THE PENALTY FREE NITSCHE METHOD AND
NONCONFORMING FINITE ELEMENTS FOR THE SIGNORINI
PROBLEM
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Abstract. We design and analyse a Nitsche method for contact problems. Compared to the seminal work of Chouly and Hild [10] (A Nitsche-based method for unilateral contact problems: numerical analysis. SIAM J. Numer. Anal. 51 (2013), no. 2) our method is constructed by expressing the contact conditions in a nonlinear function for the displacement variable instead of the lateral forces. The contact condition is then imposed using the nonsymmetric variant of Nitsche’s method that does not require a penalty term for stability. Nonconforming piecewise affine elements are considered for the bulk discretization. We prove optimal error estimates in the energy norm.

1. Introduction. We consider the Signorini problem, find $u$ such that

$$
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_D \\
\partial_n u &= 0 \quad \text{on } \Gamma_N \\
u &\leq 0, \quad \partial_n u \leq 0, \quad u \partial_n u = 0 \quad \text{on } \Gamma_C,
\end{align*}
$$

where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a convex polygonal (polyhedral) domain with boundary $\partial \Omega$ and $\Gamma_D \cup \Gamma_N \cup \Gamma_C = \partial \Omega$. We assume that $\Gamma_C$ coincides with one of the sides of the polygon. We write $\partial_n u := n \cdot \nabla u$, where $n$ denotes the outwards pointing normal of $\partial \Omega$.

It is well known that this problem admits a unique solution $u \in H^1(\Omega)$. This follows from the theory of Stampacchia applied to the corresponding variational inequality (see for instance [16]). We will also assume the additional regularity $u \in H^{\frac{3}{2}+\nu}(\Omega)$, $0 < \nu \leq \frac{1}{2}$. There exists a large body of litterature treating finite element methods for contact problems. In general however, it has proven difficult to prove optimal error estimates without making assumptions on the regularity of the exact solution or the contact zone. In the pioneering work of Scarpini and Vivaldi [25] $O(h^{\frac{3}{4}})$ convergence was proved in the energy norm for solutions in $H^2(\Omega)$. Brezzi, Hager and Raviart [8] then proved $O(h)$ convergence under the additional condition that the solution was in $W^{1,\infty}(\Omega)$ or that the number of points where the contact condition changes from binding to non-binding is finite. These initial works were followed by a series of papers where the scope was widened and sharper estimates obtained [23, 15, 5, 4, 28, 27, 11]. Discretization of (1.1) is usually performed on the variational inequality or using a penalty method. The first case however leads to some nontrivial choices in the construction of the discretization spaces in order to satisfy the nonpenetration condition and it has proved difficult to obtain optimal error estimates [19]. The latter case, leads to the usual consistency and conditioning issues of penalty methods. A detailed analysis the penalty method was recently performed by Chouly and Hild [11]. Another approach proposed by Hild and Renard [18] is to use a stabilized Lagrange-multiplier in the spirit of Barbosa and Hughes [3], using the

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reformulation of the contact condition
\[ \partial_n u = -\gamma^{-1}|u - \gamma \partial_n u|_+ \] (1.2)
where \([x]_\pm = \pm \max(0, \pm x)\), proposed by Alart and Curnier \[2\] in an augmented Lagrangian framework. Using the close relationship between the Barbosa and Hughes method and Nitsche’s method \[24\] as discussed by Stenberg \[26\], this method was then further developed in the elegant Nitsche type formulation introduced by Chouly, Hild and Renard \[10\] \[12\]. In these works optimal error estimates for solutions in \(H^{\frac{1}{2}+\nu}(\Omega)\) to the above model problem were obtained for the first time. Their method was proposed in a nonsymmetric and a symmetric version similar to Nitsche’s method for the imposition of boundary conditions; it has however been observed that in their framework, there was no equivalent to the penalty–free non–symmetric Nitsche method proposed in \[9\]. Our aim in this work is to fill this gap, rather adding a piece to the puzzle than pretending to propose a method superior to the previous variants.

The penalty free Nitsche method can be interpreted as a Lagrange multiplier method where the multiplier and the corresponding test function has been replaced by the normal flux of the solution variable and of its test function, respectively. To design this method for contact problems we take a slightly different approach than in \[10\]. Instead of working on the formulation (1.2) for the lateral forces we use a similar relation on the displacement:
\[ u = -\gamma[\partial_n u - \gamma^{-1}u]_+ \] (1.3)
Setting \(P_\gamma(u) = \gamma \partial_n u - u\) we may write this relation as
\[ u = -[P_\gamma(u)]_+ \] (1.4)
It is straightforward to show that this is equivalent to the contact condition of equation (1.1). First assume that (1.3) holds. Then by construction \(u \leq 0\). For \(u = 0\) we see that \([\partial_n u]_+ = 0\) so in this case \(\partial_n u \leq 0\). On the other hand if \(u \neq 0\) and \(\partial_n u > 0\) then \(u = -\gamma(\partial_n u - \gamma^{-1}u) < u\), which is a contradiction. Similarly if \(\partial_n u < 0\) and \(u \neq 0\) then \(u = -\gamma(\partial_n u - \gamma^{-1}u) > u\). On the other hand if \(u \leq 0\) and \(u\partial_n u = 0\) then (1.3) holds and similarly if \(\partial_n u \leq 0\) and \(u\partial_n u = 0\) then (1.3) holds.

We multiply (1.1) by a function \(v\) with zero trace on \(\Gamma_D\) and apply Green’s formula to obtain
\[ a(u, v) - \langle \partial_n u, v \rangle_{\Gamma_C} = (f, v)_\Omega, \]
where \(\langle , \rangle_\Omega\) and \(\langle , \rangle_{\Gamma_C}\) denote the \(L^2\)-scalar product on \(\Omega\) and \(\Gamma_C\) respectively and \(a(u, v) := (\nabla u, \nabla v)_\Omega\). We then add a term imposing (1.2) on the following form
\[ \langle u + \gamma[\partial_n u - \gamma^{-1}u]_+, \theta_1 \partial_n v + \theta_2 \gamma^{-1}v \rangle_{\Gamma_C}, \] (1.5)
resulting in family of Nitsche formulations defined by two parameters \(\theta_1\) and \(\theta_2\),
\[ a(u, v) - \langle \partial_n u, v \rangle_{\Gamma_C} + \theta_1 \langle \partial_n v, u \rangle_{\Gamma_C} + \theta_2 \gamma^{-1} \langle u, v \rangle_{\Gamma_C} \]
\[ + \langle \gamma[\partial_n u - \gamma^{-1}u]_+, \theta_1 \partial_n v + \theta_2 \gamma^{-1}v \rangle_{\Gamma_C} = (f, v)_\Omega. \]
Taking \(\theta_1 \in \{-1, 0, 1\}\) and \(\theta_2 = 1\) results in methods equivalent to those proposed in \[12\] on the form
\[ \langle \nabla u, \nabla v \rangle_\Omega - \langle \partial_n u, v \rangle_{\Gamma_C} \pm \langle u, \partial_n v \rangle_{\Gamma_C} + \langle \gamma^{-1}u, v \rangle_{\Gamma_C} \]
\[ + \langle [P_\gamma(u)]_+, \gamma^{-1}v \pm \partial_n v \rangle_{\Gamma_C} = (f, v)_\Omega \] (1.6)
from which we deduce that the linear part of the formulation coincides with the classical version of Nitsche’s method. It is straightforward to verify that (1.6) is equivalent with the formulation proposed in [12].

