AREA PROBLEMS INVOLVING KASNER POLYGONS

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Abstract. Sequences of polygons generated by performing iterative processes on an initial polygon have been studied extensively. One of the most popular sequences is the one sometimes referred to as Kasner polygons. Given a polygon $K$, the first Kasner descendant $K'$ of $K$ is obtained by placing the vertices of $K'$ at the midpoints of the edges of $K$.

More generally, for any fixed $m$ in $(0, 1)$ one may define a sequence of polygons $\{K^t\}_{t \geq 0}$ where each polygon $K^t$ is obtained by dividing every edge of $K^{t-1}$ into the ratio $m : (1 - m)$ in the counterclockwise (or clockwise) direction and taking these division points to be the vertices of $K^t$.

We are interested in the following problem

Let $m$ be a fixed number in $(0, 1)$ and let $n \geq 3$ be a fixed integer. Further, let $K$ be a convex $n$-gon and denote by $K'$, the first $m$-Kasner descendant of $K$, that is, the vertices of $K'$ divide the edges of $K$ into the ratio $m : (1 - m)$. What can be said about the ratio between the area of $K'$ and the area of $K$, when $K$ varies in the class of convex $n$-gons?

We provide a complete answer to this question.

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1. Introduction

Start with a fixed number $m$ in $(0, 1)$ and a convex $n$-gon $K$. Let $K'$ be the convex $n$-gon whose vertices divide the edges of $K$ into the ratio $m : (1 - m)$ in the counterclockwise direction. We say $K'$ is the first $m$-Kasner descendant of $K$. In general, we may construct a sequence of polygons $\{K^t\}_{t \geq 0}$ where $K^0 = K$, $K^1 = K'$ and $K^{t+1}$ is the first $m$-Kasner descendant of $K^t$.

Kasner noticed that if $n = 4$ and $m = 1/2$ then $K'$ is always a parallelogram. In [5] he and his students managed to characterize those $n$-gons $P$ which have the property that $P = K'$ for some convex $n$-gon $K$ and $m = 1/2$.

It is easy to notice that all $n$-gons in the sequence $\{K^t\}$ defined above have the same centroid. This was proved repeatedly; see e. g. [3] for a matrix proof or [10] for a proof using Fourier series.
It has been shown by Lükő [8] that the sequence \( \{K^t\} \) converges to an (affine) regular \( n \)-gon, thus proving a conjecture of Fejes Tóth [9]. More on Kasner polygons can be found in [1, 2, 4, 6, 7].

In this paper we study the following:

**Problem 1.1.** Let \( m \) be a fixed number in \((0, 1)\) and let \( n \geq 3 \) be a fixed integer. Let \( K \) be a convex \( n \)-gon and denote by \( K' \), the first \( m \)-Kasner descendant of \( K \). What can be said about \( \Delta(K')/\Delta(K) \), the ratio between the area of \( K' \) and the area of \( K \), when \( K \) varies in the class of convex \( n \)-gons?

**The Main Technique.** Throughout the entire paper we use the wedge product of two vectors to express areas. This operation, also known as exterior product, is defined as follows. For any two vectors \( v = (a, b) \) and \( u = (c, d) \) let the wedge product of \( v \) and \( u \) be given by \( v \wedge u := (ad - bc)/2 \).

It is easy to see that the wedge product represents the signed area of the triangle determined by the vectors \( v \) and \( u \), where the \( \pm \) sign depends on whether the angle between \( v \) and \( u \) - measured in the counterclockwise direction from \( v \) towards \( u \) - is smaller than or greater than 180°.

The following properties of the wedge product are simple consequences of the definition:

- anti-commutativity: \( v \wedge u = -u \wedge v \) and in particular \( v \wedge v = 0 \).
- linearity: \( (\alpha v + \beta u) \wedge w = \alpha v \wedge w + \beta u \wedge w \).

### 2. The Triangle Case

Let \( m \) be a fixed number in \((0, 1)\) and let \( K = ABC \) be an arbitrary triangle. Construct points \( M, N \) and \( P \) on the sides \( AB, BC \) and \( AC \), such that \( AM : MB = BN : NC = CP : PA = m : (1-m) \). We call triangle \( K' = MNP \) to be the first \( m \)-Kasner descendant of \( K \).

**Theorem 2.1.** With the notations above we have that

\[
\frac{\Delta(K')}{\Delta(K)} = 1 - 3m(1-m).
\]

**Proof.** Denote \( \overrightarrow{AB} = v_1, \overrightarrow{BC} = v_2 \) and \( \overrightarrow{CA} = v_3 \) as in figure [1]. Obviously, \( v_1 + v_2 + v_3 = 0 \). After an appropriate scaling we may assume that the area of \( ABC \) is equal to 1, that is

\[
\Delta(K) = v_1 \wedge v_2 \wedge v_3 \wedge v_1 = 1.
\]
Figure 1. A triangle and its first $m$-Kasner descendant

It is easy to see that $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = (1-m)\mathbf{v}_1 + m \mathbf{v}_2$ and $\overrightarrow{NP} = \overrightarrow{NC} + \overrightarrow{CP} = (1-m)\mathbf{v}_2 + m \mathbf{v}_3$. It follows that $\Delta(K') = \overrightarrow{MN} \wedge \overrightarrow{NP} = ((1-m)\mathbf{v}_1 + m \mathbf{v}_2) \wedge ((1-m)\mathbf{v}_2 + m \mathbf{v}_3) =$

$= (1-m)^2(\mathbf{v}_1 \wedge \mathbf{v}_2) + m(1-m)(\mathbf{v}_1 \wedge \mathbf{v}_3) + m^2(\mathbf{v}_2 \wedge \mathbf{v}_3) =$

$= (1-m)^2 + m(1-m)(-1) + m^2 = 1 - 3m(1-m).

3. The Quadrilateral Case

Let $m$ be a fixed number in $(0, 1)$ and let $K = ABCD$ be an arbitrary quadrilateral. Construct points $M$, $N$, $P$ and $Q$ on the sides $AB$, $BC$, $CD$ and $DA$, such that $AM : MB = BN : NC = CP : PD = DQ : QA = m : (1-m)$. We call quadrilateral $K' = MNPQ$ to be the first $m$-Kasner descendant of $K$.

**Theorem 3.1.** With the notations above we have that

$$\frac{\Delta(K')}{\Delta(K)} = 1 - 2m(1-m).$$

**Proof.** Denote $\overrightarrow{AB} = \mathbf{v}_1$, $\overrightarrow{BC} = \mathbf{v}_2$, $\overrightarrow{CD} = \mathbf{v}_3$ and $\overrightarrow{DA} = \mathbf{v}_4$ as in figure 2. Obviously, $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. One can express the area of $ABCD$ in a couple of ways as below.

$$\Delta(K) = \Delta(ABC) + \Delta(CDA) = \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge \mathbf{v}_4 \quad \text{and}$$

$$\Delta(K) = \Delta(BCD) + \Delta(CDA) = \mathbf{v}_2 \wedge \mathbf{v}_3 + \mathbf{v}_4 \wedge \mathbf{v}_1.$$
On the other hand

\[
\Delta(MNP) = \overrightarrow{MN} \wedge \overrightarrow{NP} = ((1-m)v_1 + mv_2) \wedge ((1-m)v_2 + mv_3) = \\
= (1-m)^2(v_1 \wedge v_2) + m(1-m)(v_1 \wedge v_3) + m^2(v_2 \wedge v_3).
\]

\[
\Delta(PQM) = \overrightarrow{PQ} \wedge \overrightarrow{QM} = ((1-m)v_3 + mv_4) \wedge ((1-m)v_4 + mv_1) = \\
= (1-m)^2(v_3 \wedge v_4) + m(1-m)(v_3 \wedge v_1) + m^2(v_4 \wedge v_1).
\]

Adding the two equalities above term by term and using (3) and (4) we obtain

\[
\Delta(K') = (1-m)^2((v_1 \wedge v_2) + (v_3 \wedge v_4)) + m^2((v_2 \wedge v_3) + (v_4 \wedge v_1)) = \\
= (1-m)^2\Delta(K) + m^2\Delta(K) = (1 - 2m(1-m))\Delta(K).
\]

We have seen that the ratio between the area of a convex \emph{n}-gon and the area of its first \emph{m}-Kasner descendant is constant if \( n \leq 4 \). This is not true anymore if the initial polygon has at least five sides. In this later case we will be interested in the range of values the ratio \( \Delta(K')/\Delta(K) \) takes when \( K \) belongs to the class of convex \emph{n}-gons. We are investigating this question in the following sections.
4. The Pentagon Case - A First Attempt

Let \( m \) be a fixed number in \((0, 1)\) and let \( K = ABCDE \) be an arbitrary convex pentagon. Construct points \( M, N, P, Q \) and \( R \) on the sides \( AB, BC, CD, DE \) and \( EA \) such that \( AM : MB = BN : NC = CP : PD = DQ : QE = ER : RA = m : (1 - m) \). As before, we call the pentagon \( K' = MNPQR \) to be the first \( m \)-Kasner descendant of \( K \).

**Theorem 4.1.** With the notations above we have that

\[
1 - 2m(1 - m) < \frac{\Delta(K')}{\Delta(K)} < 1 - m(1 - m).
\]

Moreover, both lower and upper bounds are the best possible.

For reasons which will become clear soon, we postpone the proof of theorem 4.1 until the next section. Let us first try to approach this problem the same way we proved theorems 2.1 and 3.1. As before, denote \( \overrightarrow{AB} = \mathbf{v}_1, \overrightarrow{BC} = \mathbf{v}_2, \overrightarrow{CD} = \mathbf{v}_3, \overrightarrow{DE} = \mathbf{v}_4 \) and \( \overrightarrow{EA} = \mathbf{v}_5 \) as in figure 3. Obviously, \( \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = 0 \).

![Figure 3](image.png)

**Figure 3.** A convex pentagon and its first two \( m \)-Kasner descendants

Let us introduce a couple of notations which are going to be useful later. For every \( 1 \leq i, j \leq 5 \) denote \( a_{ij} = \mathbf{v}_i \wedge \mathbf{v}_j \). Moreover, let

\[
S := a_{12} + a_{23} + a_{34} + a_{45} + a_{51},
\]

\[
T := a_{13} + a_{24} + a_{35} + a_{41} + a_{52}.
\]
It is easy to see that \( \Delta(MBN) = \overrightarrow{MB} \wedge \overrightarrow{BN} = (1-m)\mathbf{v}_1 \wedge m\mathbf{v}_2 = m(1-m)a_{12} \). Similar expressions can be found for the areas of the other four triangles \( NCP, \ PDQ, \ QER \) and \( RAM \). Using (8) it follows immediately that

\[
(8) \quad \Delta(K') = \Delta(K) - m(1-m)(a_{12} + a_{23} + a_{34} + a_{45} + a_{51}) = \Delta(K) - m(1-m)S.
\]

Notice that there are several different ways in which \( \Delta(K) \) can be expressed in terms of the \( a_{ij} \)-s. For instance, we have that

\[
\Delta(K) = \Delta(ABC) + \Delta(ACD) + \Delta(ADE) = \overrightarrow{AB} \wedge \overrightarrow{BC} + \overrightarrow{AC} \wedge \overrightarrow{CD} + \overrightarrow{DE} \wedge \overrightarrow{EA} = \mathbf{v}_1 \wedge \mathbf{v}_2 + (\mathbf{v}_1 + \mathbf{v}_2) \wedge \mathbf{v}_3 + \mathbf{v}_4 \wedge \mathbf{v}_5 = a_{12} + a_{13} + a_{23} + a_{45}.
\]

Using equalities (8) and (9) we easily derive the following

\[
\frac{\Delta(K')}{\Delta(K)} = 1 - m(1-m) \cdot \frac{a_{12} + a_{23} + a_{34} + a_{45} + a_{51}}{a_{12} + a_{13} + a_{23} + a_{45}}.
\]

**Observation.** Proving theorem 4.1 reduces to showing that

\[
1 < \frac{a_{12} + a_{23} + a_{34} + a_{45} + a_{51}}{a_{12} + a_{13} + a_{23} + a_{45}} < 2
\]

and that these inequalities cannot be improved. This is somewhat of an awkward task. The reason is that the \( a_{ij} \)-s are not independent quantities. Indeed, on one hand we have that \( \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = 0 \). This implies for instance that \( \mathbf{v}_1 \wedge (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5) = a_{12} + a_{13} + a_{14} + a_{15} = 0 \).

