Alternative formulation of the macroscopic field equations in a linear magneto-dielectric medium: Field equations

Michael E. Crenshaw
Charles M. Bowden Research Laboratory, US Army Combat Capabilities Development Command (DEVCOM) - Aviation and Missile Center, Redstone Arsenal, AL 35898, USA

We derive an alternative formulation of the field equations for macroscopic electromagnetic fields in a linear magneto-dielectric medium as an identity of the Maxwell–Minkowski equations, complementing a variety of other representations including the Ampère, Chu, Lorentz, and Minkowski formulations of continuum electrodynamics. In the new formulation of the macroscopic field equations, the material properties are carried as a renormalization of the temporal and spatial coordinates instead of as independent material constants. The new representation of the field equations raises some interesting physical issues with relativity and boundary conditions.

I. INTRODUCTION
The equations of motion for macroscopic electromagnetic fields in a transparent linear magneto-dielectric medium are not unique. In *A Dynamical Theory of the Electromagnetic Field*, Maxwell cast his theory of electromagnetism in terms of 20 quantities and 20 equations [1]. A decade later, Maxwell had reformulated his theory and the 28 “fundamental equations ... are scattered throughout more than twenty pages of text spanning two chapters” [2] of *A Treatise on Electricity and Magnetism*. The vector description of electrodynamics was later introduced by Heaviside. In a polarizable, magnetizable linear medium, the familiar vector field equations [3–6],

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0 \quad (1.1a) \]

\[ \nabla \cdot \mathbf{B} = 0 \quad (1.1b) \]

\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (1.1c) \]

\[ \nabla \cdot \mathbf{D} = 0, \quad (1.1d) \]

are known as the Maxwell–Minkowski equations, the Maxwell–Heaviside equations, or the macroscopic Maxwell equations, although this is often shortened to simply ‘the Maxwell equations.’ Here, the macroscopic fields, \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \) and \( \mathbf{H} \), are functions of position, \( \mathbf{r} \), and time, \( t \). The transparent linear medium is treated as being source free in order to focus on the fundamental physical issues.

There are other formulations of the macroscopic Maxwell equations that are used to emphasize various features of continuum electrodynamics. In the Chu formalism of electrodynamics [7–9], for example,

\[ \nabla \times \mathbf{E}^c + \frac{1}{c} \frac{\partial \mathbf{H}^c}{\partial t} = -\frac{1}{c} \frac{\partial \mathbf{M}^c}{\partial t} \quad (1.2a) \]

\[ \nabla \cdot \mathbf{E}^c = -\nabla \cdot \mathbf{P}^c, \quad (1.2b) \]

\[ \nabla \times \mathbf{H}^c - \frac{1}{c} \frac{\partial \mathbf{D}^c}{\partial t} = \frac{1}{c} \frac{\partial \mathbf{M}^c}{\partial t} \quad (1.2c) \]

\[ \nabla \cdot \mathbf{B}^c = -\nabla \cdot \mathbf{M}^c, \quad (1.2d) \]

the material response is separated from the electric field \( \mathbf{E}^c \) and the magnetic field \( \mathbf{H}^c \). The Ampère and Lorentz formulations of the field equations are also extant in the recent scientific literature [7–9].

In the current work, we derive another formulation of the Maxwell equations of motion of macroscopic fields in a transparent linear magneto-dielectric medium as a carefully derived identity of the Maxwell–Minkowski equations. Mathematical identities are ordinarily mundane, but it turns out that the new representation of the field equations raises some interesting physical issues with relativity and boundary conditions.

