Quantum $\varphi$-synchronization in coupled optomechanical system with periodic modulation

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Based on the concepts of quantum synchronization and quantum phase synchronization proposed by A. Mari et al. in Phys. Rev. Lett. 111, 103605 (2013), we introduce and characterize the measure of a more generalized quantum synchronization called quantum $\varphi$-synchronization under which the pairs of variables have the same amplitude and possess same $\varphi$ phase shift. Naturally, quantum synchronization and quantum anti-synchronization become special cases of quantum $\varphi$-synchronization. Their relations and differences are also discussed. To illustrate these theories, we investigate the $\varphi$-synchronization and quantum phase synchronization phenomena of two coupled optomechanical systems with periodic modulation and show that quantum $\varphi$-synchronization is more general as a measure of synchronization. We also show the phenomenon of quantum anti-synchronization when $\varphi = \pi$.

I. INTRODUCTION

As a collective dynamic behavior in complex systems, synchronization was first proposed by Huygens in the 17th century [1–3]. He noticed that the oscillations of two pendulum clocks with a common support tend to synchronize with each other [4]. Since then, synchronization has been widely studied and applied in classical physics. Furthermore, with the development of quantum mechanics, the concept of quantum synchronization was proposed and widely applied in the fields, such as cavity quantum electrodynamics [5, 6], atomic ensembles [7–9], van der Pol (VdP) oscillators [6, 10–12], Bose-Einstein condensation [13], superconducting circuit systems [14, 15], and so on.

In recent years, there has been a growing interest in exploiting synchronization [16] for significant applications in microscale and nano-scale systems [17]. For example, synchronization of two anharmonic nanomechanical oscillators had been experimentally implemented [18] and the measurements of synchronization in two nanomechanical beam oscillators coupled by a mechanical element have been explored [19]. In addition, the relationship between quantum synchronization and the collective behavior of classical systems is also widely concerned, such as quantum synchronization of van der Pol oscillators with trapped ions [20], quantum-classical transition of correlations of two coupled cavities [21]. Besides, the correlation between the subsystems in the system with quantum synchronization, such as entanglement and mutual information, have been discussed as the main influencing factors [22–24].

Another reason for synchronization drawing much more attention recently is the generalization of its classical concept, such as complete synchronization [25], phase synchronization [26, 27], lag synchronization [28], and generalized synchronization [29], into the continuous-variable quantum systems. After Mari et al. introduced the concept of quantum complete synchronization and quantum phase synchronization [30], some interesting efforts have been devoted to enhance the quantum synchronization and quantum phase synchronization by manipulating the modulation [31, 32], the ways of coupling between two subsystems [30, 33–35], or introducing nonlinearity [36, 37]. Furthermore, the concepts of quantum generalized synchronization, time-delay synchronization as well as in-phase and anti-phase synchronization have also been mentioned in [38, 39]. However, other than the quantum complete synchronization under which the pairs of variables have same amplitude and phase, the concept of quantum anti-synchronization corresponding to classical anti-synchronization has not been proposed yet. Moreover, a more generalized quantum synchronization can be defined as “the pairs of variables have the same amplitude and possess same $\varphi$ phase shift” (hereafter referred to as quantum $\varphi$-synchronization), i.e., for $\varphi = \pi$, the pairs of variables, such as positions and momenta, will always have a $\pi$ phase difference with each other [39]. This type of quantum $\varphi$-synchronization is called quantum anti-synchronization. Hence, one will naturally ask how to define and measure the quantum $\varphi$-synchronization?

To shed light on this question, in this work we give the definition of quantum $\varphi$-synchronization for the continuous-variable quantum systems by combining the concept of quantum synchronization and the phenomenon of transition from in-phase to anti-phase synchronization [39]. The paper is organized as follows. In Sec. II, we first reexamine the definitions of quantum complete synchronization and phase synchronization. Based on these concepts, the definition of quantum $\varphi$-synchronization is given, by which the quantum synchronization and quantum anti-synchronization can be treated as special cases of quantum $\varphi$-synchronization. The $\varphi$-synchronization of a coupled optomechanical system with periodic modulation is studied to illustrate our theory in Sec. III. In Sec. IV, a brief discussion and summary are given.

