LOCAL LANGLANDS CORRESPONDENCE, LOCAL FACTORS, AND ZETA INTEGRALS IN ANALYTIC FAMILIES

by

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Abstract. — We study the variation of the local Langlands correspondence for \( GL_n \) in characteristic-zero families. We establish an existence and uniqueness theorem for a correspondence in families, as well as a recognition theorem for when a given pair of Galois- and reductive-group- representations can be identified as local Langlands correspondents. The results, which can be used to study local-global compatibility questions along eigenvarieties, are largely analogous to those of Emerton, Helm, and Moss on the local Langlands correspondence over local rings of mixed characteristic. We apply the theory to the interpolation of local zeta integrals and of \( L \)- and \( \varepsilon \)-factors.

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1. Introduction

The aim of this work is to study the variation in characteristic-zero families of the local Langlands correspondence for \( GL_n \), and of local \( L \)-functions, \( \varepsilon \)- and \( \gamma \)-factors, and zeta integrals.

Our results on the local Langlands correspondence are analogous to those of Emerton, Helm, and Moss, who in an important series of papers [9–13, 19, 20] studied the variation of such objects in families over complete local rings of mixed characteristics; we refer to those works collectively as [Mixed]. The proofs use in a crucial way some of their ideas (but certainly not all: our context is less delicate). We hope nevertheless that they will be found to be a useful contribution to the literature:
families over characteristic-zero bases arise naturally in the context of eigenvarieties, and there are also many examples of Galois-modules over local $\mathbb{Z}_p$-algebras $R$ which are flat as $R[1/p]$-modules but not as $R$-modules;\(^{(1)}\) such families lie outside the framework of \([\text{Mixed}]\). (Readers interested in \([\text{Mixed}]\) may still find this paper a useful point of entry before diving into the subtleties of that context.) Moreover we pay special attention to questions of rationality in the coefficients, whereas the base rings in \([\text{Mixed}]\) (except in \([9]\)) are assumed to have algebraically closed residue field.

In §1.1 we describe our main results and the organisation of the paper. In §1.2 we sketch the typical applications that we have in mind.

1.1. The results. — Let $F$ be a non-archimedean local field. Let $W_F$ (resp. $W'_F$) the Weil (resp. Weil–Deligne) group of $F$, and let $G = G_n := \text{GL}_n(F)$. Representations of $W_F$ and $W'_F$ will be tacitly understood to be Frobenius-semisimple; representations of $G$ will be understood to be smooth and admissible. Let $K$ be a field of characteristic zero and Noetherian (except in \([9]\)) the category of reduced Noetherian schemes over $K$.

In §2 we recall basic facts from the theory of Bernstein–Zelevinsky and its classification of representations of $G_n$ in terms of supercuspidal supports and multisegments of such. We then define the generic local Langlands correspondence

$$r' \mapsto \pi_{\text{gen}}(r')$$

after Breuil–Schneider: it is a map from Frobenius-semisimple $n$-dimensional Weil–Deligne representations (up to isomorphism) to those indecomposable representation of $G_n$ which are generic (i.e. admit a unique Whittaker functional); it agrees with the Tate-normalised Langlands correspondence $\pi(\cdot)$ for those $r'$ such that $\pi(r')$ is generic, and it is compatible with automorphisms of the coefficient field. In §3 we define the Bernstein ‘variety’ over $\mathfrak{X}_n/\mathbb{Q}$: it is a scheme locally of finite type, whose base-change to $\mathbb{C}$ has the usual Bernstein centre of $G_n$ as ring of functions. The points of $\mathfrak{X}_n$ are naturally in bijection with Galois orbits of supercuspidal supports for $G_n$. We show in Theorem 3.2.1 that $\mathfrak{X}_{n,K}$ coarsely pro-represents the functor on $\text{Noeth}_K$ which sends $X$ to the set of isomorphism classes of rank-$n$ representations of $W_F$ over $X$. This result (the semisimple local Langlands correspondence in families) is essentially due to Helm \([12]\). In §3.3 we define an extension $\mathfrak{X}_n \to \mathfrak{X}_n$ of the Bernstein variety, whose points are in bijection with Galois orbits of multisegments of supercuspidal supports. We then show that $\mathfrak{X}'_n$ coarsely pro-represents the functor of families of Weil–Deligne representations with locally constant monodromy.

In §4 we define, for any object $X$ of $\text{Noeth}_K$, the key notion of a co-Whittaker $\mathcal{O}_X[G_n]$-module, after Helm: when $X = \text{Spec} \mathbb{C}$ this singles out the generic $G_n$-representations. Given an $n$-dimensional Weil–Deligne representation $r'$ over $X$, we show that there exists a torsion-free co-Whittaker module $\pi(r')$ whose special fibre over each generic point (in the sense of algebraic geometry) $x \in X$ satisfies

$$\pi(r')|_x \cong \pi(r'|_x).$$

Moreover such $\pi(r')$ is unique up to twisting by a line bundle with trivial $G_n$-action and it satisfies (1.1.1) at all points $x$ lying in a unique irreducible component of $X$ (in fact a weaker condition suffices): see Theorem 4.3.1. The construction of $\pi(r')$ is based on the special but almost universal case $X = \mathfrak{X}_n$.

We then prove a recognition theorem (Theorem 4.3.3): given a rank-$n$ Weil–Deligne representation $r'$, a finitely-generated $G_n$-representation $V$ over $X$, and a dense subset $\Sigma \subset X$ such that $V|_x = \pi(r'|_x) = \pi_{\text{gen}}(r'|_x)$, we show that there exists an open $U \supset X$ containing $\Sigma$ such that $V|_U \cong \pi(r'|_U)$, up to tensoring with a line bundle. (The open $U$ is the locus where $V$ is co-Whittaker.) Our results hold more generally for representations of the group $\prod_{v \in S} \text{GL}_n(F_0,v)$, where the product ranges over a finite set of non-archimedean completions of a number field $F_0$.

Finally, in §5 we show that when $r'_1, r'_2$ are Weil–Deligne representation with locally constant monodromy over $X$ of ranks $n_1 \geq n_2$ respectively, and $\pi_i = \pi(r'_i)$, the local Rankin–Selberg zeta integrals for $\pi_1 \times \pi_2$, divided by the corresponding local $L$-value, interpolate to a regular function

\(^{(1)}\)A first example occurs when $R = \mathbb{T}_m$ is a non-Gorenstein local Hecke algebra as in \([18]\) (or a corresponding big Hecke algebra), and we consider the first homology (possibly completed) of a modular curve localised at $\mathfrak{m}$. 

on $X$. The key ingredient is a result of Cogdell and Piatetski-Shapiro [6], which in our language is seen as the case where $X$ is a cover of a connected component of $X_{\kappa,C}$. We use these results to provide an interpolation of local $\gamma$- and $\varepsilon$-factors in families.

The same principles apply to many other integrals of use in the representation theory of $G_n$; as a sample we show that the standard invariant inner product on unitarisable generic representations of $G_n$ can also be interpolated in families parametrised by Weil–Deligne representations with locally constant monodromy.

1.2. Intended applications. — The typical situation where our results apply is roughly the following. Let $p$ be a rational prime. Let $G$ be a reductive group over $\mathbb{Q}$ such that $G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p)$. Let $X = \text{Spec} R$ where $R$ is either the coordinate ring of an affine open of an eigenvariety for the group $G$, or $R$ is a localisation of $R^0[1/p]$ where $R^0$ is the base of a Hida family for $G$. Let $\Sigma \subset X$ be the set of classical points and assume it is dense (otherwise we replace $X$ by the Zariski closure of $X$). One can often construct a family $\rho$ of $n$-dimensional representations of the Galois group of $\mathbb{Q}_p$ over $X$ and, by global methods, a finitely generated torsion-free $\Theta_X[G_n]$-module $\Pi$.

Local-global compatibility along eigenvarieties. — In the above situation, one often knows that (i) $\Pi_{|x}$ is generic (e.g. because it is the local component of a global cuspidal automorphic representation), and (ii) points $x \in \Sigma$ satisfy local-global compatibility, that is that the Weil–Deligne representation $r_{|x} := WD(\rho_{|x})$ is associated with $\Pi_{|x}$ by the local Langlands correspondence $\pi$. As recalled in Lemma 2.1.1, $\rho$ itself admits a Weil–Deligne representation $r' = WD(\rho)$. Then the recognition theorem 4.3.3 allows to identify, up to twisting with a line bundle, the $\Theta_X[G_n]$-modules $\Pi$ and $\pi(r')$ over an open subset $X' \subset X$ containing $\Sigma$; in other words, the family $\Pi_{X'}$ satisfies local-global compatibility. By the uniqueness property this result can be glued to cover cases where $X$ is a more general rigid space.

Interpolation of local integrals. — Suppose moreover that one wishes to interpolate some local zeta integrals $Z_x : \Pi_{|x} \to \mathbb{Q}_p(x)$, say defined in a Whittaker model of $\Pi_{|x}$. Then one can do so over $X'$ by the local-global compatibility and the results of §5. A typical situation where the interpolation of local integrals is useful is the construction of $p$-adic $L$-functions (e.g. as in [7]). There, one starts with an integral representation of the form

$$Z(\phi_{|x}) = L(\Pi_x) \prod_v Z_v(\phi_{v|x}),$$

where now $\Pi$ is a family of (usually, $p$-adic Jacquet modules of) automorphic representations over $X$, $Z(\phi_{|x})$ is a global integral of $\phi \in \Pi$, and $Z_v(\phi_{v|x})$ are local integrals on $\Pi_{v|x}$ (here $\phi = \bigotimes_v \phi_v$ is a decomposition). After interpolating $x \mapsto Z(\phi_{|x})$ or its modification through a Hecke operator at $p$, one may either (i) choose $\phi$ carefully at all $v \nmid p$, so that $Z_v(\phi_{v|x})$ is explicitly computable to be a constant or some simple function on $X$, or (ii) interpolate the $Z_v(\phi_{v|x})$ for $v | p$ along the lines of §5, and define

$$L_p(\Pi) := \frac{Z(\phi)}{\prod_{v \mid p} Z_v(\phi_v)}$$

for any $\phi \in \Pi$. (3) The approach (ii) is more flexible and less laborious, as it does not require ad hoc choices and explicit calculations. When $L_p(\Pi)$ is expected to enjoy some integrality or holomorphy properties, however, additional work may be required to establish them.

An example. — For a specific example of all of the above, see [8].

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(2) More generally, the same ideas apply to the following case. Let $F_0 \subset F_1$ be an extension of number fields, let $S_0$ be a finite set of finite places of $F_0$ disjoint from those above $p$, and let $S_1$ be the set of places of $F_1$ above $F_0$. Then we may consider a reductive group $G$ over $F_0$ such that $\prod_{v \in S_0} G_{F_0,v} \cong \prod_{v \in S} GL_n/F_1,v$.

(3) As $\Pi$ will usually interpolate the tensor product of $\Pi_{|x}$ and of the finite-dimensional Jacquet modules (or ordinary parts) of $\Pi_{|x}$ for automorphic representations $\Pi_{|x}$, the vector $\phi_v$ for $v | p$ is restricted. The product of suitably modified $p$-adic local integrals $\prod_{v \mid p} Z_v(\phi_{v|x})$ will be the interpolation factor at $x$.
Related works. — The works [17, 22], as well the earlier [25, Appendix B], implicitly deal with the correspondence defined here, studying local-global compatibility for specific eigenvarieties, or Hida families, by different methods.

1.3. Acknowledgements. — The debt this work owes to [Mixed] is clear. I would like to thank David Helm for guidance at an early stage and Olivier Fouquet, Gil Moss, Robert Pollack and Eric Urban for useful conversations. The author is supported by a public grant of the Fondation Mathématique Jacques Hadamard.

2. Generic local Langlands correspondence

In this section, we set up the framework and recall various versions of the local Langlands correspondence, most importantly (a variant of) one due to Breuil–Schneider, which will later be shown to exhibit a reasonable behaviour in families.

2.1. Weil–Deligne representations. — Let $F$ be a non-archimedean local field with residue field of cardinality $q$, and let $G_F$ (resp. $W_F$, $I_F$, $W_F'$) be its absolute Galois group (resp. Weil group, inertia group, Weil–Deligne group). We denote by $\phi \in W_F$ a geometric Frobenius and, if $r$ is a representation of $W_F$, by $r(i)$ the representation $r \otimes |^i$, where $|^i: W_F / I_F \rightarrow \mathbb{Q}_p$ is the homomorphism characterised by $|\phi| = q^{-1}$.

Recall that a representation of $W_F'$, called a Weil–Deligne representation, with values in a $\mathbb{Q}$-algebra $B$ ‘is’ a pair $(r, N)$ where $r: W_F \rightarrow B^\times$ is a group homomorphism, trivial on some open subgroup of $W_F$, and $N \in B$ is a nilpotent element satisfying $N r(\phi) = q r(\phi) N$. The classification of Weil–Deligne representations with values in $B = M_n(K)$, for an algebraically closed field $K$ of characteristic zero, is well known. The indecomposable objects are of the form

$$\text{Sp}(r, m) = r \oplus \ldots \oplus r(m-1)$$

for some irreducible continuous $W_F$-representation $r$ and an integer $m \geq 1$, with $N$ mapping $r(i)$ isomorphically onto $r(i + 1)$ for all $i < m - 1$. Any Weil–Deligne representation is a direct sum $r' = \bigoplus \text{Sp}(r_i, m_i)$ of indecomposable representations. We say that $r$ or $r' = (r, N)$ is (Frobenius-)semisimple if $r(\phi)$ is a semisimple element of $GL_n(K)$.

