EXPONENTIAL ERGODICITY FOR DAMPING HAMILTONIAN DYNAMICS WITH STATE-DEPENDENT AND NON-LOCAL COLLISIONS

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Abstract. In this paper, we investigate the exponential ergodicity in a Wasserstein-type distance for a damping Hamiltonian dynamics with state-dependent and non-local collisions, which indeed is a special case of piecewise deterministic Markov processes while is very popular in numerous modelling situations including stochastic algorithms. The approach adopted in this work is based on a combination of the refined basic coupling and the refined reflection coupling for non-local operators. In a certain sense, the main result developed in the present paper is a continuation of the counterpart in [2] on exponential ergodicity of stochastic Hamiltonian systems with Lévy noises and a complement of [5] upon exponential ergodicity for Andersen dynamics with constant jump rate functions.

Keywords: damping Hamiltonian dynamics; non-local collision; exponential ergodicity; Wasserstein-type distance; coupling

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1. Introduction

1.1. Background. Piecewise deterministic Markov processes (PDMPs for short) constitute a very natural class of non-diffusive stochastic processes, where the mathematical framework was built by Mark H. A. Davis in [11]. Roughly speaking, the PDMP is a process which jumps at some random time and moves continuously between two adjacent random times; see [12, 17] for more details. According to [11, Section 3], the probability law of a PDMP with the state space $E$ is determined by the following three ingredients: (i) a vector field $\Xi$, generating a deterministic flow; (ii) a jump rate function $J : E \to [0, \infty)$, giving the law of the random times between jumps; (iii) a jump measure $Q : E \times E \to (0, \infty)$ (i.e., for each fixed $A \in \mathcal{B}(E)$, $E \ni x \mapsto Q(x, A)$ is a measurable function, and, for each fixed $x \in E$, $\mathcal{B}(E) \ni A \mapsto Q(x, A)$ is a probability measure), giving the transition probability kernel of its jumps. The class of PDMPs is more general than compound Poisson processes and basic queues, and includes also jump processes over vector fields. PDMPs have a great variety of applications such as in biology (cellular mass), physics (polymers length), computer science (TCP window size process), reliability (workload and repairable systems), mathematical finance, to name a few; see, for instance, an excellent comprehensive survey paper [22] on recent progresses of PDMPs and related open problems. Understanding the ergodic properties of these models from all areas above, in particular the distance under which (or the rate at which) they stabilize towards equilibrium, has in turn increased the interest in the long-time behavior of PDMPs; see [3, 9, 10, 13] and references therein for the recent study.

In this paper, we consider a special class of PDMPs $(X_t, V_t)_{t \geq 0}$ on the state space $\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$ and associated with the following infinitesimal generator

\[
(\mathcal{L}f)(x, v) = \left(\langle \nabla_x f(x, v), v \rangle - \langle \nabla_v f(x, v), \gamma v + \nabla U(x) \rangle \right)
+ J(x, v) \int_{\mathbb{R}^d} \left( f(x, u) - f(x, v) \right) \varphi(u) \, du
= : (\mathcal{L}_1, \gamma f)(x, v) + (\mathcal{L}_2 f)(x, v), \quad f \in C_b^1(\mathbb{R}^{2d}),
\]

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where $\gamma > 0$, $U: \mathbb{R}^d \to \mathbb{R}$ is smooth, $J: \mathbb{R}^{2d} \to (0, \infty)$, and $\varphi(-)$, which is radial (i.e., $\varphi(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^d$), is a probability density function on $\mathbb{R}^d$. In (1.1), $C_0^1(\mathbb{R}^{2d})$ means the collection of bounded real-valued functions $f(x, v)$ on $\mathbb{R}^{2d}$, which are differentiable in $x$ and $v$, respectively, and $\nabla_x f(x, v)$ and $\nabla_v f(x, v)$ denote the first order gradients of $f(x, v)$ with respect to the variable $x$ and the variable $v$, respectively.

Now, we make some detailed expositions on the quantities involved in (1.1). More precisely, $(v, -\gamma v - \nabla U(x))$ is the vector field generating the damping Hamiltonian flow, where $\gamma$ means the friction intensity that ensures a damped-driven Hamiltonian and $-\gamma v$ stands for the damping force; $J : \mathbb{R}^{2d} \to (0, \infty)$ is the jump rate; $\varphi(u)$ du represents the jump measure. In terminology, $\mathcal{L}_{1,\gamma}$ is called the Liouville operator associated with the damping Hamiltonian flow generated by the vector field $(x, -\gamma v - \nabla U(x))$, and $\mathcal{L}_2$ is the so-called non-local collision operator. In particular, if $\varphi(u)$ is the density function of the standard normal distribution and $J(x, v) = \lambda$ for all $x, v \in \mathbb{R}^d$, $\mathcal{L}_2$ is called the complete momentum randomization operator; see, for example, [27]. It is worthwhile to emphasize that, in statistical physics, the damping Hamiltonian system has been applied widely to model many vibration phenomena (e.g., the generalized Duffing oscillator); see e.g., [21, 25]. In the past two decades, great progresses upon long term behaviors (e.g., ergodicity and large deviation) have been made for stochastic damping Hamiltonian systems; see, for instance, [8, 14, 21, 28] and references within for more details.

1.2. Main result. The purpose of this paper is to study the exponential ergodicity of the PDMP $(X_t, V_t)_{t \geq 0}$ whose generator $\mathcal{L}$ is given by (1.1). Before we state our main result, we first present the assumptions. First of all, we assume that

$$(H_0) \quad \text{For any } \beta \in \mathbb{R}, \text{ there exists a constant } K_{\beta, U} > 0 \text{ such that for all } x, x' \in \mathbb{R}^d,$$

$$|\beta(x - x') + \nabla U(x') - \nabla U(x)| \leq K_{\beta, U}|x - x'|.$$ 

In particular, $\nabla U$ is Lipschitz continuous under $(H_0)$.

For the jump rate $J$ and the probability density $\varphi$ of the jump measure, we assume that

$$(A_1) \quad J : \mathbb{R}^{2d} \to (0, \infty) \text{ is uniformly bounded between two positive constants, i.e., there exist constants } \lambda_1, \lambda_2 > 0 \text{ such that } \lambda_1 \leq J(x, v) \leq \lambda_2 \text{ for all } (x, v) \in \mathbb{R}^{2d}. \text{ Moreover, } J \text{ is globally Lipschitz continuous on } \mathbb{R}^{2d}, \text{i.e., there exists a constant } \lambda_J > 0 \text{ such that for all } (x, v), (x', v') \in \mathbb{R}^{2d},$$

$$|J(x, v) - J(x', v')| \leq \lambda_J(|x - x'| + |v - v'|).$$

$$(A_2) \quad \text{For any } \alpha, \kappa > 0, \text{ there exist } c_*(\alpha, \kappa), c^*(\alpha, \kappa) > 0 \text{ such that for all } z \in \mathbb{R}^d,$$

$$c_*(\alpha, \kappa) \leq A_{\alpha, \kappa}(z) := \int_{\mathbb{R}^d} \psi_\alpha(z)(u) \, du \quad \text{and} \quad 1 - A_{\alpha, \kappa}(z) \leq c^*(\alpha, \kappa)|z|,$$

where for all $\xi, u \in \mathbb{R}^d$,

$$\psi_\xi(u) := \varphi(u) \land \varphi(u + \xi),$$

and, for the threshold $\kappa > 0$, the truncation counterpart of $z \in \mathbb{R}^d$ is defined by

$$z_\kappa = \frac{\kappa \land |z|}{|z|} z \mathbf{1}_{\{z \neq 0\}} + 0 \mathbf{1}_{\{z = 0\}}.$$ 

Since $A_{\alpha, \kappa}(0) = \int_{\mathbb{R}^d} \psi_0(u) \, du = \int_{\mathbb{R}^d} \varphi(u) \, du = 1$, in some sense (1.3) indicates the non-degenerate property and the continuity of the probability density $\varphi$.

Besides all the assumptions above, we further need the following Lyapunov condition:

$$(B_1) \quad \text{There exist a } C^1\text{-function } \mathcal{W} : \mathbb{R}^{2d} \to [1, \infty) \text{ and constants } c_0, C_0 > 0 \text{ such that}$$

$$\lim_{|x| + |v| \to \infty} \mathcal{W}(x, v) = \infty$$

and for all $(x, v) \in \mathbb{R}^{2d}$,

$$\langle \mathcal{L} \mathcal{W} \rangle(x, v) \leq -c_0 \mathcal{W}(x, v) + C_0.$$
There exists a constant $c^* > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$
\phi_{(B_2)} \quad (1.8) \quad \int_{\mathbb{R}^d} \mathcal{W}(x, u) \varphi(u) \, du \leq c^* \inf_{v \in \mathbb{R}^d} \mathcal{W}(x, v), \quad \int_{\mathbb{R}^d} \mathcal{W}(x, u) \Psi_{\xi}(u) \, du \leq c^* \inf_{v \in \mathbb{R}^d} \mathcal{W}(x, v) |\xi|,
$$

where for all $\xi, u \in \mathbb{R}^d$,

$$
\phi_{(B_2)} \quad (1.9) \quad \Psi_{\xi}(u) := \varphi(u) - \psi_{\xi}(u)
$$

with $\psi_{\xi}(u)$ being introduced in $\phi_{(1.4)}$.

Let $\mathcal{P}(\mathbb{R}^{2d})$ be the set of probability measures on $\mathbb{R}^{2d}$. For $\mu, \nu \in \mathcal{P}(\mathbb{R}^{2d})$, define the quasi-Wasserstein distance between $\mu$ and $\nu$ induced by a distance-like function $\Phi : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to [0, \infty)$ (see e.g. $\phi_{(16)}$ Definition 4.3) as below

$$
\phi_{(B_2)} \quad (1.10) \quad W_{\Phi}(\mu, \nu) = \inf_{\Pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \Phi(x, y) \Pi(dx, dy),
$$

where $\mathcal{C}(\mu, \nu)$ stands for the collection of all couplings of $\mu$ and $\nu$. In particular, $W_{\Phi}$ goes back to the classical Wasserstein distance when $\Phi$ is a metric function. Note that $W_{\Phi}(\mu, \nu) = 0$ if and only if $\mu = \nu$, since $\Phi$ is a distance-like function. Moreover, the space

$$
\phi_{(B_2)} \quad \mathcal{P}_{\Phi}(\mathbb{R}^{2d}) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^{2d}) : \int_{\mathbb{R}^{2d}} \Phi(x, 0) \mu(dx) < \infty \right\}
$$

is complete under $W_{\Phi}$, i.e., each $W_{\Phi}$-Cauchy sequence in $\mathcal{P}_{\Phi}(\mathbb{R}^{2d})$ converges with respect to $W_{\Phi}$.

