The Static Potential to $O(\alpha^2)$ in Lattice Perturbation Theory

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We present a calculation of Wilson loops, and the static inter-quark potential to $O(\alpha^2)$ in lattice perturbation theory. This is carried out with the Wilson, Symanzik-Weisz, and Iwasaki gauge actions and the Wilson, Sheikholeslami-Wohlert, and Kogut-Susskind dynamical fermion action for small Wilson loops, and with the Wilson gauge action and each of the dynamical quark actions in the case of the static potential.

1. INTRODUCTION

The calculation of Wilson loops and derived quantities (such as the potential and static quark mass renormalisation) is perhaps one of the most mature problems in lattice perturbation theory, with many calculations throughout the last two decades.

Nevertheless, as the lattice community moves to non-perturbative simulations with (massive) improved dynamical quark actions it is necessary to extend the coverage of the literature correspondingly.

Further, while the calculation of the static potential has been well understood in the literature\textsuperscript{[1–3]}, and in particular for quenched SU(2) gauge theory\textsuperscript{[4]}, we present results for the static potential in SU(3) both with and without dynamical fermions.

2. Perturbative Wilson Loops

The bare operator for the Wilson loop contains all orders in the coupling due to the exponentiation of the gauge field in a link. This operator must be expanded and truncated to $O(\alpha^2)$.

$$W(\mathcal{C}) = \langle \mathcal{O}_W \rangle$$

$$\mathcal{O}_W(\mathcal{C}) = \frac{1}{N} \text{Tr} \prod_{l \in \mathcal{C}} U_{\mu_l}(x_l) = \prod_{l} e^{igA_{\mu_l}(x_l)T^a}$$

To $O(\alpha^2)$ there are non-trivial contributions from two, three and four insertions of the gluon field operator at all links in the perimeter of the loop.

We can derive both the static inter-quark potential and the static quark mass renormalisation from the Wilson loop,

$$V(\vec{R}, am) = \lim_{T \to \infty} -\frac{\partial}{\partial T} \log W(\vec{R}, T)$$

$$= V_{\text{phys}}(\vec{R}, am) + V_{\text{self}}(am)$$

$$V_{\text{self}} = \lim_{|\vec{R}| \to \infty} V(\vec{R})$$

The temporal component of the internal loop momentum was analytically integrated out to pick out the pole structure, and then the remaining integrals were performed in the infinite volume limit using VEGAS. In the case of the potential and the $V_{\text{self}}$ the limits in coordinate space produce additional delta functions in momentum space which were integrated over analytically.

3. Small Wilson Loops

For pure gauge, we write the Wilson loop as,

$$W_{PG} = 1 - g^2 N^2 - 1 \frac{1}{N} \text{WT}$$

$$- g^4 [(N^2 - 1) W_1 + \frac{1}{N^2} W_2]$$

In the dynamical case we write,

$$W = W_{PG} - g^4 n_f \frac{N^2 - 1}{N} X_f^{(0)} + X_f^{(1)} c_{sw} + X_f^{(2)} e_{sw}$$

$$= W_{PG} - g^4 \frac{N^2 - 1}{N} n_f X_f^{(KS)}$$

for Wilson type fermions and,
for Kogut-Susskind fermions.

We obtain new results for non-planar Wilson loops in the pure gauge case in Table 1, and for various dynamical fermion actions in Table 2. The results for planar small Wilson loops in pure gauge theory are well known [1, 2, 5, 6].

| Loop      | $W_T$   | $W_1$   | $W_2$   |
|-----------|---------|---------|---------|
|           | 0.1961(1) | 0.002624(1) | 0.005284(5) |
|           | 0.2134(1) | 0.001845(1) | 0.003911(5) |

The extraction of $\alpha_s$ with massive dynamical quarks in a “partially quenched” analysis, where the scale is set before extrapolation in the quark mass, is somewhat subtle since it depends on both the perturbative plaquette, and on the connection to $\overline{MS}$. In Figure 1 we show the mass dependence of the perturbative plaquette with the Wilson gauge action and both (improved) Wilson quark and Kogut-Susskind quark actions with massive dynamical quarks. This shows a smooth connection to the quenched case as the mass is increased across the threshold $ma \simeq 1$. These results combined with the (lattice) mass dependent connection between the lattice and $\overline{MS}$ schemes discussed in [7] allow a calculation of $\alpha_s(m_q, \mu)$.

4. Potential

We define the lattice coordinate space potential coupling

$$\alpha_R(aR, am) = -\frac{aR}{C_F} \left( V(aR, am) - V_{self}(am) \right)$$

$$V(aR, am) = \alpha_L V_1(aR) + \alpha_2^2 \left( V_{PG}^P(aR) + n_f V_F^P(aR, am) \right)$$

$\alpha_R(aR, am)$ is constant up to discretisation errors at tree level. At one loop it runs logarithmically with $R$, with pure gauge and fermionic contributions. In pure gauge we can identify a lattice tadpole at one loop, shown in Figure 2, which is exactly proportional to the tree level contribution. The improvement of lattice potentials through the use of a “lattice-R” arises at this order through the obvious connection to the tree level one.

Figure 1. Mass dependence of the fermionic contributions to the plaquette for Wilson, Clover and Kogut-Susskind fermions. The stand-alone points correspond to the $am = 0$ limit.
Table 2  
Fermionic contribution to loops (various Quark & Gluon actions)

| Glue   | Loop | $X_{nf}^{(0)}(am=0)$  | $X_{nf}^{(1)}(am=0)$ | $X_{nf}^{(2)}(am=0)$ | $X_{nf}^{(KS)}(am=0)$ |
|--------|------|----------------------|----------------------|----------------------|----------------------|
| Wilson | 1 x 1| $-6.96(2) \times 10^{-4}$ | $2.02(2) \times 10^{-3}$ | $-5.963(1) \times 10^{-4}$ | $-6.129(3) \times 10^{-4}$ |
| Iwasaki| 1 x 1| $-1.47(1) \times 10^{-4}$ | $7.8(1) \times 10^{-7}$ | $-1.698(1) \times 10^{-4}$ | $-1.296(2) \times 10^{-4}$ |
| Wilson | 1 x 1| $-1.487(4) \times 10^{-3}$ | $3.62(5) \times 10^{-5}$ | $9.836(3) \times 10^{-4}$ | $1.337(1) \times 10^{-3}$ |
| Wilson | 2 x 1| $-1.653(6) \times 10^{-3}$ | $5.04(6) \times 10^{-5}$ | $1.1624(3) \times 10^{-3}$ | $1.445(1) \times 10^{-3}$ |
| Wilson | 1 x 2| $-1.326(3) \times 10^{-3}$ | $5.67(5) \times 10^{-5}$ | $1.4759(3) \times 10^{-3}$ | $1.179(1) \times 10^{-3}$ |
| Wilson | 2 x 2| $-2.38(1) \times 10^{-3}$ | $1.61(1) \times 10^{-4}$ | $3.3421(6) \times 10^{-3}$ | $2.056(2) \times 10^{-5}$ |

work obtains an understanding of discretisation effects on the short distance potential. This is essential in order to discern dynamical quark effects on the short distance potential. Figure 2. The one-loop potential. The lattice tadpole contribution is exactly proportional to the tree level potential. The one loop terms run with the appropriate part of the logarithmic beta function.

Figure 3. Fermionic contributions to the coordinate space coupling with the Wilson gauge action and the Clover, Wilson, and Kogut Susskind actions. Decoupling is observed in the massive case. $n_f$ is two for Wilson and Clover and four for Kogut-Susskind.

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