Abstract

With the extensive application of submodularity, its generalizations are constantly being proposed. However, most of them are tailored for special problems. In this paper, we focus on quasi-submodularity, a universal generalization, which satisfies weaker properties than submodularity but still enjoys favorable performance in optimization. Similar to the diminishing return property of submodularity, we first define a corresponding property called the single sub-crossing, then we propose two algorithms for unconstrained quasi-submodular function minimization and maximization, respectively. The proposed algorithms return the reduced lattices in $O(n)$ iterations, and guarantee the objective function values are strictly monotonically increased or decreased after each iteration. Moreover, any local and global optima are definitely contained in the reduced lattices. Experimental results verify the effectiveness and efficiency of the proposed algorithms on lattice reduction.

1 Introduction

Given a ground set $N = \{1, 2, \cdots, n\}$, a set function $F : 2^N \to \mathbb{R}$ is said to be submodular \cite{7} if \( \forall X, Y \subseteq N \),
\[
F(X) + F(Y) \geq F(X \cap Y) + F(X \cup Y).
\]
An equivalent definition is given as following, i.e., \( \forall A \subseteq B \subseteq N, i \in N \setminus B \),
\[
F(i|A) \geq F(i|B),
\]
where $F(i|A) \triangleq F(A + i) - F(A)$ is called the marginal gain of $i$ with respect to $A$. It implies that submodular functions capture the diminishing return property. To facilitate our presentation, we use $F(A + i)$ to refer to $F(A \cup \{i\})$, and $F(A - i)$ to refer to $F(A \setminus \{i\})$.

With the wide application of submodularity, it has many generalizations. For example, Singh et al. \cite{21} formulate multiple sensor placement and multimodal feature selection as bisubmodular function maximization, where the objectives have multiple set arguments. Golovin and Krause \cite{10} introduce the concept of adaptive submodularity to make a sequence of adaptive decisions with uncertain responses. Feige \cite{5} proposes maximizing subadditive functions on welfare problems to capture the complement free property of the utility functions. However, all the mentioned generalizations of submodularity enjoy benefits in special application scenarios (multiset selection, adaptive decision, and complement free allocation).

In this paper, we study a universal generalization. Submodularity is often viewed as the discrete analogue of convexity \cite{15}. One of the most important generalizations of convexity is quasi-convexity
Proposition 1. Quasi-convex functions satisfy some weaker properties, but still benefit much from the optimization perspective. More specifically, quasi-convex constraints can be easily transformed to convex constraints via sublevel sets, and quasi-convex optimization problems can be solved through a series of convex feasibility problems using bisection methods [2]. Considering the celebrated analogue between submodularity and convexity, a natural question is whether submodularity has similar generalizations which satisfy weaker properties but still has some good properties in optimization? In this paper, we positively answer this question and refer to this generalization as quasi-submodularity.

As aforementioned, quasi-submodularity is a weaker property than submodularity. Similar to the diminishing return property of submodular functions, we first define a corresponding property called single sub-crossing. Then we propose two algorithms for unconstrained quasi-submodular minimization and maximization, respectively. Our theoretical analyses show that the proposed algorithms strictly increase or decrease the objective function values after each iteration. The output reduced lattices can be obtained in \( \mathcal{O}(n) \) iterations, which contain all the local and global optima of the optimization problems. The theoretical and experimental results indicate that although quasi-submodularity is a weaker property than submodularity, it enjoys favorable performance in optimization.

The rest of the paper is organized as follows. In Section 2, we introduce the concept of quasi-submodularity and define the single sub-crossing property. In Section 3 and Section 4, we present the efficient algorithms and theoretical analyses for unconstrained quasi-submodular function minimization and maximization, respectively. Section 5 provides some discussion. Experimental results in Section 6 verify the effectiveness of the proposed algorithms on lattice reduction. Some related works are discussed in Section 7. Section 8 gives some conclusions about our work.

## 2 Quasi-Submodularity

It is well known that the term semi-modular is taken from lattice theory [4]. A lattice is a partially ordered set, which contains the supremum and infimum of each element pair. Here, we introduce a very useful lattice.

**Definition 1** (Set Interval Lattice). Given two ground sets \( A, B \), a set interval lattice \( \mathcal{L} = [A, B] \) is defined as \( \{U \mid A \subseteq U \subseteq B\} \). \( \mathcal{L} \) is not empty if and only if \( A \subseteq B \).

In the set interval lattice, the partially order relation is defined as the set inclusion \( \subseteq \). A set \( S \in \mathcal{L} \) iff \( A \subseteq S \subseteq B \). Obviously, \( \forall A, B \in \mathcal{L} \), we have \( A \cap B, A \cup B \in \mathcal{L} \), thus \( \mathcal{L} \) is a lattice.

