Gravitational Decoupling in Cosmology

Francisco X. Linares Cedeño
Departamento de Física, DCI,
Campus León, Universidad de Guanajuato,
37150, León, Guanajuato, México.

Ernesto Contreras †
School of Physical Sciences & Nanotechnology,
Yachay Tech University, 100119 Urcuquí, Ecuador

Through the Minimal Geometric Deformation approach, in the present work we show how the Gravitational Decoupling formalism must be reformulated in order to decouple sources in cosmological scenarios. In particular, we use the formalism in Friedmann-Robertson-Walker and Kantowski-Sachs universes. We demonstrate that the gravitational decoupling leads to modifications of well known cosmological solutions. For instance, the appearance of an effective spatial curvature in the Friedmann-Robertson-Walker metric, as well as the incorporation of several kind of matter components in the Kantowski-Sachs case. Thus, we found that it is possible to source curvature and matter terms from geometry, which in a cosmological context may be useful to address the dark matter and dark energy problems.

I. INTRODUCTION

Undoubtedly, the Minimal Geometric Deformation approach (MGD), and its extensions, have been consolidated as a powerful and efficient way to study the decoupling of gravitational sources. This have been verified in several problems within the context of relativistic astrophysics [1–43].

It is well known that, due to the non-linearity of the Einstein Field Equations (EFE), obtaining new and relevant solutions is in general a difficult task, even for static and spherically symmetric spacetimes. Moreover, if a set of solutions is known, it is not true that a linear combination of them leads to new solutions of the EFE, this is, the superposition principle is not valid for the theory of General Relativity. Nonetheless, in the framework of MGD, given two gravitational sources $A$ and $B$, where $A$ corresponds to a well known solution of the EFE, it is possible to use it to seed solutions for $B$, which is the source of a new set of equations. Let us explain this with more detail. Suppose that certain well known solution of the EFE has a line element parameterized as

$$ds^2 = -e^{\nu(r)}dt^2 + \frac{dr^2}{\mu(r)} + r^2d\Omega^2,$$

in presence of a perfect fluid $T^\mu_\nu = \text{diag}(\rho, p, p, p)$, and where the gravitational potentials $\mu$ and $\nu$ are functions only of the radial coordinate $r$. Now, in order to extend the isotropic solution to anisotropic domains by means of the MGD, we implement the following protocol: first, we have to consider a more general energy-momentum tensor $T^{\text{tot}}_{\mu\nu}$. Then, we introduce a deformation in the $g^{rr}$ component of the metric in the following way

$$\mu(r) \rightarrow e^{-\lambda(r)} = \mu(r) + \alpha f(r),$$

where $\alpha$ is a constant measuring the strength of the geometric deformation induced by the decoupling function $f$. Finally, we impose that the following line element

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda}dr^2 + r^2d\Omega^2,$$

is a solution of the EFE

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T^{\text{tot}}_{\mu\nu},$$

where $\kappa^2 = 8\pi G$, and we assume that the total energy-momentum tensor is given by

$$T^{\text{tot}}_{\mu\nu} = T_{\mu\nu} + \alpha \theta_{\mu\nu},$$

where $T^\mu_\nu$ is the perfect fluid mentioned before, and $\theta^\mu_\nu = \text{diag}(\rho^\theta, p^\theta, p^\theta, p^\theta)$ is the anisotropic sector induced by the decoupling function $f$. Within the framework of the MGD approach, the perfect fluid $T^\mu_\nu$ and the anisotropic sector $\theta^\mu_\nu$ interact only gravitationally [13, 17, 20], this is

$$\nabla_\mu T^\mu_\nu = \nabla_\mu \theta^\mu_\nu = 0.$$

What follows is to compare terms. After some algebraic computations, we obtain two sets of differential equations, one for the perfect fluid

$$\rho = \frac{\rho \mu^\prime + \mu - 1}{\kappa^2 r^2},$$

$$p = \frac{\rho \mu \mu^\prime + \mu - 1}{\kappa^2 r^2},$$

$$p = \frac{\rho \mu^\prime \mu + 2\rho \mu \mu^\prime + \rho \mu^2 + 2\mu^\prime}{4\kappa^2 r},$$