For each $t \geq 0$, let $P_t((x, v), \cdot)$ be the transition probability kernel of the Markov process $(X_t, V_t)_{t \geq 0}$ with initial value $(X_0, V_0) = (x, v)$ associated with the generator $\mathcal{L}$. Furthermore, we shall write $\mu P_t$ to mean the distribution of $(X_t, V_t)$ with initial distribution $\mu \in \mathcal{P}(\mathbb{R}^{2d})$.

The main result of this paper is stated as follows.

**Theorem 1.1.** Assume that $(H_0), (A_1), (A_2), (B_1)$ and $(B_2)$ hold, and that the following inequality

$$
\phi_{(B_2)} \quad (1.10) \quad \beta \geq 4K_{\beta, U}
$$

is solvable in the interval $(0, \gamma^2/4]$, where $\gamma$ was given in $\phi_{(1.4)}$ and $K_{\beta, U}$ was given in $(H_0)$. Then, the PDMP $(X_t, V_t)_{t \geq 0}$ corresponding to the operator $\mathcal{L}$ in $\phi_{(1.4)}$ is exponentially ergodic in the sense that there exist a unique invariant probability measure $\mu \in \mathcal{P}_{\Phi}(\mathbb{R}^{2d})$ and a constant $\lambda^* > 0$ such that for any $\nu \in \mathcal{P}_{\Phi}(\mathbb{R}^{2d})$ and $t \geq 0$,

$$
\phi_{(B_2)} \quad (1.11) \quad W_{\Phi}(\nu P_t, \mu) \leq C(\mu, \nu) e^{-\lambda^* t},
$$

where for all $(x, v), (x', v') \in \mathbb{R}^{2d}$,

$$
\phi_{(B_2)} \quad (1.12) \quad \Phi((x, v), (x', v')) := (|x - x'| + |v - v'|) \wedge 1)(W(x, v) + W(x', v'))
$$

and $C(\mu, \nu)$ is a positive function depending on $\mu$ and $\nu$ (independent of $t$).

To illustrate the effectiveness of Theorem $\phi_{(1.1)}$ we consider the following example.

**Example 1.2.** Assume that Assumption $(A_1)$ holds. Let $U(x) = \theta |x|^2$ with

$$
\phi_{(B_2)} \quad \frac{\gamma^2}{8} \geq \theta > \frac{(\lambda_1 + \gamma)^2(\lambda_2 - \lambda_1)^2}{4(2\lambda_1 \lambda_2 - \lambda_1^2 + 4\lambda_2 \gamma + 3\gamma^2)},
$$

and $\varphi(x) = \varphi_1(x) := c_{d, \beta_1}(1 + |x|)^{-d - \beta_1}$ with $\beta_1 > 0$ or $\varphi(x) = \varphi_2(x) := c_{d, \beta_2} \exp(-|x|^2)$ with $\beta_2 > 0$. Then, the conclusion of Theorem $\phi_{(1.1)}$ holds with $W(x, v) = (1 + |x|^2 + |v|^2)$ and the previously defined $\varphi_2$ or $\varphi_1$ when $\beta_1 > 2$, and with $W(x, v) = (1 + |x|^2 + |v|^2)^{(\beta_1 - \epsilon)/2}$ for any $\epsilon \in (0, \beta_1)$ and the foregoing $\varphi_1$ when $\beta_1 \in (0, 2]$. 

1.3. Comments. Recently, plenty of interests have grown concerning the application of PDMPs to sample from a target distribution (for example, algorithms are referred to as PDMP Monte Carlo (PDMP-MC) methods). Therefore, much more efforts are devoted to the ergodicity and the other long time behaviour of the PDMPs; see e.g. [1] [3] [5] [18] and references therein. Our work is related to the existing result [7] and the recent one [5]. In [7], the exponential ergodicity for a randomized Hamiltonian Monte Carlo (also called Hybrid Monte Carlo) was treated under the same conditions that imply geometric ergodicity of the solution to underdamped Langevin equations. The proof of [7] is based on a Foster–Lyapunov drift condition, a minorization condition and Harris’ theorem. Via a coupling approach, the convergence to equilibrium of Andersen dynamics (which becomes exact randomized Hamiltonian Monte Carlo when the associated molecular system consists of only one particle) was handled in [5]. As in [5,6], we herein also adopt the probabilistic coupling method, whereas the setting is significantly different from those in [5,7]. For example,

(i) The jump rate function in [5,6,7] is a constant function and moreover the jump measure is the standard normal distribution, though the non-local collisions involved in the PDMPs in [5,6,7] are much more general. Moreover, the exponential ergodicity in a Wasserstein sense of Andersen dynamics was addressed in [5] nevertheless the position component was confined in a high-dimensional torus. According to the private communications with Nawaf Bou-Rabee, the issue on ergodicity of Andersen dynamics, where not only the velocity component but also the position component are supported on the whole Euclidean space, is highly non-trivial. Furthermore, we would like to emphasize that the exponential ergodicity of Andersen dynamics surviving on the whole Euclidean space was investigated in [6], where the semi-metric inducing the Wasserstein-type distance admits the following form: for all \((x,v),(x',v') \in \mathbb{R}^d\),

\[
\Phi((x,v),(x',v')) = ((|x-x'|+|v-v'|) \wedge 1)(|x-x'|^2 + |v-v'|^2)
\]

while the counterpart designed in Theorem 1.1 is a multiplicative type distance-like function (see (1.12) for more details) so the quasi-metric involved in [6] is essentially different from the one we exploited in Theorem 1.1.

(ii) In the present paper, most importantly, we focus on the state-dependent jump rate function. Additionally, we can not only deal with (sub-)Gaussian probability measures but also the probability measures with heavy tails such like \(\varphi(u) du = c_{d,\beta}(1 + |u|)^{-d-\beta} du\) for \(\beta > 0\). Due to the appearance of the state-dependent jump rate function, compared with [5,6,7], some additional sacrifices need to be paid. Throughout the paper, the price to pay is that we will require that the constant \(\gamma\) in the operator (1.1) is positive; that is, we merely work on the damping Hamiltonian flow in our paper. On the other hand, the main results in [5,6] require the constant jump rate function \(J(x,v)\) is large enough while here in Theorem 1.1 we do not need such kind condition even for the setting of non-constant jump rate functions. Therefore, from these points of view above, the results of [7,5] and our paper complement each other.

The approach of our paper is also motivated partly by our previous work [2] on exponential ergodicity of stochastic Hamiltonian systems with Lévy noises. However, in contrast to [2] the non-local collision operator in the present setting is not only highly degenerate but also state-dependent so much more delicate work are to be implemented. In particular, we shall adopt a combination of the refined basic coupling and the refined reflection coupling (rather than the refined basic coupling exploited merely in [2]) in order to include more general probability measures (e.g., (sup-)Gaussian or (sub-)Gaussian probability measures and probability measures with heavy tails). So, in a certain sense, Theorem 1.1 is a continuation of the corresponding main result in [2] on exponential ergodicity of stochastic Hamiltonian systems with Lévy noises. Furthermore, we emphasize that the process under investigation in this paper has some essentially different properties from stochastic Hamiltonian systems with Lévy noises under consideration in [2]. For example, under some regular conditions the process associated with stochastic Hamiltonian systems with Lévy noises can possess the strong Feller property; see [26]. Nonetheless, since
the non-local collision operator \( \mathcal{L}_2 \) in (1.1) is a bounded operator on \( B_b(\mathbb{R}^{2d}) \) under Assumption (A_1), the PDMP \((X_t, V_t)_{t \geq 0}\) corresponding to the operator \( \mathcal{L} \) in (1.1) can never enjoy the strong Feller property.

The rest of the paper is arranged as follows. In the next section, we construct a coupling operator and examine the existence of the associated coupling process. Section 3 is devoted to the Feller property. In the last section, we present some sufficient conditions to guarantee that Assumptions and the technical condition (1.10) involved in Theorem 1.1 are verifiable.

## 2. Coupling operator and coupling process

We start with some notations. Let \( I_{d \times d} \) be the \( d \times d \) identity matrix, and \( \mathbb{R}^d \otimes \mathbb{R}^d \) be the set of all \( d \times d \) matrices. For \( x \in \mathbb{R}^d \), we write \( x \otimes x = xx^* \in \mathbb{R}^d \oslash \mathbb{R}^d \) with \( x^* \) being its transpose. For \( x \in \mathbb{R}^d \), define the following orthogonal matrix

\[
\Pi_x = \begin{pmatrix} I_{d \times d} - 2 \frac{x}{|x|} \otimes \frac{x}{|x|} \end{pmatrix} = \mathbb{I}_{\{x \neq 0\}} - I_{d \times d} \mathbb{I}_{\{x = 0\}} \in \mathbb{R}^d \otimes \mathbb{R}^d.
\]

For any \( a, b \in \mathbb{R} \), let \( a^+ = \max\{a, 0\} \), i.e., the positive part of the number \( a \), and \( a \wedge b = \min\{a, b\} \).

Fix \( \alpha, \kappa > 0 \). For \( y = (x, v), (x', v') \in \mathbb{R}^{2d} \) and \( f \in C^1_b(\mathbb{R}^{2d}) \), define the following operator

\[
(\mathcal{L}_{\gamma, \alpha, \kappa} f)(y) = (\mathcal{L}_{\gamma, f})(y) + (\mathcal{L}_{2, \alpha, \kappa} f)(y),
\]

where

\[
(\mathcal{L}_{\gamma, f})(y) := \langle \nabla_x f(y), v \rangle + \langle \nabla_x f(y), v' \rangle - \langle \nabla_v f(y), \gamma v + \nabla U(x) \rangle - \langle \nabla_v f(y), \gamma v' + \nabla U(x') \rangle
\]

and

\[
(\mathcal{L}_{2, \alpha, \kappa} f)(y) := (J(x, v) \wedge J(x', v')) \times \left\{ \int_{\mathbb{R}^d} (f((x, u), (x, u + \alpha(x - x')_\kappa)) - f(y)) \psi_{\alpha(x-x')_\kappa}(u) \, du \right. \\
\left. + \int_{\mathbb{R}^d} (f((x, u), (x, \Pi_{(x-x')_\kappa} u)) - f(y)) \Psi_{\alpha(x-x')_\kappa}(u) \, du \right\}
\]

\[
+ (J(x, v) - J(x', v'))^+ \int_{\mathbb{R}^d} (f((x, u), (x', v')) - f(y)) \psi(u) \, du
\]

\[
+ (J(x', v') - J(x, v))^+ \int_{\mathbb{R}^d} (f((x, u), (x', v)) - f(y)) \psi(u) \, du,
\]

where, for \( z \in \mathbb{R}^d \), \( (z)_\kappa \) was defined as in (1.4), and \( \psi_\kappa(\cdot) \) and \( \Psi_\kappa(\cdot) \) were introduced in (1.5) and (1.3), respectively. It is easy to see that the last two items on the right hand side of (2.4) vanish once the jump rate \( J \) is a constant function (i.e., \( J(x, v) = \lambda \) for all \( (x, v) \in \mathbb{R}^{2d} \) and some \( \lambda > 0 \)).