II. MEASURE OF QUANTUM SYNCHRONIZATION AND QUANTUM $\varphi$-SYNCHRONIZATION

Unlike the synchronization in classical system, the complete synchronization in quantum system can not be defined...
straightforwardly, since the differences between the variables in the two subsystem must be strict to the limits brought by the Heisenberg principle. To address this issue, Mari et al. proposed the measure criterion of quantum complete synchronization for continuous variable (CV) systems [30]

$$S_c = \frac{1}{\langle q_- (t)^2 + p_- (t)^2 \rangle},$$  \hspace{1cm} (1)

where $q_- (t) = \frac{1}{\sqrt{2}}[q_1 (t) - q_2 (t)]$ and $p_- (t) = \frac{1}{\sqrt{2}}[p_1 (t) - p_2 (t)]$ are error operators. In order to study purely quantum-mechanical effects, the changes of variables are generally taken as

$$q_- (t) \rightarrow \delta q_- = q_- (t) - \langle q_- (t) \rangle,$$
$$p_- (t) \rightarrow \delta p_- = p_- (t) - \langle p_- (t) \rangle.$$  \hspace{1cm} (2)

Then the contribution of the classical systematic error brought by the mean values $\langle q_- (t) \rangle$ and $\langle p_- (t) \rangle$ in $S_c$ can be dropped, and $S_c$ will be replaced by the pure quantum synchronization measure

$$S_q = \frac{1}{\langle \delta q_- (t)^2 + \delta p_- (t)^2 \rangle},$$  \hspace{1cm} (3)

This is obviously not strict, unless the mean values of $q_- (t)$ and $p_- (t)$ are exactly zero, i.e., $\langle q_- (t) \rangle = 0$ and $\langle p_- (t) \rangle = 0$.

Mari et al. have explained that if the averaged phase-space trajectories (limit cycles) of the two systems are constant but different from each other, a classical systematic error can be easily excluded from the measure of synchronization [30] and mean-value synchronization is regarded as a necessary condition of pure quantum synchronization [37]. So, it is more reasonable and rigorous to study pure quantum synchronization based on mean-value synchronization. Similarly, we can generalize the definition of quantum complete synchronization to the quantum $\varphi$-synchronization

$$S_\varphi = \frac{1}{\langle q_\varphi (t)^2 + p_\varphi (t)^2 \rangle},$$  \hspace{1cm} (4)

which doesn’t require mean-value synchronization. The $\varphi$-error operators are defined as $q_\varphi (t) = \frac{1}{\sqrt{2}}[q_1 (t) - q_2 (t)]$ and $p_\varphi (t) = \frac{1}{\sqrt{2}}[p_1 (t) - p_2 (t)]$ with

$$q_\varphi (t) = q_j (t) \cos (\varphi_j) + p_j (t) \sin (\varphi_j),$$
$$p_\varphi (t) = p_j (t) \cos (\varphi_j) - q_j (t) \sin (\varphi_j),$$  \hspace{1cm} (5)

where the phase $\varphi_j = \arctan [(p_j (t))/q_j (t)]$. The upper limit of $S_\varphi$ is also given by the Heisenberg principle

$$S_\varphi \leq \frac{1}{\langle q_\varphi (t)^2 + p_\varphi (t)^2 \rangle} \leq \frac{1}{2 \sqrt{(q_\varphi (t)^2)(p_\varphi (t)^2)}} \leq \frac{1}{2 \sqrt{(q_\varphi (t)^2) - \langle q_\varphi (t)^2 \rangle \langle (p_\varphi (t)^2) - \langle p_\varphi (t)^2 \rangle \rangle}} \leq \frac{1}{\sqrt{\frac{1}{2}[q_\varphi (t), p_\varphi (t)]^2 + \frac{1}{2}[p_\varphi (t), p_\varphi (t)]^2}} = 1.$$  \hspace{1cm} (6)

This means that the closer $S_\varphi$ is to 1, the better the quantum $\varphi$-synchronization. Again, let’s take the changes of variables

$$q_\varphi (t) \rightarrow \delta q_\varphi (t) = q_\varphi (t) - \langle q_\varphi (t) \rangle,$$
$$p_\varphi (t) \rightarrow \delta p_\varphi (t) = p_\varphi (t) - \langle p_\varphi (t) \rangle.$$  \hspace{1cm} (7)

