Combinatorial Batch Codes with Redundancy

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Abstract

A combinatorial batch code with redundancy $r$ and parameters $(n,k,m,t)$ can be represented as a system $C$ of $m$ (not necessarily distinct) subsets of an underlying $n$-element set $F$, so that each $k$-subset of $F$ can be covered by every $(m-r)$-subset $K$ of $C$ while taking no more than $t$ elements of $F$ with each set in $K$. The sum of the cardinalities of the sets in $C$ is the weight of the code. We focus on the case $t = 1$, and determine the minimal weight for several ranges of the parameters. For certain parameter values, the existence of combinatorial batch codes with redundancy of specified maximum weight is related to the existence of graphs with a specified minimum girth.
1 Introduction

We study a class of combinatorial objects that we call combinatorial batch codes with redundancy. These are motivated by a data retrieval problem in which a collection of items (such as files) are stored, with possible duplication, on a collection of servers. After the items are stored, a demand will be made for some $k$-subset of the items, where $k$ is fixed in advance. The goal is to store as few total copies of the items as possible, while still being able to retrieve any $k$-subset of the items without taking too many from any one server. The study of such problems was initiated by Ishai, Kushilevitz, Ostrovsky, and Sahai [6], who proposed a class of combinatorial batch codes that provide solutions to the following particular data retrieval problem.

Question 1 (Ishai et al. [6]). Suppose a collection of $n$ items is to be stored over a set of $m$ servers. Can the items be stored so that any $k$ items are simultaneously accessible by taking at most $t$ items from each server? If so, what amount $N$ of total storage is needed? An optimal solution has the smallest total storage for given parameters $n$, $k$, $m$, and $t$.

We generalize the question by adding a requirement of redundancy. At each moment, we allow some of the servers to be unavailable (for example, they may be down for maintenance). If we place a bound, $r$, on the number of servers that may be unavailable at the same time, we are faced with a problem of retrieving each $k$-subset of the items from every collection of $m - r$ servers. To ensure this is possible, intuitively, it will be necessary to store some “redundant” copies of items. We thus study the following particular question.

Question 2. Suppose a collection of $n$ items is to be stored over a set of $m$ servers. At each moment, some number $r$ of the servers may be unavailable. Can the items be stored so that any $k$ items are simultaneously accessible from each collection of $m - r$ servers, while taking at most $t$ items from each server? What amount $N$ of total storage is needed? An optimal solution has the smallest total storage for given parameters $n$, $k$, $m$, $t$, and $r$.

In Section 2 we generalize previous definitions to include the redundancy parameter $r$, and derive several construction lemmas paralleling the results of Bujtás and Tuza [4]. The following two sections establish the optimal value of $N$ for certain parameter ranges, which are illustrated in Figure 1.
Theorem 8 Theorem 13 Theorem 12

Theorem 9 for $k = m - r$

$k$ $m$ $(k-1)\left(\frac{m}{r+k-1}\right)$ $n$

Figure 1: Ranges of the parameter $n$ addressed here, in terms of $k$, $m$, and $r$, always assuming $t = 1$ and that the conditions of Lemma 3 are met. Theorem 8 applies when $k \leq n \leq m$, and Theorem 9 applies when $n > m$ and $k = m - r$. Theorem 13 applies to certain values of $n$ less than $(k-1)\left(\frac{m}{r+k-1}\right)$, while Theorem 12 applies when $(k-1)\left(\frac{m}{r+k-1}\right) \leq n$. When $n = (k-1)\left(\frac{m}{r+k-1}\right)$, the constructions in the latter two theorems are the same.

Section 3 we study the extremal cases when $n$ is large, or small, compared to $m$. Theorem 8 characterizes $N$ when $k \leq n \leq m$. Theorem 12 characterizes $N$ when $n \geq (k-1)\left(\frac{m}{r+k-1}\right)$. Section 4 studies the “gap” between these results. We obtain a result characterizing $N$ for certain values of $n$ as small as $\frac{k-1}{r+k-1}\left(\frac{m}{r+k-2}\right)$.

In Section 5 we discuss a natural lower bound for $N(n,k,m;r)$ and discuss some nontrivial cases when the lower bound is achieved. These lower bounds are related to the existence of graphs with a lower bound on their girth.

Solutions to Question 1 are provided by combinatorial batch codes, which were introduced by Paterson, Stinson, and Wei [9] and have been the subject of several subsequent papers [2, 3, 4, 5]. Although we will not directly use combinatorial batch codes here, we state their definition for comparison with our definition of combinatorial batch codes with redundancy in the next section.

Definition 1 (Paterson et al. [9]). A combinatorial batch code with parameters $(n,k,m,t)$, abbreviated CBC or CBC$(n,k,m,t)$, is a multifamily $\mathcal{B} = \{B_1, \ldots, B_m\}$ of $m$ subsets, called servers, of a set $X = \{x_1, \ldots, x_n\}$ of $n$ items, called files, such that for each $Y \subseteq X$ with $|Y| \leq k$ there exists subsets $C_i \subseteq B_i$ for which $|C_i| \leq t$ and $Y = C_1 \cup C_2 \cup \cdots \cup C_m$. 

3
The weight of a CBC $\mathcal{B}$ is the value

$$N(\mathcal{B}) = |B_1| + |B_2| + \cdots + |B_m|.$$ 

A CBC that obtains the minimal weight (as a function of the other parameters) is optimal, and the minimum value is denoted $N(n, k, m, t)$.

Previous work has characterized $N(n, k, m, 1)$ for several ranges of parameters. In many cases, our theorems yield these previous results as a special case when we set the redundancy parameter $r$ to zero. We note this after each theorem, as appropriate.

## 2 Combinatorial batch codes with redundancy

To address Question 2, we generalize combinatorial batch codes to include redundancy. We introduce a new parameter, $r$, to measure the number of servers that may be inaccessible at one time. As usual, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. The codes studied in the following definition have also been investigated by Silberstein [10] under the name “erasure batch codes”.