In economic fields, Milgrom and Shannon [16] first propose the concept of quasi-supermodularity in comparative statics. Quasi-supermodularity captures the monotonic property of the solution set as the problem parameters change, and has been proved useful in game theory [13], parametric cuts [17], and discrete convex analysis [18]. Following [16], we give the definition of quasi-submodularity.

**Definition 2** (QSB). A set function \( F : 2^N \to \mathbb{R} \) is quasi-submodular function if \( \forall X, Y \subseteq N \), both of the following are satisfied

\[
F(X \cap Y) \geq F(X) \Rightarrow F(Y) \geq F(X \cup Y), \tag{1}
\]

\[
F(X \cap Y) > F(X) \Rightarrow F(Y) > F(X \cup Y).
\]

The following proposition implies that quasi-submodularity is a generalization of submodularity.

**Proposition 1.** Any submodular function is quasi-submodular function, but not vice versa.

**Proof.** Suppose \( F : 2^N \to \mathbb{R} \) is a submodular function, and \( F \) is not a quasi-submodular function. Then we have \( F(X \cap Y) \geq F(X), F(Y) < F(X \cup Y) \), or \( F(X \cap Y) > F(X), F(Y) \leq F(X \cup Y) \). Both of the two cases lead to \( F(X) + F(Y) < F(X \cap Y) + F(X \cup Y) \), which contradicts the definition of submodularity.

A counterexample is given to prove a quasi-submodular function may not be a submodular function. Suppose \( N = \{1, 2\}, F(\emptyset) = 1, F(\{1\}) = 0, F(\{2\}) = 1.5, \) and \( F(\{1, 2\}) = 1 \). It is easy to check that \( F \) satisfies the definition of QSB. But \( F \) is not a submodular function, since \( F(\{1\}) + F(\{2\}) < F(\emptyset) + F(\{1, 2\}) \). Actually, \( F \) is a supermodular function. \( \square \)
Similar to the diminishing return property of submodular functions, we define a corresponding property for quasi-submodularity, and name it as single sub-crossing.

**Definition 3 (SSBC).** A set function \( F : 2^N \rightarrow \mathbb{R} \) satisfies the single sub-crossing property if \( \forall A \subseteq B \subseteq N, i \in N \setminus B, \) both of the following are satisfied

\[
\begin{align*}
F(A) &\geq F(B) \Rightarrow F(A + i) \geq F(B + i), \\
F(A) &> F(B) \Rightarrow F(A + i) > F(B + i).
\end{align*}
\]

As mentioned before, submodularity and diminishing return property are equivalent definitions. Analogously, quasi-submodularity and single sub-crossing property are also equivalent.

**Proposition 2.** Any quasi-submodular function satisfies the single sub-crossing property, vice versa.

**Proof.** Suppose \( F : 2^N \rightarrow \mathbb{R} \) is a quasi-submodular function. \( \forall A \subseteq B \subseteq N, i \in N \setminus B, \) let \( X = B, Y = A + i \) in (1). It is obvious that \( F \) satisfies the SSBC property.

On the other hand, suppose \( F \) satisfies the SSBC property. \( \forall X, Y \subseteq N, \) we denote \( Y \setminus X = \{i_1, i_2, \cdots, i_k\}. \) Based on the SSBC property, if \( F(X \cap Y) \geq (>) F(X) \), then we have \( F(X \cap Y + i_1) \geq (>) F(X + i_1). \) Similarly, we have \( F(X \cap Y + i_1 + i_2) \geq (>) F(X + i_1 + i_2). \) Repeating the operation until \( i_k \) is added, we get \( F(Y) \geq (>) F(X \cup Y). \)

Note that in the proof above, if we exchange \( X \) and \( Y \), i.e., let \( X = A + i, Y = B \), we will get

\[
\begin{align*}
F(A) &\geq F(A + i) \Rightarrow F(B) \geq F(B + i), \\
F(A) &> F(A + i) \Rightarrow F(B) > F(B + i).
\end{align*}
\]

We can rewrite it using the marginal gain notation, i.e., \( \forall A \subseteq B \subseteq N, i \in N \setminus B, \)

\[
F(i|A) \leq (>) 0 \Rightarrow F(i|B) \leq (>) 0.
\]

Note that although \( X \) and \( Y \) are symmetric and interchangeable in (1), we get a different representation of the SSBC property. Actually, (3) is a weaker condition than (1). The proposed algorithms work on the weaker notion (3), and the results also hold for quasi-submodularity.

### 3 Unconstrained Quasi-Submodular Function Minimization

In this section, we concern general unconstrained quasi-submodular minimization problems, where the objective functions are given in form of value oracle. Generally, we do not make any additional assumptions (such as nonnegative, monotone, symmetric, etc) except quasi-submodularity.

