ON THE QUANTUM SYMMETRY OF RATIONAL FIELD THEORIES

J. Fuchs, A. Ganchev, H. P. Vecsernyés

The aim of this talk is to describe a possible understanding of the quantum symmetry of two-dimensional ($D = 2$) rational quantum field theories (or $D = 1$ chiral rational conformal field theories). We start by briefly sketching the operator-algebraic approach to relativistic quantum field theory (for a review, see for example [1,2]) and in particular the Doplicher–Haag–Roberts program for the description of the superselection sectors ([3] for $D > 2$ and [4] for $D = 2$). The category $C_A$ of localized endomorphisms of the observable algebra is introduced. This is a strict monoidal, rigid category which is symmetric in $D > 2$, i.e. one has permutation statistics, but only braided in $D = 2$, i.e., in $D = 2$ one has generically braid group statistics. We restrict our attention to $D = 2$ and to the rational case, i.e., when $C_A$ has a finite number of simple objects. In the case of chiral conformal field theories the corresponding category has been described in [5]. Doplicher and Roberts [6] have completed the DHR program in $D > 2$, showing that $C_A$ is equivalent to the category of finite-dimensional representations of some compact Lie group – the group of “internal” symmetries of the theory.

The fact that for $D = 2$ the category $C_A$ is a braided one has lead various people to argue that the internal symmetries are given by quantum groups. Considering rational theories one has to restrict oneself to quantum groups at roots of unity. For generic values of the deformation parameter the quantum groups of Drinfeld and Jimbo and Faddeev–Reshetikhin–Takhtadzhyan are indeed deformations of the group algebra or universal enveloping algebra of ordinary simple Lie groups or algebras, and in fact the representation theory remains unchanged. On the other hand, at $q$ a root of unity much of the similarity with the undeformed case breaks down. Quantum groups at roots of unity are not semisimple, and as a consequence their category of representations contains indecomposable (i.e., reducible, but not fully reducible) representations. Though this is well known, a number of papers simply ignored this fact. The careful analysis shows that

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\(^1\) There is a vast literature on the subject which we will not refer to – actually most of the works have been in the framework of rational conformal field theory.
in order to write down quantum group covariant vertex operators one is forced to include also the indecomposable representations (for some of the references see [7]); these are known only for the $su(2)$ case, and even in this case their analysis required a large amount of work.

An alternative is to look from the start for quantum symmetries that are described by a simple Hopf algebra. The first important step in this direction was made by Mack and Schomerus [8] who realized that in order to “truncate” the quantum groups to contain only “good” representations one has to weaken the coproduct structure, namely relax the requirement that the coproduct preserves the unity. This can only be achieved if the coassociativity of the coproduct is relaxed as well, resulting in a weak version of Drinfeld’s quasi-Hopf algebras. For related or other approaches see also [9, 10, 11, 12, 13]. Here we will restrict our attention to a certain class of weak quasi-Hopf algebras $H$ (called rational Hopf algebras) that serve as the quantum symmetry of rational theories with fully braided sectors, or equivalently with a maximally extended observable or chiral algebra [14]. The important new idea of [14] is to use amplimorphisms [15] instead of representations of $H$. This is a very powerful tool that allows to mimic on a finite-dimensional object (the rational Hopf algebra) in a straightforward way most of the information contained in the infinite-dimensional observable or chiral algebra. In particular, amplimorphisms possess left inverses, which leads to conditional expectations, Markov traces, and a characterization of $H$ by a set of rational numbers – the statistics weights. Having the right definition one can perform the Doplicher-Roberts reconstruction of the quantum symmetry from the category [16, 13] (see also [10, 12]).

1. Haag-Kastler nets. The algebraic approach to relativistic quantum field theory starts with a few basic principles – it is quantum, so the observables form a $^*$-algebra of bounded operators; all measurements are localized in space-time, so for every bounded (diamond) region $O$ of space-time there is an algebra $A(O)$ of ‘measurements performed in $O$’; the theory is relativistic, so we have Poincaré covariance and, most important, a causal structure (local commutativity), i.e. $A(O) \subseteq A(O')'$ where $A'$ denotes the commutant while $O'$ is the causal complement, i.e., points space-like to $O$. The correspondence $O \mapsto A(O)$ is a net ($\{O\}$ directed by inclusion). The inductive limit is the quasi-local algebra of observables $A = \bigcup_O A(O)$.

