A field theory in Randers-Finsler spacetime

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Abstract – Finsler geometry is a natural arena to investigate the physics of spacetimes with local Lorentz violation. The directional dependence of the Finsler metric provides a way to encode the Lorentz-violating effects into the geometric structure of the spacetime. Here, a classical field theory is proposed in a special Finsler geometry, the so-called Randers-Finsler spacetime, where the Lorentz violation is produced by a background vector field. By promoting the Randers-Finsler metric to a differential operator, a Finsler-invariant action for the scalar, gauge and fermions are proposed. The theory contains nonlocal terms, as in the Very Special Relativity based theories. By expanding the Lagrangian, minimal and nonminimal Standard Model Extension terms arise, revealing a perturbative Lorentz violation. For a CPT-even term, the Carroll-Field-Jackiw and derivative extensions are obtained.

Introduction. – At Planck scale, several theories including string theory [1], noncommutative geometry [2], Horava gravity [3], loop quantum gravity [4], doubly special relativity [5] and the very special relativity [6,7] suggest that some low-energy symmetries, as the CPT and Lorentz symmetry, may be violated. A theoretical framework to study the effects of reminiscent Lorentz-CPT breaking at intermediate scales is the so-called Standard Model Extension (SME) [8]. Lorentz-violating effects studies have been carried out in a broad class of phenomena, ranging from muons, neutrinos, photons, hydrogen atom, among others. For a comprehensive review of Lorentz-violation tests, see ref. [9].

The violation of the local Lorentz violation on gravity requires a modification on the local geometry of the spacetime. In the context of the SME, local Lorentz-violating coefficients arise due to vacuum expectation values (VEV) of self-interacting tensor fields in an Einstein-Cartan theory [10]. Another approach suggests that the modification of the dispersion relation of particles, an ubiquitous feature of Lorentz-violating theories, arises due to an anisotropic Finsler geometry.

In the Finsler geometry, the spacetime interval is measured with a general function, called Finsler function [11]. The Riemann geometry is a special case for which the Finsler function is quadratic on the vector components. The modified dispersion relations (MDR) are the starting point to propose a local Lorentz-violating Finsler spacetime [12–15]. For instance, the DSR [16] and VSR [17], whose curved extension is the so-called Bogoslovsky spacetime [18]. In the SME set up, the point particle classical Lagrangians and mass shell can be described by a class of Finsler functions, called SME-based Finsler geometries [19]. Each Lorentz-violating coefficient provides a different SME-based Finsler structure, as the b-space [19,20], bipartite [21,22] and other spacetimes [23–26].

A special Finsler structure is given by the Rander spacetime, where the anisotropy is driven by a background vector [27]. The Randers interval has the usual quadratic Lorentz interval added with a perturbative linear projection of the 4-velocity into the background vector [27]. In the context of the SME, the Randers spacetime can be regarded as a classical description of a fermion subjected to a CPT-odd Lorentz-violating coefficient [19]. The particle dynamics in the Randers spacetime is analogous to the Lorentzian invariant dynamics under the influence of a background electromagnetic vector field $a^\mu$. The effects of the Randers spacetime anisotropy have been studied in cosmology [28,29] and astrophysics [30].

The directional dependence of the Finsler metric leads naturally to a field theory living in the tangent bundle as the base space [31,32]. In this work we propose an
alternative approach, where the fields live only on the spacetime. Since the Finsler metric depends on the momentum in the MDR, by replacing the momentum by the covariant derivative, the Finsler metric turns into a differential operator acting on the fields. A similar approach was carried out for the electromagnetic field in flat spacetimes [33]. Assuming a minimal coupling prescription to the Finsler metric, nonlocal operators, as those found in VSR [6] and in Bogoslovsky spacetime [18] are found. The perturbative character of the Randers anisotropy allows us to expand the nonlocal operators which leads to Lorentz-violating terms as found in the SME [8] and in the Carroll-Field-Jackiw model higher derivative terms as found in the nonminimal SME [34,35].

This work is organized as follows. In the next section, the definition of the Randers spacetime and the dynamics of point particles are reviewed. In the third section we propose and study the dynamics of scalar, gauge and fermion fields on the Randers spacetime. Final comments and perspectives are outlined in the last section.

**Randers-Finsler spacetime.** – In this section we review the description and the main properties of the Randers-Finsler spacetime. We show how the background vector is included into the geometric structure of the spacetime and this leads to modification of the massive particle dynamics and mass shell.

