STABILIZATION INDICES OF POTENTIALLY MUMFORD CURVES

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Abstract. Let $X$ be a smooth projective curve over a complete discretely valued field $K$. Let $L/K$ be the minimal extension such that $X \times_K L$ has a semi-stable model, and write $e(L/K)$ for the ramification index of $L/K$. Let $e(X)$ be the so-called “stabilization index” of $X$, defined by Halle and Nicaise as the lcm of the multiplicities of the “principal” irreducible components of a minimal regular snc-model of $X$.

It is known that if $L/K$ is tame, then $e(X) = e(L/K)$. If one drops the tameness assumption, but instead assumes that $X$ has index one and potentially multiplicative reduction, Halle and Nicaise ask if the equality $e(X) = e(L/K)$ still holds. We prove that $e(X)$ divides $e(L/K)$ in this situation, but we give examples, in every residue characteristic, of $X$ with $K$-rational points and potentially multiplicative reduction such that $e(X) \neq e(L/K)$.

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1. Introduction

Let $K$ be a complete discretely valued field with valuation $v_K$, uniformizer $\pi_K$, valuation ring $\mathcal{O}_K$ and algebraically closed residue field $k$. This paper investigates the extent to which the minimal regular model of a smooth projective curve $X$ over $K$ sheds light on the minimal extension $L/K$ necessary for $X$ to admit a semi-stable model. We answer a question of Halle and Nicaise ([HN16, Question 10.1.1]) in the negative, showing that, even when $X$ has potentially multiplicative reduction and $K$-rational points, the so-called stabilization index of $X$ need not be equal to $[L : K]$. However, we show that the stabilization index of $X$ does divide $[L : K]$ whenever $X$ has potentially multiplicative reduction.

Let us start by defining the terms above. Let $X/K$ be a smooth, projective, geometrically connected curve. A normal model $\mathcal{X}$ of $X$ is a flat, normal, proper $\mathcal{O}_K$-curve with...
Theorem 1.2. For every prime $p$, there exists $K$ as above with characteristic $p$ residue field, and a smooth, projective, geometrically connected $K$-curve $X$ with potentially multiplicative reduction and a $K$-rational point, such that $e(X) \neq [L : K]$, where $L/K$ is the minimal extension such that $X \times_K L$ admits a stable model.

Our second theorem shows that $e(X)$ and $[L : K]$ are still related, even without the index one assumption.

Theorem 1.3. If $X/K$ is a smooth, projective geometrically connected $K$-curve of genus $g \geq 1$ with potentially multiplicative reduction, and $L/K$ is defined as in Theorem 1.2, then $e(X) \mid [L : K]$. 

If $X$ admits a semi-stable model $\mathcal{X}$ where every irreducible component of the special fiber $\mathcal{X}_k$ has genus zero, then $X$ is said to have multiplicative reduction, or to be a Mumford curve. If the same is true after a base change $L/K$, then $X$ is said to have potentially multiplicative reduction.

If $g(X) \geq 2$ or $X$ is an elliptic curve, and if the minimal extension $L/K$ over which $X \times_K L$ admits a semi-stable model is tame, then by [HN16, Proposition 4.2.2.4], $[L : K] = e(X)$. In general, however, $[L : K]$ need not equal $e(X)$. For instance [HN16, Examples 4.2.2.5, 4.2.2.6] (see also [Lor10, p. 46, Footnote 5]), show that one can have $([L : K], e(X)) = (2, 6)$ or $(6, 2)$. Both examples assume $k$ has characteristic 2. In the same book, Halle and Nicaise ask ([HN16, Question 10.1.1]) whether $X$ having potentially multiplicative reduction and index one (i.e., having a rational point over two different extensions of $K$ of relatively prime degree) is sufficient to guarantee that $[L : K] = e(X)$. Our first theorem answers this question in the negative.

Remark 1.1. Note that if $L'/K$ is a larger Galois extension, then $\text{Gal}(L'/K)$ does not act faithfully on the special fiber. More specifically, the action of $\text{Gal}(L'/L)$ is trivial because each smooth point on the special fiber $\mathcal{X}_k$ is a specialization of an $L$-rational point by Hensel’s lemma, and the smooth points are dense in $\mathcal{X}_k$ since the model is semi-stable, in particular reduced.

generic fiber isomorphic to $X$. Each irreducible component $\mathcal{V}$ of the special fiber of $\mathcal{X}$ has a multiplicity, given by the degree of the divisor of $\pi_K$ on $\mathcal{V}$.

A normal model $\mathcal{X}$ of $X$ is called regular if it is a regular scheme. If the special fiber of a regular model $\mathcal{X}$ has strict normal crossings, then $\mathcal{X}$ is called a regular snc-model. It is well-known that $X$ possesses a regular snc-model, and that there is a unique minimal such model if $g(X) \geq 1$ ([Liu02, Lemma 10.1.8]).

Suppose that $\mathcal{X}$ is a regular snc-model of $X$ with special fiber $\mathcal{X}_k$. An irreducible component $\mathcal{V}$ of $\mathcal{X}_k$ is called principal if $\mathcal{V}^{\text{red}}$ has genus at least 1 or if $\mathcal{V}$ contains at least 3 singular points of $\mathcal{X}_k^{\text{red}}$. The stability index $e(\mathcal{X})$ is the lcm of the multiplicities of the principal components. We define $e(X) = e(\mathcal{X})$, where $\mathcal{X}$ is any minimal regular snc-model of $X$ (as was mentioned above, there is only one such model when $g(X) \geq 1$).

A normal model $\mathcal{X}$ of $X$ is called semi-stable if its special fiber $\mathcal{X}_k$ is reduced and has only ordinary double points for singularities. If the same is true after a base change $L/K$, then by [HN16, Proposition 4.2.2.4], $\mathcal{X}_k^{\text{red}}$. The stability index $e(\mathcal{X})$ is the lcm of the multiplicities of the principal components. We define $e(X) = e(\mathcal{X})$, where $\mathcal{X}$ is any minimal regular snc-model of $X$ (as was mentioned above, there is only one such model when $g(X) \geq 1$).

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A normal model $\mathcal{X}$ of $X$ is called semi-stable if its special fiber $\mathcal{X}_k$ is reduced and has only ordinary double points for singularities. It is well-known that, if $g(X) \geq 2$ or $X$ is an elliptic curve, then there exists a minimal finite extension $L/K$ such $X_L := X \times_K L$ has a semi-stable model, and furthermore that $L/K$ is Galois. If $g(X) \geq 2$, there is a minimal such model $\mathcal{X}_L^{\text{st}}$ called the stable model ([DM69, Corollary 2.7]). The group $\text{Gal}(L/K)$ acts on $\mathcal{X}_L^{\text{st}}$, and this action is faithful on the special fiber, see, e.g., [Ray99, Proposition 2.2.2].
Remark 1.4. By [HN16, Example 4.2.2.5] mentioned above, Theorem 1.3 does not hold without assuming potentially multiplicative reduction.

1.1. Motivation. The immediate motivation for our work is the question of Halle and Nicaise mentioned above. Let us place this in context — see [HN16, §10.1] for more details. Recall that $K$ is a complete discretely valued field with algebraically closed residue field $k$. Suppose $A/K$ is an abelian variety. A central motivation of [HN16] is to define a “stabilization index” $e(A)$ in a natural way. The stabilization index $e(A)$ should be an integer such that for all $d$ prime to both $e(A)$ and $\text{char}(k)$, both the Néron component group $\Phi(A)$ and the motivic class of the identity component of the Néron model of $A$ behave well under base change by the unique tame extension $K(d)/K$ in a precise sense, leading to rationality results for certain motivic zeta functions. If $A$ acquires semi-stable reduction over a tamely ramified extension, or if $A$ acquires multiplicative reduction over any extension, then setting $e(A) = [L : K]$ works, where $L/K$ is the minimal such extension. Alternatively, if $A = \text{Jac}(X)$ where $X$ is a smooth projective curve with index one, then setting $e(A)$ equal to the stabilization index $e(X)$ works. As we have mentioned, if $X$ has semi-stable reduction over a tame extension $L/K$ (and thus $\text{Jac}(X)$ does as well), then $e(X) = [L : K]$, so both definitions of $e(A)$ coincide for $A = \text{Jac}(X)$. This leads one to ask if they also coincide when $A = \text{Jac}(X)$ for $X$ an index one curve with potentially multiplicative reduction, but not necessarily realized over a tame extension of $K$. Theorem 1.2 shows that this is not true in general.

Alternatively, for an arbitrary abelian variety $A$, one can try to define $e(A)$ to be the lcm of denominators of the “jumps” of $A$. These jumps are certain rational numbers encoding how the Lie algebras of the Néron models of various base changes of $A$ fit together, see [HN16, §6]. If $e(A)$ is defined this way, Halle and Nicaise show that $e(A) = e(X)$ when $A = \text{Jac}(X)$ for any smooth projective $K$-curve $X$. Combining this with Theorem 1.2, we see that there exists a curve $X_0$ with potentially multiplicative reduction such that if $A = \text{Jac}(X_0)$, then $e(A) \neq [L : K]$, where $L/K$ is the minimal extension over which $A$ (equivalently $X_0$) has semi-stable reduction.

In fact, the jumps make sense for semi-abelian varieties. So letting $T$ be the torus uniformizing $A = \text{Jac}(X_0)$, we have that $e(T) = e(A)$ where $e(T)$ is the lcm of the denominators of the jumps for $T$, since the jumps depend only on Lie algebras. So $e(T) \neq [L : K]$. This answers [HN16, Question 10.1.2] negatively. We mention that Overkamp ([Ove19, Corollary 2.2.2 and its remark]) also constructs a torus answering [HN16, Question 10.1.2] negatively. His construction uses completely different methods, and does not involve Jacobians of curves.

At a more basic level, this paper continues the study originated by Saito in [Sai87] and continued, e.g., in [Lor10], [Hal10], and [Nic13], of what one can say about the relationship between the minimal extension $L/K$ over which a curve $X$ attains stable reduction, the stable model $X^{st}$ of $X \times_K L$, and the minimal regular snc-model $\mathcal{X}$ of $X$. Let $p = \text{char}(k)$. Saito originally showed that $p \nmid [L : K]^2$ if and only if $p \nmid e(X)^2$, in which case we have already seen that $[L : K] = e(X)$. In [Lor10, Question 1.4], Lorenzini asked about a natural generalization of Saito’s criterion. Namely, if $\text{char}(k) = p$, then do we have $v_p(e(X)) \leq v_p([L : K])$? Theorem 1.3 answers this question positively in the case of potentially multiplicative reduction, and our examples also show that the inequality can be strict.

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1 equivalent to condition (1) of [Sai87, Theorem 3] by [Sai87, Theorem 1] and [Abb00, Theorem 1.6]
2 equivalent to condition (2) of [Sai87, Theorem 3]
1.2. Techniques and outline. One strategy to establish a link between regular models and stable models is as follows: If $X$ is a smooth, projective, geometrically connected $K$-curve with stable model $X^{st}$ over $L$, then the resulting quotient curve $X^{st}/\text{Gal}(L/K)$ is a normal model of $X$ with quotient singularities. So obtaining information about $X$ from $L/K$ and $X^{st}$ is tantamount to resolving quotient singularities. Our proofs of Theorems 1.2 and 1.3 rely on explicitly resolving such singularities when $X \times_K L$ has multiplicative reduction.

Throughout the paper, we exploit Mumford’s result that curves with potentially multiplicative reduction, when viewed as analytic spaces, can be realized as quotients of subsets of the projective line by Schottky groups; the details of this story are reviewed in §2.2. Indeed, all of our examples for Theorem 1.2 are presented by specifying the relevant Schottky group explicitly. Mumford’s perspective is useful because it ultimately allows us to view the quotient singularities arising from stable models as lying on certain models of $\mathbb{P}^1_K$, rather than on higher genus curves. This is useful because on models of $\mathbb{P}^1_K$, explicit resolution of singularities is well-understood, see in particular [OW20, Theorem 7.8] by Wewers and the first author.

The main consequence of [OW20, Theorem 7.8] that we use in this paper is Proposition 2.27, which states that, given a normal model $\mathcal{X}$ of $\mathbb{P}^1_K$ with special fiber $\mathcal{X}_k$, every principal component of the special fiber of the minimal regular snc-resolution of $\mathcal{X}$ has multiplicity dividing the lcm of the multiplicities of the components of $\mathcal{X}_k$. That is, the exceptional divisor introduces no principal components with “bad” multiplicity. The proof of Proposition 2.27 (as well as that of [OW20, Theorem 7.8]) uses “Mac Lane valuations”, which are a convenient notation for expressing so-called “geometric valuations” of $K(x)$, which correspond to irreducible components of normal models of $\mathbb{P}^1_K$. In short, we can explicitly write down the valuations corresponding to the irreducible components of the exceptional divisor, and it is fairly straightforward to see that the desired divisibilities of multiplicities hold.

To go from Mumford’s (analytic) perspective on curves with potentially multiplicative reduction to explicit resolution of singularities on (algebraic) models of the projective line, we frequently pass between Berkovich-analytic, rigid-analytic, formal, valuation-theoretic, and algebraic perspectives on curves. The important “dictionaries” here are discussed in §2.1. In §2.2, we give a detailed description of those aspects of the theory of Mumford curves we use in the paper from the Berkovich-analytic perspective. In §2.3, we give a brief introduction to Mac Lane valuations, and prove Proposition 2.27.

In §3, if $L/K$ is a Galois extension, we give a sufficient criterion for a Mumford curve over $L$ to be defined as a curve (not necessarily a Mumford curve) over $K$. This is of independent interest, and is perhaps well-known, but we could not find a suitable reference. In any case, it is necessary to ensure that our examples, which are presented as Mumford curves over $L$, in fact descend to curves over $K$. In §4, we present our examples, proving Theorem 1.2, and in §5, we prove Theorem 1.3.

Notation 1.5. Throughout the paper, we adopt the following conventions: we denote by $K$ a complete discretely valued field with valuation $v_K$, valuation ring $\mathcal{O}_K$, uniformizer $\pi_K$, and algebraically closed residue field $k$. The valuation $v_K$ is normalized so that $v_K(K^\times) = \mathbb{Z}$. By a $K$-curve we mean a smooth, projective, geometrically connected algebraic curve over $K$. If $X$ is such a $K$-curve, we denote by $X^{an}$ its Berkovich analytification.

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3A good example of this process in action is the proof of Proposition 4.3.
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2. Preliminaries

2.1. Models of curves and analytic geometry. Let $X$ be a $K$-curve. To $X$, we associate its Berkovich analytification $X^{\text{an}}$, using a functorial procedure that does not require base changing to the algebraic closure of $K$. As a set, $X^{\text{an}}$ consists of pairs $(\xi, |\cdot|_{\xi})$ where $\xi$ is a point of $X$ and $|\cdot|_{\xi} : \kappa(\xi) \rightarrow \mathbb{R}_{\geq 0}$ is an absolute value on the residue field of $\xi$ extending the absolute value of $K$.

The set $X^{\text{an}}$ can be endowed with the structure of a locally ringed space that makes it a $K$-analytic space in the sense of Berkovich’s theory [Ber90]. In analogy with scheme theory, Berkovich’s theory associates with a complete non-archimedean field $K$ its analytic spectrum $\mathcal{M}(K)$. Saying that $X$ is a $K$-analytic space is a shorthand for fixing a morphism of analytic spaces $X \rightarrow \mathcal{M}(K)$.

The points of the $K$-analytic curve $X^{\text{an}}$ can be classified as follows:
- If $\xi$ is a closed point of $X$, then the associated residue field $\kappa(\xi)$ is a finite extension of $K$, giving rise to a unique corresponding point in $X^{\text{an}}$. Every point of $X^{\text{an}}$ arising in this way is called of type 1.
- If $x = (\xi, |\cdot|_{\xi})$ is a point of $X^{\text{an}}$ such that $\xi$ is the generic point of $X$, then we can consider the function $v_x : K(X) \rightarrow \mathbb{R} \cup \{\infty\}$ $f \mapsto -\log |f|_{\xi}$.

It is a real valued valuation on the function field of $X$, extending the discrete valuation of $K$. Conversely, every such valuation gives rise to a point of $X^{\text{an}}$ which is not of type 1. In this context, we call $x$ the completion of $K(X)$ with respect to $v_x$, we denote by $\mathcal{H}(x)$ its residue field, and we distinguish three different cases:
- If the transcendence degree of $\mathcal{H}(x)$ over $k$ is equal to 1, then we say that $x$ is of type 2.
- If the dimension of the $\mathbb{Q}$-vector space $\left| \mathcal{H}(x)^{\times} \right|_{|K^{\times}|} \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to 1, then we say that $x$ is of type 3.
- In the remaining cases, we say that $x$ is of type 4.