Of central importance is the vacuum representation $\pi_0$. By the ‘electron behind the moon’ argument, $\pi_0$ is faithful, so one can identify $A$ with $\pi_0(A)$. We will also make the assumption of Haag duality in the vacuum sector, i.e., $A(O) = A(O')'$.

2. The category of localized endomorphisms. In order to proceed, one has to choose a class of physical representations describing the ‘charged excitations’ of $\pi_0$. The DHR criterion selects those $\pi$ that are localized in a bounded region $O$, i.e. satisfy $\pi|_{A(O)} \simeq \pi_0|_{A(O)}$. Let $V : \mathcal{H}_\pi \to \mathcal{H}_0$ be the corresponding unitary and define $\rho(A) = V \pi(A) V^{-1}$, $A \in A$. From Haag duality it is immediate that
\( \rho \) is a localized endomorphism of \( \mathcal{A} \). Conversely, every localized endomorphism defines a DHR representation by \( \pi = \pi_0 \circ \rho \).

The space of intertwiners between \( \rho_1 \) and \( \rho_2 \) is \( (\rho_1|\rho_2) = \{ T \in \mathcal{A} : T \rho_1(A) = \rho_2(A)T, A \in \mathcal{A} \} \). The localized endomorphisms are the objects and the intertwiners the arrows of the category \( \mathcal{C}_\mathcal{A} \) of (physical) representations (or of localized endomorphisms) of \( \mathcal{A} \). The gain in introducing endomorphisms is the fact that they can be composed, and thus we can define the product of representations, \( \pi_1 \times \pi_2 = \pi_0 \circ \rho_1 \circ \rho_2 \). If \( T_i \in (\rho_i|\rho_i') \) for \( i = 1, 2 \), their product is \( T_1 \times T_2 = T_1 \rho_1(T_2) = \rho_1'(T_2)T_1 \in (\rho_1 \times \rho_2 | \rho_1' \times \rho_2') \). This product turns \( \mathcal{C}_\mathcal{A} \) into a strict monoidal category. Local commutativity and the transportability of the endomorphisms ensure that \( \mathcal{C}_\mathcal{A} \) is a braided category. Transporting two endomorphisms \( \rho_i \) by unitaries \( U_i \) to \( \tilde{\rho}_i \) that have space-like separated supports, one can show that locality implies \( \tilde{\rho}_1 \tilde{\rho}_2 = \tilde{\rho}_2 \tilde{\rho}_1 \). Then the statistics operator \( \varepsilon(\rho_1, \rho_2) \in (\rho_2 \rho_1|\rho_2 \rho_1) \) is defined as \( (U_1 \times U_2)(U_2 \times U_1)^{-1} \). The statistics operator is “almost” independent of the transporters \( U_i \) – for \( D > 2 \) it is completely independent and \( \varepsilon(\rho_1, \rho_2) \) is unique, hence its square is the identity and braid group statistics reduces to permutation statistics. For \( D = 2 \) where the space-like complement of a point has two disconnected components, \( \varepsilon(\rho_1, \rho_2) \) depends only the relative left/right positions of \( \rho_i \), and in general \( \varepsilon \) is different from its inverse so that genuine braid groups arise.

Two endomorphisms are equivalent if the corresponding representations are unitarily equivalent. The set of equivalence classes \( [\rho] \) are the superselection sectors of the theory. Irreducible sectors \( [\rho_i] \) correspond to ‘elementary’ particles (the simple objects of \( \mathcal{C}_\mathcal{A} \)) and are characterized by their ‘charges’ \( r \). We restrict ourselves to rational theories, i.e., theories for which the set of simple objects of \( \mathcal{C}_\mathcal{A} \) is finite. In general the product of two irreducible endomorphisms is reducible. Decomposing it into irreducible components, let \( N_{rs}^t \) is \( \dim(\rho_r \rho_s|\rho_t) \). These numbers, the fusion rules, are independent of the representative, so we can write \( [\rho_i][\rho_s] = \sum_t N_{rs}^t[\rho_t] \).

