Weak solutions for a system of quasilinear elliptic equations

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(Received: 5 June 2020. Accepted: 1 August 2020. Published online: 4 August 2020.)

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Abstract

A system of quasilinear elliptic equations on an unbounded domain is considered. The existence of a sequence of radially symmetric weak solutions is proved via variational methods.

Keywords: sequence of solutions, elliptic problem, \(p\)-Laplacian, variational methods.

2020 Mathematics Subject Classification: 34B10, 35J20, 35J50.

1. Introduction

We consider the following problem

\[
\begin{cases}
-\Delta_p u + |u|^{p-2}u = \lambda \alpha_1(x)f_1(v) & \text{in } \mathbb{R}^N, \\
-\Delta_q v + |v|^{q-2}v = \lambda \alpha_2(x)f_2(u) & \text{in } \mathbb{R}^N, \\
u, v \in W^{1,p}(\mathbb{R}^N),
\end{cases}
\]

(1)

where \(p, q > N > 1\). We assume that \(f_1, f_2 : \mathbb{R} \to \mathbb{R}\) are continuous functions, \(\alpha_1(x), \alpha_2(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) are nonnegative (not identically zero) radially symmetric maps, and \(\lambda\) is a real parameter. Also \(\Delta_p u := \text{div}(\nabla u|^{p-2}\nabla u)\) denotes the \(p\)-Laplacian operator.

Partial differential equations are used to model a wide variety of physically significant problems arising in different areas such as physics, engineering and other applied disciplines (see [7, 11, 12, 18, 21–24, 48]). Sobolev spaces play an important role in the theory of partial differential equations as well as Orlicz-Morrey space and \(\dot{B}^{-1,1}_\infty\) space (see [2, 8–10, 32–34]). Laplace equation is the prototype for linear elliptic equations. This equation has a non-linear counterpart, the so-called \(p\)-Laplace equation (see [1, 6, 13, 14, 19, 21–24, 48]).

Here, by inspiration of [20], we prove the existence of a sequence of radially symmetric weak solutions for (1) in the unbounded domain \(\mathbb{R}^N\).

The solution of (1) belongs to the product space

\[W^{1,(p,q)}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N)\]

equipped with the norm \(\|(u,v)\|_{(p,q)} = \|u\|_p + \|v\|_q\).

Definition 1.1. For fixed \(\lambda\), \((u,v) : \mathbb{R}^N \to \mathbb{R}\) is said to be a weak solution of (1), if \((u,v) \in W^{1,(p,q)}(\mathbb{R}^N)\) and for every \((z,w) \in W^{1,(p,q)}(\mathbb{R}^N)\)

\[-\int_{\mathbb{R}^N} \nabla u(x)|^{p-2}\nabla u(x)\cdot \nabla z(x)dx - \int_{\mathbb{R}^N} |\nabla u(x)|^{q-2}\nabla u(x)\cdot \nabla w(x)dx + \int_{\mathbb{R}^N} |u(x)|^{p-2}u(x)z(x)dx + \int_{\mathbb{R}^N} |v(x)|^{q-2}v(x)w(x)dx
- \lambda \int_{\mathbb{R}^N} \alpha_1(x)f_1(v(x))z(x)dx - \lambda \int_{\mathbb{R}^N} \alpha_2(x)f_2(u(x))w(x)dx = 0,\]

where

\[\|(u,v)\|_{W^{1,(p,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |\nabla v(x)|^q dx + \int_{\mathbb{R}^N} |v(x)|^q dx\right)^{\frac{1}{q}}.\]

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Note that the critical points of a energy functional are exactly the weak solutions of (1).

Morrey’s theorem, implies the continuous embedding

\[ W^{1,(p,q)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N), \]

which says that there exists \( c \) (depends on \( p, q, N \)), such that \( \| (u,v) \|_\infty \leq c \| (u,v) \|_{W^{1,(p,q)}(\mathbb{R}^N)} \), for every \( (u,v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N) \), where \( \| (u,v) \|_\infty := \max\{\|u\|_\infty, \|v\|_\infty\} \). Since in the low-dimensional case, every function \( (u,v) \in W^{1,(p,q)}(\mathbb{R}^N) \) admits a continuous representation (see [4, p.166]). In the sequel we will replace \( (u,v) \) by this element.

We need the following notations (see [5] or [17] for more details):

1. \( O(N) \) stands for the orthogonal group of \( \mathbb{R}^N \).
2. \( B(0, s) \) denotes the open \( N \)-dimensional ball of center zero, radius \( s > 0 \) and standard Lebesgue measure, \( \text{meas}(B(0, s)) \).
3. \( \| \alpha \|_{B(0,\frac{s}{2})} := \int_{B(0,\frac{s}{2})} \alpha(x) dx \).

