Notes on of Seiberg-Witten map on manifold with flat scalar curvature

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Abstract

In this paper, we focus on the moduli space of Seiberg-Witten equation on non-compact manifold with periodic end. Suppose that the scalar curvature on the periodic end is identically zero and the topological conditions: the first de-Rham cohomology and the self-dual cohomology restricting on the periodic end vanish. Then, we will show that the moduli space of the perturbed Seiberg-Witten equation is compact.

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1 Introduction

The Seiberg-Witten invariant, or monopole invariant has been introduced by Witten \cite{Witten1992} in the 90s of the last century and found rapidly spectacular applications in 4-dimensional differential topology. It is an invariant of a smooth, closed, oriented 4-manifold $X$. When $b^+(X) > 1$, the invariant is well-defined and can be regarded as a map

$$SW : Spin^c(X) \to \mathbb{Z},$$

where $Spin^c(X)$ denotes the set of equivalent spin$^c$ structures of $X$. The construction of Seiberg-Witten invariant needs the compactness of the moduli space of Seiberg-Witten equation. For closed manifold, it is well known that the moduli space is always compact. However for non-comapct manifold, e.g. manifold with cylindrical end, the moduli space fails to be compact. For compact oriented 4-manifold with boundary, a natural way to construct a non-compact 4-manifold is to add a cylindrical end for this compact oriented 4-manifold to the boundary. A natural direction to develop the Seiberg-Witten theory on non-compact manifolds is to study the set of 4-manifolds with cylindric ends and the relations between the 4-dimensional Seiberg-Witten invariants for such manifolds and the Seiberg-Witten Floer homology for oriented closed 3-manifolds. Fundamental contributions in this direction are the monographs of Nicolascu \cite{Nicolascu2006} and Froshov \cite{Froshov1,Froshov2}.
In general, for non-compact manifold, the elliptic operators fail to be Fredholm. End-periodic elliptic operators on noncompact manifolds with periodic end were first studied by Taubes [19]. It is worth noting that one can study the topology on a certain kind of compact smooth 4-manifolds by using the Seiberg-Witten equations on periodic end:

1. Jianfeng Lin [13] constructed an obstruction for homology $S^1 \times S^3$ smooth oriented manifold, by using the Seiberg-Witten equations on periodic ends.

2. Konno and Taniguchi [12] gave a 10/8-type inequalities for some end-periodic 4-manifolds which have positive scalar curvature metrics on the ends.

In this paper, we prove the fundamental analytic result of the Seiberg-Witten on 4-manifold with periodic end. Our main concern is to study the Seiberg-Witten equation on an oriented non-compact Riemannian manifold $(X, g)$ with periodic end. The following theorem can summarize the main result of this paper.

**Theorem 1.1** Given a non-compact oriented spin$^c$ Riemannian 4-manifold $(X, g)$ with periodic end modelled on $\tilde{Y}$. Suppose that the scalar curvature on the periodic end is identically zero, the first deRham cohomology restricting on the periodic end vanishes, i.e. $b_1(\tilde{Y})$ and $b^+(Y) = 0$, where $Y$ denotes the associated closed manifold $\tilde{Y}/\mathbb{Z}$. Then the Seiberg-Witten moduli space is compact in $L^2_1$. Moreover, there exists a positive number $w$, such that preimage of bounded image of Seiberg-Witten map is bounded in $L^2_{k,w}$ for any $k \geq 2$.

Notice that if one replaces the flat scalar curvature with the positive scalar curvature, then compact theorem can be deduced by Jiangfeng Lin’s work [13, Section 4]. Moreover, if $Y$ admits a Riemannian metric whose scalar curvature function is non-negative and not identically zero. Then by the conformal deformation, e.g. Kazdan and Warner’s work [11], one gets a metric of positive scalar curvature.

**Theorem 1.2** Given a non-compact oriented spin$^c$ Riemannian 4-manifold $(X, g)$ satisfying the conditions of Theorem 1.1. We assume that the modelled manifold $Y$ in Theorem 1.1, and does not admit a metric positive scalar curvature, then:

1. $\tilde{Y} \cong \mathbb{R} \times O_5^3$, where $O_5^3$ denotes Hantzsche-Wendt manifold, which is the only oriented flat three manifold with vanishing first Betti number;

2. $Y$ must be either $O_4^{14}$, $O_5^{15}$, $O_6^{16}$, $O_7^{17}$, $O_8^{28}$ or $O_7^{27}$ from Table 4.

Furthermore, choosing a spin$^c$ structure $\mathfrak{s}$ of $(X, g)$ such that it coincides the spin$^c$ structure of $\tilde{Y}$ on the periodic end, we have that the moduli space of the Seiberg-Witten equation is compact with respect to $\mathfrak{s}$.

The structure of this paper is the following: In Section 2, we review the preliminary of 4-manifold with periodic end and fredholmness for some periodic-end elliptic operator; In Section 3, we show the proof of Theorem 1.1 In Section 4, we show the proof of Theorem 1.2.

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## 2 Manifold with the periodic ends and Fredholmness of Dirac operator

We recall the definition of $n$-manifold with periodic end and periodic structure. The original definition for such this notion was firstly given by Taubes [19], here we give a brief one.
Let $Y$ be an oriented compact manifold endowed with a primitive cohomology class $\gamma \in H^1(Y,\mathbb{Z})$. This data induces an infinite cyclic $\mathbb{Z}$ covering $\pi: \tilde{Y} \to Y$. Let $\tau_0: Y \to S^1$ be a smooth map representing the homotopy class corresponding to $\gamma \in H^1(Y,\mathbb{Z})$ under the standard identification $H^1(Y,\mathbb{Z}) \to [Y, S^1]$. We fix a lift $\tau: \tilde{Y} \to \mathbb{R}$ of this function, which satisfies

$$\tau(T(p)) = \tau(p) + 1,$$

where $T$ stands for the positive generator of $\text{Aut}(\tilde{Y}) \cong \mathbb{Z}$. Choose an oriented, connected submanifold $N \subset Y$, which is Poincaré dual to $\gamma$, and cut $Y$ open along $N$ to obtain a cobordism $W$ with boundary $\partial W = -N \cup N$.

**Definition 2.1** An oriented differentiable $n$-manifold with one periodic end is a connected $n$-manifold $X$ endowed with the following data:

1. an oriented compact manifold $K$ with boundary $\partial K = N$,
2. An orientation preserving diffeomorphism $\phi: X \to K \cup \{\iota: N \to N\}$, where $\iota$ is the equivalent relation induced by the diffeomorphism $\iota: N \to N$. We have a canonical $\mathbb{Z}$-covering $\tilde{Y} \to Y$, where

$$\tilde{Y} = \cdots \cup \iota_{i:N \to N} W \cup \iota_{j:N \to N} W \cdots$$

is obtained by gluing in an obvious way a family of copies of $W$ parameterized by $\mathbb{Z}$.