II. MAXWELLIAN CONTINUUM ELECTRODYNAMICS
We define a simple linear medium as an idealized model of a medium that can be treated as being at rest in the local coordinate frame and as having a real, time-independent linear permittivity \( \varepsilon(\mathbf{r}) \) and a real, time-independent linear permeability \( \mu(\mathbf{r}) \) corresponding to the frequency of a monochromatic field, or to the center frequency of a quasimonochromatic field [10]. As the field enters the medium from the vacuum, the field imparts optically induced forces to the material. However, it takes an intense light field applied for a long time for the material to be accelerated to relativistic speeds. Then the non-relativistic constitutive relations

\[ \mathbf{D} = \varepsilon(\mathbf{r}) \mathbf{E} \quad (2.1a) \]

\[ \mathbf{B} = \mu(\mathbf{r}) \mathbf{H} \quad (2.1b) \]

are a valid limiting case and very good approximation. (Relativistic corrections are of the order of \( |\mathbf{v}|/c \).) The constitutive relations, Eqs. (2.1), are explicit axioms of the theory.
We substitute the usual constitutive relations, Eqs. (2.1), into the Maxwell–Minkowski equations, Eqs. (1.1), to obtain

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial (\varepsilon \mathbf{E})}{\partial t} = 0 \quad (2.2a) \]
\[ \nabla \cdot (\mu \mathbf{H}) = 0 \quad (2.2b) \]
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial (\mu \mathbf{H})}{\partial t} = 0 \quad (2.2c) \]
\[ \nabla \cdot (\varepsilon \mathbf{E}) = 0 \quad (2.2d) \]

for the dynamics of macroscopic fields in a simple linear medium. Let

\[ n_e(r) = \sqrt{\varepsilon} \quad (2.3a) \]
\[ n_m(r) = \sqrt{\mu} \quad (2.3b) \]

and substitute the time-independent parameters \( n_e(r) \) and \( n_m(r) \) into Eqs. (2.2). We algebraically obtain

\[ n_m \nabla \times \mathbf{H} - \frac{n_m}{c} \frac{\partial (n_e n_m \mathbf{E})}{\partial t} = 0 \quad (2.4a) \]
\[ \nabla \cdot (n_m n_m \mathbf{H}) = 0 \quad (2.4b) \]
\[ n_e \nabla \times \mathbf{E} + \frac{n_e}{c} \frac{\partial (n_m n_m \mathbf{H})}{\partial t} = 0 \quad (2.4c) \]
\[ \nabla \cdot (n_e n_e \mathbf{E}) = 0 \quad (2.4d) \]

Next we use common vector identities \[3,4\] to commute the material parameters, \( n_e(r) \) and \( n_m(r) \), with the vector differential operators obtaining

\[ \frac{\nabla \times (n_m \mathbf{H})}{n_m} + \frac{n_e}{c} \frac{\partial (n_m n_e \mathbf{E})}{\partial t} = \frac{\nabla n_m}{n_m n_m} \times (n_m \mathbf{H}) \quad (2.5a) \]
\[ \frac{\nabla \cdot (n_m \mathbf{H})}{n_m} = \frac{\nabla n_m}{n_m n_m} \cdot (n_m \mathbf{H}) \quad (2.5b) \]
\[ \frac{\nabla \times (-n_e \mathbf{E})}{n_m} - \frac{n_e}{c} \frac{\partial (n_m \mathbf{H})}{\partial t} = \frac{\nabla n_e}{n_m n_e} \times (-n_e \mathbf{E}) \quad (2.5c) \]
\[ \frac{\nabla \cdot (-n_e \mathbf{E})}{n_m} = \frac{\nabla n_e}{n_m n_e} \cdot (-n_e \mathbf{E}) \quad (2.5d) \]

We reparametrize the macroscopic electric and magnetic fields

\[ \mathbf{\Pi} = -n_e(r) \mathbf{E} \quad (2.6a) \]
\[ \beta = n_m(r) \mathbf{H} \quad (2.6b) \]

and substitute the relations, Eqs. (2.6), into Eqs. (2.5) such that

\[ \frac{\nabla \times \beta - \frac{n_e}{c} \frac{\partial \mathbf{\Pi}}{\partial t}}{n_m} = \frac{\nabla n_m}{n_m n_m} \times \beta \quad (2.7a) \]
\[ \frac{\nabla \cdot \beta}{n_m} = - \frac{\nabla n_m}{n_m n_m} \cdot \beta \quad (2.7b) \]
\[ \frac{\nabla \times \mathbf{\Pi} - \frac{n_e}{c} \frac{\partial \beta}{\partial t}}{n_m} = \frac{\nabla n_e}{n_m n_e} \times \mathbf{\Pi} \quad (2.7c) \]
\[ \frac{\nabla \cdot \mathbf{\Pi}}{n_m} = - \frac{\nabla n_e}{n_m n_e} \cdot \mathbf{\Pi} \quad (2.7d) \]