**Definition 1.2.**

- A function \( h : \mathbb{R}^N \to \mathbb{R} \) is radially symmetric if \( h(gx) = h(x) \), for every \( g \in O(N) \) and \( x \in \mathbb{R}^N \).
- Let \( G \) be a topological group. A continuous map \( \xi : G \times X \to X : (g,x) \to \xi(g,u) := gu \), is called the action of \( G \) on the Banach space \( (X,\|\cdot\|_X) \) if
  \[ 1u = u, \quad (gm)u = g(mu), \quad u \mapsto gu \text{ is linear}. \]
- The action is said to be isometric if \( \|gu\|_X = \|u\|_X \), for every \( g \in G \).
- The space of \( G \)-invariant points is defined by
  \[ \text{Fix}(G) := \{ u \in X : gu = u, \text{for all } g \in G \}. \]
- A map \( m : X \to \mathbb{R} \) is said to be \( G \)-invariant if \( m(gu) = m \) for every \( g \in G \).

The following theorem is important to study the critical point of the functional.

**Theorem 1.1.** [27] Assume that the action of the topological group \( G \) on the Banach space \( X \) is isometric. If \( J \in C^1(X : \mathbb{R}) \) is \( G \)-invariant and if \( u \) is a critical point of \( J \) restricted to \( \text{Fix}(G) \), then \( u \) is a critical point of \( J \).

The action of the group \( O(N) \) on \( W^{1,p}(\mathbb{R}^N) \) can be defined by \( (gu)(x) := u(g^{-1}x) \), for every \( g \in W^{1,p}(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \). A computation shows that this group acts linearly and isometrically, which means \( \|u\| = \|gu\| \), for every \( g \in O(N) \) and \( u \in W^{1,p}(\mathbb{R}^N) \).

**Definition 1.3.** The subspace of radially symmetric functions of \( W^{1,(p,q)}(\mathbb{R}^N) \) is defined by

\[ X := W^{1,(p,q)}(\mathbb{R}^N) \]
\[ := \{ (u,v) \in W^{1,(p,q)}(\mathbb{R}^N) : (g_1u,g_2v) = (u,v), \text{ for all } (g_1, g_2) \in O(N) \times O(N) \}, \]

and endowed by the norm

\[ \| (u,v) \|_{W^{1,(p,q)}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^p dx \right)^{1/p} + \left( \int_{\mathbb{R}^N} |\nabla v(x)|^q dx + \int_{\mathbb{R}^N} |v(x)|^q dx \right)^{1/q}. \]

In what follows: \( \| (u,v) \| \) denotes \( \| (u,v) \|_{W^{1,(p,q)}(\mathbb{R}^N)} \).

The following crucial embedding result due to Kristály and principally based on a Strauss-type estimation (see [46]) (Also see [15, Theorem 3.1], and [16, 47] for related subjects).

**Theorem 1.2.** The embedding \( W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \), is compact whenever \( 2 \leq N < p < +\infty \).

Here we consider the following functionals:

- \( F_i(\xi) := \int_0^\xi f_i(t) dt \) for every \( \xi \in \mathbb{R} \).
- \( \Phi(u,v) := \frac{\|u\|_p^p}{p} + \frac{\|v\|_q^q}{q} \) for every \((u,v) \in X\).
- \( \Psi(u,v) := \int_{\mathbb{R}^N} \alpha_1 F_1(u(x)) dx + \int_{\mathbb{R}^N} \alpha_2 F_2(u(x)) dx \), for every \((u,v) \in X\).
By standard arguments [5], we can show that $\Phi$ is Gâteaux differentiable, coercive and sequentially lower semicontinuous whose Gâteaux derivative at the point $(u, v) \in X$ is the functional $\Phi'(u, v) \in X^*$ given by

$$
\Phi'(u, v)(z, w) = \left( \int_{\mathbb{R}^N} |\nabla u|^p - 2 \nabla u . \nabla wdx + \int_{\mathbb{R}^N} |u|^p - 2 u wdx \right) + \left( \int_{\mathbb{R}^N} |\nabla v|^q - 2 \nabla v . \nabla wdx + \int_{\mathbb{R}^N} |v|^q - 2 v wdx \right),
$$

for every $(z, w) \in X$. Also standard arguments show that the functional $\Psi$, are well defined, sequentially weakly upper semicontinuous and Gâteaux differentiable whose Gâteaux derivative at the point $(u, v) \in X$ is given by,

$$
\Psi'(u, v)(z, w) = \int_{\mathbb{R}^N} \alpha_1(x)f_1(u(x))dx + \int_{\mathbb{R}^N} \alpha_2(x)f_2(v(x))dx.
$$

## 2. Weak solutions

First we recall the following theorem [3, Theorem 2.1].

