Unimodular trees versus Einstein trees

Enrique Álvarez1,2,a, Sergio González-Martín1,2,b, Carmelo P. Martín3,c

1 Instituto de Física Teórica, IFT-UAM/CSIC, Universidad Autónoma, 28049 Madrid, Spain
2 Departamento de Física Teórica, Universidad Autónoma de Madrid, 28049 Madrid, Spain
3 Departamento de Física Teórica I Facultad de Ciencias Físicas, Universidad Complutense de Madrid (UCM), Av. Complutense S/N (Ciudad Univ.), 28040 Madrid, Spain

Received: 26 July 2016 / Accepted: 15 September 2016 / Published online: 12 October 2016
© The Author(s) 2016. This article is published with open access at Springerlink.com

Abstract
The maximally helicity violating tree-level scattering amplitudes involving three, four or five gravitons are worked out in Unimodular Gravity. They are found to coincide with the corresponding amplitudes in General Relativity. This a remarkable result, insofar as both the propagators and the vertices are quite different in the two theories.

1 Introduction

Unimodular gravity is an interesting truncation of General Relativity, where the spacetime metric is restricted to be unimodular,

\[ g \equiv \det g_{\mu\nu} = -1. \] (1)

It is convenient to implement the truncation through the (non invertible) map

\[ g_{\mu\nu} \rightarrow |g|^{-1/n} g_{\mu\nu}. \] (2)

The resulting theory is not Diff invariant anymore, but only TDiff invariant. Transverse diffeomorphisms are those such that their generator is transverse, that is,

\[ \partial_\mu \xi^\mu = 0. \] (3)

The ensuing action of Unimodular Gravity (cf. [1] for a recent review with references to previous literature), reads

\[ S_{UG} \equiv \int \! d^n x \, L_{UG} \]
\[ = M_p^{n-2} \int |g|^{1/n} \left( R + \frac{(n-1)(n-2)}{4n^2} g^{\mu\nu} \nabla_\mu g \nabla_\nu g \right). \] (4)

for the gauge choice of [1].

2 Feynman rules

The graviton propagator in Unimodular Gravity (cf. Appendix A) reads

\[ P_{\mu\nu,\rho\sigma}^{UG} = \frac{i}{2k^2} (\eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma}) - \frac{i}{k^2} \frac{\alpha^2 n^2 - n + 2}{\alpha^2 n^2 (n - 2)} \eta_{\mu\nu} \eta_{\rho\sigma} \]
\[ + \frac{2i}{n - 2} \left( k_\rho k_\sigma \eta_{\mu\nu} + k_\mu k_\nu \eta_{\rho\sigma} \right) - \frac{2i}{n - 2} \frac{k_\mu k_\nu k_\rho k_\sigma}{k^6} \] (5)

It can easily be shown using the Bianchi identities that the classical equations of motion (EM) of Unimodular Gravity coincide with those of General Relativity with an arbitrary cosmological constant. The main difference at this level between the two theories is that a constant value for the matter potential energy does not weight at all, which solves part of the cosmological constant problem (namely why the cosmological constant is not much bigger that observed). This property is preserved by quantum corrections [2].

A natural question to ask at this stage is whether the S-matrix would be the same for Unimodular Gravity as for General Relativity. Although the S-matrix elements have been studied by several authors in the case of General Relativity [4–8], we are not aware of any results concerning the computation of S-matrix elements in Unimodular Gravity. The propagators as well as the vertices are quite different in the two theories, so that the answer to the question we asked at the beginning of this paragraph is not immediate.

In the present paper we shall carry out the calculation of the maximally helicity violating three-, four-, and five-graviton amplitudes at tree level and find complete agreement between the two theories, a fact that we find remarkable.
Recall that the usual General Relativity graviton propagator in the de Donder gauge,

\[ P_{\mu\nu;\rho}^{GR} = \frac{i}{2k^2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} - \frac{2}{n-2} \eta_{\mu\nu} \eta_{\rho\sigma} \right). \tag{6} \]

has only simple poles at \( k^2 = 0 \). In the unimodular propagator, by contrast, there appear double and triple poles in addition to the simple ones. This is a technical complication and the main reason why we cannot, a priori, apply some of the recent useful techniques [19] to reduce the computation of the diagrams. In Appendix B we shall show that no gauge choice in Unimodular Gravity can yield a propagator of the form

\[ P_{\mu\nu;\rho} = \frac{i}{2k^2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} - f_1(k^2) \eta_{\mu\nu} \eta_{\rho\sigma} \right) + f_2(k^2)(k\rho k_\sigma + k_\mu k_\nu k_{\rho\sigma}) + f_3(k^2) k_\mu k_\nu k_\rho k_\sigma, \tag{7} \]

\( f_3(k^2) \) having no pole at \( k^2 = 0 \), if the Newtonian potential is to be obtained in the nonrelativistic limit. Actually, we shall see that the triple pole term in (5) is needed to retrieve the correct nonrelativistic static limit.

