Four Blocks Cycles $C(k, 1, 1, 1)$ in Digraphs

Zahraa Mohsen $^{1,2}$

Abstract

A four blocks cycle $C(k_1, k_2, k_3, k_4)$ is an oriented cycle formed by the union of four internally disjoint directed paths of lengths $k_1, k_2, k_3$, and $k_4$ respectively. El Mminy $^2$ proved that if $D$ is a digraph having a spanning out-tree $T$ with no subdivisions of $C(k, 1, 1, 1)$, then the chromatic number of $D$ is at most $8k$. In this paper, we will improve this bound to $18k$.

1 Introduction

Throughout this paper, all the graphs are considered to be simple, that is with no loops and no multiple edges. By giving an orientation to each edge of a graph $G$, we obtain an oriented graph called a digraph, denoted by $D$. Reciprocally, the graph obtained from a digraph $D$ by ignoring the directions of its arcs is called the underlying graph of $D$, and denoted by $G(D)$. The chromatic number of a digraph $D$, denoted by $\chi(D)$, is the chromatic number of its underlying graph. In $^3$, Cohen et al. proved that for any two digraphs $D_1$ and $D_2$, we have $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.

Let $D$ be a digraph. An out-tree $T$ of $D$ is a sub-digraph of $D$ whose vertices are of in-degree 1 except for one vertex of in-degree 0, which is called the root of $T$.

Let $T$ be a spanning out-tree of a digraph $D$ rooted at $r$. For a vertex $x$ of $D$, there is a unique $rx$-directed path in $T$, denoted by $T_{[r,x]}$. The level of $x$ with respect to $T$, denoted by $l_T(x)$, is the length of this path. For a non-negative integer $i$, denote $L_i(T)$ to be the set of all the vertices having a level $i$ in $T$.

For $x \in V(T)$, the ancestors of $x$ are the vertices that belong to $T_{[r,x]}$. For an ancestor $y$ of $x$, we will write $y \leq_T x$ and we will denote by $T_{[y,x]}$ the directed path in $T$ from $y$ to $x$. For two vertices $x$ and $y$ of $T$, the least common ancestor, $z$, of $x$ and $y$ is the common ancestor of $x$ and $y$ having the highest level in $T$. Let $D'$ be a sub-digraph of a digraph $D$ and $T$ be a spanning out-tree of $D$. Let $x \in V(D')$, $x$ is said to be a minimal in $D'$ for $\leq_T$ if $\forall v \in V(D')$ satisfying $v \leq_T x$, we have $x \leq_T v$. Moreover, $x$ is said to be a maximal in $D'$ for $\leq_T$ if $\forall v \in V(D')$ satisfying $x \leq_T v$, we have $v \leq_T x$.

An arc $(x,y)$ of $D$ is said to be forward (resp. backward) with respect to $T$ if $l_T(x) < l_T(y)$ (resp. $l_T(x) > l_T(y)$). For two adjacent vertices $x$ and $y$, we denote by $xy$ the arc between $x$ and $y$ if its orientation is unknown.

A maximal out-tree $T$ of $D$ is a spanning out-tree for which for any backward arc with respect to $T$, say $(x,y)$, there exists a $yx$-directed path in $T$. We can easily see that for a maximal out-tree $T$ of a digraph $D$, $L_i(T)$ is stable in $D$ for all $i \geq 0$.

A digraph $D$ is said to be strong if for any two vertices $u$ and $v$ of $D$, there is a directed path from $u$ to $v$. One can easily notice that every strong digraph $D$ has a spanning out-tree. In deed, suppose not and let $T$ be an out-tree of $D$ with maximal number of vertices. Then there exists a vertex $x \in V(D) - V(T)$. Since $D$ is strong then there exists a vertex $y \in V(D)$ such that $(y, x) \in A(D)$. Let $T' = T \cup (y, x)$, an out-tree of $D$ with $V(T') > V(T)$, a contradiction.

In $^2$, El Mminy proved that any digraph $D$ that has a spanning out-tree admits a maximal out-tree. Consequently, every strong digraph admits a maximal out-tree.

