T-EQUIVARIANT DISC POTENTIAL AND SYZ MIRROR CONSTRUCTION

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ABSTRACT. We develop a $G$-equivariant Lagrangian Floer theory by counting pearly trees in the Borel construction $L_G$. We apply the construction to smooth moment-map fibers of toric semi-Fano varieties and obtain the $T$-equivariant Landau-Ginzburg mirrors. We also apply this to the typical $S^1$-invariant SYZ singular fiber, which is the single-pinched torus, and compute its $S^1$-equivariant disc potential.

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1. Introduction

Teleman [Tel14] conjectured that the mirror of a Hamiltonian $G$-space $X$ has a holomorphic fibration to the space of conjugacy classes in the Langlands dual $G^\vee_C$. Moreover, the mirrors of the symplectic quotients $X//_\lambda G$ for $\lambda \in g^*/Ad_C^\vee$ are related to the fibers. This agrees with the work of Hori-Vafa [HV] on mirror symmetry via gauged linear sigma models.

An important step towards understanding this conjecture is to construct the mirror holomorphic fibration. In this paper, we develop an equivariant version of the SYZ mirror construction. For the concrete computations we focus on the case $G = T$, and study the $T$-equivariant Floer theory for torus fibers of an SYZ fibration preserved by $T$. In general, the $G$-equivariant disc potential $W^G$ of a $G$-invariant Lagrangian gives a fibration of the equivariant mirror space to $H^2(BG; C)$.

For instance, consider the well-known Landau-Ginzburg mirror of a compact Fano toric $d$-fold $X$ [HV, Giv95, Giv98, LLY99]. It is a pair $((\mathbb{C}^*)^d, W)$, where $W : (\mathbb{C}^*)^d \to \mathbb{C}$ is a
holomorphic function of the form

\[ W(z) = \sum_{i=1}^{m} q^{\alpha_i} z^{\nu_i}, \]

\[ \nu_1, \ldots, \nu_m \in N \cong \mathbb{Z}^d \]

are primitive generators of the one-dimensional cones of the fan \( \Sigma \) defining \( X \). \( z^{\nu_i} \) denote the corresponding Laurent monomials, and \( q^{\alpha_i}, \ldots, q^{\alpha_m} \) are Kähler parameters associated to certain curves classes \( \alpha_i, \in H_2(X, \mathbb{Z}) \).

It was shown [Giv98] that the equivariant quantum cohomology \( \text{QH}^*_T(X) \) of \( X \) is isomorphic to the Jacobian ring of \( T \). The equivariant part \( \text{equiv} = \sum_{i=1}^{m} \exp(g_i(q(q))) \text{Te}^{\omega(\beta)} \exp(v_i \cdot (x^1, \ldots, x^d)) \lambda_j + \sum_{j=1}^{\ell} (u_j \cdot (x^1, \ldots, x^d)) \lambda_j \)

where \( \beta_i \) are the basic disc classes bounded by the toric fiber, \( g_i(q(q)) \) is given by the inverse mirror map in Equation (3.7), and \( q^\lambda = \text{Te}^{\omega(\lambda)} \) are Kähler parameters and \( T \) is the formal Novikov variable.

In the Fano case \( g_i = 0 \). Taking \( \ell = d, u_j \) to be the standard basis, and \( x^i = \log z_i \) for \( i = 1, \ldots, d \), the above expression for \( \text{equiv} \) agrees with \( W_{\lambda} \).

In relation with the Teleman’s conjecture, we can also understand the above expression in the following way. The equivariant part \( \sum_{j=1}^{\ell} (u_j \cdot (x^1, \ldots, x^d)) \lambda_j \) defines a fibration \( (\mathbb{C}^x)^d \rightarrow (\mathbb{C}_c^x)^* \). Then the non-equivariant part \( W \) defined on \( (\mathbb{C}^x)^d \) can be understood as a family of potentials on the fibers.

For instance, take \( X = \mathbb{C}^d \). The non-equivariant part is simply \( W = z_1 + \ldots + z_d \). Suppose \( X/\lambda T^\ell \) is a semi-Fano toric manifold. A fiber of the above fibration is given by \( z_1 = c_j \) for some constants \( c_j \) and \( j = 1, \ldots, \ell \). Then \( W \) restricted to fibers give the mirror family of \( X/\lambda T^\ell \).

**Remark 1.2.** The mirror map \( c(\lambda) \) is crucial to precisely identify which fiber corresponds to \( X/\lambda T^\ell \). By the beautiful work of Woodward-Xu [WX], the mirror map can be understood as the change from the gauged Floer theory of \( X \) to the Fukaya category downstairs \( \text{Fuk}(X/\lambda T^\ell) \).
Gauged Floer theory is formulated in terms of vortex equations. It would be very interesting to investigate the relation with our formulation.

In general (non-toric case), we can show that the equivariant disc potential always take the form

\[ W(b) + \sum_{i=1}^{\ell} h_i(b) \lambda_i \]

for weak bounding cochains \( b \) and \( L \) having minimal Maslov index zero. Then we have the fibration \( (h_1, \ldots, h_\ell) \) of the space of weak bounding cochains over \( (t_\ell^C)^* \).

Equivariant Lagrangian Floer theory has seen major recent developments. In the exact setting and \( G = \mathbb{Z}_2 \), Seidel-Smith [SS10] provided an approach to understanding \( G \)-equivariant Floer theory by combining Lagrangian Floer theory and family Morse theory [Hut08] on \( E \mathbb{G} \to B \mathbb{G} \). Viterbo [Vit99] illustrated some related ideas for \( G = S^1 \).

Hendricks-Lipshitz-Sarkar [HLS16a, HLS16b] developed a homotopy coherent method to build up a \( G \)-equivariant Floer theory. Fukaya [Fuk17] and Daemi-Fukaya [DF17] used \( G \)-equivariant Kuranishi structure to tackle the \( G \)-equivariant transversality problem, and make a formulation using differential forms. There are other interesting works related to this subject [Sei15, LP16].

In this paper we use Morse chains on the Borel construction \( L_T \) for \( L \subset X \), a \( T \)-invariant compact Lagrangian submanifold in a symplectic manifold \( (X, \omega) \) preserved by a \( T \)-action. This is closest to the approach of Seidel-Smith, although the Lagrangians under our consideration are not exact. They bound non-constant pseudo-holomorphic discs of non-negative Maslov index. These are known as quantum corrections in mirror symmetry, which contribute non-trivial terms to \( W_\lambda \) and make it very interesting.

Since there are only finitely many generators for each finite dimensional approximation of \( L_T \), it better serves for computations and manifests the equivariant parameters \( \lambda_i \) appeared in the disc potential \( W_\lambda \). Assuming that \( L \) has minimal Maslov index zero, we construct a curved \( A_\infty \)-algebra

\[
\left( C^{\bullet}_{\text{Morse}}(L, \Lambda_0) \otimes_{\Lambda_0} \Lambda_0[\lambda_1, \ldots, \lambda_\ell], m^T_k \right)
\]

by counting pearly trees [BC07, FOOO09a] in \( L_T \), where \( \Lambda_0[\lambda_1, \ldots, \lambda_\ell] = H^*_T(\text{pt}; \Lambda_0) \), and \( \Lambda_0 \) is the Novikov ring

\[
\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^A \mid a_i \in \mathbb{C}, A_i \in \mathbb{R}_{\geq 0}; A_i \text{ increases to } +\infty \right\}.
\]

We denote by \( \Lambda_+ \subset \Lambda_0 \) the maximal ideal with positive \( T \)-valuation, and also by \( \Lambda \) the Novikov field where the exponentials \( A_i \in \mathbb{R} \).

We will show that (Theorem 2.15)

\[
m^T_k(x_1, \ldots, x_i \otimes \lambda_j, \ldots, x_k) = \lambda_j \cdot m^T_k(x_1, \ldots, x_i \otimes 1, \ldots, x_k)
\]

and hence we may treat \( \Lambda_0[\lambda_1, \ldots, \lambda_\ell] \) as a coefficient ring. Note that the terms \( m^T_k(x_1 \otimes 1, \ldots, x_k \otimes 1) \) are still series in the equivariant parameters \( \lambda \) and receive non-trivial contributions from pearly trees in \( L_T \). For this, we use the construction of homotopy unit developed by Fukaya-Oh-Ohta-Ono [FOOO09b, Chapter 7] and Charest-Woodward [CW] in a different setup.
We take the boundary deformations $b = \sum_{i=1}^{\ell} x^i x_i$ for $x^i \in \Lambda_+$, where $x_i$ are degree one critical points of $L$. We will see that if $(L, b)$ is weakly unobstructed, then the $T$-equivariant theory $(L^T, b)$ will also be weakly unobstructed over $\Lambda[\lambda_1, \ldots, \lambda_\ell]$, namely

$$m_T^0 = W \cdot 1_L \otimes 1_{B^T} + \sum_{i=1}^{\ell} h_i \cdot 1_L \otimes \lambda_i = \left( W + \sum_{i=1}^{\ell} h_i \lambda_i \right) \cdot 1_L,$$

where $h_i$ consists of contributions from pseudo-holomorphic discs of Maslov index zero. As a consequence, the usual disc potential $W$ equals to the non-equivariant limit $(\lambda_i \to 0)$ of $W_\lambda$.

As a consequence, suppose that we have a Lagrangian torus fibration which admits a free $T^\ell$-action. When the fibers are unobstructed, the non-equivariant disc potential is zero by definition. However, from the above consideration, we still have non-trivial equivariant terms and so the equivariant disc potential for a smooth torus fiber is non-zero:

$$W^T = \sum_{i=1}^{\ell} h_i \lambda_i.$$

In Section 4, we study the $S^1$-equivariant disc potential for the immersed two-sphere $S^2$ with a single nodal point. This is also known as the pinched two-torus which is the most typical singular fiber in an SYZ fibration. Even when $S^2$ does not bound any non-constant discs of Maslov index zero, it still has a non-trivial equivariant disc potential from the contribution of constant polygons at the nodal point. The corresponding moduli have non-trivial obstructions. The gluing technique via isomorphism between smooth and pinch tori in [HKL] is crucial in the computation of the explicit expression of the equivariant disc potential.

**Theorem 1.3** (Theorem 4.6). The equivariant disc potential of the immersed sphere $S^2$ is

$$W = \log(1 - uv) = -\sum_{j=1}^{\infty} \frac{(uv)^j}{j}$$

where $(u, v) \in \Lambda_0^2 - \{\text{val}(uv) > 0\}$ are the formal deformations corresponding to the degree one immersed generators of $S^2$.

The organization of this paper is as follows. We develop a Morse model for the equivariant Lagrangian Floer theory in Section 2. We apply this to formulate and compute the $T$-equivariant disc potential for toric semi-Fano manifolds in Section 3. In Section 4, we compute the $S^1$-equivariant disc potential for the immersed two-sphere $S^2$.

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2. A Morse model for equivariant Lagrangian Floer theory

There are several approaches to G-equivariant Lagrangian Floer theory for a pair of G-invariant Lagrangians in the existing literature. For $G = \mathbb{Z}_2$, Seidel and Smith [SS10] used Floer homology coupled with Morse theory on $EG$ to define G-equivariant Lagrangian Floer homology of exact Lagrangians. In [HLS16a, HLS16b], Hendricks, Lipshitz, and
Sarkar used a homotopy theoretic method to define $G$-equivariant Lagrangian Floer homology for $G$ a compact Lie group. More recently, in [DF17], Daemi and Fukaya defined an equivariant de Rham model using $G$-equivariant Kuranishi structures developed in [FOOO09b, Fuk17].

In this section, we develop a Morse model for the $G$-equivariant Lagrangian Floer theory focusing on a single $G$-invariant Lagrangian. The Lagrangian we study here can be immersed with clean self-intersection. The underlying cochain complexes are finite dimensional over the cohomology ring of $BG$, and the $A_\infty$ operations are given by counting pearl trees in the Borel construction $L_G$. This suits better for our purpose of computing disc potentials and constructing SYZ mirrors. The $A_\infty$-algebra we construct will be unital. This uses the homotopy unit construction in [FOOO09b], which has also been adapted to the stabilizing divisor perturbation scheme by Charest-Woodward [CW] for their non-equivariant Morse model.

The novelty of our construction is that we define the $G$-equivariant Lagrangian Floer theory as the (ordinary) Lagrangian Floer theory of $L_G$ as a Lagrangian submanifold of a certain symplectic manifold. This avoids the issue of equivariant transversality. Since the Lagrangians we study here bounds non-constant pseudo-holomorphic discs, we use the machinery of [FOOO09b] to handle the obstructions. On the other hand, for computing the equivariant disc potential of toric Fano manifolds, virtual technique is not necessary.

2.1 The non-equivariant singular chain model. We begin by recalling the $A_\infty$-algebra associated to a Lagrangian submanifold $L$ of a symplectic manifold constructed by Fukaya-Oh-Ohta-Ono [FOOO09b]. We refer to [FOOO09b] for a review of $A_\infty$-algebras and Kuranishi structures.

Let $(X,\omega_X)$ be a symplectic manifold of real dimension $2d$ and let $J_X$ be a compatible almost complex structure. We will always assume $X$ is convex or geometrically bounded at infinity if it is non-compact. Let $L \subset X$ be a closed, connected, and relatively spin Lagrangian submanifold. In [FOOO09b], a countably generated subcomplex $C^*(L;\Lambda_0)$ of the singular chain complex $S^*(L;\Lambda_0)$ (regarded as a cochain complex) and an $A_\infty$-algebra structure $(C^*(L;\Lambda_0),m)$ were constructed. We briefly describe $C^*(L;\Lambda_0)$ below, which we will use in constructing the equivariant Morse model.

Let $\beta \in H^2_2(X,L;\mathbb{Z})$ and denote by $\mathcal{M}(\beta,L,J_X)$ the moduli space of $J_X$-holomorphic stable bordered maps $u: (D^2,\partial D^2) \to (X,L)$ of genus 0 representing $\beta$ and by $\mathcal{M}_{k+1}(\beta,L,J_X)$ the moduli space with $k+1$ marked points $z_0,\ldots,z_k$ on $\partial D^2$, ordered counter-clockwise. Let $H^{\text{eff}}_2(X,L) \subset H^2_2(X,L;\mathbb{Z})$ denote the effective cones of $J_X$-holomorphic disc classes, i.e.

$$H^{\text{eff}}_2(X,L) = \{ \beta \in H^2_2(X,L;\mathbb{Z}) | \mathcal{M}(\beta,L,J_X) \neq \emptyset \}.$$

We denote by $\beta_0$ the constant disc class.

For $\beta \in H^{\text{eff}}_2(X,L)$, we define

$$\|\beta\| = \begin{cases} \sup\{n | \exists \beta_1,\ldots,\beta_n \in H^{\text{eff}}_2(X,L) \setminus \{\beta_0\}, \text{ with } \sum_{i=1}^n \beta_i = \beta \} + |\omega(\beta)| - 1, & \text{if } \beta \neq \beta_0; \\ -1, & \text{if } \beta = \beta_0. \end{cases}$$

Let $\delta: \{1,\ldots,k\} \to \mathbb{Z}_{\geq 0}$. For a pair $(\delta,\beta)$ with $\beta \in H^{\text{eff}}_2(X,L)$, we define

$$\| (\delta, \beta) \| = \begin{cases} \max_{i \in \{1,\ldots,k\}} \{ \delta(i) \} + \|\beta\| + k & \text{if } k \neq 0; \\ \|\beta\| & \text{if } k = 0. \end{cases}$$
Let $\bar{P} = (P_1, \ldots, P_k)$ be a $k$-tuple of singular chains. We denote by $\mathcal{M}_{k+1}(\beta; L; \bar{P})$ the fiber product (in the sense of Kuranishi structures) of $\mathcal{M}_{k+1}(\beta; L, J_X)$ with smooth singular chains $P_i$ on $L$, $i = 1, \ldots, k$,

$$\mathcal{M}_{k+1}(\beta; L; \bar{P}) = \mathcal{M}_{k+1}(\beta; L, J_X) \times_{L^k} \prod P_i.$$ 

In [FOOO09b], countable sets $\mathcal{X}_g(L)$ of smooth singular chains on $L$ were constructed, and a system of multisections $s_{\beta, k, \beta, \beta}$ for $\mathcal{M}_{k+1}^0(\beta; L; \bar{P})$, $\|\partial, \beta\| = g$, were chosen inductively on the generation $g$. Here the decoration $\partial$ records the generations of the inputs $P_i \in \mathcal{X}_{\partial(i)}(L)$.

At each inductive step, new multisections $s_{\beta, k, \beta, \beta}$ are chosen to be transversal to zero sections (of the obstruction bundles) and extend the multisections previously chosen for the boundary strata $\partial \mathcal{M}_{k+1}(\beta; L; \bar{P})$. Moreover the zero locus

$$\mathcal{M}_{k+1}^0(\beta; L; \bar{P})^{\beta, k, \beta, \beta} := s_{\beta, k, \beta, \beta}^{-1}(0)$$

is triangulated extending the triangulation on the boundary strata. The new simplices in the triangulation are then regarded as elements of $\mathcal{X}_g(L)$. Additional singular simplices are then added to $\mathcal{X}_g(L)$ so that

$$C_{(g)}(L; \Lambda_0) = \bigoplus_{\gamma \leq g} \Lambda_0 \cdot \mathcal{X}_\gamma(L)$$

remains a subcomplex of $S^\bullet(L; \Lambda_0)$ isomorphic on cohomology.

The $A_{\infty}$ maps $m_k : C^\bullet(L; \Lambda_0)^{\otimes k} \to C^\bullet(L; \Lambda_0)$ are defined by

$$m_k(P_1, \ldots, P_k) = \sum_{\beta \in \text{H}^W(X, L)} T^{\omega(\beta)} m_{k, \beta}(P_1, \ldots, P_k)$$

(2.1)

where

$$m_{k, \beta}(P_1, \ldots, P_k) = (\text{ev}_0)_* \left( \mathcal{M}_{k+1}(\beta; L; \bar{P})^{\beta} \right)$$

(2.2)

are maps of degree $2 - k - \mu_L(\beta)$.

**Remark 2.1.** In order to define $\mathbb{Z}$-graded cohomology theory, the conventional grading of the maps $m_k$ are $2 - k$. This means beyond the Calabi-Yau case, one has to define the Novikov ring $\Lambda_0$ using an extra grading parameter in order to compensate for the Maslov indices (see [FOOO09b]). In this paper, we work on the chain level and do not follow this convention. The grading of $m_{k, \beta}$ is crucial in understanding the vanishing of certain terms when doing computations in the Morse model.

The singular chain model $(C^\bullet(L; \Lambda_0), m)$ does not have a strict unit in general. [FOOO09b, Chapter 7] showed that the fundamental cycle $e$ of $L$ is a homotopy unit. We briefly describe the key ideas below and refer the readers to [FOOO09b, Chapter 3.3] for the precise definition of a homotopy unit and to [FOOO09b, Chapter 7.3] for the details of the homotopy unit construction.

We enlarge $(C^\bullet(L; \Lambda_0), m)$ to a homotopy equivalent unital $A_{\infty}$-algebra $(C^\bullet(L; \Lambda_0)^+, m^+)$ by adding a degree 0 generator $e^+$ serving as the strict unit and a degree $-1$ generator $f$ serving as a homotopy between $e$ and $e^+$

$$C^\bullet(L; \Lambda_0)^+ = C^\bullet(L; \Lambda_0) \oplus \Lambda_0 \cdot e^+ \oplus \Lambda_0 \cdot f.$$

The $A_{\infty}$ operations $m^+_k$ are defined as follows.

The restriction of $m^+_k$ to $C^\bullet(L; \Lambda_0)$ agrees with $m$. 

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*Note: The text above is a transcription of the provided image, formatted into a clear and readable document.*
\(e^+\) is the strict unit, i.e.
\[
m_2^+(e^+, x) = (-1)^{|x|}m_2^+(x, e^+) = x,
\]
for \(x \in C^*(L; \Lambda_0)^+\), and
\[
m_k^+(\ldots, e^+, \ldots) = 0
\]
for \(k \neq 2\).

