From superconducting fluctuations to the bosonic limit in the response functions above the critical temperature

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The density, current, and spin response functions are investigated above the critical temperature for a system of three-dimensional fermions interacting via an attractive short-range potential, as the strength of this potential is varied from weak to strong coupling. In the strong-coupling (bosonic) limit, we identify the dominant diagrammatic contributions for a “dilute” system of composite bosons which form as bound-fermion pairs, by giving appropriate prescriptions for mapping bosonic onto fermionic diagrams. We then extrapolate these contributions to the weak-coupling limit and compare them with the ordinary (Aslamazov-Larkin, Maki-Thompson, and density-of-states) terms occurring in the theory of superconducting fluctuations for a clean system above the critical temperature. Specifically, we show that in the strong-coupling limit, at the zeroth order in the diluteness parameter for the composite bosons, the Aslamazov-Larkin term represents formally the dominant contribution to the density and current response functions, while the Maki-Thompson and density-of-states terms are strongly suppressed. Corrections to the Aslamazov-Larkin term are further identified via the above mapping prescriptions at the next order in the diluteness parameter for the composite bosons, where the residual mutual interaction appears explicitly. The spin response function is also examined, and it is found to be correctly suppressed in the strong-coupling limit only when appropriate sets of diagrams are considered simultaneously, thus providing a criterion for grouping diagrammatic contributions to the response functions.

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I. INTRODUCTION

Response functions constitute an essential tool for connecting experimentally measurable quantities with the theoretical description of a condensed-matter system. Specifically, knowledge of density, current, spin, and heat response functions allows one to test the relevance of the degrees of freedom which are selected for an approximate description of a complex system. In particular, for superconductors the current response function plays a special role since the Meissner effect can be demonstrated by examining its behavior

Within the standard BCS (weak-coupling) theory, the transverse current response function below the critical temperature is represented as a particle-hole bubble in terms of normal and anomalous single-particle propagators. Above the critical temperature, the non-interacting Fermi gas expression is correspondingly obtained, with no sign of superconductivity being evidenced when approaching the transition from above.

In the weak-coupling limit, precursor effects of superconductivity above the critical temperature have been considered, by introducing pairing fluctuations in the Fermi gas due to the same attractive interaction which is responsible for the formation of the superconducting state below the critical temperature. In this way, the so-called Aslamazov-Larkin (AL), Maki-Thompson (MT), as well as the density-of-state (DOS) contributions have been evaluated and tested against experimental data, for superconducting samples of reduced dimensionality and for strongly anisotropic cuprate superconductors in the overdoped region.

No corresponding analysis has, however, been performed in the strong-coupling limit, where composite bosons form due to the strong fermionic attraction. Purpose of this paper is to provide this analysis, by setting up a formal classification of the diagrammatic structure for the response functions that holds specifically in the strong-coupling limit. In this way, one is able to cover the whole interaction range from weak to strong coupling, by merging the two alternative approaches (holding separately in the weak- and strong-coupling regimes) through the intermediate-coupling region. This merging appears altogether nontrivial, in that two different small parameters (namely, the Ginzburg and gas diluteness parameters) control the theory in weak and strong coupling.

The intermediate-coupling region might be specifically relevant for cuprate superconductors, for which the pairing is likely to be in an intermediate regime between overlapping Cooper pairs and non-overlapping composite bosons. In fact, the small value of the (superconducting) coherence length and the presence of a pseudogap above the critical temperature in the underdoped region suggested a crossover scenario, from a weak-coupling regime with Cooper pairs forming and condensing at the critical temperature within a BCS description, toward a strong-coupling regime whereby preformed (composite) bosons exist above the superconducting critical temper-
In this context, it appears relevant to study how the response functions evolve toward the strong-coupling limit, by specifically examining how the response of the original Fermi system can be interpreted in terms of the response of an effective Bose system. This evolution of the response functions rests on the property that the fluctuation propagator, which constitutes the building block of fluctuation theory in the weak-coupling limit above the superconducting critical temperature, acquires the form of the propagator for composite bosons in the strong-coupling limit.

The dominant diagrammatic contributions to the response functions in the strong-coupling (bosonic) limit will be selected by relying on the diluteness condition of the system (which is automatically satisfied in the strong-coupling limit), in a similar fashion to what was done in Ref. 16 for the selection of the fermionic self-energy. In that reference, the diluteness condition was exploited to determine the bosonic propagator entering the fermionic self-energy, where the bosonic propagator couples with a fermionic propagator. In this paper, we apply the diluteness condition to the physical response functions, for which a description in terms of bosons will naturally emerge in the strong-coupling limit.