Herein we will consider the method obtained when \( \theta_1 = 1 \) and \( \theta_2 = 0 \) in which case the term imposing the contact condition reduces to

\[
\langle u + \gamma [\partial_n u - \gamma^{-1} u]_+, \partial_n v \rangle_{\Gamma_C}.
\]

Observe that the two terms only differ by the exclusion of the last term which corresponds to a penalty and in that sense the latter variant is penalty free.

It follows that the penalty free version leads to the following formal restatement of (1.1) for smooth \( u \)

\[
(\nabla u, \nabla v)_\Omega - \langle \partial_n u, v \rangle_{\Gamma_C} + \langle (P_\gamma(u))_+, \partial_n v \rangle_{\Gamma_C} = (f, v)_\Omega.
\] (1.7)

Observe that the linear part of the system is equivalent to that proposed in [9] for Dirichlet boundary conditions, but that here this is used to enforce the condition (1.4) on \( u \).

For the discretization of (1.7) we will use the Crouzeix–Raviart nonconforming piecewise affine element with midpoint continuity on element edges (or continuity of averages over faces in three dimensions). As we shall see below, this element is advantageous for the formulation proposed, since the necessary stability results are relatively straightforward to prove. The nonconforming finite element space has been analysed for the Signorini problem by Hua and Wang [21]. They prove optimal convergence up to a logarithmic factor for \( H^2(\Omega) \) solutions under the assumption that the number of points where the constraint changes from binding to nonbinding is finite. In this work we prove the same optimal results for solutions in \( H^{3/2+\nu}(\Omega) \), \( \nu > 0 \) as those obtained in [10, 12].

To handle the nonconformity error we need to make an additional mild assumption on the source term: the trace of \( f \) must be well defined in the vicinity of the contact boundary \( \Gamma_C \). To make this precise, let

\[
\Omega_{t_C} := \{ x \in \bar{\Omega} : x = y - n_y t, \text{ where } y \in \Gamma_C \text{ and } 0 \leq t \leq t_C \},
\]

where \( n_y \) denotes the outward pointing normal on \( \Gamma_C \) at the point \( y \). For a fixed \( t \), we define

\[
\partial_t \Omega := \{ x \in \bar{\Omega} : x = y - n_y t, \text{ where } y \in \Gamma_C \}.
\]

Observe that for any function \( v \in H^s(\Omega_{t_C}) \) with \( s > \frac{1}{2} \) there holds

\[
\|v\|_{\Omega_{t_C}} \lesssim t_C^{\frac{1}{2}} \sup_{0 \leq t \leq t_C} \|v\|_{\partial_t \Omega}.
\] (1.8)

We introduce the norm \( \|u\|_{L^2_\infty(\Omega)} := \|u\|_{L^2(\Omega)} + \sup_{0 \leq t \leq t_C} \|v\|_{\partial_t \Omega} \) and assume that

\[
\exists t_C > 0 \text{ such that } \|f\|_{L^2_\infty(\Omega)} < \infty.
\] (1.9)

2. The nonconforming finite element method. To simplify the analysis below we will work with the nonconforming finite element space proposed by Crouzeix and Raviart in [19]. Let \( \{ T_h \} \) denote a family of shape regular and quasi uniform tessellations of \( \Omega \) into nonoverlapping simplices, such that for any two different simplices \( \kappa, \kappa' \in T_h \), \( \kappa \cap \kappa' \) consists of either the empty set, a common face or edge, or
a common vertex. The diameter of a simplex $\kappa$ will be denoted $h_\kappa$ and the outward pointing normal $n_\kappa$. The family $\{\mathcal{T}_h\}_h$ is indexed by the maximum element size of $\mathcal{T}_h$, $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$. We denote the set of element faces in $\mathcal{T}_h$ by $\mathcal{F}$ and let $\mathcal{F}_i$ denote the set of interior faces and $\mathcal{F}_\Gamma$ the set of faces in some $\Gamma \subset \partial \Omega$. We will assume that the mesh is fitted to the subsets of $\partial \Omega$ representing the boundary conditions $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, so that the boundaries of these subsets coincide with the boundaries of subsets of element faces. To each face $F$ we associate a unit normal vector, $n_F$. For interior faces its orientation is arbitrary, but fixed. On the boundary $\partial \Omega$ we identify $n_F$ with the outward pointing normal of $\partial \Omega$. The subscript on the normal is dropped in cases where it follows from the context.

We define the jump over interior faces $F \in \mathcal{F}_i$ by

$$\|v\|_F := \lim_{\epsilon \to 0^+} (v(x|_F - \epsilon n_F) - v(x|_F + \epsilon n_F))$$

and for faces on the boundary, $F \in \partial \Omega$, we let $\|v\|_F := v|_F$. Similarly we define the average of a function over an interior face $F$ by

$$\{v\}_F := \frac{1}{2} \lim_{\epsilon \to 0^+} (v(x|_F - \epsilon n_F) + v(x|_F + \epsilon n_F))$$

and for $F$ on the boundary we define $\{v\}_F := v|_F$. The classical nonconforming space of piecewise affine finite element functions (see [13]) then reads

$$V_h := \{ v_h \in L^2(\Omega) : \int_F \|v_h\| \ ds = 0, \ \forall F \in \mathcal{F}_i \cup \mathcal{F}_\Gamma \text{ and } v_h|_\kappa \in P_1(\kappa), \ \forall \kappa \in \mathcal{T}_h \}$$

where $P_1(\kappa)$ denotes the set of polynomials of degree less than or equal to one restricted to the element $\kappa$.

The finite element method takes the form: find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = L(v_h), \ \forall v_h \in V_h \quad (2.1)$$

where $L(v_h) := (f, v_h)_\Omega$ and

$$A_h(u_h, v_h) := a_h(u_h, v_h) + \langle u_h + [P_1(u_h)]_+, \partial_n v_h \rangle_{\Gamma_C} \quad (2.2)$$

with $P_1(u_h) = \gamma \partial_n u_h - u_h$ and $\gamma > 0$ a parameter to determine. The linear form $a_h(\cdot, \cdot)$ coincides with the consistent part of Nitsche’s method,

$$a_h(u_h, v_h) := a(u_h, v_h) - \langle \partial_n u_h, v_h \rangle_{\Gamma_C}$$

where we have redefined $a(u, v) := \sum_{\kappa \in \mathcal{T}_h} \langle \nabla u_h, \nabla v_h \rangle_K$. To see the effect of the nonlinear term, let $\Gamma_C^+ \subset \Gamma_C^-$ denote the part of the contact zone where $\gamma[\partial_n u - \gamma^{-1} u]_+ > 0$ and $\Gamma_C^0 = \Gamma_C \setminus \Gamma_C^+$. We may then write the form $A(\cdot, \cdot)$

$$a(u_h, v_h) - \langle \partial_n u, v_h \rangle_{\Gamma_C^+} + \langle \partial_n v, u_h \rangle_{\Gamma_C^0} + \langle \gamma \partial_n u, \partial_n v \rangle_{\Gamma_C^+}.$$ 

