A constructive proof of the convergence of Kalantari’s bound on polynomial zeros

Matt Hohertz
Department of Mathematics, Rutgers University
mrh163@math.rutgers.edu

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Abstract

In his 2006 paper, Jin proves that Kalantari’s bounds on polynomial zeros, indexed by \( m \geq 2 \) and called \( L_m \) and \( U_m \) respectively, become sharp as \( m \to \infty \). That is, given a degree \( n \) polynomial \( p(z) \) not vanishing at the origin and an error tolerance \( \varepsilon > 0 \), Jin proves that there exists an \( m \) such that

\[
\frac{L_m}{\rho_{\text{min}}} \geq 1 - \varepsilon,
\]

where \( \rho_{\text{min}} := \min_{\rho, p(\rho) = 0} |\rho| \). In this paper we derive a formula that yields such an \( m \), thereby constructively proving Jin’s theorem. In fact, we prove the stronger theorem that this convergence is uniform in a sense, its rate depending only on \( n \) and a few other parameters. We also give experimental results that suggest an optimal \( m \) of (asymptotically) \( O\left(\frac{1}{\varepsilon^d}\right) \) for some \( d \ll 2 \). A proof of these results would show that Jin’s method runs in \( O\left(\frac{n}{\varepsilon^d}\right) \) time, making it efficient for isolating high-degree polynomial zeros.
Keywords: polynomial roots, bounds on zeros, power series, polynomial reciprocals, analytic combinatorics, meromorphic functions

1 Introduction

We briefly review the bounds proved by Kalantari [2004, 2008] on the moduli of polynomial roots.

Let \( p(z) = a_n z^n + \cdots + a_1 z + a_0, a_n, a_0 \neq 0 \), be an arbitrary polynomial of degree \( n \) and \( \rho \) be an arbitrary zero of \( p(z) \). Moreover, let \( r_m \) be, as in Hohertz and Kalantari [2020], the unique positive root of \( p(x) := x^{n+1} + x - 1 \), a root that increases to 1 as \( m \to \infty \) and necessarily lies on the interval \([\frac{1}{2}, 1]\). Finally, let

\[
D_m(z) := \begin{vmatrix}
p'(z) & p''(z) & \cdots & p^{(n-1)}(z) & p^{(n)}(z) \\
p(z) & p'(z) & \cdots & p^{(n-1)}(z) & p^{(n)}(z) \\
0 & p(z) & \cdots & p^{(n-1)}(z) & p^{(n)}(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p(z) & p'(z)
\end{vmatrix}
\]  \quad (1)

and

\[
\hat{D}_{m,j}(z) := \begin{vmatrix}
p''(z) & p'''(z) & \cdots & p^{(j-1)}(z) \\
p'(z) & p''(z) & \cdots & p^{(j)}(z) \\
p(z) & p'(z) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p'(z)
\end{vmatrix}
\]  \quad (2)

Kalantari (see Kalantari [2004, 2008], and Jin [2006]) proves that

\[
|\rho| \geq \frac{r_m}{\gamma_m} := L_m
\]  \quad (3)

for \( m \geq 2 \), where

\[
\gamma_m := \max_{k=m+1, \ldots, m+n} \frac{\hat{D}_{m+1,k+1}(0)}{D_{m+1}(0)}
\]  \quad (4)

Using the equation

\[
(-1)^m \frac{\hat{D}_{m+1,k+1}(0)}{D_{m+1}(0)} = \sum_{j=\max\{0, k-n\}}^m a_{k-j} b_j,
\]  \quad (5)

where \( b_j \) is the coefficient of \( z^j \) in the Maclaurin series of \( \frac{1}{p(z)} \), Jin [2006] proves that \( L_m \) converges, as \( m \to \infty \), to \( \rho_{\min} := \min_{p(z), \rho(\rho) = 0} |\rho| \) and thus is asymptotically sharp. In effect, he proves the following theorem:

**Theorem 1 (equivalent to Theorem 3.1, Jin [2006])** For every polynomial \( p(z) \) and positive \( \epsilon \) close to zero, there exists a positive number \( m_\epsilon \) such that \( m \geq m_\epsilon \) implies

\[
\frac{L_m}{\rho_{\min}} \geq 1 - \epsilon.
\]  \quad (6)

\(^1\)Clearly, one obtains a dual upper bound \( U_m \) on \( |\rho| \) by applying the previous formulas to the reciprocal polynomial of \( p(z) \) and reciprocating both sides of the inequality of Equation (3).

