Signature of Non-Abelian to Abelian Transition in Spin Systems Through Geometric Phase

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Abelian and Non-Abelian evolution of a quantum system manifests differently in the geometric phase acquired by the system under such evolutions. In this work we develop and study, using dressed state techniques, an experimentally realizable spin system which allows us to transit smoothly from non-Abelian to Abelian evolutions by changing externally controllable parameters. The study provides insights into the underlying physical phenomenon governing such a transition allowing us greater control on phase generation in a quantum system. The robustness of geometric phase against fluctuations of the external parameters of the system has also been studied in this work. The noise analysis has direct consequence on the present search for fault tolerant quantum gates using geometric phase.

I. INTRODUCTION

Geometric phase originates due to cyclic time evolution of an Hamiltonian. This phase can be distinguished from the usual dynamical phase by its dependence on the quantum level structure of the system and the form of the time evolution of the involved Hamiltonian. Although geometric phase depends on the quantum level structure, it is independent of the energy eigenvalues. Therefore it is largely conceived as immune to external perturbations and hence a good quantum computation structure, it is independent of the energy eigenvalues. Therefore it is largely conceived as immune to external perturbations and hence a good quantum computation system, these quantum gates non-Abelian regime in the same spin system by slowly breaking the symmetry of the system. Analysis of such a system not only provides insights into the underlying physical processes governing each of the limits but also effectively probes the limits of the adiabatic theorem [13,14] and the relation of adiabaticity and non-abelian behaviour. This is relevant as most of the quantum states are operated in the super-adiabatic regime [13]. Our analysis also provides room for identifying the relevant parameters which influence the phase fluctuation in different regimes.

II. THEORY

In [16], Berry formulated the form of geometric phase under adiabatic approximation as

\[ \mathcal{G}_n = i \oint \langle \psi_n | \nabla_r | \psi_n \rangle \, dR \]

for the \( n \)th eigenstate. The quantity \( \gamma_n = \langle \psi_n | \nabla_r | \psi_n \rangle \) is called the gauge of the evolution because it remains invariant under any similarity transformation, except those involving the variable of the evolution themselves. This definition of the 'scalar' gauge holds only for non-degenerate levels. For degenerate levels, the definition is generalized to a matrix gauge \( \gamma_{mn} = i \langle \psi_m | \nabla_r | \psi_n \rangle \), where \( m \) and \( n \) belong to the degenerate subspace [17].

The adiabatic form of geometric phase can also be derived from the adiabatic theorem. The probability amplitude of an eigenstate belonging to a time dependent Hamiltonian varies as

\[ \dot{C}_m = -C_m \langle \psi_m | \dot{H} | \psi_m \rangle - \sum_{n \neq m} C_n \frac{\langle \psi_m | \hat{H} | \psi_n \rangle}{E_n - E_m} e^{i(\xi_n - \xi_m)}, \]

where the states \( m \) and \( n \) are non-degenerate and \( \xi_n \) and \( \xi_m \) are the dynamical phases. The above equation governs the time dependence of the amplitudes of the

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states, beyond the dynamical contribution. Under adiabatic approximation, \( \langle \psi_m | H | \psi_n \rangle \ll (E_n - E_m) \) and hence the second term in Eq. 2 can be neglected in comparison to the first term. Thus for adiabatic evolution, there is no 'mixing' of the different eigenstates. However, as a consequence of the time dependence of the Hamiltonian, there is an additional phase, on top of the dynamical phase, governed by \( \langle \psi | \psi \rangle \). Under the conditions of implicit time dependence, this term leads to the underlying gauge of geometric phase as derived by Berry. In the following sections we expand the ideas of Abelian and non-Abelian evolution and their corresponding gauges.

**Abelian:** Abelian evolution corresponds to evolution without any population transfer. For a non-degenerate set of levels, under adiabatic condition, all evolutions are Abelian, since adiabaticity guarantees lack of population transfer or mixings. Degenerate levels can also have Abelian evolutions, if the underlying gauge matrix corresponding to the degenerate subspace is diagonal.