Equations (2.7) are identities of the macroscopic Maxwell equations, Eqs. (2.2), for a linear magneto-dielectric medium. As long as we use the common and well-established model of a monochromatic/quasimonochromatic field propagating through a simple linear medium, Eqs. (2.7) constitute a valid alternative formulation of the macroscopic Maxwell field equations with the same regime of applicability including the same material parameters, the same boundary conditions, and the same limiting cases. It is straightforward, although cumbersome, to apply this procedure to more complex models of the medium.

We write a new timelike coordinate

\[ x^0 = \frac{ct}{n_e} \quad (2.8) \]

and note that \( n_e \) is not explicitly a function of time for the simplified model of a linear medium that we are treating here. We substitute Eq. (2.8) into Eqs. (2.7) to produce

\[ \frac{\nabla \times \beta}{n_m} + \frac{\partial \mathbf{\Pi}}{\partial x^0} = \frac{1}{n_m} \frac{\nabla n_m}{n_m} \times \beta \quad (2.9a) \]
\[ \frac{\nabla \cdot \beta}{n_m} = - \frac{1}{n_m} \frac{\nabla n_m}{n_m} \cdot \beta \quad (2.9b) \]
\[ \frac{\nabla \times \mathbf{\Pi}}{n_m} - \frac{\partial \beta}{\partial x^0} = \frac{1}{n_m} \frac{\nabla n_e}{n_e} \times \mathbf{\Pi} \quad (2.9c) \]
\[ \frac{\nabla \cdot \mathbf{\Pi}}{n_m} = - \frac{1}{n_m} \frac{\nabla n_e}{n_e} \cdot \mathbf{\Pi} \quad (2.9d) \]

We introduce the notation

\[ \nabla = \left( \frac{1}{n_m} \frac{\partial}{\partial x^0}, \frac{1}{n_m} \frac{\partial}{\partial y}, \frac{1}{n_m} \frac{\partial}{\partial z} \right) \quad (2.10) \]

such that

\[ \nabla \times \beta + \frac{\partial \mathbf{\Pi}}{\partial x^0} = \frac{\nabla n_m}{n_m} \times \beta \quad (2.11a) \]
plifies to for a linear, isotropic, homogeneous medium. Each linear, isotropic, homogeneous medium will have a different set of macroscopic field equations.

Like Eqs. (2.11), the set of Eqs. (2.11) is a systematically and rigorously derived identity of the macroscopic Maxwell equations, Eqs. (2.2), and constitutes a valid alternative formulation of the Maxwell field equations for a simple linear magneto-dielectric medium. Each linear, isotropic, homogeneous medium will have a different set of macroscopic field equations.

Although we can retain the spatial dependence of $n_m$ and $n_e$, we will adopt the limit of an arbitrarily large, transparent, linear, isotropic, homogeneous, magneto-dielectric medium in order to proceed with the fundamental physical issues without unnecessarily complicated formulas. Although a real-world material cannot be of infinite extent, an arbitrarily large medium can be treated as infinite in the sense that light cannot reach the boundaries of the medium in the finite time that it takes to perform an experiment. For this limit, we introduce new spatial variables

\[ \bar{x} = n_m x \]  
\[ \bar{y} = n_m y \]  
\[ \bar{z} = n_m z \]

such that Eqs. (2.11) become

\[ \bar{\nabla} \times \bar{\beta} + \frac{\partial \bar{\Pi}}{\partial \bar{x}^0} = 0 \]  
\[ \bar{\nabla} \cdot \bar{\beta} = 0 \]  
\[ \bar{\nabla} \times \bar{\Pi} - \frac{\partial \bar{\beta}}{\partial \bar{x}^0} = 0 \]  
\[ \bar{\nabla} \cdot \bar{\Pi} = 0. \]

where the material Laplacian operator, Eq. (2.10), simplifies to

\[ \bar{\nabla} = \left( \frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{z}} \right) \]

for a linear, isotropic, homogeneous medium. Each linear, isotropic, homogeneous medium will have a different set of macroscopic field equations for each set of magnetic and dielectric material characteristics.