Remark 2.1. If $x \in X^{\text{an}}$ is a type 2 point, then by Abhyankhar’s inequality [Bou06, §6, 10.3] we have that $\left| \mathcal{H}(x)^{\times} \right|_{|K^{\times}|}$ is a finite group. Since $|K^{\times}| \cong \mathbb{Z}$ and $|\mathcal{H}(x)^{\times}| \subset \mathbb{R}_{\geq 0}$ is torsion free, then $|\mathcal{H}(x)^{\times}|$ is also abstractly isomorphic to $\mathbb{Z}$, that is, $x$ is an absolute value induced by a discrete valuation on $\mathcal{H}(x)$. Thus we can think of type 2 points as discrete valuations on $K(X)$ restricting to $v_K$ on $K$ whose residue fields have transcendence degree 1 over $k$. Such valuations are called geometric valuations. In §2.3, we recall a useful notation for such valuations developed by Mac Lane.

In this paper, we only deal with points of type 1 and type 2. Points of type 2 are especially important for the following reason: if $\mathcal{X}$ is a model of $X$ with special fiber $\mathcal{X}_k$, reduction modulo $\pi_K$ induces a natural surjective and anti-continuous map

$$\text{red}_\mathcal{X} : X^{\text{an}} \rightarrow \mathcal{X}_k$$
(that is, the inverse image of an open subset of $\mathcal{X}_k$ is closed in $X^{an}$), called the specialization map (or reduction map) associated with $\mathcal{X}$. If $\mathcal{X}$ is normal and $\eta$ is the generic point of an irreducible component of $\mathcal{X}_k$, then $\text{red}_{\mathcal{X}}^{-1}(\eta)$ consists of a single type 2 point of $X^{an}$. This construction was first investigated in [BL85] in the setting of rigid analytic geometry and is at the heart of the description of the structure of non-archimedean $K$-analytic curves. The behavior of $\text{red}_{\mathcal{X}}^{-1}$ on generic points of $\mathcal{X}_k$ can be made precise as follows: if $V$ is an irreducible component of $X^{an}$ with generic point $\eta_V$, then the associated type 2 point $\text{red}_{\mathcal{X}}^{-1}(\eta_V)$ is the valuation

\begin{equation}
 f \mapsto \frac{1}{m_V} \text{ord}_V(f),
\end{equation}

where $m_V$ is the multiplicity of $V$.

One can show that the map associating with $\mathcal{X}$ the set of type 2 points corresponding to the irreducible components of $\mathcal{X}_k$ is actually a bijective correspondence. More precisely, we have the following result.

**Proposition 2.3.** The correspondence

\[ \mathcal{X} \mapsto V_\mathcal{X} = \{ \text{red}_{\mathcal{X}}^{-1}(\eta) \mid \eta \text{ generic point of a component of } \mathcal{X}_k \}, \]

induces an isomorphism between the partially ordered set of isomorphism classes of normal (algebraic) $O_K$-models of $\mathcal{X}$, ordered by domination, and the partially ordered set of non-empty finite subsets of $X^{an}$ containing only type 2 points (or equivalently via Remark 2.1, non-empty finite sets of geometric valuations on $K(X)$), ordered by inclusion.

**Proof.** [Rüt14, Corollary 3.18], and [FT19, Theorem 4.1] for the same statement in the context of Berkovich curves. \(\square\)

**Definition 2.4.**

(a) Let $x \in X^{an}$ be a type 2 point. Then, the correspondence of Proposition 2.3 assigns to the singleton $\{x\}$ a model $\mathcal{X}_1$ with irreducible special fiber $\mathcal{X}_{1,k}$. We define the multiplicity of $x$, denoted by $m(x)$, to be the multiplicity of $\mathcal{X}_{1,k}$. If $S$ is a set of type 2 points of $X^{an}$ containing $x$, then the corresponding model $\mathcal{X}_2$ is obtained from $\mathcal{X}_1$ via a sequence of blow-ups followed by a sequence of blow-downs that do not affect the component $\overline{V}_x$ containing the strict transform of $\mathcal{X}_{1,k}$. In particular, the multiplicity of $\overline{V}_x$ in $\mathcal{X}_2$ is again $m(x)$, so that the multiplicity of $x$ can be read off from any model arising from a set $S$ as above.

(b) If $p$ is a closed point of the special fiber $\mathcal{X}_k$, its inverse image $\text{red}_{\mathcal{X}}^{-1}(p)$ is called the formal fiber of $p$, and it is an open subset of $X^{an}$. As such, it can not be an affinoid domain but enjoys similar properties, belonging to the wider class of semi-affinoid spaces\(^4\).

(c) In the specific case of the projective line we have the following: a type 2 point $x \in \mathbb{P}_K^{1,an}$ of multiplicity 1 is a sup-norm on a $K$-rational closed disc. In other words, there exist $a \in K$ and $\rho \in |K^\times|$ such that $x$ is a valuation of the form

\[ \eta_{a,\rho}(f) := \sup_{k \in \mathbb{T}(a,\rho)} \{|f(z)|\}. \]

\(^4\)this terminology was introduced by Martin in [Mar17] to refer to a special class of analytic spaces studied by Berthelot in [Ber96], see also [dJ95, §7].
where $B(a, \rho) = \{ z \in K : v_K(z - a) \geq \rho \}$. In fact, there exists a closed point $p$ of $V_x$ whose formal fiber is a Berkovich open disc centered in a $K$-point with radius in the value group of $K$. Then, $x$ is the boundary point of such a disc, and it follows from [Ber90, §2.5] that it coincides with the sup-norm $\eta_{a, \rho}$.

**Definition 2.5.** If $X$ has a semi-stable model $\mathcal{X}$, then the dual graph of the special fiber $\mathcal{X}_k$ is canonically represented by a subset $\Sigma_X$ of the analytification $X^{an}$ as follows: the set of vertices of $\Sigma_X$ is the collection of points $\text{red}_{\mathcal{X}}^{-1}(\xi)$ for $\xi$ varying over the generic points of all the irreducible components of $\mathcal{X}_k$, while the set of edges is given by those intervals contained in the formal fibers $\text{red}_{\mathcal{X}}^{-1}(x)$ that join two vertices of $\Sigma_X$. One can see that this happens only when $x$ is a closed point of $\mathcal{X}_k$ that belongs to two different irreducible components. There is a continuous map $\rho_X : X^{an} \to \Sigma_X$ which makes $\Sigma_X$ into a strong deformation retract of $X^{an}$. If the genus of $X$ is at least 1, then there is a minimal subset of $X^{an}$ that supports a graph of the form $\Sigma_X$ for some model $\mathcal{X}$. We denote any such graph by $\Sigma_X$ and we call it the skeleton of $X^{an}$. By extension we will refer to it as the skeleton of $X$ as well. If the genus of $X$ is at least 2, then $\Sigma_X$ is the skeleton associated with the stable model of $X$.

2.2. **Mumford curves.** A central tool for our strategy of proof of our main result is the so-called Schottky uniformization of a Mumford curve $X$. In §2.2.1 we start by reviewing Mumford’s uniformization theorem for these curves (Theorem 2.10), using the point of view of Berkovich analytification outlined in the previous section. Among other things, Mumford’s theorem establishes the existence of a uniformizing analytic morphism $p : \mathcal{D} \to X^{an}$, which is a universal covering at the level of topological spaces. The idea of applying Berkovich geometry to this framework is classical, being already proposed by Berkovich in [Ber90, Section 4.4], and related to results established in the context of rigid-analytic geometry, for example by Gerritzen and van der Put [GvdP80] and Lütkebohmert [Lüt16]. However, for our purpose, we need to establish results that are specific to our situation and that are new, to the best of our knowledge. In Proposition 2.13 we build a regular semi-stable model of $X$ that will be used as a basis for other constructions. Then, we explain how given a semi-stable model $\mathcal{X}'$ of $X$ we can lift the uniformization map $p : \mathcal{D} \to X^{an}$ to a morphism of formal models $\varpi : \mathcal{D}' \to \mathcal{X}'$ (see Lemma 2.15 and the preceding construction). In the final part of §2.2.1 we establish properties of this lifting and comparison results between certain local rings of $\mathcal{D}'$ and of $\mathcal{X}$. This allows us to move freely from the language of Berkovich curves and that of (formal) models in the rest of the paper, getting in this way the best of both worlds. In §2.2.2 we consider automorphisms of $X$ that are semi-linear with respect to a Galois sub-extension of the field where $X$ is defined. In particular, Proposition 2.22 establishes conditions for these automorphisms to lift to the universal cover of $X$, which will be used at the beginning of §3 and §5.

2.2.1. **Uniformization of Mumford curves.** Let $g \in \mathbb{N}_{\geq 1}$. Recall that a Mumford curve over $K$ is an algebraic $K$-curve whose Jacobian has split totally degenerate reduction. Equivalently, it is a $K$-curve with a semi-stable model over $\mathcal{O}_K$ such that the irreducible components of the special fiber all have genus zero. Mumford showed that these are precisely the non-archimedean curves that admit a uniformization as in Theorem 2.10 below. Before stating this theorem, we need some definitions.
Figure 1. A Schottky figure adapted to a pair \((\gamma_1, \gamma_2)\).

**Definition 2.6.** Let \(\gamma_1, \ldots, \gamma_g \in \text{PGL}_2(K)\). Let \(\mathcal{B} = (D^+(\gamma_i), D^+(\gamma_i^{-1}))_{1 \leq i \leq g}\) be a set consisting of \(2g\) pairwise disjoint closed discs in \(\mathbb{P}^1_{\text{an}}\). For each \(\gamma \in \{\gamma_1^{\pm 1}, \ldots, \gamma_g^{\pm 1}\}\) we define an open disc \(D^-(\gamma)\) by setting

\[
D^-(\gamma) := \gamma(\mathbb{P}^1_K \setminus D^+(\gamma^{-1})).
\]

The set \(\mathcal{B}\) is called a **Schottky figure** adapted to \((\gamma_1, \ldots, \gamma_g)\) if, for each \(\gamma \in \{\gamma_1^{\pm 1}, \ldots, \gamma_g^{\pm 1}\}\) we have that \(D^-(\gamma)\) is a maximal open disc inside \(D^+(\gamma)\). A subgroup \(\Gamma\) of \(\text{PGL}_2(K)\) is called a **Schottky group** if there exist a generating set \(\{\gamma_1, \ldots, \gamma_g\}\) for \(\Gamma\) and a Schottky figure adapted to the \(g\)-tuple \((\gamma_1, \ldots, \gamma_g)\).

**Notation 2.8.** Let \(\mathcal{B}\) be a Schottky figure adapted to a \(g\)-tuple \((\gamma_1, \ldots, \gamma_g)\). The **fundamental domain** associated with \(\mathcal{B}\) is

\[
F := \mathbb{P}^1_{\text{an}} \setminus \bigcup_{i=1}^g (D^-(\gamma_i) \cup D^+(\gamma_i^{-1})).
\]

The **discontinuity set** and the **limit set** of the Schottky group \(\Gamma\) are respectively

\[
\mathcal{D} := \bigcup_{\gamma \in \Gamma} \gamma F \quad \text{and} \quad \mathcal{L} := \mathbb{P}^1_{\text{an}} \setminus \mathcal{D}.
\]

The **skeleton** \(\Sigma_{\mathcal{D}}\) of \(\mathcal{D}\) is the intersection of \(\mathcal{D}\) with the convex envelope of \(\mathcal{L}\):

\[
\Sigma_{\mathcal{D}} := \mathcal{D} \cap \bigcup_{x,y \in \mathcal{L}} [x,y].
\]

The **skeleton** \(\Sigma_F\) of \(F\) is the subset \(F \cap \Sigma_{\mathcal{D}}\) of \(F\). Note that both \(\mathcal{D}\) and \(F\) retract continuously on their skeletons as in Definition 2.5.
Remark 2.9. The discontinuity set and the limit set depend only on $\Gamma$, and not on the choice of a Schottky figure. Moreover, the limit set coincides with the set of limit points of orbits under the action of $\Gamma$, that is $\ell \in \mathbb{P}^{1,\text{an}}_K$ belongs to $\mathcal{L}$ if and only if there exist $x \in \mathbb{P}^{1,\text{an}}_K$ and a sequence $(\gamma_i)_{i \in \mathbb{N}}$ of elements of $\Gamma$ such that $\lim_{i \to \infty} \gamma_i(x) = \ell$. The points of the limit set are $K$-rational (see [Ber90, §4.4]).

The following theorem describes Mumford’s uniformization in the setting of Berkovich geometry.

Theorem 2.10. Let $\Gamma$ be a Schottky group with $g$ generators. The action of $\Gamma$ on $\mathcal{D}$ is free and proper and the quotient $\Gamma \backslash \mathcal{D}$ is the analytification of a Mumford curve $X^{\text{an}}$ of genus $g$. Conversely, given a Mumford curve $X$ of genus $g$ there exists a Schottky group $\Gamma$ of rank $g$ such that $X^{\text{an}} = \Gamma \backslash \mathcal{D}$, where $\mathcal{D}$ is the discontinuity set of $\Gamma$.

The quotient map $p: \mathcal{D} \to X^{\text{an}}$ is a universal covering of $X^{\text{an}}$ respecting the skeleta: if $\Sigma_{\mathcal{D}}$, $\Sigma_F$ and $\Sigma_X$ denote the skeleta of $\mathcal{D}$, $F$ and $X$ respectively, we have

$$p^{-1}(\Sigma_X) = \Sigma_{\mathcal{D}} \quad \text{and} \quad p(\Sigma_{\mathcal{D}}) = p(\Sigma_F) = \Sigma_X.$$

**Figure 2.** The fundamental domain $F$ (on the left) of the Schottky group $\Gamma$ is the complement of $2g$ discs in $\mathbb{P}^{1,\text{an}}_K$. The group $\Gamma$ identifies the ends of the skeleton $\Sigma_F$, so that the corresponding curve (on the right) retracts on its skeleton $\Sigma_X$.

Remark 2.11. Originally, Theorem 2.10 was proved by Mumford [Mum72] using tools of formal algebraic geometry. The central objects in Mumford’s paper are a certain formal scheme $\mathcal{D}$ whose generic fiber is isomorphic to $\mathcal{D}$, and a uniformization map $\mathcal{D} \to \mathcal{X}$, where $\mathcal{X}$ is the stable model of $X$. The relation between these formal schemes and Berkovich analytifications will be established in the second part of the present section.

The uniformization map $p: \mathcal{D} \to X^{\text{an}}$ of Theorem 2.10 is a main player of this paper. In the following, whenever a Mumford curve is given we always assume that it comes with the choice of a map $p$ as well.

Example 2.12 (Tate curves). If $g = 1$ in the theory above, one starts with the datum of an element $\gamma \in \text{PGL}_2(K)$ and of two disjoint closed discs $D^+(\gamma)$ and $D^+(\gamma^{-1})$ in such a
way that $\gamma(\mathbb{P}_K^{1an} \setminus D^+(\gamma^{-1}))$ is a maximal open disc inside $D^+(\gamma)$. Since $\gamma$ is loxodromic, up to conjugation it is represented by a matrix of the form $\begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$ for some $q \in \mathbb{L}$ satisfying $0 < |q| < 1$. In other words, up to a change of coordinate in $\mathbb{P}_K^1$, the transformation $\gamma$ is the multiplication by $q$ and hence the limit set $\mathcal{L}$ consists only of the two points $0$ and $\infty$. The quotient curve obtained from applying Theorem 2.10 is an elliptic curve, whose set of $K$-points is isomorphic to the multiplicative group $K^\times/q\mathbb{Z}$.

If $g \geq 2$ the skeleton $\Sigma_X$ of $X^{an}$ coincides with the dual graph of the special fiber $\mathcal{X}_k$ of the stable model of $X$ by [Ber90, 4.3]. In our setting, since $K$ is a discretely valued field, we have the following stronger result:

**Proposition 2.13.** Let $X$ be a Mumford curve of genus at least 2, or of genus 1 with $X(K) \neq \emptyset$. Then the set $\mathcal{P}$ of points of multiplicity 1 contained in the skeleton $\Sigma_X$ of $X^{an}$ is finite. Moreover, the model associated with $\mathcal{P}$ under the correspondence given by Proposition 2.3 is semi-stable.

