MODULI SPACES OF TROPICAL CURVES OF HIGHER GENUS WITH MARKED POINTS AND HOMOTOPY COLIMITS

DMITRY N. KOZLOV

Abstract. The main characters of this paper are the moduli spaces $TM_{g,n}$ of rational tropical curves of genus $g$ with $n$ marked points, with $g \geq 2$. We reduce the study of the homotopy type of these spaces to the analysis of compact spaces $X_{g,n}$, which in turn possess natural representations as a homotopy colimits of diagrams of topological spaces over combinatorially defined generalized simplicial complexes $\Delta_g$, with the latter being interesting on their own right.

We use these homotopy colimit representations to describe a CW complex decomposition for each $X_{g,n}$. Furthermore, we use these developments, coupled with some standard tools for working with homotopy colimits, to perform an in-depth analysis of special cases of genus 2 and 3, gaining a complete understanding of the moduli spaces $X_{2,0}$, $X_{2,1}$, $X_{2,2}$, and $X_{3,0}$, as well as a partial understanding of other cases, resulting in several open questions and in further conjectures.

1. Moduli spaces of tropical curves

Tropical geometry is a fairly recent new field within the broader context of algebraic geometry. During the time which elapsed since its inception, tropical geometry has already developed its language and its methods, and has furthermore found numerous applications; we refer the interested reader to [St02, Chapter 9], and more recently to [DFS07, DY07, Mi06], for both applications and the general background information.

Some spaces arising in tropical geometry are of interest from the point of view of algebraic topology as well. Often these have natural definitions and fit well in more general structures. One such instance is furnished by the moduli spaces of rational tropical curves of genus $g$ with $n$ marked points $TM_{g,n}$, which were introduced by Mikhalkin in [Mi07], see also [Ko08a] for a purely topological definition.

Prior to this work, only the case of genus 1 has been studied systematically, see [Ko08a, Ko08b], where, e.g., the homology groups with coefficients in $\mathbb{Z}_2$ were computed for this family of spaces. In this paper, we present an in-depth analysis of the moduli spaces of rational tropical curves of higher genus.

As a first step, we complement the known shrinking bridges strong deformation retraction, leading from $TM_{g,n}$ to $TM^b_{g,n}$ as described in [Ko08a, Section 3], by a further new simplification. On the intuitive level, that newly discovered strong deformation retraction increases the edge lengths proportionally, to reach the length of the longest one, stopping when the edges which are strictly shorter than the longest one form a forest. This process is described in detail in Section 2 where

2000 Mathematics Subject Classification. Primary: 57xx, secondary 14Mxx, 55xx.

Key words and phrases. Tropical geometry, combinatorial algebraic topology, moduli spaces, homotopy colimit, metric graphs.
we reduce the space $TM^b_{g,n}$ to $TM^e_{g,n}$ without changing the homotopy type (here $e$ stands for “equalisation”).

By global scaling of edge lengths, we can replace $TM^e_{g,n}$ by a compact space $X_{g,n}$, and then proceed with finding more structure in that new space. Hereafter, our main structural achievements are the representation of $X_{g,n}$ as a homotopy colimit of a diagram of topological spaces in Section 3, and the derivation from that representation of a CW complex decomposition of $X_{g,n}$ in Section 4.

There are several byproducts of that development. The main one is probably the discovery in Section 3 of a family of generalized simplicial complexes $\Delta_g$, where $g$ is any natural number. The vertices of $\Delta_g$ are indexed by the isomorphism classes of stable graphs of genus $g$, and, more generally, the simplices are indexed by filtrations by forests of these graphs.

Another byproduct is the introduction of cubical complexes (or, in some terminology, of generalized cubical complexes) $C(G, \pi)$ in Section 4. Each such complex is defined using a filtered by forests stable graph as the input data. The interesting question of connections between combinatorial properties of the input graphs and geometry of the corresponding cubical complexes arises in a natural way.

We then use in Section 5 and in Section 6 the previous developments to analyze the cases of genus 2 and 3. In the case when genus is equal to 2, we completely understand the topology of moduli spaces when the number of marked points is 0, 1, or 2. Furthermore, using an Euler characteristic formula derived using some standard enumeration under group action techniques, we can show that these moduli spaces are almost never contractible.

Due to explosion in complexity, much less can be said in the case of genus equal to 3. The main results here are the collapsibility of the generalized simplicial complex $\Delta_3$, and the determination of the asymptotics (with respect to the number of marked points $n$) of the Euler characteristic of $X_{g,n}$. The latter leads then to a conjecture concerning all genera $g$.

2. An “equalizing” deformation retraction

To start with, a few words on the terminology are in place. For a poset $P$, we denote the order complex by $\Delta(P)$. For a graph $G$ we denote by $\Delta(G)$ the corresponding CW complex, cf. [Ko08a]. For a generalized simplicial complex $K$, we denote by $\mathcal{F}(K)$ its face poset. Recall that $\Delta(\mathcal{F}(K)) \cong \text{Bd} K$, see e.g., [Ko07, p. 160, (10.4)].

We shall always work with finite undirected connected graphs only. Furthermore, graphs satisfying the following additional properties will play special role in this paper.

**Definition 2.1.** A graph $G$ is called **stable** if

- it has no bridges,
- no vertex of $G$ has valency 2, unless it is adjacent to a loop.

Due to the space constraints, we do not define the spaces $TM_{g,n}$ and $TM^b_{g,n}$ here, but rather refer to [Mi07] and [Ko08a].

2.1. Filtered graphs

**Definition 2.2.** Given a finite set $S$, an **ordered set partition** of $S$ is an ordered tuple $\pi = (S_1, \ldots, S_t)$, such that $S$ is a disjoint union of the subsets $S_i$. 

A standard situation in which ordered set partitions arise is when we have a function \( \varphi : S \to \mathbb{R} \), and we let \( S_i \) be the non-empty preimages \( S_i := \varphi^{-1}(x_i) \), \( x_i \in \mathbb{R} \), for \( i = 1, \ldots, m \), such that \( x_1 < x_2 < \cdots < x_m \).

**Definition 2.3.**

1. A **filtered graph** is a pair \((G, \pi)\), where \( G \) is a graph and \( \pi = (E_1, \ldots, E_m) \) is an ordered set partition of \( E(G) \). We shall call \( m \) the **depth** of the filtration (or of the filtered graph).
2. For such a filtered graph \((G, \pi)\) we shall say that \( G \) is **filtered by forests** if the subgraph induced by the edges \( E_1 \cup \cdots \cup E_{m-1} \) is a forest (i.e., contains no cycles). By convention, this condition is considered to be satisfied in the case \( m = 1 \).

We remark that condition that \( E_1 \cup \cdots \cup E_{m-1} \) induces a forest is equivalent to the condition that \( E_1 \cup \cdots \cup E_j \) induces a forest for all \( j = 0, \ldots, m - 1 \).

**Definition 2.4.** A graph homomorphism \( \varphi : G \to G' \) is called a **filtered graph homomorphism** if for all \( e_1, e_2 \in E(G) \) we have the implication

\[
\rho(e_1) \leq \rho(e_2) \Rightarrow \rho'(\varphi(e_1)) \leq \rho'(\varphi(e_2)).
\]

In other words, the function \( \varphi \) preserves the (non-strict) partial orders on \( E(G) \) and \( E(G') \) induced by the ordered set partitions \( \pi \) and \( \pi' \).

Now, the objects of \( \text{FGraphs} \) are precisely all filtered graphs, and morphisms are filtered graph homomorphisms. As usual in category theory we call the invertible morphisms the **isomorphisms**. This gives the notion of **isomorphic filtered graphs**. Being isomorphic is an equivalence relation, and thus we have a notion of **isomorphism classes** of filtered graphs. Also, for every filtered graph \((G, \pi)\) we get an automorphism group \( \text{Aut}(G, \pi) \), which consists of all isomorphisms of \((G, \pi)\) with itself.

Before proceeding we would like to remark that omitting the order, and considering pairs \((G, \pi)\), where \( G \) is a graph, and \( \pi \) is a usual set partition of \( E(G) \), will yield a parallel concept of **partitioned graphs**. A **partitioned graph homomorphism** between \((G, \pi)\) and \((G', \pi')\) is a graph homomorphism \( \varphi : G \to G' \), such that for any two edges \( e_1, e_2 \in E(G) \) the following condition is satisfied: if \( e_1 \) and \( e_2 \) belong to the same block of \( \pi \), then \( \varphi(e_1) \) and \( \varphi(e_2) \) belong to the same block of \( \pi' \). Accordingly, taking partitioned graphs as objects, and partitioned graph homomorphisms as morphisms, we get a category \( \text{PGraphs} \). Furthermore, dropping the order of blocks in an ordered set partition induces a forgetful functor from \( \text{FGraphs} \) to \( \text{PGraphs} \).

To an arbitrary metric graph \((G, l_G)\), the length function \( l_G : E(G) \to (0, \infty) \) at hand allows us to associate a filtered graph \((G, \pi(G))\). This is done by considering an ordered partition \( \pi(G) = (E_1, \ldots, E_m) \) of the set \( E(G) \) defined as follows:

- for every \( 1 \leq i \leq m \), all the edges in \( E_i \) have the same length, which we call \( l(E_i) \);
- we have \( 0 < l(E_1) < \cdots < l(E_m) \).
If this associated filtered graph is actually filtered by forests, then we shall say that the original metric graph is filtered by forests as well.

**Definition 2.5.** The topological space $TM_{g,n}^b$ is the subspace of $TM_{g,n}^b$ consisting of the points whose representative metric graphs are filtered by forests.

Our next goal is to show that as far as topology is concerned, it is enough to consider the smaller space $TM_{g,n}^c$.

### 2.2. The strong deformation retraction from $TM_{g,n}^b$ to $TM_{g,n}^c$.

We start by describing an explicit deformation $\Phi : TM_{g,n}^b \times [0, 1] \to TM_{g,n}^b$. Let $x$ be a point in $TM_{g,n}^b$, represented by a metric graph $(G, l_G)$, the corresponding ordered partition $\pi(G) = (E_1, \ldots, E_m)$, and the marking function $p_G : [n] \to \Delta(G)$. Let $k$ denote the maximal index, such that the graph induced by $E_1 \cup \cdots \cup E_k$ is a forest, i.e., has no cycles. If $E_1$ has a cycle, then we set $k := 0$.