\(f\) is the homotopy between \(e\) and \(e^+\) in the following sense. Let \(\bar{a} = \{a_1, \ldots, a_{|\bar{a}|}\} \subset \{1, \ldots, k\}, a_1 < \ldots < a_{|\bar{a}|}\). For \(\bar{P} = (P_1, \ldots, P_{k-|\bar{a}|})\) with \(P_i \in \mathcal{X}_0(i)(L)\), let \(\bar{P}^+\) be the \(k\)-tuple obtained by inserting \(e\) into the \(a_1, \ldots, a_{|\bar{a}|}\)-th places of \(\bar{P}\). We assume \(L \in \mathcal{X}_0(L)\) and set \(\delta^+ = g_i^+\), if \(\bar{P}^+ = (P_1^+, \ldots, P_k^+)\) and \(P_i^+ \in \mathcal{X}_0(L)\). We have the map
\[
\text{forget}_i : \mathcal{M}_{k+1}(\beta; L; \bar{P}^+) \to \mathcal{M}_{k+1-|\bar{a}|}(\beta; L; \bar{P}),
\]
which forgets the \(a_1, \ldots, a_{|\bar{a}|}\)-th marked points (and stabilizes).

Let \(\bar{a}_1 \ldots \bar{a}_2 = \bar{a}\) be a splitting of \(\bar{a}\). Denote by \(\bar{P}_\beta\) the \((k - |\bar{a}|)\)-tuple given by removing \(e\) from the \(a_1, \ldots, a_{|\bar{a}|}\)-th places of \(\bar{P}^+\). For the choices of perturbations for \(\mathcal{M}_{k+1}(\beta; L; \bar{P}^+)\), we have the perturbation \(s_{\delta^+, k, \beta, \bar{P}}\) obtained by just inserting \(e\) into the \(a_1, \ldots, a_{|\bar{a}|}\)-th place and the perturbations pulled back via the forgetful maps \(\text{forget}_{i, \bar{a}^2} : \mathcal{M}_{k+1}(\beta; L; \bar{P}^+) \to \mathcal{M}_{k+1-|\bar{a}|}(\beta; L; \bar{P})\), and Moreover, we have a perturbation \(s_{\delta^+, k, \beta, \bar{P}}\) (transversal to the zero section) on \([0, 1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L; \bar{P}^+)\) interpolating between them.

The \(m_{k, \beta}^\pm\) operations, \((k, \beta) \neq (1, \beta_0)\), with inputs \(f\) inserted into \(a_1, \ldots, a_{|\bar{a}|}\)-th place of \(\bar{P}\) are defined by
\[
(e_{0})_*\left( ([0, 1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L; \bar{P}^+))^{s_{\delta^+, k, \beta, \bar{P}}^+/\sim} \right).
\]
(2.6)

Here \(\sim\) is the equivalence relation collapsing fibers of the forgetful maps (see [FOOO09b, Definition 7.3.28]). The singular simplices which appear in (2.6) are also added during the inductive construction of \(\mathcal{X}_0(L)\).

We also set
\[
m_{1, \beta_0}^+(f) = e^+ - e
\]
and therefore
\[
m_1^+(f) = e^+ - e + h,
\]
(2.7)

\[
h = \sum_{\beta \neq \beta_0} T^{a_{\times}(\beta)} (e_{0})_*\left( ([0, 1] \times \mathcal{M}_{2}(\beta; L; e))^{s^+/\sim} \right).
\]

Here \(h\) has positive \(T\)-valuation. The terms
\[
\sum_{\mu L(\beta) \geq 2} (e_{0})_\ast\left( ([0, 1] \times \mathcal{M}_{2}(\beta; L; e))^{s^+/\sim} \right)
\]
are of degrees at most \(-2\) and therefore are degenerate singular chains.

**Remark 2.2.** As a heuristic, we can think of the \(m_{k}^\pm\) operations with inputs \(e^+\) inserted into \(a_1, \ldots, a_{|\bar{a}|}\)-th places of \(\bar{P}\) are given by the moduli space \(\mathcal{M}_{k+1}(\beta; L; \bar{P}^+)\) equipped with multisection \(s_{\delta}^+\) pulled back from \(\mathcal{M}_{k+1-|\bar{a}|}(\beta; L; \bar{P})\) (which may not be transverse to the zero section). The singular chain
\[
(e_{0})_\ast(\mathcal{M}_{k+1}(\beta; L; \bar{P}^+))^{s_{\delta}^+}.
\]
is degenerate for \((k, \beta) \neq (1, \beta_0)\), which are not zero as singular chains. This is the reason the
equivalence relation in (2.6) was introduced and the \(m_k^+\) operations with \(e^+\) as inputs are formally
defined via (2.3) and (2.4).

2.2 The non-equivariant Morse model with a strict unit. Let us now turn to a
construction of a Morse model for Lagrangian Floer theory of \(L\) using pearly trees. Pearly
trees were developed by Biran and Cornea [BC07, BC09] for monotone Lagrangians (the
idea of such a complex appeared in Oh [Oh96]). It has many important applications, in-
cluding the proof of homological mirror symmetry for Fermat Calabi-Yau hypersurfaces
by Sheridan [She15].

A Morse model of Lagrangian Floer theory was constructed in [FOOO09a] based on
their singular chain model. We follow their construction, with a modification by adding
degenerate chains to remove their technical assumption that the unstable submanifolds
of \(f\) give a simplicial triangulation of \(L\) (which does not hold in general, e.g., the standard
Morse function on \(S^2\)).

Moreover, we enlarge the \(A_{\infty}\)-algebra to have a strict unit by adapting the homotopy
unit construction in [FOOO09b, Chapter 7]. In Charest-Woodward [CW], for a rational
symplectic manifold with stabilizing divisors, a unital Morse model was constructed. We
will adapt their construction in our setting.

Let \(f : L \to \mathbb{R}\) be Morse function. Let \(V\) be a negative pseudo-gradient vector field
for \(f\), i.e. \(df(V)|_p \leq 0\), with equality if and only if \(p \in \text{Crit}(f)\); and for each \(p \in \text{Crit}(f)\),
\(V\) coincides with the negative gradient vector field for the Euclidean metric on a Morse
chart of \(p\). Let \(\Phi_t\) be the flow of \(V\). For each \(p \in \text{Crit}(f)\), we denote by \(W^s(f; p)\) and
\(W^u(f; p)\) its stable and unstable manifolds respectively. Namely,

\[
W^s(f; p) = \left\{ q \in L \left| \lim_{t \to +\infty} \Phi_t(q) = p \right. \right\},
\]

and

\[
W^u(f; p) = \left\{ q \in L \left| \lim_{t \to -\infty} \Phi_t(q) = p \right. \right\}.
\]

We define the degree \(|p|\) of \(p\) by

\[
|p| := d - \text{ind}(p),
\]

where \(\text{ind}(p)\) is the Morse index of \(p\). Then

\[
|p| = \dim W^s(f; p) = \text{codim} W^u(f; p).
\]

We assume that \(V\) satisfies the Smale conditions and call such a pair \((f, V)\) Morse-Smale.
For \(p, q \in \text{Crit}(f)\), let \(\mathcal{M}(p, q)\) be the moduli space of flow lines from \(p\) to \(q\). By the
Morse-Smale condition, we have

\[
\dim \mathcal{M}(p, q) = |q| - |p| - 1.
\]

It is well known that (see e.g. [Hut08]) \(W^u(f; p)\) and \(\mathcal{M}(p, q)\) have natural compactifi-
cations into manifolds with corners \(\overline{W}^u(f; p)\) and \(\overline{\mathcal{M}}(p, q)\) whose codimension \(k\) stratum
consists of \(k\)-times broken flow lines

\[
\overline{W}^u(f; p)_k = \bigsqcup_{r_0=p, r_1, \ldots, r_k \text{ distinct}} \overline{\mathcal{M}}(r_0, r_1) \times \cdots \times \overline{\mathcal{M}}(r_{k-1}, r_k) \times \overline{W}^u(f; r_k),
\]

and

\[
\overline{\mathcal{M}}(p, q)_k = \bigsqcup_{r_0=p, r_1, \ldots, r_k, r_{k+1}=q \text{ distinct}} \overline{\mathcal{M}}(r_0, r_1) \times \cdots \times \overline{\mathcal{M}}(r_k, r_{k+1}).
\]
In particular, we have
\[ \partial \overline{W}^u(f; p) = \sum_r \overline{M}(p, r) \times \overline{W}^u(f; r). \] (2.8)

Let \( C^*(f; \Lambda_0) \) be the Morse cochain complex
\[ C^*(f; \Lambda_0) = \bigoplus_{p \in \text{Crit}(f)} \Lambda_0 \cdot p, \]
whose grading is given by \(|p|\).

In [FOOO09a], a (non-unital) \( A_{\infty} \)-algebra structure was constructed on \( C^*(f; \Lambda_0) \) assuming \( L \) has a triangulation whose simplices are the closure \( \text{Im}(\overline{W}^u(f; p) \to L) \) of \( W^u(f; p) \). To establish an isomorphism between Morse and singular cohomology, we have to associate to each critical point \( p \) of degree \(|p|\) a singular chain \( \Delta_p \in S^{\bullet}(L; \mathbb{Z}). \)

A natural candidate for \( \Delta_p \) is a fundamental chain for \( \overline{W}^u(f; p) \). Suppose we choose such a chain \( \Delta_p \). From (2.8), we can see that in general \( \overline{W}^u(f; p) \) has boundary components of the form \( \overline{M}(p, r) \times \overline{W}^u(f; r) \) with \(|r| \geq |p| + 2\). Since \( \dim \overline{M}(p, r) \geq 1 \), and the image of \( \overline{M}(p, r) \times \overline{W}^u(f; r) \) in \( L \) is supported on \( \overline{W}^u(f; r) \), the facet of \( \overline{M}(p, r) \times \overline{W}^u(f; r) \) is a degenerate chain on \( L \). Thus, for the assignment \( p \mapsto \Delta_p \) to be a chain map, the standard approach is to mod out by degenerate chains. On the other hand, \( \mathcal{X}_X(L) \) are constructed including degenerate chains since the \( \mathfrak{m}_k \) product of degenerate chains may no longer be a degenerate chain.

We overcome this difficulty by adding certain degenerate chains to the fundamental chains (of the compactified unstable submanifolds) and define the map \( \iota : C^*(f; \mathbb{Z}) \to S^*\mathfrak{m}_k \), \( \iota(p) = \Delta_p \) accordingly to make it a chain map which induces an isomorphism on cohomology. See Figure 1.

![Figure 1. An example of the added degenerate chain.](image)

We choose cubical singular chains \( d_p \in C_{d-|p|}(\overline{W}^u(f; p); \mathbb{Z}) \) and \( m_{p,q} \in C_{|q|-|p|-1}(\overline{M}(p, q); \mathbb{Z}) \) representing the fundamental classes of \( \overline{W}^u(f; p) \) and \( \overline{M}(p, q) \), respectively, satisfying
\[ \partial d_p = \sum_r (-1)^{|r|-|p|-1} m_{p,r} \times d_r, \] (2.9)
and
\[ \partial m_{p,q} = (-1)^{|r|-|p|-1} m_{p,r} \times m_{r,q}. \] (2.10)
To regard $d_p$ as a singular chain in $S^{|p|}(L; \Lambda_0)$, we choose triangulations for (the domain of) $d_p$ and $m_{p,q}$ inductively on $\text{ind}(p) = d - |p|$. For $\text{ind}(p) = 0$, $d_p$ and $m_{p,q}$ are 0-simplices. Suppose we have chosen triangulations for $d_p$ and $m_{p,q}$ for $\text{ind}(p) < \ell$. For $\text{ind}(p) = \ell$, $m_{p,q}$ with $|q| = |p| + 1$ are 0-simplices. Suppose we have chosen triangulation for $m_{p,q}$ for $|q| - |p| = 1, \ldots , m', m' < m$. For $|q| - |p| = m$, we first triangulate $\partial m_{p,q}$ as follows: each $m_{p,r} \times m_{r,q}$ in $\partial m_{p,q}$ is a product of simplicial complexes. We choose linear orders on vertices of $m_{p,r}$ and $m_{r,q}$ (compatible with their orientations). Then, there exists a unique triangulation $m_{p,r} \times m_{r,q}$ of $m_{p,r} \times m_{r,q}$ such that the vertices of $m_{p,r} \times m_{r,q}$ are pairs $(x, y)$, where $x$ is a vertex of $m_{p,r}$ and $y$ is a vertex of $m_{r,q}$, and an $n$-simplex in $m_{p,r} \times m_{r,q}$ is defined by a set $\{(x_0, y_0), \ldots , (x_n, y_n)\}$ such that $x_0 \leq \ldots \leq x_n$, $y_0 \leq \ldots \leq y_n$.

For each $m_{p,r} \times d_r$ in the boundary of $d_p$, let $Cm_{p,r}$ be the (simplicial) cone over $m_{p,r}$ with vertices

$$\{\text{vertices of } Cm_{p,r}\} = \{\text{vertices of } m_{p,r}\} \cup \{\ast\},$$

and simplices

$$\{n\text{-simplices of } Cm_{p,r}\} = \{n\text{-simplices of } m_{p,r}\} \cup \{\text{cones of } n - 1\text{-simplices of } m_{p,r} \text{ with the vertex } \{\ast\}\},$$

for $n \geq 1$. In particular, this means $\{\ast\}$ is the vertex of the cone. As a singular chain on $L$, the map $Cm_{p,r} \to L$ is defined by composing the contraction $Cm_{p,r} \to m_{p,r}$ with the map $m_{p,r} \to L$. Note that for two simplicial complexes $A$ and $B$, we have $C(A \otimes B) = CA \otimes CB$.

We construct $\Delta_p$ inductively on $\text{ind}(p)$ as follows: For $\text{ind}(p) = 0$, we set $\Delta_p = d_p$. Suppose we have constructed $\Delta_p$ for $\text{ind}(p) < \ell$. For $\text{ind}(p) = \ell$, we define

$$d_p^\bullet = d_p \bigcup_{m_{p,r} \otimes d_r} m_{p,r} \otimes \Delta_r,$$

and set

$$\Delta_p = d_p^\bullet \bigcup_{m_{p,r} \otimes \Delta_r} Cm_{p,r} \otimes \Delta_r. \tag{2.11}$$

By construction, we have

$$\partial \Delta_p = \sum_{|r| = |p| + 1} m_{p,r} \times \Delta_r. \tag{2.12}$$

**Theorem 2.3.** The assignment $p \mapsto \Delta_p$ (with $\Delta_0$ defined as in (2.11)) defines a chain map $\iota : C^\ast(f; \mathbb{Z}) \to S^\ast(L; \mathbb{Z})$ via $\iota(p) = \Delta_p$, which induces an isomorphism on cohomology $H^\ast(C^\ast(f; \mathbb{Z}); \delta) \cong H^\ast(L; \mathbb{Z})$. Here $\delta$ denotes the Morse differential.

**Proof.** The assertion that $\iota$ is a chain map follows from (2.12). It is a standard procedure to prove that $\iota$ induces an isomorphism on cohomology, see for instance [HL99]. Denote by $\mathcal{M}(\Delta, p)$ the moduli space of flow lines from an $n$-simplex $\Delta$ to a critical point $p$, i.e.

$$\mathcal{M}(\Delta, p) = \Delta \times_L W^s(f; p).$$

It is easy to see that $\dim \mathcal{M}(\Delta, p) = n - \text{ind}(p)$. We have a well-known chain map $A : S^\ast(L; \mathbb{Z}) \to C^\ast(f; \mathbb{Z})$ defined by

$$A(\Delta) = \sum_{\text{ind}(p) = n} \# \mathcal{M}(\Delta, p) \cdot p,$$
where $\Delta$ is an $n$-simplex. When $\Delta$ is a degenerate $n$-simplex and $\text{ind}(p) = n$, the moduli space $\mathcal{M}(\Delta, p)$ is empty. Clearly, $\Pi \circ i = id : C^\bullet(f; \mathbb{Z}) \to C^\bullet(f; \mathbb{Z})$. We also have a chain homotopy $G : S^\bullet(L; \mathbb{Z}) \to S^{\bullet-1}(L; \mathbb{Z})$ between $i \circ \Pi$ and $id : S^\bullet(L; \mathbb{Z}) \to S^\bullet(L; \mathbb{Z})$ defined by $G(\Delta) = \bigcup_{t \geq 0} \Phi_t(\Delta)$, triangulated in a compatible way with $\Delta$ and $\overline{W}^u(\Delta_{A(\Delta)})$.

Below we explain the construction of unit in our setting. We always assume $f$ has a unique maximum point $1^*_L$, so that $\Delta_1 = e$ is the fundamental cycle.

Let

$$C^\bullet(f; \Lambda_0^+) = C^\bullet(f; \Lambda_0) \oplus \Lambda_0 \cdot 1^*_L \tag{2.13}$$

with $|1^*_L| = 0$ and $|1^*_L| = -1$. The superscripts $\triangledown, \cdot$ and $\ast$ are borrowed from [CW]. We extend the Morse differential $\delta$ to $C^\bullet(f; \Lambda_0^+)$ by setting

$$\delta(1^*_L) = 0, \quad \delta(1^*_L) = (-1)^d(1^*_L - 1^*_L). \tag{2.14}$$

We now construct a unital $A_{\infty}$-algebra structure on $C^\bullet(f; \Lambda_0^+)$. Let

$$\mathcal{X}_{-1}(L) = \{\Delta_p | p \in \text{Crit}(f)\}.$$

For $g \geq 0$, suppose $\mathcal{X}_{g}^e(L)$ had been constructed for $g < g$. The perturbations $\mathcal{s}_{0,\ell,\beta,\bar{\beta}}$ and $\mathcal{s}^+_{0,\ell,\beta,\bar{\beta}^+}$ in Section 2.1 can be chosen to have the following property:

**Property 2.4.** Each face simplex $\tau$ of a simplex in the triangulation of

$$(ev_0)_* \left( M_{\ell+1}(\beta; L; P_1, \ldots, P_r) \right) \mathcal{s}_{0,\ell,\beta,\bar{\beta}}^+ \tag{2.15}$$

$$\|\mathcal{d}^\bullet, \beta\| = g,$$

or in the triangulation of

$$(ev_0)_* \left[ [0, 1]^{[\mathcal{d}]} \times M_{k+1}(\beta; L; \bar{\beta}^{\ast}) \mathcal{s}^+_{\mathcal{d}^{\ast},\ell,\beta,\bar{\beta}^+} \right],$$

$$\|\mathcal{d}^{\ast}, \beta\| = g$$

is transversal to the stable submanifold $W^s(f; p)$ for all $p \in \text{Crit}(f)$. Moreover, for each $\tau$ of dimension at most $d$, there exists at most one critical point $p(\tau) \in \text{Crit}(f)$ such that $W^s(f; p(\tau))$ is of complementary dimension to $\tau$ and intersects $\tau$ at a unique point.