If \( T_{pq}^{\alpha} \), \( \alpha = 1, \ldots, N_{pq}^u \), is a basis of \( (\rho_p \rho_q|\rho_u) \), etc., then we can decompose \( \rho_p \rho_q \rho_r \) in two different ways, thereby obtaining two bases for the intertwiner space \( (\rho_p \rho_q \rho_r|\rho_s) \). The change of basis is described by the fusing matrix \( F \), i.e., \( F \) is defined by \( T_{pq}^{\alpha} T_{pq}^{\beta} = \sum_{s, \gamma, \delta} F_{\alpha \beta, \gamma \delta}^s T_{uv}^{\gamma \delta} \rho_p(T_{qr}^{s \delta}) \). The fusing matrices play a role completely analogous to \( 6j \)-symbols and in particular satisfy the pentagon equation (see e.g. [8]). One has also a braiding matrix describing the relation between \( \rho_p \rho_q \rho_r \) and \( \rho_q \rho_p \rho_r \).

Charge conjugation means that there is an involutive map \( \rho \mapsto \overline{\rho} \) of the sectors, such that \( \dim(\overline{\rho}|\rho) = 1 \). Let \( R \in (\overline{\rho}|\rho) \), \( R^* R = 1 \). Then one can introduce the left inverse \( \Phi \) of \( \rho \), defined by \( \Phi(.) = R \overline{\rho}(.) R \). Left inverses allow to define Markov traces which give important numerical characteristics of the sectors. For example the trace of the statistics operator of an irreducible sector \( [\rho_r] \) is the statistics parameter \( \lambda_r = \Phi_r(\varepsilon(\rho_r, \rho_r)) \), a complex number. The statistical dimension is \( d_r = |\lambda_r|^{-1} \), the statistical phase is \( \omega_r = d_r \lambda_r \), and the statistical weight is \( w_r = \text{arg} \omega_r / 2 \pi i \). The dimension \( d_r \) is the square root of the index of the
there is coassociator, i.e. an element \( \varphi \) only a projector. The coproduct allows to define products of representations by \( 1 \) important to note that in general \( \Delta \) is not unit preserving, i.e., \( \Delta(\mathcal{H}) \). (The *-operation on \( \mathcal{H} \) is the usual hermitian conjugation of matrices. The multiplication in

\[ \text{finite basis} \]

3. Modular fusion algebras (MFA). A rational fusion ring is a ring with a finite basis \( \{ \phi_r \} \) in which the structure constants are nonnegative integers, i.e., \( \phi_r \ast \phi_s = \sum_t N_{rs}^t \phi_t \) with \( N_{rs}^t \in \mathbb{Z}_{\geq 0} \), which is commutative, i.e., \( N_{rs}^t = N_{sr}^t \), and associative, i.e., \( \sum_u N_{pu}^r N_{sr}^u = \sum_v N_{pv}^r N_{vr}^u \). Moreover there is a conjugation, i.e., a permutation \( r \mapsto \overline{r} \) of the labels which squares to the identity and is an automorphism of the ring, i.e., \( N_{rs}^{s' \bar{r}} = N_{rs}^t \); it is implemented by \( C_{rs} = N_{rs}^0 \) (we denote by 0 the label of the identity element of the ring), i.e., \( \phi_{\overline{r}} = \sum_s C_{rs} \phi_s \), hence two conjugate elements fuse into the identity with multiplicity one.

The fusion rules matrices \( (N_p)_{qr} = N_{pq}^r \) are normal and commuting, and hence can be simultaneously diagonalized. If the diagonalization matrix \( S \) can be symmetrized and moreover one can find a diagonal matrix \( T \) such that they generate the modular group, one says that the fusion algebra is modular [18]. According to the remarks above, the sectors of a two-dimensional rational field theory with maximally extended observable algebra form a MFA.