For an oriented end periodic manifold $X$, we can find a smooth function, which is still denoted by $\tau$, such that on the end periodic part $\text{End}(X)$, it coincides with the above function. Moreover, we can let $\tau$ satisfies $\|\tau\|_{C^1}, \|\tau\|_{C^2} \leq C_0$, for the same constant $C_0$.

**Definition 2.2** Let $X$ be a manifold with one periodic end. A Riemannian metric $g$ on $X$ is called end-periodic if its restriction to $\text{End}(X)$ coincides with the pull-back of a metric on $Y$. End-periodic spin, spin$^c$ structure of $X$ and vector bundle over $X$ are defined similarly.

For a 3 or 4 dimensional spin$^c$ manifold, we have the following theorem given by Maier [14].

**Theorem 2.1 (Maier)** Let $(M, g)$ be a Riemannian spin$^c$-manifold with fixed spin$^c$-structure.

(i) If $\dim M = 4$ then for the generic connection on the canonical bundle there are no nontrivial negative (positive) harmonic spinor provided Index of Dirac operator is no less (more) than zero.

(ii) If $\dim M = 3$ there are no nontrivial harmonic spinors for the generic connection.

Combining the theorem given by Taubes, we have the following corollary.

**Corollary 1** Let $(X, s, g)$ be an oriented spin$^c$ Riemannian manifold with periodic end modelled on a closed oriented spin$^c$ 4 manifold $(Y, s_0, g_0)$. Suppose that the index of the Dirac operator with respect to the spin$^c$ structure $s_0$ vanishes. Then, for any periodic end metric $g$, we can find a generic periodic end spin$^c$ connection $A$, such that the associated Dirac operator $D_{A,g}$ on $X$ is Fredholm.
3 Proof of Theorem 1.1

3.1 Gauge fixing

Recall that $W$ denotes the unit model on the periodic part, i.e. $W/ \sim = Y$ and $W_k$ denotes the $k$-th copy of $W$. The main target of this subsection is to show the gauge-fixing on oriented spin$^c$ non-compact 4-manifold modelled on $Y$ such that $b_1(W) = 0$ and $b^+(Y) = 0$. Before the gauge-fixing, we give a definition of gauge group and review the work of Veloso.

Definition 3.1 Let $(X, g)$ be a Riemannian spin$^c$ 4-manifold with periodic ends, we define

$$G_{k+1, w} = \{ u \in L^2_{k+1, w}(X, \mathbb{C}) \| u \| = 1, \ u^{-1} du \in L^2_{k, w}(\Omega^1(X)) \}.$$

Proposition 3.1 Let $(X, g)$ be a Riemannian spin$^c$ 4-manifold with periodic ends, such that $H^1(W, \mathbb{Z})$ is of torsion.

1. One has a short exact sequence

$$1 \to \exp(L^2_{w, k+1} \oplus F^c) \to G_{k+1, w} \to H^1(X, 2i\pi \mathbb{Z}) \to 0,$$

where $F^c = i\mathbb{R}(a_c)$ for $a_c \equiv 1$ outside a compact subset of $X$. We have a morphism $p : G_{k+1, w} \to H^1(X, 2i\pi \mathbb{Z})$, given by

$$u \mapsto [u^{-1} du].$$

Using the definition, we obtain

$$\ker(p) = \exp(L^2_{k+1, w} \oplus F^c).$$

2. Putting $G = \{ u \in G_{w, k+1} | u^{-1} du \in \text{Harm}^1_{w} \}$, the restriction $p_G$ of $p$ to $G$ gives the short exact sequence

$$1 \to S^1 \to G \to H^1(X, 2i\pi \mathbb{Z}) \to 0.$$

Theorem 3.1 (Veloso [20, Proposition 3.1.1]) Let $(X, g)$ be an oriented Riemannian 4-manifold satisfying Assumption 1. Then, we have the Hodge decomposition

$$L^2_{k, w}(\Omega^1) = \text{Har}^1_{w} \oplus \text{Im}(d : L^2_{k+1, w}(\Omega^0) \oplus F \to L^2_{k, 2}(\Omega^1)) \oplus \text{Im}(d^* : L^2_{k+1, w}(\Omega^+ \to L^2_{k, 2}(\Omega^1)).$$

Moreover, the we have the following:

1. $\text{Har}^1_{w} \cong \frac{\ker(d, L^2_{k+1, w}(\Omega^1) \to L^2_{k-1, w}(\Omega^2))}{\text{Im}(d, L^2_{k+1, w}(\Omega^0) \oplus F \to L^2_{k+1, w}(\Omega^1))}.$

2. $\text{con} : \frac{\ker(d, L^2_{k+1, w}(\Omega^1) \to L^2_{k-1, w}(\Omega^2))}{\text{Im}(d, L^2_{k+1, w}(\Omega^0) \oplus F \to L^2_{k+1, w}(\Omega^1))} \hookrightarrow H^1_{dR}(X, \mathbb{R})$ is injective.

3. $\text{Im}(\text{con}) = \text{Im}(H^1_{c}(X, \mathbb{R}) \to H^1(X, \mathbb{R})) = H^1(X, \mathbb{R})$ and $\text{Har}^1_{w} \cong H^1(X, \mathbb{R})$ hold.

Let $\{\gamma_1, \ldots, \gamma_b\}$ be a basis for $H_1(X, \mathbb{Z})/\text{Tor}$ and $\{\beta_1, \ldots, \beta_b\}$ represent their Poincaré duals. Let $u_n \in G_{w, k+1}$ satisfy

$$i \int_X \beta_j \wedge (A_n - u_n^{-1} du_n) \in [0, 2\pi],$$

for $j = 1, \ldots , b$. The following lemma holds.

Lemma 3.1 (c.f. [10, Theorem 5.1.1]) Let $M$ be a compact manifold. For the one-form $a \in \Omega^1(M)$ satisfying the constraints $i \int_X \beta_j \wedge a \in [0, 2\pi]$ for $j = 1, \ldots , b$, there are constants $C_1, C_2$ such that

$$\|a\|_{L^2_1}^2 \leq C_1 \| (d^* a, da) \|_{L^2_2}^2 + C_2.$$
We need the below lemma for the preparation of showing the compactness of the moduli space. The proof is similar to [13, Lemma 3.1], here we omit it.

**Lemma 3.2** The quotient space $B_{k, \delta} = C_{k, \delta} \backslash \mathcal{G}_{k+1, \delta}$ is a Hausdorff space.