*Proof.* The curve $X$ has semi-stable reduction by virtue of being a Mumford curve, and we fix a minimal semi-stable model $\mathcal{X}$ of $X$ with special fiber $\mathcal{X}_k$. Then, Definition 2.5 ensures that the skeleton $\Sigma_X$ is non-empty and that there is a point of multiplicity 1 on $\Sigma_X$ for every irreducible component of $\mathcal{X}_k$. Let $\mathcal{X}$ be the minimal desingularization of $\mathcal{X}$. The model $\mathcal{X}$ is the minimal regular snc-model of $X$, and it is obtained by blowing up $\mathcal{X}$ at the closed double points of its closed double points that are singular in $\mathcal{X}_k$. Let $p$ be such a double point. Its formal fiber $\text{red}_{\mathcal{X}}^{-1}(p)$ is an open annulus of thickness $n$ contained in $X^{an}$. Then the model $\mathcal{X}$ needs to be blown-up exactly $[n/2]$ times at the point $p$ in order to resolve the singularity (see [Liu02, Example 8.3.53]). This procedure corresponds to changing a model by adding all the multiplicity 1 points in $\Sigma_X \cap \text{red}_{\mathcal{X}}^{-1}(p)$ to the corresponding set of type 2 points. As a result, $\widetilde{\mathcal{X}}$ is semi-stable, and we have a bijection between the irreducible components of $\widetilde{\mathcal{X}}_k$ and the points of multiplicity 1 in $\Sigma_X$. In other words, the semi-stable model $\mathcal{X}$ is the one associated with the set $\mathcal{P}$. \hfill $\Box$

Mumford’s construction of the formal model $\mathcal{D}$ of $\mathfrak{O}$ (see Remark 2.11) can be explained using the theory of formal analytic varieties [BL85, §1], which we briefly outline. First, we recall that an affinoid space $U$ has an associated canonical affine formal model, which is the formal spectrum $\mathcal{U} = \text{Spf}(\mathcal{O}_U(U)^\circ)$ where $\mathcal{O}_U(U)^\circ \subset \mathcal{O}_U(U)$ is the subring of functions bounded by 1 on $U$. This association gives rise to a canonical reduction map $\text{red}_U : U \rightarrow \mathcal{U}_k$. Given an analytic space $\mathfrak{Z}$, one denotes by a formal affinoid covering of $\mathfrak{Z}$ any admissible affinoid covering $\mathfrak{U} = \{U_i\}_{i \in I}$ such that $U_i \cap U_j$ is a finite union of subdomains of $\mathfrak{Z}$ of the form $\text{red}_U^{-1}(\widetilde{V})$ for a Zariski open $\widetilde{V}$ in the canonical reduction $\widetilde{U}_i$ of $U_i$. With every formal affinoid covering of $\mathfrak{Z}$ one can associate a formal model $\mathcal{Z}$ of $\mathfrak{Z}$, whose special fiber $\mathcal{Z}_k$ is obtained by pasting the canonical reductions $\widetilde{U}_i$ along the open subsets whose inverse image under the reduction map is $U_i \cap U_j$. This in turn gives rise to a reduction map $\text{red}_\mathfrak{Z} : \mathfrak{Z} \rightarrow \mathcal{Z}_k$ that generalizes the one defined in §2.1. The inverse image topology on $\mathfrak{Z}$ induced by the model $\mathcal{Z}$ is the one whose open sets are inverse images under the map $\text{red}_\mathfrak{Z}$ of Zariski open subsets of $\mathcal{Z}_k$. Let us consider the analytic space $\mathfrak{D}$ and its formal affinoid covering $\{U_e\}_{e \in E}$ where $E$ is the set of edges of the tree $\Sigma_\mathfrak{D}$ and $U_e$ is the affinoid subdomain of $\mathfrak{D}$ consisting of all the points in $\mathfrak{D}$ that retract to $e$ under the map $\rho_\mathfrak{X}$ introduced in Definition 2.5. The
formal model of $\mathcal{O}$ associated with this covering coincides with the formal scheme $\mathcal{D}$ defined by Mumford.

**Remark 2.14.** Let $\mathfrak{X}$ be a smooth $K$-analytic curve. From an infinite subset of type 2 points in $\mathfrak{X}$ it is not always possible to construct a formal model of $\mathfrak{X}$ and therefore there is not an obvious generalization of Proposition 2.3 in this context. However, given a normal formal model $\mathcal{X}'$ of $X^\text{an}$, a generic point $\eta \in \mathcal{X}'$, and the reduction map $\text{red}_{\mathcal{X}} : \mathfrak{X} \to \mathcal{X}'$ constructed above, one has that $\text{red}_{\mathcal{X}}^{-1}(\eta)$ consists of a unique type 2 point (which by Remark 2.1 is the same thing as a geometric valuation on $K(\mathcal{X})$). Hence, to any formal model one can associate a possibly infinite set of type 2 points in $\mathfrak{X}$ (equivalently, a possibly infinite set of geometric valuations on $K(\mathcal{X})$).

Let us now generalize the construction of the model $\mathcal{D}$. Let $\mathcal{X}'$ be a semi-stable model of $X$, and consider the formal affinoid covering $\mathfrak{U} = \{V_i\}_{i \in I}$ on $X^\text{an}$ containing all the affinoid domains whose canonical reduction $\tilde{V}_i$ is a connected Zariski open subset of the special fiber $X_k$ of $\mathcal{X}'$. Then, we can build a formal affinoid covering of $\mathcal{D}$ as follows. If $V_i \in \mathfrak{U}$ is a contractible affinoid domain, then $p^{-1}(V_i)$ is a disjoint union $\coprod_{j \in J_i} U_{ij}$ of affinoid domains of $\mathcal{D}$ such that $U_{ij} \cong V_j$ for every $j \in J_i$. If $V_i$ is not contractible, then it follows from the semi-stability of $\mathcal{X}'$ that it can be decomposed as a union of contractible affinoids and affinoids that retract on a loop. We can then suppose without loss of generality that the affinoid $V_i$ admits a continuous retraction on a loop or, in other words, that its canonical reduction is an irreducible affine curve with a unique nodal singularity. In this case, the space $p^{-1}(V_i)$ is connected, but not affinoid. However, $p^{-1}(V_i)$ is a union $\bigcup_{j \in J_i} U_{ij}$ of affinoids satisfying the following properties:

- The projection $p(U_{ij})$ is equal to $V_i$;
- The boundary of $U_{ij}$ in $\mathcal{D}$ consists of two distinct points.

The collection $\{U_{ij}\}_{i \in I, j \in J_i}$ is a formal affinoid covering of $\mathcal{D}$, and then gives rise via Bosch–Lütkebohmert’s theory to a formal model of $\mathcal{D}$, that we denote by $\mathcal{D}'$. Thanks to Remark 2.14, one can associate a set of type 2 points $Q_{\mathcal{D}'} \subset \mathcal{D}$ with the formal model $\mathcal{D}'$. If we denote by $Q_{\mathfrak{X}'}$ the set of type 2 points of $X^\text{an}$ associated with $\mathfrak{X}'$, then we have that $p^{-1}(Q_{\mathfrak{X}'}) = Q_{\mathcal{D}'}$. This follows from the definition once we remark that $Q_{\mathfrak{X}'}$ (respectively $Q_{\mathcal{D}'}$) consists precisely of the boundary points of the affinoid domains in the covering defining the formal model $\mathfrak{X}'$ (respectively $\mathcal{D}'$).

**Lemma 2.15.** Let $\mathfrak{X}'$ be a semi-stable model of a Mumford curve $X$, and let $\mathcal{D}'$ be the model of $\mathcal{O}$ constructed from $\mathfrak{X}'$ as above. Then, the uniformization map $p : \mathcal{D} \to X^\text{an}$ of Theorem 2.10 is continuous for the inverse image topologies induced by $\mathcal{D}'$ and $\mathfrak{X}'$ respectively, and extends to a local isomorphism of formal schemes $\varpi : \mathcal{D}' \to \mathfrak{X}'$.

**Proof.** Let $V \subset X^\text{an}$ be of the form $\text{red}_{\mathcal{X}'}^{-1}(\tilde{V})$ for some Zariski open subset $\tilde{V}$ of $\mathcal{X}'_k$. Then, by decomposing $\tilde{V}$ into its connected components, $V$ can be written as a union of elements $V_i$ of the formal affinoid covering $\mathfrak{U}$ associated with the formal model $\mathfrak{X}'$. By construction of the formal model $\mathcal{D}'$, the preimage $p^{-1}(V) = \bigcup p^{-1}(V_i)$ is an infinite union of affinoid domains of the form $\text{red}_{\mathcal{D}'}^{-1}(\tilde{U})$ for some connected Zariski open subset $\tilde{U}$ of $\mathcal{D}'_k$, hence $p$ is continuous for the inverse image topology.

Let us show that this continuity is enough to extend $p$ to a morphism of formal models $\varpi : \mathcal{D}' \to \mathfrak{X}'$. First of all, the functoriality of the reduction ensures the existence of a
morphism of special fibers \( p_k : \mathcal{D}_k' \to \mathcal{X}_k' \) compatible with the reduction maps as illustrated in the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{p} & X^{an} \\
\downarrow \text{red}_x & & \downarrow \text{red}_\mathcal{X} \\
\mathcal{D}_k' & \xrightarrow{p_k} & \mathcal{X}_k'.
\end{array}
\]

An affine Zariski open \( \text{Spec}(\mathcal{A}) \) of \( \mathcal{X}_k' \) can always be completed to get an affine formal subscheme \( \text{Spf}(\mathcal{A}) \subset \mathcal{X}' \). The Zariski open \( p_k^{-1}(\text{Spec}(\mathcal{A})) \) can be covered by affine opens \( \text{Spec}(\mathcal{B}_i) \subset \mathcal{D}_k' \), which lift to formal affine subschemes \( \text{Spf}(\mathcal{B}_i) \subset \mathcal{D}' \). By gluing all the induced morphisms \( \text{Spf}(\mathcal{B}_i) \to \text{Spf}(\mathcal{A}) \), we can construct a morphism \( \varpi : \mathcal{D}' \to \mathcal{X}' \) that lifts \( p_k \). The continuity for the inverse image topology then ensures that the generic fiber of \( \varpi \) is the uniformization map \( p : \mathcal{D} \to X^{an} \), as required.

Since \( p \) is a local isomorphism of analytic spaces, after possibly replacing \( \mathfrak{U} \) with a union of smaller affinoid subsets that are open for the inverse image topology we can suppose that \( p_{|\mathfrak{U}} \) is an isomorphism into its image. Hence, \( \varpi \) is a local isomorphism. \( \square \)

**Remark 2.16.** The construction of the formal scheme \( \mathcal{D}' \) can be associated with any semi-stable model \( \mathcal{X}' \) and is of particular interest in some special cases. When the genus of \( X \) is at least 2 and \( \mathcal{X} \) is the stable model of \( X \), one gets the formal model \( \mathcal{D} \) studied by Mumford (see Remark 2.11). When \( \mathcal{X}' \) is the model considered in Proposition 2.13, that is, the minimal desingularization of \( \mathcal{X} \), one gets that \( \mathcal{D}' \) is the minimal desingularization of \( \mathcal{D} \). In fact, the minimal desingularization of a semi-stable model is obtained by blowing up \( \mathcal{X} \) a finite number of times at its singular double points \( P_1, \ldots, P_n \). As a result, if \( \mathcal{Q} \) is the set of type 2 points of \( X^{an} \) associated with \( \mathcal{X} \), then the set \( \mathcal{Q}' \) of type 2 points of \( X^{an} \) associated with \( \mathcal{X}' \) is obtained from \( \mathcal{Q} \) by adding a finite number of type 2 points lying on the formal fibers \( \text{red}^{-1}(P_i) \). Since \( \mathcal{D}' \) induces by construction the set \( p^{-1}(\mathcal{Q}') \) of type 2 points of \( \mathcal{D} \), it is a regular model. Removing any subset of type 2 points from \( p^{-1}(\mathcal{Q}') \) would create a singularity, hence \( \mathcal{D}' \) is the minimal desingularization of \( \mathcal{D} \). Thanks to Lemma 2.15, in all these instances we can extend \( p : \mathcal{D} \to X^{an} \) to a local isomorphism of formal schemes \( \varpi' : \mathcal{D}' \to \mathcal{X}' \).

**Proposition 2.17.** Let \( X \) be a Mumford curve, \( \mathcal{X}' \) a semi-stable model of \( X \), and \( \mathcal{D}' \) the corresponding formal model of \( \mathcal{D} \). Then, for every closed point \( x \in \mathcal{D}' \), the map \( \varpi' : \mathcal{D}' \to \mathcal{X}' \) that lifts \( p \) induces an isomorphism of local rings

\[
\hat{\mathcal{O}}_{\mathcal{D}',x} \cong \hat{\mathcal{O}}_{\mathcal{X}',\varpi'(x)}.
\]

**Proof.** Let \( \mathfrak{U} \) be the formal affinoid covering of \( \mathcal{D} \) associated with the formal model \( \mathcal{D}' \). Let \( U \) be an affinoid in this covering containing \( \text{red}^{-1}(x) \). Since \( \mathcal{X}' \) is semi-stable, then it dominates the minimal semi-stable model, so that \( \mathfrak{U} \) is a refinement of the smallest affinoid covering of \( \mathcal{D} \) made of discs and annuli. In particular, \( U \) is contained in a fundamental domain for the action of the Schottky group, hence \( p(U) \cong U \). By construction, we have that the structure sheaf of \( \mathcal{D}' \) is obtained as the sheaf \( \mathcal{O}^\circ \) of functions bounded by 1, and the lift \( \varpi' \) is compatible with this operation, hence it induces an isomorphism

\[
\varpi'(\text{Spf}(\mathcal{O}^\circ(U))) \cong \text{Spf}(\mathcal{O}^\circ(U)),
\]

which in turn induces an isomorphism of the completed local rings \( \hat{\mathcal{O}}_{\mathcal{D}',x} \cong \hat{\mathcal{O}}_{\mathcal{X}',\varpi'(x)}. \) \( \square \)
Lemma 2.18. Let $\mathcal{E}$ be a subset of $\mathbb{P}_K^{1,\text{an}}$ consisting of type 1 points and let $\mathcal{Y}$ be a formal scheme whose generic fiber is $\mathcal{Y}_K \cong \mathbb{P}_K^{1,\text{an}} \setminus \mathcal{E}$. Let $\Sigma$ be a finite set of irreducible components of $\mathcal{Y}$, let $S$ be the corresponding set of type 2 points of $\mathcal{Y}_K$, and let $\mathcal{Y}_S$ be the normal model of $\mathbb{P}_K^{1,\text{an}}$ corresponding to $S$ via Proposition 2.3. Let $\mathcal{Y}_\Sigma$ be the set of closed points of $\mathcal{Y}$ that lie only on components in $\Sigma$ and let $\varphi$ be the injection from $\mathcal{Y}_\Sigma$ to the set of closed points of $\mathcal{Y}_S$ induced by the inclusion $\mathcal{Y}_K \subset \mathbb{P}_K^{1,\text{an}}$. Then for every point $x \in \mathcal{Y}_\Sigma$ there is an isomorphism of local rings

$$\hat{\mathcal{O}}_{\mathcal{Y},x} \cong \hat{\mathcal{O}}_{\mathcal{Y}_S,\varphi(x)}.$$

Proof. Let us fix a point $x \in \mathcal{Y}_\Sigma$. By definition, its formal fiber $V_x = \text{red}_{\mathbb{P}_K^{1,\text{an}}}(x)$ is a subset of $\mathbb{P}_K^{1,\text{an}} \setminus \mathcal{E}$. As a result, the inclusion $\mathcal{Y}_K \subset \mathbb{P}_K^{1,\text{an}}$ induces an equality between the ring of analytic functions on $V_x$ and the ring of analytic functions on the formal fiber $V_{\varphi(x)} := \text{red}_{\mathbb{P}_K^{1,\text{an}}}(\varphi(x))$. The local ring $\hat{\mathcal{O}}_{\mathcal{Y},x}$ is a reduced special $R$-algebra whose generic fiber is the semi-affinoid analytic space $V_x$. We can then apply [Mar17, Theorem 2.1] to show that $\hat{\mathcal{O}}_{\mathcal{Y},x}$ is isomorphic to the subring of bounded functions of $\mathcal{O}_{V_x}(V_x)$. The same holds for the local ring $\hat{\mathcal{O}}_{\mathcal{Y}_S,\varphi(x)}$. \hfill $\Box$

2.2.2. Lifting automorphisms of curves. Let $X$ be a $K$-curve, and let $X^{\text{an}}$ be its Berkovich analytification. The analytic curve $X^{\text{an}}$ is a path connected and locally contractible topological space by [Ber90, Corollary 4.3.3]. As such, it has a universal cover $p: \mathcal{Y} \rightarrow X^{\text{an}}$, uniquely determined up to deck transformations of the universal covering space $\mathcal{Y}$. The cover $p$ is a local homeomorphism, hence we can identify for every $y \in \mathcal{Y}$ a sufficiently small open neighborhood $V$ of $y$ with its image $p(V)$ in $X^{\text{an}}$ and define on $\mathcal{Y}$ the unique analytic structure that makes the universal cover $p$ into a local isomorphism of locally ringed spaces. The analytic curves $\mathcal{Y}$ and $X^{\text{an}}$ are isomorphic if and only if $X^{\text{an}}$ is simply connected. Note that, in general, $\mathcal{Y}$ will not be compact, and therefore not the analytification of a projective $K$-curve.