Very recently, Iyer et al. [12] propose a discrete Majorization-Minimization like submodular function minimization algorithm. In [12], for each submodular function, a tight modular upper bound is established at the current working set, then this bound is minimized as the surrogate function of the objective function. But for quasi-submodular function, there is no known superdifferential, and it can be verified that the upper bounds in [12] are no longer bounds for quasi-submodular functions. Actually, without submodularity, quasi-submodularity is sufficient to perform lattice reduction. Consequently, we design the following algorithm.

**Algorithm 1** Unconstrained Quasi-Submodular Function Minimization (UQSFMin)

**Input:** Quasi-submodular function \( F, N = \{1, 2, \ldots, n\}, X_0 \subseteq N, t \leftarrow 0. \)

**Output:** \( X_t \) as a local optimum of \( \min_{X \subseteq N} F(X). \)

1: At Iteration \( t, \) find \( U_t = \{u_i \in N \setminus X_t \mid F(u_i|X_t) < 0\}, Y_t \leftarrow X_t \cup U_t. \)
2: Find \( D_t = \{d_i \in X_t \mid F(d_i|Y_t - d_i) > 0\}. X_{t+1} \leftarrow Y_t \setminus D_t. \)
3: If \( X_{t+1} = X_t \) (iff \( U_t = \emptyset \) and \( D_t = \emptyset \)), stop and output \( X_t. \)
4: \( t \leftarrow t + 1. \) Back to Step 1.

\( X \) is a local minimum means \( \forall i \in X, F(X - i) \geq F(X), \) and \( \forall j \in N \setminus X, F(X + j) \geq F(X). \)

Algorithm 1 has several nice theoretical guarantees. First, the objective function values are strictly decreased after each iteration, as the following lemma states.
Lemma 1. After each iteration of Algorithm 1, the objective function value of the working set is strictly monotonically decreased, i.e., \( \forall t, F(X_{t+1}) < F(X_t) \).

Proof. We prove \( F(Y_t^i) < F(X_t^i), F(X_{t+1}^i) < F(Y_t^i) \) can be proved using a similar approach. Define \( U^k = \arg\min_{U \subseteq U_t : |U| = k} F(X_t \cup U) \), and \( Y^k_t = X_t \cup U^k \). According to the algorithm, \( \forall u_t \in U \setminus U^k, F(u_t | X_t) < 0 \). Since \( X_t \subseteq Y_t^k \), and \( u_t \notin Y_t^k \), based on the SSBC property, we have \( F(u_t | Y_t^k) < 0 \). This implies \( F(Y_t^{k+1}) \leq F(X_t \cup (U^k + u_t)) < F(X_t \cup U^k) = F(Y_t^k) \). Note that \( F(Y_t^k) = \min_{u_t \in U_t} F(X_t + u_t) < F(Y_t^j) = F(Y_t^j) \). We then have \( F(Y_t^j) = F(Y_t^{j+1}) < F(Y_t^{j-1}) < \cdots < F(Y_t^0) = F(X_t) \). □

If we start Algorithm 1 from \( X_0 = \emptyset \), after one iteration, we will get \( X_1 = A \triangleq \{ i \mid F(i | \emptyset) < 0 \} \). Similarly, if we start from \( X_0 = N \), we will get \( X_1 = B \triangleq \{ i \mid F(i | N - i) \leq 0 \} \). Based on the SSBC property, we have \( \forall i \in N, F(i | \emptyset) < 0 \Rightarrow F(i | N - i) < 0 \), i.e., \( A \subseteq B \). Thus the reduced lattice \( L = [A, B] \subseteq [\emptyset, N] \) is not empty, and we show that it contains all the global minima.

Lemma 2. Any global minimum of \( F(X) \) is contained in the lattice \( L = [A, B] \), i.e., \( \forall X_* \in \arg\min_{X \subseteq N} F(X), A \subseteq X_* \subseteq B \).

Proof. We prove \( A \subseteq X_* \), \( X_* \subseteq B \) can be proved in a similar way. Suppose \( A \nsubseteq X_* \), i.e., \( \exists a \in A, a \notin X_* \). According to the definition of \( A \), \( F(a | \emptyset) < 0 \). Since \( \emptyset \nsubseteq X_* \), based on the SSBC property, we have \( F(a | X_*) < 0 \), which implies \( F(X_* + a) < F(X_*) \). This contradicts the optimality of \( X_* \). □

If we start Algorithm 1 from \( Q_0 = \emptyset \), suppose we get \( Q_t \) after \( t \) iterations. It is easy to check that, in each iteration, \( Q_t \) only merges elements. So we get a chain \( \emptyset = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_t \subseteq \cdots \subseteq Q_* \), where \( Q_* \) is the final output when the algorithm terminates. Similarly, if we start from \( S_0 = N \), we can get another chain \( S_* \subseteq \cdots \subseteq S_t \subseteq \cdots \subseteq S_1 \subseteq S_0 = N \). We then prove that the endpoint sets of the two chains form a lattice, which contains all the local minima of \( F \).