**Non-Abelian:** On the contrary, non-Abelian evolutions inherently introduces mixing of states or population transfer between the states. For a degenerate subspace, if the off-diagonal elements of the gauge matrix are nonzero, then the evolution is considered as non-Abelian. However, degeneracy itself doesn’t guarantee non-zero off diagonal elements. A subspace of non-degenerate levels, under certain evolutions can have non-zero off diagonal elements. However, conditions imposed by adiabaticity doesn’t leave room for population transfer in such cases and the off-diagonal elements have no physical significance under such conditions. This can be easily demonstrated from Eq. 2 as the coupling term drops off because of finite strength of the oscillatory function and the relatively smaller coupling strength in the adiabatic limit.

**Non-Abelian to Abelian Transition:** The primary goal of this work is to see how a system responds if it is taken in a continuous manner from non-Abelian to Abelian evolution and vice versa. To achieve this, we need an evolution with degeneracies and focus on non-Abelian degenerate subspaces. Now, if we can introduce non-degeneracy into the system, without changing the underlying geometry and hence the gauge matrix of the evolution, then we can observe the physical significance of the off-diagonal elements slowly diminishing and vanishing in the Abelian regime. Thus the primary goal is to study the dynamics of the off-diagonal elements with respect to the symmetry breaking field. This also allows us to probe the Adiabatic theorem the limits of which, under the influence of a symmetry breaking field has long been debated.

**III. SYSTEM**

The system we work with is a spin system interacting with electro-magnetic fields. Even though we are interested in the \( D_{3/2} \) state of Ba\(^{+}\) ion interacting with a rotating electric field gradient \cite{18}, all the discussion made here holds true for any spin \( 3/2 \) system. We choose the electric field to be zero to avoid monopolar interaction. The dipole moment of this state is zero because of the definite parity of the states and hence we choose the electric quadrupole moment. Electric quadrupole moment interact with electric field gradient. To generate geometric phases, the electric field gradient is taken to be time dependent. The time dependence is such that the principle axes describe a conical path about the degeneracy point as shown in Fig. 1. The spin 3/2 interacting with the electric field gradient maintains a time reversal symmetry and hence there are Kramer’s degeneracies in the system, i.e., \( |\pm 1/2 \rangle \) and \( |\pm 3/2 \rangle \) states form two pairs of degenerate subspaces. The Hamiltonian of quadrupole interaction has \( S_z^2 \) terms and hence couples the \( |\pm 1/2 \rangle \) substates. Thus the \( |\pm 1/2 \rangle \) subspace undergoes non-Abelian evolution. The Hamiltonian however cannot couple \( |\pm 3/2 \rangle \) states, which have a \( \Delta m = 3 \) and thus cannot be coupled by the quadratic terms in angular momentum operator. Hence the \( |\pm 3/2 \rangle \) subspace undergoes Abelian evolution. This however is true only for the quadrupole approximation which limits the field expansion to \( S_z^2 \) terms only.

To induce the non-Abelian to Abelian transition, we now apply a time dependent magnetic field along with the electric field gradient. The magnetic field lifts the degeneracy of the system and hence drives the system away from non-Abelian behaviour. However, for a true transition from non-Abelian to Abelian, the underlying gauge is required to be the same in the presence of the magnetic field. This is achieved by making the magnetic field rotate along with the electric field gradient. This preserves the 'geometry' of the system even in the presence of the magnetic field as the quantization axis remains unchanged. 'Geometry' in this context implies the transformation which connects the diagonal basis of the Hamiltonian with the stationary basis. Because of this constancy of 'geometry', the underlying gauge of the evolution remains invariant. In the following sections a detailed theoretical description of the system's evolution under the influence of the time varying field will be derived.

**Non-Abelian Regime:** The quadrupole moment of the spin 3/2 system interacting with an electric field gradient gives rise to the non-Abelian regime for the \( |\pm 1/2 \rangle \) subspace. The Hamiltonian for a quadrupole moment-electric field gradient interaction is given as

\[
H_Q = \frac{1}{6} Q_{ij} \partial E_i \partial E_j.
\]
where $Q_{ij}$ is the $ij$th component of the quadrupole moment and is defined for spin systems as $Q_{ij} = c\left(\frac{1}{2}(S_i S_j + S_j S_i) - \frac{1}{3} \hat{S}^2 \delta_{ij}\right)$, $\frac{\partial E}{\partial x}$ is the $ij$th component of the electric field gradient tensor.