We now concern ourselves with lifting automorphisms from a Berkovich curve to its universal cover. We start by remarking that for $K$-curves the analytification functor is fully faithful, inducing a canonical isomorphism $\text{Aut}(X) \cong \text{Aut}(X^{\text{an}})$ that will allow us to identify the automorphisms of a curve with those of its analytification. If $\delta \in \text{Aut}(X)$ is a $K$-linear automorphism, we call a lift to the universal cover any automorphism $\tilde{\delta} \in \text{Aut}(\mathcal{Y})$ that makes the following diagram commute

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{p} & X^{\text{an}} \\
\delta \downarrow & & \downarrow \delta \\
\mathcal{Y} & \xrightarrow{p} & X^{\text{an}}.
\end{array}$$

Proposition 2.19. Let $X$ be a smooth connected $K$-curve, and let $\delta \in \text{Aut}(X)$ be a $K$-linear automorphism of $X$. Then, there exists a lift $\tilde{\delta} \in \text{Aut}(\mathcal{Y})$ of $\delta$ to the universal cover $\mathcal{Y}$ of $X^{\text{an}}$.

Proof. Let us fix $x \in X^{\text{an}}$ and choose $y, y' \in \mathcal{Y}$ such that $p(y) = x$ and $p(y') = \delta(x)$. Then, the map $p : (\mathcal{Y}, y') \rightarrow (X^{\text{an}}, \delta(x))$ is a covering and $\delta \circ p : (\mathcal{Y}, y) \rightarrow (X^{\text{an}}, \delta(x))$ is a local homeomorphism of pointed topological spaces. The space $\mathcal{Y}$ is compact, and therefore by [Ber90, §3.2] it is also path-connected and locally path-connected. The continuous map $\delta \circ p$
can then be lifted uniquely to a continuous map \( \tilde{\delta} : (\mathcal{Y}, y') \to (\mathcal{Y}, y) \). To show that this is a bijection, observe that we can repeat the same argument above and lift \( \delta^{-1} \circ p : (\mathcal{Y}, y) \to (X^{an}, x) \) to a continuous map \( (\mathcal{Y}, y) \to (\mathcal{Y}, y') \) which is clearly the inverse of \( \tilde{\delta} \). Finally, we can upgrade the homeomorphism \( \tilde{\delta} \) to an automorphism of analytic spaces. Since both \( \delta \circ p \) and \( p \) are local isomorphisms of analytic spaces, every point \( z \in \mathcal{Y} \) has a neighborhood \( \mathcal{Y}_z \) such that \( \tilde{\delta} \) is an isomorphism when restricted to \( \mathcal{Y}_z \). Hence \( \tilde{\delta} \) is a homeomorphism that is also a local isomorphism, and therefore it is an automorphism of the analytic space \( \mathcal{Y} \).

\[ \square \]

Let \( L/K \) be a finite Galois extension and let \( X \) be a Mumford curve over \( L \). By Schottky uniformization (Theorem 2.10), the universal cover \( \mathcal{Y} \) is isomorphic to the space \( \mathbb{P}^{1, an}_L \setminus \mathcal{L} \), where \( \mathcal{L} \) is the limit set of the Schottky group associated with \( X \). Note that we can consider any \( L \)-analytic space as a \( K \)-analytic space by post-composing with the canonical map \( \mathcal{M}(L) \to \mathcal{M}(K) \). For the scope of this paper, we are interested in relaxing the \( L \)-linearity requirement for an automorphism of \( L \)-analytic spaces to a weaker condition that reflects compatibility with the structure of the Galois group \( \text{Gal}(L/K) \), as made precise in the following definition.

**Definition 2.20.** Let \( X \) be a \( L \)-analytic space, and let \( \sigma \in \text{Gal}(L/K) \). A \( K \)-linear automorphism \( \delta \) of \( X \) is said to be \( \sigma \)-semilinear if it fits in a commutative diagram of the form:

\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{M}(L) \\
\delta & \downarrow & \sigma \\
\mathcal{X} & \longrightarrow & \mathcal{M}(L)
\end{array}
\]

Here and throughout, by abuse of notation, we use \( \sigma : \mathcal{M}(L) \to \mathcal{M}(L) \) to mean the map \( (\sigma^{-1})^* \). This has the consequence that if \( \mathcal{X} \) is in fact defined over \( K \), then the \( \sigma \)-action on \( \mathcal{X} \) that acts by \( \sigma \) on the coordinates of \( L \)-points is \( \sigma \)-semilinear.

A \( K \)-linear automorphism of \( \mathcal{X} \) is said to be \( L/K \)-semilinear if it is \( \sigma \)-semilinear for some \( \sigma \in \text{Gal}(L/K) \). In particular, every \( L \)-linear automorphism is \( L/K \)-semilinear.

Let \( \delta \) be a \( \sigma \)-semilinear automorphism of \( X^{an} \). Since by definition \( \delta \) is \( K \)-linear, we can apply Proposition 2.19 to lift it to a \( K \)-linear automorphism \( \tilde{\delta} \) of the universal covering space \( \mathcal{Y} \), in such a way that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & X^{an} \\
\uparrow \delta & \downarrow \delta & \downarrow \delta \\
\mathcal{Y} & \longrightarrow & X^{an}
\end{array}
\]

From this, we automatically obtain that the automorphism \( \tilde{\delta} \) is \( L/K \)-semilinear.

**Proposition 2.22.** Every \( \sigma \)-semilinear automorphism of \( \mathcal{Y} \) that descends to a \( \sigma \)-semilinear automorphism of \( X^{an} \) can be extended to a \( \sigma \)-semilinear automorphism of \( \mathbb{P}^{1, an}_L \).

**Proof.** A \( \sigma \)-semilinear automorphism \( \tilde{\delta} \) of \( \mathcal{Y} \) that descends to a \( \sigma \)-semilinear automorphism of \( X^{an} \) fits in the commutative Diagram 2.21. As a result, \( \tilde{\delta} \) normalizes the action of the
Schottky group, that is: for every \( \gamma \in \Gamma \) we have that \( \tilde{\delta}^{-1} \gamma \tilde{\delta} \in \Gamma \). Let us fix an \( L \)-isomorphism \( \mathcal{Y} \cong \mathbb{P}^1_{\mathcal{L}} \setminus \mathcal{L} \), and pick an element \( \ell \) in the limit set \( \mathcal{L} \). By definition, one can write \( \ell = \lim_{i \to \infty} \gamma_i(x) \) for some \( x \in \mathcal{Y} \). By the normalization property, we have \( \tilde{\delta} \circ \gamma_i = \gamma'_i \circ \tilde{\delta} \) for some \( \gamma'_i \in \Gamma \). Then we have \( \lim_{i \to \infty} \tilde{\delta}(\gamma_i(x)) = \lim_{i \to \infty} \gamma'_i(\tilde{\delta}(x)) \). This limit exists by virtue of the existence of the limit \( \lim_{i \to \infty} \gamma_i(x) \), and thus the limit on the right determines a point \( \ell' \) of the limit set \( \mathcal{L} \). We can then extend the \( \sigma \)-semilinear automorphism \( \tilde{\delta} \) to \( \mathbb{P}^1_{\mathcal{L}} \) by setting \( \tilde{\delta}(\ell) = \ell' \in \mathcal{L} \) for every \( \ell \in \mathcal{L} \). \( \square \)

2.3. Mac Lane valuations.

2.3.1. Valuation theory. Recall from Remark 2.1 that a geometric valuation on \( K[x] \) is a discrete valuation on \( K(x) \) that restricts to the valuation \( v_K \) on \( K \) and whose residue field is finitely generated over \( k \) with transcendence degree 1. There is a partial order on geometric valuations given by \( v \leq w \) if and only if \( v(f) \leq w(f) \) for all \( f \in K[x] \). The infimum of a set \( S \) of geometric valuations, if it exists, is the maximal geometric valuation \( v \) such that \( v \leq w \) for all \( w \in S \).

By [FGMN15, Corollary 7.4] and [Mac36, Theorem 8.1], or [Rütt14, Theorem 4.31], every geometric valuation \( v \) with \( v \geq v_0 \) can be written as a so-called Mac Lane valuation in the form

\[
v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]
\]

where \( n \geq 0 \), the \( \varphi_i \) are monic polynomials in \( K[x] \) of increasing degree and the \( \lambda_i \) are positive rational numbers. The meaning of the notation is this: to calculate \( v(f) \), write out the \( \varphi_n \)-adic expansion

\[
f = a_e \varphi_n^e + a_{e-1} \varphi_n^{e-1} + \cdots + a_0.
\]

Then recursively calculate \( v(f) = \inf_{0 \leq i \leq e} v_{n-1}(a_i) + i \lambda_n \), where \( v_{n-1} \) is shorthand for the valuation

\[
[v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}].
\]

In general, if \( v \) is given as in (2.23) and \( 0 \leq i \leq n \), we write \( v_i \) for the valuation \( [v_0, \ldots, v_i(\varphi_i) = \lambda_i] \).

Not all choices of the \( \varphi_j \) and \( \lambda_j \) give rise to a geometric valuation (or a valuation at all), but if \( v \) as in (2.23) is a geometric valuation, then increasing \( \lambda_n \) results in another geometric valuation. For fuller background, see [Mac36] or [Rütt14]. When we use the term Mac Lane valuation, it is assumed that we are talking about a geometric valuation.

2.3.2. Relationship with normal models. Let \( \mathcal{Y} \) be a normal model of \( \mathbb{P}^1_K \), and choose a rational function \( x \) on \( \mathbb{P}^1_K \) generating the function field. Recall from Proposition 2.3 that there is a corresponding set \( S \) of geometric valuations on \( K(x) \). We say that the valuations \( v \in S \) are included in \( \mathcal{Y} \). If \( S = \{v\} \), we call the corresponding model the \( v \)-model of \( \mathbb{P}^1_K \). If \( v \) and \( w \) are included in \( \mathcal{Y} \), we say that \( v \) is adjacent to \( w \) if \( v \prec w \) and there is no \( z \) included in \( \mathcal{Y} \) with \( v \prec z \prec w \), or if \( w \prec v \) and there is no \( z \) included in \( \mathcal{Y} \) with \( w \prec z \prec v \).

One can always make a linear fractional change of variable in \( x \) such that every \( v \in S \) satisfies \( v(x) \geq 0 \) (this is tantamount to making sure the function \( x \) does not have a pole at any generic point of the special fiber). This is equivalent to \( v \geq v_0 \), where \( v_0 \) is the Gauss valuation given by \( v_0(\sum_i a_i x^i) = \inf_i (v_K(x_i)) \). In particular, we may assume that \( S \) consists of Mac Lane valuations.
**Lemma 2.24.** Let $\mathcal{V}$ be an irreducible component of the special fiber of $\mathcal{Y}$ corresponding to a Mac Lane valuation $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$, and let $m$ be the lcm of the denominators of all the $\lambda_i$, when written in lowest terms. Then the multiplicity $m_{\mathcal{V}}$ of $\mathcal{V}$ in the special fiber equals $m$.

*Proof.* The value group of $v$ is $\frac{1}{m}\mathbb{Z}$, so the lemma follows from (2.2). □

**Lemma 2.25.** Every regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ is inf-closed. That is, if $w$ and $w'$ are twovaluations included in $\mathcal{Y}$, then inf${\{w, w'\}}$ is also included in $\mathcal{Y}$.

*Proof.* Let $v = \inf\{w, w'\}$, and write

$$v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n].$$

We may assume $v \neq w$ and $v \neq w'$. It is not possible that both $w(\varphi_n) > \lambda_n$ and $w'(\varphi_n) > \lambda_n$ because then $v$ would not equal inf${\{w, w'\}}$, as $\lambda_n$ could be increased. So by [Rüt14, Proposition 4.35], at least one of $w$ or $w'$, say $w$, is of the form

$$w = [v_0, w_1(\varphi_1) = \lambda_1, \ldots, w_n(\varphi_n) = \lambda_n, \ldots, w_m(\varphi_m) = \lambda_m]$$

with $m > n$. By [OW20, Theorem 7.8], $v$ is included in the minimal regular resolution of the $w$-model of $\mathbb{P}^1$. Since $\mathcal{Y}$ is a regular resolution of the $w$-model of $\mathbb{P}^1_K$, we thus have that $v$ is included in $\mathcal{Y}$. □

**Lemma 2.26.** Let $\mathcal{X}$ be a normal model of $\mathbb{P}^1_K$ corresponding to a set $v^1, v^2, \ldots, v^r$ of Mac Lane valuations, where

$$v^j = [v^j_0, v^j_1(\varphi_1^j) = \lambda_1^j, \ldots, v^j_{n_j}(\varphi_{n_j}^j) = \lambda_{n_j}^j]$$

and $v^j_0 = v_0$ for all $j$. If $\mathcal{X}' \to \mathcal{X}$ is the minimal regular resolution, then every principal component of the special fiber of $\mathcal{X}'$ corresponds to some $v^j_i$, for $j \in \{1, \ldots, r\}$ and $i \in \{0, \ldots, n_j\}$.

*Proof.* Let $\mathcal{X}_j$ be the $v^j$-model of $\mathbb{P}^1_K$, and let $\mathcal{X}'_j \to \mathcal{X}_j$ be its minimal regular resolution. By [OS19, Lemma 5.3], the set $S$ of Mac Lane valuations corresponding to $\mathcal{X}'$ is $\bigcup_{j=1}^r S_j$, where $S_j$ is the set of Mac Lane valuations corresponding to the $\mathcal{X}'_j$.

Suppose $v \in S \setminus \bigcup_{j=1}^r \{v^j_i\}$ corresponds to a component $\mathcal{V}$ of the special fiber of $\mathcal{X}'$. By [OW20, Lemma 7.1], $\mathcal{V}$ has genus zero. The set $S$ satisfies [KW20, Assumption 3.4(a)] automatically, and by Lemma 2.25, it also satisfies [KW20, Assumption 3.4(b)]. By [KW20, Proposition 3.5], the only irreducible components intersecting $\mathcal{V}$ correspond to valuations $w$ adjacent to $v$ in $S$. By [KW20, Proposition 2.25] there is only one such $w$ with $w < v$. We claim there cannot exist $w_1 \neq w_2$ adjacent to $v$ in $S$ with $v < w_1$ and $v < w_2$, and thus $\mathcal{V}$ is not principal.

To prove the claim, write $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$, and assume that $v > w$ with $w$ adjacent to $v$ in $S$. By [Rüt14, Proposition 4.35], $w$ is of the form

$$[v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \mu_n, v_{n+1}(\varphi_{n+1}) = \lambda_{n+1}, \ldots, v_m(\varphi_m) = \lambda_m]$$

with $\mu_n \geq \lambda_n$ and $m \geq n$. Since $w \in S$, it is in $S_j$ for some $j$. By [OW20, Theorem 7.8], $w_n \in S_j$, and thus $w_n \in S$. Since $w$ is adjacent to $v$ and $v \geq w_n \leq w$, either $w = w_n$ or $v = w_n$ (with $m > n$). If $v = w_n$ with $m > n$, then [OW20, Theorem 7.8] implies that $v = w_n$ is in fact $v^j_i$, a contradiction. So $w = w_n$, and the fact that $w$ is adjacent to $v$ in $S$ uniquely determines $\mu_n$, and thus $w$. This proves the claim. □
Proposition 2.27. Let \( \mathcal{X} \) be a normal model of \( \mathbb{P}^1_K \), and let \( \mathcal{X}' \) be its minimal regular resolution. If \( m \) is the lcm of the multiplicities of the components of the special fiber of \( \mathcal{X} \), then the stability index of \( \mathcal{X}' \) divides \( m \).