In the Randers-Finsler spacetime, the interval of two nearby time-like events \( x^\mu, \dot{x}^\mu + \dot{x}^{a\nu} dt \) are measured by the so-called Finsler function, \( d^R \mathcal{L} := F_R(x, \dot{x}) dt \), where [19,27]

\[
d^R \mathcal{L} := \left( \sqrt{-g\mu\nu(x)\dot{x}^\mu\dot{x}^\nu + \zeta a_\mu(x)\dot{x}^\mu} \right) dt. \tag{1}
\]

The Randers spacetime has the local Lorentz-invariant LLI interval \( \alpha(x, \dot{x}) := \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu} \) and a linear term \( \beta(x, \dot{x}) := \zeta a_\mu(x)\dot{x}^\mu \) which drives the Lorentz violation. Both \( g_{\mu\nu}(x) \) and \( a_\mu(x) \) are considered background geometric tensors defining the metric properties of the anisotropic spacetime. The background Randers vector \( a \) has its indexes raised and lowered with the background Lorentzian metric \( g_{\mu\nu} \). In this article the Lorentz-violating term \( \beta \) is assumed to be small compared to the LLI term \( \alpha \).

That leads to a perturbative Lorentz violation where \( 0 \leq \zeta^2 g^\mu\nu a_\mu a_\nu \ll 1 \) for a space-like \( a_\mu \) or \( \zeta^2 g^\mu\nu a_\mu a_\nu \ll -1 \) for a time-like \( a_\mu \). In addition, by assuming \( \zeta^2 g^\mu\nu a_\mu a_\nu \ll \pm 1 \), the existence of past and future time-like vectors is ensured, as discussed in details in refs. [15,36].

The modified interval 1 can be rewritten in terms of an anisotropic or Finsler metric, by \( d^R \mathcal{L} := \sqrt{-g_F^{\mu\nu}(x, \dot{x})\dot{x}^\mu\dot{x}^\nu} dt \) [11,37], where the relation between the Finsler function \( F(x, \dot{x}) \) and the Finsler metric is given by

\[
g_F^{\mu\nu}(x, \dot{x}) = -\frac{\partial F^2(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}. \tag{2}
\]

In the Randers-Finsler spacetime, the Finsler metric has the form [11]

\[
g_F^{\mu\nu}(x, \dot{x}) = \frac{F}{\alpha}g_{\mu\nu} - \frac{\beta}{\alpha}a_\mu a_\nu + 2\zeta u_\mu a_\nu + \zeta^2 a_\mu a_\nu, \tag{3}
\]

where \( u_\mu := \frac{\dot{x}^\mu}{\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}} \) is the Lorentzian 4-velocity and \( u^\mu := g^\mu\nu u_\nu \).

The Randers-Finsler Lagrangian for a massive point particle is given by [27]

\[
L_R := -m \left( \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu + \zeta a_\mu(x)\dot{x}^\mu} \right), \tag{4}
\]

from which the canonical momentum covector has the form [19,38]

\[
P^\mu := \frac{\partial L_R}{\partial \dot{x}^\mu} = P_\mu - m\zeta a_\mu, \tag{5}
\]

where \( P_\mu := m g_{\mu\nu} u^\nu \) is the local Lorentz-invariant momentum. The covariant Randers-Finsler canonical momentum \( P^\mu \) is given by \( P^\mu := g_F^{\mu\nu}(x, \dot{x}) P_R^{\nu} \), where the contravariant momentum is given by

\[
P_R^\mu := m \frac{\dot{x}^\mu}{F(x, \dot{x})} \tag{6}
\]

By introducing a Finslerian unit 4-velocity \( U^\mu := \frac{\dot{x}^\mu}{F(x, \dot{x})} \), also known as the distinguished section [11], the contravariant Randers momentum takes the familiar form

\[
P_R^\mu := m U^\mu. \tag{7}
\]

Accordingly, the covariant Randers momentum can be rewritten as \( P_R^\mu = m\omega_\mu \), where \( \omega_\mu \) is the so-called Hilbert form [11]. Note that local Lorentz-invariant object, as the momentum \( P_\mu \), are raised and lowered with the metric \( g_{\mu\nu} \) whereas the Randers-Finsler objects, as \( P_R^\mu \), are raised and lowered with the Finsler metric \( g_F(x, \dot{x}) \).