Informally speaking, the deformation $\Phi(x, -)$ should proceed as follows:

- the lengths of the edges in $E_{k+1} \cup \cdots \cup E_{m-1}$ should increase so that the differences $l(E_m) - l(E_i)$, for $i = k + 1, \ldots, m - 1$, decrease proportionally, and eventually the lengths $l(E_i)$ become equal to $l(E_m)$ at time $1$;
- the lengths of the edges in $E_1 \cup \cdots \cup E_k$ should increase proportionally to the length $l(E_{k+1})$;
- the positions of the points in the image of the marking function $p_G$ should also change proportionally to the increase of the lengths of the corresponding edges.

Formally, for $t \in [0, 1]$, the point $\Phi(x, t)$ is given by the representative $(G, l_{G}^t, p_{G}^t)$ which we now describe. To start with, the graph itself (without the metric information taken into account) is isomorphic to the graph $G$. The length function $l_{G}^t$ is different, though for $t < 1$ the corresponding ordered partition $\pi(G)^t = (E_1, \ldots, E_m)$ is the same, and we denote by $l_{G}^t$ the value of $l_{G}^t$ on the edges in $E_i$; for $t = 1$ the corresponding ordered partition is $\pi(G)^1 = (E_1, \ldots, E_k, E_{k+1} \cup \cdots \cup E_m)$, and we also use the notation $l_{G}^1$, keeping in mind that $l_{G}^1_{k+1} = \cdots = l_{G}^1_m$.

More specifically, the values $l_{G}^t_i$ are given by the formulae:

\begin{align*}
(2.1) & \quad l_{G}^t_i := l_i + t(l_m - l_i), \quad \text{for } k + 1 \leq i \leq m, \\
(2.2) & \quad l_{G}^t_i := l_i(1 + t(l_m/l_{k+1} - 1)), \quad \text{for } 1 \leq i \leq k.
\end{align*}

Furthermore, if the point $p_G(i)$, for $i \in [n]$, is a vertex of $G$, then $p_{G}^t(i) = p_G(i)$, for all $t \in [0, 1]$; else $p_G(i)$ is an internal point of some edge $e$ of $G$, in this case, the point $p_{G}^t(i)$ belongs to the same edge, and has the same position relative to the length $l_{G}^t(e)$.

**Theorem 2.6.** The function $\Phi : TM_{g,n}^b \times [0, 1] \to TM_{g,n}^b$ defined above is a strong deformation retraction from $TM_{g,n}^b$ to $TM_{g,n}^c$.

**Proof.** First, let the point $x \in TM_{g,n}^b$ be represented by a metric graph $(G, l_G)$, and let the corresponding ordered partition be $\pi(G) = (E_1, \ldots, E_m)$. When $k$ is the maximal index such that $E_1 \cup \cdots \cup E_k$ has no cycles, it follows from (2.1) that $l_{G}^1_{k+1} = \cdots = l_{G}^1_m$, and hence $\pi(G)^1 = (E_1, \ldots, E_k, E_{k+1} \cup \cdots \cup E_m)$, in particular, the point $\Phi(x, 1)$ lies in $TM_{g,n}^c$. 

On the other hand, if that point $x$ lied in $TM_{g,n}^\varepsilon$, to start with, then for the corresponding ordered partition $\pi(G) = (E_1, \ldots, E_m)$ we would have that $E_1 \cup \cdots \cup E_{m-1}$ has no cycles, hence $k = m - 1$. It then follows from the formulae (2.1) and (2.2) that $\Phi(x, t) = x$, for all $t \in [0, 1]$.

It remains to show that the map $\Phi$ is continuous. Let a point $y \in TM_{g,n}^\varepsilon$ be represented by a metric graph $(G, l_G)$. Choose a point $x \in TM_{g,n}^b$ and a number $t \in [0, 1]$, such that $\Phi(x, t) = y$. We know that the point $x$ is represented by a metric graph $(G, l_G^x)$, and denote the corresponding ordered partition of $E(G)$ by $\pi(G) = (E_1, \ldots, E_m)$. Let $k$ as before denote the maximal index such that the graph $E_1 \cup \cdots \cup E_k$ has no cycles, and we set $l_i := l_i(E_i)$, for $i = 1, \ldots, m$.

Let $\varepsilon > 0$ be an arbitrary, sufficiently small number (say, much smaller than all the non-zero distances between vertices and marked points on $G$, in 

we used the terminology “the admissible range”), and consider the $\varepsilon$-neighborhood $N_\varepsilon(y)$. To show continuity of $\Phi$ we need to find a $\delta$-neighborhood

$$
\bar{N}_\delta := N_\delta(x) \times ((t - \delta, t + \delta) \cap [0, 1])
$$

of $(x, t) \in TM_{g,n}^b \times [0, 1]$, such that $\Phi(\bar{N}_\delta) \subseteq N_\varepsilon(x)$. We claim, that to do this, it is enough to choose $\delta > 0$ so that the inequality

$$
\delta \cdot \frac{l_m}{l_{k+1}} < \varepsilon
$$

is satisfied. Note that the inequality does not depend on $t$.

Indeed, pick a point $\hat{x} \in N_\delta(x)$. By our definition of $TM_{g,n}^b$ as a topological space, which is given in detail in 

Section 3, it can be represented by the metric graph $(\hat{G}, l_{\hat{G}})$ together with the marking function $\hat{p}_G$, such that

- the subgraph induced by the set $\Sigma$ of the edges whose length is less than $\delta$ has no cycles;
- shrinking all the edges from $\Sigma$ inside $\hat{G}$ yields a graph isomorphic to $G$;
- the lengths of all the other edges of $\hat{G}$ differ from the lengths of the corresponding edges of $G$ by at most $\delta$;
- the positions of the marked points, after the dilations of the edges they belong to, have also changed by at most $\delta$.

It is now clear, that the point $\Phi(\hat{x}, \hat{t})$ lies in the $\varepsilon$-neighborhood of $y$, for any $\hat{t} \in [0, 1]$. Indeed, the point $\Phi(\hat{x}, \hat{t})$ is represented by the graph isomorphic to $\hat{G}$, with edges stretched by at most the factor $l_m/l_{k+1}$. This shows that the edges of length less that $\delta$ are precisely the ones which map to the edges of lengths less than $\varepsilon$. After they are shrunk we end up with a graph which is isomorphic to $G$. Also, the lengths of other edges in the graph representing $\Phi(\hat{x}, \hat{t})$ differ from the lengths of the corresponding edges in $G$ by at most $\delta l_m/l_{k+1}$, which by our construction is less than $\varepsilon$. Finally, the positions of the marked points also did not change by more than $\varepsilon$. Summarizing, we can conclude that $\Phi(\bar{N}_\delta) \subseteq N_\varepsilon(x)$, and hence the function $\Phi$ is continuous. $\square$

The space $TM_{g,n}^\varepsilon$ can be further simplified by normalizing the lengths of the longest edges.

**Definition 2.7.** Let $X_{g,n}$ denote the subset of $TM_{g,n}^\varepsilon$ consisting of points whose representing metric graph $G$ has longest edges of length 1.
In our terminology above, if \( \pi(G) = (E_1, \ldots, E_m) \) is the ordered partition corresponding to \( G \), then the condition in Definition 2.7 says that \( l(E_m) = 1 \). Scaling metric graphs so that their maximal edge length becomes equal to 1 yields homeomorphisms

\[
TM^e_{g,n} \cong X_{g,n} \times (0, \infty) \cong X_{g,n} \times \mathbb{R},
\]

for all integers \( g \geq 1, \ n \geq 0 \). Since (2.4) implies that the tropical moduli space \( TM_{g,n} \) is homotopy equivalent to \( X_{g,n} \). From now on, we shall only work with the latter.

**Remark 2.8.** In [Ko08a] and [Ko08b] the author defined and studied two further variations of the space \( TM_{g,n} \). These are: \( MG_{g,n} \) - the moduli spaces of all metric graphs of genus \( g \) with \( n \) marked points, and \( MG^e_{g,n} \) - the subspace with the additional condition that the marked points must be vertices. We notice here that Theorem 2.6 and its proof hold ad verbatim in these generalized situations as well.

### 3. A stratification and homotopy colimit presentation

In this section we shall describe how the tropical moduli space can be replaced, up to homotopy equivalence, by a manageable compact space, which has a nice presentation as a homotopy colimit over a certain combinatorially defined generalized simplicial complex.

#### 3.1. The generalized simplicial complex of filtered by forests stable graphs of genus \( g \)

**Definition 3.1.** For an integer \( g \geq 2 \), let \( \Sigma_g \) denote the set of all isomorphism classes of filtered by forests stable graphs \((G, \pi)\), where \( G \) has genus \( g \).

Recall, that since the depth of isomorphic filtered graphs must be the same, the notion of depth is well-defined for the isomorphism classes as well.

**Definition 3.2.** For an arbitrary integer \( g \geq 2 \) the generalized simplicial complex \( \Delta_g \) is defined as follows:

- the simplices: the \( m \)-simplices of \( \Delta_g \) are indexed by the elements of \( \Sigma_g \) of depth \( m + 1 \);
- the boundary relation: for an \( m \)-simplex \( \sigma \in \Delta_g \), \( m \geq 1 \), let \((G, \pi)\) be a representative of the indexing element, say \( \pi = (E_1, \ldots, E_{m+1}) \), then, the representatives indexing the simplices on the boundary of \( \sigma \) are obtained by
  - shrinking the edges from \( E_1 \) in \( G \), and replacing \( \pi \) with \((E_2, \ldots, E_{m+1})\), or
  - keeping the graph \( G \) intact, and merging two neighboring blocks \( E_i \) and \( E_{i+1} \), for \( i = 1, \ldots, m \), in \( \pi \).

It is easy to see that the boundary relation in the generalized simplicial complex \( \Delta_g \) is well-defined by the description above. Indeed, if \((G, \pi)\) is a filtered by forests stable graph, where \( \pi = (E_1, \ldots, E_{m+1}) \), then

- merging blocks \( E_i \) and \( E_{i+1} \), for \( i = 1, \ldots, m - 1 \), does not change the union of all blocks without the last one, hence the subgraph induced by that union remains being a forest;
- merging blocks \( E_m \) and \( E_{m+1} \) replaces the union \( E_1 \cup \cdots \cup E_m \) with the union \( E_1 \cup \cdots \cup E_{m-1} \), which is a forest as well;
shrinking a subposet inside a forest still yields a forest, hence \( E_2 \cup \cdots \cup E_m \) induces a forest in \( G/E_1 \).