In our case, because of a suitable choice of the principle axes, only the $\frac{\partial E}{\partial x}$ component of the electric field gradient tensor contributes to the Hamiltonian of the system. Thus, in the non-Abelian scenario, we obtain an effective Hamiltonian given by,

$$H_{NA} = c(S_z^2 - \frac{1}{3} S^2),$$

where $S_z = S \sin \theta \cos \phi + S_y \sin \theta \sin \phi + S \cos \theta$, $c$ being the strength of interaction and $\phi = \omega t$, $\omega$ being the rotational frequency of the electric field gradient.

The eigenstates of this Hamiltonian are doubly degenerate. The two doubly degenerate subspaces consists of $|\pm \frac{1}{2}\rangle$ corresponding to eigenvalue $c$ and $|\pm \frac{3}{2}\rangle$ corresponding to eigenvalue $-c$. Now to obtain the gauge matrices corresponding to these sets of states through the relation $\gamma_{mn} = i \langle \psi_m | \nabla_{\tau} | \psi_n \rangle$, the wavefunctions in the stationary frame are required. However, for the ease of calculation, we use the Wigner D matrices to obtain the wavefunctions in the stationary basis from the rotating basis, which is also the diagonal basis for the Hamiltonian. The Wigner D matrices are the transformations which connect these two bases. The respective gauge matrices are-

$$\gamma_{\pm3/2} = \begin{pmatrix} \frac{1}{2} \cos \theta & 0 \\ 0 & -\frac{3}{2} \cos \theta \end{pmatrix}$$

and

$$\gamma_{\pm1/2} = \begin{pmatrix} \frac{1}{2} \cos \theta & \sin \theta \\ \sin \theta & -\frac{1}{2} \cos \theta \end{pmatrix}.$$

From the matrices, we can see that for the subspace $|\pm \frac{1}{2}\rangle$, we have non-zero off-diagonal elements. Hence the degenerate subspace $|\pm \frac{1}{2}\rangle$ follows non-Abelian evolution. In the non-Abelian regime, the eigengauge is given by the eigenvalues of the gauge matrix, $\pm \frac{1}{2} \sqrt{4 - 3 \cos^2 \theta}$.

**Abelian Regime:** To transfer the system from non-Abelian regime to Abelian regime, we apply a degeneracy lifting magnetic field. However, to make any comparison between the two situations, we require the 'geometry' of the system to remain invariant in the presence of the magnetic field. More precisely, the connection between the diagonal basis and the stationary frame, which is the Wigner D matrices, should remain the same in the presence or absence of magnetic field.

The effective Hamiltonian in the Abelian regime is given by

$$H_A = c(S_z^2 - \frac{1}{3} S^2) - bS_z'.$$  \hspace{1cm} (5)

The eigenvalue of the $|\pm \frac{1}{2}\rangle$ subspace now becomes $-c \mp \frac{1}{2} b$. However, the gauge matrix does not change, as the geometry is kept invariant.

In the Abelian configuration, the adiabatic theorem leads us to conclude that the off diagonal elements of the gauge matrix corresponding to these set of states, do not contribute in the physical manifestation of the phase. In this regime, the underlying gauge is simply $\pm \frac{1}{2} \cos \theta$ corresponding to the two states.

**Non-Abelian to Abelian Transition:** In the two regimes of evolution, the states, $|\pm 1/2\rangle$, are governed by two different underlying gauges given by $\pm \frac{1}{2} \sqrt{4 - 3 \cos^2 \theta}$ for non-Abelian and $\pm \frac{1}{2} \cos \theta$ for Abelian. The physical manifestation of the gauges is obtained through phase dependent energy shifts given by $A_n \omega$. Thus the variation of the energy level shifts, on top of the energy eigenvalue, while going from non-Abelian to Abelian is the primary signature of such a transition.

In the true adiabatic regime ($\omega \rightarrow 0$), even the smallest value of $b$ will drive the system from non-Abelian to Abelian. However, for finite values of $\omega$, the system is governed by two timescales, one depends on $\omega$, the rotational frequency of the fields and the other depends on $b$, which determines the splitting between the $|\pm 1/2\rangle$ states. For $\omega$ finite, Abelian regime can only be achieved for $b \gg \omega$.