4. Rational Hopf algebras (RHA). The quantum symmetry of a rational relativistic quantum field theory in \( D = 2 \) is a rational (i.e., finite-dimensional), semi-simple, quasi-triangular, weak quasi-Hopf *-algebra with invertible monodromy matrix – for short a Rational Hopf Algebra (RHA). Let us explain one by one the elements in the definition.

- Let \( \hat{\mathcal{H}} \) be the finite set of irreducible representations of \( \mathcal{H} \), i.e., for every \( r \in \hat{\mathcal{H}} \) we have \( D_r : \mathcal{H} \rightarrow M_r = \text{End}(V_r) \), where \( V_r = \mathbb{C}^{n_r} \) and \( M_r = \text{Mat}(n_r \times n_r, \mathbb{C}) \) for some \( n_r \in \mathbb{N} \). Hence \( \mathcal{H} \) is a finite sum of full matrix algebras,

\[ H = \bigoplus_{r \in \hat{\mathcal{H}}} M_r. \tag{1} \]

The multiplication in \( H \) is the ordinary multiplication of matrices. The *-operation is the usual hermitian conjugation of matrices.

- \( H \) is endowed with a coproduct \( \Delta : H \rightarrow H \otimes H \), which is a *-monomorphism. (The *-operation on \( H \otimes H \) is defined by \( (a \otimes b)^* = a^* \otimes b^* \).) It is important to note that in general \( \Delta \) is not unit preserving, i.e., \( \Delta(1) \) is in general only a projector. The coproduct allows to define products of representations by \( (D_1 \times D_2)(a) = (D_1 \otimes D_2)(\Delta(a)) \). The coproduct is quasi-coassociative. Thus there is coassociator, i.e. an element \( \varphi \in H^{\otimes 3} \) such that

\[ (\Delta \otimes \text{id}) \circ \Delta(a) \cdot \varphi = \varphi \cdot (\text{id} \otimes \Delta) \circ \Delta(a) \quad \text{for all } a \in H, \tag{2} \]
which serves as the natural isomorphism between the two ways of bracketing a triple product of representations. Since in general \( \Delta(1) \neq 1 \otimes 1 \), one cannot ask for \( \varphi \) to be unitary, but only to be a partial isometry with domain \((id \otimes \Delta) \circ \Delta(H)\) and range \((\Delta \otimes id) \circ \Delta(H)\).

- There is a special one-dimensional representation \( \epsilon : H \to \mathbb{C} \), called the co-unit (we will denote it also by \( D_0 \)), which is a unit preserving *-homomorphism, and there are unitary elements \( \rho, \lambda \in H \) such that \( (\epsilon \otimes id) \circ \Delta(a) = \rho a \rho^* \), \((id \otimes \epsilon) \circ \Delta(a) = \lambda a \lambda^* \) for all \( a \in H \). The latter serve as natural isomorphisms between \( D_0 \times D_p \), respectively \( D_p \times D_0 \), and \( D_D \).
- For these structures there are also two compatibility constraints. First, the triangle identity \( (id \otimes \epsilon \otimes id)(\varphi) = (\lambda \otimes 1) \cdot \Delta(1) \cdot (1 \otimes \rho^*) \) expresses the fact that \( D_p \times (D_0 \times D_q) \to D_p \times D_q \to (D_p \times D_0) \times D_q \) can be also obtained by applying the coassociator. Second, the pentagon identity

\[
(\Delta \otimes id \otimes id)(\varphi) \cdot (id \otimes id \otimes \Delta)(\varphi) = (\varphi \otimes 1) \cdot (id \otimes \Delta \otimes id)(\varphi) \cdot (1 \otimes \varphi) \tag{3}
\]

expresses the equality of the two possible ways to get from \(((D_p \times D_q) \times D_r) \times D_s\) to \(D_p \times (D_q \times (D_r \times D_s))\).