The mean values of $q_\varphi (t)$ and $p_\varphi (t)$ are zero when the average amplitude and period of the two variables are the same. From Eq. (5), $S_\varphi$ equals to the pure quantum $\varphi$-synchronization measure $S_\varphi$ mathematically

$$S_\varphi = \frac{1}{\langle \delta q_\varphi (t)^2 + \delta p_\varphi (t)^2 \rangle} = \frac{1}{2 \langle \delta q_\varphi (t)^2 + \langle \delta q_\varphi (t)^2 \rangle \rangle} \langle \delta q_\varphi (t)^2 + \delta p_\varphi (t)^2 \rangle$$

$$+ 2(\delta p_1 \delta q_2 - \delta q_1 \delta p_2) \sin \varphi - 2(\delta p_1 \delta p_2 + \delta q_1 \delta q_2) \cos \varphi \rangle^{-1},$$  \hspace{1cm} (8)

where $\varphi = \varphi_2 - \varphi_1$ can be determined by the final steady state. Now let’s explain the relationship between quantum $\varphi$-synchronization and quantum complete synchronization, quantum phase synchronization. We can see from Eq. (8) that the definition of $S_\varphi$ can be reduced into (a) quantum synchronization: if $\varphi = 0$, then $S_\varphi = S_q$; (b) quantum phase synchronization: if $\langle \delta q_\varphi (t)^2 \rangle = \langle \delta p_\varphi (t)^2 \rangle$, then $S_\varphi = S_q$; (c) if $\varphi = \pi$, $S_\varphi = S_r$, where $S_r$ can be defined as quantum anti-synchronization. Therefore, quantum synchronization and quantum anti-synchronization are the special cases of quantum $\varphi$-synchronization. But the definition of quantum phase synchronization is slightly different [30]

$$S_p = \frac{1}{2 \langle \delta p_\varphi (t)^2 \rangle^{-1} = \frac{1}{\langle \delta p_\varphi (t)^2 \rangle + \langle \delta p_\varphi (t)^2 \rangle}.$$  \hspace{1cm} (9)

By comparing with Eq. (6), $S_p$ is free from the constraints of the Heisenberg uncertainty. Unlike $S_\varphi$, the measure of quantum phase synchronization $S_p$ can exceed 1. To illustrate these definitions, we next compare quantum $\varphi$-synchronization with quantum synchronization and quantum phase synchronization in coupled optomechanical system with periodic modulation.

### III. QUANTUM SYNCHRONIZATION, QUANTUM PHASE SYNCHRONIZATION AND QUANTUM $\varphi$-SYNCHRONIZATION IN COUPLED OPTOMECANICAL SYSTEM WITH PERIODIC MODULATION

To examine the relations and differences between quantum synchronization, quantum phase synchronization and quantum $\varphi$-synchronization, we consider a coupled optomechanical system with periodic modulation [35, 36]. Two subsystems coupled by optical fibers [40], consisting of a mechanical oscillator coupled with a Fabry-Perot cavity and driven by a time-periodic modulated filed (see Fig. 1). Then the Hamil-
tonian of the whole coupled system can be written as

\[ H = \sum_{j=1}^{2} \left\{-\Delta_j [1 + A_j \cos(\omega_j t)] a_j^\dagger a_j + \frac{\omega_j}{2} \left(p_j^2 + q_j^2\right) - ga_j^\dagger a_j q_j + iE(a_j^\dagger - a_j)\right\} + \lambda \left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \tag{10}\]

where \(a\) and \(a^\dagger\) are the creation and annihilation operators, \(q_j\) and \(p_j\) are the position and momentum operators of mechanical oscillator with frequency \(\omega_j\) in the \(j\)th subsystem, respectively \([41, 42]\). \(\lambda\) is the optical coupling strength and \(E\) is the intensity of the driving field. \(\Delta_j\) is the optical detuning, which is modulated with a common frequency \(\omega_j\) and amplitude \(A_j\). \(g\) is the optomechanical coupling constant. To solve time evolution of the dynamical operators \(O = q_j, p_j, a_j\) of the system, we consider the dissipation effects in the Heisenberg picture and utilize the quantum Langevin equation \([43]\). From Eq. (10), the evolution equation of the operators can be written as:

\begin{align*}
\frac{\partial}{\partial t} q_j &= \omega_j p_j, \\
\frac{\partial}{\partial t} p_j &= -\omega_j q_j - \gamma p_j + ga_j^\dagger a_j + \xi_j, \\
\frac{\partial}{\partial t} a_j &= -\left(\kappa - i\Delta_j [1 + A_j \cos(\omega_j t)]\right) a_j + ig \langle a_j \rangle q_j + E \\
&\quad - i\lambda (a_3^\dagger a_j + \sqrt{2}\kappa a_j^\dagger a_j), \tag{11}
\end{align*}