If $X$ is a scheme and $V$ is a locally free $\mathcal{O}_X$-module, a Weil–Deligne representation on $M$ is one valued in $B = \text{End}_{\mathcal{O}(X)}(V(X))$. If $X = \text{Spec} K$ is the spectrum of a field, a Weil–Deligne representation on $M$ induces a monodromy filtration on $V$ [14, (1.5.5)]. Suppose that $K$ is isomorphic to a subfield of $\mathbb{C}$; then a semisimple Weil–Deligne is said to be pure of weight $w \in \mathbb{Z}$ if all eigenvalues $\lambda$ of $\phi$ on $\text{Gr}_i(V)$ are $q$-Weil numbers of weight $w + i$ (that is, $q^n \lambda$ is an algebraic integer for some $n$, and for all $i: K \rightarrow \mathbb{C}$ we have $i(\lambda) = q^{(w+i)/2}$).

Grothendieck’s $\ell$-adic monodromy theorem in $p$-adic families. — We recall how to associate a Weil–Deligne representation to certain families of representations of $\text{Gal}(\overline{F}/F)$. This result is not used in the rest of the paper but it is useful for applying the theory developed here. Let $p$ be a prime different from the residue characteristic of $F$. We say that $A$ is a $p$-adic ring if either (i) $A$ is an affinoid algebra over $\mathbb{Q}_p$, or (ii) there is a complete Noetherian local domain $A^\circ$ of characteristic $p$, with fraction field $\mathcal{K}$, such that $A$ is a subring of $\mathcal{K}$ containing $A^\circ[1/p]$. If $A$ is a $p$-adic ring, we say that a representation $\rho$ of $\text{Gal}(\overline{F}/F)$ on a finite locally free $A$-module $M$ is continuous if, respectively, (i) $\rho$ is continuous for the topology induced from the topology of $M$, or (ii) there is a Galois-stable finite $A^\circ$-submodule $M^\circ$ such that $M^\circ \otimes_{A^\circ} A = M$ and the restriction of $\rho$ to $M^\circ$ is continuous for the topology inherited from $A^\circ$.

Lemma 2.1.1. — Let $A$ be a $p$-adic ring, let $M$ be a locally free $A$-module, and let $p: \text{Gal}(\overline{F}/F) \rightarrow \text{Aut}_A(M)$ be a representation. Fix a nontrivial homomorphism $t_p: \text{Gal}(\overline{F}/F) \rightarrow \mathbb{Q}_p$. If $\rho$ is continuous in the sense of the previous paragraph, it uniquely determines a Weil–Deligne representation

$$r' = (r, N) := \text{WD}(\rho)$$

on $M$ such that $\rho$ coincides with $t \mapsto \exp(t_p(g) N)$ on some open subgroup of $I_F$.

Proof. — See [1, Lemma 7.8.14] for case (i) and [9, Proposition 4.1.6] for case (ii).
2.2. Bernstein–Zelevinsky theory and local Langlands correspondences. — We let \( G = G_n = \text{GL}_n(F) \).

Let \( P \subset G \) be a parabolic subgroup with Levi \( M = \prod_{i=1}^t G_{d_i} \). We recall the Langlands classification (due to Bernstein–Zelevinsky) and correspondence (proved by Harris–Taylor and Henniart), according to two different normalisations, the unitary and the rational; we will use a subscript \( ? \in \{ u, r \} \) (respectively) to distinguish them; our discussion is based on [5, §3]. There are correspondingly two notions of induction \( \text{Ind}_P^G \) from a parabolic subgroup \( P \subset G \) with Levi \( M \cong \prod P_G \); \( \text{Ind}_P^G \) is the usual, unitarily normalised induction of complex representations \( \sigma = \bigotimes \sigma_i \), and

\[
\text{Ind}_P^G(\sigma_1 \otimes \ldots \otimes \sigma_t) := \bigcap_{i=1}^t \sigma_i |_{P} \bigotimes \bigotimes_{i=1}^t \sigma_i |_{P} \bigotimes \bigotimes_{i=1}^t \sigma_i |_{P}.
\]

The rational induction \( \text{Ind}_P^G \) is equivariant for the action of \( \text{Aut}(\mathbb{C}/\mathbb{Q}) \), see [5, §3.4.3].

A supercuspidal support for \( G \) over a field \( K \) is an equivalence class of pairs \( (M, \sigma) \) where \( M \subset G \) is a Levi subgroup and \( \sigma \) is a supercuspidal representation of \( M \) over \( K \), up to the relation of \( G \)-conjugacy of both. Equivalently, it is a multiset \( \{ \sigma_1, \ldots, \sigma_m \} \) where each \( \sigma_i \) is a supercuspidal representation of \( G_d \), for some partition \( n = \sum d_i \).

Let \( \pi \) be a smooth admissible irreducible representation of \( G \) over \( \mathbb{C} \). For each choice (unitary or rational) of normalisation, the set of supercuspidal representations \( \sigma = \sigma_1 \otimes \ldots \otimes \sigma_r \) of \( G \)-subgroups \( M = \prod_{i=1}^t \text{GL}_{d_i} \) such that \( \pi \) is a Jordan–Hölder constituent of the parabolic induction \( \text{Ind}_P^G \sigma \) (resp. \( \text{Ind}_P^G \sigma \)), for some parabolic \( P \subset G \) with Levi \( M \), is an orbit under the action of \( G \)-conjugacy. That is, it is a supercuspidal support, called the unitary (resp. rational) supercuspidal support of \( \pi \). The relation \( \sigma \) is the rational supercuspidal support of \( \pi \) is also equivariant with respect to automorphisms of \( \mathbb{C} \).

Theorem of Bernstein–Zelevinsky. — A segment for \( G_d \) over \( K \) is a set of representations of \( G_d \) of the form \( \Delta = \Delta(\sigma, m) = \{ \sigma, \ldots, \sigma(m-1) \} \), with \( \sigma \) a supercuspidal representation of \( G_d \) over \( K \) (so \( n = md \)). The generalised Steinberg representation

\[ \pi_\pi(\Delta) := \sigma \text{St}_\Delta \]

is the unique generic representation with unitary (resp. rational) supercuspidal support \( \Delta \). It is the unique irreducible quotient of \( \text{Ind}_P^G(\sigma \otimes \ldots \otimes \sigma(m-1)) \) and the unique irreducible subrepresentation of \( \text{Ind}_P^G(\sigma(m-1) \otimes \ldots \otimes \sigma) \). The essentially square-integrable irreducible representations of \( G \) are exactly those of the form \( \pi_\pi(\Delta) \) for some segment \( \Delta \).

Let \( s = \{ \Delta_1, \ldots, \Delta_t \} \) be a multisegment (that is a multiset of segments), and let \( \sigma \) be its supercuspidal support, that is the multiset of supercuspidal representations \( \sigma_j(\sigma) \) of \( G_d \), occurring in the \( \Delta_j \). We say that \( \Delta_i = \Delta(\sigma_i, m_i) \) precedes \( \Delta_j = \Delta(\sigma_j, m_j) \) if none of them contains the other and \( \sigma_j = \sigma_i(t+1) \) for some \( 0 \leq t \leq m_i - 1 \). Suppose that the \( \Delta_i \) in \( s \) are ordered so that for \( i < j, \Delta_i \) does not precede \( \Delta_j \). Then

\[ \pi(\pi(\Delta)) := \gamma \text{Ind}_P^G(\pi(\Delta_1), \ldots, \pi(\Delta_t)), \]

the induction with respect to any suitable parabolic \( P \), is independent of the particular choice of ordering (among those satisfying the condition mentioned) and of \( P \) up to isomorphism; it has a unique irreducible subrepresentation, which is the unique generic irreducible representation with supercuspidal support \( \sigma \), and a unique irreducible quotient denoted \( \pi(\sigma) \).

Suppose that the \( \Delta_i \) in \( s \) are ordered so that for all \( i < j, \Delta_j \) does not precede \( \Delta_i \). Then the representation

\[ \pi(\pi(\Delta)) := \gamma \text{Ind}_P^G(\pi(\Delta_1), \ldots, \pi(\Delta_t)) \]

is independent of the particular choice of ordering (among those satisfying the condition mentioned) up to isomorphism; it has a unique irreducible quotient, which is the generic representation with supercuspidal support \( \sigma \), and a unique irreducible subrepresentation, which is the representation \( \pi(\sigma) \) of the previous paragraph.

The (unitary or rational) local Langlands correspondences \( \pi_u, \pi \) (denoted collectively as \( \pi_\pi \)) are bijections between equivalence classes of Frobenius-semisimple \( n \)-dimensional Weil–Deligne representations over \( \mathbb{C} \) and of smooth admissible irreducible representations of \( G_n \) over \( \mathbb{C} \).
rational normalisation is also known as geometric or Tate normalisation.) They are related by
\[ \pi(r') = \pi_0(r') \mid \mathfrak{m}, \]
and if \( r'\ast \) denotes the dual representation to \( r' \) then
\[ \pi_0(r'\ast) \equiv \pi_0(r')^\vee \]
(small dual). Each of \( \pi_0, \pi \) restricts to a bijection between irreducible Weil–Deligne representations (thus of the form \( r' = (r, 0) \)) and supercuspidal \( G_n \)-representations, and the whole correspondence for \( G_n \) is deduced from such restricted version for all \( m \leq n \) as follows. Let
\[ r' = \bigoplus \mathrm{Sp}(r_i, m_i). \]
Let \( \sigma_{\gamma,i} := \pi_\gamma((r_i, 0)) \), and consider the multisegments
\[ s_{\gamma,r'} := \{ \Delta(\sigma_{\gamma,i}, m_i) \} \]
Then
\[ \pi_\gamma(r') = \pi(s_{\gamma,r'}). \]
The rational correspondence \( \pi \) is compatible with automorphisms of \( \mathbb{C} \).

The correspondence \( r' \mapsto \pi_\text{gen}(s_{\gamma,r'}) \) is the one defined by Breuil–Schneider in [2, pp. 162–164] and recalled in [9, §4.2]. We will be more concerned with the correspondence
\[ \pi_\text{gen}: r' \mapsto \pi_\text{gen}(s_{\gamma,r'}), \]
which we call the generic local Langlands correspondence, related to \( \pi_\text{gen} \) by \( \pi_\text{gen}(r') = \pi_\text{gen}(r'\ast) \). By [2, Lemma 4.2] and the argument at the end of [9, §4.2], it descends to a map still denoted by \( \pi_\text{gen} \) between Weil–Deligne representations over \( K \) and \( G \)-representations defined over \( K \), for any\(^{4}\) characteristic zero field \( K \). (The same is true for the rational Langlands correspondence \( r' \mapsto \pi(r') \).)

One can show that \( \pi_\text{gen} \) is a bijection onto those \( G \)-representations which are indecomposable and of Whittaker type in the sense of Definition 4.1.2 below.

Finally, the correspondence \( \pi_\gamma \) induces a bijection (semisimple Langlands correspondence)
\[ \pi_\gamma,ss: r \mapsto (M, \sigma) \]
between Frobenius-semisimple \( n \)-dimensional representations of \( \mathbb{W}_F \) over \( \mathbb{C} \) and supercuspidal supports for \( G_n \) over \( \mathbb{C} \). The correspondence \( \pi_{\text{ss}} \) is equivariant with respect to automorphisms of \( \mathbb{C} \) and descends to a map among objects defined over any characteristic zero field \( K \).

To summarise the relation between the correspondences we use: if \( r' = (r, N) \) is a Weil–Deligne representation over \( K \), then
\[ \pi(r') \subset \pi_{\text{gen}}(r'), \]
with equality if and only if \( \pi(r') \) is generic, and the supercuspidal support of both is \( \pi_{\text{ss}}(r) \).

Galois representations. — Let \( p \) be a prime different from the residue characteristic of \( F, K \) a topological field extension of \( \mathbb{Q}_p, p: \mathbb{G}_F \to \mathrm{GL}_n(K) \) a continuous representation admitting a Weil–Deligne representation \( r' = (r, N) \). Then for \( * = \text{gen}, \text{ss}, \emptyset \), we define
\[ \pi_*(p) := \pi_*(r'). \]

3. Langlands correspondence for the Bernstein varieties

We introduce the rational Bernstein variety, a scheme over \( \mathbb{Q} \) whose ring of regular functions over \( \mathbb{C} \) is the usual Bernstein centre of \( G = G_n \mathrm{GL}_n(F) \). Then we identify it with the coarse moduli space of Frobenius-semisimple representations of \( \mathbb{W}_F \). Finally, we prove an analogous result for Weil–Deligne representations with locally constant monodromy, in terms of an extension of the Bernstein variety.

3.1. The Bernstein variety. — The Bernstein centre \( \mathfrak{Z}_{n,\mathbb{Q}} \) of \( G \) is the centre (endomorphism ring of the identity functor) of the category of smooth \( G \)-modules with coefficients in \( \overline{\mathbb{Q}} \). Bernstein

\(^{4}\)In loc. cit., the coefficient field is supposed to contain \( \mathbb{Q}_p \), but this is not necessary for any of their statements or proofs, until one wants to pass from Weil–Deligne representations to Galois representations.
and Deligne gave an explicit description of it: $\mathfrak{X}_{n,Q}$ is the ring of regular functions of the ‘Bernstein variety’ $\mathfrak{X}_{n,Q}$, an infinite disjoint union of affine varieties over $\overline{Q}$ which we now turn to describing.