Proof. Let \( S \) be the set of Mac Lane valuations corresponding to the model \( \mathcal{X} \). If \( w \) is a Mac Lane valuation corresponding to a principal component of \( \mathcal{X}' \), then Lemma 2.26 implies that there exists a valuation \( v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n] \in S \) such that \( w = v_i \) for some \( i \leq n \). Now, the multiplicity \( m_w \) of the component corresponding to \( v \) is the lcm of the denominators of the \( \lambda_j, j \leq n \). Since the multiplicity \( m_w \) of the component corresponding to \( w \) is the lcm of the denominators of the \( \lambda_j \) for \( j \leq i \), we have \( m_w | m_v \) as desired. \( \square \)

Corollary 2.28. Let \( L/K \) be a finite Galois extension, write \( \mathbb{P}^1_L = \text{Proj} L[x_0, x_1] \), and let \( x = x_1/x_0 \). Suppose \( \text{Gal}(L/K) \) acts on \( \mathbb{P}^1_L \) in the standard manner via its action on \( L \), leaving \( x \) constant. Let \( Y \) be a semi-stable model of \( \mathbb{P}^1_L \) over \( \mathcal{O}_L \) such that the action of \( \text{Gal}(L/K) \) extends to \( Y \). Then the stability index of the minimal regular resolution \( \mathcal{X}' \) of \( \mathcal{X} := Y/\text{Gal}(L/K) \) divides \( [L : K] \).

Proof. The scheme \( \mathcal{X} \) is a model of \( \mathbb{P}^1_K \) such that all irreducible components of its special fiber have multiplicity dividing \( [L : K] \). The corollary now follows from Proposition 2.27. \( \square \)

3. Descent and fields of definition

In §3.1, we give a criterion for a Mumford curve over a Galois extension \( L/K \) to descend to a curve over \( K \). In §3.2, we discuss what it means for a type 2 point of the analytification of a Mumford curve \( X \) over \( L \) to be defined over a subfield of \( L \), and we give some consequences for the \( \text{Gal}(L/K) \)-action on certain semistable models of \( X \).

We take the \( \text{Gal}(L/K) \)-action on \( \mathbb{P}^{1,\text{an}}_L \) to be the usual action on points.

3.1. Descent of Mumford curves. The following proposition shows that a Mumford curve over \( L \) descends to \( K \) if and only if its Schottky group is Galois invariant, after possibly performing a change of parameter.

Proposition 3.1. Let \( L/K \) be a Galois extension. Let \( Y \) be a Mumford curve over \( L \). Denote by \( \Gamma \subset \text{PGL}_2(L) \) the corresponding Schottky group and by \( \mathcal{L} \subset \mathbb{P}^{1,\text{an}}_L \) the corresponding limit set, in such a way that \( Y \) is the quotient of \( \mathbb{P}^{1,\text{an}}_L \setminus \mathcal{L} \) by the action of \( \Gamma \). Under the natural actions of \( \Gamma \) and \( \text{Gal}(L/K) \) on \( \mathbb{P}^{1,\text{an}}_L \), we consider the following three statements:

(a) There exists a \( K \)-curve \( X \) such that \( X_L \cong Y \);

(b) After possibly conjugating \( \Gamma \) by an element of \( \text{PGL}_2(L) \), for every \( \sigma \in \text{Gal}(L/K) \) one has \( \sigma^{-1}\Gamma \sigma = \Gamma \) (the composition is viewed inside the group of \( L/K \)-semilinear automorphisms of \( \mathbb{P}^1_L \));

(c) After possibly conjugating \( \Gamma \) by an element of \( \text{PGL}_2(L) \), for every \( \sigma \in \text{Gal}(L/K) \) and \( \gamma = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Gamma \) one has \( \begin{bmatrix} \sigma(a_1) & \sigma(a_2) \\ \sigma(a_3) & \sigma(a_4) \end{bmatrix} \in \Gamma \).

Then (b) \( \iff \) (c) \( \implies \) (a), and if \( Y \) has genus 1, then all three are equivalent. Moreover, if (b) holds, then every \( \sigma \in \text{Gal}(L/K) \) satisfies \( \sigma(\mathcal{L}) = \mathcal{L} \).
Proof. $(b) \implies (a)$: First we show that condition $(b)$ implies that, up to reparametrization, the limit set $\mathcal{L}$ is globally fixed by the elements of the Galois group $\text{Gal}(L/K)$, proving the last sentence in the statement of the proposition. More precisely, let us assume $(b)$ and fix a parameter of $\mathbb{P}^{1,\text{an}}_L$ that makes the Schottky group $\Gamma$ invariant by $\text{Gal}(L/K)$-conjugation. If we write a point $\ell$ of $\mathcal{L}$ as $\ell = \lim_{i \to \infty} \gamma_i(x)$ for a certain $x \in \mathbb{P}^{1,\text{an}}_L$, then we deduce from the condition $\sigma \gamma \sigma^{-1} \in \Gamma$ for all $\gamma \in \Gamma$ that $\sigma(\ell) = \lim_{i \to \infty} \gamma_i'(\sigma(x))$ for some sequence $(\gamma_i')_{i \in \mathbb{N}}$ of elements of $\Gamma$, which means that $\sigma(\ell)$ is also in the limit set. By applying the same argument to $\sigma^{-1}$ one gets that $\sigma(\mathcal{L}) = \mathcal{L}$. This ensures that the arithmetic action of $\text{Gal}(L/K)$ on $\mathbb{P}^{1,\text{an}}_L$ restricts to an action on the universal cover $\mathcal{D}$ of $Y$:

$$\psi: \text{Gal}(L/K) \times \mathcal{D} \to \mathcal{D}.$$ 

Let us denote by $\psi_\sigma$ the automorphism of $\mathcal{D}$ defined by $\psi_\sigma(x) = \psi(\sigma, x)$. Since $\psi$ comes from the natural arithmetic action, we have that $\psi_\sigma$ is obtained as the pullback of $\sigma$ in the sense that it fits in the following Cartesian diagram:

$$\begin{array}{ccc}
\mathcal{D}^\sigma & \xrightarrow{\psi_\sigma} & \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{M}(L) & \xrightarrow{\sigma} & \mathcal{M}(L).
\end{array}$$

Moreover, using the assumption $(b)$ we have that $\psi_\sigma(\gamma(x)) = \gamma'(\psi_\sigma(x))$ for some $\gamma' \in \Gamma$, which ensures that $\psi_\sigma$ induces a $\sigma$-semilinear automorphism $\varphi_\sigma$ of the Mumford curve $Y$. From the fact that the uniformization map is $L$-linear, we obtain that the map $\varphi_\sigma$ can be obtained as the pullback of $\sigma$ in the sense that it fits in the following Cartesian diagram:

$$\begin{array}{ccc}
Y^\sigma & \xrightarrow{\varphi_\sigma} & Y \\
\downarrow & & \downarrow \\
\text{Spec}(L) & \xrightarrow{\sigma} & \text{Spec}(L).
\end{array}$$

In particular, we have that $Y^\sigma \cong Y$ for all $\sigma \in \text{Gal}(L/K)$. Let $\tau \in \text{Gal}(L/K)$, and let $\varphi^\sigma_\tau: Y^{\sigma \tau} \to Y^\sigma$ be the Cartesian pullback via $\text{Spec}(L) \xrightarrow{\tau} \text{Spec}(L)$. By compatibility with the arithmetic action, we have that $\varphi_{\sigma \tau} = \varphi_\sigma \circ \varphi^\sigma_\tau$. Using Weil’s criterion ([Wei56, Theorem 1] in the slightly weaker form provided by [Kœc04, Theorem (1.9)]), we get that the field of definition of $Y$ is contained in $K$, which corresponds to proving $(a)$.

The equivalence between $(b)$ and $(c)$ is a direct computation left to the reader.

$(a) \implies (c)$ in genus 1: Assume $(a)$ and consider the base change $X_L$, which is a Tate curve associated to a Schottky group with a single generator $\gamma_1$. After a change of variable we may assume $\gamma_1 = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ with $|\beta| < 1$. The corresponding limit set is $\{0, \infty\}$. The base change of $X$ induces a Galois action on $X_L$. For every $\sigma \in \text{Gal}(L/K)$, its action on $X_L$ can be lifted to a $\sigma$-semilinear automorphism of $\mathbb{P}^{1,\text{an}}_L \setminus \{0, \infty\}$ by Proposition 2.19. Let us call this extension $\tilde{\tau}$ and denote by $\tilde{\sigma}$ the automorphism of $\mathbb{P}^{1,\text{an}}_L$ induced by the purely arithmetic action of $\sigma$. Note that, by virtue of $\tilde{\tau}$ being a lifting of an automorphism of $X_L$, we have that $\tilde{\sigma} \Gamma \tilde{\sigma}^{-1} = \Gamma$.
Thus, the composition $\tilde{\sigma}^{-1} \circ \tilde{\tau}$ is an $L$-linear automorphism $\gamma_\sigma \in \text{PGL}_2(L)$ satisfying

$$
\gamma_\sigma \Gamma \gamma_\sigma^{-1} = \tilde{\sigma}^{-1} \tilde{\tau} \Gamma \tilde{\tau}^{-1} \tilde{\sigma} = \tilde{\sigma}^{-1} \Gamma \tilde{\sigma}.
$$

Since $\Gamma$ is generated by a unique element $\gamma_1$, this relation translates to

$$
\tilde{\sigma}^{-1} \gamma_1 \tilde{\sigma} = \gamma_\sigma \gamma_1 \gamma_\sigma^{-1},
$$

where $\epsilon = \pm 1$. This results in $\tilde{\sigma}^{-1} \gamma_1 \tilde{\sigma}$ being conjugate by an element of $\text{PGL}_2(L)$ to $\gamma_1$ (if $\epsilon = -1$ this is true because $\gamma_1$ is conjugate to $\gamma_1^{-1}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Since the ratio of the eigenvalues of an element of $\text{PGL}_2$ is conjugation-invariant, $\sigma(\beta) \in \{\beta, \beta^{-1}\}$. But $\sigma$ preserves absolute values, so $\sigma(\beta) \neq \beta^{-1}$, implying that $\sigma(\beta) = \beta$. By repeating the argument with every $\sigma \in \text{Gal}(L/K)$ we conclude that $\beta \in K$, which implies $(c)$, since $\gamma_1$ is a generator. □

**Remark 3.2.** It would be interesting to determine whether the implication $(a) \implies (c)$ in Proposition 3.1 holds for all Mumford curves. Here is a general framework one could use. Let $\Gamma$ be a Schottky group, and let $X$, $Y$, $L$, and $K$ be as in Proposition 3.1(a). Write $N(\Gamma)$ for the normalizer of $\Gamma$ in $\text{PGL}_2(L)$, and write $N^{sl}(\Gamma)$ for the normalizer of $\Gamma$ in $\text{PGL}_2(L) \rtimes \text{Gal}(L/K)$, with the standard action for the semidirect product — this latter group is the set of $(L/K)$-semilinear actions on $\mathbb{P}^1_L$.

We then have the following pushout diagram of exact sequences of groups:

$$
\begin{array}{cccccc}
1 & \rightarrow & N(\Gamma) & \rightarrow & N^{sl}(\Gamma) & \rightarrow & \text{Gal}(L/K) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & N(\Gamma)/\Gamma & \rightarrow & N^{sl}(\Gamma)/\Gamma & \rightarrow & \text{Gal}(L/K) & \rightarrow & 1
\end{array}
$$

Since $Y = \Gamma \backslash (\mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L})$, Proposition 2.19 and (2.21) show that any $L$-semilinear automorphism of $X_L \cong Y$ lifts to an $L$-semilinear automorphism of $\mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L}$, and this lift is unique up to translation by $\Gamma$. So the semilinear $\text{Gal}(L/K)$-action on $Y$ lifts to a semilinear $\text{Gal}(L/K)$-action on $\mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L}$ up to translation by $\Gamma$, which in turn lifts to a semilinear $\text{Gal}(L/K)$-action on $\mathbb{P}^1_L$ up to translation by $\Gamma$ using Proposition 2.22. In other words, if part $(a)$ of Proposition 3.1 holds, then the bottom exact sequence of (3.3) admits a section from $\text{Gal}(L/K)$. If we could show that the top exact sequence of (3.3) were to admit a section from $\text{Gal}(L/K)$, then an application of Hilbert’s Theorem 90 (see Lemma 5.4 below) would show that, up to a change of variables, the $\text{Gal}(L/K)$-action on $Y$ comes from descending the purely arithmetic $\text{Gal}(L/K)$-action on $\mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L}$. The statement that this action descends is precisely part $(b)$ of Proposition 3.1.

**Remark 3.4.** A Mumford curve over $L$ descends to a Mumford curve over $K$ if and only if its Schottky group is defined over $K$ (after possibly changing a parameter). Our examples in §4 are Mumford curves that descend to curves over $K$, but not to Mumford curves.

---

5As an aside, one can show that $N^{sl}(\Gamma)$ consists of the ordered pairs $(\gamma, \sigma)$ such that $\gamma \Gamma \gamma^{-1} = \sigma^{-1} \Gamma \sigma$. That is, conjugation of $\Gamma$ by $\gamma$ yields the same group as acting on all of the entries of the matrices in $\Gamma$ by $\sigma$.

6In the proof of $(a) \implies (c)$ in Proposition 3.1, the fact that $\sigma(\beta) = \beta$ shows that a section of the top exact sequence can be given explicitly by $\sigma \mapsto (1, \sigma) \in N^{sl}(\Gamma)$ as in the previous footnote.
Remark 3.5. Proposition 3.1 can be applied to study curves that are isomorphic to Mumford curves after base change, as follows. Let \( \Gamma \subset \operatorname{PGL}_2(L) \) be a Schottky group satisfying all the conditions of Proposition 3.1, let \( Y \) be the associated Mumford curve over \( L \), and let \( X \) be a curve over \( K \) such that \( X_L \cong Y \). Then, the Galois cohomology set \( H^1(\operatorname{Gal}(L/K), \operatorname{Aut}(X_L)) \) classifies all isomorphy classes of \( \mathcal{K} \)-curves that become isomorphic to \( Y \) after base change to \( L \), also called \( K \)-forms of \( Y \). In general, it is not easy to compute \( H^1(\operatorname{Gal}(L/K), \operatorname{Aut}(X_L)) \), but one can use the uniformization property to find some of its elements, as follows. For every element \( \gamma \in \operatorname{PGL}_2(L) \), the group \( \Gamma' := \gamma^{-1}\Gamma \gamma \) is again a Schottky group. If \( \Gamma' \) satisfies the condition \( \sigma^{-1}\Gamma' \sigma = \Gamma' \), then the arithmetic action of \( \operatorname{Gal}(L/K) \) on \( \mathbb{P}^1_L \) descends to an action on \( Y \), as in the proof of part \((b) \implies (a)\) of Proposition 3.1. As such, it defines a curve \( X' \) over \( K \), which is not isomorphic to \( X \), in general. More in detail, the automorphism \( \gamma \) induces a cocycle \( \xi_\gamma \in Z^1(\operatorname{Gal}(L/K), \operatorname{PGL}_2(L)) \) via the formula \( \xi_\gamma(\sigma) = \gamma^\sigma \gamma^{-1} \). Since we have that
\[
\gamma^\sigma \gamma^{-1} \Gamma (\gamma^\sigma \gamma^{-1})^{-1} = \gamma^\sigma \Gamma'(\gamma^\sigma)^{-1} = \Gamma,
\]
using the fact that both \( \Gamma \) and \( \Gamma' \) are \( \operatorname{Gal}(L/K) \)-equivariant, the automorphism \( \xi_\gamma(\sigma) \) normalizes \( \Gamma \) and descends to an automorphism \( \xi_\gamma(\sigma) \) of \( X_L \). This process generates a cocycle \( \xi \in Z^1(\operatorname{Gal}(L/K), \operatorname{Aut}(X_L)) \), whose cohomology class \( [\xi] \in H^1(\operatorname{Gal}(L/K), \operatorname{Aut}(X_L)) \) can be nontrivial, even though \( \xi_\gamma \) is necessarily cohomologous to the trivial cocycle by Hilbert’s Theorem 90. The forms of a Mumford curve arising in this way are said to be arithmetic. Examples of nontrivial arithmetic forms of Mumford curves will be given in detail in §4.

We note that not all \( K \)-forms of Mumford curves are arithmetic. For example, non-trivial principal homogeneous spaces for Tate curves (see e.g. [Sil16, §X.3]) are twists of Tate curves but correspond to cocycles that do not lift to \( Z^1(\operatorname{Gal}(L/K), \operatorname{PGL}_2(L)) \).