On the other hand, it can be easily shown that for any four distinct indices \( i, \ j, \ k \) and \( l \) in \( \{1, 2, 3, 4, 5\} \) we have the following equality, known as Plücker’s identity

\[
(10) \quad a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0.
\]

Indeed, it is easy to see that among the four vectors \( \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k \) and \( \mathbf{v}_l \) there are two which are independent; say \( \mathbf{v}_i \) and \( \mathbf{v}_j \) are those vectors. Then \( \mathbf{v}_k \) and \( \mathbf{v}_l \) can be expressed as linear combinations of \( \mathbf{v}_i \) and \( \mathbf{v}_j \). Suppose that \( \mathbf{v}_k = \alpha \mathbf{v}_i + \beta \mathbf{v}_j \) and \( \mathbf{v}_l = \lambda \mathbf{v}_i + \mu \mathbf{v}_j \). It follows immediately that

\[
a_{ik} = \mathbf{v}_i \wedge (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) = \beta a_{ij}, \quad a_{il} = \mathbf{v}_i \wedge (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = \mu a_{ij}, \quad a_{jk} = \mathbf{v}_j \wedge (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) = -\alpha a_{ij}, \quad a_{jl} = \mathbf{v}_j \wedge (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = -\lambda a_{ij}.
\]

and finally, \( a_{kl} = (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) \wedge (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = (\alpha \mu - \beta \lambda)a_{ij} \). Substituting these equalities into the left side of (10) we obtain the desired identity.
While it seems that this line of attack is destined to failure, one can still derive an interesting fact. Let $K'' := STUVW$ be the second $m$-Kasner descendant of $K$ - see figure 3. We would like to see whether there is a relationship linking the areas of $K$, $K'$ and $K''$.

For $1 \leq i, j \leq 5$ denote $v'_i = (1 - m)v_i + mv_{i+1}$, $a'_{ij} = v'_i \wedge v'_j$ and $S' := a'_{12} + a'_{23} + a'_{34} + a'_{45} + a'_{51}$.

First notice that

$$a'_{i,i+1} = [(1 - m)v_i + mv_{i+1}] \wedge [(1 - m)v_{i+1} + mv_{i+2}] = (1 - m)^2a_{i,i+1} + m(1 - m)a_{i,i+2} + m^2a_{i+1,i+2}.$$ 

Using notations (6) and (7) it follows that

$$S' = \sum_{i=1}^{5} a'_{i,i+1} = (1 - m)^2S + m(1 - m)T + m^2S = (1 - 2m(1 - m))S + m(1 - m)T.$$ 

A reasoning similar to the one which led us to equality (8) can be used to show that

$$\Delta(K'') = \Delta(K') - m(1 - m)S'.$$

Also, relation (9) can be rewritten as $a_{13} = \Delta(K) - a_{12} - a_{23} - a_{34}$. If one expresses the area of $K$ as $\Delta(K) = \Delta(BCD) + \Delta(BDE) + \Delta(BEA) = a_{23} + a_{24} + a_{34} + a_{51}$ we get that $a_{24} = \Delta(K) - a_{23} - a_{34} - a_{51}$. In an analogous manner one can obtain expressions for $a_{35}$, $a_{41}$ and $a_{52}$. By adding these relations term by term and taking into account (7) we obtain that

$$T = a_{13} + a_{24} + a_{35} + a_{41} + a_{52} = 5\Delta(K) - 3S.$$ 

By eliminating the quantities $S'$, $S$ and $T$ between equalities (8), (11), (12) and (13) we finally obtain a linear relationship linking the areas of $K$, $K'$ and $K''$.

$$\Delta(K'') = (2 - 5r)\Delta(K') - (1 - 5r + 5r^2)\Delta(K), \quad \text{where } r := m(1 - m).$$

In other words, if we know the area of the initial pentagon and the area of its first $m$-Kasner descendant we can compute the area of any of the $m$-Kasner descendants, $K^t$, for any $t \geq 2$.

Similar recurrence relationships are valid for polygons with more than five sides. In general, for convex $n$-gons, the recurrence involves $\lceil (n+1)/2 \rceil$ consecutive $m$-Kasner descendants. We omit the details.
5. The Pentagon Case - A Second (and Successful) Approach

Let $K = ABCDE$ be an arbitrary convex pentagon and let $m$ be a fixed constant in $(0, 1)$. Denote $r := m(1 - m)$. After an eventual relabeling of the vertices we may assume that

\[ \Delta(ABC) = \min\{\Delta(ABC), \Delta(BCD), \Delta(CDE), \Delta(DEA), \Delta(EAB)\} \]

In the literature, such triangles formed by three consecutive vertices of a convex polygon are sometimes called *ears*. Assumption (15) above fixes the ear of least area. Denote the intersection of $AD$ and $CE$ by $O$. Then define $v_1 = \overrightarrow{OA}$, $v_2 = \overrightarrow{OC}$. After an appropriate scaling, we may assume that $v_1 \wedge v_2 = \Delta(AOC) = 1$.

Since $E$, $O$, and $C$ are collinear and $D$, $O$, and $A$ are collinear, we can write $\overrightarrow{DO} = a \cdot \overrightarrow{OA} = a v_1$ and $\overrightarrow{EO} = b \cdot \overrightarrow{OC} = b v_2$, with $a, b > 0$ (see figure 4).

Using the triangle rule, we obtain that $\overrightarrow{CD} = -a v_1 - v_2$, $\overrightarrow{DE} = a v_1 - b v_2$, and $\overrightarrow{EA} = v_1 + b v_2$.

We know that every vector in the plane can be written as a linear combination of any two independent vectors. Set $\overrightarrow{OB} = v_3 = c v_1 + d v_2$ - refer again to figure 4. We also know that $\overrightarrow{AB} = v_3 - v_1$ and $\overrightarrow{BC} = v_2 - v_3$. We have that

\[ \Delta(OAB) = v_1 \wedge v_3 = v_1 \wedge (c v_1 + d v_2) = d, \]
\[ \Delta(OBC) = v_3 \wedge v_2 = (c v_1 + d v_2) \wedge v_2 = c. \]
After similar calculations, we can write the areas of various triangles in pentagon $ABCDE$ in terms of the positive constants $a, b, c, d$ as shown below: $\Delta(OCD) = a\mathbf{v}_1 \wedge \mathbf{v}_2 = a$, $\Delta(OEA) = a\mathbf{v}_1 \wedge b\mathbf{v}_2 = ab$, $\Delta(OED) = \mathbf{v}_1 \wedge b\mathbf{v}_2 = b$. We can now compute the total area of the pentagon.

$$\Delta(ABCDE) = \Delta(OAB) + \Delta(OBC) + \Delta(OCD) + \Delta(ODE) + \Delta(OEA),$$

that is,

$$\Delta(K) = \Delta(ABCDE) = a + b + c + d + ab. \quad (16)$$

Next, we compute the areas of the ears of the pentagon.

$$\Delta(ABC) = \mathbf{AB} \wedge \mathbf{BC} = (\mathbf{v}_3 - \mathbf{v}_1) \wedge (\mathbf{v}_2 - \mathbf{v}_3) = c + d - 1, \quad (17)$$

$$\Delta(BCD) = \mathbf{BC} \wedge \mathbf{CD} = (\mathbf{v}_2 - \mathbf{v}_3) \wedge (-a\mathbf{v}_1 - \mathbf{v}_2) = a - ad + c, \quad (18)$$

$$\Delta(CDE) = \mathbf{CD} \wedge \mathbf{DE} = (-a\mathbf{v}_1 - \mathbf{v}_2) \wedge (a\mathbf{v}_1 - b\mathbf{v}_2) = ab + a, \quad (19)$$

$$\Delta(DEA) = \mathbf{DE} \wedge \mathbf{EA} = (a\mathbf{v}_1 - b\mathbf{v}_2) \wedge (\mathbf{v}_1 + b\mathbf{v}_2) = ab + b.$$

It follows that

$$\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEA) + \Delta(EAB) = 2(a + b + c + d + ab) - 1 - ad - bc. \quad (20)$$

Consider now $K'$, the first $m$-Kasner descendant of the initial pentagon. We did not include $K'$ in figure 4 in order to keep things clear. However, it is easy to see from figure 3 that the area of $K'$ is the difference between the area of $K$ and the sum of the areas of the ears of the pentagon multiplied by a factor of $r = m(1 - m)$. This means that

$$\Delta(K') = \Delta(K) - r(\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEA) + \Delta(EAB)) \quad (21)$$

which after making use of (16) and (20) becomes $\Delta(K') = (1 - 2r)\Delta + r(1 + ad + bc)$ and finally

$$\frac{\Delta(K')}{\Delta(K)} = 1 - 2r + r \frac{1 + ad + bc}{a + b + c + d + ab}. \quad (22)$$

In order to prove theorem 4.1 it is enough to show that

**Claim 5.1.** With the notations from the present section we have

$$0 < \frac{1 + ad + bc}{a + b + c + d + ab} < 1 \quad (23)$$
and none of these inequalities can be improved.

Proof. It is clear the ratio is greater than 0 as \(a, b, c\) and \(d\) are all positive. To show that the ratio can be arbitrarily close to 0 take \(a = b = n\) and \(c = d = 1\). Then,

\[
\frac{1 + ad + bc}{a + b + c + d + ab} = \frac{2n + 1}{n^2 + 2n + 2} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

To show that the ratio can be arbitrarily close to 1 take \(a = n, b = 1/n\) and \(c = d = 1\). For these choices

\[
\frac{1 + ad + bc}{a + b + c + d + ab} = \frac{n^2 + n + 1}{n^2 + 3n + 1} \rightarrow 1 \text{ as } n \rightarrow \infty.
\]

Remains to show that the ratio is always less than 1. Recall that assumption (15) stated that triangle \(ABC\) is the ear of the smallest area. Refer first to equality (17). Since \(\Delta(ABC) > 0\) it follows that \(c + d > 1\). On the other hand \(\Delta(ABC) \leq \Delta(BCD)\) which after using (17) and (18) gives that \(c + d - 1 \leq a - ad + c \iff (a + 1)(1 - d) \geq 0 \iff d \leq 1\). Finally \(\Delta(ABC) \leq \Delta(EAB)\) which after using (17) and (19) implies that \(c + d - 1 \leq b - bc + d \iff (b + 1)(1 - c) \geq 0 \iff c \leq 1\).