The contravariant Randers momentum (6) can also be written as

\[
P_R^\mu = \frac{\alpha(x, P)}{F_R(x, P)} P^\mu, \tag{8}
\]

where \( P^\mu := m\dot{x}/\alpha(x, \dot{x}) \) is the local Lorentz-invariant particle momentum. By using the Randers-Finsler function in eq. (1), we obtain

\[
P_R^\mu = \frac{1}{1 + \frac{\zeta^2}{\alpha}} P^\mu \tag{9}
\]

\[
= \left( 1 - \zeta \frac{a \cdot P}{\sqrt{-P_\mu P^\mu}} + \cdots \right) P^\mu.
\]
The modified mass shell (MDR) in the Finsler spacetime is taken using the Finsler metric. Since \( g_{\mu
u}(x, U) U^\nu U^\mu = -1 \) and \( g_{\mu
u}^F(x, mU) = g_{\mu
u}^F(x, U) \), the MDR takes the covariant form \([12,18,31,32]\)

\[
g_{\mu
u}^F(x, P^R) P^R \mu P^R \nu = -m^2. \tag{10}
\]

In the Randers spacetime, the MDR has the form

\[
(g_{\mu
u} + \zeta^2 a_\mu a_\nu) P^R \mu P^R \nu - 2\xi m a_\mu P^R \nu = -m^2. \tag{11}
\]

It is worthwhile to mention that whereas the MDR in eq. (11) has a clear interpretation as a perturbative modification of the Lorentz-invariant MDR, as expected to occur for the Lorentz symmetry breaking, the Finslerian version of the MDR in eq. (10) has a geometric interpretation. The symmetries of the Finsler metric lead to the symmetries of the particles in this local Lorentz-violating spacetime. The MDR in eq. (10) is left invariant under observer local Lorentz transformations and under particle deformed transformations built with the Finsler Killing vectors, as shown in refs. \([38,39]\).

For \( P^R \mu = (E, \vec{P}) \), the MDR (11) leads to

\[
(1 - \zeta^2 a_\mu a_\mu) E^2 + 2m\zeta a_\mu E - |\vec{P}|^2 + 2m\zeta (\vec{a} \cdot \vec{P}) - \zeta^2 (\vec{a} \cdot \vec{P})^2 - m^2 = 0, \tag{12}
\]

for which the group velocity \( v_1 = \frac{\partial E}{\partial P^R} \) in the UV limit, i.e., for \( |\vec{P}| \to \infty \), behaves as \( v_1 \approx \frac{1}{\sqrt{1 - \zeta^2 a_\mu a_\mu}} \). Thus, the Randers spacetime allows superluminal velocities in the ultrarelativistic regime \([40]\). Causality issues are common features of Lorentz-violating models in the UV regime \([41]\). A well-known exception is the Bogoslovsky spacetime, where a background vector provides a nonperturbative Finsler modification of the spacetime interval \([18]\). Since the Lorentz-violating models are usually understood as effective models valid up to some cut off momentum \( |\vec{P}|_M \), it is expected that the complete quantum spacetime be free of these issues. In this work we consider the Randers-Finsler spacetime as a low-energy description of spacetime featuring local Lorentz violation.

The particle equation of motion (EOM) is given by

\[
\ddot{x}^\mu + G^\mu(x, \dot{x}) = 0,
\]

where the inertial force is defined by \( G^\mu = \frac{\partial E}{\partial P^R} \), and the Finsler Christoffel symbol is \( \Gamma^{\mu}_{\rho \sigma}(x, \dot{x}) := \frac{g^{\mu\nu}(x, \dot{x})}{2} \left[ \partial_\rho g_{\sigma \nu}(x, \dot{x}) + \partial_\sigma g_{\rho \nu}(x, \dot{x}) - \partial_\nu g_{\rho \sigma}(x, \dot{x}) \right] \). Let us seek for a covariant derivative for which the EOM can be rewritten as \( P_{\mu R} = 0 \) and preserves the MDR equation (10). Since the geometry, and hence the physics, is dependent on both the position and velocities, consider the so-called horizontal derivative \( \delta_\mu := \frac{\partial}{\partial x^\mu} - N^\mu \frac{\partial}{\partial v^\mu} \) and the vertical derivative \( \partial_\mu := F(x, \dot{x}) \frac{\partial}{\partial v^\mu} \), where \( N^\mu := \frac{\partial E}{\partial x^\mu} \) \([11]\). Let us consider the Cartan connection \( \omega^{\mu}_{\nu \rho} := \Gamma^{\mu}_{\rho \sigma} dx^\sigma + C^{\mu}_{\nu \rho} dv^\mu \),