where \(\kappa\) is the radiation loss coefficient \([44, 45]\) and \(\gamma\) is mechanical damping rate, respectively. \(a_j^\dagger\) and \(\xi_j\) are input bath operators and satisfy standard correlation:

\[ \langle a_j^\dagger(t) a^\dagger(t') + a_j(t) a_j(t')\rangle = \delta(t-t') \quad \text{and} \quad \langle \xi_j(t) \xi_j(t') + \xi_j^\dagger(t') \xi_j^\dagger(t)\rangle = \gamma (2\hbar_{\text{bath}} + 1)^2 \delta(t-t') \quad \text{under the Markovian approximation} \ [41, 42], \]

where \(\hbar_{\text{bath}} = 1/\exp(\hbar \omega_j/k_B T) - 1\) is the mean occupation number of the mechanical baths which gauges the temperature \(T\) of the system \([46-48]\). To uncover the effects of average error and quantum fluctuation on quantum synchronization as well as the correlation between mean-value synchronization and quantum synchronization, we use the mean field approximation to solve the quantum Langevin equation \([33, 49-51]\). Namely, the operators are decomposed into an average value and a small fluctuation, i.e.

\[ O(t) = \langle O(t) \rangle + \delta O(t). \tag{12}\]

Then, Eq. (11) can be divided into two different sets of equations, one for the mean value

\[ \frac{\partial}{\partial t} \langle a_j \rangle = \omega_j \langle p_j \rangle, \]

\[ \frac{\partial}{\partial t} \langle q_j \rangle = -\omega_j \langle q_j \rangle - \gamma \langle p_j \rangle + g\langle a_j \rangle^2, \]

\[ \frac{\partial}{\partial t} \langle a_j \rangle = -\left(\kappa - i\Delta_j [1 + A_j \cos(\omega_j t)]\right) \langle a_j \rangle + ig \langle a_j \rangle \langle q_j \rangle + E \\
&\quad - i\lambda (a_3^\dagger \langle a_j \rangle + \sqrt{2}\kappa \langle a_j \rangle^\dagger \langle a_j \rangle), \tag{13}\]

and the other for the fluctuation:

\[ \frac{\partial}{\partial t} \langle q_j \rangle = -\omega_j \langle q_j \rangle - \gamma \langle p_j \rangle + g\langle a_j \rangle^2 \langle \delta a_j \rangle + \langle q_j \rangle^\dagger \langle \delta a_j \rangle + \xi_j, \]

\[ \frac{\partial}{\partial t} \langle a_j \rangle = -\left(\kappa - i\Delta_j [1 + A_j \cos(\omega_j t)]\right) \langle a_j \rangle + ig \langle a_j \rangle \langle q_j \rangle \\
&\quad + \langle q_j \rangle \langle \delta a_j \rangle - i\lambda \langle \delta a_{3-j} \rangle + \sqrt{2}\kappa \langle \delta a_j \rangle^\dagger \langle \delta a_j \rangle. \tag{14}\]

By define \(u = (\langle q_j \rangle, \langle p_j \rangle, \langle q_1 \rangle, \langle q_2 \rangle, \langle q_3 \rangle, \langle q_4 \rangle, \langle q_5 \rangle, \langle q_6 \rangle, \langle q_7 \rangle)^T\) with \(\delta q_j = \frac{1}{\sqrt{2}} (\delta a_j^\dagger - \delta a_j), \delta q_j = \frac{1}{\sqrt{2}} (\delta a_j^\dagger + \delta a_j)\), Eq. (14) can be simplified to:

\[ \frac{\partial}{\partial t} \mu = Mu + n, \tag{15}\]

where \(n = (0, \xi_1, \sqrt{2}\kappa_{x1}^m, \sqrt{2}\kappa_{y1}^m, 0, \xi_2, \sqrt{2}\kappa_{x2}^m, \sqrt{2}\kappa_{y2}^m)^T\) is the noise vector with \(x_1^m = \frac{1}{\sqrt{2}} (a^{m+} + a^{m-}), \gamma_1^m = \frac{1}{\sqrt{2}} (a^{m+} - a^{m-})\). \(M\) is a time-dependent coefficient matrix:

\[ M = \begin{pmatrix} M_1 & M_0 \\ M_0 & M_2 \end{pmatrix}, \tag{16}\]

with

\[ M_1 = \begin{pmatrix} 0 & \omega_j & 0 \\ -\omega_j & -\gamma & \sqrt{2}g \text{Re} \langle a_j \rangle \\ -\sqrt{2}g \text{Im} \langle a_j \rangle & 0 & -\kappa \end{pmatrix}, \]

\[ M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \]

and

\[ M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}. \]

where \(F_j = \Delta_j [1 + A_j \cos(\omega_j t)] + g \langle q_j \rangle\). In order to study the contribution of quantum fluctuation to quantum synchronization, we consider the following covariance matrix:

\[ V_{ij} = \frac{1}{2} (u_i u_j + u_j u_i). \tag{17}\]

The evolution of \(V\) over time is governed by \([49, 52-54]\):

\[ \frac{\partial}{\partial t} V = MV + VM^T + N. \tag{18}\]

The noise matrix \(N = \text{diag}(0, \gamma(2\hbar_{\text{bath}} + 1), \kappa, \kappa, 0, \gamma(2\hbar_{\text{bath}} + 1), \kappa, \kappa)\) satisfying \(N_j \delta(t-t') = \frac{1}{\sqrt{2}} (\xi_j(t) \xi_j(t') + \xi_j^\dagger(t') \xi_j^\dagger(t)).\)

Hence, Eq. (3), Eq. (8) and Eq. (9) can be rewritten in terms of \(V_{ij}\):

\[ S_{q} = \frac{1}{2} \begin{pmatrix} V_{11} + V_{22} + V_{55} + V_{66} - V_{15} - V_{51} - V_{62} - V_{26} \end{pmatrix}^{-1}, \]

\[ S_{q}^e = \frac{1}{2} \begin{pmatrix} V_{11} + V_{22} + V_{55} + V_{66} + 2V_{25} \sin \varphi - 2V_{16} \sin \varphi \\ -2V_{26} \cos \varphi - 2V_{15} \cos \varphi \end{pmatrix}^{-1}, \]

\[ S_{p} = [V_{11} (\sin \varphi)^2 + V_{22} (\cos \varphi)^2 + V_{55}(\sin \varphi)^2 + V_{66}(\cos \varphi)^2 + 2V_{12} \sin \varphi \cos \varphi - 2V_{15} \sin \varphi \sin \varphi + 2V_{16} \sin \varphi \cos \varphi + 2V_{25} \cos \varphi \sin \varphi - 2V_{26} \cos \varphi \cos \varphi - 2V_{56} \cos \varphi \sin \varphi - \varphi)^2], \tag{19}\]

FIG. 1. Schematic illustration of coupled optomechanical system with periodic modulation.
and its evolution can be derived by solving Eq. (13), Eq. (15) and Eq. (18).

As we discussed in the last section, if $\varphi = 0$, which requires the condition of mean-value complete synchronization (As shown in Fig. 2), i.e., $\langle \dot{q}_1(t) \rangle = \langle \dot{q}_2(t) \rangle = 0$, $\langle \dot{p}_1(t) \rangle = \langle \dot{p}_2(t) \rangle = 0$, the measures of quantum $\varphi$-synchronization and quantum synchronization are equivalent, i.e., $S^\varphi_q = S_q$. However, if the mean-value synchronization is incomplete, quantum synchronization $S_q$ and quantum $\varphi$-synchronization are different as shown in Fig. 3. In this case, the mean-value synchronization is not complete, the definition of quantum $\varphi$-synchronization ($\varphi \approx 0.64$) is more rigorous and reasonable. Because quantum $\varphi$-synchronization does not require mean-value synchronization and can give the measure of synchronization for any arbitrary $\varphi \in [0, 2\pi]$. By comparing Fig. 2 and Fig. 3, we can see that the measure of quantum $\varphi$-synchronization is more general than the measure of the quantum synchronization.