An inertial class of supercuspidal supports over $\overline{Q}$ is an equivalence class $[M,s]$ (which we sometimes denote just by $[s]$) of pairs where $M$ is a Levi and $s$ is a $\widehat{T}_M(\overline{Q})$-orbit of supercuspidal representations of $M$, up to $G$-conjugation of both. Here $\widehat{T}_M(\overline{Q})$ is the space of unramified characters of $M$ valued in $\overline{Q}^\times$, which is the set of $\overline{Q}$-points of a split torus $\widehat{T}_M$ described as follows. Let $M_0 \subset M$ be the intersection of the kernels of all unramified characters of $M$ (it is also the subgroup generated by all compact subgroups of $M$); then $M/M_0$ is a free abelian group of finite rank and

$$\widehat{T}_M = \text{Spec} \overline{Q}[M/M_0].$$

Let $H_\sigma$ be the (finite) group of unramified characters of $M$ stabilising any (hence every) $\sigma$ in $s$. As $s$ is a principal homogeneous space for $\widehat{T}_M/H_\sigma$, it also acquires the structure of an affine algebraic variety. Let $W(M) := N_G(M)/M$ and $W(M,s) := \text{Stab}_{W(M)}(s)$, then

$$\mathfrak{X}_s := s/W(M,s)$$

is an algebraic variety whose points are in bijection with supercuspidal supports in the inertial class $[M,s]$. The Bernstein ‘variety’ is

$$\mathfrak{X}_{n,Q} = \prod \mathfrak{X}_s$$

where $[M,s]$ runs over inertial classes. The points of $\mathfrak{X}_s$ are in bijection with supercuspidal supports over $\overline{Q}$.

**Lemma 3.1.1.** — There is a scheme $\mathfrak{X}_n$, locally of finite type over $Q$, whose closed points $x$ are in bijection with $\text{Gal}(\overline{Q}/Q)$-orbits of supercuspidal supports over $\overline{Q}$; the base change $\mathfrak{X}_n \times_Q \overline{Q}$ is identified with $\mathfrak{X}_{n,Q}$. The centre $Z_{n,Q}$ of the category of smooth representations of $G_n$ over $Q$-vector spaces is naturally identified with $\mathcal{O}(\mathfrak{X}_{n,Q})$.

**Proof.** — The orbits of the obvious Galois action on $\mathfrak{X}_{n,Q}$ are finite; moreover if $x \in \mathfrak{X}_{n,Q}$ then the Galois orbit of the connected component of $x$ is the union of the connected components of the Galois-conjugates of $x$. It follows that each Galois orbit of connected components $\mathfrak{X}_{n,Q}$ is an affine scheme of finite type over $\overline{Q}$. Then the Galois action descends each $\mathfrak{X}_{n,Q}$ to an algebraic variety $\mathfrak{X}_i$ over $Q$ and the whole $\mathfrak{X}_{n,Q}$ to a scheme $\mathfrak{X}_n = \prod \mathfrak{X}_i$ over $Q$.

Finally, the centre $Z_n$ of the category of smooth representations of $G$ over $Q$ is naturally identified with the subring of $\mathfrak{X}_{n,Q}$ consisting of $\text{Gal}(\overline{Q}/Q)$-invariant elements (for the action $\sigma.z(x) := z(\pi^\sigma)$), and hence with $\mathcal{O}(\mathfrak{X}_{n,Q})$.

**Inertial local Langlands correspondence.** — Define an inertial type for $F$ to be a representation of $I_F$ which extends to $W_F$. Then, the inertial local Langlands correspondence $\pi_I$ is a bijection between inertial classes on the $G$-side and inertial types on the Galois side; $\pi_I^{-1}$ sends the inertial class $[M,s]$ of a supercuspidal support $(M,\sigma)$ to the restriction of $\pi_{ss}^{-1}((M,\sigma))$ to $I_F$.

**3.2. Semisimple local Langlands correspondence in families.** — The following result is essentially due to Helm [12]. It morally shows that the interpolation of the correspondence $\pi_{ss}$ of (2.2.4) in families is trivial in the sense that there is a common moduli space parametrising the isomorphism classes of objects on both sides.

**Theorem 3.2.1.** — Let $K$ be a field of characteristic zero. The Bernstein variety $\mathfrak{X}_{n,K}$ is a coarse (pro)-moduli scheme for the functor $\Phi_{n,K}$ which associates to any reduced Noetherian $K$-scheme $X$ the set of isomorphism classes of Frobenius-semisimple representations of $W_F$ on locally free $\mathcal{O}_X$-modules of rank $n$, in such a way that the map

$$\Phi_{n,K} \to \mathfrak{X}_{n,K}$$

induces $\pi_{ss}^{-1}$ on geometric points.

The proof will occupy the rest of this subsection.
It suffices to show the result for \( K \) algebraically closed, since by the compatibility with \( \pi_{ss} \) and the rationality properties of \( \pi_{ss} \) the map \( \Phi : n \mapsto X_{n,K} \) descends to \( K \). We thus assume \( K = \overline{K} \) for the rest of this subsection.

Let \( R_n^\square \) be the coordinate ring of the variety of semisimple matrices in \( GL_n(K) \). Let \( R_n := R_n^\square \) be the coordinate ring of the scheme parametrising conjugacy classes of semisimple automorphisms of an \( n \)-dimensional vector space; we have \( R_n = K[a_1, \ldots, a_{n-1}, a_n^{\pm 1}] \) where the \( a_i \) correspond to the coefficients of the characteristic polynomial of an automorphism; we also have \( \text{Spec } R_n \cong (G_m^n)/S_n \), where the isomorphism \( K[a_1, \ldots, a_{n-1}, a_n^{\pm 1}] \cong K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^S_n \) is given by \( \prod_{i=1}^n (X - x_i) = X^n + \sum_{\sigma} a_\sigma X^\sigma \).

Let us start by making explicit Bernstein–Deligne's description of \( X_{n,K} = \prod X_{[a]} \). We may fix a supercuspidal support \((M, \sigma^\natural)\) in the class \([M, s]\) such that
\[
\sigma^\natural = \bigotimes_{i=1}^s \sigma_i^{\otimes m_i},
\]
where the \( \sigma_i \) are pairwise inertially inequivalent representations of \( GL_d(F) \) and \( \sum m_i d_i = n \). Then \( X_{[a]} \) is explicitly described as follows. For \( x \in G_m \) denote by \( \chi_{x,d} \) the unramified character of \( GL_d(F) \) given by \( g \mapsto x^d \chi_{x,d} \). Let \( f_i \in \mathbb{N} \) be such that
\[
\mu_{f_i} \subset G_m
\]
is the stabiliser of \( \sigma_i \) under the action \( x \circ \sigma_i = \sigma_i \chi_{x,d_i} \). Then
\[
X_{[a]} \cong \prod_{i=1}^s G_m^{m_i}/S_{m_i}
\]
with the point which is the orbit of \( ((x_{1,1}, \ldots, x_{1,m_1}), \ldots, (x_{s,1}, \ldots, x_{s,m_s})) \) corresponding to
\[
\bigotimes_{i=1}^s \bigotimes_{j=1}^{m_i} \sigma_j^{f_i}.
\]
Let
\[
X_{\square} := \prod_{i=1}^s \text{Spec } R_n^\square \to X_{[a]}
\]
be given by the natural quotients \( \text{Spec } R_n^\square \to \text{Spec } R_n \), and we will construct a \( W_F \)-representation over \( X_{\square} \), which will be universal in some sense to be specified. We start by studying families of \( W_F \)-representations.

**Lemma 3.2.2.** — Let \( K \) be an algebraically closed field of characteristic zero, let \( A \) be a \( K \)-algebra and let \( M \) be an \( A[W_F] \)-module, finite and locally free as an \( A \)-module. Then there is an \( A[W_F] \)-module decomposition
\[
M = \bigoplus_{[\tau]} \text{Ind}_{W_F}^{W_{[\tau]}} M_{[\tau]},
\]
whose summands are uniquely determined. Here \( [\tau] \) runs over the \( W_F \)-conjugacy classes of irreducible \( I_F \)-representations occurring in \( M \), \( W_{[\tau]} \subset W_F \) is the stabiliser of the orbit \( [\tau] \), and we may choose
\[
M_{[\tau]} = \text{Hom}_{I_F}(\tau, M) \otimes \tau
\]
for any \( \tau \in [\tau] \).

**Proof.** — By the finiteness of \( M \), there is a finite index subgroup \( I_F^\natural \subset I_F \) acting trivially on \( M \); let \( T_F := I_F/I_F^\natural \). We have \( A[T_F] \cong K[T_F] \otimes A \) and the finite-group algebra \( K[T_F] \cong \bigoplus_{\tau} \text{End }_r(\tau) \) is semisimple. This induces a decomposition \( M = \bigoplus_{\tau} M_\tau \) as \( A[I_F] \)-modules, where \( \tau \) runs over equivalence classes of irreducible \( T_F \)-representations and \( M_\tau = \text{Hom}_{I_F}(\tau, M) \otimes \tau \).

By definition, the finite cyclic group \( W_F/W_{[\tau]} \) acts on \( \bigoplus_{\tau \in [\tau]} M_\tau \) by permuting the summands.

The description of \( M_{[\tau]} \) follows. □

Suppose chosen an extension \( \overline{\tau} \) of \( \tau \) to \( W_{[\tau]} \); this is equivalent to specifying, for any element \( \phi_{[\tau]} \in W_{[\tau]} \) whose class generates \( W_{[\tau]}/I_F \), an \( I_F \)-isomorphism \( \tau \cong \tau^{\phi_{[\tau]}} \). Let \( M_{[\tau]} \) be as in (3.2.5). Then \( \text{Hom}_{W_{[\tau]}}(\overline{\tau}, M) \) is an \( A \)-module isomorphic to \( \text{Hom}_{I_F}(\tau, M) \), carrying an unramified
representation of $W_\tau$, so that we may write
\[ M_\tau = \text{Hom}_{I_F}(\tau, M) \otimes \tilde{\tau} \]
as the tensor product of an unramified representation and $\tilde{\tau}$.

**Universal inertially-framed representation.** — Fix a representative $\tau$ for every $W_F$-conjugacy class $[\tau]$ of irreducible $I_F$-representations. An inertial framing\(^{(5)}\) of an $A$-representation $r$ of $I_F$ or $W_F$ is a choice, for every $[\tau]$, of a basis of $\text{Hom}_{I_F}(\tau, r)$.

Let $[M, s]$ be the inertial class fixed above and let $\nu = \pi^{-1}_J([M, s])$. We shall construct an inertially-framed Frobenius-semisimple $W_F$-representation $r_\nu$ over $\mathcal{X}_{n,s}$ which is universal among those such representation whose restriction to $I_F$ is $\nu$.

Decompose $\nu = \bigoplus \text{Hom}_{I_F}(\tau, \nu) \otimes \tau$, and let $n_{[\nu]} := \dim \text{Hom}(\tau, \nu)$, which is independent of $\tau \in [\tau]$. If we fix a supercuspidal support $(M, \sigma^\tau)$ as in (3.2.1), we obtain a representation $r^\nu = \pi_{ss}^{-1}((M, \sigma^\tau))$ of $W_F$ described as follows.

Let $\tilde{\tau}_i^F$ be the irreducible $d_i$-dimensional representation of $W_F$ associated with $\sigma_i$ by the local Langlands correspondence for $GL_d(F)$. By the compatibility of local Langlands with twisting, $\mu_{f_i}$, (3.2.2) is also the stabiliser of $\tilde{\tau}_i^F$ under the action of $G_m$ by unramified twists. It follows that we can write $\tilde{\tau}_i^F = \text{Ind}_{W_i}^{W_F} \tilde{\tau}_i$, where $W_i$ is the Weil group of the unramified extension of $F$ of degree $f_i$ and $\tilde{\tau}_i$ is a representation of $W_i$ whose restriction to $I_F$ is irreducible. Then
\[ r^\nu = \bigoplus_i (\tilde{\tau}_i^F)^{m_i} = \bigoplus_i \text{Ind}_{W_i}^{W_F} (K^{m_i} \otimes \tilde{\tau}_i) \]
where $K^{m_i}$ is a trivial $W_i$-module.

Define the representation
\[ r_\nu := \bigoplus_i \text{Ind}_{W_i}^{W_F} ((R_{m_i}^\square)^{m_i} \otimes \tilde{\tau}_i) \]
over $\mathcal{X}_{n,s} = \text{Spec } \bigotimes_i R_{m_i}^\square$, where each $(R_{m_i}^\square)^{m_i}$ is an unramified $W_F$-module on which a geometric Frobenius $\phi_i$ acts by the universal semisimple $m_i \times m_i$-matrix over $R_{m_i}^\square$. It is clear that $r_\nu$ is universal among those $W_F$-representations $r$ such that $r|_{I_F} \cong \nu$.