where \( \Gamma^{\mu}_{\rho \sigma}(x, \dot{x}) := \frac{g^{\mu \nu}(x, \dot{x})}{2} \left[ \partial_\rho g_{\sigma \nu}(x, \dot{x}) + \partial_\sigma g_{\rho \nu}(x, \dot{x}) - \partial_\nu g_{\rho \sigma}(x, \dot{x}) \right] \) and \( C^{\mu}_{\nu \rho}(x, \dot{x}) := \frac{1}{2} \partial_\eta g^{\mu \eta}(x, \dot{x}) \) is the so-called Cartan tensor, which measures the dependence of the Finsler metric \([11]\). The Cartan horizontal derivative of the Finsler metric tensor, defined as \( g^{\mu \nu}_{F\rho} := \nabla_\rho g^{\mu \nu} = \delta^{\mu}_{\rho} g^{\nu} - \Gamma^{\mu}_{\rho \sigma} g_{\sigma \nu} - \Gamma^{\eta}_{\rho \nu} g_{\eta \mu} \), vanishes, i.e., \( g^{\mu \nu}_{F\rho} \equiv 0 \). Thus, a free particle experiences anisotropic local Lorentz-violating inertial forces while preserving the MDR (10).

By applying the tetrad formalism, we can rewrite the Randers metric (3) as \( g^{F\mu}_{\nu} = E^{F\mu}_{\nu}(x, \dot{x}) E^{F\nu}_{\rho}(x, \dot{x}) \eta_{\rho \sigma} \), where the tetrads are given by

\[
E^{F\mu}_{\nu}(x, \dot{x}) = \sqrt{\frac{F}{\alpha}} \left\{ E^{\mu}_{\nu} + \left( \frac{\alpha}{F} \right)^2 \left[ \frac{\beta}{2\alpha} n^\mu u_\mu + u^\mu a_\mu \right] + a^\mu u_\mu + a^\mu a_\mu \right\}.
\]

The tetrad \( E^{F\mu}_{\nu}(x, \dot{x}) \) can be understood as a deformation of the local Lorentz-invariant tetrad \( E^{\mu}_{\nu}(x, \dot{x}) \) by the background field \( a_\mu \).

A Randers field theory. – Once we have analysed the dynamics of a particle in the Randers-Finsler spacetime, let us propose a dynamics for fields. Like for the particle action, we are interested in actions obtained from Randers-Finsler symmetric tensors, as the Finsler metric. In flat Minkowski (Lorentz-invariant) spacetime, the field momentum comes from the directional derivative of the field, which leads to the well-known identification \( P_\mu = -i\partial_\mu \). In curved local Lorentz-invariant spacetimes, the partial derivative is replaced by the covariant derivative due to the nonminimal effects, leading to \( P_\mu = -i\nabla_\mu \). In this work we propose a minimal coupling prescription in Finsler spacetimes of the form \( P^{F\mu} = -i\nabla_\mu \). Such choice provides a covariant horizontal momentum operator. Assuming the fields have only position dependence, the momentum operator has its origin on position variations. Further, we assume an approach where the direction dependence of the geometry becomes a momentum dependence of the metric. Thus, the Finsler metric can be regarded as a differential operator, where \( g^{\mu}_{\nu}(x, y) \to g^{\mu}_{\nu}(x, \nabla) \) and \( y^\mu \to \nabla^\mu F \). The relation \( P^{F\mu} = \frac{\partial}{\partial x^\mu} \) and the homogeneity of the Finsler metric allow us to write the Finsler metric as the operator \( g^{\mu}_{\nu}(x, \nabla) \). Since \( g^{\mu}_{\nu}(x, P^F) P^{F\mu} P^{F\nu} \) is a Finslerian invariant scalar, the minimal prescription adopted here allows us to define a Finsler-invariant differential operator \( g^{F\mu}_{\nu}(x, \nabla) \nabla^\mu F \nabla^\nu F \). This approach of considering the Finsler metric as an operator is similar to the noncanonical kinetic terms \([42]\) and was previously employed in the gauge vector field in a flat spacetime \([33]\).

Unlike the field theories defined on the tangent bundle \( TM \) \([32]\), the Finsler metric operator procedure enables us to propose a field theory defined on the spacetime \( M \) itself. Although this approach can be in principle be applied to any Finsler spacetime, we concentrate on the Randers-Finsler geometry.
Scalar field. We propose a Randers-invariant action for the scalar field given by

\[ S_\Phi := -\frac{1}{2} \int_M \left\{ d^4x \left[ \nabla_\mu \Phi K^{\mu \nu}(x, \nabla) \nabla_\nu \Phi + m^2 \sqrt{-g^{\mu \nu}(x, \nabla) \Phi^2} \right] \right\}, \tag{13} \]

where \( K^{\mu \nu} := \sqrt{-g^{\mu \nu}(x, \nabla) g^{F \mu \nu}(x, \nabla)}. \) For \( \zeta = 0, \) i.e., for a local Lorentz symmetric spacetime, the action (13) yields to a minimal coupling of the scalar field in a curved spacetime.