**Theorem 2.1.** Let $X$ be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, set

$$
\varphi(r) := \inf_{\Phi(u) < r} \sup_{\Psi(v) < r} \frac{\Psi(v) - \Phi(u)}{r - \Phi(u)}, \quad \gamma := \liminf_{r \to +\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).
$$

Then the following properties hold:

(a) for every $r > \inf_X \Phi$ and every $\lambda \in [0, \frac{1}{\delta}]$, the restriction of the functional $I_\lambda := \Phi - \lambda \Psi$ to $\Phi^{-1}([-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of $I_\lambda$ in $X$.

(b) if $\gamma < +\infty$, then for each $\lambda \in [0, \frac{1}{\delta}]$, the following alternative holds either:

(b1) $I_\lambda$ possesses a global minimum, or

(b2) there is a sequence $\{u_n\}$ of critical points (local minima) of $I_\lambda$ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.

(c) if $\delta < +\infty$, then for each $\lambda \in [0, \frac{1}{\delta}]$, the following alternative holds either:

(c1) there is a global minimum of $\Phi$ which is a local minimum of $I_\lambda$, or

(c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $I_\lambda$ which weakly converges to a global minimum of $\Phi$, with $\lim_{n \to +\infty} \Phi(u_n) = \inf_{u \in X} \Phi(u)$.

For fixed $D > 0$, set

$$
m(D) := \text{meas}(B(0, D)) = D^N \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})},
$$

where $\Gamma$ is the Gamma function defined by $\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} dz$ for all $t > 0$. Moreover,

$$
\Omega := \max \left\{ \frac{m(D)}{p\lambda B_1}\|\alpha_2\|_{B(0, \frac{\pi}{2})}, \frac{m(D)}{q\lambda B_2}\|\alpha_1\|_{B(0, \frac{\pi}{2})} \right\} > 0,
$$

where $\sigma(N, p) := 2^{p-N}(2^N - 1)$, $c = \frac{2N}{2^{2-N}}$, $m_1, m_0$ are upper and lower bounds for $M(t)$ in (1) and

$$
g(p, N) := \frac{1 + 2^{N+p}N \int_0^1 t^{N-1}(1-t)^p dt}{2^N}.
$$

Assume $\| \cdot \|$ denotes the norm of $L^1(\Omega)$ and $F(\xi) := F_1(\xi) + F_2(\xi)$.

**Theorem 2.2.** Let $f_1 : \mathbb{R} \to \mathbb{R}$ be two continuous and radially symmetric functions. Set

$$
A := \liminf_{\xi_1, \xi_2 \to +\infty} \frac{\max_{|\xi_1| \leq \xi_1} F_1(t_1)}{|\xi_1|^p} + \frac{\max_{|\xi_2| \leq \xi_2} F_1(t_2)}{|\xi_2|^q}, \quad B_1 := \limsup_{\xi_2 \to +\infty} \frac{F_1(\xi_2)}{|\xi_2|^p}, \quad \text{and} \quad B_2 := \limsup_{\xi_1 \to +\infty} \frac{F_2(\xi_1)}{|\xi_1|^q},
$$

where $B := B_1 + B_2$, $\xi = (\xi_1, \xi_2)$. If $\inf_{(\xi_1, \xi_2) \geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ and $A < \Omega m_0 B$, where $\Omega$ is given by (3), for every

$$
\lambda \in \Lambda := \left[ \Omega, \frac{1}{\|\alpha_2\|_{L^1(\Omega)} + q \|\alpha_1\|_{L^1(\Omega)}} A \right],
$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) in $X$. 

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Proof. For fixed $\lambda \in \Lambda$, we consider $\Phi, \Psi$ and $I_\lambda$ as in the last section. Knowing that $\Phi$ and $\Psi$ satisfy the regularity assumptions in Theorem 2.1. In order to study the critical points of $I_\lambda$ in $X$, we show that $\lambda < \frac{1}{\gamma} < +\infty$, where $\gamma = \liminf_{r \to +\infty} \phi(r)$. Let $\{t_n\}$ be a sequence of positive numbers such that $\lim_{n \to \infty} t_n = +\infty$, $r_1 := \frac{r^\Phi}{\rho^\Phi}$ and $r_2 := \frac{r^\Psi}{\rho^\Psi}$, for all $n \in \mathbb{N}$. Set $r_n := \min\{r_1, r_2\}$. Considering Theorem 1.2 (by relation (2)), a computation shows that

$$
\Phi^{-1}([\infty, r_n]) = \{(z, w) \in X : \Phi(z, w) < r_n\} = \{(z, w) \in X : \frac{||z||_p}{p} + \frac{||w||_q}{q} < r_n\} \subset \{(z, w) \in X ; ||(z, w)||_\infty < t_n\},
$$

where $t_n = \min\{t_1, t_2\}$.