\(^2\)resp., \( U_m \) to \( \rho_{\max} := \max_{p(z), \rho(\rho) = 0} |\rho| \).
Jin’s proof, however, does not provide an algorithm for finding \( m_\varepsilon \). In this paper we provide such an algorithm in the form of an equation, thus proving the following theorem:

\textbf{Theorem II} For every positive integer \( n \) and triple \((\varepsilon, \alpha, \beta)\) of positive reals with \( \varepsilon \) close to zero, there exists a positive number \( m_\varepsilon \) such that \( m \geq m_\varepsilon \) implies

\[ \frac{L_m}{\rho_{\min}} \geq 1 - \varepsilon \]  

(7)

for any polynomial \( p(z) \) with degree \( n \), \( \alpha := \max\{|a_0|, \cdots, |a_n|\} \), and \( \beta := \max\{1, \rho_{\min}^n \cdot \sum_{j=0}^{n-1} \rho_{\min}^j\} \).

Note that this theorem implies Theorem I.

We acknowledge a few drawbacks of our formula: in particular, the value of \( m_\varepsilon \) it provides, though sufficient, does not appear to be optimal. We therefore devote a section of this paper to experimental results on optimal \( m_\varepsilon \); in particular, we conjecture that \( m_\varepsilon = O\left(\frac{1}{\varepsilon^d}\right) \) for some \( d \ll 2 \). Since Jin’s method for calculating \( L_m \) runs in \( O(mn) \) time, the truth of our conjecture would imply that roots of degree \( n \) polynomials could be bounded with error tolerance \( \varepsilon \) in \( O\left(\frac{m}{\varepsilon^d}\right) \) time.

(In this paper, \( p(z) \), \( n \), \( \alpha \), etc., retain the definitions they are assigned in this introduction unless otherwise specified.)

\section{Main results}

From here on, we assume \( a_0 = p(0) = 1 \) and \( n > 2 \) without loss of generality.

By the proof of Theorem 3.1 of Jin [2006],

\[ \gamma_m \leq \max_{k=1+m, \cdots, n+m} \left[ \alpha \beta Q(k) \right]^{1/k} \cdot \rho_{\min}^{-1}, \]  

(8)

where

\[ \alpha := \max\{|1, |a_1|, \cdots, |a_n|\}, \]  

(9)

\[ \beta := \max\{1, \rho_{\min}^n \cdot \sum_{j=0}^{n-1} \rho_{\min}^j\}, \]  

(10)

and \( Q(k) \) is a monotonically increasing function\(^3\) such that

\[ |b_k| \leq Q(k) \cdot \rho_{\min}^{-k}. \]  

(11)

\textbf{Lemma 1} Let \( d_k(n) := \binom{n+k-1}{k} \) be the number of \( k \)-multisets of members of \( \{1, \ldots, n\} \). Then

\[ d_k(n) \leq \binom{n+2}{k} \frac{n-1}{\Gamma(n)}. \]  

(12)

\textbf{Proof} This is the first case of Lemma 2 of Grinshpan [2010]. \hfill \blacksquare

\(^3\)In Jin [2006], \( Q(k) \) is a polynomial with positive coefficients, of degree one less than the maximum multiplicity of the roots of \( p(z) \). However, the proof of his Theorem 3.1 requires only that \( Q(k) \) be increasing.
**Theorem III**  Let \( p(z) \) be the degree \( n \) polynomial

\[
p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + 1,
\]

and let \( \rho_{\text{min}} \) be the least of the moduli of the roots of \( p(z) \). Moreover, let

\[
\frac{1}{p(z)} = 1 + b_1 z + \cdots
\]

be the Maclaurin series of \( \frac{1}{p(z)} \). Then

\[
|b_k| \leq \rho_{\text{min}}^{-k} \frac{(k + \frac{n}{2})^{n-1}}{\Gamma(n)}.
\]

**Proof**  Clearly,

\[
b_k = \frac{1}{k!} \left[ \frac{d^k}{dz^k} \left( \frac{1}{p(z)} \right) \right]_{z=0}.
\]