Since we are going to focus on the three-, four-, and five-point amplitudes, we also need the three- and four-graviton vertices. These are obtained from the second and third order expansion of the Lagrangian around flat space (cf. Appendix C) and can be expressed in a condensed form, with a parameter \( n \) that gives the General Relativity vertex for \( n = 2 \) and the Unimodular Gravity one for \( n = 4 \). With the convention of all momenta being incoming the expression for the three-graviton vertex reads

\[ V_{\mu;\rho;\sigma;\alpha;\beta;\lambda}^{\mu\nu;\rho\sigma;\alpha;\beta} = i \kappa^2 S \left\{ \frac{(2 + n)(p_3.p_4)}{4n^4} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} \eta^{\lambda} g^{\lambda} - \frac{(2 + n)(p_3.p_4)}{4n^3} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} \right. \]

\[ + \frac{(2 + n)(p_3.p_4)}{2n^2} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} - \frac{(2 + n)(p_3.p_4)}{n^2} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} \]

\[ - \frac{(2 + n)(p_3.p_4)}{2n} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} + \frac{(2 + n)(p_3.p_4)}{n} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} \]

\[ - \frac{(2 + n)(p_3.p_4)}{n} g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} g^{\lambda} \]

\[ + (p_3.p_4) \eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\sigma\rho} \eta^{\lambda}. \tag{8} \]
In the case of a massless particle, the condition $\det p = 0$ implies that their last symbol is 2 $P_1$ instead of 4 $P_0$, where $P_0$ and $P_1$ are the four-momenta of the two massless particles.

$\epsilon^{\mu} = \epsilon_{\mu} = -1, \quad \epsilon^{+} = \epsilon^{-} = 0, \quad \epsilon^{\pm} = \epsilon_{\pm} = \epsilon_{\pm} = 0$. Therefore, the gluon polarization spinors are given by

$$\epsilon^{-}_{aa} = \epsilon^{+}_{aa} = \epsilon^{+}_{aa} - \epsilon^{-}_{aa}.$$ (13)

Henceforth, with the appropriate choice of the reference spinors we get the following rules:

1. For the four-graviton amplitudes, by choosing $r_1 = r_2 = p_4$ and $r_3 = r_4 = p_1$ we get the extra relations

$$\epsilon^{+}_1 \cdot p_4 = 0, \quad \epsilon^{+}_2 \cdot p_4 = 0, \quad \epsilon^{+}_3 \cdot p_1 = 0, \quad \epsilon^{+}_4 \cdot p_1 = 0, \quad \epsilon^{+}_i \cdot \epsilon^{+}_j = 0 \text{ except for } \epsilon^{+}_2 \cdot \epsilon^{+}_3.$$ (14)

2. For the five-graviton amplitudes, we choose now $r_1 = r_2 = p_5$ and $r_3 = r_4 = r_5 = p_1$ and we get

$$\epsilon^{+}_1 \cdot p_5 = 0, \quad \epsilon^{+}_2 \cdot p_5 = 0, \quad \epsilon^{+}_3 \cdot p_1 = 0, \quad \epsilon^{+}_4 \cdot p_1 = 0, \quad \epsilon^{+}_5 \cdot p_1 = 0, \quad \epsilon^{+}_i \cdot \epsilon^{+}_j = 0 \text{ except for } \epsilon^{+}_2 \cdot \epsilon^{+}_3 \text{ and } \epsilon^{+}_2 \cdot \epsilon^{+}_4.$$ (15)

### 4 Three-graviton amplitudes

The fact that Unimodular Gravity perturbatively expanded around Minkowski spacetime is Lorentz invariant and that the gravitational polarizations are the same as in General Relativity leads, by repeating the standard analysis [3], to the conclusion that the on-shell three-point amplitudes vanish on-shell for

**3 Spinor helicity formalism for massless particles**

Although we are not using the spinor helicity formalism explicitly, we can take advantage of some useful relationships that can be derived from it and will greatly simplify the calculations.