Although the above procedure is a priori complementary to the selection of fluctuation diagrams in the weak-coupling limit, it yet results into the same diagrams for the current (and density) response functions as far as the dominant contribution (over and above the free fermion contribution) is concerned. Specifically, the AL diagram turns out to yield the dominant contribution to the current (and density) response functions both in the weak-coupling limit (where it represents the main fluctuation effects close to the critical temperature) and in the strong-coupling limit (where it corresponds to a free-boson response). Corrections to the AL diagram will also be identified in the strong-coupling regime at the next-to-leading order in the diluteness parameter, thus including interaction effects between composite bosons. Although the calculation of these corrections to the response functions remains still to be implemented numerically, their possible relevance to recent results reported in Ref. 17 will be considered below.

Besides providing a detailed analysis of the current (and density) response functions, we will also examine the spin response function. We shall verify that the diagrams selected in the strong-coupling limit for the current (and density) response functions, give an identically vanishing contribution to the spin response function, since they correspond to spinless (composite) bosons. For obtaining a non vanishing contribution in the interesting intermediate-coupling region, therefore, the diagram for the spin response function have to be selected in the weak-coupling region, paying, however, attention that their contribution vanishes in the strong-coupling limit. To this end, it will be shown how certain diagrammatic contributions have to be included simultaneously in an appropriate way.

We shall specifically consider a Fermi system with an attractive (point-contact) interaction in a three-dimensional continuum and above the superconducting critical temperature. No lattice or impurities effects will be taken into account. Consideration of the broken-symmetry case below the critical temperature is postponed to future work.

The plan of the paper is as follows. Section II discusses the current response function at the leading and next-to-leading order in the diluteness parameter for composite bosons. Section III considers the density and spin response functions. Section IV gives our conclusions.

II. CURRENT RESPONSE FUNCTION

In this Section, we identify the dominant diagrammatic contributions to the current response function for a system of fermions with an attractive interparticle interaction in the strong-coupling limit. The leading contribution turns out to coincide formally with the AL diagram, occurring in the standard theory of superconducting fluctuations above the critical temperature. Next-to-leading diagrams in the bosonic diluteness parameter are also considered, to include the effects of the residual interaction between the composite bosons in the strong-coupling limit. Additional diagrams (such as the MT and DOS contributions), which are usually considered in superconducting fluctuation theory, are further shown to be irrelevant in the strong-coupling limit.

The systematic procedure for selecting the contributions to the response functions which are dominant in the strong-coupling limit rests on certain integrals (that contain products of fermionic single-particle Green’s functions) acquiring a particularly simple form in the strong-coupling limit, and on the standard classification of bosonic diagrams in the dilute limit. In this way, the contributions to the response functions are organized in powers of the diluteness parameter as well as of the (inverse of the) fermionic chemical potential. The property of the fermionic chemical potential of being the largest energy scale in the strong-coupling limit, in fact, considerably simplifies dealing with this limit.

The identification of the dominant diagrammatic contributions to the response functions for a system of fermions in the strong-coupling limit could proceed in two ways. Either, one may consider all possible fermionic diagrams and estimate their relative contributions by relying on the simplifying features mentioned above; or, one may start directly from the bosonic diagrams for a system of true bosons and construct the corresponding diagrams for composite bosons, where remnants of the underlying fermionic degrees of freedom explicitly appear. We shall develop below the latter approach, which has by construction the advantage of identifying the important contributions to the response functions in the
strong-coupling limit.
Detailed knowledge of the fermionic attractive interaction is not required for studying the evolution from weak to strong coupling. One may accordingly consider a “contact” potential $v_0 \delta(\mathbf{r})$, where $v_0$ is a negative constant. A suitable regularization is required in this case to remove divergences in the diagrammatic structure. In three dimensions, it is common practice to introduce the fermionic scattering length $a_F$ defined via (we set $\hbar$ and Boltzmann’s constant equal to unity throughout)

$$
\frac{m}{4\pi a_F} = \frac{1}{v_0} + \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{m}{k^2},
$$

where $\mathbf{k}$ is a wave vector and $m$ the fermionic mass. The ultraviolet divergence on the right-hand side of Eq. (2.1) is compensated by letting $v_0 \to 0^-$ in a suitable way, keeping $a_F$ finite. This is achieved by introducing an ultraviolet cutoff $k_0$ and choosing $v_0$ such that

$$
v_0 = -\frac{2\pi^2}{mk_0} - \frac{\pi^3}{ma_Fk_0^3},
$$

with $k_0 \to \infty$ eventually. The evolution from weak to strong coupling can thus be tuned by varying the scattering length $a_F$, which is negative in the weak-coupling regime (where a bound-state has not yet appeared in the associated two-body problem) and positive in the strong-coupling (bosonic) regime (where $a_F$ coincides with the bound-state radius).