For convenience, we denote all irreducible constants by the same notation $C$ for the rest of this subsection. Let $W_m$ denote the $m$th copy of $W$ and $W_{i,j}$ denote

$$W_i \cup_Z W_{m+1} \cup_Z \cdots \cup_Z W_j$$

for $j > i$, where $Z$ denotes one of the boundary of $W$. Since $b_1(\tilde{Y}) = 1$, we find a compact manifold $W'_{m, m+6}$ with smooth boundary $Z'$ such that

$$W_{m,m+6} \subset W'_{m,m+6} \subset W'_{m-1,m+7},$$

and $b_1(W'_{m,m+6}) = 0$. Given a sequence of solutions $\{(A_n, \phi_n)\}$, for each $m \geq 0, n \geq 1$, we set $\phi_{m,n} : W'_{m,m+6} \to S^1$ be a gauge transformation with the following properties:

1. $d^*(A_n - A_0 - u_{m,n}^{-1} du_{m,n}) = 0$,
2. $(A_n - A_0 - u_{m,n}^{-1} du_{m,n}, \nu_m) = 0$ for the normal vector field $\nu_m$ on $\partial W'_{m,m+6}$,
3. $u_{m+2,n}(o_{m+4}) = u_{m,n}(o_{m+4})$ for any $m \geq 1$, where $o_m \in W_m$ corresponding to a fixed point $v \in \text{int}(W)$.

Similarly, on $K \cup W_{1,3}$, we have a gauge transformation $u_0 : K \cup W_{1,3} \to S^1$ with the following properties:

1. $d^*(A_n - A_0 - u_{0,n}^{-1} du_{0,n}) = 0$,
2. $(A_n - A_0 - u_{0,n}^{-1} du_{0,n}, \nu_0) = 0$ for the normal vector field $\nu_0$ on $\partial K \cup W_{0,3}$,
3. $u_{1,n}(o_3) = u_{0,n}(o_3)$.

Such $u_{m,n}$ can be found by solving the Laplace equation with Neumann boundary condition [21, Theorem 2.3]. We denote by

$$A_{m,n} = A_n - u_{m,n}^{-1} du_{m,n}, \phi_{m,n} = u_{m,n} \phi_n,$$

for each integer $m \geq 0$.

**Lemma 3.3** There exists a constant $C$ and $\delta$ such that the following bounded result holds,

$$\| (A_{m,n} - A_0, \phi_{m,n}) \|_{L^2_{k+1}(W_{m,m+6})} \leq C,$$

for all $m,n \geq 1$, where $C$ is some uniform constant and $C(\phi_n)$ is some constant depending on $\phi_n$.

**Proof** The idea follows exactly the same as the proof of [10, Theorem 5.1.1]. By the transformation, we regard $(A_{m,n}, \phi_{m,n})$ as a solution on $W'_{0,6}$, with following bounds:

$$\| \phi_{m,n} \|_{L^2_{k}(W'_{0,6})} \leq C, \| \nabla A_{m,n}, \phi_{m,n} \|_{L^2_{k}(W'_{0,6})} \leq C, \| F_{A_{m,n}} \|_{L^2_{k}(W'_{0,6})} \leq C.$$

By the Neumann-boundary, we get a sequence with bounded $L^2$-norm on a fixed $W'_{m,m+6}$ for some $m \geq 1$, then by the regularity of the elliptic operator, one gets the desired result.

Now, we prove the following compactness proposition.
Proposition 3.2 Let \((X, g)\) be an oriented Riemannian 4-manifold with periodic end modeled in \(\tilde{Y}\), such that \(b_1(Y) = 0\). Suppose that \(\{A_n\} \subset L^2_\omega(A)\) for \(k \geq 2\) and \(\omega > 0\), is a sequence of spin\(^c\) connections with a uniform bound of the curvature, i.e. there exists a constant \(C\), such that

\[\|F_{A_n}\|^2_{L^2(X)} \leq C.\]

Then we can find a sequence of gauge transformations \(\{u_n\}\) in \(\mathcal{G}_{k+1, \delta}\) such that \(\{u_n(A_n)\}\) has a uniform \(L^2\)-norm.

Proof Let \(a_{m,n} = A_{m,n} - A_0\) and \(a_n = A_n - A_0\). First, we give some necessary estimates:

1. For \(m \geq 1\) we have

\[\|A_{m,n} - A_0\|_{L^2_\omega(W_{m,m+6})}^2 \leq \|A_{m,n} - A_0\|_{L^2_\omega(W_{m,m+6})}^2 \leq C\|\{d(A_{m,n} - A_0), d^*(A_{m,n} - A_0)\}\|_{L^2(W_{m,m+6})}^2,\]

together with Seiberg-Witten equation, we get

\[\|A_{m,n} - A_0\|_{L^2_\omega(W_{m,m+6})}^2 \leq C\|da_{m,n}\|_{L^2(W_{m,m+6})}^2 = C\|da_n\|_{L^2(W_{m,m+6})}^2.\]

2. On \(K \cup W_{1,3}\), we have

\[\|A_{m,n} - A_0\|_{L^2_\omega(K \cup W_{1,3})}^2 \leq C_1\|da_n\|_{L^2_\omega(W_{1,3})}^2 + C_2,\]

where we used Lemma 3.1 for the first estimate and perturbed Seiberg-Witten equation for the second one.

We need to glue the local gauge transformations to a global one. The idea is to follow the method in Jianfeng Lin’s work [13]. For \(m \geq 0\) and \(n \geq 1\), consider the function

\[\xi_{m+4,n} : W_{m+4} \to [0, 1]\]

with the property that

\[\xi_{m+4,n}(0_{m+4}) = 0, \quad e^{i\xi_{m+4,n}(u_{m,n}|_{W_{m+4}})} = u_{m+2,n}|_{W_{m+4}}.\]

We have \(d\xi_{m+4,n} = A_{m+2,n}|_{W_{m+4}} - A_{m,n}|_{W_{m+4}}\). By the above lemma, there exists a constant \(C(a_n)\) depending on the \(L^2_\omega\)-norm of \(a_n\) such that

\[\|\xi_{m,n}\|_{L^2_\omega(W_{m+2})} \leq C(a_n)e^{-\delta m},\]

for any \(k, m, n \geq 1\). For \(k = 0\), one deduces that

\[\|d\xi_{m,n}\|_{L^2_\omega(W_{m+4})}^2 \leq C(\|A_{m+2,n} - A_0\|_{L^2(W_{m+4})}^2 + \|A_{m,n} - A_0\|_{L^2(W_{m+4})}^2) \leq C\|da_n\|_{L^2_\omega(W_{m,m+8})}^2.\]

We regard \(\tau : W_m \to [0, 1]\) as a function, which is zero near the left-boundary and one near the right-boundary for each \(m \geq 1\). We have similar bounds for \(\tau \xi_{m,n}\) and \((1 - \tau) \xi_{m,n}\). We construct the following gauge transformations

\[u'_{m,n}(x) = \begin{cases} e^{i(\tau - 1)\xi_{m+2,n}} & x \in W_{m+2}, \\ 1 & x \in W_{m+3}, \\ e^{i\tau \xi_{m+4,n}} & x \in W_{m+4}. \end{cases}\]