The last three inequalities \((c + d > 1, c \leq 1 \text{ and } d \leq 1)\) imply that

\[1 + ad + bc \leq 1 + a + b < c + d + a + b < a + b + c + d + ab.\]

This proves that the ratio is always less that 1. This proves the claim and with it theorem 4.1. \(\Box\)

6. The Hexagon Case

As in the previous sections we start by fixing a constant \(m\) in \((0, 1)\) and considering \(K = ABCDEF\) an arbitrary convex hexagon. As before, \(K'\) denotes the first \(m\)-Kasner descendant of \(K\). The main result of this section is given in the following

**Theorem 6.1.** With the notations above we have that

\[
1 - 2m(1 - m) < \frac{\Delta(K')}{{\Delta(K)}} < 1.
\]

Moreover, both lower and upper bounds are the best possible.
concurrent anymore. By continuity, any inequality which is valid in latter case is also valid in the former. Let $M = AD \cap BE$, $N = AD \cap CF$, $P = CF \cap BE$. Denote $v_1 = \overrightarrow{MN}$, $v_2 = \overrightarrow{MP}$ and $v_3 = \overrightarrow{NP}$. It follows that $v_3 = v_2 - v_1$ - see figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Setup for the convex hexagon problem}
\end{figure}

With appropriate scaling we may assume that $\Delta(MNP) = v_1 \wedge v_2 = v_1 \wedge v_3 = v_2 \wedge v_3 = 1$. Since $A, M, N, D$ are collinear, $\overrightarrow{AM} = av_1$, $\overrightarrow{ND} = dv_1$ with $a, d > 0$. Similarly, $\overrightarrow{BM} = bv_2$, $\overrightarrow{CN} = cv_3$, $\overrightarrow{PE} = ev_2$, $\overrightarrow{PF} = fv_3$ with $b, c, e, f$ positive constants.

We try to express $\Delta(K)$, the area of the hexagon in terms of $a, b, c, d, e, f$. We begin by computing the areas of the triangles determined by one side and two long diagonals.

\[
\begin{align*}
\Delta(ABM) &= \overrightarrow{AB} \wedge \overrightarrow{BM} = (av_1 - bv_2) \wedge bv_2 = ab(v_1 \wedge v_2) = ab. \\
\Delta(CDN) &= \overrightarrow{CD} \wedge \overrightarrow{CN} = (cv_3 + dv_1) \wedge cv_3 = cd(v_1 \wedge v_3) = cd. \\
\Delta(EFP) &= \overrightarrow{PE} \wedge \overrightarrow{PF} = ev_2 \wedge fv_3 = ef(v_2 \wedge v_3) = ef. \\
\Delta(BCP) &= \overrightarrow{BC} \wedge \overrightarrow{BP} = (v_1 + bv_2 - cv_3) \wedge (b + 1)v_2 = \\
&= (b + 1)(v_1 \wedge v_2) - c(b + 1)(v_3 \wedge v_2) = 1 + b + c + bc. \\
\Delta(DEM) &= 1 + d + e + de \quad \text{and} \quad \Delta(FAN) = 1 + f + a + fa \quad \text{follow similarly.}
\end{align*}
\]

Since $\Delta(K) = \Delta(ABM) + \Delta(CDN) + \Delta(EFP) + \Delta(BCP) + \Delta(DEM) + \Delta(FAN) - 2\Delta(MNP)$ by using the expressions above and the fact that $\Delta(MNP) = 1$ we get that
Next let us compute the areas of the ears of the hexagon \( ABCDEF \). Only the computation for the first triangle is shown in detail; the others can be obtained via circular permutations.

\[
\Delta(ABC) = \overrightarrow{AB} \wedge \overrightarrow{BC} = (av_1 - bv_2) \wedge (v_1 + bv_2 - cv_3) = \\
= ab(v_1 \wedge v_2) - ac(v_1 \wedge v_3) - b(v_2 \wedge v_1) + bc(v_2 \wedge v_3) = \\
= ab - ac + b + bc = b(1 + a + c) - ac.
\]

\[
\Delta(BCD) = c(1 + b + d) - bd; \quad \Delta(CDE) = d(1 + c + e) - ce.
\]

\[
\Delta(DEF) = e(1 + d + f) - df; \quad \Delta(EFA) = f(1 + e + a) - ea.
\]

\[
\Delta(FAB) = a(1 + b + f) - fb.
\]

At this point let us introduce a few simplifying notations

\[
S := a + b + c + d + e + f, \quad T := ab + bc + cd + de + ef + fa
\]

\[
U := ac + bd + ce + df + ea + fb.
\]

It follows that the sum of the areas of all the ears

\[
\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEF) + \Delta(EFA) + \Delta(FAB) = S + 2T - U
\]

while using \((25)\) the area of the initial hexagon can be written as \(\Delta(K) = 1 + S + T\).

If \(K'\) denotes the first \(m\)-Kasner descendant of the hexagon \(K\), then a reasoning identical to the one that lead to equality \((21)\) implies that

\[
\Delta(K') = \Delta(K) - r[\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEF) + \Delta(EFA) + \Delta(FAB)].
\]

Using the last five equalities after a few straightforward algebraic manipulations we obtain that

\[
\frac{\Delta'}{\Delta} = (1 - 2r) + r \cdot \frac{2 + S + U}{1 + S + T}.
\]

It follows that theorem 6.1 will be proved as soon as we can show that
Claim 6.2. With the notations from the current section we have

\[(30) \quad 0 < \frac{2 + S + U}{1 + S + T} < 2\]

and none of the above inequalities can be improved.

Proof. It is obvious that the ratio is greater than 0 as \(a, b, c, d, e, f\) are all positive. To show it can be arbitrarily close to 0 take \(a = b = c = d = 1\) and \(e = f = n\). It is easy to check that for these choices the resulting hexagon is convex for all values of \(n \geq 1\). Moreover,

\[
\frac{2 + S + U}{1 + S + T} = \frac{6n + 8}{n^2 + 4n + 8} \to 0 \quad \text{as} \quad n \to \infty.
\]

To prove that the ratio can be arbitrarily close to 2 take \(a = b = c = d = e = f\). Again, it is simple to verify that the resulting hexagon is convex for any value of \(a > 0\). We have that,

\[
\frac{2 + S + U}{1 + S + T} = \frac{2 + 6a + 6a^2}{1 + 6a + 6a^2} \to 2 \quad \text{as} \quad a \to 0.
\]

Finally, since \(\Delta(K') < \Delta(K)\), from (29) it follows that the ratio \((2 + S + U)/(1 + S + T) < 2\).

This proves the claim and with it theorem 6.1. \(\square\)

The following result is going to be needed later. By the symmetry of figure 5, we may assume that \(a = \min\{a, b, c, d, e, f\}\). Then the following is true

\[(31) \quad 2 \cdot \overrightarrow{FA} \wedge \overrightarrow{AB} \leq \overrightarrow{EF} \wedge \overrightarrow{AB} + \overrightarrow{FA} \wedge \overrightarrow{BC}.
\]

It is easy to check that \(\overrightarrow{FA} \wedge \overrightarrow{AB} = a + ab + af - bf\), \(\overrightarrow{EF} \wedge \overrightarrow{AB} = ae - af + bf\) and \(\overrightarrow{FA} \wedge \overrightarrow{BC} = 1 + c + f - ab + ac + bf\). Then, after some algebra, inequality (31) becomes equivalent to

\[
0 \leq 1 - 2a + c + f - 3ab + ac + ae - 3af + 4bf.
\]

Since \(a = \min\{a, b, c, d, e, f\}\) we may express \(b = a + x_1\), \(c = a + x_2\), \(e = a + x_3\) and \(f = a + x_4\), where the \(x_i\)-s are nonnegative numbers. The last inequality is then equivalent to

\[
0 \leq 1 + x_2 + x_4 + a(x_1 + x_3 + x_3 + x_4) + 4x_1x_4
\]

which is obviously true. This proves inequality (31).
7. The Case of the Convex $n$-gon when $n \geq 7$

Let $m$ be a fixed constant in $(0, 1)$ and let $K = A_1A_2 \ldots A_n$ be a convex $n$-gon, with $n \geq 7$. Let $K' = B_1B_2 \ldots B_n$ be the first $m$-Kasner descendant of $K$, that is, for every $i = 1 \ldots n$ point $B_i$ is lies along side $A_iA_{i+1}$ such that $A_iB_i : B_iA_{i+1} = m : (1 - m)$. The main result of this section is given by the following

**Theorem 7.1.** With the above notations we have that

\[
1 - 2m(1 - m) < \frac{\Delta(K')}{\Delta(K)} < 1
\]

and none of the above inequalities can be improved.

While this result was to be expected (given the statement of theorem 6.1), a rigorous proof still requires some work and inspiration. We are going to need a couple of intermediate results.

**Lemma 7.2.** Consider a positively oriented convex $n$-gon $K = A_1A_2A_3 \ldots A_n$, $n \geq 6$, and denote $\overrightarrow{A_iA_{i+1}} = v_i$. Then there exists four consecutive sides $v_i$, $v_{i+1}$, $v_{i+2}$ and $v_{i+3}$ such that

\[
v_{i+1} \wedge v_{i+2} \leq v_i \wedge v_{i+2} + v_{i+1} \wedge v_{i+3}.
\]

**Proof.** The case when $K$ is a hexagon has already been proved at the end of the previous section. In fact, after an appropriate relabeling, ($ABCDEF$ becomes $A_3A_4A_5A_6A_1A_2$) inequality (31) states that the stronger inequality $2v_2 \wedge v_3 \leq v_1 \wedge v_3 + v_2 \wedge v_4$ holds true. Recall that $v_2 \wedge v_3 = \Delta(A_2A_3A_4) > 0$ by convexity.

Suppose now that $n \geq 7$ and denote by $a_{ij} = v_i \wedge v_j$ for all $1 \leq i, j \leq n$. We need to show that

\[
a_{i+1,i+2} \leq a_{i,i+2} + a_{i+1,i+3} \quad \text{for some } i = 1 \ldots n.
\]

**Case 1.** Suppose that $v_i \wedge v_{i+3} \geq 0$ for all $i = 1 \ldots n$.

Since $a_{i,i+3} \geq 0$, it follows that $a_{i,i+2} > 0$ for all $i = 1 \ldots n$. Recall that $a_{i,i+1} > 0$ by convexity.

In particular

\[
a_{13} > 0, a_{14} \geq 0, a_{24} > 0, a_{25} \geq 0, a_{35} > 0, a_{46} \geq 0.
\]
Suppose for the sake of contradiction that inequality (34) does not hold for \(i = 1\) or \(i = 3\). Given this means that \(a_{13} + a_{24} < a_{23}\) from which \(0 < a_{24} < a_{23}\). Similarly, \(a_{35} + a_{46} < a_{45}\), that is, \(0 < a_{35} < a_{45}\). Multiplying the last two inequalities term by term we obtain that \(a_{24}a_{35} < a_{23}a_{45}\).

But Plücker’s identity (10) applied to the indices 2, 3, 4 and 5 gives \(a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} = 0\).