**Remark 2.1.** As shown in Lemma 2.2 below, for any \( \gamma, \alpha, \kappa > 0 \), \( \mathcal{L}_{\gamma, \alpha, \kappa} \) is a coupling operator of \( \mathcal{L} \). Indeed, for the operator \( \mathcal{L}_{2, \alpha, \kappa} \), we adopt the synchronous coupling as showed in (2.5). The coupling operator \( \mathcal{L}_{2, \alpha, \kappa} \) associated with \( \mathcal{L}_2 \) is indeed built based on a combination of the refined basic coupling and the refined reflection coupling as well as the independent coupling:

\[
((x, v), (x', v')) \Rightarrow \begin{cases} 
((x, u), (x', u + \alpha(x - x')_\kappa)), \\
(x, u), \Pi_{(x-x')_\kappa} u), \\
((x, u), (x', v')), \\
(x, v), (x', u)) \\
((x, v) \wedge J(x', v')) \psi_{\alpha(x-x')_\kappa}(u) \, du \\
(J(x, v) \wedge J(x', v')) \Psi_{\alpha(x-x')_\kappa}(u) \, du \\
(J(x, v) - J(x', v'))^+ \varphi(u) \, du \\
+ (J(x', v') - J(x, v))^+ \varphi(u) \, du.
\end{cases}
\]

See e.g. [19] or [21] for more details. In particular, concerning the coupling counterpart of \( \mathcal{L}_{2, \alpha, \kappa} \), the velocity components change accordingly while the position components remain unchanged. More precisely, the velocity component \((v, v')\) changes into \((u, u + \alpha(x - x')_\kappa)\) with the maximum common intensity measure \((J(x, v) \wedge J(x', v')) \psi_{\alpha(x-x')_\kappa}(u) \, du\); the velocity component \((v, v')\)
moves to the point \((u, \Pi(x-x_\kappa), u)\) with the intensity measure \((J(x, v) \wedge J(x', v'))\psi_{\alpha(x-x_\kappa)}(u)\, du;\) the velocity component \((v, v')\) changes to \((u, v')\) and \((v, u)\) with the remainder intensity measures \((J(x, v) - J(x', v'))^+ \varphi(u)\, du\) and \((J(x', v') - J(x, v))^+ \varphi(u)\, du\), respectively, to guarantee the marginal property of the coupling operator \(\mathcal{L}_{2, \alpha, \kappa}\) defined by \((2.3)\). Moreover, it is worthy to stress that the construction above heavily depends on the radial property of \(\varphi\).

**Lemma 2.2.** For any \(\gamma, \alpha, \kappa > 0\), the operator \(\mathcal{L}_{\gamma, \alpha, \kappa}\), defined in \((2.2)\), is a coupling operator of \(\mathcal{L}\), introduced in \((1.1)\).

**Proof.** For simplicity, we shall write \(\mathcal{L}_{\gamma, \alpha, \kappa}, \mathcal{L}_{1, \gamma}, \) and \(\mathcal{L}_{2, \alpha, \kappa}\) as \(\mathcal{L}, \mathcal{L}_1\) and \(\mathcal{L}_2\), respectively. To demonstrate that \(\mathcal{L}\) is a coupling operator, we only need to verify that \(\mathcal{L}_1\) and \(\mathcal{L}_2\) are coupling operators corresponding to \(\mathcal{L}_{1, \gamma}\) and \(\mathcal{L}_2\), respectively. To achieve this, it is sufficient to prove that for any \(f \in C^1_0(\mathbb{R}^{2d})\) so that \(f(y) = g(x, v) + h(x', v')\) with some \(h, g \in C^1_0(\mathbb{R}^{2d})\) and for any \(y = ((x, v), (x', v'))\),

\[
(\mathcal{L}_1 f)(y) = (\mathcal{L}_{1, \gamma} g)(x, v) + (\mathcal{L}_{1, \gamma} h)(x', v'),
\]

and

\[
(\mathcal{L}_2 f)(y) = (\mathcal{L}_2 g)(x, v) + (\mathcal{L}_2 h)(x', v').
\]

It is trivial to see that \((2.5)\) holds true. On the other hand, according to the definition of \(\mathcal{L}_2\), we deduce that

\[
(\mathcal{L}_2 f)(y) = (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (g(x, u) - g(x, v)) \psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (g(x, u) - g(x, v)) \Psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x, v) - J(x', v'))^+ \int_{\mathbb{R}^d} (g(x, u) - g(x, v)) \psi(u)\, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', u + \alpha(x - x_\kappa)) - h(x', v')) \psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', \Pi_{(x-x_\kappa)} u) - h(x', v')) \Psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x', v') - J(x, v))^+ \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \psi(u)\, du
\]

\[
= (\mathcal{L}_2 g)(x, v)
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', \Pi_{(x-x_\kappa)} u) - h(x', v')) \Psi_{\alpha(x-x_\kappa)}(u)\, du
\]

\[
+ (J(x', v') - J(x, v))^+ \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \psi(u)\, du,
\]

where in the second identity we took advantage of the definition of \(\Psi\), and used the basic identity: \(a \wedge b + (a - b)^+ = a\) for any \(a, b \in \mathbb{R}\), as well as substituted the variable \(u + \alpha(x - x_\kappa)\) with the variable \(u\). Note that the matrix \(\Pi\), defined in \((2.1)\), is an orthogonal matrix and its inverse \(\Pi^{-1}\) is equal to \(\Pi\). Thus, we find

\[
u + \alpha(x - x_\kappa) = \Pi_{(x-x_\kappa)}^{-1}(\Pi_{(x-x_\kappa)} u + \alpha \Pi_{(x-x_\kappa)} (x - x_\kappa))
\]

\[
= \Pi_{(x-x_\kappa)}(\Pi_{(x-x_\kappa)} u - \alpha(x - x_\kappa)).
\]
This, along with the radial property of \( \varphi \) and \( \Pi^{-1} = \Pi \), gives us that for any mapping \( \Theta : \mathbb{R}^d \to \mathbb{R} \),

\[
\int_{\mathbb{R}^d} \Theta(\Pi(x-x')u) \Psi_{\alpha(x-x')}(u) \, du = \int_{\mathbb{R}^d} \Theta(\Pi(x-x')u) \Psi_{-\alpha(x-x')}(\Pi(x-x')u) \, du
\]

\[
= \int_{\mathbb{R}^d} \Theta(u) \Psi_{-\alpha(x-x')}(u) \, du.
\]

The identity above enables us to obtain

\[
(J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', \Pi(x-x')u) - h(x', v')) \Psi_{\alpha(x-x')}(u) \, du
\]

\[
= (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \Psi_{-\alpha(x-x')}(u) \, du.
\]

Consequently, we have

\[
(L_2 f)(y) = (L_2 g)(x, v)
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \Psi_{-\alpha(x-x')}(u) \, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \Psi_{-\alpha(x-x')}(u) \, du
\]

\[
+ (J(x', v') - J(x, v)) \int_{\mathbb{R}^d} (h(x', u) - h(x', v')) \Psi_{\alpha(x-x')}(u) \, du
\]

\[
= (L_2 g)(x, v) + (L_2 h)(x', v'),
\]

where in the second identity we used again the definition of \( \Psi \) and the fact: \( a \wedge b + (a-b)^+ = a \) for any \( a, b \in \mathbb{R} \). Therefore, (2.6) is now available. \( \square \)

Before we end this section, we address the issue on the existences of a Markovian coupling process associated with the coupling operator \( L_{\gamma, \alpha, \kappa} \). To achieve this goal, we set on \( \mathbb{R}^{4d} \) a vector field \( \Xi := \langle v, v', -\gamma v - \nabla U(x), -\gamma v' - \nabla U(x') \rangle \) and a jump measure

\[
Q(x, x', v, v', dy, dy', du, du') = (J(x, v) \wedge J(x', v')) \delta_{y = x, y' = x', u' = u + \alpha(x-x')}(u) \Psi_{\alpha(x-x')}(u) \, du
\]

\[
+ (J(x, v) \wedge J(x', v')) \delta_{y = x, y' = x', u = \Pi(x-x')u} \Psi_{\alpha(x-x')}(u) \, du
\]

\[
+ (J(x, v) - J(x', v')) \delta_{y = x, y' = x', u' = u'} \varphi(u) \, du
\]

\[
+ (J(x', v') - J(x, v)) \delta_{y = x, y' = x', u = u} \varphi(u) \, du.
\]

Under (A_1), it is clear that, for any \( (x, x', v, v') \in \mathbb{R}^{4d} \), \( Q(x, x', v, v', dy, dy', du, du') \) is a finite measure \( \mathbb{R}^{4d} \). Furthermore, we set a jump rate function \( \beta(x, x', v, v') = Q(x, x', v, v', \mathbb{R}^{4d}) \) and define a normalized jump measure

\[
\overline{Q}(x, x', v, v', dy, dy', du, du') = \beta(x, x', v, v')^{-1}Q(x, x', v, v', dy, dy', du, du').
\]

Subsequently, according to [11, Section 3], there exists an \( \mathbb{R}^{4d} \)-valued PDMP \( \{(X_t, V_t), (X'_t, V'_t)\}_{t \geq 0} \) corresponding to the triplet \( (\Xi, \beta(x, x', v, v'), \overline{Q}(x, x', v, v', dy, dy', du, du')) \). Obviously, the generator of \( \{(X_t, V_t), (X'_t, V'_t)\}_{t \geq 0} \) above is nothing else but the coupling operator \( L_{\gamma, \alpha, \kappa} \). This proves the existence of a Markovian coupling process associated with the coupling operator \( L_{\gamma, \alpha, \kappa} \) and we therefore reach our desired goal.

3. Proof of Theorem 1.1

Throughout this section, we shall write \( y = ((x, v), (x', v')) \) for all \( (x, v), (x', v') \in \mathbb{R}^{2d} \). For the parameters \( \alpha, \alpha_0 > 0 \) (whose precise values are to be given later), we introduce the following abbreviated notations:

\[
z := x - x', \quad w := v - v', \quad q := z + \alpha^{-1}w, \quad r(y) := \alpha_0|z| + |q|.
\]
For any $\varepsilon > 0$ and $(x,v), (x',v') \in \mathbb{R}^{2d}$, set

$$F(y) := f(r(y)), \quad G(y) := 1 + \varepsilon W(x,v) + \varepsilon W(x',v'),$$

where the $C^2$-function $f : [0,\infty) \to [0,\infty)$ satisfies $f(0) = 0$, $f' \geq 0$ and $f'' \leq 0$, and $W$ is the Lyapunov function given in (B$_1$).