Lemma 3. Any local minimum of \( F(X) \) is contained in the lattice \( L = [Q_+, S_] \).

Proof. Given a local minimum \( P \). In the proof of Lemma 2, we use singleton elements to construct contradictions, so we have \( Q_1 \subseteq P \subseteq S_1 \). Suppose \( Q_t \subseteq P \subseteq S_t \), we then prove \( Q_{t+1} \subseteq P \subseteq S_{t+1} \). First, we suppose \( Q_{t+1} \nsubseteq P \). Because \( Q_{t+1} = Q_t \cup U_t, \exists u \in U_t, u \notin P \). According to the definition of \( U_t, F(u | Q_t) < 0 \). Since \( Q_t \subseteq P \), based on the SSBC property, we have \( F(u | P) < 0 \). This indicates \( F(P + u) < F(P) \), which contradicts the local optimality of \( P \). Hence \( Q_{t+1} \subseteq P \). And \( P \subseteq S_{t+1} \) can be proved in a similar way. □

Moreover, the two endpoint sets \( Q_+ \) and \( S_+ \) are local minima.

Lemma 4. \( Q_+ \) and \( S_+ \) are local minima of \( F(X) \).

Proof. We prove for \( Q_+, S_+ \) can be proved similarly. According to the algorithm, \( \forall i \in N \setminus Q_+ \), \( F(i | Q_+) \geq 0 \). If \( \exists j \in Q_+ \), such that \( F(j | Q_+ - j) > 0 \), then we can suppose \( j \) was added into \( Q_+ \) at a previous iteration \( t \). Since \( Q_{t-1} - j \subseteq Q_+ - j \), based on the SSBC property, we have \( F(j | Q_{t-1} - j) > 0 \). This contradicts the proof of Lemma 1. □

Because a global minimum is also a local minimum, Lemma 3 results in the following theorem.

Theorem 1. Any global minimum of \( F(X) \) is contained in the lattice \( L = [Q_+, S_+] \), i.e., \( \forall X_* \in \arg\min_{X \subseteq N} F(X), Q_+ \subseteq X_* \subseteq S_+ \).

4 Unconstrained Quasi-Submodular Function Maximization

Unconstrained submodular function minimization problems can be exactly optimized in polynomial time [19]. Yet unconstrained submodular maximization is NP-hard [6]. The best approximation ratio for unconstrained nonnegative submodular maximization is 1/2 [3], which matches the known
hardness result [6]. As a strict superset of submodular case, general unconstrained quasi-submodular maximization is definitely NP-hard.

Iyer et al. [12] also propose a discrete Minorization-Maximization like submodular maximization algorithm. They employ the permutation based subdiﬀerential [7] to construct tight modular lower bounds, and maximize the lower bounds as surrogate functions. With diﬀerent permutation strategies, their algorithm actually mimics several existing approximation algorithms, which means their algorithm does not really reduce the lattices in optimization. In addition, for quasi-submodular cases, it also can be veriﬁed that the lower bounds in [12] are no longer bounds, and quasi-submodular functions have no known subdiﬀerential. Thus, even generalizing their algorithm is impossible.

We ﬁnd Buchbinder et al. [3] propose a simple linear time approximation method. The algorithm maintains two working sets, $S_1$ and $S_2$, and $S_1 \subseteq S_2$. At the start, $S_1 = \emptyset$ and $S_2 = N$. Then at each iteration, one element $i \in S_2 \setminus S_1$ is queried to compute its marginal gains over the two working sets, i.e., $F(i|S_1)$ and $F(i|S_2 - i)$. If $F(i|S_1) + F(i|S_2 - i) \geq 0$, then $S_1 \leftarrow S_1 + i$, otherwise $S_2 \leftarrow S_2 - i$. After $n$ iterations, the algorithm outputs $S_1 = S_2$. This algorithm is eﬃcient and achieves an approximation ratio of $1/3$. However, the approximate algorithm may mistakenly remove a certain element $e \in X_s$ from $S_2$, or add an element $u \not\in X_s$ into $S_1$. Here, $X_s$ is referred to as a global maximum. Consequently, the working lattices of their algorithm may not contain the global optima.