**Definition 2.** A combinatorial batch code with redundancy $r$ with parameters $(n, k, m, t)$ (abbreviated $r$-CBC, $r$-CBC$(n, k, m, t)$, or $r$-CBC$(n, k, m)$ when $t = 1$) is a multifamily $\mathcal{B} = \{B_1, \ldots, B_m\}$ of $m$ subsets of $[n]$ such that for each $Y \subseteq [n]$ with $|Y| \leq k$ and $J \subseteq [m]$ with $|J| \geq m - r$, there exists subsets $C_j \subseteq B_j$ for each $j \in J$ such that $|C_j| \leq t$ and $Y = \bigcup_{j \in J} C_j$. Informally put, this means that for each collection $Y$ of $k$ or fewer files, and each collection $J$ of $m - r$ or more servers, it is possible to obtain all the files in $Y$ from the servers in $J$ while taking no more than $t$ from each server.

The weight of an $r$-CBC $\mathcal{B}$ is

$$N(\mathcal{B}) = |B_1| + |B_2| + \cdots + |B_m|,$$

and an $r$-CBC that obtains the minimal $N$ (as a function of $n$, $k$, $m$, $t$, and $r$) is optimal. We denote this minimal value as $N(n, k, m, t; r)$, or simply $N(n, k, m; r)$ if $t = 1$.

We may represent an $r$-CBC$(n, k, m)$ $\mathcal{B}$ by a $m \times n$ incidence matrix $A$ such that row $i$ of $A$ represents the set $B_i \subseteq [n]$. We let $A_j$ denote the subset of $[m]$ represented by column $j$ of $A$. We let $N(A)$ be the number of 1s that appear in $A$, and say that $A$ is optimal if $N(A) = N(n, k, m; r)$. 

4
In this paper, we study the case $t = 1$ exclusively. The next lemma establishes the basic relations between the remaining parameters that are required for the existence of an $r$-CBC. For the remainder of the paper, we will always assume that our parameters satisfy the inequalities stated in the lemma.

**Lemma 3.** There is an $r$-CBC with parameters $(m, k, n, 1)$, where $k \geq 1$, if and only if $r < m$ and $k \leq \min\{n, m - r\}$.

*Proof.* For the forward direction, it is enough to note that an $m \times n$ matrix of all 1s will represent an $r$-CBC with the stated parameters, under the stated conditions. There will be $m - r$ servers available, each containing every file, and thus we can retrieve any collection of $k$ files for each $k \leq \min\{n, m - r\}$.

For the reverse direction, we show that the conditions are necessary. If $r \geq m$ then it would be possible for every server to be down, which cannot yield an $r$-CBC when $k \geq 1$. The inequality $n < k$ is impossible because one cannot retrieve more than the total number of files. The inequality $k > m - r$ is impossible because there will be $m - r$ available servers, and with $t = 1$ we may only take one file from each server.

Our first theorem extends Theorem 3 of Bujitás and Tuza [5] to combinatorial batch codes with redundancy. To achieve this, we use the following extension of Hall’s Marriage Theorem, which is also stated by Silberstein [10, Theorem 5].

**Theorem 4.** Let $\{A_1, \ldots, A_n\}$ be a family of subsets of $M$ and $r \geq 0$. For each $r$-subset $M' \subseteq M$, the following are equivalent:

(i) There exist distinct elements $a_1, \ldots, a_n$ such that $a_i \in A_i \setminus M'$ for each $i \in [n]$.

(ii) $\left| \bigcup_{j \in J} A_j \right| \geq r + c$ for every $c$-subset $J$ of $[n]$.

**Theorem 5.** Suppose that $A$ is an $m \times n$ matrix with values in $\{0, 1\}$. For each $j \leq n$, let $A_j$ be the subset of $[m]$ determined by column $j$ of $A$. The following are equivalent:

(i) The matrix $A$ represents an $r$-CBC($n, k, m$).
(ii) For every \( c \in [k] \) and every \( c \)-subset \( J \) of \([n]\), \( \left| \bigcup_{j \in J} A_j \right| \geq r + c \).

Informally put, for each \( c \in [k] \), each collection of \( c \) columns of \( A \) spans at least \( r + c \) rows.

(iii) For every \( d \)-subset \( I \) of \([m]\), with \( r \leq d \leq r+k-1 \), \( |\{i : A_i \subseteq I\}| \leq d-r \).

Informally put, for each collection of \( d \) rows of \( A \), with \( r \leq d \leq r+k-1 \), the number of columns whose 1s are completely contained by these rows is at most \( d-r \).

Proof. The implication from (ii) to (iii) follows from the definition of an \( r \)-CBC with \( t = 1 \). The implication from (iii) to (ii) is a direct application of Theorem 4. Therefore, it suffices to prove that (ii) and (iii) are equivalent.

First, assume that \( A \) satisfies condition (ii). Choose \( d \) with \( r \leq d < r+k \) and let \( I \) be a \( d \)-subset of \([m]\). Let \( J = \{i : A_i \subseteq I\} \) and let \( w = |J| \). We want to show that \( w \leq d-r \). Suppose otherwise: then \( w > d-r \), that is, \( d < r+w \). So we have a collection of more than \( d-r \) columns that are contained in at most \( d \) rows, where \( r \leq d < r+k \). So, if we let \( c = d-r+1 \), we have a collection of \( c \) columns that are contained in fewer than \( c+r \) rows. Because \( r \leq d \), we have \( c > 0 \). Because \( d < r+k \), we have \( c \leq k \). Thus \( c \in [k] \). This contradicts (ii), which states that each collection of \( c \) rows must span at least \( d = c+r \) columns. Thus we have \( w \leq d-r \), as desired.

Now assume that \( A \) satisfies condition (iii). First, we verify that \( A \) satisfies condition (ii) in the special case \( c = 1 \). This follows from the special case of (iii) with \( d = r \), which says that for every \( r \)-subset \( I \) of \([m]\) we have \( |\{i : A_i \subseteq I\}| \leq 0 \). Thus there is no column with fewer than \( r+1 \) ones, which is precisely the statement of (ii) in the case \( c = 1 \).

It remains to prove condition (ii) for \( c \geq 2 \). To this end, choose \( c \in [k] \) with \( c \geq 2 \) and let \( J \) be a \( c \)-subset of \([n]\). Let \( I = \bigcup_{j \in J} A_j \) and let \( d = |I| \). Note that, by the previous paragraph, each \( A_j \) contains at least \( r+1 \) ones, and thus \( d \geq r+1 \).