As a special case, the vertices of \( \Delta_g \) are indexed by the isomorphism classes of filtered by forests stable graphs \((G, \pi)\), where \( \pi \) is a 1-tuple \((E(G))\), which unwind all conditions, simply translates to considering the isomorphism classes of stable graphs of genus \( g \). The edges of \( \Delta_g \) correspond to taking graphs like that, choosing a forest, and then identifying those forests which map to each other under graph automorphisms. Vertices of every simplex of \( \Delta_g \) can be linearly ordered by the numbers of vertices in their indexing stable graphs. This yields a standard orientation on all the simplices of \( \Delta_g \), though we will not need that orientation in our analysis.

For \( g = 1 \) we make the convention that the generalized simplicial complex \( \Delta_1 \) consists of a single point, which corresponds to the graph with one vertex and a loop at that vertex. This is consistent with our definition of a stable graph.

For \( g = 2 \) we have two isomorphism classes of stable graphs of genus \( g \): a graph consisting of one vertex with two loops attached, and a graph consisting of two vertices connected by three edges. In the first case there are no non-empty forests. In the second case, a non-empty forest is given by taking one of the edges, and this choice is unique up to graph automorphisms. Thus, the generalized simplicial complex \( \Delta_2 \) is simply a 1-simplex, see Figure 5.1.

The case \( g = 3 \) is more complicated, see Figures 6.2 and 6.3. It will be shown in Section 6 that the complex \( \Delta_3 \) is collapsible.

Let us now look at some properties of generalized simplicial complex \( \Delta_g \) valid for all \( g \). First of all \( \Delta_g \) connected. To see this, let \( v_l \) denote the vertex which is indexed by the graph \( L \) with one vertex and \( g \) loops. Let \( w \) be some other vertex of \( \Delta_g \). The edges connecting \( w \) with \( v_l \) correspond to various spanning trees in the graph \( H \), with two trees \( T_1 \) and \( T_2 \) corresponding to the same edge if and only if there exists a graph automorphism of \( H \) which transforms \( T_1 \) to \( T_2 \). Since at least one spanning tree always exists, there is at least one edge connecting \( w \) with \( v_l \), in particular, the complex \( \Delta_g \) is connected. The example on the left hand side of Figure 5.1 shows that \( \Delta_g \) is a simplicial complex only in cases \( g = 1, 2 \), while the example on the right hand side of Figure 5.1 shows that there could be arbitrary many edges between vertices for higher genera.

**Proposition 3.3.** For arbitrary integer \( g \geq 1 \), the generalized simplicial complex \( \Delta_g \) is pure. It has dimension 0 for \( g = 1 \), and \( 2g - 3 \) for \( g \geq 2 \).

**Proof.** The statement is true for \( g = 1 \), so we assume that \( g \geq 2 \). Consider a maximal simplex \( \sigma \) of \( \Delta_g \), say it is represented by a filtered by forests stable graph \((G, \pi)\), where \( \pi = (E_1, \ldots, E_m) \). By our construction, the dimension of \( \sigma \) is \( m - 1 \). Since \( \sigma \) is maximal, and we are always allowed to split the first \( m - 1 \) blocks of \( \pi \) into smaller ones, we may assume that \(|E_1| = \cdots = |E_{m-1}| = 1\). Furthermore, since we are allowed to split of an edge from \( E_m \) as long as its union with the other blocks forms a forest, we may assume that \( E_1 \cup \cdots \cup E_{m-1} \) induces a spanning forest.

We now claim that all vertices of \( G \) have valency 3. Assume this is not the case, and take a vertex \( w \) which has valency at least 4. We consider two cases.

**Case 1.** There exists a loop adjacent to \( w \). Let \( l \) denote that loop, and let \( e_1, \ldots, e_t \) denote the other edges adjacent to \( w \). Here, by our assumptions, we have \( t \geq 2 \),
and loops (other than $l$) will appear twice in that list. Let $G'$ be a new graph, which is obtained from $G$ by
- replacing the vertex $w$ with new vertices $w_1$ and $w_2$;
- connecting the edges $e_1, \ldots, e_t$ to $w_1$, $e_t$ to $w_2$, and replacing the loop $l$ with two new edges $l_1$ and $l_2$, both connecting $w_1$ and $w_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.2}
\caption{The graph transformation used in Case 1 of the proof of Proposition 3.3}
\end{figure}

This transformation is shown graphically on Figure 3.2. We see that shrinking the edge $l_2$ in the graph $G'$ will yield a graph isomorphic to $G$, and that $G$ and $G'$ have the same genus. Furthermore, $G'$ has no bridges and no vertices of valency 2, so it is a stable graph. Setting $\pi' := (\{l_2\}, E_1, \ldots, E_m)$, we get a filtered stable graph $(G', \pi')$. The graph $G'/(\{l_2\}, E_1, \ldots, E_{m-1})$ is isomorphic to $G/(E_1, \ldots, E_{m-1})$, hence has genus $g$, implying that $\{l_2\}, E_1, \ldots, E_{m-1}$ induces a forest. Summarizing, we conclude that $(G', \pi')$ is a filtered by forests stable graph which is indexing a simplex $\tau$, such that $\dim \tau = \dim \sigma + 1$, and $\sigma \subset \tau$, contradicting the fact that $\sigma$ is a maximal simplex.

**Case 2.** All the edges adjacent to $w$ are not loops. Denote these edges $e_1, \ldots, e_t$, and assume that $e_i$ connects $w$ to a vertex $v_i$, for $i = 1, \ldots, t$. Let $C_1, \ldots, C_p$ denote...
the connected components of the graph obtained from $G$ by removing the vertex $w$ and all the adjacent edges. It is important to remark that, for all $i = 1, \ldots, p$, the valency of $w$ in the subgraph of $G$ induced by $C_i \cup \{w\}$ is at least 2, as otherwise the single edge adjacent to $w$ in $C_i \cup \{w\}$ would have been a bridge in the graph $G$. Let $e_1$ be an edge adjacent to $w$ in $C_1 \cup \{w\}$, and let $e_2$ be an edge adjacent to $w$ in $C_p \cup \{w\}$, see the left hand side of Figure 3.3; we might have $p = 1$, but that does not change the argument. Let now $G'$ be a new graph, which is obtained from $G$ by

- replacing the vertex $w$ with new vertices $w_1$ and $w_2$;
- connecting the edges $e_1$ and $e_2$ to $w_1$, $e_3, \ldots, e_t$ to $w_2$, and adding a new edge $e$ connecting $w_1$ and $w_2$.

Figure 3.3. The graph transformation used in Case 2 of the proof of Proposition 3.3

This transformation is shown on Figure 3.3. Set $\pi' := (\{e\}, E_1, \ldots, E_m)$. Essentially with the same proof as in the first case, we see that $(G', \pi')$ is filtered by forests stable graph, and that shrinking $e$ will yield $(G, \pi)$. The only fact which needs to be verified is that $e$ is not a bridge in $G'$. Indeed, on one hand, every vertex in $V(G') \setminus \{w_1, w_2\} = V(G) \setminus \{w\}$ is in the same connected component of $G' \setminus \{e\}$ as $w_2$, since it is in some $C_i$ and the valency of $w$ in $C_i \cup \{w\}$ is at least 2, hence at least one of these edges is different from $e_1, e_2$ (when $p = 1$, both $e_1$ and $e_2$ are adjacent to $w$, but then the valency of $w$ in $C_1 \cup \{w\}$ is at least 4). On the other hand $w_1$ is connected to two other vertices in $G' \setminus \{e\}$, hence is also in the same connected component of $G' \setminus \{e\}$ as $w_2$. We conclude again that $(G', \pi')$ is a filtered by forests stable graph which is indexing a simplex $\tau$, such that $\dim \tau = \dim \sigma + 1$, and $\sigma \subset \tau$, contradicting the fact that $\sigma$ is a maximal simplex.

Figure 3.4. The deformation of the graph $G$.

It is now easy to finish the proof. Assume $G$ has $v$ vertices and $e$ edges. Then we have $g = e - v + 1$. On the other hand, we have $2e = 3v$. It follows that
v = 2g − 2 and e = 3g − 3. Since \( E_1 \cup \cdots \cup E_{m-1} \) induces a spanning forest, we have \( m-1 = v-1 \), and it follows that \( \dim \sigma = 2g-3 \).

We are now ready for the main result of this subsection.

**Theorem 3.4.** For arbitrary integer \( g \geq 1 \), the generalized simplicial complex \( \Delta_g \) is homeomorphic to the space \( X_{g,0} \).

**Proof.** We shall define a map \( \rho : X_{g,0} \rightarrow \Delta_g \). Let a point \( x \in X_{g,0} \) be represented by a filtered by forests stable metric graph \((G, l_G)\), with the corresponding ordered partition \( \pi(G) = (E_1, \ldots, E_m) \). The point \( \rho(x) \) belongs to the simplex of \( \Delta_g \) indexed by the isomorphism class of \((G, \pi(G))\). Its vertices \( \{v_1, \ldots, v_m\} \) are indexed by the graphs \( G_i \), for \( i = 1, \ldots, m \), where \( G_i \) is obtained from \( G \) by shrinking all the edges in the set \( E_1 \cup \cdots \cup E_{i-1} \). The coordinate \( d_i \) of the vertex \( v_i \) is given by \( d_i := l_i - l_{i-1} \), where \( l_i = l(E_i) \), \( l_m = 1 \), and by convention \( l_0 := 0 \). Clearly, we have \( d_i > 0 \), for all \( i = 1, \ldots, m \), and \( d_1 + \cdots + d_m = l(E_m) = 1 \).

Let us check that the described map \( \rho : x \mapsto d_1 v_1 + \cdots + d_m v_m \) is indeed a homeomorphism \( \rho : X_{g,0} \rightarrow \Delta_g \). First, it is clearly bijective as the \( m \)-tuple transformation \((l_1, \ldots, l_m) \mapsto (l_1, l_2 - l_1, \ldots, l_m - l_{m-1}) \) has the inverse \((d_1, \ldots, d_m) \mapsto (d_1, d_1 + d_2, \ldots, d_1 + \cdots + d_m) \), which defines \( \rho^{-1} : \Delta_g \rightarrow X_{g,0} \). Second, both \( \rho \) and its inverse are continuous. Indeed, for \( x \in X_{g,0} \) represented by \((G, l_G)\), with the corresponding ordered partition \( \pi(G) = (E_1, \ldots, E_m) \), the representatives for the points in a small neighborhood are obtained by a combination of the following 3 steps:

1. changing edge lengths without splitting blocks: for some \( 1 \leq i \leq m-1 \), the edges in \( E_i \) get the length \( l_i + \varepsilon \) instead of \( l_i \), where \( \varepsilon \) is a small, not necessarily positive number;
2. splitting of a block: the set \( E_i \) gets split into a disjoint union of sets \( A \) and \( B \), with edges in \( A \) keeping the length \( l_i \), and edges in \( B \) getting the length \( l_i + \varepsilon \), where \( \varepsilon \) is a small positive number;
3. adding a block of short edges: we replace some of the vertices of \( G \) indexed by the graph obtained from \( G \) by shrinking \( E_1 \cup \cdots \cup E_{i-1} \cup A \) and edges in \( B \) getting the length \( l_i + \varepsilon \), whereas the vertex \( v_i \) gets coordinate \( \varepsilon \), whereas the vertex \( v_1 \) gets coordinate \( d_1 - \varepsilon \).