Moreover, the quantum $\varphi$-synchronization also can be related to the quantum phase synchronization. As shown in Fig. 4(a), both quantum $\varphi$-synchronization $S^\varphi_q$ and quantum phase synchronization $S_p$ first decrease and then increase as the increase of optical coupling strength $\lambda$, and the changing trend of $S_p$ and $S^\varphi_q$ with $\lambda$ is accordant. When $\lambda = 0.016$, both $S^\varphi_q = 0.58$ and $S_p = 0.36$ are minimized. This means that $\langle \delta q^2_1(t)^2 \rangle$ is approximately proportional to $\langle \delta p^2_1(t)^2 \rangle$ ($\langle \delta q^2_2(t)^2 \rangle$ is greater than $\langle \delta p^2_2(t)^2 \rangle$). In this case, the definition of $\varphi$-synchronization is accordance with that of the phase synchronization. When $\langle \delta q^2_1(t)^2 \rangle = \langle \delta p^2_1(t)^2 \rangle$, the two definitions are the same. However, if $\langle \delta q^2_1(t)^2 \rangle$ has no linear relation with $\langle \delta p^2_1(t)^2 \rangle$, the definitions of $\varphi$-synchronization and phase synchronization are quite different as shown in Fig. 4(b). Since the quantum $\varphi$-synchronization and quantum phase synchronization of the system are extremely critical to the modulation frequency, so we set a fixed value of frequency $\omega_c = 3$. In Fig. 4(b), the quantum $\varphi$-synchronization $S^\varphi_q$ becomes worse when the modulation amplitude $A_c$ increase. While the quantum phase synchronization $S_p$ is significantly enhanced. It is beneficial to increase the modulation amplitude to enhance the quantum phase synchronization but not for the quantum $\varphi$-synchronization. This difference is due to the fact that the quantum $\varphi$-synchronization takes both $\langle \delta q^2_1(t)^2 \rangle$ and $\langle \delta p^2_1(t)^2 \rangle$ into consideration, while the quantum phase synchronization only considers $\langle \delta p^2_1(t)^2 \rangle$.

When $\varphi = \pi$, the $\varphi$-error operators becomes $q^2(t) = \frac{1}{2}[q_1(t) + q_2(t)]$ and $p^2(t) = \frac{1}{4}[p_1(t) + p_2(t)]$. The quantum $\varphi$-synchronization becomes quantum anti-synchronization, i.e.,

$$S^\varphi_q \equiv S^\pi_q = \frac{1}{\langle \delta q^2_1(t)^2 + \delta p^2_1(t)^2 \rangle} = \left(\frac{1}{2} \langle \delta q_1 + \delta q_2 \rangle^2 \right)^{-1}.$$ 

(20)

We can also find this phenomenon of quantum anti-synchronization in coupled optomechanical system under certain parameters. As shown in Fig. 5, quantum anti-synchronization is that when the mean-value is anti-synchronization and quantum $\varphi$-synchronization is not zero.
FIG. 5. The evolution of the quantum synchronization $\tilde{S}_q$ (green solid line), and the mean value $\langle q_1 \rangle$&$\langle p_1 \rangle$ (red solid line), $\langle q_2 \rangle$&$\langle p_2 \rangle$ (blue solid line) when the system is stable with (a) $\lambda = 0.3$ and $A_0 = 1.5$, $\omega_c = 2$, (b) $\lambda = 0.2$ and $A_0 = 1$, $\omega_c = 2$. The other parameters are the same as in Fig. 2.

IV. CONCLUSIONS

In summary, we have introduced and characterized a more generalized concept called quantum $\varphi$-synchronization. It can be defined as the pairs of variables have the same amplitude and possess same $\varphi$ phase shift. The measure of the quantum complete $\varphi$-synchronization has also been defined without the need of mean-value synchronization. Therefore, the quantum synchronization and quantum anti-synchronization can be treated as the special cases of quantum $\varphi$-synchronization. Besides, the quantum phase synchronization can also be related with the quantum $\varphi$-synchronization. As an example, we have investigated the quantum $\varphi$-synchronization and quantum phase synchronization phenomena of two coupled optomechanical systems with periodic modulation. It has been shown that quantum $\varphi$-synchronization is more general as a measure of synchronization than the quantum synchronization. We have showed the different affections of the optical coupling coefficient and the modulation amplitude on the quantum phase synchronization and the quantum $\varphi$-synchronization. These two definitions of synchronization are only accordant with each in the case that $\langle \delta q^2(i)^2 \rangle$ is approximately proportional to $\langle \delta p^2(i)^2 \rangle$. Based on quantum $\varphi$-synchronization, the quantum anti-synchronization phenomenon are also been defined and observed for $\varphi = \pi$ under some parameters.

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