The four momentum $p^\mu$ for an on-shell particle is written in terms of two commuting Weyl spinors as

$$p_{\alpha \dot{\alpha}} = \tilde{\sigma}_{\mu \alpha \dot{\alpha}} p^\mu = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} + \mu_\alpha \bar{\mu}_{\dot{\alpha}}.$$ (10)

In the case of a massless particle, the condition $\det(p_{\alpha \dot{\alpha}}) = 0$ implies

$$p_{\alpha \dot{\alpha}} = \tilde{\sigma}_{\mu \alpha \dot{\alpha}} p^\mu = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}.$$ (11)

On the other hand, the polarization tensor of the graviton can be written in terms of the gluon ones as

$$\epsilon^{\mu \nu} = \epsilon^{\mu}_a \epsilon^{\nu}_a \rightarrow \epsilon^{\alpha \beta}_{\alpha \beta} = \epsilon^{\alpha \beta}_{\alpha \beta} \text{ and } \epsilon^{\alpha \beta}_{\alpha \beta} = \epsilon^{\alpha \beta}_{\alpha \beta}.$$ (12)

1 We have compared our vertices with those of [17, 18], in their notation, and in addition to the error pointed out in [17] in the four vertex, we claim that their last symbol is 2 $P_1$ instead of 4 $P_0$. 

2. For the five-graviton amplitudes, we choose now $r_1 = r_2 = p_5$ and $r_3 = r_4 = r_5 = p_1$ and we get
real momenta. Now, let us stress that little group scaling operates in Unimodular Gravity exactly in the manner as in General Relativity. Hence, it is plain that for conserved complex momenta the on-shell nonvanishing three-point amplitudes are the same in Unimodular Gravity as in General Relativity but, perhaps, for a global constant. By explicit computation of the corresponding Feynman diagrams we have found that the constant in question is the same in the two theories, as in fact the classical Newton constant is indeed the same in the two theories. Let us notice that the on-shell three-point functions for complex momenta are the elementary objects in the recursive construction of the amplitudes in theories like Yang–Mills and General Relativity with or without SUSY.

5 Four-graviton tree amplitudes

Let us recall that our goal is to compute the tree diagrams both in Unimodular Gravity and General Relativity in order to see whether there is any difference between the two theories. This is relevant for the physical content of the theories because these amplitudes give us information on the tree-level S-matrix.

We shall focus on the maximally helicity violating (MHV) diagrams with three, four and five external gravitons because they are the simplest nontrivial ones.

There are only three types of diagrams—those corresponding to the well-known s, t, and u channels, respectively—that involve four external gravitons to be worked out explicitly. The diagram that is a pure four vertex vanishes because no nonvanishing contribution to the amplitude diagram can be constructed out of two momenta entering the vertex and the four graviton polarizations satisfying the equations displayed in Sect. 3. The s-, t-, and u-channel diagrams are shown in the next figures where all gravitons are outgoing (Figs. 1, 2, 3).

The explicit result is

\[
\mathcal{A}_s(1^{-2} ; 3^{+4} ) = \frac{i k^2 (e_1, p_2)^2 (e_2, e_3)(e_4, p_2)^2}{s^2} \\
= \frac{i k^2}{4} \langle 12 \rangle \langle 34 \rangle^2, \tag{26}
\]

\[
\mathcal{A}_t(1^{-2} ; 3^{+4} ) = \frac{i k^2 (e_1, p_2)^2 (e_2, e_3)(e_4, p_2)^2}{s^2} \\
= \frac{i k^2}{4} \langle 12 \rangle \langle 34 \rangle^2, \tag{27}
\]

\[
\mathcal{A}_u(1^{-2} ; 3^{+4} ) = \frac{i k^2 (e_1, p_2)^2 (e_2, e_3)(e_4, p_2)^2}{s^2} \\
= \frac{i k^2}{4} \langle 12 \rangle \langle 34 \rangle^2, \tag{28}
\]

where as usual \( s = p_1 + p_2 \) and \( u = p_1 + p_3 \).

These amplitudes are diagram to diagram exactly the same that the ones for General Relativity. The complete amplitude is therefore

\[
\mathcal{A}(1^{-2} ; 3^{+4} ) = \frac{i k^2}{4} \langle 12 \rangle \langle 34 \rangle^2, \tag{29}
\]

in agreement with the result presented for General Relativity in [7].
6 Five-point diagrams

When computing the diagrams with five external gravitons there are three sets of diagrams. The one that is purely a five vertex vanishes identically. Indeed, no nonvanishing contribution to the amplitude diagram can be built from two momenta entering the vertex and the five graviton polarizations introduced in Sect. 3. Let us consider the others in turn.

6.1 Three vertices

There are 15 different diagrams that involve three vertices of the type shown in Fig. 4; this we shall denote by \( \mathcal{A}(1^-, 2^-, 3^+; 4^+, 5^+) \), the others will be analogously represented by using the obvious notation.

Let us write this one as an example; the full set of amplitudes can be found in Appendix D.