Given an oriented path $P$ (resp. oriented cycle $C$), a block is a maximal directed subpath of $P$ (resp. $C$). We denote by $P(k_1, k_2, \cdots, k_n)$ (resp. $C(k_1, k_2, \cdots, k_n)$) the oriented path (resp. oriented cycle) formed of $n$ blocks of lengths $k_1, k_2, \cdots$, and $k_n$ respectively. Moreover, for a block $B$ of an oriented cycle $C$ which is directed from $x$ to $y$, we denote by $x$ (resp. $y$) the source (resp. the sink) of $B$, and we write $B$ as $C_{[x,y]}$. 

$^1$Lebanese University, KALMA Laboratory, Baalbeck.

$^2$University of Paris, IMJ Laboratory, Paris.

---

1 Introduction

Throughout this paper, all the graphs are considered to be simple, that is with no loops and no multiple edges. By giving an orientation to each edge of a graph $G$, we obtain an oriented graph called a digraph, denoted by $D$. Reciprocally, the graph obtained from a digraph $D$ by ignoring the directions of its arcs is called the underlying graph of $D$, and denoted by $G(D)$. The chromatic number of a digraph $D$, denoted by $\chi(D)$, is the chromatic number of its underlying graph. In $^3$, Cohen et al. proved that for any two digraphs $D_1$ and $D_2$, we have $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.

Let $D$ be a digraph. An out-tree $T$ of $D$ is a sub-digraph of $D$ whose vertices are of in-degree 1 except for one vertex of in-degree 0, which is called the root of $T$.

Let $T$ be a spanning out-tree of a digraph $D$ rooted at $r$. For a vertex $x$ of $D$, there is a unique $rx$-directed path in $T$, denoted by $T_{[r,x]}$. The level of $x$ with respect to $T$, denoted by $l_T(x)$, is the length of this path. For a non-negative integer $i$, denote $L_i(T)$ to be the set of all the vertices having a level $i$ in $T$.

For $x \in V(T)$, the ancestors of $x$ are the vertices that belong to $T_{[r,x]}$. For an ancestor $y$ of $x$, we will write $y \leq_T x$ and we will denote by $T_{[y,x]}$ the directed path in $T$ from $y$ to $x$. For two vertices $x$ and $y$ of $T$, the least common ancestor, $z$, of $x$ and $y$ is the common ancestor of $x$ and $y$ having the highest level in $T$. Let $D'$ be a sub-digraph of a digraph $D$ and $T$ be a spanning out-tree of $D$. Let $x \in V(D')$, $x$ is said to be a minimal in $D'$ for $\leq_T$ if $\forall v \in V(D')$ satisfying $v \leq_T x$, we have $x \leq_T v$. Moreover, $x$ is said to be a maximal in $D'$ for $\leq_T$ if $\forall v \in V(D')$ satisfying $x \leq_T v$, we have $v \leq_T x$.

An arc $(x,y)$ of $D$ is said to be forward (resp. backward) with respect to $T$ if $l_T(x) < l_T(y)$ (resp. $l_T(x) > l_T(y)$). For two adjacent vertices $x$ and $y$, we denote by $xy$ the arc between $x$ and $y$ if its orientation is unknown.

A maximal out-tree $T$ of $D$ is a spanning out-tree for which for any backward arc with respect to $T$, say $(x,y)$, there exists a $yx$-directed path in $T$. We can easily see that for a maximal out-tree $T$ of a digraph $D$, $L_i(T)$ is stable in $D$ for all $i \geq 0$.

A digraph $D$ is said to be strong if for any two vertices $u$ and $v$ of $D$, there is a directed path from $u$ to $v$. One can easily notice that every strong digraph $D$ has a spanning out-tree. In deed, suppose not and let $T$ be an out-tree of $D$ with maximal number of vertices. Then there exists a vertex $x \in V(D) - V(T)$. Since $D$ is strong then there exists a vertex $y \in V(D)$ such that $(y, x) \in A(D)$. Let $T' = T \cup (y, x)$, an out-tree of $D$ with $V(T') > V(T)$, a contradiction.

In $^2$, El Mminy proved that any digraph $D$ that has a spanning out-tree admits a maximal out-tree. Consequently, every strong digraph admits a maximal out-tree.