Combining the last two relations it follows that \(a_{25}a_{34} < 0\) which contradicts (35).

**Case 2.** Suppose that \(v_i \wedge v_{i+3} < 0\) for some \(i\) in \(\{1, 2, \ldots, n\}\). With no loss of generality say \(v_{n-2} \wedge v_1 < 0\). Then all the vectors \(v_1, v_2, \ldots, v_{n-2}\) belong to the same half-plane - see figure 6.

If \(n \geq 8\), this means that all the vectors \(v_i\), with \(1 \leq i \leq 6\) belong to the same half-plane. This implies that \(v_i \wedge v_j > 0\) for all \(1 \leq i < j \leq 6\) and therefore all the conditions from (35) are satisfied. Now we can just repeat the reasoning from case 1 to obtain the desired conclusion.

![Figure 6. Lemma 7.2, Case 2, \(n \geq 8\)](image)

It remains to see what happens if \(n = 7\). We still have all the vectors \(v_i\), with \(1 \leq i \leq 5\) lying in the same half-plane - see figure 7.

![Figure 7. Lemma 7.2, Case 2, \(n = 7\)](image)
This means that \( v_i \wedge v_j > 0 \) for all \( 1 \leq i < j \leq 5 \). If \( v_4 \wedge v_6 \geq 0 \) then all the inequalities from (35) are satisfied and we are done. If \( v_4 \wedge v_6 < 0 \) this implies that we have six consecutive vectors - \( v_6, v_7, v_1, v_2, v_3, v_4 \), lying in the same half-plane. But this case has been dealt with a bit earlier.

□

We need one more result before we can proceed with the proof of theorem 7.1

**Lemma 7.3.** Let \( m \) in \((0, 1)\) be a fixed constant and let \( K = A_1A_2 \ldots A_nA_{n+1} \) be a positively oriented convex \((n+1)\)-gon, \( n \geq 6 \). Let \( K' = B_1B_2 \ldots B_nB_{n+1} \) be the first \( m \)-Kasner descendant of \( K \). Then there exists a convex \( n \)-gon \( L \), obtained by removing a certain vertex of \( K \), such that

\[
\frac{\Delta(K')}{\Delta(K)} \geq \frac{\Delta(L')}{\Delta(L)}
\]

where \( L' \) is the first \( m \)-Kasner descendant of \( L \).

**Proof.** As before, denote \( A_iA_{i+1} = v_i \) and \( v_i \wedge v_j = a_{ij} \) for all \( 1 \leq i, j \leq n + 1 \). By Lemma 7.2 we may assume that \( a_{23} \leq a_{13} + a_{24} \).

Let \( L \) be obtained from \( K \) after removing vertex \( A_3 \), that is \( L = A_1A_2A_4 \ldots A_nA_{n+1} \). Let point \( C \) on \( A_2A_4 \) such that \( A_2C : CA_4 = m : (1 - m) \) - see figure 8. Then, the first \( m \)-Kasner descendant of \( L \) is \( L' = B_1CB_4B_5 \ldots B_nB_{n+1} \). It is easy to see that \( \Delta(L) = \Delta(K) - \Delta(A_2A_3A_4) = \Delta(K) - a_{23} \).

On the other hand, the area of \( K' \) exceeds the area of \( L' \) by the area of the non-convex pentagon \( P = B_1B_2B_3B_4C \), hence \( \Delta(L') = \Delta(K') - \Delta(P) \).

![Figure 8. Figure for lemma 7.3](image-url)
We first compute $\Delta(P)$. It is easy to see that

$$\Delta(P) = \Delta(B_1B_2C) + \Delta(B_2B_3C) + \Delta(B_3B_4C). \tag{37}$$

We have that

$$\Delta(B_1B_2C) = \overrightarrow{B_1B_2} \wedge \overrightarrow{B_2C} = (1 - m)v_1 + mv_2 \wedge mv_3 = m(1 - m)a_{13} + m^2a_{23}. \tag{38}$$

$$\Delta(B_2B_3C) = \overrightarrow{B_3C} \wedge \overrightarrow{B_2C} = (1 - m)v_2 \wedge mv_3 = m(1 - m)a_{23}. \tag{39}$$

$$\Delta(B_3B_4C) = \overrightarrow{CB_3} \wedge \overrightarrow{B_3B_4} = (1 - m)v_2 \wedge ((1 - m)v_3 + mv_4) = (1 - m)^2a_{23} + m(1 - m)a_{24}. \tag{40}$$

Combining the last three equalities into (37) we obtain that $\Delta(P) = m(1 - m)(a_{13} + a_{24} - a_{23}) + a_{23}$ and after using our assumption $a_{23} \leq a_{13} + a_{24}$ we have that

$$\Delta(P) \geq a_{23}. \tag{41}$$

Finally, using (41) we obtain that

$$\frac{\Delta(L')}{\Delta(L)} = \frac{\Delta(K') - \Delta(P)}{\Delta(K) - a_{23}} \leq \frac{\Delta(K')}{\Delta(K)}$$

which proves the lemma.

We are now in position to prove the main result of this section. Below we give a more precise formulation of theorem [7.1]. As above, given $m$ in $(0,1)$ and a convex polygon $K$, $K'$ denotes the first $m$-Kasner descendant of $K$.

**Theorem 7.4.** i. For any $m$ in $(0,1)$ and for any convex $n$-gon $K$ with $n \geq 6$, we have that

$$1 - 2m(1 - m) < \frac{\Delta(K')}{\Delta(K)} < 1. \tag{39}$$

ii. For any $m$ in $(0,1)$, for any positive integer $n \geq 6$ and for any $\epsilon > 0$ there exists a convex $n$-gon $K$ such that

$$\frac{\Delta(K')}{\Delta(K)} < 1 - 2m(1 - m) + \epsilon. \tag{40}$$

iii. For any $m$ in $(0,1)$, for any positive integer $n \geq 6$ and for any $\epsilon > 0$ there exists a convex $n$-gon $K$ such that

$$\frac{\Delta(K')}{\Delta(K)} > 1 - \epsilon. \tag{41}$$
Proof. i. The second inequality in (39) is trivial since \( \text{int}(K') \subset \text{int}(K) \). For the first inequality we are going to do induction on \( n \). We have already shown in theorem 6.1 that the first inequality is true if \( n = 6 \). Let \( K \) be a convex \((n+1)\)-gon, \( n \geq 6 \). Then according to lemma 7.3 one can remove a vertex of \( K \) such that the resulting \( n \)-gon \( L \) has the property stated in (36). Coupling this with the induction hypothesis we obtain that
\[
\Delta(K') \Delta(K) \geq \Delta(L') \Delta(L) > 1 - 2(1 - \epsilon) + 2 \epsilon.
\]

ii. Start with a triangle of unit area, \( M A_1 A_2 \). Cut of a small triangle \( M A_3 A_n \) of area \( \epsilon^2 \) as shown in figure 9. Then replace the segment \( A_n A_3 \) by a small circular arc along which place the remaining vertices \( A_4, A_5, \ldots, A_{n-1} \) - as in the figure below.

![Figure 9. Main theorem, part ii](image)

We claim that the polygon \( K = A_1 A_2 \ldots A_n \) defined above has the property (40). Denote \( m(1 - m) = r \) and let \( K' \) be the first \( m \)-Kasner descendant of \( K \). We have
\[
2(1 - \epsilon) = \Delta(A_n A_1 A_2) + \Delta(A_1 A_2 A_3) \leq \sum_{i=1}^{n} \Delta(A_{i-1} A_i A_{i+1}) = \frac{\Delta(K) - \Delta(K')}{r}
\]
which after we divide by \( \Delta(K) < 1 \) and rearrange the terms becomes
\[
\frac{\Delta(K')}{\Delta(K)} < 1 - 2r + 2r \epsilon \leq 1 - 2r + \frac{\epsilon}{2} = 1 - 2m(1 - m) + \frac{\epsilon}{2}
\]
since \( r = m(1 - m) \leq 1/4 \). This proves part ii.

iii. We will use induction. We already proved that there are hexagons which satisfy (41). Let \( Q = A_1 A_2 \ldots A_n \) be a positively oriented convex \( n \)-gon, \( n \geq 6 \), for which \( \Delta(Q')/\Delta(Q) > 1 - \epsilon \). Without loss of generality assume that \( a_{n-1,1} > 0 \). Construct a point \( A_{n+1} \) such that \( A_n A_{n+1} = \lambda(v_{n-1} + v_n) \),
where $\lambda < \min\{1/2, a_{n,1}/(a_{n,1} + a_{n-1,1})\}$. Then, $P = A_1A_2A_3 \ldots A_nA_{n+1}$ is a positively oriented convex $(n+1)$-gon as shown in figure 10 below.

![Figure 10. Main theorem, part iii](image)

We claim that: $\overrightarrow{A_{n-1}A_n} \wedge \overrightarrow{A_{n+1}A_1} + \overrightarrow{A_nA_{n+1}} \wedge \overrightarrow{A_1A_2} \geq \overrightarrow{A_nA_{n+1}} \wedge \overrightarrow{A_{n+1}A_1}$.

Indeed, this is equivalent to

$$v_{n+1} \wedge (v_n - \lambda(v_{n-1} + v_n)) + \lambda(v_{n-1} + v_n) \wedge v_1 \geq \lambda(v_{n-1} + v_n) \wedge (v_n - \lambda(v_{n-1} + v_n)) \iff$$

$$\iff a_{n-1,n} - \lambda a_{n-1,n} + \lambda a_{n-1,1} + \lambda a_{n,1} \geq \lambda a_{n-1,1} \iff a_{n-1,n} \geq \lambda(2a_{n-1,n} - a_{n-1,1} - a_{n,1})$$

which is true since we have $a_{n-1,1} > 0$ by assumption, $a_{n,1} > 0$ by convexity and $\lambda \leq 1/2$. It follows that the hypotheses from lemma 7.3 are valid for polygon $P$ and vertex $A_{n+1}$, that is, we have constructed a convex $n+1$-gon $P$ for which

$$\frac{\Delta(P')}{\Delta(P)} \geq \frac{\Delta(Q')}{\Delta(Q)} > 1 - \epsilon.$$

This completes the proof of theorem 7.4.

□

Conclusions and Further Research.

In the present paper we provide a complete answer regarding the ratio between the area of a convex polygon and the area of its first $m$-Kasner descendent. It would be interesting to extend these results to the ratio between $\Delta(K)$, the area of the original polygon, and $\Delta(K^t)$, the area
of its $t$-th $m$-Kasner descendant. Same questions can be asked if instead of areas one considers perimeters.

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AREA PROBLEMS INVOLVING KASNER POLYGONS

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Abstract. Sequences of polygons generated by performing iterative processes on an initial polygon have been studied extensively. One of the most popular sequences is the one sometimes referred to as Kasner polygons. Given a polygon \( K \), the first Kasner descendant \( K' \) of \( K \) is obtained by placing the vertices of \( K' \) at the midpoints of the edges of \( K \).

More generally, for any fixed \( m \) in \((0, 1)\) one may define a sequence of polygons \( \{K^t\}_{t \geq 0} \) where each polygon \( K^t \) is obtained by dividing every edge of \( K^{t-1} \) into the ratio \( m : (1 - m) \) in the counterclockwise (or clockwise) direction and taking these division points to be the vertices of \( K^t \).