**IV. DRESSED STATE CALCULATIONS**

Unlike the systems studied so far, the system we constructed above allows us to move continuously between the non-Abelian and Abelian regimes. In the previous section we saw that by moving the system from non-Abelian to Abelian regime, the off-diagonal elements loses their physical significance. Continuous tunability of our system allows us to investigate the dynamics of the off diagonal elements with respect to symmetry breaking field, in this case the magnetic field.
For studying the dynamics of the off diagonal elements, we begin at the basic equation Eq. (2) governing the evolution of the two states. Here we work with the \(|\pm 1/2\rangle\) subspaces. We assume that \(\omega\) is small enough compared to \(\epsilon\) so that we can neglect the coupling of the \(|\pm 1/2\rangle\) subspaces with \(|\pm 3/2\rangle\) subspace. By plugging in the values of the variables, the governing equation for the \(|\pm 1/2\rangle\) substates is obtained as

\[
\left( \begin{array}{c} \dot{C}_1 \\ \dot{C}_2 \end{array} \right) = i \left( \begin{array}{cc} \frac{\omega}{2} \cos \theta & \omega \sin \theta e^{ibt} \\ \omega \sin \theta e^{-ibt} & -\frac{\omega}{2} \cos \theta \end{array} \right) \left( \begin{array}{c} C_1 \\ C_2 \end{array} \right). \tag{6}
\]

For \(b = 0\), this equation governs the behaviour in the non-Abelian regime.

The matrix

\[
H = \left( \begin{array}{cc} \frac{\omega}{2} \cos \theta & \omega \sin \theta e^{ibt} \\ \omega \sin \theta e^{-ibt} & -\frac{\omega}{2} \cos \theta \end{array} \right) \tag{7}
\]

is like an effective Hamiltonian governing the evolution of the states \(|\pm \frac{1}{2}\rangle\) for a given value of \(\omega\) and \(b\).

**Application of Dressed State Method:** The dressed state approach, which is a derivative of the Floquet Theorem of differential equations with periodic coefficients, takes into account the full time dependence and allows us to obtain the true eigenvalues, considering all the effects of the time dependence. Even though it was first developed to deal with atom photon interaction, it can be generalized to any equation with periodic coefficients. For a detailed description of the mathematical algorithm applied here to obtain the eigenvalues of Eq. (7), please refer to [19].

The eigenvalue obtained using the dressed state algorithm provides the complete picture including the effect of phase dependent energy shifts of the level. It also allows us to obtain the complete dependence of the geometric phase on \(b\) and \(\omega\) and thus letting us probe not only the non-Abelian and Abelian limit but the entire behaviour of the system.

To apply the dressed state method, we assume an ansatz of the form

\[
\left( \begin{array}{c} \dot{C}_1 \\ \dot{C}_2 \end{array} \right) = \left( \begin{array}{c} \alpha_1(t) e^{-i\omega_+ t} \\ \alpha_2(t) e^{-i\omega_- t} \end{array} \right). \tag{8}
\]

Now by inserting Eq. (8) into Eq. (6) we obtain the following equation for \(\alpha_1\) and \(\alpha_2\)

\[
i \left( \begin{array}{c} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{array} \right) = \left( \begin{array}{cc} -\omega \sin \theta e^{ibt} & \omega \sin \theta e^{i(b+\omega_-)t} \\ \frac{\omega}{2} \cos \theta - \omega_- & \frac{\omega}{2} \cos \theta - \omega_+ \end{array} \right) \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right). \tag{9}
\]

Now if we choose \(\omega_{\pm} = \mp \frac{\omega}{2}\), then the above \(2 \times 2\) matrix becomes time independent and all the information about the time dependence of the system becomes encoded in behaviour of \(\alpha_1(t)\) and \(\alpha_2(t)\). Such a choice of the values of \(\omega_{\pm}\) converts the above equation into a time independent problem, with an effective Hamiltonian given by

\[
H_D = \left( \begin{array}{cc} -\frac{\omega}{2} \cos \theta + \frac{b}{\omega} & -\frac{\omega}{2} \sin \theta \\ -\omega \sin \theta & \frac{\omega}{2} \cos \theta - \frac{b}{\omega} \end{array} \right). \tag{10}
\]