By the general Leibniz Product Rule,

\[
\frac{d^k}{dz^k} \left( \frac{1}{p(z)} \right) = \frac{1}{a_n} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \ldots, k_n} \prod_{i=1}^n \frac{d^{k_i}}{dz^{k_i}} (z - z_i)^{-1}
\]

\[
= \frac{(-1)^k k!}{a_n} \sum_{k_1+\cdots+k_n=k} \prod_{i=1}^n \frac{1}{(z - z_i)^{k_i+1}}
\]

Dividing both sides by \( k! \), setting \( z = 0 \), and taking absolute values and applying the Triangle Inequality, we obtain

\[
|b_k| \leq \sum_{k_1+\cdots+k_n=k} \prod_{i=1}^n \frac{1}{|z_i|^{k_i+1}}
\]

\[
\leq \rho_{\text{min}}^{-k},
\]

Since the sum is taken over all size \( k \) multisets of \( \{1, \ldots, n\} \), this last inequality reduces to

\[
|b_k| \leq \binom{n + k - 1}{k} \rho_{\text{min}}^{-k},
\]

from which it follows, by Lemma II that

\[
|b_k| \leq \frac{(k + \frac{n}{2})^{n-1}}{\Gamma(n)} \cdot \rho_{\text{min}}^{-k}.
\]

By Theorem III, we may set

\[
Q(k) := \frac{(k + \frac{n}{2})^{n-1}}{\Gamma(n)},
\]

which, as required, is manifestly an increasing function of \( k \). We therefore seek integral \( k \) from \( 1 + m \) to \( n + m \) that maximizes

\[
\left[ \alpha \beta \cdot \frac{(k + \frac{n}{2})^{n-1}}{\Gamma(n)} \right]^{1/k} \rho_{\text{min}}^{-1}.
\]
Lemma 2 Let \( c_1, c_2, c_3 > 0 \). Then, for large \( m \), the function

\[
f(m) := \left[ c_1 \cdot (m + c_2)^{c_3} \right]^{1/m}
\]

is decreasing.

PROOF We take the logarithm of \( f(m) \), multiply both sides by \( m \), and derive:

\[
\log(f(m)) = \frac{\log(c_1) + c_3 \cdot \log(m + c_2)}{m}
\]

\[
m \cdot \log(f(m)) = \log(c_1) + c_3 \cdot \log(m + c_2)
\]

\[
\frac{m \cdot f'(m)}{f(m)} + \log(f(m)) = \frac{c_3}{m + c_2}
\]

Thus

\[
\frac{m \cdot f'(m)}{f(m)} = \frac{\log(c_1) + c_3 \cdot \log(m + c_2)}{m}
\]

\[
f'(m) = \frac{f(m)}{m} \left( \frac{c_3}{m + c_2} - \frac{\log(c_1) + c_3 \cdot \log(m + c_2)}{m} \right).
\]

Asymptotically, the term \(-c_3 \cdot \log(m + c_2)\) dominates, so that \( f'(m) \) is negative for large \( m \).

Lemma 3 For large \( m \),

\[
r_m \geq (m + 2)^{-\frac{1}{m+1}}.
\]

PROOF Since \( f_m(x) \) is increasing for small positive \( x \), it suffices to show that

\[
\left( (m + 2)^{-\frac{1}{m+1}} \right)^{m+1} + (m + 2)^{-\frac{1}{m+1}} - 1 \leq 0.
\]

Setting \( u(t) := (t + 2)^{-\frac{1}{t+1}} \) and \( f(u) := u^{t+1} + u - 1 \), it suffices to prove that

1. \( \frac{df}{dt} > 0 \) and

2. \( \lim_{t \to \infty} f(t) = 0 \).

