On Periodic and Chaotic Orbits in a Rational Planar System

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Abstract

By folding an autonomous system of rational equations in the plane to a scalar difference equation, we show that the rational system has coexisting periodic orbits of all possible periods as well as stable aperiodic orbits for certain parameter ranges.

The orbits of autonomous rational systems may exhibit behavior such as limit cycles and chaos that are not seen in linear systems. It is difficult to prove the existence of such solutions for nonlinear planar systems. In particular, the existing literature seems to lack sufficient demonstrations of the occurrence of complex behavior in rational systems. In this paper, we investigate the semilinear rational system

\begin{align}
x_{n+1} &= ax_n + by_n + c \\
y_{n+1} &= \frac{a'x_n + b'y_n + c'}{a''x_n + b''y_n + c''}
\end{align}

where all parameters are real numbers. To avoid reductions to linear systems or to triangular systems, we assume that

\[ b \neq 0, \quad |a'| + |a''|, |a''| + |b''|, |a'| + |b'| + |c'| > 0. \]

System (1) in the case of non-negative parameters encompasses the following 136 system types in [1] (and an equal number of mirror systems obtained by switching \( x \) and \( y \))

\[(7, l), (22, l), (25, l), (40, l) \quad \text{where} \quad 3 \leq l \leq 49, \ l \neq 4, 5, 7, 9, 10, 13, 19, 20, 22, 25, 28, 40 \] (3)

In this paper we study some of the global properties of (1) through the general process of folding. This concept appears, though not by this name, in diverse areas from control theory to the study of chaos in differential systems; see [4] for details and references and also for a general definition of folding as an algorithmic process. When folded to a second-order rational difference equation, certain configurations of parameters appear that are not readily apparent through other known methods. We use these configurations to simplify the system and obtain sufficient conditions for the occurrence of cycles and chaos.

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1 The main result

Following [4], we solve (1a) for $y_n$ to obtain

$$y_n = \frac{1}{b}(x_{n+1} - ax_n - c)$$

(4)

Now

$$x_{n+2} = ax_{n+1} + by_{n+1} + c = c + ax_{n+1} + \frac{ba'x_n + bb'y_n + bc'}{a''x_n + b''y_n + c''}$$

Using (1b) and (4) to eliminate $y_n$ yields

$$x_{n+2} = c + ax_{n+1} + \frac{ba'x_n + b'(x_{n+1} - ax_n - c) + bc'}{a''x_n + (b''/b)(x_{n+1} - ax_n - c) + c''}$$

Combining terms and simplifying we obtain the rational, second-order equation

$$x_{n+2} = \frac{ab''x_{n+1}^2 + aD_{ab}''x_{n+1}x_n + (aD_{cb}'' + bb' + cb'')x_{n+1} + (bD_{ab}' + cD_{cb}'')x_n + bD_{cb}' + cD_{cb}''}{b''x_{n+1} + D_{ab}''x_n + D_{cb}''}$$

(5)

where

$$D_{ab}' = a'b - ab', \quad D_{ab}'' = a''b - ab'', \quad D_{cb}' = bc' - b'c, \quad D_{cb}'' = bc'' - b''c$$

(6)

We refer to the pair of equations (4) and (5) as a folding of (1). Note that (4) is a passive equation in the sense that it yields $y_n$ without further iterations once a solution $\{x_n\}$ of (5) is known. In this sense, we may think of (5) as a reduction of (1) to a scalar difference equation. If $(x_0, y_0)$ is an initial point of an orbit of (1) then the corresponding solution of (5) with initial values

$$x_0 \text{ and } x_1 = ax_0 + by_0 + c$$

(7)

yields the x-component of the orbit $\{(x_n, y_n)\}$ and the y-component is given (passively) by (4).

Equation (5) is a rational equation of the type studied in [2]. The existence of solutions for (5) is a nontrivial issue. We consider the special case of (5) where in addition to conditions (2) the equalities $D_{ab}', D_{ab}'', D_{cb}'' = 0$ hold, i.e.,

$$a'b = ab', \quad a''b = ab'', \quad b'c = bc''$$

(8)

In this case, (5) reduces to the first-order recursion

$$x_{n+2} = \frac{ab''x_{n+1}^2 + (bb' + cb'')x_{n+1} + bD_{cb}'}{b''x_{n+1}}$$

which by substituting $r_n = x_{n+1}$ may be written as

$$r_{n+1} = ar_n + q + \frac{s}{r_n},$$

(9)

where $q = c + \frac{bb'}{b''}, \quad s = \frac{bD_{cb}'}{b''}, \quad r_0 = x_1$. 

2
A comprehensive study of Equation (9) appears in [2]. In particular, the following is proved which we state here as a lemma.