It was discussed in Ref. [13] that the explicit form (2.2) for $v_0$ considerably simplifies the structure of the associated many-body perturbation theory. It was shown there that the effects of the interaction survive only in the particle-particle ladder depicted in Fig. 1(a), while connections among different ladders (as well as other links required to form the fermionic self-energy) are provided by the bare single-particle fermionic Green’s functions. In the context of the correlation functions considered in the present paper, a current (or density or spin) vertex made by fermionic single-particle Green’s functions (cf. Fig. 1(b)) is further required to connect the external (electromagnetic) coupling with the structure of the fermionic two-particle Green’s function.

The general expression of the particle-particle ladder of Fig. 1(a) for any temperature and value of $a_F$ reads

$$
\Gamma_0(q) = -\left\{ \frac{m}{4\pi a_F} + \int \frac{d\mathbf{k}}{(2\pi)^3} \right\}^{-1}
$$

with the four-vector notation $q \equiv (q, \omega_q)$, where $\mathbf{q}$ is a wave vector, $\omega_q = 2
\nu \beta^{-1}$ ($\nu$ integer) a bosonic Matsubara frequency, $\beta = 1/T$ the inverse temperature, and $\xi(\mathbf{k}) = k^2/(2m) - \mu$ ($\mu$ being the fermionic chemical potential). This expression acquires a particularly simple form in the strong- and weak-coupling limits.

In the strong-coupling limit, $\mu$ approaches the value $-\epsilon_0/2$ where $\epsilon_0 = (ma_F^2)^{-1}$ is the binding energy of the associated two-body problem. As $\epsilon_0$ increases without bound in strong coupling, at any finite temperature we may consider the limit $\beta \mu \to -\infty$ in Eq. (2.3), thus obtaining the polar structure

$$
\Gamma_0(q) \approx -\frac{8\pi}{m^2 a_F} \frac{1}{\omega_q} - \frac{1}{\omega_q} + \mu_B
$$

where $\mu_B = 2\mu + \epsilon_0$. Apart from the residue being different from unity, this expression has the form of a free propagator for (composite) bosons with mass $m_B = 2m$ and chemical potential $\mu_B$. Note that Eq. (2.4) holds provided $|\omega_q| \ll \epsilon_0$ and $q^2/(4m) \ll \epsilon_0$, which can be satisfied for all relevant values of $\omega_q$ and $q$ when $\epsilon_0$ is sufficiently large.

In the weak-coupling limit, on the other hand, the chemical potential is (slightly) smaller than the Fermi energy $\epsilon_F = k_F^2/(2m)$ ($k_F$ being the Fermi wave vector) for temperatures much smaller than $\epsilon_F$ itself. In this case, the particle-particle ladder (2.3) acquires the form characteristic of superconducting fluctuation theory

$$
\Gamma_0(q) \approx \frac{1}{N_0} \frac{1}{T - T_c} + \frac{1}{T - T_c} + \frac{1}{2m} + \frac{1}{|\omega_q|}
$$

where $N_0$ is the free-fermion density of states at the Fermi level (per spin component), $(T - T_c) \ll T_c$ where $T_c$ is the BCS critical temperature, $\gamma = \pi/(8T_c)$, and
η = \frac{7 \zeta(3)}{48 \pi^2} \left( \frac{k_F}{m T_c} \right)^2 \tag{2.6}

in three dimensions (\zeta(3) \approx 1.202 being the Riemann zeta function of argument 3).

On physical grounds, one expects the response functions in the strong-coupling limit of the original Fermi system to be expressed entirely in terms of composite-boson structures, namely, bosonic propagators and vertices. As anticipated in the Introduction, the evolution of the response functions from strong to weak coupling discussed in the present paper rests on the fact that the particle-particle ladder (which in the strong-coupling limit has the form \[2.3\]) of a composite-boson propagator becomes itself the building block of fluctuation theory in the weak-coupling limit [cf. Eq. (2.5)].