This corroborates the naive idea that the method should impose a Dirichlet condition on $\Gamma_C^0$, here using the penalty free Nitsche method, and a Neumann condition on $\Gamma_C^+$, here in the form of a penalty term. Observe that the continuity of the form that is obvious in the formulation (2.2) (by the continuity of $[\cdot]_+$, see more details below) is no longer clear in this latter expression.
For comparison, in the method of Chouly, Hild and Renard, the form \( A(\cdot, \cdot) \) takes the form

\[
a(u_h, v_h) - \langle \partial_n u, v_h \rangle_{\Gamma_C^0} + \theta_1 \langle \partial_v u, u_h \rangle_{\Gamma_C^0} + \langle u_h, \gamma^{-1} v_h \rangle_{\Gamma_C^0} + \theta_1 \langle \gamma \partial_n u, \partial_n v \rangle_{\Gamma_C^0},
\]

where \( \theta \) takes the values \(-1\) or \(1\) for the symmetric and nonsymmetric versions respectively. Clearly in this case the Dirichlet condition on \( \Gamma_0^C \) is imposed using the classical Nitsche method and the Neumann condition on \( \Gamma_+^C \) is imposed either weakly or with an additional penalty term (in the symmetric case, this term has the wrong sign and does not stabilize the boundary condition).

2.1. Preliminary results. For the analysis below we will use some elementary tools that we collect here. We will use the notation \( a \lesssim b \) for \( a \leq C b \) where \( C \) is a constant independent of \( h \).

The following norms on \( H^{\frac{1}{2}+\nu}(\Omega) + V_h \) will be used below simplify to simplify the notation,

\[
\| v \|_{h, \Omega} := \left( \sum_{\kappa \in T_h} \| v \|_{\kappa}^2 \right)^{\frac{1}{2}}, \quad \| v \|_{h, \Gamma^C} := \left( \sum_{F \in F_{\Gamma^C}} \| v \|_F^2 \right)^{\frac{1}{2}},
\]

the broken \( H^1 \)-norms,

\[
\| v \|_{1, h} := \| \nabla v \|_{h, \Omega} + \| v \|_{h, \Omega}
\]

and

\[
\| v \|_{1, C} := \| v \|_{1, h} + \gamma^\frac{1}{2} \| \partial_n v \|_{h, \Gamma^C} + \gamma^{-\frac{1}{2}} \| v \|_{h, \Gamma^C}.
\]

We recall, for future reference, the following inequalities:

- **Poincaré inequality**, there exists \( \alpha > 0 \) such that
  \[
  \alpha \| v \|_{1, h}^2 \leq \| \nabla v \|_h^2 \quad \forall v \in V_h + H^1(\Omega). \quad (2.3)
  \]

- **Inverse inequality,**
  \[
  \| v \|_{H^1(\kappa)} \leq C_I h_{\kappa}^{-1} \| v \|_{L^2(\kappa)} \quad \forall v \in P_1(\kappa). \quad (2.4)
  \]

- **Trace inequalities,**
  \[
  \| v \|_{L^2(\partial \kappa)} \leq C_T \left( h_{\kappa}^{-\frac{1}{2}} \| v \|_{L^2(\kappa)} + h_{\kappa}^{\frac{1}{2}} \| v \|_{H^1(\kappa)} \right) \quad \forall v \in H^1(\kappa) \quad (2.5)
  \]
  and
  \[
  \| v \|_{L^2(\partial \kappa)} \leq C_I h_{\kappa}^{-\frac{1}{2}} \| v \|_{L^2(\kappa)} \quad \forall v \in P_1(\kappa). \quad (2.6)
  \]

For the analysis below we also need a quasi-interpolation operator that maps piecewise linear nonconforming functions into the space of piecewise linear conforming functions. Let \( I_{cl} : V_h \mapsto V_h \cap H^1(\Omega) \) denote a quasi interpolation operator \([20, 21, 22]\) such that

\[
\| I_{cl} v_h - v_h \|_{\Omega} + h \| \nabla (I_{cl} v_h - v_h) \|_{h} \lesssim \| h^{\frac{1}{2}} \| v_h \|_{\Omega} \lesssim h \| \nabla v_h \|_{h}. \quad (2.7)
\]
Stability is based on the fact that we can construct a function which is zero in the bulk of the domain and with a certain value of the flux on the boundary. We make this precise in the following lemma.

**Lemma 2.1.** Let \( r : \Gamma_C \mapsto \mathbb{R} \) be a face-wise constant function such that \( r|_F \in \mathbb{R} \) for all \( F \in \mathcal{F}_C \). There exists \( v_h \in V_h \) such that

\[
\partial_n v_h|_F = r|_F \in \mathbb{R} \quad \text{for } F \in \mathcal{F}_C,
\]

\[
\int_F \{v_h\} \, ds = 0 \quad \text{for } F \in \mathcal{F}_i \cup \mathcal{F}_D \cup \mathcal{F}_N.
\]

and

\[
\|v_h\|_\Omega \lesssim h^{\frac{3}{2}} \|r\|_{\Gamma_C}.
\]

**Proof.** For a given simplex \( \kappa \) with one face in \( \mathcal{F}_C \), assume that \( x_1, \ldots, x_d \) are the vertices in \( \Gamma_C \) and \( x_0 \) is the vertex in the bulk. Define \( v_\kappa \in P_1(\kappa) \) by \( v_\kappa(x_i) = 1 \), \( i = 1, \ldots, d \) and \( v_h(x_0) = 1 - d \). Then it follows that for \( F \subset \partial \kappa \cap \Omega \)

\[
\int_F v_\kappa \, dx = 0
\]

and \( \nabla v_\kappa := |\nabla v_h|_{n_{\partial \Omega}} \), where \( n_{\partial \Omega} \) is the normal to \( \Omega \) on \( \partial \kappa \cap \partial \Omega \) and \( |\nabla v_h| = c_\kappa h_\kappa^{-1} \) where \( c_\kappa \) is a positive constant that depends only on the shape regularity of \( \kappa \). It follows that

\[
v_h := \sum_{\kappa \in T_h} v_\kappa \in V_h.
\]

We conclude by multiplying \( v_h \) in each element with \( h_\kappa c_\kappa^{-1} r_F \). Then by construction (2.8) and (2.9) are satisfied. The stability (2.10) is a consequence of an inverse trace inequality,

\[
\|v_h\|_\Omega \lesssim \left( \sum_{F \in \mathcal{F}_C} h_\kappa \|h_\kappa c_\kappa^{-1} r_F\|^2_F \right)^{\frac{1}{2}} \lesssim h^{\frac{3}{2}} \|r\|_{\Gamma_C}.
\]

The nonlinearity satisfies the following monotonicity and continuity properties.