By contrast, we want to reduce the lattices after each iteration while avoid taking erroneous steps. Fortunately, we ﬁnd that if we simultaneously maintain two working sets at each iteration, and take steps in a “crossover” method, quasi-submodularity can provide theoretical guarantees that the output lattices deﬁnitely contain all the global maxima. Hence, we propose the following algorithm.

**Algorithm 2** Unconstrained Quasi-Submodular Function Maximization (UQSFMax)

**Input:** Quasi-submodular function $F$, $N = \{1, 2, \ldots, n\}$, $X_0 \leftarrow \emptyset$, $Y_0 \leftarrow N$, $t \leftarrow 0$.

**Output:** Lattice $[X_t, Y_t]$.

1. At iteration $t$, ﬁnd $U_t = \{u_i \in Y_t \setminus X_t \mid F(u_i|Y_t - u_i) > 0\}$. $X_{t+1} \leftarrow X_t \cup U_t$.
2. Find $D_t = \{d_i \in Y_t \setminus X_t \mid F(d_i|X_t) < 0\}$. $Y_{t+1} \leftarrow Y_t \setminus D_t$.
3. If $X_{t+1} = X_t$ and $Y_{t+1} = Y_t$, stop and output $[X_t, Y_t]$.
4. $t \leftarrow t + 1$. Back to Step 1.

$X$ is a local maximum means $\forall i \in X, F(X - i) \leq F(X)$, and $\forall j \in N \setminus X, F(X + j) \leq F(X)$.

To ensure the result lattice is not empty, we prove that after each iteration Algorithm 2 maintains a nonempty lattice as the following lemma shows.

**Lemma 5.** At each iteration of Algorithm 2, the lattice $[X_t, Y_t]$ is not empty, i.e., $\forall t, X_t \subseteq Y_t$.

**Proof.** According to the deﬁnition, we have $X_0 \subseteq Y_0$. Suppose $X_t \subseteq Y_t$, we then prove $X_{t+1} \subseteq Y_{t+1}$. Because $U_t, D_t \subseteq Y_t \setminus X_t$, if we prove $U_t \cap D_t = \emptyset$, $X_{t+1} \subseteq Y_{t+1}$ will be satisﬁed.

According to the algorithm, $\forall u_i \in U_t, F(u_i|Y_t - u_i) > 0$. Since $X_t \subseteq Y_t - u_i$, and $u_i \not\in Y_t - u_i$ based on the SSBC property, we have $F(u_i|X_t) > 0$, which implies $u_i \not\in D_t$.

Algorithm 2 also has several very favorable theoretical guarantees. First, the objective function values are strictly increased after each iteration, as the following lemma states.

**Lemma 6.** After each iteration of Algorithm 2, the objective function value of endpoint sets of lattice $[X_t, Y_t]$ is strictly monotonically increased, i.e., $\forall t, F(X_{t+1}) > F(X_t)$ or $F(Y_{t+1}) > F(Y_t)$.

**Proof.** We prove $F(X_{t+1}) > F(X_t)$, $F(Y_{t+1}) > F(Y_t)$ can be proved using a similar approach. Deﬁne $U^k \in \arg \max_{U \subseteq U_t} F(X_t \cup U)$, and $X^k_t = X_t \cup U^k$. According to the algorithm, $\forall u_i \in U_t \setminus U^k$, $F(u_i|Y_t - u_i) > 0$. Since $X^k_t \subseteq Y_t - u_i$, and $u_i \not\in Y_t - u_i$ based on the SSBC property, we have $F(u_i|X^k_t) > 0$. This indicates $F(X_{t+1}^k) \geq F(X_t \cup (U^k + u_i)) > F(X_t \cup U^k) = F(X^k_t)$. Note that $F(X^k_t) = \max_{u_i \in U_t} F(X_t + u_i) > F(X_t) = F(X^0_t)$. We then have $F(X_{t+1}) = F(X_t^{[U_t]}_t) > F(X_t^{[U_t]-1}) > \cdots > F(X^0_t) = F(X_t)$. 


After the first iteration of Algorithm 2, we get $X_1 = C \triangleleft N \setminus B = \{i \mid F(i)N - i > 0\}$, and $Y_1 = D \triangleleft N \setminus A = \{i \mid F(i) \geq 0\}$. Based on Lemma 5, we have $C \subseteq D$. Thus the reduced lattice $L = [C, D] \subseteq [0, N]$ is not empty, and we show that it contains all the global maxima.

**Lemma 7.** Any global maximum of $F(X)$ is contained in the lattice $L = [C, D]$, i.e., $\forall X_* \in \arg \max_{X \subseteq N} F(X), C \subseteq X_* \subseteq D$.