On the side of \( \Delta_g \) these 3 steps can be interpreted as follows. Step (1) corresponds to the case when \( v_1, \ldots, v_m \) stay the same, whereas coordinates \( d_i \) and \( d_{i+1} \) change to \( d_i + \varepsilon \) and \( d_{i+1} - \varepsilon \). Step (2) corresponds to the situation where we add a new vertex \( w \) indexed by the graph obtained from \( G \) by shrinking \( E_1 \cup \cdots \cup E_{i-1} \cup A \); this vertex gets coordinate \( \varepsilon \), whereas the vertex \( v_i \) gets coordinate \( d_i - \varepsilon \). Finally, step (3) corresponds to the situation, where we add a new vertex \( w \) indexed by the graph obtained from \( G \) by shrinking \( E \); this vertex gets coordinate \( \varepsilon \), whereas the vertex \( v_1 \) gets coordinate \( d_1 - \varepsilon \).

**3.2. A diagram over the face poset of \( \Delta_g \).**

For a finite set \( S \), let \( X_{g,S} \) denote the moduli space of graphs, which is just like \( X_{g,n} \), but the labels for the marked points are drawn from the set \( S \), i.e., the labeling function is \( p_G : S \rightarrow \Delta[G] \). In particular, \( X_{g,n} = X_{g,[n]} \). For any injective set map \( \iota : S \rightarrow T \), we get an induced map \( \iota : X_{g,T} \rightarrow X_{g,S} \), which takes every metric graph \( G \) to itself, but changes the old labeling function \( p_G : T \rightarrow \Delta(G) \) to the new one \( q_G : S \rightarrow \Delta(G) \) defined by \( q_G(s) := p_G(\iota(s)) \), for \( s \in S \).
Composition of two injective maps \( \iota_1 : S_1 \hookrightarrow S_2 \) and \( \iota_2 : S_2 \hookrightarrow S_3 \) corresponds to the composition of the induced maps \( \iota_1 : X_{g,S_2} \rightarrow X_{g,S_1} \) and \( \iota_2 : X_{g,S_3} \rightarrow X_{g,S_2} \), as we have \( \iota_2 \circ \iota_1 = \iota_1 \circ \iota_2 \). This means that the map \( \lambda : S \mapsto X_{g,S} \) yields a contravariant functor from the category \( \text{Inj} \) of finite sets and injective set maps to the category \( \text{Top} \) of topological spaces and continuous maps. The category \( \text{Inj} \) has an initial element \( \emptyset \), hence, for every finite set \( S \) we get the unique induced map \( \lambda(\emptyset \mapsto S) : X_{g,S} \rightarrow X_{g,0} \). Clearly, the induced map \( \lambda(\emptyset \mapsto S) \) is simply the forgetful map which "forgets" all the labels.

We have described above and proved in Theorem 3.1 that the space \( X_{g,0} \) has a natural structure of a generalized simplicial complex. This can also be viewed as a stratification of \( X_{g,0} \) by open simplices. It is then a standard construction to consider the stratification on \( X_{g,n} \) induced by that forgetful map: the strata are simply the preimages of the strata of \( X_{g,0} \) under \( \lambda(\emptyset \mapsto S) \).

To formulate the second main result of this section we need to recall some terminology of diagrams and homotopy colimits. We shall only consider simplified version which we need here, and refer the reader to [Ko07, Chapter 15] for a more complete coverage of the general situation.

**Definition 3.5.** Let \( P \) be a poset. A **diagram of topological spaces** over \( P \) is a functor \( \mathcal{D} : P \rightarrow \text{Top} \).

We shall now consider a diagram over the face poset \( \mathcal{F}(\Delta_g) \), to this end we define a functor \( \mathcal{D}_{g,n} : \mathcal{F}(\Delta_g) \rightarrow \text{Top} \). Let \((G, \pi)\) be a filtered by forests stable graph of genus \( g \), and let \( \sigma \in \Delta_0 \) be the simplex represented by \((G, \pi)\). We set

\[
\mathcal{D}_{g,n}(\sigma) := \Delta(G)^n / \text{Aut } (G, \pi),
\]

where \( \Delta(G)^n = \Delta(G) \times \cdots \times \Delta(G) \) denotes the \( n \)-fold direct product of the topological spaces \( \Delta(G) \), and the group action is the diagonal one:

\[
g : (x_1, \ldots, x_n) \mapsto (gx_1, \ldots, gx_n),
\]

for all \( x_1, \ldots, x_n \in \Delta(G) \), \( g \in \text{Aut } (G, \pi) \). Clearly, the points of \( \Delta(G)^n / \text{Aut } (G, \pi) \) encode all possible marking functions \( p_G : [n] \rightarrow \Delta(G) \) up to the action of the automorphism group, with the topology being precisely the one with which we have earlier endowed the space of marked graphs.

Furthermore, let \( \tilde{\sigma} \) be a simplex on the boundary of \( \sigma \), such that \( \dim \tilde{\sigma} + 1 = \dim \sigma \). This means that \( \sigma \) covers \( \tilde{\sigma} \) in the poset \( \mathcal{F}(\Delta_g) \). To define the corresponding map \( \mathcal{D}_{g,n}(\sigma) \rightarrow \mathcal{D}_{g,n}(\tilde{\sigma}) \) we need to consider two cases. To fix the notations, assume that \( \pi = (E_1, \ldots, E_m) \).

**Case 1.** The simplex \( \tilde{\sigma} \) is represented by a filtered by forests stable graph \((G, \tau)\), where the ordered partition \( \tau \) is obtained from \( \pi \) by merging the blocks \( E_i \) and \( E_{i+1} \), for some \( i = 1, \ldots, m-1 \). In this case \( \text{Aut } (G, \pi) \) is a subgroup of \( \text{Aut } (G, \tau) \), therefore the induced quotient map

\[
q : \Delta(G)^n / \text{Aut } (G, \pi) \rightarrow \Delta(G)^n / \text{Aut } (G, \tau),
\]

\[
q : [\underline{x}]_{\text{Aut } (G, \pi)} \mapsto [\underline{x}]_{\text{Aut } (G, \tau)},
\]

is well-defined. Here \( \underline{x} = (x_1, \ldots, x_n) \), and \( [\underline{x}]_{\Gamma} \) denotes the orbit of \( \underline{x} \) with respect to the \( \Gamma \)-action. We now simply set \( \mathcal{D}_{g,n}(\sigma > \tilde{\sigma}) = \mathcal{D}_{g,n}(\sigma) \rightarrow \mathcal{D}_{g,n}(\tilde{\sigma}) \) to be the map \( q \).
Case 2. The simplex \( \hat{\sigma} \) is represented by a filtered by forests stable graph \((H, \tau)\), where the graph \( H \) is obtained from the graph \( G \) by shrinking the edges from \( E_1 \), and the \( \tau = (E_2, \ldots, E_m) \) is the corresponding ordered partition. Let \( \psi : \Delta(G) \to \Delta(H) \) denote the induced quotient map, and let \( \psi^n : \Delta(G)^n \to \Delta(H)^n \) its \( n \)-fold direct product, defined by \( \psi^n(x_1, \ldots, x_n) := (\psi(x_1), \ldots, \psi(x_n)) \). Let \( \Delta(E_1) \) denote the topological union of the edges from \( E_1 \), we have \( \Delta(G/E_1) = \Delta(G)/\Delta(E_1) \). Since the subspace \( \Delta(E_1) \subseteq \Delta(G) \) is invariant under the action of \( \text{Aut}(G, \pi) \), every automorphism of \((G, \pi)\) induces an automorphism of \((H, \tau)\). We let \( \iota : \text{Aut}(G, \pi) \to \text{Aut}(H, \tau) \) denote the corresponding map, which is actually a group homomorphism. We are now in a position to define

\[
q : \Delta(G)^n/\text{Aut}(G, \pi) \to \Delta(H)^n/\text{Aut}(H, \tau),
\]

where we use the same notations as in (3.2). The map \( q \) is well-defined, since for all \( g \in \text{Aut}(G, \pi) \), and all \( x \in \Delta(G)^n \), we have

\[
\psi^n(g(x)) = \iota(g)(\psi^n(x)).
\]

Also in this case we set \( D_{g,n}(\sigma > \tilde{\sigma}) : D_{g,n}(\sigma) \to D_{g,n}(\tilde{\sigma}) \) to be the map \( q \).

### 3.3. The homotopy colimit presentation.

**Definition 3.6.** Given a diagram \( D \) of topological spaces over a poset \( P \), the homotopy colimit of \( D \), denoted \( \text{hocolim}\ D \), is the quotient space

\[
\text{hocolim}\ D = \coprod_{\sigma = (v_0 > \cdots > v_n)} (\sigma \times D(v_0))/\sim,
\]

where the disjoint union is taken over all chains in \( P \). The equivalence relation \( \sim \) is generated by: for \( \tau = (v_0 > \cdots > v_i > \cdots > v_n) \), considered as a simplex of \( \Delta(P) \), let \( f_i : \tau_i \to \sigma \) be the inclusion map, then

- for \( i > 0 \), \( \tau_i \times D(v_0) \) is identified with the subset of \( \sigma \times D(v_0) \), by the map induced by \( f_i \);
- for \( \tau_0 = (v_1 > \cdots > v_n) \), we have \( f_0(\alpha) \times x \sim \alpha \times D(v_0 > v_1)(x) \), for any \( \alpha \in \tau_0 \), and \( x \in D(v_0) \).