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*fran2012@fisica.ugto.mx
†econtreras@yachaytech.edu.ec
and a second one for the anisotropic sector,

\[
\begin{align*}
\rho^0 &= -\frac{rf' + f}{k^2r^2}, \\
p_r^0 &= \frac{rfv' + f}{k^2r^2}, \\
p_{\perp}^0 &= -\frac{r(f'v'' + 2fv' + rf'' + 2fv')}{4k^2r},
\end{align*}
\]

where primes indicate derivative with respect to the radial coordinate \( r \). For a given solution of Eq. 7 for the gravitational potentials \( \{\nu, \mu\} \), another solution can be found by solving the second set of equations (8) involving the unknowns \( \{f, \rho^0, p^0_r, p^0_\perp\} \), this is, three equations with four unknowns. In order to completely determine the system, extra conditions have to be implemented. Some of the cases are listed below:

- **Interior solutions.** In this case, the mimic constraint for the radial pressure as an extra condition, namely \( p = p^0_r \), have been used [17, 32].
- **Hairy Black Hole.** In this case, it is usual to impose suitable Equations of States (EoS) in the anisotropic sector [21, 31].
- **Inverse problem.** The constraint is simply \( \tilde p_\perp - \tilde p_r = p_\perp^0 - p_r^0 \), where \( p_\perp, p_r \) corresponds to the components of \( T^\mu_\nu \). In contrast to the standard procedure, in this case it is assumed that a solution of Eq. (4) is given, and the goal is to explore both, the isotropic and decoupler sector. This problem has been worked out in (3 + 1) and (2 + 1)-dimensions [28].

We note that once the system (8) is solved, the solution of Eq. (4) is given by \( \{\nu, \lambda, \tilde p, \tilde p_r, \tilde p_\perp\} \), where \( \lambda \) can be determined by using both, the decoupling equation (2) and the total energy-momentum tensor \( T^\mu_\nu \), which is defined as

\[
\begin{align*}
\tilde \rho &= \rho + \alpha \rho^0, \\
\tilde p_r &= p + \alpha p^0_r, \\
\tilde p_\perp &= p + \alpha p^0_\perp.
\end{align*}
\]

We want to emphasize that the decoupling of gravitational sources in General Relativity is a highly non-trivial theoretical problem. In this sense, the MGD is not just a technique to solve the EFE. Moreover, the power of this formalism lies in the fact that it allows to solve the problem of decoupling in a direct and systematic way.

In summary, the MGD approach have been focused so far on astrophysical systems, and it has generated interesting results by extending well known solutions including anisotropic corrections. Thus, it will be interesting to explore the consequences on cosmological scenarios when considering the presence of a geometric deformation as that discussed above.

Current cosmological and astrophysical observations indicate that the most accepted cosmological model is the so-called \( \Lambda \)CDM [44–49], which offers an accurate phenomenological description of the dynamics and evolution of the Universe. According to this model, only \( \sim 4\% \) of total matter-energy content is constituted by ordinary matter made of the known fundamental particles. An-other \( \sim 26\% \) is attributed to Cold Dark Matter (CDM), a non-relativistic particle whose interaction is mostly gravitational; the remaining \( \sim 70\% \) belongs to Dark Energy, responsible of the current accelerated expansion of the Universe, and which enters in the EFE as a cosmological constant \( \Lambda \).