The wave equation

\[ \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial x^0} \left( \frac{\partial \mathbf{A}}{\partial x^0} \right) = 0 \]  

is easily formed by substituting

\[ \beta = \nabla \times \mathbf{A} \]  
\[ \Pi = \frac{\partial \mathbf{A}}{\partial x^0} = 0 \]

into the variant Maxwell–Ampère Law, Eq.(2.13a). There is a different wave equation for each simple linear material, just like there is in the Maxwell–Minkowski formulation

\[ \nabla \times (\nabla \times \mathbf{A}) + \frac{n^2}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \]  

III. SIGNIFICANCE

There is an extensive body of work in classical continuum electrodynamics and in the relationship between continuum electrodynamics and other physical principles. Equations (2.11) are obviously inconsistent with extant treatments of Fresnel boundary conditions, special relativity, conservation, Lorentz invariance, and other well-known physical principles in a linear medium. Then, it is straightforward to use physical arguments from the prior work to “prove” that Eqs. (2.11) are false. However, such a proof would be a post hoc ergo propter hoc logical fallacy. Significantly, Eqs. (2.11) and (2.2) are identities and disproving Eqs. (2.11) by physical arguments would disprove the Maxwell–Minkowski field equations in a dielectric, Eqs. (2.2), thereby negating the basis on which the physical arguments are formulated.

Absent the explicit identification of a consequential error in the derivation, Eqs. (2.11) and (2.13) can be treated as proven. There are no provable errors in the derivation and we should very much like to identify the source and remedy of the inconsistencies. This process has begun and we can report success with the Fresnel relations [11] and special relativity in a dielectric [12].

1) In continuum dynamics, boundary conditions are derived by the simultaneous application of conservation of energy and conservation of linear momentum at the boundary. This procedure fails to work for continuum electrodynamics because the electromagnetic energy and momentum are both quadratic in the fields. For the electromagnetic boundary conditions and the Fresnel relations, the established textbook derivation procedure is based on the application of Stokes’s theorem and the divergence theorem to the Maxwell–Minkowski equations [3–6]. The textbook derivation procedure is specific to the Maxwell–Minkowski formulation of continuum electrodynamics, Eqs. (2.2), and isn’t based on conservation
laws: We need a general procedure. In Ref. [11], the dielectric boundary conditions and Fresnel relations were derived from the wave equation and conservation of energy. The boundary conditions and Fresnel relations that are derived from the general procedure are the same as those derived from the Maxwell–Minkowski equations and are equally consistent with experiments.  

2) The field equations, Eqs. (2.13), have the same form as the Maxwell equations in the vacuum with \( x^0 \rightarrow x^1 \). Letting \( c \rightarrow c/n \), the material Lorentz factor for a linear medium

\[
\gamma_d = \frac{1}{\sqrt{1 - n^2v^2/c^2}} \quad (3.1)
\]

is inconsistent with special relativity. If Eqs. (2.13) are proven wrong in this manner, then Eqs. (2.2) are proven wrong and we have a significant do-over.

In adapting Einstein's special theory of relativity to a dielectric medium, Laue [13, 14] applied the relativistic velocity sum rule to a dielectric material moving uniformly in a Laboratory Frame of Reference. In Einstein–Laue dielectric special relativity, the speed of light in the dielectric depends on the velocity of the dielectric relative to the Laboratory Frame and the Lorentz factor is unchanged from Einstein's special relativity. In contrast, Ref. [12] derives the Rosen [15] theory of dielectric special relativity for inertial reference frames translating at constant speed in an arbitrarily large region of space in which the speed of light is \( c/n \) in the local rest frame. In this case, the speed of light at the location of the observer in the dielectric is independent of the motion of the dielectric in accordance with the principle of relativity but the material Lorentz factor, Eq. (3.1), is material dependent confirming the phenomenological factor shown in Eq. (3.1) and in Ref. [15].