Such perturbations are possible since there are finitely many simplexes in a triangulation and finitely many critical points. Denote by $\mathcal{X}^e_g(L)$ the set of these singular simplexes $\tau$. Define $\Pi(\tau) = e\Delta_{p(\tau)}$ where $e = \pm 1$ is given by

$$\tau \times L W^s(f; p(\tau)) = eW^u(f; p(\tau)) \cap W^s(f; p(\tau)),$$

if there exists a unique $p(\tau)$ such that $W^s(f; p(\tau))$ intersects $\tau$ at a unique point, and $\Pi(\tau) = 0$, otherwise. In particular, we have $\Pi(\tau) = 0$ whenever $\tau$ is a degenerate simplex. In order for $C^\bullet_{e=}(L; \Lambda_0) \to S^\bullet(L; \Lambda_0)$ to be a quasi-isomorphism, the following singular simplexes $G(\tau)$ are added to $\mathcal{X}^e_g(L)$ to obtain $\mathcal{X}^e_g(L)$. For each $\tau \in \mathcal{X}^e_g(L)$, let $G(\tau) = \bigcup_{t \geq 0} \Phi_t(\tau)$, triangulated in compatible way with $\tau$ and $\Pi(\tau)$. For the chain $G(\tau)$, define $G(G(\tau)) = 0$. We also put $\Pi(\Delta_p) = \Delta_p$ and $G(\Delta_p) = 0$ for $\Delta_p \in \mathcal{X}_{-1}(L)$. The maps $\Pi : C^\bullet(L; \Lambda_0) \to C^\bullet_{(-1)}(L; \Lambda_0)$ and $G : C^\bullet(L; \Lambda_0) \to C^\bullet_{(-1)}(L; \Lambda_0)$ satisfy

$$\Pi(\tau) - \tau = (-1)^d(\partial G(\tau)) + G(\partial \tau), \tag{2.15}$$

for $\tau \in C^\bullet(L; \Lambda_0)$. Let

$$C^\bullet_{(-1)}(L; \Lambda_0)^+ = C^\bullet_{(-1)}(L; \Lambda_0) \oplus \Lambda_0 \cdot e^+ \oplus \Lambda_0 \cdot f.$$
We recall that \( e^+ \) is the unit and \( f \) is the homotopy between \( e^+ \) and the fundamental class \( e \). We extend the coboundary operator \( \partial \) to \( C^\bullet(L; \Lambda_0)^+ \) by setting
\[
\partial e^+ = 0, \quad \partial f = (-1)^d (e^+ - e).
\]
We then extend \( \Pi \) and \( G \) to maps \( \Pi : C^\bullet(L; \Lambda_0)^+ \to C^\bullet(-1)(L; \Lambda_0)^+ \) and \( G : C^\bullet(L; \Lambda_0)^+ \to C^\bullet(L; \Lambda_0)^+ \) respectively by setting
\[
\Pi(e^+) = e, \quad \Pi(f) = 0,
\]
and
\[
G(e^+) = -f, \quad G(f) = 0,
\]
so that they satisfy (2.15) for \( \tau \in C^\bullet(L; \Lambda_0)^+ \).

Homological perturbation can then be applied to reduce the \( A_{\infty} \)-algebra structure on \( C^\bullet(L; \Lambda_0)^+ \) to \( C^\bullet(-1)(L; \Lambda_0)^+ \). We identify the latter with \( C^\bullet(f; \Lambda_0)^+ \) via the map \( \iota : C^\bullet(f; \Lambda_0)^+ \to C^\bullet(L; \Lambda_0)^+ \)
\[
\iota(p) = \Delta_p, \quad \iota(1) = e^+, \quad \iota(1) = f. \quad (2.16)
\]
We will denote the resulting \( A_{\infty} \)-algebra by \( (C^\bullet(f; \Lambda_0)^+, \bar{m}) \).

Similar to Theorem 5.1 in [FOOO09a], we have

**Theorem 2.5.** \( (C^\bullet(f; \Lambda_0)^+, \bar{m}) \) is a unital \( A_{\infty} \)-algebra. \( \tilde{f} \) is a unital homotopy equivalence between \( (C^\bullet(f; \Lambda_0)^+, \bar{m}) \) and \( (C^\bullet(L; \Lambda_0)^+, m^+) \).

Explicitly, the operations \( \bar{m}_k \) are given in terms of maps \( m_T \) and \( f_T \) associated to decorated planar rooted trees.

**Definition 2.6.** A decorated planar rooted tree is a quintet \( \Gamma = (T, \iota, v_0, V_{\text{tad}}, \eta) \) consisting of
- \( T \) is a tree;
- \( \iota : T \to D^2 \) is an embedding into the unit disc;
- \( v_0 \) is the root vertex and \( \iota(v_0) \in \partial D^2 \);
- \( V_{\text{tad}} \) is the set of interior vertices with valency \( 1 \);
- \( \eta : V(\Gamma)_{\text{int}} \to \mathbb{Z}_{\geq 0} \).

where \( V(\Gamma) \) is the set of vertices; \( V(\Gamma)_{\text{ext}} = \iota^{-1}(\partial D^2) \) is the set of exterior vertices and \( V(\Gamma)_{\text{int}} = V(\Gamma) \setminus V(\Gamma)_{\text{ext}} \) is the set of interior vertices. For \( k \geq 0 \), denote by \( \Gamma_{k+1} \) the set of isotopy classes represented by \( \Gamma = (T, \iota, v_0, \eta) \) with \( |V(\Gamma)_{\text{ext}}| = k + 1 \) and \( \eta(\iota(v)) > 0 \) if the valency \( \ell(v) \) of \( v \) is 1 or 2. In other words, the elements of \( \Gamma_{k+1} \) are stable.

Note that there is a unique decorated planar rooted ribbon tree \( \Gamma_0 \) not having interior vertices, which is contained in \( \Gamma_2 \). For this tree \( \Gamma_0 \), we define
\[
m_{\Gamma_0} := (-1)^d \iota : C^\bullet(-1)(L; \Lambda_0)^+ \to C^\bullet(-1)(L; \Lambda_0)^+,
\]
and we also define
\[
f_{\Gamma_0} : C^\bullet(-1)(L; \Lambda_0)^+ \to C^\bullet(L; \Lambda_0)^+
\]
by the inclusion.

For each \( k \in \mathbb{N} \), \( \Gamma_{k+1} \) contains a unique element that has a single interior vertex \( v \), which is denoted by \( \Gamma_{k+1} \). We fix a labeling \( \{\beta_0, \beta_1, \ldots\} \) of elements of \( H^2_{\text{eff}}(X, L) \), with \( \beta_0 \) the constant disc class and define
\[
m_{\Gamma_{k+1}} = \Pi \circ m_{k, \beta_0}(\iota^+),
\]
and
\[
f_{\Gamma_{k+1}} = G \circ m_{k, \beta_0}(\iota^+),
\]
For general $\Gamma$, cut it at the vertex $v_1$ closest to $v_0$ so that $\Gamma$ is decomposed into $\Gamma^{(1)}, \ldots, \Gamma^{(\ell)}$ and an interval adjacent to $v_0$ in the counter-clockwise order. We define

$$m_\Gamma = \Pi \circ m_{\ell, \beta_{\eta(v_1)}} \circ (f_{\Gamma(1)} \otimes \cdots \otimes f_{\Gamma(\ell)})$$

and

$$f_\Gamma = G \circ m_{\ell, \beta_{\eta(v_1)}} \circ (f_{\Gamma(1)} \otimes \cdots \otimes f_{\Gamma(\ell)}).$$

Let

$$\tilde{m}_{k, \beta} = \sum_{\Gamma \in \Gamma_{k+1}} T^{\omega_X(\beta)} m_{\Gamma}.$$

Finally, we define

$$\tilde{m}_k = \sum_{\beta} \tilde{m}_{k, \beta}.$$

**Remark 2.7.** Since the images of degenerate singular chains to $C^*(f; \Lambda_0)^+$ under the projection are zero, it is in fact not necessary to collapse the fibers of the forgetful maps as in (2.6) for $T_L$ to be a strict unit.

The maps $m_\Gamma$ restricted to $C^*(f; \Lambda_0)$ are given by counting pearl trees as depicted in Figure 2. For a decorated tree $\Gamma \in \Gamma_{k+1}$, the exterior vertices $v_0, \ldots, v_k$ are labelled respecting the counter-clockwise orientation. Each edge is oriented in the direction from the $k$ input vertices $v_1, \ldots, v_k$ to the root vertex $v_0$. We denote by $e_i$ the edge attached to $v_i$ and by $v^\pm(e)$ the vertices such that $e$ is the edge from $v^- (e)$ to $v^+ (e)$. For $p_1, \ldots, p_k, q \in \text{Crit}(f)$, consider the moduli space $\mathcal{M}_\Gamma(f; p_1, \ldots, p_k, q)$ consisting of the following configurations:

- for each interior vertex $v$, a bordered stable map $u_v$ representing the class $\beta_{\eta(v)}$ with $\ell(v)$ boundary marked points. We denote by $p(e, v)$ the marked point corresponding to the edge $e$ attached to $v$;
- for $i = 1, \ldots, k$, the input edge $e_i$ corresponds to a flow line $\gamma_i$ from $p_i$ to $u_{v^+(e_i)}(p(e_i, v^+(e_i)))$;
- the output edge $e_0$ corresponds to a flow line $\gamma_0$ from $u_{v^-(e_0)}(p(e_0, v^-(e_0)))$ to $q$;
- an interior edge $e$ corresponds to a flow line $\gamma_e$ from $u_{v^-(e)}(p(e, v^-(e)))$ to $u_{v^+(e)}(p(e, v^+(e)))$.

**Figure 2. Pearl trees**
The virtual dimension of $\mathcal{M}_T(f; p_1, \ldots, p_k, q)$ is given by
\[
\dim \mathcal{M}_T(f; p_1, \ldots, p_k, q) = k - 2 + \mu(\Gamma) - \sum_{i=1}^{k} |p_i| + |q|,
\]
where $\mu(\Gamma) = \sum_{v \in V_{\text{ext}}} \mu(\beta_{\eta(v)})$. The maps $m_T : C^*(f; \Lambda_0)^{\otimes k} \to C^*(f; \Lambda_0)$ are given by
\[
m_T(p_1, \ldots, p_k) = \sum_{q \in \text{Crit}(f)} \# \mathcal{M}_T(f; p_1, \ldots, p_k, q) \cdot q,
\]
where $\# \mathcal{M}_T(f; p_1, \ldots, p_k, q)$ is the signed count of the moduli space of virtual dimension 0.

Let us now consider the weak Maurer-Cartan equation for $(C^*(f; \Lambda_0)^+, \tilde{m})$. For a unital $A_\infty$-algebra $(A, m)$ over $\Lambda_0$ with the strict unit $e_A$. The weak Maurer-Cartan equation for $b \in A$ with $b \equiv 0 \mod \Lambda_+ \cdot A$ is given by
\[
m^b_0(1) = m_0(1) + m_1(b) + m_2(b, b) + \ldots \in \Lambda_0 \cdot e_A.
\]
Note that the requirement that $b$ has positive $T$-valuation ensures the convergence of $m_0^b(1)$. We denote by
\[
\mathcal{MC}(A) = \{ b \in A^{\text{odd}} | b \text{ has positive } T\text{-valuation}; m_0(1) \in \Lambda_0 \cdot e_A \},
\]
the space of odd solution to the weak Maurer-Cartan solution.

We say an $A_\infty$-algebra $A$ is \textit{weakly unobstructed} if $\mathcal{MC}(A)$ is nonempty, in which case we have $(m^b_0)^2 = 0$ for any $b \in \mathcal{MC}(A)$, thus defining a cohomology theory $H^*(A; m^b_0)$.

For any $b \in \mathcal{MC}(A)$, we put
\[
e^b := 1 + b + b \otimes b + \ldots.
\]
We define the deformation the $A_\infty$ structure $m$ by $b$
\[
m^b_k(x_1, \ldots, x_k) = m(e^b, x_1, e^b, x_2, e^b, \ldots, e^b, x_k, e^b).
\]
Note that $m^b_0(1) = m(e^b)$.

The following lemma concerns the weakly unobstructedness of $(C^*(f; \Lambda_0)^+, \tilde{m})$. This technique of finding weak bounding cochains in the case of homotopy unit was introduced in [FOOO09b, Chapter 7] and [CW, Lemma 2.44].

**Lemma 2.8.** Let $b \in C^1(f; \Lambda_+)$. Suppose $\tilde{m}_0^b = W(b)1^*_L$ and the minimal Maslov index of $L$ is nonnegative. Then there exists $b^+ \in C^1(f; \Lambda_+)^+$ such that $\tilde{m}_0^{b^+}(1) = W^+(b)1^*_L$, i.e., $(C^*(f; \Lambda_0)^+, \tilde{m})$ is weakly unobstructed. In particular, if the minimal Maslov index of $L$ is at least two, then, $W^+(b) = W(b)$.

**Proof.** By (2.7), we have
\[
\tilde{m}_1(1^*_L) = 1^*_L - I^*_L + \Pi(h),
\]
where
\[
\Pi(h) = \Pi \left( \sum_{\beta \neq \beta_0, \mu(\beta) = 0} T^{\omega(\beta)}(ev_0)_* \left( ([0, 1] \times \mathcal{M}_2(\beta; L; e))^{\#} / \sim \right) \right)
\]
is a multiple of $1^*_L$ with $T$-positive valuation. The vanishing of the contribution of higher Maslov index discs follows from the fact that the output singular chains of are of degrees at most $-2$, whose projection to $C^*(f; \Lambda_0)^+$ vanishes. Hence we can write (2.2) as
\[
\tilde{m}_1(1^*_L) = 1^*_L - (1 - h)1^*_L,
\]
where \( h \in \Lambda_+ \).

Let’s tentatively take \( b^\prime = b + 1_L^r \). We have

\[
\tilde{m}_0^b(1) = \tilde{m}_0(1) + \tilde{m}_1(b + 1_L^r) + \tilde{m}_2(b + 1_L^r, b + 1_L^r) + \ldots = W(b)1^r_L + \sum_{\mu(\beta) = 0} \tilde{m}_{1,\beta}^b(1^r_L).
\]

The second equality is due to the vanishing of the \( \tilde{m}_k \) operations with more than one \( 1_L^r \) as an input since the outputs are of degrees at most \(-2\). Also, notice that the terms \( \tilde{m}_{1,\beta}^b(1^r_L) \) are of degree 0 and therefore multiples of \( 1^r_L \).

Using the notation of (2.19), we can write

\[
\sum_{\mu(\beta) = 0} \tilde{m}_{1,\beta}^b(1^r_L) = 1^r_L - (1 - h(b))1^r_L = \sum_{\mu(\beta) = 0, \ell_1 + \ell_2 \geq 1} \tilde{m}_{1 + \ell_1 + \ell_2, \beta}^b\left( b, \ldots, b, 1^r_L, b, \ldots, b \right)_{\ell_1, \ell_2}.
\]

Since there exist no gradient flow line from \( b \) to the maximum \( 1^r \), the contribution of the terms

\[
\sum_{\ell_1 + \ell_2 \geq 1} \tilde{m}_{1 + \ell_1 + \ell_2, \beta}^b\left( b, \ldots, b, 1^r_L, b, \ldots, b \right)_{\ell_1, \ell_2}
\]

are zero, and therefore \( h(b) \in \Lambda_+ \).

Now, set \( b^+ = b + \frac{W}{1 - h(b)}1_L^r \). We have

\[
\tilde{m}_0^{b^+}(1) = W(b)1^r_L + \frac{W}{1 - h(b)}(1^r_L - (1 - h(b))1^r_L) = \frac{W(b)}{1 - h(b)}1^r_L =: W^\psi(b)1^r_L. \quad (2.20)
\]

This means \( b^+ \in MC(C^*(f; \Lambda_0)^+) \).

If the minimal Maslov index of \( L \) is at least two, then, \( h(b) = 0 \) and therefore \( W^\psi(b) = W(b) \).

**Definition 2.9.** Let \( x_1, \ldots, x_d \) be a basis of \( C^1(f; \mathbb{Q}) \), and write \( b = \sum_{i=1}^d x^i x_i, x^i \in \Lambda_+ \). If \( \tilde{m}_0 = W(x^1, \ldots, x^d)1^r_L \), then we will call \( W^\psi(x^1, \ldots, x^d) \) the disc potential of \( (C^*(f; \Lambda_0)^+, \tilde{m}) \).

### 2.3 The equivariant Morse model

Let \( G \) be a compact Lie group. We consider the finite dimensional approximations \( EG(N) \) and \( BG(N) \) of \( EG \) and \( BG \) as smooth manifolds and the maps

\[
EG(0) = G \hookrightarrow EG(1) \hookrightarrow EG(2) \hookrightarrow \ldots \quad (2.21)
\]

and

\[
BG(0) = pt \hookrightarrow BG(1) \hookrightarrow BG(2) \hookrightarrow \ldots \quad (2.22)
\]

as smooth embeddings. We also consider \( T^*EG(N) \) as a symplectic manifold equipped with the canonical symplectic form. Let \( g \) be the Lie algebra of \( G \) and let \( \mu_N : T^*EG(N) \to g^* \) be a moment map for the Hamiltonian \( G \)-action on \( T^*EG(N) \) lifted from the \( G \)-action on \( EG(N) \). Since \( G \) acts on \( T^*EG(N) \) freely, we have

\[
T^*EG(N) \cong \mu_N^{-1}(0)/G \cong T^*BG(N),
\]

canonically, as symplectic manifolds. We denote by \( I_{T^*BG(N)} \) the almost complex structure induced by a \( G \)-invariant compatible almost complex structure on \( T^*EG(N) \).

Let \( (X, \omega_X) \) be a symplectic manifold with a symplectic \( G \)-action and let \( I_X \) be a \( G \)-invariant almost complex structure compatible with \( \omega_X \). Let \( L \subset X \) be a \( G \)-invariant, relatively spin, connected, and closed Lagrangian submanifold.

Let \( X_N = X \times_G \mu_N^{-1}(0) \) and note that \( \pi : X_N \to T^*BG(N) \) is a fiber bundle with fibers \( X \). We also denote \( X_G = \lim_{\to} X_N \). Let \( i : X \to X_N \) be the inclusion of a fiber.
By construction, $X_N$ is endowed with a symplectic form $\omega_{X_N}$ and a compatible almost complex structure $J_{X_N}$ satisfying

$$\omega_{X_N}|_{L_{N-1}} = \omega_{X_{N-1}}, \quad i^*\omega_{X_N} = \omega_X,$$

and

$$J_{X_N}|_{L_{N-1}} = J_{X_{N-1}}, \quad D\iota \circ J_X = J_X \circ D\iota, \quad D\pi \circ J_X = J_{T^*BG(N)} \circ D\pi. \quad (2.24)$$

Let $L_N = L \times_G E\Gamma(N)$ be the finite dimensional approximations of the Borel construction $L_G = L \times_G E\Gamma$ and note that $L_N \to BG(N)$ is a fiber bundle with fibers $L$. (2.21) and (2.22) induce sequences of inclusions

$$X_0 = X \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots$$

and

$$L_0 = L \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \ldots$$

such that $L_N$ is a Lagrangian submanifold of $(X_N, \omega_{X_N})$. It is easy to see that $L_N$ are relatively spin. Since $H^2(T^*BG(N), BG(N)) = 0$, we have $H^2(X_N, L_N; \mathbb{Z}/2) = H^2(X, L; \mathbb{Z}/2)$. Then the choice of relative spin structure on $L$ gives us a canonical choice of relative spin structure on $L_N$.

**Proposition 2.10 (effective cone).** The induced map $\iota_* : H_2(X, L; \mathbb{Z}) \to H_2(X_N, L_N; \mathbb{Z})$ restricts to a bijection

$$\iota_* : H_2^{\text{eff}}(X, L) \sim H_2^{\text{eff}}(X_N, L_N). \quad (2.27)$$

**Proof.** Suppose $u : (D^2, \partial D^2) \to (X_N, L_N)$ be a $J_{X_N}$-holomorphic disc. Then, $\pi \circ u$ is a $J_{T^*BG(N)}$-holomorphic disc with Lagrangian boundary condition $BG(N)$. Since $BG(N) \subset T^*BG(N)$ is an exact Lagrangian, it does not bound non-trivial pseudo-holomorphic discs. This means $\pi \circ u$ is necessarily constant and $\text{Im}(u)$ is contained in a fiber of $\pi$ over $BG(N)$. \hfill $\square$

**Corollary 2.11 (Maslov index).** The Maslov index of $\beta \in H_2^{\text{eff}}(X, L)$ is equal to the Maslov index of $\iota_*\beta \in H_2^{\text{eff}}(X_N, L_N)$.