We can now establish the coarse moduli property of $\mathcal{X}_{n,K}$. Recall that this amounts to showing that:

- there is a natural transformation of functors $\Phi_{n,K} \to \mathcal{X}_{n,K}$, that is: for every isomorphism class of families of representations of $W_F$ over a test scheme $X$, there is a map $f: X \to \mathcal{X}_{n,K}$, functorially in $X$;
- for each algebraically closed field $\Omega \supset K$, the above transformation induces a bijection $\Phi_n(\Omega) \to \mathcal{X}_{n,K}(\Omega)$ (and moreover we assert that this bijection is $\pi_{ss}^{-1}$).

First, for each inertial type $\nu$, let $\Phi_\nu \subset \Phi_n$ be the subfunctor of those classes of representations $r$ such that $r|_{I_F}$ is everywhere equivalent to $\nu$. Suppose that $r$ is an $n$-dimensional family of representations of $W_F$ over a reduced Noetherian scheme $X$. As the restriction $r|_{I_F}$ is locally constant on $X$ and $X$ has finitely many connected components, $r$ defines an element of a finite union $\coprod_i \Phi_i(X) \subset \Phi(X)$. Cover $X$ with connected Zariski-open subsets $U_i: X_i \hookrightarrow X$ such that over each $X_i$ the representation $\iota_i^* r|_{I_F}$ is isomorphic to a fixed inertial type $\nu$ and moreover it admits an inertial framing. By the universal property of $r_\nu$ there is a unique map $f_i: X_i \to \mathcal{X}_{n,K}$ such that $f_i^* r_\nu \cong \iota_i^* r$. The compositions of $f_i$ with the projection $\mathcal{X}_{n,K} \to \mathcal{X}_{n,K}$ are independent of the choice of inertial framing and glue to the desired map $f: X \to \mathcal{X}_{n,K}$.

Finally, we show that when $X = \text{Spec } \Omega$, the image of the supercuspidal support $x = f(X) \in \mathcal{X}_{n,K}(\Omega)$ under $\pi_{ss}$ is $r$. In fact, suppose that $x$ is represented by the orbit of
\[ ((x_{1,1}, \ldots, x_{1,m_1}), \ldots, (x_{s,1}, \ldots, x_{s,m_s})) \in \prod_{i=1}^s G_{m_i}^{m_i}, \]
and keep the notation introduced before. Then for any choice of inertial framing the $i$th factor of $f^* r_\nu$ is the tensor product of $\tilde{\tau}_i^F$ and an $m_i$-dimensional unramified representation over $\Omega$ on

\(^{(5)}\)Analogous to the notion of pseudo-framing in [12].
which \( \phi = \phi^f_i \) acts on by a matrix with eigenvalues \( x_{i,1}^f, \ldots, x_{i,m_i}^f \). By the compatibility of \( \pi_{ss} \) with twisting, this is the \( W_F \)-representation corresponding under \( \pi_{ss} \) to the supercuspidal support \( x \) as described in (3.2.4).

This completes the proof of Theorem 3.2.1.

3.3. Monodromy and the extended Bernstein variety. — We start by discussing monodromy operators in families, mostly borrowing from [1, §7.8.1].

Partitions. — A partition of length \( \ell \) of an integer \( m \) is an ordered sequence of integers \( t = (t_j) \) such that \( n = \sum_{j=1}^{\ell} t_j \). Two partitions \( t, t' \) are said to be equivalent if there is a \( \sigma \in S_\ell \) such that \( t_{\sigma(j)} = t_j \) for all \( j \); the subgroup of self-equivalences of a partition \( t \) is denoted by \( W_t \subset S_\ell \). A multypartition is a sequence \( t = (t_i) \) of partitions \( m_i = \sum_{j=1}^{\ell} t_{i,j} \); two multypartitions \( t = (t_i) \), \( t' = (t'_i) \) are said to be equivalent if \( t_i \) is equivalent to \( t'_i \) for all \( i \); the group of self-equivalences of \( t = (t_i) \) is \( W_t = \prod_i W_{t_i} \). If \( t \) is a (multi)partition, we denote by \([t]\) its equivalence class; if \([t]\) is class of (multi)partitions, a decreasing representative \( t \) is one satisfying \( t_{(i),j} \geq t_{(i),j+1} \) for all \( i \) and \( j \).

Nilpotent operators. — Let \( K \) be a field and let \( N \) be a nilpotent operator on a vectors spaces \( V \) of dimension \( m \). Then \( N \) can be decomposed into Jordan blocks, \( N \sim J_{t_1} \oplus \cdots \oplus J_{t_\ell} \) for a partition \( t_1 + \cdots t_\ell = m \) whose class

\[ [t] = [t](N) \]

is uniquely determined. Let now \( N' \) be another nilpotent operator on a \( K \)-vector space \( V' \) of the same dimension \( m \), with associated partition \([t']\). We write

\[ N \preceq N', \text{ resp. } N \sim N' \]

if \([t] \preceq [t']\) (resp. \([t] = [t']\)) in the sense that, if \( t \) and \( t' \) are decreasing representatives of \([t]\) and \([t']\) then for all \( i \),

\[ \sum_{j=1}^{i} t_j \leq \sum_{j=1}^{i} t'_j; \]

equivalently, if \( \text{rk} N' \leq \text{rk} N^m \) (resp. \( \text{rk} N^m = \text{rk} N^m \)) for all \( i \leq m \). The association \( N \mapsto [t] \), hence the relations \( \preceq, \sim \), are compatible with field extensions.

Semicontinuity of monodromy. — Let \( K \) be a characteristic zero field, \( X \) a reduced Noetherian scheme over \( K \), and let \((r,N)\) be a Weil–Deligne representation on a locally free \( \mathcal{O}_X \)-module \( M \) of rank \( n \). As in the proof of Lemma 3.2.2, there exists a finite set of irreducible representations \( \tau \) of \( I_F \) over \( K \) such that \( M \otimes K = \bigoplus \tau \otimes \text{Hom}_{I_F}(\tau, M) \), and \( \text{Hom}_{I_F}(\tau, M) \) is locally free. The monodromy operator \( N \) leaves invariant the summands and, as it commutes with \( I_F \), induces nilpotent operators \( N_{x,\tau} \) on each of \( \text{Hom}_{I_F}(M, \tau) \).

Let \( x, y \) be points of \( X \) and let \( N_x = (N_{r,x}), N_y = (N_{r,y}) \). We say that \( N_x \preceq_{I_F} N_y \) if for all \( \tau \), \( N_{x,\tau} \preceq_{I_F} N_{y,\tau} \), and that \( N_x \sim_{I_F} N_y \) if for all \( \tau \), \( N_{r,x} \sim_{r,y} N_{r,y} \). We say that \( N \) is locally constant on \( X \) if \( N_x \sim_{I_F} N_y \) whenever \( x \) and \( y \) belong to the same connected component of \( X \).

Proposition 3.3.1. — The map

\[ \text{rk} N : x \mapsto \text{rk} N_x := (\text{rk} N^m_{x,\tau}) \]

is lower semicontinuous in the sense that, for each \( x \in X \), there is an open neighbourhood \( U \subset X \) of \( x \) such that \( N_x \preceq_{I_F} N_y \) for all \( y \in U \). Moreover the set \( D \) of discontinuity points of \( \text{rk} N \) is closed and nowhere dense in \( X \), and it contains no \( x \in X \) such that \((r_x, N_x)\) is pure.

The restriction of \((r, N)\) to the open dense \( X - D \) has locally constant monodromy.

(We say that \( \text{rk} N \) is discontinuous at \( x \in X \) if for every open neighbourhood \( U \subset X \) of \( x \) there is \( y \in U \) such that \( \text{rk} N_x \sim_{I_F} \text{rk} N_y \).)

Proof. — The lower semicontinuity of \( \text{rk} N \) follows from the upper semicontinuity of the ranks of the coherent \( \mathcal{O}_X \)-modules \( \text{Ker} N^m_{x,\tau} \) for all \( i, \tau \).

For the second assertion, we first note that the set \( D \) of discontinuity points of the function

\[ r := \text{rk} M^m \]

is the union, for all irreducible components \( X_i \subset X \), of the set \( D_i \) of discontinuity points
of $r_i := r|_{X_i}$. If $r_i$ is the rank of $M'$ at the generic point of $X_i$, each of $D_i := \{r_i(x) \leq r_i - 1\}$ is closed in $X_i$, hence in $X$. Therefore $D$ is closed in $X$ and, as it contains no generic points of irreducible components, it is nowhere dense. By [24, Theorem 4.1 (2)], $D$ contains no $x \in X$ such that $(r_x, N_x)$ is pure. Finally, the restriction of $\text{rk} N$ to $X - D$ is continuous, equivalently the monodromy is locally constant.

The extended Bernstein variety. — We now define an extension of the Bernstein variety whose points are in bijection with Galois orbits of multisegments for $G$. It is naturally a coarse moduli space for Weil–Deligne representations with locally constant monodromy.

Fix an inertial class $[s]$, which we can write in the form

$$[s] = \otimes_{i=1}^{s} \mathfrak{G}_i \otimes \mathfrak{m}_i,$$

where the $\mathfrak{g}_i$ are pairwise inequivalent inertial classes of supercuspidal representations of $G_d(F)$.

Let $[t] = ([t_i])$ be a multipartition of $(m_i)$ and, after picking any $t \in [t]$ (whose choice won’t matter), let

$$\mathfrak{X}[s][t] := \left( \prod_{i=1}^{s} \mathfrak{X}[s_i][t_i] \right)/W_t,$$

a scheme over $\overline{Q}$ with a closed immersion

$$f_{[t]} : \mathfrak{X}[s][t] \to \mathfrak{X}[s]$$

$$\sigma_{1,1}, \ldots, \sigma_{1,t_1}, \ldots, \sigma_{s,1}, \ldots, \sigma_{s,t_s} \mapsto \otimes_{i=1}^{s} \otimes_{j=1}^{t} \Delta(\sigma_{i,j}, t_{i,j})$$

where $\Delta(\sigma, t) := \otimes_{m=1}^{t} \sigma(m)$ When each $t_i = 1 + \ldots + 1$, we have $t_i = m_i$ and the above map recovers the isomorphism (3.2.3).

The action of $\tau \in \text{Gal}(\overline{Q}/Q)$ on $\mathfrak{X}[s][t]$ sends $\mathfrak{X}[s][t]$ to $\mathfrak{X}[s][t]$ and the (finite) union $\bigsqcup_{t} \mathfrak{X}[s][t]$ descends to a $Q$-subscheme of $\mathfrak{X}[s]$. Then

$$\mathfrak{X}'_{n, \overline{Q}} := \bigsqcup_{[s],[t]} \mathfrak{X}[s][t]$$

descends to a scheme

$$\mathfrak{X}'_{n, Q}$$

over $Q$: the extended Bernstein variety.

Theorem 3.3.2. — The extended Bernstein variety $\mathfrak{X}'_{n, Q}$ is a coarse (pro)-moduli scheme for the functor $\Phi'_{n, K}$ which associates to any reduced Noetherian $K$-scheme $X$ the set of isomorphism classes of Frobenius-semisimple Weil–Deligne representations on locally free $\mathcal{O}_X$-modules of rank $n$ with locally constant monodromy, in such a way that the map $\Phi'_{n, K} \to \mathfrak{X}'_{n, K}$ induces the local Langlands correspondence $\pi^{-1}$ on geometric points. This map fits into a commutative diagram

$$\begin{array}{ccc}
\Phi'_{n, K} & \longrightarrow & \mathfrak{X}'_{n, K} \\
\downarrow & & \downarrow \\
\Phi_{n, K} & \longrightarrow & \mathfrak{X}_{n, K}
\end{array}$$

where the bottom horizontal arrow is given by Theorem 3.2.1.

Proof. — We may reduce to the case $K$ algebraically closed and to testing the coarse moduli property on connected schemes $X/K$. But it is easy to see that if $(r, N)$ is a Weil–Deligne representation with constant monodromy over $X$ and the map $f : X \to \mathfrak{X}_{n, K}$ induced by $r$ via Theorem 3.2.1 has image in $\mathfrak{X}[s]$, then in fact $f$ has values in $\mathfrak{X}[s][t]$ for $[t] = [t](N)$, so that we may uniquely lift $f$ to $f' : X \to \mathfrak{X}[s][t] \subset \mathfrak{X}'_{n, K}$. Moreover the isomorphism classes of Weil–Deligne representations over $K$ are in bijection with points of $\mathfrak{X}'_{n, K}$: explicitly, the point of $\mathfrak{X}[s][t] \subset \mathfrak{X}'_{n, K}$ corresponding to the supercuspidal support $\otimes_{i=1}^{s} \otimes_{j=1}^{t} \Delta(\sigma_{i,j}, t_{i,j})$ corresponds to the Weil–Deligne representation

$$\otimes_{i=1}^{s} \otimes_{j=1}^{t} \text{Sp}(\pi^{-1}(\sigma_{i,j}), t_{i,j}).$$

\[\square\]
Let $X_{[a],[t]} := \prod_{s=1}^{\ast} X_{[a]} \to X_{[a],[t]}$. There is a ‘universal’ Weil–Deligne representation $(r,N)_{[a],[t]}$ over $X_{[a],[t]}$ given by

$$(3.3.4) \quad (r,N)_{[a],[t]} := \bigoplus_i \bigoplus_j \text{Ind}^{W_F}_{W_F'}((R^i)_{t,j} \otimes \text{Sp}(\overline{r},t_i,j))$$

with the notation of (3.2.6). Its fibre over any point of $X_{[a],[t]}$ over the supercuspidal support $\otimes_{s=1}^{\ast} \otimes_{t,j} \Delta(\sigma_i,t_i,j)$ is isomorphic to (3.3.3).