3.2. Fields of definition of type 2 points.

Definition 3.6. Let \( L/K \) be a finite Galois extension. Let us fix a parameter on \( \mathbb{P}^{1,\text{an}}_L \), so that the natural arithmetic action of \( \operatorname{Gal}(L/K) \) is well defined. If \( x \) is a point of \( \mathbb{P}^{1,\text{an}}_L \) of type 2, we say that \( x \) is defined over \( K \) if it is of the form \( \eta_{a,r} \) for \( a \in K \) and \( r \in |K^\times| \) (see Definition 2.4(c)). We can extend this definition to Mumford curves: if \( Y \) is a Mumford curve over \( L \), and we denote by \( p: \mathbb{P}^{1,\text{an}}_L \setminus \mathcal{L} \to Y^{\text{an}} \) the universal cover coming from Schottky uniformization, a point \( y \in Y^{\text{an}} \) of type 2 is defined over \( K \) if there exists a point in the pre-image \( p^{-1}(y) \) that is defined over \( K \). Note that this notion depends on the choice of a Schottky group associated with \( Y \).

Remark 3.7. A point of \( Y \) defined over any intermediate extension \( M \) of \( L/K \) has multiplicity 1. It is sufficient to prove this for a type 2 point \( x \) of \( \mathbb{P}^{1,\text{an}}_L \), as the projection \( p \) is a local isomorphism and the multiplicity of a point only depends on the analytic structure. To do this, consider the special fiber of the model of \( \mathbb{P}^1_M \) corresponding to the singleton \( \{x\} \). It consists of a unique component \( \mathcal{X}_k \), whose closed points are smooth if, and only if, their formal fibers are Berkovich discs centered in \( M \)-rational points with \( M \)-rational radius. Since this is the case when \( x \) is defined over \( M \), we have that the curve \( \mathcal{X}_k \) is reduced, which is equivalent to the fact that \( x \) has multiplicity 1.

Let \( X \) be an arithmetic \( K \)-form of a Mumford curve \( Y \) defined over \( L \) as in Remark 3.5. Then, there is a Schottky group \( \Gamma \) associated with \( X_L \) such that the action of \( \operatorname{Gal}(L/K) \) on \( X_L^{\text{an}} \) inherited from the base change is compatible with the arithmetic action on \( \mathbb{P}^{1,\text{an}}_L \) via
the uniformization map \( p: \mathbb{P}^{1,\text{an}}_L \setminus \mathcal{L} \to X^\text{an}_L \). For the rest of this section, we deal only with arithmetic forms and we fix a \( \Gamma \) as above. In this way, we can write about points defined over subfields of \( L \) (in the sense of Definition 3.6) without ambiguity.

**Lemma 3.8.** Let \( X \) be an arithmetic \( K \)-form of a Mumford \( L \)-curve, and let \( x \in X^\text{an}_L \) be a point of type 2 defined over a field \( M \) with \( K \subseteq M \subseteq L \). Let \( \mathcal{X} \) be a formal model of \( X^\text{an}_L \) whose associated set of type 2 points contains \( x \). Denote by \( \overline{V} \) the component of \( \mathcal{X} \) corresponding to \( x \) and by \( P \) a smooth point of \( \overline{V} \). Then, the Galois group \( \text{Gal}(L/M) \) acting on \( \mathcal{X} \) fixes \( \overline{V} \) pointwise, the image of \( P \) in the quotient model \( \mathcal{X}/\text{Gal}(L/M) \) is a smooth point, and the image of \( \overline{V} \) has multiplicity one.

**Proof.** Let \( \pi: X^\text{an}_L \to X^\text{an}_M \) and \( \tilde{\pi}: \mathbb{P}^{1,\text{an}}_L \to \mathbb{P}^{1,\text{an}}_M \) be the canonical projection maps. The fact that the point \( x \) is defined over \( M \) yields the following two consequences:

(a) There is a point \( y \in p^{-1}(x) \) defined over \( M \). In particular, \( y \) is fixed by every element of \( \text{Gal}(L/M) \);

(b) The multiplicity of \( \tilde{\pi}(y) \) is equal to 1 (Remark 3.7).

Since the action of \( \text{Gal}(L/K) \) on \( X_L \) descends from the purely arithmetic action on \( \mathbb{P}^{1,\text{an}}_L \), property (a) results in the fact that \( \overline{V} \) is fixed by \( \text{Gal}(L/M) \). Moreover, since \( L/M \) is totally ramified and \( y \) is defined over \( M \), the group \( \text{Gal}(L/M) \) acts with full inertia on the completed residue field \( \mathcal{H}(y) \). As a result, the action of \( \text{Gal}(L/M) \) on the component \( \overline{V} \) is trivial and the image of the smooth point \( P \in \overline{V} \) in the quotient model \( \mathcal{X}/\text{Gal}(L/M) \) is a smooth point. Finally, property (b) ensures that the image of \( \overline{V} \) in the quotient model \( \mathcal{X}/\text{Gal}(L/M) \) has multiplicity 1. \( \square \)

**Lemma 3.9.** Let \( X \) be an arithmetic \( K \)-form of a Mumford curve over \( L \), let \( \pi: X^\text{an}_L \to X^\text{an}_M \) be the canonical projection induced by the base change, and let \( x \in X^\text{an}_L \) be a point of type 2 defined over a field \( M \) with \( K \subseteq M \subseteq L \). Then we have that the multiplicity \( m(\pi(x)) \) divides the degree \([M : K]\).

**Proof.** We have that the multiplicity of a type 2 point \( y \) in a \( K \)-analytic curve is given by the index \([\mathcal{H}(y)^\times] : [K^\times]\) [Duc14, Proposition 6.5.2 (1)]. Consider now the projection morphism \( \pi': X^\text{an}_L \to X^\text{an}_M \). Since \( x \) is defined over \( M \), we can apply Lemma 3.8 to show that the point \( \pi'(x) \in X^\text{an}_M \) has multiplicity 1, that is to say \( |\mathcal{H}(\pi'(x))^\times| = |M^\times| \), and hence \( |\mathcal{H}(\pi(x))^\times| \subset |M^\times| \). This implies that

\[
m(\pi(x)) = \left[\frac{|\mathcal{H}(\pi(x))^\times|}{|K^\times|}ight] = \frac{[M^\times] : [K^\times]}{[|M^\times| : |\mathcal{H}(\pi(x))^\times|]},
\]

which proves the statement. \( \square \)

### 4. Proof of Theorem 1.2

**4.1. Generalities.** In this subsection, we set up our examples for Theorem 1.2. Our notation is as follows: \( L/K \) is a finite Galois extension, and \( \Gamma \subseteq \text{PGL}_2(L) \) is a Schottky group of rank \( g \geq 2 \) with generators \( \gamma_1, \ldots, \gamma_g \) and associated Schottky figure

\[
(D^+(\gamma_1), D^+(\gamma_1^{-1}), \ldots, D^+(\gamma_g), D^+(\gamma_g^{-1}))
\]

(Definition 2.6). Write \( \mathcal{O} = \mathbb{P}^{1,\text{an}}_L \setminus \mathcal{L} \) where \( \mathcal{L} \) is the limit set of \( \Gamma \), let \( Y^\text{an} = \Gamma \setminus \mathcal{O} \), and let \( Y \) be the corresponding projective curve over \( L \), which is the genus \( g \) Mumford curve
Lemma 2.15. The quotient

Let \( \mathcal{Y} \) be the semi-stable model of \( Y \), and let \( \mathcal{Y}' \) be the semi-stable model of \( Y \) associated to the skeleton \( \Sigma_Y \) of \( Y \). By Proposition 2.17, \( \hat{\mathcal{Y}} \) is a formal model of \( \mathcal{D} \) coming from the uniformization map \( \mathcal{D} \to Y^{an} \). Recall that the set of irreducible components of the special fiber of \( \mathcal{Y}' \) is in natural one-to-one correspondence with \( S_Y \).

Let \( \varphi' : \mathcal{D}' \to \hat{\mathcal{Y}} \) be the morphism of formal schemes coming from the uniformization map \( \mathcal{D} \to Y^{an} \) as in Lemma 2.15. Recall that \( \mathcal{D}' \) is a formal model of \( \mathcal{D} \). We write \( \Sigma_{\mathcal{D}} \) for the skeleton of \( \mathcal{D} \) (Theorem 2.10), and we write \( S_{\mathcal{D}} \) for the set of multiplicity 1 points on \( \Sigma_{\mathcal{D}} \).

Let \( F \subseteq \mathcal{D} \) be a fundamental domain for \( \Gamma \) associated to the Schottky figure above, as in Notation 2.8. Write \( \Sigma_F \) for its skeleton, and write \( S_F \subseteq S_{\mathcal{D}} \) for the set of multiplicity 1 points on \( \Sigma_F \). The group \( \Gamma \) acts on \( S_{\mathcal{D}} \). By the definition of a fundamental domain, no two elements of \( S_F \) lie in a common \( \Gamma \)-orbit of \( S_{\mathcal{D}} \). So the map \( S_F \to S_Y := \Gamma \backslash S_{\mathcal{D}} \) induced by \( S_F \to S_F \to S_Y \) is injective, and Theorem 2.10 shows it is in fact bijective.

The group \( \text{Gal}(L/K) \) acts on \( \Sigma_Y \) via descent of its action on \( \Sigma_{\mathcal{D}} \), and thus it acts on \( \mathcal{Y}' \). We pick a semi-stable blow-down \( \mathcal{Y}'' \) of \( \mathcal{Y}' \); in the examples, \( \mathcal{Y}'' \) will be chosen strategically. We make the further assumption that the action of \( \text{Gal}(L/K) \) on \( \mathcal{Y}' \) extends to \( \mathcal{Y}'' \). Let \( \mathcal{D}'' \to \hat{\mathcal{Y}}'' \) be the morphism of formal schemes coming from the uniformization map \( \mathcal{D} \to Y^{an} \) as in Lemma 2.15. The quotient \( \mathcal{Y}'' / \text{Gal}(L/K) \) is a normal model \( \mathcal{X} \) of \( X \) with special fiber \( \mathcal{X}_k \).

Lemma 4.1. Let \( x \in \mathcal{X}_k \subseteq \mathcal{X} \), let \( y \) be a preimage of \( x \) under \( \mathcal{Y}'' \to \mathcal{X} \), and let \( \bar{y} \) be the unique preimage of \( y \) under \( \mathcal{D}'' \to \hat{\mathcal{Y}}'' \to \mathcal{Y}'' \). Let \( \bar{y} \) be the \( \Gamma \)-orbit of \( \bar{y} \). Then \( \hat{\mathcal{Y}}'' \) maps isomorphically onto its image in \( \mathcal{Y}'' / H \), as in the diagram below.

\[
\begin{array}{ccc}
\hat{y} \in \mathcal{D}'' & \xrightarrow{H} & \mathcal{Z} \ni z \\
\downarrow & & \downarrow \text{id} \\
y \in \hat{\mathcal{Y}}'' & \xrightarrow{H} & \mathcal{Y}'' / H \ni y^H
\end{array}
\]

Then \( \hat{\mathcal{O}}_{\mathcal{Y}'' / H, y^H} \cong \hat{\mathcal{O}}_{\mathcal{Z}, z} \). In particular, if \( \text{Gal}(L/K)\bar{y} \subseteq \overline{F} \), then taking \( H = \text{Gal}(L/K) \) gives \( \hat{\mathcal{O}}_{\mathcal{X}, x} \cong \hat{\mathcal{O}}_{\mathcal{Z}, z} \).

Proof. By Proposition 2.17, \( \hat{\mathcal{O}}_{\mathcal{Y}'' / \bar{y}} \cong \hat{\mathcal{O}}_{\mathcal{D}'' / \bar{y}} \). The Galois \( \text{Gal}(L/K) \)-action on \( \mathcal{Y}'' \) comes from the Galois \( \text{Gal}(L/K) \)-action on \( \mathcal{D}'' \) by descent, thus the same is true for the \( H \)-actions. By assumption, the \( H \)-orbit of \( \bar{y} \) lies in the reduction \( \overline{F} \) of \( F \). Since \( F \to Y^{an} \) is an isomorphism onto its image, the formal neighborhood \( U \) of the \( H \)-orbit of \( \bar{y} \) maps isomorphically onto its image in \( \mathcal{Y}'' \), and this map respects the \( H \)-action. The lemma follows.

Proposition 4.2. Let \( x \in \mathcal{X}_k \), and maintain the notation of Lemma 4.1. Assume the Galois \( \text{Gal}(L/K) \)-orbit of \( \bar{y} \) lies in \( \overline{F} \). Let \( V \) be an irreducible component of \( \mathcal{X}_k \) containing \( x \), let \( V'' \) be the irreducible component of \( \mathcal{D}'' \) in the preimage of \( V \) lying in \( \overline{F} \), and let \( W \) be the image of \( V'' \) in \( \mathcal{Z} \). Then \( m_V = m_W \).
Proof. If \( z \) is as in Lemma 4.1, then by that lemma, \( \hat{O}_{\mathcal{X},x} \cong \hat{O}_{\mathcal{Z},z} \). Since \( x \in V \) and \( z \in W \), and the complete local rings of \( x \) and \( z \) detect the multiplicities of the irreducible components they lie on, we conclude that \( m_V = m_W \).

Let \( \mathcal{X}^{\text{reg}} \to \mathcal{X} \) be the minimal regular snc-resolution of \( \mathcal{X} \). Write \( \mathcal{X}_k^{\text{reg}} \) for the special fiber of \( \mathcal{X}^{\text{reg}} \).

**Proposition 4.3.** Let \( x \in \mathcal{X}_k \), and maintain the notation of Lemma 4.1. Assume the Gal\((L/K)\)-orbit of \( \bar{y} \) lies in \( \mathcal{F} \). Let \( d \) be the lcm of the multiplicities of the irreducible components of \( \mathcal{X}_k \) passing through \( x \). If \( E \) is a principal component of \( \mathcal{X}_k^{\text{reg}} \) that lies on the exceptional divisor of \( \mathcal{X}^{\text{reg}} \to \mathcal{X} \) and whose image is \( x \), then \( m_E \) divides \( d \).

**Proof.** By Lemma 4.1, \( \hat{O}_{\mathcal{X},x} \cong \hat{O}_{\mathcal{Z},z} \). By Proposition 4.2, \( d \) is the lcm of the multiplicities of the irreducible components of the special fiber of \( \mathcal{Z} \) passing through \( z \). Let \( S \) be the set of Mac Lane valuations corresponding to these components, and let \( \mathcal{Z}_S \) be the normal model of \( \mathcal{F}_k \) corresponding to \( S \). By Remark 2.18, we have \( \hat{O}_{\mathcal{Z},z} \cong \hat{O}_{\mathcal{Z}_S,\varphi(z)} \), where \( \varphi \) is the injection in Remark 2.18. So \( d \) is also the lcm of the multiplicities of the irreducible components of the special fiber of \( \mathcal{Z}_S \) passing through \( \varphi(z) \). Thus it suffices to show that any principal component of the exceptional divisor above \( \varphi(z) \) on the minimal regular resolution of \( \mathcal{Z}_S \) has multiplicity dividing \( d \). This follows from Proposition 2.27.

**Proposition 4.4.** Let \( x \in \mathcal{X}_k \), and maintain the notation of Proposition 4.2. Assume that \( x \) lies on only one irreducible component \( V \) of \( \mathcal{X}_k \), and that the corresponding type 2 point is defined over a field \( M \) such that \( K \subseteq M \subseteq L \) and Gal\((M/K) \cong \mathbb{Z}/p \). Assume further that Gal\((M/K) \) acts nontrivially on the preimage of \( V \) in \( \mathcal{Y}^n/\text{Gal}(L/M) \). If \( E \) is a principal component of \( \mathcal{X}_k^{\text{reg}} \) that lies on the exceptional divisor of \( \mathcal{X}^{\text{reg}} \to \mathcal{X} \) and whose image is \( x \), then \( m_E \) divides \( p \).