Given an oriented path $P$ (resp. oriented cycle $C$), a block is a maximal directed subpath of $P$ (resp. $C$). We denote by $P(k_1, k_2, \cdots, k_n)$ (resp. $C(k_1, k_2, \cdots, k_n)$) the oriented path (resp. oriented cycle) formed of $n$ blocks of lengths $k_1, k_2, \cdots$, and $k_n$ respectively. Moreover, for a block $B$ of an oriented cycle $C$ which is directed from $x$ to $y$, we denote by $x$ (resp. $y$) the source (resp. the sink) of $B$, and we write $B$ as $C_{[x,y]}$. 

$^1$Lebanese University, KALMA Laboratory, Baalbeck.

$^2$University of Paris, IMJ Laboratory, Paris.
Moreover, $x$ and $y$ are called the ends of $B$. In general, for an oriented cycle $C$ with $s$ blocks, we will denote by $B_i$’s the blocks of $C$, such that $B_{2i}$ has ends $x_{i+1}$ and $y_i$ for all $1 \leq i \leq \frac{s}{2} - 1$ and $B_{2i+1}$ has ends $x_{i+1}$ and $y_{i+1}$ for all $0 \leq i \leq \frac{s}{2} - 1$.

A subdivision of a digraph $F$ is a digraph $F'$ obtained by replacing each arc $(x, y)$ of $F$ by an $xy$-directed path of length at least 1.

Let $k_1, k_2, k_3,$ and $k_4$ be positive integers. Cohen et al. proved that any strong digraph with no subdivisions of $C(1, 1, 1, 1)$ has chromatic number less than 24. El Mniny proved that for any $k$ positive integer, any digraph with a spanning out-tree with no subdivision of $C(k, 1, 1, 1)$ has a chromatic number at most $8^k k$. In this paper, we are going to improve the bound established in [2] by proving that if $D$ is a digraph having a spanning out-tree $T$ with no subdivisions of $C(k, 1, 1, 1)$, then the chromatic number of $D$ is at most $18k$.

In our proof, we used for the first time in such investigations what we call a wheel in order to study the chromatic number of the digraphs. A wheel is a graph made up of a chordless cycle and a vertex adjacent to at least three vertices of the cycle, such vertex is called universal. In [1], Thomassen et al. proved that every graph with no wheel as a subgraph is 3-colorable. This result allowed us to get lower bounds for the chromatic number of the studied digraph, $D$, by proving that some subdigraphs of $D$ contain no wheels.

2 Main Result

This section is devoted to prove our main result dealing with the existence of four blocks cycles in digraphs having a spanning out-tree.

Let $k$ be a positive integer and let $D$ be a digraph with a spanning out-tree. $D$ admits a maximal out-tree, say $T$.

We used the same partition of $D$ introduced by El Mniny. However we found less bounds for the chromatic number of each part, sometimes by using cycles and sometimes by using the wheels that facilitated our study of such digraph.

For $i = 0, ..., k - 1$, let $V_i := \cup_{\alpha \geq 0} L_{i+\alpha k}(T)$. Define $D_i$ to be the subdigraph of $D$ induced by $V_i$, and then partition the arcs of $D_i$ as follows:

- $A_1 := \{ (x, y) | x \leq_T y \}$
- $A_2 := \{ (x, y) | y \leq_T x \}$
- $A_3 := A(D_i) \setminus (A_1 \cup A_2)$.

For $0 \leq i \leq k - 1$ and $j = 1, 2, 3$, let $D_i^j$ be the spanning subdigraph of $D_i$ whose arc-set is $A_j$.

Let $C$ be a cycle of $D$ of $s$ blocks, say $B_1, \cdots, B_s$ with $s \geq 4$. We will use the same notations for the ends of the blocks of $C$ as introduced in the introduction such that the $x_i$’s (resp. $y_i$’s) are the sources (resp. sinks) of the blocks of $C$. Without loss of generality, we will suppose that $x_1$ is minimal in $\{ x_i \}_{1 \leq i \leq s/2}$ for $\leq_T$. Let $0 \leq i \leq k - 1$, $C$ is said to be a mixed cycle if all its blocks are induced by arcs in $D_i^1$ except for the block $B_1$ that contains a vertex $z_1 \neq y_1$ such that the arcs of $C_{[x_1, z_1]}$ belong to $T$ and the arcs of $C_{[z_1, y_1]}$ belong to $D_i^1$. $B_1$ is called the mixed block of $C$.