**Lemma 2.2.** Let \( a, b \in \mathbb{R} \) then there holds

\[
([a]_+ - [b]_+)^2 \leq ([a]_+ - [b]_+)(a - b),
\]

\[
|[a]_+ - [b]_+| \leq |a - b|.
\]

**Proof.** Developing the left hand side of the expression we have

\[
[a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \leq [a]_+ a + [b]_+ b - a[b]_+ - [a]_+ b = ([a]_+ - [b]_+)(a - b).
\]
The second claim is trivially true in case both \( a \) and \( b \) are positive or negative. If \( a \) is negative and \( b \) positive then
\[
|[a]_+ - [b]_+| = |b| - |a|
\]
and similarly if \( b \) is negative and \( a \) positive
\[
|[a]_+ - [b]_+| = |a| - |b|.
\]

\[\square\]

**Lemma 2.3. (Continuity of \( A_h \))** Let \( v_1, v_2 \in H^{\frac{1}{2}+\nu} + V_h \) and \( w_h \in V_h \). Then there holds
\[
|A_h(v_1, w_h) - A_h(v_2, w_h)| \leq \|v_1 - v_2\|_{1,C}||w_h||_{1,C} \lesssim \Theta(h)^2 \|v_1 - v_2\|_{\Omega}||w_h||_{\Omega}.
\]

**Proof.** By the Cauchy–Schwarz inequality we have
\[
A_h(v_1 - v_2, w_h) \leq \|v_1 - v_2\|_{1,C}||w_h||_{1,C}.
\]

For the nonlinear term the following bound holds as a consequence of the third inequality of Lemma 2.2 and the inequalities (2.4)–(2.6):
\[
\langle \gamma(\partial_n v_1 - \gamma^{-1}v_1)_+ - [\partial_n v_2 - \gamma^{-1}v_2]_+, \partial_n w_h \rangle_{G,C} \\
\leq \langle (\gamma \frac{1}{2} \partial_n v_1 - \gamma^{-1/2}(v_1 - v_2)_+ \gamma \frac{1}{2} \partial_n w_h \rangle_{G,C} \\
\lesssim \|v_1 - v_2\|_{1,C}||w_h||_{1,C} \\
\lesssim \Theta(h)^2 \|u_1 - u_2\|_{\Omega}||w_h||_{\Omega}
\]

with \( \Theta(h) := 1 + h^{-1}(C_I + C_I \gamma^{-\frac{1}{2}} h^{-\frac{1}{2}} + C_I \gamma^{-\frac{1}{2}} h^{-\frac{1}{2}}) \). \[\square\]

### 3. Existence and uniqueness of discrete solutions

In this section we will prove that the finite dimensional nonlinear system (2.1) admits a unique solution under suitable assumptions on the parameter \( \gamma \). First, with \( N_V := \dim V_h \) define the mapping \( G : \mathbb{R}^{N_V} \rightarrow \mathbb{R}^{N_V} \) by
\[
(G(U), V)_{\mathbb{R}^{N_V}} := A_h(u_h, v_h) - L(v_h), \tag{3.1}
\]
where \( U = \{ u_i \} \), with \( u_i \) denoting the degrees of freedom of \( V_h \) associated with the Crouzeix-Raviart basis functions \( \{ \varphi_i \}^{N_V}_{i=1} \) and similarly \( V = \{ v_i \} \) denotes the vector of degrees of freedom associated with the test function \( v_h \). The nonlinear system associated to (2.1) may then be written, find \( U \in \mathbb{R}^{N_V} \) such that \( G(U) = 0 \).

Let us next prove a positivity result for the formulation (2.1) that will be useful when proving existence and uniqueness.

**Proposition 3.1.** Assume that \( \gamma = \gamma_0 \) with \( \gamma_0 \) large enough. Then, for \( u_1, u_2 \in V_h \), there exists \( v_h \in V_h \) such that
\[
\alpha \|u_1 - u_2\|_{1,h}^2 + \gamma^{-1} \|u_1 - u_2 + [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+\|_{G,C} \]
\[
\lesssim A_h(u_1, v_h) - A_h(u_2, v_h). \tag{3.2}
\]
Moreover, for $\gamma_0$ large enough, there exists $B \in \mathbb{R}^{N \times N}$ such that for $X$ with $|X|_{\mathbb{R}^{N \times N}}$ large enough

$$(G(X), BX)_{\mathbb{R}^{N\times N}} > 0$$

and there exists $b_1, b_2 > 0$ associated to $B$ such that

$$b_1|X|_{\mathbb{R}^{N\times N}} \leq |BX|_{\mathbb{R}^{N\times N}} \leq b_2|X|_{\mathbb{R}^{N\times N}}.$$

Proof. Let $w_h := u_1 - u_2$. Observe that by Lemma 2.1 we can choose $x_h(w_h) \in V_h$ such that

$$\partial_n x_h|_F = \gamma^{-1}|F|^{-1} \int_F w_h \, ds =: \gamma^{-1} \bar{w}|_F,$$

for $F \in \mathcal{F}_{\Gamma_C}$ (3.3)

and

$$\int_F \{x_h\} \, ds = 0 \text{ for } F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_D} \cup \mathcal{F}_{\Gamma_N}.$$

It follows using integration by parts that for all $y_h \in V_h$ there holds

$$\langle \nabla y_h, \nabla x_h \rangle - \langle \partial_n y_h, x_h \rangle_{\Gamma_C} = 0.$$

Now taking $v_h = w_h + x_h$ leads to

$$A_h(u_1, v_h) - A(u_2, v_h) = \|\nabla w_h\|_h^2 + \langle \nabla w_h, \nabla x_h \rangle_h - \langle \partial_n w_h, x_h \rangle_{\Gamma_C} + \langle \gamma^{-1} \bar{w}, w_h \rangle$$

$$+ \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \partial_n w_h + \gamma^{-1} \bar{w} \rangle_{\Gamma_C}$$

$$= \|\nabla w_h\|^2_h + \langle \gamma^{-1} \bar{w}, w_h \rangle$$

$$+ \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \partial_n w_h - \gamma^{-1} w_h \rangle_{\Gamma_C}$$

$$+ \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \gamma^{-1} (w_h + \bar{w}) \rangle_{\Gamma_C}$$

$$= \|\nabla w_h\|^2_h + \langle \gamma^{-1} w_h, w_h \rangle$$

$$+ \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \partial_n w_h - \gamma^{-1} w_h \rangle_{\Gamma_C}$$

$$+ 2 \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \gamma^{-1} w_h \rangle_{\Gamma_C}$$

Applying the monotonicity

$$\gamma^{-1}\|\{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +\|_{\Gamma_C}^2 \leq \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \partial_n w_h - \gamma^{-1} w_h \rangle_{\Gamma_C}$$

we see that

$$\gamma^{-1}\|\{P_\gamma(u_1)\} + - [P_\gamma(u_2)] + + w_h\|_{\Gamma_C}^2 \leq \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \partial_n w_h - \gamma^{-1} w_h \rangle_{\Gamma_C}$$

$$+ \langle \gamma^{-1}w_h, w_h \rangle_{\Gamma_C} + 2 \langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] +, \gamma^{-1} w_h \rangle_{\Gamma_C}.$$

Then, using the arithmetic-geometric inequality together with the approximation properties of the piecewise constant approximation $\bar{w}$ and an elementwise trace inequality to get the bound

$$\langle \{P_\gamma(u_1)\} + - [P_\gamma(u_2)] + + w_h, \gamma^{-1} (w_h + \bar{w}) \rangle_{\Gamma_C}$$

$$\leq \frac{1}{2} \gamma^{-1}\|\{P_\gamma(u_1)\} + - [P_\gamma(u_2)] + + w_h\|_{\Gamma_C}^2 + \frac{1}{2} \gamma^{-1} Ch \|\nabla w_h\|_h^2$$

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we finally obtain
\[
(1 - \frac{1}{2} \gamma^{-1} Ch) \| \nabla w_h \|^2_h + \frac{1}{2} \gamma^{-1} \| [P_\gamma (u_1)]_+ - [P_\gamma (u_2)]_+ + w_h \|^2_C \\
\leq A_h (u_1, v_h) - A_h (u_2, v_h).
\]

We conclude by choosing \( \gamma > Ch \).