Since $\Phi(0,0) = \Psi(0,0) = 0$, by a computation one can show

$$
\varphi(r_n) = \inf_{\Phi(u,v) < r_n} \left( \frac{\sup_{\Phi(z,w) < r_n} \Psi(z,w) - \Psi(u,v)}{r_n - \Phi(u,v)} \right) \leq (p \rho^\Phi ||\alpha_2||_1 + q \rho^\Psi ||\alpha_1||_1) A.
$$

Hence

$$
\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq (p \rho^\Phi ||\alpha_2||_1 + q \rho^\Psi ||\alpha_1||_1) A < +\infty.
$$

Now, we show that $I_\lambda$ is unbounded from below. Let $\{d_{1n}\}$ and $\{d_{2n}\}$ be two sequences of positive numbers such that $\lim_{n \to +\infty} d_{1n} = \lim_{n \to +\infty} d_{2n} = +\infty$ and

$$
B_1 = \lim_{n \to +\infty} F_1(d_{2n}) d_{2n}^p, \quad B_2 = \lim_{n \to +\infty} F_2(d_{1n}) d_{1n}^p.
$$

Define $\{(H_{1n}, H_{2n})\} \subset X$ by

$$
H_{1n}(x) := \begin{cases}
0 & \mathbb{R}^N \setminus B(0, D) \\
d_{1n} & B(0, \frac{D}{2}) \\
2d_{1n} (D - |x|) & B(0, D) \setminus B(0, \frac{D}{2}),
\end{cases}
$$

for every $n \in \mathbb{N}$ and $i = 1, 2$. By a similar argument and computations in [5, P.1017] one can show that

$$
||H_{2n}||_p = d_{2n}^p m(D) \left( \frac{\sigma(N, p)}{D^p} + g(q, N) \right) \quad \text{and} \quad ||H_{1n}||_p = d_{1n}^p m(D) \left( \frac{\sigma(N, p)}{D^p} + g(p, N) \right).
$$

Condition (i), implies

$$
\int_{\mathbb{R}^N} \alpha_1(x) F_1(H_{2n}(x)) dx \geq \int_{B(0, \frac{D}{2})} \alpha_1(x) F_1(d_{2n}) dx = F_1(d_{2n}) ||\alpha_1||_{B(0, \frac{D}{2})}, \quad \text{and}
$$

$$
\int_{\mathbb{R}^N} \alpha_2(x) F_2(H_{1n}(x)) dx \geq \int_{B(0, \frac{D}{2})} \alpha_2(x) F_2(d_{1n}) dx = F_2(d_{1n}) ||\alpha_2||_{B(0, \frac{D}{2})},
$$

for every $n \in \mathbb{N}$. Then

$$
I_\lambda(H_{1n}, H_{2n}) = \Phi(H_{1n}, H_{2n}) - \lambda \Psi(H_{1n}, H_{2n})
$$

$$
= ||H_{1n}||_p^p \frac{p}{p} + \frac{||H_{2n}||_q^q}{q} - \lambda \int_{\mathbb{R}^N} \alpha_1(x) F_1(H_{2n}(x)) dx - \lambda \int_{\mathbb{R}^N} \alpha_2(x) F_2(H_{1n}(x)) dx
$$

$$
\leq \frac{d_{1n}^p m(D) \left( \frac{\sigma(N, p)}{D^p} + g(p, N) \right)}{p} + \frac{d_{2n}^q m(D) \left( \frac{\sigma(N, q)}{D^q} + g(q, N) \right)}{q}
$$

$$
- \lambda \left( F_1(d_{2n}) ||\alpha_1||_{B(0, \frac{D}{2})} + F_2(d_{1n}) ||\alpha_2||_{B(0, \frac{D}{2})} \right).
$$

If $B < +\infty (B_1, B_2 < +\infty)$, the conditions (5) implies that there exists $N_1$ such that for all $n \geq N_1$ we have $F_1(d_{2n}) > c B_1 d_{2n}^p$, and there exists $N_2$ such that for all $n \geq N_2$ we have $F_2(d_{1n}) > c B_2 d_{1n}^q$.