Nonetheless, despite the success of the \( \Lambda \)CDM model, the true nature of dark matter and dark energy remains unknown, and this fact has motivated many alternative models. Particularly, there are theoretical proposals in which the law of gravity changes at large scales, such as \( MOND \) (MOdified Newtonian Dynamics), where the acceleration at galactic scale obey a different law of grav-

\( \Lambda \)CDM [44–49], \( f(R) \) theories, where higher order terms of the Ricci scalar modify the equations of motion (53–59), \( Brane \)world models, where the EFE are generalized with new tensors arising from an extra spatial dimension (61). There are also some proposals based on what is called Modified Gravity, in which geometric extensions of the theory of General Relativity are proposed to explain the late time acceleration (65–68), and which have been applied to the realm of dark matter as well (69–73).

Thus, it is our main goal to reformulate the MGD approach in order to extend well-known cosmological solutions, to new ones including a new geometric source. The protocol presented in this work could be important in cosmology, since it could give us hints about a geometric origin of the components of the Universe.

This work is organized as follows. Section II is devoted to the reformulation of the MGD in a Friedmann-Robertson-Walker geometry. We adapt the MGD to extent solutions of Kantowski-Sachs spacetimes in Section III. The summary and perspectives of the work are given in the last Section.

**II. GRAVITATIONAL DECOUPLING FOR A FRIEDMANN-ROBERTSON-WALKER SPACETIME**

In this section, we implement the MGD approach to decouple an anisotropic metric from an isotropic sector given by the well known Friedmann-Robertson-Walker (FRW) metric, which in spherical coordinates is given by

\[
ds^2 = -dt^2 + \frac{a^2(t)}{1 - kr^2}dr^2 + a^2(t)r^2d\Omega^2. \tag{10}
\]

Such line element can be written as

\[
ds^2 = -e^\nu dt^2 + \frac{dr^2}{\mu(r, t)} + R^2(r, t)d\Omega^2, \tag{11}
\]
where
\[ e^\nu = 1 , \] (12a)
\[ \mu(r, t) = \frac{a^2(t)}{1 - kr^2} , \] (12b)
\[ R(r, t) = a(t)r . \] (12c)

Note that with the parameterization (11), the FRW metric looks formally like the line element of the isotropic sector in Eq. (4), which is used as a seed for the MGD approach described in the previous section. However, in contrast with Eq. (1), the metric functions (11) depend also on the cosmic time \( t \) through the scale factor \( a(t) \). This feature leads to a system of differential equations arising from Eq. (2) (3) and (4) that cannot be successfully decoupled. In order to overcome the above mentioned difficulty, we reformulate the MGD by proposing the following change: instead of considering the deformation given by Eq. (4), we will consider a more general transformation
\[ \mu \rightarrow e^{-\lambda} = \tilde{\mu}(t, r) , \] (13)
where \( \tilde{\mu} \) contains the information from the isotropic sector through the function \( \mu \), and from the decoupler function \( f \), which in general will be a function of both, the radial coordinate \( r \) and the cosmic time \( t \). In this case, a suitable choice for \( \tilde{\mu} \) is
\[ \tilde{\mu}(r, t) = \frac{a^2(t)}{1 - kr^2 + af(t, r)} , \] (14)
from where Eq. (4) reads
\[ ds^2 = -dt^2 + \frac{a^2(t)}{1 - kr^2 + af(t, r)} dr^2 + a^2(t) r^2 d\Omega^2 . \] (15)