*Proof.* We prove $C \subseteq X_*, X_* \subseteq D$ can be proved in a similar way. Suppose $C \not\subseteq X_*$, i.e., $\exists c \in C, c \not\in X_*$. According to the definition, $F(c|N - c) > 0$. Since $X_* \subseteq N - c$, based on the SSBC property, we have $F(c|X_*) > 0$, that is $F(X_* + c) > F(X_*)$. This contradicts the optimality of $X_*$. □

At each iteration of Algorithm 2, $X_t$ only merges elements and $Y_t$ only removes elements. Thus we have $X_t \subseteq X_{t+1}$ and $Y_{t+1} \subseteq Y_t$, i.e., $\forall t. [X_{t+1}, Y_{t+1}] \subseteq [X_t, Y_t]$. We denote the output lattice of Algorithm 2 as $[X_*, Y_*]$. Then $[X_*, Y_*]$ is the smallest lattice in the chain which consists of the working lattices: $[X_+, Y_+], \cdots, [X_t, Y_t], \cdots, [X_1, Y_1] \subseteq [X_0, Y_0] = [0, N]$. Based on Lemma 5, $[X_+, Y_*]$ is not empty, then we prove that it contains all the global maxima of $F$.

**Theorem 2.** Suppose Algorithm 2 outputs lattice $[X_+, Y_*]$. Any global maximum of $F(X)$ is contained in the lattice $L = [X_+, Y_*]$, i.e., $\forall X_* \in \arg \max_{X \subseteq N} F(X), X_* \subseteq X_* \subseteq Y_*$.

*Proof.* Based on Lemma 7, we have $X_1 \subseteq X_* \subseteq Y_1$. Suppose $X_t \subseteq X_* \subseteq Y_t$, we then prove $X_{t+1} \subseteq X_* \subseteq Y_{t+1}$. First, we suppose $X_{t+1} \not\subseteq X_*$. Because $X_{t+1} = X_t \cup U_t$, so $\exists u \in U_t, u \not\in X_*$. According to the definition of $U_t$, $F(u|Y_t - u) > 0$. Since $X_* \subseteq Y_{t+1} - u \subseteq Y_t - u$, based on the SSBC property, we have $F(u|X_*) > 0$. This implies $F(X_* + u) > F(X_*)$, which contradicts the optimality of $X_*$. Hence $X_{t+1} \subseteq X_*$. And $X_* \subseteq Y_{t+1}$ can be proved in a similar way. □

Note that the proofs of Lemma 7 and Theorem 2 also work for local maximum cases, since we use singleton elements to construct contractions.

**Lemma 8.** Any local maximum of $F(X)$ is contained in the lattice $L = [X_+, Y_*]$. 

Lemma 8 indicates that if $X_+(Y_*)$ is a local maximum, it is the local maximum which contains the least (most) number of elements. Unfortunately, finding a local maximum for submodular functions is hard [8], let alone quasi-submodular cases. Nonetheless, Algorithm 2 provides an efficient strategy for search interval reduction, which is helpful because the reduction is on the exponential power. In the experimental section, we show the reduction can be quite surprising. Moreover, when an objective function has a unique local maximum, which is also the global maximum $X_+ = Y_*$, our algorithm can find it quickly.

**Theorem 3.** Algorithm 2 terminates in $O(n)$ iterations. The time complexity is $O(n^2)$.

*Proof.* After each iteration, at least one element is removed from the current working lattice, so it takes $O(n)$ iterations to terminate. At each iteration, all the elements in the current working lattice need to be queried once. Hence, the total complexity of Algorithm 2 is $O(n^2)$. □

5 Discussions

In Algorithm 1, $Q_+$ and $S_+$ are local minima. While in Algorithm 2, $X_*$ and $Y_*$ may not be local maxima. Whether is it possible to find a lattice for quasi-submodular maximization, where the endpoint sets are local maxima? We give an example to show that such a lattice may not exist. Suppose $N = \{1, 2\}$, $F(\emptyset) = (N) = 1$, and $F(\{1\}) = F(\{2\}) = 1.5$. It is easy to check that $F$ is submodular, thus quasi-submodular. The set of local maxima is $\{\{1\}, \{2\}\}$. There is no local maximum which contains or is contained by all the other local maxima, since $\{1\}$ and $\{2\}$ are not comparable under the set inclusion relation.
6 Experimental Results

In this section, we experimentally verify the effectiveness and efficiency of our proposed algorithms. We implement our algorithms using the SFO toolbox [13] and Matlab. All experiments are run on a single core Intel i5 2.8 GHz CPU with 4GB RAM.