Up to now we have that \( H \) is a weak quasi-bialgebra while the category \( \mathcal{C}_H \) of its representations is a monoidal category. To define contragredient representations (making \( \mathcal{C}_H \) rigid) one passes from a bialgebra to a Hopf algebra – \( H \) is endowed with an antipode, a linear *-anti-automorphism \( S : H \to H \), and non-zero elements \( l, r \in H \) such that \( a^{(1)} \cdot l \cdot S(a^{(2)}) = l \cdot \epsilon(a), S(a^{(1)}) \cdot r \cdot a^{(2)} = \epsilon(a) \cdot r \) for all \( a \in H \). (We use Sweedler type notation, i.e. write \( \Delta(a) = a^{(1)} \otimes a^{(2)} \) etc.) Again there are compatibility constraints, namely the square identities \( S(\lambda) \cdot S(\varphi_1) \cdot r \cdot \varphi_2 \cdot l \cdot S(\varphi_3) \cdot S(\rho^*) = 1 = \lambda^* \cdot \varphi_3^* \cdot l \cdot S(\varphi_2^*) \cdot r \cdot \varphi_1^* \cdot \rho \).

- Finally we want \( \mathcal{C}_H \) to be braided, i.e., the coproduct is quasi-cocommutative. Thus there is an element \( \mathcal{R} \in H \otimes H \) such that \( \Delta'(a) \cdot \mathcal{R} = \mathcal{R} \cdot \Delta(a) \) for all \( a \in H \). Here \( \Delta' \equiv \tau \circ \Delta \) (\( \tau \) permutes the factors). Again because of the weakness property one requires only that \( \mathcal{R} \) is a partial isometry. Compatibility requires the hexagon identities

\[
\varphi_{231} \cdot (\Delta \otimes id)(\mathcal{R}) \cdot \varphi_{123} = \mathcal{R}_{13} \cdot \varphi_{132} \cdot \mathcal{R}_{23}, \tag{4}
\]
\[
\varphi_{312}^* \cdot (id \otimes \Delta)(\mathcal{R}) \cdot \varphi_{123}^* = \mathcal{R}_{13} \cdot \varphi_{213}^* \cdot \mathcal{R}_{12}. \tag{5}
\]

Here we use the notation \( \varphi \equiv \varphi_{123} = \sum_i \varphi_{1,i} \otimes \varphi_{2,i} \otimes \varphi_{3,i} = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \); similarly, \( \varphi_{231} := \varphi_2 \otimes \varphi_3 \otimes \varphi_1 \), \( \mathcal{R}_{13} := \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2 \), etc.

5. **Gauge freedom.** We should not distinguish between two RHAs that possess the same category of representations. This leads one to consider the following gauge transformations, or twistings, of \( H \). Set \( \mathcal{U}_2 := \{ U \in H \otimes H \mid U U^* = 1 \otimes 1 \} \). For \( U \in \mathcal{U}_2 \) define the twisted RHA to be given by \( (H, \epsilon, \Delta_U, \rho_U, \lambda_U, \mathcal{R}_U, \varphi_U, S, l_U, r_U) \), where \( \Delta_U(a) := U \Delta(a) U^* \), \( \rho_U = \epsilon(U_1) U_2 \rho \), \( \lambda_U = U_1 \epsilon(U_2) \lambda \), \( l_U = U_1 l S(U_2) \), \( r_U = S(U_1^*) r U_2 \), \( \mathcal{R}_U = U_{21} \mathcal{R} U_{12} \) and \( \varphi_U = U_{12} [\Delta \otimes id](U) \cdot \varphi [(id \otimes \Delta)(U^*)] U_{23}^* \).
6. Amplimorphisms, monodromy matrix, statistics parameters. An amplimorphism of $H$ is a *-algebra monomorphism from $H$ to $M_n(H)$ (the $n \times n$ matrices with entries in $H$). One can define subobjects, direct sums, and an associative product $(\mu \times \nu)^{i_1j_1,i_2j_2}(a) := \mu^{i_1j_2}(\nu^{j_1i_2}(a))$ of amplimorphisms. Any non-zero representation $D$ of $H$ of dimension $m$ defines a special amplimorphism $\mu_D : H \to M_m(H)$ via