From now on, the implementation of the MGD is straightforward. Considering Eq. (15) as a solution of the EFE, we obtain
\[ G_{00} = 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] - \alpha \left[ \frac{f + rf'}{r^2 a^2} + \left( \frac{\dot{a}}{a} \right) \frac{\dot{f}}{1 - kr^2 + af} \right] = \kappa^2 \tilde{\rho} , \] (16a)
\[ G_{01} = -\frac{\alpha \dot{f}}{r \left[ 1 - kr^2 + af(t) \right]} = 0 , \] (16b)
\[ G_{11} = -\frac{a^2}{(1 - kr^2 + af)} \left[ \left( \frac{\dot{a}^2}{a^2} + \frac{2\dot{a}}{a} + \frac{k}{a^2} - \alpha \frac{f}{r^2 a^2} \right) \right] = \kappa^2 \tilde{\rho} , \] (16c)
\[ G_{22} = -r^2 \dot{a}^2 - 2r^2 \dot{a} - r^2 k + \frac{r^2 f'}{2} \]
\[ + \alpha r a^2 \left[ 2(1 - kr^2 + af)(\dot{a} f + 3\dot{a} \dot{f}) + 3\alpha af^2 \right] \]
\[ = \kappa^2 \tilde{\rho} a^2 r^2 , \] (16d)
where dots and primes denote derivatives with respect to cosmic time \( t \) and radial coordinate \( r \) respectively. According to Eq. (16a), we define
\[ \tilde{\rho} = \rho + \alpha \rho^0 , \quad \tilde{\rho}_r = p + \alpha \rho^0 , \quad \tilde{\rho}_\perp = p + \alpha \rho^0 . \] (17)

Notice that Eq. (16a) imposes a constraint on \( \rho \) given by \( \dot{f} = 0 \Rightarrow f(t, r) = f(r) \). Then, the EFE (10) reduce to
\[
3 \left[ \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - \alpha \left[ \frac{f + rf'}{r^2 a^2} \right] = \kappa^2 \tilde{\rho} , \quad \text{(18a)}
\]
\[
- \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{2\dot{a}}{a} + \frac{k}{a^2} \right] + \alpha \frac{f}{r^2 a^2} = \kappa^2 \tilde{\rho}_r , \quad \text{(18b)}
\]
\[
- \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{2\dot{a}}{a} + \frac{k}{a^2} \right] + \alpha \frac{f'}{2r a^2} = \kappa^2 \tilde{\rho}_\perp . \quad \text{(18c)}
\]

The above equations can be rewritten in terms of two sets of differential equations: one describing an isotropic system sourced by the perfect fluid \( p^\theta \), and the other set corresponding to a new set of equations sourced by \( \theta_{\mu
u} \). Thus, for the perfect fluid we have
\[ H^2 = \frac{\kappa^2 \rho}{3 - \frac{k}{a^2}} , \quad \dot{H} = -\frac{\kappa^2}{2} (\rho + p) + \frac{k}{a^2} , \] (19a)
where \( H = \dot{a}/a \) is the Hubble parameter, and
\[ -\frac{rf' + f}{a^2 r^2} = \kappa^2 \rho^\theta , \quad \frac{f}{a^2 r^2} = \kappa^2 \rho^\theta_r , \quad \frac{f'}{2a^2 r} = \kappa^2 \rho^\theta_\perp , \] (20a)
for the anisotropic system. It is worth mentioning that Eq. (20d) induces a matter content for the anisotropic sector satisfying the following EoS
\[ p^\theta_\text{tot} = -\rho^\theta , \] (21)
where \( p^\theta_\text{tot} = p^\theta_r + 2p^\theta_\perp \). On the other hand, the conservation equations (20) lead to
\[ 0 = \dot{\rho} + 3H (\rho + p) , \quad \text{(22a)}
\]
\[ 0 = \dot{\rho}^\theta + H (3\rho^\theta + p^\theta_\text{tot}) . \quad \text{(22b)}
\]
with $\rho^0_0$ the current value of the anisotropic energy density. The above expression allow us to find the decoupling function $f$ by integration of Eq. (20a), from where we obtain

$$f(r) = -\frac{\kappa^2 \rho^0}{3} r^2,$$

(24)

where we have setted the integration constant in such way that $f(0) = 0$. The previous expression for $f(r)$ leads to the following radial and perpendicular components for the anisotropic pressure

$$p^0_r = p^0_\perp = -\frac{1}{3} \rho^0,$$

(25)

which clearly satisfies the EoS (21). We can see that the anisotropic energy density (23) has the cosmological evolution of a spatial curvature term. Therefore, the anisotropic sector $\theta_{\mu\nu}$ will contribute to the spatial geometry of the Universe.