6.1 Submodular Functions

For submodular functions, UQSFMin reduces to MMin-I & II [12], so we concern unconstrained maximization problems. We test our algorithms on three submodular functions. The test functions are non-monotone, so the solutions are non-trivial. The first one is Iwata’s test function [8]. Another one is the COM (concave over modular) function \( \sqrt{w_1(X)} + w_2(N \setminus X) \), where \( w_1 \) and \( w_2 \) are randomly generated in \([0, 1]^n\) [12]. The third one is half-product which is useful in formulation of scheduling problems and physical energy models [1]. Actually, half-product is supermodular function and its minimization is NP-Hard [1]. We maximize the negative half-product.

We concern the approximation ratio of an maximization algorithm. We compare the approximation ratio and running time of UQSFMax with MMax [12]. For MMax, we consider the following variants: random permutation (RP), randomized local search (RLS), and randomized bi-directional greedy (RG). For UQSFMax, we use it as the preprocessing steps of RP, RLS and RG, and denote the corresponding combined methods as URP, URLS, and URG.

We set the ground set cardinality \( n = 100 \) for half-product and \( n = 5000 \) for Iwata’s function and COM function. Note that in such an input scale, the exact branch-and-bound algorithm [9] cannot terminate because of its exponential time complexity. Actually, since the reduced lattices of UQSFMax are quite small (contain about 10 elements), we enumerate all the subsets in the lattices to obtain the exact optima.

Table 1: Approximation ratio and running time.

| Algorithm                      | RP | URP | RLS | URLS | RG | URG |
|--------------------------------|----|-----|-----|------|----|-----|
| Approximation ratio for Iwata function | 0.94 | 1.00 | 0.99 | 1.00 | 0.98 | 1.00 |
| Approximation ratio for COM function     | 0.99 | 1.00 | 0.99 | 1.00 | 0.99 | 1.00 |
| Approximation ratio for half-product     | 0.96 | 0.97 | 0.95 | 0.94 | 0.96 | 0.99 |
| Running Time for Iwata function (seconds) | 96.18 | 2.42 | 240.62 | 2.47 | 194.30 | 2.41 |
| Running Time for COM function (seconds)  | 43.85 | 7.01 | 194.52 | 6.91 | 366.43 | 7.16 |
| Running Time for half-product (seconds)  | 0.35 | 0.22 | 0.98 | 0.52 | 9.96 | 4.59 |

According to the comparison results, we find that using our UQSFMax as the preprocessing steps of other approximation methods can reach comparable or better approximation performance while improve the efficiency, since the UQSFMax can efficiently reduce the search interval ranges of other approximation algorithms, and the reduced lattices definitely contain all the local and global optima as shown in the previous theoretical analysis.

Table 2: Average lattice reduction rate of UQSFMax.

| Function       | Iwata function | COM function | half-product |
|----------------|----------------|--------------|--------------|
| Average lattice reduction rate | 99.9%           | 99.5%         | 51.2%         |

We also record the average lattice reduction rate of our algorithm. This result also matches the running time. For example, the average lattice reduction rate for half-product is 51.2%, and the running time of URG (combined methods) is about a half of the running time of RG (MMax).
6.2 General Quasi-Submodular Functions

There exist many non-submodular quasi-submodular functions \[22\]. Since there is no existing exact method for quasi-submodular function optimization, we propose a simple quasi-submodular function for test: \( F_i(X) = \prod_{i \in X} w(i), \) and \( F_i(\emptyset) = 1, \) where \( w(i) \neq 0 \) is randomly chosen in \( \mathbb{R} \). It is easy to check that \( F_i \) is a non-submodular and non-supermodular quasi-submodular function. Suppose \( N = \{1, 2, 3, 4\} \) and \( w = [0.5, 2, 3, 4] \). Because \( F_i(\{2, 3\}) + F_i(\{3, 4\}) = 6 + 12 < 24 + 3 = F_i(\{2, 3, 4\}) + F_i(\{3\}), \) \( F_i \) is non-submodular and non-supermodular. And because \( F_i(\{1\}) + F_i(\{2\}) = 1 + 6 > 3 + 2 = F_i(\{1, 2\}) + F_i(\{2\}), \) \( F_i \) is non-supermodular. \( \forall X \subseteq N, \) we have \( F_i(X) > 0. \) Suppose \( F(X \cup Y) \geq (>) F(X). \) Since \( F(X) = F(X \cap Y) \cdot F(X \setminus Y), \) we have \( 1 \geq (>) F(X \setminus Y). \) This implies that \( F(X \cup Y) = F(Y) \cdot F(X \setminus Y) \leq (>) F(Y). \) Thus \( F_i \) is quasi-submodular.