We have

\[
\mathcal{A}(1^-, 2^-, 3^+; 4^+, 5^+) = -ik^3(\epsilon_1, p_1)^2(\epsilon_2, q_2)^2(\epsilon_3, p_3)^2(\epsilon_4, p_5)^2 (p_1 + p_2)^2(p_4 + p_5)^2 \\
+ 2ik^3(\epsilon_1, p_1)^2(\epsilon_2, q_2)^2(\epsilon_4, p_4)(\epsilon_5)(p_1)(p_5)(p_5)
\]

6.2 The four vertex

The rest of the a priori nonvanishing diagrams are those that involve one three vertex and one four vertex \( \mathcal{A}(1^-, 2^-, 3^+, 4^+, 5^+) \) as shown in Fig. 5.

Explicit computation shows that all the 10 different diagrams do vanish.

7 Conclusions

It has been shown that the MHV three-, four-, and five-graviton tree amplitudes give the same contribution both in General Relativity and Unimodular Gravity. This result holds for each diagram independently and not only for the whole amplitude. Therefore we can conclude that, at least at tree level and for three, four or five external legs, the MHV contribution to the S-matrix for pure Unimodular Gravity without coupling to other fields is the same in the two theories.

A remarkable fact is that all the terms that involve the double and triple poles in the propagator of Unimodular Gravity (5) do not contribute to any diagram we have computed in pure Unimodular Gravity. We have explicitly checked this by introducing an arbitrary coefficient in front of each piece and then verify that the final result is independent of the arbitrary coefficient we have introduced. That the contributions coming from those higher order poles go away is not trivial and we did not find any reason to expect it before computing the diagrams. Indeed, on the one hand, the triple pole summand in the propagator is needed to recover the Newtonian potential—see Appendix B—and, on the other hand, in Unimodular Gravity, one obtains the following nonzero result:

\[
k_\alpha k_\beta V_{\mu\nu,\rho\sigma}^{\alpha\beta} \epsilon_{1\mu}(p) \epsilon_{2\rho}(q) = ik(\epsilon_1 \cdot q)
\]
eral Relativity yields a vanishing result as a consequence of invariance under the full Diffeomorphism group.

As a straightforward consequence, and since the BFCW recursion relations [19] can be applied to the diagrams of General Relativity [4], our results suggest that BFCW (or a similar recurrence) can be applied to Unimodular Gravity as well. This would be remarkable because of the existence of higher order poles in the propagator. Work on these issues is ongoing, and we expect to report on this soon.

Acknowledgments We are grateful for useful correspondence with W.T. Giele, L. Dixon, and M. Spradlin. We also acknowledge useful discussions with Paolo Benincasa. This work has been partially supported by the European Union FP7 ITN INVISIBLES (Marie Curie Actions, PITN- GA-2011- 289442) and (HPRN-CT-2000-00148); COST action MP1405 (Quantum Structure of Spacetime), COST action MP1210 (The String Theory Universe) as well as by FPA2012-31880 (MICINN, Spain), FPA2014-54154-P (MICINN, Spain), and S2009ESP-1473 (CA Madrid). This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 690575. This project has also received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 674896. The authors acknowledge the support of the Spanish MINECO Centro de Excelencia Severo Ochoa Programme under grant SEV-2012-0249. Each tree-level diagram workout in this paper has been computed in two independent ways, one using the computer algebra systems FORM [20] and the other Mathematica’s xAct [21] package.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP3.

Appendix A: Feynman rules

In order to obtain the Feynman rules for Unimodular Gravity, let us start from the action

\[
S_{UG} = -\frac{2}{\kappa^2} \int d^nx \sqrt{\eta} \left( R + \frac{(n-1)(n-2)}{4n^2} \nabla_{\mu}g \nabla_{\nu}g \right)
\]

(31)

with \( \kappa^2 = 32\pi G \).

The propagator is obtained by inverting the second order expansion of the Lagrangian around flat spacetime—once properly gauge-fixed—presented in [1]. This reads

\[
\mathcal{L} = \frac{1}{4} h_{\mu\nu} \partial^2 h_{\mu\nu} - \frac{1}{4n} \partial^2 h + \left( -f \partial^2 f + \frac{\alpha}{2} f \partial^2 h + \frac{\alpha}{2} h \partial^2 f \right)
\]

\[
- \frac{1}{2} \left( \partial_{\mu} e^{(0,0)}(0,0) \partial_{\nu} e^{(0,0)}(0,0) + 2 \partial_{\mu} h_{\nu} - \frac{1}{n} \partial_{\mu} h \partial_{\nu} e^{(0,0)}(0,0) \right),
\]

(32)