We are interested in the following problem

Let \( m \) be a fixed number in \((0, 1)\) and let \( n \geq 3 \) be a fixed integer. Further, let \( K \) be a convex \( n \)-gon and denote by \( K' \), the first \( m \)-Kasner descendant of \( K \), that is, the vertices of \( K' \) divide the edges of \( K \) into the ratio \( m : (1 - m) \). What can be said about the ratio between the area of \( K' \) and the area of \( K \), when \( K \) varies in the class of convex \( n \)-gons?

We provide a complete answer to this question.

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1. Introduction

Start with a fixed number \( m \) in \((0, 1)\) and a convex \( n \)-gon \( K \). Let \( K' \) be the convex \( n \)-gon whose vertices divide the edges of \( K \) into the ratio \( m : (1 - m) \) in the counterclockwise direction. We say \( K' \) is the first \( m \)-Kasner descendant of \( K \). In general, we may construct a sequence of polygons \( \{K^t\}_{t \geq 0} \) where \( K^0 = K, K^1 = K' \) and \( K^{t+1} \) is the first \( m \)-Kasner descendant of \( K^t \).

Kasner noticed that if \( n = 4 \) and \( m = 1/2 \) then \( K' \) is always a parallelogram. In [5] he and his students managed to characterize those \( n \)-gons \( P \) which have the property that \( P = K' \) for some convex \( n \)-gon \( K \) and \( m = 1/2 \).

It is easy to notice that all \( n \)-gons in the sequence \( \{K^t\} \) defined above have the same centroid. This was proved repeatedly; see e. g. [3] for a matrix proof or [10] for a proof using Fourier series.
It has been shown by Lüko \[8\] that the sequence \(\{K^t\}\) converges to an (affine) regular \(n\)-gon, thus proving a conjecture of Fejes Tóth \[9\]. More on Kasner polygons can be found in \[1, 2, 4, 6, 7\].

In this paper we study the following:

**Problem 1.1.** Let \(m\) be a fixed number in \((0, 1)\) and let \(n \geq 3\) be a fixed integer. Let \(K\) be a convex \(n\)-gon and denote by \(K',\) the first \(m\)-Kasner descendant of \(K\). What can be said about \(\Delta(K')/\Delta(K)\), the ratio between the area of \(K'\) and the area of \(K\), when \(K\) varies in the class of convex \(n\)-gons?

**The Main Technique.** Throughout the entire paper we use the **wedge product** of two vectors to express areas. This operation, also known as exterior product, is defined as follows. For any two vectors \(v = (a, b)\) and \(u = (c, d)\) let the wedge product of \(v\) and \(u\) be given by \(v \wedge u := (ad - bc)/2\).

It is easy to see that the wedge product represents the signed area of the triangle determined by the vectors \(v\) and \(u\), where the \(\pm\) sign depends on whether the angle between \(v\) and \(u\) - measured in the counterclockwise direction from \(v\) towards \(u\) - is smaller than or greater than 180°.

The following properties of the wedge product are simple consequences of the definition:

- anti-commutativity: \(v \wedge u = -u \wedge v\) and in particular \(v \wedge v = 0\).
- linearity: \((\alpha v + \beta u) \wedge w = \alpha v \wedge w + \beta u \wedge w\).

**2. The Triangle Case**

Let \(m\) be a fixed number in \((0, 1)\) and let \(K = ABC\) be an arbitrary triangle. Construct points \(M, N\) and \(P\) on the sides \(AB, BC\) and \(AC\), such that \(AM : MB = BN : NC = CP : PA = m : (1-m)\).

We call triangle \(K' = MNP\) to be the **first \(m\)-Kasner descendant of \(K\)**.

**Theorem 2.1.** With the notations above we have that

\[
\frac{\Delta(K')}{\Delta(K)} = 1 - 3m(1-m).
\]

**Proof.** Denote \(\overrightarrow{AB} = v_1, \overrightarrow{BC} = v_2\) and \(\overrightarrow{CA} = v_3\) as in figure\[1\] Obviously, \(v_1 + v_2 + v_3 = 0\). After an appropriate scaling we may assume that the area of \(ABC\) is equal to 1, that is

\[
\Delta(K) = v_1 \wedge v_2 = v_2 \wedge v_3 = v_3 \wedge v_1 = 1.
\]
It is easy to see that $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = (1-m)v_1 + mv_2$ and $\overrightarrow{NP} = \overrightarrow{NC} + \overrightarrow{CP} = (1-m)v_2 + mv_3$.

It follows that

$$\Delta(K') = \overrightarrow{MN} \land \overrightarrow{NP} = ((1-m)v_1 + mv_2) \land ((1-m)v_2 + mv_3) =$$

$$= (1-m)^2(v_1 \land v_2) + m(1-m)(v_1 \land v_3) + m^2(v_2 \land v_3) =$$

$$= (1-m)^2 + m(1-m)(-1) + m^2 = 1 - 3m(1-m).$$

\[\square\]

### 3. The Quadrilateral Case

Let $m$ be a fixed number in $(0, 1)$ and let $K = ABCD$ be an arbitrary quadrilateral. Construct points $M, N, P$ and $Q$ on the sides $AB, BC, CD$ and $DA$, such that $AM : MB = BN : NC = CP : PD = DQ : QA = m : (1-m)$. We call quadrilateral $K' = MNPQ$ to be the first $m$-Kasner descendant of $K$.

**Theorem 3.1.** With the notations above we have that

$$\frac{\Delta(K')}{\Delta(K)} = 1 - 2m(1-m).$$

**Proof.** Denote $\overrightarrow{AB} = v_1, \overrightarrow{BC} = v_2, \overrightarrow{CD} = v_3$ and $\overrightarrow{DA} = v_4$ as in figure 2. Obviously, $v_1 + v_2 + v_3 + v_4 = \mathbf{0}$. One can express the area of $ABCD$ in a couple of ways as below.

$$\Delta(K) = \Delta(ABC) + \Delta(CDA) = v_1 \land v_2 + v_3 \land v_4 \quad \text{and}$$

$$\Delta(K) = \Delta(BCD) + \Delta(CDA) = v_2 \land v_3 + v_4 \land v_1.$$
Figure 2. A convex quadrilateral and its first $m$-Kasner descendant

On the other hand

\[
\Delta(MNP) = \overrightarrow{MN} \wedge \overrightarrow{NP} = ((1 - m)v_1 + mv_2) \wedge ((1 - m)v_2 + mv_3) = \\
= (1 - m)^2(v_1 \wedge v_2) + m(1 - m)(v_1 \wedge v_3) + m^2(v_2 \wedge v_3).
\]

\[
\Delta(PQM) = \overrightarrow{PQ} \wedge \overrightarrow{QM} = ((1 - m)v_3 + mv_4) \wedge ((1 - m)v_4 + mv_1) = \\
= (1 - m)^2(v_3 \wedge v_4) + m(1 - m)(v_3 \wedge v_1) + m^2(v_4 \wedge v_1).
\]

Adding the two equalities above term by term and using (3) and (4) we obtain

\[
\Delta(K') = (1 - m)^2((v_1 \wedge v_2) + (v_3 \wedge v_4)) + m^2((v_2 \wedge v_3) + (v_4 \wedge v_1)) = \\
= (1 - m)^2\Delta(K) + m^2\Delta(K) = (1 - 2m(1 - m))\Delta(K).
\]

□

We have seen that the ratio between the area of a convex $n$-gon and the area of its first $m$-Kasner descendant is constant if $n \leq 4$. This is not true anymore if the initial polygon has at least five sides. In this later case we will be interested in the range of values the ratio \(\Delta(K')/\Delta(K)\) takes when $K$ belongs to the class of convex $n$-gons. We are investigating this question in the following sections.
Let $m$ be a fixed number in $(0, 1)$ and let $K = ABCDE$ be an arbitrary convex pentagon. Construct points $M, N, P, Q$ and $R$ on the sides $AB, BC, CD, DE$ and $EA$ such that $AM : MB = BN : NC = CP : PD = DQ : QE = ER : RA = m : (1 - m)$. As before, we call the pentagon $K' = MNPQR$ to be the first $m$-Kasner descendant of $K$.

**Theorem 4.1.** With the notations above we have that

\[ 1 - 2m(1 - m) < \frac{\Delta(K')}{\Delta(K)} < 1 - m(1 - m). \]

Moreover, both lower and upper bounds are the best possible.

For reasons which will become clear soon, we postpone the proof of theorem 4.1 until the next section. Let us first try to approach this problem the same way we proved theorems 2.1 and 3.1. As before, denote $\overrightarrow{AB} = v_1, \overrightarrow{BC} = v_2, \overrightarrow{CD} = v_3, \overrightarrow{DE} = v_4$ and $\overrightarrow{EA} = v_5$ as in figure 3. Obviously, $v_1 + v_2 + v_3 + v_4 + v_5 = 0$.

\[ S := a_{12} + a_{23} + a_{34} + a_{45} + a_{51}. \]

\[ T := a_{13} + a_{24} + a_{35} + a_{41} + a_{52}. \]
It is easy to see that $\Delta(MBN) = \overrightarrow{MB} \land \overrightarrow{BN} = (1-m)\mathbf{v}_1 \land m\mathbf{v}_2 = m(1-m)a_{12}$. Similar expressions can be found for the areas of the other four triangles $NCP$, $PDQ$, $QER$ and $RAM$. Using (8) it follows immediately that

$$\Delta(K') = \Delta(K) - m(1-m)(a_{12} + a_{23} + a_{34} + a_{45} + a_{51}) = \Delta(K) - m(1-m)S.$$  

Notice that there are several different ways in which $\Delta(K)$ can be expressed in terms of the $a_{ij}$-s. For instance, we have that

$$\Delta(K) = \Delta(ABC) + \Delta(ACD) + \Delta(ADE) = \overrightarrow{AB} \land \overrightarrow{BC} + \overrightarrow{AC} \land \overrightarrow{CD} + \overrightarrow{DE} \land \overrightarrow{EA} =$$

$$\mathbf{v}_1 \land \mathbf{v}_2 + (\mathbf{v}_1 + \mathbf{v}_2) \land \mathbf{v}_3 + \mathbf{v}_4 \land \mathbf{v}_5 = a_{12} + a_{13} + a_{23} + a_{45}.$$  

Using equalities (8) and (9) we easily derive the following

$$\frac{\Delta(K')}{\Delta(K)} = 1 - m(1-m) \cdot \frac{a_{12} + a_{23} + a_{34} + a_{45} + a_{51}}{a_{12} + a_{13} + a_{23} + a_{45}}.$$  

**Observation.** Proving theorem 4.1 reduces to showing that

$$1 < \frac{a_{12} + a_{23} + a_{34} + a_{45} + a_{51}}{a_{12} + a_{13} + a_{23} + a_{45}} < 2$$

and that these inequalities cannot be improved. This is somewhat of an awkward task. The reason is that the $a_{ij}$-s are not independent quantities. Indeed, on one hand we have that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = \mathbf{0}$. This implies for instance that $\mathbf{v}_1 \land (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5) = a_{12} + a_{13} + a_{14} + a_{15} = 0$.