For simplicity, we will denote the disc classes $\beta \in H_2^{\text{eff}}(X, L)$ and $\iota_*\beta \in H_2^{\text{eff}}(X_N, L_N)$ both by $\beta$ and their Maslov index by $\mu(\beta)$.

**Corollary 2.12 (regularity).** A $J_{X_N}$-holomorphic disc $u : (D^2, \partial D^2) \to (X_N, L_N)$ is regular if it is regular as a disc in the corresponding fiber of $\pi$.

**Proof.** Let $E := u^*TX_N$ and $F := (\partial u)^*TL_N$. Denote by $A^0(E, F)$ the space of smooth global sections of $E$ with boundary values in $F$ and by $A^1(E)$ the space of smooth global $(0, 1)$-forms with coefficient in $E$. Consider the two term elliptic complex

$$A^0(E, F) \xrightarrow{\partial} A^1(E). \quad (2.28)$$

Here $\partial$ is the Cauchy-Riemann map linearized at $u$. Pulling-back the following exact sequences

$$0 \to TX \to TX_N \to TT^*BG(N) \to 0,$$

$$0 \to TL \to TL_N \to TBG(N) \to 0.$$

via $u$, we can choose splittings

$$E = E_1 \oplus E_2 := u^*TX \oplus (\pi \circ u)^*TT^*BG(N),$$
By Proposition 2.10, \(\pi \circ u\) is a constant map and hence \(\delta_2\) is surjective. Therefore, \(\delta\) is surjective if and only if \(\tilde{\delta}_1\) is surjective.

Let \(\{(f_N, V_N)\}_{N \geq 0}\) be a sequence of pairs with each \((f_N, V_N)\) a Morse-Smale pair on \(L_N\). Let \((C^*(f_N; \Lambda_0)^+, \hat{m}^N)\) denote be the Morse model for \(L_N\) constructed following Section 2.2. We denote by \(m^N\), \(m^{N,+}\) the \(A_{\infty}\) structure on \(C^*(L_N; \Lambda_0)\) and \(C^*(L_N; \Lambda_0)^+\), respectively, and by \(\hat{m}^N\) the maps associated to the decorated planar rooted trees \(\Gamma\) in the definition of \(m^N\).

The Morse models \((C^*(f_N; \Lambda_0)^+, \hat{m}^N)\) are constructed using the machinery of [FOOO09b]. Let’s recall that for an element \(u \in \mathcal{M}_{k+1}(\beta; L_N; J_X)\), a Kuranishi chart around \(u\) consists of \((V, E, \Gamma, \psi, s)\) where \(\Gamma\) is the finite automorphism group of \(u\) acting on the bundle \(E \to V\). We take \(V\) as a manifold with corners, \(s\) is a \(\Gamma\)-equivariant smooth section of \(E \to V\), and \(\psi\) is a homeomorphism from \(s^{-1}(0) / \Gamma\) to a neighborhood of \(u\) in \(\mathcal{M}_{k+1}(\beta; L_N; J_X)\). By Proposition 2.10, \(u\) is contained in a fiber of \(X_N \to T^*BG(N)\) over \(BG(N)\). For every \(N \geq 0\) and \(p \in BG(N)\), we choose a contractible neighborhood \(U\) of \(p\) in \(BG(N)\) and a trivialization of \(\pi^{-1}(U) \cong X \times U\) (and similarly \(L_N|_U \cong L \times U\)) such that it respects the inclusion \(BG(N) \subset BG(N + 1)\). Then \(\mathcal{M}_{k+1}(\beta; L_N; J_X)\) can also be trivialized as \(\mathcal{M}_{k+1}(\beta; L; J_X) \times U\). Therefore, we can take the Kuranishi charts of \(\mathcal{M}_{k+1}(\beta; L_N; J_X)\) as the product of the Kuranishi charts of \(\mathcal{M}_{k+1}(\beta; L; J_X)\) and \(U\). This ensures that the (stable smooth) discs parametrized by \(V\) are contained in the fibers over \(BG(N)\).

We now describe a Morse model for equivariant Lagrangian Floer theory.

**Condition 2.13.** We find Morse-Smale pairs \(\{(f_N, V_N)\}\) satisfying the following conditions.

1. For \(N \geq 1\), we have \(\left. (f_N, V_N) \right|_{L_{N-1}} = (f_{N-1}, V_{N-1})\). This implies \(\text{Crit}(f_{N-1}) \subset \text{Crit}(f_N)\), and
   \[
   W^u(f_N; p) \times_{L_N} L_{N-1} = W^u(f_{N-1}; p),
   \]
   for \(p \in \text{Crit}(f_{N-1})\).
2. For \(N \geq 1\), and \(q \in L_N \setminus L_{N-1}\), we have \(\Phi_t(q) \notin L_{N-1}\) for all \(t \geq 0\). This implies
   \[
   W^u(f_N; p) \times_{L_N} L_{N-1} = \emptyset
   \]
   for \(p \in \text{Crit}(f_N) \setminus \text{Crit}(f_{N-1})\), and \(W^s(f_N; p)\) coincides with the image of \(W^s(f_{N-1}; p)\) in \(L_N\) for \(p \in \text{Crit}(f_{N-1})\).
3. For every \(\ell \geq 0\), there exists an integer \(N(\ell) > 0\) such that \(|p| > \ell\) for all \(N \geq N(\ell)\) and \(p \in \text{Crit}(f_N) \setminus \text{Crit}(f_{N-1})\).
4. The Morse function \(f_N\) has a unique maximum \(1^*_{L_n}\) and the inclusion of critical points \(\text{Crit}(f_N) \hookrightarrow \text{Crit}(f_{N+1})\) sends \(1^*_{L_{N+1}}\) to \(1^*_{L_n}\). Thus, \(C^*(f_N; \Lambda_0)^+ \subset C^*(f_{N+1}; \Lambda_0)^+\). We will abuse notation and denote \(1^*_{L_n}\), \(1^v_{L_n}\), and \(1^\circ_{L_n}\) simply by \(1^*\), \(1^v\), and \(1^\circ\).

We remark that typical Morse functions on classifying spaces \(BG\) satisfy the above properties when restricted to \(BG(N)\). We can use a partition of unity on \(BG\) together with Morse functions on \(L\) to construct Morse functions on the total space of the fibration \(L_G \to BG\) so that their restrictions to \(L_N \to BG(N)\) satisfy the above properties.
Proposition 2.14. Let $\mathcal{C}_N^{-1} : C^\bullet(f_N; \Delta_0) \to C_N^\bullet(f_N; \Delta_0)$ be the inclusion map. Under Condition 2.13, the countable sets $\mathcal{X}_S(L_N)$ of smooth singular simplices on $L_N$, perturbations $s_{\beta, \ell, \beta, \bar{\beta}} \parallel (\delta, \beta) = g$, and $s_{\beta, \ell, \beta, \bar{\beta}}^+ \parallel (\delta^+, \beta) = g$, in Section 2.1 and 2.2 can be chosen such that the resulting $A$-algebras $(C^\bullet(f_N; \Delta_0)^+, \tilde{m}^N)$ satisfy the following property:

$$i_N^{-1}\left(\mathfrak{m}_N^{-1}(p_1, \ldots, p_k)|_{L_{N-1}}\right) = m_N^+(p_1, \ldots, p_k)|_{L_{N-1}} \quad (2.29)$$

for $p_1, \ldots, p_k \in C^\bullet(f_N; \Delta_0)^+$ and $\Gamma \in \Gamma_{k+1}$. The RHS means that we only consider the outputs of $m_N^+$ in $C^\bullet(f_N; \Delta_0)^+$

Proof. We proceed by induction on $N$. For $N = 0$, the statement of the proposition is void. Suppose we have constructed $A$-algebras $(C^\bullet(f_N; \Delta_0)^+, \tilde{m}^N)$, $N < N$, satisfying (2.29). Let $\mathcal{X}_S(L_N) \subset \mathcal{X}_S'(L_N)$ be the subset

$$\mathcal{X}_S'(L_N) = \{P \in \mathcal{X}_S(L_N)|P \times_L L_{N-1} \neq \emptyset\}.$$

Denote by $\Delta_{p,N}$ the singular simplex chosen to represent $p \in \text{Crit}(f_N)$ as in (2.11). For $g = -1$, we have

$$\mathcal{X}_{-1}^{-1}(L_N) = \{\Delta_{p,N}|p \in \text{Crit}(f_N-1)\},$$

by Condition 2.13. We define $r_{-1} : \mathcal{X}_{-1}^{-1}(L_N) \rightarrow \mathcal{X}_{-1}(L_N)$ by $r_{-1}(\Delta_{p,N}) = \Delta_{p,N-1}$.

Let $\bar{a} = \{a_1, \ldots, a_{|\bar{a}|}\} \subset \{1, \ldots, k\}$, $a_1 < \ldots < a_{|\bar{a}|}$. For $p_{1, \ldots, p_{k-|\bar{a}|}} \in \text{Crit}(f_N)$, let $\bar{\Delta}_N = (\Delta_{p_1,N}, \ldots, \Delta_{p_{k-|\bar{a}|},N})$, and let $\bar{\Delta}_N^+$ be the $k$-tuple obtained by inserting $e = a_1$ into the $a_1, \ldots, a_{|\bar{a}|}$-th places of $\bar{\Delta}_N$. Since the discs are contained in the fibers of $X_N \rightarrow T^*BG(N)$ over $BG(N)$, by Condition 2.13, we have

$$\mathcal{M}_{k+1-|\bar{a}|}(\beta; L_N; \bar{\Delta}_N) \times_{L_N} L_{N-1} = \mathcal{M}_{k+1}(\beta; L_{N-1}; \bar{\Delta}_N^-),$$

and

$$\left([0,1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L_N; \bar{\Delta}_N^+)ight) \times_{L_N} L_{N-1} = [0,1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L_{N-1}; \bar{\Delta}_N^-),$$

as compact subsets of $\mathcal{M}_{k+1-|\bar{a}|}(\beta; L_N; \bar{\Delta}_N^-)$ and $[0,1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L_N; \bar{\Delta}_N^-)$, respectively. We can therefore endow the LHS with the Kuranishi structures of the RHS.

For $g \geq 0$, suppose $\mathcal{X}_S'(L)$ had been constructed for $g' < g$. We construct the set of singular simplices $\mathcal{X}_S'(L)$, and choose perturbations $s_{\beta, \ell, \beta, \bar{\beta}} \parallel (\delta, \beta) = g$, and $s_{\beta, \ell, \beta, \bar{\beta}}^+ \parallel (\delta^+, \beta) = g$, with following additional properties:

1. There exists a bijection $r_g : \mathcal{X}_S'(L) \rightarrow \mathcal{X}_S'(L_{N-1})$.
2. Let $\tilde{P}_N = (P_{1,N}, \ldots, P_{\ell,N})$, $P_{1,N} \in \mathcal{X}_{0,(i)}(L_N)$, and $\delta(i) \leq g$, we have

$$\mathcal{M}_{\ell+1}(\beta; L_N; \tilde{P}_N) \times_{L_N} L_{N-1} = \mathcal{M}_{\ell+1}(\beta; L_{N-1}; P_{1,N-1}, \ldots, P_{\ell,N-1}), \quad (2.30)$$

where $P_{\ell,N-1} = r_{\delta(i)}(P_{\ell,N}) \in \mathcal{X}_{\delta(i)}(L_{N-1})$.
3. Let $\tilde{P}_N^+ = (P_{1,N}^+, \ldots, P_{\ell,N}^+(e\beta))$, $P_{1,N}^+ \in \mathcal{X}_{0,(i)}^+(L_N)$ with $\delta(i) \leq g$, we have

$$\left([0,1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L_N; \tilde{P}_N^+)\right) \times_{L_N} L_{N-1} = [0,1]|\bar{a}| \times \mathcal{M}_{k+1}(\beta; L_{N-1}; P_{1,N-1}^+, \ldots, P_{k,N-1}^+), \quad (2.31)$$