We want to show that \( |I| \geq r+c \). Suppose otherwise; then we have \( r+1 \leq d \leq r+c-1 \), so \( r \leq d \leq r+k-1 \), as \( c \leq k \). Thus, by (iii), we have

\[
|\{i : A_i \subseteq I\}| \leq d-r \leq (r+c-1) - r = c-1.
\]

However, we also have \( J \subseteq \{i : A_i \subseteq I\} \), so \( |\{i : A_i \subseteq I\}| \geq c \). This is a contradiction, so we conclude \( d \geq r+c \), as desired. \(\square\)
Lemma 6. Let $A$ be a matrix that represents an $r$-CBC$(n, k, m)$. Then the number of $1$s in each column of $A$ is at least $r + 1$, and if $A$ is optimal, at most $r + k$.

Proof. By setting $c = 1$ in Theorem 5 (ii), we see that each column of $A$ has cardinality at least $r + 1$.

Now let $A$ represent an $r$-CBC$(n, k, m)$, and assume without loss of generality that $A_1$ has cardinality greater than $r + k$. We will show that $A$ is not optimal. Remove an element from $A_1$ and call the resulting matrix $A'$. Thus $|A'_1| + 1 = |A_1| > r + k$ and $A'_j = A_j$ for $2 \leq j \leq n$.

To show $A'$ represents an $r$-CBC$(n, k, m)$, let $J$ be a $c$-subset of $[n]$ with $c \leq k$. If $1 \notin J$, then

$$\left| \bigcup_{j \in J} A'_j \right| = \left| \bigcup_{j \in J} A_j \right| \geq r + c.$$ 

If $1 \in J$, then

$$\left| \bigcup_{j \in J} A'_j \right| \geq |A'_1| = |A_1| - 1 \geq r + k \geq r + c.$$ 

Thus, by Theorem 5, $A'$ represents an $r$-CBC$(n, k, m)$. By construction, $N(A') = N(A) - 1$, and thus $A$ is not optimal.

The lemma allows us to bound the weight of an optimal $r$-CBC.

Corollary 7. The following inequalities hold for $N(n, k, m; r)$:

$$(r + 1)n \leq N(n, k, m; r) \leq (r + k)n.$$ 

The corollary allows us to prove $N(n, 1, m; r) = (r+1)n$. Assuming $k = 1$, form an $r$-CBC $A$ with a matrix that has $n$ columns each of cardinality $r + 1$. This matrix $A$ is an $r$-CBC with $N(A) = (r + 1)n$, and by the corollary it is impossible to have a smaller value of $N$.

3 Extremal Results

In this section, we establish the exact value of $N(m, k, n; r)$ for several families of parameter values. We first consider $r$-CBCs where there are at least as many servers as files, that is, $n \leq m$. 


**Theorem 8.** Let \( n \leq m \). Then
\[
N(n, k, m; r) = (r + 1)n.
\]

**Proof.** Let \( A \) be the \( m \times n \) matrix with columns given by
\[
A_j = \{ j + x \pmod{m} : x = 0, 1, \ldots, r \}.
\]
See Figure 2 for an example. Let \( J \subset [n] \) with \( |J| = c, c \leq k \). Then \( J = \{ j_1, j_2, \ldots, j_c \} \) with \( j_i < j_{i+1} \) for each \( i \in [c-1] \). We consider two cases, based on the size of the gaps between consecutive entries of \( J \) (considering \( j_c \) and \( j_1 \) consecutive in \( J \)). Let \( g_i = j_{i+1} - j_i \) for each \( i \in [c-1] \) and \( g_c = m + j_1 - j_c \).

**Case 1:** Suppose \( g_i > r \) for some \( i \in [c] \). If \( i = c \), then \( m + j_1 - j_c > r \), so \( j_c + r - m < j_1 \). If \( j_c + r \leq m \), then \( A_{j_c} \cap (J - \{ j_c \}) = \emptyset \). Similarly, if \( j_c + r > m \), then since \( j_c + r - m < j_1 \), \( A_{j_c} = \{ j_c, j_c+1, \ldots, m, 1, \ldots, j_c+r-m \} \) and \( A_{j_c} \cap (J - \{ j_c \}) = \emptyset \). Thus
\[
\bigcup_{j \in J} A_j \geq |A_{j_c} \cup (J - \{ j_c \})| = r + 1 + c - 1 = r + c.
\]
If \( i < c \), then \( j_i + r < j_{i+1} \leq n \leq m \), so \( A_{j_i} = \{ j_i, j_i+1, \ldots, j_i + r \} \) and \( j_{i+1} > j_i + r \), and thus \( A_{j_i} \cap (J - \{ j_i \}) = \emptyset \). It follows that
\[
\bigcup_{j \in J} A_j \geq |A_{j_i} \cup (J - \{ j_i \})| = r + 1 + c - 1 = r + c.
\]
Thus, by Theorem 5, \( A \) is an \( r \)-CBC.

**Case 2:** Suppose \( g_i \leq r \) for all \( i \in [c] \). Suppose \( \bigcup_{j \in J} A_j \neq [m] \). Then there are \( j_i, j_{i+1} \) such that \( A_{j_i} \cap A_{j_{i+1}} = \emptyset \). Thus either \( j_i + r < m \), and so \( j_i + r < j_{i+1} \) and \( g_i > r \), a contradiction, or \( j_i + r \geq m \). In the second case, \( A_{j_i} = \{ j_i, j_i+1, \ldots, m, 1, 2, \ldots, j_i + r \pmod{m} \} \) and \( A_{j_{i+1}} = \{ j_{i+1}, j_{i+1}+1, \ldots, j_{i+1} + r \} \), where \( j_{i+1} + r < j_i \), but this is also a contradiction. Thus \( \bigcup_{j \in J} A_j = [m] \) and
\[
\bigcup_{j \in J} A_j \geq m \geq r + k \geq r + c.
\]
Therefore, by Theorem 5, \( A \) is an \( r \)-CBC.

Now, because \( A \) is an \( r \)-CBC and \( \sum_{j \in [n]} |A_j| = (r + 1)n \), \( A \) is optimal by Corollary 7. \( \square \)
A = \[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Figure 2: The matrix $A$ constructed as in the proof of Theorem 8 for $m = 6, n = 4, r = 3$, and $k \leq 3$.

When $r = 0$, Theorem 8 reduces to $N(n, k, m; 0) = n$, which is Theorem 3 of Paterson, Stinson, and Wei [9].

We next consider $r$-CBCs with at least as many files as servers, that is, $n \geq m$, and a maximal number of files being retrieved, $k = m - r$.