The Hamiltonian in Eq. (10) is the dressed form of the effective Hamiltonian given by Eq. (7). The advantage is that we converted the time dependent problem into an effective time independent problem thus allowing us to capture the complete behaviour of the system through the eigenvalues of the dressed Hamiltonian. The eigenvalues of \(H_D\) are given by

\[
\pm \sqrt{4 \sin^2 \theta^2 + \cos^2 \theta^2 + \left( \frac{b}{\omega} \right)^2 - 2 \cos \theta \frac{b}{\omega}}. \tag{11}
\]

The complete solution for \(C_1\) and \(C_2\) is given by

\[
\left( \begin{array}{c} C_1 \\ C_2 \end{array} \right) = \left( \begin{array}{c} C_1(0) e^{-i\lambda t} \\ C_2(0) e^{+i\lambda t} \end{array} \right), \tag{11}
\]

where,

\[
\lambda = \frac{\omega}{2} \sqrt{4 \sin^2 \theta^2 + \cos^2 \theta^2 + \left( \frac{b}{\omega} \right)^2 - 2 \cos \theta \frac{b}{\omega}}. \tag{12}
\]

Here \(\lambda\) represents the phase dependent energy shift of the levels. As can be seen, this shift is of the form \(\gamma_n \omega\). In the pure non-Abelian or Abelian regime, the value of \(\gamma_n\) is independent of \(b\) or \(\omega\). However, in intermediate region, this gauge of the system depends on both the value of the degeneracy lifting field as well as the frequency of evolution.

To obtain the non-Abelian limit, we put \(b = 0\) and obtain the familiar non-Abelian gauge eigenvalues given by

\[
\pm \sqrt{4 - 3 \cos^2 \theta}.
\]

We can reach the Abelian limit by putting \(\omega \to 0\) for any value of \(b \neq 0\). However, physically, the exact value of \(\omega\) required to reach the Abelian limit, depends on the value of \(b\) and hence a more suitable limit for the Abelian
regime is $\frac{b}{\omega} \gg 1$. In this limit, the gauge tends to
\begin{equation}
\pm \frac{1}{2} \cos \theta.
\end{equation}

It should be mentioned here that the choice of $\omega_0 = \pm \frac{c}{b}$ is also a valid choice of the ansatz. However, this choice represents the opposite sense of rotation of the fields. In principle, these two choices physically correspond to a difference of $\pi$ of the angle between $\omega$ and $b$.

V. PERTURBATIVE ANALYSIS OF DIFFERENT CONTRIBUTIONS

In the previous section, we have obtained the complete behavior of the system. However, the dressed state approach did not reveal the underlying physical phenomenon governing the transition region. The physical processes controlling the two limits is however known. Now to obtain the physics of the transition region, we approach perturbatively from the two extremes and try to figure out the physical processes driving the system away from the two limits.

In this section our goal is to capture the response of the system due to small changes in system parameter($b$ or $\omega$) from its two extreme limits. We apply perturbation theory to achieve this. The key point is to choose unperturbed Hamiltonian in the two regimes. As the non-Abelian and Abelian regimes are very different in nature one should not expect to use the same unperturbed Hamiltonian to describe both. We work with the dressed state Hamiltonian, where the problem is reduced to a time independent situation. We use this ‘dressed’ Hamiltonian, to identify the unperturbed Hamiltonians governing the behaviour of the system in the two limits. Other than the unperturbed Hamiltonian, whatever is left, is treated as the perturbation.