Before identifying the relevant bosonic diagrams for the current response function, it is useful to establish a procedure to map a given bosonic diagram onto a corresponding set of fermionic diagrams. To this end, we proceed in a heuristic way and formulate the following prescriptions: (i) Remove from the given bosonic diagram the two outer vertices representing the bosonic coupling to the external field, thus obtaining a bosonic diagram “open” at its ends; (ii) Replace the bare bosonic propagators by the particle-particle ladders given by Eq. \[2.3\]; (iii) Connect the ensuing (fermionic) diagram to the fermionic vertex of Fig. 1(b) representing the fermionic coupling to the external field; (iv) Connect eventually the remaining dangling ends of the particle-particle ladders among themselves, in accordance with their spin structure.

In this way, besides the fermionic diagrams which correctly reproduce the value of the original bosonic diagram in the strong-coupling limit, additional fermionic diagrams may result which do not have a bosonic analogue in the strong-coupling limit and whose value is accordingly suppressed in this limit. These additional diagrams will consistently be dismissed when mapping the original bosonic diagrams onto the associated fermionic diagrams.

### A. Leading diagrams

For a system of noninteracting bosons, the current response function is depicted diagrammatically in Fig. 2(a). This diagram represents the leading contribution to the current response function also for a system of bosons interacting via a (repulsive) finite-range potential at sufficiently low density.

With the prescriptions listed above, the fermionic diagrams of Figs. 2(b) and 2(c) are generated from the bosonic diagram of Fig. 2(a), with a degeneracy factor of 2 each, due to the fermionic spin multiplicity. An additional diagram, which corresponds to a self-energy decoration of both bare fermionic propagators in the fermionic particle-hole bubble, is also generated according to the above prescriptions. Since this diagram does not have a bosonic analogue in the strong-coupling limit, it will not be considered in the following according to the above discussion.) Although the two diagrams 2(b) and 2(c) are topologically not equivalent, their expressions coincide for particle-particle ladders corresponding to a point-contact potential. We thus consider only one of these diagrams (say, diagram 2(b)) with a multiplicity factor of 4.

![Diagram](image-url)
1)πβ^{-1} (n integer) being a fermionic Matsubara frequency. Symmetry arguments show that \( J(q, Q) \) is directed along \( (2q + Q) \), allowing us to set

\[
J(q, Q) = \frac{(2q + Q)}{2m} C(q, Q).
\]

The (scalar) factor \( C(q, Q) \) can be readily evaluated in the strong-coupling limit for vanishing external four-vector \( (Q = 0) \). In this limit, the Fermi functions originating from the sum over \( \omega_n \) in Eq. (2.7) vanish exponentially like \( \exp(-\beta|\mu|) \), yielding

\[
C(q, Q = 0) \approx -\frac{m^{3/2}}{16\pi} \frac{1}{\sqrt{2|\mu|}}
\]

at the leading order in \( |\omega_n/\mu| \) and \( q^2/(2m|\mu|) \). With these approximations, and using the expression (2.3) for the particle-particle ladder in the strong-coupling limit, the value of the diagram of Fig. 2(b) for \( Q = 0 \) becomes (the “static” limit with \( \Omega_v = 0 \) and \( Q \to 0 \) is implied):

\[
\chi_f(Q = 0) \cong -4 \frac{1}{m^2} \frac{m^3}{(16\pi)^2} \frac{1}{2|\mu|} \left( \frac{8\pi}{m^2a_F} \right)^2
\times \frac{1}{\beta} \sum \omega_v \int \frac{dq}{(2\pi)^3} \frac{q q}{(i\omega_v - q^2/4m + \mu_B)^2}
\]

where the overall minus sign complies with the definition of the current response function \( \chi_j \), the factor of 4 represents the degeneracy of the diagram, and the remaining factors stem from Eqs. (2.9) and (2.4), in the order. Apart from the degeneracy factor of 4, expression (2.10) coincides with the \( Q = 0 \) limit of the current response function for a system of (composite) bosons with mass \( m_B = 2m \) and chemical potential \( \mu_B \), when for \( |\mu| \) one uses the value \( (2ma_F^2)^{-1} \) which holds in the strong-coupling limit. This response function then equals \( -n_B/m_B \), where the bosonic density \( n_B = n/2 \) is half the original fermionic density \( n \). The degeneracy factor of 4 in Eq. (2.10) restores eventually the correct value \( -n/m \) for the diagonal component of the fermionic current response function, in accordance with the f-sum rule \( \widetilde{\sigma} \). This is an explicit check that the heuristic prescriptions formulated above lead indeed to a meaningful mapping between bosonic and fermionic diagrams.