$$\mu_D := (id \otimes D) \circ \Delta.$$  

The braiding of amplimorphisms is described by the statistics operators. For special amplimorphisms $\mu_1, \mu_2$ corresponding to representations $D_1$ and $D_2$, the statistics operator $\epsilon(\mu_1; \mu_2) = [(id \otimes D_2 \otimes D_1)(\varphi \cdot \tau_{23} \mathcal{R}_{23} \cdot \varphi^*)$ ($\tau$ interchanges the tensor product factors of the underlying representation spaces) is an intertwiner between $\mu_2 \times \mu_1$ and $\mu_1 \times \mu_2$. For special amplimorphisms with non-zero $D$, there is a partial isometry $P_\mu \in (\mu_D \times \mu_D)^{id}$ given by $P_\mu^{ij} := (\text{tr} D(rr^*))^{-1/2} \varphi_1 \cdot D^{ij}(\varphi_3 r^* S(\varphi_2)), i,j = 1, \ldots, \dim D$. A standard left inverse $\Phi_\mu : M_m(H) \to H$ of a special amplimorphism $\mu : H \to M_m(H)$ is then defined as $\Phi_\mu(A) = P^{*}_\mu \cdot \bar{\mu}_D(A) \cdot P_\mu$ for all $A \in M_m(H)$. $\Phi_\mu$ is a unit preserving positive linear map satisfying $\Phi_\mu(\mu(a) \cdot B \cdot \mu(c)) = a \cdot \Phi_\mu(B) \cdot c$ for all $a, c \in H$ and all $B \in M_m(H)$.

The statistics parameter matrix $\Lambda_\mu \in M_m(H)$ and the statistics parameter $\lambda_\mu \in H$ of an amplimorphism $\mu_P : H \to M_m(H)$ are defined as $\Lambda_\mu = \Phi_\mu(\epsilon), \lambda_\mu = \Phi_\mu(\Lambda_\mu)$, where $\epsilon \equiv \epsilon(\mu; \mu)$. The statistics parameter $\lambda_\mu$ is an element of the center of $H$ and depends only on the equivalence class of $\mu$. For an irreducible $\mu = \mu_{D_r}$, $r \in \hat{H}$, $\lambda_\mu$ takes the form $\lambda_\mu = \frac{\delta_r}{d_r} \cdot \mu_r(1)$, and hence $\lambda_r = \frac{\delta_r}{d_r} \cdot 1$.

Now we can explain the last part of the definition of a RHA, namely the invertibility of the monodromy matrix $Y \in M_{|\hat{H}|}(H)$. $Y$ is defined by $Y_{rs} := d_r d_s \cdot \Phi_r \Phi_s(\epsilon(\nu_r; \nu_s) \cdot \epsilon(\nu_s; \nu_r)), r, s \in \hat{H}$. One can show that $Y_{rs} = y_{rs} \cdot 1$ with $y_{rs} \in \mathbb{C}$. As in the case of rational field theory one can show that $Y$ is invertible iff $|\sigma|^2 = \sum_{r \in \hat{H}} d_r^2$ where $\sigma := \sum_{r \in \hat{H}} d_r^2 \omega_r^{-1}$ (if the monodromy matrix is degenerate, then the algebra $H$ is said to be degenerate, too). Moreover in the nondegenerate case the matrices

$$S_{rs} := \frac{1}{|\sigma|} \cdot y_{rs}, \quad T_{rs} := \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \cdot \delta_{rs} \omega_r$$

provide a unitary representation of the modular group, and

$$c = \frac{4i}{\pi} \log \frac{\sigma}{|\sigma|} \in [0, 8)$$

plays the role of the ‘central charge’ of $H$, which should be equal (mod 8) to the Virasoro central charge of any conformal field theory model that has $H$ as its quantum symmetry.