**Theorem 9.** If $m \leq n$ and $k = m - r$, then

$$N(n, m - r, m; r) = mn - m(m - r - 1).$$

**Proof.** Let $A$ be the $m \times n$ matrix with columns given by

$$A_j = \{j + x \pmod{m} : x = 0, 1, \ldots, r\} \text{ for } j \leq m.$$ 

and $A_j = \{1, 2, \ldots, m\}$ for $m < j \leq n$. See Fig. 3 for an example. We first show that $A$ is an $r$-CBC. Let $c \leq m - r$ and let $J \subseteq [n]$ with $|J| = c$. Then, $|\bigcup_{j \in J} A_j| = m \geq r + c$ if $J \cap \{m + 1, \ldots, n\} \neq \emptyset$. So, suppose $J \cap \{m + 1, \ldots, n\} = \emptyset$. Then, we are only considering the first $m$ columns of $A$, so let $A'$ be the $m \times m$ matrix with columns given by $\{A_1, \ldots, A_m\}$. We can use the argument from the proof of Theorem 8 to show that $A'$ is an $r$-CBC, and thus satisfies Theorem 5. Therefore $A$ is an $r$-CBC. Note that in $A$, for each $d \in [m]$, $|\{i : d \in A_i\}| = n - m + r + 1$.

Because $A$ is an $r$-CBC with $N(A) = m(n - m + r + 1)$, we know that $N(n, m - r, m; r) \leq m(n - m + r + 1)$. Suppose that $N(n, m - r, m; r) < m(n - m + r + 1)$. Let $B$ be an $r$-CBC with $N(B) = N(n, m - r, m; r)$. Then there must be some $d \in [m]$ for which $|\{i : d \in B_i\}| < n - m + r + 1$. Let $J \subseteq [n] \setminus \{i : d \in B_i\}$ with $|J| = m - r$. Such a $J$ exists because

$$|n \setminus \{i : d \in B_i\}| > n - (n - m + r + 1) = m - r - 1.$$
Figure 3: The matrix $A$ constructed as in the proof of Theorem 9 when $m = 6$, $n = 7$, $r = 3$, and $k = 3$.

Then $| \bigcup_{j \in J} B_j | < (m - r) + r$, because $d \notin B_j$ for any $j \in J$. Therefore, by Theorem 5 $B$ is not an $r$-CBC.

We can thus conclude that $A$ is optimal and

$$N(n, m - r, m; r) = m(n - m + r + 1).$$

When $r = 0$, Theorem 9 reduces to $N(n, m, m; 0) = mn - m(m - 1)$, which appears as Theorem 4 of Paterson, Stinson, and Wei [9].

We now examine what occurs when $n$ is large. We first prove two technical lemmas.

**Lemma 10.** Let $A$ be an $r$-CBC, and assume there are $i, j \in [n]$ with $A_i \subseteq A_j$. Let $R$ be a nonempty set with $R \subseteq A_j \setminus A_i$. Replacing columns $A_i$ and $A_j$ with $A'_i = A_i \cup R$ and $A'_j = A_j \setminus R$, respectively, produces a matrix $A'$ which is an $r$-CBC with the same weight as $A$.

**Proof.** Assume that $A$, $A'$, $i$, and $j$ are as in the statement of the theorem. We wish to prove that $A'$ satisfies condition (iii) of Theorem 5. To this end, assume that $I$ is a $d$-subset of $[m]$. It is sufficient to show that the number $\sigma'$ of values of $k$ for which $A'_k \subseteq I$ is no more than the number $\sigma$ of values of $k$ for which $A_k \subseteq I$.

The proof has several cases. If $A_j \subseteq I$, then $A_i \subseteq I$ as well, because $A_i \subseteq A_j$. Thus $A'_i$ and $A'_j$ are also subsets of $I$. Thus if $A_j \subseteq I$ then $\sigma = \sigma'$.

If $A_i \not\subseteq I$ then, because $A_i \subseteq A_j$, we have that $A_j \not\subseteq I$. Thus neither of $A'_i$ and $A'_j$ is a subset of $I$, so again $\sigma = \sigma'$.

Because $A_i \subseteq A_j$, there is only one additional case, which occurs when $A_i \not\subseteq I$ but $A_j \not\subseteq I$. There are two subcases: if $A_j \setminus I \subseteq R$, then $A'_j \not\subseteq I$ and $A'_i \not\subseteq I$. Otherwise, if $A_j \setminus I \not\subseteq R$, then neither $A'_i$ nor $A'_j$ is a subset of $I$. Thus, in both subcases, we have $\sigma' \leq \sigma$. So $A'$ is an $r$-CBC and since $|A_i| + |A_j| = |A'_i| + |A'_j|$, it follows that $A'$ has the same weight as $A$. 

Lemma 11. For every matrix $A$ representing an optimal $r$-CBC($n, k, m$), there exists a matrix $A'$ also representing an optimal $r$-CBC($n, k, m$) for which one of the two properties hold:

(i) $r + 1 \leq |A'_j| \leq r + k - 1$ for each $j \leq n$, or

(ii) $r + k - 1 \leq |A'_j| \leq r + k$ for each $j \leq n$.

Proof. By Lemma 6, if $A$ represents an optimal $r$-CBC then all columns of $A$ have cardinality at most $r + k$. Assume $A$ does not satisfy properties (i) or (ii). Then $A$ contains columns $C$ and $C'$ of cardinality $j$ ($r + 1 \leq j \leq r + k - 2$) and $r + k$, respectively. Observe that if an $r$-CBC has a column of cardinality $r + k$, one can replace the column with any other column of cardinality $r + k$, and still satisfy the $r$-CBC property. So we can assume that $C \subseteq C'$. By our previous result, we can then produce another $r$-CBC with one fewer column of cardinality $r + k$, but with the same weight. Proceed inductively until either the resulting $r$-CBC has no columns of cardinality $r + k$ or there are no columns of cardinality less than $r + k - 1$. In these cases, we produce either a $r$-CBC of type (i) or (ii), respectively. At every step, the weight of the $r$-CBC remains unchanged, and therefore the final $r$-CBC is optimal. \hfill \Box