Then for every $n \geq N_\varepsilon := \max\{N_1, N_2\}$,

$$
I_\lambda(H_{1n}, H_{2n}) \leq \frac{d_{1n}^p m(D) \left( \frac{\sigma(N, p)}{D^p} + g(p, N) \right)}{p} + \frac{d_{2n}^q m(D) \left( \frac{\sigma(N, q)}{D^q} + g(q, N) \right)}{q} - \lambda \varepsilon \left( d_{1n}^p B_1 ||\alpha_1||_{B(0, \frac{D}{2})} + d_{2n}^q B_2 ||\alpha_2||_{B(0, \frac{D}{2})} \right).
$$
converges weakly to zero in $I$ of critical points of $W$ where $I$ one in the Theorem 2.2 implies $\lim_{n \to +\infty} I_A(H_{1n}, H_{2n}) = -\infty$. If at least one of the $B_1$ or $B_2$ are $+\infty$. Let $B_1 = +\infty$, and consider $M_1 > \Omega$, then by (5) there exists $N_{M_1}$ such that for every $n > N_{M_1}$, we have $F_1(d_{1n}) > M_1d_{1n}^p$. Moreover, for every $n > N_{M_1}$,

$$I_A(H_{1n}, H_{2n}) \leq d_{1n}^p m(D) \left( \frac{\sigma(N,p)}{p} + g(p, N) \right) + d_{2n}^p m(D) \left( \frac{\sigma(N,q)}{q} + g(q, N) \right) - \lambda \left( d_{1n}^p M_1 \| \alpha_2 \|_{B(0, \frac{1}{p^*})} + d_{2n}^p M_1 \| \alpha_1 \|_{B(0, \frac{1}{q})} \right).$$

This implies that $\lim_{n \to +\infty} I_A(H_{1n}, H_{2n}) = -\infty$.

Now, Theorem 2.1(b) implies, the functional $I_A$ admits an unbounded sequence $\{ u_n \} \subset X$ of critical points. Considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$-invariant functional $I_A : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which are unbounded in $W^{1,p}(\mathbb{R}^N)$.

Here we prove our second result which says that under different conditions the problem (1) has a sequence of weak solutions, which converges weakly to zero.

**Theorem 2.3.** Let $f_1 : \mathbb{R} \to \mathbb{R}$ be two continuous and radially symmetric functions. Set

$$A' := \liminf_{(\xi_1, \xi_2) \to 0^+} \frac{\max_{\xi_1 \leq \xi_2} F_2(t_1)}{\| \xi_1 \|^p}, \quad B' := \limsup_{\xi_2 \to 0^+} \frac{F_1(\xi_2)}{\| \xi_2 \|^p}, \quad B'_1 := \limsup_{\xi_2 \to 0^+} \frac{F_1(\xi_2)}{\| \xi_2 \|^p}, \quad \text{and} \quad B'_2 := \limsup_{\xi_1 \to 0^+} \frac{F_2(\xi_1)}{\| \xi_1 \|^p},$$

where $B' := B'_1 + B'_2$, $\xi = (\xi_1, \xi_2)$. If $\inf_{(\xi_1, \xi_2) \geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ and $A' < \Omega m_0 B'$, where $\Omega$ is given by (3), for every

$$\lambda \in \Lambda' := \left[ \Omega \left( \frac{\delta}{\delta + \| \alpha_2 \|_{1} + q \| \alpha_1 \|_{1}} \right) A' \right],$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) in $X$.

**Proof.** For fixed $\lambda \in \Lambda'$, we consider $\Phi$, $\Psi$ and $I_A$ as in Section 2. Knowing that $\Phi$ and $\Psi$ satisfy the regularity assumptions in Theorem 2.1, we show that $\lambda < \frac{1}{\delta}$. We know that $\inf X \Phi = 0$. Set $\delta := \liminf_{r \to 0^+} \varphi(r)$. A computation similar to the one in the Theorem 2.2 implies $\delta < +\infty$ and if $\lambda \in \Lambda'$ then $\lambda < \frac{1}{\delta}$. A computation (similar in the Theorem 2.2) shows that $I_A(H_{1n}, H_{2n}) < 0$ for $n$ large enough and thus zero is not a local minimum of $I_A$. Therefore, there exists a sequence $\{ u_n \} \subset X$ of critical points of $I_A$ which converges weakly to zero in $X$ as $\lim_{n \to +\infty} \Phi(u_n) = 0$.

Again, considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$-invariant functional $I_A : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which converges weakly to zero in $W^{1,p}(\mathbb{R}^N)$.

**Acknowledgment**

This work is supported by I.N.D.A.M - G.N.A.M.P.A. 2019 and the “RUDN University Program 5-100”.

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