**Proof.** Let \( H = \text{Gal}(L/M) \). By Lemma 3.8, the preimage \( y^H \) of \( x \) in \( \mathcal{Y}^n/H \) is smooth. We may assume \( x \) is singular in \( \mathcal{X} \) (otherwise \( E \) would not exist). Since \( x \) is the image of the smooth point \( y^H \) after taking the quotient by the action of Gal\((M/K) \) on \( \mathcal{Y}^n/H \), we have that Gal\((M/K) \) fixes \( y^H \). Since the irreducible component of the special fiber of \( \mathcal{Y}^n/H \) on which \( y^H \) lies has genus 0, and Gal\((M/K) \) acts nontrivially on this component, the Hurwitz formula for wildly ramified curves shows that the induced Gal\((M/K)\)-action on the special fiber of Spec\((\hat{O}_{\mathcal{Y}^n/H,y^H}) \) has ramification jump 1. This means that \( x \) is a weak wild arithmetic quotient singularity with stabilizer \( \mathbb{Z}/p \) in the language of [OW20, Definition 3.7]. By [OW20, Corollary 6.5, Definition 6.6, and Corollary 7.13(i)], the unique principal component of the exceptional divisor of the minimal resolution of the singularity at \( x \) has multiplicity \( p \), proving the proposition.

4.2. An example \((g = 2, p = 2)\). Let \( k \) be an algebraically closed field of characteristic 2, and let \( K = \text{Frac}(W(k)) \). Let \( L = K[a,b]/(a^2-2,b^2+1) \). Write \( \sqrt{2} \) and \( i \) for the respective images of \( a \) and \( b \) in \( L \). The group Gal\((L/K) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) is generated by \( \sigma \) and \( \tau \) acting nontrivially on \( \sqrt{2} \) and \( i \), respectively. In this subsection, as a specific example of the setup from §4.1, we construct a genus 2 curve \( X/K \) with potentially multiplicative reduction, a \( K \)-rational point, and stabilization index \( e(X) = 2 \), but for which \( L/K \) (of degree 4) is the minimal extension over which \( X \) attains semi-stable reduction. This proves the \( p = 2 \) case of Theorem 1.2.
Let $\Gamma$ be the subgroup of $PGL_2(L)$ generated by

$$\gamma_1 := PAP^{-1}, \ \gamma_2 := QAQ^{-1},$$

where

$$P = \begin{bmatrix} -\sqrt{2} & -\sqrt{2} \\ -1 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} -1 - i & 1 - i \\ -1 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Proposition 4.5.**

(i) The group $\Gamma$ above is a Schottky group of rank 2, with Schottky figure given by

$$(D^+(\gamma_1), D^+(\gamma_1^{-1}), D^+(\gamma_2), D^+(\gamma_2^{-1})) = (\bar{B}(\sqrt{2}, 2), \bar{B}(\sqrt{2}, 2), \bar{B}(1 + i, 3/2), \bar{B}(1 - i, 3/2)),$$

and corresponding open subdisks

$$(D^-(\gamma_1), D^-(\gamma_1^{-1}), D^-(\gamma_2), D^-(\gamma_2^{-1})) = (B(\sqrt{2}, 2), B(\sqrt{2}, 2), B(1 + i, 3/2), B(1 - i, 3/2)),$$

where $B(a, r)$ (resp. $\bar{B}(a, r)$) is the open (resp. closed) disk with center $a$ and radius $|2^r|$.

(ii) If $Y^an = \Gamma \setminus \mathcal{D}$ as in §4.1 and $Y$ is the corresponding projective curve over $L$, then the standard action of $Gal(L/K)$ on $\mathcal{D} \subseteq \mathbb{P}^1_{L^an}$ descends to $Y^an$, and thus to $Y$. In particular, $Y$ descends to a curve $X := Y/(Gal(L/K))$ defined over $K$.

**Proof.** One computes straightforwardly that $\gamma(\mathbb{P}^1_L \setminus D^+(\gamma^{-1})) = D^-(\gamma)$ for $\gamma \in \{\gamma_{1}^{\pm 1}, \gamma_{2}^{\pm 1}\}$. Since each $D^-(\gamma)$ is a maximal open subdisk of $D^+(\gamma)$, part (i) follows from Definition 2.6.

To prove part (ii), one first calculates explicitly that, as automorphisms of $\mathbb{P}^1_L$, we have $\sigma \gamma_1 \sigma^{-1} = \gamma_1^{-1}$ and $\tau \gamma_2 \tau^{-1} = \gamma_2^{-1}$, whereas $\sigma$ commutes with $\gamma_2$ and $\tau$ commutes with $\gamma_1$ (conceptually, this is because the fixed points of $\gamma_1$, resp. $\gamma_2$, are $\pm \sqrt{2}$, resp. $\pm 1 + i$, and $A$ is defined over $K$). Thus conjugating by $\sigma$, resp. $\tau$, represents reversing the direction of the “flow” of $\gamma_1$, resp. $\gamma_2$). So $\Gamma$ satisfies Proposition 3.1(b), and thus $Y$ descends to a curve $X$ defined over $K$ such that $X \times_K L \equiv Y$. In fact, since $\sigma$ and $\tau$ normalize $\Gamma$ directly (without resorting to conjugation by an element of $PGL_2(L)$), we have that $Gal(L/K)$ acts on $Y$ via the descent of its standard action on $\mathbb{P}^1_{L^an} \setminus \mathcal{L}$, and $X = Y/Gal(L/K)$.

The curve $X = Y/(Gal(L/K))$ is an arithmetic $K$-form of $Y$, consistent with the notation in §4.1.

**Lemma 4.6.** The curve $X$ has a $K$-rational point. In particular, it has index 1.

**Proof.** Since the $L$-rational point $0 \in \mathbb{A}^1_{L^an} \subseteq \mathbb{P}^1_{L^an}$ does not lie in any of the $D^-(\gamma)$, it does not lie in $\mathcal{L}$ (see Notation 2.8). So $0 \in \mathcal{D}$, and its image under $\mathcal{D} \to Y^an \to X^an$ is a $K$-rational type 1 point of $X^an$. Thus $X$ has a $K$-rational point.

Now, we define $\mathcal{Y}^st$, $\mathcal{Y}'$, $F$, $\bar{F}$, $\Sigma_Y$, $S_Y$, $\Sigma_F$, and $S_F$ as in §4.1. The skeleton $\Sigma_F$ is the skeleton connecting the type 2 points $\eta_{\sqrt{2}}, \eta_{1+i,3/2}, \eta_{-\sqrt{2}},$ and $\eta_{1-3/2}$ of $\mathbb{P}^1_{L^an}$, but not including the latter two points. We have

$$|\sqrt{2} - (-\sqrt{2})| = |2^{3/2}|, \ \ |(1 + i) - (1 - i)| = |2^1|, \ \ |\pm \sqrt{2} - (1 \pm i)| = |2^{3/4}|.$$ 

It follows that the set $S_F$ of multiplicity 1 points on $\Sigma_F$ consists of the points $\eta_{a,r}$, where the $(a, r)$ correspond to the black dots in Figure 3. The skeletons $\Sigma_F$ and $\Sigma_Y$ are pictured in Figure 3. Note that $\Sigma_Y$ is the dual graph of the special fiber of $\mathcal{Y}'$. 24
Proposition 4.7. The extension $L/K$ is the minimal extension over which $X$ acquires semi-stable reduction.

Proof. By construction, $Y \cong X \times_K L$ has semi-stable reduction. If $X$ attained semi-stable reduction over a Galois subextension $K \subseteq M \subseteq L$, then the Gal($L/K$)-action on the special fiber $Y^\text{st}$ of the stable model $Y^\text{st}$ of $Y$ would factor through Gal($M/K$) (see Remark 1.1), so it suffices to show that Gal($L/K$) acts faithfully on $Y^\text{st}$. From the right-hand part of Figure 3, one sees that $Y^\text{st}$ is constructed from $Y^\prime$ by contracting all components of $Y^\text{st}$ except those corresponding to the images of $\eta_{1+\sqrt{2}, 5/4}$ and $\eta_{1+i, 1}$. But $\sigma$ acts nontrivially on the first component (since it switches $\eta_{1\pm\sqrt{2}, 7/4}$), and $\tau$ acts nontrivially on the second component (since it switches $\eta_{1\pm i, 5/4}$). So Gal($L/K$) acts faithfully on $Y^\text{st}$. \hfill $\square$

Now we define $\mathcal{Y}''$ by letting $\mathcal{Y} \to \mathcal{Y}''$ be the morphism given by blowing down the components corresponding to the vertices labeled $(\pm \sqrt{2}, 7/4)$, $(1 \pm i, 5/4)$, $(\sqrt{2}, 3/4)$, and $(\sqrt{2}, 5/4)$ in the right-hand part of Figure 3. Then $\mathcal{Y}''$ is also a semi-stable model of $X_L$ on which Gal($L/K$) acts, consistent with the assumptions on $\mathcal{Y}''$ in §4.1. As in §4.1, we set $\mathcal{X} := \mathcal{Y}''/\text{Gal}(L/K)$ and $\mathcal{X}_k$ is its special fiber.

Lemma 4.8. All components of $\mathcal{X}_k$ have multiplicity dividing 2.

Proof. Our construction of $\mathcal{Y}''$ blew down all the components with minimal field of definition $L$. So the type 2 points corresponding to the irreducible components of the special fiber of $\mathcal{Y}''$ are each defined over either $K(\sqrt{2})$ or $K(i)$. By Lemma 3.9, the images of these components in $\mathcal{X}$ have multiplicity dividing 2. \hfill $\square$
Write $V_1, V_2$ for the irreducible components of $X_k$ corresponding to the vertices $(\sqrt{2}, 2), (1 + i, 3/2)$ in Figure 3, respectively. Write $U$ for the union of all of the other irreducible components of $X_k$.

As in §4.1, let $\text{pr}: X_{\text{reg}} \rightarrow X$ be the minimal regular snc-resolution of $X$. Write $X_{\text{reg}}^k$ for the special fiber of $X_{\text{reg}}$. Recall also from §4.1 that we have the cover $D'' \rightarrow Y''$.

**Proposition 4.9.** If $E$ is a principal component of $X_{\text{reg}}^k$ that is exceptional for $\text{pr}: X_{\text{reg}} \rightarrow X$, then the multiplicity of $E$ divides 2.

**Proof.** Let $x$ be the image of $E$ in $X$. First assume $x \in U$. Then, if $\tilde{y} \in \mathcal{F} \subseteq D''$ lies above $x$ as in Lemma 4.1, we have $\text{Gal}(L/K)\tilde{y}$ lies in $\mathcal{F}$, so by Proposition 4.3 and Lemma 4.8, the multiplicity of $E$ divides 2.

Now, assume $x \notin U$. Then we are in the situation of Proposition 4.4, with $M = K(\sqrt{2})$ or $M = K(i)$, depending on whether $x \notin V_1$ or $V_2$. Observe that, in the first case, $\text{Gal}(M/K)$ acts non-trivially on $Y''/\langle \tau \rangle$ above $V_1$, since it switches $\eta_{\pm \sqrt{2}, 7/4}$, and in the second case, $\text{Gal}(M/K)$ acts nontrivially on $Y''/\langle \sigma \rangle$ above $V_2$, since it switches $\eta_{1, \pm i, 5/4}$. So the proposition follows from Proposition 4.4.

**Corollary 4.10.** Every principal component of $X_{\text{reg}}^k$ has multiplicity dividing 2.

**Proof.** This follows immediately from Lemma 4.8 and Proposition 4.9.

In particular, $e(X_{\text{reg}})$ divides 2 (in fact, it equals 2). As mentioned in [HN16, top of p. 46], this implies that the stabilization index $e(X)$ divides 2. Since $[L : K] = 4$, this completes the example.

### 4.3. A class of examples ($g = 2p$, any $p$).

Let $k$ be an algebraically closed field of characteristic $p$. We fix a $p^2$-th root of unity $\zeta$ in $\text{Frac}(W(k))$, we let $K = \text{Frac}(W(k))(\zeta^p)$, and we fix a uniformizer $\pi_K$ of $K$. Moreover, we choose an element $s \in K$ such that $|s| = 1$ and $|s - 1| = 1$ and we set $\alpha = s + \sqrt[4]{\pi_K}$. We consider the finite Galois extension $L = K(\zeta, \alpha)$ of $K$. Since $L$ is obtained from $K$ by adjoining $p$-th roots, we can use Kummer theory to establish that the Galois group $\text{Gal}(L/K)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, generated by automorphisms $\sigma$ and $\tau$ such that

\[
\begin{align*}
\sigma(\zeta) &= \zeta^{p+1}, & \tau(\zeta) &= \zeta, \\
\sigma(\alpha) &= \alpha, & \tau(\alpha) &= s + \sqrt[p]{\pi_K}.
\end{align*}
\]

In the rest of the section, we construct a curve $X$ over $K$ with potentially multiplicative reduction, a $K$-rational point, stabilization index $e(X) = p$, and such that $L/K$ (of degree $p^2$) is the minimal extension over which $X$ acquires semi-stable reduction. This completes the proof of Theorem 1.2.

Let $\beta, \beta' \in K$ be such that $|\beta|, |\beta'| \leq |\pi_K|^4$. For $i = 0, \ldots, p - 1$ we define $A_i$ to be the following element of $PGL_2(L)$:

\[
A_i := \begin{bmatrix}
(1 - \beta \pi_K) \cdot \sigma^i(\zeta) & (\beta - 1) \pi_K \cdot \sigma^i(\zeta^2) \\
1 - \beta & (\beta - \pi_K) \cdot \sigma^i(\zeta)
\end{bmatrix}.
\]

Similarly, we define $B_i$ ($i = 0, \ldots, p - 1$) to be the following elements of $PGL_2(L)$:

\[
B_i := \begin{bmatrix}
(1 - \beta' \pi_K) \cdot \tau^i(\alpha) & (\beta' - 1) \pi_K \cdot \tau^i(\alpha^2) \\
1 - \beta' & (\beta' - \pi_K) \cdot \tau^i(\alpha)
\end{bmatrix}.
\]
Finally, we set
\[ D^+(A_i) = \overline{B} (\sigma^i(\zeta), |\pi K|^{1+\frac{1}{p}}), \]
and
\[ D^+(B_i) = \overline{B} (\tau^i(\alpha), |\pi K|^{1+\frac{1}{p}}), \]
Proposition 4.11. The subgroup $\Gamma$ of $\text{PGL}_2(L)$ generated by $A_0, \ldots, A_{p-1}, B_0, \ldots, B_{p-1}$ is a Schottky group of rank $2p$ satisfying all the conditions of Proposition 3.1. As a result, the Mumford curve uniformized by $\Gamma$ is defined over $K$.

Proof. To prove that $\Gamma$ is a Schottky group, let us show that the $4p$-uple
\[ (D^+(A_i), D^+(B_i), D^+(A_i^{-1}), D^+(B_i^{-1}))_{i=0, \ldots, p-1} \]
is a Schottky figure associated to $\Gamma$ and adapted to the corresponding set of generators. First of all, we have that $|\alpha - \tau(\alpha)| = |\pi K|^{\frac{1}{p}}|1 - \zeta^p| = |\pi K|^{\frac{1}{p}+1}$, and similarly that $|\sigma(\zeta) - \zeta| = |\zeta||1 - \zeta^p| = |\pi K|$, so these discs are disjoint. Moreover, we can explicitly compute the image of these discs under the associated loxodromic transformation. For example, by writing $A_0 = PAP^{-1}$ with $P = \begin{bmatrix} -\pi K\zeta & \zeta \\ -1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix}$, we see that the complement of a disc of the form $\overline{B}(\pi K\zeta, |\beta|\rho)$ with $|\beta|\rho < 1$ and $\rho > 1$ is transformed by $A_0$ as follows:
\[ \mathbb{P}^1 \setminus \overline{B}(\pi K\zeta, |\beta|\rho) \xrightarrow{P^{-1}} B(0, |\beta|^{-1}\rho^{-1}) \xrightarrow{A} B(0, \rho^{-1}) \xrightarrow{P} B(\zeta, \rho^{-1}). \]
In particular, we have that $A_0(\mathbb{P}^1 \setminus D^+(A_0^{-1})) = D^-(A_0).$ In the same way, by replacing $\zeta$ with $\alpha$, and $\beta$ with $\beta'$, we find that $B_0(\mathbb{P}^1 \setminus D^+(B_0^{-1})) = D^-(B_0).$ By repeating this computation for the other discs above, we show that they all satisfy the equation (2.7), and hence they define a Schottky figure adapted to $(A_0, \ldots, A_{p-1}, B_0, \ldots, B_{p-1})$.