Theorem 12. Let $r \geq 0$ and $k \leq m - r$ be integers. If $n \geq (k - 1)\binom{m}{r + k - 1}$, then

$$N(n, k, m; r) = (r + k)n - (k - 1)\binom{m}{r + k - 1}.\]$$

Proof. Following Corollary 7 of Section 2, we showed that $N(n, 1, m; r) = (r + 1)n$, so the result holds when $k = 1$. We may thus assume that $k \geq 2$. Set $M = \binom{m}{r + k - 1}$. Let $\{A_i \mid 1 \leq i \leq (k - 1)M\}$ consist of $k - 1$ copies of each possible subset of $[m]$ with cardinality $r + k - 1$, and let $\{A_i \mid (k - 1)M + 1 \leq i \leq n\}$ be a set of any $n - (k - 1)M$ subsets of $[m]$ with cardinality $r + k$. Let $A$ be the $m \times n$ matrix defined by the union of these two sets. For an example, see Figure 4. It follows that

$$N(A) = (r + k - 1)(k - 1)M + (r + k)[n - (k - 1)M]$$

$$= (r + k)n - (k - 1)M.$$ 

Let $c \leq k$ and $J \subseteq [n]$ with $|J| = c$. If there is some $j' \in J$ with $j' \geq (k - 1)M + 1$, then $|\bigcup_{j \in J} A_j| \geq |A_{j'}| = r + k \geq r + c$. Suppose
then that \( J \subseteq \{1, \ldots, (k - 1)M\} \). Then, for each \( j \), \(|A_j| = r + k - 1\), so if \( c < k \), \( \bigcup_{j \in I} A_j \geq r + k - 1 \geq r + c \). Suppose then that \( c = k \).

Consider \( j_1, j_2 \in J \) with \( A_{j_1} \neq A_{j_2} \). Such a pair of sets must exist since there are only \( k - 1 \) copies of each subset of \([m]\) with cardinality \( r + k - 1 \).

Thus \( \bigcup_{j \in J} A_j \geq |A_{j_1} \cup A_{j_2}| \geq r + k - 1 + 1 = r + c \). Thus, in all cases, Theorem 5 (ii) is satisfied and \( A \) is an \( r \)-CBC.

Let \( B \) be an optimal \( r \)-CBC\((n, k, m)\). Without loss of generality, we can assume \( B \) is of type (i) or (ii) as outlined in Lemma 11. Suppose that \( B \) is of type (i), so that \( r + 1 \leq |B_j| \leq r + k - 1 \) for each \( j \leq n \). Since \( B \) is an \( r \)-CBC, every \((r + k - 1)\)-subset of \([m]\) contains at most \( k - 1 \) columns of \( B \), meaning that \( n \leq (k - 1)M \) and thus \( n = (k - 1)M \). Let \( C \) be the set of ordered pairs

\[
C = \{(B_k, I) \mid k \in [n], \ B_k \subseteq I \subseteq [m], \ |I| = r + k - 1 \}.
\]

Observe that for each \( k \in [n] \), since \(|B_k| \leq r + k - 1 \), there is at least one such ordered pair in \( C \) including \( B_k \). Therefore \(|C| \geq n \). However, for each \( I \subseteq [m] \) with \(|I| = r + k - 1 \), there are at most \( k - 1 \) columns of \( B \) which \( I \) contains.

So \(|C| \leq (k - 1)M \). Therefore \(|C| = (k - 1)M \) and hence each column of \( B \) is contained in exactly one subset of cardinality \( r + k - 1 \). So each column of \( B \) has cardinality \( r + k - 1 \), and it follows that \( N(B) = (r + k)n - (k - 1)M \).

Now, suppose that \( B \) is of type (ii), that is, \( r + k - 1 \leq |B_j| \leq r + k \) for each \( j \leq n \). Because \( B \) is an \( r \)-CBC, the maximal number of columns of \( B \) with cardinality \( r + k - 1 \) is \((k - 1)M \). Therefore

\[
N(B) \geq (r + k - 1)(k - 1)M + (r + k)[n - (k - 1)M] = (r + k)n - (k - 1)M.
\]

If \( r = 0 \), the previous result simplifies to \( N(n, k, m; 0) = kn - (k - 1) \binom{m}{k-1} \), which is Theorem 8 of Paterson, Stinson, and Wei [9].

4 Narrowing the gap

In the previous section, for a fixed \( m, k, \) and \( r \), we established \( N(n, k, m; r) \) if \( n \leq m \) and \( n \geq (k - 1) \binom{m}{r + k - 1} \). In this section, we address the “gap” between these results, and establish \( N(n, m, k; r) \) for values immediately below the latter interval.
Figure 4: The matrix $A$ constructed as in the proof of Theorem 12 when $m = 4, k = 2, r = 1$, and $n = 8 \geq (k - 1)\binom{m}{r+k-1}$.

We begin by identifying $N(n, 2, m; r)$ for all possible parameters $n$, $m$, and $r$. If $n \geq \binom{m}{r+1}$, then by Theorem 12 $N(n, 2, m; r) = (r+2)n - \binom{m}{r+1}$. Suppose that $n \leq \binom{m}{r+1}$. Let $A$ be an $m \times n$ matrix whose columns are distinct $(r+1)$-subsets of $[m]$. Then $A$ is an $r$-CBC$(n, m)$ and $N(A) = (r+1)n$. So, by Corollary 7 $N(n, 2, m; r) = (r+1)n$. Having completed the case $k = 2$ we assume that $k \geq 3$ for the remainder of this section.

**Theorem 13.** Let $r \geq 0$, $k \geq 3$, and $m \geq r + k$. If

$$(k - 1)\binom{m}{r+k-1} - (m - r - k + 1) \cdot F(k, m, r) \leq n \leq (k - 1)\binom{m}{r+k-1}$$

(1)

for an appropriate constant $F(k, m, r) \leq \frac{k-1}{r+k-2}\binom{m}{r+k-2}$, then

$$N(n, k, m; r) = (r + k - 1)n - \left\lfloor \frac{(k - 1)\binom{m}{r+k-1} - n}{m - r - k + 1} \right\rfloor.$$  

Before proving the theorem, we first prove some technical lemmas that are in line with the argument of Bujítás and Tuza [5]. After proving Theorem 13, we follow up with some concepts from design theory, then show that the existence of a design with certain parameters significantly reduces the complexity of the lower bound in Theorem 13.