where $P_{\ell,N-1}^+ = r_{\delta(i)}(P_{\ell,N}^+) \in \mathcal{X}_{\delta(i)}(L_{N-1})$. In particular, if $P_{1,N}^+ = \Delta_{1,N}^+$, then $P_{\ell,N-1}^+ = \Delta_{1,N-1}^+$. 

```
We choose perturbation $s$. It is easy to see that these properties together would imply that the resulting perturbation for the boundary strata $\tau$ of $\hat{\lambda}$, which are of complementary dimension to the stable submanifold $W^s(f_{N-1}; p)$ in $L_{N-1}$ and intersects $W^s(f_{N-1}; p)$ at a unique point, and simplices in the triangulation of $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \tilde{P}_{N-1})^{s_{\beta,\ell,p_{N-1}}}$ which are of complementary dimension to the stable submanifold $W^s(f_N; p)$ in $L_N$ and intersects $W^s(f_N; p)$ at a unique point. Moreover, the intersection points for the corresponding simplices have the same orientation.

Let $C\left(\mathcal{U} \setminus \partial \mathcal{V}, \mathcal{V}ight)$ be the direct limit

$$C_G^\ast(L; \Lambda_0) = \lim_{\rightarrow} C\left(f_N; \Lambda_0\right)^+.$$
For $k \geq 0$ and $\Gamma \in \Gamma_{k+1}$, we define the map $m^G_{\Gamma} : C^*_G(L; \Lambda_0)^{\otimes k} \to C^*_G(L; \Lambda_0)$ as follows. Let $p_1, \ldots, p_k \in C^*_G(L; \Lambda_0)$ and suppose $p_1, \ldots, p_k \in C^*(f_{N-1}; \Lambda_0)^+$ for some $N_0$. Set $\ell = \sum_{i=1}^k |p_i| + 2 - k - \mu(\Gamma)$, and let $N(\ell) \geq N_0$ be as in Condition 2.13 (3). We define $m^G_{\Gamma}$ by

$$m^G_{\Gamma}(p_1, \ldots, p_k) = m^N_{\Gamma}(p_1, \ldots, p_k). \quad (2.32)$$

By Proposition 2.14, (2.32) is defined independent of the choice of $N(\ell)$. We then define the maps $m^G_k : C^*_G(L; \Lambda_0)^{\otimes k} \to C^*_G(L; \Lambda_0)$ by

$$m^G_k = \sum_{\beta} \sum_{\Gamma \in \Gamma_{k+1}, \beta = \sum \beta_{(i)}} T^\omega_{\Gamma}(\beta) m^G_{\Gamma}. \quad (2.33)$$

It is easy to see that $(C^*_G(L; \Lambda_0), m^G)$ is an $A_{\infty}$-algebra with a strict unit $1^G$. For fixed inputs, the $A_{\infty}$-relations can be checked on $(C^*(f^N_\Lambda; \Lambda_0), \hat{m}^N)$ for sufficiently large $N$. $(C^*_G(L; \Lambda_0), m^G)$ is independent of various choices made (up to homotopy equivalence) since $(C^*(f^N_\Lambda; \Lambda_0), \hat{m}^N)$ are independent of choices (see Theorem 4.6.1 in [FOOO09b]).

$(C^*_G(L; \Lambda_0), m^G)$ will be our Morse model for the $G$-equivariant Lagrangian Floer theory of $L$.

2.4 Homotopy partial units and the equivariant disc potentials. The main goal of this paper is to recover the equivariant toric super potentials

$$W_\Lambda = W + \sum_{i=1}^d x^i \lambda_i$$

as the disc potential of a $T = (S^1)^d$-equivariant Lagrangian Floer theory $(C^*_T(L; \Lambda_0), m^T)$ of the regular moment map fibers $L$ of $2d$-dimensional compact semi-Fano toric manifolds. The terms $\lambda_1, \ldots, \lambda_d$ are the equivariant parameters generating the cohomology ring

$$H^*_T(\text{pt}) = H^*(BT; \mathbb{C}) = \mathbb{C}[\lambda_1, \ldots, \lambda_d].$$

Clearly, arbitrary choices of Morse functions $f_N$ would not suffice for our purpose since the equivariant parameters do not manifest as elements of $C^*_T(L; \Lambda_0)$. Moreover, since $\lambda_1, \ldots, \lambda_d$ have cohomological degree 2, the expression of $W_\Lambda$ suggests that the deformation $m^T_k$ is, a priori, obstructed. For these reasons, we define in this section a homotopy equivalent alternative to our equivariant Morse model and its equivariant disc potential which is more suitable for applications.

To begin with, we fix a Morse function $f$ on $L$ with a unique maximum point and set $f_0 := f$. We choose a sequence of Morse functions $f_N$ on $L_N$ for $N \geq 1$ of the form

$$f_N = \pi^*_N \varphi_N + \phi_N$$

where

- $\varphi_N$ is taken to be a perfect Morse functions on $BG(N)$. This is typical when $G$ is a torus or $U(n)$.
- $\phi_N$ is a (generically) fiberwise Morse function over $BG(N)$ such that $\phi_N$ restricted to the fiber $L = \pi^{-1}_N(\{\lambda\})$ over each critical point $\lambda \in \text{Crit}(\varphi_N)$ agrees with $f$. 

With such choices of Morse functions, we have

$$C^\bullet(f_N; \Lambda_0) = C^\bullet(f; \Lambda_0) \otimes H^\bullet(BG(N); \Lambda_0).$$

and

$$C^\bullet_C(L; \Lambda_0) = C^\bullet(f; \Lambda_0) \otimes H^\bullet_C(pt; \Lambda_0)$$

It is well known that $H^\bullet_C(pt) = H^\bullet_C(pt)^W$ where $T \subseteq G$ is a maximal torus and $W$ is the Weyl group. This means $H^\bullet_C(pt)$ is a polynomial ring with generators in even degrees.

Let $\mathbf{1}^*_f$ be the maximum point of $f$. We enlarge $C^\bullet(f_N; \Lambda_0)$ fiberwise over $\text{Crit}(\varphi_N)$ to $C^\bullet(f_N; \Lambda_0)^\dagger$

$$C^\bullet(f_N; \Lambda_0)^\dagger = C^\bullet(f; \Lambda_0)^\dagger \otimes H^\bullet(BG(N); \Lambda_0).$$

For $\lambda \in \text{Crit}(\varphi_N)$, we put

$$\lambda^* = \mathbf{1}^* \otimes \lambda, \quad \lambda^\dagger = \mathbf{1}^\dagger \otimes \lambda,$$

We also denote the maximum point of $\text{Crit}(\varphi_N)$ by $\mathbf{1}_{BG(N)}$ and put

$$\mathbf{1}^* = \mathbf{1}^*_f \otimes \mathbf{1}_{BG(N)}, \quad \mathbf{1}^\dagger = \mathbf{1}^\dagger_f \otimes \mathbf{1}_{BG(N)}.$$

Then

$$C^\bullet(f_N; \Lambda_0)^\dagger = C^\bullet(f_N; \Lambda_0)^\dagger \oplus \left( \bigoplus_{\lambda \neq \mathbf{1}_{BG(N)}} \Lambda_0 \cdot \lambda^\dagger \right) \oplus \left( \bigoplus_{\lambda \neq \mathbf{1}_{BG(N)}} \Lambda_0 \cdot \lambda^* \right).$$

We enlarge $(C^\bullet(f_N; \Lambda_0)^\dagger, \tilde{m}^N)$ to an homotopy equivalent $A_\infty$-algebra $(C^\bullet(f_N; \Lambda_0)^\dagger, \tilde{m}^N, \tilde{m}^N)$ as follows.

Let

$$C^\bullet(L_N; \Lambda_0)^\dagger = C^\bullet(L_N; \Lambda_0)^\dagger \oplus \left( \bigoplus_{\lambda \neq \mathbf{1}_{BG(N)}} \Lambda_0 \cdot \Delta^\lambda \right) \oplus \left( \bigoplus_{\lambda \neq \mathbf{1}_{BG(N)}} \Lambda_0 \cdot \Delta^\lambda^* \right).$$

Using the idea of the homotopy unit construction, we can construct an $A_\infty$-structure $m^N, \tilde{m}^N$ on $C^\bullet(L_N; \Lambda_0)^\dagger$ with the following properties.

The restricton of $m^{N, \dagger}_k$ to $C^\bullet(L_N; \Lambda_0)^\dagger$ agrees with $m^{N, \dagger}_k$. In particular, $\Delta^\dagger = e^\dagger$ is the strict unit, and $\Delta^\dagger = f$ is the homotopy between $\Delta^\dagger$ and $\Delta^\dagger$.

The $m^{N, \dagger}_k$ operations with $\Delta^\dagger$ as an input are given by

$$m^{N, \dagger}_2(\Delta^\dagger, P) = m^{N, \dagger}_{2, \beta_0}(\Delta^\dagger, P), \quad m^{N, \dagger}_{2, \beta_0}(P, \Delta^\dagger) = m^{N, \dagger}_{2}(P, \Delta^\dagger)$$

for $P \in C^\bullet(L_N; \Lambda_0)^\dagger$, and

$$m^{N, \dagger}_{k}(\ldots, \Delta^\dagger, \ldots) = 0$$

for $k \neq 2$.

$\Delta^\dagger$ is the homotopy between $\Delta^\dagger$ and $\Delta^\dagger$ in the following sense.

Let $\bar{a} = \{a_1, \ldots, a_{|\bar{a}|}\} \subset \{1, \ldots, k\}$, $a_1 < \ldots < a_{|\bar{a}|}$. For $\bar{P} = (P_{i_1}, \ldots, P_{i_{k-|\bar{a}|}})$ with $P_i \in \mathcal{C}_\alpha(i)(L_N)$, let $\bar{P}^\dagger$ be the $k$-tuple obtained by inserting $\Delta^\dagger$ into the $a_1, \ldots, a_{|\bar{a}|}$-th places of $\bar{P}$. We put $\bar{a}^\dagger = \mathcal{G}^\dagger_{\bar{a}}$, if $\bar{P}^\dagger = (P^\dagger_{i_1}, \ldots, P^\dagger_k)$ and $P^\dagger_i \in \mathcal{C}_{\bar{a}^\dagger}(L)$.

Let $\bar{a}^\dagger \mathbin{\llbracket} \bar{a}^\dagger = \bar{a}$ be a splitting of $\bar{a}$. Denote by $\bar{P}^\dagger$ the $(k - |\bar{a}^\dagger|)$-tuple given by removing $\Delta^\dagger$ from the $a_1, \ldots, a_{|\bar{a}^\dagger|}$-th places of $\bar{P}^\dagger$. For the choices of perturbations for $\mathcal{M}_{k+1}(\bar{P}; L_N, \bar{P}^\dagger)$. We have the perturbation $s^\dagger_{\bar{a}, \bar{a}^\dagger, \bar{P}^\dagger}$ obtained by just inserting $\Delta^\dagger$ into the $a_1, \ldots, a_{|\bar{a}|}$-th place, and the perturbations $s^\dagger$ pulled back via forget $\bar{a}, \bar{a}^\dagger : \mathcal{M}_{k+1}(\bar{P}; L_N, \bar{P}^\dagger) \to$
of fiberwise Kuranishi charts for the disc moduli (see Section 2.3) ensures that $H^\cup$ product of $\lambda^\gamma$, and therefore $W$ since $(\partial)$. We again use homological perturbation to reduce $m_{\lambda^*}^{N,\lambda}$ for $m_{\lambda^*}^{N,\lambda}$, with inputs $\Delta_{\lambda^*}$ inserted into $a_1, \ldots, a_{|\lambda^*|=1}$-th place of $\bar{P}$ are defined by

\[(\text{ev}_0)_* \left( \left([0,1]|^{|\lambda^*|} \times \mathcal{M}_{k+1}(\beta; L_N; \bar{P}) \right)^{s_\lambda^*} \lambda^\beta_\gamma / \sim \right) \quad (2.37)\]

Here $\sim$ is once again the equivalence relation collapsing fibers of the forgetful maps. Since $W^\mu(f_N; \lambda^*)$ is the restriction of the bundle $L_N \to BG(N)$ over $W^\mu(f_N, \lambda)$, our choice of fiberwise Kuranishi charts for the disc moduli (see Section 2.3) ensures that

\[(\text{ev}_0)_* \left( \mathcal{M}_{k+1}(\beta; L_N; \bar{P})^{s_\lambda^*} \right)^{\lambda^\gamma} \quad (2.38)\]

are degenerate singular chains which become zero in the quotient.

The definitions of $m_{\lambda^*}^{N,\lambda}$ operations, $(k, \beta) \neq (1, \beta_0)$, with distinct $\Delta_{\lambda^*}$ and $1^*$ as inputs are similar to what we have described above, and hence omitted.

We also set

\[m_{1,\beta_0}^{N,\lambda}(\Delta_{\lambda^*}) = \Delta_{\lambda^*} - \Delta_{\lambda^*},\]

and therefore

\[m_{1,\beta_0}^{N,\lambda}(\Delta_{\lambda^*}) = \Delta_{\lambda^*} - \Delta_{\lambda^*} + h_\lambda, \quad (2.39)\]

Here $h_\lambda$ has positive $T$-valuation.

We again use homological perturbation to reduce $(C^\bullet(L_N; \Lambda_0), m^{N,\lambda})$ to a unital $A^\infty$-algebra $(C^\bullet(f_N; \Lambda_0)^\gamma, \tilde{m}^{N,\lambda})$ as in Section 2.2. Aside from the obvious properties inherited from $(C^\bullet(L_N; \Lambda_0), m^{N,\lambda})$, it has the following additional property: Let $x = \sum x_j \otimes \lambda_j \in C^\bullet(f_N; \Lambda_0)$, and denote $\lambda \cdot x = \sum x_j \otimes (\lambda \cup \lambda_j)$. Here $\cup$ is the cup product of $H^\bullet(BG(N))$. Then we have

\[\tilde{m}_{2,\beta_0}^{N,\lambda}(\lambda, x) = \tilde{m}_{2,\beta_0}^{N,\lambda}(\lambda, x) = \lambda \cdot x, \quad (2.40)\]

and therefore

\[\tilde{m}_{2,\beta_0}^{N,\lambda}(\lambda, x) = (-1)^{|x|} \tilde{m}_{2,\beta_0}^{N,\lambda}(x, \lambda), \quad (2.41)\]

Finally, let

\[C^\bullet_G(L; \Lambda_0)^\gamma = \lim_{\rightarrow} C^\bullet(f_N; \Lambda_0)^\gamma = C^\bullet(f; \Lambda_0)^\gamma \otimes H^\bullet_G(\text{pt}; \Lambda_0).\]

We define our preferred equivariant Morse model $(C^\bullet_G(L; \Lambda_0)^\gamma, m^{G,\lambda})$ using the finite-dimensional approximations $(C^\bullet(f_N; \Lambda_0)^\gamma, \tilde{m}^{N,\lambda})$ as in Section 2.3. It has the following properties:

The restriction of $m^{G,\lambda}$ to $C^\bullet(G; \Lambda_0)$ agrees with $m^{G,\lambda}$. We have

\[m_{2,\beta_0}^{G,\lambda}(\lambda, x) = (-1)^{|x|} m_{2,\beta_0}^{G,\lambda}(x, \lambda) = (-1)^{|x|} m_{2,\beta_0}^{G,\lambda}(x, \lambda) \quad (2.43)\]

for $x \in C^\bullet_G(L; \Lambda_0)^\gamma$, and
for $k \neq 2$. Furthermore, if $x \in C^\bullet(f; \Lambda_0) \otimes H^\bullet_G(pt; \Lambda_0)$, then
\[ m_2^{G,\dagger}(\lambda^\vee, x) = \lambda \cdot x. \] (2.45)

The elements $1^\star$ and $\lambda^\star$ satisfy
\[ m_1^{G,\dagger}(1^\star) = 1^\vee - (1 - h)1^\star, \] (2.46)
\[ m_1^{G,\dagger}(\lambda^\star) = \lambda^\vee - (1 - h_\lambda)\lambda^\star, \] (2.47)
for some $h, h_\lambda \in \Lambda_+$. Since the properties $\lambda^\vee$ satisfy are similar to that of a strict unit, we will call $\lambda^\vee$ partial units and $\lambda^\star$ homotopy partial units.

The following theorem shows that the $A_\infty$-subalgebra $(C^\bullet_G(L; \Lambda_0), m^G)$ can be defined over the graded coefficient ring $H^\bullet_G(pt; \Lambda_0)$, namely
\[ (C^\bullet_G(L; \Lambda_0), m^G) = (C^\bullet(f; H^\bullet_G(pt; \Lambda_0)), \hat{m}^G), \] (2.48)
It is important to point out that the RHS cannot be determined by $(C^\bullet(f; H^\bullet_G(pt; \Lambda_0)), \hat{m})$ since the latter does not reflect the geometry of $L_G$.

**Theorem 2.15.** Let $x_1, \ldots, x_k \in C^\bullet_G(L; \Lambda_0)^\dagger$. Let $\ell \in \{1, \ldots, k\}$ and suppose $x_\ell = \sum x_i \otimes \lambda_j \in C^\bullet_G(L; \Lambda_0)$. We have
\[ m_k^{G,\dagger}(x_1, \ldots, x_k) = (-1)^\ell \sum \lambda_j \cdot m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes 1, x_{\ell+1}, \ldots, x_k). \] (2.49)

**Proof.** By (2.45), we have
\[ m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes \lambda_j, x_{\ell+1}, \ldots, x_k) = m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, m_2^{G,\dagger}(\lambda^\vee, x_\ell \otimes 1), x_{\ell+1}, \ldots, x_k), \] (2.43)
and $A_\infty$-relations, we have
\[ m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, m_2^{G,\dagger}(\lambda^\vee, x_\ell \otimes 1), x_{\ell+1}, \ldots, x_k) = (-1)^{\ell-1}m_k^{G,\dagger}(m_2^{G,\dagger}(\lambda^\vee, x_1), \ldots, x_{\ell-1}, x_\ell \otimes 1, x_{\ell+1}, \ldots, x_k) \]
\[ = (-1)^\ell m_k^{G,\dagger}(\lambda^\vee, m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes 1, x_{\ell+1}, \ldots, x_k)). \]

If $k \geq 2$, we have $m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes 1, x_{\ell+1}, \ldots, x_k) \in C^\bullet_G(L; \Lambda_0)$. For $k = 1$, since $x_\ell \in C^\bullet(f; \Lambda_0) \otimes H^\bullet_G(pt; \Lambda_0)$, we also have $m_1^{G,\dagger}(x) \in C^\bullet_G(L; \Lambda_0)$. Therefore, by (2.45),
\[ m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes \lambda_j, x_{\ell+1}, \ldots, x_k) = (-1)^\ell \lambda_j \cdot m_k^{G,\dagger}(x_1, \ldots, x_{\ell-1}, x_\ell \otimes 1, x_{\ell+1}, \ldots, x_k). \]

Let’s now consider a modified version of weak Maurer-Cartan equation for $(C^\bullet_G(L; \Lambda_0)^\dagger, m^{G,\dagger})$. Let $b \in C^\bullet_G(L; \Lambda_+)^{t,odd}$. Let $m_k^{G,\dagger,b}$ be the deformation of $m_k^{G,\dagger}$ by $b$. Suppose
\[ m_0^{G,\dagger,b}(1) = \Lambda_0 \cdot 1^\vee \oplus \left( \bigoplus_{\lambda \neq 1} \Lambda_0 \cdot \lambda^\vee \right). \] (2.50)

From (2.43), (2.44), we have $(m_1^{G,\dagger,b})^2 = 0$, which gives rise to a well-defined cohomology theory $H^\bullet(C^\bullet_G(L; \Lambda_0)^\dagger, m_1^{G,\dagger,b})$. We therefore define the weak Maurer-Cartan space of $C^\bullet_G(L; \Lambda_0)^\dagger$ to be
\[ MC \left( C^\bullet_G(L; \Lambda_0)^\dagger \right) = \left\{ b \in C^\bullet_G(L; \Lambda_+)^{t,odd} \mid b \text{ satisfies } (2.50) \right\}. \]
**Definition 2.16.** We say that \((C^*_G(L; \Lambda_0)^\dagger, m^{G, \dagger})\) is weakly unobstructed if MC \((C^*_G(L; \Lambda_0)^\dagger)\) is nonempty.

Similar to Lemma 2.8, we have

**Lemma 2.17.** Let \(b \in C^1_G(L; \Lambda_+).\) Suppose \(m_0^{G, \dagger, b}(1) = W(b)1^* + \sum_{\lambda \neq 1} \phi_\lambda(b)\lambda^*,\) and the minimal Maslov index of \(L\) is nonnegative, then there exists \(b^1 \in C^1_G(L; \Lambda_+)\) such that \(m_0^{G, \dagger, b^1}(1) = W^\vee(b)1^\vee + \sum_{\lambda \neq 1} \phi_\lambda^\vee(b)\lambda^\vee,\) i.e., \((C^*_G(L; \Lambda_0)^\dagger, m^{G, \dagger})\) is weakly unobstructed. In particular, if the minimal Maslov index of \(L\) is at least two, then \(W^\vee = W(b), \phi_\lambda^\vee(b) = \phi_\lambda(b).\)

**Proof.** Let’s tentatively take \(b' = b + W(b)1^* + \sum_{\lambda \neq 1} \phi_\lambda(b)m_2^{G, \dagger}(\lambda^\vee, 1^*).\) We also put \(b_1 = b + W(b)1^*\) and \(b_2 = W(b)1^* + \sum_{\lambda \neq 1} \phi_\lambda(b)m_2^{G, \dagger}(\lambda^\vee, 1^*).\) Since

\[
m_1^{G, \dagger}(m_2^{G, \dagger}(\lambda^\vee, 1^*)) = -m_2^{G, \dagger}(\lambda^\vee, m_1^{G, \dagger}(1^*)) = - (\lambda^\vee - (1 - h)\lambda^*),
\]

we have

\[
m_0^{G, \dagger, b'}(1) = W(b)1^* + \sum_{\lambda \neq 1} \phi_\lambda(b)\lambda^* + W(b)(1^\vee - (1 - h)1^*) - \sum_{\lambda \neq 1} \phi_\lambda(b)(\lambda^\vee - (1 - h)\lambda^*) + \sum_{k=2} \left(m_k^{G, \dagger}(b_1, \ldots, b_1) + m_k^{G, \dagger}(b_2, \ldots, b_2) + \sum_{\lambda \neq 1} \phi_\lambda(b) \sum m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots) \right).
\]

Here \(\sum m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots)\) is the summation over the terms which have both \(b\) and \(m_2^{G, \dagger}(\lambda^\vee, 1^*)\) as inputs.

By the proof of Lemma 2.8, we have

\[
W(b)(1^\vee - (1 - h)1^*) + \sum_{k=2} m_k^{G, \dagger}(b_1, \ldots, b_1) = W(b)(1^\vee - (1 - h(b))1^*)
\]

for some \(h(b) \in \Lambda_+.\) We also have \(\sum_{k=2} m_k^{G, \dagger}(b_2, \ldots, b_2) = 0\) by the proof of Lemma 2.17.

Let’s now consider the terms \(m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots).\) By (2.43), (2.44) and \(A_{\infty}\)-relations, we have

\[
m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots) = (-1)^{sgn} m_2^{G, \dagger}(\lambda^\vee, m_2^{G, \dagger}(\lambda^\vee, m_2^{G, \dagger}(\lambda^\vee, \ldots, b, 1^*, \ldots)))
\]