On the other hand, it can be easily shown that for any four distinct indices $i$, $j$, $k$ and $l$ in \{1, 2, 3, 4, 5\} we have the following equality, known as Plücker's identity

$$a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0.$$  

Indeed, it is easy to see that among the four vectors $\mathbf{v}_i$, $\mathbf{v}_j$, $\mathbf{v}_k$ and $\mathbf{v}_l$ there are two which are independent; say $\mathbf{v}_i$ and $\mathbf{v}_j$ are those vectors. Then $\mathbf{v}_k$ and $\mathbf{v}_l$ can be expressed as linear combinations of $\mathbf{v}_i$ and $\mathbf{v}_j$. Suppose that $\mathbf{v}_k = \alpha \mathbf{v}_i + \beta \mathbf{v}_j$ and $\mathbf{v}_l = \lambda \mathbf{v}_i + \mu \mathbf{v}_j$. It follows immediately that

$$a_{ik} = \mathbf{v}_i \land (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) = \beta a_{ij},$$

$$a_{il} = \mathbf{v}_i \land (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = \mu a_{ij},$$

$$a_{jk} = \mathbf{v}_j \land (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) = -\alpha a_{ij},$$

$$a_{jl} = \mathbf{v}_j \land (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = -\lambda a_{ij}$$

and finally,

$$a_{kl} = (\alpha \mathbf{v}_i + \beta \mathbf{v}_j) \land (\lambda \mathbf{v}_i + \mu \mathbf{v}_j) = (\alpha \mu - \beta \lambda) a_{ij}.$$  

Substituting these equalities into the left side of (10) we obtain the desired identity.
While it seems that this line of attack is destined to failure, one can still derive an interesting fact. Let \( K'' := STUVW \) be the second \( m \)-Kasner descendant of \( K \) - see figure 3. We would like to see whether there is a relationship linking the areas of \( K, K' \) and \( K'' \).

For \( 1 \leq i, j \leq 5 \) denote \( \mathbf{v}_i' = (1 - m)\mathbf{v}_i + m\mathbf{v}_{i+1} \), \( a'_{ij} = \mathbf{v}_i' \wedge \mathbf{v}_j' \) and \( S' := a'_{12} + a'_{23} + a'_{34} + a'_{45} + a'_{51} \). First notice that

\[
a'_{i,i+1} = [(1 - m)\mathbf{v}_i + m\mathbf{v}_{i+1}] \wedge [(1 - m)\mathbf{v}_{i+1} + m\mathbf{v}_{i+2}] = (1 - m)^2a_{i,i+1} + m(1 - m)a_{i,i+2} + m^2a_{i+1,i+2}.
\]

Using notations (6) and (7) it follows that

\[
S' = \sum_{i=1}^{5} a'_{i,i+1} = (1 - m)^2S + m(1 - m)T + m^2S = (1 - 2m(1 - m))S + m(1 - m)T.
\]

A reasoning similar to the one which led us to equality (8) can be used to show that

\[
\Delta(K'') = \Delta(K') - m(1 - m)S'.
\]

Also, relation (9) can be rewritten as \( a_{13} = \Delta(K) - a_{12} - a_{23} - a_{34} \). If one expresses the area of \( K \) as \( \Delta(K) = \Delta(BCD) + \Delta(BDE) + \Delta(BEA) = a_{23} + a_{24} + a_{34} + a_{51} \) we get that \( a_{24} = \Delta(K) - a_{23} - a_{34} - a_{51} \). In an analogous manner one can obtain expressions for \( a_{35}, a_{41} \) and \( a_{52} \). By adding these relations term by term and taking into account (7) we obtain that

\[
T = a_{13} + a_{24} + a_{35} + a_{41} + a_{52} = 5\Delta(K) - 3S.
\]

By eliminating the quantities \( S', S \) and \( T \) between equalities (8), (11), (12) and (13) we finally obtain a linear relationship linking the areas of \( K, K' \) and \( K'' \).

\[
\Delta(K'') = (2 - 5r)\Delta(K') - (1 - 5r + 5r^2)\Delta(K), \quad \text{where } r := m(1 - m).
\]

In other words, if we know the area of the initial pentagon and the area of its first \( m \)-Kasner descendant we can compute the area of any of the \( m \)-Kasner descendants, \( K'^t \), for any \( t \geq 2 \). Similar recurrence relationships are valid for polygons with more than five sides. In general, for convex \( n \)-gons, the recurrence involves \( \lceil (n+1)/2 \rceil \) consecutive \( m \)-Kasner descendants. We omit the details.
5. The Pentagon Case - A Second (and Successful) Approach

Let $K = ABCDE$ be an arbitrary convex pentagon and let $m$ be a fixed constant in $(0, 1)$. Denote $r := m(1 - m)$. After an eventual relabeling of the vertices we may assume that

\[(15) \quad \Delta(ABC) = \min\{\Delta(ABC), \Delta(BCD), \Delta(CDE), \Delta(DEA), \Delta(EAB)\}\]

In the literature, such triangles formed by three consecutive vertices of a convex polygon are sometimes called ears. Assumption (15) above fixes the ear of least area. Denote the intersection of $AD$ and $CE$ by $O$. Then define $v_1 = \overrightarrow{OA}$, $v_2 = \overrightarrow{OC}$. After an appropriate scaling, we may assume that $v_1 \wedge v_2 = \Delta(AOC) = 1$.

Since $E, O,$ and $C$ are collinear and $D, O,$ and $A$ are collinear, we can write $\overrightarrow{DO} = a \cdot \overrightarrow{OA} = a v_1$ and $\overrightarrow{EO} = b \cdot \overrightarrow{OC} = b v_2$, with $a, b > 0$ (see figure 4).

\[a v_1 - b v_2\]
\[v_1 + b v_2\]
\[v_3 - v_1\]
\[v_3\]
\[v_2\]
\[v_3 - v_2\]
\[a v_1\]
\[b v_2\]
\[D\]
\[E\]
\[B\]
\[C\]
\[A\]

**Figure 4.** A better setup for the convex pentagon problem

Using the triangle rule, we obtain that $\overrightarrow{CD} = -a v_1 - v_2$, $\overrightarrow{DE} = a v_1 - b v_2$, and $\overrightarrow{EA} = v_1 + b v_2$.

We know that every vector in the plane can be written as a linear combination of any two independent vectors. Set $\overrightarrow{OB} = v_3 = c v_1 + d v_2$ - refer again to figure 4. We also know that $\overrightarrow{AB} = v_3 - v_1$ and $\overrightarrow{BC} = v_2 - v_3$. We have that

\[
\Delta(OAB) = v_1 \wedge v_3 = v_1 \wedge (c v_1 + d v_2) = d,
\]

\[
\Delta(OBC) = v_3 \wedge v_2 = (c v_1 + d v_2) \wedge v_2 = c.
\]
After similar calculations, we can write the areas of various triangles in pentagon \( ABCDE \) in terms of the positive constants \( a, b, c, d \) as shown below: \( \Delta(OCD) = a \mathbf{v}_1 \wedge \mathbf{v}_2 = a, \ \Delta(OEA) = a \mathbf{v}_1 \wedge b \mathbf{v}_2 = ab, \ \Delta(ODE) = \mathbf{v}_1 \wedge b \mathbf{v}_2 = b. \) We can now compute the total area of the pentagon.

\[
\Delta(ABCDE) = \Delta(OAB) + \Delta(OBC) + \Delta(OCD) + \Delta(ODE) + \Delta(OEA), \text{ that is,}
\]

\[
(16) \quad \Delta(K) = \Delta(ABCDE) = a + b + c + d + ab.
\]

Next, we compute the areas of the ears of the pentagon.

\[
\Delta(ABC) = \mathbf{AB} \wedge \mathbf{BC} = (\mathbf{v}_3 - \mathbf{v}_1) \wedge (\mathbf{v}_2 - \mathbf{v}_3) = c + d - 1,
\]

\[
\Delta(BCD) = \mathbf{BC} \wedge \mathbf{CD} = (\mathbf{v}_2 - \mathbf{v}_3) \wedge (-a \mathbf{v}_1 - \mathbf{v}_2) = a - ad + c,
\]

\[
\Delta(CDE) = \mathbf{CD} \wedge \mathbf{DE} = (-a \mathbf{v}_1 - \mathbf{v}_2) \wedge (a \mathbf{v}_1 - b \mathbf{v}_2) = ab + a,
\]

\[
\Delta(DEA) = \mathbf{DE} \wedge \mathbf{EA} = (a \mathbf{v}_1 - b \mathbf{v}_2) \wedge (\mathbf{v}_1 + b \mathbf{v}_2) = ab + b,
\]

\[
\Delta(EAB) = \mathbf{EA} \wedge \mathbf{AB} = (\mathbf{v}_1 + b \mathbf{v}_2) \wedge (\mathbf{v}_3 - \mathbf{v}_1) = b - bc + d.
\]

It follows that

\[
(20) \quad \Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEA) + \Delta(EAB) = 2(a + b + c + d + ab) - 1 - ad - bc.
\]

Consider now \( K' \), the first \( m \)-Kasner descendant of the initial pentagon. We did not include \( K' \) in figure 4 in order to keep things clear. However, it is easy to see from figure 3 that the area of \( K' \) is the difference between the area of \( K \) and the sum of the areas of the ears of the pentagon multiplied by a factor of \( r = m(1 - m) \). This means that

\[
(21) \quad \Delta(K') = \Delta(K) - r(\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEA) + \Delta(EAB))
\]

which after making use of (16) and (20) becomes \( \Delta(K') = (1 - 2r)\Delta + r(1 + ad + bc) \) and finally

\[
(22) \quad \frac{\Delta(K')}{\Delta(K)} = 1 - 2r + r \frac{1 + ad + bc}{a + b + c + d + ab}.
\]

In order to prove theorem 4.1 it is enough to show that

**Claim 5.1.** With the notations from the present section we have

\[
(23) \quad 0 < \frac{1 + ad + bc}{a + b + c + d + ab} < 1
\]
and none of these inequalities can be improved.

Proof. It is clear the ratio is greater than 0 as $a, b, c$ and $d$ are all positive. To show that the ratio can be arbitrarily close to 0 take $a = b = n$ and $c = d = 1$. Then,

$$\frac{1 + ad + bc}{a + b + c + d + ab} = \frac{2n + 1}{n^2 + 2n + 2} \to 0 \text{ as } n \to \infty.$$ 

To show that the ratio can be arbitrarily close to 1 take $a = n, b = 1/n$ and $c = d = 1$. For these choices

$$\frac{1 + ad + bc}{a + b + c + d + ab} = \frac{n^2 + n + 1}{n^2 + 3n + 1} \to 1 \text{ as } n \to \infty.$$ 

Remains to show that the ratio is always less than 1. Recall that assumption (15) stated that triangle $ABC$ is the ear of the smallest area. Refer first to equality (17). Since $\Delta(ABC) > 0$ it follows that $c + d > 1$. On the other hand $\Delta(ABC) \leq \Delta(BCD)$ which after using (17) and (18) gives that $c + d - 1 \leq a - ad + c \iff (a + 1)(1 - d) \geq 0 \iff d \leq 1$. Finally $\Delta(ABC) \leq \Delta(EAB)$ which after using (17) and (19) implies that $c + d - 1 \leq b - bc + d \iff (b + 1)(1 - c) \geq 0 \iff c \leq 1$.