**Lemma 14.** Let $A$ represent an $r$-CBC and for each $i > r$, let $\ell_i$ denote the number of columns of $A$ with cardinality $i$. Then

$$\sum_{i=r+1}^{r+k-1} \ell_i \binom{m-i}{r+k-1-i} \leq (k - 1)\binom{m}{r+k-1}. $$
Proof. We count in two ways the number $a$ of pairs $(R, A_j)$ where $R \subseteq [m]$, $|R| = r + k - 1$, $j \in [n]$, and $A_j \subseteq R$. By Theorem 5, every $(r + k - 1)$-subset $R$ of $[m]$ contains at most $k - 1$ columns of $A$. Therefore $a \leq (k - 1) \binom{m}{r + k - 1}$. Furthermore, for each column of $A$ with cardinality $i$, there are $\binom{m - i}{r + k - 1 - i}$ $(r + k - 1)$-subsets $R$ of $[m]$ containing it. Therefore

$$a = \sum_{\substack{R \subseteq [m] \\ |R| = r + k - 1}} |\{j : A_j \subseteq R\}| = \sum_{i=r+1}^{r+k-1} \ell_i \binom{m - i}{r + k - 1 - i}.$$ 

The result follows.

The next result identifies the maximum number of columns of cardinality $r + k - 1$ that can be appended to an $r$-CBC to obtain a larger $r$-CBC.

**Lemma 15.** Let $A$ represent an $r$-CBC$(n, k, m)$. Then $A$ can be extended to an $r$-CBC$(n + t, k, m)$ $A'$ with $t$ additional columns of cardinality $r + k - 1$ if and only if

$$t \leq (k - 1) \binom{m}{r + k - 1} - \sum_{i=r+1}^{r+k-1} \ell_i \binom{m - i}{r + k - 1 - i},$$

where for each $i > r$, $\ell_i$ denotes the number of columns in $A$ of cardinality $i$.

Proof. It is sufficient to show the result holds when $A$ has no column with cardinality greater than $r + k - 1$.

Suppose such an extension is possible and let $\ell'_i$, for $r + 1 \leq i \leq r + k - 1$, denote the number of columns in $A'$ of cardinality $i$. Observe that $\ell'_{r+k-1} = \ell_{r+k-1} + t$ and $\ell'_i = \ell_i$ for all other $i$. By Lemma 14, we have that

$$\sum_{i=r+1}^{r+k-1} \ell'_i \binom{m - i}{r + k - 1 - i} \leq (k - 1) \binom{m}{r + k - 1},$$

and therefore

$$t + \sum_{i=r+1}^{r+k-1} \ell_i \binom{m - i}{r + k - 1 - i} \leq (k - 1) \binom{m}{r + k - 1}.$$ 

Now suppose that (2) holds. Let $C$ be the set of all possible columns of cardinality $r + k - 1$. Let $C \in C$. Since there are at most $k - 1$ columns
of $A$ contained in $C \in \mathcal{C}$, we can define $t_C \geq 0$ so that there are $k - 1 - t_C$ columns of $A$ contained in $C$. Hence we can append up to $t_C$ copies of $C$ to $A$, and the resulting matrix will be an $r$-CBC. Then the lemma follows if we show that $t \leq \sum_{C \in \mathcal{C}} t_C$.

Recall that each column of $A$ with cardinality $i$ is contained in $\binom{m-i}{r+k-i-1}$ columns of $\mathcal{C}$. Therefore by an argument similar to the one above,

$$\sum_{i=r+1}^{r+k-1} \ell_i \left( \binom{m-i}{r+k-1-i} \right) = \sum_{C \in \mathcal{C}} (k - 1 - t_C) = (k - 1) \binom{m}{r+k-1} - \sum_{C \in \mathcal{C}} t_C.$$ 

The result follows. \hfill \square

We will require Lemma 1 of Bujitás and Tuza [3], which we restate here.

**Lemma 16** (Bujitás and Tuza [3]). For any three integers $i$, $p$, and $m$ satisfying $1 \leq i \leq p \leq m - 1$, the following inequality holds:

$$\left\lfloor \frac{\binom{m-i}{p-i} - 1}{m-p} \right\rfloor \geq p - i.$$

**Definition 17.** For parameters $k \geq 3$, $r \geq 0$, and $m \geq k + r$, let $F(k, m, r)$ be the largest $n$ such that an $r$-CBC($n, k, m$) exists in which each column has cardinality $r + k - 2$. Such an $r$-CBC($n, k, m$) and $F(k, m, r)$ are closely related to packing designs and packing numbers [8], which will be discussed at the end of the section.

**Lemma 18.** Letting $F(k, m, r)$ be as in Definition 17, we have

$$F(k, m, r) \leq \frac{k - 1}{r + k - 1} \binom{m}{r + k - 2}.$$ 

**Proof.** Let $P$ be an $r$-CBC($n, k, m$) with $n = F(m, k, r)$ and suppose the columns of $P$ have cardinality $r + k - 2$. We enumerate the set

$$\mathcal{C} = \{(C, I) \mid C \text{ is a column of } P, C \subseteq I \subseteq [m], |I| = r + k - 1\}$$

in two ways. For any given column $C$ of $P$, we have that $|C| = r + k - 2$ and so there are $m - (r + k - 2)$ possible subsets $I \subseteq [m]$ for which $|I| = r + k - 1$ and $C \subseteq I$. Therefore $|\mathcal{C}| = F(m, k, r) \cdot (m - r - k + 2)$. 

15
For any given $I \subseteq [m]$ with $|I| = r + k - 1$, since $P$ is an $r$-CBC$(n, k, m)$, there are at most $k - 1$ columns $C$ of $P$ for which $C \subset I$. Therefore $|C| \leq \binom{m}{r+k-1} \cdot (k-1)$. So $F(k, m, r) \cdot (m - (r + k - 2)) \leq \binom{m}{r+k-1} \cdot (k-1)$. The result follows.

We now prove Theorem 13. Setting $F(k, m, r)$ take the value from Definition 17.