On \(W_{1,3}\), we define the function

\[\xi_{3,n} : K \cup W_{0,3} \to [0, 1]\]

by the condition

\[\xi_{3,n}(0_3) = 0, \quad e^{i\xi_{3,n}(u_0,n|_{W_3})} = u_{1,n}|_{W_3}.\]
We have the gauge transformation

\[ u'_0, n = \begin{cases} 1 \\ e^{i\tau \xi, n} \end{cases} W_{1,2} \]

By the Sobolev multiplication theorem, there exists a constant \( C'(a_n) \) depending on the \( L^2 \)-norm of \( a_n \) such that

1. \( \| u'_{m,n}^{-1} du'_{m,n} \|_{L^2_{k+1}(W_{m+2,m+4})} \leq C'(a_n) e^{-\delta m} \) for any \( m \geq 1 \),

2. \( \| 1 - u'_{m,n} \|_{L^2_{k+1}(W_{m+2,m+4})} \leq C'(a_n) e^{-\delta m} \) for any \( m \geq 1 \).

We glue them together by

\[ \{ u_{m,n} \cdot u'_{m,n} | W_{m+2,m+4} | m = 1, 3, 5, \ldots \} \]

and

\[ u_{0,n} \cdot u'_{0,n} | W_5 = u_{1,n} \cdot u'_{1,n} | W_5, \]

to get a gauge transformation \( u_n : X \to S^1 \). We still need to show the uniform bound with \( L^2 \)-norm. It suffices to give an estimate for the gauge transformation,

\[ \| u'_{m,n} \|_{L^2_{k+1}(W_{m+2,m+4})} \leq C(\| d\xi_{m,n} \|_{L^2_{k}(W_{m+4})} + \| d\xi_{m-1,n} \|_{L^2_{k}(W_{m+2})}) \leq C \| da_n \|_{L^2_{k}(W_{m+2,m+4})}. \]

We set

\[ (A'_{m,n}, \phi_{m,n}) = \begin{cases} u'_{0,n}(A_{0,n}, \phi_{0,n}) & \text{on } K \cup W_{1,3} \\
'_{m,n}(A_{m,n}, \phi_{m,n}) & \text{in } W_{m+2,m+4} \text{ for odd numbers } m. \end{cases} \]

Combining with the above estimates, one has the desired result.

By the construction, we have that \( \{ u'_{m,n}^{-1} du_n \} \) also has a uniform \( L^2 \)-bound. To prove the compactness of the moduli space, we still the following proposition.

**Proposition 3.3** Let \( (X, g) \) be an oriented spin\(^ c\) Riemannian 4-manifold with period end modelled on \( Y \) such that \( b_1(Y) = 0 \) and \( b^+(Y) = 0 \). Suppose that \( \{ A_n \} \subset L^2_{k}(A) \) is a sequence of spin\(^ c\) connection with bounded \( L^2 \) curvature, i.e.

\[ \| F A \|_{L^2_{k}(X)} \leq C, \]

for some uniform constant \( C \). If we have a reference connection \( A_0 \). Then we can find a sequence of gauge transformations \( \{ u_n \} \) in \( \mathcal{G}_{k+1,8} \) such that \( d^\ast (u_n^* A_n - A_0) = 0 \) and \( \{ u_n(A_n) \} \) has a uniform \( L^2 \)-norm.

**Proof** Let \( a_n = A_n - A_0 \). By Hodge decomposition Theorem 3.1, we can rewrite \( a_n \) as

\[ a_n = d^\ast b_n + c_n + df_n, \]

where \( c_n \in \text{Harm}^1 \), \( b_n \in L^2_{k+1,w}(i\Omega^+) \) and \( f_n \in L^2_{k+1,w}(i\Omega^0) \oplus F \). We set \( u_n = e^{f_n} \). Clearly, we have

\[ d^\ast (u_n^* A_n - A_0) = 0. \]

It suffices to show that \( \{ d^\ast b_n \} \) and \( \{ c_n \} \) have a uniform \( L^2 \)-bound. Clearly, \( \{ c_n \} \) has a uniform bound, since \( \text{Harm}^1 \) is of finite dimension. We use the Fredholmness of \( \Delta^+ : L^2_2(\Omega^+) \to L^2_2(\Omega^+) \) to get the estimate on \( \{ b_n \} \). The Fredholmness of \( \Delta^+ \) implies that

\[ \| b_n \|_{L^2_2(X)} \leq C \| \Delta^+ b_n \|_{L^2_2}, \]

where we used that \( b_n \perp \ker(\Delta^+) \). Together with the inequalities

\[ \| d^\ast b_n \|_{L^2_2} \leq C \| b_n \|_{L^2_2}, \]

and

\[ \| \Delta^+ b_n \|_{L^2_2} = \| d^\ast d^\ast b_n \|_{L^2_2} = \| d^\ast a_n \|_{L^2_2} \leq \| da_n \|_{L^2}, \]

we proved our result.

\[ \square \]
3.2 Proof of compactness

In this subsection, we give the proof of our main theorem. For the argument convenience, we first summarize the Sobolev embedding and multiplication theorems. Their proofs are similar to [18], here we omit them.

**Proposition 3.4** Let $E$ be an end–periodic bundle over $X$. There is a continuous inclusion
\[
L^p_{j,\delta}(X, E) \to L^q_{i,\delta'}(X, E),
\]
for $j \geq i$, $\delta \geq \delta'$, $p \leq q$ and $j - \frac{n}{p} \geq i - \frac{n}{q}$, where $n = \dim(X)$. Moreover, this embedding is compact, if $j > i$, $\delta > \delta'$ and $j - \frac{n}{p} > i - \frac{n}{q}$.

**Proposition 3.5** Let $E$ and $F$ be an end–periodic bundle over $X$. Suppose that $\delta + \delta_1 \geq \delta_2$, $j, l \geq i$ and $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{r}$, for $\delta, \delta_1 \delta_2 \geq 0$ and $p, q, r > 1$. Then the multiplication
\[
L^p_{j,\delta}(X, E) \times L^q_{l,\delta}(X, F) \to L^r_{i,\delta_2}(X, E \otimes F),
\]
is a continuous map, if one of the following cases holds
\begin{enumerate}
\item $j - \frac{n}{p} + l - \frac{n}{q} \geq i - \frac{n}{r}$, $j - \frac{n}{p} < 0$, $l - \frac{n}{q} < 0$;
\item $\min\{j - \frac{n}{p}, l - \frac{n}{q}\} \geq i - \frac{n}{r}$, either $(j - \frac{n}{p})(l - \frac{n}{q}) = 0$, or $\max\{l - \frac{n}{q}, j - \frac{n}{p}\} > 0$;
\end{enumerate}
where $n = \dim(X)$.