Remark that $x_1$ is the smallest for $\leq_T$ in $C$.

Lemma 2.1. If $D$ contains a mixed cycle then it contains a subdivision of $C(k, 1, 1, 1)$.

Proof. Let $C$ be a mixed cycle of $D$. We will proceed by induction on the number of blocks of $C$, $s$.

For $s = 4$, if $l(C[x_1, z_1]) \geq k$ then $C$ is a subdivision of $C(k, 1, 1, 1)$. Else let $z$ be the minimal for $\leq_T$ in $C - C[x_1, z_1]$. We will study three cases depending on the position of $z$ on $C$. In deed, if $z \in C_{[x_1, y_1]}$ then we replace in $C$, $C_{[z, y_1]}$ by $T_{[z, y_1]}$ and get a subdivision of $C(k, 1, 1, 1)$. Else if $z \in C_{[x_1, y_2]}$ then $C_{[z, y_1]} \cup C_{[x_1, y_2]} \cup T_{[z, y_2]} \cup C_{[z,y_2]}$ is a subdivision of $C(k, 1, 1, 1)$. Finally if $z = x_2$, in this case we will introduce $z'$ to be a minimal for $\leq_T$ in $C - C[z, z_1] - \{ x_2 \}$. If $z' \in C_{[x_2, y_1]}$ or $C_{[x_2, y_2]}$ then replace in $C$, $C_{[z, z_1]}$ by $T_{[x_2, z_2]}$ and get a subdivision of $C(k, 1, 1, 1)$. Else if $z' \in C_{[x_2, y_3]}$ then replace in $C$, $C_{[x_2, y_1]} \cup C_{[x_2, y_2]} \cup T_{[x_2, z_2]}$ and get a subdivision of $C(k, 1, 1, 1)$. The only case left to study is if $z' \in C_{[z, y_1]}$, then replace in $C$, $C_{[x_2, y_1]} \cup C_{[z, z_1]}$ by $T_{[x_2, z_2]}$ and get a subdivision of $C(k, 1, 1, 1)$.

Suppose it is true up to $s - 2$ and let’s prove it for $s$, $s \geq 6$. Notice that $z_1 \leq_T x_i$ for all $1 \leq i \leq \frac{s}{2}$. Let $i \neq 1$ be the integer such that $t_T(x_i)$ is minimal. If $i \geq 3$, then $C_{[z_1, y_1]} \cup C_{[x_2, y_2]} \cup \cdots \cup C_{[x_{i-1}, y_{i-1}]} \cup T_{[x_i, x_i]}$
contains a mixed cycle of $D$ of blocks less than or equal to $s - 2$. Otherwise $i = 2$, we can get a mixed cycle of $D$ of blocks less than or equal to $s - 2$ by replacing in $C$, $C_{\{y_1, y_2\}} \cup C_{\{x_2, y_1\}}$ by $T_{x_1, x_2}$. In both cases, using the induction hypothesis, we get that $D$ contains a subdivision of $C(k, 1, 1, 1)$. □

Consequently, if $D$ contains a cycle $C$ of $s$ blocks, $s \geq 4$, whose all arcs are in $D_i^1$, $0 \leq i \leq k - 1$, then $D$ contains a subdivision of $C(k, 1, 1, 1)$.

Let $C$ be a cycle of $D$ of $s$ blocks, say $B_1, \cdots, B_s$. We will use the same notations for the ends of the blocks of $C$ as introduced in the introduction such that the $y_i$’s (resp. $x_i$’s) are the sources (resp. sinks) of the blocks of $C$. Let $0 \leq i \leq k - 1$, $C$ is said to be back-mixed if $s \geq 6$ and all its blocks are induced by arcs in $D_i^2$ except for $B_s$ that contains a vertex $z_1$ such that the arcs of $C_{\{z_1, x_1\}}$ belong to $T$, the arcs of $C_{\{y_2, z_1\}}$ belong to $D_i^2$, $z_1 \neq y_2$, and $z_1$ is a minimal in $C$ for $\leq T$. $B_s$ is called the back-mixed block of $C$.