Writing now the action as

\[
S = \int d^n x \Psi^A F_{AB} \Psi^B
\]

(33)

where

\[
F_{AB} = G_{AB} \partial^2 + J_{AB} \partial_\mu \partial_\nu,
\]

(34)

\[
\Psi^A = \begin{pmatrix} h_{\mu\nu} \\ f \end{pmatrix},
\]

(35)

and the different matrices involved read

\[
G_{AB} = \left( \begin{array}{cc}
\frac{1}{4} K_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma} & \frac{a}{2} g_{\mu\nu} & -\frac{1}{8} g_{\mu\nu} \\
\frac{1}{8} g_{\rho\sigma} & -1 & 0 \\
-\frac{1}{8} g_{\rho\sigma} & 0 & \frac{1}{4}
\end{array} \right),
\]

(36)

\[
J_{AB} = \left( \begin{array}{ccc}
0 & 0 & \frac{1}{4} \left( g_{\mu\nu} g_{\rho\sigma} + g_{\sigma\rho} g_{\mu\nu} \right) \\
0 & 0 & 0 \\
\frac{1}{4} \left( g_{\rho\sigma} g_{\mu\nu} + g_{\mu\nu} g_{\sigma\rho} \right) & 0 & 0
\end{array} \right).
\]

(37)

We have introduced the tensors

\[
T_{\mu\nu\rho\sigma} = \frac{1}{4} \left( g_{\mu\rho} \delta^{(a)} g_{\nu\sigma} + g_{\mu\sigma} \delta^{(a)} g_{\nu\rho} + g_{\nu\rho} \delta^{(a)} g_{\mu\sigma} + g_{\nu\sigma} \delta^{(a)} g_{\mu\rho} \right),
\]

(38)

\[
K_{\mu\nu\rho\sigma} = \frac{1}{2} \left( g_{\mu\rho} \delta^{(a)} g_{\nu\sigma} + g_{\mu\sigma} \delta^{(a)} g_{\nu\rho} \right).
\]

(39)

Appendix B: The Unimodular Gravity free propagator and Newton’s law

In Unimodular Gravity the graviton field \( h_{\mu\nu} \) couples to the traceless part, \( \hat{T}^{\mu\nu} \), of the energy-momentum tensor \( \hat{\alpha} \) la Rosenfeld or, what is the same, the traceless part of the graviton field, \( \hat{h}_{\mu\nu} \), couples to the energy-momentum tensor defined \( \hat{\alpha} \) la Rosenfeld. Indeed,

\[
-\frac{\kappa}{2} \int d^4 x \ h_{\mu\nu} \hat{T}^{\mu\nu} = -\frac{\kappa}{2} \int d^4 x \ \hat{h}_{\mu\nu} T^{\mu\nu},
\]

(40)

where

\[
\hat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} T \eta^{\mu\nu} \quad \text{and} \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4} h \eta_{\mu\nu},
\]

(41)
with $T = T^\mu_\mu$ and $h = h^\mu_\mu$.

The Newtonian potential can be obtained [22] from the tree-level one-graviton exchange, with transfer momentum $k_\mu$, between two very massive scalar particles by taking the so-called static limit: $k_\mu = (0, \vec{k})$. Let $A_{12}$ denote the amplitude for the one-graviton exchange between two scalar particles with masses $M_1$ and $M_2$, respectively. In Unimodular Gravity—see Eq. (40)—we have

$$A_{12} = -i \frac{k^2}{4} T^1_{\mu\nu}(p_1, p'_1) (\hat{h}^{\mu\nu}(k) \hat{h}^{\rho\sigma}(-k)) T^2_{\rho\sigma}(p_2, p'_2),$$

where $k = p_1 - p'_1 = p_2 - p'_2$ and $p_i^2 = M_i^2$, $i = 1, 2$. In the previous equation $(\hat{h}^{\mu\nu}(k) \hat{h}^{\rho\sigma}(-k))$ denotes the free two-point function of the traceless graviton field and $T^i_{\mu\nu}(p_i, p'_i)$, $i = 1, 2$, denote the lowest order contribution to the on-shell matrix elements of the energy-momentum tensor between (on-shell) states with momentum $p_i$ and $p'_i$, $i = 1, 2$:

$$T^i_{\mu\nu}(p_i, p'_i) = p_i \mu p'_i \nu + p_i \nu p'_i \mu + \frac{1}{2} k^2 \eta_{\mu\nu}.$$  