With the two propositions, we have the following corollary.

**Corollary 2** For any $i > 2$, $\delta > 0$ and $\dim(X) = 4$, the multiplication map
\[
L^2_{i,\delta}(X, E) \times L^2_{i,\delta}(X, F) \to L^2_{i-1,\delta}(X, E \otimes F),
\]
is compact.

We define the analytic energy as
\[
E^a(A, \phi) = \frac{1}{4} \int_X |da|^2 + 4|F^+_\partial_0 da|^2 + 4|
abla_A \phi|^2 + |\phi|^4 + \text{Scal} |\phi|^2,
\]
where $a = A - A_0$, and
\[
E_{A_0} = \frac{1}{2} \int_X F^+_\partial_0 \wedge F^+_\partial_0.
\]
We have the relationship
\[
E^a(A, \phi) - E_{A_0} = \|SW(A, \phi)\|_{L^2}.
\]
The idea to show the compactness is to follow the method of Kronheimer and Mrowka [10, Section 5]. As flat scalar curvature is a special case for non-negative scalar curvature. To prove Main Theorem, we can consider the non-compact manifold with periodic end satisfying the following assumption without loss of generality.

**Assumption 1** Let $(X, g)$ be an oriented spin$^c$ Riemannian 4-manifold with periodic end modeled on $Y$. Suppose that $X$ admits an ASD connection $A_0$, the scalar curvature of $Y$ is nonnegative, i.e. $\text{Scal}_Y \geq 0$,
\[
c^2_1(Y) - \sigma(Y) = 0
\]
and $b^+(Y) = 0$. Furthermore, we assume that $b^1(W) = 0$.

We still need the following theorem before the proving Main Theorem.
Theorem 3.2 (Veloso [20, Theorem 2.3.5]) Let $(X, g)$ be a Riemannian 4-manifold with periodic ends such that $b^{+}(W) = 0$ and $b^{-}(Y) = 0$. Then, the operator $\Delta_{+} : L_{k+2}^{2}(\Omega^{+}(X)) \to L_{k}^{2}(\Omega^{+}(X))$ is Fredholm for any integer $k \geq 0$.

Using the technique of Veloso [20, Lemma 4.3.4], we can prove the following lemma.

Theorem 3.3 Let $(A_{n}, \phi_{n})$ be a sequence in $\mathcal{C}_{k,\delta}$ solving the perturbed Seiberg-Witten equation. Suppose that the perturbation is supported outside the periodic end. Then, we can find a sequence of gauge-transformations $\{u_{n}\}$, such that the sequence of $\{u_{n}(A_{n}, \phi_{n}, \nabla A_{n}, \phi_{n})\}$ is bounded in $L_{k}^{2}$, hence admits a subsequence which is weakly convergent.

Proof We can find a sequence of gauge-transformations $\{u_{n}\}$ such that the result of Proposition 3.2 holds. For convenience, we still denote the transformed configuration sequence by $\{A_{n}, \phi_{n}\}$ By the continuous map $L_{k,w}^{2} \to L^{2}$ for any positive number $w$, we have the $L^{2}$-bound for $SW(A_{n}, \phi_{n})$. We have that

$$
E^{an}(A_{n}, \phi_{n}) = \frac{1}{4} \int_{X} |d\alpha_{n}|^{2} + |\nabla A_{n} \phi_{n}|^{2} + \frac{1}{4} \int_{X < 0} (|\phi_{n}|^{2} + \frac{\text{Scal}}{2})^{2} - \int_{X \leq 0} \frac{\text{Scal}^{2}}{16} \phi
$$

$$
\geq \frac{1}{4} \int_{X} |d\alpha_{n}|^{2} + |\nabla A_{n} \phi_{n}|^{2} + \frac{1}{4} \int_{X \leq 0} (|\phi_{n}|^{2} + \frac{\text{Scal}}{2})^{2} - \int_{X \leq 0} \frac{\text{Scal}^{2}}{16} \phi
$$

$$
- \int_{X} |(d\alpha_{n}, F^{+}_{A_{n}})| + \frac{1}{4} \int_{X \geq 0} |\phi_{n}|^{4} + \text{Scal} |\phi_{n}|^{2}.
$$

Considering the $\epsilon$-Cauchy-Schwarz inequality, we get

$$
E^{an}(A_{n}, \phi_{n}) \geq \frac{1}{4} \int_{X} |d\alpha_{n}|^{2} + |\nabla A_{n} \phi_{n}|^{2} + \frac{1}{4} \int_{X \leq 0} (|\phi_{n}|^{2} + \frac{\text{Scal}}{2})^{2} - \int_{X \leq 0} \frac{\text{Scal}^{2}}{16} \phi

- \epsilon \int_{X \leq 0} |d\alpha_{n}|^{2} - C \epsilon \int_{X \leq 0} |F^{+}_{A_{n}}|^{2} + \frac{1}{4} \int_{X \geq 0} |\phi_{n}|^{4} + \text{Scal} |\phi_{n}|^{2}

\geq (\frac{1}{4} - \epsilon) \int_{X} |d\alpha_{n}|^{2} + \int_{X} |\nabla A_{n} \phi|^{2} + \frac{1}{4} \int_{X \leq 0} (|\phi_{n}|^{2} + \frac{\text{Scal}}{2})^{2} - \int_{X \leq 0} \frac{\text{Scal}^{2}}{16} \phi

- \frac{C}{4} \int_{X \leq 0} |\phi_{n}|^{4} + \text{Scal} |\phi_{n}|^{2}.
$$

Since the scalar curvature is non-negative on $X \geq 0$, we have the bounds on $\|d\alpha_{n}\|^{2}_{L^{2}(X)}$, $\|\nabla A_{n} \phi_{n}\|^{2}_{L^{2}(X)}$ and $\int_{X\geq 0} |\phi_{n}|^{4}$. By the inequality

$$
\|\phi_{n}\|_{L^{2}(X \leq 0)} \leq \|\phi_{n}\|^{2} + \frac{\text{Scal}}{2} \|L^{2}(X \leq 0) + \|\text{Scal}^{2}\|_{L^{2}(X \leq 0)}
$$

it is clear that $\|\phi_{n}\|_{L^{2}(X)}$ is also bounded. Similarly, we have that

$$
\|d\alpha_{n}\|^{2}_{L^{2}(X)} \leq C
$$

To show $L^{2}$-bound for $\{a_{n}\}$, we need the gauge transformations, to get the bound. By Proposition 3.2, we have a uniform $L_{k}^{2}$-bound for $\{A_{n}\}$. By the Sobolev embedding $L_{k}^{2} \hookrightarrow L^{4}$, we get

$$
\|A_{n} \cdot \phi_{n}\|_{L^{4}} \leq \|A_{n}\|_{L^{4}} \|\phi_{n}\|_{L^{4}} \leq \|A_{n}\|_{L_{k}^{2}} \|\phi_{n}\|_{L^{4}}
$$

hence we have a bound by the identity

$$
\psi_{A_{n}} \phi_{n} = \psi_{A_{n}} \phi_{n} - A_{n} \cdot \phi_{n}.
$$

Since $\psi_{A_{n}}$ is Fredholm, and thus

$$
\|\phi_{n}\|_{L^{2}} \leq C \cdot (\|\psi_{A_{n}} \phi_{n}\|_{L^{2}} + \|\phi_{n}\|_{L^{4}}),
$$

which proves the $L^{2}$-bound for the spinor field part.