Remark that $x_1$ is the smallest for $\leq T$ in $C - C_{\{z_1, x_1\}}$.

**Lemma 2.2.** If $D$ contains a back-mixed cycle then it contains a subdivision of $C(k, 1, 1, 1)$.

**Proof.** Let $C$ be a back-mixed cycle of $D$. We will proceed by induction on the number of blocks of $C$, $s$. Notice that $x_1 \leq_T x_i$ for all $1 \leq i \leq \frac{s}{2}$.

For $s = 6$. Let $i \neq 1$ be the integer such that $i = 1$ is minimal. If $i = 2$, then $C_{\{y_2, x_2\}} \cup C_{\{y_2, x_3\}} \cup C_{\{y_1, x_3\}} \cup C_{\{y_3, x_3\}} \cup T_{x_1, x_2}$ contains a subdivision of $C(k, 1, 1, 1)$. Else $i = 3$, then $C_{\{y_1, x_1\}} \cup C_{\{y_1, x_2\}} \cup C_{\{y_2, x_2\}} \cup C_{\{y_3, x_2\}} \cup C_{\{y_3, x_3\}} \cup T_{x_1, x_3}$ contains a subdivision of $C(k, 1, 1, 1)$.

Suppose it is true up to $s - 2$, and let’s prove it for $s \geq 8$. Let $i \neq 1$ be the integer such that $i = 1$ is minimal. If $i > 3$, then $C_{\{y_2, x_2\}} \cup C_{\{y_2, x_3\}} \cup \cdots \cup C_{\{y_{i-1}, x_i\}} \cup T_{x_1, x_i}$ contains a back-mixed cycle of $D$ with blocks less than or equal to $s - 2$, and so $D$ contains a subdivision of $C(k, 1, 1, 1)$. Else if $i = 2$, then replace in $C$, $C_{\{y_1, x_1\}} \cup C_{\{y_1, x_2\}}$ by $T_{x_1, x_2}$, this contains a back-mixed cycle of $D$ of blocks less than or equal $s - 2$, and so $D$ contains a subdivision of $C(k, 1, 1, 1)$. Finally if $i = 3$, notice that $C_{\{y_1, x_1\}} \cup C_{\{y_1, x_2\}} \cup C_{\{y_2, x_2\}} \cup C_{\{y_3, x_2\}} \cup C_{\{y_3, x_3\}} \cup T_{x_1, x_3}$ contains a subdivision of $C(k, 1, 1, 1)$.

Consequently, if $D$ contains a cycle $C$ of $s$ blocks, $s \geq 6$, whose all arcs are in $D_i^2$, $0 \leq i \leq k - 1$, then $D$ contains a subdivision of $C(k, 1, 1, 1)$.

Let $C$ be a cycle of $D$ of $4$ blocks, say $B_1, \cdots, B_4$. We will use the same notations for the ends of the blocks of $C$ as introduced in the introduction such that the $y_i$’s (resp. $x_i$’s) are the sources (resp. sinks) of the blocks of $C$. $C$ is said to be a bad $4$-blocks cycle if it is either a $C(1, 1, 1, 1)$ (named bad of type 1) or a $C(2, 1, 1, 1)$ such that $B_1 = (y_1, z_1) \cup (z_1, x_1)$ and $x_1 \leq T x_2 \leq_T z_1 \leq_T y_2 \leq_T y_1$ (named bad of type 2). Else $C$ is said to be a good $4$-blocks cycle.

**Lemma 2.3.** Let $C$ be a cycle with $4$-blocks in $D_i^2$, $0 \leq i \leq k - 1$. If $D$ contains no subdivision of $C(k, 1, 1, 1)$, then $C$ is a bad cycle.