**Perturbation in the Abelian Limit:** The Abelian limit corresponds to the situation where $b \gg \omega$. In this condition, the phase dependent energy shift is given by $\frac{b}{\frac{1}{2} \cos \theta}$. The effective time independent Hamiltonian which can describe this system, including the phase dependent energy shifts, can be written as

\begin{equation}
H^A_D = \left( \begin{array}{cc}
\frac{1}{2} \cos \theta \omega - \frac{b}{2} & 0 \\
0 & -\frac{1}{2} \cos \theta \omega + \frac{b}{2}
\end{array} \right)
\end{equation}

Now to study the deviation of the system from the Abelian limit, we rewrite the total dressed Hamiltonian as

\begin{equation}
\mathcal{H}_D = H^A_D + \delta H^A_D.
\end{equation}

where $\delta H^A_D = \mathcal{H}_D - H^A_D$ is the perturbing Hamiltonian. The form of $\delta H^A_D$ comes out as

\begin{equation}
\delta H^A_D = \left( \begin{array}{cc}
0 & \omega \sin \theta \\
\omega \sin \theta & 0
\end{array} \right).
\end{equation}

Now we calculate the terms of the perturbation series using this $\delta H^A_D$. The first order contribution of this perturbation being zero, the leading term of the perturbation series is the second order contribution which is

\begin{equation}
E''_D = -\frac{\sin^2 \theta}{\cos \theta - \frac{b}{\omega}}.
\end{equation}

We can thus have a handle on the underlying physical processes governing the level shift of the system from the Abelian behaviour. As the perturbation series reveals, the ‘non-Abelian’ perturbation does not effect the energy levels of the unperturbed states. However, it causes a population transfer between the eigenstates as is given by a non-zero second order term. We also notice that decreasing the value of $\frac{b}{\omega}$, which we know takes the system away from Abelian behaviour, also increases the

**FIG. 2:** Figure demonstrates the non-Abelian to Abelian transition. Figure (a) depicts the three dimensional dependence of the evolution gauge on $b$ and $\omega$. Figures (b),(c) and (d) show the behaviour of the system moving away from the non-Abelian point for angular frequencies of 1, 10 and 100 Hz. As can be seen, with increase of rotational frequency, the transition of the system is much more slow. The non-Abelian behaviour is ‘retained’ for higher values of magnetic field for a higher angular frequency. The figure also demonstrates that by selecting appropriate values of $b$ and $\omega$, we can bring control the behaviour of the system precisely, with the phase acquired defined only by the pair of values of $b$ and $\omega$. For these graphs, the value of $\theta$ is kept as 57.3'. Magnetic field $b$ and angular frequencies $\omega$ are expressed in Hz.
coupling between the two states.

This perturbation however fails when $\frac{b}{\omega}$ approaches $\cos \theta$. This is because for $b = \omega \cos \theta$, the second order contribution has a singularity. This point indicates a deviation of the guiding physics from the Abelian behaviour. For $b > \omega |\cos \theta|$, the Hamiltonian decomposition used above holds true.

**Perturbation in the Non-Abelian Limit:** The non-Abelian limit corresponds to $b \ll \omega$. At the non-Abelian point, that is $b = 0$, the phase dependent energy shift is given by $\frac{b}{\omega} \sqrt{4 - 3 \cos^2 \theta}$. As in the Abelian limit, in the non-Abelian limit, the Hamiltonian is given by

$$H_{D\,NA} = \left( \frac{b}{2} \cos \theta + \sqrt{3 \cos \theta} \right)$$  \hspace{1cm} (17)

We again write the total dressed Hamiltonian as

$$H_D = H_{D\,NA} + \delta H_{D\,NA}$$

where $\delta H_{D\,NA} = H_D - H_{D\,NA}$ is the perturbing Hamiltonian in the Non-Abelian limit. $\delta H_{D\,NA}$ is given as

$$\delta H_{D\,NA} = \left( -\frac{b}{2} 0 \frac{b}{2} \right).$$ \hspace{1cm} (18)

Now we calculate the perturbing terms using this $\delta H_{D\,NA}$. We find now that unlike the Abelian limit, in this case, the first order perturbation is non-zero whereas the second order contribution is zero. The first order contribution is given as

$$E_D' = \frac{b \cos \theta}{\omega 2\sqrt{4 - 3 \cos^2 \theta}}.$$ \hspace{1cm} (19)

This series illuminate the fact that in the deviation of the system from non-Abelian behaviour, level shift plays the major role. The increasing energy gap between the levels however leads to a reduced rate of population transfer and thus drives the system away from the non-Abelian behaviour.

**VI. SENSITIVITY OF GEOMETRIC PHASE TO PARAMETER FLUCTUATIONS**

Having explored both the regions perturbatively, we can now analyze the influence of fluctuations of different external parameters on the geometric phase.