We compare UQSFMax with MMax \[12\] on maximizing \( F_i \). The ground set cardinality \( n \) is set to be 2000, and \( w(i) \) is randomly generated in \( [-2, 0] \cup (0, 2] \). For each algorithm, we record its approximation ratio and running time.

| Algorithm         | RP | RLS | RG | UQSFMax |
|-------------------|----|-----|----|---------|
| Approximation ratio for \( F_i \) | 0.00 | 0.00 | 1.00 | 1.00 |
| Running time for \( F_i \) (seconds) | 4.23 | 9.66 | 47.68 | 0.43 |

The approximation ratios of RP and RLS are both zeros, because the optima are quite big numbers. RG also obtains the optima. For this specific simple function, this is not surprising. This example shows that our proposed algorithms can work for general quasi-submodular function optimization problems, which cannot be handled by existing submodular optimization algorithms.

7 Related Work

In this section, we introduce some related work of quasi-submodularity.

Quasi-submodularity stems from economic fields. Milgrom and Shannon \[16\] find supermodularity is not necessary for the description of complementarity, and they first propose the definition of quasi-supermodularity. Moreover, they find that the maximizer of a quasi-supermodular function is monotone as the parameter changes \[16\]. In combinatorial optimization, for quasi-submodular functions, this property means the set of minimizers has a nested structure, which is the foundation of MMin \[12\] and the proposed UQSFMin algorithm.

**Theorem 4 (Reformulated from \[16\]).** Given a quasi-submodular function \( F : 2^N \rightarrow \mathbb{R}. \forall A, B \subseteq N, A \subseteq B, \exists S_A \in \arg \min_{S \subseteq A} F(S), S_B \in \arg \min_{S \subseteq B} F(S), \) s.t. \( S_A \subseteq S_B. \)

**Proof.** Suppose \( S_A \in \arg \min_{S \subseteq A} F(S), S_B \in \arg \min_{S \subseteq B} F(S). \) We have \( F(S_A) \leq F(S_A \cap S_B) \) because of \( S_A \subseteq A, S_A \cap S_B \subseteq A \) and \( F(S_A) = \min_{S \subseteq A} F(S). \) According to quasi-submodularity, we have \( F(S_A \cup S_B) \leq F(S_B). \) Denote \( S_B = S_A \cup S_B. \) It is obvious that \( S_B \in \arg \min_{S \subseteq B} F(S) \) and \( S_A \subseteq S_B. \)

Based on the theorem above, suppose we start from \( X = \emptyset, \) if \( \exists i \in N \setminus X, F(X + i) < F(X), \) then we can set \( X \leftarrow X + i. \) Because this theorem ensures that there exists a chain structure of minimizers. This is a general principle. First, it works in submodular cases, for submodularity is a strict subset of quasi-submodularity. Moreover, when the supergradients in \[12\] are not supergradients for quasi-submodular functions, this principle can also hold.

In \[16\], only quasi-submodular minimization (quasi-supermodular maximization) is considered. Whether there is a corresponding principle for quasi-submodular maximization is not clear.

Another related direction is discrete convex analysis \[17, 18\], which departs further from combinatorial optimization. In this paper, we consider set functions, i.e., functions defined on \( \{0, 1\}^n \). While in \[18\], quasi L-convex function, which is defined on \( \mathbb{Z}^n \), is proposed.
In [18], quasi L-convex function is a kind of integer-valued function. When we restrict its domain from $\mathbb{Z}^n$ to $\{0, 1\}^n$, quasi L-convex function reduces to quasi-submodular function. Meanwhile, their results based on $\mathbb{Z}^n$ domain extension reduces to trivial cases in combinatorial optimization. Hence, we view quasi L-convexity [18] as a generalization of quasi-submodularity based on domain extension, i.e., extending the domain from $\{0, 1\}^n$ to $\mathbb{Z}^n$.

There are some applications related to quasi-submodularity.

In rent seeking game, every contestant tends to maximize his probability of winning for a rent by adjusting his bidding. The payoff function of each contestant is quasi-submodular on his bidding and the total bidding of all the contestants (also called "aggregator"). Rent seeking is a kind of aggregative quasi-submodular game [20], where each player’s payoff function is quasi-submodular. We refer readers to [20] for more details and examples of aggregative quasi-submodular games.

In minimum cut problem with parametrized arc capacities, submodularity implies nested structural properties [11]. Granot et al. [11] find that quasi-submodularity also leads to the same properties. But how to employ the properties to find an efficient max flow update for quasi-submodular function is open at present [11].

8 Conclusions

In this paper, we focus on a universal generalization of submodularity called quasi-submodularity. We propose two effective and efficient algorithms for unconstrained quasi-submodular function optimization. The theoretical analyses and experimental results demonstrate that although quasi-submodularity is a weaker property than submodularity, it has some good properties in optimization.

In our future work, we would like to make our algorithms exact for quasi-submodular function minimization and approximate for quasi-submodular function maximization, and try to incorporate the constrained optimization into our framework.

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