**Proof.** Suppose to the contrary that $C$ is a good $4$-blocks cycle. In deed, consider without loss of generality $x_1$ to be a minimal in $C$ for $\leq_T$. Let $z$ minimal in $C - \{x_1\}$ for $\leq_T$. If $z \in C_{\{y_1, x_1\}}$ for $i = 1$ or $2$ then replace in $C$, $(z, x_1)$ by $T_{x_1, x_2}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z = x_2$. Let $z'$ be a minimal in $C - \{x_1, x_2\}$ for $\leq_T$. If $z' \in C_{\{y_2, x_2\}}$ for $i = 1$ or $2$ then replace $(z', x_2)$ by $T_{x_2, z'}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Suppose that $y_1$ and $y_2$ can not be both minimal in $C - \{x_1, x_2\}$ for $\leq_T$. Since $z'$ is bad of type 1, a contradiction. Moreover if without loss of generality $y_1$ is minimal in $C - \{x_1, x_2\}$ for $\leq_T$, then let $z''$ be a maximal in $C - \{y_2\}$ for $\leq_T$ such that $z'' \leq_T y_2$. If $z'' \in C_{\{y_2, x_2\}}$ for $i = 1$ or $2$ then replace in $C$, $(y_2, z'')$ by $T_{x_2, z''}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z'' = y_1$ or $z'' = x_2$ and so $C$ is bad of type 1, a contradiction. Hence $z' \notin C_{\{y_2, x_2\}}$ for $i = 1, 2$. If without loss of generality $z' \in C_{\{y_1, x_2\}}$ for $i = 1$ or $2$ then replace in $C$, $(y_2, z'')$ by $T_{x_2, z''}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z'' = y_2$ and let $z'''$ be a maximal in $C - \{y_2\}$ with $z''' \leq_T y_2$. Notice that $z'''$ can not be $y_2$ since else $C$ is bad of type 2, where $x_1 \leq_T x_2 \leq_T z' \leq_T y_2 \leq_T y_1$, a contradiction. Hence $z''' \in C_{\{y_1, x_2\}}$ or $C_{\{y_1, z''\}}$ and
Lemma 2.4. If $D$ has no subdivision of $C(k,1,1,1)$ then $\chi(D^3_i) \leq 2$, for all $0 \leq i \leq k - 1$.

Proof. We claim that the underlying graph of $D^3_i$ is bipartite. In deed, suppose to the contrary that $D^3_i$ contains an odd cycle $C = x_1 \ldots x_l$. Without loss of generality, suppose that $x_1$ is with minimal level for $T$ in $C$. Now we will study two cases:

Case 1: Neither $x_l$ is an ancestor of $x_2$ nor $x_2$ is an ancestor of $x_l$. In this case $l_T(x_1) = l_T(x_2)$, since else let $y$ be their least common ancestor, so $T_{[y,x_2]} \cup T_{[y,x_1]} \cup (x_1,x_l) \cup (x_2,x_l)$ is a subdivision of $C(k,1,1,1)$, a contradiction. Thus $t \geq 5$. Similarly, we can see that both $T_{[y,x_1]}$ and $T_{[y,x_2]}$ have lengths less than $k$.

Notice that $x_2$ is an ancestor of $x_{l-1}$. If not, let $z$ be the least common ancestor of $x_2$ and $x_{l-1}$.

If $x_1 \notin T_{[z,x_{l-1}]}$ then $T_{[z,x_{l-1}]} \cup T_{[z,x_2]} \cup (x_1,x_2) \cup (x_1,x_l) \cup x_{l-1}x_l$ is a subdivision of $C(k,1,1,1)$, a contradiction. Else, $T_{[y,x_1]} \cup x_{l-1}x_l \cup T_{[y,x_2]} \cup (x_1,x_2) \cup T_{[x_1,x_{l-1}]}$ is a subdivision of $C(k,1,1,1)$, a contradiction.

As well, $x_l$ is an ancestor of $x_3$. Thus, $(x_2,x_3)$ and $(x_1,x_{l-1})$ are arcs in $D^3_i$. Then $T_{[x_2,x_3]} \cup T_{[x_2,x_{l-1}]} \cup (x_2,x_3)$ and $(x_1,x_{l-1})$ is a subdivision of $C(k,1,1,1)$, a contradiction.

Case 2: Without loss of generality, suppose that $x_2 \leq_T x_1$.