Notice that for a general, but constant EoS $\omega$, the energy density for different components of the Universe evolve as function of the scale factor $a$ as $\rho_i = \rho_{0,i} a^{-3(\omega+1)}$, where $i$ labels different matter content each one with a characteristic EoS, for example baryons ($\omega = 0$), cold dark matter ($\omega = 0$), photons ($\omega = 1/3$), neutrinos ($\omega = 1/3$), cosmological constant ($\omega = -1$), and spatial curvature ($\omega = -1/3$), which is precisely the relationship between the radial and perpendicular pressures with the anisotropic energy density, as we show above in Eq. (24). This result is a consequence of the constraint (104), which forces to the decoupling function $f$ to be independent of the cosmic time $t$. This imposes the unique form of the EoS (21), which combined with Eq. (22b) leads to the evolution of the anisotropic energy density given by Eq. (23).

This is the case for a FRW metric, where the anisotropic sector $\theta_{\mu\nu}$ has the specific behavior of a spatial curvature term. Nonetheless, it could be different for other spacetimes. In the next Section, we will consider another line element, which will lead to different behaviors of the anisotropic sector.

### III. GRAVITATIONAL DECOUPLING FOR KANTOWSKI–SACHS COSMOLOGY

Now, we implement the gravitational decoupling formalism in the context of Kanstowski-Sachs (KS) cosmology [86]. Let us start with the KS line element parameterized as

$$ds^2 = -dt^2 + F(t)^2 dr^2 + S(t)^2 d\Omega^2.$$

(26)

Note that the KS metric can be cast in the form of Eq. (23) whenever we identify

$$e^{-\lambda} \rightarrow F(t)^2, \quad r \rightarrow S(t).$$

(27a)

Using Eq. (26), the EFE (1) reads

$$\kappa^2 \ddot{\rho} = \frac{2 S \ddot{S} + F \dot{S}^2 + F}{F S^2},$$

(28a)

$$\kappa^2 \ddot{p}_r = -\frac{2 \dot{S} \ddot{S} + \dot{S}^2 + 1}{S^2},$$

(28b)

$$\kappa^2 \ddot{p}_\perp = -\frac{S \dddot{F} + \dot{F} \ddot{S} + F \dddot{S}}{FS},$$

(28c)

where $\ddot{\rho}, \ddot{p}_r, \ddot{p}_\perp$ correspond to the components of the total energy-momentum tensor $T^\mu_{\nu}^{\text{tot}}$ given by Eq. (5), and which can be splitted as Eq. (11). Following the MGD strategy, the standard KS matter content $T^\mu_{\nu} = \text{diag}(-\rho, p_r, p_\perp, p_\perp)$ will be the source of a well-known solution of KS metric, which is given by

$$ds^2 = -dt^2 + (R(t))^2 dr^2 + S(t)^2 d\Omega^2.$$

(29)

We assume that the $g^{rr}$ component of the line elements (20a) and (20b) are connected by

$$F^2 = \frac{R^2}{1 + \alpha f},$$

(30)

with $f$ the decoupling function. Since the gravitational potentials of the KS line element do not depend on the radial coordinate $r$, the decoupling function $f$ can be set to be a function only of the cosmic time $t$. This is different from the FRW case, where the general assumption was $f(t, r)$. Thus, the EFE can be rewritten in two sets of equations: one set corresponding to the well-known KS solution

$$\kappa^2 \rho = H^2_S + 2 H_S H_R + \frac{1}{S^2},$$

(31a)

$$\kappa^2 p_r = -\left(2 H_S + 3 H_S + \frac{1}{S^2}\right),$$

(31b)

$$\kappa^2 p_\perp = -\left(H_S + H_R + H_S + H_R + H_S H_R\right)$$

(31c)

