$ | 8                                                 | $\pi_{W_2}\left(\left\{\left[y_1\right],\left[d_{5,13,1}\right],\left[y_1\right],\left[d_{5,13,2}\right],\left[y_1\right],\left[d_{5,12,1}\right],\left[y_1\right],\left[d_{5,11,1}\right],\left[y_1\right],\left[d_{5,12,1}\right]\right\}\right)$ |

Using the notation of [11] we write $V_1 = \langle y_1, \ldots, y_{14} \rangle$, $[y_i]_{\nu} = \rho(h_{13A})_{ij}(y_i)$, $\nu = e^{2\pi i / 13}$, $d_{i,j,1} = \rho(h_{13A})_{ij}(\pi_{W_k}(V^2_1))$ for $i = 5, 6$ and $j = 1 \ldots 12$, $d_{i,j,1} = \rho(h_{13A})_{ij}(h_{1/12} \cdot \delta_0)$ for $j = 1 \ldots 13$, $d_{i,j,k} = \rho(h_{13A})_{ij}(h_{1/12} \cdot \delta_k)$ for $k = 1, 2$ and $j = 1 \ldots 13$, $d_{i,j,k} = \rho(h_{13A})_{ij}(h_{1/12} \cdot \delta_k)$ for $k = 1, 2$ and $j = 13$.

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