Let $i$ the smallest integer greater than 2 satisfying $l_T(x_i) > l_T(x_{i-1})$. We will study the following cases:

If $i = 3$. Consider the least common ancestor of $x_1$ and $x_3$, $y$, then $T_{[y,x_1]} \cup (x_1,x_2) \cup T_{[y,x_3]} \cup (x_2,x_3) \cup T_{[x_2,x_1]}$ is a subdivision of $C(k,1,1,1)$, a contradiction. Notice that $D^3_i$ has no path of type $P(1,2)$ satisfying the same properties of the $x_1x_2x_3$ which is a path of type $P(1,2)$ with $x_2 \leq_T x_1$, since else we can find similarly a subdivision of $C(k,1,1,1)$ in $D$, a contradiction.

If $i = t$, let $z$ the least common ancestor of $x_1$ and $x_{t-2}$, then $T_{[z,x_2]} \cup (x_1,x_1) \cup T_{[z,x_t]} \cup (x_{t-1},x_{t-2}) \cup (x_{t-1},x_t)$ is a subdivision of $C(k,1,1,1)$, a contradiction.

If $4 \leq i < t$. Then $l_T(x_i) > l_T(x_{i-2})$, since else we can consider the least common ancestor of $x_1$ and $x_i$ to find a subdivision of $C(k,1,1,1)$, a contradiction. Hence $l_T(x_i) > l_T(x_{i-2})$. Notice that $x_{i-2} \leq_T x_i$, since else let $y$ be their least common ancestor, and so $T_{[y,x_{i-2}]} \cup T_{[y,x_i]} \cup (x_{i-1},x_i) \cup (x_{i-2},x_{i-1})$ is a subdivision of $C(k,1,1,1)$, a contradiction. Notice that here $i = 4$, since else one can notice that $(x_{i-1},x_i) \cup (x_{i-1},x_{i-2}) \cup (x_{i-2},x_{i-3})$ is a path $P(1,2)$ as in case $i = 3$, a contradiction. Notice that neither $x_4 \leq_T x_t$ nor $x_t \leq_T x_4$, since else we can combine the directed path in $T$ between $x_4$ and $x_t$ and the oriented path in $C$ between $x_4$ and $x_t$ and find a subdivision of $C(k,1,1,1)$, a contradiction. Moreover, using the least common ancestor of $x_1$ and $x_3$, we can prove that $l_T(x_t) = l_T(x_1)$ and the length of the directed paths in $T$ from their least common ancestor to each is less than $k$. Now, using the least common ancestor of $x_4$ and $x_t$ we can show similarly that $l_T(x_1) = l_T(x_3)$. Notice that $t \geq 7$.

Denote by $y$ the least common ancestor of $x_5$ and $x_t$. Notice here that neither $x_1$ nor $x_3$ is ancestor of $x_1$. Also $y \neq x_t$, since else we can find a subdivision of $C(k,1,1,1)$, a contradiction. Besides, $y \neq x_3$, since else we can use the least common ancestor of $x_1$ and $x_3$ to find a subdivision of $C(k,1,1,1)$, a contradiction.

We claim also that neither $x_1$ nor $x_3$ is ancestor of $x_5$. In deed, if $x_1 \leq_T x_5$, we'll consider the least common ancestor of $x_1$ and $x_t$ to find a subdivision of $C(k,1,1,1)$, a contradiction. Similarly, if $x_3 \leq_T x_5$, we'll consider the least common ancestor of $x_4$ and $x_t$ and the least common ancestor of $x_1$ and $x_3$ to find a subdivision of $C(k,1,1,1)$, a contradiction.

Hence denote by $z$ the least common ancestor of $x_1$ and $x_3$. If $y \notin T_{[z,x_1]}$, $i = 1,3$, then $T_{[z,x_3]} \cup (x_3,x_4) \cup x_4x_5 \cup T_{[y,x_3]} \cup T_{[y,x_1]} \cup T_{[z,x_1]} \cup (x_1,x_t)$ is a subdivision of $C(k,1,1,1)$, a contradiction. Else, $y$ is in particular the least common ancestor of $x_2$ and $x_5$ and $l(T_{[y,x_2]}) \geq k$, then $T_{[y,x_2]} \cup T_{[y,x_5]} \cup x_5x_4 \cup (x_3,x_4) \cup (x_2,x_3)$ is a subdivision of $C(k,1,1,1)$, a contradiction.