Now, for very massive particles and for $k_\mu = (0, \vec{k})$, we have

$$\frac{1}{2M_i} T^i_{\mu\nu}(p_i, p'_i) = M_i \eta_{\mu0} \eta_{\nu0}, \quad i = 1, 2,$$

so that, in the static limit, one gets

$$\frac{1}{2M_1 2M_2} A_{12} = -i \frac{k^2}{4} \frac{M_1 M_2}{\vec{k}} \hat{h}^{\mu\nu}(0) \hat{h}^{\rho\sigma}(0),$$

with $k_\mu = (0, \vec{k})$. It is the right hand side of the previous equation which must be equal to the Newtonian potential in Fourier space $V_{Nw}(\vec{k})$, where

$$V_{Nw}(\vec{k}) = -\frac{k^2}{8} \frac{M_1 M_2}{\vec{k}}.$$  

Let us make the following ansatz for the free graviton two-point function, $(\hat{h}^{\mu\nu}(k) \hat{h}^{\rho\sigma}(-k))$, in Unimodular Gravity:

$$\langle \hat{h}^{\mu\nu}(k) \hat{h}^{\rho\sigma}(-k) \rangle = \frac{i}{2k^2} \left( \eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\nu} \eta_{\rho\sigma} \right)$$

$$-i \frac{a(k^2)}{2k^2} \eta_{\mu\nu} \eta_{\rho\sigma}$$

$$+ i \frac{b(k^2)}{(k^2)^2} (k_\mu k_\sigma \eta_{\nu\rho} + k_\mu k_\rho \eta_{\nu\sigma})$$

$$+ i c(k^2) \frac{(k^2)}{(k^2)^3} k_\mu k_\nu k_\rho k_\sigma,$$

where $a(k^2)$, $b(k^2)$, and $c(k^2)$ are arbitrary real functions. This ansatz is the most general expression consistent with Lorentz covariance, boson symmetry, the fact that $h^{\mu\nu}$ is a symmetric tensor and that when one replaces in the free two-point function the tensor

$$\frac{1}{2} \left( \eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\nu} \eta_{\rho\sigma} \right)$$

with the following sum over polarizations:

$$\sum_{\lambda = \pm 2} c^{(\lambda)} e^{i (\lambda)}$$

only a simple pole factor $1/k^2$ multiplies this sum, as befits the unitarity and the fact that the classical action of the theory is quadratic in the derivatives.

From Eq. (47), one obtains after a little algebra

$$\langle \hat{h}^{\mu\nu}(k) \hat{h}^{\rho\sigma}(-k) \rangle = \frac{i}{2k^2} \left( \eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} + \left( - \frac{1}{2} + \frac{c(k^2)}{8} \right) \eta_{\mu\nu} \eta_{\rho\sigma} \right)$$

$$+ i \frac{c(k^2)}{(k^2)^2} (k_\mu k_\sigma \eta_{\nu\rho} + k_\mu k_\rho \eta_{\nu\sigma}) + \frac{c(k^2)}{(k^2)^3} k_\mu k_\nu k_\rho k_\sigma.$$

Substituting the previous result in Eq. (45)—recall that $k_\mu = (0, \vec{k})$—one gets

$$-i \frac{\kappa^2}{4} M_1 M_2 \langle \hat{h}^{\mu\nu}(0) \hat{h}^{\rho\sigma}(0) \rangle = -\frac{\kappa^2}{8} M_1 M_2 \left( \frac{3}{2} + \frac{c(-k^2)}{8} \right) \frac{1}{\vec{k}^2}.$$  

This expression will match the Newtonian potential in (46) if, and only if, $c(-k^2) = -4$, which, by Lorentz invariance, leads to

$$c(k^2) = -4,$$

whatever the value of $k_\mu$. In summary, we need a triple pole in the $k_\mu k_\nu k_\rho k_\sigma$ contribution to the two-point function in (47) to get the Newtonian potential right. This is what actually happens when one works out the propagator of Unimodular Gravity by using the BRST technique explained in [1]. Notice that the propagator in (5) yields the Newtonian potential, since the coefficient multiplying the contribution

$$\frac{k_\mu k_\nu k_\rho k_\sigma}{k^6}$$

is $-4$, at $n = 4$.  