Now,

\[
\frac{df}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial t}
\]

where

\[
\frac{\partial f}{\partial u} = (t + 1)u^{t} + 1
\]

\[
= (t + 1)(t + 2)^{-\frac{t}{t+1}} + 1
\]

and

\[
\frac{\partial f}{\partial t} = u^{t+1} \cdot \log(u)
\]

\[
= \frac{1}{t+2} \cdot \frac{\log(t+2)}{t+1}
\]
To calculate $\frac{\partial u}{\partial t}$, we take the logarithm of both sides of the equation
\[ u = (t + 2)^{-\frac{1}{m}}, \]  
(40)

obtaining
\[ \log u = -\frac{\log(t + 2)}{t + 1}, \]  
(41)
\[ \frac{1}{u} \frac{\partial u}{\partial t} = \frac{(t + 1)^{-\frac{1}{m}} - t - 1}{(t + 1)^{2}}. \]  
(42)

As $t \to \infty$, $\frac{\partial f}{\partial u}$ is positive, and $\frac{\partial f}{\partial t} \to 0$. Thus $\frac{df}{dt} > 0$ for sufficiently large $t$.

As for the second prong,
\[ f(t) = \frac{1}{t + 2} + (t + 2)^{-\frac{1}{m}} - 1 \]  
(43)
\[ \to 0 + 1 - 1 = 0. \]  
(44)

Suppose $m$ sufficiently large that the conclusions of Lemmas 2 and 3 apply. By Lemma 2, the quantity of Equation (26) is maximized for $k = 1 + m$. Thus
\[ \gamma_m \leq \frac{\alpha \beta (m + 1 + \frac{2}{n})^{n-1}}{\Gamma(n)} \cdot \rho_{\min}^{-\frac{1}{m}}, \]  
(45)

implying, by Lemma 3, that
\[ L_m := \frac{r_m}{\gamma_m} \geq \left[ \frac{\Gamma(n)}{\alpha \beta (m + 2) (m + 1 + \frac{2}{n})^{n-1}} \right]^{\frac{1}{m}} \cdot \rho_{\min}, \]  
(46)
\[ \frac{L_m}{\rho_{\min}} \geq \left[ \frac{\Gamma(n)}{\alpha \beta (m + 1 + \frac{2}{n})^{n}} \right]^{\frac{1}{m}} =: g(m; n, \alpha, \beta). \]  
(47)

Therefore, Equation (7) of Theorem II holds provided that $m \geq \max \{m_\varepsilon, m_\ell\}$, where $m_\varepsilon$ solves the equation
\[ g(m_\varepsilon; n, \alpha, \beta) = 1 - \varepsilon \]  
(48)
and $m_\ell$ is the least $m$ that is “sufficiently large” for the conclusions of Lemmas 2 and 3 to hold. Since $g(m; n, \alpha, \beta)$ is the reciprocal of the $m^{th}$ root of a polynomial in $m$, it has a limit of 1 as $m \to \infty$; and since both $m_\varepsilon$ and $m_\ell$ depend only on $n, \alpha, \beta,$ and $\varepsilon$, Theorem II is proved. (We hereafter assume $m_\varepsilon \geq m_\ell$.)

3 Discussion

We anticipate three primary objections to our method:

1. Equation (48) does not yield an obvious asymptotic bound for $m_\varepsilon$. Indeed, we have not yet succeeded in rigorously proving such a bound. However, the experimental results we detail in Section 4 suggest that $m_\varepsilon = O\left(\frac{C}{n^d}\right)$ for some constant $C \leq 1.1$ and $d \in (1.28, 1.42)$. Note that this estimate does not depend on $n$. 

2. The function $g(m; n, \alpha, \beta)$ approximates $\frac{1}{\rho_{\min}}$ yet depends on $\beta$, which is itself a function of $\rho_{\min}$. This apparent circularity can be overcome by replacing $\rho_{\min}$ in the formula for $\beta$ with the modulus of a known root, or a known upper bound of $\rho_{\min}$. In particular, this allows the estimate $\beta = n$ if at least one root of $p(z)$ is known to lie in $\mathbb{D}$.