Given two diagrams \( D_1 \) and \( D_2 \) over the same poset \( P \), a diagram map \( F \) is a collection of maps \( F(x) : D_1(x) \to D_2(x) \), for all \( x \in P \), which commute with the diagram structure maps, i.e., for all \( x, y \in P \), such that \( x > y \), we have

\[
D_2(x > y)(F(x)) = F(y)(D_1(x > y)(x)).
\]

It is a standard fact, that a diagram map induces a continuous map between the corresponding homotopy colimits \( F : \text{hocolim}\ D_1 \to \text{hocolim}\ D_2 \).

In particular, taking \( D_2(x) \) to be a point, for all \( x \in P \), we get a diagram of topological spaces, whose colimit is the order complex \( \Delta(P) \). Setting \( F(x) \) to be the map which takes everything to one point, we certainly get a diagram map. Thus, we arrive at a map \( p : \text{hocolim}\ D_1 \to \text{hocolim}\ D_2 \cong \Delta(P) \). In these circumstances, the space \( \Delta(P) \) is called the base space, the map \( p \) is called the base projection map, and the preimages of points under \( p \) are the fibers. We refer the reader to [Ko07, Subsection 15.2.2] for further details.

In our specific case we see that the base space is \( X_{g,0} \cong \Delta_g \cong \text{Bd} \Delta_g \cong \Delta(\mathcal{F}(\Delta_g)) \). It turns out that the homotopy colimit of the diagram \( D_{g,n} \) describes
precisely the tropical moduli space which we are studying, and that the base projection map is the natural one.

**Theorem 3.7.** The space \( X_{g,n} \) is homeomorphic to the homotopy colimit of the diagram \( \mathcal{D}_{g,n} \), whereas the map \( \lambda(\emptyset \mapsto S) : X_{g,n} \to \Delta_g \) is the base projection map. In particular, the moduli space of the rational tropical curves with \( n \) marked points \( TM_{g,n} \) is homotopy equivalent to \( \text{hocolim} \mathcal{D}_{g,n} \).

**Proof.** This is pretty much straightforward from our construction. For a point \( x \in \text{hocolim} \mathcal{D}_{g,n} \), its image \( p(x) \) under the base projection map is represented by a filtered by forests stable metric graph (\( (G, l_G) \)), whereas the position of \( x \) within its fiber \( \Delta(G)^n / \text{Aut}(G, \pi) \) encodes all the ways to mark \( n \) points, modulo the action of the automorphism group.

This gives a map \( \rho : \text{hocolim} \mathcal{D}_{g,n} \to X_{g,n} \). Clearly, that map is bijective. To see that both \( \rho \) and its inverse are continuous we need to see that small perturbations of a point \( x \) in \( \text{hocolim} \mathcal{D}_{g,n} \), resp. in \( X_{g,n} \), causes small perturbations of \( \rho(x) \), resp. of \( \rho^{-1}(x) \). There are \( 3 \) possibilities for a small perturbation of \( x \in \text{hocolim} \mathcal{D}_{g,n} \):

1. we stay in the fiber, i.e., the base metric graph does not change;
2. the base metric graph changes, but we stay within the same simplex in \( \Delta_g \);
3. we move to an adjacent simplex of higher dimension in \( \Delta_g \).

In \( X_{g,n} \) these perturbations correspond to the following. Perturbation (1) corresponds the graph being fixed and the marks moving on that graph. In this case the topology is the same as we mentioned before when discussing the fibers. Perturbation (2) corresponds to rescaling of the blocks of edges of equal lengths, together with moving of the marks on the rescaled graph. That topology is the same in this case as well, follows from our proof of the homeomorphism \( \Delta_g \cong X_{0,g} \).

When analyzing perturbation (3) we distinguish two cases. In the first case, no new edges are added, but the blocks of edges of equal lengths are getting split. Here the topology is the same due to construction of the attachment map in (3.2). Finally, in the second case, we add a number of very short edges, by replacing some of the vertices with trees. In that situation the topology is the same due to construction of the attachment map in (3.3). \( \square \)

**Remark 3.8.** With the benefit of hindsight, one may now interpret Theorem 2.6 and its proof, using the language of homotopy colimits. This is because already the space \( TM_{g,n} \) can be represented as a result of a gluing construction, similar to homotopy colimit, whose base would be not \( \Delta_g \), but rather a difference between two other generalized simplicial complexes \( \Delta \setminus \Delta' \). The strong deformation retraction in the proof of Theorem 2.6 can then be thought of as corresponding to the retraction of all not closed cells in \( \Delta \setminus \Delta' \), resulting in the homotopy colimit with the base \( \Delta_g \).

The last theorem of this section describes the standard way the spaces which are presented as homotopy colimits can be simplified, while preserving their homotopy type. We shall that technique in Section 5.

**Theorem 3.9.** (Homotopy Lemma)

Let \( \mathcal{F} : \mathcal{D}_1 \to \mathcal{D}_2 \) be a diagram map between diagrams of spaces over \( \Delta \), such that for each \( v \in \Delta^{(0)} \), the map \( \mathcal{F}(v) : \mathcal{D}_1(v) \to \mathcal{D}_2(v) \) is a homotopy equivalence. Then the induced map \( \text{hocolim} \mathcal{F} : \text{hocolim} \mathcal{D}_1 \to \text{hocolim} \mathcal{D}_2 \) is a homotopy equivalence as well.

Again, we refer to [Ko07, Chapter 15] for the proof and further information.
4. CW structure on $X_{g,n}$ and its Euler characteristic

We shall now describe CW structure on the spaces $X_{g,n}$ derived from the homotopy colimit representation from Theorem 3.7.

4.1. Cubical complexes associated to filtered stable graphs.

To start with, we associate cubical complexes to all simplices of $\Delta_g$. For a filtered stable graph $(G, \pi)$ we let $S(G, \pi)$ denote the 1-dimensional simplicial complex obtained from $\Delta(G)$ by inserting the middle point into every edge of $G$ which is flipped by some element of the group $\text{Aut}(G, \pi)$; this of course should include all the loops of $G$.

**Definition 4.1.** Let $\sigma$ be a simplex of $\Delta_g$ represented by a filtered stable graph $(G, \pi)$ of genus $g$. We set

$$C(G, \pi) := S(G, \pi)^n / \text{Aut}(G, \pi),$$

where the action of the group $\text{Aut}(G, \pi)$ on the direct product $S(G, \pi)^n$ is the diagonal one.

We shall also use the notation $C(\sigma)$. Since $S(G, \pi)$ is a subdivision of $\Delta(G)$, the comparison of the definitions (4.1) and (4.1) shows that $C(G, \pi)$ is homeomorphic to $D_{g,n}(G, \pi)$. We shall think of each cube of $S(G, \pi)^n$ as specifying where every label $i = 1, \ldots, n$ should lie. We shall call the corresponding part of $S(G, \pi)^n$ the **allowed locus** of $i$, and note that it is either a vertex of $S(G, \pi)$, or an edge or a half-edge of $G$. The same is true for $C(G, \pi)$, but of course modulo the $\text{Aut}(G, \pi)$-action. See Figure 4.1 for examples.

![Figure 4.1](image)

Figure 4.1. Here we show three examples of the subdivisions $S(G, \pi)$, in each one we have $\pi = (E(G))$. We depict the original vertices of $G$ as filled-in and the added subdivision points as hollow. On the right hand side we show examples of two different presentations of the same cube in $C(G, \pi)$; here the fattened-up half-edge and a vertex indicate the allowed loci of the marked points.

Importantly, the space $C(G, \pi)$ has the structure of a cubical complex. This is because $S(G, \pi)$ is a cubical complex, and the $\text{Aut}(G, \pi)$-action on $S(G, \pi)^n$ has the following property: if the cube of $S(G, \pi)^n$ is preserved by an element $g \in \text{Aut}(G, \pi)$, then it must be fixed by it pointwise. Indeed, if $g\tau = \tau$, for some $g \in \text{Aut}(G, \pi)$, and a cube $\tau$ from $S(G, \pi)^n$, then for all $i = 1, \ldots, n$, the allowed locus of $i$ is preserved by $g$. Clearly, if the allowed locus of $i$ is a vertex of $S(G, \pi)$, or a half-edge of $G$, then it must be preserved pointwise. If, on the other hand,
the allowed locus is an edge $e$ of $G$ and it is not preserved pointwise by $g$, then $g$ must flip $e$, hence, by our construction, the edge $e$ should have been subdivided in $S(G, \pi)$, yielding a contradiction.

4.2. The CW structure on $X_{g,n}$.

The indexing of cells. The cells are indexed by pairs $(\sigma, \tau)$, where $\sigma$ is a simplex of the generalized simplicial complex $\Delta_g$, and $\tau$ is a cube of $C(\sigma)$. We shall denote such a cell $c(\sigma, \tau)$. The open cell $c(\sigma, \tau)$ consists of all the points of $X_{g,n}$, which are indexed by filtered by forests stable graphs $(G, \pi)$ representing the simplex $\sigma$ of $\Delta_g$, with $n$ marked points, such that the point marked $i$ belongs to the corresponding allowed locus prescribed by the cube $\tau$, for $i = 1, \ldots, n$, modulo the action of the group $\text{Aut}(G, \pi)$. The dimension of the cell $c(\sigma, \tau)$ is equal to $\dim c + \dim \tau$.

For example, the cell depicted on the right hand side of Figure 4.1 is indexed by the pair $(\sigma, \tau)$, where $\sigma$ is a vertex, indexed by $(G, \pi)$, where $G$ is the graph with 4 vertices and 6 edges shown there, and $\pi = (E(G))$, and $\tau$ is a 1-dimensional cube. In particular, this cell has dimension 1. Of course, the two presentations of this cell may correspond to different cells, if the ordered partition $\pi$ is chosen differently.

The attachment maps. Consider the cell $c(\sigma, \tau)$. Assume the simplex $\sigma$ is represented by a filtered by forests stable graph $(G, \pi)$, where $\pi = (E_1, \ldots, E_t)$. To describe the attachment of $c(\sigma, \tau)$ we need to tell how to glue in the cells from the boundary $\partial c(\sigma, \tau)$; for this it is enough to take the cells of dimension $\dim c + \dim \tau - 1$ from $(\partial \sigma) \times \tau$ and $\sigma \times (\partial \tau)$. The case when we take the cells from $\sigma \times (\partial \tau)$ is easy, as we simply glue along the attachment map of the cell $\tau$ in the cubical complex $C(\sigma)$.

Let us consider the case when we take a cell $\lambda \times \tau$ from $(\partial \sigma) \times \tau$ of dimension $\dim c + \dim \tau - 1$. There are two distinguished cases.