For the second claim, first consider equation (3.2) with \( u_1 = u_h, \ u_2 = 0 \),
\[
\alpha \| u_h \|^2_{1,h} + \gamma^{-1} \| u_h - [P_\gamma (u_h)]_+ \|^2_C \lesssim A_h (u_h, u_h + x_h (u_h)). \tag{3.5}
\]
Let the positive constants \( c_h \) and \( C_h \) denote the square roots of the smallest and the largest eigenvalues respectively of the matrix given by \( \langle \varphi, \varphi \rangle_\Omega \), \( 1 \leq i, j \leq N \) such that
\[
c_h |U|_{\mathbb{R}^N} \leq \| u_h \|_\Omega \leq C_h |U|_{\mathbb{R}^N}.
\]
Let \( B \) denote the transformation matrix such that the finite element function corresponding to the vector \( BU \) is the function \( u_h + x_h (u_h) \). First we show that for \( \gamma \) sufficiently large there are constants \( b_1 \) and \( b_2 \) such that \( b_1 |U|_{\mathbb{R}^N} \leq |BU|_{\mathbb{R}^N} \leq b_2 |U|_{\mathbb{R}^N} \).

This can be seen by observing that
\[
\| u_h \|_\Omega \leq \| u_h + x_h \|_\Omega + \| x_h \|_\Omega \leq C_h |BU|_{\mathbb{R}^N} + C \gamma^{-1} h \| u_h \|_\Omega
\]
so that
\[
c_h (1 - C \gamma^{-1} h) |U|_{\mathbb{R}^N} \leq (1 - C \gamma^{-1} h) \| u_h \|_\Omega \leq C_h |BU|_{\mathbb{R}^N}.
\]

Similarly we may prove the upper bound using \( c_h |BU|_{\mathbb{R}^N} \leq \| u_h + x_h \|_\Omega \) so that
\[
c_h |BU|_{\mathbb{R}^N} \leq \| u_h \|_\Omega + \| x_h \|_\Omega \leq \| u_h \|_\Omega + C \gamma^{-1} h \| u_h \|_\Omega \leq C_h (1 + C \gamma^{-1} h) |U|_{\mathbb{R}^N}.
\]

Then there holds using (3.5),
\[
(G (U), BU)_{\mathbb{R}^N} = A_h (u_h, u_h + x_h (u_h)) - L (u_h + x_h (u_h)) \\
\geq \alpha \| u_h \|^2_{1,h} - \frac{C^2}{2} \| f \|^2_\Omega \geq \alpha \| u_h \|^2_{1,h} - \frac{C^2}{2} \| f \|^2_\Omega,
\]
where \( C_\star \) is the constant such that \( L (u_h + x_h (u_h)) \leq C_\star \| f \|_\Omega \| u_h \|_{1,h} \) and \( \lambda_1 \) is the smallest eigenvalue of the matrix defined by \( \langle \nabla \varphi_i, \nabla \varphi_j \rangle_h + \langle \varphi_i, \varphi_j \rangle_\Omega \), \( 1 \leq i, j \leq N \).

We conclude that for
\[
|U|_{\mathbb{R}^N} > \frac{C_\star}{\alpha \lambda_1^2} \| f \|_\Omega
\]
there holds
\[
(G (U), BU)_{\mathbb{R}^N} > 0.
\]

\( \square \)

**Proposition 3.2.** The formulation (2.1) admits a unique solution for \( \gamma = \gamma_0 h \), with \( \gamma_0 \) large enough.
For the first term we have the problem
\begin{equation}
(1.1)
\end{equation}
perturbation arguments. By combining the techniques of the uniqueness argument above with the Galerkin

If \(\gamma = \gamma_0 h\). Assume that \(u_1\) and \(u_2\) both are solutions to (2.1), then for \(v_h\) chosen as in the Proposition,
\begin{equation}
\alpha\|u_1 - u_2\|^2_{1,h} \lesssim A_h(u_1, v_h) - A_h(u_2, v_h) = (f, v_h)_\Omega - (f, v_h)_\Omega = 0.
\end{equation}

4. A priori error estimates. A priori error estimates may now be derived by combining the techniques of the uniqueness argument above with the Galerkin perturbation arguments.

Theorem 4.1. Assume that \(u \in H^{1/2 + \nu}(\Omega)\), with \(0 < \nu \leq \frac{1}{2}\) is the solution of the problem (1.1). Assume that \(u_h\) denotes the solution of (2.1), where \(\gamma = \gamma_0 h\). If \(\gamma_0\) is chosen sufficiently large and \(h \leq \tilde{t}_C\), where \(\tilde{t}_C\) is the constant of assumption (1.9), then there holds, with \(c := u - u_h\),
\begin{equation}
\alpha^2\|c\|^2_{1,h} + \gamma^{-\frac{1}{2}}\|P_n u_h\| + u_h \|v_h\|_{V_C} \lesssim \inf_{v_h \in V_h} (\|u - v_h\|_{1,C} + h^{\frac{1}{2}}\|\partial_n (u - v_h)\|_{L^2(\Omega)})
+ hf\|f\|_{L^2(\Omega)}.
\end{equation}

Proof. Using the definition of the form \(a(\cdot, \cdot)\) we have
\begin{equation}
\|\nabla c\|^2_h \leq a(c, c) = a(c, u - u_h) + a(c, v_h - u_h).
\end{equation}

For the first term we have
\begin{equation}
a(c, u - u_h) \leq \frac{1}{2}\|\nabla c\|^2_h + \frac{1}{2}\|\nabla (u - v_h)\|^2_h.
\end{equation}
Considering the second term we see that

$$a(e, v_h - u_h) = \langle \{\partial_n u\}, [v_h - u_h] \rangle_{F \setminus F_C} + \langle \{\partial_n e, v_h - u_h\} \rangle_{F_C}$$

$$- \langle \{\partial_n(v_h - u_h), e\} \rangle_{F_C}$$

$$- \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u_h) \rangle_{F_C}$$

(4.3)

Using that

$$\langle \{\partial_n e, v_h - u_h\} \rangle_{F_C} - \langle \{\partial_n(v_h - u_h), e\} \rangle_{F_C} = \langle \{\partial_n e, v_h - u\} \rangle_{F_C} - \langle \{\partial_n(v_h - u), e\} \rangle_{F_C}$$

and

$$\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u_h) \rangle_{F_C} = \langle [P_\gamma u]_+ + [P_\gamma u_h]_+, \partial_n e \rangle_{F_C}$$

we arrive at the identity

$$a(e, v_h - u_h) = \langle \{\partial_n u\}, [v_h - u_h] \rangle_{F \setminus F_C} + \langle \{\partial_n e, v_h - u\} \rangle_{F_C}$$

$$- \langle \{\partial_n(v_h - u), e\} + ([P_\gamma u]_+ + [P_\gamma u_h]_+) \rangle_{F_C}$$

$$- \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, P_\gamma(u - u_h) \rangle_{F_C}$$

$$- \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, e \rangle_{F_C}$$

(4.4)