It is worth noting that, when the expression (2.3) for the particle-particle ladder (valid in the weak-coupling limit close to \( \Sigma \) ) is used in diagram 2(b) and the expression (2.8) is also retained, one recovers the standard Aslamazov-Larkin contribution to the current response function \( \chi_j \) which represents the leading fluctuation contribution in the weak-coupling limit. This is a nontrivial result because the weak- and strong-coupling regimes admit entirely different classifications schemes based, respectively, on the Ginzburg and diluteness (gas) parameters.

![Diagram](image-url)
Green’s function. Diagrams 2(b) and 2(c) then result by connecting, respectively, diagrams 3(b) and 3(c) with the fermionic vertex of Fig. 1(b), while connecting diagram 2(d) with the fermionic vertex of Fig. 1(b) yields the Maki-Thompson diagram to be considered in Section II.C.

B. Next-to-leading diagrams

The leading contribution to the current response function considered in Section II.A corresponds to non-interacting composite bosons. The residual interaction between composite bosons should, however, play an important role in the not-too-extreme strong-coupling regime. For this reason, it is relevant to introduce the effects of the interaction between composite bosons in the physical response functions. This leads us to search for nontrivial corrections to the AL diagram, by examining diagrams of higher-order in the bosonic diluteness parameter $n_B^{1/3} a_B$, where $a_B$ is the scattering length associated with the residual interaction between composite bosons.

A bosonic diagram, which is next-to-leading with respect to diagram 2(a), is depicted in Fig. 4(a), where the dark square in the middle represents a (symmetrized) bosonic interaction. The presence of an additional bosonic cycle in diagram 4(a) with respect to diagram 2(a) accounts, in fact, for an additional power in the bosonic density. A further bosonic diagram of the same order in the density can be obtained from diagram 2(a), by dressing either one of the two bosonic propagators with a low-density self-energy, as in the theory of the interacting dilute Bose system. The physical interplay of these two diagrams in the context of the density and spin response functions will be addressed in Section IV.

The bosonic diagram of Fig. 4(a) can be mapped into a corresponding set of diagrams for the fermionic response function(s), according to the rules developed in Ref. 16 for the interaction vertex and to the heuristic prescriptions stated above. In this way, one ends up with the two fermionic diagrams of Figs. 4(b) and 4(c), with a degeneracy factor of 8 and 4, in the order, having also taken into account that expressions of topologically not equivalent diagrams may coincide for a fermionic point-contact potential. We will verify below that, while diagram 4(b) has a meaningful strong-coupling limit in terms of composite-boson propagators, diagram 4(c) lacks a bosonic representation and yields consistently a subleading contribution in this limit. For these reasons, one may retain diagram 4(b) and disregard diagram 4(c) to follow the evolution from strong to weak coupling.

To verify that to diagram 4(b) there corresponds a meaningful strong-coupling limit, we evaluate the central part of this diagram for $Q = 0$ and obtain:

\[
\frac{1}{\beta} \sum_{\omega_n} \int \frac{d\omega''}{(2\pi)^3} \mathcal{G}^0(-\omega'') \mathcal{G}^0(\omega'' + q')
\times \mathcal{G}^0(-\omega' + q - q') \mathcal{G}^0(\omega' + q') \approx \frac{(ma_F)^3}{16\pi} \quad (2.11)
\]

where use has been made of the relation $2|\mu| \approx \epsilon_0 = (ma_F)^{-1}$ that holds in this limit. The diagram 4(b) thus contains the factors

\[
-\left( -\frac{m^2 a_F}{8\pi} \right)^2 \left( -\frac{8\pi}{m^2 a_F} \right)^4 \frac{(ma_F)^3}{16\pi} = -\frac{4\pi a_F}{m}
\quad (2.12)
\]

which arise, respectively, from the current vertex (cf. Eq. (2.9)), from the residue of the particle-particle ladder (cf. Eq. (2.4)), and from the expression (2.11), while the overall minus sign originates from the three fermionic
loops. In this way, the strength $v(0) = 4\pi a_P/m$ of the residual interaction between composite bosons discussed in Ref. 13 is correctly reconstructed, and diagram 4(b) is proved to give a faithful representation of the bosonic diagram of Fig. 4(a).