4. Langlands correspondence in families

4.1. Co-Whittaker modules. — In this subsection, $F$ denotes a global field, and $S$ a finite set of non-archimedean places of $F$. We denote by $W_{F,S} := \prod_{\mathfrak{p} \in S} W_{F_{\mathfrak{p}}}$, $G_v = G_{n,v} := \text{GL}_n(F_v)$, $G_S = G_{n,S} := \prod_{v \in S} G_{n,v}$. A Weil–Deligne representation of $W_{F,S}$ is a collection $r_{v}^{\ast} = (r_{v}^{\ast})_{v \in S}$ where $r_v$ is a Weil–Deligne representation of $W_{F,v}$. We denote by $\pi_{gen,v}$ the local Langlands correspondence (2.2.3) for the field $F_v$.

Let $G$ be a locally compact group, $K$ a field of characteristic zero and $X$ a Noetherian $K$-scheme. An $\mathcal{O}_X[G]$-module $V$ is said to be smooth if every $v \in V$ is stabilised by some compact open subgroup of $G$, and admissible if for every compact open subgroup $U \subset G$, the $\mathcal{O}_X$-module $V^U$ is finitely generated.

We recall some formalism introduced in [7]. Let $\mu_{\mathbb{Q}}$ be the ind-scheme of roots of unity over $\mathbb{Q}$. We define the space of additive characters of $F_v$ of level 0 to be

$$(4.1.1) \quad \Psi_v := \text{Hom}(F_v/\mathcal{O}_F,v) - \text{Hom}(F_v/\omega_v^{-1}\mathcal{O}_F,v,\mu_{\mathbb{Q}}),$$

where $\omega_v \in F_v$ is a uniformiser and we regard $\text{Hom}(F_v/\mathcal{O}_F,v)$ as a profinite group scheme over $\mathbb{Q}$.(6) The scheme $\Psi_v$ is a torsor for the action of $\mathcal{O}_F^\times$ (viewed as a constant profinite group scheme over $\mathbb{Q}$) by $a.\psi_v(x) := \psi_v(ax)$. We denote by $\psi_{\text{univ},v} : F_v \to \mathcal{O}(\Psi_v)^\times$ the universal additive character of level 0 of $F_v$.

Let $N_v \subset G_v$ be the upper-triangular unipotent subgroup, and extend any additive character $\psi_v$ of $F_v$ to a character of $N_v$, still denoted by $\psi_v$, by $u \mapsto \psi_v(u_12 + \ldots + u_{n-1,n})$. For a smooth admissible $\mathcal{O}_X[G_v]$-module $V$, define

$$(V \otimes \mathcal{O}(\Psi_v))^{(n)} := \text{Hom}_{N_v}(V, \mathcal{O}(\Psi_v)(\psi_{\text{univ},v})),$$

where $\mathcal{O}(\Psi_v)(\psi_{\text{univ},v})$ is the 1-dimensional $\mathcal{O}(\Psi_v)[N_v]$-module with action by $\psi_{\text{univ},v}$. One verifies that the right-hand side is an $\mathcal{O}_X \otimes \psi_v$-module on which the action of $\mathcal{O}_F^\times$, given by $c.f(y) = cf(c^{-1}y)$, is trivial. Hence $(V \otimes \mathcal{O}(\Psi_v))^{(n)}$ descends to an $\mathcal{O}_X$-module $V^{(n)}$. This defines, independently of choices, a functor “top derivative” (with respect to $G_v$)

$$V \mapsto V^{(n)}$$

descending to an $\mathcal{O}_X$-submodule such that $\mathfrak{J}(V)^{(n)} = V^{(n)}$.

(6) If $F_v = \mathbb{Q}_\ell$, then $\text{Hom}(F_v/\mathcal{O}_F,v) = T_\ell \mathbb{Q}_\ell$, the $\ell$-adic Tate module of roots of unity.
Lemma 4.1.1. — For a smooth admissible $\mathcal{O}_X[G_S]$-module $V$, the following are equivalent:
1. Every nonzero $\mathcal{O}_X[G_S]$-module quotient $W$ of $V$ is non-degenerate (i.e. it satisfies $W^{(n)} \neq 0$).
2. The space $\mathcal{O}(V)$ generates $V$ over $\mathcal{O}_X[G_S]$.
Moreover if $V$ is finitely generated over $\mathcal{O}_X[G_S]$, the set of $x \in X$ such that $V \otimes K(x)$ satisfies the above conditions is open.

Proof. — We may reduce to the case where $X$ is the spectrum of a local ring; then the equivalence is proved in [9, Lemma 6.3.2]. The set of $x \in X$ such that $V \otimes K(x)$ satisfies both conditions is open as it is the complement of the support of the coherent $\mathcal{O}_X$-module $\mathcal{O}$ defined as follows: let $U \subset G_S$ be an open compact subgroup such that $V_U$ generates $V$ over $\mathcal{O}_X[G_S]$. Then $\mathcal{O}$ is the sheaf of $U$-invariant vectors of the quotient of $V$ by the $\mathcal{O}_X[G_S]$-span of $3(V)$.

The following is [13, Definition 2.1] (after [11, Definition 6.1]).

Definition 4.1.2. — A smooth admissible $\mathcal{O}_X[G_S]$-module $V$ is said to be
- of Whittaker type if $V^{(n)}$ is a locally free $\mathcal{O}_X$-module of rank 1;
- strictly of Whittaker type if $V^{(n)} \cong \mathcal{O}_X$.
- (strictly) co-Whittaker if it is (strictly) of Whittaker type and it satisfies the conditions of Lemma 4.1.1 (that is, $\mathcal{O}(V)$ generates $V$ over $\mathcal{O}_X[G_S]$).

Lemma 4.1.3. — Let $V$ be a smooth admissible finitely generated $\mathcal{O}_X[G_S]$-module such that $(V \otimes K(x))^{(n)} \neq 0$ for all $x$ in a dense subset $\Sigma \subset X$. The set of $x' \in X$ such that $V \otimes K(x')$ is co-Whittaker is open.

Proof. — By the semicontinuity of the rank, the set $U_r \subset X$ of those $x$ such that $V^{(n)} \otimes K(x)$ has rank $\leq r$ is open for all $r$; by assumption, $U_0 = \emptyset$ and at all points of the open $U := U_1$, $V^{(n)}$ has fibre-rank 1 so that $(V^{(n)})^*_r$ is an invertible sheaf. The openness of the other two conditions in Definition 4.1.2 is Lemma 4.1.1.

Definition 4.1.4. — If $V$ is a finitely generated $\mathcal{O}_X[G_S]$-module of Whittaker type, its maximal co-Whittaker submodule $V'$ is the $\mathcal{O}_X[G_S]$-span of $3(V)$.

By the previous Lemma, a fibre $V_x$ is co-Whittaker if and only if $V_{x'} = V_x$, i.e. $x$ belongs to the open complement of the support of $V/V'$.

Lemma 4.1.5. — Let $V$ be a torsion-free co-Whittaker module over $X$. Then the natural map $\mathcal{O}_X \to \operatorname{End}_{\mathcal{O}_X[G_S]} V$ is an isomorphism.

Proof. — We may reduce to the case where $X$ is the spectrum of a local ring; this is a consequence of [9, Proposition 6.3.4 (3)].

Lemma 4.1.6. — Let $V_1$, $V_2$ be torsion-free co-Whittaker modules over $X$ and suppose that there is a dense subset $\Sigma \subset X$ such that
$$V_1 \otimes K(x) \cong V_2 \otimes K(x)$$
for all $x \in \Sigma$. Then $L := \operatorname{Hom}_{\mathcal{O}_X[G_S]}(V_2, V_1)$ is an invertible sheaf and
$$V_1 \cong V_2 \otimes L$$
as $\mathcal{O}_X[G_S]$-modules.

Proof. — We may reduce to the case where $X$ is affine, treated in [9, Lemma 6.3.7].

The previous result can be generalised.

Lemma 4.1.7. — Let $V_1$, $V_2$ be smooth admissible torsion-free $\mathcal{O}_X[G_S]$-modules over $X$ such that $V_2$ is co-Whittaker, $3(V_1)$ generates $V_1$ over $\mathcal{O}_X[G_S]$, and $V_1^{(n)}$ is locally free of rank $r$ (a locally constant function on $X$). Suppose that there is a dense subset $\Sigma \subset X$ such that
$$V_1 \otimes K(x) \cong V_2^{\otimes r(x)} \otimes K(x)$$
for all $x \in \Sigma$. Then $H := \text{Hom}_{\sigma[X]}(V_2, V_1)$ is a locally free $\sigma_X$-module of rank $r$, and the natural map

$$V_2 \otimes H \xrightarrow{\cong} V_1$$

is an isomorphism of $\sigma_X[G_S]$-modules.

**Proof.** — We may reduce to the case $X = \text{Spec} A$ is affine and connected (so that $r \in \mathbb{N}$) and the $V_i^{(n)}$ are both free over $A$. We have $\prod_{x \in \Sigma} V_i \otimes K(x) \cong \prod_{x \in \Sigma} V_i^{(n)} \otimes K(x)$ by the argument in the proof of [9, Lemma 6.3.7], we deduce $V_j \otimes \mathcal{H} \cong V_j^{(n)} \otimes \mathcal{H}$ by torsion-freeness, both $V_1$ and $V_2^{(n)}$ embed in this space. We deduce that $V_1^{(n)}$ and $V_2^{(n)}$ are both free $A$-submodules of $V_1^{(n)} \otimes \mathcal{H}$ of the same rank $r$, hence $V_1^{(n)} \cong \gamma V_2^{(n)}$ for some $\gamma \in \text{GL}_r(\mathcal{H})$, and $\mathcal{H}(V_1) \cong \mathcal{H}(\gamma V_2^{(n)})$; taking the $A[G]$-spans of both sides, $\gamma$ yields $V_1 \cong V_2^{(n)}$. As $\text{End}_{A[G]}(V_2) \cong A$ by Lemma 4.1.5, we may write $H \cong A^r$ and $V_i \cong V_2 \otimes H$.

4.2. The universal co-Whittaker module. — In this subsection, $F$ is a local field and $G = \text{GL}_n(F)$; we write $\Psi := \Psi_{\text{gen}}$. We explain the construction of a co-Whittaker module over the Bernstein variety $\mathcal{X}_n$ for $G$, which will be universal in some sense. We will use two constructions enjoying complementary rationality properties. The first one is taken from Bushnell–Henniart [3] and Helm [11]. Consider the $\mathcal{O}(\Psi)[G]$-module

$$\mathcal{W} := \text{c-Ind}_G^H \Psi_{\text{univ}}.$$

The action of the Bernstein centre allows us to consider it as a $\mathcal{F}_n \otimes \mathcal{O}(\Psi)[G]$-module.

The second construction proceeds “by hand” over (each connected component of) $\mathcal{X}_n$. First consider the component $\mathcal{X}_{n,1} \subset \mathcal{X}_n$ corresponding to the trivial supersingular support; we have $\mathcal{X}_{n,1} \cong G_m^n/S_n$. Cover $G_m^n$ with open sets $U_w$ for $w \in S_n$, such that $(\chi_1, \ldots, \chi_n)$ belongs to $U_w$ if and only if for all $i < j$, $\chi_{w i}$ does not follow $\chi_{w j}$. Then $\mathcal{W}_w := 1_F^{\mathcal{X}_n}(\chi_{w 1} \otimes \cdots \otimes \chi_{w n})$ define a co-Whittaker module over $U_w$, containing a unique line of spherical vectors of which we may choose a generator $V_w^G$ corresponding to the function in the induced representation taking the value $1$ at $g = 1$. For $w, w' \in S_n$, there is an intertwining map $T_{w, w'}: \mathcal{W}_w \to \mathcal{W}_{w'}$ over $U_w \cap U_{w'}$, which we may normalise so as to send $V_w^G$ to $V_{w'}^G$. The data $(\mathcal{W}_w, T_{w, w'})$ define a faithfully flat descent datum corresponding to a module $\mathcal{W}_{n,1}$ over $\mathcal{X}_{n,1}$.

Now let $(M, s)$ be any supersingular support and choose for it a representative of the form $\sigma = \otimes \sigma_i^{n_i}$ with the $\sigma_i$ pairwise inequivalent representations of $G_{d_i}$. If the set of unramified characters of $G_{d_i}$, stabilising $\sigma_i$, consists of those valued in $\mu_{d_i}$, we have $\mathcal{X}_s \cong \prod_i \mathcal{X}_{m_i,1}$ where $\mathcal{X}_{m_{i,1}} = \mathcal{X}_{m_i,1}/H_f$, with $H_f$, the image in $\mathcal{X}_{m_i,1}$ of $\mu_{d_i}^{\otimes} \subset G_{m_i}$. Then

$$\mathcal{W}_s := \bigotimes_i (\mathcal{W}_{m_{i,1}} \otimes \sigma_i)$$

defines a co-Whittaker module over $\prod_i \mathcal{X}_{m_{i,1}}$, which descends to $\mathcal{X}_s$ and is independent of the choice of representative $s \in S$. We let $\mathcal{W}'$ be the $\mathcal{O}(\mathcal{X}_n, G)$-module whose restriction to $\mathcal{X}_s$ is $\mathcal{W}_s$. One can verify point-by-point that $\mathcal{W}'$ is a co-Whittaker module. Let $L := \mathcal{W}'^{(n)}$, then the top derivative of $\mathcal{W}' \otimes L$ is free of rank one over $\mathcal{X}_n$. By Lemma 4.1.6 (applied to each connected component of $\mathcal{X}_n \otimes \sigma(\Psi)$), there is an isomorphism $\alpha: \mathcal{W}' \otimes L \cong \mathcal{W}_s^{(n)}$ over $\mathcal{X}_s$.