where the inner most term \(m_k^{G, \dagger}(\ldots, b, 1^*, \ldots)\) has only \(1^*\) and \(b\) as inputs. By degree reason \(m_k^{G, \dagger}(\ldots, b, 1^*, \ldots) = 0\) when it has more than one \(1^*\) as inputs. This means \(m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots) = 0\) when it has two \(m_2^{G, \dagger}(\lambda^\vee, 1^*)\) or both \(W(b)1^*\) and \(m_2^{G, \dagger}(\lambda^\vee, 1^*)\) as inputs. Therefore,

\[
\sum m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots) = \sum m_k^{G, \dagger}(b, \ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), b, \ldots, b) = (-1)^{sgn} \sum m_2^{G, \dagger}(\lambda^\vee, m_2^{G, \dagger}(b, \ldots, b, 1^*, b, \ldots, b)).
\]

Recall from the proof of Lemma 2.8 that \(m_k^{G, \dagger}(b, \ldots, b, 1^*, b, \ldots, b)\) is a multiple of \(1^*\) with positive \(T\)-valuation. Since \(m_2^{G, \dagger}(\lambda^\vee, 1^*) = \lambda^*,\) the RHS above is a multiple of \(\lambda^*\) with positive \(T\)-valuation. We can therefore write

\[
- \sum_{\lambda \neq 1} \phi_\lambda(b)(\lambda^\vee - (1 - h(b))\lambda^*) + \sum_{k \geq 2, \lambda \neq 1} \phi_\lambda(b)m_k^{G, \dagger}(\ldots, b, m_2^{G, \dagger}(\lambda^\vee, 1^*), \ldots)
\]

\[
= - \sum_{\lambda \neq 1} \phi_\lambda(b)(\lambda^\vee - (1 - h(b))\lambda^*)
\]
for some \(h_\lambda(b) \in \Lambda_+\).

Finally, let
\[
b\dagger = b + \frac{W(b)}{1 - h(b)} \mathbf{1}^* + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{h_\lambda(b) - 1} m_2^{G,\dagger}(\lambda^\vee, \mathbf{1}^*). \tag{2.51}
\]

We have
\[
m_0^{G,\dagger}(1) = W(b) \mathbf{1}^* + \sum_{\lambda \neq 1} \phi_\lambda(b) \mathbf{1}^* + \frac{W(b)}{1 - h(b)} (1^\vee - (1 - h(b)) \mathbf{1}^*) + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{1 - h_\lambda(b)} (\lambda^\vee - (1 - h_\lambda(b)) \mathbf{1}^*)
\]
\[
= \frac{W(b)}{1 - h(b)} 1^\vee + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{1 - h_\lambda(b)} \lambda^\vee =: W^\vee(b) 1^\vee + \sum_{\lambda \neq 1} \phi_\lambda^\vee(b) \lambda^\vee.
\]

Since \(W(b), \phi_\lambda(b) \in \Lambda_+\), we get \(b\dagger \in MC(G(L; \Lambda_0)^\dagger)\).

If the minimal Maslov index of \(L\) is at least two, then, \(h(b) = 0, h_\lambda(b) = 0\), and therefore \(W^\vee(b) = W(b), \phi_\lambda^\vee(b) = \phi_\lambda(b)\). \(\square\)

Equation (2.46) implies \([1^*] = [1^\vee]/(1 - h) \in HF_G(L; \Lambda_0)\) (in the weakly unobstructed case so that the equivariant Floer cohomology is well-defined). Since \(h \in \Lambda_+\), \(1^*\) is a cohomological unit. This is important, for instance when we consider isomorphisms of objects in the Fukaya category.

**Corollary 2.18.** In the setting of Lemma 2.17, if \(b \in C^1(f; \Lambda_0)\) and \(m_{0}^{\dagger}(1) \in \Lambda_0 1^*,\) then \((G^*(L; \Lambda_0)^\dagger, m_0^{G,\dagger})\) is weakly unobstructed.

**Proof.** If \(b \in C^1(f; \Lambda_0)\) and \(m_{0}^{\dagger}(1) \in \Lambda_0 1^*\), then \(m_0^{G,\dagger}(1)\) is of the form \(m_0^{G,\dagger}(1) = W(b) 1^* + \sum_{\lambda \neq 1} \phi_\lambda(b) \lambda^\vee\). \(\square\)

Here we remark that if \(b \in C^1_G(L; \Lambda_0)\), then in fact \(b \in C^1(f; \Lambda_0)\), since the generators of \(H^*_G(\text{pt}; \Lambda_0)\) are of (even) degrees at least 2 (other than the fundamental class).

**Definition 2.19.** Let \(x_1, \ldots, x_d\) be a basis of \(C^1_G(L; \mathbb{Q})\), and write \(b = \sum_{i=1}^{d} x^i x_i, x^i \in \Lambda_+\).

If \(m_0^{G,\dagger}(1) = W(x_1, \ldots, x_d) 1^* + \sum_{\lambda \neq 1} \phi_\lambda(x_1, \ldots, x_d) \lambda^\vee\), then we will call \(W^\vee(x_1, \ldots, x_d) + \sum_{\lambda \neq 1} \phi_\lambda^\vee(x_1, \ldots, x_d) \lambda\) the equivariant disc potential of \((G^*(L; \Lambda_0)^\dagger, m_0^{G})\).

### 2.5 Equivariant Floer theory for Lagrangian immersions

In this section we describe a generalization of the equivariant Morse model to immersed Lagrangians with clean self-intersections. The singular chain model of Lagrangian Floer theory for immersed Lagrangians with transverse self-intersections has been developed by Akaho-Joyce [AJ10]. Their construction can be generalized to the case of clean self-intersections when combined with Lagrangian Floer theory for a pair of cleanly intersecting Lagrangians in [FOOO09b]. We can then apply the homotopy unit construction and homological perturbation to obtain a unital Morse model as in Section 2.1 and 2.2.

As we will explain below, in the Morse model, we count pearly trees, where the vertices of trees are stable polygons (versus stable discs in the smooth case). The stable polygons have corners and marked points along the boundary edges. The evaluation map of these special points are targeted at the product of copies of the clean intersection (where the corners sit in) and copies of the normalization \(\tilde{L}\) of the immersion. Morse flow lines in the clean intersection are attached to vertices of the polygon, while flow lines in the normalization are attached to smooth points of the polygon.

Let \(L \subset X\) be a closed, connected, relatively spin, immersed Lagrangian submanifold with clean self-intersections. We denote by \(i : \tilde{L} \rightarrow X\) this immersion, and by \(\mathcal{I} \subset L\) the
self-intersection. \( \tilde{L} \times_i \tilde{L} \) has a connected component being \( R_0 := \tilde{L} \) itself (as the diagonal), and other connected components \( R_j \) for \( j = 1, \ldots, 2r \) corresponding to branch jumps at the immersed loci (where each clean intersection corresponds to two ways of branch jumps).

Let \( \alpha : \{0, \ldots, k\} \to \{0, 1, \ldots, 2r\} \) which are labellings for branch jumps at boundary marked points of a disc. The moduli spaces in consideration are denoted by \( \mathcal{M}_{k+1}(\alpha; \beta; L, J_X) \). The elements of \( \mathcal{M}_{k+1}(\alpha; \beta; L, J_X) \) are stable discs in class \( \beta \) with the specified branch jumps \( \alpha \) at the boundary marked points (where \( \alpha(j) = 0 \) means the \( j \)-th marked point is evaluated to \( \tilde{L} \)).

We have the evaluations maps at the boundary input marked points
\[
ev_i : \mathcal{M}_{k+1}(\alpha; \beta; L, J_X) \to R_{\alpha(i)}
\]
for \( i = 1, \ldots, k \), where \( R_{\alpha_i} \) can be identified with the corresponding immersed loci of \( L \). We also have the evaluation at the output marked point
\[
ev_0 : \mathcal{M}_{k+1}(\alpha; \beta; L, J_X) \to R_{\sigma(\alpha(i))}
\]
where \( \sigma \) reverses the labels of branch jump (and \( \sigma(0) = 0 \)). As in the smooth case, the generations of singular chains \( X_g(L) \) are inductively constructed and are elements of \( S^*(\tilde{L} \times_i \tilde{L}; \mathbb{Q}) \). We again apply the homotopy unit construction so the resulting \( A_\infty \)-algebra \( (C^*(L; \Lambda_0)^+, m) \) is unital.

We construct the non-equivariant Morse model for Lagrangian immersion similarly as previous sections. Let \( f \) be a Morse function on \( L \times_i \tilde{L} \), which has a unique maximum in each connected component \( R_j \). Moreover \( f|_{\tilde{L}} \) is taken such that its critical points are away from (preimages of) the immersed loci. Let \( C^*(f; \Lambda_0) \) be the cochain complex generated by the critical points of \( f \).

We have \( C^*(f; \Lambda_0) \to C^*(\tilde{L} \times_i \tilde{L}; \Lambda_0) \) by mapping each critical point to its unstable singular chain. As in Section 2.2, we use homological perturbation to obtain an \( A_\infty \)-algebra \( (C^*(f; \Lambda_0), (\tilde{m})) \). The \( A_\infty \) operations are counting pearly trees: the interior vertices of a tree are decorated by stable polygons. The edges of a tree correspond to gradient flow lines of the Morse function \( f \) (on one of the connected components \( R_j \)). The connected component \( R_i \) that the flow line is contained in has to match with the branch jump label of the corner of the disc that the flow line is attached to, or otherwise the fiber product is empty. See Figure 6 for some examples.

For the equivariant Morse model, we consider a Lagrangian immersion \( \iota : L \to X \) which is \( G \)-equivariant. We have the equivariant space \( L_G = L \times_G EG \to BG \) which has the self-clean intersection \( \mathcal{I}_G \subset L_G \), the normalization \( \iota : \tilde{L}_G \to L_G \), and \( L_G \times_i \tilde{L}_G \). Then the \( G \)-equivariant Lagrangian Floer theory \( (C^*(L_G; \Lambda_{0, nov}), m^G) \) (also \( (C^*(L_G; \Lambda_{0, nov})^+, m^{G, +}) \)) is defined using a family Morse model on the finite dimensional approximations of \( L_G \to BG \) as in Section 2.2 and 2.4. This give rise to interesting equivariant disc potentials even in the case when \( L \) is unobstructed as we will see in Section 4.

3. The \( T \)-equivariant disc potentials of toric manifolds

In this section, we study the equivariant Morse model in the case of a torus \( T = (S^1)^{\ell} \) acting on a closed, connected, relative spin Lagrangian submanifold \( L = (S^1)^{\ell} \times P \) of product type of a 2d-dimensional symplectic \( T \)-manifold \( X \), such that \( T \) acts freely on the first factor of \( L \) and trivially on the second factor. We choose a suitable choice of Morse-Smale pairs in order to construct \( (C^*_T(L; \Lambda_0), m^{T, +}) \). When applied to the regular moment map fibers \( L = (S^1)^d \) of a compact semi-Fano toric manifold, we recover the equivariant toric superpotential \( W_\Lambda \) as the equivariant disc potential of \( (C^*_T(L; \Lambda_0), m^{T, +}) \).
3.1 Morse theory on the approximation spaces. We begin by describing our choice of Morse-Smale pairs \((f_N, V_N)\) on \(L_N\).

The models we choose for the universal bundle \(ET\) and the classifying space \(BT\) are \(ET = (S^\infty)^\ell\) and \(BT = (\mathbb{C}P^\infty)^\ell\), respectively. We also have the finite dimensional approximations \(ET(N) = (S^{2N+1})^\ell\) and \(BT(N) = (\mathbb{C}P^N)^\ell\). Since the \(T\) acts trivially on \(P\), we have \(L_T = (S^1)^\ell_T \times (P \times (\mathbb{C}P^\infty)^\ell)\) and \(L_N = (S^1)^\ell_N \times (P \times (\mathbb{C}P^N)^\ell)\).

For \(N \geq 1\), let \([z_{1,0}, \ldots, z_{i,\ell}]\) be the homogeneous coordinates on the \(i\)-th component of \(BT(N) = (\mathbb{C}P^N)^\ell\). Let \(\pi_N : L_N \to (\mathbb{C}P^N)^\ell\) be the projection map. Let \(\{U_{j_i \ldots j_\ell}\}\) be the open cover of \((\mathbb{C}P^N)^\ell\) defined by

\[
U_{j_i \ldots j_\ell} = \left\{ [z_{1,0}, \ldots, z_{i,\ell}] \mid z_{j_i} \neq 0 \right\}.
\]

Set \(\tilde{U}_{j_i \ldots j_\ell} = \pi_N^{-1}(U_{j_i \ldots j_\ell}) \cong L \times (\mathbb{C}^N)^\ell\). We will be working with the atlas \(\{\tilde{U}_{j_i \ldots j_\ell}\}\) for \(L_N\) with local coordinates

\[
\left( \theta_1^{(j_i \ldots j_\ell)}, \ldots, \theta_\ell^{(j_i \ldots j_\ell)}, \bar{p}^{(j_i \ldots j_\ell)} \right), \left( z_i^{(j_i \ldots j_\ell)}, \ldots, \bar{z}_{j_i}^{(j_i \ldots j_\ell)} = 1, \ldots, z_i^{(j_i \ldots j_\ell)} \right)_{i=1,\ldots,\ell}
\]

on \(\tilde{U}_{j_i \ldots j_\ell}\), where \(\theta_i^{(j_i \ldots j_\ell)} \in [0,2\pi)\) are angular coordinates on \((S^1)^\ell\), \(\bar{p}^{(j_i \ldots j_\ell)}\) is any coordinate system on \(P\), and the term under “\(~\)" is omitted. Set \(\hat{\theta}_{i_j}^{(j_i \ldots j_\ell)} = \text{Arg}(z_{i_j}^{(j_i \ldots j_\ell)})\). The transition map \(\tilde{U}_{j_i \ldots j_\ell} \to \tilde{U}_{j_i' \ldots j_\ell'}\) is given by

\[
\left( \theta_1^{(j_i \ldots j_\ell)}, \ldots, \theta_\ell^{(j_i \ldots j_\ell)}, \bar{p}^{(j_i \ldots j_\ell)} \right), \left( z_i^{(j_i \ldots j_\ell)}, \ldots, \bar{z}_{j_i}^{(j_i \ldots j_\ell)} = 1, \ldots, z_i^{(j_i \ldots j_\ell)} \right)_{i=1,\ldots,\ell} \\
\left( \theta_1^{(j_i \ldots j_\ell)} + \theta_{i_j}^{(j_i \ldots j_{\ell'})}, \ldots, \theta_\ell^{(j_i \ldots j_\ell)} + \theta_{i_j}^{(j_i \ldots j_{\ell'})}, \bar{p}^{(j_i \ldots j_\ell)} \right), \left( \frac{z_i^{(j_i \ldots j_\ell)}}{z_{i_{j_i}}^{(j_i \ldots j_{\ell'})}}, \ldots, \frac{z_i^{(j_i \ldots j_\ell)}}{z_{i_{j_i}}^{(j_i \ldots j_{\ell'})}} = 1, \ldots, \frac{z_i^{(j_i \ldots j_\ell)}}{z_{i_{j_i}}^{(j_i \ldots j_{\ell'})}} \right)_{i=1,\ldots,\ell}.
\]

This pearl tree is allowed
This pearl tree does not exist

Figure 3. In the figure, \(\beta_0\) is a constant polygon. The pearly tree shown on the right actually does not exist, due to inconsistency of Lagrangian boundary labels.
Let’s fix the inclusion \((\mathbb{CP}^N)^\ell \hookrightarrow (\mathbb{CP}^{N+1})^\ell\) to be
\[
([z_{0}, \ldots, z_{N}]) \mapsto ([z_{0}, \ldots, z_{N}, 0]) .
\]
(3.1)
Which in turn, fixes the inclusions \(X_N \hookrightarrow X_{N+1}\) and \(L_N \hookrightarrow L_{N+1}\).

We have a perfect Morse function \(\varphi : (\mathbb{CP}^\infty)^\ell \to \mathbb{R}\)
\[
\varphi([z_{0}, \ldots, z_{1}]) = -\sum_{i=1}^{\ell} \sum_{k=0}^{\infty} k|z_{i,k}|^2
\]
decreasing along the finite dimensional strata. The critical points of \(\varphi\) are of the form
\[
\text{Crit}(\varphi) = \{([0, \ldots, 0, z_{i,j}] : (j_1 \ldots j_\ell) \in \mathbb{Z}_{\geq 0}^\ell \}.
\]
with degrees given by \(\sum_{i=1}^{\ell} 2j_i\). We denote the degree 2 critical points with \(j_i = 1\), and \(j_k = 0\) for \(k \neq i\) by \(\lambda_i\). We also denote the critical point where \(\varphi\) attains the maximum by \(1_{BT}\), i.e.
\[
1_{BT} = ([z_{0}, \ldots, 0, 0], \ldots, j_{j_1 \ldots j_\ell}) .
\]
We set \(\varphi_N = \varphi|_{BT(N)}\). Note that \(\varphi_N\) is a perfect Morse function on \((\mathbb{CP}^N)^\ell\). We will abuse notation and denote the degree two critical points and the maximum of \(\varphi_N\) again by \(1_{BT}\) and \(\lambda_i\), respectively. Note that we have
\[
H^\ast(BT(N); \mathbb{Z}) = H^\ast((\mathbb{CP}^N)^\ell; \mathbb{Z}) = \mathbb{Z}[\lambda_1, \ldots, \lambda_\ell]/(\lambda_i^{N+1}),
\]
and
\[
H^\ast_{\ast}(pt; \mathbb{Z}) = \mathbb{Z}[\lambda_1, \ldots, \lambda_\ell].
\]
On the other hand, let \(f_P\) be a Morse function on \(P\) with a unique maximum \(1_P\), and let \(f_{(S^1)^\ell} : (S^1)^\ell \to \mathbb{R}\) be the perfect Morse function
\[
f_{(S^1)^\ell}(\theta_1, \ldots, \theta_\ell) = \sum_{i=1}^{\ell} \cos(\theta_i).
\]
The critical points of \(f_{(S^1)^\ell}\) are of the form
\[
\text{Crit} \left( f_{(S^1)^\ell} \right) = \{ (\theta_1, \ldots, \theta_\ell) \in L | \theta_i = 0 \text{ or } \theta_i = \pi \} .
\]
Let \(f = (f_{(S^1)^\ell}, f_P)\) be the Morse function on \(L\). We denote by \(1_{L}\) the critical point where \(f\) attains the maximum, i.e. \(1_L = ((0, \ldots, 0), 1_P)\). We also denote by \(x_1, \ldots, x_\ell\) the degree 1 critical points, and \(x_i \wedge x_j, 1 \leq i < j \leq \ell\) the degree 2 critical points of \(f\) of the following forms
\[
x_i = ((0, \ldots, \theta_i = \pi, \ldots, 0), 1_P),
\]
and
\[
x_i \wedge x_j = ((0, \ldots, \theta_i = \pi, \ldots, \theta_j = \pi, \ldots, 0), 1_P).
\]

Let \(\{D_j\}\) be the open cover of \(\mathbb{CP}^\infty\) given by
\[
D_j = \left\{ ([z_{0}^{(j)}, \ldots, z_{N}^{(j)})] \in U_j : \left| z_{k}^{(j)} \right| < 2 \text{ for all } k \right\} .
\]
(3.2)
Let \(\{\rho^{(j)}\}\) be a smooth partition of unity subordinate to \(\{D_j\}\), and denote by \(\rho^{(j)}_i\) the post-composition of \(\rho^{(j)}\) with the projection from \(L_T\) to the \(i\)-th component of \((\mathbb{CP}^\infty)^\ell\). We define \(\rho^{(j_1 \ldots j_\ell)} : L_T \to \mathbb{R}\) by
\[
\rho^{(j_1 \ldots j_\ell)} = \prod_{i=1}^{\ell} \rho^{(j_i)}_i .
\]
Let \( \rho_N^{(j_1, \ldots, j_k)} = \rho^{(j_1, \ldots, j_k)}_{\partial B_N} \), then \( \{ \rho_N^{(j_1, \ldots, j_k)} \} \) is a smooth partition of unity on \( L_N \).

Let \( \phi_N^{(j_1, \ldots, j_k)} : \mathcal{U}_{j_1, \ldots, j_k} \to \mathbb{R} \) be the fiber-wise Morse function
\[
\phi_N^{(j_1, \ldots, j_k)} \left( \left( \theta_1^{(j_1, \ldots, j_k)}, \ldots, \theta_\ell^{(j_1, \ldots, j_k)}, p^{(j_1, \ldots, j_k)} \right), \left( z_{1,0}^{(j_1, \ldots, j_k)}, \ldots, z_{1,N}^{(j_1, \ldots, j_k)} \right) \right) = f \left( \theta_1^{(j_1, \ldots, j_k)}, \ldots, \theta_\ell^{(j_1, \ldots, j_k)}, p^{(j_1, \ldots, j_k)} \right),
\]
and set
\[
\phi_N = \sum_{(j_1, \ldots, j_k)} \rho_N^{(j_1, \ldots, j_k)} \phi_N^{(j_1, \ldots, j_k)}.
\]

Notice that \( \phi_N = \phi_N^{(j_1, \ldots, j_k)} \) in a neighborhood of \( \pi_N^{-1}(\lambda), \lambda \in \text{Crit}(\phi_N) \). Then, for \( \epsilon > 0 \) sufficiently small, the function \( f_N : L_N \to \mathbb{R} \) defined by
\[
f_N := \epsilon \phi_N + \pi_N^* \phi_N
\]
is a Morse function, with critical points
\[
\text{Crit}(f_N) = \left\{ (x, \lambda) | x \in \text{Crit} \left( \phi_N |_{\pi_N^{-1}(\lambda)} \right), \lambda \in \text{Crit}(\phi_N) \right\} = \text{Crit}(f) \times \text{Crit}(\phi_N).
\]
The degree of \( (x, \lambda) \in \text{Crit}(f_N) \) is given by
\[
|(x, \lambda)| = |x| + |\lambda|,
\]
where \( |x| \) and \( |\lambda| \) are the degrees of \( x \) and \( \lambda \) as critical points of \( f \) and \( \phi_N \), respectively.