In the Randers spacetime, by means of the identification \( y^\mu \to \nabla^\mu, \) the contravariant metric has the form \[ g^{\mu \nu}(x, \nabla) = \frac{g^{\mu \nu}}{1 + \zeta^2 a \cdot \nabla^\nu} - \frac{\zeta^2}{1 + \zeta^2 a \cdot \nabla^\nu} (a^\mu \nabla^\nu + a^\nu \nabla^\mu) + \frac{\zeta^4}{(1 + \zeta^2 a \cdot \nabla^\nu)^3} [a \cdot \nabla + a^2] \nabla^\nu \nabla^\nu, \tag{14} \]

where \( a \cdot \nabla := a^\mu \nabla^\mu. \) Then, defining a dimensionless background field \( b^\mu := \zeta a^\mu, \) the Randers action for the scalar field can be rewritten as

\[ S_\Phi = -\frac{1}{2} \int_M d^4x \sqrt{-g} \left\{ g^{\mu \nu} \nabla_\mu \Phi (1 + b \cdot \nabla)^2 \nabla_\nu \Phi +(1 + \zeta^2 a \cdot \nabla)^4 m^2 \Phi^2 + \zeta^2 a \cdot \nabla \Phi (1 + b \cdot \nabla)^2 (g^{F \mu \nu} b^\rho) \right\}
\times \nabla_\rho \nabla_\nu \Phi + \zeta^2 g^{\mu \rho} g^{\nu \sigma} \nabla_\mu \Phi \left[ (\frac{\zeta^2 (b \cdot \nabla)^2}{1 + \zeta^2 (b \cdot \nabla)^2}) \nabla_\nu \nabla_\rho \nabla_\sigma \Phi \right]. \tag{15} \]

The action exhibits nonlocal dynamical terms similar to those of VSR [6]. The perturbative character of the Randers spacetime allows us to rewrite the Randers Lagrangian as

\[ L_\Phi^F = L_{\Phi} + \zeta L_{\Phi}^L + \zeta^2 L_{\Phi}^L + \cdots, \tag{16} \]

where \( L_{\Phi} = \frac{1}{4} \int d^4x \{ F_{\mu \nu}(K^{5 \mu \nu}) \nabla_\nu \Phi - 5m^2 \sqrt{-g} b^\rho \nabla_\rho \Phi^2 \}, \tag{17} \]

where the mass dimension-five Lorentz-violating operator \((K^{5 \mu \nu})\) has the form

\[ (K^{5 \mu \nu}) := -\frac{\sqrt{-g}}{4} \left\{ 3g^{\mu \nu} b^\rho - 2(g^{\rho \mu} b^\nu + g^{\rho \nu} b^\mu) \right\} \nabla_\rho. \tag{18} \]

The second-order terms form the Lorentz-violating Lagrangian

\[ L_{\Phi}^L = \nabla_\mu \Phi (K^{5 \mu \nu} \nabla_\nu \Phi - \frac{15m^2 \sqrt{-g}}{8} b^\rho b^\sigma \nabla_\rho \nabla_\sigma \Phi^2), \tag{19} \]

where the mass dimension-six operator \((K^{6 \mu \nu})\) is defined as

\[ (K^{6 \mu \nu}) := -\frac{\sqrt{-g}}{4} \left\{ \frac{3}{8} g^{\mu \rho} b^\sigma - \frac{b^2}{2} g^{\mu \sigma} \right\} \nabla_\rho \nabla_\sigma. \tag{20} \]

It is worthwhile to say that the last term in the first-order perturbed Lagrangian (17), for a covariantly constant background vector \( b^\nu, \) provides a total derivative term which can be dropped from the action. The operators \((K^{5 \mu \nu})\) and \((K^{6 \mu \nu})\) in a flat background metric have the same form of the nonminimal Standard Model Extension for dimension-5 and -6 Lorentz-violating operators.