where we have defined $H_S \equiv \dot{S}/S$ and $H_R \equiv \dot{R}/R$, and the other set containing the information of the decoupler sector

$$\kappa^2 \dot{\rho} = -\frac{\dot{f} H_S}{1 + \alpha f},$$

(32a)

$$\kappa^2 \dot{p}_r = 0,$$

(32b)

$$\kappa^2 \dot{p}_\perp = \frac{\dot{f}}{2(1 + \alpha f)} - \frac{3 \alpha f^2}{4(1 + \alpha f)^2}$$

$$+ \frac{\dot{f}}{2(1 + \alpha f)} (2 H_R + H_S),$$

(32c)

where we observe that there will be not contribution from the radial component of the anisotropic pressure. In fact, this behaviour coincides formally with the matter sector of the Florides interior solution [87], which represents an anisotropic Schwarzschild interior solution with vanishing radial pressure. This is quite interesting since the
KS metric possess a very well-known property under the interchange $r \leftrightarrow t$, which maps from a Schwarzschild interior solution to an anisotropic (KS) cosmological solution, and vice-versa \textcolor{red}{[88]} (some applications of this property in different contexts can be found at \textcolor{red}{[88–91]}). Therefore, Eq. (32) can be interpreted as the mapping of the anisotropic interior Schwarzschild solution \textcolor{red}{[87]} to a cosmological KS background in presence of a MGD.

On the other hand, the conservation of the energy-momentum tensor leads to

$$0 = \rho + H_R (3 \rho + p_r) + 2 H_S p_\perp,$$

$$0 = \rho^\theta + 3 H_R \rho^\theta + 2 H_S p_\perp^\theta - \frac{3 \alpha \dot{f}}{2(1 + \alpha f)} \rho^\theta. \quad (33b)$$

Contrary to the FRW case, where the conservation of $\theta_{\mu\nu}$ led to an unique expression for the anisotropic energy density (see Eq. (23)), this time we need to provide an EoS in order to solve the system (32). Below we show the most general solutions for some particular cases of interest for several matter content:

**Dust**

Let us impose the dust condition, namely

$$p^\theta = p_\perp^\theta = 0,$$ \quad (34)

from where Eq. (32) leads to

$$f(t) = \frac{4}{\alpha^3 \left(-2 \sqrt{2} c_1 \int \frac{dt}{R^\perp(t) S(t)} + c_2\right)^2} - \frac{1}{\alpha}. \quad (35)$$

Now, replacing the above result into Eq. (32), the density for the decoupler sector will have the form

$$\rho^\theta = -\frac{\sqrt{2} c_1 H_S}{2 \pi \alpha R^2 S \left(c_2 - 2 \sqrt{2} c_1 \int \frac{dt}{R^\perp(t) S(t)}\right)}. \quad (36)$$

**Barotropic fluid**

Another possibility is to consider a barotropic EoS

$$p_{\perp^\theta} = \omega \rho^\theta.$$ \quad (37)

Then, the geometric deformation function $f$ and the anisotropic energy density $\rho^\theta$ are respectively given by

$$f(t) = \frac{4}{\alpha^3 \left(c_2 - 2 \sqrt{2} c_1 \int \frac{dt}{R^\perp(t) S(t)}\right)^2} - \frac{1}{\alpha}, \quad (38a)$$

$$\rho^\theta = \frac{c_1 S^{-2(\omega + 1)} \dot{S}}{\pi \alpha R^2 \left(4 c_1 \int \frac{dt}{R^\perp(t) S(t)} dt - 2 \sqrt{2} c_2\right)}. \quad (38b)$$