Observe now that the following relation holds using monotonicity and the elementary relation $a^2 + b^2 + 2ab = (a + b)^2$, with $a = \gamma^{-1/2}(u - u_h)$ and $b = \gamma^{-1/2}([P_\gamma u]_+ - [P_\gamma u_h]_+),$

$$-\gamma^{-1}\|e\|_{F_C}^2 - \gamma^{-1}\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, P_\gamma e \rangle_{F_C} - \gamma^{-1}\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, 2e \rangle_{F_C}$$

$$\leq -\gamma^{-1}\|e\|_{F_C}^2 - \gamma^{-1}\| [P_\gamma u]_+ - [P_\gamma u_h]_+ \|_{F_C}^2 - \gamma^{-1}\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, 2e \rangle_{F_C}$$

$$\leq -\|\gamma^{-\frac{1}{2}}([P_\gamma u_h]_+ + u_h)\|_{F_C}^2.$$

We deduce the following bound

$$a(e, v_h - u_h) \leq \langle \{\partial_n u\}, [v_h - u_h] \rangle_{F \setminus F_C} + \langle \{\partial_n e, v_h - u\} \rangle_{F_C}$$

$$- \langle \{\partial_n(v_h - u), e\} + ([P_\gamma u]_+ + [P_\gamma u_h]_+) \rangle_{F_C}$$

$$- \gamma^{-\frac{1}{2}}([P_\gamma u_h]_+ + u_h) \|_{F_C}^2 + \gamma^{-1} \langle e, [P_\gamma u]_+ - [P_\gamma u_h]_+ \rangle_{F_C}$$

(4.5)

Choosing now $x_h \in V_h$ as in Lemma 2.1 but with $\partial_n x_h|_F = \gamma^{-1}(\bar{u} - \bar{u}_h)|_F = \gamma^{-1}\hat{e}|_F$ on faces $F \subset F_C$ we obtain

$$a_h(e, x_h) - \langle \{\partial_n u\}, [x_h] \rangle_{F} + \gamma^{-1}\|\hat{e}\|_{F_C}^2 + \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \gamma^{-1}\hat{e} \rangle_{F_C} = 0.$$

Note that using orthogonality on the faces (1.4) we have

$$\gamma^{-1}\|\hat{e}\|_{F_C}^2 = \gamma^{-1}\|\hat{e}\|_{F_C}^2 - \gamma^{-1}\|\hat{e} - e\|_{F_C}^2.$$
and once again using orthogonality and also the contact condition
\[
\langle [P_h u]_+ - [P_h u_h]_+, \gamma^{-1} \bar{e} \rangle_{\Gamma_C} = \langle [P_h u]_+ - [P_h u_h]_+ + e, \gamma^{-1} \bar{e} \rangle_{\Gamma_C}
- \langle [P_h u_h]_+ + u_h, \gamma^{-1} \bar{e} - \gamma^{-1} e \rangle_{\Gamma_C}
- \gamma^{-1} ||e - e||^2_{\Gamma_C}.
\]

For the last term in the right hand side we may add and subtract \(v_h - \bar{v}_h\) and use the triangle inequality followed by the interpolation properties of the projection onto piecewise constants and a trace inequality to obtain
\[
\gamma^{-1} ||e - e||^2_{\Gamma_C} \leq C (\gamma^{-1} ||u - v_h||^2_{\Gamma_C} + \gamma^{-1} h^{-1} h^2 \|\nabla (v_h - u_h)\|_h^2)
\leq C (||u - v_h||^2_{1, C} + \gamma^{-1} h^{-1} h^2 \|\nabla e\|_h^2) \quad (4.6)
\]

As a consequence
\[
\gamma^{-1} ||e||^2_{\Gamma_C} + \langle [P_h u]_+ - [P_h u_h]_+, \gamma^{-1} e \rangle_{\Gamma_C} \leq \frac{1}{4} \gamma^{-\frac{1}{2}} \langle [P_h u_h]_+ + u_h \rangle^2_{\Gamma_C}
+ C (||u - v_h||^2_{\Gamma_C} + \gamma^{-1} h^{-1} h^2 \|\nabla e\|_h^2)
- a_h(e, x_h) + \langle \{\partial_h u\}, \{x_h\}\rangle_F. \quad (4.7)
\]

Collecting the results of equations (4.1), (4.2), (4.5) and (4.7) and applying the Poincaré inequality (2.3) leads to
\[
\alpha \left(\frac{1}{2} - C \frac{h}{\gamma}\right) ||e||^2_{1,h} + \frac{1}{2\gamma} ||([P_h u]_+ + u_h)||^2_{\Gamma_C} \leq - a(e, x_h) + \langle \partial_h e, v_h - u \rangle_{\Gamma_C}
- \langle [P_h u_h]_+ + u_h, \partial_h (v_h - u) \rangle_{\Gamma_C}
+ \langle \{\partial_h u\}, \{v_h - u_h\}\rangle_F
+ \langle \{\partial_h e\}, \{x_h\}\rangle_F
+ C \left(1 + \frac{h}{\gamma}\right) ||u - v_h||^2_{1,\Gamma_C}. \quad (4.8)
\]

Observe that \(a(u_h, x_h) - \langle \{\partial_h u_h\}, \{x_h\}\rangle_F = 0\) using integration by parts and the construction of \(x_h\). Then, once again by integration by parts, we have
\[
a(e, x_h) - \langle \{\partial_h e\}, \{x_h\}\rangle_F = (-\Delta u, x_h)_{\Omega_C} \leq ||\Delta u||_{\Omega_C} ||x_h||_{\Omega_C},
\]

where \(\Omega_C\) is the set of elements with one face on \(\Gamma_C\). Let \(h_C > 0\) be the largest value such that \(\partial_h C \cap \partial_h C \neq \emptyset\) and assume that \(h_C \leq t_C\). Observe that by the construction of \(x_h\) and adding and subtracting \(v_h\) there holds
\[
||x_h||_{\Omega_C} \lesssim h^\frac{1}{2} h^{-1} ||\bar{e}||_{\Gamma_C} \lesssim h^\frac{1}{2} h^{-1} ||e||_{\Gamma_C}
\lesssim h^\frac{1}{2} (h^{-1}) (||u - v_h||_{\Gamma_C} + ||u_h - v_h||_{\Gamma_C}).
\]