In conclusion, we have a uniform $L^{2}$-bound for $(A_{n}, \phi_{n})$ and $L^{2}$-bound for $(\nabla A_{n} \phi_{n}, |\phi_{n}|^{2})$. □
We have the following lemma.

**Lemma 3.4 (Veloso [20, Lemma 4.4.1])** Let \( \{(A_n, \phi_n)\} \) be the above sequence and \((a_\infty, \phi_\infty)\) be the weak limit in \( L^2_{\text{loc}} \). Then, the following sequence

1. \( \text{SW}(A_n, \phi_n) \to \text{SW}(a_\infty) \),
2. \( \nabla A_n \phi_n \to \nabla A_\infty \phi_\infty \),
3. \( \|\phi_n\|^2 \to \|\phi_\infty\|^2 \),

converge in \( L^2_{\text{loc}} \). Moreover, the \( L^2 \)-weak limit of \( \{(\nabla A_n \phi_n, |\phi_n|^2)\} \) is \( (\nabla A_\infty \phi_\infty, |\phi_\infty|^2) \).

We show the following theorem for the compactness under \( L^2 \)-norm.

**Theorem 3.4** Let \( \{(A_n, \phi_n)\} \) be a sequence solving the perturbed Seiberg-Witten in Theorem 3.3. Suppose that \( \{(A_n, \phi_n)\} \) satisfies the following properties: \( d^*(A_n - A_0) = 0 \), for some reference connection \( A_0 \) and \( \{A_n\} \) has a uniform \( L^2 \)-bound. Then we have that there is subsequence converging strongly in \( L^2 \), still denoted by \( \{(A_n, \phi_n)\} \).

**Proof** For the convenience, we set \( a_n = A_n - A_0 \) and \( a_\infty = A_\infty - A_0 \). The idea is similar to the technique given by Veloso in his thesis [20, Lemma 4.5.2]. As \( w > 0 \) for any \( k \geq 2 \), one get a compact map

\[
L^2_{k-1} \to L^2.
\]

It follows that \( \text{SW}(A_n, \phi_n) \) has convergent subsequence in \( L^2 \)-space by using the \( L^2_{k-1} \)-bound for \( \text{SW}(A_n, \phi_n) \). We set the limit \( (\phi, \omega) \), by [20, Lemma 4.4.1], \( \text{SW}(A_n, \phi_n) \) converges in \( L^1_{\text{loc}} \) to \( \text{SW}(a_\infty, \phi_\infty) \), where \( (a_\infty, \phi_\infty) \) denotes the weak \( L^2 \)-limit. Therefore, we get that \( \text{SW}(A_n, \phi_n) \) converges to \( \text{SW}(a_\infty, \phi_\infty) \) in \( L^2 \). By the identity formula, we have

\[
\lim_{n \to \infty} E^{an}(A_n, \phi_n) = E^{an}(a_\infty, \phi_\infty).
\]

Since we have the identity,

\[
E^{an}(A_n, \phi_n) + \int_X \|\phi_n\|^2 + \frac{1}{4} \left( \frac{\|\phi_n\|^4}{4} + \int_{X < 0} \text{Scal} \right) = \frac{1}{4} \left( \|\phi_n\|^2 + \text{Scal} \left/ \|\phi_n\|^2 \right. \right) + \frac{1}{4} \left( \frac{\|\phi_n\|^4}{4} + \int_{X > 0} \text{Scal} \right)
\]

By the weak \( L^2 \)-convergence of \( (A_n, \phi_n) \), it follows that

\[
E^{an}(A_n, \phi_n) + \int_X \|\phi_n\|^2 + \frac{1}{4} \left( \frac{\|\phi_n\|^4}{4} + \int_{X < 0} \text{Scal} \right) + \frac{1}{4} \left( \frac{\|\phi_n\|^4}{4} + \int_{X > 0} \text{Scal} \right)
\]

Therefore,

\[
\frac{1}{4} \|\phi_n\|^2_{L^2(X)} + \frac{1}{4} \|\phi_n\|^4_{L^4(X > 0)} + \int_{X > 0} \text{Scal} \|\phi_n\|^2 + \frac{1}{4} \left( \frac{\|\phi_n\|^4}{4} + \int_{X > 0} \text{Scal} \right)
\]

converges to

\[
\frac{1}{4} \|\phi_n\|^2_{L^2(X)} + \frac{1}{4} \|\phi_\infty\|^4_{L^4(X > 0)} + \int_{X > 0} \text{Scal} \|\phi_\infty\|^2 + \frac{1}{4} \left( \frac{\|\phi_\infty\|^4}{4} + \int_{X > 0} \text{Scal} \right)
\]

Applying the argument of Theorem 3.3, one can find a subsequence such that the terms

\[
\|\phi_n\|^4_{L^4(X > 0)}, \|\text{Scal} \phi_n\|^2_{L^2(X > 0)}, \|\nabla A_n \phi_n\|^2_{L^2(X)}, \|\phi_n\|^2 + \text{Scal} / 2 \right/ L^2
\]

converge. By the weak convergence, one obtains the following inequalities,
\begin{itemize}
\item $\lim(\|da_n\|_{L^2(X)}) \geq \|da_\infty\|_{L^2(X)}$, $\lim(\|\phi_n\|_{L^4(X^{>0})}) \geq \|\phi_\infty\|_{L^4(X^{>0})}$,
\item $\lim(\|\text{Sca}l^*\phi_n\|_{L^2(X^{>0})}^2) \geq \|\text{Sca}l^*\phi_\infty\|_{L^2(X^{>0})}^2$, $\lim(\|\nabla A_n \phi_n\|_{L^4(X)}) \geq \|\nabla A_\infty \phi_\infty\|_{L^4(X)}$,
\item $\lim(\|\phi_n\|^2 + \text{Sca}l/2\|\phi_n\|_{L^2(X^{<0})}) \geq \|\phi_\infty\|^2 + \text{Sca}l/2\|\phi_\infty\|_{L^2(X^{<0})}$.
\end{itemize}