The Einstein–Laue theory of dielectric special relativity is experimentally proven by the Fizeau [16] water tube experiment. An observer in the vacuum, such as Fizeau [16], will measure the speed of light in a dielectric to be dependent on the velocity of the dielectric relative to the Laboratory Frame of Reference. We can never place a matter-based observer, no matter how small, in a continuous dielectric because the model dielectric is continuous at all length scales and will always be displaced. Ref. [12] shows how to relate measurements in the vacuum to events in the dielectric so that the Rosen theory of dielectric special relativity is also experimentally proven.

IV. CONCLUSION

The essential point of this communication is that the variant formulation of macroscopic field equations, Eqs. (2.11), is a rigorously derived and valid identity of the macroscopic Maxwell–Minkowski equations for a simple magneto-dielectric medium. Consequently, any physical principles that are inconsistent with Eqs. (2.11) do not prove an error in the derivation of Eqs. (2.11) but, instead, indicate either i) an error in the application of the fundamental physical principles, which are intrinsic to the vacuum, to a linear magneto-dielectric medium, ii) an inconsistency between the macroscopic Maxwell field equations and the fundamental physical principles, or iii) both. Having proven Eqs. (2.11) to be identities of the macroscopic Maxwell–Minkowski equations for a simple linear medium, we can use these results in future work without further argument about their validity.

[1] T. K. Simpson, Maxwell on the Electromagnetic Field: A Guided Study, Rutgers Univ. Press, New Brunswick (1997).
[2] N. Wheeler, Theories of Maxwellian Design, http://www.reed.edu/physics/faculty/wheeler/documents/Electrodynamics/Miscellaneous-Essays/Maxwellian-Theories.pdf (Jul 2017).
[3] J. D. Jackson, Classical Electrodynamics, 2nd Ed., John Wiley, New York (1975).
[4] J. B. Marion, Classical Dynamics of Particles and Systems, Academic, New York (1970).
[5] D. J. Griffiths, Introduction to Electrodynamics, Prentice-Hall, Englewood Cliffs (1981).
[6] A. Zangwill, Modern Electrodynamics, Cambridge Univ. Press, Cambridge (2011).
[7] P. Penfield, Jr. and H. A. Haus, Electrodynamics of Moving Media, MIT Press, Cambridge (1967).
[8] B. A. Kemp, Resolution of the Abraham–Minkowski debate: Implications for the electromagnetic wave theory of light in matter, J. Appl. Phys., 109, 111101 (2011).
[9] B. A. Kemp, Macroscopic Theory of Optical Momentum, Progress in Optics, 20, 437–488 (2015).
[10] M. E. Crenshaw, Application of axiomatic formal theory to the Abraham–Minkowski controversy, http://arxiv.org/pdf/1502.05997.pdf (Aug 2019).
[11] M. E. Crenshaw, Representation independent boundary conditions for a piecewise-homogenous linear magneto-dielectric medium, AIP Advances 9, 075102 (2019).
[12] M. E. Crenshaw, Reconciliation of the Rosen and Laue theories of special relativity in a linear dielectric medium, Am. J. Phys. 87, 296–300 (2019).
[13] G. Weinstein, Albert Einstein and the Fizeau 1851 Water Tube Experiment, http://arxiv.org/pdf/1204.3390.pdf (2012).
[14] M. Laue, Ann. Phys. (Berlin) 23, 989–990 (1907).
[15] N. Rosen, Am. J. Phys. 20, 161–164 (1952).
[16] H. Fizeau, On the hypotheses relating to the luminous aether, and an experiment which appears to demonstrate that the motion of bodies alters the velocity with which light propagates itself in their interior, Philos. Mag. 2, 568–573 (1851).