Case 1. The simplex $\lambda$ is indexed by the filtered by forests stable graphs $(G', \pi')$ obtained from $(G, \pi)$ by merging blocks $E_i$ and $E_{i+1}$, for some $i = 1, \ldots, t$, in $\pi$. Then, we have $\text{Aut}(G, \pi) \leq \text{Aut}(G', \pi')$, which induces the quotient map

$$q : S(G, \pi)^n / \text{Aut}(G, \pi) \to S(G', \pi')^n / \text{Aut}(G', \pi'),$$

defined by $q : [x] \mapsto [x]$, where we identify the spaces $S(G, \pi)$ and $S(G', \pi')$. This is precisely the gluing map from that part of boundary of $c(\sigma, \tau)$ to $c(\lambda, \tau)$ which we are looking for.

Case 2. The simplex $\lambda$ is indexed by the filtered by forests stable graphs $(G', \pi')$ obtained from $(G, \pi)$ by shrinking the edges $E_1$. In that case, we have a shrinking map $\psi^n : S(G, \pi)^n \to S(G', \pi')^n$, and the map $\iota : \text{Aut}(G, \pi) \hookrightarrow \text{Aut}(G', \pi')$, which together induce the quotient cubical map

$$q : S(G, \pi)^n / \text{Aut}(G, \pi) \to S(G', \pi')^n / \text{Aut}(G', \pi'),$$

defined by $q : [x] \mapsto [\psi^n(x)]$. This map is well-defined due to the identity $\psi^n$, and provides us with the required gluing map from that part of boundary of $c(\sigma, \tau)$ to $c(\lambda, \tau)$.

This cell structure allows us to write down the chain complex $(C_*(X_{g,n}; \mathbb{Z}_2); \partial)$ of vector spaces over $\mathbb{Z}_2$ for computing $H_*(X_{g,n}; \mathbb{Z}_2)$. The vector space $C_*(X_{g,n}; \mathbb{Z}_2)$ is generated by all the $t$-cells, and the algebraic boundary maps are induced by the topological boundary described above, with every individual gluing contributing the cell one is glued onto with coefficient 1, if the cell has the right dimension. The
degenerate gluings yield contribution 0. We remark that the gluing is degenerate if and only if we are gluing a cell of dimension $\dim \sigma + \dim \tau - 1$ from $(\partial \sigma) \times \tau$, we are in Case 2 above, and some of the $n$ marked points happen to lie on one of the edges from the set $E_1$.

4.3. The colimit of $D_{g,n}$.

It is curious, that while the homotopy colimit of the diagram $D_{g,n}$ is a rather complicated space, its colimit is in fact quite simple.

**Proposition 4.2.** The colimit of the diagram $D_{g,n}$ is a point.

**Proof.** One can describe the space colim $D_{g,n}$ explicitly as follows: it is the union of all the spaces in the diagram $D_{g,n}$ modulo a certain equivalence relation. This relation is generated by the elementary relations, which say that for $\alpha, \alpha' \in F(\Delta_g)$, such that $\alpha > \alpha'$, and $x \in D_{g,n}(\alpha)$, we have $x \sim D_{g,n}(\alpha > \alpha')(x)$.

Let us now fix $\alpha$ to be the vertex of $\Delta_g$ indexed by the graph with one vertex and $g$ loops, and let $w$ denote that unique vertex. The $n$-tuple $\bar{w} = (w, \ldots, w)$ indexes a vertex in the cubical complex $C(\alpha)$, and hence also a point in colim $D_{g,n}$, we shall show that every other point $x \in \cup_{x \in \Delta_g} D_{g,n}(\sigma)$ is equivalent to $\bar{w}$. Using the described cubical structures on $C(\sigma)$, it is actually enough to show that any cube in any $C(\sigma)$ is equivalent to $\bar{w}$.

Let $c$ be a cube of some $C(G, \pi)$, and assume that $c$ is indexed by the $n$-tuple $(a_1, \ldots, a_n)$, where each $a_i$ is either a vertex or an edge of $S(G, \pi)$. To start with, setting all edges of $G$ equal corresponds to a map in the diagram $D_{g,n}$, and of course the cube $c$ will be equivalent to its image under this map. Therefore, we may replace $c$ with its image, and assume that all the edges of $G$ have the same length.

Assume now that some $a_i$ is either a barycenter of a loop $l$ of $\Delta(G)$, or a half-edge lying on a loop $l$ of $G$. In that case, there exists a filtered by forests stable graph $(G', \pi')$, such that shrinking the shortest edges of $G'$ will yield $(G, \pi)$. For example, it can be obtained by any admissible “unlooping” of $l$. That shrinking corresponds to a surjective map in the diagram $D_{g,n}$, and we can replace the cube $c$ with any of its preimage cubes in $C(G', \pi')$, which is of course equivalent to $c$. Repeating this procedure, we can see that we may assume that no $a_i$ is a part of a loop of $G$.

Assume now, that some $a_i$ is an edge of $S(G, \pi)$, and let $e$ denote the underlying edge of $G$. Let $(G', \pi')$ be obtained from $(G, \pi)$ by letting $e$ be slightly shorter than the other edges. There is cube $c' \in C(G', \pi')$, indexed by $(a'_1, \ldots, a'_n)$ which is mapped to $c$ by the corresponding map $f$ of $D_{g,n}$, hence $c \sim c'$. Let $c''$ be a cube of $C(G', \pi')$ obtained from $c'$ by replacing $a'_i$ by any of its endpoints. We can shrink $e$ in $G'$ obtaining $(G'', \pi'')$. Under the map of the diagram $D_{g,n}$ corresponding to this shrinking, the cubes $c'$ and $c''$ map to the same cube, hence they are equivalent. We conclude that $c$ is equivalent to $f(c'')$. Repeating this procedure, we can see that we may assume that all $a_i$'s correspond to vertices of $S(G, \pi)$.

Clearly, the argument of the last paragraph can be also used to show that $c$ is equivalent to $c' = (a'_1, \ldots, a'_n)$, where the $n$-tuple $(a'_1, \ldots, a'_n)$ is obtained from $(a_1, \ldots, a_n)$ by replacing any $a_i$ by any neighboring vertex of $S(G, \pi)$. Repeating this, we end up with the cube where all the $a_i$’s are the same vertices of $S(G, \pi)$ an correspond to a vertex of $G$. Taking the appropriate shrinking map to $\alpha$ we conclude that all the cubes are equivalent to the vertex $\bar{w}$, and hence the entire colimit is just a point. \qed
4.4. General formula for the generating function of the numbers of cells of $X_{g,n}$.

We shall now use the cell structure of $X_{g,n}$ to derive formulae for its Euler characteristic. For this we will need the following weighted version of Burnside Lemma, see e.g., [Bj92, Exercise 14.4.5, page 313].

**Lemma 4.3.** (Weighted Burnside Lemma).
Let $\Gamma$ be a group acting on a set $X$, and let $w : X \to \Omega$ be a $\Gamma$-invariant map, where $(\Omega, +)$ is an abelian group, then we have

\[
\sum_{g \in O(\Gamma, X)} w(o) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \sum_{x \in \text{Fix}(g)} w(x),
\]

where $O(\Gamma, X)$ denotes the set of orbits of the $\Gamma$-action on $X$.

For a finite-dimensional CW complex $K$ and a variable $x$, we let $P(K)(x)$ denote the generating function for the numbers of cells, i.e., $P(K) := \sum_{d=0}^{\dim K} n_d x^d$, where $n_d$ denotes the number of $d$-cells in $K$. When the choice of the variable is clear, we shall simply write $P(K)$. Note that $P(K)(-1)$ equals to the (nonreduced) Euler characteristic of $K$.

Let now $X$ be the set of the cubes of $S(G, \pi)^n$, and let $\Gamma$ be the group $\text{Aut}(G, \pi)$ with the standard $\Gamma$-action on $X$. Let us furthermore set $w(\sigma) := x^{\dim \sigma}$ for all cubes $\sigma$ of $S(G, \pi)^n$. Since $\sum_{d=0}^{\dim K} n_d x^d = \sum_{\sigma \in K} x^{\dim \sigma}$, for any CW complex $K$, we get $P(C(G, \pi)) = \sum_{\sigma \in C(G, \pi)} w(\sigma)$. Substituting this data into (4.2) we obtain

\[
P(C(G, \pi)) = \frac{1}{|\text{Aut}(G, \pi)|} \sum_{\gamma \in \text{Aut}(G, \pi)} P(\text{Fix}_{C(G, \pi)}(\gamma)),
\]

where $\text{Fix}_{C(G, \pi)}(\gamma)$ denotes the subcomplex of $C(G, \pi)$ fixed by the group element $\gamma$. It follows from our construction that $\text{Fix}_{C(G, \pi)}(\gamma) = (\text{Fix}_{S(G, \pi)}(\gamma))^n$, where $\text{Fix}_{S(G, \pi)}(\gamma)$ denotes the subcomplex of $S(G, \pi)$ fixed by the group element $\gamma$. Since $P(K^n) = P(K)^n$ for an arbitrary finite CW complex $K$, the equation (4.3) translates to equation

\[
P(C(G, \pi)) = \frac{1}{|\text{Aut}(G, \pi)|} \sum_{\gamma \in \text{Aut}(G, \pi)} P(\text{Fix}_{S(G, \pi)}(\gamma))^n.
\]

In particular, substituting $x = -1$ into (4.4) we obtain the equation for the corresponding Euler characteristic

\[
\chi(C(G, \pi)) = \frac{1}{|\text{Aut}(G, \pi)|} \sum_{\gamma \in \text{Aut}(G, \pi)} \chi(\text{Fix}_{S(G, \pi)}(\gamma))^n.
\]

Summing with appropriate signs over all the isomorphism classes of filtered by forests stable graphs, equations (4.4) and (4.5) yield equations

\[
P(X_{g,n}) = \sum_{\sigma \in \Delta} (-1)^{\dim \sigma} P(C(\sigma)) = \frac{1}{|\text{Aut}(G, \pi)|} \sum_{\sigma \in \Delta} (-1)^{\dim \sigma} \sum_{\gamma \in \text{Aut}(G, \pi)} P(\text{Fix}_{S(\sigma)}(\gamma))^n.
\]
and
\[ \chi(X_{g,n}) = \frac{1}{|\text{Aut}(G,\pi)|} \sum_{\sigma \in \Delta_g} (-1)^{\dim \sigma} \sum_{\gamma \in \text{Aut}(G,\pi)} \chi(\text{Fix}_{S(G,\pi)}(\gamma))^n. \]

These formulae will come in handy in our analysis of cases of small genus.