Consider, for example, $p(z) = 3z^7 - z^2 + 2$. By Viète’s formulas, $p(z)$ has at least one zero in $\mathbb{D}$; thus we may estimate $\beta$ with $n = 7$ to obtain the equation

$$\left[\frac{\Gamma(7)}{\left(\frac{7}{2} \cdot 7 \cdot (m_{\varepsilon} + 1 + \frac{1}{2})\right)^{m_{\varepsilon} + 1}}\right]^{\frac{1}{m_{\varepsilon} + 1}} = 1 - \varepsilon$$

(49)

for $m_{\varepsilon}$. For $\varepsilon = 0.05$, the solution is $m_{\varepsilon} = 828$; indeed, $L_{828} \approx 0.876282$ and $\rho_{\min} \approx 0.88$, yielding

$$\frac{L_{828}}{\rho_{\min}} \approx 0.995817 \gg 0.95.$$  

(50)

3. The function $g(m; n, \alpha, \beta)$ tends to overestimate the minimum $m$ necessary for error tolerance $\varepsilon$. Returning to the previous example, for $p(z) = 3z^7 - z^2 + 2$ only $m = 20$ calculations, and not $m_{\varepsilon} = 828$, are required to approximate $\rho_{\min} \approx 0.88$ to 95% accuracy. In the next section we suggest a formula for an $m$ (generally less than $m_{\varepsilon}$) that tends to suffice in practice.

4 Experimental results

**Conjecture I** *Equation (7) holds for $m = m_c$, where*

$$m_c = O\left(\frac{1}{\varepsilon^c}\right)$$

(51)

*for some fixed $1 < c \ll 2$. In particular, $m_c$ does not depend on $n$.*

To arrive at Conjecture I we performed two experiments:

1. Took the average value of $m_c$ for 100 polynomials with random degree between 2 and 256, constant term 1, and non-constant coefficients of random maximum absolute value between 2 and 256, setting for each polynomial $\varepsilon = 0.2 \cdot 2^{-j}$ for some random integer $j$ between 0 and 7 inclusive.

2. For each $j = 0, \ldots, 19$, took the average value of $m_c$ for 200 polynomials with degree 10 and coefficients of absolute value no more than 99, setting $\varepsilon = 0.2 \cdot (0.8)^j$.

In Experiment II we performed linear regression on $m_c$ with independent variables $\alpha$, $n$, and $\varepsilon$; our models and the corresponding values of $r^2$ are shown in Table I. Most strikingly, the results of Experiment II suggest no correlation between $m_c$ and the degree of polynomial, attributing less than 1% of variation in $\log m_c$ to changes in variables other than $\varepsilon$. Using these results, we might infer the following approximate formula for $m_c$ (ignoring variables for degree and $\alpha$, whose correlation coefficients are small):

$$m_c = \frac{C}{\varepsilon^d},$$

(52)

where $C \in (1.071929, 1.095269)$ and $d \in (1.286622, 1.287167)$.  

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By contrast, linear regression on the results of Experiment 2 yields the formula of Equation (52) with $C \approx 0.540619$ and $d \approx 1.412172$ ($r^2 = 0.9961$). Taking $C$ from Experiment 1 and $d$ from Experiment 2, we define

$$m_{exp} := \frac{1.095269}{e^{1.412172}}$$

as an approximation of suitable $m$ for purposes of the next section.

5 Examples

In Hohertz and Kalantari [2020] we introduced the Collatz polynomials $P_N(z)$, defined as

$$P_N(z) := \sum_{j=0}^{h(N)} c^j(N) \cdot z^j$$

where

$$c(N) := \begin{cases} \frac{3N+1}{2}, & N \text{ odd} \\ N, & N \text{ even,} \end{cases}$$

$$c^{j+1}(N) := c\left(c^j(N)\right), \text{ and } h(N) := \min\{j : c^j(N) = 1\}.$$ Defining $M(N) := \max_{j \geq 0} c^j(N)$, we have that

$$\alpha := \frac{M(N)}{N},$$

$$\beta \approx 2^{h(N)},$$

and

$$n = h(N).$$

Using the expected value $h(N) = \frac{2}{\log(3)} \cdot \log N \approx 6.952118 \cdot \log N$ proposed in Applegate and Lagarias [2002] and the value $\alpha = 8N$ conjectured in Silva [1999] to bound $\frac{M(N)}{N}$ above, we obtain the formula

$$\left[\frac{\Gamma(6.952118 \cdot \log N)}{8N \cdot (2m_{exp} + 2 + 6.952118 \cdot \log N \cdot 6.952118 \log N)}\right]^{\frac{1}{m_{exp}+1}} = 1 - \epsilon.$$

For $P_5(z) = 5 + 8z + 4z^2 + 2z^3 + z^4$ and $\epsilon = 0.05$, Equation (59) has solution $m_e = 1518$ (rounded up to the nearest integer). This value gives the estimate $L_{1434} \approx 0.995717$, within 0.5% of $\rho_{min} = |−1| = 1$. On the other hand, $[m_{exp}] = 76$; the corresponding estimate is $L_{76} \approx 0.947933$, just over 5% less than $\rho_{min}$.