The statistics parameters and the monodromy matrix are independent of the gauge freedom described above, while the statistics operators are invariant up to unitary equivalence.
7. **Polynomial equations.** Now we consider in more detail the structure of RHAs. The product of two irreducible representations in general is reducible, i.e., we have \( D_p \times D_q = \bigoplus_r N^r_{pq} D_r \) with \( N^r_{pq} = \dim(D_p \times D_q) | D_r) \). In terms of the representation spaces one has \( V_p \otimes V_q \supseteq \bigoplus_r (D_p \times D_q | D_r) \otimes V_r \), and hence

\[
n_p n_q \geq \sum r N^r_{pq} n_r.
\]  

This becomes an equality only if \( \Delta(1) = 1 \otimes 1 \). A function \( \hat{H} \ni r \mapsto n_r \in \mathbb{N} \) satisfying the inequality (12) is called a weak dimension function (finding such \( n_r \) is easy, and in fact there are infinitely many solutions).

A basis in the intertwiner space \( (D_{p_1} \times D_{p_2} | D_r) \) is given by Clebsch–Gordan coefficients \([p_1 p_2 r]_{i_1 i_2 k} \alpha \), where \( p_j, r \in \hat{H}, \alpha \in \{1, 2, \ldots, N^r_{p_1 p_2}\} \), and \( i_j \in \{1, 2, \ldots, n_{p_j}\} \), etc. They contain the same information as the coproduct; indeed, on matrix units \( e_{ij}^{(r)} \in M_t \subseteq \hat{H} \), the coproduct \( \Delta \) acts as \( \Delta(e_{ij}^{(r)}) = \sum [k_1 k_2 i]_{\alpha} [p_1 p_2 r]^{* k_1 k_2 i} \alpha \) with \( p_j \in \hat{H}, k_j \in \{1, \ldots, n_{p_j}\}, \alpha \in \{1, \ldots, N^r_{p_1 p_2}\} \).

We denote by (10) and the sum being over all labels that appear in the expression. From the requirements that \( \varphi \) is a partial isometry it follows that for fixed \( p, q, r \) and \( t \) for which \( F^{(pq)}_{\alpha u} \) are non-vanishing, \( F^{(pq)}_{\alpha u, \gamma v, \beta} \) is a unitary matrix in the (multi-)indices \( (\alpha, u, \beta) \) and \( (\gamma, v, \delta) \), i.e.

\[
\sum w \sum \sum F^{(pq)}_{\alpha u, \gamma v, \beta} F^{(pq)}_{\gamma v, \mu u, w} = \delta_{\alpha \gamma} \delta_{\beta \delta} \delta_{uv}
\]  

for \( N^r_{pq} N^s_{ur} > 0 \) and \( \alpha \in \{1, 2, \ldots, N^r_{pq}\}, \beta \in \{1, 2, \ldots, N^s_{ur}\} \).

The cocommutator \( \mathcal{R} \) intertwines \( \Delta \) and \( \Delta' \); thus it can be written as

\[
\mathcal{R} = \sum \sum \sum R^{(pq)}_{\alpha u, \gamma v, \beta} F^{(pq)}_{\gamma v, \mu u, w} F^{(pq)}_{\alpha u, \gamma v, \beta} = \delta_{\alpha \gamma} \delta_{\beta \delta} \delta_{uv}
\]

with \( R^{(pq)}_{\alpha u, \gamma v, \beta} \in \bigoplus \) and again a sum over ‘everything’. Because \( \mathcal{R} \) is a partial isometry, for fixed \( p, q, r \) and \( t, R^{(pq)}_{\alpha u, \gamma v, \beta} \) is (if non-vanishing) a unitary matrix in the indices \( \alpha \) and \( \beta \).