The last three inequalities ($c + d > 1, c \leq 1$ and $d \leq 1$) imply that

$$1 + ad + bc \leq 1 + a + b < c + d + a + b < a + b + c + d + ab.$$ 

This proves that the ratio is always less than 1. This proves the claim and with it theorem 4.1. □

6. The Hexagon Case

As in the previous sections we start by fixing a constant $m$ in $(0, 1)$ and considering $K = ABCDEF$ an arbitrary convex hexagon. As before, $K'$ denotes the first $m$-Kasner descendant of $K$. The main result of this section is given in the following

**Theorem 6.1.** With the notations above we have that

$$(24) \quad 1 - 2m(1 - m) < \frac{\Delta(K')}{\Delta(K)} < 1.$$ 

Moreover, both lower and upper bounds are the best possible.

We need a setup similar to the one used for pentagons. Suppose first that the long diagonals, $AD, BE,$ and $CF$ are not concurrent. If these diagonals do have a common point, then perturb the position of one of the vertices by an arbitrarily small amount so that the diagonals are not
concurrent anymore. By continuity, any inequality which is valid in latter case is also valid in the
former. Let \( M = AD \cap BE, N = AD \cap CF, P = CF \cap BE \). Denote \( \mathbf{v}_1 = \overrightarrow{MN}, \mathbf{v}_2 = \overrightarrow{MP} \) and
\( \mathbf{v}_3 = \overrightarrow{NP} \). It follows that \( \mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \) - see figure 5.

\[
\begin{align*}
\Delta(ABM) &= \overrightarrow{AB} \wedge \overrightarrow{BM} = (a\mathbf{v}_1 - b\mathbf{v}_2) \wedge b\mathbf{v}_2 = ab(\mathbf{v}_1 \wedge \mathbf{v}_2) = ab. \\
\Delta(CDN) &= \overrightarrow{CD} \wedge \overrightarrow{CN} = (c\mathbf{v}_3 + d\mathbf{v}_1) \wedge c\mathbf{v}_3 = cd(\mathbf{v}_1 \wedge \mathbf{v}_3) = cd. \\
\Delta(EFP) &= \overrightarrow{PE} \wedge \overrightarrow{PF} = e\mathbf{v}_2 \wedge f\mathbf{v}_3 = ef(\mathbf{v}_2 \wedge \mathbf{v}_3) = ef. \\
\Delta(BCP) &= \overrightarrow{BC} \wedge \overrightarrow{BP} = (\mathbf{v}_1 + b\mathbf{v}_2 - c\mathbf{v}_3) \wedge (b + 1)\mathbf{v}_2 = \\
&= (b + 1)(\mathbf{v}_1 \wedge \mathbf{v}_2) - c(b + 1)(\mathbf{v}_3 \wedge \mathbf{v}_2) = 1 + b + c + bc. \\
\Delta(DEM) &= 1 + d + e + de \quad \text{and} \quad \Delta(FAN) = 1 + f + a + fa \quad \text{follow similarly.}
\end{align*}
\]

Since \( \Delta(K) = \Delta(ABM) + \Delta(CDN) + \Delta(EFP) + \Delta(BCP) + \Delta(DEM) + \Delta(FAN) - 2\Delta(MNP) \)
by using the expressions above and the fact that \( \Delta(MNP) = 1 \) we get that

\textbf{Figure 5.} Setup for the convex hexagon problem

With appropriate scaling we may assume that \( \Delta(MNP) = \mathbf{v}_1 \wedge \mathbf{v}_2 = \mathbf{v}_1 \wedge \mathbf{v}_3 = \mathbf{v}_2 \wedge \mathbf{v}_3 = 1 \).
Since \( A, M, N, D \) are collinear, \( \overrightarrow{AM} = a\mathbf{v}_1, \overrightarrow{ND} = d\mathbf{v}_1 \) with \( a, d > 0 \). Similarly, \( \overrightarrow{BM} = b\mathbf{v}_2, \overrightarrow{CN} = c\mathbf{v}_3, \overrightarrow{PE} = e\mathbf{v}_2, \overrightarrow{PF} = f\mathbf{v}_3 \) with \( b, c, e, f \) positive constants.

We try to express \( \Delta(K) \), the area of the hexagon in terms of \( a, b, c, d, e, f \). We begin by
computing the areas of the triangles determined by one side and two long diagonals.
\[ \Delta(K) = 1 + (a + b + c + d + e + f) + (ab + bc + cd + de + ef + fa). \]

Next let us compute the areas of the ears of the hexagon \( ABCDEF \). Only the computation for the first triangle is shown in detail; the others can be obtained via circular permutations.

\[ \Delta(ABC) = \overrightarrow{AB} \wedge \overrightarrow{BC} = (a \mathbf{v}_1 - b \mathbf{v}_2) \wedge (\mathbf{v}_1 + b \mathbf{v}_2 - c \mathbf{v}_3) = \]
\[ = ab(\mathbf{v}_1 \wedge \mathbf{v}_2) - ac(\mathbf{v}_1 \wedge \mathbf{v}_3) - b(\mathbf{v}_2 \wedge \mathbf{v}_1) + bc(\mathbf{v}_2 \wedge \mathbf{v}_3) = \]
\[ = ab - ac + b + bc = b(1 + a + c) - ac. \]

\[ \Delta(BCD) = c(1 + b + d) - bd; \quad \Delta(CDE) = d(1 + c + e) - ce. \]

\[ \Delta(DEF) = e(1 + d + f) - df; \quad \Delta(EFA) = f(1 + e + a) - ea. \]

\[ \Delta(FAB) = a(1 + b + f) - fb. \]

At this point let us introduce a few simplifying notations

\[ S := a + b + c + d + e + f, \quad T := ab + bc + cd + de + ef + fa \]

\[ U := ac + bd + ce + df + ea + fb. \]

It follows that the sum of the areas of all the ears

\[ \Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEF) + \Delta(EFA) + \Delta(FAB) = S + 2T - U \]

while using (25) the area of the initial hexagon can be written as \( \Delta(K) = 1 + S + T. \)

If \( K' \) denotes the first \( m \)-Kasner descendant of the hexagon \( K \), then a reasoning identical to the one that lead to equality (21) implies that

\[ \Delta(K') = \Delta(K) - r [\Delta(ABC) + \Delta(BCD) + \Delta(CDE) + \Delta(DEF) + \Delta(EFA) + \Delta(FAB)]. \]

Using the last five equalities after a few straightforward algebraic manipulations we obtain that

\[ \frac{\Delta'}{\Delta} = (1 - 2r) + r \cdot \frac{2 + S + U}{1 + S + T}. \]

It follows that theorem 6.1 will be proved as soon as we can show that
Claim 6.2. With the notations from the current section we have

\[
0 < \frac{2 + S + U}{1 + S + T} < 2
\]

and none of the above inequalities can be improved.

Proof. It is obvious that the ratio is greater than 0 as \(a, b, c, d, e, f\) are all positive. To show it can be arbitrarily close to 0 take \(a = b = c = d = 1\) and \(e = f = n\). It is easy to check that for these choices the resulting hexagon is convex for all values of \(n \geq 1\). Moreover,

\[
\frac{2 + S + U}{1 + S + T} = \frac{6n + 8}{n^2 + 4n + 8} \to 0 \text{ as } n \to \infty.
\]

To prove that the ratio can be arbitrarily close to 2 take \(a = b = c = d = e = f\). Again, it is simple to verify that the resulting hexagon is convex for any value of \(a > 0\). We have that,

\[
\frac{2 + S + U}{1 + S + T} = \frac{2 + 6a + 6a^2}{1 + 6a + 6a^2} \to 2 \text{ as } a \to 0.
\]

Finally, since \(\Delta(K') < \Delta(K)\), from (29) it follows that the ratio \((2 + S + U)/(1 + S + T) < 2\).

This proves the claim and with it theorem 6.1. □

The following result is going to be needed later. By the symmetry of figure 5, we may assume that \(a = \min\{a, b, c, d, e, f\}\). Then the following is true

\[
2 \cdot \overrightarrow{FA} \wedge \overrightarrow{AB} \leq \overrightarrow{EF} \wedge \overrightarrow{AB} + \overrightarrow{FA} \wedge \overrightarrow{BC}.
\]

It is easy to check that \(\overrightarrow{FA} \wedge \overrightarrow{AB} = a + ab + af - bf\), \(\overrightarrow{EF} \wedge \overrightarrow{AB} = ae - af + bf\) and \(\overrightarrow{FA} \wedge \overrightarrow{BC} = 1 + c + f - ab + ac + bf\). Then, after some algebra, inequality (31) becomes equivalent to

\[
0 \leq 1 - 2a + c + f - 3ab + ac + ae - 3af + 4bf.
\]

Since \(a = \min\{a, b, c, d, e, f\}\) we may express \(b = a + x_1\), \(c = a + x_2\), \(e = a + x_3\) and \(f = a + x_4\), where the \(x_i\)-s are nonnegative numbers. The last inequality is then equivalent to

\[
0 \leq 1 + x_2 + x_4 + a(x_1 + x_3 + x_3 + x_4) + 4x_1x_4
\]

which is obviously true. This proves inequality (31).
7. The Case of the Convex \( n \)-gon when \( n \geq 7 \)

Let \( m \) be a fixed constant in \((0, 1)\) and let \( K = A_1A_2\ldots A_n \) be a convex \( n \)-gon, with \( n \geq 7 \). Let \( K' = B_1B_2\ldots B_n \) be the first \( m \)-Kasner descendant of \( K \), that is, for every \( i = 1\ldots n \) point \( B_i \) is lies along side \( A_iA_{i+1} \) such that \( A_iB_i : B_iA_{i+1} = m : (1-m) \). The main result of this section is given by the following

**Theorem 7.1.** With the above notations we have that

\[
1 - 2m(1-m) < \frac{\Delta(K')}{\Delta(K)} < 1
\]

and none of the above inequalities can be improved.

While this result was to be expected (given the statement of theorem 6.1), a rigorous proof still requires some work and inspiration. We are going to need a couple of intermediate results.

**Lemma 7.2.** Consider a positively oriented convex \( n \)-gon \( K = A_1A_2A_3\ldots A_n \), \( n \geq 6 \), and denote \( \overrightarrow{A_iA_{i+1}} = v_i \). Then there exists four consecutive sides \( v_i \), \( v_{i+1} \), \( v_{i+2} \) and \( v_{i+3} \) such that

\[
v_{i+1} \wedge v_{i+2} \leq v_i \wedge v_{i+2} + v_{i+1} \wedge v_{i+3}.
\]

**Proof.** The case when \( K \) is a hexagon has already been proved at the end of the previous section. In fact, after an appropriate relabeling, \((ABCDEF) \) becomes \( A_3A_4A_5A_6A_1A_2 \) inequality (31) states that the stronger inequality \( 2v_2 \wedge v_3 \leq v_1 \wedge v_3 + v_2 \wedge v_4 \) holds true. Recall that \( v_2 \wedge v_3 = \Delta(A_2A_3A_4) > 0 \) by convexity.

Suppose now that \( n \geq 7 \) and denote by \( a_{ij} = v_i \wedge v_j \) for all \( 1 \leq i, j \leq n \). We need to show that

\[
a_{i+1,i+2} \leq a_{i,i+2} + a_{i+1,i+3} \quad \text{for some} \quad i = 1\ldots n.
\]

**Case 1.** Suppose that \( v_i \wedge v_{i+3} \geq 0 \) for all \( i = 1\ldots n \)

Since \( a_{i,i+3} \geq 0 \), it follows that \( a_{i,i+2} \geq 0 \) for all \( i = 1\ldots n \). Recall that \( a_{i,i+1} > 0 \) by convexity.