Consistent with Conjecture [4] and the results of Section 4, $m_{exp}$ appears to bound $\rho_{min}$ well for the polynomials $P_N$. Indeed, on the interval $N \in [2,703]$, the degree of $P_N$ ranges from 1 to 108, yet $\frac{L_{max}}{\rho_{min}} < 0.95$ for only six values of $N$ (it performs worst for $N = 137$, for which $\frac{L_{max}}{\rho_{min}} \approx 0.927254$).

6 Conclusion

All of our experiments suggest the independence of the value $m$ in Jin’s method from the polynomial degree $n$: this striking conjecture would have powerful implications if true and warrants further study. In particular, we

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4 that is, defined at least for those positive integers for which the Collatz trajectory eventually terminates - conjectured to be all of them.
encourage research into proof or disproof of Conjecture \(I\) as well as an asymptotic bound on the minimum \(m\) required for error tolerance \(\varepsilon\) (or a value for the exponent \(d\) if Conjecture \(I\) holds). Finally, acknowledging the theoretical significance of our Equation (48), we nevertheless hope to sharpen the resulting \(m_\varepsilon\), for which we would like to find a closed formula.

7 Tables

Table 1: Results of Experiment \([1]\) Average \(m_c\) for polynomials with random values of \(\alpha, n,\) and \(\varepsilon\).

| VARIABLE MEANINGS | LINEAR MODELS |
|-------------------|---------------|
| \(u\) \(v\) \(w\) | \(\log m_c = \) |
| \(\alpha\) \(\alpha\) \(\alpha\) | \(6.0405 - 0.0025u\) 0.0063 |
| DEGREE \(\alpha\) DEGREE \(\alpha\) DEGREE \(\alpha\) | \(5.9261 - 0.0015v\) 0.0028 |
| \(\log \varepsilon\) \(\log \varepsilon\) \(\log \varepsilon\) | \(0.0910 - 1.2866w\) 0.9958 |
| \(\alpha\) DEGREE \(\alpha\) DEGREE \(\alpha\) DEGREE \(\alpha\) | \(6.2239 - 0.0025u - 0.0015v\) 0.0090 |
| \(\log \varepsilon\) \(\log \varepsilon\) \(\log \varepsilon\) \(\log \varepsilon\) | \(0.0728 + 0.0001u - 1.2871w\) 0.9958 |
| \(\alpha\) DEGREE \(\log \varepsilon\) \(\log \varepsilon\) \(\log \varepsilon\) | \(0.0886 + 0.00002v - 1.2867w\) 0.9958 |
| \(\alpha\) DEGREE \(\log \varepsilon\) \(\log \varepsilon\) \(\log \varepsilon\) | \(0.0695 + 0.0001u + 0.00002v - 1.2872w\) 0.9958 |

Table 2: Results of Experiment \([2]\) Average \(m_c\) for degree 10 polynomials with coefficients \(|a_i| \leq 99\) and \(\varepsilon = 0.2 \cdot (0.8)^j\).

| \(j\) | \(m_{c_{avg}}\) | \(j\) | \(m_{c_{avg}}\) | \(j\) | \(m_{c_{avg}}\) | \(j\) | \(m_{c_{avg}}\) |
|-------|-------------|-------|-------------|-------|-------------|-------|-------------|
| 0     | 4.085       | 5     | 27.525      | 10    | 138.140     | 15    | 586.480     |
| 1     | 6.090       | 6     | 38.270      | 11    | 186.495     | 16    | 773.515     |
| 2     | 9.105       | 7     | 53.895      | 12    | 250.010     | 17    | 1018.430    |
| 3     | 13.125      | 8     | 74.775      | 13    | 333.670     | 18    | 1336.235    |
| 4     | 19.050      | 9     | 101.935     | 14    | 443.225     | 19    | 1748.345    |
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