Let us now rewrite the pentagon identity (13) in terms of the F-matrices that are defined by (14). Using the orthogonality of the Clebsch–Gordan coefficients, we obtain

\[
\sum_{\sigma=1} F^{(pq)}_{\alpha u, \beta y, \mu z} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} = \sum_{w \in H} \sum_{k=1} \sum_{\lambda=1} \sum_{\eta=1} F^{(pq)_{\alpha u, \beta y, \mu z}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(pq)_{\alpha u}}_{\gamma v, \gamma w, \sigma} F^{(ur)_{\alpha u}}_{\gamma v, \gamma w, \sigma} \]
Similarly, with (12) the hexagon identities read
\[
\sum_{u \in \hat{H}} N_{ur}^t N_{pq}^u \sum_{\delta, \lambda = 1} F_{\beta w \gamma, \kappa u \delta}^{(rpq)_t} R_{\kappa u \lambda, \mu \nu}^{(urw)_t} = \sum_{a=1}^{N_r^w} \sum_{\delta=1}^{N_p^w} R_{a \alpha, \beta}^{(pr)_w} F_{\alpha \omega \gamma, \delta \nu}^{(pq)_t} R_{\gamma \delta, \mu}^{(qr)_w},
\]
\[
\sum_{u \in \hat{H}} N_{ur}^t N_{pq}^u \sum_{\delta, \lambda = 1} F_{\beta w \gamma, \kappa u \delta}^{(rpq)_t} R_{\kappa u \lambda, \mu \nu}^{(urw)_t} = \sum_{a=1}^{N_r^w} \sum_{\delta=1}^{N_p^w} R_{a \alpha, \beta}^{(pr)_w} F_{\alpha \omega \gamma, \delta \nu}^{(pq)_t} R_{\gamma \delta, \mu}^{(qr)_w}.
\]

8. Outlook. The essential information of \(C_A\) consists of the fusion rules \(\{N_{pq}^r\}\) and the fusing and braiding matrices \(\{F_{pqr}^{(t)}, R_{pq}^{(r)}\}\) (i.e., the category can be reconstructed from this information [5]). As we see from the results above, given these data together with a weak dimension function \(\{n_r\}\), one can construct a RHA \(H\) such that its category \(C_H\) will be equivalent to \(C_A\) (see [16, 12, 13, 10, 11] for details). Thus with the correct definition the Doplicher–Roberts reconstruction becomes ‘almost a tautology’ for rational theories.

Consider now the character rings. The character ring \([C_A]\) is a MFA characterized by the fusion rules \(\{N_{pq}^r\}\) and the statistical weights \(\{\omega_r\}\). (The dimensions \(\{d_r\}\) are determined by the fusion rules, while from the statistical phases and the fusion rules one easily recovers the \(S\) and \(T\) matrices of the modular group.) Obviously also every RHA (more precisely, every equivalence class [RHA] of RHAs modulo twisting and modulo the choice of weak dimension function) defines a MFA. An interesting but highly nontrivial problem is to analyze the map
\[
[RHA] \rightarrow \text{MFA}.
\]

Is this map one to one? In other words, are the statistical phases (and, of course, the fusion rules) enough to distinguish between different [RHA], or equivalently between different categories \(C_A\)? Most likely the answer to this question is yes.

Is this map onto? This is a much more difficult question. Ideally one would like, given \(\{N_{pq}^r\}\) and \(\{\omega_r\}\), to have an ansatz for the fusing and braiding matrices \(\{F^{(t)}, R^{(r)}\}\) in terms of statistical phases and dimensions that solve the polynomial equations. A positive answer to this question will mean that one can reconstruct the braid group representation of a conformal model from its modular properties.

We have explored this problem only in a tiny corner of the space of all MFAs. All possible MFAs of dimension \(\leq 3\) have been classified. Taking only the fusion rule data \(\{N_{pq}^r\}\) of these MFAs one can find the general solution to the corresponding polynomial equations. Referring to [20] for the details, here we only mention that for \(|\hat{H}| \leq 3\) the map in question is an isomorphism.

A positive answer to the posed question will mean that the classification of MFAs (a formidable problem in itself) will lead to a classification of the Moore–Seiberg categories \(C_A\). The next natural question will be how far is a classification of MFAs from a classification of, say, rational conformal field theories.
It is known that there are different models (having for example different Virasoro central charges) that share the same braiding properties and hence Moore-Seiberg categories. Hence one can ask what additional information is necessary to distinguish between them. In particular, are the conformal weights (i.e., not just their fractional parts which are the statistical weights) already sufficient?

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