 Springer
Appendix C: Expansion of the Unimodular Gravity Lagrangian

Starting from the action (31) we perform a background field expansion of the metric around Minkowski, \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \), so it can be written as

\[
S_{UG} = -\frac{2}{\kappa^2} \int d^n x \left( L_0 + \kappa L_1 + \kappa^2 L_2 + \kappa^3 L_3 + \cdots \right). 
\]

(54)

Keeping \( n \) free at this point it is worth to notice that this expansion will reduce to the General Relativity one taking \( n = 2 \). As we are expanding Minkowski the first two terms vanish and the others read\(^2\)

\[
L_2 = \frac{1}{4} h^{\mu\nu} \partial_\mu h_\nu - \frac{n + 2}{2n} \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu + \frac{1}{2} \left( \partial_\mu h^{\mu\nu} \right) \left( \partial_\nu h_\nu \right) - \frac{1}{2n} \left( \partial_\mu h^{\mu\nu} \right) \left( \partial_\nu h_\nu \right), 
\]

\[
L_3 = -\frac{3}{4} h^{\mu\nu\rho\sigma} \partial_\mu h_\nu \partial_\rho h_\sigma + \frac{1}{4n} \left( 3n - 2n^2 \right) h^{\mu\nu} \partial_\mu \partial_\nu h - \frac{1}{n} \left( 3n - 2n^2 \right) h^{\mu\nu} \partial_\mu \partial_\nu h, 
\]

\[
L_3 = \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu - \frac{n + 2}{2n} \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu + \frac{1}{2} \left( \partial_\mu h^{\mu\nu} \right) \left( \partial_\nu h_\nu \right) - \frac{1}{2n} \left( \partial_\mu h^{\mu\nu} \right) \left( \partial_\nu h_\nu \right), 
\]

(55)

Integrating by parts and discarding total derivatives the cubic term can be written as

\[
L_3 = \frac{n + 2}{4n^3} h^{\mu\nu} \partial_\mu h_\nu - \frac{n + 2}{2n^2} h^{\mu\nu} \partial_\mu h_\nu + \frac{3}{2} h^{\mu\nu} \partial_\mu h_\nu h^{\nu\rho} \partial_\rho h - \frac{3}{2} \frac{1}{4n} h^{\mu\nu} \partial_\mu h_\nu h^{\nu\rho} \partial_\rho h. 
\]

\[
-\frac{1}{4n} \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu \partial_\nu h - \frac{n + 2}{2n} \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu + \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu h^{\nu\rho} \partial_\rho h - \frac{1}{2} h^{\mu\nu} \partial_\mu h_\nu h^{\nu\rho} \partial_\rho h. 
\]

(56)

Appendix D: The full set of five-graviton tree diagrams

We have

\[
A(1^{-}, 2^{-}; 4^{+}; 3^{+}, 5^{+}) = \frac{ik^3(e_1, p_2)^2(e_2, e_3)^2(e_4, p_2)^2(e_5, p_3)(e_5, p_4)}{(p_1 + p_2)^2(p_3 + p_4)^2}, 
\]

(56)

For the General Relativity expansion we find a discrepancy with the expansion given in (16) for the third order lagrangian; the term proportional to \( h \nabla^\mu h \nabla_\mu h \) has the opposite sign.
\[
- \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_2 + p_5)^2} \\
- \frac{i k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)^2(e_5p_3)}{(p_1 + p_3)^2(p_2 + p_5)^2} \\
+ \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_4p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
- \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)^2(e_4p_3)(e_5p_2)^2}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
+ \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_4p_2)(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
- \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)^2(e_4p_3)(e_5p_2)^2}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
+ \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
= \mathcal{A}(1^-, 4^+; 2^-; 3^+, 5^+) \\
= \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_5)^2} \\
- \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_2 + p_5)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)(e_5p_3)}{(p_1 + p_4)^2(p_2 + p_5)^2} \\
- \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_3 + p_5)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_4p_3)(e_5p_2)^2}{(p_1 + p_4)^2(p_3 + p_5)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_3 + p_5)^2} \\
- \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_3 + p_5)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_3 + p_5)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_4)^2(p_3 + p_5)^2} \tag{58}
\]

\[
\mathcal{A}(1^-, 3^+; 4^+; 2^-; 3^+, 5^+) \\
= \frac{i k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_5p_2)^2}{(p_1 + p_3)^2(p_2 + p_4)^2} \\
- \frac{i k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_5p_2)}{(p_1 + p_3)^2(p_2 + p_4)^2} \\
- \frac{i k^3(e_1p_3)^2(e_2e_4)^2(e_4p_2)(e_3p_2)(e_5p_2)^2}{(p_1 + p_3)^2(p_3 + p_4)^2} \\
- \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_4p_2)(e_5p_2)^2}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
+ \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_4p_2)(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_4 + p_5)^2} \\
- \frac{2k^3(e_1p_3)^2(e_2e_4)^2(e_3p_2)(e_4p_3)(e_5p_2)(e_5p_3)}{(p_1 + p_3)^2(p_4 + p_5)^2} \tag{59}
\]

\[
\mathcal{A}(1^-, 3^+; 5^+; 2^-; 3^+, 4^+) \\
= \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{q^2(p_2 + p_5)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_3)^2(e_4t_2)^2(e_5p_2)^2}{q^2(p_2 + p_5)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_3)(e_2e_4)(e_3p_4)(e_4t_2)(e_5p_2)^2}{q^2(p_2 + p_5)^2} \tag{60}
\]