In particular

\[
a_{13} > 0, \ a_{14} \geq 0, \ a_{24} > 0, \ a_{25} \geq 0, \ a_{35} > 0, \ a_{46} \geq 0.
\]
Suppose for the sake of contradiction that inequality (34) does not hold for $i = 1$ or $i = 3$. Given (35) this means that $a_{13} + a_{24} < a_{23}$ from which $0 < a_{24} < a_{23}$. Similarly, $a_{35} + a_{46} < a_{45}$, that is, $0 < a_{35} < a_{45}$. Multiplying the last two inequalities term by term we obtain that $a_{24}a_{35} < a_{23}a_{45}$.

But Plücker’s identity (10) applied to the indices 2, 3, 4 and 5 gives $a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} = 0$.

Combining the last two relations it follows that $a_{25}a_{34} < 0$ which contradicts (35).

**Case 2.** Suppose that $v_i \wedge v_{i+3} < 0$ for some $i$ in $\{1, 2, \ldots, n\}$. With no loss of generality say $v_{n-2} \wedge v_1 < 0$. Then all the vectors $v_1, v_2 \ldots, v_{n-2}$ belong to the same half-plane - see figure 6.

If $n \geq 8$, this means that all the vectors $v_i$, with $1 \leq i \leq 6$ belong to the same half-plane. This implies that $v_i \wedge v_j > 0$ for all $1 \leq i < j \leq 6$ and therefore all the conditions from (35) are satisfied. Now we can just repeat the reasoning from case 1 to obtain the desired conclusion.

![Figure 6. Lemma 7.2, Case 2, $n \geq 8$](image6)

It remains to see what happens if $n = 7$. We still have all the vectors $v_i$, with $1 \leq i \leq 5$ lying in the same half-plane - see figure 7.

![Figure 7. Lemma 7.2, Case 2, $n = 7$](image7)
This means that $v_i \wedge v_j > 0$ for all $1 \leq i < j \leq 5$. If $v_4 \wedge v_6 \geq 0$ then all the inequalities from (35) are satisfied and we are done. If $v_4 \wedge v_6 < 0$ this implies that we have six consecutive vectors - $v_6, v_7, v_1, v_2, v_3, v_4$, lying in the same half-plane. But this case has been dealt with a bit earlier. 

\[ \square \]

We need one more result before we can proceed with the proof of theorem 7.1

**Lemma 7.3.** Let $m$ in $(0, 1)$ be a fixed constant and let $K = A_1 A_2 \ldots A_n A_{n+1}$ be a positively oriented convex $(n+1)$-gon, $n \geq 6$. Let $K' = B_1 B_2 \ldots B_n B_{n+1}$ be the first $m$-Kasner descendant of $K$. Then there exists a convex $n$-gon $L$, obtained by removing a certain vertex of $K$, such that

\[ \frac{\Delta(K')}{\Delta(K)} \geq \frac{\Delta(L')}{\Delta(L)} \tag{36} \]

where $L'$ is the first $m$-Kasner descendant of $L$.

**Proof.** As before, denote $A_i A_{i+1} = v_i$ and $v_i \wedge v_j = a_{ij}$ for all $1 \leq i, j \leq n+1$. By Lemma 7.2 we may assume that $a_{23} \leq a_{13} + a_{24}$.

Let $L$ be obtained from $K$ after removing vertex $A_3$, that is $L = A_1 A_2 A_4 \ldots A_n A_{n+1}$. Let point $C$ on $A_2 A_4$ such that $A_2 C : CA_4 = m : (1 - m)$ - see figure 8. Then, the first $m$-Kasner descendant of $L$ is $L' = B_1 C B_4 B_5 \ldots B_n B_{n+1}$. It is easy to see that $\Delta(L) = \Delta(K) - \Delta(A_2 A_3 A_4) = \Delta(K) - a_{23}$.

On the other hand, the area of $K'$ exceeds the area of $L'$ by the area of the non-convex pentagon $P = B_1 B_2 B_3 B_4 C$, hence $\Delta(L') = \Delta(K') - \Delta(P)$.

\[ \text{Figure 8. Figure for lemma 7.3} \]
We first compute $\Delta(P)$. It is easy to see that

\begin{equation}
\Delta(P) = \Delta(B_1B_2C) + \Delta(B_2B_3C) + \Delta(B_3B_4C).
\end{equation}

We have that

\[ \Delta(B_1B_2C) = \overrightarrow{B_1B_2} \wedge \overrightarrow{B_2C} = ((1-m)v_1 + mv_2) \wedge mv_3 = m(1-m)a_{13} + m^2a_{23}. \]

\[ \Delta(B_2B_3C) = \overrightarrow{B_2C} \wedge \overrightarrow{B_3C} = (1-m)v_2 \wedge mv_3 = m(1-m)a_{23}. \]

\[ \Delta(B_3B_4C) = \overrightarrow{CB_3} \wedge \overrightarrow{B_3B_4} = (1-m)v_2 \wedge ((1-m)v_3 + mv_4) = (1-m)^2a_{23} + m(1-m)a_{24}. \]

Combining the last three equalities into (37) we obtain that $\Delta(P) = m(1-m)(a_{13} + a_{24} - a_{23}) + a_{23}$ and after using our assumption $a_{23} \leq a_{13} + a_{24}$ we have that

\begin{equation}
\Delta(P) \geq a_{23}.
\end{equation}

Finally, using (38) we obtain that

\[ \frac{\Delta(L')}{\Delta(L)} = \frac{\Delta(K') - \Delta(P)}{\Delta(K) - a_{23}} \leq \frac{\Delta(K')}{\Delta(K)} \]

which proves the lemma.

We are now in position to prove the main result of this section. Below we give a more precise formulation of theorem 7.1. As above, given $m$ in $(0, 1)$ and a convex polygon $K$, $K'$ denotes the first $m$-Kasner descendant of $K$.

**Theorem 7.4.** i. For any $m$ in $(0, 1)$ and for any convex $n$-gon $K$ with $n \geq 6$, we have that

\begin{equation}
1 - 2m(1-m) < \frac{\Delta(K')}{\Delta(K)} < 1.
\end{equation}

ii. For any $m$ in $(0, 1)$, for any positive integer $n \geq 6$ and for any $\epsilon > 0$ there exists a convex $n$-gon $K$ such that

\begin{equation}
\frac{\Delta(K')}{\Delta(K)} < 1 - 2m(1-m) + \epsilon.
\end{equation}

iii. For any $m$ in $(0, 1)$, for any positive integer $n \geq 6$ and for any $\epsilon > 0$ there exists a convex $n$-gon $K$ such that

\begin{equation}
\frac{\Delta(K')}{\Delta(K)} > 1 - \epsilon.
\end{equation}
Proof. i. The second inequality in (39) is trivial since $\text{int}(K') \subset \text{int}(K)$. For the first inequality we are going to do induction on $n$. We have already shown in theorem 6.1 that the first inequality is true if $n = 6$. Let $K$ be a convex $(n+1)$-gon, $n \geq 6$. Then according to lemma 7.3 one can remove a vertex of $K$ such that the resulting $n$-gon $L$ has the property stated in (36). Coupling this with the induction hypothesis we obtain that

$$\Delta(K')/\Delta(K) \geq \Delta(L')/\Delta(L) > 1 - 2m(1 - m).$$

ii. Start with a triangle of unit area, $MA_1A_2$. Cut of a small triangle $MA_3A_n$ of area $\epsilon^2$ as shown in figure 9. Then replace the segment $A_nA_3$ by a small circular arc along which place the remaining vertices $A_4, A_5, \ldots, A_{n-1}$ - as in the figure below.

![Figure 9. Main theorem, part ii](image)

We claim that the polygon $K = A_1A_2 \ldots A_n$ defined above has the property (40). Denote $m(1 - m) = r$ and let $K'$ be the first $m$-Kasner descendant of $K$. We have

$$2(1 - \epsilon) = \Delta(A_nA_1A_2) + \Delta(A_1A_2A_3) < \sum_{i=1}^{n} \Delta(A_{i-1}A_iA_{i+1}) = \frac{\Delta(K) - \Delta(K')}{r}$$

which after we divide by $\Delta(K) < 1$ and rearrange the terms becomes

$$\frac{\Delta(K')}{\Delta(K)} < 1 - 2r + 2r\epsilon \leq 1 - 2r + \frac{\epsilon}{2} = 1 - 2m(1 - m) + \frac{\epsilon}{2}$$

since $r = m(1 - m) \leq 1/4$. This proves part ii.

iii. We will use induction. We already proved that there are hexagons which satisfy (41). Let $Q = A_1A_2 \ldots A_n$ be a positively oriented convex $n$-gon, $n \geq 6$, for which $\Delta(Q')/\Delta(Q) > 1 - \epsilon$. Without loss of generality assume that $a_{n-1,1} > 0$. Construct a point $A_{n+1}$ such that $A_nA_{n+1} = \lambda(v_{n-1} + v_n)$,
where $\lambda < \min\{1/2, a_{n,1}/(a_{n,1} + a_{n-1,1})\}$. Then, $P = A_1 A_2 A_3 \ldots A_n A_{n+1}$ is a positively oriented convex $(n+1)$-gon as shown in figure 10 below.

We claim that:

$\overrightarrow{A_{n-1}A_n} \land \overrightarrow{A_{n+1}A_1} + \overrightarrow{A_nA_{n+1}} \land \overrightarrow{A_1A_2} \geq \overrightarrow{A_nA_{n+1}} \land \overrightarrow{A_{n+1}A_1}$.

Indeed, this is equivalent to

$v_{n+1} \land (v_n - \lambda(v_{n-1} + v_n)) + \lambda(v_{n-1} + v_n) \land v_1 \geq \lambda(v_{n-1} + v_n) \land (v_n - \lambda(v_{n-1} + v_n)) \Leftrightarrow$

$\Leftrightarrow a_{n-1,n} - \lambda a_{n-1,n} + \lambda a_{n-1,1} + \lambda a_{n,1} \geq \lambda a_{n-1,n} \Leftrightarrow a_{n-1,n} \geq \lambda(2a_{n-1,n} - a_{n-1,1} - a_{n,1})$

which is true since we have $a_{n-1,1} > 0$ by assumption, $a_{n,1} > 0$ by convexity and $\lambda \leq 1/2$. It follows that the hypotheses from lemma 7.3 are valid for polygon $P$ and vertex $A_{n+1}$, that is, we have constructed a convex $n+1$-gon $P$ for which

$$\frac{\Delta(P')}{\Delta(P)} \geq \frac{\Delta(Q')}{\Delta(Q)} > 1 - \epsilon.$$ 

This completes the proof of theorem 7.4.

Conclusions and Further Research.

In the present paper we provide a complete answer regarding the ratio between the area of a convex polygon and the area of its first $m$-Kasner descendent. It would be interesting to extend these results to the ratio between $\Delta(K)$, the area of the original polygon, and $\Delta(K')$, the area
of its $t$-th $m$-Kasner descendant. Same questions can be asked if instead of areas one considers
perimeters.

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