**Lemma 3.1.** For $F, G$, given in (3.2),

$$(3.3) \quad \mathcal{L}_{\gamma,\alpha,\kappa}(FG)(y) = G(y)(\mathcal{L}_{1,\gamma}F)(y) + F(y)(\mathcal{L}_{\gamma,\alpha,\kappa}G)(y) + \Pi(y),$$

where

$$\Pi(y) := (J(x,v) \land J(x',v')) \times \int_{\mathbb{R}^d} \left( F((x,u),(x',u + \alpha(z)\kappa)) - F(y) \right) G((x,u),(x',u + \alpha(z)\kappa)) \psi_{\alpha(z)\kappa}(u) \, du$$

$$+ (J(x,v) \land J(x',v')) \times \int_{\mathbb{R}^d} \left( F((x,u),(x',\Pi(z)\kappa u)) - F(y) \right) G((x,u),(x',\Pi(z)\kappa u)) \Psi_{\alpha(z)\kappa}(u) \, du$$

$$(3.4) \quad + \left( J(x,v) - J(x',v') \right)^+ \int_{\mathbb{R}^d} \left( F((x,u),(x',v')) - F(y) \right) G((x,u),(x',v')) \psi(u) \, du$$

$$+ \left( J(x',v') - J(x,v) \right)^+ \int_{\mathbb{R}^d} \left( F((x,u),(x',u)) - F(y) \right) G((x,v),(x',u)) \psi(u) \, du.$$}

**Proof.** Apparently, the chain rule yields that for all $y \in \mathbb{R}^{2d}$,

$$(3.5) \quad \left( \mathcal{L}_{1,\gamma}(FG) \right)(y) = F(y)(\mathcal{L}_{1,\gamma}G)(y) + G(y)(\mathcal{L}_{1,\gamma}F)(y).$$

Next, by invoking the addition-subtraction strategy and taking the definition of the operator $\mathcal{L}_{2,\alpha,\kappa}$ into account, we derive that for all $y \in \mathbb{R}^{2d}$,

$$(3.6) \quad \left( \mathcal{L}_{2,\alpha,\kappa}(FG) \right)(y) = \Pi(y)$$

$$+ \left( J(x,v) \land J(x',v') \right) F(y)$$

$$\times \left\{ \int_{\mathbb{R}^d} \left( G((x,u),(x',u + \alpha(z)\kappa)) - G_{\varepsilon}(y) \right) \psi_{\alpha(z)\kappa}(u) \, du \right.\right.$$}

$$+ \left. \int_{\mathbb{R}^d} \left( G((x,u),(x',\Pi(z)\kappa u)) - G(y) \right) \Psi_{\alpha(z)\kappa}(u) \, du \right\}$$

$$+ \left( J(x,v) - J(x',v') \right)^+ F(y) \int_{\mathbb{R}^d} \left( G((x,u),(x',v')) - G(y) \right) \psi(u) \, du$$

$$+ \left( J(x',v') - J(x,v) \right)^+ F(y) \int_{\mathbb{R}^d} \left( G((x,v),(x',u)) - G(y) \right) \psi(u) \, du$$

$$= F(y)(\mathcal{L}_{2,\alpha,\kappa}G)(y) + \Pi(y),$$

where the remainder term $\Pi(\cdot)$ was introduced in (3.4). Thus, recalling $\mathcal{L}_{\gamma,\alpha,\kappa} = \mathcal{L}_{1,\gamma} + \mathcal{L}_{2,\alpha,\kappa}$ and combining (3.5) with (3.6) enables us to derive (3.3). \(\square\)

In the following, we assume that Assumption (B$_1$) holds. Let $W(x,v)$ and $c_0, C_0$ be the function and the constants in (B$_1$). Define the following two sets

$$(3.7) \quad \mathcal{A} = \{ y \in \mathbb{R}^{2d} : 4C_0 \geq c_0 W(x,v) + c_0 W(x',v') \}, \quad \Gamma = \{ y \in \mathbb{R}^{2d} : r(y) \geq R_0 \},$$

where

$$(3.8) \quad R_0 = R_0(\alpha, \alpha_0) := \sup \{ r(y) : y \in \mathcal{A} \}. $$

Due to (1.6) (i.e., $\lim_{|x|, |V| \to \infty} W(x,v) = \infty$), there is an $R^* > 0$ (independent of $\alpha, \alpha$ but dependent on $c_0$ and $C_0$) such that $|x| + |v| \leq R^*$ for all $(x, v) \in \mathcal{A}_0$, where

$$\mathcal{A}_0 := \{ (x,v) \in \mathbb{R}^{2d} : W(x,v) \leq 4C_0/c_0 \}. $$
It is trivial to see that $\mathcal{A}$ is a subset of the product space $\mathcal{A}_0 \times \mathcal{A}_0$. Hence, we find that
\[
R_0 := \sup_{y \in \mathcal{A}} r(y) \leq (1 + \alpha_0 + \alpha^{-1}) \sup_{y \in \mathcal{A}} (|x| + |v| + |x'| + |v'|)
\leq 2(1 + \alpha_0 + \alpha^{-1}) \sup_{(x,v) \in \mathcal{A}_0} (|x| + |v|)
\leq 2R'(1 + \alpha_0 + \alpha^{-1}).
\]
As a consequence, $R_0$ can be bounded by the number $R'(1 + \alpha_0 + \alpha^{-1})$ up to an absolute constant independent of $\alpha_0, \alpha$. On the other hand, it follows from the definitions of $\mathcal{A}$ and $\Gamma$ that $\mathcal{A} \subset \Gamma^c$.

Now, we set for all $y \in \mathbb{R}^d$,
\[
\widehat{F}(r(y)) := f(r(y) \wedge R_0),
\]
where $f(.)$ was given in (3.2).

**Lemma 3.2.** Under Assumption (B1), for all $y \in \mathcal{A}^c \cap \Gamma = \Gamma$,
\[
(3.9) \quad (\mathcal{L}_{\gamma,a,\kappa}(\hat{G})) (y) \leq -\frac{c_0\varepsilon}{1+2\varepsilon} \hat{F}(y)G(y).
\]

**Proof.** For $y \in \mathcal{A}^c \cap \Gamma$ (in particular, $r(y) \geq R_0$), $f(r(y) \wedge R_0) = f(R_0)$ and $f'_-(r(y) \wedge R_0) = 0$, where $f'_-$ means the left derivative of $f$. The chain rule shows that
\[
(\mathcal{L}_{1,\gamma}(\hat{F})) (y) = f'_-(r(y) \wedge R_0) (\mathcal{L}_{1,\gamma}r)(y) = 0
\]
and so
\[
(3.10) \quad G(y)(\mathcal{L}_{1,\gamma}\hat{F})(y) = 0.
\]
Next, in addition to $f' > 0$ on $[0, \infty)$ and the positive properties of $J$ and $W$, we find that $\Pi(y) \leq 0$ once $r(y) \geq R_0$.

On the other hand, by applying Lemma 2.2 and taking (1.7) into consideration, we have
\[
(3.11) \quad \hat{F}(y)(\mathcal{L}_{\gamma,a,\kappa}G)(y) = \varepsilon f(r(y) \wedge R_0)((\mathcal{L}W)(x,v) + (\mathcal{L}W)(x',v'))
\leq \varepsilon f(r(y) \wedge R_0)(-c_0W(x,v) - c_0W(x',v') + 2C_0).
\]
Thus, combining (3.10) with (3.11) and making use of Lemma 3.1 leads to
\[
(\mathcal{L}_{\gamma,a,\kappa}(\hat{G})) (y) \leq \varepsilon f(R_0)(-c_0W(x,v) - c_0W(x',v') + 2C_0)
\]
in case of $r(y) \geq R_0$. Subsequently, for all $y \in \mathcal{A}^c$, we obviously have
\[
\frac{c_0}{2} (W(x,v) + W(x',v')) \geq 2C_0.
\]
Accordingly, we arrive at
\[
(3.12) \quad (\mathcal{L}_{\gamma,a,\kappa}(\hat{G})) (y) \leq -\frac{c_0\varepsilon}{2} f(R_0)(W(x,v) + W(x',v')).
\]
Additionally, due to $W \geq 1$, we evidently have
\[
(3.13) \quad G(y) = 1 + \varepsilon (W(x,v) + W(x',v')) \leq (1/2 + \varepsilon)(W(x,v) + W(x',v')).
\]
As a result, concerning the case $r(y) \geq R_0$, we obtain from (3.12) that
\[
(\mathcal{L}_{\gamma,a,\kappa}(\hat{G})) (y) \leq -\frac{c_0\varepsilon}{1+2\varepsilon} f(R_0)G(y) = -\frac{c_0\varepsilon}{1+2\varepsilon} \hat{F}(y)G(y).
\]
Hence, the desired assertion (3.9) follows. \qed

Now, for any $a_0 > 0$ (which will be fixed later), we take the function $f$ in (3.2) to be
\[
(3.14) \quad f(s) = \frac{1}{a_0} (1 - e^{-a_0s}), \quad s \geq 0,
\]
which definitely satisfies that $f(0) = 0$, $f' > 0$ and $f'' < 0$ on $[0, \infty)$. Moreover, simple calculations yield the following two crucial estimates:
\[
(3.15) \quad f(s) - f(t) \leq f'(t)(s-t), \quad s, t \geq 0
\]
and
\[
(3.16) \quad f(s) - f(t) \leq \frac{1}{a_0} f'(t), \quad s, t \geq 0;
\]
see, for instance, [5] Lemma 5.2 for more details.