\[
\mathcal{A}(1^-, 5^+; 2^-; 3^+, 3^+) \\
= \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_3)(e_4p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_3)^2(e_4p_3)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_3)(e_2e_4)(e_3p_4)(e_4p_2)(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \tag{61}
\]

\[
\mathcal{A}(1^-, 4^+; 5^+; 2^-; 3^+, 3^+) \\
= \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_3)(e_4p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_3)^2(e_4p_3)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
- \frac{i k^3(e_1p_4)^2(e_2e_4)^2(e_3p_2)^2(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \\
+ \frac{2k^3(e_1p_4)^2(e_2e_3)(e_2e_4)(e_3p_4)(e_4p_2)(e_5p_2)^2}{(p_1 + p_4)^2(p_2 + p_3)^2} \tag{62}
\]
5. S. Ananth, S. Theisen, KLT relations from the Einstein-Hilbert Lagrangian. Phys. Lett. B 652, 128 (2007). doi:10.1016/j.physletb.2007.07.003, arXiv:0706.1778
6. F.A. Berends, W.T. Giele, H. Kuijf, On relations between multigluon and multigraviton scattering. Phys. Lett. B 211, 91 (1988). doi:10.1016/0370-2693(88)90813-1
7. F. Cachazo, P. Svrcek, Tree level recursion relations in general relativity. arXiv:hep-th/0502160
8. P. Benincasa, F. Cachazo, Consistency conditions on the S-matrix of massless particles. arXiv:0705.4305 [hep-th]
9. J.M. Henn, J.C. Plefka, Scattering amplitudes in gauge theories. Lect. Notes Phys. 883, 1 (2014). doi:10.1007/978-3-642-54022-6
10. F.A. Berends, W. Giele, Nucl. Phys. B 294, 700 (1987). doi:10.1016/0550-3213(87)90604-3
11. N. Arkani-Hamed, J. Kaplan, On tree amplitudes in gauge theory and gravity. JHEP 0804, 076 (2008). doi:10.1088/1126-6708/2008/04/076, arXiv:0801.2385 [hep-th]
12. N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng, T. Sondergaard, Proof of gravity and Yang-Mills amplitude relations. JHEP 1009, 067 (2010). doi:10.1007/JHEP09(2010)067, arXiv:1007.3111 [hep-th]
13. H. Elvang, D.Z. Freedman, Note on graviton MHV amplitudes. JHEP 0805, 096 (2008). doi:10.1088/1126-6708/2008/05/096, arXiv:0710.1270 [hep-th]
14. N.E.J. Bjerrum-Bohr, D.C. Dunbar, H. Ita, W.B. Perkins, K. Risager, MHV-vertices for gravity amplitudes. JHEP 0601, 009 (2006). doi:10.1088/1126-6708/2006/01/009, arXiv:hep-th/0509016
15. J. Bedford, A. Brandhuber, B.J. Spence, G. Travaglini, A Recursion relation for gravity amplitudes. Nucl. Phys. B 721, 98 (2005). doi:10.1016/j.nuclphysb.2005.016, arXiv:hep-th/0502146
16. M.H. Goroff, A. Sagnotti, The ultraviolet behavior of Einstein gravity. Nucl. Phys. B 266, 709 (1986). doi:10.1016/0550-3213(86)90193-8
17. S. Sannan, Gravity as the limit of the type II superstring theory. Phys. Rev. D 34, 1749 (1986). doi:10.1103/PhysRevD.34.1749
18. B.S. DeWitt, Quantum theory of gravity. 3. Applications of the covariant theory. Phys. Rev. 162, 1239 (1967). doi:10.1103/PhysRev.162.1239
19. R. Britto, F. Cachazo, B. Feng, E. Witten, Direct proof of tree-level recursion relation in Yang-Mills theory. Phys. Rev. Lett. 94, 181602 (2005). doi:10.1103/PhysRevLett.94.181602, arXiv:hep-th/0501052
20. J. Kuipers, T. Ueda, J.A.M. Vermaseren, J. Vollinga, FORM version 4.0. Comput. Phys. Commun. 184, 1453 (2013). doi:10.1016/j.cpc.2012.12.028, arXiv:1203.6543 [cs.SC]
21. J.M. Martín-García et al., xAct: efficient tensor computer algebra for Mathematica (2002–2013). http://xact.es/
22. J.F. Donoghue, Phys. Rev. D 50, 3874 (1994). doi:10.1103/PhysRevD.50.3874, arXiv:gr-qc/9405057