Let \( V_p \) be a negative pseudo-gradient vector field for \( f_P \). Let \( V_N \) be the vector field given by
\[
V_N|_{\mathcal{U}_{j_1, \ldots, j_k}} = 4 \sum_{i=1}^\ell \sum_{j \neq j_i} (j - j_i) \left| z_{i,j}^{(j_1, \ldots, j_k)} \right|^2 \frac{\partial}{\partial z_{i,j}^{(j_1, \ldots, j_k)}} + \frac{\partial}{\partial z_{i,j}^{(j_1, \ldots, j_k)}}
\]
\[
+ \sum_{i=1}^\ell \rho_{N,i}^{(j_i)} \sin \left( \theta_i^{(j_1, \ldots, j_k)} \right) + \sum_{j \neq j_i} \rho_{N,j}^{(j_j)} \sin \left( \theta_i^{(j_1, \ldots, j_k)} + \theta_{i,j}^{(j_1, \ldots, j_k)} \right) \frac{\partial}{\partial \theta_i^{(j_1, \ldots, j_k)}}.
\]
Here \( \rho_{N,i}^{(j_i)} = \rho_i^{(j_i)} |_{\partial B_N} \). Then
\[
\gamma_N = V_N + V_P
\]
is a negative pseudo-gradient for \( f_N \). For \( N = 0 \), we set \( f_0 = f \), and \( \gamma_0 = \sum_{i=1}^\ell \sin(\theta_i) \frac{\partial}{\partial \theta_i} + V_P \).

**Proposition 3.1.** For generic choices of \( V_P \), the pairs \( \{(f_N, \gamma_N)\} \) are Morse-Smale and satisfy Condition 2.13.

For \( p, q \in \text{Crit}(f_N) \), let \( \mathcal{M}(p, q) \) be the moduli space of gradient flow lines from \( p \) to \( q \), modulo reparametrizations. Let \( (C^*(f_N; Z), d_N) \) to be the Morse cochain complex
\[
C^*(f_N; Z) := \bigoplus_{p \in \text{Crit}(f_N)} Z \cdot p
\]
equipped with the differential \( d_N : C^k(f_N; Z) \to C^{k+1}(f_N; Z) \)
\[
d_N(p) = \sum_{q \in \text{Crit}(f_N) \atop |q| = k+1} \sharp \mathcal{M}(p, q) \cdot q.
\]
We have
\[
C^*(f_N; Z) = C^*(f; Z) \otimes Z[\lambda_1, \ldots, \lambda_\ell]/(\lambda_1^{N+1}). \tag{3.3}
\]
For convenience, we will write the elements of \( \text{Crit}(f_N) \) as \( x \otimes \lambda \), where \( x \in \text{Crit}(f) \) and \( \lambda \in \text{Crit}(\varphi_N) \). The following result will be used in the computation of the equivariant disc potentials.

**Proposition 3.2.** For \( N \geq 1 \) and \( x_1, \ldots, x_\ell \), we have

\[
d_N(x_i \otimes 1_{BT}) = 1_L \otimes \lambda_i.
\]

In particular, there exists a unique gradient flow line from \( x_i \otimes 1_{BT} \) to \( 1_L \otimes \lambda_i \).

**Proof.** The possible outputs of \( d_N(x_i \otimes 1_{BT}) \) is of degree 2 critical points of the form \( x_j \land x_k \otimes 1_{BT} \) and \( 1_L \otimes \lambda_j \). It is easy to see that \( \mathcal{M}(x_i \otimes 1_{BT}, x_j \land x_k \otimes 1_{BT}) = \emptyset \) if both \( j \neq i \) and \( k \neq i \), and \( \mathcal{M}(x_i \otimes 1_{BT}, x_j \land x_k \otimes 1_{BT}) \) consists of two points of opposite orientation if either \( j = i \) or \( k = i \).

Suppose \( \Phi: \mathbb{R} \to \mathbb{L}_N \) is a flow line from \( x_i \otimes 1_{BT} \) to \( 1_L \otimes \lambda_j \). Its projection \( \pi_N \circ \Phi: \mathbb{R} \to (\mathbb{C}P^N)^\ell \) is a flow line for the vector field \( V \) on \( (\mathbb{C}P^N)^\ell \) given by

\[
V|_{U_{\lambda_1 \ldots \lambda_\ell}} = 4 \sum_{i=1}^{\ell} \sum_{j \neq i} (j - i)|z_{ij}(x_i \land x_j)|^2 \frac{\partial}{\partial z_{ij}(x_i \land x_j)}
\]

from \( 1_{BT} \to \lambda_j \), whose image is contained in \( U_{0 \ldots 0} \). This means the image of \( \Phi \) is contained in \( \tilde{U}_{0 \ldots 0} \), and we can therefore write

\[
\Phi(t) = \left( \left( \theta_i^{(0 \ldots 0)}(t), \ldots, \theta_\ell^{(0 \ldots 0)}(t) \right), \left( z_{\mu,1}^{(0 \ldots 0)}(t), \ldots, z_{\mu,N}^{(0 \ldots 0)}(t) \right) \right)_{\mu = 1, \ldots, \ell},
\]

in terms of the coordinates on \( \tilde{U}_{0 \ldots 0} \). For \( \Phi(t) \) to have the correct asymptotics, we must have \( z_{\mu,v}^{(0 \ldots 0)}(t) = 0 \) for \( \mu \neq i \) and \( v \neq 1 \); \( \theta_\mu^{(0 \ldots 0)}(t) = 0 \) for \( \mu \neq i \); \( z_{\mu,1}^{(0 \ldots 0)}(t) = e^{2t + i\vartheta} \) where \( \vartheta \in [0, 2\pi) \); \( \theta_i^{(0 \ldots 0)}(t) \) satisfies

\[
\lim_{t \to -\infty} \theta_i^{(0 \ldots 0)}(t) = \pi,
\]

and

\[
\lim_{t \to +\infty} \theta_i^{(0 \ldots 0)}(t) + \delta_{ij} \vartheta = 0,
\]

in addition to

\[
\frac{d\theta_i^{(0 \ldots 0)}(t)}{dt} = a(t) \sin \left( \theta_i^{(0 \ldots 0)}(t) \right) + b(t) \sin \left( \theta_i^{(0 \ldots 0)}(t) + \vartheta \right).
\]

Here \( a(t) = \rho_1^{(0)}|_{U_0} (e^{2t + i\vartheta}) \) and \( b(t) = \rho_1^{(1)}|_{U_0} (e^{2t + i\vartheta}) \), and \( \{\rho_1^{(0)}, \rho_1^{(1)}\} \) is the partition of unity for \( \mathbb{C}P^1 \).

For existence of a flow line when \( i = j \), we note that the flow line with \( \theta_i^{(0 \ldots 0)}(t) = \pi \) and \( \vartheta = 0 \) has the desired asymptotics. As for uniqueness, assume without loss of generality that \( \Phi(0) \) is in a neighborhood of \( x_i \otimes 1_{BT} \) such that \( \rho_N^{(0)}(t) = 1 \) and \( \rho_N^{(1)}(t) = 0 \). In this neighborhood, we have explicitly

\[
\theta_i^{(0 \ldots 0)}(t) = 2 \cot^{-1} \left( e^{-t} \cot \left( \frac{\theta_i^{(0 \ldots 0)}(0)}{2} \right) \right).
\]

It is easy to see that

\[
\lim_{t \to -\infty} \theta_i^{(0 \ldots 0)}(t) = \pi,
\]
if \( \theta_i^{(0...0)}(0) = \pi \), and

\[
\lim_{t \to -\infty} \theta_i^{(0...0)}(t) = 0,
\]

otherwise. Solving for

\[
\frac{d\theta_i^{(0...0)}(t)}{dt} = 0
\]
gives

\[
\tan\theta_i^{(0...0)}(t) = \frac{b(t) \sin(\theta)}{a(t) + b(t) \cos(\theta)}.
\]

This means we have

\[
\lim_{t \to +\infty} \theta_i^{(0...0)}(t) + \delta_{ij} \vartheta = \pi,
\]

unless \( \theta = 0 \) and \( i = j \), in which case

\[
\lim_{t \to +\infty} \theta_i^{(0...0)}(t) + \delta_{ij} \vartheta = 0.
\]

\[\square\]

3.2 Computing the T-equivariant disc potentials. By applying the construction in Section 2.3 and 2.4 with the choice of Morse-Smale pairs made in Section 3.1, one obtains

\[(C^\bullet(L; \Lambda_0)^\dagger, m^{T,t}) = (C^\bullet(f; \Lambda_0)^+ \otimes \Lambda_0[\lambda_1, \ldots, \lambda_l], m^{T,t})\]

associated to \( L \) equipped with the \( T \)-action. In this section, we compute the equivariant disc potential of \((C^\bullet(L; \Lambda_0)^\dagger, m^{T,t})\) assuming that every holomorphic disc bounded by \( L \) has non-negative Maslov index.

For simplicity of notations, we will suppress “-” and denote the unique maximum \( 1^*_L \) on \( L \) by \( 1_L \). Let \( x_1, \ldots, x_L, y_1, \ldots, y_L \) be a basis of \( C^1(f; \mathbb{Q}) \), and put \( x_i := x_i \otimes 1_{BT} \) and \( y_i := y_i \otimes 1_{BT} \). We also denote \( \lambda_i := 1_L \otimes \lambda_i \), and \( 1_{LT} := 1_L \otimes 1_{BT} \).

Let \( b = \sum_{i=1}^L x_i \otimes x_i + \sum_{i=1}^L y_i \otimes y_i \), where \( x_i, y_i \in \Lambda^+ \). We consider the boundary deformation of \( m_0^{T,t} \) by \( b \)

\[
m_0^{T,b}(1) = m_0^{T,b}(1) = m_0^T(1) + m_1^T(b) + m_2^T(b, b) + \cdots.
\]

The first equality above follows from the fact that the restriction of \( m^{T,t} \) to \( C^*_T(L; \Lambda_0) \) agrees with \( m^T \).

We compute the obstruction \( m_0^{T,b}(1) \) by counting pearly trees in (finite dimensional approximations of) \( L_T \subset X_T \). Since \( m_{k,b}^{T}(b, \ldots, b) \) is of degree \( 2 - \mu(\beta) \), the outputs of \( m_0^{T,b}(1) \) have degree either 0 or 2 depending on the Maslov indices of the contributing disc classes. The possible outputs are of the following forms:

- (Degree zero) \( 1_{LT} \),
- (Degree two) \( p \otimes 1_{BT} \), where \( p \in \text{Crit}(f) \) is a degree 2 critical point,
- (Degree two) \( \lambda_i \).

Notice that all the critical points above are contained in \( \text{Crit}(f_1) \). Thus, \( m_0^{T,b}(1) \) can be computed by counting pearly trees in \( L_1 \).

---

\(^1\)To be more precise, we should perform the calculation on the approximation spaces \( L_N \subset X_N \) and then take its limit (see (2.32), (2.33) and (2.34)). By abuse of notations, the \( A_X \)-operations on \( L_T \) are regarded as the \( A_X \)-operations on the approximation spaces.
Proposition 3.3. Suppose $T = (S^1)^\ell$ acts on $(X,\omega_X)$ preserving $\omega_X$. Let $L \subset X$ be a $T$-invariant closed Lagrangian submanifold of product type $L = (S^1)^\ell \times P$ such that the $T$-action on $L$ is standard, i.e., $T$ acts freely on $(S^1)^\ell$ and trivially on $P$. Suppose $L$ has non-negative minimal Maslov index, then

$$m_0^{TB}(1) = \tilde{m}_0^b(1) \otimes 1_{BT} + \sum_{i=1}^\ell h_i(\vec{x},\vec{y})\lambda_i$$  \hspace{1cm} (3.4)

where $\tilde{m}$ is the $A_\infty$-structure of $(C^*; \Lambda_0, \tilde{m})$, and $h_i(\vec{x},\vec{y}) \in \Lambda_0$. Moreover, if $\tilde{m}_0^b(1) = W(\vec{x},\vec{y})1_L$ for some $W(\vec{x},\vec{y}) \in \Lambda_+$, then $(C^*_{\Gamma}(L;\Lambda_0)^b, m^{TB})$ is weakly unobstructed.

Proof. The first two types of outputs are contributed by pearly trees contained in the fiber $L$ over the critical point $1_{BT}$, and coincide with $\tilde{m}_0^b(1)$ by our construction of $m^T$. Thus, we have the expression (3.4).

The last assertion follows from Corollary 2.18, namely, if $\tilde{m}_0^b(1) = W(\vec{x},\vec{y})1_L$, then

$$m_0^{TB}(1) = W(\vec{x},\vec{y})1_L + \sum_{i=1}^\ell h_i(\vec{x},\vec{y})\lambda_i.$$  \hspace{1cm} (3.5)

$\square$

In the following, we shall compute $\tilde{m}_0^b(1)$ and $h_i(x^1,\ldots,x^\ell)$ explicitly under additional assumptions. We begin by simplifying $h_i(\vec{x},\vec{y})$ under the condition that the minimal Maslov index of $L$ is at least 2.

Lemma 3.4. In the situation of Proposition 3.3, if, in addition, the minimal Maslov index of $L$ is at least 2, then $h_i(\vec{x},\vec{y}) = x^i$ in (3.4).

Proof. The terms $h_i(\vec{x},\vec{y})$ are contributed by disc classes of Maslov index 0 by degree reason. Since $L$ has minimal Maslov index at least two, the only contribution comes from the trivial disc class $\beta_0$, hence the computation of the terms $m^T_k(b,\ldots,b)$ reduces to $m^T_1(b,\ldots,b)$, where $\Gamma \in \Gamma_{k+1}$ is a stable planar rooted tree with all interior vertices decorated by $\beta_0$. In other words, the configurations $\Gamma$ we are counting are Morse flow trees in $L_1$.

By Proposition 3.2, we have $m^T_1(x_i) = \lambda_i$. We will show momentarily that the coefficients of $\lambda_i$ in the terms $m^T_k(x_{i1},\ldots,x_{ik})$ are zero for $k \geq 2$ and $(i_1,\ldots,i_k) \in \{1,\ldots,\ell\}^k$. First, let’s consider the case when $i_j \neq i$ for some $j$. In the proof of Proposition 3.2, we showed that there exists no flow line from $x_i$ to $\lambda_i$. This means for generic small perturbations, the moduli spaces $\mathcal{M}_\Gamma(f_{i1};x_{i1},\ldots,x_{ik};\lambda_i)$ (which are responsible for the coefficient of $\lambda_i$ in $m^T_k(x_{i1},\ldots,x_{ik})$) are empty.

Now, we consider $m^T_k(x_i,\ldots,x_i)$, $k \geq 2$. Since rotations on the $i$-th circle factor of $L$ commutes with the structure group of $L_1$, we have a global $S^1$-action on $L_1$ rotating the $i$-th circle factor of the fibers $L$. To achieve transversality for the moduli spaces $\mathcal{M}_\Gamma(f_{i1};x_{i1},\ldots,x_{ik};\lambda_i)$, we can perturb the the flow lines from the first $k - 1$ inputs using the $S^1$-action. We have the unique flow line $\gamma$ from $x_i$ to $\lambda_i$ as in Proposition 3.2. For a perturbed flow line from the first $k - 1$ inputs to intersect with $\gamma$ (in order to form a flow tree), its projection in $BT(1)$ must coincide with $\pi(\gamma)$. However, over $\pi(\gamma)$, the flow lines from the first $k - 1$ inputs are simply the rotations of $\gamma$ by $S^1$, which do not intersect with $\gamma$. Thus, $\mathcal{M}_\Gamma(f_{i1};x_{i1},\ldots,x_{ik};\lambda_i)$ are empty for generic small perturbations.

Similarly, since the fiber bundle $P_\Gamma = P \times BT$ is trivial, there exists no flow line from $y_j$ to $\lambda_i$, and therefore the coefficients of $\lambda_i$ in the terms $m^T_k(\ldots,y_j,\ldots,b)$ are zero.
Thus, in the setting of Lemma 3.4, we have
\[ m_0^{T,b}(1) = m_0^b(1) \otimes 1_{BT} + \sum_{i=1}^{\ell} x^i \lambda_i. \]

In [FOOO10], it was shown that the moment map fibers of a compact toric manifold are weakly unobstructed in the de Rham model. The following is the corresponding statement in the Morse model (we restrict ourselves to semi-Fano case to ensure that minimal Maslov index is 2).

**Lemma 3.5.** Let \( L \) be a regular moment map fiber of a compact semi-Fano toric manifold. We have
\[ r m_0^b(1) = r m_0^b(1) \cup L T + \ell \sum_{i=1}^{\ell} x^i \lambda_i. \]

**Proof.** The only possible outputs of \( m_0^b(1) \) are multiples of \( 1_L \) and degree two critical points of \( f \) (which are of the form \( x_j \wedge x_k \)), contributed by Maslov index zero and two disc classes, respectively. Since \( L \) has minimal Maslov index two, only Morse flow trees contribute to a degree two critical points. We have \( m_1(x_i) = 0 \), since \( f \) is a perfect Morse function on \( L \). For \( m_k(x_{i_1}, \ldots, x_{i_k}) \), \( k \geq 2 \), if there are two repeated inputs \( x_i \), then by perturbing unstable hypertori \( \Delta x_i \) using the \( S^1 \)-action rotating the \( i \)-th circle factor of \( L \), the perturbed hypertori do not intersect and hence \( m_k(x_{i_1}, \ldots, x_{i_k}) = 0 \). In the case of distinct inputs, we have
\[ \sum_{\sigma \in S_k} \tilde{m}_k \left( x_{l(1)}, \ldots, x_{l(k)} \right) = 0, \quad k \geq 2, \]
due to the orientations on the corresponding moduli spaces. 

**Remark 3.6.** Before proceeding, a remark is in order addressing the perturbations used in the proof of Lemma 3.4, and 3.5. Recall that our Morse model is derived from the singular chain model. In the singular chain model, we do not perturb the input singular chains when choosing perturbation for a fiber product \( M_{k+1}(\beta; L N; \vec{P}) \) since the singular chains are fixed during the inductive construction and doing so would destroy the \( \Lambda_x \)-structure. Instead, we realize the perturbation of input singular chains by perturbing the respective evaluation maps.

Combining the above lemmas, to show that the equivariant toric superpotentials coincide with the equivariant disc potentials, it remains to compute the (non-equivariant) disc potential.

Recall that the holomorphic disc classes are generated by the basic disc classes \( \beta_i \) for \( 1 \leq i \leq m \) [CO06]. Moreover, by [FOOO10], a stable disc class of Maslov index two must be of the form \( \beta_i + \alpha \) for some curve class \( \alpha \) with \( c_1(\alpha) = 0 \). Let \( n_1(\beta_i + \alpha) \) be the degree of the virtual fundamental class \( \text{ev}_* [M_1(\beta_i + \alpha)] \). In the Morse model, it is given by counting the number of times \( \text{ev}_* [M_1(\beta_i + \alpha)] \) hits the maximal point.

**Theorem 3.7** (Equivariant toric superpotential). For compact semi-Fano toric manifolds \( X \) of dimension \( 2d \), let \( \ell v_1, \ldots, v_m \in \mathbb{Z}^d \) be the primitive generators of the one-dimensional cones of the fan \( \Sigma \) defining \( X \). We have
\[ m_0^{T,b}(1) = W^{\text{disc}}(x^1, \ldots, x^d) 1_{LT} + \sum_{i=1}^{d} x^i \lambda_i. \]
where
\[ W^{\text{disc}} = \sum_{i=1}^{m} \left( \sum_{\alpha \in \Gamma(\alpha)} n_1(\beta_i + \alpha) T^{\omega_x(\alpha)} \right) T^{\omega_x(\beta_i)} \exp(p_i \cdot (x^1, \ldots, x^d)) \] (3.6)

In particular in the Fano case,
\[ m_0^{T,b} = W^\text{HV}(e^{x^1}, \ldots, e^{x^d}) \mathbf{1}_{L_T} + \sum_{i=1}^{d} x^i \lambda_i, \]
where $W^\text{HV}$ is the Hori-Vafa superpotential [HV, HKK+03].

**Proof.** Recall from Proposition 3.3 that $W^{\text{disc}} = \tilde{m}_0^b(1)$. By degree reason, a stable rooted tree $\Gamma \in \Gamma_{k+1}$, contributing to the term $\tilde{m}_k(b, \ldots, b)$, $k \geq 1$, must have exactly one interior vertex decorated by a Maslov index two disc class and the remaining interior vertices are decorated by the trivial disc class $\beta_0$.

We take the perfect Morse function $f$ on $L$ such that the boundaries of the finitely many basic holomorphic discs passing through the maximal point do not intersect with $\Delta_{x_j} \cap \Delta_{x_i}$ for any $\mu \neq \nu$, where $\Delta_{x_j}$ denotes the unstable hypertorus of the degree-one critical point $x_j$. This means the only $\Gamma \in \Gamma_{k+1}$, contributing to $\tilde{m}_k(b, \ldots, b)$ has exactly one interior vertex, which is decorated by $\beta_i + \alpha$. It remains to consider contributions from the moduli spaces of the form $\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_{i_1}}, \ldots, \Delta_{x_{i_k}})$, $(\mu_1, \ldots, \mu_k) \in \{1, \ldots, d\}^k$. Since $L$ has minimal Maslov index two and $\beta_i + \alpha$ has Maslov index two, the moduli spaces $\mathcal{M}_{k+1}(\beta_i + \alpha; L; J_X)$ have no (non-constant) disc bubbles. We have $\mathcal{M}_{k+1}(\beta_i + \alpha; L; J_X) = \mathcal{M}_1(\beta_i + \alpha; L; J_X) \times \hat{C}_k$ via the forgetful map, where $\hat{C}_k$ is certain iterated blow up of $C_k = \{(t_1, \ldots, t_k)|0 \leq t_1 \leq \ldots \leq t_k \leq 1\}$, defined in [FOOO10, Lemma 11.8] (see also [FOOO16, Remark 4.1.1]). The output evaluation of the fundamental class $[\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_{i_1}}, \ldots, \Delta_{x_{i_k}})]$ intersects the maximal point only if the boundary of the basic holomorphic disc, which represents $\beta_i$, passes through the maximal point, intersects all the input hypertori $\Delta_{x_{i_j}}$.