**Lemma 1** Let \( a = s = 1 \) and \( q < 0 \) in (9).

(a) If \( q > -2 \) then all solutions of (9) with \( r_0 > 0 \) are well-defined, positive and bounded.
(b) If \( -\sqrt{3}/2 < q < -\sqrt{2} \) then (9) has an asymptotically stable 2-cycle \( \{t_1, t_2\} \) where

\[
\begin{align*}
  t_1 &= \frac{-q - \sqrt{q^2 - 2}}{2}, \\
  t_2 &= \frac{-q + \sqrt{q^2 - 2}}{2}
\end{align*}
\]

(c) If \( q = -\sqrt{3} \) then (9) has a stable solution of period 3

\[
r_0 = \frac{2}{\sqrt{3}} \left(1 + \cos \frac{\pi}{9}\right), \quad r_1 = r_0 - \sqrt{3} + \frac{1}{r_0}, \quad r_2 = r_1 - \sqrt{3} + \frac{1}{r_1}
\]

(d) If \( -2 < q \leq -\sqrt{3} \) then solutions of (9) with \( r_0 > 0 \) include cycles of all possible periods.

(e) For \( -2 < q < -\sqrt{3} \) orbits of (9) with \( r_0 > 0 \) are positive, bounded and chaotic in the sense of Li and Yorke (see [3]).

Now we have the following.

**Proposition 2** Assume that conditions (2) and (8) hold with \( a = 1, b'' = bD'_{cb}, c + bb'/b'' = q < 0 \).

(a) If \( q > -2 \) then all solutions of (1) with \( x_0 + by_0 + c > 0 \) are well-defined and bounded.

(b) If \( -\sqrt{5}/2 < q < -\sqrt{2} \) then (1) has an asymptotically stable 2-cycle \( \{(x_1, y_1), (x_2, y_2)\} \) where \( y_i \) is given by (4) and

\[
\begin{align*}
  x_1 &= \frac{-q - \sqrt{q^2 - 2}}{2}, \\
  x_2 &= \frac{-q + \sqrt{q^2 - 2}}{2}
\end{align*}
\]

(c) If \( bb' = -b''(c + \sqrt{3}) \) and \( x_0 + by_0 + c = 2(1 + \cos \pi/9)/\sqrt{3} \) then the points \( (x_i, y_i), i = 1, 2, 3 \) constitute a stable orbit of period 3 for (1) where \( y_i \) is given by (4) and

\[
\begin{align*}
  x_1 &= \frac{2}{\sqrt{3}} \left(1 + \cos \frac{\pi}{9}\right), \\
  x_2 &= x_1 - \sqrt{3} + \frac{1}{x_1}, \\
  x_3 &= x_2 - \sqrt{3} + \frac{1}{x_2}
\end{align*}
\]

(d) If \( -2 < q \leq -\sqrt{3} \) then orbits of (1) with \( x_0 + by_0 + c > 0 \) include cycles of all possible periods.

(e) For \( -2 < c + bb'/b'' < -\sqrt{3} \) orbits of (1) with \( x_0 + by_0 + c > 0 \) are bounded and exhibit chaotic behavior.

**Proof.** Let \( p \) be the minimal (or prime period) of a solution \( \{r_n\} \) of (9) with \( r_0 > 0 \). Then the sequence \( \{x_n\} \) also has minimal period \( p \) and by (4) \( \{y_n\} \) has period \( p \). It follows that the orbit \( \{(x_n, y_n)\} \) has minimal period \( p \). Now with \( r_0 = x_0 + by_0 + c > 0 \) statements (a)-(e) are true by the above Lemma.
We point out that the hypotheses of Proposition 2 are sufficient (but clearly not necessary) for proving that the system is capable of generating periodic and complex trajectories. We obtain additional information about the orbits of (1) from results concerning (5) in a future paper.

Proposition 2 applies to several of the 136 system types listed in (3) that satisfy conditions (8), thus settling the existence of periodic orbits or occurrence of complex behavior for those special cases. For example, the following system

\[
\begin{align*}
x_{n+1} &= x_n + 2y_n - 2 \\
y_{n+1} &= \frac{0.75x_n + 1.5y_n}{3x_n + 6y_n - 6}
\end{align*}
\]

which is type (40,49) satisfies Part (b) of Proposition 2 \((q = -1.5)\) and therefore, has an asymptotically stable 2-cycle \(\{(1,0.75),(0.5,1.25)\}\) (a limit cycle). Changing some of the parameters in the above system yields the following which satisfies Parts (d) and (f) of Proposition 2 with \(q \approx -1.83\)

\[
\begin{align*}
x_{n+1} &= x_n + 2y_n - 2 \\
y_{n+1} &= \frac{0.25x_n + 0.5y_n + 1}{3x_n + 6y_n - 6}
\end{align*}
\]

This system has periodic orbits of all periods (depending on initial points) and exhibits Li-Yorke type chaos. It is also worth mentioning that the rather different type (40,37) system below also satisfies Parts (d) and (f) of Proposition 2

\[
\begin{align*}
x_{n+1} &= x_n + 1.5y_n - 1.8 \\
y_{n+1} &= \frac{1}{1.5x_n + 2.25y_n - 2.7}
\end{align*}
\]

References

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