To develop a general framework, let us assume that $b' = b + \delta b$ and $\omega' = \omega + \delta \omega$, where $\delta b$ and $\delta \omega$ are the fluctuations of $b$ and $\omega$ respectively. The phase at the two extremes, that is the Abelian and non-Abelian limits are independent of the values of $b$ and $\omega$. However in the intermediate regions, it is dependent on these parameters. In general, we can write the phase as follows

$$\gamma(b, \omega) = \gamma_0 + \gamma'(b, \omega)$$ \hspace{1cm} (20)

where $\gamma_0$ can be either the Abelian or non-Abelian phase and $\gamma'$ is the perturbative deviation in each limit. $\gamma_0$ is independent of $\omega$ and $b$ in both the limits.

Now to study the effect of fluctuation of each parameter on the geometric phase, we apply the above mentioned substitution and obtain

$$\gamma(b', \omega') = \gamma_0 + \gamma'(b, \omega) + \frac{\partial \gamma'}{\partial \omega} \delta \omega + \frac{\partial \gamma'}{\partial b} \delta b$$ \hspace{1cm} (21)

The effects of the fluctuations of these parameters on the geometric phase are obtained through $\frac{\partial \gamma'}{\partial \omega}$ and $\frac{\partial \gamma'}{\partial b}$.

In the Abelian limit, we have

$$\frac{\partial \gamma'}{\partial \omega} = \frac{\sin^2 \theta}{b}$$ \hspace{1cm} (22)

and

$$\frac{\partial \gamma'}{\partial b} = \frac{\omega \sin^2 \theta}{b^2}.$$ \hspace{1cm} (23)

In the above equations it is assumed $\frac{b}{\omega} \gg 1$ which holds true in the Abelian regime.

For Non-Abelian limit,

$$\frac{\partial \gamma'}{\partial \omega} = \frac{b \cos \theta}{2\omega^2 \sqrt{4 - 3 \cos^2 \theta}}$$ \hspace{1cm} (24)
and
\[ \frac{\partial \gamma}{\partial b} = \frac{\cos \theta}{2\omega\sqrt{4 - 3\cos^2 \theta}}. \] (25)

From the above four equations, depicting the effect of fluctuation of parameters on the geometric phase, it is evident that the role of $\omega$ and $b$ are interchanged in the non-Abelian and the Abelian limits. While the non-Abelian limit is more sensitive to magnetic field fluctuations, the Abelian limit on the other hand is more sensitive to fluctuations of the angular frequency.

Experimentally usually magnetic field noise is one of the biggest source of dephasing for quantum systems. From that point of view, we can say that the phase fluctuations due to magnetic field noise will be much higher in the non-Abelian than as compared to the Abelian regime and thus the Abelian limit is much more robust against fluctuation of magnetic field. However, if in a certain situation, robustness against rotational frequency is required, then non-Abelian limit is a much better choice than Abelian limit.

VII. DISCUSSION AND CONCLUSION

In this work we have performed an extensive study of a system as it is continuously moved from Abelian regime to Non-Abelian regime. Although it was known that Abelian regime is signified by non transfer of population and non-Abelian system by population transfer, we have for the first time shown the dynamics of the system with respect to a symmetry breaking field, driving the system from one regime to another.

The dressed state approach revealed the exact dynamics of the system and at the same time allowed us to probe the underlying mechanisms a play using the method of perturbation.

The perturbative approach also helped us to gain insight into robustness of geometric phase to external parameter fluctuations. As can be seen the non-Abelian limit is more susceptible to magnetic field fluctuations whereas Abelian limit is prone to fluctuations arising from angular frequency fluctuations. Thus a detailed understanding of Abelian and non-Abelian evolutions and the behaviour off the system in between can in turn lead to better designing of architecture for implementation of quantum computation protocols. The two field system provides a greater handle on phase engineering requirements for quantum technology purposes.

VIII. ACKNOWLEDGEMENTS

The authors would like to thank Mr. Sanjib Ghosh for extensive discussion both on the scientific as well as presentation aspect of this paper. We would also like to thank CQT for financial support.

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