In the remainder of the paper, we shall fix the threshold \( \kappa \) in the coupling operator \( \widetilde{L}_{\gamma, \alpha, \kappa} \), defined in [2.2], as below
\[
\kappa = R_0/\alpha_0.
\]

**Lemma 3.3.** Under Assumptions \((A_1), (A_2), (B_1)\) and \((B_2)\), for all \( y \in \Gamma^c \),
\[
\left( \widetilde{L}_{\gamma, \alpha, \kappa}(FG) \right)(y) \leq G(y) f'_-(r(y)) \left( \widetilde{L}_{\gamma, \alpha, \kappa} \right)(y) - \lambda_1 c_s(\alpha, \kappa) f'_-(r(y)) |q|
\]
\[
+ \frac{1}{a_0} \left( \lambda_2 (\alpha^* \kappa) + 2 \lambda_3 (1 + \alpha)(1 \vee \alpha^*) \right) f'_-(r(y)) G(y) |z|
\]
\[
+ \frac{2 \alpha}{a_0} \lambda_3 (1 \vee \alpha^*) f'_-(r(y)) G(y) |q|
\]
\[
+ \varepsilon f(r(y)) \left( - c_0 W(x, v) + W(x', v') + 2 C_0 \right).
\]

**Proof.** For \( y \in \Gamma^c \) (i.e., \( r(y) < R_0 \)), the chain rule yields
\[
(3.18) \quad \left( \widetilde{L}_{\gamma, \alpha, \kappa} \right)(y) = f'_-(r(y)) \left( \widetilde{L}_{\gamma, \alpha, \kappa} \right)(y).
\]
By virtue of \((3.11)\), we readily have
\[
(3.19) \quad \widetilde{F}(y) \left( \widetilde{L}_{\gamma, \alpha, \kappa} G \right)(y) \leq \varepsilon f(r(y)) \left( - c_0 W(x, v) - c_0 W(x', v') + 2 C_0 \right).
\]

Write down the four terms on the right hand side of \((3.14)\) as \( \Upsilon_1(y), \Upsilon_2(y), \Upsilon_3(y) \) and \( \Upsilon_4(y) \), respectively. Below, we intend to quantify \( \Upsilon_i(y), i = 1, 2, 3, 4 \), separately. According to the definition of \( r(\cdot) \), we deduce
\[
\Upsilon_1(y) = (J(x, v) \wedge J(x', v')) \left( f(\alpha_0 + (1 - (1 \wedge \kappa/|z|)) |z|) - f(r(y)) \right)
\]
\[
\times \int_{\mathbb{R}^d} G((x, u), (x', u + \alpha(z)_\kappa)) \psi_{\alpha(z)\kappa}(u) \, (du)
\]
\[
\leq (J(x, v) \wedge J(x', v')) f'_-(r(y))
\]
\[
\times \int_{\mathbb{R}^d} G((x, u), (x', u + \alpha(z)_\kappa)) \psi_{\alpha(z)\kappa}(u) \, (du)(\alpha_0 |z| - r(y))
\]
\[
= - |q| (J(x, v) \wedge J(x', v')) f'_-(r(y)) \int_{\mathbb{R}^d} G((x, u), (x', u + \alpha(z)_\kappa)) \psi_{\alpha(z)\kappa}(u) \, (du)
\]
\[
\leq - \lambda_1 c_s(\alpha, \kappa) f'_-(r(y)) |q|,
\]
where in the first inequality we used the fact \( \kappa = R_0/\alpha_0 > r(y)/\alpha_0 \geq |z| \), in the identity we utilized \( r(y) = \alpha_0 |z| + |q| \), and the last inequality is available due to \((B_2)\) and \( G \geq 1 \).

Next, by invoking \((3.16)\), together with \( f' > 0 \) on \([0, \infty)\), we obtain that
\[
\Upsilon_2(y) \leq \frac{1}{a_0} (J(x, v) \wedge J(x', v')) f'_-(r(y)) \int_{\mathbb{R}^d} G((x, u), (x', \Pi(z)_\kappa u)) \Psi_{\alpha(z)\kappa}(u) \, (du)
\]
\[
\leq \frac{\lambda_2}{a_0} f'_-(r(y)) \left\{ 1 - A_{\alpha, \kappa}(z) + \varepsilon \int_{\mathbb{R}^d} W(x, u) \Psi_{\alpha(z)\kappa}(u) \, (du) 
\right.
\]
\[
+ \varepsilon \int_{\mathbb{R}^d} W(x', u) \Psi_{\alpha(z)\kappa}(u) \, (du) \right\}
\]
\[
= \frac{\lambda_2}{a_0} f'_-(r(y)) \left\{ 1 - A_{\alpha, \kappa}(z) + \varepsilon \int_{\mathbb{R}^d} W(x, u) \Psi_{\alpha(z)\kappa}(u) \, (du) 
\right.
\]
\[
+ \varepsilon \int_{\mathbb{R}^d} W(x', u) \Psi_{-\alpha(z)\kappa}(u) \, (du) \right\}
\]
\[
\begin{align*}
&\leq \frac{\lambda_2}{a_0}f'_L(r(y))\left\{1 - A_{a_0,k}(z) + \varepsilon c^{**}\alpha(\kappa \wedge |z|)\left(\inf_{v \in \mathbb{R}^d} W(x, v) + \inf_{v' \in \mathbb{R}^d} W(x', v')\right)\right\} \\
&\leq \frac{\lambda_2}{a_0} \left( c^{*}(\alpha, \kappa) \lor (c^{**}\alpha) \right) f'_L(r(y)) G(y) |z|,
\end{align*}
\]

where the identity is due to (2.77), the last two inequality holds true owing to (B_2), and the last display follows from (A_2). Once more, using (3.16) yields

\[
\Upsilon_3(y) + \Upsilon_4(y) \leq \frac{f'_L(r(y))}{a_0} \left( J(x, v) - J(x', v') \right)^+ \left(1 + \varepsilon W(x, v') + \varepsilon \int_{\mathbb{R}^d} W(x, u) \varphi(u) \, du\right) \\
+ \frac{f'_L(r(y))}{a_0} \left( J(x', v') - J(x, v) \right)^+ \left(1 + \varepsilon W(x, v) + \varepsilon \int_{\mathbb{R}^d} W(x', u) \varphi(u) \, du\right) \\
\leq \frac{\lambda_f f'_L(r(y))}{a_0} \left( (1 + \alpha) |z| + \alpha |q| \right) \\
\times \left( 2 + \varepsilon W(x, v) + \varepsilon W(x', v') + c^{**} \varepsilon \left( \inf_{v \in \mathbb{R}^d} W(x, v) + \inf_{v' \in \mathbb{R}^d} W(x', v')\right) \right) \\
\leq \frac{2\lambda_f}{a_0} (1 \lor c^{**}) \left( (1 + \alpha) |z| + \alpha |q| \right) f'_L(r(y)) G(y),
\]

where in the second inequality we exploited (1.2) and (B_2) as well as \( w = \alpha(q - z) \).

Consequently, we complete the proof of Lemma 3.3 by combining all the estimates above for \( \Upsilon_i(y) \) \((1 \leq i \leq 4)\) with (3.18) and (3.19).

From now on, we assume that the inequality (1.10) is solvable on the interval \((0, \gamma^2/4]\). Then, there exists \( \beta \in (0, \gamma^2/4]\) solving the inequality \( \beta \geq 4K_{\beta,U} \). Due to \( \beta \in (0, \gamma^2/4]\), there exists an \( \alpha > 0 \) such that \( \beta = \alpha \gamma - \alpha^2 \). Hence, the inequality

\[
\alpha \gamma - \alpha^2 \geq 4K_{\alpha(\gamma - \alpha),U}
\]

is also solvable. That is,

\[
(3.20) \quad \alpha^{-1} \gamma - 1 \geq 4\alpha^{-2} K_{\alpha(\gamma - \alpha),U}.
\]

In the sequel, we settle out the parameters involved in (2.4), (3.1) and (3.2), respectively. More precisely, for the positive constant \( \alpha \), a solution to (3.20), we shall stipulate

\[
\alpha_0 = \frac{\gamma}{\alpha} - 1, \quad \kappa = \frac{R_0}{a_0}, \quad a_0 = \frac{4K_0}{\alpha_0} \lor \left( \frac{1}{\lambda_1 c_3(\alpha, \kappa)} \lor \frac{2}{c_0} \right) \lambda_j(1 \lor c^{**}) + 4 \left( \frac{\lambda_1 c_3(\alpha, \kappa)}{4\lambda_j(1 \lor c^{**})} - 1 \right) \right) \land \left( \frac{a_0 R_0}{8C_0 (e^\alpha R_0 - 1)} \min\{\alpha, \lambda_1 c_3(\alpha, \kappa)\} \right),
\]

where \( R_0 \) was defined by (3.8), and

\[
K_0 := \lambda_2 \left( c^{*}(\alpha, \kappa) \lor (c^{**}\alpha) \right) + 2\lambda_f (1 + \alpha) (1 \lor c^{**}).
\]

According to the prescribed value of \( a_0 \), it is evident to see that the value of \( \varepsilon \) set in (3.21) is positive. Seemingly, the parameters set in (3.21) are a little bit weird and complicated while the precise alternatives will become more and more clear by tracking the proof of Lemma 3.3 below.

**Lemma 3.4.** Assume that (H_0), (A_1), (A_2), (B_1) and (B_2) hold, and that the inequality (1.10) is solvable. Then for all \( y \in \Gamma_c \),

\[
(3.22) \quad (\mathcal{L}_{\gamma,\alpha,k}(\widetilde{F}G))(y) \leq -\lambda^* \widetilde{F}(y) G(y),
\]

where

\[
\lambda^* := \frac{c_0 \varepsilon}{2(1 + 2\varepsilon)} \land \left( \frac{a_0 R_0}{4(e^\alpha R_0 - 1)} \min\{\alpha, \lambda_1 c_3(\alpha, \kappa)\} (1 + 4\varepsilon C_0/c_0)^{-1} \right).
\]
Proof. A direct calculation yields
\[
\langle \mathcal{L}_{1,r}(y), (z, w) \rangle = \frac{\alpha_0}{|z|} q + \frac{1}{|q|} (q, -(\alpha^{-1}y - 1)w + \alpha^{-1}(\nabla U(x') - \nabla U(x)) \rangle
\]
\[
= -\alpha_0 \alpha |z| + \frac{\alpha_0 \alpha}{|z|} (q, -(\gamma - \alpha)|q|) + \frac{1}{\alpha|q|} (q, \alpha(\gamma - \alpha)z + \nabla U(x') - \nabla U(x)) \leq ( -\alpha_0 \alpha + \alpha^{-1}K_{a(\gamma - \alpha), U}) |z| + (\alpha + \alpha_0 \alpha - \gamma)|q|
\]
which, in the last identity we utilized the identity \( w = \alpha(q - z) \), in the inequality we employed (H\(_0\)), and in the last identity we took the fact that \( \alpha + \alpha_0 \alpha - \gamma = 0 \) due to \( (3.21) \) into account. Plugging the previous inequality back into (3.17) implies that for all \( y \in \Gamma^c \),
\[
\langle \mathcal{L}_{\gamma,a,\kappa}(FG) \rangle(y) \leq \left( -\alpha_0 \alpha + \alpha^{-1}K_{a(\gamma - \alpha), U} + \frac{K_0}{\alpha_0} \right) G(y)f'_-(r(y)) |z| - \lambda_1 c_\gamma(\alpha, \kappa)f'_-(r(y)) |q|
\]
\[
- \lambda_1 c_\gamma(\alpha, \kappa)f'_-(r(y)) |q| + \frac{2\alpha}{\alpha_0} \lambda_j(1 + c^{\text{**}})G(y)f'_-(r(y)) |q|
\]
\[
+ \varepsilon f(r(y)) \left( -c_\gamma W(x, v) - c_\gamma W(x', v') + 2C_0 \right).
\]
In terms of (3.20) and (3.21), we obviously have
\[
\alpha^{-1}K_{a(\gamma - \alpha), U} \leq \frac{\alpha}{4} \frac{(\gamma - \alpha)}{\alpha - 1} \leq \frac{\alpha_0 \alpha}{4}, \quad \frac{K_0}{\alpha_0} \leq \frac{\alpha_0 \alpha}{4}.
\]
Then, (3.28) is reduced into
\[
\langle \mathcal{L}_{\gamma,a,\kappa}(FG) \rangle(y) \leq \frac{1}{2} \alpha_0 \alpha G(y)f'_-(r(y)) |z| - \lambda_1 c_\gamma(\alpha, \kappa)f'_-(r(y)) |q|
\]
\[
+ \frac{2\alpha}{\alpha_0} \lambda_j(1 + c^{\text{**}})G(y)f'_-(r(y)) |q|
\]
\[
+ \varepsilon f(r(y)) \left( -c_\gamma W(x, v) - c_\gamma W(x', v') + 2C_0 \right).
\]
In what follows, we aim to show that (3.22) is verifiable for two separate cases.