**Definition 4.2.1.** — The universal co-Whittaker module over $\mathcal{X}_n$ is the $\mathcal{O}(\mathcal{X}_n, G)$-module $\mathcal{W} = \mathcal{W}_s$ arising by descent from the data $(\mathcal{W}', \mathcal{W}^{(n)}, \alpha)$ described above.

4.3. Local Langlands correspondence in families. — We readopt the notation of §4.1.

**Theorem 4.3.1.** — Let $K$ be a field of characteristic zero and let $r'_S = (r_S, N_S)$ be an $n$-dimensional Weil–Deligne representation of $W_{F, S}$ over a reduced Noetherian $K$-scheme $X$. Then:

1. There exists a unique torsion-free strictly co-Whittaker $\mathcal{O}_X[G_S]$-module

$$\pi(r'_S)$$

such that for every irreducible component $X_\eta \subset X$ with generic point $\eta$, we have

$$\pi(r'_S) \otimes_{\mathcal{O}_X, \eta} K(\eta) \cong \otimes_{v \in S} \pi_v(r'_{\eta_v}),$$
where \( \pi_{\text{gen}}(r_{v,n}') \) is the generic Langlands correspondence (2.2.3).

2. Moreover for every \( x \in X \), there is a surjection

\[ \otimes_{v \in S} \pi_{\text{gen},v}(r_{v,n}') \to \pi(r_{S}') \otimes K(x), \]

which is an isomorphism if for all \( v \in S \), the \( \max_{\eta} N_{v,n} \) for the partial order (3.3.1) exists and it is \( \sim N_{v,x} \); here \( \eta \) runs through the minimal primes of \( \mathcal{O}_{X,x} \).

**Proof.** — Uniqueness follows from the Lemma 4.1.6. For existence, we reduce to the case where \( S = \{v\} \) consists of a single prime: if \( \pi(r_v') \) is as required by the statement with \( S = \{v\} \), then the maximal \( \mathcal{O}_X \)-torsion-free quotient of \( \otimes_{v \in S} \pi(r_v') \) is as required by the statement in the general case.

We thus fix \( S = \{v\} \) and omit the associated subscripts from the notation. Let \( \mathfrak{W} \) be the universal co-Whittaker module over \( \mathfrak{X}_n \) of Definition 4.2.1. By Theorem 4.3.3, the \( \mathcal{O}_X \)-representation \( r \) of \( W_F \) defines a map \( \alpha : X \to \mathfrak{X}_n \), with image in a union of connected components of \( \mathfrak{X}_n \); as those are affine, \( \alpha \) factors through \( \text{Spec} \mathcal{O}(X) \). Let \( \mathfrak{W}_X := \alpha^* \mathfrak{W} \) and let \( V \) be the image of the natural map

\[ \mathfrak{W}_X \to \prod_{\eta} \pi_{\text{gen}}(r_{\eta}'). \]

By [11, Lemma 6.4] and its proof, \( V \) satisfies the required property.

Concretely, the fibre of \( V \) at \( x \) is described as follows. Let \( \pi_{\text{gen}}(r_{\eta}') := \text{Ker} (\mathfrak{W}_X \to \pi_{\text{gen}}(r_{\eta}')) \). Then

\[ V_x \cong \mathfrak{W}_X \otimes \mathcal{O}_{X,x}/(p_x, (\pi_{\text{gen}}(r_{\eta}')_\eta)) \cong \mathfrak{W}_{\alpha(x)}/((\pi_{\text{gen}}(r_{\eta}')_\eta) \otimes K(x)_\eta), \]

where in the second term \( \eta \) ranges through the generic points of irreducible components of \( X \) and in the third term it ranges over the generic points of those irreducible components containing \( x \). Then the assertion of part 2 follows from this description.

**Definition 4.3.2.** — The association

\[ r_S' \mapsto \pi(r_S') \]

of Theorem 4.3.1 is called the local Langlands correspondence in families. When \( X = \text{Spec} A \) is the spectrum of a \( p \)-adic ring as defined above Lemma 2.1.1, and \( \rho_S \) is a continuous representation of \( \prod_{v \in S} \text{Gal}(\mathcal{F}_v/F_v) \) over \( X \), we define

\[ \pi(\rho_S) := \pi(r_S'), \]

where \( r_S' = (r_v') \) with \( r_v' = \text{WD}(\rho) \).

**Theorem 4.3.3.** — Let \( K \) be a field of characteristic zero and let \( X \) be a reduced Noetherian scheme over \( K \). Let \( r_S' = (r_S, N_S) \) be an \( n \)-dimensional Weil–Deligne representation of \( W_{F,S} \) over \( X \), and let \( V \) be a smooth admissible finitely generated torsion-free \( \mathcal{O}_X[G_S] \)-module such that \( V \otimes K(x) \cong \pi(r_{S,x}') \) for all \( x \) in a dense set \( \Sigma \) of points of \( X \). Then there exist an open subset \( U \subset X \) containing \( \Sigma \) and an invertible sheaf \( L \) over \( U \) such that

\[ V|_U \cong L^{-1} \otimes \pi(r_{S}')|_U \]

as \( \mathcal{O}_U[G] \)-modules. Explicitly, \( U \) can be taken to be the maximal open subset of \( X \) such that \( V|_U \) is co-Whittaker, and \( L = V|_{U}^{(n)} = \text{Hom} (V|_{U}, \pi(r_{S}')|_{U}) \).

**Proof.** — This follows from Lemma 4.1.6 using Lemma 4.1.3.

5. \( L \)-functions and zeta integrals in analytic families

In this section, \( F \) is a non-archimedean local field, \( G_n := \text{GL}_n(F) \), and \( X \) is a reduced Noetherian scheme over a field \( K \) of characteristic zero.

5.1. \( L \)-functions. — The following result is obvious.

**Proposition 5.1.1.** — Let \( (r, N) \) be a Weil–Deligne representation of \( W_F \) over \( X \). Assume that \( N \) is locally constant on \( X \). Then

\[ L^{-1}((r, N)) := \det(1 - \phi((\text{Ker} N)^{1r})) \in \mathcal{O}(X), \]
defines a function satisfying
\[
L^{-1}((r, N))(x) = L(0, (r_x, N_x))^{-1}
\]
for all \(x \in X\).

**Remark 5.1.2.** — By Proposition 3.3.1, any Weil–Deligne representation \(r' = (r, N)\) over \(X\) has locally constant monodromy over a dense open subset of \(X\) containing all pure specialisations.

If \((r, N)\) and \((r, N')\) are Weil–Deligne representations over \(X\) such that \(N \leq_{r'} N'\) (that is \(N_x \leq_{r_x} N'_x\) for all \(x \in X\)), and \(X' \subset X\) is a locally closed subset over which both \(N\) and \(N'\) are locally constant, then \(L^{-1}((r, N)|_{X'})\) divides \(L^{-1}((r, N)|_{X'})\) in \(\mathcal{O}(X')\).

Suppose that \(\nu = \pi_{r,0}^{-1}([s])\) is an inertial type and \(\mathfrak{X}_\nu = \mathfrak{X}_[s] \subset \mathfrak{X}_{n,\overline{Q}}\) is the corresponding component of the Bernstein variety, and let \(\mathfrak{X}_\nu^{\square}\) be the \(\mathfrak{X}_\nu\)-scheme carrying the universal \(W_F\)-representation \(r_\nu\) of type \(\nu\). Then \(L^{-1}((r_\nu, 0)) \in \mathcal{O}(\mathfrak{X}_\nu^{\square})\) descends to \(\mathfrak{X}_\nu\); by glueing we obtain a regular function over all of \(\mathfrak{X}_{n,\overline{Q}}\) which is Galois-equivalent, hence descends to a function
\[
L_{s_n}^{-1} \in \mathcal{O}(\mathfrak{X}_{n,\overline{Q}})
\]

Similarly, we define
\[
L^{-1} \in \mathcal{O}(\mathfrak{X}'_{n,\overline{Q}})
\]
by first descending \(L^{-1}((r, N)|_{[s],[t]}))\) via \(\mathfrak{X}_[s],[t] \rightarrow \mathfrak{X}_{[s],[t]}\) (with notation as in (3.3.4)), then glueing to \(\mathfrak{X}'_{n,\overline{Q}}\) and finally observing Galois-equivalence to descend to \(\mathfrak{X}'_{n,\overline{Q}}\).

In the situation of Proposition 5.2.3, suppose that for each \(i = 1, 2\), \(V_i = \pi((r_i, N_i))\) for some Weil–Deligne representations \((r_i, N_i)\) over \(X\) and that \(N_i\) is locally constant on \(X\). In this case, \(\pi((r, N))\) interpolates the generic Langlands correspondence \(\pi_{gen}\) at all \(x \in X\) by Theorem 4.3.1. By the compatibility of the local Langlands correspondence with \(L\)-functions, for each \(x \in \mathfrak{X}_{n,\mathbb{C}}\) corresponding to a \(W_F\)-representation \(r\), we have
\[
L^{-1}(x) = L(0, \pi_n((r, 0)))^{-1} = L((1 - n)/2, \pi((r, 0)))^{-1}.
\]

**Rankin–Selberg \(L\)-functions.** — Let \(n_1, n_2 \geq 1\). We aim to define Rankin–Selberg \(L\)-functions \(L_{s_{n_1}}^{-1}(\cdot, \mathcal{R}S)\) on \(\mathfrak{X}_{n_1} \times \mathfrak{X}_{n_2}\) and \(L^{-1}(\cdot, \mathcal{R}S)\) on \(\mathfrak{X}_n' \times \mathfrak{X}_n''\). We define the latter and the former is easier. After base-change to \(\overline{Q}\) we may restrict to components \(\mathfrak{X}_{[s_1],[t_1]} \times \mathfrak{X}_{[s_2],[t_2]}\) corresponding to inertial types \(\nu = \pi_{r,0}^{-1}([s])\) dimensions \(n_1, n_2\) and classes of multipartitions \([t_1],[t_2]\). Then we have a map \(\mathfrak{X}_1^{\square}_{[s_1],[t_1]} \times \mathfrak{X}_2^{\square}_{[s_2],[t_2]} \rightarrow \mathfrak{X}'_{n_1,n_2}\) given, via Theorem 3.3.2, by the representation \((r, N)|_{[s_1],[t_1]} \otimes (r, N)|_{[s_2],[t_2]}\) (with the notation of (3.3.4)). It is clear that this map factors through \(\mathfrak{X}_{[s_1],[t_1]} \times \mathfrak{X}_{[s_2],[t_2]}\) and that the map obtained by glueing to all of \(\mathfrak{X}'_{n_1,\overline{Q}} \times \mathfrak{X}_n''\overline{Q}\) is Galois-equivalent and hence descends to \(\mathfrak{X}'_{n_1,\mathbb{C}} \times \mathfrak{X}_n'\overline{Q}\to \mathfrak{X}'_{n_1,n_2}\).

We can now define
\[
L_{s_{n_1}}^{-1}(\cdot, \mathcal{R}S) = \mathcal{R}S^* L_{s_{n_1}}^{-1},
\]
\[
L^{-1}(\cdot, \mathcal{R}S') = \mathcal{R}S'^* L^{-1}.
\]

By construction, the latter interpolates the usual \(L\)-functions at all points of \(\mathfrak{X}'_{n_1,\mathbb{C}} \times \mathfrak{X}_n'\overline{Q}\).

For \(i = 1, 2\) let \(V_i\) be a finitely generated \(\mathcal{O}_X[G_{n_i}]\)-module of Whittaker type. Then \(V_i\) has a maximal co-Whittaker \(\mathcal{O}_X[G_{n_i}]\)-submodule \(V_i^{\mathcal{T}}\) (cf. Definition 4.1.4). By Theorem 3.2.1, there are maps \(\alpha_i\colon X \to \mathfrak{X}_{n_i}\) and surjections \(\alpha_i^* \mathfrak{M}_{n_i} \rightarrow (V_i^{\mathcal{T}})^{(n_i)\mathcal{O}_X} \otimes V_i^{\mathcal{T}}.\) Let
\[
L_{s_{n_1}}^{-1}((1 - n_1 n_2)/2, V_1 \times V_2) := (\alpha_1 \times \alpha_2^*)^* L_{s_{n_1}}^{-1}(\cdot, \mathcal{R}S).
\]

Let \(V_2(T)\) be the twist of \(V_2\) by \(\chi_T \circ \det\), where \(\chi_T\colon F^\times \rightarrow Q[T\pm 1]^\times\) is the unramified character \(x \mapsto T^{\nu(F(x))}\). Then we denote
\[
L_{s_{n_1}}^{-1}((1 - n_1 n_2)/2, V_1 \times V_2, T) := L_{s_{n_1}}^{-1}((1 - n_1 n_2)/2, V_1 \times V_2(T))
\]
and
\[
L_{s_{n_1}}^{-1}(m + (1 - n_1 n_2)/2, V_1 \times V_2) := L_{s_{n_1}}^{-1}((1 - n_1 n_2)/2, V_1 \times V_2, q^{-m})
\]
We apply the same procedure to obtain the following result.
Proposition 5.1.3. — Let \( V_i = \pi_i((r_i, N_i)) \) for some Weil–Deligne representations with locally constant monodromy over \( X \). Then for each \( m \in (1 - n_1 n_2)/2 + \mathbb{Z} \) there is an unique element

\[
L^{-1}(m, V_1 \times V_2) \in \mathcal{O}(X)
\]

whose value at any \( x \in X \) equals \( L(m, V_{1,x} \times V_{2,x})^{-1} \).