Let \(w_h = u_h - v_h\), then by adding and subtracting \(I_{cf} w_h\) and applying the local trace inequality (2.6) and the standard global trace inequality for functions in \(H^1(\Omega)\) we obtain
\[
||w_h||_{\Gamma_C} \leq ||w_h - I_{cf} w_h||_{\Gamma_C} + ||I_{cf} w_h||_{\Gamma_C}
\lesssim h^{-\frac{1}{2}} ||w_h - I_{cf} w_h||_{1,h} + ||w_h - I_{cf} w_h||_{1,h} + ||w_h||_{1,h}.
\]
Applying the discrete interpolation estimate (2.7), we then have
\[
\|w_h\|_{H^1} \lesssim \|w_h\|_{1,h}
\]
from which it follows that
\[
(h^{\gamma-1})\|w_h - v_h\|_{H^1} \lesssim (h^{\gamma-1})(\|e\|_{1,h} + \|u - v_h\|_{1,h}).
\]
For the factor \(\|\Delta u\|_{\Omega_C}\) we use (1.8) to obtain the bound
\[
\|\Delta u\|_{\Omega_C} \lesssim h^{\frac{3}{2}} \sup_{0 \leq t \leq \gamma C} \|\Delta u\|_{\partial \Omega} \leq h^{\frac{3}{2}} \|f\|_{L^2(\Omega)}.
\]
It follows that
\[
a_h(e, v_h) - \langle \{\partial_n e\}, [x_h]\rangle \lesssim h\|f\|_{L^2(\Omega)}(h^{\gamma-1})(\|e\|_{1,h} + \|u - v_h\|_{1,h}).
\]
(4.9)
For the remaining terms of (1.8) we have by first adding and subtracting \(v_h\) and using the mean value property of the space \(V_h\) and then applying the Cauchy-Schwarz inequality followed by the arithmetic-geometric inequality,
\[
\langle \partial_n e, v_h - u \rangle_{H^1_c} - \langle [P_\gamma u_h]_+ + u_h, \partial_n (v_h - u) \rangle_{H^1_c} + \langle \{\partial_n u\}, [v_h - u]\rangle
\]
\[
= \langle \partial_n (u - v_h), v_h - u \rangle_{H^1_c} + \langle \partial_n (v_h - u_h), v_h - u \rangle_{H^1_c}
\]
\[
- \langle [P_\gamma u_h]_+ + u_h, \partial_n (v_h - u) \rangle_{H^1_c}
\]
\[
+ \langle \{\partial_n (u - v_h)\}, [v_h - u]\rangle
\]
\[
\leq C\varepsilon^{-1}\|u - v_h\|_{1,h}^2 + \gamma\|\partial_n (u - v_h)\|_{H^1_c}^2 + \frac{1}{2}\gamma^{-1}\|\gamma^0 u_h\|_{1,h}^2 + u_h^2_{1,h}
\]
\[
+ \varepsilon(\gamma\|\partial_n (v_h - u_h)\|_{H^1_c}^2 + \gamma^{-1}\|v_h - u_h\|_{H^1_c}^2).
\]
(4.10)
Using the zero average property of the nonconforming space, elementwise trace inequalities and a triangular inequality we obtain
\[
\gamma\|\partial_n (v_h - u_h)\|_{H^1_c}^2 + \gamma^{-1}\|v_h - u_h\|_{H^1_c}^2 \lesssim C\gamma_0\|v_h - u_h\|_{1,h}^2
\]
\[
\leq 2C\gamma_0(\|e\|_{1,h}^2 + \|v_h - u_h\|_{1,h}^2).
\]
Observe that \(C\gamma_0\) is constant for \(\gamma_0 = \gamma/h\) fixed, but it can not be made small by choosing \(\gamma_0\) large (or small). Instead we choose \(\varepsilon < \alpha/(16C\gamma_0)\) to obtain the bound
\[
\langle \partial_n e, v_h - u \rangle_{H^1_c} - \langle [P_\gamma u_h]_+ + u_h, \partial_n (v_h - u) \rangle_{H^1_c} + \langle \{\partial_n u\}, [v_h - u]\rangle \leq C\|u - v_h\|_{1,h}^2 + \gamma\|\partial_n (u - v_h)\|_{H^1_c}^2 + \frac{1}{4\gamma}\|\gamma^0 u_h\|_{1,h}^2 + \frac{\alpha}{8}\|e\|_{1,h}^2.
\]
(4.11)
Collecting the above bounds (4.8), (4.9) and (4.11), choosing \(h\gamma^{-1}\) and \(\varepsilon\) small enough (i.e. \(\gamma_0\) large enough) we conclude that for all \(v_h \in V_h\)
\[
\alpha_0^2\|e\|_{1,h} + \gamma^{-\frac{1}{2}}\|\gamma^0 u_h\|_{1,h} \lesssim (\|u - v_h\|_{1,h}^2 + h^{\frac{3}{2}}\|\partial_n (u - v_h)\|_{H^1_c})
\]
\[
+ \frac{h}{\alpha_0^2}\|f\|_{L^2(\Omega)}.
\]
Corollary 4.2. Under the assumptions of Theorem 4.1 there holds
\[
\alpha \frac{1}{2} \| e \|_{1,H} + \gamma^{-\frac{1}{2}} \| [P_1 u_h]_+ + u_h \|_{\Gamma_C} \lesssim \frac{h^{\frac{1}{2} + \nu}}{\alpha} \| \partial_\nu n(u - i_h u) \|_{H^{\frac{1}{2} + \nu}(\Gamma)} + \frac{h}{\alpha} \| f \|_{L^2(\Omega)}.
\] (4.12)

Proof. This is immediate from the best approximation result of Theorem 4.1 and the existence of an optimal approximation of \( u \) in \( V_h \). Since the Crouzeix-Raviart space contains the \( H^1 \)-conforming space of piecewise affine functions we may take the standard Lagrange interpolant \( i_h u \) for which there holds (see [14, 12]).

\[
\| u - i_h u \|_{1,C} + h^{\frac{1}{2}} \| \partial_\nu n(u - i_h u) \|_{\Gamma_C} \lesssim h^{\frac{1}{2} + \nu} \| u \|_{H^{\frac{1}{2} + \nu}(\Gamma)}.
\]

\[\square\]

5. Numerical example. Here we will consider two examples on the unit square, \( \Omega = [0,1]^2 \). We have used the package FreeFEM++ for the computations [17]. We let \( \Gamma_D = [0,1] \times \{1\} \), \( \Gamma_N = \{0\} \times [0,1] \cup \{1\} \times [0,1] \) and \( \Gamma_C = [0,1] \times \{0\} \). In all cases we use a fixed point iteration to compute the solution and we iterate until the relative \( H^1 \)-error of the increment if smaller than \( 10^{-5} \).

In the graphics below the \( H^1 \)-error is marked with squares, the \( L^2 \)-error with circles and finally the residual quantity \( \| u_h + [P_1 u_h]_+ \|_{\Gamma_C} \) by triangles. Dotted lines are reference lines with slopes \( O(h^0) \) (upper) and \( O(h^2) \) (lower).

5.1. Problem with known solution. We first consider an example where the exact solution is known,
\[
u(x,y) := -\cos(\pi/2 \, y) \, \sin^2(\pi \, x)
\]
with the right hand side
\[
f = \frac{\pi^2}{4} \cos(\pi/2 \, y) \sin^2(\pi \, x) - 2\pi^2 \cos(\pi/2 \, y) \cos(2\pi \, x).
\]
Observe that this actually is a solution to a linear Neumann problem, but we can still use it as a solution to the nonlinear problem. The contact takes place in the set \( \{0,1\} \). We solve it on a sequence of Union Jack style meshes (see the left plot of Figure 5.1 for an example) with \( h/\sqrt{2} \in 2^{-i/2} \) for \( i = 0 \). The result is presented in the left graphic of Figure 5.2. As expected we observe first order convergence of the relative \( H^1 \)-error and second order convergence of the relative \( L^2 \)-error. As expected the residual quantity has a convergence close to \( O(h^{\frac{1}{2}}) \).