Case (i): \( y \in A \cap \Gamma^c \) (i.e., \( y \in A \)). For such case, we in particular have \( G(y) \leq 1 + 4 \varepsilon C_0/c_0 \). In the light of the precise value of \( \varepsilon \) given in (3.21), we obtain that
\[
\frac{2\alpha}{\alpha_0} \lambda_j(1 + c^{\text{**}}) \left( 1 + 4 \varepsilon C_0/c_0 \right) \leq \frac{1}{2} \lambda_1 c_\gamma(\alpha, \kappa).
\]
Thus, from (3.21) and \( G \geq 1 \), we find that for all \( y \in A \),
\[
\langle \mathcal{L}_{\gamma,a,\kappa}(FG) \rangle(y) \leq \left( -\frac{1}{2} \alpha_0 \alpha G(y)f'_-(r(y)) |z| - \lambda_1 c_\gamma(\alpha, \kappa)f'_-(r(y)) |q|
\]
\[
+ \frac{2\alpha}{\alpha_0} \lambda_j(1 + c^{\text{**}}) \left( 1 + 4 \varepsilon C_0/c_0 \right) f'_-(r(y)) |q| + 2C_0 \varepsilon f(r(y)) \right.
\]
\[
\leq -\frac{1}{2} \alpha_0 \alpha f'_-(r(y)) |z| - \lambda_1 c_\gamma(\alpha, \kappa)f'_-(r(y)) |q| + 2C_0 \varepsilon f(r(y)) \right.
\]
\[
\leq -\frac{1}{2} \min\{\alpha, \lambda_1 c_\gamma(\alpha, \kappa)\} f'_-(r(y)) r(y) + 2C_0 \varepsilon f(r(y)) \right.
\]
where in the last display we used the fact that \( r(y) = \alpha_0 |z| + |q| \). Subsequently, combining the following facts that
\[
f'(s)s = \frac{a_0 s}{e^{a_0 s} - 1} f(s) \leq f(s), \quad s \geq 0
\]
and that $s \mapsto \frac{a_0 s}{\epsilon_0 + 1}$ is decreasing on $[0, \infty)$, leads to
\begin{align}
(\mathcal{L}^\gamma_\gamma,\alpha,\kappa(\bar{FG}))(y) & \leq -\frac{1}{2} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} \int f'(r(y))r(y) + \frac{2C_0 \varepsilon (e^{a_0 R_0} - 1)}{a_0 R_0} f'(r(y))r(y) \\
& \leq -\frac{1}{4} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} f'(r(y))r(y) \\
& \leq -\frac{a_0 R_0}{4(e^{a_0 R_0} - 1)} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} f(r(y)) \\
& \leq -\frac{a_0 R_0}{4(e^{a_0 R_0} - 1)} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} (1 + 4\varepsilon C_0/a_0)^{-1} G(y)f(r(y)),
\end{align}
(3.27)
where in the first inequality we invoked the basic fact $r(y) \leq R_0$ due to $y \in \mathcal{A}$, in the second inequality we employed the fact
\[
2\varepsilon C_0 (e^{a_0 R_0} - 1) \leq \frac{1}{4} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\}
\]
by taking the alternative of $\varepsilon$, given in (3.21), into consideration, and in the last inequality we applied (3.13).

**Case (ii):** $y \in \mathcal{A}^c \cap \Gamma^c$. Concerning this setting, we have $r(y) < R_0$ and
\[
(3.28) c_0 W(x, v) + c_0 W(x', v') \geq 4C_0.
\]
From (3.23), it is obvious to see that
\[
\frac{2\alpha}{a_0} \lambda_1 (1 + c^*) \leq \frac{1}{2} \lambda_1 c_*(\alpha, \kappa).
\]
Thus, we derive from (3.24) and (3.28) that for all $y \in \mathcal{A}^c \cap \Gamma^c$,
\begin{align}
(\mathcal{L}^\gamma_\gamma,\alpha,\kappa(\bar{FG}))(y) & \leq -\frac{1}{2} a_0 \alpha G(y) f'(r(y))|z| - \lambda_1 c_*(\alpha, \kappa) f'(r(y))|q| \\
& + \frac{2\alpha}{a_0} \lambda_1 (1 + c^*) f'(r(y))|q| \\
& + \frac{2\alpha}{a_0} \lambda_1 (1 + c^*) \varepsilon (W(x, v) + W(x', v')) f'(r(y))|q| \\
& - \frac{c_0 \varepsilon}{2} (W(x, v) + W(x', v')) f(r(y)) \\
& \leq -\frac{1}{2} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} f'(r(y))r(y) \\
& + \frac{c_0 \varepsilon}{4} (W(x, v) + W(x', v')) f'(r(y))|q| \\
& - \frac{c_0 \varepsilon}{2} (W(x, v) + W(x', v')) f(r(y)) \\
& \leq -\frac{c_0 \varepsilon}{4} (W(x, v) + W(x', v')) f(r(y)) \\
& \leq -\frac{c_0 \varepsilon}{2(1 + 2\varepsilon)} G(y)f(r(y)),
\end{align}
(3.29)
where in the first inequality we utilized the fact that $G(y) = 1 + \varepsilon(W(x, v) + W(x', v'))$, in the second inequality we exploited $G \geq 1$, $r(y) = a_0|z| + |q|$ and
\[
\frac{2\alpha}{a_0} \lambda_1 (1 + c^*) \leq \frac{c_0}{4}
\]
with the help of (3.21), in the third inequality we used the basic fact that $q \leq r(y)$ and (3.26), and in the last inequality we took advantage of (3.13) again.

Consequently, the assertion (3.22) follows immediately by combining (3.27) with (3.29).

Next, combining Lemma 3.2 with Lemma 3.4 we arrive at the following proposition.
Proposition 3.5. Assume that $(H_0), (A_1), (A_2), (B_1)$ and $(B_2)$ hold, and that the inequality (1.10) is solvable. Then, for all $y \in \mathbb{R}^{4d},$
\[
(\tilde{\mathcal{L}}_{\gamma,\alpha,\kappa}(\tilde{F}G))(y) \leq -\lambda^*\tilde{F}(y)G(y),
\]
where
\[
\lambda^* := \frac{c_0\varepsilon}{2(1 + 2\varepsilon)} \wedge \left( \frac{a_0 R_0}{4(\varepsilon c_0 R_0 - 1)} \min\{\alpha, \lambda_1 c_*(\alpha, \kappa)\} (1 + 4\varepsilon c_0/c_0)^{-1} \right).
\]

Now, we are in position to complete the proof of Theorem 1.1. Although, with Proposition 3.5 at hand, the proof of Theorem 1.1 is more or less standard, we herein provide an outline to make the content self-contained.

Proof of Theorem 1.1. Let $(Y_t)_{t \geq 0} = ((X_t, V_t), (X'_t, V'_t))_{t \geq 0}$ be the coupling process associated with the coupling operator $\tilde{\mathcal{L}}_{\gamma,\alpha,\kappa}$ as mentioned at the end of Section 2 and let $\tilde{E}y$ be the expectation operator under the probability measure $\tilde{P}y$, the distribution of $(Y_t)_{t \geq 0}$ with the initial point $y$. Then, we deduce from Proposition 3.5 that
\[
(\tilde{F}G)(Y_t) \leq e^{-\lambda^*t}.
\]
Note that $(\tilde{F}G)(y)$ is comparable with the quasi-distance function
\[
\Phi(y) := (|x - x'| + |v - v'|) \wedge 1 (W(x, v) + W(x', v'));
\]
that is, there exist constants $c_1, c_2 > 0$ such that for all $y \in \mathbb{R}^{4d},$
\[
c_1\Phi(y) \leq (\tilde{F}G)(y) \leq c_2\Phi(y).
\]
This, together with (3.30), yields that there is a constant $c_3 > 0$ such that for all $y \in \mathbb{R}^{4d}$ and $t > 0,$
\[
W_\Phi(\delta_{(x,v)}P_t, \delta_{(x',v')}P_t) \leq c_3\Phi(y)e^{-\lambda^*t},
\]
which further implies that the semigroup $(P_t)_{t \geq 0}$ exhibits the Feller property, and moreover, via the Banach fixed point theorem, has a unique invariant probability measure $\mu \in \mathcal{P}_\Phi(\mathbb{R}^{2d});$ see, for instance, [16, Corollary 4.11] for more details. Now, for any $\nu \in \mathcal{P}_\Phi(\mathbb{R}^{2d}),$ integrating with respect to $\pi \in \mathcal{C}(\nu, \mu)$ on both sides of (3.31) yields
\[
\int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} W_\Phi(\delta_{(x,v)}P_t, \delta_{(x',v')}P_t) \pi(dy) \leq c_3e^{-\lambda^*t} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \Phi(y)\pi(dy),
\]
which, combining the basic fact that
\[
W_\Phi(\nu P_t, \mu P_t) \leq \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} W_\Phi(\delta_{(x,v)}P_t, \delta_{(x',v')}P_t) \pi(dy),
\]
and taking infimum with respect to all $\pi \in \mathcal{C}(\nu, \mu)$ leads to
\[
W_\Phi(\nu P_t, \mu P_t) \leq c_3e^{-\lambda^*t}W_\Phi(\nu, \mu).
\]
Thus, (1.11) follows by taking the invariance of the invariant probability measure $\mu$ into consideration. We therefore complete the corresponding proof. \hfill $\square$

4. SUFFICIENT CONDITIONS ON ASSUMPTIONS AND (1.10)

In this section, we aim to provide some sufficient conditions or concrete examples to demonstrate that all the assumptions and the technical condition (1.10) are verifiable.