5. The case of genus 2

In this and the next section we will analyse the cases of small genus \( g = 2 \) and \( g = 3 \), and make some conjectures which should hold for all \( g \) and \( n \).

5.1. The cases \( n = 0 \) and \( n = 1 \).

Let us start with considering the space \( TM_{2,n} \). As was shown in [Ko08a], this space is homotopy equivalent to \( TM_{b,2,n} \). Furthermore, by Theorem 2.6, there is a strong deformation retraction from \( TM_{b,2,n} \) to \( TM_{e,2,n} \); it is symbolically shown on Figure 5.1 with arrows.

As in general case, it is now sufficient to understand the space \( X_{2,n} \). According to Theorem 5.7, that space is a homotopy colimit shown on the left hand side of Figure 5.2. The groups on that figure are as follows:

- the group \( \Gamma_1 \) consists of the flips of the loops, with a possible swap of the two loops; it has cardinality 8 and is isomorphic to the wreath product of \( S_2 \) with \( S_2 \);
- the group \( \Gamma_2 \) is generated by reflections of the graph with respect to the horizontal and vertical axes; it has cardinality 4 and is isomorphic to the direct product of \( S_2 \) with \( S_2 \);
- the group \( \Gamma_3 \) is generated by \( S_3 \) permuting the edges and the reflection with respect to the horizontal axis; it has cardinality 12 and is isomorphic to the direct product of \( S_3 \) with \( S_2 \).

We again used the hollow circles to denote the points inserted in the midpoints of edges, as is prescribed by the group actions; in fact here, we simply need to subdivide all the edges. As elaborated on in Section 4 these subdivisions induce
a CW structure on the homotopy colimit. For $n = 0$, this structure coincides with the generalized simplicial complex $\Delta_g$, so here we simply get a 1-simplex.

![Diagram](image1)

**Figure 5.2.** On the left hand side we show the diagram $D_{2,n}$, and the right hand side we show the special case $n = 1$.

For $n = 1$ we get the diagram of spaces shown on the right hand side of Figure 5.2. In that diagram, the map corresponding to the diagonal arrow pointing southwest shrinks the edge $t_1$ to the point $v_1$, and maps the edge $t_2$ homeomorphically to the edge $e_1$, whereas the map corresponding to the diagonal arrow pointing southeast takes the vertex $e_3$ to $v_4$, and maps both edges $t_1$ and $t_2$ homeomorphically to the edge $e_2$. The homotopy colimit of this diagram is the cell complex of dimension 2, shown on Figure 5.3. Clearly, that complex is collapsible in the sense of [Co73], in particular, it is contractible.

![Diagram](image2)

**Figure 5.3.** The CW structure on $X_{2,1}$.

5.2. The case $n = 2$.

For shorthand notations, let $A \xleftarrow{f} C \xrightarrow{g} B$ denote the spaces and maps in the diagram on the left hand side of Figure 5.2 for $n = 2$, with the space $A$ corresponding the space in the southwest corner of that diagram. We start by understanding the cubical structure of the spaces $A, B,$ and $C$, all of which are 2-dimensional pure cubical complexes.
The case of the cubical complex $A$ is illustrated on Figure 5.4. Here the filtered by forests stable graph $(G, \pi)$ consists of one vertex and two loops of length 1. The subdivided space $S(G, \pi)$ is shown on the left hand side of Figure 5.4 where also notations for vertices and edges are fixed. Using these notations we describe the cubes of $A$. Each one is described by two letters, corresponding to the allowed loci of the points marked 1 and 2 (in that order). Thus, the complex $A$ has 5 vertices: $ww$, $ww'$, $wv$, $vw$, $vv'$; 6 edges: $we_1$, $e_1w$, $ve_1$, $e_1v$, $ve'_1$, $e'_1v$; 3 2-cubes: $e_1e_1$, $e_1e_2$, $e_1e'_1$; where we of course only list one of the equivalent descriptions for each cube, for example $vv' = v'v$, and $e'_1v = e_1v' = e_2v' = e'_2v$.

![Figure 5.4. The cubical complex $A$.](image)

We see from the Figure 5.4 that the 2-cubes $e_1e_1$ and $e_1e_2$ have common boundary, and hence form a 2-sphere; and that the 2-cube $e_1e'_1$ is attached to that 2-sphere along two neighboring sides. We conclude that $A$ is homotopy equivalent to a 2-sphere.

Next we describe the cubical complex $B$. The corresponding subdivided filtered by forests stable graph is shown on the left hand side of Figure 5.5. With the notations there, the complex $B$ has 6 vertices: $ww$, $ww'$, $wv$, $vw$, $vv'$; 8 edges: $we_1$, $e_1w$, $we_2$, $e_2w$, $ve_1$, $e_1v$, $ve'_1$, $e'_1v$; 4 2-cubes: $e_1e_1$, $e_1e_2$, $e_1e'_1$, $e_1e'_2$; where we again list only one of the equivalent descriptions for each cube. The 4 2-cubes can be glued to each other in the order on the Figure 6.4 forming a cubical complex homeomorphic to the 2-sphere. That complex may also conveniently be visualised as follows: take the double pyramid with a 2-cube base (also known as a 3-dimensional crosspolytope, which is the polytope dual to the 3-cube) and delete the 4 edges of the base, each time merging the adjacent triangles to form 2-cubes.
Finally, we describe the cubical complex \( C \). The corresponding subdivided filtered by forests stable graph is shown in the middle of Figure 5.5. With the notations there, the complex \( C \) has
11 vertices: \( uu, uw, wv, vu, vv, vv', vv, vv, vv' \);
18 edges: \( ud_1, d_1u, we_1, e_1u, vd_1, d_1v, wd_1, d_1w, wd_2, d_2w, wv_1, e_1w, we_2, e_2w, ve_1, e_1v, ve_1'e_1, e_1'vv' \);
10 2-cubes: \( d_1d_1, d_1e_1, d_1e_2, d_1d_2, e_1d_1, e_1d_2, e_1e_1, e_1e_2, e_1'e_1, e_1'e_2 \).

As Figures 5.6 and 5.7 illustrate, it is helpful to divide the 10 constituting 2-cubes into 3 groups: \( e_1e_1 \) and \( e_1e_2 \) and \( e_1'e_1 \), and the remaining 6 2-cubes. The 2-cubes in each group, when glued, form a 2-cube, and all these 3 new 2-cubes have a common boundary. Thus, we conclude that the complex \( C \) is homeomorphic to a 2-sphere with a disc glued in, which in turn, is homotopy equivalent to a wedge of two 2-spheres \( S^2 \vee S^2 \).

As part of the analysis of the maps \( f : C \to A \) and \( g : C \to B \), we show in Table 5.1 what their values on 2-cubes; there we use * to denote lower-dimensional cubes. We invite the reader to get the geometric intuition for what these maps actually do to the spaces.

We are now ready to prove the following result.

\textbf{Theorem 5.1.} The moduli space of tropical curves of genus 2 with two marked points \( TM_{2,2} \) is contractible.

\textbf{Proof.} Let us describe subcomplexes of \( A, B, \) and \( C \):

- the complex \( \tilde{A} \) is obtained from \( A \) by deleting the interior of the 2-cell \( e_1e_1 \);
- the complex \( \tilde{B} \) is obtained from \( A \) by deleting the interior of the 2-cell \( e_1e_1' \);
- the complex \( \tilde{C} \) is obtained from \( A \) by deleting the interiors of the 2-cells \( e_1e_1 \) and \( d_1d_1 \).
It follows from our detailed descriptions of the cubical structures on $A$, $B$, and $C$, that we have inclusions $f(\bar{C}) \subseteq \bar{A}$ and $g(\bar{C}) \subseteq \bar{B}$. It follows from the standard properties of maps of quotients, see e.g., [Ri75, Satz 11.7, p. 115], that the induced maps $\tilde{f} : C/\bar{C} \to A/\bar{A}$ and $\tilde{g} : C/\bar{C} \to B/\bar{B}$ are well-defined by $\tilde{f}(\{x\}) := [f(x)]$ and $\tilde{g}(\{x\}) := [g(x)]$. Furthermore, the quotient maps $q^A : A \to A/\bar{A}$, $q^B : B \to B/\bar{B}$,
and \(q^C : C \rightarrow C/\tilde{C}\), induce a diagram map \(q\) from \(A \xleftarrow{f} C \xrightarrow{g} B\) to \(A/\tilde{A} \xleftarrow{\tilde{f}} C/\tilde{C} \xrightarrow{\tilde{g}} B/\tilde{B}\).

As is clear from our explicit description, all three complexes \(\tilde{A}, \tilde{B}\), and \(\tilde{C}\), are contractible. Since inclusions of CW subcomplexes are cofibrations, see e.g., [Hat02, Proposition 0.17], it follows that the maps \(q^A, q^B,\) and \(q^C\), are homotopy equivalences. Using Theorem 3.9 (Homotopy Lemma), we conclude that \(q\) induces a homotopy equivalence between the homotopy colimits of these two diagrams.

![Diagram](image)

**Figure 5.8.** On the left hand side we show the diagram \(A/\tilde{A} \xleftarrow{\tilde{f}} C/\tilde{C} \xrightarrow{\tilde{g}} B/\tilde{B}\), whereas on the right hand side we show how its homotopy colimit can be obtained by a simple self-identification on a cone over \(S^2\).