Hence $D^3_i$ contains no odd cycle then $\chi(D^3_i) \leq 2$.

Now we are ready to prove our main result:

Theorem 2.1. Let $k$ be a positive integer and $D$ a digraph with a spanning out-tree with no subdivisions of $C(k,1,1,1)$ then the chromatic number of $D$ is at most $18k$.
Claim 2: $\chi(D^2_i) \leq 3$, for all $0 \leq i \leq k - 1$.

Proof. Suppose to the contrary that $\chi(D^2_i) > 3$, then $D^2_i$ contains a wheel of cycle say $C$ and a universal vertex $x$. If $l_T(x)$ is a cycle of 4 blocks or more, since else $D$ contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $C$ is a cycle of 2 blocks. Denote by $z_1$ (resp. $z_2$) the vertex of $C$ with in-degree (resp. out-degree) zero.

We will consider the same partition for the arcs of $D$ used in the beginning of this section.

Claim 1: $\chi(D^2_l) \leq 3$, for all $0 \leq i \leq k - 1$.

Proof. Suppose to the contrary that $\chi(D^2_l) > 3$, then $D^2_l$ contains a wheel of cycle say $C$ and a universal vertex $x$. If $l_T(x)$ is a cycle of 4 blocks or more, since else $D$ contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $C$ is a cycle of 2 blocks. Denote by $z_1$ (resp. $z_2$) the vertex of $C$ with in-degree (resp. out-degree) zero.

We will reach a contradiction finding a subdivision of $C(k, 1, 1, 1)$ by studying the order of the levels of $x$ and its neighbors. Denote by $x_1, x_2, x_3$ three neighbors of $x$ on $C$ such that $l_T(x_1) < l_T(x_2) < l_T(x_3)$.

If $l_T(x_1) < l_T(x_2)$, then since $z_1 \notin \{x_1, x_2, x_3\}$, $C[z_1, x_2] \cup (x_2, x) \cup (x, x_3) \cup C[z_1, x_3]$ is cycle with 4 blocks in $D^1_i$ and so $D$ contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $l_T(x_2) < l_T(x)$. If $x_1$ and $x_2$ belong to the same block on $C$, then $(C - C[x_1, x_2]) \cup (x_1, x) \cup (x_2, x)$ is a cycle with 4 blocks in $D^1_i$, and so $D$ contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Otherwise, $C[z_1, x_2] \cup (x_2, x) \cup (x_1, x)$ is a cycle with 4 blocks in $D^1_i$ and so $D$ contains a subdivision of $C(k, 1, 1, 1)$, a contradiction.

With the fact that $D_i = D^1_l \cup D^2_l \cup D^3_l$ then $\chi(D_i) \leq 2.3.3 = 18$ for all $i \in \{0, \ldots, k - 1\}$. Consequently, as $V(D_i)$, $0 \leq i \leq k - 1$ form a partition of $V(D)$, we obtain a proper 18-coloring of $D$ by giving to each $D_i$ 18 distinct colors. This implies the hoped result.
Acknowledgment The author would like to thank A. El Sahili and M. Mortada for their valuable remarks. The author would like also to acknowledge the National Council for Scientific Research of Lebanon (CNRS-L) and the Agence Universitaire de la Francophonie in cooperation with Lebanese University for granting a doctoral fellowship to Zahraa Mohsen.

References

[1] Carsten Thomassen, Bjarne Toft, Non-separating induced cycles in graphs, Journal of Combinatorial Theory, Series B, Volume 31, Issue 2, 1981, Pages 199-224.

[2] Darine Al-Mniny, Subdivisions of four blocks cycles in digraphs with large chromatic number, Discrete Applied Mathematics, Volume 305, 2021, Pages 71-75.

[3] Cohen, N, Havet, F, Lochet, W, Nisse, N. Subdivisions of oriented cycles in digraphs with large chromatic number. J Graph Theory. 2018; 89: 439–456.