Proof. — Define first

\[
L^{-1}((1 - n_1 n_2)/2, V_1 \times V_2) := L^{-1}((r_1, N_1) \otimes (r_2, N_2)) = (a'_1 \times a'_2)^*L^{-1}(\cdot, \text{RS})
\]

where \( a'_i : X \to X'_{n_i} \) are the maps given by Theorem 3.3.2. Let

\[
L^{-1}((1 - n_1 n_2)/2, V_1 \times V_2, T) := L^{-1}((1 - n_1 n_2)/2, V_1 \times V_2(T)),
\]

then for any \( k \in \mathbb{Z} \) the desired element is

\[
L^{-1}(k + (1 - n_1 n_2)/2, V_1 \times V_2) := L^{-1}((1 - n_1 n_2)/2, V_1 \times V_2, q^{-k}).
\]

\[\Box\]

5.2. Zeta integrals. — We denote by \( N_n \subset G_n \) the upper-triangular unipotent subgroup, by \( A_n \subset G_n \) the diagonal torus, and by \( K_n \subset G_n \) the maximal compact subgroup \( \text{GL}_n(\mathcal{O}_F) \). The Iwasawa decomposition asserts \( G_n = N_n A_n K_n \). We fix the following choices of measures. On \( F \), we take the Haar measure \( dx \) assigning volume 1 to \( \mathcal{O}_F \) (then \( |d_F|^{1/2}dx \) is the self-dual Haar measure for additive characters of level zero, where \( d_F \in F \) is a generator of the different ideal). On \( F^\times \), we take the measure \( d^\times x = \zeta_F(1)\frac{d}{|d|} \). On \( G_n \), we take the measure \( dg = \zeta_F(1)\prod_{i > 1} \frac{dx_{ij}}{|dx_{ij}|} \) if \( g = (x_{ij}) \). On \( N_n \), we take the measure \( dn = \prod_{i,j} dx_{ij} \) if \( n = (x_{ij}) \). On \( A_n \), we take the measure \( da = \prod_{i} d^\times x_{ii} \) if \( a = (x_{ij}) \). On the quotient \( N_n \backslash G_n \) we take the quotient measure.

Let \( X \) be a Noetherian \( K \)-scheme and let \( V \) be an \( \mathcal{O}_X[G_n] \)-module of Whittaker type. Let \( \psi : F \to \mathcal{O}(\Psi)^\times \) be the universal additive character of level zero. (We remove subscripts from the notation of (4.1.1). The choice of the universal \( \psi \) allows for specialising our treatment to any additive character of \( F \) of level zero, and of course the restriction on the level could be removed.)

We denote by

\[
\mathcal{W}(V, \psi)
\]

the image of the natural map

\[
V(n) \otimes_{\mathcal{O}_X} V \otimes_K \mathcal{O}(\Psi) \to \text{Ind}_{N_n}^{G_n} \psi.
\]

If \( V' \subset V \) is the maximal co-Whittaker submodule of \( V \) (cf. Definition 4.1.4), then \( \mathcal{W}(V, \psi) = \mathcal{W}(V', \psi) \) and it is strictly co-Whittaker.

Definition of the integrals. — For \( i = 1, 2 \), let \( V_i \) be an \( \mathcal{O}_X[G_{n_i}] \)-module of Whittaker type as above, with \( n_2 \leq n_1 \). Let \( W_1 \in \mathcal{W}(V_1, \psi), W_2 \in \mathcal{W}(V_2, \psi^{-1}) \), and \( m \in (1 - n_1 n_2)/2 + \mathbb{Z} \), and when \( n_2 = n_1 = n \), let \( \Phi \in \mathcal{S}(F^\times, \mathcal{O}(X)) \) be a Schwartz function with values in \( \mathcal{O}(X) \). Denote by \( G_n := \{ v_F(\det g) = j \} \subset G_n \). Define, in the respective cases \( n_2 < n_1 \) and \( 0 \leq k \leq n_1 - n_2 - 1 \),

\[
I^j_k(W_1, W_2, m) := \int_{M_{n_2}(F)}\int_{N_{n_2}\backslash G_{n_2}} W_1 \left( \begin{pmatrix} g & x \\ I_k & I_{n_1 - n_2 - k} \end{pmatrix} \right) W_2(g)q_F^{-j(m - \frac{1 + n_2}{2})}dgdx,
\]

\[
I^j(W_1, W_2, \Phi, m) := \int_{N_n\backslash G_n} W_1(g)W_2(g)\Phi(e_n g)q_F^{-jm}dg,
\]

\[
I(j)(W_1, W_2, (\Phi), m, T) := \sum_j I^j_k(W_1, W_2, (\Phi), m)T^j,
\]

where we use the notation \( I(j)(W_1, W_2, (\Phi), m, T) \) with parenthetical \( \langle \Phi, (k) \rangle \) whenever we want to treat uniformly the two cases \( n_2 < n_1, n_2 = n_1 \). Our normalisation is such that when \( T = q^{-s} \) and for a suitable choice of \( X \), the integral \( I(j)(W_1, W_2, (\Phi), m, T) \) is the zeta integral denoted by \( I(s + m; W_1, W_2, (\Phi)) \) in [6, §3.2.1]. If moreover \( X = \text{Spec} \mathcal{C} \), it is known that \( I(j)(W_1, W_2, (\Phi), m, q^{-s}) \) converges in some right half-plane.

Lemma 5.2.1. — The series \( I(j)(W_1, W_2, (\Phi), m, T) \) belongs to \( \mathcal{A}[T][T^{-1}] \otimes \mathcal{O}(\Psi) \).
Proof. —  This is [19, Lemma 3.1] if \( n_2 < n_1 \). The case \( n_2 = n_1 \) is similar and proved as follows. By smoothness there exists a compact open \( K'_n \subset K_n \) such that \( W_i(gk) = W_i(g) \) for all \( g \in G_n, k \in K'_n \). Then, by the Iwasawa decomposition, each integral \( I(W_1, W_2, \Phi, m) \) equals a finite sum of integrals of the from

\[
(5.2.1) \quad \int_{A'_n} W'_i(a) W'_2(a) \Phi'(e_n a) \, da,
\]

where \( A'_n = A_n \cap G'_n \), and the \( W'_i \), \( \Phi' \) are translates of the \( W_i \) and \( \Phi \) by elements of \( K_n \). We show that for \( j \ll 0 \), these integrals vanish. We use coordinates \( a = \alpha(a_1, \ldots, a_n) := \text{diag}(a_1 \cdots a_n, \ldots, a_{n-1} a_n, a_n) \) on \( A_n \), with \( (a_1, \ldots, a_n) \in (F^\infty)^n \). By a standard argument (see e.g. [20, Lemma 3.2]), there is a constant \( C \), depending on the level of \( W'_i \), such that each \( W'_i(a) = 0 \) unless \( v_F(a_k) \geq -C \) for all \( 1 \leq j \leq n - 1 \). As \( \Phi \) is a Schwartz function, we have \( \Phi(e_n a) = \Phi((0, \ldots, 0, a_n)) = 0 \) unless \( v_F(a_n) \geq -C_n \) for some constant \( C_n \). With \( C' = \max\{C, C_n\} \), it follows that the integrand of \( (5.2.1) \) vanishes on \( A'_n \) whenever \( j < -\binom{n+1}{2} C' \).

Lemma 5.2.2. — Let \( n_2 \leq n_1 \) and let \( \mathcal{X}_{[s],[t]} \subset \mathcal{X}_{n_1,\mathbb{Q}} \) be connected components of the extended Bernstein varieties. Let \( A_i := \mathcal{O}(\mathcal{X}_{[s],[t]}^i) \) and \( A := A_1 \times A_2 \), let \( V_i = \pi((r, N)_{[s],[t]}^i) \), and let \( W_1 \in \mathcal{W}(V_1, \psi) \otimes A_2, W_2 \in A_1 \otimes \mathcal{W}(V_2, \psi^{-1}), m \in (1 - n_2)/2 + \mathbb{Z} \).

Then the Laurent series

\[
L^{-1}(m, V_1 \times V_2, T) \cdot I(k)(W_1, W_2, (\Phi, m, T))
\]

belongs to \( A[T] \otimes \mathcal{O}(\Psi) \). Its value at \( T = 1 \) is denoted by

\[
L^{-1}(m, V_1 \times V_2) \cdot I(k)(W_1, W_2, (\Phi, m, T)).
\]

Proof. — We may prove the result after base-change to \( \mathbb{C} \) (which we don’t signal in the notation). Moreover, if \( n_2 < n_1 \) we may reduce to the case \( k = 0 \) as in [16, §2.7]. Recall that Spec \( A_i \) is the quotient by a finite group of

\[
\text{Spec } \widetilde{A}_i := \mathbb{C}^{\sum \ell_{i,k}}/m, \mathbb{Q}
\]

if \( t_i = (t_{i,r}) \) with \( t_{i,r} \) a partition of length \( \ell_{i,r} \). Then we may also prove the result after pullback to Spec \( A_i \). Similarly to the construction of the universal co-Whittaker module preceding Definition 4.2.1, we may cover Spec \( A_i \) with open sets \( U_j \) such that over \( U_j \supset (\chi_1, \ldots, \chi_{1}, \chi_2, \ldots, \chi_{t}, \ldots, \chi_{s+1}) \) we have

\[
V_i|_{U_j} \cong I_j^i(\Delta(\sigma_1 \chi_{1,1}, t_{1,1}), \ldots, \Delta(\sigma_1 \chi_{1,t_1}, t_{1,t_1}), \ldots, \Delta(\sigma_s \chi_{s,1}, t_{s,1}), \ldots, \Delta(\sigma_s \chi_{s,t_s}, t_{s,t_s})).
\]

(Namely, \( U_j \) is the locus where the ordering of the segments occurring in the above representation satisfies the condition detailed above (2.2.1).) Then the desired result is proved by Cogdell and Piatetski-Shapiro in [6, paragraph after Proposition 4.1]. (In their notation, Spec \( \widetilde{A}_1 \), resp. Spec \( \widetilde{A}_2 \), is \( D_n \), resp. \( D_n \), where \( \pi \) and \( \sigma \) are specialisations of \( V_1 \) and \( V_2 \).

Proposition 5.2.3. — Let \( X \) be a Noetherian \( K \)-scheme, and for \( i = 1, 2 \), let \( V_i \) be an \( \mathcal{O}_X[G_n] \)-module of Whittaker type as above, with \( n_2 < n_1 \). Let \( W_1 \), resp. \( W_2 \), be sections of \( \mathcal{W}(V_1, \psi) \), resp. \( \mathcal{W}(V_2, \psi^{-1}) \); let \( m \in (1 - n_2)/2 + \mathbb{Z} \).

Then the Laurent series

\[
L^{-1}_{ss}(m, V_1 \times V_2, T) \cdot I(k)(W_1, W_2, (\Phi, m, T))
\]

is a section of \( \mathcal{O}_X[T] \otimes \mathcal{O}(\Psi) \). Its value at \( T = 1 \) is denoted by

\[
L^{-1}_{ss}(m, V_1 \times V_2) \cdot I(k)(W_1, W_2, (\Phi, m, T)).
\]

Example. — Suppose that \( X = \text{Spec } \mathbb{C}[q^{\pm s}] \) where \( s \) is an indeterminate, that \( V_0 \circ \pi((r_1, N_1)) \) are given irreducible generic representations of \( G_n \), over \( \mathbb{C} \), and that \( V_1 = V_0 \circ A, V_2 = V_0 \cdot |s|^a \), the twist of \( V_0 \) by the unramified character \( |s|^a : F^\times \to \mathbb{C}[q^{\pm s}]^* \) sending a uniformiser to \( q^{-s} \). Let \( W_i \) be Whittaker functions for \( V_0 \circ \pi \) and let \( W_1 = W_0, W_2 = W_0 \cdot |s|^a \). Let \( V_0^{ss} = \pi((r_1, 0)) \). Then \( I(W_1, W_2, m) \) is the zeta integral commonly denoted by \( I(W_0^n, W_0^{ss} \cdot s + m) \) and \( L^{-1}_{ss}(m, V_1 \times V_2) = L(s + m, V_1^{ss} \times V_2^{ss})^{-1} \), and with \( s' = s + n \) the proposition says that

\[
L(s', V_1^{ss} \times V_2^{ss})^{-1} \cdot I(W_1^{w_0}, W_2^{w_0}, m') \in \mathbb{C}[q^{\pm s'}].
\]
By the theory of Rankin–Selberg zeta integrals \cite{16} this statement (for all \( W_1^\circ, W_2^\circ \)) is equivalent to the assertion that \( L(s', V_1^{(s,s)} \times V_2^{(s,s)})^{-1} \) divides \( L(s', V_1^\circ \times V_2^\circ)^{-1} \) in \( C[q^{1,s}]. \)

**Proof.** — We may assume \( X = \Spec A \) is affine. After possibly replacing \( A \) with \( A \otimes_K \overline{K} \) and \( V_i \) with its maximal co-Whittaker submodule we may assume that \( K \) is algebraically closed and each \( V_i \) is co-Whittaker. We may also assume that \( \Spec A \) is connected and, by tensoring with an invertible \( A \)-module, that \( V_1^{(i,n)} \) is free. Let \( \alpha_i : \Spec A \to X_{n_i,K} \) be as described before the proposition, and suppose that \( \alpha_i \) has image in the base-change from \( \overline{Q} \) to \( K \) of a component \( X_{[a]} = \Spec A_i \). Let \( \mathfrak{M}_i \) be the restriction of \( \mathfrak{M}_{n_i} \) to \( X_{[a_i]} \), then there is a surjection \( \mathfrak{M}_i \otimes A \to V_i \) inducing an isomorphism \( \mathfrak{M}(\mathfrak{M}_i, \psi) \otimes_{A, \alpha_i} A \cong \mathfrak{M}(V_i, \psi) \). Thus we reduce to the case where \( A \) is a module over \( A_1 \otimes A_2 \) and \( V_i = \mathfrak{M}_i \otimes A_i, A \), and it is enough to treat the case \( A = A_1 \otimes A_2 \). This follows from the previous lemma.