Proof of Theorem 13. Suppose that $m = r + k$. Then $N(n, k, m; r) = mn - m(r - 1)$ by Theorem 9 and our formula gives

$$N = (m-1)n - \lfloor (k-1)m - n \rfloor = mn - km + m = mn - (m - r)m + m = mn - (m - r - 1)m.$$ 

So the formula holds. We now assume that $m > r + k$. Let $P$ be an $r$-CBC$(n, k, m)$ with $n = F(m, k, r)$ and suppose the columns of $P$ have cardinality $r + k - 2$. Let $\mathcal{C}$ be the set of all columns contained in $[m]$ with cardinality $r + k - 1$. Let

$$x = \left\lfloor \frac{(k-1)\binom{m}{r+k-1} - n}{m - r - k + 1} \right\rfloor.$$ 

It follows from the restriction on $n$ that $0 \leq x \leq F(k, m, r)$. Let $B$ be an $r$-CBC$(x, k, m)$ consisting of $x$ columns from $P$. By Lemma 15, $B$ can be extended by appending up to $(k-1)\binom{m}{r+k-1} - x(m - r - k + 2)$ columns from $\mathcal{C}$, and

$$(k-1)\binom{m}{r+k-1} - x(m - r - k + 2)$$

$$= (k-1)\binom{m}{r+k-1} - \left\lfloor \frac{(k-1)\binom{m}{r+k-1} - n}{m - r - k + 1} \right\rfloor (m - r - k + 2)$$

$$\geq (k-1)\binom{m}{r+k-1} - \left\lfloor \frac{(k-1)\binom{m}{r+k-1} - n}{m - r - k + 1} \right\rfloor + (k-1)\binom{m}{r+k-1} - n.$$
\[ (k-1) \left( \frac{m}{r+k-1} \right) - \left[ \frac{(k-1)(m) - n}{m-r-k+1} \right] - (k-1) \left( \frac{m}{r+k-1} \right) + n \]

\[ = n - x. \]

Let \( B' \) be an extension of \( B \) obtained by appending the appropriate \( n - x \) columns from \( C \). Then \( N(B') = (r + k - 1)n - x \) and hence \( N(n, k, m; r) \leq (r + k - 1)n - x \).

Let \( A \) be an optimal \( r \)-CBC(\( n, k, m \)). In what follows, we show that \( N(A) \geq (r + k - 1)n - x \). By Lemma 11, we may assume that \( A \) is type (i) or (ii). If \( A \) is type (ii), then \( \ell_i \geq r + k - 1 \) for each \( j \in [n] \), and therefore \( N(A) \geq (r + k - 1)n \).

Suppose now that \( A \) is type (i); that is \( r + 1 \leq \ell_i \leq r + k - 1 \) for each \( j \in [n] \). It is sufficient to show that

\[ N(A) = \sum_{i=r+1}^{r+k-1} \ell_i (r + k - 1)n - \sum_{i=r+1}^{r+k-1} (r + k - 1 - i)\ell_i \geq (r + k - 1)n - x, \]

where again \( \ell_i (r + 1 \leq i \leq r + k - 1) \) denotes the number of columns of \( A \) with cardinality \( i \). Hence it is sufficient to show that

\[ \sum_{i=r+1}^{r+k-1} (r + k - 1 - i)\ell_i \leq x. \] (3)

By Lemma 14, we have that

\[ \sum_{i=r+1}^{r+k-1} \ell_i \left( \frac{m - i}{r + k - 1 - i} \right) \leq (k-1) \left( \frac{m}{r+k-1} \right). \]

Since \( \ell_{r+k-1} = n - (\ell_{r+1} \cdots + \ell_{r+k-2}) \), we can substitute:

\[ n - (\ell_{r+1} \cdots + \ell_{r+k-2}) + \sum_{i=r+1}^{r+k-2} \ell_i \left( \frac{m - i}{r + k - 1 - i} \right) \leq (k-1) \left( \frac{m}{r+k-1} \right). \]

We can move the \( n \) over to the right side of the inequality, incorporate the \( \ell_i \)s in the summation, and divide both sides by \( m - r - k + 1 \) (which is at least 1):

\[ \sum_{i=r+1}^{r+k-2} \ell_i \left( \frac{m - i}{m - r - k + 1} \right) - \frac{1}{m - r - k + 1} \leq (k-1) \left( \frac{m}{r+k-1} \right) - \frac{n}{m - r - k + 1}. \]
Taking the floor on the left for each term will produce something smaller than the floor of the term on the right, and the result follows. \[\square\]

We now classify when equality is reached in Lemma 18, which in turn minimizes and simplifies the lower bound of the interval (1) in Theorem 13.

**Definition 19.** Let $X$ be a set and $\mathcal{B}$ be a family of subsets of $X$. Recall that the ordered pair $(X, \mathcal{B})$ is a $t$-$(v, k, \lambda)$ design if $|X| = v$, $|\mathcal{B}| = k$ for each $B \in \mathcal{B}$, and each $t$-subset of $X$ is contained in exactly $\lambda$ sets in $\mathcal{B}$. Moreover, in such a design, it is known that $|\mathcal{B}| = \binom{v}{t} \cdot \lambda / \binom{k}{t}$. See Khosrovshahi and Laue [7] for more information about $t$-designs.

**Definition 20.** Let $X$ be a set and $\mathcal{B}$ be a family of subsets of $X$. A $t$-$(v, k, \lambda)$ packing design if $|X| = v$, $|\mathcal{B}| = k$ for each $B \in \mathcal{B}$, and each $t$-subset of $X$ is contained in at most $\lambda$ sets in $\mathcal{B}$. The packing number $D_\lambda(v, k, t)$ is the number of blocks in a maximum $t$-$(v, k, \lambda)$ packing design. Therefore $D_\lambda(v, k, t) \leq \binom{v}{t} \cdot \lambda / \binom{k}{t}$ with equality when a $t$-$(v, k, \lambda)$ design exists. For more information on packing designs, see Mills and Mullin [8].

There is a connection between an $r$-CBC and the complement of a packing design with appropriate parameters satisfying an additional property.

**Construction 21.** Let $g = m - (r + k)$ and $\mathcal{D}$ be a maximal $(g + 1)$-$(m, g + 2, k - 1)$ packing design with vertex set $[m]$, block set $\mathcal{B}$, with the additional property (P):

(P): no block in $\mathcal{B}$ appears more than $k - 2$ times.

Observe that $|\mathcal{B}| \leq \frac{k-1}{r+k-1} \binom{m}{r+k-1}$. Let $A$ be a matrix whose columns are the complements of the sets in $\mathcal{B}$, and thus each of the $\frac{k-1}{r+k-1} \binom{m}{r+k-1}$ columns of $A$ has cardinality $r + k - 2$.