Proposition 4.1. Suppose that the function $r \mapsto \varphi(r)$ is bounded and decreasing on $(0, \infty)$ such that $\varphi(r) > 0$ for all $r > 0$. Then, Assumption $(A_2)$ holds.
Proof. Since \( \varphi(\cdot) \) is decreasing on \((0, \infty)\), we deduce that for any \( z \in \mathbb{R}^d \),
\[
\int_{\mathbb{R}^d} \psi_z(u) \, du \geq \int_{\mathbb{R}^d} (\varphi(|u|) \wedge \varphi(|u| + |z|)) \, du \geq \int_{\mathbb{R}^d} \varphi(|u| + |z|) \, du = \int_{\{ |u| \geq |z| \}} \varphi(u) \, du,
\]
which implies that for \( r > 0 \),
\[
\inf_{z \in \mathbb{R}^d : |z| \leq r} \int_{\mathbb{R}^d} \psi_z(u) \, du \geq \int_{\{ |u| \geq r \}} \varphi(u) \, du > 0
\]
and that for all \( z \in \mathbb{R}^d \),
\[
1 - \int_{\mathbb{R}^d} \psi_z(u) \, du \leq \int_{\{ |u| \geq |z| \}} \varphi(u) \, du = c_0 \int_0^{|z|} r^{d-1} \varphi(r) \, dr \leq c_1 |z|
\]
with some constants \( c_0, c_1 > 0 \). Therefore, Assumption (A_2) holds. \( \square \)

The Lyapunov function \( W \) satisfying (1.6) and (1.7) is imposed to analyze the long-time behavior of the process \((X_t, V_t)_{t \geq 0}\). Below, we build examples to demonstrate that (1.6) and (1.7) are valid. Suppose that \( U(x) \geq 0 \) for all \( x \in \mathbb{R}^d \). Let
\[
W_0(x, v) = 1 + 2U(x) + \theta_0|x|^2 + |v|^2 + \theta^* \langle x, v \rangle, \quad x, v \in \mathbb{R}^d,
\]
where
\[
\theta_0 := \frac{1}{4}(\lambda_1 + \gamma)^2, \quad \theta^* := \frac{(\lambda_1 + \gamma)^2}{2(\lambda_2 + \gamma)}
\]
with \( \lambda_1 \) and \( \lambda_2 \) being given in Assumption (A_1). Due to \( \lambda_1 \leq \lambda_2 \), it is easy to see from (4.2) that
\[
(\theta^*)^2 = \frac{(\lambda_1 + \gamma)^4}{4(\lambda_2 + \gamma)^2} \leq \theta_0.
\]

By the inequality that \( 2ab \leq \varepsilon a^2 + \beta^2/\varepsilon \) for all \( a, b \in \mathbb{R} \) and \( \varepsilon > 0 \), we obtain that for all \( x, v \in \mathbb{R}^d \),
\[
(4\theta_0 - (\theta^*)^2) \left( \frac{1}{8} |x|^2 + \frac{1}{(\theta^*)^2 + 4\theta_0} |v|^2 \right) \leq \theta_0 |x|^2 + |v|^2 + \theta^* \langle x, v \rangle
\]
\[
\leq \left( 1 \lor \theta_0 + \frac{\theta^*}{2} \right) (|x|^2 + |v|^2)
\]
so, in view of \( U(x) \geq 0 \), \( W_0 \geq 1 \) and \( \lim_{|x| + |v| \to \infty} W_0(x, v) = \infty \).

**Proposition 4.2.** Assume that (A_1) holds, \( \int_{\mathbb{R}^d} |u|^\beta \varphi(u) \, du < \infty \) for some \( \beta \in (0, 2) \), and that \( U(x) \geq 0 \) for all \( x \in \mathbb{R}^d \) satisfying that there exist constants
\[
(4.4) \quad c^* > c^*_0 := \frac{(\lambda_1 + \gamma)^2(\lambda_2 - \lambda_1)}{4(2\lambda_1\lambda_2 - \lambda_1^2 + 4\lambda_2\gamma + 3\gamma^2)}
\]
and \( c^**, C_0^* \geq 0 \) such that for all \( x \in \mathbb{R}^d \),
\[
\langle x, \nabla U(x) \rangle \geq c^* |x|^2 + c^** U(x) - C_0^*
\]
(4.5)

Then, there exist constants \( c_0, C_0 > 0 \) so that
\[
\langle \mathcal{L}W \rangle(x, v) \leq -c_0 W(x, v) + C_0,
\]
(4.6)

where \( W(x, v) := W_0(x, v)^{\beta/2} \) with \( W_0 \) being defined in (1.1). Hence, the Assumption (B_1) holds for \( W \).

**Proof.** For any \( \rho > 0 \), let
\[
c_{\rho,*} = \int_{\{ |u| \leq \rho \}} \varphi(u) \, du,
\]
and
\[
\Theta_{1,\rho} = (1 - c_{\rho,*})\lambda_2 \theta^* (1 + 2\theta_0^{\beta/2} (2/\beta - 1)), \quad \Theta_{2,\rho} = 2(1 - c_{\rho,*}) \lambda_2 \theta_0^{\beta/2} ((2/\beta - 1)\theta_0 + 1),
\]
\[
\Theta_{3,\rho} = (1 - c_{\rho,*}) \lambda_2 (1 + 2\theta_0^{\beta/2} (2/\beta - 1)), \quad \Theta_{4,\rho} = 4(1 - c_{\rho,*}) \lambda_2 \theta_0^{\beta/2} (2/\beta - 1).
\]
Since $\varphi(u) \, du$ is a probability measure, in addition to $c^* > c^*_0$, defined in (4.3), there exists a constant $\rho > 0$ sufficiently large such that

$$\Theta_{2,\rho} < c^* \theta^*, \quad \Theta_{3,\rho} < \lambda_1 + 2\gamma - \theta^*, \quad \Theta_{4,\rho} < c^{**} \theta^*$$

and

$$c^* \theta^* - \Theta_{2,\rho} > \frac{(C_{0}^{**} + \Theta_{1,\rho})^2}{4(\lambda_1 + 2\gamma - \theta^* - \Theta_{3,\rho})} =: c_{0,\rho}.$$ 

where $C_{0}^{**} := \frac{1}{2(\lambda_2 + \gamma)}(\lambda_1 + \gamma)^2(\lambda_2 - \lambda_1)$. Below, we shall choose $\rho > 0$ large enough such that (4.7) and (4.8) hold simultaneously.

Obviously, for all $x, v \in \mathbb{R}^d$,

$$\nabla_x W_0(x, v) = 2\nabla U(x) + 2\theta_0 x + \theta^* v \quad \text{and} \quad \nabla_v W_0(x, v) = 2v + \theta^* x.$$ 

By the chain rule, for $W(x, v) = W_0(x, v)^{\beta/2}$, it follows from (4.1), (4.3) and (4.9) that

$$\mathcal{L} W(x, v) \leq \frac{\beta}{2} W_0(x, v)^{\frac{\beta - 2}{2}} ((2\theta_0 - \theta^* \gamma)(x, v) - (2\gamma - \theta^*) |v|^2 - c^* \theta^* |x|^2 - c^{**} \theta^* U(x) + \theta^* C_0^*)$$

$$+ J(x, v) \int_{\{|u| \leq \rho\}} (W_0(x, u)^{\beta/2} - W_0(x, v)^{\beta/2}) \varphi(u) \, du$$

$$+ J(x, v) \int_{\{|u| > \rho\}} (W_0(x, u)^{\beta/2} - W_0(x, v)^{\beta/2}) \varphi(u) \, du$$

$$= : \Upsilon_1(x, v) + J(x, v) \Upsilon_2(x, v) + J(x, v) \Upsilon_3(x, v).$$

Since the function $x \mapsto x^{\beta/2}$ with $\beta \in (0, 2]$ is concave on $[0, \infty)$, the mean value theorem enables us to obtain that

$$\Upsilon_2(x, v) \leq \frac{\beta}{2} W_0(x, v)^{\frac{\beta - 2}{2}} \int_{\{|u| \leq \rho\}} (W_0(x, u) - W_0(x, v)) \varphi(u) \, du$$

$$\leq \frac{\beta}{2 \rho^2} - \frac{\beta c_{0,\rho}}{2} W_0(x, v)^{\frac{\beta - 2}{2}} (|v|^2 + \theta^* (x, v)),$$

where in the second inequality we utilized the fact that $\varphi(\cdot)$ is a radial function and meanwhile used $W_0 \geq 1$ and $\beta \in (0, 2]$.

On the other hand, by the basic inequality $(a + b)^{\ell} \leq a^{\ell} + b^{\ell}$ for all $a, b \geq 0$ and $\ell \in (0, 1]$, we deduce from (4.3) and $\beta \in (0, 2]$ that

$$\Upsilon_3(x, v)$$

$$\leq \int_{\{|u| > \rho\}} \left( (1 + 2U(x))^{\beta/2} + (\theta_0 |x|^2 + |u|^2 + \theta^* (x, u))^{\beta/2} - (1 + 2U(x))^{\beta/2} \right) \varphi(u) \, du$$

$$\leq (\theta_0 |x|^2)^{\beta/2} (1 - c_{\rho,\ast}) + \int_{\{|u| > \rho\}} |u|^2 \varphi(u) \, du + (\theta^* |x|)^{\beta/2} \int_{\{|u| > \rho\}} |u|^2 \varphi(u) \, du$$

$$\leq (\theta_0 |x|^2)^{\beta/2} (1 - c_{\rho,\ast}) + \int_{\{|u| > \rho\}} |u|^2 \varphi(u) \, du + (\theta^* |x|)^{\beta/2} \left( \int_{\{|u| > \rho\}} |u|^2 \varphi(u) \, du \right)^{1/2}$$

$$\leq 2(\theta_0 |x|^2)^{\beta/2} (1 - c_{\rho,\ast}) + C_{\rho,\ast}$$

with

$$C_{\rho,\ast} := \left( 1 + \frac{(\theta^*/\theta_0^{1/2})^\beta}{4(1 - c_{\rho,\ast})} \right) \int_{\{|u| > \rho\}} |u|^2 \varphi(u) \, du,$$

where in the third inequality we employed Jensen’s inequality and in the last inequality we exploited Young’s inequality. Again, via Young’s inequality, for $\beta \in (0, 2]$ we arrive at

$$|x|^\beta = W_0(x, v)^{\frac{\beta - 2}{2}} W_0(x, v)^{\frac{2 - \beta}{2}} |x|^3 \leq \frac{\beta}{2} W_0(x, v)^{\frac{\beta - 2}{2}} ((2/\beta - 1) W_0(x, v) + |x|^2).$$
Plugging this back into (4.11) yields
\[
(4.13) \quad \Upsilon_3(x,v) \leq \frac{\beta}{2} W_0(x,v) \frac{\partial^2}{\partial x^2} \theta^{3/2}(1 - c_{\rho,*}) ((4/\beta - 2) W_0(x,v) + 2|x|^2) + C_{\rho,*}.
\]