By our construction and previous developments the homotopy colimit of \(A \xleftarrow{f} C \xrightarrow{g} B\) is homotopy equivalent to the tropical module space \(TM_{2,2}\). On the other hand, the spaces \(A/\tilde{A}\) and \(B/\tilde{B}\) are each homeomorphic to a 2-sphere, whereas the space \(C/\tilde{C}\) is homeomorphic to a wedge of 2-spheres \(S^2 \vee S^2\). The map \(\tilde{f}\) shrinks one of the spheres in the wedge to a point and maps the other one homeomorphically to \(S^2 \cong A/\tilde{A}\), while the map \(\tilde{g}\) maps both 2-spheres in \(C\) homeomorphically to \(S^2 \cong B/\tilde{B}\). Hence the homotopy colimit of \(A/\tilde{A} \xleftarrow{\tilde{f}} C/\tilde{C} \xrightarrow{\tilde{g}} B/\tilde{B}\) is homeomorphic

| the 2-cube of \(C\) | image under \(f\) | image under \(g\) |
|---------------------|-----------------|-----------------|
| \(e_1e_1\)         | \(e_1e_1\)     | \(e_1e_1\)     |
| \(e_1e_2\)         | \(e_1e_2\)     | \(e_1e_2\)     |
| \(e_1'e_1\)        | \(e_1'e_1\)    | \(e_1'e_1\)    |
| \(e_1'e_2\)        | \(e_1'e_2\)    | \(e_1'e_2\)    |
| \(d_1d_1\)         | *               | \(e_1e_1\)     |
| \(d_1e_1\)         | *               | \(e_1'e_1\)    |
| \(d_1'e_2\)        | *               | \(e_1'e_2\)    |
| \(d_1d_2\)         | *               | \(e_1'e_2\)    |
| \(e_1'd_1\)        | *               | \(e_1'e_1\)    |
| \(e_1'd_2\)        | *               | \(e_1'e_2\)    |

Table 5.1. The values of \(f\) and \(g\) on the 2-cubes of \(C\).
to the space obtained from a cone over $S^2$ by taking an edge connecting the apex to one of the points in the base, dividing this edge in the middle and identifying the two halves with each other by a reflection about the middle point, see Figure 5.8. In particular, it is contractible, and we have therefore proved the theorem. □

5.3. The Euler characteristic of the space $X_{2,n}$ and the proof that this space is almost never contractible.

Let us now calculate the Euler characteristic, and more generally, the generating function for the number of cells of $X_{2,n}$. This can be done using the formula (4.6). For this, we need to calculate $P(\text{Fix}_{S(G,\pi)}(\gamma))$, for all filtered by forests stable graphs of genus 2, and all $\gamma \in \text{Aut}(G,\pi)$. That information is summarized in tables on Figures 5.9, 5.10, and 5.11 corresponding to the three different filtered by forests stable graphs of genus 2. On Figure 5.11 we use the convention that

\[
0^n = \begin{cases} 
1, & \text{for } n = 0, \\
0, & \text{for } n \geq 1.
\end{cases}
\]

\[
\begin{array}{|c|c|c|}
\hline
\gamma & \text{Fix}_{S(G,\pi)}(\gamma) & P(\text{Fix}_{S(G,\pi)}(\gamma)) \\
\hline
\text{id} & & 3 + 4x \\
\hline
\alpha_1, \alpha_2 & & 3 + 2x \\
\hline
\beta, \alpha_1\beta, \alpha_1\alpha_2\beta & & 1 \\
\hline
\end{array}
\]

Figure 5.9. Both loops of the graph on the left hand side have length 1. The elements $\alpha_1$ and $\alpha_2$ are reflections of the corresponding loops, and $\beta$ is swapping the loops.

We conclude that

$$P(X_{2,n}) = \frac{1}{24} (3 \cdot (3 + 4x)^n - 4 \cdot (5 + 6x)^n + 6 \cdot (3 + 2x)^n - 3^n + 4 \cdot 2^n + 12 + 4 \cdot 0^n),$$

and hence

$$\chi(X_{2,n}) = \frac{1}{24} (-3^n + (-1)^{n+1} + 2^{n+2} + 18 + 4 \cdot 0^n),$$

for all nonnegative integers $n$. For small values of $n$ we get

$$\chi(X_{2,0}) = \chi(X_{2,1}) = \chi(X_{2,2}) = \chi(X_{2,3}) = 1, \quad \chi(X_{2,4}) = 0, \quad \chi(X_{2,5}) = -4.$$

In particular, we conclude that $X_{2,n}$ is not contractible for all $n \geq 4$. At present, it is unknown whether $X_{2,3}$ is contractible or not.
Figure 5.10. The uppermost and the lowest edge of the graph on the left hand side have equal length, while the middle edge is shorter. The element $\alpha$ corresponds to the vertical reflection, whereas the element $\beta$ swaps the two long edges.

| $\gamma$ | $\text{Fix}_{S(G,\pi)}(\gamma)$ | $\mathcal{P}(\text{Fix}_{S(G,\pi)}(\gamma))$ |
|----------|----------------------------------|------------------------------------------|
| $\text{id}$ | ![image](image) | $5 + 6x$ |
| $\alpha$ | ![image](image) | $3 + 2x$ |
| $\beta$ | ![image](image) | $3 + 2x$ |
| $\alpha\beta$ | ![image](image) | $1$ |

Figure 5.11. In the graph on the left hand side all the three edges are of length 1. Again, the element $\alpha$ corresponds to the vertical reflection. In addition, we have a permutation action of $S_3$ on the edges.

| $\gamma$ | $\text{Fix}_{S(G,\pi)}(\gamma)$ | $\mathcal{P}(\text{Fix}_{S(G,\pi)}(\gamma))$ |
|----------|----------------------------------|------------------------------------------|
| $\text{id}$ | ![image](image) | $5 + 6x$ |
| $\alpha$ | ![image](image) | $3 + 2x$ |
| $(12), (13), (23)$ | ![image](image) | $2$ |
| $(12), (13), (23)$ | ![image](image) | $3 + 2x$ |
| $\alpha(123), \alpha(132)$ | ![image](image) | $0$ |
| $\alpha(12), \alpha(13), \alpha(23)$ | ![image](image) | $1$ |
6. The case of genus 3

6.1. Collapsibility of $\Delta_3$.

As was shown in Proposition 3.3, the generalized simplicial complex $\Delta_3$ is pure, and it has dimension 4. There are eight isomorphism classes of stable graphs of genus 3. These are shown on Figure 6.1 and correspond to eight vertices of $\Delta_3$.

![Figure 6.1](image)

Figure 6.1. The eight isomorphism classes of stable graphs of genus 3, corresponding to the eight vertices of $\Delta_3$.

**Theorem 6.1.** The generalized simplicial complex $\Delta_3 \cong X_{3,0}$ is collapsible.

**Proof.** As a first simplification, we notice that the vertex $G_2$ belongs to a unique 3-simplex. This 3-simplex is indexed by the filtered by forests stable graph $(G_7, \pi)$, with $\pi = (\{e_1\}, \{e_2\}, \{e_3\}, E(G_7) \setminus \{e_1, e_2, e_3\})$, where $e_1$ is one of the vertical edges, $e_2$ is another vertical edge, non-adjacent to $e_1$, and $e_3$ is a horizontal edge connecting the two. Therefore, deletion of the vertex $G_2$ along with all the adjacent simplices corresponds to a (non-elementary) simplicial collapse. Let us denote by $X$ the thus obtained generalized simplicial complex. It is enough to show that this complex is collapsible.

We notice that the closed star of the vertex $G_1$ constitutes the entire complex $X$. Hence, it is enough to show that the generalized simplicial complexes $\text{dl}_X(G_1)$,
Figure 6.2. The deletion of the vertex $G_1$ from the complex $X$.

the deletion of $G_1$ from $X$, and $\text{lk}_X(G_1)$, the link of $G_1$ in $X$, are both collapsible. These complexes are shown on Figures 6.2 and 6.3 from which it is apparent that there are various ways to collapse each one to a point. □

6.2. The asymptotics of Euler characteristic of $X_{3,n}$, and the conjectural asymptotics for all $X_{g,n}$.

Let us now show that $\Omega(\chi(X_{3,n})) = 4^n$, that is $\Omega(\chi(X_{3,n}))/4^n$ converges to a constant, as $n$ goes to infinity. We use the formula (4.7). The nontrivial contributions from different cells of $\Delta_3$ are summarized on Figure 6.4, where we list the contributions below corresponding cells. Summing up we see that the final answer is $1/48$.

It was shown in [Ko08b] that for $n \geq 1$, we have $TM_{1,n+1} \simeq (S^1)^n/\mathbb{Z}_2$, where the $\mathbb{Z}_2$-action is a simultaneous reflection on each factor circle $S^1$. Taking the points fixed by that reflection as vertices, and two semicircles as edges we induce a $\mathbb{Z}_2$-invariant cubical structure on $(S^1)^n$ with $\mathcal{P}((S^1)^n) = (2 + 2x)^n$. Using equation (4.2) we obtain the equality $\mathcal{P}((S^1)^n/\mathbb{Z}_2) = \frac{1}{2}((2 + 2x)^n + 2^n)$. In particular, we have $\chi(TM_{1,n}) = 2^{n-2}$, for $n \geq 2$.

Emboldened by the settled cases $g = 1, 2, 3$, we arrive at the following conjecture.

**Conjecture 6.2.** Assume that $g$ be a fixed arbitrary positive integer. We have 

\begin{equation}
\Omega(\chi(TM_{g,n})) = (g+1)^n,
\end{equation}

where $\chi(TM_{g,n})$ is considered as a function of $n$, and the asymptotics is taken for $n \rightarrow \infty$.

It would also be interesting to find an interpretation for the actual coefficient of $(g+1)^n$, the values which we have computed are $1/4, -1/24, 1/48$. 


7. Conclusion and open questions

Though having gained a substantial advance in our understanding of the structure of the spaces $X_{g,n}$, and thus of the homotopy type of the moduli spaces of rational tropical curves $TM_{g,n}$, quite a few questions remain open.
For example, it seems natural to conjecture that $\Delta_g$ is not always contractible, and it would be interesting to find out whether that phenomenon occurs already for $g = 4$. At the moment, not much is known about $\Delta_4$ beyond the fact that it has 43 vertices, whose indexing stable graphs of genus 4 are shown on Figure 7.1.

Furthermore, adding the marked points into the fray, makes for a lot more questions. As a first concrete one, we would like to know whether $X_{g,3}$ is contractible or not. In general, it certainly feels like the spaces $X_{g,n}$ should always never be contractible, so settling Conjecture 6.2 seems useful.

In Section 5 we could understand the homotopy type of the space $X_{2,2}$ by coming up with some deformations of the spaces in the diagram $D_{g,n}$, and using the Homotopy Lemma. The natural question here is of course whether that technique could be turned into a system and help us understand other instances of $g$ and $n$.

Figure 7.1. The 43 isomorphism classes of stable graphs of genus 4, corresponding to the 43 vertices of $\Delta_4$.

Acknowledgments. The author would like to thank Eva-Maria Feichtner for the valuable discussions. He would also like to thank the University of Bremen for the support of the work on this project within the framework of AG CALTOP. Part of this research has been done while the author was visiting the Mathematical Research Institute at Oberwolfach, which is hereby gratefully acknowledged.
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Department of Mathematics, University of Bremen, 28334 Bremen, Federal Republic of Germany

E-mail address: dfk@math.uni-bremen.de