**Lemma 22.** The matrix $A$ in Construction 27 is an $r$-CBC.

**Proof.** Let $1 \leq c \leq k$ and $J \subseteq [n]$ have cardinality $c$. If $c \leq k - 2$, then $\left| \bigcup_{j \in J} A_j \right| \geq r + k - 2 \geq r + c$. If $c = k - 1$ then, by property (P), not all the sets $A_j$ ($j \in J$) are equal, so $\left| \bigcup_{j \in J} A_j \right| \geq r + k - 1$. Therefore, to prove $A$ is an $r$-CBC, we need to show that if $|J| = k$, then $\left| \bigcup_{j \in J} A_j \right| \geq r + k$.

Assume that $A$ is not an $r$-CBC. Then there exists $J \subseteq [n]$ of cardinality $k$ such that $\left| \bigcup_{j \in J} A_j \right| < r + k$. Let $B_j \in \mathcal{B}$ ($j \in J$) be the complements of
Then \(\left|\bigcap_{j \in J} B_j\right| > m - (r + k)\), so \(\left|\bigcap_{j \in J} B_j\right| \geq g + 1\). So there exists a \((g+1)\)-subset of \([m]\) which is contained in \(k\) blocks of a \((g+1)-(m, g+2, k-1)\) packing design, which is a contradiction. Therefore \(\bigcup_{j \in J} A_j\geq r + k\) for every \(k\)-subset \(J\) of \([m]\).

If the packing design used in Construction 21 is, in fact, a design, then we are able to restate Theorem 13 in a simplified form.

**Corollary 23.** Let \(r \geq 0, k \geq 3, m \geq r + k, g = m - (r + k)\), and suppose there exists a \((g+1)-(m, g+2, k-1)\) design with property (P). If

\[
\frac{k - 1}{r + k - 1} \binom{m}{r + k - 2} \leq n \leq (k - 1) \binom{m}{r + k - 1},
\]

then

\[
N(n, k, m; r) = (r + k - 1)n - \left\lfloor \frac{(k - 1) \binom{m}{r + k - 1} - n}{m - r - k + 1} \right\rfloor.
\]

Let \(r = 0\) and \(B\) be the set of all \((g + 2)\)-subsets of \([m]\). Then \(([m], B)\) is a \((g + 1)-(m, g + 2, k - 1)\) design with property (P) (in fact, each block is distinct). Therefore the hypotheses of Corollary 23 are satisfied. Thus Theorem 1 of Bujitás and Tuza [3] is a special case of Corollary 23.

## 5 Achieving the trivial minimum

We close with an inverse problem involving \(r\)-CBCs. By Corollary 14 an \(r\)-CBC\((n, k, m)\) must have weight at least \((r + 1)n\). Given parameters \(k, m,\) and \(r\), let \(n(k, m; r)\) be the maximum value of \(n\) such that \(N(n, k, m; r) = (r + 1)n\). If there are no such \(r\)-CBCs for all \(n \geq k\), we say \(n(k, m; r)\) does not exist. In this section, we construct \(r\)-CBCs whose weight is \((r + 1)n\) and identify \(n(k, m; r)\) in some special cases.

Observe that if \(k = 1\), then any matrix in which each column has cardinality \(r + 1\) is sufficient, so \(n(1, m; r) = \infty\). We now consider when \(k \geq 2\).

Suppose that \(r = 0\). Then a CBC\((n, k, m)\) has weight \(n\) if and only if \(m \geq n\) and each column is distinct. So \(n(k, m; 0)\) exists (and equals \(m\)) if and only if \(m \geq n\).

For larger \(r\), this family of \(r\)-CBCs can be quite complex. In what follows, we give a correspondence between 1-CBCs with weight \(2n\) and graphs on \(m\) vertices with appropriate girth.
Theorem 24. Let \( 2 \leq k < m \leq n \). Then \( A \) is a 1-CBC\((n, k, m)\) with weight \( 2n \) if and only if \( A \) is the incidence matrix for a simple graph with \( m \) vertices and girth at least \( k + 1 \).

Proof. Let \( G \) be a simple graph with girth at least \( k + 1 \) and let \( A \) be its incidence matrix. Let \( e_j \) denote the edge corresponding with column \( A_j \). Observe that \(|A_j| = 2\) for each \( j \in [n] \) and hence \( N(A) = 2n \).

Let \( J \) be a \( c \)-subset of \([n]\) with \( c \leq k \) and \( G' \) be the graph induced by the edge set \( \{e_j \mid j \in J\} \). Then \( G' \) has no cycles, and therefore \( G' \) is a forest. Hence \( G' \) has \( a + c \) incident vertices, where \( a \geq 1 \) is the number of connected components of \( G' \). Therefore \(|\bigcup_{j \in J} A_j| = a + c \geq 1 + c \). So \( A \) is a 1-CBC\((n, k, m)\).

Suppose \( A \) is a 1-CBC\((n, k, m)\) for which \( N(A) = 2n \). Then \(|A_j| = 2\) for each \( j \in [n] \), and therefore can be interpreted as the incidence matrix of a graph \( G \). If \( G \) is not simple, then \( G \) has two parallel edges, say which correspond to \( A_p \) and \( A_q \). Then \(|A_p \cup A_q| = 2 < 3\), which contradicts \( A \) being a 1-CBC.

Assume that \( G \) does not have girth at least \( k + 1 \). Then there exists a cycle in \( G \) with \( c \leq k \) edges. Let \( J \subseteq [n] \) index the edges of the cycle. So \(|\bigcup_{j \in J} A_j| = c < 1 + c\), a contradiction. Therefore the graph \( G \) has girth at least \( k + 1 \). \( \square \)

Corollary 25. For all \( m \geq 1 \), \( n(2, m; 1) = \binom{m}{2} \) and \( n(3, m; 1) = \lfloor m^2 / 4 \rfloor \).

Proof. A graph with \( m \) vertices and girth at least 3 is a simple graph, meaning a complete graph maximizes the number of edges. So \( n(2, m; 1) = \binom{m}{2} \).

A graph with \( m \) vertices and girth at least 4 is a triangle-free graph. By Turán’s theorem, a triangle-free graph on \( m \) vertices with a maximum number of edges is a complete bipartite graph whose parts are sizes \( \lfloor m/2 \rfloor \) and \( \lceil m/2 \rceil \). So \( n(3, m; 1) = \lfloor m/2 \rfloor \cdot \lceil m/2 \rceil = \lfloor m^2 / 4 \rfloor \). \( \square \)

The maximum number of edges in a graph on \( m \) vertices and girth \( k + 1 \geq 4 \) is known only for certain pairs of \( m \) and \( k \), but no other infinite families are known at this time \( \Pi \).
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