Further, the ratio of diagram 4(c) to diagram 4(b) can be estimated to be of the order

$$\frac{m}{a_P} \left( \frac{na_P^3}{\partial n/\partial \mu} \right)^2,$$

which is indeed much smaller than unity in the low-density limit ($n_B^{1/3} a_B \ll 1$), provided the compressibility $(\partial n/\partial \mu)$ does not vanish.

It is again interesting to mention that diagram 4(b) (with a degeneracy factor of 8) can alternatively be obtained in fermionic language by: Considering the fermionic effective two-particle interaction depicted in Figs. 3(b) and 3(c) to act twice in the two-fermion Green’s function; Connecting the ensuing four diagrams with the fermionic vertex of Fig. 1(b); Recognizing the equivalence of these four diagrams; Summing eventually over the spin components. Diagram 4(c), on the other hand is not reducible in the (fermionic) two-particle channel and corresponds to a choice of the fermionic effective two-particle interaction different from 3(b) and 3(c).

In this way, we have identified the next-to-leading contributions to the dominant (AL) diagram, which take into account correlation effects among composite bosons in the strong-coupling limit.

### C. Subleading diagrams

We finally consider additional diagrams (besides diagram 4(c) considered in Section II.B) which are subleading in the strong-coupling limit. A noticeable example is the Maki-Thompson diagram, which is obtained by connecting the effective two-particle interaction of Fig. 3(d) with the external coupling of Fig. 1(b). This diagram has been extensively studied in the weak-coupling limit in the context of the theory of superconducting fluctuations.

Since this diagram contains only one particle-particle ladder, it is expected to have no analogue in bosonic language (at least for the normal phase) and, consequently, not to contribute to the response functions in the strong-coupling limit. Upon evaluating the MT diagram for the current response function in the strong-coupling limit at $Q = 0$, however, one obtains the finite value $n/m$ (including spin multiplicity).

This apparent contradiction can be overcome by considering also the density-of-state (DOS) diagram depicted in Fig. 5(a) (with a multiplicity factor of 4), obtained by making the fermionic self-energy insertion of Fig. 5(b) into the bare fermionic particle-hole bubble. In the strong-coupling limit, diagram 5(a) gives, in fact, the contribution $-n/m$ to the $Q = 0$ current response function, thus cancelling exactly the contribution of the MT diagram.

![Diagram 5](image)

**FIG. 5.** (a) DOS diagram for the response functions; (b) Self-energy diagram for a dilute Fermi gas.

This example suggests that diagrams for the response functions may need to be grouped into suitable sets, in order to get a meaningful strong-coupling limit. The grouping procedure appears to be especially relevant for the spin response function, that ought to vanish in the strong-coupling limit for spinless composite bosons, as discussed in the next Section.

### III. DENSITY AND SPIN RESPONSE FUNCTIONS

In this Section, we complement the analysis of the strong-coupling limit by analyzing the density and spin response functions. We begin by considering the standard AL, MT, and DOS diagrams of the theory of superconducting fluctuations. We next consider a specific example to show that a whole set of diagrams needs be associated with a given subleading diagram, for the spin response function to be exponentially suppressed in the strong-coupling limit, as required on physical grounds for spinless composite bosons.

The contribution to the density response function from the AL diagram contains two (scalar) factors of the type...
\[ D(q, Q) = \frac{1}{\beta} \sum_{\omega_n} \int \frac{dk}{(2\pi)^3} \, g^0(-k) \, g^0(k + q) \times g^0(k + q + Q), \] (3.1)

which can be readily evaluated in the strong-coupling limit for \( Q = 0 \), to give

\[ D(q, Q = 0) \approx -\frac{m^2 a_F}{8\pi}. \] (3.2)

This factor thus cancels the residue of the particle-particle ladder in the strong-coupling limit, yielding for the density response function the following expression:

\[ \chi_n(Q) \equiv -\frac{4}{\beta} \sum_{\omega_\nu} \int \frac{dq}{(2\pi)^3} \, \frac{1}{\nu \omega_\nu + \frac{q^2}{4m} + \mu_B} \times \frac{1}{\nu \omega_\nu + i\Omega_\nu - \frac{(\nu + Q)^2}{4m} + \mu_B}. \] (3.3)

Here, the minus sign is due to the definition of the density response function and the factor of 4 accounts for the degeneracy of the diagram. In the “static” (\( \Omega_\nu = 0 \) and \( Q \to 0 \)) and “dynamic” (\( Q = 0 \) and \( \Omega_\nu \to 0 \)) limits this expression correctly produces the values \(-4\partial n_B/\partial \mu_B = -\partial n/\partial \mu = 0\), in the order.