Vector gauge field. Let us propose a Finslerian dynamics for the Abelian gauge field such that the field strength is defined using Cartan horizontal covariant derivatives instead of the Levi-Civita covariant derivative, i.e.,

\[ F_{\mu \nu}^R := \nabla_\mu A_\nu - \nabla_\nu A_\mu - \Gamma_{\mu \nu}^\rho A_\rho = A_{\nu, \mu} - A_{\mu, \nu}, \tag{21} \]

where \( \nabla_\mu A_\nu := \partial_\mu A_\nu - \Gamma_{\mu \nu}^\rho A_\rho. \) Assuming a torsion-free horizontal covariant derivative, i.e., \( \Gamma_{\mu \nu}^\rho = \Gamma_{\nu \mu}^\rho, \) we obtain \( F_{\mu \nu}^R = F_{\mu \nu}. \) Therefore, the electric and magnetic components of the field strength and the gauge symmetry are preserved in the Randers spacetime.

A gauge Finslerian extension of the Maxwell action has the form

\[ S_A^F := -\frac{1}{4} \int d^4x \{ F_{\mu \nu} K^{F \mu \nu \rho \sigma}(x, \nabla) F_{\rho \sigma} \}, \tag{22} \]

where \( K^{F \mu \nu \rho \sigma} := \frac{\sqrt{-g_F}}{4} g^{F \mu \nu}(x, \nabla) g^{F \rho \sigma}(x, \nabla). \) For \( \zeta = 0, \) the tensor \( K^{F \mu \nu \rho \sigma} \) turns into \( \frac{\sqrt{-g_F}}{4} g^{F \rho \sigma}(x, \nabla) \) we obtain the usual Maxwell term \( L_A = -\frac{\sqrt{-g_F}}{4} g^{F \rho \sigma}(x, \nabla) F_{\rho \sigma}. \) In a flat Finsler spacetime, a similar approach was employed in ref. [33].

In the Randers spacetime, the Finsler gauge action (21) yields to the Finsler gauge Lagrangian

\[ L_A^F = -\frac{\sqrt{-g_F}}{4} F_{\mu \nu} \left\{ (1 + \zeta \partial \cdot \nabla)^2 g^{\mu \rho} g^{\nu \sigma} - 2\zeta^2 \left[ g^{\mu \rho} g^{\nu \lambda} g^{\sigma \xi} \right] (1 + \zeta \partial \cdot \nabla)^2 \right\} F_{\rho \sigma}. \tag{23} \]

Let us analyse the gauge Lagrangian in eq. (22) term by term. The first term,

\[ L_A^1 = -\frac{\sqrt{-g_F}}{4} F_{\mu \nu}(1 + \zeta \partial \cdot \nabla)^2 g^{\mu \rho} g^{\nu \sigma} F_{\rho \sigma}, \tag{24} \]

can be expanded in powers of \( \zeta \) as

\[ L_A^1 = -\frac{\sqrt{-g_F}}{4} F_{\mu \nu} F_{\rho \sigma} + \zeta F_{\mu \nu} (K^{5 \mu \nu})_{\rho \sigma} + \cdots, \tag{25} \]
where the zeroth-order term consists of the usual Maxwell Lagrangian and the first-order and second-order Lorentz-violating Lagrangian \(\mathcal{L}_{\text{ALV}}\) terms are, respectively, 
\[
(k_{(5)}^{\mu \nu \rho})_{\mu \nu \rho} = -\sqrt{-g}g^{\mu \nu}g^{\rho \sigma}b^\lambda \nabla_\lambda \quad \text{and} \quad (\tilde{k}_{(6)}^{\mu \nu \rho \sigma})_{\mu \nu \rho \sigma} = -\sqrt{-g}g^{\mu \nu}g^{\rho \sigma}g^{\lambda \xi} \nabla_\lambda \nabla_\xi
\]
The second term,
\[
\mathcal{L}_A := \sqrt{-g}\frac{\xi}{2} \partial_{\mu} \left[ g^{\mu \nu}g^{\rho \sigma}g^{\lambda \xi} \left( 1 + \frac{b}{\sqrt{-g}} \right)^2 \nabla_\lambda \nabla_\xi \right] F_{\rho \sigma},
\]
can be expanded as
\[
\mathcal{L}_A := \xi^2 F_{\mu \nu} \left( k_{(5)}^{\mu \nu \rho} F_{\rho \sigma} + F_{\mu \nu} \tilde{k}_{(6)}^{\mu \nu \rho \sigma} F_{\rho \sigma} \right) + \cdots,
\]
where 
\[
(k_{(5)}^{\mu \nu \rho})_{\mu \nu \rho} := \frac{\sqrt{-g}}{2} g^{\mu \nu}g^{\rho \sigma} \left( g^{\lambda \beta} b^\lambda + b^\beta b^\gamma \right) \nabla_\lambda \quad \text{and} \quad \tilde{k}_{(6)}^{\mu \nu \rho \sigma} := \frac{\sqrt{-g}}{2} g^{\mu \nu}g^{\rho \sigma} \left( \nabla_\nu b^\rho + \nabla_\rho b^\nu \right).
\]
It is worth noting that \(\tilde{k}_{(6)}^{\mu \nu \rho \sigma} = 0\) for a covariantly constant background vector, i.e., for a Randers spacetime of Berwald type. For a background flat spacetime and constant background vector, the mass dimension-five operator \((\tilde{k}_{(5)}^{\mu \nu \rho})_{\mu \nu \rho}\) belongs to the nonminimal SME [34,33].