**Polotropic fluid**

A more interesting case arises when considering a polytropic fluid, which EoS have the form

$$p_{\perp^\theta} = \omega (\rho^\theta)^\beta,$$ \quad (39)

where $\beta = (n + 1)/n$. For ultracompact objects, such a neutron stars, the polytropic index $n$ takes values from 0.5 to 1 for stiff EoS, or $n = 1.5, 2$ for softer ones (see for instance \textcolor{red}{[92–101]}). In a cosmological context, several scenarios have been tested for polytropic fluid, from primordial to late time Universe \textcolor{red}{[92–101]}. When considering the case $n = 1$ we have

$$f(t) = -\frac{1}{\alpha} \left[1 - e^{H(t)}\right], \quad (40)$$

where

$$H = \int \frac{1}{R(t)^2 S(t)} \left(c_1 - \int \frac{dt}{\omega S(t)^2 + 2 \pi \alpha R^2 S(t) \dot{S}}\right) dt.$$ \quad (41)

In all the previous expressions, $c_1$ and $c_2$ are integration constants. It is straightforward to see that the general cases for the barotropic and polytropic solutions recover the most simple case of dust when $\omega = 0$.

**Cold Dark Matter**

The standard KS cosmology requires $p_r = p_\perp = 0$ in order to have a CDM component, which evolution is given by $\rho_{CDM} = \rho_{CDM,0}/R^3$ after solving Eq. (44a). In our case for the decoupler sector, the condition $p_{\perp^\theta} = 0$ is automatically satisfied by Eq. (32), and we have only to impose that $p_{\perp^\theta} = f = 0$, i.e., a constant deformation function $f$ will contribute as cold dark matter, as can be seen from Eq. (33b). The dust condition applies for cold dark matter, in whose case we can ask for $f$ in Eq. (33) to be constant by making the integration constant $c_1 = 0$. Nonetheless, this will lead to a null anisotropic energy density in Eq. (36). The solution to obtain $\rho_{CDM}^\theta = \rho_{CDM,0}^\theta/R^3$ from the dust condition is in fact a more complicated one, in which an integro-differential equation relating the gravitational potentials $R$ and $S$ is obtained

$$\frac{dS(t)}{dt} + \left(\beta_1 + \beta_2 \int \frac{dt}{R^2(t) S(t)}\right) S^2(t) \frac{R(t)}{R(t)} = 0, \quad (42)$$

where $\beta_1$ and $\beta_2$ are constants. Thus, for a given solution $R(t)$ and $S(t)$ from the standard KS sector satisfying the above equation, the anisotropic energy density $\rho^\theta$ will behave as a cold dark matter component.

**Cosmological Constant**

Different from the FRW case, the MGD for the KS cosmology allow us to propose several EoS, as we have
shown above. An interesting question could be: there exist a MGD able to induce an anisotropic fluid behaving as a cosmological constant? In other words, what is the value of the deformation function $f$ such that $p_\perp^0 = -\rho^0 = -\Lambda/k^2$? The latter condition can be replaced in Eq. (33b), from where we obtain
\[
\alpha \lbrack 1 - c_1 e^{|f(2H\theta(t)-(4/3)H\delta S(t)|dt} \rbrack , \quad (43)
\]
this is, the above expression leads to a Kantowski-Sachs Universe with cosmological constant by means of a purely geometrical source.

**IV. CONCLUSIONS**

In this work we were able to find new general, analytical and exact cosmological solutions to the Einstein field equations by applying the Minimal Geometric Deformation approach. Particularly, we were focused on two cosmological metrics: the Friedmann-Robertson-Walker spacetime and a Kantowski-Sachs Universe, where an ansatz for the MGD in the spatial part of both metrics were proposed. This allowed us to use the Gravitational Decoupling formalism to find anisotropic extensions to the well-known solutions of these cosmological scenarios. Specifically, the set of Eqs. (24), (35), (38a), (40), (43) constitutes different realizations of the deformation function $f$ sourcing the decoupler sector. The major physical implications of this analysis are the following:

**$\Lambda$CDM**: the spatial curvature term of the FRW line element gets modified as $[1 - kr^2 + \alpha f(r)]^{-1}$, with $f(r)$ given by $f(r) = -(\kappa^2 \rho_0^0/3)r^2$ according to Eq. (24). Therefore, an *effective* spatial curvature term can be defined as
\[
k \rightarrow k_{eff} \equiv k + \frac{\kappa^2 \rho_0^0}{3} . \quad (44)
\]
Given the current values of the energy density parameters for both, total matter ($\Omega_{M,0} = 0.315 \pm 0.007$) and cosmological constant ($\Omega_{\Lambda,0} = 0.685 \pm 0.007$) [49], Eq. (44) opens the possibility of a degeneration between the spatial curvature $k$ and the current value of the anisotropic energy density $\rho_0^0$, in such a way that
\[
1 = (\Omega_{M,0} + \Omega_{\Lambda,0} + \Omega_{k_{eff,0}}) , \quad \text{with } \Omega_{k_{eff}} \approx 0 . \quad (45)
\]
Therefore, measurements of $\Omega_{k_{eff}}$ could be in fact indicating a non-flat spatial geometry with $k \neq 0$ countered by the anisotropic term, with $\rho_0^0 = -3/\kappa^2 (\rho_0^0 = 3/\kappa^2)$ for a spherical (hyperbolic) space. Moreover, from the current value of the spatial curvature parameter $\Omega_k = 0.001 \pm 0.002$ [49], we obtain a small value of the anisotropic energy density, which in units of the critical density $\rho_{c,0} \equiv 3H_0^2/\kappa^2$ is of the order $\rho_0^0 \approx 10^{-3}\rho_{c,0}$.

**Kantowski-Sachs**: Contrary to the FRW case, where the EoS sets the behavior of the anisotropic component in an unique form, we found that a Kantowski-Sachs Universe allows to have a variety of matter components induced from the decoupler sector. By specifying a particular solution for $S(t), R(t)$ in a standard KS model, its extended version with a new geometric component arising from the decoupling function $f$ can be computed. We showed that it is possible to map $f$ to a matter-like terms, such as dust, barotropic and polytropic fluids, cold dark matter, and to a cosmological constant as well. Even when the most accepted cosmological model is $\Lambda$CDM, the Universe we observe is in a non-linear phase of structure formation and of accelerating expansion, features attributed to new forms of matter (Cold Dark Matter) and energy (Cosmological Constant). Thus, within anisotropic Universes like those provided by Kantowski-Sachs models, in combination with the MGD, it is possible to address the question of where these new components of the Universe come from: they arise as geometrical effects of spacetime.

Before concluding this work, we would like to point out that the decoupler source $\theta_{\mu\nu}$ could represent the coupling with scalar or vector fields [13, 18]. Moreover, this new matter content could encode the information of a new gravitational sector $X$ of extended theories of gravitation, whose Modified Einstein-Hilbert action $S_{\text{MEH}}$ can be expressed as
\[
S_{\text{MEH}} = S_{\text{EH}} + \int d^4x \sqrt{-g} \mathcal{L}_X . \quad (46)
\]
where $S_{\text{EH}}$ is the standard Einstein-Hilbert action, and $\mathcal{L}_X$ is the Lagrangian density of the $X$–gravitational sector, which could be given by theories beyond general relativity, such as $f(r)$, Lovelock gravity [42], Einstein–Aether gravity, among others [30]. This new gravitational sector can be encoded in $\theta_{\mu\nu}$ as follows
\[
\theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_X}{\delta g_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_X}{\delta g_{\mu\nu}} - g_{\mu\nu} \mathcal{L}_X . \quad (47)
\]
Therefore, the Gravitational Decoupling formalism through the Minimal Geometric Deformation approach promises to be useful not only to explore geometrical aspect of the large scale Universe, but also to source matter and energy components which could drive the process of structures formation and late time acceleration. This will be interesting to analyze with some deep in future studies.

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[2] J. Ovalle, Int. J. Mod. Phys. D18, 837 (2009).