5.2. Problem with unknown solution. Here we propose the problem obtained by setting
\[
f = (2\pi N)^2 \cos(2\pi N \, x), \quad N \in \{3, 5\}.
\] (5.1)
We solve the problem on a mesh with \( h = 2\sqrt{2} \cdot 10^{-3} \) (a 500 × 500 mesh), using the nonsymmetric Nitsche method from [12] and piecewise quadratic conforming approximation to obtain a reference solution. We report the contour lines of the solution for \( N = 5 \) in the right plot of Figure 5.1. Then we solve the problem for \( h/\sqrt{2} \in 2^{-i/2} \) for \( i = 0 \) and compute the same quantities as in the previous case. The convergences are reported in the right graphic of Figure 5.2. The cases \( N = 3 \) and \( N = 5 \) are distinguished by the use of white and black markers respectively, similar convergence orders were observed in both cases. First order convergence is observed for the error in the \( H^1 \)-norm and second order convergence in the \( L^2 \)-error. As before the convergence of \( \| u_h + [P_1 u_h]_+ \|_{\Gamma_C} \) is approximately \( O(h^{\frac{1}{2}}) \).
Figure 5.1. Left: example of a computational mesh. Right: the fine mesh solution using (5.1) with $N = 5$.

Figure 5.2. Convergence plots of the two numerical examples. Left: the problem from Section 5.1. Right: from Section 5.2. Dotted lines are reference curves. Upper $O(h)$, lower $O(h^2)$. Square markers - $H^1$-error; circle markers - $L^2$-error; triangle markers - satisfaction of the contact condition, $\|u_h + [P_h(u_h)]_+\|_{L^2}$. In the right plot, white markers indicate $N = 3$ and black markers $N = 5$.

6. Conclusion. We have proved that the nonsymmetric Nitsche method of [9] may be applied in the framework of [10, 12] for the approximation of unilateral contact problems. An optimal error estimate for a method using a nonconforming finite element space was derived combining tools from the inf-sup analysis of [9] with the monotonicity argument of [10, 12]. The theoretical results were illustrated in two numerical examples. Herein we only considered the simplified case of the Signorini problem based on Poisson’s equation, but the extension to elasticity may be feasible using the results from [7]. Another natural question is if the above analysis can be extended to the case of standard conforming elements. The difficulty here is to handle the non-local character of the function necessary for the stability argument, adding a layer of terms that must be estimated. Numerical experiments not reported here.
indicate that the conforming method also performs well.

REFERENCES

[1] Y. Achdou, C. Bernardi, and F. Coquel. A priori and a posteriori analysis of finite volume discretizations of Darcy’s equations. Numer. Math., 96(1):17–42, 2003.

[2] P. Alart and A. Curnier. A mixed formulation for frictional contact problems prone to Newton like solution methods. Comput. Methods Appl. Mech. Engrg., 92(3):353–375, 1991.

[3] H. J. C. Barbosa and T. J. R. Hughes. Boundary Lagrange multipliers in finite element methods: error analysis in natural norms. Numer. Math., 62(1):1–15, 1992.

[4] Z. Belhachmi and F. Ben Belgacem. Quadratic finite element approximation of the Signorini problem. Math. Comp., 72(241):89–104, 2003.

[5] F. Ben Belgacem. Numerical simulation of some variational inequalities arisen from unilateral contact problems by the finite element methods. SIAM J. Numer. Anal., 37(4):1198–1216, 2000.

[6] F. Ben Belgacem and Y. Renard. Hybrid finite element methods for the Signorini problem. Math. Comp., 72(243):1117–1145, 2003.

[7] T. Boiveau and E. Burman. A penalty-free Nitsche method for the weak imposition of boundary conditions in compressible and incompressible elasticity. IMA J. Numer. Anal., 36(2):770–795, 2016.

[8] F. Brezzi, W. W. Hager, and P.-A. Raviart. Error estimates for the finite element solution of variational inequalities. Numer. Math., 28(4):431–443, 1977.

[9] E. Burman. A penalty-free nonsymmetric Nitsche-type method for the weak imposition of boundary conditions. SIAM J. Numer. Anal., 50(4):1959–1981, 2012.

[10] F. Chouly and P. Hild. A Nitsche-based method for unilateral contact problems: numerical analysis. SIAM J. Numer. Anal., 51(2):1295–1307, 2013.

[11] F. Chouly and P. Hild. On convergence of the penalty method for unilateral contact problems. Appl. Numer. Math., 65:27–40, 2013.

[12] F. Chouly, P. Hild, and Y. Renard. Symmetric and non-symmetric variants of Nitsche’s method for contact problems in elasticity: theory and numerical experiments. Math. Comp., 84(293):1089–1112, 2015.

[13] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 7(R-3):33–75, 1973.

[14] T. Dupont and R. Scott. Polynomial approximation of functions in Sobolev spaces. Math. Comp., 34(150):441–463, 1980.

[15] R. Glowinski and P. Le Tallec. Augmented Lagrangian and operator-splitting methods in non-linear mechanics, volume 9 of SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[16] J. Haslinger, I. Hlaváček, and J. Nečas. Numerical methods for unilateral problems in solid mechanics. In Handbook of numerical analysis, Vol. IV, Handb. Numer. Anal., IV, pages 313–485. North-Holland, Amsterdam, 1996.

[17] F. Hecht. New development in freefem++. J. Numer. Math., 20(3-4):251–265, 2012.

[18] P. Hild and Y. Renard. A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. Numer. Math., 115(1):101–129, 2010.

[19] P. Hild and Y. Renard. An improved a priori error analysis for finite element approximations of Signorini’s problem. SIAM J. Numer. Anal., 50(5):2400–2419, 2012.

[20] R. H. W. Hoppe and B. Wohlmuth. Element-oriented and edge-oriented local error estimators for nonconforming finite element methods. RAIRO Modél. Math. Anal. Numér., 30(2):237–263, 1996.

[21] D. Hua and L. Wang. The nonconforming finite element method for Signorini problem. J. Comput. Math., 25(1):67–80, 2007.

[22] O. A. Karakashian and F. Pascal. A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. SIAM J. Numer. Anal., 41(6):2374–2399, 2003.

[23] N. Kikuchi and J. T. Oden. Contact problems in elasticity: a study of variational inequalities and finite element methods, volume 8 of SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[24] J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Sem. Univ. Hamburg, 36:9–15, 1971.
[25] F. Scarpini and M. A. Vivaldi. Error estimates for the approximation of some unilateral problems. *RAIRO Anal. Numér.*, 11(2):197–208, 221, 1977.

[26] R. Stenberg. On some techniques for approximating boundary conditions in the finite element method. *J. Comput. Appl. Math.*, 63(1-3):139–148, 1995.

[27] B. I. Wohlmuth, A. Popp, M. W. Gee, and W. A. Wall. An abstract framework for a priori estimates for contact problems in 3D with quadratic finite elements. *Comput. Mech.*, 49(6):735–747, 2012.

[28] P. Wriggers and G. Zavarise. A formulation for frictionless contact problems using a weak form introduced by Nitsche. *Computational Mechanics*, 41(3):407–420, 2007.