Combining with Radon Riesz Theorem, we establish the following:
\begin{itemize}
\item $\lim(da_n) = da_\infty$ in $L^2(X)$, $\lim(\phi_n) = \phi_\infty$ in $L^4(X^{>0})$,
\item $\lim(\nabla A_n \phi_n) = \nabla A_\infty \phi_\infty$ in $L^2(X)$,
\item $\lim(|\phi_n|^2) = |\phi_\infty|^2$ in $L^2(X^{<0})$.
\end{itemize}

We need to show the convergence of $\{a_n\}$ in $L^2$-norm. For each $a_n$, Theorem 3.1 provides a decomposition
$$a_n = d^*b_n + c_n,$$
where $\{b_n\} \subset \text{Im}(d^* : L^2_k+1(\Omega^+) \to L^2_k(\Omega^1))$ and $\{c_n\} \subset \text{Harm}^1_w$. We also have a similar decomposition for $a_\infty$, i.e.
$$a_\infty = d^*b_\infty + c_\infty.$$  
Obviously $\{c_n\}$ is uniformly bounded in the finite dimensional space $\text{Harm}^1_m$, hence the there exists a subsequence converge to the (weak)limit $c_\infty$. Clearly, one has that
$$\|\Delta_+(b_n - b_\infty)\|_{L^2} = \|d^*d^*(b_n - b_\infty)\|_{L^2} = \|d^*(a_n - a_\infty)\|_{L^2} \to 0,$$
where we use the Fredholmness of $\Delta_+$ by Theorem 3.2. Combining with the estimate
$$\|b_n - b_\infty\|_{L^2} \leq \|\Delta_+(b_n - b_\infty)\|_{L^2} + \|\text{pr}_{\text{ker}(\Delta_+)}(b_n - b_\infty)\|_{L^2}.$$  
Since $b_n \perp \text{ker}(\Delta^+)$ and $b_\infty \perp \text{ker}(\Delta^+)$, we deduce that there exists a convergent subsequence of $\{d^*b_n\}$. By the estimate,
$$\|A_n - A_\infty\|_{L^2} \leq \|d^*(b_n - b_\infty)\|_{L^2} + C\|c - c_\infty\|_{L^2},$$
we have that $A_n \rightarrow A_\infty$ in $L^2$. Furthermore, we have the following convergence:
\begin{itemize}
\item The convergence of $|\phi_n|^2 \rightarrow |\phi_\infty|^2$ in $L^2(X^{>0})$ and the convergence of $\phi_n \rightarrow \phi_\infty$ in $L^4(X^{>0})$ imply that $\phi_n \rightarrow \phi_\infty$ in $L^4(X)$.
\item The convergence of $\nabla A_n \phi_n \rightarrow \nabla A_\infty \phi_\infty$ in $L^2$ implies that
$$\nabla A_0 \phi_n + a_n \cdot \phi_n \to \nabla A_0 \phi_\infty + a_\infty \cdot \phi_\infty$$
in $L^2$.
\item The convergence of $\phi_n \rightarrow \phi_\infty$ in $L^4$ and the convergence of $a_n \rightarrow a_\infty$, together with the Sobolev embedding $L^2 \rightarrow L^4$ show that $a_n \cdot \phi_n \rightarrow a_\infty \cdot \phi_\infty$, which implies that $\nabla A_0 \phi_n \rightarrow \nabla A_0 \phi_\infty$ in $L^2$.
\item By the Fredholmness of $\mathcal{D}_{A_0}$, we have
$$\|\phi_n - \phi_\infty\|_{L^2} \leq C(\|\mathcal{D}_{A_0}(\phi_n - \phi_\infty)\|_{L^2} + \|\phi_n - \phi_\infty\|_{L^4}),$$
which means that $\phi_n \rightarrow \phi_\infty$ in $L^2$. Therefore, we proved the theorem.
\end{itemize}
By the similar method of proof of [20, Theorem 4.6.1], we have the theorem below

**Theorem 3.5** Let \((X, g)\) be an oriented 4-manifold satisfying Assumption 1. Choose a real number \(w > 0\) and an integer \(k \geq 2\). Let \(\{(A_n, \phi_n)\}\) be a sequence with uniform bounded Seiberg-Witten image, i.e. there exists a constant \(C\) such that

\[
\|\text{SW}(A_n, \phi_n)\|_{L^2_{k-1}} \leq C
\]

for all \(n \geq 1\). Moreover, we suppose that \(\{(A_n, \phi_n)\}\) is convergent in \(L^2_1\) and \(d^*(A_n - A_0) = 0\). Then, there exists a subsequence of \(\{(A_n, \phi_n)\}\) which is bounded in \(L^2_{k,w}\).

Theorem 1.1 is a special case of Theorem 3.5, therefore we proved Theorem 1.1.

## 4 Proof of Theorem 1.2

Let \((Y, g)\) be a closed oriented 4 Riemannian manifold with flat scalar curvature. Suppose that \(Y\) admits a flat scalar curvature but does not admit any positive scalar curvature. We consider the Ricci flow [8] with initial condition

\[
\begin{align*}
\frac{\partial g(t)}{\partial t} &= -\text{Ric}_{g(t)}, \\
g(0) &= g.
\end{align*}
\]

By the straightforward calculus, c.f. [16, Theorem 1.8], one has that

\[
\frac{\partial \text{Scal}^{(r)}(t)}{\partial t} = -\Delta \text{Scal}^{(r)}(t) + 2|\text{Ric}^{(r)}(t)|^2.
\]

By the maximal principle [16, Corollary 2.10], it is known that \(\text{Scal}^{(r)}(t) \geq 0\). Since \(Y\) does not admit any metric with positive scalar curvature, one deduces that \(\text{Scal}^{(r)}(t) = 0\), \(\text{Ric}^{(r)}(t) = 0\), for short time \(t \in [0, T]\). This implies that \((Y, g)\) is Ricci flat. By the work of Fischer and Wolf [5], we get an isometric decomposition

\[
Y \cong \Gamma' \backslash T^{b_1}(Y) \times M_0^{4-b_1}(Y).
\]

where \(M_0^{4-b_1}(Y)\) is a closed connected Ricci-flat manifold with dimension \(4 - b_1(Y)\) and \(\Gamma'\) is a finite group of fixed point free isometries of \(T^{b_1}(Y) \times M_0^{4-b_1}(Y)\). Since \(b_1(Y) = 1\), we have that \(M_0^2\) is a closed Ricci flat manifold, hence Riemannian flat by the below lemma.

**Lemma 4.1** Let \((M, g)\) be a Riemannian three manifold. Suppose that \((M, g)\) is Ricci flat. Then \((M, g)\) is Riemannian flat.