The third term, 
\[
\mathcal{L}_B := \frac{\sqrt{-g}}{2} \xi^2 F_{\mu \nu} \left[ g^{\mu \nu}g^{\rho \sigma}g^{\lambda \xi} \left( 1 + \frac{b}{\sqrt{-g}} \right)^2 \nabla_\lambda \nabla_\xi \right] F_{\rho \sigma},
\]
can be rewritten as
\[
\mathcal{L}_B := \xi^2 F_{\mu \nu} (k_{(5)}^{(6)})_{\mu \nu \rho} F_{\rho \sigma},
\]
where 
\[
(k_{(5)}^{(6)})_{\mu \nu \rho} := -\frac{\sqrt{-g}}{2} b^\beta b^\gamma g^{\mu \nu}g^{\rho \sigma} g^{\lambda \xi} \nabla_\lambda \nabla_\xi,
\]
for a flat background metric and vector, is a dimension-six Lorentz coefficient of the nonminimal SME [34].

The Randers geometry also allows the following gauge-invariant action coupling:
\[
\tilde{S}_F := \xi \int d^4x [e^{\mu \nu \rho} \tilde{A}_\mu \tilde{K}_\nu F_{\rho \sigma}],
\]
where \(\tilde{K}_\nu := \sqrt{-g}^{-1} g^{\nu \rho}(\nabla_\alpha a_\rho)\). Expanding the action in eq. (29), we obtain the Lorentz-violating Lagrangian
\[
\tilde{L}_F := \xi e^{\mu \nu \rho \sigma} \tilde{A}_\mu \tilde{A}_\rho F_{\nu \sigma} + \xi^2 e^{\mu \nu \rho \sigma} \tilde{A}_\mu \nabla_\lambda \nabla_\sigma F_{\nu \rho} + \xi^2 e^{\mu \nu \rho \sigma} \tilde{A}_\mu \tilde{A}_\rho \nabla_\lambda \nabla_\sigma F_{\nu \rho} + \cdots,
\]
i.e., the so-called Carroll-Field-Jackiw Lagrangian [44] and corrections.

Fermionic field. In order to describe fermions, we adopt the tetrad formalism, whereby the Randers gamma matrices are defined as \(\gamma^{\mu \nu}(x, P) := E_\mu(x, P) \gamma^\lambda\) and the tetrads are defined in eq. (3).

As done for the Finsler metric, the Finsler gamma matrices are seen as operators, defined as
\[
\tilde{\gamma}^{\mu \nu}(D) := \sqrt{-g}^{-1} g^{\mu \nu} \gamma^{F \mu}(D).
\]
A Finsler-invariant Dirac action has the form
\[
\mathcal{S}_\Psi := \int_M d^4 x [\bar{\Psi} (\gamma^{F \mu}(D) D_\mu - m) \Psi + \text{H.C.}],
\]
produces the Carroll-Field-Jackiw term and corrections. As perspectives we point out the analysis of bounds for the coupling proposed based on tests for the gauge [34], and fermions [35]. Here we only deal with massive particles for which the spacetime interval is time-like. However, some Lorentz-violating theories are plagued with causal and instabilities problems, mainly at UV and massless regime (light cone) [15,41]. The inclusion space-like and null intervals and the corresponding study of the causal structure of the fields defined here, as performed in ref. [15], are another relevant perspective. The presence of high-order derivatives also demands further studies on the Cauchy problem of this fields. The stability analysis of the fields and the gravitational sector are also important future developments.