To prove that this group is fixed under the action of $\text{Gal}(L/K)$ we note that $\sigma(A_i) = A_{i+1}$ for $i = 0, \ldots, p - 2, \sigma(A_{p-1}) = A_0$ and $\tau(A_i) = A_i$ for $i = 0, \ldots, p - 1$, and a similar relation holds for the $B_i$'s. Hence, $\Gamma$ satisfies condition (c) of Proposition 3.1, which implies conditions (b) and (a) of that proposition. Condition (a) is what we seek. \hfill \Box

With the Schottky figure found in the proof of Proposition 4.11 we can associate a fundamental domain for the action of $\Gamma$:
\[ F = \mathbb{P}_L^{1,\text{an}} \setminus \bigcup_{i=0}^{p-1} (D^-(A_i) \cup D^-(B_i) \cup D^+(A_i^{-1}) \cup D^+(B_i^{-1})) , \]
whose skeleton is depicted in Figure 4. The vertices $v_i$ in that figure can be described very concretely as sup-norms of closed discs that we denoted by $\eta_{\alpha, \rho}$ in Definition 2.4(c). Namely, if we set $|\pi K| = r$ we have
\[ v_1 = \eta_{0,1} \quad v_2 = \eta_{0,r} \quad v_3 = \eta_{\zeta, r} \quad v_4 = \eta_{\pi K\zeta, r^2} \quad v_5 = \eta_{\alpha, r^{1+\frac{1}{p}}} \quad v_6 = \eta_{\pi K\alpha, r^{2+\frac{1}{p}}}. \]
The fundamental domain $F$ has $4p$ boundary points, one for each disc that is a connected component of the complement $\mathbb{P}_L^{1,\text{an}} \setminus F$. In the figure, we have labeled these boundary points with the corresponding discs.

Let us call $Y$ the Mumford curve uniformized by $\Gamma$. By the second part of Theorem 2.10, the skeleton $\Sigma_Y$ of $Y^{\text{an}}$ is obtained by pairwise identifying the ends of the skeleton of of the fundamental domain $F$, as depicted in Figure 5. Since $Y$ has stable reduction, we have that
Figure 4. The skeleton $\Sigma_F$ of the fundamental domain $F$

$\Sigma_Y$ is also the dual graph of the special fiber $\mathcal{Y}_k^{st}$ of the stable model $\mathcal{Y}^{st}$ of $Y$. We deduce that $\mathcal{Y}_k^{st}$ consists of six irreducible components. For every vertex $v_i$ of such a graph, we denote by $V_i$ the corresponding irreducible component of $\mathcal{Y}_k^{st}$.

Figure 5. The dual graph of $\mathcal{Y}_k^{st}$

Since the points $v_i$ are fixed by the action of $\text{Gal}(L/K)$ on $\mathbb{P}^{1,\text{an}}_L$, the irreducible components $V_i$ are also fixed by the action of $\text{Gal}(L/K)$ on $\mathcal{Y}_k^{st}$. This action cyclically permutes the double points of the intersections $V_3 \cap V_4$ and $V_5 \cap V_6$.

**Remark 4.12.** Proposition 4.11 shows that there is a $K$-curve $X$ whose base change to $L$ is isomorphic to $Y$. By construction, the $K$-form $X$ of $Y$ is arithmetic in the sense of §3. The fundamental domain $F$ is especially nice, since $g(F) = F$ for every $g \in \text{Gal}(L/K)$.

**Lemma 4.13.** The curve $X$ has a $K$-rational point, and the extension $L/K$ is the minimal one over which $X$ acquires semi-stable reduction.
In this section, we adopt notation analogous to that of §4. Specifically, let $X$ be a curve over $K$ of genus $\geq 1$ with potentially multiplicative reduction, realized over a minimal Galois extension $L/K$. Write $Y = X \times_K L$. Then $Y$ is a Mumford curve with analytification $\Gamma \setminus \mathcal{O}$, where $\Gamma$ is the associated Schottky group and $\mathcal{O} = \mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L}$ with $\mathcal{L}$ the limit set of $\Gamma$. Let $\mathcal{Y}$ be model of $Y$ associated to the full set of multiplicity 1 points in the skeleton of $\mathcal{O}$. Then, we define $\mathcal{Y} \to \mathcal{X}$ to be the blow down obtained by contracting the components with minimal field of definition equal to $L$. From the labeling above, we deduce that none of the components $V_i$ for $i = 1, \ldots, 6$ gets contracted in this way, and hence $\mathcal{Y}$ is an admissible blow-up of $\mathcal{Y}^\text{st}$, and in particular it is again a semi-stable model of $Y$. As in §4.1, we set $\mathcal{X} = \mathcal{Y}/\text{Gal}(L/K)$.

Lemma 4.14 (cf. Lemma 4.8). All components of $\mathcal{X}_k$ have multiplicity dividing $p$.

Proof. This follows from Lemma 3.9, as $\mathcal{Y}$ contains only components whose minimal field of definition is strictly contained in $L$. □

Proposition 4.15 (cf. Proposition 4.9). Any principal component $E$ of $\mathcal{X}_k^\text{reg}$ that is exceptional in the desingularization of the model $\mathcal{X}$ has multiplicity dividing $p$.

Proof. Let $x \in \mathcal{X}$ be the image of $E$ under the desingularization $\mathcal{X}^\text{reg} \to \mathcal{X}$. Since the fundamental domain $F$ is fixed by the action of $\text{Gal}(L/K)$ on $\mathbb{P}^1_{L,\text{an}}$, if $\tilde{y} \in \mathcal{D}^\text{tr}$ lies above $x$ as in Lemma 4.1, then $\text{Gal}(L/K)\tilde{y}$ lies in the reduction $\overline{F}$ of $F$. So by Proposition 4.3 and Lemma 4.14, the multiplicity of $E$ divides $p$. □

Corollary 4.16 (cf. Corollary 4.10). Every principal component of $\mathcal{X}_k^\text{reg}$ has multiplicity dividing $p$.

Proof. This is a direct consequence of Lemma 4.14 and Proposition 4.15. □

5. Divisibility of the stabilization index

In this section, we adopt notation analogous to that of §4. Specifically, let $X$ be a curve over $K$ of genus $\geq 1$ with potentially multiplicative reduction, realized over a minimal Galois extension $L/K$. Write $Y = X \times_K L$. Then $Y$ is a Mumford curve with analytification $\Gamma \setminus \mathcal{O}$, where $\Gamma$ is the associated Schottky group and $\mathcal{O} = \mathbb{P}^1_{L,\text{an}} \setminus \mathcal{L}$ with $\mathcal{L}$ the limit set of $\Gamma$. Let $\mathcal{Y}$ be model of $Y$ associated to the full set of multiplicity 1 points in the skeleton of $\mathcal{O}$, as in Proposition 2.13. The morphism $\varpi: \mathcal{D} \to \mathcal{Y}$ is the morphism of formal schemes coming from the uniformization map $\mathcal{O} \to Y_{\text{ar}}$ as in Lemma 2.15. Note that the $L/K$-semilinear action of $\text{Gal}(L/K)$ on $Y$ extends to $\mathcal{Y}$, because $\mathcal{Y}$ is defined canonically in terms of $Y$. If $g \in \text{Gal}(L/K)$, write $\sigma_g$ for the corresponding automorphism of $\mathcal{Y}$.

Now, let $\mathcal{X} = \mathcal{Y}/\text{Gal}(L/K)$, write $\mathcal{X}_k$ for its special fiber, and let $\mathcal{X}^\text{reg} \to \mathcal{X}$ be its minimal regular snc-resolution. The special fiber of $\mathcal{X}^\text{reg}$ is denoted $\mathcal{X}_k^\text{reg}$.
Define $e(\mathcal{X}^{\text{reg}})$ to be the lcm of the multiplicities of the principal components of $\mathcal{X}^{\text{reg}}$. By [HN16, top of p. 46], $e(X) \mid e(\mathcal{X}^{\text{reg}})$. The goal of this section is to prove Theorem 1.3, by showing that $e(\mathcal{X}^{\text{reg}}) \mid [L : K]$, which implies $e(X) \mid [L : K]$. We start with some preparatory results.

**Lemma 5.1.** Every $L/K$-semilinear automorphism of $Y$ lifts to an $L/K$-semilinear automorphism of $\mathcal{D}$, and any such lift in turn extends to a unique $L/K$-semilinear automorphism on both $\mathbb{P}^1_L$ and on $\mathcal{D}'$.

**Proof.** The first assertion follows from Proposition 2.19 and (2.21). The second assertion follows from Proposition 2.22 and the fact that $\mathcal{D}'$ is canonically constructed from $Y$. □

**Remark 5.2.** Compare Lemma 5.1 above with Remark 3.2, where the goal is to lift an entire $L/K$-semilinear group action, not just a single automorphism. This is not always possible, but we give a sufficient criterion for the lifting of such an action in the following proposition.

**Proposition 5.3.** Suppose there is a closed point $y$ on the special fiber of $\mathcal{Y}'$ that is preserved by the action of $\text{Gal}(L/K)$. Then the canonical $\text{Gal}(L/K)$-action on $\mathcal{Y}'$ lifts to an $L/K$-semilinear $\text{Gal}(L/K)$-action on $\mathcal{D}'$ as well as on $\mathbb{P}^1_L$. Furthermore, the lift can be chosen to preserve an arbitrary point of $\mathcal{D}'$ above $y$.

**Proof.** By Proposition 2.22, proving the first statement reduces to showing the $\text{Gal}(L/K)$-action on $\mathcal{Y}'$ lifts to $\mathcal{D}'$. Pick a point $\tilde{y}$ of $\mathcal{D}'$ lying above $y$. For every $\sigma \in \text{Gal}(L/K)$, choose a lift $\tilde{\sigma}$ of the action of $\sigma$ to $\mathcal{D}'$ as in Lemma 5.1. After composing $\tilde{\sigma}$ with an element of $\Gamma$, we may assume that $\tilde{\sigma}$ fixes $\tilde{y}$. Indeed, since $\Gamma$ acts freely on $\mathcal{D}'$, this property determines $\tilde{\sigma}$ uniquely.

Now, for all $\sigma, \tau \in \text{Gal}(L/K)$, we have $\tilde{\sigma} \tilde{\tau}$ preserves $\tilde{y}$. Since $\tilde{\sigma} \tilde{\tau}$ is a lift of $\sigma \tau$, the uniqueness of lifts preserving $\tilde{y}$ implies that $\tilde{\sigma} \tilde{\tau} = \tilde{\sigma} \tilde{\tau}$. Thus the entire $\text{Gal}(L/K)$-action lifts to $\mathcal{D}'$, proving the first statement. The point $\tilde{y}$ is fixed by the lift, proving the second statement. □

**Lemma 5.4.** Every $L/K$-semilinear action on $\mathbb{P}^1_L$ is, up to a change of coordinate, the purely arithmetic action given by fixing the coordinate and acting on the coefficients.

**Proof.** This is essentially the proof of [OW20, Corollary 4.11]; we reproduce it here and slightly correct it. Consider, equivalently, an $L/K$-semilinear action $\rho$ of $\text{Gal}(L/K)$ on the function field $L(x)$. It is represented by a cocycle in $H^1(\text{Gal}(L/K), PGL_2(L))$, where $\sigma \in \text{Gal}(L/K)$ is sent to $\alpha \in PGL_2(L)$ such that $\rho(\sigma)(x) = \alpha(x)$. By Hilbert’s Theorem 90 (see, e.g., [Ser79, X, Proposition 3]), this cohomology set injects into $H^2(G, L^*)$, which is trivial by [Ser97, Corollary and Example (c) on p. 80]. So the action is given by a coboundary, and thus has the form $\sigma(x) = xB^\sigma B^{-1}$ where $B \in PGL_2(L)$ is independent of $\sigma$. Letting $y = xB^{-1}$, we see that $g(y) = x(B^\sigma)^{-1}B^\sigma B^{-1} = y$ for all $\sigma \in \text{Gal}(L/K)$. This proves the lemma. □

Recall that we have the following diagram, where $p_1$ is the quotient map and $p_2$ is the resolution:

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{p_1} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X}^{\text{reg}} & \xrightarrow{p_2} & \mathcal{X} = \mathcal{Y}'/\text{Gal}(L/K)
\end{array}
$$

To prove Theorem 1.3, it suffices to show the following result:
Theorem 5.5. Let $x$ be a point on the special fiber of $\mathcal{X}$. Then every principal irreducible component of the special fiber $\mathcal{X}_k^{\text{reg}}$ of $\mathcal{X}^{\text{reg}}$ meeting $p_2^{-1}(x)$ has multiplicity dividing $[L : K]$.

In other words, we can restrict attention to a formal neighborhood of $x \in \mathcal{X}$. For the rest of §5, fix such a point $x$.

Proposition 5.6. In order to prove Theorem 5.5, it suffices to assume that $p_1^{-1}(x)$ consists of a single point. That is, that $\text{Gal}(L/K)$ acts on $p_1^{-1}(x)$ with full inertia.

Proof. Pick a point $y \in p_1^{-1}(x)$, let $H \subseteq \text{Gal}(L/K)$ be the subgroup fixing $y$, and let $K' = L^H$.

Let $p_1'$ be the quotient morphism $\mathcal{Y}' \to \mathcal{Y}'/H =: \mathcal{Z}$, let $z = p_1'(y)$, and let $Z = X \times_K K'$. Then $\mathcal{Z}$ is a model of $Z$ and the morphism $p_1$ factors as

$$\mathcal{Y}' \xrightarrow{p_1'} \mathcal{Z} \xrightarrow{p_1^Z} \mathcal{X}.$$ 

Since $(p_1')^{-1}(z)$ equals the singleton \{y\} by construction, we may assume that Theorem 5.5 holds for $(\mathcal{Z}, z)$. We must prove Theorem 5.5 for $(\mathcal{X}, x)$.

By assumption, the morphism $p_1''$ is étale at $z$. By étale base change, the map $\mathcal{Z}^{\text{reg}} := \mathcal{X}^{\text{reg}} \times_{\mathcal{X}} \mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$ induced from $p_2$ gives a resolution of the singularity at $z$, and $\mathcal{Z}^{\text{reg}} \to \mathcal{X}^{\text{reg}}$ is étale along $r^{-1}(z)$. Suppose $U$ is a principal irreducible component of $\mathcal{X}_k^{\text{reg}}$ meeting $p_2^{-1}(x)$, and $V$ is an irreducible component of $r^{-1}(z)$ lying above $U$. Note that $V$ is principal as well. Since the restriction $V \to U$ of $\mathcal{Z}^{\text{reg}} \to \mathcal{X}^{\text{reg}}$ is generically étale, the multiplicity $m_U$ (which is calculated with respect to the uniformizer $\pi_K$) is $[K' : K]$ times the multiplicity $m_V$ (which is calculated with respect to the uniformizer $\pi_{K'}$). By Theorem 5.5 applied to $(\mathcal{Z}, z)$, we have $m_V \mid [L : K']$. Thus $m_U$ divides $[L : K'][K' : K] = [L : K]$.

Proof of Theorem 5.5. After applying Proposition 5.6, Lemma 5.3 applies, taking $y$ to be single point $p_1^{-1}(x)$. So the action of $\text{Gal}(L/K)$ on $\mathcal{Y}$ lifts to an action on $\mathcal{D}'$ and on $\mathbb{P}^1_L$. By Lemma 5.4, a coordinate on $\mathbb{P}^1_L$ can be chosen so that $\text{Gal}(L/K)$ fixes this coordinate.

By Lemma 5.3, we may assume there is a closed point $\tilde{y} \in \mathcal{D}'$ above $y$ that is fixed by the lift of the $\text{Gal}(L/K)$-action. Let $\Sigma$ be the set of irreducible components of $\mathcal{D}'$ passing through $\tilde{y}$, and let $\mathcal{D}_\Sigma$ and $\varphi$ be the model of $\mathbb{P}^1_L$ associated to $\Sigma$ and the injection on closed points as in Remark 2.18. The same remark gives us that $\hat{\mathcal{O}}_{\mathcal{D}', \tilde{y}} \cong \hat{\mathcal{O}}_{\mathcal{D}_\Sigma, \varphi(\tilde{y})}$.

Furthermore, since $\text{Gal}(L/K)$ fixes $\tilde{y}$, it also acts on $\mathcal{D}_\Sigma$, and the Galois action on $\hat{\mathcal{O}}_{\mathcal{D}, \tilde{y}}$ is isomorphic to that on $\hat{\mathcal{O}}_{\mathcal{D}_\Sigma, \varphi(\tilde{y})}$. Applying Corollary 2.28 to $\mathcal{D}_\Sigma$, all principal components of the minimal regular resolution of the singularity of $\mathcal{D}_\Sigma/\text{Gal}(L/K)$ at the image of $\varphi(\tilde{y})$ under the quotient have multiplicity dividing $[L : K]$. Thus the same is true for $\mathcal{D}'/\text{Gal}(L/K)$ at the image of $\tilde{y}$, and thus in turn for $\mathcal{Y}'/(\text{Gal}(L/K))$ at $x$. This proves the proposition. □
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