Concerning the spin correlation function, the contributions to \( \chi_{zz} \) from the AL diagrams 2(b) and 2(c) cancel each other identically for all coupling strengths (these diagrams, on the other hand, do not contribute to \( \chi_{xx} \) and \( \chi_{yy} \) owing to their spin structure). This is consistent with our previous result that, in the strong-coupling limit, the AL diagram gives an appropriate description of a system of composite bosons. One may further verify that the spin response function vanishes identically also for the corrections 4(b) to the AL diagram, a result which is also expected since this diagram was selected in the strong-coupling limit.

The contributions to the \( (Q = 0) \) density response function from the MT diagram 3(d) and the DOS diagram 5(a) do not cancel each other in the strong-coupling limit, contrary to the case of the current response function treated in Section II.C. Rather, each of these diagrams gives the same finite contribution \(-ma^2_F n_B = -n/(4\mu_\nu)\), which however vanishes as \( |\mu_\nu| \) increases in the strong-coupling limit. In the strong-coupling limit, both MT and DOS diagrams are thus irrelevant also for the density response function.

The contributions to the spin correlation function \( \chi_{zz} \) from the MT and DOS diagrams cancel instead each other for \( Q = 0 \) in the strong-coupling limit, since the MT contribution acquires an extra minus sign with respect to the DOS contribution. In particular, the spin response function, obtained by considering these two diagrams simultaneously, vanishes exponentially like \( \exp(-\beta |\mu|) \) when approaching the strong-coupling limit, due to the behavior of the Fermi functions in this limit. This is precisely what is expected on physical grounds, since a non-vanishing contribution to the spin response for spin-less composite bosons results only when the temperature is comparable with their binding energy and the composite bosons break apart. In this context, it is interesting to mention that the progressive vanishing of the spin susceptibility upon approaching the strong-coupling limit has been confirmed by Monte Carlo data for the negative-\( U \) Hubbard model, even though the predicted exponential behavior cannot be fully confirmed from the limited set of Monte Carlo data.

The above examples concerning the spin response for the AL, MT, and DOS diagrams (plus the correction 4(b) to the AL diagram) suggest that: (i) Diagrams selected in the strong-coupling regime according to the diluteness condition by considering the current response function, cannot be used to describe the spin response function, since they would yield a vanishing spin response function for all couplings. This implies that additional diagrams have unavoidably to be considered for a full description of the weak-coupling regime; (ii) These additional diagrams introduced in the weak-coupling regime (for instance, by counting powers of the Ginzburg parameter as in the theory of superconducting fluctuations) are necessarily subleading in the strong-coupling limit, as far as the current and density response are concerned. However, there is a priori no guarantee that they also result in an exponentially vanishing spin response function in the strong-coupling limit, as required on physical grounds. To make sure that this happens, suitable sets of diagrams need to be grouped in an appropriate way.

As a specific example, let us consider diagram 4(c), which we concluded in Section II.B to be subleading as far as the current response is concerned. This diagram alone yields a contribution to the spin response function which is not exponentially vanishing in the strong-coupling limit. Additional diagrams have thus to be associated with diagram 4(c), to obtain the correct exponential behavior of the spin response function in the strong-coupling limit. To this end, we consider the two contributions to the thermodynamic potential depicted schematically in Figs. 6(a) and 6(b) and perform all possible \( (Q = 0) \) magnetic-field insertions in the fermionic single-particle propagators, as to get the “static” spin susceptibility (no additional contributions are obtained by making magnetic-field insertions inside the particle-particle ladder in the strong-coupling limit). In this way, two sets of six diagrams each result, which include, by construction, diagram 4(c) (counted twice, due to the equivalence of two diagrams for the case of a point-contact potential) plus decorations of the AL, MT, and DOS diagrams. In the strong-coupling limit (when all terms proportional to the Fermi functions are exponentially suppressed), it can be indeed be shown that the contributions to the spin response function \( \chi_{zz} \) from the six diagrams obtained from Fig. 6(a) (as well from the six diagrams obtained from Fig. 6(b)) add up to zero.