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REFERENCES

[1] Kostelecký V. A. and Samuel S., Phys. Rev. D, 39 (1989) 683; Phys. Rev. Lett., 63 (1989) 224; 66 (1991) 1811.
[2] Carroll S. M., Harvey J. A., Kostelecký V. A., Lane C. D. and Okamoto T., Phys. Rev. Lett., 87 (2001) 141601.
[3] Hořava P., Phys. Rev. D, 79 (2009) 084008.
[4] Alfaro J., Morales-Tecotl H. A. and Urrutia L. F., Phys. Rev. Lett., 84 (2000) 2318; Phys. Rev. D, 65 (2002) 103509.
[5] Magueijo J. and Smolin L., Phys. Rev. Lett., 88 (2002) 190403.
[6] Cohen A. G. and Glashow S. L., Phys. Rev. Lett., 97 (2006) 021601.
[7] Maluf R. V., Silva J. E. G., Cruz W. T. and Almeida C. A. S., Phys. Lett. B, 738 (2014) 341.
[8] Colladay D. and Kostelecký V. A., Phys. Rev. D, 55 (1997) 6760.
[9] Kostelecký V. A. and Russell N., Rev. Mod. Phys., 83 (2011) 11.
[10] Kostelecký V. A., Phys. Rev. D, 69 (2004) 105009.
[11] Bao D., Chern S. and Shen Z., An Introduction to Riemann-Finsler Geometry (Springer) 1991.
[12] Girelli F., Liberati S. and Sindoni L., Phys. Rev. D, 75 (2007) 064015.
[13] Raetzel D., Rivera S. and Schuller F. P., Phys. Rev. D, 83 (2011) 044047.
[14] Pfeiffer C., Int. J. Geom. Methods Mod. Phys., 16 (2019) 1941004.
[15] Javaloyes M. A. and Sánchez M., RACSAM, 114 (2020) 30.
[16] Amelino-Camelia G., Barcaroli L., Gibon J. K., Liberati S. and Loreti N., Phys. Rev. D, 90 (2014) 125030.
[17] Gibbons G. W., Gomis J. and Pope C. N., Phys. Rev. D, 76 (2007) 081701.
[18] Bogoslovsky G. Y. and Goenner H. F., Gen. Relativ. Gravit., 31 (1999) 1565.
[19] Kostelecký V. A., Phys. Lett. B, 701 (2011) 137.
[20] Foster J. and Lehnert R., Phys. Lett. B, 746 (2015) 164.
[21] Kostelecký V. A., Russell N. and Tso R., Phys. Lett. B, 716 (2012) 470.
[22] Silva J. E. G. and Almeida C. A. S., Phys. Lett. B, 731 (2014) 74.
[23] Colladay D. and McDonald P., Phys. Rev. D, 85 (2012) 044042; 92 (2015) 085031.
[24] Russell N., Phys. Rev. D, 91 (2015) 045008.
[25] Schreck M., Phys. Rev. D, 91 (2015) 105001.
[26] Schreck M., Eur. Phys. J. C, 75 (2015) 187.
[27] Randers G., Phys. Rev., 59 (1941) 195.
[28] Chang Z. and Li X., Phys. Lett. B, 676 (2009) 173.
[29] Basilakos S., Kouretsis A. P., Saridakis E. N. and Stavrinos P. C., Phys. Rev. D, 88 (2013) 123510.
[30] Chang Z. and Li X., Phys. Lett. B, 668 (2008) 453; Li X. and Chang Z., Phys. Lett. B, 692 (2010) 1.
[31] Vaca R. S. I., Class. Quantum. Grav., 28 (2001) 215001.
[32] Pfeiffer C. and Wohlfarth M. N. R., Phys. Rev. D, 85 (2012) 064009.
[33] Ivan Y., Lämmertz J. C. and Perlick V., Phys. Rev. D, 90 (2014) 124057.
[34] Kostelecký V. A. and Mewes M., Phys. Rev. D, 80 (2009) 015020.
[35] Kostelecký V. A. and Mewes M., Phys. Rev. D, 88 (2013) 096006.
[36] Hohmann M., Pfeiffer C. and Voicu N., Phys. Rev. D, 100 (2019) 064035.
[37] Djian C. and Farran H., Geometry of Pseudo-Finsler Submanifolds (Kluwer Academic Publishers) 2000.
[38] Chang Z. and Li X., Phys. Lett. B, 663 (2008) 103.
[39] Silva J. E. G., Maluf R. V. and Almeida C. A. S., Phys. Lett. B, 798 (2019) 135009.
[40] Silva J. E. G., Maluf R. V. and Almeida C. A. S., Phys. Lett. B, 766 (2017) 263.
[41] Kostelecký V. A. and Lehnert R., Phys. Rev. D, 63 (2001) 065008.
[42] Babichev E., Mukhanov V. and Vilkman A., JHEP, 02 (2008) 101.
[43] Schreck M., Phys. Rev. D, 89 (2014) 105019.
[44] Carroll S. M., Field G. B. and Jackiw R., Phys. Rev. D, 41 (1990) 1231.