**Proposition 5.2.4.** — In the situation of Proposition 5.2.3, suppose that \( V_i = \pi((r_i, N_i)) \) for a pair of families \((r_i, N_i)\) of Weil–Deligne representations over \( X \) with locally constant monodromy. Then the Laurent series

\[
L^{-1}(m, V_1 \times V_2, T) \cdot I(W_1, W_2, (\Phi), m, T)
\]

is a section of \( \mathcal{O}_X[T] \otimes \mathcal{O}(\Psi) \). Its value at \( T = 1 \) is denoted by

\[
L^{-1}(m, V_1 \times V_2) \cdot I(W_1, W_2, (\Phi), m).
\]

**Proof.** — Similarly to the previous proof, we may reduce to the case where \( K \) is algebraically closed and \( X = \Spec A \) is affine and connected. Then, by Theorem 3.3.2, for each \( i \) we have a map \( \alpha_i' : X \to X_{[a_i], \lbrack t_i \rbrack} \) such that \( V_i \cong \pi((r, N)_{[a_i] \times \lbrack t_i \rbrack}) \otimes \mathcal{O}(X_{[a_i] \times \lbrack t_i \rbrack}) \). Thus we again reduce to Lemma 5.2.2.

We give an application to local invariant inner products.

**Lemma 5.2.5.** — Let \( V \) be a smooth admissible irreducible and generic representation of \( G_n \) over \( \mathbb{C} \). Suppose that \( V \) is essentially unitarisable, that is a twist of \( V \) is identified with the space of smooth vectors in a unitary representation. Then there is a \( G_n \)-invariant bilinear pairing

\[
(\ , )_V : \mathfrak{M}(V, \psi) \otimes \mathfrak{M}(V^\vee, \psi^{-1}) \to \mathbb{C}
\]
given by the absolutely convergent integral

\[
\langle W, W^\vee \rangle_V = (\vol(K_n)L(1, V \times V^\vee))^{-1} \cdot \int_{\Gamma_n \\backslash G_{n,1}} W((\gamma_1))W^\vee((\gamma_1)) \, dg
\]

If \( W, W^\vee \) are unramified and normalised by \( W(1) = W^\vee(1) = 1 \), then \( \langle W, W^\vee \rangle = 1 \).

**Proof.** — We may reduce to the case where \( V \) is unitarisable, then \( V^\vee \cong \overline{\mathfrak{M}} \) and \( W \mapsto \overline{\mathfrak{M}} \) defines an isomorphism \( \mathfrak{M}(V^\vee, \psi^{-1}) \cong \overline{\mathfrak{M}}(V, \psi) \). Then \( (\ , )_V \) is identified with the pairing of \cite[§1]{15}. The second statement is proved in Proposition 2.3 *ibid.*

For a scheme \( X/{\mathbb{Q}} \), denote by \( \mathfrak{O}_X(j) \) the unramified 1-dimensional representation of \( W_F \) sending a geometric Frobenius to \( q_F^j \in \mathfrak{O}(X)^\times \), and for a Weil–Deligne representation \( r' \) denote by \( r'^* \) its dual. When \( X = \Spec \mathbb{C} \), by (2.2.2) we have \( \pi(r'^* (1 - n)) = \pi(r') \).

**Proposition 5.2.6.** — Let \( r' := (r, N) \) be an \( n \)-dimensional Weil–Deligne representation with locally constant monodromy over \( X \), let \( r'^*(1 - n) := \Hom_{\mathfrak{O}_X}(r', \mathfrak{O}_X(1 - n)) \). Let \( V := \pi(r') \), \( V^\vee := \pi(r'^* (1 - n)) \), so that for each \( x \in X \) the representation \( (V^\vee)_x \) is the contragredient of \( V_x \). Then there is a pairing

\[
(\ , ) : \mathfrak{M}(V, \psi) \otimes \mathfrak{O}_X \mathfrak{M}(V^\vee, \psi^{-1}) \to \mathfrak{O}_X
\]
such that for each complex geometric point \( x \in X(\mathbb{C}) \) such that \( V_x \) is essentially unitarisable and any \( W \in \mathfrak{M}(V, \psi) \), \( W^\vee \in \mathfrak{M}(V^\vee, \psi^{-1}) \), we have

\[
\langle W, W^\vee \rangle(x) = \langle W_x, W^\vee_x \rangle_{V_x}.
\]
\textbf{Proof.} — Let \( \Phi \in \mathcal{S}(F^n, \mathbb{Q}) \) be a Schwartz function. By [26, proof of Proposition 3.1] (after [15]), for each \( x \in X(\mathbb{C}) \) we have
\[
\hat{\Phi}(0) \cdot \int_{N_{a-1} \backslash G_{a-1}} W_x((q_{a})) W_x^\vee((q_{a})) \, dg = I(W_x, W_x, \Phi, 1).
\]
Then we may choose any \( \Phi \) such that \( \hat{\Phi}(0) = 1 \) and define
\[
\langle W, W^\vee \rangle := \text{vol}(K_n)^{-1} L^{-1}(1, V \times V^\vee) \cdot I(W, W^\vee, \Phi, 1),
\]
where the right-hand side is provided by Proposition 5.2.4. \( \square \)

\textbf{5.3. Local constants.} — In this final subsection we interpolate \( \gamma \)- and \( \varepsilon \)-factors.

Recall that the Deligne–Langlands \( \gamma \)-factor of a complex Weil–Deligne representation \( r' = (r, N) \) of \( W_{F'} \), with respect to a nontrivial \( \psi : F \to \mathbb{C}^\times \), is
\[
\gamma(r, \psi) = \varepsilon((r, N), \psi) \frac{L((r, N)^* (1))}{L(r, N)}
\]
Unlike \( L \)- and \( \varepsilon \)-factors, it is independent of the monodromy as the notation suggest; its interpolation will therefore be particularly simple.

If \( r \) a representation of \( W_F \), we will denote by \( L_{ss}(r) \) the inverse (possibly equal to \( \infty \)) of the function \( L^{-1}((r, 0)) \) defined in Proposition 5.1.1.

\textbf{Theorem 5.3.1.} — Let \( X \) be an object of \( \text{Nth}_X \) and let \( r \) be a representation of \( W_F \) on a locally free \( \mathcal{O}_X \)-module of rank \( n \). Then there is a meromorphic function
\[
\gamma(r) \in \mathcal{H}(X \times \Psi) \cup \{ \infty \}
\]
such that for all complex geometric points \( (x, \psi) \) of \( X \times \Psi \) we have
\[
\gamma(r)(x, \psi) = \gamma(r_x, \psi).
\]
The divisor of \( \gamma(r) \) is the product of \( \Psi \) and of
\begin{equation}
(\gamma(r))_X = (L_{ss}(r^*(1))/L_{ss}(r)).
\end{equation}
\textbf{Proof.} — By Theorem 3.2.1, it suffices to construct a function \( \gamma \) on the product \( \mathcal{X}_n \times \Psi \) such that \( \gamma(x, \psi) = \gamma(r_x, \psi) \) for any complex geometric point \( (x, \psi) \).

If \( n = 1 \), the explicit formulas for \( L(r) \) (Proposition 5.1.1) and
\[
\varepsilon(r, \psi) = \begin{cases} 
1 & \text{if } r \text{ is unramified} \\
\int_{\mathcal{O}_{F'} \setminus \mathcal{O} \times (r_{-1})^{-1}} \psi(t) \, dt & \text{if } r \text{ is ramified of conductor } f
\end{cases}
\]
clearly interpolate over (any connected component of) \( \mathcal{X}_1 \times \Psi \).

Assume now \( n \geq 2 \). We use the argument of Moss [20]. Let \( \mathfrak{M} \) be the universal co-Whittaker module over \( \mathcal{X}_n \times \Psi \) (Definition 4.2.1). We consider zeta integrals \( I_k(W, m) := I_k(W, 1, m) \) for the product of \( \mathfrak{M} \) (or its contragredient) and the trivial representation of \( G_1 \) over \( \mathcal{X}_n \times \Psi \). Define
\[
w_{n-1,1} := \begin{pmatrix} (-1)^n & 0 & \cdots & 0 \\
0 & 0 & \cdots & (-1)^{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 \end{pmatrix}
\]
and, for \( W \in \mathfrak{M} \), \( W^*(g) := W(g^{-1}) \), which is a Whittaker function on the contragredient. By the local functional equation of [16] and the compatibility of the Langlands correspondence with \( \gamma \)-factors, the function \( \gamma \) on \( \mathcal{X}_n \times \Psi \) is characterised, if it exists, by the property
\begin{equation}
I_{n-2}(w_{n-1,1} W \cdot 1) = \gamma \cdot I_0(W, 0)
\end{equation}
for some (equivalently for all) \( W \in \mathfrak{M} \).

Let \( W_0 \in \mathfrak{M} \) be such that \( I_0(W, 0) = 1 \) (e.g. we may take any \( W_0 \) whose restriction to \( G_1 \times \{I_{n-1}\} \) equals a suitable constant multiple of the characteristic function of a compact subgroup of \( F^\times \)). Then
\[
\gamma := I((w_{n-1,1} W_0)^t, 1)
\]
satisfies (5.3.2), hence it is the desired function. □

**Corollary 5.3.2.** — Let $X$ be an object of $\text{Noeth}_K$ and let $r' = (r, N)$ be a representation of $W_F'$ on a locally free $\mathcal{O}_X$-module $M$, with locally constant monodromy. Then there is a unique section

$$\varepsilon(r') \in \mathcal{O}(X \times \Psi)^{\times}$$

such that for all complex geometric points $(x, \psi)$ of $X \times \Psi$ we have

$$\varepsilon(r')(x, \psi) = \varepsilon(r'_x, \psi).$$

For a related result proved by a different method, see [4].

**Proof.** — By Theorem 3.3.2 it suffices to consider the case of the Weil–Deligne representation $r' = (r, N) = (r, N)_{\mathfrak{s},[\mathfrak{t}]}$ over $X^{\square}_{\mathfrak{s},[\mathfrak{t}]}$ of (3.3.4), and show that the resulting function $\varepsilon(r')$ descends to $X_{\mathfrak{s},[\mathfrak{t}]}$. By Theorem 5.3.1 and Proposition (5.1.1), we may define

$$\varepsilon(r') := \gamma(r) \frac{L(r')}{L(r^s(1))} = \gamma(r) \frac{L^{-1}(r^s(1))}{L^{-1}(r')}.$$  

That $\varepsilon(r')$ has neither zeros nor poles is, by Theorem 5.3.1, equivalent to the same statement for the ratio

$$(5.3.3) \quad \frac{L(r')}{L(r^s(1))} \cdot \frac{L(r^s(1))}{L(r)}.$$  

We assert that in fact $(5.3.3) = \text{det}(-\phi)^{r'} / \text{Ker}(N)^{r'}$, which is well-defined as $\text{Ker}(N)$ is locally free. The assertion can be checked on geometric points and reduced to the case of a Speh representation $\text{Sp}(r, m)$, which is easy to verify. □

Special cases or variants of the following corollary were proved in [21] and [23].

**Corollary 5.3.3.** — Let $X$ be a connected noetherian scheme over $K$ and let $r'$ be a Weil–Deligne representation on a locally free sheaf over $X$. Let $X^{\text{pure}} \subset |X|$ be the set of those $x$ such that the monodromy filtration on $r^s_x$ is pure.

Suppose that there is an isomorphism $r' \cong r^s(1)$. Then the function

$$\varepsilon : X^{\text{pure}} \to \{\pm 1\}$$

$$x \mapsto \varepsilon(r'_x) := \varepsilon(r'_x, \psi) \quad (\text{for any } \psi : F \to K(x)^{\times})$$

is constant.

**Proof.** — This follows from Corollary 5.3.2 and Proposition 3.3.1. □

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