To summarize, we have argued that diagrams which have a meaningful strong-coupling limit as far as the cur-
rent and density response are concerned, yield an \textit{identically vanishing} contribution to the spin response function. Other diagrams that do not have a meaningful strong-coupling limit, on the other hand, give contributions to the spin response function in the strong-coupling limit which instead vanish \textit{exponentially} like $\exp(-\beta|\mu|)$ in the correct way, provided these diagrams are grouped into suitable sets, as shown explicitly by the examples considered above.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{(a)-(b) Diagrams for the thermodynamic potential, from which the contributions of Figs. 4(b) and 4(c) to the static spin susceptibilities can be derived.}
\end{figure}

### IV. DISCUSSION AND CONCLUDING REMARKS

In this paper, we have examined the evolution from weak to strong coupling of the response functions for a three dimensional (clean) Fermi system with an attractive interaction above its critical temperature. While in the weak-coupling limit the standard analysis of superconducting fluctuations applies, we have shown that in the strong-coupling limit the original fermionic response functions become identical to the response functions of a system of composite bosons. We have, in fact, verified that only those fermionic diagrams, to which there corresponds a meaningful representation in terms of composite bosons, contribute to the strong-coupling limit. The AL, MT, and DOS diagrams of superconducting fluctuation theory have been analyzed among others. We have also argued that the analysis of the spin response function may serve as a constraint to select sets of diagrams for the current and density response functions, which are relevant for weak coupling but are suppressed for strong coupling.

It is evident from our analysis that many diagrams contributing to the weak-coupling limit are suppressed in the strong-coupling limit. Consistently, by selecting the relevant diagrams for the response functions starting only from the strong-coupling limit, one might miss important contributions to the weak-coupling limit. For this reason, our analysis in the strong-coupling limit must be supplemented by the standard criterion of superconducting fluctuation theory for selecting suitable sets of diagrams in the weak-coupling limit. This is especially true for the spin response function, which vanishes for a system of spinless bosons: Extrapolating to the weak-coupling limit only diagrams which contribute in the bosonic limit to the current and density response functions, would result into a vanishing spin response function for all coupling strengths.

Controlling the two (weak- and strong-coupling) limits separately may prove especially important for describing the intermediate (crossover) region, for which no controlled theory can be specifically formulated. One reasonable strategy to approach the crossover region is then to \textit{interpolate} between theories which are controlled in the two (weak- and strong-coupling) limits, which can be done by including all dominant diagrams in either one of the two limits and then evaluating them over the whole coupling range. This contrasts somewhat with what was found in Ref. [16] for the fermionic self-energy, for which a single approximation selected in the strong-coupling regime proved also sufficient to describe the weak-coupling region. For the response functions, at the leading order one may include the AL diagram (which is dominant both in the strong- and weak-coupling limit) plus the MT and DOS diagrams (which are relevant to the weak-coupling limit but are strongly suppressed in the strong-coupling limit). At the next-to-leading order, the effect of the residual interaction between composite bosons can be included considering the corrections to the AL diagrams discussed in Section II.B.

In this context, it is interesting to comment on the recent results reported in Ref. [17] regarding the temperature dependence of the density and spin susceptibilities for a two-dimensional negative-U Hubbard model, calculated via the AL, MT and DOS diagrams, and then compared with available Monte Carlo results for $U = -4t$ ($t$ being the nearest-neighbor hopping). These authors find a remarkable agreement between their calculation and the Monte Carlo data for the spin susceptibility, \textit{provided} the mass term in the particle-particle ladder (2.3) is replaced by a mass term with the characteristic tem-
perature dependence of the Kosterlitz-Thouless theory (this replacement should amount to inserting self-energy corrections in the bosonic propagators of the AL diagram). For the density susceptibility, however, this replacement alone proved not sufficient to reproduce the Monte Carlo data. The discussion presented in Section II.B indeed suggests that modifications of the AL diagram obtained by considering bosonic self-energy corrections to the particle-particle ladder should also be accompanied by the inclusion of an additional diagram (namely, diagram 4(b) for the density response function), which in the strong-coupling limit accounts for the residual bosonic interaction at the same order in the diluteness parameter. Numerical calculations including this additional diagram have not yet been performed.

In this paper, we have considered the response functions in the normal phase above the critical temperature. It would certainly be interesting to extend this analysis below the superconducting critical temperature and study the continuous evolution of the response functions from the weak-coupling limit of (BCS) superconductivity to the strong-coupling limit where Bose-Einstein condensation takes place. In this case, a description in terms of Bogoliubov quasi-particles may be appropriate for a dilute system of composite bosons (at least close to zero temperature), with the superfluid density being affected at finite temperature by sound modes in the strong-coupling limit and by pair-breaking effects in the weak-coupling limit. Which of these two effects dominate in the intermediate (crossover) region is a challenging question, which can be addressed only by numerical calculations of a suitable set of diagrams. Work along these lines is in progress.

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