**Proof** The first Bianchi identity of the Riemann curvature implies that it can be identified with a symmetric mapping from the space of two forms to itself via

\[
\omega_{ij} \mapsto \text{Riem}^{kl}_{ij} \omega_{kl},
\]

where we used the Einstein convention. The mapping

\[
\phi : \text{Riem}_{ijkl} \mapsto g^{ik} \text{Riem}_{ijkl}
\]

has the properties:

\[
\phi \circ \otimes = \text{Id} \text{ on } (0,2) \text{ tensors}, \otimes \circ \phi = \text{Id} \text{ on } (0,4) \text{ tensors},
\]

where \(\otimes\) denotes the Kulkarni-Nomizu product [1, Definition 1.110] with metric \(g\). At each point \(p \in M\) the the Riemannian curvature space has the same dimension 3 with the Ricci curvature space, which implies that the Riemannian curvature is determined by Ricci curvature. \(\square\)
A crystallographic group on $\Gamma$ can be split as $\{1\} \times \Gamma_2$, where $\Gamma_2$ acts freely on $M_0^3$. Obviously $b_1(\Gamma_2, M_0^3) = 0$, by Wolf's work [23, Theorem 3.5.5], there is only one choice for such a manifold $M_0^3 = \mathbb{R}^3/\Gamma_2^1$, i.e. $\Gamma_2^1$ is represented by $\{a, b, c, t_1, t_2, t_3\}$ satisfying the relations
\[
cba = t_1t_3, \quad a^2 = t_1, \quad at_2a^{-1}t_2^{-1} = 1, \quad at_3a^{-1}t_3 = 1, \\
bt_1b^{-1}t_1 = 1, \quad b^2 = t_2, \quad bt_3b^{-1}t_3 = 1, \\
ct_1c^{-1}t_1, \quad ct_2c^{-1}t_2 = 1, \quad c^2 = t_3.
\]
One also has that $H_1(M_0^3, \mathbb{Z}) \cong (\mathbb{Z}_4)^2$. In such a case, the associated $Y$ is of class $O_4^1$ in Table 4, i.e. Hantsche-Wendt manifold, and the periodic end of $(X, g)$ becomes cylindrical end, i.e. modelling on $\mathbb{R}^3/\mathbb{Z}_4$. In general, for complete flat Riemannian $n$-manifold $(M, g)$, it is known that its universal covering is the Euclid space $\mathbb{R}^n$ and $M \cong \Gamma \backslash \mathbb{R}^n$, where $\pi_1(M) \cong \Gamma$. When $(M, g)$ is a closed Riemannian flat, one has that it is finitely covered by $\mathbb{T}^n$, i.e. $M \cong \Gamma / \mathbb{T}^n$. Equip $\mathbb{R}^n$ with an metric, we have the isometric group $\text{Isom}(n) \cong O(n) \rtimes \mathbb{R}^n$.

The fundamental theorem determines a short exact sequence
\[
1 \to \Gamma \cap \mathbb{R}^n \to \Gamma \to \Gamma / \mathbb{R}^n \to 1,
\]
where $\Gamma \cap \mathbb{R}^n$ is a torsion free abelian group of rank $n$ and $\Gamma / \mathbb{R}^n \cong \mathbb{Z}^n$.

**Definition 4.1** A crystallographic group on $\mathbb{R}^n$ to a discrete uniform subgroup of $\text{Isom}(n)$.

For crystallographic group, we have the two celebrated theorems.

**Theorem 4.1 (Bieberbach [2])** If $\Gamma < \text{Isom}(n)$ is a crystallographic group, then $\Gamma \cap \mathbb{R}^n \cong \mathbb{Z}^n$ is a normal subgroup of finite index, and any minimal set of generators of $\Gamma \cap \mathbb{R}^n$ is a vector space basis of $\mathbb{R}^n$ relative to which the $O(n)$-components of the elements of $\Gamma$ have all integral entries.

**Theorem 4.2 (Bieberbach [3])** For each $n > 0$, there are only a finite number of isomorphism classes of crystallographic groups on $\mathbb{R}^n$. Two crystallographic groups are isomorphic if and only if they are conjugate by an element of the affine group $\text{Aff}(n)$.

**Proof** (of Theorem 1.2) In dimensions 4, there are 74 homeomorphism equivalence classes of closed flat manifolds. The classification of closed flat 4-manifolds was first achieved by Brown et al. [4] in their classification of all the isomorphism classes of 4-dimensional crystallographic groups. Brown et al. found 74 homeomorphism equivalence classes of closed flat 4-manifolds, where 27 classes are orientable manifolds and 47 classes are nonorientable manifolds. The classification of the closed flat 4-manifold are given by Hillman [9].

If we put a further restriction on the first Betti number $b_1 = 1$, there are only 19 possible classes in orientable manifolds. All these 19 classes are fibration over $S^1$, whose fibers are oriented closed 3-manifold with the monodromy representation $T$. We put a table of these 19 classes at the end of this section, the ordering is inherited from Hillman [9]. The manifolds are ordered inversely with respect to their first Betti number. We only list the first homology group, fibering, the monodromy representation and whether it is spin or not, where $y$ denotes yes and $n$ denotes no in the below table. For non-compact manifold with periodic end, we need to classify the $\mathbb{Z}$-covering of these 19 classes up to isomorphism. In general, the $\mathbb{Z}$-coverings of a topological space $Y$ are one-by-one corresponding to the primitive of the group $\text{Hom}(\pi_1(Y), \mathbb{Z}) \cong \text{Hom}(H_1(Y, \mathbb{Z}), \mathbb{Z})$. Since the first Betti number is 1, one deduces that for each class among these 19 classes $\mathbb{Z}$-covering is unique up to isomorphism, i.e. the line $\mathbb{R}$ product with the fiber. Moreover we need the vanishing first Betti number for these $\mathbb{Z}$-covering, hence only 6 classes remain.
| manifold | fibering | $H_1$ | $T$ | spin |
|----------|----------|-------|-----|------|
| $O_4^1$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus (\mathbb{Z}_2)^4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^0$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y, z \mapsto z$ | y |
| $O_4^1$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y, z \mapsto z$ | y |
| $O_4^2$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus (\mathbb{Z}_2)^2$ | $x \mapsto x^{-1}$ | y |
| $O_4^3$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^4$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus (\mathbb{Z}_2)^4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^5$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^6$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y, z \mapsto z$ | y |
| $O_4^7$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^8$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y, z \mapsto z$ | y |
| $O_4^9$  | $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |
| $O_4^{10}$| $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y, z \mapsto z$ | y |
| $O_4^{11}$| $O_2^1 \times S^1$ | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $x \mapsto x^{-1}, y \mapsto y$ | y |

Table 1: Orientable flat 4-Manifold with $b_1 = 1$

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