Suppose the output evaluation of the fundamental class $[\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_{i_1}}, \ldots, \Delta_{x_{i_k}})]$ intersects the maximal point. Let $r_j$ be the number of times $\Delta_{x_j}$ appears in the inputs. We denote by $X_{i_1, \ldots, j_{\ell}, r_{\ell}}$ the copies of $\Delta_{x_j}$ in the order the appear in the inputs. Let $X_j^{(1)}, \ldots, X_j^{(r_j)}$ disjoint small perturbations of $\Delta_{x_j}$ along the the direction of $\partial \beta_i$. We also denote by $X_{i_1, \ldots, j_{\ell}, r_{\ell}}$ the corresponding perturbation of the ordered copies of $\Delta_{x_j}$, Let’s consider all possible configurations of perturbed input hypertori $X_{i_1, \ldots, r_{\ell}}$, \{1, \ldots, r_{\ell}\} = \{1, \ldots, r\}. Then, each fiber $\text{forget}^{-1}(p)$ contains $\prod_{j=1}^{d} m_j^{r_j}$ many discs intersecting these configurations, where $m_j$ is the multiplicity of $\partial \beta_i \cap \Delta_{x_j}$. This can be understood as follows: The number of intersection points of some discs with $r_j$ copies of $\Delta_{x_j}$ is $m_j r_j$. $\prod_{j=1}^{d} m_j^{r_j}$ is the number of ways to choose one intersection point on each copy of $\Delta_{x_j}$. Such a choice uniquely determines a configuration for which a disc in $\text{forget}^{-1}(p)$ intersects (at the boundary marked points). Conversely, the boundary marked points of a disc in $\text{forget}^{-1}(p)$ intersecting a configuration give rise to such a choice (see Figure 4 for an illustration).

We denote by $E$ the obstruction bundle of $\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_{i_1}}, \ldots, \Delta_{x_{i_k}})$ and let $s : \mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_{i_1}}, \ldots, \Delta_{x_{i_k}}) \rightarrow E^{\otimes k}$ be a section corresponding to the summation of
all possible configurations of perturbed input hypertori described above. We choose the multisection $s : \mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{x_p}, \ldots, \Delta_{x_k}) \to E^\otimes k/S_k$ to be $s/S_k$, resulting in the coefficient $\frac{n_1(\beta_i + \alpha) \prod_{j=1}^d m_j^{r_j}}{k!}$. Summing over all the possible inputs of $k$ hypertori and the $\prod_{j=1}^d m_j^{r_j}$ orderings among each them, we obtain the term $n_1(\beta_i + \alpha) \exp(v_i \cdot (x^1, \ldots, x^d))$.

**Figure 4.** The circles correspond to a choice of intersection points which determines the configuration of perturbed hypertori.

**Remark 3.8.** In above we have chosen the perturbation for the fiber product to be the average of all configurations of the perturbed unstable hypertori of $x_i$. Indeed, we can choose other perturbations, which gives different expressions of the (non-equivariant) disc potential.

Take $\mathbb{P}^1$ as an example. Let $L$ be the equator. Let’s perturb $x$ in the counterclockwise order with respect to the left hemisphere. Then the non-equivariant disc potential will read

$$W = T^{A/2} \left( \frac{1}{1 - x} + (1 - x) \right)$$

instead of the well-known expression $T^{A/2} (e^x + e^{-x})$.

In the Fano case, $n_1(\beta_i + \alpha) = 0$ whenever $\alpha \neq 0$ by dimension reason. Moreover $n_1(\beta_i) = 1$ [CO06]. More generally, the (non-equivariant) disc potential $W_{\text{disc}}$ for compact semi-Fano toric manifold has been computed by [CLLT] using Seidel representations. The coefficients are given by the inverse mirror map. We recall it in the following theorem.

**Theorem 3.9 ([CLLT]).** Let $X$ be a compact semi-Fano toric manifold. Then

$$\sum_{\alpha \in C^1(\alpha) = 0} n_1(\beta_i + \alpha) T^{\omega(\alpha)} = \exp(g_i(\tilde{q}(q))),$$

where

$$g_i(\tilde{q}) := \sum_C \frac{(-1)^{(D_i \cdot C)}((-D_i \cdot C) - 1)!}{\prod_{j \neq i} (D_j \cdot C)!} \tilde{q}^C,$$

(3.7)

the summation is over all effective curve classes $C \in H^2_{\text{eff}}(X)$ satisfying $-K_X \cdot C = 0, D_i \cdot C < 0$ and $D_j \cdot C \geq 0$ for all $j \neq i$. 


and \( \tilde{q} = \tilde{q}(q) \) is the inverse of the mirror map \( q = q(\tilde{q}) \).

A similar result also holds for toric semi-Fano Gorenstein orbifold [CCL+]. However, we stick with the manifold case for simplicity. Moreover, compactness of \( X \) can be replaced by requiring \( X \) to be semi-projective, so that the disc moduli spaces are compact.

A toric manifold \( X \) is said to be semi-projective if it has a torus-fixed point, and the natural map \( X \to \text{Spec} H^0(X, O_X) \) is projective [CLS11, Section 7.2]. Combining, we get the following.

**Corollary 3.10.** For a toric fiber of a semi-projective and semi-Fano toric manifold, the equivariant disc potential equals to

\[
W_{\text{equiv}} = \sum_{i=1}^{m} \exp(g_i(\tilde{q}(q))) T^{\omega_X(\beta_i)} \exp(v_i \cdot (x^1, \ldots, x^d)) + \sum_{i=1}^{d} x^i \lambda_i
\]

where \( g_i(\tilde{q}(q)) \) is given by the inverse mirror map in Equation (3.7).

4. **\( S^1 \)-equivariant disc potential for the immersed 2-sphere**

In this section, we consider the \( S^1 \)-invariant immersed two-sphere \( S^2 \) with a single nodal point. This is a typical singular fiber in a Lagrangian torus fibration, which has an important application in the SYZ mirror construction.

First, we make the following observation which is important for finding isomorphisms between objects in the Fukaya category.

**Lemma 4.1.** Let \( L_1, L_2 \) be two graded Lagrangians in the Fukaya category that cleanly intersect with each other, and \( \alpha \in CF^0(L_1, L_2) \). Suppose that \( CF^j(L_1, L_2) = 0 \) for all \( j < 0 \). For any \( b_i \in CF^1(L_i, \Lambda_+) \) and \( w_i \in \Lambda_+ \),

\[
m_1^{b_1 + w_1, 1_i^{b_2 + w_2}}(\alpha) = m_1^{b_1, b_2}(\alpha).
\]

Similarly, if further \( CF^j(L_2, L_1) = 0 \) for all \( j < 0 \) and \( \beta \in CF^0(L_2, L_1) \), then

\[
m_2^{b_1 + w_1, 1_i^{b_2 + w_2}}(\beta, \alpha) = m_2^{b_1, b_2}(\beta, \alpha).
\]
Proof. This is simply due to degree reason. The degree of $1_{L_i}$ is $(-1)$. The extra terms are $m_k^{b_1, b_2, . . . , b_k}(1_{L_1}, . . . , 1_{L_i}, \alpha, 1_{L_{i+1}}, . . . , 1_{L_k}) \in CF^*(L_1, L_2)$ for $k \geq 2$, which has degree $2 - k + (-1) \cdot (k - 1) = 3 - 2k < 0$ and hence must vanish. The extra terms in $m_2^{\beta_1 + w_1 \cdot 1_{i_1}, \beta_2 + w_2 \cdot 1_{i_2}}(\alpha, \beta)$ also vanish for the same degree reason.

By the above lemma, it suffices to check $m_1^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\alpha) = 0$, $m_1^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\beta) = 0$, $m_2^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\alpha) = 1_{L_1}^{\gamma_1} + m_1^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\gamma_1)$ for some $\gamma_1$, $m_2^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\beta) = 1_{L_2}^{\gamma_2} + m_1^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\gamma_2)$ for some $\gamma_2$.

An isomorphism pair between $(L_1, b_1 + w_1 \cdot 1_{i_1})$ and $(L_2, b_2 + w_2 \cdot 1_{i_2})$, where $b_1 + w_1 \cdot 1_{i_1}$ are weak bounding cochains, consists of $\alpha \in CF^0(L_1, L_2)$ and $\beta \in CF^0(L_2, L_1)$ satisfying $m_1^{b_1 + w_1 \cdot 1_{i_1}, b_2 + w_2 \cdot 1_{i_2}}(\alpha, \beta) = 0$.

The above statements have a direct $G$-equivariant analog by replacing $m_p$ by $m_p^G$ and $w_i = w_i^0 + \sum_\lambda \varphi_\lambda \cdot \lambda$ where the summation is over degree two equivariant parameters $\lambda$, $w_i^0, \varphi_\lambda \in \Lambda$.

We consider the following immersed sphere $S^2$ in $X := \{(a, b, z) \in \mathbb{C}^2 \times \mathbb{C}^\times \mid ab = 1 + z\}$.

Equip $X$ with the symplectic form inherited from the standard symplectic form on $\mathbb{C}^3$. Let $\Pi: X \to \mathbb{C}^\times$ be the projection to the $z$-component. Regarding $S^1$ as the unit circle in the complex plane $\mathbb{C}$, the space $X$ has a fiberwise Hamiltonian $S^1$-action given by $\eta \cdot (a, b, z) \to (\eta \cdot a, \eta^{-1} \cdot b, z)$. One obtains the special Lagrangian torus fibration $\left(\sqrt{|a|^2 - |b|^2}^2, |z|\right)$ (4.1) with respect to the holomorphic volume form $\Omega_X = (da \wedge db)/z$, see [HL82, Gol01, Gro01]. In particular, a singular fiber which will be denoted by $S$ occurs over the point $(|a|^2 - |b|^2 = 0, |z| = 1)$ and intersects transversally at the point $(0, 0, -1)$.

Following the construction in Section 2.3, we construct the Borel model for $S$ and its finite dimensional approximations. In this case, the clean intersection $F \subset S$ is simply a point and invariant under the $S^1$-action. Setting $T := S^1$ and $\tilde{L} := S^2$, we then have the immersion $\tilde{L}_N := \tilde{L} \times T EG(N) \to X_N$ (4.2) cleanly intersecting at $\mathcal{I}_T(N) = BT(N) = CP^N$ for each $N \in \mathbb{N}$. Each finite dimensional approximation can be understood by toric geometry topologically. For instance, the domain of the lowest dimensional approximation (4.2) is regarded as the Hirzebruch surface $F_1 \to CP^1$ whose fiber is $\tilde{L}$, see Figure 6 (b).

We fix a perfect family Morse theory on $\tilde{L}_T \to BT$, which has generators $1_{e_2} \otimes \lambda^\ell$ and $pt_{e_2} \otimes \lambda^\ell$ for $\ell \geq 0$. Recall that $L$ is special with respect to $\Omega_X$ in $X$. We take a holomorphic volume form $\Omega_{BG(N)}$ on the base $T^*BG(N)$ which makes the zero section is special. We
take a positive linear combination $\Omega_N$ of $\Omega_X$ and $\Omega_{BG(N)}$ as a holomorphic volume form of the $X$-bundle over $T^*BG(N)$.

**Lemma 4.2.** The Lagrangian $\bar{L}_N$ in (4.2) is special with respect to $\Omega_N$. In particular, $\bar{L}_N$ is $\mathbb{Z}$-graded.

![Comparison of three Lagrangians and constant disc contribution](image)

**Figure 6.** Comparison of three Lagrangians and constant disc contribution.

Following Section 2.5, we produce the $T$-equivariant $A_X$-algebra $(C^*(L_T, \Lambda_0), m_0)$ associated to $L$. Note that there are immersed generators $U \otimes \lambda^\ell_T, V \otimes \lambda^\ell_T$ of $L_N$ where $\lambda^\ell_T$ for $\ell \geq 0$ labels critical points of the Morse function on $I_T = \mathbb{CP}^\ell$. By abuse of notations, degree one immersed generators of $L_T$ will be simply denoted by $U, V$.

**Lemma 4.3.** The formally deformed immersed Lagrangian $(S_T, b = uU + vV)$ is weakly unobstructed.

**Proof.** Since $S_T$ has minimal Maslov index $\geq 2$, $m_0^{S_T, b}$ can be expressed as

$$m_0^{S_T, b}(1) = m_0^{S_T}(1) \otimes 1_{BT} + W(u, v) \cdot 1_S \otimes \lambda$$

by the same argument in Proposition 3.3. Unlike the smooth case, extra care for $m_0^{S_T, b}$ is necessary because constant discs might bubble off at the immersed loci. Consider the anti-symplectic involution $\iota: X_T \rightarrow X_T$ given by $(a, b, z) \mapsto (\overline{b}, \overline{a}, \overline{z})$, which acts on $X_N$ fiberwise and swaps two immersed generators $U$ and $V$. In [HKL], by taking orientations respecting the involution, the constant disc contributions are canceled out. Thus $(S, b = uU + vV)$ is unobstructed, that is, $m_0^{S_T, b}(1) = 0$. Therefore,

$$m_0^{S_T, b}(1) = W(u, v) \cdot 1_S \otimes \lambda.$$

More precisely, the $1_S \otimes \lambda$ appeared above is $\lambda^\ast$ in the notation of Section 2.4. By Lemma 2.17, the above $b$ can be modified (by adding terms involving $1^\ast$) to $b'$ such that $m_0^{S_T, b'}(1) = W^\ast(u, v) \cdot \lambda^\ast$. By abuse of notations, we shall simply replace $b$ by $b'$ and denote it by $b$ again, $\lambda^\ast$ by $\lambda$, $W^\ast$ by $W$, and $1^\ast$ by 1.
Remark 4.4. This line of thoughts is similar to the work in [HKL]. In this example by the maximal principle we do not have non-constant holomorphic discs. However, there are still very interesting non-trivial contributions coming from constant polygons with corners at $U, V$. Note that for each constant polygon, $U$ and $V$ corners must occur in pair (to go back to the original branch). This is the reason that the Floer theory is convergent for $\{ (u,v) \in A^2_0 : \text{val}(uv) > 0 \}$.

In order to compute $W(u,v)$, we compare $S$ with the $T$-equivariant theory of Chekanov and Clifford tori $L^{(1)}$ and $L^{(2)}$ respectively. For the special Lagrangian torus fibration (4.1), the Lagrangian torus over a point in the chamber $|z| < 1$ (resp. $|z| > 1$) is called Chekanov (resp. Clifford) type. Since any pair of distinct fibers in (4.1) does not intersect, they cannot be isomorphic. We apply Lagrangian isotopies to Chekanov torus and Clifford torus without intersecting the wall $z = 0$ to make them intersect as in Figure 6 (a). Those isotoped Lagrangians are still $Z$-graded and called Chekanov torus and Clifford torus respectively.

The equivariant theory for the tori $L^{(i)}$ is understood in the same way as in the toric case above, except that in this local case $L^{(i)}$’s do not bound any non-constant holomorphic disc.

We equip $L^{(i)}$ with the non-trivial spin structure along the $T$-orbit direction. This will give extra systematic contributions to the orientation bundle of the moduli space of strips bounded by $L^{(i)}$ and other Lagrangians. The reason for doing this will be seen below.

Fix a perfect Morse function on each $L^{(i)}$, such that the (compactified) unstable submanifolds of the degree one generators $X^{(i)}, Y^{(i)}$ are the hypertori chosen in [HKL]. The unstable submanifold of $X^{(i)}$ is transverse to the $T = S^1$ orbits in $L^{(i)}$. We also have a perfect family Morse theory for $L^{(i)}_T$ as above, whose critical points are labeled by $\eta \otimes (\lambda^{(i)})^t$ where $\eta = 1^{(i)}, X^{(i)}, Y^{(i)}, X^{(i)} \wedge Y^{(i)}$. The formal deformations are taken as $b^{(i)} = x^{(i)}X^{(i)} + y^{(i)}Y^{(i)}$ (where $X^{(i)}, Y^{(i)}$ denotes $X^{(i)} \otimes 1^{(i)}, Y^{(i)} \otimes 1^{(i)}$). By Proposition 3.3, the equivariant disc potential equals to

$$W^{(i)} = x^{(i)}\lambda^{(i)}.$$ 

Since $L^{(i)}$ does not bound any non-constant Maslov-zero disc, $(W^{(i)})^\circ = W^{(i)}$ and again we will denote $(\lambda^{(i)})^\circ$ by $\lambda^{(i)}$.

Lemma 4.5. The three formally deformed Lagrangians $(L^{(1)}, b^{(1)}), (L^{(2)}, b^{(2)})$, and $(S, b)$ with a non-standard spin structure along the $T$-orbit direction are isomorphic if

$$\begin{align*}
uv &= 1 - \exp x^{(i)}, \quad \text{for } i = 1, 2 \\
u &= \exp y^{(1)}, \\
v &= \exp (-y^{(2)}).
\end{align*}$$

(4.3)

Proof. $L^{(i)}$ intersects with $S$ at two circles $C^{(i)}_j$ for $j = 1, 2$, which are invariant under the $T$-action (which is free on $C^{(i)}_j$). Thus we have the equivariant spaces $(C^{(i)}_j)_T$, which are the clean intersections (denoted by $a$ and $b$) between $L^{(i)}_T$ and $S_T$ in $X_T$. We have the morphisms $a_k \otimes \lambda^{(i)}_{C^{(i)}_1}, \beta_{k+1} \otimes \lambda^{(i)}_{C^{(i)}_2}$ for $k = 0, 1$ from $L^{(i)}$ to $S$, and $\beta_k \otimes \lambda^{(i)}_{C^{(i)}_2}, \beta_{k+1} \otimes \lambda^{(i)}_{C^{(i)}_1}$ for $k = 0, 1$ from $S$ to $L^{(i)}$ where $a_k$’s and $\beta_k$’s are depicted in Figure 6 (a) and the subscripts denote the degrees. They are given by critical points of a perfect family Morse theory on
Theorem 4.6. The equivariant potential function of the immersed sphere $S^2$ is

$$W = \log(1 - uv) = - \sum_{j=1}^{\infty} \frac{(uv)^j}{j}.$$

Proof. Consider the $A_\infty$ equation

$$((m_1^T)^{b, b^{(1)}}, b^{(1)})^2(\alpha_0) = \pm (m_2^T)^{b, b^{(1)}}, b^{(1)})(m_0^T)^{L^{(1)}, b^{(1)}}, (\alpha_0) \pm (m_2^T)^{b, b^{(1)}}, b^{(1)})(\alpha_0, (m_0^T)^{S, b}).$$

(4.3) makes (LHS) zero. Hence

$$x^{(1)} \cdot (m_2^T)^{b, b^{(1)}}, b^{(1)}(\lambda^{(1)}, \alpha_0) \pm W(u, v) \cdot (m_2^T)^{b, b^{(1)}}, b^{(1)}(\alpha_0, \lambda) = 0.$$

On the other hand, we have

$$(m_2^T)^{b, b^{(1)}}, b^{(1)}(\lambda^{(1)}, \alpha_0) = (m_2^T)^{b, b^{(1)}}, b^{(1)}(\alpha_0, \lambda) = \alpha_0 \otimes \lambda_{c_1}$$

(recall that $\lambda$ and $\lambda^{(1)}$ are partial units). Hence we have

$$W(u, v) = \pm x^{(1)}.$$

By $\exp x^{(1)} = 1 - uv$, we have

$$W = \log(1 - uv) = - \sum_{j=1}^{\infty} \frac{(uv)^j}{j}.$$

Remark 4.7. Note that if we did not take the non-trivial spin structure, then the gluing formula reads $\exp x^{(1)} = -1 + uv$. However, the left hand side lies in $1 + \Lambda_+$, while the right hand side lies in $-1 + \Lambda_+$, and hence there is no common intersection. It means the formal deformations of $L_T^{(1)}$ in the trivial spin structure can be isomorphic with the formal deformations of $S_T$. This is why we take the torus $L^{(i)}$ equipped with the non-trivial spin structure in the very beginning.

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