Now, with the help of (4.10) and (4.13) and by taking the expression of $W_0$, given in (4.1), we deduce from $W_0 \geq 1$ and $\beta \in (0,2]$ that
\[
(\mathcal{L}W)(x,v) \leq \frac{\beta}{2} W_0(x,v) \frac{\partial^2}{\partial x^2} \left\{ (2\theta - \theta^* \gamma)(x,v) - (2\gamma - \theta^*)|v|^2 - c' \theta^* |x|^2 - c'' \theta^* U(x) \right\}
+ \frac{\beta}{2} W_0(x,v) \frac{\partial^2}{\partial x^2} J(x,v)
\times \left\{ - c_{\rho,*} |v|^2 + \theta^*(x,v)) + 2\theta_0^2 (1 - c_{\rho,*}) ((2/\beta - 1)(2U(x) + \theta_0|x|^2 + |v|^2 + \theta^*(x,v)) + |x|^2) \right\}
+ C_{\rho,**}
\leq \frac{\beta}{2} W_0(x,v) \frac{\partial^2}{\partial x^2} \left\{ - (c^* \theta^* - \Theta_{4,\rho}) U(x) - (c^* \theta^* - \Theta_{2,\rho}) |x|^2 - (\lambda_1 + 2\gamma - \theta^* - \Theta_{3,\rho}) |v|^2
+ (C_{0,x}(x,v) + \Xi(x,v)) \langle x,v \rangle \right\} + C_{\rho,**},
\]
where $C_{0,x}(x,v) := 2\theta_0 - \theta^*(J(x,v) + \gamma)$, $\Xi(x,v) := (1 - c_{\rho,*}) J(x,v) \theta^*(1 + 2\theta_0^2 (2/\beta - 1))$ and $C_{\rho,**} := \theta^* C_{0,**} + \lambda_2 \rho^2 + \lambda_2 C_{\rho,**} + 2\lambda_2 \theta_0^2 (2/\beta - 1)$ with $C_{\rho,*}$ being introduced in (4.12).

Note that
\[
0 \leq C_{0,x}(x,v) + \Xi(x,v) \leq 2\theta_0 - \theta^*(\lambda_1 + \gamma) + (1 - c_{\rho,*}) \lambda_2 \theta^*(1 + 2\theta_0^2 (2/\beta - 1)) = C_{0,**} + \Theta_{1,\rho}
\leq \frac{1}{2}(c^* \theta^* - \Theta_{2,\rho} + c_{0,\rho}**) + (\lambda_1 + 2\gamma - \theta^* - \Theta_{3,\rho}) \frac{2c_{0,\rho}**}{c^* \theta^* - \Theta_{2,\rho} + c_{0,\rho}**},
\]
where $c_{0,\rho}** > 0$ was defined by (4.8). This yields that
\[
(\mathcal{L}W)(x,v) \leq \frac{\beta}{2} W_0(x,v) \frac{\partial^2}{\partial x^2} \left\{ - (c^* \theta^* - \Theta_{4,\rho}) U(x) - \frac{1}{2} (c^* \theta^* - \Theta_{2,\rho} - c_{0,\rho}**) |x|^2
- (\lambda_1 + 2\gamma - \theta^* - \Theta_{3,\rho}) \frac{c^* \theta^* - \Theta_{2,\rho} - c_{0,\rho}**}{c^* \theta^* - \Theta_{2,\rho} + c_{0,\rho}**} |v|^2 \right\} + C_{\rho,**},
\]
Consequently, (4.16) follows by combining (4.13) with (4.7) and (4.8). \qed

**Proposition 4.3.** Assume that the function $r \mapsto \varphi(r)$ is bounded and decreasing on $(0, \infty)$ so that $\varphi(r) > 0$ for all $r > 0$ and $\int_{\mathbb{R}^d} |u|^2 \varphi(u) du < \infty$ for some $\beta \in (0,2]$, and that there exists a constant $c_{0,**} > 0$ such that for all $\xi \in \mathbb{R}^d$,
\[
(4.14) \quad \int_{\mathbb{R}^d} |u|^2 \varphi(u) du \leq c_{0,**} |\xi|.
\]
Then, Assumption (B2) holds for $W(x,v) = W_0(x,v)^{\beta/2}$, where $W_0$ was defined in (4.1).

**Proof.** Via the inequality that $(a + b + c)^\ell \leq a^\ell + b^\ell + c^\ell \leq 3(a + b + c)^\ell$ for all $a, b, c \geq 0$ and $\ell \in (0,1)$, we infer from Jensen’s inequality and Young’s inequality that
\[
\int_{\mathbb{R}^d} W(x,u) \varphi(u) du \leq (1 + 2U(x) + \theta_0 |x|^2)^{\beta/2} + \int_{\mathbb{R}^d} |u|^2 \varphi(u) du + (\theta^* |x|)^{\beta/2} \int_{\mathbb{R}^d} |u|^2 \varphi(u) du
\leq (1 + 2U(x) + \theta_0 |x|^2)^{\beta/2} + (\theta_0 |x|^2)^{\beta/2} + C_{**}
\leq 3 \left( 1 + (C_{**})^{2/\beta} \vee (2\theta_0) \right)^{\beta/2} (1 + 2U(x) + |x|^2)^{\beta/2},
\]

where

\[ C^{**} := \left(1 + \frac{1}{4(\theta^*/\theta_0^{1/2})^2}\right) \int_{\mathbb{R}^d} |u|^{\beta} \varphi(u) \, du < \infty \]

by taking \( \int_{\mathbb{R}^d} |u|^{\beta} \varphi(u) \, du < \infty \) into consideration. Thus, we arrive at

\[ \int_{\mathbb{R}^d} \mathcal{W}(x, u) \varphi(u) \, du \leq 3 \left(1 + (C^{**})^{2/\beta} \right)^{\beta/2} \inf_{v \in \mathbb{R}^d} \left(1 + 2U(x) + |x|^2 + |v|^2\right)^{\beta/2}. \]

Therefore, the first inequality in (1.8) holds true for \( \mathcal{W} \).

Next, by taking advantage of the inequality that \((a + b)\ell \leq a^\ell + b^\ell\) for all \(a, b \geq 0\) and \(\ell \in (0, 1]\), and by invoking Hölder’s inequality, we find that

\[
\int_{\mathbb{R}^d} \mathcal{W}(x, u) \Psi_\xi(u) \, du \leq \left(1 + 2U(x) + \theta_0 |x|^2\right)^{\beta/2} \int_{\mathbb{R}^d} \Psi_\xi(u) \, du + \int_{\mathbb{R}^d} |u|^{\beta} \Psi_\xi(u) \, du \]

\[ + \left(\theta^* |x|\right)^{\beta/2} \left(\int_{\mathbb{R}^d} |u|^{\beta} \Psi_\xi(u) \, du\right)^{1/2} \left(\int_{\mathbb{R}^d} \Psi_\xi(u) \, du\right)^{1/2}. \]

Then, applying Proposition 4.1 and taking (4.14) into account yield that for some constant \(C_{**} > 0\),

\[ \int_{\mathbb{R}^d} \mathcal{W}(x, u) \Psi_\xi(u) \, du \leq C_{**} \left(1 + 2U(x) + |x|^2\right)^{\beta/2} |\xi|. \]

Consequently, the second inequality in (1.8) is proved thanks to (4.3) again.

By summing up the previous analysis, we make a conclusion that Assumption \((B_2)\) is provable for \(\mathcal{W}(x, v) = \mathcal{W}_0(x, v)^{\beta/2}\).

Examples for the probability density function \(\varphi\) that satisfies all the assumptions in Propositions 4.2 and 4.3 are \(\varphi(x) = \varphi_1(x) := c_{d, \beta_1} (1 + |x|)^{-d - \beta_1}\) with \(\beta_1 > 0\) and \(c_{d, \beta_1} > 0\) or \(\varphi(x) = \varphi_2(x) := c_{d, \beta_2} \exp(-|x|^{2\gamma})\) with \(\beta_2 > 0\) and \(c_{d, \beta_2} \geq 0\).

Finally, we intend to validate the condition (1.10).

**Proposition 4.4.** Assume \(U(x) = \theta |x|^2\) for all \(\theta > 0\) and \(x \in \mathbb{R}^d\). Then, the inequality (1.10) is provable in case of \(\gamma \geq 2\sqrt{2} \theta\).

**Proof.** Due to \(\gamma \geq 2\sqrt{2} \theta\), for any

\[ \alpha \in \left(0, \left(\gamma - \sqrt{\gamma^2 - 8\theta}\right)/2\right) \cup \left(\left(\gamma + \sqrt{\gamma^2 - 8\theta}\right)/2, +\infty\right), \]

we have \(\alpha(\gamma - \alpha) \leq 2\theta\). On the other hand, for all

\[ \frac{1}{2} \left(\gamma - \sqrt{\gamma^2 - 32\theta/5}\right) < \alpha < \frac{1}{2} \left(\gamma + \sqrt{\gamma^2 - 32\theta/5}\right) \]

provided \(\gamma \geq \sqrt{32\theta/5}\), it holds that \(\alpha^2 - \gamma \alpha + \theta \gamma/5 \leq 0\). Hence, for \(\gamma \geq 2\sqrt{2} \theta\), we find that there exists an \(\alpha > 0\) satisfying simultaneously

\[ \alpha(\gamma - \alpha) \leq 2\theta \]

and

\[ 5\alpha^2 - 5\gamma \alpha + 8\theta \leq 0. \]

Thanks to \(U(x) = \theta |x|^2\), \(x \in \mathbb{R}^d\), we obviously obtain that for all \(x, x' \in \mathbb{R}^d\),

\[ \nabla U(x) - \nabla U(x') = 2\theta (x - x'). \]

Thus, for \(\beta := \alpha(\gamma - \alpha) \leq 2\theta\) due to (1.13), we deduce that for all \(x, x' \in \mathbb{R}^d\),

\[ |\beta(x - x') + \nabla U(x') - \nabla U(x)| = (2\theta - \beta)|x - x'|. \]

Therefore, Assumption \((H_0)\) is satisfied for \(K_{\beta, \theta} = 2\theta - \beta\).
Next, it is clear that $\beta \leq 2\theta \leq \gamma^2/4$. On the other hand, with (4.16) at hand, we find that for $\alpha > 0$ solving (4.15) and (4.16),

$$\beta = \gamma \alpha - \alpha^2 \geq 4(2\theta - \alpha(\gamma - \alpha)) = 4K_\alpha(\gamma - \alpha), U = 4K_{\beta, U}$$

by recalling $K_{\beta, U} = 2\theta - \beta$ with $\beta = \alpha(\gamma - \alpha)$. Consequently, we can reach a conclusion that the inequality (1.10) is solvable. □

With all the propositions above at hand, we can easily verify Example 1.2, and so the detail is omitted here to save space.

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